Stacks Project

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Part 1

Preliminaries

CHAPTER 1

Introduction

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1.1. Overview

0001 Besides the book by Laumon and Moret-Bailly, see [**LMB00**], and the work (in progress) by Fulton et al, we think there is a place for an open source textbook on algebraic stacks and the algebraic geometry that is needed to define them. The Stacks Project attempts to do this by building the foundations starting with commutative algebra and proceeding via the theory of schemes and algebraic spaces to a comprehensive foundation for the theory of algebraic stacks.

We expect this material to be read online as a key feature are the hyperlinks giving quick access to internal references spread over many different pages. If you use an embedded pdf or dvi viewer in your browser, the cross file links should work.

This project is a collaborative effort and we encourage you to help out. Please email any typos or errors you find while reading or any suggestions, additional material, or examples you have to stacks.project@gmail.com. You can download a tarball containing all source files, extract, run make, and use a dvi or pdf viewer locally. Please feel free to edit the LaTeX files and email your improvements.

1.2. Attribution

06LB The scope of this work is such that it is a daunting task to attribute correctly and succinctly all of those mathematicians whose work has led to the development of the theory we try to explain here. We hope eventually to generate enough community interest to find contributors willing to write sections with historical remarks for each and every chapter.

Those who contributed to this work are listed on the title page of the book version of this work and online. Here we would like to name a selection of major contributions:

(1) Jarod Alper contributed a chapter discussing the literature on algebraic stacks, see Guide to Literature, Section 112.1.
(2) Bhargav Bhatt wrote the initial version of a chapter on étale morphisms of schemes, see Étale Morphisms, Section 41.1.
(3) Bhargav Bhatt wrote the initial version of More on Algebra, Section 15.90.
(4) Kiran Kedlaya contributed the initial writeup of Descent, Section 35.4.
(5) The initial versions of
    (a) Algebra, Section 10.28,
    (b) Injectives, Section 19.2, and
    (c) the chapter on fields, see Fields, Section 9.1.

    are from The CRing Project, courtesy of Akhil Mathew et al.
(6) Alex Perry wrote the material on projective modules, Mittag-Leffler modules, including the proof of Algebra, Theorem 10.95.6.

(7) Alex Perry wrote the chapter on deformation theory a la Schlessinger and Rim, see Formal Deformation Theory, Section 90.1.
(8) Thibaut Pugin, Zachary Maddock and Min Lee took notes for a course which formed the basis for a chapter on étale cohomology and a chapter on the trace formula. See Étale Cohomology, Section 59.1 and The Trace Formula, Section 64.1.
(9) David Rydh has contributed many helpful comments, pointed out several mistakes, helped out in an essential way with the material on residual gerbes, and was the originator for the material in More on Groupoids in Spaces, Sections 79.12 and 79.15.
(10) Burt Totaro contributed Examples, Sections 110.65, 110.66, and Properties of Stacks, Section 100.12.
(11) The chapter on pro-étale cohomology, see Pro-étale Cohomology, Section 61.1, is taken from a paper by Bhargav Bhatt and Peter Scholze.
(12) Bhargav Bhatt contributed Examples, Sections 110.72 and 110.76.
(13) Ofer Gabber found mistakes, contributed corrections and he contributed Varieties, Lemma 33.7.17, Formal Spaces, Lemma 87.14.7, the material in More on Groupoids, Section 40.15, the main result of Properties of Spaces, Section 66.17, and the proof of More on Flatness, Proposition 38.25.13.
(14) János Kollár contributed Algebra, Lemma 10.119.2 and Local Cohomology, Proposition 51.8.7.
(15) Kiran Kedlaya wrote the initial version of More on Algebra, Section 15.91.
(16) Matthew Emerton, Toby Gee, and Brandon Levin contributed some results on thickenings, in particular More on Morphisms of Stacks, Lemmas 106.3.7, 106.3.8, and 106.3.9.
(17) Lena Min Ji wrote the initial version of More on Algebra, Section 15.126.
(18) Matthew Emerton and Toby Gee wrote the initial versions of Geometry of Stacks, Sections 107.3 and 107.5.

1.3. Other chapters

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(3) Set Theory
(4) Categories
(5) Topology
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(7) Sites and Sheaves
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(40) More on Groupoid Schemes
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CHAPTER 2

Conventions

0002 **2.1. Comments**

0003 The philosophy behind the conventions used in writing these documents is to choose those conventions that work.

2.2. Set theory

0004 We use Zermelo-Fraenkel set theory with the axiom of choice. See [**Kun83**]. We do not use universes (different from SGA4). We do not stress set-theoretic issues, but we make sure everything is correct (of course) and so we do not ignore them either.

2.3. Categories

0005 A category $\mathcal{C}$ consists of a set of objects and, for each pair of objects, a set of morphisms between them. In other words, it is what is called a "small" category in other texts. We will use "big" categories (categories whose objects form a proper class) as well, but only those that are listed in Categories, Remark 4.2.2.

2.4. Algebra

0006 In these notes a ring is a commutative ring with a 1. Hence the category of rings has an initial object $\mathbf{Z}$ and a final object $\{0\}$ (this is the unique ring where $1 = 0$). Modules are assumed unitary. See [**Eis95**].

2.5. Notation

055X The natural integers are elements of $\mathbf{N} = \{1, 2, 3, \ldots\}$. The integers are elements of $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The field of rational numbers is denoted $\mathbf{Q}$. The field of real numbers is denoted $\mathbf{R}$. The field of complex numbers is denoted $\mathbf{C}$.

2.6. Other chapters

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CHAPTER 3

Set Theory

0007

3.1. Introduction

0008 We need some set theory every now and then. We use Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as described in [**Kun83**] and [**Jec02**].

3.2. Everything is a set

0009 Most mathematicians think of set theory as providing the basic foundations for mathematics. So how does this really work? For example, how do we translate the sentence "X is a scheme" into set theory? Well, we just unravel the definitions: A scheme is a locally ringed space such that every point has an open neighbourhood which is an affine scheme. A locally ringed space is a ringed space such that every stalk of the structure sheaf is a local ring. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A topological space is a pair (X, τ) consisting of a set X and a set of subsets $\tau \subset \mathcal{P}(X)$ satisfying the axioms of a topology. And so on and so forth.

So how, given a set S would we recognize whether it is a scheme? The first thing we look for is whether the set S is an ordered pair. This is defined (see [**Jec02**], page 7) as saying that S has the form $(a, b) := \{\{a\}, \{a, b\}\}$ for some sets a, b. If this is the case, then we would take a look to see whether a is an ordered pair (c, d). If so we would check whether $d \subset \mathcal{P}(c)$, and if so whether d forms the collection of sets for a topology on the set c. And so on and so forth.

So even though it would take a considerable amount of work to write a complete formula $\phi_{scheme}(x)$ with one free variable x in set theory that expresses the notion "x is a scheme", it is possible to do so. The same thing should be true for any mathematical object.

3.3. Classes

000A Informally we use the notion of a *class*. Given a formula $\phi(x, p_1, \ldots, p_n)$, we call

$$C = \{x : \phi(x, p_1, \ldots, p_n)\}$$

a *class*. A class is easier to manipulate than the formula that defines it, but it is not strictly speaking a mathematical object. For example, if R is a ring, then we may consider the class of all R-modules (since after all we may translate the sentence "M is an R-module" into a formula in set theory, which then defines a class). A *proper class* is a class which is not a set.

In this way we may consider the category of R-modules, which is a "big" category—in other words, it has a proper class of objects. Similarly, we may consider the "big" category of schemes, the "big" category of rings, etc.

3.4. Ordinals

05N1 A set T is *transitive* if $x \in T$ implies $x \subset T$. A set α is an *ordinal* if it is transitive and well-ordered by $\in$. In this case, we define $\alpha + 1 = \alpha \cup \{\alpha\}$, which is another ordinal called the *successor* of α. An ordinal α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$. The smallest ordinal is $\emptyset$ which is also denoted 0. If α is not 0, and not a successor ordinal, then α is called a *limit ordinal* and we have

$$\alpha = \bigcup\nolimits_{\gamma \in \alpha} \gamma.$$

The first limit ordinal is ω and it is also the first infinite ordinal. The first uncountable ordinal ω_1 is the set of all countable ordinals. The collection of all ordinals is a proper class. It is well-ordered by $\in$ in the following sense: any nonempty set (or even class) of ordinals has a least element. Given a set A of ordinals, we define the *supremum* of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well-ordered set $(S, <)$, there is a unique ordinal α such that $(S, <) \cong (\alpha, \in)$; this is called the *order type* of the well-ordered set.

3.5. The hierarchy of sets

000B We define by transfinite recursion $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set), and for a limit ordinal α,

$$V_\alpha = \bigcup\nolimits_{\beta < \alpha} V_\beta.$$

Note that each V_α is a transitive set.

000C **Lemma 3.5.1.** *Every set is an element of V_α for some ordinal α.*

Proof. See [**Jec02**, Lemma 6.3]. □

In [**Kun83**, Chapter III] it is explained that this lemma is equivalent to the axiom of foundation. The *rank* of a set S is the least ordinal α such that $S \in V_{\alpha+1}$. By a *partial universe* we shall mean a suitably large set of the form V_α which will be clear from the context.

3.6. Cardinality

000D The *cardinality* of a set A is the least ordinal α such that there exists a bijection between A and α. We sometimes use the notation $\alpha = |A|$ to indicate this. We say an ordinal α is a *cardinal* if and only if it occurs as the cardinality of some set A—in other words, if $\alpha = |A|$. We use the greek letters κ, λ for cardinals. The first infinite cardinal is ω, and in this context it is denoted $\aleph_0$. A set is *countable* if its cardinality is $\leq \aleph_0$. If α is an ordinal, then we denote α^+ the least cardinal $> \alpha$. You can use this to define $\aleph_1 = \aleph_0^+$, $\aleph_2 = \aleph_1^+$, etc, and in fact you can define $\aleph_\alpha$ for any ordinal α by transfinite recursion. We note the equality $\aleph_1 = \omega_1$.

The *addition* of cardinals κ, λ is denoted $\kappa \oplus \lambda$; it is the cardinality of $\kappa \amalg \lambda$. The *multiplication* of cardinals κ, λ is denoted $\kappa \otimes \lambda$; it is the cardinality of $\kappa \times \lambda$. If κ and λ are infinite cardinals, then $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$. The *exponentiation* of cardinals κ, λ is denoted κ^λ; it is the cardinality of the set of (set) maps from λ to κ. Given any set K of cardinals, the *supremum* of K is $\sup_{\kappa \in K} \kappa = \bigcup_{\kappa \in K} \kappa$, which is also a cardinal.

3.7. Cofinality

000E A *cofinal subset* S of a well-ordered set T is a subset $S \subset T$ such that $\forall t \in T \exists s \in S(t \leq s)$. Note that a subset of a well-ordered set is a well-ordered set (with induced ordering). Given an ordinal α, the *cofinality* $\text{cf}(\alpha)$ of α is the least ordinal β which occurs as the order type of some cofinal subset of α. The cofinality of an ordinal is always a cardinal. Hence alternatively we can define the cofinality of α as the least cardinality of a cofinal subset of α.

05N2 **Lemma 3.7.1.** *Suppose that $T = \text{colim}_{\alpha<\beta} T_\alpha$ is a colimit of sets indexed by ordinals less than a given ordinal β. Suppose that $\varphi : S \to T$ is a map of sets. Then φ lifts to a map into T_α for some $\alpha < \beta$ provided that β is not a limit of ordinals indexed by S, in other words, if β is an ordinal with $cf(\beta) > |S|$.*

Proof. For each element $s \in S$ pick a $\alpha_s < \beta$ and an element $t_s \in T_{\alpha_s}$ which maps to $\varphi(s)$ in T. By assumption $\alpha = \sup_{s \in S} \alpha_s$ is strictly smaller than β. Hence the map $\varphi_\alpha : S \to T_\alpha$ which assigns to s the image of t_s in T_α is a solution. □

The following is essentially Grothendieck's argument for the existence of ordinals with arbitrarily large cofinality which he used to prove the existence of enough injectives in certain abelian categories, see [**Gro57**].

05N3 **Proposition 3.7.2.** *Let κ be a cardinal. Then there exists an ordinal whose cofinality is bigger than κ.*

Proof. If κ is finite, then $\omega = \text{cf}(\omega)$ works. Let us thus assume that κ is infinite. Consider the smallest ordinal α whose cardinality is strictly greater than κ. We claim that $\text{cf}(\alpha) > \kappa$. Note that α is a limit ordinal, since if $\alpha = \beta + 1$, then $|\alpha| = |\beta|$ (because α and β are infinite) and this contradicts the minimality of α. (Of course α is also a cardinal, but we do not need this.) To get a contradiction suppose $S \subset \alpha$ is a cofinal subset with $|S| \leq \kappa$. For $\beta \in S$, i.e., $\beta < \alpha$, we have $|\beta| \leq \kappa$ by minimality of α. As α is a limit ordinal and S cofinal in α we obtain $\alpha = \bigcup_{\beta \in S} \beta$. Hence $|\alpha| \leq |S| \otimes \kappa \leq \kappa \otimes \kappa \leq \kappa$ which is a contradiction with our choice of α. □

3.8. Reflection principle

000F Some of this material is in the chapter of [**Kun83**] called "Easy consistency proofs".

Let $\phi(x_1, \ldots, x_n)$ be a formula of set theory. Let us use the convention that this notation implies that all the free variables in ϕ occur among $x_1, \ldots, x_n$. Let M be a set. The formula $\phi^M(x_1, \ldots, x_n)$ is the formula obtained from $\phi(x_1, \ldots, x_n)$ by replacing all the $\forall x$ and $\exists x$ by $\forall x \in M$ and $\exists x \in M$, respectively. So the formula $\phi(x_1, x_2) = \exists x(x \in x_1 \wedge x \in x_2)$ is turned into $\phi^M(x_1, x_2) = \exists x \in M(x \in x_1 \wedge x \in x_2)$. The formula ϕ^M is called the *relativization of ϕ to M*.

000G **Theorem 3.8.1.** *Suppose given $\phi_1(x_1, \ldots, x_n), \ldots, \phi_m(x_1, \ldots, x_n)$ a* **finite** *collection of formulas of set theory. Let M_0 be a set. There exists a set M such that $M_0 \subset M$ and $\forall x_1, \ldots, x_n \in M$, we have*

$$\forall i = 1, \ldots, m,\ \phi_i^M(x_1, \ldots, x_n) \Leftrightarrow \forall i = 1, \ldots, m,\ \phi_i(x_1, \ldots, x_n).$$

In fact we may take $M = V_\alpha$ for some limit ordinal α.

Proof. See [**Jec02**, Theorem 12.14] or [**Kun83**, Theorem 7.4]. □

We view this theorem as saying the following: Given any $x_1, \ldots, x_n \in M$ the formulas hold with the bound variables ranging through all sets if and only if they hold for the bound variables ranging through elements of V_α. This theorem is a meta-theorem because it deals with the formulas of set theory directly. It actually says that given the finite list of formulas $\phi_1, \ldots, \phi_m$ with at most free variables $x_1, \ldots, x_n$ the sentence

$$\forall M_0\ \exists M,\ M_0 \subset M\ \forall x_1, \ldots, x_n \in M$$
$$\phi_1(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m(x_1, \ldots, x_n) \leftrightarrow \phi_1^M(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m^M(x_1, \ldots, x_n)$$

is provable in ZFC. In other words, whenever we actually write down a finite list of formulas ϕ_i, we get a theorem.

It is somewhat hard to use this theorem in "ordinary mathematics" since the meaning of the formulas $\phi_i^M(x_1, \ldots, x_n)$ is not so clear! Instead, we will use the idea of the proof of the reflection principle to prove the existence results we need directly.

3.9. Constructing categories of schemes

000H We will discuss how to apply this to produce, given an initial set of schemes, a "small" category of schemes closed under a list of natural operations. Before we do so, we introduce the size of a scheme. Given a scheme S we define

$$\text{size}(S) = \max(\aleph_0, \kappa_1, \kappa_2),$$

where we define the cardinal numbers κ_1 and κ_2 as follows:

(1) We let κ_1 be the cardinality of the set of affine opens of S.
(2) We let κ_2 be the supremum of all the cardinalities of all $\Gamma(U, \mathcal{O}_S)$ for all $U \subset S$ affine open.

000I **Lemma 3.9.1.** *For every cardinal κ, there exists a set A such that every element of A is a scheme and such that for every scheme S with size$(S) \leq \kappa$, there is an element $X \in A$ such that $X \cong S$ (isomorphism of schemes).*

Proof. Omitted. Hint: think about how any scheme is isomorphic to a scheme obtained by glueing affines. □

We denote *Bound* the function which to each cardinal κ associates

046U (3.9.1.1) $$Bound(\kappa) = \max\{\kappa^{\aleph_0}, \kappa^+\}.$$

We could make this function grow much more rapidly, e.g., we could set $Bound(\kappa) = \kappa^\kappa$, and the result below would still hold. For any ordinal α, we denote Sch_α the full subcategory of category of schemes whose objects are elements of V_α. Here is the result we are going to prove.

000J **Lemma 3.9.2.** *With notations size, Bound and Sch_α as above. Let S_0 be a set of schemes. There exists a limit ordinal α with the following properties:*

000K (1) *We have $S_0 \subset V_\alpha$; in other words, $S_0 \subset \text{Ob}(Sch_\alpha)$.*
000L (2) *For any $S \in \text{Ob}(Sch_\alpha)$ and any scheme T with size$(T) \leq Bound(\text{size}(S))$, there exists a scheme $S' \in \text{Ob}(Sch_\alpha)$ such that $T \cong S'$.*
000M (3) *For any countable[1] diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_\alpha$, the limit $\lim_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*

[1]Both the set of objects and the morphism sets are countable. In fact you can prove the lemma with $\aleph_0$ replaced by any cardinal whatsoever in (3) and (4).

000N (4) *For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_\alpha$, the colimit $\operatorname{colim}_\mathcal{I} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*

Proof. We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) For any $S \in \mathrm{Ob}(Sch_{f(\alpha)})$ and any scheme T with $\mathrm{size}(T) \leq Bound(\mathrm{size}(S))$, there exists a scheme $S' \in \mathrm{Ob}(Sch_\beta)$ such that $T \cong S'$.
(3) For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_{f(\alpha)}$, if the limit $\lim_\mathcal{I} F$ or the colimit $\operatorname{colim}_\mathcal{I} F$ exists in Sch, then it is isomorphic to a scheme in Sch_β.

To see β exists, we argue as follows. Since $\mathrm{Ob}(Sch_{f(\alpha)})$ is a set, we see that $\kappa = \sup_{S \in \mathrm{Ob}(Sch_{f(\alpha)})} Bound(\mathrm{size}(S))$ exists and is a cardinal. Let A be a set of schemes obtained starting with κ as in Lemma 3.9.1. There is a set $CountCat$ of countable categories such that any countable category is isomorphic to an element of $CountCat$. Hence in (3) above we may assume that $\mathcal{I}$ is an element in $CountCat$. This means that the pairs $(\mathcal{I}, F)$ in (3) range over a set. Thus, there exists a set B whose elements are schemes such that for every $(\mathcal{I}, F)$ as in (3), if the limit or colimit exists, then it is isomorphic to an element in B. Hence, if we pick any β such that $A \cup B \subset V_\beta$ and $\beta > \max\{\alpha + 1, f(\alpha)\}$, then (1)–(3) hold. Since every nonempty collection of ordinals has a least element, we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick β_0 such that $S_0 \subset V_{\beta_0}$. By construction $f(\beta) \geq \beta$ and we see that also $S_0 \subset V_{f(\beta_0)}$. Moreover, as f is nondecreasing, we see $S_0 \subset V_{f(\beta)}$ is true for any $\beta \geq \beta_0$. Next, choose any ordinal $\beta_1 > \beta_0$ with cofinality $\mathrm{cf}(\beta_1) > \omega = \aleph_0$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that $\alpha = f(\beta_1)$ is a solution to the problem posed in the lemma.

The first property of the lemma holds by our choice of $\beta_1 > \beta_0$ above.

Since β_1 is a limit ordinal (as its cofinality is infinite), we get $f(\beta_1) = \sup_{\beta < \beta_1} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_1\} \subset f(\beta_1)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_1)} = \bigcup\nolimits_{\beta < \beta_1} V_{f(\beta)}.$$

Now, let $S \in \mathrm{Ob}(Sch_\alpha)$. We define $\beta(S)$ to be the least ordinal β such that $S \in \mathrm{Ob}(Sch_{f(\beta)})$. By the above we see that always $\beta(S) < \beta_1$. Since $\mathrm{Ob}(Sch_{f(\beta+1)}) \subset \mathrm{Ob}(Sch_\alpha)$, we see by construction of f above that the second property of the lemma is satisfied.

Suppose that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(Sch_\alpha)$ is a countable collection. Consider the function $\omega \to \beta_1$, $n \mapsto \beta(S_n)$. Since the cofinality of β_1 is $> \omega$, the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(Sch_{f(\beta)})$. It follows that any functor $F : \mathcal{I} \to Sch_\alpha$ factors through one of the subcategories $Sch_{f(\beta)}$. Thus, if there exists a scheme X that is the colimit or limit of the diagram F, then, by construction of f, we see X is isomorphic to an object of $Sch_{f(\beta+1)}$ which is a subcategory of Sch_α. This proves the last two assertions of the lemma. □

000O **Remark 3.9.3.** The lemma above can also be proved using the reflection principle. However, one has to be careful. Namely, suppose the sentence $\phi_{scheme}(X)$ expresses the property "X is a scheme", then what does the formula $\phi_{scheme}^{V_\alpha}(X)$ mean? It is true that the reflection principle says we can find α such that for all $X \in V_\alpha$ we have $\phi_{scheme}(X) \leftrightarrow \phi_{scheme}^{V_\alpha}(X)$ but this is entirely useless. It is only by combining two such statements that something interesting happens. For example suppose $\phi_{red}(X, Y)$ expresses the property "X, Y are schemes, and Y is the reduction of X" (see Schemes, Definition 26.12.5). Suppose we apply the reflection principle to the pair of formulas $\phi_1(X, Y) = \phi_{red}(X, Y)$, $\phi_2(X) = \exists Y, \phi_1(X, Y)$. Then it is easy to see that any α produced by the reflection principle has the property that given $X \in \mathrm{Ob}(Sch_\alpha)$ the reduction of X is also an object of Sch_α (left as an exercise).

000P **Lemma 3.9.4.** *Let S be an affine scheme. Let $R = \Gamma(S, \mathcal{O}_S)$. Then the size of S is equal to* $\max\{\aleph_0, |R|\}$.

Proof. There are at most $\max\{|R|, \aleph_0\}$ affine opens of $\mathrm{Spec}(R)$. This is clear since any affine open $U \subset \mathrm{Spec}(R)$ is a finite union of principal opens $D(f_1) \cup \ldots \cup D(f_n)$ and hence the number of affine opens is at most $\sup_n |R|^n = \max\{|R|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. On the other hand, we see that $\Gamma(U, \mathcal{O}) \subset R_{f_1} \times \ldots \times R_{f_n}$ and hence $|\Gamma(U, \mathcal{O})| \leq \max\{\aleph_0, |R_{f_1}|, \ldots, |R_{f_n}|\}$. Thus it suffices to prove that $|R_f| \leq \max\{\aleph_0, |R|\}$ which is omitted. □

000Q **Lemma 3.9.5.** *Let S be a scheme. Let $S = \bigcup_{i \in I} S_i$ be an open covering. Then* $size(S) \leq \max\{|I|, \sup_i\{size(S_i)\}\}$.

Proof. Let $U \subset S$ be any affine open. Since U is quasi-compact there exist finitely many elements $i_1, \ldots, i_n \in I$ and affine opens $U_i \subset U \cap S_i$ such that $U = U_1 \cup U_2 \cup \ldots \cup U_n$. Thus

$$|\Gamma(U, \mathcal{O}_U)| \leq |\Gamma(U_1, \mathcal{O})| \otimes \ldots \otimes |\Gamma(U_n, \mathcal{O})| \leq \sup_i\{\mathrm{size}(S_i)\}$$

Moreover, it shows that the set of affine opens of S has cardinality less than or equal to the cardinality of the set

$$\coprod_{n \in \omega} \coprod_{i_1, \ldots, i_n \in I} \{\text{affine opens of } S_{i_1}\} \times \ldots \times \{\text{affine opens of } S_{i_n}\}.$$

Each of the sets inside the disjoint union has cardinality at most $\sup_i\{\mathrm{size}(S_i)\}$. The index set has cardinality at most $\max\{|I|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. Hence by [**Jec02**, Lemma 5.8] the cardinality of the coproduct is at most $\max\{\aleph_0, |I|\} \otimes \sup_i\{\mathrm{size}(S_i)\}$. The lemma follows. □

04T6 **Lemma 3.9.6.** *Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Then we have* $size(X \times_S Y) \leq \max\{size(X), size(Y)\}$.

Proof. Let $S = \bigcup_{k \in K} S_k$ be an affine open covering. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \mathrm{size}(X), \mathrm{size}(Y)$. For each $i \in I$ there exists a finite set K_i of $k \in K$ such that $f(U_i) \subset \bigcup_{k \in K_i} S_k$. For each $j \in J$ there exists a finite set K_j of $k \in K$ such that $g(V_j) \subset \bigcup_{k \in K_j} S_k$. Hence $f(X), g(Y)$ are contained in $S' = \bigcup_{k \in K'} S_k$ with $K' = \bigcup_{i \in I} K_i \cup \bigcup_{j \in J} K_j$. Note that the cardinality of K' is at most $\max\{\aleph_0, |I|, |J|\}$. Applying Lemma 3.9.5 we see that it suffices to prove that $\mathrm{size}(f^{-1}(S_k) \times_{S_k} g^{-1}(S_k)) \leq \max\{\mathrm{size}(X), \mathrm{size}(Y))\}$ for $k \in K'$. In other words, we may assume that S is affine.

Assume S affine. Let $X = \bigcup_{i\in I} U_i$, $Y = \bigcup_{j\in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. Again by Lemma 3.9.5 it suffices to prove the lemma for the products $U_i \times_S V_j$. By Lemma 3.9.4 we see that it suffices to show that

$$|A \otimes_C B| \leq \max\{\aleph_0, |A|, |B|\}.$$

We omit the proof of this inequality. □

04T7 **Lemma 3.9.7.** *Let S be a scheme. Let $f : X \to S$ be locally of finite type with X quasi-compact. Then size$(X) \leq$ size(S).*

Proof. We can find a finite affine open covering $X = \bigcup_{i=1,\ldots n} U_i$ such that each U_i maps into an affine open S_i of S. Thus by Lemma 3.9.5 we reduce to the case where both S and X are affine. In this case by Lemma 3.9.4 we see that it suffices to show

$$|A[x_1, \ldots, x_n]| \leq \max\{\aleph_0, |A|\}.$$

We omit the proof of this inequality. □

In Algebra, Lemma 10.107.13 we will show that if $A \to B$ is an epimorphism of rings, then $|B| \leq \max(|A|, \aleph_0)$. The analogue for schemes is the following lemma.

04VA **Lemma 3.9.8.** *Let $f : X \to Y$ be a monomorphism of schemes. If at least one of the following properties holds, then size$(X) \leq$ size(Y):*

(1) *f is quasi-compact,*
(2) *f is locally of finite presentation,*
(3) *add more here as needed.*

But the bound does not hold for monomorphisms which are locally of finite type.

Proof. Let $Y = \bigcup_{j\in J} V_j$ be an affine open covering of Y with $|J| \leq \text{size}(Y)$. By Lemma 3.9.5 it suffices to bound the size of the inverse image of V_j in X. Hence we reduce to the case that Y is affine, say $Y = \text{Spec}(B)$. For any affine open $\text{Spec}(A) \subset X$ we have $|A| \leq \max(|B|, \aleph_0) = \text{size}(Y)$, see remark above and Lemma 3.9.4. Thus it suffices to show that X has at most size(Y) affine opens. This is clear if X is quasi-compact, whence case (1) holds. In case (2) the number of isomorphism classes of B-algebras A that can occur is bounded by size(B), because each A is of finite type over B, hence isomorphic to an algebra $B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in B[x_1, \ldots, x_n]$. However, as $X \to Y$ is a monomorphism, there is a unique morphism $\text{Spec}(A) \to X$ over $Y = \text{Spec}(B)$ if there is one, hence the number of affine opens of X is bounded by the number of these isomorphism classes.

To prove the final statement of the lemma consider the ring $B = \prod_{n\in\mathbf{N}} \mathbf{F}_2$ and set $Y = \text{Spec}(B)$. For every ultrafilter $\mathcal{U}$ on $\mathbf{N}$ we obtain a maximal ideal $\mathfrak{m}_\mathcal{U}$ with residue field $\mathbf{F}_2$; the map $B \to \mathbf{F}_2$ sends the element (x_n) to $\lim_\mathcal{U} x_n$. Details omitted. The morphism of schemes $X = \coprod_\mathcal{U} \text{Spec}(\mathbf{F}_2) \to Y$ is a monomorphism as all the points are distinct. However the cardinality of the set of affine open subschemes of X is equal to the cardinality of the set of ultrafilters on $\mathbf{N}$ which is $2^{2^{\aleph_0}}$. We conclude as $|B| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$. □

000R **Lemma 3.9.9.** *Let α be an ordinal as in Lemma 3.9.2 above. The category Sch_α satisfies the following properties:*

(1) *If $X, Y, S \in \mathrm{Ob}(Sch_\alpha)$, then for any morphisms $f : X \to S$, $g : Y \to S$ the fibre product $X \times_S Y$ in Sch_α exists and is a fibre product in the category of schemes.*
(2) *Given any at most countable collection $S_1, S_2, \ldots$ of elements of $\mathrm{Ob}(Sch_\alpha)$, the coproduct $\coprod_i S_i$ exists in $\mathrm{Ob}(Sch_\alpha)$ and is a coproduct in the category of schemes.*
(3) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any open immersion $U \to S$, there exists a $V \in \mathrm{Ob}(Sch_\alpha)$ with $V \cong U$.*
(4) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any closed immersion $T \to S$, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(5) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any finite type morphism $T \to S$, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(6) *Suppose S is a scheme which has an open covering $S = \bigcup_{i \in I} S_i$ such that there exists a $T \in \mathrm{Ob}(Sch_\alpha)$ with (a) $\mathrm{size}(S_i) \leq \mathrm{size}(T)^{\aleph_0}$ for all $i \in I$, and (b) $|I| \leq \mathrm{size}(T)^{\aleph_0}$. Then S is isomorphic to an object of Sch_α.*
(7) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any morphism $f : T \to S$ locally of finite type such that T can be covered by at most $\mathrm{size}(S)^{\aleph_0}$ open affines, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$. For example this holds if T can be covered by at most $|\mathbf{R}| = 2^{\aleph_0} = \aleph_0^{\aleph_0}$ open affines.*
(8) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any monomorphism $T \to S$ which is either locally of finite presentation or quasi-compact, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(9) *Suppose that $T \in \mathrm{Ob}(Sch_\alpha)$ is affine. Write $R = \Gamma(T, \mathcal{O}_T)$. Then any of the following schemes is isomorphic to a scheme in Sch_α:*
   (a) *For any ideal $I \subset R$ with completion $R^* = \lim_n R/I^n$, the scheme $\mathrm{Spec}(R^*)$.*
   (b) *For any finite type R-algebra R', the scheme $\mathrm{Spec}(R')$.*
   (c) *For any localization $S^{-1}R$, the scheme $\mathrm{Spec}(S^{-1}R)$.*
   (d) *For any prime $\mathfrak{p} \subset R$, the scheme $\mathrm{Spec}(\overline{\kappa(\mathfrak{p})})$.*
   (e) *For any subring $R' \subset R$, the scheme $\mathrm{Spec}(R')$.*
   (f) *Any scheme of finite type over a ring of cardinality at most $|R|^{\aleph_0}$.*
   (g) *And so on.*

Proof. Statements (1) and (2) follow directly from the definitions. Statement (3) follows as the size of an open subscheme U of S is clearly smaller than or equal to the size of S. Statement (4) follows from (5). Statement (5) follows from (7). Statement (6) follows as the size of S is $\leq \max\{|I|, \sup_i \mathrm{size}(S_i)\} \leq \mathrm{size}(T)^{\aleph_0}$ by Lemma 3.9.5. Statement (7) follows from (6). Namely, for any affine open $V \subset T$ we have $\mathrm{size}(V) \leq \mathrm{size}(S)$ by Lemma 3.9.7. Thus, we see that (6) applies in the situation of (7). Part (8) follows from Lemma 3.9.8.

Statement (9) is translated, via Lemma 3.9.4, into an upper bound on the cardinality of the rings R^*, $S^{-1}R$, $\overline{\kappa(\mathfrak{p})}$, R', etc. Perhaps the most interesting one is the ring R^*. As a set, it is the image of a surjective map $R^{\mathbf{N}} \to R^*$. Since $|R^{\mathbf{N}}| = |R|^{\aleph_0}$, we see that it works by our choice of $Bound(\kappa)$ being at least $\kappa^{\aleph_0}$. Phew! (The cardinality of the algebraic closure of a field is the same as the cardinality of the field, or it is $\aleph_0$.) □

000S **Remark 3.9.10.** Let R be a ring. Suppose we consider the ring $\prod_{\mathfrak{p} \in \mathrm{Spec}(R)} \kappa(\mathfrak{p})$. The cardinality of this ring is bounded by $|R|^{2^{|R|}}$, but is not bounded by $|R|^{\aleph_0}$ in

general. For example if $R = \mathbf{C}[x]$ it is not bounded by $|R|^{\aleph_0}$ and if $R = \prod_{n\in\mathbf{N}} \mathbf{F}_2$ it is not bounded by $|R|^{|R|}$. Thus the "And so on" of Lemma 3.9.9 above should be taken with a grain of salt. Of course, if it ever becomes necessary to consider these rings in arguments pertaining to fppf/étale cohomology, then we can change the function *Bound* above into the function $\kappa \mapsto \kappa^{2^\kappa}$.

In the following lemma we use the notion of an fpqc covering which is introduced in Topologies, Section 34.9.

0AHK **Lemma 3.9.11.** *Let $f : X \to Y$ be a morphism of schemes. Assume there exists an fpqc covering $\{g_j : Y_j \to Y\}_{j\in J}$ such that g_j factors through f. Then size$(Y) \leq$ size(X).*

Proof. Let $V \subset Y$ be an affine open. By definition there exist $n \geq 0$ and $a : \{1, \ldots, n\} \to J$ and affine opens $V_i \subset Y_{a(i)}$ such that $V = g_{a(1)}(V_1) \cup \ldots \cup g_{a(n)}(V_n)$. Denote $h_j : Y_j \to X$ a morphism such that $f \circ h_j = g_j$. Then $h_{a(1)}(V_1) \cup \ldots \cup h_{a(n)}(V_n)$ is a quasi-compact subset of $f^{-1}(V)$. Hence we can find a quasi-compact open $W \subset f^{-1}(V)$ which contains $h_{a(i)}(V_i)$ for $i = 1, \ldots, n$. In particular $V = f(W)$.

On the one hand this shows that the cardinality of the set of affine opens of Y is at most the cardinality of the set S of quasi-compact opens of X. Since every quasi-compact open of X is a finite union of affines, we see that the cardinality of this set is at most $\sup |S|^n = \max(\aleph_0, |S|)$. On the other hand, we have $\mathcal{O}_Y(V) \subset \prod_{i=1,\ldots,n} \mathcal{O}_{Y_{a(i)}}(V_i)$ because $\{V_i \to V\}$ is an fpqc covering. Hence $\mathcal{O}_Y(V) \subset \mathcal{O}_X(W)$ because $V_i \to V$ factors through W. Again since W has a finite covering by affine opens of X we conclude that $|\mathcal{O}_Y(V)|$ is bounded by the size of X. The lemma now follows from the definition of the size of a scheme. □

In the following lemma we use the notion of an fppf covering which is introduced in Topologies, Section 34.7.

0AHL **Lemma 3.9.12.** *Let $\{f_i : X_i \to X\}_{i\in I}$ be an fppf covering of a scheme. There exists an fppf covering $\{W_j \to X\}_{j\in J}$ which is a refinement of $\{X_i \to X\}_{i\in I}$ such that size$(\coprod W_j) \leq$ size(X).*

Proof. Choose an affine open covering $X = \bigcup_{a\in A} U_a$ with $|A| \leq \text{size}(X)$. For each a we can choose a finite subset $I_a \subset I$ and for $i \in I_a$ a quasi-compact open $W_{a,i} \subset X_i$ such that $U_a = \bigcup_{i\in I_a} f_i(W_{a,i})$. Then $\text{size}(W_{a,i}) \leq \text{size}(X)$ by Lemma 3.9.7. We conclude that $\text{size}(\coprod_a \coprod_{i\in I_a} W_{i,a}) \leq \text{size}(X)$ by Lemma 3.9.5. □

3.10. Sets with group action

000T Let G be a group. We denote *G-Sets* the "big" category of G-sets. For any ordinal α, we denote *G-Sets$_\alpha$* the full subcategory of *G-Sets* whose objects are in V_α. As a notion for size of a G-set we take $\text{size}(S) = \max\{\aleph_0, |G|, |S|\}$ (where $|G|$ and $|S|$ are the cardinality of the underlying sets). As above we use the function $Bound(\kappa) = \kappa^{\aleph_0}$.

000U **Lemma 3.10.1.** *With notations G, G-Sets$_\alpha$, size, and Bound as above. Let S_0 be a set of G-sets. There exists a limit ordinal α with the following properties:*

(1) *We have $S_0 \cup \{{}_G G\} \subset \text{Ob}(G\text{-Sets}_\alpha)$.*

(2) *For any $S \in \mathrm{Ob}(G\text{-}Sets_\alpha)$ and any G-set T with $size(T) \leq Bound(size(S))$, there exists an $S' \in \mathrm{Ob}(G\text{-}Sets_\alpha)$ that is isomorphic to T.*
(3) *For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to G\text{-}Sets_\alpha$, the limit $\lim_{\mathcal{I}} F$ and colimit $\mathrm{colim}_{\mathcal{I}} F$ exist in $G\text{-}Sets_\alpha$ and are the same as in G-Sets.*

Proof. Omitted. Similar to but easier than the proof of Lemma 3.9.2 above. □

000V **Lemma 3.10.2.** *Let α be an ordinal as in Lemma 3.10.1 above. The category $G\text{-}Sets_\alpha$ satisfies the following properties:*
(1) *The G-set ${}_GG$ is an object of $G\text{-}Sets_\alpha$.*
(2) *(Co)Products, fibre products, and pushouts exist in $G\text{-}Sets_\alpha$ and are the same as their counterparts in G-Sets.*
(3) *Given an object U of $G\text{-}Sets_\alpha$, any G-stable subset $O \subset U$ is isomorphic to an object of $G\text{-}Sets_\alpha$.*

Proof. Omitted. □

3.11. Coverings of a site

000W Suppose that $\mathcal{C}$ is a category (as in Categories, Definition 4.2.1) and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying properties (1), (2), and (3) of Sites, Definition 7.6.2. We list them here:
(1) If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

For an ordinal α, we set $\mathrm{Cov}(\mathcal{C})_\alpha = \mathrm{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ, we set $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ equal to the set of elements $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_\alpha$ such that $|I| \leq \kappa$.

We recall the following notion, see Sites, Definition 7.8.2. Two families of morphisms, $\{\varphi_i : U_i \to U\}_{i\in I}$ and $\{\psi_j : W_j \to U\}_{j\in J}$, with the same target of $\mathcal{C}$ are called *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$. This defines an equivalence relation on families of morphisms having a fixed target.

000X **Lemma 3.11.1.** *With notations as above. Let $Cov_0 \subset Cov(\mathcal{C})$ be a set contained in $Cov(\mathcal{C})$. There exist a cardinal κ and a limit ordinal α with the following properties:*
(1) *We have $Cov_0 \subset Cov(\mathcal{C})_{\kappa,\alpha}$.*
(2) *The set of coverings $Cov(\mathcal{C})_{\kappa,\alpha}$ satisfies (1), (2), and (3) of Sites, Definition 7.6.2 (see above). In other words $(\mathcal{C}, Cov(\mathcal{C})_{\kappa,\alpha})$ is a site.*
(3) *Every covering in $Cov(\mathcal{C})$ is combinatorially equivalent to a covering in $Cov(\mathcal{C})_{\kappa,\alpha}$.*

Proof. To prove this, we first consider the set $\mathcal{S}$ of all sets of morphisms of $\mathcal{C}$ with fixed target. In other words, an element of $\mathcal{S}$ is a subset T of $\mathrm{Arrows}(\mathcal{C})$ such that all elements of T have the same target. Given a family $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms with fixed target, we define $Supp(\mathcal{U}) = \{\varphi \in \mathrm{Arrows}(\mathcal{C}) \mid \exists i \in I, \varphi = \varphi_i\}$. Note that two families $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are combinatorially equivalent if and only if $Supp(\mathcal{U}) = Supp(\mathcal{V})$. Next, we define

$\mathcal{S}_\tau \subset \mathcal{S}$ to be the subset $\mathcal{S}_\tau = \{T \in \mathcal{S} \mid \exists\ \mathcal{U} \in \mathrm{Cov}(\mathcal{C})\ T = Supp(\mathcal{U})\}$. For every element $T \in \mathcal{S}_\tau$, set $\beta(T)$ to equal the least ordinal β such that there exists a $\mathcal{U} \in \mathrm{Cov}(\mathcal{C})_\beta$ such that $T = \mathrm{Supp}(\mathcal{U})$. Finally, set $\beta_0 = \sup_{T \in S_\tau} \beta(T)$. At this point it follows that every $\mathcal{U} \in \mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to some element of $\mathrm{Cov}(\mathcal{C})_{\beta_0}$.

Let κ be the maximum of $\aleph_0$, the cardinality $|\mathrm{Arrows}(\mathcal{C})|$,

$$\sup_{\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})_{\beta_0}} |I|, \quad \text{and} \quad \sup_{\{U_i \to U\}_{i \in I} \in \mathrm{Cov}_0} |I|.$$

Since κ is an infinite cardinal, we have $\kappa \otimes \kappa = \kappa$. Note that obviously $\mathrm{Cov}(\mathcal{C})_{\beta_0} = \mathrm{Cov}(\mathcal{C})_{\kappa,\beta_0}$.

We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$ and for each i we have $\{W_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$, then $\{W_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})_{\kappa,\beta}$.
(3) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ and $W \to U$ is a morphism of $\mathcal{C}$, then $\{U_i \times_U W \to W\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,\beta}$.

To see β exists we note that clearly the collection of all coverings $\{W_{ij} \to U\}$ and $\{U_i \times_U W \to W\}$ that occur in (2) and (3) form a set. Hence there is some ordinal β such that V_β contains all of these coverings. Moreover, the index set of the covering $\{W_{ij} \to U\}$ has cardinality $\sum_{i \in I} |J_i| \leq \kappa \otimes \kappa = \kappa$, and hence these coverings are contained in $\mathrm{Cov}(\mathcal{C})_{\kappa,\beta}$. Since every nonempty collection of ordinals has a least element we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick an ordinal β_1 such that $\mathrm{Arrows}(\mathcal{C}) \subset V_{\beta_1}$, $\mathrm{Cov}_0 \subset V_{\beta_0}$, and $\beta_1 \geq \beta_0$. By construction $f(\beta_1) \geq \beta_1$ and we see that the same properties hold for $V_{f(\beta_1)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_1$. Next, choose any ordinal $\beta_2 > \beta_1$ with cofinality $\mathrm{cf}(\beta_2) > \kappa$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that the pair κ, $\alpha = f(\beta_2)$ is a solution to the problem posed in the lemma.

The first and third property of the lemma holds by our choices of κ, $\beta_2 > \beta_1 > \beta_0$ above.

Since β_2 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_2) = \sup_{\beta < \beta_2} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_2\} \subset f(\beta_2)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_2)} = \bigcup\nolimits_{\beta < \beta_2} V_{f(\beta)}.$$

Now, let $\mathcal{U} \in \mathrm{Cov}_{\kappa,\alpha}$. We define $\beta(\mathcal{U})$ to be the least ordinal β such that $\mathcal{U} \in \mathrm{Cov}_{\kappa, f(\beta)}$. By the above we see that always $\beta(\mathcal{U}) < \beta_2$.

We have to show properties (1), (2), and (3) defining a site hold for the pair $(\mathcal{C}, \mathrm{Cov}_{\kappa,\alpha})$. The first holds because by our choice of β_2 all arrows of $\mathcal{C}$ are contained in $V_{f(\beta_2)}$. For the third, we use that given a covering $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ we have $\beta(\mathcal{U}) < \beta_2$ and hence any base change of $\mathcal{U}$ is by construction of f contained in $\mathrm{Cov}(\mathcal{C})_{\kappa, f(\beta+1)}$ and hence in $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$.

Finally, for the second condition, suppose that $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$ and for each i we have $\mathcal{W}_i = \{W_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$. Consider the function $I \to \beta_2$, $i \mapsto \beta(\mathcal{W}_i)$. Since the cofinality of β_2 is $> \kappa \geq |I|$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\mathcal{W}_i \in \mathrm{Cov}_{\kappa,f(\beta)}$ for all $i \in I$. It follows that the covering $\{W_{ij} \to U\}_{i\in I, j\in J_i}$ is an element of $\mathrm{Cov}(\mathcal{C})_{\kappa,f(\beta+1)} \subset \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ as desired. □

000Y **Remark 3.11.2.** It is likely the case that, for some limit ordinal α, the set of coverings $\mathrm{Cov}(\mathcal{C})_\alpha$ satisfies the conditions of the lemma. This is after all what an application of the reflection principle would appear to give (modulo caveats as described at the end of Section 3.8 and in Remark 3.9.3).

3.12. Abelian categories and injectives

000Z The following lemma applies to the category of modules over a sheaf of rings on a site.

0010 **Lemma 3.12.1.** *Suppose given a big category $\mathcal{A}$ (see Categories, Remark 4.2.2). Assume $\mathcal{A}$ is abelian and has enough injectives. See Homology, Definitions 12.5.1 and 12.27.4. Then for any given set of objects $\{A_s\}_{s\in S}$ of $\mathcal{A}$, there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ with the following properties:*

1. *the inclusion functor $\mathcal{A}' \to \mathcal{A}$ is exact,*
2. *$\mathrm{Ob}(\mathcal{A}')$ is a set,*
3. *$\mathrm{Ob}(\mathcal{A}')$ contains A_s for each $s \in S$,*
4. *$\mathcal{A}'$ has enough injectives, and*
5. *an object of $\mathcal{A}'$ is injective if and only if it is an injective object of $\mathcal{A}$.*

Proof. Omitted. □

3.13. Other chapters

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Schemes

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CHAPTER 4

Categories

0011 ## 4.1. Introduction

0012 Categories were first introduced in [**EM45**]. The category of categories (which is a proper class) is a 2-category. Similarly, the category of stacks forms a 2-category. If you already know about categories, but not about 2-categories you should read Section 4.28 as an introduction to the formal definitions later on.

4.2. Definitions

0013 We recall the definitions, partly to fix notation.

0014 **Definition 4.2.1.** A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

It is customary to require all the morphism sets $\mathrm{Mor}_{\mathcal{C}}(x, y)$ to be disjoint. In this way a morphism $\phi : x \to y$ has a unique *source* x and a unique *target* y. This is not strictly necessary, although care has to be taken in formulating condition (2) above if it is not the case. It is convenient and we will often assume this is the case. In this case we say that ϕ and ψ are *composable* if the source of ϕ is equal to the target of ψ, in which case $\phi \circ \psi$ is defined. An equivalent definition would be to define a category as a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ)$ consisting of a set of objects, a set of morphisms (arrows), source, target and composition subject to a long list of axioms. We will occasionally use this point of view.

0015 **Remark 4.2.2.** Big categories. In some texts a category is allowed to have a proper class of objects. We will allow this as well in these notes but only in the following list of cases (to be updated as we go along). In particular, when we say: "Let $\mathcal{C}$ be a category" then it is understood that $\mathrm{Ob}(\mathcal{C})$ is a set.

(1) The category *Sets* of sets.
(2) The category *Ab* of abelian groups.
(3) The category *Groups* of groups.
(4) Given a group G the category G-*Sets* of sets with a left G-action.

(5) Given a ring R the category Mod_R of R-modules.
(6) Given a field k the category of vector spaces over k.
(7) The category of rings.
(8) The category of divided power rings, see Divided Power Algebra, Section 23.3.
(9) The category of schemes.
(10) The category *Top* of topological spaces.
(11) Given a topological space X the category $PSh(X)$ of presheaves of sets over X.
(12) Given a topological space X the category $Sh(X)$ of sheaves of sets over X.
(13) Given a topological space X the category $PAb(X)$ of presheaves of abelian groups over X.
(14) Given a topological space X the category $Ab(X)$ of sheaves of abelian groups over X.
(15) Given a small category $\mathcal{C}$ the category of functors from $\mathcal{C}$ to *Sets*.
(16) Given a category $\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$.
(17) Given a site $\mathcal{C}$ the category of sheaves of sets over $\mathcal{C}$.

One of the reason to enumerate these here is to try and avoid working with something like the "collection" of "big" categories which would be like working with the collection of all classes which I think definitively is a meta-mathematical object.

0016 **Remark 4.2.3.** It follows directly from the definition that any two identity morphisms of an object x of $\mathcal{A}$ are the same. Thus we may and will speak of *the* identity morphism id_x of x.

0017 **Definition 4.2.4.** A morphism $\phi : x \to y$ is an *isomorphism* of the category $\mathcal{C}$ if there exists a morphism $\psi : y \to x$ such that $\phi \circ \psi = \mathrm{id}_y$ and $\psi \circ \phi = \mathrm{id}_x$.

An isomorphism ϕ is also sometimes called an *invertible* morphism, and the morphism ψ of the definition is called the *inverse* and denoted ϕ^{-1}. It is unique if it exists. Note that given an object x of a category $\mathcal{A}$ the set of invertible elements $\mathrm{Aut}_{\mathcal{A}}(x)$ of $\mathrm{Mor}_{\mathcal{A}}(x, x)$ forms a group under composition. This group is called the *automorphism* group of x in $\mathcal{A}$.

0018 **Definition 4.2.5.** A *groupoid* is a category where every morphism is an isomorphism.

0019 **Example 4.2.6.** A group G gives rise to a groupoid with a single object x and morphisms $\mathrm{Mor}(x, x) = G$, with the composition rule given by the group law in G. Every groupoid with a single object is of this form.

001A **Example 4.2.7.** A set C gives rise to a groupoid $\mathcal{C}$ defined as follows: As objects we take $\mathrm{Ob}(\mathcal{C}) := C$ and for morphisms we take $\mathrm{Mor}(x, y)$ empty if $x \neq y$ and equal to $\{\mathrm{id}_x\}$ if $x = y$.

001B **Definition 4.2.8.** A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an *identity* functor $\mathrm{id}_{\mathcal{A}}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a *composition* functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

001C **Definition 4.2.9.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects $x, y \in \mathrm{Ob}(\mathcal{A})$ the map
$$F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called *fully faithful*.
(3) The functor F is called *essentially surjective* if for any object $y \in \mathrm{Ob}(\mathcal{B})$ there exists an object $x \in \mathrm{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

001D **Definition 4.2.10.** A *subcategory* of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{B}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$ and such that the identity morphism of an object of $\mathcal{A}$ matches the one in $\mathcal{B}$. We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$ if $\mathrm{Mor}_{\mathcal{A}}(x, y) = \mathrm{Mor}_{\mathcal{B}}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{A})$. We say $\mathcal{A}$ is a *strictly full* subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in \mathrm{Ob}(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $\mathrm{Ob}(\mathcal{B})$.

001E **Remark 4.2.11.** Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to *Sets* is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section 3.2) even though the category of sets is "big". Namely, the range of F on objects will be a set $F(\mathrm{Ob}(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(\mathrm{Ob}(\mathcal{A}))$.

001F **Example 4.2.12.** A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example 4.2.6. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

001G **Example 4.2.13.** Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects over* X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

001H **Example 4.2.14.** Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects under* X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute.

Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

001I **Definition 4.2.15.** Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation*, or a *morphism of functors* $t : F \to G$, is a collection $\{t_x\}_{x \in \mathrm{Ob}(\mathcal{A})}$ such that

(1) $t_x : F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$, and
(2) for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{array}{ccc} F(x) & \xrightarrow{t_x} & G(x) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle G(\phi)} \\ F(y) & \xrightarrow{t_y} & G(y) \end{array}$$

Sometimes we use the diagram

$$\mathcal{A} \overset{F}{\underset{G}{\Downarrow t}} \mathcal{B}$$

to indicate that t is a morphism from F to G.

Note that every functor F comes with the *identity* transformation $\mathrm{id}_F : F \to F$. In addition, given a morphism of functors $t : F \to G$ and a morphism of functors $s : E \to F$ then the *composition* $t \circ s$ is defined by the rule

$$(t \circ s)_x = t_x \circ s_x : E(x) \to G(x)$$

for $x \in \mathrm{Ob}(\mathcal{A})$. It is easy to verify that this is indeed a morphism of functors from E to G. In this way, given categories $\mathcal{A}$ and $\mathcal{B}$ we obtain a new category, namely the category of functors between $\mathcal{A}$ and $\mathcal{B}$.

02C2 **Remark 4.2.16.** This is one instance where the same thing does not hold if $\mathcal{A}$ is a "big" category. For example consider functors *Sets* $\to$ *Sets*. As we have currently defined it such a functor is a class and not a set. In other words, it is given by a formula in set theory (with some variables equal to specified sets)! It is not a good idea to try to consider all possible formulae of set theory as part of the definition of a mathematical object. The same problem presents itself when considering sheaves on the category of schemes for example. We will come back to this point later.

001J **Definition 4.2.17.** An *equivalence of categories* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{B}}$, respectively $\mathrm{id}_{\mathcal{A}}$. In this case we say that G is a *quasi-inverse* to F.

05SG **Lemma 4.2.18.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Suppose for every $X \in \mathrm{Ob}(\mathcal{B})$ we are given an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then there is a unique functor $j : \mathcal{B} \to \mathcal{A}$ such that j extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $id_{\mathcal{B}} \to F \circ j$. Moreover, j and F are quasi-inverse equivalences of categories.*

Proof. To construct $j : \mathcal{B} \to \mathcal{A}$, there are two steps. Firstly, we define the map $j : \mathrm{Ob}(\mathcal{B}) \to \mathrm{Ob}(\mathcal{A})$ that associates $j(X)$ to $X \in \mathcal{B}$. Secondly, if $X, Y \in \mathrm{Ob}(\mathcal{B})$ and $\phi : X \to Y$, we consider $\phi' := i_Y \circ \phi \circ i_X^{-1}$. There is a unique φ verifying $F(\varphi) = \phi'$,

using that F is fully faithful. We define $j(\phi) = \varphi$. We omit the verification that j is a functor. By construction the diagram

$$\begin{array}{ccc} X & \xrightarrow[i_X]{} & F(j(X)) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle F\circ j(\phi)} \\ Y & \xrightarrow{i_Y} & F(j(Y)) \end{array}$$

commutes. Hence, as each i_X is an isomorphism, $\{i_X\}_X$ is an isomorphism of functors $\mathrm{id}_{\mathcal{B}} \to F \circ j$. To conclude, we have to also prove that $j \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{A}}$. However, since F is fully faithful, in order to do this it suffices to prove this after post-composing with F, i.e., it suffices to show that $F \circ j \circ F$ is isomorphic to $F \circ \mathrm{id}_{\mathcal{A}}$ (small detail omitted). Since $F \circ j \cong \mathrm{id}_{\mathcal{B}}$ this is clear. □

02C3 **Lemma 4.2.19.** *A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in \mathrm{Ob}(\mathcal{B})$ an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then we apply Lemma 4.2.18 using that F is fully faithful. □

001K **Definition 4.2.20.** Let $\mathcal{A}$, $\mathcal{B}$ be categories. We define the *product category* $\mathcal{A} \times \mathcal{B}$ to be the category with objects $\mathrm{Ob}(\mathcal{A} \times \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and

$$\mathrm{Mor}_{\mathcal{A}\times\mathcal{B}}((x, y), (x', y')) := \mathrm{Mor}_{\mathcal{A}}(x, x') \times \mathrm{Mor}_{\mathcal{B}}(y, y').$$

Composition is defined componentwise.

4.3. Opposite Categories and the Yoneda Lemma

001L

001M **Definition 4.3.1.** Given a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

In other words $\mathrm{Mor}_{\mathcal{C}^{opp}}(x, y) = \mathrm{Mor}_{\mathcal{C}}(y, x)$. Composition in $\mathcal{C}^{opp}$ is the same as in $\mathcal{C}$ except backwards: if $\phi : y \to z$ and $\psi : x \to y$ are morphisms in $\mathcal{C}^{opp}$, in other words arrows $z \to y$ and $y \to x$ in $\mathcal{C}$, then $\phi \circ^{opp} \psi$ is the morphism $x \to z$ of $\mathcal{C}^{opp}$ which corresponds to the composition $z \to y \to x$ in $\mathcal{C}$.

001N **Definition 4.3.2.** Let $\mathcal{C}$, $\mathcal{S}$ be categories. A *contravariant* functor F from $\mathcal{C}$ to $\mathcal{S}$ is a functor $\mathcal{C}^{opp} \to \mathcal{S}$.

Concretely, a contravariant functor F is given by a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{S})$ and for every morphism $\psi : x \to y$ in $\mathcal{C}$ a morphism $F(\psi) : F(y) \to F(x)$. These should satisfy the property that, given another morphism $\phi : y \to z$, we have $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ as morphisms $F(z) \to F(x)$. (Note the reverse of order.)

02X6 **Definition 4.3.3.** Let $\mathcal{C}$ be a category.

(1) A *presheaf of sets on* $\mathcal{C}$ or simply a *presheaf* is a contravariant functor F from $\mathcal{C}$ to *Sets*.
(2) The category of presheaves is denoted $PSh(\mathcal{C})$.

Of course the category of presheaves is a proper class.

001O **Example 4.3.4.** Functor of points. For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$\begin{array}{rccc} h_U & : & \mathcal{C} & \longrightarrow & Sets \\ & & X & \longmapsto & \mathrm{Mor}_{\mathcal{C}}(X, U) \end{array}$$

which takes an object X to the set $\mathrm{Mor}_{\mathcal{C}}(X, U)$. In other words h_U is a presheaf. Given a morphism $f : X \to Y$ the corresponding map $h_U(f) : \mathrm{Mor}_{\mathcal{C}}(Y, U) \to \mathrm{Mor}_{\mathcal{C}}(X, U)$ takes ϕ to $\phi \circ f$. We will always denote this presheaf $h_U : \mathcal{C}^{opp} \to Sets$. It is called the *representable presheaf* associated to U. If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the *functor of points* of U.

Note that given a morphism $\phi : U \to V$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(\phi) : h_U \to h_V$ defined by composing with the morphism $U \to V$. This turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \longrightarrow PSh(\mathcal{C})$$

Note that the target is a "big" category, see Remark 4.2.2. On the other hand, h is an actual mathematical object (i.e. a set), compare Remark 4.2.11.

001P **Lemma 4.3.5** (Yoneda lemma)**.** *Let $U, V \in \mathrm{Ob}(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$. In other words the functor h is fully faithful. More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, F) \longrightarrow F(U), \quad s \longmapsto s_U(id_U).$$

Appeared in some form in [**Yon54**]. Used by Grothendieck in a generalized form in [**Gro95b**].

Proof. For the first statement, just take $\phi = s_U(\mathrm{id}_U) \in \mathrm{Mor}_{\mathcal{C}}(U, V)$. For the second statement, given $\xi \in F(U)$ define s by $s_V : h_U(V) \to F(V)$ by sending the element $f : V \to U$ of $h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$ to $F(f)(\xi)$. □

001Q **Definition 4.3.6.** A contravariant functor $F : \mathcal{C} \to Sets$ is said to be *representable* if it is isomorphic to the functor of points h_U for some object U of $\mathcal{C}$.

Let $\mathcal{C}$ be a category and let $F : \mathcal{C}^{opp} \to Sets$ be a representable functor. Choose an object U of $\mathcal{C}$ and an isomorphism $s : h_U \to F$. The Yoneda lemma guarantees that the pair (U, s) is unique up to unique isomorphism. The object U is called an object *representing* F. By the Yoneda lemma the transformation s corresponds to a unique element $\xi \in F(U)$. This element is called the *universal object*. It has the property that for $V \in \mathrm{Ob}(\mathcal{C})$ the map

$$\mathrm{Mor}_{\mathcal{C}}(V, U) \longrightarrow F(V), \quad (f : V \to U) \longmapsto F(f)(\xi)$$

is a bijection. Thus ξ is universal in the sense that every element of $F(V)$ is equal to the image of ξ via $F(f)$ for a unique morphism $f : V \to U$ in $\mathcal{C}$.

4.4. Products of pairs

001R

001S **Definition 4.4.1.** Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *product* of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms

$\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times y)$ making the diagram

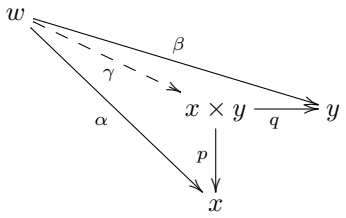

commute.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

001T **Definition 4.4.2.** We say the category $\mathcal{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having products" or "having finite products" which usually means something else (in particular it always implies there exists a final object).

4.5. Coproducts of pairs

04AN

04AO **Definition 4.5.1.** Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg y)$ and $j \in \mathrm{Mor}_{\mathcal{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg y, w)$ making the diagram

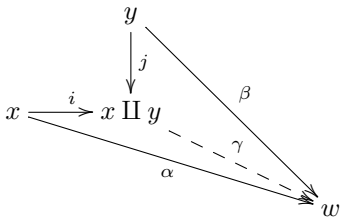

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathcal{C}}(x \amalg y, w) = \mathrm{Mor}_{\mathcal{C}}(x, w) \times \mathrm{Mor}_{\mathcal{C}}(y, w)$$

functorially in w.

04AP **Definition 4.5.2.** We say the category $\mathcal{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having coproducts" or "having finite coproducts" which usually means something else (in particular it always implies there exists an initial object in $\mathcal{C}$).

4.6. Fibre products

001U

001V **Definition 4.6.1.** Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, z)$ making the diagram

$$\begin{array}{ccc} x \times_y z & \xrightarrow{q} & z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & y \end{array}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times_y z)$ making the diagram

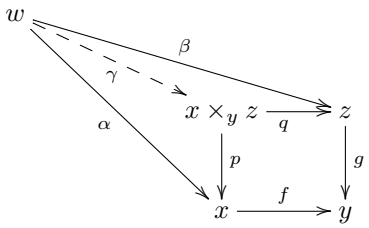

commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

08N0 **Definition 4.6.2.** We say a commutative diagram

$$\begin{array}{ccc} w & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

in a category is *cartesian* if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

001W **Definition 4.6.3.** We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(z, y)$.

001X **Definition 4.6.4.** A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable* if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $x \times_y z$ exists.

001Y **Lemma 4.6.5.** *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Let $t \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(t, z)$. As g and f are representable, we obtain commutative diagrams

$$\begin{array}{ccc} y \times_z t & \xrightarrow{q} & t \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varphi} \\ y & \xrightarrow{g} & z \end{array} \qquad \begin{array}{ccc} x \times_y (y \times_z t) & \xrightarrow{q'} & y \times_z t \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ x & \xrightarrow{f} & y \end{array}$$

with the universal property of Definition 4.6.1. We claim that $x \times_z t = x \times_y (y \times_z t)$ with morphisms $q \circ q' : x \times_z t \to t$ and $p' : x \times_z t \to x$ is a fibre product. First, it follows from the commutativity of the diagrams above that $\varphi \circ q \circ q' = g \circ f \circ p'$. To verify the universal property, let $w \in \mathrm{Ob}(\mathcal{C})$ and suppose $\alpha : w \to x$ and $\beta : w \to y$ are morphisms with $\varphi \circ \beta = g \circ f \circ \alpha$. By definition of the fibre product, there are unique morphisms δ and γ such that

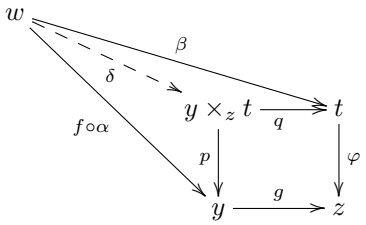

and

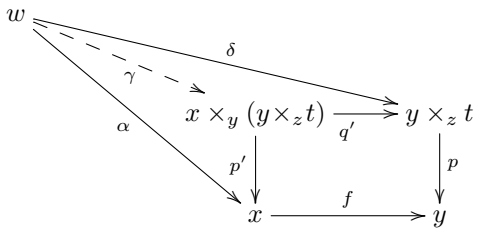

commute. Then, γ makes the diagram

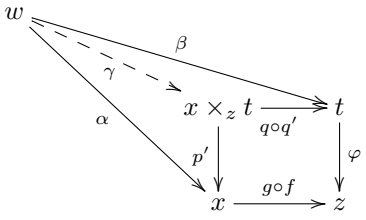

commute. To show its uniqueness, let γ' verify $q \circ q' \circ \gamma' = \beta$ and $p' \circ \gamma' = \alpha$. Because γ is unique, we just need to prove that $q' \circ \gamma' = \delta$ and $p' \circ \gamma' = \alpha$ to conclude. We supposed the second equality. For the first one, we also need to use the uniqueness of delta. Notice that δ is the only morphism verifying $q \circ \delta = \beta$ and $p \circ \delta = f \circ \alpha$. We already supposed that $q \circ (q' \circ \gamma') = \beta$. Furthermore, by definition of the fibre product, we know that $f \circ p' = p \circ q'$. Therefore:

$$p \circ (q' \circ \gamma') = (p \circ q') \circ \gamma' = (f \circ p') \circ \gamma' = f \circ (p' \circ \gamma') = f \circ \alpha.$$

Then $q' \circ \gamma' = \delta$, which concludes the proof. □

001Z **Lemma 4.6.6.** *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{aligned} w &\mapsto h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ &= (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ &= h_x(w) \times_{h_y(w)} h_z(w) \end{aligned}$$

which is representable by assumption. □

4.7. Examples of fibre products

0020 In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibre products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_Y Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Z$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category *Groups* of groups.
(2) The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

4.8. Fibre products and representability

0021 In this section we work out fibre products in the category of contravariant functors from a category to the category of sets. This will later be superseded during the discussion of sites, presheaves, sheaves. Of some interest is the notion of a "representable morphism" between such functors.

0022 **Lemma 4.8.1.** *Let $\mathcal{C}$ be a category. Let $F, G, H : \mathcal{C}^{opp} \to Sets$ be functors. Let $a : F \to G$ and $b : H \to G$ be transformations of functors. Then the fibre product $F \times_{a,G,b} H$ in the category $PSh(\mathcal{C})$ exists and is given by the formula*

$$(F \times_{a,G,b} H)(X) = F(X) \times_{a_X, G(X), b_X} H(X)$$

for any object X of $\mathcal{C}$.

Proof. Omitted. □

As a special case suppose we have a morphism $a : F \to G$, an object $U \in \mathrm{Ob}(\mathcal{C})$ and an element $\xi \in G(U)$. According to the Yoneda Lemma 4.3.5 this gives a transformation $\xi : h_U \to G$. The fibre product in this case is described by the rule

$$(h_U \times_{\xi,G,a} F)(X) = \{(f, \xi') \mid f : X \to U,\ \xi' \in F(X),\ G(f)(\xi) = a_X(\xi')\}$$

If F, G are also representable, then this is the functor representing the fibre product, if it exists, see Section 4.6. The analogy with Definition 4.6.4 prompts us to define a notion of representable transformations.

0023 **Definition 4.8.2.** Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to Sets$ be functors. We say a morphism $a : F \to G$ is *representable*, or that F *is relatively representable over* G, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable.

03KC **Lemma 4.8.3.** *Let $\mathcal{C}$ be a category. Let $a : F \to G$ be a morphism of contravariant functors from $\mathcal{C}$ to Sets. If a is representable, and G is a representable functor, then F is representable.*

Proof. Omitted. □

0024 **Lemma 4.8.4.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

1. *the diagonal $\Delta : F \to F \times F$ is representable,*
2. *for every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable,*
3. *for every pair U, V in $\mathcal{C}$ and any $\xi \in F(U)$, $\xi' \in F(V)$ the fibre product $h_U \times_{\xi,F,\xi'} h_V$ is representable.*

Proof. We will continue to use the Yoneda lemma to identify $F(U)$ with transformations $h_U \to F$ of functors.

Equivalence of (2) and (3). Let U, ξ, V, ξ' be as in (3). Both (2) and (3) tell us exactly that $h_U \times_{\xi,F,\xi'} h_V$ is representable; the only difference is that the statement (3) is symmetric in U and V whereas (2) is not.

Assume condition (1). Let U, ξ, V, ξ' be as in (3). Note that $h_U \times h_V = h_{U \times V}$ is representable. Denote $\eta : h_{U \times V} \to F \times F$ the map corresponding to the product $\xi \times \xi' : h_U \times h_V \to F \times F$. Then the fibre product $F \times_{\Delta,F\times F,\eta} h_{U\times V}$ is representable by assumption. This means there exist $W \in \mathrm{Ob}(\mathcal{C})$, morphisms $W \to U$, $W \to V$ and $h_W \to F$ such that

$$\begin{array}{ccc} h_W & \longrightarrow & h_U \times h_V \\ \downarrow & & \downarrow{\scriptstyle \xi\times\xi'} \\ F & \longrightarrow & F \times F \end{array}$$

is cartesian. Using the explicit description of fibre products in Lemma 4.8.1 the reader sees that this implies that $h_W = h_U \times_{\xi,F,\xi'} h_V$ as desired.

Assume the equivalent conditions (2) and (3). Let U be an object of $\mathcal{C}$ and let $(\xi, \xi') \in (F \times F)(U)$. By (3) the fibre product $h_U \times_{\xi,F,\xi'} h_U$ is representable. Choose an object W and an isomorphism $h_W \to h_U \times_{\xi,F,\xi'} h_U$. The two projections $\mathrm{pr}_i : h_U \times_{\xi,F,\xi'} h_U \to h_U$ correspond to morphisms $p_i : W \to U$ by Yoneda. Consider $W' = W \times_{(p_1,p_2),U\times U} U$. It is formal to show that W' represents $F \times_{\Delta,F\times F} h_U$ because

$$h_{W'} = h_W \times_{h_U \times h_U} h_U = (h_U \times_{\xi,F,\xi'} h_U) \times_{h_U \times h_U} h_U = F \times_{F\times F} h_U.$$

Thus Δ is representable and this finishes the proof. □

4.9. Pushouts

0025 The dual notion to fibre products is that of pushouts.

0026 **Definition 4.9.1.** Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. A *pushout* of f and g is an object $x \amalg_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg_y z)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(z, x \amalg_y z)$ making the diagram

$$\begin{array}{ccc} y & \xrightarrow{g} & z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ x & \xrightarrow{p} & x \amalg_y z \end{array}$$

commute, and such that the following universal property holds: For any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg_y z, w)$ making the diagram

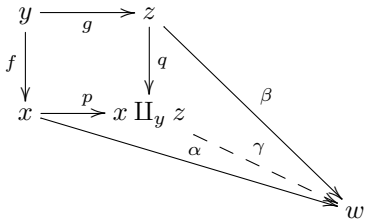

commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the pushout as the fibre product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 4.6.

08N1 **Definition 4.9.2.** We say a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \longrightarrow & w \end{array}$$

in a category is *cocartesian* if w and the morphisms $x \to w$ and $z \to w$ form a pushout of the morphisms $y \to x$ and $y \to z$.

4.10. Equalizers

0027

0028 **Definition 4.10.1.** Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $e : Z \to X$ is an *equalizer* for the pair (a, b) if $a \circ e = b \circ e$ and if (Z, e) satisfies the following universal property: For every morphism $t : W \to X$ in $\mathcal{C}$ such that $a \circ t = b \circ t$ there exists a unique morphism $s : W \to Z$ such that $t = e \circ s$.

As in the case of the fibre products above, equalizers when they exist are unique up to unique isomorphism. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.11. Coequalizers

0029

002A **Definition 4.11.1.** Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $c : Y \to Z$ is a *coequalizer* for the pair (a, b) if $c \circ a = c \circ b$ and if (Z, c) satisfies the following universal property: For every morphism $t : Y \to W$ in $\mathcal{C}$ such that $t \circ a = t \circ b$ there exists a unique morphism $s : Z \to W$ such that $t = s \circ c$.

As in the case of the pushouts above, coequalizers when they exist are unique up to unique isomorphism, and this follows from the uniqueness of equalizers upon considering the opposite category. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.12. Initial and final objects

002B

002C **Definition 4.12.1.** Let $\mathcal{C}$ be a category.

(1) An object x of the category $\mathcal{C}$ is called an *initial* object if for every object y of $\mathcal{C}$ there is exactly one morphism $x \to y$.
(2) An object x of the category $\mathcal{C}$ is called a *final* object if for every object y of $\mathcal{C}$ there is exactly one morphism $y \to x$.

In the category of sets the empty set $\emptyset$ is an initial object, and in fact the only initial object. Also, any *singleton*, i.e., a set with one element, is a final object (so it is not unique).

4.13. Monomorphisms and Epimorphisms

003A

003B **Definition 4.13.1.** Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

003C **Example 4.13.2.** In the category of sets the monomorphisms correspond to injective maps and the epimorphisms correspond to surjective maps.

08LR **Lemma 4.13.3.** *Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then*

(1) *f is a monomorphism if and only if X is the fibre product $X \times_Y X$, and*
(2) *f is an epimorphism if and only if Y is the pushout $Y \amalg_X Y$.*

Proof. Let suppose that f is a monomorphism. Let W be an object of $\mathcal{C}$ and $\alpha, \beta \in \mathrm{Mor}_{\mathcal{C}}(W, X)$ such that $f \circ \alpha = f \circ \beta$. Therefore $\alpha = \beta$ as f is monic. In addition, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

which verify the universal property with $\gamma := \alpha = \beta$. Thus X is indeed the fibre product $X \times_Y X$.

Suppose that $X \times_Y X \cong X$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes and if $W \in \mathrm{Ob}(\mathcal{C})$ and $\alpha, \beta : W \to X$ such that $f \circ \alpha = f \circ \beta$, we have a unique γ verifying

$$\gamma = \mathrm{id}_X \circ \gamma = \alpha = \beta$$

which proves that $\alpha = \beta$.

The proof is exactly the same for the second point, but with the pushout $Y \amalg_X Y = Y$. □

4.14. Limits and colimits

002D Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

002E **Definition 4.14.1.** A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_{\mathcal{I}} M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_{\mathcal{I}} M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_{\mathcal{I}} M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibre products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

002F **Definition 4.14.2.** A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_{\mathcal{I}} M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_{\mathcal{I}} M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Colimits $(\mathrm{colim}_{\mathcal{I}} M, (s_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Coproducts of pairs, pushouts, and coequalizers are examples of colimits. The colimit over an empty diagram is an initial object of $\mathcal{C}$. In the category of sets all colimits exist.

002G **Remark 4.14.3.** The index category of a (co)limit will never be allowed to have a proper class of objects. In this project it means that it cannot be one of the categories listed in Remark 4.2.2

002H **Remark 4.14.4.** We often write $\lim_i M_i$, $\operatorname{colim}_i M_i$, $\lim_{i\in\mathcal{I}} M_i$, or $\operatorname{colim}_{i\in\mathcal{I}} M_i$ instead of the versions indexed by $\mathcal{I}$. Using this notation, and using the description of limits and colimits of sets in Section 4.15 below, we can say the following. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram.

(1) The object $\lim_i M_i$ if it exists satisfies the following property

$$\operatorname{Mor}_{\mathcal{C}}(W, \lim_i M_i) = \lim_i \operatorname{Mor}_{\mathcal{C}}(W, M_i)$$

where the limit on the right takes place in the category of sets.

(2) The object $\operatorname{colim}_i M_i$ if it exists satisfies the following property

$$\operatorname{Mor}_{\mathcal{C}}(\operatorname{colim}_i M_i, W) = \lim_{i\in\mathcal{I}^{opp}} \operatorname{Mor}_{\mathcal{C}}(M_i, W)$$

where on the right we have the limit over the opposite category with value in the category of sets.

By the Yoneda lemma (and its dual) this formula completely determines the limit, respectively the colimit.

0G2U **Remark 4.14.5.** Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. In this setting a *cone* for M is given by an object W and a family of morphisms $q_i : W \to M_i$, $i \in \operatorname{Ob}(\mathcal{I})$ such that for all morphisms $\phi : i \to i'$ of $\mathcal{I}$ the diagram

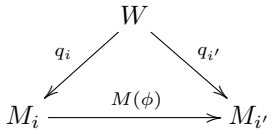

is commutative. The collection of cones forms a category with an obvious notion of morphisms. Clearly, the limit of M, if it exists, is a final object in the category of cones. Dually, a *cocone* for M is given by an object W and a family of morphisms $t_i : M_i \to W$ such that for all morphisms $\phi : i \to i'$ in $\mathcal{I}$ the diagram

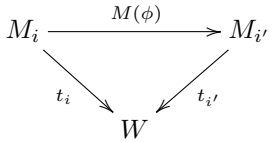

commutes. The collection of cocones forms a category with an obvious notion of morphisms. Similarly to the above the colimit of M exists if and only if the category of cocones has an initial object.

As an application of the notions of limits and colimits we define products and coproducts.

002I **Definition 4.14.6.** Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i\in I} M_i$ is by definition $\lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

002J **Definition 4.14.7.** Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i\in I} M_i$ is by definition $\operatorname{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the coproduct is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

002K **Lemma 4.14.8.** *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose colimits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : M \to N \circ H$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \operatorname{colim}_{\mathcal{I}} M \longrightarrow \operatorname{colim}_{\mathcal{J}} N$$

such that all the diagrams

$$\begin{array}{ccc} M_i & \longrightarrow & \operatorname{colim}_{\mathcal{I}} M \\ {\scriptstyle t_i}\downarrow & & \downarrow{\scriptstyle \theta} \\ N_{H(i)} & \longrightarrow & \operatorname{colim}_{\mathcal{J}} N \end{array}$$

commute.

Proof. Omitted. □

002L **Lemma 4.14.9.** *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose limits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : N \circ H \to M$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \lim_{\mathcal{J}} N \longrightarrow \lim_{\mathcal{I}} M$$

such that all the diagrams

$$\begin{array}{ccc} \lim_{\mathcal{J}} N & \longrightarrow & N_{H(i)} \\ \downarrow{\scriptstyle \theta} & & {\scriptstyle t_i}\downarrow \\ \lim_{\mathcal{I}} M & \longrightarrow & M_i \end{array}$$

commute.

Proof. Omitted. □

002M **Lemma 4.14.10.** *Let $\mathcal{I}$, $\mathcal{J}$ be index categories. Let $M : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor. Assume that $M_{i,\infty} = \operatorname{colim}_j M_{i,j}$ exists for all i. Then the resulting functor $M_{-,\infty} : \mathcal{I} \to \mathcal{C}$ has a colimit if and only if M does, and then the colimits coincide. In particular, we have*

$$\operatorname{colim}_i \operatorname{colim}_j M_{i,j} = \operatorname{colim}_{i,j} M_{i,j} = \operatorname{colim}_j \operatorname{colim}_i M_{i,j}$$

provided all the indicated colimits exist. Similar for limits.

Proof. Omitted. □

002N **Lemma 4.14.11.** *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrows(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\prod_{i\in I} M_i$ and $\prod_{a\in A} M_{t(a)}$ exist. Suppose that the equalizer of*

$$\prod_{i\in I} M_i \xrightarrow[\psi]{\phi} \prod_{a\in A} M_{t(a)}$$

exists, where the morphisms are determined by their components as follows: $p_a \circ \psi = M(a) \circ p_{s(a)}$ and $p_a \circ \phi = p_{t(a)}$. Then this equalizer is the limit of the diagram.

Proof. Omitted. □

002P **Lemma 4.14.12.** *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrows(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\coprod_{i\in I} M_i$ and $\coprod_{a\in A} M_{s(a)}$ exist. Suppose that the coequalizer of*

$$\coprod_{a\in A} M_{s(a)} \xrightarrow[\psi]{\phi} \coprod_{i\in I} M_i$$

exists, where the morphisms are determined by their components as follows: The component $M_{s(a)}$ maps via ψ to the component $M_{t(a)}$ via the morphism $M(a)$. The component $M_{s(a)}$ maps via ϕ to the component $M_{s(a)}$ by the identity morphism. Then this coequalizer is the colimit of the diagram.

Proof. Omitted. □

4.15. Limits and colimits in the category of sets

002U Not only do limits and colimits exist in *Sets* but they are also easy to describe. Namely, let $M : \mathcal{I} \to Sets$, $i \mapsto M_i$ be a diagram of sets. Denote $I = \mathrm{Ob}(\mathcal{I})$. The limit is described as

$$\lim_{\mathcal{I}} M = \{(m_i)_{i\in I} \in \prod_{i\in I} M_i \mid \forall \phi : i \to i' \text{ in } \mathcal{I}, M(\phi)(m_i) = m_{i'}\}.$$

So we think of an element of the limit as a compatible system of elements of all the sets M_i.

On the other hand, the colimit is

$$\mathrm{colim}_{\mathcal{I}} M = (\coprod_{i\in I} M_i)/\sim$$

where the equivalence relation $\sim$ is the equivalence relation generated by setting $m_i \sim m_{i'}$ if $m_i \in M_i$, $m_{i'} \in M_{i'}$ and $M(\phi)(m_i) = m_{i'}$ for some $\phi : i \to i'$. In other words, $m_i \in M_i$ and $m_{i'} \in M_{i'}$ are equivalent if there are a chain of morphisms in $\mathcal{I}$

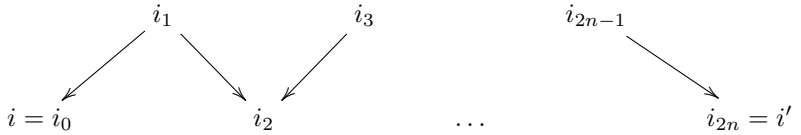

and elements $m_{i_j} \in M_{i_j}$ mapping to each other under the maps $M_{i_{2k-1}} \to M_{i_{2k-2}}$ and $M_{i_{2k-1}} \to M_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above.

This is not a very pleasant type of object to work with. But if the diagram is filtered then it is much easier to describe. We will explain this in Section 4.19.

4.16. Connected limits

04AQ A (co)limit is called connected if its index category is connected.

002S **Definition 4.16.1.** We say that a category $\mathcal{I}$ is *connected* if the equivalence relation generated by $x \sim y \Leftrightarrow \mathrm{Mor}_{\mathcal{I}}(x, y) \neq \emptyset$ has exactly one equivalence class.

Here we follow the convention of Topology, Definition 5.7.1 that connected spaces are nonempty. The following in some vague sense characterizes connected limits.

002T **Lemma 4.16.2.** *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to \mathcal{C}/X$ be a diagram in the category of objects over X. If the index category $\mathcal{I}$ is connected and the limit of M exists in $\mathcal{C}/X$, then the limit of the composition $\mathcal{I} \to \mathcal{C}/X \to \mathcal{C}$ exists and is the same.*

Proof. Let $L \to X$ be an object representing the limit in $\mathcal{C}/X$. Consider the functor

$$W \longmapsto \lim_i \mathrm{Mor}_{\mathcal{C}}(W, M_i).$$

Let (φ_i) be an element of the set on the right. Since each M_i comes equipped with a morphism $s_i : M_i \to X$ we get morphisms $f_i = s_i \circ \varphi_i : W \to X$. But as $\mathcal{I}$ is connected we see that all f_i are equal. Since $\mathcal{I}$ is nonempty there is at least one f_i. Hence this common value $W \to X$ defines the structure of an object of W in $\mathcal{C}/X$ and (φ_i) defines an element of $\lim_i \mathrm{Mor}_{\mathcal{C}/X}(W, M_i)$. Thus we obtain a unique morphism $\phi : W \to L$ such that φ_i is the composition of ϕ with $L \to M_i$ as desired. □

04AR **Lemma 4.16.3.** *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to X/\mathcal{C}$ be a diagram in the category of objects under X. If the index category $\mathcal{I}$ is connected and the colimit of M exists in $X/\mathcal{C}$, then the colimit of the composition $\mathcal{I} \to X/\mathcal{C} \to \mathcal{C}$ exists and is the same.*

Proof. Omitted. Hint: This lemma is dual to Lemma 4.16.2. □

4.17. Cofinal and initial categories

09WN In the literature sometimes the word "final" is used instead of cofinal in the following definition.

04E6 **Definition 4.17.1.** Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is cofinal in* $\mathcal{J}$ or that H is *cofinal* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exist an $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $y \to H(x)$, and
(2) given $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $y \to H(x)$ and $y \to H(x')$ there exist a sequence of morphisms

$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$

in $\mathcal{I}$ and morphisms $y \to H(x_i)$ in $\mathcal{J}$ with $y \to H(x_0)$ and $y \to H(x_{2n})$ the given morphisms such that the diagrams

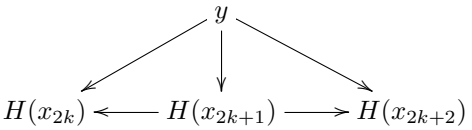

commute for $k = 0, \ldots, n-1$.

In other words, fixing an object y of $\mathcal{J}$ consider the set S of pairs (x, b) where x is an object of $\mathcal{I}$ and $b : y \to H(x)$ is a morphism. Consider the equivalence relation on S generated by $(x, b) \sim (x', b')$ if there exists a morphism $a : x \to x'$ with $b' = H(a) \circ b$. Then S should consist of exactly one equivalence class.

04E7 **Lemma 4.17.2.** *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is cofinal in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism*

$$\operatorname{colim}_{\mathcal{I}} M \circ H = \operatorname{colim}_{\mathcal{J}} M$$

if either side exists.

Proof. Omitted. □

09WP **Definition 4.17.3.** Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is initial in* $\mathcal{J}$ or that H is *initial* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exist an $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $H(x) \to y$,
(2) for any $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $H(x) \to y$, $H(x') \to y$ in $\mathcal{J}$ there exist a sequence of morphisms

$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$

in $\mathcal{I}$ and morphisms $H(x_i) \to y$ in $\mathcal{J}$ such that the diagrams

$$\begin{array}{ccccc} H(x_{2k}) & \longleftarrow & H(x_{2k+1}) & \longrightarrow & H(x_{2k+2}) \\ & \searrow & \downarrow & \swarrow & \\ & & y & & \end{array}$$

commute for $k = 0, \ldots, n-1$.

This is just the dual notion to "cofinal" functors.

002R **Lemma 4.17.4.** *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is initial in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism*

$$\lim_{\mathcal{I}} M \circ H = \lim_{\mathcal{J}} M$$

if either side exists.

Proof. Omitted. □

05US **Lemma 4.17.5.** *Let $F : \mathcal{I} \to \mathcal{I}'$ be a functor. Assume*

(1) *the fibre categories (see Definition 4.32.2) of $\mathcal{I}$ over $\mathcal{I}'$ are all connected, and*
(2) *for every morphism $\alpha' : x' \to y'$ in $\mathcal{I}'$ there exists a morphism $\alpha : x \to y$ in $\mathcal{I}$ such that $F(\alpha) = \alpha'$.*

Then for every diagram $M : \mathcal{I}' \to \mathcal{C}$ the colimit $\operatorname{colim}_{\mathcal{I}} M \circ F$ exists if and only if $\operatorname{colim}_{\mathcal{I}'} M$ exists and if so these colimits agree.

Proof. One can prove this by showing that $\mathcal{I}$ is cofinal in $\mathcal{I}'$ and applying Lemma 4.17.2. But we can also prove it directly as follows. It suffices to show that for any object T of $\mathcal{C}$ we have

$$\lim_{\mathcal{I}^{opp}} \operatorname{Mor}_{\mathcal{C}}(M_{F(i)}, T) = \lim_{(\mathcal{I}')^{opp}} \operatorname{Mor}_{\mathcal{C}}(M_{i'}, T)$$

If $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ is an element of the right hand side, then setting $f_i = g_{F(i)}$ we obtain an element $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of the left hand side. Conversely, let $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ be

an element of the left hand side. Note that on each (connected) fibre category $\mathcal{I}_{i'}$ the functor $M \circ F$ is constant with value $M_{i'}$. Hence the morphisms f_i for $i \in \mathrm{Ob}(\mathcal{I})$ with $F(i) = i'$ are all the same and determine a well defined morphism $g_{i'} : M_{i'} \to T$. By assumption (2) the collection $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ defines an element of the right hand side. □

0A2B **Lemma 4.17.6.** *Let $\mathcal{I}$ and $\mathcal{J}$ be a categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. If $\mathcal{I}$ is connected, then for a diagram $M : \mathcal{J} \to \mathcal{C}$ the colimit $\mathrm{colim}_{\mathcal{J}} M$ exists if and only if $\mathrm{colim}_{\mathcal{I} \times \mathcal{J}} M \circ p$ exists and if so these colimits are equal.*

Proof. This is a special case of Lemma 4.17.5. □

4.18. Finite limits and colimits

04AS A *finite* (co)limit is a (co)limit whose index category is finite, i.e., the index category has finitely many objects and finitely many morphisms. A (co)limit is called *nonempty* if the index category is nonempty. A (co)limit is called *connected* if the index category is connected, see Definition 4.16.1. It turns out that there are "enough" finite index categories.

05XU **Lemma 4.18.1.** *Let $\mathcal{I}$ be a category with*

(1) *$\mathrm{Ob}(\mathcal{I})$ is finite, and*
(2) *there exist finitely many morphisms $f_1, \ldots, f_m \in \mathrm{Arrows}(\mathcal{I})$ such that every morphism of $\mathcal{I}$ is a composition $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_k}$.*

Then there exists a functor $F : \mathcal{J} \to \mathcal{I}$ such that

(a) *$\mathcal{J}$ is a finite category, and*
(b) *for any diagram $M : \mathcal{I} \to \mathcal{C}$ the (co)limit of M over $\mathcal{I}$ exists if and only if the (co)limit of $M \circ F$ over $\mathcal{J}$ exists and in this case the (co)limits are canonically isomorphic.*

Moreover, $\mathcal{J}$ is connected (resp. nonempty) if and only if $\mathcal{I}$ is so.

Proof. Say $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\}$. Denote $s, t : \{1, \ldots, m\} \to \{1, \ldots, n\}$ the functions such that $f_j : x_{s(j)} \to x_{t(j)}$. We set $\mathrm{Ob}(\mathcal{J}) = \{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ Besides the identity morphisms we introduce morphisms $g_j : y_{s(j)} \to z_{t(j)}$, $j = 1, \ldots, m$ and morphisms $h_i : y_i \to z_i$, $i = 1, \ldots, n$. Since all of the nonidentity morphisms in $\mathcal{J}$ go from a y to a z there are no compositions to define and no associativities to check. Set $F(y_i) = F(z_i) = x_i$. Set $F(g_j) = f_j$ and $F(h_i) = \mathrm{id}_{x_i}$. It is clear that F is a functor. It is clear that $\mathcal{J}$ is finite. It is clear that $\mathcal{J}$ is connected, resp. nonempty if and only if $\mathcal{I}$ is so.

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Consider an object W of $\mathcal{C}$ and morphisms $q_i : W \to M(x_i)$ as in Definition 4.14.1. Then by taking $q_i : W \to M(F(y_i)) = M(F(z_i)) = M(x_i)$ we obtain a family of maps as in Definition 4.14.1 for the diagram $M \circ F$. Conversely, suppose we are given maps $qy_i : W \to M(F(y_i))$ and $qz_i : W \to M(F(z_i))$ as in Definition 4.14.1 for the diagram $M \circ F$. Since

$$M(F(h_i)) = \mathrm{id} : M(F(y_i)) = M(x_i) \longrightarrow M(x_i) = M(F(z_i))$$

we conclude that $qy_i = qz_i$ for all i. Set q_i equal to this common value. The compatibility of $q_{s(j)} = qy_{s(j)}$ and $q_{t(j)} = qz_{t(j)}$ with the morphism $M(f_j)$ guarantees that the family q_i is compatible with all morphisms in $\mathcal{I}$ as by assumption

every such morphism is a composition of the morphisms f_j. Thus we have found a canonical bijection

$$\lim_{B \in \mathrm{Ob}(\mathcal{J})} \mathrm{Mor}_{\mathcal{C}}(W, M(F(B))) = \lim_{A \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}_{\mathcal{C}}(W, M(A))$$

which implies the statement on limits in the lemma. The statement on colimits is proved in the same way (proof omitted). □

04AT **Lemma 4.18.2.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Connected finite limits exist in $\mathcal{C}$.*
(2) *Equalizers and fibre products exist in $\mathcal{C}$.*

Proof. Since equalizers and fibre products are finite connected limits we see that (1) implies (2). For the converse, let $\mathcal{I}$ be a finite connected index category. Let $F : \mathcal{J} \to \mathcal{I}$ be the functor of index categories constructed in the proof of Lemma 4.18.1. Then we see that we may replace $\mathcal{I}$ by $\mathcal{J}$. The result is that we may assume that $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\} \amalg \{y_1, \ldots, y_m\}$ with $n, m \geq 1$ such that all nonidentity morphisms in $\mathcal{I}$ are morphisms $f : x_i \to y_j$ for some i and j.

Suppose that $n > 1$. Since $\mathcal{I}$ is connected there exist indices i_1, i_2 and j_0 and morphisms $a : x_{i_1} \to y_{j_0}$ and $b : x_{i_2} \to y_{j_0}$. Consider the category

$$\mathcal{I}' = \{x\} \amalg \{x_1, \ldots, \hat{x}_{i_1}, \ldots, \hat{x}_{i_2}, \ldots x_n\} \amalg \{y_1, \ldots, y_m\}$$

with

$$\mathrm{Mor}_{\mathcal{I}'}(x, y_j) = \mathrm{Mor}_{\mathcal{I}}(x_{i_1}, y_j) \amalg \mathrm{Mor}_{\mathcal{I}}(x_{i_2}, y_j)$$

and all other morphism sets the same as in $\mathcal{I}$. For any functor $M : \mathcal{I} \to \mathcal{C}$ we can construct a functor $M' : \mathcal{I}' \to \mathcal{C}$ by setting

$$M'(x) = M(x_{i_1}) \times_{M(a), M(y_{j_0}), M(b)} M(x_{i_2})$$

and for a morphism $f' : x \to y_j$ corresponding to, say, $f : x_{i_1} \to y_j$ we set $M'(f) = M(f) \circ \mathrm{pr}_1$. Then the functor M has a limit if and only if the functor M' has a limit (proof omitted). Hence by induction we reduce to the case $n = 1$.

If $n = 1$, then the limit of any $M : \mathcal{I} \to \mathcal{C}$ is the successive equalizer of pairs of maps $x_1 \to y_j$ hence exists by assumption. □

04AU **Lemma 4.18.3.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite limits exist in $\mathcal{C}$.*
(2) *Products of pairs and equalizers exist in $\mathcal{C}$.*
(3) *Products of pairs and fibre products exist in $\mathcal{C}$.*

Proof. Since products of pairs, fibre products, and equalizers are limits with nonempty index categories we see that (1) implies both (2) and (3). Assume (2). Then finite nonempty products and equalizers exist. Hence by Lemma 4.14.11 we see that finite nonempty limits exist, i.e., (1) holds. Assume (3). If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1, \mathrm{pr}_2), A \times A, \Delta} A.$$

Thus (3) implies (2), and the lemma is proved. □

002O **Lemma 4.18.4.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Finite limits exist in $\mathcal{C}$.*
(2) *Finite products and equalizers exist.*
(3) *The category has a final object and fibre products exist.*

Proof. Since finite products, fibre products, equalizers, and final objects are limits over finite index categories we see that (1) implies both (2) and (3). By Lemma 4.14.11 above we see that (2) implies (1). Assume (3). Note that the product $A \times B$ is the fibre product over the final object. If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1,\mathrm{pr}_2),A\times A,\Delta} A.$$

Thus (3) implies (2) and the lemma is proved. □

04AV **Lemma 4.18.5.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Connected finite colimits exist in $\mathcal{C}$.*
(2) *Coequalizers and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is dual to Lemma 4.18.2. □

04AW **Lemma 4.18.6.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite colimits exist in $\mathcal{C}$.*
(2) *Coproducts of pairs and coequalizers exist in $\mathcal{C}$.*
(3) *Coproducts of pairs and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is the dual of Lemma 4.18.3. □

002Q **Lemma 4.18.7.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Finite colimits exist in $\mathcal{C}$.*
(2) *Finite coproducts and coequalizers exist in $\mathcal{C}$.*
(3) *The category has an initial object and pushouts exist.*

Proof. Omitted. Hint: This is dual to Lemma 4.18.4. □

4.19. Filtered colimits

04AX Colimits are easier to compute or describe when they are over a filtered diagram. Here is the definition.

002V **Definition 4.19.1.** We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *directed*, or *filtered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$, $y \to z$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : y \to z$ of $\mathcal{I}$ such that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *directed*, or *filtered* if $\mathrm{id} : \mathcal{I} \to \mathcal{I}$ is filtered (in other words you erase the M in part (3) above).

We observe that any diagram with filtered index category is filtered, and this is how filtered colimits usually come about. In fact, if $M : \mathcal{I} \to \mathcal{C}$ is a filtered diagram, then we can factor M as $\mathcal{I} \to \mathcal{I}' \to \mathcal{C}$ where $\mathcal{I}'$ is a filtered index category[1] such that $\mathrm{colim}_{\mathcal{I}} M$ exists if and only if $\mathrm{colim}_{\mathcal{I}'} M'$ exists in which case the colimits are canonically isomorphic.

[1] Namely, let $\mathcal{I}'$ have the same objects as $\mathcal{I}$ but where $\mathrm{Mor}_{\mathcal{I}'}(x, y)$ is the quotient of $\mathrm{Mor}_{\mathcal{I}}(x, y)$ by the equivalence relation which identifies $a, b : x \to y$ if $M(a) = M(b)$.

Suppose that $M : \mathcal{I} \to Sets$ is a filtered diagram. In this case we may describe the equivalence relation in the formula

$$\operatorname{colim}_{\mathcal{I}} M = (\coprod\nolimits_{i \in I} M_i)/\sim$$

simply as follows

$$m_i \sim m_{i'} \Leftrightarrow \exists i'', \phi : i \to i'', \phi' : i' \to i'', M(\phi)(m_i) = M(\phi')(m_{i'}).$$

In other words, two elements are equal in the colimit if and only if they "eventually become equal".

002W **Lemma 4.19.2.** *Let $\mathcal{I}$ and $\mathcal{J}$ be index categories. Assume that $\mathcal{I}$ is filtered and $\mathcal{J}$ is finite. Let $M : \mathcal{I} \times \mathcal{J} \to Sets$, $(i, j) \mapsto M_{i,j}$ be a diagram of diagrams of sets. In this case*

$$\operatorname{colim}_i \lim_j M_{i,j} = \lim_j \operatorname{colim}_i M_{i,j}.$$

In particular, colimits over $\mathcal{I}$ commute with finite products, fibre products, and equalizers of sets.

Proof. Omitted. In fact, it is a fun exercise to prove that a category is filtered if and only if colimits over the category commute with finite limits (into the category of sets). □

We give a counter example to the lemma in the case where $\mathcal{J}$ is infinite. Namely, let $\mathcal{I}$ consist of $\mathbf{N} = \{1, 2, 3, \ldots\}$ with a unique morphism $i \to i'$ whenever $i \leq i'$. Let $\mathcal{J}$ be the discrete category $\mathbf{N} = \{1, 2, 3, \ldots\}$ (only morphisms are identities). Let $M_{i,j} = \{1, 2, \ldots, i\}$ with obvious inclusion maps $M_{i,j} \to M_{i',j}$ when $i \leq i'$. In this case $\operatorname{colim}_i M_{i,j} = \mathbf{N}$ and hence

$$\lim_j \operatorname{colim}_i M_{i,j} = \prod\nolimits_j \mathbf{N} = \mathbf{N}^{\mathbf{N}}$$

On the other hand $\lim_j M_{i,j} = \prod_j M_{i,j}$ and hence

$$\operatorname{colim}_i \lim_j M_{i,j} = \bigcup\nolimits_i \{1, 2, \ldots, i\}^{\mathbf{N}}$$

which is smaller than the other limit.

0BUC **Lemma 4.19.3.** *Let $\mathcal{I}$ be a category. Let $\mathcal{J}$ be a full subcategory. Assume that $\mathcal{I}$ is filtered. Assume also that for any object i of $\mathcal{I}$, there exists a morphism $i \to j$ to some object j of $\mathcal{J}$. Then $\mathcal{J}$ is filtered and cofinal in $\mathcal{I}$.* [**KS06**, Proposition 3.2.4]

Proof. Omitted. Pleasant exercise of the notions involved. □

It turns out we sometimes need a more finegrained control over the possible conditions one can impose on index categories. Thus we add some lemmas on the possible things one can require.

09WQ **Lemma 4.19.4.** *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$ and $y \to z$. Then*

1. *If M and N are diagrams of sets over $\mathcal{I}$, then $\operatorname{colim}(M_i \times N_i) \to \operatorname{colim} M_i \times \operatorname{colim} N_i$ is surjective,*
2. *in general colimits of diagrams of sets over $\mathcal{I}$ do not commute with finite nonempty products.*

Proof. Proof of (1). Let $(\overline{m}, \overline{n})$ be an element of $\operatorname{colim} M_i \times \operatorname{colim} N_i$. Then we can find $m \in M_x$ and $n \in N_y$ for some $x, y \in \mathrm{Ob}(\mathcal{I})$ such that m maps to $\overline{m}$ and n maps to $\overline{n}$. See Section 4.15. Choose $a : x \to z$ and $b : y \to z$ in $\mathcal{I}$. Then $(M(a)(m), N(b)(n))$ is an element of $(M \times N)_z$ whose image in $\operatorname{colim}(M_i \times N_i)$ maps to $(\overline{m}, \overline{n})$ as desired.

Proof of (2). Let G be a non-trivial group and let $\mathcal{I}$ be the one-object category with endomorphism monoid G. Then $\mathcal{I}$ trivially satisfies the condition stated in the lemma. Now let G act on itself by translation and view the G-set G as a set-valued $\mathcal{I}$-diagram. Then

$$\operatorname{colim}_{\mathcal{I}} G \times \operatorname{colim}_{\mathcal{I}} G \cong G/G \times G/G$$

is not isomorphic to

$$\operatorname{colim}_{\mathcal{I}}(G \times G) \cong (G \times G)/G$$

This example indicates that you cannot just drop the additional condition Lemma 4.19.2 even if you only care about finite products. □

09WR **Lemma 4.19.5.** *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$ and $y \to z$. Let $M : \mathcal{I} \to Ab$ be a diagram of abelian groups over $\mathcal{I}$. Then the colimit of M in the category of sets surjects onto the colimit of M in the category of abelian groups.*

Proof. Recall that the colimit in the category of sets is the quotient of the disjoint union $\coprod M_i$ by relation, see Section 4.15. Similarly, the colimit in the category of abelian groups is a quotient of the direct sum $\bigoplus M_i$. The assumption of the lemma means that given $i, j \in \mathrm{Ob}(\mathcal{I})$ and $m \in M_i$ and $n \in M_j$, then we can find an object k and morphisms $a : i \to k$ and $b : j \to k$. Thus $m + n$ is represented in the colimit by the element $M(a)(m) + M(b)(n)$ of M_k. Thus the $\coprod M_i$ surjects onto the colimit. □

09WS **Lemma 4.19.6.** *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

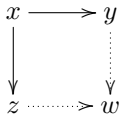

in $\mathcal{I}$ there exist an object w and dotted arrows making the diagram commute. Then $\mathcal{I}$ is either empty or a nonempty disjoint union of connected categories having the same property.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exist a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. Then $\mathcal{I} = \coprod \mathcal{I}_j$ with $\mathcal{I}_j$ nonempty as desired. □

09WT **Lemma 4.19.7.** *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

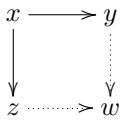

in $\mathcal{I}$ there exist an object w and dotted arrows making the diagram commute. Then

(1) *an injective morphism $M \to N$ of diagrams of sets over $\mathcal{I}$ gives rise to an injective map $\operatorname{colim} M_i \to \operatorname{colim} N_i$ of sets,*
(2) *in general the same is not the case for diagrams of abelian groups and their colimits.*

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Thus we may assume $\mathcal{I}$ is nonempty. In this case we can write $\mathcal{I} = \coprod \mathcal{I}_j$ where each $\mathcal{I}_j$ is nonempty and satisfies the same property, see Lemma 4.19.6. Since $\operatorname{colim}_{\mathcal{I}} M = \coprod_j \operatorname{colim}_{\mathcal{I}_j} M|_{\mathcal{I}_j}$ this reduces the proof of (1) to the connected case.

Assume $\mathcal{I}$ is connected and $M \to N$ is injective, i.e., all the maps $M_i \to N_i$ are injective. We identify M_i with the image of $M_i \to N_i$, i.e., we will think of M_i as a subset of N_i. We will use the description of the colimits given in Section 4.15 without further mention. Let $s, s' \in \operatorname{colim} M_i$ map to the same element of $\operatorname{colim} N_i$. Say s comes from an element m of M_i and s' comes from an element m' of $M_{i'}$. Then we can find a sequence $i = i_0, i_1, \ldots, i_n = i'$ of objects of $\mathcal{I}$ and morphisms

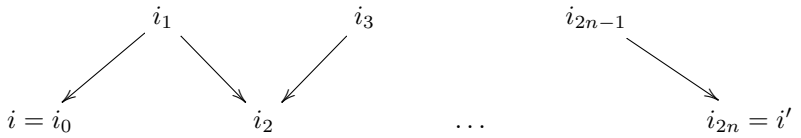

and elements $n_{i_j} \in N_{i_j}$ mapping to each other under the maps $N_{i_{2k-1}} \to N_{i_{2k-2}}$ and $N_{i_{2k-1}} \to N_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above with $n_{i_0} = m$ and $n_{i_{2n}} = m'$. We will prove by induction on n that this implies $s = s'$. The base case $n = 0$ is trivial. Assume $n \geq 1$. Using the assumption on $\mathcal{I}$ we find a commutative diagram

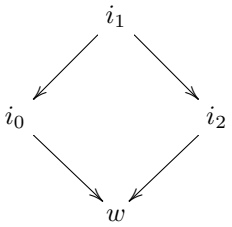

We conclude that m and n_{i_2} map to the same element of N_w because both are the image of the element n_{i_1}. In particular, this element is an element $m'' \in M_w$ which gives rise to the same element as s in $\operatorname{colim} M_i$. Then we find the chain

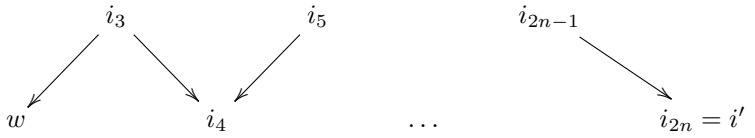

and the elements n_{i_j} for $j \geq 3$ which has a smaller length than the chain we started with. This proves the induction step and the proof of (1) is complete.

Let G be a group and let $\mathcal{I}$ be the one-object category with endomorphism monoid G. Then $\mathcal{I}$ satisfies the condition stated in the lemma because given $g_1, g_2 \in G$ we can find $h_1, h_2 \in G$ with $h_1 g_1 = h_2 g_2$. A diagram M over $\mathcal{I}$ in Ab is the same thing as an abelian group M with G-action and $\operatorname{colim}_{\mathcal{I}} M$ is the coinvariants M_G of M. Take G the group of order 2 acting trivially on $M = \mathbf{Z}/2\mathbf{Z}$ mapping into the first summand of $N = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ where the nontrivial element of G acts by $(x, y) \mapsto (x + y, y)$. Then $M_G \to N_G$ is zero. □

002X **Lemma 4.19.8.** *Let $\mathcal{I}$ be an index category, i.e., a category. Assume*

(1) *for every pair of morphisms $a : w \to x$ and $b : w \to y$ in $\mathcal{I}$ there exist an object z and morphisms $c : x \to z$ and $d : y \to z$ such that $c \circ a = d \circ b$, and*
(2) *for every pair of morphisms $a, b : x \to y$ there exists a morphism $c : y \to z$ such that $c \circ a = c \circ b$.*

Then $\mathcal{I}$ is a (possibly empty) union of disjoint filtered index categories $\mathcal{I}_j$.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exist a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the first assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. The rest is clear from the definitions. □

002Y **Lemma 4.19.9.** *Let $\mathcal{I}$ be an index category satisfying the hypotheses of Lemma 4.19.8 above. Then colimits over $\mathcal{I}$ commute with fibre products and equalizers in sets (and more generally with finite connected limits).*

Proof. By Lemma 4.19.8 we may write $\mathcal{I} = \coprod \mathcal{I}_j$ with each $\mathcal{I}_j$ filtered. By Lemma 4.19.2 we see that colimits of $\mathcal{I}_j$ commute with equalizers and fibre products. Thus it suffices to show that equalizers and fibre products commute with coproducts in the category of sets (including empty coproducts). In other words, given a set J and sets A_j, B_j, C_j and set maps $A_j \to B_j$, $C_j \to B_j$ for $j \in J$ we have to show that

$$(\coprod\nolimits_{j \in J} A_j) \times_{(\coprod_{j \in J} B_j)} (\coprod\nolimits_{j \in J} C_j) = \coprod\nolimits_{j \in J} A_j \times_{B_j} C_j$$

and given $a_j, a'_j : A_j \to B_j$ that

$$\mathrm{Equalizer}(\coprod\nolimits_{j \in J} a_j, \coprod\nolimits_{j \in J} a'_j) = \coprod\nolimits_{j \in J} \mathrm{Equalizer}(a_j, a'_j)$$

This is true even if $J = \emptyset$. Details omitted. □

4.20. Cofiltered limits

04AY Limits are easier to compute or describe when they are over a cofiltered diagram. Here is the definition.

04AZ **Definition 4.20.1.** We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *codirected* or *cofiltered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,

(2) for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $z \to x$, $z \to y$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : w \to x$ of $\mathcal{I}$ such that $M(a \circ c) = M(b \circ c)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *codirected*, or *cofiltered* if $\mathrm{id} : \mathcal{I} \to \mathcal{I}$ is cofiltered (in other words you erase the M in part (3) above).

We observe that any diagram with cofiltered index category is cofiltered, and this is how this situation usually occurs.

As an example of why cofiltered limits of sets are "easier" than general ones, we mention the fact that a cofiltered diagram of finite nonempty sets has nonempty limit (Lemma 4.21.7). This result does not hold for a general limit of finite nonempty sets.

4.21. Limits and colimits over preordered sets

002Z A special case of diagrams is given by systems over preordered sets.

00D3 **Definition 4.21.1.** Let I be a set and let $\leq$ be a binary relation on I.

(1) We say $\leq$ is a *preorder* if it is transitive (if $i \leq j$ and $j \leq k$ then $i \leq k$) and reflexive ($i \leq i$ for all $i \in I$).
(2) A *preordered set* is a set endowed with a preorder.
(3) A *directed set* is a preordered set $(I, \leq)$ such that I is not empty and such that $\forall i, j \in I$, there exists $k \in I$ with $i \leq k, j \leq k$.
(4) We say $\leq$ is a *partial order* if it is a preorder which is antisymmetric (if $i \leq j$ and $j \leq i$, then $i = j$).
(5) A *partially ordered set* is a set endowed with a partial order.
(6) A *directed partially ordered set* is a directed set whose ordering is a partial order.

It is customary to drop the $\leq$ from the notation when talking about preordered sets, that is, one speaks of the preordered set I rather than of the preordered set $(I, \leq)$. Given a preordered set I the symbol $\geq$ is defined by the rule $i \geq j \Leftrightarrow j \leq i$ for all $i, j \in I$. The phrase "partially ordered set" is sometimes abbreviated to "poset".

Given a preordered set I we can construct a category: the objects are the elements of I, there is exactly one morphism $i \to i'$ if $i \leq i'$, and otherwise none. Conversely, given a category $\mathcal{C}$ with at most one arrow between any two objects, the set $\mathrm{Ob}(\mathcal{C})$ is endowed with a preorder defined by the rule $x \leq y \Leftrightarrow \mathrm{Mor}_{\mathcal{C}}(x, y) \neq \emptyset$.

0030 **Definition 4.21.2.** Let $(I, \leq)$ be a preordered set. Let $\mathcal{C}$ be a category.

(1) A *system over I in $\mathcal{C}$*, sometimes called a *inductive system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \mathrm{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
(2) An *inverse system over I in $\mathcal{C}$*, sometimes called a *projective system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i' \leq i$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \mathrm{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i'' \leq i' \leq i$. (Note reversal of inequalities.)

We will say $(M_i, f_{ii'})$ is a (inverse) system over I to denote this. The maps $f_{ii'}$ are sometimes called the *transition maps*.

In other words a system over I is just a diagram $M : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category we associated to I above: objects are elements of I and there is a unique arrow $i \to i'$ in $\mathcal{I}$ if and only if $i \leq i'$. An inverse system is a diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$. From this point of view we could take (co)limits of any (inverse) system over I. However, it is customary to take *only colimits of systems over I* and *only limits of inverse systems over I*. More precisely: Given a system $(M_i, f_{ii'})$ over I the colimit of the system $(M_i, f_{ii'})$ is defined as

$$\operatorname{colim}_{i \in I} M_i = \operatorname{colim}_{\mathcal{I}} M,$$

i.e., as the colimit of the corresponding diagram. Given an inverse system $(M_i, f_{ii'})$ over I the limit of the inverse system $(M_i, f_{ii'})$ is defined as

$$\lim_{i \in I} M_i = \lim_{\mathcal{I}^{opp}} M,$$

i.e., as the limit of the corresponding diagram.

0CN1 **Remark 4.21.3.** Let I be a preordered set. From I we can construct a canonical partially ordered set $\overline{I}$ and an order preserving map $\pi : I \to \overline{I}$. Namely, we can define an equivalence relation $\sim$ on I by the rule

$$i \sim j \Leftrightarrow (i \leq j \text{ and } j \leq i).$$

We set $\overline{I} = I/\sim$ and we let $\pi : I \to \overline{I}$ be the quotient map. Finally, $\overline{I}$ comes with a unique partial ordering such that $\pi(i) \leq \pi(j) \Leftrightarrow i \leq j$. Observe that if I is a directed set, then $\overline{I}$ is a directed partially ordered set. Given an (inverse) system N over $\overline{I}$ we obtain an (inverse) system M over I by setting $M_i = N_{\pi(i)}$. This construction defines a functor between the category of inverse systems over I and $\overline{I}$. In fact, this is an equivalence. The reason is that if $i \sim j$, then for any system M over I the maps $M_i \to M_j$ and $M_j \to M_i$ are mutually inverse isomorphisms. More precisely, choosing a section $s : \overline{I} \to I$ of π a quasi-inverse of the functor above sends M to N with $N_{\bar{i}} = M_{s(\bar{i})}$. Finally, this correspondence is compatible with colimits of systems: if M and N are related as above and if either $\operatorname{colim}_{\overline{I}} N$ or $\operatorname{colim}_I M$ exists then so does the other and $\operatorname{colim}_{\overline{I}} N = \operatorname{colim}_I M$. Similar results hold for inverse systems and limits of inverse systems.

The upshot of Remark 4.21.3 is that while computing a colimit of a system or a limit of an inverse system, we may always assume the preorder is a partial order.

0031 **Definition 4.21.4.** Let I be a preordered set. We say a system (resp. inverse system) $(M_i, f_{ii'})$ is a *directed system* (resp. *directed inverse system*) if I is a directed set (Definition 4.21.1): I is nonempty and for all $i_1, i_2 \in I$ there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

In this case the colimit is sometimes (unfortunately) called the "direct limit". We will not use this last terminology. It turns out that diagrams over a filtered category are no more general than directed systems in the following sense.

0032 **Lemma 4.21.5.** *Let $\mathcal{I}$ be a filtered index category. There exist a directed set I and a system $(x_i, \varphi_{ii'})$ over I in $\mathcal{I}$ with the following properties:*

(1) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. If* $\operatorname{colim}_{i\in I} M(x_i)$ *exists then so does* $\operatorname{colim}_{\mathcal{I}} M$ *and the transformation*

$$\theta : \operatorname{colim}_{i\in I} M(x_i) \longrightarrow \operatorname{colim}_{\mathcal{I}} M$$

of Lemma 4.14.8 is an isomorphism.

(2) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$ in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding inverse system over I. If* $\lim_{i\in I} M(x_i)$ *exists then so does* $\lim_{\mathcal{I}^{opp}} M$ *and the transformation*

$$\theta : \lim_{\mathcal{I}^{opp}} M \longrightarrow \lim_{i\in I} M(x_i)$$

of Lemma 4.14.9 is an isomorphism.

Proof. As explained in the text following Definition 4.21.2, we may view preordered sets as categories and systems as functors. Throughout the proof, we will freely shift between these two points of view. We prove the first statement by constructing a category $\mathcal{I}_0$, corresponding to a directed set[2], and a cofinal functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$. Then, by Lemma 4.17.2, the colimit of a diagram $M : \mathcal{I} \to \mathcal{C}$ coincides with the colimit of the diagram $M \circ M_0 : \mathcal{I}_0 \to \mathcal{C}$, from which the statement follows. The second statement is dual to the first and may be proved by interpreting a limit in $\mathcal{C}$ as a colimit in $\mathcal{C}^{opp}$. We omit the details.

A category $\mathcal{F}$ is called *finitely generated* if there exists a finite set F of arrows in $\mathcal{F}$, such that each arrow in $\mathcal{F}$ may be obtained by composing arrows from F. In particular, this implies that $\mathcal{F}$ has finitely many objects. We start the proof by reducing to the case when $\mathcal{I}$ has the property that every finitely generated subcategory of $\mathcal{I}$ may be extended to a finitely generated subcategory with a unique final object.

Let ω denote the directed set of finite ordinals, which we view as a filtered category. It is easy to verify that the product category $\mathcal{I}\times\omega$ is also filtered, and the projection $\Pi : \mathcal{I} \times \omega \to \mathcal{I}$ is cofinal.

Now let $\mathcal{F}$ be any finitely generated subcategory of $\mathcal{I}\times\omega$. By using the axioms of a filtered category and a simple induction argument on a finite set of generators of $\mathcal{F}$, we may construct a cocone $(\{f_i\}, i_\infty)$ in $\mathcal{I}\times\omega$ for the diagram $\mathcal{F} \to \mathcal{I}\times\omega$. That is, a morphism $f_i : i \to i_\infty$ for every object i in $\mathcal{F}$ such that for each arrow $f : i \to i'$ in $\mathcal{F}$ we have $f_i = f_{i'} \circ f$. We can also choose i_∞ such that there are no arrows from i_∞ to an object in $\mathcal{F}$. This is possible since we may always post-compose the arrows f_i with an arrow which is the identity on the $\mathcal{I}$-component and strictly increasing on the ω-component. Now let $\mathcal{F}^+$ denote the category consisting of all objects and arrows in $\mathcal{F}$ together with the object i_∞, the identity arrow id_{i_∞} and the arrows f_i. Since there are no arrows from i_∞ in $\mathcal{F}^+$ to any object of $\mathcal{F}$, the arrow set in $\mathcal{F}^+$ is closed under composition, so $\mathcal{F}^+$ is indeed a category. By construction, it is a finitely generated subcategory of $\mathcal{I}$ which has i_∞ as unique final object. Since, by Lemma 4.17.2, the colimit of any diagram $M : \mathcal{I} \to \mathcal{C}$ coincides with the colimit of $M \circ \Pi$, this gives the desired reduction.

The set of all finitely generated subcategories of $\mathcal{I}$ with a unique final object is naturally ordered by inclusion. We take $\mathcal{I}_0$ to be the category corresponding to this set. We also have a functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$, which takes an arrow $\mathcal{F} \subset \mathcal{F}'$ in

[2]In fact, our construction will produce a directed partially ordered set.

$\mathcal{I}_0$ to the unique map from the final object of $\mathcal{F}$ to the final object of $\mathcal{F}'$. Given any two finitely generated subcategories of $\mathcal{I}$, the category generated by these two categories is also finitely generated. By our assumption on $\mathcal{I}$, it is also contained in a finitely generated subcategory of $\mathcal{I}$ with a unique final object. This shows that $\mathcal{I}_0$ is directed.

Finally, we verify that M_0 is cofinal. Since any object of $\mathcal{I}$ is the final object in the subcategory consisting of only that object and its identity arrow, the functor M_0 is surjective on objects. In particular, Condition (1) of Definition 4.17.1 is satisfied. Given an object i of $\mathcal{I}$, objects $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{I}_0$ and maps $\varphi_1 : i \to M_0(\mathcal{F}_1)$ and $\varphi_2 : i \to M_0(\mathcal{F}_2)$ in $\mathcal{I}$, we can take $\mathcal{F}_{12}$ to be a finitely generated category with a unique final object containing $\mathcal{F}_1$, $\mathcal{F}_2$ and the morphisms φ_1, φ_2. The resulting diagram commutes

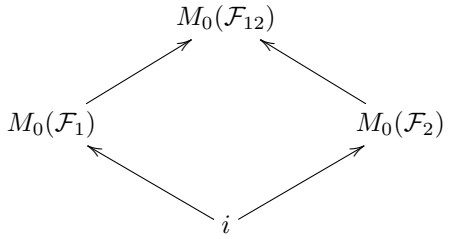

since it lives in the category $\mathcal{F}_{12}$ and $M_0(\mathcal{F}_{12})$ is final in this category. Hence also Condition (2) is satisfied, which concludes the proof. □

09P8 **Remark 4.21.6.** Note that a finite directed set $(I, \geq)$ always has a greatest object i_∞. Hence any colimit of a system $(M_i, f_{ii'})$ over such a set is trivial in the sense that the colimit equals M_{i_∞}. In contrast, a colimit indexed by a finite filtered category need not be trivial. For instance, let $\mathcal{I}$ be the category with a single object i and a single non-trivial morphism e satisfying $e = e \circ e$. The colimit of a diagram $M : \mathcal{I} \to Sets$ is the image of the idempotent $M(e)$. This illustrates that something like the trick of passing to $\mathcal{I} \times \omega$ in the proof of Lemma 4.21.5 is essential.

086J **Lemma 4.21.7.** *If $S : \mathcal{I} \to Sets$ is a cofiltered diagram of sets and all the S_i are finite nonempty, then $\lim_i S_i$ is nonempty. In other words, the limit of a directed inverse system of finite nonempty sets is nonempty.*

Proof. The two statements are equivalent by Lemma 4.21.5. Let I be a directed set and let $(S_i)_{i \in I}$ be an inverse system of finite nonempty sets over I. Let us say that a *subsystem* T is a family $T = (T_i)_{i \in I}$ of nonempty subsets $T_i \subset S_i$ such that $T_{i'}$ is mapped into T_i by the transition map $S_{i'} \to S_i$ for all $i' \geq i$. Denote $\mathcal{T}$ the set of subsystems. We order $\mathcal{T}$ by inclusion. Suppose T_α, $\alpha \in A$ is a totally ordered family of elements of $\mathcal{T}$. Say $T_\alpha = (T_{\alpha,i})_{i \in I}$. Then we can find a lower bound $T = (T_i)_{i \in I}$ by setting $T_i = \bigcap_{\alpha \in A} T_{\alpha,i}$ which is manifestly a finite nonempty subset of S_i as all the $T_{\alpha,i}$ are nonempty and as the T_α form a totally ordered family. Thus we may apply Zorn's lemma to see that $\mathcal{T}$ has minimal elements.

Let's analyze what a minimal element $T \in \mathcal{T}$ looks like. First observe that the maps $T_{i'} \to T_i$ are all surjective. Namely, as I is a directed set and T_i is finite, the intersection $T_i' = \bigcap_{i' \geq i} \mathrm{Im}(T_{i'} \to T_i)$ is nonempty. Thus $T' = (T_i')$ is a subsystem contained in T and by minimality $T' = T$. Finally, we claim that T_i is a singleton

for each i. Namely, if $x \in T_i$, then we can define $T'_{i'} = (T_{i'} \to T_i)^{-1}(\{x\})$ for $i' \geq i$ and $T'_j = T_j$ if $j \not\geq i$. This is another subsystem as we've seen above that the transition maps of the subsystem T are surjective. By minimality we see that $T = T'$ which indeed implies that T_i is a singleton. This holds for every $i \in I$, hence we see that $T_i = \{x_i\}$ for some $x_i \in S_i$ with $x_{i'} \mapsto x_i$ under the map $S_{i'} \to S_i$ for every $i' \geq i$. In other words, $(x_i) \in \lim S_i$ and the lemma is proved. □

4.22. Essentially constant systems

05PT Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Assume the index category $\mathcal{I}$ is filtered. In this case there are three successively stronger notions which pick out an object X of $\mathcal{C}$. The first is just

$$X = \operatorname{colim}_{i \in \mathcal{I}} M_i.$$

Then X comes equipped with the coprojections $M_i \to X$. A stronger condition would be to require that X is the colimit and that there exist an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that the composition $X \to M_i \to X$ is id_X. An even stronger condition is the following.

05PU **Definition 4.22.1.** Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.

(1) Assume the index category $\mathcal{I}$ is filtered and let $(X, \{M_i \to X\}_i)$ be a cocone for M, see Remark 4.14.5. We say M is *essentially constant* with *value* X if there exist an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $i \to k$ and $j \to k$ such that the morphism $M_j \to M_k$ equals the composition $M_j \to X \to M_i \to M_k$.

(2) Assume the index category $\mathcal{I}$ is cofiltered and let $(X, \{X \to M_i\}_i)$ be a cone for M, see Remark 4.14.5. We say M is *essentially constant* with *value* X if there exist an $i \in \mathcal{I}$ and a morphism $M_i \to X$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $k \to i$ and $k \to j$ such that the morphism $M_k \to M_j$ equals the composition $M_k \to M_i \to X \to M_j$.

Please keep in mind Lemma 4.22.3 when using this definition.

Which of the two versions is meant will be clear from context. If there is any confusion we will distinguish between these by saying that the first version means M is essentially constant as an *ind-object*, and in the second case we will say it is essentially constant as a *pro-object*. This terminology is further explained in Remarks 4.22.4 and 4.22.5. In fact we will often use the terminology "essentially constant system" which formally speaking is only defined for systems over directed sets.

05PV **Definition 4.22.2.** Let $\mathcal{C}$ be a category. A directed system $(M_i, f_{ii'})$ is an *essentially constant system* if M viewed as a functor $I \to \mathcal{C}$ defines an essentially constant diagram. A directed inverse system $(M_i, f_{ii'})$ is an *essentially constant inverse system* if M viewed as a functor $I^{opp} \to \mathcal{C}$ defines an essentially constant inverse diagram.

If $(M_i, f_{ii'})$ is an essentially constant system and the morphisms $f_{ii'}$ are monomorphisms, then for all $i \leq i'$ sufficiently large the morphisms $f_{ii'}$ are isomorphisms.

On the other hand, consider the system

$$\mathbf{Z}^2 \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to \ldots$$

with maps given by $(a, b) \mapsto (a + b, 0)$. This system is essentially constant with value $\mathbf{Z}$ but every transition map has a kernel.

Here is an example of a system which is not essentially constant. Let $M = \bigoplus_{n \geq 0} \mathbf{Z}$ and to let $S : M \to M$ be the shift operator $(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$. In this case the system $M \to M \to M \to \ldots$ with transition maps S has colimit 0 and the composition $0 \to M \to 0$ is the identity, but the system is not essentially constant.

The following lemma is a sanity check.

0G2V **Lemma 4.22.3.** *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. If $\mathcal{I}$ is filtered and M is essentially constant as an ind-object, then $X = \operatorname{colim} M_i$ exists and M is essentially constant with value X. If $\mathcal{I}$ is cofiltered and M is essentially constant as a pro-object, then $X = \lim M_i$ exists and M is essentially constant with value X.*

Proof. Omitted. This is a good exercise in the definitions. □

05PW **Remark 4.22.4.** Let $\mathcal{C}$ be a category. There exists a big category Ind-$\mathcal{C}$ of *ind-objects of* $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are filtered diagrams in $\mathcal{C}$, then we can define

$$\operatorname{Mor}_{\text{Ind-}\mathcal{C}}(F, G) = \lim_i \operatorname{colim}_j \operatorname{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to \text{Ind-}\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only if F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

05PX **Remark 4.22.5.** Let $\mathcal{C}$ be a category. There exists a big category Pro-$\mathcal{C}$ of *pro-objects* of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams in $\mathcal{C}$, then we can define

$$\operatorname{Mor}_{\text{Pro-}\mathcal{C}}(F, G) = \lim_j \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to \text{Pro-}\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only if F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

0G2W **Example 4.22.6.** Let $\mathcal{C}$ be a category. Let (X_n) and (Y_n) be inverse systems in $\mathcal{C}$ over $\mathbf{N}$ with the usual ordering. Picture:

$$\ldots \to X_3 \to X_2 \to X_1 \quad \text{and} \quad \ldots \to Y_3 \to Y_2 \to Y_1$$

Let $a : (X_n) \to (Y_n)$ be a morphism of pro-objects of $\mathcal{C}$. What does a amount to? Well, for each $n \in \mathbf{N}$ there should exist an $m(n)$ and a morphism $a_n : X_{m(n)} \to Y_n$. These morphisms ought to agree in the following sense: for all $n' \geq n$ there exists an $m(n', n) \geq m(n'), m(n)$ such that the diagram

$$\begin{array}{ccccc} X_{m(n,n')} & & \longrightarrow & & X_{m(n)} \\ \downarrow & & & & \downarrow{\scriptstyle a_n} \\ X_{m(n')} & \xrightarrow{a_{n'}} & Y_{n'} & \longrightarrow & Y_n \end{array}$$

commutes. After replacing $m(n)$ by $\max_{k,l\leq n}\{m(n,k), m(k,l)\}$ we see that we obtain $\ldots \geq m(3) \geq m(2) \geq m(1)$ and a commutative diagram

$$\begin{array}{ccccccc} \ldots & \longrightarrow & X_{m(3)} & \longrightarrow & X_{m(2)} & \longrightarrow & X_{m(1)} \\ & & \downarrow{\scriptstyle a_3} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_1} \\ \ldots & \longrightarrow & Y_3 & \longrightarrow & Y_2 & \longrightarrow & Y_1 \end{array}$$

Given an increasing map $m' : \mathbf{N} \to \mathbf{N}$ with $m' \geq m$ and setting $a'_i : X_{m'(i)} \to X_{m(i)} \to Y_i$ the pair (m', a') defines the same morphism of pro-systems. Conversely, given two pairs (m_1, a_1) and (m_1, a_2) as above then these define the same morphism of pro-objects if and only if we can find $m' \geq m_1, m_2$ such that $a'_1 = a'_2$.

0G2X **Remark 4.22.7.** Let $\mathcal{C}$ be a category. Let $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ be cofiltered diagrams in $\mathcal{C}$. Consider the functors $A, B : \mathcal{C} \to \mathit{Sets}$ defined by

$$A(X) = \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(F(i), X) \quad \text{and} \quad B(X) = \operatorname{colim}_j \operatorname{Mor}_{\mathcal{C}}(G(j), X)$$

We claim that a morphism of pro-systems from F to G is the same thing as a transformation of functors $t : B \to A$. Namely, given t we can apply t to the class of $\mathrm{id}_{G(j)}$ in $B(G(j))$ to get a compatible system of elements $\xi_j \in A(G(j)) = \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(F(i), G(j))$ which is exactly our definition of a morphism in Pro-$\mathcal{C}$ in Remark 4.22.5. We omit the construction of a transformation $B \to A$ given a morphism of pro-objects from F to G.

05SH **Lemma 4.22.8.** *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered (resp. cofiltered) index category $\mathcal{I}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If M is essentially constant as an ind-object (resp. pro-object), then so is $F \circ M : \mathcal{I} \to \mathcal{D}$.*

Proof. If X is a value for M, then it follows immediately from the definition that $F(X)$ is a value for $F \circ M$. □

05PY **Lemma 4.22.9.** *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered index category $\mathcal{I}$. The following are equivalent*

1. *M is an essentially constant ind-object,*
2. *there exists a cocone $(X, \{M_i \to X\}_i)$ such that for any W in $\mathcal{C}$ the map $\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \to \operatorname{Mor}_{\mathcal{C}}(W, X)$ is bijective,*
3. *$X = \operatorname{colim}_i M_i$ exists and for any W in $\mathcal{C}$ the map $\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \to \operatorname{Mor}_{\mathcal{C}}(W, X)$ is bijective, and*
4. *there exists an i in $\mathcal{I}$ and a morphism $X \to M_i$ such that for any W in $\mathcal{C}$ the map $\operatorname{Mor}_{\mathcal{C}}(W, X) \to \operatorname{colim}_{j\geq i} \operatorname{Mor}_{\mathcal{C}}(W, M_j)$ is bijective.*

In cases (2), (3), and (4) the value of the essentially constant system is X.

Proof. It is clear that (3) implies (2). Assume (2). Then $\mathrm{id}_X \in \operatorname{Mor}_{\mathcal{C}}(X, X)$ comes from a morphism $X \to M_i$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\operatorname{Mor}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(W, X)$$

are bijective for all W where the first one is induced by the morphism $X \to M_i$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_j \to X \to M_i \to M_k$ is equal to $M_j \to M_k$ for some k large enough. This proves (1) holds.

Assume (4). Let k be an object of $\mathcal{I}$. Setting $W = M_k$ we deduce there exists a unique morphism $M_k \to X$ such that there exists a j and morphisms $k \to j$ and $i \to j$ in $\mathcal{I}$ such that $M_k \to X \to M_i \to M_j$ is equal to $M_k \to M_j$. The uniqueness guarantees that we obtain a cocone $(X, \{M_k \to X\})$. In this way we see that (4) implies (2); some details omitted.

We omit the proof that (1) implies the other conditions. □

05PZ **Lemma 4.22.10.** *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with cofiltered index category $\mathcal{I}$. The following are equivalent*

(1) *M is an essentially constant pro-object,*
(2) *there exists a cone $(X, \{X \to M_i\})$ such that for any W in $\mathcal{C}$ the map $\mathrm{colim}_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_i, W) \to \mathrm{Mor}_{\mathcal{C}}(X, W)$ is bijective,*
(3) *$X = \lim_i M_i$ exists and for any W in $\mathcal{C}$ the map $\mathrm{colim}_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_i, W) \to \mathrm{Mor}_{\mathcal{C}}(X, W)$ is bijective, and*
(4) *there exists an i in $\mathcal{I}$ and a morphism $M_i \to X$ such that for any W in $\mathcal{C}$ the map $\mathrm{Mor}_{\mathcal{C}}(X, W) \to \mathrm{colim}_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_i, W)$ is bijective.*

In cases (2), (3), and (4) the value of the essentially constant system is X.

Proof. This lemma is dual to Lemma 4.22.9. □

0A1S **Lemma 4.22.11.** *Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of filtered index categories. If H is cofinal, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.*

Proof. This follows formally from Lemmas 4.22.9 and 4.17.2. □

0A2C **Lemma 4.22.12.** *Let $\mathcal{I}$ and $\mathcal{J}$ be filtered categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. Then $\mathcal{I} \times \mathcal{J}$ is filtered and a diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ p : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ is essentially constant.*

Proof. We omit the verification that $\mathcal{I} \times \mathcal{J}$ is filtered. The equivalence follows from Lemma 4.22.11 because p is cofinal (verification omitted). □

0A1T **Lemma 4.22.13.** *Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of cofiltered index categories. If H is initial, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.*

Proof. This follows formally from Lemmas 4.22.10, 4.17.4, 4.17.2, and the fact that if $\mathcal{I}$ is initial in $\mathcal{J}$, then $\mathcal{I}^{opp}$ is cofinal in $\mathcal{J}^{opp}$. □

4.23. Exact functors

0033 In this section we define exact functors.

0034 **Definition 4.23.1.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) Suppose all finite limits exist in $\mathcal{A}$. We say F is *left exact* if it commutes with all finite limits.
(2) Suppose all finite colimits exist in $\mathcal{A}$. We say F is *right exact* if it commutes with all finite colimits.
(3) We say F is *exact* if it is both left and right exact.

0035 **Lemma 4.23.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$, see Lemma 4.18.4. The following are equivalent:*

(1) *F is left exact,*
(2) *F commutes with finite products and equalizers, and*
(3) *F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.*

Proof. Lemma 4.14.11 shows that (2) implies (1). Suppose (3) holds. The fibre product over the final object is the product. If $a, b : A \to B$ are morphisms of $\mathcal{A}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1,\mathrm{pr}_2),A\times A,\Delta} A.$$

Thus (3) implies (2). Finally (1) implies (3) because the empty limit is a final object, and fibre products are limits. □

0GMN **Lemma 4.23.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite colimits exist in $\mathcal{A}$, see Lemma 4.18.7. The following are equivalent:*

(1) *F is right exact,*
(2) *F commutes with finite coproducts and coequalizers, and*
(3) *F transforms an initial object of $\mathcal{A}$ into an initial object of $\mathcal{B}$, and commutes with pushouts.*

Proof. Dual to Lemma 4.23.2. □

4.24. Adjoint functors

0036

0037 **Definition 4.24.1.** Let $\mathcal{C}$, $\mathcal{D}$ be categories. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors. We say that u is a *left adjoint* of v, or that v is a *right adjoint* to u if there are bijections

$$\mathrm{Mor}_{\mathcal{D}}(u(X), Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, v(Y))$$

functorial in $X \in \mathrm{Ob}(\mathcal{C})$, and $Y \in \mathrm{Ob}(\mathcal{D})$.

In other words, this means that there is a *given* isomorphism of functors $\mathcal{C}^{opp} \times \mathcal{D} \to Sets$ from $\mathrm{Mor}_{\mathcal{D}}(u(-), -)$ to $\mathrm{Mor}_{\mathcal{C}}(-, v(-))$. For any object X of $\mathcal{C}$ we obtain a morphism $X \to v(u(X))$ corresponding to $\mathrm{id}_{u(X)}$. Similarly, for any object Y of $\mathcal{D}$ we obtain a morphism $u(v(Y)) \to Y$ corresponding to $\mathrm{id}_{v(Y)}$. These maps are called the *adjunction maps*. The adjunction maps are functorial in X and Y, hence we obtain morphisms of functors

$$\eta : \mathrm{id}_{\mathcal{C}} \to v \circ u \quad \text{(unit)} \quad \text{and} \quad \epsilon : u \circ v \to \mathrm{id}_{\mathcal{D}} \quad \text{(counit)}.$$

Moreover, if $\alpha : u(X) \to Y$ and $\beta : X \to v(Y)$ are morphisms, then the following are equivalent

(1) α and β correspond to each other via the bijection of the definition,
(2) β is the composition $X \to v(u(X)) \xrightarrow{v(\alpha)} v(Y)$, and
(3) α is the composition $u(X) \xrightarrow{u(\beta)} u(v(Y)) \to Y$.

In this way one can reformulate the notion of adjoint functors in terms of adjunction maps.

0A8B **Lemma 4.24.2.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If for each $y \in \mathrm{Ob}(\mathcal{D})$ the functor $x \mapsto \mathrm{Mor}_{\mathcal{D}}(u(x), y)$ is representable, then u has a right adjoint.*

Proof. For each y choose an object $v(y)$ and an isomorphism $\mathrm{Mor}_{\mathcal{C}}(-, v(y)) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y)$ of functors. By Yoneda's lemma (Lemma 4.3.5) for any morphism $g : y \to y'$ the transformation of functors

$$\mathrm{Mor}_{\mathcal{C}}(-, v(y)) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y') \to \mathrm{Mor}_{\mathcal{C}}(-, v(y'))$$

corresponds to a unique morphism $v(g) : v(y) \to v(y')$. We omit the verification that v is a functor and that it is right adjoint to u. □

0FWV **Lemma 4.24.3.** *Let u be a left adjoint to v as in Definition 4.24.1.* Bhargav Bhatt, private communication.

(1) *If $v \circ u$ is fully faithful, then u is fully faithful.*
(2) *If $u \circ v$ is fully faithful, then v is fully faithful.*

Proof. Proof of (2). Assume $u \circ v$ is fully faithful. Say we have X, Y in $\mathcal{D}$. Then the natural composite map

$$\mathrm{Mor}(X, Y) \to \mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y)))$$

is a bijection, so v is at least faithful. To show full faithfulness, we must show that the second map above is injective. But the adjunction between u and v says that

$$\mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y))) \to \mathrm{Mor}(u(v(X)), Y)$$

is a bijection, where the first map is natural one and the second map comes from the counit $u(v(Y)) \to Y$ of the adjunction. So this says that $\mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y)))$ is also injective, as wanted. The proof of (1) is dual to this. □

07RB **Lemma 4.24.4.** *Let u be a left adjoint to v as in Definition 4.24.1. Then*

(1) *u is fully faithful $\Leftrightarrow$ $id \cong v \circ u \Leftrightarrow \eta : id \to v \circ u$ is an isomorphism,*
(2) *v is fully faithful $\Leftrightarrow$ $u \circ v \cong id \Leftrightarrow \epsilon : u \circ v \to id$ is an isomorphism.*

Proof. Proof of (1). Assume u is fully faithful. We will show $\eta_X : X \to v(u(X))$ is an isomorphism. Let $X' \to v(u(X))$ be any morphism. By adjointness this corresponds to a morphism $u(X') \to u(X)$. By fully faithfulness of u this corresponds to a unique morphism $X' \to X$. Thus we see that post-composing by η_X defines a bijection $\mathrm{Mor}(X', X) \to \mathrm{Mor}(X', v(u(X)))$. Hence η_X is an isomorphism. If there exists an isomorphism $\mathrm{id} \cong v \circ u$ of functors, then $v \circ u$ is fully faithful. By Lemma 4.24.3 we see that u is fully faithful. By the above this implies η is an isomorphism. Thus all 3 conditions are equivalent (and these conditions are also equivalent to $v \circ u$ being fully faithful).

Part (2) is dual to part (1). □

0038 **Lemma 4.24.5.** *Let u be a left adjoint to v as in Definition 4.24.1.*

(1) *Suppose that $M : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\mathrm{colim}_{\mathcal{I}} M$ exists in $\mathcal{C}$. Then $u(\mathrm{colim}_{\mathcal{I}} M) = \mathrm{colim}_{\mathcal{I}} u \circ M$. In other words, u commutes with (representable) colimits.*
(2) *Suppose that $M : \mathcal{I} \to \mathcal{D}$ is a diagram, and suppose that $\lim_{\mathcal{I}} M$ exists in $\mathcal{D}$. Then $v(\lim_{\mathcal{I}} M) = \lim_{\mathcal{I}} v \circ M$. In other words v commutes with representable limits.*

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, see Remark 4.14.4. So

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(u(\mathrm{colim}_{i\in\mathcal{I}} M_i), Y) &= \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_{i\in\mathcal{I}} M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{D}}(u(M_i), Y) \end{aligned}$$

proves that $u(\mathrm{colim}_{i\in\mathcal{I}} M_i)$ is the colimit we are looking for. A similar argument works for the other statement. □

0039 **Lemma 4.24.6.** *Let u be a left adjoint of v as in Definition 4.24.1.*

(1) *If $\mathcal{C}$ has finite colimits, then u is right exact.*
(2) *If $\mathcal{D}$ has finite limits, then v is left exact.*

Proof. Obvious from the definitions and Lemma 4.24.5. □

0GLL **Lemma 4.24.7.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a left adjoint to the functor $v : \mathcal{D} \to \mathcal{C}$. Let $\eta_X : X \to v(u(X))$ be the unit and $\epsilon_Y : u(v(Y)) \to Y$ be the counit. Then*

$$u(X) \xrightarrow{u(\eta_X)} u(v(u(X))) \xrightarrow{\epsilon_{u(X)}} u(X) \quad \text{and} \quad v(Y) \xrightarrow{\eta_{v(Y)}} v(u(v(Y))) \xrightarrow{v(\epsilon_Y)} v(Y)$$

are the identity morphisms.

Proof. Omitted. □

0B65 **Lemma 4.24.8.** *Let $u_1, u_2 : \mathcal{C} \to \mathcal{D}$ be functors with right adjoints $v_1, v_2 : \mathcal{D} \to \mathcal{C}$. Let $\beta : u_2 \to u_1$ be a transformation of functors. Let $\beta^\vee : v_1 \to v_2$ be the corresponding transformation of adjoint functors. Then*

$$\begin{array}{ccc} u_2 \circ v_1 & \xrightarrow[\beta]{} & u_1 \circ v_1 \\ {\scriptstyle \beta^\vee}\downarrow & & \downarrow \\ u_2 \circ v_2 & \longrightarrow & id \end{array}$$

is commutative where the unlabeled arrows are the counit transformations.

Proof. This is true because $\beta^\vee_D : v_1 D \to v_2 D$ is the unique morphism such that the induced maps $\mathrm{Mor}(C, v_1 D) \to \mathrm{Mor}(C, v_2 D)$ is the map $\mathrm{Mor}(u_1 C, D) \to \mathrm{Mor}(u_2 C, D)$ induced by $\beta_C : u_2 C \to u_1 C$. Namely, this means the map

$$\mathrm{Mor}(u_1 v_1 D, D') \to \mathrm{Mor}(u_2 v_1 D, D')$$

induced by $\beta_{v_1 D}$ is the same as the map

$$\mathrm{Mor}(v_1 D, v_1 D') \to \mathrm{Mor}(v_1 D, v_2 D')$$

induced by $\beta^\vee_{D'}$. Taking $D' = D$ we find that the counit $u_1 v_1 D \to D$ precomposed by $\beta_{v_1 D}$ corresponds to $\beta^\vee_D$ under adjunction. This exactly means that the diagram commutes when evaluated on D. □

0DV0 **Lemma 4.24.9.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $v : \mathcal{A} \to \mathcal{B}$ and $v' : \mathcal{B} \to \mathcal{C}$ be functors with left adjoints u and u' respectively. Then*

(1) *The functor $v'' = v' \circ v$ has a left adjoint equal to $u'' = u \circ u'$.*
(2) *Given X in $\mathcal{A}$ we have*

0DV1 (4.24.9.1) $$\epsilon^v_X \circ u(\epsilon^{v'}_{v(X)}) = \epsilon^{v''}_X : u''(v''(X)) \to X$$

Where ϵ is the counit of the adjunctions.

Proof. Let us unwind the formula in (2) because this will also immediately prove (1). First, the counit of the adjunctions for the pairs (u, v) and (u', v') are maps $\epsilon^v_X : u(v(X)) \to X$ and $\epsilon^{v'}_Y : u'(v'(Y)) \to Y$, see discussion following Definition 4.24.1. With u'' and v'' as in (1) we unwind everything

$$u''(v''(X)) = u(u'(v'(v(X)))) \xrightarrow{u(\epsilon^{v'}_{v(X)})} u(v(X)) \xrightarrow{\epsilon^v_X} X$$

to get the map on the left hand side of (4.24.9.1). Let us denote this by $\epsilon^{v''}_X$ for now. To see that this is the counit of an adjoint pair (u'', v'') we have to show that given Z in $\mathcal{C}$ the rule that sends a morphism $\beta : Z \to v''(X)$ to $\alpha = \epsilon^{v''}_X \circ u''(\beta) : u''(Z) \to X$ is a bijection on sets of morphisms. This is true because, this is the composition of the rule sending β to $\epsilon^{v'}_{v(X)} \circ u'(\beta)$ which is a bijection by assumption on (u', v') and then sending this to $\epsilon^v_X \circ u(\epsilon^{v'}_{v(X)} \circ u'(\beta))$ which is a bijection by assumption on (u, v). □

4.25. A criterion for representability

0AHM The following lemma is often useful to prove the existence of universal objects in big categories, please see the discussion in Remark 4.25.2.

0AHN **Lemma 4.25.1.** *Let $\mathcal{C}$ be a big[3] category which has limits. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

(1) *F commutes with limits,*
(2) *there exist a family $\{x_i\}_{i \in I}$ of objects of $\mathcal{C}$ and for each $i \in I$ an element $f_i \in F(x_i)$ such that for $y \in \mathrm{Ob}(\mathcal{C})$ and $g \in F(y)$ there exist an i and a morphism $\varphi : x_i \to y$ with $F(\varphi)(f_i) = g$.*

Then F is representable, i.e., there exists an object x of $\mathcal{C}$ such that

$$F(y) = \mathrm{Mor}_{\mathcal{C}}(x, y)$$

functorially in y.

Proof. Let $\mathcal{I}$ be the category whose objects are the pairs (x_i, f_i) and whose morphisms $(x_i, f_i) \to (x_{i'}, f_{i'})$ are maps $\varphi : x_i \to x_{i'}$ in $\mathcal{C}$ such that $F(\varphi)(f_i) = f_{i'}$. Set

$$x = \lim_{(x_i, f_i) \in \mathcal{I}} x_i$$

(this will not be the x we are looking for, see below). The limit exists by assumption. As F commutes with limits we have

$$F(x) = \lim_{(x_i, f_i) \in \mathcal{I}} F(x_i).$$

Hence there is a universal element $f \in F(x)$ which maps to $f_i \in F(x_i)$ under F applied to the projection map $x \to x_i$. Using f we obtain a transformation of functors

$$\xi : \mathrm{Mor}_{\mathcal{C}}(x, -) \longrightarrow F(-)$$

see Section 4.3. Let y be an arbitrary object of $\mathcal{C}$ and let $g \in F(y)$. Choose $x_i \to y$ such that f_i maps to g which is possible by assumption. Then F applied to the maps

$$x \longrightarrow x_i \longrightarrow y$$

[3] See Remark 4.2.2.

(the first being the projection map of the limit defining x) sends f to g. Hence the transformation ξ is surjective.

In order to find the object representing F we let $e : x' \to x$ be the equalizer of all self maps $\varphi : x \to x$ with $F(\varphi)(f) = f$. Since F commutes with limits, it commutes with equalizers, and we see there exists an $f' \in F(x')$ mapping to f in $F(x)$. Since ξ is surjective and since f' maps to f we see that also $\xi' : \mathrm{Mor}_{\mathcal{C}}(x', -) \to F(-)$ is surjective. Finally, suppose that $a, b : x' \to y$ are two maps such that $F(a)(f') = F(b)(f')$. We have to show $a = b$. Consider the equalizer $e' : x'' \to x'$. Again we find $f'' \in F(x'')$ mapping to f'. Choose a map $\psi : x \to x''$ such that $F(\psi)(f) = f''$. Then we see that $e \circ e' \circ \psi : x \to x$ is a morphism with $F(e \circ e' \circ \psi)(f) = f$. Hence $e \circ e' \circ \psi \circ e = e$. Since e is a monomorphism, this implies that e' is an epimorphism, thus $a = b$ as desired. □

0AHP **Remark 4.25.2.** The lemma above is often used to construct the free something on something. For example the free abelian group on a set, the free group on a set, etc. The idea, say in the case of the free group on a set E is to consider the functor

$$F : \mathit{Groups} \to \mathit{Sets}, \quad G \longmapsto \mathrm{Map}(E, G)$$

This functor commutes with limits. As our family of objects we can take a family $E \to G_i$ consisting of groups G_i of cardinality at most $\max(\aleph_0, |E|)$ and set maps $E \to G_i$ such that every isomorphism class of such a structure occurs at least once. Namely, if $E \to G$ is a map from E to a group G, then the subgroup G' generated by the image has cardinality at most $\max(\aleph_0, |E|)$. The lemma tells us the functor is representable, hence there exists a group F_E such that $\mathrm{Mor}_{\mathit{Groups}}(F_E, G) = \mathrm{Map}(E, G)$. In particular, the identity morphism of F_E corresponds to a map $E \to F_E$ and one can show that F_E is generated by the image without imposing any relations.

Another typical application is that we can use the lemma to construct colimits once it is known that limits exist. We illustrate it using the category of topological spaces which has limits by Topology, Lemma 5.14.1. Namely, suppose that $\mathcal{I} \to \mathit{Top}$, $i \mapsto X_i$ is a functor. Then we can consider

$$F : \mathit{Top} \longrightarrow \mathit{Sets}, \quad Y \longmapsto \lim_{\mathcal{I}} \mathrm{Mor}_{\mathit{Top}}(X_i, Y)$$

This functor commutes with limits. Moreover, given any topological space Y and an element $(\varphi_i : X_i \to Y)$ of $F(Y)$, there is a subspace $Y' \subset Y$ of cardinality at most $|\coprod X_i|$ such that the morphisms φ_i map into Y'. Namely, we can take the induced topology on the union of the images of the φ_i. Thus it is clear that the hypotheses of the lemma are satisfied and we find a topological space X representing the functor F, which precisely means that X is the colimit of the diagram $i \mapsto X_i$.

0AHQ **Theorem 4.25.3** (Adjoint functor theorem)**.** *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of big categories. Assume $\mathcal{C}$ has limits, G commutes with them, and for every object y of $\mathcal{D}$ there exists a set of pairs $(x_i, f_i)_{i \in I}$ with $x_i \in \mathrm{Ob}(\mathcal{C})$, $f_i \in \mathrm{Mor}_{\mathcal{D}}(y, G(x_i))$ such that for any pair (x, f) with $x \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{D}}(y, G(x))$ there are an i and a morphism $h : x_i \to x$ such that $f = G(h) \circ f_i$. Then G has a left adjoint F.*

Proof. The assumptions imply that for every object y of $\mathcal{D}$ the functor $x \mapsto \mathrm{Mor}_{\mathcal{D}}(y, G(x))$ satisfies the assumptions of Lemma 4.25.1. Thus it is representable by an object, let's call it $F(y)$. An application of Yoneda's lemma (Lemma 4.3.5)

turns the rule $y \mapsto F(y)$ into a functor which by construction is an adjoint to G. We omit the details. □

4.26. Categorically compact objects

0FWW A little bit about "small" objects of a category.

0FWX **Definition 4.26.1.** Let $\mathcal{C}$ be a big[4] category. An object X of $\mathcal{C}$ is called a *categorically compact* if we have

$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{colim}_i M_i) = \mathrm{colim}_i \mathrm{Mor}_{\mathcal{C}}(X, M_i)$$

for every filtered diagram $M : \mathcal{I} \to \mathcal{C}$ such that $\mathrm{colim}_i M_i$ exists.

Often this definition is made only under the assumption that $\mathcal{C}$ has all filtered colimits.

0FWY **Lemma 4.26.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be big categories having filtered colimits. Let $\mathcal{C}' \subset \mathcal{C}$ be a small full subcategory consisting of categorically compact objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is a filtered colimit of objects of $\mathcal{C}'$. Then every functor $F' : \mathcal{C}' \to \mathcal{D}$ has a unique extension $F : \mathcal{C} \to \mathcal{D}$ commuting with filtered colimits.*

Proof. For every object X of $\mathcal{C}$ we may write X as a filtered colimit $X = \mathrm{colim}\, X_i$ with $X_i \in \mathrm{Ob}(\mathcal{C}')$. Then we set

$$F(X) = \mathrm{colim}\, F'(X_i)$$

in $\mathcal{D}$. We will show below that this construction does not depend on the choice of the colimit presentation of X.

Suppose given a morphism $\alpha : X \to Y$ of $\mathcal{C}$ and $X = \mathrm{colim}_{i \in I} X_i$ and $Y = \mathrm{colim}_{j \in J} Y_i$ are written as filtered colimit of objects in $\mathcal{C}'$. For each $i \in I$ since X_i is a categorically compact object of $\mathcal{C}$ we can find a $j \in J$ and a commutative diagram

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ \downarrow & & \downarrow \alpha \\ Y_j & \longrightarrow & Y \end{array}$$

Then we obtain a morphism $F'(X_i) \to F'(Y_j) \to F(Y)$ where the second morphism is the coprojection into $F(Y) = \mathrm{colim}\, F'(Y_j)$. The arrow $\beta_i : F'(X_i) \to F(Y)$ does not depend on the choice of j. For $i \leq i'$ the composition

$$F'(X_i) \to F'(X_{i'}) \xrightarrow{\beta_{i'}} F(Y)$$

is equal to β_i. Thus we obtain a well defined arrow

$$F(\alpha) : F(X) = \mathrm{colim}\, F(X_i) \to F(Y)$$

by the universal property of the colimit. If $\alpha' : Y \to Z$ is a second morphism of $\mathcal{C}$ and $Z = \mathrm{colim}\, Z_k$ is also written as filtered colimit of objects in $\mathcal{C}'$, then it is a pleasant exercise to show that the induced morphisms $F(\alpha) : F(X) \to F(Y)$ and $F(\alpha') : F(Y) \to F(Z)$ compose to the morphism $F(\alpha' \circ \alpha)$. Details omitted.

In particular, if we are given two presentations $X = \mathrm{colim}\, X_i$ and $X = \mathrm{colim}\, X'_{i'}$ as filtered colimits of systems in $\mathcal{C}'$, then we get mutually inverse arrows $\mathrm{colim}\, F'(X_i) \to \mathrm{colim}\, F'(X'_{i'})$ and $\mathrm{colim}\, F'(X'_{i'}) \to \mathrm{colim}\, F'(X_i)$. In other words, the value $F(X)$ is

[4] See Remark 4.2.2.

well defined independent of the choice of the presentation of X as a filtered colimit of objects of $\mathcal{C}'$. Together with the functoriality of F discussed in the previous paragraph, we find that F is a functor. Also, it is clear that $F(X) = F'(X)$ if $X \in \mathrm{Ob}(\mathcal{C}')$.

The uniqueness statement in the lemma is clear, provided we show that F commutes with filtered colimits (because this statement doesn't make sense otherwise). To show this, suppose that $X = \mathrm{colim}_{\lambda\in\Lambda} X_\lambda$ is a filtered colimit of $\mathcal{C}$. Since F is a functor we certainly get a map

$$\mathrm{colim}_\lambda F(X_\lambda) \longrightarrow F(X)$$

On the other hand, write $X = \mathrm{colim}\, X_i$ as a filtered colimit of objects of $\mathcal{C}'$. As above, for each $i \in I$ we can choose a $\lambda \in \Lambda$ and a commutative diagram

$$\begin{array}{ccc} X_i & \longrightarrow & X_\lambda \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

As above this determines a well defined morphism $F'(X_i) \to \mathrm{colim}_\lambda F(X_\lambda)$ compatible with transition morphisms and hence a morphism

$$F(X) = \mathrm{colim}_i F'(X_i) \longrightarrow \mathrm{colim}_\lambda F(X_\lambda)$$

This morphism is inverse to the morphism above (details omitted) and proves that $F(X) = \mathrm{colim}_\lambda F(X_\lambda)$ as desired. □

4.27. Localization in categories

04VB The basic idea of this section is given a category $\mathcal{C}$ and a set of arrows S to construct a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that all elements of S become invertible in $S^{-1}\mathcal{C}$ and such that F is universal among all functors with this property. References for this section are [**GZ67**, Chapter I, Section 2] and [**Ver96**, Chapter II, Section 2].

04VC **Definition 4.27.1.** Let $\mathcal{C}$ be a category. A set of arrows S of $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties:

LMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

LMS2 Every solid diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle t}\downarrow & & \vdots\,{\scriptstyle s} \\ Z & \overset{f}{\dashrightarrow} & W \end{array}$$

with $t \in S$ can be completed to a commutative dotted square with $s \in S$.

LMS3 For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$ there exists an $s \in S$ with source Y such that $s \circ f = s \circ g$.

A set of arrows S of $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties:

RMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

RMS2 Every solid diagram

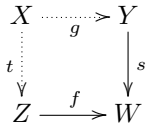

with $s \in S$ can be completed to a commutative dotted square with $t \in S$.

RMS3 For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.

A set of arrows S of $\mathcal{C}$ is called a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system. In other words, this means that MS1, MS2, MS3 hold, where MS1 = LMS1 + RMS1, MS2 = LMS2 + RMS2, and MS3 = LMS3 + RMS3. (That said, of course LMS1 = RMS1 = MS1.)

These conditions are useful to construct the categories $S^{-1}\mathcal{C}$ as follows.

Left calculus of fractions. Let $\mathcal{C}$ be a category and let S be a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.27.2):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X \to Y', s : Y \to Y')$ with $s \in S$ up to equivalence. (The equivalence is defined below. Think of the equivalence class of a pair (f, s) as $s^{-1}f : X \to Y$.)
(3) Two pairs $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exist a third pair $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ of $\mathcal{C}$ fitting into the commutative diagram

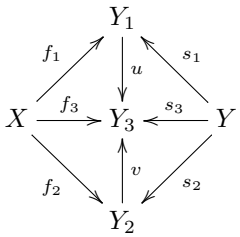

(4) The composition of the equivalence classes of the pairs $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$ where h and $u \in S$ are chosen to fit into a commutative diagram

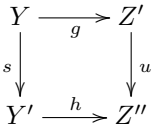

which exists by assumption.
(5) The identity morphism $X \to X$ in $S^{-1}\mathcal{C}$ is the equivalence class of the pair $(\mathrm{id} : X \to X, \mathrm{id} : X \to X)$.

04VD **Lemma 4.27.2.** *Let $\mathcal{C}$ be a category and let S be a left multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*

(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative (and the identity morphisms satisfy the identity axioms), and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. Proof of (1). Let us say two pairs $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are elementary equivalent if there exists a morphism $a : Y_1 \to Y_2$ of $\mathcal{C}$ such that $a \circ f_1 = f_2$ and $a \circ s_1 = s_2$. Diagram:

$$\begin{array}{ccccc} X & \xrightarrow[f_1]{} & Y_1 & \xleftarrow[s_1]{} & Y \\ \| & & \downarrow{\scriptstyle a} & & \| \\ X & \xrightarrow[f_2]{} & Y_2 & \xleftarrow[s_2]{} & Y \end{array}$$

Let us denote this property by saying $p_1 E p_2$. Note that pEp and $aEb, bEc \Rightarrow aEc$. (Despite its name, E is not an equivalence relation.) Part (1) claims that the relation $p \sim p' \Leftrightarrow \exists q : pEq \wedge p'Eq$ (where q is supposed to be a pair satisfying the same conditions as p and p') is an equivalence relation. A simple formal argument, using the properties of E above, shows that it suffices to prove $p_3 E p_1, p_3 E p_2 \Rightarrow p_1 \sim p_2$. Thus suppose that we are given a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & {\scriptstyle f_1}\nearrow & \uparrow{\scriptstyle a_{31}} & \nwarrow{\scriptstyle s_1} & \\ X & \xrightarrow[f_3]{} & Y_3 & \xleftarrow[s_3]{} & Y \\ & {\scriptstyle f_2}\searrow & \downarrow{\scriptstyle a_{32}} & \swarrow{\scriptstyle s_2} & \\ & & Y_2 & & \end{array}$$

with $s_i \in S$. First we apply LMS2 to get a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow[s_2]{} & Y_2 \\ {\scriptstyle s_1}\downarrow & & \vdots{\scriptstyle a_{24}} \\ Y_1 & \xrightarrow{a_{14}} & Y_4 \end{array}$$

with $a_{24} \in S$. Then, we have

$$a_{14} \circ a_{31} \circ s_3 = a_{14} \circ s_1 = a_{24} \circ s_2 = a_{24} \circ a_{32} \circ s_3.$$

Hence, by LMS3, there exists a morphism $s_{44} : Y_4 \to Y_4'$ such that $s_{44} \in S$ and $s_{44} \circ a_{14} \circ a_{31} = s_{44} \circ a_{24} \circ a_{32}$. Hence, after replacing Y_4, a_{14} and a_{24} by Y_4', $s_{44} \circ a_{14}$ and $s_{44} \circ a_{24}$, we may assume that $a_{14} \circ a_{31} = a_{24} \circ a_{32}$ (and we still have $a_{24} \in S$ and $a_{14} \circ s_1 = a_{24} \circ s_2$). Set

$$f_4 = a_{14} \circ f_1 = a_{14} \circ a_{31} \circ f_3 = a_{24} \circ a_{32} \circ f_3 = a_{24} \circ f_2$$

and $s_4 = a_{14} \circ s_1 = a_{24} \circ s_2$. Then, the diagram

$$\begin{array}{ccccc} X & \xrightarrow[f_1]{} & Y_1 & \xleftarrow[s_1]{} & Y \\ \| & & \downarrow{\scriptstyle a_{14}} & & \| \\ X & \xrightarrow[f_4]{} & Y_4 & \xleftarrow[s_4]{} & Y \end{array}$$

commutes, and we have $s_4 \in S$ (by LMS1). Thus, $p_1 E p_4$, where $p_4 = (f_4, s_4)$. Similarly, $p_2 E p_4$. Combining these, we find $p_1 \sim p_2$.

Proof of (2). Let $p = (f : X \to Y', s : Y \to Y')$ and $q = (g : Y \to Z', t : Z \to Z')$ be pairs as in the definition of composition above. To compose we choose a diagram

$$\begin{array}{ccc} Y & \xrightarrow[g]{} & Z' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle u_2} \\ Y' & \xrightarrow{h_2} & Z_2 \end{array}$$

with $u_2 \in S$. We first show that the equivalence class of the pair $r_2 = (h_2 \circ f : X \to Z_2, u_2 \circ t : Z \to Z_2)$ is independent of the choice of (Z_2, h_2, u_2). Namely, suppose that (Z_3, h_3, u_3) is another choice with corresponding composition $r_3 = (h_3 \circ f : X \to Z_3, u_3 \circ t : Z \to Z_3)$. Then by LMS2 we can choose a diagram

$$\begin{array}{ccc} Z' & \xrightarrow[u_3]{} & Z_3 \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle u_{34}} \\ Z_2 & \xrightarrow{h_{24}} & Z_4 \end{array}$$

with $u_{34} \in S$. We have $h_2 \circ s = u_2 \circ g$ and similarly $h_3 \circ s = u_3 \circ g$. Now,

$$u_{34} \circ h_3 \circ s = u_{34} \circ u_3 \circ g = h_{24} \circ u_2 \circ g = h_{24} \circ h_2 \circ s.$$

Hence, LMS3 shows that there exist a Z_4' and an $s_{44} : Z_4 \to Z_4'$ such that $s_{44} \circ u_{34} \circ h_3 = s_{44} \circ h_{24} \circ h_2$. Replacing Z_4, h_{24} and u_{34} by Z_4', $s_{44} \circ h_{24}$ and $s_{44} \circ u_{34}$, we may assume that $u_{34} \circ h_3 = h_{24} \circ h_2$. Meanwhile, the relations $u_{34} \circ u_3 = h_{24} \circ u_2$ and $u_{34} \in S$ continue to hold. We can now set $h_4 = u_{34} \circ h_3 = h_{24} \circ h_2$ and $u_4 = u_{34} \circ u_3 = h_{24} \circ u_2$. Then, we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow[h_2 \circ f]{} & Z_2 & \xleftarrow[u_2 \circ t]{} & Z \\ \| & & \downarrow{\scriptstyle h_{24}} & & \| \\ X & \xrightarrow{h_4 \circ f} & Z_4 & \xleftarrow{u_4 \circ t} & Z \\ \| & & \uparrow{\scriptstyle u_{34}} & & \| \\ X & \xrightarrow[h_3 \circ f]{} & Z_3 & \xleftarrow[u_3 \circ t]{} & Z \end{array}$$

Hence we obtain a pair $r_4 = (h_4 \circ f : X \to Z_4, u_4 \circ t : Z \to Z_4)$ and the above diagram shows that we have $r_2 E r_4$ and $r_3 E r_4$, whence $r_2 \sim r_3$, as desired. Thus it now makes sense to define $p \circ q$ as the equivalence class of all possible pairs r obtained as above.

To finish the proof of (2) we have to show that given pairs p_1, p_2, q such that $p_1 E p_2$ then $p_1 \circ q = p_2 \circ q$ and $q \circ p_1 = q \circ p_2$ whenever the compositions make sense. To do this, write $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ and let $a : Y_1 \to Y_2$ be a morphism of $\mathcal{C}$ such that $f_2 = a \circ f_1$ and $s_2 = a \circ s_1$. First assume that $q = (g : Y \to Z', t : Z \to Z')$. In this case choose a commutative diagram as the one on the left

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle u} \\ Y_2 & \xrightarrow{h} & Z'' \end{array} \quad \Rightarrow \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle u} \\ Y_1 & \xrightarrow{h \circ a} & Z'' \end{array}$$

(with $u \in S$), which implies the diagram on the right is commutative as well. Using these diagrams we see that both compositions $q \circ p_1$ and $q \circ p_2$ are the equivalence class of $(h \circ a \circ f_1 : X \to Z'', u \circ t : Z \to Z'')$. Thus $q \circ p_1 = q \circ p_2$. The proof of the other case, in which we have to show $p_1 \circ q = p_2 \circ q$, is omitted. (It is similar to the case we did.)

Proof of (3). We have to prove associativity of composition. Consider a solid diagram

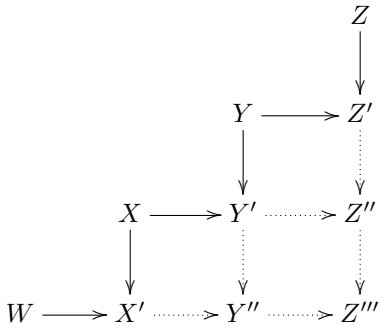

(whose vertical arrows belong to S) which gives rise to three composable pairs. Using LMS2 we can choose the dotted arrows making the squares commutative and such that the vertical arrows are in S. Then it is clear that the composition of the three pairs is the equivalence class of the pair $(W \to Z''', Z \to Z''')$ gotten by composing the horizontal arrows on the bottom row and the vertical arrows on the right column.

We leave it to the reader to check the identity axioms. □

0BM1 **Remark 4.27.3.** The motivation for the construction of $S^{-1}\mathcal{C}$ is to "force" the morphisms in S to be invertible by artificially creating inverses to them (at the cost of some existing morphisms possibly becoming identified with each other). This is similar to the localization of a commutative ring at a multiplicative subset, and more generally to the localization of a noncommutative ring at a right denominator set (see [**Lam99**, Section 10A]). This is more than just a similarity: The construction of $S^{-1}\mathcal{C}$ (or, more precisely, its version for additive categories $\mathcal{C}$) actually generalizes the latter type of localization. Namely, a noncommutative ring can be viewed as a pre-additive category with a single object (the morphisms being the elements of the ring); a multiplicative subset of this ring then becomes a set S of morphisms satisfying LMS1 (aka RMS1). Then, the conditions RMS2 and RMS3 for this category and this subset S translate into the two conditions ("right permutable" and "right reversible") of a right denominator set (and similarly for LMS and left denominator sets), and $S^{-1}\mathcal{C}$ (with a properly defined additive structure) is the one-object category corresponding to the localization of the ring.

0BM2 **Definition 4.27.4.** Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any morphism $f : X \to Y'$ in $\mathcal{C}$ and any morphism $s : Y \to Y'$ in S, we denote by $s^{-1}f$ the equivalence class of the pair $(f : X \to Y', s : Y \to Y')$. This is a morphism from X to Y in $S^{-1}\mathcal{C}$.

This notation is suggestive, and the things it suggests are true: Given any morphism $f : X \to Y'$ in $\mathcal{C}$ and any two morphisms $s : Y \to Y'$ and $t : Y' \to Y''$ in S, we have $(t \circ s)^{-1}(t \circ f) = s^{-1}f$. Also, for any $f : X \to Y'$ and $g : Y' \to Z'$ in $\mathcal{C}$

and all $s : Z \to Z'$ in S, we have $s^{-1}(g \circ f) = \left(s^{-1}g\right) \circ \left(\mathrm{id}_{Y'}^{-1} f\right)$. Finally, for any $f : X \to Y'$ in $\mathcal{C}$, all $s : Y \to Y'$ in S, and $t : Z \to Y$ in S, we have $(s \circ t)^{-1} f = \left(t^{-1}\mathrm{id}_Y\right) \circ \left(s^{-1} f\right)$. This is all clear from the definition. We can "write any finite collection of morphisms with the same target as fractions with common denominator".

04VE **Lemma 4.27.5.** *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ (indexed by i), we can find an element $s : Y \to Y'$ of S and a family of morphisms $f_i : X_i \to Y'$ of $\mathcal{C}$ such that each g_i is the equivalence class of the pair $(f_i : X_i \to Y', s : Y \to Y')$.*

Proof. For each i choose a representative $(X_i \to Y_i, s_i : Y \to Y_i)$ of g_i. The lemma follows if we can find a morphism $s : Y \to Y'$ in S such that for each i there is a morphism $a_i : Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, then we can do this by completing the square

$$\begin{array}{ccc} Y & \xrightarrow[s_2]{} & Y_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle t_2} \\ Y_1 & \xrightarrow{a_1} & Y' \end{array}$$

with $t_2 \in S$ as is possible by Definition 4.27.1. Then $s = t_2 \circ s_2 \in S$ works. If we have $n > 2$ morphisms, then we use the above trick to reduce to the case of $n - 1$ morphisms, and we win by induction. □

There is an easy characterization of equality of morphisms if they have the same denominator.

04VF **Lemma 4.27.6.** *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X \to Y', s : Y \to Y')$ and $(g : X \to Y', s : Y \to Y')$. The following are equivalent*

(1) *$A = B$*
(2) *there exists a morphism $t : Y' \to Y''$ in S with $t \circ f = t \circ g$, and*
(3) *there exists a morphism $a : Y' \to Y''$ such that $a \circ f = a \circ g$ and $a \circ s \in S$.*

Proof. We are going to use that $S^{-1}\mathcal{C}$ is a category (Lemma 4.27.2) and we will use the notation of Definition 4.27.4 as well as the discussion following that definition to identify some morphisms in $S^{-1}\mathcal{C}$. Thus we write $A = s^{-1}f$ and $B = s^{-1}g$.

If $A = B$ then $(\mathrm{id}_{Y'}^{-1} s) \circ A = (\mathrm{id}_{Y'}^{-1} s) \circ B$. We have $(\mathrm{id}_{Y'}^{-1} s) \circ A = \mathrm{id}_{Y'}^{-1} f$ and $(\mathrm{id}_{Y'}^{-1} s) \circ B = \mathrm{id}_{Y'}^{-1} g$. The equality of $\mathrm{id}_{Y'}^{-1} f$ and $\mathrm{id}_{Y'}^{-1} g$ means by definition that there exists a commutative diagram

$$\begin{array}{ccccc} & & Y' & & \\ & {\scriptstyle f}\nearrow & \downarrow{\scriptstyle u} & \nwarrow{\scriptstyle \mathrm{id}_{Y'}} & \\ X & \xrightarrow{h} & Z & \xleftarrow{t} & Y' \\ & {\scriptstyle g}\searrow & \uparrow{\scriptstyle v} & \swarrow{\scriptstyle \mathrm{id}_{Y'}} & \\ & & Y' & & \end{array}$$

with $t \in S$. In particular $u = v = t \in S$ and $t \circ f = t \circ g$. Thus (1) implies (2).

The implication (2) $\Rightarrow$ (3) is immediate. Assume a is as in (3). Denote $s' = a \circ s \in S$. Then $\mathrm{id}_{Y''}^{-1} s'$ is an isomorphism in the category $S^{-1}\mathcal{C}$ (with inverse $(s')^{-1}\mathrm{id}_{Y''}$). Thus to check $A = B$ it suffices to check that $\mathrm{id}_{Y''}^{-1} s' \circ A = \mathrm{id}_{Y''}^{-1} s' \circ B$. We compute using the rules discussed in the text following Definition 4.27.4 that $\mathrm{id}_{Y''}^{-1} s' \circ A = \mathrm{id}_{Y''}^{-1}(a \circ s) \circ s^{-1} f = \mathrm{id}_{Y''}^{-1}(a \circ f) = \mathrm{id}_{Y''}^{-1}(a \circ g) = \mathrm{id}_{Y''}^{-1}(a \circ s) \circ s^{-1} g = \mathrm{id}_{Y''}^{-1} s' \circ B$ and we see that (1) is true. □

05Q0 **Remark 4.27.7.** Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. Given an object Y of $\mathcal{C}$ we denote Y/S the category whose objects are $s : Y \to Y'$ with $s \in S$ and whose morphisms are commutative diagrams

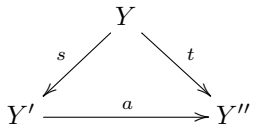

where $a : Y' \to Y''$ is arbitrary. We claim that the category Y/S is filtered (see Definition 4.19.1). Namely, LMS1 implies that $\mathrm{id}_Y : Y \to Y$ is in Y/S; hence Y/S is nonempty. LMS2 implies that given $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ we can find a diagram

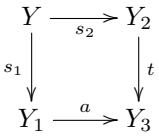

with $t \in S$. Hence $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ both have maps to $t \circ s_2 : Y \to Y_3$ in Y/S. Finally, given two morphisms a, b from $s_1 : Y \to Y_1$ to $s_2 : Y \to Y_2$ in Y/S we see that $a \circ s_1 = b \circ s_1$; hence by LMS3 there exists a $t : Y_2 \to Y_3$ in S such that $t \circ a = t \circ b$. Now the combined results of Lemmas 4.27.5 and 4.27.6 tell us that

05Q1 (4.27.7.1) $$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Mor}_{\mathcal{C}}(X, Y')$$

This formula expressing morphism sets in $S^{-1}\mathcal{C}$ as a filtered colimit of morphism sets in $\mathcal{C}$ is occasionally useful.

04VG **Lemma 4.27.8.** *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$.*

1. *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_Y : Y \to Y)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
2. *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
3. *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. Parts (1) and (2) are clear. (In (2), the inverse of $Q(s)$ is the equivalence class of the pair (id_Y, s).) To see (3) just set $H(X) = G(X)$ and set $H((f : X \to Y', s : Y \to Y')) = G(s)^{-1} \circ G(f)$. Details omitted. □

05Q2 **Lemma 4.27.9.** *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite colimits.*

Proof. Let $\mathcal{I}$ be a finite category and let $\mathcal{I} \to \mathcal{C}$, $i \mapsto X_i$ be a functor whose colimit exists. Then using (4.27.7.1), the fact that Y/S is filtered, and Lemma 4.19.2 we have

$$\begin{aligned}\mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(\mathrm{colim}\, X_i), Q(Y)) &= \mathrm{colim}_{(s:Y\to Y')\in Y/S} \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}\, X_i, Y') \\ &= \mathrm{colim}_{(s:Y\to Y')\in Y/S} \lim_i \mathrm{Mor}_{\mathcal{C}}(X_i, Y') \\ &= \lim_i \mathrm{colim}_{(s:Y\to Y')\in Y/S} \mathrm{Mor}_{\mathcal{C}}(X_i, Y') \\ &= \lim_i \mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(X_i), Q(Y))\end{aligned}$$

and this isomorphism commutes with the projections from both sides to the set $\mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(X_j), Q(Y))$ for each $j \in \mathrm{Ob}(\mathcal{I})$. Thus, $Q(\mathrm{colim}\, X_i)$ satisfies the universal property for the colimit of the functor $i \mapsto Q(X_i)$; hence, it is this colimit, as desired. □

05Q3 **Lemma 4.27.10.** *Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{array}{ccc} Q(X) & \xrightarrow{a} & Q(X') \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(f')} \\ Q(Y) & \xrightarrow{b} & Q(Y') \end{array}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exist a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & X'' & \xleftarrow{s} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{h} & Y'' & \xleftarrow{t} & Y' \end{array}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = s^{-1}g$, $b = t^{-1}h$.

Proof. We choose maps and objects in the following way: First write $a = s^{-1}g$ for some $s : X' \to X''$ in S and $g : X \to X''$. By LMS2 we can find $t : Y' \to Y''$ in S and $f'' : X'' \to Y''$ such that

$$\begin{array}{ccccc} X & \xrightarrow{g} & X'' & \xleftarrow{s} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & & Y'' & \xleftarrow{t} & Y' \end{array}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X'' \xrightarrow{f''} Y'' \xleftarrow{t} Y'$$

by postcomposing both t and f'' by a morphism $d : Y'' \to Y'''$ with the property that $d \circ t \in S$. According to Remark 4.27.7 we may after such a replacement assume that there exists a morphism $h : Y \to Y''$ such that $b = t^{-1}h$ holds[5]. At this point we have everything as in the lemma except that we don't know that the left square of the diagram commutes. But the definition of composition in $S^{-1}\mathcal{C}$ shows that

[5]Here is a more down-to-earth way to see this: Write $b = q^{-1}i$ for some $q : Y' \to Z$ in S and some $i : Y \to Z$. By LMS2 we can find $r : Y'' \to Y'''$ in S and $j : Z \to Y'''$ such that $j \circ q = r \circ t$. Now, set $d = r$ and $h = j \circ i$.

$b \circ Q(f)$ is the equivalence class of the pair $(h \circ f : X \to Y'', t : Y' \to Y'')$ (since b is the equivalence class of the pair $(h : Y \to Y'', t : Y' \to Y'')$, while $Q(f)$ is the equivalence class of the pair $(f : X \to Y, \mathrm{id} : Y \to Y)$), while $Q(f') \circ a$ is the equivalence class of the pair $(f'' \circ g : X \to Y'', t : Y' \to Y'')$ (since a is the equivalence class of the pair $(g : X \to X'', s : X' \to X'')$, while $Q(f')$ is the equivalence class of the pair $(f' : X' \to Y', \mathrm{id} : Y' \to Y')$). Since we know that $b \circ Q(f) = Q(f') \circ a$, we thus conclude that the equivalence classes of the pairs $(h \circ f : X \to Y'', t : Y' \to Y'')$ and $(f'' \circ g : X \to Y'', t : Y' \to Y'')$ are equal. Hence using Lemma 4.27.6 we can find a morphism $d : Y'' \to Y'''$ such that $d \circ t \in S$ and $d \circ h \circ f = d \circ f'' \circ g$. Hence we make one more replacement of the kind described above and we win. □

Right calculus of fractions. Let $\mathcal{C}$ be a category and let S be a right multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.27.11):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X' \to Y, s : X' \to X)$ with $s \in S$ up to equivalence. (The equivalence is defined below. Think of the equivalence class of a pair (f, s) as $fs^{-1} : X \to Y$.)
(3) Two pairs $(f_1 : X_1 \to Y, s_1 : X_1 \to X)$ and $(f_2 : X_2 \to Y, s_2 : X_2 \to X)$ are said to be equivalent if there exist a third pair $(f_3 : X_3 \to Y, s_3 : X_3 \to X)$ and morphisms $u : X_3 \to X_1$ and $v : X_3 \to X_2$ of $\mathcal{C}$ fitting into the commutative diagram

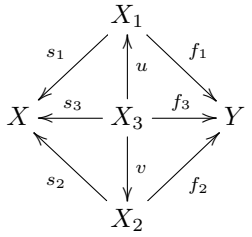

(4) The composition of the equivalence classes of the pairs $(f : X' \to Y, s : X' \to X)$ and $(g : Y' \to Z, t : Y' \to Y)$ is defined as the equivalence class of a pair $(g \circ h : X'' \to Z, s \circ u : X'' \to X)$ where h and $u \in S$ are chosen to fit into a commutative diagram

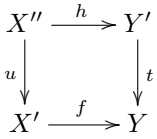

which exists by assumption.
(5) The identity morphism $X \to X$ in $S^{-1}\mathcal{C}$ is the equivalence class of the pair $(\mathrm{id} : X \to X, \mathrm{id} : X \to X)$.

04VH **Lemma 4.27.11.** *Let $\mathcal{C}$ be a category and let S be a right multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative (and the identity morphisms satisfy the identity axioms), and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. This lemma is dual to Lemma 4.27.2. It follows formally from that lemma by replacing $\mathcal{C}$ by its opposite category in which S is a left multiplicative system. □

0BM3 **Definition 4.27.12.** Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any morphism $f : X' \to Y$ in $\mathcal{C}$ and any morphism $s : X' \to X$ in S, we denote by fs^{-1} the equivalence class of the pair $(f : X' \to Y, s : X' \to X)$. This is a morphism from X to Y in $S^{-1}\mathcal{C}$.

Identities similar (actually, dual) to the ones in Definition 4.27.4 hold. We can "write any finite collection of morphisms with the same source as fractions with common denominator".

04VI **Lemma 4.27.13.** *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X \to Y_i$ of morphisms of $S^{-1}\mathcal{C}$ (indexed by i), we can find an element $s : X' \to X$ of S and a family of morphisms $f_i : X' \to Y_i$ of $\mathcal{C}$ such that g_i is the equivalence class of the pair $(f_i : X' \to Y_i, s : X' \to X)$.*

Proof. This lemma is the dual of Lemma 4.27.5 and follows formally from that lemma by replacing all categories in sight by their opposites. □

There is an easy characterization of equality of morphisms if they have the same denominator.

04VJ **Lemma 4.27.14.** *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X' \to Y, s : X' \to X)$ and $(g : X' \to Y, s : X' \to X)$. The following are equivalent*

(1) *$A = B$,*
(2) *there exists a morphism $t : X'' \to X'$ in S with $f \circ t = g \circ t$, and*
(3) *there exists a morphism $a : X'' \to X'$ with $f \circ a = g \circ a$ and $s \circ a \in S$.*

Proof. This is dual to Lemma 4.27.6. □

05Q4 **Remark 4.27.15.** Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. Given an object X of $\mathcal{C}$ we denote S/X the category whose objects are $s : X' \to X$ with $s \in S$ and whose morphisms are commutative diagrams

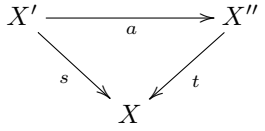

where $a : X' \to X''$ is arbitrary. The category S/X is cofiltered (see Definition 4.20.1). (This is dual to the corresponding statement in Remark 4.27.7.) Now the combined results of Lemmas 4.27.13 and 4.27.14 tell us that

05Q5 (4.27.15.1) $$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{opp}} \mathrm{Mor}_{\mathcal{C}}(X', Y)$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

04VK **Lemma 4.27.16.** *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$.*

(1) *The rules* $X \mapsto X$ *and* $(f : X \to Y) \mapsto (f : X \to Y, id_X : X \to X)$ *define a functor* $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.
(2) *For any* $s \in S$ *the morphism* $Q(s)$ *is an isomorphism in* $S^{-1}\mathcal{C}$.
(3) *If* $G : \mathcal{C} \to \mathcal{D}$ *is any functor such that* $G(s)$ *is invertible for every* $s \in S$, *then there exists a unique functor* $H : S^{-1}\mathcal{C} \to \mathcal{D}$ *such that* $H \circ Q = G$.

Proof. This lemma is the dual of Lemma 4.27.8 and follows formally from that lemma by replacing all categories in sight by their opposites. □

05Q6 **Lemma 4.27.17.** *Let* $\mathcal{C}$ *be a category and let* S *be a right multiplicative system of morphisms of* $\mathcal{C}$. *The localization functor* $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ *commutes with finite limits.*

Proof. This is dual to Lemma 4.27.9. □

05Q7 **Lemma 4.27.18.** *Let* $\mathcal{C}$ *be a category. Let* S *be a right multiplicative system. If* $f : X \to Y$, $f' : X' \to Y'$ *are two morphisms of* $\mathcal{C}$ *and if*

$$\begin{array}{ccc} Q(X) & \xrightarrow{a} & Q(X') \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(f')} \\ Q(Y) & \xrightarrow{b} & Q(Y') \end{array}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, *then there exist a morphism* $f'' : X'' \to Y''$ *in* $\mathcal{C}$ *and a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{s} & X'' & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{t} & Y'' & \xrightarrow{h} & Y' \end{array}$$

in $\mathcal{C}$ *with* $s, t \in S$ *and* $a = gs^{-1}$, $b = ht^{-1}$.

Proof. This lemma is dual to Lemma 4.27.10. □

Multiplicative systems and two sided calculus of fractions. If S is a multiplicative system then left and right calculus of fractions give canonically isomorphic categories.

04VL **Lemma 4.27.19.** *Let* $\mathcal{C}$ *be a category and let* S *be a multiplicative system. The category of left fractions and the category of right fractions* $S^{-1}\mathcal{C}$ *are canonically isomorphic.*

Proof. Denote $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ the two categories of fractions. By the universal properties of Lemmas 4.27.8 and 4.27.16 we obtain functors $\mathcal{C}_{left} \to \mathcal{C}_{right}$ and $\mathcal{C}_{right} \to \mathcal{C}_{left}$. By the uniqueness statement in the universal properties, these functors are each other's inverse. □

05Q8 **Definition 4.27.20.** Let $\mathcal{C}$ be a category and let S be a multiplicative system. We say S is *saturated* if, in addition to MS1, MS2, MS3, we also have

MS4 Given three composable morphisms f, g, h, if $fg, gh \in S$, then $g \in S$.

Note that a saturated multiplicative system contains all isomorphisms. Moreover, if f, g, h are composable morphisms in a category and fg, gh are isomorphisms, then g is an isomorphism (because then g has both a left and a right inverse, hence is invertible).

05Q9 **Lemma 4.27.21.** *Let $\mathcal{C}$ be a category and let S be a multiplicative system. Denote $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor. The set*

$$\hat{S} = \{f \in Arrows(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}$$

is equal to

$$S' = \{f \in Arrows(\mathcal{C}) \mid \text{there exist } g, h \text{ such that } gf, fh \in S\}$$

and is the smallest saturated multiplicative system containing S. In particular, if S is saturated, then $\hat{S} = S$.

Proof. It is clear that $S \subset S' \subset \hat{S}$ because elements of S' map to morphisms in $S^{-1}\mathcal{C}$ which have both left and right inverses. Note that S' satisfies MS4, and that $\hat{S}$ satisfies MS1. Next, we prove that $S' = \hat{S}$.

Let $f \in \hat{S}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism in $S^{-1}\mathcal{C}$. (We may use both left fractions and right fractions to describe morphisms in $S^{-1}\mathcal{C}$, see Lemma 4.27.19.) The relation $\mathrm{id}_X = s^{-1}gf$ in $S^{-1}\mathcal{C}$ means there exists a commutative diagram

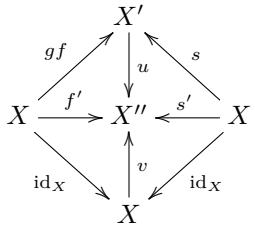

for some morphisms f', u, v and $s' \in S$. Hence $ugf = s' \in S$. Similarly, using that $\mathrm{id}_Y = fht^{-1}$ one proves that $fhw \in S$ for some w. We conclude that $f \in S'$. Thus $S' = \hat{S}$. Provided we prove that $S' = \hat{S}$ is a multiplicative system it is now clear that this implies that $S' = \hat{S}$ is the smallest saturated system containing S.

Our remarks above take care of MS1 and MS4, so to finish the proof of the lemma we have to show that LMS2, RMS2, LMS3, RMS3 hold for $\hat{S}$. Let us check that LMS2 holds for $\hat{S}$. Suppose we have a solid diagram

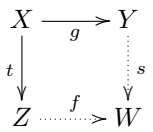

with $t \in \hat{S}$. Pick a morphism $a : Z \to Z'$ such that $at \in S$. Then we can use LMS2 for S to find a commutative diagram

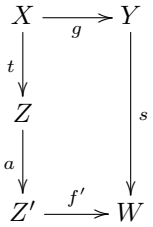

and setting $f = f' \circ a$ we win. The proof of RMS2 is dual to this. Finally, suppose given a pair of morphisms $f, g : X \to Y$ and $t \in \hat{S}$ with target X such that $ft = gt$.

Then we pick a morphism b such that $tb \in S$. Then $ftb = gtb$ which implies by LMS3 for S that there exists an $s \in S$ with source Y such that $sf = sg$ as desired. The proof of RMS3 is dual to this. □

4.28. Formal properties

003D In this section we discuss some formal properties of the 2-category of categories. This will lead us to the definition of a (strict) 2-category later.

Let us denote $\mathrm{Ob}(Cat)$ the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(Cat)$ we have the "small" category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as

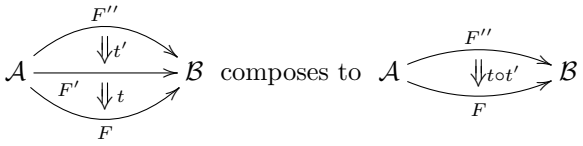

is called *vertical* composition. We will use the usual symbol $\circ$ for this. Next, we will define *horizontal* composition. In order to do this we explain a bit more of the structure at hand.

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : \mathrm{Ob}(\mathrm{Fun}(\mathcal{B}, \mathcal{C})) \times \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{B})) \longrightarrow \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words – forgetting about transformations of functors – we see that Cat forms a category. How does this structure interact with the morphisms between functors?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_Gt$. It is given by the formula $({}_Gt)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that ${}_G(\mathrm{id}_F) = \mathrm{id}_{G \circ F}$ and that ${}_G(t_1 \circ t_2) = {}_Gt_1 \circ {}_Gt_2$. Of course we also have that ${}_{\mathrm{id}_\mathcal{A}}t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by $(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that $(\mathrm{id}_G)_F = \mathrm{id}_{G \circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\mathrm{id}_\mathcal{B}} = s$.

These constructions satisfy the additional properties

$${}_{G_1}({}_{G_2}t) = {}_{G_1 \circ G_2}t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_Hs)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{array}{ccc} G \circ F & \xrightarrow{{}_Gt} & G \circ F' \\ {\scriptstyle s_F}\downarrow & & \downarrow{\scriptstyle s_{F'}} \\ G' \circ F & \xrightarrow[{}_{G'}t]{} & G' \circ F' \end{array}$$

in other words ${}_{G'}t \circ s_F = s_{F'} \circ {}_Gt$. To prove this we just consider what happens on any object $x \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{array}{ccc} G(F(x)) & \xrightarrow{G(t_x)} & G(F'(x)) \\ {\scriptstyle s_{F(x)}}\downarrow & & \downarrow{\scriptstyle s_{F'(x)}} \\ G'(F(x)) & \xrightarrow[G'(t_x)]{} & G'(F'(x)) \end{array}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

003E **Definition 4.28.1.** Given a diagram as in the left hand side of:

$$\mathcal{A} \overset{F}{\underset{F'}{\Downarrow t}} \mathcal{B} \overset{G}{\underset{G'}{\Downarrow s}} \mathcal{C} \quad \text{gives} \quad \mathcal{A} \overset{G\circ F}{\underset{G'\circ F'}{\Downarrow s\star t}} \mathcal{C}$$

we define the *horizontal* composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_Gt$.

Now we see that we may recover our previously constructed transformations ${}_Gt$ and s_F as ${}_Gt = \mathrm{id}_G \star t$ and $s_F = s \star \mathrm{id}_F$. Furthermore, all of the rules we found above are consequences of the properties stated in the lemma that follows.

003F **Lemma 4.28.2.** *The horizontal and vertical compositions have the following properties*

(1) $\circ$ *and* $\star$ *are associative,*
(2) *the identity transformations* id_F *are units for* $\circ$*,*
(3) *the identity transformations of the identity functors* $id_{id_\mathcal{A}}$ *are units for* $\star$ *and* $\circ$*, and*
(4) *given a diagram*

$$\mathcal{A} \overset{F}{\underset{F'}{\overset{\Downarrow t}{\underset{\Downarrow t'}{}}}} \mathcal{B} \overset{G}{\underset{G'}{\overset{\Downarrow s}{\underset{\Downarrow s'}{}}}} \mathcal{C} \qquad (F'' \text{ below } F',\ G'' \text{ below } G')$$

we have $(s' \circ s) \star (t' \circ t) = (s' \star t') \circ (s \star t)$.

Proof. The last statement turns using our previous notation into the following equation

$$s'_{F''} \circ {}_{G'}t' \circ s_{F'} \circ {}_Gt = (s' \circ s)_{F''} \circ {}_G(t' \circ t).$$

According to our result above applied to the middle composition we may rewrite the left hand side as $s'_{F''} \circ s_{F''} \circ {}_G t' \circ {}_G t$ which is easily shown to be equal to the right hand side. □

Another way of formulating condition (4) of the lemma is that composition of functors and horizontal composition of transformation of functors gives rise to a functor

$$(\circ, \star) : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$

whose source is the product category, see Definition 4.2.20.

4.29. 2-categories

003G We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Section 4.28 and Example 4.30.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example.

003H **Definition 4.29.1.** A (strict) 2-*category* $\mathcal{C}$ consists of the following data

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Mor}_{\mathcal{C}}(x, y)$. The objects of $\mathrm{Mor}_{\mathcal{C}}(x, y)$ will be called 1-*morphisms* and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called 2-*morphisms* and denoted $t : F' \to F$. The composition of 2-morphisms in $\mathrm{Mor}_{\mathcal{C}}(x, y)$ will be called *vertical* composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a functor
$$(\circ, \star) : \mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(x, z).$$
The image of the pair of 1-morphisms (F, G) on the left hand side will be called the *composition* of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the *horizontal* composition and denoted $t \star s$.

These data are to satisfy the following rules:

(1) The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.
(2) Horizontal composition of 2-morphisms is associative.
(3) The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. The only example we have so far is that of the 2-category whose objects are a given collection of categories, 1-morphisms are functors between these categories, and 2-morphisms are natural transformations of functors, see Section 4.28. As far as this text is concerned all 2-categories will be sub 2-categories of this example. Here is what it means to be a sub 2-category.

02X7 **Definition 4.29.2.** Let $\mathcal{C}$ be a 2-category. A *sub* 2-*category* $\mathcal{C}'$ of $\mathcal{C}$, is given by a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathrm{Ob}(\mathcal{C})$ and sub categories $\mathrm{Mor}_{\mathcal{C}'}(x, y)$ of the categories $\mathrm{Mor}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{C}')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

003J **Remark 4.29.3.** Big 2-categories. In many texts a 2-category is allowed to have a class of objects (but hopefully a "class of classes" is not allowed). We will allow these "big" 2-categories as well, but only in the following list of cases (to be updated as we go along):

(1) The 2-category of categories *Cat*.
(2) The $(2,1)$-category of categories *Cat*.
(3) The 2-category of groupoids *Groupoids*; this is a $(2,1)$-category.
(4) The 2-category of fibred categories over a fixed category.
(5) The $(2,1)$-category of fibred categories over a fixed category.
(6) The 2-category of categories fibred in groupoids over a fixed category; this is a $(2,1)$-category.
(7) The 2-category of stacks over a fixed site.
(8) The $(2,1)$-category of stacks over a fixed site.
(9) The 2-category of stacks in groupoids over a fixed site; this is a $(2,1)$-category.
(10) The 2-category of stacks in setoids over a fixed site; this is a $(2,1)$-category.
(11) The 2-category of algebraic stacks over a fixed scheme; this is a $(2,1)$-category.

See Definition 4.30.1. Note that in each case the class of objects of the 2-category $\mathcal{C}$ is a proper class, but for all objects $x, y \in \mathrm{Ob}(C)$ the category $\mathrm{Mor}_{\mathcal{C}}(x,y)$ is "small" (according to our conventions).

The notion of equivalence of categories that we defined in Section 4.2 extends to the more general setting of 2-categories as follows.

003L **Definition 4.29.4.** Two objects x, y of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Sometimes we need to say what it means to have a functor from a category into a 2-category.

003N **Definition 4.29.5.** Let $\mathcal{A}$ be a category and let $\mathcal{C}$ be a 2-category.

(1) A *functor* from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a "usual" functor into the category formed out of 2-category by forgetting all the 2-morphisms.
(2) A *weak functor*, or a *pseudo functor* φ from $\mathcal{A}$ into the 2-category $\mathcal{C}$ is given by the following data
   (a) a map $\varphi : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$,
   (b) for every pair $x, y \in \mathrm{Ob}(\mathcal{A})$, and every morphism $f : x \to y$ a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$,
   (c) for every $x \in \mathrm{Ob}(A)$ a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$, and
   (d) for every pair of composable morphisms $f : x \to y$, $g : y \to z$ of $\mathcal{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$.

   These data are subject to the following conditions:
   (a) the 2-morphisms α_x and $\alpha_{g,f}$ are all isomorphisms,

(b) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{\mathrm{id}_y, f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$:

$$\varphi(x) \underset{\varphi(f)}{\overset{\varphi(f)}{\Downarrow \mathrm{id}_{\varphi(f)}}} \varphi(y) \underset{\varphi(\mathrm{id}_y)}{\overset{\mathrm{id}_{\varphi(y)}}{\Downarrow \alpha_y}} \varphi(y) \;=\; \varphi(x) \underset{\varphi(\mathrm{id}_y)\circ\varphi(f)}{\overset{\varphi(f)}{\Downarrow \alpha_{\mathrm{id}_y, f}}} \varphi(y)$$

(c) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{f, \mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x$,

(d) for any triple of composable morphisms $f : w \to x$, $g : x \to y$, and $h : y \to z$ of $\mathcal{A}$ we have

$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h, g\circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h\circ g, f}$$

in other words the following diagram with objects 1-morphisms and arrows 2-morphisms commutes

$$\begin{array}{ccc} \varphi(h \circ g \circ f) & \xrightarrow[\alpha_{h\circ g, f}]{} & \varphi(h \circ g) \circ \varphi(f) \\ {\scriptstyle \alpha_{h, g\circ f}}\downarrow & & \downarrow{\scriptstyle \alpha_{h,g} \star \mathrm{id}_{\varphi(f)}} \\ \varphi(h) \circ \varphi(g \circ f) & \xrightarrow{\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}} & \varphi(h) \circ \varphi(g) \circ \varphi(f) \end{array}$$

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. Finally, there are the notions of *functor between* 2*-categories*, and *pseudo functor between* 2*-categories*. This last notion leads us into 3-category territory. We would like to avoid having to define this at almost any cost!

4.30. (2, 1)-categories

02X8 Some 2-categories have the property that all 2-morphisms are isomorphisms. These will play an important role in the following, and they are easier to work with.

003I **Definition 4.30.1.** A (strict) $(2,1)$-*category* is a 2-category in which all 2-morphisms are isomorphisms.

003K **Example 4.30.2.** The 2-category *Cat*, see Remark 4.29.3, can be turned into a $(2,1)$-category by only allowing isomorphisms of functors as 2-morphisms.

In fact, more generally any 2-category $\mathcal{C}$ produces a $(2,1)$-category by considering the sub 2-category $\mathcal{C}'$ with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of $\mathcal{C}$. In this situation we will say "*let* $\mathcal{C}'$ *be the* $(2,1)$*-category associated to* $\mathcal{C}$" or similar. For example, the $(2,1)$-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

003M **Remark 4.30.3.** Thus there are variants of the construction of Example 4.30.2 above where we look at the 2-category of groupoids, or categories fibred in groupoids over a fixed category, or stacks. And so on.

4.31. 2-fibre products

003O In this section we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{array}{ccc} w & \longrightarrow & y \\ \downarrow & & \downarrow \\ x & \longrightarrow & z \end{array}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a "weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{array}{ccc} w & \xrightarrow[a]{} & x \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ y & \xrightarrow[g]{} & z. \end{array}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The 2-*category of* 2-*commutative diagrams over* f *and* g is the 2-category defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is an invertible 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w', a', b', ϕ') to (w, a, b, ϕ) are given by $(k : w' \to w, \alpha : a' \to a \circ k, \beta : b' \to b \circ k)$ such that

$$\begin{array}{ccc} f \circ a' & \xrightarrow[\mathrm{id}_f \star \alpha]{} & f \circ a \circ k \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi \star \mathrm{id}_k} \\ g \circ b' & \xrightarrow[\mathrm{id}_g \star \beta]{} & g \circ b \circ k \end{array}$$

is commutative,
(3) given a second 1-morphism $(k', \alpha', \beta') : (w'', a'', b'', \phi'') \to (w', \alpha', \beta', \phi')$ the composition of 1-morphisms is given by the rule

$$(k, \alpha, \beta) \circ (k', \alpha', \beta') = (k \circ k', (\alpha \star \mathrm{id}_{k'}) \circ \alpha', (\beta \star \mathrm{id}_{k'}) \circ \beta'),$$

(4) a 2-morphism between 1-morphisms (k_i, α_i, β_i), $i = 1, 2$ with the same source and target is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{array}{ccc} a' & \xrightarrow[\alpha_1]{} & a \circ k_1 \\ & {\scriptstyle \alpha_2}\searrow & \downarrow{\scriptstyle \mathrm{id}_a \star \delta} \\ & & a \circ k_2 \end{array} \qquad \begin{array}{ccc} b \circ k_1 & \xleftarrow[\beta_1]{} & b' \\ {\scriptstyle \mathrm{id}_b \star \delta}\downarrow & \swarrow{\scriptstyle \beta_2} & \\ b \circ k_2 & & \end{array}$$

commute,

(5) vertical composition of 2-morphisms is given by vertical composition of the morphisms δ in $\mathcal{C}$, and
(6) horizontal composition of the diagram

$$(w'', a'', b'', \phi'') \xrightarrow[(k_2', \alpha_2', \beta_2')]{(k_1', \alpha_1', \beta_1')} (w', a', b', \phi') \xrightarrow[(k_2, \alpha_2, \beta_2)]{(k_1, \alpha_1, \beta_1)} (w, a, b, \phi)$$

is given by the diagram

$$(w'', a'', b'', \phi'') \xrightarrow[(k_2 \circ k_2', (\alpha_2 \star \mathrm{id}_{k_2'}) \circ \alpha_2', (\beta_2 \star \mathrm{id}_{k_2'}) \circ \beta_2')]{(k_1 \circ k_1', (\alpha_1 \star \mathrm{id}_{k_1'}) \circ \alpha_1', (\beta_1 \star \mathrm{id}_{k_1'}) \circ \beta_1')} (w, a, b, \phi)$$

Note that if $\mathcal{C}$ is actually a $(2,1)$-category, the morphisms α and β in (2) above are automatically also isomorphisms[6]. In addition the 2-category of 2-commutative diagrams is also a $(2,1)$-category if $\mathcal{C}$ is a $(2,1)$-category.

003P **Definition 4.31.1.** A *final object* of a $(2,1)$-category $\mathcal{C}$ is an object x such that

(1) for every $y \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $y \to x$, and
(2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Likely, in the more general case of 2-categories there are different flavours of final objects. We do not want to get into this and hence we only define 2-fibre products in the $(2,1)$-case.

003Q **Definition 4.31.2.** Let $\mathcal{C}$ be a $(2,1)$-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. A *2-fibre product of f and g* is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in \mathrm{Ob}(\mathcal{C})$, and denote the required morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times_z y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times_z y, y)$ making the diagram

$$\begin{array}{ccc} x \times_z y & \xrightarrow{p} & x \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ y & \xrightarrow{g} & z \end{array}$$

2-commute and we will denote the given invertible 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $b \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ with a given 2-isomorphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times_z y)$ making the diagram

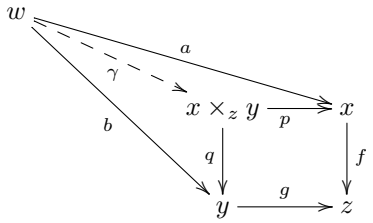

[6] In fact it seems in the 2-category case that one could define another 2-category of 2-commutative diagrams where the direction of the arrows α, β is reversed, or even where the direction of only one of them is reversed. This is why we restrict to $(2,1)$-categories later on.

2-commute such that for suitable choices of $a \to p \circ \gamma$ and $b \to q \circ \gamma$ the diagram

$$\begin{array}{ccc} f \circ a & \longrightarrow & f \circ p \circ \gamma \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi \star \mathrm{id}_\gamma} \\ g \circ b & \longrightarrow & g \circ q \circ \gamma \end{array}$$

commutes. Moreover γ is unique up to isomorphism. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

003R **Example 4.31.3.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ as follows:

(1) an object of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(b)} \\ F(A') & \xrightarrow{f'} & G(B') \end{array}$$

is commutative.

Moreover, we define functors $p : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

02X9 **Lemma 4.31.4.** *In the $(2,1)$-category of categories 2-fibre products exist and are given by the construction of Example 4.31.3.*

Proof. Let us check the universal property: let $\mathcal{W}$ be a category, let $a : \mathcal{W} \to \mathcal{A}$ and $b : \mathcal{W} \to \mathcal{B}$ be functors, and let $t : F \circ a \to G \circ b$ be an isomorphism of functors.

Consider the functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ given by $W \mapsto (a(W), b(W), t_W)$. (Check this is a functor omitted.) Moreover, consider $\alpha : a \to p \circ \gamma$ and $\beta : b \to q \circ \gamma$ obtained from the identities $p \circ \gamma = a$ and $q \circ \gamma = b$. Then it is clear that (γ, α, β) is a morphism from (W, a, b, t) to $(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$.

Let $(k, \alpha', \beta') : (W, a, b, t) \to (\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$ be a second such morphism. For an object W of $\mathcal{W}$ let us write $k(W) = (a_k(W), b_k(W), t_{k,W})$. Hence $p(k(W)) = a_k(W)$ and so on. The map α' corresponds to functorial maps $\alpha' : a(W) \to a_k(W)$. Since we are working in the $(2,1)$-category of categories, in fact each of the maps $a(W) \to a_k(W)$ is an isomorphism. We can use these (and their counterparts $b(W) \to b_k(W)$) to get isomorphisms

$$\delta_W : \gamma(W) = (a(W), b(W), t_W) \longrightarrow (a_k(W), b_k(W), t_{k,W}) = k(W).$$

It is straightforward to show that δ defines a 2-isomorphism between γ and k in the 2-category of 2-commutative diagrams as desired. □

06RL **Remark 4.31.5.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Another, slightly more symmetrical, construction of a 2-fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is as follows. An object is a quintuple (A, B, C, a, b) where A, B, C are objects of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and where $a : F(A) \to C$ and $b : G(B) \to C$ are isomorphisms. A morphism $(A, B, C, a, b) \to (A', B', C', a', b')$ is given by a triple of morphisms $A \to A', B \to B', C \to C'$ compatible with the morphisms a, b, a', b'. We can prove directly that this leads to a 2-fibre product. However, it is easier to observe that the functor $(A, B, C, a, b) \mapsto (A, B, b^{-1} \circ a)$ gives an equivalence from the category of quintuples to the category constructed in Example 4.31.3.

02XA **Lemma 4.31.6.** *Let*

$$\begin{array}{ccccc} & & \mathcal{Y} & & \\ & & \downarrow{\scriptstyle I} & \searrow{\scriptstyle K} & \\ \mathcal{X} & \xrightarrow{H} & \mathcal{Z} & & \mathcal{B} \\ & \searrow{\scriptstyle L} & & \searrow{\scriptstyle M} & \downarrow{\scriptstyle G} \\ & & \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

be a 2-commutative diagram of categories. A choice of isomorphisms $\alpha : G \circ K \to M \circ I$ and $\beta : M \circ H \to F \circ L$ determines a morphism

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

of 2-fibre products associated to this situation.

Proof. Just use the functor

$$(X, Y, \phi) \longmapsto (L(X), K(Y), \alpha_Y^{-1} \circ M(\phi) \circ \beta_X^{-1})$$

on objects and

$$(a, b) \longmapsto (L(a), K(b))$$

on morphisms. □

02XB **Lemma 4.31.7.** *Assumptions as in Lemma 4.31.6.*

(1) *If K and L are faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is faithful.*
(2) *If K and L are fully faithful and M is faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is fully faithful.*
(3) *If K and L are equivalences and M is fully faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is an equivalence.*

Proof. Let (X, Y, ϕ) and (X', Y', ϕ') be objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Set $Z = H(X)$ and identify it with $I(Y)$ via ϕ. Also, identify $M(Z)$ with $F(L(X))$ via α_X and identify $M(Z)$ with $G(K(Y))$ via β_Y. Similarly for $Z' = H(X')$ and $M(Z')$. The map on morphisms is the map

$$\begin{array}{c} \mathrm{Mor}_{\mathcal{X}}(X, X') \times_{\mathrm{Mor}_{\mathcal{Z}}(Z, Z')} \mathrm{Mor}_{\mathcal{Y}}(Y, Y') \\ \downarrow \\ \mathrm{Mor}_{\mathcal{A}}(L(X), L(X')) \times_{\mathrm{Mor}_{\mathcal{C}}(M(Z), M(Z'))} \mathrm{Mor}_{\mathcal{B}}(K(Y), K(Y')) \end{array}$$

Hence parts (1) and (2) follow. Moreover, if K and L are equivalences and M is fully faithful, then any object (A, B, ϕ) is in the essential image for the following reasons: Pick X, Y such that $L(X) \cong A$ and $K(Y) \cong B$. Then the fully faithfulness

of M guarantees that we can find an isomorphism $H(X) \cong I(Y)$. Some details omitted. □

02XC **Lemma 4.31.8.** *Let*

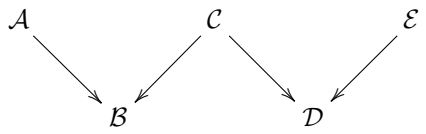

be a diagram of categories and functors. Then there is a canonical isomorphism

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{A} \times_{\mathcal{B}} (\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$$

of categories.

Proof. Just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, (C, E, \psi), \phi)$$

if you know what I mean. □

Henceforth we do not write the parentheses when dealing with fibre products of more than 2 categories.

04S7 **Lemma 4.31.9.** *Let*

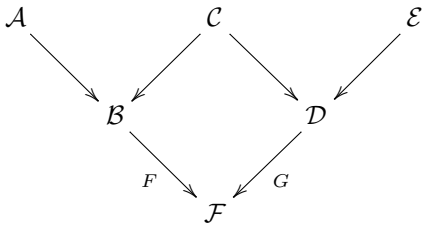

be a commutative diagram of categories and functors. Then there is a canonical functor

$$pr_{02} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \longrightarrow \mathcal{A} \times_{\mathcal{F}} \mathcal{E}$$

of categories.

Proof. If we write $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ as $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E}$ then we can just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, E, G(\psi) \circ F(\phi))$$

if you know what I mean. □

02XD **Lemma 4.31.10.** *Let*

$$\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C} \leftarrow \mathcal{D}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{C}} \mathcal{D} \cong \mathcal{A} \times_{\mathcal{B}} \mathcal{D}$$

of categories.

Proof. Omitted. □

We claim that this means you can work with these 2-fibre products just like with ordinary fibre products. Here are some further lemmas that actually come up later.

02XE **Lemma 4.31.11.** *Let*

$$\begin{array}{ccc} \mathcal{C}_3 & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{C}_1 \times \mathcal{C}_2 & \xrightarrow{G_1 \times G_2} & \mathcal{S} \times \mathcal{S} \end{array}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism $\mathcal{C}_3 \cong \mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$.

Proof. We may assume that $\mathcal{C}_3$ is the category $(\mathcal{C}_1 \times \mathcal{C}_2) \times_{\mathcal{S} \times \mathcal{S}} \mathcal{S}$ constructed in Example 4.31.3. Hence an object is a triple $((X_1, X_2), S, \phi)$ where $\phi = (\phi_1, \phi_2) : (G_1(X_1), G_2(X_2)) \to (S, S)$ is an isomorphism. Thus we can associate to this the triple $(X_1, X_2, \phi_2^{-1} \circ \phi_1)$. Conversely, if (X_1, X_2, ψ) is an object of $\mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$, then we can associate to this the triple $((X_1, X_2), G_2(X_2), (\psi, \mathrm{id}_{G_2(X_2)}))$. We claim these constructions given mutually inverse functors. We omit describing how to deal with morphisms and showing they are mutually inverse. □

02XF **Lemma 4.31.12.** *Let*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{C} & \xrightarrow{G_1 \times G_2} & \mathcal{S} \times \mathcal{S} \end{array}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism

$$\mathcal{C}' \cong (\mathcal{C} \times_{G_1, \mathcal{S}, G_2} \mathcal{C}) \times_{(p,q), \mathcal{C} \times \mathcal{C}, \Delta} \mathcal{C}.$$

Proof. An object of the right hand side is given by $((C_1, C_2, \phi), C_3, \psi)$ where $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism and $\psi = (\psi_1, \psi_2) : (C_1, C_2) \to (C_3, C_3)$ is an isomorphism. Hence we can associate to this the triple $(C_3, G_1(C_1), (G_1(\psi_1^{-1}), \phi^{-1} \circ G_2(\psi_2^{-1})))$ which is an object of $\mathcal{C}'$. Details omitted. □

04Z1 **Lemma 4.31.13.** *Let* $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ *and* $\mathcal{C} \to \mathcal{D}$ *be functors between categories. Then the diagram*

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \longrightarrow & \mathcal{A} \times_{\mathcal{D}} \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}/\mathcal{D}}} & \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \end{array}$$

is a 2-fibre product diagram.

Proof. Omitted. □

04YR **Lemma 4.31.14.** *Let*

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product of categories. Then the diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is 2-cartesian.

Proof. This is a purely 2-category theoretic statement, valid in any $(2,1)$-category with 2-fibre products. Explicitly, it follows from the following chain of equivalences:

$$\begin{aligned}\mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} (\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) &= \mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} ((\mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \times_{\mathcal{V}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{V})) \\ &= \mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \\ &= \mathcal{X} \times_{\mathcal{Y}} \mathcal{V} = \mathcal{U}\end{aligned}$$

see Lemmas 4.31.8 and 4.31.10. □

4.32. Categories over categories

02XG In this section we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

003Y **Definition 4.32.1.** Let $\mathcal{C}$ be a category. The 2-*category of categories over* $\mathcal{C}$ is the 2-category defined as follows:

(1) Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

In this 2-category we define horizontal and vertical composition exactly as is done for *Cat* in Section 4.28. The axioms of a 2-category are satisfied for the same reason that the hold in *Cat*. To see this one can also use that the axioms hold in *Cat* and verify things such as "vertical composition of 2-morphisms over $\mathcal{C}$ gives another 2-morphism over $\mathcal{C}$". This is clear.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category, and some notions of lifting associated to this situation.

02XH **Definition 4.32.2.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The *fibre category* over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects
$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$
and morphisms
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\phi \in \mathrm{Mor}_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$
(2) A *lift* of an object $U \in \mathrm{Ob}(\mathcal{C})$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. We will also sometime say that x *lies over* U.
(3) Similarly, a *lift* of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that ϕ *lies over* f.

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

Here is the obligatory lemma describing the 2-fibre product in the $(2,1)$-category of categories over $\mathcal{C}$.

0040 **Lemma 4.32.3.** *Let $\mathcal{C}$ be a category. The $(2,1)$-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*
(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that*
   (a) *a and b induce the same morphism $U \to U'$, and*
   (b) *the diagram*

$$\begin{array}{ccc} F(x) & \xrightarrow{f} & G(y) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(b)} \\ F(x') & \xrightarrow{f'} & G(y') \end{array}$$

   is commutative.

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_{\mathcal{W}} : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors over $\mathcal{C}$. The desired functor $\gamma : \mathcal{W} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by $W \mapsto (p_{\mathcal{W}}(W), X(W), Y(W), t_W)$. Details omitted; compare with Lemma 4.31.4. □

0H2D **Example 4.32.4.** The constructions of 2-fibre products of categories over categories given in Lemma 4.32.3 and of categories in Lemma 4.31.4 (as in Example 4.31.3) produce non-equivalent outputs in general. Namely, let $\mathcal{S}$ be the groupoid category with one object and two arrows, and let $\mathcal{X}$ be the discrete category with one object. Taking the 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ as categories yields the discrete category with two objects. However, if we view all of these as categories over $\mathcal{S}$, the 2-fiber product $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ as categories over $\mathcal{S}$ is the discrete category with one object. The difference is that (in the notation of Lemma 4.32.3), we were allowed to choose any comparison isomorphism f in the first situation, but could only choose the identity arrow in the second situation.

02XI **Lemma 4.32.5.** *Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be morphisms of categories over $\mathcal{C}$. For any object U of $\mathcal{C}$ we have the following identity of fibre categories*

$$(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U = \mathcal{X}_U \times_{\mathcal{S}_U} \mathcal{Y}_U$$

Proof. Omitted. □

4.33. Fibred categories

02XJ A very brief discussion of fibred categories is warranted.

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of "fibre product $V \times_U x$" (or a *base change* of x via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into "$V \times_U x$" should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$ such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{array}{ccccc} z & \longrightarrow & ? & \longrightarrow & x \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ p(z) & \longrightarrow & V & \xrightarrow{f} & U \end{array}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

02XK **Definition 4.33.1.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A *strongly cartesian morphism*, or more precisely a *strongly $\mathcal{C}$-cartesian morphism* is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in \mathrm{Ob}(\mathcal{S})$ the map

$$\mathrm{Mor}_{\mathcal{S}}(z, y) \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 4.3.5, given $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$ with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), U)} \mathrm{Mor}_{\mathcal{C}}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

02XL **Lemma 4.33.2.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.*

(1) *The composition of two strongly cartesian morphisms is strongly cartesian.*
(2) *Any isomorphism of $\mathcal{S}$ is strongly cartesian.*
(3) *Any strongly cartesian morphism φ such that $p(\varphi)$ is an isomorphism, is an isomorphism.*

Proof. Proof of (1). Let $\varphi : y \to x$ and $\psi : z \to y$ be strongly cartesian. Let t be an arbitrary object of $\mathcal{S}$. Then we have

$$\begin{aligned} &\mathrm{Mor}_{\mathcal{S}}(t, z) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, y) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(y)) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \end{aligned}$$

hence $z \to x$ is strongly cartesian.

Proof of (2). Let $y \to x$ be an isomorphism. Then $p(y) \to p(x)$ is an isomorphism too. Hence $\mathrm{Mor}_{\mathcal{C}}(p(z), p(y)) \to \mathrm{Mor}_{\mathcal{C}}(p(z), p(x))$ is a bijection. Hence $\mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y))$ is bijective to $\mathrm{Mor}_{\mathcal{S}}(z, x)$. Hence the

displayed map of Definition 4.33.1 is a bijection as $y \to x$ is an isomorphism, and we conclude that $y \to x$ is strongly cartesian.

Proof of (3). Assume $\varphi : y \to x$ is strongly cartesian with $p(\varphi) : p(y) \to p(x)$ an isomorphism. Applying the definition with $z = x$ shows that $(\mathrm{id}_x, p(\varphi)^{-1})$ comes from a unique morphism $\chi : x \to y$. We omit the verification that χ is the inverse of φ. □

09WU **Lemma 4.33.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be composable functors between categories. Let $x \to y$ be a morphism of $\mathcal{A}$. If $x \to y$ is strongly $\mathcal{B}$-cartesian and $F(x) \to F(y)$ is strongly $\mathcal{C}$-cartesian, then $x \to y$ is strongly $\mathcal{C}$-cartesian.*

Proof. This follows directly from the definition. □

06N4 **Lemma 4.33.4.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. Assume*

(1) *$x \to y$ is strongly cartesian,*
(2) *$p(x) \times_{p(y)} p(z)$ exists, and*
(3) *there exists a strongly cartesian morphism $a : w \to z$ in $\mathcal{S}$ with $p(w) = p(x) \times_{p(y)} p(z)$ and $p(a) = pr_2 : p(x) \times_{p(y)} p(z) \to p(z)$.*

Then the fibre product $x \times_y z$ exists and is isomorphic to w.

Proof. Since $x \to y$ is strongly cartesian there exists a unique morphism $b : w \to x$ such that $p(b) = \mathrm{pr}_1$. To see that w is the fibre product we compute

$$\begin{aligned}
&\mathrm{Mor}_{\mathcal{S}}(t, w) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(z))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(w)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(z))} (\mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z))) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{S}}(t, y)} \mathrm{Mor}_{\mathcal{S}}(t, y) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{S}}(t, y)} \mathrm{Mor}_{\mathcal{S}}(t, x)
\end{aligned}$$

as desired. The first equality holds because $a : w \to z$ is strongly cartesian and the last equality holds because $x \to y$ is strongly cartesian. □

02XM **Definition 4.33.5.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over $\mathcal{C}$* if given any $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$\begin{array}{ccc}
f^*x & \xrightarrow[f^*\phi]{} & f^*x' \\
\downarrow & & \downarrow \\
x & \xrightarrow{\phi} & x'
\end{array}$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

02XN **Definition 4.33.6.** Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A *choice of pullbacks*[7] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.
(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a *pullback functor* (associated to the choices $f^*x \to x$ made above).

Of course we may always assume our choice of pullbacks has the property that $\mathrm{id}_U^* x = x$, although in practice this is a useless property without imposing further assumptions on the pullbacks.

02XO **Lemma 4.33.7.** *Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. Assume given a choice of pullbacks for $p : \mathcal{S} \to \mathcal{C}$.*

(1) *For any pair of composable morphisms $f : V \to U$, $g : W \to V$ there is a unique isomorphism*

$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$

as functors $\mathcal{S}_U \to \mathcal{S}_W$ such that for every $y \in \mathrm{Ob}(\mathcal{S}_U)$ the following diagram commutes

$$\begin{array}{ccc} g^*f^*y & \longrightarrow & f^*y \\ {\scriptstyle (\alpha_{g,f})_y}\uparrow & & \downarrow \\ (f \circ g)^*y & \longrightarrow & y \end{array}$$

(2) *If $f = id_U$, then there is a canonical isomorphism $\alpha_U : id \to (id_U)^*$ as functors $\mathcal{S}_U \to \mathcal{S}_U$.*
(3) *The quadruple $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{g,f}, \alpha_U)$ defines a pseudo functor from $\mathcal{C}^{opp}$ to the $(2,1)$-category of categories, see Definition 4.29.5.*

Proof. In fact, it is clear that the commutative diagram of part (1) uniquely determines the morphism $(\alpha_{g,f})_y$ in the fibre category $\mathcal{S}_W$. It is an isomorphism since both the morphism $(f \circ g)^*y \to y$ and the composition $g^*f^*y \to f^*y \to y$ are strongly cartesian morphisms lifting $f \circ g$ (see discussion following Definition 4.33.1 and Lemma 4.33.2). In the same way, since $\mathrm{id}_x : x \to x$ is clearly strongly cartesian over id_U (with $U = p(x)$) we see that there exists an isomorphism $(\alpha_U)_x : x \to (\mathrm{id}_U)^*x$. (Of course we could have assumed beforehand that $f^*x = x$ whenever f is an identity morphism, but it is better for the sake of generality not to assume this.) We omit the verification that $\alpha_{g,f}$ and α_U so obtained are transformations of functors. We also omit the verification of (3). □

[7] This is probably nonstandard terminology. In some texts this is called a "cleavage" but it conjures up the wrong image. Maybe a "cleaving" would be a better word. A related notion is that of a "splitting", but in many texts a "splitting" means a choice of pullbacks such that $g^*f^* = (f \circ g)^*$ for any composable pair of morphisms. Compare also with Definition 4.36.2.

042G **Lemma 4.33.8.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is fibred over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$.*

Proof. Denote $p_i : \mathcal{S}_i \to \mathcal{C}$ the given functors. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. We claim that in this case F maps strongly cartesian morphisms to strongly cartesian morphisms. Namely, suppose that $\varphi : y \to x$ is strongly cartesian in $\mathcal{S}_1$. Set $f : V \to U$ equal to $p_1(\varphi)$. Suppose that $z' \in \mathrm{Ob}(\mathcal{S}_2)$, with $W = p_2(z')$, and we are given $g : W \to V$ and $\psi' : z' \to F(x)$ such that $p_2(\psi') = f \circ g$. Then

$$\psi = j \circ G(\psi') : G(z') \to G(F(x)) \to x$$

is a morphism in $\mathcal{S}_1$ with $p_1(\psi) = f \circ g$. Hence by assumption there exists a unique morphism $\xi : G(z') \to y$ lying over g such that $\psi = \varphi \circ \xi$. This in turn gives a morphism

$$\xi' = F(\xi) \circ i^{-1} : z' \to F(G(z')) \to F(y)$$

lying over g with $\psi' = F(\varphi) \circ \xi'$. We omit the verification that ξ' is unique. □

The conclusion from Lemma 4.33.8 is that equivalences map strongly cartesian morphisms to strongly cartesian morphisms. But this may not be the case for an arbitrary functor between fibred categories over $\mathcal{C}$. Hence we define the 2-category of fibred categories as follows.

02XP **Definition 4.33.9.** Let $\mathcal{C}$ be a category. The 2-*category of fibred categories over* $\mathcal{C}$ is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.32.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Note the condition on 1-morphisms. Note also that this is a true 2-category and not a $(2,1)$-category. Hence when taking 2-fibre products we first pass to the associated $(2,1)$-category.

02XQ **Lemma 4.33.10.** *Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.32.3.*

Proof. Basically what one has to show here is that given $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ morphisms of fibred categories over $\mathcal{C}$, then the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Lemma 4.32.3 is fibred. Let us show that $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ has plenty of strongly cartesian morphisms. Namely, suppose we have (U, x, y, ϕ) an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. And suppose $f : V \to U$ is a morphism in $\mathcal{C}$. Choose strongly cartesian morphisms $a : f^*x \to x$ in $\mathcal{X}$ lying over f and $b : f^*y \to y$ in $\mathcal{Y}$ lying over f. By assumption $F(a)$ and $G(b)$ are strongly cartesian. Since $\phi : F(x) \to G(y)$ is an isomorphism, by the uniqueness of strongly cartesian morphisms we find a unique isomorphism $f^*\phi : F(f^*x) \to G(f^*y)$ such

that $G(b) \circ f^*\phi = \phi \circ F(a)$. In other words $(a, b) : (V, f^*x, f^*y, f^*\phi) \to (U, x, y, \phi)$ is a morphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. We omit the verification that this is a strongly cartesian morphism (and that these are in fact the only strongly cartesian morphisms). □

02XR **Lemma 4.33.11.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a fibred category and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is a fibred category.*

Proof. Suppose that $\varphi : x' \to x$ is strongly cartesian with respect to p. We claim that φ is strongly cartesian with respect to p' also. Set $g = p'(\varphi)$, so that $g : V'/U \to V/U$ for some morphisms $f : V \to U$ and $f' : V' \to U$. Let $z \in \mathrm{Ob}(\mathcal{S})$. Set $p'(z) = (W \to U)$. To show that φ is strongly cartesian for p' we have to show

$$\mathrm{Mor}_{\mathcal{S}}(z, x') \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U)} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V'/U),$$

given by $\psi' \longmapsto (\varphi \circ \psi', p'(\psi'))$ is bijective. Suppose given an element (ψ, h) of the right hand side, then in particular $g \circ h = p(\psi)$, and by the condition that φ is strongly cartesian we get a unique morphism $\psi' : z \to x'$ with $\psi = \varphi \circ \psi'$ and $p(\psi') = h$. OK, and now $p'(\psi') : W/U \to V/U$ is a morphism whose corresponding map $W \to V$ is h, hence equal to h as a morphism in $\mathcal{C}/U$. Thus ψ' is a unique morphism $z \to x'$ which maps to the given pair (ψ, h). This proves the claim.

Finally, suppose given $g : V'/U \to V/U$ and x with $p'(x) = V/U$. Since $p : \mathcal{S} \to \mathcal{C}$ is a fibred category we see there exists a strongly cartesian morphism $\varphi : x' \to x$ with $p(\varphi) = g$. By the same argument as above it follows that $p'(\varphi) = g : V'/U \to V/U$. And as seen above the morphism φ is strongly cartesian. Thus the conditions of Definition 4.33.5 are satisfied and we win. □

09WV **Lemma 4.33.12.** *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred over $\mathcal{B}$ and $\mathcal{B}$ is fibred over $\mathcal{C}$, then $\mathcal{A}$ is fibred over $\mathcal{C}$.*

Proof. This follows from the definitions and Lemma 4.33.3. □

06N5 **Lemma 4.33.13.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$ with $x \to y$ strongly cartesian. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists, $p(x \times_y z) = p(x) \times_{p(y)} p(z)$, and $x \times_y z \to z$ is strongly cartesian.*

Proof. Pick a strongly cartesian morphism $\mathrm{pr}_2^* z \to z$ lying over $\mathrm{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$. Then $\mathrm{pr}_2^* z = x \times_y z$ by Lemma 4.33.4. □

08NF **Lemma 4.33.14.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. There exist 1-morphisms of fibred categories over $\mathcal{C}$*

$$\mathcal{X} \underset{w}{\overset{u}{\rightleftarrows}} \mathcal{X}' \xrightarrow{v} \mathcal{Y}$$

such that $F = v \circ u$ and such that

(1) *$u : \mathcal{X} \to \mathcal{X}'$ is fully faithful,*
(2) *w is left adjoint to u, and*
(3) *$v : \mathcal{X}' \to \mathcal{Y}$ is a fibred category.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : y \to F(x)$ is a morphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b) : U \to U'$ and such that $f' \circ b = F(a) \circ f$.

Let us make a choice of pullbacks for both p and q and let us use the same notation to indicate them. Let (U, x, y, f) be an object and let $h : V \to U$ be a morphism. Consider the morphism $c : (V, h^*x, h^*y, h^*f) \to (U, x, y, f)$ coming from the given strongly cartesian maps $h^*x \to x$ and $h^*y \to y$. We claim c is strongly cartesian in $\mathcal{X}'$ over $\mathcal{C}$. Namely, suppose we are given an object (W, x', y', f') of $\mathcal{X}'$, a morphism $(a, b) : (W, x', y', f') \to (U, x, y, f)$ lying over $W \to U$, and a factorization $W \to V \to U$ of $W \to U$ through h. As $h^*x \to x$ and $h^*y \to y$ are strongly cartesian we obtain morphisms $a' : x' \to h^*x$ and $b' : y' \to h^*y$ lying over the given morphism $W \to V$. Consider the diagram

$$\begin{array}{ccccc} y' & \longrightarrow & h^*y & \longrightarrow & y \\ {\scriptstyle f'}\downarrow & & {\scriptstyle h^*f}\downarrow & & {\scriptstyle f}\downarrow \\ F(x') & \longrightarrow & F(h^*x) & \longrightarrow & F(x) \end{array}$$

The outer rectangle and the right square commute. Since F is a 1-morphism of fibred categories the morphism $F(h^*x) \to F(x)$ is strongly cartesian. Hence the left square commutes by the universal property of strongly cartesian morphisms. This proves that $\mathcal{X}'$ is fibred over $\mathcal{C}$.

The functor $u : \mathcal{X} \to \mathcal{X}'$ is given by $x \mapsto (p(x), x, F(x), \mathrm{id})$. This is fully faithful. The functor $\mathcal{X}' \to \mathcal{Y}$ is given by $(U, x, y, f) \mapsto y$. The functor $w : \mathcal{X}' \to \mathcal{X}$ is given by $(U, x, y, f) \mapsto x$. Each of these functors is a 1-morphism of fibred categories over $\mathcal{C}$ by our description of strongly cartesian morphisms of $\mathcal{X}'$ over $\mathcal{C}$. Adjointness of w and u means that

$$\mathrm{Mor}_{\mathcal{X}}(x, x') = \mathrm{Mor}_{\mathcal{X}'}((U, x, y, f), (p(x'), x', F(x'), \mathrm{id})),$$

which follows immediately from the definitions.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a fibred category. Let $c : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Set $V = q(y')$ and let $h = q(c) : V \to U$. Let $a : h^*x \to x$ and $b : h^*y \to y$ be the strongly cartesian morphisms covering h. Since F is a 1-morphism of fibred categories we may identify $h^*F(x) = F(h^*x)$ with strongly cartesian morphism $F(a) : F(h^*x) \to F(x)$. By the universal property of $b : h^*y \to y$ there is a morphism $c' : y' \to h^*y$ in $\mathcal{Y}_V$ such that $c = b \circ c'$. We claim that

$$(a, c) : (V, h^*x, y', h^*f \circ c') \longrightarrow (U, x, y, f)$$

is strongly cartesian in $\mathcal{X}'$ over $\mathcal{Y}$. To see this let (W, x_1, y_1, f_1) be an object of $\mathcal{X}'$, let $(a_1, b_1) : (W, x_1, y_1, f_1) \to (U, x, y, f)$ be a morphism and let $b_1 = c \circ b_1'$ for some morphism $b_1' : y_1 \to y'$. Then

$$(a_1', b_1') : (W, x_1, y_1, f_1) \longrightarrow (V, h^*x, y', h^*f \circ c')$$

(where $a_1' : x_1 \to h^*x$ is the unique morphism lying over the given morphism $q(b_1') : W \to V$ such that $a_1 = a \circ a_1'$) is the desired morphism. □

4.34. Inertia

04Z2 Given fibred categories $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ over a category $\mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ we have the diagonal morphism

$$\Delta = \Delta_{\mathcal{S}/\mathcal{S}'} : \mathcal{S} \longrightarrow \mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$$

in the $(2,1)$-category of fibred categories over $\mathcal{C}$.

034H **Lemma 4.34.1.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Consider the category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ over $\mathcal{C}$ whose*

(1) *objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ is an automorphism with $F(\alpha) = id$,*
(2) *morphisms $(x, \alpha) \to (y, \beta)$ are given by morphisms $\phi : x \to y$ such that*

$$\begin{array}{ccc} x & \xrightarrow{\phi} & y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ x & \xrightarrow{\phi} & y \end{array}$$

commutes, and
(3) *the functor $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ is given by $(x, \alpha) \mapsto p(x)$.*

Then

(1) *there is an equivalence*

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$$

in the $(2,1)$-category of categories over $\mathcal{C}$, and
(2) *$\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ is a fibred category over $\mathcal{C}$.*

Proof. Note that (2) follows from (1) by Lemmas 4.33.10 and 4.33.8. Thus it suffices to prove (1). We will use without further mention the construction of the 2-fibre product from Lemma 4.33.10. In particular an object of $\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$ is a triple $(x, y, (\iota, \kappa))$ where x and y are objects of $\mathcal{S}$, and $(\iota, \kappa) : (x, x, \mathrm{id}_{F(x)}) \to (y, y, \mathrm{id}_{F(y)})$ is an isomorphism in $\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$. This just means that $\iota, \kappa : x \to y$ are isomorphisms and that $F(\iota) = F(\kappa)$. Consider the functor

$$I_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$$

which to an object (x, α) of the left hand side assigns the object $(x, x, (\alpha, \mathrm{id}_x))$ of the right hand side and to a morphism ϕ of the left hand side assigns the morphism (ϕ, ϕ) of the right hand side. We claim that a quasi-inverse to that morphism is given by the functor

$$\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S} \longrightarrow I_{\mathcal{S}/\mathcal{S}'}$$

which to an object $(x, y, (\iota, \kappa))$ of the left hand side assigns the object $(x, \kappa^{-1} \circ \iota)$ of the right hand side and to a morphism $(\phi, \phi') : (x, y, (\iota, \kappa)) \to (z, w, (\lambda, \mu))$ of the left hand side assigns the morphism ϕ. Indeed, the endo-functor of $I_{\mathcal{S}/\mathcal{S}'}$ induced by composing the two functors above is the identity on the nose, and the endo-functor induced on $\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$ is isomorphic to the identity via the natural isomorphism

$$(\mathrm{id}_x, \kappa) : (x, x, (\kappa^{-1} \circ \iota, \mathrm{id}_x)) \longrightarrow (x, y, (\iota, \kappa)).$$

Some details omitted. □

034I **Definition 4.34.2.** Let $\mathcal{C}$ be a category.

(1) Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. The *relative inertia of $\mathcal{S}$ over $\mathcal{S}'$* is the fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ of Lemma 4.34.1.
(2) By the *inertia fibred category $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$* we mean $\mathcal{I}_{\mathcal{S}} = \mathcal{I}_{\mathcal{S}/\mathcal{C}}$.

Note that there are canonical 1-morphisms

042H (4.34.2.1) $$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}} \longrightarrow \mathcal{S}$$

of fibred categories over $\mathcal{C}$. In terms of the description of Lemma 4.34.1 these simply map the object (x, α) to the object x and the morphism $\phi : (x, \alpha) \to (y, \beta)$ to the morphism $\phi : x \to y$. There is also a *neutral section*

04Z3 (4.34.2.2) $$e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \quad \text{and} \quad e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}}$$

defined by the rules $x \mapsto (x, \mathrm{id}_x)$ and $(\phi : x \to y) \mapsto \phi$. This is a right inverse to (4.34.2.1). Given a 2-commutative square

$$\begin{array}{ccc} \mathcal{S}_1 & \xrightarrow[G]{} & \mathcal{S}_2 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \mathcal{S}_1' & \xrightarrow{G'} & \mathcal{S}_2' \end{array}$$

there are *functoriality maps*

04Z4 (4.34.2.3) $$\mathcal{I}_{\mathcal{S}_1/\mathcal{S}_1'} \longrightarrow \mathcal{I}_{\mathcal{S}_2/\mathcal{S}_2'} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2}$$

defined by the rules $(x, \alpha) \mapsto (G(x), G(\alpha))$ and $\phi \mapsto G(\phi)$. In particular there is always a comparison map

04Z5 (4.34.2.4) $$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{I}_{\mathcal{S}}$$

and all the maps above are compatible with this.

04Z6 **Lemma 4.34.3.** *Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of categories fibred over a category $\mathcal{C}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{S}/\mathcal{S}'} & \xrightarrow[(4.34.2.4)]{} & \mathcal{I}_{\mathcal{S}} \\ {\scriptstyle F\circ(4.34.2.1)}\downarrow & & \downarrow{\scriptstyle (4.34.2.3)} \\ \mathcal{S}' & \xrightarrow{e} & \mathcal{I}_{\mathcal{S}'} \end{array}$$

is a 2-fibre product.

Proof. Omitted. □

4.35. Categories fibred in groupoids

003S In this section we explain how to think about categories fibred in groupoids and we see how they are basically the same as functors with values in the $(2, 1)$-category of groupoids.

003T **Definition 4.35.1.** Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. We say that $\mathcal{S}$ is *fibred in groupoids* over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

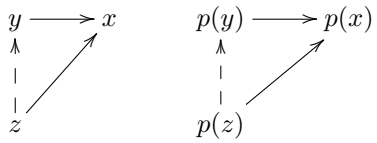

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

03WP (4.35.1.1)

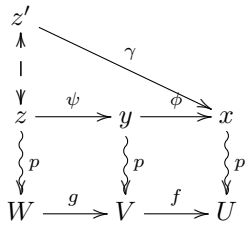

where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

It should be clear from this discussion that a category fibred in groupoids is very closely related to a fibred category. Here is the result.

003V **Lemma 4.35.2.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and*
(2) *all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 4.33.7 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

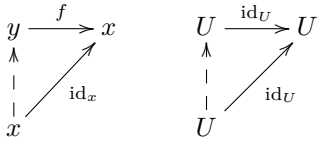

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = \mathrm{id}_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = \mathrm{id}_y$. Then $fgh = f : y \to x$. We have $fg = \mathrm{id}_x$, so $h = f$. Condition (2) of Definition 4.35.1 says exactly that every morphism of

$\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 4.35.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 4.35.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 4.35.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) $\Rightarrow$ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 4.33.7. □

03WQ **Lemma 4.35.3.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ defined as follows*

(1) $\mathrm{Ob}(\mathcal{S}') = \mathrm{Ob}(\mathcal{S})$, *and*
(2) *for $x, y \in \mathrm{Ob}(\mathcal{S}')$ the set of morphisms between x and y in $\mathcal{S}'$ is the set of strongly cartesian morphisms between x and y in $\mathcal{S}$.*

Let $p' : \mathcal{S}' \to \mathcal{C}$ be the restriction of p to $\mathcal{S}'$. Then $p' : \mathcal{S}' \to \mathcal{C}$ is fibred in groupoids.

Proof. Note that the construction makes sense since by Lemma 4.33.2 the identity morphism of any object of $\mathcal{S}$ is strongly cartesian, and the composition of strongly cartesian morphisms is strongly cartesian. The first lifting property of Definition 4.35.1 follows from the condition that in a fibred category given any morphism $f : V \to U$ and x lying over U there exists a strongly cartesian morphism $\varphi : y \to x$ lying over f. Let us check the second lifting property of Definition 4.35.1 for the category $p' : \mathcal{S}' \to \mathcal{C}$ over $\mathcal{C}$. To do this we argue as in the discussion following Definition 4.35.1. Thus in Diagram 4.35.1.1 the morphisms ϕ, ψ and γ are strongly cartesian morphisms of $\mathcal{S}$. Hence γ and $\phi \circ \psi$ are strongly cartesian morphisms of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ and having the same target in $\mathcal{S}$. By the discussion following Definition 4.33.1 this means these two arrows are isomorphic as desired (here we use also that any isomorphism in $\mathcal{S}$ is strongly cartesian, by Lemma 4.33.2 again). □

003U **Example 4.35.4.** A homomorphism of groups $p : G \to H$ gives rise to a functor $p : \mathcal{S} \to \mathcal{C}$ as in Example 4.2.12. This functor $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in \mathrm{Ob}(\mathcal{C})$ is the category associated to the kernel of p as in Example 4.2.6.

Given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts. One example is the functor from Example 4.35.4 when $G \to H$ is not surjective. Here is another example.

02C4 **Example 4.35.5.** Let $\mathrm{Ob}(\mathcal{C}) = \{A, B, T\}$ and $\mathrm{Mor}_{\mathcal{C}}(A, B) = \{f\}$, $\mathrm{Mor}_{\mathcal{C}}(B, T) = \{g\}$, $\mathrm{Mor}_{\mathcal{C}}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See

the diagram below for a picture of this category. Now let $\mathrm{Ob}(\mathcal{S}) = \{A', B', T'\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', B') = \emptyset$, $\mathrm{Mor}_{\mathcal{S}}(B', T') = \{g'\}$, $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $\mathrm{Mor}_{\mathcal{S}}(A', B') = \{f'_1, f'_2\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\} = \{g'f'_1\} = \{g'f'_2\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.

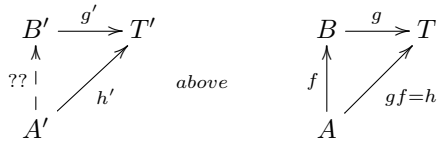

Later we would like to make assertions such as "any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or "any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with.

02XS **Definition 4.35.6.** Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Definition 4.33.9) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in groupoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2, 1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

0041 **Lemma 4.35.7.** *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.32.3.*

Proof. By Lemma 4.33.10 the fibre product as described in Lemma 4.32.3 is a fibred category. Hence it suffices to prove that the fibre categories are groupoids, see Lemma 4.35.2. By Lemma 4.32.5 it is enough to show that the 2-fibre product of groupoids is a groupoid, which is clear (from the construction in Lemma 4.31.4 for example). □

0H2E **Remark 4.35.8.** Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. Let $p : \mathcal{S} \to \mathcal{C}$ be the given functor. We claim the 2-fibre product of Lemma 4.35.7 is canonically equivalent (as a category) to the one in Example 4.31.3. Objects of the former are quadruples (U, x, y, α) where $p(\alpha) = \mathrm{id}_U$ (see Lemma 4.32.3) and objects of the latter are triples (x, y, α) (see Example 4.31.3). The equivalence between the two categories is given by the rules $(U, x, y, \alpha) \mapsto (x, y, \alpha)$ and $(x, y, \alpha) \mapsto (p(f(x)), x, y', \alpha')$ where $\alpha' = g(\gamma)^{-1} \circ \alpha$ and $\gamma : y' \to y$ is a lift of the arrow $p(\alpha) : p(f(x)) \to p(g(y))$. Details omitted.

003Z **Lemma 4.35.9.** *Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G : \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$.*

(1) *Then G is faithful (resp. fully faithful, resp. an equivalence) if and only if for each $U \in \mathrm{Ob}(\mathcal{C})$ the induced functor $G_U : \mathcal{S}_U \to \mathcal{S}'_U$ is faithful (resp. fully faithful, resp. an equivalence).*
(2) *If G is an equivalence, then G is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{C}$.*

Proof. Let x, y be objects of $\mathcal{S}$ lying over the same object U. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{S}}(x,y) & \xrightarrow{\quad G \quad} & \mathrm{Mor}_{\mathcal{S}'}(G(x), G(y)) \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p'} & \\ & \mathrm{Mor}_{\mathcal{C}}(U, U) & \end{array}$$

From this diagram it is clear that if G is faithful (resp. fully faithful) then so is each G_U.

Suppose G is an equivalence. For every object x' of $\mathcal{S}'$ there exists an object x of $\mathcal{S}$ such that $G(x)$ is isomorphic to x'. Suppose that x' lies over U' and x lies over U. Then there is an isomorphism $f : U' \to U$ in $\mathcal{C}$, namely, p' applied to the isomorphism $x' \to G(x)$. By the axioms of a category fibred in groupoids there exists an arrow $f^*x \to x$ of $\mathcal{S}$ lying over f. Hence there exists an isomorphism $\alpha : x' \to G(f^*x)$ such that $p'(\alpha) = \mathrm{id}_{U'}$ (this time by the axioms for $\mathcal{S}'$). All in all we conclude that for every object x' of $\mathcal{S}'$ we can choose a pair $(o_{x'}, \alpha_{x'})$ consisting of an object $o_{x'}$ of $\mathcal{S}$ and an isomorphism $\alpha_{x'} : x' \to G(o_{x'})$ with $p'(\alpha_{x'}) = \mathrm{id}_{p'(x')}$. From this point on we proceed as usual (see proof of Lemma 4.2.19) to produce an inverse functor $F : \mathcal{S}' \to \mathcal{S}$, by taking $x' \mapsto o_{x'}$ and $\varphi' : x' \to y'$ to the unique arrow $\varphi_{\varphi'} : o_{x'} \to o_{y'}$ with $\alpha_{y'}^{-1} \circ G(\varphi_{\varphi'}) \circ \alpha_{x'} = \varphi'$. With these choices F is a functor over $\mathcal{C}$. We omit the verification that $G \circ F$ and $F \circ G$ are 2-isomorphic to the respective identity functors (in the 2-category of categories fibred in groupoids over $\mathcal{C}$).

Suppose that G_U is faithful (resp. fully faithful) for all $U \in \mathrm{Ob}(\mathcal{C})$. To show that G is faithful (resp. fully faithful) we have to show for any objects $x, y \in \mathrm{Ob}(\mathcal{S})$ that G induces an injection (resp. bijection) between $\mathrm{Mor}_{\mathcal{S}}(x,y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$. Set $U = p(x)$ and $V = p(y)$. It suffices to prove that G induces an injection (resp. bijection) between morphism $x \to y$ lying over f to morphisms $G(x) \to G(y)$ lying over f for any morphism $f : U \to V$. Now fix $f : U \to V$. Denote $f^*y \to y$ a pullback. Then also $G(f^*y) \to G(y)$ is a pullback. The set of morphisms from x to y lying over f is bijective to the set of morphisms between x and f^*y lying over id_U. (By the second axiom of a category fibred in groupoids.) Similarly the set of morphisms from $G(x)$ to $G(y)$ lying over f is bijective to the set of morphisms between $G(x)$ and $G(f^*y)$ lying over id_U. Hence the fact that G_U is faithful (resp. fully faithful) gives the desired result.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in \mathrm{Ob}(\mathcal{S}')$ then $z' \in \mathrm{Ob}(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in \mathrm{Ob}(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. $\square$

04Z7 **Lemma 4.35.10.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids. Let $G : \mathcal{S} \to \mathcal{S}'$ be a functor over $\mathcal{C}$. Then G is fully faithful if and only if the diagonal*

$$\Delta_G : \mathcal{S} \longrightarrow \mathcal{S} \times_{G,\mathcal{S}',G} \mathcal{S}$$

is an equivalence.

Proof. By Lemma 4.35.9 it suffices to look at fibre categories over an object U of $\mathcal{C}$. An object of the right hand side is a triple (x, x', α) where $\alpha : G(x) \to G(x')$ is a morphism in $\mathcal{S}'_U$. The functor Δ_G maps the object x of $\mathcal{S}_U$ to the triple $(x, x, \mathrm{id}_{G(x)})$. Note that (x, x', α) is in the essential image of Δ_G if and only if $\alpha = G(\beta)$ for some morphism $\beta : x \to x'$ in $\mathcal{S}_U$ (details omitted). Hence in order for Δ_G to be an equivalence, every α has to be the image of a morphism $\beta : x \to x'$, and also every two distinct morphisms $\beta, \beta' : x \to x'$ have to give distinct morphisms $G(\beta), G(\beta')$. This proves the lemma. □

03YT **Lemma 4.35.11.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_i$, $i = 1, 2, 3, 4$ be categories fibred in groupoids over $\mathcal{C}$. Suppose that $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{S}_3 \to \mathcal{S}_4$ are equivalences over $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_2, \mathcal{S}_3) \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_4), \quad \alpha \longmapsto \psi \circ \alpha \circ \varphi$$

is an equivalence of categories.

Proof. This is a generality and holds in any 2-category. □

042I **Lemma 4.35.12.** *Let $\mathcal{C}$ be a category. If $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then so is the inertia fibred category $\mathcal{I}_\mathcal{S} \to \mathcal{C}$.*

Proof. Clear from the construction in Lemma 4.34.1 or by using (from the same lemma) that $I_\mathcal{S} \to \mathcal{S} \times_{\Delta,\mathcal{S}\times_\mathcal{C}\mathcal{S},\Delta} \mathcal{S}$ is an equivalence and appealing to Lemma 4.35.7. □

02XT **Lemma 4.35.13.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is fibred in groupoids.*

Proof. We have already seen in Lemma 4.33.11 that p' is a fibred category. Hence it suffices to prove the fibre categories are groupoids, see Lemma 4.35.2. For $V \in \mathrm{Ob}(\mathcal{C})$ we have

$$\mathcal{S}_V = \coprod\nolimits_{f:V\to U} \mathcal{S}_{(f:V\to U)}$$

where the left hand side is the fibre category of p and the right hand side is the disjoint union of the fibre categories of p'. Hence the result. □

09WW **Lemma 4.35.14.** *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred in groupoids over $\mathcal{B}$ and $\mathcal{B}$ is fibred in groupoids over $\mathcal{C}$, then $\mathcal{A}$ is fibred in groupoids over $\mathcal{C}$.*

Proof. One can prove this directly from the definition. However, we will argue using the criterion of Lemma 4.35.2. By Lemma 4.33.12 we see that $\mathcal{A}$ is fibred over $\mathcal{C}$. To finish the proof we show that the fibre category $\mathcal{A}_U$ is a groupoid for U in $\mathcal{C}$. Namely, if $x \to y$ is a morphism of $\mathcal{A}_U$, then its image in $\mathcal{B}$ is an isomorphism as $\mathcal{B}_U$ is a groupoid. But then $x \to y$ is an isomorphism, for example by Lemma 4.33.2 and the fact that every morphism of $\mathcal{A}$ is strongly $\mathcal{B}$-cartesian (see Lemma 4.35.2). □

06N6 **Lemma 4.35.15.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists and $p(x \times_y z) = p(x) \times_{p(y)} p(z)$.*

Proof. Follows from Lemma 4.33.13. □

06N7 **Lemma 4.35.16.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. There exists a factorization $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ by 1-morphisms of categories fibred in groupoids over $\mathcal{C}$ such that $\mathcal{X} \to \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and such that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{Y}$.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : F(x) \to y$ is an isomorphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b)$ and such that $f' \circ F(a) = b \circ f$. In other words $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ with the construction of the 2-fibre product from Lemma 4.32.3. By Lemma 4.35.7 we see that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{C}$ and that $\mathcal{X}' \to \mathcal{Y}$ is a morphism of categories over $\mathcal{C}$. As functor $\mathcal{X} \to \mathcal{X}'$ we take $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ on objects and $(a : x \to x') \mapsto (a, F(a))$ on morphisms. It is clear that the composition $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ equals F. We omit the verification that $\mathcal{X} \to \mathcal{X}'$ is an equivalence of fibred categories over $\mathcal{C}$.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a category fibred in groupoids. Let $b : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Because $\mathcal{X}$ is fibred in groupoids over $\mathcal{C}$ we can find a morphism $a : x' \to x$ lying over $U' = q(y') \to q(y) = U$. Since $\mathcal{Y}$ is fibred in groupoids over $\mathcal{C}$ and since both $F(x') \to F(x)$ and $y' \to y$ lie over the same morphism $U' \to U$ we can find $f' : F(x') \to y'$ lying over $\mathrm{id}_{U'}$ such that $f \circ F(a) = b \circ f'$. Hence we obtain $(a, b) : (U', x', y', f') \to (U, x, y, f)$. This verifies the first condition (1) of Definition 4.35.1. To see (2) let $(a, b) : (U', x', y', f') \to (U, x, y, f)$ and $(a', b') : (U'', x'', y'', f'') \to (U, x, y, f)$ be morphisms of $\mathcal{X}'$ and let $b'' : y' \to y''$ be a morphism of $\mathcal{Y}$ such that $b' \circ b'' = b$. We have to show that there exists a unique morphism $a'' : x' \to x''$ such that $f'' \circ F(a'') = b'' \circ f'$ and such that $(a', b') \circ (a'', b'') = (a, b)$. Because $\mathcal{X}$ is fibred in groupoids we know there exists a unique morphism $a'' : x' \to x''$ such that $a' \circ a'' = a$ and $p(a'') = q(b'')$. Because $\mathcal{Y}$ is fibred in groupoids we see that $F(a'')$ is the unique morphism $F(x') \to F(x'')$ such that $F(a') \circ F(a'') = F(a)$ and $q(F(a'')) = q(b'')$. The relation $f'' \circ F(a'') = b'' \circ f'$ follows from this and the given relations $f \circ F(a) = b \circ f'$ and $f \circ F(a') = b' \circ f''$. □

06N8 **Lemma 4.35.17.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume we have a 2-commutative diagram*

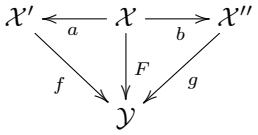

where a and b are equivalences of categories over $\mathcal{C}$ and f and g are categories fibred in groupoids. Then there exists an equivalence $h : \mathcal{X}'' \to \mathcal{X}'$ of categories over $\mathcal{Y}$ such that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{C}$.

If the diagram above actually commutes, then we can arrange it so that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{Y}$.

Proof. We will show that both $\mathcal{X}'$ and $\mathcal{X}''$ over $\mathcal{Y}$ are equivalent to the category fibred in groupoids $\mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ over $\mathcal{Y}$, see proof of Lemma 4.35.16. Choose a quasi-inverse $b^{-1} : \mathcal{X}'' \to \mathcal{X}$ in the 2-category of categories over $\mathcal{C}$. Since the right triangle of the diagram is 2-commutative we see that

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow[b^{-1}]{} & \mathcal{X}'' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{Y} & \longleftarrow & \mathcal{Y} \end{array}$$

is 2-commutative. Hence we obtain a 1-morphism $c : \mathcal{X}'' \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ by the universal property of the 2-fibre product. Moreover c is a morphism of categories over $\mathcal{Y}$ (!) and an equivalence (by the assumption that b is an equivalence, see Lemma 4.31.7). Hence c is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{Y}$ by Lemma 4.35.9.

We still have to construct a 2-isomorphism between $c \circ b$ and the functor $d : \mathcal{X} \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$, $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ constructed in the proof of Lemma 4.35.16. Let $\alpha : F \to g \circ b$ and $\beta : b^{-1} \circ b \to \mathrm{id}$ be 2-isomorphisms between 1-morphisms of categories over $\mathcal{C}$. Note that $c \circ b$ is given by the rule

$$x \mapsto (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

on objects. Then we see that

$$(\beta_x, \alpha_x) : (p(x), x, F(x), \mathrm{id}_{F(x)}) \longrightarrow (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

is a functorial isomorphism which gives our 2-morphism $d \to b \circ c$. Finally, if the diagram commutes then α_x is the identity for all x and we see that this 2-morphism is a 2-morphism in the 2-category of categories over $\mathcal{Y}$. □

4.36. Presheaves of categories

02XU In this section we compare the notion of fibred categories with the closely related notion of a "presheaf of categories". The basic construction is explained in the following example.

02XV **Example 4.36.1.** Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to Cat$ is a functor to the 2-category of categories, see Definition 4.29.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. Let us check that this is indeed

a fibred category. Given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then we claim $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a strongly cartesian lift of f. We have to show a h in the diagram on the left determines (h, ν) on the right:

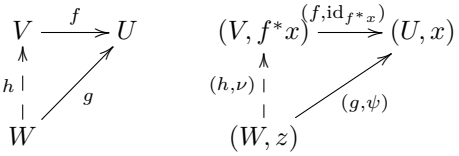

Just take $\nu = \psi$ which works because $f \circ h = g$ and hence $g^*x = h^*f^*x$. Moreover, this is the only lift making the diagram (on the right) commute.

02XW **Definition 4.36.2.** Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to Cat$ is a functor to the 2-category of categories. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the fibred category constructed in Example 4.36.1. A *split fibred category* is a fibred category isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

02XX **Lemma 4.36.3.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a fibred category over $\mathcal{C}$. Then $\mathcal{S}$ is split if and only if for some choice of pullbacks (see Definition 4.33.6) the pullback functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.*

Proof. This is immediate from the definitions. □

004A **Lemma 4.36.4.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. There exists a contravariant functor $F : \mathcal{C} \to Cat$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ in the 2-category of fibred categories over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.*

Proof. Let us make a choice of pullbacks (see Definition 4.33.6). By Lemma 4.33.7 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_U).

At this point we need to check that p' makes $\mathcal{S}'$ into a fibred category over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ which maps strongly cartesian morphisms to strongly cartesian morphisms. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 4.33.7 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

4.37. Presheaves of groupoids

0048 In this section we compare the notion of categories fibred in groupoids with the closely related notion of a "presheaf of groupoids". The basic construction is explained in the following example.

0049 **Example 4.37.1.** This example is the analogue of Example 4.36.1, for "presheaves of groupoids" instead of "presheaves of categories". The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \mathit{Groupoids}$ is a functor to the category of groupoids, see Definition 4.29.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned}\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x)\end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}_F$ is fibred in groupoids over $\mathcal{C}$, as we have already seen in Example 4.36.1 that $\mathcal{S}_F$ is a fibred category, see Lemma 4.35.2. But we can also prove conditions (1), (2) of Definition 4.35.1 directly as follows: (1) Lifts of morphisms exist since given $f : V \to U$ in $\mathcal{C}$ and (U, x) an object of $\mathcal{S}_F$ over U, then $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a lift of f. (2) Suppose given solid diagrams as follows

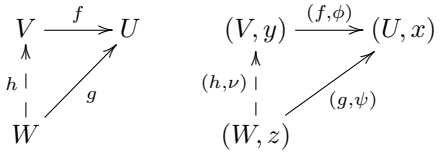

Then for the dotted arrows we have $\nu = (h^*\phi)^{-1} \circ \psi$ so given h there exists a ν which is unique by uniqueness of inverses.

04TL **Definition 4.37.2.** Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \mathit{Groupoids}$ is a functor to the 2-category of groupoids. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the category fibred in groupoids constructed in Example 4.37.1. A *split category fibred in groupoids* is a category fibred in groupoids isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

02XY **Lemma 4.37.3.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a contravariant functor $F : \mathcal{C} \to \mathit{Groupoids}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.*

Proof. Make a choice of pullbacks (see Definition 4.33.6). By Lemmas 4.33.7 and 4.35.2 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e.,

$x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^* x_1 \to f_2^* x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_U).

At this point we need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^* f_i^* x_i = (f_i \circ g)^* x_i$ from Lemma 4.35.2 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^* x_1$ to $(f_2 \circ g)^* x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

We will see an alternative proof of this lemma in Section 4.42.

4.38. Categories fibred in sets

0042

02Y0 **Definition 4.38.1.** A category is called *discrete* if the only morphisms are the identity morphisms.

A discrete category has only one interesting piece of information: its set of objects. Thus we sometime confuse discrete categories with sets.

0043 **Definition 4.38.2.** Let $\mathcal{C}$ be a category. A *category fibred in sets*, or a *category fibred in discrete categories* is a category fibred in groupoids all of whose fibre categories are discrete.

We want to clarify the relationship between categories fibred in sets and presheaves (see Definition 4.3.3). To do this it makes sense to first make the following definition.

04S8 **Definition 4.38.3.** Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in sets over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.35.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in sets.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category. Here is the obligatory lemma on the existence of 2-fibre products.

0047 **Lemma 4.38.4.** *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.32.3 returns a category fibred in sets if one starts out with such.*

Proof. Omitted. □

04TM **Example 4.38.5.** This example is the analogue of Examples 4.36.1 and 4.37.1 for presheaves instead of "presheaves of categories". The output will be a category fibred in sets instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to Sets$ is a presheaf. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^* : F(U) \to F(V)$. We construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) = \{f \in \mathrm{Mor}_{\mathcal{C}}(V, U) \mid f^*x = y\}$$

Composition is inherited from composition in $\mathcal{C}$ which works as $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. As every fibre category $\mathcal{S}_{F,U}$ is discrete with underlying set $F(U)$ and we have already see in Example 4.37.1 that $\mathcal{S}_F$ is a category fibred in groupoids, we conclude that $\mathcal{S}_F$ is fibred in sets.

02Y2 **Lemma 4.38.6.** *Let $\mathcal{C}$ be a category. The only 2-morphisms between categories fibred in sets are identities. In other words, the 2-category of categories fibred in sets is a category. Moreover, there is an equivalence of categories*

$$\left\{\begin{array}{c}\textit{the category of presheaves}\\ \textit{of sets over } \mathcal{C}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\textit{the category of categories}\\ \textit{fibred in sets over } \mathcal{C}\end{array}\right\}$$

The functor from left to right is the construction $F \to \mathcal{S}_F$ discussed in Example 4.38.5. The functor from right to left assigns to $p : \mathcal{S} \to \mathcal{C}$ the presheaf of objects $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$.

Proof. The first assertion is clear, as the only morphisms in the fibre categories are identities.

Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in \mathrm{Ob}(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 4.35.2. It follows that we may think of the assignments $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$. □

Here is an important example of a category fibred in sets.

0044 **Example 4.38.7.** Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider the representable presheaf $h_X = \mathrm{Mor}_{\mathcal{C}}(-, X)$ (see Example 4.3.4). On the other hand, consider the category $p : \mathcal{C}/X \to \mathcal{C}$ from Example 4.2.13. The fibre category $(\mathcal{C}/X)_U$ has as objects morphisms $h : U \to X$, and only identities as morphisms. Hence we see that under the correspondence of Lemma 4.38.6 we have

$$h_X \longleftrightarrow \mathcal{C}/X.$$

In other words, the category $\mathcal{C}/X$ is canonically equivalent to the category $\mathcal{S}_{h_X}$ associated to h_X in Example 4.38.5.

For this reason it is tempting to define a "representable" object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition for use since we prefer to have a notion which is invariant under equivalences. To make this precise we study exactly which categories fibred in groupoids are equivalent to categories fibred in sets.

4.39. Categories fibred in setoids

04S9

02XZ **Definition 4.39.1.** Let us call a category a *setoid*[8] if it is a groupoid where every object has exactly one automorphism: the identity.

If C is a set with an equivalence relation $\sim$, then we can make a setoid $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and $\mathrm{Mor}_{\mathcal{C}}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $\mathrm{Mor}_{\mathcal{C}}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setoid category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism – not equivalence) from the procedure just described.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes (and adds identity morphisms). In terms of sets endowed with an equivalence relation this corresponds to taking the quotient by the equivalence relation.

04SA **Definition 4.39.2.** Let $\mathcal{C}$ be a category. A *category fibred in setoids* is a category fibred in groupoids all of whose fibre categories are setoids.

Below we will clarify the relationship between categories fibred in setoids and categories fibred in sets.

02Y1 **Definition 4.39.3.** Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in setoids over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.35.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in setoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category.
Here is the obligatory lemma on the existence of 2-fibre products.

04SB **Lemma 4.39.4.** *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.32.3 returns a category fibred in setoids if one starts out with such.*

Proof. Omitted. □

0045 **Lemma 4.39.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a category over $\mathcal{C}$.*

(1) *If $\mathcal{S} \to \mathcal{S}'$ is an equivalence over $\mathcal{C}$ with $\mathcal{S}'$ fibred in sets over $\mathcal{C}$, then*
   (a) *$\mathcal{S}$ is fibred in setoids over $\mathcal{C}$, and*
   (b) *for each $U \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Ob}(\mathcal{S}_U) \to \mathrm{Ob}(\mathcal{S}'_U)$ identifies the target as the set of isomorphism classes of the source.*
(2) *If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $can : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$.*

[8] A set on steroids!?

Proof. Let us prove (2). An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in \mathrm{Ob}(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and $x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. Some details omitted. □

Thus categories fibred in setoids are exactly the categories fibred in groupoids which are equivalent to categories fibred in sets. Moreover, an equivalence of categories fibred in sets is an isomorphism by Lemma 4.38.6.

04SC **Lemma 4.39.6.** *Let $\mathcal{C}$ be a category. The construction of Lemma 4.39.5 part (2) gives a functor*

$$F : \left\{ \begin{array}{c} \text{the 2-category of categories} \\ \text{fibred in setoids over } \mathcal{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{the category of categories} \\ \text{fibred in sets over } \mathcal{C} \end{array} \right\}$$

(see Definition 4.29.5). This functor is an equivalence in the following sense:

1. *for any two 1-morphisms $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ there exists a unique 2-isomorphism $f \to g$,*
2. *for any morphism $h : F(\mathcal{S}_1) \to F(\mathcal{S}_2)$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = h$, and*
3. *any category fibred in sets $\mathcal{S}$ is equal to $F(\mathcal{S})$.*

In particular, defining $F_i \in PSh(\mathcal{C})$ by the rule $F_i(U) = \mathrm{Ob}(\mathcal{S}_{i,U})/\cong$, we have

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{PSh(\mathcal{C})}(F_1, F_2)$$

More precisely, given any map $\phi : F_1 \to F_2$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Lemma 4.38.6 the target of F is a category hence the assertion makes sense. The construction of Lemma 4.39.5 part (2) assigns to $\mathcal{S}$ the category fibred in sets whose value over U is the set of isomorphism classes in $\mathcal{S}_U$. Hence it is clear that it defines a functor as indicated. Let $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ be as in (1). For each object U of $\mathcal{C}$ and each object x of $\mathcal{S}_{1,U}$ we see that $f(x) \cong g(x)$ by assumption. As $\mathcal{S}_2$ is fibred in setoids there exists a unique isomorphism $t_x : f(x) \to g(x)$ in $\mathcal{S}_{2,U}$. Clearly the rule $x \mapsto t_x$ gives the desired 2-isomorphism $f \to g$. We omit the proofs of (2) and (3). To see the final assertion use Lemma 4.38.6 to see that the right hand side is equal to $\mathrm{Mor}_{Cat/\mathcal{C}}(F(\mathcal{S}_1), F(\mathcal{S}_2))$ and apply (1) and (2) above. □

Here is another characterization of categories fibred in setoids among all categories fibred in groupoids.

042J **Lemma 4.39.7.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. The following are equivalent:*

1. *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, and*
2. *the canonical 1-morphism $\mathcal{I}_{\mathcal{S}} \to \mathcal{S}$, see (4.34.2.1), is an equivalence (of categories over $\mathcal{C}$).*

Proof. Assume (2). The category $\mathcal{I}_\mathcal{S}$ has objects (x, α) where $x \in \mathcal{S}$, say with $p(x) = U$, and $\alpha : x \to x$ is a morphism in $\mathcal{S}_U$. Hence if $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence over $\mathcal{C}$ then every pair of objects (x, α), (x, α') are isomorphic in the fibre category of $\mathcal{I}_\mathcal{S}$ over U. Looking at the definition of morphisms in $\mathcal{I}_\mathcal{S}$ we conclude that α, α' are conjugate in the group of automorphisms of x. Hence taking $\alpha' = \mathrm{id}_x$ we conclude that every automorphism of x is equal to the identity. Since $\mathcal{S} \to \mathcal{C}$ is fibred in groupoids this implies that $\mathcal{S} \to \mathcal{C}$ is fibred in setoids. We omit the proof of (1) $\Rightarrow$ (2). □

04SD **Lemma 4.39.8.** *Let $\mathcal{C}$ be a category. The construction of Lemma 4.39.6 which associates to a category fibred in setoids a presheaf is compatible with products, in the sense that the presheaf associated to a 2-fibre product $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ is the fibre product of the presheaves associated to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. The lemma just says that

$$\mathrm{Ob}((\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_U)/\cong \quad \text{equals} \quad \mathrm{Ob}(\mathcal{X}_U)/\cong \times_{\mathrm{Ob}(\mathcal{Y}_U)/\cong} \mathrm{Ob}(\mathcal{Z}_U)/\cong$$

the proof of which we omit. (But note that this would not be true in general if the category $\mathcal{Y}_U$ is not a setoid.) □

4.40. Representable categories fibred in groupoids

04SE Here is our definition of a representable category fibred in groupoids. As promised this is invariant under equivalences.

0046 **Definition 4.40.1.** Let $\mathcal{C}$ be a category. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called *representable* if there exist an object X of $\mathcal{C}$ and an equivalence $j : \mathcal{S} \to \mathcal{C}/X$ (in the 2-category of groupoids over $\mathcal{C}$).

The usual abuse of notation is to say that X *represents* $\mathcal{S}$ and not mention the equivalence j. We spell out what this entails.

02Y3 **Lemma 4.40.2.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids.*

(1) *$\mathcal{S}$ is representable if and only if the following conditions are satisfied:*
   (a) *$\mathcal{S}$ is fibred in setoids, and*
   (b) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.*
(2) *If $\mathcal{S}$ is representable the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$, is uniquely determined up to isomorphism.*

Proof. The first assertion follows immediately from Lemma 4.39.5. For the second, suppose that $j' : \mathcal{S} \to \mathcal{C}/X'$ is a second such pair. Choose a 1-morphism $t' : \mathcal{C}/X' \to \mathcal{S}$ such that $j' \circ t' \cong \mathrm{id}_{\mathcal{C}/X'}$ and $t' \circ j' \cong \mathrm{id}_\mathcal{S}$. Then $j \circ t' : \mathcal{C}/X' \to \mathcal{C}/X$ is an equivalence. Hence it is an isomorphism, see Lemma 4.38.6. Hence by the Yoneda Lemma 4.3.5 (via Example 4.38.7 for example) it is given by an isomorphism $X' \to X$. □

04SF **Lemma 4.40.3.** *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-}isomorphism = \mathrm{Mor}_\mathcal{C}(X, Y)$$

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 4.38.7 we have $\mathcal{C}/X = \mathcal{S}_{h_X}$ and $\mathcal{C}/Y = \mathcal{S}_{h_Y}$. By Lemma 4.39.6 we have

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y)$$

By the Yoneda Lemma 4.3.5 we have $\mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. □

4.41. The 2-Yoneda lemma

0GWH Let $\mathcal{C}$ be a category. The 2-category of fibred categories over $\mathcal{C}$ was constructed/defined in Definition 4.33.9. If $\mathcal{S}$, $\mathcal{S}'$ are fibred categories over $\mathcal{C}$ then

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category. Here is the 2-category analogue of the Yoneda lemma in the setting of fibred categories.

0GWI **Lemma 4.41.1** (2-Yoneda lemma for fibred categories)**.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.33.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and
(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 4.33.7.

We omit the verification that this is an inverse to the functor of the lemma. □

Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ is a "full" sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.35.6). Hence if $\mathcal{S}$, $\mathcal{S}'$ are fibred in groupoids over $\mathcal{C}$ then

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category (see Definition 4.32.1). These are all groupoids, see remarks following Definition 4.35.6. Here is the 2-category analogue of the Yoneda lemma.

004B **Lemma 4.41.2** (2-Yoneda lemma)**.** *Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.33.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and

(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^* x \xrightarrow{(\alpha_{g,f})_x} g^* f^* x \to f^* x$$

where $\alpha_{g,f}$ is as in Lemma 4.35.2.

We omit the verification that this is an inverse to the functor of the lemma. □

076J **Remark 4.41.3.** We can use the 2-Yoneda lemma to give an alternative proof of Lemma 4.37.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in \mathrm{Ob}(\mathcal{C})$ let

$$F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}).$$

If $f : U \to V$ the induced functor $\mathcal{C}/U \to \mathcal{C}/V$ induces the morphism $F(f) : F(V) \to F(U)$. Clearly F is a functor. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 4.37.1. There is an obvious functor $G : \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(U)$, to $x(\mathrm{id}_U) \in \mathcal{S}$. Now Lemma 4.41.2 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G is an equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 4.35.9.

4.42. Representable 1-morphisms

02Y4 Let $\mathcal{C}$ be a category. In this section we explain what it means for a 1-morphism between categories fibred in groupoids over $\mathcal{C}$ to be representable.

Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{C}/U \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. We want to describe the 2-fibre product

$$\begin{array}{ccc} (\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \mathcal{C}/U & \xrightarrow{G} & \mathcal{Y} \end{array}$$

Let $y = G(\mathrm{id}_U) \in \mathcal{Y}_U$. Make a choice of pullbacks for $\mathcal{Y}$ (see Definition 4.33.6). Then G is isomorphic to the functor $(f : V \to U) \mapsto f^* y$, see Lemma 4.41.2 and its proof. We may think of an object of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ as a quadruple $(V, f : V \to U, x, \phi)$, see Lemma 4.32.3. Using the description of G above we may think of ϕ as an isomorphism $\phi : f^* y \to F(x)$ in $\mathcal{Y}_V$.

02Y5 **Lemma 4.42.1.** *In the situation above the fibre category of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ over an object $f : V \to U$ of $\mathcal{C}/U$ is the category described as follows:*

(1) *objects are pairs (x, ϕ), where $x \in \mathrm{Ob}(\mathcal{X}_V)$, and $\phi : f^* y \to F(x)$ is a morphism in $\mathcal{Y}_V$,*
(2) *the set of morphisms between (x, ϕ) and (x', ϕ') is the set of morphisms $\psi : x \to x'$ in $\mathcal{X}_V$ such that $F(\psi) = \phi' \circ \phi^{-1}$.*

Proof. See discussion above. □

02Y6 **Lemma 4.42.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Let $G : \mathcal{C}/U \to \mathcal{Y}$ be a 1-morphism. Then*

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is a category fibred in groupoids.

Proof. We have already seen in Lemma 4.35.7 that the composition

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U \longrightarrow \mathcal{C}$$

is a category fibred in groupoids. Then the lemma follows from Lemma 4.35.13. □

02Y7 **Definition 4.42.3.** Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. We say F is *representable*, or that $\mathcal{X}$ *is relatively representable over* $\mathcal{Y}$, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $G : \mathcal{C}/U \to \mathcal{Y}$ the category fibred in groupoids

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is representable.

02Y8 **Lemma 4.42.4.** *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. If F is representable then every one of the functors*

$$F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$$

between fibre categories is faithful.

Proof. Clear from the description of fibre categories in Lemma 4.42.1 and the characterization of representable fibred categories in Lemma 4.40.2. □

02Y9 **Lemma 4.42.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Make a choice of pullbacks for $\mathcal{Y}$. Assume*

(1) *each functor $F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
(2) *for each U and each $y \in \mathcal{Y}_U$ the presheaf*

$$(f : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathcal{X}_V, \phi : f^*y \to F(x)\}/\cong$$

is a representable presheaf on $\mathcal{C}/U$.

Then F is representable.

Proof. Clear from the description of fibre categories in Lemma 4.42.1 and the characterization of representable fibred categories in Lemma 4.40.2. □

Before we state the next lemma we point out that the 2-category of categories fibred in groupoids is a $(2,1)$-category, and hence we know what it means to say that it has a final object (see Definition 4.31.1). And it has a final object namely $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Thus we define 2-*products* of categories fibred in groupoids over $\mathcal{C}$ as the 2-fibre products

$$\mathcal{X} \times \mathcal{Y} := \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}.$$

With this definition in place the following lemma makes sense.

02YA **Lemma 4.42.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

(1) *The diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable.*

(2) *For every U in $\mathcal{C}$, any $G : \mathcal{C}/U \to \mathcal{S}$ is representable.*

Proof. Suppose the diagonal is representable, and let U, G be given. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $G' : \mathcal{C}/V \to \mathcal{S}$. Note that $\mathcal{C}/U \times \mathcal{C}/V = \mathcal{C}/U \times V$ is representable. Hence the fibre product

$$\begin{array}{ccc} (\mathcal{C}/U \times V) \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{C}/U \times V & \xrightarrow{(G,G')} & \mathcal{S} \times \mathcal{S} \end{array}$$

is representable by assumption. This means there exists $W \to U \times V$ in $\mathcal{C}$, such that

$$\begin{array}{ccc} \mathcal{C}/W & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{C}/U \times \mathcal{C}/V & \longrightarrow & \mathcal{S} \times \mathcal{S} \end{array}$$

is cartesian. This implies that $\mathcal{C}/W \cong \mathcal{C}/U \times_{\mathcal{S}} \mathcal{C}/V$ (see Lemma 4.31.11 and Remark 4.35.8) as desired.

Assume (2) holds. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $(G, G') : \mathcal{C}/V \to \mathcal{S} \times \mathcal{S}$. We have to show that $\mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ is representable. What we know is that $\mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$ is representable, say by $a : W \to V$ in $\mathcal{C}/V$. The equivalence

$$\mathcal{C}/W \to \mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$$

followed by the second projection to $\mathcal{C}/V$ gives a second morphism $a' : W \to V$. Consider $W' = W \times_{(a,a'),V\times V} V$. There exists an equivalence

$$\mathcal{C}/W' \cong \mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$$

namely

$$\begin{aligned} \mathcal{C}/W' &\cong \mathcal{C}/W \times_{(\mathcal{C}/V\times\mathcal{C}/V)} \mathcal{C}/V \\ &\cong \big(\mathcal{C}/V \times_{(G,\mathcal{S},G')} \mathcal{C}/V\big) \times_{(\mathcal{C}/V\times\mathcal{C}/V)} \mathcal{C}/V \\ &\cong \mathcal{C}/V \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \end{aligned}$$

(for the last isomorphism see Lemma 4.31.12 and Remark 4.35.8) which proves the lemma. □

Bibliographic notes: Parts of this have been taken from Vistoli's notes [**Vis04**].

4.43. Monoidal categories

0FFJ Let $\mathcal{C}$ be a category. Suppose we are given a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

We often want to know whether $\otimes$ satisfies an associative rule and whether there is a unit for $\otimes$.

An *associativity constraint* for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

such that for all objects X, Y, Z, W the diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes (Z \otimes W)) & \longrightarrow & (X \otimes Y) \otimes (Z \otimes W) & \longrightarrow & ((X \otimes Y) \otimes Z) \otimes W \\ \downarrow & & & & \uparrow \\ X \otimes ((Y \otimes Z) \otimes W) & & \longrightarrow & & (X \otimes (Y \otimes Z)) \otimes W \end{array}$$

is commutative where every arrow is determined by a suitable application of ϕ and functoriality of $\otimes$. Given an associativity constraint there are well defined functors

$$\mathcal{C} \times \ldots \times \mathcal{C} \longrightarrow \mathcal{C}, \quad (X_1, \ldots, X_n) \longmapsto X_1 \otimes \ldots \otimes X_n$$

for all $n \geq 1$.

Let ϕ be an associativity constraint. A *unit* for $(\mathcal{C}, \otimes, \phi)$ is an object $\mathbf{1}$ of $\mathcal{C}$ together with functorial isomorphisms

$$\mathbf{1} \otimes X \to X \quad \text{and} \quad X \otimes \mathbf{1} \to X$$

such that for all objects X, Y the diagram

$$\begin{array}{ccc} X \otimes (\mathbf{1} \otimes Y) & \xrightarrow[\phi]{} & (X \otimes \mathbf{1}) \otimes Y \\ & \searrow \quad \swarrow & \\ & X \otimes Y & \end{array}$$

is commutative where the diagonal arrows are given by the isomorphisms introduced above.

An equivalent definition would be that a unit is a pair $(\mathbf{1}, 1)$ where $\mathbf{1}$ is an object of $\mathcal{C}$ and $1 : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ is an isomorphism such that the functors $L : X \mapsto \mathbf{1} \otimes X$ and $R : X \mapsto X \otimes \mathbf{1}$ are equivalences. Certainly, given a unit as above we get the isomorphism $1 : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ for free and L and R are equivalences as they are isomorphic to the identity functor. Conversely, given $(\mathbf{1}, 1)$ such that L and R are equivalences, we obtain functorial isomorphisms $l : \mathbf{1} \otimes X \to X$ and $r : X \otimes \mathbf{1} \to X$ characterized by $L(l) = 1 \otimes \mathrm{id}_X$ and $R(r) = \mathrm{id}_X \otimes 1$. Then we can use r and l in the notion of unit as above.

A unit is unique up to unique isomorphism if it exists (exercise).

0FFK **Definition 4.43.1.** A triple $(\mathcal{C}, \otimes, \phi)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, and ϕ is an associativity constraint is called a *monoidal category* if there exists a unit $\mathbf{1}$.

We always write $\mathbf{1}$ to denote a unit of a monoidal category; as it is determined up to unique isomorphism there is no harm in choosing one. From now on we no longer write the brackets when taking tensor products in monoidal categories and we always identify $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ with X. Moreover, we will say "let $\mathcal{C}$ be a monoidal category" with $\otimes, \phi, \mathbf{1}$ understood.

0FFL **Definition 4.43.2.** Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal categories. A *functor of monoidal categories* $F : \mathcal{C} \to \mathcal{C}'$ is given by a functor F as indicated and an isomorphism

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

functorial in X and Y such that for all objects X, Y, and Z the diagram

$$\begin{array}{ccccc} F(X) \otimes (F(Y) \otimes F(Z)) & \longrightarrow & F(X) \otimes F(Y \otimes Z) & \longrightarrow & F(X \otimes (Y \otimes Z)) \\ \downarrow & & & & \downarrow \\ (F(X) \otimes F(Y)) \otimes F(Z) & \longrightarrow & F(X \otimes Y) \otimes F(Z) & \longrightarrow & F((X \otimes Y) \otimes Z) \end{array}$$

commutes and such that $F(\mathbf{1})$ is a unit in $\mathcal{C}'$.

By our conventions about units, we may always assume $F(\mathbf{1}) = \mathbf{1}$ if F is a functor of monoidal categories. As an example, if $A \to B$ is a ring homomorphism, then the functor $M \mapsto M \otimes_A B$ is functor of monoidal categories from Mod_A to Mod_B.

0FFM **Lemma 4.43.3.** *Let $\mathcal{C}$ be a monoidal category. Let X be an object of $\mathcal{C}$. The following are equivalent*

(1) *the functor $L : Y \mapsto X \otimes Y$ is an equivalence,*
(2) *the functor $R : Y \mapsto Y \otimes X$ is an equivalence,*
(3) *there exists an object X' such that $X \otimes X' \cong X' \otimes X \cong \mathbf{1}$.*

Proof. Assume (1). Choose X' such that $L(X') = \mathbf{1}$, i.e., $X \otimes X' \cong \mathbf{1}$. Denote L' and R' the functors corresponding to X'. The equation $X \otimes X' \cong \mathbf{1}$ implies $L \circ L' \cong \text{id}$. Thus L' must be the quasi-inverse to L (which exists by assumption). Hence $L' \circ L \cong \text{id}$. Hence $X' \otimes X \cong \mathbf{1}$. Thus (3) holds.

The proof of (2) $\Rightarrow$ (3) is dual to what we just said.

Assume (3). Then it is clear that L' and L are quasi-inverse to each other and it is clear that R' and R are quasi-inverse to each other. Thus (1) and (2) hold. □

0FFN **Definition 4.43.4.** Let $\mathcal{C}$ be a monoidal category. An object X of $\mathcal{C}$ is called *invertible* if any (or all) of the equivalent conditions of Lemma 4.43.3 hold.

Observe that if $F : \mathcal{C} \to \mathcal{C}'$ is a functor of monoidal categories, then F sends invertible objects to invertible objects.

0FFP **Definition 4.43.5.** Given a monoidal category $(\mathcal{C}, \otimes, \phi)$ and an object X a *left dual* is an object Y together with morphisms $\eta : \mathbf{1} \to X \otimes Y$ and $\epsilon : Y \otimes X \to \mathbf{1}$ such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta \otimes 1} & X \otimes Y \otimes X \\ & \searrow^{1} & \downarrow{\scriptstyle 1 \otimes \epsilon} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{1 \otimes \eta} & Y \otimes X \otimes Y \\ & \searrow^{1} & \downarrow{\scriptstyle \epsilon \otimes 1} \\ & & Y \end{array}$$

commute. In this situation we say that X is a *right dual* of Y.

Observe that if $F : \mathcal{C} \to \mathcal{C}'$ is a functor of monoidal categories, then $F(Y)$ is a left dual of $F(X)$ if Y is a left dual of X.

0FFQ **Lemma 4.43.6.** *Let $\mathcal{C}$ be a monoidal category. If Y is a left dual to X, then*

$$\text{Mor}(Z' \otimes X, Z) = \text{Mor}(Z', Z \otimes Y) \quad \textit{and} \quad \text{Mor}(Y \otimes Z', Z) = \text{Mor}(Z', X \otimes Z)$$

functorially in Z and Z'.

Proof. Consider the maps

$$\mathrm{Mor}(Z' \otimes X, Z) \to \mathrm{Mor}(Z' \otimes X \otimes Y, Z \otimes Y) \to \mathrm{Mor}(Z', Z \otimes Y)$$

where we use η in the second arrow and the sequence of maps

$$\mathrm{Mor}(Z', Z \otimes Y) \to \mathrm{Mor}(Z' \otimes X, Z \otimes Y \otimes X) \to \mathrm{Mor}(Z' \otimes X, Z)$$

where we use ϵ in the second arrow. A straightforward calculation using the properties of η and ϵ shows that the compositions of these are mutually inverse. Similarly for the other equality. □

0FFR **Remark 4.43.7.** Lemma 4.43.6 says in particular that $Z \mapsto Z \otimes Y$ is the right adjoint of $Z' \mapsto Z' \otimes X$. In particular, uniqueness of adjoint functors guarantees that a left dual of X, if it exists, is unique up to unique isomorphism. Conversely, assume the functor $Z \mapsto Z \otimes Y$ is a right adjoint of the functor $Z' \mapsto Z' \otimes X$, i.e., we're given a bijection

$$\mathrm{Mor}(Z' \otimes X, Z) \longrightarrow \mathrm{Mor}(Z', Z \otimes Y)$$

functorial in both Z and Z'. The unit of the adjunction produces maps

$$\eta_Z : Z \to Z \otimes X \otimes Y$$

functorial in Z and the counit of the adjoint produces maps

$$\epsilon_{Z'} : Z' \otimes Y \otimes X \to Z'$$

functorial in Z'. In particular, we find $\eta = \eta_{\mathbf{1}} : \mathbf{1} \to X \otimes Y$ and $\epsilon = \epsilon_{\mathbf{1}} : Y \otimes X \to \mathbf{1}$. As an exercise in the relationship between units, counits, and the adjunction isomorphism, the reader can show that we have

$$(\epsilon \otimes \mathrm{id}_Y) \circ \eta_Y = \mathrm{id}_Y \quad \text{and} \quad \epsilon_X \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X$$

However, this isn't enough to show that $(\epsilon \otimes \mathrm{id}_Y) \circ (\mathrm{id}_Y \otimes \eta) = \mathrm{id}_Y$ and $(\mathrm{id}_X \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X$, because we don't know in general that $\eta_Y = \mathrm{id}_Y \otimes \eta$ and we don't know that $\epsilon_X = \epsilon \otimes \mathrm{id}_X$. For this it would suffice to know that our adjunction isomorphism has the following property: for every W, Z, Z' the diagram

$$\begin{array}{ccc} \mathrm{Mor}(Z' \otimes X, Z) & \longrightarrow & \mathrm{Mor}(Z', Z \otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}_W \otimes -} & & \downarrow{\scriptstyle \mathrm{id}_W \otimes -} \\ \mathrm{Mor}(W \otimes Z' \otimes X, W \otimes Z) & \longrightarrow & \mathrm{Mor}(W \otimes Z', W \otimes Z \otimes Y) \end{array}$$

If this holds, we will say *the adjunction is compatible with the given tensor structure.* Thus the requirement that $Z \mapsto Z \otimes Y$ be the right adjoint of $Z' \mapsto Z' \otimes X$ compatible with the given tensor structure is an equivalent formulation of the property of being a left dual.

0FFS **Lemma 4.43.8.** *Let $\mathcal{C}$ be a monoidal category. If Y_i, $i = 1, 2$ are left duals of X_i, $i = 1, 2$, then $Y_2 \otimes Y_1$ is a left dual of $X_1 \otimes X_2$.*

Proof. Follows from uniqueness of adjoints and Remark 4.43.7. □

A *commutativity constraint* for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\psi : X \otimes Y \longrightarrow Y \otimes X$$

such that the composition

$$X \otimes Y \xrightarrow{\psi} Y \otimes X \xrightarrow{\psi} X \otimes Y$$

is the identity. We say ψ is *compatible* with a given associativity constraint ϕ if for all objects X, Y, Z the diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\phi} & (X \otimes Y) \otimes Z & \xrightarrow{\psi} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle \psi} & & & & \downarrow{\scriptstyle \phi} \\ X \otimes (Z \otimes Y) & \xrightarrow{\phi} & (X \otimes Z) \otimes Y & \xrightarrow{\psi} & (Z \otimes X) \otimes Y \end{array}$$

commutes.

0FFW **Definition 4.43.9.** A quadruple $(\mathcal{C}, \otimes, \phi, \psi)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is a functor, ϕ is an associativity constraint, and ψ is a commutativity constraint compatible with ϕ is called a *symmetric monoidal category* if there exists a unit.

To be sure, if $(\mathcal{C}, \otimes, \phi, \psi)$ is a symmetric monoidal category, then $(\mathcal{C}, \otimes, \phi)$ is a monoidal category.

0FN8 **Lemma 4.43.10.** *Let $(\mathcal{C}, \otimes, \phi, \psi)$ be a symmetric monoidal category. Let X be an object of $\mathcal{C}$ and let Y, $\eta : \mathbf{1} \to X \otimes Y$, and $\epsilon : Y \otimes X \to \mathbf{1}$ be a left dual of X as in Definition 4.43.5. Then $\eta' = \psi \circ \eta : \mathbf{1} \to Y \otimes X$ and $\epsilon' = \epsilon \circ \psi : X \otimes Y \to \mathbf{1}$ makes X into a left dual of Y.*

Proof. Omitted. Hint: pleasant exercise in the definitions. □

0FFY **Definition 4.43.11.** Let $\mathcal{C}$ and $\mathcal{C}'$ be symmetric monoidal categories. A *functor of symmetric monoidal categories* $F : \mathcal{C} \to \mathcal{C}'$ is given by a functor F as indicated and an isomorphism

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

functorial in X and Y such that F is a functor of monoidal categories and such that for all objects X and Y the diagram

$$\begin{array}{ccc} F(X) \otimes F(Y) & \longrightarrow & F(X \otimes Y) \\ \downarrow & & \downarrow \\ F(Y) \otimes F(X) & \longrightarrow & F(Y \otimes X) \end{array}$$

commutes.

0GWJ **Remark 4.43.12.** Let $\mathcal{C}$ be a monoidal category. We say $\mathcal{C}$ has an *internal hom* if for every pair of objects X, Y of $\mathcal{C}$ there is an object $hom(X, Y)$ of $\mathcal{C}$ such that we have

$$\mathrm{Mor}(X, hom(Y, Z)) = \mathrm{Mor}(X \otimes Y, Z)$$

functorially in X, Y, Z. By the Yoneda lemma the bifunctor $(X, Y) \mapsto hom(X, Y)$ is determined up to unique isomorphism if it exists. Given an internal hom we obtain canonical maps

(1) $hom(X, Y) \otimes X \to Y$,
(2) $hom(Y, Z) \otimes hom(X, Y) \to hom(X, Z)$,
(3) $Z \otimes hom(X, Y) \to hom(X, Z \otimes Y)$,
(4) $Y \to hom(X, Y \otimes X)$, and
(5) $hom(Y, Z) \otimes X \to hom(hom(X, Y), Z)$ in case $\mathcal{C}$ is symmetric monoidal.

Namely, the map in (1) is the image of $\mathrm{id}_{hom(X,Y)}$ by $\mathrm{Mor}(hom(X,Y), hom(X,Y)) \to \mathrm{Mor}(hom(X,Y) \otimes X, Y)$. To construct the map in (2) by the defining property of $hom(X,Z)$ we need to construct a map

$$hom(Y,Z) \otimes hom(X,Y) \otimes X \longrightarrow Z$$

and such a map exists since by (1) we have maps $hom(X,Y) \otimes X \to Y$ and $hom(Y,Z) \otimes Y \to Z$. To construct the map in (3) by the defining property of $hom(X, Z \otimes Y)$ we need to construct a map

$$Z \otimes hom(X,Y) \otimes X \to Z \otimes Y$$

for which we use $\mathrm{id}_Z \otimes a$ where a is the map in (1). To construct the map in (4) we note that we already have the map $Y \otimes hom(X,X) \to hom(X, Y \otimes X)$ by (3). Thus it suffices to construct a map $\mathbf{1} \to hom(X,X)$ and for this we take the element in $\mathrm{Mor}(\mathbf{1}, hom(X,X))$ corresponding to the canonical isomorphism $\mathbf{1} \otimes X \to X$ in $\mathrm{Mor}(\mathbf{1} \otimes X, X)$. Finally, we come to (5). By the universal property of $hom(hom(X,Y), Z)$ it suffices to construct a map

$$hom(Y,Z) \otimes X \otimes hom(X,Y) \longrightarrow Z$$

We do this by swapping the last two tensor products using the commutativity constraint and then using the maps $hom(X,Y) \otimes X \to Y$ and $hom(Y,Z) \otimes Y \to Z$.

4.44. Categories of dotted arrows

0H17 We discuss certain "categories of dotted arrows" in $(2,1)$-categories. These will appear when formulating various lifting criteria for algebraic stacks, see for example Morphisms of Stacks, Section 101.39 and More on Morphisms of Stacks, Section 106.8.

0H18 **Definition 4.44.1.** Let $\mathcal{C}$ be a $(2,1)$-category. Consider a 2-commutative solid diagram

0H19 (4.44.1.1)

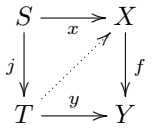

in $\mathcal{C}$. Fix a 2-isomorphism

$$\gamma : y \circ j \to f \circ x$$

witnessing the 2-commutativity of the diagram. Given (4.44.1.1) and γ, a *dotted arrow* is a triple (a, α, β) consisting of a morphism $a \colon T \to X$ and 2-isomorphisms $\alpha : a \circ j \to x$, $\beta : y \to f \circ a$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_j)$, in other words such that

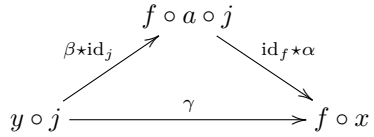

is commutative. A *morphism of dotted arrows* $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ such that $\alpha = \alpha' \circ (\theta \star \mathrm{id}_j)$ and $\beta' = (\mathrm{id}_f \star \theta) \circ \beta$.

In the situation of Definition 4.44.1, there is an associated *category of dotted arrows*. This category is a groupoid. It may depend on γ in general. The next two lemmas say that categories of dotted arrows are well-behaved with respect to base change and composition for f.

0H1A **Lemma 4.44.2.** *Let $\mathcal{C}$ be a $(2,1)$-category. Assume given a 2-commutative diagram*

$$\begin{array}{ccccc} S & \xrightarrow{x'} & X' & \xrightarrow{q} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{y'} & Y' & \xrightarrow{g} & Y \end{array}$$

in $\mathcal{C}$, where the right square is 2-cartesian with respect to a 2-isomorphism ϕ: $g \circ p \to f \circ q$. Choose a 2-arrow $\gamma' : y' \circ j \to p \circ x'$. Set $x = q \circ x'$, $y = g \circ y'$ and let $\gamma : y \circ j \to f \circ x$ be the 2-isomorphism $\gamma = (\phi \star id_{x'}) \circ (id_g \star \gamma')$. Then the category $\mathcal{D}'$ of dotted arrows for the left square and γ' is equivalent to the category $\mathcal{D}$ of dotted arrows for the outer rectangle and γ.

Proof. There is a functor $\mathcal{D}' \to \mathcal{D}$ which is $(a, \alpha, \beta) \mapsto (q \circ a, \mathrm{id}_q \star \alpha, (\phi \star \mathrm{id}_a) \circ (\mathrm{id}_g \star \beta))$ on objects and $\theta \mapsto \mathrm{id}_q \star \theta$ on arrows. Checking that this functor $\mathcal{D}' \to \mathcal{D}$ is an equivalence follows formally from the universal property for 2-fibre products as in Section 4.31. Details omitted. □

0H1B **Lemma 4.44.3.** *Let $\mathcal{C}$ be a $(2,1)$-category. Assume given a solid 2-commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{x} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & & Y \\ & \searrow^{z} & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

in $\mathcal{C}$. Choose a 2-isomorphism γ: $z \circ j \to g \circ f \circ x$. Let $\mathcal{D}$ be the category of dotted arrows for the outer rectangle and γ. Let $\mathcal{D}'$ be the category of dotted arrows for the solid square

$$\begin{array}{ccc} S & \xrightarrow{} & Y \\ {\scriptstyle j}\downarrow & \nearrow_{f\circ x} & \downarrow{\scriptstyle g} \\ T & \xrightarrow{z} & Z \end{array}$$

and γ. Then $\mathcal{D}$ is equivalent to a category $\mathcal{D}''$ which has the following property: there is a functor $\mathcal{D}'' \to \mathcal{D}'$ which turns $\mathcal{D}''$ into a category fibred in groupoids over $\mathcal{D}'$ and whose fibre categories are isomorphic to categories of dotted arrows for certain solid squares of the form

$$\begin{array}{ccc} S & \xrightarrow{x} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \xrightarrow{y} & Y \end{array}$$

and some choices of 2-isomorphism $y \circ j \to f \circ x$.

Proof. Construct the category $\mathcal{D}''$ whose objects are tuples $(a, \alpha, \beta, b, \eta)$ where (a, α, β) is an object of $\mathcal{D}$ and $b\colon T \to Y$ is a 1-morphism and $\eta\colon b \to f \circ a$ is a 2-isomorphism. Morphisms $(a, \alpha, \beta, b, \eta) \to (a', \alpha', \beta', b', \eta')$ in $\mathcal{D}''$ are pairs (θ_1, θ_2), where $\theta_1\colon a \to a'$ defines an arrow $(a, \alpha, \beta) \to (a', \alpha', \beta')$ in $\mathcal{D}$ and $\theta_2\colon b \to b'$ is a 2-isomorphism with the compatibility condition $\eta' \circ \theta_2 = (\mathrm{id}_f \star \theta_1) \circ \eta$.

There is a functor $\mathcal{D}'' \to \mathcal{D}'$ which is $(a, \alpha, \beta, b, \eta) \mapsto (b, (\mathrm{id}_f \star \alpha) \circ (\eta \star \mathrm{id}_j), (\mathrm{id}_g \star \eta^{-1}) \circ \beta)$ on objects and $(\theta_1, \theta_2) \mapsto \theta_2$ on arrows. Then $\mathcal{D}'' \to \mathcal{D}'$ is fibred in groupoids.

If (y, δ, ϵ) is an object of $\mathcal{D}'$, write $\mathcal{D}_{y,\delta}$ for the category of dotted arrows for the last displayed diagram with $y \circ j \to f \circ x$ given by δ. There is a functor $\mathcal{D}_{y,\delta} \to \mathcal{D}''$ given by $(a, \alpha, \eta) \mapsto (a, \alpha, (\mathrm{id}_g \star \eta) \circ \epsilon, y, \eta)$ on objects and $\theta \mapsto (\theta, \mathrm{id}_y)$ on arrows. This exhibits an isomorphism from $\mathcal{D}_{y,\delta}$ to the fibre category of $\mathcal{D}'' \to \mathcal{D}'$ over (y, δ, ϵ).

There is also a functor $\mathcal{D} \to \mathcal{D}''$ which is $(a, \alpha, \beta) \mapsto (a, \alpha, \beta, f \circ a, \mathrm{id}_{f \circ a})$ on objects and $\theta \mapsto (\theta, \mathrm{id}_f \star \theta)$ on arrows. This functor is fully faithful and essentially surjective, hence an equivalence. Details omitted. □

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CHAPTER 5

Topology

004C

5.1. Introduction

004D Basic topology will be explained in this document. A reference is [**Eng77**].

5.2. Basic notions

004E The following is a list of basic notions in topology. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on topology before continuing.

004F (1) X is a *topological space*,

004G (2) $x \in X$ is a *point*,

0B12 (3) $E \subset X$ is a *locally closed* subset,

004H (4) $x \in X$ is a *closed point*,

08ZA (5) $E \subset X$ is a *dense* subset,

004I (6) $f : X_1 \to X_2$ is *continuous*,

0BBW (7) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is *upper semi-continuous* if $\{x \in X \mid f(x) < a\}$ is open for all $a \in \mathbf{R}$,

0BBX (8) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is *lower semi-continuous* if $\{x \in X \mid f(x) > a\}$ is open for all $a \in \mathbf{R}$,

(9) a continuous map of spaces $f : X \to Y$ is *open* if $f(U)$ is open in Y for $U \subset X$ open,

(10) a continuous map of spaces $f : X \to Y$ is *closed* if $f(Z)$ is closed in Y for $Z \subset X$ closed,

004J (11) a *neighbourhood of* $x \in X$ is any subset $E \subset X$ which contains an open subset that contains x,

09R7 (12) the *induced topology* on a subset $E \subset X$,

004K (13) $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an *open covering of* U (note: we allow any U_i to be empty and we even allow, in case U is empty, the empty set for I),

0GM1 (14) a *subcover* of a covering as in (13) is an open covering $\mathcal{U}' : U = \bigcup_{i \in I'} U_i$ where $I' \subset I$,

004L (15) the open covering $\mathcal{V}$ is a *refinement* of the open covering $\mathcal{U}$ (if $\mathcal{V} : U = \bigcup_{j \in J} V_j$ and $\mathcal{U} : U = \bigcup_{i \in I} U_i$ this means each V_j is completely contained in one of the U_i),

004M (16) $\{E_i\}_{i \in I}$ *is a fundamental system of neighbourhoods of* x *in* X,

004N (17) a topological space X is called *Hausdorff* or *separated* if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$,

08ZB (18) the *product* of two topological spaces,
08ZC (19) the *fibre product* $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
0B30 (20) the *discrete topology* and the *indiscrete topology* on a set,
(21) etc.

5.3. Hausdorff spaces

08ZD The category of topological spaces has finite products.

08ZE **Lemma 5.3.1.** *Let X be a topological space. The following are equivalent:*
(1) *X is Hausdorff,*
(2) *the diagonal $\Delta(X) \subset X \times X$ is closed.*

Proof. We suppose that X is Hausdorff. Let $(x, y) \notin \Delta(X)$, i.e., $x \neq y$. There are U and V disjoint open sets of X such that $x \in U$ and $y \in V$. This implies that $U \times V \subset (X \times X) \setminus \Delta(X)$. This shows that $(X \times X) \setminus \Delta(X)$ is an open set of $X \times X$ which is equivalent to say that the diagonal $\Delta(X) \subset X \times X$ is closed in $X \times X$. The converse is similar: The complement $(X \times X) \setminus \Delta(X)$ consist precisely of $(x, y) \in X \times X$ with $x \neq y$. Thus, if $\Delta(X) \subset X \times X$ is closed, then, by the definition of the product topology, for every such (x, y), there are opens $U, V \subset X$ with $(x, y) \in U \times V$ and $(U \times V) \cap \Delta(X) = \emptyset$. In other words, with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. □

08ZF **Lemma 5.3.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. If Y is Hausdorff, then the graph of f is closed in $X \times Y$.*

Proof. The graph is the inverse image of the diagonal under the map $X \times Y \to Y \times Y$. Thus the lemma follows from Lemma 5.3.1. □

08ZG **Lemma 5.3.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $s : Y \to X$ be a continuous map such that $f \circ s = id_Y$. If X is Hausdorff, then $s(Y)$ is closed.*

Proof. This follows from Lemma 5.3.1 as $s(Y) = \{x \in X \mid x = s(f(x))\}$. □

08ZH **Lemma 5.3.4.** *Let $X \to Z$ and $Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, then $X \times_Z Y$ is closed in $X \times Y$.*

Proof. This follows from Lemma 5.3.1 as $X \times_Z Y$ is the inverse image of $\Delta(Z)$ under $X \times Y \to Z \times Z$. □

5.4. Separated maps

0CY0 Just the definition and some simple lemmas.

0CY1 **Definition 5.4.1.** A continuous map $f : X \to Y$ of topological spaces is called *separated* if and only if the diagonal $\Delta : X \to X \times_Y X$ is a closed map.

0CY2 **Lemma 5.4.2.** *Let $f : X \to Y$ be continuous map of topological spaces. The following are equivalent:*
(1) *f is separated,*
(2) *$\Delta(X) \subset X \times_Y X$ is a closed subset,*
(3) *given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'.*

Proof. If f is separated, by Definition 5.4.1, Δ is a closed map. The fact that X is closed in X gives us that $\Delta(X)$ is closed in $X \times_Y X$. Thus (1) implies (2).

Assune $\Delta(X) \subset X \times_Y X$ is a closed subset and denote U the complementary open. This means we have an open set $W \subset X \times X$ such that $W \cap (X \times_Y X) = U$. However, by definition of the product topology, if $(x, x') \in W \cap (X \times_Y X)$, we have V and V' open sets of X such that $x \in V$, $x' \in V'$ and $V \times V' \subset W$. If we had $V \cap V' \neq \emptyset$, we would have $z \in V \cap V'$. However, $(z, z) \in X \times_Y X$, so $(z, z) \in (V \times V') \cap (X \times_Y X) \subset U$, which is absurd. Therefore $V \cap V' = \emptyset$, and we have two disjoint open neighborhoods for x and x'. It proves that (2) implies (3).

Finally, we suppose that given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'. Let F be a closed set of X. We will show that $\Delta(F)$ is a closed subset of $X \times_Y X$. Let $(x, x') \in X \times_Y X$ be a point not contained in $\Delta(F)$. Then either $x \neq x'$ or $x \notin F$. In the first case, we choose disjoint open neighbourhoods $V, V' \subset X$ of x, x' and we see that $(V \times V') \cap X \times_Y X$ is an open neighbourhood of (x, x') not meeting $\Delta(F)$. In the second case, we see that $((X \setminus F) \times X) \cap X \times_Y X$ is an open neighbourhood of (x, x') not meeting $\Delta(F)$. We have shown that (3) implies (1). □

0CY3 **Lemma 5.4.3.** *Let $f : X \to Y$ be continuous map of topological spaces. If X is Hausdorff, then f is separated.*

Proof. Clear from Lemmas 5.4.2 and 5.3.1 as $\Delta(X)$ closed in $X \times X$ implies $\Delta(X)$ closed in $X \times_Y X$. □

0CY4 **Lemma 5.4.4.** *Let $f : X \to Y$ and $Y' \to Y$ be continuous maps of topological spaces. If f is separated, then $f' : Y' \times_Y X \to Y'$ is separated.*

Proof. Follows from characterization (2) of Lemma 5.4.2. Namely, with $X' = Y' \times_Y X$ the diagonal $\Delta(X')$ in the fibre product $X' \times_{Y'} X'$ is the inverse image of $\Delta(X)$ in $X \times_Y X$. □

5.5. Bases

004O Basic material on bases for topological spaces.

004P **Definition 5.5.1.** Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *base for the topology on X* or a *basis for the topology on X* if the following conditions hold:

(1) Every element $B \in \mathcal{B}$ is open in X.
(2) For every open $U \subset X$ and every $x \in U$, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

The following lemma is sometimes used to define a topology.

0D5P **Lemma 5.5.2.** *Let X be a set and let $\mathcal{B}$ be a collection of subsets. Assume that $X = \bigcup_{B \in \mathcal{B}} B$ and that given $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Then there is a unique topology on X such that $\mathcal{B}$ is a basis for this topology.*

Proof. Let $\sigma(\mathcal{B})$ be the set of subsets of X which can be written as unions of elements of $\mathcal{B}$. We claim $\sigma(\mathcal{B})$ is a topology. Namely, the empty set is an element of $\sigma(\mathcal{B})$ (as an empty union) and X is an element of $\sigma(\mathcal{B})$ (as the union of all elements

of $\mathcal{B}$). It is clear that $\sigma(\mathcal{B})$ is preserved under unions. Finally, if $U, V \in \sigma(\mathcal{B})$ then write $U = \bigcup_{i\in I} U_i$ and $V = \bigcup_{j\in J} V_j$ with $U_i, V_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. Then

$$U \cap V = \bigcup\nolimits_{i\in I, j\in J} U_i \cap V_j$$

The assumption in the lemma tells us that $U_i \cap V_j \in \sigma(\mathcal{B})$ hence we see that $U \cap V$ is too. Thus $\sigma(\mathcal{B})$ is a topology. Properties (1) and (2) of Definition 5.5.1 are immediate for this topology. To prove the uniqueness of this topology let $\mathcal{T}$ be a topology on X such that $\mathcal{B}$ is a base for it. Then of course every element of $\mathcal{B}$ is in $\mathcal{T}$ by (1) of Definition 5.5.1 and hence $\sigma(\mathcal{B}) \subset \mathcal{T}$. Conversely, part (2) of Definition 5.5.1 tells us that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$, i.e., $\mathcal{T} \subset \sigma(\mathcal{B})$. This finishes the proof. □

004Q **Lemma 5.5.3.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each V_j is an element of the basis $\mathcal{B}$.*

Proof. If $x \in U = \bigcup_{i\in I} U_i$, there is an $i_x \in I$ such that $x \in U_{i_x}$. Thus we have a $B_{i_x} \in \mathcal{B}$ verifying $x \in B_{i_x} \subset U_{i_x}$. Set $J = \{i_x | x \in U\}$ and for $j = i_x \in J$ set $V_j = B_{i_x}$. This gives the desired open covering of U by $\{V_j\}_{j\in J}$. □

08ZI **Definition 5.5.4.** Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *subbase for the topology on X* or a *subbasis for the topology on X* if the finite intersections of elements of $\mathcal{B}$ form a basis for the topology on X.

In particular every element of $\mathcal{B}$ is open.

08ZJ **Lemma 5.5.5.** *Let X be a set. Given any collection $\mathcal{B}$ of subsets of X there is a unique topology on X such that $\mathcal{B}$ is a subbase for this topology.*

Proof. By convention $\bigcap_\emptyset B = X$. Thus we can apply Lemma 5.5.2 to the set of finite intersections of elements from $\mathcal{B}$. □

0D5Q **Lemma 5.5.6.** *Let X be a topological space. Let $\mathcal{B}$ be a collection of opens of X. Assume $X = \bigcup_{U\in\mathcal{B}} U$ and for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W\in\mathcal{B}, W\subset U\cap V} W$. Then there is a continuous map $f : X \to Y$ of topological spaces such that*

(1) *for $U \in \mathcal{B}$ the image $f(U)$ is open,*
(2) *for $U \in \mathcal{B}$ we have $f^{-1}(f(U)) = U$, and*
(3) *the opens $f(U)$, $U \in \mathcal{B}$ form a basis for the topology on Y.*

Proof. Define an equivalence relation $\sim$ on points of X by the rule

$$x \sim y \Leftrightarrow (\forall U \in \mathcal{B} : x \in U \Leftrightarrow y \in U)$$

Let Y be the set of equivalence classes and $f : X \to Y$ the natural map. Part (2) holds by construction. The assumptions on $\mathcal{B}$ exactly mirror the assumptions in Lemma 5.5.2 on the set of subsets $f(U)$, $U \in \mathcal{B}$. Hence there is a unique topology on Y such that (3) holds. Then (1) is clear as well. □

5.6. Submersive maps

0405 If X is a topological space and $E \subset X$ is a subset, then we usually endow E with the *induced topology*.

09R8 **Lemma 5.6.1.** *Let X be a topological space. Let Y be a set and let $f : Y \to X$ be an injective map of sets. The induced topology on Y is the topology characterized by each of the following statements:*

(1) *it is the weakest topology on Y such that f is continuous,*
(2) *the open subsets of Y are $f^{-1}(U)$ for $U \subset X$ open,*
(3) *the closed subsets of Y are the sets $f^{-1}(Z)$ for $Z \subset X$ closed.*

Proof. The set $\mathcal{T} = \{f^{-1}(U) | U \subset X \text{ open}\}$ is a topology on Y. Firstly, $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(X) = Y$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Now let $\{V_i\}_{i\in I}$ be a collection of open subsets where $V_i \in \mathcal{T}$ and write $V_i = f^{-1}(U_i)$ where U_i is an open subset of X, then

$$\bigcup\nolimits_{i\in I} V_i = \bigcup\nolimits_{i\in I} f^{-1}(U_i) = f^{-1}\left(\bigcup\nolimits_{i\in I} U_i\right)$$

So $\bigcup_{i\in I} V_i \in \mathcal{T}$ as $\bigcup_{i\in I} U_i$ is open in X. Now let $V_1, V_2 \in \mathcal{T}$. We have U_1, U_2 open in X such that $V_1 = f^{-1}(U_1)$ and $V_2 = f^{-1}(U_2)$. Then

$$V_1 \cap V_2 = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $U_1 \cap U_2$ is open in X.

Any topology on Y such that f is continuous contains $\mathcal{T}$ according to the definition of a continuous map. Thus $\mathcal{T}$ is indeed the weakest topology on Y such that f is continuous. This proves that (1) and (2) are equivalent.

The equivalence of (2) and (3) follows from the equality $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. □

Dually, if X is a topological space and $X \to Y$ is a surjection of sets, then Y can be endowed with the *quotient topology*.

08ZK **Lemma 5.6.2.** *Let X be a topological space. Let Y be a set and let $f : X \to Y$ be a surjective map of sets. The quotient topology on Y is the topology characterized by each of the following statements:*

(1) *it is the strongest topology on Y such that f is continuous,*
(2) *a subset V of Y is open if and only if $f^{-1}(V)$ is open,*
(3) *a subset Z of Y is closed if and only if $f^{-1}(Z)$ is closed.*

Proof. The set $\mathcal{T} = \{V \subset Y | f^{-1}(V) \text{ is open}\}$ is a topology on Y. Firstly $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(Y) = X$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Let $(V_i)_{i\in I}$ be a family of elements $V_i \in \mathcal{T}$. Then

$$\bigcup\nolimits_{i\in I} f^{-1}(V_i) = f^{-1}\left(\bigcup\nolimits_{i\in I} V_i\right)$$

Thus $\bigcup_{i\in I} V_i \in \mathcal{T}$ as $\bigcup_{i\in I} f^{-1}(V_i)$ is open in X. Furthermore if $V_1, V_2 \in \mathcal{T}$ then

$$f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $f^{-1}(V_1) \cap f^{-1}(V_2)$ is open in X.

Finally a topology on Y such that f is continuous is included in $\mathcal{T}$ according to the definition of a continuous function, so $\mathcal{T}$ is the strongest topology on Y such that f is continuous. It proves that (1) and (2) are equivalent.

Finally, (2) and (3) equivalence follows from $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. □

Let $f : X \to Y$ be a continuous map of topological spaces. In this case we obtain a factorization $X \to f(X) \to Y$ of maps of sets. We can endow $f(X)$ with the quotient topology coming from the surjection $X \to f(X)$ or with the induced topology coming from the injection $f(X) \to Y$. The map

$$(f(X), \text{quotient topology}) \longrightarrow (f(X), \text{induced topology})$$

is continuous.

0406 **Definition 5.6.3.** Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say f is a *strict map of topological spaces* if the induced topology and the quotient topology on $f(X)$ agree (see discussion above).
(2) We say f is *submersive*[1] if f is surjective and strict.

Thus a continuous map $f : X \to Y$ is submersive if f is a surjection and for any $T \subset Y$ we have T is open or closed if and only if $f^{-1}(T)$ is so. In other words, Y has the quotient topology relative to the surjection $X \to Y$.

02YB **Lemma 5.6.4.** *Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

(1) $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$,
(2) *$T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,*
(3) *$T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and*
(4) *$T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. If $x \in X$, and $x \notin \overline{f^{-1}(T)}$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since f is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting T. Hence $x \notin f^{-1}(\overline{T})$. This proves (1). Part (2) is an easy consequence of (1). Part (3) is obvious from the fact that f is open and surjective. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)} = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(\overline{T}) \to \overline{T}$ we see that T is open in $\overline{T}$, i.e., T is locally closed. □

0AAU **Lemma 5.6.5.** *Let $f : X \to Y$ be surjective, closed, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

(1) $\overline{T} = f(\overline{f^{-1}(T)})$,
(2) *$T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,*
(3) *$T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and*
(4) *$T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. Then $T \subset f(\overline{f^{-1}(T)}) \subset \overline{T}$ is a closed subset, hence we get (1). Part (2) is obvious from the fact that f is closed and surjective. Part (3) follows from (2) applied to the complement of T. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)}$ is open. Since the map $\overline{f^{-1}(T)} \to \overline{T}$ is surjective by (1) we can apply part (3) to the map $\overline{f^{-1}(T)} \to \overline{T}$ induced by f to conclude that T is open in $\overline{T}$, i.e., T is locally closed. □

[1] This is very different from the notion of a submersion between differential manifolds! It is probably a good idea to use "strict and surjective" in stead of "submersive".

5.7. Connected components

004R

004S **Definition 5.7.1.** Let X be a topological space.

(1) We say X is *connected* if X is not empty and whenever $X = T_1 \amalg T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.
(2) We say $T \subset X$ is a *connected component* of X if T is a maximal connected subset of X.

The empty space is not connected.

0376 **Lemma 5.7.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.*

Proof. Let $A \subset f(E)$ an open and closed subset of $f(E)$. Because f is continuous, $f^{-1}(A)$ is an open and closed subset of E. As E is connected, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = E$. However, $A \subset f(E)$ implies that $A = f(f^{-1}(A))$. Indeed, if $x \in f(f^{-1}(A))$ then there is $y \in f^{-1}(A)$ such that $f(y) = x$ and because $y \in f^{-1}(A)$, we have $f(y) \in A$ i.e. $x \in A$. Reciprocally, if $x \in A$, $A \subset f(E)$ implies that there is $y \in E$ such that $f(y) = x$. Therefore $y \in f^{-1}(A)$, and then $x \in f(f^{-1}(A))$. Thus $A = \emptyset$ or $A = f(E)$ proving that $f(E)$ is connected. □

004T **Lemma 5.7.3.** *Let X be a topological space.*

(1) *If $T \subset X$ is connected, then so is its closure.*
(2) *Any connected component of X is closed (but not necessarily open).*
(3) *Every connected subset of X is contained in a unique connected component of X.*
(4) *Every point of X is contained in a unique connected component, in other words, X is the disjoint union of its connected components.*

Proof. Let $\overline{T}$ be the closure of the connected subset T. Suppose $\overline{T} = T_1 \amalg T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \amalg (T \cap T_2)$. Hence T equals one of the two, say $T = T_1 \cap T$. Thus $\overline{T} \subset T_1$. This implies (1) and (2).

Let A be a nonempty set of connected subsets of X such that $\Omega = \bigcap_{T \in A} T$ is nonempty. We claim $E = \bigcup_{T \in A} T$ is connected. Namely, E is nonempty as it contains Ω. Say $E = E_1 \amalg E_2$ with E_i closed in E. We may assume E_1 meets Ω (after renumbering). Then each $T \in A$ meets E_1 and hence must be contained in E_1 as T is connected. Hence $E \subset E_1$ which proves the claim.

Let $W \subset X$ be a nonempty connected subset. If we apply the result of the previous paragraph to the set of all connected subsets of X containing W, then we see that E is a connected component of X. This implies existence and uniqueness in (3).

Let $x \in X$. Taking $W = \{x\}$ in the previous paragraph we see that x is contained in a unique connected component of X. Any two distinct connected components must be disjoint (by the result of the second paragraph).

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbf{N}} \{0, 1\}$ with the product topology. Its connected components are singletons, which are not open. □

0FIY **Remark 5.7.4.** Let X be a topological space and $x \in X$. Let $Z \subset X$ be the connected component of X passing through x. Consider the intersection E of all [**Eng77**, Example 6.1.24]

open and closed subsets of X containing x. It is clear that $Z \subset E$. In general $Z \neq E$. For example, let $X = \{x, y, z_1, z_2, \ldots\}$ with the topology with the following basis of opens, $\{z_n\}$, $\{x, z_n, z_{n+1}, \ldots\}$, and $\{y, z_n, z_{n+1}, \ldots\}$ for all n. Then $Z = \{x\}$ and $E = \{x, y\}$. We omit the details.

0377 **Lemma 5.7.5.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that*

(1) *all fibres of f are connected, and*
(2) *a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.*

Then f induces a bijection between the sets of connected components of X and Y.

Proof. Let $T \subset Y$ be a connected component. Note that T is closed, see Lemma 5.7.3. The lemma follows if we show that $f^{-1}(T)$ is connected because any connected subset of X maps into a connected component of Y by Lemma 5.7.2. Suppose that $f^{-1}(T) = Z_1 \amalg Z_2$ with Z_1, Z_2 closed. For any $t \in T$ we see that $f^{-1}(\{t\}) = Z_1 \cap f^{-1}(\{t\}) \amalg Z_2 \cap f^{-1}(\{t\})$. By (1) we see $f^{-1}(\{t\})$ is connected we conclude that either $f^{-1}(\{t\}) \subset Z_1$ or $f^{-1}(\{t\}) \subset Z_2$. In other words $T = T_1 \amalg T_2$ with $f^{-1}(T_i) = Z_i$. By (2) we conclude that T_i is closed in Y. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired. □

0378 **Lemma 5.7.6.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, (b) all fibres of f are connected. Then f induces a bijection between the sets of connected components of X and Y.*

Proof. This is a special case of Lemma 5.7.5. □

07VB **Lemma 5.7.7.** *Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Assume that (a) Y is connected, (b) f is open and closed, and (c) there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ is a finite set. Then X has at most $|f^{-1}(y)|$ connected components. Hence any connected component T of X is open and closed, and $f(T)$ is a nonempty open and closed subset of Y, which is therefore equal to Y.*

Proof. If the topological space X has at least N connected components for some $N \in \mathbf{N}$, we find by induction a decomposition $X = X_1 \amalg \ldots \amalg X_N$ of X as a disjoint union of N nonempty open and closed subsets $X_1, \ldots, X_N$ of X. As f is open and closed, each $f(X_i)$ is a nonempty open and closed subset of Y and is hence equal to Y. In particular the intersection $X_i \cap f^{-1}(y)$ is nonempty for each $1 \leq i \leq N$. Hence $f^{-1}(y)$ has at least N elements. □

04MC **Definition 5.7.8.** A topological space is *totally disconnected* if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected but not discrete.

08ZL **Lemma 5.7.9.** *Let X be a topological space. Let $\pi_0(X)$ be the set of connected components of X. Let $X \to \pi_0(X)$ be the map which sends $x \in X$ to the connected component of X passing through x. Endow $\pi_0(X)$ with the quotient topology. Then $\pi_0(X)$ is a totally disconnected space and any continuous map $X \to Y$ from X to a totally disconnected space Y factors through $\pi_0(X)$.*

Proof. By Lemma 5.7.5 the connected components of $\pi_0(X)$ are the singletons. We omit the proof of the second statement. □

04MD **Definition 5.7.10.** A topological space X is called *locally connected* if every point $x \in X$ has a fundamental system of connected neighbourhoods.

04ME **Lemma 5.7.11.** *Let X be a topological space. If X is locally connected, then*

(1) *any open subset of X is locally connected, and*
(2) *the connected components of X are open.*

So also the connected components of open subsets of X are open. In particular, every point has a fundamental system of open connected neighbourhoods.

Proof. For all $x \in X$ let write $\mathcal{N}(x)$ the fundamental system of connected neighbourhoods of x and let $U \subset X$ be an open subset of X. Then for all $x \in U$, U is a neighbourhood of x, so the set $\{V \in \mathcal{N}(x) | V \subset U\}$ is not empty and is a fundamental system of connected neighbourhoods of x in U. Thus U is locally connected and it proves (1).

Let $x \in \mathcal{C} \subset X$ where $\mathcal{C}$ is the connected component of x. Because X is locally connected, there exists $\mathcal{N}$ a connected neighbourhood of x. Therefore by the definition of a connected component, we have $\mathcal{N} \subset \mathcal{C}$ and then $\mathcal{C}$ is a neighbourhood of x. It implies that $\mathcal{C}$ is a neighbourhood of each of his point, in other words $\mathcal{C}$ is open and (2) is proven. □

5.8. Irreducible components

004U

004V **Definition 5.8.1.** Let X be a topological space.

(1) We say X is *irreducible*, if X is not empty, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$.
(2) We say $Z \subset X$ is an *irreducible component* of X if Z is a maximal irreducible subset of X.

An irreducible space is obviously connected.

0379 **Lemma 5.8.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.*

Proof. Clearly we may assume $E = X$ (i.e., X irreducible) and $f(E) = Y$ (i.e., f surjective). First, $Y \neq \emptyset$ since $X \neq \emptyset$. Next, assume $Y = Y_1 \cup Y_2$ with Y_1, Y_2 closed. Then $X = X_1 \cup X_2$ with $X_i = f^{-1}(Y_i)$ closed in X. By assumption on X, we must have $X = X_1$ or $X = X_2$, hence $Y = Y_1$ or $Y = Y_2$ since f is surjective. □

004W **Lemma 5.8.3.** *Let X be a topological space.*

(1) *If $T \subset X$ is irreducible so is its closure in X.*
(2) *Any irreducible component of X is closed.*
(3) *Any irreducible subset of X is contained in an irreducible component of X.*
(4) *Every point of X is contained in some irreducible component of X, in other words, X is the union of its irreducible components.*

Proof. Let $\overline{T}$ be the closure of the irreducible subset T. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence T equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$. This proves (1). Part (2) follows immediately from (1) and the definition of irreducible components.

Let $T \subset X$ be irreducible. Consider the set A of irreducible subsets $T \subset T' \subset X$. Note that A is nonempty since $T \in A$. There is a partial ordering on A coming from inclusion: $T' \leq T'' \Leftrightarrow T' \subset T''$. Choose a maximal totally ordered subset $A' \subset A$, and let $E = \bigcup_{T' \in A'} T'$. We claim that E is irreducible. Namely, suppose that $E = Z_1 \cup Z_2$ is a union of two closed subsets of E. For each $T' \in A'$ we have either $T' \subset Z_1$ or $T' \subset Z_2$, by irreducibility of T'. Suppose that for some $T'_0 \in A'$ we have $T'_0 \not\subset Z_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T' \subset Z_2$ for all $T' \in A'$. Hence $E = Z_2$. This proves (3). Part (4) is an immediate consequence of (3) as a singleton space is irreducible. □

0G2Y **Lemma 5.8.4.** *Let X be a topological space and suppose $X = \bigcup_{i=1,\ldots,n} X_i$ where each X_i is an irreducible closed subset of X and no X_i is contained in the union of the other members. Then each X_i is an irreducible component of X and each irreducible component of X is one of the X_i.*

Proof. Let Y be an irreducible component of X. Write $Y = \bigcup_{i=1,\ldots,n}(Y \cap X_i)$ and note that each $Y \cap X_i$ is closed in Y since X_i is closed in X. By irreducibility of Y we see that Y is equal to one of the $Y \cap X_i$, i.e., $Y \subset X_i$. By maximality of irreducible components we get $Y = X_i$.

Conversely, take one of the X_i. By Lemma 5.8.3 it is contained in an irreducible component Y of X. By the previous paragraph $Y = X_j$ for some j. So $X_i \subset X_j$ and since the original union does not have redundant members, $X_i = X_j$, which is an irreducible component. □

0GM2 **Lemma 5.8.5.** *Let $f : X \to Y$ be a surjective, continuous map of topological spaces. If X has a finite number, say n, of irreducible components, then Y has $\leq n$ irreducible components.*

Proof. Say $X_1, \ldots, X_n$ are the irreducible components of X. By Lemmas 5.8.2 and 5.8.3 the closure $Y_i \subset Y$ of $f(X_i)$ is irreducible. Since f is surjective, we see that Y is the union of the Y_i. We may choose a minimal subset $I \subset \{1, \ldots, n\}$ such that $Y = \bigcup_{i \in I} Y_i$. Then we may apply Lemma 5.8.4 to see that the Y_i for $i \in I$ are the irreducible components of Y. □

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of X.

004X **Definition 5.8.6.** Let X be a topological space.

(1) Let $Z \subset X$ be an irreducible closed subset. A *generic point* of Z is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$.
(2) The space X is called *Kolmogorov*, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of X which contains exactly one of the two points.
(3) The space X is called *quasi-sober* if every irreducible closed subset has a generic point.
(4) The space X is called *sober* if every irreducible closed subset has a unique generic point.

A topological space X is Kolmogorov, quasi-sober, resp. sober if and only if the map $x \mapsto \overline{\{x\}}$ from X to the set of irreducible closed subsets of X is injective, surjective, resp. bijective. Hence we see that a topological space is sober if and only if it is quasi-sober and Kolmogorov.

0B31 **Lemma 5.8.7.** *Let X be a topological space and let $Y \subset X$.*

(1) *If X is Kolmogorov then so is Y.*
(2) *Suppose Y is locally closed in X. If X is quasi-sober then so is Y.*
(3) *Suppose Y is locally closed in X. If X is sober then so is Y.*

Proof. Proof of (1). Suppose X is Kolmogorov. Let $x, y \in Y$ with $x \neq y$. Then $\overline{\{x\}} \cap Y = \overline{\{x\}} \neq \overline{\{y\}} = \overline{\{y\}} \cap Y$. Hence $\overline{\{x\}} \cap Y \neq \overline{\{y\}} \cap Y$. This shows that Y is Kolmogorov.

Proof of (2). Suppose X is quasi-sober. It suffices to consider the cases Y is closed and Y is open. First, suppose Y is closed. Let Z be an irreducible closed subset of Y. Then Z is an irreducible closed subset of X. Hence there exists $x \in Z$ with $\overline{\{x\}} = Z$. It follows $\overline{\{x\}} \cap Y = Z$. This shows Y is quasi-sober. Second, suppose Y is open. Let Z be an irreducible closed subset of Y. Then $\overline{Z}$ is an irreducible closed subset of X. Hence there exists $x \in \overline{Z}$ with $\overline{\{x\}} = \overline{Z}$. If $x \notin Y$ we get the contradiction $Z = Z \cap Y \subset \overline{Z} \cap Y = \overline{\{x\}} \cap Y = \emptyset$. Therefore $x \in Y$. It follows $Z = \overline{Z} \cap Y = \overline{\{x\}} \cap Y$. This shows Y is quasi-sober.

Proof of (3). Immediately from (1) and (2). □

06N9 **Lemma 5.8.8.** *Let X be a topological space and let $(X_i)_{i \in I}$ be a covering of X.*

(1) *Suppose X_i is locally closed in X for every $i \in I$. Then, X is Kolmogorov if and only if X_i is Kolmogorov for every $i \in I$.*
(2) *Suppose X_i is open in X for every $i \in I$. Then, X is quasi-sober if and only if X_i is quasi-sober for every $i \in I$.*
(3) *Suppose X_i is open in X for every $i \in I$. Then, X is sober if and only if X_i is sober for every $i \in I$.*

Proof. Proof of (1). If X is Kolmogorov then so is X_i for every $i \in I$ by Lemma 5.8.7. Suppose X_i is Kolmogorov for every $i \in I$. Let $x, y \in X$ with $\overline{\{x\}} = \overline{\{y\}}$. There exists $i \in I$ with $x \in X_i$. There exists an open subset $U \subset X$ such that X_i is a closed subset of U. If $y \notin U$ we get the contradiction $x \in \overline{\{x\}} \cap U = \overline{\{y\}} \cap U = \emptyset$. Hence $y \in U$. It follows $y \in \overline{\{y\}} \cap U = \overline{\{x\}} \cap U \subset X_i$. This shows $y \in X_i$. It follows $\overline{\{x\}} \cap X_i = \overline{\{y\}} \cap X_i$. Since X_i is Kolmogorov we get $x = y$. This shows X is Kolmogorov.

Proof of (2). If X is quasi-sober then so is X_i for every $i \in I$ by Lemma 5.8.7. Suppose X_i is quasi-sober for every $i \in I$. Let Y be an irreducible closed subset of X. As $Y \neq \emptyset$ there exists $i \in I$ with $X_i \cap Y \neq \emptyset$. As X_i is open in X it follows $X_i \cap Y$ is non-empty and open in Y, hence irreducible and dense in Y. Thus $X_i \cap Y$ is an irreducible closed subset of X_i. As X_i is quasi-sober there exists $x \in X_i \cap Y$ with $X_i \cap Y = \overline{\{x\}} \cap X_i \subset \overline{\{x\}}$. Since $X_i \cap Y$ is dense in Y and Y is closed in X it follows $Y = \overline{X_i \cap Y} \cap Y \subset \overline{X_i \cap Y} \subset \overline{\{x\}} \subset Y$. Therefore $Y = \overline{\{x\}}$. This shows X is quasi-sober.

Proof of (3). Immediately from (1) and (2). □

0B32 **Example 5.8.9.** Let X be an indiscrete space of cardinality at least 2. Then X is quasi-sober but not Kolmogorov. Moreover, the family of its singletons is a covering of X by discrete and hence Kolmogorov spaces.

0B33 **Example 5.8.10.** Let Y be an infinite set, furnished with the topology whose closed sets are Y and the finite subsets of Y. Then Y is Kolmogorov but not quasi-sober. However, the family of its singletons (which are its irreducible components) is a covering by discrete and hence sober spaces.

0B34 **Example 5.8.11.** Let X and Y be as in Example 5.8.9 and Example 5.8.10. Then, $X \amalg Y$ is neither Kolmogorov nor quasi-sober.

0B35 **Example 5.8.12.** Let Z be an infinite set and let $z \in Z$. We furnish Z with the topology whose closed sets are Z and the finite subsets of $Z \setminus \{z\}$. Then Z is sober but its subspace $Z \setminus \{z\}$ is not quasi-sober.

004Y **Example 5.8.13.** Recall that a topological space X is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case X is irreducible if and only if X is a singleton. Similarly, any subset of X is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

004Z **Lemma 5.8.14.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) Y is irreducible, (b) f is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then X is irreducible.*

Proof. Suppose $X = Z_1 \cup Z_2$ with Z_i closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = X \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = X \setminus Z_1$. To get a contradiction assume that U_1 and U_2 are both nonempty. By (b) we see that $f(U_i)$ is open. By (a) we have Y irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point y which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of X_y which is a contradiction. □

037A **Lemma 5.8.15.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then f induces a bijection between irreducible components.*

Proof. We point out that assumption (b) implies that f is surjective (see Definition 5.8.1). Let $T \subset Y$ be an irreducible component. Note that T is closed, see Lemma 5.8.3. The lemma follows if we show that $f^{-1}(T)$ is irreducible because any irreducible subset of X maps into an irreducible component of Y by Lemma 5.8.2. Note that $f^{-1}(T) \to T$ satisfies the assumptions of Lemma 5.8.14. Hence we win. □

The construction of the following lemma is sometimes called the "soberification".

0A2N **Lemma 5.8.16.** *Let X be a topological space. There is a canonical continuous map* [**GD71**, Page 68 ff]

$$c : X \longrightarrow X'$$

from X to a sober topological space X' which is universal among continuous maps from X to sober topological spaces. Moreover, the assignment $U' \mapsto c^{-1}(U')$ is a bijection between opens of X' and X which commutes with finite intersections and arbitrary unions. The image $c(X)$ is a Kolmogorov topological space and the map $c : X \to c(X)$ is universal for maps of X into Kolmogorov spaces.

Proof. Let X' be the set of irreducible closed subsets of X and let

$$c : X \to X', \quad x \mapsto \overline{\{x\}}.$$

For $U \subset X$ open, let $U' \subset X'$ denote the set of irreducible closed subsets of X which meet U. Then $c^{-1}(U') = U$. In particular, if $U_1 \neq U_2$ are open in X, then $U_1' \neq U_2'$. Hence c induces a bijection between the subsets of X' of the form U' and the opens of X.

Let U_1, U_2 be open in X. Suppose that $Z \in U_1'$ and $Z \in U_2'$. Then $Z \cap U_1$ and $Z \cap U_2$ are nonempty open subsets of the irreducible space Z and hence $Z \cap U_1 \cap U_2$ is nonempty. Thus $(U_1 \cap U_2)' = U_1' \cap U_2'$. The rule $U \mapsto U'$ is also compatible with arbitrary unions (details omitted). Thus it is clear that the collection of U' form a topology on X' and that we have a bijection as stated in the lemma.

Next we show that X' is sober. Let $T \subset X'$ be an irreducible closed subset. Let $U \subset X$ be the open such that $X' \setminus T = U'$. Then $Z = X \setminus U$ is irreducible because of the properties of the bijection of the lemma. We claim that $Z \in T$ is the unique generic point. Namely, any open of the form $V' \subset X'$ which does not contain Z must come from an open $V \subset X$ which misses Z, i.e., is contained in U.

Finally, we check the universal property. Let $f : X \to Y$ be a continuous map to a sober topological space. Then we let $f' : X' \to Y$ be the map which sends the irreducible closed $Z \subset X$ to the unique generic point of $\overline{f(Z)}$. It follows immediately that $f' \circ c = f$ as maps of sets, and the properties of c imply that f' is continuous. We omit the verification that the continuous map f' is unique. We also omit the proof of the statements on Kolmogorov spaces. □

0GM3 **Lemma 5.8.17.** *Let X be a connected topological space with a finite number of irreducible components $X_1, \ldots, X_n$. If $n > 1$ there is an $1 \leq j \leq n$ such that $X' = \bigcup_{i \neq j} X_i$ is connected.*

Proof. This is a graph theory problem. Let Γ be the graph with vertices $V = \{1, \ldots, n\}$ and an edge between i and j if and only if $X_i \cap X_j$ is nonempty. Connectedness of X means that Γ is connected. Our problem is to find $1 \leq j \leq n$ such that $\Gamma \setminus \{j\}$ is still connected. You can do this by choosing $j, j' \in E$ with maximal distance and then j works (choose a leaf!). Details omitted. □

5.9. Noetherian topological spaces

0050

0051 **Definition 5.9.1.** A topological space is called *Noetherian* if the descending chain condition holds for closed subsets of X. A topological space is called *locally Noetherian* if every point has a neighbourhood which is Noetherian.

0052 **Lemma 5.9.2.** *Let X be a Noetherian topological space.*

(1) *Any subset of X with the induced topology is Noetherian.*
(2) *The space X has finitely many irreducible components.*
(3) *Each irreducible component of X contains a nonempty open of X.*

Proof. Let $T \subset X$ be a subset of X. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of T. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of T_i stabilizes. Thus T is Noetherian.

Let A be the set of closed subsets of X which do not have finitely many irreducible components. Assume that A is not empty to arrive at a contradiction. The set A is partially ordered by inclusion: $\alpha \leq \alpha' \Leftrightarrow Z_\alpha \subset Z_{\alpha'}$. By the descending

chain condition we may find a smallest element of A, say Z. As Z is not a finite union of irreducible components, it is not irreducible. Hence we can write $Z = Z' \cup Z''$ and both are strictly smaller closed subsets. By construction $Z' = \bigcup Z'_i$ and $Z'' = \bigcup Z''_j$ are finite unions of their irreducible components (Lemma 5.8.3). Hence $Z = \bigcup Z'_i \cup \bigcup Z''_j$ is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of Z into its irreducible components (Lemma 5.8.4), a contradiction.

Let $Z \subset X$ be an irreducible component of X. Let $Z_1, \ldots, Z_n$ be the other irreducible components of X. Consider $U = Z \setminus (Z_1 \cup \ldots \cup Z_n)$. This is not empty since otherwise the irreducible space Z would be contained in one of the other Z_i. Because $X = Z \cup Z_1 \cup \ldots Z_n$ (see Lemma 5.8.3), also $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ and hence open in X. Thus Z contains a nonempty open of X. □

04Z8 **Lemma 5.9.3.** *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is Noetherian, then $f(X)$ is Noetherian.*
(2) *If X is locally Noetherian and f open, then $f(X)$ is locally Noetherian.*

Proof. In case (1), suppose that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ is a descending chain of closed subsets of $f(X)$ (as usual with the induced topology as a subset of Y). Then $f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots$ is a descending chain of closed subsets of X. Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ stabilizes also. In case (2), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subset X$ of x which is Noetherian. Then $f(E) \subset f(X)$ is a neighbourhood which is Noetherian by part (1). □

0053 **Lemma 5.9.4.** *Let X be a topological space. Let $X_i \subset X$, $i = 1, \ldots, n$ be a finite collection of subsets. If each X_i is Noetherian (with the induced topology), then $\bigcup_{i=1,\ldots,n} X_i$ is Noetherian (with the induced topology).*

Proof. Let $\{F_m\}_{m \in \mathbf{N}}$ a decreasing sequence of closed subsets of $X' = \bigcup_{i=1,\ldots,n} X_i$ with the induced topology. Then we can find a decreasing sequence $\{G_m\}_{m \in \mathbf{N}}$ of closed subsets of X verifying $F_m = G_m \cap X'$ for all m (small detail omitted). As X_i is noetherian and $\{G_m \cap X_i\}_{m \in \mathbf{N}}$ a decreasing sequence of closed subsets of X_i, there exists $m_i \in \mathbf{N}$ such that for all $m \geq m_i$ we have $G_m \cap X_i = G_{m_i} \cap X_i$. Let $m_0 = \max_{i=1,\ldots,n} m_i$. Then clearly

$$F_m = G_m \cap X' = G_m \cap (X_1 \cup \ldots \cup X_n) = (G_m \cap X_1) \cup \ldots (G_m \cap X_n)$$

stabilizes for $m \geq m_0$ and the proof is complete. □

02HZ **Example 5.9.5.** Any nonempty, Kolmogorov Noetherian topological space has a closed point (combine Lemmas 5.12.8 and 5.12.13). Let $X = \{1, 2, 3, \ldots\}$. Define a topology on X with opens $\emptyset$, $\{1, 2, \ldots, n\}$, $n \geq 1$ and X. Thus X is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 28.5.9.

04MF **Lemma 5.9.6.** *Let X be a locally Noetherian topological space. Then X is locally connected.*

Proof. Let $x \in X$. Let E be a neighbourhood of x. We have to find a connected neighbourhood of x contained in E. By assumption there exists a neighbourhood E' of x which is Noetherian. Then $E \cap E'$ is Noetherian, see Lemma 5.9.2. Let

$E \cap E' = Y_1 \cup \ldots \cup Y_n$ be the decomposition into irreducible components, see Lemma 5.9.2. Let $E'' = \bigcup_{x \in Y_i} Y_i$. This is a connected subset of $E \cap E'$ containing x. It contains the open $E \cap E' \setminus (\bigcup_{x \notin Y_i} Y_i)$ of $E \cap E'$ and hence it is a neighbourhood of x in X. This proves the lemma. □

5.10. Krull dimension

0054

0055 **Definition 5.10.1.** Let X be a topological space.

(1) A *chain of irreducible closed subsets* of X is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with Z_i closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n-1$.
(2) The *length* of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of X is the integer n.
(3) The *dimension* or more precisely the *Krull dimension* $\dim(X)$ of X is the element of $\{-\infty, 0, 1, 2, 3, \ldots, \infty\}$ defined by the formula:

$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$

Thus $\dim(X) = -\infty$ if and only if X is the empty space.
(4) Let $x \in X$. The *Krull dimension of X at x* is defined as

$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$

the minimum of $\dim(U)$ where U runs over the open neighbourhoods of x in X.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then x has a fundamental system of open neighbourhoods U with $\dim(U) = \dim_x(X)$.

0B7I **Lemma 5.10.2.** *Let X be a topological space. Then $\dim(X) = \sup \dim_x(X)$ where the supremum runs over the points x of X.*

Proof. It is clear that $\dim(X) \geq \dim_x(X)$ for all $x \in X$ (see discussion following Definition 5.10.1). Thus an inequality in one direction. For the converse, let $n \geq 0$ and suppose that $\dim(X) \geq n$. Then we can find a chain of irreducible closed subsets $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$. Pick $x \in Z_0$. For every open neighbourhood U of x we get a chain of irreducible closed subsets

$$Z_0 \cap U \subset Z_1 \cap U \subset \ldots \subset Z_n \cap U$$

in U. Namely, the sets $U \cap Z_i$ are irreducible closed in U and the inclusions are strict (details omitted; hint: the closure of $U \cap Z_i$ is Z_i). In this way we see that $\dim_x(X) \geq n$ which proves the other inequality. □

0056 **Example 5.10.3.** The Krull dimension of the usual Euclidean space $\mathbf{R}^n$ is 0.

0057 **Example 5.10.4.** Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n+1)$-element topological space of Krull dimension n.

0058 **Definition 5.10.5.** Let X be a topological space. We say that X is *equidimensional* if every irreducible component of X has the same dimension.

5.11. Codimension and catenary spaces

02I0 We only define the codimension of irreducible closed subsets.

02I3 **Definition 5.11.1.** Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. The *codimension* of Y in X is the supremum of the lengths e of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in X starting with Y. We will denote this $\text{codim}(Y, X)$.

The codimension is an element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$. If $\text{codim}(Y, X) < \infty$, then every chain can be extended to a maximal chain (but these do not all have to have the same length).

02I4 **Lemma 5.11.2.** *Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then*

$$\textit{codim}(Y, X) = \textit{codim}(Y \cap U, U)$$

Proof. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

02I5 **Example 5.11.3.** Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1]$, $(1 - 1/n, 1]$ for $n \in \mathbf{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset$, $\{0\}$, $[0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\text{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

02I1 **Definition 5.11.4.** Let X be a topological space. We say X is *catenary* if for every pair of irreducible closed subsets $T \subset T'$ we have $\text{codim}(T, T') < \infty$ and every maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

has the same length (equal to the codimension).

02I2 **Lemma 5.11.5.** *Let X be a topological space. The following are equivalent:*

(1) *X is catenary,*
(2) *X has an open covering by catenary spaces.*

Moreover, in this case any locally closed subspace of X is catenary.

Proof. Suppose that X is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

02I6 **Lemma 5.11.6.** *Let X be a topological space. The following are equivalent:*

(1) *X is catenary, and*
(2) *for every pair of irreducible closed subsets $Y \subset Y'$ we have $\textit{codim}(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have*

$$\textit{codim}(Y, Y'') = \textit{codim}(Y, Y') + \textit{codim}(Y', Y'').$$

Proof. Let suppose that X is catenary. According to Definition 5.11.4, for every pair of irreducible closed subsets $Y \subset Y'$ we have $\text{codim}(Y, Y') < \infty$. Let $Y \subset Y' \subset Y''$ be a triple of irreducible closed subsets of X. Let

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y'$$

be a maximal chain of irreducible closed subsets between Y and Y' where $e_1 = \text{codim}(Y, Y')$. Let also

$$Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

be a maximal chain of irreducible closed subsets between Y' and Y'' where $e_2 = \text{codim}(Y', Y'')$. As the two chains are maximal, the concatenation

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

is maximal too (between Y and Y'') and its length equals to $e_1 + e_2$. As X is catenary, each maximal chain has the same length equals to the codimension. Thus the point (2) that $\text{codim}(Y, Y'') = e_1+e_2 = \text{codim}(Y, Y')+\text{codim}(Y', Y'')$ is verified.

For the reciprocal, we show by induction that : if $Y = Y_1 \subset ... \subset Y_n = Y'$, then $\text{codim}(Y, Y') = \text{codim}(Y_1, Y_2) + ... + \text{codim}(Y_{n-1}, Y_n)$. Therefore, it forces maximal chains to have the same length. □

5.12. Quasi-compact spaces and maps

0059 The phrase "compact" will be reserved for Hausdorff topological spaces. And many spaces occurring in algebraic geometry are not Hausdorff.

005A **Definition 5.12.1.** Quasi-compactness.

(1) We say that a topological space X is *quasi-compact* if every open covering of X has a finite subcover.
(2) We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
(3) We say a subset $Z \subset X$ is *retrocompact* if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called *compact* if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. The notion of quasi-compactness of a map is very different from the notion of a "proper map", since there we require (besides closedness and separatedness) the inverse image of any quasi-compact subset of the target to be quasi-compact, whereas in the definition above we only consider quasi-compact *open* sets.

005B **Lemma 5.12.2.** *A composition of quasi-compact maps is quasi-compact.*

Proof. This is immediate from the definition. □

005C **Lemma 5.12.3.** *A closed subset of a quasi-compact topological space is quasi-compact.*

Proof. Let $E \subset X$ be a closed subset of the quasi-compact space X. Let $E = \bigcup V_j$ be an open covering. Choose $U_j \subset X$ open such that $V_j = E \cap U_j$. Then $X = (X \setminus E) \cup \bigcup U_j$ is an open covering of X. Hence $X = (X \setminus E) \cup U_{j_1} \cup \ldots \cup U_{j_n}$ for some n and indices j_i. Thus $E = V_{j_1} \cup \ldots \cup V_{j_n}$ as desired. □

08YB **Lemma 5.12.4.** *Let X be a Hausdorff topological space.*

(1) *If $E \subset X$ is quasi-compact, then it is closed.*
(2) *If $E_1, E_2 \subset X$ are disjoint quasi-compact subsets then there exists opens $E_i \subset U_i$ with $U_1 \cap U_2 = \emptyset$.*

Proof. Proof of (1). Let $x \in X$, $x \notin E$. For every $e \in E$ we can find disjoint opens V_e and U_e with $e \in V_e$ and $x \in U_e$. Since $E \subset \bigcup V_e$ we can find finitely many $e_1, \ldots, e_n$ such that $E \subset V_{e_1} \cup \ldots \cup V_{e_n}$. Then $U = U_{e_1} \cap \ldots \cap U_{e_n}$ is an open neighbourhood of x which avoids $V_{e_1} \cup \ldots \cup V_{e_n}$. In particular it avoids E. Thus E is closed.

Proof of (2). In the proof of (1) we have seen that given $x \in E_1$ we can find an open neighbourhood $x \in U_x$ and an open $E_2 \subset V_x$ such that $U_x \cap V_x = \emptyset$. Because E_1 is quasi-compact we can find a finite number $x_i \in E_1$ such that $E_1 \subset U = U_{x_1} \cup \ldots \cup U_{x_n}$. We take $V = V_{x_1} \cap \ldots \cap V_{x_n}$ to finish the proof. □

08YC **Lemma 5.12.5.** *Let X be a quasi-compact Hausdorff space. Let $E \subset X$. The following are equivalent: (a) E is closed in X, (b) E is quasi-compact.*

Proof. The implication (a) ⇒ (b) is Lemma 5.12.3. The implication (b) ⇒ (a) is Lemma 5.12.4. □

The following is really a reformulation of the quasi-compact property.

005D **Lemma 5.12.6.** *Let X be a quasi-compact topological space. If $\{Z_\alpha\}_{\alpha \in A}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in A} Z_\alpha$ is nonempty.*

Proof. We suppose that $\bigcap_{\alpha \in A} Z_\alpha = \emptyset$. So we have $\bigcup_{\alpha \in A}(X \setminus Z_\alpha) = X$ by complementation. As the subsets Z_α are closed, $\bigcup_{\alpha \in A}(X \setminus Z_\alpha)$ is an open covering of the quasi-compact space X. Thus there exists a finite subset $J \subset A$ such that $X = \bigcup_{\alpha \in J}(X \setminus Z_\alpha)$. The complementary is then empty, which means that $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$. It proves there exists a finite subcollection of $\{Z_\alpha\}_{\alpha \in J}$ verifying $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$, which concludes by contraposition. □

04Z9 **Lemma 5.12.7.** *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is quasi-compact, then $f(X)$ is quasi-compact.*
(2) *If f is quasi-compact, then $f(X)$ is retrocompact.*

Proof. If $f(X) = \bigcup V_i$ is an open covering, then $X = \bigcup f^{-1}(V_i)$ is an open covering. Hence if X is quasi-compact then $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$ for some $i_1, \ldots, i_n \in I$ and hence $f(X) = V_{i_1} \cup \ldots \cup V_{i_n}$. This proves (1). Assume f is quasi-compact, and let $V \subset Y$ be quasi-compact open. Then $f^{-1}(V)$ is quasi-compact, hence by (1) we see that $f(f^{-1}(V)) = f(X) \cap V$ is quasi-compact. Hence $f(X)$ is retrocompact. □

005E **Lemma 5.12.8.** *Let X be a topological space. Assume that*

(1) *X is nonempty,*
(2) *X is quasi-compact, and*
(3) *X is Kolmogorov.*

Then X has a closed point.

Proof. Consider the set

$$\mathcal{T} = \{Z \subset X \mid Z = \overline{\{x\}} \text{ for some } x \in X\}$$

of all closures of singletons in X. It is nonempty since X is nonempty. Make $\mathcal{T}$ into a partially ordered set using the relation of inclusion. Suppose Z_α, $\alpha \in A$ is a totally ordered subset of $\mathcal{T}$. By Lemma 5.12.6 we see that $\bigcap_{\alpha\in A} Z_\alpha \neq \emptyset$. Hence there exists some $x \in \bigcap_{\alpha\in A} Z_\alpha$ and we see that $Z = \overline{\{x\}} \in \mathcal{T}$ is a lower bound for the family. By Zorn's lemma there exists a minimal element $Z \in \mathcal{T}$. As X is Kolmogorov we conclude that $Z = \{x\}$ for some x and $x \in X$ is a closed point. □

08ZM **Lemma 5.12.9.** *Let X be a quasi-compact Kolmogorov space. Then the set X_0 of closed points of X is quasi-compact.*

Proof. Let $X_0 = \bigcup U_{i,0}$ be an open covering. Write $U_{i,0} = X_0 \cap U_i$ for some open $U_i \subset X$. Consider the complement Z of $\bigcup U_i$. This is a closed subset of X, hence quasi-compact (Lemma 5.12.3) and Kolmogorov. By Lemma 5.12.8 if Z is nonempty it would have a closed point which contradicts the fact that $X_0 \subset \bigcup U_i$. Hence $Z = \emptyset$ and $X = \bigcup U_i$. Since X is quasi-compact this covering has a finite subcover and we conclude. □

005F **Lemma 5.12.10.** *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha\in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.12.3 and assumption (1). By assumption (2) there exist quasi-compact opens $U, V \subset X$ with $B = S \cap U$ and $C = S \cap V$ (details omitted). Then $U \cap V \cap S = \emptyset$. Hence $\bigcap_\alpha U \cap V \cap Z_\alpha = \emptyset$. By assumption (3) the intersection $U \cap V$ is quasi-compact. By Lemma 5.12.6 for some $\alpha' \in A$ we have $U \cap V \cap Z_{\alpha'} = \emptyset$. Since $X \setminus (U \cup V)$ is disjoint from S and closed in X hence quasi-compact, we can use the same lemma to see that $Z_{\alpha''} \subset U \cup V$ for some $\alpha'' \in A$. Then $Z_\alpha = Z_{\alpha'} \cap Z_{\alpha''}$ is contained in $U \cup V$ and disjoint from $U \cap V$. Hence $Z_\alpha = U \cap Z_\alpha \amalg V \cap Z_\alpha$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

08ZN **Lemma 5.12.11.** *Let X be a topological space. Assume X is quasi-compact and Hausdorff. For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.*

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha\in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not,

then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.12.3. By Lemma 5.12.4 there exist disjoint opens $U, V \subset X$ with $B \subset U$ and $C \subset V$. Then $X \setminus U \cup V$ is closed in X hence quasi-compact (Lemma 5.12.3). It follows that $(X \setminus U \cup V) \cap Z_\alpha = \emptyset$ for some α by Lemma 5.12.6. In other words, $Z_\alpha \subset U \cup V$. Thus $Z_\alpha = Z_\alpha \cap V \amalg Z_\alpha \cap U$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

04PL **Lemma 5.12.12.** *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For a subset $T \subset X$ the following are equivalent:

(a) *T is an intersection of open and closed subsets of X, and*
(b) *T is closed in X and is a union of connected components of X.*

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \notin T$. Let $x \in C \subset X$ be the connected component of X containing x. By Lemma 5.12.10 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets V_α of X which contain C. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a V_α. As T is a union of connected components of X we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since T is quasi-compact as a closed subset of a quasi-compact space (see Lemma 5.12.3) we deduce that $T \cap V_\alpha = \emptyset$ for some α, see Lemma 5.12.6. For this α we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of X which contains T and not x. The lemma follows. □

04ZA **Lemma 5.12.13.** *Let X be a Noetherian topological space.*

(1) *The space X is quasi-compact.*
(2) *Any subset of X is retrocompact.*

Proof. Suppose $X = \bigcup U_i$ is an open covering of X indexed by the set I which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of I inductively in the following way: Choose i_{n+1} such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that X is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of X is Noetherian by Lemma 5.9.2. □

04ZB **Lemma 5.12.14.** *A quasi-compact locally Noetherian space is Noetherian.*

Proof. The conditions imply immediately that X has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 5.9.4. □

08ZP **Lemma 5.12.15** (Alexander subbase theorem)**.** *Let X be a topological space. Let $\mathcal{B}$ be a subbase for X. If every covering of X by elements of $\mathcal{B}$ has a finite refinement, then X is quasi-compact.*

Proof. Assume there is an open covering of X which does not have a finite refinement. Using Zorn's lemma we can choose a maximal open covering $X = \bigcup_{i \in I} U_i$ which does not have a finite refinement (details omitted). In other words, if $U \subset X$ is any open which does not occur as one of the U_i, then the covering $X = U \cup \bigcup_{i \in I} U_i$

does have a finite refinement. Let $I' \subset I$ be the set of indices such that $U_i \in \mathcal{B}$. Then $\bigcup_{i \in I'} U_i \neq X$, since otherwise we would get a finite refinement covering X by our assumption on $\mathcal{B}$. Pick $x \in X$, $x \notin \bigcup_{i \in I'} U_i$. Pick $i \in I$ with $x \in U_i$. Pick $V_1, \ldots, V_n \in \mathcal{B}$ such that $x \in V_1 \cap \ldots \cap V_n \subset U_i$. This is possible as $\mathcal{B}$ is a subbasis for X. Note that V_j does not occur as a U_i. By maximality of the chosen covering we see that for each j there exist $i_{j,1}, \ldots, i_{j,n_j} \in I$ such that $X = V_j \cup U_{i_{j,1}} \cup \ldots \cup U_{i_{j,n_j}}$. Since $V_1 \cap \ldots \cap V_n \subset U_i$ we conclude that $X = U_i \cup \bigcup U_{i_{j,l}}$ a contradiction. □

5.13. Locally quasi-compact spaces

08ZQ Recall that a neighbourhood of a point need not be open.

0068 **Definition 5.13.1.** A topological space X is called *locally quasi-compact*[2] if every point has a fundamental system of quasi-compact neighbourhoods.

The term *locally compact space* in the literature often refers to a space as in the following lemma.

08ZR **Lemma 5.13.2.** *A Hausdorff space is locally quasi-compact if and only if every point has a quasi-compact neighbourhood.*

Proof. Let X be a Hausdorff space. Let $x \in X$ and let $x \in E \subset X$ be a quasi-compact neighbourhood. Then E is closed by Lemma 5.12.4. Suppose that $x \in U \subset X$ is an open neighbourhood of x. Then $Z = E \setminus U$ is a closed subset of E not containing x. Hence we can find a pair of disjoint open subsets $W, V \subset E$ of E such that $x \in V$ and $Z \subset W$, see Lemma 5.12.4. It follows that $\overline{V} \subset E$ is a closed neighbourhood of x contained in $E \cap U$. Also $\overline{V}$ is quasi-compact as a closed subset of E (Lemma 5.12.3). In this way we obtain a fundamental system of quasi-compact neighbourhoods of x. □

0CQN **Lemma 5.13.3** (Baire category theorem)**.** *Let X be a locally quasi-compact Hausdorff space. Let $U_n \subset X$, $n \geq 1$ be dense open subsets. Then $\bigcap_{n \geq 1} U_n$ is dense in X.*

Proof. After replacing U_n by $\bigcap_{i=1,\ldots,n} U_i$ we may assume that $U_1 \supset U_2 \supset \ldots$. Let $x \in X$. We will show that x is in the closure of $\bigcap_{n \geq 1} U_n$. Thus let E be a neighbourhood of x. To show that $E \cap \bigcap_{n \geq 1} U_n$ is nonempty we may replace E by a smaller neighbourhood. After replacing E by a smaller neighbourhood, we may assume that E is quasi-compact.

Set $x_0 = x$ and $E_0 = E$. Below, we will inductively choose a point $x_i \in E_{i-1} \cap U_i$ and a quasi-compact neighbourhood E_i of x_i with $E_i \subset E_{i-1} \cap U_i$. Because X is Hausdorff, the subsets $E_i \subset X$ are closed (Lemma 5.12.4). Since the E_i are also nonempty we conclude that $\bigcap_{i \geq 1} E_i$ is nonempty (Lemma 5.12.6). Since $\bigcap_{i \geq 1} E_i \subset E \cap \bigcap_{n \geq 1} U_n$ this proves the lemma.

The base case $i = 0$ we have done above. Induction step. Since E_{i-1} is a neighbourhood of x_{i-1} we can find an open $x_{i-1} \in W \subset E_{i-1}$. Since U_i is dense in X we

[2] This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.

see that $W \cap U_i$ is nonempty. Pick any $x_i \in W \cap U_i$. By definition of locally quasi-compact spaces we can find a quasi-compact neighbourhood E_i of x_i contained in $W \cap U_i$. Then $E_i \subset E_{i-1} \cap U_i$ as desired. □

09UV **Lemma 5.13.4.** *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then there exists an open covering $X = \bigcup_{i \in I} V_i$ such that $\overline{V_i} \subset U_i$ for all i.*

Proof. Let $x \in X$. Choose an $i(x) \in I$ such that $x \in U_{i(x)}$. Since $X \setminus U_{i(x)}$ and $\{x\}$ are disjoint closed subsets of X, by Lemmas 5.12.3 and 5.12.4 there exists an open neighbourhood U_x of x whose closure is disjoint from $X \setminus U_{i(x)}$. Thus $\overline{U_x} \subset U_{i(x)}$. Since X is quasi-compact, there is a finite list of points $x_1, \ldots, x_m$ such that $X = U_{x_1} \cup \ldots \cup U_{x_m}$. Setting $V_i = \bigcup_{i=i(x_j)} U_{x_j}$ the proof is finished. □

09UW **Lemma 5.13.5.** *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Suppose given an integer $p \geq 0$ and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open covering $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$. Then there exists an open covering $X = \bigcup_{j \in J} V_j$ and a map $\alpha : J \to I$ such that $\overline{V_j} \subset U_{\alpha(j)}$ and such that each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k.*

Proof. Since X is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result for I finite by induction on p. The base case $p = 0$ is immediate by taking a covering as in Lemma 5.13.4 refining the open covering $X = \bigcup W_{i_0, k}$.

Induction step. Assume the lemma proven for $p-1$. For all p-tuples $i'_0, \ldots, i'_{p-1}$ of I let $U_{i'_0} \cap \ldots \cap U_{i'_{p-1}} = \bigcup W_{i'_0 \ldots i'_{p-1}, k}$ be a common refinement of the coverings $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$ for those $(p+1)$-tuples such that $\{i'_0, \ldots, i'_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). (There are finitely many of these as I is finite.) By induction there exists a solution for these opens, say $X = \bigcup V_j$ and $\alpha : J \to I$. At this point the covering $X = \bigcup_{j \in J} V_j$ and α satisfy $\overline{V_j} \subset U_{\alpha(j)}$ and each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k if there is a repetition in $\alpha(j_0), \ldots, \alpha(j_p)$. Of course, we may and do assume that J is finite.

Fix $i_0, \ldots, i_p \in I$ pairwise distinct. Consider $(p+1)$-tuples $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for any k. Let N be the number of such $(p+1)$-tuples. We will show how to decrease N. Since

$$\overline{V_{j_0}} \cap \ldots \cap \overline{V_{j_p}} \subset U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$$

we find a finite set $K = \{k_1, \ldots, k_t\}$ such that the LHS is contained in $\bigcup_{k \in K} W_{i_0 \ldots i_p, k}$. Then we consider the open covering

$$V_{j_0} = (V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})) \cup (\bigcup\nolimits_{k \in K} V_{j_0} \cap W_{i_0 \ldots i_p, k})$$

The first open on the RHS intersects $V_{j_1} \cap \ldots \cap V_{j_p}$ in the empty set and the other opens $V_{j_0, k}$ of the RHS satisfy $V_{j_0, k} \cap V_{j_1} \ldots \cap V_{j_p} \subset W_{\alpha(j_0) \ldots \alpha(j_p), k}$. Set $J' = J \amalg K$. For $j \in J$ set $V'_j = V_j$ if $j \neq j_0$ and set $V'_{j_0} = V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})$. For $k \in K$ set $V'_k = V_{j_0, k}$. Finally, the map $\alpha' : J' \to I$ is given by α on J and maps every element of K to i_0. A simple check shows that N has decreased by one under this replacement. Repeating this procedure N times we arrive at the situation where $N = 0$.

To finish the proof we argue by induction on the number M of $(p+1)$-tuples $i_0, \ldots, i_p \in I$ with pairwise distinct entries for which there exists a $(p+1)$-tuple $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0)\ldots\alpha(j_p),k}$ for any k. To do this, we claim that the operation performed in the previous paragraph does not increase M. This follows formally from the fact that the map $\alpha' : J' \to I$ factors through a map $\beta : J' \to J$ such that $V'_{j'} \subset V_{\beta(j')}$. □

09UX **Lemma 5.13.6.** *Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that*

(1) *$Z \subset \bigcup U_i$, and*
(2) *for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.*

Then there exist opens V_i of X such that we have $Z \subset \bigcup V_i$, for all i we have $\overline{V_i} \subset U_i$, and we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). Because X is locally quasi-compact and Z is quasi-compact, we can find a neighbourhood $Z \subset E$ which is quasi-compact, i.e., E is quasi-compact and contains an open neighbourhood of Z in X. If we prove the result after replacing X by E, then the result follows. Hence we may assume X is quasi-compact.

We prove the result in case I is finite and X is quasi-compact by induction on p. The base case is $p = 0$. In this case we have $X = (X \setminus Z) \cup \bigcup W_i$. By Lemma 5.13.4 we can find a covering $X = V \cup \bigcup V_i$ by opens $V_i \subset W_i$ and $V \subset X \setminus Z$ with $\overline{V_i} \subset W_i$ for all i. Then we see that we obtain a solution of the problem posed by the lemma.

Induction step. Assume the lemma proven for $p-1$. Set $W_{j_0\ldots j_{p-1}}$ equal to the intersection of all $W_{i_0\ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0\ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0\ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0\ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0\ldots i_p}$$

is a closed subset of X contained in $U_{i_0} \cap \ldots \cap U_{i_p}$ not meeting Z. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

0CY5 **Lemma 5.13.7.** *Let X be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of Z have disjoint open neighbourhoods in X. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that*

(1) *$Z \subset \bigcup U_i$, and*
(2) *for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.*

Then there exist opens V_i of X such that

(1) *$Z \subset \bigcup V_i$,*

(2) $V_i \subset U_i$ *for all* i,
(3) $\overline{V_i} \cap Z \subset U_i$ *for all* i, *and*
(4) $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ *for all* $(p+1)$*-tuples* $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result in case I is finite by induction on p.

The base case is $p = 0$. For $z \in Z \cap U_i$ and $z' \in Z \setminus U_i$ there exist disjoint opens $z \in V_{z,z'}$ and $z' \in W_{z,z'}$ of X. Since $Z \setminus U_i$ is quasi-compact (Lemma 5.12.3), we can choose a finite number $z'_1, \ldots, z'_r$ such that $Z \setminus U_i \subset W_{z,z'_1} \cup \ldots \cup W_{z,z'_r}$. Then we see that $V_z = V_{z,z'_1} \cap \ldots \cap V_{z,z'_r} \cap U_i$ is an open neighbourhood of z contained in U_i with the property that $\overline{V_z} \cap Z \subset U_i$. Since z and i where arbitrary and since Z is quasi-compact we can find a finite list $z_1, i_1, \ldots, z_t, i_t$ and opens $V_{z_j} \subset U_{i_j}$ with $\overline{V_{z_j}} \cap Z \subset U_{i_j}$ and $Z \subset \bigcup V_{z_j}$. Then we can set $V_i = W_i \cap (\bigcup_{j:i=i_j} V_{z_j})$ to solve the problem for $p = 0$.

Induction step. Assume the lemma proven for $p - 1$. Set $W_{j_0 \ldots j_{p-1}}$ equal to the intersection of all $W_{i_0 \ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0 \ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0 \ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0 \ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0 \ldots i_p}$$

is a closed subset of X not meeting Z by our property (3) of the opens V_i. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

5.14. Limits of spaces

08ZS The category of topological spaces has products. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow $\prod_{i \in I} X_i$ with the *product topology*. As a basis for the topology we use sets of the form $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i.

The category of topological spaces has equalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the equalizer of a and b is the subset $\{x \in X \mid a(x) = b(x)\} \subset X$ endowed with the induced topology.

08ZT **Lemma 5.14.1.** *The category of topological spaces has limits and the forgetful functor to sets commutes with them.*

Proof. This follows from the discussion above and Categories, Lemma 4.14.11. It follows from the description above that the forgetful functor commutes with limits. Another way to see this is to use Categories, Lemma 4.24.5 and use that the forgetful functor has a left adjoint, namely the functor which assigns to a set the corresponding discrete topological space. □

0A2P **Lemma 5.14.2.** *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let $X = \lim X_i$ be the limit with projection maps $f_i : X \to X_i$.*

(1) *Any open of X is of the form $\bigcup_{j\in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset X_j$.*
(2) *Any quasi-compact open of X is of the form $f_i^{-1}(U_i)$ for some i and some $U_i \subset X_i$ open.*

Proof. The construction of the limit given above shows that $X \subset \prod X_i$ with the induced topology. A basis for the topology of $\prod X_i$ are the opens $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i. Say $i_1, \ldots, i_n \in \mathrm{Ob}(\mathcal{I})$ are the objects such that $U_{i_j} \neq X_{i_j}$. Then

$$X \cap \prod U_i = f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

For a general limit of topological spaces these form a basis for the topology on X. However, if $\mathcal{I}$ is cofiltered as in the statement of the lemma, then we can pick a $j \in \mathrm{Ob}(\mathcal{I})$ and morphisms $j \to i_l$, $l = 1, \ldots, n$. Let

$$U_j = (X_j \to X_{i_1})^{-1}(U_{i_1}) \cap \ldots \cap (X_j \to X_{i_n})^{-1}(U_{i_n})$$

Then it is clear that $X \cap \prod U_i = f_j^{-1}(U_j)$. Thus for any open W of X there is a set A and a map $\alpha : A \to \mathrm{Ob}(\mathcal{I})$ and opens $U_a \subset X_{\alpha(a)}$ such that $W = \bigcup f_{\alpha(a)}^{-1}(U_a)$. Set $J = \mathrm{Im}(\alpha)$ and for $j \in J$ set $U_j = \bigcup_{\alpha(a)=j} U_a$ to see that $W = \bigcup_{j\in J} f_j^{-1}(U_j)$. This proves (1).

To see (2) suppose that $\bigcup_{j\in J} f_j^{-1}(U_j)$ is quasi-compact. Then it is equal to $f_{j_1}^{-1}(U_{j_1}) \cup \ldots \cup f_{j_m}^{-1}(U_{j_m})$ for some $j_1, \ldots, j_m \in J$. Since $\mathcal{I}$ is cofiltered, we can pick a $i \in \mathrm{Ob}(\mathcal{I})$ and morphisms $i \to j_l$, $l = 1, \ldots, m$. Let

$$U_i = (X_i \to X_{j_1})^{-1}(U_{j_1}) \cup \ldots \cup (X_i \to X_{j_m})^{-1}(U_{j_m})$$

Then our open equals $f_i^{-1}(U_i)$ as desired. □

0A2Q **Lemma 5.14.3.** *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let X be a topological space such that*

(1) *$X = \lim X_i$ as a set (denote f_i the projection maps),*
(2) *the sets $f_i^{-1}(U_i)$ for $i \in \mathrm{Ob}(\mathcal{I})$ and $U_i \subset X_i$ open form a basis for the topology of X.*

Then X is the limit of the X_i as a topological space.

Proof. Follows from the description of the limit topology in Lemma 5.14.2. □

08ZU **Theorem 5.14.4** (Tychonov)**.** *A product of quasi-compact spaces is quasi-compact.*

Proof. Let I be a set and for $i \in I$ let X_i be a quasi-compact topological space. Set $X = \prod X_i$. Let $\mathcal{B}$ be the set of subsets of X of the form $U_i \times \prod_{i'\in I, i'\neq i} X_{i'}$ where $U_i \subset X_i$ is open. By construction this family is a subbasis for the topology on X. By Lemma 5.12.15 it suffices to show that any covering $X = \bigcup_{j\in J} B_j$ by elements B_j of $\mathcal{B}$ has a finite refinement. We can decompose $J = \coprod J_i$ so that if $j \in J_i$, then $B_j = U_j \times \prod_{i'\neq i} X_{i'}$ with $U_j \subset X_i$ open. If $X_i = \bigcup_{j\in J_i} U_j$, then there is a finite refinement and we conclude that $X = \bigcup_{j\in J} B_j$ has a finite refinement. If this is not the case, then for every i we can choose an point $x_i \in X_i$ which is not in $\bigcup_{j\in J_i} U_j$. But then the point $x = (x_i)_{i\in I}$ is an element of X not contained in $\bigcup_{j\in J} B_j$, a contradiction. □

The following lemma does not hold if one drops the assumption that the spaces X_i are Hausdorff, see Examples, Section 110.4.

08ZV **Lemma 5.14.5.** *Let $\mathcal{I}$ be a category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact and Hausdorff, then $\lim X_i$ is quasi-compact.*

Proof. Recall that $\lim X_i$ is a subspace of $\prod X_i$. By Theorem 5.14.4 this product is quasi-compact. Hence it suffices to show that $\lim X_i$ is a closed subspace of $\prod X_i$ (Lemma 5.12.3). If $\varphi : j \to k$ is a morphism of $\mathcal{I}$, then let $\Gamma_\varphi \subset X_j \times X_k$ denote the graph of the corresponding continuous map $X_j \to X_k$. By Lemma 5.3.2 this graph is closed. It is clear that $\lim X_i$ is the intersection of the closed subsets

$$\Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l \subset \prod X_i$$

Thus the result follows. □

The following lemma generalizes Categories, Lemma 4.21.7 and partially generalizes Lemma 5.12.6.

0A2R **Lemma 5.14.6.** *Let $\mathcal{I}$ be a cofiltered category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact, Hausdorff, and nonempty, then $\lim X_i$ is nonempty.*

Proof. In the proof of Lemma 5.14.5 we have seen that $X = \lim X_i$ is the intersection of the closed subsets

$$Z_\varphi = \Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l$$

inside the quasi-compact space $\prod X_i$ where $\varphi : j \to k$ is a morphism of $\mathcal{I}$ and $\Gamma_\varphi \subset X_j \times X_k$ is the graph of the corresponding morphism $X_j \to X_k$. Hence by Lemma 5.12.6 it suffices to show any finite intersection of these subsets is nonempty. Assume $\varphi_t : j_t \to k_t$, $t = 1, \ldots, n$ is a finite collection of morphisms of $\mathcal{I}$. Since $\mathcal{I}$ is cofiltered, we can pick an object j and a morphism $\psi_t : j \to j_t$ for each t. For each pair t, t' such that either (a) $j_t = j_{t'}$, or (b) $j_t = k_{t'}$, or (c) $k_t = k_{t'}$ we obtain two morphisms $j \to l$ with $l = j_t$ in case (a), (b) or $l = k_t$ in case (c). Because $\mathcal{I}$ is cofiltered and since there are finitely many pairs (t, t') we may choose a map $j' \to j$ which equalizes these two morphisms for all such pairs (t, t'). Pick an element $x \in X_{j'}$ and for each t let x_{j_t}, resp. x_{k_t} be the image of x under the morphism $X_{j'} \to X_j \to X_{j_t}$, resp. $X_{j'} \to X_j \to X_{j_t} \to X_{k_t}$. For any index $l \in \mathrm{Ob}(\mathcal{I})$ which is not equal to j_t or k_t for some t we pick an arbitrary element $x_l \in X_l$ (using the axiom of choice). Then $(x_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ is in the intersection

$$Z_{\varphi_1} \cap \ldots \cap Z_{\varphi_n}$$

by construction and the proof is complete. □

5.15. Constructible sets

04ZC

005G **Definition 5.15.1.** Let X be a topological space. Let $E \subset X$ be a subset of X.

(1) We say E is *constructible*[3] in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.

[3] In the second edition of EGA I [**GD71**] this was called a "globally constructible" set and a the terminology "constructible" was used for what we call a locally constructible set.

(2) We say E is *locally constructible* in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is constructible in V_i.

005H **Lemma 5.15.2.** *The collection of constructible sets is closed under finite intersections, finite unions and complements.*

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose $U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. □

005I **Lemma 5.15.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.*

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the definition of constructible sets. □

005J **Lemma 5.15.4.** *Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 5.15.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. □

09YD **Lemma 5.15.5.** *Let $U \subset X$ be a retrocompact open. Let $E \subset U$. If E is constructible in U, then E is constructible in X.*

Proof. Suppose that $V, W \subset U$ are retrocompact open in U. Then V, W are retrocompact open in X (Lemma 5.12.2). Hence $V \cap (U \setminus W) = V \cap (X \setminus W)$ is constructible in X. We conclude since every constructible subset of U is a finite union of subsets of the form $V \cap (U \setminus W)$. □

053W **Lemma 5.15.6.** *Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.*

Proof. If E is constructible in X, then by Lemma 5.15.4 we see that $E \cap V_j$ is constructible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then $E = \bigcup E \cap V_j$ is a finite union of constructible sets by Lemma 5.15.5 and hence constructible. □

09YE **Lemma 5.15.7.** *Let X be a topological space. Let $Z \subset X$ be a closed subset such that $X \setminus Z$ is quasi-compact. Then for a constructible set $E \subset X$ the intersection $E \cap Z$ is constructible in Z.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap Z$ is retrocompact in Z by Lemma 5.15.3. To show this let $W \subset Z$ be open and quasi-compact. The subset $W' = W \cup (X \setminus Z)$ is quasi-compact, open, and $W = Z \cap W'$. Hence $V \cap Z \cap W = V \cap Z \cap W'$ is a closed subset of the quasi-compact

open $V \cap W'$ as V is retrocompact in X. Thus $V \cap Z \cap W$ is quasi-compact by Lemma 5.12.3. □

09YF **Lemma 5.15.8.** *Let X be a topological space. Let $T \subset X$ be a subset. Suppose*

(1) *T is retrocompact in X,*
(2) *quasi-compact opens form a basis for the topology on X.*

Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in T.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap T$ is retrocompact in T by Lemma 5.15.3. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of T with the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap T \cap W$ is quasi-compact. □

09YG **Lemma 5.15.9.** *Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If E is constructible in Z, then E is constructible in X.*

Proof. Suppose that $V \subset Z$ is retrocompact open in Z. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of X. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in Z (Lemma 5.12.3) and V is retrocompact in Z. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in X. In this way we see that $\tilde{V}$ is retrocompact in X. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in X. We conclude since every constructible subset of Z is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. □

09YH **Lemma 5.15.10.** *Let X be a topological space. Every constructible subset of X is retrocompact.*

Proof. Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with U_i, V_i retrocompact open in X. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V^c \cap W$ is quasi-compact if U, V are retrocompact open and W is quasi-compact open. This is true because $U \cap V^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma 5.12.3 applies. □

Question: Does the following lemma also hold if we assume X is a quasi-compact topological space? Compare with Lemma 5.15.7.

09YI **Lemma 5.15.11.** *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. For E, E' constructible in X, the intersection $E \cap E'$ is constructible in E.*

Proof. Combine Lemmas 5.15.8 and 5.15.10. □

09YJ **Lemma 5.15.12.** *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. Let E be constructible in X and $F \subset E$ constructible in E. Then F is constructible in X.*

Proof. Observe that any retrocompact subset T of X has a basis for the induced topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma 5.15.10). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in E by Lemma 5.15.11. Hence $F \cap E_i$ is constructible in E_i by Lemma 5.15.11. Thus it suffices to prove the lemma in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 5.15.9 to the first inclusion and Lemma 5.15.5 to the second. □

0F2K **Lemma 5.15.13.** *Let X be a quasi-compact topological space having a basis consisting of quasi-compact opens such that the intersection of any two quasi-compact opens is quasi-compact. Let $T \subset X$ be a locally closed subset such that T is quasi-compact and T^c is retrocompact in X. Then T is constructible in X.*

Proof. Note that T is quasi-compact and open in $\overline{T}$. Using our basis of quasi-compact opens we can write $T = U \cap \overline{T}$ where U is quasi-compact open in X. Then $V = U \setminus T = U \cap T^c$ is retrocompact in U as T^c is retrocompact in X. Hence V is quasi-compact. Since the intersection of any two quasi-compact opens is quasi-compact any quasi-compact open of X is retrocompact. Thus $T = U \cap V^c$ with U and $V = U \setminus T$ retrocompact opens of X. A fortiori, T is constructible in X. □

09YK **Lemma 5.15.14.** *Let X be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then E is constructible in X if and only if $E \cap E_j$ is constructible in E_j for each $j = 1, \ldots, m$.*

Proof. Combine Lemmas 5.15.11 and 5.15.12. □

005K **Lemma 5.15.15.** *Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent*

(1) *The intersection $E \cap Z$ contains an open dense subset of Z.*
(2) *The intersection $E \cap Z$ is dense in Z.*

If Z has a generic point ξ, then this is also equivalent to

(3) *We have $\xi \in E$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. Assume (2). Note that $E \cap Z$ is a finite union of locally closed subsets Z_i of Z. Since Z is irreducible, one of the Z_i must be dense in Z. Then this Z_i is dense open in Z as it is open in its closure. Hence (1) holds.

Suppose that $\xi \in Z$ is a generic point. If the equivalent conditions (1) and (2) hold, then $\xi \in E$. Conversely, if $\xi \in E$ then $\xi \in E \cap Z$ and hence $E \cap Z$ is dense in Z. □

5.16. Constructible sets and Noetherian spaces

053X

005L **Lemma 5.16.1.** *Let X be a Noetherian topological space. The constructible sets in X are precisely the finite unions of locally closed subsets of X.*

Proof. This follows immediately from Lemma 5.12.13. □

053Y **Lemma 5.16.2.** *Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in Y, then $f^{-1}(E)$ is constructible in X.*

Proof. Follows immediately from Lemma 5.16.1 and the definition of a continuous map. □

053Z **Lemma 5.16.3.** *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:*

(1) *E is constructible in X, and*
(2) *for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of Z or is not dense in Z.*

Proof. Assume E is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in Z by Lemma 5.16.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets T_i of Z. Clearly if none of the T_i is open in Z, then $E \cap Z$ is not dense in Z. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $\mathcal{S}$ of closed subsets Y of X such that $E \cap Y$ is not constructible in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.9.2. If $r > 1$, then each $Y_i \cap E$ is constructible in Y_i and hence a finite union of locally closed subsets of Y_i. Thus $E \cap Y$ is a finite union of locally closed subsets of Y too and we conclude that $E \cap Y$ is constructible in Y by Lemma 5.16.1. This is a contradiction and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open V of Y or (b) is not dense in Y. In case (a) we see, by minimality of Y, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.16.1. This is a contradiction and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of Y we see that $E \cap Y'$ is a finite union of locally closed subsets of Y' and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.16.1. This contradiction finishes the proof of the lemma. □

0540 **Lemma 5.16.4.** *Let X be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in X. The following are equivalent:*

(1) *E is a neighbourhood of x, and*
(2) *for every irreducible closed subset Y of X which contains x the intersection $E \cap Y$ is dense in Y.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the set $\mathcal{S}$ of closed subsets Y of X containing x such that $E \cap Y$ is not a neighbourhood of x in Y. If $\mathcal{S} \neq \emptyset$, then it has a minimal element Y as X is Noetherian. Suppose $Y = Y_1 \cup Y_2$ with two smaller nonempty closed subsets Y_1, Y_2. If $x \in Y_i$ for $i = 1, 2$, then $Y_i \cap E$ is a neighbourhood of x in Y_i and we conclude $Y \cap E$ is a neighbourhood of x in Y which is a contradiction. If $x \in Y_1$ but $x \notin Y_2$ (say), then $Y_1 \cap E$ is a neighbourhood of x in Y_1 and hence also in Y, which is a contradiction as well. We conclude that Y is irreducible closed. By assumption (2) we see that $E \cap Y$ is dense in Y. Thus $E \cap Y$ contains an open V of Y, see Lemma 5.16.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of x in Y which is a contradiction. If $x \notin V$, then

$Y' = Y \setminus V$ is a proper closed subset of Y containing x. By minimality of Y we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of x in Y'. But then $V' \cup V$ is an open neighbourhood of x in Y contained in E, a contradiction. This contradiction finishes the proof of the lemma. □

0541 **Lemma 5.16.5.** *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:*

(1) *E is open in X, and*
(2) *for every irreducible closed subset Y of X the intersection $E \cap Y$ is either empty or contains a nonempty open of Y.*

Proof. This follows formally from Lemmas 5.16.3 and 5.16.4. □

5.17. Characterizing proper maps

005M We include a section discussing the notion of a proper map in usual topology. We define a continuous map of topological spaces to be *proper* if it is universally closed and separated. Although this matches well with the definition of a proper morphism in algebraic geometry, this is different from the definition in Bourbaki. With our definition of a proper map of topological spaces, the proper base change theorem (Cohomology, Theorem 20.18.2) holds without any further assumptions. Furthermore, given a morphism $f : X \to Y$ of finite type schemes over $\mathbf{C}$ one has: f is proper as a morphism of schemes if and only if the continuous map $f : X(\mathbf{C}) \to Y(\mathbf{C})$ on $\mathbf{C}$-points with the classical topology is proper. This is explained in [**Gro71**, Exp. XII, Prop. 3.2(v)] which also has a footnote pointing out that they take properness in topology to be Bourbaki's notion with separatedness added on.

We find it useful to have names for three distinct concepts: separated, universally closed, and both of those together (i.e., properness). For a continuous map $f : X \to Y$ of locally compact Hausdorff spaces the word "proper" has long been used for the notion "$f^{-1}(\text{compact}) = \text{compact}$" and this is equivalent to universal closedness for such nice spaces. In fact, we will see the preimage condition formulated for clarity using the word "quasi-compact" is equivalent to universal closedness in general, if one includes the assumption of the map being closed. See also [**Lan93**, Exercises 22-26 in Chapter II] but beware that Lang uses "proper" as a synonym for "universally closed", like Bourbaki does.

005N **Lemma 5.17.1** (Tube lemma)**.** *Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.*

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of X and $V_{(a,b)}$ of Y such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix b and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}$. Hence

$$A \times \{b\} \subset (U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}) \times (V_{(a_1,b)} \cap \ldots \cap V_{(a_n,b)}) \subset W.$$

Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b \subset W$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \ldots \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. □

The notation in the following definition may be slightly different from what you are used to.

005O **Definition 5.17.2.** Let $f : X \to Y$ be a continuous map between topological spaces.

(1) We say that the map f is *closed* if the image of every closed subset is closed.
(2) We say that the map f is *Bourbaki-proper*[4] if the map $Z \times X \to Z \times Y$ is closed for any topological space Z.
(3) We say that the map f is *quasi-proper* if the inverse image $f^{-1}(V)$ of every quasi-compact subset $V \subset Y$ is quasi-compact.
(4) We say that f is *universally closed* if the map $f' : Z \times_Y X \to Z$ is closed for any continuous map $g : Z \to Y$.
(5) We say that f is *proper* if f is separated and universally closed.

The following lemma is useful later.

005P **Lemma 5.17.3.** *A topological space X is quasi-compact if and only if the projection map $Z \times X \to Z$ is closed for any topological space Z.*

Combination of [**Bou71**, I, p. 75, Lemme 1] and [**Bou71**, I, p. 76, Corrolaire 1].

Proof. (See also remark below.) If X is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of U_i cover X. Let Z be the subset of the power set $\mathcal{P}(I)$ of I consisting of I and all nonempty finite subsets of I. Define a topology on Z with as a basis for the topology the following sets:

(1) All subsets of $Z \setminus \{I\}$.
(2) For every finite subset K of I the set $U_K := \{J \subset I \mid J \in Z,\ K \subset J\})$.

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$$M = \{(J, x) \mid J \in Z,\ x \in \textstyle\bigcap_{i \in J} U_i^c)\}$$

If $(J, x) \notin M$, then $x \in U_i$ for some $i \in J$. Hence $U_{\{i\}} \times U_i \subset Z \times X$ is an open subset containing (J, x) and not intersecting M. Hence M is closed. The projection of M to Z is $Z - \{I\}$ which is not closed. Hence $Z \times X \to Z$ is not closed.

Assume X is quasi-compact. Let Z be a topological space. Let $M \subset Z \times X$ be closed. Let $z \in Z$ be a point which is not in $\mathrm{pr}_1(M)$. By the Tube Lemma 5.17.1 there exists an open $U \subset Z$ such that $U \times X$ is contained in the complement of M. Hence $\mathrm{pr}_1(M)$ is closed. □

005Q **Remark 5.17.4.** Lemma 5.17.3 is a combination of [**Bou71**, I, p. 75, Lemme 1] and [**Bou71**, I, p. 76, Corollaire 1].

005R **Theorem 5.17.5.** *Let $f : X \to Y$ be a continuous map between topological spaces. The following conditions are equivalent:*

(1) *The map f is quasi-proper and closed.*
(2) *The map f is Bourbaki-proper.*
(3) *The map f is universally closed.*
(4) *The map f is closed and $f^{-1}(y)$ is quasi-compact for any $y \in Y$.*

In [**Bou71**, I, p. 75, Theorem 1] you can find: (2) ⇔ (4). In [**Bou71**, I, p. 77, Proposition 6] you can find: (2) ⇒ (1).

Proof. (See also the remark below.) If the map f satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

[4] This is the terminology used in [**Bou71**]. Sometimes this property may be called "universally closed" in the literature.

Assume map f satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let $g : Z \to Y$ be a continuous map of topological spaces and consider the diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

During the proof we will use that $Z \times_Y X \to Z \times X$ is a homeomorphism onto its image, i.e., that we may identify $Z \times_Y X$ with the corresponding subset of $Z \times X$ with the induced topology. The image of $f' : Z \times_Y X \to Z$ is $\mathrm{Im}(f') = \{z : g(z) \in f(X)\}$. Because $f(X)$ is closed, we see that $\mathrm{Im}(f')$ is a closed subspace of Z. Consider a closed subset $P \subset Z \times_Y X$. Let $z \in Z$, $z \notin f'(P)$. If $z \notin \mathrm{Im}(f')$, then $Z \setminus \mathrm{Im}(f')$ is an open neighbourhood which avoids $f'(P)$. If z is in $\mathrm{Im}(f')$ then $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}$ and $f^{-1}\{g(z)\}$ is quasi-compact by assumption. Because P is a closed subset of $Z \times_Y X$, we have a closed P' of $Z \times X$ such that $P = P' \cap Z \times_Y X$. Since $(f')^{-1}\{z\}$ is a subset of $P^c = P'^c \cup (Z \times_Y X)^c$, and since $(f')^{-1}\{z\}$ is disjoint from $(Z \times_Y X)^c$ we see that $(f')^{-1}\{z\}$ is contained in P'^c. We may apply the Tube Lemma 5.17.1 to $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X$. This gives $V \times U$ containing $(f')^{-1}\{z\}$ where U and V are open sets in X and Z respectively and $V \times U$ has empty intersection with P'. Then the set $V \cap g^{-1}(Y - f(U^c))$ is open in Z since f is closed, contains z, and has empty intersection with the image of P. Thus $f'(P)$ is closed. In other words, the map f is universally closed.

The implication (3) $\Rightarrow$ (2) is trivial. Namely, given any topological space Z consider the projection morphism $g : Z \times Y \to Y$. Then it is easy to see that f' is the map $Z \times X \to Z \times Y$, in other words that $(Z \times Y) \times_Y X = Z \times X$. (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume f satisfies (2). We will prove it satisfies (1). Note that f is closed as f can be identified with the map $\{pt\} \times X \to \{pt\} \times Y$ which is assumed closed. Choose any quasi-compact subset $K \subset Y$. Let Z be any topological space. Because $Z \times X \to Z \times Y$ is closed we see the map $Z \times f^{-1}(K) \to Z \times K$ is closed (if T is closed in $Z \times f^{-1}(K)$, write $T = Z \times f^{-1}(K) \cap T'$ for some closed $T' \subset Z \times X$). Because K is quasi-compact, $K \times Z \to Z$ is closed by Lemma 5.17.3. Hence the composition $Z \times f^{-1}(K) \to Z \times K \to Z$ is closed and therefore $f^{-1}(K)$ must be quasi-compact by Lemma 5.17.3 again. □

005S **Remark 5.17.6.** Here are some references to the literature. In [**Bou71**, I, p. 75, Theorem 1] you can find: (2) $\Leftrightarrow$ (4). In [**Bou71**, I, p. 77, Proposition 6] you can find: (2) $\Rightarrow$ (1). Of course, trivially we have (1) $\Rightarrow$ (4). Thus (1), (2) and (4) are equivalent. The equivalence of (3) and (4) is [**Lan93**, Chapter II, Exercise 25].

08YD **Lemma 5.17.7.** *Let $f : X \to Y$ be a continuous map of topological spaces. If X is quasi-compact and Y is Hausdorff, then f is universally closed.*

Proof. Since every point of Y is closed, we see from Lemma 5.12.3 that the closed subset $f^{-1}(y)$ of X is quasi-compact for all $y \in Y$. Thus, by Theorem 5.17.5 it suffices to show that f is closed. If $E \subset X$ is closed, then it is quasi-compact (Lemma 5.12.3), hence $f(E) \subset Y$ is quasi-compact (Lemma 5.12.7), hence $f(E)$ is closed in Y (Lemma 5.12.4). □

08YE **Lemma 5.17.8.** *Let $f : X \to Y$ be a continuous map of topological spaces. If f is bijective, X is quasi-compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to prove f is closed, because this implies that f^{-1} is continuous. If $T \subset X$ is closed, then T is quasi-compact by Lemma 5.12.3, hence $f(T)$ is quasi-compact by Lemma 5.12.7, hence $f(T)$ is closed by Lemma 5.12.4. □

5.18. Jacobson spaces

005T

005U **Definition 5.18.1.** Let X be a topological space. Let X_0 be the set of closed points of X. We say that X is *Jacobson* if every closed subset $Z \subset X$ is the closure of $Z \cap X_0$.

Note that a topological space X is Jacobson if and only if every nonempty locally closed subset of X has a point closed in X.

Let X be a Jacobson space and let X_0 be the set of closed points of X with the induced topology. Clearly, the definition implies that the morphism $X_0 \to X$ induces a bijection between the closed subsets of X_0 and the closed subsets of X. Thus many properties of X are inherited by X_0. For example, the Krull dimensions of X and X_0 are the same.

005V **Lemma 5.18.2.** *Let X be a topological space. Let X_0 be the set of closed points of X. Suppose that for every point $x \in X$ the intersection $X_0 \cap \overline{\{x\}}$ is dense in $\overline{\{x\}}$. Then X is Jacobson.*

Proof. Let Z be closed subset of X and U be and open subset of X such that $U \cap Z$ is nonempty. Then for $x \in U \cap Z$ we have that $\overline{\{x\}} \cap U$ is a nonempty subset of $Z \cap U$, and by hypothesis it contains a point closed in X as required. □

02I7 **Lemma 5.18.3.** *Let X be a Kolmogorov topological space with a basis of quasi-compact open sets. If X is not Jacobson, then there exists a non-closed point $x \in X$ such that $\{x\}$ is locally closed.*

Proof. As X is not Jacobson there exists a closed set Z and an open set U in X such that $Z \cap U$ is nonempty and does not contain points closed in X. As X has a basis of quasi-compact open sets we may replace U by an open quasi-compact neighborhood of a point in $Z \cap U$ and so we may assume that U is quasi-compact open. By Lemma 5.12.8, there exists a point $x \in Z \cap U$ closed in $Z \cap U$, and so $\{x\}$ is locally closed but not closed in X. □

005W **Lemma 5.18.4.** *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Then X is Jacobson if and only if each U_i is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.*

Proof. Let X be a topological space. Let X_0 be the set of closed points of X. Let $U_{i,0}$ be the set of closed points of U_i. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each U_i is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., x is closed in U_i. Let $\overline{\{x\}}$ be the closure in X. Consider $\overline{\{x\}} \cap U_j$. If $x \notin U_j$, then $\overline{\{x\}} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of U_j containing x. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \amalg T'$. Then

T, T' are closed subsets of U_j and T contains x. As U_j is Jacobson we see that the closed points of U_j are dense in T. Because $T = \{x\} \amalg T'$ this can only be the case if x is closed in U_j. Hence $\overline{\{x\}} \cap U_j = \{x\}$. We conclude that $\overline{\{x\}} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each U_i is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in Z.

Conversely, assume that X is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence also $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in Z. Thus U_i is Jacobson as desired. □

005X **Lemma 5.18.5.** *Let X be Jacobson. The following types of subsets $T \subset X$ are Jacobson:*

(1) *Open subspaces.*
(2) *Closed subspaces.*
(3) *Locally closed subspaces.*
(4) *Unions of locally closed subspaces.*
(5) *Constructible sets.*
(6) *Any subset $T \subset X$ which locally on X is a union of locally closed subsets.*

In each of these cases closed points of T are closed in X.

Proof. Let X_0 be the set of closed points of X. For any subset $T \subset X$ we let $(*)$ denote the property:

(*) Every nonempty locally closed subset of T has a point closed in X.

Note that always $X_0 \cap T \subset T_0$. Hence property $(*)$ implies that T is Jacobson. In addition it clearly implies that every closed point of T is closed in X.

Suppose that $T = \bigcup_i T_i$ with T_i locally closed in X. Take $A \subset T$ a locally closed nonempty subset in T, then there exists a T_i such that $A \cap T_i$ is nonempty, it is locally closed in T_i and so in X. As X is Jacobson A has a point closed in X. □

07JU **Lemma 5.18.6.** *A finite Jacobson space is discrete. A Jacobson space with finitely many closed points is discrete.*

Proof. If X is finite then the set $X_0 \subset X$ of closed points is finite. Assume X_0 is is finite and X is Jacobson. Then $\overline{X_0} = X$ by the Jacobson property. Now $X_0 = \{x_1, \ldots, x_n\} = \bigcup_{i=1,\ldots,n} \{x_i\}$ is a finite union of closed sets, hence closed, so $X = \overline{X_0} = X_0$. Every point is closed, and by finiteness, every point is open. □

005Z **Lemma 5.18.7.** *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\textit{finite unions loc. closed subsets of } X\} \leftrightarrow \{\textit{finite unions loc. closed subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. We just prove that the correspondence $E \mapsto E \cap X_0$ is injective. Indeed if $E \neq E'$ then without loss of generality $E \setminus E'$ is nonempty, and it is a finite union of locally closed sets (details omitted). As X is Jacobson, we see that $(E \setminus E') \cap X_0 = E \cap X_0 \setminus E' \cap X_0$ is not empty. □

005Y **Lemma 5.18.8.** *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\text{constructible subsets of } X\} \leftrightarrow \{\text{constructible subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. From Lemma 5.18.7 above, we just have to see that if U is open in X then $U \cap X_0$ is retrocompact in X_0 if and only if U is retrocompact in X. This follows if we prove that for U open in X then $U \cap X_0$ is quasi-compact if and only if U is quasi-compact. From Lemma 5.18.5 it follows that we may replace X by U and assume that $U = X$. Finally notice that any collection of opens $\mathcal{U}$ of X cover X if and only if they cover X_0, using the Jacobson property of X in the closed $X \setminus \bigcup \mathcal{U}$ to find a point in X_0 if it were nonempty. □

5.19. Specialization

0060

0061 **Definition 5.19.1.** Let X be a topological space.

(1) If $x, x' \in X$ then we say x is a *specialization* of x', or x' is a *generalization* of x if $x \in \overline{\{x'\}}$. Notation: $x' \rightsquigarrow x$.
(2) A subset $T \subset X$ is *stable under specialization* if for all $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.
(3) A subset $T \subset X$ is *stable under generalization* if for all $x \in T$ and every generalization $x' \rightsquigarrow x$ we have $x' \in T$.

0062 **Lemma 5.19.2.** *Let X be a topological space.*

(1) *Any closed subset of X is stable under specialization.*
(2) *Any open subset of X is stable under generalization.*
(3) *A subset $T \subset X$ is stable under specialization if and only if the complement T^c is stable under generalization.*

Proof. Let F be a closed subset of X, if $y \in F$ then $\{y\} \subset F$, so $\overline{\{y\}} \subset \overline{F} = F$ as F is closed. Thus for all specialization x of y, we have $x \in F$.

Let $x, y \in X$ such that $x \in \overline{\{y\}}$ and let T be a subset of X. Saying that T is stable under specialization means that $y \in T$ implies $x \in T$ and reciprocally saying that T is stable under generalization means that $x \in T$ implies $y \in T$. Therefore (3) is proven using contraposition.

The second property follows from (1) and (3) by considering the complement. □

0EES **Lemma 5.19.3.** *Let $T \subset X$ be a subset of a topological space X. The following are equivalent*

(1) *T is stable under specialization, and*
(2) *T is a (directed) union of closed subsets of X.*

Proof. Suppose that T is stable under specialization, then for all $y \in T$ we have $\overline{\{y\}} \subset T$. Thus $T = \bigcup_{y \in T} \overline{\{y\}}$ which is an union of closed subsets of X. Reciprocally, suppose that $T = \bigcup_{i \in I} F_i$ where F_i are closed subsets of X. If $y \in T$ then there exists $i \in I$ such that $y \in F_i$. As F_i is closed, we have $\overline{\{y\}} \subset F_i \subset T$, which proves that T is stable under specialization. □

0063 **Definition 5.19.4.** Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say that *specializations lift along f* or that f is *specializing* if given $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$.
(2) We say that *generalizations lift along f* or that f is *generalizing* if given $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ there exists a generalization $x' \rightsquigarrow x$ of x in X such that $f(x') = y'$.

0064 **Lemma 5.19.5.** *Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. If specializations lift along both f and g then specializations lift along $g \circ f$. Similarly for "generalizations lift along".*

Proof. Let $z' \rightsquigarrow z$ be a specialization in Z and let $x' \in X$ such as $g \circ f(x') = z'$. Then because specializations lift along g, there exists a specialization $f(x') \rightsquigarrow y$ of $f(x')$ in Y such that $g(y) = z$. Likewise, because specializations lift along f, there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$. It provides a specialization $x' \rightsquigarrow x$ of x' in X such that $g \circ f(x) = z$. In other words, specialization lift along $g \circ f$. □

0065 **Lemma 5.19.6.** *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If specializations lift along f, and if $T \subset X$ is stable under specialization, then $f(T) \subset Y$ is stable under specialization.*
(2) *If generalizations lift along f, and if $T \subset X$ is stable under generalization, then $f(T) \subset Y$ is stable under generalization.*

Proof. Let $y' \rightsquigarrow y$ be a specialization in Y where $y' \in f(T)$ and let $x' \in T$ such that $f(x') = y'$. Because specialization lift along f, there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$. But T is stable under specialization so $x \in T$ and then $y \in f(T)$. Therefore $f(T)$ is stable under specialization.

The proof of (2) is identical, using that generalizations lift along f. □

0066 **Lemma 5.19.7.** *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If f is closed then specializations lift along f.*
(2) *If f is open, X is a Noetherian topological space, each irreducible closed subset of X has a generic point, and Y is Kolmogorov then generalizations lift along f.*

Proof. Assume f is closed. Let $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ be given. Consider the closed subset $T = \overline{\{x'\}}$ of X. Then $f(T) \subset Y$ is a closed subset, and $y' \in f(T)$. Hence also $y \in f(T)$. Hence $y = f(x)$ with $x \in T$, i.e., $x' \rightsquigarrow x$.

Assume f is open, X Noetherian, every irreducible closed subset of X has a generic point, and Y is Kolmogorov. Let $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ be given. Consider $T = f^{-1}(\{y'\}) \subset X$. Take an open neighbourhood $x \in U \subset X$ of x. Then $f(U) \subset Y$ is open and $y \in f(U)$. Hence also $y' \in f(U)$. In other words, $T \cap U \neq \emptyset$. This proves that $x \in \overline{T}$. Since X is Noetherian, T is Noetherian (Lemma 5.9.2). Hence it has a decomposition $T = T_1 \cup \ldots \cup T_n$ into irreducible components. Then correspondingly $\overline{T} = \overline{T_1} \cup \ldots \cup \overline{T_n}$. By the above $x \in \overline{T_i}$ for some i. By assumption there exists a generic point $x' \in \overline{T_i}$, and we see that $x' \rightsquigarrow x$. As $x' \in \overline{T}$ we see that $f(x') \in \overline{\{y'\}}$. Note that $f(\overline{T_i}) = f(\overline{\{x'\}}) \subset \overline{\{f(x')\}}$. If $f(x') \neq y'$, then

since Y is Kolmogorov $f(x')$ is not a generic point of the irreducible closed subset $\overline{\{y'\}}$ and the inclusion $\overline{\{f(x')\}} \subset \overline{\{y'\}}$ is strict, i.e., $y' \notin f(\overline{T_i})$. This contradicts the fact that $f(T_i) = \{y'\}$. Hence $f(x') = y'$ and we win. □

06NA **Lemma 5.19.8.** *Suppose that $s, t : R \to U$ and $\pi : U \to X$ are continuous maps of topological spaces such that*

(1) *π is open,*
(2) *U is sober,*
(3) *s, t have finite fibres,*
(4) *generalizations lift along s, t,*
(5) *$(t, s)(R) \subset U \times U$ is an equivalence relation on U and X is the quotient of U by this equivalence relation (as a set).*

Then X is Kolmogorov.

Proof. Properties (3) and (5) imply that a point x corresponds to an finite equivalence class $\{u_1, \ldots, u_n\} \subset U$ of the equivalence relation. Suppose that $x' \in X$ is a second point corresponding to the equivalence class $\{u'_1, \ldots, u'_m\} \subset U$. Suppose that $u_i \rightsquigarrow u'_j$ for some i, j. Then for any $r' \in R$ with $s(r') = u'_j$ by (4) we can find $r \rightsquigarrow r'$ with $s(r) = u_i$. Hence $t(r) \rightsquigarrow t(r')$. Since $\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))$ we conclude that every element of $\{u'_1, \ldots, u'_m\}$ is the specialization of an element of $\{u_1, \ldots, u_n\}$. Thus $\overline{\{u_1\}} \cup \ldots \cup \overline{\{u_n\}}$ is a union of equivalence classes, hence of the form $\pi^{-1}(Z)$ for some subset $Z \subset X$. By (1) we see that Z is closed in X and in fact $Z = \overline{\{x\}}$ because $\pi(\overline{\{u_i\}}) \subset \overline{\{x\}}$ for each i. In other words, $x \rightsquigarrow x'$ if and only if some lift of x in U specializes to some lift of x' in U, if and only if every lift of x' in U is a specialization of some lift of x in U.

Suppose that both $x \rightsquigarrow x'$ and $x' \rightsquigarrow x$. Say x corresponds to $\{u_1, \ldots, u_n\}$ and x' corresponds to $\{u'_1, \ldots, u'_m\}$ as above. Then, by the results of the preceding paragraph, we can find a sequence

$$\ldots \rightsquigarrow u'_{j_3} \rightsquigarrow u_{i_3} \rightsquigarrow u'_{j_2} \rightsquigarrow u_{i_2} \rightsquigarrow u'_{j_1} \rightsquigarrow u_{i_1}$$

which must repeat, hence by (2) we conclude that $\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}$, i.e., $x = x'$. Thus X is Kolmogorov. □

02JF **Lemma 5.19.9.** *Let $f : X \to Y$ be a morphism of topological spaces. Suppose that Y is a sober topological space, and f is surjective. If either specializations or generalizations lift along f, then* $\dim(X) \geq \dim(Y)$.

Proof. Assume specializations lift along f. Let $Z_0 \subset Z_1 \subset \ldots Z_e \subset Y$ be a chain of irreducible closed subsets of X. Let $\xi_e \in X$ be a point mapping to the generic point of Z_e. By assumption there exists a specialization $\xi_e \rightsquigarrow \xi_{e-1}$ in X such that ξ_{e-1} maps to the generic point of Z_{e-1}. Continuing in this manner we find a sequence of specializations

$$\xi_e \rightsquigarrow \xi_{e-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

with ξ_i mapping to the generic point of Z_i. This clearly implies the sequence of irreducible closed subsets

$$\overline{\{\xi_0\}} \subset \overline{\{\xi_1\}} \subset \ldots \overline{\{\xi_e\}}$$

is a chain of length e in X. The case when generalizations lift along f is similar. □

0542 **Lemma 5.19.10.** *Let X be a Noetherian sober topological space. Let $E \subset X$ be a subset of X.*

(1) *If E is constructible and stable under specialization, then E is closed.*
(2) *If E is constructible and stable under generalization, then E is open.*

Proof. Let E be constructible and stable under generalization. Let $Y \subset X$ be an irreducible closed subset with generic point $\xi \in Y$. If $E \cap Y$ is nonempty, then it contains ξ (by stability under generalization) and hence is dense in Y, hence it contains a nonempty open of Y, see Lemma 5.16.3. Thus E is open by Lemma 5.16.5. This proves (2). To prove (1) apply (2) to the complement of E in X. □

5.20. Dimension functions

02I8 It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 5.8.6). Thus the definition below can be improved by considering the sober topological space associated to X. Since the underlying topological space of a scheme is sober we do not bother with this improvement.

02I9 **Definition 5.20.1.** Let X be a topological space.

(1) Let $x, y \in X$, $x \neq y$. Suppose $x \rightsquigarrow y$, that is y is a specialization of x. We say y is an *immediate specialization* of x if there is no $z \in X \setminus \{x, y\}$ with $x \rightsquigarrow z$ and $z \rightsquigarrow y$.
(2) A map $\delta : X \to \mathbf{Z}$ is called a *dimension function*[5] if
   (a) whenever $x \rightsquigarrow y$ and $x \neq y$ we have $\delta(x) > \delta(y)$, and
   (b) for every immediate specialization $x \rightsquigarrow y$ in X we have $\delta(x) = \delta(y) + 1$.

It is clear that if δ is a dimension function, then so is $\delta + t$ for any $t \in \mathbf{Z}$. Here is a fun lemma.

02IA **Lemma 5.20.2.** *Let X be a topological space. If X is sober and has a dimension function, then X is catenary. Moreover, for any $x \rightsquigarrow y$ we have*

$$\delta(x) - \delta(y) = \mathit{codim}\left(\overline{\{y\}}, \overline{\{x\}}\right).$$

Proof. Suppose $Y \subset Y' \subset X$ are irreducible closed subsets. Let $\xi \in Y$, $\xi' \in Y'$ be their generic points. Then we see immediately from the definitions that $\mathrm{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty$. In fact the first inequality is an equality. Namely, suppose

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'$$

is any maximal chain of irreducible closed subsets. Let $\xi_i \in Y_i$ denote the generic point. Then we see that $\xi_i \rightsquigarrow \xi_{i+1}$ is an immediate specialization. Hence we see that $e = \delta(\xi) - \delta(\xi')$ as desired. This also proves the last statement of the lemma. □

02IB **Lemma 5.20.3.** *Let X be a topological space. Let δ, δ' be two dimension functions on X. If X is locally Noetherian and sober then $\delta - \delta'$ is locally constant on X.*

Proof. Let $x \in X$ be a point. We will show that $\delta - \delta'$ is constant in a neighbourhood of x. We may replace X by an open neighbourhood of x in X which is Noetherian. Hence we may assume X is Noetherian and sober. Let $Z_1, \ldots, Z_r$ be the irreducible components of X passing through x. (There are finitely many as X is Noetherian, see Lemma 5.9.2.) Let $\xi_i \in Z_i$ be the generic point. Note

[5] This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).

$Z_1 \cup \ldots \cup Z_r$ is a neighbourhood of x in X (not necessarily closed). We claim that $\delta - \delta'$ is constant on $Z_1 \cup \ldots \cup Z_r$. Namely, if $y \in Z_i$, then

$$\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i)$$

by Lemma 5.20.2. Similarly for δ'. Whence the result. □

02IC **Lemma 5.20.4.** *Let X be locally Noetherian, sober and catenary. Then any point has an open neighbourhood $U \subset X$ which has a dimension function.*

Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 5.11.5 and that a Noetherian topological space has finitely many irreducible components, see Lemma 5.9.2. In the proof of Lemma 5.20.3 we saw how to construct such a function. Namely, we first replace X by a Noetherian open neighbourhood of x. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of X. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace X by

$$X \setminus \left(\bigcup\nolimits_{x \notin Z_i} Z_i \cup \bigcup\nolimits_{x \notin Z_{ijk}} Z_{ijk} \right)$$

so that we may assume $x \in Z_i$ for all i and $x \in Z_{ijk}$ for all i, j, k. For $y \in X$ choose any i such that $y \in Z_i$ and set

$$\delta(y) = -\text{codim}(\overline{\{y\}}, Z_i) + \text{codim}(\overline{\{x\}}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of i. Namely, suppose that $y \in Z_{ijk}$ for some i, j, k. Then we have (using Lemma 5.11.6)

$$\begin{aligned} \delta(y) &= -\text{codim}(\overline{\{y\}}, Z_i) + \text{codim}(\overline{\{x\}}, Z_i) \\ &= -\text{codim}(\overline{\{y\}}, Z_{ijk}) - \text{codim}(Z_{ijk}, Z_i) + \text{codim}(\overline{\{x\}}, Z_{ijk}) + \text{codim}(Z_{ijk}, Z_i) \\ &= -\text{codim}(\overline{\{y\}}, Z_{ijk}) + \text{codim}(\overline{\{x\}}, Z_{ijk}) \end{aligned}$$

which is symmetric in i and j. We omit the proof that it is a dimension function. □

02ID **Remark 5.20.5.** Combining Lemmas 5.20.3 and 5.20.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, \mathbf{Z})$.

5.21. Nowhere dense sets

03HM

03HN **Definition 5.21.1.** Let X be a topological space.

(1) Given a subset $T \subset X$ the *interior* of T is the largest open subset of X contained in T.
(2) A subset $T \subset X$ is called *nowhere dense* if the closure of T has empty interior.

03HO **Lemma 5.21.2.** *Let X be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.*

Proof. It suffices to prove the lemma for two nowhere dense sets as the result in general will follow by induction. Let $A, B \subset X$ be nowhere dense subsets. We have $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Hence, if $U \subset \overline{A \cup B}$ is an open subset of X, then $U \setminus U \cap \overline{B}$ is an open subset of U and of X and contained in $\overline{A}$ and hence empty. Similarly $U \setminus U \cap \overline{A}$ is empty. Thus $U = \emptyset$ as desired. □

03J0 **Lemma 5.21.3.** *Let X be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If T is nowhere dense in U, then T is nowhere dense in X.*

Proof. Assume T is nowhere dense in U. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of T in X. Say $x \in V \subset \overline{T}$ with $V \subset X$ open in X. Note that $\overline{T} \cap U$ is the closure of T in U. Hence the interior of $\overline{T} \cap U$ being empty implies $V \cap U = \emptyset$. Thus x cannot be in the closure of U, a fortiori cannot be in the closure of T, a contradiction. □

03HP **Lemma 5.21.4.** *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in U_i for all i, then T is nowhere dense in X.*

Proof. Denote $\overline{T}_i$ the closure of $T \cap U_i$ in U_i. We have $\overline{T} \cap U_i = \overline{T}_i$. Taking the interior commutes with intersection with opens, thus

$$(\text{interior of } \overline{T}) \cap U_i = \text{interior of } (\overline{T} \cap U_i) = \text{interior in } U_i \text{ of } \overline{T}_i$$

By assumption the last of these is empty. Hence T is nowhere dense in X. □

03HQ **Lemma 5.21.5.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If f is a homeomorphism of X onto a closed subset of Y and T is nowhere dense in X, then also $f(T)$ is nowhere dense in Y.*

Proof. Because $f(X)$ is closed in Y and f is a homeomorphism of X onto $f(X)$, we see that the closure of $f(T)$ in Y equals $f(\overline{T})$. Hence if $V \subset Y$ is open and contained in the closure of $f(T)$, then $U = f^{-1}(V)$ is open and contained in $\overline{T}$. Hence $U = \emptyset$, which in turn shows that $V = \emptyset$ as desired. □

03HR **Lemma 5.21.6.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If f is open and T is a closed nowhere dense subset of Y, then also $f^{-1}(T)$ is a closed nowhere dense subset of X. If f is surjective and open, then T is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.*

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of T, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of T.) □

5.22. Profinite spaces

08ZW Here is the definition.

08ZX **Definition 5.22.1.** A topological space is *profinite* if it is homeomorphic to a limit of a diagram of finite discrete spaces.

This is not the most convenient characterization of a profinite space.

08ZY **Lemma 5.22.2.** *Let X be a topological space. The following are equivalent*

(1) *X is a profinite space, and*
(2) *X is Hausdorff, quasi-compact, and totally disconnected.*

If this is true, then X is a cofiltered limit of finite discrete spaces.

Proof. Assume (1). Choose a diagram $i \mapsto X_i$ of finite discrete spaces such that $X = \lim X_i$. As each X_i is Hausdorff and quasi-compact we find that X is quasi-compact by Lemma 5.14.5. If $x, x' \in X$ are distinct points, then x and x' map to distinct points in some X_i. Hence x and x' have disjoint open neighbourhoods, i.e., X is Hausdorff. In exactly the same way we see that X is totally disconnected.

Assume (2). Let $\mathcal{I}$ be the set of finite disjoint union decompositions $X = \coprod_{i \in I} U_i$ with U_i nonempty open (and closed) for all $i \in I$. For each $I \in \mathcal{I}$ there is a continuous map $X \to I$ sending a point of U_i to i. We define a partial ordering: $I \leq I'$ for $I, I' \in \mathcal{I}$ if and only if the covering corresponding to I' refines the covering corresponding to I. In this case we obtain a canonical map $I' \to I$. In other words we obtain an inverse system of finite discrete spaces over $\mathcal{I}$. The maps $X \to I$ fit together and we obtain a continuous map

$$X \longrightarrow \lim_{I \in \mathcal{I}} I$$

We claim this map is a homeomorphism, which finishes the proof. (The final assertion follows too as the partially ordered set $\mathcal{I}$ is directed: given two disjoint union decompositions of X we can find a third refining both.) Namely, the map is injective as X is totally disconnected and hence $\{x\}$ is the intersection of all open and closed subsets of X containing x (Lemma 5.12.11) and the map is surjective by Lemma 5.12.6. By Lemma 5.17.8 the map is a homeomorphism. □

0ET8 **Lemma 5.22.3.** *A limit of profinite spaces is profinite.*

Proof. Let $i \mapsto X_i$ be a diagram of profinite spaces over the index category $\mathcal{I}$. Let us use the characterization of profinite spaces in Lemma 5.22.2. In particular each X_i is Hausdorff, quasi-compact, and totally disconnected. By Lemma 5.14.1 the limit $X = \lim X_i$ exists. By Lemma 5.14.5 the limit X is quasi-compact. Let $x, x' \in X$ be distinct points. Then there exists an i such that x and x' have distinct images x_i and x_i' in X_i under the projection $X \to X_i$. Then x_i and x_i' have disjoint open neighbourhoods in X_i. Taking the inverse images of these opens we conclude that X is Hausdorff. Similarly, x_i and x_i' are in distinct connected components of X_i whence necessarily x and x' must be in distinct connected components of X. Hence X is totally disconnected. This finishes the proof. □

08ZZ **Lemma 5.22.4.** *Let X be a profinite space. Every open covering of X has a refinement by a finite covering $X = \coprod U_i$ with U_i open and closed.*

Proof. Write $X = \lim X_i$ as a limit of an inverse system of finite discrete spaces over a directed set I (Lemma 5.22.2). Denote $f_i : X \to X_i$ the projection. For every point $x = (x_i) \in X$ a fundamental system of open neighbourhoods is the collection $f_i^{-1}(\{x_i\})$. Thus, as X is quasi-compact, we may assume we have an open covering

$$X = f_{i_1}^{-1}(\{x_{i_1}\}) \cup \ldots \cup f_{i_n}^{-1}(\{x_{i_n}\})$$

Choose $i \in I$ with $i \geq i_j$ for $j = 1, \ldots, n$ (this is possible as I is a directed set). Then we see that the covering

$$X = \coprod\nolimits_{t \in X_i} f_i^{-1}(\{t\})$$

refines the given covering and is of the desired form. □

0900 **Lemma 5.22.5.** *Let X be a topological space. If X is quasi-compact and every connected component of X is the intersection of the open and closed subsets containing it, then $\pi_0(X)$ is a profinite space.*

Proof. We will use Lemma 5.22.2 to prove this. Since $\pi_0(X)$ is the image of a quasi-compact space it is quasi-compact (Lemma 5.12.7). It is totally disconnected by construction (Lemma 5.7.9). Let $C, D \subset X$ be distinct connected components of X. Write $C = \bigcap U_\alpha$ as the intersection of the open and closed subsets of X containing C. Any finite intersection of U_α's is another. Since $\bigcap U_\alpha \cap D = \emptyset$ we conclude that $U_\alpha \cap D = \emptyset$ for some α (use Lemmas 5.7.3, 5.12.3 and 5.12.6) Since U_α is open and closed, it is the union of the connected components it contains, i.e., U_α is the inverse image of some open and closed subset $V_\alpha \subset \pi_0(X)$. This proves that the points corresponding to C and D are contained in disjoint open subsets, i.e., $\pi_0(X)$ is Hausdorff. □

5.23. Spectral spaces

08YF The material in this section is taken from [**Hoc69**] and [**Hoc67**]. In his thesis Hochster proves (among other things) that the spectral spaces are exactly the topological spaces that occur as the spectrum of a ring.

08YG **Definition 5.23.1.** A topological space X is called *spectral* if it is sober, quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology. A continuous map $f : X \to Y$ of spectral spaces is called *spectral* if the inverse image of a quasi-compact open is quasi-compact.

In other words a continuous map of spectral spaces is spectral if and only if it is quasi-compact (Definition 5.12.1).

Let X be a spectral space. The *constructible topology* on X is the topology which has as a subbase of opens the sets U and U^c where U is a quasi-compact open of X. Note that since X is spectral an open $U \subset X$ is retrocompact if and only if U is quasi-compact. Hence the constructible topology can also be characterized as the coarsest topology such that every constructible subset of X is both open and closed (see Section 5.15 for definitions and properties of constructible sets). It follows that a subset of X is open, resp. closed in the constructible topology if and only if it is a union, resp. intersection of constructible subsets. Since the collection of quasi-compact opens is a basis for the topology on X we see that the constructible topology is stronger than the given topology on X.

0901 **Lemma 5.23.2.** *Let X be a spectral space. The constructible topology is Hausdorff, totally disconnected, and quasi-compact.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is sober, there is an open subset U containing exactly one of the two points x, y. Say $x \in U$. We may replace U by a quasi-compact open neighbourhood of x contained in U. Then U and U^c are open and closed in the constructible topology. Hence X is Hausdorff in the constructible topology because $x \in U$ and $y \in U^c$ are disjoint opens in the constructible topology. The existence of U also implies x and y are in distinct connected components in the constructible topology, whence X is totally disconnected in the constructible topology.

Let $\mathcal{B}$ be the collection of subsets $B \subset X$ with B either quasi-compact open or closed with quasi-compact complement. If $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$. It suffices to show every covering $X = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$ has a finite refinement, see Lemma 5.12.15. Taking complements we see that we have to show that any family $\{B_i\}_{i \in I}$ of elements of $\mathcal{B}$ such that $B_{i_1} \cap \ldots \cap B_{i_n} \neq \emptyset$ for all n and all $i_1, \ldots, i_n \in I$ has a common point of intersection. We may and do assume $B_i \neq B_{i'}$ for $i \neq i'$.

To get a contradiction assume $\{B_i\}_{i \in I}$ is a family of elements of $\mathcal{B}$ having the finite intersection property but empty intersection. An application of Zorn's lemma shows that we may assume our family is maximal (details omitted). Let $I' \subset I$ be those indices such that B_i is closed and set $Z = \bigcap_{i \in I'} B_i$. This is a closed subset of X which is nonempty by Lemma 5.12.6. If Z is reducible, then we can write $Z = Z' \cup Z''$ as a union of two closed subsets, neither equal to Z. This means in particular that we can find a quasi-compact open $U' \subset X$ meeting Z' but not Z''. Similarly, we can find a quasi-compact open $U'' \subset X$ meeting Z'' but not Z'. Set $B' = X \setminus U'$ and $B'' = X \setminus U''$. Note that $Z'' \subset B'$ and $Z' \subset B''$. If there exist a finite number of indices $i_1, \ldots, i_n \in I$ such that $B' \cap B_{i_1} \cap \ldots \cap B_{i_n} = \emptyset$ as well as a finite number of indices $j_1, \ldots, j_m \in I$ such that $B'' \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$ then we find that $Z \cap B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$. However, the set $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m}$ is quasi-compact hence we would find a finite number of indices $i'_1, \ldots, i'_l \in I'$ with $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} \cap B_{i'_1} \cap \ldots \cap B_{i'_l} = \emptyset$, a contradiction. Thus we see that we may add either B' or B'' to the given family contradicting maximality. We conclude that Z is irreducible. However, this leads to a contradiction as well, as now every nonempty (by the same argument as above) open $Z \cap B_i$ for $i \in I \setminus I'$ contains the unique generic point of Z. This contradiction proves the lemma. □

0A2S **Lemma 5.23.3.** *Let $f : X \to Y$ be a spectral map of spectral spaces. Then*

(1) *f is continuous in the constructible topology,*
(2) *the fibres of f are quasi-compact, and*
(3) *the image is closed in the constructible topology.*

Proof. Let X' and Y' denote X and Y endowed with the constructible topology which are quasi-compact Hausdorff spaces by Lemma 5.23.2. Part (1) says $X' \to Y'$ is continuous and follows immediately from the definitions. Part (3) follows as $f(X')$ is a quasi-compact subset of the Hausdorff space Y', see Lemma 5.12.4. We have a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of continuous maps of topological spaces. Since Y' is Hausdorff we see that the fibres X'_y are closed in X'. As X' is quasi-compact we see that X'_y is quasi-compact (Lemma 5.12.3). As $X'_y \to X_y$ is a surjective continuous map we conclude that X_y is quasi-compact (Lemma 5.12.7). □

0G1J **Lemma 5.23.4.** *Let X and Y be spectral spaces. Let $f : X \to Y$ be a continuous map. Then f is spectral if and only if f is continuous in the constructible topology.*

Proof. The only if part of this is Lemma 5.23.3. Assume f is continuous in the constructible topology. Let $V \subset Y$ be quasi-compact open. Then V is open and

closed in the constructible topology. Hence $f^{-1}(V)$ is open and closed in the constructible topology. Hence $f^{-1}(V)$ is quasi-compact in the constructible topology as X is quasi-compact in the constructible topology by Lemma 5.23.2. Since the identity $f^{-1}(V) \to f^{-1}(V)$ is surjective and continuous from the constructible topology to the usual topology, we conclude that $f^{-1}(V)$ is quasi-compact in the topology of X by Lemma 5.12.7. This finishes the proof. □

0902 **Lemma 5.23.5.** *Let X be a spectral space. Let $E \subset X$ be closed in the constructible topology (for example constructible or closed). Then E with the induced topology is a spectral space.*

Proof. Let $Z \subset E$ be a closed irreducible subset. Let η be the generic point of the closure $\overline{Z}$ of Z in X. To prove that E is sober, we show that $\eta \in E$. If not, then since E is closed in the constructible topology, there exists a constructible subset $F \subset X$ such that $\eta \in F$ and $F \cap E = \emptyset$. By Lemma 5.15.15 this implies $F \cap \overline{Z}$ contains a nonempty open subset of $\overline{Z}$. But this is impossible as $\overline{Z}$ is the closure of Z and $Z \cap F = \emptyset$.

Since E is closed in the constructible topology, it is quasi-compact in the constructible topology (Lemmas 5.12.3 and 5.23.2). Hence a fortiori it is quasi-compact in the topology coming from X. If $U \subset X$ is a quasi-compact open, then $E \cap U$ is closed in the constructible topology, hence quasi-compact (as seen above). It follows that the quasi-compact open subsets of E are the intersections $E \cap U$ with U quasi-compact open in X. These form a basis for the topology. Finally, given two $U, U' \subset X$ quasi-compact opens, the intersection $(E \cap U) \cap (E \cap U') = E \cap (U \cap U')$ and $U \cap U'$ is quasi-compact as X is spectral. This finishes the proof. □

0903 **Lemma 5.23.6.** *Let X be a spectral space. Let $E \subset X$ be a subset closed in the constructible topology (for example constructible).*

(1) *If $x \in \overline{E}$, then x is the specialization of a point of E.*
(2) *If E is stable under specialization, then E is closed.*
(3) *If $E' \subset X$ is open in the constructible topology (for example constructible) and stable under generalization, then E' is open.*

Proof. Proof of (1). Let $x \in \overline{E}$. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. The intersection $U_i \cap E$ is nonempty for all i. Since the subsets $U_i \cap E$ are closed in the constructible topology we see that $\bigcap (U_i \cap E)$ is nonempty by Lemma 5.23.2 and Lemma 5.12.6. Since $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Thus x is a specialization of a point of E.

Part (2) is immediate from (1).

Proof of (3). Assume E' is as in (3). The complement of E' is closed in the constructible topology (Lemma 5.15.2) and closed under specialization (Lemma 5.19.2). Hence the complement is closed by (2), i.e., E' is open. □

0904 **Lemma 5.23.7.** *Let X be a spectral space. Let $x, y \in X$. Then either there exists a third point specializing to both x and y, or there exist disjoint open neighbourhoods containing x and y.*

Proof. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. Let $\{V_j\}$ be the set of quasi-compact open

neighbourhoods of y. A finite intersection of the V_j is another one. If $U_i \cap V_j$ is empty for some i, j we are done. If not, then the intersection $U_i \cap V_j$ is nonempty for all i and j. The sets $U_i \cap V_j$ are closed in the constructible topology on X. By Lemma 5.23.2 we see that $\bigcap(U_i \cap V_j)$ is nonempty (Lemma 5.12.6). Since X is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Similarly, $\bigcap V_j$ is the set of generalizations of y. Thus any element of $\bigcap(U_i \cap V_j)$ specializes to both x and y. □

0905 **Lemma 5.23.8.** *Let X be a spectral space. The following are equivalent:*

(1) *X is profinite,*
(2) *X is Hausdorff,*
(3) *X is totally disconnected,*
(4) *every quasi-compact open is closed,*
(5) *there are no nontrivial specializations between points,*
(6) *every point of X is closed,*
(7) *every point of X is the generic point of an irreducible component of X,*
(8) *the constructible topology equals the given topology on X, and*
(9) *add more here.*

Proof. Lemma 5.22.2 shows the implication (1) $\Rightarrow$ (3). Irreducible components are closed, so if X is totally disconnected, then every point is closed. So (3) implies (6). The equivalence of (6) and (5) is immediate, and (6) $\Leftrightarrow$ (7) holds because X is sober. Assume (5). Then all constructible subsets of X are closed (Lemma 5.23.6), in particular all quasi-compact opens are closed. So (5) implies (4). Since X is sober, for any two points there is a quasi-compact open containing exactly one of them, hence (4) implies (2). Parts (4) and (8) are equivalent by the definition of the constructible topology. It remains to prove (2) implies (1). Suppose X is Hausdorff. Every quasi-compact open is also closed (Lemma 5.12.4). This implies X is totally disconnected. Hence it is profinite, by Lemma 5.22.2. □

0906 **Lemma 5.23.9.** *If X is a spectral space, then $\pi_0(X)$ is a profinite space.*

Proof. Combine Lemmas 5.12.10 and 5.22.5. □

0907 **Lemma 5.23.10.** *The product of two spectral spaces is spectral.*

Proof. Let X, Y be spectral spaces. Denote $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let $Z \subset X \times Y$ be a closed irreducible subset. Then $p(Z) \subset X$ is irreducible and $q(Z) \subset Y$ is irreducible. Let $x \in X$ be the generic point of the closure of $p(X)$ and let $y \in Y$ be the generic point of the closure of $q(Y)$. If $(x, y) \notin Z$, then there exist opens $x \in U \subset X$, $y \in V \subset Y$ such that $Z \cap U \times V = \emptyset$. Hence Z is contained in $(X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since Z is irreducible, we see that either $Z \subset (X \setminus U) \times Y$ or $Z \subset X \times (Y \setminus V)$. In the first case $p(Z) \subset (X \setminus U)$ and in the second case $q(Z) \subset (Y \setminus V)$. Both cases are absurd as x is in the closure of $p(Z)$ and y is in the closure of $q(Z)$. Thus we conclude that $(x, y) \in Z$, which means that (x, y) is the generic point for Z.

A basis of the topology of $X \times Y$ are the opens of the form $U \times V$ with $U \subset X$ and $V \subset Y$ quasi-compact open (here we use that X and Y are spectral). Then $U \times V$ is quasi-compact as the product of quasi-compact spaces is quasi-compact. Moreover, any quasi-compact open of $X \times Y$ is a finite union of such quasi-compact

rectangles $U \times V$. It follows that the intersection of two such is again quasi-compact (since X and Y are spectral). This concludes the proof. □

09XU **Lemma 5.23.11.** *Let $f : X \to Y$ be a continuous map of topological spaces. If*

(1) *X and Y are spectral,*
(2) *f is spectral and bijective, and*
(3) *generalizations (resp. specializations) lift along f.*

Then f is a homeomorphism.

Proof. Since f is spectral it defines a continuous map between X and Y in the constructible topology. By Lemmas 5.23.2 and 5.17.8 it follows that $X \to Y$ is a homeomorphism in the constructible topology. Let $U \subset X$ be quasi-compact open. Then $f(U)$ is constructible in Y. Let $y \in Y$ specialize to a point in $f(U)$. By the last assumption we see that $f^{-1}(y)$ specializes to a point of U. Hence $f^{-1}(y) \in U$. Thus $y \in f(U)$. It follows that $f(U)$ is open, see Lemma 5.23.6. Whence f is a homeomorphism. To prove the lemma in case specializations lift along f one shows instead that $f(Z)$ is closed if $X \setminus Z$ is a quasi-compact open of X. □

09XV **Lemma 5.23.12.** *The inverse limit of a directed inverse system of finite sober topological spaces is a spectral topological space.*

Proof. Let I be a directed set. Let X_i be an inverse system of finite sober spaces over I. Let $X = \lim X_i$ which exists by Lemma 5.14.1. As a set $X = \lim X_i$. Denote $p_i : X \to X_i$ the projection. Because I is directed we may apply Lemma 5.14.2. A basis for the topology is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since an open covering of $p_i^{-1}(U_i)$ is in particular an open covering in the profinite topology, we conclude that $p_i^{-1}(U_i)$ is quasi-compact. Given $U_i \subset X_i$ and $U_j \subset X_j$, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some $k \geq i, j$ and open $U_k \subset X_k$. Finally, if $Z \subset X$ is irreducible and closed, then $p_i(Z) \subset X_i$ is irreducible and therefore has a unique generic point ξ_i (because X_i is a finite sober topological space). Then $\xi = \lim \xi_i$ is a generic point of Z (it is a point of Z as Z is closed). This finishes the proof. □

09XW **Lemma 5.23.13.** *Let W be the topological space with two points, one closed, the other not. A topological space is spectral if and only if it is homeomorphic to a subspace of a product of copies of W which is closed in the constructible topology.*

Proof. Write $W = \{0, 1\}$ where 0 is a specialization of 1 but not vice versa. Let I be a set. The space $\prod_{i \in I} W$ is spectral by Lemma 5.23.12. Thus we see that a subspace of $\prod_{i \in I} W$ closed in the constructible topology is a spectral space by Lemma 5.23.5.

For the converse, let X be a spectral space. Let $U \subset X$ be a quasi-compact open. Consider the continuous map

$$f_U : X \longrightarrow W$$

which maps every point in U to 1 and every point in $X \setminus U$ to 0. Taking the product of these maps we obtain a continuous map

$$f = \prod f_U : X \longrightarrow \prod\nolimits_U W$$

By construction the map $f : X \to Y$ is spectral. By Lemma 5.23.3 the image of f is closed in the constructible topology. If $x', x \in X$ are distinct, then since X is

sober either x' is not a specialization of x or conversely. In either case (as the quasi-compact opens form a basis for the topology of X) there exists a quasi-compact open $U \subset X$ such that $f_U(x') \neq f_U(x)$. Thus f is injective. Let $Y = f(X)$ endowed with the induced topology. Let $y' \rightsquigarrow y$ be a specialization in Y and say $f(x') = y'$ and $f(x) = y$. Arguing as above we see that $x' \rightsquigarrow x$, since otherwise there is a U such that $x \in U$ and $x' \notin U$, which would imply $f_U(x') \not\rightsquigarrow f_U(x)$. We conclude that $f : X \to Y$ is a homeomorphism by Lemma 5.23.11. □

09XX **Lemma 5.23.14.** *A topological space is spectral if and only if it is a directed inverse limit of finite sober topological spaces.*

Proof. One direction is given by Lemma 5.23.12. For the converse, assume X is spectral. Then we may assume $X \subset \prod_{i \in I} W$ is a subset closed in the constructible topology where $W = \{0, 1\}$ as in Lemma 5.23.13. We can write

$$\prod\nolimits_{i \in I} W = \lim\nolimits_{J \subset I \text{ finite}} \prod\nolimits_{j \in J} W$$

as a cofiltered limit. For each J, let $X_J \subset \prod_{j \in J} W$ be the image of X. Then we see that $X = \lim X_J$ as sets because X is closed in the product with the constructible topology (detail omitted). A formal argument (omitted) on limits shows that $X = \lim X_J$ as topological spaces. □

0A2T **Lemma 5.23.15.** *Let X be a topological space and let $c : X \to X'$ be the universal map from X to a sober topological space, see Lemma 5.8.16.*

(1) *If X is quasi-compact, so is X'.*
(2) *If X is quasi-compact, has a basis of quasi-compact opens, and the intersection of two quasi-compact opens is quasi-compact, then X' is spectral.*
(3) *If X is Noetherian, then X' is a Noetherian spectral space.*

Proof. Let $U \subset X$ be open and let $U' \subset X'$ be the corresponding open, i.e., the open such that $c^{-1}(U') = U$. Then U is quasi-compact if and only if U' is quasi-compact, as pulling back by c is a bijection between the opens of X and X' which commutes with unions. This in particular proves (1).

Proof of (2). It follows from the above that X' has a basis of quasi-compact opens. Since c^{-1} also commutes with intersections of pairs of opens, we see that the intersection of two quasi-compact opens X' is quasi-compact. Finally, X' is quasi-compact by (1) and sober by construction. Hence X' is spectral.

Proof of (3). It is immediate that X' is Noetherian as this is defined in terms of the acc for open subsets which holds for X. We have already seen in (2) that X' is spectral. □

5.24. Limits of spectral spaces

0A2U Lemma 5.23.14 tells us that every spectral space is a cofiltered limit of finite sober spaces. Every finite sober space is a spectral space and every continuous map of finite sober spaces is a spectral map of spectral spaces. In this section we prove some lemmas concerning limits of systems of spectral topological spaces along spectral maps.

0A2V **Lemma 5.24.1.** *Let $\mathcal{I}$ be a category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.*

(1) *Given subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is quasi-compact.*
(2) *The space $X = \lim X_i$ is quasi-compact.*

Proof. The limit $Z = \lim Z_i$ exists by Lemma 5.14.1. Denote X_i' the space X_i endowed with the constructible topology and Z_i' the corresponding subspace of X_i'. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X_j' \to X_i'$. Thus $f_a|_{Z_j} : Z_j' \to Z_i'$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 5.23.2 and 5.12.3). Thus $Z' = \lim Z_i$ is quasi-compact by Lemma 5.14.5. The maps $Z_i' \to Z_i$ are continuous, hence $Z' \to Z$ is continuous and a bijection on underlying sets. Hence Z is quasi-compact as the image of the surjective continuous map $Z' \to Z$ (Lemma 5.12.7). □

0A2W **Lemma 5.24.2.** *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.*

(1) *Given nonempty subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is nonempty.*
(2) *If each X_i is nonempty, then $X = \lim X_i$ is nonempty.*

Proof. Denote X_i' the space X_i endowed with the constructible topology and Z_i' the corresponding subspace of X_i'. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X_j' \to X_i'$. Thus $f_a|_{Z_j} : Z_j' \to Z_i'$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 5.23.2 and 5.12.3). By Lemma 5.14.6 the space $\lim Z_i'$ is nonempty. Since $\lim Z_i' = \lim Z_i$ as sets we conclude. □

0A2X **Lemma 5.24.3.** *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $E, F \subset X_i$ be subsets with E closed in the constructible topology and F open in the constructible topology. Then $p_i^{-1}(E) \subset p_i^{-1}(F)$ if and only if there is a morphism $a : j \to i$ in $\mathcal{I}$ such that $f_a^{-1}(E) \subset f_a^{-1}(F)$.*

Proof. Observe that

$$p_i^{-1}(E) \setminus p_i^{-1}(F) = \lim_{a:j\to i} f_a^{-1}(E) \setminus f_a^{-1}(F)$$

Since f_a is a spectral map, it is continuous in the constructible topology hence the set $f_a^{-1}(E) \setminus f_a^{-1}(F)$ is closed in the constructible topology. Hence Lemma 5.24.2 applies to show that the LHS is nonempty if and only if each of the spaces of the RHS is nonempty. □

0A2Y **Lemma 5.24.4.** *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $E \subset X$ be a constructible subset. Then there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a constructible subset $E_i \subset X_i$ such that $p_i^{-1}(E_i) = E$. If E is open, resp. closed, we may choose E_i open, resp. closed.*

Proof. Assume E is a quasi-compact open of X. By Lemma 5.14.2 we can write $E = p_i^{-1}(U_i)$ for some i and some open $U_i \subset X_i$. Write $U_i = \bigcup U_{i,\alpha}$ as a union of quasi-compact opens. As E is quasi-compact we can find $\alpha_1, \ldots, \alpha_n$ such that $E = p_i^{-1}(U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n})$. Hence $E_i = U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n}$ works.

Assume E is a constructible closed subset. Then E^c is quasi-compact open. So $E^c = p_i^{-1}(F_i)$ for some i and quasi-compact open $F_i \subset X_i$ by the result of the previous paragraph. Then $E = p_i^{-1}(F_i^c)$ as desired.

If E is general we can write $E = \bigcup_{l=1,\ldots,n} U_l \cap Z_l$ with U_l constructible open and Z_l constructible closed. By the result of the previous paragraphs we may write $U_l = p_{i_l}^{-1}(U_{l,i_l})$ and $Z_l = p_{j_l}^{-1}(Z_{l,j_l})$ with $U_{l,i_l} \subset X_{i_l}$ constructible open and $Z_{l,j_l} \subset X_{j_l}$ constructible closed. As $\mathcal{I}$ is cofiltered we may choose an object k of $\mathcal{I}$ and morphism $a_l : k \to i_l$ and $b_l : k \to j_l$. Then taking $E_k = \bigcup_{l=1,\ldots,n} f_{a_l}^{-1}(U_{l,i_l}) \cap f_{b_l}^{-1}(Z_{l,j_l})$ we obtain a constructible subset of X_k whose inverse image in X is E. □

0A2Z **Lemma 5.24.5.** *Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Then the inverse limit $X = \lim X_i$ is a spectral topological space and the projection maps $p_i : X \to X_i$ are spectral.*

Proof. The limit $X = \lim X_i$ exists (Lemma 5.14.1) and is quasi-compact by Lemma 5.24.1.

Denote $p_i : X \to X_i$ the projection. Because $\mathcal{I}$ is cofiltered we can apply Lemma 5.14.2. Hence a basis for the topology on X is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since a basis for the topology of X_i is given by the quasi-compact open, we conclude that a basis for the topology on X is given by $p_i^{-1}(U_i)$ with $U_i \subset X_i$ quasi-compact open. A formal argument shows that

$$p_i^{-1}(U_i) = \lim_{a:j\to i} f_a^{-1}(U_i)$$

as topological spaces. Since each f_a is spectral the sets $f_a^{-1}(U_i)$ are closed in the constructible topology of X_j and hence $p_i^{-1}(U_i)$ is quasi-compact by Lemma 5.24.1. Thus X has a basis for the topology consisting of quasi-compact opens.

Any quasi-compact open U of X is of the form $U = p_i^{-1}(U_i)$ for some i and some quasi-compact open $U_i \subset X_i$ (see Lemma 5.24.4). Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact open, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some k and quasi-compact open $U_k \subset X_k$. Namely, choose k and morphisms $k \to i$ and $k \to j$ and let U_k be the intersection of the pullbacks of U_i and U_j to X_k. Thus we see that the intersection of two quasi-compact opens of X is quasi-compact open.

Finally, let $Z \subset X$ be irreducible and closed. Then $p_i(Z) \subset X_i$ is irreducible and therefore $Z_i = \overline{p_i(Z)}$ has a unique generic point ξ_i (because X_i is a spectral space). Then $f_a(\xi_j) = \xi_i$ for $a : j \to i$ in $\mathcal{I}$ because $\overline{f_a(Z_j)} = Z_i$. Hence $\xi = \lim \xi_i$ is a point of X. Claim: $\xi \in Z$. Namely, if not we can find a quasi-compact open containing ξ disjoint from Z. This would be of the form $p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. Then $\xi_i \in U_i$ but $p_i(Z) \cap U_i = \emptyset$ which contradicts $\xi_i \in \overline{p_i(Z)}$. So $\xi \in Z$ and hence $\overline{\{\xi\}} \subset Z$. Conversely, every $z \in Z$ is in the closure of ξ. Namely, given a quasi-compact open neighbourhood U of z we write $U = p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. We see that $p_i(z) \in U_i$ hence $\xi_i \in U_i$ hence $\xi \in U$. Thus ξ is a generic point of Z. We omit the proof that ξ is the unique generic point of Z (hint: show that a second generic point has to be equal to ξ by showing that it has to map to ξ_i in X_i since by spectrality of X_i the irreducible Z_i has a unique generic point). This finishes the proof. □

0A30 **Lemma 5.24.6.** *Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.*

(1) *Given any quasi-compact open $U \subset X$ there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a quasi-compact open $U_i \subset X_i$ such that $p_i^{-1}(U_i) = U$.*
(2) *Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact opens such that $p_i^{-1}(U_i) \subset p_j^{-1}(U_j)$ there exist $k \in \mathrm{Ob}(\mathcal{I})$ and morphisms $a : k \to i$ and $b : k \to j$ such that $f_a^{-1}(U_i) \subset f_b^{-1}(U_j)$.*
(3) *If $U_i, U_{1,i}, \ldots, U_{n,i} \subset X_i$ are quasi-compact opens and $p_i^{-1}(U_i) = p_i^{-1}(U_{1,i}) \cup \ldots \cup p_i^{-1}(U_{n,i})$ then $f_a^{-1}(U_i) = f_a^{-1}(U_{1,i}) \cup \ldots \cup f_a^{-1}(U_{n,i})$ for some morphism $a : j \to i$ in $\mathcal{I}$.*
(4) *Same statement as in (3) but for intersections.*

Proof. Part (1) is a special case of Lemma 5.24.4. Part (2) is a special case of Lemma 5.24.3 as quasi-compact opens are both open and closed in the constructible topology. Parts (3) and (4) follow formally from (1) and (2) and the fact that taking inverse images of subsets commutes with taking unions and intersections. □

0A31 **Lemma 5.24.7.** *Let W be a subset of a spectral space X. The following are equivalent:*

(1) *W is an intersection of constructible sets and closed under generalizations,*
(2) *W is quasi-compact and closed under generalizations,*
(3) *there exists a quasi-compact subset $E \subset X$ such that W is the set of points specializing to E,*
(4) *W is an intersection of quasi-compact open subsets,*
0ANZ (5) *there exists a nonempty set I and quasi-compact opens $U_i \subset X$, $i \in I$ such that $W = \bigcap U_i$ and for all $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$.*

In this case we have (a) W is a spectral space, (b) $W = \lim U_i$ as topological spaces, and (c) for any open U containing W there exists an i with $U_i \subset U$.

Proof. Let $W \subset X$ satisfy (1). Then W is closed in the constructible topology, hence quasi-compact in the constructible topology (by Lemmas 5.23.2 and 5.12.3), hence quasi-compact in the topology of X (because opens in X are open in the constructible topology). Thus (2) holds.

It is clear that (2) implies (3) by taking $E = W$.

Let X be a spectral space and let $E \subset W$ be as in (3). Since every point of W specializes to a point of E we see that an open of W which contains E is equal to W. Hence since E is quasi-compact, so is W. If $x \in X$, $x \notin W$, then $Z = \overline{\{x\}}$ is disjoint from W. Since W is quasi-compact we can find a quasi-compact open U with $W \subset U$ and $U \cap Z = \emptyset$. We conclude that (4) holds.

If $W = \bigcap_{j \in J} U_j$ then setting I equal to the set of finite subsets of J and $U_i = U_{j_1} \cap \ldots \cap U_{j_r}$ for $i = \{j_1, \ldots, j_r\}$ shows that (4) implies (5). It is immediate that (5) implies (1).

Let I and U_i be as in (5). Since $W = \bigcap U_i$ we have $W = \lim U_i$ by the universal property of limits. Then W is a spectral space by Lemma 5.24.5. Let $U \subset X$ be an open neighbourhood of W. Then $E_i = U_i \cap (X \setminus U)$ is a family of constructible subsets of the spectral space $Z = X \setminus U$ with empty intersection. Using that the

spectral topology on Z is quasi-compact (Lemma 5.23.2) we conclude from Lemma 5.12.6 that $E_i = \emptyset$ for some i. □

0AP0 **Lemma 5.24.8.** *Let X be a spectral space. Let $E \subset X$ be a constructible subset. Let $W \subset X$ be the set of points of X which specialize to a point of E. Then $W \setminus E$ is a spectral space. If $W = \bigcap U_i$ with U_i as in Lemma 5.24.7 (5) then $W \setminus E = \lim(U_i \setminus E)$.*

Proof. Since E is constructible, it is quasi-compact and hence Lemma 5.24.7 applies to W. If E is constructible, then E is constructible in U_i for all $i \in I$. Hence $U_i \setminus E$ is spectral by Lemma 5.23.5. Since $W \setminus E = \bigcap(U_i \setminus E)$ we have $W \setminus E = \lim U_i \setminus E$ by the universal property of limits. Then $W \setminus E$ is a spectral space by Lemma 5.24.5. □

5.25. Stone-Čech compactification

0908 The Stone-Čech compactification of a topological space X is a map $X \to \beta(X)$ from X to a Hausdorff quasi-compact space $\beta(X)$ which is universal for such maps. We prove this exists by a standard argument using the following simple lemma.

0909 **Lemma 5.25.1.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that $f(X)$ is dense in Y and that Y is Hausdorff. Then the cardinality of Y is at most the cardinality of $P(P(X))$ where P is the power set operation.*

Proof. Let $S = f(X) \subset Y$. Let $\mathcal{D}$ be the set of all closed domains of Y, i.e., subsets $D \subset Y$ which equal the closure of its interior. Note that the closure of an open subset of Y is a closed domain. For $y \in Y$ consider the set

$$I_y = \{T \subset S \mid \text{ there exists } D \in \mathcal{D} \text{ with } T = S \cap D \text{ and } y \in D\}.$$

Since S is dense in Y for every closed domain D we see that $S \cap D$ is dense in D. Hence, if $D \cap S = D' \cap S$ for $D, D' \in \mathcal{D}$, then $D = D'$. Thus $I_y = I_{y'}$ implies that $y = y'$ because the Hausdorff condition assures us that we can find a closed domain containing y but not y'. The result follows. □

Let X be a topological space. By Lemma 5.25.1, there is a set I of isomorphism classes of continuous maps $f : X \to Y$ which have dense image and where Y is Hausdorff and quasi-compact. For $i \in I$ choose a representative $f_i : X \to Y_i$. Consider the map

$$\prod f_i : X \longrightarrow \prod\nolimits_{i \in I} Y_i$$

and denote $\beta(X)$ the closure of the image. Since each Y_i is Hausdorff, so is $\beta(X)$. Since each Y_i is quasi-compact, so is $\beta(X)$ (use Theorem 5.14.4 and Lemma 5.12.3).

Let us show the canonical map $X \to \beta(X)$ satisfies the universal property with respect to maps to Hausdorff, quasi-compact spaces. Namely, let $f : X \to Y$ be such a morphism. Let $Z \subset Y$ be the closure of $f(X)$. Then $X \to Z$ is isomorphic to one of the maps $f_i : X \to Y_i$, say $f_{i_0} : X \to Y_{i_0}$. Thus f factors as $X \to \beta(X) \to \prod Y_i \to Y_{i_0} \cong Z \to Y$ as desired.

090A **Lemma 5.25.2.** *Let X be a Hausdorff, locally quasi-compact space. There exists a map $X \to X^*$ which identifies X as an open subspace of a quasi-compact Hausdorff space X^* such that $X^* \setminus X$ is a singleton (one point compactification). In particular, the map $X \to \beta(X)$ identifies X with an open subspace of $\beta(X)$.*

Proof. Set $X^* = X \amalg \{\infty\}$. We declare a subset V of X^* to be open if either $V \subset X$ is open in X, or $\infty \in V$ and $U = V \cap X$ is an open of X such that $X \setminus U$ is quasi-compact. We omit the verification that this defines a topology. It is clear that $X \to X^*$ identifies X with an open subspace of X.

Since X is locally quasi-compact, every point $x \in X$ has a quasi-compact neighbourhood $x \in E \subset X$. Then E is closed (Lemma 5.12.4 part (1)) and $V = (X \setminus E) \amalg \{\infty\}$ is an open neighbourhood of ∞ disjoint from the interior of E. Thus X^* is Hausdorff.

Let $X^* = \bigcup V_i$ be an open covering. Then for some i, say i_0, we have $\infty \in V_{i_0}$. By construction $Z = X^* \setminus V_{i_0}$ is quasi-compact. Hence the covering $Z \subset \bigcup_{i \neq i_0} Z \cap V_i$ has a finite refinement which implies that the given covering of X^* has a finite refinement. Thus X^* is quasi-compact.

The map $X \to X^*$ factors as $X \to \beta(X) \to X^*$ by the universal property of the Stone-Čech compactification. Let $\varphi : \beta(X) \to X^*$ be this factorization. Then $X \to \varphi^{-1}(X)$ is a section to $\varphi^{-1}(X) \to X$ hence has closed image (Lemma 5.3.3). Since the image of $X \to \beta(X)$ is dense we conclude that $X = \varphi^{-1}(X)$. □

5.26. Extremally disconnected spaces

08YH The material in this section is taken from [**Gle58**] (with a slight modification as in [**Rai59**]). In Gleason's paper it is shown that in the category of quasi-compact Hausdorff spaces, the "projective objects" are exactly the extremally disconnected spaces.

08YI **Definition 5.26.1.** A topological space X is called *extremally disconnected* if the closure of every open subset of X is open.

If X is Hausdorff and extremally disconnected, then X is totally disconnected (this isn't true in general). If X is quasi-compact, Hausdorff, and extremally disconnected, then X is profinite by Lemma 5.22.2, but the converse does not hold in general. For example the p-adic integers $\mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$ is a profinite space which is not extremally disconnected. Namely, if $U \subset \mathbf{Z}_p$ is the set of nonzero elements whose valuation is even, then U is open but its closure is $U \cup \{0\}$ which is not open.

08YJ **Lemma 5.26.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. Assume f is surjective and $f(E) \neq Y$ for all proper closed subsets $E \subset X$. Then for $U \subset X$ open the subset $f(U)$ is contained in the closure of $Y \setminus f(X \setminus U)$.*

Proof. Pick $y \in f(U)$ and let $V \subset Y$ be any open neighbourhood of y. We will show that V intersects $Y \setminus f(X \setminus U)$. Note that $W = U \cap f^{-1}(V)$ is a nonempty open subset of X, hence $f(X \setminus W) \neq Y$. Take $y' \in Y$, $y' \notin f(X \setminus W)$. It is elementary to show that $y' \in V$ and $y' \in Y \setminus f(X \setminus U)$. □

08YK **Lemma 5.26.3.** *Let X be an extremally disconnected space. If $U, V \subset X$ are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are disjoint too.*

Proof. By assumption $\overline{U}$ is open, hence $V \cap \overline{U}$ is open and disjoint from U, hence empty because $\overline{U}$ is the intersection of all the closed subsets of X containing U. This means the open $\overline{V} \cap \overline{U}$ avoids V hence is empty by the same argument. □

08YL **Lemma 5.26.4.** *Let $f : X \to Y$ be a continuous map of Hausdorff quasi-compact topological spaces. If Y is extremally disconnected, f is surjective, and $f(Z) \neq Y$ for every proper closed subset Z of X, then f is a homeomorphism.*

Proof. By Lemma 5.17.8 it suffices to show that f is injective. Suppose that $x, x' \in X$ are distinct points with $y = f(x) = f(x')$. Choose disjoint open neighbourhoods $U, U' \subset X$ of x, x'. Observe that f is closed (Lemma 5.17.7) hence $T = f(X \setminus U)$ and $T' = f(X \setminus U')$ are closed in Y. Since X is the union of $X \setminus U$ and $X \setminus U'$ we see that $Y = T \cup T'$. By Lemma 5.26.2 we see that y is contained in the closure of $Y \setminus T$ and the closure of $Y \setminus T'$. On the other hand, by Lemma 5.26.3, this intersection is empty. In this way we obtain the desired contradiction. □

08YM **Lemma 5.26.5.** *Let $f : X \to Y$ be a continuous surjective map of Hausdorff quasi-compact topological spaces. There exists a quasi-compact subset $E \subset X$ such that $f(E) = Y$ but $f(E') \neq Y$ for all proper closed subsets $E' \subset E$.*

Proof. We will use without further mention that the quasi-compact subsets of X are exactly the closed subsets (Lemma 5.12.5). Consider the collection $\mathcal{E}$ of all quasi-compact subsets $E \subset X$ with $f(E) = Y$ ordered by inclusion. We will use Zorn's lemma to show that $\mathcal{E}$ has a minimal element. To do this it suffices to show that given a totally ordered family E_λ of elements of $\mathcal{E}$ the intersection $\bigcap E_\lambda$ is an element of $\mathcal{E}$. It is quasi-compact as it is closed. For every $y \in Y$ the sets $E_\lambda \cap f^{-1}(\{y\})$ are nonempty and closed, hence the intersection $\bigcap E_\lambda \cap f^{-1}(\{y\}) = \bigcap (E_\lambda \cap f^{-1}(\{y\}))$ is nonempty by Lemma 5.12.6. This finishes the proof. □

08YN **Proposition 5.26.6.** *Let X be a Hausdorff, quasi-compact topological space. The following are equivalent*

(1) *X is extremally disconnected,*
(2) *for any surjective continuous map $f : Y \to X$ with Y Hausdorff quasi-compact there exists a continuous section, and*
(3) *for any solid commutative diagram*

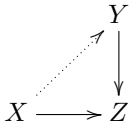

of continuous maps of quasi-compact Hausdorff spaces with $Y \to Z$ surjective, there is a dotted arrow in the category of topological spaces making the diagram commute.

Proof. It is clear that (3) implies (2). On the other hand, if (2) holds and $X \to Z$ and $Y \to Z$ are as in (3), then (2) assures there is a section to the projection $X \times_Z Y \to X$ which implies a suitable dotted arrow exists (details omitted). Thus (3) is equivalent to (2).

Assume X is extremally disconnected and let $f : Y \to X$ be as in (2). By Lemma 5.26.5 there exists a quasi-compact subset $E \subset Y$ such that $f(E) = X$ but $f(E') \neq X$ for all proper closed subsets $E' \subset E$. By Lemma 5.26.4 we find that $f|_E : E \to X$ is a homeomorphism, the inverse of which gives the desired section.

Assume (2). Let $U \subset X$ be open with complement Z. Consider the continuous surjection $f : \overline{U} \amalg Z \to X$. Let σ be a section. Then $\overline{U} = \sigma^{-1}(\overline{U})$ is open. Thus X is extremally disconnected. □

090B **Lemma 5.26.7.** *Let $f : X \to X$ be a surjective continuous selfmap of a Hausdorff topological space. If f is not id_X, then there exists a proper closed subset $E \subset X$ such that $X = E \cup f(E)$.*

Proof. Pick $p \in X$ with $f(p) \neq p$. Choose disjoint open neighbourhoods $p \in U$, $f(p) \in V$ and set $E = X \setminus U \cap f^{-1}(V)$. Then $p \notin E$ hence E is a proper closed subset. If $x \in X$, then either $x \in E$, or if not, then $x \in U \cap f^{-1}(V)$. Writing $x = f(y)$ (possible as f is surjective). If $y \in U \cap f^{-1}(V)$ then we would have $x = f(y) \in V$ which is a contradiction with $x \in U$. Hence $y \in E$ and $x \in f(E)$. □

090C **Example 5.26.8.** We can use Proposition 5.26.6 to see that the Stone-Čech compactification $\beta(X)$ of a discrete space X is extremally disconnected. Namely, let $f : Y \to \beta(X)$ be a continuous surjection where Y is quasi-compact and Hausdorff. Then we can lift the map $X \to \beta(X)$ to a continuous (!) map $X \to Y$ as X is discrete. By the universal property of the Stone-Čech compactification we see that we obtain a factorization $X \to \beta(X) \to Y$. Since $\beta(X) \to Y \to \beta(X)$ equals the identity on the dense subset X we conclude that we get a section. In particular, we conclude that the Stone-Čech compactification of a discrete space is totally disconnected, whence profinite (see discussion following Definition 5.26.1 and Lemma 5.22.2).

Using the supply of extremally disconnected spaces given by Example 5.26.8 we can prove that every quasi-compact Hausdorff space has a "projective cover" in the category of quasi-compact Hausdorff spaces.

090D **Lemma 5.26.9.** *Let X be a quasi-compact Hausdorff space. There exists a continuous surjection $X' \to X$ with X' quasi-compact, Hausdorff, and extremally disconnected. If we require that every proper closed subset of X' does not map onto X, then X' is unique up to isomorphism.*

Proof. Let $Y = X$ but endowed with the discrete topology. Let $X' = \beta(Y)$. The continuous map $Y \to X$ factors as $Y \to X' \to X$. This proves the first statement of the lemma by Example 5.26.8.

By Lemma 5.26.5 we can find a quasi-compact subset $E \subset X'$ surjecting onto X such that no proper closed subset of E surjects onto X. Because X' is extremally disconnected there exists a continuous map $f : X' \to E$ over X (Proposition 5.26.6). Composing f with the map $E \to X'$ gives a continuous selfmap $f|_E : E \to E$. Observe that $f|_E$ has to be surjective as otherwise the image would be a proper closed subset surjecting onto X. Hence $f|_E$ has to be id_E as otherwise Lemma 5.26.7 shows that E isn't minimal. Thus the id_E factors through the extremally disconnected space X'. A formal, categorical argument (using the characterization of Proposition 5.26.6) shows that E is extremally disconnected.

To prove uniqueness, suppose we have a second $X'' \to X$ minimal cover. By the lifting property proven in Proposition 5.26.6 we can find a continuous map $g : X' \to X''$ over X. Observe that g is a closed map (Lemma 5.17.7). Hence $g(X') \subset X''$ is a closed subset surjecting onto X and we conclude $g(X') = X''$ by minimality of X''. On the other hand, if $E \subset X'$ is a proper closed subset, then $g(E) \neq X''$ as E does not map onto X by minimality of X'. By Lemma 5.26.4 we see that g is an isomorphism. □

090E **Remark 5.26.10.** Let X be a quasi-compact Hausdorff space. Let κ be an infinite cardinal bigger or equal than the cardinality of X. Then the cardinality of the minimal quasi-compact, Hausdorff, extremally disconnected cover $X' \to X$ (Lemma 5.26.9) is at most 2^{2^κ}. Namely, choose a subset $S \subset X'$ mapping bijectively to X. By minimality of X' the set S is dense in X'. Thus $|X'| \leq 2^{2^\kappa}$ by Lemma 5.25.1.

5.27. Miscellany

0067 The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

0069 **Lemma 5.27.1.** *Let X be a topological space which*

(1) *has a basis of the topology consisting of quasi-compact opens, and*
(2) *has the property that the intersection of any two quasi-compact opens is quasi-compact.*

Then

(1) *X is locally quasi-compact,*
(2) *a quasi-compact open $U \subset X$ is retrocompact,*
(3) *any quasi-compact open $U \subset X$ has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and all U_j and $U_j \cap U_{j'}$ quasi-compact,*
(4) *add more here.*

Proof. Omitted. □

06RM **Definition 5.27.2.** Let X be a topological space. We say $x \in X$ is an *isolated point* of X if $\{x\}$ is open in X.

5.28. Partitions and stratifications

09XY Stratifications can be defined in many different ways. We welcome comments on the choice of definitions in this section.

09XZ **Definition 5.28.1.** Let X be a topological space. A *partition* of X is a decomposition $X = \coprod X_i$ into locally closed subsets X_i. The X_i are called the *parts* of the partition. Given two partitions of X we say one *refines* the other if the parts of one are unions of parts of the other.

Any topological space X has a partition into connected components. If X has finitely many irreducible components $Z_1, \ldots, Z_r$, then there is a partition with parts $X_I = \bigcap_{i \in I} Z_i \setminus (\bigcup_{i \notin I} Z_i)$ whose indices are subsets $I \subset \{1, \ldots, r\}$ which refines the partition into connected components.

09Y0 **Definition 5.28.2.** Let X be a topological space. A *good stratification* of X is a partition $X = \coprod X_i$ such that for all $i, j \in I$ we have

$$X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}.$$

Given a good stratification $X = \coprod_{i \in I} X_i$ we obtain a partial ordering on I by setting $i \leq j$ if and only if $X_i \subset \overline{X_j}$. Then we see that

$$\overline{X_j} = \bigcup_{i \leq j} X_i$$

However, what often happens in algebraic geometry is that one just has that the left hand side is a subset of the right hand side in the last displayed formula. This leads to the following definition.

09Y1 **Definition 5.28.3.** Let X be a topological space. A *stratification* of X is given by a partition $X = \coprod_{i \in I} X_i$ and a partial ordering on I such that for each $j \in I$ we have

$$\overline{X_j} \subset \bigcup_{i \leq j} X_i$$

The parts X_i are called the *strata* of the stratification.

We often impose additional conditions on the stratification. For example, stratifications are particularly nice if they are *locally finite*, which means that every point has a neighbourhood which meets only finitely many strata. More generally we introduce the following definition.

0BDS **Definition 5.28.4.** Let X be a topological space. Let I be a set and for $i \in I$ let $E_i \subset X$ be a subset. We say the collection $\{E_i\}_{i \in I}$ is *locally finite* if for all $x \in X$ there exists an open neighbourhood U of x such that $\{i \in I | E_i \cap U \neq \emptyset\}$ is finite.

09Y2 **Remark 5.28.5.** Given a locally finite stratification $X = \coprod X_i$ of a topological space X, we obtain a family of closed subsets $Z_i = \bigcup_{j \leq i} X_j$ of X indexed by I such that

$$Z_i \cap Z_j = \bigcup_{k \leq i,j} Z_k$$

Conversely, given closed subsets $Z_i \subset X$ indexed by a partially ordered set I such that $X = \bigcup Z_i$, such that every point has a neighbourhood meeting only finitely many Z_i, and such that the displayed formula holds, then we obtain a locally finite stratification of X by setting $X_i = Z_i \setminus \bigcup_{j<i} Z_j$.

09Y3 **Lemma 5.28.6.** *Let X be a topological space. Let $X = \coprod X_i$ be a finite partition of X. Then there exists a finite stratification of X refining it.*

Proof. Let $T_i = \overline{X_i}$ and $\Delta_i = T_i \setminus X_i$. Let S be the set of all intersections of T_i and Δ_i. (For example $T_1 \cap T_2 \cap \Delta_4$ is an element of S.) Then $S = \{Z_s\}$ is a finite collection of closed subsets of X such that $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on S by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s'<s} Z_{s'}$ to get the desired stratification. □

09Y4 **Lemma 5.28.7.** *Let X be a topological space. Suppose $X = T_1 \cup \ldots \cup T_n$ is written as a union of constructible subsets. There exists a finite stratification $X = \coprod X_i$ with each X_i constructible such that each T_k is a union of strata.*

Proof. By definition of constructible subsets, we can write each T_i as a finite union of $U \cap V^c$ with $U, V \subset X$ retrocompact open. Hence we may assume that $T_i = U_i \cap V_i^c$ with $U_i, V_i \subset X$ retrocompact open. Let S be the finite set of closed subsets of X consisting of $\emptyset, X, U_i^c, V_i^c$ and finite intersections of these. If $Z \in S$, then Z is constructible in X (Lemma 5.15.2). Moreover, $Z \cap Z' \in S$ for all $Z, Z' \in S$. Define a partial ordering on S by inclusion. For $Z \in S$ set $X_Z = Z \setminus \bigcup_{Z' < Z,\ Z' \in S} Z'$ to get a stratification $X = \coprod_{Z \in S} X_Z$ satisfying the properties stated in the lemma. □

09Y5 **Lemma 5.28.8.** *Let X be a Noetherian topological space. Any finite partition of X can be refined by a finite good stratification.*

Proof. Let $X = \coprod X_i$ be a finite partition of X. Let Z be an irreducible component of X. Since $X = \bigcup \overline{X_i}$ with finite index set, there is an i such that $Z \subset \overline{X_i}$. Since X_i is locally closed this implies that $Z \cap X_i$ contains an open of Z. Thus $Z \cap X_i$ contains an open U of X (Lemma 5.9.2). Write $X_i = U \amalg X_i^1 \amalg X_i^2$ with $X_i^1 = (X_i \setminus U) \cap \overline{U}$ and $X_i^2 = (X_i \setminus U) \cap \overline{U}^c$. For $i' \neq i$ we set $X_{i'}^1 = X_{i'} \cap \overline{U}$ and $X_{i'}^2 = X_{i'} \cap \overline{U}^c$. Then

$$X \setminus U = \coprod X_l^k$$

is a partition such that $\overline{U} \setminus U = \bigcup X_l^1$. Note that $X \setminus U$ is closed and strictly smaller than X. By Noetherian induction we can refine this partition by a finite good stratification $X \setminus U = \coprod_{\alpha \in A} T_\alpha$. Then $X = U \amalg \coprod_{\alpha \in A} T_\alpha$ is a finite good stratification of X refining the partition we started with. □

5.29. Colimits of spaces

0B1W The category of topological spaces has coproducts. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow the set $\coprod_{i \in I} X_i$ with the *coproduct topology*. As a basis for this topology we use sets of the form U_i where $U_i \subset X_i$ is open.

The category of topological spaces has coequalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the coequalizer of a and b is the coequalizer $Y/\sim$ in the category of sets endowed with the quotient topology (Section 5.6).

0B1X **Lemma 5.29.1.** *The category of topological spaces has colimits and the forgetful functor to sets commutes with them.*

Proof. This follows from the discussion above and Categories, Lemma 4.14.12. Another proof of existence of colimits is sketched in Categories, Remark 4.25.2. It follows from the above that the forgetful functor commutes with colimits. Another way to see this is to use Categories, Lemma 4.24.5 and use that the forgetful functor has a right adjoint, namely the functor which assigns to a set the corresponding chaotic (or indiscrete) topological space. □

5.30. Topological groups, rings, modules

0B1Y This is just a short section with definitions and elementary properties.

0B1Z **Definition 5.30.1.** A *topological group* is a group G endowed with a topology such that multiplication $G \times G \to G$, $(x, y) \mapsto xy$ and inverse $G \to G$, $x \mapsto x^{-1}$ are continuous. A *homomorphism of topological groups* is a homomorphism of groups which is continuous.

If G is a topological group and $H \subset G$ is a subgroup, then H with the induced topology is a topological group. If G is a topological group and $G \to H$ is a surjection of groups, then H endowed with the quotient topology is a topological group.

0BMC **Example 5.30.2.** Let E be a set. We can endow the set of self maps $\mathrm{Map}(E, E)$ with the compact open topology, i.e., the topology such that given $f : E \to E$ a fundamental system of neighbourhoods of f is given by the sets $U_S(f) = \{f' : E \to E \mid f'|_S = f|_S\}$ where $S \subset E$ is finite. With this topology the action

$$\mathrm{Map}(E, E) \times E \longrightarrow E, \quad (f, e) \longmapsto f(e)$$

is continuous when E is given the discrete topology. If X is a topological space and $X \times E \to E$ is a continuous map, then the map $X \to \mathrm{Map}(E, E)$ is continuous. In other words, the compact open topology is the coarsest topology such that the "action" map displayed above is continuous. The composition

$$\mathrm{Map}(E, E) \times \mathrm{Map}(E, E) \to \mathrm{Map}(E, E)$$

is continuous as well (as is easily verified using the description of neighbourhoods above). Finally, if $\mathrm{Aut}(E) \subset \mathrm{Map}(E, E)$ is the subset of invertible maps, then the inverse $i : \mathrm{Aut}(E) \to \mathrm{Aut}(E)$, $f \mapsto f^{-1}$ is continuous too. Namely, say $S \subset E$ is finite, then $i^{-1}(U_S(f^{-1})) = U_{f^{-1}(S)}(f)$. Hence $\mathrm{Aut}(E)$ is a topological group as in Definition 5.30.1.

0B20 **Lemma 5.30.3.** *The category of topological groups has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of groups.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let G_i, $i \in I$ be a collection of topological groups. Take the usual product $G = \prod G_i$ with the product topology. Since $G \times G = \prod (G_i \times G_i)$ as a topological space (because products commutes with products in any category), we see that multiplication on G is continuous. Similarly for the inverse map. Let $a, b : G \to H$ be two homomorphisms of topological groups. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of groups endowed with the induced topology. □

0BR1 **Lemma 5.30.4.** *Let G be a topological group. The following are equivalent*

(1) *G as a topological space is profinite,*
(2) *G is a limit of a diagram of finite discrete topological groups,*
(3) *G is a cofiltered limit of finite discrete topological groups.*

Proof. We have the corresponding result for topological spaces, see Lemma 5.22.2. Combined with Lemma 5.30.3 we see that it suffices to prove that (1) implies (3).

We first prove that every neighbourhood E of the neutral element e contains an open subgroup. Namely, since G is the cofiltered limit of finite discrete topological spaces (Lemma 5.22.2), we can choose a continuous map $f : G \to T$ to a finite discrete space T such that $f^{-1}(f(\{e\})) \subset E$. Consider

$$H = \{g \in G \mid f(gg') = f(g') \text{ for all } g' \in G\}$$

This is a subgroup of G and contained in E. Thus it suffices to show that H is open. Pick $t \in T$ and set $W = f^{-1}(\{t\})$. Observe that $W \subset G$ is open and closed, in particular quasi-compact. For each $w \in W$ there exist open neighbourhoods $e \in U_w \subset G$ and $w \in U'_w \subset W$ such that $U_w U'_w \subset W$. By quasi-compactness we can find $w_1, \ldots, w_n$ such that $W = \bigcup U'_{w_i}$. Then $U_t = U_{w_1} \cap \ldots \cap U_{w_n}$ is an open neighbourhood of e such that $f(gw) = t$ for all $w \in W$. Since T is finite we see that $\bigcap_{t \in T} U_t \subset H$ is an open neighbourhood of e. Since $H \subset G$ is a subgroup it follows that H is open.

Suppose that $H \subset G$ is an open subgroup. Since G is quasi-compact we see that the index of H in G is finite. Say $G = Hg_1 \cup \ldots \cup Hg_n$. Then $N = \bigcap_{i=1,\ldots,n} g_i H g_i^{-1}$

is an open normal subgroup contained in H. Since N also has finite index we see that $G \to G/N$ is a surjection to a finite discrete topological group.

Consider the map

$$G \longrightarrow \lim_{N \subset G \text{ open and normal}} G/N$$

We claim that this map is an isomorphism of topological groups. This finishes the proof of the lemma as the limit on the right is cofiltered (the intersection of two open normal subgroups is open and normal). The map is continuous as each $G \to G/N$ is continuous. The map is injective as G is Hausdorff and every neighbourhood of e contains an N by the arguments above. The map is surjective by Lemma 5.12.6. By Lemma 5.17.8 the map is a homeomorphism. □

0BR2 **Definition 5.30.5.** A topological group is called a *profinite group* if it satisfies the equivalent conditions of Lemma 5.30.4.

If $G_1 \to G_2 \to G_3 \to \ldots$ is a system of topological groups then the colimit $G = \operatorname{colim} G_n$ as a topological group (Lemma 5.30.6) is in general different from the colimit as a topological space (Lemma 5.29.1) even though these have the same underlying set. See Examples, Section 110.78.

0B21 **Lemma 5.30.6.** *The category of topological groups has colimits and colimits commute with the forgetful functor to the category of groups.*

Proof. We will use the argument of Categories, Remark 4.25.2 to prove existence of colimits. Namely, suppose that $\mathcal{I} \to \mathit{Top}$, $i \mapsto G_i$ is a functor into the category $\mathit{TopGroup}$ of topological groups. Then we can consider

$$F : \mathit{TopGroup} \longrightarrow \mathit{Sets}, \quad H \longmapsto \lim_{\mathcal{I}} \operatorname{Mor}_{\mathit{TopGroup}}(G_i, H)$$

This functor commutes with limits. Moreover, given any topological group H and an element $(\varphi_i : G_i \to H)$ of $F(H)$, there is a subgroup $H' \subset H$ of cardinality at most $|\coprod G_i|$ (coproduct in the category of groups, i.e., the free product on the G_i) such that the morphisms φ_i map into H'. Namely, we can take the induced topology on the subgroup generated by the images of the φ_i. Thus it is clear that the hypotheses of Categories, Lemma 4.25.1 are satisfied and we find a topological group G representing the functor F, which precisely means that G is the colimit of the diagram $i \mapsto G_i$.

To see the statement on commutation with the forgetful functor to groups we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a group the corresponding chaotic (or indiscrete) topological group. □

0B22 **Definition 5.30.7.** A *topological ring* is a ring R endowed with a topology such that addition $R \times R \to R$, $(x, y) \mapsto x + y$ and multiplication $R \times R \to R$, $(x, y) \mapsto xy$ are continuous. A *homomorphism of topological rings* is a homomorphism of rings which is continuous.

In the Stacks project rings are commutative with 1. If R is a topological ring, then $(R, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring and $R' \subset R$ is a subring, then R' with the induced topology is a topological ring. If R is a topological ring and $R \to R'$ is a surjection of rings, then R' endowed with the quotient topology is a topological ring.

0B23 **Lemma 5.30.8.** *The category of topological rings has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of rings.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let R_i, $i \in I$ be a collection of topological rings. Take the usual product $R = \prod R_i$ with the product topology. Since $R \times R = \prod(R_i \times R_i)$ as a topological space (because products commutes with products in any category), we see that addition and multiplication on R are continuous. Let $a, b : R \to R'$ be two homomorphisms of topological rings. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of rings endowed with the induced topology. □

0B24 **Lemma 5.30.9.** *The category of topological rings has colimits and colimits commute with the forgetful functor to the category of rings.*

Proof. The exact same argument as used in the proof of Lemma 5.30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to rings we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a ring the corresponding chaotic (or indiscrete) topological ring. □

0B25 **Definition 5.30.10.** Let R be a topological ring. A *topological module* is an R-module M endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous. A *homomorphism of topological modules* is a homomorphism of modules which is continuous.

If R is a topological ring and M is a topological module, then $(M, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring, M is a topological module and $M' \subset M$ is a submodule, then M' with the induced topology is a topological module. If R is a topological ring, M is a topological module, and $M \to M'$ is a surjection of modules, then M' endowed with the quotient topology is a topological module.

0B26 **Lemma 5.30.11.** *Let R be a topological ring. The category of topological modules over R has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of R-modules.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let M_i, $i \in I$ be a collection of topological modules over R. Take the usual product $M = \prod M_i$ with the product topology. Since $M \times M = \prod(M_i \times M_i)$ as a topological space (because products commutes with products in any category), we see that addition on M is continuous. Similarly for multiplication $R \times M \to M$. Let $a, b : M \to M'$ be two homomorphisms of topological modules over R. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of modules endowed with the induced topology. □

0B27 **Lemma 5.30.12.** *Let R be a topological ring. The category of topological modules over R has colimits and colimits commute with the forgetful functor to the category of modules over R.*

Proof. The exact same argument as used in the proof of Lemma 5.30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to R-modules we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a module the corresponding chaotic (or indiscrete) topological module. □

5.31. Other chapters

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CHAPTER 6

Sheaves on Spaces

006A

6.1. Introduction

006B Basic properties of sheaves on topological spaces will be explained in this document. A reference is [**God73**].

This will be superseded by the discussion of sheaves over sites later in the documents. But perhaps it makes sense to briefly define some of the notions here.

6.2. Basic notions

006C The following is a list of basic notions in topology.

(1) Let X be a topological space. The phrase: "Let $U = \bigcup_{i \in I} U_i$ be an open covering" means the following: I is a set and for each $i \in I$ we are given an open subset $U_i \subset X$ such that U is the union of the U_i. It is allowed to have $I = \emptyset$ in which case there are no U_i and $U = \emptyset$. It is also allowed, in case $I \neq \emptyset$ to have any or all of the U_i be empty.
(2) etc, etc.

6.3. Presheaves

006D

006E **Definition 6.3.1.** Let X be a topological space.

(1) A *presheaf $\mathcal{F}$ of sets on X* is a rule which assigns to each open $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X* is a rule which assigns to each open $U \subset X$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps, i.e., whenever $V \subset U \subset X$ are open the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_V^U} & & \downarrow{\scriptstyle \rho_V^U} \\ \mathcal{F}(V) & \xrightarrow{\varphi} & \mathcal{G}(V) \end{array}$$

commutes.
(3) The category of presheaves of sets on X will be denoted $PSh(X)$.

The elements of the set $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. For every $V \subset U$ the map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map*. We will use the notation $s|_V := \rho_V^U(s)$ if $s \in \mathcal{F}(U)$. This notation is consistent with the notion of restriction of functions from topology because if $W \subset V \subset U$ and s is a section of $\mathcal{F}$ over

U then $s|_W = (s|_V)|_W$ by the property of the restriction maps expressed in the definition above.

Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, -)$ or by $H^0(U, -)$. In other words, the following equalities are tautological

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F}).$$

In this chapter we will not use this notation, but in others we will.

006F **Definition 6.3.2.** Let X be a topological space. Let A be a set. The *constant presheaf with value* A is the presheaf that assigns the set A to every open $U \subset X$, and such that all restriction mappings are id_A.

6.4. Abelian presheaves

006G In this section we briefly point out some features of the category of presheaves that allow one to define presheaves of abelian groups.

006H **Example 6.4.1.** Let X be a topological space. Consider a rule $\mathcal{F}$ that associates to every open subset of X a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps ρ^U_V. The resulting structure is a presheaf of sets on X. It is a final object in the category of presheaves of sets on X, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

006I **Lemma 6.4.2.** *Let X be a topological space. The category of presheaves of sets on X has products (see Categories, Definition 4.14.6). Moreover, the set of sections of the product $\mathcal{F} \times \mathcal{G}$ over an open U is the product of the sets of sections of $\mathcal{F}$ and $\mathcal{G}$ over U.*

Proof. Namely, suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on the topological space X. Consider the rule $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$, denoted $\mathcal{F} \times \mathcal{G}$. If $V \subset U \subset X$ are open then define the restriction mapping

$$(\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

by mapping $(s, t) \mapsto (s|_V, t|_V)$. Then it is immediately clear that $\mathcal{F} \times \mathcal{G}$ is a presheaf. Also, there are projection maps $p : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $q : \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. We leave it to the reader to show that for any third presheaf $\mathcal{H}$ we have $\mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G}) = \mathrm{Mor}(\mathcal{H}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{H}, \mathcal{G})$. □

Recall that if $(A, + : A \times A \to A, - : A \to A, 0 \in A)$ is an abelian group, then the zero and the negation maps are uniquely determined by the addition law. In other words, it makes sense to say "let $(A, +)$ be an abelian group".

006J **Lemma 6.4.3.** *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets. Consider the following types of structure on $\mathcal{F}$:*

(1) *For every open U the structure of an abelian group on $\mathcal{F}(U)$ such that all restriction maps are abelian group homomorphisms.*
(2) *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ (see Example 6.4.1) satisfying all the axioms of $+, -, 0$ in a usual abelian group.*

(3) *A map of presheaves* $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$*, a map of presheaves* $- : \mathcal{F} \to \mathcal{F}$ *and a map* $0 : * \to \mathcal{F}$ *such that for each open* $U \subset X$ *the quadruple* $(\mathcal{F}(U), +, -, 0)$ *is an abelian group,*
(4) *A map of presheaves* $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *such that for every open* $U \subset X$ *the map* $+ : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ *defines the structure of an abelian group.*

There are natural bijections between the collections of types of data (1) - (4) above.

Proof. Omitted. □

The lemma says that to give an abelian group object $\mathcal{F}$ in the category of presheaves is the same as giving a presheaf of sets $\mathcal{F}$ such that all the sets $\mathcal{F}(U)$ are endowed with the structure of an abelian group and such that all the restriction mappings are group homomorphisms. For most algebra structures we will take this approach to (pre)sheaves of such objects, i.e., we will define a (pre)sheaf of such objects to be a (pre)sheaf $\mathcal{F}$ of sets all of whose sets of sections $\mathcal{F}(U)$ are endowed with this structure compatibly with the restriction mappings.

006K **Definition 6.4.4.** Let X be a topological space.

(1) A *presheaf of abelian groups on* X or an *abelian presheaf over* X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ_V^U are homomorphisms of abelian groups, see Lemma 6.4.3 above.
(2) A *morphism of abelian presheaves over* X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

006L **Example 6.4.5.** Let X be a topological space. For each $x \in X$ suppose given an abelian group M_x. For $U \subset X$ open we set

$$\mathcal{F}(U) = \bigoplus\nolimits_{x \in U} M_x.$$

We denote a typical element in this abelian group by $\sum_{i=1}^n m_{x_i}$, where $x_i \in U$ and $m_{x_i} \in M_{x_i}$. (Of course we may always choose our representation such that $x_1, \ldots, x_n$ are pairwise distinct.) We define for $V \subset U \subset X$ open a restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ by mapping an element $s = \sum_{i=1}^n m_{x_i}$ to the element $s|_V = \sum_{x_i \in V} m_{x_i}$. We leave it to the reader to verify that this is a presheaf of abelian groups.

6.5. Presheaves of algebraic structures

006M Let us clarify the definition of presheaves of algebraic structures. Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. Typically F is a "forgetful" functor. For an object $M \in \mathrm{Ob}(\mathcal{C})$ we often call $F(M)$ the *underlying set* of the object M. If $M \to M'$ is a morphism in $\mathcal{C}$ we call $F(M) \to F(M')$ the *underlying map of sets*. In fact, we will often not distinguish between an object and its underlying set, and similarly for morphisms. So we will say a map of sets $F(M) \to F(M')$ is a *morphism of algebraic structures*, if it is equal to $F(f)$ for some morphism $f : M \to M'$ in $\mathcal{C}$.

In analogy with Definition 6.4.4 above a "presheaf of objects of $\mathcal{C}$" could be defined by the following data:

(1) a presheaf of sets $\mathcal{F}$, and
(2) for every open $U \subset X$ a choice of an object $A(U) \in \mathrm{Ob}(\mathcal{C})$

subject to the following conditions (using the phraseology above)

(1) for every open $U \subset X$ the set $\mathcal{F}(U)$ is the underlying set of $A(U)$, and
(2) for every $V \subset U \subset X$ open the map of sets $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of algebraic structures.

In other words, for every $V \subset U$ open in X the restriction mappings ρ_V^U is the image $F(\alpha_V^U)$ for some unique morphism $\alpha_V^U : A(U) \to A(V)$ in the category $\mathcal{C}$. The uniqueness is forced by the condition that F is faithful; it also implies that $\alpha_W^U = \alpha_W^V \circ \alpha_V^U$ whenever $W \subset V \subset U$ are open in X. The system $(A(-), \alpha_V^U)$ is what we will define as a presheaf with values in $\mathcal{C}$ on X, compare Sites, Definition 7.2.2. We recover our presheaf of sets $(\mathcal{F}, \rho_V^U)$ via the rules $\mathcal{F}(U) = F(A(U))$ and $\rho_V^U = F(\alpha_V^U)$.

006N **Definition 6.5.1.** Let X be a topological space. Let $\mathcal{C}$ be a category.

(1) A *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$* is given by a rule which assigns to every open $U \subset X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$* is given by a morphism $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ compatible with restriction morphisms.

006O **Definition 6.5.2.** Let X be a topological space. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \mathit{Sets}$ be a faithful functor. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The presheaf of sets $U \mapsto F(\mathcal{F}(U))$ is called the *underlying presheaf of sets of $\mathcal{F}$*.

It is customary to use the same letter $\mathcal{F}$ to denote the underlying presheaf of sets, and this makes sense according to our discussion preceding Definition 6.5.1. In particular, the phrase "let $s \in \mathcal{F}(U)$" or "let s be a section of $\mathcal{F}$ over U" signifies that $s \in F(\mathcal{F}(U))$.

This notation and these definitions apply in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field,* etc and *morphisms between these.*

6.6. Presheaves of modules

006P Suppose that $\mathcal{O}$ is a presheaf of rings on X. We would like to define the notion of a presheaf of $\mathcal{O}$-modules over X. In analogy with Definition 6.4.4 we are tempted to define this as a presheaf of sets $\mathcal{F}$ such that for every open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an $\mathcal{O}(U)$-module compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). However, it is customary (and equivalent) to define it as in the following definition.

006Q **Definition 6.6.1.** Let X be a topological space, and let $\mathcal{O}$ be a presheaf of rings on X.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle \mathrm{id} \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$

commutes.
(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.
(4) The category of presheaves of $\mathcal{O}$-modules is denoted $PMod(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of $\mathcal{F}$*. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the *tensor product presheaf*. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

006R **Lemma 6.6.2.** *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

6.7. Sheaves

006S In this section we explain the sheaf condition.

006T **Definition 6.7.1.** Let X be a topological space.

(1) A *sheaf $\mathcal{F}$ of sets on X* is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.
(3) The category of sheaves of sets on X is denoted $Sh(X)$.

006U **Remark 6.7.2.** There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf over the empty set $\emptyset \subset X$. It is necessary, and we already did if you read the definition right. Namely, note that the empty set is covered by the empty open covering, and hence the "collection of sections s_i" from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in. In other words, if you read the definition right you automatically deduce that $\mathcal{F}(\emptyset) =$ *a final object*, which in the case of a sheaf of sets is a singleton. If you do not like this argument, then you can just require that $\mathcal{F}(\emptyset) = \{*\}$.

In particular, this condition will then ensure that if $U, V \subset X$ are open and *disjoint* then

$$\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V).$$

(Because the fibre product over a final object is a product.)

006V **Example 6.7.3.** Let X, Y be topological spaces. Consider the rule $\mathcal{F}$ which associates to the open $U \subset X$ the set

$$\mathcal{F}(U) = \{f : U \to Y \mid f \text{ is continuous}\}$$

with the obvious restriction mappings. We claim that $\mathcal{F}$ is a sheaf. To see this suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to Y$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to Y$ is a map such that its restriction to U_i agrees with the continuous map f_i. Hence clearly f is continuous!

We can use the result of the example to define constant sheaves. Namely, suppose that A is a set. Endow A with the discrete topology. Let $U \subset X$ be an open subset. Then we have

$$\{f : U \to A \mid f \text{ continuous}\} = \{f : U \to A \mid f \text{ locally constant}\}.$$

Thus the rule which assigns to an open all locally constant maps into A is a sheaf.

006W **Definition 6.7.4.** Let X be a topological space. Let A be a set. The *constant sheaf with value* A denoted $\underline{A}$, or $\underline{A}_X$ is the sheaf that assigns to an open $U \subset X$ the set of all locally constant maps $U \to A$ with restriction mappings given by restrictions of functions.

006X **Example 6.7.5.** Let X be a topological space. Let $(A_x)_{x \in X}$ be a family of sets A_x indexed by points $x \in X$. We are going to construct a sheaf of sets Π from this data. For $U \subset X$ open set

$$\Pi(U) = \prod\nolimits_{x \in U} A_x.$$

For $V \subset U \subset X$ open define a restriction mapping by the following rule: An element $s = (a_x)_{x \in U} \in \Pi(U)$ restricts to $s|_V = (a_x)_{x \in V}$. It is obvious that this defines a presheaf of sets. We claim this is a sheaf. Namely, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i \in \Pi(U_i)$ are such that s_i and s_j agree over $U_i \cap U_j$. Write $s_i = (a_{i,x})_{x \in U_i}$. The compatibility condition implies that $a_{i,x} = a_{j,x}$ in the set A_x whenever $x \in U_i \cap U_j$. Hence there exists a unique element $s = (a_x)_{x \in U}$ in $\Pi(U) = \prod_{x \in U} A_x$ with the property that $a_x = a_{i,x}$ whenever $x \in U_i$ for some i. Of course this element s has the property that $s|_{U_i} = s_i$ for all i.

006Y **Example 6.7.6.** Let X be a topological space. Suppose for each $x \in X$ we are given an abelian group M_x. Consider the presheaf $\mathcal{F} : U \mapsto \bigoplus_{x \in U} M_x$ defined in Example 6.4.5. This is not a sheaf in general. For example, if X is an infinite set with the discrete topology, then the sheaf condition would imply that $\mathcal{F}(X) = \prod_{x \in X} \mathcal{F}(\{x\})$ but by definition we have $\mathcal{F}(X) = \bigoplus_{x \in X} M_x = \bigoplus_{x \in X} \mathcal{F}(\{x\})$. And an infinite direct sum is in general different from an infinite direct product.

However, if X is a topological space such that every open of X is quasi-compact, then $\mathcal{F}$ *is* a sheaf. This is left as an exercise to the reader.

6.8. Abelian sheaves

006Z

0070 **Definition 6.8.1.** Let X be a topological space.

(1) An *abelian sheaf on X* or *sheaf of abelian groups on X* is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i \in I}$ to the element

$$(s_{i_0}|_{U_{i_0} \cap U_{i_1}} - s_{i_1}|_{U_{i_0} \cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

6.9. Sheaves of algebraic structures

0071 Let us clarify the definition of sheaves of certain types of structures. First, let us reformulate the sheaf condition. Namely, suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X. The sheaf condition can be reformulated as follows. Let $U = \bigcup_{i \in I} U_i$ be an open covering. Consider the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

Here the left map is defined by the rule $s \mapsto \prod_{i \in I} s|_{U_i}$. The two maps on the right are the maps

$$\prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_0}|_{U_{i_0} \cap U_{i_1}} \text{ resp. } \prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_1}|_{U_{i_0} \cap U_{i_1}}.$$

The sheaf condition exactly says that the left arrow is the equalizer of the right two. This generalizes immediately to the case of presheaves with values in a category as long as the category has products.

0072 **Definition 6.9.1.** Let X be a topological space. Let $\mathcal{C}$ be a category with products. A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X is a *sheaf* if for every open covering the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. A good example to keep in mind is the case where $\mathcal{C}$ is the category of abelian groups and F is the forgetful functor. Consider a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X. We would like to reformulate the condition above in terms of the underlying presheaf of sets (Definition 6.5.2). Note that the underlying presheaf of sets is a sheaf of sets if and only if all the diagrams

$$F(\mathcal{F}(U)) \longrightarrow \prod_{i \in I} F(\mathcal{F}(U_i)) \rightrightarrows \prod_{(i_0,i_1) \in I \times I} F(\mathcal{F}(U_{i_0} \cap U_{i_1}))$$

of sets – after applying the forgetful functor F – are equalizer diagrams! Thus we would like $\mathcal{C}$ to have products and equalizers and we would like F to commute with them. This is equivalent to the condition that $\mathcal{C}$ has limits and that F commutes with them, see Categories, Lemma 4.14.11. But this is not yet good enough (see Example 6.9.4); we also need F to *reflect isomorphisms*. This property means that given a morphism $f : A \to A'$ in $\mathcal{C}$, then f is an isomorphism if (and only if) $F(f)$ is a bijection.

0073 **Lemma 6.9.2.** *Suppose the category $\mathcal{C}$ and the functor $F : \mathcal{C} \to Sets$ have the following properties:*

(1) *F is faithful,*
(2) *$\mathcal{C}$ has limits and F commutes with them, and*
(3) *the functor F reflects isomorphisms.*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Proof. Assume that $\mathcal{F}$ is a sheaf. Then $\mathcal{F}(U)$ is the equalizer of the diagram above and by assumption we see $F(\mathcal{F}(U))$ is the equalizer of the corresponding diagram of sets. Hence $F(\mathcal{F})$ is a sheaf of sets.

Assume that $F(\mathcal{F})$ is a sheaf. Let $E \in \mathrm{Ob}(\mathcal{C})$ be the equalizer of the two parallel arrows in Definition 6.9.1. We get a canonical morphism $\mathcal{F}(U) \to E$, simply because $\mathcal{F}$ is a presheaf. By assumption, the induced map $F(\mathcal{F}(U)) \to F(E)$ is an isomorphism, because $F(E)$ is the equalizer of the corresponding diagram of sets. Hence we see $\mathcal{F}(U) \to E$ is an isomorphism by condition (3) of the lemma. □

The lemma in particular applies to *sheaves of groups, rings, algebras over a fixed ring, modules over a fixed ring, vector spaces over a fixed field,* etc. In other words, these are presheaves of groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc such that the underlying presheaf of sets is a sheaf.

0074 **Example 6.9.3.** Let X be a topological space. For each open $U \subset X$ consider the $\mathbf{R}$-algebra $\mathcal{C}^0(U) = \{f : U \to \mathbf{R} \mid f \text{ is continuous}\}$. There are obvious restriction mappings that turn this into a presheaf of $\mathbf{R}$-algebras over X. By Example 6.7.3 it is a sheaf of sets. Hence by the Lemma 6.9.2 it is a sheaf of $\mathbf{R}$-algebras over X.

0075 **Example 6.9.4.** Consider the category of topological spaces *Top*. There is a natural faithful functor $Top \to Sets$ which commutes with products and equalizers. But it does not reflect isomorphisms. And, in fact it turns out that the analogue of Lemma 6.9.2 is wrong. Namely, suppose $X = \mathbf{N}$ with the discrete topology. Let A_i, for $i \in \mathbf{N}$ be a discrete topological space. For any subset $U \subset \mathbf{N}$ define $\mathcal{F}(U) = \prod_{i \in U} A_i$ with the discrete topology. Then this is a presheaf of topological spaces whose underlying presheaf of sets is a sheaf, see Example 6.7.5. However, if each A_i

has at least two elements, then this is not a sheaf of topological spaces according to Definition 6.9.1. The reader may check that putting the *product topology* on each $\mathcal{F}(U) = \prod_{i \in U} A_i$ does lead to a sheaf of topological spaces over X.

6.10. Sheaves of modules

0076

0077 **Definition 6.10.1.** Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 6.6.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\mathit{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

6.11. Stalks

0078 Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section 4.21 for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. Given a pair (U, s) we sometimes denote s_x the element of $\mathcal{F}_x$ corresponding to the equivalence class of (U, s). We sometimes use the phrase "image of s in $\mathcal{F}_x$" to denote s_x. For example, given two pairs (U, s) and (U', s') we sometimes say "s is equal to s' in $\mathcal{F}_x$" to indicate that $s_x = s'_x$. Other authors use the terminology "germ of s at x".

An obvious consequence of this definition is that for any open $U \subset X$ there is a canonical map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

defined by $s \mapsto \prod_{x \in U}(U, s)$. Think about it!

0079 **Lemma 6.11.1.** *Let $\mathcal{F}$ be a sheaf of sets on the topological space X. For every open $U \subset X$ the map*

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Suppose that $s, s' \in \mathcal{F}(U)$ map to the same element in every stalk $\mathcal{F}_x$ for all $x \in U$. This means that for every $x \in U$, there exists an open $V^x \subset U$, $x \in V^x$ such that $s|_{V^x} = s'|_{V^x}$. But then $U = \bigcup_{x \in U} V^x$ is an open covering. Thus by the uniqueness in the sheaf condition we see that $s = s'$. □

007A **Definition 6.11.2.** Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is *separated* if for every open $U \subset X$ the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Another observation is that the construction of the stalk $\mathcal{F}_x$ is functorial in the presheaf $\mathcal{F}$. In other words, it gives a functor

$$PSh(X) \longrightarrow Sets,\ \mathcal{F} \longmapsto \mathcal{F}_x.$$

This functor is called the *stalk functor*. Namely, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ by the rule $(U, s) \mapsto (U, \varphi(s))$. To see that this works we have to check that if $(U, s) = (U', s')$ in $\mathcal{F}_x$ then also $(U, \varphi(s)) = (U', \varphi(s'))$ in $\mathcal{G}_x$. This is clear since φ is compatible with the restriction mappings.

007B **Example 6.11.3.** Let X be a topological space. Let A be a set. Denote temporarily A_p the constant presheaf with value A (p for presheaf – not for point). There is a canonical map of presheaves $A_p \to \underline{A}$ into the constant sheaf with value A. For every point we have canonical bijections $A = (A_p)_x = \underline{A}_x$, where the second map is induced by functoriality from the map $A_p \to \underline{A}$.

007C **Example 6.11.4.** Suppose $X = \mathbf{R}^n$ with the Euclidean topology. Consider the presheaf of $\mathcal{C}^\infty$ functions on X, denoted $\mathcal{C}^\infty_{\mathbf{R}^n}$. In other words, $\mathcal{C}^\infty_{\mathbf{R}^n}(U)$ is the set of $\mathcal{C}^\infty$-functions $f : U \to \mathbf{R}$. As in Example 6.7.3 it is easy to show that this is a sheaf. In fact it is a sheaf of $\mathbf{R}$-vector spaces.

Next, let $x \in X = \mathbf{R}^n$ be a point. How do we think of an element in the stalk $\mathcal{C}^\infty_{\mathbf{R}^n, x}$? Such an element is given by a $\mathcal{C}^\infty$-function f whose domain contains x. And a pair of such functions f, g determine the same element of the stalk if they agree in a neighbourhood of x. In other words, an element of $\mathcal{C}^\infty_{\mathbf{R}^n, x}$ is the same thing as what is sometimes called a *germ of a $\mathcal{C}^\infty$-function at x*.

007D **Example 6.11.5.** Let X be a topological space. Let A_x be a set for each $x \in X$. Consider the sheaf $\mathcal{F} : U \mapsto \prod_{x \in U} A_x$ of Example 6.7.5. We would just like to point out here that the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is in general *not* equal to the set A_x. Of course there is a map $\mathcal{F}_x \to A_x$, but that is in general the best you can say. For example, suppose $x = \lim x_n$ with $x_n \neq x_m$ for all $n \neq m$ and suppose that $A_y = \{0, 1\}$ for all $y \in X$. Then $\mathcal{F}_x$ maps onto the (infinite) set of tails of sequences of 0s and 1s. Namely, every open neighbourhood of x contains almost all of the x_n. On the other hand, if every neighbourhood of x contains a point y such that $A_y = \emptyset$, then $\mathcal{F}_x = \emptyset$.

6.12. Stalks of abelian presheaves

007E We first deal with the case of abelian groups as a model for the general case.

007F **Lemma 6.12.1.** *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of abelian groups on X. There exists a unique structure of an abelian group on $\mathcal{F}_x$ such that for every $U \subset X$ open, $x \in U$ the map $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism. Moreover,*

$$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$$

holds in the category of abelian groups.

Proof. We define addition of a pair of elements (U, s) and (V, t) as the pair $(U \cap V, s|_{U\cap V} + t|_{U\cap V})$. The rest is easy to check. □

What is crucial in the proof above is that the partially ordered set of open neighbourhoods is a directed set (Categories, Definition 4.21.1). Namely, the coproduct of two abelian groups A, B is the direct sum $A \oplus B$, whereas the coproduct in the category of sets is the disjoint union $A \amalg B$, showing that colimits in the category of abelian groups do not agree with colimits in the category of sets in general.

6.13. Stalks of presheaves of algebraic structures

007G The proof of Lemma 6.12.1 will work for any type of algebraic structure such that directed colimits commute with the forgetful functor.

007H **Lemma 6.13.1.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

(1) *F is faithful, and*
(2) *directed colimits exist in $\mathcal{C}$ and F commutes with them.*

Let X be a topological space. Let $x \in X$. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then

$$\mathcal{F}_x = \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

exists in $\mathcal{C}$. Its underlying set is equal to the stalk of the underlying presheaf of sets of $\mathcal{F}$. Furthermore, the construction $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor from the category of presheaves with values in $\mathcal{C}$ to $\mathcal{C}$.

Proof. Omitted. □

By the very definition, all the morphisms $\mathcal{F}(U) \to \mathcal{F}_x$ are morphisms in the category $\mathcal{C}$ which (after applying the forgetful functor F) turn into the corresponding maps for the underlying sheaf of sets. As usual we will not distinguish between the morphism in $\mathcal{C}$ and the underlying map of sets, which is permitted since F is faithful.

This lemma applies in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field.*

6.14. Stalks of presheaves of modules

007I

007J **Lemma 6.14.1.** *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. The canonical map $\mathcal{O}_x \times \mathcal{F}_x \to \mathcal{F}_x$ coming from the multiplication map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ defines a $\mathcal{O}_x$-module structure on the abelian group $\mathcal{F}_x$.*

Proof. Omitted. □

007K **Lemma 6.14.2.** *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Omitted. □

6.15. Algebraic structures

007L In this section we mildly formalize the notions we have encountered in the sections above.

007M **Definition 6.15.1.** A *type of algebraic structure* is given by a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to Sets$ with the following properties

(1) F is faithful,
(2) $\mathcal{C}$ has limits and F commutes with limits,
(3) $\mathcal{C}$ has filtered colimits and F commutes with them, and
(4) F reflects isomorphisms.

We make this definition to point out the properties we will use in a number of arguments below. But we will not actually study this notion in any great detail, since we are prohibited from studying "big" categories by convention, except for those listed in Categories, Remark 4.2.2. Among those the following have the required properties.

007N **Lemma 6.15.2.** *The following categories, endowed with the obvious forgetful functor, define types of algebraic structures:*

(1) *The category of pointed sets.*
(2) *The category of abelian groups.*
(3) *The category of groups.*
(4) *The category of monoids.*
(5) *The category of rings.*
(6) *The category of R-modules for a fixed ring R.*
(7) *The category of Lie algebras over a fixed field.*

Proof. Omitted. □

From now on we will think of a (pre)sheaf of algebraic structures and their stalks, in terms of the underlying (pre)sheaf of sets. This is allowable by Lemmas 6.9.2 and 6.13.1.

In the rest of this section we point out some results on algebraic structures that will be useful in the future.

007O **Lemma 6.15.3.** *Let $(\mathcal{C}, F)$ be a type of algebraic structure.*

(1) *$\mathcal{C}$ has a final object 0 and $F(0) = \{*\}$.*
(2) *$\mathcal{C}$ has products and $F(\prod A_i) = \prod F(A_i)$.*
(3) *$\mathcal{C}$ has fibre products and $F(A \times_B C) = F(A) \times_{F(B)} F(C)$.*
(4) *$\mathcal{C}$ has equalizers, and if $E \to A$ is the equalizer of $a, b : A \to B$, then $F(E) \to F(A)$ is the equalizer of $F(a), F(b) : F(A) \to F(B)$.*
(5) *$A \to B$ is a monomorphism if and only if $F(A) \to F(B)$ is injective.*
(6) *if $F(a) : F(A) \to F(B)$ is surjective, then a is an epimorphism.*
(7) *given $A_1 \to A_2 \to A_3 \to \ldots$, then $\operatorname{colim} A_i$ exists and $F(\operatorname{colim} A_i) = \operatorname{colim} F(A_i)$, and more generally for any filtered colimit.*

Proof. Omitted. The only interesting statement is (5) which follows because $A \to B$ is a monomorphism if and only if $A \to A \times_B A$ is an isomorphism, and then applying the fact that F reflects isomorphisms. □

007P **Lemma 6.15.4.** *Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $A, B, C \in \mathrm{Ob}(\mathcal{C})$. Let $f : A \to B$ and $g : C \to B$ be morphisms of $\mathcal{C}$. If $F(g)$ is injective, and $\mathrm{Im}(F(f)) \subset \mathrm{Im}(F(g))$, then f factors as $f = g \circ t$ for some morphism $t : A \to C$.*

Proof. Consider $A \times_B C$. The assumptions imply that $F(A \times_B C) = F(A) \times_{F(B)} F(C) = F(A)$. Hence $A = A \times_B C$ because F reflects isomorphisms. The result follows. □

007Q **Example 6.15.5.** The lemma will be applied often to the following situation. Suppose that we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$. Suppose $C \to D$ is injective on underlying sets, and suppose that the composition $A \to B \to D$ has image on underlying sets in the image of $C \to D$. Then we get a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$.

007R **Example 6.15.6.** Let $F : \mathcal{C} \to \mathit{Sets}$ be a type of algebraic structures. Let X be a topological space. Suppose that for every $x \in X$ we are given an object $A_x \in \mathrm{Ob}(\mathcal{C})$. Consider the presheaf Π with values in $\mathcal{C}$ on X defined by the rule $\Pi(U) = \prod_{x \in U} A_x$ (with obvious restriction mappings). Note that the associated presheaf of sets $U \mapsto F(\Pi(U)) = \prod_{x \in U} F(A_x)$ is a sheaf by Example 6.7.5. Hence Π is a sheaf of algebraic structures of type $(\mathcal{C}, F)$. This gives many examples of sheaves of abelian groups, groups, rings, etc.

6.16. Exactness and points

007S In any category we have the notion of epimorphism, monomorphism, isomorphism, etc.

007T **Lemma 6.16.1.** *Let X be a topological space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.*

1. *The map φ is a monomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.*
2. *The map φ is an epimorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.*
3. *The map φ is an isomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective.*

Proof. Omitted. □

It follows that in the category of sheaves of sets the notions epimorphism and monomorphism can be described as follows.

007U **Definition 6.16.2.** Let X be a topological space.

(1) A presheaf $\mathcal{F}$ is called a *subpresheaf* of a presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open $U \subset X$ such that the restriction maps of $\mathcal{G}$ induce the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$. We sometimes indicate this by the notation $\mathcal{F} \subset \mathcal{G}$.
(2) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(3) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U open in X.
(4) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(5) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if for every open U of X and every section s of $\mathcal{G}(U)$ there exists an open covering $U = \bigcup U_i$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ for all i.

0H7H **Lemma 6.16.3.** *Let X be a topological space.*

(1) *Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.*
(2) *Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps which are surjective (resp. injective) on all the stalks.*

Proof. Omitted. □

007W **Lemma 6.16.4.** *let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $\mathcal{F}$, $\mathcal{G}$ are sheaves on X with values in $\mathcal{C}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of the underlying sheaves of sets. If for all points $x \in X$ the map $\mathcal{F}_x \to \mathcal{G}_x$ is a morphism of algebraic structures, then φ is a morphism of sheaves of algebraic structures.*

Proof. Let U be an open subset of X. Consider the diagram of (underlying) sets

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \prod_{x \in U} \mathcal{F}_x \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

By assumption, and previous results, all but the left vertical arrow are morphisms of algebraic structures. In addition the bottom horizontal arrow is injective, see Lemma 6.11.1. Hence we conclude by Lemma 6.15.4, see also Example 6.15.5 □

Short exact sequences of abelian sheaves, etc will be discussed in the chapter on sheaves of modules. See Modules, Section 17.3.

6.17. Sheafification

007X In this section we explain how to get the sheafification of a presheaf on a topological space. We will use stalks to describe the sheafification in this case. This is different from the general procedure described in Sites, Section 7.10, and perhaps somewhat easier to understand.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. For every open $U \subset X$ we define

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

(*) For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

Note that $(*)$ is a condition for each $u \in U$, and that given $u \in U$ the truth of this condition depends only on the values s_v for v in any open neighbourhood of u. Thus it is clear that, if $V \subset U \subset X$ are open, the projection maps

$$\prod_{u \in U} \mathcal{F}_u \longrightarrow \prod_{v \in V} \mathcal{F}_v$$

maps elements of $\mathcal{F}^\#(U)$ into $\mathcal{F}^\#(V)$. Using these maps as the restriction mappings, we turn $\mathcal{F}^\#$ into a presheaf of sets on X.

Furthermore, the map $\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$ described in Section 6.11 clearly has image in $\mathcal{F}^\#(U)$. In addition, if $V \subset U \subset X$ are open then we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}^\#(U) & \longrightarrow & \prod_{u \in U} \mathcal{F}_u \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}^\#(V) & \longrightarrow & \prod_{v \in V} \mathcal{F}_v \end{array}$$

where the vertical maps are induced from the restriction mappings. Thus we see that there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^\#$.

In Example 6.7.5 we saw that the rule $\Pi(\mathcal{F}) : U \mapsto \prod_{u \in U} \mathcal{F}_u$ is a sheaf, with obvious restriction mappings. And by construction $\mathcal{F}^\#$ is a subpresheaf of this. In other words, we have morphisms of presheaves

$$\mathcal{F} \to \mathcal{F}^\# \to \Pi(\mathcal{F}).$$

In addition the rule that associates to $\mathcal{F}$ the sequence above is clearly functorial in the presheaf $\mathcal{F}$. This notation will be used in the proofs of the lemmas below.

007Y **Lemma 6.17.1.** *The presheaf $\mathcal{F}^\#$ is a sheaf.*

Proof. It is probably better for the reader to find their own explanation of this than to read the proof here. In fact the lemma is true for the same reason as why the presheaf of continuous function is a sheaf, see Example 6.7.3 (and this analogy can be made precise using the "espace étalé").

Anyway, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i = (s_{i,u})_{u \in U_i} \in \mathcal{F}^\#(U_i)$ such that s_i and s_j agree over $U_i \cap U_j$. Because $\Pi(\mathcal{F})$ is a sheaf, we find an element $s = (s_u)_{u \in U}$ in $\prod_{u \in U} \mathcal{F}_u$ restricting to s_i on U_i. We have to check property $(*)$. Pick $u \in U$. Then $u \in U_i$ for some i. Hence by $(*)$ for s_i, there exists a V open, $u \in V \subset U_i$ and a $\sigma \in \mathcal{F}(V)$ such that $s_{i,v} = (V, \sigma)$ in $\mathcal{F}_v$ for all $v \in V$. Since $s_{i,v} = s_v$ we get $(*)$ for s. □

007Z **Lemma 6.17.2.** *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $x \in X$. Then $\mathcal{F}_x = \mathcal{F}^\#_x$.*

Proof. The map $\mathcal{F}_x \to \mathcal{F}_x^\#$ is injective, since already the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ is injective. Namely, there is a canonical map $\Pi(\mathcal{F})_x \to \mathcal{F}_x$ which is a left inverse to the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$, see Example 6.11.5. To show that it is surjective, suppose that $\overline{s} \in \mathcal{F}_x^\#$. We can find an open neighbourhood U of x such that $\overline{s}$ is the equivalence class of (U, s) with $s \in \mathcal{F}^\#(U)$. By definition, this means there exists an open neighbourhood $V \subset U$ of x and a section $\sigma \in \mathcal{F}(V)$ such that $s|_V$ is the image of σ in $\Pi(\mathcal{F})(V)$. Clearly the class of (V, σ) defines an element of $\mathcal{F}_x$ mapping to $\overline{s}$. □

0080 **Lemma 6.17.3.** *Let $\mathcal{F}$ be a presheaf of sets on X. Any map $\mathcal{F} \to \mathcal{G}$ into a sheaf of sets factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. Clearly, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^\# & \longrightarrow & \Pi(\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^\# & \longrightarrow & \Pi(\mathcal{G}) \end{array}$$

So it suffices to prove that $\mathcal{G} = \mathcal{G}^\#$. To see this it suffices to prove, for every point $x \in X$ the map $\mathcal{G}_x \to \mathcal{G}_x^\#$ is bijective, by Lemma 6.16.1. And this is Lemma 6.17.2 above. □

This lemma really says that there is an adjoint pair of functors: $i : Sh(X) \to PSh(X)$ (inclusion) and $\# : PSh(X) \to Sh(X)$ (sheafification). The formula is that

$$\mathrm{Mor}_{PSh(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$$

which says that sheafification is a left adjoint of the inclusion functor. See Categories, Section 4.24.

0081 **Example 6.17.4.** See Example 6.11.3 for notation. The map $A_p \to \underline{A}$ induces a map $A_p^\# \to \underline{A}$. It is easy to see that this is an isomorphism. In words: The sheafification of the constant presheaf with value A is the constant sheaf with value A.

0082 **Lemma 6.17.5.** *Let X be a topological space. A presheaf $\mathcal{F}$ is separated (see Definition 6.11.2) if and only if the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is injective.*

Proof. This is clear from the construction of $\mathcal{F}^\#$ in this section. □

0H7I **Lemma 6.17.6.** *Let X be a topological space. The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).*

Proof. Omitted. □

6.18. Sheafification of abelian presheaves

0083 The following strange looking lemma is likely unnecessary, but very convenient to deal with sheafification of presheaves of algebraic structures.

0084 **Lemma 6.18.1.** *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $U \subset X$ be open. There is a canonical fibre product diagram*

$$\begin{array}{ccc} \mathcal{F}^\#(U) & \longrightarrow & \Pi(\mathcal{F})(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \Pi(\mathcal{F})_x \end{array}$$

where the maps are the following:

1. *The left vertical map has components $\mathcal{F}^\#(U) \to \mathcal{F}^\#_x = \mathcal{F}_x$ where the equality is Lemma 6.17.2.*
2. *The top horizontal map comes from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 6.17.*
3. *The right vertical map has obvious component maps $\Pi(\mathcal{F})(U) \to \Pi(\mathcal{F})_x$.*
4. *The bottom horizontal map has components $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ which come from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 6.17.*

Proof. It is clear that the diagram commutes. We have to show it is a fibre product diagram. The bottom horizontal arrow is injective since all the maps $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ are injective (see beginning proof of Lemma 6.17.2). A section $s \in \Pi(\mathcal{F})(U)$ is in $\mathcal{F}^\#$ if and only if $(*)$ holds. But $(*)$ says that around every point the section s comes from a section of $\mathcal{F}$. By definition of the stalk functors, this is equivalent to saying that the value of s in every stalk $\Pi(\mathcal{F})_x$ comes from an element of the stalk $\mathcal{F}_x$. Hence the lemma. □

0085 **Lemma 6.18.2.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Then there exists a unique structure of abelian sheaf on $\mathcal{F}^\#$ such that $\mathcal{F} \to \mathcal{F}^\#$ is a morphism of abelian presheaves. Moreover, the following adjointness property holds*

$$\mathrm{Mor}_{PAb(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}^\#, \mathcal{G}).$$

Proof. Recall the sheaf of sets $\Pi(\mathcal{F})$ defined in Section 6.17. All the stalks $\mathcal{F}_x$ are abelian groups, see Lemma 6.12.1. Hence $\Pi(\mathcal{F})$ is a sheaf of abelian groups by Example 6.15.6. Also, it is clear that the map $\mathcal{F} \to \Pi(\mathcal{F})$ is a morphism of abelian presheaves. If we show that condition $(*)$ of Section 6.17 defines a subgroup of $\Pi(\mathcal{F})(U)$ for all open subsets $U \subset X$, then $\mathcal{F}^\#$ canonically inherits the structure of abelian sheaf. This is quite easy to do by hand, and we leave it to the reader to find a good simple argument. The argument we use here, which generalizes to presheaves of algebraic structures is the following: Lemma 6.18.1 show that $\mathcal{F}^\#(U)$ is the fibre product of a diagram of abelian groups. Thus $\mathcal{F}^\#$ is an abelian subgroup as desired.

Note that at this point $\mathcal{F}^\#_x$ is an abelian group by Lemma 6.12.1 and that $\mathcal{F}_x \to \mathcal{F}^\#_x$ is a bijection (Lemma 6.17.2) and a homomorphism of abelian groups. Hence $\mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism of abelian groups. This will be used below without further mention.

To prove the adjointness property we use the adjointness property of sheafification of presheaves of sets. For example if $\psi : \mathcal{F} \to i(\mathcal{G})$ is morphism of presheaves then we obtain a morphism of sheaves $\psi' : \mathcal{F}^\# \to \mathcal{G}$. What we have to do is to check that this is a morphism of abelian sheaves. We may do this for example by noting that it is true on stalks, by Lemma 6.17.2, and then using Lemma 6.16.4 above. □

6.19. Sheafification of presheaves of algebraic structures

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0087 **Lemma 6.19.1.** *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$ on X. Then there exists a sheaf $\mathcal{F}^\#$ with values in $\mathcal{C}$ and a morphism $\mathcal{F} \to \mathcal{F}^\#$ of presheaves with values in $\mathcal{C}$ with the following properties:*

(1) *The map $\mathcal{F} \to \mathcal{F}^\#$ identifies the underlying sheaf of sets of $\mathcal{F}^\#$ with the sheafification of the underlying presheaf of sets of $\mathcal{F}$.*
(2) *For any morphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf with values in $\mathcal{C}$ there exists a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. The proof is the same as the proof of Lemma 6.18.2, with repeated application of Lemma 6.15.4 (see also Example 6.15.5). The main idea however, is to define $\mathcal{F}^\#(U)$ as the fibre product in $\mathcal{C}$ of the diagram

$$\begin{array}{ccc} & & \Pi(\mathcal{F})(U) \\ & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \Pi(\mathcal{F})_x \end{array}$$

compare Lemma 6.18.1. □

6.20. Sheafification of presheaves of modules

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0089 **Lemma 6.20.1.** *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}^\# \times \mathcal{F}^\# & \longrightarrow & \mathcal{F}^\# \end{array}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : Mod(\mathcal{O}^\#) \to PMod(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : PMod(\mathcal{O}) \to Mod(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{Mod(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. In Section 6.6 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

008A **Lemma 6.20.2.** *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 6.6.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

008B **Lemma 6.20.3.** *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Follows directly from Lemma 6.14.2 and the fact that taking stalks commutes with sheafification. □

6.21. Continuous maps and sheaves

008C Let $f : X \to Y$ be a continuous map of topological spaces. We will define the pushforward and pullback functors for presheaves and sheaves.

Let $\mathcal{F}$ be a presheaf of sets on X. We define the *pushforward* of $\mathcal{F}$ by the rule

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subset Y$. Given $V_1 \subset V_2 \subset Y$ open the restriction map is given by the commutativity of the diagram

$$\begin{array}{ccc} f_*\mathcal{F}(V_2) & = & \mathcal{F}(f^{-1}(V_2)) \\ \downarrow & & \downarrow \text{restriction for } \mathcal{F} \\ f_*\mathcal{F}(V_1) & = & \mathcal{F}(f^{-1}(V_1)) \end{array}$$

It is clear that this defines a presheaf of sets. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* : PSh(X) \longrightarrow PSh(Y).$$

008D **Lemma 6.21.1.** *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. This immediately follows from the fact that if $V = \bigcup V_j$ is an open covering in Y, then $f^{-1}(V) = \bigcup f^{-1}(V_j)$ is an open covering in X. □

As a consequence we obtain a functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

This is compatible with composition in the following strong sense.

008E **Lemma 6.21.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal (on both presheaves and sheaves of sets).*

Proof. This is because $(g \circ f)_*\mathcal{F}(W) = \mathcal{F}((g \circ f)^{-1}W)$ and $(g_* \circ f_*)\mathcal{F}(W) = \mathcal{F}(f^{-1}g^{-1}W)$ and $(g \circ f)^{-1}W = f^{-1}g^{-1}W$. □

Let $\mathcal{G}$ be a presheaf of sets on Y. The *pullback presheaf* $f_p\mathcal{G}$ of a given presheaf $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words it should be a presheaf $f_p\mathcal{G}$ on X such that

$$\mathrm{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

By the Yoneda lemma this determines the pullback uniquely. It turns out that it actually exists.

008F **Lemma 6.21.3.** *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V} \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Proof. The colimit is over the partially ordered set consisting of open subsets $V \subset Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, since if V, V' are in it then so is $V \cap V'$. Furthermore, if $U_1 \subset U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f_p\mathcal{G}(U_2)$ is a subsystem of the one defining $f_p\mathcal{G}(U_1)$ and we obtain a restriction map (for example by applying the generalities in Categories, Lemma 4.14.8).

Note that the construction of the colimit is clearly functorial in $\mathcal{G}$, and similarly for the restriction mappings. Hence we have defined f_p as a functor.

A small useful remark is that there exists a canonical map $\mathcal{G}(U) \to f_p\mathcal{G}(f^{-1}(U))$, because the system of open neighbourhoods of $f(f^{-1}(U))$ contains the element U. This is compatible with restriction mappings. In other words, there is a canonical map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$.

Let $\mathcal{F}$ be a presheaf of sets on X. Suppose that $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of presheaves of sets. The corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$.

Another small useful remark is that there exists a canonical map $c_{\mathcal{F}} : f_pf_*\mathcal{F} \to \mathcal{F}$. Namely, let $U \subset X$ open. For every open neighbourhood $V \supset f(U)$ in Y there exists a map $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$, namely the restriction map on $\mathcal{F}$. And this is compatible with the restriction mappings between values of $\mathcal{F}$ on f^{-1} of varying

opens containing $f(U)$. Thus we obtain a canonical map $f_p f_* \mathcal{F}(U) \to \mathcal{F}(U)$. Another trivial verification shows that these maps are compatible with restriction maps and define a map $c_\mathcal{F}$ of presheaves of sets.

Suppose that $\varphi : \mathcal{G} \to f_*\mathcal{F}$ is a map of presheaves of sets. Consider $f_p\varphi : f_p\mathcal{G} \to f_p f_*\mathcal{F}$. Postcomposing with $c_\mathcal{F}$ gives the desired map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$. We omit the verification that this construction is inverse to the construction in the other direction given above. □

008G **Lemma 6.21.4.** *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathcal{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This you can see as follows

$$\begin{aligned}(f_p\mathcal{G})_x &= \operatorname{colim}_{x\in U} f_p\mathcal{G}(U) \\ &= \operatorname{colim}_{x\in U} \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(x)\in V} \mathcal{G}(V) \\ &= \mathcal{G}_{f(x)}\end{aligned}$$

Here we have used Categories, Lemma 4.14.10, and the fact that any V open in Y containing $f(x)$ occurs in the third description above. Details omitted. □

Let $\mathcal{G}$ be a sheaf of sets on Y. The *pullback sheaf* $f^{-1}\mathcal{G}$ is defined by the formula

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#.$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves. In other words,

$$\operatorname{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Namely, we have

$$\begin{aligned}\operatorname{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \operatorname{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \operatorname{Mor}_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \operatorname{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})\end{aligned}$$

For the first equality we use that sheafification is a left adjoint to the inclusion of sheaves in presheaves. For the second equality we use that f_p is a left adjoint to f_* on presheaves. We will return to this statement in the proof of Lemma 6.21.8.

008H **Lemma 6.21.5.** *Let $x \in X$. Let $\mathcal{G}$ be a sheaf of sets on Y. There is a canonical bijection of stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This is a combination of Lemmas 6.17.2 and 6.21.4. □

008I **Lemma 6.21.6.** *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)_p \cong f_p \circ g_p$ on presheaves.*

Proof. To see this use that adjoint functors are unique up to unique isomorphism, and Lemma 6.21.2. □

008J **Definition 6.21.7.** Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. An *f-map* $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of

maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subset Y$ such that

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow[\xi_V]{} & \mathcal{F}(f^{-1}V) \\ \text{restriction of } \mathcal{G} \big\downarrow & & \big\downarrow \text{restriction of } \mathcal{F} \\ \mathcal{G}(V') & \xrightarrow{\xi_{V'}} & \mathcal{F}(f^{-1}V') \end{array}$$

commutes for all $V' \subset V \subset Y$ open.

In the literature we sometimes find this defined alternatively as in part (4) of Lemma 6.21.8 but as the lemma shows there is really no difference.

008K **Lemma 6.21.8.** *Let $f : X \to Y$ be a continuous map. There are bijections between the following four sets*

(1) *the set of maps $\mathcal{G} \to f_*\mathcal{F}$,*
(2) *the set of maps $f^{-1}\mathcal{G} \to \mathcal{F}$,*
(3) *the set of f-maps $\xi : \mathcal{G} \to \mathcal{F}$, and*
(4) *the set of all collections of maps $\xi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all $U \subset X$ and $V \subset Y$ open such that $f(U) \subset V$ compatible with all restriction maps,*

functorially in $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$.

Proof. A map of sheaves $a : \mathcal{G} \to f_*\mathcal{F}$ is by definition a rule which to each open V of Y assigns a map $a_V : \mathcal{G}(V) \to f_*\mathcal{F}(V)$ and we have $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Thus at least the "data" corresponds exactly to what you need for an f-map ξ from $\mathcal{G}$ to $\mathcal{F}$. To show that the sets (1) and (3) are in bijection we observe that a is a map of sheaves if and only if corresponding family of maps a_V satisfy the condition in Definition 6.21.7.

Recall that $f^{-1}\mathcal{G}$ is the sheafification of $f_p\mathcal{G}$. By the universal property of sheafification a map of sheaves $b : f^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map of presheaves $b_p : f_p\mathcal{G} \to \mathcal{F}$ where f_p is the functor defined earlier in the section. To give such a map b_p you need to specify for each open U of X a map

$$b_{p,U} : \operatorname{colim}_{f(U) \subset V} \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. We may and do view $b_{p,U}$ as a collection of maps $b_{p,U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all V open in Y with $f(U) \subset V$. These maps have to be compatible with all possible restriction mappings you can think of. In other words, we see that b_p corresponds to a collection of maps as in (4). Of course, conversely such a collection defines a map b_p and in turn a map $b : f^{-1}\mathcal{G} \to \mathcal{F}$.

To finish the proof of the lemma you have to show that by "forgetting structure" the rule that to a collection $\xi_{U,V}$ as in (4) associates the f-map ξ with $\xi_V = \xi_{f^{-1}(V),V}$ is bijective. To do this, if ξ is a usual f-map then we just define $\tilde{\xi}_{U,V}$ to be the composition of $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ by the restruction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ which makes sense exactly because $f(U) \subset V$, i.e., $U \subset f^{-1}(V)$. This finishes the proof. □

It is sometimes convenient to think about f-maps instead of maps between sheaves either on X or on Y. We define composition of f-maps as follows.

008L **Definition 6.21.9.** Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. The

composition of φ and ψ is the $(g \circ f)$-map $\varphi \circ \psi$ defined by the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{H}(W) & \xrightarrow{(\varphi\circ\psi)_W} & \mathcal{F}(f^{-1}g^{-1}W) \\ & {\scriptstyle\psi_W}\searrow \quad \nearrow{\scriptstyle\varphi_{g^{-1}W}} & \\ & \mathcal{G}(g^{-1}W) & \end{array}$$

We leave it to the reader to verify that this works. Another way to think about this is to think of $\varphi \circ \psi$ as the composition

$$\mathcal{H} \xrightarrow{\psi} g_*\mathcal{G} \xrightarrow{g_*\varphi} g_*f_*\mathcal{F} = (g \circ f)_*\mathcal{F}$$

Now, doesn't it seem that thinking about f-maps is somehow easier?

Finally, given a continuous map $f : X \to Y$, and an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ there is a natural map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

for all $x \in X$. The image of a representative (V, s) of an element in $\mathcal{G}_{f(x)}$ is mapped to the element in $\mathcal{F}_x$ with representative $(f^{-1}V, \varphi_V(s))$. We leave it to the reader to see that this is well defined. Another way to state it is that it is the unique map such that all diagrams

$$\begin{array}{ccc} \mathcal{F}(f^{-1}V) & \longrightarrow & \mathcal{F}_x \\ {\scriptstyle\varphi_V}\uparrow & & \uparrow{\scriptstyle\varphi_x} \\ \mathcal{G}(V) & \longrightarrow & \mathcal{G}_{f(x)} \end{array}$$

(for $f(x) \in V \subset Y$ open) commute.

008M **Lemma 6.21.10.** *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. Let $x \in X$ be a point. The map on stalks $(\varphi \circ \psi)_x : \mathcal{H}_{g(f(x))} \to \mathcal{F}_x$ is the composition*

$$\mathcal{H}_{g(f(x))} \xrightarrow{\psi_{f(x)}} \mathcal{G}_{f(x)} \xrightarrow{\varphi_x} \mathcal{F}_x$$

Proof. Immediate from Definition 6.21.9 and the definition of the map on stalks above. □

6.22. Continuous maps and abelian sheaves

008N Let $f : X \to Y$ be a continuous map. We claim there are functors

$$\begin{aligned} f_* : PAb(X) &\longrightarrow PAb(Y) \\ f_* : Ab(X) &\longrightarrow Ab(Y) \\ f_p : PAb(Y) &\longrightarrow PAb(X) \\ f^{-1} : Ab(Y) &\longrightarrow Ab(X) \end{aligned}$$

with similar properties to their counterparts in Section 6.21. To see this we argue in the following way.

Each of the functors will be constructed in the same way as the corresponding functor in Section 6.21. This works because all the colimits in that section are directed colimits (but we will work through it below).

First off, given an abelian presheaf $\mathcal{F}$ on X and an abelian presheaf $\mathcal{G}$ on Y we define

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \end{aligned}$$

as abelian groups. The restriction mappings are the same as the restriction mappings for presheaves of sets (and they are all homomorphisms of abelian groups).

The assignments $\mathcal{F} \mapsto f_*\mathcal{F}$ and $\mathcal{G} \to f_p\mathcal{G}$ are functors on the categories of presheaves of abelian groups. This is clear, as (for example) a map of abelian presheaves $\mathcal{G}_1 \to \mathcal{G}_2$ gives rise to a map of directed systems $\{\mathcal{G}_1(V)\}_{f(U)\subset V} \to \{\mathcal{G}_2(V)\}_{f(U)\subset V}$ all of whose maps are homomorphisms and hence gives rise to a homomorphism of abelian groups $f_p\mathcal{G}_1(U) \to f_p\mathcal{G}_2(U)$.

The functors f_* and f_p are adjoint on the category of presheaves of abelian groups, i.e., we have

$$\operatorname{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

To prove this, note that the map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves. Hence if $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves, then the corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$ is also a map of abelian presheaves. For the other direction we point out that the map $c_{\mathcal{F}} : f_pf_*\mathcal{F} \to \mathcal{F}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves as well (since it is made out of restriction mappings of $\mathcal{F}$ which are all homomorphisms). Hence given a map of abelian presheaves $\varphi : \mathcal{G} \to f_*\mathcal{F}$ the map $c_{\mathcal{F}} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves as well. Since these constructions $\psi \mapsto f_*\psi$ and $\varphi \mapsto c_{\mathcal{F}} \circ f_p\varphi$ are inverse to each other as constructions on maps of presheaves of sets we see they are also inverse to each other on maps of abelian presheaves.

If $\mathcal{F}$ is an abelian sheaf on Y, then $f_*\mathcal{F}$ is an abelian sheaf on X. This is true because of the definition of an abelian sheaf and because this is true for sheaves of sets, see Lemma 6.21.1. This defines the functor f_* on the category of abelian sheaves.

We define $f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#$ as before. Adjointness of f_* and f^{-1} follows formally as in the case of presheaves of sets. Here is the argument:

$$\begin{aligned} \operatorname{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \operatorname{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \operatorname{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \operatorname{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

008O **Lemma 6.22.1.** *Let $f : X \to Y$ be a continuous map.*

1. *Let $\mathcal{G}$ be an abelian presheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f_p\mathcal{G})_x$ of Lemma 6.21.4 is an isomorphism of abelian groups.*
2. *Let $\mathcal{G}$ be an abelian sheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$ of Lemma 6.21.5 is an isomorphism of abelian groups.*

Proof. Omitted. □

Given a continuous map $f : X \to Y$ and sheaves of abelian groups $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of abelian groups* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with

homomorphisms of abelian groups) or we can simply say that it is the same as a map of abelian sheaves $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The group of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the groups $\mathrm{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are abelian group homomorphisms.

6.23. Continuous maps and sheaves of algebraic structures

008P Let $(\mathcal{C}, F)$ be a type of algebraic structure. For a topological space X let us introduce the notation:

(1) $PSh(X, \mathcal{C})$ will be the category of presheaves with values in $\mathcal{C}$.
(2) $Sh(X, \mathcal{C})$ will be the category of sheaves with values in $\mathcal{C}$.

Let $f : X \to Y$ be a continuous map of topological spaces. The same arguments as in the previous section show there are functors

$$\begin{aligned} f_* : PSh(X, \mathcal{C}) &\longrightarrow PSh(Y, \mathcal{C}) \\ f_* : Sh(X, \mathcal{C}) &\longrightarrow Sh(Y, \mathcal{C}) \\ f_p : PSh(Y, \mathcal{C}) &\longrightarrow PSh(X, \mathcal{C}) \\ f^{-1} : Sh(Y, \mathcal{C}) &\longrightarrow Sh(X, \mathcal{C}) \end{aligned}$$

constructed in the same manner and with the same properties as the functors constructed for abelian (pre)sheaves. In particular there are commutative diagrams

$$\begin{array}{ccc} PSh(X, \mathcal{C}) & \xrightarrow{f_*} & PSh(Y, \mathcal{C}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ PSh(X) & \xrightarrow{f_*} & PSh(Y) \end{array} \qquad \begin{array}{ccc} Sh(X, \mathcal{C}) & \xrightarrow{f_*} & Sh(Y, \mathcal{C}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ Sh(X) & \xrightarrow{f_*} & Sh(Y) \end{array}$$

$$\begin{array}{ccc} PSh(Y, \mathcal{C}) & \xrightarrow{f_p} & PSh(X, \mathcal{C}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ PSh(Y) & \xrightarrow{f_p} & PSh(X) \end{array} \qquad \begin{array}{ccc} Sh(Y, \mathcal{C}) & \xrightarrow{f^{-1}} & Sh(X, \mathcal{C}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ Sh(Y) & \xrightarrow{f^{-1}} & Sh(X) \end{array}$$

The main formulas to keep in mind are the following

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \mathrm{colim}_{f(U) \subset V} \mathcal{G}(V) \\ f^{-1}\mathcal{G} &= (f_p\mathcal{G})^\# \\ (f_p\mathcal{G})_x &= \mathcal{G}_{f(x)} \\ (f^{-1}\mathcal{G})_x &= \mathcal{G}_{f(x)} \end{aligned}$$

Each of these formulas has the property that they hold in the category $\mathcal{C}$ and that upon taking underlying sets we get the corresponding formula for presheaves of sets. In addition we have the adjointness properties

$$\begin{aligned} \mathrm{Mor}_{PSh(X,\mathcal{C})}(f_p\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{PSh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) \\ \mathrm{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

To prove these, the main step is to construct the maps

$$i_{\mathcal{G}} : \mathcal{G} \longrightarrow f_*f_p\mathcal{G}$$

and

$$c_{\mathcal{F}} : f_pf_*\mathcal{F} \longrightarrow \mathcal{F}$$

which occur in the proof of Lemma 6.21.3 as morphisms of presheaves with values in $\mathcal{C}$. This may be safely left to the reader since the constructions are exactly the same as in the case of presheaves of sets.

Given a continuous map $f : X \to Y$ and sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of algebraic structures* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with morphisms in $\mathcal{C}$) or we can simply say that it is the same as a map of sheaves of algebraic structures $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are homomorphisms of algebraic structures.

008Q **Lemma 6.23.1.** *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose given sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map of underlying sheaves of sets. If for every $V \subset Y$ open the map of sets $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ is the effect of a morphism in $\mathcal{C}$ on underlying sets, then φ comes from a unique f-morphism between sheaves of algebraic structures.*

Proof. Omitted. □

6.24. Continuous maps and sheaves of modules

008R The case of sheaves of modules is more complicated. The reason is that the natural setting for defining the pullback and pushforward functors, is the setting of ringed spaces, which we will define below. First we state a few obvious lemmas.

008S **Lemma 6.24.1.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subset Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}V) \times \mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_*\mathcal{O}$-module on $f_*\mathcal{F}$. □

008T **Lemma 6.24.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_p\mathcal{O} \times f_p\mathcal{G} \longrightarrow f_p\mathcal{G}$$

which turns $f_p\mathcal{G}$ into a presheaf of $f_p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subset X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f_p\mathcal{O}(U) \times f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{O}(V) \times \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(U)\subset V} (\mathcal{O}(V) \times \mathcal{G}(V)) \\ &\to \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= f_p\mathcal{G}(U). \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits, see Categories, Lemma 4.19.2. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_p\mathcal{O}$-module on $f_p\mathcal{G}$. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : PMod(\mathcal{O}_X) &\longrightarrow PMod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) &\longrightarrow PMod(f_p\mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

008U **Lemma 6.24.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f_p\mathcal{O}$-modules. Then*

$$\operatorname{Mor}_{PMod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{PMod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $i_{\mathcal{O}} : \mathcal{O} \to f_*f_p\mathcal{O}$ (defined first in the proof of Lemma 6.21.3).

Proof. Note that we have

$$\operatorname{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

according to Section 6.22. So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$(\psi : f_p\mathcal{G} \to \mathcal{F}) \longmapsto (f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*\mathcal{F})$$

and in the other direction by the rule

$$(\varphi : \mathcal{G} \to f_*\mathcal{F}) \longmapsto (c_{\mathcal{F}} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F})$$

where $i_{\mathcal{G}}$ and $c_{\mathcal{F}}$ are as in Section 6.22. Hence, using the functoriality of f_* and f_p we see that it suffices to check that the maps $i_{\mathcal{G}} : \mathcal{G} \to f_* f_p \mathcal{G}$ and $c_{\mathcal{F}} : f_p f_* \mathcal{F} \to \mathcal{F}$ are compatible with module structures, which we leave to the reader. □

008V **Lemma 6.24.4.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p, f_p f_* \mathcal{O}} f_p \mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(f_* \mathcal{O})}(\mathcal{G}, f_* \mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we use the map $c_{\mathcal{O}} : f_p f_* \mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned} \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p, f_p f_* \mathcal{O}} f_p \mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{PMod(f_p f_* \mathcal{O})}(f_p \mathcal{G}, \mathcal{F}_{f_p f_* \mathcal{O}}) \\ &= \mathrm{Mor}_{PMod(f_* \mathcal{O})}(\mathcal{G}, f_*(\mathcal{F}_{f_p f_* \mathcal{O}})) \\ &= \mathrm{Mor}_{PMod(f_* \mathcal{O})}(\mathcal{G}, f_* \mathcal{F}). \end{aligned}$$

The first equality is Lemma 6.6.2. The second equality is Lemma 6.24.3. The third equality is given by the equality $f_*(\mathcal{F}_{f_p f_* \mathcal{O}}) = f_* \mathcal{F}$ of abelian sheaves which is $f_* \mathcal{O}$-linear. Namely, $\mathrm{id}_{f_* \mathcal{O}}$ corresponds to $c_{\mathcal{O}}$ under the adjunction described in the proof of Lemma 6.21.3 and thus $\mathrm{id}_{f_* \mathcal{O}} = f_* c_{\mathcal{O}} \circ i_{f_* \mathcal{O}}$. □

008W **Lemma 6.24.5.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_* \mathcal{F}$, as defined in Lemma 6.24.1 is a sheaf of $f_* \mathcal{O}$-modules.*

Proof. Obvious from the definition and Lemma 6.21.1. □

008X **Lemma 6.24.6.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f_p and sheafification. Thus the lemma is a combination of Lemma 6.24.2 and Lemma 6.20.1. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$\begin{aligned} f_* : Mod(\mathcal{O}_X) &\longrightarrow Mod(f_* \mathcal{O}_X) \\ f^{-1} : Mod(\mathcal{O}_Y) &\longrightarrow Mod(f^{-1} \mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

008Y **Lemma 6.24.7.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_* \mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we think of $f_* \mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_* f^{-1} \mathcal{O}$.

Proof. Argue by the equalities

$$\begin{aligned} \mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

where the second is Lemmas 6.24.3 and the first is by Lemma 6.20.1. □

008Z **Lemma 6.24.8.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned} \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ &= \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

which are a combination of Lemma 6.20.2 and 6.24.7. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a (pre)sheaf of rings on X and let $\mathcal{O}_Y$ be a (pre)sheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : PMod(\mathcal{O}_X) &\longrightarrow PMod(f_*\mathcal{O}_X) \\ f_* : Mod(\mathcal{O}_X) &\longrightarrow Mod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) &\longrightarrow PMod(f_p\mathcal{O}_Y) \\ f^{-1} : Mod(\mathcal{O}_Y) &\longrightarrow Mod(f^{-1}\mathcal{O}_Y) \end{aligned}$$

Clearly, usually the pair of functors (f_*, f^{-1}) on sheaves of modules are not adjoint, because their target categories do not match. Namely, as we saw above, it works only if by some miracle the sheaves of rings $\mathcal{O}_X, \mathcal{O}_Y$ satisfy the relations $\mathcal{O}_X = f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$. This is almost never true in practice. We interrupt the discussion to define the correct notion of morphism for which a suitable adjoint pair of functors on sheaves of modules exists.

6.25. Ringed spaces

0090 Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We are supposed to think of the sheaf of rings $\mathcal{O}_X$ as a sheaf of functions on X. And if $f : X \to Y$ is a "suitable" map, then by composition a function on Y turns into a function on X. Thus there should be a natural f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$, see Definition 6.21.7 and Lemma 6.21.8. For a precise example, see Example 6.25.2 below. Here is the relevant abstract definition.

0091 **Definition 6.25.1.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$.

0092 **Example 6.25.2.** Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions $\mathcal{C}^0_X$ on X and $\mathcal{C}^0_Y$ on Y, see

Example 6.9.3. We claim that there is a natural f-map $f^\sharp : \mathcal{C}^0_Y \to \mathcal{C}^0_X$ associated to f. Namely, we simply define it by the rule

$$\begin{array}{ccc} \mathcal{C}^0_Y(V) & \longrightarrow & \mathcal{C}^0_X(f^{-1}V) \\ h & \longmapsto & h \circ f \end{array}$$

Strictly speaking we should write $f^\sharp(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is a family of maps as in Definition 6.21.7 and compatible with the $\mathbf{R}$-algebra structures. Hence it is an f-map of sheaves of $\mathbf{R}$-algebras, see Lemma 6.23.1.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are $\mathcal{C}^\infty$-manifolds and $f : M \to N$ is an infinitely differentiable map, then f induces a canonical morphism of ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$. The construction (which is identical to the above) is left to the reader.

It may not be completely obvious how to compose morphisms of ringed spaces hence we spell it out here.

0093 **Definition 6.25.3.** Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then we define the *composition of morphisms of ringed spaces* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of f-maps defined in Definition 6.21.9.

6.26. Morphisms of ringed spaces and modules

0094 We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed spaces.

0095 **Definition 6.26.1.** Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of the module structure given in Lemma 6.24.5.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_X$-modules defined by the formula

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

where the ring map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the map corresponding to $f^\sharp$, and where the module structure is given by Lemma 6.24.6.

Thus we have defined functors

$$\begin{array}{ccc} f_* : \mathit{Mod}(\mathcal{O}_X) & \longrightarrow & \mathit{Mod}(\mathcal{O}_Y) \\ f^* : \mathit{Mod}(\mathcal{O}_Y) & \longrightarrow & \mathit{Mod}(\mathcal{O}_X) \end{array}$$

The final result on these functors is that they are indeed adjoint as expected.

0096 **Lemma 6.26.2.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(\mathcal{O}_X)}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(f^{-1}\mathcal{O}_Y)}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_Y}) \\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 6.20.2 and 6.24.7. □

0097 **Lemma 6.26.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal. There is a canonical isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. The result on pushforwards is a consequence of Lemma 6.21.2 and our definitions. The result on pullbacks follows from this by the same argument as in the proof of Lemma 6.21.6. □

Given a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, a sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ on Y, the notion of an *f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of sheaves of modules* makes sense. We can just define it as an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of abelian sheaves (see Definition 6.21.7 and Lemma 6.21.8) such that for all open $V \subset Y$ the map

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

is an $\mathcal{O}_Y(V)$-module map. Here we think of $\mathcal{F}(f^{-1}V)$ as an $\mathcal{O}_Y(V)$-module via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{Mod(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Mod(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F})$. See above.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, and $x \in X$ the induced map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is an $\mathcal{O}_{Y,f(x)}$-module map where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{F}_x$ comes from the $\mathcal{O}_{X,x}$-module structure via the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Here is a related lemma.

0098 **Lemma 6.26.4.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $x \in X$. Then*

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

as $\mathcal{O}_{X,x}$-modules where the tensor product on the right uses $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Proof. This follows from Lemma 6.20.3 and the identification of the stalks of pullback sheaves at x with the corresponding stalks at $f(x)$. See the formulae in Section 6.23 for example. □

6.27. Skyscraper sheaves and stalks

0099

009A **Definition 6.27.1.** Let X be a topological space.

(1) Let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We call $i_{x,*}A$ the *skyscraper sheaf at x with value A*.
(2) If in (1) above A is an abelian group then we think of $i_{x,*}A$ as a sheaf of abelian groups on X.
(3) If in (1) above A is an algebraic structure then we think of $i_{x,*}A$ as a sheaf of algebraic structures.
(4) If $(X, \mathcal{O}_X)$ is a ringed space, then we think of $i_x : \{x\} \to X$ as a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ and if A is a $\mathcal{O}_{X,x}$-module, then we think of $i_{x,*}A$ as a sheaf of $\mathcal{O}_X$-modules.
(5) We say a sheaf of sets $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and a set A such that $\mathcal{F} \cong i_{x,*}A$.
(6) We say a sheaf of abelian groups $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.
(7) We say a sheaf of algebraic structures $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and an algebraic structure A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of algebraic structures.
(8) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we say $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point $x \in X$ and a $\mathcal{O}_{X,x}$-module A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of $\mathcal{O}_X$-modules.

009B **Lemma 6.27.2.** *Let X be a topological space, $x \in X$ a point, and A a set. For any point $x' \in X$ the stalk of the skyscraper sheaf at x with value A at x' is*

$$(i_{x,*}A)_{x'} = \begin{cases} A & \text{if} \quad x' \in \overline{\{x\}} \\ \{*\} & \text{if} \quad x' \notin \overline{\{x\}} \end{cases}$$

A similar description holds for the case of abelian groups, algebraic structures and sheaves of modules.

Proof. Omitted. □

009C **Lemma 6.27.3.** *Let X be a topological space, and let $x \in X$ a point. The functors $\mathcal{F} \mapsto \mathcal{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula*

$$\mathrm{Mor}_{Sets}(\mathcal{F}_x, A) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, i_{x,*}A).$$

A similar statement holds for the case of abelian groups, algebraic structures. In the case of sheaves of modules we have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

Proof. Omitted. Hint: The stalk functor can be seen as the pullback functor for the morphism $i_x : \{x\} \to X$. Then the adjointness follows from adjointness of i_x^{-1} and $i_{x,*}$ (resp. i_x^* and $i_{x,*}$ in the case of sheaves of modules). □

6.28. Limits and colimits of presheaves

009D Let X be a topological space. Let $\mathcal{I} \to PSh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) For any open $U \subset X$ we have

$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U)$$

and

$$(\mathrm{colim}_i \mathcal{F}_i)(U) = \mathrm{colim}_i \mathcal{F}_i(U).$$

(3) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact (see Categories, Definition 4.23.1).
(4) Let $x \in X$. We always have

$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

The proofs are all easy.

6.29. Limits and colimits of sheaves

009E Let X be a topological space. Let $\mathcal{I} \to Sh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) The inclusion functor $i : Sh(X) \to PSh(X)$ commutes with limits. In other words, we may compute the limit in the category of sheaves as the limit in the category of presheaves. In particular, for any open $U \subset X$ we have

$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U).$$

(3) The inclusion functor $i : Sh(X) \to PSh(X)$ does not commute with colimits in general (not even with finite colimits – think surjections). The colimit is computed as the sheafification of the colimit in the category of presheaves:

$$\mathrm{colim}_i \mathcal{F}_i = \Big(U \mapsto \mathrm{colim}_i \mathcal{F}_i(U)\Big)^{\#}.$$

(4) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact.
(5) Let $x \in X$. We always have

$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

(6) The sheafification functor $^{\#} : PSh(X) \to Sh(X)$ commutes with all colimits, and with finite limits. But it does not commute with all limits.

The proofs are all easy. Here is an example of what is true for directed colimits of sheaves.

009F **Lemma 6.29.1.** *Let X be a topological space. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of sets over I, see Categories, Section 4.21. Let $U \subset X$ be an open subset. Consider the canonical map*

$$\Psi : \mathrm{colim}_i \mathcal{F}_i(U) \longrightarrow (\mathrm{colim}_i \mathcal{F}_i)(U)$$

(1) *If all the transition maps are injective then Ψ is injective for any open U.*
(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 6.11.2). By the discussion above we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Lemma 6.17.5 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s)$ and $\varphi_{i'i_j}(s')$ have the same restriction to U_j. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s')$ agree on the opens U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s')$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over the overlaps $U_j \cap U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). In particular we see that U is quasi-compact and hence by (2) we have injectivity of Ψ. Let s be an element of the target of Ψ. By assumption there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, with $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \cap U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \cap U_{j'}$. Choose an index $i \in I$ which is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

009G **Example 6.29.2.** Let $X = \{s_1, s_2, \xi_1, \xi_2, \xi_3, \ldots\}$ as a set. Declare a subset $U \subset X$ to be open if $s_1 \in U$ or $s_2 \in U$ implies U contains all of the ξ_i. Let $U_n = \{\xi_n, \xi_{n+1}, \ldots\}$, and let $j_n : U_n \to X$ be the inclusion map. Set $\mathcal{F}_n = j_{n,*}\underline{\mathbf{Z}}$. There are transition maps $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_n$. Note that $\mathcal{F}_{n,\xi_m} = 0$ if $m < n$ because $\{\xi_m\}$ is an open subset of X which misses U_n. Hence we see that $\mathcal{F}_{\xi_n} = 0$ for all n. On the other hand the stalk $\mathcal{F}_{s_i}$, $i = 1, 2$ is the colimit

$$M = \operatorname{colim}_n \prod\nolimits_{m \geq n} \mathbf{Z}$$

which is not zero. We conclude that the sheaf $\mathcal{F}$ is the direct sum of the skyscraper sheaves with value M at the closed points s_1 and s_2. Hence $\Gamma(X, \mathcal{F}) = M \oplus M$. On the other hand, the reader can verify that $\operatorname{colim}_n \Gamma(X, \mathcal{F}_n) = M$. Hence some condition is necessary in part (4) of Lemma 6.29.1 above.

There is a version of the previous lemma dealing with sheaves on a diagram of spectral spaces. To state it we introduce some notation. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces over $\mathcal{I}$ such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.

0A32 **Lemma 6.29.3.** *In the situation described above, let $i \in \mathrm{Ob}(\mathcal{I})$ and let $\mathcal{G}$ be a sheaf on X_i. For $U_i \subset X_i$ quasi-compact open we have*

$$p_i^{-1}\mathcal{G}(p_i^{-1}(U_i)) = \mathrm{colim}_{a:j\to i} f_a^{-1}\mathcal{G}(f_a^{-1}(U_i))$$

Proof. Let us prove the canonical map $\mathrm{colim}_{a:j\to i} f_a^{-1}\mathcal{G}(f_a^{-1}(U_i)) \to p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ is injective. Let s, s' be sections of $f_a^{-1}\mathcal{G}$ over $f_a^{-1}(U_i)$ for some $a : j \to i$. For $b : k \to j$ let $Z_k \subset f_{a\circ b}^{-1}(U_i)$ be the closed subset of points x such that the image of s and s' in the stalk $(f_{a\circ b}^{-1}\mathcal{G})_x$ are different. If Z_k is nonempty for all $b : k \to j$, then by Topology, Lemma 5.24.2 we see that $\lim_{b:k\to j} Z_k$ is nonempty too. Then for $x \in \lim_{b:k\to j} Z_k \subset X$ (observe that $\mathcal{I}/j \to \mathcal{I}$ is initial) we see that the image of s and s' in the stalk of $p_i^{-1}\mathcal{G}$ at x are different too since $(p_i^{-1}\mathcal{G})_x = (f_{b\circ a}^{-1}\mathcal{G})_{p_k(x)}$ for all $b : k \to j$ as above. Thus if the images of s and s' in $p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ are the same, then Z_k is empty for some $b : k \to j$. This proves injectivity.

Surjectivity. Let s be a section of $p_i^{-1}\mathcal{G}$ over $p_i^{-1}(U_i)$. By Topology, Lemma 5.24.5 the set $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X. By construction of the pullback sheaf, we can find an open covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$, opens $V_{l,i} \subset X_i$, sections $s_{l,i} \in \mathcal{G}(V_{l,i})$ such that $p_i(W_l) \subset V_{l,i}$ and $p_i^{-1}s_{l,i}|_{W_l} = s|_{W_l}$. Because X and X_i are spectral and $p_i^{-1}(U_i)$ is quasi-compact open, we may assume L is finite and W_l and $V_{l,i}$ quasi-compact open for all l. Then we can apply Topology, Lemma 5.24.6 to find $a : j \to i$ and open covering $f_a^{-1}(U_i) = \bigcup_{l\in L} W_{l,j}$ by quasi-compact opens whose pullback to X is the covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$ and such that moreover $W_{l,j} \subset f_a^{-1}(V_{l,i})$. Write $s_{l,j}$ the restriction of the pullback of $s_{l,i}$ by f_a to $W_{l,j}$. Then we see that $s_{l,j}$ and $s_{l',j}$ restrict to elements of $(f_a^{-1}\mathcal{G})(W_{l,j}\cap W_{l',j})$ which pullback to the same element $(p_i^{-1}\mathcal{G})(W_l \cap W_{l'})$, namely, the restriction of s. Hence by injectivity, we can find $b : k \to j$ such that the sections $f_b^{-1}s_{l,j}$ glue to a section over $f_{a\circ b}^{-1}(U_i)$ as desired. □

Next, in addition to the cofiltered system X_i of spectral spaces, assume given

(1) a sheaf $\mathcal{F}_i$ on X_i for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$ on X.

0A33 **Lemma 6.29.4.** *In the situation described above, let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be a quasi-compact open. Then*

$$\mathrm{colim}_{a:j\to i} \mathcal{F}_j(f_a^{-1}(U_i)) = \mathcal{F}(p_i^{-1}(U_i))$$

Proof. Recall that $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X, see Topology, Lemma 5.24.5. Hence Lemma 6.29.1 applies and we have

$$\mathcal{F}(p_i^{-1}(U_i)) = \mathrm{colim}_{a:j\to i} p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)).$$

A formal argument shows that

$$\mathrm{colim}_{a:j\to i} \mathcal{F}_j(f_a^{-1}(U_i)) = \mathrm{colim}_{a:j\to i} \mathrm{colim}_{b:k\to j} f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$

Thus it suffices to show that

$$p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)) = \mathrm{colim}_{b:k\to j} f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$

This is Lemma 6.29.3 applied to $\mathcal{F}_j$ and the quasi-compact open $f_a^{-1}(U_i)$. □

6.30. Bases and sheaves

009H Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

009I **Definition 6.30.1.** Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.

(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

As in the case of usual presheaves we use the terminology of sections, restrictions of sections, etc. In particular, we may define the *stalk* of $\mathcal{F}$ at a point $x \in X$ by the colimit

$$\mathcal{F}_x = \mathrm{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

As in the case of the stalk of a presheaf on X this limit is directed. The reason is that the collection of $U \in \mathcal{B}$, $x \in U$ is a fundamental system of open neighbourhoods of x.

It is easy to make examples to show that the notion of a presheaf on X is very different from the notion of a presheaf on a basis for the topology on X. This does not happen in the case of sheaves. A much more useful notion therefore, is the following.

009J **Definition 6.30.2.** Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a presheaf of sets on $\mathcal{B}$ which satisfies the following additional property: Given any $U \in \mathcal{B}$, and any covering $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$, and any coverings $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds:

(**) For any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$, $\forall k \in I_{ij}$

$$s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A *morphism of sheaves of sets on $\mathcal{B}$* is simply a morphism of presheaves of sets.

First we explain that it suffices to check the sheaf condition $(**)$ on a cofinal system of coverings. In the situation of the definition, suppose $U \in \mathcal{B}$. Let us temporarily denote $\mathrm{Cov}_{\mathcal{B}}(U)$ the set of all coverings of U by elements of $\mathcal{B}$. Note that $\mathrm{Cov}_{\mathcal{B}}(U)$ is preordered by refinement. A subset $C \subset \mathrm{Cov}_{\mathcal{B}}(U)$ is a cofinal system, if for every $\mathcal{U} \in \mathrm{Cov}_{\mathcal{B}}(U)$ there exists a covering $\mathcal{V} \in C$ which refines $\mathcal{U}$.

009K **Lemma 6.30.3.** *With notation as above. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. For each $U \in \mathcal{B}$, and each $\mathcal{U} : U = \bigcup U_i$ in $C(U)$, let coverings*

$\mathcal{U}_{ij} : U_i \cap U_j = \bigcup U_{ijk}$, $U_{ijk} \in \mathcal{B}$ be given. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ the sheaf condition $(**)$ holds (for the given coverings $\mathcal{U}_{ij}$).*

Proof. We have to show that (2) implies (1). Suppose that $U \in \mathcal{B}$, and that $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an arbitrary covering by elements of $\mathcal{B}$. Because the system $C(U)$ is cofinal we can find an element $\mathcal{V} : U = \bigcup_{j \in J} V_j$ in $C(U)$ which refines $\mathcal{U}$. This means there exists a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$.

Note that if $s, s' \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = s'|_{U_i}$, then

$$s|_{V_j} = (s|_{U_{\alpha(j)}})|_{V_j} = (s'|_{U_{\alpha(j)}})|_{V_j} = s'|_{V_j}$$

for all j. Hence by the uniqueness in $(**)$ for the covering $\mathcal{V}$ we conclude that $s = s'$. Thus we have proved the uniqueness part of $(**)$ for our arbitrary covering $\mathcal{U}$.

Suppose furthermore that $U_i \cap U_{i'} = \bigcup_{k \in I_{ii'}} U_{ii'k}$ are arbitrary coverings by $U_{ii'k} \in \mathcal{B}$. Let us try to prove the existence part of $(**)$ for the system $(\mathcal{U}, \mathcal{U}_{ij})$. Thus let $s_i \in \mathcal{F}(U_i)$ and suppose we have

$$s_i|_{U_{ii'k}} = s_{i'}|_{U_{ii'k}}$$

for all i, i', k. Set $t_j = s_{\alpha(j)}|_{V_j}$, where $\mathcal{V}$ and α are as above.

There is one small kink in the argument here. Namely, let $\mathcal{V}_{jj'} : V_j \cap V_{j'} = \bigcup_{l \in J_{jj'}} V_{jj'l}$ be the covering given to us by the statement of the lemma. It is not a priori clear that

$$t_j|_{V_{jj'l}} = t_{j'}|_{V_{jj'l}}$$

for all j, j', l. To see this, note that we do have

$$t_j|_W = t_{j'}|_W \text{ for all } W \in \mathcal{B}, W \subset V_{jj'l} \cap U_{\alpha(j)\alpha(j')k}$$

for all $k \in I_{\alpha(j)\alpha(j')}$, by our assumption on the family of elements s_i. And since $V_j \cap V_{j'} \subset U_{\alpha(j)} \cap U_{\alpha(j')}$ we see that $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$ agree on the members of a covering of $V_{jj'l}$ by elements of $\mathcal{B}$. Hence by the uniqueness part proved above we finally deduce the desired equality of $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$. Then we get the existence of an element $t \in \mathcal{F}(U)$ by property $(**)$ for $(\mathcal{V}, \mathcal{V}_{jj'})$.

Again there is a small snag. We know that t restricts to t_j on V_j but we do not yet know that t restricts to s_i on U_i. To conclude this note that the sets $U_i \cap V_j$, $j \in J$ cover U_i. Hence also the sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$ cover U_i. We leave it to the reader to see that t and s_i restrict to the same section of $\mathcal{F}$ on any $W \in \mathcal{B}$ which is contained in one of the open sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$. Hence by the uniqueness part seen above we win. □

009L **Lemma 6.30.4.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every triple $U, U', U'' \in \mathcal{B}$ with $U' \subset U$ and $U'' \subset U$ we have $U' \cap U'' \in \mathcal{B}$. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*

(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ and for every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathcal{F}(U)$ which restricts to s_i on U_i.*

Proof. This is a reformulation of Lemma 6.30.3 above in the special case where the coverings $\mathcal{U}_{ij}$ each consist of a single element. But also this case is much easier and is an easy exercise to do directly. □

009M **Lemma 6.30.5.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $U \in \mathcal{B}$. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. The map*

$$\mathcal{F}(U) \to \prod\nolimits_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

(*) *For any $x \in U$ there exists a $V \in \mathcal{B}$, with $x \in V \subset U$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = (V, \sigma)$ in $\mathcal{F}_y$.*

Proof. First note that the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective by the uniqueness in the sheaf condition of Definition 6.30.2. Let (s_x) be any element on the right hand side which satisfies $(*)$. Clearly this means we can find a covering $U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from certain $\sigma_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$ the sections σ_i and σ_j agree in the stalk $\mathcal{F}_y$. Hence there exists an element $V_{ijy} \in \mathcal{B}$, $y \in V_{ijy}$ such that $\sigma_i|_{V_{ijy}} = \sigma_j|_{V_{ijy}}$. Thus the sheaf condition $(**)$ of Definition 6.30.2 applies to the system of σ_i and we obtain a section $s \in \mathcal{F}(U)$ with the desired property. □

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. There is a natural restriction functor from the category of sheaves of sets on X to the category of sheaves of sets on $\mathcal{B}$. It turns out that this is an equivalence of categories. In down to earth terms this means the following.

009N **Lemma 6.30.6.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. There exists a unique sheaf of sets $\mathcal{F}^{ext}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. We first construct a presheaf $\mathcal{F}^{ext}$ with the desired property. Namely, for an arbitrary open $U \subset X$ we define $\mathcal{F}^{ext}(U)$ as the set of elements $(s_x)_{x \in U}$ such that $(*)$ of Lemma 6.30.5 holds. It is clear that there are restriction mappings that turn $\mathcal{F}^{ext}$ into a presheaf of sets. Also, by Lemma 6.30.5 we see that $\mathcal{F}(U) = \mathcal{F}^{ext}(U)$ whenever U is an element of the basis $\mathcal{B}$. To see $\mathcal{F}^{ext}$ is a sheaf one may argue as in the proof of Lemma 6.17.1. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

009O **Lemma 6.30.7.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote $Sh(\mathcal{B})$ the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X) \longrightarrow Sh(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.6 above. Checking the obvious functorialities is left to the reader. □

This ends the discussion of sheaves of sets on a basis $\mathcal{B}$. Let $(\mathcal{C}, F)$ be a type of algebraic structure. At the end of this section we would like to point out that the constructions above work for sheaves with values in $\mathcal{C}$. Let us briefly define the relevant notions.

009P **Definition 6.30.8.** Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) A *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ and whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a morphism of algebraic structures $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.
(3) Given a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ we say that $U \mapsto F(\mathcal{F}(U))$ is the underlying presheaf of sets.
(4) A *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ whose underlying presheaf of sets is a sheaf.

At this point we can define the *stalk* at $x \in X$ of a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \mathrm{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It exists as an object of $\mathcal{C}$ because of our assumptions on $\mathcal{C}$. Also, we see that the underlying set of $\mathcal{F}_x$ is the stalk of the underlying presheaf of sets on $\mathcal{B}$.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf with values in $\mathcal{C}$. The analogue of Lemma 6.30.6 need some care. Here it is.

009Q **Lemma 6.30.9.** *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. There exists a unique sheaf $\mathcal{F}^{ext}$ with values in $\mathcal{C}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. By the conditions imposed on the pair $(\mathcal{C}, F)$ it suffices to come up with a presheaf $\mathcal{F}^{ext}$ which does the correct thing on the level of underlying presheaves of sets. Thus our first task is to construct a suitable object $\mathcal{F}^{ext}(U)$ for all open $U \subset X$. We could do this by imitating Lemma 6.18.1 in the setting of presheaves on $\mathcal{B}$. However, a slightly different method (but basically equivalent) is the following: Define it as the directed colimit

$$\mathcal{F}^{ext}(U) := \mathrm{colim}_{\mathcal{U}} FIB(\mathcal{U})$$

over all coverings $\mathcal{U} : U = \bigcup_{i \in I} U_i$ by $U_i \in \mathcal{B}$ of the fibre product

$$\begin{array}{ccc} FIB(\mathcal{U}) & \longrightarrow & \prod_{x \in U} \mathcal{F}_x \\ \downarrow & & \downarrow \\ \prod_{i \in I} \mathcal{F}(U_i) & \longrightarrow & \prod_{i \in I} \prod_{x \in U_i} \mathcal{F}_x \end{array}$$

By the usual arguments, see Lemma 6.15.4 and Example 6.15.5 it suffices to show that this construction on underlying sets is the same as the definition using $(**)$ above. Details left to the reader. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}_x^{ext}$$

as objects in $\mathcal{C}$ in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

009R **Lemma 6.30.10.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Denote $Sh(\mathcal{B}, \mathcal{C})$ the category of sheaves with values in $\mathcal{C}$ on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X, \mathcal{C}) \longrightarrow Sh(\mathcal{B}, \mathcal{C})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.9 above. Checking the obvious functorialities is left to the reader. □

Finally we come to the case of (pre)sheaves of modules on a basis. We will use the easy fact that the category of presheaves of sets on a basis has products and that they are described by taking products of values on elements of the bases.

009S **Definition 6.30.11.** Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{B}$.

(1) A *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$* is a presheaf of abelian groups on $\mathcal{B}$ together with a morphism of presheaves of sets $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in \mathcal{B}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ turns the group $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$* is a morphism of abelian presheaves on $\mathcal{B}$ which induces an $\mathcal{O}(U)$-module homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathcal{B}$.
(3) Suppose that $\mathcal{O}$ is a sheaf of rings on $\mathcal{B}$. A *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$* is a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ whose underlying presheaf of abelian groups is a sheaf.

We can define the *stalk* at $x \in X$ of a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \operatorname{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It is a $\mathcal{O}_x$-module.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf of $\mathcal{O}$-modules. The analogue of Lemma 6.30.6 is as follows.

009T **Lemma 6.30.12.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{B}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{B}$. Let $\mathcal{O}^{ext}$ be the sheaf of rings on X extending $\mathcal{O}$ and let $\mathcal{F}^{ext}$ be the abelian sheaf on X extending $\mathcal{F}$, see Lemma 6.30.9. There exists a canonical map*

$$\mathcal{O}^{ext} \times \mathcal{F}^{ext} \longrightarrow \mathcal{F}^{ext}$$

which agrees with the given map over elements of $\mathcal{B}$ and which endows $\mathcal{F}^{ext}$ with the structure of an $\mathcal{O}^{ext}$-module.

Proof. It suffices to construct the multiplication map on the level of presheaves of sets. Perhaps the easiest way to see this is to prove directly that if $(f_x)_{x\in U}$, $f_x \in \mathcal{O}_x$ and $(m_x)_{x\in U}$, $m_x \in \mathcal{F}_x$ satisfy $(*)$, then the element $(f_x m_x)_{x\in U}$ also satisfies $(*)$. Then we get the desired result, because in the proof of Lemma 6.30.6 we construct the extension in terms of families of elements of stalks satisfying $(*)$. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as $\mathcal{O}_x$-modules in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x, or simply because it is true on the underlying sets.

009U **Lemma 6.30.13.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on X. Denote $Mod(\mathcal{O}|_{\mathcal{B}})$ the category of sheaves of $\mathcal{O}|_{\mathcal{B}}$-modules on $\mathcal{B}$. There is an equivalence of categories*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}|_{\mathcal{B}})$$

which assigns to a sheaf of $\mathcal{O}$-modules on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.12 above. Checking the obvious functorialities is left to the reader. □

Finally, we address the question of the relationship of this with continuous maps. This is now very easy thanks to the work above. First we do the case where there is a basis on the target given.

009V **Lemma 6.30.14.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a morphism*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. This is trivial because the collection of maps amounts to a morphism between the restrictions of $\mathcal{G}$ and $f_*\mathcal{F}$ to $\mathcal{B}$. By Lemma 6.30.10 this is the same as giving a morphism from $\mathcal{G}$ to $f_*\mathcal{F}$, which by Lemma 6.21.8 is the same as an f-map. See also Lemma 6.23.1 and the discussion preceding it for how to deal with the case of sheaves of algebraic structures. □

Here is the analogue for ringed spaces.

009W **Lemma 6.30.15.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

(where $\mathcal{F}(f^{-1}V)$ has a module structure using $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$) compatible with restriction mappings. Then there is a unique f-map (see discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. Same as the proof of the corresponding lemma for sheaves of algebraic structures above. □

009X **Lemma 6.30.16.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a morphism*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and the discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Let us first proves this for sheaves of sets. Fix $V \subset Y$ open. Pick $s \in \mathcal{G}(V)$. We are going to construct an element $\varphi_V(s) \in \mathcal{F}(f^{-1}V)$. We can define a value $\varphi(s)_x$ in the stalk $\mathcal{F}_x$ for every $x \in f^{-1}V$ by picking a $U \in \mathcal{B}_X$ with $x \in U \subset f^{-1}V$ and setting $\varphi(s)_x$ equal to the equivalence class of $(U, \varphi_V^U(s))$ in the stalk. Clearly, the family $(\varphi(s)_x)_{x \in f^{-1}V}$ satisfies condition $(*)$ because the maps φ_V^U for varying U are compatible with restrictions in the sheaf $\mathcal{F}$. Thus, by the proof of Lemma 6.30.6 we see that $(\varphi(s)_x)_{x \in f^{-1}V}$ corresponds to a unique element $\varphi_V(s)$ of $\mathcal{F}(f^{-1}V)$. Thus we have defined a set map $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. The compatibility between φ_V and φ_V^U follows from Lemma 6.30.5.

We leave it to the reader to show that the construction of φ_V is compatible with restriction mappings as we vary $V \in \mathcal{B}_Y$. Thus we may apply Lemma 6.30.14 above to "glue" them to the desired f-map.

Finally, we note that the map of sheaves of sets so constructed satisfies the property that the map on stalks

$$\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is the colimit of the system of maps φ_V^U as $V \in \mathcal{B}_Y$ varies over those elements that contain $f(x)$ and $U \in \mathcal{B}_X$ varies over those elements that contain x. In particular, if $\mathcal{G}$ and $\mathcal{F}$ are the underlying sheaves of sets of sheaves of algebraic structures, then we see that the maps on stalks is a morphism of algebraic structures. Hence we conclude that the associated map of sheaves of underlying sets $f^{-1}\mathcal{G} \to \mathcal{F}$ satisfies the assumptions of Lemma 6.23.1. We conclude that $f^{-1}\mathcal{G} \to \mathcal{F}$ is a morphism of sheaves with values in $\mathcal{C}$. And by adjointness this means that φ is an f-map of sheaves of algebraic structures. □

009Y **Lemma 6.30.17.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. Here the $\mathcal{O}_Y(V)$-module structure on $\mathcal{F}(U)$ comes from the $\mathcal{O}_X(U)$-module structure via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(U)$. Then there is a unique f-map of sheaves of modules (see Definition 6.21.7 and the discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Similar to the above and omitted. □

6.31. Open immersions and (pre)sheaves

009Z Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X. In Section 6.21 we have defined functors j_* and j^{-1} such that j_* is right adjoint to j^{-1}. It turns out that for an open immersion there is a left adjoint for j^{-1}, which we will denote $j_!$. First we point out that j^{-1} has a particularly simple description in the case of an open immersion.

00A0 **Lemma 6.31.1.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.*

1. *Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j_p\mathcal{G}$ (see Section 6.21) is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
2. *Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
3. *For any point $u \in U$ and any sheaf $\mathcal{G}$ on X we have a canonical identification of stalks*
$$j^{-1}\mathcal{G}_u = (\mathcal{G}|_U)_u = \mathcal{G}_u.$$
4. *On the category of presheaves of U we have $j_p j_* = id$.*
5. *On the category of sheaves of U we have $j^{-1}j_* = id$.*

The same description holds for (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures, and (pre)sheaves of modules.

Proof. The colimit in the definition of $j_p\mathcal{G}(V)$ is over collection of all $W \subset X$ open such that $V \subset W$ ordered by reverse inclusion. Hence this has a largest element, namely V. This proves (1). And (2) follows because the assignment $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open is clearly a sheaf if $\mathcal{G}$ is a sheaf. Assertion (3) follows from (2) since the collection of open neighbourhoods of u which are contained in U is cofinal in the collection of all open neighbourhoods of u in X. Parts (4) and (5) follow by computing $j^{-1}j_*\mathcal{F}(V) = j_*\mathcal{F}(V) = \mathcal{F}(V)$.

The exact same arguments work for (pre)sheaves of abelian groups and (pre)sheaves of algebraic structures. □

00A1 **Definition 6.31.2.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

1. Let $\mathcal{G}$ be a presheaf of sets, abelian groups or algebraic structures on X. The presheaf $j_p\mathcal{G}$ described in Lemma 6.31.1 is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.
2. Let $\mathcal{G}$ be a sheaf of sets on X, abelian groups or algebraic structures on X. The sheaf $j^{-1}\mathcal{G}$ is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.

(3) If $(X, \mathcal{O})$ is a ringed space, then the pair $(U, \mathcal{O}|_U)$ is called the *open subspace of* $(X, \mathcal{O})$ *associated to* U.
(4) If $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules then $\mathcal{G}|_U$ together with the multiplication map $\mathcal{O}|_U \times \mathcal{G}|_U \to \mathcal{G}|_U$ (see Lemma 6.24.6) is called the *restriction of* $\mathcal{G}$ *to* U.

We leave a definition of the restriction of presheaves of modules to the reader. Ok, so in this section we will discuss a left adjoint to the restriction functor. Here is the definition in the case of (pre)sheaves of sets.

00A2 **Definition 6.31.3.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be a presheaf of sets on U. We define the *extension of* $\mathcal{F}$ *by the empty set* $j_{p!}\mathcal{F}$ to be the presheaf of sets on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.
(2) Let $\mathcal{F}$ be a sheaf of sets on U. We define the *extension of* $\mathcal{F}$ *by the empty set* $j_!\mathcal{F}$ to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.

00A3 **Lemma 6.31.4.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of presheaves of U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of U we have $j^{-1}j_! = id$.*

Proof. To map $j_{p!}\mathcal{F}$ into $\mathcal{G}$ it is enough to map $\mathcal{F}(V) \to \mathcal{G}(V)$ whenever $V \subset U$ compatibly with restriction mappings. And by Lemma 6.31.1 the same description holds for maps $\mathcal{F} \to \mathcal{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk. Statements (4) and (5) follow by computing the value of the sheaf on any open of U. □

Note that if $\mathcal{F}$ is a sheaf of abelian groups on U, then in general $j_!\mathcal{F}$ as defined above, is not a sheaf of abelian groups, for example because some of its stalks are empty (hence not abelian groups for sure). Thus we need to modify the definition of $j_!$ depending on the type of sheaves we consider. The reason for choosing the empty set in the definition of the extension by the empty set, is that it is the initial object in the category of sets. Thus in the case of abelian groups we use 0 (and more generally for sheaves with values in any abelian category).

00A4 **Definition 6.31.5.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be an abelian presheaf on U. We define the *extension* $j_{p!}\mathcal{F}$ *of* $\mathcal{F}$ *by* 0 to be the abelian presheaf on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} 0 & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(2) Let $\mathcal{F}$ be an abelian sheaf on U. We define the *extension* $j_!\mathcal{F}$ *of* $\mathcal{F}$ *by* 0 to be the sheafification of the abelian presheaf $j_{p!}\mathcal{F}$.

(3) Let $\mathcal{C}$ be a category having an initial object e. Let $\mathcal{F}$ be a presheaf on U with values in $\mathcal{C}$. We define the *extension* $j_{p!}\mathcal{F}$ *of* $\mathcal{F}$ *by* e to be the presheaf on X with values in $\mathcal{C}$ defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} e & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(4) Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Let $\mathcal{F}$ be a sheaf of algebraic structures on U (of the give type). We define the *extension* $j_!\mathcal{F}$ *of* $\mathcal{F}$ *by* e to be the sheafification of the presheaf $j_{p!}\mathcal{F}$ defined above.

(5) Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the presheaf of $\mathcal{O}$-modules which is equal to $j_{p!}\mathcal{F}$ as an abelian presheaf endowed with the multiplication map $\mathcal{O} \times j_{p!}\mathcal{F} \to j_{p!}\mathcal{F}$.

(6) Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the $\mathcal{O}$-module which is equal to $j_!\mathcal{F}$ as an abelian sheaf endowed with the multiplication map $\mathcal{O} \times j_!\mathcal{F} \to j_!\mathcal{F}$.

It is true that one can define $j_!$ in the setting of sheaves of algebraic structures (see below). However, it depends on the type of algebraic structures involved what the resulting object is. For example, if $\mathcal{O}$ is a sheaf of rings on U, then $j_{!,rings}\mathcal{O} \neq j_{!,abelian}\mathcal{O}$ since the initial object in the category of rings is $\mathbf{Z}$ and the initial object in the category of abelian groups is 0. In particular the functor $j_!$ *does not commute with taking underlying sheaves of sets*, in contrast to what we have seen so far! We separate out the case of (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures and (pre)sheaves of modules as usual.

00A5 **Lemma 6.31.6.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Consider the functors of restriction and extension by* 0 *for abelian (pre)sheaves.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Ab(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be an abelian sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} 0 & \textit{if} \quad x \notin U \\ \mathcal{F}_x & \textit{if} \quad x \in U \end{cases}$$

(4) *On the category of abelian presheaves of U we have $j_p j_{p!} = id$.*
(5) *On the category of abelian sheaves of U we have $j^{-1} j_! = id$.*

Proof. Omitted. □

00A6 **Lemma 6.31.7.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Consider the functors of restriction and extension by e for (pre)sheaves of algebraic structure defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X,\mathcal{C})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} e & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of presheaves of algebraic structures on U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of algebraic structures on U we have $j^{-1} j_! = id$.*

Proof. Omitted. □

00A7 **Lemma 6.31.8.** *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. Consider the functors of restriction and extension by 0 for (pre)sheaves of modules defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{PMod(\mathcal{O})}(j_{p!}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{PMod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Mod(\mathcal{O})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Mod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of sheaves of $\mathcal{O}|_U$-modules on U we have $j^{-1} j_! = id$.*

Proof. Omitted. □

Note that by the lemmas above, both the functors j_* and $j_!$ are fully faithful embeddings of the category of sheaves on U into the category of sheaves on X. It is only true for the functor $j_!$ that one can easily describe the essential image of this functor.

00A8 **Lemma 6.31.9.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Sh(U) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \emptyset$ for all $x \in X \setminus U$.

Proof. Fully faithfulness follows formally from $j^{-1}j_! = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$j_!j^{-1}\mathcal{G} \to \mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

00A9 **Lemma 6.31.10.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Ab(U) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

00AA **Lemma 6.31.11.** *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. The functor*

$$j_! : Sh(U, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = e$ for all $x \in X \setminus U$.

Proof. Omitted. □

00AB **Lemma 6.31.12.** *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. The functor*

$$j_! : Mod(\mathcal{O}|_U) \longrightarrow Mod(\mathcal{O})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

00AC **Remark 6.31.13.** Let $j : U \to X$ be an open immersion of topological spaces as above. Let $x \in X$, $x \notin U$. Let $\mathcal{F}$ be a sheaf of sets on U. Then $j_!\mathcal{F}_x = \emptyset$ by Lemma 6.31.4. Hence $j_!$ does not transform a final object of $Sh(U)$ into a final object of $Sh(X)$ unless $U = X$. According to our conventions in Categories, Section 4.23 this means that the functor $j_!$ is not left exact as a functor between the categories of sheaves of sets. It will be shown later that $j_!$ on abelian sheaves is exact, see Modules, Lemma 17.3.4.

6.32. Closed immersions and (pre)sheaves

00AD Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In Section 6.21 we have defined functors i_* and i^{-1} such that i_* is right adjoint to i^{-1}.

00AE **Lemma 6.32.1.** *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. Let $\mathcal{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathcal{F}$ are described as follows*

$$i_*\mathcal{F}_x = \begin{cases} \{*\} & \text{if} \quad x \notin Z \\ \mathcal{F}_x & \text{if} \quad x \in Z \end{cases}$$

where $\{*\}$ denotes a singleton set. Moreover, $i^{-1}i_* = id$ on the category of sheaves of sets on Z. Moreover, the same holds for abelian sheaves on Z, resp. sheaves of algebraic structures on Z where $\{*\}$ has to be replaced by 0, resp. a final object of the category of algebraic structures.

Proof. If $x \notin Z$, then there exist arbitrarily small open neighbourhoods U of x which do not meet Z. Because $\mathcal{F}$ is a sheaf we have $\mathcal{F}(i^{-1}(U)) = \{*\}$ for any such U, see Remark 6.7.2. This proves the first case. The second case comes from the fact that for $z \in Z$ any open neighbourhood of z is of the form $Z \cap U$ for some open U of X. For the statement that $i^{-1}i_* = \text{id}$ consider the canonical map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$. This is an isomorphism on stalks (see above) and hence an isomorphism.

For sheaves of abelian groups, and sheaves of algebraic structures you argue in the same manner. □

00AF **Lemma 6.32.2.** *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Sh(Z) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \{*\}$ for all $x \in X \setminus Z$.

Proof. Fully faithfulness follows formally from $i^{-1}i_* = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$\mathcal{G} \to i_*i^{-1}\mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

00AG **Lemma 6.32.3.** *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

00AH **Lemma 6.32.4.** *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure with final object 0. The functor*

$$i_* : Sh(Z, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

00AI **Remark 6.32.5.** Let $i : Z \to X$ be a closed immersion of topological spaces as above. Let $x \in X$, $x \notin Z$. Let $\mathcal{F}$ be a sheaf of sets on Z. Then $(i_*\mathcal{F})_x = \{*\}$ by Lemma 6.32.1. Hence if $\mathcal{F} = * \amalg *$, where $*$ is the singleton sheaf, then $i_*\mathcal{F}_x = \{*\} \neq i_*(*)_x \amalg i_*(*)_x$ because the latter is a two point set. According to our conventions in Categories, Section 4.23 this means that the functor i_* is not right exact as a functor between the categories of sheaves of sets. In particular, it cannot have a right adjoint, see Categories, Lemma 4.24.6.

On the other hand, we will see later (see Modules, Lemma 17.6.3) that i_* on abelian sheaves is exact, and does have a right adjoint, namely the functor that associates to an abelian sheaf on X the sheaf of sections supported in Z.

00AJ **Remark 6.32.6.** We have not discussed the relationship between closed immersions and ringed spaces. This is because the notion of a closed immersion of ringed spaces is best discussed in the setting of quasi-coherent sheaves, see Modules, Section 17.13.

6.33. Glueing sheaves

00AK In this section we glue sheaves defined on the members of a covering of X. We first deal with maps.

04TN **Lemma 6.33.1.** *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of sets on X. Given a collection*

$$\varphi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. For each open subset $U \subset X$ define

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U), \quad s \mapsto \varphi_U(s)$$

where $\varphi_U(s)$ is the unique section verifying

$$(\varphi_U(s))|_{U \cap U_i} = (\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}).$$

Existence and uniqueness of such a section follows from the sheaf axioms due to the fact that

$$\begin{aligned} ((\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}))|_{U \cap U_i \cap U_j} &= (\varphi_i)_{U \cap U_i \cap U_j}(s|_{U \cap U_i \cap U_j}) \\ &= (\varphi_j)_{U \cap U_i \cap U_j}(s|_{U \cap U_i \cap U_j}) \\ &= ((\varphi_j)_{U \cap U_j}(s|_{U \cap U_j}))|_{U \cap U_i \cap U_j}. \end{aligned}$$

This family of maps gives us indeed a map of sheaves: Let $V \subset U \subset X$ be open subsets then

$$(\varphi_U(s))|_V = \varphi_V(s|_V)$$

since for each $i \in I$ the following holds

$$\begin{aligned}(\varphi_U(s))|_{V\cap U_i} &= ((\varphi_U(s))|_{U\cap U_i})|_{V\cap U_i} \\ &= ((\varphi_i)_{U\cap U_i}(s|_{U\cap U_i}))|_{V\cap U_i} \\ &= (\varphi_i)_{V\cap U_i}(s|_{V\cap U_i}) \\ &= \varphi_V(s_V)|_{V\cap U_i}.\end{aligned}$$

Furthermore, its restriction to each U_i agrees with φ_i since given $U \subset X$ open subset and $s \in \mathcal{F}(U \cap U_i)$ then

$$\begin{aligned}\varphi_{U\cap U_i}(s) &= \varphi_{U\cap U_i}(s)|_{U\cap U_i} \\ &= (\varphi_i)_{U\cap U_i}(s|_{U\cap U_i}) \\ &= (\varphi_i)_{U\cap U_i}(s).\end{aligned}$$

□

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on the topological space X the rule

$$U \longmapsto \mathrm{Mor}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules, Section 17.22.

Let X be a topological space. Let $X = \bigcup_{i\in I} U_i$ be an open covering. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \longrightarrow \mathcal{F}_j|_{U_i\cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i|_{U_i\cap U_j\cap U_k} & \xrightarrow{\varphi_{ik}} & \mathcal{F}_k|_{U_i\cap U_j\cap U_k} \\ & {\scriptstyle\varphi_{ij}}\searrow \quad \nearrow{\scriptstyle\varphi_{jk}} & \\ & \mathcal{F}_j|_{U_i\cap U_j\cap U_k} & \end{array}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $X = \bigcup U_i$.

00AL **Lemma 6.33.2.** *Let X be a topological space. Let $X = \bigcup_{i\in I} U_i$ be an open covering. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $X = \bigcup U_i$ there exists a sheaf of sets $\mathcal{F}$ on X together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{U_i\cap U_j} & \xrightarrow{\varphi_i} & \mathcal{F}_i|_{U_i\cap U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\varphi_{ij}} \\ \mathcal{F}|_{U_i\cap U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{U_i\cap U_j} \end{array}$$

are commutative.

Proof. First proof. In this proof we give a formula for the set of sections of $\mathcal{F}$ over an open $W \subset X$. Namely, we define

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subset W$ are defined by the restricting each of the s_i to $W' \cap U_i$. The sheaf condition for $\mathcal{F}$ follows immediately from the sheaf condition for each of the $\mathcal{F}_i$.

We still have to prove that $\mathcal{F}|_{U_i}$ maps isomorphically to $\mathcal{F}_i$. Let $W \subset U_i$. In this case the condition in the definition of $\mathcal{F}(W)$ implies that $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. And the commutativity of the diagrams in the definition of a glueing data assures that we may start with *any* section $s \in \mathcal{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$.

Second proof (sketch). Let $\mathcal{B}$ be the set of opens $U \subset X$ such that $U \subset U_i$ for some $i \in I$. Then $\mathcal{B}$ is a base for the topology on X. For $U \in \mathcal{B}$ we pick $i \in I$ with $U \subset U_i$ and we set $\mathcal{F}(U) = \mathcal{F}_i(U)$. Using the isomorphisms φ_{ij} we see that this prescription is "independent of the choice of i". Using the restriction mappings of $\mathcal{F}_i$ we find that $\mathcal{F}$ is a sheaf on $\mathcal{B}$. Finally, use Lemma 6.30.6 to extend $\mathcal{F}$ to a unique sheaf $\mathcal{F}$ on X. □

00AM **Lemma 6.33.3.** *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $(\mathcal{F}_i, \varphi_{ij})$ be a glueing data of sheaves of abelian groups, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules for some sheaf of rings $\mathcal{O}$ on X. Then the construction in the proof of Lemma 6.33.2 above leads to a sheaf of abelian groups, resp. sheaf of algebraic structures, resp. sheaf of $\mathcal{O}$-modules.*

Proof. This is true because in the construction the set of sections $\mathcal{F}(W)$ over an open W is given as the equalizer of the maps

$$\prod_{i \in I} \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}_i(W \cap U_i \cap U_j)$$

And in each of the cases envisioned this equalizer gives an object in the relevant category whose underlying set is the object considered in the cited lemma. □

00AN **Lemma 6.33.4.** *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathcal{F}$ the following collection of glueing data*

$$(\mathcal{F}|_{U_i}, (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \to (\mathcal{F}|_{U_j})|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup U_i$ defines an equivalence of categories between $Sh(X)$ and the category of glueing data. A similar statement holds for abelian sheaves, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules.

Proof. The functor is fully faithful by Lemma 6.33.1 and essentially surjective (via an explicitly given quasi-inverse functor) by Lemma 6.33.2. □

This lemma means that if the sheaf $\mathcal{F}$ was constructed from the glueing data $(\mathcal{F}_i, \varphi_{ij})$ and if $\mathcal{G}$ is a sheaf on X, then a morphism $f : \mathcal{F} \to \mathcal{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathcal{F}_i \longrightarrow \mathcal{G}|_{U_i}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathcal{G} \to \mathcal{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathcal{G}|_{U_i} \longrightarrow \mathcal{F}_i$$

compatible with the glueing maps φ_{ij}.

6.34. Other chapters

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CHAPTER 7

Sites and Sheaves

00UZ

7.1. Introduction

00V0 The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [**AGV71**]. Our notion of a site differs from that in [**AGV71**]; what we call a site is called a category endowed with a pretopology in [**AGV71**, Exposé II, Définition 1.3]. The reason we do this is that in algebraic geometry it is often convenient to work with a given class of coverings, for example when defining when a property of schemes is local in a given topology, see Descent, Section 35.15. Our exposition will closely follow [**Art62**]. We will not use universes.

7.2. Presheaves

00V1 Let $\mathcal{C}$ be a category. A *presheaf of sets* is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to *Sets* (see Categories, Remark 4.2.11). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$. The elements of this set are called the *sections* of $\mathcal{F}$ over U. For every morphism $f : V \to U$ the map $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map* and is often denoted $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Another way of expressing this is to say that $f^*(s)$ is the *pullback* of s via f. Functoriality means that $g^*f^*(s) = (f \circ g)^*(s)$. Sometimes we use the notation $s|_V := f^*(s)$. This notation is consistent with the notion of restriction of functions from topology because if $W \to V \to U$ are morphisms in $\mathcal{C}$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the functorial nature of $\mathcal{F}$. Of course we have to be careful since it may very well happen that there is more than one morphism $V \to U$ and it is certainly not going to be the case that the corresponding pullback maps are equal.

00V2 **Definition 7.2.1.** A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$.

Note that for any object U of $\mathcal{C}$ the functor of points h_U, see Categories, Example 4.3.4 is a presheaf. These are called the *representable presheaves*. These presheaves have the pleasing property that for any presheaf $\mathcal{F}$ we have

090F (7.2.1.1) $$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

This is the Yoneda lemma (Categories, Lemma 4.3.5).

Similarly, we can define the notion of a presheaf of abelian groups, rings, etc. More generally we may define a presheaf with values in a category.

00V3 **Definition 7.2.2.** Let $\mathcal{C}$, $\mathcal{A}$ be categories. A *presheaf* $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^{opp} \to \mathcal{A}$. A *morphism* of presheaves $\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a transformation of functors from $\mathcal{F}$ to $\mathcal{G}$.

These form the objects and morphisms of the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.

00V4 **Remark 7.2.3.** As already pointed out we may consider the category of presheaves with values in any of the "big" categories listed in Categories, Remark 4.2.2. These will be "big" categories as well and they will be listed in the above mentioned remark as we go along.

7.3. Injective and surjective maps of presheaves

00V5

00V6 **Definition 7.3.1.** Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

00V7 **Lemma 7.3.2.** *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if it is a monomorphism of $PSh(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is a monomorphism. Let $U \in \mathrm{Ob}(\mathcal{C})$; we need to show that φ_U is injective. So let $a, b \in \mathcal{F}(U)$ be such that $\varphi_U(a) = \varphi_U(b)$; we need to check that $a = b$. Under the isomorphism (7.2.1.1), the elements a and b of $\mathcal{F}(U)$ correspond to two natural transformations $a', b' \in \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. Similarly, under the analogous isomorphism $\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{G}) = \mathcal{G}(U)$, the two equal elements $\varphi_U(a)$ and $\varphi_U(b)$ of $\mathcal{G}(U)$ correspond to the two natural transformations $\varphi \circ a', \varphi \circ b' \in \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{G})$, which therefore must also be equal. So $\varphi \circ a' = \varphi \circ b'$, and thus $a' = b'$ (since φ is monic), whence $a = b$. This finishes (1).

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if it is an epimorphism of $PSh(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is an epimorphism.

For any two morphisms $f : A \to B$ and $g : A \to C$ in the category *Sets*, we let $\mathrm{inl}_{f,g}$ and $\mathrm{inr}_{f,g}$ denote the two canonical maps from B and C to $B \coprod_A C$. (Here, the pushout is evaluated in *Sets*.)

Now, we define a presheaf $\mathcal{H}$ of sets on $\mathcal{C}$ by setting $\mathcal{H}(U) = \mathcal{G}(U) \coprod_{\mathcal{F}(U)} \mathcal{G}(U)$ (where the pushout is evaluated in *Sets* and induced by the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$) for every $U \in \mathrm{Ob}(\mathcal{C})$; its action on morphisms is defined in the obvious way (by the functoriality of pushout). Then, there are two natural transformations $i_1 : \mathcal{G} \to \mathcal{H}$ and $i_2 : \mathcal{G} \to \mathcal{H}$ whose components at an object $U \in \mathrm{Ob}(\mathcal{C})$ are given by the maps $\mathrm{inl}_{\varphi_U, \varphi_U}$ and $\mathrm{inr}_{\varphi_U, \varphi_U}$, respectively. The definition of a pushout shows that $i_1 \circ \varphi = i_2 \circ \varphi$, whence $i_1 = i_2$ (since φ is an epimorphism). Thus, for every $U \in \mathrm{Ob}(\mathcal{C})$, we have $\mathrm{inl}_{\varphi_U, \varphi_U} = \mathrm{inr}_{\varphi_U, \varphi_U}$. Thus, φ_U must be surjective (since a simple combinatorial argument shows that if $f : A \to B$ is a morphism in *Sets*, then $\mathrm{inl}_{f,f} = \mathrm{inr}_{f,f}$ if and only if f is surjective). In other words, φ is surjective, and (2) is proven.

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is both injective and surjective if and only if it is an isomorphism of $PSh(\mathcal{C})$. This time, the "if" direction is straightforward. To prove the "only if" direction, it suffices to observe that if φ is both injective and surjective, then φ_U is an invertible map for every $U \in \mathrm{Ob}(\mathcal{C})$, and the inverses of these maps for all U can be combined to a natural transformation $\mathcal{G} \to \mathcal{F}$ which is an inverse to φ. □

00V8 **Definition 7.3.3.** We say $\mathcal{F}$ is a *subpresheaf* of $\mathcal{G}$ if for every object $U \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{F}(U)$ is a subset of $\mathcal{G}(U)$, compatibly with the restriction mappings.

In other words, the inclusion maps $\mathcal{F}(U) \to \mathcal{G}(U)$ glue together to give an (injective) morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

00V9 **Lemma 7.3.4.** *Let $\mathcal{C}$ be a category. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{C}$. There exists a unique subpresheaf $\mathcal{G}' \subset \mathcal{G}$ such that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ and such that the first map is surjective.*

Proof. To prove existence, just set $\mathcal{G}'(U) = \varphi_U(\mathcal{F}(U))$ for every $U \in \mathrm{Ob}(C)$ (and inherit the action on morphisms from $\mathcal{G}$), and prove that this defines a subpresheaf of $\mathcal{G}$ and that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ with the first map being surjective. Uniqueness is straightforward. □

00VA **Definition 7.3.5.** Notation as in Lemma 7.3.4. We say that $\mathcal{G}'$ is the *image of* φ.

7.4. Limits and colimits of presheaves

00VB Let $\mathcal{C}$ be a category. Limits and colimits exist in the category $PSh(\mathcal{C})$. In addition, for any $U \in \mathrm{Ob}(\mathcal{C})$ the functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with limits and colimits. Perhaps the easiest way to prove these statements is the following. Given a diagram $\mathcal{F} : \mathcal{I} \to PSh(\mathcal{C})$ define presheaves

$$\mathcal{F}_{\lim} : U \longmapsto \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \text{ and } \mathcal{F}_{\mathrm{colim}} : U \longmapsto \mathrm{colim}_{i \in \mathcal{I}} \mathcal{F}_i(U)$$

There are clearly projection maps $\mathcal{F}_{\lim} \to \mathcal{F}_i$ and canonical maps $\mathcal{F}_i \to \mathcal{F}_{\mathrm{colim}}$. These maps satisfy the requirements of the maps of a limit (resp. colimit) of Categories, Definition 4.14.1 (resp. Categories, Definition 4.14.2). Indeed, they clearly form a cone, resp. a cocone, over $\mathcal{F}$. Furthermore, if $(\mathcal{G}, q_i : \mathcal{G} \to \mathcal{F}_i)$ is another system (as in the definition of a limit), then we get for every U a system of maps $\mathcal{G}(U) \to \mathcal{F}_i(U)$ with suitable functoriality requirements. And thus a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$. It is easy to verify these are compatible as we vary U and arise from the desired map $\mathcal{G} \to \mathcal{F}_{\lim}$. A similar argument works in the case of the colimit.

7.5. Functoriality of categories of presheaves

00VC Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In this case we denote

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. Note that by the previous section this functor commutes with all limits and colimits.

For $V \in \mathrm{Ob}(\mathcal{D})$ let $\mathcal{I}_V^u$ denote the category with

053L (7.5.0.1)
$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{I}_V^u) & = & \{(U, \phi) \mid U \in \mathrm{Ob}(\mathcal{C}), \phi : V \to u(U)\} \\ \mathrm{Mor}_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) & = & \{f : U \to U' \text{ in } \mathcal{C} \mid u(f) \circ \phi = \phi'\} \end{array}$$

We sometimes drop the superscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define a left adjoint to the functor u^p. Before we do so we prove a few technical lemmas.

00X4 **Lemma 7.5.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{C}$ has fibre products and equalizers, and that u commutes with them. Then the categories $(\mathcal{I}_V)^{opp}$ satisfy the hypotheses of Categories, Lemma 4.19.8.*

Proof. There are two conditions to check.

First, suppose we are given three objects $\phi : V \to u(U)$, $\phi' : V \to u(U')$, and $\phi'' : V \to u(U'')$ and morphisms $a : U' \to U$, $b : U'' \to U$ such that $u(a) \circ \phi' = \phi$ and $u(b) \circ \phi'' = \phi$. We have to show there exists another object $\phi''' : V \to u(U''')$ and morphisms $c : U''' \to U'$ and $d : U''' \to U''$ such that $u(c) \circ \phi''' = \phi'$, $u(d) \circ \phi''' = \phi''$ and $a \circ c = b \circ d$. We take $U''' = U' \times_U U''$ with c and d the projection morphisms. This works as u commutes with fibre products; we omit the verification.

Second, suppose we are given two objects $\phi : V \to u(U)$ and $\phi' : V \to u(U')$ and morphisms $a, b : (U, \phi) \to (U', \phi')$. We have to find a morphism $c : (U'', \phi'') \to (U, \phi)$ which equalizes a and b. Let $c : U'' \to U$ be the equalizer of a and b in the category $\mathcal{C}$. As u commutes with equalizers and since $u(a) \circ \phi = u(b) \circ \phi = \phi'$ we obtain a morphism $\phi'' : V \to u(U'')$. □

00X3 **Lemma 7.5.2.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume*

(1) *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
(2) *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then the index categories $(\mathcal{I}_V^u)^{opp}$ are filtered (see Categories, Definition 4.19.1).

Proof. The assumptions imply that the assumptions of Lemma 7.5.1 are satisfied (see the discussion in Categories, Section 4.18). By Categories, Lemma 4.19.8 we see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $\mathrm{Ob}(\mathcal{I}_V)$. By assumption $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Clearly the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. □

Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\overline{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$ by setting $\overline{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$ we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \longrightarrow Sets, \quad (U, \phi) \longmapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) = \mathrm{colim}_{\mathcal{I}_V^{opp}} \mathcal{F}_V$$

As a colimit we obtain for each $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$ by Categories, Lemma 4.14.8. It is the unique map so that for all $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{c(\phi)} & u_p\mathcal{F}(V) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{F}(U) & \xrightarrow{c(\phi \circ g)} & u_p\mathcal{F}(V') \end{array}$$

commutes. The uniqueness of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$.

00VD **Lemma 7.5.3.** *There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).*

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_V \to \mathcal{F}'_V$, and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D})$$

00VE **Lemma 7.5.4.** *The functor u_p is a left adjoint to the functor u^p. In other words the formula*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. We will leave it to the reader to verify they are each others inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^pu_p\mathcal{F} \to u^p\mathcal{G}$. Lemma 7.5.3 says that there is a map $\mathcal{F} \to u^pu_p\mathcal{F}$. The composition of the two gives the desired map. (The good thing about this construction is that it is clearly functorial in everything in sight.)

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $u_p\beta : u_p\mathcal{F} \to u_pu^p\mathcal{G}$. Let $V \in \mathrm{Ob}(\mathcal{D})$. We claim that the functor $u^p\mathcal{G}_V$ on $\mathcal{I}_V$ has a canonical map to the constant functor with value $\mathcal{G}(V)$. Namely, for every object (U, ϕ) of $\mathcal{I}_V$, the value of $u^p\mathcal{G}_V$ on this object is $\mathcal{G}(u(U))$ which maps to $\mathcal{G}(V)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_pu^p\mathcal{G}(V) \to \mathcal{G}(V)$. Another trivial verification shows that this is functorial in V leading to a map of presheaves $u_pu^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_pu^p\mathcal{G} \to \mathcal{G}$ is the desired map. □

00VF **Remark 7.5.5.** Suppose that $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_Y \to \mathcal{A}$ has a colimit in $\mathcal{A}$. In this case it is clear that there are functors u^p and u_p, defined in exactly the same way as above, on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

04D2 **Lemma 7.5.6.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. For any object U of $\mathcal{C}$ we have $u_ph_U = h_{u(U)}$.*

Proof. By adjointness of u_p and u^p we have

$$\mathrm{Mor}_{PSh(\mathcal{D})}(u_p h_U, \mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, u^p\mathcal{G}) = u^p\mathcal{G}(U) = \mathcal{G}(u(U))$$

and hence by Yoneda's lemma we see that $u_p h_U = h_{u(U)}$ as presheaves. □

7.6. Sites

00VG Our notion of a site uses the following type of structures.

0396 **Definition 7.6.1.** Let $\mathcal{C}$ be a category, see Conventions, Section 2.3. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

00VH **Definition 7.6.2.** A *site*[1] is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called *coverings of* $\mathcal{C}$, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Clarifications. In axiom (1) we require there should be a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that $I = \{i\}$ is a singleton set and such that the morphism $U_i \to U$ is equal to the morphism $V \to U$ given in (1). In the following we often denote $\{V \to U\}$ a family of morphisms with fixed target whose index set is a singleton. In axiom (3) we require the existence of the covering for some choice of the fibre products $U_i \times_U V$ for $i \in I$.

00VI **Remark 7.6.3.** (On set theoretic issues – skip on a first reading.) The main reason for introducing sites is to study the category of sheaves on a site, because it is the generalization of the category of sheaves on a topological space that has been so important in algebraic geometry. In order to avoid thinking about things like "classes of classes" and so on, we will not allow sites to be "big" categories, in contrast to what we do for categories and 2-categories.

Suppose that $\mathcal{C}$ is a category and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying (1), (2) and (3) above. We will not allow this as a site either, mainly because we are going to take limits over coverings. However, there are several natural ways to replace $\mathrm{Cov}(\mathcal{C})$ by a set of coverings or a slightly different structure that give rise to the same category of sheaves. For example:

(1) In Sets, Section 3.11 we show how to pick a suitable set of coverings that gives the same category of sheaves.
(2) Another thing we can do is to take the associated topology (see Definition 7.48.2). The resulting topology on $\mathcal{C}$ has the same category of sheaves. Two topologies have the same categories of sheaves if and only if they are equal, see Theorem 7.50.2. A topology on a category is given by a

[1] This notation differs from that of [**AGV71**], as explained in the introduction.

choice of sieves on objects. The collection of all possible sieves and even all possible topologies on $\mathcal{C}$ is a set.
(3) We could also slightly modify the notion of a site, see Remark 7.48.4 below, and end up with a canonical set of coverings.

Each of these solutions has some minor drawback. For the first, one has to check that constructions later on do not depend on the choice of the set of coverings. For the second, one has to learn about topologies and redo many of the arguments for sites. For the third, see the last sentence of Remark 7.48.4.

Our approach will be to work with sites as in Definition 7.6.2 above. Given a category $\mathcal{C}$ with a proper class of coverings as above, we will replace this by a set of coverings producing a site using Sets, Lemma 3.11.1. It is shown in Lemma 7.8.8 below that the resulting category of sheaves (the topos) is independent of this choice. We leave it to the reader to use one of the other two strategies to deal with these issues if he/she so desires.

00VJ **Example 7.6.4.** Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X_{Zar})$ if and only if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in X_{Zar} we have $U \times V = U \cap V$. Note that in particular the empty set has to be an element of X_{Zar} since otherwise this would not work in general. Furthermore, it is equally important, as we will see later, to allow the *empty covering of the empty set as a covering*! We turn X_{Zar} into a site by choosing a suitable set of coverings $\mathrm{Cov}(X_{Zar})_{\kappa,\alpha}$ as in Sets, Lemma 3.11.1. Presheaves and sheaves (as defined below) on the site X_{Zar} agree exactly with the usual notion of a presheaves and sheaves on a topological space, as defined in Sheaves, Section 6.1.

00VK **Example 7.6.5.** Let G be a group. Consider the category G-*Sets* whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. An important example is ${}_GG$ which is the G-set whose underlying set is G and action given by left multiplication. This category has fiber products, see Categories, Section 4.7. We declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a covering if $\bigcup_{i\in I} \varphi_i(U_i) = U$. This gives a class of coverings on G-*Sets* which is easily seen to satisfy conditions (1), (2), and (3) of Definition 7.6.2. The result is not a site since both the collection of objects of the underlying category and the collection of coverings form a proper class. We first replace by G-*Sets* by a full subcategory G-$Sets_\alpha$ as in Sets, Lemma 3.10.1. After this the site $(G\text{-}Sets_\alpha, \mathrm{Cov}_{\kappa,\alpha'}(G\text{-}Sets_\alpha))$ gotten by suitably restricting the collection of coverings as in Sets, Lemma 3.11.1 will be denoted $\mathcal{T}_G$.

As a special case, if the group G is countable, then we can let $\mathcal{T}_G$ be the category of countable G-sets and coverings those jointly surjective families of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ such that I is countable.

07GE **Example 7.6.6.** Let $\mathcal{C}$ be a category. There is a canonical way to turn this into a site where $\{f : V \to U \mid f \text{ is an isomorphism}\}$ are the coverings of U. Sheaves on this site are the presheaves on $\mathcal{C}$. This corresponding topology is called the *chaotic* or *indiscrete topology*.

7.7. Sheaves

00VL Let $\mathcal{C}$ be a site. Before we introduce the notion of a sheaf with values in a category we explain what it means for a presheaf of sets to be a sheaf. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$ and let $\{U_i \to U\}_{i\in I}$ be an element of $\mathrm{Cov}(\mathcal{C})$. By assumption all the fibre products $U_i \times_U U_j$ exist in $\mathcal{C}$. There are two natural maps

$$\prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

which we will denote pr_i^*, $i = 0, 1$ as indicated in the displayed equation. Namely, an element of the left hand side corresponds to a family $(s_i)_{i\in I}$, where each s_i is a section of $\mathcal{F}$ over U_i. For each pair $(i_0, i_1) \in I \times I$ we have the projection morphisms

$$\mathrm{pr}_{i_0}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_0} \text{ and } \mathrm{pr}_{i_1}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_1}.$$

Thus we may pull back either the section s_{i_0} via the first of these maps or the section s_{i_1} via the second. Explicitly the maps we referred to above are

$$\mathrm{pr}_0^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_0}^{(i_0,i_1),*}(s_{i_0})\right)_{(i_0,i_1)\in I\times I}$$

and

$$\mathrm{pr}_1^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_1}^{(i_0,i_1),*}(s_{i_1})\right)_{(i_0,i_1)\in I\times I}.$$

Finally consider the natural map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i), \quad s \longmapsto (s|_{U_i})_{i\in I}$$

where we have used the notation $s|_{U_i}$ to indicate the pullback of s via the map $U_i \to U$. It is clear from the functorial nature of $\mathcal{F}$ and the commutativity of the fibre product diagrams that $\mathrm{pr}_0^*((s|_{U_i})_{i\in I}) = \mathrm{pr}_1^*((s|_{U_i})_{i\in I})$.

00VM **Definition 7.7.1.** Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

00VN (7.7.1.1)
$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Loosely speaking this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for all pairs $(i, j) \in I \times I$ then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

04B3 **Remark 7.7.2.** If the covering $\{U_i \to U\}_{i\in I}$ is the empty family (this means that $I = \emptyset$), then the sheaf condition signifies that $\mathcal{F}(U) = \{*\}$ is a singleton set. This is because in (7.7.1.1) the second and third sets are empty products in the category of sets, which are final objects in the category of sets, hence singletons.

00VO **Example 7.7.3.** Let X be a topological space. Let X_{Zar} be the site constructed in Example 7.6.4. The notion of a sheaf on X_{Zar} coincides with the notion of a sheaf on X introduced in Sheaves, Definition 6.7.1.

00VP **Example 7.7.4.** Let X be a topological space. Let us consider the site X'_{Zar} which is the same as the site X_{Zar} of Example 7.6.4 except that we disallow the empty covering of the empty set. In other words, we do allow the covering $\{\emptyset \to \emptyset\}$ but we do not allow the covering whose index set is empty. It is easy to show that this still defines a site. However, we claim that the sheaves on X'_{Zar} are different from the sheaves on X_{Zar}. For example, as an extreme case consider the situation where $X = \{p\}$ is a singleton. Then the objects of X'_{Zar} are $\emptyset, X$ and every covering of $\emptyset$ can be refined by $\{\emptyset \to \emptyset\}$ and every covering of X by $\{X \to X\}$. Clearly, a sheaf on this is given by any choice of a set $\mathcal{F}(\emptyset)$ and any choice of a set $\mathcal{F}(X)$, together with any restriction map $\mathcal{F}(X) \to \mathcal{F}(\emptyset)$. Thus sheaves on X'_{Zar} are the same as usual sheaves on the two point space $\{\eta, p\}$ with open sets $\{\emptyset, \{\eta\}, \{p, \eta\}\}$. In general sheaves on X'_{Zar} are the same as sheaves on the space $X \amalg \{\eta\}$, with opens given by the empty set and any set of the form $U \cup \{\eta\}$ for $U \subset X$ open.

00VQ **Definition 7.7.5.** The category $Sh(\mathcal{C})$ of sheaves of sets is the full subcategory of the category $PSh(\mathcal{C})$ whose objects are the sheaves of sets.

Let $\mathcal{A}$ be a category. If products indexed by I, and $I \times I$ exist in $\mathcal{A}$ for any I that occurs as an index set for covering families then Definition 7.7.1 above makes sense, and defines a notion of a sheaf on $\mathcal{C}$ with values in $\mathcal{A}$. Note that the diagram in $\mathcal{A}$

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram if and only if for every object X of $\mathcal{A}$ the diagram of sets

$$\mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)) \longrightarrow \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_i)) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_{i_0} \times_U U_{i_1}))$$

is an equalizer diagram.

Suppose $\mathcal{A}$ is arbitrary. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{A}$. Choose any object $X \in \mathrm{Ob}(\mathcal{A})$. Then we get a presheaf of sets $\mathcal{F}_X$ defined by the rule

$$\mathcal{F}_X(U) = \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)).$$

From the above it follows that a good definition is obtained by requiring all the presheaves $\mathcal{F}_X$ to be sheaves of sets.

00VR **Definition 7.7.6.** Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a *sheaf* if for all objects X of $\mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ (defined above) is a sheaf.

7.8. Families of morphisms with fixed target

00VS This section is meant to introduce some notions regarding families of morphisms with the same target.

00VT **Definition 7.8.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another.

(1) A *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, or simply a *morphism from $\mathcal{U}$ to $\mathcal{V}$* is given by a morphism $U \to V$, a map

of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is commutative.

(2) In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a *refinement* of the family $\mathcal{V}$.

A trivial but important remark is that if $\mathcal{V} = \{V_j \to V\}_{j \in J}$ is the *empty family of maps*, i.e., if $J = \emptyset$, then no family $\mathcal{U} = \{U_i \to V\}_{i \in I}$ with $I \neq \emptyset$ can refine $\mathcal{V}$!

00VU **Definition 7.8.2.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with fixed target.

(1) We say $\mathcal{U}$ and $\mathcal{V}$ are *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$.

(2) We say $\mathcal{U}$ and $\mathcal{V}$ are *tautologically equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ and for all $i \in I$ and $j \in J$ commutative diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ & \searrow \quad \swarrow & \\ & U & \end{array} \qquad \begin{array}{ccc} V_j & \longrightarrow & U_{\beta(j)} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

with isomorphisms as horizontal arrows.

00VV **Lemma 7.8.3.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target.*

(1) *If $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent then they are tautologically equivalent.*

(2) *If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent then $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

(3) *The relation "being combinatorially equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(4) *The relation "being tautologically equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(5) *The relation "$\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{U}$" is an equivalence relation on all families of morphisms with fixed target.*

Proof. Omitted. □

In the following lemma, given a category $\mathcal{C}$, a presheaf $\mathcal{F}$ on $\mathcal{C}$, a family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that all fibre products $U_i \times_U U_{i'}$ exist, we say that *the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$* holds if the diagram (7.7.1.1) is an equalizer diagram.

00VW **Lemma 7.8.4.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target. Assume that the fibre products $U_i \times_U U_{i'}$ and $V_j \times_U V_{j'}$ exist. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent, then for any presheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ is equivalent to the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$.*

Proof. First, note that if $\varphi : A \to B$ is an isomorphism in the category $\mathcal{C}$, then $\varphi^* : \mathcal{F}(B) \to \mathcal{F}(A)$ is an isomorphism. Let $\beta : J \to I$ be a map and let $\chi_j : V_j \to U_{\beta(j)}$ be isomorphisms over U which are assumed to exist by hypothesis. Let us show that the sheaf condition for $\mathcal{V}$ implies the sheaf condition for $\mathcal{U}$. Suppose given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$$

in $\mathcal{F}(U_i \times_U U_{i'})$ for all pairs $(i, i') \in I \times I$. Then we can define $s_j = \chi_j^* s_{\beta(j)}$. For any pair $(j, j') \in J \times J$ the morphism $\chi_j \times_{\mathrm{id}_U} \chi_{j'} : V_j \times_U V_{j'} \to U_{\beta(j)} \times_U U_{\beta(j')}$ is an isomorphism as well. Hence by transport of structure we see that

$$s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$$

as well. The sheaf condition w.r.t. $\mathcal{V}$ implies there exists a unique s such that $s|_{V_j} = s_j$ for all $j \in J$. By the first remark of the proof this implies that $s|_{U_i} = s_i$ for all $i \in \mathrm{Im}(\beta)$ as well. Suppose that $i \in I$, $i \notin \mathrm{Im}(\beta)$. For such an i we have isomorphisms $U_i \to V_{\alpha(i)} \to U_{\beta(\alpha(i))}$ over U. This gives a morphism $U_i \to U_i \times_U U_{\beta(\alpha(i))}$ which is a section of the projection. Because s_i and $s_{\beta(\alpha(i))}$ restrict to the same element on the fibre product we conclude that $s_{\beta(\alpha(i))}$ pulls back to s_i via $U_i \to U_{\beta(\alpha(i))}$. Thus we see that also $s_i = s|_{U_i}$ as desired. □

0G1K **Lemma 7.8.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $id : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If $\mathcal{F}(U) \to \prod_{j \in J} \mathcal{F}(V_j)$ is injective then $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective.*

Proof. Omitted. □

0G1L **Lemma 7.8.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $id : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If*

1. *the fibre products $U_i \times_U U_{i'}$, $U_i \times_U V_j$, $V_j \times_U V_{j'}$ exist,*
2. *$\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{V}$, and*
3. *for every $i \in I$ the map $\mathcal{F}(U_i) \to \prod_{j \in J} \mathcal{F}(V_j \times_U U_i)$ is injective.*

Then $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{U}$.

Proof. By Lemma 7.8.5 the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective. Suppose given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Set $s_j = f_j^*(s_{\alpha(j)}) \in \mathcal{F}(V_j)$. Since the morphisms f_j are morphisms over U we obtain induced morphisms $f_{jj'} : V_j \times_U V_{j'} \to U_{\alpha(i)} \times_U U_{\alpha(i')}$ compatible with the $f_j, f_{j'}$ via the projection maps. It follows that

$$s_j|_{V_j \times_U V_{j'}} = f_{jj'}^*(s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = f_{jj'}^*(s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = s_{j'}|_{V_j \times_U V_{j'}}$$

for all $j, j' \in J$. Hence, by the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$, we get a section $s \in \mathcal{F}(U)$ which restricts to s_j on each V_j. We are done if we show s restricts to s_i on U_i for any $i \in I$. Since $\mathcal{F}$ satisfies (3) it suffices to show that s and s_i restrict to the same element over $U_i \times_U V_j$ for all $j \in J$. To see this we use

$$s|_{U_i \times_U V_j} = s_j|_{U_i \times_U V_j} = (\mathrm{id} \times f_j)^* s_{\alpha(j)}|_{U_i \times_U U_{\alpha(j)}} = (\mathrm{id} \times f_j)^* s_i|_{U_i \times_U U_{\alpha(j)}} = s_i|_{U_i \times_U V_j}$$

as desired. □

00VX **Lemma 7.8.7.** *Let $\mathcal{C}$ be a category. Let Cov_i, $i = 1, 2$ be two sets of families of morphisms with fixed target which each define the structure of a site on $\mathcal{C}$.*

(1) *If every $\mathcal{U} \in Cov_1$ is tautologically equivalent to some $\mathcal{V} \in Cov_2$, then $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also, every $\mathcal{U} \in Cov_2$ is tautologically equivalent to some $\mathcal{V} \in Cov_1$ then the category of sheaves are equal.*
(2) *Suppose that for each $\mathcal{U} \in Cov_1$ there exists a $\mathcal{V} \in Cov_2$ such that $\mathcal{V}$ refines $\mathcal{U}$. In this case $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also for every $\mathcal{U} \in Cov_2$ there exists a $\mathcal{V} \in Cov_1$ such that $\mathcal{V}$ refines $\mathcal{U}$, then the categories of sheaves are equal.*

Proof. Part (1) follows directly from Lemma 7.8.4 and the definitions.

Proof of (2). Let $\mathcal{F}$ be a sheaf of sets for the site $(\mathcal{C}, \mathrm{Cov}_2)$. Let $\mathcal{U} \in \mathrm{Cov}_1$, say $\mathcal{U} = \{U_i \to U\}_{i \in I}$. By assumption we may choose a refinement $\mathcal{V} \in \mathrm{Cov}_2$ of $\mathcal{U}$, say $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and refinement given by $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Observe that $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{V}$ and for the coverings $\{V_j \times_U U_i \to U_i\}_{j \in J}$ as these are in Cov_2. Hence $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{U}$ by Lemma 7.8.6. □

00VY **Lemma 7.8.8.** *Let $\mathcal{C}$ be a category. Let $Cov(\mathcal{C})$ be a proper class of coverings satisfying conditions (1), (2) and (3) of Definition 7.6.2. Let $Cov_1, Cov_2 \subset Cov(\mathcal{C})$ be two subsets of $Cov(\mathcal{C})$ which endow $\mathcal{C}$ with the structure of a site. If every covering $\mathcal{U} \in Cov(\mathcal{C})$ is combinatorially equivalent to a covering in Cov_1 and combinatorially equivalent to a covering in Cov_2, then $Sh(\mathcal{C}, Cov_1) = Sh(\mathcal{C}, Cov_2)$.*

Proof. This is clear from Lemmas 7.8.7 and 7.8.3 above as the hypothesis implies that every covering $\mathcal{U} \in \mathrm{Cov}_1 \subset \mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to an element of Cov_2, and similarly with the roles of Cov_1 and Cov_2 reversed. □

7.9. The example of G-sets

00VZ As an example, consider the site $\mathcal{T}_G$ of Example 7.6.5. We will describe the category of sheaves on $\mathcal{T}_G$. The answer will turn out to be independent of the choices made in defining $\mathcal{T}_G$. In fact, during the proof we will need only the following properties of the site $\mathcal{T}_G$:

(a) $\mathcal{T}_G$ is a full subcategory of G-*Sets*,
(b) $\mathcal{T}_G$ contains the G-set ${}_GG$,
(c) $\mathcal{T}_G$ has fibre products and they are the same as in G-*Sets*,
(d) given $U \in \mathrm{Ob}(\mathcal{T}_G)$ and a G-invariant subset $O \subset U$, there exists an object of $\mathcal{T}_G$ isomorphic to O, and
(e) any surjective family of maps $\{U_i \to U\}_{i \in I}$, with $U, U_i \in \mathrm{Ob}(\mathcal{T}_G)$ is combinatorially equivalent to a covering of $\mathcal{T}_G$.

These properties hold by Sets, Lemmas 3.10.2 and 3.11.1.

Remark that the map

$$\mathrm{Hom}_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $R_g : s \mapsto sg$ (i.e. right multiplication). Note that $R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$ so the opposite is necessary.

This implies that for every presheaf $\mathcal{F}$ on $\mathcal{T}_G$ the value $\mathcal{F}({}_GG)$ inherits the structure of a G-set as follows: $g \cdot s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ defined by $\mathcal{F}(R_g)(s)$. This is a left action because

$$(g_1 g_2) \cdot s = \mathcal{F}(R_{g_1 g_2})(s) = \mathcal{F}(R_{g_2} \circ R_{g_1})(s) = \mathcal{F}(R_{g_1})(\mathcal{F}(R_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

Here we've used that $\mathcal{F}$ is contravariant. Note that if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{T}_G$ then we get a map $\mathcal{F}({}_G G) \to \mathcal{G}({}_G G)$ which is compatible with the G-actions we have just defined. All in all we have constructed a functor

$$PSh(\mathcal{T}_G) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}({}_G G).$$

We leave it to the reader to verify that this construction has the pleasing property that the representable presheaf h_U is mapped to something canonically isomorphic to U. In a formula $h_U({}_G G) = \mathrm{Hom}_G({}_G G, U) \cong U$.

Suppose that S is a G-set. We define a presheaf $\mathcal{F}_S$ by the formula[2]

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S).$$

This is clearly a presheaf. On the other hand, suppose that $\{U_i \to U\}_{i \in I}$ is a covering in $\mathcal{T}_G$. This implies that $\coprod_i U_i \to U$ is surjective. Thus it is clear that the map

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S) \longrightarrow \prod \mathcal{F}_S(U_i) = \prod \mathrm{Mor}_{G\text{-}Sets}(U_i, S)$$

is injective. And, given a family of G-equivariant maps $s_i : U_i \to S$, such that all the diagrams

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle s_j} \\ U_i & \xrightarrow{s_i} & S \end{array}$$

commute, there is a unique G-equivariant map $s : U \to S$ such that s_i is the composition $U_i \to U \to S$. Namely, we just define $s(u) = s_i(u_i)$ where $i \in I$ is any index such that there exists some $u_i \in U_i$ mapping to u under the map $U_i \to U$. The commutativity of the diagrams above implies exactly that this construction is well defined. All in all we have constructed a functor

$$G\text{-}Sets \longrightarrow Sh(\mathcal{T}_G), \quad S \longmapsto \mathcal{F}_S.$$

We now have the following diagram of categories and functors

$$\begin{array}{ccc} PSh(\mathcal{T}_G) & \xrightarrow{\mathcal{F} \mapsto \mathcal{F}({}_G G)} & G\text{-}Sets \\ & \nwarrow \quad \swarrow_{S \mapsto \mathcal{F}_S} & \\ & Sh(\mathcal{T}_G) & \end{array}$$

It is immediate from the definitions that $\mathcal{F}_S({}_G G) = \mathrm{Mor}_G({}_G G, S) = S$, the last equality by evaluation at 1. This almost proves the following.

00W0 **Proposition 7.9.1.** *The functors $\mathcal{F} \mapsto \mathcal{F}({}_G G)$ and $S \mapsto \mathcal{F}_S$ define quasi-inverse equivalences between $Sh(\mathcal{T}_G)$ and G-Sets.*

Proof. We have already seen that composing the functors one way around is isomorphic to the identity functor. In the other direction, for any sheaf $\mathcal{H}$ there is a natural map of sheaves

$$can : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}({}_G G)}.$$

[2]It may appear this is the representable presheaf defined by S. This may not be the case because S may not be an object of $\mathcal{T}_G$ which was chosen to be a sufficiently large set of G-sets.

Namely, for any object U of $\mathcal{T}_G$ we let can_U be the map

$$\begin{array}{ccc} \mathcal{H}(U) & \longrightarrow & \mathcal{F}_{\mathcal{H}({}_GG)}(U) = \mathrm{Mor}_G(U, \mathcal{H}({}_GG)) \\ s & \longmapsto & (u \mapsto \alpha_u^* s). \end{array}$$

Here $\alpha_u : {}_GG \to U$ is the map $\alpha_u(g) = gu$ and $\alpha_u^* : \mathcal{H}(U) \to \mathcal{H}({}_GG)$ is the pullback map. A trivial but confusing verification shows that this is indeed a map of presheaves. We have to show that *can* is an isomorphism. We do this by showing can_U is an isomorphism for all $U \in \mathrm{Ob}(\mathcal{T}_G)$. We leave the (important but easy) case that $U = {}_GG$ to the reader. A general object U of $\mathcal{T}_G$ is a disjoint union of G-orbits: $U = \coprod_{i \in I} O_i$. The family of maps $\{O_i \to U\}_{i \in I}$ is tautologically equivalent to a covering in $\mathcal{T}_G$ (by the properties of $\mathcal{T}_G$ listed at the beginning of this section). Hence by Lemma 7.8.4 the sheaf $\mathcal{H}$ satisfies the sheaf property with respect to $\{O_i \to U\}_{i \in I}$. The sheaf property for this covering implies $\mathcal{H}(U) = \prod_i \mathcal{H}(O_i)$. Hence it suffices to show that can_U is an isomorphism when U consists of a single G-orbit. Let $u \in U$ and let $H \subset G$ be its stabilizer. Clearly, $\mathrm{Mor}_G(U, \mathcal{H}({}_GG)) = \mathcal{H}({}_GG)^H$ equals the subset of H-invariant elements. On the other hand consider the covering $\{{}_GG \to U\}$ given by $g \mapsto gu$ (again it is just combinatorially equivalent to some covering of $\mathcal{T}_G$, and again this doesn't matter). Note that the fibre product $({}_GG) \times_U ({}_GG)$ is equal to $\{(g, gh), g \in G, h \in H\} \cong \coprod_{h \in H} {}_GG$. Hence the sheaf property for this covering reads as

$$\mathcal{H}(U) \longrightarrow \mathcal{H}({}_GG) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{h \in H} \mathcal{H}({}_GG).$$

The two maps pr_i^* into the factor $\mathcal{H}({}_GG)$ differ by multiplication by h. Now the result follows from this and the fact that *can* is an isomorphism for $U = {}_GG$. □

7.10. Sheafification

00W1 In order to define the sheafification we study the zeroth Čech cohomology group of a covering and its functoriality properties.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i \in I} \in \prod\nolimits_i \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \ \forall i, j \in I\}.$$

As we will see later, this is the zeroth Čech cohomology of $\mathcal{F}$ over U with respect to the covering $\mathcal{U}$. A small remark is that we can define $H^0(\mathcal{U}, \mathcal{F})$ as soon as all the morphisms $U_i \to U$ are representable, i.e., $\mathcal{U}$ need not be a covering of the site. There is a canonical map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams

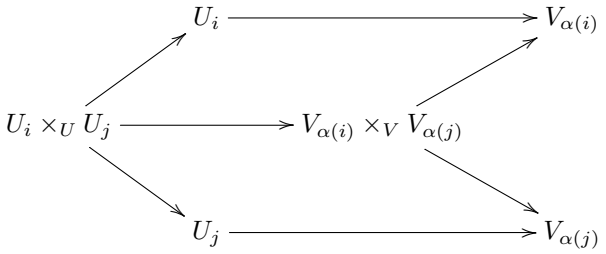

This in turn produces a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

By construction, a presheaf $\mathcal{F}$ is a sheaf if and only if for every covering $\mathcal{U}$ of $\mathcal{C}$ the natural map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ is bijective. We will use this notion to prove the following simple lemma about limits of sheaves.

00W2 **Lemma 7.10.1.** *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\lim_{\mathcal{I}} \mathcal{F}$ exists and is equal to the limit in the category of presheaves.*

Proof. Let $\lim_i \mathcal{F}_i$ be the limit as a presheaf. We will show that this is a sheaf and then it will trivially follow that it is a limit in the category of sheaves. To prove the sheaf property, let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be a covering. Let $(s_j)_{j \in J}$ be an element of $H^0(\mathcal{V}, \lim_i \mathcal{F}_i)$. Using the projection maps we get elements $(s_{j,i})_{j \in J}$ in $H^0(\mathcal{V}, \mathcal{F}_i)$. By the sheaf property for $\mathcal{F}_i$ we see that there is a unique $s_i \in \mathcal{F}_i(V)$ such that $s_{j,i} = s_i|_{V_j}$. Let $\phi : i \to i'$ be a morphism of the index category. We would like to show that $\mathcal{F}(\phi) : \mathcal{F}_i \to \mathcal{F}_{i'}$ maps s_i to $s_{i'}$. We know this is true for the sections $s_{i,j}$ and $s_{i',j}$ for all j and hence by the sheaf property for $\mathcal{F}_{i'}$ this is true. At this point we have an element $s = (s_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of $(\lim_i \mathcal{F}_i)(V)$. We leave it to the reader to see this element has the required property that $s_j = s|_{V_j}$. □

00W3 **Example 7.10.2.** A particular example is the limit over the empty diagram. This gives the final object in the category of (pre)sheaves. It is the presheaf that associates to each object U of $\mathcal{C}$ a singleton set, with unique restriction mappings and moreover this presheaf is a sheaf. We often denote this sheaf by $*$.

Let $\mathcal{J}_U$ be the category of all coverings of U. In other words, the objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set. Note that $\mathrm{Ob}(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. According to the remarks above the construction $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{F})$ is a contravariant functor on $\mathcal{J}_U$. We define

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{opp}} H^0(\mathcal{U}, \mathcal{F})$$

See Categories, Section 4.14 for a discussion of limits and colimits. We point out that later we will see that $\mathcal{F}^+(U)$ is the zeroth Čech cohomology of $\mathcal{F}$ over U.

Before we say more about the structure of the colimit, we turn the collection of sets $\mathcal{F}^+(U)$, $U \in \mathrm{Ob}(\mathcal{C})$ into a presheaf. Namely, let $V \to U$ be a morphism of $\mathcal{C}$. By the axioms of a site there is a functor[3]

$$\mathcal{J}_U \longrightarrow \mathcal{J}_V, \quad \{U_i \to U\} \longmapsto \{U_i \times_U V \to V\}.$$

Note that the projection maps furnish a functorial morphism of coverings $\{U_i \times_U V \to V\} \to \{U_i \to U\}$ and hence, by the construction above, a functorial map of sets $H^0(\{U_i \to U\}, \mathcal{F}) \to H^0(\{U_i \times_U V \to V\}, \mathcal{F})$. In other words, there is a transformation of functors from $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to Sets$ to the composition $\mathcal{J}_U^{opp} \to \mathcal{J}_V^{opp} \xrightarrow{H^0(-,\mathcal{F})} Sets$. Hence by generalities of colimits we obtain a canonical map $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. In terms of the description of the set $\mathcal{F}^+(U)$ above, it just takes the element associated with $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$ to the element associated with $(s_i|_{V \times_U U_i}) \in H^0(\{U_i \times_U V \to V\}, \mathcal{F})$.

[3]This construction actually involves a choice of the fibre products $U_i \times_U V$ and hence the axiom of choice. The resulting map does not depend on the choices made, see below.

00W4 **Lemma 7.10.3.** *The constructions above define a presheaf $\mathcal{F}^+$ together with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*

Proof. All we have to do is to show that given morphisms $W \to V \to U$ the composition $\mathcal{F}^+(U) \to \mathcal{F}^+(V) \to \mathcal{F}^+(W)$ equals the map $\mathcal{F}^+(U) \to \mathcal{F}^+(W)$. This can be shown directly by verifying that, given a covering $\{U_i \to U\}$ and $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$, we have canonically $W \times_U U_i \cong W \times_V (V \times_U U_i)$, and $s_i|_{W \times_U U_i}$ corresponds to $(s_i|_{V \times_U U_i})|_{W \times_V (V \times_U U_i)}$ via this isomorphism. □

More indirectly, the result of Lemma 7.10.6 shows that we may pullback an element s as above via any morphism from any covering of W to $\{U_i \to U\}$ and we will always end up with the same element in $\mathcal{F}^+(W)$.

00W5 **Lemma 7.10.4.** *The association $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^+)$ is a functor.*

Proof. Instead of proving this we state exactly what needs to be proven. Let $\mathcal{F} \to \mathcal{G}$ be a map of presheaves. Prove the commutativity of:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ \end{array}$$

□

The next two lemmas imply that the colimits above are colimits over a directed set.

00W6 **Lemma 7.10.5.** *Given a pair of coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ of a given object U of the site $\mathcal{C}$, there exists a covering which is a common refinement.*

Proof. Since $\mathcal{C}$ is a site we have that for every i the family $\{V_j \times_U U_i \to U_i\}_j$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j}$ is a covering of U. Clearly this covering refines both given coverings. □

00W7 **Lemma 7.10.6.** *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.*

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha : I \to J$ and maps $f_i : U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta : I \to J$ and maps $g_i : U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram

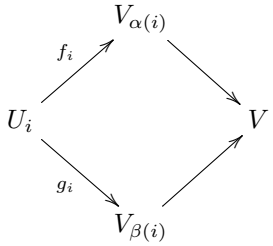

is commutative for every $i \in I$. Hence f and g factor through the fibre product

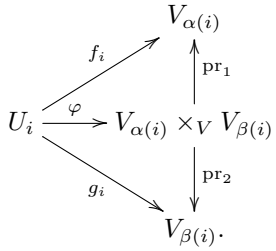

Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i,$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal. □

00W8 **Remark 7.10.7.** In particular this lemma shows that if $\mathcal{U}$ is a refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical identification $H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{V}, \mathcal{F})$.

From these two lemmas, and the fact that $\mathcal{J}_U$ is nonempty, it follows that the diagram $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to \mathit{Sets}$ is filtered, see Categories, Definition 4.19.1. Hence, by Categories, Section 4.19 the colimit $\mathcal{F}^+(U)$ may be described in the following straightforward manner. Namely, every element in the set $\mathcal{F}^+(U)$ arises from an element $s \in H^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U}$ of U. Given a second element $s' \in H^0(\mathcal{U}', \mathcal{F})$ then s and s' determine the same element of the colimit if and only if there exists a covering $\mathcal{V}$ of U and refinements $f : \mathcal{V} \to \mathcal{U}$ and $f' : \mathcal{V} \to \mathcal{U}'$ such that $f^*s = (f')^*s'$ in $H^0(\mathcal{V}, \mathcal{F})$. Since the trivial covering $\{\mathrm{id}_U\}$ is an object of $\mathcal{J}_U$ we get a canonical map $\mathcal{F}(U) \to \mathcal{F}^+(U)$.

00W9 **Lemma 7.10.8.** *The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. Namely, let $\{U_i \to U\}$ be a covering such that s arises from the element $(s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$. According to Lemma 7.10.6 we may consider the covering $\{U_i \to U_i\}$ and the (obvious) morphism of coverings $\{U_i \to U_i\} \to \{U_i \to U\}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And indeed, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

00WA **Definition 7.10.9.** We say that a presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is *separated* if, for all coverings of $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

00WB **Theorem 7.10.10.** *With $\mathcal{F}$ as above*

00WC (1) *The presheaf $\mathcal{F}^+$ is separated.*

00WD (2) *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\mathcal{F} \to \mathcal{F}^+$ is injective.*

00WE (3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

00WF (4) *The presheaf $\mathcal{F}^{++}$ is always a sheaf.*

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three

coverings of U: the covering $\{U_i \to U\}$ above, a covering $\mathcal{U}$ for s as in Lemma 7.10.8, and a similar covering $\mathcal{U}'$ for s'. By Lemma 7.10.5, we can find a common refinement, say $\{W_j \to U\}$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering we see that s, s' map to the same element of $H^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $H^0(\mathcal{U}', \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}$ is a covering, and let (s_i) be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Choose coverings (as in Lemma 7.10.8) $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have the equality there. Hence $(s_{ij}) \in H^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. We leave it to the reader to verify that $s|_{U_i} = s_i$.

Proof of (3). This is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to H^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

00WG **Definition 7.10.11.** Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

00WH **Proposition 7.10.12.** *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. By Lemma 7.10.4 we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \mathcal{F}^{++} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ & \longrightarrow & \mathcal{G}^{++} \end{array}$$

and by Theorem 7.10.10 the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 7.10.8 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

It is clear from this result that the functor $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^\#)$ is unique up to unique isomorphism of functors. Actually, let us temporarily denote $i : Sh(\mathcal{C}) \to PSh(\mathcal{C})$ the functor of inclusion. The result above actually says that

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}).$$

In other words, the functor of sheafification is the left adjoint to the inclusion functor i.

0H7J **Example 7.10.13.** Let $\mathcal{C}$ be a site. Let E be a set. The *constant sheaf* $\underline{E}$ *with value* E is the sheafification of the presheaf given by the constant functor with value E. It is characterized by the rule $\mathrm{Mor}_{Sh(\mathcal{C})}(\underline{E}, \mathcal{F}) = \mathrm{Mor}_{Sets}(E, \Gamma(\mathcal{C}, \mathcal{F}))$ for every sheaf $\mathcal{F}$ on $\mathcal{C}$ where the notation is from Definition 7.45.1.

We finish this section with a couple of lemmas.

00WI **Lemma 7.10.14.** *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\operatorname{colim}_{\mathcal{I}} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.*

Proof. Since the sheafification functor is a left adjoint it commutes with all colimits, see Categories, Lemma 4.24.5. Hence, since $PSh(\mathcal{C})$ has colimits, we deduce that $Sh(\mathcal{C})$ has colimits (which are the sheafifications of the colimits in presheaves). □

00WJ **Lemma 7.10.15.** *The functor $PSh(\mathcal{C}) \to Sh(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}^\#$ is exact.*

Proof. Since it is a left adjoint it is right exact, see Categories, Lemma 4.24.6. On the other hand, by Lemmas 7.10.5 and Lemma 7.10.6 the colimits in the construction of $\mathcal{F}^+$ are really over the directed set $\mathrm{Ob}(\mathcal{J}_U)$ where $\mathcal{U} \geq \mathcal{U}'$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Hence by Categories, Lemma 4.19.2 we see that $\mathcal{F} \to \mathcal{F}^+$ commutes with finite limits (as a functor from presheaves to presheaves). Then we conclude using Lemma 7.10.1. □

00WK **Lemma 7.10.16.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. Denote $\theta^2 : \mathcal{F} \to \mathcal{F}^\#$ the canonical map of $\mathcal{F}$ into its sheafification. Let U be an object of $\mathcal{C}$. Let $s \in \mathcal{F}^\#(U)$. There exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{F}(U_i)$ such that*

(1) *$s|_{U_i} = \theta^2(s_i)$, and*
(2) *for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{C}$ such that the pullbacks of s_i and s_j to each U_{ijk} agree.*

Conversely, given any covering $\{U_i \to U\}$, elements $s_i \in \mathcal{F}(U_i)$ such that (2) holds, then there exists a unique section $s \in \mathcal{F}^\#(U)$ such that (1) holds.

Proof. Omitted. □

0H6Y **Lemma 7.10.17.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets on $\mathcal{C}$. Denote $\mathcal{B}$ the set of $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is bijective. If every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$, then $\mathcal{F}^\# \to \mathcal{G}^\#$ is an isomorphism.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let us prove that $\mathcal{F}^\#(U) \to \mathcal{G}^\#(U)$ is surjective. To do this we will use Lemma 7.10.16 without further mention. For any $s \in \mathcal{G}^\#(U)$ there exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{G}(U_i)$ such that

(1) $s|_{U_i}$ is the image of s_i via $\mathcal{G} \to \mathcal{G}^\#$, and
(2) for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{D}$ such that the pullbacks of s_i and s_j to each U_{ijk} agree.

By assumption, for each i we may choose a covering $\{U_{i,a} \to U_i\}$ with $U_{i,a} \in \mathcal{B}$. Then $\{U_{i,a} \to U\}$ is a covering. Denoting $s_{i,a}$ the image of s_i in $\mathcal{G}(U_{i,a})$ we see that the pullbacks of $s_{i,a}$ and $s_{j,b}$ to the members of the covering

$$\{(U_{i,a} \times_U U_{j,b}) \times_{U_i \times_U U_j} U_{ijk} \to U_{i,a} \times_U U_{j,b}\}$$

agree. Hence we may assume that $U_i \in \mathcal{B}$. Repeating the argument, we may also assume $U_{ijk} \in \mathcal{B}$ for all i, j, k (details omitted). Then since $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ is bijective by our definition of $\mathcal{B}$ in the statement of the lemma, we get unique $t_i \in \mathcal{F}(U_i)$ mapping to s_i. The pullbacks of t_i and t_j to each U_{ijk} agree in $\mathcal{F}(U_{ijk})$ because $U_{ijk} \in \mathcal{B}$ and because we have the agreement for t_i and t_j. Then the lemma tells us there exists a unique $t \in \mathcal{F}^\#(U)$ such that $t|_{U_i}$ is the image of t_i. It follows that t maps to s. We omit the proof that $\mathcal{F}^\#(U) \to \mathcal{G}^\#(U)$ is injective. □

7.11. Injective and surjective maps of sheaves

00WL

00WM **Definition 7.11.1.** Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

00WN **Lemma 7.11.2.** *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of the category $Sh(\mathcal{C})$. A map of sheaves is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

086K **Lemma 7.11.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves of sets. Then the diagram*

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \longrightarrow \mathcal{G}$$

represents $\mathcal{G}$ as a coequalizer.

Proof. Let $\mathcal{H}$ be a sheaf of sets and let $\varphi : \mathcal{F} \to \mathcal{H}$ be a map of sheaves equalizing the two maps $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F}$. Let $\mathcal{G}' \subset \mathcal{G}$ be the presheaf image of the map $\mathcal{F} \to \mathcal{G}$. As the product $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ may be computed in the category of presheaves we see that it is equal to the presheaf product $\mathcal{F} \times_{\mathcal{G}'} \mathcal{F}$. Hence φ induces a unique map of presheaves $\psi' : \mathcal{G}' \to \mathcal{H}$. Since $\mathcal{G}$ is the sheafification of $\mathcal{G}'$ by Lemma 7.11.2 we conclude that ψ' extends uniquely to a map of sheaves $\psi : \mathcal{G} \to \mathcal{H}$. We omit the verification that φ is equal to the composition of ψ and the given map. □

7.12. Representable sheaves

00WO Let $\mathcal{C}$ be a category. The canonical topology is the finest topology such that all representable presheaves are sheaves (it is formally defined in Definition 7.47.12 but we will not need this). This topology is not always the topology associated to the structure of a site on $\mathcal{C}$. We will give a collection of coverings that generates this topology in case $\mathcal{C}$ has fibered products. First we give the following general definition.

00WP **Definition 7.12.1.** Let $\mathcal{C}$ be a category. We say that a family $\{U_i \to U\}_{i \in I}$ is an *effective epimorphism* if all the morphisms $U_i \to U$ are representable (see Categories, Definition 4.6.4), and for any $X \in \mathrm{Ob}(\mathcal{C})$ the sequence

$$\mathrm{Mor}_{\mathcal{C}}(U, X) \longrightarrow \prod_{i \in I} \mathrm{Mor}_{\mathcal{C}}(U_i, X) \rightrightarrows \prod_{(i,j) \in I^2} \mathrm{Mor}_{\mathcal{C}}(U_i \times_U U_j, X)$$

is an equalizer diagram. We say that a family $\{U_i \to U\}$ is a *universal effective epimorphism* if for any morphism $V \to U$ the base change $\{U_i \times_U V \to V\}$ is an effective epimorphism.

The class of families which are universal effective epimorphisms satisfies the axioms of Definition 7.6.2. If $\mathcal{C}$ has fibre products, then the associated topology is the canonical topology. (In this case, to get a site argue as in Sets, Lemma 3.11.1.)

Conversely, suppose that $\mathcal{C}$ is a site such that all representable presheaves are sheaves. Then clearly, all coverings are universal effective epimorphisms. Thus the following definition is the "correct" one in the setting of sites.

00WQ **Definition 7.12.2.** We say that the topology on a site $\mathcal{C}$ is *weaker than the canonical topology*, or that the topology is *subcanonical* if all the coverings of $\mathcal{C}$ are universal effective epimorphisms.

A representable sheaf is a representable presheaf which is also a sheaf. Since it is perhaps better to avoid this terminology when the topology is not subcanonical, we only define it formally in that case.

00WR **Definition 7.12.3.** Let $\mathcal{C}$ be a site whose topology is subcanonical. The Yoneda embedding h (see Categories, Section 4.3) presents $\mathcal{C}$ as a full subcategory of the category of sheaves of $\mathcal{C}$. In this case we call sheaves of the form h_U with $U \in \mathrm{Ob}(\mathcal{C})$ *representable sheaves* on $\mathcal{C}$. Notation: Sometimes, the representable sheaf h_U associated to U is denoted $\underline{U}$.

Note that we have in the situation of the definition

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$, since it holds for presheaves, see (7.2.1.1). In general the presheaves h_U are not sheaves and to get a sheaf you have to sheafify them. In this case we still have

090I (7.12.3.1) $$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$. Namely, the first equality holds by the adjointness property of $\#$ and the second is (7.2.1.1).

00WT **Lemma 7.12.4.** *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism of presheaves of sets*

$$\coprod_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification.

Proof. By Lemma 7.11.2 above we have to show that $\coprod_{i \in I} h_{U_i}^\# \to h_U^\#$ is an epimorphism. Let $\mathcal{F}$ be a sheaf of sets. A morphism $h_U^\# \to \mathcal{F}$ corresponds to a section $s \in \mathcal{F}(U)$. Hence the injectivity of $\mathrm{Mor}(h_U^\#, \mathcal{F}) \to \prod_i \mathrm{Mor}(h_{U_i}^\#, \mathcal{F})$ follows directly from the sheaf property of $\mathcal{F}$. □

The next lemma says, in the case the topology is weaker than the canonical topology, that every sheaf is made up out of representable sheaves in a way.

00WS **Lemma 7.12.5.** *Let $\mathcal{C}$ be a site. Let $E \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object of $\mathcal{C}$ has a covering by elements of E. Let $\mathcal{F}$ be a sheaf of sets. There exists a diagram of sheaves of sets*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

which represents $\mathcal{F}$ as a coequalizer, such that $\mathcal{F}_i$, $i = 0, 1$ are coproducts of sheaves of the form $h_U^\#$ with $U \in E$.

Proof. First we show there is an epimorphism $\mathcal{F}_0 \to \mathcal{F}$ of the desired type. Namely, just take

$$\mathcal{F}_0 = \coprod\nolimits_{U \in E, s \in \mathcal{F}(U)} (h_U)^\# \longrightarrow \mathcal{F}$$

Here the arrow restricted to the component corresponding to (U, s) maps the element $\mathrm{id}_U \in h_U^\#(U)$ to the section $s \in \mathcal{F}(U)$. This is an epimorphism according to Lemma 7.11.2 and our condition on E. To construct $\mathcal{F}_1$ first set $\mathcal{G} = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ and then construct an epimorphism $\mathcal{F}_1 \to \mathcal{G}$ as above. See Lemma 7.11.3. □

0GLW **Lemma 7.12.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$. Then there exists a diagram $\mathcal{I} \to \mathcal{C}$, $i \mapsto U_i$ such that*

$$\mathcal{F} = \mathrm{colim}_{i \in \mathcal{I}} h_{U_i}^\#$$

Moreover, if $E \subset \mathrm{Ob}(\mathcal{C})$ is a subset such that every object of $\mathcal{C}$ has a covering by elements of E, then we may assume U_i is an element of E for all $i \in \mathrm{Ob}(\mathcal{I})$.

Proof. Let $\mathcal{I}$ be the category whose objects are pairs (U, s) with $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$ and whose morphisms $(U, s) \to (U', s')$ are morphisms $f : U \to U'$ in $\mathcal{C}$ with $f^* s' = s$. For each object (U, s) of $\mathcal{I}$ the element s defines by the Yoneda lemma a map $s : h_U \to \mathcal{F}$ of presheaves. Hence by the universal property of sheafification a map $h_U^\# \to \mathcal{F}$. These maps are immediately seen to be compatible with the morphisms in the category $\mathcal{I}$. Hence we obtain a map $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ of presheaves (where the colimit is taken in the category of presheaves) and a map $\mathrm{colim}_{(U,s)} (h_U)^\# \to \mathcal{F}$ of sheaves (where the colimit is taken in the category of sheaves). Since sheafification is the left adjoint to the inclusion functor $Sh(\mathcal{C}) \to PSh(\mathcal{C})$ (Proposition 7.10.12) we have $\mathrm{colim}(h_U)^\# = (\mathrm{colim}\, h_U)^\#$ by Categories, Lemma 4.24.5. Thus it suffices to show that $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ is an isomorphism of presheaves. To see this we show that for every object X of $\mathcal{C}$ the map $\mathrm{colim}_{(U,s)} h_U(X) \to \mathcal{F}(X)$ is bijective. Namely, it has an inverse sending the element $t \in \mathcal{F}(X)$ to the element of the set $\mathrm{colim}_{(U,s)} h_U(X)$ corresponding to (X, t) and $\mathrm{id}_X \in h_X(X)$.

We omit the proof of the final statement. □

7.13. Continuous functors

00WU

00WV **Definition 7.13.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every $\{V_i \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1) $\{u(V_i) \to u(V)\}_{i \in I}$ is in $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Recall that given a functor u as above, and a presheaf of sets $\mathcal{F}$ on $\mathcal{D}$ we have defined $u^p \mathcal{F}$ to be simply the presheaf $\mathcal{F} \circ u$, in other words

$$u^p \mathcal{F}(V) = \mathcal{F}(u(V))$$

for every object V of $\mathcal{C}$.

00WW **Lemma 7.13.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathcal{F}$ is a sheaf on $\mathcal{D}$ then $u^p \mathcal{F}$ is a sheaf as well.*

Proof. Let $\{V_i \to V\}$ be a covering. By assumption $\{u(V_i) \to u(V)\}$ is a covering in $\mathcal{D}$ and $u(V_i \times_V V_j) = u(V_i) \times_{u(V)} u(V_j)$. Hence the sheaf condition for $u^p\mathcal{F}$ and the covering $\{V_i \to V\}$ is precisely the same as the sheaf condition for $\mathcal{F}$ and the covering $\{u(V_i) \to u(V)\}$. □

In order to avoid confusion we sometimes denote

$$u^s : Sh(\mathcal{D}) \longrightarrow Sh(\mathcal{C})$$

the functor u^p restricted to the subcategory of sheaves of sets. Recall that u^p has a left adjoint $u_p : PSh(\mathcal{C}) \to PSh(\mathcal{D})$, see Section 7.5.

00WX **Lemma 7.13.3.** *In the situation of Lemma 7.13.2. The functor $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^\#$ is a left adjoint to u^s.*

Proof. Follows directly from Lemma 7.5.4 and Proposition 7.10.12. □

Here is a technical lemma.

00WY **Lemma 7.13.4.** *In the situation of Lemma 7.13.2. For any presheaf $\mathcal{G}$ on $\mathcal{C}$ we have $(u_p\mathcal{G})^\# = (u_p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{D}$ we have

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{D})}(u_s(\mathcal{G}^\#), \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, u^s\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}^\#, u^p\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, u^p\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}((u_p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

04D3 **Lemma 7.13.5.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. For any object U of $\mathcal{C}$ we have $u_s h_U^\# = h_{u(U)}^\#$.*

Proof. Follows from Lemmas 7.5.6 and 7.13.4. □

00WZ **Remark 7.13.6.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let us use the definition of tautologically equivalent families of maps, see Definition 7.8.2 to (slightly) weaken the conditions defining continuity. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let us call u *quasi-continuous* if for every $\mathcal{V} = \{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1') the family of maps $\{u(V_i) \to u(V)\}_{i\in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

We are going to see that Lemmas 7.13.2 and 7.13.3 hold in case u is quasi-continuous as well.

We first remark that the morphisms $u(V_i) \to u(V)$ are representable, since they are isomorphic to representable morphisms (by the first condition). In particular, the family $u(\mathcal{V}) = \{u(V_i) \to u(V)\}_{i\in I}$ gives rise to a zeroth Čech cohomology group $H^0(u(\mathcal{V}), \mathcal{F})$ for any presheaf $\mathcal{F}$ on $\mathcal{D}$. Let $\mathcal{U} = \{U_j \to u(V)\}_{j\in J}$ be an element of $\mathrm{Cov}(\mathcal{D})$ tautologically equivalent to $\{u(V_i) \to u(V)\}_{i\in I}$. Note that $u(\mathcal{V})$ is a

refinement of $\mathcal{U}$ and vice versa. Hence by Remark 7.10.7 we see that $H^0(u(\mathcal{V}), \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}(u(V)) = H^0(u(\mathcal{V}), \mathcal{F})$ because of the sheaf property expressed in terms of zeroth Čech cohomology groups. We conclude that $u^p\mathcal{F}$ is a sheaf if $\mathcal{F}$ is a sheaf, since $H^0(\mathcal{V}, u^p\mathcal{F}) = H^0(u(\mathcal{V}), \mathcal{F})$ which we just observed is equal to $\mathcal{F}(u(V)) = u^p\mathcal{F}(V)$. Thus Lemma 7.13.2 holds. Lemma 7.13.3 follows immediately.

7.14. Morphisms of sites

00X0

00X1 **Definition 7.14.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Notice how the functor u goes in the direction *opposite* the morphism f. If $f \leftrightarrow u$ is a morphism of sites then we use the notation $f^{-1} = u_s$ and $f_* = u^s$. The functor f^{-1} is called the *pullback functor* and the functor f_* is called the *pushforward functor*. As in topology we have the following adjointness property

$$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$

The motivation for this definition comes from the following example.

00X2 **Example 7.14.2.** Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Consider the functor $u : Y_{Zar} \to X_{Zar}$, $V \mapsto f^{-1}(V)$. This functor is clearly continuous because inverse images of open coverings are open coverings. (Actually, this depends on how you chose sets of coverings for X_{Zar} and Y_{Zar}. But in any case the functor is quasi-continuous, see Remark 7.13.6.) It is easy to verify that the functor u^s equals the usual pushforward functor f_* from topology. Hence, since u_s is an adjoint and since the usual topological pullback functor f^{-1} is an adjoint as well, we get a canonical isomorphism $f^{-1} \cong u_s$. Since f^{-1} is exact we deduce that u_s is exact. Hence u defines a morphism of sites $f : X_{Zar} \to Y_{Zar}$, which we may denote f as well since we've already seen the functors u_s, u^s agree with their usual notions anyway.

0EWI **Example 7.14.3.** Let $\mathcal{C}$ be a category. Let

$$\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$$

be two sets of families of morphisms with fixed target which turn $\mathcal{C}$ into a site. Denote $\mathcal{C}_\tau$ the site corresponding to $\mathrm{Cov}(\mathcal{C})$ and $\mathcal{C}_{\tau'}$ the site corresponding to $\mathrm{Cov}'(\mathcal{C})$. We claim the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

Namely, observe that $\mathrm{id} : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is continuous as every τ'-covering is a τ-covering. Thus the functor $\epsilon_* = \mathrm{id}^s$ is the identity functor on underlying presheaves. Hence the left adjoint ϵ^{-1} of ϵ_* sends a τ'-sheaf $\mathcal{F}$ to the τ-sheafification of $\mathcal{F}$ (by the universal property of sheafification). Finite limits of τ'-sheaves agree with finite limits of presheaves (Lemma 7.10.1) and τ-sheafification commutes with finite limits (Lemma 7.10.15). Thus ϵ^{-1} is left exact. Since ϵ^{-1} is a left adjoint it is also right exact (Categories, Lemma 4.24.6). Thus ϵ^{-1} is exact and we have checked all the conditions of Definition 7.14.1.

03CB **Lemma 7.14.4.** *Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v : \mathcal{C}_3 \to \mathcal{C}_1$ is continuous and defines a morphism of sites $\mathcal{C}_1 \to \mathcal{C}_3$.*

Proof. It is immediate from the definitions that $u \circ v$ is a continuous functor. In addition, we clearly have $(u \circ v)^p = v^p \circ u^p$, and hence $(u \circ v)^s = v^s \circ u^s$. Hence functors $(u \circ v)_s$ and $u_s \circ v_s$ are both left adjoints of $(u \circ v)^s$. Therefore $(u \circ v)_s \cong u_s \circ v_s$ and we conclude that $(u \circ v)_s$ is exact as a composition of exact functors. □

03CC **Definition 7.14.5.** Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ be morphisms of sites given by continuous functors $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$. The *composition* $g \circ f$ is the morphism of sites corresponding to the functor $u \circ v$.

In this situation we have $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (see proof of Lemma 7.14.4).

00X5 **Lemma 7.14.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume all the categories $(\mathcal{I}_V^u)^{opp}$ of Section 7.5 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof. Since u_s is the left adjoint of u^s we see that u_s is right exact, see Categories, Lemma 4.24.6. Hence it suffices to show that u_s is left exact. In other words we have to show that u_s commutes with finite limits. Because the categories $\mathcal{I}_Y^{opp}$ are filtered we see that u_p commutes with finite limits, see Categories, Lemma 4.19.2 (this also uses the description of limits in *PSh*, see Section 7.4). And since sheafification commutes with finite limits as well (Lemma 7.10.15) we conclude because $u_s = \# \circ u_p$. □

00X6 **Proposition 7.14.7.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume furthermore the following:*

(1) *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
(2) *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. This follows from Lemmas 7.5.2 and 7.14.6. □

00X7 **Remark 7.14.8.** The conditions of Proposition 7.14.7 above are equivalent to saying that u is left exact, i.e., commutes with finite limits. See Categories, Lemmas 4.18.4 and 4.23.2. It seems more natural to phrase it in terms of final objects and fibre products since this seems to have more geometric meaning in the examples.

Lemma 7.22.4 will provide another way to prove a continuous functor gives rise to a morphism of sites.

00X8 **Remark 7.14.9.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Analogously to Definition 7.14.1 we say that a *quasi-morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a quasi-continuous functor $u : \mathcal{C} \to \mathcal{D}$ (see Remark 7.13.6) such that u_s is exact. The analogue of Proposition 7.14.7 in this setting is obtained by replacing the word "continuous" by the word "quasi-continuous", and replacing the word "morphism" by "quasi-morphism". The proof is literally the same.

In Definition 7.14.1 the condition that u_s be exact cannot be omitted. For example, the conclusion of the following lemma need not hold if one only assumes that u is continuous.

08H2 **Lemma 7.14.10.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Given any object V of $\mathcal{D}$ there exists a covering $\{V_j \to V\}$ such that for every j there exists a morphism $V_j \to u(U_j)$ for some object U_j of $\mathcal{C}$.*

Proof. Since $f^{-1} = u_s$ is exact we have $f^{-1}* = *$ where $*$ denotes the final object of the category of sheaves (Example 7.10.2). Since $f^{-1}* = u_s*$ is the sheafification of u_p* we see there exists a covering $\{V_j \to V\}$ such that $(u_p*)(V_j)$ is nonempty. Since $(u_p*)(V_j)$ is a colimit over the category $\mathcal{I}^u_{V_j}$ whose objects are morphisms $V_j \to u(U)$ the lemma follows. □

7.15. Topoi

00X9 Here is a definition of a topos which is suitable for our purposes. Namely, a topos is the category of sheaves on a site. In order to specify a topos you just specify the site. The real difference between a topos and a site lies in the definition of morphisms. Namely, it turns out that there are lots of morphisms of topoi which do not come from morphisms of the underlying sites.

00XA **Definition 7.15.1** (Topoi)**.** A *topos* is the category $Sh(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *morphism of topoi* f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ is given by a pair of functors $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that
   (a) we have
$$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
   bifunctorially, and
   (b) the functor f^{-1} commutes with finite limits, i.e., is left exact.

(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $g : Sh(\mathcal{E}) \to Sh(\mathcal{D})$ the *composition* $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Suppose that $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is an equivalence of (possibly "big") categories. If $\mathcal{S}_1$, $\mathcal{S}_2$ are topoi, then setting $f_* = \alpha$ and f^{-1} equal to a quasi-inverse of α gives a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ of topoi. Moreover this morphism is an equivalence in the 2-category of topoi (see Section 7.36). Thus it makes sense to say "$\mathcal{S}$ is a topos" if $\mathcal{S}$ is equivalent to the category of sheaves on a site (and not necessarily equal to the category of sheaves on a site). We will occasionally use this abuse of notation.

The *empty topos* is topos of sheaves on the site $\mathcal{C}$, where $\mathcal{C}$ is the empty category. We will sometimes write $\emptyset$ for this site. This is a site which has a unique sheaf (since $\emptyset$ has no objects). Thus $Sh(\emptyset)$ is equivalent to the category having a single object and a single morphism.

The *punctual topos* is the topos of sheaves on the site $\mathcal{C}$ which has a single object pt and one morphism id_{pt} and whose only covering is the covering $\{\mathrm{id}_{pt}\}$. We will simply write pt for this site. It is clear that the category of sheaves = the category of presheaves = the category of sets. In a formula $Sh(pt) = Sets$.

Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Note that f_* commutes with all limits and that f^{-1} commutes with all colimits, see Categories, Lemma 4.24.5. In particular, the condition on f^{-1} in the definition above guarantees that f^{-1} is exact. Morphisms of topoi are often constructed using either Lemma 7.21.1 or the following lemma.

00XC **Lemma 7.15.2.** *Given a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$ the pair of functors $(f^{-1} = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 7.14.1. □

00XD **Remark 7.15.3.** There are many sites that give rise to the topos $Sh(pt)$. A useful example is the following. Suppose that S is a set (of sets) which contains at least one nonempty element. Let $\mathcal{S}$ be the category whose objects are elements of S and whose morphisms are arbitrary set maps. Assume that $\mathcal{S}$ has fibre products. For example this will be the case if $S = \mathcal{P}(\text{infinite set})$ is the power set of any infinite set (exercise in set theory). Make $\mathcal{S}$ into a site by declaring surjective families of maps to be coverings (and choose a suitable sufficiently large set of covering families as in Sets, Section 3.11). We claim that $Sh(\mathcal{S})$ is equivalent to the category of sets.

We first prove this in case S contains $e \in S$ which is a singleton. In this case, there is an equivalence of topoi $i : Sh(pt) \to Sh(\mathcal{S})$ given by the functors

05UW (7.15.3.1) $$i^{-1}\mathcal{F} = \mathcal{F}(e), \quad i_*E = (U \mapsto \mathrm{Mor}_{Sets}(U, E))$$

Namely, suppose that $\mathcal{F}$ is a sheaf on $\mathcal{S}$. For any $U \in \mathrm{Ob}(\mathcal{S}) = S$ we can find a covering $\{\varphi_u : e \to U\}_{u \in U}$, where φ_u maps the unique element of e to $u \in U$. The sheaf condition implies in this case that $\mathcal{F}(U) = \prod_{u \in U} \mathcal{F}(e)$. In other words $\mathcal{F}(U) = \mathrm{Mor}_{Sets}(U, \mathcal{F}(e))$. Moreover, this rule is compatible with restriction mappings. Hence the functor

$$i_* : Sets = Sh(pt) \longrightarrow Sh(\mathcal{S}), \quad E \longmapsto (U \mapsto \mathrm{Mor}_{Sets}(U, E))$$

is an equivalence of categories, and its inverse is the functor i^{-1} given above.

If $\mathcal{S}$ does not contain a singleton, then the functor i_* as defined above still makes sense. To show that it is still an equivalence in this case, choose any nonempty $\tilde{e} \in S$ and a map $\varphi : \tilde{e} \to \tilde{e}$ whose image is a singleton. For any sheaf $\mathcal{F}$ set

$$\mathcal{F}(e) := \mathrm{Im}(\mathcal{F}(\varphi) : \mathcal{F}(\tilde{e}) \longrightarrow \mathcal{F}(\tilde{e}))$$

and show that this is a quasi-inverse to i_*. Details omitted.

00XB **Remark 7.15.4.** (Set theoretical issues related to morphisms of topoi. Skip on a first reading.) A morphism of topoi as defined above is not a set but a class. In other words it is given by a mathematical formula rather than a mathematical object. Although we may contemplate the collection of all morphisms between two given topoi, it is not a good idea to introduce it as a mathematical object. On the other hand, suppose $\mathcal{C}$ and $\mathcal{D}$ are given sites. Consider a functor $\Phi : \mathcal{C} \to Sh(\mathcal{D})$. Such a thing is a set, in other words, it is a mathematical object. We may, in succession, ask the following questions on Φ.

(1) Is it true, given a sheaf $\mathcal{F}$ on $\mathcal{D}$, that the rule $U \mapsto \mathrm{Mor}_{Sh(\mathcal{D})}(\Phi(U), \mathcal{F})$ defines a sheaf on $\mathcal{C}$? If so, this defines a functor $\Phi_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$.
(2) Is it true that Φ_* has a left adjoint? If so, write Φ^{-1} for this left adjoint.
(3) Is it true that Φ^{-1} is exact?

If the last question still has the answer "yes", then we obtain a morphism of topoi (Φ_*, Φ^{-1}). Moreover, given any morphism of topoi (f_*, f^{-1}) we may set $\Phi(U) = f^{-1}(h_U^\#)$ and obtain a functor Φ as above with $f_* \cong \Phi_*$ and $f^{-1} \cong \Phi^{-1}$ (compatible with adjoint property). The upshot is that by working with the collection of Φ instead of morphisms of topoi, we (a) replaced the notion of a morphism of topoi by a mathematical object, and (b) the collection of Φ forms a class (and not a collection of classes). Of course, more can be said, for example one can work out

more precisely the significance of conditions (2) and (3) above; we do this in the case of points of topoi in Section 7.32.

00XE **Remark 7.15.5.** (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (see Remark 7.14.9) gives rise to a morphism of topoi f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ exactly as in Lemma 7.15.2.

7.16. G-sets and morphisms

04D4 Let $\varphi : G \to H$ be a homomorphism of groups. Choose (suitable) sites $\mathcal{T}_G$ and $\mathcal{T}_H$ as in Example 7.6.5 and Section 7.9. Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor which assigns to a H-set U the G-set U_φ which has the same underlying set but G action defined by $g \cdot \xi = \varphi(g)\xi$ for $g \in G$ and $\xi \in U$. It is clear that u commutes with finite limits and is continuous[4]. Applying Proposition 7.14.7 and Lemma 7.15.2 we obtain a morphism of topoi

$$f : Sh(\mathcal{T}_G) \longrightarrow Sh(\mathcal{T}_H)$$

associated with φ. Using Proposition 7.9.1 we see that we get a pair of adjoint functors

$$f_* : G\text{-}Sets \longrightarrow H\text{-}Sets, \quad f^{-1} : H\text{-}Sets \longrightarrow G\text{-}Sets.$$

Let's work out what are these functors in this case.

We first work out a formula for f_*. Recall that given a G-set S the corresponding sheaf $\mathcal{F}_S$ on $\mathcal{T}_G$ is given by the rule $\mathcal{F}_S(U) = \mathrm{Mor}_G(U, S)$. And on the other hand, given a sheaf $\mathcal{G}$ on $\mathcal{T}_H$ the corresponding H-set is given by the rule $\mathcal{G}({}_H H)$. Hence we see that

$$f_*S = \mathrm{Mor}_{G\text{-}Sets}(({}_H H)_\varphi, S).$$

If we work this out a little bit more then we get

$$f_*S = \{a : H \to S \mid a(gh) = ga(h)\}$$

with left H-action given by $(h \cdot a)(h') = a(h'h)$ for any element $a \in f_*S$.

Next, we explicitly compute f^{-1}. Note that since the topology on $\mathcal{T}_G$ and $\mathcal{T}_H$ is subcanonical, all representable presheaves are sheaves. Moreover, given an object V of $\mathcal{T}_H$ we see that $f^{-1}h_V$ is equal to $h_{u(V)}$ (see Lemma 7.13.5). Hence we see that $f^{-1}S = S_\varphi$ for representable sheaves. Since every sheaf on $\mathcal{T}_H$ is a coproduct of representable sheaves we conclude that this is true in general. Hence we see that for any H-set T we have

$$f^{-1}T = T_\varphi.$$

The adjunction between f^{-1} and f_* is evidenced by the formula

$$\mathrm{Mor}_{G\text{-}Sets}(T_\varphi, S) = \mathrm{Mor}_{H\text{-}Sets}(T, f_*S)$$

with f_*S as above. This can be proved directly. Moreover, it is then clear that (f^{-1}, f_*) form an adjoint pair and that f^{-1} is exact. So alternatively to the above the morphism of topoi $f : G\text{-}Sets \to H\text{-}Sets$ can be defined directly in this manner.

[4]Set theoretical remark: First choose $\mathcal{T}_H$. Then choose $\mathcal{T}_G$ to contain $u(\mathcal{T}_H)$ and such that every covering in $\mathcal{T}_H$ corresponds to a covering in $\mathcal{T}_G$. This is possible by Sets, Lemmas 3.10.1, 3.10.2 and 3.11.1.

7.17. Quasi-compact objects and colimits

090G To be able to use the same language as in the case of topological spaces we introduce the following terminology.

090H **Definition 7.17.1.** Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is *quasi-compact* if given a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists another covering $\mathcal{V} = \{V_j \to U\}_{j\in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $V_j \to U_{\alpha(j)}$ (see Definition 7.8.1) such that the image of α is a finite subset of I.

Of course the usual notion is sufficient to conclude that U is quasi-compact.

0D05 **Lemma 7.17.2.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. Consider the following conditions*

(1) *U is quasi-compact,*
(2) *for every covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists a finite covering $\{V_j \to U\}_{j=1,\ldots,m}$ of $\mathcal{C}$ refining $\mathcal{U}$, and*
(3) *for every covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ there exists a finite subset $I' \subset I$ such that $\{U_i \to U\}_{i\in I'}$ is a covering in $\mathcal{C}$.*

Then we always have (3) $\Rightarrow$ (2) $\Rightarrow$ (1) but the reverse implications do not hold in general.

Proof. The implications are immediate from the definitions. Let $X = [0, 1] \subset \mathbf{R}$ as a topological space (with the usual ϵ-δ topology). Let $\mathcal{C}$ be the category of open subspaces of X with inclusions as morphisms and usual open coverings (compare with Example 7.6.4). However, then we change the notion of covering in $\mathcal{C}$ to exclude all finite coverings, except for the coverings of the form $\{U \to U\}$. It is easy to see that this will be a site as in Definition 7.6.2. In this site the object $X = U = [0, 1]$ is quasi-compact in the sense of Definition 7.17.1 but U does not satisfy (2). We leave it to the reader to make an example where (2) holds but not (3). □

Here is the topos theoretic meaning of a quasi-compact object.

0D06 **Lemma 7.17.3.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent*

(1) *U is quasi-compact, and*
(2) *for every surjection of sheaves $\coprod_{i\in I} \mathcal{F}_i \to h_U^\#$ there is a finite subset $J \subset I$ such that $\coprod_{i\in J} \mathcal{F}_i \to h_U^\#$ is surjective.*

Proof. Assume (1) and let $\coprod_{i\in I} \mathcal{F}_i \to h_U^\#$ be a surjection. Then id_U is a section of $h_U^\#$ over U. Hence there exists a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ a section s_a of $\coprod_{i\in I} \mathcal{F}_i$ over U_a mapping to id_U. By the construction of coproducts as sheafification of coproducts of presheaves (Lemma 7.10.14), for each a there exists a covering $\{U_{ab} \to U_a\}_{b\in B_a}$ and for all $b \in B_a$ an $\iota(b) \in I$ and a section s_b of $\mathcal{F}_{\iota(b)}$ over U_{ab} mapping to $\mathrm{id}_U|_{U_{ab}}$. Thus after replacing the covering $\{U_a \to U\}_{a\in A}$ by $\{U_{ab} \to U\}_{a\in A, b\in B_a}$ we may assume we have a map $\iota : A \to I$ and for each $a \in A$ a section s_a of $\mathcal{F}_{\iota(a)}$ over U_a mapping to id_U. Since U is quasi-compact, there is a covering $\{V_c \to U\}_{c\in C}$, a map $\alpha : C \to A$ with finite image, and $V_c \to U_{\alpha(c)}$ over U. Then we see that $J = \mathrm{Im}(\iota \circ \alpha) \subset I$ works because $\coprod_{c\in C} h_{V_c}^\# \to h_U^\#$ is

surjective (Lemma 7.12.4) and factors through $\coprod_{i\in J}\mathcal{F}_i \to h_U^\#$. (Here we use that the composition $h_{V_c}^\# \to h_{U_{\alpha(c)}} \xrightarrow{s_{\alpha(c)}} \mathcal{F}_{\iota(\alpha(c))} \to h_U^\#$ is the map $h_{V_c}^\# \to h_U^\#$ coming from the morphism $V_c \to U$ because $s_{\alpha(c)}$ maps to $\mathrm{id}_U|_{U_{\alpha(c)}}$.)

Assume (2). Let $\{U_i \to U\}_{i\in I}$ be a covering. By Lemma 7.12.4 we see that $\coprod_{i\in I} h_{U_i}^\# \to h_U^\#$ is surjective. Thus we find a finite subset $J \subset I$ such that $\coprod_{j\in J} h_{U_j}^\# \to h_U^\#$ is surjective. Then arguing as above we find a covering $\{V_c \to U\}_{c\in C}$ of U in $\mathcal{C}$ and a map $\iota : C \to J$ such that id_U lifts to a section of s_c of $h_{U_{\iota(c)}}^\#$ over V_c. Refining the covering even further we may assume $s_c \in h_{U_{\iota(c)}}(V_c)$ mapping to id_U. Then $s_c : V_c \to U_{\iota(c)}$ is a morphism over U and we conclude. □

The lemma above motivates the following definition.

0D07 **Definition 7.17.4.** An object $\mathcal{F}$ of a topos $Sh(\mathcal{C})$ is *quasi-compact* if for any surjective map $\coprod_{i\in I}\mathcal{F}_i \to \mathcal{F}$ of $Sh(\mathcal{C})$ there exists a finite subset $J \subset I$ such that $\coprod_{i\in J}\mathcal{F}_i \to \mathcal{F}$ is surjective. A topos $Sh(\mathcal{C})$ is said to be *quasi-compact* if its final object $*$ is a quasi-compact object.

By Lemma 7.17.3 if the site $\mathcal{C}$ has a final object X, then $Sh(\mathcal{C})$ is quasi-compact if and only if X is quasi-compact.

0GMP **Lemma 7.17.5.** *Let $\mathcal{C}$ be a site.*

(1) *If $U \to V$ is a morphism of $\mathcal{C}$ such that $h_U^\# \to h_V^\#$ is surjective and U is quasi-compact, then V is quasi-compact.*
(2) *If $\mathcal{F} \to \mathcal{G}$ is a surjection of sheaves of sets and $\mathcal{F}$ is quasi-compact, then $\mathcal{G}$ is quasi-compact.*

Proof. Omitted. □

0GMQ **Lemma 7.17.6.** *Let $\mathcal{C}$ be a site. If $n \geq 1$ and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are quasi-compact sheaves on $\mathcal{C}$, then $\coprod_{i=1,\ldots,n}\mathcal{F}_i$ is quasi-compact.*

Proof. Omitted. □

The following two lemmas form the analogue of Sheaves, Lemma 6.29.1 for sites.

0738 **Lemma 7.17.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Let $U \in \mathrm{Ob}(\mathcal{C})$. Consider the canonical map*

$$\Psi : \mathrm{colim}_i \mathcal{F}_i(U) \longrightarrow (\mathrm{colim}_i \mathcal{F}_i)(U)$$

With the terminology introduced above:

(1) *If all the transition maps are injective then Ψ is injective for any U.*
(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of coverings $\{U_j \to U\}_{j\in J}$ with J finite and $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \mathrm{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 7.10.9). By Lemma 7.10.14 we have $(\mathcal{F}')^\# = \mathrm{colim}_i \mathcal{F}_i$. By Theorem 7.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. This means we can choose a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ an index $i_a \in I$, $i_a \geq i$, $i_a \geq i'$ such that $\varphi_{ii_a}(s) = \varphi_{i'i_a}(s')$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b\in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i'' \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on V_b for all $b \in B$ and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. There exists a covering $\{U_a \to U\}_{a\in A}$ and for each $a \in A$ an index $i_a \in I$ and a section $s_a \in \mathcal{F}_{i_a}(U_a)$ such that $s|_{U_a}$ comes from s_a for all $a \in A$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b\in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. By (1) the sections $s_b = \varphi_{i_{\alpha(b)}i}(s_{\alpha(b)})|_{V_b}$ agree over $V_b \times_U V_{b'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). By Lemma 7.17.2 the object U is quasi-compact, hence Ψ is injective by (2). To prove surjectivity, let s be an element of the target of Ψ. By assumption there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$, with $U_j \times_U U_{j'}$ quasi-compact for all $1 \leq j, j' \leq m$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \times_U U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \times_U U_{j'}$. Choose an index $i \in I$ which is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

0GMR **Lemma 7.17.8.** *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Consider the canonical map*

$$\Psi : \operatorname{colim}_i \Gamma(\mathcal{C}, \mathcal{F}_i) \longrightarrow \Gamma(\mathcal{C}, \operatorname{colim}_i \mathcal{F}_i)$$

We have the following:

(1) *If all the transition maps are injective then Ψ is injective.*
(2) *If $Sh(\mathcal{C})$ is quasi-compact, then Ψ is injective.*
(3) *If $Sh(\mathcal{C})$ is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *Assume there exists a set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ with the following properties:*
   (a) *for every surjection $\mathcal{F} \to *$ there exists a $\mathcal{K} \in S$ and a map $\mathcal{K} \to \mathcal{F}$ such that $\mathcal{K} \to *$ is surjective,*
   (b) *for $\mathcal{K} \in S$ the product $\mathcal{K} \times \mathcal{K}$ is quasi-compact.*

   Then Ψ is bijective.

Proof. Proof of (1). Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 7.10.9). By Lemma 7.10.14 we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Theorem 7.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Proof of (2). Assume $Sh(\mathcal{C})$ is quasi-compact. Recall that $\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}(*, \mathcal{F})$ for all $\mathcal{F}$ in $Sh(\mathcal{C})$. Let $a_i, b_i : * \to \mathcal{F}_i$ and for $i' \geq i$ denote $a_{i'}, b_{i'} : * \to \mathcal{F}_{i'}$

the composition with the transition maps of the system. Set $a = \operatorname{colim}_{i' \geq i} a_{i'}$ and similarly for b. For $i' \geq i$ denote

$$E_{i'} = \text{Equalizer}(a_{i'}, b_{i'}) \subset * \quad \text{and} \quad E = \text{Equalizer}(a, b) \subset *$$

By Categories, Lemma 4.19.2 we have $E = \operatorname{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Hence, if $E = *$, i.e., if $a = b$, then because $*$ is quasi-compact, we see that $E_{i'} = *$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. This proves (2).

Proof of (3). Assume $Sh(\mathcal{C})$ is quasi-compact and all transition maps are injective. Let $a : * \to \operatorname{colim} \mathcal{F}_i$ be a map. Then $E_i = a^{-1}(\mathcal{F}_i) \subset *$ is a subsheaf and we have $\operatorname{colim} E_i = *$ (by the reference above). Hence for some i we have $E_i = *$ and we see that the image of a is contained in $\mathcal{F}_i$ as desired.

Proof of (4). Let $S \subset \text{Ob}(Sh(\mathcal{C}))$ satisfy (4)(a), (b). Applying (4)(a) to $\text{id} : * \to *$ we find there exists a $\mathcal{K} \in S$ such that $\mathcal{K} \to *$ is surjective. The maps $\mathcal{K} \times \mathcal{K} \to \mathcal{K} \to *$ are surjective. By (4)(b) and Lemma 7.17.5 we conclude that $\mathcal{K}$ and $Sh(\mathcal{C})$ are quasi-compact. Thus Ψ is injective by (2). Set $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$. Let $s : * \to \mathcal{F}$ be a global section of the colimit. Since $\coprod \mathcal{F}_i \to \mathcal{F}$ is surjective, we see that the projection

$$\coprod\nolimits_{i \in I} * \times_{s,\mathcal{F}} \mathcal{F}_i \to *$$

is surjective. By (4)(a) we obtain $\mathcal{K} \in S$ and a map $\mathcal{K} \to \coprod_{i \in I} * \times_{s,\mathcal{F}} \mathcal{F}_i$ with $\mathcal{K} \to *$ surjective. Since $\mathcal{K}$ is quasi-compact we obtain a factorization $\mathcal{K} \to \coprod_{i' \in I'} * \times_{s,\mathcal{F}} \mathcal{F}_{i'}$ for some finite subset $I' \subset I$. Let $i \in I$ be an upper bound for the finite subset I'. The transition maps define a map $\coprod_{i' \in I'} \mathcal{F}_{i'} \to \mathcal{F}_i$. This in turn produces a map $\mathcal{K} \to * \times_{s,\mathcal{F}} \mathcal{F}_i$. In other words, we obtain $\mathcal{K} \in S$ with $\mathcal{K} \to *$ surjective and a commutative diagram

$$\begin{array}{ccccc} \mathcal{K} \times \mathcal{K} & \rightrightarrows & \mathcal{K} & \longrightarrow & * \\ & & \downarrow & & \downarrow{\scriptstyle s} \\ & & \mathcal{F}_i & \longrightarrow & \mathcal{F} = \operatorname{colim} \mathcal{F}_i \end{array}$$

Observe that the top row of this diagram is a coequalizer. Hence it suffices to show that after increasing i the two induced maps $a_i, b_i : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_i$ are equal. This is done shown in the next paragraph using the exact same argument as in the proof of (2) and we urge the reader to skip the rest of the proof.

For $i' \geq i$ denote $a_{i'}, b_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_{i'}$ the composition of a_i, b_i with the transition maps of the system. Set $a = \operatorname{colim}_{i' \geq i} a_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}$ and similarly for b. We have $a = b$ by the commutativity of the diagram above. For $i' \geq i$ denote

$$E_{i'} = \text{Equalizer}(a_{i'}, b_{i'}) \subset \mathcal{K} \times \mathcal{K} \quad \text{and} \quad E = \text{Equalizer}(a, b) \subset \mathcal{K} \times \mathcal{K}$$

By Categories, Lemma 4.19.2 we have $E = \operatorname{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Since $a = b$ we have $E = \mathcal{K} \times \mathcal{K}$. As $\mathcal{K} \times \mathcal{K}$ is quasi-compact by (4)(b), we see that $E_{i'} = \mathcal{K} \times \mathcal{K}$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. □

0GMS **Remark 7.17.9.** Let $\mathcal{C}$ be a site. There are several ways to ensure that the hypotheses of part (4) of Lemma 7.17.8 are satisfied. Here are a few.

(1) Assume there exists a set $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ with the following properties:

(a) for every surjection $\mathcal{F} \to *$ there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\mathcal{F}(U_j)$ nonempty and $\coprod h^\#_{U_j} \to *$ surjective,
(b) for $U, U' \in \mathcal{B}$ the sheaf $h^\#_U \times h^\#_{U'}$ is quasi-compact.

(2) Assume there exists a set $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ with the following properties:
(a) there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\coprod h^\#_{U_j} \to *$ surjective,
(b) for $U \in \mathcal{B}$ any covering of U can be refined by a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$, and
(c) for $U, U' \in \mathcal{B}$ there exist $m \geq 0$, $U_1, \ldots, U_m \in \mathcal{B}$, and morphisms $U_j \to U$ and $U_j \to U'$ such that $\coprod h^\#_{U_j} \to h^\#_U \times h^\#_{U'}$ is surjective.

(3) Suppose that
(a) $Sh(\mathcal{C})$ is quasi-compact,
(b) every object of $\mathcal{C}$ has a covering whose members are quasi-compact objects,
(c) if U and U' are quasi-compact, then the sheaf $h^\#_U \times h^\#_{U'}$ is quasi-compact.

In cases (1) and (2) we set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h^\#_U$ for $U \in \mathcal{B}$. In case (3) we set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h^\#_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact.

Later we will need a bound on what can happen with colimits as follows.

0GS0 **Lemma 7.17.10.** *Let $\mathcal{C}$ be a site. Let β be an ordinal. Let $\beta \to Sh(\mathcal{C})$, $\alpha \mapsto \mathcal{F}_\alpha$ be a system of sheaves over β. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the canonical map*

$$\mathrm{colim}_{\alpha<\beta} \mathcal{F}_\alpha(U) \longrightarrow (\mathrm{colim}_{\alpha<\beta} \mathcal{F}_\alpha)(U)$$

If the cofinality of β is large enough, then this map is bijective for all U.

Proof. The left hand side is the value on U of the colimit $\mathcal{F}_{\mathrm{colim}}$ taken in the category of presheaves, see Section 7.4. Recall that $\mathrm{colim}_{\alpha<\beta} \mathcal{F}_\alpha$ is the sheafification $\mathcal{F}^\#_{\mathrm{colim}}$ of $\mathcal{F}_{\mathrm{colim}}$, see Lemma 7.10.14. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an element of the set $\mathrm{Cov}(\mathcal{C})$ of coverings of $\mathcal{C}$. If the cofinality of β is larger than the cardinality of I, then we claim

$$H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}}) = \mathrm{colim}\, H^0(\mathcal{U}, \mathcal{F}_\alpha) = \mathrm{colim}\, \mathcal{F}_\alpha(U) = \mathcal{F}_{\mathrm{colim}}(U)$$

The second and third equality signs are clear. For the first, say $s = (s_i) \in H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}})$. Then for each i the element s_i comes from an element $s_{i,\alpha_i} \in \mathcal{F}_{\alpha_i}(U_i)$ for some $\alpha_i < \beta$. By the assumption on cofinality, we can choose $\alpha_i = \alpha$ independent of i. Then s_i and s_j map to the same element of $\mathcal{F}_{\alpha_{i,j}}(U_i \times_U U_j)$ for some $\alpha_{i,j} < \beta$. Since the cardinality if $I \times I$ is also less than the cofinality of β, we see that we may after increasing α assume $\alpha_{i,j} = \alpha$ for all i, j. This proves that the natural map $\mathrm{colim}\, H^0(\mathcal{U}, \mathcal{F}_\alpha) \to H^0(\mathcal{U}, \mathcal{F}_{\mathrm{colim}})$ is surjective. A very similar argument shows that it is injective. In particular, we see that $\mathcal{F}_{\mathrm{colim}}$ satisfies the sheaf condition for $\mathcal{U}$. Thus if the cofinality of β is larger than the supremum of the cardinalities of the set of index sets I of coverings, then we conclude. □

7.18. Colimits of sites

0EXI We need an analogue of Lemma 7.17.7 in the case that the site is the limit of an inverse system of sites. For simplicity we only explain the construction in case the index sets of coverings are finite.

0A34 **Situation 7.18.1.** Here we are given

(1) a cofiltered index category $\mathcal{I}$,
(2) for $i \in \mathrm{Ob}(\mathcal{I})$ a site $\mathcal{C}_i$ such that every covering in $\mathcal{C}_i$ has a finite index set,
(3) for a morphism $a : i \to j$ in $\mathcal{I}$ a morphism of sites $f_a : \mathcal{C}_i \to \mathcal{C}_j$ given by a continuous functor $u_a : \mathcal{C}_j \to \mathcal{C}_i$,

such that $f_a \circ f_b = f_c$ whenever $c = a \circ b$ in $\mathcal{I}$.

09YL **Lemma 7.18.2.** *In Situation 7.18.1 we can construct a site $(\mathcal{C}, Cov(\mathcal{C}))$ as follows*

(1) *as a category $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$, and*
(2) *$Cov(\mathcal{C})$ is the union of the images of $Cov(\mathcal{C}_i)$ by $u_i : \mathcal{C}_i \to \mathcal{C}$.*

Proof. Our definition of composition of morphisms of sites implies that $u_b \circ u_a = u_c$ whenever $c = a \circ b$ in $\mathcal{I}$. The formula $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$ means that $\mathrm{Ob}(\mathcal{C}) = \operatorname{colim} \mathrm{Ob}(\mathcal{C}_i)$ and $\mathrm{Arrows}(\mathcal{C}) = \operatorname{colim} \mathrm{Arrows}(\mathcal{C}_i)$. Then source, target, and composition are inherited from the source, target, and composition on $\mathrm{Arrows}(\mathcal{C}_i)$. In this way we obtain a category. Denote $u_i : \mathcal{C}_i \to \mathcal{C}$ the obvious functor. Remark that given any finite diagram in $\mathcal{C}$ there exists an i such that this diagram is the image of a diagram in $\mathcal{C}_i$.

Let $\{U^t \to U\}$ be a covering of $\mathcal{C}$. We first prove that if $V \to U$ is a morphism of $\mathcal{C}$, then $U^t \times_U V$ exists. By our remark above and our definition of coverings, we can find an i, a covering $\{U_i^t \to U_i\}$ of $\mathcal{C}_i$ and a morphism $V_i \to U_i$ whose image by u_i is the given data. We claim that $U^t \times_U V$ is the image of $U_i^t \times_{U_i} V_i$ by u_i. Namely, for every $a : j \to i$ in $\mathcal{I}$ the functor u_a is continuous, hence $u_a(U_i^t \times_{U_i} V_i) = u_a(U_i^t) \times_{u_a(U_i)} u_a(V_i)$. In particular we can replace i by j, if we so desire. Thus, if W is another object of $\mathcal{C}$, then we may assume $W = u_i(W_i)$ and we see that

$$\begin{aligned}
&\mathrm{Mor}_{\mathcal{C}}(W, u_i(U_i^t \times_{U_i} V_i)) \\
&= \operatorname{colim}_{a:j\to i} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t \times_{U_i} V_i)) \\
&= \operatorname{colim}_{a:j\to i} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t)) \times_{\mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i))} \mathrm{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(V_i)) \\
&= \mathrm{Mor}_{\mathcal{C}}(W, U^t) \times_{\mathrm{Mor}_{\mathcal{C}}(W,U)} \mathrm{Mor}_{\mathcal{C}}(W, V)
\end{aligned}$$

as filtered colimits commute with finite limits (Categories, Lemma 4.19.2). It also follows that $\{U^t \times_U V \to V\}$ is a covering in $\mathcal{C}$. In this way we see that axiom (3) of Definition 7.6.2 holds.

To verify axiom (2) of Definition 7.6.2 let $\{U^t \to U\}_{t\in T}$ be a covering of $\mathcal{C}$ and for each t let $\{U^{ts} \to U^t\}$ be a covering of $\mathcal{C}$. Then we can find an i and a covering $\{U_i^t \to U_i\}_{t\in T}$ of $\mathcal{C}_i$ whose image by u_i is $\{U^t \to U\}$. Since T is **finite** we may choose an $a : j \to i$ in $\mathcal{I}$ and coverings $\{U_j^{ts} \to u_a(U_i^t)\}$ of $\mathcal{C}_j$ whose image by u_j gives $\{U^{ts} \to U^t\}$. Then we conclude that $\{U^{ts} \to U\}$ is a covering of $\mathcal{C}$ by an application of axiom (2) to the site $\mathcal{C}_j$.

We omit the proof of axiom (1) of Definition 7.6.2. □

0A35 **Lemma 7.18.3.** *In Situation 7.18.1 let $u_i : \mathcal{C}_i \to \mathcal{C}$ be as constructed in Lemma 7.18.2. Then u_i defines a morphism of sites $f_i : \mathcal{C} \to \mathcal{C}_i$. For $U_i \in \mathrm{Ob}(\mathcal{C}_i)$ and sheaf $\mathcal{F}$ on $\mathcal{C}_i$ we have*

09YM (7.18.3.1) $$f_i^{-1}\mathcal{F}(u_i(U_i)) = \operatorname{colim}_{a:j\to i} f_a^{-1}\mathcal{F}(u_a(U_i))$$

Proof. It is immediate from the arguments in the proof of Lemma 7.18.2 that the functors u_i are continuous. To finish the proof we have to show that $f_i^{-1} := u_{i,s}$ is an exact functor $Sh(\mathcal{C}_i) \to Sh(\mathcal{C})$. In fact it suffices to show that f_i^{-1} is left exact, because it is right exact as a left adjoint (Categories, Lemma 4.24.6). We first prove (7.18.3.1) and then we deduce exactness.

For an arbitrary object V of $\mathcal{C}$ we can pick a $a : j \to i$ and an object $V_j \in \mathrm{Ob}(\mathcal{C})$ with $V = u_j(V_j)$. Then we can set

$$\mathcal{G}(V) = \mathrm{colim}_{b:k\to j} f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j))$$

The value $\mathcal{G}(V)$ of the colimit is independent of the choice of $b : j \to i$ and of the object V_j with $u_j(V_j) = V$; we omit the verification. Moreover, if $\alpha : V \to V'$ is a morphism of $\mathcal{C}$, then we can choose $b : j \to i$ and a morphism $\alpha_j : V_j \to V_j'$ with $u_j(\alpha_j) = \alpha$. This induces a map $\mathcal{G}(V') \to \mathcal{G}(V)$ by using the restrictions along the morphisms $u_b(\alpha_j) : u_b(V_j) \to u_b(V_j')$. A check shows that $\mathcal{G}$ is a presheaf (omitted). In fact, $\mathcal{G}$ satisfies the sheaf condition. Namely, any covering $\mathcal{U} = \{U^t \to U\}$ in $\mathcal{C}$ comes from a finite level. Say $\mathcal{U}_j = \{U_j^t \to U_j\}$ is mapped to $\mathcal{U}$ by u_j for some $a : j \to i$ in $\mathcal{I}$. Then we have

$$H^0(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{b:k\to j} H^0(u_b(\mathcal{U}_j), f_{b\circ a}^{-1}\mathcal{F}) = \mathrm{colim}_{b:k\to j} f_{b\circ a}^{-1}\mathcal{F}(u_b(U_j)) = \mathcal{G}(U)$$

as desired. The first equality holds because filtered colimits commute with finite limits (Categories, Lemma 4.19.2). By construction $\mathcal{G}(U)$ is given by the right hand side of (7.18.3.1). Hence (7.18.3.1) is true if we can show that $\mathcal{G}$ is equal to $f_i^{-1}\mathcal{F}$.

In this paragraph we check that $\mathcal{G}$ is canonically isomorphic to $f_i^{-1}\mathcal{F}$. We strongly encourage the reader to skip this paragraph. To check this we have to show there is a bijection $\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$ functorial in the sheaf $\mathcal{H}$ on $\mathcal{C}$ where $f_{i,*} = u_i^p$. A map $\mathcal{G} \to \mathcal{H}$ is the same thing as a compatible system of maps

$$\varphi_{a,b,V_j} : f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j)) \longrightarrow \mathcal{H}(u_j(V_j))$$

for all $a : j \to i$, $b : k \to j$ and $V_j \in \mathrm{Ob}(\mathcal{C}_j)$. The compatibilities force the maps φ_{a,b,V_j} to be equal to $\varphi_{a\circ b,\mathrm{id},u_b(V_j)}$. Given $a : j \to i$, the family of maps $\varphi_{a,\mathrm{id},V_j}$ corresponds to a map of sheaves $\varphi_a : f_a^{-1}\mathcal{F} \to f_{j,*}\mathcal{H}$. The compatibilities between the $\varphi_{a,\mathrm{id},u_a(V_i)}$ and the $\varphi_{\mathrm{id},\mathrm{id},V_i}$ implies that φ_a is the adjoint of the map φ_{id} via

$$\mathrm{Mor}_{Sh(\mathcal{C}_j)}(f_a^{-1}\mathcal{F}, f_{j,*}\mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{a,*}f_{j,*}\mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$$

Thus finally we see that the whole system of maps φ_{a,b,V_j} is determined by the map $\varphi_{id} : \mathcal{F} \to f_{i,*}\mathcal{H}$. Conversely, given such a map $\psi : \mathcal{F} \to f_{i,*}\mathcal{H}$ we can read the argument just given backwards to construct the family of maps φ_{a,b,V_j}. This finishes the proof that $\mathcal{G} = f_i^{-1}\mathcal{F}$.

Assume (7.18.3.1) holds. Then the functor $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(U)$ commutes with finite limits because finite limits of sheaves are computed in the category of presheaves (Lemma 7.10.1), the functors f_a^{-1} commutes with finite limits, and filtered colimits commute with finite limits. To see that $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(V)$ commutes with finite limits for a general object V of $\mathcal{C}$, we can use the same argument using the formula for $f_i^{-1}\mathcal{F}(V) = \mathcal{G}(V)$ given above. Thus f_i^{-1} is left exact and the proof of the lemma is complete. □

09YN **Lemma 7.18.4.** *In Situation 7.18.1 assume given*

(1) *a sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,*

(2) *for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of sheaves on $\mathcal{C}_j$*

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$ on the site $\mathcal{C}$ of Lemma 7.18.2. Let $i \in \mathrm{Ob}(\mathcal{I})$ and $X_i \in Ob(\mathcal{C}_i)$. Then

$$\operatorname{colim}_{a:j\to i} \mathcal{F}_j(u_a(X_i)) = \mathcal{F}(u_i(X_i))$$

Proof. A formal argument shows that

$$\operatorname{colim}_{a:j\to i} \mathcal{F}_i(u_a(X_i)) = \operatorname{colim}_{a:j\to i} \operatorname{colim}_{b:k\to j} f_b^{-1}\mathcal{F}_j(u_{a\circ b}(X_i))$$

By (7.18.3.1) we see that the inner colimit is equal to $f_j^{-1}\mathcal{F}_j(u_i(X_i))$ hence we conclude by Lemma 7.17.7. □

0EXJ **Lemma 7.18.5.** *In Situation 7.18.1 assume we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then*

$$\mathcal{F} = \operatorname{colim} f_i^{-1} f_{i,*}\mathcal{F}$$

where the transition maps are $f_j^{-1}\varphi_a$ for $a : j \to i$ where $\varphi_a : f_a^{-1} f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$ is a canonical map satisfying a cocycle condition as in Lemma 7.18.4.

Proof. For the morphism

$$\varphi_a : f_a^{-1} f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$$

we choose the adjoint to the identity map

$$f_{i,*}\mathcal{F} \to f_{a,*} f_{j,*}\mathcal{F}$$

Hence φ_a is the counit for the adjunction given by $(f_a^{-1}, f_{a,*})$. We must prove that for all $a : j \to i$ and $b : k \to i$ with composition $c = a \circ b$ we have $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$. This follows from Categories, Lemma 4.24.9. Lastly, we must prove that the map given by adjunction

$$\operatorname{colim}_{i\in I} f_i^{-1} f_{i,*}\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. For an object U of $\mathcal{C}$ we need to show the map

$$(\operatorname{colim}_{i\in I} f_i^{-1}\mathcal{F}_i)(U) \to \mathcal{F}(U)$$

is bijective. Choose an i and an object U_i of $\mathcal{C}_i$ with $u_i(U_i) = U$. Then the left hand side is equal to

$$(\operatorname{colim}_{i\in I} f_i^{-1}\mathcal{F}_i)(U) = \operatorname{colim}_{a:j\to i} f_{j,*}\mathcal{F}(u_a(U_i))$$

by Lemma 7.18.4. Since $u_j(u_a(U_i)) = U$ we have $f_{j,*}\mathcal{F}(u_a(U_i)) = \mathcal{F}(U)$ for all $a : j \to i$ by definition. Hence the value of the colimit is $\mathcal{F}(U)$ and the proof is complete. □

7.19. More functoriality of presheaves

00XF In this section we revisit the material of Section 7.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. It turns out that this functor not only has a left adjoint (namely u_p) but also a right adjoint.

Namely, for any $V \in \mathrm{Ob}(\mathcal{D})$ we define a category ${}_V\mathcal{I} = {}^u_V\mathcal{I}$. Its objects are pairs $(U, \psi : u(U) \to V)$. Note that the arrow is in the opposite direction from the arrow we used in defining the category $\mathcal{I}^u_V$ in Section 7.5. A morphism $(U, \psi) \to (U', \psi')$ is given by a morphism $\alpha : U \to U'$ such that $\psi = \psi' \circ u(\alpha)$. In addition, given

any presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ we introduce the functor ${}_V\mathcal{F} : {}_V\mathcal{I}^{opp} \to Sets$, which is defined by the rule ${}_V\mathcal{F}(U, \psi) = \mathcal{F}(U)$. We define

$$ {}_pu(\mathcal{F})(V) := \lim_{{}_V\mathcal{I}^{opp}} {}_V\mathcal{F} $$

As a limit there are projection maps $c(\psi) : {}_pu(\mathcal{F})(V) \to \mathcal{F}(U)$ for every object (U, ψ) of ${}_V\mathcal{I}$. In fact,

$$ {}_pu(\mathcal{F})(V) = \left\{ \begin{array}{c} \text{collections } s_{(U,\psi)} \in \mathcal{F}(U) \\ \forall \beta : (U_1, \psi_1) \to (U_2, \psi_2) \text{ in } {}_V\mathcal{I} \\ \text{we have } \beta^* s_{(U_2,\psi_2)} = s_{(U_1,\psi_1)} \end{array} \right\} $$

where the correspondence is given by $s \mapsto s_{(U,\psi)} = c(\psi)(s)$. We leave it to the reader to define the restriction mappings ${}_pu(\mathcal{F})(V) \to {}_pu(\mathcal{F})(V')$ associated to any morphism $V' \to V$ of $\mathcal{D}$. The resulting presheaf will be denoted ${}_pu\mathcal{F}$.

00XG **Lemma 7.19.1.** *There is a canonical map ${}_pu\mathcal{F}(u(U)) \to \mathcal{F}(U)$, which is compatible with restriction maps.*

Proof. This is just the projection map $c(\mathrm{id}_{u(U)})$ above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors ${}_V\mathcal{F} \to {}_V\mathcal{F}'$, and hence to a map of presheaves ${}_pu\mathcal{F} \to {}_pu\mathcal{F}'$. In other words, we have defined a functor

$$ {}_pu : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D}) $$

00XH **Lemma 7.19.2.** *The functor ${}_pu$ is a right adjoint to the functor u^p. In other words the formula*

$$ \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) $$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. This is proved in exactly the same way as the proof of Lemma 7.5.4. We note that the map $u^p{}_pu\mathcal{F} \to \mathcal{F}$ from Lemma 7.19.1 is the map that is used to go from the right to the left.

Alternately, think of a presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ as a presheaf $\mathcal{F}'$ on $\mathcal{C}^{opp}$ with values in $Sets^{opp}$, and similarly on $\mathcal{D}$. Check that $({}_pu\mathcal{F})' = u_p(\mathcal{F}')$, and that $(u^p\mathcal{G})' = u^p(\mathcal{G}')$. By Remark 7.5.5 we have the adjointness of u_p and u^p for presheaves with values in $Sets^{opp}$. The result then follows formally from this. □

Thus given a functor $u : \mathcal{C} \to \mathcal{D}$ of categories we obtain a sequence of functors

$$ u_p, u^p, {}_pu $$

between categories of presheaves where in each consecutive pair the first is left adjoint to the second.

09VQ **Lemma 7.19.3.** *Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of categories. Assume that v is right adjoint to u. Then we have*

(1) *$u^p h_V = h_{v(V)}$ for any V in $\mathcal{D}$,*
(2) *the category $\mathcal{I}^v_U$ has an initial object,*
(3) *the category ${}^u_V\mathcal{I}$ has a final object,*
(4) *${}_pu = v^p$, and*
(5) *$u^p = v_p$.*

Proof. Proof of (1). Let V be an object of $\mathcal{D}$. We have $u^p h_V = h_{v(V)}$ because $u^p h_V(U) = \mathrm{Mor}_{\mathcal{D}}(u(U), V) = \mathrm{Mor}_{\mathcal{C}}(U, v(V))$ by assumption.

Proof of (2). Let U be an object of $\mathcal{C}$. Let $\eta : U \to v(u(U))$ be the map adjoint to the map $\mathrm{id} : u(U) \to u(U)$. Then we claim $(u(U), \eta)$ is an initial object of $\mathcal{I}^v_U$. Namely, given an object $(V, \phi : U \to v(V))$ of $\mathcal{I}^v_U$ the morphism ϕ is adjoint to a map $\psi : u(U) \to V$ which then defines a morphism $(u(U), \eta) \to (V, \phi)$.

Proof of (3). Let V be an object of $\mathcal{D}$. Let $\xi : u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id} : v(V) \to v(V)$. Then we claim $(v(V), \xi)$ is a final object of ${}^u_V\mathcal{I}$. Namely, given an object $(U, \psi : u(U) \to V)$ of ${}^u_V\mathcal{I}$ the morphism ψ is adjoint to a map $\phi : U \to v(V)$ which then defines a morphism $(U, \psi) \to (v(V), \xi)$.

Hence for any presheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} v^p\mathcal{F}(V) &= \mathcal{F}(v(V)) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(h_{v(V)}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) \\ &= {}_pu\mathcal{F}(V) \end{aligned}$$

which proves part (4). Part (5) follows by the uniqueness of adjoint functors. □

7.20. Cocontinuous functors

00XI There is another way to construct morphisms of topoi. This involves using cocontinuous functors between sites defined as follows.

00XJ **Definition 7.20.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j\in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i\in I}$ refines the covering $\{V_j \to u(U)\}_{j\in J}$.

Note that $\{u(U_i) \to u(U)\}_{i\in I}$ is in general *not* a covering of the site $\mathcal{D}$.

00XK **Lemma 7.20.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then ${}_pu\mathcal{F}$ is a sheaf on $\mathcal{D}$, which we will denote ${}_su\mathcal{F}$.*

Proof. Let $\{V_j \to V\}_{j\in J}$ be a covering of the site $\mathcal{D}$. We have to show that

$$ {}_pu\mathcal{F}(V) \longrightarrow \prod {}_pu\mathcal{F}(V_j) \rightrightarrows \prod {}_pu\mathcal{F}(V_j \times_V V_{j'}) $$

is an equalizer diagram. Since ${}_pu$ is right adjoint to u^p we have

$${}_pu\mathcal{F}(V) = \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}((u^p h_V)^\#, \mathcal{F})$$

Hence it suffices to show that

07GF (7.20.2.1) $$ \coprod u^p h_{V_j \times_V V_{j'}} \rightrightarrows \coprod u^p h_{V_j} \longrightarrow u^p h_V $$

becomes a coequalizer diagram after sheafification. (Recall that a coproduct in the category of sheaves is the sheafification of the coproduct in the category of presheaves, see Lemma 7.10.14.)

We first show that the second arrow of (7.20.2.1) becomes surjective after sheafification. To do this we use Lemma 7.11.2. Thus it suffices to show a section s of $u^p h_V$ over U lifts to a section of $\coprod u^p h_{V_j}$ on the members of a covering of U.

Note that s is a morphism $s : u(U) \to V$. Then $\{V_j \times_{V,s} u(U) \to u(U)\}$ is a covering of $\mathcal{D}$. Hence, as u is cocontinuous, there is a covering $\{U_i \to U\}$ such that $\{u(U_i) \to u(U)\}$ refines $\{V_j \times_{V,s} u(U) \to u(U)\}$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i : u(U_i) \to V_j$ for some j, i.e., $s|_{U_i}$ is in the image of $u^p h_{V_j}(U_i) \to u^p h_V(U_i)$ as desired.

Let $s, s' \in (\coprod u^p h_{V_j})^\#(U)$ map to the same element of $(u^p h_V)^\#(U)$. To finish the proof of the lemma we show that after replacing U by the members of a covering that s, s' are the image of the same section of $\coprod u^p h_{V_j \times_V V_{j'}}$ by the two maps of (7.20.2.1). We may first replace U by the members of a covering and assume that $s \in u^p h_{V_j}(U)$ and $s' \in u^p h_{V_{j'}}(U)$. A second such replacement guarantees that s and s' have the same image in $u^p h_V(U)$ instead of in the sheafification. Hence $s : u(U) \to V_j$ and $s' : u(U) \to V_{j'}$ are morphisms of $\mathcal{D}$ such that

$$\begin{array}{ccc} u(U) & \xrightarrow[s']{} & V_{j'} \\ {\scriptstyle s}\downarrow & & \downarrow \\ V_j & \longrightarrow & V \end{array}$$

is commutative. Thus we obtain $t = (s, s') : u(U) \to V_j \times_V V_{j'}$, i.e., a section $t \in u^p h_{V_j \times_V V_{j'}}(U)$ which maps to s, s' as desired. □

00XL **Lemma 7.20.3.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functor $Sh(\mathcal{D}) \to Sh(\mathcal{C})$, $\mathcal{G} \mapsto (u^p\mathcal{G})^\#$ is a left adjoint to the functor ${}_su$ introduced in Lemma 7.20.2 above. Moreover, it is exact.*

Proof. Let us prove the adjointness property as follows

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, {}_su\mathcal{F}). \end{aligned}$$

Thus it is a left adjoint and hence right exact, see Categories, Lemma 4.24.6. We have seen that sheafification is left exact, see Lemma 7.10.15. Moreover, the inclusion $i : Sh(\mathcal{D}) \to PSh(\mathcal{D})$ is left exact by Lemma 7.10.1. Finally, the functor u^p is left exact because it is a right adjoint (namely to u_p). Thus the functor is the composition ${}^\# \circ u^p \circ i$ of left exact functors, hence left exact. □

We finish this section with a technical lemma.

00XM **Lemma 7.20.4.** *In the situation of Lemma 7.20.3. For any presheaf $\mathcal{G}$ on $\mathcal{D}$ we have $(u^p\mathcal{G})^\# = (u^p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p(\mathcal{G}^\#))^\#, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_su\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

09W7 **Remark 7.20.5.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g : u(U) \to V$ and $f : W \to V$ in $\mathcal{D}$ we can consider the functor

$$\mathcal{C}^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_{\mathcal{C}}(T, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(T),V)} \mathrm{Mor}_{\mathcal{D}}(u(T), W)$$

If this functor is representable, denote $U \times_{g,V,f} W$ the corresponding object of $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are sites. Consider the property P: for every covering $\{f_j : V_j \to V\}$ of $\mathcal{D}$ and any morphism $g : u(U) \to V$ we have

(1) $U \times_{g,V,f_i} V_i$ exists for all i, and
(2) $\{U \times_{g,V,f_i} V_i \to U\}$ is a covering of $\mathcal{C}$.

Please note the similarity with the definition of continuous functors. If u has P then u is cocontinuous (details omitted). Many of the cocontinuous functors we will encounter satisfy P.

7.21. Cocontinuous functors and morphisms of topoi

00XN It is clear from the above that a cocontinuous functor u gives a morphism of topoi in the same direction as u. Thus this is in the opposite direction from the morphism of topoi associated (under certain conditions) to a continuous u as in Definition 7.14.1, Proposition 7.14.7, and Lemma 7.15.2.

00XO **Lemma 7.21.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functors $g_* = {}_su$ and $g^{-1} = (u^p)^\#$ define a morphism of topoi g from $Sh(\mathcal{C})$ to $Sh(\mathcal{D})$.*

Proof. This is exactly the content of Lemma 7.20.3. □

03L5 **Lemma 7.21.2.** *Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Then $v \circ u$ is cocontinuous and we have $h = g \circ f$ where $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, resp. $g : Sh(\mathcal{D}) \to Sh(\mathcal{E})$, resp. $h : Sh(\mathcal{C}) \to Sh(\mathcal{E})$ is the morphism of topoi associated to u, resp. v, resp. $v \circ u$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\{E_i \to v(u(U))\}$ be a covering of U in $\mathcal{E}$. By assumption there exists a covering $\{D_j \to u(U)\}$ in $\mathcal{D}$ such that $\{v(D_j) \to v(u(U))\}$ refines $\{E_i \to v(u(U))\}$. Also by assumption there exists a covering $\{C_l \to U\}$ in $\mathcal{C}$ such that $\{u(C_l) \to u(U)\}$ refines $\{D_j \to u(U)\}$. Then it is true that $\{v(u(C_l)) \to v(u(U))\}$ refines the covering $\{E_i \to v(u(U))\}$. This proves that $v \circ u$ is cocontinuous. To prove the last assertion it suffices to show that ${}_sv \circ {}_su = {}_s(v \circ u)$. It suffices to prove that ${}_pv \circ {}_pu = {}_p(v \circ u)$, see Lemma 7.20.2. Since ${}_pu$, resp. ${}_pv$, resp. ${}_p(v \circ u)$ is right adjoint to u^p, resp. v^p, resp. $(v \circ u)^p$ it suffices to prove that $u^p \circ v^p = (v \circ u)^p$. And this is direct from the definitions. □

00XP **Example 7.21.3.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subspace. Recall that we have sites X_{Zar} and U_{Zar}, see Example 7.6.4. Recall that we have the functor $u : X_{Zar} \to U_{Zar}$ associated to j which is continuous and gives rise to a morphism of sites $U_{Zar} \to X_{Zar}$, see Example 7.14.2. This also gives a morphism of topoi (j_*, j^{-1}). Next, consider the functor $v : U_{Zar} \to X_{Zar}$, $V \mapsto v(V) = V$ (just the same open but now thought of as an object of X_{Zar}). This functor is cocontinuous. Namely, if $v(V) = \bigcup_{j \in J} W_j$ is an open covering in X, then each W_j must be a subset of U and hence is of the form $v(V_j)$, and trivially $V = \bigcup_{j \in J} V_j$ is an open covering in U. We conclude by Lemma 7.21.1 above that there is a morphism of topoi associated to v

$$Sh(U) \longrightarrow Sh(X)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = j^{-1}$ and that ${}_sv = j_*$, in other words, that this is the same morphism of topoi as the one given above. Perhaps the easiest way to see this is to realize that for any sheaf $\mathcal{G}$ on X we have $v^p\mathcal{G}(V) = \mathcal{G}(V)$ which according to Sheaves, Lemma 6.31.1 is a description of $j^{-1}\mathcal{G}$ (and hence sheafification is superfluous in this case). The equality of ${}_sv$ and j_* follows by uniqueness of adjoint functors (but may also be computed directly).

00XQ **Example 7.21.4.** This example is a slight generalization of Example 7.21.3. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that f is open. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Recall that we have the functor $u : Y_{Zar} \to X_{Zar}$ associated to f which is continuous and gives rise to a morphism of sites $X_{Zar} \to Y_{Zar}$, see Example 7.14.2. This also gives a morphism of topoi (f_*, f^{-1}). Next, consider the functor $v : X_{Zar} \to Y_{Zar}$, $U \mapsto v(U) = f(U)$. This functor is cocontinuous. Namely, if $f(U) = \bigcup_{j\in J} V_j$ is an open covering in Y, then setting $U_j = f^{-1}(V_j) \cap U$ we get an open covering $U = \bigcup U_j$ such that $f(U) = \bigcup f(U_j)$ is a refinement of $f(U) = \bigcup V_j$. We conclude by Lemma 7.21.1 above that there is a morphism of topoi associated to v

$$Sh(X) \longrightarrow Sh(Y)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = f^{-1}$ and that ${}_sv = f_*$, in other words, that this is the same morphism of topoi as the one given above. For any sheaf $\mathcal{G}$ on Y we have $v^p\mathcal{G}(U) = \mathcal{G}(f(U))$. On the other hand, we may compute $u_p\mathcal{G}(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) = \mathcal{G}(f(U))$ because clearly $(f(U), U \subset f^{-1}(f(U)))$ is an initial object of the category $\mathcal{I}^u_U$ of Section 7.5. Hence $u_p = v^p$ and we conclude $f^{-1} = u_s = (v^p\ )^\#$. The equality of ${}_sv$ and f_* follows by uniqueness of adjoint functors (but may also be computed directly).

In the first Example 7.21.3 the functor v is also continuous. But in the second Example 7.21.4 it is generally not continuous because condition (2) of Definition 7.13.1 may fail. Hence the following lemma applies to the first example, but not to the second.

00XR **Lemma 7.21.5.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous, and*
(b) *u is continuous.*

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. Then

(1) *sheafification in the formula $g^{-1} = (u^p\ )^\#$ is unnecessary, in other words $g^{-1}(\mathcal{G})(U) = \mathcal{G}(u(U))$,*
(2) *g^{-1} has a left adjoint $g_! = (u_p\ )^\#$, and*
(3) *g^{-1} commutes with arbitrary limits and colimits.*

Proof. By Lemma 7.13.2 for any sheaf $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G}$ is a sheaf on $\mathcal{C}$. And then we see the adjointness by the following string of equalities

$$\begin{aligned} \operatorname{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) \\ &= \operatorname{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Mor}_{Sh(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

The statement on limits and colimits follows from the discussion in Categories, Section 4.24. □

In the situation of Lemma 7.21.5 above we see that we have a sequence of adjoint functors

$$g_!,\ g^{-1},\ g_*.$$

The functor $g_!$ is *not* exact in general, because it does not transform a final object of $Sh(\mathcal{C})$ into a final object of $Sh(\mathcal{D})$ in general. See Sheaves, Remark 6.31.13. On the other hand, in the topological setting of Example 7.21.3 the functor $j_!$ is exact on abelian sheaves, see Modules, Lemma 17.3.4. The following lemma gives the generalization to the case of sites.

00XS **Lemma 7.21.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_!$ above commutes with fibre products and equalizers (and more generally with finite connected limits).

Proof. Assume (a), (b), and (c). We have $g_! = (u_p\ )^\#$. Recall (Lemma 7.10.1) that limits of sheaves are equal to the corresponding limits as presheaves. And sheafification commutes with finite limits (Lemma 7.10.15). Thus it suffices to show that u_p commutes with fibre products and equalizers. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Section 7.5 commute with fibre products and equalizers. This follows from Lemma 7.5.1 and Categories, Lemma 4.19.9. □

The following lemma deals with a case that is even more like the morphism associated to an open immersion of topological spaces.

00XT **Lemma 7.21.7.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Let X be an object of $\mathcal{C}$. In Lemmas 7.20.2 and 7.21.5 we have seen that sheafification is not necessary for the functors $g^{-1} = (u^p\ )^\#$ and $g_* = ({}_pu\ )^\#$. We may compute $(g^{-1}g_*\mathcal{F})(X) = g_*\mathcal{F}(u(X)) = \lim \mathcal{F}(Y)$. Here the limit is over the category of pairs $(Y, u(Y) \to u(X))$ where the morphisms $u(Y) \to u(X)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the limit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. This proves that $g^{-1}g_*\mathcal{F} = \mathcal{F}$.

On the other hand, $(g^{-1}g_!\mathcal{F})(X) = g_!\mathcal{F}(u(X)) = (u_p\mathcal{F})^\#(u(X))$, and $u_p\mathcal{F}(u(X)) = \operatorname{colim} \mathcal{F}(Y)$. Here the colimit is over the category of pairs $(Y, u(X) \to u(Y))$ where the morphisms $u(X) \to u(Y)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the colimit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. Thus for every $X \in \mathrm{Ob}(\mathcal{C})$ we have $u_p\mathcal{F}(u(X)) = \mathcal{F}(X)$. Since u is cocontinuous and continuous any covering of $u(X)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(X_i) \to u(X)\}$ of $\mathcal{D}$ where $\{X_i \to X\}$ is a covering in $\mathcal{C}$. This implies that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ also, since in the colimit defining the value of $(u_p\mathcal{F})^+$ on $u(X)$ we may restrict to the cofinal system of coverings $\{u(X_i) \to u(X)\}$ as above.

Hence we see that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(u_p\mathcal{F})^\#(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Finally, here is a case that does not have any corresponding topological example. We will use this lemma to see what happens when we enlarge a "partial universe" of schemes keeping the same topology. In the situation of the lemma, the morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ identifies $Sh(\mathcal{C})$ as a subtopos of $Sh(\mathcal{D})$ (Section 7.43) and moreover, the given embedding has a retraction.

00XU **Lemma 7.21.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous,*
(c) *u is fully faithful,*
(d) *fibre products exist in $\mathcal{C}$ and u commutes with them, and*
(e) *there exist final objects $e_\mathcal{C} \in \mathrm{Ob}(\mathcal{C})$, $e_\mathcal{D} \in \mathrm{Ob}(\mathcal{D})$ such that $u(e_\mathcal{C}) = e_\mathcal{D}$.*

Let $g_!, g^{-1}, g_*$ be as above. Then, u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ with $f_* = g^{-1}$, $f^{-1} = g_!$. The composition

$$Sh(\mathcal{C}) \xrightarrow{g} Sh(\mathcal{D}) \xrightarrow{f} Sh(\mathcal{C})$$

is isomorphic to the identity morphism of the topos $Sh(\mathcal{C})$. Moreover, the functor f^{-1} is fully faithful.

Proof. By assumption the functor u satisfies the hypotheses of Proposition 7.14.7. Hence u defines a morphism of sites and hence a morphism of topoi f as in Lemma 7.15.2. The formulas $f_* = g^{-1}$ and $f^{-1} = g_!$ are clear from the lemma cited and Lemma 7.21.5. We have $f_* \circ g_* = g^{-1} \circ g_* \cong \mathrm{id}$, and $g^{-1} \circ f^{-1} = g^{-1} \circ g_! \cong \mathrm{id}$ by Lemma 7.21.7.

We still have to show that f^{-1} is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(Sh(\mathcal{C}))$. We have to show that the map

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$$

is bijective. But the right hand side is equal to

$$\begin{aligned}\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, f_*f^{-1}\mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}f^{-1}\mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})\end{aligned}$$

(the first equality by adjunction) which proves what we want. □

00XV **Example 7.21.9.** Let X be a topological space. Let $i : Z \to X$ be the inclusion of a subset (with induced topology). Consider the functor $u : X_{Zar} \to Z_{Zar}$, $U \mapsto u(U) = Z \cap U$. At first glance it may appear that this functor is cocontinuous as well. After all, since Z has the induced topology, shouldn't any covering of $U \cap Z$ it come from a covering of U in X? Not so! Namely, what if $U \cap Z = \emptyset$? In that case, the empty covering is a covering of $U \cap Z$, and the empty covering can only be refined by the empty covering. Thus we conclude that u cocontinuous $\Rightarrow$ every nonempty open U of X has nonempty intersection with Z. But this is not sufficient. For example, if $X = \mathbf{R}$ the real number line with the usual topology, and $Z = \mathbf{R} \setminus \{0\}$, then there is an open covering of Z, namely $Z = \{x < 0\} \cup \bigcup_n \{1/n < x\}$ which cannot be refined by the restriction of any open covering of X.

7.22. Cocontinuous functors which have a right adjoint

00XW Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of the underlying categories such that v is right adjoint to u. In this case, if v is continuous, then u is cocontinuous (Lemma 7.22.4). If u is cocontinuous, then it is often (but not always, see Example 7.22.5) the case that v is continuous, and if so, then v defines a morphism of sites whose associated morphism of topoi is the same as that defined by u.

00XX **Lemma 7.22.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{C}$ be functors. Assume that u is cocontinuous and that v is a right adjoint to u. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to u, see Lemma 7.21.1. Then*

(1) *for a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $g_*\mathcal{F}$ is equal to the presheaf $v^p\mathcal{F}$, in other words, $(g_*\mathcal{F})(V) = \mathcal{F}(v(V))$, and*
(2) *for a sheaf $\mathcal{G}$ on $\mathcal{D}$ we have $g^{-1}\mathcal{G} = (v_p\mathcal{G})^\#$.*

Proof. For $\mathcal{F}$ as in (1) we have

$$g_*\mathcal{F} = {}_su\mathcal{F} = {}_pu\mathcal{F} = v^p\mathcal{F} = \mathcal{F} \circ v$$

The first equality is Lemma 7.21.1. The second equality is Lemma 7.20.2. The third equality is Lemma 7.19.3. The final equality is the definition of v^p in Section 7.5. This proves (1). For $\mathcal{G}$ as in (2) we have

$$g^{-1}\mathcal{G} = (u^p\mathcal{G})^\# = (v_p\mathcal{G})^\#$$

The first equality is Lemma 7.21.1. The second equality is Lemma 7.19.3. □

00XY **Lemma 7.22.2.** *Notation and assumptions as in Lemma 7.22.1. If in addition v is continuous then v defines a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ whose associated morphism of topoi is equal to g.*

Proof. We will use the results of Lemma 7.22.1 without further mention. To prove that v defines a morphism of sites f as in the statement of the lemma, we have to show that v_s is an exact functor (see Definition 7.14.1). Since $v_s\mathcal{G} = (v_p\mathcal{G})^\# = g^{-1}\mathcal{G}$ this follows from the fact that g is a morphism of topoi. Then we see that $f^{-1} = v_s = g^{-1}$ and we find that $f = g$ as morphisms of topoi. □

0EWJ **Example 7.22.3.** This example continues the discussion of Example 7.14.3 from which we borrow the notation $\mathcal{C}, \tau, \tau', \epsilon$. Observe that the identity functor $v : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is a continuous functor and the identity functor $u : \mathcal{C}_\tau \to \mathcal{C}_{\tau'}$ is a cocontinuous functor. Moreover u is left adjoint to v. Hence the results of Lemmas 7.22.1 and 7.22.2 apply and we conclude v defines a morphism of sites, namely

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

whose corresponding morphism of topoi is the same as the morphism of topoi associated to the cocontinuous functor u.

09VR **Lemma 7.22.4.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites and let $v : \mathcal{D} \to \mathcal{C}$ be a continuous functor. Assume v has a left adjoint $u : \mathcal{C} \to \mathcal{D}$. Then*

(1) *u is cocontinuous,*
(2) *the results of Lemmas 7.22.1 and 7.22.2 hold.*

In particular, v defines a morphism of sites $f : \mathcal{C} \to \mathcal{D}$.

Proof. Let U be an object of $\mathcal{C}$ and let $\{V_j \to u(U)\}_{j\in J}$ be a covering in $\mathcal{D}$. Then $\{v(V_j) \to v(u(U))\}_{i\in I}$ is a covering in $\mathcal{C}$. Via the adjunction map $U \to v(u(U))$ we can base change this to a covering $\{W_j \to U\}_{j\in J}$ with $W_j = v(V_j) \times_{v(u(U))} U$. Denoting $p_j : W_j \to v(V_j)$ the first projection, we obtain maps

$$u(W_j) \xrightarrow{u(p_j)} u(v(V_j)) \longrightarrow V_j$$

where the second arrow is the adjunction map. This determines a morphism $\{u(W_j) \to u(U)\} \to \{V_j \to u(U)\}$ of families of maps with fixed target, showing that u is indeed cocontinuous. The other statements are immediate from Lemmas 7.22.1 and 7.22.2. □

0H6Z **Example 7.22.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of the underlying categories such that v is right adjoint to u. Lemma 7.22.4 shows that if v is continuous, then u is cocontinous. Conversely, if u is cocontinuous, then we can't conclude that v is continuous. We will give an example of this phenomenon using the big étale and smooth sites of a scheme, but presumably there is an elementary example as well. Namely, consider a scheme S and the sites $(Sch/S)_{étale}$ and $(Sch/S)_{smooth}$. We may assume these sites have the same underlying category, see Topologies, Remark 34.11.1. Let $u = v = \mathrm{id}$. Then u as a functor from $(Sch/S)_{étale}$ to $(Sch/S)_{smooth}$ is cocontinuous as every smooth covering of a scheme can be refined by an étale covering, see More on Morphisms, Lemma 37.38.7. Conversely, the functor v from $(Sch/S)_{smooth}$ to $(Sch/S)_{étale}$ is not continuous as a smooth covering is not an étale covering in general.

7.23. Cocontinuous functors which have a left adjoint

08NG It may happen that a cocontinuous functor u has a left adjoint w.

08NH **Lemma 7.23.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$, see Lemmas 7.21.1 and 7.21.5.*

1. *If $w : \mathcal{D} \to \mathcal{C}$ is a left adjoint to u, then*
   (a) *$g_!\mathcal{F}$ is the sheaf associated to the presheaf $w^p\mathcal{F}$, and*
   (b) *$g_!$ is exact.*
2. *if w is a continuous left adjoint, then $g_!$ has a left adjoint.*
3. *If w is a cocontinuous left adjoint, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. Recall that $g_!\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$. Hence (1)(a) follows from the fact that $u_p = w^p$ by Lemma 7.19.3.

To see (1)(b) note that $g_!$ commutes with all colimits as $g_!$ is a left adjoint (Categories, Lemma 4.24.5). Let $i \mapsto \mathcal{F}_i$ be a finite diagram in $Sh(\mathcal{C})$. Then $\lim \mathcal{F}_i$ is computed in the category of presheaves (Lemma 7.10.1). Since w^p is a right adjoint (Lemma 7.5.4) we see that $w^p \lim \mathcal{F}_i = \lim w^p\mathcal{F}_i$. Since sheafification is exact (Lemma 7.10.15) we conclude by (1)(a).

Assume w is continuous. Then $g_! = (w^p\ )^\# = w^s$ but sheafification isn't necessary and one has the left adjoint w_s, see Lemmas 7.13.2 and 7.13.3.

Assume w is cocontinuous. The equality $g_! = h^{-1}$ follows from (1)(a) and the definitions. The equality $g^{-1} = h_*$ follows from the equality $g_! = h^{-1}$ and uniqueness of adjoint functor. Alternatively one can deduce it from Lemma 7.22.1. □

7.24. Existence of lower shriek

09YW In this section we discuss some cases of morphisms of topoi f for which f^{-1} has a left adjoint $f_!$.

09YX **Lemma 7.24.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be two sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $E \subset \mathrm{Ob}(\mathcal{D})$ be a subset such that*

(1) *for $V \in E$ there exists a sheaf $\mathcal{G}$ on $\mathcal{C}$ such that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{F})$ functorially for $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(2) *every object of $\mathcal{D}$ has a covering by objects of E.*

Then f^{-1} has a left adjoint $f_!$.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the sheaf $\mathcal{G}_V$ corresponding to $V \in E$ is defined up to unique isomorphism by the formula $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F})$. Recall that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F})$. Denote $i_V : h_V^\# \to f^{-1}\mathcal{G}_V$ the map corresponding to id in $\mathrm{Mor}(\mathcal{G}_V, \mathcal{G}_V)$. Functoriality in (1) implies that the bijection is given by

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F}) \to \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F}), \quad \varphi \mapsto f^{-1}\varphi \circ i_V$$

For any $V_1, V_2 \in E$ there is a canonical map

$$\mathrm{Mor}_{Sh(\mathcal{D})}(h_{V_2}^\#, h_{V_1}^\#) \to \mathrm{Hom}_{Sh(\mathcal{C})}(\mathcal{G}_{V_2}, \mathcal{G}_{V_1}), \quad \varphi \mapsto f_!(\varphi)$$

which is characterized by $f^{-1}(f_!(\varphi)) \circ i_{V_2} = i_{V_1} \circ \varphi$. Note that $\varphi \mapsto f_!(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $h_V^\# \mapsto \mathcal{G}_V$ and $\varphi \mapsto f_!\varphi$ is a functor from the full subcategory of $Sh(\mathcal{D})$ whose objects are the $h_V^\#$.

Let J be a set and let $J \to E$, $j \mapsto V_j$ be a map. Then we have a functorial bijection

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod \mathcal{G}_{V_j}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(\coprod h_{V_j}^\#, f^{-1}\mathcal{F})$$

using the product of the bijections above. Hence we can extend the functor $f_!$ to the full subcategory of $Sh(\mathcal{D})$ whose objects are coproducts of $h_V^\#$ with $V \in E$.

Given an arbitrary sheaf $\mathcal{H}$ on $\mathcal{D}$ we choose a coequalizer diagram

$$\mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \longrightarrow \mathcal{H}$$

where $\mathcal{H}_i = \coprod h_{V_{i,j}}^\#$ is a coproduct with $V_{i,j} \in E$. This is possible by assumption (2), see Lemma 7.12.5 (for those worried about set theoretical issues, note that the construction given in Lemma 7.12.5 is canonical). Define $f_!(\mathcal{H})$ to be the sheaf on $\mathcal{C}$ which makes

$$f_!\mathcal{H}_1 \rightrightarrows f_!\mathcal{H}_0 \longrightarrow f_!\mathcal{H}$$

a coequalizer diagram. Then

$$\begin{aligned}
\mathrm{Mor}(f_!\mathcal{H}, \mathcal{F}) &= \mathrm{Equalizer}(\ \mathrm{Mor}(f_!\mathcal{H}_0, \mathcal{F}) \rightrightarrows \mathrm{Mor}(f_!\mathcal{H}_1, \mathcal{F})\ ) \\
&= \mathrm{Equalizer}(\ \mathrm{Mor}(\mathcal{H}_0, f^{-1}\mathcal{F}) \rightrightarrows \mathrm{Mor}(\mathcal{H}_1, f^{-1}\mathcal{F})\ ) \\
&= \mathrm{Hom}(\mathcal{H}, f^{-1}\mathcal{F})
\end{aligned}$$

Hence we see that we can extend $f_!$ to the whole category of sheaves on $\mathcal{D}$. □

7.25. Localization

00XZ Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. See Categories, Example 4.2.13 for the definition of the category $\mathcal{C}/U$ of objects over U. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor

$$j_U : \mathcal{C}/U \longrightarrow \mathcal{C}.$$

This is clearly cocontinuous and continuous. Hence by the results of the previous sections we obtain a morphism of topoi

$$j_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})$$

given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

00Y0 **Definition 7.25.1.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The site $\mathcal{C}/U$ is called the *localization of the site $\mathcal{C}$ at the object U*.
(2) The morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ is called the *localization morphism*.
(3) The functor j_{U*} is called the *direct image functor*.
(4) For a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*.
(5) For a sheaf $\mathcal{G}$ on $\mathcal{C}/U$ the sheaf $j_{U!}\mathcal{G}$ is called the *extension of $\mathcal{G}$ by the empty set*.

The restriction $j_U^{-1}\mathcal{F}$ is the sheaf defined by the rule $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$ as expected. The extension by the empty set also has a very easy description in this case; here it is.

03CD **Lemma 7.25.2.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}/U$. Then $j_{U!}(\mathcal{G}^\#)$ is the sheaf associated to the presheaf*

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings.

Proof. By Lemma 7.21.5 we have $j_{U!}(\mathcal{G}^\#) = ((j_U)_p\mathcal{G}^\#)^\#$. By Lemma 7.13.4 this is equal to $((j_U)_p\mathcal{G})^\#$. Hence it suffices to prove that $(j_U)_p$ is given by the formula above for any presheaf $\mathcal{G}$ on $\mathcal{C}/U$. OK, and by the definition in Section 7.5 we have

$$(j_U)_p\mathcal{G}(V) = \mathrm{colim}_{(W/U, V \to W)} \mathcal{G}(W)$$

Now it is clear that the category of pairs $(W/U, V \to W)$ has an object $O_\varphi = (\varphi : V \to U, \mathrm{id} : V \to V)$ for every $\varphi : V \to U$, and moreover for any object there is a unique morphism from one of the O_φ into it. The result follows. □

03HU **Lemma 7.25.3.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let X/U be an object of $\mathcal{C}/U$. Then we have $j_{U!}(h^\#_{X/U}) = h^\#_X$.*

Proof. Denote $p : X \to U$ the structure morphism of X. By Lemma 7.25.2 we see $j_{U!}(h^\#_{X/U})$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \{\psi : V \to X \mid p \circ \psi = \varphi\}$$

This is clearly the same thing as $\mathrm{Mor}_{\mathcal{C}}(V, X)$. Hence the lemma follows. □

We have $j_{U!}(*) = h_U^\#$ by either of the two lemmas above. Hence for every sheaf $\mathcal{G}$ over $\mathcal{C}/U$ there is a canonical map of sheaves $j_{U!}\mathcal{G} \to h_U^\#$. This characterizes sheaves in the essential image of $j_{U!}$.

00Y1 **Lemma 7.25.4.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ gives an equivalence of categories*

$$Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})/h_U^\#$$

Proof. Let us denote objects of $\mathcal{C}/U$ as pairs (X, a) where X is an object of $\mathcal{C}$ and $a : X \to U$ is a morphism of $\mathcal{C}$. Similarly, objects of $Sh(\mathcal{C})/h_U^\#$ are pairs $(\mathcal{F}, \varphi)$. The functor $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ sends $\mathcal{G}$ to the pair $(j_{U!}\mathcal{G}, \gamma)$ where γ is the composition of $j_{U!}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}* = h_U^\#$.

Let us construct a functor from $Sh(\mathcal{C})/h_U^\#$ to $Sh(\mathcal{C}/U)$. Suppose that $(\mathcal{F}, \varphi)$ is given. For an object (X, a) of $\mathcal{C}/U$ we consider the set $\mathcal{F}_\varphi(X, a)$ of elements $s \in \mathcal{F}(X)$ which under φ map to the image of $a \in \mathrm{Mor}_\mathcal{C}(X, U) = h_U(X)$ in $h_U^\#(X)$. It is easy to see that $(X, a) \mapsto \mathcal{F}_\varphi(X, a)$ is a sheaf on $\mathcal{C}/U$. Clearly, the rule $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ defines a functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$.

Consider also the functor $PSh(\mathcal{C})/h_U \to PSh(\mathcal{C}/U)$, $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ where $\mathcal{F}_\varphi(X, a)$ is defined as the set of elements of $\mathcal{F}(X)$ mapping to $a \in h_U(X)$. We claim that the diagram

$$\begin{array}{ccc} PSh(\mathcal{C})/h_U & \longrightarrow & PSh(\mathcal{C}/U) \\ \downarrow & & \downarrow \\ Sh(\mathcal{C})/h_U^\# & \longrightarrow & Sh(\mathcal{C}/U) \end{array}$$

commutes, where the vertical arrows are given by sheafification. To see this[5], it suffices to prove that the construction commutes with the functor $\mathcal{F} \mapsto \mathcal{F}^+$ of Lemmas 7.10.3 and 7.10.4 and Theorem 7.10.10. Commutation with $\mathcal{F} \mapsto \mathcal{F}^+$ follows from the fact that given (X, a) the categories of coverings of (X, a) in $\mathcal{C}/U$ and coverings of X in $\mathcal{C}$ are canonically identified.

Next, let $PSh(\mathcal{C}/U) \to PSh(\mathcal{C})/h_U$ send $\mathcal{G}$ to the pair $(j_{U!}^{PSh}\mathcal{G}, \gamma)$ where $j_{U!}^{PSh}\mathcal{G}$ the presheaf defined by the formula in Lemma 7.25.2 and γ is the composition of $j_{U!}^{PSh}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}^{PSh}* = h_U$ (obvious from the formula). Then it is immediately clear that the diagram

$$\begin{array}{ccc} PSh(\mathcal{C}/U) & \longrightarrow & PSh(\mathcal{C})/h_U \\ \downarrow & & \downarrow \\ Sh(\mathcal{C}/U) & \longrightarrow & Sh(\mathcal{C})/h_U^\# \end{array}$$

[5] An alternative is to describe $\mathcal{F}_\varphi$ by the cartesian diagram

$$\begin{array}{ccc} \mathcal{F}_\varphi & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{F}|_{\mathcal{C}/U} & \longrightarrow & h_U|_{\mathcal{C}/U} \end{array} \quad \text{for presheaves and} \quad \begin{array}{ccc} \mathcal{F}_\varphi & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{F}|_{\mathcal{C}/U} & \longrightarrow & h_U^\#|_{\mathcal{C}/U} \end{array}$$

for sheaves and use that restriction to $\mathcal{C}/U$ commutes with sheafification.

commutes, where the vertical arrows are sheafification. Putting everything together it suffices to show there are functorial isomorphisms $(j_{U!}^{PSh}\mathcal{G})_\gamma = \mathcal{G}$ for $\mathcal{G}$ in $PSh(\mathcal{C}/U)$ and $j_{U!}^{PSh}\mathcal{F}_\varphi = \mathcal{F}$ for $(\mathcal{F}, \varphi)$ in $PSh(\mathcal{C})/h_U$. The value of the presheaf $(j_{U!}^{PSh}\mathcal{G})_\gamma$ on (X, a) is the fibre of the map

$$\coprod_{a':X\to U} \mathcal{G}(X, a') \to \mathrm{Mor}_{\mathcal{C}}(X, U)$$

over a which is $\mathcal{G}(X, a)$. This proves the first equality. The value of the presheaf $j_{U!}^{PSh}\mathcal{F}_\varphi$ is on X is

$$\coprod_{a:X\to U} \mathcal{F}_\varphi(X, a) = \mathcal{F}(X)$$

because given a set map $S \to S'$ the set S is the disjoint union of its fibres. □

Lemma 7.25.4 says the functor $j_{U!}$ is the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

where the first arrow is an equivalence.

04BB **Lemma 7.25.5.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ commutes with fibre products and equalizers (and more generally finite connected limits). In particular, if $\mathcal{F} \subset \mathcal{F}'$ in $Sh(\mathcal{C}/U)$, then $j_{U!}\mathcal{F} \subset j_{U!}\mathcal{F}'$.*

Proof. Via Lemma 7.25.4 and the fact that an equivalence of categories commutes with all limits, this reduces to the fact that the functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ commutes with fibre products and equalizers. Alternatively, one can prove this directly using the description of $j_{U!}$ in Lemma 7.25.2 using that sheafification is exact. (Also, in case $\mathcal{C}$ has fibre products and equalizers, the result follows from Lemma 7.21.6.) □

0E8E **Lemma 7.25.6.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ reflects injections and surjections.*

Proof. We have to show $j_{U!}$ reflects monomorphisms and epimorphisms, see Lemma 7.11.2. Via Lemma 7.25.4 this reduces to the fact that the functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ reflects monomorphisms and epimorphisms. □

03EE **Lemma 7.25.7.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $j_{U!}j_U^{-1}\mathcal{F} = \mathcal{F} \times h_U^\#$.*

Proof. This is clear from the description of $j_{U!}$ in Lemma 7.25.2. □

03EH **Lemma 7.25.8.** *Let $\mathcal{C}$ be a site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/V & \xrightarrow{\quad j \quad} & \mathcal{C}/U \\ & {\scriptstyle j_V}\searrow \quad \swarrow{\scriptstyle j_U} & \\ & \mathcal{C} & \end{array}$$

of continuous and cocontinuous functors. The functor $j : \mathcal{C}/V \to \mathcal{C}/U$, $(a : W \to V) \mapsto (f \circ a : W \to U)$ is identified with the functor $j_{V/U} : (\mathcal{C}/U)/(V/U) \to \mathcal{C}/U$ via the identification $(\mathcal{C}/U)/(V/U) = \mathcal{C}/V$. Moreover we have $j_{V!} = j_{U!} \circ j_!$, $j_V^{-1} = j^{-1} \circ j_U^{-1}$, and $j_{V*} = j_{U*} \circ j_*$.

Proof. The commutativity of the diagram is immediate. The agreement of j with $j_{V/U}$ follows from the definitions. By Lemma 7.21.2 we see that the following diagram of morphisms of topoi

04IK (7.25.8.1)
$$\begin{array}{ccc} Sh(\mathcal{C}/V) & \xrightarrow{\quad j \quad} & Sh(\mathcal{C}/U) \\ & {\scriptstyle j_V} \searrow \quad \swarrow {\scriptstyle j_U} & \\ & Sh(\mathcal{C}) & \end{array}$$

is commutative. This proves that $j_V^{-1} = j^{-1} \circ j_U^{-1}$ and $j_{V*} = j_{U*} \circ j_*$. The equality $j_{V!} = j_{U!} \circ j_!$ follows formally from adjointness properties. □

04IL **Lemma 7.25.9.** *Notation $\mathcal{C}$, $f : V \to U$, j_U, j_V, and j as in Lemma 7.25.8. Via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.25.4 we have*

(1) *the functor j^{-1} has the following description*
$$j^{-1}(\mathcal{H} \xrightarrow{\varphi} h_U^\#) = (\mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \to h_V^\#).$$

(2) *the functor $j_!$ has the following description*
$$j_!(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (\mathcal{H} \xrightarrow{h_f \circ \varphi} h_U^\#)$$

Proof. Proof of (2). Recall that the identification $Sh(\mathcal{C}/V) \to Sh(\mathcal{C})/h_V^\#$ sends $\mathcal{G}$ to $j_{V!}\mathcal{G} \to j_{V!}(*) = h_V^\#$ and similarly for $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$. Thus $j_!\mathcal{G}$ is mapped to $j_{U!}(j_!\mathcal{G}) \to j_{U!}(*) = h_U^\#$ and (2) follows because $j_{U!}j_! = j_{V!}$ by Lemma 7.25.8.

The reader can now prove (1) by using that j^{-1} is the right adjoint to $j_!$ and using that the rule in (1) is the right adjoint to the rule in (2). Here is a direct proof. Suppose that $\varphi : \mathcal{H} \to h_U^\#$ is an object of $Sh(\mathcal{C})/h_U^\#$. By the proof of Lemma 7.25.4 this corresponds to the sheaf $\mathcal{H}_\varphi$ on $\mathcal{C}/U$ defined by the rule
$$(a : W \to U) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = a\}$$
on $\mathcal{C}/U$. The pullback $j^{-1}\mathcal{H}_\varphi$ to $\mathcal{C}/V$ is given by the rule
$$(a : W \to V) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = f \circ a\}$$
by the description of $j^{-1} = j_{U/V}^{-1}$ as the restriction of $\mathcal{H}_\varphi$ to $\mathcal{C}/V$. On the other hand, applying the rule to the object
$$\mathcal{H}' = \mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \xrightarrow{\quad \varphi' \quad} h_V^\#$$
of $Sh(\mathcal{C})/h_V^\#$ we get $\mathcal{H}'_{\varphi'}$ given by
$$\begin{aligned}(a : W \to V) \longmapsto & \{s' \in \mathcal{H}'(W) \mid \varphi'(s') = a\} \\ = & \{(s, a') \in \mathcal{H}(W) \times h_V^\#(W) \mid a' = a \text{ and } \varphi(s) = f \circ a'\}\end{aligned}$$
which is exactly the same rule as the one describing $j^{-1}\mathcal{H}_\varphi$ above. □

0494 **Remark 7.25.10.** Localization and presheaves. Let $\mathcal{C}$ be a category. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^{-1}, j_{U*} and $j_{U!}$ have not been defined for presheaves. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Example 7.6.6). Hence we also obtain a functor

$$j_U^{-1} : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{C}/U)$$

and functors

$$j_{U*}, j_{U!} : PSh(\mathcal{C}/U) \longrightarrow PSh(\mathcal{C})$$

which are right, left adjoint to j_U^{-1}. By Lemma 7.25.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ commutes with fibre products and equalizers.

09W8 **Remark 7.25.11.** Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. The cocontinuous functors $\mathcal{C}/U \to \mathcal{C}$ and $j : \mathcal{C}/U \to \mathcal{C}/V$ (Lemma 7.25.8) satisfy property P of Remark 7.20.5. For example, if we have objects (X/U), (W/V), a morphism $g : j(X/U) \to (W/V)$, and a covering $\{f_i : (W_i/V) \to (W/V)\}$ then $(X \times_W W_i/U)$ is an avatar of $(X/U) \times_{g,(W/V),f_i} (W_i/V)$ and the family $\{(X \times_W W_i/U) \to (X/U)\}$ is a covering of $\mathcal{C}/U$.

7.26. Glueing sheaves

04TP This section is the analogue of Sheaves, Section 6.33.

04TQ **Lemma 7.26.1.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{C}$. Given a collection*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{G}|_{\mathcal{C}/U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\varphi_{ij} : \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \to \mathcal{G}|_{\mathcal{C}/U_i \times_U U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F}|_{\mathcal{C}/U} \longrightarrow \mathcal{G}|_{\mathcal{C}/U}$$

whose restriction to each $\mathcal{C}/U_i$ agrees with φ_i.

Proof. The restrictions used in the lemma are those of Lemma 7.25.8. Let V/U be an object of $\mathcal{C}/U$. Set $V_i = U_i \times_U V$ and denote $\mathcal{V} = \{V_i \to V\}$. Observe that $(U_i \times_U U_j) \times_U V = V_i \times_V V_j$. Then we have $\mathcal{F}|_{\mathcal{C}/U_i}(V_i/U_i) = \mathcal{F}(V_i)$ and $\mathcal{F}|_{\mathcal{C}/U_i \times_U U_j}(V_i \times_V V_j/U_i \times_U U_j) = \mathcal{F}(V_i \times_V V_j)$ and similarly for $\mathcal{G}$. Thus we can define φ on sections over V as the dotted arrows in the diagram

$$\begin{array}{ccccccc}
\mathcal{F}(V) & = & H^0(\mathcal{V}, \mathcal{F}) & \longrightarrow & \prod \mathcal{F}(V_i) & \rightrightarrows & \prod \mathcal{F}(V_i \times_V V_j) \\
 & & \vdots & & \prod \varphi_i \downarrow & & \prod \varphi_{ij} \downarrow \\
\mathcal{G}(V) & = & H^0(\mathcal{V}, \mathcal{G}) & \longrightarrow & \prod \mathcal{G}(V_i) & \rightrightarrows & \prod \mathcal{G}(V_i \times_V V_j)
\end{array}$$

The equality signs come from the sheaf condition; see Section 7.10 for the notation $H^0(\mathcal{V}, -)$. We omit the verification that these maps are compatible with the restriction maps. □

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on a site $\mathcal{C}$ the rule

$$U \longmapsto \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U})$$

defines a sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules on Sites, Section 18.27.

0BWQ **Lemma 7.26.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be sheaves on $\mathcal{C}$. There is a canonical bijection*

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries.

Proof. The lemma says that the functors $- \times \mathcal{G}$ and $\mathcal{H}om(\mathcal{G}, -)$ are adjoint to each other. To show this, we use the notion of unit and counit, see Categories, Section 4.24. The unit

$$\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F} \times \mathcal{G})$$

sends $s \in \mathcal{F}(U)$ to the map $\mathcal{G}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/U} \times \mathcal{G}|_{\mathcal{C}/U}$ which over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(V) \times \mathcal{G}(V), \quad t \longmapsto (s|_V, t).$$

The counit

$$\epsilon_{\mathcal{H}} : \mathcal{H}om(\mathcal{G}, \mathcal{H}) \times \mathcal{G} \longrightarrow \mathcal{H}$$

is the evaluation map. It is given by the rule

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{H}|_{\mathcal{C}/U}) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (\varphi, s) \longmapsto \varphi(s).$$

Then for each $\varphi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$, the corresponding morphism $\mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$ is given by mapping each section $s \in \mathcal{F}(U)$ to the morphism of sheaves on $\mathcal{C}/U$ which on sections over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{H}(V), \quad t \longmapsto \varphi(s|_V, t).$$

Conversely, for each $\psi : \mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$, the corresponding morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is given by

$$\mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (s, t) \longmapsto \psi(s)(t)$$

on sections over an object U. We omit the details of the proof showing that these constructions are mutually inverse. □

0D7X **Lemma 7.26.3.** *Let $\mathcal{C}$ be a site and $U \in \mathrm{Ob}(\mathcal{C})$. Then $\mathcal{H}om(h_U^\#, \mathcal{F}) = j_*(\mathcal{F}|_{\mathcal{C}/U})$ for $\mathcal{F}$ in $Sh(\mathcal{C})$.*

Proof. This can be shown by directly constructing an isomorphism of sheaves. Instead we argue as follows. Let $\mathcal{G}$ be a sheaf on $\mathcal{C}$. Then

$$\begin{aligned} \mathrm{Mor}(\mathcal{G}, j_*(\mathcal{F}|_{\mathcal{C}/U})) &= \mathrm{Mor}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}) \\ &= \mathrm{Mor}(j_!(\mathcal{G}|_{\mathcal{C}/U}), \mathcal{F}) \\ &= \mathrm{Mor}(\mathcal{G} \times h_U^\#, \mathcal{F}) \\ &= \mathrm{Mor}(\mathcal{G}, \mathcal{H}om(h_U^\#, \mathcal{F})) \end{aligned}$$

and we conclude by the Yoneda lemma. Here we used Lemmas 7.26.2 and 7.25.7. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on $\mathcal{C}/U_i$. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j} \longrightarrow \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} & \xrightarrow{\varphi_{ik}} & \mathcal{F}_k|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} \\ & {\scriptstyle\varphi_{ij}}\searrow \quad \nearrow{\scriptstyle\varphi_{jk}} & \\ & \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j\times_U U_k} & \end{array}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $\{U_i \to U\}_{i\in I}$.

04TR **Lemma 7.26.4.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $\{U_i \to U\}_{i\in I}$ there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} & \xrightarrow{\varphi_i} & \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\varphi_{ij}} \\ \mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j} \end{array}$$

are commutative.

Proof. Let us describe how to construct the sheaf $\mathcal{F}$ on $\mathcal{C}/U$. Let $a : V \to U$ be an object of $\mathcal{C}/U$. Then

$$\mathcal{F}(V/U) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \times_U V/U_i) \mid \varphi_{ij}(s_i|_{U_i\times_U U_j\times_U V}) = s_j|_{U_i\times_U U_j\times_U V}\}$$

We omit the construction of the restriction mappings. We omit the verification that this is a sheaf. We omit the construction of the isomorphisms φ_i, and we omit proving the commutativity of the diagrams of the lemma. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. Associated to $\mathcal{F}$ we have its *canonical glueing data* given by the restrictions $\mathcal{F}|_{\mathcal{C}/U_i}$ and the canonical isomorphisms

$$\left(\mathcal{F}|_{\mathcal{C}/U_i}\right)|_{\mathcal{C}/U_i\times_U U_j} = \left(\mathcal{F}|_{\mathcal{C}/U_j}\right)|_{\mathcal{C}/U_i\times_U U_j}$$

coming from the fact that the composition of the functors $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_i \to \mathcal{C}/U$ and $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_j \to \mathcal{C}/U$ are equal.

04TS **Lemma 7.26.5.** *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. The category $Sh(\mathcal{C}/U)$ is equivalent to the category of glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}/U$ the canonical glueing data.*

Proof. In Lemma 7.26.1 we saw that the functor is fully faithful, and in Lemma 7.26.4 we proved that it is essentially surjective (by explicitly constructing a quasi-inverse functor). □

Let $\mathcal{C}$ be a site. We are going to discuss a version of glueing sheaves on the entire site $\mathcal{C}$. For each object U in $\mathcal{C}$, let $\mathcal{F}_U$ be a sheaf on $\mathcal{C}/U$. Recall that there is a functor $j_f : \mathcal{C}/V \to \mathcal{C}/U$ associated to each morphism $f : V \to U$ in $\mathcal{C}$, given by $(a : W \to V) \mapsto (f \circ a : W \to U)$. For each such f, let

$$c_f : j_f^{-1}\mathcal{F}_U \to \mathcal{F}_V$$

be an isomorphism of sheaves. Assume that given any two arrows $f : V \to U$ and $g : W \to V$ in $\mathcal{C}$, the composition $c_g \circ j_g^{-1} c_f$ is equal to $c_{f \circ g}$. We will call such a collection of data $(\mathcal{F}_U, c_f)$ an *absolute glueing data for sheaves of sets on* $\mathcal{C}$. A morphism of absolute glueing data $(\mathcal{F}_U, c_f) \to (\mathcal{G}_U, c'_f)$ is given by a collection (φ_U) of morphisms of sheaves $\varphi_U : \mathcal{F}_U \to \mathcal{G}_U$, such that

$$\begin{array}{ccc} j_f^{-1}\mathcal{F}_U & \xrightarrow{c_f} & \mathcal{F}_V \\ {\scriptstyle j_f^{-1}\varphi_U}\downarrow & & \downarrow{\scriptstyle \varphi_V} \\ j_f^{-1}\mathcal{G}_U & \xrightarrow{c'_f} & \mathcal{G}_V \end{array}$$

commutes for every morphism $f : V \to U$ in $\mathcal{C}$.

Associated to any sheaf $\mathcal{F}$ on $\mathcal{C}$ is its *canonical absolute glueing data* $(\mathcal{F}|_{\mathcal{C}/U}, c_f)$, where the canonical isomorphisms $c_f : j_f^{-1}\mathcal{F}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/V}$ for $f : V \to U$ come from the relation $j_V = j_U \circ j_f$ as in Lemma 7.25.8. Any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{C}$ induces a morphism $(\varphi|_{\mathcal{C}/U})$ of canonical absolute glueing data.

0GWK **Lemma 7.26.6.** *Let $\mathcal{C}$ be a site. The category $Sh(\mathcal{C})$ is equivalent to the category of absolute glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}$ the canonical absolute glueing data.*

Proof. Given an absolute glueing data $(\mathcal{F}_U, c_f)$, we construct a sheaf $\mathcal{F}$ on $\mathcal{C}$ by setting $\mathcal{F}(U) = \mathcal{F}_U(U)$, where restriction along $f : V \to U$ given by the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_U(U) & \longrightarrow & \mathcal{F}_U(V) & \xrightarrow{c_f} & \mathcal{F}_V(V) \\ \| & & & & \| \\ \mathcal{F}(U) & & \longrightarrow & & \mathcal{F}(V) \end{array}$$

The compatibility condition $c_g \circ j_g^{-1} c_f = c_{f \circ g}$ ensures that $\mathcal{F}$ is a presheaf, and also ensures that the maps $c_f : \mathcal{F}_U(V) \to \mathcal{F}(V)$ define an isomorphism $\mathcal{F}_U \to \mathcal{F}|_{\mathcal{C}/U}$ for each U. Since each $\mathcal{F}_U$ is a sheaf, this implies that $\mathcal{F}$ is a sheaf as well. The functor $(\mathcal{F}_U, c_f) \mapsto \mathcal{F}$ just constructed is quasi-inverse to the functor which takes a sheaf on $\mathcal{C}$ to its canonical glueing data. Further details omitted. □

0GWL **Remark 7.26.7.** There is a variant of Lemma 7.26.6 which comes up in algebraic geometry. Namely, suppose that $\mathcal{C}$ is a site with all fibre products and for each $U \in \mathrm{Ob}(\mathcal{C})$ we are given a full subcategory $U_\tau \subset \mathcal{C}/U$ with the following properties

(1) U/U is in U_τ,
(2) for V/U in U_τ and covering $\{V_j \to V\}$ of $\mathcal{C}$ we have V_j/U in U_τ and
(3) for a morphism $U' \to U$ of $\mathcal{C}$ and V/U in U_τ the base change $V' = V \times_U U'$ is in U'_τ.

In this setting U_τ is a site for all U in $\mathcal{C}$ and the base change functor $U_\tau \to U'_\tau$ defines a morphism $f_\tau : U_\tau \to U'_\tau$ of sites for all morphisms $f : U' \to U$ of $\mathcal{C}$. The glueing statement we obtain then reads as follows: A sheaf $\mathcal{F}$ on $\mathcal{C}$ is given by the following data:

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$ a sheaf $\mathcal{F}_U$ on U_τ,
(2) for every $f : U' \to U$ in $\mathcal{C}$ a map $c_f : f_\tau^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the following conditions:

(a) given $f : U' \to U$ and $g : U'' \to U'$ in $\mathcal{C}$ the composition $c_g \circ g_\tau^{-1} c_f$ is equal to $c_{f\circ g}$, and
(b) if $f : U' \to U$ is in U_τ then c_f is an isomorphism.

If we ever need this we will precisely state and prove this here. (Note that this result is slightly different from the statements above as we are not requiring all the maps c_f to be isomorphisms!)

7.27. More localization

04IM In this section we prove a few lemmas on localization where we impose some additional hypotheses on the site on or the object we are localizing at.

03HT **Lemma 7.27.1.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. If the topology on $\mathcal{C}$ is subcanonical, see Definition 7.12.2, and if $\mathcal{G}$ is a sheaf on $\mathcal{C}/U$, then*

$$j_{U!}(\mathcal{G})(V) = \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

in other words sheafification is not necessary in Lemma 7.25.2.

Proof. Let $\mathcal{V} = \{V_i \to V\}_{i\in I}$ be a covering of V in the site $\mathcal{C}$. We are going to check the sheaf condition for the presheaf $\mathcal{H}$ of Lemma 7.25.2 directly. Let $(s_i, \varphi_i)_{i\in I} \in \prod_i \mathcal{H}(V_i)$, This means $\varphi_i : V_i \to U$ is a morphism in $\mathcal{C}$, and $s_i \in \mathcal{G}(V_i \xrightarrow{\varphi_i} U)$. The restriction of the pair (s_i, φ_i) to $V_i \times_V V_j$ is the pair $(s_i|_{V_i\times_V V_j/U}, \mathrm{pr}_1 \circ \varphi_i)$, and likewise the restriction of the pair (s_j, φ_j) to $V_i \times_V V_j$ is the pair $(s_j|_{V_i\times_V V_j/U}, \mathrm{pr}_2 \circ \varphi_j)$. Hence, if the family (s_i, φ_i) lies in $\check{H}^0(\mathcal{V}, \mathcal{H})$, then we see that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{pr}_2 \circ \varphi_j$. The condition that the topology on $\mathcal{C}$ is weaker than the canonical topology then implies that there exists a unique morphism $\varphi : V \to U$ such that φ_i is the composition of $V_i \to V$ with φ. At this point the sheaf condition for $\mathcal{G}$ guarantees that the sections s_i glue to a unique section $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$. Hence $(s, \varphi) \in \mathcal{H}(V)$ as desired. □

03CE **Lemma 7.27.2.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has products of pairs of objects. Then*

(1) *the functor j_U has a continuous right adjoint, namely the functor $v(X) = X \times U/U$,*
(2) *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}$ whose associated morphism of topoi equals $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$, and*
(3) *we have $j_{U*}\mathcal{F}(X) = \mathcal{F}(X \times U/U)$.*

Proof. The functor v being right adjoint to j_U means that given Y/U and X we have

$$\mathrm{Mor}_{\mathcal{C}}(Y, X) = \mathrm{Mor}_{\mathcal{C}/U}(Y/U, X \times U/U)$$

which is clear. To check that v is continuous let $\{X_i \to X\}$ be a covering of $\mathcal{C}$. By the third axiom of a site (Definition 7.6.2) we see that

$$\{X_i \times_X (X \times U) \to X \times_X (X \times U)\} = \{X_i \times U \to X \times U\}$$

is a covering of $\mathcal{C}$ also. Hence v is continuous. The other statements of the lemma follow from Lemmas 7.22.1 and 7.22.2. □

09W9 **Lemma 7.27.3.** *Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Let j be as in Lemma 7.25.8. Then*

(1) *the functor $j : \mathcal{C}/U \to \mathcal{C}/V$ has a continuous right adjoint, namely the functor $v : (X/V) \mapsto (X \times_V U/U)$,*
(2) *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}/V$ whose associated morphism of topoi equals j, and*
(3) *we have $j_*\mathcal{F}(X/V) = \mathcal{F}(X \times_V U/U)$.*

Proof. Follows from Lemma 7.27.2 since j may be viewed as a localization functor by Lemma 7.25.8. □

A fundamental property of an open immersion is that the restriction of the push-forward and the restriction of the extension by the empty set produces back the original sheaf. This is not always true for the functors associated to j_U above. It is true when U is a "subobject of the final object".

00Y2 **Lemma 7.27.4.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. This is a special case of Lemma 7.21.7 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. □

0EYV **Lemma 7.27.5.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

be a commutative diagram of $\mathcal{C}$. The morphisms of Lemma 7.25.8 produce commutative diagrams

$$\begin{array}{ccc} \mathcal{C}/U' & \xrightarrow{j_{U'/U}} & \mathcal{C}/U \\ {\scriptstyle j_{U'/V'}}\downarrow & & \downarrow{\scriptstyle j_{U/V}} \\ \mathcal{C}/V' & \xrightarrow{j_{V'/V}} & \mathcal{C}/V \end{array} \quad \text{and} \quad \begin{array}{ccc} Sh(\mathcal{C}/U') & \xrightarrow{j_{U'/U}} & Sh(\mathcal{C}/U) \\ {\scriptstyle j_{U'/V'}}\downarrow & & \downarrow{\scriptstyle j_{U/V}} \\ Sh(\mathcal{C}/V') & \xrightarrow{j_{V'/V}} & Sh(\mathcal{C}/V) \end{array}$$

of continuous and cocontinuous functors and of topoi. Moreover, if the initial diagram of $\mathcal{C}$ is cartesian, then we have $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$.

Proof. The commutativity of the left square in the first statement of the lemma is immediate from the definitions. It implies the commutativity of the diagram of topoi by Lemma 7.21.2. Assume the diagram is cartesian. By the uniqueness of adjoint functors, to show $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$ is equivalent to showing

$j_{U/V}^{-1} \circ j_{V'/V!} = j_{U'/U!} \circ j_{U'/V'}^{-1}$. Via the identifications of Lemma 7.25.4 we may think of our diagram of topoi as

$$\begin{array}{ccc} Sh(\mathcal{C})/h_{U'}^\# & \longrightarrow & Sh(\mathcal{C})/h_U^\# \\ \downarrow & & \downarrow \\ Sh(\mathcal{C})/h_{V'}^\# & \longrightarrow & Sh(\mathcal{C})/h_V^\# \end{array}$$

and we know how to interpret the functors j^{-1} and $j_!$ by Lemma 7.25.9. Thus we have to show given $\mathcal{F} \to h_{V'}^\#$ that

$$\mathcal{F} \times_{h_{V'}^\#} h_{U'}^\# = \mathcal{F} \times_{h_V^\#} h_U^\#$$

as sheaves with map to $h_U^\#$. This is true because $h_{U'} = h_{V'} \times_{h_V} h_U$ and hence also

$$h_{U'}^\# = h_{V'}^\# \times_{h_V^\#} h_U^\#$$

as sheafification is exact. □

7.28. Localization and morphisms

04I8 The following lemma is important in order to understand relation between localization and morphisms of sites and topoi.

03CF **Lemma 7.28.1.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$ and set $U = u(V)$. Then the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/U$ determines a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The morphism f' fits into a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow[j_U]{} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.25.4 the functor $(f')^{-1}$ is described by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} h_U^\#).$$

Finally, we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. It is clear that u' is continuous, and hence we get functors $f'_* = (u')^s = (u')^p$ (see Sections 7.5 and 7.13) and an adjoint $(f')^{-1} = (u')_s = ((u')_p)^\#$. The assertion $f'_* j_U^{-1} = j_V^{-1} f_*$ follows as

$$(j_V^{-1} f_* \mathcal{F})(V'/V) = f_* \mathcal{F}(V') = \mathcal{F}(u(V')) = (j_U^{-1}\mathcal{F})(u(V')/U) = (f'_* j_U^{-1}\mathcal{F})(V'/V)$$

which holds even for presheaves. What isn't clear a priori is that $(f')^{-1}$ is exact, that the diagram commutes, and that the description of $(f')^{-1}$ holds.

Let $\mathcal{H}$ be a sheaf on $\mathcal{D}/V$. Let us compute $j_{U!}(f')^{-1}\mathcal{H}$. We have

$$\begin{aligned} j_{U!}(f')^{-1}\mathcal{H} &= ((j_U)_p(u'_p\mathcal{H})^\#)^\# \\ &= ((j_U)_p u'_p\mathcal{H})^\# \\ &= (u_p(j_V)_p\mathcal{H})^\# \\ &= f^{-1}j_{V!}\mathcal{H} \end{aligned}$$

The first equality by unwinding the definitions. The second equality by Lemma 7.13.4. The third equality because $u \circ j_V = j_U \circ u'$. The fourth equality by Lemma 7.13.4 again. All of the equalities above are isomorphisms of functors, and hence we may interpret this as saying that the following diagram of categories and functors is commutative

$$\begin{array}{ccccc} Sh(\mathcal{C}/U) & \xrightarrow[j_{U!}]{} & Sh(\mathcal{C})/h_U^\# & \longrightarrow & Sh(\mathcal{C}) \\ \uparrow{\scriptstyle (f')^{-1}} & & \uparrow{\scriptstyle f^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_{V!}} & Sh(\mathcal{D})/h_V^\# & \longrightarrow & Sh(\mathcal{D}) \end{array}$$

The middle arrow makes sense as $f^{-1}h_V^\# = (h_{u(V)})^\# = h_U^\#$, see Lemma 7.13.5. In particular this proves the description of $(f')^{-1}$ given in the statement of the lemma. Since by Lemma 7.25.4 the left horizontal arrows are equivalences and since f^{-1} is exact by assumption we conclude that $(f')^{-1} = u'_s$ is exact. Namely, because it is a left adjoint it is already right exact (Categories, Lemma 4.24.5). Hence we only need to show that it transforms a final object into a final object and commutes with fibre products (Categories, Lemma 4.23.2). Both are clear for the induced functor $f^{-1} : Sh(\mathcal{D})/h_V^\# \to Sh(\mathcal{C})/h_U^\#$. This proves that f' is a morphism of sites.

We still have to verify that $(f')^{-1}j_V^{-1} = j_U^{-1}f^{-1}$. To see this use the formula above and the description in Lemma 7.25.7. Namely, combined these give, for any sheaf $\mathcal{G}$ on $\mathcal{D}$, that

$$j_{U!}(f')^{-1}j_V^{-1}\mathcal{G} = f^{-1}j_{V!}j_V^{-1}\mathcal{G} = f^{-1}(\mathcal{G} \times h_V^\#) = f^{-1}\mathcal{G} \times h_U^\# = j_{U!}j_U^{-1}f^{-1}\mathcal{G}.$$

Since the functor $j_{U!}$ induces an equivalence $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ we conclude. □

The following lemma is a special case of the more general Lemma 7.28.1 above.

03EF **Lemma 7.28.2.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a functor. Let $V \in \mathrm{Ob}(\mathcal{D})$. Set $U = u(V)$. Assume that*

(1) *$\mathcal{C}$ and $\mathcal{D}$ have all finite limits,*
(2) *u is continuous, and*
(3) *u commutes with finite limits.*

There exists a commutative diagram of morphisms of sites

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow[j_U]{} & \mathcal{C} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the right vertical arrow corresponds to u, the left vertical arrow corresponds to the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/u(V)$ and the horizontal arrows correspond to the functors $\mathcal{C} \to \mathcal{C}/U$, $X \mapsto X \times U$ and $\mathcal{D} \to \mathcal{D}/V$, $Y \mapsto Y \times V$ as

in Lemma 7.27.2. Moreover, the associated diagram of morphisms of topoi is equal to the diagram of Lemma 7.28.1. In particular we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. Note that u satisfies the assumptions of Proposition 7.14.7 and hence induces a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ by that proposition. It is clear that u induces a functor u' as indicated. It is clear that this functor also satisfies the assumptions of Proposition 7.14.7. Hence we get a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The diagram commutes by our definition of composition of morphisms of sites (see Definition 7.14.5) and because

$$u(Y \times V) = u(Y) \times u(V) = u(Y) \times U$$

which shows that the diagram of categories and functors opposite to the diagram of the lemma commutes. □

At this point we can localize a site, we know how to relocalize, and we can localize a morphism of sites at an object of the site downstairs. If we combine these then we get the following kind of diagram.

04IN **Lemma 7.28.3.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f_c}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

We have $f_c = f' \circ j_{U/u(V)}$ where $f' : Sh(\mathcal{C}/u(V)) \to Sh(\mathcal{D}/V)$ is as in Lemma 7.28.1 and $j_{U/u(V)} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C}/u(V))$ is as in Lemma 7.25.8. Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.25.4 the functor $(f_c)^{-1}$ is described by the rule

$$(f_c)^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, h^\#_{u(V)}, c} h_U^\# \to h_U^\#).$$

Finally, given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{array}{ccc} U' & \xrightarrow{c'} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C}/U') & \xrightarrow{j_{U'/U}} & Sh(\mathcal{C}/U) \\ {\scriptstyle f_{c'}}\downarrow & & \downarrow{\scriptstyle f_c} \\ Sh(\mathcal{D}/V') & \xrightarrow{j_{V'/V}} & Sh(\mathcal{D}/V). \end{array}$$

commutes.

Proof. This lemma proves itself, and is more a collection of things we know at this stage of the development of theory. For example the commutativity of the first square follows from the commutativity of Diagram (7.25.8.1) and the commutativity

of the diagram in Lemma 7.28.1. The description of f_c^{-1} follows on combining Lemma 7.25.9 with Lemma 7.28.1. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{h^\#_{u(V)},c} h^\#_U \times_{h^\#_U} h^\#_{U'} = f^{-1}(\mathcal{H} \times_{h^\#_V} h^\#_{V'}) \times_{h^\#_{u(V'),c'}} h^\#_{U'}$$

which is formal using that $f^{-1}h^\#_V = h^\#_{u(V)}$ and $f^{-1}h^\#_{V'} = h^\#_{u(V')}$, see Lemma 7.13.5. □

In the following lemma we find another kind of functoriality of localization, in case the morphism of topoi comes from a cocontinuous functor. This is a kind of diagram which is different from the diagram in Lemma 7.28.1, and in particular, in general the equality $f'_* j_U^{-1} = j_V^{-1} f_*$ seen in Lemma 7.28.1 does not hold in the situation of the following lemma.

03EG **Lemma 7.28.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let U be an object of $\mathcal{C}$, and set $V = u(U)$. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the left vertical arrow is $u' : \mathcal{C}/U \to \mathcal{D}/V$, $U'/U \mapsto V'/V$. Then u' is cocontinuous also and we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}) \end{array}$$

where f (resp. f') corresponds to u (resp. u').

Proof. The commutativity of the first diagram is clear. It implies the commutativity of the second diagram provided we show that u' is cocontinuous.

Let U'/U be an object of $\mathcal{C}/U$. Let $\{V_j/V \to u(U')/V\}_{j\in J}$ be a covering of $u(U')/V$ in $\mathcal{D}/V$. Since u is cocontinuous there exists a covering $\{U'_i \to U'\}_{i\in I}$ such that the family $\{u(U'_i) \to u(U')\}$ refines the covering $\{V_j \to u(U')\}$ in $\mathcal{D}$. In other words, there exists a map of index sets $\alpha : I \to J$ and morphisms $\phi_i : u(U'_i) \to V_{\alpha(i)}$ over U'. Think of U'_i as an object over U via the composition $U'_i \to U' \to U$. Then $\{U'_i/U \to U'/U\}$ is a covering of $\mathcal{C}/U$ such that $\{u(U'_i)/V \to u(U')/V\}$ refines $\{V_j/V \to u(U')/V\}$ (use the same α and the same maps ϕ_i). Hence $u' : \mathcal{C}/U \to \mathcal{D}/V$ is cocontinuous. □

0D5R **Lemma 7.28.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let V be an object of $\mathcal{D}$. Let ${}^u_V\mathcal{I}$ be the category introduced in Section 7.19. We have a commutative diagram*

$$\begin{array}{ccc} {}^u_V\mathcal{I} & \xrightarrow{j} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array} \quad \text{where} \quad \begin{array}{c} j : (U, \psi) \mapsto U \\ u' : (U, \psi) \mapsto (\psi : u(U) \to V) \end{array}$$

Declare a family of morphisms $\{(U_i, \psi_i) \to (U, \psi)\}$ of ${}^u_V\mathcal{I}$ to be a covering if and only if $\{U_i \to U\}$ is a covering in $\mathcal{C}$. Then

(1) *${}^u_V\mathcal{I}$ is a site,*
(2) *j is continuous and cocontinuous,*
(3) *u' is cocontinuous,*
(4) *we get a commutative diagram of topoi*

$$\begin{array}{ccc} Sh({}^u_V\mathcal{I}) & \xrightarrow{j} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}) \end{array}$$

where f (resp. f') corresponds to u (resp. u'), and
(5) *we have $f'_* j^{-1} = j_V^{-1} f_*$.*

Proof. Parts (1), (2), (3), and (4) are straightforward consequences of the definitions and the fact that the functor j commutes with fibre products. We omit the details. To see (5) recall that f_* is given by ${}_s u = {}_p u$. Hence the value of $j_V^{-1} f_* \mathcal{F}$ on V'/V is the value of ${}_p u\mathcal{F}$ on V' which is the limit of the values of $\mathcal{F}$ on the category ${}^u_{V'}\mathcal{I}$. Clearly, there is an equivalence of categories

$${}^u_{V'}\mathcal{I} \to {}^{u'}_{V'/V}\mathcal{I}$$

Since the value of $f'_* j^{-1}\mathcal{F}$ on V'/V is given by the limit of the values of $j^{-1}\mathcal{F}$ on the category ${}^{u'}_{V'/V}\mathcal{I}$ and since the values of $j^{-1}\mathcal{F}$ on objects of ${}^u_V\mathcal{I}$ are just the values of $\mathcal{F}$ (by Lemma 7.21.5 as j is continuous and cocontinuous) we see that (5) is true. □

The following two results are of a slightly different nature.

0FN1 **Lemma 7.28.6.** *Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{v'} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

Assume

(1) *u, u', v, and v' are cocontinuous giving rise to morphisms of topoi f, f', g, and g' by Lemma 7.21.1,*
(2) *$v \circ u' = u \circ v'$,*
(3) *v and v' are continuous as well as cocontinuous, and*
(4) *for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^u_{v(V')}\mathcal{I}$ given by v is cofinal.*

Then $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.

Proof. The categories ${}^{u'}_{V'}\mathcal{I}$ and ${}^u_{v(V')}\mathcal{I}$ are defined in Section 7.19. The functor in condition (4) sends the object $\psi : u'(U') \to V'$ of ${}^{u'}_{V'}\mathcal{I}$ to the object $v(\psi) : uv'(U') = vu'(U') \to v(V')$ of ${}^u_{v(V')}\mathcal{I}$. Recall that g^{-1} is given by v^p (Lemma 7.21.5) and f_* is given by ${}_s u = {}_p u$. Hence the value of $g^{-1} f_* \mathcal{F}$ on V' is the value of ${}_p u\mathcal{F}$ on $v(V')$ which is the limit

$$\lim_{u(U) \to v(V') \in \mathrm{Ob}({}^u_{v(V')}\mathcal{I}^{opp})} \mathcal{F}(U)$$

By the same reasoning, the value of $f'_*(g')^{-1}\mathcal{F}$ on V' is given by the limit

$$\lim_{u'(U')\to V'\in \mathrm{Ob}(^{u'}_{V'}\mathcal{I}^{opp})} \mathcal{F}(v'(U'))$$

Thus assumption (4) and Categories, Lemma 4.17.4 show that these agree and the first equality of the lemma is proved. The second equality follows from the first by uniqueness of adjoints. □

0FN2 **Lemma 7.28.7.** *Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{v'} & \mathcal{C} \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

With notation as in Sections 7.14 and 7.21 assume

(1) *u and u' are continuous giving rise to morphisms of sites f and f',*
(2) *v and v' are cocontinuous giving rise to morphisms of topoi g and g',*
(3) *$u \circ v = v' \circ u'$, and*
(4) *v and v' are continuous as well as cocontinuous.*

Then[6] *$f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.*

Proof. Namely, we have

$$f'_*(g')^{-1}\mathcal{F} = (u')^p((v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition and the second by Lemma 7.21.5. We have

$$g^{-1}f_*\mathcal{F} = (v^p u^p \mathcal{F})^\# = ((u')^p(v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition, the second because $u \circ v = v' \circ u'$, the third because we already saw that $(u')^p(v')^p\mathcal{F}$ is a sheaf. This proves $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and the equality $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ follows by uniqueness of left adjoints. □

7.29. Morphisms of topoi

039Z In this section we show that any morphism of topoi is equivalent to a morphism of topoi which comes from a morphism of sites. Please compare with [**AGV71**, Exposé IV, Proposition 4.9.4].

03A0 **Lemma 7.29.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(1) *u is cocontinuous,*
(2) *u is continuous,*
(3) *given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, then there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$,*
(4) *given $U', U \in \mathrm{Ob}(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$ there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$, and*
(5) *given $V \in \mathrm{Ob}(\mathcal{D})$ there exists a covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i\in I}$.*

[6] In this generality we don't know $f \circ g'$ is equal to $g \circ f'$ as morphisms of topoi (there is a canonical 2-arrow from the first to the second which may not be an isomorphism).

Then the morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{D})$$

associated to the cocontinuous functor u by Lemma 7.21.1 is an equivalence.

Proof. Assume u satisfies properties (1) – (5). We will show that the adjunction mappings

$$\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G} \quad \text{and} \quad g^{-1}g_*\mathcal{F} \longrightarrow \mathcal{F}$$

are isomorphisms.

Note that Lemma 7.21.5 applies and we have $g^{-1}\mathcal{G}(U) = \mathcal{G}(u(U))$ for any sheaf $\mathcal{G}$ on $\mathcal{D}$. Next, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let V be an object of $\mathcal{D}$. By definition we have $g_*\mathcal{F}(V) = \lim_{u(U)\to V} \mathcal{F}(U)$. Hence

$$g^{-1}g_*\mathcal{F}(U) = \lim_{U', u(U')\to u(U)} \mathcal{F}(U')$$

where the morphisms $\psi : u(U') \to u(U)$ need not be of the form $u(\alpha)$. The category of such pairs (U', ψ) has a final object, namely (U, id), which gives rise to the map from the limit into $\mathcal{F}(U)$. Let $(s_{(U',\psi)})$ be an element of the limit. We want to show that $s_{(U',\psi)}$ is uniquely determined by the value $s_{(U,\mathrm{id})} \in \mathcal{F}(U)$. By property (4) given any (U', ψ) there exists a covering $\{U'_i \to U'\}$ such that the compositions $u(U'_i) \to u(U') \to u(U)$ are of the form $u(c_i)$ for some $c_i : U'_i \to U$ in $\mathcal{C}$. Hence

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s_{(U,\mathrm{id})}).$$

Since $\mathcal{F}$ is a sheaf it follows that indeed $s_{(U',\psi)}$ is determined by $s_{(U,\mathrm{id})}$. This proves uniqueness. For existence, assume given any $s \in \mathcal{F}(U)$, $\psi : u(U') \to u(U)$, $\{f_i : U'_i \to U'\}$ and $c_i : U'_i \to U$ such that $\psi \circ u(f_i) = u(c_i)$ as above. We claim there exists a (unique) element $s_{(U',\psi)} \in \mathcal{F}(U')$ such that

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s).$$

Namely, a priori it is not clear the elements $c_i^*(s)|_{U'_i \times_{U'} U'_j}$ and $c_j^*(s)|_{U'_i \times_{U'} U'_j}$ agree, since the diagram

$$\begin{array}{ccc} U'_i \times_{U'} U'_j & \xrightarrow[\mathrm{pr}_2]{} & U'_j \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle c_j} \\ U'_i & \xrightarrow{c_i} & U \end{array}$$

need not commute. But condition (3) of the lemma guarantees that there exist coverings $\{f_{ijk} : U'_{ijk} \to U'_i \times_{U'} U'_j\}_{k\in K_{ij}}$ such that $c_i \circ \mathrm{pr}_1 \circ f_{ijk} = c_j \circ \mathrm{pr}_2 \circ f_{ijk}$. Hence

$$f_{ijk}^*\left(c_i^* s|_{U'_i \times_{U'} U'_j}\right) = f_{ijk}^*\left(c_j^* s|_{U'_i \times_{U'} U'_j}\right)$$

Hence $c_i^*(s)|_{U'_i \times_{U'} U'_j} = c_j^*(s)|_{U'_i \times_{U'} U'_j}$ by the sheaf condition for $\mathcal{F}$ and hence the existence of $s_{(U',\psi)}$ also by the sheaf condition for $\mathcal{F}$. The uniqueness guarantees that the collection $(s_{(U',\psi)})$ so obtained is an element of the limit with $s_{(U,\psi)} = s$. This proves that $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Let V be an object of $\mathcal{D}$. Then we see that

$$g_*g^{-1}\mathcal{G}(V) = \lim_{U,\psi:u(U)\to V} \mathcal{G}(u(U))$$

By the preceding paragraph we see that the value of the sheaf $g_*g^{-1}\mathcal{G}$ on an object V of the form $V = u(U)$ is equal to $\mathcal{G}(u(U))$. (Formally, this holds because we have $g^{-1}g_*g^{-1} \cong g^{-1}$, and the description of g^{-1} given at the beginning of the

proof; informally just by comparing limits here and above.) Hence the adjunction mapping $\mathcal{G} \to g_* g^{-1}\mathcal{G}$ has the property that it is a bijection on sections over any object of the form $u(U)$. Since by axiom (5) there exists a covering of V by objects of the form $u(U)$ we see easily that the adjunction map is an isomorphism. $\square$

It will be convenient to give cocontinuous functors as in Lemma 7.29.1 a name.

03CG **Definition 7.29.2.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$* is a cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions and conclusions of Lemma 7.29.1.

03CH **Lemma 7.29.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. For every object U of $\mathcal{C}$ we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ \downarrow & & \downarrow u \\ \mathcal{D}/u(U) & \xrightarrow{j_{u(U)}} & \mathcal{D} \end{array}$$

as in Lemma 7.28.4. The left vertical arrow is a special cocontinuous functor. Hence in the commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ \downarrow & & \downarrow u \\ Sh(\mathcal{D}/u(U)) & \xrightarrow{j_{u(U)}} & Sh(\mathcal{D}) \end{array}$$

the vertical arrows are equivalences.

Proof. We have seen the existence and commutativity of the diagrams in Lemma 7.28.4. We have to check hypotheses (1) – (5) of Lemma 7.29.1 for the induced functor $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. This is completely mechanical.

Property (1). This is Lemma 7.28.4.

Property (2). Let $\{U_i'/U \to U'/U\}_{i \in I}$ be a covering of U'/U in $\mathcal{C}/U$. Because u is continuous we see that $\{u(U_i')/u(U) \to u(U')/u(U)\}_{i \in I}$ is a covering of $u(U')/u(U)$ in $\mathcal{D}/u(U)$. Hence (2) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (3). Let $a, b : U''/U \to U'/U$ in $\mathcal{C}/U$ be morphisms such that $u(a) = u(b)$ in $\mathcal{D}/u(U)$. Because u satisfies (3) we see there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$. This gives a covering $\{f_i : U_i''/U \to U''/U\}$ in $\mathcal{C}/U$ such that $a \circ f_i = b \circ f_i$. Hence (3) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (4). Let $U''/U, U'/U \in \mathrm{Ob}(\mathcal{C}/U)$ and a morphism $c : u(U'')/u(U) \to u(U')/u(U)$ in $\mathcal{D}/u(U)$ be given. Because u satisfies property (4) there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ and morphisms $c_i : U_i'' \to U'$ such that $u(c_i) = c \circ u(f_i)$. We think of U_i'' as an object over U via the composition $U_i'' \to U'' \to U$. It may not be true that c_i is a morphism over U! But since $u(c_i)$ is a morphism over $u(U)$ we may apply property (3) for u and find coverings $\{f_{ik} : U_{ik}'' \to U_i''\}$ such that $c_{ik} = c_i \circ f_{ik} : U_{ik}'' \to U'$ are morphisms over U. Hence $\{f_i \circ f_{ik} : U_{ik}''/U \to U''/U\}$ is a covering in $\mathcal{C}/U$ such that $u(c_{ik}) = c \circ u(f_{ik})$. Hence (4) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (5). Let $h : V \to u(U)$ be an object of $\mathcal{D}/u(U)$. Because u satisfies property (5) there exists a covering $\{c_i : u(U_i) \to V\}$ in $\mathcal{D}$. By property (4) we can find coverings $\{f_{ij} : U_{ij} \to U_i\}$ and morphisms $c_{ij} : U_{ij} \to U$ such that $u(c_{ij}) = h \circ c_i \circ u(f_{ij})$. Hence $\{u(U_{ij})/u(U) \to V/u(U)\}$ is a covering in $\mathcal{D}/u(U)$ of the desired shape and we conclude that (5) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. □

03A1 **Lemma 7.29.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{C}' \subset Sh(\mathcal{C})$ be a full subcategory (with a set of objects) such that*

(1) *$h_U^\# \in \mathrm{Ob}(\mathcal{C}')$ for all $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *$\mathcal{C}'$ is preserved under fibre products in $Sh(\mathcal{C})$.*

Declare a covering of $\mathcal{C}'$ to be any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ of maps such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is a surjective map of sheaves. Then

(1) *$\mathcal{C}'$ is a site (after choosing a set of coverings, see Sets, Lemma 3.11.1),*
(2) *representable presheaves on $\mathcal{C}'$ are sheaves (i.e., the topology on $\mathcal{C}'$ is subcanonical, see Definition 7.12.2),*
(3) *the functor $v : \mathcal{C} \to \mathcal{C}'$, $U \mapsto h_U^\#$ is a special cocontinuous functor, hence induces an equivalence $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$,*
(4) *for any $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ we have $g^{-1}h_{\mathcal{F}} = \mathcal{F}$, and*
(5) *for any $U \in \mathrm{Ob}(\mathcal{C})$ we have $g_* h_U^\# = h_{v(U)} = h_{h_U^\#}$.*

Proof. Warning: Some of the statements above may look be a bit confusing at first; this is because objects of $\mathcal{C}'$ can also be viewed as sheaves on $\mathcal{C}$! We omit the proof that the coverings of $\mathcal{C}'$ as described in the lemma satisfy the conditions of Definition 7.6.2.

Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a surjective family of morphisms of sheaves. Let $\mathcal{G}$ be another sheaf. Part (2) of the lemma says that the equalizer of

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{i \in I} \mathcal{F}_i, \mathcal{G}) \rightrightarrows \mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{(i_0, i_1) \in I \times I} \mathcal{F}_{i_0} \times_{\mathcal{F}} \mathcal{F}_{i_1}, \mathcal{G})$$

is $\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})$. This is clear (for example use Lemma 7.11.3).

To prove (3) we have to check conditions (1) – (5) of Lemma 7.29.1. The fact that v is cocontinuous is equivalent to the description of surjective maps of sheaves in Lemma 7.11.2. The functor v is continuous because $U \mapsto h_U^\#$ commutes with fibre products, and transforms coverings into coverings (see Lemma 7.10.15, and Lemma 7.12.4). Properties (3), (4) of Lemma 7.29.1 are statements about morphisms $f : h_{U'}^\# \to h_U^\#$. Such a morphism is the same thing as an element of $h_U^\#(U')$. Hence (3) and (4) are immediate from the construction of the sheafification. Property (5) of Lemma 7.29.1 is Lemma 7.12.5. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ the equivalence of topoi associated with v by Lemma 7.29.1.

Let $\mathcal{F}$ be as in part (4) of the lemma. For any $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$g^{-1}h_{\mathcal{F}}(U) = h_{\mathcal{F}}(v(U)) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$$

The first equality by Lemma 7.21.5. Thus part (4) holds.

Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then

$$\begin{aligned} g_* h_U^\# (\mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C}')}(h_{\mathcal{F}}, g_* h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1} h_{\mathcal{F}}, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, h_U^\#) \\ &= \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}, h_U^\#) \end{aligned}$$

as desired (where the third equality was shown above). □

Using this we can massage any topos to live over a site having all finite limits.

03CI **Lemma 7.29.5.** *Let $Sh(\mathcal{C})$ be a topos. Let $\{\mathcal{F}_i\}_{i \in I}$ be a set of sheaves on $\mathcal{C}$. There exists an equivalence of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ induced by a special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ of sites such that*

1. *$\mathcal{C}'$ has a subcanonical topology,*
2. *a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_j} \to h_V$ is surjective,*
3. *$\mathcal{C}'$ has fibre products and a final object (i.e., $\mathcal{C}'$ has all finite limits),*
4. *every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable, and*
5. *each $g_* \mathcal{F}_i$ is a representable sheaf.*

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$, the given sheaves $\mathcal{F}_i$ and the final sheaf $*$ (see Example 7.10.2). We are going to inductively define full subcategories

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_2 \subset \ldots \subset Sh(\mathcal{C})$$

Namely, given $\mathcal{C}_n$ let $\mathcal{C}_{n+1}$ be the full subcategory consisting of all fibre products and subsheaves of objects of $\mathcal{C}_n$. (Note that $\mathcal{C}_{n+1}$ has a set of objects.) Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{G}_j \to \mathcal{G}\}_{j \in J}$ of morphisms of objects of $\mathcal{C}'$ such that $\coprod \mathcal{G}_j \to \mathcal{G}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. Apply Lemma 7.29.4. □

Here is the goal of the current section.

03A2 **Lemma 7.29.6.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow[v]{} \mathcal{C}' \xleftarrow[u]{} \mathcal{D}$$

such that

1. *the functor v is a special cocontinuous functor,*
2. *the functor u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$, and*
3. *the morphism of topoi f agrees with the composition of morphisms of topoi*

$$Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}') \longrightarrow Sh(\mathcal{D})$$

where the first arrow comes from v via Lemma 7.29.1 and the second arrow from u via Lemma 7.15.2.

This statement is closely related to [**AGV71**, Proposition 4.9.4. Exposé IV]. In order to get the whole result, one should also use [**AGV71**, Remarque 4.7.4, Exposé IV].

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ and all $f^{-1} h_V^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $V \in \mathrm{Ob}(\mathcal{D})$. Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any

family $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ such that $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. The functor $u : \mathcal{D} \to \mathcal{C}'$ is given by $V \mapsto f^{-1}h_V^\#$.

Part (1) follows from Lemma 7.29.4.

Proof of (2) and (3) of the lemma. The functor u commutes with fibre products as both $V \mapsto h_V^\#$ and f^{-1} do. Moreover, since f^{-1} is exact and commutes with arbitrary colimits we see that it transforms a covering into a surjective family of morphisms of sheaves. Hence u is continuous. To see that it defines a morphism of sites we still have to see that u_s is exact. In order to do this we will show that $g^{-1} \circ u_s = f^{-1}$. Namely, then since g^{-1} is an equivalence and f^{-1} is exact we will conclude. Because g^{-1} is adjoint to g_*, and u_s is adjoint to u^s, and f^{-1} is adjoint to f_* it also suffices to prove that $u^s \circ g_* = f_*$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$ and let V be an object of $\mathcal{D}$. Then

$$\begin{aligned}
(u^s g_*\mathcal{F})(V) &= (g_*\mathcal{F})(f^{-1}h_V^\#) \\
&= \mathrm{Mor}_{Sh(\mathcal{C}')}(h_{f^{-1}h_V^\#}, g_*\mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}h_{f^{-1}h_V^\#}, \mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}h_V^\#, \mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f_*\mathcal{F}) \\
&= f_*\mathcal{F}(V)
\end{aligned}$$

The first equality because $u^s = u^p$. The second equality is the Yoneda lemma. The third equality by adjointness of g^{-1} and g_*. The fourth equality is by Lemma 7.29.4 (4). The fifth equality by adjointness of f^{-1} and f_*. The sixth equality by the Yoneda lemma. Hence $u^s g_*\mathcal{F} = f_*\mathcal{F}$ and this finishes the proof of the lemma. □

03CJ **Remark 7.29.7.** Notation and assumptions as in Lemma 7.29.6. If the site $\mathcal{D}$ has a final object and fibre products then the functor $u : \mathcal{D} \to \mathcal{C}'$ satisfies all the assumptions of Proposition 7.14.7. Namely, in addition to the properties mentioned in the lemma u also transforms the final object of $\mathcal{D}$ into the final object of $\mathcal{C}'$. This is clear from the construction of u. Hence, if we first apply Lemmas 7.29.5 to $\mathcal{D}$ and then Lemma 7.29.6 to the resulting morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D}')$ we obtain the following statement: Any morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ fits into a commutative diagram

$$\begin{array}{ccc}
Sh(\mathcal{C}) & \xrightarrow{f} & Sh(\mathcal{D}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle e} \\
Sh(\mathcal{C}') & \xrightarrow{f'} & Sh(\mathcal{D}')
\end{array}$$

where the following properties hold:

(1) the morphisms e and g are equivalences given by special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$,
(2) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have fibre products, final objects and have subcanonical topologies,
(3) the morphism $f' : \mathcal{C}' \to \mathcal{D}'$ comes from a morphism of sites corresponding to a functor $u : \mathcal{D}' \to \mathcal{C}'$ to which Proposition 7.14.7 applies, and

(4) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

It is often useful to replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}'$.

03CK **Remark 7.29.8.** Notation and assumptions as in Lemma 7.29.6. Suppose that in addition the original morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D})$ is an equivalence. Then the construction in the proof of Lemma 7.29.6 gives two functors

$$\mathcal{C} \to \mathcal{C}' \leftarrow \mathcal{D}$$

which are both special cocontinuous functors. Hence in this case we can actually factor the morphism of topoi as a composition

$$Sh(\mathcal{C}) \to Sh(\mathcal{C}') = Sh(\mathcal{D}') \leftarrow Sh(\mathcal{D})$$

as in Remark 7.29.7, but with the middle morphism an identity.

7.30. Localization of topoi

04GY We repeat some of the material on localization to the apparently more general case of topoi. In reality this is not more general since we may always enlarge the underlying sites to assume that we are localizing at objects of the site.

04GZ **Lemma 7.30.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the category $Sh(\mathcal{C})/\mathcal{F}$ is a topos. There is a canonical morphism of topoi*

$$j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \longrightarrow Sh(\mathcal{C})$$

which is a localization as in Section 7.25 such that

(1) *the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$, and*
(2) *the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.*

Proof. Apply Lemma 7.29.5. This means we may assume $\mathcal{C}$ is a site with subcanonical topology, and $\mathcal{F} = h_U = h_U^\#$ for some $U \in \mathrm{Ob}(\mathcal{C})$. Hence the material of Section 7.25 applies. In particular, there is an equivalence $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ such that the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

is equal to $j_{U!}$, see Lemma 7.25.4. Denote $a : Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$ the inverse functor, so $j_{\mathcal{F}!} = j_{U!} \circ a$, $j_{\mathcal{F}}^{-1} = a^{-1} \circ j_U^{-1}$, and $j_{\mathcal{F},*} = j_{U,*} \circ a$. The description of $j_{\mathcal{F}!}$ follows from the above. The description of $j_{\mathcal{F}}^{-1}$ follows from Lemma 7.25.7. □

04H0 **Lemma 7.30.2.** *In the situation of Lemma 7.30.1, the functor $j_{\mathcal{F},*}$ is the one associates to $\varphi : \mathcal{G} \to \mathcal{F}$ the sheaf*

$$U \longmapsto \{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}.$$

Proof. For any $\varphi : \mathcal{G} \to \mathcal{F}$, let us use the notation $\mathcal{G}/\mathcal{F}$ to denote the corresponding object of $Sh(\mathcal{C})/\mathcal{F}$. We have

$$(j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F}))(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F})) = \mathrm{Mor}_{Sh(\mathcal{C})/\mathcal{F}}(j_{\mathcal{F}}^{-1}h_U^\#, (\mathcal{G}/\mathcal{F})).$$

By Lemma 7.30.1 this set is the fiber over the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ under the map of sets

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) \xrightarrow{\varphi\circ} \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}).$$

By the adjunction in Lemma 7.26.2, we have

$$\begin{aligned}\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}), \\ \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{F})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}),\end{aligned}$$

and under the adjunction, the map $\varphi\circ$ becomes the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}), \quad \psi \longmapsto \varphi|_{\mathcal{C}/U} \circ \psi,$$

the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ becomes $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$. Therefore $(j_{\mathcal{F},*}\mathcal{G}/\mathcal{F})(U)$ is isomorphic to the fiber of $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$ under the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \xrightarrow{\varphi|_{\mathcal{C}/U}\circ} \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}),$$

which is $\{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U$ such that α is a right inverse to $\varphi|_U\}$ as desired. □

0791 **Lemma 7.30.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $\mathcal{C}/\mathcal{F}$ be the category of pairs (U, s) where $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$. Let a covering in $\mathcal{C}/\mathcal{F}$ be a family $\{(U_i, s_i) \to (U, s)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then $j : \mathcal{C}/\mathcal{F} \to \mathcal{C}$ is a continuous and cocontinuous functor of sites which induces a morphism of topoi $j : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})$. In fact, there is an equivalence $Sh(\mathcal{C}/\mathcal{F}) = Sh(\mathcal{C})/\mathcal{F}$ which turns j into $j_{\mathcal{F}}$.*

Proof. We omit the verification that $\mathcal{C}/\mathcal{F}$ is a site and that j is continuous and cocontinuous. By Lemma 7.21.5 there exists a morphism of topoi j as indicated, with $j^{-1}\mathcal{G}(U, s) = \mathcal{G}(U)$, and there is a left adjoint $j_!$ to j^{-1}. A morphism $\varphi : * \to j^{-1}\mathcal{G}$ on $\mathcal{C}/\mathcal{F}$ is the same thing as a rule which assigns to every pair (U, s) a section $\varphi(s) \in \mathcal{G}(U)$ compatible with restriction maps. Hence this is the same thing as a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$. We conclude that $j_!* = \mathcal{F}$. In particular, for every $\mathcal{H} \in Sh(\mathcal{C}/\mathcal{F})$ there is a canonical map

$$j_!\mathcal{H} \to j_!* = \mathcal{F}$$

i.e., we obtain a functor $j'_! : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})/\mathcal{F}$. An inverse to this functor is the rule which assigns to an object $\varphi : \mathcal{G} \to \mathcal{F}$ of $Sh(\mathcal{C})/\mathcal{F}$ the sheaf

$$a(\mathcal{G}/\mathcal{F}) : (U, s) \longmapsto \{t \in \mathcal{G}(U) \mid \varphi(t) = s\}$$

We omit the verification that $a(\mathcal{G}/\mathcal{F})$ is a sheaf and that a is inverse to $j'_!$. □

04IP **Definition 7.30.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$.

(1) The topos $Sh(\mathcal{C})/\mathcal{F}$ is called the *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$*.
(2) The morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ of Lemma 7.30.1 is called the *localization morphism*.

We are going to show that whenever the sheaf $\mathcal{F}$ is equal to $h_U^\#$ for some object U of the site, then the localization of the topos is equal to the category of sheaves on the localization of the site at U. Moreover, we are going to check that any functorialities are compatible with this identification.

04IQ **Lemma 7.30.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$. Then $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ constructed in Lemma 7.30.1 agrees with the morphism*

of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ constructed in Section 7.25 via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.25.4.

Proof. We have seen in Lemma 7.25.4 that the composition $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{U!}$. The functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{\mathcal{F}!}$ by Lemma 7.30.1. Hence $j_{\mathcal{F}!} = j_{U!}$ via the identification. So $j_{\mathcal{F}}^{-1} = j_U^{-1}$ (by adjointness) and so $j_{\mathcal{F},*} = j_{U,*}$ (by adjointness again). □

04IR **Lemma 7.30.6.** *Let $\mathcal{C}$ be a site. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{G} & \xrightarrow{\quad j \quad} & Sh(\mathcal{C})/\mathcal{F} \\ & {\scriptstyle j_{\mathcal{G}}} \searrow \quad \swarrow {\scriptstyle j_{\mathcal{F}}} & \\ & Sh(\mathcal{C}) & \end{array}$$

where $j = j_{\mathcal{G}/\mathcal{F}}$ is the localization of the topos $Sh(\mathcal{C})/\mathcal{F}$ at the object $\mathcal{G}/\mathcal{F}$. In particular we have

$$j^{-1}(\mathcal{H} \to \mathcal{F}) = (\mathcal{H} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G})$$

and

$$j_!(\mathcal{E} \xrightarrow{e} \mathcal{G}) = (\mathcal{E} \xrightarrow{s \circ e} \mathcal{F}).$$

Proof. The description of j^{-1} and $j_!$ comes from the description of those functors in Lemma 7.30.1. The equality of functors $j_{\mathcal{G}!} = j_{\mathcal{F}!} \circ j_!$ is clear from the description of these functors (as forgetful functors). By adjointness we also obtain the equalities $j_{\mathcal{G}}^{-1} = j^{-1} \circ j_{\mathcal{F}}^{-1}$, and $j_{\mathcal{G},*} = j_{\mathcal{F},*} \circ j_*$. □

04IS **Lemma 7.30.7.** *Assume $\mathcal{C}$ and $s : \mathcal{G} \to \mathcal{F}$ are as in Lemma 7.30.6. If $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and $s : \mathcal{G} \to \mathcal{F}$ comes from a morphism $V \to U$ of $\mathcal{C}$ then the diagram in Lemma 7.30.6 is identified with diagram (7.25.8.1) via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.25.4.*

Proof. This is true because the descriptions of j^{-1} agree. See Lemma 7.25.9 and Lemma 7.30.6. □

7.31. Localization and morphisms of topoi

04IT This section is the analogue of Section 7.28 for morphisms of topoi.

04H1 **Lemma 7.31.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow{j_{\mathcal{F}}} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

The morphism f' is characterized by the property that

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F})$$

and we have $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$.

Proof. Since the statement is about topoi and does not refer to the underlying sites we may change sites at will. Hence by the discussion in Remark 7.29.7 we may assume that f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Proposition 7.14.7 between sites having all finite limits and subcanonical topologies, and such that $\mathcal{G} = h_V$ for some object V of $\mathcal{D}$. Then $\mathcal{F} = f^{-1}h_V = h_{u(V)}$ by Lemma 7.13.5. By Lemma 7.28.1 we obtain a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}), \end{array}$$

and we have $f'_* j_U^{-1} = j_V^{-1} f_*$. By Lemma 7.30.5 we may identify $j_{\mathcal{F}}$ and j_U and $j_{\mathcal{G}}$ and j_V. The description of $(f')^{-1}$ is given in Lemma 7.28.1. □

04IU **Lemma 7.31.2.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Set $U = u(V)$. Set $\mathcal{G} = h_V^\#$, and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$ (see Lemma 7.13.5). Then the diagram of morphisms of topoi of Lemma 7.31.1 agrees with the diagram of morphisms of topoi of Lemma 7.28.1 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 7.30.5.*

Proof. This is not a complete triviality as the choice of morphism of sites giving rise to f made in the proof of Lemma 7.31.1 may be different from the morphisms of sites given to us in the lemma. But in both cases the functor $(f')^{-1}$ is described by the same rule. Hence they agree and the associated morphism of topoi is the same. Some details omitted. □

04IV **Lemma 7.31.3.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G} \in Sh(\mathcal{D})$, $\mathcal{F} \in Sh(\mathcal{C})$ and $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow{j_{\mathcal{F}}} & Sh(\mathcal{C}) \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

We have $f_s = f' \circ j_{\mathcal{F}/f^{-1}\mathcal{G}}$ where $f' : Sh(\mathcal{C})/f^{-1}\mathcal{G} \to Sh(\mathcal{D})/\mathcal{F}$ is as in Lemma 7.31.1 and $j_{\mathcal{F}/f^{-1}\mathcal{G}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})/f^{-1}\mathcal{G}$ is as in Lemma 7.30.6. The functor $(f_s)^{-1}$ is described by the rule

$$(f_s)^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, f^{-1}\mathcal{G}, s} \mathcal{F} \to \mathcal{F}).$$

Finally, given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$ and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{s'} & f^{-1}\mathcal{G}' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f^{-1}b} \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F}' & \xrightarrow{j_{\mathcal{F}'/\mathcal{F}}} & Sh(\mathcal{C})/\mathcal{F} \\ \downarrow{\scriptstyle f_{s'}} & & \downarrow{\scriptstyle f_s} \\ Sh(\mathcal{D})/\mathcal{G}' & \xrightarrow{j_{\mathcal{G}'/\mathcal{G}}} & Sh(\mathcal{D})/\mathcal{G}. \end{array}$$

commutes.

Proof. The commutativity of the first square follows from the commutativity of the diagram in Lemma 7.30.6 and the commutativity of the diagram in Lemma 7.31.1. The description of f_s^{-1} follows on combining the descriptions of $(f')^{-1}$ in Lemma 7.31.1 with the description of $(j_{\mathcal{F}/f^{-1}\mathcal{G}})^{-1}$ in Lemma 7.30.6. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{f^{-1}\mathcal{G},s} \mathcal{F} \times_{\mathcal{F}} \mathcal{F}' = f^{-1}(\mathcal{H} \times_{\mathcal{G}} \mathcal{G}') \times_{f^{-1}\mathcal{G}',s'} \mathcal{F}'$$

which is formal. □

04IW **Lemma 7.31.4.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Let $c : U \to u(V)$ be a morphism. Set $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$. Let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be the map induced by c. Then the diagram of morphisms of topoi of Lemma 7.28.3 agrees with the diagram of morphisms of topoi of Lemma 7.31.3 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 7.30.5.*

Proof. This follows on combining Lemmas 7.30.7 and 7.31.2. □

7.32. Points

00Y3

00Y4 **Definition 7.32.1.** Let $\mathcal{C}$ be a site. A *point of the topos $Sh(\mathcal{C})$* is a morphism of topoi p from $Sh(pt)$ to $Sh(\mathcal{C})$.

We will define a point of a site in terms of a functor $u : \mathcal{C} \to Sets$. It will turn out later that u will define a morphism of sites which gives rise to a point of the topos associated to $\mathcal{C}$, see Lemma 7.32.8.

Let $\mathcal{C}$ be a site. Let $p = u$ be a functor $u : \mathcal{C} \to Sets$. This curious language is introduced because it seems funny to talk about neighbourhoods of functors; so we think of a "point" p as a geometric thing which is given by a categorical datum, namely the functor u. The fact that p is actually equal to u does not matter. A *neighbourhood* of p is a pair (U, x) with $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in u(U)$. A *morphism of neighbourhoods* $(V, y) \to (U, x)$ is given by a morphism $\alpha : V \to U$ of $\mathcal{C}$ such that $u(\alpha)(y) = x$. Note that the category of neighbourhoods isn't a "big" category.

We define the *stalk* of a presheaf $\mathcal{F}$ at p as

04EH (7.32.1.1) $$\mathcal{F}_p = \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}(U).$$

The colimit is over the opposite of the category of neighbourhoods of p. In other words, an element of $\mathcal{F}_p$ is given by a triple (U, x, s), where (U, x) is a neighbourhood of p and $s \in \mathcal{F}(U)$. Equality of triples is the equivalence relation generated by $(U, x, s) \sim (V, y, \alpha^* s)$ when α is as above.

Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets, then we get a canonical map of stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. Thus we obtain a *stalk functor*

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_p.$$

We have defined the stalk functor using any functor $p = u : \mathcal{C} \to Sets$. No conditions are necessary for the definition to work[7]. On the other hand, it is probably better not to use this notion unless p actually is a point (see definition below), since in general the stalk functor does not have good properties.

00Y5 **Definition 7.32.2.** Let $\mathcal{C}$ be a site. A *point p of the site $\mathcal{C}$* is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is left exact.

The conditions should be familiar since they are modeled after those of Definitions 7.13.1 and 7.14.1. Note that (3) implies that $*_p = \{*\}$, see Example 7.10.2. Hence $u(U) \neq \emptyset$ for at least some U (because the empty colimit produces the empty set). We will show below (Lemma 7.32.7) that this does give rise to a point of the topos $Sh(\mathcal{C})$. Before we do so, we prove some lemmas for general functors u.

00Y6 **Lemma 7.32.3.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. There are functorial isomorphisms $(h_U)_p = u(U)$ for $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. An element of $(h_U)_p$ is given by a triple (V, y, f), where $V \in \mathrm{Ob}(\mathcal{C})$, $y \in u(V)$ and $f \in h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$. Two such (V, y, f), (V', y', f') determine the same object if there exists a morphism $\phi : V \to V'$ such that $u(\phi)(y) = y'$ and $f' \circ \phi = f$, and in general you have to take chains of identities like this to get the correct equivalence relation. In any case, every (V, y, f) is equivalent to the element $(U, u(f)(y), \mathrm{id}_U)$. If ϕ exists as above, then the triples (V, y, f), (V', y', f') determine the same triple $(U, u(f)(y), \mathrm{id}_U) = (U, u(f')(y'), \mathrm{id}_U)$. This proves that the map $u(U) \to (h_U)_p$, $x \mapsto$ class of (U, x, id_U) is bijective. □

Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. In analogy with the constructions in Section 7.5 given a set E we define a presheaf u^pE by the rule

04EI (7.32.3.1) $$U \longmapsto u^pE(U) = \mathrm{Mor}_{Sets}(u(U), E) = \mathrm{Map}(u(U), E).$$

This defines a functor $u^p : Sets \to PSh(\mathcal{C})$, $E \mapsto u^pE$.

00Y7 **Lemma 7.32.4.** *For any functor $u : \mathcal{C} \to Sets$. The functor u^p is a right adjoint to the stalk functor on presheaves.*

Proof. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Let E be a set. A morphism $\mathcal{F} \to u^pE$ is given by a compatible system of maps $\mathcal{F}(U) \to \mathrm{Map}(u(U), E)$, i.e., a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. And by definition of $\mathcal{F}_p$ a map $\mathcal{F}_p \to E$ is given by a rule associating with each triple (U, x, σ) an element in E such that equivalent triples map to the same element, see discussion surrounding Equation (7.32.1.1). This also means a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. □

In analogy with Section 7.13 we have the following lemma.

[7] One should try to avoid the case where $u(U) = \emptyset$ for all U.

00Y8 **Lemma 7.32.5.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. Suppose that for every covering $\{U_i \to U\}$ of $\mathcal{C}$*

(1) *the map $\coprod u(U_i) \to u(U)$ is surjective, and*
(2) *the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.*

Then we have

(1) *the presheaf $u^p E$ is a sheaf for all sets E, denote it $u^s E$,*
(2) *the stalk functor $Sh(\mathcal{C}) \to Sets$ and the functor $u^s : Sets \to Sh(\mathcal{C})$ are adjoint, and*
(3) *we have $\mathcal{F}_p = \mathcal{F}_p^\#$ for every presheaf of sets $\mathcal{F}$.*

Proof. The first assertion is immediate from the definition of a sheaf, assumptions (1) and (2), and the definition of $u^p E$. The second is a restatement of the adjointness of u^p and the stalk functor (Lemma 7.32.4) restricted to sheaves. The third assertion follows as, for any set E, we have

$$\mathrm{Map}(\mathcal{F}_p, E) = \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p E) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, u^s E) = \mathrm{Map}(\mathcal{F}_p^\#, E)$$

by the adjointness property of sheafification. □

In particular Lemma 7.32.5 holds when $p = u$ is a point. In this case we think of the sheaf $u^s E$ as the "skyscraper" sheaf with value E at p.

00Y9 **Definition 7.32.6.** Let p be a point of the site $\mathcal{C}$ given by the functor u. For a set E we define $p_* E = u^s E$ the sheaf described in Lemma 7.32.5 above. We sometimes call this a *skyscraper sheaf*.

In particular we have the following adjointness property of skyscraper sheaves and stalks:

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, p_* E) = \mathrm{Map}(\mathcal{F}_p, E)$$

This motivates the notation $p^{-1}\mathcal{F} = \mathcal{F}_p$ which we will sometimes use.

00YA **Lemma 7.32.7.** *Let $\mathcal{C}$ be a site.*

(1) *Let p be a point of the site $\mathcal{C}$. Then the pair of functors (p_*, p^{-1}) introduced above define a morphism of topoi $Sh(pt) \to Sh(\mathcal{C})$.*
(2) *Let $p = (p_*, p^{-1})$ be a point of the topos $Sh(\mathcal{C})$. Then the functor $u : U \mapsto p^{-1}(h_U^\#)$ gives rise to a point p' of the site $\mathcal{C}$ whose associated morphism of topoi $(p'_*, (p')^{-1})$ is equal to p.*

Proof. Proof of (1). By the above the functors p_* and p^{-1} are adjoint. The functor p^{-1} is required to be exact by Definition 7.32.2. Hence the conditions imposed in Definition 7.15.1 are all satisfied and we see that (1) holds.

Proof of (2). Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then $\coprod (h_{U_i})^\# \to h_U^\#$ is surjective, see Lemma 7.12.4. Since p^{-1} is exact (by definition of a morphism of topoi) we conclude that $\coprod u(U_i) \to u(U)$ is surjective. This proves part (1) of Definition 7.32.2. Sheafification is exact, see Lemma 7.10.15. Hence if $U \times_V W$ exists in $\mathcal{C}$, then

$$h^\#_{U \times_V W} = h^\#_U \times_{h^\#_V} h^\#_W$$

and we see that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ since p^{-1} is exact. This proves part (2) of Definition 7.32.2. Let $p' = u$, and let $\mathcal{F}_{p'}$ be the stalk functor defined by Equation (7.32.1.1) using u. There is a canonical comparison map $c : \mathcal{F}_{p'} \to \mathcal{F}_p = p^{-1}\mathcal{F}$. Namely, given a triple (U, x, σ) representing an element ξ of $\mathcal{F}_{p'}$ we think of σ

as a map $\sigma : h_U^\# \to \mathcal{F}$ and we can set $c(\xi) = p^{-1}(\sigma)(x)$ since $x \in u(U) = p^{-1}(h_U^\#)$. By Lemma 7.32.3 we see that $(h_U)_{p'} = u(U)$. Since conditions (1) and (2) of Definition 7.32.2 hold for p' we also have $(h_U^\#)_{p'} = (h_U)_{p'}$ by Lemma 7.32.5. Hence we have

$$(h_U^\#)_{p'} = (h_U)_{p'} = u(U) = p^{-1}(h_U^\#)$$

We claim this bijection equals the comparison map $c : (h_U^\#)_{p'} \to p^{-1}(h_U^\#)$ (verification omitted). Any sheaf on $\mathcal{C}$ is a coequalizer of maps of coproducts of sheaves of the form $h_U^\#$, see Lemma 7.12.5. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ and the functor p^{-1} commute with arbitrary colimits (as they are both left adjoints). We conclude c is an isomorphism for every sheaf $\mathcal{F}$. Thus the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ is isomorphic to p^{-1} and we in particular see that it is exact. This proves condition (3) of Definition 7.32.2 holds and p' is a point. The final assertion has already been shown above, since we saw that $p^{-1} = (p')^{-1}$. □

Actually a point always corresponds to a morphism of sites as we show in the following lemma.

04EL **Lemma 7.32.8.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let S_0 be an infinite set such that $u(U) \subset S_0$ for all $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be the site constructed out of the powerset $S = \mathcal{P}(S_0)$ in Remark 7.15.3. Then*

(1) *there is an equivalence $i : Sh(pt) \to Sh(\mathcal{S})$,*
(2) *the functor $u : \mathcal{C} \to \mathcal{S}$ induces a morphism of sites $f : \mathcal{S} \to \mathcal{C}$, and*
(3) *the composition*

$$Sh(pt) \to Sh(\mathcal{S}) \to Sh(\mathcal{C})$$

is the morphism of topoi (p_*, p^{-1}) of Lemma 7.32.7.

Proof. Part (1) we saw in Remark 7.15.3. Moreover, recall that the equivalence associates to the set E the sheaf i_*E on $\mathcal{S}$ defined by the rule $V \mapsto \mathrm{Mor}_{Sets}(V, E)$. Part (2) is clear from the definition of a point of $\mathcal{C}$ (Definition 7.32.2) and the definition of a morphism of sites (Definition 7.14.1). Finally, consider f_*i_*E. By construction we have

$$f_*i_*E(U) = i_*E(u(U)) = \mathrm{Mor}_{Sets}(u(U), E)$$

which is equal to $p_*E(U)$, see Equation (7.32.3.1). This proves (3). □

Contrary to what happens in the topological case it is not always true that the stalk of the skyscraper sheaf with value E is E. Here is what is true in general.

05UX **Lemma 7.32.9.** *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. For any set E there are canonical maps*

$$E \longrightarrow (p_*E)_p \longrightarrow E$$

whose composition is id_E.

Proof. There is always an adjunction map $(p_*E)_p = p^{-1}p_*E \to E$. This map is an isomorphism when $E = \{*\}$ because p_* and p^{-1} are both left exact, hence transform the final object into the final object. Hence given $e \in E$ we can consider

the map $i_e : \{*\} \to E$ which gives

$$\begin{array}{ccc} p^{-1}p_*\{*\} & \xrightarrow[p^{-1}p_*i_e]{} & p^{-1}p_*E \\ \cong\downarrow & & \downarrow \\ \{*\} & \xrightarrow{i_e} & E \end{array}$$

whence the map $E \to (p_*E)_p = p^{-1}p_*E$ as desired. □

05UY **Lemma 7.32.10.** *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. The functor $p_* : Sets \to Sh(\mathcal{C})$ has the following properties: It commutes with arbitrary limits, it is left exact, it is faithful, it transforms surjections into surjections, it commutes with coequalizers, it reflects injections, it reflects surjections, and it reflects isomorphisms.*

Proof. Because p_* is a right adjoint it commutes with arbitrary limits and it is left exact. The fact that $p^{-1}p_*E \to E$ is a canonically split surjection implies that p_* is faithful, reflects injections, reflects surjections, and reflects isomorphisms. By Lemma 7.32.7 we may assume that p comes from a point $u : \mathcal{C} \to Sets$ of the underlying site $\mathcal{C}$. In this case the sheaf p_*E is given by

$$p_*E(U) = \mathrm{Mor}_{Sets}(u(U), E)$$

see Equation (7.32.3.1) and Definition 7.32.6. It follows immediately from this formula that p_* transforms surjections into surjections and coequalizers into coequalizers. □

7.33. Constructing points

05UZ In this section we give criteria for when a functor from a site to the category of sets defines a point of that site.

0F4E **Lemma 7.33.1.** *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. If the category of neighbourhoods of p is cofiltered, then the stalk functor (7.32.1.1) is left exact.*

Proof. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a finite diagram of sheaves. We have to show that the stalk of the limit of this system agrees with the limit of the stalks. Let $\mathcal{F}$ be the limit of the system as a *presheaf*. According to Lemma 7.10.1 this is a sheaf and it is the limit in the category of sheaves. Hence we have to show that $\mathcal{F}_p = \lim_{\mathcal{I}} \mathcal{F}_{i,p}$. Recall also that $\mathcal{F}$ has a simple description, see Section 7.4. Thus we have to show that

$$\lim_i \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}_i(U) = \mathrm{colim}_{\{(U,x)\}^{opp}} \lim_i \mathcal{F}_i(U).$$

This holds, by Categories, Lemma 4.19.2, because the opposite of the category of neighbourhoods is filtered by assumption. □

00YB **Lemma 7.33.2.** *Let $\mathcal{C}$ be a site. Assume that $\mathcal{C}$ has a final object X and fibred products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that*

1. *$u(X)$ is a singleton set, and*
2. *for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is bijective.*

Then the the category of neighbourhoods of p is cofiltered and consequently the stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ commutes with finite limits.

Proof. Please note the analogy with Lemma 7.5.2. The assumptions on $\mathcal{C}$ imply that $\mathcal{C}$ has finite limits. See Categories, Lemma 4.18.4. Assumption (1) implies that the category of neighbourhoods is nonempty. Suppose (U, x) and (V, y) are neighbourhoods. Then $u(U \times V) = u(U \times_X V) = u(U) \times_{u(X)} u(V) = u(U) \times u(V)$ by (2). Hence there exists a neighbourhood $(U \times_X V, z)$ mapping to both (U, x) and (V, y). Let $a, b : (V, y) \to (U, x)$ be two morphisms in the category of neighbourhoods. Let W be the equalizer of $a, b : V \to U$. As in the proof of Categories, Lemma 4.18.4 we may write W in terms of fibre products:

$$W = (V \times_{a,U,b} V) \times_{(pr_1,pr_2),V\times V,\Delta} V$$

The bijectivity in (2) guarantees there exists an element $z \in u(W)$ which maps to $((y, y), y)$. Then $(W, z) \to (V, y)$ equalizes a, b as desired. The "consequently" clause is Lemma 7.33.1. □

00YC **Proposition 7.33.3.** *Let $\mathcal{C}$ be a site. Assume that finite limits exist in $\mathcal{C}$. (I.e., $\mathcal{C}$ has fibre products, and a final object.) A point p of such a site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to Sets$ such that*

(1) *u commutes with finite limits, and*
(2) *if $\{U_i \to U\}$ is a covering, then $\coprod_i u(U_i) \to u(U)$ is surjective.*

Proof. Suppose first that p is a point (Definition 7.32.2) given by a functor u. Condition (2) is satisfied directly from the definition of a point. By Lemma 7.32.3 we have $(h_U)_p = u(U)$. By Lemma 7.32.5 we have $(h_U^\#)_p = (h_U)_p$. Thus we see that u is equal to the composition of functors

$$\mathcal{C} \xrightarrow{h} PSh(\mathcal{C}) \xrightarrow{\#} Sh(\mathcal{C}) \xrightarrow{()_p} Sets$$

Each of these functors is left exact, and hence we see u satisfies (1).

Conversely, suppose that u satisfies (1) and (2). In this case we immediately see that u satisfies the first two conditions of Definition 7.32.2. And its stalk functor is exact, because it is a left adjoint by Lemma 7.32.5 and it commutes with finite limits by Lemma 7.33.2. □

00YD **Remark 7.33.4.** In fact, let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has a final object X and fibre products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that

(1) $u(X) = \{*\}$ a singleton, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is surjective.
(3) for every covering $\{U_i \to U\}$ the map $\coprod u(U_i) \to u(U)$ is surjective.

Then, in general, p is **not** a point of $\mathcal{C}$. An example is the category $\mathcal{C}$ with two objects $\{U, X\}$ and exactly one non-identity arrow, namely $U \to X$. We endow $\mathcal{C}$ with the trivial topology, i.e., the only coverings are $\{U \to U\}$ and $\{X \to X\}$. A sheaf $\mathcal{F}$ is the same thing as a presheaf and consists of a triple $(A, B, A \to B)$: namely $A = \mathcal{F}(X)$, $B = \mathcal{F}(U)$ and $A \to B$ is the restriction mapping corresponding to $U \to X$. Note that $U \times_X U = U$ so fibre products exist. Consider the functor $u = p$ with $u(X) = \{*\}$ and $u(U) = \{*_1, *_2\}$. This satisfies (1), (2), and (3), but the corresponding stalk functor (7.32.1.1) is the functor

$$(A, B, A \to B) \longmapsto B \amalg_A B$$

which isn't exact. Namely, consider $(\emptyset, \{1\}, \emptyset \to \{1\}) \to (\{1\}, \{1\}, \{1\} \to \{1\})$ which is an injective map of sheaves, but is transformed into the noninjective map of sets

$$\{1\} \amalg \{1\} \longrightarrow \{1\} \amalg_{\{1\}} \{1\}$$

by the stalk functor.

00YE **Example 7.33.5.** Let X be a topological space. Let X_{Zar} be the site of Example 7.6.4. Let $x \in X$ be a point. Consider the functor

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \{*\} & \text{if} \quad x \in U \end{cases}$$

This functor commutes with product and fibred products, and turns coverings into surjective families of maps. Hence we obtain a point p of the site X_{Zar}. It is immediately verified that the stalk functor agrees with the stalk at x defined in Sheaves, Section 6.11.

04EJ **Example 7.33.6.** Let X be a topological space. What are the points of the topos $Sh(X)$? To see this, let X_{Zar} be the site of Example 7.6.4. By Lemma 7.32.7 a point of $Sh(X)$ corresponds to a point of this site. Let p be a point of the site X_{Zar} given by the functor $u : X_{Zar} \to Sets$. We are going to use the characterization of such a u in Proposition 7.33.3. This implies immediately that $u(\emptyset) = \emptyset$ and $u(U \cap V) = u(U) \times u(V)$. In particular we have $u(U) = u(U) \times u(U)$ via the diagonal map which implies that $u(U)$ is either a singleton or empty. Moreover, if $U = \bigcup U_i$ is an open covering then

$$u(U) = \emptyset \Rightarrow \forall i,\ u(U_i) = \emptyset \quad \text{and} \quad u(U) \neq \emptyset \Rightarrow \exists i,\ u(U_i) \neq \emptyset.$$

We conclude that there is a unique largest open $W \subset X$ with $u(W) = \emptyset$, namely the union of all the opens U with $u(U) = \emptyset$. Let $Z = X \setminus W$. If $Z = Z_1 \cup Z_2$ with $Z_i \subset Z$ closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ so $\emptyset = u(W) = u(X \setminus Z_1) \times u(X \setminus Z_2)$ and we conclude that $u(X \setminus Z_1) = \emptyset$ or that $u(X \setminus Z_2) = \emptyset$. This means that $X \setminus Z_1 = W$ or that $X \setminus Z_2 = W$. In other words, Z is irreducible. Now we see that u is described by the rule

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad Z \cap U = \emptyset \\ \{*\} & \text{if} \quad Z \cap U \neq \emptyset \end{cases}$$

Note that for any irreducible closed $Z \subset X$ this functor satisfies assumptions (1), (2) of Proposition 7.33.3 and hence defines a point. In other words we see that points of the site X_{Zar} are in one-to-one correspondence with irreducible closed subsets of X. In particular, if X is a sober topological space, then points of X_{Zar} and points of X are in one to one correspondence, see Example 7.33.5.

00YF **Example 7.33.7.** Consider the site $\mathcal{T}_G$ described in Example 7.6.5 and Section 7.9. The forgetful functor $u : \mathcal{T}_G \to Sets$ commutes with products and fibred products and turns coverings into surjective families. Hence it defines a point of $\mathcal{T}_G$. We identify $Sh(\mathcal{T}_G)$ and G-$Sets$. The stalk functor

$$p^{-1} : Sh(\mathcal{T}_G) = G\text{-}Sets \longrightarrow Sets$$

is the forgetful functor. The pushforward p_* is the functor

$$Sets \longrightarrow Sh(\mathcal{T}_G) = G\text{-}Sets$$

which maps a set S to the G-set $\mathrm{Map}(G, S)$ with action $g \cdot \psi = \psi \circ R_g$ where R_g is right multiplication. In particular we have $p^{-1}p_*S = \mathrm{Map}(G, S)$ as a set and the maps $S \to \mathrm{Map}(G, S) \to S$ of Lemma 7.32.9 are the obvious ones.

08RH **Example 7.33.8.** Let $\mathcal{C}$ be a category endowed with the chaotic topology (Example 7.6.6). For every object U_0 of $\mathcal{C}$ the functor $u : U \mapsto \mathrm{Mor}_{\mathcal{C}}(U_0, U)$ defines a point p of $\mathcal{C}$. Namely, conditions (1) and (2) of Definition 7.32.2 are immediate as the only coverings are given by identity maps. Condition (2) holds because $\mathcal{F}_p = \mathcal{F}(U_0)$ and since the topology is discrete taking sections over U_0 is an exact functor.

7.34. Points and morphisms of topoi

05V0 In this section we make a few remarks about points and morphisms of topoi.

0F4F **Lemma 7.34.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $v : \mathcal{D} \to Sets$ be a functor and set $w = v \circ u$. Denote q, resp., p the stalk functor (7.32.1.1) associated to v, resp. w. Then $(u_p\mathcal{F})_q = \mathcal{F}_p$ functorially in the presheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. This is a simple categorical fact. We have

$$\begin{aligned}(u_p\mathcal{F})_q &= \mathrm{colim}_{(V,y)} \mathrm{colim}_{U,\phi:V\to u(U)} \mathcal{F}(U) \\ &= \mathrm{colim}_{(V,y,U,\phi:V\to u(U))} \mathcal{F}(U) \\ &= \mathrm{colim}_{(U,x)} \mathcal{F}(U) \\ &= \mathcal{F}_p\end{aligned}$$

The first equality holds by the definition of u_p and the definition of the stalk functor. Observe that $y \in v(V)$. In the second equality we simply combine colimits. To see the third equality we apply Categories, Lemma 4.17.5 to the functor F of diagram categories defined by the rule

$$F((V, y, U, \phi : V \to u(U))) = (U, v(\phi)(y)).$$

This makes sense because $w(U) = v(u(U))$. Let us check the hypotheses of Categories, Lemma 4.17.5. Observe that F has a right inverse, namely $(U, x) \mapsto (u(U), x, U, \mathrm{id} : u(U) \to u(U))$. Again this makes sense because $x \in w(U) = v(u(U))$. On the other hand, there is always a morphism

$$(V, y, U, \phi : V \to u(U)) \longrightarrow (u(U), v(\phi)(y), U, \mathrm{id} : u(U) \to u(U))$$

in the fibre category over (U, x) which shows the fibre categories are connected. The fourth and final equality is clear. □

05V1 **Lemma 7.34.2.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let q be a point of $\mathcal{D}$ given by the functor $v : \mathcal{D} \to Sets$, see Definition 7.32.2. Then the functor $v \circ u : \mathcal{C} \to Sets$ defines a point p of $\mathcal{C}$ and moreover there is a canonical identification*

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

First proof Lemma 7.34.2. Note that since u is continuous and since v defines a point, it is immediate that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.32.2. Let us prove the displayed equality. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then

$$(f^{-1}\mathcal{F})_q = (u_s\mathcal{F})_q = (u_p\mathcal{F})_q = \mathcal{F}_p$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 7.32.5, and the third by Lemma 7.34.1. Hence now we see that p also satisfies condition (3) of Definition 7.32.2 because it is a composition of exact functors. This finishes the proof. □

Second proof Lemma 7.34.2. By Lemma 7.32.8 we may factor (q_*, q^{-1}) as

$$Sh(pt) \xrightarrow{i} Sh(\mathcal{S}) \xrightarrow{h} Sh(\mathcal{D})$$

where the second morphism of topoi comes from a morphism of sites $h : \mathcal{S} \to \mathcal{D}$ induced by the functor $v : \mathcal{D} \to \mathcal{S}$ (which makes sense as $\mathcal{S} \subset Sets$ is a full subcategory containing every object in the image of v). By Lemma 7.14.4 the composition $v \circ u : \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $g : \mathcal{S} \to \mathcal{C}$. In particular, the functor $v \circ u : \mathcal{C} \to \mathcal{S}$ is continuous which by the definition of the coverings in $\mathcal{S}$, see Remark 7.15.3, means that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.32.2. On the other hand, we see that

$$g_* i_* E(U) = i_* E(v(u(U)) = \mathrm{Mor}_{Sets}(v(u(U)), E)$$

by the construction of i in Remark 7.15.3. Note that this is the same as the formula for which is equal to $(v \circ u)^p E$, see Equation (7.32.3.1). By Lemma 7.32.5 the functor $g_* i_* = (v \circ u)^p = (v \circ u)^s$ is right adjoint to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_q$. Hence we see that the stalk functor p^{-1} is canonically isomorphic to $i^{-1} \circ g^{-1}$. Hence it is exact and we conclude that p is a point. Finally, as we have $g = f \circ h$ by construction we see that $p^{-1} = i^{-1} \circ h^{-1} \circ f^{-1} = q^{-1} \circ f^{-1}$, i.e., we have the displayed formula of the lemma. □

05V2 **Lemma 7.34.3.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $q : Sh(pt) \to Sh(\mathcal{D})$ be a point. Then $p = f \circ q$ is a point of the topos $Sh(\mathcal{C})$ and we have a canonical identification*

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is immediate from the definitions and the fact that we can compose morphisms of topoi. □

7.35. Localization and points

04EK In this section we show that points of a localization $\mathcal{C}/U$ are constructed in a simple manner from the points of $\mathcal{C}$.

04H2 **Lemma 7.35.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$ and let $x \in u(U)$. The functor*

$$v : \mathcal{C}/U \longrightarrow Sets, \quad (\varphi : V \to U) \longmapsto \{y \in u(V) \mid u(\varphi)(y) = x\}$$

defines a point q of the site $\mathcal{C}/U$ such that the diagram

$$\begin{array}{ccc} & & Sh(pt) \\ & \overset{q}{\swarrow} & \downarrow p \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \end{array}$$

commutes. In other words $\mathcal{F}_p = (j_U^{-1}\mathcal{F})_q$ for any sheaf on $\mathcal{C}$.

Proof. Choose S and $\mathcal{S}$ as in Lemma 7.32.8. We may identify $Sh(pt) = Sh(\mathcal{S})$ as in that lemma, and we may write $p = f : Sh(\mathcal{S}) \to Sh(\mathcal{C})$ for the morphism of topoi induced by u. By Lemma 7.28.1 we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{S}/u(U)) & \xrightarrow[j_{u(U)}]{} & Sh(\mathcal{S}) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}), \end{array}$$

where p' is given by the functor $u' : \mathcal{C}/U \to \mathcal{S}/u(U)$, $V/U \mapsto u(V)/u(U)$. Consider the functor $j_x : \mathcal{S} \cong \mathcal{S}/x$ obtained by assigning to a set E the set E endowed with the constant map $E \to u(U)$ with value x. Then j_x is a fully faithful cocontinuous functor which has a continuous right adjoint $v_x : (\psi : E \to u(U)) \mapsto \psi^{-1}(\{x\})$. Note that $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$, and $v_x \circ u' = v$. These observations imply that we have the following commutative diagram of topoi

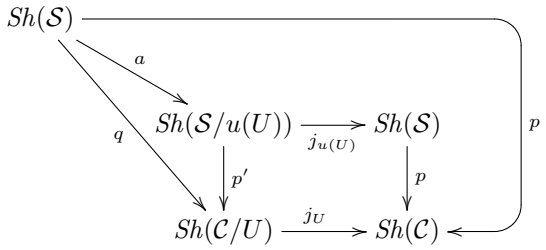

Namely:

(1) The morphism $a : Sh(\mathcal{S}) \to Sh(\mathcal{S}/u(U))$ is the morphism of topoi associated to the cocontinuous functor j_x, which equals the morphism associated to the continuous functor v_x, see Lemma 7.21.1 and Section 7.22.
(2) The composition $p \circ j_{u(U)} \circ a = p$ since $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$.
(3) The composition $p' \circ a$ gives a morphism of topoi. Moreover, it is the morphism of topoi associated to the continuous functor $v_x \circ u' = v$. Hence v does indeed define a point q of $\mathcal{C}/U$ which fits into the diagram above by construction.

This ends the proof of the lemma. □

04H3 **Lemma 7.35.2.** *Let $\mathcal{C}$, p, u, U be as in Lemma 7.35.1. The construction of Lemma 7.35.1 gives a one to one correspondence between points q of $\mathcal{C}/U$ lying over p and elements x of $u(U)$.*

Proof. Let q be a point of $\mathcal{C}/U$ given by the functor $v : \mathcal{C}/U \to Sets$ such that $j_U \circ q = p$ as morphisms of topoi. Recall that $u(V) = p^{-1}(h_V^\#)$ for any object V of $\mathcal{C}$, see Lemma 7.32.7. Similarly $v(V/U) = q^{-1}(h_{V/U}^\#)$ for any object V/U of $\mathcal{C}/U$. Consider the following two diagrams

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, V) \\ \downarrow & & \downarrow \\ \mathrm{Mor}_{\mathcal{C}/U}(W/U, U/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, U) \end{array} \qquad \begin{array}{ccc} h_{V/U}^\# & \longrightarrow & j_U^{-1}(h_V^\#) \\ \downarrow & & \downarrow \\ h_{U/U}^\# & \longrightarrow & j_U^{-1}(h_U^\#) \end{array}$$

The right hand diagram is the sheafification of the diagram of presheaves on $\mathcal{C}/U$ which maps W/U to the left hand diagram of sets. (There is a small technical point to make here, namely, that we have $(j_U^{-1}h_V)^\# = j_U^{-1}(h_V^\#)$ and similarly for h_U, see Lemma 7.20.4.) Note that the left hand diagram of sets is cartesian. Since sheafification is exact (Lemma 7.10.15) we conclude that the right hand diagram is cartesian.

Apply the exact functor q^{-1} to the right hand diagram to get a cartesian diagram

$$\begin{array}{ccc} v(V/U) & \longrightarrow & u(V) \\ \downarrow & & \downarrow \\ v(U/U) & \longrightarrow & u(U) \end{array}$$

of sets. Here we have used that $q^{-1} \circ j^{-1} = p^{-1}$. Since U/U is a final object of $\mathcal{C}/U$ we see that $v(U/U)$ is a singleton. Hence the image of $v(U/U)$ in $u(U)$ is an element x, and the top horizontal map gives a bijection $v(V/U) \to \{y \in u(V) \mid y \mapsto x \text{ in } u(U)\}$ as desired. □

04H4 **Lemma 7.35.3.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$. For any sheaf $\mathcal{G}$ on $\mathcal{C}/U$ we have*

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ associated to elements $x \in u(U)$ as in Lemma 7.35.1.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \coprod_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 7.25.2. Also, the stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 7.32.5. Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \coprod\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

To each element (V, y, φ, s) of this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by our construction of the points q. □

04H5 **Remark 7.35.4.** Warning: The result of Lemma 7.35.3 has no analogue for $j_{U,*}$.

7.36. 2-morphisms of topoi

04I9 This is a brief section concerning the notion of a 2-morphism of topoi.

04IA **Definition 7.36.1.** Let $f, g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be two morphisms of topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$.

Pictorially we sometimes represent t as follows:

$$Sh(\mathcal{C}) \;\underset{g}{\overset{f}{\Downarrow t}}\; Sh(\mathcal{D})$$

Note that since f^{-1} is adjoint to f_* and g^{-1} is adjoint to g_* we see that t induces also a transformation of functors $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) uniquely characterized by the condition that the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}) & = & \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) \\ {\scriptstyle t\circ -}\downarrow & & \downarrow{\scriptstyle -\circ t} \\ \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, g_*\mathcal{F}) & = & \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}\mathcal{G}, \mathcal{F}) \end{array}$$

commutes. Because of set theoretic difficulties (see Remark 7.15.4) we do not obtain a 2-category of topoi. But we can still define horizontal and vertical composition and show that the axioms of a strict 2-category listed in Categories, Section 4.29 hold. Namely, vertical composition of 2-morphisms is clear (just compose transformations of functors), composition of 1-morphisms has been defined in Definition 7.15.1, and horizontal composition of

$$Sh(\mathcal{C}) \underset{g}{\overset{f}{\Downarrow t}} Sh(\mathcal{D}) \underset{g'}{\overset{f'}{\Downarrow s}} Sh(\mathcal{E})$$

is defined by the transformation of functors $s \star t$ introduced in Categories, Definition 4.28.1. Explicitly, $s \star t$ is given by

$$f'_*f_*\mathcal{F} \xrightarrow{f'_*t} f'_*g_*\mathcal{F} \xrightarrow{s} g'_*g_*\mathcal{F} \quad \text{or} \quad f'_*f_*\mathcal{F} \xrightarrow{s} g'_*f_*\mathcal{F} \xrightarrow{g'_*t} g'_*g_*\mathcal{F}$$

(these maps are equal). Since these definitions agree with the ones in Categories, Section 4.28 it follows from Categories, Lemma 4.28.2 that the axioms of a strict 2-category hold with these definitions.

7.37. Morphisms between points

00YG

00YH **Lemma 7.37.1.** *Let $\mathcal{C}$ be a site. Let $u, u' : \mathcal{C} \to Sets$ be two functors, and let $t : u' \to u$ be a transformation of functors. Then we obtain a canonical transformation of stalk functors $t_{stalk} : \mathcal{F}_{p'} \to \mathcal{F}_p$ which agrees with t via the identifications of Lemma 7.32.3.*

Proof. Omitted. □

00YI **Definition 7.37.2.** Let $\mathcal{C}$ be a site. Let p, p' be points of $\mathcal{C}$ given by functors $u, u' : \mathcal{C} \to Sets$. A *morphism* $f : p \to p'$ is given by a transformation of functors

$$f_u : u' \to u.$$

Note how the transformation of functors goes the other way. This makes sense, as we will see later, by thinking of the morphism f as a kind of 2-arrow pictorially as follows:

$$Sets = Sh(pt) \underset{p'}{\overset{p}{\Downarrow f}} Sh(\mathcal{C})$$

Namely, we will see later that f_u induces a canonical transformation of functors $p_* \to p'_*$ between the skyscraper sheaf constructions.

This is a fairly important notion, and deserves a more complete treatment here. List of desiderata

(1) Describe the automorphisms of the point of $\mathcal{T}_G$ described in Example 7.33.7.
(2) Describe $\mathrm{Mor}(p, p')$ in terms of $\mathrm{Mor}(p_*, p'_*)$.
(3) Specialization of points in topological spaces. Show that if $x' \in \overline{\{x\}}$ in the topological space X, then there is a morphism $p \to p'$, where p (resp. p') is the point of X_{Zar} associated to x (resp. x').

7.38. Sites with enough points

00YJ

00YK **Definition 7.38.1.** Let $\mathcal{C}$ be a site.

(1) A family of points $\{p_i\}_{i\in I}$ is called *conservative* if every map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is an isomorphism on all the fibres $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$ is an isomorphism.
(2) We say that $\mathcal{C}$ *has enough points* if there exists a conservative family of points.

It turns out that you can then check "exactness" at the stalks.

00YL **Lemma 7.38.2.** *Let $\mathcal{C}$ be a site and let $\{p_i\}_{i\in I}$ be a conservative family of points. Then*

(1) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ injective implies φ injective.*
(2) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ surjective implies φ surjective.*
(3) *Given any pair of maps of sheaves $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{1,p_i} = \varphi_{2,p_i}$ implies $\varphi_1 = \varphi_2$.*
(4) *Given a finite diagram $\mathcal{G} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{F}$ and morphisms $q_j : \mathcal{F} \to \mathcal{G}_j$ then $(\mathcal{F}, q_j)$ is a limit of the diagram if and only if for each i the stalk $(\mathcal{F}_{p_i}, (q_j)_{p_i})$ is one.*
(5) *Given a finite diagram $\mathcal{F} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{G}$ and morphisms $e_j : \mathcal{F}_j \to \mathcal{G}$ then $(\mathcal{G}, e_j)$ is a colimit of the diagram if and only if for each i the stalk $(\mathcal{G}_{p_i}, (e_j)_{p_i})$ is one.*

Proof. We will use over and over again that all the stalk functors commute with any finite limits and colimits and hence with products, fibred products, etc. We will also use that injective maps are the monomorphisms and the surjective maps are the epimorphisms. A map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism. Hence (1). Similarly, $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism. Hence (2). The maps $a, b : \mathcal{F} \to \mathcal{G}$ are equal if and only if $\mathrm{Equalizer}(a, b : \mathcal{F} \to \mathcal{G}) \to \mathcal{F}$ is an isomorphism[8]. Hence (3). The assertions (4) and (5) follow immediately from the definitions and the remarks at the start of this proof. □

00YM **Lemma 7.38.3.** *Let $\mathcal{C}$ be a site and let $\{(p_i, u_i)\}_{i\in I}$ be a family of points. The family is conservative if and only if for every sheaf $\mathcal{F}$ and every $U \in \mathrm{Ob}(\mathcal{C})$ and*

[8]The equalizer can be expressed in terms of products and fibre products, see proof of Categories, Lemma 4.18.4.

every pair of distinct sections $s, s' \in \mathcal{F}(U)$, $s \neq s'$ there exists an i and $x \in u_i(U)$ such that the triples (U, x, s) and (U, x, s') define distinct elements of $\mathcal{F}_{p_i}$.

Proof. Suppose that the family is conservative and that $\mathcal{F}$, U, and s, s' are as in the lemma. The sections s, s' define maps $a, a' : (h_U)^\# \to \mathcal{F}$ which are distinct. Hence, by Lemma 7.38.2 there is an i such that $a_{p_i} \neq a'_{p_i}$. Recall that $(h_U)^\#_{p_i} = u_i(U)$, by Lemmas 7.32.3 and 7.32.5. Hence there exists an $x \in u_i(U)$ such that $a_{p_i}(x) \neq a'_{p_i}(x)$ in $\mathcal{F}_{p_i}$. Unwinding the definitions you see that (U, x, s) and (U, x, s') are as in the statement of the lemma.

To prove the converse, assume the condition on the existence of points of the lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves which is an isomorphism at all the stalks. We have to show that ϕ is both injective and surjective, see Lemma 7.11.2. Injectivity is an immediate consequence of the assumption. To show surjectivity we have to show that $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism (Categories, Lemma 4.13.3). Since this map is clearly surjective, it suffices to check injectivity which follows as $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is injective on all stalks by assumption. □

In the following lemma the points $q_{i,x}$ are exactly all the points of $\mathcal{C}/U$ lying over the point p_i according to Lemma 7.35.2.

04H6 **Lemma 7.38.4.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. let $\{(p_i, u_i)\}_{i \in I}$ be a family of points of $\mathcal{C}$. For $x \in u_i(U)$ let $q_{i,x}$ be the point of $\mathcal{C}/U$ constructed in Lemma 7.35.1. If $\{p_i\}$ is a conservative family of points, then $\{q_{i,x}\}_{i \in I, x \in u_i(U)}$ is a conservative family of points of $\mathcal{C}/U$. In particular, if $\mathcal{C}$ has enough points, then so does every localization $\mathcal{C}/U$.*

Proof. We know that $j_{U!}$ induces an equivalence $j_{U!} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$, see Lemma 7.25.4. Moreover, we know that $(j_{U!}\mathcal{G})_{p_i} = \coprod_x \mathcal{G}_{q_{i,x}}$, see Lemma 7.35.3. Hence the result follows formally. □

The following lemma tells us we can check the existence of points locally on the site.

06UL **Lemma 7.38.5.** *Let $\mathcal{C}$ be a site. Let $\{U_i\}_{i \in I}$ be a family of objects of $\mathcal{C}$. Assume*

(1) *$\coprod h_{U_i}^\# \to *$ is a surjective map of sheaves, and*
(2) *each localization $\mathcal{C}/U_i$ has enough points.*

Then $\mathcal{C}$ has enough points.

Proof. For each $i \in I$ let $\{p_j\}_{j \in J_i}$ be a conservative family of points of $\mathcal{C}/U_i$. For $j \in J_i$ denote $q_j : Sh(pt) \to Sh(\mathcal{C})$ the composition of p_j with the localization morphism $Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C})$. Then q_j is a point, see Lemma 7.34.3. We claim that the family of points $\{q_j\}_{j \in \coprod J_i}$ is conservative. Namely, let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{q_j} \to \mathcal{G}_{q_j}$ is an isomorphism for all $j \in \coprod J_i$. Let W be an object of $\mathcal{C}$. By assumption (1) there exists a covering $\{W_a \to W\}$ and morphisms $W_a \to U_{i(a)}$. Since $(\mathcal{F}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{F}_{q_j}$ and $(\mathcal{G}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{G}_{q_j}$ by Lemma 7.34.3 we see that $\mathcal{F}|_{U_{i(a)}} \to \mathcal{G}|_{U_{i(a)}}$ is an isomorphism since the family of points $\{p_j\}_{j \in J_{i(a)}}$ is conservative. Hence $\mathcal{F}(W_a) \to \mathcal{G}(W_a)$ is bijective for each a. Similarly $\mathcal{F}(W_a \times_W W_b) \to \mathcal{G}(W_a \times_W W_b)$ is bijective for each a, b. By the sheaf condition this shows that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective, i.e., $\mathcal{F} \to \mathcal{G}$ is an isomorphism. □

0F4G **Lemma 7.38.6.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $\{(q_i, v_i)\}_{i \in I}$ be a conservative family of points of $\mathcal{D}$. If each functor $u_i = v_i \circ u$ defines a point of $\mathcal{C}$, then u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$.*

Proof. Denote p_i the stalk functor (7.32.1.1) on $PSh(\mathcal{C})$ corresponding to the functor u_i. We have

$$(f^{-1}\mathcal{F})_{q_i} = (u_s\mathcal{F})_{q_i} = (u_p\mathcal{F})_{q_i} = \mathcal{F}_{p_i}$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 7.32.5, and the third by Lemma 7.34.1. Hence if p_i is a point, then pulling back by f and then taking stalks at q_i is an exact functor. Since the family of points $\{q_i\}$ is conservative, this implies that f^{-1} is an exact functor and we see that f is a morphism of sites by Definition 7.14.1. □

7.39. Criterion for existence of points

00YN This section corresponds to Deligne's appendix to [**AGV71**, Exposé VI]. In fact it is almost literally the same.

Let $\mathcal{C}$ be a site. Suppose that $(I, \geq)$ is a directed set, and that $(U_i, f_{ii'})$ is an inverse system over I, see Categories, Definition 4.21.2. Given the data $(I, \geq, U_i, f_{ii'})$ we define

$$u : \mathcal{C} \longrightarrow Sets, \quad u(V) = \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(U_i, V)$$

Let $\mathcal{F} \mapsto \mathcal{F}_p$ be the stalk functor associated to u as in Section 7.32. It is direct from the definition that actually

$$\mathcal{F}_p = \operatorname{colim}_i \mathcal{F}(U_i)$$

in this special case. Note that u commutes with all finite limits (I mean those that are representable in $\mathcal{C}$) because each of the functors $V \mapsto \operatorname{Mor}_{\mathcal{C}}(U_i, V)$ do, see Categories, Lemma 4.19.2.

We say that a system $(I, \geq, U_i, f_{ii'})$ is a *refinement* of $(J, \geq, V_j, g_{jj'})$ if $J \subset I$, the ordering on J induced from that of I and $V_j = U_j$, $g_{jj'} = f_{jj'}$ (in words, the inverse system over J is induced by that over I). Let u be the functor associated to $(I, \geq, U_i, f_{ii'})$ and let u' be the functor associated to $(J, \geq, V_j, g_{jj'})$. This induces a transformation of functors

$$u' \longrightarrow u$$

simply because the colimits for u' are over a subsystem of the systems in the colimits for u. In particular we get an associated transformation of stalk functors $\mathcal{F}_{p'} \to \mathcal{F}_p$, see Lemma 7.37.1.

00YO **Lemma 7.39.1.** *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. Let $\{W_k \to W\}$ be a finite covering of $\mathcal{C}$. Let $f \in u'(W)$. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. There exists a $j_0 \in J$ such that f is defined by $f' : V_{j_0} \to W$. For $j \geq j_0$ we set $V_{j,k} = V_j \times_{f' \circ f_{jj_0}, W} W_k$. Then $\{V_{j,k} \to V_j\}$ is a finite covering in the site $\mathcal{C}$. Hence $\mathcal{F}(V_j) \subset \prod_k \mathcal{F}(V_{j,k})$. By Categories, Lemma 4.19.2 once again we see that

$$\mathcal{F}_{p'} = \operatorname{colim}_j \mathcal{F}(V_j) \longrightarrow \prod\nolimits_k \operatorname{colim}_j \mathcal{F}(V_{j,k})$$

is injective. Hence there exists a k such that s and s' have distinct image in $\mathrm{colim}_j \mathcal{F}(V_{j,k})$. Let $J_0 = \{j \in J, j \geq j_0\}$ and $I = J \amalg J_0$. We order I so that no element of the second summand is smaller than any element of the first, but otherwise using the ordering on J. If $j \in I$ is in the first summand then we use V_j and if $j \in I$ is in the second summand then we use $V_{j,k}$. We omit the definition of the transition maps of the inverse system. By the above it follows that s, s' have distinct image in $\mathcal{F}_p$. Moreover, the restriction of f' to $V_{j,k}$ factors through W_k by construction. □

00YP **Lemma 7.39.2.** *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and such that for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$, and any $f \in u'(W)$ the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. Let E be the set of pairs $(\{W_k \to W\}, f \in u'(W))$. Consider pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ such that

(1) $(I, \geq, U_i, g_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$,
(2) s, s' map to distinct elements of $\mathcal{F}_p$, and
(3) for every pair $(\{W_k \to W\}, f \in u'(W)) \in E'$ we have that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

We order such pairs by inclusion in the first factor and by refinement in the second. Denote $\mathcal{S}$ the class of all pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ as above. We claim that the hypothesis of Zorn's lemma holds for $\mathcal{S}$. Namely, suppose that $(E'_a, (I_a, \geq, U_i, f_{ii'}))_{a \in A}$ is a totally ordered subset of $\mathcal{S}$. Then we can define $E' = \bigcup_{a \in A} E'_a$ and we can set $I = \bigcup_{a \in A} I_a$. We claim that the corresponding pair $(E', (I, \geq, U_i, f_{ii'}))$ is an element of $\mathcal{S}$. Conditions (1) and (3) are clear. For condition (2) you note that

$$u = \mathrm{colim}_{a \in A} u_a \text{ and correspondingly } \mathcal{F}_p = \mathrm{colim}_{a \in A} \mathcal{F}_{p_a}$$

The distinctness of the images of s, s' in this stalk follows from the description of a directed colimit of sets, see Categories, Section 4.19. We will simply write $(E', (I, \ldots)) = \bigcup_{a \in A} (E'_a, (I_a, \ldots))$ in this situation.

OK, so Zorn's Lemma would apply if $\mathcal{S}$ was a set, and this would, combined with Lemma 7.39.1 above easily prove the lemma. It doesn't since $\mathcal{S}$ is a class. In order to circumvent this we choose a well ordering on E. For $e \in E$ set $E'_e = \{e' \in E \mid e' \leq e\}$. Using transfinite recursion we construct pairs $(E'_e, (I_e, \ldots)) \in \mathcal{S}$ such that $e_1 \leq e_2 \Rightarrow (E'_{e_1}, (I_{e_1}, \ldots)) \leq (E'_{e_2}, (I_{e_2}, \ldots))$. Let $e \in E$, say $e = (\{W_k \to W\}, f \in u'(W))$. If e has a predecessor $e-1$, then we let $(I_e, \ldots)$ be a refinement of $(I_{e-1}, \ldots)$ as in Lemma 7.39.1 with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. If e does not have a predecessor, then we let $(I_e, \ldots)$ be a refinement of $\bigcup_{e' < e} (I_{e'}, \ldots)$ with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. Finally, the union $\bigcup_{e \in E} I_e$ will be a solution to the problem posed in the lemma. □

00YQ **Proposition 7.39.3.** *Let $\mathcal{C}$ be a site. Assume that*

(1) *finite limits exist in $\mathcal{C}$, and*
(2) *every covering $\{U_i \to U\}_{i \in I}$ has a refinement by a finite covering of $\mathcal{C}$.*

Then $\mathcal{C}$ has enough points.

[**AGV71**, Exposé VI, Appendix by Deligne, Proposition 9.0]

Proof. We have to show that given any sheaf $\mathcal{F}$ on $\mathcal{C}$, any $U \in \mathrm{Ob}(\mathcal{C})$, and any distinct sections $s, s' \in \mathcal{F}(U)$, there exists a point p such that s, s' have distinct image in $\mathcal{F}_p$. See Lemma 7.38.3. Consider the system $(J, \geq, V_j, g_{jj'})$ with $J = \{1\}$, $V_1 = U$, $g_{11} = \mathrm{id}_U$. Apply Lemma 7.39.2. By the result of that lemma we get a system $(I, \geq, U_i, f_{ii'})$ refining our system such that $s_p \neq s'_p$ and such that moreover for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$ the map $\coprod_k u(W_k) \to u(W)$ is surjective. Since every covering of $\mathcal{C}$ can be refined by a finite covering we conclude that $\coprod_k u(W_k) \to u(W)$ is surjective for *any* covering $\{W_k \to W\}$ of the site $\mathcal{C}$. This implies that $u = p$ is a point, see Proposition 7.33.3 (and the discussion at the beginning of this section which guarantees that u commutes with finite limits). □

0DW0 **Lemma 7.39.4.** *Let $\mathcal{C}$ be a site. Let I be a set and for $i \in I$ let U_i be an object of $\mathcal{C}$ such that*

(1) *$\coprod h_{U_i}$ surjects onto the final object of $Sh(\mathcal{C})$, and*
(2) *$\mathcal{C}/U_i$ satisfies the hypotheses of Proposition 7.39.3.*

Then $\mathcal{C}$ has enough points.

Proof. By assumption (2) and the proposition $\mathcal{C}/U_i$ has enough points. The points of $\mathcal{C}/U_i$ give points of $\mathcal{C}$ via the procedure of Lemma 7.34.2. Thus it suffices to show: if $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on $\mathcal{C}$ such that $\phi|_{\mathcal{C}/U_i}$ is an isomorphism for all i, then ϕ is an isomorphism. By assumption (1) for every object W of $\mathcal{C}$ there is a covering $\{W_j \to W\}_{j \in J}$ such that for $j \in J$ there is an $i \in I$ and a morphism $f_j : W_j \to U_i$. Then the maps $\mathcal{F}(W_j) \to \mathcal{G}(W_j)$ are bijective and similarly for $\mathcal{F}(W_j \times_W W_{j'}) \to \mathcal{G}(W_j \times_W W_{j'})$. The sheaf condition tells us that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective as desired. □

7.40. Weakly contractible objects

090J A *weakly contractible* object of a site is one that satisfies the equivalent conditions of the following lemma.

090K **Lemma 7.40.1.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following conditions are equivalent*

(1) *For every covering $\{U_i \to U\}$ there exists a map of sheaves $h_U^\# \to \coprod h_{U_i}^\#$ right inverse to the sheafification of $\coprod h_{U_i} \to h_U$.*
(2) *For every surjection of sheaves of sets $\mathcal{F} \to \mathcal{G}$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.*

Proof. Assume (1) and let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. For $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ and $t_i \in \mathcal{F}(U_i)$ mapping to $s|_{U_i}$, see Definition 7.11.1. Think of t_i as a map $t_i : h_{U_i}^\# \to \mathcal{F}$ via (7.12.3.1). Then precomposing $\coprod t_i : \coprod h_{U_i}^\# \to \mathcal{F}$ with the map $h_U^\# \to \coprod h_{U_i}^\#$ we get from (1) we obtain a section $t \in \mathcal{F}(U)$ mapping to s. Thus (2) holds.

Assume (2) holds. Let $\{U_i \to U\}$ be a covering. Then $\coprod h_{U_i}^\# \to h_U^\#$ is surjective (Lemma 7.12.4). Hence by (2) there exists a section s of $\coprod h_{U_i}^\#$ mapping to the section id_U of $h_U^\#$. This section corresponds to a map $h_U^\# \to \coprod h_{U_i}^\#$ which is right inverse to the sheafification of $\coprod h_{U_i} \to h_U$ which proves (1). □

090L **Definition 7.40.2.** Let $\mathcal{C}$ be a site.

(1) We say an object U of $\mathcal{C}$ is *weakly contractible* if the equivalent conditions of Lemma 7.40.1 hold.
(2) We say a site has *enough weakly contractible objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ with U_i weakly contractible for all i.
(3) More generally, if P is a property of objects of $\mathcal{C}$ we say that $\mathcal{C}$ has *enough P objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ such that U_i has P for all i.

The small étale site of $\mathbf{A}^1_{\mathbf{C}}$ does not have any weakly contractible objects. On the other hand, the small pro-étale site of any scheme has enough contractible objects.

7.41. Exactness properties of pushforward

04D5 Let f be a morphism of topoi. The functor f_* in general is only left exact. There are many additional conditions one can impose on this functor to single out particular classes of morphisms of topoi. We collect them here and note some of the logical dependencies. Some parts of the following lemma are purely category theoretical (i.e., they do not depend on having a morphism of topoi, just having a pair of adjoint functors is enough).

04D6 **Lemma 7.41.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Consider the following properties (on sheaves of sets):*

(1) *f_* is faithful,*
(2) *f_* is fully faithful,*
(3) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(4) *f_* transforms surjections into surjections,*
(5) *f_* commutes with coequalizers,*
(6) *f_* commutes with pushouts,*
(7) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(8) *f_* reflects injections,*
(9) *f_* reflects surjections,*
(10) *f_* reflects bijections, and*
(11) *for any surjection $\mathcal{F} \to f^{-1}\mathcal{G}$ there exists a surjection $\mathcal{G}' \to \mathcal{G}$ such that $f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}$ factors through $\mathcal{F} \to f^{-1}\mathcal{G}$.*

Then we have the following implications

(a) *(2) $\Rightarrow$ (1),*
(b) *(3) $\Rightarrow$ (1),*
(c) *(7) $\Rightarrow$ (1), (2), (3), (8), (9), (10).*
(d) *(3) $\Leftrightarrow$ (9),*
(e) *(6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4),*
(f) *(4) $\Leftrightarrow$ (11),*
(g) *(9) $\Rightarrow$ (8), (10), and*
(h) *(2) $\Leftrightarrow$ (7).*

Picture

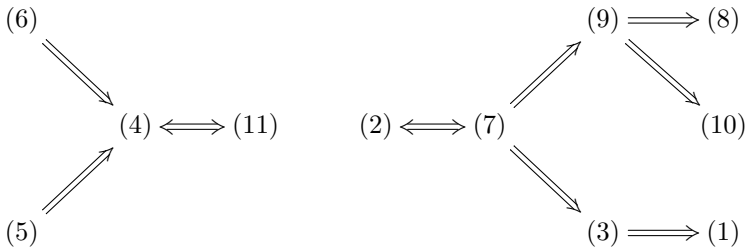

Proof. Proof of (a): This is immediate from the definitions.

Proof of (b). Suppose that $a, b : \mathcal{F} \to \mathcal{F}'$ are maps of sheaves on $\mathcal{C}$. If $f_*a = f_*b$, then $f^{-1}f_*a = f^{-1}f_*b$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \rightrightarrows & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \rightrightarrows & f^{-1}f_*\mathcal{F}' \end{array}$$

If the bottom two arrows are equal and the vertical arrows are surjective then the top two arrows are equal. Hence (b) follows.

Proof of (c). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

If (7) holds, then the vertical arrows are isomorphisms. Hence if f_*a is injective (resp. surjective, resp. bijective) then the bottom arrow is injective (resp. surjective, resp. bijective) and hence the top arrow is injective (resp. surjective, resp. bijective). Thus we see that (7) implies (8), (9), (10). It is clear that (7) implies (3). The implications (7) $\Rightarrow$ (2), (1) follow from (a) and (h) which we will see below.

Proof of (d). Assume (3). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (3) this implies that a is surjective. This means that (9) holds. Assume (9). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. It suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

Proof of (e). We use Categories, Lemma 4.13.3 without further mention. If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}' \to \mathcal{F}'$ is an isomorphism. Hence (6) implies that

$$f_*\mathcal{F}' \amalg_{f_*\mathcal{F}} f_*\mathcal{F}' = f_*(\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}') \longrightarrow f_*\mathcal{F}'$$

is an isomorphism also. And this in turn implies that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence we see that (6) implies (4). If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}'$ is the coequalizer of the two projections $\mathcal{F} \times_{\mathcal{F}'} \mathcal{F} \to \mathcal{F}$ by Lemma 7.11.3. Hence if (5) holds, then $f_*\mathcal{F}'$ is the coequalizer of the two projections

$$f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F}) = f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which clearly means that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence (5) implies (4) as well.

Proof of (f). Assume (4). Let $\mathcal{F} \to f^{-1}\mathcal{G}$ be a surjective map of sheaves on $\mathcal{C}$. By (4) we see that $f_*\mathcal{F} \to f_*f^{-1}\mathcal{G}$ is surjective. Let $\mathcal{G}'$ be the fibre product

$$\begin{array}{ccc} f_*\mathcal{F} & \longrightarrow & f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{G}' & \longrightarrow & \mathcal{G} \end{array}$$

so that $\mathcal{G}' \to \mathcal{G}$ is surjective also. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & f^{-1}\mathcal{G} \end{array}$$

and we see the required result. Conversely, assume (11). Let $a : \mathcal{F} \to \mathcal{F}'$ be surjective map of sheaves on $\mathcal{C}$. Consider the fibre product diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Because the lower horizontal arrow is surjective and by (11) we can find a surjection $\gamma : \mathcal{G}' \to f_*\mathcal{F}'$ such that $f^{-1}\gamma$ factors through $\mathcal{F}'' \to f^{-1}f_*\mathcal{F}'$:

$$\begin{array}{ccccc} & & \mathcal{F} & \longrightarrow & \mathcal{F}' \\ & & \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Pushing this down using f_* we get a commutative diagram

$$\begin{array}{ccccc} & & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{F}' \\ & & \uparrow & & \uparrow \\ f_*f^{-1}\mathcal{G}' & \longrightarrow & f_*\mathcal{F}'' & \longrightarrow & f_*f^{-1}f_*\mathcal{F}' \\ \uparrow & & & & \uparrow \\ \mathcal{G}' & & \longrightarrow & & f_*\mathcal{F}' \end{array}$$

which proves that (4) holds.

Proof of (g). Assume (9). We use Categories, Lemma 4.13.3 without further mention. Let $a : \mathcal{F} \to \mathcal{F}'$ be a map of sheaves on $\mathcal{C}$ such that f_*a is injective. This means that $f_*\mathcal{F} \to f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} = f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F})$ is an isomorphism. Thus by (9) we see that $\mathcal{F} \to \mathcal{F} \times_{\mathcal{F}'} \mathcal{F}$ is surjective, i.e., an isomorphism. Thus a is injective, i.e., (8) holds. Since (10) is trivially equivalent to (8) + (9) we are done with (g).

Proof of (h). This is Categories, Lemma 4.24.4. □

Here is a condition on a morphism of sites which guarantees that the functor f_* transforms surjective maps into surjective maps.

04D7 **Lemma 7.41.2.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for any object U of $\mathcal{C}$ and any covering $\{V_j \to u(U)\}$ in $\mathcal{D}$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that the map of sheaves*

$$\coprod h^\#_{u(U_i)} \to h^\#_{u(U)}$$

factors through the map of sheaves

$$\coprod h^\#_{V_j} \to h^\#_{u(U)}.$$

Then f_* transforms surjective maps of sheaves into surjective maps of sheaves.

Proof. Let $a : \mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves on $\mathcal{D}$. Let U be an object of $\mathcal{C}$ and let $s \in f_*\mathcal{G}(U) = \mathcal{G}(u(U))$. By assumption there exists a covering $\{V_j \to u(U)\}$ and sections $s_j \in \mathcal{F}(V_j)$ with $a(s_j) = s|_{V_j}$. Now we may think of the sections s, s_j and a as giving a commutative diagram of maps of sheaves

$$\begin{array}{ccc} \coprod h^\#_{V_j} & \xrightarrow[\coprod s_j]{} & \mathcal{F} \\ \downarrow & & \downarrow a \\ h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

By assumption there exists a covering $\{U_i \to U\}$ such that we can enlarge the commutative diagram above as follows

$$\begin{array}{ccccc} & & \coprod h^\#_{V_j} & \xrightarrow[\coprod s_j]{} & \mathcal{F} \\ & \nearrow & \downarrow & & \downarrow a \\ \coprod h^\#_{u(U_i)} & \longrightarrow & h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

Because $\mathcal{F}$ is a sheaf the map from the left lower corner to the right upper corner corresponds to a family of sections $s_i \in \mathcal{F}(u(U_i))$, i.e., sections $s_i \in f_*\mathcal{F}(U_i)$. The commutativity of the diagram implies that $a(s_i)$ is equal to the restriction of s to U_i. In other words we have shown that f_*a is a surjective map of sheaves. □

04D8 **Example 7.41.3.** Assume $f : \mathcal{D} \to \mathcal{C}$ satisfies the assumptions of Lemma 7.41.2. Then it is in general not the case that f_* commutes with coequalizers or pushouts. Namely, suppose that f is the morphism of sites associated to the morphism of topological spaces $X = \{1, 2\} \to Y = \{*\}$ (see Example 7.14.2), where Y is a singleton space, and $X = \{1, 2\}$ is a discrete space with two points. A sheaf $\mathcal{F}$ on X is given by a pair (A_1, A_2) of sets. Then $f_*\mathcal{F}$ corresponds to the set $A_1 \times A_2$. Hence if $a = (a_1, a_2), b = (b_1, b_2) : (A_1, A_2) \to (B_1, B_2)$ are maps of sheaves on X, then the coequalizer of a, b is (C_1, C_2) where C_i is the coequalizer of a_i, b_i, and the coequalizer of f_*a, f_*b is the coequalizer of

$$a_1 \times a_2, b_1 \times b_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

which is in general different from $C_1 \times C_2$. Namely, if $A_2 = \emptyset$ then $A_1 \times A_2 = \emptyset$, and hence the coequalizer of the displayed arrows is $B_1 \times B_2$, but in general $C_1 \neq B_1$. A similar example works for pushouts.

The following lemma gives a criterion for when a morphism of sites has a functor f_* which reflects injections and surjections. Note that this also implies that f_* is faithful and that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is always surjective.

04D9 **Lemma 7.41.4.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for every object V of $\mathcal{D}$ there exist objects U_i of $\mathcal{C}$ and morphisms $u(U_i) \to V$ such that $\{u(U_i) \to V\}$ is a covering of $\mathcal{D}$. In this case the functor $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ reflects injections and surjections.*

Proof. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be maps of sheaves on $\mathcal{D}$. By assumption for every object V of $\mathcal{D}$ we get $\mathcal{F}(V) \subset \prod \mathcal{F}(u(U_i)) = \prod f_*\mathcal{F}(U_i)$ by the sheaf condition for some $U_i \in \mathrm{Ob}(\mathcal{C})$ and similarly for $\mathcal{G}$. Hence it is clear that if $f_*\alpha$ is injective, then α is injective. In other words f_* reflects injections.

Suppose that $f_*\alpha$ is surjective. Then for $V, U_i, u(U_i) \to V$ as above and a section $s \in \mathcal{G}(V)$, there exist coverings $\{U_{ij} \to U_i\}$ such that $s|_{u(U_{ij})}$ is in the image of $\mathcal{F}(u(U_{ij}))$. Since $\{u(U_{ij}) \to V\}$ is a covering (as u is continuous and by the axioms of a site) we conclude that s is locally in the image. Thus α is surjective. In other words f_* reflects surjections. □

08LS **Example 7.41.5.** We construct a morphism $f : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Lemma 7.41.4. Namely, let $\varphi : G \to H$ be a morphism of finite groups. Consider the sites $\mathcal{D} = \mathcal{T}_G$ and $\mathcal{C} = \mathcal{T}_H$ of countable G-sets and H-sets and coverings countable families of jointly surjective maps (Example 7.6.5). Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor described in Section 7.16 and $f : \mathcal{T}_G \to \mathcal{T}_H$ the corresponding morphism of sites. If φ is injective, then every countable G-set is, as a G-set, the quotient of a countable H-set (this fails if φ isn't injective). Thus f satisfies the hypothesis of Lemma 7.41.4. If the sheaf $\mathcal{F}$ on $\mathcal{T}_G$ corresponds to the G-set S, then the canonical map

$$f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}$$

corresponds to the map

$$\mathrm{Map}_G(H, S) \longrightarrow S, \quad a \longmapsto a(1_H)$$

If φ is injective but not surjective, then this map is surjective (as it should according to Lemma 7.41.4) but not injective in general (for example take $G = \{1\}$, $H = \{1, \sigma\}$, and $S = \{1, 2\}$). Moreover, the functor f_* does not commute with coequalizers or pushouts (for $G = \{1\}$ and $H = \{1, \sigma\}$).

7.42. Almost cocontinuous functors

04B4 Let $\mathcal{C}$ be a site. The category $PSh(\mathcal{C})$ has an initial object, namely the presheaf which assigns the empty set to each object of $\mathcal{C}$. Let us denote this presheaf by $\emptyset$. It follows from the properties of sheafification that the sheafification $\emptyset^\#$ of $\emptyset$ is an initial object of the category $Sh(\mathcal{C})$ of sheaves on $\mathcal{C}$.

04B5 **Definition 7.42.1.** Let $\mathcal{C}$ be a site. We say an object U of $\mathcal{C}$ is *sheaf theoretically empty* if $\emptyset^\# \to h_U^\#$ is an isomorphism of sheaves.

The following lemma makes this notion more explicit.

04B6 **Lemma 7.42.2.** *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent:*

(1) *U is sheaf theoretically empty,*
(2) *$\mathcal{F}(U)$ is a singleton for each sheaf $\mathcal{F}$,*
(3) *$\emptyset^\#(U)$ is a singleton,*
(4) *$\emptyset^\#(U)$ is nonempty, and*
(5) *the empty family is a covering of U in $\mathcal{C}$.*

Moreover, if U is sheaf theoretically empty, then for any morphism $U' \to U$ of $\mathcal{C}$ the object U' is sheaf theoretically empty.

Proof. For any sheaf $\mathcal{F}$ we have $\mathcal{F}(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F})$. Hence, we see that (1) and (2) are equivalent. It is clear that (2) implies (3) implies (4). If every covering of U is given by a nonempty family, then $\emptyset^+(U)$ is empty by definition of the plus construction. Note that $\emptyset^+ = \emptyset^\#$ as $\emptyset$ is a separated presheaf, see Theorem 7.10.10. Thus we see that (4) implies (5). If (5) holds, then $\mathcal{F}(U)$ is a singleton for every sheaf $\mathcal{F}$ by the sheaf condition for $\mathcal{F}$, see Remark 7.7.2. Thus (5) implies (2) and (1) – (5) are equivalent. The final assertion of the lemma follows from Axiom (3) of Definition 7.6.2 applied the empty covering of U. □

04B7 **Definition 7.42.3.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. We say u is *almost cocontinuous* if for every object U of $\mathcal{C}$ and every covering $\{V_j \to u(U)\}_{j\in J}$ there exists a covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ such that for each i in I we have at least one of the following two conditions

(1) $u(U_i)$ is sheaf theoretically empty, or
(2) the morphism $u(U_i) \to u(U)$ factors through V_j for some $j \in J$.

The motivation for this definition comes from a closed immersion $i : Z \to X$ of topological spaces. As discussed in Example 7.21.9 the continuous functor $X_{Zar} \to Z_{Zar}$, $U \mapsto Z \cap U$ is not cocontinuous. But it is almost cocontinuous in the sense defined above. We know that i_* while not exact on sheaves of sets, is exact on sheaves of abelian groups, see Sheaves, Remark 6.32.5. And this holds in general for continuous and almost cocontinuous functors.

04B8 **Lemma 7.42.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ such that $\mathcal{G}(V)$ is a singleton whenever V is sheaf theoretically empty. Then $(u^p\mathcal{G})^\# = u^p(\mathcal{G}^\#)$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that $(u^p\mathcal{G})^\#(U) = u^p(\mathcal{G}^\#)(U)$. It suffices to show that $(u^p\mathcal{G})^+(U) = u^p(\mathcal{G}^+)(U)$ since $\mathcal{G}^+$ is another presheaf for which the assumption of the lemma holds. We have

$$u^p(\mathcal{G}^+)(U) = \mathcal{G}^+(u(U)) = \mathrm{colim}_{\mathcal{V}}\, \check{H}^0(\mathcal{V}, \mathcal{G})$$

where the colimit is over the coverings $\mathcal{V}$ of $u(U)$ in $\mathcal{D}$. On the other hand, we see that

$$u^p(\mathcal{G})^+(U) = \mathrm{colim}_{\mathcal{U}}\, \check{H}^0(u(\mathcal{U}), \mathcal{G})$$

where the colimit is over the category of coverings $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ and $u(\mathcal{U}) = \{u(U_i) \to u(U)\}_{i\in I}$. The condition that u is continuous means that each $u(\mathcal{U})$ is a covering. Write $I = I_1 \amalg I_2$, where

$$I_2 = \{i \in I \mid u(U_i) \text{ is sheaf theoretically empty}\}$$

Then $u(\mathcal{U})' = \{u(U_i) \to u(U)\}_{i \in I_1}$ is still a covering of because each of the other pieces can be covered by the empty family and hence can be dropped by Axiom (2) of Definition 7.6.2. Moreover, $\check{H}^0(u(\mathcal{U}), \mathcal{G}) = \check{H}^0(u(\mathcal{U})', \mathcal{G})$ by our assumption on $\mathcal{G}$. Finally, the condition that u is almost cocontinuous implies that for every covering $\mathcal{V}$ of $u(U)$ there exists a covering $\mathcal{U}$ of U such that $u(\mathcal{U})'$ refines $\mathcal{V}$. It follows that the two colimits displayed above have the same value as desired. □

04B9 **Lemma 7.42.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Then $u^s = u^p : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. Let $\mathcal{I}$ be a finite connected index category. Let $\mathcal{I} \to Sh(\mathcal{D})$, $i \mapsto \mathcal{G}_i$ by a diagram. We know that the colimit of this diagram is the sheafification of the colimit in the category of presheaves, see Lemma 7.10.14. Denote colim^{Psh} the colimit in the category of presheaves. Since $\mathcal{I}$ is finite and connected we see that $\text{colim}_i^{Psh} \mathcal{G}_i$ is a presheaf satisfying the assumptions of Lemma 7.42.4 (because a finite connected colimit of singleton sets is a singleton). Hence that lemma gives

$$\begin{aligned} u^s(\text{colim}_i \mathcal{G}_i) &= u^s((\text{colim}_i^{Psh} \mathcal{G}_i)^\#) \\ &= (u^p(\text{colim}_i^{Psh} \mathcal{G}_i))^\# \\ &= (\text{colim}_i^{PSh} u^p(\mathcal{G}_i))^\# \\ &= \text{colim}_i u^s(\mathcal{G}_i) \end{aligned}$$

as desired. □

04BA **Lemma 7.42.6.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. If u is almost cocontinuous then f_* commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. This is a special case of Lemma 7.42.5. □

7.43. Subtopoi

08LT Here is the definition.

08LU **Definition 7.43.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is called an *embedding* if f_* is fully faithful.

According to Lemma 7.41.1 this is equivalent to asking the adjunction map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ to be an isomorphism for every sheaf $\mathcal{F}$ on $\mathcal{D}$.

08LV **Definition 7.43.2.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is a *subtopos* if there exists an embedding of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ such that E is equal to the essential image of the functor f_*.

The subtopoi constructed in the following lemma will be dubbed "open" in the definition later on.

08LW **Lemma 7.43.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{F}$ is a subobject of the final object of $Sh(\mathcal{C})$, and*
(2) *the topos $Sh(\mathcal{C})/\mathcal{F}$ is a subtopos of $Sh(\mathcal{C})$.*

Proof. We have seen in Lemma 7.30.1 that $Sh(\mathcal{C})/\mathcal{F}$ is a topos. In fact, we recall the proof. First we apply Lemma 7.29.5 to see that we may assume $\mathcal{C}$ is a site with a subcanonical topology, fibre products, a final object X, and an object U with $\mathcal{F} = h_U$. The proof of Lemma 7.30.1 shows that the morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ is equal (modulo certain identifications) to the localization morphism $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$.

Assume (2). This means that $j_U^{-1} j_{U,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism for all sheaves $\mathcal{G}$ on $\mathcal{C}/U$. For any object Z/U of $\mathcal{C}/U$ we have

$$(j_{U,*}h_{Z/U})(U) = \mathrm{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U)$$

by Lemma 7.27.2. Setting $\mathcal{G} = h_{Z/U}$ in the equality above we obtain

$$\mathrm{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U) = \mathrm{Mor}_{\mathcal{C}/U}(U, Z/U)$$

for all Z/U. By Yoneda's lemma (Categories, Lemma 4.3.5) this implies $U \times_X U = U$. By Categories, Lemma 4.13.3 $U \to X$ is a monomorphism, in other words (1) holds.

Assume (1). Then $j_U^{-1} j_{U,*} = \mathrm{id}$ by Lemma 7.27.4. □

08LX **Definition 7.43.4.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *open subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos $Sh(\mathcal{C})/\mathcal{F}$ described in Lemma 7.43.3.

This means there is a bijection between the collection of open subtopoi of $Sh(\mathcal{C})$ and the set of subobjects of the final object of $Sh(\mathcal{C})$. Given an open subtopos there is a "closed" complement.

08LY **Lemma 7.43.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$. The full subcategory of sheaves $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism is a subtopos of $Sh(\mathcal{C})$.*

Proof. We apply Lemma 7.29.5 to see that we may assume $\mathcal{C}$ is a site with the properties listed in that lemma. In particular $\mathcal{C}$ has a final object X (so that $* = h_X$) and an object U with $\mathcal{F} = h_U$.

Let $\mathcal{D} = \mathcal{C}$ as a category but a covering is a family $\{V_j \to V\}$ of morphisms such that $\{V_i \to V\} \cup \{U \times_X V \to V\}$ is a covering. By our choice of $\mathcal{C}$ this means exactly that

$$h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. We claim that $\mathcal{D}$ is a site, i.e., the coverings satisfy the conditions (1), (2), (3) of Definition 7.6.2. Condition (1) holds. For condition (2) suppose that $\{V_i \to V\}$ and $\{V_{ij} \to V_i\}$ are coverings of $\mathcal{D}$. Then the composition

$$\coprod \left(h_{U \times_X V_i} \amalg \coprod h_{V_{ij}} \right) \longrightarrow h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. Since each of the morphisms $U \times_X V_i \to V$ factors through $U \times_X V$ we see that

$$h_{U \times_X V} \amalg \coprod h_{V_{ij}} \longrightarrow h_V$$

is surjective, i.e., $\{V_{ij} \to V\}$ is a covering of V in $\mathcal{D}$. Condition (3) follows similarly as a base change of a surjective map of sheaves is surjective.

Note that the (identity) functor $u : \mathcal{C} \to \mathcal{D}$ is continuous and commutes with fibre products and final objects. Hence we obtain a morphism $f : \mathcal{D} \to \mathcal{C}$ of

sites (Proposition 7.14.7). Observe that f_* is the identity functor on underlying presheaves, hence fully faithful. To finish the proof we have to show that the essential image of f_* is the full subcategory $E \subset Sh(\mathcal{C})$ singled out in the lemma. To do this, note that $\mathcal{G} \in \mathrm{Ob}(Sh(\mathcal{C}))$ is in E if and only if $\mathcal{G}(U \times_X V)$ is a singleton for all objects V of $\mathcal{C}$. Thus such a sheaf satisfies the sheaf property for all coverings of $\mathcal{D}$ (argument omitted). Conversely, if $\mathcal{G}$ satisfies the sheaf property for all coverings of $\mathcal{D}$, then $\mathcal{G}(U \times_X V)$ is a singleton, as in $\mathcal{D}$ the object $U \times_X V$ is covered by the empty covering. □

08LZ **Definition 7.43.6.** Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *closed subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos described in Lemma 7.43.5.

All right, and now we can define what it means to have a closed immersion and an open immersion of topoi.

08M0 **Definition 7.43.7.** Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi.

(1) We say f is an *open immersion* if f is an embedding and the essential image of f_* is an open subtopos.
(2) We say f is a *closed immersion* if f is an embedding and the essential image of f_* is a closed subtopos.

08M1 **Lemma 7.43.8.** *Let $i : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a closed immersion of topoi. Then i_* is fully faithful, transforms surjections into surjections, commutes with coequalizers, commutes with pushouts, reflects injections, reflects surjections, and reflects bijections.*

Proof. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$ and let $E \subset Sh(\mathcal{C})$ be the full subcategory consisting of those $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism. By Lemma 7.43.5 the functor i_* is isomorphic to the inclusion functor $\iota : E \to Sh(\mathcal{C})$.

Let $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ be the localization functor (Lemma 7.30.1). Note that E can also be described as the collection of sheaves $\mathcal{G}$ such that $j_{\mathcal{F}}^{-1}\mathcal{G} = *$.

Let $a, b : \mathcal{G}_1 \to \mathcal{G}_2$ be two morphism of E. To prove ι commutes with coequalizers it suffices to show that the coequalizer of a, b in $Sh(\mathcal{C})$ lies in E. This is clear because the coequalizer of two morphisms $* \to *$ is $*$ and because $j_{\mathcal{F}}^{-1}$ is exact. Similarly for pushouts.

Thus i_* satisfies properties (5), (6), and (7) of Lemma 7.41.1 and hence the morphism i satisfies all properties mentioned in that lemma, in particular the ones mentioned in this lemma. □

7.44. Sheaves of algebraic structures

00YR In Sheaves, Section 6.15 we introduced a type of algebraic structure to be a pair $(\mathcal{A}, s)$, where $\mathcal{A}$ is a category, and $s : \mathcal{A} \to Sets$ is a functor such that

(1) s is faithful,
(2) $\mathcal{A}$ has limits and s commutes with limits,
(3) $\mathcal{A}$ has filtered colimits and s commutes with them, and
(4) s reflects isomorphisms.

For such a type of algebraic structure we saw that a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ on a space X is a sheaf if and only if the associated presheaf of sets is a sheaf. Moreover, we worked out the notion of stalk, and given a continuous map $f : X \to Y$ we defined adjoint functors pushforward and pullback on sheaves of algebraic structures which agrees with pushforward and pullback on the underlying sheaves of sets. In addition extending a sheaf of algebraic structures from a basis to all opens of a space, works as expected.

Part of this material still works in the setting of sites and sheaves. Let $(\mathcal{A}, s)$ be a type of algebraic structure. Let $\mathcal{C}$ be a site. Let us denote $PSh(\mathcal{C}, \mathcal{A})$, resp. $Sh(\mathcal{C}, \mathcal{A})$ the category of presheaves, resp. sheaves with values in $\mathcal{A}$ on $\mathcal{C}$.

(α) A presheaf with values in $\mathcal{A}$ is a sheaf if and only if its underlying presheaf of sets is a sheaf. See the proof of Sheaves, Lemma 6.9.2.
(β) Given a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ the presheaf $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf. This is true since the colimits in the sheafification process are filtered, and even colimits over directed sets (see Section 7.10, especially the proof of Lemma 7.10.15) and since s commutes with filtered colimits.
(γ) We get the following commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C}, \mathcal{A}) & \underset{\#}{\rightleftarrows} & PSh(\mathcal{C}, \mathcal{A}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ Sh(\mathcal{C}) & \rightleftarrows & PSh(\mathcal{C}) \end{array}$$

(δ) We have $\mathcal{F} = \mathcal{F}^\#$ if and only if $\mathcal{F}$ is a sheaf of algebraic structures.
(ϵ) The functor $^\#$ is adjoint to the inclusion functor:

$$\mathrm{Mor}_{PSh(\mathcal{C},\mathcal{A})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C},\mathcal{A})}(\mathcal{G}^\#, \mathcal{F})$$

The proof is the same as the proof of Proposition 7.10.12.
(ζ) The functor $\mathcal{F} \mapsto \mathcal{F}^\#$ is left exact. The proof is the same as the proof of Lemma 7.10.15.

00YS **Definition 7.44.1.** Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a functor $u : \mathcal{C} \to \mathcal{D}$. We define the *pushforward* functor for presheaves of algebraic structures by the rule $u^p\mathcal{F}(U) = \mathcal{F}(uU)$, and for sheaves of algebraic structures by the same rule, namely $f_*\mathcal{F}(U) = \mathcal{F}(uU)$.

The problem comes with trying the define the pullback. The reason is that the colimits defining the functor u_p in Section 7.5 may not be filtered. Thus the axioms above are not enough in general to define the pullback of a (pre)sheaf of algebraic structures. Nonetheless, in almost all cases the following lemma is sufficient to define pushforward, and pullback of (pre)sheaves of algebraic structures.

00YT **Lemma 7.44.2.** *Suppose the functor $u : \mathcal{C} \to \mathcal{D}$ satisfies the hypotheses of Proposition 7.14.7, and hence gives rise to a morphism of sites $f : \mathcal{D} \to \mathcal{C}$. In this case the pullback functor f^{-1} (resp. u_p) and the pushforward functor f_* (resp. u^p) extend to an adjoint pair of functors on the categories of sheaves (resp. presheaves) of algebraic structures. Moreover, these functors commute with taking the underlying sheaf (resp. presheaf) of sets.*

Proof. We have defined $f_* = u^p$ above. In the course of the proof of Proposition 7.14.7 we saw that all the colimits used to define u_p are filtered under the

assumptions of the proposition. Hence we conclude from the definition of a type of algebraic structure that we may define u_p by exactly the same colimits as a functor on presheaves of algebraic structures. Adjointness of u_p and u^p is proved in exactly the same way as the proof of Lemma 7.5.4. The discussion of sheafification of presheaves of algebraic structures above then implies that we may define $f^{-1}(\mathcal{F}) = (u_p\mathcal{F})^\#$. □

We briefly discuss a method for dealing with pullback and pushforward for a general morphism of sites, and more generally for any morphism of topoi.

Let $\mathcal{C}$ be a site. In the case $\mathcal{A} = Ab$, we may think of an abelian (pre)sheaf on $\mathcal{C}$ as a quadruple $(\mathcal{F}, +, 0, i)$. Here the data are

(D1) $\mathcal{F}$ is a sheaf of sets,
(D2) $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets,
(D3) $0 : * \to \mathcal{F}$ is a morphism from the singleton sheaf (see Example 7.10.2) to $\mathcal{F}$, and
(D4) $i : \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets.

These data have to satisfy the following axioms

(A1) $+$ is associative and commutative,
(A2) 0 is a unit for $+$, and
(A3) $+ \circ (1, i) = 0 \circ (\mathcal{F} \to *)$.

Compare Sheaves, Lemma 6.4.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites. Note that since f^{-1} is exact we have $f^{-1}* = *$ and $f^{-1}(\mathcal{F} \times \mathcal{F}) = f^{-1}\mathcal{F} \times f^{-1}\mathcal{F}$. Thus we can define $f^{-1}\mathcal{F}$ simply as the quadruple $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. The axioms are going to be preserved because f^{-1} is a functor which commutes with finite limits. Finally it is not hard to check that f_* and f^{-1} are adjoint as usual.

In [**AGV71**] this method is used. They introduce something called an "*espèce the structure algébrique ≪définie par limites projectives finie≫*". For such an espèce you can use the method described above to define a pair of adjoint functors f^{-1} and f_* as above. This clearly works for most algebraic structures that one encounters in practice. Instead of formalizing this construction we simply list those algebraic structures for which this method works (to be verified case by case). In fact, this method works for any morphism of topoi.

00YV **Proposition 7.44.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f = (f^{-1}, f_*)$ be a morphism of topoi from $Sh(\mathcal{D}) \to Sh(\mathcal{C})$. The method introduced above gives rise to an adjoint pair of functors (f^{-1}, f_*) on sheaves of algebraic structures compatible with taking the underlying sheaves of sets for the following types of algebraic structures:*

(1) *pointed sets,*
(2) *abelian groups,*
(3) *groups,*
(4) *monoids,*
(5) *rings,*
(6) *modules over a fixed ring, and*
(7) *lie algebras over a fixed field.*

Moreover, in each of these cases the results above labeled (α), (β), (γ), (δ), (ϵ), and (ζ) hold.

Proof. The final statement of the proposition holds simply since each of the listed categories, endowed with the obvious forgetful functor, is indeed a type of algebraic structure in the sense explained at the beginning of this section. See Sheaves, Lemma 6.15.2.

Proof of (2). We think of a sheaf of abelian groups as a quadruple $(\mathcal{F}, +, 0, i)$ as explained in the discussion preceding the proposition. If $(\mathcal{F}, +, 0, i)$ lives on $\mathcal{C}$, then its pullback is defined as $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. If $(\mathcal{G}, +, 0, i)$ lives on $\mathcal{D}$, then its pushforward is defined as $(f_*\mathcal{G}, f_*+, f_*0, f_*i)$. This works because $f_*(\mathcal{G} \times \mathcal{G}) = f_*\mathcal{G} \times f_*\mathcal{G}$. Adjointness follows from adjointness of the set based functors, since

$$\mathrm{Mor}_{Ab(\mathcal{C})}((\mathcal{F}_1, +, 0, i), (\mathcal{F}_2, +, 0, i)) = \left\{ \begin{array}{c} \varphi \in \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}_1, \mathcal{F}_2) \\ \varphi \text{ is compatible with } +, 0, i \end{array} \right\}$$

Details left to the reader.

This method also works for sheaves of rings by thinking of a sheaf of rings (with unit) as a sextuple $(\mathcal{O}, +, 0, i, \cdot, 1)$ satisfying a list of axioms that you can find in any elementary algebra book.

A sheaf of pointed sets is a pair $(\mathcal{F}, p)$, where $\mathcal{F}$ is a sheaf of sets, and $p : * \to \mathcal{F}$ is a map of sheaves of sets.

A sheaf of groups is given by a quadruple $(\mathcal{F}, \cdot, 1, i)$ with suitable axioms.

A sheaf of monoids is given by a pair $(\mathcal{F}, \cdot)$ with suitable axiom.

Let R be a ring. An sheaf of R-modules is given by a quintuple $(\mathcal{F}, +, 0, i, \{\lambda_r\}_{r \in R})$, where the quadruple $(\mathcal{F}, +, 0, i)$ is a sheaf of abelian groups as above, and $\lambda_r : \mathcal{F} \to \mathcal{F}$ is a family of morphisms of sheaves of sets such that $\lambda_r \circ 0 = 0$, $\lambda_r \circ + = + \circ (\lambda_r, \lambda_r)$, $\lambda_{r+r'} = + \circ \lambda_r \times \lambda_{r'} \circ (\mathrm{id}, \mathrm{id})$, $\lambda_{rr'} = \lambda_r \circ \lambda_{r'}$, $\lambda_1 = \mathrm{id}$, $\lambda_0 = 0 \circ (\mathcal{F} \to *)$. □

We will discuss the category of sheaves of modules over a sheaf of rings in Modules on Sites, Section 18.10.

00YU **Remark 7.44.4.** Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor which gives rise to a morphism of sites $\mathcal{C} \to \mathcal{D}$. Note that even in the case of abelian groups we have not defined a pullback functor for presheaves of abelian groups. Since all colimits are representable in the category of abelian groups, we certainly may define a functor u_p^{ab} on abelian presheaves by the same colimits as we have used to define u_p on presheaves of sets. It will also be the case that u_p^{ab} is adjoint to u^p on the categories of abelian presheaves. However, it will not always be the case that u_p^{ab} agrees with u_p on the underlying presheaves of sets.

7.45. Pullback maps

06UM It sometimes happens that a site $\mathcal{C}$ does not have a final object. In this case we define the global section functor as follows.

06UN **Definition 7.45.1.** The *global sections* of a presheaf of sets $\mathcal{F}$ over a site $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(*, \mathcal{F})$$

where $*$ is the final object in the category of presheaves on $\mathcal{C}$, i.e., the presheaf which associates to every object a singleton.

Of course the same definition applies to sheaves as well. Here is one way to compute global sections.

0792 **Lemma 7.45.2.** *Let $\mathcal{C}$ be a site. Let $a, b : V \to U$ be objects of $\mathcal{C}$ such that*

$$h_V^\# \rightrightarrows h_U^\# \longrightarrow *$$

is a coequalizer in $Sh(\mathcal{C})$. Then $\Gamma(\mathcal{C}, \mathcal{F})$ is the equalizer of $a^*, b^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Proof. Since $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ this is clear from the definitions. □

Now, let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then for any sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a pullback map

$$f^{-1} : \Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, f^{-1}\mathcal{F})$$

Namely, as f^{-1} is exact it transforms $*$ into $*$. Hence a global section s of $\mathcal{F}$ over $\mathcal{C}$, which is a map of sheaves $s : * \to \mathcal{F}$, can be pulled back to $f^{-1}s : * = f^{-1}* \to f^{-1}\mathcal{F}$.

We can generalize this a bit by considering a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$. Then we compose the construction above with the obvious map $\Gamma(\mathcal{D}, f^{-1}\mathcal{F}) \to \Gamma(\mathcal{D}, \mathcal{G})$ to get a map

$$\Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, \mathcal{G})$$

This map is sometimes also called a pullback map.

A slightly more general construction which occurs frequently in nature is the following. Suppose that we have a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{f} & Sh(\mathcal{C}) \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle g} & \\ & Sh(\mathcal{B}) & \end{array}$$

Next, suppose that we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then there is a *pullback map*

$$f^{-1} : g_*\mathcal{F} \longrightarrow h_* f^{-1}\mathcal{F}$$

Namely, it is just the map coming from the identification $g_* f_* f^{-1}\mathcal{F} = h_* f^{-1}\mathcal{F}$ together with g_* applied to the canonical map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$. If g is the identity, then this map on global sections agrees with the pullback map above.

In the situation of the previous paragraph, suppose we have a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$, then we compose the pullback map above with h_* applied to $f^{-1}\mathcal{F} \to \mathcal{G}$ to get a map

$$g_*\mathcal{F} \longrightarrow h_*\mathcal{G}$$

Restricting to sections over an object of $\mathcal{B}$ one recovers the "pullback map" on global sections discussed above (with suitable choices of sites).

An even more general situation is where we have a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{f} & Sh(\mathcal{C}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Sh(\mathcal{B}) & \xrightarrow{e} & Sh(\mathcal{A}) \end{array}$$

and a sheaf $\mathcal{G}$ on $\mathcal{C}$. Then there is a *base change map*

$$e^{-1}g_*\mathcal{G} \longrightarrow h_*f^{-1}\mathcal{G}.$$

Namely, this map is adjoint to a map $g_*\mathcal{G} \to e_*h_*f^{-1}\mathcal{G} = (e \circ h)_*f^{-1}\mathcal{G}$ which is the pullback map just described.

0F6X **Remark 7.45.3.** Consider a commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{B}') & \xrightarrow[k]{} & Sh(\mathcal{B}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{C}') & \xrightarrow{l} & Sh(\mathcal{C}) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Sh(\mathcal{D}') & \xrightarrow{m} & Sh(\mathcal{D}) \end{array}$$

of topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} m^{-1} \circ (g \circ f)_* &= m^{-1} \circ g_* \circ f_* \\ &\to g'_* \circ l^{-1} \circ f_* \\ &\to g'_* \circ f'_* \circ k^{-1} \\ &= (g' \circ f')_* \circ k^{-1} \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0F6Y **Remark 7.45.4.** Consider a commutative diagram

$$\begin{array}{ccccc} Sh(\mathcal{C}'') & \xrightarrow[g']{} & Sh(\mathcal{C}') & \xrightarrow{g} & Sh(\mathcal{C}) \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}'') & \xrightarrow{h'} & Sh(\mathcal{D}') & \xrightarrow{h} & Sh(\mathcal{D}) \end{array}$$

of ringed topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} (h \circ h')^{-1} \circ f_* &= (h')^{-1} \circ h^{-1} \circ f_* \\ &\to (h')^{-1} \circ f'_* \circ g^{-1} \\ &\to f''_* \circ (g')^{-1} \circ g^{-1} \\ &= f''_* \circ (g \circ g')^{-1} \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

7.46. Comparison with SGA4

0CMZ Our notation for the functors u^p and u_p from Section 7.5 and u^s and u_s from Section 7.13 is taken from [**Art62**, pages 14 and 42]. Having made these choices, the notation for the functor ${}_pu$ in Section 7.19 and ${}_su$ in Section 7.20 seems reasonable. In this section we compare our notation with that of SGA4.

Presheaves: Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. The functor u^p is denoted u^* in [**AGV71**, Exposee I, Section 5]. The functor u_p is denoted $u_!$ in

[**AGV71**, Exposee I, Proposition 5.1]. The functor ${}_pu$ is denoted u_* in [**AGV71**, Exposee I, Proposition 5.1]. In other words, we have

$$u_p, u^p, {}_pu \quad (SP) \quad \text{versus} \quad u_!, u^*, u_* \quad (SGA4)$$

The reader should be cautioned that different notation is used for these functors in different parts of SGA4.

Sheaves and continuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a continuous functor (Definition 7.13.1). The functor u^s is denoted u_s in [**AGV71**, Exposee III, 1.11]. The functor u_s is denoted u^s in [**AGV71**, Exposee III, Proposition 1.2]. In other words, we have

$$u_s, u^s \quad (SP) \quad \text{versus} \quad u^s, u_s \quad (SGA4)$$

When u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (Definition 7.14.1) we see that the associated morphism of topoi (Lemma 7.15.2) is the same as that in [**AGV71**, Exposee IV, (4.9.1.1)].

Sheaves and cocontinuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a cocontinuous functor (Definition 7.20.1). The functor ${}_su$ (Lemma 7.20.2) is denoted u_* in [**AGV71**, Exposee III, Proposition 2.3]. The functor $(u^p)^\#$ is denoted u^* in [**AGV71**, Exposee III, Proposition 2.3]. In other words, we have

$$(u^p)^\#, {}_su \quad (SP) \quad \text{versus} \quad u^*, u_* \quad (SGA4)$$

Thus the morphism of topoi associated to u in Lemma 7.21.1 is the same as that in [**AGV71**, Exposee IV, 4.7].

Morphisms of Topoi: If f is a morphism of topoi given by the functors (f^{-1}, f_*) then the functor f^{-1} is denoted f^* in [**AGV71**, Exposee IV, Definition 3.1]. We will use f^{-1} to denote pullback of sheaves of sets or more generally sheaves of algebraic structure (Section 7.44). We will use f^* to denote pullback of sheaves of modules for a morphism of ringed topoi (Modules on Sites, Definition 18.13.1).

7.47. Topologies

00YW In this section we define what a topology on a category is as defined in [**AGV71**]. One can develop all of the machinery of sheaves and topoi in this language. A modern exposition of this material can be found in [**KS06**]. However, the case of most interest for algebraic geometry is the topology defined by a site on its underlying category. Thus we strongly suggest the first time reader **skip this section and all other sections of this chapter**!

00YX **Definition 7.47.1.** Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S *on* U is a subpresheaf $S \subset h_U$.

In other words, a sieve on U picks out for each object $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of the set of all morphisms $T \to U$. In fact, the only condition on the collection of subsets $S(T) \subset h_U(T) = \mathrm{Mor}_{\mathcal{C}}(T, U)$ is the following rule

00YY (7.47.1.1)
$$\left.\begin{array}{c}(\alpha : T \to U) \in S(T) \\ g : T' \to T\end{array}\right\} \Rightarrow (\alpha \circ g : T' \to U) \in S(T')$$

A good mental picture to keep in mind is to think of the map $S \to h_U$ as a "morphism from S to U".

00YZ **Lemma 7.47.2.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

(1) *The collection of sieves on U is a set.*
(2) *Inclusion defines a partial ordering on this set.*
(3) *Unions and intersections of sieves are sieves.*
00Z0 (4) *Given a family of morphisms $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ with target U there exists a unique smallest sieve S on U such that each $U_i \to U$ belongs to $S(U_i)$.*
(5) *The sieve $S = h_U$ is the maximal sieve.*
(6) *The empty subpresheaf is the minimal sieve.*

Proof. By our definition of subpresheaf, the collection of all subpresheaves of a presheaf $\mathcal{F}$ is a subset of $\prod_{U\in \mathrm{Ob}(\mathcal{C})} \mathcal{P}(\mathcal{F}(U))$. And this is a set. (Here $\mathcal{P}(A)$ denotes the powerset of A.) Hence the collection of sieves on U is a set.

The partial ordering is defined by: $S \leq S'$ if and only if $S(T) \subset S'(T)$ for all $T \to U$. Notation: $S \subset S'$.

Given a collection of sieves S_i, $i \in I$ on U we can define $\bigcup S_i$ as the sieve with values $(\bigcup S_i)(T) = \bigcup S_i(T)$ for all $T \in \mathrm{Ob}(\mathcal{C})$. We define the intersection $\bigcap S_i$ in the same way.

Given $\{U_i \to U\}_{i\in I}$ as in the statement, consider the morphisms of presheaves $h_{U_i} \to h_U$. We simply define S as the union of the images (Definition 7.3.5) of these maps of presheaves.

The last two statements of the lemma are obvious. □

00Z1 **Definition 7.47.3.** Let $\mathcal{C}$ be a category. Given a family of morphisms $\{f_i : U_i \to U\}_{i\in I}$ of $\mathcal{C}$ with target U we say the sieve S on U described in Lemma 7.47.2 part (4) is the *sieve on U generated by the morphisms f_i*.

00Z2 **Definition 7.47.4.** Let $\mathcal{C}$ be a category. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Let $S \subset h_U$ be a sieve. We define the *pullback of S by f* to be the sieve $S \times_U V$ of V defined by the rule

$$(\alpha : T \to V) \in (S \times_U V)(T) \Leftrightarrow (f \circ \alpha : T \to U) \in S(T)$$

We leave it to the reader to see that this is indeed a sieve (hint: use Equation 7.47.1.1). We also sometimes call $S \times_U V$ the *base change* of S by $f : V \to U$.

00Z3 **Lemma 7.47.5.** *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let S be a sieve on U. If $f : V \to U$ is in S, then $S \times_U V = h_V$ is maximal.*

Proof. Trivial from the definitions. □

00Z4 **Definition 7.47.6.** Let $\mathcal{C}$ be a category. A *topology on $\mathcal{C}$* is given by a rule which assigns to every $U \in \mathrm{Ob}(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U satisfying the following conditions

(1) For every morphism $f : V \to U$ in $\mathcal{C}$, and every element $S \in J(U)$ the pullback $S \times_U V$ is an element of $J(V)$.
(2) If S and S' are sieves on $U \in \mathrm{Ob}(\mathcal{C})$, if $S \in J(U)$, and if for all $f \in S(V)$ the pullback $S' \times_U V$ belongs to $J(V)$, then S' belongs to $J(U)$.
(3) For every $U \in \mathrm{Ob}(\mathcal{C})$ the maximal sieve $S = h_U$ belongs to $J(U)$.

In this case, the sieves belonging to $J(U)$ are called the *covering sieves*.

00Z5 **Lemma 7.47.7.** *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

(1) *Finite intersections of elements of $J(U)$ are in $J(U)$.*

(2) *If $S \in J(U)$ and $S' \supset S$, then $S' \in J(U)$.*

Proof. Let $S, S' \in J(U)$. Consider $S'' = S \cap S'$. For every $V \to U$ in $S(V)$ we have

$$S' \times_U V = S'' \times_U V$$

simply because $V \to U$ already is in S. Hence by the second axiom of the definition we see that $S'' \in J(U)$.

Let $S \in J(U)$ and $S' \supset S$. For every $V \to U$ in $S(V)$ we have $S' \times_U V = h_V$ by Lemma 7.47.5. Thus $S' \times_U V \in J(V)$ by the third axiom. Hence $S' \in J(U)$ by the second axiom. □

00Z6 **Definition 7.47.8.** Let $\mathcal{C}$ be a category. Let J, J' be two topologies on $\mathcal{C}$. We say that J is *finer* or *stronger* than J' if and only if for every object U of $\mathcal{C}$ we have $J'(U) \subset J(U)$. In this case we also say that J' is *coarser* or *weaker* than J.

In other words, any covering sieve of J' is a covering sieve of J. There exists a finest topology on $\mathcal{C}$, namely that topology where any sieve is a covering sieve. This is called the *discrete topology* of $\mathcal{C}$. There also exists a coarsest topology. Namely, the topology where $J(U) = \{h_U\}$ for all objects U. This is called the *chaotic* or *indiscrete topology*.

00Z7 **Lemma 7.47.9.** *Let $\mathcal{C}$ be a category. Let $\{J_i\}_{i \in I}$ be a set of topologies.*

(1) *The rule $J(U) = \bigcap J_i(U)$ defines a topology on $\mathcal{C}$.*
(2) *There is a coarsest topology finer than all of the topologies J_i.*

Proof. The first part is direct from the definitions. The second follows by taking the intersection of all topologies finer than all of the J_i. □

At this point we can define without any motivation what a sheaf is.

00Z8 **Definition 7.47.10.** Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say that $\mathcal{F}$ is a *sheaf* on $\mathcal{C}$ if for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every covering sieve S of U the canonical map

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

is bijective.

Recall that the left hand side of the displayed formula equals $\mathcal{F}(U)$. In other words, $\mathcal{F}$ is a sheaf if and only if a section of $\mathcal{F}$ over U is the same thing as a compatible collection of sections $s_{T,\alpha} \in \mathcal{F}(T)$ parametrized by $(\alpha : T \to U) \in S(T)$, and this for every covering sieve S on U.

00Z9 **Lemma 7.47.11.** *Let $\mathcal{C}$ be a category. Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of presheaves of sets on $\mathcal{C}$. For each $U \in \mathrm{Ob}(\mathcal{C})$ denote $J(U)$ the set of sieves S with the following property: For every morphism $V \to U$, the maps*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_V, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}_i)$$

are bijective for all $i \in I$. Then J defines a topology on $\mathcal{C}$. This topology is the finest topology in which all of the $\mathcal{F}_i$ are sheaves.

Proof. If we show that J is a topology, then the last statement of the lemma immediately follows. The first and third axioms of a topology are immediately verified. Thus, assume that we have an object U, and sieves S, S' of U such that

$S \in J(U)$, and for all $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. We have to show that $S' \in J(U)$. In other words, we have to show that for any $f : W \to U$, the maps

$$\mathcal{F}_i(W) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_W, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$$

are bijective for all $i \in I$. Pick an element $i \in I$ and pick an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$. We will construct a section $s \in \mathcal{F}_i(W)$ mapping to φ.

Suppose $\alpha : V \to W$ is an element of $S \times_U W$. According to the definition of pullbacks we see that the composition $f \circ \alpha : V \to W \to U$ is in S. Hence $S' \times_U V$ is in $J(W)$ by assumption on the pair of sieves S, S'. Now we have a commutative diagram of presheaves

$$\begin{array}{ccc} S' \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S' \times_U W & \longrightarrow & h_W \end{array}$$

The restriction of φ to $S' \times_U V$ corresponds to an element $s_{V,\alpha} \in \mathcal{F}_i(V)$. This we see from the definition of J, and because $S' \times_U V$ is in $J(W)$. We leave it to the reader to check that the rule $(V, \alpha) \mapsto s_{V,\alpha}$ defines an element $\psi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U W, \mathcal{F}_i)$. Since $S \in J(U)$ we see immediately from the definition of J that ψ corresponds to an element s of $\mathcal{F}_i(W)$.

We leave it to the reader to verify that the construction $\varphi \mapsto s$ is inverse to the natural map displayed above. □

00ZA **Definition 7.47.12.** Let $\mathcal{C}$ be a category. The finest topology on $\mathcal{C}$ such that all representable presheaves are sheaves, see Lemma 7.47.11, is called the *canonical topology* of $\mathcal{C}$.

7.48. The topology defined by a site

00ZB Suppose that $\mathcal{C}$ is a category, and suppose that $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are sets of coverings that define the structure of a site on $\mathcal{C}$. In this situation it can happen that the categories of sheaves (of sets) for $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are the same, see for example Lemma 7.8.7.

It turns out that the category of sheaves on $\mathcal{C}$ with respect to some topology J determines and is determined by the topology J. This is a nontrivial statement which we will address later, see Theorem 7.50.2.

Accepting this for the moment it makes sense to study the topology determined by a site.

00ZC **Lemma 7.48.1.** *Let $\mathcal{C}$ be a site with coverings $Cov(\mathcal{C})$. For every object U of $\mathcal{C}$, let $J(U)$ denote the set of sieves S on U with the following property: there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ so that the sieve S' generated by the f_i (see Definition 7.47.3) is contained in S.*

(1) *This J is a topology on $\mathcal{C}$.*
(2) *A presheaf $\mathcal{F}$ is a sheaf for this topology (see Definition 7.47.10) if and only if it is a sheaf on the site (see Definition 7.7.1).*

Proof. To prove the first assertion we just note that axioms (1), (2) and (3) of the definition of a site (Definition 7.6.2) directly imply the axioms (3), (2) and (1) of the definition of a topology (Definition 7.47.6). As an example we prove J has property (2). Namely, let U be an object of $\mathcal{C}$, let S, S' be sieves on U such that $S \in J(U)$, and such that for every $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. By definition of $J(U)$ we can find a covering $\{f_i : U_i \to U\}$ of the site such that S the image of $h_{U_i} \to h_U$ is contained in S. Since each $S' \times_U U_i$ is in $J(U_i)$ we see that there are coverings $\{U_{ij} \to U_i\}$ of the site such that $h_{U_{ij}} \to h_{U_i}$ is contained in $S' \times_U U_i$. By definition of the base change this means that $h_{U_{ij}} \to h_U$ is contained in the subpresheaf $S' \subset h_U$. By axiom (2) for sites we see that $\{U_{ij} \to U\}$ is a covering of U and we conclude that $S' \in J(U)$ by definition of J.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf in the topology J. We will show that $\mathcal{F}$ is a sheaf on the site as well. Let $\{f_i : U_i \to U\}_{i \in I}$ be a covering of the site. Let $s_i \in \mathcal{F}(U_i)$ be a family of sections such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. We have to show that there exists a unique section $s \in \mathcal{F}(U)$ restricting back to the s_i on the U_i. Let $S \subset h_U$ be the sieve generated by the f_i. Note that $S \in J(U)$ by definition. In stead of constructing s, by the sheaf condition in the topology, it suffices to construct an element

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

Take $\alpha \in S(T)$ for some object $T \in \mathcal{U}$. This means exactly that $\alpha : T \to U$ is a morphism which factors through f_i for some $i \in I$ (and maybe more than 1). Pick such an index i and a factorization $\alpha = f_i \circ \alpha_i$. Define $\varphi(\alpha) = \alpha_i^* s_i$. If i', $\alpha = f_i \circ \alpha'_{i'}$ is a second choice, then $\alpha_i^* s_i = (\alpha'_{i'})^* s_{i'}$ exactly because of our condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus $\varphi(\alpha)$ is well defined. We leave it to the reader to verify that φ, which in turn determines s is correct in the sense that s restricts back to s_i.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. We will show that $\mathcal{F}$ is a sheaf for the topology J as well. Let U be an object of $\mathcal{C}$. Let S be a covering sieve on U with respect to the topology J. Let

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

We have to show there is a unique element in $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$ which restricts back to φ. By definition there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ such that $f_i : U_i \in U$ belongs to $S(U_i)$. Hence we can set $s_i = \varphi(f_i) \in \mathcal{F}(U_i)$. Then it is a pleasant exercise to see that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus we obtain the desired section s by the sheaf condition for $\mathcal{F}$ on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. Details left to the reader. □

00ZD **Definition 7.48.2.** Let $\mathcal{C}$ be a site with coverings $\mathrm{Cov}(\mathcal{C})$. The *topology associated to* $\mathcal{C}$ is the topology J constructed in Lemma 7.48.1 above.

Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ be two coverings defining the structure of a site on $\mathcal{C}$. It may very well happen that the topologies defined by these are the same. If this happens then we say $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ *define the same topology* on $\mathcal{C}$. And if this happens then the categories of sheaves are the same, by Lemma 7.48.1.

It is usually the case that we only care about the topology defined by a collection of coverings, and we view the possibility of choosing different sets of coverings as a tool to study the topology.

00ZE **Remark 7.48.3.** Enlarging the class of coverings. Clearly, if $\mathrm{Cov}(\mathcal{C})$ defines the structure of a site on $\mathcal{C}$ then we may add to $\mathcal{C}$ any set of families of morphisms with fixed target tautologically equivalent (see Definition 7.8.2) to elements of $\mathrm{Cov}(\mathcal{C})$ without changing the topology.

00ZF **Remark 7.48.4.** Shrinking the class of coverings. Let $\mathcal{C}$ be a site. Consider the set

$$\mathcal{S} = P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$$

where $P(\mathrm{Arrows}(\mathcal{C}))$ is the power set of the set of morphisms, i.e., the set of all sets of morphisms. Let $\mathcal{S}_\tau \subset \mathcal{S}$ be the subset consisting of those $(T, U) \in \mathcal{S}$ such that (a) all $\varphi \in T$ have target U, (b) the collection $\{\varphi\}_{\varphi \in T}$ is tautologically equivalent (see Definition 7.8.2) to some covering in $\mathrm{Cov}(\mathcal{C})$. Clearly, considering the elements of $\mathcal{S}_\tau$ as the coverings, we do not get exactly the notion of a site as defined in Definition 7.6.2. The structure $(\mathcal{C}, \mathcal{S}_\tau)$ we get satisfies slightly modified conditions. The modified conditions are:

(0') $\mathrm{Cov}(\mathcal{C}) \subset P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$,
(1') If $V \to U$ is an isomorphism then $(\{V \to U\}, U) \in \mathrm{Cov}(\mathcal{C})$.
(2') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and for $f : U' \to U$ in T we are given $(T_f, U') \in \mathrm{Cov}(\mathcal{C})$, then setting $T' = \{f \circ f' \mid f \in T,\ f' \in T_f\}$, we get $(T', U) \in \mathrm{Cov}(\mathcal{C})$.
(3') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and $g : V \to U$ is a morphism of $\mathcal{C}$ then
  (a) $U' \times_{f,U,g} V$ exists for $f : U' \to U$ in T, and
  (b) setting $T' = \{\mathrm{pr}_2 : U' \times_{f,U,g} V \to V \mid f : U' \to U \in T\}$ for some choice of fibre products we get $(T', V) \in \mathrm{Cov}(\mathcal{C})$.

And it is easy to verify that, given a structure satisfying (0') – (3') above, then after suitably enlarging $\mathrm{Cov}(\mathcal{C})$ (compare Sets, Section 3.11) we get a site. Obviously there is little difference between this notion and the actual notion of a site, at least from the point of view of the topology. There are two benefits: because of condition (0') above the coverings automatically form a set, and because of (0') the totality of all structures of this type forms a set as well. The price you pay for this is that you have to keep writing "tautologically equivalent" everywhere.

7.49. Sheafification in a topology

00ZG In this section we explain the analogue of the sheafification construction in a topology.

Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets. For every $U \in \mathrm{Ob}(\mathcal{C})$ we define

$$L\mathcal{F}(U) = \mathrm{colim}_{S \in J(U)^{opp}} \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

as a colimit. Here we think of $J(U)$ as a partially ordered set, ordered by inclusion, see Lemma 7.47.2. The transition maps in the system are defined as follows. If $S \subset S'$ are in $J(U)$, then $S \to S'$ is a morphism of presheaves. Hence there is a natural restriction mapping

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

Thus we see that $S \mapsto \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is a directed system as in Categories, Definition 4.21.2 provided we reverse the ordering on $J(U)$ (which is what the superscript ^{opp} is supposed to indicate). In particular, since $h_U \in J(U)$ there is a canonical map

$$\ell : \mathcal{F}(U) \longrightarrow L\mathcal{F}(U)$$

coming from the identification $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. In addition, the colimit defining $L\mathcal{F}(U)$ is directed since for any pair of covering sieves S, S' on U the sieve $S \cap S'$ is a covering sieve too, see Lemma 7.47.2.

Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $S \in J(U)$. There is a commutative diagram

$$\begin{array}{ccc} S \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S & \longrightarrow & h_U \end{array}$$

We can use the left vertical map to get canonical restriction maps

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}).$$

Base change $S \mapsto S \times_U V$ induces an order preserving map $J(U) \to J(V)$. And the restriction maps define a transformation of functors as in Categories, Lemma categories-lemma-functorial-colimit. Hence we get a natural restriction map

$$L\mathcal{F}(U) \longrightarrow L\mathcal{F}(V).$$

00ZH **Lemma 7.49.1.** *In the situation above.*

(1) *The assignment $U \mapsto L\mathcal{F}(U)$ combined with the restriction mappings defined above is a presheaf.*
(2) *The maps ℓ glue to give a morphism of presheaves $\ell : \mathcal{F} \to L\mathcal{F}$.*
(3) *The rule $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\ell} L\mathcal{F})$ is a functor.*
(4) *If $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, then $L\mathcal{F}$ is a subpresheaf of $L\mathcal{G}$.*
(5) *The map $\ell : \mathcal{F} \to L\mathcal{F}$ has the following property: For every section $s \in L\mathcal{F}(U)$ there exists a covering sieve S on U and an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ such that $\ell(\varphi)$ equals the restriction of s to S.*

Proof. Omitted. □

00ZI **Definition 7.49.2.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. We say that a presheaf of sets $\mathcal{F}$ is *separated* if for every object U and every covering sieve S on U the canonical map $\mathcal{F}(U) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is injective.

00ZJ **Theorem 7.49.3.** *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets.*

(1) *The presheaf $L\mathcal{F}$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $L\mathcal{F}$ is a sheaf and the map of presheaves $\mathcal{F} \to L\mathcal{F}$ is injective.*
(3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to L\mathcal{F}$ is an isomorphism.*
(4) *The presheaf $LL\mathcal{F}$ is always a sheaf.*

Proof. Part (3) is trivial from the definition of L and the definition of a sheaf (Definition 7.47.10). Part (4) follows formally from the others.

We sketch the proof of (1). Suppose S is a covering sieve of the object U. Suppose that $\varphi_i \in L\mathcal{F}(U)$, $i = 1, 2$ map to the same element in $\mathrm{Mor}_{PSh(\mathcal{C})}(S, L\mathcal{F})$. We may find a single covering sieve S' on U such that both φ_i are represented by elements $\varphi_i \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F})$. We may assume that $S' = S$ by replacing both S and S' by $S' \cap S$ which is also a covering sieve, see Lemma 7.47.2. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. Then we have $S \times_U V = h_V$, see Lemma 7.47.5. Thus the restrictions of φ_i via $V \to U$ correspond to sections $s_{i,V,\alpha}$ of $\mathcal{F}$ over V. The assumption is that there exist a covering sieve $S_{V,\alpha}$ of V such that $s_{i,V,\alpha}$ restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S_{V,\alpha}, \mathcal{F})$. Consider the sieve S'' on U defined by the rule

00ZK
$$
\begin{aligned}
(f : T \to U) \in S''(T) \quad \Leftrightarrow \quad & \exists\ V,\ \alpha : V \to U,\ \alpha \in S(V), \\
& \exists\ g : T \to V,\ g \in S_{V,\alpha}(T), \\
& f = \alpha \circ g
\end{aligned}
\tag{7.49.3.1}
$$

By axiom (2) of a topology we see that S'' is a covering sieve on U. By construction we see that φ_1 and φ_2 restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S'', L\mathcal{F})$ as desired.

We sketch the proof of (2). Assume that $\mathcal{F}$ is a separated presheaf of sets on $\mathcal{C}$ with respect to the topology J. Let S be a covering sieve of the object U of $\mathcal{C}$. Suppose that $\varphi \in \mathrm{Mor}_{\mathcal{C}}(S, L\mathcal{F})$. We have to find an element $s \in L\mathcal{F}(U)$ restricting to φ. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. The value $\varphi(\alpha) \in L\mathcal{F}(V)$ is given by a covering sieve $S_{V,\alpha}$ of V and a morphism of presheaves $\varphi_{V,\alpha} : S_{V,\alpha} \to \mathcal{F}$. As in the proof above, define a covering sieve S'' on U by Equation (7.49.3.1). We define

$$\varphi'' : S'' \longrightarrow \mathcal{F}$$

by the following simple rule: For every $f : T \to U$, $f \in S''(T)$ choose V, α, g as in Equation (7.49.3.1). Then set

$$\varphi''(f) = \varphi_{V,\alpha}(g).$$

We claim this is independent of the choice of V, α, g. Consider a second such choiceV', α', g'. The restrictions of $\varphi_{V,\alpha}$ and $\varphi_{V',\alpha'}$ to the intersection of the following covering sieves on T

$$(S_{V,\alpha} \times_{V,g} T) \cap (S_{V',\alpha'} \times_{V',g'} T)$$

agree. Namely, these restrictions both correspond to the restriction of φ to T (via f) and the desired equality follows because $\mathcal{F}$ is separated. Denote the common restriction ψ. The independence of choice follows because $\varphi_{V,\alpha}(g) = \psi(\mathrm{id}_T) = \varphi_{V',\alpha'}(g')$. OK, so now φ'' gives an element $s \in L\mathcal{F}(U)$. We leave it to the reader to check that s restricts to φ. □

00ZL **Definition 7.49.4.** Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := LL\mathcal{F}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

00ZM **Proposition 7.49.5.** *Let $\mathcal{C}$ be a category endowed with a topology. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. Same as the proof of Proposition 7.10.12. □

7.50. Topologies and sheaves

00ZN

00ZO **Lemma 7.50.1.** *Let $\mathcal{C}$ be a category endowed with a topology J. Let U be an object of $\mathcal{C}$. Let S be a sieve on U. The following are equivalent*

(1) *The sieve S is a covering sieve.*
(2) *The sheafification $S^\# \to h_U^\#$ of the map $S \to h_U$ is an isomorphism.*

Proof. First we make a couple of general remarks. We will use that $S^\# = LLS$, and $h_U^\# = LLh_U$. In particular, by Lemma 7.49.1, we see that $S^\# \to h_U^\#$ is injective. Note that $\mathrm{id}_U \in h_U(U)$. Hence it gives rise to sections of Lh_U and $h_U^\# = LLh_U$ over U which we will also denote id_U.

Suppose S is a covering sieve. It clearly suffices to find a morphism $h_U \to S^\#$ such that the composition $h_U \to h_U^\#$ is the canonical map. To find such a map it suffices to find a section $s \in S^\#(U)$ which restricts to id_U. But since S is a covering sieve, the element $\mathrm{id}_S \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, S)$ gives rise to a section of LS over U which restricts to id_U in Lh_U. Hence we win.

Suppose that $S^\# \to h_U^\#$ is an isomorphism. Let $1 \in S^\#(U)$ be the element corresponding to id_U in $h_U^\#(U)$. Because $S^\# = LLS$ there exists a covering sieve S' on U such that 1 comes from a

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', LS).$$

This in turn means that for every $\alpha : V \to U$, $\alpha \in S'(V)$ there exists a covering sieve $S_{V,\alpha}$ on V such that $\varphi(\alpha)$ corresponds to a morphism of presheaves $S_{V,\alpha} \to S$. In other words $S_{V,\alpha}$ is contained in $S \times_U V$. By the second axiom of a topology we see that S is a covering sieve. □

00ZP **Theorem 7.50.2.** *Let $\mathcal{C}$ be a category. Let J, J' be topologies on $\mathcal{C}$. The following are equivalent*

(1) *$J = J'$,*
(2) *sheaves for the topology J are the same as sheaves for the topology J'.*

Proof. It is a tautology that if $J = J'$ then the notions of sheaves are the same. Conversely, Lemma 7.50.1 characterizes covering sieves in terms of the sheafification functor. But the sheafification functor $PSh(\mathcal{C}) \to Sh(\mathcal{C}, J)$ is the left adjoint of the inclusion functor $Sh(\mathcal{C}, J) \to PSh(\mathcal{C})$. Hence if the subcategories $Sh(\mathcal{C}, J)$ and $Sh(\mathcal{C}, J')$ are the same, then the sheafification functors are the same and hence the collections of covering sieves are the same. □

00ZQ **Lemma 7.50.3.** *Assumption and notation as in Theorem 7.50.2. Then $J \subset J'$ if and only if every sheaf for the topology J' is a sheaf for the topology J.*

Proof. One direction is clear. For the other direction suppose that $Sh(\mathcal{C}, J') \subset Sh(\mathcal{C}, J)$. By formal nonsense this implies that if $\mathcal{F}$ is a presheaf of sets, and $\mathcal{F} \to \mathcal{F}^\#$, resp. $\mathcal{F} \to \mathcal{F}^{\#,\prime}$ is the sheafification wrt J, resp. J' then there is a canonical map $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ equals the canonical map $\mathcal{F} \to \mathcal{F}^{\#,\prime}$. Of course, $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ identifies the second sheaf as the sheafification of the first with respect to the topology J'. Apply this to the map $S \to h_U$ of

Lemma 7.50.1. We get a commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & S^\# & \longrightarrow & S^{\#,\prime} \\ \downarrow & & \downarrow & & \downarrow \\ h_U & \longrightarrow & h_U^\# & \longrightarrow & h_U^{\#,\prime} \end{array}$$

And clearly, if S is a covering sieve for the topology J then the middle vertical map is an isomorphism (by the lemma) and we conclude that the right vertical map is an isomorphism as it is the sheafification of the one in the middle wrt J'. By the lemma again we conclude that S is a covering sieve for J' as well. □

7.51. Topologies and continuous functors

00ZR Explain how a continuous functor gives an adjoint pair of functors on sheaves.

7.52. Points and topologies

00ZS Recall from Section 7.32 that given a functor $p = u : \mathcal{C} \to Sets$ we can define a stalk functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \mathcal{F} \longmapsto \mathcal{F}_p.$$

00ZT **Definition 7.52.1.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. A *point* p of the topology is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering sieve S on U the map $S_p \to (h_U)_p$ is surjective.
(2) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ is exact.

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CHAPTER 8

Stacks

0266 ## 8.1. Introduction

0267 In this very short chapter we introduce stacks, and stacks in groupoids. See [**DM69**], and [**Vis04**].

8.2. Presheaves of morphisms associated to fibred categories

02Z9 Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Suppose that $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U. We are going to define a functor

$$Mor(x, y) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

In other words this will be a presheaf on $\mathcal{C}/U$, see Sites, Definition 7.2.2. Make a choice of pullbacks as in Categories, Definition 4.33.6. Then, for $f : V \to U$ we set

$$Mor(x, y)(f : V \to U) = \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

Let $f' : V' \to U$ be a second object of $\mathcal{C}/U$. We also have to define the restriction map corresponding to a morphism $g : V'/U \to V/U$ in $\mathcal{C}/U$, in other words $g : V' \to V$ and $f' = f \circ g$. This will be a map

$$\mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Mor}_{\mathcal{S}_{V'}}(f'^*x, f'^*y), \quad \phi \longmapsto \phi|_{V'}$$

This map will basically be g^*, except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\mathrm{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation $\alpha_{g,f}$ of Categories, Lemma 4.33.7. In a formula, the restriction map is described by

$$\phi|_{V'} = (\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x.$$

Of course, nobody thinks of this restriction map in this way. We will only do this once in order to verify the following lemma.

026A **Lemma 8.2.1.** *This actually does give a presheaf.*

Proof. Let $g : V'/U \to V/U$ be as above and similarly $g' : V''/U \to V'/U$ be morphisms in $\mathcal{C}/U$. So $f' = f \circ g$ and $f'' = f' \circ g' = f \circ g \circ g'$. Let $\phi \in \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$. Then we have

$$\begin{aligned}
& (\alpha_{g\circ g',f})_y^{-1} \circ (g \circ g')^*\phi \circ (\alpha_{g\circ g',f})_x \\
= \; & (\alpha_{g\circ g',f})_y^{-1} \circ (\alpha_{g',g})_{f^*y}^{-1} \circ (g')^*g^*\phi \circ (\alpha_{g',g})_{f^*x} \circ (\alpha_{g\circ g',f})_x \\
= \; & (\alpha_{g',f'})_y^{-1} \circ (g')^*(\alpha_{g,f})_y^{-1} \circ (g')^*g^*\phi \circ (g')^*(\alpha_{g,f})_x \circ (\alpha_{g',f'})_x \\
= \; & (\alpha_{g',f'})_y^{-1} \circ (g')^*\Big((\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x\Big) \circ (\alpha_{g',f'})_x
\end{aligned}$$

which is what we want, namely $\phi|_{V''} = (\phi|_{V'})|_{V''}$. The first equality holds because $\alpha_{g',g}$ is a transformation of functors, and hence

$$\begin{array}{ccc} (g \circ g')^* f^* x & \xrightarrow[(g\circ g')^*\phi]{} & (g \circ g')^* f^* y \\ {\scriptstyle (\alpha_{g',g})_{f^*x}}\downarrow & & \downarrow{\scriptstyle (\alpha_{g',g})_{f^*y}} \\ (g')^* g^* f^* x & \xrightarrow{(g')^* g^* \phi} & (g')^* g^* f^* y \end{array}$$

commutes. The second equality holds because of property (d) of a pseudo functor since $f' = f \circ g$ (see Categories, Definition 4.29.5). The last equality follows from the fact that $(g')^*$ is a functor. □

From now on we often omit mentioning the transformations $\alpha_{g,f}$ and we simply identify the functors $g^* \circ f^*$ and $(f \circ g)^*$. In particular, given $g : V'/U \to V/U$ the restriction mappings for the presheaf $Mor(x, y)$ will sometimes be denoted $\phi \mapsto g^*\phi$. We formalize the construction in a definition.

02ZB **Definition 8.2.2.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Given an object U of $\mathcal{C}$ and objects x, y of the fibre category, the *presheaf of morphisms from x to y* is the presheaf

$$(f : V \to U) \longmapsto \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$$

described above. It is denoted $Mor(x, y)$. The subpresheaf $Isom(x, y)$ whose value over V is the set of isomorphisms $f^*x \to f^*y$ in the fibre category $\mathcal{S}_V$ is called the *presheaf of isomorphisms from x to y*.

If $\mathcal{S}$ is fibred in groupoids then of course $Isom(x, y) = Mor(x, y)$, and it is customary to use the *Isom* notation.

042V **Lemma 8.2.3.** *Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism of fibred categories over the category $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}((\mathcal{S}_1)_U)$. Then F defines a canonical morphism of presheaves*

$$Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$

on $\mathcal{C}/U$.

Proof. By Categories, Definition 4.33.9 the functor F maps strongly cartesian morphisms to strongly cartesian morphisms. Hence if $f : V \to U$ is a morphism in $\mathcal{C}$, then there are canonical isomorphisms $\alpha_V : f^*F(x) \to F(f^*x)$, $\beta_V : f^*F(y) \to F(f^*y)$ such that $f^*F(x) \to F(f^*x) \to F(x)$ is the canonical morphism $f^*F(x) \to F(x)$, and similarly for β_V. Thus we may define

$$\begin{array}{ccc} Mor_{\mathcal{S}_1}(x, y)(f : V \to U) & = & \mathrm{Mor}_{\mathcal{S}_{1,V}}(f^*x, f^*y) \\ & & \downarrow \\ Mor_{\mathcal{S}_2}(F(x), F(y))(f : V \to U) & = & \mathrm{Mor}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y)) \end{array}$$

by $\phi \mapsto \beta_V^{-1} \circ F(\phi) \circ \alpha_V$. We omit the verification that this is compatible with the restriction mappings. □

02ZA **Remark 8.2.4.** Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. In this case we can prove Lemma 8.2.1 using Categories, Lemma 4.36.4 which says that $\mathcal{S} \to \mathcal{C}$ is equivalent to the category associated to a contravariant functor $F : \mathcal{C} \to \mathit{Groupoids}$.

In the case of the fibred category associated to F we have $g^* \circ f^* = (f \circ g)^*$ on the nose and there is no need to use the maps $\alpha_{g,f}$. In this case the lemma is (even more) trivial. Of course then one uses that the $Mor(x, y)$ presheaf is unchanged when passing to an equivalent fibred category which follows from Lemma 8.2.3.

04SI **Lemma 8.2.5.** *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Let $U \in \mathrm{Ob}(\mathcal{C})$ and let $x, y \in \mathrm{Ob}(\mathcal{S}_U)$. Denote $x, y : \mathcal{C}/U \to \mathcal{S}$ also the corresponding 1-morphisms, see Categories, Lemma 4.41.2. Then*

(1) *the 2-fibre product $\mathcal{S} \times_{\mathcal{S} \times \mathcal{S}, (x,y)} \mathcal{C}/U$ is fibred in setoids over $\mathcal{C}/U$, and*
(2) *$Isom(x, y)$ is the presheaf of sets corresponding to this category fibred in setoids, see Categories, Lemma 4.39.6.*

Proof. Omitted. Hint: Objects of the 2-fibre product are $(a : V \to U, z, (\alpha, \beta))$ where $\alpha : z \to a^*x$ and $\beta : z \to a^*y$ are isomorphisms in $\mathcal{S}_V$. Thus the relationship with $Isom(x, y)$ comes by assigning to such an object the isomorphism $\beta \circ \alpha^{-1}$. □

8.3. Descent data in fibred categories

02ZC In this section we define the notion of a descent datum in the abstract setting of a fibred category. Before we do so we point out that this is completely analogous to descent data for quasi-coherent sheaves (Descent, Section 35.2) and descent data for schemes over schemes (Descent, Section 35.34).

We will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

026B **Definition 8.3.1.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$. Assume all the fibre products $U_i \times_U U_j$, and $U_i \times_U U_j \times_U U_k$ exist.

(1) A *descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$* is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$, an isomorphism $\varphi_{ij} : \mathrm{pr}_0^* X_i \to \mathrm{pr}_1^* X_j$ in $\mathcal{S}_{U_i \times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\mathrm{pr}_{02}^* \varphi_{ik}} & \mathrm{pr}_2^* X_k \\ & {\scriptstyle \mathrm{pr}_{01}^* \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} & \\ & \mathrm{pr}_1^* X_j & \end{array}$$

in the category $\mathcal{S}_{U_i \times_U U_j \times_U U_k}$ commutes. This is called the *cocycle condition*.

(2) A *morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : X_i \to X'_i$ in $\mathcal{S}_{U_i}$ such that all the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^* X_j \\ {\scriptstyle \mathrm{pr}_0^* \psi_i} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1^* \psi_j} \\ \mathrm{pr}_0^* X'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^* X'_j \end{array}$$

in the categories $\mathcal{S}_{U_i \times_U U_j}$ commute.

(3) The category of descent data relative to $\mathcal{U}$ is denoted $DD(\mathcal{U})$.

The fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist if each of the morphisms $f_i : U_i \to U$ is *representable*, see Categories, Definition 4.6.4. Recall that in a site one of the conditions for a covering $\{U_i \to U\}$ is that each of the morphisms is representable, see Sites, Definition 7.6.2 part (3). In fact the main interest in the definition above is where $\mathcal{C}$ is a site and $\{U_i \to U\}$ is a covering of $\mathcal{C}$. However, a descent datum is just an abstract gadget that can be defined as above. This is useful: for example, given a fibred category over $\mathcal{C}$ one can look at the collection of families with respect to which descent data are effective, and try to use these as the family of coverings for a site.

026C **Remarks 8.3.2.** Two remarks on Definition 8.3.1 are in order. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\{f_i : U_i \to U\}_{i \in I}$, and (X_i, φ_{ij}) be as in Definition 8.3.1.

(1) There is a diagonal morphism $\Delta : U_i \to U_i \times_U U_i$. We can pull back φ_{ii} via this morphism to get an automorphism $\Delta^*\varphi_{ii} \in \mathrm{Aut}_{U_i}(X_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \to U_i \times_U U_i \times_U U_i$ we deduce that $\Delta^*\varphi_{ii} \circ \Delta^*\varphi_{ii} = \Delta^*\varphi_{ii}$; thus $\Delta^*\varphi_{ii} = \mathrm{id}_{X_i}$.
(2) There is a morphism $\Delta_{13} : U_i \times_U U_j \to U_i \times_U U_j \times_U U_i$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^*\varphi_{ji}) \circ \varphi_{ij} = \mathrm{id}_{\mathrm{pr}_0^* X_i}$, where $\sigma : U_i \times_U U_j \to U_j \times_U U_i$ is the switching morphism.

02ZD **Lemma 8.3.3.** *(Pullback of descent data.) Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be a families of morphisms of $\mathcal{C}$ with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$, and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha : I \to J$, $h : U \to V$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.*

(1) *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to V\}$. The system*
$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum relative to $\mathcal{U}$.
(2) *This construction defines a functor between descent data relative to $\mathcal{V}$ and descent data relative to $\mathcal{U}$.*
(3) *Given a second $\alpha' : I \to J$, $h' : U \to V$ and $g_i' : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*

Proof. Omitted. □

02ZE **Definition 8.3.4.** With $\mathcal{U} = \{U_i \to U\}_{i \in I}$, $\mathcal{V} = \{V_j \to V\}_{j \in J}$, $\alpha : I \to J$, $h : U \to V$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 8.3.3 the functor
$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$
constructed in that lemma is called the *pullback functor* on descent data.

Given $h : U \to V$, if there exists a morphism $\tilde{h} : \mathcal{U} \to \mathcal{V}$ covering h then $\tilde{h}^*$ is independent of the choice of $\tilde{h}$ as we saw in Lemma 8.3.3. Hence we will sometimes simply write h^* to indicate the pullback functor.

026E **Definition 8.3.5.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with target U. Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

(1) Given an object X of $\mathcal{S}_U$ the *trivial descent datum* is the descent datum (X, id_X) with respect to the family $\{\mathrm{id}_U : U \to U\}$.
(2) Given an object X of $\mathcal{S}_U$ we have a *canonical descent datum* on the family of objects f_i^*X by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i : U_i \to U\} \to \{\mathrm{id}_U : U \to U\}$. We denote this descent datum (f_i^*X, can).
(3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \to U\}$ is called *effective* if there exists an object X of $\mathcal{S}_U$ such that (X_i, φ_{ij}) is isomorphic to (f_i^*X, can).

Note that the rule that associates to $X \in \mathcal{S}_U$ its canonical descent datum relative to $\mathcal{U}$ defines a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U}).$$

A descent datum is effective if and only if it is in the essential image of this functor. Let us make explicit the canonical descent datum as follows.

026D **Lemma 8.3.6.** *In the situation of Definition 8.3.5 part (2) the maps $can_{ij} : pr_0^*f_i^*X \to pr_1^*f_j^*X$ are equal to $(\alpha_{pr_1,f_j})_X \circ (\alpha_{pr_0,f_i})_X^{-1}$ where $\alpha_{.,.}$ is as in Categories, Lemma 4.33.7 and where we use the equality $f_i \circ pr_0 = f_j \circ pr_1$ as maps $U_i \times_U U_j \to U$.*

Proof. Omitted. □

0GEA **Lemma 8.3.7.** *Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $id : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. If*

(1) *for $0 \leq p \leq 3$ and $0 \leq q \leq 3$ with $p + q \geq 2$ and $i_1, \ldots, i_p \in I$ and $j_1, \ldots, j_q \in J$ the fibre products $U_{i_1} \times_U \ldots \times_U U_{i_p} \times_U V_{j_1} \times_U \ldots \times_U V_{j_q}$ exist,*
(2) *the functor $\mathcal{S}_U \to DD(\mathcal{V})$ is an equivalence,*
(3) *for every $i \in I$ the functor $\mathcal{S}_{U_i} \to DD(\mathcal{V}_i)$ is fully faithful, and*
(4) *for every $i, i' \in I$ the functor $\mathcal{S}_{U_i \times_U U_{i'}} \to DD(\mathcal{V}_{ii'})$ is faithful.*

Here $\mathcal{V}_i = \{U_i \times_U V_j \to U_i\}_{j \in J}$ and $\mathcal{V}_{ii'} = \{U_i \times_U U_{i'} \times_U V_j \to U_i \times_U U_{i'}\}_{j \in J}$. Then $\mathcal{S}_U \to DD(\mathcal{U})$ is an equivalence.

Proof. Condition (1) guarantees we have enough fibre products so that the statement makes sense. We will show that the functor $\mathcal{S}_U \to DD(\mathcal{U})$ is essentially surjective. Suppose given a descent datum $(X_i, \varphi_{ii'})$ relative to $\mathcal{U}$. By Lemma 8.3.3 we can pull this back to a descent datum $(X_j, \varphi_{jj'})$ for $\mathcal{V}$. By assumption (2) this descent datum is effective, hence we get an object X of $\mathcal{S}_U$ such that the pullback of the trivial descent datum (X, id_X) by the morphism $\mathcal{V} \to \{U \to U\}$ is isomorphic to $(X_j, \varphi_{jj'})$. Next, observe that we have a diagram

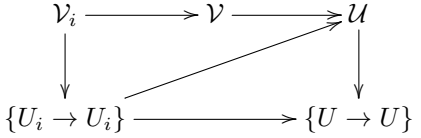

of morphisms of families of maps with fixed target of $\mathcal{C}$. This diagram does not commute, but by Lemma 8.3.3 the pullback functors on descent data one gets are canonically isomorphic. Hence (X, id_X) and (X_i, id_{X_i}) pull back to isomorphic objects in $DD(\mathcal{V}_i)$. Hence by assumption (3) we obtain an isomorphism $(U_i \to U)^*X \to X_i$ in the category $\mathcal{S}_{U_i}$. We omit the verification that these arrows are compatible with the morphisms $\varphi_{ii'}$; hint: use the faithfulness of the functors in condition (4). We also omit the verification that the functor $\mathcal{S}_U \to DD(\mathcal{U})$ is fully faithful. □

8.4. Stacks

0268 Here is the definition of a stack. It mixes the notion of a fibred category with the notion of descent.

026F **Definition 8.4.1.** Let $\mathcal{C}$ be a site. A *stack* over $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category, see Categories, Definition 4.33.5,
(2) for any $U \in \mathrm{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$ the presheaf $Mor(x, y)$ (see Definition 8.2.2) is a sheaf on the site $\mathcal{C}/U$, and
(3) for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

We find the formulation above the most convenient way to think about a stack. Namely, given a category over $\mathcal{C}$ in order to verify that it is a stack you proceed to check properties (1), (2) and (3) in that order. Certainly properties (2) and (3) do not make sense if the category isn't fibred. Without (2) we cannot prove that the descent in (3) is unique up to unique isomorphism and functorial.

The following lemma provides an alternative definition.

02ZF **Lemma 8.4.2.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{S}$ is a stack over $\mathcal{C}$, and*
(2) *for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the functor*
$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$
which associates to an object its canonical descent datum is an equivalence.

Proof. Omitted. □

04TU **Lemma 8.4.3.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a stack over the site $\mathcal{C}$. Let $\mathcal{S}'$ be a subcategory of $\mathcal{S}$. Assume*

(1) *if $\varphi : y \to x$ is a strongly cartesian morphism of $\mathcal{S}$ and x is an object of $\mathcal{S}'$, then y is isomorphic to an object of $\mathcal{S}'$,*
(2) *$\mathcal{S}'$ is a full subcategory of $\mathcal{S}$, and*
(3) *if $\{f_i : U_i \to U\}$ is a covering of $\mathcal{C}$, and x an object of $\mathcal{S}$ over U such that f_i^*x is isomorphic to an object of $\mathcal{S}'$ for each i, then x is isomorphic to an object of $\mathcal{S}'$.*

Then $\mathcal{S}' \to \mathcal{C}$ is a stack.

Proof. Omitted. Hints: The first condition guarantees that $\mathcal{S}'$ is a fibred category. The second condition guarantees that the *Isom*-presheaves of $\mathcal{S}'$ are sheaves (as they are identical to their counter parts in $\mathcal{S}$). The third condition guarantees that the

descent condition holds in $\mathcal{S}'$ as we can first descend in $\mathcal{S}$ and then (3) implies the resulting object is isomorphic to an object of $\mathcal{S}'$. □

042W **Lemma 8.4.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack over $\mathcal{C}$.*

Proof. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. By Categories, Lemma 4.33.8 we see that $\mathcal{S}_1$ is fibred if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$. Hence we may assume that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fibred. Moreover, the proof of Categories, Lemma 4.33.8 shows that F and G map strongly cartesian morphisms to strongly cartesian morphisms, i.e., F and G are 1-morphisms of fibred categories over $\mathcal{C}$. This means that given $U \in \mathrm{Ob}(\mathcal{C})$, and $x, y \in \mathcal{S}_{1,U}$ then the presheaves

$$Mor_{\mathcal{S}_1}(x, y), Mor_{\mathcal{S}_1}(F(x), F(y)) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

are identified, see Lemma 8.2.3. Hence the first is a sheaf if and only if the second is a sheaf. Finally, we have to show that if every descent datum in $\mathcal{S}_1$ is effective, then so is every descent datum in $\mathcal{S}_2$. To do this, let $(X_i, \varphi_{ii'})$ be a descent datum in $\mathcal{S}_2$ relative the covering $\{U_i \to U\}$ of the site $\mathcal{C}$. Then $(G(X_i), G(\varphi_{ii'}))$ is a descent datum in $\mathcal{S}_1$ relative the covering $\{U_i \to U\}$. Let X be an object of $\mathcal{S}_{1,U}$ such that the descent datum (f_i^*X, can) is isomorphic to $(G(X_i), G(\varphi_{ii'}))$. Then $F(X)$ is an object of $\mathcal{S}_{2,U}$ such that the descent datum $(f_i^*F(X), can)$ is isomorphic to $(F(G(X_i)), F(G(\varphi_{ii'})))$ which in turn is isomorphic to the original descent datum $(X_i, \varphi_{ii'})$ using i. □

The 2-category of stacks over $\mathcal{C}$ is defined as follows.

02ZG **Definition 8.4.5.** Let $\mathcal{C}$ be a site. The 2-*category of stacks over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.33.9) defined as follows:

(1) Its objects will be stacks $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

026G **Lemma 8.4.6.** *Let $\mathcal{C}$ be a site. The $(2,1)$-category of stacks over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.*

Proof. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of stacks over $\mathcal{C}$ as defined above. The category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Categories, Lemma 4.32.3 is a fibred category according to Categories, Lemma 4.33.10. (This is where we use that f and g preserve strongly cartesian morphisms.) It remains to show that the morphism presheaves are sheaves and that descent relative to coverings of $\mathcal{C}$ is effective.

Recall that an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by a quadruple (U, x, y, ϕ). It lies over the object U of $\mathcal{C}$. Next, let (U, x', y', ϕ') be second object lying over U. Recall that $\phi : f(x) \to g(y)$, and $\phi' : f(x') \to g(y')$ are isomorphisms in the category $\mathcal{S}_U$.

Let us use these isomorphisms to identify $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. With this identifications it is clear that

$$Mor((U, x, y, \phi), (U, x', y', \phi')) = Mor(x, x') \times_{Mor(z,z')} Mor(y, y')$$

as presheaves. However, as the fibred product in the category of presheaves preserves sheaves (Sites, Lemma 7.10.1) we see that this is a sheaf.

Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a covering of the site $\mathcal{C}$. Let (X_i, χ_{ij}) be a descent datum in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ relative to $\mathcal{U}$. Write $X_i = (U_i, x_i, y_i, \phi_i)$ as above. Write $\chi_{ij} = (\varphi_{ij}, \psi_{ij})$ as in the definition of the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ (see Categories, Lemma 4.32.3). It is clear that (x_i, φ_{ij}) is a descent datum in $\mathcal{X}$ and that (y_i, ψ_{ij}) is a descent datum in $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are stacks these descent data are effective. Thus we get $x \in \mathrm{Ob}(\mathcal{X}_U)$, and $y \in \mathrm{Ob}(\mathcal{Y}_U)$ with $x_i = x|_{U_i}$, and $y_i = y|_{U_i}$ compatibly with descent data. Set $z = f(x)$ and $z' = g(y)$ which are both objects of $\mathcal{S}_U$. The morphisms ϕ_i are elements of $Isom(z, z')(U_i)$ with the property that $\phi_i|_{U_i \times_U U_j} = \phi_j|_{U_i \times_U U_j}$. Hence by the sheaf property of $Isom(z, z')$ we obtain an isomorphism $\phi : z = f(x) \to z' = g(y)$. We omit the verification that the canonical descent datum associated to the object (U, x, y, ϕ) of $(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U$ is isomorphic to the descent datum we started with. □

04WQ **Lemma 8.4.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism. Then the following are equivalent*

(1) *F is fully faithful,*
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every $x, y \in \mathrm{Ob}(\mathcal{S}_{1,U})$ the map*

$$F : Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$

is an isomorphism of sheaves on $\mathcal{C}/U$.

Proof. Assume (1). For U, x, y as in (2) the displayed map F evaluates to the map $F : \mathrm{Mor}_{\mathcal{S}_{1,V}}(x|_V, y|_V) \to \mathrm{Mor}_{\mathcal{S}_{2,V}}(F(x|_V), F(y|_V))$ on an object V of $\mathcal{C}$ lying over U. Now, since F is fully faithful, the corresponding map $\mathrm{Mor}_{\mathcal{S}_1}(x|_V, y|_V) \to \mathrm{Mor}_{\mathcal{S}_2}(F(x|_V), F(y|_V))$ is a bijection. Morphisms in the fibre category $\mathcal{S}_{1,V}$ are exactly those morphisms between $x|_V$ and $y|_V$ in $\mathcal{S}_1$ lying over id_V. Similarly, morphisms in the fibre category $\mathcal{S}_{2,V}$ are exactly those morphisms between $F(x|_V)$ and $F(y|_V)$ in $\mathcal{S}_2$ lying over id_V. Thus we find that F induces a bijection between these also. Hence (2) holds.

Assume (2). Suppose given objects U, V of $\mathcal{C}$ and $x \in \mathrm{Ob}(\mathcal{S}_{1,U})$ and $y \in \mathrm{Ob}(\mathcal{S}_{1,V})$. To show that F is fully faithful, it suffices to prove it induces a bijection on morphisms lying over a fixed $f : U \to V$. Choose a strongly Cartesian $f^*y \to y$ in $\mathcal{S}_1$ lying above f. This results in a bijection between the set of morphisms $x \to y$ in $\mathcal{S}_1$ lying over f and $\mathrm{Mor}_{\mathcal{S}_{1,U}}(x, f^*y)$. Since F preserves strongly Cartesian morphisms as a 1-morphism in the 2-category of stacks over $\mathcal{C}$, we also get a bijection between the set of morphisms $F(x) \to F(y)$ in $\mathcal{S}_2$ lying over f and $\mathrm{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$. Since F induces a bijection $\mathrm{Mor}_{\mathcal{S}_{1,U}}(x, f^*y) \to \mathrm{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$ we conclude (1) holds. □

046N **Lemma 8.4.8.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism which is fully faithful. Then the following are equivalent*

(1) *F is an equivalence,*

(2) *for every* $U \in \mathrm{Ob}(\mathcal{C})$ *and for every* $x \in \mathrm{Ob}(\mathcal{S}_{2,U})$ *there exists a covering* $\{f_i : U_i \to U\}$ *such that* $f_i^* x$ *is in the essential image of the functor* $F : \mathcal{S}_{1,U_i} \to \mathcal{S}_{2,U_i}$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. To see that (2) implies (1) we have to show that every x as in (2) is in the essential image of the functor F. To do this choose a covering as in (2), $x_i \in \mathrm{Ob}(\mathcal{S}_{1,U_i})$, and isomorphisms $\varphi_i : F(x_i) \to f_i^* x$. Then we get a descent datum for $\mathcal{S}_1$ relative to $\{f_i : U_i \to U\}$ by taking

$$\varphi_{ij} : x_i|_{U_i \times_U U_j} \longrightarrow x_j|_{U_i \times_U U_j}$$

the arrow such that $F(\varphi_{ij}) = \varphi_j^{-1} \circ \varphi_i$. This descent datum is effective by the axioms of a stack, and hence we obtain an object x_1 of $\mathcal{S}_1$ over U. We omit the verification that $F(x_1)$ is isomorphic to x over U. □

03ZZ **Remark 8.4.9.** (Cutting down a "big" stack to get a stack.) Let $\mathcal{C}$ be a site. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is functor from a "big" category to $\mathcal{C}$, i.e., suppose that the collection of objects of $\mathcal{S}$ forms a proper class. Finally, suppose that $p : \mathcal{S} \to \mathcal{C}$ satisfies conditions (1), (2), (3) of Definition 8.4.1. In general there is no way to replace $p : \mathcal{S} \to \mathcal{C}$ by a equivalent category such that we obtain a stack. The reason is that it can happen that a fibre categories $\mathcal{S}_U$ may have a proper class of isomorphism classes of objects. On the other hand, suppose that

(4) for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a set $S_U \subset \mathrm{Ob}(\mathcal{S}_U)$ such that every object of $\mathcal{S}_U$ is isomorphic in $\mathcal{S}_U$ to an element of S_U.

In this case we can find a full subcategory $\mathcal{S}_{small}$ of $\mathcal{S}$ such that, setting $p_{small} = p|_{\mathcal{S}_{small}}$, we have

(a) the functor $p_{small} : \mathcal{S}_{small} \to \mathcal{C}$ defines a stack, and
(b) the inclusion $\mathcal{S}_{small} \to \mathcal{S}$ is fully faithful and essentially surjective.

(Hint: For every $U \in \mathrm{Ob}(\mathcal{C})$ let $\alpha(U)$ denote the smallest ordinal such that $\mathrm{Ob}(\mathcal{S}_U) \cap V_{\alpha(U)}$ surjects onto the set of isomorphism classes of $\mathcal{S}_U$, and set $\alpha = \sup_{U \in \mathrm{Ob}(\mathcal{C})} \alpha(U)$. Then take $\mathrm{Ob}(\mathcal{S}_{small}) = \mathrm{Ob}(\mathcal{S}) \cap V_\alpha$. For notation used see Sets, Section 3.5.)

8.5. Stacks in groupoids

02ZH Among stacks those which are fibred in groupoids are somewhat easier to comprehend. We redefine them as follows.

02ZI **Definition 8.5.1.** A *stack in groupoids* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.35.1),
(2) for all $U \in \mathrm{Ob}(\mathcal{C})$, for all $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

Usually the hardest part to check is the third condition. Here is the lemma comparing this with the notion of a stack.

02ZJ **Lemma 8.5.2.** *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a category over* $\mathcal{C}$. *The following are equivalent*

(1) $\mathcal{S}$ *is a stack in groupoids over* $\mathcal{C}$,

(2) $\mathcal{S}$ *is a stack over* $\mathcal{C}$ *and all fibre categories are groupoids, and*
(3) $\mathcal{S}$ *is fibred in groupoids over* $\mathcal{C}$ *and is a stack over* $\mathcal{C}$*.*

Proof. Omitted, but see Categories, Lemma 4.35.2. □

03YI **Lemma 8.5.3.** *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a stack. Let* $p' : \mathcal{S}' \to \mathcal{C}$ *be the category fibred in groupoids associated to* $\mathcal{S}$ *constructed in Categories, Lemma 4.35.3. Then* $p' : \mathcal{S}' \to \mathcal{C}$ *is a stack in groupoids.*

Proof. Recall that the morphisms in $\mathcal{S}'$ are exactly the strongly cartesian morphisms of $\mathcal{S}$, and that any isomorphism of $\mathcal{S}$ is such a morphism. Hence descent data in $\mathcal{S}'$ are exactly the same thing as descent data in $\mathcal{S}$. Now apply Lemma 8.4.2. Some details omitted. □

042X **Lemma 8.5.4.** *Let* $\mathcal{C}$ *be a site. Let* $\mathcal{S}_1$*,* $\mathcal{S}_2$ *be categories over* $\mathcal{C}$*. Suppose that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are equivalent as categories over* $\mathcal{C}$*. Then* $\mathcal{S}_1$ *is a stack in groupoids over* $\mathcal{C}$ *if and only if* $\mathcal{S}_2$ *is a stack in groupoids over* $\mathcal{C}$*.*

Proof. Follows by combining Lemmas 8.5.2 and 8.4.4. □

The 2-category of stacks in groupoids over $\mathcal{C}$ is defined as follows.

02ZK **Definition 8.5.5.** Let $\mathcal{C}$ be a site. The 2*-category of stacks in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in groupoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in groupoids over $\mathcal{C}$ is a (strict) $(2, 1)$-category.

02ZL **Lemma 8.5.6.** *Let* $\mathcal{C}$ *be a category. The* 2*-category of stacks in groupoids over* $\mathcal{C}$ *has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.*

Proof. This is clear from Categories, Lemma 4.35.7 and Lemmas 8.5.2 and 8.4.6. □

8.6. Stacks in setoids

042Y This is just a brief section saying that a stack in sets is the same thing as a sheaf of sets. Please consult Categories, Section 4.39 for notation.

042Z **Definition 8.6.1.** Let $\mathcal{C}$ be a site.

(1) A *stack in setoids* over $\mathcal{C}$ is a stack over $\mathcal{C}$ all of whose fibre categories are setoids.
(2) A *stack in sets*, or a *stack in discrete categories* is a stack over $\mathcal{C}$ all of whose fibre categories are discrete.

From the discussion in Section 8.5 this is the same thing as a stack in groupoids whose fibre categories are setoids (resp. discrete). Moreover, it is also the same thing as a category fibred in setoids (resp. sets) which is a stack.

0430 **Lemma 8.6.2.** *Let $\mathcal{C}$ be a site. Under the equivalence*

$$\left\{\begin{array}{c}\text{the category of presheaves}\\ \text{of sets over } \mathcal{C}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{the category of categories}\\ \text{fibred in sets over } \mathcal{C}\end{array}\right\}$$

of Categories, Lemma 4.38.6 the stacks in sets correspond precisely to the sheaves.

Proof. Omitted. Hint: Show that effectivity of descent corresponds exactly to the sheaf condition. □

0432 **Lemma 8.6.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a category fibred in setoids over $\mathcal{C}$. Then $\mathcal{S}$ is a stack in setoids if and only if the unique equivalent category $\mathcal{S}'$ fibred in sets (see Categories, Lemma 4.39.5) is a stack in sets. In other words, if and only if the presheaf*

$$U \longmapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$$

is a sheaf.

Proof. Omitted. □

0431 **Lemma 8.6.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in setoids over $\mathcal{C}$.*

Proof. By Categories, Lemma 4.39.5 we see that a category $\mathcal{S}$ over $\mathcal{C}$ is fibred in setoids over $\mathcal{C}$ if and only if it is equivalent over $\mathcal{C}$ to a category fibred in sets. Hence we see that $\mathcal{S}_1$ is fibred in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred in setoids over $\mathcal{C}$. Hence now the lemma follows from Lemma 8.6.3. □

The 2-category of stacks in setoids over $\mathcal{C}$ is defined as follows.

0433 **Definition 8.6.5.** Let $\mathcal{C}$ be a site. The 2-*category of stacks in setoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in setoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in setoids over $\mathcal{C}$ is a (strict) $(2,1)$-category.

0434 **Lemma 8.6.6.** *Let $\mathcal{C}$ be a site. The 2-category of stacks in setoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.*

Proof. This is clear from Categories, Lemmas 4.35.7 and 4.39.4 and Lemmas 8.5.2 and 8.4.6. □

05UI **Lemma 8.6.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}, \mathcal{T}$ be stacks in groupoids over $\mathcal{C}$ and let $\mathcal{R}$ be a stack in setoids over $\mathcal{C}$. Let $f : \mathcal{T} \to \mathcal{S}$ and $g : \mathcal{R} \to \mathcal{S}$ be 1-morphisms. If f is faithful, then the 2-fibre product*

$$\mathcal{T} \times_{f,\mathcal{S},g} \mathcal{R}$$

is a stack in setoids over $\mathcal{C}$.

Proof. Immediate from the explicit description of the 2-fibre product in Categories, Lemma 4.32.3. □

05UJ **Lemma 8.6.8.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a stack in groupoids over $\mathcal{C}$ and let $\mathcal{S}_i$, $i = 1, 2$ be stacks in setoids over $\mathcal{C}$. Let $f_i : \mathcal{S}_i \to \mathcal{S}$ be 1-morphisms. Then the 2-fibre product*

$$\mathcal{S}_1 \times_{f_1, \mathcal{S}, f_2} \mathcal{S}_2$$

is a stack in setoids over $\mathcal{C}$.

Proof. This is a special case of Lemma 8.6.7 as f_2 is faithful. □

06DV **Lemma 8.6.9.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. Assume

(1) *for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
(2) *G' is faithful,*

then G is faithful.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.32.3. By Categories, Lemma 4.35.9 the faithfulness of G, G' can be checked on fibre categories. Suppose that y, y' are objects of $\mathcal{T}_1$ over the object U of $\mathcal{C}$. Let $\alpha, \beta : y \to y'$ be morphisms of $(\mathcal{T}_1)_U$ such that $G(\alpha) = G(\beta)$. Our object is to show that $\alpha = \beta$. Considering instead $\gamma = \alpha^{-1} \circ \beta$ we see that $G(\gamma) = \mathrm{id}_{G(y)}$ and we have to show that $\gamma = \mathrm{id}_y$. By assumption we can find a covering $\{U_i \to U\}$ such that $G(y)|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$. Since it suffices to show that $\gamma|_{U_i} = \mathrm{id}$ for each i, we may therefore assume that we have $f : F(x) \to G(y)$ for some object x of $\mathcal{S}_2$ over U and morphisms f of $(\mathcal{S}_1)_U$. In this case we get a morphism

$$(1, \gamma) : (U, x, y, f) \longrightarrow (U, x, y, f)$$

in the fibre category of $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ over U whose image under G' in $\mathcal{S}_1$ is id_x. As G' is faithful we conclude that $\gamma = \mathrm{id}_y$ and we win. □

05W9 **Lemma 8.6.10.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ \downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If

(1) *$F : \mathcal{S}_2 \to \mathcal{S}_1$ is fully faithful,*
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
(3) *$\mathcal{T}_2$ is a stack in setoids.*

then $\mathcal{T}_1$ is a stack in setoids.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.32.3. Pick $U \in \mathrm{Ob}(\mathcal{C})$ and $y \in \mathrm{Ob}((\mathcal{T}_1)_U)$. We have to show that the sheaf $Aut(y)$ on $\mathcal{C}/U$ is trivial. To to this we may replace U by the members of a covering of U. Hence by assumption (2) we may assume that there exists an object $x \in \mathrm{Ob}((\mathcal{S}_2)_U)$ and an isomorphism $f : F(x) \to G(y)$. Then $y' = (U, x, y, f)$ is an object of $\mathcal{T}_2$ over U which is mapped to y under the projection $\mathcal{T}_2 \to \mathcal{T}_1$. Because F is fully faithful by (1) the map $Aut(y') \to Aut(y)$ is surjective, use the explicit description of morphisms in $\mathcal{T}_2$ in Categories, Lemma 4.32.3. Since by (3) the sheaf $Aut(y')$ is trivial we get the result of the lemma. □

0CKJ **Lemma 8.6.11.** *Let $\mathcal{C}$ be a site. Let $F : \mathcal{S} \to \mathcal{T}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume that*

(1) *$\mathcal{T}$ is a stack in groupoids over $\mathcal{C}$,*
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$ the functor $\mathcal{S}_U \to \mathcal{T}_U$ of fibre categories is faithful,*
(3) *for each U and each $y \in \mathrm{Ob}(\mathcal{T}_U)$ the presheaf*

$$(h : V \to U) \longmapsto \{(x, f) \mid x \in \mathrm{Ob}(\mathcal{S}_V), f : F(x) \to f^*y \text{ over } V\}/\cong$$

is a sheaf on $\mathcal{C}/U$.

Then $\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$.

Proof. We have to prove descent for morphisms and descent for objects.

Descent for morphisms. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let x, x' be objects of $\mathcal{S}$ over U. For each i let $\alpha_i : x|_{U_i} \to x'|_{U_i}$ be a morphism over U_i such that α_i and α_j restrict to the same morphism $x|_{U_i \times_U U_j} \to x'|_{U_i \times_U U_j}$. Because $\mathcal{T}$ is a stack in groupoids, there is a morphism $\beta : F(x) \to F(x')$ over U whose restriction to U_i is $F(\alpha_i)$. Then we can think of $\xi = (x, \beta)$ and $\xi' = (x', \mathrm{id}_{F(x')})$ as sections of the presheaf associated to $y = F(x')$ over U in assumption (3). On the other hand, the restrictions of ξ and ξ' to U_i are $(x|_{U_i}, F(\alpha_i))$ and $(x'|_{U_i}, \mathrm{id}_{F(x'|_{U_i})})$. These are isomorphic to each other by the morphism α_i. Thus ξ and ξ' are isomorphic by assumption (3). This means there is a morphism $\alpha : x \to x'$ over U with $F(\alpha) = \beta$. Since F is faithful on fibre categories we obtain $\alpha|_{U_i} = \alpha_i$.

Descent of objects. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let (x_i, φ_{ij}) be a descent datum for $\mathcal{S}$ with respect to the given covering. Because $\mathcal{T}$ is a stack in groupoids, there is an object y in $\mathcal{T}_U$ and isomorphisms $\beta_i : F(x_i) \to y|_{U_i}$ such that $F(\varphi_{ij}) = \beta_j|_{U_i \times_U U_j} \circ (\beta_i|_{U_i \times_U U_j})^{-1}$. Then (x_i, β_i) are sections of the presheaf associated to y over U defined in assumption (3). Moreover, φ_{ij} defines an isomorphism from the pair $(x_i, \beta_i)|_{U_i \times_U U_j}$ to the pair $(x_j, \beta_j)|_{U_i \times_U U_j}$. Hence by assumption (3) there exists a pair (x, β) over U whose restriction to U_i is isomorphic to (x_i, β_i). This means there are morphisms $\alpha_i : x_i \to x|_{U_i}$ with $\beta_i = \beta|_{U_i} \circ F(\alpha_i)$. Since F is faithful on fibre categories a calculation shows that $\varphi_{ij} = \alpha_j|_{U_i \times_U U_j} \circ (\alpha_i|_{U_i \times_U U_j})^{-1}$. This finishes the proof. □

8.7. The inertia stack

036X Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories over the category $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Recall that we have defined in Categories, Definition 4.34.2 a *relative inertia fibred category* $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ as the category whose objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ with

$F(\alpha) = \mathrm{id}_{F(x)}$. There is also an absolute version, namely the *inertia* $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$. These inertia categories are actually stacks over $\mathcal{C}$ provided that $\mathcal{S}$ and $\mathcal{S}'$ are stacks.

036Y **Lemma 8.7.1.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be stacks over the site $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of stacks over $\mathcal{C}$.*

(1) *The inertia $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$ are stacks over $\mathcal{C}$.*
(2) *If $\mathcal{S}, \mathcal{S}'$ are stacks in groupoids over $\mathcal{C}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*
(3) *If $\mathcal{S}, \mathcal{S}'$ are stacks in setoids over $\mathcal{C}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*

Proof. The first three assertions follow from Lemmas 8.4.6, 8.5.6, and 8.6.6 and the equivalence in Categories, Lemma 4.34.1 part (1). □

04ZM **Lemma 8.7.2.** *Let $\mathcal{C}$ be a site. If $\mathcal{S}$ is a stack in groupoids, then the canonical 1-morphism $\mathcal{I}_{\mathcal{S}} \to \mathcal{S}$ is an equivalence if and only if $\mathcal{S}$ is a stack in setoids.*

Proof. Follows directly from Categories, Lemma 4.39.7. □

8.8. Stackification of fibred categories

02ZM The result of the procedure in the following lemma will be called the *stackification* of a fibred category over a site.

02ZN **Lemma 8.8.1.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. There exists a stack $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.33.9) such that*

(1) *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the map*
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.*

Moreover the stack $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof by naive method. In this proof method we proceed in stages:

First, given x lying over U and any object y of $\mathcal{S}$, we say that two morphisms $a, b : x \to y$ of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ are *locally equal* if there exists a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}$ such that the compositions
$$f_i^* x \to x \xrightarrow{a} y, \quad f_i^* x \to x \xrightarrow{b} y$$
are equal. This gives an equivalence relation $\sim$ on arrows of $\mathcal{S}$. If $b \sim b'$ then $a \circ b \circ c \sim a \circ b' \circ c$ (verification omitted). Hence we can quotient out by this equivalence relation to obtain a new category $\mathcal{S}^1$ over $\mathcal{C}$ together with a morphism $G^1 : \mathcal{S} \to \mathcal{S}^1$.

One checks that G^1 preserves strongly cartesian morphisms and that $\mathcal{S}^1$ is a fibred category over $\mathcal{C}$. Checks omitted. Thus we reduce to the case where locally equal morphisms are equal.

Next, we add morphisms as follows. Given x lying over U and any object y of lying over V a *locally defined morphism from x to y* is given by

(1) a morphism $f : U \to V$,
(2) a covering $\{f_i : U_i \to U\}$ of U, and
(3) morphisms $a_i : f_i^*x \to y$ with $p(a_i) = f \circ f_i$

with the property that the compositions

$$(f_i \times f_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times f_j)^*x \to f_j^*x \xrightarrow{a_j} y$$

are equal. Note that a usual morphism $a : x \to y$ gives a locally defined morphism $(p(a) : U \to V, \{\mathrm{id}_U\}, a)$. We say two locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ and $(g, \{g_j : U'_j \to U\}, b_j)$ are *equal* if $f = g$ and the compositions

$$(f_i \times g_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times g_j)^*x \to g_j^*x \xrightarrow{b_j} y$$

are equal (this is the right condition since we are in the situation where locally equal morphisms are equal). To compose locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ from x to y and $(g, \{g_j : V_j \to V\}, b_j)$ from y to z lying over W, just take $g \circ f : U \to W$, the covering $\{U_i \times_V V_j \to U\}$, and as maps the compositions

$$x|_{U_i \times_V V_j} \xrightarrow{\mathrm{pr}_0^* a_i} y|_{V_j} \xrightarrow{b_j} z$$

We omit the verification that this is a locally defined morphism.

One checks that $\mathcal{S}^2$ with the same objects as $\mathcal{S}$ and with locally defined morphisms as morphisms is a category over $\mathcal{C}$, that there is a functor $G^2 : \mathcal{S} \to \mathcal{S}^2$ over $\mathcal{C}$, that this functor preserves strongly cartesian objects, and that $\mathcal{S}^2$ is a fibred category over $\mathcal{C}$. Checks omitted. This reduces one to the case where the morphism presheaves of $\mathcal{S}$ are all sheaves, by checking that the effect of using locally defined morphisms is to take the sheafification of the (separated) morphisms presheaves.

Finally, in the case where the morphism presheaves are all sheaves we have to add objects in order to make sure descent conditions are effective in the end result. The simplest way to do this is to consider the category $\mathcal{S}'$ whose objects are pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of $\mathcal{C}$ and $\xi = (X_i, \varphi_{ii'})$ is a descent datum relative $\mathcal{U}$. Suppose given two such data $(\mathcal{U}, \xi) = (\{f_i : U_i \to U\}, x_i, \varphi_{ii'})$ and $(\mathcal{V}, \eta) = (\{g_j : V_j \to V\}, y_j, \psi_{jj'})$. We define

$$\mathrm{Mor}_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$$

as the set of (f, a_{ij}), where $f : U \to V$ and

$$a_{ij} : x_i|_{U_i \times_V V_j} \longrightarrow y_j$$

are morphisms of $\mathcal{S}$ lying over $U_i \times_V V_j \to V_j$. These have to satisfy the following condition: for any $i, i' \in I$ and $j, j' \in J$ set $W = (U_i \times_U U_{i'}) \times_V (V_j \times_V V_{j'})$. Then

$$\begin{array}{ccc} x_i|_W & \xrightarrow[a_{ij}|_W]{} & y_j|_W \\ {\scriptstyle \varphi_{ii'}|_W}\downarrow & & \downarrow{\scriptstyle \psi_{jj'}|_W} \\ x_{i'}|_W & \xrightarrow{a_{i'j'}|_W} & y_{j'}|_W \end{array}$$

commutes. At this point you have to verify the following things:

(1) there is a well defined composition on morphisms as above,
(2) this turns $\mathcal{S}'$ into a category over $\mathcal{C}$,
(3) there is a functor $G : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$,
(4) for x, y objects of $\mathcal{S}$ we have $\mathrm{Mor}_{\mathcal{S}}(x, y) = \mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$,

(5) any object of $\mathcal{S}'$ locally comes from an object of $\mathcal{S}$, i.e., part (2) of the lemma holds,
(6) G preserves strongly cartesian morphisms,
(7) $\mathcal{S}'$ is a fibred category over $\mathcal{C}$, and
(8) $\mathcal{S}'$ is a stack over $\mathcal{C}$.

This is all not hard but there is a lot of it. Details omitted. □

Less naive proof. Here is a less naive proof. By Categories, Lemma 4.36.4 there exists an equivalence of fibred categories $\mathcal{S} \to \mathcal{S}'$ where $\mathcal{S}'$ is a split fibred category, i.e., one in which the pullback functors compose on the nose. Obviously the lemma for $\mathcal{S}'$ implies the lemma for $\mathcal{S}$. Hence we may think of $\mathcal{S}$ as a presheaf in categories.

Consider the 2-category Cat temporarily as a category by forgetting about 2-morphisms. Let us think of a category as a quintuple $(\text{Ob}, \text{Arrows}, s, t, \circ)$ as in Categories, Section 4.2. Consider the forgetful functor

$$forget : Cat \to Sets \times Sets, \quad (\text{Ob}, \text{Arrows}, s, t, \circ) \mapsto (\text{Ob}, \text{Arrows}).$$

Then $forget$ is faithful, Cat has limits and $forget$ commutes with them, Cat has directed colimits and $forget$ commutes with them, and $forget$ reflects isomorphisms. We can sheafify presheaves with values in Cat, and by an argument similar to the one in the first part of Sites, Section 7.44 the result commutes with $forget$. Applying this to $\mathcal{S}$ we obtain a sheafification $\mathcal{S}^\#$ which has a sheaf of objects and a sheaf of morphisms both of which are the sheafifications of the corresponding presheaves for $\mathcal{S}$. In this case it is quite easy to see that the map $\mathcal{S} \to \mathcal{S}^\#$ has the properties (1) and (2) of the lemma.

However, the category $\mathcal{S}^\#$ may not yet be a stack since, although the presheaf of objects is a sheaf, the descent condition may not yet be satisfied. To remedy this we have to add more objects. But the argument above does reduce us to the case where $\mathcal{S} = \mathcal{S}_F$ for some sheaf(!) $F : \mathcal{C}^{opp} \to Cat$ of categories. In this case consider the functor $F' : \mathcal{C}^{opp} \to Cat$ defined by

(1) The set $\text{Ob}(F'(U))$ is the set of pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of U and $\xi = (x_i, \varphi_{ii'})$ is a descent datum relative to $\mathcal{U}$.
(2) A morphism in $F'(U)$ from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$ is an element of
$$\operatorname{colim} \text{Mor}_{DD(\mathcal{W})}(a^*\xi, b^*\eta)$$
where the colimit is over all common refinements $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$. This colimit is filtered (verification omitted). Hence composition of morphisms in $F(U)$ is defined by finding a common refinement and composing in $DD(\mathcal{W})$.
(3) Given $h : V \to U$ and an object $(\mathcal{U}, \xi)$ of $F'(U)$ we set $F'(h)(\mathcal{U}, \xi)$ equal to $(V \times_U \mathcal{U}, \text{pr}_1^*\xi)$. More precisely, if $\mathcal{U} = \{U_i \to U\}$ and $\xi = (x_i, \varphi_{ii'})$, then $V \times_U \mathcal{U} = \{V \times_U U_i \to V\}$ which comes with a canonical morphism $\text{pr}_1 : V \times_U \mathcal{U} \to \mathcal{U}$ and $\text{pr}_1^*\xi$ is the pullback of ξ with respect to this morphism (see Definition 8.3.4).
(4) Given $h : V \to U$, objects $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ and a morphism between them, represented by $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$, and $\alpha : a^*\xi \to b^*\eta$, then $F'(h)(\alpha)$ is represented by $a' : V \times_U \mathcal{W} \to V \times_U \mathcal{U}$, $b' : V \times_U \mathcal{W} \to V \times_U \mathcal{V}$, and the pullback α' of the morphism α via the map $V \times_U \mathcal{W} \to \mathcal{W}$. This works since pullbacks in $\mathcal{S}_F$ commute on the nose.

There is a map $F \to F'$ given by associating to an object x of $F(U)$ the object $(\{U \to U\}, (x, triv))$ of $F'(U)$. At this point you have to check that the corresponding functor $\mathcal{S}_F \to \mathcal{S}_{F'}$ has properties (1) and (2) of the lemma, and finally that $\mathcal{S}_{F'}$ is a stack. Details omitted. □

0435 **Lemma 8.8.2.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the stack and 1-morphism constructed in Lemma 8.8.1. This construction has the following universal property: Given a stack $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{X}$ of fibred categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{F} & \mathcal{X} \\ & {\scriptstyle G}\searrow \quad \nearrow{\scriptstyle H} & \\ & \mathcal{S}' & \end{array}$$

is 2-commutative.

Proof. Omitted. Hint: Suppose that $x' \in \mathrm{Ob}(\mathcal{S}'_U)$. By the result of Lemma 8.8.1 there exists a covering $\{U_i \to U\}_{i \in I}$ such that $x'|_{U_i} = G(x_i)$ for some $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$. Moreover, there exist coverings $\{U_{ijk} \to U_i \times_U U_j\}$ and isomorphisms $\alpha_{ijk} : x_i|_{U_{ijk}} \to x_j|_{U_{ijk}}$ with $G(\alpha_{ijk}) = \mathrm{id}_{x'|_{U_{ijk}}}$. Set $y_i = F(x_i)$. Then you can check that

$$F(\alpha_{ijk}) : y_i|_{U_{ijk}} \to y_j|_{U_{ijk}}$$

agree on overlaps and therefore (as $\mathcal{X}$ is a stack) define a morphism $\beta_{ij} : y_i|_{U_i \times_U U_j} \to y_j|_{U_i \times_U U_j}$. Next, you check that the β_{ij} define a descent datum. Since $\mathcal{X}$ is a stack these descent data are effective and we find an object y of $\mathcal{X}_U$ agreeing with $G(x_i)$ over U_i. The hint is to set $H(x') = y$. □

04W9 **Lemma 8.8.3.** *Notation and assumptions as in Lemma 8.8.2. There is a canonical equivalence of categories*

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{X}) = \mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}', \mathcal{X})$$

given by the constructions in the proof of the aforementioned lemma.

Proof. Omitted. □

04Y1 **Lemma 8.8.4.** *Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of fibred categories over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. Let us denote $\mathcal{X}', \mathcal{Y}', \mathcal{Z}'$ the stackifications and $\mathcal{W}$ the stackification of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. By construction of 2-fibre products there is a canonical 1-morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$. As the second 2-fibre product is a stack (see Lemma 8.4.6) this 1-morphism induces a 1-morphism $h : \mathcal{W} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ by the universal property of stackification, see Lemma 8.8.2. Now h is a morphism of stacks, and we may check that it is an equivalence using Lemmas 8.4.7 and 8.4.8.

Thus we first prove that h induces isomorphisms of Mor-sheaves. Let ξ, ξ' be objects of $\mathcal{W}$ over $U \in \mathrm{Ob}(\mathcal{C})$. We want to show that

$$h : Mor(\xi, \xi') \longrightarrow Mor(h(\xi), h(\xi'))$$

is an isomorphism. To do this we may work locally on U (see Sites, Section 7.26). Hence by construction of $\mathcal{W}$ (see Lemma 8.8.1) we may assume that ξ, ξ' actually

come from objects (x, z, α) and (x', z', α') of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U. By the same lemma once more we see that in this case $Mor(\xi, \xi')$ is the sheafification of

$$V/U \longmapsto \mathrm{Mor}_{\mathcal{X}_V}(x|_V, x'|_V) \times_{\mathrm{Mor}_{\mathcal{Y}_V}(f(x)|_V, f(x')|_V)} \mathrm{Mor}_{\mathcal{Z}_V}(z|_V, z'|_V)$$

and that $Mor(h(\xi), h(\xi'))$ is equal to the fibre product

$$Mor(i(x), i(x')) \times_{Mor(j(f(x)), j(f(x')))} Mor(k(z), k(z'))$$

where $i : \mathcal{X} \to \mathcal{X}'$, $j : \mathcal{Y} \to \mathcal{Y}'$, and $k : \mathcal{Z} \to \mathcal{Z}'$ are the canonical functors. Thus the first displayed map of this paragraph is an isomorphism as sheafification is exact (and hence the sheafification of a fibre product of presheaves is the fibre product of the sheafifications).

Finally, we have to check that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of h. Write such an object as a triple (x', z', α). Then x' locally comes from an object of $\mathcal{X}$, z' locally comes from an object of $\mathcal{Z}$, and having made suitable replacements for x', z' the morphism α of $\mathcal{Y}'_U$ locally comes from a morphism of $\mathcal{Y}$. In other words, we have shown that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$, hence a fortiori it is locally in the essential image of h. □

06NS **Lemma 8.8.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ be a fibred category over $\mathcal{C}$. The stackification of the inertia fibred category $\mathcal{I}_{\mathcal{X}}$ is inertia of the stackification of $\mathcal{X}$.*

Proof. This follows from the fact that stackification is compatible with 2-fibre products by Lemma 8.8.4 and the fact that there is a formula for the inertia in terms of 2-fibre products of categories over $\mathcal{C}$, see Categories, Lemma 4.34.1. □

8.9. Stackification of categories fibred in groupoids

02ZO Here is the result.

02ZP **Lemma 8.9.1.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. There exists a stack in groupoids $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of categories fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.35.6) such that*

(1) *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the map*

$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$

induced by G identifies the right hand side with the sheafification of the left hand side, and

(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.*

Moreover the stack in groupoids $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof. Apply Lemma 8.8.1. The result will be a stack in groupoids by applying Lemma 8.5.2. □

0436 **Lemma 8.9.2.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the stack in groupoids and 1-morphism constructed in Lemma 8.9.1. This construction has the following universal property:*

Given a stack in groupoids $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{X}$ of categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram

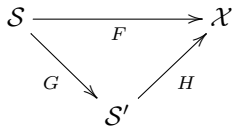

is 2-commutative.

Proof. This is a special case of Lemma 8.8.2. □

04Y2 **Lemma 8.9.3.** *Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of categories fibred in groupoids over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. This is a special case of Lemma 8.8.4. □

8.10. Inherited topologies

06NT It turns out that a fibred category over a site inherits a canonical topology from the underlying site.

06NU **Lemma 8.10.1.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $Cov(\mathcal{S})$ be the set of families $\{x_i \to x\}_{i \in I}$ of morphisms in $\mathcal{S}$ with fixed target such that (a) each $x_i \to x$ is strongly cartesian, and (b) $\{p(x_i) \to p(x)\}_{i \in I}$ is a covering of $\mathcal{C}$. Then $(\mathcal{S}, Cov(\mathcal{S}))$ is a site.*

Proof. We have to check the three conditions of Sites, Definition 7.6.2.

(1) If $x \to y$ is an isomorphism of $\mathcal{S}$, then it is strongly cartesian by Categories, Lemma 4.33.2 and $p(x) \to p(y)$ is an isomorphism of $\mathcal{C}$. Thus $\{p(x) \to p(y)\}$ is a covering of $\mathcal{C}$ whence $\{x \to y\} \in \mathrm{Cov}(\mathcal{S})$.
(2) If $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}(\mathcal{S})$ and for each i we have $\{y_{ij} \to x_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{S})$, then each composition $y_{ij} \to x$ is strongly cartesian by Categories, Lemma 4.33.2 and $\{p(y_{ij}) \to p(x)\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$. Hence also $\{y_{ij} \to x\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{S})$.
(3) Suppose $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}(\mathcal{S})$ and $y \to x$ is a morphism of $\mathcal{S}$. As $\{p(x_i) \to p(x)\}$ is a covering of $\mathcal{C}$ we see that $p(x_i) \times_{p(x)} p(y)$ exists. Hence Categories, Lemma 4.33.13 implies that $x_i \times_x y$ exists, that $p(x_i \times_x y) = p(x_i) \times_{p(x)} p(y)$, and that $x_i \times_x y \to y$ is strongly cartesian. Since also $\{p(x_i) \times_{p(x)} p(y) \to p(y)\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we conclude that $\{x_i \times_x y \to y\}_{i \in I} \in \mathrm{Cov}(\mathcal{S})$

This finishes the proof. □

Note that if $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then the coverings of the site $\mathcal{S}$ in Lemma 8.10.1 are characterized by

$$\{x_i \to x\} \in \mathrm{Cov}(\mathcal{S}) \Leftrightarrow \{p(x_i) \to p(x)\} \in \mathrm{Cov}(\mathcal{C})$$

because every morphism of $\mathcal{S}$ is strongly cartesian.

06NV **Definition 8.10.2.** Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. We say $(\mathcal{S}, \mathrm{Cov}(\mathcal{S}))$ as in Lemma 8.10.1 is the *structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$*. We sometimes indicate this by saying that $\mathcal{S}$ *is endowed with the topology inherited from $\mathcal{C}$*.

In particular we obtain a topos of sheaves $Sh(\mathcal{S})$ in this situation. It turns out that this topos is functorial with respect to 1-morphisms of fibred categories.

06NW **Lemma 8.10.3.** *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. Then F is a continuous and cocontinuous functor between the structure of sites inherited from $\mathcal{C}$. Hence F induces a morphism of topoi $f : Sh(\mathcal{X}) \to Sh(\mathcal{Y})$ with $f_* = {}_sF = {}_pF$ and $f^{-1} = F^s = F^p$. In particular $f^{-1}(\mathcal{G})(x) = \mathcal{G}(F(x))$ for a sheaf $\mathcal{G}$ on $\mathcal{Y}$ and object x of $\mathcal{X}$.*

Proof. We first prove that F is continuous. Let $\{x_i \to x\}_{i\in I}$ be a covering of $\mathcal{X}$. By Categories, Definition 4.33.9 the functor F transforms strongly cartesian morphisms into strongly cartesian morphisms, hence $\{F(x_i) \to F(x)\}_{i\in I}$ is a covering of $\mathcal{Y}$. This proves part (1) of Sites, Definition 7.13.1. Moreover, let $x' \to x$ be a morphism of $\mathcal{X}$. By Categories, Lemma 4.33.13 the fibre product $x_i \times_x x'$ exists and $x_i \times_x x' \to x'$ is strongly cartesian. Hence $F(x_i \times_x x') \to F(x')$ is strongly cartesian. By Categories, Lemma 4.33.13 applied to $\mathcal{Y}$ this means that $F(x_i \times_x x') = F(x_i) \times_{F(x)} F(x')$. This proves part (2) of Sites, Definition 7.13.1 and we conclude that F is continuous.

Next we prove that F is cocontinuous. Let $x \in \mathrm{Ob}(\mathcal{X})$ and let $\{y_i \to F(x)\}_{i\in I}$ be a covering in $\mathcal{Y}$. Denote $\{U_i \to U\}_{i\in I}$ the corresponding covering of $\mathcal{C}$. For each i choose a strongly cartesian morphism $x_i \to x$ in $\mathcal{X}$ lying over $U_i \to U$. Then $F(x_i) \to F(x)$ and $y_i \to F(x)$ are both a strongly cartesian morphisms in $\mathcal{Y}$ lying over $U_i \to U$. Hence there exists a unique isomorphism $F(x_i) \to y_i$ in $\mathcal{Y}_{U_i}$ compatible with the maps to $F(x)$. Thus $\{x_i \to x\}_{i\in I}$ is a covering of $\mathcal{X}$ such that $\{F(x_i) \to F(x)\}_{i\in I}$ is isomorphic to $\{y_i \to F(x)\}_{i\in I}$. Hence F is cocontinuous, see Sites, Definition 7.20.1.

The final assertion follows from the first two, see Sites, Lemmas 7.21.1, 7.20.2, and 7.21.5. □

0CN0 **Lemma 8.10.4.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}/x \to \mathcal{C}/U$ where $\mathcal{X}$ is endowed with the topology inherited from $\mathcal{C}$.*

Proof. Here $\mathcal{C}/U$ is the localization of the site $\mathcal{C}$ at the object U and similarly for $\mathcal{X}/x$. It follows from Categories, Definition 4.35.1 that the rule $x'/x \mapsto p(x')/p(x)$ defines an equivalence of categories $\mathcal{X}/x \to \mathcal{C}/U$. Whereupon it follows from Definition 8.10.2 that coverings of x' in $\mathcal{X}/x$ are in bijective correspondence with coverings of $p(x')$ in $\mathcal{C}/U$. □

06NX **Lemma 8.10.5.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. If F turns $\mathcal{X}$ into a category fibred in groupoids over $\mathcal{Y}$, then $\mathcal{X}$ is a stack in groupoids over $\mathcal{Y}$ (with topology inherited from $\mathcal{C}$).*

Proof. Let us prove descent for objects. Let $\{y_i \to y\}$ be a covering of $\mathcal{Y}$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{X}$ with respect to this covering. Then (x_i, φ_{ij}) is also a descent datum with respect to the covering $\{q(y_i) \to q(y)\}$ of $\mathcal{C}$. As $\mathcal{X}$ is a stack in groupoids we obtain an object x over $q(y)$ and isomorphisms $\psi_i : x|_{q(y_i)} \to x_i$ over $q(y_i)$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \psi_j|_{q(y_i)\times_{q(y)} q(y_j)} \circ \psi_i^{-1}|_{q(y_i)\times_{q(y)} q(y_j)}.$$

Consider the sheaf $I = Isom_{\mathcal{Y}}(F(x), y)$ on $\mathcal{C}/p(x)$. Note that $s_i = F(\psi_i) \in I(q(x_i))$ because $F(x_i) = y_i$. Because $F(\varphi_{ij}) = \mathrm{id}$ (as we started with a descent datum over $\{y_i \to y\}$) the displayed formula shows that $s_i|_{q(y_i)\times_{q(y)}q(y_j)} = s_j|_{q(y_i)\times_{q(y)}q(y_j)}$. Hence the local sections s_i glue to $s : F(x) \to y$. As F is fibred in groupoids we see that x is isomorphic to an object x' with $F(x') = y$. We omit the verification that x' in the fibre category of $\mathcal{X}$ over y is a solution to the problem of descent posed by the descent datum (x_i, φ_{ij}). We also omit the proof of the sheaf property of the *Isom*-presheaves of $\mathcal{X}/\mathcal{Y}$. □

09WX **Lemma 8.10.6.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ be a stack. Endow $\mathcal{X}$ with the topology inherited from $\mathcal{C}$ and let $q : \mathcal{Y} \to \mathcal{X}$ be a stack. Then $\mathcal{Y}$ is a stack over $\mathcal{C}$. If p and q define stacks in groupoids, then $\mathcal{Y}$ is a stack in groupoids over $\mathcal{C}$.*

Proof. We check the three conditions in Definition 8.4.1 to prove that $\mathcal{Y}$ is a stack over $\mathcal{C}$. By Categories, Lemma 4.33.12 we find that $\mathcal{Y}$ is a fibred category over $\mathcal{C}$. Thus condition (1) holds.

Let U be an object of $\mathcal{C}$ and let y_1, y_2 be objects of $\mathcal{Y}$ over U. Denote $x_i = q(y_i)$ in $\mathcal{X}$. Consider the map of presheaves

$$q : Mor_{\mathcal{Y}/\mathcal{C}}(y_1, y_2) \longrightarrow Mor_{\mathcal{X}/\mathcal{C}}(x_1, x_2)$$

on $\mathcal{C}/U$, see Lemma 8.2.3. Let $\{U_i \to U\}$ be a covering and let φ_i be a section of the presheaf on the left over U_i such that φ_i and φ_j restrict to the same section over $U_i \times_U U_j$. We have to find a morphism $\varphi : y_1 \to y_2$ restricting to φ_i. Note that $q(\varphi_i) = \psi|_{U_i}$ for some morphism $\psi : x_1 \to x_2$ over U because the second presheaf is a sheaf (by assumption). Let $y_{12} \to y_2$ be the strongly $\mathcal{X}$-cartesian morphism of $\mathcal{Y}$ lying over ψ. Then φ_i corresponds to a morphism $\varphi_i' : y_1|_{U_i} \to y_{12}|_{U_i}$ over $x_1|_{U_i}$. In other words, φ_i' now define local sections of the presheaf

$$Mor_{\mathcal{Y}/\mathcal{X}}(y_1, y_{12})$$

over the members of the covering $\{x_1|_{U_i} \to x_1\}$. By assumption these glue to a unique morphism $y_1 \to y_{12}$ which composed with the given morphism $y_{12} \to y_2$ produces the desired morphism $y_1 \to y_2$.

Finally, we show that descent data are effective. Let $\{f_i : U_i \to U\}$ be a covering of $\mathcal{C}$ and let (y_i, φ_{ij}) be a descent datum relative to this covering (Definition 8.3.1). Setting $x_i = q(y_i)$ and $\psi_{ij} = q(\varphi_{ij})$ we obtain a descent datum (x_i, ψ_{ij}) for the covering in $\mathcal{X}$. By assumption on $\mathcal{X}$ we may assume $x_i = x|_{U_i}$ and the ψ_{ij} equal to the canonical descent datum (Definition 8.3.5). In this case $\{x|_{U_i} \to x\}$ is a covering and we can view (y_i, φ_{ij}) as a descent datum relative to this covering. By our assumption that $\mathcal{Y}$ is a stack over $\mathcal{C}$ we see that it is effective which finishes the proof of condition (3).

The final assertion follows because $\mathcal{Y}$ is a stack over $\mathcal{C}$ and is fibred in groupoids by Categories, Lemma 4.35.14. □

8.11. Gerbes

06NY Gerbes are a special kind of stacks in groupoids.

06NZ **Definition 8.11.1.** A *gerbe* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is a stack in groupoids over $\mathcal{C}$ (see Definition 8.5.1),

(2) for $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\mathcal{S}_{U_i}$ is nonempty, and
(3) for $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{S}_{U_i}$.

In other words, a gerbe is a stack in groupoids such that any two objects are locally isomorphic and such that objects exist locally.

06P0 **Lemma 8.11.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a gerbe over $\mathcal{C}$.*

Proof. Assume $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$. By Lemma 8.5.4 we see $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be equivalences of categories over $\mathcal{C}$. Given $U \in \mathrm{Ob}(\mathcal{C})$ we see that there exists a covering $\{U_i \to U\}$ such that $(\mathcal{S}_1)_{U_i}$ is nonempty. Applying F we see that $(\mathcal{S}_2)_{U_i}$ is nonempty. Given $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}((\mathcal{S}_2)_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $G(x)|_{U_i} \cong G(y)|_{U_i}$ in $(\mathcal{S}_1)_{U_i}$. By Categories, Lemma 4.35.9 this implies $x|_{U_i} \cong y|_{U_i}$ in $(\mathcal{S}_2)_{U_i}$. □

We want to generalize the definition of gerbes a bit. Namely, let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over a site $\mathcal{C}$. We want to say what it means for $\mathcal{X}$ to be a gerbe over $\mathcal{Y}$. By Section 8.10 the category $\mathcal{Y}$ inherits the structure of a site from $\mathcal{C}$. A naive guess is: Just require that $\mathcal{X} \to \mathcal{Y}$ *is* a gerbe in the sense above. Except the notion so obtained is not invariant under replacing $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$; this is even the case for the property of being fibred in groupoids over $\mathcal{Y}$. However, it turns out that we can replace $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$ which is fibred in groupoids over $\mathcal{Y}$, and then the property of being a gerbe over $\mathcal{Y}$ is independent of this choice. Here is the precise formulation.

06P1 **Lemma 8.11.3.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. The following are equivalent*

(1) *For some (equivalently any) factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids, the map $F' : \mathcal{X}' \to \mathcal{Y}$ is a gerbe (with the topology on $\mathcal{Y}$ inherited from $\mathcal{C}$).*
(2) *The following two conditions are satisfied*
   (a) *for $y \in \mathrm{Ob}(\mathcal{Y})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i such that $F(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$, and*
   (b) *for $U \in \mathrm{Ob}(\mathcal{C})$, $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$, and $b : F(x) \to F(x')$ in $\mathcal{Y}_U$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$.*

Proof. By Categories, Lemma 4.35.16 there exists a factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids. By Categories, Lemma 4.35.17 given any two such factorizations $F = F' \circ a = F'' \circ b$ we have that $\mathcal{X}'$ is equivalent to $\mathcal{X}''$ as categories over $\mathcal{Y}$. Hence Lemma 8.11.2 guarantees that the condition (1) is independent of the choice of the factorization. Moreover, this means that we may assume $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ as in the proof of Categories, Lemma 4.35.16

Let us prove that (a) and (b) imply that $\mathcal{X}' \to \mathcal{Y}$ is a gerbe. First of all, by Lemma 8.10.5 we see that $\mathcal{X}' \to \mathcal{Y}$ is a stack in groupoids. Next, let y be an

object of $\mathcal{Y}$ lying over $U \in \mathrm{Ob}(\mathcal{C})$. By (a) we can find a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i and isomorphisms $f_i : F(x_i) \to y|_{U_i}$ in $\mathcal{Y}_{U_i}$. Then $(U_i, x_i, y|_{U_i}, f_i)$ are objects of $\mathcal{X}'_{U_i}$, i.e., the second condition of Definition 8.11.1 holds. Finally, let (U, x, y, f) and (U, x', y, f') be objects of $\mathcal{X}'$ lying over the same object $y \in \mathrm{Ob}(\mathcal{Y})$. Set $b = (f')^{-1} \circ f$. By condition (b) we can find a covering $\{U_i \to U\}$ and isomorphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$. Then

$$(a_i, \mathrm{id}) : (U, x, y, f)|_{U_i} \to (U, x', y, f')|_{U_i}$$

is a morphism in $\mathcal{X}'_{U_i}$ as desired. This proves that (2) implies (1).

To prove that (1) implies (2) one reads the arguments in the preceding paragraph backwards. Details omitted. □

06P2 **Definition 8.11.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if the equivalent conditions of Lemma 8.11.3 are satisfied.

This definition does not conflict with Definition 8.11.1 when $\mathcal{Y} = \mathcal{C}$ because in this case we may take $\mathcal{X}' = \mathcal{X}$ in part (1) of Lemma 8.11.3. Note that conditions (2)(a) and (2)(b) of Lemma 8.11.3 are quite close in spirit to conditions (2) and (3) of Definition 8.11.1. Namely, (2)(a) says that the map of presheaves of isomorphism classes of objects becomes a surjection after sheafification. Moreover, (2)(b) says that

$$\mathit{Isom}_{\mathcal{X}}(x, x') \longrightarrow \mathit{Isom}_{\mathcal{Y}}(F(x), F(x'))$$

is a surjection of sheaves on $\mathcal{C}/U$ for any U and $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$.

06P3 **Lemma 8.11.5.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-fibre product of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.32.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$.

Let y' be an object of $\mathcal{Y}'$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $x_i \in \mathcal{X}_{U_i}$ with isomorphisms $\alpha_i : G(y')|_{U_i} \to F(x_i)$. Then $(U_i, y'|_{U_i}, x_i, \alpha_i)$ is an object of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U_i whose image in $\mathcal{Y}'$ is $y'|_{U_i}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x'_1, x'_2 be objects of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U, and let $b' : F'(x'_1) \to F'(x'_2)$ be a morphism in $\mathcal{Y}'_U$. Write $x'_i = (U, y'_i, x_i, \alpha_i)$. Note that $F'(x'_i) = x_i$ and $G'(x'_i) = y'_i$. By assumption there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x_1|_{U_i} \to x_2|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = G(b')|_{U_i}$. Then $(b'|_{U_i}, a_i)$ is a morphism $x'_1|_{U_i} \to x'_2|_{U_i}$ as required in (2)(b). □

06R3 **Lemma 8.11.6.** *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Z}$.

Let z be an object of $\mathcal{Z}$ lying over the object U of $\mathcal{C}$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and objects $y_i \in \mathcal{Y}_{U_i}$ such that $G(y_i) \cong z|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and objects $x_{ij} \in \mathcal{X}_{U_{ij}}$ such that $F(x_{ij}) \cong y_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(x_{ij}) \cong z|_{U_{ij}}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $c : (G \circ F)(x_1) \to (G \circ F)(x_2)$ be a morphism in $\mathcal{Z}_U$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and morphisms $b_i : F(x_1)|_{U_i} \to F(x_2)|_{U_i}$ in $\mathcal{Y}_{U_i}$ such that $G(b_i) = c|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a_{ij} : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ in $\mathcal{X}_{U_{ij}}$ such that $F(a_{ij}) = b_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(a_{ij}) = c|_{U_{ij}}$ as required in (2)(b). □

06P4 **Lemma 8.11.7.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(\mathcal{Y}_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $G : \mathcal{Y}'_{U_i} \to \mathcal{Y}_{U_i}$ and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.32.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Y}$.

Let y be an object of $\mathcal{Y}$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $y'_i \in \mathcal{Y}'_{U_i}$ with $G(y'_i) \cong y|_{U_i}$. By (2)(a) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and objects x'_{ij} of $\mathcal{X}'$ over U_{ij} with $F'(x'_{ij})$ isomorphic to the restriction of y'_i to U_{ij}. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $G'(x'_{ij})$ are objects of $\mathcal{X}$ over U_{ij} whose images in $\mathcal{Y}$ are isomorphic to the restrictions $y|_{U_{ij}}$. This proves (2)(a) for $\mathcal{X} \to \mathcal{Y}$.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $b : F(x_1) \to F(x_2)$ be a morphism in $\mathcal{Y}_U$. By assumption we may choose a covering $\{U_i \to U\}$ and objects y'_i of $\mathcal{Y}'$ over U_i such that there exist isomorphisms $\alpha_i : G(y'_i) \to F(x_1)|_{U_i}$. Then we get objects

$$x'_{1i} = (U_i, y'_i, x_1|_{U_i}, \alpha_i) \quad \text{and} \quad x'_{2i} = (U_i, y'_i, x_2|_{U_i}, b|_{U_i} \circ \alpha_i)$$

of $\mathcal{X}'$ over U_i. The identity morphism on y'_i is a morphism $F'(x'_{1i}) \to F'(x'_{2i})$. By (2)(b) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a'_{ij} : x'_{1i}|_{U_{ij}} \to x'_{2i}|_{U_{ij}}$ such that $F'(a'_{ij}) = \mathrm{id}_{y'_i}|_{U_{ij}}$. Unwinding the definition of morphisms in $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ we see that $G'(a'_{ij}) : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ are the morphisms we're looking for, i.e., (2)(b) holds for $\mathcal{X} \to \mathcal{Y}$. □

Gerbes all of whose automorphism sheaves are abelian play an important role in algebraic geometry.

0CJY **Lemma 8.11.8.** *Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site $\mathcal{C}$. Assume that for all $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$ the sheaf of groups $Aut(x) = Isom(x, x)$ on $\mathcal{C}/U$ is abelian. Then there exist*

(1) *a sheaf $\mathcal{G}$ of abelian groups on $\mathcal{C}$,*
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$ and every $x \in \mathrm{Ob}(\mathcal{S}_U)$ an isomorphism $\mathcal{G}|_U \to Aut(x)$*

such that for every U and every morphism $\varphi : x \to y$ in $\mathcal{S}_U$ the diagram

$$\begin{array}{ccc} \mathcal{G}|_U & = & \mathcal{G}|_U \\ \downarrow & & \downarrow \\ Aut(x) & \xrightarrow{\alpha \mapsto \varphi \circ \alpha \circ \varphi^{-1}} & Aut(y) \end{array}$$

is commutative.

Proof. Let x, y be two objects of $\mathcal{S}$ with $U = p(x) = p(y)$.

If there is a morphism $\varphi : x \to y$ over U, then it is an isomorphism and then we indeed get an isomorphism $Aut(x) \to Aut(y)$ sending α to $\varphi \circ \alpha \circ \varphi^{-1}$. Moreover, since we are assuming $Aut(x)$ is commutative, this isomorphism is independent of the choice of φ by a simple computation: namely, if ψ is a second such map, then

$$\varphi \circ \alpha \circ \varphi^{-1} = \psi \circ \psi^{-1} \circ \varphi \circ \alpha \circ \varphi^{-1} = \psi \circ \alpha \circ \psi^{-1} \circ \varphi \circ \varphi^{-1} = \psi \circ \alpha \circ \psi^{-1}$$

The upshot is a canonical isomorphism of sheaves $Aut(x) \to Aut(y)$. Furthermore, if there is a third object z and a morphism $y \to z$ (and hence also a morphism $x \to z$), then the canonical isomorphisms $Aut(x) \to Aut(y)$, $Aut(y) \to Aut(z)$, and $Aut(x) \to Aut(z)$ are compatible in the sense that

$$\begin{array}{ccc} Aut(x) & \longrightarrow & Aut(z) \\ & \searrow \quad \nearrow & \\ & Aut(y) & \end{array}$$

commutes.

If there is no morphism from x to y over U, then we can choose a covering $\{U_i \to U\}$ such that there exist morphisms $x|_{U_i} \to y|_{U_i}$. This gives canonical isomorphisms

$$Aut(x)|_{U_i} \longrightarrow Aut(y)|_{U_i}$$

which agree over $U_i \times_U U_j$ (by canonicity). By glueing of sheaves (Sites, Lemma 7.26.1) we get a unique isomorphism $Aut(x) \to Aut(y)$ whose restriction to any U_i is the canonical isomorphism of the previous paragraph. Similarly to the above these canonical isomorphisms satisfy a compatibility if we have a third object over U.

What if the fibre category of $\mathcal{S}$ over U is empty? Well, in this case we can find a covering $\{U_i \to U\}$ and objects x_i of $\mathcal{S}$ over U_i. Then we set $\mathcal{G}_i = Aut(x_i)$. By the above we obtain canonical isomorphisms

$$\varphi_{ij} : \mathcal{G}_i|_{U_i \times_U U_j} \longrightarrow \mathcal{G}_j|_{U_i \times_U U_j}$$

whose restrictions to $U_i \times_U U_j \times_U U_k$ satisfy the cocycle condition explained in Sites, Section 7.26. By Sites, Lemma 7.26.4 we obtain a sheaf $\mathcal{G}$ over U whose restriction to U_i gives $\mathcal{G}_i$ in a manner compatible with the glueing maps φ_{ij}.

If $\mathcal{C}$ has a final object U, then this finishes the proof as we can take $\mathcal{G}$ equal to the sheaf we just constructed. In the general case we need to verify that the sheaves $\mathcal{G}$ constructed over varying U are compatible in a canonical manner. This is omitted. □

8.12. Functoriality for stacks

04WA In this section we study what happens if we want to change the base site of a stack. This section can be skipped on a first reading.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{D}$ be a category over $\mathcal{D}$. In this situation we denote $u^p\mathcal{S}$ the category over $\mathcal{C}$ defined as follows

(1) An object of $u^p\mathcal{S}$ is a pair (U, y) consisting of an object U of $\mathcal{C}$ and an object y of $\mathcal{S}_{u(U)}$.
(2) A morphism $(a, \beta) : (U, y) \to (U', y')$ is given by a morphism $a : U \to U'$ of $\mathcal{C}$ and a morphism $\beta : y \to y'$ of $\mathcal{S}$ such that $p(\beta) = u(a)$.

Note that with these definitions the fibre category of $u^p\mathcal{S}$ over U is equal to the fibre category of $\mathcal{S}$ over $u(U)$.

04WB **Lemma 8.12.1.** *In the situation above, if $\mathcal{S}$ is a fibred category over $\mathcal{D}$ then $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Proof. Please take a look at the discussion surrounding Categories, Definitions 4.33.1 and 4.33.5 before reading this proof. Let $(a, \beta) : (U, y) \to (U', y')$ be a morphism of $u^p\mathcal{S}$. We claim that (a, β) is strongly cartesian if and only if β is strongly cartesian. First, assume β is strongly cartesian. Consider any second morphism $(a_1, \beta_1) : (U_1, y_1) \to (U', y')$ of $u^p\mathcal{S}$. Then

$$\begin{aligned}
&\mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U, y)) \\
&= \mathrm{Mor}_{\mathcal{C}}(U_1, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U))} \mathrm{Mor}_{\mathcal{S}}(y_1, y) \\
&= \mathrm{Mor}_{\mathcal{C}}(U_1, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U))} \mathrm{Mor}_{\mathcal{S}}(y_1, y') \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U)) \\
&= \mathrm{Mor}_{\mathcal{S}}(y_1, y') \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \mathrm{Mor}_{\mathcal{C}}(U_1, U) \\
&= \mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U', y')) \times_{\mathrm{Mor}_{\mathcal{C}}(U_1, U')} \mathrm{Mor}_{\mathcal{C}}(U_1, U)
\end{aligned}$$

the second equality as β is strongly cartesian. Hence we see that indeed (a, β) is strongly cartesian. Conversely, suppose that (a, β) is strongly cartesian. Choose a strongly cartesian morphism $\beta' : y'' \to y'$ in $\mathcal{S}$ with $p(\beta') = u(a)$. Then bot $(a, \beta) : (U, y) \to (U, y')$ and $(a, \beta') : (U, y'') \to (U, y)$ are strongly cartesian and lift a. Hence, by the uniqueness of strongly cartesian morphisms (see discussion in Categories, Section 4.33) there exists an isomorphism $\iota : y \to y''$ in $\mathcal{S}_{u(U)}$ such that $\beta = \beta' \circ \iota$, which implies that β is strongly cartesian in $\mathcal{S}$ by Categories, Lemma 4.33.2.

Finally, we have to show that given (U', y') and $U \to U'$ we can find a strongly cartesian morphism $(U, y) \to (U', y')$ in $u^p\mathcal{S}$ lifting the morphism $U \to U'$. This follows from the above as by assumption we can find a strongly cartesian morphism $y \to y'$ lifting the morphism $u(U) \to u(U')$. □

04WC **Lemma 8.12.2.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack over $\mathcal{C}$.*

Proof. We have seen in Lemma 8.12.1 that $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$. Moreover, in the proof of that lemma we have seen that a morphism (a, β) of $u^p\mathcal{S}$ is strongly cartesian if and only β is strongly cartesian in $\mathcal{S}$. Hence, given a morphism $a : U \to U'$ of $\mathcal{C}$, not only do we have the equalities $(u^p\mathcal{S})_U = \mathcal{S}_U$ and $(u^p\mathcal{S})_{U'} = \mathcal{S}_{U'}$, but via these equalities the pullback functors agree; in a formula $a^*(U', y') = (U, u(a)^* y')$.

Having said this, let $\mathcal{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$. As u is continuous we see that $\mathcal{V} = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and that $u(U_i \times_U U_j) = u(U_i) \times_{u(U)} u(U_j)$ and similarly for the triple fibre products $U_i \times_U U_j \times_U U_k$. As we have the identifications of fibre categories and pullbacks we see that descend data relative to $\mathcal{U}$ are identical to descend data relative to $\mathcal{V}$. Since by assumption we have effective descent in $\mathcal{S}$ we conclude the same holds for $u^p\mathcal{S}$. □

04WD **Lemma 8.12.3.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack in groupoids over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$.*

Proof. This follows immediately from Lemma 8.12.2 and the fact that all fibre categories are groupoids. □

04WE **Definition 8.12.4.** Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{S}$ be a fibred category over $\mathcal{D}$. In this setting we write $f_*\mathcal{S}$ for the fibred category $u^p\mathcal{S}$ defined above. We say that $f_*\mathcal{S}$ is the *pushforward of $\mathcal{S}$ along f*.

By the results above we know that $f_*\mathcal{S}$ is a stack (in groupoids) if $\mathcal{S}$ is a stack (in groupoids). It is harder to define the pullback of a stack (and we'll need additional assumptions for our particular construction – feel free to write up and submit a more general construction). We do this in several steps.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. In this setting we define a category $u_{pp}\mathcal{S}$ as follows:

(1) An object of $u_{pp}\mathcal{S}$ is a triple $(U, \phi : V \to u(U), x)$ where $U \in \mathrm{Ob}(\mathcal{C})$, the map $\phi : V \to u(U)$ is a morphism in $\mathcal{D}$, and $x \in \mathrm{Ob}(\mathcal{S}_U)$.

(2) A morphism

$$(U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2)$$

of $u_{pp}\mathcal{S}$ is given by a (a, b, α) where $a : U_1 \to U_2$ is a morphism of $\mathcal{C}$, $b : V_1 \to V_2$ is a morphism of $\mathcal{D}$, and $\alpha : x_1 \to x_2$ is morphism of $\mathcal{S}$, such that $p(\alpha) = a$ and the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{b} & V_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ u(U_1) & \xrightarrow{u(a)} & u(U_2) \end{array}$$

commutes in $\mathcal{D}$.

We think of $u_{pp}\mathcal{S}$ as a category over $\mathcal{D}$ via

$$p_{pp} : u_{pp}\mathcal{S} \longrightarrow \mathcal{D}, \quad (U, \phi : V \to u(U), x) \longmapsto V.$$

The fibre category of $u_{pp}\mathcal{S}$ over an object V of $\mathcal{D}$ does not have a simple description.

04WF **Lemma 8.12.5.** *In the situation above assume*

(1) $p : \mathcal{S} \to \mathcal{C}$ *is a fibred category,*
(2) $\mathcal{C}$ *has nonempty finite limits, and*
(3) $u : \mathcal{C} \to \mathcal{D}$ *commutes with nonempty finite limits.*

Consider the set $R \subset Arrows(u_{pp}\mathcal{S})$ *of morphisms of the form*

$$(a, id_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α *strongly cartesian. Then* R *is a right multiplicative system.*

Proof. According to Categories, Definition 4.27.1 we have to check RMS1, RMS2, RMS3. Condition RMS1 holds as a composition of strongly cartesian morphisms is strongly cartesian, see Categories, Lemma 4.33.2.

To check RMS2 suppose we have a morphism

$$(a, b, \alpha) : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

of $u_{pp}\mathcal{S}$ and a morphism

$$(c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with γ strongly cartesian from R. In this situation set $U_1' = U_1 \times_U U'$, and denote $a' : U_1' \to U'$ and $c' : U_1' \to U_1$ the projections. As $u(U_1') = u(U_1) \times_{u(U)} u(U')$ we see that $\phi_1' = (\phi_1, \phi') : V_1 \to u(U_1')$ is a morphism in $\mathcal{D}$. Let $\gamma_1 : x_1' \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\gamma_1) = \phi_1'$ (which exists because $\mathcal{S}$ is a fibred category over $\mathcal{C}$). Then as $\gamma : x' \to x$ is strongly cartesian there exists a unique morphism $\alpha' : x_1' \to x'$ with $p(\alpha') = a'$. At this point we see that

$$(a', b, \alpha') : (U_1, \phi_1 : V_1 \to u(U_1'), x_1') \longrightarrow (U, \phi : V \to u(U'), x')$$

is a morphism and that

$$(c', \mathrm{id}_{V_1}, \gamma_1) : (U_1', \phi_1' : V_1 \to u(U_1'), x_1') \longrightarrow (U_1, \phi : V_1 \to u(U_1), x_1)$$

is an element of R which form a solution of the existence problem posed by RMS2.

Finally, suppose that

$$(a, b, \alpha), (a', b', \alpha') : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

are two morphisms of $u_{pp}\mathcal{S}$ and suppose that

$$(c, \mathrm{id}_V, \gamma) : (U, \phi : V \to u(U), x) \longrightarrow (U', \phi : V \to u(U'), x')$$

is an element of R which equalizes the morphisms (a, b, α) and (a', b', α'). This implies in particular that $b = b'$. Let $d : U_2 \to U_1$ be the equalizer of a, a' which exists (see Categories, Lemma 4.18.3). Moreover, $u(d) : u(U_2) \to u(U_1)$ is the equalizer of $u(a), u(a')$ hence (as $b = b'$) there is a morphism $\phi_2 : V_1 \to u(U_2)$ such that $\phi_1 = u(d) \circ \phi_1$. Let $\delta : x_2 \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\delta) = u(d)$. Now we claim that $\alpha \circ \delta = \alpha' \circ \delta$. This is true because γ is strongly cartesian, $\gamma \circ \alpha \circ \delta = \gamma \circ \alpha' \circ \delta$, and $p(\alpha \circ \delta) = p(\alpha' \circ \delta)$. Hence the arrow

$$(d, \mathrm{id}_{V_1}, \delta) : (U_2, \phi_2 : V_1 \to u(U_2), x_2) \longrightarrow (U_1, \phi_1 : V_1 \to u(U_1), x_1)$$

is an element of R and equalizes (a, b, α) and (a', b', α'). Hence R satisfies RMS3 as well. □

04WG **Lemma 8.12.6.** *With notation and assumptions as in Lemma 8.12.5. Set* $u_p\mathcal{S} = R^{-1}u_{pp}\mathcal{S}$*, see Categories, Section 4.27. Then* $u_p\mathcal{S}$ *is a fibred category over* $\mathcal{D}$*.*

Proof. We use the description of $u_p\mathcal{S}$ given just above Categories, Lemma 4.27.11. Note that the functor $p_{pp} : u_{pp}\mathcal{S} \to \mathcal{D}$ transforms every element of R to an identity morphism. Hence by Categories, Lemma 4.27.16 we obtain a canonical functor $p_p : u_p\mathcal{S} \to \mathcal{D}$ extending the given functor. This is how we think of $u_p\mathcal{S}$ as a category over $\mathcal{D}$.

First we want to characterize the $\mathcal{D}$-strongly cartesian morphisms in $u_p\mathcal{S}$. A morphism $f : X \to Y$ of $u_p\mathcal{S}$ is the equivalence class of a pair $(f' : X' \to Y, r : X' \to X)$ with $r \in R$. In fact, in $u_p\mathcal{S}$ we have $f = (f', 1) \circ (r, 1)^{-1}$ with obvious notation. Note that an isomorphism is always strongly cartesian, as are compositions of strongly cartesian morphisms, see Categories, Lemma 4.33.2. Hence f is strongly cartesian if and only if $(f', 1)$ is so. Thus the following claim completely characterizes strongly cartesian morphisms. Claim: A morphism

$$(a, b, \alpha) : X_1 = (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2) = X_2$$

of $u_{pp}\mathcal{S}$ has image $f = ((a, b, \alpha), 1)$ strongly cartesian in $u_p\mathcal{S}$ if and only if α is a strongly cartesian morphism of $\mathcal{S}$.

Assume α strongly cartesian. Let $X = (U, \phi : V \to u(U), x)$ be another object, and let $f_2 : X \to X_2$ be a morphism of $u_p\mathcal{S}$ such that $p_p(f_2) = b \circ b_1$ for some $b_1 : U \to U_1$. To show that f is strongly cartesian we have to show that there exists a unique morphism $f_1 : X \to X_1$ in $u_p\mathcal{S}$ such that $p_p(f_1) = b_1$ and $f_2 = f \circ f_1$ in $u_p\mathcal{S}$. Write $f_2 = (f_2' : X' \to X_2, r : X' \to X)$. Again we can write $f_2 = (f_2', 1) \circ (r, 1)^{-1}$ in $u_p\mathcal{S}$. Since $(r, 1)$ is an isomorphism whose image in $\mathcal{D}$ is an identity we see that finding a morphism $f_1 : X \to X_1$ with the required properties is the same thing as finding a morphism $f_1' : X' \to X_1$ in $u_p\mathcal{S}$ with $p(f_1') = b_1$ and $f_2' = f \circ f_1'$. Hence we may assume that f_2 is of the form $f_2 = ((a_2, b_2, \alpha_2), 1)$ with $b_2 = b \circ b_1$. Here is a picture

$$\begin{array}{ccc} & & (U_1, V_1 \to u(U_1), x_1) \\ & & \downarrow{\scriptstyle (a,b,\alpha)} \\ (U, V \to u(U), x) & \xrightarrow{(a_2,b_2,\alpha_2)} & (U_2, V_2 \to u(U_2), x_2) \end{array}$$

Now it is clear how to construct the morphism f_1. Namely, set $U' = U \times_{U_2} U_1$ with projections $c : U' \to U$ and $a_1 : U' \to U_1$. Pick a strongly cartesian morphism $\gamma : x' \to x$ lifting the morphism c. Since $b_2 = b \circ b_1$, and since $u(U') = u(U) \times_{u(U_2)} u(U_1)$ we see that $\phi' = (\phi, \phi_1 \circ b_1) : V \to u(U')$. Since α is strongly cartesian, and $a \circ a_1 = a_2 \circ c = p(\alpha_2 \circ \gamma)$ there exists a morphism $\alpha_1 : x' \to x_1$ lifting a_1 such that $\alpha \circ \alpha_1 = \alpha_2 \circ \gamma$. Set $X' = (U', \phi' : V \to u(U'), x')$. Thus we see that

$$f_1 = ((a_1, b_1, \alpha_1) : X' \to X_1, (c, \mathrm{id}_V, \gamma) : X' \to X) : X \longrightarrow X_1$$

works, in fact the diagram

$$\begin{array}{ccc} (U', \phi' : V \to u(U'), x') & \xrightarrow{(a_1,b_1,\alpha_1)} & (U_1, V_1 \to u(U_1), x_1) \\ {\scriptstyle (c,\mathrm{id}_V,\gamma)}\downarrow & & \downarrow{\scriptstyle (a,b,\alpha)} \\ (U, V \to u(U), x) & \xrightarrow{(a_2,b_2,\alpha_2)} & (U_2, V_2 \to u(U_2), x_2) \end{array}$$

is commutative by construction. This proves existence.

Next we prove uniqueness, still in the special case $f = ((a, b, \alpha), 1)$ and $f_2 = ((a_2, b_2, \alpha_2), 1)$. We strongly advise the reader to skip this part. Suppose that

$g_1, g_1' : X \to X_1$ are two morphisms of $u_p\mathcal{S}$ such that $p_p(g_1) = p_p(g_1') = b_1$ and $f_2 = f \circ g_1 = f \circ g_1'$. Our goal is to show that $g_1 = g_1'$. By Categories, Lemma 4.27.13 we may represent g_1 and g_1' as the equivalence classes of $(f_1 : X' \to X_1, r : X' \to X)$ and $(f_1' : X' \to X_1, r : X' \to X)$ for some $r \in R$. By Categories, Lemma 4.27.14 we see that $f_2 = f \circ g_1 = f \circ g_1'$ means that there exists a morphism $r' : X'' \to X'$ in $u_{pp}\mathcal{S}$ such that $r' \circ r \in R$ and

$$(a, b, \alpha) \circ f_1 \circ r' = (a, b, \alpha) \circ f_1' \circ r' = (a_2, b_2, \alpha_2) \circ r'$$

in $u_{pp}\mathcal{S}$. Note that now g_1 is represented by $(f_1 \circ r', r \circ r')$ and similarly for g_1'. Hence we may assume that

$$(a, b, \alpha) \circ f_1 = (a, b, \alpha) \circ f_1' = (a_2, b_2, \alpha_2).$$

Write $r = (c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x')$, $f_1 = (a_1, b_1, \alpha_1)$, and $f_1' = (a_1', b_1, \alpha_1')$. Here we have used the condition that $p_p(g_1) = p_p(g_1')$. The equalities above are now equivalent to $a \circ a_1 = a \circ a_1' = a_2 \circ c$ and $\alpha \circ \alpha_1 = \alpha \circ \alpha_1' = \alpha_2 \circ \gamma$. It need not be the case that $a_1 = a_1'$ in this situation. Thus we have to precompose by one more morphism from R. Namely, let $U'' = \mathrm{Eq}(a_1, a_1')$ be the equalizer of a_1 and a_1' which is a subobject of U'. Denote $c' : U'' \to U'$ the canonical monomorphism. Because of the relations among the morphisms above we see that $V \to u(U')$ maps into $u(U'') = u(\mathrm{Eq}(a_1, a_1')) = \mathrm{Eq}(u(a_1), u(a_1'))$. Hence we get a new object $(U'', \phi'' : V \to u(U''), x'')$, where $\gamma' : x'' \to x'$ is a strongly cartesian morphism lifting γ. Then we see that we may precompose f_1 and f_1' with the element $(c', \mathrm{id}_V, \gamma')$ of R. After doing this, i.e., replacing $(U', \phi' : V \to u(U'), x')$ with $(U'', \phi'' : V \to u(U''), x'')$, we get back to the previous situation where in addition we now have that $a_1 = a_1'$. In this case it follows formally from the fact that α is strongly cartesian (!) that $\alpha_1 = \alpha_1'$. This shows that $g_1 = g_1'$ as desired.

We omit the proof of the fact that for any strongly cartesian morphism of $u_p\mathcal{S}$ of the form $((a, b, \alpha), 1)$ the morphism α is strongly cartesian in $\mathcal{S}$. (We do not need the characterization of strongly cartesian morphisms in the rest of the proof, although we do use it later in this section.)

Let $(U, \phi : V \to u(U), x)$ be an object of $u_p\mathcal{S}$. Let $b : V' \to V$ be a morphism of $\mathcal{D}$. Then the morphism

$$(\mathrm{id}_U, b, \mathrm{id}_x) : (U, \phi \circ b : V' \to u(U), x) \longrightarrow (U, \phi : V \to u(U), x)$$

is strongly cartesian by the result of the preceding paragraphs and we win. □

04WH **Lemma 8.12.7.** *With notation and assumptions as in Lemma 8.12.6. If $\mathcal{S}$ is fibred in groupoids, then $u_p\mathcal{S}$ is fibred in groupoids.*

Proof. By Lemma 8.12.6 we know that $u_p\mathcal{S}$ is a fibred category. Let $f : X \to Y$ be a morphism of $u_p\mathcal{S}$ with $p_p(f) = \mathrm{id}_V$. We are done if we can show that f is invertible, see Categories, Lemma 4.35.2. Write f as the equivalence class of a pair $((a, b, \alpha), r)$ with $r \in R$. Then $p_p(r) = \mathrm{id}_V$, hence $p_{pp}((a, b, \alpha)) = \mathrm{id}_V$. Hence $b = \mathrm{id}_V$. But any morphism of $\mathcal{S}$ is strongly cartesian, see Categories, Lemma 4.35.2 hence we see that $(a, b, \alpha) \in R$ is invertible in $u_p\mathcal{S}$ as desired. □

04WI **Lemma 8.12.8.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be categories over $\mathcal{C}$ and $\mathcal{D}$. Assume that*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a fibred category,*
(2) *$q : \mathcal{T} \to \mathcal{D}$ is a fibred category,*

(3) $\mathcal{C}$ *has nonempty finite limits, and*
(4) $u : \mathcal{C} \to \mathcal{D}$ *commutes with nonempty finite limits.*

Then we have a canonical equivalence of categories

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. In this proof we use the notation x/U to denote an object x of $\mathcal{S}$ which lies over U in $\mathcal{C}$. Similarly y/V denotes an object y of $\mathcal{T}$ which lies over V in $\mathcal{D}$. In the same vein $\alpha/a : x/U \to x'/U'$ denotes the morphism $\alpha : x \to x'$ with image $a : U \to U'$ in $\mathcal{C}$.

Let $G : u_p\mathcal{S} \to \mathcal{T}$ be a 1-morphism of fibred categories over $\mathcal{D}$. Denote $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ the composition of G with the canonical (localization) functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. Then consider the functor $H : \mathcal{S} \to u^p\mathcal{T}$ given by

$$H(x/U) = (U, G'(U, \mathrm{id}_{u(U)} : u(U) \to u(U), x))$$

on objects and by

$$H((\alpha, a) : x/U \to x'/U') = G'(a, u(a), \alpha)$$

on morphisms. Since G transforms strongly cartesian morphisms into strongly cartesian morphisms, we see that if α is strongly cartesian, then $H(\alpha)$ is strongly cartesian. Namely, we've seen in the proof of Lemma 8.12.6 that in this case the map $(a, u(a), \alpha)$ becomes strongly cartesian in $u_p\mathcal{S}$. Clearly this construction is functorial in G and we obtain a functor

$$A : \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T}) \longrightarrow \mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T})$$

Conversely, let $H : \mathcal{S} \to u^p\mathcal{T}$ be a 1-morphism of fibred categories. Recall that an object of $u^p\mathcal{T}$ is a pair (U, y) with $y \in \mathrm{Ob}(\mathcal{T}_{u(U)})$. We denote $\mathrm{pr} : u^p\mathcal{T} \to \mathcal{T}$ the functor $(U, y) \mapsto y$. In this case we define a functor $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ by the rules

$$G'(U, \phi : V \to u(U), x) = \phi^*\mathrm{pr}(H(x))$$

on objects and we let

$$G'((a, b, \alpha) : (U, \phi : V \to u(U), x) \to (U', \phi' : V' \to u(U'), x')) = \beta$$

be the unique morphism $\beta : \phi^*\mathrm{pr}(H(x)) \to (\phi')^*\mathrm{pr}(H(x'))$ such that $q(\beta) = b$ and the diagram

$$\begin{array}{ccc} \phi^*\mathrm{pr}(H(x)) & \xrightarrow[\beta]{} & (\phi')^*\mathrm{pr}(H(x')) \\ \downarrow & & \downarrow \\ \mathrm{pr}(H(x)) & \xrightarrow{\mathrm{pr}(H(a,\alpha))} & \mathrm{pr}(H(x')) \end{array}$$

Such a morphism exists and is unique because $\mathcal{T}$ is a fibred category.

We check that $G'(r)$ is an isomorphism if $r \in R$. Namely, if

$$(a, \mathrm{id}_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α strongly cartesian is an element of the right multiplicative system R of Lemma 8.12.5 then $H(\alpha)$ is strongly cartesian, and $\mathrm{pr}(H(\alpha))$ is strongly cartesian, see proof of Lemma 8.12.1. Hence in this case the morphism β has $q(\beta) = \mathrm{id}_V$ and is strongly cartesian. Hence β is an isomorphism by Categories, Lemma 4.33.2. Thus by Categories, Lemma 4.27.16 we obtain a canonical extension $G : u_p\mathcal{S} \to \mathcal{T}$.

Next, let us prove that G transforms strongly cartesian morphisms into strongly cartesian morphisms. Suppose that $f : X \to Y$ is a strongly cartesian. By the characterization of strongly cartesian morphisms in $u_p\mathcal{S}$ we can write f as $((a, b, \alpha) : X' \to Y, r : X' \to Y)$ where $r \in R$ and α strongly cartesian in $\mathcal{S}$. By the above it suffices to show that $G'(a, b\alpha)$ is strongly cartesian. As before the condition that α is strongly cartesian implies that $\mathrm{pr}(H(a, \alpha)) : \mathrm{pr}(H(x)) \to \mathrm{pr}(H(x'))$ is strongly cartesian in $\mathcal{T}$. Since in the commutative square above now all arrows except possibly β is strongly cartesian it follows that also β is strongly cartesian as desired. Clearly the construction $H \mapsto G$ is functorial in H and we obtain a functor

$$B : \mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) \longrightarrow \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

To finish the proof of the lemma we have to show that the functors A and B are mutually quasi-inverse. We omit the verifications. □

04WJ **Definition 8.12.9.** Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $\mathcal{S}$ be a stack over $\mathcal{C}$. In this setting we write $f^{-1}\mathcal{S}$ for the stackification of the fibred category $u_p\mathcal{S}$ over $\mathcal{D}$ constructed above. We say that $f^{-1}\mathcal{S}$ is the *pullback of $\mathcal{S}$ along f*.

Of course, if $\mathcal{S}$ is a stack in groupoids, then $f^{-1}\mathcal{S}$ is a stack in groupoids by Lemmas 8.9.1 and 8.12.7.

04WK **Lemma 8.12.10.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be stacks. Then we have a canonical equivalence of categories*

$$\mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, f_*\mathcal{T}) = \mathrm{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. For $i = 1, 2$ an i-morphism of stacks is the same thing as a i-morphism of fibred categories, see Definition 8.4.5. By Lemma 8.12.8 we have already

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

Hence the result follows from Lemma 8.8.3 as $u^p\mathcal{T} = f_*\mathcal{T}$ and $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$. □

04WR **Lemma 8.12.11.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category, and let $\mathcal{S} \to \mathcal{S}'$ be the stackification of $\mathcal{S}$. Then $f^{-1}\mathcal{S}'$ is the stackification of $u_p\mathcal{S}$.*

Proof. Omitted. Hint: This is the analogue of Sites, Lemma 7.13.4. □

The following lemma tells us that the 2-category of stacks over Sch_{fppf} is a "full 2-sub category" of the 2-category of stacks over Sch'_{fppf} provided that Sch'_{fppf} contains Sch_{fppf} (see Topologies, Section 34.12).

04WS **Lemma 8.12.12.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor satisfying the assumptions of Sites, Lemma 7.21.8. Let $f : \mathcal{D} \to \mathcal{C}$ be the corresponding morphism of sites. Then*

(1) *for every stack $p : \mathcal{S} \to \mathcal{C}$ the canonical functor $\mathcal{S} \to f_* f^{-1}\mathcal{S}$ is an equivalence of stacks,*
(2) *given stacks $\mathcal{S}, \mathcal{S}'$ over $\mathcal{C}$ the construction f^{-1} induces an equivalence*
$$\mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}')$$
of morphism categories.

Proof. Note that by Lemma 8.12.10 we have an equivalence of categories
$$\mathrm{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}') = \mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, f_* f^{-1}\mathcal{S}')$$
Hence (2) follows from (1).

To prove (1) we are going to use Lemma 8.4.8. This lemma tells us that we have to show that $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is fully faithful and that all objects of $f_* f^{-1}\mathcal{S}$ are locally in the essential image.

We quickly describe the functor can, see proof of Lemma 8.12.8. To do this we introduce the functor $c'' : \mathcal{S} \to u_{pp}\mathcal{S}$ defined by $c''(x/U) = (U, \mathrm{id} : u(U) \to u(U), x)$, and $c''(\alpha/a) = (a, u(a), \alpha)$. We set $c' : \mathcal{S} \to u_p\mathcal{S}$ equal to the composition of c'' and the canonical functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. We set $c : \mathcal{S} \to f^{-1}\mathcal{S}$ equal to the composition of c' and the canonical functor $u_p\mathcal{S} \to f^{-1}\mathcal{S}$. Then $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is the functor which to x/U associates the pair $(U, c(x))$ and to α/a the morphism $(a, c(\alpha))$.

Fully faithfulness. To prove this we are going to use Lemma 8.4.7. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $x, y \in \mathcal{S}_U$. First off, as u is fully faithful, we have
$$\mathrm{Mor}_{(f_* f^{-1}\mathcal{S})_U}(can(x), can(y)) = \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U)}}(c(x), c(y))$$
directly from the definition of f_*. Similar holds after pulling back to any U'/U. Because $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$, and since u is continuous and cocontinuous the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U')}}(c(x|_{U'}), c(y|_{U'}))$$
is the sheafification of the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(u_p\mathcal{S})_{u(U')}}(c'(x|_{U'}), c'(y|_{U'}))$$
Hence to finish the proof of fully faithfulness it suffices to show that for any U and x, y the map
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) \longrightarrow \mathrm{Mor}_{(u_p\mathcal{S})_U}(c'(x), c'(y))$$
is bijective. A morphism $f : x \to y$ in $u_p\mathcal{S}$ over $u(U)$ is given by an equivalence class of diagrams
$$\begin{array}{ccc} (U', \phi : u(U) \to u(U'), x') & \xrightarrow[(a,b,\alpha)]{} & (U, \mathrm{id} : u(U) \to u(U), y) \\ \downarrow{\scriptstyle (c, \mathrm{id}_{u(U)}, \gamma)} & & \\ (U, \mathrm{id} : u(U) \to u(U), x) & & \end{array}$$
with γ strongly cartesian and $b = \mathrm{id}_{u(U)}$. But since u is fully faithful we can write $\phi = u(c')$ for some morphism $c' : U \to U'$ and then we see that $a \circ c' = \mathrm{id}_U$ and $c \circ c' = \mathrm{id}_{U'}$. Because γ is strongly cartesian we can find a morphism $\gamma' : x \to x'$ lifting c' such that $\gamma \circ \gamma' = \mathrm{id}_x$. By definition of the equivalence classes defining morphisms in $u_p\mathcal{S}$ it follows that the morphism
$$(U, \mathrm{id} : u(U) \to u(U), x) \xrightarrow[(\mathrm{id}, \mathrm{id}, \alpha \circ \gamma')]{} (U, \mathrm{id} : u(U) \to u(U), y)$$

of $u_{pp}\mathcal{S}$ induces the morphism f in $u_p\mathcal{S}$. This proves that the map is surjective. We omit the proof that it is injective.

Finally, we have to show that any object of $f_*f^{-1}\mathcal{S}$ locally comes from an object of $\mathcal{S}$. This is clear from the constructions (details omitted). □

8.13. Stacks and localization

04WT Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. We want to understand stacks over $\mathcal{C}/U$ as stacks over $\mathcal{C}$ together with a morphism towards U. The following lemma is the reason why this is easier to do when the presheaf h_U is a sheaf.

04WU **Lemma 8.13.1.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then $j_U : \mathcal{C}/U \to \mathcal{C}$ is a stack over $\mathcal{C}$ if and only if h_U is a sheaf.*

Proof. Combine Lemma 8.6.3 with Categories, Example 4.38.7. □

Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. We denote $j : \mathcal{C}/U \to \mathcal{C}$ the localization functor.

Construction A. Let $p : \mathcal{S} \to \mathcal{C}/U$ be a stack over the site $\mathcal{C}/U$. We define a stack $j_!p : j_!\mathcal{S} \to \mathcal{C}$ as follows:

(1) As a category $j_!\mathcal{S} = \mathcal{S}$, and
(2) the functor $j_!p : j_!\mathcal{S} \to \mathcal{C}$ is just the composition $j \circ p$.

We omit the verification that this is a stack (hint: Use that h_U is a sheaf to glue morphisms to U). There is a canonical functor

$$j_!\mathcal{S} \longrightarrow \mathcal{C}/U$$

namely the functor p which is a 1-morphism of stacks over $\mathcal{C}$.

Construction B. Let $q : \mathcal{T} \to \mathcal{C}$ be a stack over $\mathcal{C}$ which is endowed with a morphism of stacks $p : \mathcal{T} \to \mathcal{C}/U$ over $\mathcal{C}$. In this case it is automatically the case that $p : \mathcal{T} \to \mathcal{C}/U$ is a stack over $\mathcal{C}/U$.

04WV **Lemma 8.13.2.** *Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. Constructions A and B above define mutually inverse (!) functors of 2-categories*

$$\left\{\begin{array}{c}\text{2-category of}\\ \text{stacks over } \mathcal{C}/U\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{2-category of pairs } (\mathcal{T}, p) \text{ consisting}\\ \text{of a stack } \mathcal{T} \text{ over } \mathcal{C} \text{ and a morphism}\\ p : \mathcal{T} \to \mathcal{C}/U \text{ of stacks over } \mathcal{C}\end{array}\right\}$$

Proof. This is clear. □

8.14. Other chapters

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CHAPTER 9

Fields

09FA

9.1. Introduction

09FB In this chapter, we shall discuss the theory of fields. Recall that a *field* is a nonzero ring in which all nonzero elements are invertible. Equivalently, the only two ideals of a field are (0) and (1) since any nonzero element is a unit. Consequently fields will be the simplest cases of much of the theory developed later.

The theory of field extensions has a different feel from standard commutative algebra since, for instance, any morphism of fields is injective. Nonetheless, it turns out that questions involving rings can often be reduced to questions about fields. For instance, any domain can be embedded in a field (its quotient field), and any *local ring* (that is, a ring with a unique maximal ideal; we have not defined this term yet) has associated to it its residue field (that is, its quotient by the maximal ideal). A knowledge of field extensions will thus be useful.

9.2. Basic definitions

09FC Because we have placed this chapter before the chapter discussing commutative algebra we need to introduce some of the basic definitions here before we discuss these in greater detail in the algebra chapters.

09FD **Definition 9.2.1.** A *field* is a nonzero ring where every nonzero element is invertible. Given a field a *subfield* is a subring that is itself a field.

For a field k, we write k^* for the subset $k \setminus \{0\}$. This generalizes the usual notation R^* that refers to the group of invertible elements in a ring R.

09FE **Definition 9.2.2.** A *domain* or an *integral domain* is a nonzero ring where 0 is the only zerodivisor.

9.3. Examples of fields

09FF To get started, let us begin by providing several examples of fields. The reader should recall that if R is a ring and $I \subset R$ an ideal, then R/I is a field precisely when I is a maximal ideal.

09FG **Example 9.3.1** (Rational numbers)**.** The rational numbers form a field. It is called the *field of rational numbers* and denoted $\mathbf{Q}$.

09FH **Example 9.3.2** (Prime fields)**.** If p is a prime number, then $\mathbf{Z}/(p)$ is a field, denoted $\mathbf{F}_p$. Indeed, (p) is a maximal ideal in $\mathbf{Z}$. Thus, fields may be finite: $\mathbf{F}_p$ contains p elements.

09FI **Example 9.3.3.** In a principal ideal domain, an ideal generated by an irreducible element is maximal. Now, if k is a field, then the polynomial ring $k[x]$ is a principal ideal domain. It follows that if $P \in k[x]$ is an irreducible polynomial (that is, a nonconstant polynomial that does not admit a factorization into terms of smaller degrees), then $k[x]/(P)$ is a field. It contains a copy of k in a natural way. This is a very general way of constructing fields. For instance, the complex numbers $\mathbf{C}$ can be constructed as $\mathbf{R}[x]/(x^2+1)$.

09FJ **Example 9.3.4** (Quotient fields)**.** Recall that, given a domain A, there is an imbedding $A \to F$ into a field F constructed from A in exactly the same manner that $\mathbf{Q}$ is constructed from $\mathbf{Z}$. Formally the elements of F are (equivalence classes of) fractions a/b, $a, b \in A$, $b \neq 0$. As usual $a/b = a'/b'$ if and only if $ab' = ba'$. The field F is called the *quotient field*, or *field of fractions*, or *fraction field* of A. The quotient field has the following universal property: given an injective ring map $\varphi : A \to K$ to a field K, there is a unique map $\psi : F \to K$ making

$$\begin{array}{ccc} F & \xrightarrow{} & K \\ \uparrow & \overset{\psi}{\nearrow} & \\ A & \overset{\varphi}{\nearrow} & \end{array}$$

commute. Indeed, it is clear how to define such a map: we set $\psi(a/b) = \varphi(a)\varphi(b)^{-1}$ where injectivity of φ assures that $\varphi(b) \neq 0$ if $b \neq 0$.

09FK **Example 9.3.5** (Field of rational functions)**.** If k is a field, then we can consider the field $k(x)$ of rational functions over k. This is the quotient field of the polynomial ring $k[x]$. In other words, it is the set of quotients F/G for $F, G \in k[x]$, $G \neq 0$ with the obvious equivalence relation.

09FL **Example 9.3.6.** Let X be a Riemann surface. Let $\mathbf{C}(X)$ denote the set of meromorphic functions on X. Then $\mathbf{C}(X)$ is a ring under multiplication and addition of functions. It turns out that in fact $\mathbf{C}(X)$ is a field. Namely, if a nonzero function $f(z)$ is meromorphic, so is $1/f(z)$. For example, let S^2 be the Riemann sphere; then we know from complex analysis that the ring of meromorphic functions $\mathbf{C}(S^2)$ is the field of rational functions $\mathbf{C}(z)$.

9.4. Vector spaces

09FM One reason fields are so nice is that the theory of modules over fields (i.e. vector spaces), is very simple.

09FN **Lemma 9.4.1.** *If k is a field, then every k-module is free.*

Proof. Indeed, by linear algebra we know that a k-module (i.e. vector space) V has a *basis* $\mathcal{B} \subset V$, which defines an isomorphism from the free vector space on $\mathcal{B}$ to V. □

09FP **Lemma 9.4.2.** *Every exact sequence of modules over a field splits.*

Proof. This follows from Lemma 9.4.1 as every vector space is a projective module. □

This is another reason why much of the theory in future chapters will not say very much about fields, since modules behave in such a simple manner. Note that Lemma 9.4.2 is a statement about the *category* of k-modules (for k a field),

because the notion of exactness is inherently arrow-theoretic, i.e., makes use of purely categorical notions, and can in fact be phrased within a so-called *abelian category*.

Henceforth, since the study of modules over a field is linear algebra, and since the ideal theory of fields is not very interesting, we shall study what this chapter is really about: *extensions* of fields.

9.5. The characteristic of a field

09FQ In the category of rings, there is an *initial object* $\mathbf{Z}$: any ring R has a map from $\mathbf{Z}$ into it in precisely one way. For fields, there is no such initial object. Nonetheless, there is a family of objects such that every field can be mapped into in exactly one way by exactly one of them, and in no way by the others.

Let F be a field. Think of F as a ring to get a ring map $f : \mathbf{Z} \to F$. The image of this ring map is a domain (as a subring of a field) hence the kernel of f is a prime ideal in $\mathbf{Z}$. Hence the kernel of f is either (0) or (p) for some prime number p.

In the first case we see that f is injective, and in this case we think of $\mathbf{Z}$ as a subring of F. Moreover, since every nonzero element of F is invertible we see that it makes sense to talk about $p/q \in F$ for $p, q \in \mathbf{Z}$ with $q \neq 0$. Hence in this case we may and we do think of $\mathbf{Q}$ as a subring of F. One can easily see that this is the smallest subfield of F in this case.

In the second case, i.e., when $\mathrm{Ker}(f) = (p)$ we see that $\mathbf{Z}/(p) = \mathbf{F}_p$ is a subring of F. Clearly it is the smallest subfield of F.

Arguing in this way we see that every field contains a smallest subfield which is either $\mathbf{Q}$ or finite equal to $\mathbf{F}_p$ for some prime number p.

09FR **Definition 9.5.1.** The *characteristic* of a field F is 0 if $\mathbf{Z} \subset F$, or is a prime p if $p = 0$ in F. The *prime subfield of* F is the smallest subfield of F which is either $\mathbf{Q} \subset F$ if the characteristic is zero, or $\mathbf{F}_p \subset F$ if the characteristic is $p > 0$.

It is easy to see that if $E \subset F$ is a subfield, then the characteristic of E is the same as the characteristic of F.

09FS **Example 9.5.2.** The characteristic of $\mathbf{F}_p$ is p, and that of $\mathbf{Q}$ is 0.

9.6. Field extensions

09FT In general, though, we are interested not so much in fields by themselves but in field *extensions*. This is perhaps analogous to studying not rings but *algebras* over a fixed ring. The nice thing for fields is that the notion of a "field over another field" just recovers the notion of a field extension, by the next result.

09FU **Lemma 9.6.1.** *If F is a field and R is a nonzero ring, then any ring homomorphism $\varphi : F \to R$ is injective.*

Proof. Indeed, let $a \in \mathrm{Ker}(\varphi)$ be a nonzero element. Then we have $\varphi(1) = \varphi(a^{-1}a) = \varphi(a^{-1})\varphi(a) = 0$. Thus $1 = \varphi(1) = 0$ and R is the zero ring. □

09FV **Definition 9.6.2.** If F is a field contained in a field E, then E is said to be a *field extension* of F. We shall write E/F to indicate that E is an extension of F.

So if F, F' are fields, and $F \to F'$ is any ring-homomorphism, we see by Lemma 9.6.1 that it is injective, and F' can be regarded as an extension of F, by a slight abuse of language. Alternatively, a field extension of F is just an F-algebra that happens to be a field. This is completely different than the situation for general rings, since a ring homomorphism is not necessarily injective.

Let k be a field. There is a *category* of field extensions of k. An object of this category is an extension E/k, that is a (necessarily injective) morphism of fields

$$k \to E,$$

while a morphism between extensions E/k and E'/k is a k-algebra morphism $E \to E'$; alternatively, it is a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

The set of morphisms from $E \to E'$ in the category of extensions of k will be denoted by $\mathrm{Mor}_k(E, E')$.

09FW **Definition 9.6.3.** A *tower* of fields $E_n/E_{n-1}/\ldots/E_0$ consists of a sequence of extensions of fields E_n/E_{n-1}, E_{n-1}/E_{n-2}, ..., E_1/E_0.

Let us give a few examples of field extensions.

09FX **Example 9.6.4.** Let k be a field, and $P \in k[x]$ an irreducible polynomial. We have seen that $k[x]/(P)$ is a field (Example 9.3.3). Since it is also a k-algebra in the obvious way, it is an extension of k.

09FY **Example 9.6.5.** If X is a Riemann surface, then the field of meromorphic functions $\mathbf{C}(X)$ (Example 9.3.6) is an extension field of $\mathbf{C}$, because any element of $\mathbf{C}$ induces a meromorphic — indeed, holomorphic — constant function on X.

Let F/k be a field extension. Let $S \subset F$ be any subset. Then there is a *smallest* subextension of F (that is, a subfield of F containing k) that contains S. To see this, consider the family of subfields of F containing S and k, and take their intersection; one checks that this is a field. By a standard argument one shows, in fact, that this is the set of elements of F that can be obtained via a finite number of elementary algebraic operations (addition, multiplication, subtraction, and division) involving elements of k and S.

09FZ **Definition 9.6.6.** Let k be a field. If F/k is an extension of fields and $S \subset F$, we write $k(S)$ for the smallest subfield of F containing k and S. We will say that S *generates the field extension* $k(S)/k$. If $S = \{\alpha\}$ is a singleton, then we write $k(\alpha)$ instead of $k(\{\alpha\})$. We say F/k is a *finitely generated field extension* if there exists a finite subset $S \subset F$ with $F = k(S)$.

For instance, $\mathbf{C}$ is generated by i over $\mathbf{R}$.

09G0 **Exercise 9.6.7.** Show that $\mathbf{C}$ does not have a countable set of generators over $\mathbf{Q}$.

Let us now classify extensions generated by one element.

09G1 **Lemma 9.6.8** (Classification of simple extensions)**.** *If a field extension F/k is generated by one element, then it is k-isomorphic either to the rational function field $k(t)/k$ or to one of the extensions $k[t]/(P)$ for $P \in k[t]$ irreducible.*

We will see that many of the most important cases of field extensions are generated by one element, so this is actually useful.

Proof. Let $\alpha \in F$ be such that $F = k(\alpha)$; by assumption, such an α exists. There is a morphism of rings

$$k[t] \to F$$

sending the indeterminate t to α. The image is a domain, so the kernel is a prime ideal. Thus, it is either (0) or (P) for $P \in k[t]$ irreducible.

If the kernel is (P) for $P \in k[t]$ irreducible, then the map factors through $k[t]/(P)$, and induces a morphism of fields $k[t]/(P) \to F$. Since the image contains α, we see easily that the map is surjective, hence an isomorphism. In this case, $k[t]/(P) \simeq F$.

If the kernel is trivial, then we have an injection $k[t] \to F$. One may thus define a morphism of the quotient field $k(t)$ into F; given a quotient $R(t)/Q(t)$ with $R(t), Q(t) \in k[t]$, we map this to $R(\alpha)/Q(\alpha)$. The hypothesis that $k[t] \to F$ is injective implies that $Q(\alpha) \neq 0$ unless Q is the zero polynomial. The quotient field of $k[t]$ is the rational function field $k(t)$, so we get a morphism $k(t) \to F$ whose image contains α. It is thus surjective, hence an isomorphism. □

0H7K **Lemma 9.6.9.** *Let k be a field and let E/k and F/k be field extensions. Then there exists a common field extension M/k, i.e., an extension field such that there exist maps $E \to M$ and $F \to M$ of extensions of k.*

Proof. We only prove this when E is a finitely generated field extension of k; the general case follows from this by a Zorn's lemma type argument (details omitted).

First, suppose that E is a simple extension of k. By Lemma 9.6.8 this means either $E = k(t)$ is the rational function field or $E = k[t]/(P)$ for some irreducible polynomial $P \in k[t]$. In the first case, we take $M = F(t)$ the rational function field with obvious maps $E \to M$ and $F \to M$. In the second case, we choose an irreducible factor Q of the image of P in $F[t]$ and we take $M = F[t]/(Q)$ with obvious maps $E \to M$ and $F \to M$.

If $E = k(\alpha_1, \ldots, \alpha_n)$, then by induction on n we can find an extension M/k and maps $F \to M$ and $k(\alpha_1, \ldots, \alpha_{n-1}) \to M$. By the simple case discussed in the previous paragraph, we can find an extension $M'/k(\alpha_1, \ldots, \alpha_{n-1})$ and maps $M \to M'$ and $k(\alpha_1, \ldots, \alpha_n) \to M'$. Then M' viewed as an extension of k works. □

9.7. Finite extensions

09G2 If F/E is a field extension, then evidently F is also a vector space over E (the scalar action is just multiplication in F).

09G3 **Definition 9.7.1.** Let F/E be an extension of fields. The dimension of F considered as an E-vector space is called the *degree* of the extension and is denoted $[F : E]$. If $[F : E] < \infty$ then F is said to be a *finite* extension of E.

09G4 **Example 9.7.2.** The field $\mathbf{C}$ is a two dimensional vector space over $\mathbf{R}$ with basis $1, i$. Thus $\mathbf{C}$ is a finite extension of $\mathbf{R}$ of degree 2.

09G5 **Lemma 9.7.3.** *Let $K/E/F$ be a tower of algebraic field extensions. If K is finite over F, then K is finite over E.*

Proof. Direct from the definition. □

Let us now consider the degree in the most important special example, that given by Lemma 9.6.8, in the next two examples.

09G6 **Example 9.7.4** (Degree of a rational function field)**.** If k is any field, then the rational function field $k(t)$ is *not* a finite extension. For example the elements $\{t^n, n \in \mathbf{Z}\}$ are linearly independent over k.

In fact, if k is uncountable, then $k(t)$ is *uncountably* dimensional as a k-vector space. To show this, we claim that the family of elements $\{1/(t-\alpha), \alpha \in k\} \subset k(t)$ is linearly independent over k. A nontrivial relation between them would lead to a contradiction: for instance, if one works over $\mathbf{C}$, then this follows because $\frac{1}{t-\alpha}$, when considered as a meromorphic function on $\mathbf{C}$, has a pole at α and nowhere else. Consequently any sum $\sum c_i \frac{1}{t-\alpha_i}$ for the $c_i \in k^*$, and $\alpha_i \in k$ distinct, would have poles at each of the α_i. In particular, it could not be zero.

Amusingly, this leads to a quick proof of the Hilbert Nullstellensatz over the complex numbers. For a slightly more general result, see Algebra, Theorem 10.35.11.

0BU1 **Lemma 9.7.5.** *A finite extension of fields is a finitely generated field extension. The converse is not true.*

Proof. Let F/E be a finite extension of fields. Let $\alpha_1, \ldots, \alpha_n$ be a basis of F as a vector space over E. Then $F = E(\alpha_1, \ldots, \alpha_n)$ hence F/E is a finitely generated field extension. The converse is not true as follows from Example 9.7.4. □

09G7 **Example 9.7.6** (Degree of a simple algebraic extension)**.** Consider a monogenic field extension E/k of the form discussed in Example 9.6.4. In other words, $E = k[t]/(P)$ for $P \in k[t]$ an irreducible polynomial. Then the degree $[E : k]$ is just the degree $d = \deg(P)$ of the polynomial P. Indeed, say

09G8 (9.7.6.1) $$P = a_d t^d + a_{d-1} t^{d-1} + \ldots + a_0.$$

with $a_d \neq 0$. Then the images of $1, t, \ldots, t^{d-1}$ in $k[t]/(P)$ are linearly independent over k, because any relation involving them would have degree strictly smaller than that of P, and P is the element of smallest degree in the ideal (P).

Conversely, the set $S = \{1, t, \ldots, t^{d-1}\}$ (or more properly their images) spans $k[t]/(P)$ as a vector space. Indeed, we have by (9.7.6.1) that $a_d t^d$ lies in the span of S. Since a_d is invertible, we see that t^d is in the span of S. Similarly, the relation $tP(t) = 0$ shows that the image of t^{d+1} lies in the span of $\{1, t, \ldots, t^d\}$ — by what was just shown, thus in the span of S. Working upward inductively, we find that the image of t^n for $n \geq d$ lies in the span of S.

This confirms the observation that $[\mathbf{C} : \mathbf{R}] = 2$, for instance. More generally, if k is a field, and $\alpha \in k$ is not a square, then the irreducible polynomial $x^2 - \alpha \in k[x]$ allows one to construct an extension $k[x]/(x^2 - \alpha)$ of degree two. We shall write this as $k(\sqrt{\alpha})$. Such extensions will be called *quadratic,* for obvious reasons.

The basic fact about the degree is that it is *multiplicative in towers.*

09G9 **Lemma 9.7.7** (Multiplicativity)**.** *Suppose given a tower of fields $F/E/k$. Then*

$$[F : k] = [F : E][E : k]$$

Proof. Let $\alpha_1, \ldots, \alpha_n \in F$ be an E-basis for F. Let $\beta_1, \ldots, \beta_m \in E$ be a k-basis for E. Then the claim is that the set of products $\{\alpha_i\beta_j, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a k-basis for F. Indeed, let us check first that they span F over k.

By assumption, the $\{\alpha_i\}$ span F over E. So if $f \in F$, there are $a_i \in E$ with

$$f = \sum\nolimits_i a_i\alpha_i,$$

and, for each i, we can write $a_i = \sum b_{ij}\beta_j$ for some $b_{ij} \in k$. Putting these together, we find

$$f = \sum\nolimits_{i,j} b_{ij}\alpha_i\beta_j,$$

proving that the $\{\alpha_i\beta_j\}$ span F over k.

Suppose now that there existed a nontrivial relation

$$\sum\nolimits_{i,j} c_{ij}\alpha_i\beta_j = 0$$

for the $c_{ij} \in k$. In that case, we would have

$$\sum\nolimits_i \alpha_i \left(\sum\nolimits_j c_{ij}\beta_j\right) = 0,$$

and the inner terms lie in E as the β_j do. Now E-linear independence of the $\{\alpha_i\}$ shows that the inner sums are all zero. Then k-linear independence of the $\{\beta_j\}$ shows that the c_{ij} all vanish. □

We sidetrack to a slightly tangential definition.

09GA **Definition 9.7.8.** A field K is said to be a *number field* if it has characteristic 0 and the extension $K/\mathbf{Q}$ is finite.

Number fields are the basic objects in algebraic number theory. We shall see later that, for the analog of the integers $\mathbf{Z}$ in a number field, something kind of like unique factorization still holds (though strict unique factorization generally does not!).

9.8. Algebraic extensions

09GB An important class of extensions are those where every element generates a finite extension.

09GC **Definition 9.8.1.** Consider a field extension F/E. An element $\alpha \in F$ is said to be *algebraic* over E if α is the root of some nonzero polynomial with coefficients in E. If all elements of F are algebraic then F is said to be an *algebraic extension* of E.

By Lemma 9.6.8, the subextension $E(\alpha)$ is isomorphic either to the rational function field $E(t)$ or to a quotient ring $E[t]/(P)$ for $P \in E[t]$ an irreducible polynomial. In the latter case, α is algebraic over E (in fact, the proof of Lemma 9.6.8 shows that we can pick P such that α is a root of P); in the former case, it is not.

09GD **Example 9.8.2.** The field $\mathbf{C}$ is algebraic over $\mathbf{R}$. Namely, if $\alpha = a + ib$ in $\mathbf{C}$, then $\alpha^2 - 2a\alpha + a^2 + b^2 = 0$ is a polynomial equation for α over $\mathbf{R}$.

09GE **Example 9.8.3.** Let X be a compact Riemann surface, and let $f \in \mathbf{C}(X) - \mathbf{C}$ any nonconstant meromorphic function on X (see Example 9.3.6). Then it is known that $\mathbf{C}(X)$ is algebraic over the subextension $\mathbf{C}(f)$ generated by f. We shall not prove this.

09GF **Lemma 9.8.4.** *Let $K/E/F$ be a tower of field extensions.*

(1) *If $\alpha \in K$ is algebraic over F, then α is algebraic over E.*
(2) *If K is algebraic over F, then K is algebraic over E.*

Proof. This is immediate from the definitions. □

We now show that there is a deep connection between finiteness and being algebraic.

09GG **Lemma 9.8.5.** *A finite extension is algebraic. In fact, an extension E/k is algebraic if and only if every subextension $k(\alpha)/k$ generated by some $\alpha \in E$ is finite.*

In general, it is very false that an algebraic extension is finite.

Proof. Let E/k be finite, say of degree n. Choose $\alpha \in E$. Then the elements $1, \alpha, \ldots, \alpha^n$ are linearly dependent over k, or we would necessarily have $[E : k] > n$. A relation of linear dependence now gives the desired polynomial that α must satisfy.

For the last assertion, note that a monogenic extension $k(\alpha)/k$ is finite if and only if α is algebraic over k, by Examples 9.7.4 and 9.7.6. So if E/k is algebraic, then each $k(\alpha)/k$, $\alpha \in E$, is a finite extension, and conversely. □

We can extract a lemma of the last proof (really of Examples 9.7.4 and 9.7.6): a monogenic extension is finite if and only if it is algebraic. We shall use this observation in the next result.

09GH **Lemma 9.8.6.** *Let k be a field, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be elements of some extension field such that each α_i is algebraic over k. Then the extension $k(\alpha_1, \ldots, \alpha_n)/k$ is finite. That is, a finitely generated algebraic extension is finite.*

Proof. Indeed, each extension $k(\alpha_1, \ldots, \alpha_{i+1})/k(\alpha_1, \ldots, \alpha_i)$ is generated by one element and algebraic, hence finite. By multiplicativity of degree (Lemma 9.7.7) we obtain the result. □

The set of complex numbers that are algebraic over $\mathbf{Q}$ are simply called the *algebraic numbers.* For instance, $\sqrt{2}$ is algebraic, i is algebraic, but π is not. It is a basic fact that the algebraic numbers form a field, although it is not obvious how to prove this from the definition that a number is algebraic precisely when it satisfies a nonzero polynomial equation with rational coefficients (e.g. by polynomial equations).

09GI **Lemma 9.8.7.** *Let E/k be a field extension. Then the elements of E algebraic over k form a subextension of E/k.*

Proof. Let $\alpha, \beta \in E$ be algebraic over k. Then $k(\alpha, \beta)/k$ is a finite extension by Lemma 9.8.6. It follows that $k(\alpha + \beta) \subset k(\alpha, \beta)$ is a finite extension, which implies that $\alpha + \beta$ is algebraic by Lemma 9.8.5. Similarly for the difference, product and quotient of α and β. □

Many nice properties of field extensions, like those of rings, will have the property that they will be preserved by towers and composita.

09GJ **Lemma 9.8.8.** *Let E/k and F/E be algebraic extensions of fields. Then F/k is an algebraic extension of fields.*

Proof. Choose $\alpha \in F$. Then α is algebraic over E. The key observation is that α is algebraic over a finitely generated subextension of k. That is, there is a finite set $S \subset E$ such that α is algebraic over $k(S)$: this is clear because being algebraic means that a certain polynomial in $E[x]$ that α satisfies exists, and as S we can take the coefficients of this polynomial. It follows that α is algebraic over $k(S)$. In particular, the extension $k(S, \alpha)/k(S)$ is finite. Since S is a finite set, and $k(S)/k$ is algebraic, Lemma 9.8.6 shows that $k(S)/k$ is finite. Using multiplicativity (Lemma 9.7.7) we find that $k(S, \alpha)/k$ is finite, so α is algebraic over k. □

The method of proof in the previous argument — that being algebraic over E was a property that *descended* to a finitely generated subextension of E — is an idea that recurs throughout algebra. It often allows one to reduce general commutative algebra questions to the Noetherian case for example.

09GK **Lemma 9.8.9.** *Let E/F be an algebraic extension of fields. Then the cardinality $|E|$ of E is at most* $\max(\aleph_0, |F|)$.

Proof. Let S be the set of nonconstant polynomials with coefficients in F. For every $P \in S$ the set of roots $r(P, E) = \{\alpha \in E \mid P(\alpha) = 0\}$ is finite (details omitted). Moreover, the fact that E is algebraic over F implies that $E = \bigcup_{P \in S} r(P, E)$. It is clear that S has cardinality bounded by $\max(\aleph_0, |F|)$ because it is a countable union of finite products of copies of F. Thus so does E. □

0BID **Lemma 9.8.10.** *Let E/F be a finite or more generally an algebraic extension of fields. Any subring $F \subset R \subset E$ is a field.*

Proof. Let $\alpha \in R$ be nonzero. Then $1, \alpha, \alpha^2, \ldots$ are contained in R. By Lemma 9.8.5 we find a nontrivial relation $a_0 + a_1\alpha + \ldots + a_d\alpha^d = 0$ with $a_i \in F$. We may assume $a_0 \neq 0$ because if not we can divide the relation by α to decrease d. Then we see that

$$a_0 = \alpha(-a_1 - \ldots - a_d\alpha^{d-1})$$

which proves that the inverse of α is the element $a_0^{-1}(-a_1 - \ldots - a_d\alpha^{d-1})$ of R. □

0BMD **Lemma 9.8.11.** *Let E/F an algebraic extension of fields. Any F-algebra map $f : E \to E$ is an automorphism.*

Proof. If E/F is finite, then $f : E \to E$ is an F-linear injective map (Lemma 9.6.1) of finite dimensional vector spaces, and hence bijective. In general we still see that f is injective. Let $\alpha \in E$ and let $P \in F[x]$ be a polynomial such that $P(\alpha) = 0$. Let $E' \subset E$ be the subfield of E generated by the roots $\alpha = \alpha_1, \ldots, \alpha_n$ of P in E. Then E' is finite over F by Lemma 9.8.6. Since f preserves the set of roots, we find that $f|_{E'} : E' \to E'$. Hence $f|_{E'}$ is an isomorphism by the first part of the proof and we conclude that α is in the image of f. □

9.9. Minimal polynomials

09GL Let E/k be a field extension, and let $\alpha \in E$ be algebraic over k. Then α satisfies a (nontrivial) polynomial equation in $k[x]$. Consider the set of polynomials $P \in k[x]$ such that $P(\alpha) = 0$; by hypothesis, this set does not just contain the zero polynomial. It is easy to see that this set is an *ideal*. Indeed, it is the kernel of the map

$$k[x] \to E, \quad x \mapsto \alpha$$

Since $k[x]$ is a PID, there is a *generator* $P \in k[x]$ of this ideal. If we assume P monic, without loss of generality, then P is uniquely determined.

09GM **Definition 9.9.1.** The polynomial P above is called the *minimal polynomial* of α over k.

The minimal polynomial has the following characterization: it is the monic polynomial, of smallest degree, that annihilates α. Any nonconstant multiple of P will have larger degree, and only multiples of P can annihilate α. This explains the name *minimal*.

Clearly the minimal polynomial is *irreducible*. This is equivalent to the assertion that the ideal in $k[x]$ consisting of polynomials annihilating α is prime. This follows from the fact that the map $k[x] \to E, x \mapsto \alpha$ is a map into a domain (even a field), so the kernel is a prime ideal.

09GN **Lemma 9.9.2.** *The degree of the minimal polynomial is* $[k(\alpha) : k]$.

Proof. This is just a restatement of the argument in Lemma 9.6.8: the observation is that if P is the minimal polynomial of α, then the map

$$k[x]/(P) \to k(\alpha), \quad x \mapsto \alpha$$

is an isomorphism as in the aforementioned proof, and we have counted the degree of such an extension (see Example 9.7.6). □

So the observation of the above proof is that if $\alpha \in E$ is algebraic, then $k(\alpha) \subset E$ is isomorphic to $k[x]/(P)$.

9.10. Algebraic closure

09GP The "fundamental theorem of algebra" states that $\mathbf{C}$ is algebraically closed. A beautiful proof of this result uses Liouville's theorem in complex analysis, we shall give another proof (see Lemma 9.23.1).

09GQ **Definition 9.10.1.** A field F is said to be *algebraically closed* if every algebraic extension E/F is trivial, i.e., $E = F$.

This may not be the definition in every text. Here is the lemma comparing it with the other one.

09GR **Lemma 9.10.2.** *Let F be a field. The following are equivalent*

(1) *F is algebraically closed,*
(2) *every irreducible polynomial over F is linear,*
(3) *every nonconstant polynomial over F has a root,*
(4) *every nonconstant polynomial over F is a product of linear factors.*

Proof. If F is algebraically closed, then every irreducible polynomial is linear. Namely, if there exists an irreducible polynomial of degree > 1, then this generates a nontrivial finite (hence algebraic) field extension, see Example 9.7.6. Thus (1) implies (2). If every irreducible polynomial is linear, then every irreducible polynomial has a root, whence every nonconstant polynomial has a root. Thus (2) implies (3).

Assume every nonconstant polynomial has a root. Let $P \in F[x]$ be nonconstant. If $P(\alpha) = 0$ with $\alpha \in F$, then we see that $P = (x - \alpha)Q$ for some $Q \in F[x]$ (by

division with remainder). Thus we can argue by induction on the degree that any nonconstant polynomial can be written as a product $c\prod(x-\alpha_i)$.

Finally, suppose that every nonconstant polynomial over F is a product of linear factors. Let E/F be an algebraic extension. Then all the simple subextensions $F(\alpha)/F$ of E are necessarily trivial (because the only irreducible polynomials are linear by assumption). Thus $E = F$. We see that (4) implies (1) and we are done. □

Now we want to define a "universal" algebraic extension of a field. Actually, we should be careful: the algebraic closure is *not* a universal object. That is, the algebraic closure is not unique up to *unique* isomorphism: it is only unique up to isomorphism. But still, it will be very handy, if not functorial.

09GS **Definition 9.10.3.** Let F be a field. An *algebraic closure* of F is a field $\overline{F}$ containing F such that:

(1) $\overline{F}$ is algebraic over F.
(2) $\overline{F}$ is algebraically closed.

If F is algebraically closed, then F is its own algebraic closure. We now prove the basic existence result.

09GT **Theorem 9.10.4.** *Every field has an algebraic closure.*

The proof will mostly be a red herring to the rest of the chapter. However, we will want to know that it is *possible* to embed a field inside an algebraically closed field, and we will often assume it done.

Proof. Let F be a field. By Lemma 9.8.9 the cardinality of an algebraic extension of F is bounded by $\max(\aleph_0, |F|)$. Choose a set S containing F with $|S| > \max(\aleph_0, |F|)$. Let's consider triples (E, σ_E, μ_E) where

(1) E is a set with $F \subset E \subset S$, and
(2) $\sigma_E : E \times E \to E$ and $\mu_E : E \times E \to E$ are maps of sets such that (E, σ_E, μ_E) defines the structure of a field extension of F (in particular $\sigma_E(a, b) = a +_F b$ for $a, b \in F$ and similarly for μ_E), and
(3) E/F is an algebraic field extension.

The collection of all triples (E, σ_E, μ_E) forms a set I. For $i \in I$ we will denote $E_i = (E_i, \sigma_i, \mu_i)$ the corresponding field extension to F. We define a partial ordering on I by declaring $i \leq i'$ if and only if $E_i \subset E_{i'}$ (this makes sense as E_i and $E_{i'}$ are subsets of the same set S) and we have $\sigma_i = \sigma_{i'}|_{E_i \times E_i}$ and $\mu_i = \mu_{i'}|_{E_i \times E_i}$, in other words, $E_{i'}$ is a field extension of E_i.

Let $T \subset I$ be a totally ordered subset. Then it is clear that $E_T = \bigcup_{i \in T} E_i$ with induced maps $\sigma_T = \bigcup \sigma_i$ and $\mu_T = \bigcup \mu_i$ is another element of I. In other words every totally order subset of I has a upper bound in I. By Zorn's lemma there exists a maximal element (E, σ_E, μ_E) in I. We claim that E is an algebraic closure. Since by definition of I the extension E/F is algebraic, it suffices to show that E is algebraically closed.

To see this we argue by contradiction. Namely, suppose that E is not algebraically closed. Then there exists an irreducible polynomial P over E of degree > 1, see Lemma 9.10.2. By Lemma 9.8.5 we obtain a nontrivial finite extension

$E' = E[x]/(P)$. Observe that E'/F is algebraic by Lemma 9.8.8. Thus the cardinality of E' is $\leq \max(\aleph_0, |F|)$. By elementary set theory we can extend the given injection $E \subset S$ to an injection $E' \to S$. In other words, we may think of E' as an element of our set I contradicting the maximality of E. This contradiction completes the proof. □

09GU **Lemma 9.10.5.** *Let F be a field. Let $\overline{F}$ be an algebraic closure of F. Let M/F be an algebraic extension. Then there is a morphism of F-extensions $M \to \overline{F}$.*

Proof. Consider the set I of pairs (E, φ) where $F \subset E \subset M$ is a subextension and $\varphi : E \to \overline{F}$ is a morphism of F-extensions. We partially order the set I by declaring $(E, \varphi) \leq (E', \varphi')$ if and only if $E \subset E'$ and $\varphi'|_E = \varphi$. If $T = \{(E_t, \varphi_t)\} \subset I$ is a totally ordered subset, then $\bigcup \varphi_t : \bigcup E_t \to \overline{F}$ is an element of I. Thus every totally ordered subset of I has an upper bound. By Zorn's lemma there exists a maximal element (E, φ) in I. We claim that $E = M$, which will finish the proof. If not, then pick $\alpha \in M$, $\alpha \notin E$. The α is algebraic over E, see Lemma 9.8.4. Let P be the minimal polynomial of α over E. Let P^φ be the image of P by φ in $\overline{F}[x]$. Since $\overline{F}$ is algebraically closed there is a root β of P^φ in $\overline{F}$. Then we can extend φ to $\varphi' : E(\alpha) = E[x]/(P) \to \overline{F}$ by mapping x to β. This contradicts the maximality of (E, φ) as desired. □

09GV **Lemma 9.10.6.** *Any two algebraic closures of a field are isomorphic.*

Proof. Let F be a field. If M and $\overline{F}$ are algebraic closures of F, then there exists a morphism of F-extensions $\varphi : M \to \overline{F}$ by Lemma 9.10.5. Now the image $\varphi(M)$ is algebraically closed. On the other hand, the extension $\varphi(M) \subset \overline{F}$ is algebraic by Lemma 9.8.4. Thus $\varphi(M) = \overline{F}$. □

9.11. Relatively prime polynomials

09GW Let K be an algebraically closed field. Then the ring $K[x]$ has a very simple ideal structure as we saw in Lemma 9.10.2. In particular, every polynomial $P \in K[x]$ can be written as

$$P = c(x - \alpha_1) \ldots (x - \alpha_n),$$

where c is the constant term and the $\alpha_1, \ldots, \alpha_n \in k$ are the roots of P (counted with multiplicity). Clearly, the only irreducible polynomials in $K[x]$ are the linear polynomials $c(x - \alpha)$, $c, \alpha \in K$ (and $c \neq 0$).

09GX **Definition 9.11.1.** If k is any field, we say that two polynomials in $k[x]$ are *relatively prime* if they generate the unit ideal in $k[x]$.

Continuing the discussion above, if K is an algebraically closed field, two polynomials in $K[x]$ are relatively prime if and only if they have no common roots. This follows because the maximal ideals of $K[x]$ are of the form $(x - \alpha)$, $\alpha \in K$. So if $F, G \in K[x]$ have no common root, then (F, G) cannot be contained in any $(x - \alpha)$ (as then they would have a common root at α).

If k is *not* algebraically closed, then this still gives information about when two polynomials in $k[x]$ generate the unit ideal.

09GY **Lemma 9.11.2.** *Two polynomials in $k[x]$ are relatively prime precisely when they have no common roots in an algebraic closure $\overline{k}$ of k.*

Proof. The claim is that any two polynomials P, Q generate (1) in $k[x]$ if and only if they generate (1) in $\overline{k}[x]$. This is a piece of linear algebra: a system of linear equations with coefficients in k has a solution if and only if it has a solution in any extension of k. Consequently, we can reduce to the case of an algebraically closed field, in which case the result is clear from what we have already proved. □

9.12. Separable algebraic extensions

09GZ In characteristic p something funny happens with irreducible polynomials over fields. We explain this in the following lemma.

09H0 **Lemma 9.12.1.** *Let F be a field. Let $P \in F[x]$ be an irreducible polynomial over F. Let $P' = dP/dx$ be the derivative of P with respect to x. Then one of the following two cases happens*

(1) *P and P' are relatively prime, or*
(2) *P' is the zero polynomial.*

The second case can only happen if F has characteristic $p > 0$. In this case $P(x) = Q(x^q)$ where $q = p^f$ is a power of p and $Q \in F[x]$ is an irreducible polynomial such that Q and Q' are relatively prime.

Proof. Note that P' has degree $< \deg(P)$. Hence if P and P' are not relatively prime, then $(P, P') = (R)$ where R is a polynomial of degree $< \deg(P)$ contradicting the irreducibility of P. This proves we have the dichotomy between (1) and (2).

Assume we are in case (2) and $P = a_d x^d + \ldots + a_0$. Then $P' = d a_d x^{d-1} + \ldots + a_1$. In characteristic 0 we see that this forces $a_d, \ldots, a_1 = 0$ which would mean P is constant a contradiction. Thus we conclude that the characteristic p is positive. In this case the condition $P' = 0$ forces $a_i = 0$ whenever p does not divide i. In other words, $P(x) = P_1(x^p)$ for some nonconstant polynomial P_1. Clearly, P_1 is irreducible as well. By induction on the degree we see that $P_1(x) = Q(x^q)$ as in the statement of the lemma, hence $P(x) = Q(x^{pq})$ and the lemma is proved. □

09H1 **Definition 9.12.2.** Let F be a field. Let K/F be an extension of fields.

(1) We say an irreducible polynomial P over F is *separable* if it is relatively prime to its derivative.
(2) Given $\alpha \in K$ algebraic over F we say α is *separable* over F if its minimal polynomial is separable over F.
(3) If K is an algebraic extension of F, we say K is *separable*[1] over F if every element of K is separable over F.

By Lemma 9.12.1 in characteristic 0 every irreducible polynomial is separable, every algebraic element in an extension is separable, and every algebraic extension is separable.

09H2 **Lemma 9.12.3.** *Let $K/E/F$ be a tower of algebraic field extensions.*

(1) *If $\alpha \in K$ is separable over F, then α is separable over E.*
(2) *if K is separable over F, then K is separable over E.*

[1] For nonalgebraic extensions this definition does not make sense and is not the correct one. We refer the reader to Algebra, Sections 10.42 and 10.44.

Proof. We will use Lemma 9.12.1 without further mention. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Then $P' = Q'R + QR'$. Thus if $Q' = 0$, then Q divides P and P' hence $P' = 0$ by the lemma. This proves (1). Part (2) follows immediately from (1) and the definitions. □

09H3 **Lemma 9.12.4.** *Let F be a field. An irreducible polynomial P over F is separable if and only if P has pairwise distinct roots in an algebraic closure of F.*

Proof. Suppose that $\alpha \in \overline{F}$ is a root of both P and P'. Then $P = (x - \alpha)Q$ for some polynomial Q. Taking derivatives we obtain $P' = Q + (x - \alpha)Q'$. Thus α is a root of Q. Hence we see that if P and P' have a common root, then P does not have pairwise distinct roots. Conversely, if P has a repeated root, i.e., $(x-\alpha)^2$ divides P, then α is a root of both P and P'. Combined with Lemma 9.11.2 this proves the lemma. □

09H4 **Lemma 9.12.5.** *Let F be a field and let $\overline{F}$ be an algebraic closure of F. Let $p > 0$ be the characteristic of F. Let P be a polynomial over F. Then the set of roots of P and $P(x^p)$ in $\overline{F}$ have the same cardinality (not counting multiplicity).*

Proof. Clearly, α is a root of $P(x^p)$ if and only if α^p is a root of P. In other words, the roots of $P(x^p)$ are the roots of $x^p - \beta$, where β is a root of P. Thus it suffices to show that the map $\overline{F} \to \overline{F}$, $\alpha \mapsto \alpha^p$ is bijective. It is surjective, as $\overline{F}$ is algebraically closed which means that every element has a pth root. It is injective because $\alpha^p = \beta^p$ implies $(\alpha - \beta)^p = 0$ because the characteristic is p. And of course in a field $x^p = 0$ implies $x = 0$. □

Let F be a field and let P be an irreducible polynomial over F. Then we know that $P = Q(x^q)$ for some separable irreducible polynomial Q (Lemma 9.12.1) where q is a power of the characteristic p (and if the characteristic is zero, then $q = 1$[2] and $Q = P$). By Lemma 9.12.5 the number of roots of P and Q in any algebraic closure of F is the same. By Lemma 9.12.4 this number is equal to the degree of Q.

09H5 **Definition 9.12.6.** Let F be a field. Let P be an irreducible polynomial over F. The *separable degree* of P is the cardinality of the set of roots of P in any algebraic closure of F (see discussion above). Notation $\deg_s(P)$.

The separable degree of P always divides the degree and the quotient is a power of the characteristic. If the characteristic is zero, then $\deg_s(P) = \deg(P)$.

09H6 **Situation 9.12.7.** Here F be a field and K/F is a finite extension generated by elements $\alpha_1, \ldots, \alpha_n \in K$. We set $K_0 = F$ and

$$K_i = F(\alpha_1, \ldots, \alpha_i)$$

to obtain a tower of finite extensions $K = K_n/K_{n-1}/\ldots/K_0 = F$. Denote P_i the minimal polynomial of α_i over K_{i-1}. Finally, we fix an algebraic closure $\overline{F}$ of F.

Let F, K, α_i, and $\overline{F}$ be as in Situation 9.12.7. Suppose that $\varphi : K \to \overline{F}$ is a morphism of extensions of F. Then we obtain maps $\varphi_i : K_i \to \overline{F}$. In particular, we can take the image of $P_i \in K_{i-1}[x]$ by φ_{i-1} to get a polynomial $P_i^\varphi \in \overline{F}[x]$.

[2] A good convention for this chapter is to set $0^0 = 1$.

09H7 **Lemma 9.12.8.** *In Situation 9.12.7 the correspondence*

$$\mathrm{Mor}_F(K, \overline{F}) \longrightarrow \{(\beta_1, \ldots, \beta_n) \text{ as below}\}, \quad \varphi \longmapsto (\varphi(\alpha_1), \ldots, \varphi(\alpha_n))$$

is a bijection. Here the right hand side is the set of n-tuples $(\beta_1, \ldots, \beta_n)$ of elements of $\overline{F}$ such that β_i is a root of P_i^φ.

Proof. Let $(\beta_1, \ldots, \beta_n)$ be an element of the right hand side. We construct a map of fields corresponding to it by induction. Namely, we set $\varphi_0 : K_0 \to \overline{F}$ equal to the given map $K_0 = F \subset \overline{F}$. Having constructed $\varphi_{i-1} : K_{i-1} \to \overline{F}$ we observe that $K_i = K_{i-1}[x]/(P_i)$. Hence we can set φ_i equal to the unique map $K_i \to \overline{F}$ inducing φ_{i-1} on K_{i-1} and mapping x to β_i. This works precisely as β_i is a root of P_i^φ. Uniqueness implies that the two constructions are mutually inverse. □

09H8 **Lemma 9.12.9.** *In Situation 9.12.7 we have* $|\mathrm{Mor}_F(K, \overline{F})| = \prod_{i=1}^n \deg_s(P_i)$.

Proof. This follows immediately from Lemma 9.12.8. Observe that a key ingredient we are tacitly using here is the well-definedness of the separable degree of an irreducible polynomial which was observed just prior to Definition 9.12.6. □

We now use the result above to characterize separable field extensions.

09H9 **Lemma 9.12.10.** *Assumptions and notation as in Situation 9.12.7. If each P_i is separable, i.e., each α_i is separable over K_{i-1}, then*

$$|\mathrm{Mor}_F(K, \overline{F})| = [K : F]$$

and the field extension K/F is separable. If one of the α_i is not separable over K_{i-1}, then $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$.

Proof. If α_i is separable over K_{i-1} then $\deg_s(P_i) = \deg(P_i) = [K_i : K_{i-1}]$ (last equality by Lemma 9.9.2). By multiplicativity (Lemma 9.7.7) we have

$$[K : F] = \prod [K_i : K_{i-1}] = \prod \deg(P_i) = \prod \deg_s(P_i) = |\mathrm{Mor}_F(K, \overline{F})|$$

where the last equality is Lemma 9.12.9. By the exact same argument we get the strict inequality $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$ if one of the α_i is not separable over K_{i-1}.

Finally, assume again that each α_i is separable over K_{i-1}. We will show K/F is separable. Let $\gamma = \gamma_1 \in K$ be arbitrary. Then we can find additional elements $\gamma_2, \ldots, \gamma_m$ such that $K = F(\gamma_1, \ldots, \gamma_m)$ (for example we could take $\gamma_2 = \alpha_1, \ldots, \gamma_{n+1} = \alpha_n$). Then we see by the last part of the lemma (already proven above) that if γ is not separable over F we would have the strict inequality $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$ contradicting the very first part of the lemma (already prove above as well). □

09HA **Lemma 9.12.11.** *Let K/F be a finite extension of fields. Let $\overline{F}$ be an algebraic closure of F. Then we have*

$$|\mathrm{Mor}_F(K, \overline{F})| \leq [K : F]$$

with equality if and only if K is separable over F.

Proof. This is a corollary of Lemma 9.12.10. Namely, since K/F is finite we can find finitely many elements $\alpha_1, \ldots, \alpha_n \in K$ generating K over F (for example we can choose the α_i to be a basis of K over F). If K/F is separable, then each α_i is separable over $F(\alpha_1, \ldots, \alpha_{i-1})$ by Lemma 9.12.3 and we get equality by Lemma 9.12.10. On the other hand, if we have equality, then no matter how we choose

$\alpha_1, \ldots, \alpha_n$ we get that α_1 is separable over F by Lemma 9.12.10. Since we can start the sequence with an arbitrary element of K it follows that K is separable over F. □

09HB **Lemma 9.12.12.** *Let E/k and F/E be separable algebraic extensions of fields. Then F/k is a separable extension of fields.*

Proof. Choose $\alpha \in F$. Then α is separable algebraic over E. Let $P = x^d + \sum_{i<d} a_i x^i$ be the minimal polynomial of α over E. Each a_i is separable algebraic over k. Consider the tower of fields

$$k \subset k(a_0) \subset k(a_0, a_1) \subset \ldots \subset k(a_0, \ldots, a_{d-1}) \subset k(a_0, \ldots, a_{d-1}, \alpha)$$

Because a_i is separable algebraic over k it is separable algebraic over $k(a_0, \ldots, a_{i-1})$ by Lemma 9.12.3. Finally, α is separable algebraic over $k(a_0, \ldots, a_{d-1})$ because it is a root of P which is irreducible (as it is irreducible over the possibly bigger field E) and separable (as it is separable over E). Thus $k(a_0, \ldots, a_{d-1}, \alpha)$ is separable over k by Lemma 9.12.10 and we conclude that α is separable over k as desired. □

09HC **Lemma 9.12.13.** *Let E/k be a field extension. Then the elements of E separable over k form a subextension of E/k.*

Proof. Let $\alpha, \beta \in E$ be separable over k. Then β is separable over $k(\alpha)$ by Lemma 9.12.3. By Lemma 9.12.10 (applied with $n = 2$, $\alpha_1 = \alpha$, and $\alpha_2 = \beta$) we see that $k(\alpha, \beta)$ is separable over k. □

9.13. Linear independence of characters

0CKK Here is the statement.

0CKL **Lemma 9.13.1.** *Let L be a field. Let G be a monoid, for example a group. Let $\chi_1, \ldots, \chi_n : G \to L$ be pairwise distinct homomorphisms of monoids where L is regarded as a monoid by multiplication. Then $\chi_1, \ldots, \chi_n$ are L-linearly independent: if $\lambda_1, \ldots, \lambda_n \in L$ not all zero, then $\sum \lambda_i \chi_i(g) \neq 0$ for some $g \in G$.*

Proof. If $n = 1$ this is true because $\chi_1(e) = 1$ if $e \in G$ is the neutral (identity) element. We prove the result by induction for $n > 1$. Suppose that $\lambda_1, \ldots, \lambda_n \in L$ not all zero. If $\lambda_i = 0$ for some, then we win by induction on n. Since we want to show that $\sum \lambda_i \chi_i(g) \neq 0$ for some $g \in G$ we may after dividing by $-\lambda_n$ assume that $\lambda_n = -1$. Then the only way we get in trouble is if

$$\chi_n(g) = \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(g)$$

for all $g \in G$. Fix $h \in G$. Then we would also get

$$\begin{aligned} \chi_n(h)\chi_n(g) &= \chi_n(hg) \\ &= \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(hg) \\ &= \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(h)\chi_i(g) \end{aligned}$$

Multiplying the previous relation by $\chi_n(h)$ and subtracting we obtain

$$0 = \sum\nolimits_{i=1,\ldots,n-1} \lambda_i (\chi_n(h) - \chi_i(h))\chi_i(g)$$

for all $g \in G$. Since $\lambda_i \neq 0$ we conclude that $\chi_n(h) = \chi_i(h)$ for all i by induction. The choice of h above was arbitrary, so we conclude that $\chi_i = \chi_n$ for $i \leq n-1$ which contradicts the assumption that our characters χ_i are pairwise distinct. □

0EM9 **Lemma 9.13.2.** *Let L be a field. Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in L$ pairwise distinct elements of L. Then there exists an $e \geq 0$ such that $\sum_{i=1,\ldots,n} \alpha_i^e \neq 0$.*

Proof. Apply linear independence of characters (Lemma 9.13.1) to the monoid homomorphisms $\mathbf{Z}_{\geq 0} \to L$, $e \mapsto \alpha_i^e$. □

0CKM **Lemma 9.13.3.** *Let K/F and L/F be field extensions. Let $\sigma_1, \ldots, \sigma_n : K \to L$ be pairwise distinct morphisms of F-extensions. Then $\sigma_1, \ldots, \sigma_n$ are L-linearly independent: if $\lambda_1, \ldots, \lambda_n \in L$ not all zero, then $\sum \lambda_i \sigma_i(\alpha) \neq 0$ for some $\alpha \in K$.*

Proof. Apply Lemma 9.13.1 to the restrictions of σ_i to the groups of units. □

0CKN **Lemma 9.13.4.** *Let K/F and L/F be field extensions with K/F finite separable and L algebraically closed. Then the map*

$$K \otimes_F L \longrightarrow \prod\nolimits_{\sigma \in \mathrm{Hom}_F(K,L)} L, \quad \alpha \otimes \beta \mapsto (\sigma(\alpha)\beta)_\sigma$$

is an isomorphism of L-algebras.

Proof. Choose a basis $\alpha_1, \ldots, \alpha_n$ of K as a vector space over F. By Lemma 9.12.11 (and a tiny omitted argument) the set $\mathrm{Hom}_F(K, L)$ has n elements, say $\sigma_1, \ldots, \sigma_n$. In particular, the two sides have the same dimension n as vector spaces over L. Thus if the map is not an isomorphism, then it has a kernel. In other words, there would exist $\mu_j \in L$, $j = 1, \ldots, n$ not all zero, with $\sum \alpha_j \otimes \mu_j$ in the kernel. In other words, $\sum \sigma_i(\alpha_j)\mu_j = 0$ for all i. This would mean the $n \times n$ matrix with entries $\sigma_i(\alpha_j)$ is not invertible. Thus we can find $\lambda_1, \ldots, \lambda_n \in L$ not all zero, such that $\sum \lambda_i \sigma_i(\alpha_j) = 0$ for all j. Now any element $\alpha \in K$ can be written as $\alpha = \sum \beta_j \alpha_j$ with $\beta_j \in F$ and we would get

$$\sum \lambda_i \sigma_i(\alpha) = \sum \lambda_i \sigma_i(\sum \beta_j \alpha_j) = \sum \beta_j \sum \lambda_i \sigma_i(\alpha_j) = 0$$

which contradicts Lemma 9.13.3. □

9.14. Purely inseparable extensions

09HD Purely inseparable extensions are the opposite of the separable extensions defined in the previous section. These extensions only show up in positive characteristic.

09HE **Definition 9.14.1.** Let F be a field of characteristic $p > 0$. Let K/F be an extension.

(1) An element $\alpha \in K$ is *purely inseparable* over F if there exists a power q of p such that $\alpha^q \in F$.
(2) The extension K/F is said to be *purely inseparable* if and only if every element of K is purely inseparable over F.

If we have a field extension L/M (with no condition on the characteristic of M), then we will say the extension is *purely inseparable* if either $L = M$ or the characteristic of M is a prime number p and L/M is purely inseparable in the sense defined above.

Observe that a purely inseparable extension is necessarily algebraic. Let F be a field of characteristic $p > 0$. An example of a purely inseparable extension is gotten by adjoining the pth root of an element $t \in F$ which does not yet have one. Namely, the lemma below shows that $P = x^p - t$ is irreducible, and hence

$$K = F[x]/(P) = F[t^{1/p}]$$

is a field. And K is purely inseparable over F because every element

$$a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{(p-1)/p}, \quad a_i \in F$$

of K has pth power equal to

$$(a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{(p-1)/p})^p = a_0^p + a_1^p t + \ldots + a_{p-1}^p t^{p-1} \in F$$

This situation occurs for the field $\mathbf{F}_p(t)$ of rational functions over $\mathbf{F}_p$.

09HF **Lemma 9.14.2.** *Let p be a prime number. Let F be a field of characteristic p. Let $t \in F$ be an element which does not have a pth root in F. Then the polynomial $x^p - t$ is irreducible over F.*

Proof. To see this, suppose that we have a factorization $x^p - t = fg$. Taking derivatives we get $f'g + fg' = 0$. Note that neither $f' = 0$ nor $g' = 0$ as the degrees of f and g are smaller than p. Moreover, $\deg(f') < \deg(f)$ and $\deg(g') < \deg(g)$. We conclude that f and g have a factor in common. Thus if $x^p - t$ is reducible, then it is of the form $x^p - t = cf^n$ for some irreducible f, $c \in F^*$, and $n > 1$. Since p is a prime number this implies $n = p$ and f linear, which would imply $x^p - t$ has a root in F. Contradiction. □

We will see that taking pth roots is a very important operation in characteristic p.

09HG **Lemma 9.14.3.** *Let E/k and F/E be purely inseparable extensions of fields. Then F/k is a purely inseparable extension of fields.*

Proof. Say the characteristic of k is p. Choose $\alpha \in F$. Then $\alpha^q \in E$ for some p-power q. Whereupon $(\alpha^q)^{q'} \in k$ for some p-power q'. Hence $\alpha^{qq'} \in k$. □

09HH **Lemma 9.14.4.** *Let E/k be a field extension. Then the elements of E purely-inseparable over k form a subextension of E/k.*

Proof. Let p be the characteristic of k. Let $\alpha, \beta \in E$ be purely inseparable over k. Say $\alpha^q \in k$ and $\beta^{q'} \in k$ for some p-powers q, q'. If q'' is a p-power, then $(\alpha + \beta)^{q''} = \alpha^{q''} + \beta^{q''}$. Hence if $q'' \geq q, q'$, then we conclude that $\alpha + \beta$ is purely inseparable over k. Similarly for the difference, product and quotient of α and β. □

09HI **Lemma 9.14.5.** *Let E/F be a finite purely inseparable field extension of characteristic $p > 0$. Then there exists a sequence of elements $\alpha_1, \ldots, \alpha_n \in E$ such that we obtain a tower of fields*

$$E = F(\alpha_1, \ldots, \alpha_n) \supset F(\alpha_1, \ldots, \alpha_{n-1}) \supset \ldots \supset F(\alpha_1) \supset F$$

such that each intermediate extension is of degree p and comes from adjoining a pth root. Namely, $\alpha_i^p \in F(\alpha_1, \ldots, \alpha_{i-1})$ is an element which does not have a pth root in $F(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 1, \ldots, n$.

Proof. By induction on the degree of E/F. If the degree of the extension is 1 then the result is clear (with $n = 0$). If not, then choose $\alpha \in E$, $\alpha \notin F$. Say $\alpha^{p^r} \in F$ for some $r > 0$. Pick r minimal and replace α by $\alpha^{p^{r-1}}$. Then $\alpha \notin F$, but $\alpha^p \in F$. Then $t = \alpha^p$ is not a pth power in F (because that would imply $\alpha \in F$, see Lemma 9.12.5 or its proof). Thus $F \subset F(\alpha)$ is a subextension of degree p (Lemma 9.14.2). By induction we find $\alpha_1, \ldots, \alpha_n \in E$ generating $E/F(\alpha)$ satisfying the conclusions of the lemma. The sequence $\alpha, \alpha_1, \ldots, \alpha_n$ does the job for the extension E/F. □

030K **Lemma 9.14.6.** *Let E/F be an algebraic field extension. There exists a unique subextension $E/E_{sep}/F$ such that E_{sep}/F is separable and E/E_{sep} is purely inseparable.*

Proof. If the characteristic is zero we set $E_{sep} = E$. Assume the characteristic is $p > 0$. Let E_{sep} be the set of elements of E which are separable over F. This is a subextension by Lemma 9.12.13 and of course E_{sep} is separable over F. Given an α in E there exists a p-power q such that α^q is separable over F. Namely, q is that power of p such that the minimal polynomial of α is of the form $P(x^q)$ with P separable algebraic, see Lemma 9.12.1. Hence E/E_{sep} is purely inseparable. Uniqueness is clear. □

030L **Definition 9.14.7.** Let E/F be an algebraic field extension. Let E_{sep} be the subextension found in Lemma 9.14.6.

(1) The integer $[E_{sep} : F]$ is called the *separable degree* of the extension. Notation $[E : F]_s$.
(2) The integer $[E : E_{sep}]$ is called the *inseparable degree*, or the *degree of inseparability* of the extension. Notation $[E : F]_i$.

Of course in characteristic 0 we have $[E : F] = [E : F]_s$ and $[E : F]_i = 1$. By multiplicativity (Lemma 9.7.7) we have

$$[E : F] = [E : F]_s [E : F]_i$$

even in case some of these degrees are infinite. In fact, the separable degree and the inseparable degree are multiplicative too (see Lemma 9.14.9).

09HJ **Lemma 9.14.8.** *Let K/F be a finite extension. Let $\overline{F}$ be an algebraic closure of F. Then $[K : F]_s = |\operatorname{Mor}_F(K, \overline{F})|$.*

Proof. We first prove this when K/F is purely inseparable. Namely, we claim that in this case there is a unique map $K \to \overline{F}$. This can be seen by choosing a sequence of elements $\alpha_1, \ldots, \alpha_n \in K$ as in Lemma 9.14.5. The irreducible polynomial of α_i over $F(\alpha_1, \ldots, \alpha_{i-1})$ is $x^p - \alpha_i^p$. Applying Lemma 9.12.9 we see that $|\operatorname{Mor}_F(K, \overline{F})| = 1$. On the other hand, $[K : F]_s = 1$ in this case hence the equality holds.

Let's return to a general finite extension K/F. In this case choose $F \subset K_s \subset K$ as in Lemma 9.14.6. By Lemma 9.12.11 we have $|\operatorname{Mor}_F(K_s, \overline{F})| = [K_s : F] = [K : F]_s$. On the other hand, every field map $\sigma' : K_s \to \overline{F}$ extends to a unique field map $\sigma : K \to \overline{F}$ by the result of the previous paragraph. In other words $|\operatorname{Mor}_F(K, \overline{F})| = |\operatorname{Mor}_F(K_s, \overline{F})|$ and the proof is done. □

09HK **Lemma 9.14.9** (Multiplicativity)**.** *Suppose given a tower of algebraic field extensions $K/E/F$. Then*

$$[K : F]_s = [K : E]_s [E : F]_s \quad \text{and} \quad [K : F]_i = [K : E]_i [E : F]_i$$

Proof. We first prove this in case K is finite over F. Since we have multiplicativity for the usual degree (by Lemma 9.7.7) it suffices to prove one of the two formulas. By Lemma 9.14.8 we have $[K : F]_s = |\operatorname{Mor}_F(K, \overline{F})|$. By the same lemma, given any $\sigma \in \operatorname{Mor}_F(E, \overline{F})$ the number of extensions of σ to a map $\tau : K \to \overline{F}$ is $[K : E]_s$. Namely, via $E \cong \sigma(E) \subset \overline{F}$ we can view $\overline{F}$ as an algebraic closure of E. Combined with the fact that there are $[E : F]_s = |\operatorname{Mor}_F(E, \overline{F})|$ choices for σ we obtain the result.

We omit the proof if the extensions are infinite. □

9.15. Normal extensions

09HL Let $P \in F[x]$ be a nonconstant polynomial over a field F. We say P *splits completely into linear factors over* F or *splits completely over* F if there exist $c \in F^*$, $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in F$ such that

$$P = c(x - \alpha_1) \ldots (x - \alpha_n)$$

in $F[x]$. Normal extensions are defined as follows.

09HM **Definition 9.15.1.** Let E/F be an algebraic field extension. We say E is *normal* over F if for all $\alpha \in E$ the minimal polynomial P of α over F splits completely into linear factors over E.

As in the case of separable extensions, it takes a bit of work to establish the basic properties of this notion.

09HN **Lemma 9.15.2.** *Let $K/E/F$ be a tower of algebraic field extensions. If K is normal over F, then K is normal over E.*

Proof. Let $\alpha \in K$. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Hence, if P splits completely over K, then so does Q. □

09HP **Lemma 9.15.3.** *Let F be a field. Let M/F be an algebraic extension. Let $M/E_i/F$, $i \in I$ be subextensions with E_i/F normal. Then $\bigcap E_i$ is normal over F.*

Proof. Direct from the definitions. □

0EXK **Lemma 9.15.4.** *Let E/F be a normal algebraic field extension. Then the subextension $E/E_{sep}/F$ of Lemma 9.14.6 is normal.*

Proof. If the characteristic is zero, then $E_{sep} = E$, and the result is clear. If the characteristic is $p > 0$, then E_{sep} is the set of elements of E which are separable over F. Then if $\alpha \in E_{sep}$ has minimal polynomial P write $P = c(x - \alpha)(x - \alpha_2) \ldots (x - \alpha_d)$ with $\alpha_2, \ldots, \alpha_d \in E$. Since P is a separable polynomial and since α_i is a root of P, we conclude $\alpha_i \in E_{sep}$ as desired. □

09HQ **Lemma 9.15.5.** *Let E/F be an algebraic extension of fields. Let $\overline{F}$ be an algebraic closure of F. The following are equivalent*

(1) *E is normal over F, and*
(2) *for every pair $\sigma, \sigma' \in \operatorname{Mor}_F(E, \overline{F})$ we have $\sigma(E) = \sigma'(E)$.*

Proof. Let $\mathcal{P}$ be the set of all minimal polynomials over F of all elements of E. Set

$$T = \{\beta \in \overline{F} \mid P(\beta) = 0 \text{ for some } P \in \mathcal{P}\}$$

It is clear that if E is normal over F, then $\sigma(E) = T$ for all $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Thus we see that (1) implies (2).

Conversely, assume (2). Pick $\beta \in T$. We can find a corresponding $\alpha \in E$ whose minimal polynomial $P \in \mathcal{P}$ annihilates β. Because $F(\alpha) = F[x]/(P)$ we can find an element $\sigma_0 \in \mathrm{Mor}_F(F(\alpha), \overline{F})$ mapping α to β. By Lemma 9.10.5 we can extend σ_0 to a $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Whence we see that β is in the common image of all embeddings $\sigma : E \to \overline{F}$. It follows that $\sigma(E) = T$ for any σ. Fix a σ. Now let $P \in \mathcal{P}$. Then we can write

$$P = (x - \beta_1) \ldots (x - \beta_n)$$

for some n and $\beta_i \in \overline{F}$ by Lemma 9.10.2. Observe that $\beta_i \in T$. Thus $\beta_i = \sigma(\alpha_i)$ for some $\alpha_i \in E$. Thus $P = (x - \alpha_1) \ldots (x - \alpha_n)$ splits completely over E. This finishes the proof. □

0BR3 **Lemma 9.15.6.** *Let E/F be an algebraic extension of fields. If E is generated by $\alpha_i \in E$, $i \in I$ over F and if for each i the minimal polynomial of α_i over F splits completely in E, then E/F is normal.*

Proof. Let P_i be the minimal polynomial of α_i over F. Let $\alpha_i = \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,d_i}$ be the roots of P_i over E. Given two embeddings $\sigma, \sigma' : E \to \overline{F}$ over F we see that

$$\{\sigma(\alpha_{i,1}), \ldots, \sigma(\alpha_{i,d_i})\} = \{\sigma'(\alpha_{i,1}), \ldots, \sigma'(\alpha_{i,d_i})\}$$

because both sides are equal to the set of roots of P_i in $\overline{F}$. The elements $\alpha_{i,j}$ generate E over F and we find that $\sigma(E) = \sigma'(E)$. Hence E/F is normal by Lemma 9.15.5. □

0BME **Lemma 9.15.7.** *Let $L/M/K$ be a tower of algebraic extensions.*

(1) *If M/K is normal, then any automorphism τ of L/K induces an automorphism $\tau|_M : M \to M$.*
(2) *If L/K is normal, then any K-algebra map $\sigma : M \to L$ extends to an automorphism of L.*

Proof. Choose an algebraic closure $\overline{L}$ of L (Theorem 9.10.4).

Let τ be as in (1). Then $\tau(M) = M$ as subfields of $\overline{L}$ by Lemma 9.15.5 and hence $\tau|_M : M \to M$ is an automorphism.

Let $\sigma : M \to L$ be as in (2). By Lemma 9.10.5 we can extend σ to a map $\tau : L \to \overline{L}$, i.e., such that

$$\begin{array}{ccc} L & \xrightarrow{\tau} & \overline{L} \\ \uparrow & \nearrow_{\sigma} & \uparrow \\ M & \longleftarrow & K \end{array}$$

is commutative. By Lemma 9.15.5 we see that $\tau(L) = L$. Hence $\tau : L \to L$ is an automorphism which extends σ. □

09HR **Definition 9.15.8.** Let E/F be an extension of fields. Then $\mathrm{Aut}(E/F)$ or $\mathrm{Aut}_F(E)$ denotes the automorphism group of E as an object of the category of F-extensions. Elements of $\mathrm{Aut}(E/F)$ are called *automorphisms of E over F* or *automorphisms of E/F*.

Here is a characterization of normal extensions in terms of automorphisms.

09HS **Lemma 9.15.9.** *Let E/F be a finite extension. We have*

$$|Aut(E/F)| \leq [E : F]_s$$

with equality if and only if E is normal over F.

Proof. Choose an algebraic closure $\overline{F}$ of F. Recall that $[E : F]_s = |\mathrm{Mor}_F(E, \overline{F})|$. Pick an element $\sigma_0 \in \mathrm{Mor}_F(E, \overline{F})$. Then the map

$$\mathrm{Aut}(E/F) \longrightarrow \mathrm{Mor}_F(E, \overline{F}), \quad \tau \longmapsto \sigma_0 \circ \tau$$

is injective. Thus the inequality. If equality holds, then every $\sigma \in \mathrm{Mor}_F(E, \overline{F})$ is gotten by precomposing σ_0 by an automorphism. Hence $\sigma(E) = \sigma_0(E)$. Thus E is normal over F by Lemma 9.15.5.

Conversely, assume that E/F is normal. Then by Lemma 9.15.5 we have $\sigma(E) = \sigma_0(E)$ for all $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Thus we get an automorphism of E over F by setting $\tau = \sigma_0^{-1} \circ \sigma$. Whence the map displayed above is surjective. □

0BR4 **Lemma 9.15.10.** *Let L/K be an algebraic normal extension of fields. Let E/K be an extension of fields. Then either there is no K-embedding from L to E or there is one $\tau : L \to E$ and every other one is of the form $\tau \circ \sigma$ where $\sigma \in Aut(L/K)$.*

Proof. Given τ replace L by $\tau(L) \subset E$ and apply Lemma 9.15.7. □

9.16. Splitting fields

09HT The following lemma is a useful tool for constructing normal field extensions.

09HU **Lemma 9.16.1.** *Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. There exists a smallest field extension E/F such that P splits completely over E. Moreover, the field extension E/F is normal and unique up to (nonunique) isomorphism.*

Proof. Choose an algebraic closure $\overline{F}$. Then we can write $P = c(x-\beta_1)\ldots(x-\beta_n)$ in $\overline{F}[x]$, see Lemma 9.10.2. Note that $c \in F^*$. Set $E = F(\beta_1, \ldots, \beta_n)$. Then it is clear that E is minimal with the requirement that P splits completely over E.

Next, let E' be another minimal field extension of F such that P splits completely over E'. Write $P = c(x - \alpha_1)\ldots(x - \alpha_n)$ with $c \in F$ and $\alpha_i \in E'$. Again it follows from minimality that $E' = F(\alpha_1, \ldots, \alpha_n)$. Moreover, if we pick any $\sigma : E' \to \overline{F}$ (Lemma 9.10.5) then we immediately see that $\sigma(\alpha_i) = \beta_{\tau(i)}$ for some permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Thus $\sigma(E') = E$. This implies that E' is a normal extension of F by Lemma 9.15.5 and that $E \cong E'$ as extensions of F thereby finishing the proof. □

09HV **Definition 9.16.2.** Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. The field extension E/F constructed in Lemma 9.16.1 is called the *splitting field of P over F*.

09DT **Lemma 9.16.3.** *Let E/F be a finite extension of fields. There exists a unique smallest finite extension K/E such that K is normal over F.*

Proof. Choose generators $\alpha_1, \ldots, \alpha_n$ of E over F. Let $P_1, \ldots, P_n$ be the minimal polynomials of $\alpha_1, \ldots, \alpha_n$ over F. Set $P = P_1 \ldots P_n$. Observe that $(x-\alpha_1)\ldots(x-\alpha_n)$ divides P, since each $(x-\alpha_i)$ divides P_i. Say $P = (x-\alpha_1)\ldots(x-\alpha_n)Q$. Let K/E be the splitting field of P over E. We claim that K is the splitting field of P over F as well (which implies that K is normal over F). This is clear because K/E is generated by the roots of Q over E and E is generated by the roots of $(x-\alpha_1)\ldots(x-\alpha_n)$ over F, hence K is generated by the roots of P over F.

Uniqueness. Suppose that K'/E is a second smallest extension such that K'/F is normal. Choose an algebraic closure $\overline{F}$ and an embedding $\sigma_0 : E \to \overline{F}$. By Lemma 9.10.5 we can extend σ_0 to $\sigma : K \to \overline{F}$ and $\sigma' : K' \to \overline{F}$. By Lemma 9.15.3 we see that $\sigma(K) \cap \sigma'(K')$ is normal over F. By minimality we conclude that $\sigma(K) = \sigma'(K')$. Thus $\sigma^{-1} \circ \sigma' : K' \to K$ gives an isomorphism of extensions of E. □

0BMF **Definition 9.16.4.** Let E/F be a finite extension of fields. The field extension K/E constructed in Lemma 9.16.3 is called the *normal closure E over F*.

One can construct the normal closure inside any given normal extension.

0BMG **Lemma 9.16.5.** *Let L/K be an algebraic normal extension.*

(1) *If $L/M/K$ is a subextension with M/K finite, then there exists a tower $L/M'/M/K$ with M'/K finite and normal.*
(2) *If $L/M'/M/K$ is a tower with M/K normal and M'/M finite, then there exists a tower $L/M''/M'/M/K$ with M''/M finite and M''/K normal.*

Proof. Proof of (1). Let M' be the smallest subextension of L/K containing M which is normal over K. By Lemma 9.16.3 this is the normal closure of M/K and is finite over K.

Proof of (2). Let $\alpha_1, \ldots, \alpha_n \in M'$ generate M' over M. Let $P_1, \ldots, P_n$ be the minimal polynomials of $\alpha_1, \ldots, \alpha_n$ over K. Let $\alpha_{i,j}$ be the roots of P_i in L. Let $M'' = M(\alpha_{i,j})$. It follows from Lemma 9.15.6 (applied with the set of generators $M \cup \{\alpha_{i,j}\}$) that M'' is normal over K. □

The following lemma can sometimes be used to prove properties of the normal closure.

0EXL **Lemma 9.16.6.** *Let L/K be a finite extension. Let M/L be the normal closure of L over K. Then there is a surjective map*

$$L \otimes_K L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras where the number of tensors can be taken $[L : K]_s \leq [L : K]$.

Proof. Choose an algebraic closure $\overline{K}$ of K. Set $n = [L : K]_s = |\operatorname{Mor}_K(L, \overline{K})|$ with equality by Lemma 9.14.8. Say $\operatorname{Mor}_K(L, \overline{K}) = \{\sigma_1, \ldots, \sigma_n\}$. Let $M' \subset \overline{K}$ be the K-subalgebra generated by $\sigma_i(L)$, $i = 1, \ldots, n$. Then M' is a field since any K-subalgebra of $\overline{K}$ is a field. Any K-algebra map from M' to $\overline{K}$ permutes the σ_i so sends M' into and onto M'. By construction the field M' is generated by conjugates of elements of $\sigma_1(L)$. Having said this it follows from Lemma 9.15.5 that M' is normal over K and that it is the smallest normal subextension of $\overline{K}$

containing $\sigma_1(L)$. By uniqueness of normal closure we have $M \cong M'$. Finally, there is a surjective map

$$L \otimes_K L \otimes_K \ldots \otimes_K L \longrightarrow M', \quad \lambda_1 \otimes \ldots \otimes \lambda_n \longmapsto \sigma_1(\lambda_1) \ldots \sigma_n(\lambda_n)$$

and note that $n \leq [L : K]$ by definition. □

9.17. Roots of unity

09HW Let F be a field. For an integer $n \geq 1$ we set

$$\mu_n(F) = \{\zeta \in F \mid \zeta^n = 1\}$$

This is called the *group of nth roots of unity* or *nth roots of* 1. It is an abelian group under multiplication with neutral element given by 1. Observe that in a field the number of roots of a polynomial of degree d is always at most d. Hence we see that $|\mu_n(F)| \leq n$ as it is defined by a polynomial equation of degree n. Of course every element of $\mu_n(F)$ has order dividing n. Moreover, the subgroups

$$\mu_d(F) \subset \mu_n(F), \quad d|n$$

each have at most d elements. This implies that $\mu_n(F)$ is cyclic.

09HX **Lemma 9.17.1.** *Let A be an abelian group of exponent dividing n such that $\{x \in A \mid dx = 0\}$ has cardinality at most d for all $d|n$. Then A is cyclic of order dividing n.*

Proof. The conditions imply that $|A| \leq n$, in particular A is finite. The structure of finite abelian groups shows that $A = \mathbf{Z}/e_1\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/e_r\mathbf{Z}$ for some integers $1 < e_1|e_2|\ldots|e_r$. This would imply that $\{x \in A \mid e_1 x = 0\}$ has cardinality e_1^r. Hence $r = 1$. □

Applying this to the field $\mathbf{F}_p$ we obtain the celebrated result that the group $(\mathbf{Z}/p\mathbf{Z})^*$ is a cyclic group. More about this in the section on finite fields.

One more observation is often useful: If F has characteristic $p > 0$, then $\mu_{p^n}(F) = \{1\}$. This is true because raising to the pth power is an injective map on fields of characteristic p as we have seen in the proof of Lemma 9.12.5. (Of course, it also follows from the statement of that lemma itself.)

9.18. Finite fields

09HY Let F be a finite field. It is clear that F has positive characteristic as we cannot have an injection $\mathbf{Q} \to F$. Say the characteristic of F is p. The extension $\mathbf{F}_p \subset F$ is finite. Hence we see that F has $q = p^f$ elements for some $f \geq 1$.

Let us think about the group of units F^*. This is a finite abelian group, so it has some exponent e. Then $F^* = \mu_e(F)$ and we see from the discussion in Section 9.17 that F^* is a cyclic group of order $q - 1$. (A posteriori it follows that $e = q - 1$ as well.) In particular, if $\alpha \in F^*$ is a generator then it clearly is true that

$$F = \mathbf{F}_p(\alpha)$$

In other words, the extension $F/\mathbf{F}_p$ is generated by a single element. Of course, the same thing is true for any extension of finite fields E/F (because E is already generated by a single element over the prime field).

9.19. Primitive elements

09HZ Let E/F be a finite extension of fields. An element $\alpha \in E$ is called a *primitive element of E over F* if $E = F(\alpha)$.

030N **Lemma 9.19.1** (Primitive element)**.** *Let E/F be a finite extension of fields. The following are equivalent*

(1) *there exists a primitive element for E over F, and*
(2) *there are finitely many subextensions $E/K/F$.*

Moreover, (1) and (2) hold if E/F is separable.

Proof. Let $\alpha \in E$ be a primitive element. Let P be the minimal polynomial of α over F. Next, let $E/K/F$ be a subextension. Let Q be the minimal polynomial of α over K. Observe that $\deg(Q) = [E : K]$. Writing $Q = x^d + \sum_{i<d} a_i x^i$ we claim that K is equal to $L = F(a_0, \ldots, a_{d-1})$. Indeed α has degree d over L and $L \subset K$. Hence $[E : L] = [E : K]$ and it follows that $[K : L] = 1$, i.e., $K = L$. Thus it suffices to show there are at most finitely many possibilities for the polynomial Q. This is clear because we have a factorization $P = QR$ in $K[x]$ in particular in $E[x]$. Since we have unique factorization in $E[x]$ there are at most finitely many monic factors of P in $E[x]$.

If F is a finite field (equivalently E is a finite field), then E/F has a primitive element by the discussion in Section 9.18. Next, assume F is infinite and there are at most finitely many proper subfields $E/K/F$. List them, say $K_1, \ldots, K_N$. Then each $K_i \subset E$ is a proper sub F-vector space. As F is infinite we can find a vector $\alpha \in E$ with $\alpha \notin K_i$ for all i (a vector space can never be equal to a finite union of proper subvector spaces; details omitted). Then α is a primitive element for E over F.

Having established the equivalence of (1) and (2) we now turn to the final statement of the lemma. Choose an algebraic closure $\overline{F}$ of F. Enumerate the elements $\sigma_1, \ldots, \sigma_n \in \mathrm{Mor}_F(E, \overline{F})$. Since E/F is separable we have $n = [E : F]$ by Lemma 9.12.11. Note that if $i \neq j$, then

$$V_{ij} = \mathrm{Ker}(\sigma_i - \sigma_j : E \longrightarrow \overline{F})$$

is not equal to E. Hence arguing as in the preceding paragraph we can find $\alpha \in E$ with $\alpha \notin V_{ij}$ for all $i \neq j$. It follows that $|\mathrm{Mor}_F(F(\alpha), \overline{F})| \geq n$. On the other hand $[F(\alpha) : F] \leq [E : F]$. Hence equality by Lemma 9.12.11 and we conclude that $E = F(\alpha)$. □

9.20. Trace and norm

0BIE Let L/K be a finite extension of fields. By Lemma 9.4.1 we can choose an isomorphism $L \cong K^{\oplus n}$ of K-modules. Of course $n = [L : K]$ is the degree of the field extension. Using this isomorphism we get for a K-algebra map

$$L \longrightarrow \mathrm{Mat}(n \times n, K), \quad \alpha \longmapsto \text{matrix of multiplication by } \alpha$$

Thus given $\alpha \in L$ we can take the trace and the determinant of the corresponding matrix. Of course these quantities are independent of the choice of the basis chosen above. More canonically, simply thinking of L as a finite dimensional K-vector space we have $\mathrm{Trace}_K(\alpha : L \to L)$ and the determinant $\det_K(\alpha : L \to L)$.

0BIF **Definition 9.20.1.** Let L/K be a finite extension of fields. For $\alpha \in L$ we define the *trace* $\mathrm{Trace}_{L/K}(\alpha) = \mathrm{Trace}_K(\alpha : L \to L)$ and the *norm* $\mathrm{Norm}_{L/K}(\alpha) = \det_K(\alpha : L \to L)$.

It is clear from the definition that $\mathrm{Trace}_{L/K}$ is K-linear and satisfies $\mathrm{Trace}_{L/K}(\alpha) = [L : K]\alpha$ for $\alpha \in K$. Similarly $\mathrm{Norm}_{L/K}$ is multiplicative and $\mathrm{Norm}_{L/K}(\alpha) = \alpha^{[L:K]}$ for $\alpha \in K$. This is a special case of the more general construction discussed in Exercises, Exercises 111.22.6 and 111.22.7.

0BIG **Lemma 9.20.2.** *Let L/K be a finite extension of fields. Let $\alpha \in L$ and let P be the minimal polynomial of α over K. Then the characteristic polynomial of the K-linear map $\alpha : L \to L$ is equal to P^e with $e \deg(P) = [L : K]$.*

Proof. Choose a basis $\beta_1, \ldots, \beta_e$ of L over $K(\alpha)$. Then e satisfies $e \deg(P) = [L : K]$ by Lemmas 9.9.2 and 9.7.7. Then we see that $L = \bigoplus K(\alpha)\beta_i$ is a direct sum decomposition into α-invariant subspaces hence the characteristic polynomial of $\alpha : L \to L$ is equal to the characteristic polynomial of $\alpha : K(\alpha) \to K(\alpha)$ to the power e.

To finish the proof we may assume that $L = K(\alpha)$. In this case by Cayley-Hamilton we see that α is a root of the characteristic polynomial. And since the characteristic polynomial has the same degree as the minimal polynomial, we find that equality holds. □

0BIH **Lemma 9.20.3.** *Let L/K be a finite extension of fields. Let $\alpha \in L$ and let $P = x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of α over K. Then*

$$\mathit{Norm}_{L/K}(\alpha) = (-1)^{[L:K]} a_d^e \quad \text{and} \quad \mathit{Trace}_{L/K}(\alpha) = -e a_1$$

where $ed = [L : K]$.

Proof. Follows immediately from Lemma 9.20.2 and the definitions. □

0BII **Lemma 9.20.4.** *Let L/K be a finite extension of fields. Let V be a finite dimensional vector space over L. Let $\varphi : V \to V$ be an L-linear map. Then*

$$\mathit{Trace}_K(\varphi : V \to V) = \mathit{Trace}_{L/K}(\mathit{Trace}_L(\varphi : V \to V))$$

and

$$\det_K(\varphi : V \to V) = \mathit{Norm}_{L/K}(\det_L(\varphi : V \to V))$$

Proof. Choose an isomorphism $V = L^{\oplus n}$ so that φ corresponds to an $n \times n$ matrix. In the case of traces, both sides of the formula are additive in φ. Hence we can assume that φ corresponds to the matrix with exactly one nonzero entry in the (i, j) spot. In this case a direct computation shows both sides are equal.

In the case of norms both sides are zero if φ has a nonzero kernel. Hence we may assume φ corresponds to an element of $\mathrm{GL}_n(L)$. Both sides of the formula are multiplicative in φ. Since every element of $\mathrm{GL}_n(L)$ is a product of elementary matrices we may assume that φ either looks like

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \quad \text{or} \quad E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

(because we may also permute the basis elements if we like). In both cases the formula is easy to verify by direct computation. □

0BIJ **Lemma 9.20.5.** *Let $M/L/K$ be a tower of finite extensions of fields. Then*

$$\mathit{Trace}_{M/K} = \mathit{Trace}_{L/K} \circ \mathit{Trace}_{M/L} \quad \text{and} \quad \mathit{Norm}_{M/K} = \mathit{Norm}_{L/K} \circ \mathit{Norm}_{M/L}$$

Proof. Think of M as a vector space over L and apply Lemma 9.20.4. □

The trace pairing is defined using the trace.

0BIK **Definition 9.20.6.** Let L/K be a finite extension of fields. The *trace pairing* for L/K is the symmetric K-bilinear form

$$Q_{L/K} : L \times L \longrightarrow K, \quad (\alpha, \beta) \longmapsto \mathrm{Trace}_{L/K}(\alpha\beta)$$

It turns out that a finite extension of fields is separable if and only if the trace pairing is nondegenerate.

0BIL **Lemma 9.20.7.** *Let L/K be a finite extension of fields. The following are equivalent:*

(1) *L/K is separable,*
(2) *$\mathit{Trace}_{L/K}$ is not identically zero, and*
(3) *the trace pairing $Q_{L/K}$ is nondegenerate.*

Proof. It is clear that (3) implies (2). If (2) holds, then pick $\gamma \in L$ with $\mathrm{Trace}_{L/K}(\gamma) \neq 0$. Then if $\alpha \in L$ is nonzero, we see that $Q_{L/K}(\alpha, \gamma/\alpha) \neq 0$. Hence $Q_{L/K}$ is nondegenerate. This proves the equivalence of (2) and (3).

Suppose that K has characteristic p and $L = K(\alpha)$ with $\alpha \notin K$ and $\alpha^p \in K$. Then $\mathrm{Trace}_{L/K}(1) = p = 0$. For $i = 1, \ldots, p-1$ we see that $x^p - \alpha^{pi}$ is the minimal polynomial for α^i over K and we find $\mathrm{Trace}_{L/K}(\alpha^i) = 0$ by Lemma 9.20.3. Hence for this kind of purely inseparable degree p extension we see that $\mathrm{Trace}_{L/K}$ is identically zero.

Assume that L/K is not separable. Then there exists a subfield $L/K'/K$ such that L/K' is a purely inseparable degree p extension as in the previous paragraph, see Lemmas 9.14.6 and 9.14.5. Hence by Lemma 9.20.5 we see that $\mathrm{Trace}_{L/K}$ is identically zero.

Assume on the other hand that L/K is separable. By induction on the degree we will show that $\mathrm{Trace}_{L/K}$ is not identically zero. Thus by Lemma 9.20.5 we may assume that L/K is generated by a single element α (use that if the trace is nonzero then it is surjective). We have to show that $\mathrm{Trace}_{L/K}(\alpha^e)$ is nonzero for some $e \geq 0$. Let $P = x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of α over K. Then P is also the characteristic polynomial of the linear maps $\alpha : L \to L$, see Lemma 9.20.2. Since L/k is separable we see from Lemma 9.12.4 that P has d pairwise distinct roots $\alpha_1, \ldots, \alpha_d$ in an algebraic closure $\overline{K}$ of K. Thus these are the eigenvalues of $\alpha : L \to L$. By linear algebra, the trace of α^e is equal to $\alpha_1^e + \ldots + \alpha_d^e$. Thus we conclude by Lemma 9.13.2. □

Let K be a field and let $Q : V \times V \to K$ be a bilinear form on a finite dimensional vector space over K. Say $\dim_K(V) = n$. Then Q defines a linear map $Q : V \to V^*$, $v \mapsto Q(v, -)$ where $V^* = \mathrm{Hom}_K(V, K)$ is the dual vector space. Hence a linear map

$$\det(Q) : \wedge^n(V) \longrightarrow \wedge^n(V)^*$$

If we pick a basis element $\omega \in \wedge^n(V)$, then we can write $\det(Q)(\omega) = \lambda\omega^*$, where ω^* is the dual basis element in $\wedge^n(V)^*$. If we change our choice of ω into $c\omega$ for some

$c \in K^*$, then ω^* changes into $c^{-1}\omega^*$ and therefore λ changes into $c^2\lambda$. Thus the class of λ in $K/(K^*)^2$ is well defined and is called the *discriminant of* Q. Unwinding the definitions we see that

$$\lambda = \det(Q(v_i, v_j)_{1 \leq i,j \leq n})$$

if $\{v_1, \ldots, v_n\}$ is a basis for V over K. Observe that the discriminant is nonzero if and only if Q is nondegenerate.

0BIM **Definition 9.20.8.** Let L/K be a finite extension of fields. The *discriminant of* L/K is the discriminant of the trace pairing $Q_{L/K}$.

By the discussion above and Lemma 9.20.7 we see that the discriminant is nonzero if and only if L/K is separable. For $a \in K$ we often say "the discriminant is a" when it would be more correct to say the discriminant is the class of a in $K/(K^*)^2$.

0BIN **Exercise 9.20.9.** Let L/K be an extension of degree 2. Show that exactly one of the following happens

(1) the discriminant is 0, the characteristic of K is 2, and L/K is purely inseparable obtained by taking a square root of an element of K,
(2) the discriminant is 1, the characteristic of K is 2, and L/K is separable of degree 2,
(3) the discriminant is not a square, the characteristic of K is not 2, and L is obtained from K by taking the square root of the discriminant.

9.21. Galois theory

09DU Here is the definition.

09I0 **Definition 9.21.1.** A field extension E/F is called *Galois* if it is algebraic, separable, and normal.

It turns out that a finite extension is Galois if and only if it has the "correct" number of automorphisms.

09I1 **Lemma 9.21.2.** *Let E/F be a finite extension of fields. Then E is Galois over F if and only if $|Aut(E/F)| = [E : F]$.*

Proof. Assume $|\mathrm{Aut}(E/F)| = [E : F]$. By Lemma 9.15.9 this implies that E/F is separable and normal, hence Galois. Conversely, if E/F is separable then $[E : F] = [E : F]_s$ and if E/F is in addition normal, then Lemma 9.15.9 implies that $|\mathrm{Aut}(E/F)| = [E : F]$. □

Motivated by the lemma above we introduce the Galois group as follows.

09DV **Definition 9.21.3.** If E/F is a Galois extension, then the group $\mathrm{Aut}(E/F)$ is called the *Galois group* and it is denoted $\mathrm{Gal}(E/F)$.

If L/K is an infinite Galois extension, then one should think of the Galois group as a topological group. We will return to this in Section 9.22.

09I2 **Lemma 9.21.4.** *Let $K/E/F$ be a tower of algebraic field extensions. If K is Galois over F, then K is Galois over E.*

Proof. Combine Lemmas 9.15.2 and 9.12.3. □

0EXM **Lemma 9.21.5.** *Let L/K be a finite separable extension of fields. Let M be the normal closure of L over K (Definition 9.16.4). Then M/K is Galois.*

Proof. The subextension $M/M_{sep}/K$ of Lemma 9.14.6 is normal by Lemma 9.15.4. Since L/K is separable we have $L \subset M_{sep}$. By minimality $M = M_{sep}$ and the proof is done. □

Let G be a group acting on a field K (by field automorphisms). We will often use the notation

$$K^G = \{x \in K \mid \sigma(x) = x \ \forall \sigma \in G\}$$

and we will call this the *fixed field* for the action of G on K.

09I3 **Lemma 9.21.6.** *Let K be a field. Let G be a finite group acting faithfully on K. Then the extension K/K^G is Galois, we have $[K : K^G] = |G|$, and the Galois group of the extension is G.*

Proof. Given $\alpha \in K$ consider the orbit $G \cdot \alpha \subset K$ of α under the group action. Consider the polynomial

$$P = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \beta) \in K[x]$$

The key to the whole lemma is that this polynomial is invariant under the action of G and hence has coefficients in K^G. Namely, for $\tau \in G$ we have

$$P^\tau = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \tau(\beta)) = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \beta) = P$$

because the map $\beta \mapsto \tau(\beta)$ is a permutation of the orbit $G \cdot \alpha$. Thus $P \in K^G[x]$. Since also $P(\alpha) = 0$ as α is an element of its orbit we conclude that the extension K/K^G is algebraic. Moreover, the minimal polynomial Q of α over K^G divides the polynomial P just constructed. Hence Q is separable (by Lemma 9.12.4 for example) and we conclude that K/K^G is separable. Thus K/K^G is Galois. To finish the proof it suffices to show that $[K : K^G] = |G|$ since then G will be the Galois group by Lemma 9.21.2.

Pick finitely many elements $\alpha_i \in K$, $i = 1, \ldots, n$ such that $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, n$ implies σ is the neutral element of G. Set

$$L = K^G(\{\sigma(\alpha_i); 1 \leq i \leq n, \sigma \in G\}) \subset K$$

and observe that the action of G on K induces an action of G on L. We will show that L has degree $|G|$ over K^G. This will finish the proof, since if $L \subset K$ is proper, then we can add an element $\alpha \in K$, $\alpha \notin L$ to our list of elements $\alpha_1, \ldots, \alpha_n$ without increasing L which is absurd. This reduces us to the case that K/K^G is finite which is treated in the next paragraph.

Assume K/K^G is finite. By Lemma 9.19.1 we can find $\alpha \in K$ such that $K = K^G(\alpha)$. By the construction in the first paragraph of this proof we see that α has degree at most $|G|$ over K. However, the degree cannot be less than $|G|$ as G acts faithfully on $K^G(\alpha) = L$ by construction and the inequality of Lemma 9.15.9. □

09DW **Theorem 9.21.7** (Fundamental theorem of Galois theory)**.** *Let L/K be a finite Galois extension with Galois group G. Then we have $K = L^G$ and the map*

$$\{\textit{subgroups of } G\} \longrightarrow \{\textit{subextensions } L/M/K\}, \quad H \longmapsto L^H$$

is a bijection whose inverse maps M to $Gal(L/M)$. The normal subgroups H of G correspond exactly to those subextensions M with M/K Galois.

Proof. By Lemma 9.21.4 given a subextension $L/M/K$ the extension L/M is Galois. Of course L/M is also finite (Lemma 9.7.3). Thus $|\mathrm{Gal}(L/M)| = [L : M]$ by Lemma 9.21.2. Conversely, if $H \subset G$ is a finite subgroup, then $[L : L^H] = |H|$ by Lemma 9.21.6. It follows formally from these two observations that we obtain a bijective correspondence as in the theorem.

If $H \subset G$ is normal, then L^H is fixed by the action of G and we obtain a canonical map $G/H \to \mathrm{Aut}(L^H/K)$. This map has to be injective as $\mathrm{Gal}(L/L^H) = H$. Hence $|G/H| = [L^H : K]$ and L^H is Galois by Lemma 9.21.2.

Conversely, assume that $K \subset M \subset L$ with M/K Galois. By Lemma 9.15.7 we see that every element $\tau \in \mathrm{Gal}(L/K)$ induces an element $\tau|_M \in \mathrm{Gal}(M/K)$. This induces a homomorphism of Galois groups $\mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$ whose kernel is H. Thus H is a normal subgroup. □

0BMH **Lemma 9.21.8.** *Let $L/M/K$ be a tower of fields. Assume L/K and M/K are finite Galois. Then we obtain a short exact sequence*

$$1 \to \mathit{Gal}(L/M) \to \mathit{Gal}(L/K) \to \mathit{Gal}(M/K) \to 1$$

of finite groups.

Proof. Namely, by Lemma 9.15.7 we see that every element $\tau \in \mathrm{Gal}(L/K)$ induces an element $\tau|_M \in \mathrm{Gal}(M/K)$ which gives us the homomorphism on the right. The map on the left identifies the left group with the kernel of the right arrow. The sequence is exact because the sizes of the groups work out correctly by multiplicativity of degrees in towers of finite extensions (Lemma 9.7.7). One can also use Lemma 9.15.7 directly to see that the map on the right is surjective. □

9.22. Infinite Galois theory

0BMI The Galois group comes with a canonical topology.

0BMJ **Lemma 9.22.1.** *Let E/F be a Galois extension. Endow $\mathit{Gal}(E/F)$ with the coarsest topology such that*

$$\mathit{Gal}(E/F) \times E \longrightarrow E$$

is continuous when E is given the discrete topology. Then

1. *for any topological space X and map $X \to \mathit{Gal}(E/F)$ such that the action $X \times E \to E$ is continuous the induced map $X \to \mathit{Gal}(E/F)$ is continuous,*
2. *this topology turns $\mathit{Gal}(E/F)$ into a profinite topological group.*

Proof. Throughout this proof we think of E as a discrete topological space. Recall that the compact open topology on the set of self maps $\mathrm{Map}(E, E)$ is the universal topology such that the action $\mathrm{Map}(E, E) \times E \to E$ is continuous. See Topology, Example 5.30.2 for a precise statement. The topology of the lemma on $\mathrm{Gal}(E/F)$ is the induced topology coming from the injective map $\mathrm{Gal}(E/F) \to \mathrm{Map}(E, E)$. Hence the universal property (1) follows from the corresponding universal property of the compact open topology. Since the set of invertible self maps $\mathrm{Aut}(E)$ endowed with the compact open topology forms a topological group, see Topology, Example 5.30.2, and since $\mathrm{Gal}(E/F) = \mathrm{Aut}(E/F) \to \mathrm{Map}(E, E)$ factors through $\mathrm{Aut}(E)$ we obtain a topological group. In other words, we are using the injection

$$\mathrm{Gal}(E/F) \subset \mathrm{Aut}(E)$$

to endow $\mathrm{Gal}(E/F)$ with the induced structure of a topological group (see Topology, Section 5.30) and by construction this is the coarsest structure of a topological group such that the action $\mathrm{Gal}(E/F) \times E \to E$ is continuous.

To show that $\mathrm{Gal}(E/F)$ is profinite we argue as follows (our argument is necessarily nonstandard because we have defined the topology before showing that the Galois group is an inverse limit of finite groups). By Topology, Lemma 5.30.4 it suffices to show that the underlying topological space of $\mathrm{Gal}(E/F)$ is profinite. For any subset $S \subset E$ consider the set

$$G(S) = \{f : S \to E \mid \begin{matrix} f(\alpha) \text{ is a root of the minimal polynomial} \\ \text{of } \alpha \text{ over } F \text{ for all } \alpha \in S \end{matrix}\}$$

Since a polynomial has only a finite number of roots we see that $G(S)$ is finite for all $S \subset E$ finite. If $S \subset S'$ then restriction gives a map $G(S') \to G(S)$. Also, observe that if $\alpha \in S \cap F$ and $f \in G(S)$, then $f(\alpha) = \alpha$ because the minimal polynomial is linear in this case. Consider the profinite topological space

$$G = \lim\nolimits_{S \subset E \text{ finite}} G(S)$$

Consider the canonical map

$$c : \mathrm{Gal}(E/F) \longrightarrow G, \quad \sigma \longmapsto (\sigma|_S : S \to E)_S$$

This is injective and unwinding the definitions the reader sees the topology on $\mathrm{Gal}(E/F)$ as defined above is the induced topology from G. An element $(f_S) \in G$ is in the image of c exactly if (A) $f_S(\alpha) + f_S(\beta) = f_S(\alpha+\beta)$ and (M) $f_S(\alpha)f_S(\beta) = f_S(\alpha\beta)$ whenever this makes sense (i.e., $\alpha, \beta, \alpha+\beta, \alpha\beta \in S$). Namely, this means $\lim f_S : E \to E$ will be an F-algebra map and hence an automorphism by Lemma 9.8.11. The conditions (A) and (M) for a given triple (S, α, β) define a closed subset of G and hence $\mathrm{Gal}(E/F)$ is homeomorphic to a closed subset of a profinite space and therefore profinite itself. □

0BMK **Lemma 9.22.2.** *Let $L/M/K$ be a tower of fields. Assume both L/K and M/K are Galois. Then there is a canonical surjective continuous homomorphism $c : Gal(L/K) \to Gal(M/K)$.*

Proof. By Lemma 9.15.7 given $\tau : L \to L$ in $\mathrm{Gal}(L/K)$ the restriction $\tau|_M : M \to M$ is an element of $\mathrm{Gal}(M/K)$. This defines the homomorphism c. Continuity follows from the universal property of the topology: the action

$$\mathrm{Gal}(L/K) \times M \longrightarrow M, \quad (\tau, x) \longmapsto \tau(x) = c(\tau)(x)$$

is continuous as $M \subset L$ and the action $\mathrm{Gal}(L/K) \times L \to L$ is continuous. Hence continuity of c by part (1) of Lemma 9.22.1. Lemma 9.15.7 also shows that the map is surjective. □

Here is a more standard way to think about the Galois group of an infinite Galois extension.

0BU2 **Lemma 9.22.3.** *Let L/K be a Galois extension with Galois group G. Let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$ with L_λ/K finite Galois with Galois group G_λ. Define a partial ordering on Λ by the rule $\lambda \geq \lambda'$ if and only if $L_\lambda \supset L_{\lambda'}$. Then*

(1) *Λ is a directed partially ordered set,*
(2) *L_λ is a system of K-extensions over Λ and $L = \mathrm{colim}\, L_\lambda$,*

(3) *G_λ is an inverse system of finite groups over Λ, the transition maps are surjective, and*
$$G = \lim_{\lambda\in\Lambda} G_\lambda$$
as a profinite group, and
(4) *each of the projections $G \to G_\lambda$ is continuous and surjective.*

Proof. Every subfield of L containing K is separable over K (follows immediately from the definition). Let $S \subset L$ be a finite subset. Then $K(S)/K$ is finite and there exists a tower $L/E/K(S)/K$ such that E/K is finite Galois, see Lemma 9.16.5. Hence $E = L_\lambda$ for some $\lambda \in \Lambda$. This certainly implies the set Λ is not empty. Also, given $\lambda_1, \lambda_2 \in \Lambda$ we can write $L_{\lambda_i} = K(S_i)$ for finite sets $S_1, S_2 \subset L$ (Lemma 9.7.5). Then there exists a $\lambda \in \Lambda$ such that $K(S_1 \cup S_2) \subset L_\lambda$. Hence $\lambda \geq \lambda_1, \lambda_2$ and Λ is directed (Categories, Definition 4.21.4). Finally, since every element in L is contained in L_λ for some $\lambda \in \Lambda$, it follows from the description of filtered colimits in Categories, Section 4.19 that $\operatorname{colim} L_\lambda = L$.

If $\lambda \geq \lambda'$ in Λ, then we obtain a canonical surjective map $G_\lambda \to G_{\lambda'}$, $\sigma \mapsto \sigma|_{L_{\lambda'}}$ by Lemma 9.21.8. Thus we get an inverse system of finite groups with surjective transition maps.

Recall that $G = \mathrm{Aut}(L/K)$. By Lemma 9.22.2 the restriction $\sigma|_{L_\lambda}$ of a $\sigma \in G$ to L_λ is an element of G_λ. Moreover, this procedure gives a continuous surjection $G \to G_\lambda$. Since the transition mappings in the inverse system of G_λ are given by restriction also, it is clear that we obtain a canonical continuous map
$$G \longrightarrow \lim_{\lambda\in\Lambda} G_\lambda$$
Continuity by definition of limits in the category of topological groups; recall that these limits commute with the forgetful functor to the categories of sets and topological spaces by Topology, Lemma 5.30.3. On the other hand, since $L = \operatorname{colim} L_\lambda$ it is clear that any element of the inverse limit (viewed as a set) defines an automorphism of L. Thus the map is bijective. Since the topology on both sides is profinite, and since a bijective continuous map of profinite spaces is a homeomorphism (Topology, Lemma 5.17.8), the proof is complete. □

0BML **Theorem 9.22.4** (Fundamental theorem of infinite Galois theory)**.** *Let L/K be a Galois extension. Let $G = Gal(L/K)$ be the Galois group viewed as a profinite topological group (Lemma 9.22.1). Then we have $K = L^G$ and the map*
$$\{\text{closed subgroups of } G\} \longrightarrow \{\text{subextensions } L/M/K\}, \quad H \longmapsto L^H$$
is a bijection whose inverse maps M to $Gal(L/M)$. The finite subextensions M correspond exactly to the open subgroups $H \subset G$. The normal closed subgroups H of G correspond exactly to subextensions M Galois over K.

Proof. We will use the result of finite Galois theory (Theorem 9.21.7) without further mention. Let $S \subset L$ be a finite subset. There exists a tower $L/E/K$ such that $K(S) \subset E$ and such that E/K is finite Galois, see Lemma 9.16.5. In other words, we see that L/K is the union of its finite Galois subextensions. For such an E, by Lemma 9.22.2 the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K)$ is surjective and continuous, i.e., the kernel is open because the topology on $\mathrm{Gal}(E/K)$ is discrete. In particular we see that no element of $L \setminus K$ is fixed by $\mathrm{Gal}(L/K)$ as $E^{\mathrm{Gal}(E/K)} = K$. This proves that $L^G = K$.

By Lemma 9.21.4 given a subextension $L/M/K$ the extension L/M is Galois. It is immediate from the definition of the topology on G that the subgroup $\mathrm{Gal}(L/M)$ is closed. By the above applied to L/M we see that $L^{\mathrm{Gal}(L/M)} = M$.

Conversely, let $H \subset G$ be a closed subgroup. We claim that $H = \mathrm{Gal}(L/L^H)$. The inclusion $H \subset \mathrm{Gal}(L/L^H)$ is clear. Suppose that $g \in \mathrm{Gal}(L/L^H)$. Let $S \subset L$ be a finite subset. We will show that the open neighbourhood $U_S(g) = \{g' \in G \mid g'(s) = g(s)\}$ of g meets H. This implies that $g \in H$ because H is closed. Let $L/E/K$ be a finite Galois subextension containing $K(S)$ as in the first paragraph of the proof and consider the homomorphism $c : \mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K)$. Then $L^H \cap E = E^{c(H)}$. Since g fixes L^H it fixes $E^{c(H)}$ and hence $c(g) \in c(H)$ by finite Galois theory. Pick $h \in H$ with $c(h) = c(g)$. Then $h \in U_S(g)$ as desired.

At this point we have established the correspondence between closed subgroups and subextensions.

Assume $H \subset G$ is open. Arguing as above we find that H contains $\mathrm{Gal}(L/E)$ for some large enough finite Galois subextension E and we find that L^H is contained in E whence finite over K. Conversely, if M is a finite subextension, then M is generated by a finite subset S and the corresponding subgroup is the open subset $U_S(e)$ where $e \in G$ is the neutral element.

Assume that $K \subset M \subset L$ with M/K Galois. By Lemma 9.22.2 there is a surjective continuous homomorphism of Galois groups $\mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$ whose kernel is $\mathrm{Gal}(L/M)$. Thus $\mathrm{Gal}(L/M)$ is a normal closed subgroup.

Finally, assume $N \subset G$ is normal and closed. For any $L/E/K$ as in the first paragraph of the proof, the image $c(N) \subset \mathrm{Gal}(E/K)$ is a normal subgroup. Hence $L^N = \bigcup E^{c(N)}$ is a union of Galois extensions of K (by finite Galois theory) whence Galois over K. □

0BMM **Lemma 9.22.5.** *Let $L/M/K$ be a tower of fields. Assume L/K and M/K are Galois. Then we obtain a short exact sequence*

$$1 \to Gal(L/M) \to Gal(L/K) \to Gal(M/K) \to 1$$

of profinite topological groups.

Proof. This is a reformulation of Lemma 9.22.2. □

9.23. The complex numbers

09I4 The fundamental theorem of algebra states that the field of complex numbers is an algebraically closed field. In this section we discuss this briefly.

The first remark we'd like to make is that you need to use a little bit of input from calculus in order to prove this. We will use the intuitively clear fact that every odd degree polynomial over the reals has a real root. Namely, let $P(x) = a_{2k+1}x^{2k+1} + \ldots + a_0 \in \mathbf{R}[x]$ for some $k \geq 0$ and $a_{2k+1} \neq 0$. We may and do assume $a_{2k+1} > 0$. Then for $x \in \mathbf{R}$ very large (positive) we see that $P(x) > 0$ as the term $a_{2k+1}x^{2k+1}$ dominates all the other terms. Similarly, if $x \ll 0$, then $P(x) < 0$ by the same reason (and this is where we use that the degree is odd). Hence by the intermediate value theorem there is an $x \in \mathbf{R}$ with $P(x) = 0$.

A conclusion we can draw from the above is that $\mathbf{R}$ has no nontrivial odd degree field extensions, as elements of such extensions would have odd degree minimal polynomials.

Next, let $K/\mathbf{R}$ be a finite Galois extension with Galois group G. Let $P \subset G$ be a 2-sylow subgroup. Then $K^P/\mathbf{R}$ is an odd degree extension, hence by the above $K^P = \mathbf{R}$, which in turn implies $G = P$. (All of these arguments rely on Galois theory of course.) Thus G is a 2-group. If G is nontrivial, then we see that $\mathbf{C} \subset K$ as $\mathbf{C}$ is (up to isomorphism) the only degree 2 extension of $\mathbf{R}$. If G has more than 2 elements we would obtain a quadratic extension of $\mathbf{C}$. This is absurd as every complex number has a square root.

The conclusion: $\mathbf{C}$ is algebraically closed. Namely, if not then we'd get a nontrivial finite extension $K/\mathbf{C}$ which we could assume normal (hence Galois) over $\mathbf{R}$ by Lemma 9.16.3. But we've seen above that then $K = \mathbf{C}$.

09I5 **Lemma 9.23.1** (Fundamental theorem of algebra)**.** *The field* $\mathbf{C}$ *is algebraically closed.*

Proof. See discussion above. □

9.24. Kummer extensions

09I6 Let K be a field. Let $n \geq 2$ be an integer such that K contains a primitive nth root of 1. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^n = a$. Then L/K is Galois. If $G = \mathrm{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mu_n(K), \quad \sigma \longmapsto \sigma(b)/b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing n as a subgroup of the cyclic group $\mu_n(K)$. Kummer theory gives a converse.

09DX **Lemma 9.24.1** (Kummer extensions)**.** *Let L/K be a Galois extension of fields whose Galois group is $\mathbf{Z}/n\mathbf{Z}$. Assume moreover that the characteristic of K is prime to n and that K contains a primitive nth root of 1. Then $L = K[z]$ with $z^n \in K$.*

Proof. Let $\zeta \in K$ be a primitive nth root of 1. Let σ be a generator of $\mathrm{Gal}(L/K)$. Consider $\sigma : L \to L$ as a K-linear operator. Note that $\sigma^n - 1 = 0$ as a linear operator. Applying linear independence of characters (Lemma 9.13.1), we see that there cannot be a polynomial over K of degree $< n$ annihilating σ. Hence the minimal polynomial of σ as a linear operator is $x^n - 1$. Since ζ is a root of $x^n - 1$ by linear algebra there is a $0 \neq z \in L$ such that $\sigma(z) = \zeta z$. This z satisfies $z^n \in K$ because $\sigma(z^n) = (\zeta z)^n = z^n$. Moreover, we see that $z, \sigma(z), \ldots, \sigma^{n-1}(z) = z, \zeta z, \ldots \zeta^{n-1} z$ are pairwise distinct which guarantees that z generates L over K. Hence $L = K[z]$ as required. □

0EXN **Lemma 9.24.2.** *Let K be a field with algebraic closure $\overline{K}$. Let p be a prime different from the characteristic of K. Let $\zeta \in \overline{K}$ be a primitive pth root of 1. Then $K(\zeta)/K$ is a Galois extension of degree dividing $p - 1$.*

Proof. The polynomial $x^p - 1$ splits completely over $K(\zeta)$ as its roots are $1, \zeta, \zeta^2, \ldots, \zeta^{p-1}$. Hence $K(\zeta)/K$ is a splitting field and hence normal. The extension is separable as $x^p - 1$ is a separable polynomial. Thus the extension is Galois. Any automorphism

of $K(\zeta)$ over K sends ζ to ζ^i for some $1 \leq i \leq p-1$. Thus the Galois group is a subgroup of $(\mathbf{Z}/p\mathbf{Z})^*$. $\square$

0EXP **Lemma 9.24.3.** *Let K be a field. Let L/K be a finite extension of degree e which is generated by an element α with $a = \alpha^e \in K$. Then any sub extension $L/L'/K$ is generated by α^d for some $d|e$.*

Proof. Observe that for $d|e$ the subfield $K(\alpha^d)$ has $[K(\alpha^d) : K] = e/d$ and $[L : K(\alpha^d)] = d$ and that both extensions $K(\alpha^d)/K$ and $L/K(\alpha^d)$ are extensions as in the lemma.

We will use induction on the pair of integers $([L : L'], [L' : K])$ ordered lexicographically. Let p be a prime number dividing e and set $d = e/p$. If $K(\alpha^d)$ is contained in L', then we win by induction, because then it suffices to prove the lemma for $L/L'/K(\alpha^d)$. If not, then $[L'(\alpha^d) : L'] = p$ and by induction hypothesis we have $L'(\alpha^d) = K(\alpha^i)$ for some $i|d$. If $i \neq 1$ we are done by induction. Thus we may assume that $[L : L'] = p$.

If e is not a power of p, then we can do this trick again with a second prime number and we win. Thus we may assume e is a power of p.

If the characteristic of K is p and e is a pth power, then L/K is purely inseparable. Hence L/L' is purely inseparable of degree p and hence $\alpha^p \in L'$. Thus $L' = K(\alpha^p)$ and this case is done.

The final case is where e is a power of p, the characteristic of K is not p, L/L' is a degree p extension, and $L = L'(\alpha^{e/p})$. Claim: this can only happen if $e = p$ and $L' = K$. The claim finishes the proof.

First, we prove the claim when K contains a primitive pth root of unity ζ. In this case the degree p extension $K(\alpha^{e/p})/K$ is Galois with Galois group generated by the automorphism $\alpha^{e/p} \mapsto \zeta\alpha^{e/p}$. On the other hand, since L is generated by $\alpha^{e/p}$ and L' we see that the map

$$K(\alpha^{e/p}) \otimes_K L' \longrightarrow L$$

is an isomorphism of K-algebras (look at dimensions). Thus L has an automorphism σ of order p over K sending $\alpha^{e/p}$ to $\zeta\alpha^{e/p}$. Then $\sigma(\alpha) = \zeta'\alpha$ for some eth root of unity ζ' (as α^e is in K). Then on the one hand $(\zeta')^{e/p} = \zeta$ and on the other hand ζ' has to be a pth root of 1 as σ has order p. Thus $e/p = 1$ and the claim has been shown.

Finally, suppose that K does not contain a primitive pth root of 1. Choose a primitive pth root ζ in some algebraic closure $\overline{L}$ of L. Consider the diagram

$$\begin{array}{ccc} K(\zeta) & \longrightarrow & L(\zeta) \\ \uparrow & & \uparrow \\ K & \longrightarrow & L \end{array}$$

By Lemma 9.24.2 the vertical extensions have degree prime to p. Hence $[L(\zeta) : K(\zeta)]$ is divisible by e. On the other hand, $L(\zeta)$ is generated by α over $K(\zeta)$ and hence $[L(\zeta) : K(\zeta)] \leq e$. Thus $[L(\zeta) : K(\zeta)] = e$. Similarly we have $[K(\alpha^{e/p}, \zeta) : K(\zeta)] = p$ and $[L(\zeta) : L'(\zeta)] = p$. Thus the fields $K(\zeta), L'(\zeta), L(\zeta)$ and the element α fall into the case discussed in the previous paragraph we conclude $e = p$ as desired. $\square$

9.25. Artin-Schreier extensions

09I7 Let K be a field of characteristic $p > 0$. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^p - x = a$. Then L/K is Galois. If $G = \mathrm{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mathbf{Z}/p\mathbf{Z}, \quad \sigma \longmapsto \sigma(b) - b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing p as a subgroup of $\mathbf{Z}/p\mathbf{Z}$. The theory of Artin-Schreier extensions gives a converse.

09DY **Lemma 9.25.1** (Artin-Schreier extensions)**.** *Let L/K be a Galois extension of fields of characteristic $p > 0$ with Galois group $\mathbf{Z}/p\mathbf{Z}$. Then $L = K[z]$ with $z^p - z \in K$.*

Proof. Let σ be a generator of $\mathrm{Gal}(L/K)$. Consider $\sigma : L \to L$ as a K-linear operator. Observe that $\sigma^p - 1 = 0$ as a linear operator. Applying linear independence of characters (Lemma 9.13.1), there cannot be a polynomial of degree $< p$ annihilating σ. We conclude that the minimal polynomial of σ is $x^p - 1 = (x-1)^p$. This implies that there exists $w \in L$ such that $(\sigma - 1)^{p-1}(w) = y$ is nonzero. Then $\sigma(y) = y$, i.e., $y \in K$. Thus $z = y^{-1}(\sigma - 1)^{p-2}(w)$ satisfies $\sigma(z) = z + 1$. Since $z \notin K$ we have $L = K[z]$. Moreover since $\sigma(z^p - z) = (z+1)^p - (z+1) = z^p - z$ we see that $z^p - z \in K$ and the proof is complete. □

9.26. Transcendence

030D We recall the standard definitions.

030E **Definition 9.26.1.** Let K/k be a field extension.

(1) A collection of elements $\{x_i\}_{i \in I}$ of K is called *algebraically independent* over k if the map
$$k[X_i; i \in I] \longrightarrow K$$
which maps X_i to x_i is injective.
(2) The field of fractions of a polynomial ring $k[x_i; i \in I]$ is denoted $k(x_i; i \in I)$.
(3) A *purely transcendental extension* of k is any field extension K/k isomorphic to the field of fractions of a polynomial ring over k.
(4) A *transcendence basis* of K/k is a collection of elements $\{x_i\}_{i \in I}$ which are algebraically independent over k and such that the extension $K/k(x_i; i \in I)$ is algebraic.

09I8 **Example 9.26.2.** The field $\mathbf{Q}(\pi)$ is purely transcendental because π isn't the root of a nonzero polynomial with rational coefficients. In particular, $\mathbf{Q}(\pi) \cong \mathbf{Q}(x)$.

030F **Lemma 9.26.3.** *Let E/F be a field extension. A transcendence basis of E over F exists. Any two transcendence bases have the same cardinality.*

Proof. Let A be an algebraically independent subset of E. Let G be a subset of E containing A that generates E/F. We claim we can find a transcendence basis B such that $A \subset B \subset G$. To prove this, consider the collection $\mathcal{B}$ of algebraically independent subsets whose members are subsets of G that contain A. Define a partial ordering on $\mathcal{B}$ using inclusion. Then $\mathcal{B}$ contains at least one element A. The union of the elements of a totally ordered subset T of $\mathcal{B}$ is an algebraically independent subset of E over F since any algebraic dependence relation would

have occurred in one of the elements of T (since polynomials only involve finitely many variables). The union also contains A and is contained in G. By Zorn's lemma, there is a maximal element $B \in \mathcal{B}$. Now we claim E is algebraic over $F(B)$. This is because if it wasn't then there would be an element $f \in G$ transcendental over $F(B)$ since $F(G) = E$. Then $B \cup \{f\}$ would be algebraically independent contradicting the maximality of B. Thus B is our transcendence basis.

Let B and B' be two transcendence bases. Without loss of generality, we can assume that $|B'| \leq |B|$. Now we divide the proof into two cases: the first case is that B is an infinite set. Then for each $\alpha \in B'$, there is a finite set $B_\alpha \subset B$ such that α is algebraic over $F(B_\alpha)$ since any algebraic dependence relation only uses finitely many indeterminates. Then we define $B^* = \bigcup_{\alpha \in B'} B_\alpha$. By construction, $B^* \subset B$, but we claim that in fact the two sets are equal. To see this, suppose that they are not equal, say there is an element $\beta \in B \setminus B^*$. We know β is algebraic over $F(B')$ which is algebraic over $F(B^*)$. Therefore β is algebraic over $F(B^*)$, a contradiction. So $|B| \leq |\bigcup_{\alpha \in B'} B_\alpha|$. Now if B' is finite, then so is B so we can assume B' is infinite; this means

$$|B| \leq |\bigcup\nolimits_{\alpha \in B'} B_\alpha| = |B'|$$

because each B_α is finite and B' is infinite. Therefore in the infinite case, $|B| = |B'|$.

Now we need to look at the case where B is finite. In this case, B' is also finite, so suppose $B = \{\alpha_1, \ldots, \alpha_n\}$ and $B' = \{\beta_1, \ldots, \beta_m\}$ with $m \leq n$. We perform induction on m: if $m = 0$ then E/F is algebraic so $B = \emptyset$ so $n = 0$. If $m > 0$, there is an irreducible polynomial $f \in F[x, y_1, \ldots, y_n]$ such that $f(\beta_1, \alpha_1, \ldots, \alpha_n) = 0$ and such that x occurs in f. Since β_1 is not algebraic over F, f must involve some y_i so without loss of generality, assume f uses y_1. Let $B^* = \{\beta_1, \alpha_2, \ldots, \alpha_n\}$. We claim that B^* is a basis for E/F. To prove this claim, we see that we have a tower of algebraic extensions

$$E/F(B^*, \alpha_1)/F(B^*)$$

since α_1 is algebraic over $F(B^*)$. Now we claim that B^* (counting multiplicity of elements) is algebraically independent over F because if it weren't, then there would be an irreducible $g \in F[x, y_2, \ldots, y_n]$ such that $g(\beta_1, \alpha_2, \ldots, \alpha_n) = 0$ which must involve x making β_1 algebraic over $F(\alpha_2, \ldots, \alpha_n)$ which would make α_1 algebraic over $F(\alpha_2, \ldots, \alpha_n)$ which is impossible. So this means that $\{\alpha_2, \ldots, \alpha_n\}$ and $\{\beta_2, \ldots, \beta_m\}$ are bases for E over $F(\beta_1)$ which means by induction, $m = n$. □

030G **Definition 9.26.4.** Let K/k be a field extension. The *transcendence degree* of K over k is the cardinality of a transcendence basis of K over k. It is denoted $\mathrm{trdeg}_k(K)$.

030H **Lemma 9.26.5.** *Let $L/K/k$ be field extensions. Then*

$$\mathit{trdeg}_k(L) = \mathit{trdeg}_K(L) + \mathit{trdeg}_k(K).$$

Proof. Choose a transcendence basis $A \subset K$ of K over k. Choose a transcendence basis $B \subset L$ of L over K. Then it is straightforward to see that $A \cup B$ is a transcendence basis of L over k. □

09I9 **Example 9.26.6.** Consider the field extension $\mathbf{Q}(e, \pi)$ formed by adjoining the numbers e and π. This field extension has transcendence degree at least 1 since both e and π are transcendental over the rationals. However, this field extension might

have transcendence degree 2 if e and π are algebraically independent. Whether or not this is true is unknown and whence the problem of determining $\mathrm{trdeg}(\mathbf{Q}(e, \pi))$ is open.

09IA **Example 9.26.7.** Let F be a field and $E = F(t)$. Then $\{t\}$ is a transcendence basis since $E = F(t)$. However, $\{t^2\}$ is also a transcendence basis since $F(t)/F(t^2)$ is algebraic. This illustrates that while we can always decompose an extension E/F into an algebraic extension E/F' and a purely transcendental extension F'/F, this decomposition is not unique and depends on choice of transcendence basis.

09IB **Example 9.26.8.** Let X be a compact Riemann surface. Then the function field $\mathbf{C}(X)$ (see Example 9.3.6) has transcendence degree one over $\mathbf{C}$. In fact, *any* finitely generated extension of $\mathbf{C}$ of transcendence degree one arises from a Riemann surface. There is even an equivalence of categories between the category of compact Riemann surfaces and (non-constant) holomorphic maps and the opposite of the category of finitely generated extensions of $\mathbf{C}$ of transcendence degree 1 and morphisms of $\mathbf{C}$-algebras. See [**For91**].

There is an algebraic version of the above statement as well. Given an (irreducible) algebraic curve in projective space over an algebraically closed field k (e.g. the complex numbers), one can consider its "field of rational functions": basically, functions that look like quotients of polynomials, where the denominator does not identically vanish on the curve. There is a similar anti-equivalence of categories (Algebraic Curves, Theorem 53.2.6) between smooth projective curves and non-constant morphisms of curves and finitely generated extensions of k of transcendence degree one. See [**Har77**].

037I **Definition 9.26.9.** Let K/k be a field extension.

(1) The *algebraic closure of k in K* is the subfield k' of K consisting of elements of K which are algebraic over k.
(2) We say k is *algebraically closed in K* if every element of K which is algebraic over k is contained in k.

0G1M **Lemma 9.26.10.** *Let k'/k be a finite extension of fields. Let $k'(x_1, \ldots, x_r)/k(x_1, \ldots, x_r)$ be the induced extension of purely transcendental extensions. Then $[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$.*

Proof. By multiplicativity of degrees of extensions (Lemma 9.7.7) it suffices to prove this when k' is generated by a single element $\alpha \in k'$ over k. Let $f \in k[T]$ be the minimal polynomial of α over k. Then $k'(x_1, \ldots, x_r)$ is generated by $\alpha, x_1, \ldots, x_r$ over k and hence $k'(x_1, \ldots, x_r)$ is generated by α over $k(x_1, \ldots, x_r)$. Thus it suffices to show that f is still irreducible as an element of $k(x_1, \ldots, x_r)[T]$. We only sketch the proof. It is clear that f is irreducible as an element of $k[x_1, \ldots, x_r, T]$ for example because f is monic as a polynomial in T and any putative factorization in $k[x_1, \ldots, x_r, T]$ would lead to a factorization in $k[T]$ by setting x_i equal to 0. By Gauss' lemma we conclude. □

037J **Lemma 9.26.11.** *Let K/k be a finitely generated field extension. The algebraic closure of k in K is finite over k.*

Proof. Let $x_1, \ldots, x_r \in K$ be a transcendence basis for K over k. Then $n = [K : k(x_1, \ldots, x_r)] < \infty$. Suppose that $k \subset k' \subset K$ with k'/k finite. In this case

$[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$, see Lemma 9.26.10. Hence

$$[k' : k] = [k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] \leq [K : k(x_1, \ldots, x_r)] = n.$$

In other words, the degrees of finite subextensions are bounded and the lemma follows. □

9.27. Linearly disjoint extensions

09IC Let k be a field, K and L field extensions of k. Suppose also that K and L are embedded in some larger field Ω.

09ID **Definition 9.27.1.** Consider a diagram

09IE (9.27.1.1)
$$\begin{array}{ccc} L & \longrightarrow & \Omega \\ \uparrow & & \uparrow \\ k & \longrightarrow & K \end{array}$$

of field extensions. The *compositum of K and L in Ω* written KL is the smallest subfield of Ω containing both L and K.

It is clear that KL is generated by the set $K \cup L$ over k, generated by the set K over L, and generated by the set L over K.

Warning: The (isomorphism class of the) composition depends on the choice of the embeddings of K and L into Ω. For example consider the number fields $K = \mathbf{Q}(2^{1/8}) \subset \mathbf{R}$ and $L = \mathbf{Q}(2^{1/12}) \subset \mathbf{R}$. The compositum inside $\mathbf{R}$ is the field $\mathbf{Q}(2^{1/24})$ of degree 24 over $\mathbf{Q}$. However, if we embed $K = \mathbf{Q}[x]/(x^8 - 2)$ into $\mathbf{C}$ by mapping x to $2^{1/8}e^{2\pi i/8}$, then the compositum $\mathbf{Q}(2^{1/12}, 2^{1/8}e^{2\pi i/8})$ contains $i = e^{2\pi i/4}$ and has degree 48 over $\mathbf{Q}$ (we omit showing the degree is 48, but the existence of i certainly proves the two composita are not isomorphic).

09IF **Definition 9.27.2.** Consider a diagram of fields as in (9.27.1.1). We say that K and L are *linearly disjoint over k in Ω* if the map

$$K \otimes_k L \longrightarrow KL, \quad \sum x_i \otimes y_i \longmapsto \sum x_i y_i$$

is injective.

The following lemma does not seem to fit anywhere else.

030M **Lemma 9.27.3.** *Let E/F be a normal algebraic field extension. There exist subextensions $E/E_{sep}/F$ and $E/E_{insep}/F$ such that*

(1) *$F \subset E_{sep}$ is Galois and $E_{sep} \subset E$ is purely inseparable,*
(2) *$F \subset E_{insep}$ is purely inseparable and $E_{insep} \subset E$ is Galois,*
(3) *$E = E_{sep} \otimes_F E_{insep}$.*

Proof. We found the subfield E_{sep} in Lemma 9.14.6. We set $E_{insep} = E^{\mathrm{Aut}(E/F)}$. Details omitted. □

9.28. Review

037H In this section we give a quick review of what has transpired above.

Let K/k be a field extension. Let $\alpha \in K$. Then we have the following possibilities:

(1) The element α is transcendental over k.
(2) The element α is algebraic over k. Denote $P(T) \in k[T]$ its *minimal polynomial*. This is a monic polynomial $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ with coefficients in k. It is irreducible and $P(\alpha) = 0$. These properties uniquely determine P, and the integer d is called the *degree of α over k*. There are two subcases:
   (a) The polynomial $\mathrm{d}P/\mathrm{d}T$ is not identically zero. This is equivalent to the condition that $P(T) = \prod_{i=1,\ldots,d}(T - \alpha_i)$ for pairwise distinct elements $\alpha_1, \ldots, \alpha_d$ in the algebraic closure of k. In this case we say that α is *separable* over k.
   (b) The $\mathrm{d}P/\mathrm{d}T$ is identically zero. In this case the characteristic p of k is > 0, and P is actually a polynomial in T^p. Clearly there exists a largest power $q = p^e$ such that P is a polynomial in T^q. Then the element α^q is separable over k.

030J **Definition 9.28.1.** Algebraic field extensions.

(1) A field extension K/k is called *algebraic* if every element of K is algebraic over k.
(2) An algebraic extension k'/k is called *separable* if every $\alpha \in k'$ is separable over k.
(3) An algebraic extension k'/k is called *purely inseparable* if the characteristic of k is $p > 0$ and for every element $\alpha \in k'$ there exists a power q of p such that $\alpha^q \in k$.
(4) An algebraic extension k'/k is called *normal* if for every $\alpha \in k'$ the minimal polynomial $P(T) \in k[T]$ of α over k splits completely into linear factors over k'.
(5) An algebraic extension k'/k is called *Galois* if it is separable and normal.

The following lemma does not seem to fit anywhere else.

031V **Lemma 9.28.2.** *Let K be a field of characteristic $p > 0$. Let L/K be a separable algebraic extension. Let $\alpha \in L$.*

(1) *If the coefficients of the minimal polynomial of α over K are pth powers in K then α is a pth power in L.*
(2) *More generally, if $P \in K[T]$ is a polynomial such that (a) α is a root of P, (b) P has pairwise distinct roots in an algebraic closure, and (c) all coefficients of P are pth powers, then α is a pth power in L.*

Proof. It follows from the definitions that (2) implies (1). Assume P is as in (2). Write $P(T) = \sum_{i=0}^{d} a_i T^{d-i}$ and $a_i = b_i^p$. The polynomial $Q(T) = \sum_{i=0}^{d} b_i T^{d-i}$ has distinct roots in an algebraic closure as well, because the roots of Q are the pth roots of the roots of P. If α is not a pth power, then $T^p - \alpha$ is an irreducible polynomial over L (Lemma 9.14.2). Moreover Q and $T^p - \alpha$ have a root in common in an algebraic closure $\overline{L}$. Thus Q and $T^p - \alpha$ are not relatively prime, which implies $T^p - \alpha | Q$ in $L[T]$. This contradicts the fact that the roots of Q are pairwise distinct. $\square$

9.29. Other chapters

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CHAPTER 10

Commutative Algebra

00AO

10.1. Introduction

00AP Basic commutative algebra will be explained in this document. A reference is [**Mat70a**].

10.2. Conventions

00AQ A ring is commutative with 1. The zero ring is a ring. In fact it is the only ring that does not have a prime ideal. The Kronecker symbol δ_{ij} will be used. If $R \to S$ is a ring map and $\mathfrak{q}$ a prime of S, then we use the notation "$\mathfrak{p} = R \cap \mathfrak{q}$" to indicate the prime which is the inverse image of $\mathfrak{q}$ under $R \to S$ even if R is not a subring of S and even if $R \to S$ is not injective.

10.3. Basic notions

00AR The following is a list of basic notions in commutative algebra. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on algebra before continuing.

00AS (1) R is a *ring*,

00AT (2) $x \in R$ is *nilpotent*,

00AU (3) $x \in R$ is a *zerodivisor*,

00AV (4) $x \in R$ is a *unit*,

00AW (5) $e \in R$ is an *idempotent*,

00AX (6) an idempotent $e \in R$ is called *trivial* if $e = 1$ or $e = 0$,

00AY (7) $\varphi : R_1 \to R_2$ is a *ring homomorphism*,

00AZ (8) $\varphi : R_1 \to R_2$ is *of finite presentation*, or R_2 *is a finitely presented* R_1*-algebra*, see Definition 10.6.1,

00B0 (9) $\varphi : R_1 \to R_2$ is *of finite type*, or R_2 *is a finite type* R_1*-algebra*, see Definition 10.6.1,

00B1 (10) $\varphi : R_1 \to R_2$ is *finite*, or R_2 *is a finite* R_1*-algebra*,

00B2 (11) R is a *(integral) domain*,

00B3 (12) R is *reduced*,

00B4 (13) R is *Noetherian*,

00B5 (14) R is a *principal ideal domain* or a *PID*,

00B6 (15) R is a *Euclidean domain*,

00B7 (16) R is a *unique factorization domain* or a *UFD*,

00B8 (17) R is a *discrete valuation ring* or a *dvr*,

00B9 (18) K is a *field*,
00BA (19) L/K is a *field extension*,
00BB (20) L/K is an *algebraic field extension*,
00BC (21) $\{t_i\}_{i\in I}$ is a *transcendence basis* for L over K,
00BD (22) the *transcendence degree* $\mathrm{trdeg}(L/K)$ of L over K,
00BE (23) the field k is *algebraically closed*,
00BF (24) if L/K is algebraic, and Ω/K an extension with Ω algebraically closed, then there exists a ring map $L \to \Omega$ extending the map on K,
00BG (25) $I \subset R$ is an *ideal*,
00BH (26) $I \subset R$ is *radical*,
00BI (27) if I is an ideal then we have its *radical* $\sqrt{I}$,
00BJ (28) $I \subset R$ is *nilpotent* means that $I^n = 0$ for some $n \in \mathbf{N}$,
0543 (29) $I \subset R$ is *locally nilpotent* means that every element of I is nilpotent,
00BK (30) $\mathfrak{p} \subset R$ is a *prime ideal*,
00BL (31) if $\mathfrak{p} \subset R$ is prime and if $I, J \subset R$ are ideal, and if $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.
00BM (32) $\mathfrak{m} \subset R$ is a *maximal ideal*,
00BN (33) any nonzero ring has a maximal ideal,
00BO (34) the *Jacobson radical* of R is $\mathrm{rad}(R) = \bigcap_{\mathfrak{m}\subset R} \mathfrak{m}$ the intersection of all the maximal ideals of R,
00BP (35) the ideal (T) *generated* by a subset $T \subset R$,
00BQ (36) the *quotient ring* R/I,
00BR (37) an ideal I in the ring R is prime if and only if R/I is a domain,
00BS (38) an ideal I in the ring R is maximal if and only if the ring R/I is a field,
00BT (39) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_2$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R_1,
00BU (40) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_1$ is an ideal, then $\varphi(I) \cdot R_2$ (sometimes denoted $I \cdot R_2$, or IR_2) is the ideal of R_2 generated by $\varphi(I)$,
00BV (41) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $\mathfrak{p} \subset R_2$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of R_1,
00BW (42) M is an *R-module*,
055Y (43) for $m \in M$ the *annihilator* $I = \{f \in R \mid fm = 0\}$ of m in R,
00BX (44) $N \subset M$ is an *R-submodule*,
00BY (45) M is a *Noetherian R-module*,
00BZ (46) M is a *finite R-module*,
00C0 (47) M is a *finitely generated R-module*,
00C1 (48) M is a *finitely presented R-module*,
00C2 (49) M is a *free R-module*,
0516 (50) if $0 \to K \to L \to M \to 0$ is a short exact sequence of R-modules and K, M are free, then L is free,
00C3 (51) if $N \subset M \subset L$ are R-modules, then $L/M = (L/N)/(M/N)$,
00C4 (52) S is a *multiplicative subset of R*,
00C5 (53) the *localization* $R \to S^{-1}R$ of R,
00C6 (54) if R is a ring and S is a multiplicative subset of R then $S^{-1}R$ is the zero ring if and only if S contains 0,

00C7 (55) if R is a ring and if the multiplicative subset S consists completely of nonzerodivisors, then $R \to S^{-1}R$ is injective,

(56) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and S is a multiplicative subset of R_1, then $\varphi(S)$ is a multiplicative subset of R_2,

00C8 (57) if S, S' are multiplicative subsets of R, and if SS' denotes the set of products $SS' = \{r \in R \mid \exists s \in S, \exists s' \in S', r = ss'\}$ then SS' is a multiplicative subset of R,

00C9 (58) if S, S' are multiplicative subsets of R, and if $\overline{S}$ denotes the image of S in $(S')^{-1}R$, then $(SS')^{-1}R = \overline{S}^{-1}((S')^{-1}R)$,

00CA (59) the *localization* $S^{-1}M$ of the R-module M,

00CB (60) the functor $M \mapsto S^{-1}M$ preserves injective maps, surjective maps, and exactness,

00CC (61) if S, S' are multiplicative subsets of R, and if M is an R-module, then $(SS')^{-1}M = S^{-1}((S')^{-1}M)$,

00CD (62) if R is a ring, I an ideal of R, and S a multiplicative subset of R, then $S^{-1}I$ is an ideal of $S^{-1}R$, and we have $S^{-1}R/S^{-1}I = \overline{S}^{-1}(R/I)$, where $\overline{S}$ is the image of S in R/I,

00CE (63) if R is a ring, and S a multiplicative subset of R, then any ideal I' of $S^{-1}R$ is of the form $S^{-1}I$, where one can take I to be the inverse image of I' in R,

00CF (64) if R is a ring, M an R-module, and S a multiplicative subset of R, then any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subset M$, where one can take N to be the inverse image of N' in M,

00CG (65) if $S = \{1, f, f^2, \ldots\}$ then $R_f = S^{-1}R$ and $M_f = S^{-1}M$,

00CH (66) if $S = R \setminus \mathfrak{p} = \{x \in R \mid x \notin \mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, then it is customary to denote $R_\mathfrak{p} = S^{-1}R$ and $M_\mathfrak{p} = S^{-1}M$,

00CI (67) a *local ring* is a ring with exactly one maximal ideal,

03C0 (68) a *semi-local ring* is a ring with finitely many maximal ideals,

00CJ (69) if $\mathfrak{p}$ is a prime in R, then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$,

00CK (70) the *residue field*, denoted $\kappa(\mathfrak{p})$, of the prime $\mathfrak{p}$ in the ring R is the field of fractions of the domain $R/\mathfrak{p}$; it is equal to $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R/\mathfrak{p}$,

00CL (71) given R and M_1, M_2 the *tensor product* $M_1 \otimes_R M_2$,

0F0K (72) given matrices A and B in a ring R of sizes $m \times n$ and $n \times m$ we have $\det(AB) = \sum \det(A_S)\det({}_SB)$ in R where the sum is over subsets $S \subset \{1, \ldots, n\}$ of size m and A_S is the $m \times m$ submatrix of A with columns corresponding to S and ${}_SB$ is the $m \times m$ submatrix of B with rows corresponding to S,

(73) etc.

10.4. Snake lemma

07JV The snake lemma and its variants are discussed in the setting of abelian categories in Homology, Section 12.5.

07JW **Lemma 10.4.1.** *Given a commutative diagram* [**CE56**, III, Lemma 3.3]

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W \end{array}$$

of abelian groups with exact rows, there is a canonical exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$$

Moreover: if $X \to Y$ is injective, then the first map is injective; if $V \to W$ is surjective, then the last map is surjective.

Proof. The map $\partial : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ is defined as follows. Take $z \in \mathrm{Ker}(\gamma)$. Choose $y \in Y$ mapping to z. Then $\beta(y) \in V$ maps to zero in W. Hence $\beta(y)$ is the image of some $u \in U$. Set $\partial z = \overline{u}$, the class of u in the cokernel of α. Proof of exactness is omitted. □

10.5. Finite modules and finitely presented modules

0517 Just some basic notation and lemmas.

0518 **Definition 10.5.1.** Let R be a ring. Let M be an R-module.

(1) We say M is a *finite R-module*, or a *finitely generated R-module* if there exist $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$ such that every element of M is an R-linear combination of the x_i. Equivalently, this means there exists a surjection $R^{\oplus n} \to M$ for some $n \in \mathbf{N}$.
(2) We say M is a *finitely presented R-module* or an *R-module of finite presentation* if there exist integers $n, m \in \mathbf{N}$ and an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Informally, M is a finitely presented R-module if and only if it is finitely generated and the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a *presentation* of M.

07JX **Lemma 10.5.2.** *Let R be a ring. Let $\alpha : R^{\oplus n} \to M$ and $\beta : N \to M$ be module maps. If $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$, then there exists an R-module map $\gamma : R^{\oplus n} \to N$ such that $\alpha = \beta \circ \gamma$.*

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $R^{\oplus n}$. Let $x_i \in N$ be an element with $\alpha(e_i) = \beta(x_i)$ which exists by assumption. Set $\gamma(a_1, \ldots, a_n) = \sum a_i x_i$. By construction $\alpha = \beta \circ \gamma$. □

0519 **Lemma 10.5.3.** *Let R be a ring. Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules.

(1) *If M_1 and M_3 are finite R-modules, then M_2 is a finite R-module.*
(2) *If M_1 and M_3 are finitely presented R-modules, then M_2 is a finitely presented R-module.*
(3) *If M_2 is a finite R-module, then M_3 is a finite R-module.*
(4) *If M_2 is a finitely presented R-module and M_1 is a finite R-module, then M_3 is a finitely presented R-module.*

(5) *If M_3 is a finitely presented R-module and M_2 is a finite R-module, then M_1 is a finite R-module.*

Proof. Proof of (1). If $x_1, \ldots, x_n$ are generators of M_1 and $y_1, \ldots, y_m \in M_2$ are elements whose images in M_3 are generators of M_3, then $x_1, \ldots, x_n, y_1, \ldots, y_m$ generate M_2.

Part (3) is immediate from the definition.

Proof of (5). Assume M_3 is finitely presented and M_2 finite. Choose a presentation

$$R^{\oplus m} \to R^{\oplus n} \to M_3 \to 0$$

By Lemma 10.5.2 there exists a map $R^{\oplus n} \to M_2$ such that the solid diagram

$$\begin{array}{ccccccccc} & & R^{\oplus m} & \longrightarrow & R^{\oplus n} & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \vdots & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

commutes. This produces the dotted arrow. By the snake lemma (Lemma 10.4.1) we see that we get an isomorphism

$$\mathrm{Coker}(R^{\oplus m} \to M_1) \cong \mathrm{Coker}(R^{\oplus n} \to M_2)$$

In particular we conclude that $\mathrm{Coker}(R^{\oplus m} \to M_1)$ is a finite R-module. Since $\mathrm{Im}(R^{\oplus m} \to M_1)$ is finite by (3), we see that M_1 is finite by part (1).

Proof of (4). Assume M_2 is finitely presented and M_1 is finite. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M_2 \to 0$. Choose a surjection $R^{\oplus k} \to M_1$. By Lemma 10.5.2 there exists a factorization $R^{\oplus k} \to R^{\oplus n} \to M_2$ of the composition $R^{\oplus k} \to M_1 \to M_2$. Then $R^{\oplus k+m} \to R^{\oplus n} \to M_3 \to 0$ is a presentation.

Proof of (2). Assume that M_1 and M_3 are finitely presented. The argument in the proof of part (1) produces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{\oplus n} & \longrightarrow & R^{\oplus n+m} & \longrightarrow & R^{\oplus m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with surjective vertical arrows. By the snake lemma we obtain a short exact sequence

$$0 \to \mathrm{Ker}(R^{\oplus n} \to M_1) \to \mathrm{Ker}(R^{\oplus n+m} \to M_2) \to \mathrm{Ker}(R^{\oplus m} \to M_3) \to 0$$

By part (5) we see that the outer two modules are finite. Hence the middle one is finite too. By (4) we see that M_2 is of finite presentation. □

00KZ **Lemma 10.5.4.** *Let R be a ring, and let M be a finite R-module. There exists a filtration by finite R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal I_i of R.

Proof. By induction on the number of generators of M. Let $x_1, \ldots, x_r \in M$ be generators. Let $M' = Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$. □

0560 **Lemma 10.5.5.** *Let $R \to S$ be a ring map. Let M be an S-module. If M is finite as an R-module, then M is finite as an S-module.*

Proof. In fact, any R-generating set of M is also an S-generating set of M, since the R-module structure is induced by the image of R in S. □

10.6. Ring maps of finite type and of finite presentation

00F2

00F3 **Definition 10.6.1.** Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of *finite type*, or that S *is a finite type R-algebra* if there exist an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \ldots, x_n] \to S$.
(2) We say $R \to S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S$.

Informally, $R \to S$ is of finite presentation if and only if S is finitely generated as an R-algebra and the ideal of relations among the generators is finitely generated. A choice of a surjection $R[x_1, \ldots, x_n] \to S$ as in the definition is sometimes called a *presentation* of S.

00F4 **Lemma 10.6.2.** *The notions finite type and finite presentation have the following permanence properties.*

(1) *A composition of ring maps of finite type is of finite type.*
(2) *A composition of ring maps of finite presentation is of finite presentation.*
(3) *Given $R \to S' \to S$ with $R \to S$ of finite type, then $S' \to S$ is of finite type.*
(4) *Given $R \to S' \to S$, with $R \to S$ of finite presentation, and $R \to S'$ of finite type, then $S' \to S$ is of finite presentation.*

Proof. We only prove the last assertion. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $S' = R[y_1, \ldots, y_a]/I$. Say that the class $\bar{y}_i$ of y_i maps to $h_i \bmod (f_1, \ldots, f_m)$ in S. Then it is clear that $S = S'[x_1, \ldots, x_n]/(f_1, \ldots, f_m, h_1 - \bar{y}_1, \ldots, h_a - \bar{y}_a)$. □

00R2 **Lemma 10.6.3.** *Let $R \to S$ be a ring map of finite presentation. For any surjection $\alpha : R[x_1, \ldots, x_n] \to S$ the kernel of α is a finitely generated ideal in $R[x_1, \ldots, x_n]$.*

Proof. Write $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_k)$. Choose $g_i \in R[y_1, \ldots, y_m]$ which are lifts of $\alpha(x_i)$. Then we see that $S = R[x_i, y_j]/(f_l, x_i - g_i)$. Choose $h_j \in R[x_1, \ldots, x_n]$ such that $\alpha(h_j)$ corresponds to $y_j \bmod (f_1, \ldots, f_k)$. Consider the map $\psi : R[x_i, y_j] \to R[x_i]$, $x_i \mapsto x_i$, $y_j \mapsto h_j$. Then the kernel of α is the image of $(f_l, x_i - g_i)$ under ψ and we win. □

0561 **Lemma 10.6.4.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume $R \to S$ is of finite type and M is finitely presented as an R-module. Then M is finitely presented as an S-module.*

Proof. This is similar to the proof of part (4) of Lemma 10.6.2. We may assume $S = R[x_1, \ldots, x_n]/J$. Choose $y_1, \ldots, y_m \in M$ which generate M as an R-module and choose relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ which generate the kernel of $R^{\oplus m} \to M$. For any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ write

$$x_i y_j = \sum a_{ijk} y_k$$

for some $a_{ijk} \in R$. Consider the S-module N generated by $y_1, \ldots, y_m$ subject to the relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ and $x_i y_j = \sum a_{ijk} y_k$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then N has a presentation

$$S^{\oplus nm+t} \longrightarrow S^{\oplus m} \longrightarrow N \longrightarrow 0$$

By construction there is a surjective map $\varphi : N \to M$. To finish the proof we show φ is injective. Suppose $z = \sum b_j y_j \in N$ for some $b_j \in S$. We may think of b_j as a polynomial in $x_1, \ldots, x_n$ with coefficients in R. By applying the relations of the form $x_i y_j = \sum a_{ijk} y_k$ we can inductively lower the degree of the polynomials. Hence we see that $z = \sum c_j y_j$ for some $c_j \in R$. Hence if $\varphi(z) = 0$ then the vector $(c_1, \ldots, c_m)$ is an R-linear combination of the vectors $(a_{i1}, \ldots, a_{im})$ and we conclude that $z = 0$ as desired. □

10.7. Finite ring maps

0562 Here is the definition.

0563 **Definition 10.7.1.** Let $\varphi : R \to S$ be a ring map. We say $\varphi : R \to S$ is *finite* if S is finite as an R-module.

00GJ **Lemma 10.7.2.** *Let $R \to S$ be a finite ring map. Let M be an S-module. Then M is finite as an R-module if and only if M is finite as an S-module.*

Proof. One of the implications follows from Lemma 10.5.5. To see the other assume that M is finite as an S-module. Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Pick $y_1, \ldots, y_m \in M$ which generate M as an S-module. Then $x_i y_j$ generate M as an R-module. □

00GL **Lemma 10.7.3.** *Suppose that $R \to S$ and $S \to T$ are finite ring maps. Then $R \to T$ is finite.*

Proof. If t_i generate T as an S-module and s_j generate S as an R-module, then $t_i s_j$ generate T as an R-module. (Also follows from Lemma 10.7.2.) □

0D46 **Lemma 10.7.4.** *Let $\varphi : R \to S$ be a ring map.*

(1) *If φ is finite, then φ is of finite type.*
(2) *If S is of finite presentation as an R-module, then φ is of finite presentation.*

Proof. For (1) if $x_1, \ldots, x_n \in S$ generate S as an R-module, then $x_1, \ldots, x_n$ generate S as an R-algebra. For (2), suppose that $\sum r_j^i x_i = 0$, $j = 1, \ldots, m$ is a set of generators of the relations among the x_i when viewed as R-module generators of S. Furthermore, write $1 = \sum r_i x_i$ for some $r_i \in R$ and $x_i x_j = \sum r_{ij}^k x_k$ for some $r_{ij}^k \in R$. Then

$$S = R[t_1, \ldots, t_n]/(\sum r_j^i t_i,\ 1 - \sum r_i t_i,\ t_i t_j - \sum r_{ij}^k t_k)$$

as an R-algebra which proves (2). □

For more information on finite ring maps, please see Section 10.36.

10.8. Colimits

07N7 Some of the material in this section overlaps with the general discussion on colimits in Categories, Sections 4.14 – 4.21. The notion of a preordered set is defined in Categories, Definition 4.21.1. It is a slightly weaker notion than a partially ordered set.

00D4 **Definition 10.8.1.** Let $(I, \leq)$ be a preordered set. A *system* (M_i, μ_{ij}) *of R-modules over I* consists of a family of R-modules $\{M_i\}_{i \in I}$ indexed by I and a family of R-module maps $\{\mu_{ij} : M_i \to M_j\}_{i \leq j}$ such that for all $i \leq j \leq k$

$$\mu_{ii} = \mathrm{id}_{M_i} \quad \mu_{ik} = \mu_{jk} \circ \mu_{ij}$$

We say (M_i, μ_{ij}) is a *directed system* if I is a directed set.

This is the same as the notion defined in Categories, Definition 4.21.2 and Section 4.21. We refer to Categories, Definition 4.14.2 for the definition of a colimit of a diagram/system in any category.

00D5 **Lemma 10.8.2.** *Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. The colimit of the system (M_i, μ_{ij}) is the quotient R-module $(\bigoplus_{i \in I} M_i)/Q$ where Q is the R-submodule generated by all elements*

$$\iota_i(x_i) - \iota_j(\mu_{ij}(x_i))$$

where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the natural inclusion. We denote the colimit $M = \mathrm{colim}_i M_i$. We denote $\pi : \bigoplus_{i \in I} M_i \to M$ the projection map and $\phi_i = \pi \circ \iota_i : M_i \to M$.

Proof. This lemma is a special case of Categories, Lemma 4.14.12 but we will also prove it directly in this case. Namely, note that $\phi_i = \phi_j \circ \mu_{ij}$ in the above construction. To show the pair (M, ϕ_i) is the colimit we have to show it satisfies the universal property: for any other such pair (Y, ψ_i) with $\psi_i : M_i \to Y$, $\psi_i = \psi_j \circ \mu_{ij}$, there is a unique R-module homomorphism $g : M \to Y$ such that the following diagram commutes:

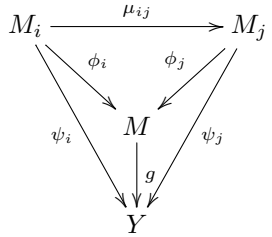

And this is clear because we can define g by taking the map ψ_i on the summand M_i in the direct sum $\bigoplus M_i$. □

00D6 **Lemma 10.8.3.** *Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. Assume that I is directed. The colimit of the system (M_i, μ_{ij}) is canonically isomorphic to the module M defined as follows:*

(1) *as a set let*

$$M = \left(\coprod_{i \in I} M_i\right) / \sim$$

where for $m \in M_i$ and $m' \in M_{i'}$ we have

$$m \sim m' \Leftrightarrow \mu_{ij}(m) = \mu_{i'j}(m') \text{ for some } j \geq i, i'$$

(2) *as an abelian group for $m \in M_i$ and $m' \in M_{i'}$ we define the sum of the classes of m and m' in M to be the class of $\mu_{ij}(m) + \mu_{i'j}(m')$ where $j \in I$ is any index with $i \leq j$ and $i' \leq j$, and*
(3) *as an R-module define for $m \in M_i$ and $x \in R$ the product of x and the class of m in M to be the class of xm in M.*

The canonical maps $\phi_i : M_i \to M$ are induced by the canonical maps $M_i \to \coprod_{i \in I} M_i$.

Proof. Omitted. Compare with Categories, Section 4.19. □

00D7 **Lemma 10.8.4.** *Let (M_i, μ_{ij}) be a directed system. Let $M = \operatorname{colim} M_i$ with $\mu_i : M_i \to M$. Then, $\mu_i(x_i) = 0$ for $x_i \in M_i$ if and only if there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.*

Proof. This is clear from the description of the directed colimit in Lemma 10.8.3. □

00D8 **Example 10.8.5.** Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities. A system $(M_a, M_b, M_c, \mu_{ab}, \mu_{ac})$ over I consists of three R-modules M_a, M_b, M_c and two R-module homomorphisms $\mu_{ab} : M_a \to M_b$ and $\mu_{ac} : M_a \to M_c$. The colimit of the system is just

$$M := \operatorname{colim}_{i \in I} M_i = \operatorname{Coker}(M_a \to M_b \oplus M_c)$$

where the map is $\mu_{ab} \oplus -\mu_{ac}$. Thus the kernel of the canonical map $M_a \to M$ is $\operatorname{Ker}(\mu_{ab}) + \operatorname{Ker}(\mu_{ac})$. And the kernel of the canonical map $M_b \to M$ is the image of $\operatorname{Ker}(\mu_{ac})$ under the map μ_{ab}. Hence clearly the result of Lemma 10.8.4 is false for general systems.

00D9 **Definition 10.8.6.** Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set I. A *homomorphism of systems* Φ from (M_i, μ_{ij}) to (N_i, ν_{ij}) is by definition a family of R-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ for all $i \leq j$.

This is the same notion as a transformation of functors between the associated diagrams $M : I \to \operatorname{Mod}_R$ and $N : I \to \operatorname{Mod}_R$, in the language of categories. The following lemma is a special case of Categories, Lemma 4.14.8.

00DA **Lemma 10.8.7.** *Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set. A morphism of systems $\Phi = (\phi_i)$ from (M_i, μ_{ij}) to (N_i, ν_{ij}) induces a unique homomorphism*

$$\operatorname{colim} \phi_i : \operatorname{colim} M_i \longrightarrow \operatorname{colim} N_i$$

such that

$$\begin{array}{ccc} M_i & \longrightarrow & \operatorname{colim} M_i \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \operatorname{colim} \phi_i} \\ N_i & \longrightarrow & \operatorname{colim} N_i \end{array}$$

commutes for all $i \in I$.

Proof. Write $M = \operatorname{colim} M_i$ and $N = \operatorname{colim} N_i$ and $\phi = \operatorname{colim} \phi_i$ (as yet to be constructed). We will use the explicit description of M and N in Lemma 10.8.2

without further mention. The condition of the lemma is equivalent to the condition that

$$\begin{array}{ccc} \bigoplus_{i\in I} M_i & \longrightarrow & M \\ {\scriptstyle \bigoplus \phi_i}\downarrow & & \downarrow{\scriptstyle \phi} \\ \bigoplus_{i\in I} N_i & \longrightarrow & N \end{array}$$

commutes. Hence it is clear that if ϕ exists, then it is unique. To see that ϕ exists, it suffices to show that the kernel of the upper horizontal arrow is mapped by $\bigoplus \phi_i$ to the kernel of the lower horizontal arrow. To see this, let $j \leq k$ and $x_j \in M_j$. Then

$$(\bigoplus \phi_i)(x_j - \mu_{jk}(x_j)) = \phi_j(x_j) - \phi_k(\mu_{jk}(x_j)) = \phi_j(x_j) - \nu_{jk}(\phi_j(x_j))$$

which is in the kernel of the lower horizontal arrow as required. □

00DB **Lemma 10.8.8.** *Let I be a directed set. Let (L_i, λ_{ij}), (M_i, μ_{ij}), and (N_i, ν_{ij}) be systems of R-modules over I. Let $\varphi_i : L_i \to M_i$ and $\psi_i : M_i \to N_i$ be morphisms of systems over I. Assume that for all $i \in I$ the sequence of R-modules*

$$L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$$

is a complex with homology H_i. Then the R-modules H_i form a system over I, the sequence of R-modules

$$\operatorname{colim}_i L_i \xrightarrow{\varphi} \operatorname{colim}_i M_i \xrightarrow{\psi} \operatorname{colim}_i N_i$$

is a complex as well, and denoting H its homology we have

$$H = \operatorname{colim}_i H_i.$$

Proof. It is clear that $\operatorname{colim}_i L_i \xrightarrow{\varphi} \operatorname{colim}_i M_i \xrightarrow{\psi} \operatorname{colim}_i N_i$ is a complex. For each $i \in I$, there is a canonical R-module morphism $H_i \to H$ (sending each $[m] \in H_i = \operatorname{Ker}(\psi_i)/\operatorname{Im}(\varphi_i)$ to the residue class in $H = \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi)$ of the image of m in $\operatorname{colim}_i M_i$). These give rise to a morphism $\operatorname{colim}_i H_i \to H$. It remains to show that this morphism is surjective and injective.

We are going to repeatedly use the description of colimits over I as in Lemma 10.8.3 without further mention. Let $h \in H$. Since $H = \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi)$ we see that h is the class mod $\operatorname{Im}(\varphi)$ of an element $[m]$ in $\operatorname{Ker}(\psi) \subset \operatorname{colim}_i M_i$. Choose an i such that $[m]$ comes from an element $m \in M_i$. Choose a $j \geq i$ such that $\nu_{ij}(\psi_i(m)) = 0$ which is possible since $[m] \in \operatorname{Ker}(\psi)$. After replacing i by j and m by $\mu_{ij}(m)$ we see that we may assume $m \in \operatorname{Ker}(\psi_i)$. This shows that the map $\operatorname{colim}_i H_i \to H$ is surjective.

Suppose that $h_i \in H_i$ has image zero on H. Since $H_i = \operatorname{Ker}(\psi_i)/\operatorname{Im}(\varphi_i)$ we may represent h_i by an element $m \in \operatorname{Ker}(\psi_i) \subset M_i$. The assumption on the vanishing of h_i in H means that the class of m in $\operatorname{colim}_i M_i$ lies in the image of φ. Hence there exists a $j \geq i$ and an $l \in L_j$ such that $\varphi_j(l) = \mu_{ij}(m)$. Clearly this shows that the image of h_i in H_j is zero. This proves the injectivity of $\operatorname{colim}_i H_i \to H$. □

00DC **Example 10.8.9.** Taking colimits is not exact in general. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities, as in Example 10.8.5. Consider the map of systems $(0, \mathbf{Z}, \mathbf{Z}, 0, 0) \to (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 1, 1)$. From

the description of the colimit in Example 10.8.5 we see that the associated map of colimits is not injective, even though the map of systems is injective on each object. Hence the result of Lemma 10.8.8 is false for general systems.

04B0 **Lemma 10.8.10.** *Let $\mathcal{I}$ be an index category satisfying the assumptions of Categories, Lemma 4.19.8. Then taking colimits of diagrams of abelian groups over $\mathcal{I}$ is exact (i.e., the analogue of Lemma 10.8.8 holds in this situation).*

Proof. By Categories, Lemma 4.19.8 we may write $\mathcal{I} = \coprod_{j \in J} \mathcal{I}_j$ with each $\mathcal{I}_j$ a filtered category, and J possibly empty. By Categories, Lemma 4.21.5 taking colimits over the index categories $\mathcal{I}_j$ is the same as taking the colimit over some directed set. Hence Lemma 10.8.8 applies to these colimits. This reduces the problem to showing that coproducts in the category of R-modules over the set J are exact. In other words, exact sequences $L_j \to M_j \to N_j$ of R modules we have to show that

$$\bigoplus\nolimits_{j \in J} L_j \longrightarrow \bigoplus\nolimits_{j \in J} M_j \longrightarrow \bigoplus\nolimits_{j \in J} N_j$$

is exact. This can be verified by hand, and holds even if J is empty. □

10.9. Localization

00CM

00CN **Definition 10.9.1.** Let R be a ring, S a subset of R. We say S is a *multiplicative subset of R* if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st$$

One can check that $S^{-1}A$ becomes a ring under these operations.

00CO **Definition 10.9.2.** This ring is called the *localization of A with respect to S*.

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the *localization map*. In general the localization map is not injective, unless S contains no zerodivisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zerodivisors in S. The localization of a ring has the following universal property.

00CP **Proposition 10.9.3.** *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

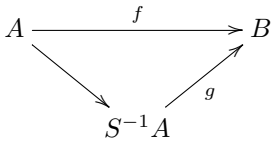

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. □

00CQ **Lemma 10.9.4.** *The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.*

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. □

07JY **Lemma 10.9.5.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$-module.

07JZ **Definition 10.9.6.** The $S^{-1}A$-module $S^{-1}M$ is called the *localization* of M at S.

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the *localization map*. It satisfies the following universal property.

07K0 **Lemma 10.9.7.** *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a

given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned}
\alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\
&= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\
&= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\
&= r\alpha(m/s) + r'\alpha(m'/s')
\end{aligned}$$

and we win. □

02C5 **Example 10.9.8.** Let A be a ring and let M be an A-module. Here are some important examples of localizations.

(1) Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the *localization of A, resp. M at $\mathfrak{p}$*.
(2) Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the *localization of A, resp. M with respect to f*. Note that $A_f = 0$ if and only if f is nilpotent in A.
(3) Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the *total quotient ring*, or the *total ring of fractions* of A.
(4) If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A. Please see Fields, Example 9.3.4.

00CR **Lemma 10.9.9.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \operatorname{colim}_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma 10.9.7. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

02C6 **Proposition 10.9.10.** *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

Proof. The map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/s')/(s/1)$ to x/ss'.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

02C7 **Proposition 10.9.11.** *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$

$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

00CS **Proposition 10.9.12.** *Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in N, which means $xt \in \mathrm{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. □

02C8 **Lemma 10.9.13.** *Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. □

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

00CT **Proposition 10.9.14.** *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. □

We now consider how submodules behave in localization.

00CU **Lemma 10.9.15.** *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.*

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. □

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition 10.9.14.

02C9 **Lemma 10.9.16.** *Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

Proof. Immediate from Lemma 10.9.15. □

10.10. Internal Hom

0581 If R is a ring, and M, N are R-modules, then

$$\mathrm{Hom}_R(M, N) = \{\varphi : M \to N\}$$

is the set of R-linear maps from M to N. This set comes with the structure of an abelian group by setting $(\varphi+\psi)(m) = \varphi(m)+\psi(m)$, as usual. In fact, $\mathrm{Hom}_R(M, N)$ is also an R-module via the rule $(x\varphi)(m) = x\varphi(m) = \varphi(xm)$.

Given maps $a : M \to M'$ and $b : N \to N'$ of R-modules, we can pre-compose and post-compose homomorphisms by a and b. This leads to the following commutative

diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M', N) & \xrightarrow[b\circ -]{} & \mathrm{Hom}_R(M', N') \\ {\scriptstyle -\circ a}\downarrow & & \downarrow{\scriptstyle -\circ a} \\ \mathrm{Hom}_R(M, N) & \xrightarrow{b\circ -} & \mathrm{Hom}_R(M, N') \end{array}$$

In fact, the maps in this diagram are R-module maps. Thus Hom_R defines an additive functor

$$\mathrm{Mod}_R^{opp} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R, \quad (M, N) \longmapsto \mathrm{Hom}_R(M, N)$$

0582 **Lemma 10.10.1.** *Exactness and* Hom_R*. Let R be a ring. Let M_1, M_2, M_3 be R-modules. Let $M_1 \to M_2$ and $M_2 \to M_3$ be R-module maps.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ is exact for all R-modules N.*
(2) *$0 \to M_1 \to M_2 \to M_3$ is exact if and only if $0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3)$ is exact for all R-modules N.*

Proof. Omitted. □

0583 **Lemma 10.10.2.** *Let R be a ring. Let M be a finitely presented R-module. Let N be an R-module.*

(1) *For $f \in R$ we have $\mathrm{Hom}_R(M, N)_f = \mathrm{Hom}_{R_f}(M_f, N_f) = \mathrm{Hom}_R(M_f, N_f)$,*
(2) *for a multiplicative subset S of R we have*

$$S^{-1}\,\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) = \mathrm{Hom}_R(S^{-1}M, S^{-1}N).$$

Proof. Part (1) is a special case of part (2). The second equality in (2) follows from Lemma 10.9.5. Choose a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} R \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R \to M \to 0.$$

By Lemma 10.10.1 this gives an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \bigoplus\nolimits_{i=1,\ldots,n} N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} N.$$

Inverting S and using Proposition 10.9.12 we get an exact sequence

$$0 \to S^{-1}\,\mathrm{Hom}_R(M, N) \to \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} S^{-1}N$$

and the result follows since $S^{-1}M$ sits in an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,m} S^{-1}R \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}R \to S^{-1}M \to 0$$

which induces (by Lemma 10.10.1) the exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \to \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} S^{-1}N$$

which is the same as the one above. □

10.11. Characterizing finite and finitely presented modules

0G8M Given a module N over a ring R, you can characterize whether or not N is a finite module or a finitely presented module in terms of the functor $\mathrm{Hom}_R(N, -)$.

0G8N **Lemma 10.11.1.** *Let R be a ring. Let N be an R-module. The following are equivalent*

(1) *N is a finite R-module,*
(2) *for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\, \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is injective.*

Proof. Assume (1) and choose generators $x_1, \ldots, x_m$ for N. If $N \to M_i$ is a module map and the composition $N \to M_i \to M$ is zero, then because $M = \mathrm{colim}_{i' \geq i} M_{i'}$ for each $j \in \{1, \ldots, m\}$ we can find a $i' \geq i$ such that x_j maps to zero in $M_{i'}$. Since there are finitely many x_j we can find a single i' which works for all of them. Then the composition $N \to M_i \to M_{i'}$ is zero and we conclude the map is injective, i.e., part (2) holds.

Assume (2). For a finite subset $E \subset N$ denote $N_E \subset N$ the R-submodule generated by the elements of E. Then $0 = \mathrm{colim}\, N/N_E$ is a filtered colimit. Hence we see that $\mathrm{id} : N \to N$ maps into N_E for some E, i.e., N is finitely generated. □

For purposes of reference, we define what it means to have a relation between elements of a module.

07N8 **Definition 10.11.2.** Let R be a ring. Let M be an R-module. Let $n \geq 0$ and $x_i \in M$ for $i = 1, \ldots, n$. A *relation* between $x_1, \ldots, x_n$ in M is a sequence of elements $f_1, \ldots, f_n \in R$ such that $\sum_{i=1,\ldots,n} f_i x_i = 0$.

00HA **Lemma 10.11.3.** *Let R be a ring and let M be an R-module. Then M is the colimit of a directed system (M_i, μ_{ij}) of R-modules with all M_i finitely presented R-modules.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto (\sum g_e f_{e,s})_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. □

0G8P **Lemma 10.11.4.** *Let R be a ring. Let N be an R-module. The following are equivalent*

(1) *N is a finitely presented R-module,*
(2) *for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\, \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is bijective.*

Proof. Assume (1) and choose an exact sequence $F_{-1} \to F_0 \to N \to 0$ with F_i finite free. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_{-1}, M)$$

functorial in the R-module M. The functors $\mathrm{Hom}_R(F_i, M)$ commute with filtered colimits as $\mathrm{Hom}_R(R^{\oplus n}, M) = M^{\oplus n}$. Since filtered colimits are exact (Lemma 10.8.8) we see that (2) holds.

Assume (2). By Lemma 10.11.3 we can write $N = \operatorname{colim} N_i$ as a filtered colimit such that N_i is of finite presentation for all i. Thus id_N factors through N_i for some i. This means that N is a direct summand of a finitely presented R-module (namely N_i) and hence finitely presented. □

10.12. Tensor products

00CV

00CW **Definition 10.12.1.** Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be *R-bilinear* if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ is also R-linear.

00CX **Lemma 10.12.2.** *Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

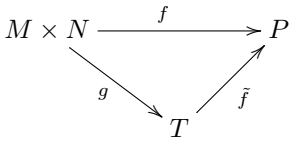

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

The R-module T which satisfies the above universal property is called the *tensor product* of R-modules M and N, denoted as $M \otimes_R N$.

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M \times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$\begin{aligned}(x + x', y) - (x, y) - (x', y),\\ (x, y + y') - (x, y) - (x, y'),\\ (ax, y) - a(x, y),\\ (x, ay) - a(x, y)\end{aligned}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence

$j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. □

00CY **Lemma 10.12.3.** *Let M, N, P be R-modules, then the bilinear maps*

$$(x, y) \mapsto y \otimes x$$
$$(x + y, z) \mapsto x \otimes z + y \otimes z$$
$$(r, x) \mapsto rx$$

induce unique isomorphisms

$$M \otimes_R N \to N \otimes_R M,$$
$$(M \oplus N) \otimes_R P \to (M \otimes_R P) \oplus (N \otimes_R P),$$
$$R \otimes_R M \to M$$

Proof. Omitted. □

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

00CZ **Lemma 10.12.4.** *Let $M_1, \ldots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping $g : M_1 \times \ldots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \ldots \times M_r \to P$ there exists a unique R-module homomorphism $f' : T \to P$ such that $f' \circ g = f$. Such a module T is unique up to unique isomorphism. We denote it $M_1 \otimes_R \ldots \otimes_R M_r$ and we denote the universal multilinear map $(m_1, \ldots, m_r) \mapsto m_1 \otimes \ldots \otimes m_r$.*

Proof. Omitted. □

00D0 **Lemma 10.12.5.** *The homomorphisms*

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P)$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ and $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ are well-defined and are isomorphisms.

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ which sends $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. □

Doing induction we see that this extends to multi-tensor products. Combined with Lemma 10.12.3 we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

00D1 **Definition 10.12.6.** An abelian group N is called an (A, B)-*bimodule* if it is both an A-module and a B-module and for all $a \in A$ and $b \in B$ the multiplication by a and b commute, so $b(an) = a(bn)$ for all $n \in N$. In this situation we usually write the B-action on the right: so for $b \in B$ and $n \in N$ the result of multiplying n by b is denoted nb. With this convention the compatibility above is that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. The shorthand ${}_AN_B$ is used to denote an (A, B)-bimodule N.

00D2 **Lemma 10.12.7.** *For A-module M, B-module P and (A, B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A, B)-bimodule structure, and moreover*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A, B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. By Lemma 10.12.5, these two modules are isomorphic as both as A-module and B-module via the same mapping. □

00DE **Lemma 10.12.8.** *For any three R-modules M, N, P,*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. □

00DD **Lemma 10.12.9** (Tensor products commute with colimits)**.** *Let (M_i, μ_{ij}) be a system over the preordered set I. Let N be an R-module. Then*

$$\mathrm{colim}(M_i \otimes N) \cong (\mathrm{colim}\, M_i) \otimes N.$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = \mathrm{colim}_i M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. First proof. The functor $M' \mapsto M' \otimes_R N$ is left adjoint to the functor $N' \mapsto \mathrm{Hom}_R(N, N')$ by Lemma 10.12.8. Thus $M' \mapsto M' \otimes_R N$ commutes with all colimits, see Categories, Lemma 4.24.5.

Second direct proof. Let $P = \mathrm{colim}(M_i \otimes N)$ with coprojections $\lambda_i : M_i \otimes N \to P$. Let $M = \mathrm{colim}\, M_i$ with coprojections $\mu_i : M_i \to M$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{array}{ccc} M_i \otimes N & \xrightarrow[\mu_i \otimes 1]{} & M \otimes N \\ {\scriptstyle \mu_{ij} \otimes 1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ M_j \otimes N & \xrightarrow{\mu_j \otimes 1} & M \otimes N \end{array}$$

By Lemma 10.8.7 these maps induce a unique homomorphism $\psi : P \to M \otimes N$ such that $\mu_i \otimes 1 = \psi \circ \lambda_i$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that $\widehat{\phi} \circ (\mu_i \times 1) = \lambda_i \circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M \otimes N \to P$. From the commutative diagram below:

$$\begin{array}{ccccccc}
M_i \times N & \xrightarrow{g_i} & M_i \otimes N & \xrightarrow[\mathrm{id}]{} & M_i \otimes N & & \\
\downarrow{\scriptstyle \mu_i \times \mathrm{id}} & & \downarrow{\scriptstyle \lambda_i} & & \downarrow{\scriptstyle \mu_i \otimes \mathrm{id}} & \searrow^{\lambda_i} & \\
M \times N & \xrightarrow{\widehat{\phi}} & P & \xrightarrow{\psi} & M \otimes N & \xrightarrow{\phi} & P
\end{array}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. □

00DF **Lemma 10.12.10.** *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

00DG (10.12.10.1) $$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. For every R-module P we apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

which is exact by Lemma 10.10.1 (1). By Lemma 10.12.8 this becomes the sequence

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

which is therefore also exact. Then using Lemma 10.10.1 (1) again, we arrive at the desired exact sequence. □

00DH **Remark 10.12.11.** However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

00DI **Example 10.12.12.** Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

00DJ **Remark 10.12.13.** For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later in Section 10.39.

05BS **Lemma 10.12.14.** *Let R be a ring. Let M and N be R-modules.*

(1) *If N and M are finite, then so is $M \otimes_R N$.*
(2) *If N and M are finitely presented, then so is $M \otimes_R N$.*

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma 10.12.10. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma 10.5.3. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma 10.5.3. □

00DK **Lemma 10.12.15.** *Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f'(a/s, m) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. □

00DL **Lemma 10.12.16.** *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st$$

Proof. We may use Lemma 10.12.7 and Lemma 10.12.15 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. □

10.13. Tensor algebra

00DM Let R be a ring. Let M be an R-module. We define the *tensor algebra of M over R* to be the noncommutative R-algebra

$$\mathrm{T}(M) = \mathrm{T}_R(M) = \bigoplus\nolimits_{n\geq 0} \mathrm{T}^n(M)$$

with $\mathrm{T}^0(M) = R$, $\mathrm{T}^1(M) = M$, $\mathrm{T}^2(M) = M \otimes_R M$, $\mathrm{T}^3(M) = M \otimes_R M \otimes_R M$, and so on. Multiplication is defined by the rule that on pure tensors we have

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_m$$

and we extend this by linearity.

We define the *exterior algebra $\wedge(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\wedge^n(M)$ is denoted $x_1 \wedge \ldots \wedge x_n$. These elements generate $\wedge^n(M)$, they are R-linear in each x_i and they are zero when two of the x_i are equal (i.e., they are alternating as functions of $x_1, x_2, \ldots, x_n$). The multiplication on $\wedge(M)$ is graded commutative, i.e., every $x \in M$ and $y \in M$ satisfy $x \wedge y = -y \wedge x$.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\wedge(M)$ is free over R with basis the elements

$$x_{i_1} \wedge \ldots \wedge x_{i_r}$$

with $0 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.

We define the *symmetric algebra $Sym(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes y - y \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted just $x_1 \ldots x_n$. These elements generate $\mathrm{Sym}^n(M)$, these are R-linear in each x_i and $x_1 \ldots x_n = x'_1 \ldots x'_n$ if the sequence of elements $x_1, \ldots, x_n$ is a permutation of the sequence $x'_1, \ldots, x'_n$. Thus we see that $\mathrm{Sym}(M)$ is commutative.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\mathrm{Sym}(M) = R[x_1, \ldots, x_n]$ is a polynomial algebra.

00DN **Lemma 10.13.1.** *Let R be a ring. Let M be an R-module. If M is a free R-module, so is each symmetric and exterior power.*

Proof. Omitted, but see above for the finite free case. □

00DO **Lemma 10.13.2.** *Let R be a ring. Let $M_2 \to M_1 \to M \to 0$ be an exact sequence of R-modules. There are exact sequences*

$$M_2 \otimes_R Sym^{n-1}(M_1) \to Sym^n(M_1) \to Sym^n(M) \to 0$$

and similarly

$$M_2 \otimes_R \wedge^{n-1}(M_1) \to \wedge^n(M_1) \to \wedge^n(M) \to 0$$

Proof. Omitted. □

00DP **Lemma 10.13.3.** *Let R be a ring. Let M be an R-module. Let x_i, $i \in I$ be a given system of generators of M as an R-module. Let $n \geq 2$. There exists a canonical exact sequence*

$$\bigoplus_{1\leq j_1<j_2\leq n} \bigoplus_{i_1,i_2\in I} T^{n-2}(M) \oplus \bigoplus_{1\leq j_1<j_2\leq n} \bigoplus_{i\in I} T^{n-2}(M) \to T^n(M) \to \wedge^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ in the first summand maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$
$$+ \quad \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

and $m_1 \otimes \ldots \otimes m_{n-2}$ in the second summand maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_i \otimes \ldots \otimes m_{n-2}}_{\text{with } x_i \text{ and } x_i \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

There is also a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \to T^n(M) \to Sym^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$
$$- \quad \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

Proof. Omitted. □

0H1C **Lemma 10.13.4.** *Let $A \to B$ be a ring map. Let M be a B-module. Let $n > 1$. The kernel of the A-linear map $M \otimes_A \ldots \otimes_A M \to \wedge^n_B(M)$ is generated as an A-module by the elements $m_1 \otimes \ldots \otimes m_n$ with $m_i = m_j$ for $i \neq j$, $m_1, \ldots, m_n \in M$ and the elements $m_1 \otimes \ldots \otimes bm_i \otimes \ldots \otimes m_n - m_1 \otimes \ldots \otimes bm_j \otimes \ldots \otimes m_n$ for $i \neq j$, $m_1, \ldots, m_n \in M$, and $b \in B$.*

Proof. Omitted. □

00DQ **Lemma 10.13.5.** *Let R be a ring. Let M_i be a directed system of R-modules. Then $\operatorname{colim}_i T(M_i) = T(\operatorname{colim}_i M_i)$ and similarly for the symmetric and exterior algebras.*

Proof. Omitted. Hint: Apply Lemma 10.12.9. □

0C6F **Lemma 10.13.6.** *Let R be a ring and let $S \subset R$ be a multiplicative subset. Then $S^{-1}T_R(M) = T_{S^{-1}R}(S^{-1}M)$ for any R-module M. Similar for symmetric and exterior algebras.*

Proof. Omitted. Hint: Apply Lemma 10.12.16. □

10.14. Base change

05G3 We formally introduce base change in algebra as follows.

05G4 **Definition 10.14.1.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The *base change* of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The *base change* of the S-module M is the S'-module $M \otimes_R R'$.

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

05G5 **Lemma 10.14.2.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be a ring map and let $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$ be the base changes.*

(1) *If M is a finite S-module, then the base change M' is a finite S'-module.*
(2) *If M is an S-module of finite presentation, then the base change M' is an S'-module of finite presentation.*
(3) *If $R \to S$ is of finite type, then the base change $R' \to S'$ is of finite type.*
(4) *If $R \to S$ is of finite presentation, then the base change $R' \to S'$ is of finite presentation.*

Proof. Proof of (1). Take a surjective, S-linear map $S^{\oplus n} \to M \to 0$. By Lemma 10.12.3 and 10.12.10 the result after tensoring with R' is a surjection $S'^{\oplus n} \to M' \to 0$, so M' is a finitely generated S'-module. Proof of (2). Take a presentation $S^{\oplus m} \to S^{\oplus n} \to M \to 0$. By Lemma 10.12.3 and 10.12.10 the result after tensoring with R' gives a finite presentation $S'^{\oplus m} \to S'^{\oplus n} \to M' \to 0$, of the S'-module M'. Proof of (3). This follows by the remark preceding the lemma as we can take I to be finite by assumption. Proof of (4). This follows by the remark preceding the lemma as we can take I and J to be finite by assumption. □

Let $\varphi : R \to S$ be a ring map. Given an S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the *restriction* of N to R.

05DQ **Lemma 10.14.3.** *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$\mathrm{Hom}_R(M, N_R) = \mathrm{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. □

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

08YP **Lemma 10.14.4.** *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto \mathrm{Hom}_R(S, M)$ are adjoint functors. In a formula*

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \mathrm{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \mathrm{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. □

08YQ **Lemma 10.14.5.** *Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have*

$$\mathrm{Hom}_R(M \otimes_S N, P) = \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Lemmas 10.14.4 and 10.12.8. Namely, we have

$$\begin{aligned}\mathrm{Hom}_R(M \otimes_S N, P) &= \mathrm{Hom}_S(M \otimes_S N, \mathrm{Hom}_R(S, P)) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, P))) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))\end{aligned}$$

as desired. □

10.15. Miscellany

00DR The proofs in this section should not refer to any results except those from the section on basic notions, Section 10.3.

07K1 **Lemma 10.15.1.** *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in J \setminus \mathfrak{p}$. Their product is an element of $IJ \subset \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

00DS **Lemma 10.15.2** (Prime avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

(1) *$J \not\subset I_i$ for $i = 1, \ldots, r$, and*
(2) *all but two of I_i are prime ideals.*

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 10.15.1) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

0EHL **Lemma 10.15.3.** *Let R be a ring. Let $x \in R$, $I \subset R$ an ideal, and $\mathfrak{p}_i$, $i = 1, \ldots, r$ be prime ideals. Suppose that $x + I \not\subset \mathfrak{p}_i$ for $i = 1, \ldots, r$. Then there exists a $y \in I$ such that $x + y \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. After reordering we may assume $x \notin \mathfrak{p}_i$ for $i < s$ and $x \in \mathfrak{p}_i$ for $i \geq s$. If $s = r + 1$ then we are done. If not, then we can find $y \in I$ with $y \notin \mathfrak{p}_s$. Choose $f \in \bigcap_{i<s} \mathfrak{p}_i$ with $f \notin \mathfrak{p}_s$. Then $x + fy$ is not contained in $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Thus we win by induction on s. □

00DT **Lemma 10.15.4** (Chinese remainder)**.** *Let R be a ring.*

(1) *If $I_1, \ldots, I_r$ are ideals such that $I_a + I_b = R$ when $a \neq b$, then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and $R/(I_1 I_2 \ldots I_r) \cong R/I_1 \times \ldots \times R/I_r$.*
(2) *If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are pairwise distinct maximal ideals then $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and the above applies.*

Proof. Let us first prove $I_1 \cap \ldots \cap I_r = I_1 \ldots I_r$ as this will also imply the injectivity of the induced ring homomorphism $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$. The inclusion

$I_1 \cap \ldots \cap I_r \supset I_1 \ldots I_r$ is always fulfilled since ideals are closed under multiplication with arbitrary ring elements. To prove the other inclusion, we claim that the ideals

$$I_1 \ldots \hat{I}_i \ldots I_r, \quad i = 1, \ldots, r$$

generate the ring R. We prove this by induction on r. It holds when $r = 2$. If $r > 2$, then we see that R is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_{r-1}$, $i = 1, \ldots, r-1$. Hence I_r is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 1, \ldots, r-1$. Applying the same argument with the reverse ordering on the ideals we see that I_1 is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 2, \ldots, r$. Since $R = I_1 + I_r$ by assumption we see that R is the sum of the ideals displayed above. Therefore we can find elements $a_i \in I_1 \ldots \hat{I}_i \ldots I_r$ such that their sum is one. Multiplying this equation by an element of $I_1 \cap \ldots \cap I_r$ gives the other inclusion. It remains to show that the canonical map $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$ is surjective. For this, consider its action on the equation $1 = \sum_{i=1}^r a_i$ we derived above. On the one hand, a ring morphism sends 1 to 1 and on the other hand, the image of any a_i is zero in R/I_j for $j \neq i$. Therefore, the image of a_i in R/I_i is the identity. So given any element $(\bar{b}_1, \ldots, \bar{b}_r) \in R/I_1 \times \ldots \times R/I_r$, the element $\sum_{i=1}^r a_i \cdot b_i$ is an inverse image in R.

To see (2), by the very definition of being distinct maximal ideals, we have $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and so the above applies. □

07DQ **Lemma 10.15.5.** *Let R be a ring. Let $n \geq m$. Let A be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.*

(1) *For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f1_{m \times m}$.*
(2) *If $f \in R$ and $BA = f1_{m \times m}$ for some $m \times n$ matrix B, then $f^m \in J$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$, we denote by E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i \in \{1,\ldots,n\}} R \longrightarrow \bigoplus\nolimits_{i \in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I) 1_{m \times m}$. The $m \times m$ minors of A are the determinants $\det A_I$ for all the $I \subset \{1, \ldots, n\}$ with $|I| = m$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f1_{m \times m} = BA$ then by the Cauchy-Binet formula (72) we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. □

080R **Lemma 10.15.6.** *Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R, written in block form as*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 has size $m \times m$. Let B be the adjugate (transpose of cofactor) matrix to A_1. Then

$$AB = \begin{pmatrix} f1_{m \times m} \\ C \end{pmatrix}$$

where $f = \det(A_1)$ and c_{ij} is (up to sign) the determinant of the $m \times m$ minor of A corresponding to the rows $1, \ldots, \hat{j}, \ldots, m, i$.

Proof. Since the adjugate has the property $A_1B = BA_1 = f$ the first block of the expression for AB is correct. Note that

$$c_{ij} = \sum\nolimits_k a_{ik}b_{kj} = \sum(-1)^{j+k}a_{ik}\det(A_1^{jk})$$

where A_1^{ij} means A_1 with the jth row and kth column removed. This last expression is the row expansion of the determinant of the matrix in the statement of the lemma. □

05WI **Lemma 10.15.7.** *Let R be a nonzero ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.*

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$ (Lemma 10.5.2). It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace R by the localization R_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Proposition 10.9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

0FJ7 **Lemma 10.15.8.** *Let R be a nonzero ring. Let $n, m \geq 0$ be integers. If $R^{\oplus n}$ is isomorphic to $R^{\oplus m}$ as R-modules, then $n = m$.*

Proof. Immediate from Lemma 10.15.7. □

10.16. Cayley-Hamilton

05G6

00DX **Lemma 10.16.1.** *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(xid_{n\times n} - A)$). Then $P(A) = 0$ in $Mat(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

(1) If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.
(2) If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.
(3) Thus we may first reduce to the case $R = \mathbf{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbf{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

□

05BT **Lemma 10.16.2.** *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in R^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ R^{\oplus n} & \longrightarrow & M \end{array}$$

is commutative where $A = (a_{ij})$. By Lemma 10.16.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

05G7 **Lemma 10.16.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in I^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ I^{\oplus n} & \longrightarrow & M \end{array}$$

is commutative where $A = (a_{ij})$. By Lemma 10.16.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. □

As a fun example application we prove the following surprising lemma.

05G8 **Lemma 10.16.4.** *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

First proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Lemma 10.16.3 to M as an R'-module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi))\varphi$, and hence φ is invertible. □

Second proof. We perform induction on the number of generators of M over R. If M is generated by one element, then $M \cong R/I$ for some ideal $I \subset R$. In this case we may replace R by R/I so that $M = R$. In this case $\varphi : R \to R$ is given by multiplication on M by an element $r \in R$. The surjectivity of φ forces r invertible, since φ must hit 1, which implies that φ is invertible.

Now assume that we have proven the lemma in the case of modules generated by $n - 1$ elements, and are examining a module M generated by n elements. Let

A mean the ring $R[t]$, and regard the module M as an A-module by letting t act via φ; since M is finite over R, it is finite over $R[t]$ as well, and since we're trying to prove φ injective, a set-theoretic property, we might as well prove the endomorphism $t : M \to M$ over A injective. We have reduced our problem to the case our endomorphism is multiplication by an element of the ground ring. Let $M' \subset M$ denote the sub-A-module generated by the first $n-1$ of the generators of M, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi|_{M'}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \bmod M'} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0, \end{array}$$

where the restriction of φ to M' and the map induced by φ on the quotient M/M' are well-defined since φ is multiplication by an element in the base, and M' and M/M' are A-modules in their own right. By the case $n = 1$ the map $M/M' \to M/M'$ is an isomorphism. A diagram chase implies that $\varphi|_{M'}$ is surjective hence by induction $\varphi|_{M'}$ is an isomorphism. This forces the middle column to be an isomorphism by the snake lemma. □

10.17. The spectrum of a ring

00DY We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

00DZ **Definition 10.17.1.** Let R be a ring.

(1) The *spectrum* of R is the set of prime ideals of R. It is usually denoted $\mathrm{Spec}(R)$.
(2) Given a subset $T \subset R$ we let $V(T) \subset \mathrm{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
(3) Given an element $f \in R$ we let $D(f) \subset \mathrm{Spec}(R)$ be the set of primes not containing f.

00E0 **Lemma 10.17.2.** *Let R be a ring.*

(1) *The spectrum of a ring R is empty if and only if R is the zero ring.*
(2) *Every nonzero ring has a maximal ideal.*
(3) *Every nonzero ring has a minimal prime ideal.*
(4) *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
(5) *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
(6) *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (27), then $V(I) = V(\sqrt{I})$.*
(7) *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*
(8) *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
(9) *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
(10) *If $(I_a)_{a \in A}$ is a set of ideals of R then $\bigcap_{a \in A} V(I_a) = V(\bigcup_{a \in A} I_a)$.*
(11) *If $f \in R$, then $D(f) \amalg V(f) = \mathrm{Spec}(R)$.*
(12) *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*
(13) *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*

(14) *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*
(15) *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*
(16) *If $f_i \in R$ for $i \in I$, then $\bigcup_{i \in I} D(f_i)$ is the complement of $V(\{f_i\}_{i \in I})$ in* $\operatorname{Spec}(R)$.
(17) *If $f \in R$ and $D(f) = \operatorname{Spec}(R)$, then f is a unit.*

Proof. We address each part in the corresponding item below.

(1) This is a direct consequence of (2) or (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of $\mathfrak{A}$. Then $\bigcup_{I \in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I \in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I \in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I \in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.
(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.
(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I \subset \mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.
(8) I is not the unit ideal if and only if I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.
(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.
(10) $\mathfrak{p} \in \bigcap_{a \in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\bigcup_{a \in A} I_a)$
(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.
(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(a) = \emptyset$. Now, as

shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.
(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.
(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
(16) $\mathfrak{p} \in \bigcup_{i \in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\{f_i\}_{i \in I})$
(17) If $D(f) = \operatorname{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 10.17.1 form the closed subsets of a topology on $\operatorname{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

00E1 **Definition 10.17.3.** Let R be a ring. The topology on $\operatorname{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $\operatorname{Spec}(R)$.

It should be clear from context whether we consider $\operatorname{Spec}(R)$ just as a set or as a topological space.

00E2 **Lemma 10.17.4.** *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\operatorname{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (41) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\operatorname{Spec}(R'') \longrightarrow \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R)$$

equals $\operatorname{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

00E3 **Lemma 10.17.5.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of* Spec *a homeomorphism*

$$\operatorname{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \operatorname{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\operatorname{Spec}(R)$*. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p} = \mathfrak{p}(S^{-1}R)$.*

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (54) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (62) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to

$\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (55). This proves that the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 10.17.4. Finally, let $D(g) \subset \mathrm{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\mathrm{Spec}(S^{-1}R)$. Hence by Lemma 10.17.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

00E4 **Lemma 10.17.6.** *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R_f) \longrightarrow D(f) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 10.17.5. □

It is not the case that every "affine open" of a spectrum is a standard open. See Example 10.27.4.

00E5 **Lemma 10.17.7.** *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R/I) \longrightarrow V(I) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (51) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f + I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

00E8 **Lemma 10.17.8.** *Let R be a ring. The space* $\mathrm{Spec}(R)$ *is quasi-compact.*

Proof. It suffices to prove that any covering of $\mathrm{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathrm{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i \in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 10.17.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i \in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathrm{Spec}(R) = \cup_{i \in J} D(f_i)$. □

04PM **Lemma 10.17.9.** *Let R be a ring. The topology on $X = \mathrm{Spec}(R)$ has the following properties:*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

Proof. The spectrum of a ring is quasi-compact, see Lemma 10.17.8. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma 10.17.6) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. □

10.18. Local rings

07BH Local rings are the bread and butter of algebraic geometry.

07BI **Definition 10.18.1.** A *local ring* is a ring with exactly one maximal ideal. If R is a local ring, then the maximal ideal is often denoted $\mathfrak{m}_R$ and the field $R/\mathfrak{m}_R$ is called the *residue field* of the local ring R. We often say "let $(R, \mathfrak{m})$ be a local ring" or "let $(R, \mathfrak{m}, \kappa)$ be a local ring" to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "*local ring map* $\varphi : R \to S$" means that φ is a local homomorphism of local rings.

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

The localization $R_\mathfrak{p}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Namely, by Lemma 10.17.5 every prime ideal of $R_\mathfrak{p}$ is contained in the prime ideal $\mathfrak{p}R_\mathfrak{p}$ (hence this is a maximal ideal and the only maximal ideal of $R_\mathfrak{p}$). The residue field of $R_\mathfrak{p}$ is denoted $\kappa(\mathfrak{p})$; we call it the *residue field of* $\mathfrak{p}$; by Proposition 10.9.14 we may identify $\kappa(\mathfrak{p})$ with the field of fractions of the domain $R/\mathfrak{p}$. Via the composition

$$\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(R_\mathfrak{p}) \to \mathrm{Spec}(R)$$

the unique point of the source maps to the point $\mathfrak{p}$ of the target.

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and consider the prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R. Since $\varphi(\mathfrak{p}) \subset \mathfrak{q}$ the induced ring map

$$R_\mathfrak{p} \to S_\mathfrak{q}, \quad r/g \mapsto \varphi(r)/\varphi(g)$$

is a local ring map and we obtain an induced map of residue fields $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$.

0H70 **Example 10.18.2.** If R is a local ring and $\mathfrak{p} \subset R$ is a non-maximal prime ideal, then $R \to R_\mathfrak{p}$ is not a local homomorphism.

00E9 **Lemma 10.18.3.** *Let R be a ring. The following are equivalent:*

1. *R is a local ring,*
2. *$\mathrm{Spec}(R)$ has exactly one closed point,*
3. *R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and*
4. *R is not the zero ring and for every $x \in R$ either x or $1 - x$ is invertible or both.*

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem (Lemma 10.15.4). This element x is not invertible and neither is $1 - x$ which is a contradiction. Thus (4) and (1) are equivalent. □

07BJ **Lemma 10.18.4.** *Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:*

(1) φ *is a local ring map,*
(2) $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$,
(3) $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, *and*
(4) *for any* $x \in R$, *if* $\varphi(x)$ *is invertible in* S, *then* x *is invertible in* R.

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containing the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Finally, (4) is the contrapositive of (2) by Lemma 10.18.3. □

00E6 **Remark 10.18.5.** A fundamental commutative diagram associated to a ring map $\varphi : R \to S$ and a prime $\mathfrak{p} \subset R$ is the following

$$\begin{array}{ccccccc}
\kappa(\mathfrak{p}) \otimes_R S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} & \longleftarrow & S_\mathfrak{p} & \longleftarrow & S & \longrightarrow & S/\mathfrak{p}S \longrightarrow (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} & \longleftarrow & R_\mathfrak{p} & \longleftarrow & R & \longrightarrow & R/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p})
\end{array}$$

In this diagram the outer left and outer right columns are identical. On spectra the horizontal maps induce homeomorphisms onto their images and the squares induce fibre squares of topological spaces (see Lemmas 10.17.5 and 10.17.7). This shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring. If there does exist a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, i.e., with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ then we can extend the diagram to the following diagram

$$\begin{array}{ccccccc}
\kappa(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} & \longleftarrow & S_\mathfrak{q} & \longleftarrow & S & \longrightarrow & S/\mathfrak{q} \longrightarrow \kappa(\mathfrak{q}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\kappa(\mathfrak{p}) \otimes_R S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} & \longleftarrow & S_\mathfrak{p} & \longleftarrow & S & \longrightarrow & S/\mathfrak{p}S \longrightarrow (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} & \longleftarrow & R_\mathfrak{p} & \longleftarrow & R & \longrightarrow & R/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p})
\end{array}$$

In this diagram it is still the case that the outer left and outer right columns are identical and that on spectra the horizontal maps induce homeomorphisms onto their image.

00E7 **Lemma 10.18.6.** *Let* $\varphi : R \to S$ *be a ring map. Let* $\mathfrak{p}$ *be a prime of* R. *The following are equivalent*

(1) $\mathfrak{p}$ *is in the image of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$,
(2) $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
(3) $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \neq 0$,
(4) $(S/\mathfrak{p}S)_\mathfrak{p} \neq 0$, *and*
(5) $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

Proof. We have already seen the equivalence of the first two in Remark 10.18.5. The others are just reformulations of this. □

10.19. The Jacobson radical of a ring

0AMD We recall that the *Jacobson radical* $\mathrm{rad}(R)$ of a ring R is the intersection of all maximal ideals of R. If R is local then $\mathrm{rad}(R)$ is the maximal ideal of R.

0AME **Lemma 10.19.1.** *Let R be a ring with Jacobson radical $\text{rad}(R)$. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *$I \subset \text{rad}(R)$, and*
(2) *every element of $1 + I$ is a unit in R.*

In this case every element of R which maps to a unit of R/I is a unit.

Proof. If $f \in \text{rad}(R)$, then $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence $1 + f \notin \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Thus the closed subset $V(1+f)$ of $\text{Spec}(R)$ is empty. This implies that $1 + f$ is a unit, see Lemma 10.17.2.

Conversely, assume that $1 + f$ is a unit for all $f \in I$. If $\mathfrak{m}$ is a maximal ideal and $I \not\subset \mathfrak{m}$, then $I + \mathfrak{m} = R$. Hence $1 = f + g$ for some $g \in \mathfrak{m}$ and $f \in I$. Then $g = 1 + (-f)$ is not a unit, contradiction.

For the final statement let $f \in R$ map to a unit in R/I. Then we can find $g \in R$ mapping to the multiplicative inverse of f mod I. Then $fg = 1 \bmod I$. Hence fg is a unit of R by (2) which implies that f is a unit. □

0B7C **Lemma 10.19.2.** *Let $\varphi : R \to S$ be a ring map such that the induced map $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. Then an element $x \in R$ is a unit if and only if $\varphi(x) \in S$ is a unit.*

Proof. If x is a unit, then so is $\varphi(x)$. Conversely, if $\varphi(x)$ is a unit, then $\varphi(x) \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Spec}(S)$. Hence $x \notin \varphi^{-1}(\mathfrak{q}) = \text{Spec}(\varphi)(\mathfrak{q})$ for all $\mathfrak{q} \in \text{Spec}(S)$. Since $\text{Spec}(\varphi)$ is surjective we conclude that x is a unit by part (17) of Lemma 10.17.2. □

10.20. Nakayama's lemma

07RC We quote from [**Mat70a**]: "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

00DV **Lemma 10.20.1** (Nakayama's lemma)**.** *Let R be a ring with Jacobson radical $\text{rad}(R)$. Let M be an R-module. Let $I \subset R$ be an ideal.* [**Mat70a**, 1.M Lemma (NAK) page 11]

00DW (1) *If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset \text{rad}(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1 + I$ such that $fM \subset N$ and $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \text{rad}(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \text{rad}(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \text{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*

(12) *If $\{x_\alpha\}_{\alpha\in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ (since $M = IM$). In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$ (since the matrix A is entrywise congruent to the $m \times m$ identity matrix modulo I). By Lemma 10.15.5 (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m\times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f\delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f\delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Lemma 10.19.1 an element of $1 + \mathrm{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

0GLX **Lemma 10.20.2.** *Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S + I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.*[1]

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum (a_{ij}/s_{ij}) x_j$ in $S^{-1}M$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Multiplying by the product $s \in S$ of the s_{ij} we see that $s y_i = \sum a'_{ij} x_j$ in $S^{-1}M$ for some $a'_{ij} \in R$. This in turn means there exist $t_i \in S$ such that $t_i s y_i = \sum t_i a'_{ij} x_j$ in M. Thus if $t \in S$ is the product of the t_i, then we see that y_i is in the R_{st}-submodule generated by $x_1, \ldots, x_r$ of M_{st}. Hence $x_1, \ldots, x_r$ generates M_{st}.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Lemma 10.20.1 we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. □

0E8M **Lemma 10.20.3.** *Let $A \to B$ be a local homomorphism of local rings. Assume*

(1) *B is finite as an A-module,*
(2) *$\mathfrak{m}_B$ is a finitely generated ideal,*
(3) *$A \to B$ induces an isomorphism on residue fields, and*
(4) *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.*

Then $A \to B$ is surjective.

[1] Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \setminus \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \notin \mathfrak{p}$.

Proof. To show that $A \to B$ is surjective, we view it as a map of A-modules and apply Lemma 10.20.1 (6). We conclude it suffices to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. As $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. View $\mathfrak{m}_A B \to \mathfrak{m}_B$ as a map of B-modules and apply Lemma 10.20.1 (6). We conclude it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. This follows from assumption (4). □

10.21. Open and closed subsets of spectra

04PN It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

00EC **Lemma 10.21.1.** *Let R be a ring. Let $e \in R$ be an idempotent. In this case*

$$\operatorname{Spec}(R) = D(e) \amalg D(1-e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$\begin{aligned} D(e) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e \neq 0 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e = 1 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Similarly we have

$$\begin{aligned} D(1-e) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid 1-e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e \neq 1 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid e = 0 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Since the image of e in any residue field is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $\operatorname{Spec}(R)$. □

00ED **Lemma 10.21.2.** *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\operatorname{Spec}(R_1) \to \operatorname{Spec}(R)$ and $\operatorname{Spec}(R_2) \to \operatorname{Spec}(R)$. The induced map*

$$\operatorname{Spec}(R_1) \amalg \operatorname{Spec}(R_2) \longrightarrow \operatorname{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 10.21.1 combined with Lemma 10.17.6. □

We reprove the following lemma later after introducing a glueing lemma for functions. See Section 10.24.

00EE **Lemma 10.21.3.** *Let R be a ring. For each $U \subset \operatorname{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \operatorname{Spec}(R)$ and idempotents $e \in R$.*

Proof. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact by Lemma 10.17.8, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Lemma 10.17.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \amalg V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$, see Lemma 10.17.2. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $0 = xy = x(1 - x)$ as $I^N J^N = (0)$. Thus $x = x^2$ is idempotent and contained in $I^N \subset I$. The idempotent $y = 1 - x$ is contained in $J^N \subset J$. This shows that the idempotent x maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that x maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \notin \mathfrak{p}$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1 e_2' \notin \mathfrak{p}$ or $e_1' e_2 \notin \mathfrak{p}$, by Lemma 10.17.2. It is easy to see this gives the desired prime. □

00EF **Lemma 10.21.4.** *Let R be a nonzero ring. Then $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents.*

Proof. Obvious from Lemma 10.21.3 and the definition of a connected topological space. □

00EH **Lemma 10.21.5.** *Let $I \subset R$ be a finitely generated ideal of a ring R such that $I = I^2$. Then*

1. *there exists an idempotent $e \in R$ such that $I = (e)$,*
2. *$R/I \cong R_{e'}$ for the idempotent $e' = 1 - e \in R$, and*
3. *$V(I)$ is open and closed in $\mathrm{Spec}(R)$.*

Proof. By Nakayama's Lemma 10.20.1 there exists an element $f = 1 + i$, $i \in I$ such that $fI = 0$. Then $f^2 = f + fi = f$ is an idempotent. Consider the idempotent $e = 1 - f = -i \in I$. For $j \in I$ we have $ej = j - fj = j$ hence $I = (e)$. This proves (1).

Parts (2) and (3) follow from (1). Namely, we have $V(I) = V(e) = \mathrm{Spec}(R) \setminus D(e)$ which is open and closed by either Lemma 10.21.1 or Lemma 10.21.3. This proves (3). For (2) observe that the map $R \to R_{e'}$ is surjective since $x/(e')^n = x/e' = xe'/(e')^2 = xe'/e' = x/1$ in $R_{e'}$. The kernel of the map $R \to R_{e'}$ is the set of elements of R annihilated by a positive power of e'. Since e' is idempotent this is the ideal of elements annihilated by e' which is the ideal $I = (e)$ as $e + e' = 1$ is a pair of orthogonal idempotents. This proves (2). □

10.22. Connected components of spectra

00EB Connected components of spectra are not as easy to understand as one may think at first. This is because we are used to the topology of locally connected spaces, but the spectrum of a ring is in general not locally connected.

04PP **Lemma 10.22.1.** *Let R be a ring. Let $T \subset \mathrm{Spec}(R)$ be a subset of the spectrum. The following are equivalent*

(1) *T is closed and is a union of connected components of $\mathrm{Spec}(R)$,*
(2) *T is an intersection of open and closed subsets of $\mathrm{Spec}(R)$, and*
(3) *$T = V(I)$ where $I \subset R$ is an ideal generated by idempotents.*

Moreover, the ideal in (3) if it exists is unique.

Proof. By Lemma 10.17.9 and Topology, Lemma 5.12.12 we see that (1) and (2) are equivalent. Assume (2) and write $T = \bigcap U_\alpha$ with $U_\alpha \subset \mathrm{Spec}(R)$ open and closed. Then $U_\alpha = D(e_\alpha)$ for some idempotent $e_\alpha \in R$ by Lemma 10.21.3. Then setting $I = (1 - e_\alpha)$ we see that $T = V(I)$, i.e., (3) holds. Finally, assume (3). Write $T = V(I)$ and $I = (e_\alpha)$ for some collection of idempotents e_α. Then it is clear that $T = \bigcap V(e_\alpha) = \bigcap D(1 - e_\alpha)$.

Suppose that I is an ideal generated by idempotents. Let $e \in R$ be an idempotent such that $V(I) \subset V(e)$. Then by Lemma 10.17.2 we see that $e^n \in I$ for some $n \geq 1$. As e is an idempotent this means that $e \in I$. Hence we see that I is generated by exactly those idempotents e such that $T \subset V(e)$. In other words, the ideal I is completely determined by the closed subset T which proves uniqueness. □

00EG **Lemma 10.22.2.** *Let R be a ring. A connected component of $\mathrm{Spec}(R)$ is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.*

Proof. Let $\mathfrak{p}$ be a prime of R. By Lemma 10.17.9 we have see that the hypotheses of Topology, Lemma 5.12.10 are satisfied for the topological space $\mathrm{Spec}(R)$. Hence the connected component of $\mathfrak{p}$ in $\mathrm{Spec}(R)$ is the intersection of open and closed subsets of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Lemma 10.21.3. Any idempotent e which is not in this collection clearly maps to 1 in R/I. □

10.23. Glueing properties

00EN In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

00HN **Lemma 10.23.1.** *Let R be a ring.*

(1) *For an element x of an R-module M the following are equivalent*
   (a) *$x = 0$,*
   (b) *x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$,*
   (c) *x maps to zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R.*

   In other words, the map $M \to \prod_\mathfrak{m} M_\mathfrak{m}$ is injective.
(2) *Given an R-module M the following are equivalent*
   (a) *M is zero,*
   (b) *$M_\mathfrak{p}$ is zero for all $\mathfrak{p} \in \mathrm{Spec}(R)$,*
   (c) *$M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$ of R.*
(3) *Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent*
   (a) *$M_1 \to M_2 \to M_3$ is exact,*
   (b) *for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,*

(c) *for every maximal ideal* $\mathfrak{m}$ *of* R *the localization* $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ *is exact.*

(4) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*
   (a) f *is injective,*
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ *is injective for all primes* $\mathfrak{p}$ *of* R*,*
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ *is injective for all maximal ideals* $\mathfrak{m}$ *of* R*.*

(5) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*
   (a) f *is surjective,*
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ *is surjective for all primes* $\mathfrak{p}$ *of* R*,*
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ *is surjective for all maximal ideals* $\mathfrak{m}$ *of* R*.*

(6) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*
   (a) f *is bijective,*
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ *is bijective for all primes* $\mathfrak{p}$ *of* R*,*
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ *is bijective for all maximal ideals* $\mathfrak{m}$ *of* R*.*

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. By Lemma 10.17.2 we see I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \mathrm{Ker}(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. By Proposition 10.9.12 we have that $H_\mathfrak{p}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). □

00EO **Lemma 10.23.2.** *Let* R *be a ring. Let* M *be an* R*-module. Let* S *be an* R*-algebra. Suppose that* $f_1, \ldots, f_n$ *is a finite list of elements of* R *such that* $\bigcup D(f_i) = \mathrm{Spec}(R)$*, in other words* $(f_1, \ldots, f_n) = R$*.*

(1) *If each* $M_{f_i} = 0$ *then* $M = 0$*.*
(2) *If each* M_{f_i} *is a finite* R_{f_i}*-module, then* M *is a finite* R*-module.*
(3) *If each* M_{f_i} *is a finitely presented* R_{f_i}*-module, then* M *is a finitely presented* R*-module.*
(4) *Let* $M \to N$ *be a map of* R*-modules. If* $M_{f_i} \to N_{f_i}$ *is an isomorphism for each* i *then* $M \to N$ *is an isomorphism.*
(5) *Let* $0 \to M'' \to M \to M' \to 0$ *be a complex of* R*-modules. If* $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ *is exact for each* i*, then* $0 \to M'' \to M \to M' \to 0$ *is exact.*
(6) *If each* R_{f_i} *is Noetherian, then* R *is Noetherian.*
(7) *If each* S_{f_i} *is a finite type* R_{f_i}*-algebra, then* S *is a finite type* R*-algebra.*
(8) *If each* S_{f_i} *is of finite presentation over* R_{f_i}*, then* S *is a finitely presented* R*-algebra.*

Proof. We prove each of the parts in turn.

(1) By Proposition 10.9.10 this implies $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, so we conclude by Lemma 10.23.1.

(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it is surjective by Lemma 10.23.1 and M is finitely generated.

(3) By (2) we have a short exact sequence

$$0 \to K \to R^n \to M \to 0$$

Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$ by Lemma 10.5.3. By (2) this implies that K is a finite R-module and therefore M is finitely presented.

(4) By Proposition 10.9.10 the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 10.23.1.

(5) By Proposition 10.9.10 the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 10.23.1.

(6) We will show that every ideal of R has a finite generating set: For this, let $I \subset R$ be an arbitrary ideal. By Proposition 10.9.12 each $I_{f_i} \subset R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).

(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by (1) and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, \ldots, X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, \ldots, X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, \ldots, X_n]$, so an application of (2) finishes the proof.

□

00EP **Lemma 10.23.3.** *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_n$ is a finite list of elements of S such that $\bigcup D(g_i) = \operatorname{Spec}(S)$ in other words $(g_1, \ldots, g_n) = S$.*

(1) *If each S_{g_i} is of finite type over R, then S is of finite type over R.*
(2) *If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.*

Proof. Choose $h_1, \ldots, h_n \in S$ such that $\sum h_i g_i = 1$.

Proof of (1). For each i choose a finite list of elements $x_{i,j} \in S_{g_i}$, $j = 1, \ldots, m_i$ which generate S_{g_i} as an R-algebra. Write $x_{i,j} = y_{i,j}/g_i^{n_{i,j}}$ for some $y_{i,j} \in S$ and some $n_{i,j} \geq 0$. Consider the R-subalgebra $S' \subset S$ generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ and $y_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Since localization is exact (Proposition 10.9.12), we see that $S'_{g_i} \to S_{g_i}$ is injective. On the other hand, it is surjective by our choice of $y_{i,j}$. The elements $g_1, \ldots, g_n$ generate the unit ideal in S' as $h_1, \ldots, h_n \in S'$. Thus $S' \to S$ viewed as an S'-module map is an isomorphism by Lemma 10.23.2.

Proof of (2). We already know that S is of finite type. Write $S = R[x_1, \ldots, x_m]/J$ for some ideal J. For each i choose a lift $g'_i \in R[x_1, \ldots, x_m]$ of g_i and we choose a lift $h'_i \in R[x_1, \ldots, x_m]$ of h_i. Then we see that

$$S_{g_i} = R[x_1, \ldots, x_m, y_i]/(J_i + (1 - y_i g'_i))$$

where J_i is the ideal of $R[x_1, \ldots, x_m, y_i]$ generated by J. Small detail omitted. By Lemma 10.6.3 we may choose a finite list of elements $f_{i,j} \in J$, $j = 1, \ldots, m_i$ such that the images of $f_{i,j}$ in J_i and $1 - y_i g'_i$ generate the ideal $J_i + (1 - y_i g'_i)$. Set

$$S' = R[x_1, \ldots, x_m]/\left(\sum h'_i g'_i - 1, f_{i,j}; i = 1, \ldots, n, j = 1, \ldots, m_i\right)$$

There is a surjective R-algebra map $S' \to S$. The classes of the elements $g'_1, \ldots, g'_n$ in S' generate the unit ideal and by construction the maps $S'_{g'_i} \to S_{g_i}$ are injective. Thus we conclude as in part (1). □

10.24. Glueing functions

00EI In this section we show that given an open covering

$$\mathrm{Spec}(R) = \bigcup\nolimits_{i=1}^{n} D(f_i)$$

by standard opens, and given an element $h_i \in R_{f_i}$ for each i such that $h_i = h_j$ as elements of $R_{f_i f_j}$ then there exists a unique $h \in R$ such that the image of h in R_{f_i} is h_i. This result can be interpreted in two ways:

(1) The rule $D(f) \mapsto R_f$ is a sheaf of rings on the standard opens, see Sheaves, Section 6.30.
(2) If we think of elements of R_f as the "algebraic" or "regular" functions on $D(f)$, then these glue as would continuous, resp. differentiable functions on a topological, resp. differentiable manifold.

00EK **Lemma 10.24.1.** *Let R be a ring. Let $f_1, \ldots, f_n$ be elements of R generating the unit ideal. Let M be an R-module. The sequence*

$$0 \to M \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^{n} M_{f_i} \xrightarrow{\beta} \bigoplus\nolimits_{i,j=1}^{n} M_{f_i f_j}$$

is exact, where $\alpha(m) = (m/1, \ldots, m/1)$ and $\beta(m_1/f_1^{e_1}, \ldots, m_n/f_n^{e_n}) = (m_i/f_i^{e_i} - m_j/f_j^{e_j})_{(i,j)}$.

Proof. It suffices to show that the localization of the sequence at any maximal ideal $\mathfrak{m}$ is exact, see Lemma 10.23.1. Since $f_1, \ldots, f_n$ generate the unit ideal, there is an i such that $f_i \notin \mathfrak{m}$. After renumbering we may assume $i = 1$. Note that $(M_{f_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i}$ and $(M_{f_i f_j})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i f_j}$, see Proposition 10.9.11. In particular $(M_{f_1})_{\mathfrak{m}} = M_{\mathfrak{m}}$ and $(M_{f_1 f_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i}$, because f_1 is a unit. Note that the maps in the sequence are the canonical ones coming from Lemma 10.9.7 and the identity

map on M. Having said all of this, after replacing R by $R_{\mathfrak{m}}$, M by $M_{\mathfrak{m}}$, and f_i by their image in $R_{\mathfrak{m}}$, and f_1 by $1 \in R_{\mathfrak{m}}$, we reduce to the case where $f_1 = 1$.

Assume $f_1 = 1$. Injectivity of α is now trivial. Let $m = (m_i) \in \bigoplus_{i=1}^n M_{f_i}$ be in the kernel of β. Then $m_1 \in M_{f_1} = M$. Moreover, $\beta(m) = 0$ implies that m_1 and m_i map to the same element of $M_{f_1 f_i} = M_{f_i}$. Thus $\alpha(m_1) = m$ and the proof is complete. □

00EJ **Lemma 10.24.2.** *Let R be a ring, and let $f_1, f_2, \ldots f_n \in R$ generate the unit ideal in R. Then the following sequence is exact:*

$$0 \longrightarrow R \longrightarrow \bigoplus\nolimits_i R_{f_i} \longrightarrow \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

where the maps $\alpha : R \longrightarrow \bigoplus_i R_{f_i}$ and $\beta : \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$ are defined as

$$\alpha(x) = \left(\frac{x}{1}, \ldots, \frac{x}{1}\right) \text{ and } \beta\left(\frac{x_1}{f_1^{r_1}}, \ldots, \frac{x_n}{f_n^{r_n}}\right) = \left(\frac{x_i}{f_i^{r_i}} - \frac{x_j}{f_j^{r_j}} \text{ in } R_{f_i f_j}\right).$$

Proof. Special case of Lemma 10.24.1. □

The following we have already seen above, but we state it explicitly here for convenience.

00EM **Lemma 10.24.3.** *Let R be a ring. If $\operatorname{Spec}(R) = U \amalg V$ with both U and V open then $R \cong R_1 \times R_2$ with $U \cong \operatorname{Spec}(R_1)$ and $V \cong \operatorname{Spec}(R_2)$ via the maps in Lemma 10.21.2. Moreover, both R_1 and R_2 are localizations as well as quotients of the ring R.*

Proof. By Lemma 10.21.3 we have $U = D(e)$ and $V = D(1-e)$ for some idempotent e. By Lemma 10.24.2 we see that $R \cong R_e \times R_{1-e}$ (since clearly $R_{e(1-e)} = 0$ so the glueing condition is trivial; of course it is trivial to prove the product decomposition directly in this case). The lemma follows. □

0565 **Lemma 10.24.4.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Let M be an R-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if*

$$M \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} M, \quad m \longmapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$ we have $m = 0$. This clearly implies the displayed map is injective. Conversely, suppose the displayed map is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$. If $e_i = 1$ for all i, then we immediately conclude that $m = 0$ from the injectivity of the displayed map. Next, we prove this holds for any such data by induction on $e = \sum e_i$. The base case is $e = n$, and we have just dealt with this. If some $e_i > 1$, then set $m' = f_i m$. By induction we see that $m' = 0$. Hence we see that $f_i m = 0$, i.e., we may take $e_i = 1$ which decreases e and we win. □

The following lemma is better stated and proved in the more general context of flat descent. However, it makes sense to state it here since it fits well with the above.

00EQ **Lemma 10.24.5.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Suppose we are given the following data:*

(1) *For each i an R_{f_i}-module M_i.*
(2) *For each pair i, j an $R_{f_i f_j}$-module isomorphism $\psi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$.*

which satisfy the "cocycle condition" that all the diagrams

$$\begin{array}{ccc} (M_i)_{f_j f_k} & \xrightarrow{\psi_{ik}} & (M_k)_{f_i f_j} \\ & {\scriptstyle \psi_{ij}} \searrow \quad \nearrow {\scriptstyle \psi_{jk}} & \\ & (M_j)_{f_i f_k} & \end{array}$$

commute (for all triples i, j, k). Given this data define

$$M = \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_i \longrightarrow \bigoplus\nolimits_{1 \leq i,j \leq n} (M_i)_{f_j}\right)$$

where $(m_1, \ldots, m_n)$ maps to the element whose (i, j)th entry is $m_i/1 - \psi_{ji}(m_j/1)$. Then the natural map $M \to M_i$ induces an isomorphism $M_{f_i} \to M_i$. Moreover $\psi_{ij}(m/1) = m/1$ for all $m \in M$ (with obvious notation).

Proof. To show that $M_{f_1} \to M_1$ is an isomorphism, it suffices to show that its localization at every prime $\mathfrak{p}'$ of R_{f_1} is an isomorphism, see Lemma 10.23.1. Write $\mathfrak{p}' = \mathfrak{p}R_{f_1}$ for some prime $\mathfrak{p} \subset R$, $f_1 \notin \mathfrak{p}$, see Lemma 10.17.6. Since localization is exact (Proposition 10.9.12), we see that

$$\begin{aligned} (M_{f_1})_{\mathfrak{p}'} &= M_{\mathfrak{p}} \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_{i,\mathfrak{p}} \longrightarrow \bigoplus\nolimits_{1 \leq i,j \leq n} ((M_i)_{f_j})_{\mathfrak{p}}\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_{i,\mathfrak{p}} \longrightarrow \bigoplus\nolimits_{1 \leq i,j \leq n} (M_{i,\mathfrak{p}})_{f_j}\right) \end{aligned}$$

Here we also used Proposition 10.9.11. Since f_1 is a unit in $R_{\mathfrak{p}}$, this reduces us to the case where $f_1 = 1$ by replacing R by $R_{\mathfrak{p}}$, f_i by the image of f_i in $R_{\mathfrak{p}}$, M by $M_{\mathfrak{p}}$, and f_1 by 1.

Assume $f_1 = 1$. Then $\psi_{1j} : (M_1)_{f_j} \to M_j$ is an isomorphism for $j = 2, \ldots, n$. If we use these isomorphisms to identify $M_j = (M_1)_{f_j}$, then we see that $\psi_{ij} : (M_1)_{f_i f_j} \to (M_1)_{f_i f_j}$ is the canonical identification. Thus the complex

$$0 \to M_1 \to \bigoplus\nolimits_{1 \leq i \leq n} (M_1)_{f_i} \longrightarrow \bigoplus\nolimits_{1 \leq i,j \leq n} (M_1)_{f_i f_j}$$

is exact by Lemma 10.24.1. Thus the first map identifies M_1 with M in this case and everything is clear. □

10.25. Zerodivisors and total rings of fractions

02LV The local ring at a minimal prime has the following properties.

00EU **Lemma 10.25.1.** *Let $\mathfrak{p}$ be a minimal prime of a ring R. Every element of the maximal ideal of $R_{\mathfrak{p}}$ is nilpotent. If R is reduced then $R_{\mathfrak{p}}$ is a field.*

Proof. If some element x of $\mathfrak{p}R_{\mathfrak{p}}$ is not nilpotent, then $D(x) \neq \emptyset$, see Lemma 10.17.2. This contradicts the minimality of $\mathfrak{p}$. If R is reduced, then $\mathfrak{p}R_{\mathfrak{p}} = 0$ and hence it is a field. □

00EW **Lemma 10.25.2.** *Let R be a reduced ring. Then*

(1) *R is a subring of a product of fields,*
(2) *$R \to \prod_{\mathfrak{p}\ minimal} R_{\mathfrak{p}}$ is an embedding into a product of fields,*

(3) $\bigcup_{\mathfrak{p}\ minimal} \mathfrak{p}$ *is the set of zerodivisors of* R.

Proof. By Lemma 10.25.1 each of the rings $R_\mathfrak{p}$ is a field. In particular, the kernel of the ring map $R \to R_\mathfrak{p}$ is $\mathfrak{p}$. By Lemma 10.17.2 we have $\bigcap_\mathfrak{p} \mathfrak{p} = (0)$. Hence (2) and (1) are true. If $xy = 0$ and $y \neq 0$, then $y \notin \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Hence $x \in \mathfrak{p}$. Thus every zerodivisor of R is contained in $\bigcup_{\mathfrak{p}\ \text{minimal}} \mathfrak{p}$. Conversely, suppose that $x \in \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Then x maps to zero in $R_\mathfrak{p}$, hence there exists $y \in R$, $y \notin \mathfrak{p}$ such that $xy = 0$. In other words, x is a zerodivisor. This finishes the proof of (3) and the lemma. □

The total ring of fractions $Q(R)$ of a ring R was introduced in Example 10.9.8.

02LW **Lemma 10.25.3.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzerodivisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.*

Proof. If $x \in S^{-1}R$ is a nonzerodivisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzerodivisor in R. Whence the lemma. □

We can apply glueing results to prove something about total rings of fractions $Q(R)$ which we introduced in Example 10.9.8.

02LX **Lemma 10.25.4.** *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ by Lemma 10.15.2. Hence $\mathrm{Spec}(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $\mathrm{Spec}(R)$, see Lemma 10.17.5). Therefore $\mathrm{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $\mathrm{Spec}(A_i) = \{\mathfrak{q}_i\}$, see Lemma 10.24.3. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. □

10.26. Irreducible components of spectra

00ER We show that irreducible components of the spectrum of a ring correspond to the minimal primes in the ring.

00ES **Lemma 10.26.1.** *Let R be a ring.*

(1) *For a prime $\mathfrak{p} \subset R$ the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$. In a formula $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.*
(2) *The irreducible closed subsets of* $\mathrm{Spec}(R)$ *are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a prime.*
(3) *The irreducible components (see Topology, Definition 5.8.1) of* $\mathrm{Spec}(R)$ *are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a minimal prime.*

Proof. Note that if $\mathfrak{p} \in V(I)$, then $I \subset \mathfrak{p}$. Hence, clearly $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular $V(\mathfrak{p})$ is the closure of a singleton and hence irreducible. The second assertion implies the third. To show the second, let $V(I) \subset \mathrm{Spec}(R)$ with I a radical ideal. If I is not prime, then choose $a, b \in R$, $a, b \notin I$ with $ab \in I$. In this case $V(I, a) \cup V(I, b) = V(I)$, but neither $V(I, b) = V(I)$ nor $V(I, a) = V(I)$, by Lemma 10.17.2. Hence $V(I)$ is not irreducible. □

In other words, this lemma shows that every irreducible closed subset of $\mathrm{Spec}(R)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p}$. Since $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ we see that each irreducible closed subset has a unique generic point, see Topology, Definition 5.8.6. In particular, $\mathrm{Spec}(R)$ is a sober topological space. We record this fact in the following lemma.

090M **Lemma 10.26.2.** *The spectrum of a ring is a spectral space, see Topology, Definition 5.23.1.*

Proof. Formally this follows from Lemma 10.26.1 and Lemma 10.17.9. See also discussion above. □

00ET **Lemma 10.26.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime.*

(1) *the set of irreducible closed subsets of $\mathrm{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with primes $\mathfrak{q} \subset R_\mathfrak{p}$.*
(2) *The set of irreducible components of $\mathrm{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with minimal primes $\mathfrak{q} \subset R_\mathfrak{p}$.*

Proof. Follows from Lemma 10.26.1 and the description of $\mathrm{Spec}(R_\mathfrak{p})$ in Lemma 10.17.5 which shows that $\mathrm{Spec}(R_\mathfrak{p})$ corresponds to primes $\mathfrak{q}$ in R with $\mathfrak{q} \subset \mathfrak{p}$. □

00EV **Lemma 10.26.4.** *Let R be a ring. Let $\mathfrak{p}$ be a minimal prime of R. Let $W \subset \mathrm{Spec}(R)$ be a quasi-compact open not containing the point $\mathfrak{p}$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $D(f) \cap W = \emptyset$.*

Proof. Since W is quasi-compact we may write it as a finite union of standard affine opens $D(g_i)$, $i = 1, \ldots, n$. Since $\mathfrak{p} \notin W$ we have $g_i \in \mathfrak{p}$ for all i. By Lemma 10.25.1 each g_i is nilpotent in $R_\mathfrak{p}$. Hence we can find an $f \in R$, $f \notin \mathfrak{p}$ such that for all i we have $fg_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f)$ works. □

04MG **Lemma 10.26.5.** *Let R be a ring. Let $X = \mathrm{Spec}(R)$ as a topological space. The following are equivalent*

(1) *X is profinite,*
(2) *X is Hausdorff,*
(3) *X is totally disconnected.*
(4) *every quasi-compact open of X is closed,*
(5) *there are no nontrivial inclusions between its prime ideals,*
(6) *every prime ideal is a maximal ideal,*
(7) *every prime ideal is minimal,*
(8) *every standard open $D(f) \subset X$ is closed, and*
(9) *add more here.*

Proof. First proof. It is clear that (5), (6), and (7) are equivalent. It is clear that (4) and (8) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (7) $\Rightarrow$ (4) follows from Lemma 10.26.4. Assume (4) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of R. Choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$ (if needed switch $\mathfrak{p}$ with $\mathfrak{p}'$). Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (4) the open $D(f)$ is also closed. Hence $\mathfrak{p}$ and $\mathfrak{p}'$ are in disjoint open neighbourhoods whose union is X. Thus X is Hausdorff and totally disconnected. Thus (4) $\Rightarrow$ (2) and (3). If (3) holds then there cannot be any specializations between points of $\mathrm{Spec}(R)$ and we see that (5) holds. If X is Hausdorff then every point is closed, so (2) implies (6). Thus (2), (3), (4), (5), (6), (7) and (8) are equivalent. Any profinite space is Hausdorff, so (1) implies (2). If X

satisfies (2) and (3), then X (being quasi-compact by Lemma 10.17.8) is profinite by Topology, Lemma 5.22.2.

Second proof. Besides the equivalence of (4) and (8) this follows from Lemma 10.26.2 and purely topological facts, see Topology, Lemma 5.23.8. □

10.27. Examples of spectra of rings

00EX In this section we put some examples of spectra.

00EY **Example 10.27.1.** In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x]/(x^2-4))$. Let $\mathfrak{p}$ be an arbitrary prime in X. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]/(x^2-4)$ be the natural ring map. Then, $\phi^{-1}(\mathfrak{p})$ is a prime in $\mathbf{Z}$. If $\phi^{-1}(\mathfrak{p}) = (2)$, then since $\mathfrak{p}$ contains 2, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4,2) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4,2)$. Any prime in $(\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ corresponds to a prime in $(\mathbf{Z}/2\mathbf{Z})[x]$ containing (x^2). Such primes will then contain x. Since $(\mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x)$ is a field, (x) is a maximal ideal. Since any prime contains (x) and (x) is maximal, the ring contains only one prime (x). Thus, in this case, $\mathfrak{p} = (2,x)$. Now, if $\phi^{-1}(\mathfrak{p}) = (q)$ for $q > 2$, then since $\mathfrak{p}$ contains q, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4,q) \cong (\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4,q)$. Any prime in $(\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ corresponds to a prime in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x^2-4) = (x-2)(x+2)$. Hence, these primes must contain either $x-2$ or $x+2$. Since $(\mathbf{Z}/q\mathbf{Z})[x]$ is a PID, all nonzero primes are maximal, and so there are precisely 2 primes in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x-2)(x+2)$, namely $(x-2)$ and $(x+2)$. In conclusion, there exist two primes $(q, x-2)$ and $(q, x+2)$ since $2 \neq -2 \in \mathbf{Z}/(q)$. Finally, we treat the case where $\phi^{-1}(\mathfrak{p}) = (0)$. Notice that $\mathfrak{p}$ corresponds to a prime ideal in $\mathbf{Z}[x]$ that contains $(x^2-4) = (x-2)(x+2)$. Hence, $\mathfrak{p}$ contains either $(x-2)$ or $(x+2)$. Hence, $\mathfrak{p}$ corresponds to a prime in $\mathbf{Z}[x]/(x-2)$ or one in $\mathbf{Z}[x]/(x+2)$ that intersects $\mathbf{Z}$ only at 0, by assumption. Since $\mathbf{Z}[x]/(x-2) \cong \mathbf{Z}$ and $\mathbf{Z}[x]/(x+2) \cong \mathbf{Z}$, this means that $\mathfrak{p}$ must correspond to 0 in one of these rings. Thus, $\mathfrak{p} = (x-2)$ or $\mathfrak{p} = (x+2)$ in the original ring.

00EZ **Example 10.27.2.** In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x])$. Fix $\mathfrak{p} \in X$. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]$ and notice that $\phi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(\mathbf{Z})$. If $\phi^{-1}(\mathfrak{p}) = (q)$ for q a prime number $q > 0$, then $\mathfrak{p}$ corresponds to a prime in $(\mathbf{Z}/(q))[x]$, which must be generated by a polynomial that is irreducible in $(\mathbf{Z}/(q))[x]$. If we choose a representative of this polynomial with minimal degree, then it will also be irreducible in $\mathbf{Z}[x]$. Hence, in this case $\mathfrak{p} = (q, f_q)$ where f_q is an irreducible polynomial in $\mathbf{Z}[x]$ that is irreducible when viewed in $(\mathbf{Z}/(q)[x])$. Now, assume that $\phi^{-1}(\mathfrak{p}) = (0)$. In this case, $\mathfrak{p}$ must be generated by nonconstant polynomials which, since $\mathfrak{p}$ is prime, may be assumed to be irreducible in $\mathbf{Z}[x]$. By Gauss' lemma, these polynomials are also irreducible in $\mathbf{Q}[x]$. Since $\mathbf{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducibles f, g generating $\mathfrak{p}$, then $1 = af + bg$ for $a, b \in \mathbf{Q}[x]$. Multiplying through by a common denominator, we see that $m = \bar{a}f + \bar{b}g$ for $\bar{a}, \bar{b} \in \mathbf{Z}[x]$ and nonzero $m \in \mathbf{Z}$. This is a contradiction. Hence, $\mathfrak{p}$ is generated by one irreducible polynomial in $\mathbf{Z}[x]$.

00F0 **Example 10.27.3.** In this example we describe $X = \mathrm{Spec}(k[x,y])$ when k is an arbitrary field. Clearly (0) is prime, and any principal ideal generated by an irreducible polynomial will also be a prime since $k[x,y]$ is a unique factorization domain. Now assume $\mathfrak{p}$ is an element of X that is not principal. Since $k[x,y]$

is a Noetherian UFD, the prime ideal $\mathfrak{p}$ can be generated by a finite number of irreducible polynomials $(f_1, \ldots, f_n)$. Now, I claim that if f, g are irreducible polynomials in $k[x, y]$ that are not associates, then $(f, g) \cap k[x] \neq 0$. To do this, it is enough to show that f and g are relatively prime when viewed in $k(x)[y]$. In this case, $k(x)[y]$ is a Euclidean domain, so by applying the Euclidean algorithm and clearing denominators, we obtain $p = af + bg$ for $p, a, b \in k[x]$. Thus, assume this is not the case, that is, that some nonunit $h \in k(x)[y]$ divides both f and g. Then, by Gauss's lemma, for some $a, b \in k(x)$ we have $ah|f$ and $bh|g$ for $ah, bh \in k[x]$. By irreducibility, $ah = f$ and $bh = g$ (since $h \notin k(x)$). So, back in $k(x)[y]$, f, g are associates, as $\frac{a}{b}g = f$. Since $k(x)$ is the fraction field of $k[x]$, we can write $g = \frac{r}{s}f$ for elements $r, s \in k[x]$ sharing no common factors. This implies that $sg = rf$ in $k[x, y]$ and so s must divide f since $k[x, y]$ is a UFD. Hence, $s = 1$ or $s = f$. If $s = f$, then $r = g$, implying $f, g \in k[x]$ and thus must be units in $k(x)$ and relatively prime in $k(x)[y]$, contradicting our hypothesis. If $s = 1$, then $g = rf$, another contradiction. Thus, we must have f, g relatively prime in $k(x)[y]$, a Euclidean domain. Thus, we have reduced to the case $\mathfrak{p}$ contains some irreducible polynomial $p \in k[x] \subset k[x, y]$. By the above, $\mathfrak{p}$ corresponds to a prime in the ring $k[x, y]/(p) = k(\alpha)[y]$, where α is an element algebraic over k with minimum polynomial p. This is a PID, and so any prime ideal corresponds to (0) or an irreducible polynomial in $k(\alpha)[y]$. Thus, $\mathfrak{p}$ is of the form (p) or (p, f) where f is a polynomial in $k[x, y]$ that is irreducible in the quotient $k[x, y]/(p)$.

00F1 **Example 10.27.4.** Consider the ring

$$R = \{f \in \mathbf{Q}[z] \text{ with } f(0) = f(1)\}.$$

Consider the map

$$\varphi : \mathbf{Q}[A, B] \to R$$

defined by $\varphi(A) = z^2 - z$ and $\varphi(B) = z^3 - z^2$. It is easily checked that $(A^3 - B^2 + AB) \subset \mathrm{Ker}(\varphi)$ and that $A^3 - B^2 + AB$ is irreducible. Assume that φ is surjective; then since R is an integral domain (it is a subring of an integral domain), $\mathrm{Ker}(\varphi)$ must be a prime ideal of $\mathbf{Q}[A, B]$. The prime ideals which contain $(A^3 - B^2 + AB)$ are $(A^3 - B^2 + AB)$ itself and any maximal ideal (f, g) with $f, g \in \mathbf{Q}[A, B]$ such that f is irreducible mod g. But R is not a field, so the kernel must be $(A^3 - B^2 + AB)$; hence φ gives an isomorphism $R \to \mathbf{Q}[A, B]/(A^3 - B^2 + AB)$.

To see that φ is surjective, we must express any $f \in R$ as a $\mathbf{Q}$-coefficient polynomial in $A(z) = z^2 - z$ and $B(z) = z^3 - z^2$. Note the relation $zA(z) = B(z)$. Let $a = f(0) = f(1)$. Then $z(z - 1)$ must divide $f(z) - a$, so we can write $f(z) = z(z - 1)g(z) + a = A(z)g(z) + a$. If $\deg(g) < 2$, then $h(z) = c_1 z + c_0$ and $f(z) = A(z)(c_1 z + c_0) + a = c_1 B(z) + c_0 A(z) + a$, so we are done. If $\deg(g) \geq 2$, then by the polynomial division algorithm, we can write $g(z) = A(z)h(z) + b_1 z + b_0$ ($\deg(h) \leq \deg(g) - 2$), so $f(z) = A(z)^2 h(z) + b_1 B(z) + b_0 A(z)$. Applying division to $h(z)$ and iterating, we obtain an expression for $f(z)$ as a polynomial in $A(z)$ and $B(z)$; hence φ is surjective.

Now let $a \in \mathbf{Q}$, $a \neq 0, \frac{1}{2}, 1$ and consider

$$R_a = \{f \in \mathbf{Q}[z, \frac{1}{z - a}] \text{ with } f(0) = f(1)\}.$$

This is a finitely generated $\mathbf{Q}$-algebra as well: it is easy to check that the functions $z^2 - z$, $z^3 - z$, and $\frac{a^2 - a}{z - a} + z$ generate R_a as an $\mathbf{Q}$-algebra. We have the following

inclusions:

$$R \subset R_a \subset \mathbf{Q}[z, \frac{1}{z-a}], \quad R \subset \mathbf{Q}[z] \subset \mathbf{Q}[z, \frac{1}{z-a}].$$

Recall (Lemma 10.17.5) that for a ring T and a multiplicative subset $S \subset T$, the ring map $T \to S^{-1}T$ induces a map on spectra $\mathrm{Spec}(S^{-1}T) \to \mathrm{Spec}(T)$ which is a homeomorphism onto the subset

$$\{\mathfrak{p} \in \mathrm{Spec}(T) \mid S \cap \mathfrak{p} = \emptyset\} \subset \mathrm{Spec}(T).$$

When $S = \{1, f, f^2, \ldots\}$ for some $f \in T$, this is the open set $D(f) \subset T$. We now verify a corresponding property for the ring map $R \to R_a$: we will show that the map $\theta : \mathrm{Spec}(R_a) \to \mathrm{Spec}(R)$ induced by inclusion $R \subset R_a$ is a homeomorphism onto an open subset of $\mathrm{Spec}(R)$ by verifying that θ is an injective local homeomorphism. We do so with respect to an open cover of $\mathrm{Spec}(R_a)$ by two distinguished opens, as we now describe. For any $r \in \mathbf{Q}$, let $\mathrm{ev}_r : R \to \mathbf{Q}$ be the homomorphism given by evaluation at r. Note that for $r = 0$ and $r = 1-a$, this can be extended to a homomorphism $\mathrm{ev}'_r : R_a \to \mathbf{Q}$ (the latter because $\frac{1}{z-a}$ is well-defined at $z = 1-a$, since $a \neq \frac{1}{2}$). However, ev_a does not extend to R_a. Write $\mathfrak{m}_r = \mathrm{Ker}(\mathrm{ev}_r)$. We have

$$\mathfrak{m}_0 = (z^2 - z, z^3 - z),$$
$$\mathfrak{m}_a = ((z-1+a)(z-a), (z^2-1+a)(z-a)), \text{ and}$$
$$\mathfrak{m}_{1-a} = ((z-1+a)(z-a), (z-1+a)(z^2-a)).$$

To verify this, note that the right-hand sides are clearly contained in the left-hand sides. Then check that the right-hand sides are maximal ideals by writing the generators in terms of A and B, and viewing R as $\mathbf{Q}[A, B]/(A^3 - B^2 + AB)$. Note that $\mathfrak{m}_a$ is not in the image of θ: we have

$$(z^2 - z)^2(z - a)\left(\frac{a^2 - a}{z - a} + z\right) = (z^2 - z)^2(a^2 - a) + (z^2 - z)^2(z - a)z$$

The left hand side is in $\mathfrak{m}_a R_a$ because $(z^2 - z)(z - a)$ is in $\mathfrak{m}_a$ and because $(z^2 - z)(\frac{a^2-a}{z-a} + z)$ is in R_a. Similarly the element $(z^2 - z)^2(z - a)z$ is in $\mathfrak{m}_a R_a$ because $(z^2 - z)$ is in R_a and $(z^2 - z)(z - a)$ is in $\mathfrak{m}_a$. As $a \notin \{0, 1\}$ we conclude that $(z^2 - z)^2 \in \mathfrak{m}_a R_a$. Hence no ideal I of R_a can satisfy $I \cap R = \mathfrak{m}_a$, as such an I would have to contain $(z^2 - z)^2$, which is in R but not in $\mathfrak{m}_a$. The distinguished open set $D((z - 1 + a)(z - a)) \subset \mathrm{Spec}(R)$ is equal to the complement of the closed set $\{\mathfrak{m}_a, \mathfrak{m}_{1-a}\}$. Then check that $R_{(z-1+a)(z-a)} = (R_a)_{(z-1+a)(z-a)}$; calling this localized ring R', then, it follows that the map $R \to R'$ factors as $R \to R_a \to R'$. By Lemma 10.17.5, then, these maps express $\mathrm{Spec}(R') \subset \mathrm{Spec}(R_a)$ and $\mathrm{Spec}(R') \subset \mathrm{Spec}(R)$ as open subsets; hence $\theta : \mathrm{Spec}(R_a) \to \mathrm{Spec}(R)$, when restricted to $D((z - 1 + a)(z - a))$, is a homeomorphism onto an open subset. Similarly, θ restricted to $D((z^2+z+2a-2)(z-a)) \subset \mathrm{Spec}(R_a)$ is a homeomorphism onto the open subset $D((z^2+z+2a-2)(z-a)) \subset \mathrm{Spec}(R)$. Depending on whether $z^2 + z + 2a - 2$ is irreducible or not over $\mathbf{Q}$, this former distinguished open set has complement equal to one or two closed points along with the closed point $\mathfrak{m}_a$. Furthermore, the ideal in R_a generated by the elements $(z^2+z+2a-a)(z-a)$ and $(z-1+a)(z-a)$ is all of R_a, so these two distinguished open sets cover $\mathrm{Spec}(R_a)$. Hence in order to show that θ is a homeomorphism onto $\mathrm{Spec}(R) - \{\mathfrak{m}_a\}$, it suffices to show that these one or two points can never equal $\mathfrak{m}_{1-a}$. And this is indeed the case, since $1 - a$ is a root of $z^2 + z + 2a - 2$ if and only if $a = 0$ or $a = 1$, both of which do not occur.

Despite this homeomorphism which mimics the behavior of a localization at an element of R, while $\mathbf{Q}[z, \frac{1}{z-a}]$ is the localization of $\mathbf{Q}[z]$ at the maximal ideal $(z-a)$, the ring R_a is *not* a localization of R: Any localization $S^{-1}R$ results in more units than the original ring R. The units of R are $\mathbf{Q}^\times$, the units of $\mathbf{Q}$. In fact, it is easy to see that the units of R_a are $\mathbf{Q}^*$. Namely, the units of $\mathbf{Q}[z, \frac{1}{z-a}]$ are $c(z-a)^n$ for $c \in \mathbf{Q}^*$ and $n \in \mathbf{Z}$ and it is clear that these are in R_a only if $n = 0$. Hence R_a has no more units than R does, and thus cannot be a localization of R.

We used the fact that $a \neq 0, 1$ to ensure that $\frac{1}{z-a}$ makes sense at $z = 0, 1$. We used the fact that $a \neq 1/2$ in a few places: (1) In order to be able to talk about the kernel of ev_{1-a} on R_a, which ensures that $\mathfrak{m}_{1-a}$ is a point of R_a (i.e., that R_a is missing just one point of R). (2) At the end in order to conclude that $(z-a)^{k+\ell}$ can only be in R for $k = \ell = 0$; indeed, if $a = 1/2$, then this is in R as long as $k + \ell$ is even. Hence there would indeed be more units in R_a than in R, and R_a could possibly be a localization of R.

10.28. A meta-observation about prime ideals

05K7 This section is taken from the CRing project. Let R be a ring and let $S \subset R$ be a multiplicative subset. A consequence of Lemma 10.17.5 is that an ideal $I \subset R$ maximal with respect to the property of not intersecting S is prime. The reason is that $I = R \cap \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of the ring $S^{-1}R$. It turns out that for many properties of ideals, the maximal ones are prime. A general method of seeing this was developed in [**LR08**]. In this section, we digress to explain this phenomenon.

Let R be a ring. If I is an ideal of R and $a \in R$, we define

$$(I : a) = \{x \in R \mid xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R \mid xJ \subset I\}.$$

05K8 **Lemma 10.28.1.** *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. □

Let $\mathcal{F}$ be a collection of ideals of R. We are interested in conditions that will guarantee that the maximal elements in the complement of $\mathcal{F}$ are prime.

05K9 **Definition 10.28.2.** Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an *Oka family* if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.

Let us give some examples of Oka families. The first example is the basic example discussed in the introduction to this section.

05KA **Example 10.28.3.** Let R be a ring and let S be a multiplicative subset of R. We claim that $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family. Namely, suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

05KB **Example 10.28.4.** Let R be a ring, $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \ldots, a_n$ and (I, a) is generated by $a, b_1, \ldots, b_m$ with $b_1, \ldots, b_m \in I$, then I is generated by $aa_1, \ldots, aa_n, b_1, \ldots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \ldots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

05KC **Example 10.28.5.** Let us show that the family of principal ideals of a ring R is an Oka family. Indeed, suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 10.28.1 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

05KD **Example 10.28.6.** Let R be a ring. Let κ be an infinite cardinal. The family of ideals which can be generated by at most κ elements is an Oka family. The argument is analogous to the argument in Example 10.28.4 and is omitted.

0G1N **Example 10.28.7.** Let A be a ring, $I \subset A$ an ideal, and $a \in A$ an element. There is a short exact sequence $0 \to A/(I : a) \to A/I \to A/(I, a) \to 0$ where the first arrow is given by multiplication by a. Thus if P is a property of A-modules that is stable under extensions and holds for 0, then the family of ideals I such that A/I has P is an Oka family.

05KE **Proposition 10.28.8.** *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. It follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. □

At this point we are able to turn most of the examples above into a lemma about prime ideals in a ring.

05KF **Lemma 10.28.9.** *Let R be a ring. Let S be a multiplicative subset of R. An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.*

Proof. This is the example discussed in the introduction to this section. For an alternative proof, combine Example 10.28.3 with Proposition 10.28.8. □

05KG **Lemma 10.28.10.** *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*
(2) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated*[2].

Proof. The first assertion is an immediate consequence of Example 10.28.4 and Proposition 10.28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \ldots, a_n$, then all the generators would belong to some I_α and would

[2]Later we will say that R is Noetherian.

consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By the first part this ideal is prime. □

05KH **Lemma 10.28.11.** *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being principal is prime.*
(2) *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. The first part follows from Example 10.28.5 and Proposition 10.28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by the first part. □

05KI **Lemma 10.28.12.** *Let R be a ring.*

(1) *An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.*
(2) *If R is nonzero and every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.*

Proof. Consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime, see Lemma 10.28.9. Thus, if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. □

05KJ **Remark 10.28.13.** Let R be a ring. Let κ be an infinite cardinal. By applying Example 10.28.6 and Proposition 10.28.8 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by $\aleph_0$ elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than $\aleph_0$ and let

$$R = k[\{x_n\}_{n\geq 1}, \{z_{t,n}\}_{t\in T, n\geq 0}]/(x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

0G2Z **Example 10.28.14.** Let R be a ring and $X = \mathrm{Spec}(R)$. Since closed subsets of X correspond to radical ideas of R (Lemma 10.17.2) we see that X is a Noetherian topological space if and only if we have ACC for radical ideals. This holds if and only if every radical ideal is the radical of a finitely generated ideal (details omitted). Let

Comment by Lukas Heger of November 12, 2020.

$$\mathcal{F} = \{I \subset R \mid \sqrt{I} = \sqrt{(f_1, \ldots, f_n)} \text{ for some } n \text{ and } f_1, \ldots, f_n \in R\}.$$

The reader can show that $\mathcal{F}$ is an Oka family by using the identity

$$\sqrt{I} = \sqrt{(I, a)(I : a)}$$

which holds for any ideal $I \subset R$ and any element $a \in R$. On the other hand, if we have a totally ordered chain of ideals $\{I_\alpha\}$ none of which are in $\mathcal{F}$, then the union $I = \bigcup I_\alpha$ cannot be in $\mathcal{F}$ either. Otherwise $\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}$, then $f_i^e \in I$ for some e, then $f_i^e \in I_\alpha$ for some α independent of i, then $\sqrt{I_\alpha} = \sqrt{(f_1, \ldots, f_n)}$, contradiction. Thus if the set of ideals not in $\mathcal{F}$ is nonempty, then it has maximal elements and exactly as in Lemma 10.28.10 we conclude that X is a Noetherian

topological space if and only if every prime ideal of R is equal to $\sqrt{(f_1, \ldots, f_n)}$ for some $f_1, \ldots, f_n \in R$. If we ever need this result we will carefully state and prove this result here.

10.29. Images of ring maps of finite presentation

00F5 In this section we prove some results on the topology of maps $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induced by ring maps $R \to S$, mainly Chevalley's Theorem. In order to do this we will use the notions of constructible sets, quasi-compact sets, retrocompact sets, and so on which are defined in Topology, Section 5.15.

00F6 **Lemma 10.29.1.** *Let $U \subset \mathrm{Spec}(R)$ be open. The following are equivalent:*

1. *U is retrocompact in $\mathrm{Spec}(R)$,*
2. *U is quasi-compact,*
3. *U is a finite union of standard opens, and*
4. *there exists a finitely generated ideal $I \subset R$ such that $X \setminus V(I) = U$.*

Proof. We have (1) $\Rightarrow$ (2) because $\mathrm{Spec}(R)$ is quasi-compact, see Lemma 10.17.8. We have (2) $\Rightarrow$ (3) because standard opens form a basis for the topology. Proof of (3) $\Rightarrow$ (1). Let $U = \bigcup_{i=1\ldots n} D(f_i)$. To show that U is retrocompact in $\mathrm{Spec}(R)$ it suffices to show that $U \cap V$ is quasi-compact for any quasi-compact open V of $\mathrm{Spec}(R)$. Write $V = \bigcup_{j=1\ldots m} D(g_j)$ which is possible by (2) $\Rightarrow$ (3). Each standard open is homeomorphic to the spectrum of a ring and hence quasi-compact, see Lemmas 10.17.6 and 10.17.8. Thus $U \cap V = (\bigcup_{i=1\ldots n} D(f_i)) \cap (\bigcup_{j=1\ldots m} D(g_j)) = \bigcup_{i,j} D(f_i g_j)$ is a finite union of quasi-compact opens hence quasi-compact. To finish the proof note that (4) is equivalent to (3) by Lemma 10.17.2. □

00F7 **Lemma 10.29.2.** *Let $\varphi : R \to S$ be a ring map. The induced continuous map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is quasi-compact. For any constructible set $E \subset \mathrm{Spec}(R)$ the inverse image $f^{-1}(E)$ is constructible in $\mathrm{Spec}(S)$.*

Proof. We first show that the inverse image of any quasi-compact open $U \subset \mathrm{Spec}(R)$ is quasi-compact. By Lemma 10.29.1 we may write U as a finite open of standard opens. Thus by Lemma 10.17.4 we see that $f^{-1}(U)$ is a finite union of standard opens. Hence $f^{-1}(U)$ is quasi-compact by Lemma 10.29.1 again. The second assertion now follows from Topology, Lemma 5.15.3. □

0G1P **Lemma 10.29.3.** *Let R be a ring. A subset of $\mathrm{Spec}(R)$ is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ for $f, g_1, \ldots, g_m \in R$.*

Proof. By Lemma 10.29.1 the subset $D(f)$ and the complement of $V(g_1, \ldots, g_m)$ are retro-compact open. Hence $D(f) \cap V(g_1, \ldots, g_m)$ is a constructible subset and so is any finite union of such. Conversely, let $T \subset \mathrm{Spec}(R)$ be constructible. By Topology, Definition 5.15.1, we may assume that $T = U \cap V^c$, where $U, V \subset \mathrm{Spec}(R)$ are retrocompact open. By Lemma 10.29.1 we may write $U = \bigcup_{i=1,\ldots,n} D(f_i)$ and $V = \bigcup_{j=1,\ldots,m} D(g_j)$. Then $T = \bigcup_{i=1,\ldots,n} \big(D(f_i) \cap V(g_1, \ldots, g_m)\big)$. □

00F8 **Lemma 10.29.4.** *Let R be a ring and let $T \subset \mathrm{Spec}(R)$ be constructible. Then there exists a ring map $R \to S$ of finite presentation such that T is the image of $\mathrm{Spec}(S)$ in $\mathrm{Spec}(R)$.*

Proof. The spectrum of a finite product of rings is the disjoint union of the spectra, see Lemma 10.21.2. Hence if $T = T_1 \cup T_2$ and the result holds for T_1 and T_2, then the result holds for T. By Lemma 10.29.3 we may assume that $T = D(f) \cap V(g_1, \ldots, g_m)$. In this case T is the image of the map $\mathrm{Spec}((R/(g_1, \ldots, g_m))_f) \to \mathrm{Spec}(R)$, see Lemmas 10.17.6 and 10.17.7. □

00F9 **Lemma 10.29.5.** *Let R be a ring. Let f be an element of R. Let $S = R_f$. Then the image of a constructible subset of $\mathrm{Spec}(S)$ is constructible in $\mathrm{Spec}(R)$.*

Proof. We repeatedly use Lemma 10.29.1 without mention. Let U, V be quasi-compact open in $\mathrm{Spec}(S)$. We will show that the image of $U \cap V^c$ is constructible. Under the identification $\mathrm{Spec}(S) = D(f)$ of Lemma 10.17.6 the sets U, V correspond to quasi-compact opens U', V' of $\mathrm{Spec}(R)$. Hence it suffices to show that $U' \cap (V')^c$ is constructible in $\mathrm{Spec}(R)$ which is clear. □

00FA **Lemma 10.29.6.** *Let R be a ring. Let I be a finitely generated ideal of R. Let $S = R/I$. Then the image of a constructible subset of $\mathrm{Spec}(S)$ is constructible in $\mathrm{Spec}(R)$.*

Proof. If $I = (f_1, \ldots, f_m)$, then we see that $V(I)$ is the complement of $\bigcup D(f_i)$, see Lemma 10.17.2. Hence it is constructible, by Lemma 10.29.1. Denote the map $R \to S$ by $f \mapsto \overline{f}$. We have to show that if $\overline{U}, \overline{V}$ are retrocompact opens of $\mathrm{Spec}(S)$, then the image of $\overline{U} \cap \overline{V}^c$ in $\mathrm{Spec}(R)$ is constructible. By Lemma 10.29.1 we may write $\overline{U} = \bigcup D(\overline{g_i})$. Setting $U = \bigcup D(g_i)$ we see $\overline{U}$ has image $U \cap V(I)$ which is constructible in $\mathrm{Spec}(R)$. Similarly the image of $\overline{V}$ equals $V \cap V(I)$ for some retrocompact open V of $\mathrm{Spec}(R)$. Hence the image of $\overline{U} \cap \overline{V}^c$ equals $U \cap V(I) \cap V^c$ as desired. □

00FB **Lemma 10.29.7.** *Let R be a ring. The map $\mathrm{Spec}(R[x]) \to \mathrm{Spec}(R)$ is open, and the image of any standard open is a quasi-compact open.*

Proof. It suffices to show that the image of a standard open $D(f)$, $f \in R[x]$ is quasi-compact open. The image of $D(f)$ is the image of $\mathrm{Spec}(R[x]_f) \to \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\overline{f}$ be the image of f in $\kappa(\mathfrak{p})[x]$. Recall, see Lemma 10.18.6, that $\mathfrak{p}$ is in the image if and only if $R[x]_f \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]_{\overline{f}}$ is not the zero ring. This is exactly the condition that f does not map to zero in $\kappa(\mathfrak{p})[x]$, in other words, that some coefficient of f is not in $\mathfrak{p}$. Hence we see: if $f = a_d x^d + \ldots + a_0$, then the image of $D(f)$ is $D(a_d) \cup \ldots \cup D(a_0)$. □

We prove a property of characteristic polynomials which will be used below.

00FC **Lemma 10.29.8.** *Let $R \to A$ be a ring homomorphism. Assume $A \cong R^{\oplus n}$ as an R-module. Let $f \in A$. The multiplication map $m_f : A \to A$ is R-linear and hence has a characteristic polynomial $P(T) = T^n + r_{n-1}T^{n-1} + \ldots + r_0 \in R[T]$. For any prime $\mathfrak{p} \in \mathrm{Spec}(R)$, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.*

Proof. This follows quite easily once we prove that the characteristic polynomial $\bar{P}(T) \in \kappa(\mathfrak{p})[T]$ of the multiplication map $m_{\bar{f}} : A \otimes_R \kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$ which multiplies elements of $A \otimes_R \kappa(\mathfrak{p})$ by $\bar{f}$, the image of f viewed in $\kappa(\mathfrak{p})$, is just the image of $P(T)$ in $\kappa(\mathfrak{p})[T]$. Let (a_{ij}) be the matrix of the map m_f with entries in R, using a basis $e_1, \ldots, e_n$ of A as an R-module. Then, $A \otimes_R \kappa(\mathfrak{p}) \cong (R \otimes_R \kappa(\mathfrak{p}))^{\oplus n} = \kappa(\mathfrak{p})^n$, which is an n-dimensional vector space over $\kappa(\mathfrak{p})$ with basis $e_1 \otimes 1, \ldots, e_n \otimes 1$. The

image $\bar{f} = f \otimes 1$, and so the multiplication map $m_{\bar{f}}$ has matrix $(a_{ij} \otimes 1)$. Thus, the characteristic polynomial is precisely the image of $P(T)$.

From linear algebra, we know that a linear transformation acts nilpotently on an n-dimensional vector space if and only if the characteristic polynomial is T^n (since the characteristic polynomial divides some power of the minimal polynomial). Hence, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\bar{P}(T) = T^n$. This occurs if and only if $r_i \in \mathfrak{p}$ for all $0 \leq i \leq n-1$, that is when $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$. □

00FD **Lemma 10.29.9.** *Let R be a ring. Let $f, g \in R[x]$ be polynomials. Assume the leading coefficient of g is a unit of R. There exists elements $r_i \in R$, $i = 1 \ldots, n$ such that the image of $D(f) \cap V(g)$ in $\mathrm{Spec}(R)$ is $\bigcup_{i=1,\ldots,n} D(r_i)$.*

Proof. Write $g = ux^d + a_{d-1}x^{d-1} + \ldots + a_0$, where d is the degree of g, and hence $u \in R^*$. Consider the ring $A = R[x]/(g)$. It is, as an R-module, finite free with basis the images of $1, x, \ldots, x^{d-1}$. Consider multiplication by (the image of) f on A. This is an R-module map. Hence we can let $P(T) \in R[T]$ be the characteristic polynomial of this map. Write $P(T) = T^d + r_{d-1}T^{d-1} + \ldots + r_0$. We claim that $r_0, \ldots, r_{d-1}$ have the desired property. We will use below the property of characteristic polynomials that

$$\mathfrak{p} \in V(r_0, \ldots, r_{d-1}) \Leftrightarrow \text{multiplication by } f \text{ is nilpotent on } A \otimes_R \kappa(\mathfrak{p}).$$

This was proved in Lemma 10.29.8.

Suppose $\mathfrak{q} \in D(f) \cap V(g)$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then there is a nonzero map $A \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ which is compatible with multiplication by f. And f acts as a unit on $\kappa(\mathfrak{q})$. Thus we conclude $\mathfrak{p} \notin V(r_0, \ldots, r_{d-1})$.

On the other hand, suppose that $r_i \notin \mathfrak{p}$ for some prime $\mathfrak{p}$ of R and some $0 \leq i \leq d-1$. Then multiplication by f is not nilpotent on the algebra $A \otimes_R \kappa(\mathfrak{p})$. Hence there exists a prime ideal $\overline{\mathfrak{q}} \subset A \otimes_R \kappa(\mathfrak{p})$ not containing the image of f. The inverse image of $\overline{\mathfrak{q}}$ in $R[x]$ is an element of $D(f) \cap V(g)$ mapping to $\mathfrak{p}$. □

00FE **Theorem 10.29.10** (Chevalley's Theorem)**.** *Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of $\mathrm{Spec}(S)$ in $\mathrm{Spec}(R)$ is constructible.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We may factor $R \to S$ as $R \to R[x_1] \to R[x_1, x_2] \to \ldots \to R[x_1, \ldots, x_{n-1}] \to S$. Hence we may assume that $S = R[x]/(f_1, \ldots, f_m)$. In this case we factor the map as $R \to R[x] \to S$, and by Lemma 10.29.6 we reduce to the case $S = R[x]$. By Lemma 10.29.1 suffices to show that if $T = (\bigcup_{i=1\ldots n} D(f_i)) \cap V(g_1, \ldots, g_m)$ for $f_i, g_j \in R[x]$ then the image in $\mathrm{Spec}(R)$ is constructible. Since finite unions of constructible sets are constructible, it suffices to deal with the case $n = 1$, i.e., when $T = D(f) \cap V(g_1, \ldots, g_m)$.

Note that if $c \in R$, then we have

$$\mathrm{Spec}(R) = V(c) \amalg D(c) = \mathrm{Spec}(R/(c)) \amalg \mathrm{Spec}(R_c),$$

and correspondingly $\mathrm{Spec}(R[x]) = V(c) \amalg D(c) = \mathrm{Spec}(R/(c)[x]) \amalg \mathrm{Spec}(R_c[x])$. The intersection of $T = D(f) \cap V(g_1, \ldots, g_m)$ with each part still has the same shape, with f, g_i replaced by their images in $R/(c)[x]$, respectively $R_c[x]$. Note that the image of T in $\mathrm{Spec}(R)$ is the union of the image of $T \cap V(c)$ and $T \cap D(c)$.

Using Lemmas 10.29.5 and 10.29.6 it suffices to prove the images of both parts are constructible in $\mathrm{Spec}(R/(c))$, respectively $\mathrm{Spec}(R_c)$.

Let us assume we have $T = D(f) \cap V(g_1, \ldots, g_m)$ as above, with $\deg(g_1) \leq \deg(g_2) \leq \ldots \leq \deg(g_m)$. We are going to use induction on m, and on the degrees of the g_i. Let $d_1 = \deg(g_1)$, i.e., $g_1 = cx^{d_1} + l.o.t$ with $c \in R$ not zero. Cutting R up into the pieces $R/(c)$ and R_c we either lower the degree of g_1 (and this is covered by induction) or we reduce to the case where c is invertible. If c is invertible, and $m > 1$, then write $g_2 = c'x^{d_2} + l.o.t$. In this case consider $g_2' = g_2 - (c'/c)x^{d_2-d_1}g_1$. Since the ideals $(g_1, g_2, \ldots, g_m)$ and $(g_1, g_2', g_3, \ldots, g_m)$ are equal we see that $T = D(f) \cap V(g_1, g_2', g_3 \ldots, g_m)$. But here the degree of g_2' is strictly less than the degree of g_2 and hence this case is covered by induction.

The bases case for the induction above are the cases (a) $T = D(f) \cap V(g)$ where the leading coefficient of g is invertible, and (b) $T = D(f)$. These two cases are dealt with in Lemmas 10.29.9 and 10.29.7. □

10.30. More on images

00FF In this section we collect a few additional lemmas concerning the image on Spec for ring maps. See also Section 10.41 for example.

00FG **Lemma 10.30.1.** *Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.*

Proof. By induction on the number of generators of S over R. During the proof we may replace R by R_f and S by S_f for some nonzero $f \in R$.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $h \in \mathfrak{q}$ be a nonzero element with minimal degree in x. Write $h = fx^d + a_{d-1}x^{d-1} + \ldots + a_0$ with $a_i \in R$ and $f \neq 0$. After inverting f in R and S we may assume that h is monic. We obtain a surjective R-algebra map $R[x]/(h) \to S$. We have $R[x]/(h) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ as an R-module and by minimality of d we see that $R[x]/(h)$ maps injectively into S. Thus $R[x]/(h) \cong S$ is finitely presented over R.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the R-subalgebra generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. □

00FH **Lemma 10.30.2.** *Let $R \to S$ be a finite type ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.*

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak{p} \subset R$ be the prime corresponding to ξ and let $\mathfrak{q} \subset S$ be the prime corresponding

to η. Consider the diagram

$$\begin{array}{ccccccc} \eta & \longrightarrow & E \cap Y' & \longrightarrow & Y' = \mathrm{Spec}(S/\mathfrak{q}) & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \xi & \longrightarrow & f(E) \cap X' & \longrightarrow & X' = \mathrm{Spec}(R/\mathfrak{p}) & \longrightarrow & X \end{array}$$

By Lemma 10.29.2 the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 10.30.1 combined with Chevalley's theorem (Theorem 10.29.10) we see that there exist dense opens $U \subset X$, $V \subset Y$ such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V, see Topology, Lemma 5.15.4. Hence $f(E \cap V)$ is constructible in U and contains ξ. By Topology, Lemma 5.15.15 we see that $f(E \cap V)$ contains a dense open $U' \subset U$. □

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

00FI **Lemma 10.30.3.** *Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*
2. *For any ideal $I \subset R$ the inverse image of $\sqrt{IS}$ in R is equal to $\sqrt{I}$.*
3. *For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.*
4. *For every prime $\mathfrak{p}$ of R the inverse image of $\mathfrak{p}S$ in R is $\mathfrak{p}$.*

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) as well.

Proof. If $J \subset S$ is an ideal, then $\sqrt{\varphi^{-1}(J)} = \varphi^{-1}(\sqrt{J})$. This shows that (2) and (3) are equivalent. The implication (3) $\Rightarrow$ (4) is immediate. If $I \subset R$ is a radical ideal, then Lemma 10.17.2 guarantees that $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$. Hence (4) $\Rightarrow$ (2). By Lemma 10.18.6 we have $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$ if and only if $\mathfrak{p}$ is in the image. Hence (1) $\Leftrightarrow$ (4). Thus (1), (2), (3), and (4) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over the prime $\mathfrak{p}$ of R. To see that $\mathfrak{p}'$ is in the image of $\mathrm{Spec}(R' \otimes_R S) \to \mathrm{Spec}(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}')$ is not zero, see Lemma 10.18.6. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

which is not zero as $S \otimes_R \kappa(\mathfrak{p})$ is not zero by assumption and $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$ is an extension of fields. □

00FJ **Lemma 10.30.4.** *Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The ring map $R \to S$ is injective.*
2. *The image $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains a dense set of points.*
3. *There exists a prime ideal $\mathfrak{q} \subset S$ whose inverse image in R is (0).*

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0) by Lemma 10.18.6.

Note that (0) is dense in $\mathrm{Spec}(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak{q}$ of S such that $\varphi(f) \notin \mathfrak{q}$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. □

00FK **Lemma 10.30.5.** *Let $R \subset S$ be an injective ring map. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ hits all the minimal primes.*

Proof. Let $\mathfrak{p} \subset R$ be a minimal prime. In this case $R_\mathfrak{p}$ has a unique prime ideal. Hence it suffices to show that $S_\mathfrak{p}$ is not zero. And this follows from the fact that localization is exact, see Proposition 10.9.12. □

00FL **Lemma 10.30.6.** *Let $R \to S$ be a ring map. The following are equivalent:*

1. *The kernel of $R \to S$ consists of nilpotent elements.*
2. *The minimal primes of R are in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*
3. *The image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is dense in $\mathrm{Spec}(R)$.*

Proof. Let $I = \mathrm{Ker}(R \to S)$. Note that $\sqrt{(0)} = \bigcap_{\mathfrak{q} \subset S} \mathfrak{q}$, see Lemma 10.17.2. Hence $\sqrt{I} = \bigcap_{\mathfrak{q} \subset S} R \cap \mathfrak{q}$. Thus $V(I) = V(\sqrt{I})$ is the closure of the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. This shows that (1) is equivalent to (3). It is clear that (2) implies (3). Finally, assume (1). We may replace R by R/I and S by S/IS without affecting the topology of the spectra and the map. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 10.30.5. □

0CAN **Lemma 10.30.7.** *Let $R \to S$ be a ring map. If a minimal prime $\mathfrak{p} \subset R$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, then it is the image of a minimal prime.*

Proof. Say $\mathfrak{p} = \mathfrak{q} \cap R$. Then choose a minimal prime $\mathfrak{r} \subset S$ with $\mathfrak{r} \subset \mathfrak{q}$, see Lemma 10.17.2. By minimality of $\mathfrak{p}$ we see that $\mathfrak{p} = \mathfrak{r} \cap R$. □

0H7L **Lemma 10.30.8.** *Let $A \subset B$ be an inclusion of domains induces an algebraic extension of fraction fields. If $J \subset B$ is a nonzero ideal, then $A \cap J$ is nonzero too. Thus the image of a proper closed subset of $\mathrm{Spec}(B)$ is not dense in $\mathrm{Spec}(A)$.*

Proof. Let $x \in J$ be a nonzero element. Since x is algebraic over the fraction field of A, there exists a $d \geq 1$ and $a_0, \ldots, a_d \in A$ with $a_0, a_d \neq 0$ such that $a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 = 0$ in B. Then $a_0 \in I$. □

10.31. Noetherian rings

00FM A ring R is *Noetherian* if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma 10.28.10 it suffices to check that every prime ideal of R is finitely generated.

00FN **Lemma 10.31.1.** *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i.

Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. □

0306 **Lemma 10.31.2.** *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be an ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0, j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. □

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called "absolute Noetherian reduction".

00FO **Lemma 10.31.3.** *Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.*

Proof. This is immediate from Lemma 10.31.1 and the fact that fields are Noetherian rings and that $\mathbf{Z}$ is Noetherian ring (because it is a principal ideal domain). □

00FP **Lemma 10.31.4.** *Let R be a Noetherian ring.*

(1) *Any finite R-module is of finite presentation.*
(2) *Any submodule of a finite R-module is finite.*
(3) *Any finite type R-algebra is of finite presentation over R.*

Proof. Let M be a finite R-module. By Lemma 10.5.4 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma 10.5.3. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite by Lemma 10.5.3 part (5). This proves part (2).

To see (3) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Lemma 10.31.1. □

00FQ **Lemma 10.31.5.** *If R is a Noetherian ring then* $\mathrm{Spec}(R)$ *is a Noetherian topological space, see Topology, Definition 5.9.1.*

Proof. This is because any closed subset of $\mathrm{Spec}(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma 10.17.2. And this correspondence is inclusion reversing. Thus the result follows from the definitions. □

00FR **Lemma 10.31.6.** *If R is a Noetherian ring then* $\mathrm{Spec}(R)$ *has finitely many irreducible components. In other words R has finitely many minimal primes.*

Proof. By Lemma 10.31.5 and Topology, Lemma 5.9.2 we see there are finitely many irreducible components. By Lemma 10.26.1 these correspond to minimal primes of R. □

0CY6 **Lemma 10.31.7.** *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

Proof. By Lemma 10.14.2 finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma 10.31.1. □

045I **Lemma 10.31.8.** *Let k be a field and let R be a Noetherian k-algebra. If K/k is a finitely generated field extension then $K \otimes_k R$ is Noetherian.*

Proof. Since K/k is a finitely generated field extension, there exists a finitely generated k-algebra $B \subset K$ such that K is the fraction field of B. In other words, $K = S^{-1}B$ with $S = B \setminus \{0\}$. Then $K \otimes_k R = S^{-1}(B \otimes_k R)$. Then $B \otimes_k R$ is Noetherian by Lemma 10.31.7. Finally, $K \otimes_k R = S^{-1}(B \otimes_k R)$ is Noetherian by Lemma 10.31.1. □

Here are some fun lemmas that are sometimes useful.

0BX1 **Lemma 10.31.9.** *Let R be a ring and $\mathfrak{p} \subset R$ be a prime. There exists an $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to R_\mathfrak{p}$ is injective in each of the following cases*

(1) *R is a domain,*
(2) *R is Noetherian, or*
(3) *R is reduced and has finitely many minimal primes.*

Proof. If R is a domain, then $R \subset R_\mathfrak{p}$, hence $f = 1$ works. If R is Noetherian, then the kernel I of $R \to R_\mathfrak{p}$ is a finitely generated ideal and we can find $f \in R$, $f \notin \mathfrak{p}$ such that $IR_f = 0$. For this f the map $R_f \to R_\mathfrak{p}$ is injective and f works. If R is reduced with finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then we can choose $f \in \bigcap_{\mathfrak{p}_i \not\subset \mathfrak{p}} \mathfrak{p}_i$, $f \notin \mathfrak{p}$. Indeed, if $\mathfrak{p}_i \not\subset \mathfrak{p}$ then there exist $f_i \in \mathfrak{p}_i$, $f_i \notin \mathfrak{p}$ and $f = \prod f_i$ works. For this f we have $R_f \subset R_\mathfrak{p}$ because the minimal primes of R_f correspond to minimal primes of $R_\mathfrak{p}$ and we can apply Lemma 10.25.2 (some details omitted). □

06RN **Lemma 10.31.10.** *Any surjective endomorphism of a Noetherian ring is an isomorphism.*

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f \circ f) \subset \mathrm{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. □

10.32. Locally nilpotent ideals

0AMF Here is the definition.

00IL **Definition 10.32.1.** Let R be a ring. Let $I \subset R$ be an ideal. We say I is *locally nilpotent* if for every $x \in I$ there exists an $n \in \mathbf{N}$ such that $x^n = 0$. We say I is *nilpotent* if there exists an $n \in \mathbf{N}$ such that $I^n = 0$.

0EGG **Example 10.32.2.** Let $R = k[x_n | n \in \mathbf{N}]$ be the polynomial ring in infinitely many variables over a field k. Let I be the ideal generated by the elements x_n^n for $n \in \mathbf{N}$ and $S = R/I$. Then the ideal $J \subset S$ generated by the images of x_n, $n \in \mathbf{N}$ is locally nilpotent, but not nilpotent. Indeed, since S-linear combinations of nilpotents are nilpotent, to prove that J is locally nilpotent it is enough to observe that all its generators are nilpotent (which they obviously are). On the other hand, for each $n \in \mathbf{N}$ it holds that $x_{n+1}^n \notin I$, so that $J^n \neq 0$. It follows that J is not nilpotent.

0544 **Lemma 10.32.3.** *Let $R \to R'$ be a ring map and let $I \subset R$ be a locally nilpotent ideal. Then IR' is a locally nilpotent ideal of R'.*

Proof. This follows from the fact that if $x, y \in R'$ are nilpotent, then $x + y$ is nilpotent too. Namely, if $x^n = 0$ and $y^m = 0$, then $(x + y)^{n+m-1} = 0$. □

0AMG **Lemma 10.32.4.** *Let R be a ring and let $I \subset R$ be a locally nilpotent ideal. An element x of R is a unit if and only if the image of x in R/I is a unit.*

Proof. If x is a unit in R, then its image is clearly a unit in R/I. It remains to prove the converse. Assume the image of $y \in R$ in R/I is the inverse of the image of x. Then $xy = 1 - z$ for some $z \in I$. This means that $1 \equiv z$ modulo xR. Since z lies in the locally nilpotent ideal I, we have $z^N = 0$ for some sufficiently large N. It follows that $1 = 1^N \equiv z^N = 0$ modulo xR. In other words, x divides 1 and is hence a unit. □

00IM **Lemma 10.32.5.** *Let R be a Noetherian ring. Let I, J be ideals of R. Suppose $J \subset \sqrt{I}$. Then $J^n \subset I$ for some n. In particular, in a Noetherian ring the notions of "locally nilpotent ideal" and "nilpotent ideal" coincide.*

Proof. Say $J = (f_1, \ldots, f_s)$. By assumption $f_i^{d_i} \in I$. Take $n = d_1 + d_2 + \ldots + d_s + 1$. □

00J9 **Lemma 10.32.6.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Then $R \to R/I$ induces a bijection on idempotents.*

First proof of Lemma 10.32.6. As I is locally nilpotent it is contained in every prime ideal. Hence $\mathrm{Spec}(R/I) = V(I) = \mathrm{Spec}(R)$. Hence the lemma follows from Lemma 10.21.3. □

Second proof of Lemma 10.32.6. Suppose $\overline{e} \in R/I$ is an idempotent. We have to lift $\overline{e}$ to an idempotent of R.

First, choose any lift $f \in R$ of $\overline{e}$, and set $x = f^2 - f$. Then, $x \in I$, so x is nilpotent (since I is locally nilpotent). Let now J be the ideal of R generated by x. Then, J is nilpotent (not just locally nilpotent), since it is generated by the nilpotent x.

Now, assume that we have found a lift $e \in R$ of $\overline{e}$ such that $e^2 - e \in J^k$ for some $k \geq 1$. Let $e' = e - (2e-1)(e^2-e) = 3e^2 - 2e^3$, which is another lift of $\overline{e}$ (since the idempotency of $\overline{e}$ yields $e^2 - e \in I$). Then

$$(e')^2 - e' = (4e^2 - 4e - 3)(e^2 - e)^2 \in J^{2k}$$

by a simple computation.

We thus have started with a lift e of $\overline{e}$ such that $e^2 - e \in J^k$, and obtained a lift e' of $\overline{e}$ such that $(e')^2 - e' \in J^{2k}$. This way we can successively improve the approximation (starting with $e = f$, which fits the bill for $k = 1$). Eventually, we reach a stage where $J^k = 0$, and at that stage we have a lift e of $\overline{e}$ such that $e^2 - e \in J^k = 0$, that is, this e is idempotent.

We thus have seen that if $\overline{e} \in R/I$ is any idempotent, then there exists a lift of $\overline{e}$ which is an idempotent of R. It remains to prove that this lift is unique. Indeed, let e_1 and e_2 be two such lifts. We need to show that $e_1 = e_2$.

By definition of e_1 and e_2, we have $e_1 \equiv e_2 \mod I$, and both e_1 and e_2 are idempotent. From $e_1 \equiv e_2 \mod I$, we see that $e_1 - e_2 \in I$, so that $e_1 - e_2$ is nilpotent (since I is locally nilpotent). A straightforward computation (using the idempotency of e_1 and e_2) reveals that $(e_1 - e_2)^3 = e_1 - e_2$. Using this and induction, we obtain $(e_1 - e_2)^k = e_1 - e_2$ for any positive odd integer k. Since all high enough k satisfy $(e_1 - e_2)^k = 0$ (since $e_1 - e_2$ is nilpotent), this shows $e_1 - e_2 = 0$, so that $e_1 = e_2$, which completes our proof. □

05BU **Lemma 10.32.7.** *Let A be a possibly noncommutative algebra. Let $e \in A$ be an element such that $x = e^2 - e$ is nilpotent. Then there exists an idempotent of the form $e' = e + x(\sum a_{i,j} e^i x^j) \in A$ with $a_{i,j} \in \mathbf{Z}$.*

Proof. Consider the ring $R_n = \mathbf{Z}[e]/((e^2-e)^n)$. It is clear that if we can prove the result for each R_n then the lemma follows. In R_n consider the ideal $I = (e^2 - e)$ and apply Lemma 10.32.6. □

0CAP **Lemma 10.32.8.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $n \geq 1$ be an integer which is invertible in R/I. Then*

1. *the nth power map $1 + I \to 1 + I$, $1 + x \mapsto (1 + x)^n$ is a bijection,*
2. *a unit of R is a nth power if and only if its image in R/I is an nth power.*

Proof. Let $a \in R$ be a unit whose image in R/I is the same as the image of b^n with $b \in R$. Then b is a unit (Lemma 10.32.4) and $ab^{-n} = 1 + x$ for some $x \in I$. Hence $ab^{-n} = c^n$ by part (1). Thus (2) follows from (1).

Proof of (1). This is true because there is an inverse to the map $1 + x \mapsto (1 + x)^n$. Namely, we can consider the map which sends $1 + x$ to

$$\begin{aligned}(1+x)^{1/n} &= 1 + \binom{1/n}{1} x + \binom{1/n}{2} x^2 + \binom{1/n}{3} x^3 + \ldots \\ &= 1 + \frac{1}{n} x + \frac{1-n}{2n^2} x^2 + \frac{(1-n)(1-2n)}{6n^3} x^3 + \ldots\end{aligned}$$

as in elementary calculus. This makes sense because the series is finite as $x^k = 0$ for all $k \gg 0$ and each coefficient $\binom{1/n}{k} \in \mathbf{Z}[1/n]$ (details omitted; observe that n is invertible in R by Lemma 10.32.4). □

10.33. Curiosity

02JG Lemma 10.24.3 explains what happens if $V(I)$ is open for some ideal $I \subset R$. But what if $\mathrm{Spec}(S^{-1}R)$ is closed in $\mathrm{Spec}(R)$? The next two lemmas give a partial answer. For more information see Section 10.108.

02JH **Lemma 10.33.1.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is closed. Then $S^{-1}R \cong R/I$ for some ideal $I \subset R$.*

Proof. Let $I = \mathrm{Ker}(R \to S^{-1}R)$ so that $V(I)$ contains the image. Say the image is the closed subset $V(I') \subset \mathrm{Spec}(R)$ for some ideal $I' \subset R$. So $V(I') \subset V(I)$. For $f \in I'$ we see that $f/1 \in S^{-1}R$ is contained in every prime ideal. Hence f^n maps to zero in $S^{-1}R$ for some $n \geq 1$ (Lemma 10.17.2). Hence $V(I') = V(I)$. Then this implies every $g \in S$ is invertible mod I. Hence we get ring maps $R/I \to S^{-1}R$ and $S^{-1}R \to R/I$. The first map is injective by choice of I. The second is the map $S^{-1}R \to S^{-1}(R/I) = R/I$ which has kernel $S^{-1}I$ because localization is exact. Since $S^{-1}I = 0$ we see also the second map is injective. Hence $S^{-1}R \cong R/I$. □

02JI **Lemma 10.33.2.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is closed. If R is Noetherian, or $\mathrm{Spec}(R)$ is a Noetherian topological space, or S is finitely generated as a monoid, then $R \cong S^{-1}R \times R'$ for some ring R'.*

Proof. By Lemma 10.33.1 we have $S^{-1}R \cong R/I$ for some ideal $I \subset R$. By Lemma 10.24.3 it suffices to show that $V(I)$ is open. If R is Noetherian then $\mathrm{Spec}(R)$ is a Noetherian topological space, see Lemma 10.31.5. If $\mathrm{Spec}(R)$ is a Noetherian topological space, then the complement $\mathrm{Spec}(R)\backslash V(I)$ is quasi-compact, see Topology, Lemma 5.12.13. Hence there exist finitely many $f_1, \ldots, f_n \in I$ such that $V(I) = V(f_1, \ldots, f_n)$. Since each f_i maps to zero in $S^{-1}R$ there exists a $g \in S$ such that $gf_i = 0$ for $i = 1, \ldots, n$. Hence $D(g) = V(I)$ as desired. In case S is finitely generated as a monoid, say S is generated by $g_1, \ldots, g_m$, then $S^{-1}R \cong R_{g_1 \ldots g_m}$ and we conclude that $V(I) = D(g_1 \ldots g_m)$. □

10.34. Hilbert Nullstellensatz

00FS

00FV **Theorem 10.34.1** (Hilbert Nullstellensatz)**.** *Let k be a field.*

00FW (1) *For any maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the field extension $\kappa(\mathfrak{m})/k$ is finite.*

00FX (2) *Any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of maximal ideals containing it.*

The same is true in any finite type k-algebra.

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear.

Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation by Lemmas 10.31.3 and 10.31.4. Thus the image of $\mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $\mathrm{Spec}(k[x_n])$ is constructible by Theorem 10.29.10. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

Proof of (2). Let $I \subset R$ be a radical ideal, with R of finite type over k. Let $f \in R$, $f \notin I$. We have to find a maximal ideal $\mathfrak{m} \subset R$ with $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. The ring $(R/I)_f$ is nonzero, since $1 = 0$ in this ring would mean $f^n \in I$ and since I is radical this would mean $f \in I$ contrary to our assumption on f. Thus we may choose a maximal ideal $\mathfrak{m}'$ in $(R/I)_f$, see Lemma 10.17.2. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $\kappa(\mathfrak{m}')/k$ is finite. Hence $R/\mathfrak{m}$ is a field by Fields, Lemma 9.8.10. Thus $\mathfrak{m}$ is maximal and the proof is complete. □

00FY **Lemma 10.34.2.** *Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists an $f \in R$ such that R_f is a field, and K/R_f is a finite field extension.*

Proof. By Lemma 10.30.2 there exist a nonempty open $U \subset \mathrm{Spec}(R)$ contained in the image $\{(0)\}$ of $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz (Theorem 10.34.1). □

10.35. Jacobson rings

00FZ Let R be a ring. The closed points of $\mathrm{Spec}(R)$ are the maximal ideals of R. Often rings which occur naturally in algebraic geometry have lots of maximal ideals. For example finite type algebras over a field or over $\mathbf{Z}$. We will show that these are examples of Jacobson rings.

00G0 **Definition 10.35.1.** Let R be a ring. We say that R is a *Jacobson ring* if every radical ideal I is the intersection of the maximal ideals containing it.

00G1 **Lemma 10.35.2.** *Any algebra of finite type over a field is Jacobson.*

Proof. This follows from Theorem 10.34.1 and Definition 10.35.1. □

00G2 **Lemma 10.35.3.** *Let R be a ring. If every prime ideal of R is the intersection of the maximal ideals containing it, then R is Jacobson.*

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. See Lemma 10.17.2. □

00G3 **Lemma 10.35.4.** *A ring R is Jacobson if and only if* $\mathrm{Spec}(R)$ *is Jacobson, see Topology, Definition 5.18.1.*

Proof. Suppose R is Jacobson. Let $Z \subset \mathrm{Spec}(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset \mathrm{Spec}(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $\mathrm{Spec}(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $\mathrm{Spec}(R)$. According to Lemma 10.17.2 we may assume that $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $\mathrm{Spec}(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is a radical ideal, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But Lemma 10.17.2 shows there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. □

034J **Lemma 10.35.5.** *Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold*

(1) *$\mathfrak{p}$ is not a maximal ideal,*
(2) *$f \notin \mathfrak{p}$,*
(3) *$V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, and*
(4) *$(R/\mathfrak{p})_f$ is a field.*

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Lemma 10.35.4 this means there exists an closed subset $T \subset \mathrm{Spec}(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $\mathrm{Spec}((R/I)_f)$ if $T = V(I)$, see Lemmas 10.17.7 and 10.17.6. As any ring has a maximal ideal (Lemma 10.17.2) we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as $t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds (for example combine Lemmas 10.18.3 and 10.25.1).

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $\mathrm{Spec}(R)$ contains at least one closed point as $\mathrm{Spec}(R)$ is a Jacobson topological space. □

00G4 **Lemma 10.35.6.** *The ring $\mathbf{Z}$ is a Jacobson ring. More generally, let R be a ring such that*

(1) *R is a domain,*
(2) *R is Noetherian,*
(3) *any nonzero prime ideal is a maximal ideal, and*
(4) *R has infinitely many maximal ideals.*

Then R is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. By Lemma 10.17.7 we see that $V(x)$ is homeomorphic to $\mathrm{Spec}(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $\mathrm{Spec}(R/xR)$ (Lemma 10.26.1). As R/xR is Noetherian, the topological space $\mathrm{Spec}(R/xR)$ is Noetherian (Lemma 10.31.5) and has finitely many irreducible components (Topology, Lemma 5.9.2). Thus $V(x)$ is finite as desired. □

02CC **Example 10.35.7.** Let A be an infinite set. For each $\alpha \in A$, let k_α be a field. We claim that $R = \prod_{\alpha \in A} k_\alpha$ is Jacobson. First, note that any element $f \in R$ has the form $f = ue$, with $u \in R$ a unit and $e \in R$ an idempotent (left to the reader). Hence $D(f) = D(e)$, and $R_f = R_e = R/(1-e)$ is a quotient of R. Actually, any ring with this property is Jacobson. Namely, say $\mathfrak{p} \subset R$ is a prime ideal and $f \in R$, $f \notin \mathfrak{p}$. We have to find a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. Because R_f is a quotient of R we see that any maximal ideal of R_f corresponds to a maximal ideal of R not containing f. Hence the result follows by choosing a maximal ideal of R_f containing $\mathfrak{p}R_f$.

00G5 **Example 10.35.8.** A domain R with finitely many maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$ is not a Jacobson ring, except when it is a field. Namely, in this case (0) is not the intersection of the maximal ideals $(0) \neq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n \neq 0$. In particular a discrete valuation ring, or any local ring with at least two prime ideals is not a Jacobson ring.

00GA **Lemma 10.35.9.** *Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{q})/\kappa(\mathfrak{m})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.*

Proof. Consider the diagram

$$\begin{array}{ccccc} S & \longrightarrow & S/\mathfrak{q} & \longrightarrow & \kappa(\mathfrak{q}) \\ \uparrow & & \uparrow & & \\ R & \longrightarrow & R/\mathfrak{m} & & \end{array}$$

We see that $\kappa(\mathfrak{m}) \subset S/\mathfrak{q} \subset \kappa(\mathfrak{q})$. Because the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is algebraic, any ring between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{q})$ is a field (Fields, Lemma 9.8.10). Thus $S/\mathfrak{q}$ is a field, and a posteriori equal to $\kappa(\mathfrak{q})$. □

00FT **Lemma 10.35.10.** *Suppose that k is a field and suppose that V is a nonzero vector space over k. Assume the dimension of V (which is a cardinal number) is smaller than the cardinality of k. Then for any linear operator $T : V \to V$ there exists some monic polynomial $P(t) \in k[t]$ such that $P(T)$ is not invertible.*

Proof. If not then V inherits the structure of a vector space over the field $k(t)$. But the dimension of $k(t)$ over k is at least the cardinality of k for example due to the fact that the elements $\frac{1}{t-\lambda}$ are k-linearly independent. □

Here is another version of Hilbert's Nullstellensatz.

00FU **Theorem 10.35.11.** *Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i\in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then*

(1) *for all maximal ideals $\mathfrak{m} \subset S$ the field extension $\kappa(\mathfrak{m})/k$ is algebraic, and*
(2) *S is a Jacobson ring.*

Proof. If I is finite then the result follows from the Hilbert Nullstellensatz, Theorem 10.34.1. In the rest of the proof we assume I is infinite. It suffices to prove the result for $\mathfrak{m} \subset k[\{x_i\}_{i\in I}]$ maximal in the polynomial ring on variables x_i, since S is a quotient of this. As I is infinite the set of monomials $x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$, $i_1, \ldots, i_r \in I$ and $e_1, \ldots, e_r \geq 0$ has cardinality at most equal to the cardinality of I. Because the cardinality of $I \times \ldots \times I$ is the cardinality of I, and also the cardinality of $\bigcup_{n\geq 0} I^n$ has the same cardinality. (If I is finite, then this is not true and in that case this proof only works if k is uncountable.)

To arrive at a contradiction pick $T \in \kappa(\mathfrak{m})$ transcendental over k. Note that the k-linear map $T : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{m})$ given by multiplication by T has the property that $P(T)$ is invertible for all monic polynomials $P(t) \in k[t]$. Also, $\kappa(\mathfrak{m})$ has dimension at most the cardinality of I over k since it is a quotient of the vector space $k[\{x_i\}_{i\in I}]$ over k (whose dimension is $\#I$ as we saw above). This is impossible by Lemma 10.35.10.

To show that S is Jacobson we argue as follows. If not then there exists a prime $\mathfrak{q} \subset S$ and an element $f \in S$, $f \notin \mathfrak{q}$ such that $\mathfrak{q}$ is not maximal and $(S/\mathfrak{q})_f$ is a field, see Lemma 10.35.5. But note that $(S/\mathfrak{q})_f$ is generated by at most $\#I + 1$ elements. Hence the field extension $(S/\mathfrak{q})_f/k$ is algebraic (by the first part of the proof). This implies that $\kappa(\mathfrak{q})$ is an algebraic extension of k hence $\mathfrak{q}$ is maximal by Lemma 10.35.9. This contradiction finishes the proof. □

046V **Lemma 10.35.12.** *Let k be a field. Let S be a k-algebra. For any field extension K/k whose cardinality is larger than the cardinality of S we have*

(1) *for every maximal ideal $\mathfrak{m}$ of S_K the field $\kappa(\mathfrak{m})$ is algebraic over K, and*
(2) *S_K is a Jacobson ring.*

Proof. Choose $k \subset K$ such that the cardinality of K is greater than the cardinality of S. Since the elements of S generate the K-algebra S_K we see that Theorem 10.35.11 applies. □

02CB **Example 10.35.13.** The trick in the proof of Theorem 10.35.11 really does not work if k is a countable field and I is countable too. Let k be a countable field. Let x be a variable, and let $k(x)$ be the field of rational functions in x. Consider the polynomial algebra $R = k[x, \{x_f\}_{f\in k[x]-\{0\}}]$. Let $I = (\{fx_f - 1\}_{f\in k[x]-\{0\}})$. Note that I is a proper ideal in R. Choose a maximal ideal $I \subset \mathfrak{m}$. Then $k \subset R/\mathfrak{m}$ is isomorphic to $k(x)$, and is not algebraic over k.

00G6 **Lemma 10.35.14.** *Let R be a Jacobson ring. Let $f \in R$. The ring R_f is Jacobson and maximal ideals of R_f correspond to maximal ideals of R not containing f.*

Proof. By Topology, Lemma 5.18.5 we see that $D(f) = \mathrm{Spec}(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $\mathrm{Spec}(R)$ which happen to lie in $D(f)$. Thus R_f is Jacobson by Lemma 10.35.4. □

00G7 **Example 10.35.15.** Here is a simple example that shows Lemma 10.35.14 to be false if R is not Jacobson. Consider the ring $R = \mathbf{Z}_{(2)}$, i.e., the localization of $\mathbf{Z}$ at the prime (2). The localization of R at the element 2 is isomorphic to $\mathbf{Q}$, in a formula: $R_2 \cong \mathbf{Q}$. Clearly the map $R \to R_2$ maps the closed point of $\mathrm{Spec}(\mathbf{Q})$ to the generic point of $\mathrm{Spec}(R)$.

00G8 **Example 10.35.16.** Here is a simple example that shows Lemma 10.35.14 is false if R is Jacobson but we localize at infinitely many elements. Namely, let $R = \mathbf{Z}$ and consider the localization $(R \setminus \{0\})^{-1}R \cong \mathbf{Q}$ of R at the set of all nonzero elements. Clearly the map $\mathbf{Z} \to \mathbf{Q}$ maps the closed point of $\mathrm{Spec}(\mathbf{Q})$ to the generic point of $\mathrm{Spec}(\mathbf{Z})$.

00G9 **Lemma 10.35.17.** *Let R be a Jacobson ring. Let $I \subset R$ be an ideal. The ring R/I is Jacobson and maximal ideals of R/I correspond to maximal ideals of R containing I.*

Proof. The proof is the same as the proof of Lemma 10.35.14. □

0CY7 **Lemma 10.35.18.** *Let R be a Jacobson ring. Let K be a field. Let $R \subset K$ and K is of finite type over R. Then R is a field and K/R is a finite field extension.*

Proof. First note that R is a domain. By Lemma 10.34.2 we see that R_f is a field and K/R_f is a finite field extension for some nonzero $f \in R$. Hence (0) is a maximal ideal of R_f and by Lemma 10.35.14 we conclude (0) is a maximal ideal of R. □

00GB **Proposition 10.35.19.** *Let R be a Jacobson ring. Let $R \to S$ be a ring map of finite type. Then*

1. *The ring S is Jacobson.*
2. *The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ transforms closed points to closed points.*
3. *For $\mathfrak{m}' \subset S$ maximal lying over $\mathfrak{m} \subset R$ the field extension $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})$ is finite.*

Proof. Let $\mathfrak{m}' \subset S$ be a maximal ideal and $R \cap \mathfrak{m}' = \mathfrak{m}$. Then $R/\mathfrak{m} \to S/\mathfrak{m}'$ satisfies the conditions of Lemma 10.35.18 by Lemma 10.35.17. Hence $R/\mathfrak{m}$ is a field and $\mathfrak{m}$ a maximal ideal and the induced residue field extension is finite. This proves (2) and (3).

If S is not Jacobson, then by Lemma 10.35.5 there exists a non-maximal prime ideal $\mathfrak{q}$ of S and an $g \in S$, $g \notin \mathfrak{q}$ such that $(S/\mathfrak{q})_g$ is a field. To arrive at a contradiction we show that $\mathfrak{q}$ is a maximal ideal. Let $\mathfrak{p} = \mathfrak{q} \cap R$. Then $R/\mathfrak{p} \to (S/\mathfrak{q})_g$ satisfies the conditions of Lemma 10.35.18 by Lemma 10.35.17. Hence $R/\mathfrak{p}$ is a field and the field extension $\kappa(\mathfrak{p}) \to (S/\mathfrak{q})_g = \kappa(\mathfrak{q})$ is finite, thus algebraic. Then $\mathfrak{q}$ is a maximal ideal of S by Lemma 10.35.9. Contradiction. □

00GC **Lemma 10.35.20.** *Any finite type algebra over $\mathbf{Z}$ is Jacobson.*

Proof. Combine Lemma 10.35.6 and Proposition 10.35.19. □

00GD **Lemma 10.35.21.** *Let $R \to S$ be a finite type ring map of Jacobson rings. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. Denote with a subscript $_0$ the set of closed points of a topological space.*

(1) *We have $f(E)_0 = f(E_0) = X_0 \cap f(E)$.*
(2) *A point $\xi \in X$ is in $f(E)$ if and only if $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$.*

Proof. We have a commutative diagram of continuous maps

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f(E) & \longrightarrow & X \end{array}$$

Suppose $x \in f(E)$ is closed in $f(E)$. Then $f^{-1}(\{x\}) \cap E$ is nonempty and closed in E. Applying Topology, Lemma 5.18.5 to both inclusions

$$f^{-1}(\{x\}) \cap E \subset E \subset Y$$

we find there exists a point $y \in f^{-1}(\{x\}) \cap E$ which is closed in Y. In other words, there exists $y \in Y_0$ and $y \in E_0$ mapping to x. Hence $x \in f(E_0)$. This proves that $f(E)_0 \subset f(E_0)$. Proposition 10.35.19 implies that $f(E_0) \subset X_0 \cap f(E)$. The inclusion $X_0 \cap f(E) \subset f(E)_0$ is trivial. This proves the first assertion.

Suppose that $\xi \in f(E)$. According to Lemma 10.30.2 the set $f(E) \cap \overline{\{\xi\}}$ contains a dense open subset of $\overline{\{\xi\}}$. Since X is Jacobson we conclude that $f(E) \cap \overline{\{\xi\}}$ contains a dense set of closed points, see Topology, Lemma 5.18.5. We conclude by part (1) of the lemma.

On the other hand, suppose that $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$. By Lemma 10.29.4 there exists a ring map $S \to S'$ of finite presentation such that E is the image of $Y' := \mathrm{Spec}(S') \to Y$. Then E_0 is the image of Y'_0 by the first part of the lemma applied to the ring map $S \to S'$. Thus we may assume that $E = Y$ by replacing S by S'. Suppose ξ corresponds to $\mathfrak{p} \subset R$. Consider the diagram

$$\begin{array}{ccc} S & \longrightarrow & S/\mathfrak{p}S \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\mathfrak{p} \end{array}$$

This diagram and the density of $f(Y_0) \cap V(\mathfrak{p})$ in $V(\mathfrak{p})$ shows that the morphism $R/\mathfrak{p} \to S/\mathfrak{p}S$ satisfies condition (2) of Lemma 10.30.4. Hence we conclude there exists a prime $\overline{\mathfrak{q}} \subset S/\mathfrak{p}S$ mapping to (0). In other words the inverse image $\mathfrak{q}$ of $\overline{\mathfrak{q}}$ in S maps to $\mathfrak{p}$ as desired. □

The conclusion of the lemma above is that we can read off the image of f from the set of closed points of the image. This is a little nicer in case the map is of finite presentation because then we know that images of a constructible is constructible. Before we state it we introduce some notation. Denote $\mathrm{Constr}(X)$ the set of constructible sets. Let $R \to S$ be a ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Denote with a subscript $_0$ the set of closed points of a topological space.

00GE **Lemma 10.35.22.** *With notation as above. Assume that R is a Noetherian Jacobson ring. Further assume $R \to S$ is of finite type. There is a commutative diagram*

$$\begin{array}{ccc} \mathit{Constr}(Y) & \xrightarrow{E \mapsto E_0} & \mathit{Constr}(Y_0) \\ \downarrow{\scriptstyle E \mapsto f(E)} & & \downarrow{\scriptstyle E \mapsto f(E)} \\ \mathit{Constr}(X) & \xrightarrow{E \mapsto E_0} & \mathit{Constr}(X_0) \end{array}$$

where the horizontal arrows are the bijections from Topology, Lemma 5.18.8.

Proof. Since $R \to S$ is of finite type, it is of finite presentation, see Lemma 10.31.4. Thus the image of a constructible set in X is constructible in Y by Chevalley's theorem (Theorem 10.29.10). Combined with Lemma 10.35.21 the lemma follows. □

To illustrate the use of Jacobson rings, we give the following two examples.

00GF **Example 10.35.23.** Let k be a field. The space $\mathrm{Spec}(k[x,y]/(xy))$ has two irreducible components: namely the x-axis and the y-axis. As a generalization, let

$$R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal in $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ generated by the entries of the 2×2 product matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

In this example we will describe $\mathrm{Spec}(R)$.

To prove the statement about $\mathrm{Spec}(k[x,y]/(xy))$ we argue as follows. If $\mathfrak{p} \subset k[x,y]$ is any ideal containing xy, then either x or y would be contained in $\mathfrak{p}$. Hence the minimal such prime ideals are just (x) and (y). In case k is algebraically closed, the max-Spec of these components can then be visualized as the point sets of y- and x-axis.

For the generalization, note that we may identify the closed points of the spectrum of $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}])$ with the space of matrices

$$\left\{ (X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right\}$$

at least if k is algebraically closed. Now define a group action of $\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k)$ on the space of matrices $\{(X,Y)\}$ by

$$(g_1, g_2, g_3) \times (X,Y) \mapsto ((g_1 X g_2^{-1}, g_2 Y g_3^{-1})).$$

Here, also observe that the algebraic set

$$\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \subset \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k)$$

is irreducible since it is the max spectrum of the domain

$$k[x_{11}, x_{12}, \ldots, z_{21}, z_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}, (y_{11}y_{22} - y_{12}y_{21})^{-1}, (z_{11}z_{22} - z_{12}z_{21})^{-1}].$$

Since the image of irreducible an algebraic set is still irreducible, it suffices to classify the orbits of the set $\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) | XY = 0\}$ and take their closures. From standard linear algebra, we are reduced to the following three cases:

(1) $\exists(g_1, g_2)$ such that $g_1 X g_2^{-1} = I_{2\times 2}$. Then Y is necessarily 0, which as an algebraic set is invariant under the group action. It follows that this orbit is contained in the irreducible algebraic set defined by the prime ideal $(y_{11}, y_{12}, y_{21}, y_{22})$. Taking the closure, we see that $(y_{11}, y_{12}, y_{21}, y_{22})$ is actually a component.

(2) $\exists(g_1, g_2)$ such that

$$g_1 X g_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This case occurs if and only if X is a rank 1 matrix, and furthermore, Y is killed by such an X if and only if

$$x_{11}y_{11} + x_{12}y_{21} = 0; \quad x_{11}y_{12} + x_{12}y_{22} = 0;$$

$$x_{21}y_{11} + x_{22}y_{21} = 0; \quad x_{21}y_{12} + x_{22}y_{22} = 0.$$

Fix a rank 1 X, such non zero Y's satisfying the above equations form an irreducible algebraic set for the following reason($Y = 0$ is contained the previous case): $0 = g_1 X g_2^{-1} g_2 Y$ implies that

$$g_2 Y = \begin{pmatrix} 0 & 0 \\ y'_{21} & y'_{22} \end{pmatrix}.$$

With a further $\mathrm{GL}(2, k)$-action on the right by g_3, $g_2 Y$ can be brought into

$$g_2 Y g_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus such Y's form an irreducible algebraic set isomorphic to the image of $\mathrm{GL}(2, k)$ under this action. Finally, notice that the "rank 1" condition for X's forms an open dense subset of the irreducible algebraic set $\det X = x_{11}x_{22} - x_{12}x_{21} = 0$. It now follows that all the five equations define an irreducible component $(x_{11}y_{11} + x_{12}y_{21}, x_{11}y_{12} + x_{12}y_{22}, x_{21}y_{11} + x_{22}y_{21}, x_{21}y_{12} + x_{22}y_{22}, x_{11}x_{22} - x_{12}x_{21})$ in the open subset of the space of pairs of nonzero matrices. It can be shown that the pair of equations $\det X = 0$, $\det Y = 0$ cuts $\mathrm{Spec}(R)$ in an irreducible component with the above locus an open dense subset.

(3) $\exists(g_1, g_2)$ such that $g_1 X g_2^{-1} = 0$, or equivalently, $X = 0$. Then Y can be arbitrary and this component is thus defined by $(x_{11}, x_{12}, x_{21}, x_{22})$.

00GG **Example 10.35.24.** For another example, consider $R = k[\{t_{ij}\}_{i,j=1}^n]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the entries of the product matrix $T^2 - T$, $T = (t_{ij})$. From linear algebra, we know that under the $GL(n, k)$-action defined by $g, T \mapsto gTg^{-1}$, T is classified by the its rank and each T is conjugate to some $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, which has r 1's and $n-r$ 0's. Thus each orbit of such a $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ under the group action forms an irreducible component and every idempotent matrix is contained in one such orbit. Next we will show that any two different orbits are necessarily disjoint. For this purpose we only need to cook up polynomial functions that take different values on different orbits. In characteristic 0 cases, such a function can be taken to be $f(t_{ij}) = trace(T) = \sum_{i=1}^n t_{ii}$. In positive characteristic cases, things are slightly more tricky since we might have $trace(T) = 0$

even if $T \neq 0$. For instance, $char = 3$

$$trace \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 3 = 0$$

Anyway, these components can be separated using other functions. For instance, in the characteristic 3 case, $tr(\wedge^3 T)$ takes value 1 on the components corresponding to $diag(1, 1, 1)$ and 0 on other components.

10.36. Finite and integral ring extensions

00GH Trivial lemmas concerning finite and integral ring maps. We recall the definition.

00GI **Definition 10.36.1.** Let $\varphi : R \to S$ be a ring map.

(1) An element $s \in S$ is *integral over* R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.
(2) The ring map φ is *integral* if every $s \in S$ is integral over R.

052I **Lemma 10.36.2.** *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Consider the map $\varphi : M \to M$, $x \mapsto y \cdot x$. By Lemma 10.16.2 there exists a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. □

00GK **Lemma 10.36.3.** *A finite ring map is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 10.36.2 to $M = S$ to see that y is integral over R. □

00GM **Lemma 10.36.4.** *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 10.36.3. □

02JJ **Lemma 10.36.5.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is finite,*
(2) *$R \to S$ is integral and of finite type, and*
(3) *there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.*

Proof. Clear from Lemma 10.36.4. □

00GN **Lemma 10.36.6.** *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 10.36.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 10.36.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma 10.7.3 applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 10.36.3. □

00GO **Lemma 10.36.7.** *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 10.36.4 and 10.36.3. □

0CY8 **Lemma 10.36.8.** *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \ldots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i.*

Proof. Omitted. □

00GP **Definition 10.36.9.** Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 10.36.7, is called the *integral closure* of R in S. If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

0CY9 **Lemma 10.36.10.** *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S_i'. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S_i'. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.*

Proof. This follows immediately from Lemma 10.36.8. □

0307 **Lemma 10.36.11.** *Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f' f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. □

034K **Lemma 10.36.12.** *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

(1) *x is integral over R, and*
(2) *for every prime ideal $\mathfrak{p} \subset R$ the element $x \in S_\mathfrak{p}$ is integral over $R_\mathfrak{p}$.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in $S_\mathfrak{p}$ for some $a_i \in R_\mathfrak{p}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and $\mathrm{Spec}(R)$ is quasi-compact (Lemma 10.17.8) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i}. Hence we conclude from Lemma 10.23.2 that S' is finite over R. Hence x is integral over R by Lemma 10.36.4. □

02JK **Lemma 10.36.13.** *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*
(1) *If $R \to S$ is integral so is $R' \to S'$.*
(2) *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P'_i over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

02JL **Lemma 10.36.14.** *Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*
(1) *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*
(2) *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

02JM **Lemma 10.36.15.** *Let $A \to B \to C$ be ring maps.*
(1) *If $A \to C$ is integral so is $B \to C$.*
(2) *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. □

0308 **Lemma 10.36.16.** *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. □

00GQ **Lemma 10.36.17.** *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_\mathfrak{p} \neq S_\mathfrak{p}$, see Lemma 10.18.6. The localization $R_\mathfrak{p} \to S_\mathfrak{p}$ is injective (as localization is exact) and integral by Lemma 10.36.11 or 10.36.13. Hence we may replace R, S by $R_\mathfrak{p}$, $S_\mathfrak{p}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let

$R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 10.36.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 10.20.1 we see $S' = 0$. Contradiction. □

00GR **Lemma 10.36.18.** *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 10.36.17 we see that $\mathrm{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma 10.25.1). The other assertions are immediate from this. □

00GS **Lemma 10.36.19.** *Let k be a field. Let S be a k-algebra over k.*

(1) *If S is a domain and finite dimensional over k, then S is a field.*
(2) *If S is integral over k and a domain, then S is a field.*
(3) *If S is integral over k then every prime of S is a maximal ideal (see Lemma 10.26.5 for more consequences).*

Proof. The statement on primes follows from the statement "integral + domain ⇒ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 10.36.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$ containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. □

00GT **Lemma 10.36.20.** *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(S)$ be distinct primes having the same image in $\mathrm{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark 10.18.5 the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 10.36.19. □

05DR **Lemma 10.36.21.** *Suppose $R \to S$ is finite. Then the fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ are finite.*

Proof. By the discussion in Remark 10.18.5 the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma 10.31.1. By Lemma 10.36.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma 10.31.6 there are at most finitely many. □

00GU **Lemma 10.36.22.** *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}$ be a prime of S mapping to $\mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}'$.*

Proof. We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Lemma 10.36.17 we win. □

The property expressed in the lemma above is called the "going up property" for the ring map $R \to S$, see Definition 10.41.1.

0564 **Lemma 10.36.23.** *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.*

Proof. One of the implications follows from Lemma 10.6.4. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma 10.5.3 we see that it suffices to prove that S is finitely presented as an R-module.

Pick $y_1, \ldots, y_n \in S$ such that $y_1, \ldots, y_n$ generate S as an R-module. By Lemma 10.36.2 each y_i is integral over R. Choose monic polynomials $P_i(x) \in R[x]$ with $P_i(y_i) = 0$. Consider the ring

$$S' = R[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$$

Then we see that S is of finite presentation as an S'-algebra by Lemma 10.6.2. Since $S' \to S$ is surjective, the kernel $J = \mathrm{Ker}(S' \to S)$ is finitely generated as an ideal by Lemma 10.6.3. Hence J is a finite S'-module (immediate from the definitions). Thus $S = \mathrm{Coker}(J \to S')$ is of finite presentation as an S'-module by Lemma 10.5.3. Hence, arguing as in the first paragraph, it suffices to show that S' is of finite presentation as an R-module. Actually, S' is free as an R-module with basis the monomials $x_1^{e_1} \ldots x_n^{e_n}$ for $0 \leq e_i < \deg(P_i)$. Namely, write $R \to S'$ as the composition

$$R \to R[x_1]/(P_1(x_1)) \to R[x_1, x_2]/(P_1(x_1), P_2(x_2)) \to \ldots \to S'$$

This shows that the ith ring in this sequence is free as a module over the $(i-1)$st one with basis $1, x_i, \ldots, x_i^{\deg(P_i)-1}$. The result follows easily from this by induction. Some details omitted. □

052J **Lemma 10.36.24.** *Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence*

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i(b_0 + b_1 y/x + \ldots + b_m (y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i(a_0 + a_1 x/y + \ldots + a_n (x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 10.36.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. □

10.37. Normal rings

037B We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

0309 **Definition 10.37.1.** A domain R is called *normal* if it is integrally closed in its field of fractions.

034L **Lemma 10.37.2.** *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Omitted. □

The following notion is occasionally useful when studying normality.

00GW **Definition 10.37.3.** Let R be a domain.

(1) An element g of the fraction field of R is called *almost integral over R* if there exists an element $r \in R$, $r \neq 0$ such that $rg^n \in R$ for all $n \geq 0$.
(2) The domain R is called *completely normal* if every almost integral element of the fraction field of R is contained in R.

The following lemma shows that a Noetherian domain is normal if and only if it is completely normal.

00GX **Lemma 10.37.4.** *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, then so are $u+v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Lemma 10.36.3. □

00GY **Lemma 10.37.5.** *Any localization of a normal domain is normal.*

Proof. Let R be a normal domain, and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j}a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$. □

00GZ **Lemma 10.37.6.** *A principal ideal domain is normal.*

Proof. Let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R. □

00H0 **Lemma 10.37.7.** *Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.*

(1) *If f is integral over $R[x]$ then all α_i are integral over R, and*
(2) *If f is almost integral over $R[x]$ then all α_i are almost integral over R.*

Proof. We first prove the second statement. Write $f = \alpha_0 + \alpha_1 x + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_0 + b_1 x + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring (Lemma 10.37.4) we deduce that $f - \alpha_r x^r = \alpha_0 + \alpha_1 x + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on r we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(f) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian, see Lemma 10.31.3. It is still the case that $f \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 10.37.4 the element f is almost integral over $R_0[x]$. By the second statement of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, they are integral over R_0, see Lemma 10.37.4. Of course, then they are integral over R. □

030A **Lemma 10.37.8.** *Let R be a normal domain. Then $R[x]$ is a normal domain.*

Proof. The result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by Lemma 10.37.6. Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 10.37.7 the elements α_i are integral over R and hence are in R. □

0BI0 **Lemma 10.37.9.** *Let R be a Noetherian normal domain. Then $R[[x]]$ is a Noetherian normal domain.*

Proof. The power series ring is Noetherian by Lemma 10.31.2. Let $f, g \in R[[x]]$ be nonzero elements such that $w = f/g$ is integral over $R[[x]]$. Let K be the fraction field of R. Since the ring of Laurent series $K((x)) = K[[x]][1/x]$ is a field, we can write $w = a_n x^n + a_{n+1} x^{n+1} + \ldots$ for some $n \in \mathbf{Z}$, $a_i \in K$, and $a_n \neq 0$. By Lemma 10.37.4 we see there exists a nonzero element $h = b_m x^m + b_{m+1} x^{m+1} + \ldots$ in $R[[x]]$ with $b_m \neq 0$ such that $w^e h \in R[[x]]$ for all $e \geq 1$. We conclude that $n \geq 0$ and that $b_m a_n^e \in R$ for all $e \geq 1$. Since R is Noetherian this implies that $a_n \in R$ by the same lemma. Now, if $a_n, a_{n+1}, \ldots, a_{N-1} \in R$, then we can apply the same argument to $w - a_n x^n - \ldots - a_{N-1} x^{N-1} = a_N x^N + \ldots$. In this way we see that all $a_i \in R$ and the lemma is proved. □

030B **Lemma 10.37.10.** *Let R be a domain. The following are equivalent:*

(1) *The domain R is a normal domain,*
(2) *for every prime $\mathfrak{p} \subset R$ the local ring $R_\mathfrak{p}$ is a normal domain, and*
(3) *for every maximal ideal $\mathfrak{m}$ the ring $R_\mathfrak{m}$ is a normal domain.*

Proof. We deduce (1) $\Rightarrow$ (2) from Lemma 10.37.5. The implication (2) $\Rightarrow$ (3) is immediate. The implication (3) $\Rightarrow$ (1) follows from the fact that for any domain R we have

$$R = \bigcap\nolimits_{\mathfrak{m}} R_{\mathfrak{m}}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak{m}$, whence $I = R$. □

Lemma 10.37.10 shows that the following definition is compatible with Definition 10.37.1. (It is the definition from EGA – see [**DG67**, IV, 5.13.5 and 0, 4.1.4].)

00GV **Definition 10.37.11.** A ring R is called *normal* if for every prime $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is a normal domain (see Definition 10.37.1).

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Lemma 10.23.1).

034M **Lemma 10.37.12.** *A normal ring is integrally closed in its total ring of fractions.*

Proof. Let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak{p} \subset R$ be a prime. As $R \to R_{\mathfrak{p}}$ is flat we see that $R_{\mathfrak{p}} \subset Q(R) \otimes_R R_{\mathfrak{p}}$. As $R_{\mathfrak{p}}$ is a normal domain we see that $x \otimes 1$ is an element of $R_{\mathfrak{p}}$. Hence we can find $a, f \in R$, $f \notin \mathfrak{p}$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_{\mathfrak{p}} = Q(R)_{\mathfrak{p}}$, which in turn means that there exists an $f' \in R$, $f' \notin \mathfrak{p}$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$. □

037C **Lemma 10.37.13.** *A localization of a normal ring is a normal ring.*

Proof. Omitted. □

00H1 **Lemma 10.37.14.** *Let R be a normal ring. Then $R[x]$ is a normal ring.*

Proof. Let $\mathfrak{q}$ be a prime of $R[x]$. Set $\mathfrak{p} = R \cap \mathfrak{q}$. Then we see that $R_{\mathfrak{p}}[x]$ is a normal domain by Lemma 10.37.8. Hence $(R[x])_{\mathfrak{q}}$ is a normal domain by Lemma 10.37.5. □

0CYA **Lemma 10.37.15.** *A finite product of normal rings is normal.*

Proof. It suffices to show that the product of two normal rings, say R and S, is normal. By Lemma 10.21.3 the prime ideals of $R \times S$ are of the form $\mathfrak{p} \times S$ and $R \times \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are primes of R and S respectively. Localization yields $(R \times S)_{\mathfrak{p} \times S} = R_{\mathfrak{p}}$ which is a normal domain by assumption. Similarly for S. □

030C **Lemma 10.37.16.** *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

(1) *R is a normal ring,*
(2) *R is integrally closed in its total ring of fractions, and*
(3) *R is a finite product of normal domains.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold in general, see Lemmas 10.37.12 and 10.37.15.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. By Lemmas 10.25.2 and 10.25.4 we have $Q(R) = R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_n}$, and by Lemma 10.25.1 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of n domains (see Sections 10.22 and 10.24). Each factor is of the form $R/\mathfrak{p}_i$ with field of fractions $R_{\mathfrak{p}_i}$. By Lemma 10.36.10 each map $R/\mathfrak{p}_i \to R_{\mathfrak{p}_i}$ is integrally closed. Hence R is a finite product of normal domains. □

037D **Lemma 10.37.17.** *Let $(R_i, \varphi_{ii'})$ be a directed system (Categories, Definition 10.8.1) of rings. If each R_i is a normal ring so is $R = \operatorname{colim}_i R_i$.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. Set $\mathfrak{p}_i = R_i \cap \mathfrak{p}$ (usual abuse of notation). Then we see that $R_{\mathfrak{p}} = \operatorname{colim}_i (R_i)_{\mathfrak{p}_i}$. Since each $(R_i)_{\mathfrak{p}_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b come from objects a_i, b_i over R_i, P comes from a monic polynomial $P_i \in R_i[T]$ and $P_i(a_i/b_i) = 0$. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. □

10.38. Going down for integral over normal

037E We first play around a little bit with the notion of elements integral over an ideal, and then we prove the theorem referred to in the section title.

00H2 **Definition 10.38.1.** Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. We say an element $g \in S$ is *integral over I* if there exists a monic polynomial $P = x^d + \sum_{j<d} a_j x^j$ with coefficients $a_j \in I^{d-j}$ such that $P^\varphi(g) = 0$ in S.

This is mostly used when $\varphi = \mathrm{id}_R : R \to R$. In this case the set I' of elements integral over I is called the *integral closure of I*. We will see that I' is an ideal of R (and of course $I \subset I'$).

00H3 **Lemma 10.38.2.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $A = \sum I^n t^n \subset R[t]$ be the subring of the polynomial ring generated by $R \oplus It \subset R[t]$. An element $s \in S$ is integral over I if and only if the element $st \in S[t]$ is integral over A.*

Proof. Suppose st is integral over A. Let $P = x^d + \sum_{j<d} a_j x^j$ be a monic polynomial with coefficients in A such that $P^\varphi(st) = 0$. Let $a'_j \in A$ be the degree $d - j$ part of a_j, in other words $a'_j = a''_j t^{d-j}$ with $a''_j \in I^{d-j}$. For degree reasons we still have $(st)^d + \sum_{j<d} \varphi(a''_j) t^{d-j} (st)^j = 0$. Hence $s^d + \sum_{j<d} \varphi(a''_j) s^j = 0$ and we see that s is integral over I.

Suppose that s is integral over I. Say $P = x^d + \sum_{j<d} a_j x^j$ with $a_j \in I^{d-j}$. Then we immediately find a polynomial $Q = x^d + \sum_{j<d} (a_j t^{d-j}) x^j$ with coefficients in A which proves that st is integral over A. □

00H4 **Lemma 10.38.3.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The set of elements of S which are integral over I form a R-submodule of S. Furthermore, if $s \in S$ is integral over R, and s' is integral over I, then ss' is integral over I.*

Proof. We will use Lemma 10.36.7 without further mention. Closure under addition is clear from the characterization of Lemma 10.38.2 whose notation we adopt. Any element $s \in S$ which is integral over R corresponds to the degree 0 element s of $S[t]$ which is integral over A (because $R \subset A$). Hence we see that multiplication by s on $S[t]$ preserves the property of being integral over A, $\square$

00H5 **Lemma 10.38.4.** *Suppose $\varphi : R \to S$ is integral. Suppose $I \subset R$ is an ideal. Then every element of IS is integral over I.*

Proof. Immediate from Lemma 10.38.3. $\square$

00H6 **Lemma 10.38.5.** *Let K be a field. Let $n, m \in \mathbf{N}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in K$. If the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ divides the polynomial $x^m + b_{m-1}x^{m-1} + \ldots + b_0$ in $K[x]$ then*

(1) *$a_0, \ldots, a_{n-1}$ are integral over any subring R_0 of K containing the elements $b_0, \ldots, b_{m-1}$, and*
(2) *each a_i lies in $\sqrt{(b_0, \ldots, b_{m-1})R}$ for any subring $R \subset K$ containing the elements $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$.*

Proof. Let L/K be a field extension such that we can write $x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \prod_{i=1}^m (x - \beta_i)$ with $\beta_i \in L$. See Fields, Section 9.16. Each β_i is integral over R_0. Since each a_i is a homogeneous polynomial in $\beta_1, \ldots, \beta_m$ we deduce the same for the a_i (use Lemma 10.36.7).

Choose $c_0, \ldots, c_{m-n-1} \in K$ such that

$$x^m + b_{m-1}x^{m-1} + \ldots + b_0 = (x^n + a_{n-1}x^{n-1} + \ldots + a_0)(x^{m-n} + c_{m-n-1}x^{m-n-1} + \ldots + c_0).$$

By part (1) the elements c_i are integral over R. Consider the integral extension

$$R \subset R' = R[c_0, \ldots, c_{m-n-1}] \subset K$$

By Lemmas 10.36.17 and 10.30.3 we see that $R \cap \sqrt{(b_0, \ldots, b_{m-1})R'} = \sqrt{(b_0, \ldots, b_{m-1})R}$. Thus we may replace R by R' and assume $c_i \in R$. Dividing out the radical $\sqrt{(b_0, \ldots, b_{m-1})}$ we get a reduced ring $\overline{R}$. We have to show that the images $\overline{a}_i \in \overline{R}$ are zero. And in $\overline{R}[x]$ we have the relation

$$x^m = x^m + \overline{b}_{m-1}x^{m-1} + \ldots + \overline{b}_0 = (x^n + \overline{a}_{n-1}x^{n-1} + \ldots + \overline{a}_0)(x^{m-n} + \overline{c}_{m-n-1}x^{m-n-1} + \ldots + \overline{c}_0).$$

It is easy to see that this implies $\overline{a}_i = 0$ for all i. Indeed by Lemma 10.25.1 the localization of $\overline{R}$ at a minimal prime $\mathfrak{p}$ is a field and $\overline{R}_{\mathfrak{p}}[x]$ a UFD. Thus $f = x^n + \sum \overline{a}_i x^i$ is associated to x^n and since f is monic $f = x^n$ in $\overline{R}_{\mathfrak{p}}[x]$. Then there exists an $s \in \overline{R}$, $s \notin \mathfrak{p}$ such that $s(f - x^n) = 0$. Therefore all $\overline{a}_i$ lie in $\mathfrak{p}$ and we conclude by Lemma 10.25.2. $\square$

00H7 **Lemma 10.38.6.** *Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.*

Proof. Let $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be a polynomial with coefficients in R such that $P(g) = 0$. Let $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal polynomial for g over the fraction field K of R. Then Q divides P in $K[x]$. By Lemma 10.38.5 we see the a_i are integral over R. Since R is normal this means they are in R. $\square$

00H8 **Proposition 10.38.7.** *Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}'$ be a prime of S with $\mathfrak{p}' = R \cap \mathfrak{q}'$. Then there exists a prime $\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{q}'$ such that $\mathfrak{p} = R \cap \mathfrak{q}$. In other words: the going down property holds for $R \to S$, see Definition 10.41.1.*

Proof. Let $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}'$ be as in the statement. We have to show there is a prime $\mathfrak{q}$, with $\mathfrak{q} \subset \mathfrak{q}'$ and $R \cap \mathfrak{q} = \mathfrak{p}$. This is the same as finding a prime of $S_{\mathfrak{q}'}$ mapping to $\mathfrak{p}$. According to Lemma 10.18.6 we have to show that $\mathfrak{p}S_{\mathfrak{q}'} \cap R = \mathfrak{p}$. Pick $z \in \mathfrak{p}S_{\mathfrak{q}'} \cap R$. We may write $z = y/g$ with $y \in \mathfrak{p}S$ and $g \in S$, $g \notin \mathfrak{q}'$. Written differently we have $zg = y$.

By Lemma 10.38.4 there exists a monic polynomial $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $b_i \in \mathfrak{p}$ such that $P(y) = 0$.

By Lemma 10.38.6 the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Note that not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ since that would imply $g^n = \sum_{j<n} a_j g^j \in \mathfrak{p}S \subset \mathfrak{p}'S \subset \mathfrak{q}'$ which is a contradiction.

Since $y = zg$ we see immediately from the above that $Q' = x^n + za_{n-1}x^{n-1} + \ldots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and by Lemma 10.38.5 we see that $z^j a_{n-j} \in \sqrt{(b_0, \ldots, b_{m-1})} \subset \mathfrak{p}$, $j = 1, \ldots, n$. Because not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ we conclude $z \in \mathfrak{p}$ as desired. □

10.39. Flat modules and flat ring maps

00H9 One often used result is that if $M = \operatorname{colim}_{i \in \mathcal{I}} M_i$ is a colimit of R-modules and if N is an R-module then

$$M \otimes N = \operatorname{colim}_{i \in \mathcal{I}} M_i \otimes_R N,$$

see Lemma 10.12.9. This property is usually expressed by saying that $\otimes$ *commutes with colimits*. Another often used result is that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence and if M is any R-module, then

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is still exact, see Lemma 10.12.10. Both of these properties tell us that the functor $N \mapsto M \otimes_R N$ *is right exact*. See Categories, Section 4.23 and Homology, Section 12.7. An R-module M is flat if $N \mapsto N \otimes_R M$ is also left exact, i.e., if it is exact. Here is the precise definition.

00HB **Definition 10.39.1.** Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Here is an example of how you can use the flatness condition.

0BBY **Lemma 10.39.2.** *Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.*

Proof. Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Since the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$ we conclude. □

05UT **Lemma 10.39.3.** *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\operatorname{colim}_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact, see Lemma 10.8.8. □

00HC **Lemma 10.39.4.** *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R'-module, then M' is a (faithfully) flat R-module.*

Proof. The first statement of the lemma is a particular case of the second, so it is clearly enough to prove the latter. Let $R \to R'$ be a flat ring map, and M' a flat R'-module. We need to prove that M' is a flat R-module. Let $N_1 \to N_2 \to N_3$ be an exact complex of R-modules. Then, the complex $R' \otimes_R N_1 \to R' \otimes_R N_2 \to R' \otimes_R N_3$ is exact (since R' is flat as an R-module), and so the complex $M' \otimes_{R'} (R' \otimes_R N_1) \to M' \otimes_{R'} (R' \otimes_R N_2) \to M' \otimes_{R'} (R' \otimes_R N_3)$ is exact (since M' is a flat R'-module). Since $M' \otimes_{R'} (R' \otimes_R N) \cong (M' \otimes_{R'} R') \otimes_R N \cong M' \otimes_R N$ for any R-module N functorially (by Lemmas 10.12.7 and 10.12.3), this complex is isomorphic to the complex $M' \otimes_R N_1 \to M' \otimes_R N_2 \to M' \otimes_R N_3$, which is therefore also exact. This shows that M' is a flat R-module. Tracing this argument backwards, we can show that if $R \to R'$ is faithfully flat, and if M' is faithfully flat as an R'-module, then M' is faithfully flat as an R-module. □

00HD **Lemma 10.39.5.** *Let M be an R-module. The following are equivalent:*

00HE (1) *M is flat over R.*

00HF (2) *for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.*

00HG (3) *for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

00HH (4) *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

Proof. The implications (1) implies (2) implies (3) implies (4) are all trivial. Thus we prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \operatorname{Ker}(N_2 \to N_3)$ and $Q = \operatorname{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done[3]. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

[3]Here is the argument in more detail: Assume that we know that the second and fourth arrows are injective. Lemma 10.12.10 (applied to the exact sequence $K \to N_2 \to Q \to 0$) yields that the sequence $K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to 0$ is exact. Hence, $\operatorname{Ker}(N_2 \otimes_R M \to Q \otimes_R M) = \operatorname{Im}(K \otimes_R M \to N_2 \otimes_R M)$. Since $\operatorname{Im}(K \otimes_R M \to N_2 \otimes_R M) = \operatorname{Im}(N_1 \otimes_R M \to N_2 \otimes_R M)$ (due to the surjectivity of $N_1 \otimes_R M \to K \otimes_R M$) and $\operatorname{Ker}(N_2 \otimes_R M \to Q \otimes_R M) = \operatorname{Ker}(N_2 \otimes_R M \to N_3 \otimes_R M)$ (due to the injectivity of $Q \otimes_R M \to N_3 \otimes_R M$), this becomes $\operatorname{Ker}(N_2 \otimes_R M \to N_3 \otimes_R M) = \operatorname{Im}(N_1 \otimes_R M \to N_2 \otimes_R M)$, which shows that the functor $- \otimes_R M$ is exact, whence M is flat.

Assume $K \to N$ is an injective R-module map and let $x \in \mathrm{Ker}(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes_R M$ is the colimit of R-modules of the form $K_i \otimes_R M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes_R M$. Thus we may assume that K is a finite R-module. Assume this. We regard the injection $K \to N$ as an inclusion, so that $K \subset N$.

The R-module N is the union of its finite R-submodules that contain K. Hence, $N \otimes_R M$ is the colimit of R-modules of the form $N_i \otimes_R M$ where N_i runs over all finite R-submodules of N that contain K (again since tensor product commutes with colimits). Notice that this is a colimit over a directed system (since the sum of two finite submodules of N is again finite). Hence, (by Lemma 10.8.4) the element $x \in K \otimes_R M$ maps to zero in at least one of these R-modules $N_i \otimes_R M$ (since x maps to zero in $N \otimes_R M$). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subset L' \subset R^{\oplus n}$. For any R-submodule $G \subset R^{\oplus n}$, we have a canonical map $G \otimes_R M \to M^{\oplus n}$ obtained by composing $G \otimes_R M \to R^n \otimes_R M = M^{\oplus n}$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M/L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too[4].

Thus it suffices to show that $L \otimes_R M \to M^{\oplus n}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{array}{ccccccccc} & & L' \otimes_R M & \longrightarrow & L \otimes_R M & \longrightarrow & L'' \otimes_R M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M^{\oplus n} & \longrightarrow & M^{\oplus n-1} & \longrightarrow & 0 \end{array}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of the finitely generated ideals I_α contained in it. In other words, $I = \mathrm{colim}\, I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \mathrm{colim}\, I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. □

05UU **Lemma 10.39.6.** *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings over the directed set I. Let $R = \mathrm{colim}_i R_i$.*

(1) *If M is an R-module such that M is flat as an R_i-module for all i, then M is flat as an R-module.*

[4]This becomes obvious if we identify $L' \otimes_R M$ and $L \otimes_R M$ with submodules of $M^{\oplus n}$ (which is legitimate since the maps $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective and commute with the obvious map $L' \otimes_R M \to L \otimes_R M$).

(2) *For $i \in I$ let M_i be a flat R_i-module and for $i' \geq i$ let $f_{ii'} : M_i \to M_{i'}$ be a $\varphi_{ii'}$-linear map such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$. Then $M = \operatorname{colim}_{i \in I} M_i$ is a flat R-module.*

Proof. Part (1) is a special case of part (2) with $M_i = M$ for all i and $f_{ii'} = \mathrm{id}_M$. Proof of (2). Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 10.39.5 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}' R$. Then $\mathfrak{a} = \operatorname{colim}_{i' \geq i} \mathfrak{a}' R_{i'}$. Since $\otimes$ commutes with colimits the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}' R_{i'} \otimes_{R_{i'}} M_{i'} \longrightarrow M_{i'}$$

These maps are all injective by assumption. Since colimits over I are exact by Lemma 10.8.8 we win. □

00HI **Lemma 10.39.7.** *Suppose that M is (faithfully) flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.*

Proof. For any R'-module N we have a canonical isomorphism $N \otimes_{R'} (R' \otimes_R M) = N \otimes_R M$. Hence the desired exactness properties of the functor $- \otimes_{R'} (R' \otimes_R M)$ follow from the corresponding exactness properties of the functor $- \otimes_R M$. □

00HJ **Lemma 10.39.8.** *Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.*

Proof. By Lemma 10.39.7 we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. □

0584 **Lemma 10.39.9.** *Let R be a ring. Let $S \to S'$ be a flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$.*

(1) *If M is flat over R, then M' is flat over R.*
(2) *If $S \to S'$ is faithfully flat, then M is flat over R if and only if M' is flat over R.*

Proof. Let $N \to N'$ be an injection of R-modules. By the flatness of $S \to S'$ we have

$$\operatorname{Ker}(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \operatorname{Ker}(N \otimes_R M' \to N' \otimes_R M')$$

If M is flat over R, then the left hand side is zero and we find that M' is flat over R by the second characterization of flatness in Lemma 10.39.5. If M' is flat over R then we have the vanishing of the right hand side and if in addition $S \to S'$ is faithfully flat, this implies that $\operatorname{Ker}(N \otimes_R M \to N' \otimes_R M)$ is zero which in turn shows that M is flat over R. □

039V **Lemma 10.39.10.** *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. □

Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is *trivial* if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

00HK **Lemma 10.39.11** (Equational criterion of flatness)**.** *A module M over R is flat if and only if every relation in M is trivial.*

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation in M. Let $I = (f_1, \ldots, f_n)$, and let $K = \mathrm{Ker}(R^n \to I, (a_1, \ldots, a_n) \mapsto \sum_i a_i f_i)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes x_i \in R^n \otimes_R M$ where e_i is the ith basis element of R^n. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum f_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j \right) = \sum \left(\sum f_i a_{ij} \right) \otimes y_j = 0$$

□

00HL **Lemma 10.39.12.** *Suppose that R is a ring, $0 \to M'' \to M' \to M \to 0$ a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence*

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

Proof. Let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows from the snake lemma applied to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (M'')^{(I)} & \to & (M')^{(I)} & \to & M^{(I)} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R K & \to & M' \otimes_R K & \to & M \otimes_R K & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact. □

00HM **Lemma 10.39.13.** *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.*

Proof. We will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 10.39.5. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M' \otimes_R I & \to & M \otimes_R I & \to & M'' \otimes_R I & \to & 0 \end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by Lemma 10.39.12. □

00HO **Lemma 10.39.14.** *Let R be a ring. Let M be an R-module. The following are equivalent*

(1) *M is faithfully flat, and*
(2) *M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.*

Proof. If M is faithfully flat, then $0 \to \mathrm{Ker}(\alpha) \to N \to N'$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \mathrm{Ker}(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. □

00HP **Lemma 10.39.15.** *Let M be a flat R-module. The following are equivalent:*

(1) *M is faithfully flat,*
(2) *for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,*
(3) *for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and*
(4) *for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.*

Proof. Assume M faithfully flat and $N \neq 0$. By Lemma 10.39.14 the nonzero map $1 : N \to N$ induces a nonzero map $M \otimes_R N \to M \otimes_R N$, so $M \otimes_R N \neq 0$. Thus (1) implies (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate.

Assume (4). Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \mathrm{Ker}(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. To finish the proof we will show $H = 0$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. □

00HQ **Lemma 10.39.16.** *Let $R \to S$ be a flat ring map. The following are equivalent:*

(1) *$R \to S$ is faithfully flat,*
(2) *the induced map on* Spec *is surjective, and*

(3) *any closed point* $x \in \mathrm{Spec}(R)$ *is in the image of the map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof. This follows quickly from Lemma 10.39.15, because we saw in Remark 10.18.5 that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

00HR **Lemma 10.39.17.** *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Lemma 10.39.16. □

Flatness meshes well with localization.

00HT **Lemma 10.39.18.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

(1) *The localization $S^{-1}R$ is a flat R-algebra.*
(2) *If M is an $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.*
(3) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module for all primes $\mathfrak{p}$ of R.*
(4) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.*
(5) *Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.*
(6) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_\mathfrak{q}$ is a flat $R_\mathfrak{p}$-module (with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$) for all primes $\mathfrak{q}$ of A.*
(7) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_\mathfrak{m}$ is a flat $R_\mathfrak{p}$-module (with $\mathfrak{p} = R \cap \mathfrak{m}$) for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Let us prove the last statement of the lemma. In the proof we will use repeatedly that localization is exact and commutes with tensor product, see Sections 10.9 and 10.12.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_\mathfrak{m}$ is a flat $R_\mathfrak{p}$-module for all maximal ideals $\mathfrak{m}$ of A (with $\mathfrak{p} = R \cap \mathfrak{m}$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_\mathfrak{m} \to M_\mathfrak{m}$ is injective because $(I \otimes_R M)_\mathfrak{m} = I_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{m}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Lemma 10.23.1. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_\mathfrak{p}$ is an ideal. We have to show that $I \otimes_{R_\mathfrak{p}} M_\mathfrak{q} \to M_\mathfrak{q}$ is injective. We can write $I = J_\mathfrak{p}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_\mathfrak{p}} M_\mathfrak{q} \to M_\mathfrak{q}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_\mathfrak{q}$ is a flat $R_\mathfrak{p}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

00HS **Lemma 10.39.19.** *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. By Lemma 10.39.18 the local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat. By Lemma 10.39.17 this local ring map is faithfully flat. By Lemma 10.39.16 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

The property of $R \to S$ described in the lemma is called the "going down property". See Definition 10.41.1.

090N **Lemma 10.39.20.** *Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \operatorname{colim}_i S_i$ is a faithfully flat R-algebra.*

Proof. By Lemma 10.39.3 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Lemma 10.39.16 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \operatorname{colim} S_i/\mathfrak{m}S_i$ is nonzero as well because 1 is not equal to zero. Thus the image of $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Lemma 10.39.16. □

10.40. Supports and annihilators

080S Some very basic definitions and lemmas.

00L1 **Definition 10.40.1.** Let R be a ring and let M be an R-module. The *support of* M is the set

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

0585 **Lemma 10.40.2.** *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \mathit{Supp}(M) = \emptyset.$$

Proof. Actually, Lemma 10.23.1 even shows that $\operatorname{Supp}(M)$ always contains a maximal ideal if M is not zero. □

07T7 **Definition 10.40.3.** Let R be a ring. Let M be an R-module.

(1) Given an element $m \in M$ the *annihilator of* m is the ideal

$$\operatorname{Ann}_R(m) = \operatorname{Ann}(m) = \{f \in R \mid fm = 0\}.$$

(2) The *annihilator of* M is the ideal

$$\operatorname{Ann}_R(M) = \operatorname{Ann}(M) = \{f \in R \mid fm = 0 \ \forall m \in M\}.$$

07T8 **Lemma 10.40.4.** *Let $R \to S$ be a flat ring map. Let M be an R-module and $m \in M$. Then $\mathit{Ann}_R(m)S = \mathit{Ann}_S(m \otimes 1)$. If M is a finite R-module, then $\mathit{Ann}_R(M)S = \mathit{Ann}_S(M \otimes_R S)$.*

Proof. Set $I = \operatorname{Ann}_R(m)$. By definition there is an exact sequence $0 \to I \to R \to M$ where the map $R \to M$ sends f to fm. Using flatness we obtain an exact sequence $0 \to I \otimes_R S \to S \to M \otimes_R S$ which proves the first assertion. If $m_1, \ldots, m_n$ is a set of generators of M then $\operatorname{Ann}_R(M) = \bigcap \operatorname{Ann}_R(m_i)$. Similarly $\operatorname{Ann}_S(M \otimes_R S) = \bigcap \operatorname{Ann}_S(m_i \otimes 1)$. Set $I_i = \operatorname{Ann}_R(m_i)$. Then it suffices to show that $\bigcap_{i=1,\ldots,n}(I_i S) = (\bigcap_{i=1,\ldots,n} I_i)S$. This is Lemma 10.39.2. □

00L2 **Lemma 10.40.5.** *Let R be a ring and let M be an R-module. If M is finite, then $\mathit{Supp}(M)$ is closed. More precisely, if $I = \mathit{Ann}(M)$ is the annihilator of M, then $V(I) = \mathit{Supp}(M)$.*

Proof. We will show that $V(I) = \text{Supp}(M)$.

Suppose $\mathfrak{p} \in \text{Supp}(M)$. Then $M_\mathfrak{p} \neq 0$. Choose an element $m \in M$ whose image in $M_\mathfrak{p}$ is nonzero. Then the annihilator of m is contained in $\mathfrak{p}$ by construction of the localization $M_\mathfrak{p}$. Hence a fortiori $I = \text{Ann}(M)$ must be contained in $\mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \notin \text{Supp}(M)$. Then $M_\mathfrak{p} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. By Lemma 10.9.9 there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i} x_i = 0$ for some $n_i \geq 1$. Hence $f^n M = 0$ for $n = \max\{n_i\}$ as desired. □

0BUR **Lemma 10.40.6.** *Let $R \to R'$ be a ring map and let M be a finite R-module. Then Supp$(M \otimes_R R')$ is the inverse image of Supp(M).*

Proof. Let $\mathfrak{p} \in \text{Supp}(M)$. By Nakayama's lemma (Lemma 10.20.1) we see that

$$M \otimes_R \kappa(\mathfrak{p}) = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$$

is a nonzero $\kappa(\mathfrak{p})$ vector space. Hence for every prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$ we see that

$$(M \otimes_R R')_{\mathfrak{p}'}/\mathfrak{p}'(M \otimes_R R')_{\mathfrak{p}'} = (M \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = M \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is nonzero. This implies $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$. For the converse, if $\mathfrak{p}' \subset R'$ is a prime lying over an arbitrary prime $\mathfrak{p} \subset R$, then

$$(M \otimes_R R')_{\mathfrak{p}'} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}.$$

Hence if $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$ lies over the prime $\mathfrak{p} \subset R$, then $\mathfrak{p} \in \text{Supp}(M)$. □

07Z5 **Lemma 10.40.7.** *Let R be a ring, let M be an R-module, and let $m \in M$. Then $\mathfrak{p} \in V(Ann(m))$ if and only if m does not map to zero in $M_\mathfrak{p}$.*

Proof. We may replace M by $Rm \subset M$. Then (1) $\text{Ann}(m) = \text{Ann}(M)$ and (2) m does not map to zero in $M_\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Supp}(M)$. The result now follows from Lemma 10.40.5. □

051B **Lemma 10.40.8.** *Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then Supp(M) is a closed subset of* Spec(R) *whose complement is quasi-compact.*

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \text{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 10.20.1 we see that

$$M_\mathfrak{p} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \text{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\text{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $\text{Spec}(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. □

00L3 **Lemma 10.40.9.** *Let R be a ring and let M be an R-module.*

(1) *If M is finite then the support of M/IM is Supp$(M) \cap V(I)$.*
(2) *If $N \subset M$, then Supp$(N) \subset$ Supp(M).*
(3) *If Q is a quotient module of M then Supp$(Q) \subset$ Supp(M).*
(4) *If $0 \to N \to M \to Q \to 0$ is a short exact sequence then Supp$(M) =$ Supp$(Q) \cup$ Supp(N).*

Proof. The functors $M \mapsto M_\mathfrak{p}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_\mathfrak{p} \neq 0$ and $I \subset \mathfrak{p}$ implies $(M/IM)_\mathfrak{p} = M_\mathfrak{p}/IM_\mathfrak{p} \neq 0$. This follows from Nakayama's Lemma 10.20.1. □

10.41. Going up and going down

00HU Suppose $\mathfrak{p}$, $\mathfrak{p}'$ are primes of the ring R. Let $X = \mathrm{Spec}(R)$ with the Zariski topology. Denote $x \in X$ the point corresponding to $\mathfrak{p}$ and $x' \in X$ the point corresponding to $\mathfrak{p}'$. Then we have:

$$x' \rightsquigarrow x \Leftrightarrow \mathfrak{p}' \subset \mathfrak{p}.$$

In words: x is a specialization of x' if and only if $\mathfrak{p}' \subset \mathfrak{p}$. See Topology, Section 5.19 for terminology and notation.

00HV **Definition 10.41.1.** Let $\varphi : R \to S$ be a ring map.

(1) We say a $\varphi : R \to S$ satisfies *going up* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$ there exists a prime $\mathfrak{q}'$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}'$ lies over $\mathfrak{p}'$.
(2) We say a $\varphi : R \to S$ satisfies *going down* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}'$ in S lying over $\mathfrak{p}'$ there exists a prime $\mathfrak{q}$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}$ lies over $\mathfrak{p}$.

So far we have see the following cases of this:

(1) An integral ring map satisfies going up, see Lemma 10.36.22.
(2) As a special case finite ring maps satisfy going up.
(3) As a special case quotient maps $R \to R/I$ satisfy going up.
(4) A flat ring map satisfies going down, see Lemma 10.39.19
(5) As a special case any localization satisfies going down.
(6) An extension $R \subset S$ of domains, with R normal and S integral over R satisfies going down, see Proposition 10.38.7.

Here is another case where going down holds.

0407 **Lemma 10.41.2.** *Let $R \to S$ be a ring map. If the induced map $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, then $R \to S$ satisfies going down.*

Proof. Suppose that $\mathfrak{p} \subset \mathfrak{p}' \subset R$ and $\mathfrak{q}' \subset S$ lies over $\mathfrak{p}'$. As φ is open, for every $g \in S$, $g \notin \mathfrak{q}'$ we see that $\mathfrak{p}$ is in the image of $D(g) \subset \mathrm{Spec}(S)$. In other words $S_g \otimes_R \kappa(\mathfrak{p})$ is not zero. Since $S_{\mathfrak{q}'}$ is the directed colimit of these S_g this implies that $S_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemmas 10.9.9 and 10.12.9. Hence $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R)$ as desired. □

00HW **Lemma 10.41.3.** *Let $R \to S$ be a ring map.*

(1) *$R \to S$ satisfies going down if and only if generalizations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.19.4.*
(2) *$R \to S$ satisfies going up if and only if specializations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.19.4.*

Proof. Omitted. □

00HX **Lemma 10.41.4.** *Suppose $R \to S$ and $S \to T$ are ring maps satisfying going down. Then so does $R \to T$. Similarly for going up.*

Proof. According to Lemma 10.41.3 this follows from Topology, Lemma 5.19.5 □

00HY **Lemma 10.41.5.** *Let $R \to S$ be a ring map. Let $T \subset \mathrm{Spec}(R)$ be the image of $\mathrm{Spec}(S)$. If T is stable under specialization, then T is closed.*

Proof. We give two proofs.

First proof. Let $\mathfrak{p} \subset R$ be a prime ideal such that the corresponding point of $\mathrm{Spec}(R)$ is in the closure of T. This means that for every $f \in R$, $f \notin \mathfrak{p}$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $\mathrm{Spec}(S_f)$ in $\mathrm{Spec}(R)$. Hence we conclude that $S_f \neq 0$. In other words, $1 \neq 0$ in the ring S_f. Since $S_\mathfrak{p}$ is the directed colimit of the rings S_f we conclude that $1 \neq 0$ in $S_\mathfrak{p}$. In other words, $S_\mathfrak{p} \neq 0$ and considering the image of $\mathrm{Spec}(S_\mathfrak{p}) \to \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ we see there exists a $\mathfrak{p}' \in T$ with $\mathfrak{p}' \subset \mathfrak{p}$. As we assumed T closed under specialization we conclude $\mathfrak{p}$ is a point of T as desired.

Second proof. Let $I = \mathrm{Ker}(R \to S)$. We may replace R by R/I. In this case the ring map $R \to S$ is injective. By Lemma 10.30.5 all the minimal primes of R are contained in the image T. Hence if T is stable under specialization then it contains all primes. □

00HZ **Lemma 10.41.6.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *Going up holds for $R \to S$, and*
(2) *the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed.*

Proof. It is a general fact that specializations lift along a closed map of topological spaces, see Topology, Lemma 5.19.7. Hence the second condition implies the first.

Assume that going up holds for $R \to S$. Let $V(I) \subset \mathrm{Spec}(S)$ be a closed set. We want to show that the image of $V(I)$ in $\mathrm{Spec}(R)$ is closed. The ring map $S \to S/I$ obviously satisfies going up. Hence $R \to S \to S/I$ satisfies going up, by Lemma 10.41.4. Replacing S by S/I it suffices to show the image T of $\mathrm{Spec}(S)$ in $\mathrm{Spec}(R)$ is closed. By Topology, Lemmas 5.19.2 and 5.19.6 this image is stable under specialization. Thus the result follows from Lemma 10.41.5. □

00I0 **Lemma 10.41.7.** *Let R be a ring. Let $E \subset \mathrm{Spec}(R)$ be a constructible subset.*

(1) *If E is stable under specialization, then E is closed.*
(2) *If E is stable under generalization, then E is open.*

Proof. First proof. The first assertion follows from Lemma 10.41.5 combined with Lemma 10.29.4. The second follows because the complement of a constructible set is constructible (see Topology, Lemma 5.15.2), the first part of the lemma and Topology, Lemma 5.19.2.

Second proof. Since $\mathrm{Spec}(R)$ is a spectral space by Lemma 10.26.2 this is a special case of Topology, Lemma 5.23.6. □

00I1 **Proposition 10.41.8.** *Let $R \to S$ be flat and of finite presentation. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open. More generally this holds for any ring map $R \to S$ of finite presentation which satisfies going down.*

Proof. If $R \to S$ is flat, then $R \to S$ satisfies going down by Lemma 10.39.19. Thus to prove the lemma we may assume that $R \to S$ has finite presentation and satisfies going down.

Since the standard opens $D(g) \subset \mathrm{Spec}(S)$, $g \in S$ form a basis for the topology, it suffices to prove that the image of $D(g)$ is open. Recall that $\mathrm{Spec}(S_g) \to \mathrm{Spec}(S)$ is

a homeomorphism of $\mathrm{Spec}(S_g)$ onto $D(g)$ (Lemma 10.17.6). Since $S \to S_g$ satisfies going down (see above), we see that $R \to S_g$ satisfies going down by Lemma 10.41.4. Thus after replacing S by S_g we see it suffices to prove the image is open. By Chevalley's theorem (Theorem 10.29.10) the image is a constructible set E. And E is stable under generalization because $R \to S$ satisfies going down, see Topology, Lemmas 5.19.2 and 5.19.6. Hence E is open by Lemma 10.41.7. □

037F **Lemma 10.41.9.** *Let k be a field, and let R, S be k-algebras. Let $S' \subset S$ be a sub k-algebra, and let $f \in S' \otimes_k R$. In the commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}((S \otimes_k R)_f) & \longrightarrow & \mathrm{Spec}((S' \otimes_k R)_f) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(R) & \end{array}$$

the images of the diagonal arrows are the same.

Proof. Let $\mathfrak{p} \subset R$ be in the image of the south-west arrow. This means (Lemma 10.18.6) that

$$(S' \otimes_k R)_f \otimes_R \kappa(\mathfrak{p}) = (S' \otimes_k \kappa(\mathfrak{p}))_f$$

is not the zero ring, i.e., $S' \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. The ring map $S' \otimes_k \kappa(\mathfrak{p}) \to S \otimes_k \kappa(\mathfrak{p})$ is injective. Hence also $S \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. Hence $(S \otimes_k R)_f \otimes_R \kappa(\mathfrak{p})$ is not the zero ring. Thus (Lemma 10.18.6) we see that $\mathfrak{p}$ is in the image of the south-east arrow as desired. □

037G **Lemma 10.41.10.** *Let k be a field. Let R and S be k-algebras. The map* $\mathrm{Spec}(S \otimes_k R) \to \mathrm{Spec}(R)$ *is open.*

Proof. Let $f \in S \otimes_k R$. It suffices to prove that the image of the standard open $D(f)$ is open. Let $S' \subset S$ be a finite type k-subalgebra such that $f \in S' \otimes_k R$. The map $R \to S' \otimes_k R$ is flat and of finite presentation, hence the image U of $\mathrm{Spec}((S' \otimes_k R)_f) \to \mathrm{Spec}(R)$ is open by Proposition 10.41.8. By Lemma 10.41.9 this is also the image of $D(f)$ and we win. □

Here is a tricky lemma that is sometimes useful.

00EA **Lemma 10.41.11.** *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume that*

1. *there exists a unique prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, and*
2. *either*
   (a) *going up holds for $R \to S$, or*
   (b) *going down holds for $R \to S$ and there is at most one prime of S above every prime of R.*

Then $S_{\mathfrak{p}} = S_{\mathfrak{q}}$.

Proof. Consider any prime $\mathfrak{q}' \subset S$ which corresponds to a point of $\mathrm{Spec}(S_{\mathfrak{p}})$. This means that $\mathfrak{p}' = R \cap \mathfrak{q}'$ is contained in $\mathfrak{p}$. Here is a picture

$$\begin{array}{ccccc} \mathfrak{q}' & \text{---} & ? & \text{---} & S \\ | & & | & & | \\ \mathfrak{p}' & \text{---} & \mathfrak{p} & \text{---} & R \end{array}$$

Assume (1) and (2)(a). By going up there exists a prime $\mathfrak{q}'' \subset S$ with $\mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}''$ lying over $\mathfrak{p}$. By the uniqueness of $\mathfrak{q}$ we conclude that $\mathfrak{q}'' = \mathfrak{q}$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

Assume (1) and (2)(b). By going down there exists a prime $\mathfrak{q}'' \subset \mathfrak{q}$ lying over $\mathfrak{p}'$. By the uniqueness of primes lying over $\mathfrak{p}'$ we see that $\mathfrak{q}' = \mathfrak{q}''$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

In both cases we conclude that the map $\mathrm{Spec}(S_\mathfrak{q}) \to \mathrm{Spec}(S_\mathfrak{p})$ is bijective. Clearly this means all the elements of $S - \mathfrak{q}$ are all invertible in $S_\mathfrak{p}$, in other words $S_\mathfrak{p} = S_\mathfrak{q}$. □

The following lemma is a generalization of going down for flat ring maps.

080T **Lemma 10.41.12.** *Let $R \to S$ be a ring map. Let N be a finite S-module flat over R. Endow $Supp(N) \subset \mathrm{Spec}(S)$ with the induced topology. Then generalizations lift along $Supp(N) \to \mathrm{Spec}(R)$.*

Proof. The meaning of the statement is as follows. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}' \subset S$ be a prime $\mathfrak{q}' \in \mathrm{Supp}(N)$ Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$, $\mathfrak{q} \in \mathrm{Supp}(N)$ lying over $\mathfrak{p}$. As N is flat over R we see that $N_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}'}$, see Lemma 10.39.18. As $N_{\mathfrak{q}'}$ is finite over $S_{\mathfrak{q}'}$ and not zero since $\mathfrak{q}' \in \mathrm{Supp}(N)$ we see that $N_{\mathfrak{q}'} \otimes_{S_{\mathfrak{q}'}} \kappa(\mathfrak{q}')$ is nonzero by Nakayama's Lemma 10.20.1. Thus $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p}')$ is also not zero. We conclude from Lemma 10.39.15 that $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ is nonzero. Let $J \subset S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ be the annihilator of the finite nonzero module $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$. Since J is a proper ideal we can choose a prime $\mathfrak{q} \subset S$ which corresponds to a prime of $S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})/J$. This prime is in the support of N, lies over $\mathfrak{p}$, and is contained in $\mathfrak{q}'$ as desired. □

10.42. Separable extensions

030I In this section we talk about separability for nonalgebraic field extensions. This is closely related to the concept of geometrically reduced algebras, see Definition 10.43.1.

030O **Definition 10.42.1.** Let K/k be a field extension.

(1) We say K is *separably generated over k* if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the extension $K/k(x_i; i \in I)$ is a separable algebraic extension.
(2) We say K is *separable over k* if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension K'/k is separably generated.

With this awkward definition it is not clear that a separably generated field extension is itself separable. It will turn out that this is the case, see Lemma 10.44.3.

030P **Lemma 10.42.2.** *Let K/k be a separable field extension. For any subextension $K/K'/k$ the field extension K'/k is separable.*

Proof. This is direct from the definition. □

030Q **Lemma 10.42.3.** *Let K/k be a separably generated, and finitely generated field extension. Set $r = trdeg_k(K)$. Then there exist elements $x_1, \ldots, x_{r+1}$ of K such that*

(1) *$x_1, \ldots, x_r$ is a transcendence basis of K over k,*

(2) $K = k(x_1, \ldots, x_{r+1})$, *and*
(3) x_{r+1} *is separable over* $k(x_1, \ldots, x_r)$.

Proof. Combine the definition with Fields, Lemma 9.19.1. □

04KM **Lemma 10.42.4.** *Let K/k be a finitely generated field extension. There exists a diagram*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separably generated field extension.

Proof. This lemma is only interesting when the characteristic of k is $p > 0$. Choose $x_1, \ldots, x_r$ a transcendence basis of K over k. As K is finitely generated over k the extension $k(x_1, \ldots, x_r) \subset K$ is finite. Let $K/K_{sep}/k(x_1, \ldots, x_r)$ be the subextension found in Fields, Lemma 9.14.6. If $K = K_{sep}$ then we are done. We will use induction on $d = [K : K_{sep}]$.

Assume that $d > 1$. Choose a $\beta \in K$ with $\alpha = \beta^p \in K_{sep}$ and $\beta \notin K_{sep}$. Let $P = T^n + a_1 T^{n-1} + \ldots + a_n$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. Let k'/k be a finite purely inseparable extension obtained by adjoining pth roots such that each a_i is a pth power in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Such an extension exists; details omitted. Let L be a field fitting into the diagram

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ k(x_1, \ldots, x_r) & \longrightarrow & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \end{array}$$

We may and do assume L is the compositum of K and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Let $L/L_{sep}/k'(x_1^{1/p}, \ldots, x_r^{1/p})$ be the subextension found in Fields, Lemma 9.14.6. Then L_{sep} is the compositum of K_{sep} and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. The element $\alpha \in L_{sep}$ is a zero of the polynomial P all of whose coefficients are pth powers in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$ and whose roots are pairwise distinct. By Fields, Lemma 9.28.2 we see that $\alpha = (\alpha')^p$ for some $\alpha' \in L_{sep}$. Clearly, this means that β maps to $\alpha' \in L_{sep}$. In other

words, we get the tower of fields

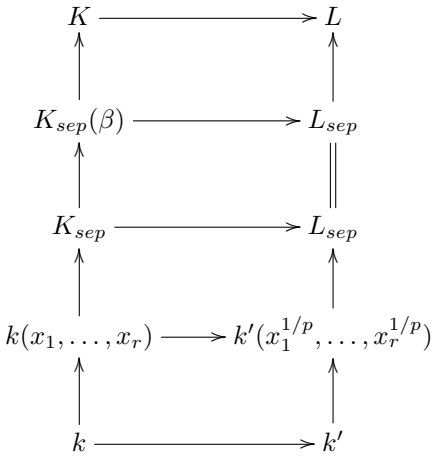

Thus this construction leads to a new situation with $[L : L_{sep}] < [K : K_{sep}]$. By induction we can find $k' \subset k''$ and $L \subset L'$ as in the lemma for the extension L/k'. Then the extensions k''/k and L'/K work for the extension K/k. This proves the lemma. □

10.43. Geometrically reduced algebras

05DS The main result on geometrically reduced algebras is Lemma 10.44.4. We suggest the reader skip to the lemma after reading the definition.

030S **Definition 10.43.1.** Let k be a field. Let S be a k-algebra. We say S is *geometrically reduced over k* if for every field extension K/k the K-algebra $K \otimes_k S$ is reduced.

Let k be a field and let S be a reduced k-algebra. To check that S is geometrically reduced it will suffice to check that $\overline{k} \otimes_k S$ is reduced (where $\overline{k}$ denotes the algebraic closure of k). In fact it is enough to check this for finite purely inseparable field extensions k'/k. See Lemma 10.44.4.

030T **Lemma 10.43.2.** *Elementary properties of geometrically reduced algebras. Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically reduced over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically reduced, then S is geometrically reduced.*
(3) *A directed colimit of geometrically reduced k-algebras is geometrically reduced.*
(4) *If S is geometrically reduced over k, then any localization of S is geometrically reduced over k.*

Proof. Omitted. The second and third property follow from the fact that tensor product commutes with colimits. □

04KN **Lemma 10.43.3.** *Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k. If R is geometrically reduced over k, then $R[x]$ is geometrically reduced over k.*

Proof. Omitted. Hints: A localization of a reduced ring is reduced, and localization commutes with tensor products. □

In the proofs of the following lemmas we will repeatedly use the following observation: Suppose that $R' \subset R$ and $S' \subset S$ are inclusions of k-algebras. Then the map $R' \otimes_k S' \to R \otimes_k S$ is injective.

00I3 **Lemma 10.43.4.** *Let k be a field. Let R, S be k-algebras.*

(1) *If $R \otimes_k S$ is nonreduced, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ is not reduced.*
(2) *If $R \otimes_k S$ contains a nonzero zerodivisor, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nonzero zerodivisor.*
(3) *If $R \otimes_k S$ contains a nontrivial idempotent, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nontrivial idempotent.*

Proof. Suppose $z \in R \otimes_k S$ is nilpotent. We may write $z = \sum_{i=1,\ldots,n} x_i \otimes y_i$. Thus we may take R' the k-subalgebra generated by the x_i and S' the k-subalgebra generated by the y_i. The second and third statements are proved in the same way. □

034N **Lemma 10.43.5.** *Let k be a field. Let S be a geometrically reduced k-algebra. Let R be any reduced k-algebra. Then $R \otimes_k S$ is reduced.*

Proof. By Lemma 10.43.4 we may assume that R is of finite type over k. Then R, as a reduced Noetherian ring, embeds into a finite product of fields (see Lemmas 10.25.4, 10.31.6, and 10.25.1). Hence we may assume R is a finite product of fields. In this case it follows from Definition 10.43.1 that $R \otimes_k S$ is reduced. □

030U **Lemma 10.43.6.** *Let k be a field. Let S be a reduced k-algebra. Let K/k be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.*

Proof. Assume $k \subset K$ is separable. By Lemma 10.43.4 we may assume that S is of finite type over k and K is finitely generated over k. Then S embeds into a finite product of fields, namely its total ring of fractions (see Lemmas 10.25.1 and 10.25.4). Hence we may actually assume that S is a domain. We choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 10.42.3. Let $P \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of x_{r+1}. It is a separable polynomial. It is easy to see that $k[x_1, \ldots, x_r] \otimes_k S = S[x_1, \ldots, x_r]$ is a domain. This implies $k(x_1, \ldots, x_r) \otimes_k S$ is a domain as it is a localization of $S[x_1, \ldots, x_r]$. The ring extension $k(x_1, \ldots, x_r) \otimes_k S \subset K \otimes_k S$ is generated by a single element x_{r+1} with a single equation, namely P. Hence $K \otimes_k S$ embeds into $F[T]/(P)$ where F is the fraction field of $k(x_1, \ldots, x_r) \otimes_k S$. Since P is separable this is a finite product of fields and we win.

At this point we do not yet know that a separably generated field extension is separable, so we have to prove the lemma in this case also. To do this suppose that $\{x_i\}_{i \in I}$ is a separating transcendence basis for K over k. For any finite set of elements $\lambda_j \in K$ there exists a finite subset $T \subset I$ such that $k(\{x_i\}_{i \in T}) \subset k(\{x_i\}_{i \in T} \cup \{\lambda_j\})$ is finite separable. Hence we see that K is a directed colimit of finitely generated and separably generated extensions of k. Thus the argument of the preceding paragraph applies to this case as well. □

07K2 **Lemma 10.43.7.** *Let k be a field and let S be a k-algebra. Assume that S is reduced and that $S_\mathfrak{p}$ is geometrically reduced for every minimal prime $\mathfrak{p}$ of S. Then S is geometrically reduced.*

Proof. Since S is reduced the map $S \to \prod_{\mathfrak{p}\text{ minimal}} S_\mathfrak{p}$ is injective, see Lemma 10.25.2. If K/k is a field extension, then the maps

$$S \otimes_k K \to (\prod S_\mathfrak{p}) \otimes_k K \to \prod S_\mathfrak{p} \otimes_k K$$

are injective: the first as $k \to K$ is flat and the second by inspection because K is a free k-module. As $S_\mathfrak{p}$ is geometrically reduced the ring on the right is reduced. Thus we see that $S \otimes_k K$ is reduced as a subring of a reduced ring. □

0C2X **Lemma 10.43.8.** *Let k'/k be a separable algebraic extension. Then there exists a multiplicative subset $S \subset k' \otimes_k k'$ such that the multiplication map $k' \otimes_k k' \to k'$ is identified with $k' \otimes_k k' \to S^{-1}(k' \otimes_k k')$.*

Proof. First assume k'/k is finite separable. Then $k' = k(\alpha)$, see Fields, Lemma 9.19.1. Let $P \in k[x]$ be the minimal polynomial of α over k. Then P is an irreducible, separable, monic polynomial, see Fields, Section 9.12. Then $k'[x]/(P) \to k' \otimes_k k'$, $\sum \alpha_i x^i \mapsto \alpha_i \otimes \alpha^i$ is an isomorphism. We can factor $P = (x - \alpha)Q$ in $k'[x]$ and since P is separable we see that $Q(\alpha) \neq 0$. Then it is clear that the multiplicative set S' generated by Q in $k'[x]/(P)$ works, i.e., that $k' = (S')^{-1}(k'[x]/(P))$. By transport of structure the image S of S' in $k' \otimes_k k'$ works.

In the general case we write $k' = \bigcup k_i$ as the union of its finite subfield extensions over k. For each i there is a multiplicative subset $S_i \subset k_i \otimes_k k_i$ such that $k_i = S_i^{-1}(k_i \otimes_k k_i)$. Then $S = \bigcup S_i \subset k' \otimes_k k'$ works. □

0C2Y **Lemma 10.43.9.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically reduced over k if and only if it is geometrically reduced over k'.*

Proof. Assume A is geometrically reduced over k'. Let K/k be a field extension. Then $K \otimes_k k'$ is a reduced ring by Lemma 10.43.6. Hence by Lemma 10.43.5 we find that $K \otimes_k A = (K \otimes_k k') \otimes_{k'} A$ is reduced.

Assume A is geometrically reduced over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 10.43.8, we see that $K \otimes_{k'} A$ is a localization of a reduced algebra, hence reduced. □

10.44. Separable extensions, continued

05DT In this section we continue the discussion started in Section 10.42.

0H71 **Lemma 10.44.1.** *Let k be a field of characteristic $p > 1$. Let K/k be a field extension generated by $x_1, \ldots, x_{n+1} \in K$ such that*

1. *$\{x_1, \ldots, x_n\}$ is a transcendence base of K/k,*
2. *for every k-linearly independent subset $\{a_1, \ldots, a_m\}$ of K the set $\{a_1^p, \ldots, a_m^p\}$ is k-linearly independent.*

Then there is $1 \leq j \leq n+1$ such that $\{x_1, \ldots, \widehat{x}_j, \ldots, x_{n+1}\}$ is a separating transcendence base for K/k.

Proof. By assumption x_{n+1} is algebraic over $k(x_1, \ldots, x_n)$ so there exists a non-zero polynomial $F \in k[X_1, \ldots, X_{n+1}]$ such that $F(x_1, \ldots, x_{n+1}) = 0$. Choose F of minimal total degree. Then F is irreducible, because at least one irreducible factor must also have the same property.

We claim that, for some i, not all powers of X_i appearing in F are multiples of p. Suppose for a contradiction that all the exponents appearing in F were multiples of p, then the set

$$\{x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}} \mid \lambda_\alpha \neq 0\} \subset K$$

is k-linearly dependent where λ_α are the coefficients of F. By assumption (2) we conclude the set

$$\{x_1^{\alpha_1/p} \ldots x_{n+1}^{\alpha_{n+1}/p} \mid \lambda_\alpha \neq 0\}$$

is also k-linearly dependent, contradicting minimality of $\deg(F)$.

Choose i for which a non-pth power of X_i appears in F. Then we see that x_i is algebraic over $L = k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$. By Fields, Lemma 9.26.3 we see that $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}$ is a transcendence base of K/k. Thus L is the fraction field of the polynomial ring over k in $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}$. By Gauss' Lemma we conclude that

$$P(T) = F(x_1, \ldots, x_{i-1}, T, x_{i+1}, \ldots, x_{n+1}) \in L[T]$$

is irreducible. By construction $P(T)$ is not contained in $L[T^p]$. Hence K/L is separable as required. □

Let p be a prime number and let k be a field of characteristic p. In this case we write $k^{1/p}$ for the extension of k gotten by adjoining pth roots of all the elements of k to k. (In other words it is the subfield of an algebraic closure of k generated by the pth roots of elements of k.)

030W **Lemma 10.44.2.** *Let k be a field of characteristic $p > 0$. Let K/k be a field extension. The following are equivalent:*

1. *K is separable over k,*
2. *for every k-linearly independent subset $\{a_1, \ldots, a_m\}$ of K the set $\{a_1^p, \ldots, a_m^p\}$ is k-linearly independent,*
3. *the ring $K \otimes_k k^{1/p}$ is reduced, and*
4. *K is geometrically reduced over k.*

Proof. The implication (1) ⇒ (4) follows from Lemma 10.43.6. The implication (4) ⇒ (3) is immediate.

Assume (3). Consider the ring homomorphism $m : K \otimes_k k^{1/p} \to K$ given by

$$\lambda \otimes \mu \to \lambda^p \mu^p$$

Note that $x^p = m(x) \otimes 1$ for all $x \in K \otimes_k k^{1/p}$. Since $K \otimes_k k^{1/p}$ is reduced we see m is injective. If $\{a_1, \ldots, a_m\} \subset K$ is k-linearly independent, then $\{a_1 \otimes 1, \ldots, a_m \otimes 1\}$ is $k^{1/p}$-linearly independent. By injectivity of m we deduce that no nontrivial k-linear combination of $a_1^p, \ldots, a_m^p$ is is zero. Hence (3) implies (2).

Assume (2). To prove (1) we may assume that K is finitely generated over k and we have to prove that K is separably generated over k. Let $\{x_1, \ldots, x_d\}$ be a transcendence base of K/k. By Fields, Lemma 9.8.6 we have $[K : K'] < \infty$ where $K' = k(x_1, \ldots, x_d)$. Choose the transcendence base such that the degree of inseparability $[K : K']_i$ is minimal. If K/K' is separable then we win. Assume

this is not the case to get a contradiction. Then there exists $x_{d+1} \in K$ which is not separable over K', and in particular $[K'(x_{d+1}) : K']_i > 1$. Then by Lemma 10.44.1 there is $1 \leq j \leq n+1$ such that $K'' = k(x_1, \ldots, \widehat{x}_j, \ldots, x_{d+1})$ satisfies $[K'(x_{d+1}) : K'']_i = 1$. By multiplicativity $[K : K'']_i < [K : K']_i$ and we obtain the contradiction. □

030X **Lemma 10.44.3.** *A separably generated field extension is separable.*

Proof. Combine Lemma 10.43.6 with Lemma 10.44.2. □

In the following lemma we will use the notion of the perfect closure which is defined in Definition 10.45.5.

030V **Lemma 10.44.4.** *Let k be a field. Let S be a k-algebra. The following are equivalent:*

(1) *$k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k,*
(2) *$k^{1/p} \otimes_k S$ is reduced,*
(3) *$k^{perf} \otimes_k S$ is reduced, where k^{perf} is the perfect closure of k,*
(4) *$\overline{k} \otimes_k S$ is reduced, where $\overline{k}$ is the algebraic closure of k, and*
(5) *S is geometrically reduced over k.*

Proof. Note that any finite purely inseparable extension k'/k embeds in k^{perf}. Moreover, $k^{1/p}$ embeds into k^{perf} which embeds into $\overline{k}$. Thus it is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and that (3) $\Rightarrow$ (1).

We prove that (1) $\Rightarrow$ (5). Assume $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k. Let K/k be an extension of fields. We have to show that $K \otimes_k S$ is reduced. By Lemma 10.43.4 we reduce to the case where K/k is a finitely generated field extension. Choose a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

as in Lemma 10.42.4. By assumption $k' \otimes_k S$ is reduced. By Lemma 10.43.6 it follows that $K' \otimes_k S$ is reduced. Hence we conclude that $K \otimes_k S$ is reduced as desired.

Finally we prove that (2) $\Rightarrow$ (5). Assume $k^{1/p} \otimes_k S$ is reduced. Then S is reduced. Moreover, for each localization $S_{\mathfrak{p}}$ at a minimal prime $\mathfrak{p}$, the ring $k^{1/p} \otimes_k S_{\mathfrak{p}}$ is a localization of $k^{1/p} \otimes_k S$ hence is reduced. But $S_{\mathfrak{p}}$ is a field by Lemma 10.25.1, hence $S_{\mathfrak{p}}$ is geometrically reduced by Lemma 10.44.2. It follows from Lemma 10.43.7 that S is geometrically reduced. □

10.45. Perfect fields

05DU Here is the definition.

030Y **Definition 10.45.1.** Let k be a field. We say k is *perfect* if every field extension of k is separable over k.

030Z **Lemma 10.45.2.** *A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a pth root.*

Proof. The characteristic zero case is clear. Assume the characteristic of k is $p > 0$. If k is perfect, then all the field extensions where we adjoin a pth root of an element of k have to be trivial, hence every element of k has a pth root. Conversely if every element has a pth root, then $k = k^{1/p}$ and every field extension of k is separable by Lemma 10.44.2. □

030R **Lemma 10.45.3.** *Let K/k be a finitely generated field extension. There exists a diagram*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separable field extension. In this situation we can assume that $K' = k'K$ is the compositum, and also that $K' = (k' \otimes_k K)_{red}$.

Proof. By Lemma 10.42.4 we can find such a diagram with K'/k' separably generated. By Lemma 10.44.3 this implies that K' is separable over k'. The compositum $k'K$ is a subextension of K'/k' and hence $k' \subset k'K$ is separable by Lemma 10.42.2. The ring $(k' \otimes_k K)_{red}$ is a domain as for some $n \gg 0$ the map $x \mapsto x^{p^n}$ maps it into K. Hence it is a field by Lemma 10.36.19. Thus $(k' \otimes_k K)_{red} \to K'$ maps it isomorphically onto $k'K$. □

046W **Lemma 10.45.4.** *For every field k there exists a purely inseparable extension k'/k such that k' is perfect. The field extension k'/k is unique up to unique isomorphism.*

Proof. If the characteristic of k is zero, then $k' = k$ is the unique choice. Assume the characteristic of k is $p > 0$. For every $n > 0$ there exists a unique algebraic extension $k \subset k^{1/p^n}$ such that (a) every element $\lambda \in k$ has a p^nth root in k^{1/p^n} and (b) for every element $\mu \in k^{1/p^n}$ we have $\mu^{p^n} \in k$. Namely, consider the ring map $k \to k^{1/p^n} = k$, $x \mapsto x^{p^n}$. This is injective and satisfies (a) and (b). It is clear that $k^{1/p^n} \subset k^{1/p^{n+1}}$ as extensions of k via the map $y \mapsto y^p$. Then we can take $k' = \bigcup k^{1/p^n}$. Some details omitted. □

046X **Definition 10.45.5.** Let k be a field. The field extension k'/k of Lemma 10.45.4 is called the *perfect closure* of k. Notation k^{perf}/k.

Note that if k'/k is any algebraic purely inseparable extension, then k' is a subextension of k^{perf}, i.e., $k^{perf}/k'/k$. Namely, $(k')^{perf}$ is isomorphic to k^{perf} by the uniqueness of Lemma 10.45.4.

00I4 **Lemma 10.45.6.** *Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.*

Proof. The first statement follows from Lemma 10.44.4. For the second statement use the first statement and Lemma 10.43.5. □

10.46. Universal homeomorphisms

0BR5 Let k'/k be an algebraic purely inseparable field extension. Then for any k-algebra R the ring map $R \to k' \otimes_k R$ induces a homeomorphism of spectra. The reason for this is the slightly more general Lemma 10.46.7 below.

0BR6 **Lemma 10.46.1.** *Let $\varphi : R \to S$ be a surjective map with locally nilpotent kernel. Then φ induces a homeomorphism of spectra and isomorphisms on residue fields. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ is surjective with locally nilpotent kernel.*

Proof. By Lemma 10.17.7 the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto the closed subset $V(\mathrm{Ker}(\varphi))$. Of course $V(\mathrm{Ker}(\varphi)) = \mathrm{Spec}(R)$ because every prime ideal of R contains every nilpotent element of R. This also implies the statement on residue fields. By right exactness of tensor product we see that $\mathrm{Ker}(\varphi)R'$ is the kernel of the surjective map $R' \to R' \otimes_R S$. Hence the final statement by Lemma 10.32.3. □

0BR7 **Lemma 10.46.2.** *Let k'/k be a field extension. The following are equivalent* [**Alp14**, Lemma 3.1.6]

(1) *for each $x \in k'$ there exists an $n > 0$ such that $x^n \in k$, and*
(2) *$k' = k$ or k and k' have characteristic $p > 0$ and either k'/k is a purely inseparable extension or k and k' are algebraic extensions of $\mathbf{F}_p$.*

Proof. Observe that each of the possibilities listed in (2) satisfies (1). Thus we assume k'/k satisfies (1) and we prove that we are in one of the cases of (2). Discarding the case $k = k'$ we may assume $k' \neq k$. It is clear that k'/k is algebraic. Hence we may assume that k'/k is a nontrivial finite extension. Let $k'/k'_{sep}/k$ be the separable subextension found in Fields, Lemma 9.14.6. We have to show that $k = k'_{sep}$ or that k is an algebraic over $\mathbf{F}_p$. Thus we may assume that k'/k is a nontrivial finite separable extension and we have to show k is algebraic over $\mathbf{F}_p$.

Pick $x \in k'$, $x \notin k$. Pick $n, m > 0$ such that $x^n \in k$ and $(x+1)^m \in k$. Let $\overline{k}$ be an algebraic closure of k. We can choose embeddings $\sigma, \tau : k' \to \overline{k}$ with $\sigma(x) \neq \tau(x)$. This follows from the discussion in Fields, Section 9.12 (more precisely, after replacing k' by the k-extension generated by x it follows from Fields, Lemma 9.12.8). Then we see that $\sigma(x) = \zeta\tau(x)$ for some nth root of unity ζ in $\overline{k}$. Similarly, we see that $\sigma(x+1) = \zeta'\tau(x+1)$ for some mth root of unity $\zeta' \in \overline{k}$. Since $\sigma(x+1) \neq \tau(x+1)$ we see $\zeta' \neq 1$. Then

$$\zeta'(\tau(x) + 1) = \zeta'\tau(x+1) = \sigma(x+1) = \sigma(x) + 1 = \zeta\tau(x) + 1$$

implies that

$$\tau(x)(\zeta' - \zeta) = 1 - \zeta'$$

hence $\zeta' \neq \zeta$ and

$$\tau(x) = (1 - \zeta')/(\zeta' - \zeta)$$

Hence every element of k' which is not in k is algebraic over the prime subfield. Since k' is generated over the prime subfield by the elements of k' which are not in k, we conclude that k' (and hence k) is algebraic over the prime subfield.

Finally, if the characteristic of k is 0, the above leads to a contradiction as follows (we encourage the reader to find their own proof). For every rational number y we similarly get a root of unity ζ_y such that $\sigma(x+y) = \zeta_y\tau(x+y)$. Then we find

$$\zeta\tau(x) + y = \zeta_y(\tau(x) + y)$$

and by our formula for $\tau(x)$ above we conclude $\zeta_y \in \mathbf{Q}(\zeta, \zeta')$. Since the number field $\mathbf{Q}(\zeta, \zeta')$ contains only a finite number of roots of unity we find two distinct rational numbers y, y' with $\zeta_y = \zeta_{y'}$. Then we conclude that

$$y - y' = \sigma(x+y) - \sigma(x+y') = \zeta_y(\tau(x+y)) - \zeta_{y'}\tau(x+y') = \zeta_y(y - y')$$

which implies $\zeta_y = 1$ a contradiction. □

0BR8 **Lemma 10.46.3.** *Let $\varphi : R \to S$ be a ring map. If*

(1) *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and*
(2) $\mathrm{Ker}(\varphi)$ *is locally nilpotent,*

then φ induces a homeomorphism on spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 10.46.2.

Proof. Assume (1) and (2). Let $\mathfrak{q}, \mathfrak{q}'$ be primes of S lying over the same prime ideal $\mathfrak{p}$ of R. Suppose $x \in S$ with $x \in \mathfrak{q}$, $x \notin \mathfrak{q}'$. Then $x^n \in \mathfrak{q}$ and $x^n \notin \mathfrak{q}'$ for all $n > 0$. If $x^n = \varphi(y)$ with $y \in R$ for some $n > 0$ then

$$x^n \in \mathfrak{q} \Rightarrow y \in \mathfrak{p} \Rightarrow x^n \in \mathfrak{q}'$$

which is a contradiction. Hence there does not exist an x as above and we conclude that $\mathfrak{q} = \mathfrak{q}'$, i.e., the map on spectra is injective. By assumption (2) the kernel $I = \mathrm{Ker}(\varphi)$ is contained in every prime, hence $\mathrm{Spec}(R) = \mathrm{Spec}(R/I)$ as topological spaces. As the induced map $R/I \to S$ is integral by assumption (1) Lemma 10.36.17 shows that $\mathrm{Spec}(S) \to \mathrm{Spec}(R/I)$ is surjective. Combining the above we see that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective. If $x \in S$ is arbitrary, and we pick $y \in R$ such that $\varphi(y) = x^n$ for some $n > 0$, then we see that the open $D(x) \subset \mathrm{Spec}(S)$ corresponds to the open $D(y) \subset \mathrm{Spec}(R)$ via the bijection above. Hence we see that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism.

To see the statement on residue fields, let $\mathfrak{q} \subset S$ be a prime lying over a prime ideal $\mathfrak{p} \subset R$. Let $x \in \kappa(\mathfrak{q})$. If we think of $\kappa(\mathfrak{q})$ as the residue field of the local ring $S_\mathfrak{q}$, then we see that x is the image of some $y/z \in S_\mathfrak{q}$ with $y \in S$, $z \in S$, $z \notin \mathfrak{q}$. Choose $n, m > 0$ such that y^n, z^m are in the image of φ. Then x^{nm} is the residue of $(y/z)^{nm} = (y^n)^m/(z^m)^n$ which is in the image of $R_\mathfrak{p} \to S_\mathfrak{q}$. Hence x^{nm} is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$. □

0EUH **Lemma 10.46.4.** *Let $\varphi : R \to S$ be a ring map. Assume*

(a) *S is generated as an R-algebra by elements x such that $x^2, x^3 \in \varphi(R)$, and*
(b) $\mathrm{Ker}(\varphi)$ *is locally nilpotent,*

Then φ induces isomorphisms on residue fields and a homeomorphism of spectra. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). The map on spectra is closed as S is integral over R, see Lemmas 10.41.6 and 10.36.22. The image is dense by Lemma 10.30.6. Thus $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. If $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ then the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is generated by elements $\alpha \in \kappa(\mathfrak{q})$ whose square and cube are in $\kappa(\mathfrak{p})$. Thus clearly $\alpha \in \kappa(\mathfrak{p})$ and we find that $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$. If $\mathfrak{q}, \mathfrak{q}'$ were two distinct primes lying over $\mathfrak{p}$, then at least one of the generators x of S as in (a) would have distinct images in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q}') = \kappa(\mathfrak{p})$. This would contradict the fact that both x^2 and x^3 do have the same image. This proves that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective hence a homeomorphism (by what was already shown).

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 10.30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 10.30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. □

0545 **Lemma 10.46.5.** *Let p be a prime number. Let $n, m > 0$ be two integers. There exists an integer a such that $(x+y)^{p^a}, p^a(x+y) \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$.*

Proof. This is clear for $p^a(x+y)$ as soon as $a \geq n, m$. In fact, pick $a \gg n, m$. Write

$$(x+y)^{p^a} = \sum\nolimits_{i,j \geq 0, i+j=p^a} \binom{p^a}{i,j} x^i y^j$$

For every $i, j \geq 0$ with $i + j = p^a$ write $i = qp^n + r$ with $r \in \{0, \ldots, p^n - 1\}$ and $j = q'p^m + r'$ with $r' \in \{0, \ldots, p^m - 1\}$. The condition $(x+y)^{p^a} \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$ holds if

$$p^{nr+mr'} \text{ divides } \binom{p^a}{i,j}$$

If $r = r' = 0$ then the divisibility holds. If $r \neq 0$, then we write

$$\binom{p^a}{i,j} = \frac{p^a}{i}\binom{p^a - 1}{i-1, j}$$

Since $r \neq 0$ the rational number p^a/i has p-adic valuation at least $a-(n-1)$ (because i is not divisible by p^n). Thus $\binom{p^a}{i,j}$ is divisible by p^{a-n+1} in this case. Similarly, we see that if $r' \neq 0$, then $\binom{p^a}{i,j}$ is divisible by p^{a-m+1}. Picking $a = np^n + mp^m + n + m$ will work. □

0BR9 **Lemma 10.46.6.** *Let k'/k be a field extension. Let p be a prime number. The following are equivalent*

1. *k' is generated as a field extension of k by elements x such that there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$, and*
2. *$k = k'$ or the characteristic of k and k' is p and k'/k is purely inseparable.*

Proof. Let $x \in k'$. If there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$ and if the characteristic is not p, then $x \in k$. If the characteristic is p, then we find $x^{p^n} \in k$ and hence x is purely inseparable over k. □

0BRA **Lemma 10.46.7.** *Let $\varphi : R \to S$ be a ring map. Let p be a prime number. Assume*

(a) *S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
(b) *$\mathrm{Ker}(\varphi)$ is locally nilpotent,*

Then φ induces a homeomorphism of spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 10.46.6. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). Note that (b) is equivalent to condition (2) of Lemma 10.46.3. Let $T \subset S$ be the set of elements $x \in S$ such that there exists an integer $n > 0$ such that $x^{p^n}, p^n x \in \varphi(R)$. We claim that $T = S$. This will prove that condition (1) of Lemma 10.46.3 holds and hence φ induces a homeomorphism on spectra. By assumption (a) it suffices to show that $T \subset S$ is an R-sub algebra. If $x \in T$ and $y \in R$, then it is clear that $yx \in T$. Suppose $x, y \in T$ and $n, m > 0$ such that $x^{p^n}, y^{p^m}, p^n x, p^m y \in \varphi(R)$. Then $(xy)^{p^{n+m}}, p^{n+m} xy \in \varphi(R)$ hence $xy \in T$. We have $x + y \in T$ by Lemma 10.46.5 and the claim is proved.

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 10.30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of

nilpotent elements, see Lemma 10.30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. Finally, the condition on residue fields follows from (a) as generators for S as an R-algebra map to generators for the residue field extensions. □

0BRB **Lemma 10.46.8.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ induces an injective map of spectra,*
(2) *φ induces purely inseparable residue field extensions.*

Then for any ring map $R \to R'$ properties (1) and (2) are true for $R' \to R' \otimes_R S$.

Proof. Set $S' = R' \otimes_R S$ so that we have a commutative diagram of continuous maps of spectra of rings

$$\begin{array}{ccc} \mathrm{Spec}(S') & \longrightarrow & \mathrm{Spec}(S) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over $\mathfrak{p} \subset R$. If there is no prime ideal of S lying over $\mathfrak{p}$, then there is no prime ideal of S' lying over $\mathfrak{p}'$. Otherwise, by Remark 10.18.5 there is a unique prime ideal $\mathfrak{r}$ of $F = S \otimes_R \kappa(\mathfrak{p})$ whose residue field is purely inseparable over $\kappa(\mathfrak{p})$. Consider the ring maps

$$\kappa(\mathfrak{p}) \to F \to \kappa(\mathfrak{r})$$

By Lemma 10.25.1 the ideal $\mathfrak{r} \subset F$ is locally nilpotent, hence we may apply Lemma 10.46.1 to the ring map $F \to \kappa(\mathfrak{r})$. We may apply Lemma 10.46.7 to the ring map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{r})$. Hence the composition and the second arrow in the maps

$$\kappa(\mathfrak{p}') \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{r})$$

induces bijections on spectra and purely inseparable residue field extensions. This implies the same thing for the first map. Since

$$\kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F = \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}) \otimes_R S = \kappa(\mathfrak{p}') \otimes_R S = \kappa(\mathfrak{p}') \otimes_{R'} R' \otimes_R S$$

we conclude by the discussion in Remark 10.18.5. □

0BRC **Lemma 10.46.9.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is integral,*
(2) *φ induces an injective map of spectra,*
(3) *φ induces purely inseparable residue field extensions.*

Then φ induces a homeomorphism from $\mathrm{Spec}(S)$ onto a closed subset of $\mathrm{Spec}(R)$ and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.

Proof. The map on spectra is closed by Lemmas 10.41.6 and 10.36.22. The properties are preserved under base change by Lemmas 10.46.8 and 10.36.13. □

0BRD **Lemma 10.46.10.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is integral,*
(2) *φ induces an bijective map of spectra,*
(3) *φ induces purely inseparable residue field extensions.*

Then φ induces a homeomorphism on spectra and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.

Proof. Follows from Lemmas 10.46.9 and 10.30.3. □

09EF **Lemma 10.46.11.** *Let $\varphi : R \to S$ be a ring map such that*

(1) *the kernel of φ is locally nilpotent, and*
(2) *S is generated as an R-algebra by elements x such that there exist $n > 0$ and a polynomial $P(T) \in R[T]$ whose image in $S[T]$ is $(T - x)^n$.*

Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is a homeomorphism and $R \to S$ induces purely inseparable extensions of residue fields. Moreover, conditions (1) and (2) remain true on arbitrary base change.*

Proof. We may replace R by $R/\operatorname{Ker}(\varphi)$, see Lemma 10.46.1. Assumption (2) implies S is generated over R by elements which are integral over R. Hence $R \subset S$ is integral (Lemma 10.36.7). In particular $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective and closed (Lemmas 10.36.17, 10.41.6, and 10.36.22).

Let $x \in S$ be one of the generators in (2), i.e., there exists an $n > 0$ be such that $(T - x)^n \in R[T]$. Let $\mathfrak{p} \subset R$ be a prime. The $\kappa(\mathfrak{p}) \otimes_R S$ ring is nonzero by the above and Lemma 10.18.6. If the characteristic of $\kappa(\mathfrak{p})$ is zero then we see that $nx \in R$ implies $1 \otimes x$ is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. Hence $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$ is an isomorphism. If the characteristic of $\kappa(\mathfrak{p})$ is $p > 0$, then write $n = p^k m$ with m prime to p. In $\kappa(\mathfrak{p}) \otimes_R S[T]$ we have

$$(T - 1 \otimes x)^n = ((T - 1 \otimes x)^{p^k})^m = (T^{p^k} - 1 \otimes x^{p^k})^m$$

and we see that $mx^{p^k} \in R$. This implies that $1 \otimes x^{p^k}$ is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. Hence Lemma 10.46.7 applies to $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. In both cases we conclude that $\kappa(\mathfrak{p}) \otimes_R S$ has a unique prime ideal with residue field purely inseparable over $\kappa(\mathfrak{p})$. By Remark 10.18.5 we conclude that φ is bijective on spectra.

The statement on base change is immediate. □

10.47. Geometrically irreducible algebras

00I2 An algebra S over a field k is geometrically irreducible if the algebra $S \otimes_k k'$ has a unique minimal prime for every field extension k'/k. In this section we develop a bit of theory relevant to this notion.

00I6 **Lemma 10.47.1.** *Let $R \to S$ be a ring map. Assume*

(a) $\mathrm{Spec}(R)$ *is irreducible,*
(b) *$R \to S$ is flat,*
(c) *$R \to S$ is of finite presentation,*
(d) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have irreducible spectra for a dense collection of primes $\mathfrak{p}$ of R.*

Then $\mathrm{Spec}(S)$ *is irreducible. This is true more generally with (b) + (c) replaced by "the map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is open".*

Proof. The assumptions (b) and (c) imply that the map on spectra is open, see Proposition 10.41.8. Hence the lemma follows from Topology, Lemma 5.8.14. □

00I7 **Lemma 10.47.2.** *Let k be a separably closed field. Let R, S be k-algebras. If R, S have a unique minimal prime, so does $R \otimes_k S$.*

Proof. Let $k \subset \overline{k}$ be a perfect closure, see Definition 10.45.5. By assumption $\overline{k}$ is algebraically closed. The ring maps $R \to R \otimes_k \overline{k}$ and $S \to S \otimes_k \overline{k}$ and $R \otimes_k S \to (R \otimes_k S) \otimes_k \overline{k} = (R \otimes_k \overline{k}) \otimes_{\overline{k}} (S \otimes_k \overline{k})$ satisfy the assumptions of Lemma 10.46.7. Hence we may assume k is algebraically closed.

We may replace R and S by their reductions. Hence we may assume that R and S are domains. By Lemma 10.45.6 we see that $R \otimes_k S$ is reduced. Hence its spectrum is reducible if and only if it contains a nonzero zerodivisor. By Lemma 10.43.4 we reduce to the case where R and S are domains of finite type over k algebraically closed.

Note that the ring map $R \to R \otimes_k S$ is of finite presentation and flat. Moreover, for every maximal ideal $\mathfrak{m}$ of R we have $(R \otimes_k S) \otimes_R R/\mathfrak{m} \cong S$ because $k \cong R/\mathfrak{m}$ by the Hilbert Nullstellensatz Theorem 10.34.1. Moreover, the set of maximal ideals is dense in the spectrum of R since $\mathrm{Spec}(R)$ is Jacobson, see Lemma 10.35.2. Hence we see that Lemma 10.47.1 applies to the ring map $R \to R \otimes_k S$ and we conclude that the spectrum of $R \otimes_k S$ is irreducible as desired. □

037K **Lemma 10.47.3.** *Let k be a field. Let R be a k-algebra. The following are equivalent*

1. *for every field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,*
2. *for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,*
3. *the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the separable algebraic closure of k, and*
4. *the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the algebraic closure of k.*

Proof. It is clear that (1) implies (2).

Assume (2) and let $\overline{k}$ is the separable algebraic closure of k. Suppose $\mathfrak{q}_i \subset R \otimes_k \overline{k}$, $i = 1, 2$ are two minimal prime ideals. For every finite subextension $\overline{k}/k'/k$ the extension k'/k is separable and the ring map $R \otimes_k k' \to R \otimes_k \overline{k}$ is flat. Hence $\mathfrak{p}_i = (R \otimes_k k') \cap \mathfrak{q}_i$ are minimal prime ideals (as we have going down for flat ring maps by Lemma 10.39.19). Thus we see that $\mathfrak{p}_1 = \mathfrak{p}_2$ by assumption (2). Since $\overline{k} = \bigcup k'$ we conclude $\mathfrak{q}_1 = \mathfrak{q}_2$. Hence $\mathrm{Spec}(R \otimes_k \overline{k})$ is irreducible.

Assume (3) and let $\overline{k}$ be the algebraic closure of k. Let $\overline{k}/\overline{k}'/k$ be the corresponding separable algebraic closure of k. Then $\overline{k}/\overline{k}'$ is purely inseparable (in positive characteristic) or trivial. Hence $R \otimes_k \overline{k}' \to R \otimes_k \overline{k}$ induces a homeomorphism on spectra, for example by Lemma 10.46.7. Thus we have (4).

Assume (4). Let k'/k be an arbitrary field extension and let $\overline{k}$ be the algebraic closure of k. We may choose a field F such that both k' and $\overline{k}$ are isomorphic to subfields of F. Then

$$R \otimes_k F = (R \otimes_k \overline{k}) \otimes_{\overline{k}} F$$

and hence we see from Lemma 10.47.2 that $R \otimes_k F$ has a unique minimal prime. Finally, the ring map $R \otimes_k k' \to R \otimes_k F$ is flat and injective and hence any minimal prime of $R \otimes_k k'$ is the image of a minimal prime of $R \otimes_k F$ (by Lemma 10.30.5 and going down). We conclude that there is only one such minimal prime and the proof is complete. □

037L **Definition 10.47.4.** Let k be a field. Let S be a k-algebra. We say S is *geometrically irreducible over k* if for every field extension k'/k the spectrum of $S \otimes_k k'$ is irreducible[5].

By Lemma 10.47.3 it suffices to check this for finite separable field extensions k'/k or for k' equal to the separable algebraic closure of k.

037M **Lemma 10.47.5.** *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically irreducible over k if and only if the spectrum of R is irreducible.*

Proof. Immediate from the remark following Definition 10.47.4. □

037N **Lemma 10.47.6.** *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically irreducible over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically irreducible, then S is geometrically irreducible.*
(3) *A directed colimit of geometrically irreducible k-algebras is geometrically irreducible.*

Proof. Let $S' \subset S$ be a subalgebra. Then for any extension k'/k the ring map $S' \otimes_k k' \to S \otimes_k k'$ is injective also. Hence (1) follows from Lemma 10.30.5 (and the fact that the image of an irreducible space under a continuous map is irreducible). The second and third property follow from the fact that tensor product commutes with colimits. □

037O **Lemma 10.47.7.** *Let k be a field. Let S be a geometrically irreducible k-algebra. Let R be any k-algebra. The map*

$$\operatorname{Spec}(R \otimes_k S) \longrightarrow \operatorname{Spec}(R)$$

induces a bijection on irreducible components.

Proof. Recall that irreducible components correspond to minimal primes (Lemma 10.26.1). As $R \to R \otimes_k S$ is flat we see by going down (Lemma 10.39.19) that any minimal prime of $R \otimes_k S$ lies over a minimal prime of R. Conversely, if $\mathfrak{p} \subset R$ is a (minimal) prime then

$$R \otimes_k S/\mathfrak{p}(R \otimes_k S) = (R/\mathfrak{p}) \otimes_k S \subset \kappa(\mathfrak{p}) \otimes_k S$$

by flatness of $R \to R \otimes_k S$. The ring $\kappa(\mathfrak{p}) \otimes_k S$ has irreducible spectrum by assumption. It follows that $R \otimes_k S/\mathfrak{p}(R \otimes_k S)$ has a single minimal prime (Lemma 10.30.5). In other words, the inverse image of the irreducible set $V(\mathfrak{p})$ is irreducible. Hence the lemma follows. □

Let us make some remarks on the notion of geometrically irreducible field extensions.

037P **Lemma 10.47.8.** *Let K/k be a field extension. If k is algebraically closed in K, then K is geometrically irreducible over k.*

Proof. Assume k is algebraically closed in K. By Definition 10.47.4 and Lemma 10.47.3 it suffices to show that the spectrum of $K \otimes_k k'$ is irreducible for every finite separable extension k'/k. Say k' is generated by $\alpha \in k'$ over k, see Fields, Lemma 9.19.1. Let $P = T^d + a_1 T^{d-1} + \ldots + a_d \in k[T]$ be the minimal polynomial

[5] An irreducible space is nonempty.

of α. Then $K \otimes_k k' \cong K[T]/(P)$. The only way the spectrum of $K[T]/(P)$ can be reducible is if P is reducible in $K[T]$. Assume $P = P_1P_2$ is a nontrivial factorization in $K[T]$ to get a contradiction. By Lemma 10.38.5 we see that the coefficients of P_1 and P_2 are algebraic over k. Our assumption implies the coefficients of P_1 and P_2 are in k which contradicts the fact that P is irreducible over k. □

0G30 **Lemma 10.47.9.** *Let K/k be a geometrically irreducible field extension. Let S be a geometrically irreducible K-algebra. Then S is geometrically irreducible over k.*

Proof. By Definition 10.47.4 and Lemma 10.47.3 it suffices to show that the spectrum of $S \otimes_k k'$ is irreducible for every finite separable extension k'/k. Since K is geometrically irreducible over k we see that $K' = K \otimes_k k'$ is a finite, separable field extension of K. Hence the spectrum of $S \otimes_k k' = S \otimes_K K'$ is irreducible as S is assumed geometrically irreducible over K. □

0G31 **Lemma 10.47.10.** *Let K/k be a field extension. The following are equivalent*

(1) *K is geometrically irreducible over k, and*
(2) *the induced extension $K(t)/k(t)$ of purely transcendental extensions is geometrically irreducible.*

Proof. Assume (1). Denote Ω an algebraic closure of $k(t)$. By Definition 10.47.4 we find that the spectrum of

$$K \otimes_k \Omega = K \otimes_k k(t) \otimes_{k(t)} \Omega$$

is irreducible. Since $K(t)$ is a localization of $K \otimes_k k(T)$ we conclude that the spectrum of $K(t) \otimes_{k(t)} \Omega$ is irreducible. Thus by Lemma 10.47.3 we find that $K(t)/k(t)$ is geometrically irreducible.

Assume (2). Let k'/k be a field extension. We have to show that $K \otimes_k k'$ has a unique minimal prime. We know that the spectrum of

$$K(t) \otimes_{k(t)} k'(t)$$

is irreducible, i.e., has a unique minimal prime. Since there is an injective map $K \otimes_k k' \to K(t) \otimes_{k(t)} k'(t)$ (details omitted) we conclude by Lemmas 10.30.5 and 10.30.7. □

0G32 **Lemma 10.47.11.** *Let $K/L/M$ be a tower of fields with L/M geometrically irreducible. Let $x \in K$ be transcendental over L. Then $L(x)/M(x)$ is geometrically irreducible.*

Proof. This follows from Lemma 10.47.10 because the fields $L(x)$ and $M(x)$ are purely transcendental extensions of L and M. □

0G33 **Lemma 10.47.12.** *Let K/k be a field extension. The following are equivalent*

(1) *K/k is geometrically irreducible, and*
(2) *every element $\alpha \in K$ separably algebraic over k is in k.*

Proof. Assume (1) and let $\alpha \in K$ be separably algebraic over k. Then $k' = k(\alpha)$ is a finite separable extension of k contained in K. By Lemma 10.47.6 the extension k'/k is geometrically irreducible. In particular, we see that the spectrum of $k' \otimes_k \overline{k}$ is irreducible (and hence if it is a product of fields, then there is exactly one factor). By Fields, Lemma 9.13.4 it follows that $\mathrm{Hom}_k(k', \overline{k})$ has one element which in turn implies that $k' = k$ by Fields, Lemma 9.12.11. Thus (2) holds.

Assume (2). Let $k' \subset K$ be the subfield consisting of elements algebraic over k. By Lemma 10.47.8 the extension K/k' is geometrically irreducible. By assumption k'/k is a purely inseparable extension. By Lemma 10.46.7 the extension k'/k is geometrically irreducible. Hence by Lemma 10.47.9 we see that K/k is geometrically irreducible. □

037Q **Lemma 10.47.13.** *Let K/k be a field extension. Consider the subextension $K/k'/k$ consisting of elements separably algebraic over k. Then K is geometrically irreducible over k'. If K/k is a finitely generated field extension, then $[k' : k] < \infty$.*

Proof. The first statement is immediate from Lemma 10.47.12 and the fact that elements separably algebraic over k' are in k' by the transitivity of separable algebraic extensions, see Fields, Lemma 9.12.12. If K/k is finitely generated, then k' is finite over k by Fields, Lemma 9.26.11. □

04KP **Lemma 10.47.14.** *Let K/k be an extension of fields. Let $\overline{k}/k$ be a separable algebraic closure. Then $Gal(\overline{k}/k)$ acts transitively on the primes of $\overline{k} \otimes_k K$.*

Proof. Let $K/k'/k$ be the subextension found in Lemma 10.47.13. Note that as $k \subset \overline{k}$ is integral all the prime ideals of $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k k'$ are maximal, see Lemma 10.36.20. By Lemma 10.47.7 the map

$$\mathrm{Spec}(\overline{k} \otimes_k K) \to \mathrm{Spec}(\overline{k} \otimes_k k')$$

is bijective because (1) all primes are minimal primes, (2) $\overline{k} \otimes_k K = (\overline{k} \otimes_k k') \otimes_{k'} K$, and (3) K is geometrically irreducible over k'. Hence it suffices to prove the lemma for the action of $\mathrm{Gal}(\overline{k}/k)$ on the primes of $\overline{k} \otimes_k k'$.

As every prime of $\overline{k} \otimes_k k'$ is maximal, the residue fields are isomorphic to $\overline{k}$. Hence the prime ideals of $\overline{k} \otimes_k k'$ correspond one to one to elements of $\mathrm{Hom}_k(k', \overline{k})$ with $\sigma \in \mathrm{Hom}_k(k', \overline{k})$ corresponding to the kernel $\mathfrak{p}_\sigma$ of $1 \otimes \sigma : \overline{k} \otimes_k k' \to \overline{k}$. In particular $\mathrm{Gal}(\overline{k}/k)$ acts transitively on this set as desired. □

10.48. Geometrically connected algebras

05DV

037R **Lemma 10.48.1.** *Let k be a separably algebraically closed field. Let R, S be k-algebras. If $\mathrm{Spec}(R)$, and $\mathrm{Spec}(S)$ are connected, then so is $\mathrm{Spec}(R \otimes_k S)$.*

Proof. Recall that $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents, see Lemma 10.21.4. Hence, by Lemma 10.43.4 we may assume R and S are of finite type over k. In this case R and S are Noetherian, and have finitely many minimal primes, see Lemma 10.31.6. Thus we may argue by induction on $n + m$ where n, resp. m is the number of irreducible components of $\mathrm{Spec}(R)$, resp. $\mathrm{Spec}(S)$. Of course the case where either n or m is zero is trivial. If $n = m = 1$, i.e., $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ both have one irreducible component, then the result holds by Lemma 10.47.2. Suppose that $n > 1$. Let $\mathfrak{p} \subset R$ be a minimal prime corresponding to the irreducible closed subset $T \subset \mathrm{Spec}(R)$. Let $T' \subset \mathrm{Spec}(R)$ be the union of the other $n - 1$ irreducible components. Choose an ideal $I \subset R$ such that $T' = V(I) = \mathrm{Spec}(R/I)$ (Lemma 10.17.7). By choosing our minimal prime carefully we may in addition arrange it so that T' is connected, see Topology, Lemma 5.8.17. Then $T \cup T' = \mathrm{Spec}(R)$ and $T \cap T' = V(\mathfrak{p} + I) = \mathrm{Spec}(R/(\mathfrak{p} + I))$ is not empty as $\mathrm{Spec}(R)$ is assumed connected. The inverse image of T in $\mathrm{Spec}(R \otimes_k S)$ is

$\mathrm{Spec}(R/\mathfrak{p} \otimes_k S)$, and the inverse of T' in $\mathrm{Spec}(R \otimes_k S)$ is $\mathrm{Spec}(R/I \otimes_k S)$. By induction these are both connected. The inverse image of $T \cap T'$ is $\mathrm{Spec}(R/(\mathfrak{p} + I) \otimes_k S)$ which is nonempty. Hence $\mathrm{Spec}(R \otimes_k S)$ is connected. □

037S **Lemma 10.48.2.** *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension k'/k the spectrum of $R \otimes_k k'$ is connected, and*
(2) *for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is connected.*

Proof. For any extension of fields k'/k the connectivity of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ having no nontrivial idempotents, see Lemma 10.21.4. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 10.43.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ having no nontrivial idempotents. For any field extension k'/k, there exists a field extension $\overline{k}'/\overline{k}$ with $k' \subset \overline{k}'$. By Lemma 10.48.1 we see that $R \otimes_k \overline{k}'$ has no nontrivial idempotents. If $R \otimes_k k'$ has a nontrivial idempotent, then also $R \otimes_k \overline{k}'$, contradiction. □

037T **Definition 10.48.3.** Let k be a field. Let S be a k-algebra. We say S is *geometrically connected over k* if for every field extension k'/k the spectrum of $S \otimes_k k'$ is connected.

By Lemma 10.48.2 it suffices to check this for finite separable field extensions k'/k.

037U **Lemma 10.48.4.** *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically connected over k if and only if the spectrum of R is connected.*

Proof. Immediate from the remark following Definition 10.48.3. □

037V **Lemma 10.48.5.** *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically connected over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically connected, then S is geometrically connected.*
(3) *A directed colimit of geometrically connected k-algebras is geometrically connected.*

Proof. This follows from the characterization of connectedness in terms of the nonexistence of nontrivial idempotents. The second and third property follow from the fact that tensor product commutes with colimits. □

The following lemma will be superseded by the more general Varieties, Lemma 33.7.4.

037W **Lemma 10.48.6.** *Let k be a field. Let S be a geometrically connected k-algebra. Let R be any k-algebra. The map*

$$R \longrightarrow R \otimes_k S$$

induces a bijection on idempotents, and the map

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on connected components.

Proof. The second assertion follows from the first combined with Lemma 10.22.2. By Lemmas 10.48.5 and 10.43.4 we may assume that R and S are of finite type over k. Then we see that also $R \otimes_k S$ is of finite type over k. Note that in this case all the rings are Noetherian and hence their spectra have finitely many connected components (since they have finitely many irreducible components, see Lemma 10.31.6). In particular, all connected components in question are open! Hence via Lemma 10.24.3 we see that the first statement of the lemma in this case is equivalent to the second. Let's prove this. As the algebra S is geometrically connected and nonzero we see that all fibres of $X = \mathrm{Spec}(R \otimes_k S) \to \mathrm{Spec}(R) = Y$ are connected and nonempty. Also, as $R \to R \otimes_k S$ is flat of finite presentation the map $X \to Y$ is open (Proposition 10.41.8). Topology, Lemma 5.7.6 shows that $X \to Y$ induces bijection on connected components. □

10.49. Geometrically integral algebras

05DW Here is the definition.

05DX **Definition 10.49.1.** Let k be a field. Let S be a k-algebra. We say S is *geometrically integral over k* if for every field extension k'/k the ring of $S \otimes_k k'$ is a domain.

Any question about geometrically integral algebras can be translated in a question about geometrically reduced and irreducible algebras.

05DY **Lemma 10.49.2.** *Let k be a field. Let S be a k-algebra. In this case S is geometrically integral over k if and only if S is geometrically irreducible as well as geometrically reduced over k.*

Proof. Omitted. □

0FWF **Lemma 10.49.3.** *Let k be a field. Let S be a k-algebra. The following are equivalent*

(1) *S is geometrically integral over k,*
(2) *for every finite extension k'/k of fields the ring $S \otimes_k k'$ is a domain,*
(3) *$S \otimes_k \overline{k}$ is a domain where $\overline{k}$ is the algebraic closure of k.*

Proof. Follows from Lemmas 10.49.2, 10.44.4, and 10.47.3. □

09P9 **Lemma 10.49.4.** *Let k be a field. Let S be a geometrically integral k-algebra. Let R be a k-algebra and an integral domain. Then $R \otimes_k S$ is an integral domain.*

Proof. By Lemma 10.43.5 the ring $R \otimes_k S$ is reduced and by Lemma 10.47.7 the ring $R \otimes_k S$ is irreducible (the spectrum has just one irreducible component), so $R \otimes_k S$ is an integral domain. □

10.50. Valuation rings

00I8 Here are some definitions.

00I9 **Definition 10.50.1.** Valuation rings.

(1) Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.
(2) Let A be a ring. We say A is a *valuation ring* if A is a local domain and if A is maximal for the relation of domination among local rings contained in the fraction field of A.

(3) Let A be a valuation ring with fraction field K. If $R \subset K$ is a subring of K, then we say A is *centered* on R if $R \subset A$.

With this definition a field is a valuation ring.

00IA **Lemma 10.50.2.** *Let K be a field. Let $A \subset K$ be a local subring. Then there exists a valuation ring with fraction field K dominating A.*

Proof. We consider the collection of local subrings of K as a partially ordered set using the relation of domination. Suppose that $\{A_i\}_{i \in I}$ is a totally ordered collection of local subrings of K. Then $B = \bigcup A_i$ is a local subring which dominates all of the A_i. Hence by Zorn's Lemma, it suffices to show that if $A \subset K$ is a local ring whose fraction field is not K, then there exists a local ring $B \subset K$, $B \neq A$ dominating A.

Pick $t \in K$ which is not in the fraction field of A. If t is transcendental over A, then $A[t] \subset K$ and hence $A[t]_{(t,\mathfrak{m})} \subset K$ is a local ring distinct from A dominating A. Suppose t is algebraic over A. Then for some nonzero $a \in A$ the element at is integral over A. In this case the subring $A' \subset K$ generated by A and ta is finite over A. By Lemma 10.36.17 there exists a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ dominates A. If $A = A'_{\mathfrak{m}'}$, then t is in the fraction field of A which we assumed not to be the case. Thus $A \neq A'_{\mathfrak{m}'}$ as desired. □

00IC **Lemma 10.50.3.** *Let A be a valuation ring. Then A is a normal domain.*

Proof. Suppose x is in the field of fractions of A and integral over A. Let A' denote the subring of K generated by A and x. Since $A \subset A'$ is an integral extension, we see by Lemma 10.36.17 that there is a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ dominates A. Since A is a valuation ring we conclude that $A = A'_{\mathfrak{m}'}$ and therefore that $x \in A$. □

00IB **Lemma 10.50.4.** *Let A be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $x \in K$. Then either $x \in A$ or $x^{-1} \in A$ or both.*

Proof. Assume that x is not in A. Let A' denote the subring of K generated by A and x. Since A is a valuation ring we see that there is no prime of A' lying over $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we see that $V(\mathfrak{m}A') = \emptyset$. Then $\mathfrak{m}A' = A'$ by Lemma 10.17.2. Hence we can write $1 = \sum_{i=0}^{d} t_i x^i$ with $t_i \in \mathfrak{m}$. This implies that $(1 - t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0$. In particular we see that x^{-1} is integral over A, and hence $x^{-1} \in A$ by Lemma 10.50.3. □

052K **Lemma 10.50.5.** *Let $A \subset K$ be a subring of a field K such that for all $x \in K$ either $x \in A$ or $x^{-1} \in A$ or both. Then A is a valuation ring with fraction field K.*

Proof. If A is not K, then A is not a field and there is a nonzero maximal ideal $\mathfrak{m}$. If $\mathfrak{m}'$ is a second maximal ideal, then choose $x, y \in A$ with $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$, $x \notin \mathfrak{m}'$, and $y \in \mathfrak{m}'$. Then neither $x/y \in A$ nor $y/x \in A$ contradicting the assumption of the lemma. Thus we see that A is a local ring. Suppose that A' is a local ring contained in K which dominates A. Let $x \in A'$. We have to show that $x \in A$. If not, then $x^{-1} \in A$, and of course $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$ which contradicts $x \in A'$. □

0AS4 **Lemma 10.50.6.** *Let I be a directed set. Let (A_i, φ_{ij}) be a system of valuation rings over I. Then $A = \operatorname{colim} A_i$ is a valuation ring.*

Proof. It is clear that A is a domain. Let $a, b \in A$. Lemma 10.50.5 tells us we have to show that either $a|b$ or $b|a$ in A. Choose i so large that there exist $a_i, b_i \in A_i$ mapping to a, b. Then Lemma 10.50.4 applied to a_i, b_i in A_i implies the result for a, b in A. □

052L **Lemma 10.50.7.** *Let L/K be an extension of fields. If $B \subset L$ is a valuation ring, then $A = K \cap B$ is a valuation ring.*

Proof. We can replace L by the fraction field F of B and K by $K \cap F$. Then the lemma follows from a combination of Lemmas 10.50.4 and 10.50.5. □

0AAV **Lemma 10.50.8.** *Let L/K be an algebraic extension of fields. If $B \subset L$ is a valuation ring with fraction field L and not a field, then $A = K \cap B$ is a valuation ring and not a field.*

Proof. By Lemma 10.50.7 the ring A is a valuation ring. If A is a field, then $A = K$. Then $A = K \subset B$ is an integral extension, hence there are no proper inclusions among the primes of B (Lemma 10.36.20). This contradicts the assumption that B is a local domain and not a field. □

088Y **Lemma 10.50.9.** *Let A be a valuation ring. For any prime ideal $\mathfrak{p} \subset A$ the quotient $A/\mathfrak{p}$ is a valuation ring. The same is true for the localization $A_\mathfrak{p}$ and in fact any localization of A.*

Proof. Use the characterization of valuation rings given in Lemma 10.50.5. □

088Z **Lemma 10.50.10.** *Let A' be a valuation ring with residue field K. Let A be a valuation ring with fraction field K. Then $C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A\}$ is a valuation ring.*

Proof. Note that $\mathfrak{m}_{A'} \subset C$ and $C/\mathfrak{m}_{A'} = A$. In particular, the fraction field of C is equal to the fraction field of A'. We will use the criterion of Lemma 10.50.5 to prove the lemma. Let x be an element of the fraction field of C. By the lemma we may assume $x \in A'$. If $x \in \mathfrak{m}_{A'}$, then we see $x \in C$. If not, then x is a unit of A' and we also have $x^{-1} \in A'$. Hence either x or x^{-1} maps to an element of A by the lemma again. □

090P **Lemma 10.50.11.** *Let A be a normal domain with fraction field K.*

1. *For every $x \in K$, $x \notin A$ there exists a valuation ring $A \subset V \subset K$ with fraction field K such that $x \notin V$.*
2. *If A is local, we can moreover choose V which dominates A.*

In other words, A is the intersection of all valuation rings in K containing A and if A is local, then A is the intersection of all valuation rings in K dominating A.

Proof. Suppose $x \in K$, $x \notin A$. Consider $B = A[x^{-1}]$. Then $x \notin B$. Namely, if $x = a_0 + a_1x^{-1} + \ldots + a_dx^{-d}$ then $x^{d+1} - a_0x^d - \ldots - a_d = 0$ and x is integral over A in contradiction with the fact that A is normal. Thus x^{-1} is not a unit in B. Thus $V(x^{-1}) \subset \mathrm{Spec}(B)$ is not empty (Lemma 10.17.2), and we can choose a prime $\mathfrak{p} \subset B$ with $x^{-1} \in \mathfrak{p}$. Choose a valuation ring $V \subset K$ dominating $B_\mathfrak{p}$ (Lemma 10.50.2). Then $x \notin V$ as $x^{-1} \in \mathfrak{m}_V$.

If A is local, then we claim that $x^{-1}B + \mathfrak{m}_AB \neq B$. Namely, if $1 = (a_0 + a_1x^{-1} + \ldots + a_dx^{-d})x^{-1} + a'_0 + \ldots + a'_dx^{-d}$ with $a_i \in A$ and $a'_i \in \mathfrak{m}_A$, then we'd get

$$(1 - a'_0)x^{d+1} - (a_0 + a'_1)x^d - \ldots - a_d = 0$$

Since $a_0' \in \mathfrak{m}_A$ we see that $1 - a_0'$ is a unit in A and we conclude that x would be integral over A, a contradiction as before. Then choose the prime $\mathfrak{p} \supset x^{-1}B + \mathfrak{m}_A B$ we find V dominating A. □

An *totally ordered abelian group* is a pair $(\Gamma, \geq)$ consisting of an abelian group Γ endowed with a total ordering $\geq$ such that $\gamma \geq \gamma' \Rightarrow \gamma + \gamma'' \geq \gamma' + \gamma''$ for all $\gamma, \gamma', \gamma'' \in \Gamma$.

00ID **Lemma 10.50.12.** *Let A be a valuation ring with field of fractions K. Set $\Gamma = K^*/A^*$ (with group law written additively). For $\gamma, \gamma' \in \Gamma$ define $\gamma \geq \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $A - \{0\} \to \Gamma$. Then $(\Gamma, \geq)$ is a totally ordered abelian group.*

Proof. Omitted, but follows easily from Lemma 10.50.4. Note that in case $A = K$ we obtain the zero group $\Gamma = \{0\}$ endowed with its unique total ordering. □

00IE **Definition 10.50.13.** Let A be a valuation ring.

(1) The totally ordered abelian group $(\Gamma, \geq)$ of Lemma 10.50.12 is called the *value group* of the valuation ring A.
(2) The map $v : A - \{0\} \to \Gamma$ and also $v : K^* \to \Gamma$ is called the *valuation* associated to A.
(3) The valuation ring A is called a *discrete valuation ring* if $\Gamma \cong \mathbf{Z}$.

Note that if $\Gamma \cong \mathbf{Z}$ then there is a unique such isomorphism such that $1 \geq 0$. If the isomorphism is chosen in this way, then the ordering becomes the usual ordering of the integers.

00IF **Lemma 10.50.14.** *Let A be a valuation ring. The valuation $v : A - \{0\} \to \Gamma_{\geq 0}$ has the following properties:*

(1) $v(a) = 0 \Leftrightarrow a \in A^*$,
(2) $v(ab) = v(a) + v(b)$,
(3) $v(a + b) \geq \min(v(a), v(b))$ *provided* $a + b \neq 0$.

Proof. Omitted. □

090Q **Lemma 10.50.15.** *Let A be a ring. The following are equivalent*

(1) *A is a valuation ring,*
(2) *A is a local domain and every finitely generated ideal of A is principal.*

Proof. Assume A is a valuation ring and let $f_1, \ldots, f_n \in A$. Choose i such that $v(f_i)$ is minimal among $v(f_j)$. Then $(f_i) = (f_1, \ldots, f_n)$. Conversely, assume A is a local domain and every finitely generated ideal of A is principal. Pick $f, g \in A$ and write $(f, g) = (h)$. Then $f = ah$ and $g = bh$ and $h = cf + dg$ for some $a, b, c, d \in A$. Thus $ac + bd = 1$ and we see that either a or b is a unit, i.e., either g/f or f/g is an element of A. This shows A is a valuation ring by Lemma 10.50.5. □

00IG **Lemma 10.50.16.** *Let $(\Gamma, \geq)$ be a totally ordered abelian group. Let K be a field. Let $v : K^* \to \Gamma$ be a homomorphism of abelian groups such that $v(a + b) \geq \min(v(a), v(b))$ for $a, b \in K$ with $a, b, a + b$ not zero. Then*

$$A = \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

is a valuation ring with value group $\mathrm{Im}(v) \subset \Gamma$, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

and with group of units

$$A^* = \{x \in K^* \mid v(x) = 0\}.$$

Proof. Omitted. □

Let $(\Gamma, \geq)$ be a totally ordered abelian group. An *ideal of* Γ is a subset $I \subset \Gamma$ such that all elements of I are ≥ 0 and $\gamma \in I$, $\gamma' \geq \gamma$ implies $\gamma' \in I$. We say that such an ideal is *prime* if $0 \notin I$ and if $\gamma + \gamma' \in I, \gamma, \gamma' \geq 0 \Rightarrow \gamma \in I$ or $\gamma' \in I$.

00IH **Lemma 10.50.17.** *Let A be a valuation ring. Ideals in A correspond $1-1$ with ideals of Γ. This bijection is inclusion preserving, and maps prime ideals to prime ideals.*

Proof. Omitted. □

00II **Lemma 10.50.18.** *A valuation ring is Noetherian if and only if it is a discrete valuation ring or a field.*

Proof. Suppose A is a discrete valuation ring with valuation $v : A \setminus \{0\} \to \mathbf{Z}$ normalized so that $\mathrm{Im}(v) = \mathbf{Z}_{\geq 0}$. By Lemma 10.50.17 the ideals of A are the subsets $I_n = \{0\} \cup v^{-1}(\mathbf{Z}_{\geq n})$. It is clear that any element $x \in A$ with $v(x) = n$ generates I_n. Hence A is a PID so certainly Noetherian.

Suppose A is a Noetherian valuation ring with value group Γ. By Lemma 10.50.17 we see the ascending chain condition holds for ideals in Γ. We may assume A is not a field, i.e., there is a $\gamma \in \Gamma$ with $\gamma > 0$. Applying the ascending chain condition to the subsets $\gamma + \Gamma_{\geq 0}$ with $\gamma > 0$ we see there exists a smallest element γ_0 which is bigger than 0. Let $\gamma \in \Gamma$ be an element $\gamma > 0$. Consider the sequence of elements γ, $\gamma - \gamma_0$, $\gamma - 2\gamma_0$, etc. By the ascending chain condition these cannot all be > 0. Let $\gamma - n\gamma_0$ be the last one ≥ 0. By minimality of γ_0 we see that $0 = \gamma - n\gamma_0$. Hence Γ is a cyclic group as desired. □

10.51. More Noetherian rings

00IJ

00IK **Lemma 10.51.1.** *Let R be a Noetherian ring. Any finite R-module is of finite presentation. Any submodule of a finite R-module is finite. The ascending chain condition holds for R-submodules of a finite R-module.*

Proof. We first show that any submodule N of a finite R-module M is finite. We do this by induction on the number of generators of M. If this number is 1, then $N = J/I \subset M = R/I$ for some ideals $I \subset J \subset R$. Thus the definition of Noetherian implies the result. If the number of generators of M is greater than 1, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ where M' and M'' have fewer generators. Note that setting $N' = M' \cap N$ and $N'' = \mathrm{Im}(N \to M'')$ gives a similar short exact sequence for N. Hence the result follows from the induction hypothesis since the number of generators of N is at most the number of generators of N' plus the number of generators of N''.

To show that M is finitely presented just apply the previous result to the kernel of a presentation $R^n \to M$.

It is well known and easy to prove that the ascending chain condition for R-submodules of M is equivalent to the condition that every submodule of M is a finite R-module. We omit the proof. □

00IN **Lemma 10.51.2** (Artin-Rees)**.** *Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2 M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. By Lemma 10.51.1 it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. □

00IO **Lemma 10.51.3.** *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that*

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \text{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Lemma 10.51.2 to $\mathrm{Im}(f) \subset N$ and note that $f : I^{n-c} M \to I^{n-c} f(M)$ is surjective. □

00IP **Lemma 10.51.4** (Krull's intersection theorem)**.** *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 10.51.2 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 10.20.1 we see that $N = 0$. □

00IQ **Lemma 10.51.5.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.*

(1) *For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*
(2) *If I is contained in the Jacobson radical of R, then $N = 0$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N, see Lemma 10.51.1. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_\mathfrak{p}$ is zero, by Lemma 10.51.4. Hence we can find $g_i \in R$, $g_i \notin \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Lemma 10.23.1. □

00IR **Remark 10.51.6.** Lemma 10.51.4 in particular implies that $\bigcap_n I^n = (0)$ when $I \subset R$ is a non-unit ideal in a Noetherian local ring R. More generally, let R be a Noetherian ring and $I \subset R$ an ideal. Suppose that $f \in \bigcap_{n \in \mathbf{N}} I^n$. Then Lemma 10.51.5 says that for every prime ideal $I \subset \mathfrak{p}$ there exists a $g \in R$, $g \notin \mathfrak{p}$ such that f maps to zero in R_g. In algebraic geometry we express this by saying that "f is zero in an open neighbourhood of the closed set $V(I)$ of $\mathrm{Spec}(R)$".

00IS **Lemma 10.51.7** (Artin-Tate)**.** *Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is of finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian we conclude that $T \subset S$ is finite over T' and we win. □

10.52. Length

00IU

02LY **Definition 10.52.1.** Let R be a ring. For any R-module M we define the *length* of M over R by the formula

$$\text{length}_R(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words it is the supremum of the lengths of chains of submodules. There is an obvious notion of when a chain of submodules is a refinement of another. This gives a partial ordering on the collection of all chains of submodules, with the smallest chain having the shape $0 = M_0 \subset M_1 = M$ if M is not zero. We note the obvious fact that if the length of M is finite, then every chain can be refined to a maximal chain. But it is not as obvious that all maximal chains have the same length (as we will see later).

02LZ **Lemma 10.52.2.** *Let R be a ring. Let M be an R-module. If $\text{length}_R(M) < \infty$ then M is a finite R-module.*

Proof. Omitted. □

00IV **Lemma 10.52.3.** *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then the length of M is the sum of the lengths of M' and M''.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\text{length}_R M \geq \text{length}_R M' + \text{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \text{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. □

00IW **Lemma 10.52.4.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. If M is an R-module and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$. In other words, if M has finite length then $\mathfrak{m}^n M = 0$ for some n.*

Proof. Assume $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. Choose $x \in M$ and $f_1, \ldots, f_n \in \mathfrak{m}$ such that $f_1 f_2 \ldots f_n x \neq 0$. The first n steps in the filtration

$$0 \subset Rf_1 \ldots f_n x \subset Rf_1 \ldots f_{n-1} x \subset \ldots \subset Rx \subset M$$

are distinct. For example, if $Rf_1 x = Rf_1 f_2 x$, then $f_1 x = g f_1 f_2 x$ for some g, hence $(1 - gf_2) f_1 x = 0$ hence $f_1 x = 0$ as $1 - gf_2$ is a unit which is a contradiction with the choice of x and $f_1, \ldots, f_n$. Hence the length is infinite. □

00IX **Lemma 10.52.5.** *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. □

00IY **Lemma 10.52.6.** *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m}M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak{m}$ vector space. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 10.52.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. □

00IZ **Lemma 10.52.7.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$, by Lemma 10.9.15. The lemma follows. □

00J0 **Lemma 10.52.8.** *Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1} M \subset \ldots \subset \mathfrak{m}M \subset M$. All of the subquotients are finitely generated R-modules to which Lemma 10.52.6 applies. We conclude by additivity, see Lemma 10.52.3. □

00J1 **Definition 10.52.9.** Let R be a ring. Let M be an R-module. We say M is *simple* if $M \neq 0$ and every submodule of M is either equal to M or to 0.

00J2 **Lemma 10.52.10.** *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *M is simple,*
2. *$\text{length}_R(M) = 1$, and*
3. *$M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 10.52.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

00J3 **Lemma 10.52.11.** *Let R be a ring. Let M be a finite length R-module. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

(1) *$n = length_R(M)$,*
(2) *each M_i/M_{i-1} is simple,*
(3) *each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
(4) *given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = length_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Lemma 10.52.10 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 10.52.3, we see $n = \text{length}_R(M)$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_\mathfrak{m}$ with successive quotients $(M_i/M_{i-1})_\mathfrak{m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R we have

$$(R/\mathfrak{m}')_\mathfrak{m} \cong \begin{cases} 0 & \text{if } \mathfrak{m} \neq \mathfrak{m}', \\ R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} & \text{if } \mathfrak{m} = \mathfrak{m}' \end{cases}$$

This we leave to the reader. □

02M0 **Lemma 10.52.12.** *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$length_A(M) = \sum\nolimits_{i=1,\ldots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] length_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $length_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$$

by B-submodules as in Lemma 10.52.11. Then each quotient M_j/M_{j-1} is isomorphic to $\kappa(\mathfrak{m}_{i(j)})$ for some $i(j) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak{m}_i)) = [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ by Lemma 10.52.6. The lemma follows by additivity of lengths (Lemma 10.52.3). □

02M1 **Lemma 10.52.13.** *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$length_A(M) length_B(B/\mathfrak{m}_A B) = length_B(M \otimes_A B).$$

In particular, if $length_B(B/\mathfrak{m}_A B) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat by Lemma 10.39.17. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_A B$. Thus the lemma follows. □

02M2 **Lemma 10.52.14.** *Let $A \to B \to C$ be flat local homomorphisms of local rings. Then*

$$\mathit{length}_B(B/\mathfrak{m}_A B)\mathit{length}_C(C/\mathfrak{m}_B C) = \mathit{length}_C(C/\mathfrak{m}_A C)$$

Proof. Follows from Lemma 10.52.13 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak{m}_A B$ □

10.53. Artinian rings

00J4 Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory.

00J5 **Definition 10.53.1.** A ring R is *Artinian* if it satisfies the descending chain condition for ideals.

00J6 **Lemma 10.53.2.** *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. □

00J7 **Lemma 10.53.3.** *If R is Artinian then R has only finitely many maximal ideals.*

Proof. Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are pairwise distinct maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak{m}_i$ are surjective). □

00J8 **Lemma 10.53.4.** *Let R be Artinian. The Jacobson radical of R is a nilpotent ideal.*

Proof. Let $I \subset R$ be the Jacobson radical. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By minimality we have $J + IJ' = J$ or $J + IJ' = J'$. In the latter case we get $J' = J + Ix$ and by Lemma 10.20.1 we obtain $J = J'$ a contradiction. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. □

00JA **Lemma 10.53.5.** *Any ring with finitely many maximal ideals and locally nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $I = \bigcap_{i=1}^n \mathfrak{m}_i$ be the Jacobson radical of R. Assume I is locally nilpotent. Let $\mathfrak{p}$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak{p} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. Since $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$ we conclude $\mathfrak{p} \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$. Hence $\mathfrak{p} \supset \mathfrak{m}_i$ for some i, and so $\mathfrak{p} = \mathfrak{m}_i$. Thus the spectrum of R is the discrete

topological space $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. By Lemma 10.24.3 applied $n-1$ times we find that $R = R_1 \times \ldots \times R_n$ where the spectrum of R_i is a singleton for each i. Thus R_i is a local ring and since it is a localization of R (by the lemma), it is one of the local rings of R as desired. □

00JB **Lemma 10.53.6.** *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Lemmas 10.53.3, 10.53.4, and 10.53.5.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak{m}$. By Lemma 10.53.4 we have $\mathfrak{m}^n = 0$ for some n. Consider the sequence $0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Lemma 10.52.6 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of sub vector spaces which would correspond to an infinite descending chain of ideals in R. □

10.54. Homomorphisms essentially of finite type

07DR Some simple remarks on localizations of finite type ring maps.

00QM **Definition 10.54.1.** Let $R \to S$ be a ring map.

(1) We say that $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.
(2) We say that $R \to S$ is *essentially of finite presentation* if S is the localization of an R-algebra of finite presentation.

07DS **Lemma 10.54.2.** *The class of ring maps which are essentially of finite type is preserved under composition. Similarly for essentially of finite presentation.*

Proof. Omitted. □

0AUF **Lemma 10.54.3.** *The class of ring maps which are essentially of finite type is preserved by base change. Similarly for essentially of finite presentation.*

Proof. Omitted. □

07DT **Lemma 10.54.4.** *Let $R \to S$ be a ring map. Assume S is an Artinian local ring with maximal ideal $\mathfrak{m}$. Then*

(1) *$R \to S$ is finite if and only if $R \to S/\mathfrak{m}$ is finite,*
(2) *$R \to S$ is of finite type if and only if $R \to S/\mathfrak{m}$ is of finite type.*
(3) *$R \to S$ is essentially of finite type if and only if the composition $R \to S/\mathfrak{m}$ is essentially of finite type.*

Proof. If $R \to S$ is finite, then $R \to S/\mathfrak{m}$ is finite by Lemma 10.7.3. Conversely, assume $R \to S/\mathfrak{m}$ is finite. As S has finite length over itself (Lemma 10.53.6) we can choose a filtration

$$0 \subset I_1 \subset \ldots \subset I_n = S$$

by ideals such that $I_i/I_{i-1} \cong S/\mathfrak{m}$ as S-modules. Thus S has a filtration by R-submodules I_i such that each successive quotient is a finite R-module. Thus S is a finite R-module by Lemma 10.5.3.

If $R \to S$ is of finite type, then $R \to S/\mathfrak{m}$ is of finite type by Lemma 10.6.2. Conversely, assume that $R \to S/\mathfrak{m}$ is of finite type. Choose $f_1, \ldots, f_n \in S$ which map to generators of $S/\mathfrak{m}$. Then $A = R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ is a ring map such that $A \to S/\mathfrak{m}$ is surjective (in particular finite). Hence $A \to S$ is finite by part (1) and we see that $R \to S$ is of finite type by Lemma 10.6.2.

If $R \to S$ is essentially of finite type, then $R \to S/\mathfrak{m}$ is essentially of finite type by Lemma 10.54.2. Conversely, assume that $R \to S/\mathfrak{m}$ is essentially of finite type. Suppose $S/\mathfrak{m}$ is the localization of $R[x_1, \ldots, x_n]/I$. Choose $f_1, \ldots, f_n \in S$ whose congruence classes modulo $\mathfrak{m}$ correspond to the congruence classes of $x_1, \ldots, x_n$ modulo I. Consider the map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ with kernel J. Set $A = R[x_1, \ldots, x_n]/J \subset S$ and $\mathfrak{p} = A \cap \mathfrak{m}$. Note that $A/\mathfrak{p} \subset S/\mathfrak{m}$ is equal to the image of $R[x_1, \ldots, x_n]/I$ in $S/\mathfrak{m}$. Hence $\kappa(\mathfrak{p}) = S/\mathfrak{m}$. Thus $A_\mathfrak{p} \to S$ is finite by part (1). We conclude that S is essentially of finite type by Lemma 10.54.2. □

The following lemma can be proven using properness of projective space instead of the algebraic argument we give here.

0AUG **Lemma 10.54.5.** *Let $\varphi : R \to S$ be essentially of finite type with R and S local (but not necessarily φ local). Then there exists an n and a maximal ideal $\mathfrak{m} \subset R[x_1, \ldots, x_n]$ lying over $\mathfrak{m}_R$ such that S is a localization of a quotient of $R[x_1, \ldots, x_n]_\mathfrak{m}$.*

Proof. We can write S as a localization of a quotient of $R[x_1, \ldots, x_n]$. Hence it suffices to prove the lemma in case $S = R[x_1, \ldots, x_n]_\mathfrak{q}$ for some prime $\mathfrak{q} \subset R[x_1, \ldots, x_n]$. If $\mathfrak{q} + \mathfrak{m}_R R[x_1, \ldots, x_n] \neq R[x_1, \ldots, x_n]$ then we can find a maximal ideal $\mathfrak{m}$ as in the statement of the lemma with $\mathfrak{q} \subset \mathfrak{m}$ and the result is clear.

Choose a valuation ring $A \subset \kappa(\mathfrak{q})$ which dominates the image of $R \to \kappa(\mathfrak{q})$ (Lemma 10.50.2). If the image $\lambda_i \in \kappa(\mathfrak{q})$ of x_i is contained in A, then $\mathfrak{q}$ is contained in the inverse image of $\mathfrak{m}_A$ via $R[x_1, \ldots, x_n] \to A$ which means we are back in the preceding case. Hence there exists an i such that $\lambda_i^{-1} \in A$ and such that $\lambda_j/\lambda_i \in A$ for all $j = 1, \ldots, n$ (because the value group of A is totally ordered, see Lemma 10.50.12). Then we consider the map

$$R[y_0, y_1, \ldots, \hat{y_i}, \ldots, y_n] \to R[x_1, \ldots, x_n]_\mathfrak{q}, \quad y_0 \mapsto 1/x_i, \quad y_j \mapsto x_j/x_i$$

Let $\mathfrak{q}' \subset R[y_0, \ldots, \hat{y_i}, \ldots, y_n]$ be the inverse image of $\mathfrak{q}$. Since $y_0 \notin \mathfrak{q}'$ it is easy to see that the displayed arrow defines an isomorphism on localizations. On the other hand, the result of the first paragraph applies to $R[y_0, \ldots, \hat{y_i}, \ldots, y_n]$ because y_j maps to an element of A. This finishes the proof. □

10.55. K-groups

00JC Let R be a ring. We will introduce two abelian groups associated to R. The first of the two is denoted $K_0'(R)$ and has the following properties[6]:

(1) For every finite R-module M there is given an element $[M]$ in $K_0'(R)$,

[6]The definition makes sense for any ring but is rarely used unless R is Noetherian.

(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K'_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K'_0(R)$ among the generators $[M]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The actual construction is a bit more annoying since one has to take care that the collection of all finitely generated R-modules is a proper class. However, this problem can be overcome by taking as set of generators of the group $K'_0(R)$ the elements $[R^n/K]$ where n ranges over all integers and K ranges over all submodules $K \subset R^n$. The generators for the subgroup of relations imposed on these elements will be the relations coming from short exact sequences whose terms are of the form R^n/K. The element $[M]$ is defined by choosing n and K such that $M \cong R^n/K$ and putting $[M] = [R^n/K]$. Details left to the reader.

00JD **Lemma 10.55.1.** *If R is an Artinian local ring then the length function defines a natural abelian group homomorphism $\text{length}_R : K'_0(R) \to \mathbf{Z}$.*

Proof. The length of any finite R-module is finite, because it is the quotient of R^n which has finite length by Lemma 10.53.6. And the length function is additive, see Lemma 10.52.3. □

The second of the two is denoted $K_0(R)$ and has the following properties:

(1) For every finite projective R-module M there is given an element $[M]$ in $K_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite projective R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The construction of this group is done as above.

We note that there is an obvious map $K_0(R) \to K'_0(R)$ which is not an isomorphism in general.

00JE **Example 10.55.2.** Note that if $R = k$ is a field then we clearly have $K_0(k) = K'_0(k) \cong \mathbf{Z}$ with the isomorphism given by the dimension function (which is also the length function).

0FJ8 **Example 10.55.3.** Let R be a PID. We claim $K_0(R) = K'_0(R) = \mathbf{Z}$. Namely, any finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 10.15.8. Given a short exact sequence of finite free modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 10.5.2). We conclude $K_0(R) = \mathbf{Z}$.

The structure theorem for modules of a PID says that any finitely generated R-module is of the form $M = R^{\oplus r} \oplus R/(d_1) \oplus \ldots \oplus R/(d_k)$. Consider the short exact sequence

$$0 \to (d_i) \to R \to R/(d_i) \to 0$$

Since the ideal (d_i) is isomorphic to R as a module (it is free with generator d_i), in $K'_0(R)$ we have $[(d_i)] = [R]$. Then $[R/(d_i)] = [(d_i)] - [R] = 0$. From this it

follows that a torsion module has zero class in $K'_0(R)$. Using the rank of the free part gives an identification $K'_0(R) = \mathbf{Z}$ and the canonical homomorphism from $K_0(R) \to K'_0(R)$ is an isomorphism.

00JF **Example 10.55.4.** Let k be a field. Then $K_0(k[x]) = K'_0(k[x]) = \mathbf{Z}$. This follows from Example 10.55.3 as $R = k[x]$ is a PID.

00JG **Example 10.55.5.** Let k be a field. Let $R = \{f \in k[x] \mid f(0) = f(1)\}$, compare Example 10.27.4. In this case $K_0(R) \cong k^* \oplus \mathbf{Z}$, but $K'_0(R) = \mathbf{Z}$.

00JH **Lemma 10.55.6.** *Let $R = R_1 \times R_2$. Then $K_0(R) = K_0(R_1) \times K_0(R_2)$ and $K'_0(R) = K'_0(R_1) \times K'_0(R_2)$*

Proof. Omitted. □

00JI **Lemma 10.55.7.** *Let R be an Artinian local ring. The map $length_R : K'_0(R) \to \mathbf{Z}$ of Lemma 10.55.1 is an isomorphism.*

Proof. Omitted. □

00JJ **Lemma 10.55.8.** *Let $(R, \mathfrak{m})$ be a local ring. Every finite projective R-module is finite free. The map $rank_R : K_0(R) \to \mathbf{Z}$ defined by $[M] \to rank_R(M)$ is well defined and an isomorphism.*

Proof. Let P be a finite projective R-module. Choose elements $x_1, \ldots, x_n \in P$ which map to a basis of $P/\mathfrak{m}P$. By Nakayama's Lemma 10.20.1 these elements generate P. The corresponding surjection $u : R^{\oplus n} \to P$ has a splitting as P is projective. Hence $R^{\oplus n} = P \oplus Q$ with $Q = \mathrm{Ker}(u)$. It follows that $Q/\mathfrak{m}Q = 0$, hence Q is zero by Nakayama's lemma. In this way we see that every finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 10.15.8. Given a short exact sequence of finite free R-modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\mathrm{rank}(M) = \mathrm{rank}(M') + \mathrm{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 10.5.2). We conclude $K_0(R) = \mathbf{Z}$. □

00JK **Lemma 10.55.9.** *Let R be a local Artinian ring. There is a commutative diagram*

$$\begin{array}{ccc} K_0(R) & \longrightarrow & K'_0(R) \\ {\scriptstyle rank_R}\downarrow & & \downarrow{\scriptstyle length_R} \\ \mathbf{Z} & \xrightarrow{length_R(R)} & \mathbf{Z} \end{array}$$

where the vertical maps are isomorphisms by Lemmas 10.55.7 and 10.55.8.

Proof. Let P be a finite projective R-module. We have to show that $\mathrm{length}_R(P) = \mathrm{rank}_R(P)\mathrm{length}_R(R)$. By Lemma 10.55.8 the module P is finite free. So $P \cong R^{\oplus n}$ for some $n \geq 0$. Then $\mathrm{rank}_R(P) = n$ and $\mathrm{length}_R(R^{\oplus n}) = n\mathrm{length}_R(R)$ by additivity of lengths (Lemma 10.52.3). Thus the result holds. □

10.56. Graded rings

00JL A *graded ring* will be for us a ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ of the underlying abelian group such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d>0} S_d$. A *graded module* will be an S-module M endowed with a direct sum decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ of the underlying abelian group such that $S_d \cdot M_e \subset M_{d+e}$. Note that for modules we do allow nonzero elements in negative degrees. We think of S as a graded S-module by setting $S_{-k} = (0)$ for $k > 0$. An element x (resp. f) of M (resp. S) is called *homogeneous* if $x \in M_d$ (resp. $f \in S_d$) for some d. A *map of graded S-modules* is a map of S-modules $\varphi : M \to M'$ such that $\varphi(M_d) \subset M'_d$. We do not allow maps to shift degrees. Let us denote $\mathrm{GrHom}_0(M, N)$ the S_0-module of homomorphisms of graded modules from M to N.

At this point there are the notions of graded ideal, graded quotient ring, graded submodule, graded quotient module, graded tensor product, etc. We leave it to the reader to find the relevant definitions, and lemmas. For example: A short exact sequence of graded modules is short exact in every degree.

Given a graded ring S, a graded S-module M and $n \in \mathbf{Z}$ we denote $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$. This is called the *twist of M by n*. In particular we get modules $S(n)$, $n \in \mathbf{Z}$ which will play an important role in the study of projective schemes. There are some obvious functorial isomorphisms such as $(M \oplus N)(n) = M(n) \oplus N(n)$, $(M \otimes_S N)(n) = M \otimes_S N(n) = M(n) \otimes_S N$. In addition we can define a graded S-module structure on the S_0-module

$$\mathrm{GrHom}(M, N) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{GrHom}_n(M, N), \quad \mathrm{GrHom}_n(M, N) = \mathrm{GrHom}_0(M, N(n)).$$

We omit the definition of the multiplication.

0EKB **Lemma 10.56.1.** *Let S be a graded ring. Let M be a graded S-module.*

(1) *If $S_+M = M$ and M is finite, then $M = 0$.*
(2) *If $N, N' \subset M$ are graded submodules, $M = N + S_+N'$, and N' is finite, then $M = N$.*
(3) *If $N \to M$ is a map of graded modules, $N/S_+N \to M/S_+M$ is surjective, and M is finite, then $N \to M$ is surjective.*
(4) *If $x_1, \ldots, x_n \in M$ are homogeneous and generate M/S_+M and M is finite, then $x_1, \ldots, x_n$ generate M.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_r$ of M over S. We may assume that y_i is homogeneous of degree d_i. After renumbering we may assume $d_r = \min(d_i)$. Then the condition that $S_+M = M$ implies $y_r = 0$. Hence $M = 0$ by induction on r. Part (2) follows by applying (1) to M/N. Part (3) follows by applying (2) to the submodules $\mathrm{Im}(N \to M)$ and M. Part (4) follows by applying (3) to the module map $\bigoplus S(-d_i) \to M$, $(s_1, \ldots, s_n) \mapsto \sum s_i x_i$. □

Let S be a graded ring. Let $d \geq 1$ be an integer. We set $S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$. We think of $S^{(d)}$ as a graded ring with degree n summand $(S^{(d)})_n = S_{nd}$. Given a graded S-module M we can similarly consider $M^{(d)} = \bigoplus_{n \in \mathbf{Z}} M_{nd}$ which is a graded $S^{(d)}$-module.

0EGH **Lemma 10.56.2.** *Let S be a graded ring, which is finitely generated over S_0. Then for all sufficiently divisible d the algebra $S^{(d)}$ is generated in degree 1 over S_0.*

Proof. Say S is generated by $f_1, \ldots, f_r \in S$ over S_0. After replacing f_i by their homogeneous parts, we may assume f_i is homogeneous of degree $d_i > 0$. Then any element of S_n is a linear combination with coefficients in S_0 of monomials $f_1^{e_1} \ldots f_r^{e_r}$ with $\sum e_i d_i = n$. Let m be a multiple of $\text{lcm}(d_i)$. For any $N \geq r$ if

$$\sum e_i d_i = Nm$$

then for some i we have $e_i \geq m/d_i$ by an elementary argument. Hence every monomial of degree Nm is a product of a monomial of degree m, namely f_i^{m/d_i}, and a monomial of degree $(N-1)m$. It follows that any monomial of degree nrm with $n \geq 2$ is a product of monomials of degree rm. Thus $S^{(rm)}$ is generated in degree 1 over S_0. □

077G **Lemma 10.56.3.** *Let $R \to S$ be a homomorphism of graded rings. Let $S' \subset S$ be the integral closure of R in S. Then*

$$S' = \bigoplus\nolimits_{d \geq 0} S' \cap S_d,$$

i.e., S' is a graded R-subalgebra of S.

Proof. We have to show the following: If $s = s_n + s_{n+1} + \ldots + s_m \in S'$, then each homogeneous part $s_j \in S'$. We will prove this by induction on $m - n$ over all homomorphisms $R \to S$ of graded rings. First note that it is immediate that s_0 is integral over R_0 (hence over R) as there is a ring map $S \to S_0$ compatible with the ring map $R \to R_0$. Thus, after replacing s by $s - s_0$, we may assume $n > 0$. Consider the extension of graded rings $R[t, t^{-1}] \to S[t, t^{-1}]$ where t has degree 0. There is a commutative diagram

$$\begin{array}{ccc} S[t,t^{-1}] & \xrightarrow{s \mapsto t^{\deg(s)} s} & S[t,t^{-1}] \\ \uparrow & & \uparrow \\ R[t,t^{-1}] & \xrightarrow{r \mapsto t^{\deg(r)} r} & R[t,t^{-1}] \end{array}$$

where the horizontal maps are ring automorphisms. Hence the integral closure C of $S[t, t^{-1}]$ over $R[t, t^{-1}]$ maps into itself. Thus we see that

$$t^m(s_n + s_{n+1} + \ldots + s_m) - (t^n s_n + t^{n+1} s_{n+1} + \ldots + t^m s_m) \in C$$

which implies by induction hypothesis that each $(t^m - t^i)s_i \in C$ for $i = n, \ldots, m-1$. Note that for any ring A and $m > i \geq n > 0$ we have $A[t, t^{-1}]/(t^m - t^i - 1) \cong A[t]/(t^m - t^i - 1) \supset A$ because $t(t^{m-1} - t^{i-1}) = 1$ in $A[t]/(t^m - t^i - 1)$. Since $t^m - t^i$ maps to 1 we see the image of s_i in the ring $S[t]/(t^m - t^i - 1)$ is integral over $R[t]/(t^m - t^i - 1)$ for $i = n, \ldots, m-1$. Since $R \to R[t]/(t^m - t^i - 1)$ is finite we see that s_i is integral over R by transitivity, see Lemma 10.36.6. Finally, we also conclude that $s_m = s - \sum_{i=n,\ldots,m-1} s_i$ is integral over R. □

10.57. Proj of a graded ring

00JM

Let S be a graded ring. A *homogeneous ideal* is simply an ideal $I \subset S$ which is also a graded submodule of S. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous parts in S then $f_i \in I$ for each i. To check that a homogeneous ideal $\mathfrak{p}$ is prime it suffices to check that if $ab \in \mathfrak{p}$ with a, b homogeneous then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

00JN **Definition 10.57.1.** Let S be a graded ring. We define $\text{Proj}(S)$ to be the set of homogeneous prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. The set $\text{Proj}(S)$ is a subset of $\text{Spec}(S)$ and we endow it with the induced topology. The topological space $\text{Proj}(S)$ is called the *homogeneous spectrum* of the graded ring S.

Note that by construction there is a continuous map

$$\text{Proj}(S) \longrightarrow \text{Spec}(S_0).$$

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Let $f \in S_d$ and assume that $d \geq 1$. We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. If M is a graded S-module, then we define the $S_{(f)}$-module $M_{(f)}$ as the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd.

00JO **Lemma 10.57.2.** *Let S be a $\mathbf{Z}$-graded ring containing a homogeneous invertible element of positive degree. Then the set $G \subset \text{Spec}(S)$ of $\mathbf{Z}$-graded primes of S (with induced topology) maps homeomorphically to $\text{Spec}(S_0)$.*

Proof. First we show that the map is a bijection by constructing an inverse. Let $f \in S_d$, $d > 0$ be invertible in S. If $\mathfrak{p}_0$ is a prime of S_0, then $\mathfrak{p}_0 S$ is a $\mathbf{Z}$-graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$. And if $ab \in \mathfrak{p}_0 S$ with a, b homogeneous, then $a^d b^d / f^{\deg(a)+\deg(b)} \in \mathfrak{p}_0$. Thus either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, in other words either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. It follows that $\sqrt{\mathfrak{p}_0 S}$ is a $\mathbf{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak{p}_0$.

To show that the map is a homeomorphism we show that the image of $G \cap D(g)$ is open. If $g = \sum g_i$ with $g_i \in S_i$, then by the above $G \cap D(g)$ maps onto the set $\bigcup D(g_i^d/f^i)$ which is open. □

For $f \in S$ homogeneous of degree > 0 we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak{p} \in \text{Proj}(S) \mid I \subset \mathfrak{p}\}.$$

We will use more generally the notation $V_+(E)$ for any set E of homogeneous elements $E \subset S$.

00JP **Lemma 10.57.3** (Topology on Proj)**.** *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.*

(1) *The sets $D_+(f)$ are open in Proj(S).*
(2) *We have $D_+(ff') = D_+(f) \cap D_+(f')$.*
(3) *Let $g = g_0 + \ldots + g_m$ be an element of S with $g_i \in S_i$. Then*

$$D(g) \cap \text{Proj}(S) = (D(g_0) \cap \text{Proj}(S)) \cup \bigcup\nolimits_{i \geq 1} D_+(g_i).$$

(4) *Let $g_0 \in S_0$ be a homogeneous element of degree 0. Then*
$$D(g_0) \cap Proj(S) = \bigcup\nolimits_{f \in S_d,\ d \geq 1} D_+(g_0 f).$$
(5) *The open sets $D_+(f)$ form a basis for the topology of $Proj(S)$.*
(6) *Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural $\mathbf{Z}$-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms*
$$D_+(f) \leftarrow \{\mathbf{Z}\text{-graded primes of } S_f\} \to \mathrm{Spec}(S_{(f)}).$$
(7) *There exists an S such that $Proj(S)$ is not quasi-compact.*
(8) *The sets $V_+(I)$ are closed.*
(9) *Any closed subset $T \subset Proj(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*
(10) *For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.*

Proof. Since $D_+(f) = \mathrm{Proj}(S) \cap D(f)$, these sets are open. This proves (1). Also (2) follows as $D(ff') = D(f) \cap D(f')$. Similarly the sets $V_+(I) = \mathrm{Proj}(S) \cap V(I)$ are closed. This proves (8).

Suppose that $T \subset \mathrm{Proj}(S)$ is closed. Then we can write $T = \mathrm{Proj}(S) \cap V(J)$ for some ideal $J \subset S$. By definition of a homogeneous ideal if $g \in J$, $g = g_0 + \ldots + g_m$ with $g_d \in S_d$ then $g_d \in \mathfrak{p}$ for all $\mathfrak{p} \in T$. Thus, letting $I \subset S$ be the ideal generated by the homogeneous parts of the elements of J we have $T = V_+(I)$. This proves (9).

The formula for $\mathrm{Proj}(S) \cap D(g)$, with $g \in S$ is direct from the definitions. This proves (3). Consider the formula for $\mathrm{Proj}(S) \cap D(g_0)$. The inclusion of the right hand side in the left hand side is obvious. For the other inclusion, suppose $g_0 \notin \mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Proj}(S)$. If all $g_0 f \in \mathfrak{p}$ for all homogeneous f of positive degree, then we see that $S_+ \subset \mathfrak{p}$ which is a contradiction. This gives the other inclusion. This proves (4).

The collection of opens $D(g) \cap \mathrm{Proj}(S)$ forms a basis for the topology since the standard opens $D(g) \subset \mathrm{Spec}(S)$ form a basis for the topology on $\mathrm{Spec}(S)$. By the formulas above we can express $D(g) \cap \mathrm{Proj}(S)$ as a union of opens $D_+(f)$. Hence the collection of opens $D_+(f)$ forms a basis for the topology also. This proves (5).

Proof of (6). First we note that $D_+(f)$ may be identified with a subset (with induced topology) of $D(f) = \mathrm{Spec}(S_f)$ via Lemma 10.17.6. Note that the ring S_f has a $\mathbf{Z}$-grading. The homogeneous elements are of the form r/f^n with $r \in S$ homogeneous and have degree $\deg(r/f^n) = \deg(r) - n\deg(f)$. The subset $D_+(f)$ corresponds exactly to those prime ideals $\mathfrak{p} \subset S_f$ which are $\mathbf{Z}$-graded ideals (i.e., generated by homogeneous elements). Hence we have to show that the set of $\mathbf{Z}$-graded prime ideals of S_f maps homeomorphically to $\mathrm{Spec}(S_{(f)})$. This follows from Lemma 10.57.2.

Let $S = \mathbf{Z}[X_1, X_2, X_3, \ldots]$ with grading such that each X_i has degree 1. Then it is easy to see that
$$\mathrm{Proj}(S) = \bigcup\nolimits_{i=1}^{\infty} D_+(X_i)$$
does not have a finite refinement. This proves (7).

Let $I \subset S$ be a graded ideal. If $\sqrt{I} \supset S_+$ then $V_+(I) = \emptyset$ since every prime $\mathfrak{p} \in \mathrm{Proj}(S)$ does not contain S_+ by definition. Conversely, suppose that $S_+ \not\subset \sqrt{I}$.

Then we can find an element $f \in S_+$ such that f is not nilpotent modulo I. Clearly this means that one of the homogeneous parts of f is not nilpotent modulo I, in other words we may (and do) assume that f is homogeneous. This implies that $IS_f \neq S_f$, in other words that $(S/I)_f$ is not zero. Hence $(S/I)_{(f)} \neq 0$ since it is a ring which maps into $(S/I)_f$. Pick a prime $\mathfrak{q} \subset (S/I)_{(f)}$. This corresponds to a graded prime of S/I, not containing the irrelevant ideal $(S/I)_+$. And this in turn corresponds to a graded prime ideal $\mathfrak{p}$ of S, containing I but not containing S_+ as desired. This proves (10) and finishes the proof. □

00JQ **Example 10.57.4.** Let R be a ring. If $S = R[X]$ with $\deg(X) = 1$, then the natural map $\text{Proj}(S) \to \text{Spec}(R)$ is a bijection and in fact a homeomorphism. Namely, suppose $\mathfrak{p} \in \text{Proj}(S)$. Since $S_+ \not\subset \mathfrak{p}$ we see that $X \notin \mathfrak{p}$. Thus if $aX^n \in \mathfrak{p}$ with $a \in R$ and $n > 0$, then $a \in \mathfrak{p}$. It follows that $\mathfrak{p} = \mathfrak{p}_0 S$ with $\mathfrak{p}_0 = \mathfrak{p} \cap R$.

If $\mathfrak{p} \in \text{Proj}(S)$, then we define $S_{(\mathfrak{p})}$ to be the ring whose elements are fractions r/f where $r, f \in S$ are homogeneous elements of the same degree such that $f \notin \mathfrak{p}$. As usual we say $r/f = r'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(rf' - r'f) = 0$. Given a graded S-module M we let $M_{(\mathfrak{p})}$ be the $S_{(\mathfrak{p})}$-module whose elements are fractions x/f with $x \in M$ and $f \in S$ homogeneous of the same degree such that $f \notin \mathfrak{p}$. We say $x/f = x'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(xf' - x'f) = 0$.

00JR **Lemma 10.57.5.** *Let S be a graded ring. Let M be a graded S-module. Let $\mathfrak{p}$ be an element of Proj(S). Let $f \in S$ be a homogeneous element of positive degree such that $f \notin \mathfrak{p}$, i.e., $\mathfrak{p} \in D_+(f)$. Let $\mathfrak{p}' \subset S_{(f)}$ be the element of* $\text{Spec}(S_{(f)})$ *corresponding to $\mathfrak{p}$ as in Lemma 10.57.3. Then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$ and compatibly $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$.*

Proof. We define a map $\psi : M_{(\mathfrak{p})} \to (M_{(f)})_{\mathfrak{p}'}$. Let $x/g \in M_{(\mathfrak{p})}$. We set

$$\psi(x/g) = (xg^{\deg(f)-1}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)}).$$

This makes sense since $\deg(x) = \deg(g)$ and since $g^{\deg(f)}/f^{\deg(g)} \notin \mathfrak{p}'$. We omit the verification that ψ is well defined, a module map and an isomorphism. Hint: the inverse sends $(x/f^n)/(g/f^m)$ to $(xf^m)/(gf^n)$. □

Here is a graded variant of Lemma 10.15.2.

00JS **Lemma 10.57.6.** *Suppose S is a graded ring, $\mathfrak{p}_i$, $i = 1, \ldots, r$ homogeneous prime ideals and $I \subset S_+$ a graded ideal. Assume $I \not\subset \mathfrak{p}_i$ for all i. Then there exists a homogeneous element $x \in I$ of positive degree such that $x \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in I$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r-1$. If $x \notin \mathfrak{p}_r$ we are done. So assume $x \in \mathfrak{p}_r$. If $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ then $I \subset \mathfrak{p}_r$ a contradiction. Pick $y \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ homogeneous and $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. □

00JT **Lemma 10.57.7.** *Let S be a graded ring. Let $\mathfrak{p} \subset S$ be a prime. Let $\mathfrak{q}$ be the homogeneous ideal of S generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal of S.*

Proof. To prove that $\mathfrak{q}$ is prime, it suffices to check that if $f, g \in S$ are homogeneous and $fg \in \mathfrak{q}$, then either f or g in $\mathfrak{q}$. Then $fg \in \mathfrak{p}$ because $\mathfrak{p} \subset \mathfrak{q}$. Since $\mathfrak{p}$ is prime

we see that either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Since f and g are homogeneous, it then is clear that either $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$. □

00JU **Lemma 10.57.8.** *Let S be a graded ring.*

(1) *Any minimal prime of S is a homogeneous ideal of S.*
(2) *Given a homogeneous ideal $I \subset S$ any minimal prime over I is homogeneous.*

Proof. The first assertion holds because the prime $\mathfrak{q}$ constructed in Lemma 10.57.7 satisfies $\mathfrak{q} \subset \mathfrak{p}$. The second because we may consider S/I and apply the first part. □

07Z2 **Lemma 10.57.9.** *Let R be a ring. Let S be a graded R-algebra. Let $f \in S_+$ be homogeneous. Assume that S is of finite type over R. Then*

(1) *the ring $S_{(f)}$ is of finite type over R, and*
(2) *for any finite graded S-module M the module $M_{(f)}$ is a finite $S_{(f)}$-module.*

Proof. Choose $f_1, \ldots, f_n \in S$ which generate S as an R-algebra. We may assume that each f_i is homogeneous (by decomposing each f_i into its homogeneous components). An element of $S_{(f)}$ is a sum of the form

$$\sum\nolimits_{e\deg(f)=\sum e_i \deg(f_i)} \lambda_{e_1\ldots e_n} f_1^{e_1} \ldots f_n^{e_n}/f^e$$

with $\lambda_{e_1\ldots e_n} \in R$. Thus $S_{(f)}$ is generated as an R-algebra by the $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with the property that $e\deg(f) = \sum e_i \deg(f_i)$. If $e_i \geq \deg(f)$ then we can write this as

$$f_1^{e_1} \ldots f_n^{e_n}/f^e = f_i^{\deg(f)}/f^{\deg(f_i)} \cdot f_1^{e_1} \ldots f_i^{e_i - \deg(f)} \ldots f_n^{e_n}/f^{e-\deg(f_i)}$$

Thus we only need the elements $f_i^{\deg(f)}/f^{\deg(f_i)}$ as well as the elements $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with $e\deg(f) = \sum e_i \deg(f_i)$ and $e_i < \deg(f)$. This is a finite list and we see that (1) is true.

To see (2) suppose that M is generated by homogeneous elements $x_1, \ldots, x_m$. Then arguing as above we find that $M_{(f)}$ is generated as an $S_{(f)}$-module by the finite list of elements of the form $f_1^{e_1} \ldots f_n^{e_n} x_j/f^e$ with $e\deg(f) = \sum e_i \deg(f_i) + \deg(x_j)$ and $e_i < \deg(f)$. □

052N **Lemma 10.57.10.** *Let R be a ring. Let R' be a finite type R-algebra, and let M be a finite R'-module. There exists a graded R-algebra S, a graded S-module N and an element $f \in S$ homogeneous of degree 1 such that*

(1) *$R' \cong S_{(f)}$ and $M \cong N_{(f)}$ (as modules),*
(2) *$S_0 = R$ and S is generated by finitely many elements of degree 1 over R, and*
(3) *N is a finite S-module.*

Proof. We may write $R' = R[x_1, \ldots, x_n]/I$ for some ideal I. For an element $g \in R[x_1, \ldots, x_n]$ denote $\tilde{g} \in R[X_0, \ldots, X_n]$ the element homogeneous of minimal degree such that $g = \tilde{g}(1, x_1, \ldots, x_n)$. Let $\tilde{I} \subset R[X_0, \ldots, X_n]$ generated by all elements $\tilde{g}$, $g \in I$. Set $S = R[X_0, \ldots, X_n]/\tilde{I}$ and denote f the image of X_0 in S. By construction we have an isomorphism

$$S_{(f)} \longrightarrow R', \quad X_i/X_0 \longmapsto x_i.$$

To do the same thing with the module M we choose a presentation

$$M = (R')^{\oplus r}/\sum\nolimits_{j\in J} R'k_j$$

with $k_j = (k_{1j}, \ldots, k_{rj})$. Let $d_{ij} = \deg(\tilde{k}_{ij})$. Set $d_j = \max\{d_{ij}\}$. Set $K_{ij} = X_0^{d_j - d_{ij}}\tilde{k}_{ij}$ which is homogeneous of degree d_j. With this notation we set

$$N = \mathrm{Coker}\left(\bigoplus\nolimits_{j\in J} S(-d_j) \xrightarrow{(K_{ij})} S^{\oplus r}\right)$$

which works. Some details omitted. □

10.58. Noetherian graded rings

00JV A bit of theory on Noetherian graded rings including some material on Hilbert polynomials.

07Z4 **Lemma 10.58.1.** *Let S be a graded ring. A set of homogeneous elements $f_i \in S_+$ generates S as an algebra over S_0 if and only if they generate S_+ as an ideal of S.*

Proof. If the f_i generate S as an algebra over S_0 then every element in S_+ is a polynomial without constant term in the f_i and hence S_+ is generated by the f_i as an ideal. Conversely, suppose that $S_+ = \sum Sf_i$. We will prove that any element f of S can be written as a polynomial in the f_i with coefficients in S_0. It suffices to do this for homogeneous elements. Say f has degree d. Then we may perform induction on d. The case $d = 0$ is immediate. If $d > 0$ then $f \in S_+$ hence we can write $f = \sum g_i f_i$ for some $g_i \in S$. As S is graded we can replace g_i by its homogeneous component of degree $d - \deg(f_i)$. By induction we see that each g_i is a polynomial in the f_i and we win. □

00JW **Lemma 10.58.2.** *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated as an ideal of S.*

Proof. It is clear that if S is Noetherian then $S_0 = S/S_+$ is Noetherian and S_+ is finitely generated. Conversely, assume S_0 is Noetherian and S_+ finitely generated as an ideal of S. Pick generators $S_+ = (f_1, \ldots, f_n)$. By decomposing the f_i into homogeneous pieces we may assume each f_i is homogeneous. By Lemma 10.58.1 we see that $S_0[X_1, \ldots X_n] \to S$ sending X_i to f_i is surjective. Thus S is Noetherian by Lemma 10.31.1. □

00JX **Definition 10.58.3.** Let A be an abelian group. We say that a function $f : n \mapsto f(n) \in A$ defined for all sufficient large integers n is a *numerical polynomial* if there exists $r \geq 0$, elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum\nolimits_{i=0}^{r} \binom{n}{i} a_i$$

for all $n \gg 0$.

The reason for using the binomial coefficients is the elementary fact that any polynomial $P \in \mathbf{Q}[T]$ all of whose values at integer points are integers, is equal to a sum $P(T) = \sum a_i \binom{T}{i}$ with $a_i \in \mathbf{Z}$. Note that in particular the expressions $\binom{T+1}{i+1}$ are of this form.

00JY **Lemma 10.58.4.** *If $A \to A'$ is a homomorphism of abelian groups and if $f : n \mapsto f(n) \in A$ is a numerical polynomial, then so is the composition.*

Proof. This is immediate from the definitions. □

00JZ **Lemma 10.58.5.** *Suppose that $f : n \mapsto f(n) \in A$ is defined for all n sufficiently large and suppose that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then f is a numerical polynomial.*

Proof. Let $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$ for all $n \gg 0$. Set $g(n) = f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$. Then $g(n) - g(n-1) = 0$ for all $n \gg 0$. Hence g is eventually constant, say equal to a_{-1}. We leave it to the reader to show that $a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ has the required shape (see remark above the lemma). □

00K0 **Lemma 10.58.6.** *If M is a finitely generated graded S-module, and if S is finitely generated over S_0, then each M_n is a finite S_0-module.*

Proof. Suppose the generators of M are m_i and the generators of S are f_i. By taking homogeneous components we may assume that the m_i and the f_i are homogeneous and we may assume $f_i \in S_+$. In this case it is clear that each M_n is generated over S_0 by the "monomials" $\prod f_i^{e_i} m_j$ whose degree is n. □

00K1 **Proposition 10.58.7.** *Suppose that S is a Noetherian graded ring and M a finite graded S-module. Consider the function*

$$\mathbf{Z} \longrightarrow K_0'(S_0), \quad n \longmapsto [M_n]$$

see Lemma 10.58.6. If S_+ is generated by elements of degree 1, then this function is a numerical polynomial.

Proof. We prove this by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n \gg 0$ and the result holds. To prove the induction step, let $x \in S_1$ be one of a minimal set of generators, such that the induction hypothesis applies to the graded ring $S/(x)$.

First we show the result holds if x is nilpotent on M. This we do by induction on the minimal integer r such that $x^r M = 0$. If $r = 1$, then M is a module over S/xS and the result holds (by the other induction hypothesis). If $r > 1$, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that the integers r', r'' are strictly smaller than r. Thus we know the result for M'' and M'. Hence we get the result for M because of the relation $[M_d] = [M'_d] + [M''_d]$ in $K_0'(S_0)$.

If x is not nilpotent on M, let $M' \subset M$ be the largest submodule on which x is nilpotent. Consider the exact sequence $0 \to M' \to M \to M/M' \to 0$ we see again it suffices to prove the result for M/M'. In other words we may assume that multiplication by x is injective.

Let $\overline{M} = M/xM$. Note that the map $x : M \to M$ is *not* a map of graded S-modules, since it does not map M_d into M_d. Namely, for each d we have the following short exact sequence

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1} \to 0$$

This proves that $[M_{d+1}] - [M_d] = [\overline{M}_{d+1}]$. Hence we win by Lemma 10.58.5. □

02CD **Remark 10.58.8.** If S is still Noetherian but S is not generated in degree 1, then the function associated to a graded S-module is a periodic polynomial (i.e., it is a numerical polynomial on the congruence classes of integers modulo n for some n).

00K2 **Example 10.58.9.** Suppose that $S = k[X_1, \ldots, X_d]$. By Example 10.55.2 we may identify $K_0(k) = K_0'(k) = \mathbf{Z}$. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module gives rise to a numerical polynomial $n \mapsto \dim_k(M_n)$.

00K3 **Lemma 10.58.10.** *Let k be a field. Suppose that $I \subset k[X_1, \ldots, X_d]$ is a nonzero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ (see Example 10.58.9) has degree $< d - 1$ (or is zero if $d = 1$).*

Proof. The numerical polynomial associated to the graded module $k[X_1, \ldots, X_d]$ is $n \mapsto \binom{n-1+d}{d-1}$. For any nonzero homogeneous $f \in I$ of degree e and any degree $n >> e$ we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$ and hence $\dim_k(I_n) \geq \binom{n-e-1+d}{d-1}$. Hence $\dim_k(M_n) \leq \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$. We win because the last expression has degree $< d - 1$ (or is zero if $d = 1$). □

10.59. Noetherian local rings

00K4 In all of this section $(R, \mathfrak{m}, \kappa)$ is a Noetherian local ring. We develop some theory on Hilbert functions of modules in this section. Let M be a finite R-module. We define the *Hilbert function* of M to be the function

$$\varphi_M : n \longmapsto \text{length}_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M)$$

defined for all integers $n \geq 0$. Another important invariant is the function

$$\chi_M : n \longmapsto \text{length}_R(M/\mathfrak{m}^{n+1} M)$$

defined for all integers $n \geq 0$. Note that we have by Lemma 10.52.3 that

$$\chi_M(n) = \sum\nolimits_{i=0}^{n} \varphi_M(i).$$

There is a variant of this construction which uses an ideal of definition.

07DU **Definition 10.59.1.** Let $(R, \mathfrak{m})$ be a local Noetherian ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ is called *an ideal of definition of R.*

Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak{m}^r \subset I$ for some r, see Lemma 10.32.5. Hence any finite R-module annihilated by a power of I has a finite length, see Lemma 10.52.8. Thus it makes sense to define

$$\varphi_{I,M}(n) = \text{length}_R(I^n M/I^{n+1} M) \quad \text{and} \quad \chi_{I,M}(n) = \text{length}_R(M/I^{n+1} M)$$

for all $n \geq 0$. Again we have that

$$\chi_{I,M}(n) = \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i).$$

00K5 **Lemma 10.59.2.** *Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \geq c_2$ we have*

$$c_1 + \chi_{I,M'}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M'}(n)$$

Proof. Since M/M' has finite length there is a $c_2 \geq 0$ such that $I^{c_2} M \subset M'$. Let $c_1 = \text{length}_R(M/M')$. For $n \geq c_2$ we have

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1} M) \\ &= c_1 + \text{length}_R(M'/I^{n+1} M) \\ &\leq c_1 + \text{length}_R(M'/I^{n+1} M') \\ &= c_1 + \chi_{I,M'}(n) \end{aligned}$$

On the other hand, since $I^{c_2}M \subset M'$, we have $I^nM \subset I^{n-c_2}M'$ for $n \geq c_2$. Thus for $n \geq c_2$ we get

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\ &= c_1 + \chi_{I,M'}(n - c_2) \end{aligned}$$

which finishes the proof. □

00K6 **Lemma 10.59.3.** *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength l and $c \geq 0$ such that*

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + l$$

and

$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$$

for all $n \geq c$.

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 10.51.2 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$ for $n \geq c$. From the short exact sequence

$$0 \to M'/M' \cap I^nM \to M/I^nM \to M''/I^nM'' \to 0$$

and additivity of lengths (Lemma 10.52.3) we obtain the equality

$$\chi_{I,M}(n-1) = \chi_{I,M''}(n-1) + \chi_{I,N}(n-c-1) + \text{length}_R(M'/N)$$

for $n \geq c$. We have $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ and similarly for the modules M'' and N. Hence we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$ for $n \geq c$. □

00K7 **Lemma 10.59.4.** *Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$ for $n \geq 1$.*

Proof. There exists an integer $c \geq 1$ such that $(I')^c \subset I$. Hence we get a surjection $M/(I')^{c(n+1)}M \to M/I^{n+1}M$. Whence the result with $a = 2c - 1$. □

00K8 **Proposition 10.59.5.** *Let R be a Noetherian local ring. Let M be a finite R-module. Let $I \subset R$ be an ideal of definition. The Hilbert function $\varphi_{I,M}$ and the function $\chi_{I,M}$ are numerical polynomials.*

Proof. Consider the graded ring $S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots = \bigoplus_{d \geq 0} I^d/I^{d+1}$. Consider the graded S-module $N = M/IM \oplus IM/I^2M \oplus \ldots = \bigoplus_{d \geq 0} I^dM/I^{d+1}M$. This pair (S, N) satisfies the hypotheses of Proposition 10.58.7. Hence the result for $\varphi_{I,M}$ follows from that proposition and Lemma 10.55.1. The result for $\chi_{I,M}$ follows from this and Lemma 10.58.5. □

09CA **Definition 10.59.6.** Let R be a Noetherian local ring. Let M be a finite R-module. The *Hilbert polynomial* of M over R is the element $P(t) \in \mathbf{Q}[t]$ such that $P(n) = \varphi_M(n)$ for $n \gg 0$.

By Proposition 10.59.5 we see that the Hilbert polynomial exists.

00K9 **Lemma 10.59.7.** *Let R be a Noetherian local ring. Let M be a finite R-module.*

(1) *The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.*
(2) *The degree of the numerical polynomial $\chi_{I,M}$ is independent of the ideal of definition I.*

Proof. Part (2) follows immediately from Lemma 10.59.4. Part (1) follows from (2) because $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ for $n \geq 1$. □

00KA **Definition 10.59.8.** Let R be a local Noetherian ring and M a finite R-module. We denote $d(M)$ the element of $\{-\infty, 0, 1, 2, \ldots\}$ defined as follows:

(1) If $M = 0$ we set $d(M) = -\infty$,
(2) if $M \neq 0$ then $d(M)$ is the degree of the numerical polynomial χ_M.

If $\mathfrak{m}^n M \neq 0$ for all n, then we see that $d(M)$ is the degree $+1$ of the Hilbert polynomial of M.

00KB **Lemma 10.59.9.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let M be a finite R-module which does not have finite length. If $M' \subset M$ is a submodule with finite colength, then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree $<$ degree of either polynomial.*

Proof. Follows from Lemma 10.59.2 by elementary calculus. □

00KC **Lemma 10.59.10.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then*

(1) *if M' does not have finite length, then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree $<$ the degree of $\chi_{I,M'}$,*
(2) $\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$, *and*
(3) $\max\{d(M'), d(M'')\} = d(M)$,

Proof. We first prove (1). Let $N \subset M'$ be as in Lemma 10.59.3. By Lemma 10.59.9 the numerical polynomial $\chi_{I,M'} - \chi_{I,N}$ has degree $<$ the common degree of $\chi_{I,M'}$ and $\chi_{I,N}$. By Lemma 10.59.3 the difference

$$\chi_{I,M}(n) - \chi_{I,M''}(n) - \chi_{I,N}(n-c)$$

is constant for $n \gg 0$. By elementary calculus the difference $\chi_{I,N}(n) - \chi_{I,N}(n-c)$ has degree $<$ the degree of $\chi_{I,N}$ which is bigger than zero (see above). Putting everything together we obtain (1).

Note that the leading coefficients of $\chi_{I,M'}$ and $\chi_{I,M''}$ are nonnegative. Thus the degree of $\chi_{I,M'} + \chi_{I,M''}$ is equal to the maximum of the degrees. Thus if M' does not have finite length, then (2) follows from (1). If M' does have finite length, then $I^n M \to I^n M''$ is an isomorphism for all $n \gg 0$ by Artin-Rees (Lemma 10.51.2). Thus $M/I^n M \to M''/I^n M''$ is a surjection with kernel M' for $n \gg 0$ and we see that $\chi_{I,M}(n) - \chi_{I,M''}(n) = \text{length}(M')$ for all $n \gg 0$. Thus (2) holds in this case also.

Proof of (3). This follows from (2) except if one of M, M', or M'' is zero. We omit the proof in these special cases. □

10.60. Dimension

00KD Please compare with Topology, Section 5.10.

0GIE **Definition 10.60.1.** Let R be a ring. A *chain of prime ideals* is a sequence $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ of prime ideals of R such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, n-1$. The *length* of this chain of prime ideals is n.

Recall that we have an inclusion reversing bijection between prime ideals of a ring R and irreducible closed subsets of $\mathrm{Spec}(R)$, see Lemma 10.26.1.

00KE **Definition 10.60.2.** The *Krull dimension* of the ring R is the Krull dimension of the topological space $\mathrm{Spec}(R)$, see Topology, Definition 5.10.1. In other words it is the supremum of the integers $n \geq 0$ such that R has a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

of length n.

00KF **Definition 10.60.3.** The *height* of a prime ideal $\mathfrak{p}$ of a ring R is the dimension of the local ring $R_\mathfrak{p}$.

00KG **Lemma 10.60.4.** *The Krull dimension of R is the supremum of the heights of its (maximal) primes.*

Proof. This is so because we can always add a maximal ideal at the end of a chain of prime ideals. □

00KH **Lemma 10.60.5.** *A Noetherian ring of dimension 0 is Artinian. Conversely, any Artinian ring is Noetherian of dimension zero.*

Proof. Assume R is a Noetherian ring of dimension 0. By Lemma 10.31.5 the space $\mathrm{Spec}(R)$ is Noetherian. By Topology, Lemma 5.9.2 we see that $\mathrm{Spec}(R)$ has finitely many irreducible components, say $\mathrm{Spec}(R) = Z_1 \cup \ldots \cup Z_r$. According to Lemma 10.26.1 each $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $\mathrm{Spec}(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the Jacobson radical $\bigcap \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. By Lemma 10.53.5 R is equal to $\prod R_{\mathfrak{p}_i}$. Since $R_{\mathfrak{p}_i}$ is also Noetherian and dimension 0, the previous arguments show that its radical $\mathfrak{p}_i R_{\mathfrak{p}_i}$ is locally nilpotent. Lemma 10.32.5 gives $\mathfrak{p}_i^n R_{\mathfrak{p}_i} = 0$ for some $n \geq 1$. By Lemma 10.52.8 we conclude that $R_{\mathfrak{p}_i}$ has finite length over R. Hence we conclude that R is Artinian by Lemma 10.53.6.

If R is an Artinian ring then by Lemma 10.53.6 it is Noetherian. All of its primes are maximal by a combination of Lemmas 10.53.3, 10.53.4 and 10.53.5. □

In the following we will use the invariant $d(-)$ defined in Definition 10.59.8. Here is a warm up lemma.

00KI **Lemma 10.60.6.** *Let R be a Noetherian local ring. Then* $\dim(R) = 0 \Leftrightarrow d(R) = 0$.

Proof. This is because $d(R) = 0$ if and only if R has finite length as an R-module. See Lemma 10.53.6. □

00KJ **Proposition 10.60.7.** *Let R be a ring. The following are equivalent:*

(1) *R is Artinian,*
(2) *R is Noetherian and* $\dim(R) = 0$,

(3) *R has finite length as a module over itself,*
(4) *R is a finite product of Artinian local rings,*
(5) *R is Noetherian and $\mathrm{Spec}(R)$ is a finite discrete topological space,*
(6) *R is a finite product of Noetherian local rings of dimension 0,*
(7) *R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,*
(8) *R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,*
(9) *R is Noetherian, has finitely many maximal ideals and its Jacobson radical ideal is nilpotent, and*
(10) *R is Noetherian and there are no strict inclusions among its primes.*

Proof. This is a combination of Lemmas 10.53.5, 10.53.6, 10.60.5, and 10.60.6. □

00KK **Lemma 10.60.8.** *Let R be a local Noetherian ring. The following are equivalent:*

00KL (1) $\dim(R) = 1$,
00KM (2) $d(R) = 1$,
00KN (3) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $V(x) = \{\mathfrak{m}\}$,*
00KO (4) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $\mathfrak{m} = \sqrt{(x)}$, and*
00KP (5) *there exists an ideal of definition generated by 1 element, and no ideal of definition is generated by 0 elements.*

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. According to Lemma 10.31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Thus the only prime containing x is $\mathfrak{m}$ and hence (3).

If (3) then $\mathfrak{m} = \sqrt{(x)}$ by Lemma 10.17.2, and hence (4). The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\mathrm{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition.

Assume (2). To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p})) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Lemma 10.60.6. But $R/(xR + \mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR + \mathfrak{p})$ and $\mathfrak{m}/(xR + \mathfrak{p})$ which gives the desired contradiction. □

00KQ **Proposition 10.60.9.** *Let R be a local Noetherian ring. Let $d \geq 0$ be an integer. The following are equivalent:*

00KR (1) $\dim(R) = d$,
00KS (2) $d(R) = d$,
00KT (3) *there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.*

Proof. This proof is really just the same as the proof of Lemma 10.60.8. We will prove the proposition by induction on d. By Lemmas 10.60.6 and 10.60.8 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. According to Lemma 10.31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d-1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_d$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d-1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{q}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR+\mathfrak{p})) \leq d-1$, and hence $\dim(R/(xR+\mathfrak{p})) \leq d-1$, by induction. Now $R/(xR+\mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR+\mathfrak{p}) \subset \mathfrak{q}_2/(xR+\mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR+\mathfrak{p})$ which gives $e-1 \leq d-1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. □

Let $(R, \mathfrak{m})$ be a Noetherian local ring. From the above it is clear that $\mathfrak{m}$ cannot be generated by fewer than $\dim(R)$ variables. By Nakayama's Lemma 10.20.1 the minimal number of generators of $\mathfrak{m}$ equals $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we have the following fundamental inequality

$$\dim(R) \leq \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

It turns out that the rings where equality holds have a lot of good properties. They are called regular local rings.

00KU **Definition 10.60.10.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

(1) A *system of parameters of* R is a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ which generates an ideal of definition of R,
(2) if there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$ then we call R a *regular local ring* and $x_1, \ldots, x_d$ a *regular system of parameters*.

The following lemmas are clear from the proofs of the lemmas and proposition above, but we spell them out so we have convenient references.

00KV **Lemma 10.60.11.** *Let R be a Noetherian ring. Let $x \in R$.*

(1) *If $\mathfrak{p}$ is minimal over (x) then the height of $\mathfrak{p}$ is 0 or 1.*

(2) *If* $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ *and* $\mathfrak{q}$ *is minimal over* $(\mathfrak{p}, x)$*, then there is no prime strictly between* $\mathfrak{p}$ *and* $\mathfrak{q}$*.*

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over x, then the only prime ideal of $R_\mathfrak{p}$ containing x is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. This is true because the primes of $R_\mathfrak{p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak{p}$, see Lemma 10.17.5. Hence Lemma 10.60.8 shows $\dim(R_\mathfrak{p}) = 1$ if x is not nilpotent in $R_\mathfrak{p}$. Of course, if x is nilpotent in $R_\mathfrak{p}$ the argument gives that $\mathfrak{p}R_\mathfrak{p}$ is the only prime ideal and we see that the height is 0.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height 1 or 0 in $R/\mathfrak{p}$. This immediately implies there cannot be a prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$. □

0BBZ **Lemma 10.60.12.** *Let* R *be a Noetherian ring. Let* $f_1, \ldots, f_r \in R$*.*

(1) *If* $\mathfrak{p}$ *is minimal over* $(f_1, \ldots, f_r)$ *then the height of* $\mathfrak{p}$ *is* $\leq r$*.*
(2) *If* $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ *and* $\mathfrak{q}$ *is minimal over* $(\mathfrak{p}, f_1, \ldots, f_r)$*, then every chain of primes between* $\mathfrak{p}$ *and* $\mathfrak{q}$ *has length at most* r*.*

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over $f_1, \ldots, f_r$, then the only prime ideal of $R_\mathfrak{p}$ containing $f_1, \ldots, f_r$ is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. This is true because the primes of $R_\mathfrak{p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak{p}$, see Lemma 10.17.5. Hence Proposition 10.60.9 shows $\dim(R_\mathfrak{p}) \leq r$.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height $\leq r$. This immediately implies the statement about chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$. □

00KW **Lemma 10.60.13.** *Suppose that* R *is a Noetherian local ring and* $x \in \mathfrak{m}$ *an element of its maximal ideal. Then* $\dim R \leq \dim R/xR + 1$*. If* x *is not contained in any of the minimal primes of* R *then equality holds. (For example if* x *is a nonzerodivisor.)*

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 10.60.9. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. □

02IE **Lemma 10.60.14.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. Suppose* $x_1, \ldots, x_d \in \mathfrak{m}$ *generate an ideal of definition and* $d = \dim(R)$*. Then* $\dim(R/(x_1, \ldots, x_i)) = d - i$ *for all* $i = 1, \ldots, d$*.*

Proof. Follows either from the proof of Proposition 10.60.9, or by using induction on d and Lemma 10.60.13. □

10.61. Applications of dimension theory

02IF We can use the results on dimension to prove certain rings have infinite spectra and to produce more Jacobson rings.

02IG **Lemma 10.61.1.** *Let* R *be a Noetherian local domain of dimension* ≥ 2*. A nonempty open subset* $U \subset \mathrm{Spec}(R)$ *is infinite.*

Proof. To get a contradiction, assume that $U \subset \mathrm{Spec}(R)$ is finite. In this case $(0) \in U$ and $\{(0)\}$ is an open subset of U (because the complement of $\{(0)\}$ is the union of the closures of the other points). Thus we may assume $U = \{(0)\}$.

Let $\mathfrak{m} \subset R$ be the maximal ideal. We can find an $x \in \mathfrak{m}$, $x \neq 0$ such that $V(x) \cup U = \mathrm{Spec}(R)$. In other words we see that $D(x) = \{(0)\}$. In particular we see that $\dim(R/xR) = \dim(R) - 1 \geq 1$, see Lemma 10.60.13. Let $\overline{y}_2, \ldots, \overline{y}_{\dim(R)} \in R/xR$ generate an ideal of definition of R/xR, see Proposition 10.60.9. Choose lifts $y_2, \ldots, y_{\dim(R)} \in R$, so that $x, y_2, \ldots, y_{\dim(R)}$ generate an ideal of definition in R. This implies that $\dim(R/(y_2)) = \dim(R) - 1$ and $\dim(R/(y_2, x)) = \dim(R) - 2$, see Lemma 10.60.14. Hence there exists a prime $\mathfrak{p}$ containing y_2 but not x. This contradicts the fact that $D(x) = \{(0)\}$. □

The rings $k[[t]]$ where k is a field, or the ring of p-adic numbers are Noetherian rings of dimension 1 with finitely many primes. This is the maximum dimension for which this can happen.

0ALV **Lemma 10.61.2.** *A Noetherian ring with finitely many primes has dimension ≤ 1.*

Proof. Let R be a Noetherian ring with finitely many primes. If R is a local domain, then the lemma follows from Lemma 10.61.1. If R is a domain, then $R_{\mathfrak{m}}$ has dimension ≤ 1 for all maximal ideals $\mathfrak{m}$ by the local case. Hence $\dim(R) \leq 1$ by Lemma 10.60.4. If R is general, then $\dim(R/\mathfrak{q}) \leq 1$ for every minimal prime $\mathfrak{q}$ of R. Since every prime contains a minimal prime (Lemma 10.17.2), this implies $\dim(R) \leq 1$. □

0ALW **Lemma 10.61.3.** *Let S be a nonzero finite type algebra over a field k. The following are equivalent*

(1) $\dim(S) = 0$,
(2) *S has finitely many primes,*
(3) *S has finitely many maximal ideals,*
(4) $\mathrm{Spec}(S)$ *satisfies one of the equivalent conditions of Lemma 10.26.5,*
(5) $\dim_k(S) < \infty$,
(6) *S is Artinian,*
(7) $\mathrm{Spec}(S)$ *is a discrete topological space,*
(8) *add more here.*

Proof. It is immediate from the definitions that (1) is equivalent to (4) by looking at part (5) of Lemma 10.26.5. Recall that $\mathrm{Spec}(S)$ is sober, Noetherian, and Jacobson, see Lemmas 10.26.2, 10.31.5, 10.35.2, and 10.35.4. If S has dimension 0, then every point defines an irreducible component and there are only a finite number of irreducible components (Topology, Lemma 5.9.2). Thus (1) implies (2). Trivially (2) implies (3). If (3) holds, then $\mathrm{Spec}(S)$ is discrete by Topology, Lemma 5.18.6 and hence the dimension of S is 0.

At this point we know that (1) – (4) are equivalent. The implication (5) $\Rightarrow$ (6) is Lemma 10.53.2. The implication (6) $\Rightarrow$ (7) follows from Proposition 10.60.7. The implication (7) $\Rightarrow$ (4) is immediate. Conversely, if S satisfies (1) – (4), then S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ all maximal. Note that $\kappa(\mathfrak{m}_i)$ is a finite extension of k by the Hilbert Nullstellensatz (Theorem 10.34.1). By Proposition 10.60.7 we also see that S is Artinian. Next, Lemma 10.53.6 tells us that $\mathrm{length}_S(S) < \infty$. Thus $\dim_k(S) < \infty$ by Lemma 10.52.12. We conclude that (1) – (7) are equivalent. (Note: another and more standard way to prove $\dim(S) = 0 \Rightarrow \dim_k(S) < \infty$ is to use Noether normalization, but we don't have this available to us yet.) □

00KX **Lemma 10.61.4.** *Noetherian Jacobson rings.*

(1) *Any Noetherian domain R of dimension 1 with infinitely many primes is Jacobson.*
(2) *Any Noetherian ring such that every prime $\mathfrak{p}$ is either maximal or contained in infinitely many prime ideals is Jacobson.*

Proof. Part (1) is a reformulation of Lemma 10.35.6.

Let R be a Noetherian ring such that every non-maximal prime $\mathfrak{p}$ is contained in infinitely many prime ideals. Assume $\mathrm{Spec}(R)$ is not Jacobson to get a contradiction. By Lemmas 10.26.1 and 10.31.5 we see that $\mathrm{Spec}(R)$ is a sober, Noetherian topological space. By Topology, Lemma 5.18.3 we see that there exists a non-maximal ideal $\mathfrak{p} \subset R$ such that $\{\mathfrak{p}\}$ is a locally closed subset of $\mathrm{Spec}(R)$. In other words, $\mathfrak{p}$ is not maximal and $\{\mathfrak{p}\}$ is an open subset of $V(\mathfrak{p})$. Consider a prime $\mathfrak{q} \subset R$ with $\mathfrak{p} \subset \mathfrak{q}$. Recall that the topology on the spectrum of $(R/\mathfrak{p})_\mathfrak{q} = R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is induced from that of $\mathrm{Spec}(R)$, see Lemmas 10.17.5 and 10.17.7. Hence we see that $\{(0)\}$ is a locally closed subset of $\mathrm{Spec}((R/\mathfrak{p})_\mathfrak{q})$. By Lemma 10.61.1 we conclude that $\dim((R/\mathfrak{p})_\mathfrak{q}) = 1$. Since this holds for every $\mathfrak{q} \supset \mathfrak{p}$ we conclude that $\dim(R/\mathfrak{p}) = 1$. At this point we use the assumption that $\mathfrak{p}$ is contained in infinitely many primes to see that $\mathrm{Spec}(R/\mathfrak{p})$ is infinite. Hence by part (1) of the lemma we see that $V(\mathfrak{p}) \cong \mathrm{Spec}(R/\mathfrak{p})$ is the closure of its closed points. This is the desired contradiction since it means that $\{\mathfrak{p}\} \subset V(\mathfrak{p})$ cannot be open. □

10.62. Support and dimension of modules

00KY Some basic results on the support and dimension of modules.

00L0 **Lemma 10.62.1.** *Let R be a Noetherian ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

First proof. By Lemma 10.5.4 it suffices to do the case $M = R/I$ for some ideal I. Consider the set S of ideals J such that the lemma does not hold for the module R/J, and order it by inclusion. To arrive at a contradiction, assume that S is not empty. Because R is Noetherian, S has a maximal element J. By definition of S, the ideal J cannot be prime. Pick $a, b \in R$ such that $ab \in J$, but neither $a \in J$ nor $b \in J$. Consider the filtration $0 \subset aR/(J \cap aR) \subset R/J$. Note that both the submodule $aR/(J \cap aR)$ and the quotient module $(R/J)/(aR/(J \cap aR))$ are cyclic modules; write them as R/J' and R/J'' so we have a short exact sequence $0 \to R/J' \to R/J \to R/J'' \to 0$. The inclusion $J \subset J'$ is strict as $b \in J'$ and the inclusion $J \subset J''$ is strict as $a \in J''$. Hence by maximality of J, both R/J' and R/J'' have a filtration as above and hence so does R/J. Contradiction. □

Second proof. For an R-module M we say $P(M)$ holds if there exists a filtration as in the statement of the lemma. Observe that P is stable under extensions and holds for 0. By Lemma 10.5.4 it suffices to prove $P(R/I)$ holds for every ideal I. If not then because R is Noetherian, there is a maximal counter example J. By Example 10.28.7 and Proposition 10.28.8 the ideal J is prime which is a contradiction. □

00L4 **Lemma 10.62.2.** *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.62.1. Then $Supp(M) = \bigcup V(\mathfrak{p}_i)$ and in particular $\mathfrak{p}_i \in Supp(M)$.*

Proof. This follows from Lemmas 10.40.5 and 10.40.9. □

00L5 **Lemma 10.62.3.** *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then $Supp(M) = \{\mathfrak{m}\}$ if and only if M has finite length over R.*

Proof. Assume that $\mathrm{Supp}(M) = \{\mathfrak{m}\}$. It suffices to show that all the primes $\mathfrak{p}_i$ in the filtration of Lemma 10.62.1 are the maximal ideal. This is clear by Lemma 10.62.2.

Suppose that M has finite length over R. Then $\mathfrak{m}^n M = 0$ by Lemma 10.52.4. Since some element of $\mathfrak{m}$ maps to a unit in $R_\mathfrak{p}$ for any prime $\mathfrak{p} \neq \mathfrak{m}$ in R we see $M_\mathfrak{p} = 0$. □

00L6 **Lemma 10.62.4.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Then $I^n M = 0$ for some $n \geq 0$ if and only if $Supp(M) \subset V(I)$.*

Proof. Indeed, $I^n M = 0$ is equivalent to $I^n \subset \mathrm{Ann}(M)$. Since R is Noetherian, this is equivalent to $I \subset \sqrt{\mathrm{Ann}(M)}$, see Lemma 10.32.5. This in turn is equivalent to $V(I) \supset V(\mathrm{Ann}(M))$, see Lemma 10.17.2. By Lemma 10.40.5 this is equivalent to $V(I) \supset \mathrm{Supp}(M)$. □

00L7 **Lemma 10.62.5.** *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.62.1. The minimal elements of the set $\{\mathfrak{p}_i\}$ are the minimal elements of $Supp(M)$. The number of times a minimal prime $\mathfrak{p}$ occurs is*

$$\#\{i \mid \mathfrak{p}_i = \mathfrak{p}\} = length_{R_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. The first statement follows because $\mathrm{Supp}(M) = \bigcup V(\mathfrak{p}_i)$, see Lemma 10.62.2. Let $\mathfrak{p} \in \mathrm{Supp}(M)$ be minimal. The support of $M_\mathfrak{p}$ is the set consisting of the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence by Lemma 10.62.3 the length of $M_\mathfrak{p}$ is finite and > 0. Next we note that $M_\mathfrak{p}$ has a filtration with subquotients $(R/\mathfrak{p}_i)_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$. These are zero if $\mathfrak{p}_i \not\subset \mathfrak{p}$ and equal to $\kappa(\mathfrak{p})$ if $\mathfrak{p}_i \subset \mathfrak{p}$ because by minimality of $\mathfrak{p}$ we have $\mathfrak{p}_i = \mathfrak{p}$ in this case. The result follows since $\kappa(\mathfrak{p})$ has length 1. □

00L8 **Lemma 10.62.6.** *Let R be a Noetherian local ring. Let M be a finite R-module. Then $d(M) = \dim(Supp(M))$ where $d(M)$ is as in Definition 10.59.8.*

Proof. Let $M_i, \mathfrak{p}_i$ be as in Lemma 10.62.1. By Lemma 10.59.10 we obtain the equality $d(M) = \max\{d(R/\mathfrak{p}_i)\}$. By Proposition 10.60.9 we have $d(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}_i)$. Trivially $\dim(R/\mathfrak{p}_i) = \dim V(\mathfrak{p}_i)$. Since all minimal primes of $\mathrm{Supp}(M)$ occur among the $\mathfrak{p}_i$ (Lemma 10.62.5) we win. □

0B51 **Lemma 10.62.7.** *Let R be a Noetherian ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then $\max\{\dim(Supp(M')), \dim(Supp(M''))\} = \dim(Supp(M))$.*

Proof. If R is local, this follows immediately from Lemmas 10.62.6 and 10.59.10. A more elementary argument, which works also if R is not local, is to use that $\mathrm{Supp}(M')$, $\mathrm{Supp}(M'')$, and $\mathrm{Supp}(M)$ are closed (Lemma 10.40.5) and that $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$ (Lemma 10.40.9). □

10.63. Associated primes

00L9 Here is the standard definition. For non-Noetherian rings and non-finite modules it may be more appropriate to use the definition in Section 10.66.

00LA **Definition 10.63.1.** Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *associated* to M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted $\mathrm{Ass}_R(M)$ or $\mathrm{Ass}(M)$.

0586 **Lemma 10.63.2.** *Let R be a ring. Let M be an R-module. Then $Ass(M) \subset Supp(M)$.*

Proof. If $m \in M$ has annihilator $\mathfrak{p}$, then in particular no element of $R \backslash \mathfrak{p}$ annihilates m. Hence m is a nonzero element of $M_\mathfrak{p}$, i.e., $\mathfrak{p} \in \mathrm{Supp}(M)$. □

02M3 **Lemma 10.63.3.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $Ass(M') \subset Ass(M)$ and $Ass(M) \subset Ass(M') \cup Ass(M'')$. Also $Ass(M' \oplus M'') = Ass(M') \cup Ass(M'')$.*

Proof. If $m' \in M'$, then the annihilator of m' viewed as an element of M' is the same as the annihilator of m' viewed as an element of M. Hence the inclusion $\mathrm{Ass}(M') \subset \mathrm{Ass}(M)$. Let $m \in M$ be an element whose annihilator is a prime ideal $\mathfrak{p}$. If there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $m' = gm \in M'$ then the annihilator of m' is $\mathfrak{p}$. If there does not exist a $g \in R$, $g \notin \mathfrak{p}$ such that $gm \in M'$, then the annilator of the image $m'' \in M''$ of m is $\mathfrak{p}$. This proves the inclusion $\mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$. We omit the proof of the final statement. □

00LB **Lemma 10.63.4.** *Let R be a ring, and M an R-module. Suppose there exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Then $Ass(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof. By induction on the length n of the filtration $\{M_i\}$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$. If $m \in M_{n-1}$ we are done by induction. If not, then m maps to a nonzero element of $M/M_{n-1} \cong R/\mathfrak{p}_n$. Hence we have $\mathfrak{p} \subset \mathfrak{p}_n$. If equality does not hold, then we can find $f \in \mathfrak{p}_n$, $f \notin \mathfrak{p}$. In this case the annihilator of fm is still $\mathfrak{p}$ and $fm \in M_{n-1}$. Thus we win by induction. □

00LC **Lemma 10.63.5.** *Let R be a Noetherian ring. Let M be a finite R-module. Then $Ass(M)$ is finite.*

Proof. Immediate from Lemma 10.63.4 and Lemma 10.62.1. □

02CE **Proposition 10.63.6.** *Let R be a Noetherian ring. Let M be a finite R-module. The following sets of primes are the same:*

1. *The minimal primes in the support of M.*
2. *The minimal primes in $Ass(M)$.*
3. *For any filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ the minimal primes of the set $\{\mathfrak{p}_i\}$.*

Proof. Choose a filtration as in (3). In Lemma 10.62.5 we have seen that the sets in (1) and (3) are equal.

Let $\mathfrak{p}$ be a minimal element of the set $\{\mathfrak{p}_i\}$. Let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i$, $m \notin M_{i-1}$. The annihilator of m is contained in $\mathfrak{p}_i = \mathfrak{p}$ and contains $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_i$. By our choice of i and $\mathfrak{p}$ we have $\mathfrak{p}_j \not\subset \mathfrak{p}$ for $j < i$ and hence we have $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$. Pick $f \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1}$, $f \notin \mathfrak{p}$. Then fm has annihilator $\mathfrak{p}$. In this way we see that $\mathfrak{p}$ is an associated prime of M. By Lemma 10.63.2 we have $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$ and hence $\mathfrak{p}$ is minimal in $\mathrm{Ass}(M)$. Thus the set of primes in (1) is contained in the set of primes of (2).

Let $\mathfrak{p}$ be a minimal element of $\mathrm{Ass}(M)$. Since $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$ there is a minimal element $\mathfrak{q}$ of $\mathrm{Supp}(M)$ with $\mathfrak{q} \subset \mathfrak{p}$. We have just shown that $\mathfrak{q} \in \mathrm{Ass}(M)$. Hence $\mathfrak{q} = \mathfrak{p}$ by minimality of $\mathfrak{p}$. Thus the set of primes in (2) is contained in the set of primes of (1). □

0587 **Lemma 10.63.7.** *Let R be a Noetherian ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \mathit{Ass}(M) = \emptyset.$$

Proof. If $M = (0)$, then $\mathrm{Ass}(M) = \emptyset$ by definition. If $M \neq 0$, pick any nonzero finitely generated submodule $M' \subset M$, for example a submodule generated by a single nonzero element. By Lemma 10.40.2 we see that $\mathrm{Supp}(M')$ is nonempty. By Proposition 10.63.6 this implies that $\mathrm{Ass}(M')$ is nonempty. By Lemma 10.63.3 this implies $\mathrm{Ass}(M) \neq \emptyset$. □

05BV **Lemma 10.63.8.** *Let R be a Noetherian ring. Let M be an R-module. Any $\mathfrak{p} \in \mathit{Supp}(M)$ which is minimal among the elements of $\mathit{Supp}(M)$ is an element of $\mathit{Ass}(M)$.*

Proof. If M is a finite R-module, then this is a consequence of Proposition 10.63.6. In general write $M = \bigcup M_\lambda$ as the union of its finite submodules, and use that $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_\lambda)$ and $\mathrm{Ass}(M) = \bigcup \mathrm{Ass}(M_\lambda)$. □

00LD **Lemma 10.63.9.** *Let R be a Noetherian ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in \mathit{Ass}(M)} \mathfrak{q}$ is the set of elements of R which are zerodivisors on M.*

Proof. Any element in any associated prime clearly is a zerodivisor on M. Conversely, suppose $x \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid xm = 0\}$. Since N is not zero it has an associated prime $\mathfrak{q}$ by Lemma 10.63.7. Then $x \in \mathfrak{q}$ and $\mathfrak{q}$ is an associated prime of M by Lemma 10.63.3. □

0B52 **Lemma 10.63.10.** *Let R is a Noetherian local ring, M a finite R-module, and $f \in \mathfrak{m}$ an element of the maximal ideal of R. Then*

$$\dim(\mathit{Supp}(M/fM)) \leq \dim(\mathit{Supp}(M)) \leq \dim(\mathit{Supp}(M/fM)) + 1$$

If f is not in any of the minimal primes of the support of M (for example if f is a nonzerodivisor on M), then equality holds for the right inequality.

Proof. (The parenthetical statement follows from Lemma 10.63.9.) The first inequality follows from $\mathrm{Supp}(M/fM) \subset \mathrm{Supp}(M)$, see Lemma 10.40.9. For the second inequality, note that $\mathrm{Supp}(M/fM) = \mathrm{Supp}(M) \cap V(f)$, see Lemma 10.40.9. It follows, for example by Lemma 10.62.2 and elementary properties of dimension, that it suffices to show $\dim V(\mathfrak{p}) \leq \dim(V(\mathfrak{p}) \cap V(f)) + 1$ for primes $\mathfrak{p}$ of R. This is a consequence of Lemma 10.60.13. Finally, if f is not contained in any minimal prime of the support of M, then the chains of primes in $\mathrm{Supp}(M/fM)$ all give rise to chains in $\mathrm{Supp}(M)$ which are at least one step away from being maximal. □

05BW **Lemma 10.63.11.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then* $\mathrm{Spec}(\varphi)(Ass_S(M)) \subset Ass_R(M)$.

Proof. If $\mathfrak{q} \in \mathrm{Ass}_S(M)$, then there exists an m in M such that the annihilator of m in S is $\mathfrak{q}$. Then the annihilator of m in R is $\mathfrak{q} \cap R$. □

05BX **Remark 10.63.12.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\mathrm{Spec}(\varphi)(\mathrm{Ass}_S(M)) \supset \mathrm{Ass}_R(M)$. For example, consider the ring map $R = k \to S = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ and $M = S$. Then $\mathrm{Ass}_R(M)$ is not empty, but $\mathrm{Ass}_S(S)$ is empty.

05DZ **Lemma 10.63.13.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. If S is Noetherian, then* $\mathrm{Spec}(\varphi)(Ass_S(M)) = Ass_R(M)$.

Proof. We have already seen in Lemma 10.63.11 that $\mathrm{Spec}(\varphi)(\mathrm{Ass}_S(M)) \subset \mathrm{Ass}_R(M)$. For the converse, choose a prime $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Let $m \in M$ be an element such that the annihilator of m in R is $\mathfrak{p}$. Let $I = \{g \in S \mid gm = 0\}$ be the annihilator of m in S. Then $R/\mathfrak{p} \subset S/I$ is injective. Combining Lemmas 10.30.5 and 10.30.7 we see that there is a prime $\mathfrak{q} \subset S$ minimal over I mapping to $\mathfrak{p}$. By Proposition 10.63.6 we see that $\mathfrak{q}$ is an associated prime of S/I, hence $\mathfrak{q}$ is an associated prime of M by Lemma 10.63.3 and we win. □

05BY **Lemma 10.63.14.** *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection* $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ *we have* $Ass_{R/I}(M) = Ass_R(M)$.

Proof. Omitted. □

0310 **Lemma 10.63.15.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p} \subset R$ be a prime.*

(1) *If $\mathfrak{p} \in Ass(M)$ then $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$.*
(2) *If $\mathfrak{p}$ is finitely generated then the converse holds as well.*

Proof. If $\mathfrak{p} \in \mathrm{Ass}(M)$ there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. As localization is exact (Proposition 10.9.12) we see that the annihilator of $m/1$ in $M_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$ hence (1) holds. Assume $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}(M_\mathfrak{p})$ and $\mathfrak{p} = (f_1, \ldots, f_n)$. Let m/g be an element of $M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}R_\mathfrak{p}$. This implies that the annihilator of m is contained in $\mathfrak{p}$. As $f_i m/g = 0$ in $M_\mathfrak{p}$ we see there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i f_i m = 0$ in M. Combined we see the annihilator of $g_1 \ldots g_n m$ is $\mathfrak{p}$. Hence $\mathfrak{p} \in \mathrm{Ass}(M)$. □

05BZ **Lemma 10.63.16.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection* $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ *we have*

(1) $Ass_R(S^{-1}M) = Ass_{S^{-1}R}(S^{-1}M)$,
(2) $Ass_R(M) \cap \mathrm{Spec}(S^{-1}R) \subset Ass_R(S^{-1}M)$, *and*
(3) *if R is Noetherian this inclusion is an equality.*

Proof. For $m \in S^{-1}M$, let $I \subset R$ and $J \subset S^{-1}R$ be the annihilators of m. Then I is the inverse image of J by the map $R \to S^{-1}R$ and $J = S^{-1}I$. The equality in (1) follows by the description of the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ in Lemma 10.17.5. For $m \in M$, let $I \subset R$ be the annihilator of m in R and let $J \subset S^{-1}R$ be the annihilator of $m/1 \in S^{-1}M$. We have $J = S^{-1}I$ which implies (2). The equality in the Noetherian case follows from Lemma 10.63.15 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_\mathfrak{p} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

05C0 **Lemma 10.63.17.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$Ass_R(M) = Ass_R(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we get the inclusion $\text{Ass}(M) \subset \text{Ass}(S^{-1}M)$ from Lemma 10.63.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element whose annihilator is a prime ideal $\mathfrak{p}$. Then the annihilator of $n \in M$ is also $\mathfrak{p}$. □

00LL **Lemma 10.63.18.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset \mathfrak{m}$ be an ideal. Let M be a finite R-module. The following are equivalent:*

1. *There exists an $x \in I$ which is not a zerodivisor on M.*
2. *We have $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in Ass(M)$.*

Proof. If there exists a nonzerodivisor x in I, then x clearly cannot be in any associated prime of M. Conversely, suppose $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$. In this case we can choose $x \in I$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$ by Lemmas 10.63.5 and 10.15.2. By Lemma 10.63.9 the element x is not a zerodivisor on M. □

0311 **Lemma 10.63.19.** *Let R be a ring. Let M be an R-module. If R is Noetherian the map*

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in Ass(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Then if $\mathfrak{p}$ is an associated prime of $Rx \subset M$ we see on the one hand that $\mathfrak{p} \in \text{Ass}(M)$ (Lemma 10.63.3) and on the other hand that $(Rx)_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero. This contradiction shows that $\text{Ass}(Rx) = \emptyset$. Hence $Rx = 0$ by Lemma 10.63.7. □

This lemma should probably be put somewhere else.

0GEC **Lemma 10.63.20.** *Let k be a field. Let S be a finite type k algebra. If $\dim(S) > 0$, then there exists an element $f \in S$ which is a nonzerodivisor and a nonunit.*

Proof. By Lemma 10.63.5 the ring S has finitely many associated prime ideals. By Lemma 10.61.3 the ring S has infinitely many maximal ideals. Hence we can choose a maximal ideal $\mathfrak{m} \subset S$ which is not an associated prime of S. By prime avoidance (Lemma 10.15.2), we can choose a nonzero $f \in \mathfrak{m}$ which is not contained in any of the associated primes of S. By Lemma 10.63.9 the element f is a nonzerodivisor and as $f \in \mathfrak{m}$ we see that f is not a unit. □

10.64. Symbolic powers

05G9 Here is the definition.

0313 **Definition 10.64.1.** Let R be a ring. Let $\mathfrak{p}$ be a prime ideal. For $n \geq 0$ the nth *symbolic power* of $\mathfrak{p}$ is the ideal $\mathfrak{p}^{(n)} = \text{Ker}(R \to R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}})$.

Note that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ but equality does not always hold.

0314 **Lemma 10.64.2.** *Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. Let $n > 0$. Then $Ass(R/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$.*

Proof. If $\mathfrak{q}$ is an associated prime of $R/\mathfrak{p}^{(n)}$ then clearly $\mathfrak{p} \subset \mathfrak{q}$. On the other hand, any element $x \in R$, $x \notin \mathfrak{p}$ is a nonzerodivisor on $R/\mathfrak{p}^{(n)}$. Namely, if $y \in R$ and $xy \in \mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ then $y \in \mathfrak{p}^n R_{\mathfrak{p}}$, hence $y \in \mathfrak{p}^{(n)}$. Hence the lemma follows. □

0BC0 **Lemma 10.64.3.** *Let $R \to S$ be flat ring map. Let $\mathfrak{p} \subset R$ be a prime such that $\mathfrak{q} = \mathfrak{p}S$ is a prime of S. Then $\mathfrak{p}^{(n)}S = \mathfrak{q}^{(n)}$.*

Proof. Since $\mathfrak{p}^{(n)} = \mathrm{Ker}(R \to R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ we see using flatness that $\mathfrak{p}^{(n)}S$ is the kernel of the map $S \to S_\mathfrak{p}/\mathfrak{p}^n S_\mathfrak{p}$. On the other hand $\mathfrak{q}^{(n)}$ is the kernel of the map $S \to S_\mathfrak{q}/\mathfrak{q}^n S_\mathfrak{q} = S_\mathfrak{q}/\mathfrak{p}^n S_\mathfrak{q}$. Hence it suffices to show that

$$S_\mathfrak{p}/\mathfrak{p}^n S_\mathfrak{p} \longrightarrow S_\mathfrak{q}/\mathfrak{p}^n S_\mathfrak{q}$$

is injective. Observe that the right hand module is the localization of the left hand module by elements $f \in S$, $f \notin \mathfrak{q}$. Thus it suffices to show these elements are nonzerodivisors on $S_\mathfrak{p}/\mathfrak{p}^n S_\mathfrak{p}$. By flatness, the module $S_\mathfrak{p}/\mathfrak{p}^n S_\mathfrak{p}$ has a finite filtration whose subquotients are

$$\mathfrak{p}^i S_\mathfrak{p}/\mathfrak{p}^{i+1} S_\mathfrak{p} \cong \mathfrak{p}^i R_\mathfrak{p}/\mathfrak{p}^{i+1} R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{p} \cong V \otimes_{\kappa(\mathfrak{p})} (S/\mathfrak{q})_\mathfrak{p}$$

where V is a $\kappa(\mathfrak{p})$ vector space. Thus f acts invertibly as desired. □

10.65. Relative assassin

05GA Discussion of relative assassins. Let $R \to S$ be a ring map. Let N be an S-module. In this situation we can introduce the following sets of primes $\mathfrak{q}$ of S:

(1) A: with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p}))$,
(2) A': with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q}$ is in the image of $\mathrm{Ass}_{S \otimes \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$ under the canonical map $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p})) \to \mathrm{Spec}(S)$,
(3) A_{fin}: with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \mathrm{Ass}_S(N/\mathfrak{p}N)$,
(4) A'_{fin}: for some prime $\mathfrak{p}' \subset R$ we have $\mathfrak{q} \in \mathrm{Ass}_S(N/\mathfrak{p}'N)$,
(5) B: for some R-module M we have $\mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R M)$, and
(6) B_{fin}: for some finite R-module M we have $\mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R M)$.

Let us determine some of the relations between these sets.

05GB **Lemma 10.65.1.** *Let $R \to S$ be a ring map. Let N be an S-module. Let A, A', A_{fin}, B, and B_{fin} be the subsets of* $\mathrm{Spec}(S)$ *introduced above.*

(1) *We always have $A = A'$.*
(2) *We always have $A_{fin} \subset A$, $B_{fin} \subset B$, $A_{fin} \subset A'_{fin} \subset B_{fin}$ and $A \subset B$.*
(3) *If S is Noetherian, then $A = A_{fin}$ and $B = B_{fin}$.*
(4) *If N is flat over R, then $A = A_{fin} = A'_{fin}$ and $B = B_{fin}$.*
(5) *If R is Noetherian and N is flat over R, then all of the sets are equal, i.e., $A = A' = A_{fin} = A'_{fin} = B = B_{fin}$.*

Proof. Some of the arguments in the proof will be repeated in the proofs of later lemmas which are more precise than this one (because they deal with a given module M or a given prime $\mathfrak{p}$ and not with the collection of all of them).

Proof of (1). Let $\mathfrak{p}$ be a prime of R. Then we have

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

the first equality by Lemma 10.63.14 and the second by Lemma 10.63.16 part (1). This prove that $A = A'$. The inclusion $A_{fin} \subset A'_{fin}$ is clear.

Proof of (2). Each of the inclusions is immediate from the definitions except perhaps $A_{fin} \subset A$ which follows from Lemma 10.63.16 and the fact that we require $\mathfrak{p} = R \cap \mathfrak{q}$ in the formulation of A_{fin}.

Proof of (3). The equality $A = A_{fin}$ follows from Lemma 10.63.16 part (3) if S is Noetherian. Let $\mathfrak{q} = (g_1, \ldots, g_m)$ be a finitely generated prime ideal of S. Say $z \in N \otimes_R M$ is an element whose annihilator is $\mathfrak{q}$. We may pick a finite submodule $M' \subset M$ such that z is the image of $z' \in N \otimes_R M'$. Then $\mathrm{Ann}_S(z') \subset \mathfrak{q} = \mathrm{Ann}_S(z)$. Since $N \otimes_R -$ commutes with colimits and since M is the directed colimit of finite R-modules we can find $M' \subset M'' \subset M$ such that the image $z'' \in N \otimes_R M''$ is annihilated by $g_1, \ldots, g_m$. Hence $\mathrm{Ann}_S(z'') = \mathfrak{q}$. This proves that $B = B_{fin}$ if S is Noetherian.

Proof of (4). If N is flat, then the functor $N \otimes_R -$ is exact. In particular, if $M' \subset M$, then $N \otimes_R M' \subset N \otimes_R M$. Hence if $z \in N \otimes_R M$ is an element whose annihilator $\mathfrak{q} = \mathrm{Ann}_S(z)$ is a prime, then we can pick any finite R-submodule $M' \subset M$ such that $z \in N \otimes_R M'$ and we see that the annihilator of z as an element of $N \otimes_R M'$ is equal to $\mathfrak{q}$. Hence $B = B_{fin}$. Let $\mathfrak{p}'$ be a prime of R and let $\mathfrak{q}$ be a prime of S which is an associated prime of $N/\mathfrak{p}'N$. This implies that $\mathfrak{p}'S \subset \mathfrak{q}$. As N is flat over R we see that $N/\mathfrak{p}'N$ is flat over the integral domain $R/\mathfrak{p}'$. Hence every nonzero element of $R/\mathfrak{p}'$ is a nonzerodivisor on $N/\mathfrak{p}'$. Hence none of these elements can map to an element of $\mathfrak{q}$ and we conclude that $\mathfrak{p}' = R \cap \mathfrak{q}$. Hence $A_{fin} = A'_{fin}$. Finally, by Lemma 10.63.17 we see that $\mathrm{Ass}_S(N/\mathfrak{p}'N) = \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p}'))$, i.e., $A'_{fin} = A$.

Proof of (5). We only need to prove $A'_{fin} = B_{fin}$ as the other equalities have been proved in (4). To see this let M be a finite R-module. By Lemma 10.62.1 there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Since N is flat we obtain a filtration by S-submodules

$$0 = N \otimes_R M_0 \subset N \otimes_R M_1 \subset \ldots \subset N \otimes_R M_n = N \otimes_R M$$

such that each subquotient is isomorphic to $N/\mathfrak{p}_i N$. By Lemma 10.63.3 we conclude that $\mathrm{Ass}_S(N \otimes_R M) \subset \bigcup \mathrm{Ass}_S(N/\mathfrak{p}_i N)$. Hence we see that $B_{fin} \subset A'_{fin}$. Since the other inclusion is part of (2) we win. □

We define the relative assassin of N over S/R to be the set $A = A'$ above. As a motivation we point out that it depends only on the fibre modules $N \otimes_R \kappa(\mathfrak{p})$ over the fibre rings. As in the case of the assassin of a module we warn the reader that this notion makes most sense when the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Noetherian, for example if $R \to S$ is of finite type.

05GC **Definition 10.65.2.** Let $R \to S$ be a ring map. Let N be an S-module. The *relative assassin of N over S/R* is the set

$$\mathrm{Ass}_{S/R}(N) = \{\mathfrak{q} \subset S \mid \mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) \text{ with } \mathfrak{p} = R \cap \mathfrak{q}\}.$$

This is the set named A in Lemma 10.65.1.

The spirit of the next few results is that they are about the relative assassin, even though this may not be apparent.

0312 **Lemma 10.65.3.** *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. If N is flat as R-module, then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_S(N/\mathfrak{p}N)$$

and if R is Noetherian then we have equality.

Proof. If $\mathfrak{p} \in \mathrm{Ass}_R(M)$ then there exists an injection $R/\mathfrak{p} \to M$. As N is flat over R we obtain an injection $R/\mathfrak{p} \otimes_R N \to M \otimes_R N$. Since $R/\mathfrak{p} \otimes_R N = N/\mathfrak{p}N$ we conclude that $\mathrm{Ass}_S(N/\mathfrak{p}N) \subset \mathrm{Ass}_S(M \otimes_R N)$, see Lemma 10.63.3. Hence the right hand side is contained in the left hand side.

Write $M = \bigcup M_\lambda$ as the union of its finitely generated R-submodules. Then also $N \otimes_R M = \bigcup N \otimes_R M_\lambda$ (as N is R-flat). By definition of associated primes we see that $\mathrm{Ass}_S(N \otimes_R M) = \bigcup \mathrm{Ass}_S(N \otimes_R M_\lambda)$ and $\mathrm{Ass}_R(M) = \bigcup \mathrm{Ass}(M_\lambda)$. Hence we may assume M is finitely generated.

Let $\mathfrak{q} \in \mathrm{Ass}_S(M \otimes_R N)$, and assume R is Noetherian and M is a finite R-module. To finish the proof we have to show that $\mathfrak{q}$ is an element of the right hand side. First we observe that $\mathfrak{q}S_\mathfrak{q} \in \mathrm{Ass}_{S_\mathfrak{q}}((M \otimes_R N)_\mathfrak{q})$, see Lemma 10.63.15. Let $\mathfrak{p}$ be the corresponding prime of R. Note that

$$(M \otimes_R N)_\mathfrak{q} = M \otimes_R N_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$$

If $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ then there exists an element $x \in \mathfrak{p}R_\mathfrak{p}$ which is a nonzerodivisor in $M_\mathfrak{p}$ (see Lemma 10.63.18). Since $N_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ we see that the image of x in $\mathfrak{q}S_\mathfrak{q}$ is a nonzerodivisor on $(M \otimes_R N)_\mathfrak{q}$. This is a contradiction with the assumption that $\mathfrak{q}S_\mathfrak{q} \in \mathrm{Ass}_S((M \otimes_R N)_\mathfrak{q})$. Hence we conclude that $\mathfrak{p}$ is one of the associated primes of M.

Continuing the argument we choose a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R, see Lemma 10.62.1. (By Lemma 10.63.4 we have $\mathfrak{p}_i = \mathfrak{p}$ for at least one i.) This gives a filtration

$$0 = M_0 \otimes_R N \subset M_1 \otimes_R N \subset \ldots \subset M_n \otimes_R N = M \otimes_R N$$

with subquotients isomorphic to $N/\mathfrak{p}_i N$. If $\mathfrak{p}_i \neq \mathfrak{p}$ then $\mathfrak{q}$ cannot be associated to the module $N/\mathfrak{p}_i N$ by the result of the preceding paragraph (as $\mathrm{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$). Hence we conclude that $\mathfrak{q}$ is associated to $N/\mathfrak{p}N$ as desired. □

05C1 **Lemma 10.65.4.** *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$Ass_S(N) = Ass_S(N \otimes_R K) = Ass_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\mathrm{Spec}(S \otimes_R K) \subset \mathrm{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.63.17 combined with Lemma 10.63.16 part (1). □

05C2 **Lemma 10.65.5.** *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. Assume N is flat as R-module. Then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

where we use Remark 10.18.5 to think of the spectra of fibre rings as subsets of $\mathrm{Spec}(S)$. *If R is Noetherian then this inclusion is an equality.*

Proof. This is equivalent to Lemma 10.65.3 by Lemmas 10.63.14, 10.39.7, and 10.65.4. □

05E0 **Remark 10.65.6.** Let $R \to S$ be a ring map. Let N be an S-module. Let $\mathfrak{p}$ be a prime of R. Then

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p})).$$

The first equality by Lemma 10.63.14 and the second by Lemma 10.63.16 part (1).

10.66. Weakly associated primes

0546 This is a variant on the notion of an associated prime that is useful for non-Noetherian ring and non-finite modules.

0547 **Definition 10.66.1.** Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *weakly associated* to M if there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the prime ideals containing the annihilator $\mathrm{Ann}(m) = \{f \in R \mid fm = 0\}$. The set of all such primes is denoted $\mathrm{WeakAss}_R(M)$ or $\mathrm{WeakAss}(M)$.

Thus an associated prime is a weakly associated prime. Here is a characterization in terms of the localization at the prime.

0566 **Lemma 10.66.2.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime of R. The following are equivalent:*

1. *$\mathfrak{p}$ is weakly associated to M,*
2. *$\mathfrak{p}R_\mathfrak{p}$ is weakly associated to $M_\mathfrak{p}$, and*
3. *$M_\mathfrak{p}$ contains an element whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$.*

Proof. Assume (1). Then there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the primes containing the annihilator $I = \{x \in R \mid xm = 0\}$ of m. As localization is exact, the annihilator of m in $M_\mathfrak{p}$ is $I_\mathfrak{p}$. Hence $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime of $R_\mathfrak{p}$ containing the annihilator $I_\mathfrak{p}$ of m in $M_\mathfrak{p}$. This implies (2) holds, and also (3) as it implies that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$.

Applying the implication (1) $\Rightarrow$ (3) to $M_\mathfrak{p}$ over $R_\mathfrak{p}$ we see that (2) $\Rightarrow$ (3).

Finally, assume (3). This means there exists an element $m/f \in M_\mathfrak{p}$ whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$. Then the annihilator $I = \{x \in R \mid xm = 0\}$ of m in M is such that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$. Clearly this means that $\mathfrak{p}$ contains I and is minimal among the primes containing I, i.e., (1) holds. □

0EMA **Lemma 10.66.3.** *For a reduced ring the weakly associated primes of the ring are the minimal primes.*

Proof. Let $(R, \mathfrak{m})$ be a reduced local ring. Suppose $x \in R$ is an element whose annihilator has radical $\mathfrak{m}$. If $\mathfrak{m} \neq 0$, then x cannot be a unit, so $x \in \mathfrak{m}$. Then in particular $x^{1+n} = 0$ for some $n \geq 0$. Hence $x = 0$. Which contradicts the assumption that the annihilator of x is contained in $\mathfrak{m}$. Thus we see that $\mathfrak{m} = 0$, i.e., R is a field. By Lemma 10.66.2 this implies the statement of the lemma. □

0548 **Lemma 10.66.4.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $\mathrm{WeakAss}(M') \subset \mathrm{WeakAss}(M)$ and $\mathrm{WeakAss}(M) \subset \mathrm{WeakAss}(M') \cup \mathrm{WeakAss}(M'')$.*

Proof. We will use the characterization of weakly associated primes of Lemma 10.66.2. Let $\mathfrak{p}$ be a prime of R. As localization is exact we obtain the short exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. Suppose that $m \in M_\mathfrak{p}$ is an element whose annihilator has radical $\mathfrak{p}R_\mathfrak{p}$. Then either the image $\overline{m}$ of m in $M''_\mathfrak{p}$ is zero

and $m \in M'_{\mathfrak{p}}$, or the radical of the annihilator of $\overline{m}$ is $\mathfrak{p}R_{\mathfrak{p}}$. This proves that $\text{WeakAss}(M) \subset \text{WeakAss}(M') \cup \text{WeakAss}(M'')$. The inclusion $\text{WeakAss}(M') \subset \text{WeakAss}(M)$ is immediate from the definitions. □

0588 **Lemma 10.66.5.** *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \textit{WeakAss}(M) = \emptyset$$

Proof. If $M = (0)$ then $\text{WeakAss}(M) = \emptyset$ by definition. Conversely, suppose that $M \neq 0$. Pick a nonzero element $m \in M$. Write $I = \{x \in R \mid xm = 0\}$ the annihilator of m. Then $R/I \subset M$. Hence $\text{WeakAss}(R/I) \subset \text{WeakAss}(M)$ by Lemma 10.66.4. But as $I \neq R$ we have $V(I) = \text{Spec}(R/I)$ contains a minimal prime, see Lemmas 10.17.2 and 10.17.7, and we win. □

0589 **Lemma 10.66.6.** *Let R be a ring. Let M be an R-module. Then*

$$\textit{Ass}(M) \subset \textit{WeakAss}(M) \subset \textit{Supp}(M).$$

Proof. The first inclusion is immediate from the definitions. If $\mathfrak{p} \in \text{WeakAss}(M)$, then by Lemma 10.66.2 we have $M_{\mathfrak{p}} \neq 0$, hence $\mathfrak{p} \in \text{Supp}(M)$. □

05C3 **Lemma 10.66.7.** *Let R be a ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in \textit{WeakAss}(M)} \mathfrak{q}$ is the set of elements of R which are zerodivisors on M.*

Proof. Suppose $f \in \mathfrak{q} \in \text{WeakAss}(M)$. Then there exists an element $m \in M$ such that $\mathfrak{q}$ is minimal over $I = \{x \in R \mid xm = 0\}$. Hence there exists a $g \in R$, $g \notin \mathfrak{q}$ and $n > 0$ such that $f^n gm = 0$. Note that $gm \neq 0$ as $g \notin I$. If we take n minimal as above, then $f(f^{n-1}gm) = 0$ and $f^{n-1}gm \neq 0$, so f is a zerodivisor on M. Conversely, suppose $f \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid fm = 0\}$. Since N is not zero it has a weakly associated prime $\mathfrak{q}$ by Lemma 10.66.5. Clearly $f \in \mathfrak{q}$ and by Lemma 10.66.4 $\mathfrak{q}$ is a weakly associated prime of M. □

05C4 **Lemma 10.66.8.** *Let R be a ring. Let M be an R-module. Any $\mathfrak{p} \in \textit{Supp}(M)$ which is minimal among the elements of $\textit{Supp}(M)$ is an element of $\textit{WeakAss}(M)$.*

Proof. Note that $\text{Supp}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$ in $\text{Spec}(R_{\mathfrak{p}})$. In particular $M_{\mathfrak{p}}$ is nonzero, and hence $\text{WeakAss}(M_{\mathfrak{p}}) \neq \emptyset$ by Lemma 10.66.5. Since $\text{WeakAss}(M_{\mathfrak{p}}) \subset \text{Supp}(M_{\mathfrak{p}})$ by Lemma 10.66.6 we conclude that $\text{WeakAss}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$, whence $\mathfrak{p} \in \text{WeakAss}(M)$ by Lemma 10.66.2. □

058A **Lemma 10.66.9.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime ideal of R which is finitely generated. Then*

$$\mathfrak{p} \in \textit{Ass}(M) \Leftrightarrow \mathfrak{p} \in \textit{WeakAss}(M).$$

In particular, if R is Noetherian, then $\textit{Ass}(M) = \textit{WeakAss}(M)$.

Proof. Write $\mathfrak{p} = (g_1, \ldots, g_n)$ for some $g_i \in R$. It is enough the prove the implication "$\Leftarrow$" as the other implication holds in general, see Lemma 10.66.6. Assume $\mathfrak{p} \in \text{WeakAss}(M)$. By Lemma 10.66.2 there exists an element $m \in M_{\mathfrak{p}}$ such that $I = \{x \in R_{\mathfrak{p}} \mid xm = 0\}$ has radical $\mathfrak{p}R_{\mathfrak{p}}$. Hence for each i there exists a smallest $e_i > 0$ such that $g_i^{e_i} m = 0$ in $M_{\mathfrak{p}}$. If $e_i > 1$ for some i, then we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease $\sum e_i$. Hence we may assume that the annihilator of $m \in M_{\mathfrak{p}}$ is $(g_1, \ldots, g_n)R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$. By Lemma 10.63.15 we see that $\mathfrak{p} \in \text{Ass}(M)$. □

05C5 **Remark 10.66.10.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\mathrm{Spec}(\varphi)(\mathrm{WeakAss}_S(M)) \subset \mathrm{WeakAss}_R(M)$ contrary to the case of associated primes (see Lemma 10.63.11). An example is to consider the ring map

$$R = k[x_1, x_2, x_3, \ldots] \to S = k[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]/(x_1y_1, x_2y_2, x_3y_3, \ldots)$$

and $M = S$. In this case $\mathfrak{q} = \sum x_i S$ is a minimal prime of S, hence a weakly associated prime of $M = S$ (see Lemma 10.66.8). But on the other hand, for any nonzero element of S the annihilator in R is finitely generated, and hence does not have radical equal to $R \cap \mathfrak{q} = (x_1, x_2, x_3, \ldots)$ (details omitted).

05C6 **Lemma 10.66.11.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then we have $\mathrm{Spec}(\varphi)(\mathit{WeakAss}_S(M)) \supset \mathit{WeakAss}_R(M)$.*

Proof. Let $\mathfrak{p}$ be an element of $\mathrm{WeakAss}_R(M)$. Then there exists an $m \in M_\mathfrak{p}$ whose annihilator $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Consider the annihilator $J = \{x \in S_\mathfrak{p} \mid xm = 0\}$ of m in $S_\mathfrak{p}$. As $IS_\mathfrak{p} \subset J$ we see that any minimal prime $\mathfrak{q} \subset S_\mathfrak{p}$ over J lies over $\mathfrak{p}$. Moreover such a $\mathfrak{q}$ corresponds to a weakly associated prime of M for example by Lemma 10.66.2. □

05C7 **Remark 10.66.12.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the associated map on spectra. Then we have

$$f(\mathrm{Ass}_S(M)) \subset \mathrm{Ass}_R(M) \subset \mathrm{WeakAss}_R(M) \subset f(\mathrm{WeakAss}_S(M))$$

see Lemmas 10.63.11, 10.66.11, and 10.66.6. In general all of the inclusions may be strict, see Remarks 10.63.12 and 10.66.10. If S is Noetherian, then all the inclusions are equalities as the outer two are equal by Lemma 10.66.9.

05E1 **Lemma 10.66.13.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the associated map on spectra. If φ is a finite ring map, then*

$$\mathit{WeakAss}_R(M) = f(\mathit{WeakAss}_S(M)).$$

Proof. One of the inclusions has already been proved, see Remark 10.66.12. To prove the other assume $\mathfrak{q} \in \mathrm{WeakAss}_S(M)$ and let $\mathfrak{p}$ be the corresponding prime of R. Let $m \in M$ be an element such that $\mathfrak{q}$ is a minimal prime over $J = \{g \in S \mid gm = 0\}$. Thus the radical of $JS_\mathfrak{q}$ is $\mathfrak{q}S_\mathfrak{q}$. As $R \to S$ is finite there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_l$ over $\mathfrak{p}$, see Lemma 10.36.21. Pick $x \in \mathfrak{q}$ with $x \notin \mathfrak{q}_i$ for $i > 1$, see Lemma 10.15.2. By the above there exists an element $y \in S$, $y \notin \mathfrak{q}$ and an integer $t > 0$ such that $yx^t m = 0$. Thus the element $ym \in M$ is annihilated by x^t, hence ym maps to zero in $M_{\mathfrak{q}_i}$, $i = 2, \ldots, l$. To be sure, ym does not map to zero in $S_\mathfrak{q}$.

The ring $S_\mathfrak{p}$ is semi-local with maximal ideals $\mathfrak{q}_i S_\mathfrak{p}$ by going up for finite ring maps, see Lemma 10.36.22. If $f \in \mathfrak{p}R_\mathfrak{p}$ then some power of f ends up in $JS_\mathfrak{q}$ hence for some $t > 0$ we see that $f^t ym$ maps to zero in $M_\mathfrak{q}$. As ym vanishes at the other maximal ideals of $S_\mathfrak{p}$ we conclude that $f^t ym$ is zero in $M_\mathfrak{p}$, see Lemma 10.23.1. In this way we see that $\mathfrak{p}$ is a minimal prime over the annihilator of ym in R and we win. □

05C8 **Lemma 10.66.14.** *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ we have $\mathit{WeakAss}_{R/I}(M) = \mathit{WeakAss}_R(M)$.*

Proof. Special case of Lemma 10.66.13. □

05C9 **Lemma 10.66.15.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ we have $\mathit{WeakAss}_R(S^{-1}M) = \mathit{WeakAss}_{S^{-1}R}(S^{-1}M)$ and*

$$\mathit{WeakAss}(M) \cap \mathrm{Spec}(S^{-1}R) = \mathit{WeakAss}(S^{-1}M).$$

Proof. Suppose that $m \in S^{-1}M$. Let $I = \{x \in R \mid xm = 0\}$ and $I' = \{x' \in S^{-1}R \mid x'm = 0\}$. Then $I' = S^{-1}I$ and $I \cap S = \emptyset$ unless $I = R$ (verifications omitted). Thus primes in $S^{-1}R$ minimal over I' correspond bijectively to primes in R minimal over I and avoiding S. This proves the equality $\mathrm{WeakAss}_R(S^{-1}M) = \mathrm{WeakAss}_{S^{-1}R}(S^{-1}M)$. The second equality follows from Lemma 10.66.2 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_\mathfrak{p} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

05CA **Lemma 10.66.16.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$\mathit{WeakAss}(M) = \mathit{WeakAss}(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we obtain $\mathrm{WeakAss}(M) \subset \mathrm{WeakAss}(S^{-1}M)$ from Lemma 10.66.4. Conversely, suppose that $n/s \in S^{-1}M$ is an element with annihilator I and $\mathfrak{p}$ a prime which is minimal over I. Then the annihilator of $n \in M$ is I and $\mathfrak{p}$ is a prime minimal over I. □

05CB **Lemma 10.66.17.** *Let R be a ring. Let M be an R-module. The map*

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in \mathit{WeakAss}(M)} M_\mathfrak{p}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Set $N = Rx \subset M$. If $\mathfrak{p}$ is a weakly associated prime of N we see on the one hand that $\mathfrak{p} \in \mathrm{WeakAss}(M)$ (Lemma 10.66.4) and on the other hand that $N_\mathfrak{p} \subset M_\mathfrak{p}$ is not zero. This contradiction shows that $\mathrm{WeakAss}(N) = \emptyset$. Hence $N = 0$, i.e., $x = 0$ by Lemma 10.66.5. □

05CC **Lemma 10.66.18.** *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$\mathit{WeakAss}_S(N) = \mathit{WeakAss}_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\mathrm{Spec}(S \otimes_R K) \subset \mathrm{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.66.16. □

0CUB **Lemma 10.66.19.** *Let K/k be a field extension. Let R be a k-algebra. Let M be an R-module. Let $\mathfrak{q} \subset R \otimes_k K$ be a prime lying over $\mathfrak{p} \subset R$. If $\mathfrak{q}$ is weakly associated to $M \otimes_k K$, then $\mathfrak{p}$ is weakly associated to M.*

Proof. Let $z \in M \otimes_k K$ be an element such that $\mathfrak{q}$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. Choose a finitely generated subextension $K/L/k$ such that $z \in M \otimes_k L$. Since $R \otimes_k L \to R \otimes_k K$ is flat we see that $J = I(R \otimes_k K)$ where $I \subset R \otimes_k L$ is the annihilator of z in the smaller ring (Lemma 10.40.4). Thus

$\mathfrak{q} \cap (R \otimes_k L)$ is minimal over I by going down (Lemma 10.39.19). In this way we reduce to the case described in the next paragraph.

Assume K/k is a finitely generated field extension. Let $x_1, \ldots, x_r \in K$ be a transcendence basis of K over k, see Fields, Section 9.26. Set $L = k(x_1, \ldots, x_r)$. Say $[K : L] = n$. Then $R \otimes_k L \to R \otimes_k K$ is a finite ring map. Hence $\mathfrak{q} \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k K$ viewed as a $R \otimes_k L$-module by Lemma 10.66.13. Since $M \otimes_k K \cong (M \otimes_k L)^{\oplus n}$ as a $R \otimes_k L$-module, we see that $\mathfrak{q} \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k L$ (for example by using Lemma 10.66.4 and induction). In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension. We may replace R by $R_{\mathfrak{p}}$, M by $M_{\mathfrak{p}}$ and $\mathfrak{q}$ by $\mathfrak{q}(R_{\mathfrak{p}} \otimes_k K)$. See Lemma 10.66.15. In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension and R is local with maximal ideal $\mathfrak{p}$. We claim that any $f \in R \otimes_k K$, $f \notin \mathfrak{p}(R \otimes_k K)$ is a nonzerodivisor on $M \otimes_k K$. Namely, let $z \in M \otimes_k K$ be an element. There is a finite R-submodule $M' \subset M$ such that $z \in M' \otimes_k K$ and such that M' is minimal with this property: choose a basis $\{t_\alpha\}$ of K as a k-vector space, write $z = \sum m_\alpha \otimes t_\alpha$ and let M' be the R-submodule generated by the m_α. If $z \in \mathfrak{p}(M' \otimes_k K) = \mathfrak{p}M' \otimes_k K$, then $\mathfrak{p}M' = M'$ and $M' = 0$ by Lemma 10.20.1 a contradiction. Thus z has nonzero image $\overline{z}$ in $M'/\mathfrak{p}M' \otimes_k K$ But $R/\mathfrak{p} \otimes_k K$ is a domain as a localization of $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$ and $M'/\mathfrak{p}M' \otimes_k K$ is a free module, hence $f\overline{z} \neq 0$. This proves the claim.

Finally, pick $z \in M \otimes_k K$ such that $\mathfrak{q}$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. For $f \in \mathfrak{p}$ there exists an $n \geq 1$ and a $g \in R \otimes_k K$, $g \notin \mathfrak{q}$ such that $gf^n z \in J$, i.e., $gf^n z = 0$. (This holds because $\mathfrak{q}$ lies over $\mathfrak{p}$ and $\mathfrak{q}$ is minimal over J.) Above we have seen that g is a nonzerodivisor hence $f^n z = 0$. This means that $\mathfrak{p}$ is a weakly associated prime of $M \otimes_k K$ viewed as an R-module. Since $M \otimes_k K$ is a direct sum of copies of M we conclude that $\mathfrak{p}$ is a weakly associated prime of M as before. □

10.67. Embedded primes

02M4 Here is the definition.

02M5 **Definition 10.67.1.** Let R be a ring. Let M be an R-module.

(1) The associated primes of M which are not minimal among the associated primes of M are called the *embedded associated primes* of M.
(2) The *embedded primes of* R are the embedded associated primes of R as an R-module.

Here is a way to get rid of these.

02M6 **Lemma 10.67.2.** *Let R be a Noetherian ring. Let M be a finite R-module. Consider the set of R-submodules*

$$\{K \subset M \mid \mathit{Supp}(K) \text{ nowhere dense in } \mathit{Supp}(M)\}.$$

This set has a maximal element K and the quotient $M' = M/K$ has the following properties

(1) $\mathit{Supp}(M) = \mathit{Supp}(M')$,

(2) M' *has no embedded associated primes,*
(3) *for any* $f \in R$ *which is contained in all embedded associated primes of* M *we have* $M_f \cong M'_f$.

Proof. We will use Lemma 10.63.5 and Proposition 10.63.6 without further mention. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes in the support of M. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the embedded associated primes of M. Then $\text{Ass}(M) = \{\mathfrak{q}_j, \mathfrak{p}_i\}$. Let

$$K = \{m \in M \mid \text{Supp}(Rm) \subset \bigcup V(\mathfrak{p}_i)\}$$

It is immediately seen to be a submodule. Since M is finite over a Noetherian ring, we know K is finite too. Hence $\text{Supp}(K)$ is nowhere dense in $\text{Supp}(M)$. Let $K' \subset M$ be another submodule with support nowhere dense in $\text{Supp}(M)$. This means that $K_{\mathfrak{q}_j} = 0$. Hence if $m \in K'$, then m maps to zero in $M_{\mathfrak{q}_j}$ which in turn implies $(Rm)_{\mathfrak{q}_j} = 0$. On the other hand we have $\text{Ass}(Rm) \subset \text{Ass}(M)$. Hence the support of Rm is contained in $\bigcup V(\mathfrak{p}_i)$. Therefore $m \in K$ and thus $K' \subset K$ as m was arbitrary in K'.

Let $M' = M/K$. Since $K_{\mathfrak{q}_j} = 0$ we know $M'_{\mathfrak{q}_j} = M_{\mathfrak{q}_j}$ for all j. Hence M and M' have the same support.

Suppose $\mathfrak{q} = \text{Ann}(\overline{m}) \in \text{Ass}(M')$ where $\overline{m} \in M'$ is the image of $m \in M$. Then $m \notin K$ and hence the support of Rm must contain one of the $\mathfrak{q}_j$. Since $M_{\mathfrak{q}_j} = M'_{\mathfrak{q}_j}$, we know $\overline{m}$ does not map to zero in $M'_{\mathfrak{q}_j}$. Hence $\mathfrak{q} \subset \mathfrak{q}_j$ (actually we have equality), which means that all the associated primes of M' are not embedded.

Let f be an element contained in all $\mathfrak{p}_i$. Then $D(f) \cap \text{supp}(K) = 0$. Hence $M_f = M'_f$ because $K_f = 0$. □

02M7 **Lemma 10.67.3.** *Let* R *be a Noetherian ring. Let* M *be a finite* R*-module. For any* $f \in R$ *we have* $(M')_f = (M_f)'$ *where* $M \to M'$ *and* $M_f \to (M_f)'$ *are the quotients constructed in Lemma 10.67.2.*

Proof. Omitted. □

02M8 **Lemma 10.67.4.** *Let* R *be a Noetherian ring. Let* M *be a finite* R*-module without embedded associated primes. Let* $I = \{x \in R \mid xM = 0\}$. *Then the ring* R/I *has no embedded primes.*

Proof. We may replace R by R/I. Hence we may assume every nonzero element of R acts nontrivially on M. By Lemma 10.40.5 this implies that $\text{Spec}(R)$ equals the support of M. Suppose that $\mathfrak{p}$ is an embedded prime of R. Let $x \in R$ be an element whose annihilator is $\mathfrak{p}$. Consider the nonzero module $N = xM \subset M$. It is annihilated by $\mathfrak{p}$. Hence any associated prime $\mathfrak{q}$ of N contains $\mathfrak{p}$ and is also an associated prime of M. Then $\mathfrak{q}$ would be an embedded associated prime of M which contradicts the assumption of the lemma. □

10.68. Regular sequences

0AUH In this section we develop some basic properties of regular sequences.

00LF **Definition 10.68.1.** Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_r$ of R is called an M*-regular sequence* if the following conditions hold:

(1) f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, r$, and

(2) the module $M/(f_1, \ldots, f_r)M$ is not zero.

If I is an ideal of R and $f_1, \ldots, f_r \in I$ then we call $f_1, \ldots, f_r$ an *M-regular sequence in I*. If $M = R$, we call $f_1, \ldots, f_r$ simply a *regular sequence* (in I).

Please pay attention to the fact that the definition depends on the order of the elements $f_1, \ldots, f_r$ (see examples below). Some papers/books drop the requirement that the module $M/(f_1, \ldots, f_r)M$ is nonzero. This has the advantage that being a regular sequence is preserved under localization. However, we will use this definition mainly to define the depth of a module in case R is local; in that case the f_i are required to be in the maximal ideal – a condition which is not preserved under going from R to a localization $R_{\mathfrak{p}}$.

00LG **Example 10.68.2.** Let k be a field. In the ring $k[x, y, z]$ the sequence $x, y(1-x), z(1-x)$ is regular but the sequence $y(1-x), z(1-x), x$ is not.

00LH **Example 10.68.3.** Let k be a field. Consider the ring $k[x, y, w_0, w_1, w_2, \ldots]/I$ where I is generated by yw_i, $i = 0, 1, 2, \ldots$ and $w_i - xw_{i+1}$, $i = 0, 1, 2, \ldots$. The sequence x, y is regular, but y is a zerodivisor. Moreover you can localize at the maximal ideal (x, y, w_i) and still get an example.

00LJ **Lemma 10.68.4.** *Let R be a local Noetherian ring. Let M be a finite R-module. Let $x_1, \ldots, x_c$ be an M-regular sequence. Then any permutation of the x_i is a regular sequence as well.*

Proof. First we do the case $c = 2$. Consider $K \subset M$ the kernel of $x_2 : M \to M$. For any $z \in K$ we know that $z = x_1 z'$ for some $z' \in M$ because x_2 is a nonzerodivisor on M/x_1M. Because x_1 is a nonzerodivisor on M we see that $x_2 z' = 0$ as well. Hence $x_1 : K \to K$ is surjective. Thus $K = 0$ by Nakayama's Lemma 10.20.1. Next, consider multiplication by x_1 on M/x_2M. If $z \in M$ maps to an element $\overline{z} \in M/x_2M$ in the kernel of this map, then $x_1 z = x_2 y$ for some $y \in M$. But then since x_1, x_2 is a regular sequence we see that $y = x_1 y'$ for some $y' \in M$. Hence $x_1(z - x_2 y') = 0$ and hence $z = x_2 y'$ and hence $\overline{z} = 0$ as desired.

For the general case, observe that any permutation is a composition of transpositions of adjacent indices. Hence it suffices to prove that

$$x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_c$$

is an M-regular sequence. This follows from the case we just did applied to the module $M/(x_1, \ldots, x_{i-2})$ and the length 2 regular sequence x_{i-1}, x_i. □

00LM **Lemma 10.68.5.** *Let R, S be local rings. Let $R \to S$ be a flat local ring homomorphism. Let $x_1, \ldots, x_r$ be a sequence in R. Let M be an R-module. The following are equivalent*

(1) *$x_1, \ldots, x_r$ is an M-regular sequence in R, and*
(2) *the images of $x_1, \ldots, x_r$ in S form a $M \otimes_R S$-regular sequence.*

Proof. This is so because $R \to S$ is faithfully flat by Lemma 10.39.17. □

061L **Lemma 10.68.6.** *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_r$ be a sequence in R whose image in $R_{\mathfrak{p}}$ forms an $M_{\mathfrak{p}}$-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_r$ in R_g forms an M_g-regular sequence.*

Proof. Set

$$K_i = \mathrm{Ker}\left(x_i : M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_{i-1})M\right).$$

This is a finite R-module whose localization at $\mathfrak{p}$ is zero by assumption. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $(K_i)_g = 0$ for all $i = 1, \ldots, r$. This g works. □

065K **Lemma 10.68.7.** *Let A be a ring. Let I be an ideal generated by a regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A.*

Proof. This follows immediately from the definitions. □

0F1T **Lemma 10.68.8.** *Let R be a ring. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Let $f_1, \ldots, f_r \in R$. If $f_1, \ldots, f_r$ is M_1-regular and M_3-regular, then $f_1, \ldots, f_r$ is M_2-regular.*

Proof. By Lemma 10.4.1, if $f_1 : M_1 \to M_1$ and $f_1 : M_3 \to M_3$ are injective, then so is $f_1 : M_2 \to M_2$ and we obtain a short exact sequence

$$0 \to M_1/f_1M_1 \to M_2/f_1M_2 \to M_3/f_1M_3 \to 0$$

The lemma follows from this and induction on r. Some details omitted. □

07DV **Lemma 10.68.9.** *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_r \in R$ and $e_1, \ldots, e_r > 0$ integers. Then $f_1, \ldots, f_r$ is an M-regular sequence if and only if $f_1^{e_1}, \ldots, f_r^{e_r}$ is an M-regular sequence.*

Proof. We will prove this by induction on r. If $r = 1$ this follows from the following two easy facts: (a) a power of a nonzerodivisor on M is a nonzerodivisor on M and (b) a divisor of a nonzerodivisor on M is a nonzerodivisor on M. If $r > 1$, then by induction applied to M/f_1M we have that $f_1, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, f_2^{e_2}, \ldots, f_r^{e_r}$ is an M-regular sequence. Thus it suffices to show, given $e > 0$, that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, \ldots, f_r$ is an M-regular sequence. We will prove this by induction on e. The case $e = 1$ is trivial. Since f_1 is a nonzerodivisor under both assumptions (by the case $r = 1$) we have a short exact sequence

$$0 \to M/f_1M \xrightarrow{f_1^{e-1}} M/f_1^eM \to M/f_1^{e-1}M \to 0$$

Suppose that $f_1, f_2, \ldots, f_r$ is an M-regular sequence. Then by induction the elements $f_2, \ldots, f_r$ are M/f_1M and $M/f_1^{e-1}M$-regular sequences. By Lemma 10.68.8 $f_2, \ldots, f_r$ is M/f_1^eM-regular. Hence $f_1^e, f_2, \ldots, f_r$ is M-regular. Conversely, suppose that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence. Then $f_2 : M/f_1^eM \to M/f_1^eM$ is injective, hence $f_2 : M/f_1M \to M/f_1M$ is injective, hence by induction(!) $f_2 : M/f_1^{e-1}M \to M/f_1^{e-1}M$ is injective, hence

$$0 \to M/(f_1, f_2)M \xrightarrow{f_1^{e-1}} M/(f_1^e, f_2)M \to M/(f_1^{e-1}, f_2)M \to 0$$

is a short exact sequence by Lemma 10.4.1. This proves the converse for $r = 2$. If $r > 2$, then we have $f_3 : M/(f_1^e, f_2)M \to M/(f_1^e, f_2)M$ is injective, hence $f_3 : M/(f_1, f_2)M \to M/(f_1, f_2)M$ is injective, and so on. Some details omitted. □

07DW **Lemma 10.68.10.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ which do not generate the unit ideal. The following are equivalent:*

(1) *any permutation of $f_1, \ldots, f_r$ is a regular sequence,*

(2) *any subsequence of $f_1, \ldots, f_r$ (in the given order) is a regular sequence, and*
(3) *$f_1x_1, \ldots, f_rx_r$ is a regular sequence in the polynomial ring $R[x_1, \ldots, x_r]$.*

Proof. It is clear that (1) implies (2). We prove (2) implies (1) by induction on r. The case $r = 1$ is trivial. The case $r = 2$ says that if $a, b \in R$ are a regular sequence and b is a nonzerodivisor, then b, a is a regular sequence. This is clear because the kernel of $a : R/(b) \to R/(b)$ is isomorphic to the kernel of $b : R/(a) \to R/(a)$ if both a and b are nonzerodivisors. The case $r > 2$. Assume (2) holds and say we want to prove $f_{\sigma(1)}, \ldots, f_{\sigma(r)}$ is a regular sequence for some permutation σ. We already know that $f_{\sigma(1)}, \ldots, f_{\sigma(r-1)}$ is a regular sequence by induction. Hence it suffices to show that f_s where $s = \sigma(r)$ is a nonzerodivisor modulo $f_1, \ldots, \hat{f}_s, \ldots, f_r$. If $s = r$ we are done. If $s < r$, then note that f_s and f_r are both nonzerodivisors in the ring $R/(f_1, \ldots, \hat{f}_s, \ldots, f_{r-1})$ (by induction hypothesis again). Since we know f_s, f_r is a regular sequence in that ring we conclude by the case of sequence of length 2 that f_r, f_s is too.

Note that $R[x_1, \ldots, x_r]/(f_1x_1, \ldots, f_ix_i)$ as an R-module is a direct sum of the modules

$$R/I_E \cdot x_1^{e_1} \ldots x_r^{e_r}$$

indexed by multi-indices $E = (e_1, \ldots, e_r)$ where I_E is the ideal generated by f_j for $1 \leq j \leq i$ with $e_j > 0$. Hence $f_{i+1}x_i$ is a nonzerodivisor on this if and only if f_{i+1} is a nonzerodivisor on R/I_E for all E. Taking E with all positive entries, we see that f_{i+1} is a nonzerodivisor on $R/(f_1, \ldots, f_i)$. Thus (3) implies (2). Conversely, if (2) holds, then any subsequence of $f_1, \ldots, f_i, f_{i+1}$ is a regular sequence in particular f_{i+1} is a nonzerodivisor on all R/I_E. In this way we see that (2) implies (3). □

10.69. Quasi-regular sequences

061M We introduce the notion of quasi-regular sequence which is slightly weaker than that of a regular sequence and easier to use. Let R be a ring and let $f_1, \ldots, f_c \in R$. Set $J = (f_1, \ldots, f_c)$. Let M be an R-module. Then there is a canonical map

061N (10.69.0.1)
$$M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^nM/J^{n+1}M$$

of graded $R/J[X_1, \ldots, X_c]$-modules defined by the rule

$$\overline{m} \otimes X_1^{e_1} \ldots X_c^{e_c} \longmapsto f_1^{e_1} \ldots f_c^{e_c} m \bmod J^{e_1+\ldots+e_c+1}M.$$

Note that (10.69.0.1) is always surjective.

061P **Definition 10.69.1.** Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_c$ of R is called *M-quasi-regular* if (10.69.0.1) is an isomorphism. If $M = R$, we call $f_1, \ldots, f_c$ simply a *quasi-regular sequence*.

So if $f_1, \ldots, f_c$ is a quasi-regular sequence, then

$$R/J[X_1, \ldots, X_c] = \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

where $J = (f_1, \ldots, f_c)$. It is clear that being a quasi-regular sequence is independent of the order of $f_1, \ldots, f_c$.

00LN **Lemma 10.69.2.** *Let R be a ring.*

(1) *A regular sequence $f_1, \ldots, f_c$ of R is a quasi-regular sequence.*

(2) *Suppose that M is an R-module and that $f_1, \ldots, f_c$ is an M-regular sequence. Then $f_1, \ldots, f_c$ is an M-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. We prove the first assertion by induction on c. We have to show that given any relation $\sum_{|I|=n} a_I f^I \in J^{n+1}$ with $a_I \in R$ we actually have $a_I \in J$ for all multi-indices I. Since any element of J^{n+1} is of the form $\sum_{|I|=n} b_I f^I$ with $b_I \in J$ we may assume, after replacing a_I by $a_I - b_I$, the relation reads $\sum_{|I|=n} a_I f^I = 0$. We can rewrite this as

$$\sum\nolimits_{e=0}^{n} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Here and below the "primed" multi-indices I' are required to be of the form $I' = (i_1, \ldots, i_{c-1}, 0)$. We will show by induction on $l \in \{0, \ldots, n\}$ that if we have a relation

$$\sum\nolimits_{e=0}^{l} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

then $a_{I',e} \in J$ for all I', e. Namely, set $J' = (f_1, \ldots, f_{c-1})$. Observe that $\sum_{|I'|=n-l} a_{I',l} f^{I'}$ is mapped into $(J')^{n-l+1}$ by f_c^l. By induction hypothesis (for the induction on c) we see that $f_c^l a_{I',l} \in J'$. Because f_c is not a zerodivisor on R/J' (as $f_1, \ldots, f_c$ is a regular sequence) we conclude that $a_{I',l} \in J'$. This allows us to rewrite the term $(\sum_{|I'|=n-l} a_{I',l} f^{I'}) f_c^l$ in the form $(\sum_{|I'|=n-l+1} f_c b_{I',l-1} f^{I'}) f_c^{l-1}$. This gives a new relation of the form

$$\left(\sum\nolimits_{|I'|=n-l+1} (a_{I',l-1} + f_c b_{I',l-1}) f^{I'} \right) f_c^{l-1} + \sum\nolimits_{e=0}^{l-2} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Now by the induction hypothesis (on l this time) we see that all $a_{I',l-1} + f_c b_{I',l-1} \in J$ and all $a_{I',e} \in J$ for $e \leq l-2$. This, combined with $a_{I',l} \in J' \subset J$ seen above, finishes the proof of the induction step.

The second assertion means that given any formal expression $F = \sum_{|I|=n} m_I X^I$, $m_I \in M$ with $\sum m_I f^I \in J^{n+1} M$, then all the coefficients m_I are in J. This is proved in exactly the same way as we prove the corresponding result for the first assertion above. □

065L **Lemma 10.69.3.** *Let $R \to R'$ be a flat ring map. Let M be an R-module. Suppose that $f_1, \ldots, f_r \in R$ form an M-quasi-regular sequence. Then the images of $f_1, \ldots, f_r$ in R' form a $M \otimes_R R'$-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_r)$, $J' = JR'$ and $M' = M \otimes_R R'$. We have to show the canonical map $\mu : R'/J'[X_1, \ldots X_r] \otimes_{R'/J'} M'/J'M' \to \bigoplus (J')^n M'/(J')^{n+1} M'$ is an isomorphism. Because $R \to R'$ is flat the sequences $0 \to J^n M \to M$ and $0 \to J^{n+1} M \to J^n M \to J^n M/J^{n+1} M \to 0$ remain exact on tensoring with R'. This first implies that $J^n M \otimes_R R' = (J')^n M'$ and then that $(J')^n M'/(J')^{n+1} M' = J^n M/J^{n+1} M \otimes_R R'$. Thus μ is the tensor product of (10.69.0.1), which is an isomorphism by assumption, with $\mathrm{id}_{R'}$ and we conclude. □

061Q **Lemma 10.69.4.** *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_c$ be a sequence in R whose image in $R_\mathfrak{p}$ forms an $M_\mathfrak{p}$-quasi-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_c$ in R_g forms an M_g-quasi-regular sequence.*

Proof. Consider the kernel K of the map (10.69.0.1). As $M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c]$ is a finite $R/J[X_1, \ldots, X_c]$-module and as $R/J[X_1, \ldots, X_c]$ is Noetherian, we see that K is also a finite $R/J[X_1, \ldots, X_c]$-module. Pick homogeneous generators $k_1, \ldots, k_t \in K$. By assumption for each $i = 1, \ldots, t$ there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i k_i = 0$. Hence $g = g_1 \ldots g_t$ works. □

061R **Lemma 10.69.5.** *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_c \in R$ be an M-quasi-regular sequence. For any i the sequence $\overline{f}_{i+1}, \ldots, \overline{f}_c$ of $\overline{R} = R/(f_1, \ldots, f_i)$ is an $\overline{M} = M/(f_1, \ldots, f_i)M$-quasi-regular sequence.*

Proof. It suffices to prove this for $i = 1$. Set $\overline{J} = (\overline{f}_2, \ldots, \overline{f}_c) \subset \overline{R}$. Then

$$\begin{aligned} \overline{J}^n \overline{M}/\overline{J}^{n+1}\overline{M} &= (J^n M + f_1 M)/(J^{n+1} M + f_1 M) \\ &= J^n M/(J^{n+1} M + J^n M \cap f_1 M). \end{aligned}$$

Thus, in order to prove the lemma it suffices to show that $J^{n+1}M + J^n M \cap f_1 M = J^{n+1}M + f_1 J^{n-1} M$ because that will show that $\bigoplus_{n \geq 0} \overline{J}^n \overline{M}/\overline{J}^{n+1}\overline{M}$ is the quotient of $\bigoplus_{n \geq 0} J^n M/J^{n+1}M \cong M/JM[X_1, \ldots, X_c]$ by X_1. Actually, we have $J^n M \cap f_1 M = f_1 J^{n-1} M$. Namely, if $m \notin J^{n-1}M$, then $f_1 m \notin J^n M$ because $\bigoplus J^n M/J^{n+1}M$ is the polynomial algebra $M/J[X_1, \ldots, X_c]$ by assumption. □

061S **Lemma 10.69.6.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_c \in \mathfrak{m}$ be an M-quasi-regular sequence. Then $f_1, \ldots, f_c$ is an M-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. Let us show that f_1 is a nonzerodivisor on M. Suppose $x \in M$ is not zero. By Krull's intersection theorem there exists an integer r such that $x \in J^r M$ but $x \notin J^{r+1}M$, see Lemma 10.51.4. Then $f_1 x \in J^{r+1}M$ is an element whose class in $J^{r+1}M/J^{r+2}M$ is nonzero by the assumed structure of $\bigoplus J^n M/J^{n+1}M$. Whence $f_1 x \neq 0$.

Now we can finish the proof by induction on c using Lemma 10.69.5. □

061T **Remark 10.69.7** (Other types of regular sequences)**.** In the paper [**Kab71**] the author discusses two more regularity conditions for sequences $x_1, \ldots, x_r$ of elements of a ring R. Namely, we say the sequence is *Koszul-regular* if $H_i(K_\bullet(R, x_\bullet)) = 0$ for $i \geq 1$ where $K_\bullet(R, x_\bullet)$ is the Koszul complex. The sequence is called *H_1-regular* if $H_1(K_\bullet(R, x_\bullet)) = 0$. One has the implications regular $\Rightarrow$ Koszul-regular $\Rightarrow$ H_1-regular $\Rightarrow$ quasi-regular. By examples the author shows that these implications cannot be reversed in general even if R is a (non-Noetherian) local ring and the sequence generates the maximal ideal of R. We introduce these notions in more detail in More on Algebra, Section 15.30.

065M **Remark 10.69.8.** Let k be a field. Consider the ring

$$A = k[x, y, w, z_0, z_1, z_2, \ldots]/(y^2 z_0 - wx, z_0 - y z_1, z_1 - y z_2, \ldots)$$

In this ring x is a nonzerodivisor and the image of y in A/xA gives a quasi-regular sequence. But it is not true that x, y is a quasi-regular sequence in A because $(x, y)/(x, y)^2$ isn't free of rank two over $A/(x, y)$ due to the fact that $wx = 0$ in $(x, y)/(x, y)^2$ but w isn't zero in $A/(x, y)$. Hence the analogue of Lemma 10.68.7 does not hold for quasi-regular sequences.

065N **Lemma 10.69.9.** *Let R be a ring. Let $J = (f_1, \ldots, f_r)$ be an ideal of R. Let M be an R-module. Set $\overline{R} = R/\bigcap_{n\geq 0} J^n$, $\overline{M} = M/\bigcap_{n\geq 0} J^n M$, and denote $\overline{f}_i$ the image of f_i in $\overline{R}$. Then $f_1, \ldots, f_r$ is M-quasi-regular if and only if $\overline{f}_1, \ldots, \overline{f}_r$ is $\overline{M}$-quasi-regular.*

Proof. This is true because $J^n M/J^{n+1}M \cong \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$. □

10.70. Blow up algebras

052P In this section we make some elementary observations about blowing up.

052Q **Definition 10.70.1.** Let R be a ring. Let $I \subset R$ be an ideal.

(1) The *blowup algebra*, or the *Rees algebra*, associated to the pair (R, I) is the graded R-algebra
$$\mathrm{Bl}_I(R) = \bigoplus\nolimits_{n\geq 0} I^n = R \oplus I \oplus I^2 \oplus \ldots$$
where the summand I^n is placed in degree n.

(2) Let $a \in I$ be an element. Denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra. Then the *affine blowup algebra* $R[\frac{I}{a}]$ is the algebra $(\mathrm{Bl}_I(R))_{(a^{(1)})}$ constructed in Section 10.57.

In other words, an element of $R[\frac{I}{a}]$ is represented by an expression of the form x/a^n with $x \in I^n$. Two representatives x/a^n and y/a^m define the same element if and only if $a^k(a^m x - a^n y) = 0$ for some $k \geq 0$.

07Z3 **Lemma 10.70.2.** *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Let $R' = R[\frac{I}{a}]$ be the affine blowup algebra. Then*

(1) *the image of a in R' is a nonzerodivisor,*
(2) *$IR' = aR'$, and*
(3) *$(R')_a = R_a$.*

Proof. Immediate from the description of $R[\frac{I}{a}]$ above. □

0BIP **Lemma 10.70.3.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal and $a \in I$. Set $J = IS$ and let $b \in J$ be the image of a. Then $S[\frac{J}{b}]$ is the quotient of $S \otimes_R R[\frac{I}{a}]$ by the ideal of elements annihilated by some power of b.*

Proof. Let S' be the quotient of $S \otimes_R R[\frac{I}{a}]$ by its b-power torsion elements. The ring map
$$S \otimes_R R[\tfrac{I}{a}] \longrightarrow S[\tfrac{J}{b}]$$
is surjective and annihilates a-power torsion as b is a nonzerodivisor in $S[\frac{J}{b}]$. Hence we obtain a surjective map $S' \to S[\frac{J}{b}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = y/b^n$ be an element of $S[\frac{J}{b}]$, i.e., $y \in J^n$. Write $y = \sum x_i s_i$ with $x_i \in I^n$ and $s_i \in S$. We map z to the class of $\sum s_i \otimes x_i/a^n$ in S'. This is well defined because an element of the kernel of the map $S \otimes_R I^n \to J^n$ is annihilated by a^n, hence maps to zero in S'. □

0G8Q **Example 10.70.4.** Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. With notation as in Definition 10.70.1 there is an isomorphism
$$P[T_1, \ldots, T_n]/(t_i T_j - t_j T_i) \longrightarrow \mathrm{Bl}_I(P)$$

sending T_i to $t_i^{(1)}$. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective one has to show: for each $e \geq 1$ the P-module I^e is generated by the monomials $t^E = t_1^{e_1} \ldots x_n^{e_n}$ for multiindices $E = (e_1, \ldots, e_n)$ of degree $|E| = e$ subject only to the relations $t_i t^E = t_j t^{E'}$ when $|E| = |E'| = e$ and $e_a + \delta_{ai} = e'_a + \delta_{aj},\ a = 1, \ldots, n$ (Kronecker delta). We omit the details.

0G8R **Example 10.70.5.** Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. Let $a = t_1$. With notation as in Definition 10.70.1 there is an isomorphism

$$P[x_2, \ldots, x_n]/(t_1 x_2 - t_2, \ldots, t_1 x_n - t_n) \longrightarrow P[\tfrac{I}{a}] = P[\tfrac{I}{t_1}]$$

sending x_i to t_i/t_1. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective, the reader can argue that the source and target of our map are t_1-torsion free and that the map is an isomorphism after inverting t_1, see Lemma 10.70.2. Alternatively, the reader can use the description of the Rees algebra in Example 10.70.4. We omit the details.

0G8S **Lemma 10.70.6.** *Let R be a ring. Let $I = (a_1, \ldots, a_n)$ be an ideal of R. Let $a = a_1$. Then there is a surjection*

$$R[x_2, \ldots, x_n]/(ax_2 - a_2, \ldots, ax_n - a_n) \longrightarrow R[\tfrac{I}{a}]$$

whose kernel is the a-power torsion in the source.

Proof. Consider the ring map $P = \mathbf{Z}[t_1, \ldots, t_n] \to R$ sending t_i to a_i. Set $J = (t_1, \ldots, t_n)$. By Example 10.70.5 we have $P[\frac{J}{t_1}] = P[x_2, \ldots, x_n]/(t_1 x_2 - t_2, \ldots, t_1 x_n - t_n)$. Apply Lemma 10.70.3 to the map $P \to A$ to conclude. □

080U **Lemma 10.70.7.** *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Set $R' = R[\frac{I}{a}]$. If $f \in R$ is such that $V(f) = V(I)$, then f maps to a nonzerodivisor in R' and $R'_f = R'_a = R_a$.*

Proof. We will use the results of Lemma 10.70.2 without further mention. The assumption $V(f) = V(I)$ implies $V(fR') = V(IR') = V(aR')$. Hence $a^n = fb$ and $f^m = ac$ for some $b, c \in R'$. The lemma follows. □

0BBI **Lemma 10.70.8.** *Let R be a ring, $I \subset R$ an ideal, $a \in I$, and $f \in R$. Set $R' = R[\frac{I}{a}]$ and $R'' = R[\frac{fI}{fa}]$. Then there is a surjective R-algebra map $R' \to R''$ whose kernel is the set of f-power torsion elements of R'.*

Proof. The map is given by sending x/a^n for $x \in I^n$ to $f^n x/(fa)^n$. It is straightforward to check this map is well defined and surjective. Since af is a nonzero divisor in R'' (Lemma 10.70.2) we see that the set of f-power torsion elements are mapped to zero. Conversely, if $x \in R'$ and $f^n x \neq 0$ for all $n > 0$, then $(af)^n x \neq 0$ for all n as a is a nonzero divisor in R'. It follows that the image of x in R'' is not zero by the description of R'' following Definition 10.70.1. □

052S **Lemma 10.70.9.** *If R is reduced then every (affine) blowup algebra of R is reduced.*

Proof. Let $I \subset R$ be an ideal and $a \in I$. Suppose x/a^n with $x \in I^n$ is a nilpotent element of $R[\frac{I}{a}]$. Then $(x/a^n)^m = 0$. Hence $a^N x^m = 0$ in R for some $N \geq 0$. After increasing N if necessary we may assume $N = me$ for some $e \geq 0$. Then

$(a^e x)^m = 0$ and since R is reduced we find $a^e x = 0$. This means that $x/a^n = 0$ in $R[\frac{I}{a}]$. □

052R **Lemma 10.70.10.** *Let R be a domain, $I \subset R$ an ideal, and $a \in I$ a nonzero element. Then the affine blowup algebra $R[\frac{I}{a}]$ is a domain.*

Proof. Suppose x/a^n, y/a^m with $x \in I^n$, $y \in I^m$ are elements of $R[\frac{I}{a}]$ whose product is zero. Then $a^N xy = 0$ in R. Since R is a domain we conclude that either $x = 0$ or $y = 0$. □

052T **Lemma 10.70.11.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $a \in I$. If a is not contained in any minimal prime of R, then $\mathrm{Spec}(R[\frac{I}{a}]) \to \mathrm{Spec}(R)$ has dense image.*

Proof. If $a^k x = 0$ for $x \in R$, then x is contained in all the minimal primes of R and hence nilpotent, see Lemma 10.17.2. Thus the kernel of $R \to R[\frac{I}{a}]$ consists of nilpotent elements. Hence the result follows from Lemma 10.30.6. □

052M **Lemma 10.70.12.** *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let $R \subset A \subset K$ be a valuation ring which dominates R. Then*

$$A = \operatorname{colim} R[\tfrac{I}{a}]$$

is a directed colimit of affine blowups $R \to R[\frac{I}{a}]$ with the following properties

(1) *$a \in I \subset \mathfrak{m}$,*
(2) *I is finitely generated, and*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero.*

Proof. Any blowup algebra $R[\frac{I}{a}]$ is a domain contained in K see Lemma 10.70.10. The lemma simply says that A is the directed union of the ones where $a \in I$ have properties (1), (2), (3). If $R[\frac{I}{a}] \subset A$ and $R[\frac{J}{b}] \subset A$, then we have

$$R[\tfrac{I}{a}] \cup R[\tfrac{J}{b}] \subset R[\tfrac{IJ}{ab}] \subset A$$

The first inclusion because $x/a^n = b^n x/(ab)^n$ and the second one because if $z \in (IJ)^n$, then $z = \sum x_i y_i$ with $x_i \in I^n$ and $y_i \in J^n$ and hence $z/(ab)^n = \sum (x_i/a^n)(y_i/b^n)$ is contained in A.

Consider a finite subset $E \subset A$. Say $E = \{e_1, \ldots, e_n\}$. Choose a nonzero $a \in R$ such that we can write $e_i = f_i/a$ for all $i = 1, \ldots, n$. Set $I = (f_1, \ldots, f_n, a)$. We claim that $R[\frac{I}{a}] \subset A$. This is clear as an element of $R[\frac{I}{a}]$ can be represented as a polynomial in the elements e_i. The lemma follows immediately from this observation. □

10.71. Ext groups

00LO In this section we do a tiny bit of homological algebra, in order to establish some fundamental properties of depth over Noetherian local rings.

00LP **Lemma 10.71.1.** *Let R be a ring. Let M be an R-module.*

(1) *There exists an exact complex*

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

with F_i free R-modules.

(2) *If R is Noetherian and M finite over R, then we can choose the complex such that F_i is finite free. In other words, we can find an exact complex*

$$\ldots \to R^{\oplus n_2} \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0.$$

Proof. Let us explain only the Noetherian case. As a first step choose a surjection $R^{n_0} \to M$. Then having constructed an exact complex of length e we simply choose a surjection $R^{n_{e+1}} \to \mathrm{Ker}(R^{n_e} \to R^{n_{e-1}})$ which is possible because R is Noetherian. □

00LQ **Definition 10.71.2.** Let R be a ring. Let M be an R-module.

(1) A (left) *resolution* $F_\bullet \to M$ of M is an exact complex

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of R-modules.
(2) A *resolution of M by free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a free R-module.
(3) A *resolution of M by finite free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a finite free R-module.

We often use the notation $F_\bullet$ to denote a complex of R-modules

$$\ldots \to F_i \to F_{i-1} \to \ldots$$

In this case we often use d_i or $d_{F,i}$ to denote the map $F_i \to F_{i-1}$. In this section we are always going to assume that F_0 is the last nonzero term in the complex. The *ith homology group of the complex $F_\bullet$* is the group $H_i = \mathrm{Ker}(d_{F,i})/\mathrm{Im}(d_{F,i+1})$. A *map of complexes* $\alpha : F_\bullet \to G_\bullet$ is given by maps $\alpha_i : F_i \to G_i$ such that $\alpha_{i-1} \circ d_{F,i} = d_{G,i-1} \circ \alpha_i$. Such a map induces a map on homology $H_i(\alpha) : H_i(F_\bullet) \to H_i(G_\bullet)$. If $\alpha, \beta : F_\bullet \to G_\bullet$ are maps of complexes, then a *homotopy* between α and β is given by a collection of maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = d_{G,i+1} \circ h_i + h_{i-1} \circ d_{F,i}$. Two maps $\alpha, \beta : F_\bullet \to G_\bullet$ are said to be *homotopic* if a homotopy between α and β exists.

We will use a very similar notation regarding complexes of the form $F^\bullet$ which look like

$$\ldots \to F^i \xrightarrow{d^i} F^{i+1} \to \ldots$$

There are maps of complexes, homotopies, etc. In this case we set $H^i(F^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$ and we call it the *ith cohomology group*.

00LR **Lemma 10.71.3.** *Any two homotopic maps of complexes induce the same maps on (co)homology groups.*

Proof. Omitted. □

00LS **Lemma 10.71.4.** *Let R be a ring. Let $M \to N$ be a map of R-modules. Let $N_\bullet \to N$ be an arbitrary resolution. Let*

$$\ldots \to F_2 \to F_1 \to F_0 \to M$$

be a complex of R-modules where each F_i is a free R-module. Then

(1) *there exists a map of complexes $F_\bullet \to N_\bullet$ such that*

$$\begin{array}{ccc} F_0 & \longrightarrow & M \\ \downarrow & & \downarrow \\ N_0 & \longrightarrow & N \end{array}$$

is commutative, and

(2) *any two maps $\alpha, \beta : F_\bullet \to N_\bullet$ as in (1) are homotopic.*

Proof. Proof of (1). Because F_0 is free we can find a map $F_0 \to N_0$ lifting the map $F_0 \to M \to N$. We obtain an induced map $F_1 \to F_0 \to N_0$ which ends up in the image of $N_1 \to N_0$. Since F_1 is free we may lift this to a map $F_1 \to N_1$. This in turn induces a map $F_2 \to F_1 \to N_1$ which maps to zero into N_0. Since $N_\bullet$ is exact we see that the image of this map is contained in the image of $N_2 \to N_1$. Hence we may lift to get a map $F_2 \to N_2$. Repeat.

Proof of (2). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : F_0 \to N_0$ is contained in the image of $N_1 \to N_0$. Hence we may lift γ_0 to a map $h_0 : F_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{F,1}$. By our choice of h_0 we see that the image of γ_1' is contained in the kernel of $N_1 \to N_0$. Since $N_\bullet$ is exact we may lift γ_1' to a map $h_1 : F_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

At this point we are ready to define the groups $\mathrm{Ext}^i_R(M, N)$. Namely, choose a resolution $F_\bullet$ of M by free R-modules, see Lemma 10.71.1. Consider the (cohomological) complex

$$\mathrm{Hom}_R(F_\bullet, N) : \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \to \mathrm{Hom}_R(F_2, N) \to \ldots$$

We define $\mathrm{Ext}^i_R(M, N)$ for $i \geq 0$ to be the ith cohomology group of this complex[7]. For $i < 0$ we set $\mathrm{Ext}^i_R(M, N) = 0$. Before we continue we point out that

$$\mathrm{Ext}^0_R(M, N) = \mathrm{Ker}(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)) = \mathrm{Hom}_R(M, N)$$

because we can apply part (1) of Lemma 10.10.1 to the exact sequence $F_1 \to F_0 \to M \to 0$. The following lemma explains in what sense this is well defined.

00LT **Lemma 10.71.5.** *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1, and $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma 10.71.4. Then the induced maps*

$$H^i(\alpha) : H^i(\mathrm{Hom}_R(F_\bullet, N)) \longrightarrow H^i(\mathrm{Hom}_R(G_\bullet, N))$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H^i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. Another map $\beta : F_\bullet \to G_\bullet$ inducing φ is homotopic to α by Lemma 10.71.4. Hence the maps $\mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(G_\bullet, N)$ are homotopic. Hence the independence result follows from Lemma 10.71.3.

Suppose that φ is an isomorphism. Let $\psi : M_2 \to M_1$ be an inverse. Choose $\beta : G_\bullet \to F_\bullet$ be a map inducing $\psi : M_2 = \mathrm{Coker}(d_{G,1}) \to M_1 = \mathrm{Coker}(d_{F,1})$, see

[7] At this point it would perhaps be more appropriate to say "an" in stead of "the" Ext-group.

Lemma 10.71.4. OK, and now consider the map $H^i(\alpha) \circ H^i(\beta) = H^i(\alpha \circ \beta)$. By the above the map $H^i(\alpha \circ \beta)$ is the *same* as the map $H^i(\mathrm{id}_{G_\bullet}) = \mathrm{id}$. Similarly for the composition $H^i(\beta) \circ H^i(\alpha)$. Hence $H^i(\alpha)$ and $H^i(\beta)$ are inverses of each other. □

00LU **Lemma 10.71.6.** *Let R be a ring. Let M be an R-module. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \\ \to \mathrm{Ext}^1_R(M, N') \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M, N'') \to \ldots \end{aligned}$$

Proof. Pick a free resolution $F_\bullet \to M$. Since each of the F_i are free we see that we get a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F_\bullet, N') \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N'') \to 0$$

Thus we get the long exact sequence from the snake lemma applied to this. □

065P **Lemma 10.71.7.** *Let R be a ring. Let N be an R-module. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \\ \to \mathrm{Ext}^1_R(M'', N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M', N) \to \ldots \end{aligned}$$

Proof. Pick sets of generators $\{m'_{i'}\}_{i' \in I'}$ and $\{m''_{i''}\}_{i'' \in I''}$ of M' and M''. For each $i'' \in I''$ choose a lift $\tilde{m}''_{i''} \in M$ of the element $m''_{i''} \in M''$. Set $F' = \bigoplus_{i' \in I'} R$, $F'' = \bigoplus_{i'' \in I''} R$ and $F = F' \oplus F''$. Mapping the generators of these free modules to the corresponding chosen generators gives surjective R-module maps $F' \to M'$, $F'' \to M''$, and $F \to M$. We obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

By the snake lemma we see that the sequence of kernels $0 \to K' \to K \to K'' \to 0$ is short exact sequence of R-modules. Hence we can continue this process indefinitely. In other words we obtain a short exact sequence of resolutions fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F'_\bullet & \to & F_\bullet & \to & F''_\bullet & \to & 0 \end{array}$$

Because each of the sequences $0 \to F'_n \to F_n \to F''_n \to 0$ is split exact (by construction) we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F''_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F'_\bullet, N) \to 0$$

by applying the $\mathrm{Hom}_R(-, N)$ functor. Thus we get the long exact sequence from the snake lemma applied to this. □

00LV **Lemma 10.71.8.** *Let R be a ring. Let M, N be R-modules. Any $x \in R$ such that either $xN = 0$, or $xM = 0$ annihilates each of the modules $\mathrm{Ext}^i_R(M, N)$.*

Proof. Pick a free resolution $F_\bullet$ of M. Since $\mathrm{Ext}^i_R(M, N)$ is defined as the cohomology of the complex $\mathrm{Hom}_R(F_\bullet, N)$ the lemma is clear when $xN = 0$. If $xM = 0$, then we see that multiplication by x on $F_\bullet$ lifts the zero map on M. Hence by Lemma 10.71.5 we see that it induces the same map on Ext groups as the zero map. □

08YR **Lemma 10.71.9.** *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $\mathrm{Ext}^i_R(M, N)$ is a finite R-module for all i.*

Proof. This holds because $\mathrm{Ext}^i_R(M, N)$ is computed as the cohomology groups of a complex $\mathrm{Hom}_R(F_\bullet, N)$ with each F_n a finite free R-module, see Lemma 10.71.1. □

10.72. Depth

00LE Here is our definition.

00LI **Definition 10.72.1.** Let R be a ring, and $I \subset R$ an ideal. Let M be a finite R-module. The *I-depth* of M, denoted $\mathrm{depth}_I(M)$, is defined as follows:

(1) if $IM \neq M$, then $\mathrm{depth}_I(M)$ is the supremum in $\{0, 1, 2, \ldots, \infty\}$ of the lengths of M-regular sequences in I,
(2) if $IM = M$ we set $\mathrm{depth}_I(M) = \infty$.

If $(R, \mathfrak{m})$ is local we call $\mathrm{depth}_{\mathfrak{m}}(M)$ simply the *depth* of M.

Explanation. By Definition 10.68.1 the empty sequence is not a regular sequence on the zero module, but for practical purposes it turns out to be convenient to set the depth of the 0 module equal to $+\infty$. Note that if $I = R$, then $\mathrm{depth}_I(M) = \infty$ for all finite R-modules M. If I is contained in the Jacobson radical of R (e.g., if R is local and $I \subset \mathfrak{m}_R$), then $M \neq 0 \Rightarrow IM \neq M$ by Nakayama's lemma. A module M has I-depth 0 if and only if M is nonzero and I does not contain a nonzerodivisor on M.

Example 10.68.2 shows depth does not behave well even if the ring is Noetherian, and Example 10.68.3 shows that it does not behave well if the ring is local but non-Noetherian. We will see depth behaves well if the ring is local Noetherian.

0AUI **Lemma 10.72.2.** *Let R be a ring, $I \subset R$ an ideal, and M a finite R-module. Then $\mathrm{depth}_I(M)$ is equal to the supremum of the lengths of sequences $f_1, \ldots, f_r \in I$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$.*

Proof. Suppose that $IM = M$. Then Lemma 10.20.1 shows there exists an $f \in I$ such that $f : M \to M$ is id_M. Hence $f, 0, 0, 0, \ldots$ is an infinite sequence of successive nonzerodivisors and we see agreement holds in this case. If $IM \neq M$, then we see that a sequence as in the lemma is an M-regular sequence and we conclude that agreement holds as well. □

00LK **Lemma 10.72.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Then $\dim(\mathit{Supp}(M)) \geq \mathit{depth}(M)$.*

Proof. The proof is by induction on $\dim(\mathrm{Supp}(M))$. If $\dim(\mathrm{Supp}(M)) = 0$, then $\mathrm{Supp}(M) = \{\mathfrak{m}\}$, whence $\mathrm{Ass}(M) = \{\mathfrak{m}\}$ (by Lemmas 10.63.2 and 10.63.7), and hence the depth of M is zero for example by Lemma 10.63.18. For the induction step we assume $\dim(\mathrm{Supp}(M)) > 0$. Let $f_1, \ldots, f_d$ be a sequence of elements of $\mathfrak{m}$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$. According to Lemma 10.72.2 it suffices to prove $\dim(\mathrm{Supp}(M)) \geq d$. We may assume $d > 0$ otherwise the lemma holds. By Lemma 10.63.10 we have $\dim(\mathrm{Supp}(M/f_1M)) = \dim(\mathrm{Supp}(M)) - 1$. By induction we conclude $\dim(\mathrm{Supp}(M/f_1M)) \geq d - 1$ as desired. □

0AUJ **Lemma 10.72.4.** *Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finite nonzero R-module such that $IM \neq M$. Then $\mathrm{depth}_I(M) < \infty$.*

Proof. Since M/IM is nonzero we can choose $\mathfrak{p} \in \text{Supp}(M/IM)$ by Lemma 10.40.2. Then $(M/IM)_\mathfrak{p} \neq 0$ which implies $I \subset \mathfrak{p}$ and moreover implies $M_\mathfrak{p} \neq IM_\mathfrak{p}$ as localization is exact. Let $f_1, \ldots, f_r \in I$ be an M-regular sequence. Then $M_\mathfrak{p}/(f_1, \ldots, f_r)M_\mathfrak{p}$ is nonzero as $(f_1, \ldots, f_r) \subset I$. As localization is flat we see that the images of $f_1, \ldots, f_r$ form a $M_\mathfrak{p}$-regular sequence in $I_\mathfrak{p}$. Since this works for every M-regular sequence in I we conclude that $\text{depth}_I(M) \leq \text{depth}_{I_\mathfrak{p}}(M_\mathfrak{p})$. The latter is $\leq \text{depth}(M_\mathfrak{p})$ which is $< \infty$ by Lemma 10.72.3. □

00LW **Lemma 10.72.5.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then depth(M) is equal to the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero.*

Proof. Let $\delta(M)$ denote the depth of M and let $i(M)$ denote the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero. We will see in a moment that $i(M) < \infty$. By Lemma 10.63.18 we have $\delta(M) = 0$ if and only if $i(M) = 0$, because $\mathfrak{m} \in \text{Ass}(M)$ exactly means that $i(M) = 0$. Hence if $\delta(M)$ or $i(M)$ is > 0, then we may choose $x \in \mathfrak{m}$ such that (a) x is a nonzerodivisor on M, and (b) $\text{depth}(M/xM) = \delta(M) - 1$. Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$ by Lemma 10.71.6:

$$\begin{aligned} 0 \to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M/xM) \\ \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M/xM) \to \ldots \end{aligned}$$

Since $x \in \mathfrak{m}$ all the maps $\text{Ext}^i_R(\kappa, M) \to \text{Ext}^i_R(\kappa, M)$ are zero, see Lemma 10.71.8. Thus it is clear that $i(M/xM) = i(M) - 1$. Induction on $\delta(M)$ finishes the proof. □

00LX **Lemma 10.72.6.** *Let R be a local Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of nonzero finite R-modules.*

(1) *depth$(N) \geq \min\{\text{depth}(N'), \text{depth}(N'')\}$*
(2) *depth$(N'') \geq \min\{\text{depth}(N), \text{depth}(N') - 1\}$*
(3) *depth$(N') \geq \min\{\text{depth}(N), \text{depth}(N'') + 1\}$*

Proof. Use the characterization of depth using the Ext groups $\text{Ext}^i(\kappa, N)$, see Lemma 10.72.5, and use the long exact cohomology sequence

$$\begin{aligned} 0 \to \text{Hom}_R(\kappa, N') \to \text{Hom}_R(\kappa, N) \to \text{Hom}_R(\kappa, N'') \\ \to \text{Ext}^1_R(\kappa, N') \to \text{Ext}^1_R(\kappa, N) \to \text{Ext}^1_R(\kappa, N'') \to \ldots \end{aligned}$$

from Lemma 10.71.6. □

090R **Lemma 10.72.7.** *Let R be a local Noetherian ring and M a nonzero finite R-module.*

(1) *If $x \in \mathfrak{m}$ is a nonzerodivisor on M, then depth$(M/xM) =$ depth$(M) - 1$.*
(2) *Any M-regular sequence $x_1, \ldots, x_r$ can be extended to an M-regular sequence of length depth(M).*

Proof. Part (2) is a formal consequence of part (1). Let $x \in R$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 10.72.6 we see that the depth drops by at most 1. On the other hand, if $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence we see that the depth drops by at least 1. □

0CN5 **Lemma 10.72.8.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. Let $x \in \mathfrak{m}$, $\mathfrak{p} \in Ass(M)$, and $\mathfrak{q}$ minimal over $\mathfrak{p} + (x)$. Then $\mathfrak{q} \in Ass(M/x^nM)$ for some $n \geq 1$.*

Proof. Pick a submodule $N \subset M$ with $N \cong R/\mathfrak{p}$. By the Artin-Rees lemma (Lemma 10.51.2) we can pick $n > 0$ such that $N \cap x^nM \subset xN$. Let $\overline{N} \subset M/x^nM$ be the image of $N \to M \to M/x^nM$. By Lemma 10.63.3 it suffices to show $\mathfrak{q} \in \mathrm{Ass}(\overline{N})$. By our choice of n there is a surjection $\overline{N} \to N/xN = R/\mathfrak{p} + (x)$ and hence $\mathfrak{q}$ is in the support of $\overline{N}$. Since $\overline{N}$ is annihilated by x^n and $\mathfrak{p}$ we see that $\mathfrak{q}$ is minimal among the primes in the support of $\overline{N}$. Thus $\mathfrak{q}$ is an associated prime of $\overline{N}$ by Lemma 10.63.8. □

0BK4 **Lemma 10.72.9.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. For $\mathfrak{p} \in Ass(M)$ we have $\dim(R/\mathfrak{p}) \geq depth(M)$.*

Proof. If $\mathfrak{m} \in \mathrm{Ass}(M)$ then there is a nonzero element $x \in M$ which is annihilated by all elements of $\mathfrak{m}$. Thus $\mathrm{depth}(M) = 0$. In particular the lemma holds in this case.

If $\mathrm{depth}(M) = 1$, then by the first paragraph we find that $\mathfrak{m} \notin \mathrm{Ass}(M)$. Hence $\dim(R/\mathfrak{p}) \geq 1$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$ and the lemma is true in this case as well.

We will prove the lemma in general by induction on $\mathrm{depth}(M)$ which we may and do assume to be > 1. Pick $x \in \mathfrak{m}$ which is a nonzerodivisor on M. Note $x \notin \mathfrak{p}$ (Lemma 10.63.9). By Lemma 10.60.13 we have $\dim(R/\mathfrak{p} + (x)) = \dim(R/\mathfrak{p}) - 1$. Thus there exists a prime $\mathfrak{q}$ minimal over $\mathfrak{p} + (x)$ with $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1$ (small argument omitted; hint: the dimension of a Noetherian local ring A is the maximum of the dimensions of $A/\mathfrak{r}$ taken over the minimal primes $\mathfrak{r}$ of A). Pick n as in Lemma 10.72.8 so that $\mathfrak{q}$ is an associated prime of M/x^nM. We may apply induction hypothesis to M/x^nM and $\mathfrak{q}$ because $\mathrm{depth}(M/x^nM) = \mathrm{depth}(M) - 1$ by Lemma 10.72.7. We find $\dim(R/\mathfrak{q}) \geq \mathrm{depth}(M/x^nM)$ and we win. □

0FCC **Lemma 10.72.10.** *Let R be a local Noetherian ring and M a finite R-module. For a prime ideal $\mathfrak{p} \subset R$ we have $depth(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) \geq depth(M)$.*

Proof. If $M_\mathfrak{p} = 0$, then $\mathrm{depth}(M_\mathfrak{p}) = \infty$ and the lemma holds. If $\mathrm{depth}(M) \leq \dim(R/\mathfrak{p})$, then the lemma is true. If $\mathrm{depth}(M) > \dim(R/\mathfrak{p})$, then $\mathfrak{p}$ is not contained in any associated prime $\mathfrak{q}$ of M by Lemma 10.72.9. Hence we can find an $x \in \mathfrak{p}$ not contained in any associated prime of M by Lemma 10.15.2 and Lemma 10.63.5. Then x is a nonzerodivisor on M, see Lemma 10.63.9. Hence $\mathrm{depth}(M/xM) = \mathrm{depth}(M) - 1$ and $\mathrm{depth}(M_\mathfrak{p}/xM_\mathfrak{p}) = \mathrm{depth}(M_\mathfrak{p}) - 1$ provided $M_\mathfrak{p}$ is nonzero, see Lemma 10.72.7. Thus we conclude by induction on $\mathrm{depth}(M)$. □

0AUK **Lemma 10.72.11.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $R \to S$ be a finite ring map. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of S. Let N be a finite S-module. Then*

$$\min_{i=1,\ldots,n} depth(N_{\mathfrak{m}_i}) = depth_\mathfrak{m}(N)$$

Proof. By Lemmas 10.36.20, 10.36.22, and Lemma 10.36.21 the maximal ideals of S are exactly the primes of S lying over $\mathfrak{m}$ and there are finitely many of them. Hence the statement of the lemma makes sense. We will prove the lemma by induction on $k = \min_{i=1,\ldots,n} \mathrm{depth}(N_{\mathfrak{m}_i})$. If $k = 0$, then $\mathrm{depth}(N_{\mathfrak{m}_i}) = 0$ for some i. By Lemma 10.72.5 this means $\mathfrak{m}_i S_{\mathfrak{m}_i}$ is an associated prime of $N_{\mathfrak{m}_i}$ and hence $\mathfrak{m}_i$

is an associated prime of N (Lemma 10.63.16). By Lemma 10.63.13 we see that $\mathfrak{m}$ is an associated prime of N as an R-module. Whence $\text{depth}_{\mathfrak{m}}(N) = 0$. This proves the base case. If $k > 0$, then we see that $\mathfrak{m}_i \notin \text{Ass}_S(N)$. Hence $\mathfrak{m} \notin \text{Ass}_R(N)$, again by Lemma 10.63.13. Thus we can find $f \in \mathfrak{m}$ which is not a zerodivisor on N, see Lemma 10.63.18. By Lemma 10.72.7 all the depths drop exactly by 1 when passing from N to N/fN and the induction hypothesis does the rest. □

10.73. Functorialities for Ext

087M In this section we briefly discuss the functoriality of Ext with respect to change of ring, etc. Here is a list of items to work out.

(1) Given $R \to R'$, an R-module M and an R'-module N' the R-module $\text{Ext}^i_R(M, N')$ has a natural R'-module structure. Moreover, there is a canonical R'-linear map $\text{Ext}^i_{R'}(M \otimes_R R', N') \to \text{Ext}^i_R(M, N')$.
(2) Given $R \to R'$ and R-modules M, N there is a natural R-module map $\text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, N \otimes_R R')$.

087N **Lemma 10.73.1.** *Given a flat ring map $R \to R'$, an R-module M, and an R'-module N' the natural map*

$$\text{Ext}^i_{R'}(M \otimes_R R', N') \to Ext^i_R(M, N')$$

is an isomorphism for $i \geq 0$.

Proof. Choose a free resolution $F_\bullet$ of M. Since $R \to R'$ is flat we see that $F_\bullet \otimes_R R'$ is a free resolution of $M \otimes_R R'$ over R'. The statement is that the map

$$\text{Hom}_{R'}(F_\bullet \otimes_R R', N') \to \text{Hom}_R(F_\bullet, N')$$

induces an isomorphism on homology groups, which is true because it is an isomorphism of complexes by Lemma 10.14.3. □

10.74. An application of Ext groups

02HN Here it is.

02HO **Lemma 10.74.1.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $N \to M$ be a homomorphism of finite R-modules. Suppose that there exists arbitrarily large n such that $N/I^nN \to M/I^nM$ is a split injection. Then $N \to M$ is a split injection.*

Proof. Assume $\varphi : N \to M$ satisfies the assumptions of the lemma. Note that this implies that $\text{Ker}(\varphi) \subset I^nN$ for arbitrarily large n. Hence by Lemma 10.51.5 we see that φ is injection. Let $Q = M/N$ so that we have a short exact sequence

$$0 \to N \to M \to Q \to 0.$$

Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Q \to 0$$

be a finite free resolution of Q. We can choose a map $\alpha : F_0 \to M$ lifting the map $F_0 \to Q$. This induces a map $\beta : F_1 \to N$ such that $\beta \circ d_2 = 0$. The extension above is split if and only if there exists a map $\gamma : F_0 \to N$ such that $\beta = \gamma \circ d_1$. In other words, the class of β in $\text{Ext}^1_R(Q, N)$ is the obstruction to splitting the short exact sequence above.

Suppose n is a large integer such that $N/I^nN \to M/I^nM$ is a split injection. This implies

$$0 \to N/I^nN \to M/I^nM \to Q/I^nQ \to 0.$$

is still short exact. Also, the sequence

$$F_1/I^nF_1 \xrightarrow{d_1} F_0/I^nF_0 \to Q/I^nQ \to 0$$

is still exact. Arguing as above we see that the map $\overline{\beta} : F_1/I^nF_1 \to N/I^nN$ induced by β is equal to $\overline{\gamma_n} \circ d_1$ for some map $\overline{\gamma_n} : F_0/I^nF_0 \to N/I^nN$. Since F_0 is free we can lift $\overline{\gamma_n}$ to a map $\gamma_n : F_0 \to N$ and then we see that $\beta - \gamma_n \circ d_1$ is a map from F_1 into I^nN. In other words we conclude that

$$\beta \in \mathrm{Im}\Big(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)\Big) + I^n \mathrm{Hom}_R(F_1, N).$$

for this n.

Since we have this property for arbitrarily large n by assumption we conclude that the image of β in the cokernel of $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ is zero by Lemma 10.51.5. Hence β is in the image of the map $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ as desired. □

10.75. Tor groups and flatness

00LY In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma 10.71.1. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\mathrm{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

00LZ **Lemma 10.75.1.** *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma 10.71.4. Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma 10.71.5. □

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \mathrm{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition 4.2.20. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M_1, N_1) & \longrightarrow & \mathrm{Tor}_i^R(M_1, N_2) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_i^R(M_2, N_1) & \longrightarrow & \mathrm{Tor}_i^R(M_2, N_2) \end{array}$$

00M0 **Lemma 10.75.2.** *Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence*

$$Tor_1^R(M, N') \to Tor_1^R(M, N) \to Tor_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

Proof. The proof of this is the same as the proof of Lemma 10.71.6. □

Consider a homological double complex of R-modules

$$\begin{array}{ccccccc} \ldots & \xrightarrow{d} & A_{2,0} & \xrightarrow{d} & A_{1,0} & \xrightarrow{d} & A_{0,0} \\ & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ \ldots & \xrightarrow{d} & A_{2,1} & \xrightarrow{d} & A_{1,1} & \xrightarrow{d} & A_{0,1} \\ & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ \ldots & \xrightarrow{d} & A_{2,2} & \xrightarrow{d} & A_{1,2} & \xrightarrow{d} & A_{0,2} \\ & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ & & \ldots & & \ldots & & \ldots \end{array}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \mathrm{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \mathrm{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge

commutative diagram

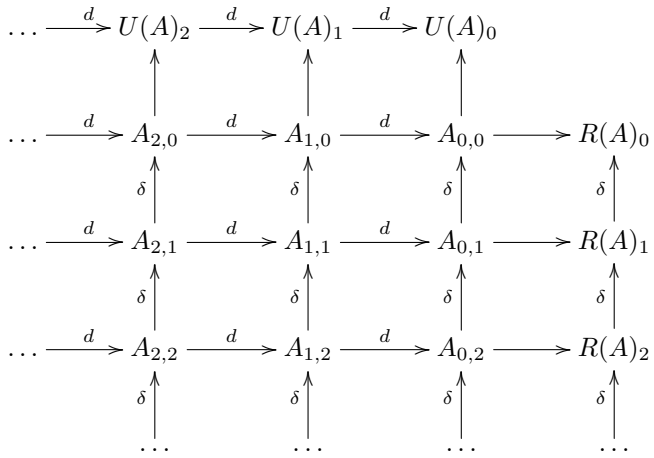

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

00M1 **Lemma 10.75.3.** *Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that*

(1) *Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.*
(2) *Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.*

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) + \delta(a_{i,1}), d(a_{i,1}) + \delta(a_{i-1,2}), \ldots, d(a_{1,i}) + \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

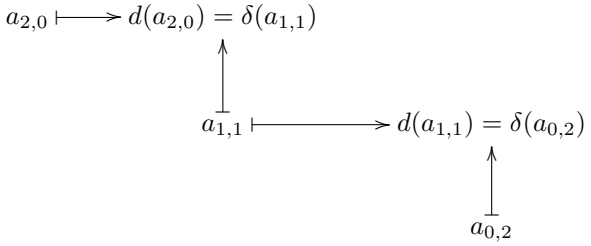

Naturally, we divide out by "trivial" zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\overline{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\overline{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\text{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \text{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \text{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{array}{ccccc} H_i(U(A)_\bullet) & \longleftarrow & \mathbf{H}_i(A) & \longrightarrow & H_i(R(A)_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ H_i(U(B)_\bullet) & \longleftarrow & \mathbf{H}_i(B) & \longrightarrow & H_i(R(B)_\bullet) \end{array}$$

This proves the functoriality. □

00M2 **Remark 10.75.4.** The isomorphism constructed above is the "correct" one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

00M3 **Lemma 10.75.5.** *Let R be a ring. For any $i \geq 0$ the functors $Mod_R \times Mod_R \to Mod_R$, $(M, N) \mapsto Tor_i^R(M, N)$ and $(M, N) \mapsto Tor_i^R(N, M)$ are canonically isomorphic.*

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \text{id}$, and
(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\text{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see

that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 10.75.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \mathrm{Coker}(A_{1,i} \to A_{0,i}) \\ &= \mathrm{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \mathrm{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \mathrm{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \mathrm{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 10.75.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma 10.71.4 we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 10.75.3. □

00M4 **Remark 10.75.6.** An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\mathrm{Tor}_i^R(M, M) \to \mathrm{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

0AZ4 **Lemma 10.75.7.** *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $\mathrm{Tor}_p^R(M, N)$ is a finite R-module for all p.*

Proof. This holds because $\mathrm{Tor}_p^R(M, N)$ is computed as the cohomology groups of a complex $F_\bullet \otimes_R N$ with each F_n a finite free R-module, see Lemma 10.71.1. □

00M5 **Lemma 10.75.8.** *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *The module M is flat over R.*
(2) *For all $i > 0$ the functor $Tor_i^R(M, -)$ is zero.*
(3) *The functor $Tor_1^R(M, -)$ is zero.*
(4) *For all ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*
(5) *For all finitely generated ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 10.75.2. We get an exact sequence

$$\mathrm{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma 10.39.5. □

00M6 **Remark 10.75.9.** The proof of Lemma 10.75.8 actually shows that

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{Ker}(I \otimes_R M \to M).$$

10.76. Functorialities for Tor

00M7 In this section we briefly discuss the functoriality of Tor with respect to change of ring, etc. Here is a list of items to work out.

(1) Given a ring map $R \to R'$, an R-module M and an R'-module N' the R-modules $\mathrm{Tor}_i^R(M, N')$ have a natural R'-module structure.
(2) Given a ring map $R \to R'$ and R-modules M, N there is a natural R-module map $\mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$.
(3) Given a ring map $R \to R'$ an R-module M and an R'-module N' there exists a natural R'-module map $\mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N')$.

00M8 **Lemma 10.76.1.** *Given a flat ring map $R \to R'$ and R-modules M, N the natural R-module map $Tor_i^R(M, N) \otimes_R R' \to Tor_i^{R'}(M \otimes_R R', N \otimes_R R')$ is an isomorphism for all i.*

Proof. Omitted. This is true because a free resolution $F_\bullet$ of M over R stays exact when tensoring with R' over R and hence $(F_\bullet \otimes_R N) \otimes_R R'$ computes the Tor groups over R'. □

The following lemma does not seem to fit anywhere else.

0BNF **Lemma 10.76.2.** *Let R be a ring. Let $M = \mathrm{colim}\, M_i$ be a filtered colimit of R-modules. Let N be an R-module. Then $Tor_n^R(M, N) = \mathrm{colim}\, Tor_n^R(M_i, N)$ for all n.*

Proof. Choose a free resolution $F_\bullet$ of N. Then $F_\bullet \otimes_R M = \mathrm{colim}\, F_\bullet \otimes_R M_i$ as complexes by Lemma 10.12.9. Thus the result by Lemma 10.8.8. □

10.77. Projective modules

05CD Some lemmas on projective modules.

05CE **Definition 10.77.1.** Let R be a ring. An R-module P is *projective* if and only if the functor $\mathrm{Hom}_R(P, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(M, -)$ is left exact for any R-module M, see Lemma 10.10.1. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, N')$ is surjective.

05CF **Lemma 10.77.2.** *Let R be a ring. Let P be an R-module. The following are equivalent*

(1) *P is projective,*
(2) *P is a direct summand of a free R-module, and*
(3) *$\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M.*

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in \mathrm{Hom}_R(P, F)$ such that $\pi \circ i = \mathrm{id}_P$. In other words $F \cong \mathrm{Ker}(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $\mathrm{Hom}_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $\mathrm{Hom}_R(F, -) = \mathrm{Hom}_R(P, -) \times \mathrm{Hom}_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\ldots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $\mathrm{Hom}_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to P)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \mathrm{Ker}(s) \oplus \mathrm{Ker}(F_0 \to P) = P \oplus \mathrm{Ker}(F_0 \to P)$$

and we win. □

0G8T **Lemma 10.77.3.** *Let R be a Noetherian ring. Let P be a finite R-module. If $\mathrm{Ext}^1_R(P, M) = 0$ for every finite R-module M, then P is projective.*

This lemma can be strengthened: There is a version for finitely presented R-modules if R is not assumed Noetherian. There is a version with M running through all finite length modules in the Noetherian case.

Proof. Choose a surjection $R^{\oplus n} \to P$ with kernel M. Since $\mathrm{Ext}^1_R(P, M) = 0$ this surjection is split and we conclude by Lemma 10.77.2. □

065Q **Lemma 10.77.4.** *A direct sum of projective modules is projective.*

Proof. This is true by the characterization of projectives as direct summands of free modules in Lemma 10.77.2. □

07LV **Lemma 10.77.5.** *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R/I-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. By Lemma 10.77.2 we can choose a set A and a direct sum decomposition $\bigoplus_{\alpha\in A} R/I = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Write $F = \bigoplus_{\alpha\in A} R$ for the free R-module on A. Choose a lift $p : F \to F$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $\bigoplus_{\alpha\in A} R/I$. Note that $p^2 - p \in \mathrm{End}_R(F)$ is a nilpotent endomorphism of F (as I is nilpotent and the matrix entries of $p^2 - p$ are in I; more precisely, if $I^n = 0$, then $(p^2 - p)^n = 0$). Hence by Lemma 10.32.7 we can modify our choice of p and assume that p is a projector. Set $P = \mathrm{Im}(p)$. □

0D47 **Lemma 10.77.6.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $\overline{P}$ be a finite projective R/I-module. Then there exists a finite projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. Recall that $\overline{P}$ is a direct summand of a free R/I-module $\bigoplus_{\alpha\in A} R/I$ by Lemma 10.77.2. As $\overline{P}$ is finite, it follows that $\overline{P}$ is contained in $\bigoplus_{\alpha\in A'} R/I$ for some $A' \subset A$ finite. Hence we may assume we have a direct sum decomposition $(R/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some n and some R/I-module $\overline{K}$. Choose a lift $p \in \mathrm{Mat}(n \times n, R)$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $(R/I)^{\oplus n}$. Note that $p^2 - p \in \mathrm{Mat}(n \times n, R)$ is nilpotent: as I is locally nilpotent and the matrix entries c_{ij} of $p^2 - p$ are in I we have $c_{ij}^t = 0$ for some $t > 0$ and then $(p^2 - p)^{tn^2} = 0$ (by looking at the matrix coefficients). Hence by Lemma 10.32.7 we can modify our choice of p and assume that p is a projector. Set $P = \mathrm{Im}(p)$. □

05CG **Lemma 10.77.7.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *I is nilpotent,*
(2) *M/IM is a projective R/I-module,*
(3) *M is a flat R-module.*

Then M is a projective R-module.

Proof. By Lemma 10.77.5 we can find a projective R-module P and an isomorphism $P/IP \to M/IM$. We are going to show that M is isomorphic to P which will finish the proof. Because P is projective we can lift the map $P \to P/IP \to M/IM$ to an R-module map $P \to M$ which is an isomorphism modulo I. Since $I^n = 0$ for some n, we can use the filtrations

$$0 = I^n M \subset I^{n-1}M \subset \ldots \subset IM \subset M$$
$$0 = I^n P \subset I^{n-1}P \subset \ldots \subset IP \subset P$$

to see that it suffices to show that the induced maps $I^a P/I^{a+1}P \to I^a M/I^{a+1}M$ are bijective. Since both P and M are flat R-modules we can identify this with the map

$$I^a/I^{a+1} \otimes_{R/I} P/IP \longrightarrow I^a/I^{a+1} \otimes_{R/I} M/IM$$

induced by $P \to M$. Since we chose $P \to M$ such that the induced map $P/IP \to M/IM$ is an isomorphism, we win. □

0H7M **Lemma 10.77.8.** *Let R be a ring. Let $I, J \subset R$ be ideals such that $I \cap J = 0$. Let P be an R-module such that P/IP is a projective R/I-module and P/JP is a projective R/J-module. Then P is a projective R-module.*

Proof. Choose a surjection $p : F \to P$ where F is a free R-module. Since P/IP is a projective R/I-module, we can choose a map $f : P/IP \to F/IF$ which is a right inverse to $p \bmod I$. Consider the map $q : F/JF \to F/(I+J)F \times_{P/(I+J)P} P/JP$ induced by the quotient map $F/JF \to F/(I+J)F$ and p. Note that q is a surjective map of R/J-modules (small detail omitted). Consider the map $f' : P/JP \to F/(I+J)F \times_{P/(I+J)P} P/JP$ induced by f and the identity map. Since P/JP is a projective R/J-module and q surjective, we can choose a map $g : P/JP \to F/JF$ such that $q \circ g = f'$. Then $f \bmod I + J = g \bmod I + J$. Since $F = F/JF \times_{F/(I+J)F} F/IF$ because $I \cap J = 0$, we conclude that we obtain a well defined homomorphism $h = (f, g) : P \to F$ of R-modules. The map $a = p \circ h : P \to P$ reduces to the identity modulo I and modulo J. Then a also induces the identity map on the submodule IP: if $x = \sum i_\alpha x_\alpha$ with $i_\alpha \in I$ and $x_\alpha \in P$, then $a(x) = \sum i_\alpha a(x_\alpha) = \sum i_\alpha x_\alpha = x$ because $a(x_\alpha) = x_\alpha + j_\alpha$ with $j_\alpha \in J$ and $i_\alpha j_\alpha = 0$. Thus $a : P \to P$ acts as the identity on IP and on P/IP and we conclude that a is an automorphism of P (small detail omitted). Thus P is a summand of F, whence projective. □

10.78. Finite projective modules

00NV

00NW **Definition 10.78.1.** Let R be a ring and M an R-module.

(1) We say that M is *locally free* if we can cover $\mathrm{Spec}(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$.
(2) We say that M is *finite locally free* if we can choose the covering such that each M_{f_i} is finite free.
(3) We say that M is *finite locally free of rank r* if we can choose the covering such that each M_{f_i} is isomorphic to $R_{f_i}^{\oplus r}$.

Note that a finite locally free R-module is automatically finitely presented by Lemma 10.23.2. Moreover, if M is a finite locally free module of rank r over a ring R and if R is nonzero, then r is uniquely determined by Lemma 10.15.8 (because at least one of the localizations R_{f_i} is a nonzero ring).

00NX **Lemma 10.78.2.** *Let R be a ring and let M be an R-module. The following are equivalent*

(1) *M is finitely presented and R-flat,*
(2) *M is finite projective,*
(3) *M is a direct summand of a finite free R-module,*
(4) *M is finitely presented and for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the localization $M_\mathfrak{p}$ is free,*
(5) *M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_\mathfrak{m}$ is free,*
(6) *M is finite and locally free,*
(7) *M is finite locally free, and*
(8) *M is finite, for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ is free, and the function*

$$\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 10.77.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak{p}$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's lemma (in the form of Lemma 10.20.2) these elements generate M_g for some $g \in R$, $g \notin \mathfrak{p}$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g$ has the following two properties: (a) $\mathrm{Ker}(\varphi)$ is a finite R_g-module (see Lemma 10.5.3) and (b) $\mathrm{Ker}(\varphi) \otimes \kappa(\mathfrak{p}) = 0$ by flatness of M_g over R_g (see Lemma 10.39.12). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\mathrm{Ker}(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Lemma 10.23.2, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Lemma 10.23.2, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 10.39.18) we conclude that (5) implies (1). At this point we have

$$\begin{array}{ccccccccc} (2) & \Longleftrightarrow & (3) & \Longrightarrow & (1) & \Longrightarrow & (7) & \Longleftrightarrow & (6) \\ & & & & \Uparrow & & \Downarrow & \searrow & \\ & & & & (5) & \Longleftarrow & (4) & & (8) \end{array}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $\mathrm{Hom}_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. By Lemma 10.10.2 we see that

$$0 \to \mathrm{Hom}_R(M, N'')_{f_i} \to \mathrm{Hom}_R(M, N)_{f_i} \to \mathrm{Hom}_R(M, N')_{f_i} \to 0$$

is equal to $0 \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact by Lemma 10.23.2.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 10.20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_{\mathfrak{p}} : R_{\mathfrak{p}}^{\oplus r} \to M_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in D(fg)$, see Lemma 10.23.1. By our choice of f the map $\Psi_{\mathfrak{p}}$ is surjective. By assumption (8) we have $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus \rho_M(\mathfrak{p})}$ and by our choice of g we have $\rho_M(\mathfrak{p}) = r$. Hence $\Psi_{\mathfrak{p}}$ determines a surjection

$R_{\mathfrak{p}}^{\oplus r} \to M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus r}$ whence is an isomorphism by Lemma 10.16.4. (Of course this last fact follows from a simple matrix argument also.) □

0FWG **Lemma 10.78.3.** *Let R be a reduced ring and let M be an R-module. Then the equivalent conditions of Lemma 10.78.2 are also equivalent to*

(9) *M is finite and the function $\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}$, $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant in the Zariski topology.*

Proof. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 10.20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. Since Ψ is surjective, it suffices to show that Ψ is injective. Since R_{fg} is reduced, it suffices to show that Ψ is injective after localization at all minimal primes $\mathfrak{p}$ of R_{fg}, see Lemma 10.25.2. However, we know that $R_{\mathfrak{p}} = \kappa(\mathfrak{p})$ by Lemma 10.25.1 and $\rho_M(\mathfrak{p}) = r$ hence $\Psi_{\mathfrak{p}} : R_{\mathfrak{p}}^{\oplus r} \to M \otimes_R \kappa(\mathfrak{p})$ is an isomorphism as a surjective map of finite dimensional vector spaces of the same dimension. □

00NY **Remark 10.78.4.** It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_{\mathfrak{m}} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

00NZ **Lemma 10.78.5.** *(Warning: see Remark 10.78.4.) Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.*

Proof. Follows from the equational criterion of flatness, see Lemma 10.39.11. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak{m}M$. By Nakayama's Lemma 10.20.1 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \ldots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak{m}M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \notin \mathfrak{m}M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \notin \mathfrak{m}M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \ldots, x_n$. By our choice of x_i we have $f_i \in \mathfrak{m}$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we see that $a_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i(x_i + (-a_{ij}/a_{nj})x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj})x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. □

00O1 **Lemma 10.78.6.** *Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.*

Proof. By Lemma 10.78.2 being finite projective over a local ring is the same thing as being finite free. Suppose that $M \otimes_R S$ is a finite free S-module. Pick $x_1, \ldots, x_r \in M$ whose images in $M/\mathfrak{m}_R M$ form a basis over $\kappa(\mathfrak{m})$. Then we see that $x_1 \otimes 1, \ldots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Lemma 10.39.17 we see that it is an isomorphism. □

02M9 **Lemma 10.78.7.** *Let R be a semi-local ring. Let M be a finite locally free module. If M has constant rank, then M is free. In particular, if R has connected spectrum, then M is free.*

Proof. Omitted. Hints: First show that $M/\mathfrak{m}_i M$ has the same dimension d for all maximal ideal $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R using the rank is constant. Next, show that there exist elements $x_1, \ldots, x_d \in M$ which form a basis for each $M/\mathfrak{m}_i M$ by the Chinese remainder theorem. Finally show that $x_1, \ldots, x_d$ is a basis for M. □

Here is a technical lemma that is used in the chapter on groupoids.

03C1 **Lemma 10.78.8.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field. Let $R \to S$ be a ring map. Let M be an S-module and let $N \subset M$ be an R-submodule. Assume*

(1) *S is semi-local and $\mathfrak{m}S$ is contained in the Jacobson radical of S,*
(2) *M is a finite free S-module, and*
(3) *N generates M as an S-module.*

Then N contains an S-basis of M.

Proof. Assume M is free of rank n. Let $I \subset S$ be the Jacobson radical. By Nakayama's Lemma 10.20.1 a sequence of elements $m_1, \ldots, m_n$ is a basis for M if and only if $\overline{m}_i \in M/IM$ generate M/IM. Hence we may replace M by M/IM, N by $N/(N \cap IM)$, R by $R/\mathfrak{m}$, and S by S/IS. In this case we see that S is a finite product of fields $S = k_1 \times \ldots \times k_r$ and $M = k_1^{\oplus n} \times \ldots \times k_r^{\oplus n}$. The fact that $N \subset M$ generates M as an S-module means that there exist $x_j \in N$ such that a linear combination $\sum a_j x_j$ with $a_j \in S$ has a nonzero component in each factor $k_i^{\oplus n}$. Because $R = k$ is an infinite field, this means that also some linear combination $y = \sum c_j x_j$ with $c_j \in k$ has a nonzero component in each factor. Hence $y \in N$ generates a free direct summand $Sy \subset M$. By induction on n the result holds for M/Sy and the submodule $\overline{N} = N/(N \cap Sy)$. In other words there exist $\overline{y}_2, \ldots, \overline{y}_n$ in $\overline{N}$ which (freely) generate M/Sy. Then $y, y_2, \ldots, y_n$ (freely) generate M and we win. □

0DVB **Lemma 10.78.9.** *Let R be ring. Let L, M, N be R-modules. The canonical map*

$$\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$$

is an isomorphism if M is finite projective.

Proof. By Lemma 10.78.2 we see that M is finitely presented as well as finite locally free. By Lemmas 10.10.2 and 10.12.16 formation of the left and right hand side of the arrow commutes with localization. We may check that our map is an isomorphism after localization, see Lemma 10.23.2. Thus we may assume M is finite free. In this case the lemma is immediate. □

10.79. Open loci defined by module maps

05GD The set of primes where a given module map is surjective, or an isomorphism is sometimes open. In the case of finite projective modules we can look at the rank of the map.

05GE **Lemma 10.79.1.** *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with N a finite R-module. Then we have the equality*

$$\begin{aligned} U &= \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is surjective}\} \\ &= \{\mathfrak{p} \subset R \mid \varphi \otimes \kappa(\mathfrak{p}) : M \otimes \kappa(\mathfrak{p}) \to N \otimes \kappa(\mathfrak{p}) \text{ is surjective}\} \end{aligned}$$

and U is an open subset of $\mathrm{Spec}(R)$. Moreover, for any $f \in R$ such that $D(f) \subset U$ the map $M_f \to N_f$ is surjective.

Proof. The equality in the displayed formula follows from Nakayama's lemma. Nakayama's lemma also implies that U is open. See Lemma 10.20.1 especially part (3). If $D(f) \subset U$, then $M_f \to N_f$ is surjective on all localizations at primes of R_f, and hence it is surjective by Lemma 10.23.1. □

05GF **Lemma 10.79.2.** *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with M finite and N finitely presented. Then*

$$U = \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is an isomorphism}\}$$

is an open subset of $\mathrm{Spec}(R)$.

Proof. Let $\mathfrak{p} \in U$. Pick a presentation $N = R^{\oplus n}/\sum_{j=1,\ldots,m} Rk_j$. Denote e_i the image in N of the ith basis vector of $R^{\oplus n}$. For each $i \in \{1, \ldots, n\}$ choose an element $m_i \in M_\mathfrak{p}$ such that $\varphi(m_i) = f_i e_i$ for some $f_i \in R$, $f_i \notin \mathfrak{p}$. This is possible as $\varphi_\mathfrak{p}$ is an isomorphism. Set $f = f_1 \ldots f_n$ and let $\psi : R_f^{\oplus n} \to M_f$ be the map which maps the ith basis vector to m_i/f_i. Note that $\varphi_f \circ \psi$ is the localization at f of the given map $R^{\oplus n} \to N$. As $\varphi_\mathfrak{p}$ is an isomorphism we see that $\psi(k_j)$ is an element of M which maps to zero in $M_\mathfrak{p}$. Hence we see that there exist $g_j \in R$, $g_j \notin \mathfrak{p}$ such that $g_j\psi(k_j) = 0$. Setting $g = g_1 \ldots g_m$, we see that ψ_g factors through N_{fg} to give a map $\chi : N_{fg} \to M_{fg}$. By construction χ is a right inverse to φ_{fg}. It follows that $\chi_\mathfrak{p}$ is an isomorphism. By Lemma 10.79.1 there is an $h \in R$, $h \notin \mathfrak{p}$ such that $\chi_h : N_{fgh} \to M_{fgh}$ is surjective. Hence φ_{fgh} and χ_h are mutually inverse maps, which implies that $D(fgh) \subset U$ as desired. □

0GWM **Lemma 10.79.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime. Let M be a finitely presented R-module. If $M_\mathfrak{p}$ is free, then there is an $f \in R$, $f \notin \mathfrak{p}$ such that M_f is a free R_f-module.*

Proof. Choose a basis $x_1, \ldots, x_n \in M_\mathfrak{p}$. We can choose an $f \in R$, $f \notin \mathfrak{p}$ such that x_i is the image of some $y_i \in M_f$. After replacing y_i by $f^m y_i$ for $m \gg 0$ we may assume $y_i \in M$. Namely, this replaces $x_1, \ldots, x_n$ by $f^m x_1, \ldots, f^m x_n$ which is still a basis as f maps to a unit in $R_\mathfrak{p}$. Hence we obtain a homomorphism $\varphi = (y_1, \ldots, y_n) : R^{\oplus n} \to M$ of R-modules whose localization at $\mathfrak{p}$ is an isomorphism. By Lemma 10.79.2 we can find an $f \in R$, $f \notin \mathfrak{p}$ such that $\varphi_\mathfrak{q}$ is an isomorphism for all primes $\mathfrak{q} \subset R$ with $f \notin \mathfrak{q}$. Then it follows from Lemma 10.23.1 that φ_f is an isomorphism and the proof is complete. □

00O0 **Lemma 10.79.4.** *Let R be a ring. Let $\varphi : P_1 \to P_2$ be a map of finite projective modules. Then*

(1) *The set U of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is injective is open and for any $f \in R$ such that $D(f) \subset U$ we have*
   (a) *$P_{1,f} \to P_{2,f}$ is injective, and*
   (b) *the module $\mathrm{Coker}(\varphi)_f$ is finite projective over R_f.*
(2) *The set W of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is surjective is open and for any $f \in R$ such that $D(f) \subset W$ we have*
   (a) *$P_{1,f} \to P_{2,f}$ is surjective, and*
   (b) *the module $\mathrm{Ker}(\varphi)_f$ is finite projective over R_f.*
(3) *The set V of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is an isomorphism is open and for any $f \in R$ such that $D(f) \subset V$ the map $\varphi : P_{1,f} \to P_{2,f}$ is an isomorphism of modules over R_f.*

Proof. To prove the set U is open we may work locally on $\mathrm{Spec}(R)$. Thus we may replace R by a suitable localization and assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$, see Lemma 10.78.2. In this case injectivity of $\varphi \otimes \kappa(\mathfrak{p})$ is equivalent to $n_1 \leq n_2$ and some $n_1 \times n_1$ minor f of the matrix of φ being invertible in $\kappa(\mathfrak{p})$. Thus $D(f) \subset U$. This argument also shows that $P_{1,\mathfrak{p}} \to P_{2,\mathfrak{p}}$ is injective for $\mathfrak{p} \in U$.

Now suppose $D(f) \subset U$. By the remark in the previous paragraph and Lemma 10.23.1 we see that $P_{1,f} \to P_{2,f}$ is injective, i.e., (1)(a) holds. By Lemma 10.78.2 to prove (1)(b) it suffices to prove that $\mathrm{Coker}(\varphi)$ is finite projective locally on $D(f)$. Thus, as we saw above, we may assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$ and that some minor of the matrix of φ is invertible in R. If the minor in question corresponds to the first n_1 basis vectors of R^{n_2}, then using the last $n_2 - n_1$ basis vectors we get a map $R^{n_2-n_1} \to R^{n_2} \to \mathrm{Coker}(\varphi)$ which is easily seen to be an isomorphism.

Openness of W and (2)(a) for $D(f) \subset W$ follow from Lemma 10.79.1. Since $P_{2,f}$ is projective over R_f we see that $\varphi_f : P_{1,f} \to P_{2,f}$ has a section and it follows that $\mathrm{Ker}(\varphi)_f$ is a direct summand of $P_{2,f}$. Therefore $\mathrm{Ker}(\varphi)_f$ is finite projective. Thus (2)(b) holds as well.

It is clear that $V = U \cap W$ is open and the other statement in (3) follows from (1)(a) and (2)(a). □

10.80. Faithfully flat descent for projectivity of modules

058B In the next few sections we prove, following Raynaud and Gruson [**GR71**], that the projectivity of modules descends along faithfully flat ring maps. The idea of the proof is to use dévissage à la Kaplansky [**Kap58**] to reduce to the case of countably generated modules. Given a well-behaved filtration of a module M, dévissage allows us to express M as a direct sum of successive quotients of the filtering submodules (see Section 10.84). Using this technique, we prove that a projective module is a direct sum of countably generated modules (Theorem 10.84.5). To prove descent of projectivity for countably generated modules, we introduce a "Mittag-Leffler" condition on modules, prove that a countably generated module is projective if and only if it is flat and Mittag-Leffler (Theorem 10.93.3), and then show that the property of being a Mittag-Leffler module descends (Lemma 10.95.1). Finally, given an arbitrary module M whose base change by a faithfully flat ring map is projective, we filter M by submodules whose successive quotients are countably generated projective modules, and then by dévissage conclude M is a direct sum of projectives, hence projective itself (Theorem 10.95.6).

We note that there is an error in the proof of faithfully flat descent of projectivity in [**GR71**]. There, descent of projectivity along faithfully flat ring maps is deduced from descent of projectivity along a more general type of ring map ([**GR71**, Example 3.1.4(1) of Part II]). However, the proof of descent along this more general type of map is incorrect. In [**Gru73**], Gruson explains what went wrong, although he does not provide a fix for the case of interest. Patching this hole in the proof of faithfully flat descent of projectivity comes down to proving that the property of being a Mittag-Leffler module descends along faithfully flat ring maps. We do this in Lemma 10.95.1.

10.81. Characterizing flatness

058C In this section we discuss criteria for flatness. The main result in this section is Lazard's theorem (Theorem 10.81.4 below), which says that a flat module is the colimit of a directed system of free finite modules. We remind the reader of the "equational criterion for flatness", see Lemma 10.39.11. It turns out that this can be massaged into a seemingly much stronger property.

058D **Lemma 10.81.1.** *Let M be an R-module. The following are equivalent:*

(1) *M is flat.*
(2) *If $f : R^n \to M$ is a module map and $x \in \mathrm{Ker}(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \mathrm{Ker}(h)$.*
(3) *Suppose $f : R^n \to M$ is a module map, $N \subset \mathrm{Ker}(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \mathrm{Ker}(h)$ and f factors through h. Then given any $x \in \mathrm{Ker}(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \mathrm{Ker}(h')$ and f factors through h'.*
(4) *If $f : R^n \to M$ is a module map and $N \subset \mathrm{Ker}(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \mathrm{Ker}(h)$.*

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness[8]. To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \mathrm{Ker}(f)$ in (4). Clearly (4) implies (2). □

058E **Lemma 10.81.2.** *Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map, then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.*

Proof. This is just a reformulation of condition (4) from Lemma 10.81.1. □

058F **Lemma 10.81.3.** *Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is*

[8] In fact, a module map $f : R^n \to M$ corresponds to a choice of elements $x_1, x_2, \ldots, x_n$ of M (namely, the images of the standard basis elements $e_1, e_2, \ldots, e_n$); furthermore, an element $x \in \mathrm{Ker}(f)$ corresponds to a relation between these $x_1, x_2, \ldots, x_n$ (namely, the relation $\sum_i f_i x_i = 0$, where the f_i are the coordinates of x). The module map h (represented as an $m \times n$-matrix) corresponds to the matrix (a_{ij}) from Lemma 10.39.11, and the y_j of Lemma 10.39.11 are the images of the standard basis vectors of R^m under g.

a surjective R-module map, then the induced map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M)$ *is surjective.*

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Lemma 10.81.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the lemma holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated, f factors through a free finite submodule of N. Thus M satisfies the condition of Lemma 10.81.2, hence is flat. □

058G **Theorem 10.81.4** (Lazard's theorem)**.** *Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.*

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$ for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $\mathrm{colim}_{e \in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 10.81.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible since every element of M can be written as $f(x_i)$ for infinitely many distinct $i \in I$ (by our choice of I). Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \mathrm{Ker}(R^{J'} \to F)$. Observe:

(1) The square

$$\begin{array}{ccc} R^{J'} & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle g} \\ R^I & \xrightarrow[f]{} & M \end{array}$$

is commutative, hence $N' \subset K = \mathrm{Ker}(f)$;

(2) $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;
(3) $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. □

10.82. Universally injective module maps

058H Next we discuss universally injective module maps, which are in a sense complementary to flat modules (see Lemma 10.82.5). We follow Lazard's thesis [**Laz69**]; also see [**Lam99**].

058I **Definition 10.82.1.** Let $f : M \to N$ be a map of R-modules. Then f is called *universally injective* if for every R-module Q, the map $f \otimes_R \mathrm{id}_Q : M \otimes_R Q \to N \otimes_R Q$ is injective. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called *universally exact* if it is exact and $M_1 \to M_2$ is universally injective.

058J **Example 10.82.2.** Examples of universally exact sequences.

(1) A split short exact sequence is universally exact since tensoring commutes with taking direct sums.
(2) The colimit of a directed system of universally exact sequences is universally exact. This follows from the fact that taking directed colimits is exact and that tensoring commutes with taking colimits. In particular the colimit of a directed system of split exact sequences is universally exact. We will see below that, conversely, any universally exact sequence arises in this way.

Next we give a list of criteria for a short exact sequence to be universally exact. They are analogues of criteria for flatness given above. Parts (3)-(6) below correspond, respectively, to the criteria for flatness given in Lemmas 10.39.11, 10.81.1, 10.81.3, and Theorem 10.81.4.

058K **Theorem 10.82.3.** *Let*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an exact sequence of R-modules. The following are equivalent:

(1) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is universally exact.*
(2) *For every finitely presented R-module Q, the sequence*

$$0 \to M_1 \otimes_R Q \to M_2 \otimes_R Q \to M_3 \otimes_R Q \to 0$$

is exact.
(3) *Given elements $x_i \in M_1$ $(i = 1, \ldots, n)$, $y_j \in M_2$ $(j = 1, \ldots, m)$, and $a_{ij} \in R$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ such that for all i*

$$f_1(x_i) = \sum\nolimits_j a_{ij} y_j,$$

there exists $z_j \in M_1$ $(j = 1, \ldots, m)$ such that for all i,

$$x_i = \sum\nolimits_j a_{ij} z_j.$$

(4) *Given a commutative diagram of R-module maps*

$$\begin{array}{ccc} R^n & \longrightarrow & R^m \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f_1} & M_2 \end{array}$$

where m and n are integers, there exists a map $R^m \to M_1$ making the top triangle commute.

(5) *For every finitely presented R-module P, the R-module map $\mathrm{Hom}_R(P, M_2) \to \mathrm{Hom}_R(P, M_3)$ is surjective.*

(6) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is the colimit of a directed system of split exact sequences of the form*

$$0 \to M_1 \to M_{2,i} \to M_{3,i} \to 0$$

where the $M_{3,i}$ are finitely presented.

Proof. Obviously (1) implies (2).

Next we show (2) implies (3). Let $f_1(x_i) = \sum_j a_{ij} y_j$ be relations as in (3). Let (d_j) be a basis for R^m, (e_i) a basis for R^n, and $R^m \to R^n$ the map given by $d_j \mapsto \sum_i a_{ij} e_i$. Let Q be the cokernel of $R^m \to R^n$. Then tensoring $R^m \to R^n \to Q \to 0$ by the map $f_1 : M_1 \to M_2$, we get a commutative diagram

$$\begin{array}{ccccccc} M_1^{\oplus m} & \longrightarrow & M_1^{\oplus n} & \longrightarrow & M_1 \otimes_R Q & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M_2^{\oplus m} & \longrightarrow & M_2^{\oplus n} & \longrightarrow & M_2 \otimes_R Q & \longrightarrow & 0 \end{array}$$

where $M_1^{\oplus m} \to M_1^{\oplus n}$ is given by

$$(z_1, \ldots, z_m) \mapsto (\textstyle\sum_j a_{1j} z_j, \ldots, \sum_j a_{nj} z_j),$$

and $M_2^{\oplus m} \to M_2^{\oplus n}$ is given similarly. We want to show $x = (x_1, \ldots, x_n) \in M_1^{\oplus n}$ is in the image of $M_1^{\oplus m} \to M_1^{\oplus n}$. By (2) the map $M_1 \otimes Q \to M_2 \otimes Q$ is injective, hence by exactness of the top row it is enough to show x maps to 0 in $M_2 \otimes Q$, and so by exactness of the bottom row it is enough to show the image of x in $M_2^{\oplus n}$ is in the image of $M_2^{\oplus m} \to M_2^{\oplus n}$. This is true by assumption.

Condition (4) is just a translation of (3) into diagram form.

Next we show (4) implies (5). Let $\varphi : P \to M_3$ be a map from a finitely presented R-module P. We must show that φ lifts to a map $P \to M_2$. Choose a presentation of P,

$$R^n \xrightarrow{g_1} R^m \xrightarrow{g_2} P \to 0.$$

Using freeness of R^n and R^m, we can construct $h_2 : R^m \to M_2$ and then $h_1 : R^n \to M_1$ such that the following diagram commutes

$$\begin{array}{ccccccccc} & & R^n & \xrightarrow{g_1} & R^m & \xrightarrow{g_2} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0. \end{array}$$

By (4) there is a map $k_1 : R^m \to M_1$ such that $k_1 \circ g_1 = h_1$. Now define $h_2' : R^m \to M_2$ by $h_2' = h_2 - f_1 \circ k_1$. Then

$$h_2' \circ g_1 = h_2 \circ g_1 - f_1 \circ k_1 \circ g_1 = h_2 \circ g_1 - f_1 \circ h_1 = 0.$$

Hence by passing to the quotient h_2' defines a map $\varphi' : P \to M_2$ such that $\varphi' \circ g_2 = h_2'$. In a diagram, we have

$$\begin{array}{ccc} R^m & \xrightarrow{g_2} & P \\ {\scriptstyle h_2'}\downarrow & \swarrow{\scriptstyle \varphi'} & \downarrow{\scriptstyle \varphi} \\ M_2 & \xrightarrow{f_2} & M_3. \end{array}$$

where the top triangle commutes. We claim that φ' is the desired lift, i.e. that $f_2 \circ \varphi' = \varphi$. From the definitions we have

$$f_2 \circ \varphi' \circ g_2 = f_2 \circ h_2' = f_2 \circ h_2 - f_2 \circ f_1 \circ k_1 = f_2 \circ h_2 = \varphi \circ g_2.$$

Since g_2 is surjective, this finishes the proof.

Now we show (5) implies (6). Write M_3 as the colimit of a directed system of finitely presented modules $M_{3,i}$, see Lemma 10.11.3. Let $M_{2,i}$ be the fiber product of $M_{3,i}$ and M_2 over M_3—by definition this is the submodule of $M_2 \times M_{3,i}$ consisting of elements whose two projections onto M_3 are equal. Let $M_{1,i}$ be the kernel of the projection $M_{2,i} \to M_{3,i}$. Then we have a directed system of exact sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0,$$

and for each i a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{1,i} & \longrightarrow & M_{2,i} & \longrightarrow & M_{3,i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

compatible with the directed system. From the definition of the fiber product $M_{2,i}$, it follows that the map $M_{1,i} \to M_1$ is an isomorphism. By (5) there is a map $M_{3,i} \to M_2$ lifting $M_{3,i} \to M_3$, and by the universal property of the fiber product this gives rise to a section of $M_{2,i} \to M_{3,i}$. Hence the sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$$

split. Passing to the colimit, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{colim} M_{1,i} & \longrightarrow & \operatorname{colim} M_{2,i} & \longrightarrow & \operatorname{colim} M_{3,i} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with exact rows and outer vertical maps isomorphisms. Hence $\operatorname{colim} M_{2,i} \to M_2$ is also an isomorphism and (6) holds.

Condition (6) implies (1) by Example 10.82.2 (2). □

The previous theorem shows that a universally exact sequence is always a colimit of split short exact sequences. If the cokernel of a universally injective map is finitely presented, then in fact the map itself splits:

058L **Lemma 10.82.4.** *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Suppose M_3 is of finite presentation. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact if and only if it is split.

Proof. A split short exact sequence is always universally exact, see Example 10.82.2. Conversely, if the sequence is universally exact, then by Theorem 10.82.3 (5) applied to $P = M_3$, the map $M_2 \to M_3$ admits a section. □

The following lemma shows how universally injective maps are complementary to flat modules.

058M **Lemma 10.82.5.** *Let M be an R-module. Then M is flat if and only if any exact sequence of R-modules*

$$0 \to M_1 \to M_2 \to M \to 0$$

is universally exact.

Proof. This follows from Lemma 10.81.3 and Theorem 10.82.3 (5). □

058N **Example 10.82.6.** Non-split and non-flat universally exact sequences.

(1) In spite of Lemma 10.82.4, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but non-split. For instance, take $R = \mathbf{Z}$, let $M_1 = \bigoplus_{n=1}^{\infty} \mathbf{Z}$, let $M_2 = \prod_{n=1}^{\infty} \mathbf{Z}$, and let M_3 be the cokernel of the inclusion $M_1 \to M_2$. Then M_1, M_2, M_3 are all flat since they are torsion-free (More on Algebra, Lemma 15.22.11), so by Lemma 10.82.5,

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact. However there can be no section $s : M_3 \to M_2$. In fact, if x is the image of $(2, 2^2, 2^3, \ldots) \in M_2$ in M_3, then any module map $s : M_3 \to M_2$ must kill x. This is because $x \in 2^n M_3$ for any $n \geq 1$, hence $s(x)$ is divisible by 2^n for all $n \geq 1$ and so must be 0.

(2) In spite of Lemma 10.82.5, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but with M_1, M_2, M_3 all non-flat. In fact if M is any non-flat module, just take the split exact sequence

$$0 \to M \to M \oplus M \to M \to 0.$$

For instance over $R = \mathbf{Z}$, take M to be any torsion module.

(3) Taking the direct sum of an exact sequence as in (1) with one as in (2), we get a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact, non-split, and such that M_1, M_2, M_3 are all non-flat.

058P **Lemma 10.82.7.** *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules, and suppose M_2 is flat. Then M_1 and M_3 are flat.*

Proof. Let $0 \to N \to N' \to N'' \to 0$ be a short exact sequence of R-modules. Consider the commutative diagram

$$\begin{array}{ccccc} M_1 \otimes_R N & \longrightarrow & M_2 \otimes_R N & \longrightarrow & M_3 \otimes_R N \\ \downarrow & & \downarrow & & \downarrow \\ M_1 \otimes_R N' & \longrightarrow & M_2 \otimes_R N' & \longrightarrow & M_3 \otimes_R N' \\ \downarrow & & \downarrow & & \downarrow \\ M_1 \otimes_R N'' & \longrightarrow & M_2 \otimes_R N'' & \longrightarrow & M_3 \otimes_R N'' \end{array}$$

(we have dropped the 0's on the boundary). By assumption the rows give short exact sequences and the arrow $M_2 \otimes N \to M_2 \otimes N'$ is injective. Clearly this implies that $M_1 \otimes N \to M_1 \otimes N'$ is injective and we see that M_1 is flat. In particular the left and middle columns give rise to short exact sequences. It follows from a diagram chase that the arrow $M_3 \otimes N \to M_3 \otimes N'$ is injective. Hence M_3 is flat. □

05CH **Lemma 10.82.8.** *Let R be a ring. Let $M \to M'$ be a universally injective R-module map. Then for any R-module N the map $M \otimes_R N \to M' \otimes_R N$ is universally injective.*

Proof. Omitted. □

05CI **Lemma 10.82.9.** *Let R be a ring. A composition of universally injective R-module maps is universally injective.*

Proof. Omitted. □

05CJ **Lemma 10.82.10.** *Let R be a ring. Let $M \to M'$ and $M' \to M''$ be R-module maps. If their composition $M \to M''$ is universally injective, then $M \to M'$ is universally injective.*

Proof. Omitted. □

05CK **Lemma 10.82.11.** *Let $R \to S$ be a faithfully flat ring map. Then $R \to S$ is universally injective as a map of R-modules. In particular $R \cap IS = I$ for any ideal $I \subset R$.*

Proof. Let N be an R-module. We have to show that $N \to N \otimes_R S$ is injective. As S is faithfully flat as an R-module, it suffices to prove this after tensoring with S. Hence it suffices to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$, $n \otimes s \mapsto n \otimes 1 \otimes s$ is injective. This is true because there is a retraction, namely, $n \otimes s \otimes s' \mapsto n \otimes ss'$. □

05CL **Lemma 10.82.12.** *Let $R \to S$ be a ring map. Let $M \to M'$ be a map of S-modules. The following are equivalent*

1. *$M \to M'$ is universally injective as a map of R-modules,*
2. *for each prime $\mathfrak{q}$ of S the map $M_\mathfrak{q} \to M'_\mathfrak{q}$ is universally injective as a map of R-modules,*
3. *for each maximal ideal $\mathfrak{m}$ of S the map $M_\mathfrak{m} \to M'_\mathfrak{m}$ is universally injective as a map of R-modules,*
4. *for each prime $\mathfrak{q}$ of S the map $M_\mathfrak{q} \to M'_\mathfrak{q}$ is universally injective as a map of $R_\mathfrak{p}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R, and*
5. *for each maximal ideal $\mathfrak{m}$ of S the map $M_\mathfrak{m} \to M'_\mathfrak{m}$ is universally injective as a map of $R_\mathfrak{p}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}$ in R.*

Proof. Let N be an R-module. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Then we have

$$(M \otimes_R N)_\mathfrak{q} = M_\mathfrak{q} \otimes_R N = M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}.$$

Moreover, the same thing holds for M' and localization is exact. Also, if N is an $R_\mathfrak{p}$-module, then $N_\mathfrak{p} = N$. Using this the equivalences can be proved in a straightforward manner.

For example, suppose that (5) holds. Let $K = \mathrm{Ker}(M \otimes_R N \to M' \otimes_R N)$. By the remarks above we see that $K_\mathfrak{m} = 0$ for each maximal ideal $\mathfrak{m}$ of S. Hence $K = 0$ by Lemma 10.23.1. Thus (1) holds. Conversely, suppose that (1) holds. Take any $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$. Take any module N over $R_\mathfrak{p}$. Then by assumption $\mathrm{Ker}(M \otimes_R N \to M' \otimes_R N) = 0$. Hence by the formulae above and the fact that $N = N_\mathfrak{p}$ we see that $\mathrm{Ker}(M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N \to M'_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = 0$. In other words (4) holds. Of course (4) $\Rightarrow$ (5) is immediate. Hence (1), (4) and (5) are all equivalent. We omit the proof of the other equivalences. □

05CM **Lemma 10.82.13.** *Let $\varphi : A \to B$ be a ring map. Let $S \subset A$ and $S' \subset B$ be multiplicative subsets such that $\varphi(S) \subset S'$. Let $M \to M'$ be a map of B-modules.*

(1) *If $M \to M'$ is universally injective as a map of A-modules, then $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules and as a map of $S^{-1}A$-modules.*
(2) *If M and M' are $(S')^{-1}B$-modules, then $M \to M'$ is universally injective as a map of A-modules if and only if it is universally injective as a map of $S^{-1}A$-modules.*

Proof. You can prove this using Lemma 10.82.12 but you can also prove it directly as follows. Assume $M \to M'$ is A-universally injective. Let Q be an A-module. Then $Q \otimes_A M \to Q \otimes_A M'$ is injective. Since localization is exact we see that $(S')^{-1}(Q \otimes_A M) \to (S')^{-1}(Q \otimes_A M')$ is injective. As $(S')^{-1}(Q \otimes_A M) = Q \otimes_A (S')^{-1}M$ and similarly for M' we see that $Q \otimes_A (S')^{-1}M \to Q \otimes_A (S')^{-1}M'$ is injective, hence $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules. This proves the first part of (1). To see (2) we can use the following two facts: (a) if Q is an $S^{-1}A$-module, then $Q \otimes_A S^{-1}A = Q$, i.e., tensoring with Q over A is the same thing as tensoring with Q over $S^{-1}A$, (b) if M is any A-module on which the elements of S are invertible, then $M \otimes_A Q = M \otimes_{S^{-1}A} S^{-1}Q$. Part (2) follows from this immediately. □

0AS5 **Lemma 10.82.14.** *Let R be a ring and let $M \to M'$ be a map of R-modules. If M' is flat, then $M \to M'$ is universally injective if and only if $M/IM \to M'/IM'$ is injective for every finitely generated ideal I of R.*

Proof. It suffices to show that $M \otimes_R Q \to M' \otimes_R Q$ is injective for every finite R-module Q, see Theorem 10.82.3. Then Q has a finite filtration $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ by submodules whose subquotients are isomorphic to cyclic modules R/I_i, see Lemma 10.5.4. Since M' is flat, we obtain a filtration

$$\begin{array}{ccccccc} M \otimes Q_1 & \longrightarrow & M \otimes Q_2 & \longrightarrow & \ldots & \longrightarrow & M \otimes Q \\ \downarrow & & \downarrow & & & & \downarrow \\ M' \otimes Q_1 & \hookrightarrow & M' \otimes Q_2 & \hookrightarrow & \ldots & \hookrightarrow & M' \otimes Q \end{array}$$

of $M' \otimes_R Q$ by submodules $M' \otimes_R Q_i$ whose successive quotients are $M' \otimes_R R/I_i = M'/I_iM'$. A simple induction argument shows that it suffices to check $M/I_iM \to M'/I_iM'$ is injective. Note that the collection of finitely generated ideals $I_i' \subset I_i$ is a directed set. Thus $M/I_iM = \operatorname{colim} M/I_i'M$ is a filtered colimit, similarly for M', the maps $M/I_i'M \to M'/I_i'M'$ are injective by assumption, and since filtered colimits are exact (Lemma 10.8.8) we conclude. □

10.83. Descent for finite projective modules

058Q In this section we give an elementary proof of the fact that the property of being a *finite* projective module descends along faithfully flat ring maps. The proof does not apply when we drop the finiteness condition. However, the method is indicative of the one we shall use to prove descent for the property of being a *countably generated* projective module—see the comments at the end of this section.

058R **Lemma 10.83.1.** *Let M be an R-module. Then M is finite projective if and only if M is finitely presented and flat.*

Proof. This is part of Lemma 10.78.2. However, at this point we can give a more elegant proof of the implication (1) ⇒ (2) of that lemma as follows. If M is finitely presented and flat, then take a surjection $R^n \to M$. By Lemma 10.81.3 applied to $P = M$, the map $R^n \to M$ admits a section. So M is a direct summand of a free module and hence projective. □

Here are some properties of modules that descend.

03C4 **Lemma 10.83.2.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. Then*

(1) *if the S-module $M \otimes_R S$ is of finite type, then M is of finite type,*
(2) *if the S-module $M \otimes_R S$ is of finite presentation, then M is of finite presentation,*
(3) *if the S-module $M \otimes_R S$ is flat, then M is flat, and*
(4) *add more here as needed.*

Proof. Assume $M \otimes_R S$ is of finite type. Let $y_1, \ldots, y_m$ be generators of $M \otimes_R S$ over S. Write $y_j = \sum x_i \otimes f_i$ for some $x_1, \ldots, x_n \in M$. Then we see that the map $\varphi : R^{\oplus n} \to M$ has the property that $\varphi \otimes \mathrm{id}_S : S^{\oplus n} \to M \otimes_R S$ is surjective. Since $R \to S$ is faithfully flat we see that φ is surjective, and M is finitely generated.

Assume $M \otimes_R S$ is of finite presentation. By (1) we see that M is of finite type. Choose a surjection $R^{\oplus n} \to M$ and denote K the kernel. As $R \to S$ is flat we see that $K \otimes_R S$ is the kernel of the base change $S^{\oplus n} \to M \otimes_R S$. As $M \otimes_R S$ is of finite presentation we conclude that $K \otimes_R S$ is of finite type. Hence by (1) we see that K is of finite type and hence M is of finite presentation.

Part (3) is Lemma 10.39.8. □

058S **Proposition 10.83.3.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is finite projective, then M is finite projective.*

Proof. Follows from Lemmas 10.83.1 and 10.83.2. □

The next few sections are about removing the finiteness assumption by using dévissage to reduce to the countably generated case. In the countably generated case, the strategy is to find a characterization of countably generated projective modules

analogous to Lemma 10.83.1, and then to prove directly that this characterization descends. We do this by introducing the notion of a Mittag-Leffler module and proving that if a module M is countably generated, then it is projective if and only if it is flat and Mittag-Leffler (Theorem 10.93.3). When M is finitely generated, this statement reduces to Lemma 10.83.1 (since, according to Example 10.91.1 (1), a finitely generated module is Mittag-Leffler if and only if it is finitely presented).

10.84. Transfinite dévissage of modules

058T In this section we introduce a dévissage technique for decomposing a module into a direct sum. The main result is that a projective module is a direct sum of countably generated modules (Theorem 10.84.5 below). We follow [**Kap58**].

058U **Definition 10.84.1.** Let M be an R-module. A *direct sum dévissage* of M is a family of submodules $(M_\alpha)_{\alpha\in S}$, indexed by an ordinal S and increasing (with respect to inclusion), such that:

(0) $M_0 = 0$;
(1) $M = \bigcup_\alpha M_\alpha$;
(2) if $\alpha \in S$ is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$;
(3) if $\alpha + 1 \in S$, then M_α is a direct summand of $M_{\alpha+1}$.

If moreover

(4) $M_{\alpha+1}/M_\alpha$ is countably generated for $\alpha + 1 \in S$,

then $(M_\alpha)_{\alpha\in S}$ is called a *Kaplansky dévissage* of M.

The terminology is justified by the following lemma.

058V **Lemma 10.84.2.** *Let M be an R-module. If $(M_\alpha)_{\alpha\in S}$ is a direct sum dévissage of M, then $M \cong \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha$.*

Proof. By property (3) of a direct sum dévissage, there is an inclusion $M_{\alpha+1}/M_\alpha \to M$ for each $\alpha \in S$. Consider the map

$$f : \bigoplus\nolimits_{\alpha+1\in S} M_{\alpha+1}/M_\alpha \to M$$

given by the sum of these inclusions. Further consider the restrictions

$$f_\beta : \bigoplus\nolimits_{\alpha+1\leq\beta} M_{\alpha+1}/M_\alpha \longrightarrow M$$

for $\beta \in S$. Transfinite induction on S shows that the image of f_β is M_β. For $\beta = 0$ this is true by (0). If $\beta + 1$ is a successor ordinal and it is true for β, then it is true for $\beta + 1$ by (3). And if β is a limit ordinal and it is true for $\alpha < \beta$, then it is true for β by (2). Hence f is surjective by (1).

Transfinite induction on S also shows that the restrictions f_β are injective. For $\beta = 0$ it is true. If $\beta+1$ is a successor ordinal and f_β is injective, then let x be in the kernel and write $x = (x_{\alpha+1})_{\alpha+1\leq\beta+1}$ in terms of its components $x_{\alpha+1} \in M_{\alpha+1}/M_\alpha$. By property (3) and the fact that the image of f_β is M_β both $(x_{\alpha+1})_{\alpha+1\leq\beta}$ and $x_{\beta+1}$ map to 0. Hence $x_{\beta+1} = 0$ and, by the assumption that the restriction f_β is injective also $x_{\alpha+1} = 0$ for every $\alpha + 1 \leq \beta$. So $x = 0$ and $f_{\beta+1}$ is injective. If β is a limit ordinal consider an element x of the kernel. Then x is already contained in the domain of f_α for some $\alpha < \beta$. Thus $x = 0$ which finishes the induction. We conclude that f is injective since f_β is for each $\beta \in S$. □

058W **Lemma 10.84.3.** *Let M be an R-module. Then M is a direct sum of countably generated R-modules if and only if it admits a Kaplansky dévissage.*

Proof. The lemma takes care of the "if" direction. Conversely, suppose $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Well-order I so that we can think of it as an ordinal. Then setting $M_i = \bigoplus_{j<i} N_j$ gives a Kaplansky dévissage $(M_i)_{i \in I}$ of M. □

058X **Theorem 10.84.4.** *Suppose M is a direct sum of countably generated R-modules. If P is a direct summand of M, then P is also a direct sum of countably generated R-modules.*

Proof. Write $M = P \oplus Q$. We are going to construct a Kaplansky dévissage $(M_\alpha)_{\alpha \in S}$ of M which, in addition to the defining properties (0)-(4), satisfies:

(5) Each M_α is a direct summand of M;
(6) $M_\alpha = P_\alpha \oplus Q_\alpha$, where $P_\alpha = P \cap M_\alpha$ and $Q_\alpha = Q \cap M_\alpha$.

(Note: if properties (0)-(2) hold, then in fact property (3) is equivalent to property (5).)

To see how this implies the theorem, it is enough to show that $(P_\alpha)_{\alpha \in S}$ forms a Kaplansky dévissage of P. Properties (0), (1), and (2) are clear. By (5) and (6) for (M_α), each P_α is a direct summand of M. Since $P_\alpha \subset P_{\alpha+1}$, this implies P_α is a direct summand of $P_{\alpha+1}$; hence (3) holds for (P_α). For (4), note that

$$M_{\alpha+1}/M_\alpha \cong P_{\alpha+1}/P_\alpha \oplus Q_{\alpha+1}/Q_\alpha,$$

so $P_{\alpha+1}/P_\alpha$ is countably generated because this is true of $M_{\alpha+1}/M_\alpha$.

It remains to construct the M_α. Write $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Choose a well-ordering of I. By transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that M_α is a direct sum of some subset of the N_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Since each M_β for $\beta < \alpha$ is a direct sum of a subset of the N_i, the same will be true of M_α. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. If not, choose the smallest $j \in I$ such that N_j is not contained in M_α. We will construct an infinite matrix $(x_{mn}), m, n = 1, 2, 3, \ldots$ such that:

(1) N_j is contained in the submodule of M generated by the entries x_{mn};
(2) if we write any entry $x_{k\ell}$ in terms of its P- and Q-components, $x_{k\ell} = y_{k\ell} + z_{k\ell}$, then the matrix (x_{mn}) contains a set of generators for each N_i for which $y_{k\ell}$ or $z_{k\ell}$ has nonzero component.

Then we define $M_{\alpha+1}$ to be the submodule of M generated by M_α and all x_{mn}; by property (2) of the matrix (x_{mn}), $M_{\alpha+1}$ will be a direct sum of some subset of the N_i. To construct the matrix (x_{mn}), let $x_{11}, x_{12}, x_{13}, \ldots$ be a countable set of generators for N_j. Then if $x_{11} = y_{11} + z_{11}$ is the decomposition into P- and Q-components, let $x_{21}, x_{22}, x_{23}, \ldots$ be a countable set of generators for the sum of the N_i for which y_{11} or z_{11} have nonzero component. Repeat this process on x_{12} to get elements $x_{31}, x_{32}, \ldots$, the third row of our matrix. Repeat on x_{21} to get the fourth

row, on x_{13} to get the fifth, and so on, going down along successive anti-diagonals as indicated below:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \ldots \\ x_{21} & x_{22} & x_{23} & \ldots & \\ x_{31} & x_{32} & \ldots & & \\ x_{41} & \ldots & & & \\ \ldots & & & & \end{pmatrix}.$$

Transfinite induction on I (using the fact that we constructed $M_{\alpha+1}$ to contain N_j for the smallest j such that N_j is not contained in M_α) shows that for each $i \in I$, N_i is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $i \in I$ there is $\alpha \in S$ such that N_i is contained in M_α. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The family $(M_\alpha)_{\alpha \in S}$ moreover satisfies the other defining properties, and also (5) and (6) above: properties (0), (2), (4), and (6) are clear by construction; property (5) is true because each M_α is by construction a direct sum of some N_i; and (3) is implied by (5) and the fact that $M_\alpha \subset M_{\alpha+1}$. □

As a corollary we get the result for projective modules stated at the beginning of the section.

058Y **Theorem 10.84.5.** *If P is a projective R-module, then P is a direct sum of countably generated projective R-modules.*

Proof. A module is projective if and only if it is a direct summand of a free module, so this follows from Theorem 10.84.4. □

10.85. Projective modules over a local ring

058Z In this section we prove a very cute result: a projective module M over a local ring is free (Theorem 10.85.4 below). Note that with the additional assumption that M is finite, this result is Lemma 10.78.5. In general we have:

0590 **Lemma 10.85.1.** *Let R be a ring. Then every projective R-module is free if and only if every countably generated projective R-module is free.*

Proof. Follows immediately from Theorem 10.84.5. □

Here is a criterion for a countably generated module to be free.

0591 **Lemma 10.85.2.** *Let M be a countably generated R-module with the following property: if $M = N \oplus N'$ with N' a finite free R-module, then any element of N is contained in a free direct summand of N. Then M is free.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. We inductively construct finite free direct summands $F_1, F_2, \ldots$ of M such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of M which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$M = F_1 \oplus \ldots \oplus F_n \oplus N$$

and let $x \in N$ be the image of x_{n+1}. Then we can find a free direct summand $F_{n+1} \subset N$ containing x by the assumption in the statement of the lemma. Of course we can replace F_{n+1} by a finite free direct summand of F_{n+1} and the induction step is complete. Then $M = \bigoplus_{i=1}^{\infty} F_i$ is free. □

0592 **Lemma 10.85.3.** *Let P be a projective module over a local ring R. Then any element of P is contained in a free direct summand of P.*

Proof. Since P is projective it is a direct summand of some free R-module F, say $F = P \oplus Q$. Let $x \in P$ be the element that we wish to show is contained in a free direct summand of P. Let B be a basis of F such that the number of basis elements needed in the expression of x is minimal, say $x = \sum_{i=1}^n a_i e_i$ for some $e_i \in B$ and $a_i \in R$. Then no a_j can be expressed as a linear combination of the other a_i; for if $a_j = \sum_{i \neq j} a_i b_i$ for some $b_i \in R$, then replacing e_i by $e_i + b_i e_j$ for $i \neq j$ and leaving unchanged the other elements of B, we get a new basis for F in terms of which x has a shorter expression.

Let $e_i = y_i + z_i, y_i \in P, z_i \in Q$ be the decomposition of e_i into its P- and Q-components. Write $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$, where t_i is a linear combination of elements in B other than $e_1, \ldots, e_n$. To finish the proof it suffices to show that the matrix (b_{ij}) is invertible. For then the map $F \to F$ sending $e_i \mapsto y_i$ for $i = 1, \ldots, n$ and fixing $B \setminus \{e_1, \ldots, e_n\}$ is an isomorphism, so that $y_1, \ldots, y_n$ together with $B \setminus \{e_1, \ldots, e_n\}$ form a basis for F. Then the submodule N spanned by $y_1, \ldots, y_n$ is a free submodule of P; N is a direct summand of P since $N \subset P$ and both N and P are direct summands of F; and $x \in N$ since $x \in P$ implies $x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$.

Now we prove that (b_{ij}) is invertible. Plugging $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$ into $\sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$ and equating the coefficients of e_j gives $a_j = \sum_{i=1}^n a_i b_{ij}$. But as noted above, our choice of B guarantees that no a_j can be written as a linear combination of the other a_i. Thus b_{ij} is a non-unit for $i \neq j$, and $1 - b_{ii}$ is a non-unit—so in particular b_{ii} is a unit—for all i. But a matrix over a local ring having units along the diagonal and non-units elsewhere is invertible, as its determinant is a unit. □

0593 **Theorem 10.85.4.** *If P is a projective module over a local ring R, then P is free.*

Proof. Follows from Lemmas 10.85.1, 10.85.2, and 10.85.3. □

10.86. Mittag-Leffler systems

0594 The purpose of this section is to define Mittag-Leffler systems and why this is a useful notion.

In the following, I will be a directed set, see Categories, Definition 4.21.1. Let $(A_i, \varphi_{ji} : A_j \to A_i)$ be an inverse system of sets or of modules indexed by I, see Categories, Definition 4.21.4. This is a directed inverse system as we assumed I directed (Categories, Definition 4.21.4). For each $i \in I$, the images $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ form a decreasing directed family of subsets (or submodules) of A_i. Let $A_i' = \bigcap_{j \geq i} \varphi_{ji}(A_j)$. Then $\varphi_{ji}(A_j') \subset A_i'$ for $j \geq i$, hence by restricting we get a directed inverse system $(A_i', \varphi_{ji}|_{A_j'})$. From the construction of the limit of an inverse system in the category of sets or modules, we have $\lim A_i = \lim A_i'$. The Mittag-Leffler condition on (A_i, φ_{ji}) is that A_i' equals $\varphi_{ji}(A_j)$ for some $j \geq i$ (and hence equals $\varphi_{ki}(A_k)$ for all $k \geq j$):

0595 **Definition 10.86.1.** Let (A_i, φ_{ji}) be a directed inverse system of sets over I. Then we say (A_i, φ_{ji}) is *Mittag-Leffler* if for each $i \in I$, the family $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$. If (A_i, φ_{ji}) is a directed inverse system

of modules over a ring R, we say that it is Mittag-Leffler if the underlying inverse system of sets is Mittag-Leffler.

0596 **Example 10.86.2.** If (A_i, φ_{ji}) is a directed inverse system of sets or of modules and the maps φ_{ji} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ji}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ji}(A_j)$ for $j \geq i$. Then $\varphi_{ji}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $\lim A_i = \lim A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

0597 **Lemma 10.86.3.** *Let (A_i, φ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ji}) is Mittag-Leffler and the A_i are nonempty, then $\lim A_i$ is nonempty.*

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$. So by Example 10.86.2 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This follows from the axiom of choice. □

The Mittag-Leffler condition will be important for us because of the following exactness property.

0598 **Lemma 10.86.4.** *Let*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\lim B_i \to \lim C_i$. So let $(c_i) \in \lim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 10.86.3 would show $\lim E_i$ is nonempty, and taking any element of $\lim E_i$ would give an element of $\lim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ki}(E_k) = \varphi_{ji}(E_j)$ for $k \geq j$. Always $\varphi_{ki}(E_k) \subset \varphi_{ji}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\varphi_{ki}(x_k) = \varphi_{ji}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ki}(a_k) = \varphi_{ji}(a_j)$. Hence

$$\varphi_{ki}(e'_k + a_k) = \varphi_{ji}(e'_j) + \varphi_{ji}(a_j) = \varphi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. □

10.87. Inverse systems

03C9 In many papers (and in this section) the term *inverse system* is used to indicate an inverse system over the partially ordered set $(\mathbf{N}, \geq)$. We briefly discuss such systems in this section. This material will be discussed more broadly in Homology, Section 12.31. Suppose we are given a ring R and a sequence of R-modules

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

with maps as indicated. By composing successive maps we obtain maps $\varphi_{ii'} : M_i \to M_{i'}$ whenever $i \geq i'$ such that moreover $\varphi_{ii''} = \varphi_{i'i''} \circ \varphi_{ii'}$ whenever $i \geq i' \geq i''$. Conversely, given the system of maps $\varphi_{ii'}$ we can set $\varphi_i = \varphi_{i(i-1)}$ and recover the maps displayed above. In this case

$$\lim M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

compare with Categories, Section 4.15. As explained in Homology, Section 12.31 this is actually a limit in the category of R-modules, as defined in Categories, Section 4.14.

03CA **Lemma 10.87.1.** *Let R be a ring. Let $0 \to K_i \to L_i \to M_i \to 0$ be short exact sequences of R-modules, $i \geq 1$ which fit into maps of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K_{i+1} & \longrightarrow & L_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & 0 \end{array}$$

If for every i there exists a $c = c(i) \geq i$ such that $\mathrm{Im}(K_c \to K_i) = \mathrm{Im}(K_j \to K_i)$ for all $j \geq c$, then the sequence

$$0 \to \lim K_i \to \lim L_i \to \lim M_i \to 0$$

is exact.

Proof. This is a special case of the more general Lemma 10.86.4. □

10.88. Mittag-Leffler modules

0599 A Mittag-Leffler module is (very roughly) a module which can be written as a directed limit whose dual is a Mittag-Leffler system. To be able to give a precise definition we need to do a bit of work.

059A **Definition 10.88.1.** Let (M_i, f_{ij}) be a directed system of R-modules. We say that (M_i, f_{ij}) is a *Mittag-Leffler directed system of modules* if each M_i is an R-module of finite presentation and if for every R-module N, the inverse system

$$(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$$

is Mittag-Leffler.

We are going to characterize those R-modules that are colimits of Mittag-Leffler directed systems of modules.

059B **Definition 10.88.2.** Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then we say g *dominates* f if for any R-module Q, we have $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$.

It is enough to check this condition for finitely presented modules.

059C **Lemma 10.88.3.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then g dominates f if and only if for any finitely presented R-module Q, we have $\mathrm{Ker}(f \otimes_R id_Q) \subset \mathrm{Ker}(g \otimes_R id_Q)$.*

Proof. Suppose $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$ for all finitely presented modules Q. If Q is an arbitrary module, write $Q = \mathrm{colim}_{i \in I} Q_i$ as a colimit of a directed system of finitely presented modules Q_i. Then $\mathrm{Ker}(f \otimes_R \mathrm{id}_{Q_i}) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_{Q_i})$ for all i. Since taking directed colimits is exact and commutes with tensor product, it follows that $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$. □

0AUM **Lemma 10.88.4.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Consider the pushout of f and g,*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M' & \xrightarrow{f'} & N' \end{array}$$

Then g dominates f if and only if f' is universally injective.

Proof. Recall that N' is $M' \oplus N$ modulo the submodule consisting of elements $(g(x), -f(x))$ for $x \in M$. From the construction of N' we have a short exact sequence

$$0 \to \mathrm{Ker}(f) \cap \mathrm{Ker}(g) \to \mathrm{Ker}(f) \to \mathrm{Ker}(f') \to 0.$$

Since tensoring commutes with taking pushouts, we have such a short exact sequence

$$0 \to \mathrm{Ker}(f \otimes \mathrm{id}_Q) \cap \mathrm{Ker}(g \otimes \mathrm{id}_Q) \to \mathrm{Ker}(f \otimes \mathrm{id}_Q) \to \mathrm{Ker}(f' \otimes \mathrm{id}_Q) \to 0$$

for every R-module Q. So f' is universally injective if and only if $\mathrm{Ker}(f \otimes \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes \mathrm{id}_Q)$ for every Q, if and only if g dominates f. □

The above definition of domination is sometimes related to the usual notion of domination of maps as the following lemma shows.

059D **Lemma 10.88.5.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Suppose $\mathrm{Coker}(f)$ is of finite presentation. Then g dominates f if and only if g factors through f, i.e. there exists a module map $h : N \to M'$ such that $g = h \circ f$.*

Proof. Consider the pushout of f and g as in the statement of Lemma 10.88.4. From the construction of the pushout it follows that $\mathrm{Coker}(f') = \mathrm{Coker}(f)$, so $\mathrm{Coker}(f')$ is of finite presentation. Then by Lemma 10.82.4, f' is universally injective if and only if

$$0 \to M' \xrightarrow{f'} N' \to \mathrm{Coker}(f') \to 0$$

splits. This is the case if and only if there is a map $h' : N' \to M'$ such that $h' \circ f' = \mathrm{id}_{M'}$. From the universal property of the pushout, the existence of such an h' is equivalent to g factoring through f. □

059E **Proposition 10.88.6.** *Let M be an R-module. Let (M_i, f_{ij}) be a directed system of finitely presented R-modules, indexed by I, such that $M = \mathrm{colim}\, M_i$. Let $f_i : M_i \to M$ be the canonical map. The following are equivalent:*

(1) *For every finitely presented R-module P and module map $f : P \to M$, there exists a finitely presented R-module Q and a module map $g : P \to Q$ such that g and f dominate each other, i.e.,* $\mathrm{Ker}(f \otimes_R id_N) = \mathrm{Ker}(g \otimes_R id_N)$ *for every R-module N.*
(2) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ dominates $f_i : M_i \to M$.*
(3) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$.*
(4) *For every R-module N, the inverse system* $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ *is Mittag-Leffler.*
(5) *For $N = \prod_{s \in I} M_s$, the inverse system* $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ *is Mittag-Leffler.*

Proof. First we prove the equivalence of (1) and (2). Suppose (1) holds and let $i \in I$. Corresponding to the map $f_i : M_i \to M$, we can choose $g : M_i \to Q$ as in (1). Since M_i and Q are of finite presentation, so is $\mathrm{Coker}(g)$. Then by Lemma 10.88.5, $f_i : M_i \to M$ factors through $g : M_i \to Q$, say $f_i = h \circ g$ for some $h : Q \to M$. Then since Q is finitely presented, h factors through $M_j \to M$ for some $j \geq i$, say $h = f_j \circ h'$ for some $h' : Q \to M_j$. In total we have a commutative diagram

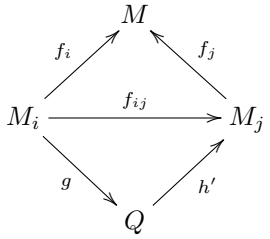

Thus f_{ij} dominates g. But g dominates f_i, so f_{ij} dominates f_i.

Conversely, suppose (2) holds. Let P be of finite presentation and $f : P \to M$ a module map. Then f factors through $f_i : M_i \to M$ for some $i \in I$, say $f = f_i \circ g'$ for some $g' : P \to M_i$. Choose by (2) a $j \geq i$ such that f_{ij} dominates f_i. We have a commutative diagram

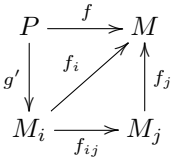

From the diagram and the fact that f_{ij} dominates f_i, we find that f and $f_{ij} \circ g'$ dominate each other. Hence taking $g = f_{ij} \circ g' : P \to M_j$ works.

Next we prove (2) is equivalent to (3). Let $i \in I$. It is always true that f_i dominates f_{ik} for $k \geq i$, since f_i factors through f_{ik}. If (2) holds, choose $j \geq i$ such that f_{ij} dominates f_i. Then since domination is a transitive relation, f_{ij} dominates f_{ik} for $k \geq i$. All M_i are of finite presentation, so $\mathrm{Coker}(f_{ik})$ is of finite presentation for $k \geq i$. By Lemma 10.88.5, f_{ij} factors through f_{ik} for all $k \geq i$. Thus (2) implies (3). On the other hand, if (3) holds then for any R-module N, $f_{ij} \otimes_R \mathrm{id}_N$ factors through $f_{ik} \otimes_R \mathrm{id}_N$ for $k \geq i$. So $\mathrm{Ker}(f_{ik} \otimes_R \mathrm{id}_N) \subset \mathrm{Ker}(f_{ij} \otimes_R \mathrm{id}_N)$ for $k \geq i$. But $\mathrm{Ker}(f_i \otimes_R \mathrm{id}_N : M_i \otimes_R N \to M \otimes_R N)$ is the union of $\mathrm{Ker}(f_{ik} \otimes_R \mathrm{id}_N)$ for $k \geq i$.

Thus $\mathrm{Ker}(f_i \otimes_R \mathrm{id}_N) \subset \mathrm{Ker}(f_{ij} \otimes_R \mathrm{id}_N)$ for any R-module N, which by definition means f_{ij} dominates f_i.

It is trivial that (3) implies (4) implies (5). We show (5) implies (3). Let $N = \prod_{s \in I} M_s$. If (5) holds, then given $i \in I$ choose $j \geq i$ such that

$$\mathrm{Im}(\mathrm{Hom}(M_j, N) \to \mathrm{Hom}(M_i, N)) = \mathrm{Im}(\mathrm{Hom}(M_k, N) \to \mathrm{Hom}(M_i, N))$$

for all $k \geq j$. Passing the product over $s \in I$ outside of the Hom's and looking at the maps on each component of the product, this says

$$\mathrm{Im}(\mathrm{Hom}(M_j, M_s) \to \mathrm{Hom}(M_i, M_s)) = \mathrm{Im}(\mathrm{Hom}(M_k, M_s) \to \mathrm{Hom}(M_i, M_s))$$

for all $k \geq j$ and $s \in I$. Taking $s = j$ we have

$$\mathrm{Im}(\mathrm{Hom}(M_j, M_j) \to \mathrm{Hom}(M_i, M_j)) = \mathrm{Im}(\mathrm{Hom}(M_k, M_j) \to \mathrm{Hom}(M_i, M_j))$$

for all $k \geq j$. Since f_{ij} is the image of $\mathrm{id} \in \mathrm{Hom}(M_j, M_j)$ under $\mathrm{Hom}(M_j, M_j) \to \mathrm{Hom}(M_i, M_j)$, this shows that for any $k \geq j$ there is $h \in \mathrm{Hom}(M_k, M_j)$ such that $f_{ij} = h \circ f_{ik}$. If $j \geq k$ then we can take $h = f_{kj}$. Hence (3) holds. □

059F **Definition 10.88.7.** Let M be an R-module. We say that M is *Mittag-Leffler* if the equivalent conditions of Proposition 10.88.6 hold.

In particular a finitely presented module is Mittag-Leffler.

059G **Remark 10.88.8.** Let M be a flat R-module. By Lazard's theorem (Theorem 10.81.4) we can write $M = \mathrm{colim}\, M_i$ as the colimit of a directed system (M_i, f_{ij}) where the M_i are free finite R-modules. For M to be Mittag-Leffler, it is enough for the inverse system of duals $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ to be Mittag-Leffler. This follows from criterion (4) of Proposition 10.88.6 and the fact that for a free finite R-module F, there is a functorial isomorphism $\mathrm{Hom}_R(F, R) \otimes_R N \cong \mathrm{Hom}_R(F, N)$ for any R-module N.

05CN **Lemma 10.88.9.** *If R is a ring and M, N are Mittag-Leffler modules over R, then $M \otimes_R N$ is a Mittag-Leffler module.*

Proof. Write $M = \mathrm{colim}_{i \in I} M_i$ and $N = \mathrm{colim}_{j \in J} N_j$ as directed colimits of finitely presented R-modules. Denote $f_{ii'} : M_i \to M_{i'}$ and $g_{jj'} : N_j \to N_{j'}$ the transition maps. Then $M_i \otimes_R N_j$ is a finitely presented R-module (see Lemma 10.12.14), and $M \otimes_R N = \mathrm{colim}_{(i,j) \in I \times J} M_i \otimes_R M_j$. Pick $(i, j) \in I \times J$. By the definition of a Mittag-Leffler module we have Proposition 10.88.6 (3) for both systems. In other words there exist $i' \geq i$ and $j' \geq j$ such that for every choice of $i'' \geq i$ and $j'' \geq j$ there exist maps $a : M_{i''} \to M_{i'}$ and $b : M_{j''} \to M_{j'}$ such that $f_{ii'} = a \circ f_{ii''}$ and $g_{jj'} = b \circ g_{jj''}$. Then it is clear that $a \otimes b : M_{i''} \otimes_R N_{j''} \to M_{i'} \otimes_R N_{j'}$ serves the same purpose for the system $(M_i \otimes_R N_j, f_{ii'} \otimes g_{jj'})$. Thus by the characterization Proposition 10.88.6 (3) we conclude that $M \otimes_R N$ is Mittag-Leffler. □

05CP **Lemma 10.88.10.** *Let R be a ring and M an R-module. Then M is Mittag-Leffler if and only if for every finite free R-module F and module map $f : F \to M$, there exists a finitely presented R-module Q and a module map $g : F \to Q$ such that g and f dominate each other, i.e., $\mathrm{Ker}(f \otimes_R id_N) = \mathrm{Ker}(g \otimes_R id_N)$ for every R-module N.*

Proof. Since the condition is clear weaker than condition (1) of Proposition 10.88.6 we see that a Mittag-Leffler module satisfies the condition. Conversely, suppose that M satisfies the condition and that $f : P \to M$ is an R-module map from a finitely presented R-module P into M. Choose a surjection $F \to P$ where F is a finite free R-module. By assumption we can find a map $F \to Q$ where Q is a finitely presented R-module such that $F \to Q$ and $F \to M$ dominate each other. In particular, the kernel of $F \to Q$ contains the kernel of $F \to P$, hence we obtain an R-module map $g : P \to Q$ such that $F \to Q$ is equal to the composition $F \to P \to Q$. Let N be any R-module and consider the commutative diagram

$$\begin{array}{ccc} F \otimes_R N & \longrightarrow & Q \otimes_R N \\ \downarrow & \nearrow & \\ P \otimes_R N & \longrightarrow & M \otimes_R N \end{array}$$

By assumption the kernels of $F \otimes_R N \to Q \otimes_R N$ and $F \otimes_R N \to M \otimes_R N$ are equal. Hence, as $F \otimes_R N \to P \otimes_R N$ is surjective, also the kernels of $P \otimes_R N \to Q \otimes_R N$ and $P \otimes_R N \to M \otimes_R N$ are equal. □

05CQ **Lemma 10.88.11.** *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. If M is a Mittag-Leffler module over S then M is a Mittag-Leffler module over R.*

Proof. Assume M is a Mittag-Leffler module over S. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules M_i. As M is Mittag-Leffler over S there exists for each i an index $j \geq i$ such that for all $k \geq j$ there is a factorization $f_{ij} = h \circ f_{ik}$ (where h depends on i, the choice of j and k). Note that by Lemma 10.36.23 the modules M_i are also finitely presented as R-modules. Moreover, all the maps f_{ij}, f_{ik}, h are maps of R-modules. Thus we see that the system (M_i, f_{ij}) satisfies the same condition when viewed as a system of R-modules. Thus M is Mittag-Leffler as an R-module. □

05CR **Lemma 10.88.12.** *Let R be a ring. Let $S = R/I$ for some finitely generated ideal I. Let M be an S-module. Then M is a Mittag-Leffler module over R if and only if M is a Mittag-Leffler module over S.*

Proof. One implication follows from Lemma 10.88.11. To prove the other, assume M is Mittag-Leffler as an R-module. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules. As I is finitely generated, the ring S is finite and finitely presented as an R-algebra, hence the modules M_i are finitely presented as R-modules, see Lemma 10.36.23. Next, let N be any S-module. Note that for each i we have $\operatorname{Hom}_R(M_i, N) = \operatorname{Hom}_S(M_i, N)$ as $R \to S$ is surjective. Hence the condition that the inverse system $(\operatorname{Hom}_R(M_i, N))_i$ satisfies Mittag-Leffler, implies that the system $(\operatorname{Hom}_S(M_i, N))_i$ satisfies Mittag-Leffler. Thus M is Mittag-Leffler over S by definition. □

05CS **Remark 10.88.13.** Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module which is Mittag-Leffler as an R-module. Then it is in general not the case that M is Mittag-Leffler as an S-module. For example suppose that S is the ring of dual numbers over R, i.e., $S = R \oplus R\epsilon$ with $\epsilon^2 = 0$. Then an S-module consists of an R-module M endowed with a square zero R-linear endomorphism $\epsilon : M \to M$. Now suppose that M_0 is an R-module which is not Mittag-Leffler.

Choose a presentation $F_1 \xrightarrow{u} F_0 \to M_0 \to 0$ with F_1 and F_0 free R-modules. Set $M = F_1 \oplus F_0$ with

$$\epsilon = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : M \longrightarrow M.$$

Then $M/\epsilon M \cong F_1 \oplus M_0$ is not Mittag-Leffler over $R = S/\epsilon S$, hence not Mittag-Leffler over S (see Lemma 10.88.12). On the other hand, $M/\epsilon M = M \otimes_S S/\epsilon S$ which would be Mittag-Leffler over S if M was, see Lemma 10.88.9.

10.89. Interchanging direct products with tensor

059H Let M be an R-module and let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. Then there is a canonical map $M \otimes_R \left(\prod_{\alpha \in A} Q_\alpha\right) \to \prod_{\alpha \in A}(M \otimes_R Q_\alpha)$ given on pure tensors by $x \otimes (q_\alpha) \mapsto (x \otimes q_\alpha)$. This map is not necessarily injective or surjective, as the following example shows.

059I **Example 10.89.1.** Take $R = \mathbf{Z}$, $M = \mathbf{Q}$, and consider the family $Q_n = \mathbf{Z}/n$ for $n \geq 1$. Then $\prod_n (M \otimes Q_n) = 0$. However there is an injection $\mathbf{Q} \to M \otimes (\prod_n Q_n)$ obtained by tensoring the injection $\mathbf{Z} \to \prod_n Q_n$ by M, so $M \otimes (\prod_n Q_n)$ is nonzero. Thus $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n)$ is not injective.

On the other hand, take again $R = \mathbf{Z}$, $M = \mathbf{Q}$, and let $Q_n = \mathbf{Z}$ for $n \geq 1$. The image of $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n) = \prod_n M$ consists precisely of sequences of the form $(a_n/m)_{n \geq 1}$ with $a_n \in \mathbf{Z}$ and m some nonzero integer. Hence the map is not surjective.

We determine below the precise conditions needed on M for the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ to be surjective, bijective, or injective for all choices of $(Q_\alpha)_{\alpha \in A}$. This is relevant because the modules for which it is injective turn out to be exactly Mittag-Leffler modules (Proposition 10.89.5). In what follows, if M is an R-module and A a set, we write M^A for the product $\prod_{\alpha \in A} M$.

059J **Proposition 10.89.2.** *Let M be an R-module. The following are equivalent:*

1. *M is finitely generated.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective.*
3. *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is surjective.*
4. *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is surjective.*

Proof. First we prove (1) implies (2). Choose a surjection $R^n \to M$ and consider the commutative diagram

$$\begin{array}{ccc} R^n \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (R^n \otimes_R Q_\alpha) \\ \downarrow & & \downarrow \\ M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha). \end{array}$$

The top arrow is an isomorphism and the vertical arrows are surjections. We conclude that the bottom arrow is a surjection.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). In fact for (1) to hold it suffices that the element $d = (x)_{x \in M}$ of M^M is in the image of the map $f : M \otimes_R R^M \to M^M$. In this case $d = \sum_{i=1}^n f(x_i \otimes a_i)$ for some $x_i \in M$

and $a_i \in R^M$. If for $x \in M$ we write $p_x : M^M \to M$ for the projection onto the x-th factor, then

$$x = p_x(d) = \sum\nolimits_{i=1}^{n} p_x(f(x_i \otimes a_i)) = \sum\nolimits_{i=1}^{n} p_x(a_i)x_i.$$

Thus $x_1, \ldots, x_n$ generate M. □

059K **Proposition 10.89.3.** *Let M be an R-module. The following are equivalent:*

(1) *M is finitely presented.*
(2) *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is bijective.*
(3) *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is bijective.*
(4) *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is bijective.*

Proof. First we prove (1) implies (2). Choose a presentation $R^m \to R^n \to M$ and consider the commutative diagram

$$\begin{array}{ccccccc}
R^m \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & R^n \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\prod_\alpha (R^m \otimes_R Q_\alpha) & \longrightarrow & \prod_\alpha (R^n \otimes_R Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) & \longrightarrow & 0.
\end{array}$$

The first two vertical arrows are isomorphisms and the rows are exact. This implies that the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective and, by a diagram chase, also injective. Hence (2) holds.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). From Proposition 10.89.2, if (4) holds we already know that M is finitely generated. So we can choose a surjection $F \to M$ where F is free and finite. Let K be the kernel. We must show K is finitely generated. For any set A, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & K \otimes_R R^A & \longrightarrow & F \otimes_R R^A & \longrightarrow & M \otimes_R R^A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} \cong & & \downarrow{\scriptstyle f_1} \cong & & \\
0 & \longrightarrow & K^A & \longrightarrow & F^A & \longrightarrow & M^A & \longrightarrow & 0.
\end{array}$$

The map f_1 is an isomorphism by assumption, the map f_2 is an isomorphism since F is free and finite, and the rows are exact. A diagram chase shows that f_3 is surjective, hence by Proposition 10.89.2 we get that K is finitely generated. □

We need the following lemma for the next proposition.

059L **Lemma 10.89.4.** *Let M be an R-module, P a finitely presented R-module, and $f : P \to M$ a map. Let Q be an R-module and suppose $x \in \mathrm{Ker}(P \otimes Q \to M \otimes Q)$. Then there exists a finitely presented R-module P' and a map $f' : P \to P'$ such that f factors through f' and $x \in \mathrm{Ker}(P \otimes Q \to P' \otimes Q)$.*

Proof. Write M as a colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Since P is finitely presented, the map $f : P \to M$ factors

through $M_j \to M$ for some $j \in I$. Upon tensoring by Q we have a commutative diagram

$$\begin{array}{ccc} & M_j \otimes Q & \\ \nearrow & & \searrow \\ P \otimes Q & \longrightarrow & M \otimes Q. \end{array}$$

The image y of x in $M_j \otimes Q$ is in the kernel of $M_j \otimes Q \to M \otimes Q$. Since $M \otimes Q = \operatorname{colim}_{i \in I}(M_i \otimes Q)$, this means y maps to 0 in $M_{j'} \otimes Q$ for some $j' \geq j$. Thus we may take $P' = M_{j'}$ and f' to be the composite $P \to M_j \to M_{j'}$. □

059M **Proposition 10.89.5.** *Let M be an R-module. The following are equivalent:*

(1) *M is Mittag-Leffler.*
(2) *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is injective.*

Proof. First we prove (1) implies (2). Suppose M is Mittag-Leffler and let x be in the kernel of $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. Write M as a colimit $M = \operatorname{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Then $M \otimes_R (\prod_\alpha Q_\alpha)$ is the colimit of $M_i \otimes_R (\prod_\alpha Q_\alpha)$. So x is the image of an element $x_i \in M_i \otimes_R (\prod_\alpha Q_\alpha)$. We must show that x_i maps to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$ for some $j \geq i$. Since M is Mittag-Leffler, we may choose $j \geq i$ such that $M_i \to M_j$ and $M_i \to M$ dominate each other. Then consider the commutative diagram

$$\begin{array}{ccc} M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) \\ \uparrow & & \uparrow \\ M_i \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (M_i \otimes_R Q_\alpha) \\ \downarrow & & \downarrow \\ M_j \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (M_j \otimes_R Q_\alpha) \end{array}$$

whose bottom two horizontal maps are isomorphisms, according to Proposition 10.89.3. Since x_i maps to 0 in $\prod_\alpha (M \otimes_R Q_\alpha)$, its image in $\prod_\alpha (M_i \otimes_R Q_\alpha)$ is in the kernel of the map $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. But this kernel equals the kernel of $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M_j \otimes_R Q_\alpha)$ according to the choice of j. Thus x_i maps to 0 in $\prod_\alpha (M_j \otimes_R Q_\alpha)$ and hence to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$.

Now suppose (2) holds. We prove M satisfies formulation (1) of being Mittag-Leffler from Proposition 10.88.6. Let $f : P \to M$ be a map from a finitely presented module P to M. Choose a set B of representatives of the isomorphism classes of finitely presented R-modules. Let A be the set of pairs (Q, x) where $Q \in B$ and $x \in \operatorname{Ker}(P \otimes Q \to M \otimes Q)$. For $\alpha = (Q, x) \in A$, we write Q_α for Q and x_α for x. Consider the commutative diagram

$$\begin{array}{ccc} M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) \\ \uparrow & & \uparrow \\ P \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P \otimes_R Q_\alpha) \end{array}$$

The top arrow is an injection by assumption, and the bottom arrow is an isomorphism by Proposition 10.89.3. Let $x \in P \otimes_R (\prod_\alpha Q_\alpha)$ be the element corresponding to $(x_\alpha) \in \prod_\alpha (P \otimes_R Q_\alpha)$ under this isomorphism. Then $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to M \otimes_R (\prod_\alpha Q_\alpha))$ since the top arrow in the diagram is injective. By Lemma 10.89.4, we get a finitely presented module P' and a map $f' : P \to P'$ such that $f : P \to M$ factors through f' and $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to P' \otimes_R (\prod_\alpha Q_\alpha))$. We have a commutative diagram

$$\begin{array}{ccc} P' \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P' \otimes_R Q_\alpha) \\ \uparrow & & \uparrow \\ P \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P \otimes_R Q_\alpha) \end{array}$$

where both the top and bottom arrows are isomorphisms by Proposition 10.89.3. Thus since x is in the kernel of the left vertical map, (x_α) is in the kernel of the right vertical map. This means $x_\alpha \in \mathrm{Ker}(P \otimes_R Q_\alpha \to P' \otimes_R Q_\alpha)$ for every $\alpha \in A$. By the definition of A this means $\mathrm{Ker}(P \otimes_R Q \to P' \otimes_R Q) \supset \mathrm{Ker}(P \otimes_R Q \to M \otimes_R Q)$ for all finitely presented Q and, since $f : P \to M$ factors through $f' : P \to P'$, actually equality holds. By Lemma 10.88.3, f and f' dominate each other. □

0AS6 **Lemma 10.89.6.** *Let M be a flat Mittag-Leffler module over R. Let F be an R-module and let $x \in F \otimes_R M$. Then there exists a smallest submodule $F' \subset F$ such that $x \in F' \otimes_R M$. Also, F' is a finite R-module.*

Proof. Since M is flat we have $F' \otimes_R M \subset F \otimes_R M$ if $F' \subset F$ is a submodule, hence the statement makes sense. Let $I = \{F' \subset F \mid x \in F' \otimes_R M\}$ and for $i \in I$ denote $F_i \subset F$ the corresponding submodule. Then x maps to zero under the map

$$F \otimes_R M \longrightarrow \prod (F/F_i \otimes_R M)$$

whence by Proposition 10.89.5 x maps to zero under the map

$$F \otimes_R M \longrightarrow \left(\prod F/F_i\right) \otimes_R M$$

Since M is flat the kernel of this arrow is $(\bigcap F_i) \otimes_R M$ which proves that $F' = \bigcap F_i$. To see that F' is a finite module, suppose that $x = \sum_{j=1,\ldots,m} f_j \otimes m_j$ with $f_j \in F'$ and $m_j \in M$. Then $x \in F'' \otimes_R M$ where $F'' \subset F'$ is the submodule generated by $f_1, \ldots, f_m$. Of course then $F'' = F'$ and we conclude the final statement holds. □

059N **Lemma 10.89.7.** *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules. Then:*

(1) *If M_2 is Mittag-Leffler, then M_1 is Mittag-Leffler.*
(2) *If M_1 and M_3 are Mittag-Leffler, then M_2 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_2 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_3 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_\alpha (M_1 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_2 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_3 \otimes Q_\alpha) & \longrightarrow & 0 \end{array}$$

with exact rows. Thus (1) and (2) follow from Proposition 10.89.5. □

0EGI **Lemma 10.89.8.** *Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. If M_1 is finitely generated and M_2 is Mittag-Leffler, then M_3 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules, since tensor product is right exact, we have a commutative diagram

$$\begin{array}{ccccccc} M_1 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_2 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_3 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_\alpha (M_1 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_2 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_3 \otimes Q_\alpha) & \longrightarrow & 0 \end{array}$$

with exact rows. By Proposition 10.89.2 the left vertical arrow is surjective. By Proposition 10.89.5 the middle vertical arrow is injective. A diagram chase shows the right vertical arrow is injective. Hence M_3 is Mittag-Leffler by Proposition 10.89.5. □

0AS7 **Lemma 10.89.9.** *If $M = \operatorname{colim} M_i$ is the colimit of a directed system of Mittag-Leffler R-modules M_i with universally injective transition maps, then M is Mittag-Leffler.*

Proof. Let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. We have to show that $M \otimes_R (\prod Q_\alpha) \to \prod M \otimes_R Q_\alpha$ is injective and we know that $M_i \otimes_R (\prod Q_\alpha) \to \prod M_i \otimes_R Q_\alpha$ is injective for each i, see Proposition 10.89.5. Since $\otimes$ commutes with filtered colimits, it suffices to show that $\prod M_i \otimes_R Q_\alpha \to \prod M \otimes_R Q_\alpha$ is injective. This is clear as each of the maps $M_i \otimes_R Q_\alpha \to M \otimes_R Q_\alpha$ is injective by our assumption that the transition maps are universally injective. □

059P **Lemma 10.89.10.** *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of R-modules, then M is Mittag-Leffler if and only if each M_i is Mittag-Leffler.*

Proof. The "only if" direction follows from Lemma 10.89.7 (1) and the fact that a split short exact sequence is universally exact. The converse follows from Lemma 10.89.9 but we can also argue it directly as follows. First note that if I is finite then this follows from Lemma 10.89.7 (2). For general I, if all M_i are Mittag-Leffler then we prove the same of M by verifying condition (1) of Proposition 10.88.6. Let $f : P \to M$ be a map from a finitely presented module P. Then f factors as $P \xrightarrow{f'} \bigoplus_{i' \in I'} M_{i'} \hookrightarrow \bigoplus_{i \in I} M_i$ for some finite subset I' of I. By the finite case $\bigoplus_{i' \in I'} M_{i'}$ is Mittag-Leffler and hence there exists a finitely presented module Q and a map $g : P \to Q$ such that g and f' dominate each other. Then also g and f dominate each other. □

05CT **Lemma 10.89.11.** *Let $R \to S$ be a ring map. Let M be an S-module. If S is Mittag-Leffler as an R-module, and M is flat and Mittag-Leffler as an S-module, then M is Mittag-Leffler as an R-module.*

Proof. We deduce this from the characterization of Proposition 10.89.5. Namely, suppose that Q_α is a family of R-modules. Consider the composition

$$\begin{array}{c} M \otimes_R \prod_\alpha Q_\alpha = M \otimes_S S \otimes_R \prod_\alpha Q_\alpha \\ \downarrow \\ M \otimes_S \prod_\alpha (S \otimes_R Q_\alpha) \\ \downarrow \\ \prod_\alpha (M \otimes_S S \otimes_R Q_\alpha) = \prod_\alpha (M \otimes_R Q_\alpha) \end{array}$$

The first arrow is injective as M is flat over S and S is Mittag-Leffler over R and the second arrow is injective as M is Mittag-Leffler over S. Hence M is Mittag-Leffler over R. □

10.90. Coherent rings

05CU We use the discussion on interchanging $\prod$ and $\otimes$ to determine for which rings products of flat modules are flat. It turns out that these are the so-called coherent rings. You may be more familiar with the notion of a coherent $\mathcal{O}_X$-module on a ringed space, see Modules, Section 17.12.

05CV **Definition 10.90.1.** Let R be a ring. Let M be an R-module.

(1) We say M is a *coherent module* if it is finitely generated and every finitely generated submodule of M is finitely presented over R.
(2) We say R is a *coherent ring* if it is coherent as a module over itself.

Thus a ring is coherent if and only if every finitely generated ideal is finitely presented as a module.

0EWV **Example 10.90.2.** A valuation ring is a coherent ring. Namely, every nonzero finitely generated ideal is principal (Lemma 10.50.15), hence free as a valuation ring is a domain, hence finitely presented.

The category of coherent modules is abelian.

05CW **Lemma 10.90.3.** *Let R be a ring.*

(1) *A finite submodule of a coherent module is coherent.*
(2) *Let $\varphi : N \to M$ be a homomorphism from a finite module to a coherent module. Then $\mathrm{Ker}(\varphi)$ is finite, $\mathrm{Im}(\varphi)$ is coherent, and $\mathrm{Coker}(\varphi)$ is coherent.*
(3) *Let $\varphi : N \to M$ be a homomorphism of coherent modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent modules.*
(4) *Given a short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ if two out of three are coherent so is the third.*

Proof. The first statement is immediate from the definition.

Let $\varphi : N \to M$ satisfy the assumptions of (2). First, $\mathrm{Im}(\varphi)$ is finite, hence coherent by (1). In particular $\mathrm{Im}(\varphi)$ is finitely presented, so applying Lemma 10.5.3 to the exact sequence $0 \to \mathrm{Ker}(\varphi) \to N \to \mathrm{Im}(\varphi) \to 0$ we see that $\mathrm{Ker}(\varphi)$ is finite. To prove that $\mathrm{Coker}(\varphi)$ is coherent, let $E \subset \mathrm{Coker}(\varphi)$ be a finite submodule, and let E' be its inverse image in M. From the exact sequence $0 \to \mathrm{Im}(\varphi) \to E' \to E \to 0$

and since $\mathrm{Ker}(\varphi)$ is finite we conclude by Lemma 10.5.3 that $E' \subset M$ is finite, hence finitely presented because M is coherent. The same exact sequence then shows that E is finitely presented, whence our claim.

Part (3) follows immediately from (1) and (2).

Let $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0$ be a short exact sequence of R-modules as in (4). It remains to prove that if M_1 and M_3 are coherent so is M_2. By Lemma 10.5.3 we see that M_2 is finite. Let $N_2 \subset M_2$ be a finite submodule. Put $N_3 = p(N_2) \subset M_3$ and $N_1 = i^{-1}(N_2) \subset M_1$. We have an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$. Clearly N_3 is finite (as a quotient of N_2), hence finitely presented (as a finite submodule of M_3). It follows by Lemma 10.5.3 (5) that N_1 is finite, hence finitely presented (as a finite submodule of M_1). We conclude by Lemma 10.5.3 (2) that N_2 is finitely presented. □

05CX **Lemma 10.90.4.** *Let R be a ring. If R is coherent, then a module is coherent if and only if it is finitely presented.*

Proof. It is clear that a coherent module is finitely presented (over any ring). Conversely, if R is coherent, then $R^{\oplus n}$ is coherent and so is the cokernel of any map $R^{\oplus m} \to R^{\oplus n}$, see Lemma 10.90.3. □

05CY **Lemma 10.90.5.** *A Noetherian ring is a coherent ring.*

Proof. By Lemma 10.31.4 any finite R-module is finitely presented. In particular any ideal of R is finitely presented. □

05CZ **Proposition 10.90.6.** *Let R be a ring. The following are equivalent* This is [**Cha60**, Theorem 2.1].

(1) *R is coherent,*
(2) *any product of flat R-modules is flat, and*
(3) *for every set A the module R^A is flat.*

Proof. Assume R coherent, and let Q_α, $\alpha \in A$ be a set of flat R-modules. We have to show that $I \otimes_R \prod_\alpha Q_\alpha \to \prod Q_\alpha$ is injective for every finitely generated ideal I of R, see Lemma 10.39.5. Since R is coherent I is an R-module of finite presentation. Hence $I \otimes_R \prod_\alpha Q_\alpha = \prod I \otimes_R Q_\alpha$ by Proposition 10.89.3. The desired injectivity follows as $I \otimes_R Q_\alpha \to Q_\alpha$ is injective by flatness of Q_α.

The implication (2) $\Rightarrow$ (3) is trivial.

Assume that the R-module R^A is flat for every set A. Let I be a finitely generated ideal in R. Then $I \otimes_R R^A \to R^A$ is injective by assumption. By Proposition 10.89.2 and the finiteness of I the image is equal to I^A. Hence $I \otimes_R R^A = I^A$ for every set A and we conclude that I is finitely presented by Proposition 10.89.3. □

10.91. Examples and non-examples of Mittag-Leffler modules

059Q We end this section with some examples and non-examples of Mittag-Leffler modules.

059R **Example 10.91.1.** Mittag-Leffler modules.

(1) Any finitely presented module is Mittag-Leffler. This follows, for instance, from Proposition 10.88.6 (1). In general, it is true that a finitely generated module is Mittag-Leffler if and only it is finitely presented. This follows from Propositions 10.89.2, 10.89.3, and 10.89.5.

(2) A free module is Mittag-Leffler since it satisfies condition (1) of Proposition 10.88.6.
(3) By the previous example together with Lemma 10.89.10, projective modules are Mittag-Leffler.

We also want to add to our list of examples power series rings over a Noetherian ring R. This will be a consequence the following lemma.

059S **Lemma 10.91.2.** *Let M be a flat R-module. The following are equivalent*

(1) *M is Mittag-Leffler, and*
(2) *if F is a finite free R-module and $x \in F \otimes_R M$, then there exists a smallest submodule F' of F such that $x \in F' \otimes_R M$.*

Proof. The implication (1) $\Rightarrow$ (2) is a special case of Lemma 10.89.6. Assume (2). By Theorem 10.81.4 we can write M as the colimit $M = \operatorname{colim}_{i \in I} M_i$ of a directed system (M_i, f_{ij}) of finite free R-modules. By Remark 10.88.8, it suffices to show that the inverse system $(\operatorname{Hom}_R(M_i, R), \operatorname{Hom}_R(f_{ij}, R))$ is Mittag-Leffler. In other words, fix $i \in I$ and for $j \geq i$ let Q_j be the image of $\operatorname{Hom}_R(M_j, R) \to \operatorname{Hom}_R(M_i, R)$; we must show that the Q_j stabilize.

Since M_i is free and finite, we can make the identification $\operatorname{Hom}_R(M_i, M_j) = \operatorname{Hom}_R(M_i, R) \otimes_R M_j$ for all j. Using the fact that the M_j are free, it follows that for $j \geq i$, Q_j is the smallest submodule of $\operatorname{Hom}_R(M_i, R)$ such that $f_{ij} \in Q_j \otimes_R M_j$. Under the identification $\operatorname{Hom}_R(M_i, M) = \operatorname{Hom}_R(M_i, R) \otimes_R M$, the canonical map $f_i : M_i \to M$ is in $\operatorname{Hom}_R(M_i, R) \otimes_R M$. By the assumption on M, there exists a smallest submodule Q of $\operatorname{Hom}_R(M_i, R)$ such that $f_i \in Q \otimes_R M$. We are going to show that the Q_j stabilize to Q.

For $j \geq i$ we have a commutative diagram

$$\begin{array}{ccc} Q_j \otimes_R M_j & \longrightarrow & \operatorname{Hom}_R(M_i, R) \otimes_R M_j \\ \downarrow & & \downarrow \\ Q_j \otimes_R M & \longrightarrow & \operatorname{Hom}_R(M_i, R) \otimes_R M. \end{array}$$

Since $f_{ij} \in Q_j \otimes_R M_j$ maps to $f_i \in \operatorname{Hom}_R(M_i, R) \otimes_R M$, it follows that $f_i \in Q_j \otimes_R M$. Hence, by the choice of Q, we have $Q \subset Q_j$ for all $j \geq i$.

Since the Q_j are decreasing and $Q \subset Q_j$ for all $j \geq i$, to show that the Q_j stabilize to Q it suffices to find a $j \geq i$ such that $Q_j \subset Q$. As an element of

$$\operatorname{Hom}_R(M_i, R) \otimes_R M = \operatorname{colim}_{j \in J}(\operatorname{Hom}_R(M_i, R) \otimes_R M_j),$$

f_i is the colimit of f_{ij} for $j \geq i$, and f_i also lies in the submodule

$$\operatorname{colim}_{j \in J}(Q \otimes_R M_j) \subset \operatorname{colim}_{j \in J}(\operatorname{Hom}_R(M_i, R) \otimes_R M_j).$$

It follows that for some $j \geq i$, f_{ij} lies in $Q \otimes_R M_j$. Since Q_j is the smallest submodule of $\operatorname{Hom}_R(M_i, R)$ with $f_{ij} \in Q_j \otimes_R M_j$, we conclude $Q_j \subset Q$. □

05D0 **Lemma 10.91.3.** *Let R be a Noetherian ring and A a set. Then $M = R^A$ is a flat and Mittag-Leffler R-module.*

Proof. Combining Lemma 10.90.5 and Proposition 10.90.6 we see that M is flat over R. We show that M satisfies the condition of Lemma 10.91.2. Let F be a free

finite R-module. If F' is any submodule of F then it is finitely presented since R is Noetherian. So by Proposition 10.89.3 we have a commutative diagram

$$\begin{array}{ccc} F' \otimes_R M & \longrightarrow & F \otimes_R M \\ \downarrow \cong & & \downarrow \cong \\ (F')^A & \longrightarrow & F^A \end{array}$$

by which we can identify the map $F' \otimes_R M \to F \otimes_R M$ with $(F')^A \to F^A$. Hence if $x \in F \otimes_R M$ corresponds to $(x_\alpha) \in F^A$, then the submodule of F' of F generated by the x_α is the smallest submodule of F such that $x \in F' \otimes_R M$. □

059T **Lemma 10.91.4.** *Let R be a Noetherian ring and n a positive integer. Then the R-module $M = R[[t_1, \ldots, t_n]]$ is flat and Mittag-Leffler.*

Proof. As an R-module, we have $M = R^A$ for a (countable) set A. Hence this lemma is a special case of Lemma 10.91.3. □

059U **Example 10.91.5.** Non Mittag-Leffler modules.

(1) By Example 10.89.1 and Proposition 10.89.5, $\mathbf{Q}$ is not a Mittag-Leffler $\mathbf{Z}$-module.
(2) We prove below (Theorem 10.93.3) that for a flat and countably generated module, projectivity is equivalent to being Mittag-Leffler. Thus any flat, countably generated, non-projective module M is an example of a non-Mittag-Leffler module. For such an example, see Remark 10.78.4.
(3) Let k be a field. Let $R = k[[x]]$. The R-module $M = \prod_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler. Namely, consider the element $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ defined by $\xi_{2^m} = x^{2^{m-1}}$ and $\xi_n = 0$ else, so
$$\xi = (0, x, 0, x^2, 0, 0, 0, x^4, 0, 0, 0, 0, 0, 0, 0, x^8, \ldots)$$
Then the annihilator of ξ in $M/x^{2^m}M$ is generated $x^{2^{m-1}}$ for $m \gg 0$. But if M was Mittag-Leffler, then there would exist a finite R-module Q and an element $\xi' \in Q$ such that the annihilator of ξ' in Q/x^lQ agrees with the annihilator of ξ in M/x^lM for all $l \geq 1$, see Proposition 10.88.6 (1). Now you can prove there exists an integer $a \geq 0$ such that the annihilator of ξ' in Q/x^lQ is generated by either x^a or x^{l-a} for all $l \gg 0$ (depending on whether $\xi' \in Q$ is torsion or not). The combination of the above would give for all $l = 2^m >> 0$ the equality $a = l/2$ or $l - a = l/2$ which is nonsensical.
(4) The same argument shows that (x)-adic completion of $\bigoplus_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler over $R = k[[x]]$ (hint: ξ is actually an element of this completion).
(5) Let $R = k[a, b]/(a^2, ab, b^2)$. Let S be the finitely presented R-algebra with presentation $S = R[t]/(at - b)$. Then as an R-module S is countably generated and indecomposable (details omitted). On the other hand, R is Artinian local, hence complete local, hence a henselian local ring, see Lemma 10.153.9. If S was Mittag-Leffler as an R-module, then it would be a direct sum of finite R-modules by Lemma 10.153.13. Thus we conclude that S is not Mittag-Leffler as an R-module.

10.92. Countably generated Mittag-Leffler modules

05D1 It turns out that countably generated Mittag-Leffler modules have a particularly simple structure.

059W **Lemma 10.92.1.** *Let M be an R-module. Write $M = \operatorname{colim}_{i \in I} M_i$ where (M_i, f_{ij}) is a directed system of finitely presented R-modules. If M is Mittag-Leffler and countably generated, then there is a directed countable subset $I' \subset I$ such that $M \cong \operatorname{colim}_{i \in I'} M_i$.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. For each x_n choose $i \in I$ such that x_n is in the image of the canonical map $f_i : M_i \to M$; let $I'_0 \subset I$ be the set of all these i. Now since M is Mittag-Leffler, for each $i \in I'_0$ we can choose $j \in I$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$ (condition (3) of Proposition 10.88.6); let I'_1 be the union of I'_0 with all of these j. Since I'_1 is a countable, we can enlarge it to a countable directed set $I'_2 \subset I$. Now we can apply the same procedure to I'_2 as we did to I'_0 to get a new countable set $I'_3 \subset I$. Then we enlarge I'_3 to a countable directed set I'_4. Continuing in this way—adding in a j as in Proposition 10.88.6 (3) for each $i \in I'_\ell$ if ℓ is odd and enlarging I'_ℓ to a directed set if ℓ is even—we get a sequence of subsets $I'_\ell \subset I$ for $\ell \geq 0$. The union $I' = \bigcup I'_\ell$ satisfies:

(1) I' is countable and directed;
(2) each x_n is in the image of $f_i : M_i \to M$ for some $i \in I'$;
(3) if $i \in I'$, then there is $j \in I'$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \in I$ with $k \geq i$. In particular $\operatorname{Ker}(f_{ik}) \subset \operatorname{Ker}(f_{ij})$ for $k \geq i$.

We claim that the canonical map $\operatorname{colim}_{i \in I'} M_i \to \operatorname{colim}_{i \in I} M_i = M$ is an isomorphism. By (2) it is surjective. For injectivity, suppose $x \in \operatorname{colim}_{i \in I'} M_i$ maps to 0 in $\operatorname{colim}_{i \in I} M_i$. Representing x by an element $\tilde{x} \in M_i$ for some $i \in I'$, this means that $f_{ik}(\tilde{x}) = 0$ for some $k \in I, k \geq i$. But then by (3) there is $j \in I', j \geq i$, such that $f_{ij}(\tilde{x}) = 0$. Hence $x = 0$ in $\operatorname{colim}_{i \in I'} M_i$. □

Lemma 10.92.1 implies that a countably generated Mittag-Leffler module M over R is the colimit of a system

$$M_1 \to M_2 \to M_3 \to M_4 \to \ldots$$

with each M_n a finitely presented R-module. To see this argue as in the proof of Lemma 10.86.3 to see that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. Suppose $R = k[x_1, x_2, x_3, \ldots]$ and $M = R/(x_i)$. Then M is finitely generated but not finitely presented, hence not Mittag-Leffler (see Example 10.91.1 part (1)). But of course you can write $M = \operatorname{colim}_n M_n$ by taking $M_n = R/(x_1, \ldots, x_n)$, hence the condition that you can write M as such a limit does not imply that M is Mittag-Leffler.

05D2 **Lemma 10.92.2.** *Let R be a ring. Let M be an R-module. Assume M is Mittag-Leffler and countably generated. For any R-module map $f : P \to M$ with P finitely generated there exists an endomorphism $\alpha : M \to M$ such that*

(1) *$\alpha : M \to M$ factors through a finitely presented R-module, and*
(2) *$\alpha \circ f = f$.*

Proof. Write $M = \operatorname{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules with I countable, see Lemma 10.92.1. The transition maps are denoted f_{ij}

and we use $f_i : M_i \to M$ to denote the canonical maps into M. Set $N = \prod_{s\in I} M_s$. Denote

$$M_i^* = \mathrm{Hom}_R(M_i, N) = \prod\nolimits_{s\in I} \mathrm{Hom}_R(M_i, M_s)$$

so that (M_i^*) is an inverse system of R-modules over I. Note that $\mathrm{Hom}_R(M, N) = \lim M_i^*$. As M is Mittag-Leffler, we find for every $i \in I$ an index $k(i) \geq i$ such that

$$E_i := \bigcap\nolimits_{i'\geq i} \mathrm{Im}(M_{i'}^* \to M_i^*) = \mathrm{Im}(M_{k(i)}^* \to M_i^*)$$

Choose and fix $j \in I$ such that $\mathrm{Im}(P \to M) \subset \mathrm{Im}(M_j \to M)$. This is possible as P is finitely generated. Set $k = k(j)$. Let $x = (0, \ldots, 0, \mathrm{id}_{M_k}, 0, \ldots, 0) \in M_k^*$ and note that this maps to $y = (0, \ldots, 0, f_{jk}, 0, \ldots, 0) \in M_j^*$. By our choice of k we see that $y \in E_j$. By Example 10.86.2 the transition maps $E_i \to E_j$ are surjective for each $i \geq j$ and $\lim E_i = \lim M_i^* = \mathrm{Hom}_R(M, N)$. Hence Lemma 10.86.3 guarantees there exists an element $z \in \mathrm{Hom}_R(M, N)$ which maps to y in $E_j \subset M_j^*$. Let z_k be the kth component of z. Then $z_k : M \to M_k$ is a homomorphism such that

$$\begin{array}{ccc} M & \xrightarrow{z_k} & M_k \\ {\scriptstyle f_j}\uparrow & \nearrow_{f_{jk}} & \\ M_j & & \end{array}$$

commutes. Let $\alpha : M \to M$ be the composition $f_k \circ z_k : M \to M_k \to M$. Then α factors through a finitely presented module by construction and $\alpha \circ f_j = f_j$. Since the image of f is contained in the image of f_j this also implies that $\alpha \circ f = f$. □

We will see later (see Lemma 10.153.13) that Lemma 10.92.2 means that a countably generated Mittag-Leffler module over a henselian local ring is a direct sum of finitely presented modules.

10.93. Characterizing projective modules

059V The goal of this section is to prove that a module is projective if and only if it is flat, Mittag-Leffler, and a direct sum of countably generated modules (Theorem 10.93.3 below).

059X **Lemma 10.93.1.** *Let M be an R-module. If M is flat, Mittag-Leffler, and countably generated, then M is projective.*

Proof. By Lazard's theorem (Theorem 10.81.4), we can write $M = \mathrm{colim}_{i\in I} M_i$ for a directed system of finite free R-modules (M_i, f_{ij}) indexed by a set I. By Lemma 10.92.1, we may assume I is countable. Now let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of R-modules. We must show that applying $\mathrm{Hom}_R(M, -)$ preserves exactness. Since M_i is finite free,

$$0 \to \mathrm{Hom}_R(M_i, N_1) \to \mathrm{Hom}_R(M_i, N_2) \to \mathrm{Hom}_R(M_i, N_3) \to 0$$

is exact for each i. Since M is Mittag-Leffler, $(\mathrm{Hom}_R(M_i, N_1))$ is a Mittag-Leffler inverse system. So by Lemma 10.86.4,

$$0 \to \lim_{i\in I} \mathrm{Hom}_R(M_i, N_1) \to \lim_{i\in I} \mathrm{Hom}_R(M_i, N_2) \to \lim_{i\in I} \mathrm{Hom}_R(M_i, N_3) \to 0$$

is exact. But for any R-module N there is a functorial isomorphism $\mathrm{Hom}_R(M, N) \cong \lim_{i \in I} \mathrm{Hom}_R(M_i, N)$, so

$$0 \to \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3) \to 0$$

is exact. □

059Y **Remark 10.93.2.** Lemma 10.93.1 does not hold without the countable generation assumption. For example, the $\mathbf{Z}$-module $M = \mathbf{Z}[[x]]$ is flat and Mittag-Leffler but not projective. It is Mittag-Leffler by Lemma 10.91.4. Subgroups of free abelian groups are free, hence a projective $\mathbf{Z}$-module is in fact free and so are its submodules. Thus to show M is not projective it suffices to produce a non-free submodule. Fix a prime p and consider the submodule N consisting of power series $f(x) = \sum a_i x^i$ such that for every integer $m \geq 1$, p^m divides a_i for all but finitely many i. Then $\sum a_i p^i x^i$ is in N for all $a_i \in \mathbf{Z}$, so N is uncountable. Thus if N were free it would have uncountable rank and the dimension of N/pN over $\mathbf{Z}/p$ would be uncountable. This is not true as the elements $x^i \in N/pN$ for $i \geq 0$ span N/pN.

059Z **Theorem 10.93.3.** *Let M be an R-module. Then M is projective if and only it satisfies:*

(1) *M is flat,*
(2) *M is Mittag-Leffler,*
(3) *M is a direct sum of countably generated R-modules.*

Proof. First suppose M is projective. Then M is a direct summand of a free module, so M is flat and Mittag-Leffler since these properties pass to direct summands. By Kaplansky's theorem (Theorem 10.84.5), M satisfies (3).

Conversely, suppose M satisfies (1)-(3). Since being flat and Mittag-Leffler passes to direct summands, M is a direct sum of flat, Mittag-Leffler, countably generated R-modules. Lemma 10.93.1 implies M is a direct sum of projective modules. Hence M is projective. □

05A0 **Lemma 10.93.4.** *Let $f : M \to N$ be universally injective map of R-modules. Suppose M is a direct sum of countably generated R-modules, and suppose N is flat and Mittag-Leffler. Then M is projective.*

Proof. By Lemmas 10.82.7 and 10.89.7, M is flat and Mittag-Leffler, so the conclusion follows from Theorem 10.93.3. □

05A1 **Lemma 10.93.5.** *Let R be a Noetherian ring and let M be a R-module. Suppose M is a direct sum of countably generated R-modules, and suppose there is a universally injective map $M \to R[[t_1, \ldots, t_n]]$ for some n. Then M is projective.*

Proof. Follows from Lemmas 10.93.4 and 10.91.4. □

10.94. Ascending properties of modules

05A2 All of the properties of a module in Theorem 10.93.3 ascend along arbitrary ring maps:

05A3 **Lemma 10.94.1.** *Let $R \to S$ be a ring map. Let M be an R-module. Then:*

(1) *If M is flat, then the S-module $M \otimes_R S$ is flat.*
(2) *If M is Mittag-Leffler, then the S-module $M \otimes_R S$ is Mittag-Leffler.*

(3) *If M is a direct sum of countably generated R-modules, then the S-module $M \otimes_R S$ is a direct sum of countably generated S-modules.*
(4) *If M is projective, then the S-module $M \otimes_R S$ is projective.*

Proof. All are obvious except (2). For this, use formulation (3) of being Mittag-Leffler from Proposition 10.88.6 and the fact that tensoring commutes with taking colimits. Alternatively, one can use the characterization of Proposition 10.89.5. □

10.95. Descending properties of modules

05A4 We address the faithfully flat descent of the properties from Theorem 10.93.3 that characterize projectivity. In the presence of flatness, the property of being a Mittag-Leffler module descends:

05A5 **Lemma 10.95.1.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is Mittag-Leffler, then M is Mittag-Leffler.*

Email from Juan Pablo Acosta Lopez dated 12/20/14.

Proof. Write $M = \operatorname{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules M_i. Using Proposition 10.88.6, we see that we have to prove that for each $i \in I$ there exists $i \leq j$, $j \in I$ such that $M_i \to M_j$ dominates $M_i \to M$.

Take N the pushout

$$\begin{array}{ccc} M_i & \longrightarrow & M_j \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

Then the lemma is equivalent to the existence of j such that $M_j \to N$ is universally injective, see Lemma 10.88.4. Observe that the tensorization by S

$$\begin{array}{ccc} M_i \otimes_R S & \longrightarrow & M_j \otimes_R S \\ \downarrow & & \downarrow \\ M \otimes_R S & \longrightarrow & N \otimes_R S \end{array}$$

Is a pushout diagram. So because $M \otimes_R S = \operatorname{colim}_{i \in I} M_i \otimes_R S$ expresses $M \otimes_R S$ as a colimit of S-modules of finite presentation, and $M \otimes_R S$ is Mittag-Leffler, there exists $j \geq i$ such that $M_j \otimes_R S \to N \otimes_R S$ is universally injective. So using that $R \to S$ is faithfully flat we conclude that $M_j \to N$ is universally injective too. □

0GVD **Lemma 10.95.2.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated, then M is countably generated.*

Proof. Say $M \otimes_R S$ is generated by the elements y_i, $i = 1, 2, 3, \ldots$. Write $y_i = \sum_{j=1,\ldots,n_i} x_{ij} \otimes s_{ij}$ for some $n_i \geq 0$, $x_{ij} \in M$ and $s_{ij} \in S$. Denote $M' \subset M$ the submodule generated by the countable collection of elements x_{ij}. Then $M' \otimes_R S \to M \otimes_R S$ is surjective as the image contains the generators y_i. Since S is faithfully flat over R we conclude that $M' = M$ as desired. □

At this point the faithfully flat descent of countably generated projective modules follows easily.

05A6 **Lemma 10.95.3.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated and projective, then M is countably generated and projective.*

Proof. Follows from Lemmas 10.83.2, 10.95.1, and 10.95.2 and Theorem 10.93.3. □

All that remains is to use dévissage to reduce descent of projectivity in the general case to the countably generated case. First, two simple lemmas.

05A7 **Lemma 10.95.4.** *Let $R \to S$ be a ring map, let M be an R-module, and let Q be a countably generated S-submodule of $M \otimes_R S$. Then there exists a countably generated R-submodule P of M such that $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ contains Q.*

Proof. Let $y_1, y_2, \ldots$ be generators for Q and write $y_j = \sum_k x_{jk} \otimes s_{jk}$ for some $x_{jk} \in M$ and $s_{jk} \in S$. Then take P be the submodule of M generated by the x_{jk}. □

05A8 **Lemma 10.95.5.** *Let $R \to S$ be a ring map, and let M be an R-module. Suppose $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is a direct sum of countably generated S-modules Q_i. If N is a countably generated submodule of M, then there is a countably generated submodule N' of M such that $N' \supset N$ and $\mathrm{Im}(N' \otimes_R S \to M \otimes_R S) = \bigoplus_{i \in I'} Q_i$ for some subset $I' \subset I$.*

Proof. Let $N'_0 = N$. We construct by induction an increasing sequence of countably generated submodules $N'_\ell \subset M$ for $\ell = 0, 1, 2, \ldots$ such that: if I'_ℓ is the set of $i \in I$ such that the projection of $\mathrm{Im}(N'_\ell \otimes_R S \to M \otimes_R S)$ onto Q_i is nonzero, then $\mathrm{Im}(N'_{\ell+1} \otimes_R S \to M \otimes_R S)$ contains Q_i for all $i \in I'_\ell$. To construct $N'_{\ell+1}$ from N'_ℓ, let Q be the sum of (the countably many) Q_i for $i \in I'_\ell$, choose P as in Lemma 10.95.4, and then let $N'_{\ell+1} = N'_\ell + P$. Having constructed the N'_ℓ, just take $N' = \bigcup_\ell N'_\ell$ and $I' = \bigcup_\ell I'_\ell$. □

05A9 **Theorem 10.95.6.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is projective, then M is projective.*

Proof. We are going to construct a Kaplansky dévissage of M to show that it is a direct sum of projective modules and hence projective. By Theorem 10.84.5 we can write $M \otimes_R S = \bigoplus_{i \in I} Q_i$ as a direct sum of countably generated S-modules Q_i. Choose a well-ordering on M. Using transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that $M_\alpha \otimes_R S$ is a direct sum of some subset of the Q_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Since each $M_\beta \otimes_R S$ for $\beta < \alpha$ is a direct sum of a subset of the Q_i, the same will be true of $M_\alpha \otimes_R S$. If $\alpha+1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. Otherwise choose the smallest $x \in M$ (with respect to the fixed well-ordering) such that $x \notin M_\alpha$. Since S is flat over R, $(M/M_\alpha) \otimes_R S = M \otimes_R S / M_\alpha \otimes_R S$, so since $M_\alpha \otimes_R S$ is a direct sum of some Q_i, the same is true of $(M/M_\alpha) \otimes_R S$. By Lemma 10.95.5, we can find a countably generated R-submodule P of M/M_α containing the image of x in M/M_α and such that $P \otimes_R S$ (which equals $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ since S is flat over R) is a direct sum of some Q_i. Since $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is projective and projectivity passes to direct summands, $P \otimes_R S$ is also projective. Thus by Lemma 10.95.3, P is projective. Finally we define $M_{\alpha+1}$ to be the preimage of P in M, so that $M_{\alpha+1}/M_\alpha = P$ is countably generated and projective. In particular M_α is a direct summand of $M_{\alpha+1}$ since projectivity of $M_{\alpha+1}/M_\alpha$ implies the sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ splits.

Transfinite induction on M (using the fact that we constructed $M_{\alpha+1}$ to contain the smallest $x \in M$ not contained in M_α) shows that each $x \in M$ is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $x \in M$ there is $\alpha \in S$ such that $x \in M_\alpha$. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The other properties are clear by construction. We conclude $M = \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$. Since each $M_{\alpha+1}/M_\alpha$ is projective by construction, M is projective. □

10.96. Completion

00M9 Suppose that R is a ring and I is an ideal. We define the *completion of R with respect to I* to be the limit

$$R^\wedge = \lim_n R/I^n.$$

An element of $R^\wedge$ is given by a sequence of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$ for all n. We will view $R^\wedge$ as an R-algebra. Similarly, if M is an R-module then we define the *completion of M with respect to I* to be the limit

$$M^\wedge = \lim_n M/I^nM.$$

An element of $M^\wedge$ is given by a sequence of elements $m_n \in M/I^nM$ such that $m_n \equiv m_{n+1} \bmod I^nM$ for all n. We will view $M^\wedge$ as an $R^\wedge$-module. From this description it is clear that there are always canonical maps

$$M \longrightarrow M^\wedge \quad \text{and} \quad M \otimes_R R^\wedge \longrightarrow M^\wedge.$$

Moreover, given a map $\varphi : M \to N$ of modules we get an induced map $\varphi^\wedge : M^\wedge \to N^\wedge$ on completions making the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ M^\wedge & \longrightarrow & N^\wedge \end{array}$$

commute. In general completion is not an exact functor, see Examples, Section 110.9. Here are some initial positive results.

0315 **Lemma 10.96.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\varphi : M \to N$ be a map of R-modules.*

(1) *If $M/IM \to N/IN$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
(2) *If $M \to N$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
(3) *If $0 \to K \to M \to N \to 0$ is a short exact sequence of R-modules and N is flat, then $0 \to K^\wedge \to M^\wedge \to N^\wedge \to 0$ is a short exact sequence.*
(4) *The map $M \otimes_R R^\wedge \to M^\wedge$ is surjective for any finite R-module M.*

Proof. Assume $M/IM \to N/IN$ is surjective. Then the map $M/I^nM \to N/I^nN$ is surjective for each $n \geq 1$ by Nakayama's lemma. More precisely, apply Lemma 10.20.1 part (11) to the map $M/I^nM \to N/I^nN$ over the ring R/I^n and the nilpotent ideal I/I^n to see this. Set $K_n = \{x \in M \mid \varphi(x) \in I^nN\}$. Thus we get short exact sequences

$$0 \to K_n/I^nM \to M/I^nM \to N/I^nN \to 0$$

We claim that the canonical map $K_{n+1}/I^{n+1}M \to K_n/I^nM$ is surjective. Namely, if $x \in K_n$ write $\varphi(x) = \sum z_j n_j$ with $z_j \in I^n$, $n_j \in N$. By assumption we can write $n_j = \varphi(m_j) + \sum z_{jk} n_{jk}$ with $m_j \in M$, $z_{jk} \in I$ and $n_{jk} \in N$. Hence

$$\varphi(x - \sum z_j m_j) = \sum z_j z_{jk} n_{jk}.$$

This means that $x' = x - \sum z_j m_j \in K_{n+1}$ maps to x mod I^nM which proves the claim. Now we may apply Lemma 10.87.1 to the inverse system of short exact sequences above to see (1). Part (2) is a special case of (1). If the assumptions of (3) hold, then for each n the sequence

$$0 \to K/I^nK \to M/I^nM \to N/I^nN \to 0$$

is short exact by Lemma 10.39.12. Hence we can directly apply Lemma 10.87.1 to conclude (3) is true. To see (4) choose generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Hence by (2) we see $(R^\wedge)^{\oplus n} \to M^\wedge$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Assertion (4) follows from this. □

0317 **Definition 10.96.2.** Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-adically complete* if the map

$$M \longrightarrow M^\wedge = \lim_n M/I^nM$$

is an isomorphism[9]. We say R is *I-adically complete* if R is I-adically complete as an R-module.

It is not true that the completion of an R-module M with respect to I is I-adically complete. For an example see Examples, Section 110.7. If the ideal is finitely generated, then the completion is complete.

05GG **Lemma 10.96.3.** *Let R be a ring. Let I be a finitely generated ideal of R. Let M be an R-module. Then*

(1) *the completion $M^\wedge$ is I-adically complete, and*
(2) *$I^nM^\wedge = \mathrm{Ker}(M^\wedge \to M/I^nM) = (I^nM)^\wedge$ for all $n \geq 1$.*

In particular $R^\wedge$ is I-adically complete, $I^nR^\wedge = (I^n)^\wedge$, and $R^\wedge/I^nR^\wedge = R/I^n$.

[**Mat78**, Theorem 15]. The slick proof given here is from an email of Bjorn Poonen dated Nov 5, 2016.

Proof. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Applying Lemma 10.96.1 part (2) to the surjection $(f_1, \ldots, f_r) : M^{\oplus r} \to I^nM$ yields a surjection

$$(M^\wedge)^{\oplus r} \xrightarrow{(f_1, \ldots, f_r)} (I^nM)^\wedge = \lim_{m \geq n} I^nM/I^mM = \mathrm{Ker}(M^\wedge \to M/I^nM).$$

On the other hand, the image of $(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \to M^\wedge$ is $I^nM^\wedge$. Thus $M^\wedge/I^nM^\wedge \simeq M/I^nM$. Taking inverse limits yields $(M^\wedge)^\wedge \simeq M^\wedge$; that is, $M^\wedge$ is I-adically complete. □

0BNG **Lemma 10.96.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to M \to N \to Q \to 0$ be an exact sequence of R-modules such that Q is annihilated by a power of I. Then completion produces an exact sequence $0 \to M^\wedge \to N^\wedge \to Q \to 0$.*

[9] This includes the condition that $\bigcap I^nM = 0$.

Proof. Say $I^cQ = 0$. Then $Q/I^nQ = Q$ for $n \geq c$. On the other hand, it is clear that $I^nM \subset M \cap I^nN \subset I^{n-c}M$ for $n \geq c$. Thus $M^\wedge = \lim M/(M \cap I^nN)$. Apply Lemma 10.87.1 to the system of exact sequences

$$0 \to M/(M \cap I^nN) \to N/I^nN \to Q \to 0$$

for $n \geq c$ to conclude. □

0318 **Lemma 10.96.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Denote $K_n = \mathrm{Ker}(M^\wedge \to M/I^nM)$. Then $M^\wedge$ is I-adically complete if and only if K_n is equal to $I^nM^\wedge$ for all $n \geq 1$.*

Taken from an unpublished note of Lenstra and de Smit.

Proof. The module $I^nM^\wedge$ is contained in K_n. Thus for each $n \geq 1$ there is a canonical exact sequence

$$0 \to K_n/I^nM^\wedge \to M^\wedge/I^nM^\wedge \to M/I^nM \to 0.$$

As $I^nM^\wedge$ maps onto $I^nM/I^{n+1}M$ we see that $K_{n+1} + I^nM^\wedge = K_n$. Thus the inverse system $\{K_n/I^nM^\wedge\}_{n\geq 1}$ has surjective transition maps. By Lemma 10.87.1 we see that there is a short exact sequence

$$0 \to \lim_n K_n/I^nM^\wedge \to (M^\wedge)^\wedge \to M^\wedge \to 0$$

Hence $M^\wedge$ is complete if and only if $K_n/I^nM^\wedge = 0$ for all $n \geq 1$. □

05GI **Lemma 10.96.6.** *Let R be a ring, let $I \subset R$ be an ideal, and let $R^\wedge = \lim R/I^n$.*

1. *any element of $R^\wedge$ which maps to a unit of R/I is a unit,*
2. *any element of $1 + I$ maps to an invertible element of $R^\wedge$,*
3. *any element of $1 + IR^\wedge$ is invertible in $R^\wedge$, and*
4. *the ideals $IR^\wedge$ and $\mathrm{Ker}(R^\wedge \to R/I)$ are contained in the Jacobson radical of $R^\wedge$.*

Proof. Let $x \in R^\wedge$ map to a unit x_1 in R/I. Then x maps to a unit x_n in R/I^n for every n by Lemma 10.32.4. Hence $y = (x_n^{-1}) \in \lim R/I^n = R^\wedge$ is an inverse to x. Parts (2) and (3) follow immediately from (1). Part (4) follows from (1) and Lemma 10.19.1. □

090S **Lemma 10.96.7.** *Let A be a ring. Let $I = (f_1, \ldots, f_r)$ be a finitely generated ideal. If $M \to \lim M/f_i^nM$ is surjective for each i, then $M \to \lim M/I^nM$ is surjective.*

Proof. Note that $\lim M/I^nM = \lim M/(f_1^n, \ldots, f_r^n)M$ as $I^n \supset (f_1^n, \ldots, f_r^n) \supset I^{rn}$. An element ξ of $\lim M/(f_1^n, \ldots, f_r^n)M$ can be symbolically written as

$$\xi = \sum\nolimits_{n\geq 0} \sum\nolimits_i f_i^n x_{n,i}$$

with $x_{n,i} \in M$. If $M \to \lim M/f_i^nM$ is surjective, then there is an $x_i \in M$ mapping to $\sum x_{n,i}f_i^n$ in $\lim M/f_i^nM$. Then $x = \sum x_i$ maps to ξ in $\lim M/I^nM$. □

090T **Lemma 10.96.8.** *Let A be a ring. Let $I \subset J \subset A$ be ideals. If M is J-adically complete and I is finitely generated, then M is I-adically complete.*

Proof. Assume M is J-adically complete and I is finitely generated. We have $\bigcap I^nM = 0$ because $\bigcap J^nM = 0$. By Lemma 10.96.7 it suffices to prove the surjectivity of $M \to \lim M/I^nM$ in case I is generated by a single element. Say $I = (f)$. Let $x_n \in M$ with $x_{n+1} - x_n \in f^nM$. We have to show there exists an $x \in M$ such that $x_n - x \in f^nM$ for all n. As $x_{n+1} - x_n \in J^nM$ and as M

is J-adically complete, there exists an element $x \in M$ such that $x_n - x \in J^n M$. Replacing x_n by $x_n - x$ we may assume that $x_n \in J^n M$. To finish the proof we will show that this implies $x_n \in I^n M$. Namely, write $x_n - x_{n+1} = f^n z_n$. Then

$$x_n = f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$$

The sum $z_n + f z_{n+1} + f^2 z_{n+2} + \ldots$ converges in M as $f^c \in J^c$. The sum $f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$ converges in M to x_n because the partial sums equal $x_n - x_{n+c}$ and $x_{n+c} \in J^{n+c} M$. □

0319 **Lemma 10.96.9.** *Let R be a ring. Let I, J be ideals of R. Assume there exist integers $c, d > 0$ such that $I^c \subset J$ and $J^d \subset I$. Then completion with respect to I agrees with completion with respect to J for any R-module. In particular an R-module M is I-adically complete if and only if it is J-adically complete.*

Proof. Consider the system of maps $M/I^n M \to M/J^{\lfloor n/c \rfloor} M$ and the system of maps $M/J^m M \to M/I^{\lfloor m/d \rfloor} M$ to get mutually inverse maps between the completions. □

031A **Lemma 10.96.10.** *Let R be a ring. Let I be an ideal of R. Let M be an I-adically complete R-module, and let $K \subset M$ be an R-submodule. The following are equivalent*

(1) *$K = \bigcap (K + I^n M)$ and*
(2) *M/K is I-adically complete.*

Proof. Set $N = M/K$. By Lemma 10.96.1 the map $M = M^\wedge \to N^\wedge$ is surjective. Hence $N \to N^\wedge$ is surjective. It is easy to see that the kernel of $N \to N^\wedge$ is the module $\bigcap (K + I^n M)/K$. □

031B **Lemma 10.96.11.** *Let R be a ring. Let I be an ideal of R. Let M be an R-module. If (a) R is I-adically complete, (b) M is a finite R-module, and (c) $\bigcap I^n M = (0)$, then M is I-adically complete.*

Proof. By Lemma 10.96.1 the map $M = M \otimes_R R = M \otimes_R R^\wedge \to M^\wedge$ is surjective. The kernel of this map is $\bigcap I^n M$ hence zero by assumption. Hence $M \cong M^\wedge$ and M is complete. □

031D **Lemma 10.96.12.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *R is I-adically complete,*
(2) *$\bigcap_{n \geq 1} I^n M = (0)$, and*
(3) *M/IM is a finite R/I-module.*

Then M is a finite R-module.

Proof. Let $x_1, \ldots, x_n \in M$ be elements whose images in M/IM generate M/IM as a R/I-module. Denote $M' \subset M$ the R-submodule generated by $x_1, \ldots, x_n$. By Lemma 10.96.1 the map $(M')^\wedge \to M^\wedge$ is surjective. Since $\bigcap I^n M = 0$ we see in particular that $\bigcap I^n M' = (0)$. Hence by Lemma 10.96.11 we see that M' is complete, and we conclude that $M' \to M^\wedge$ is surjective. Finally, the kernel of $M \to M^\wedge$ is zero since it is equal to $\bigcap I^n M = (0)$. Hence we conclude that $M \cong M' \cong M^\wedge$ is finitely generated. □

10.97. Completion for Noetherian rings

0BNH In this section we discuss completion with respect to ideals in Noetherian rings.

00MA **Lemma 10.97.1.** *Let I be an ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.*

(1) *If $K \to N$ is an injective map of finite R-modules, then the map on completions $K^\wedge \to N^\wedge$ is injective.*
(2) *If $0 \to K \to N \to M \to 0$ is a short exact sequence of finite R-modules, then $0 \to K^\wedge \to N^\wedge \to M^\wedge \to 0$ is a short exact sequence.*
(3) *If M is a finite R-module, then $M^\wedge = M \otimes_R R^\wedge$.*

Proof. Setting $M = N/K$ we find that part (1) follows from part (2). Let $0 \to K \to N \to M \to 0$ be as in (2). For each n we get the short exact sequence

$$0 \to K/(I^n N \cap K) \to N/I^n N \to M/I^n M \to 0.$$

By Lemma 10.87.1 we obtain the exact sequence

$$0 \to \lim K/(I^n N \cap K) \to N^\wedge \to M^\wedge \to 0.$$

By the Artin-Rees Lemma 10.51.2 we may choose c such that $I^n K \subset I^n N \cap K \subset I^{n-c}K$ for $n \geq c$. Hence $K^\wedge = \lim K/I^n K = \lim K/(I^n N \cap K)$ and we conclude that (2) is true.

Let M be as in (3) and let $0 \to K \to R^{\oplus t} \to M \to 0$ be a presentation of M. We get a commutative diagram

$$\begin{array}{ccccccccc} & & K \otimes_R R^\wedge & \longrightarrow & R^{\oplus t} \otimes_R R^\wedge & \longrightarrow & M \otimes_R R^\wedge & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^\wedge & \longrightarrow & (R^{\oplus t})^\wedge & \longrightarrow & M^\wedge & \longrightarrow & 0 \end{array}$$

The top row is exact, see Section 10.39. The bottom row is exact by part (2). By Lemma 10.96.1 the vertical arrows are surjective. The middle vertical arrow is an isomorphism. We conclude (3) holds by the Snake Lemma 10.4.1. □

00MB **Lemma 10.97.2.** *Let I be an ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.*

(1) *The ring map $R \to R^\wedge$ is flat.*
(2) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely generated R-modules.*

Proof. Consider $J \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$ where J is an arbitrary ideal of R. According to Lemma 10.97.1 this is identified with $J^\wedge \to R^\wedge$ and $J^\wedge \to R^\wedge$ is injective. Part (1) follows from Lemma 10.39.5. Part (2) is a reformulation of Lemma 10.97.1 part (2). □

00MC **Lemma 10.97.3.** *Let I be an ideal of a Noetherian ring R. Denote $R^\wedge$ the completion of R with respect to I. If I is contained in the Jacobson radical of R, then the ring map $R \to R^\wedge$ is faithfully flat. In particular, if $(R, \mathfrak{m})$ is a Noetherian local ring, then the completion $\lim_n R/\mathfrak{m}^n$ is faithfully flat.*

Proof. By Lemma 10.97.2 it is flat. The composition $R \to R^\wedge \to R/I$ where the last map is the projection map $R^\wedge \to R/I$ shows that any maximal ideal of R is in the image of $\mathrm{Spec}(R^\wedge) \to \mathrm{Spec}(R)$. Hence the map is faithfully flat by Lemma 10.39.15. □

031C **Lemma 10.97.4.** *Let R be a Noetherian ring. Let I be an ideal of R. Let M be an R-module. Then the completion $M^\wedge$ of M with respect to I is I-adically complete, $I^n M^\wedge = (I^n M)^\wedge$, and $M^\wedge/I^n M^\wedge = M/I^n M$.*

Proof. This is a special case of Lemma 10.96.3 because I is a finitely generated ideal. □

05GH **Lemma 10.97.5.** *Let I be an ideal of a ring R. Assume*

(1) *R/I is a Noetherian ring,*
(2) *I is finitely generated.*

Then the completion $R^\wedge$ of R with respect to I is a Noetherian ring complete with respect to $IR^\wedge$.

Proof. By Lemma 10.96.3 we see that $R^\wedge$ is I-adically complete. Hence it is also $IR^\wedge$-adically complete. Since $R^\wedge/IR^\wedge = R/I$ is Noetherian we see that after replacing R by $R^\wedge$ we may in addition to assumptions (1) and (2) assume that also R is I-adically complete.

Let $f_1, \ldots, f_t$ be generators of I. Then there is a surjection of rings $R/I[T_1, \ldots, T_t] \to \bigoplus I^n/I^{n+1}$ mapping T_i to the element $\overline{f}_i \in I/I^2$. Hence $\bigoplus I^n/I^{n+1}$ is a Noetherian ring. Let $J \subset R$ be an ideal. Consider the ideal

$$\bigoplus J \cap I^n/J \cap I^{n+1} \subset \bigoplus I^n/I^{n+1}.$$

Let $\overline{g}_1, \ldots, \overline{g}_m$ be generators of this ideal. We may choose $\overline{g}_j$ to be a homogeneous element of degree d_j and we may pick $g_j \in J \cap I^{d_j}$ mapping to $\overline{g}_j \in J \cap I^{d_j}/J \cap I^{d_j+1}$. We claim that $g_1, \ldots, g_m$ generate J.

Let $x \in J \cap I^n$. There exist $a_j \in I^{\max(0, n-d_j)}$ such that $x - \sum a_j g_j \in J \cap I^{n+1}$. The reason is that $J \cap I^n/J \cap I^{n+1}$ is equal to $\sum \overline{g}_j I^{n-d_j}/I^{n-d_j+1}$ by our choice of $g_1, \ldots, g_m$. Hence starting with $x \in J$ we can find a sequence of vectors $(a_{1,n}, \ldots, a_{m,n})_{n \geq 0}$ with $a_{j,n} \in I^{\max(0,n-d_j)}$ such that

$$x = \sum\nolimits_{n=0,\ldots,N} \sum\nolimits_{j=1,\ldots,m} a_{j,n} g_j \bmod I^{N+1}$$

Setting $A_j = \sum_{n \geq 0} a_{j,n}$ we see that $x = \sum A_j g_j$ as R is complete. Hence J is finitely generated and we win. □

0316 **Lemma 10.97.6.** *Let R be a Noetherian ring. Let I be an ideal of R. The completion $R^\wedge$ of R with respect to I is Noetherian.*

Proof. This is a consequence of Lemma 10.97.5. It can also be seen directly as follows. Choose generators $f_1, \ldots, f_n$ of I. Consider the map

$$R[[x_1, \ldots, x_n]] \longrightarrow R^\wedge, \quad x_i \longmapsto f_i.$$

This is a well defined and surjective ring map (details omitted). Since $R[[x_1, \ldots, x_n]]$ is Noetherian (see Lemma 10.31.2) we win. □

Suppose $R \to S$ is a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $S^\wedge$ be the completion of S with respect to $\mathfrak{n}$. In general $S^\wedge$ is not the $\mathfrak{m}$-adic completion of S. If $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$ then we do have $S^\wedge = \lim S/\mathfrak{m}^n S$ by Lemma 10.96.9. In some cases this even implies that $S^\wedge$ is finite over $R^\wedge$.

0394 **Lemma 10.97.7.** *Let $R \to S$ be a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $R^\wedge$, resp. $S^\wedge$ be the completion of R, resp. S with respect to $\mathfrak{m}$, resp. $\mathfrak{n}$. If $\mathfrak{m}$ and $\mathfrak{n}$ are finitely generated and $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$, then*

(1) *$S^\wedge$ is equal to the $\mathfrak{m}$-adic completion of S, and*
(2) *$S^\wedge$ is a finite $R^\wedge$-module.*

Proof. We have $\mathfrak{m}S \subset \mathfrak{n}$ because $R \to S$ is a local ring map. The assumption $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$ implies that $S/\mathfrak{m}S$ is an Artinian ring, see Lemma 10.53.2. Hence has dimension 0, see Lemma 10.60.5, hence $\mathfrak{n} = \sqrt{\mathfrak{m}S}$. This and the fact that $\mathfrak{n}$ is finitely generated implies that $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$. By Lemma 10.96.9 we see that $S^\wedge$ can be identified with the $\mathfrak{m}$-adic completion of S. As $\mathfrak{m}$ is finitely generated we see from Lemma 10.96.3 that $S^\wedge$ and $R^\wedge$ are $\mathfrak{m}$-adically complete. At this point we may apply Lemma 10.96.12 to $S^\wedge$ as an $R^\wedge$-module to conclude. □

07N9 **Lemma 10.97.8.** *Let R be a Noetherian ring. Let $R \to S$ be a finite ring map. Let $\mathfrak{p} \subset R$ be a prime and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the primes of S lying over $\mathfrak{p}$ (Lemma 10.36.21). Then*

$$R^\wedge_\mathfrak{p} \otimes_R S = (S_\mathfrak{p})^\wedge = S^\wedge_{\mathfrak{q}_1} \times \ldots \times S^\wedge_{\mathfrak{q}_m}$$

where the $(S_\mathfrak{p})^\wedge$ is the completion with respect to $\mathfrak{p}$ and the local rings $R_\mathfrak{p}$ and $S_{\mathfrak{q}_i}$ are completed with respect to their maximal ideals.

Proof. We may replace R by the localization $R_\mathfrak{p}$ and S by $S_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$. Hence we may assume that R is a local Noetherian ring and that $\mathfrak{p} = \mathfrak{m}$ is its maximal ideal. The $\mathfrak{q}_i S_{\mathfrak{q}_i}$-adic completion $S^\wedge_{\mathfrak{q}_i}$ is equal to the $\mathfrak{m}$-adic completion by Lemma 10.97.7. For every $n \geq 1$ prime ideals of $S/\mathfrak{m}^n S$ are in 1-to-1 correspondence with the maximal ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ of S (by going up for S over R, see Lemma 10.36.22). Hence $S/\mathfrak{m}^n S = \prod S_{\mathfrak{q}_i}/\mathfrak{m}^n S_{\mathfrak{q}_i}$ by Lemma 10.53.6 (using for example Proposition 10.60.7 to see that $S/\mathfrak{m}^n S$ is Artinian). Hence the $\mathfrak{m}$-adic completion $S^\wedge$ of S is equal to $\prod S^\wedge_{\mathfrak{q}_i}$. Finally, we have $R^\wedge \otimes_R S = S^\wedge$ by Lemma 10.97.1. □

05D3 **Lemma 10.97.9.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of R-modules. If M is flat over R and M/IM is a projective R/I-module, then the sequence of I-adic completions*

$$0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$$

is a split exact sequence.

Proof. As M is flat, each of the sequences

$$0 \to K/I^n K \to P/I^n P \to M/I^n M \to 0$$

is short exact, see Lemma 10.39.12 and the sequence $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is a short exact sequence, see Lemma 10.96.1. It suffices to show that we can find splittings $s_n : M/I^n M \to P/I^n P$ such that $s_{n+1} \bmod I^n = s_n$. We will construct these s_n by induction on n. Pick any splitting s_1, which exists as M/IM is a projective R/I-module. Assume given s_n for some $n > 0$. Set $P_{n+1} = \{x \in P \mid x \bmod I^n P \in \mathrm{Im}(s_n)\}$. The map $\pi : P_{n+1}/I^{n+1} P_{n+1} \to M/I^{n+1} M$ is surjective

(details omitted). As $M/I^{n+1}M$ is projective as a R/I^{n+1}-module by Lemma 10.77.7 we may choose a section $t : M/I^{n+1}M \to P_{n+1}/I^{n+1}P_{n+1}$ of π. Setting s_{n+1} equal to the composition of t with the canonical map $P_{n+1}/I^{n+1}P_{n+1} \to P/I^{n+1}P$ works. □

0DYC **Lemma 10.97.10.** *Let A be a Noetherian ring. Let $I, J \subset A$ be ideals. If A is I-adically complete and A/I is J-adically complete, then A is J-adically complete.*

Proof. Let B be the $(I+J)$-adic completion of A. By Lemma 10.97.2 B/IB is the J-adic completion of A/I hence isomorphic to A/I by assumption. Moreover B is I-adically complete by Lemma 10.96.8. Hence B is a finite A-module by Lemma 10.96.12. By Nakayama's lemma (Lemma 10.20.1 using I is in the Jacobson radical of A by Lemma 10.96.6) we find that $A \to B$ is surjective. The map $A \to B$ is flat by Lemma 10.97.2. The image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains $V(I)$ and as I is contained in the Jacobson radical of A we find $A \to B$ is faithfully flat (Lemma 10.39.16). Thus $A \to B$ is injective. Thus A is complete with respect to $I + J$, hence a fortiori complete with respect to J. □

10.98. Taking limits of modules

09B7 In this section we discuss what happens when we take a limit of modules.

0G1Q **Lemma 10.98.1.** *Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $I^n M_n = 0$. Then $M = \lim M_n$ is I-adically complete.*

Proof. We have $M \to M/I^nM \to M_n$. Taking the limit we get $M \to M^\wedge \to M$. Hence M is a direct summand of $M^\wedge$. Since $M^\wedge$ is I-adically complete by Lemma 10.96.3, so is M. □

09B8 **Lemma 10.98.2.** *Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $M_n = M_{n+1}/I^nM_{n+1}$. Set $M = \lim M_n$. Then $M/I^nM = M_n$ and M is I-adically complete.*

Proof. By Lemma 10.98.1 we see that M is I-adically complete. Since the transition maps are surjective, the maps $M \to M_n$ are surjective. Consider the inverse system of short exact sequences

$$0 \to N_n \to M \to M_n \to 0$$

defining N_n. Since $M_n = M_{n+1}/I^nM_{n+1}$ the map $N_{n+1} + I^nM \to N_n$ is surjective. Hence $N_{n+1}/(N_{n+1} \cap I^{n+1}M) \to N_n/(N_n \cap I^nM)$ is surjective. Taking the inverse limit of the short exact sequences

$$0 \to N_n/(N_n \cap I^nM) \to M/I^nM \to M_n \to 0$$

we obtain an exact sequence

$$0 \to \lim N_n/(N_n \cap I^nM) \to M^\wedge \to M$$

Since M is I-adically complete we conclude that $\lim N_n/(N_n \cap I^nM) = 0$ and hence by the surjectivity of the transition maps we get $N_n/(N_n \cap I^nM) = 0$ for all n. Thus $M_n = M/I^nM$ as desired. □

0EKC **Lemma 10.98.3.** *Let A be a Noetherian graded ring. Let $I \subset A_+$ be a homogeneous ideal. Let (N_n) be an inverse system of finite graded A-modules with $N_n = N_{n+1}/I^n N_{n+1}$. Then there is a finite graded A-module N such that $N_n = N/I^n N$ as graded modules for all n.*

Proof. Pick r and homogeneous elements $x_{1,1}, \ldots, x_{1,r} \in N_1$ of degrees $d_1, \ldots, d_r$ generating N_1. Since the transition maps are surjective, we can pick a compatible system of homogeneous elements $x_{n,i} \in N_n$ lifting $x_{1,i}$. By the graded Nakayama lemma (Lemma 10.56.1) we see that N_n is generated by the elements $x_{n,1}, \ldots, x_{n,r}$ sitting in degrees $d_1, \ldots, d_r$. Thus for $m \leq n$ we see that $N_n \to N_n/I^m N_n$ is an isomorphism in degrees $< \min(d_i) + m$ (as $I^m N_n$ is zero in those degrees). Thus the inverse system of degree d parts

$$\ldots = N_{2+d-\min(d_i),d} = N_{1+d-\min(d_i),d} = N_{d-\min(d_i),d} \to N_{-1+d-\min(d_i),d} \to \ldots$$

stabilizes as indicated. Let N be the graded A-module whose dth graded part is this stabilization. In particular, we have the elements $x_i = \lim x_{n,i}$ in N. We claim the x_i generate N: any $x \in N_d$ is a linear combination of $x_1, \ldots, x_r$ because we can check this in $N_{d-\min(d_i),d}$ where it holds as $x_{d-\min(d_i),i}$ generate $N_{d-\min(d_i)}$. Finally, the reader checks that the surjective map $N/I^n N \to N_n$ is an isomorphism by checking to see what happens in each degree as before. Details omitted. □

0EKD **Lemma 10.98.4.** *Let A be a graded ring. Let $I \subset A_+$ be a homogeneous ideal. Denote $A' = \lim A/I^n$. Let (G_n) be an inverse system of graded A-modules with G_n annihilated by I^n. Let M be a graded A-module and let $\varphi_n : M \to G_n$ be a compatible system of graded A-module maps. If the induced map*

$$\varphi : M \otimes_A A' \longrightarrow \lim G_n$$

is an isomorphism, then $M_d \to \lim G_{n,d}$ is an isomorphism for all $d \in \mathbf{Z}$.

Proof. By convention graded rings are in degrees ≥ 0 and graded modules may have nonzero parts of any degree, see Section 10.56. The map φ exists because $\lim G_n$ is a module over A' as G_n is annihilated by I^n. Another useful thing to keep in mind is that we have

$$\bigoplus\nolimits_{d \in \mathbf{Z}} \lim G_{n,d} \subset \lim G_n \subset \prod\nolimits_{d \in \mathbf{Z}} \lim G_{n,d}$$

where a subscript $_d$ indicates the dth graded part.

Injective. Let $x \in M_d$. If $x \mapsto 0$ in $\lim G_{n,d}$ then $x \otimes 1 = 0$ in $M \otimes_A A'$. Then we can find a finitely generated submodule $M' \subset M$ with $x \in M'$ such that $x \otimes 1$ is zero in $M' \otimes_A A'$. Say M' is generated by homogeneous elements sitting in degrees $d_1, \ldots, d_r$. Let $n = d - \min(d_i) + 1$. Since A' has a map to A/I^n and since $A \to A/I^n$ is an isomorphism in degrees $\leq n-1$ we see that $M' \to M' \otimes_A A'$ is injective in degrees $\leq n-1$. Thus $x = 0$ as desired.

Surjective. Let $y \in \lim G_{n,d}$. Choose a finite sum $\sum x_i \otimes f_i'$ in $M \otimes_A A'$ mapping to y. We may assume x_i is homogeneous, say of degree d_i. Observe that although A' is not a graded ring, it is a limit of the graded rings $A/I^n A$ and moreover, in any given degree the transition maps eventually become isomorphisms (see above). This gives

$$A = \bigoplus\nolimits_{d \geq 0} A_d \subset A' \subset \prod\nolimits_{d \geq 0} A_d$$

Thus we can write

$$f_i' = \sum\nolimits_{j=0,\ldots,d-d_i-1} f_{i,j} + f_i + g_i'$$

with $f_{i,j} \in A_j$, $f_i \in A_{d-d_i}$, and $g_i' \in A'$ mapping to zero in $\prod_{j \leq d-d_i} A_j$. Now if we compute $\varphi_n(\sum_{i,j} f_{i,j} x_i) \in G_n$, then we get a sum of homogeneous elements of degree $< d$. Hence $\varphi(\sum x_i \otimes f_{i,j})$ maps to zero in $\lim G_{n,d}$. Similarly, a computation shows the element $\varphi(\sum x_i \otimes g_i')$ maps to zero in $\prod_{d' \leq d} \lim G_{n,d'}$. Since we know that $\varphi(\sum x_i \otimes f_i')$ is y, we conclude that $\sum f_i x_i \in M_d$ maps to y as desired. □

10.99. Criteria for flatness

00MD In this section we prove some important technical lemmas in the Noetherian case. We will (partially) generalize these to the non-Noetherian case in Section 10.128.

00ME **Lemma 10.99.1.** *Suppose that $R \to S$ is a local homomorphism of local rings with S Noetherian. Denote $\mathfrak{m}$ the maximal ideal of R. Let M be a flat R-module and N a finite S-module. Let $u : N \to M$ be a map of R-modules. If $\overline{u} : N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective then u is injective. In this case $M/u(N)$ is flat over R.*

Proof. First we claim that $u_n : N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M$ is injective for all $n \geq 1$. We proceed by induction, the base case is that $\overline{u} = u_1$ is injective. By our assumption that M is flat over R we have a short exact sequence $0 \to M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^n M \to 0$. Also, $M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. We have a similar exact sequence $N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to N/\mathfrak{m}^{n+1}N \to N/\mathfrak{m}^n N \to 0$ for N except we do not have the zero on the left. We also have $N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = N/\mathfrak{m}N \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Thus the map u_{n+1} is injective as both u_n and the map $\overline{u} \otimes \mathrm{id}_{\mathfrak{m}^n/\mathfrak{m}^{n+1}}$ are.

By Krull's intersection theorem (Lemma 10.51.4) applied to N over the ring S and the ideal $\mathfrak{m}S$ we have $\bigcap \mathfrak{m}^n N = 0$. Thus the injectivity of u_n for all n implies u is injective.

To show that $M/u(N)$ is flat over R, it suffices to show that $\mathrm{Tor}_1^R(M/u(N), R/I) = 0$ for every ideal $I \subset R$, see Lemma 10.75.8. From the short exact sequence

$$0 \to N \xrightarrow{u} M \to M/u(N) \to 0$$

and the flatness of M we obtain an exact sequence of Tors

$$0 \to \mathrm{Tor}_1^R(M/u(N), R/I) \to N/IN \to M/IM$$

See Lemma 10.75.2. Thus it suffices to show that N/IN injects into M/IM. Note that $R/I \to S/IS$ is a local homomorphism of local rings with S/IS Noetherian, $N/IN \to M/IM$ is a map of R/I-modules, N/IN is finite over S/IS, and M/IM is flat over R/I and $u \bmod I : N/IN \to M/IM$ is injective modulo $\mathfrak{m}$. Thus we may apply the first part of the proof to $u \bmod I$ and we conclude. □

00MF **Lemma 10.99.2.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$. Then S/fS is flat over R, and f is a nonzerodivisor in S.*

Proof. Follows directly from Lemma 10.99.1. □

00MG **Lemma 10.99.3.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$. Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.*

Proof. Induction and Lemma 10.99.2. □

00MH **Lemma 10.99.4.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a nonzero finite S-module. Suppose that (a) $M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and (b) M is flat over R. Then M is free and S is flat over R.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a basis for the free module $M/\mathfrak{m}M$. Choose $x_1, \ldots, x_n \in M$ with x_i mapping to $\overline{x}_i$. Let $u : S^{\oplus n} \to M$ be the map which maps the ith standard basis vector to x_i. By Lemma 10.99.1 we see that u is injective. On the other hand, by Nakayama's Lemma 10.20.1 the map is surjective. The lemma follows. □

00MI **Lemma 10.99.5.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ be a finite complex of finite S-modules. Assume that each F_i is R-flat, and that the complex $0 \to F_e/\mathfrak{m}F_e \to F_{e-1}/\mathfrak{m}F_{e-1} \to \ldots \to F_0/\mathfrak{m}F_0$ is exact. Then $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ is exact, and moreover the module $\mathrm{Coker}(F_1 \to F_0)$ is R-flat.*

Proof. By induction on e. If $e = 1$, then this is exactly Lemma 10.99.1. If $e > 1$, we see by Lemma 10.99.1 that $F_e \to F_{e-1}$ is injective and that $C = \mathrm{Coker}(F_e \to F_{e-1})$ is a finite S-module flat over R. Hence we can apply the induction hypothesis to the complex $0 \to C \to F_{e-2} \to \ldots \to F_0$. We deduce that $C \to F_{e-2}$ is injective and the exactness of the complex follows, as well as the flatness of the cokernel of $F_1 \to F_0$. □

In the rest of this section we prove two versions of what is called the "*local criterion of flatness*". Note also the interesting Lemma 10.128.1 below.

00MJ **Lemma 10.99.6.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Let M be an R-module. If $Tor_1^R(\kappa, M) = 0$, then for every finite length R-module N we have $Tor_1^R(N, M) = 0$.*

Proof. By descending induction on the length of N. If the length of N is 1, then $N \cong \kappa$ and we are done. If the length of N is more than 1, then we can fit N into a short exact sequence $0 \to N' \to N \to N'' \to 0$ where N', N'' are finite length R-modules of smaller length. The vanishing of $\mathrm{Tor}_1^R(N, M)$ follows from the vanishing of $\mathrm{Tor}_1^R(N', M)$ and $\mathrm{Tor}_1^R(N'', M)$ (induction hypothesis) and the long exact sequence of Tor groups, see Lemma 10.75.2. □

00MK **Lemma 10.99.7** (Local criterion for flatness)**.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$. Let M be a finite S-module. If $Tor_1^R(\kappa, M) = 0$, then M is flat over R.*

Proof. Let $I \subset R$ be an ideal. By Lemma 10.39.5 it suffices to show that $I \otimes_R M \to M$ is injective. By Remark 10.75.9 we see that this kernel is equal to $\mathrm{Tor}_1^R(M, R/I)$. By Lemma 10.99.6 we see that $J \otimes_R M \to M$ is injective for all ideals of finite colength.

Choose $n >> 0$ and consider the following short exact sequence

$$0 \to I \cap \mathfrak{m}^n \to I \oplus \mathfrak{m}^n \to I + \mathfrak{m}^n \to 0$$

This is a sub sequence of the short exact sequence $0 \to R \to R^{\oplus 2} \to R \to 0$. Thus we get the diagram

$$\begin{array}{ccccc} (I \cap \mathfrak{m}^n) \otimes_R M & \longrightarrow & I \otimes_R M \oplus \mathfrak{m}^n \otimes_R M & \longrightarrow & (I + \mathfrak{m}^n) \otimes_R M \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M \oplus M & \longrightarrow & M \end{array}$$

Note that $I + \mathfrak{m}^n$ and $\mathfrak{m}^n$ are ideals of finite colength. Thus a diagram chase shows that $\mathrm{Ker}((I \cap \mathfrak{m}^n) \otimes_R M \to M) \to \mathrm{Ker}(I \otimes_R M \to M)$ is surjective. We conclude in particular that $K = \mathrm{Ker}(I \otimes_R M \to M)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_R M$ in $I \otimes_R M$. By Artin-Rees, Lemma 10.51.2 we see that K is contained in $\mathfrak{m}^{n-c}(I \otimes_R M)$ for some $c > 0$ and all $n >> 0$. Since $I \otimes_R M$ is a finite S-module (!) and since S is Noetherian, we see that this implies $K = 0$. Namely, the above implies K maps to zero in the $\mathfrak{m}S$-adic completion of $I \otimes_R M$. But the map from S to its $\mathfrak{m}S$-adic completion is faithfully flat by Lemma 10.97.3. Hence $K = 0$, as desired. □

In the following we often encounter the conditions "M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$". The following lemma gives some consequences of these conditions (it is a generalization of Lemma 10.99.6).

051C **Lemma 10.99.8.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$ then*

(1) *M/I^nM is flat over R/I^n for all $n \geq 1$, and*
(2) *for any module N which is annihilated by I^m for some $m \geq 0$ we have $Tor_1^R(N, M) = 0$.*

In particular, if I is nilpotent, then M is flat over R.

Proof. Assume M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$. Let N be an R/I-module. Choose a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{i \in I} R/I \to N \to 0$$

By the long exact sequence of Tor and the vanishing of $\mathrm{Tor}_1^R(R/I, M)$ we get

$$0 \to \mathrm{Tor}_1^R(N, M) \to K \otimes_R M \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M \to N \otimes_R M \to 0$$

But since K, $\bigoplus_{i \in I} R/I$, and N are all annihilated by I we see that

$$\begin{aligned} K \otimes_R M &= K \otimes_{R/I} M/IM, \\ (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M &= (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM, \\ N \otimes_R M &= N \otimes_{R/I} M/IM. \end{aligned}$$

As M/IM is flat over R/I we conclude that

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM \to N \otimes_{R/} M/IM \to 0$$

is exact. Combining this with the above we conclude that $\mathrm{Tor}_1^R(N, M) = 0$ for any R-module N annihilated by I.

Let us prove (2) by induction on m. The case $m = 1$ was done in the previous paragraph. For N annihilated by I^m for $m > 1$ we may choose an exact sequence $0 \to N' \to N \to N'' \to 0$ with N' and N'' annihilated by I^{m-1}. For example one can take $N' = IN$ and $N'' = N/IN$. Then the exact sequence

$$\mathrm{Tor}_1^R(N', M) \to \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}_1^R(N'', M)$$

and induction prove the vanishing we want.

Finally, we prove (1). Given $n \geq 1$ we have to show that M/I^nM is flat over R/I^n. In other words, we have to show that the functor $N \mapsto N \otimes_{R/I^n} M/I^nM$ is exact on the category of R-modules N annihilated by I^n. However, for such N we have $N \otimes_{R/I^n} M/I^nM = N \otimes_R M$. By the vanishing of Tor_1 in (2) we see that the functor $N \mapsto N \otimes_R M$ is exact on the category of N annihilated by some power of I and we conclude. □

0AS8 **Lemma 10.99.9.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module.*

(1) *If M/IM is flat over R/I and $M \otimes_R I/I^2 \to IM/I^2M$ is injective, then M/I^2M is flat over R/I^2.*
(2) *If M/IM is flat over R/I and $M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M$ is injective for $n = 1, \ldots, k$, then $M/I^{k+1}M$ is flat over R/I^{k+1}.*

Proof. The first statement is a consequence of Lemma 10.99.8 applied with R replaced by R/I^2 and M replaced by M/I^2M using that

$$\mathrm{Tor}_1^{R/I^2}(M/I^2M, R/I) = \mathrm{Ker}(M \otimes_R I/I^2 \to IM/I^2M),$$

see Remark 10.75.9. The second statement follows in the same manner using induction on n to show that $M/I^{n+1}M$ is flat over R/I^{n+1} for $n = 1, \ldots, k$. Here we use that

$$\mathrm{Tor}_1^{R/I^{n+1}}(M/I^{n+1}M, R/I^n) = \mathrm{Ker}(M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M)$$

for every n. □

00ML **Lemma 10.99.10** (Variant of the local criterion)**.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal in R. Let M be a finite S-module. If $Tor_1^R(M, R/I) = 0$ and M/IM is flat over R/I, then M is flat over R.*

Proof. First proof: By Lemma 10.99.8 we see that $\mathrm{Tor}_1^R(\kappa, M)$ is zero where κ is the residue field of R. Hence we see that M is flat over R by Lemma 10.99.7.

Second proof: Let $\mathfrak{m}$ be the maximal ideal of R. We will show that $\mathfrak{m} \otimes_R M \to M$ is injective, and then apply Lemma 10.99.7. Suppose that $\sum f_i \otimes x_i \in \mathfrak{m} \otimes_R M$ and that $\sum f_i x_i = 0$ in M. By the equational criterion for flatness Lemma 10.39.11 applied to M/IM over R/I we see there exist $\overline{a}_{ij} \in R/I$ and $\overline{y}_j \in M/IM$ such that $x_i \bmod IM = \sum_j \overline{a}_{ij}\overline{y}_j$ and $0 = \sum_i (f_i \bmod I)\overline{a}_{ij}$. Let $a_{ij} \in R$ be a lift of $\overline{a}_{ij}$ and similarly let $y_j \in M$ be a lift of $\overline{y}_j$. Then we see that

$$\begin{aligned} \sum f_i \otimes x_i &= \sum f_i \otimes x_i + \sum f_i a_{ij} \otimes y_j - \sum f_i \otimes a_{ij} y_j \\ &= \sum f_i \otimes (x_i - \sum a_{ij} y_j) + \sum (\sum f_i a_{ij}) \otimes y_j \end{aligned}$$

Since $x_i - \sum a_{ij}y_j \in IM$ and $\sum f_i a_{ij} \in I$ we see that there exists an element in $I \otimes_R M$ which maps to our given element $\sum f_i \otimes x_i$ in $\mathfrak{m} \otimes_R M$. But $I \otimes_R M \to M$ is injective by assumption (see Remark 10.75.9) and we win. □

In particular, in the situation of Lemma 10.99.10, suppose that $I = (x)$ is generated by a single element x which is a nonzerodivisor in R. Then $\mathrm{Tor}_1^R(M, R/(x)) = (0)$ if and only if x is a nonzerodivisor on M.

0523 **Lemma 10.99.11.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *M is a finite S-module, and*
(4) *for each $n \geq 1$ the module M/I^nM is flat over R/I^n.*

Then for every $\mathfrak{q} \in V(IS)$ the localization $M_\mathfrak{q}$ is flat over R. In particular, if S is local and IS is contained in its maximal ideal, then M is flat over R.

Proof. We are going to use Lemma 10.99.10. By assumption M/IM is flat over R/I. Hence it suffices to check that $\mathrm{Tor}_1^R(M, R/I)$ is zero on localization at $\mathfrak{q}$. By Remark 10.75.9 this Tor group is equal to $K = \mathrm{Ker}(I \otimes_R M \to M)$. We know that the kernel of $I/I^n \otimes_{R/I^n} M/I^nM \to M/I^nM$ is zero for all $n \geq 1$. Hence an element of K maps to zero in $I/I^n \otimes_{R/I^n} M/I^nM$. Since

$$I/I^n \otimes_{R/I^n} M/I^nM = I/I^n \otimes_R M = (I \otimes_R M)/I^{n-1}(I \otimes_R M)$$

we conclude that $K \subset I^{n-1}(I \otimes_R M)$ for all $n \geq 1$. By the Artin-Rees lemma, and more precisely Lemma 10.51.5 we conclude that $K_\mathfrak{q} = 0$, as desired. □

00MM **Lemma 10.99.12.** *Let $R \to R' \to R''$ be ring maps. Let M be an R-module. Suppose that $M \otimes_R R'$ is flat over R'. Then the natural map $Tor_1^R(M, R') \otimes_{R'} R'' \to Tor_1^R(M, R'')$ is onto.*

Proof. Let $F_\bullet$ be a free resolution of M over R. The complex $F_2 \otimes_R R' \to F_1 \otimes_R R' \to F_0 \otimes_R R'$ computes $\mathrm{Tor}_1^R(M, R')$. The complex $F_2 \otimes_R R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R''$ computes $\mathrm{Tor}_1^R(M, R'')$. Note that $F_i \otimes_R R' \otimes_{R'} R'' = F_i \otimes_R R''$. Let $K' = \mathrm{Ker}(F_1 \otimes_R R' \to F_0 \otimes_R R')$ and similarly $K'' = \mathrm{Ker}(F_1 \otimes_R R'' \to F_0 \otimes_R R'')$. Thus we have an exact sequence

$$0 \to K' \to F_1 \otimes_R R' \to F_0 \otimes_R R' \to M \otimes_R R' \to 0.$$

By the assumption that $M \otimes_R R'$ is flat over R', the sequence

$$K' \otimes_{R'} R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R'' \to M \otimes_R R'' \to 0$$

is still exact. This means that $K' \otimes_{R'} R'' \to K''$ is surjective. Since $\mathrm{Tor}_1^R(M, R')$ is a quotient of K' and $\mathrm{Tor}_1^R(M, R'')$ is a quotient of K'' we win. □

00MN **Lemma 10.99.13.** *Let $R \to R'$ be a ring map. Let $I \subset R$ be an ideal and $I' = IR'$. Let M be an R-module and set $M' = M \otimes_R R'$. The natural map $Tor_1^R(R'/I', M) \to Tor_1^{R'}(R'/I', M')$ is surjective.*

Proof. Let $F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M over R. Set $F_i' = F_i \otimes_R R'$. The sequence $F_2' \to F_1' \to F_0' \to M' \to 0$ may no longer be exact at F_1'. A free resolution of M' over R' therefore looks like

$$F_2' \oplus F_2'' \to F_1' \to F_0' \to M' \to 0$$

for a suitable free module F_2'' over R'. Next, note that $F_i \otimes_R R'/I' = F_i'/IF_i' = F_i'/I'F_i'$. So the complex $F_2'/I'F_2' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^R(M, R'/I')$. On the other hand $F_i' \otimes_{R'} R'/I' = F_i'/I'F_i'$ and similarly for F_2''. Thus the complex $F_2'/I'F_2' \oplus F_2''/I'F_2'' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^{R'}(M', R'/I')$. Since the vertical map on complexes

$$\begin{array}{ccccc} F_2'/I'F_2' & \longrightarrow & F_1'/I'F_1' & \longrightarrow & F_0'/I'F_0' \\ \downarrow & & \downarrow & & \downarrow \\ F_2'/I'F_2' \oplus F_2''/I'F_2'' & \longrightarrow & F_1'/I'F_1' & \longrightarrow & F_0'/I'F_0' \end{array}$$

clearly induces a surjection on cohomology we win. □

00MO **Lemma 10.99.14.** *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

be a commutative diagram of local homomorphisms of local Noetherian rings. Let $I \subset R$ be a proper ideal. Let M be a finite S-module. Denote $I' = IR'$ and $M' = M \otimes_S S'$. Assume that

(1) *S' is a localization of the tensor product $S \otimes_R R'$,*
(2) *M/IM is flat over R/I,*
(3) *$Tor_1^R(M, R/I) \to Tor_1^{R'}(M', R'/I')$ is zero.*

Then M' is flat over R'.

Proof. Since S' is a localization of $S \otimes_R R'$ we see that M' is a localization of $M \otimes_R R'$. Note that by Lemma 10.39.7 the module $M/IM \otimes_{R/I} R'/I' = M \otimes_R R'/I'(M \otimes_R R')$ is flat over R'/I'. Hence also $M'/I'M'$ is flat over R'/I' as the localization of a flat module is flat. By Lemma 10.99.10 it suffices to show that $\mathrm{Tor}_1^{R'}(M', R'/I')$ is zero. Since M' is a localization of $M \otimes_R R'$, the last assumption implies that it suffices to show that $\mathrm{Tor}_1^R(M, R/I) \otimes_R R' \to \mathrm{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective.

By Lemma 10.99.13 we see that $\mathrm{Tor}_1^R(M, R'/I') \to \mathrm{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective. So now it suffices to show that $\mathrm{Tor}_1^R(M, R/I) \otimes_R R' \to \mathrm{Tor}_1^R(M, R'/I')$ is surjective. This follows from Lemma 10.99.12 by looking at the ring maps $R \to R/I \to R'/I'$ and the module M. □

Please compare the lemma below to Lemma 10.101.8 (the case of a nilpotent ideal) and Lemma 10.128.8 (the case of finitely presented algebras).

00MP **Lemma 10.99.15** (Critère de platitude par fibres; Noetherian case)**.** *Let R, S, S' be Noetherian local rings and let $R \to S \to S'$ be local ring homomorphisms. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let M be an S'-module. Assume*

(1) *The module M is finite over S'.*
(2) *The module M is not zero.*
(3) *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
(4) *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. Set $I = \mathfrak{m}S \subset S$. Then we see that M/IM is a flat S/I-module because of (3). Since $\mathfrak{m} \otimes_R S' \to I \otimes_S S'$ is surjective we see that also $\mathfrak{m} \otimes_R M \to I \otimes_S M$ is surjective. Consider

$$\mathfrak{m} \otimes_R M \to I \otimes_S M \to M.$$

As M is flat over R the composition is injective and so both arrows are injective. In particular $\mathrm{Tor}_1^S(S/I, M) = 0$ see Remark 10.75.9. By Lemma 10.99.10 we conclude that M is flat over S. Note that since $M/\mathfrak{m}_{S'}M$ is not zero by Nakayama's Lemma 10.20.1 we see that actually M is faithfully flat over S by Lemma 10.39.15 (since it forces $M/\mathfrak{m}_S M \neq 0$).

Consider the exact sequence $0 \to \mathfrak{m} \to R \to \kappa \to 0$. This gives an exact sequence $0 \to \mathrm{Tor}_1^R(\kappa, S) \to \mathfrak{m} \otimes_R S \to I \to 0$. Since M is flat over S this gives an exact sequence $0 \to \mathrm{Tor}_1^R(\kappa, S) \otimes_S M \to \mathfrak{m} \otimes_R M \to I \otimes_S M \to 0$. By the above this implies that $\mathrm{Tor}_1^R(\kappa, S) \otimes_S M = 0$. Since M is faithfully flat over S this implies that $\mathrm{Tor}_1^R(\kappa, S) = 0$ and we conclude that S is flat over R by Lemma 10.99.7. □

0H7N **Lemma 10.99.16.** *Let A be a ring, let M be an A-module, and let $f \in A$. If*

(1) *f is a nonzerodivisor on A and M,*
(2) *M_f is a flat A_f-module, and*
(3) *M/fM is a flat A/fA-module,*

Then M is a flat A-module. Same with "flat" replaced by "faithfully flat".

Proof. Since $0 \to A \to A \to A/fA \to 0$ and $0 \to M \to M \to M/fM \to 0$ are exact, we find that $\mathrm{Tor}_i^A(M, A/fA) = 0$ for $i = 1$ and $i = 2$. By Lemma 10.99.8 we conclude that $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N annihilated by f (this uses the flatness of M/fM over A/fA). Given an A-module N annihilated by f we may choose a short exact sequence $0 \to N' \to F \to N \to 0$ of A-modules where F is a direct sum of copies of A/fA. From the exact sequence

$$\mathrm{Tor}_2^A(M, F) \to \mathrm{Tor}_2^A(M, N) \to \mathrm{Tor}_1^A(M, N')$$

we conclude that $\mathrm{Tor}_2^A(M, N) = 0$ for all A-modules N annihilated by f. Next, let K be an arbitrary A-module. We may break the map $f : K \to K$ into two short exact sequences

$$0 \to K[f] \to K \to K' \to 0 \quad \text{and} \quad 0 \to K' \to K \to K/fK \to 0$$

where $K' = K/K[f] \cong fK$. Applying the exact sequences of Tor we obtain exact sequences

$$\mathrm{Tor}_1^A(K[f], M) \to \mathrm{Tor}_1^A(K, M) \to \mathrm{Tor}_1^A(K', M)$$

and

$$\mathrm{Tor}_2^A(K/fK, M) \to \mathrm{Tor}_1^A(K', M) \to \mathrm{Tor}_1^A(K, M)$$

Using the vanishing of $\mathrm{Tor}_1^A(K[f], M)$ and $\mathrm{Tor}_2^A(K/fK, M)$ we conclude that $f : K \to K$ induces an injective map on $\mathrm{Tor}_1^A(M, K)$. In other words, we see that $\mathrm{Tor}_1^A(M, K)$ is zero if and only if $\mathrm{Tor}_1^A(M, K) \otimes_A A_f$ is zero. By Lemma 10.76.1 we have

$$\mathrm{Tor}_1^A(M, K) \otimes_A A_f = \mathrm{Tor}_1^{A_f}(M_f, K_f) = 0$$

The vanishing by the flatness of M_f over A_f (Lemma 10.75.8). We conclude that $\mathrm{Tor}_1^A(M, K) = 0$ for all A-modules K. Hence M is flat over A by Lemma 10.75.8.

We omit the argument for the case of faithfully flat modules. □

0H7P **Lemma 10.99.17.** *Let A be a ring, let M be an A-module. Let $I = (f_1, \ldots, f_r)$ be an ideal of A generated by $r \geq 1$ elements. If*

(1) *M_{f_i} is a flat A_{f_i}-module for $i = 1, \ldots, r$,*
(2) *M/IM is a flat A/I-module,*
(3) *$Tor_i^A(M, A/I) = 0$ for $i = 1, \ldots, r+1$.*

Then M is a flat A-module. Same with "flat" replaced by "faithfully flat".

Proof. From Lemma 10.99.8 we see that $\mathrm{Tor}_1^A(M, K) = 0$ for all A-modules K annihilated by I. Given an A-module K annihilated by I we may choose a short exact sequence $0 \to N \to F \to K \to 0$ of A-modules where F is a direct sum of copies of A/fA. We obtain an exact sequence

$$\mathrm{Tor}_i^A(M, F) \to \mathrm{Tor}_i^A(M, K) \to \mathrm{Tor}_{i-1}^A(M, N)$$

Thus using assumption (3) and induction on i we conclude that $\mathrm{Tor}_i^A(M, K) = 0$ for all A-modules K annihilated by I and $i = 1, \ldots, r+1$.

Suppose that for some $1 \leq j \leq r$ we have shown that $\mathrm{Tor}_i^A(M, K) = 0$ for all A-modules K annihilated by $f_1, \ldots, f_j$ and $i = 1, \ldots, j+1$. Let K be an A-module annihilated by $f_1, \ldots, f_{j-1}$. We may break the map $f_j : K \to K$ into two short exact sequences

$$0 \to K[f_j] \to K \to K' \to 0 \quad \text{and} \quad 0 \to K' \to K \to K/f_jK \to 0$$

where $K' = K/K[f_j] \cong f_jK$. Let $1 \leq i \leq j$. Applying the exact sequences of Tor we obtain exact sequences

$$\mathrm{Tor}_i^A(K[f_j], M) \to \mathrm{Tor}_i^A(K, M) \to \mathrm{Tor}_i^A(K', M)$$

and

$$\mathrm{Tor}_{i+1}^A(K/f_jK, M) \to \mathrm{Tor}_i^A(K', M) \to \mathrm{Tor}_i^A(K, M)$$

Using the vanishing of $\mathrm{Tor}_i^A(K[f_j], M)$ and $\mathrm{Tor}_{i+1}^A(K/f_jK, M)$ we conclude that $f_j : K \to K$ induces an injective map on $\mathrm{Tor}_i^A(M, K)$. In other words, we see that $\mathrm{Tor}_i^A(M, K)$ is zero if and only if $\mathrm{Tor}_i^A(M, K) \otimes_A A_{f_j}$ is zero. By Lemma 10.76.1 we have

$$\mathrm{Tor}_i^A(M, K) \otimes_A A_{f_j} = \mathrm{Tor}_i^{A_{f_j}}(M_{f_j}, K_{f_j}) = 0$$

The vanishing by the flatness of M_{f_j} over A_{f_j} (Lemma 10.75.8). We conclude that $\mathrm{Tor}_i^A(M, K) = 0$ for all A-modules K annihilated by $f_1, \ldots, f_{j-1}$ and $i = 1, \ldots, j$. By descending induction on j, we conclude that this holds for $j = 0$, i.e., we see that $\mathrm{Tor}_1^A(M, K) = 0$ for all A-modules K . Hence M is flat over A by Lemma 10.75.8.

We omit the argument for the case of faithfully flat modules. □

10.100. Base change and flatness

051D Some lemmas which deal with what happens with flatness when doing a base change.

00MQ **Lemma 10.100.1.** *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

be a commutative diagram of local homomorphisms of local rings. Assume that S' is a localization of the tensor product $S \otimes_R R'$. Let M be an S-module and set $M' = S' \otimes_S M$.

(1) *If M is flat over R then M' is flat over R'.*
(2) *If M' is flat over R' and $R \to R'$ is flat then M is flat over R.*

In particular we have

(3) *If S is flat over R then S' is flat over R'.*
(4) *If $R' \to S'$ and $R \to R'$ are flat then S is flat over R.*

Proof. Proof of (1). If M is flat over R, then $M \otimes_R R'$ is flat over R' by Lemma 10.39.7. If $W \subset S \otimes_R R'$ is the multiplicative subset such that $W^{-1}(S \otimes_R R') = S'$ then $M' = W^{-1}(M \otimes_R R')$. Hence M' is flat over R' as the localization of a flat module, see Lemma 10.39.18 part (5). This proves (1) and in particular, we see that (3) holds.

Proof of (2). Suppose that M' is flat over R' and $R \to R'$ is flat. By (3) applied to the diagram reflected in the northwest diagonal we see that $S \to S'$ is flat. Thus $S \to S'$ is faithfully flat by Lemma 10.39.17. We are going to use the criterion of Lemma 10.39.5 (3) to show that M is flat. Let $I \subset R$ be an ideal. If $I \otimes_R M \to M$ has a kernel, so does $(I \otimes_R M) \otimes_S S' \to M \otimes_S S' = M'$. Note that $I \otimes_R R' = IR'$ as $R \to R'$ is flat, and that

$$(I \otimes_R M) \otimes_S S' = (I \otimes_R R') \otimes_{R'} (M \otimes_S S') = IR' \otimes_{R'} M'.$$

From flatness of M' over R' we conclude that this maps injectively into M'. This concludes the proof of (2), and hence (4) is true as well. □

Here is yet another application of the local criterion of flatness.

0GEB **Lemma 10.100.2.** *Consider a commutative diagram of local rings and local homomorphisms*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

Let M be a finite S-module. Assume that

(1) *the horizontal arrows are flat ring maps*
(2) *M is flat over R,*
(3) *$\mathfrak{m}_R R' = \mathfrak{m}_{R'}$,*
(4) *R' and S' are Noetherian.*

Then $M' = M \otimes_S S'$ is flat over R'.

Proof. Since $\mathfrak{m}_R \subset R$ and $R \to R'$ is flat, we get $\mathfrak{m}_R \otimes_R R' = \mathfrak{m}_R R' = \mathfrak{m}_{R'}$ by assumption (3). Observe that M' is a finite S'-module which is flat over R by Lemma 10.39.9. Thus $\mathfrak{m}_R \otimes_R M' \to M'$ is injective. Then we get

$$\mathfrak{m}_R \otimes_R M' = \mathfrak{m}_R \otimes_R R' \otimes_{R'} M' = \mathfrak{m}_{R'} \otimes_{R'} M'$$

Thus $\mathfrak{m}_{R'} \otimes_{R'} M' \to M'$ is injective. This shows that $\mathrm{Tor}_1^{R'}(\kappa_{R'}, M') = 0$ (Remark 10.75.9). Thus M' is flat over R' by Lemma 10.99.7. □

10.101. Flatness criteria over Artinian rings

051E We discuss some flatness criteria for modules over Artinian rings. Note that an Artinian local ring has a nilpotent maximal ideal so that the following two lemmas apply to Artinian local rings.

051F **Lemma 10.101.1.** *Let $(R, \mathfrak{m})$ be a local ring with nilpotent maximal ideal $\mathfrak{m}$. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:*

(1) *$\{\overline{x}_\alpha\}_{\alpha\in A}$ forms a basis for the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and*
(2) *$\{x_\alpha\}_{\alpha\in A}$ forms a basis for M over R.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). By Nakayama's Lemma 10.20.1 the elements x_α generate M. Then one gets a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{\alpha\in A} R \to M \to 0$$

Tensoring with $R/\mathfrak{m}$ and using Lemma 10.39.12 we obtain $K/\mathfrak{m}K = 0$. By Nakayama's Lemma 10.20.1 we conclude $K = 0$. □

051G **Lemma 10.101.2.** *Let R be a local ring with nilpotent maximal ideal. Let M be an R-module. The following are equivalent*

(1) *M is flat over R,*
(2) *M is a free R-module, and*
(3) *M is a projective R-module.*

Proof. Since any projective module is flat (as a direct summand of a free module) and every free module is projective, it suffices to prove that a flat module is free. Let M be a flat module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be elements such that $\overline{x_\alpha} \in M/\mathfrak{m}M$ forms a basis over the residue field of R. By Lemma 10.101.1 the x_α are a basis for M over R and we win. □

051H **Lemma 10.101.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be a collection of elements of M. Assume*

(1) *I is nilpotent,*
(2) *$\{\overline{x}_\alpha\}_{\alpha\in A}$ forms a basis for M/IM over R/I, and*
(3) *$Tor_1^R(R/I, M) = 0$.*

Then M is free on $\{x_\alpha\}_{\alpha\in A}$ over R.

Proof. Let R, I, M, $\{x_\alpha\}_{\alpha\in A}$ be as in the lemma and satisfy assumptions (1), (2), and (3). By Nakayama's Lemma 10.20.1 the elements x_α generate M over R. The assumption $\mathrm{Tor}_1^R(R/I, M) = 0$ implies that we have a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Let $\sum f_\alpha x_\alpha = 0$ be a relation in M. By choice of x_α we see that $f_\alpha \in I$. Hence we conclude that $\sum f_\alpha \otimes x_\alpha = 0$ in $I \otimes_R M$. The map $I \otimes_R M \to I/I^2 \otimes_{R/I} M/IM$ and the fact that $\{x_\alpha\}_{\alpha\in A}$ forms a basis for M/IM implies that $f_\alpha \in I^2$! Hence we conclude that there are no relations among the images of the x_α in M/I^2M. In other words, we see that M/I^2M is free with basis the images of the x_α. Using the map $I \otimes_R M \to I/I^3 \otimes_{R/I^2} M/I^2M$ we then conclude that $f_\alpha \in I^3$! And so on. Since $I^n = 0$ for some n by assumption (1) we win. □

051I **Lemma 10.101.4.** *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *M/IM is flat over R/I, and*
(2) *$R' \otimes_R M$ is flat over R'.*

Set $I_2 = \varphi^{-1}(\varphi(I^2)R')$. Then M/I_2M is flat over R/I_2.

Proof. We may replace R, M, and R' by R/I_2, M/I_2M, and $R'/\varphi(I)^2R'$. Then $I^2 = 0$ and φ is injective. By Lemma 10.99.8 and the fact that $I^2 = 0$ it suffices to prove that $\mathrm{Tor}_1^R(R/I, M) = K = \mathrm{Ker}(I \otimes_R M \to M)$ is zero. Set $M' = M \otimes_R R'$ and $I' = IR'$. By assumption the map $I' \otimes_{R'} M' \to M'$ is injective. Hence K maps to zero in

$$I' \otimes_{R'} M' = I' \otimes_R M = I' \otimes_{R/I} M/IM.$$

Then $I \to I'$ is an injective map of R/I-modules. Since M/IM is flat over R/I the map

$$I \otimes_{R/I} M/IM \longrightarrow I' \otimes_{R/I} M/IM$$

is injective. This implies that K is zero in $I \otimes_R M = I \otimes_{R/I} M/IM$ as desired. □

051J **Lemma 10.101.5.** *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *I is nilpotent,*
(2) *$R \to R'$ is injective,*
(3) *M/IM is flat over R/I, and*
(4) *$R' \otimes_R M$ is flat over R'.*

Then M is flat over R.

Proof. Define inductively $I_1 = I$ and $I_{n+1} = \varphi^{-1}(\varphi(I_n)^2R')$ for $n \geq 1$. Note that by Lemma 10.101.4 we find that M/I_nM is flat over R/I_n for each $n \geq 1$. It is clear that $\varphi(I_{n+1}) \subset \varphi(I)^{2^n}R'$. Since I is nilpotent we see that $\varphi(I_n) = 0$ for some n. As φ is injective we conclude that $I_n = 0$ for some n and we win. □

Here is the local Artinian version of the local criterion for flatness.

051K **Lemma 10.101.6.** *Let R be an Artinian local ring. Let M be an R-module. Let $I \subset R$ be a proper ideal. The following are equivalent*

(1) *M is flat over R, and*
(2) *M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$.*

Proof. The implication (1) ⇒ (2) follows immediately from the definitions. Assume M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$. By Lemma 10.101.2 this implies that M/IM is free over R/I. Pick a set A and elements $x_\alpha \in M$ such that the images in M/IM form a basis. By Lemma 10.101.3 we conclude that M is free and in particular flat. □

It turns out that flatness descends along injective homomorphism whose source is an Artinian ring.

051L **Lemma 10.101.7.** *Let $R \to S$ be a ring map. Let M be an R-module. Assume*

(1) *R is Artinian*
(2) *$R \to S$ is injective, and*
(3) *$M \otimes_R S$ is a flat S-module.*

Then M is a flat R-module.

Proof. First proof: Let $I \subset R$ be the Jacobson radical of R. Then I is nilpotent and M/IM is flat over R/I as R/I is a product of fields, see Section 10.53. Hence M is flat by an application of Lemma 10.101.5.

Second proof: By Lemma 10.53.6 we may write $R = \prod R_i$ as a finite product of local Artinian rings. This induces similar product decompositions for both R and S. Hence we reduce to the case where R is local Artinian (details omitted).

Assume that $R \to S$, M are as in the lemma satisfying (1), (2), and (3) and in addition that R is local with maximal ideal $\mathfrak{m}$. Let A be a set and $x_\alpha \in A$ be elements such that $\overline{x}_\alpha$ forms a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. By Nakayama's Lemma 10.20.1 we see that the elements x_α generate M as an R-module. Set $N = S \otimes_R M$ and $I = \mathfrak{m}S$. Then $\{1 \otimes x_\alpha\}_{\alpha \in A}$ is a family of elements of N which form a basis for N/IN. Moreover, since N is flat over S we have $\mathrm{Tor}_1^S(S/I, N) = 0$. Thus we conclude from Lemma 10.101.3 that N is free on $\{1 \otimes x_\alpha\}_{\alpha \in A}$. The injectivity of $R \to S$ then guarantees that there cannot be a nontrivial relation among the x_α with coefficients in R. □

Please compare the lemma below to Lemma 10.99.15 (the case of Noetherian local rings), Lemma 10.128.8 (the case of finitely presented algebras), and Lemma 10.128.10 (the case of locally nilpotent ideals).

06A5 **Lemma 10.101.8** (Critère de platitude par fibres: Nilpotent case)**.** *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a nilpotent ideal and M an S'-module. Assume

(1) *The module M/IM is a flat S/IS-module.*
(2) *The module M is a flat R-module.*

Then M is a flat S-module and $S_\mathfrak{q}$ is flat over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. As M is flat over R tensoring with the short exact sequence $0 \to I \to R \to R/I \to 0$ gives a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Note that $I \otimes_R M \to IS \otimes_S M$ is surjective. Combined with the above this means both maps in

$$I \otimes_R M \to IS \otimes_S M \to M$$

are injective. Hence $\mathrm{Tor}_1^S(IS, M) = 0$ (see Remark 10.75.9) and we conclude that M is a flat S-module by Lemma 10.99.8. To finish we need to show that $S_\mathfrak{q}$ is flat over R for any prime $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero. This follows from Lemma 10.39.15 and 10.39.10. □

10.102. What makes a complex exact?

00MR Some of this material can be found in the paper [**BE73**] by Buchsbaum and Eisenbud.

00MS **Situation 10.102.1.** Here R is a ring, and we have a complex

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} R^{n_0}$$

In other words we require $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, e-1$.

00MT **Lemma 10.102.2.** *Suppose R is a ring. Let*

$$\ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots$$

be a complex of finite free R-modules. Suppose that for some i some matrix coefficient of the map φ_i is invertible. Then the displayed complex is isomorphic to the direct sum of a complex

$$\ldots \to R^{n_{i+2}} \xrightarrow{\varphi_{i+2}} R^{n_{i+1}} \to R^{n_i - 1} \to R^{n_{i-1} - 1} \to R^{n_{i-2}} \xrightarrow{\varphi_{i-2}} R^{n_{i-3}} \to \ldots$$

and the complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ where the map $R \to R$ is the identity map.

Proof. The assumption means, after a change of basis of R^{n_i} and $R^{n_{i-1}}$ that the first basis vector of R^{n_i} is mapped via φ_i to the first basis vector of $R^{n_{i-1}}$. Let e_j denote the jth basis vector of R^{n_i} and f_k the kth basis vector of $R^{n_{i-1}}$. Write $\varphi_i(e_j) = \sum a_{jk} f_k$. So $a_{1k} = 0$ unless $k = 1$ and $a_{11} = 1$. Change basis on R^{n_i} again by setting $e'_j = e_j - a_{j1} e_1$ for $j > 1$. After this change of coordinates we have $a_{j1} = 0$ for $j > 1$. Note the image of $R^{n_{i+1}} \to R^{n_i}$ is contained in the subspace spanned by e_j, $j > 1$. Note also that $R^{n_{i-1}} \to R^{n_{i-2}}$ has to annihilate f_1 since it is in the image. These conditions and the shape of the matrix (a_{jk}) for φ_i imply the lemma. □

In Situation 10.102.1 we say a complex of the form

$$0 \to \ldots \to 0 \to R \xrightarrow{1} R \to 0 \to \ldots \to 0$$

or of the form

$$0 \to \ldots \to 0 \to R$$

is *trivial*. More precisely, we say $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is trivial if either there exists an $e \geq i \geq 1$ with $n_i = n_{i-1} = 1$, $\varphi_i = \mathrm{id}_R$, and $n_j = 0$ for $j \notin \{i, i-1\}$ or $n_0 = 1$ and $n_i = 0$ for $i > 0$. The lemma above clearly says that any finite complex of finite free modules over a local ring is up to direct sums with trivial complexes the same as a complex all of whose maps have all matrix coefficients in the maximal ideal.

00MY **Lemma 10.102.3.** *In Situation 10.102.1. Suppose R is a local Noetherian ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in Ass(R)$, in other words R has depth 0. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. Pick $x \in R$, $x \neq 0$, with $\mathfrak{m}x = 0$. Let i be the biggest index such that $n_i > 0$. If $i = 0$, then the statement is true. If $i > 0$ denote f_1 the first basis vector of R^{n_i}. Since xf_1 is not mapped to zero by exactness of the complex we deduce that some matrix coefficient of the map $R^{n_i} \to R^{n_{i-1}}$ is not in $\mathfrak{m}$. Lemma 10.102.2 then allows us to decrease $n_e + \ldots + n_1$. Induction finishes the proof. □

00MU **Lemma 10.102.4.** *In Situation 10.102.1. Let R be a Artinian local ring. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. This is a special case of Lemma 10.102.3 because an Artinian local ring has depth 0. □

Below we define the rank of a map of finite free modules. This is just one possible definition of rank. It is just the definition that works in this section; there are others that may be more convenient in other settings.

00MV **Definition 10.102.5.** Let R be a ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules.

(1) The *rank* of φ is the maximal r such that $\wedge^r\varphi : \wedge^r R^m \to \wedge^r R^n$ is nonzero.
(2) We let $I(\varphi) \subset R$ be the ideal generated by the $r \times r$ minors of the matrix of φ, where r is the rank as defined above.

The rank of $\varphi : R^m \to R^n$ is 0 if and only if $\varphi = 0$ and in this case $I(\varphi) = R$.

00MW **Lemma 10.102.6.** *In Situation 10.102.1, suppose the complex is isomorphic to a direct sum of trivial complexes. Then we have*

(1) *the maps φ_i have rank $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,*
(2) *for all i, $1 \leq i \leq e-1$ we have $\text{rank}(\varphi_{i+1}) + \text{rank}(\varphi_i) = n_i$,*
(3) *each $I(\varphi_i) = R$.*

Proof. We may assume the complex is the direct sum of trivial complexes. Then for each i we can split the standard basis elements of R^{n_i} into those that map to a basis element of $R^{n_{i-1}}$ and those that are mapped to zero (and these are mapped onto by basis elements of $R^{n_{i+1}}$ if $i > 0$). Using descending induction starting with $i = e$ it is easy to prove that there are r_{i+1}-basis elements of R^{n_i} which are mapped to zero and r_i which are mapped to basis elements of $R^{n_{i-1}}$. From this the result follows. □

00MZ **Lemma 10.102.7.** *In Situation 10.102.1. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $x \in \mathfrak{m}$ be a nonzerodivisor. The complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$.*

Proof. Denote $F_\bullet$ the complex with terms $F_i = R^{n_i}$ and differential given by φ_i. Then we have a short exact sequence of complexes

$$0 \to F_\bullet \xrightarrow{x} F_\bullet \to F_\bullet/xF_\bullet \to 0$$

Applying the snake lemma we get a long exact sequence

$$H_i(F_\bullet) \xrightarrow{x} H_i(F_\bullet) \to H_i(F_\bullet/xF_\bullet) \to H_{i-1}(F_\bullet) \xrightarrow{x} H_{i-1}(F_\bullet)$$

The lemma follows. □

00N0 **Lemma 10.102.8** (Acyclicity lemma)**.** *Let R be a local Noetherian ring. Let $0 \to M_e \to M_{e-1} \to \ldots \to M_0$ be a complex of finite R-modules. Assume $\text{depth}(M_i) \geq i$. Let i be the largest index such that the complex is not exact at M_i. If $i > 0$ then $\text{Ker}(M_i \to M_{i-1})/\text{Im}(M_{i+1} \to M_i)$ has depth ≥ 1.* [**PS73**, Lemma 1.8]

Proof. Let $H = \mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ be the cohomology group in question. We may break the complex into short exact sequences $0 \to M_e \to M_{e-1} \to K_{e-2} \to 0$, $0 \to K_j \to M_j \to K_{j-1} \to 0$, for $i+2 \leq j \leq e-2$, $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$, $0 \to K_i \to M_i \to M_{i-1}$, and $0 \to B_i \to K_i \to H \to 0$. We proceed up through these complexes to prove the statements about depths, repeatedly using Lemma 10.72.6. First of all, since $\mathrm{depth}(M_e) \geq e$, and $\mathrm{depth}(M_{e-1}) \geq e-1$ we deduce that $\mathrm{depth}(K_{e-2}) \geq e-1$. At this point the sequences $0 \to K_j \to M_j \to K_{j-1} \to 0$ for $i+2 \leq j \leq e-2$ imply similarly that $\mathrm{depth}(K_{j-1}) \geq j$ for $i+2 \leq j \leq e-2$. The sequence $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$ then shows that $\mathrm{depth}(B_i) \geq i+1$. The sequence $0 \to K_i \to M_i \to M_{i-1}$ shows that $\mathrm{depth}(K_i) \geq 1$ since M_i has depth $\geq i \geq 1$ by assumption. The sequence $0 \to B_i \to K_i \to H \to 0$ then implies the result. □

00N1 **Proposition 10.102.9.** *In Situation 10.102.1, suppose R is a local Noetherian ring. The following are equivalent* [**BE73**, Corollary 1]

(1) $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ *is exact at $R^{n_e}, \ldots, R^{n_1}$, and*
(2) *for all i, $1 \leq i \leq e$ the following two conditions are satisfied:*
   (a) *$\mathrm{rank}(\varphi_i) = r_i$ where $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1} n_{e-1} + (-1)^{e-i} n_e$,*
   (b) *$I(\varphi_i) = R$, or $I(\varphi_i)$ contains a regular sequence of length i.*

Proof. If for some i some matrix coefficient of φ_i is not in $\mathfrak{m}$, then we apply Lemma 10.102.2. It is easy to see that the proposition for a complex and for the same complex with a trivial complex added to it are equivalent. Thus we may assume that all matrix entries of each φ_i are elements of the maximal ideal. We may also assume that $e \geq 1$.

Assume the complex is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $\mathfrak{q} \in \mathrm{Ass}(R)$. Note that the ring $R_\mathfrak{q}$ has depth 0 and that the complex remains exact after localization at $\mathfrak{q}$. We apply Lemmas 10.102.3 and 10.102.6 to the localized complex over $R_\mathfrak{q}$. We conclude that $\varphi_{i,\mathfrak{q}}$ has rank r_i for all i. Since $R \to \bigoplus_{\mathfrak{q} \in \mathrm{Ass}(R)} R_\mathfrak{q}$ is injective (Lemma 10.63.19), we conclude that φ_i has rank r_i over R by the definition of rank as given in Definition 10.102.5. Therefore we see that $I(\varphi_i)_\mathfrak{q} = I(\varphi_{i,\mathfrak{q}})$ as the ranks do not change. Since all of the ideals $I(\varphi_i)_\mathfrak{q}$, $e \geq i \geq 1$ are equal to $R_\mathfrak{q}$ (by the lemmas referenced above) we conclude none of the ideals $I(\varphi_i)$ is contained in $\mathfrak{q}$. This implies that $I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$ is not contained in any of the associated primes of R. By Lemma 10.15.2 we may choose $x \in I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(R)$. Observe that x is a nonzerodivisor (Lemma 10.63.9). According to Lemma 10.102.7 the complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$. By induction on e all the ideals $I(\varphi_i)/xR$ have a regular sequence of length $i-1$. This proves that $I(\varphi_i)$ contains a regular sequence of length i.

Assume (2)(a) and (2)(b) hold. We will prove that (1) holds by induction on $\dim(R)$. If $\dim(R) = 0$, then we must have $I(\varphi_i) = R$ for $1 \leq i \leq e$ by (2)(b). Since the coefficients of φ_i are contained in the maximal ideal this can happen only if $r_i = 0$ for all i. By (2)(a) we conclude that $e = 0$ and (1) holds. Assume $\dim(R) > 0$. We claim that for any prime $\mathfrak{p} \subset R$ conditions (2)(a) and (2)(b) hold for the complex $0 \to R_\mathfrak{p}^{n_e} \to R_\mathfrak{p}^{n_{e-1}} \to \ldots \to R_\mathfrak{p}^{n_0}$ with maps $\varphi_{i,\mathfrak{p}}$ over $R_\mathfrak{p}$. Namely, since $I(\varphi_i)$ contains a nonzero divisor, the image of $I(\varphi_i)$ in $R_\mathfrak{p}$ is nonzero. This implies that the rank of $\varphi_{i,\mathfrak{p}}$ is the same as the rank of φ_i: the rank as defined above

of a matrix φ over a ring R can only drop when passing to an R-algebra R' and this happens if and only if $I(\varphi)$ maps to zero in R'. Thus (2)(a) holds. Having said this we know that $I(\varphi_{i,\mathfrak{p}}) = I(\varphi_i)_{\mathfrak{p}}$ and we see that (2)(b) is preserved under localization as well. By induction on the dimension of R we may assume the complex is exact when localized at any nonmaximal prime $\mathfrak{p}$ of R. Thus $\mathrm{Ker}(\varphi_i)/\mathrm{Im}(\varphi_{i+1})$ has support contained in $\{\mathfrak{m}\}$ and hence if nonzero has depth 0. As $I(\varphi_i) \subset \mathfrak{m}$ for all i because of what was said in the first paragraph of the proof, we see that (2)(b) implies $\mathrm{depth}(R) \geq e$. By Lemma 10.102.8 we see that the complex is exact at $R^{n_e}, \ldots, R^{n_1}$ concluding the proof. □

0GLM **Remark 10.102.10.** If in Proposition 10.102.9 the equivalent conditions (1) and (2) are satisfied, then there exists a j such that $I(\varphi_i) = R$ if and only if $i \geq j$. As in the proof of the proposition, it suffices to see this when all the matrices have coefficients in the maximal ideal $\mathfrak{m}$ of R. In this case we see that $I(\varphi_j) = R$ if and only if $\varphi_j = 0$. But if $\varphi_j = 0$, then we get arbitrarily long exact complexes $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_j} \to 0 \to 0 \to \ldots \to 0$ and hence by the proposition we see that $I(\varphi_i)$ for $i > j$ has to be R (since otherwise it is a proper ideal of a Noetherian local ring containing arbitrary long regular sequences which is impossible).

10.103. Cohen-Macaulay modules

00N2 Here we show that Cohen-Macaulay modules have good properties. We postpone using Ext groups to establish the connection with duality and so on.

00N3 **Definition 10.103.1.** Let R be a Noetherian local ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if $\dim(\mathrm{Supp}(M)) = \mathrm{depth}(M)$.

A first goal will be to establish Proposition 10.103.4. We do this by a (perhaps nonstandard) sequence of elementary lemmas involving almost none of the earlier results on depth. Let us introduce some notation.

Let R be a local Noetherian ring. Let M be a Cohen-Macaulay module, and let $f_1, \ldots, f_d$ be an M-regular sequence with $d = \dim(\mathrm{Supp}(M))$. We say that $g \in \mathfrak{m}$ is *good with respect to* $(M, f_1, \ldots, f_d)$ if for all $i = 0, 1, \ldots, d-1$ we have $\dim(\mathrm{Supp}(M) \cap V(g, f_1, \ldots, f_i)) = d - i - 1$. This is equivalent to the condition that $\dim(\mathrm{Supp}(M/(f_1, \ldots, f_i)M) \cap V(g)) = d - i - 1$ for $i = 0, 1, \ldots, d-1$.

00N4 **Lemma 10.103.2.** *Notation and assumptions as above. If g is good with respect to $(M, f_1, \ldots, f_d)$, then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay with maximal regular sequence $f_1, \ldots, f_{d-1}$.*

Proof. We prove the lemma by induction on d. If $d = 0$, then M is finite and there is no case to which the lemma applies. If $d = 1$, then we have to show that $g : M \to M$ is injective. The kernel K has support $\{\mathfrak{m}\}$ because by assumption $\dim \mathrm{Supp}(M) \cap V(g) = 0$. Hence K has finite length. Hence $f_1 : K \to K$ injective implies the length of the image is the length of K, and hence $f_1 K = K$, which by Nakayama's Lemma 10.20.1 implies $K = 0$. Also, $\dim \mathrm{Supp}(M/gM) = 0$ and so M/gM is Cohen-Macaulay of depth 0.

Assume $d > 1$. Observe that g is good for $(M/f_1M, f_2, \ldots, f_d)$, as is easily seen from the definition. By induction, we have that (a) g is a nonzerodivisor on M/f_1M and (b) $M/(g, f_1)M$ is Cohen-Macaulay with maximal regular sequence $f_2, \ldots, f_{d-1}$.

By Lemma 10.68.4 we see that g, f_1 is an M-regular sequence. Hence g is a nonzerodivisor on M and $f_1, \ldots, f_{d-1}$ is an M/gM-regular sequence. $\square$

00N5 **Lemma 10.103.3.** *Let R be a Noetherian local ring. Let M be a Cohen-Macaulay module over R. Suppose $g \in \mathfrak{m}$ is such that $\dim(Supp(M) \cap V(g)) = \dim(Supp(M)) - 1$. Then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay of depth one less.*

Proof. Choose a M-regular sequence $f_1, \ldots, f_d$ with $d = \dim(\text{Supp}(M))$. If g is good with respect to $(M, f_1, \ldots, f_d)$ we win by Lemma 10.103.2. In particular the lemma holds if $d = 1$. (The case $d = 0$ does not occur.) Assume $d > 1$. Choose an element $h \in R$ such that (i) h is good with respect to $(M, f_1, \ldots, f_d)$, and (ii) $\dim(\text{Supp}(M) \cap V(h, g)) = d - 2$. To see h exists, let $\{\mathfrak{q}_j\}$ be the (finite) set of minimal primes of the closed sets $\text{Supp}(M)$, $\text{Supp}(M) \cap V(f_1, \ldots, f_i)$, $i = 1, \ldots, d-1$, and $\text{Supp}(M) \cap V(g)$. None of these $\mathfrak{q}_j$ is equal to $\mathfrak{m}$ and hence we may find $h \in \mathfrak{m}$, $h \notin \mathfrak{q}_j$ by Lemma 10.15.2. It is clear that h satisfies (i) and (ii). From Lemma 10.103.2 we conclude that M/hM is Cohen-Macaulay. By (ii) we see that the pair $(M/hM, g)$ satisfies the induction hypothesis. Hence $M/(h, g)M$ is Cohen-Macaulay and $g : M/hM \to M/hM$ is injective. By Lemma 10.68.4 we see that $g : M \to M$ and $h : M/gM \to M/gM$ are injective. Combined with the fact that $M/(g, h)M$ is Cohen-Macaulay this finishes the proof. $\square$

00N6 **Proposition 10.103.4.** *Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$. Let M be a Cohen-Macaulay module over R whose support has dimension d. Suppose that $g_1, \ldots, g_c$ are elements of $\mathfrak{m}$ such that $\dim(Supp(M/(g_1, \ldots, g_c)M)) = d - c$. Then $g_1, \ldots, g_c$ is an M-regular sequence, and can be extended to a maximal M-regular sequence.*

Proof. Let $Z = \text{Supp}(M) \subset \text{Spec}(R)$. By Lemma 10.60.13 in the chain $Z \supset Z \cap V(g_1) \supset \ldots \supset Z \cap V(g_1, \ldots, g_c)$ each step decreases the dimension at most by 1. Hence by assumption each step decreases the dimension by exactly 1 each time. Thus we may successively apply Lemma 10.103.3 to the modules $M/(g_1, \ldots, g_i)$ and the element g_{i+1}.

To extend $g_1, \ldots, g_c$ by one element if $c < d$ we simply choose an element $g_{c+1} \in \mathfrak{m}$ which is not in any of the finitely many minimal primes of $Z \cap V(g_1, \ldots, g_c)$, using Lemma 10.15.2. $\square$

Having proved Proposition 10.103.4 we continue the development of standard theory.

0C6G **Lemma 10.103.5.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $x \in \mathfrak{m}$ be a nonzerodivisor on M. Then M is Cohen-Macaulay if and only if M/xM is Cohen-Macaulay.*

Proof. By Lemma 10.72.7 we have $\text{depth}(M/xM) = \text{depth}(M) - 1$. By Lemma 10.63.10 we have $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$. $\square$

0AAD **Lemma 10.103.6.** *Let $R \to S$ be a surjective homomorphism of Noetherian local rings. Let N be a finite S-module. Then N is Cohen-Macaulay as an S-module if and only if N is Cohen-Macaulay as an R-module.*

Proof. Omitted. $\square$

0BUS **Lemma 10.103.7.** *Let R be a Noetherian local ring. Let M be a finite Cohen-Macaulay R-module. If $\mathfrak{p} \in Ass(M)$, then $\dim(R/\mathfrak{p}) = \dim(Supp(M))$ and $\mathfrak{p}$ is a minimal prime in the support of M. In particular, M has no embedded associated primes.* [**DG67**, Chapter 0, Proposition 16.5.4]

Proof. By Lemma 10.72.9 we have $\text{depth}(M) \leq \dim(R/\mathfrak{p})$. Of course $\dim(R/\mathfrak{p}) \leq \dim(\text{Supp}(M))$ as $\mathfrak{p} \in \text{Supp}(M)$ (Lemma 10.63.2). Thus we have equality in both inequalities as M is Cohen-Macaulay. Then $\mathfrak{p}$ must be minimal in $\text{Supp}(M)$ otherwise we would have $\dim(R/\mathfrak{p}) < \dim(\text{Supp}(M))$. Finally, minimal primes in the support of M are equal to the minimal elements of $\text{Ass}(M)$ (Proposition 10.63.6) hence M has no embedded associated primes (Definition 10.67.1). □

00NF **Definition 10.103.8.** Let R be a Noetherian local ring. A finite module M over R is called a *maximal Cohen-Macaulay* module if $\text{depth}(M) = \dim(R)$.

In other words, a maximal Cohen-Macaulay module over a Noetherian local ring is a finite module with the largest possible depth over that ring. Equivalently, a maximal Cohen-Macaulay module over a Noetherian local ring R is a Cohen-Macaulay module of dimension equal to the dimension of the ring. In particular, if M is a Cohen-Macaulay R-module with $\text{Spec}(R) = \text{Supp}(M)$, then M is maximal Cohen-Macaulay. Thus the following two lemmas are on maximal Cohen-Macaulay modules.

0AAE **Lemma 10.103.9.** *Let R be a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = Supp(M)$. Then any maximal chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = \dim(R)$.*

Proof. We will prove this by induction on $\dim(R)$. If $\dim(R) = 0$, then the statement is clear. Assume $\dim(R) > 0$. Then $n > 0$. Choose an element $x \in \mathfrak{p}_1$, with x not in any of the minimal primes of R, and in particular $x \notin \mathfrak{p}_0$. (See Lemma 10.15.2.) Then $\dim(R/xR) = \dim(R) - 1$ by Lemma 10.60.13. The module M/xM is Cohen-Macaulay over R/xR by Proposition 10.103.4 and Lemma 10.103.6. The support of M/xM is $\text{Spec}(R/xR)$ by Lemma 10.40.9. After replacing x by x^n for some n, we may assume that $\mathfrak{p}_1$ is an associated prime of M/xM, see Lemma 10.72.8. By Lemma 10.103.7 we conclude that $\mathfrak{p}_1/(x)$ is a minimal prime of R/xR. It follows that the chain $\mathfrak{p}_1/(x) \subset \ldots \subset \mathfrak{p}_n/(x)$ is a maximal chain of primes in R/xR. By induction we find that this chain has length $\dim(R/xR) = \dim(R) - 1$ as desired. □

0AAF **Lemma 10.103.10.** *Suppose R is a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = Supp(M)$. Then for a prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. Follows immediately from Lemma 10.103.9. □

0AAG **Lemma 10.103.11.** *Suppose R is a Noetherian local ring. Let M be a Cohen-Macaulay module over R. For any prime $\mathfrak{p} \subset R$ the module $M_\mathfrak{p}$ is Cohen-Macaulay over $R_\mathfrak{p}$.*

Proof. We may and do assume $\mathfrak{p} \neq \mathfrak{m}$ and M not zero. Choose a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_c \subset \mathfrak{p}_{c-1} \subset \ldots \subset \mathfrak{p}_1 \subset \mathfrak{m}$. If we prove the result for $M_{\mathfrak{p}_1}$ over $R_{\mathfrak{p}_1}$, then the lemma will follow by induction on c. Thus we may assume that there is no

prime strictly between $\mathfrak{p}$ and $\mathfrak{m}$. Note that $\dim(\text{Supp}(M_\mathfrak{p})) \leq \dim(\text{Supp}(M)) - 1$ because any chain of primes in the support of $M_\mathfrak{p}$ can be extended by one more prime (namely $\mathfrak{m}$) in the support of M. On the other hand, we have $\text{depth}(M_\mathfrak{p}) \geq \text{depth}(M) - \dim(R/\mathfrak{p}) = \text{depth}(M) - 1$ by Lemma 10.72.10 and our choice of $\mathfrak{p}$. Thus $\text{depth}(M_\mathfrak{p}) \geq \dim(\text{Supp}(M_\mathfrak{p}))$ as desired (the other inequality is Lemma 10.72.3). □

0AAH **Definition 10.103.12.** Let R be a Noetherian ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if $M_\mathfrak{p}$ is a Cohen-Macaulay module over $R_\mathfrak{p}$ for all primes $\mathfrak{p}$ of R.

By Lemma 10.103.11 it suffices to check this in the maximal ideals of R.

0AAI **Lemma 10.103.13.** *Let R be a Noetherian ring. Let M be a Cohen-Macaulay module over R. Then $M \otimes_R R[x_1, \ldots, x_n]$ is a Cohen-Macaulay module over $R[x_1, \ldots, x_n]$.*

Proof. By induction on the number of variables it suffices to prove this for $M[x] = M \otimes_R R[x]$ over $R[x]$. Let $\mathfrak{m} \subset R[x]$ be a maximal ideal, and let $\mathfrak{p} = R \cap \mathfrak{m}$. Let $f_1, \ldots, f_d$ be a $M_\mathfrak{p}$-regular sequence in the maximal ideal of $R_\mathfrak{p}$ of length $d = \dim(\text{Supp}(M_\mathfrak{p}))$. Note that since $R[x]$ is flat over R the localization $R[x]_\mathfrak{m}$ is flat over $R_\mathfrak{p}$. Hence, by Lemma 10.68.5, the sequence $f_1, \ldots, f_d$ is a $M[x]_\mathfrak{m}$-regular sequence of length d in $R[x]_\mathfrak{m}$. The quotient

$$Q = M[x]_\mathfrak{m}/(f_1, \ldots, f_d)M[x]_\mathfrak{m} = M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R[x]_\mathfrak{m}$$

has support equal to the primes lying over $\mathfrak{p}$ because $R_\mathfrak{p} \to R[x]_\mathfrak{m}$ is flat and the support of $M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p}$ is equal to $\{\mathfrak{p}\}$ (details omitted; hint: follows from Lemmas 10.40.4 and 10.40.5). Hence the dimension is 1. To finish the proof it suffices to find an $f \in \mathfrak{m}$ which is a nonzerodivisor on Q. Since $\mathfrak{m}$ is a maximal ideal, the field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{p})$ is finite (Theorem 10.34.1). Hence we can find $f \in \mathfrak{m}$ which viewed as a polynomial in x has leading coefficient not in $\mathfrak{p}$. Such an f acts as a nonzerodivisor on

$$M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \otimes_R R[x] = \bigoplus\nolimits_{n \geq 0} M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \cdot x^n$$

and hence acts as a nonzerodivisor on Q. □

10.104. Cohen-Macaulay rings

00N7 Most of the results of this section are special cases of the results in Section 10.103.

00N8 **Definition 10.104.1.** A Noetherian local ring R is called *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself.

Note that this is equivalent to requiring the existence of a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_d)$ has dimension 0. We will usually just say "regular sequence" and not "R-regular sequence".

02JN **Lemma 10.104.2.** *Let R be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_c \in \mathfrak{m}$ be elements. Then*

$$x_1, \ldots, x_c \text{ is a regular sequence } \Leftrightarrow \dim(R/(x_1, \ldots, x_c)) = \dim(R) - c$$

If so $x_1, \ldots, x_c$ can be extended to a regular sequence of length $\dim(R)$ and each quotient $R/(x_1, \ldots, x_i)$ is a Cohen-Macaulay ring of dimension $\dim(R) - i$.

Proof. Special case of Proposition 10.103.4. □

00N9 **Lemma 10.104.3.** *Let R be Noetherian local. Suppose R is Cohen-Macaulay of dimension d. Any maximal chain of ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = d$.*

Proof. Special case of Lemma 10.103.9. □

00NA **Lemma 10.104.4.** *Suppose R is a Noetherian local Cohen-Macaulay ring of dimension d. For any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. Follows immediately from Lemma 10.104.3. (Also, this is a special case of Lemma 10.103.10.) □

00NB **Lemma 10.104.5.** *Suppose R is a Cohen-Macaulay local ring. For any prime $\mathfrak{p} \subset R$ the ring $R_\mathfrak{p}$ is Cohen-Macaulay as well.*

Proof. Special case of Lemma 10.103.11. □

00NC **Definition 10.104.6.** A Noetherian ring R is called *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

00ND **Lemma 10.104.7.** *Suppose R is a Noetherian Cohen-Macaulay ring. Any polynomial algebra over R is Cohen-Macaulay.*

Proof. Special case of Lemma 10.103.13. □

00NE **Lemma 10.104.8.** *Let R be a Noetherian local Cohen-Macaulay ring of dimension d. Let $0 \to K \to R^{\oplus n} \to M \to 0$ be an exact sequence of R-modules. Then either $M = 0$, or $\text{depth}(K) > \text{depth}(M)$, or $\text{depth}(K) = \text{depth}(M) = d$.*

Proof. This is a special case of Lemma 10.72.6. □

00NG **Lemma 10.104.9.** *Let R be a local Noetherian Cohen-Macaulay ring of dimension d. Let M be a finite R-module of depth e. There exists an exact complex*

$$0 \to K \to F_{d-e-1} \to \ldots \to F_0 \to M \to 0$$

with each F_i finite free and K maximal Cohen-Macaulay.

Proof. Immediate from the definition and Lemma 10.104.8. □

06LC **Lemma 10.104.10.** *Let $\varphi : A \to B$ be a map of local rings. Assume that B is Noetherian and Cohen-Macaulay and that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$. Then there exists a sequence of elements $f_1, \ldots, f_{\dim(B)}$ in A such that $\varphi(f_1), \ldots, \varphi(f_{\dim(B)})$ is a regular sequence in B.*

Proof. By induction on $\dim(B)$ it suffices to prove: If $\dim(B) \geq 1$, then we can find an element f of A which maps to a nonzerodivisor in B. By Lemma 10.104.2 it suffices to find $f \in A$ whose image in B is not contained in any of the finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. By the assumption that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$ we see that $\mathfrak{m}_A \not\subset \varphi^{-1}(\mathfrak{q}_i)$. Hence we can find f by Lemma 10.15.2. □

10.105. Catenary rings

00NH Compare with Topology, Section 5.11.

00NI **Definition 10.105.1.** A ring R is said to be *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, there exists an integer bounding the lengths of all finite chains of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$ and all maximal such chains have the same length.

02IH **Lemma 10.105.2.** *A ring R is catenary if and only if the topological space* $\mathrm{Spec}(R)$ *is catenary (see Topology, Definition 5.11.4).*

Proof. Immediate from the definition and the characterization of irreducible closed subsets in Lemma 10.26.1. □

In general it is not the case that a finitely generated R-algebra is catenary if R is. Thus we make the following definition.

00NL **Definition 10.105.3.** A Noetherian ring R is said to be *universally catenary* if every R-algebra of finite type is catenary.

We restrict to Noetherian rings as it is not clear this definition is the right one for non-Noetherian rings. By Lemma 10.105.7 to check a Noetherian ring R is universally catenary, it suffices to check each polynomial algebra $R[x_1, \ldots, x_n]$ is catenary.

00NJ **Lemma 10.105.4.** *Any localization of a catenary ring is catenary. Any localization of a Noetherian universally catenary ring is universally catenary.*

Proof. Let A be a ring and let $S \subset A$ be a multiplicative subset. The description of $\mathrm{Spec}(S^{-1}A)$ in Lemma 10.17.5 shows that if A is catenary, then so is $S^{-1}A$. If $S^{-1}A \to C$ is of finite type, then $C = S^{-1}B$ for some finite type ring map $A \to B$. Hence if A is Noetherian and universally catenary, then B is catenary and we see that C is catenary too. This proves the lemma. □

0ECE **Lemma 10.105.5.** *Let A be a Noetherian universally catenary ring. Any A-algebra essentially of finite type over A is universally catenary.*

Proof. If B is a finite type A-algebra, then B is Noetherian by Lemma 10.31.1. Any finite type B-algebra is a finite type A-algebra and hence catenary by our assumption that A is universally catenary. Thus B is universally catenary. Any localization of B is universally catenary by Lemma 10.105.4 and this finishes the proof. □

0AUN **Lemma 10.105.6.** *Let R be a ring. The following are equivalent*

(1) *R is catenary,*
(2) *$R_\mathfrak{p}$ is catenary for all prime ideals $\mathfrak{p}$,*
(3) *$R_\mathfrak{m}$ is catenary for all maximal ideals $\mathfrak{m}$.*

Assume R is Noetherian. The following are equivalent

(1) *R is universally catenary,*
(2) *$R_\mathfrak{p}$ is universally catenary for all prime ideals $\mathfrak{p}$,*
(3) *$R_\mathfrak{m}$ is universally catenary for all maximal ideals $\mathfrak{m}$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 10.105.4 in both cases. The implication (2) $\Rightarrow$ (3) is immediate in both cases. Assume $R_{\mathfrak{m}}$ is catenary for all maximal ideals $\mathfrak{m}$ of R. If $\mathfrak{p} \subset \mathfrak{q}$ are primes in R, then choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m}$. Chains of primes ideals between $\mathfrak{p}$ and $\mathfrak{q}$ are in 1-to-1 correspondence with chains of prime ideals between $\mathfrak{p}R_{\mathfrak{m}}$ and $\mathfrak{q}R_{\mathfrak{m}}$ hence we see R is catenary. Assume R is Noetherian and $R_{\mathfrak{m}}$ is universally catenary for all maximal ideals $\mathfrak{m}$ of R. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over the prime $\mathfrak{p} \subset R$. Choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$ in R. Then $R_{\mathfrak{p}}$ is a localization of $R_{\mathfrak{m}}$ hence universally catenary by Lemma 10.105.4. Then $S_{\mathfrak{p}}$ is catenary as a finite type ring over $R_{\mathfrak{p}}$. Hence $S_{\mathfrak{q}}$ is catenary as a localization. Thus S is catenary by the first case treated above. □

00NK **Lemma 10.105.7.** *Any quotient of a catenary ring is catenary. Any quotient of a Noetherian universally catenary ring is universally catenary.*

Proof. Let A be a ring and let $I \subset A$ be an ideal. The description of $\mathrm{Spec}(A/I)$ in Lemma 10.17.7 shows that if A is catenary, then so is A/I. The second statement is a special case of Lemma 10.105.5. □

0AUP **Lemma 10.105.8.** *Let R be a Noetherian ring.*

(1) *R is catenary if and only if $R/\mathfrak{p}$ is catenary for every minimal prime $\mathfrak{p}$.*
(2) *R is universally catenary if and only if $R/\mathfrak{p}$ is universally catenary for every minimal prime $\mathfrak{p}$.*

Proof. If $\mathfrak{a} \subset \mathfrak{b}$ is an inclusion of primes of R, then we can find a minimal prime $\mathfrak{p} \subset \mathfrak{a}$ and the first assertion is clear. We omit the proof of the second. □

00NM **Lemma 10.105.9.** *A Noetherian Cohen-Macaulay ring is universally catenary. More generally, if R is a Noetherian ring and M is a Cohen-Macaulay R-module with $\mathrm{Supp}(M) = \mathrm{Spec}(R)$, then R is universally catenary.*

Proof. Since a polynomial algebra over R is Cohen-Macaulay, by Lemma 10.104.7, it suffices to show that a Cohen-Macaulay ring is catenary. Let R be Cohen-Macaulay and $\mathfrak{p} \subset \mathfrak{q}$ primes of R. By definition $R_{\mathfrak{q}}$ and $R_{\mathfrak{p}}$ are Cohen-Macaulay. Take a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{q}$. Next choose a maximal chain of primes $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m = \mathfrak{p}$. By Lemma 10.104.3 we have $n + m = \dim(R_{\mathfrak{q}})$. And we have $m = \dim(R_{\mathfrak{p}})$ by the same lemma. Hence $n = \dim(R_{\mathfrak{q}}) - \dim(R_{\mathfrak{p}})$ is independent of choices.

To prove the more general statement, argue exactly as above but using Lemmas 10.103.13 and 10.103.9. □

0ECF **Lemma 10.105.10.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent*

(1) *A is catenary, and*
(2) *$\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function on $\mathrm{Spec}(A)$.*

Proof. If A is catenary, then $\mathrm{Spec}(A)$ has a dimension function δ by Topology, Lemma 5.20.4 (and Lemma 10.105.2). We may assume $\delta(\mathfrak{m}) = 0$. Then we see that

$$\delta(\mathfrak{p}) = \mathrm{codim}(V(\mathfrak{m}), V(\mathfrak{p})) = \dim(A/\mathfrak{p})$$

by Topology, Lemma 5.20.2. In this way we see that (1) implies (2). The reverse implication follows from Topology, Lemma 5.20.2 as well. □

10.106. Regular local rings

00NN Regular local rings are defined in Definition 10.60.10. It is not that easy to show that all prime localizations of a regular local ring are regular. In fact, quite a bit of the material developed so far is geared towards a proof of this fact. See Proposition 10.110.5, and trace back the references.

00NO **Lemma 10.106.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring of dimension d. The graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the graded polynomial algebra $\kappa[X_1, \ldots, X_d]$.*

Proof. Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$, see Definition 10.60.10. There is a surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which maps X_i to the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$. Since $d(R) = d$ by Proposition 10.60.9 we know that the numerical polynomial $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has degree $d - 1$. By Lemma 10.58.10 we conclude that the surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. □

00NP **Lemma 10.106.2.** *Any regular local ring is a domain.*

Proof. We will use that $\bigcap \mathfrak{m}^n = 0$ by Lemma 10.51.4. Let $f, g \in R$ such that $fg = 0$. Suppose that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$, with a, b maximal. Since $fg = 0 \in \mathfrak{m}^{a+b+1}$ we see from the result of Lemma 10.106.1 that either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. Contradiction. □

00NQ **Lemma 10.106.3.** *Let R be a regular local ring and let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence, and each $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d - c$. In particular R is Cohen-Macaulay.*

Proof. Note that R/x_1R is a Noetherian local ring of dimension $\geq d-1$ by Lemma 10.60.13 with $x_2, \ldots, x_d$ generating the maximal ideal. Hence it is a regular local ring by definition. Since R is a domain by Lemma 10.106.2 x_1 is a nonzerodivisor. □

00NR **Lemma 10.106.4.** *Let R be a regular local ring. Let $I \subset R$ be an ideal such that R/I is a regular local ring as well. Then there exists a minimal set of generators $x_1, \ldots, x_d$ for the maximal ideal $\mathfrak{m}$ of R such that $I = (x_1, \ldots, x_c)$ for some $0 \leq c \leq d$.*

Proof. Say $\dim(R) = d$ and $\dim(R/I) = d - c$. Denote $\overline{\mathfrak{m}} = \mathfrak{m}/I$ the maximal ideal of R/I. Let $\kappa = R/\mathfrak{m}$. We have

$$\dim_\kappa((I + \mathfrak{m}^2)/\mathfrak{m}^2) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) - \dim(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = d - (d - c) = c$$

by the definition of a regular local ring. Hence we can choose $x_1, \ldots, x_c \in I$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent and supplement with $x_{c+1}, \ldots, x_d$ to get a minimal system of generators of $\mathfrak{m}$. The induced map $R/(x_1, \ldots, x_c) \to R/I$ is a surjection between regular local rings of the same dimension (Lemma 10.106.3). It follows that the kernel is zero, i.e., $I = (x_1, \ldots, x_c)$. Namely, if not then we would have $\dim(R/I) < \dim(R/(x_1, \ldots, x_c))$ by Lemmas 10.106.2 and 10.60.13. □

00NS **Lemma 10.106.5.** *Let R be a Noetherian local ring. Let $x \in \mathfrak{m}$. Let M be a finite R-module such that x is a nonzerodivisor on M and M/xM is free over R/xR. Then M is free over R.*

Proof. Let $m_1, \ldots, m_r$ be elements of M which map to a R/xR-basis of M/xM. By Nakayama's Lemma 10.20.1 $m_1, \ldots, m_r$ generate M. If $\sum a_i m_i = 0$ is a relation, then $a_i \in xR$ for all i. Hence $a_i = b_i x$ for some $b_i \in R$. Hence the kernel K of $R^r \to M$ satisfies $xK = K$ and hence is zero by Nakayama's lemma. □

00NT **Lemma 10.106.6.** *Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.*

Proof. Let M be a maximal Cohen-Macaulay module over R. Let $x \in \mathfrak{m}$ be part of a regular sequence generating $\mathfrak{m}$. Then x is a nonzerodivisor on M by Proposition 10.103.4, and M/xM is a maximal Cohen-Macaulay module over R/xR. By induction on $\dim(R)$ we see that M/xM is free. We win by Lemma 10.106.5. □

00NU **Lemma 10.106.7.** *Suppose R is a Noetherian local ring. Let $x \in \mathfrak{m}$ be a nonzerodivisor such that R/xR is a regular local ring. Then R is a regular local ring. More generally, if $x_1, \ldots, x_r$ is a regular sequence in R such that $R/(x_1, \ldots, x_r)$ is a regular local ring, then R is a regular local ring.*

Proof. This is true because x together with the lifts of a system of minimal generators of the maximal ideal of R/xR will give $\dim(R)$ generators of $\mathfrak{m}$. Use Lemma 10.60.13. The last statement follows from the first and induction. □

07DX **Lemma 10.106.8.** *Let $(R_i, \varphi_{ii'})$ be a directed system of local rings whose transition maps are local ring maps. If each R_i is a regular local ring and $R = \operatorname{colim} R_i$ is Noetherian, then R is a regular local ring.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal; it is the colimit of the maximal ideal $\mathfrak{m}_i \subset R_i$. We prove the lemma by induction on $d = \dim \mathfrak{m}/\mathfrak{m}^2$. If $d = 0$, then $R = R/\mathfrak{m}$ is a field and R is a regular local ring. If $d > 0$ pick an $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. For some i we can find an $x_i \in \mathfrak{m}_i$ mapping to x. Note that $R/xR = \operatorname{colim}_{i' \geq i} R_{i'}/x_i R_{i'}$ is a Noetherian local ring. By Lemma 10.106.3 we see that $R_{i'}/x_i R_{i'}$ is a regular local ring. Hence by induction we see that R/xR is a regular local ring. Since each R_i is a domain (Lemma 10.106.1) we see that R is a domain. Hence x is a nonzerodivisor and we conclude that R is a regular local ring by Lemma 10.106.7. □

10.107. Epimorphisms of rings

04VM In any category there is a notion of an *epimorphism*. Some of this material is taken from [**Laz69**] and [**Maz68**].

04VN **Lemma 10.107.1.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism,*
(2) *the two ring maps $S \to S \otimes_R S$ are equal,*
(3) *either of the ring maps $S \to S \otimes_R S$ is an isomorphism, and*
(4) *the ring map $S \otimes_R S \to S$ is an isomorphism.*

Proof. Omitted. □

04VP **Lemma 10.107.2.** *The composition of two epimorphisms of rings is an epimorphism.*

Proof. Omitted. Hint: This is true in any category. □

04VQ **Lemma 10.107.3.** *If $R \to S$ is an epimorphism of rings and $R \to R'$ is any ring map, then $R' \to R' \otimes_R S$ is an epimorphism.*

Proof. Omitted. Hint: True in any category with pushouts. □

04VR **Lemma 10.107.4.** *If $A \to B \to C$ are ring maps and $A \to C$ is an epimorphism, so is $B \to C$.*

Proof. Omitted. Hint: This is true in any category. □

This means in particular, that if $R \to S$ is an epimorphism with image $\overline{R} \subset S$, then $\overline{R} \to S$ is an epimorphism. Hence while proving results for epimorphisms we may often assume the map is injective. The following lemma means in particular that every localization is an epimorphism.

04VS **Lemma 10.107.5.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is an epimorphism, and*
(2) *$R_\mathfrak{p} \to S_\mathfrak{p}$ is an epimorphism for each prime $\mathfrak{p}$ of R.*

Proof. Since $S_\mathfrak{p} = R_\mathfrak{p} \otimes_R S$ (see Lemma 10.12.15) we see that (1) implies (2) by Lemma 10.107.3. Conversely, assume that (2) holds. Let $a, b : S \to A$ be two ring maps from S to a ring A equalizing the map $R \to S$. By assumption we see that for every prime $\mathfrak{p}$ of R the induced maps $a_\mathfrak{p}, b_\mathfrak{p} : S_\mathfrak{p} \to A_\mathfrak{p}$ are the same. Hence $a = b$ as $A \subset \prod_\mathfrak{p} A_\mathfrak{p}$, see Lemma 10.23.1. □

04VT **Lemma 10.107.6.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism and finite, and*
(2) *$R \to S$ is surjective.*

Proof. (This lemma seems to have been reproved many times in the literature, and has many different proofs.) It is clear that a surjective ring map is an epimorphism. Suppose that $R \to S$ is a finite ring map such that $S \otimes_R S \to S$ is an isomorphism. Our goal is to show that $R \to S$ is surjective. Assume S/R is not zero. The exact sequence $R \to S \to S/R \to 0$ leads to an exact sequence

$$R \otimes_R S \to S \otimes_R S \to S/R \otimes_R S \to 0.$$

Our assumption implies that the first arrow is an isomorphism, hence we conclude that $S/R \otimes_R S = 0$. Hence also $S/R \otimes_R S/R = 0$. By Lemma 10.5.4 there exists a surjection of R-modules $S/R \to R/I$ for some proper ideal $I \subset R$. Hence there exists a surjection $S/R \otimes_R S/R \to R/I \otimes_R R/I = R/I \neq 0$, contradiction. □

04VU **Lemma 10.107.7.** *A faithfully flat epimorphism is an isomorphism.*

Proof. This is clear from Lemma 10.107.1 part (3) as the map $S \to S \otimes_R S$ is the map $R \to S$ tensored with S. □

04VV **Lemma 10.107.8.** *If $k \to S$ is an epimorphism and k is a field, then $S = k$ or $S = 0$.*

Proof. This is clear from the result of Lemma 10.107.7 (as any nonzero algebra over k is faithfully flat), or by arguing directly that $R \to R \otimes_k R$ cannot be surjective unless $\dim_k(R) \leq 1$. □

04VW **Lemma 10.107.9.** *Let $R \to S$ be an epimorphism of rings. Then*

(1) *$\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective, and*
(2) *for $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$ we have $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$.*

Proof. Let $\mathfrak{p}$ be a prime of R. The fibre of the map is the spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. By Lemma 10.107.3 the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$ is an epimorphism, and hence by Lemma 10.107.8 we have either $S \otimes_R \kappa(\mathfrak{p}) = 0$ or $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})$ which proves (1) and (2). □

04VX **Lemma 10.107.10.** *Let R be a ring. Let M, N be R-modules. Let $\{x_i\}_{i\in I}$ be a set of generators of M. Let $\{y_j\}_{j\in J}$ be a set of generators of N. Let $\{m_j\}_{j\in J}$ be a family of elements of M with $m_j = 0$ for all but finitely many j. Then*

$$\sum\nolimits_{j\in J} m_j \otimes y_j = 0 \text{ in } M \otimes_R N$$

is equivalent to the following: There exist $a_{i,j} \in R$ with $a_{i,j} = 0$ for all but finitely many pairs (i, j) such that

$$m_j = \sum\nolimits_{i\in I} a_{i,j} x_i \quad \text{for all } j \in J,$$
$$0 = \sum\nolimits_{j\in J} a_{i,j} y_j \quad \text{for all } i \in I.$$

Proof. The sufficiency is immediate. Suppose that $\sum_{j\in J} m_j \otimes y_j = 0$. Consider the short exact sequence

$$0 \to K \to \bigoplus\nolimits_{j\in J} R \to N \to 0$$

where the jth basis vector of $\bigoplus_{j\in J} R$ maps to y_j. Tensor this with M to get the exact sequence

$$K \otimes_R M \to \bigoplus\nolimits_{j\in J} M \to N \otimes_R M \to 0.$$

The assumption implies that there exist elements $k_i \in K$ such that $\sum k_i \otimes x_i$ maps to the element $(m_j)_{j\in J}$ of the middle. Writing $k_i = (a_{i,j})_{j\in J}$ and we obtain what we want. □

04VY **Lemma 10.107.11.** *Let $\varphi : R \to S$ be a ring map. Let $g \in S$. The following are equivalent:*

1. *$g \otimes 1 = 1 \otimes g$ in $S \otimes_R S$, and*
2. *there exist $n \geq 0$ and elements $y_i, z_j \in S$ and $x_{i,j} \in R$ for $1 \leq i, j \leq n$ such that*
   (a) *$g = \sum_{i,j\leq n} x_{i,j} y_i z_j$,*
   (b) *for each j we have $\sum x_{i,j} y_i \in \varphi(R)$, and*
   (c) *for each i we have $\sum x_{i,j} z_j \in \varphi(R)$.*

Proof. It is clear that (2) implies (1). Conversely, suppose that $g \otimes 1 = 1 \otimes g$. Choose generators $\{s_i\}_{i\in I}$ of S as an R-module with $0, 1 \in I$ and $s_0 = 1$ and $s_1 = g$. Apply Lemma 10.107.10 to the relation $g \otimes s_0 + (-1) \otimes s_1 = 0$. We see that there exist $a_{i,j} \in R$ such that $g = \sum_i a_{i,0} s_i$, $-1 = \sum_i a_{i,1} s_i$, and for $j \neq 0, 1$ we have $0 = \sum_i a_{i,j} s_i$, and moreover for all i we have $\sum_j a_{i,j} s_j = 0$. Then we have

$$\sum\nolimits_{i,j\neq 0} a_{i,j} s_i s_j = -g + a_{0,0}$$

and for each $j \neq 0$ we have $\sum_{i\neq 0} a_{i,j} s_i \in R$. This proves that $-g + a_{0,0}$ can be written as in (2). It follows that g can be written as in (2). Details omitted. Hint: Show that the set of elements of S which have an expression as in (2) form an R-subalgebra of S. □

04VZ **Remark 10.107.12.** Let $R \to S$ be a ring map. Sometimes the set of elements $g \in S$ such that $g \otimes 1 = 1 \otimes g$ is called the *epicenter* of S. It is an R-algebra. By the construction of Lemma 10.107.11 we get for each g in the epicenter a matrix factorization

$$(g) = YXZ$$

with $X \in \mathrm{Mat}(n \times n, R)$, $Y \in \mathrm{Mat}(1 \times n, S)$, and $Z \in \mathrm{Mat}(n \times 1, S)$. Namely, let $x_{i,j}, y_i, z_j$ be as in part (2) of the lemma. Set $X = (x_{i,j})$, let y be the row vector whose entries are the y_i and let z be the column vector whose entries are the z_j. With this notation conditions (b) and (c) of Lemma 10.107.11 mean exactly that $YX \in \mathrm{Mat}(1 \times n, R)$, $XZ \in \mathrm{Mat}(n \times 1, R)$. It turns out to be very convenient to consider the triple of matrices (X, YX, XZ). Given $n \in \mathbf{N}$ and a triple (P, U, V) we say that (P, U, V) is a *n-triple associated to g* if there exists a matrix factorization as above such that $P = X$, $U = YX$ and $V = XZ$.

04W0 **Lemma 10.107.13.** *Let $R \to S$ be an epimorphism of rings. Then the cardinality of S is at most the cardinality of R. In a formula: $|S| \leq |R|$.*

Proof. The condition that $R \to S$ is an epimorphism means that each $g \in S$ satisfies $g \otimes 1 = 1 \otimes g$, see Lemma 10.107.1. We are going to use the notation introduced in Remark 10.107.12. Suppose that $g, g' \in S$ and suppose that (P, U, V) is an n-triple which is associated to both g and g'. Then we claim that $g = g'$. Namely, write $(P, U, V) = (X, YX, XZ)$ for a matrix factorization $(g) = YXZ$ of g and write $(P, U, V) = (X', Y'X', X'Z')$ for a matrix factorization $(g') = Y'X'Z'$ of g'. Then we see that

$$(g) = YXZ = UZ = Y'X'Z = Y'PZ = Y'XZ = Y'V = Y'X'Z' = (g')$$

and hence $g = g'$. This implies that the cardinality of S is bounded by the number of possible triples, which has cardinality at most $\sup_{n \in \mathbf{N}} |R|^n$. If R is infinite then this is at most $|R|$, see [**Kun83**, Ch. I, 10.13].

If R is a finite ring then the argument above only proves that S is at worst countable. In fact in this case R is Artinian and the map $R \to S$ is surjective. We omit the proof of this case. □

08YS **Lemma 10.107.14.** *Let $R \to S$ be an epimorphism of rings. Let N_1, N_2 be S-modules. Then $\mathrm{Hom}_S(N_1, N_2) = \mathrm{Hom}_R(N_1, N_2)$. In other words, the restriction functor $Mod_S \to Mod_R$ is fully faithful.*

Proof. Let $\varphi : N_1 \to N_2$ be an R-linear map. For any $x \in N_1$ consider the map $S \otimes_R S \to N_2$ defined by the rule $g \otimes g' \mapsto g\varphi(g'x)$. Since both maps $S \to S \otimes_R S$ are isomorphisms (Lemma 10.107.1), we conclude that $g\varphi(g'x) = gg'\varphi(x) = \varphi(gg'x)$. Thus φ is S-linear. □

10.108. Pure ideals

04PQ The material in this section is discussed in many papers, see for example [**Laz67**], [**Bko70**], and [**DM83**].

04PR **Definition 10.108.1.** Let R be a ring. We say that $I \subset R$ is *pure* if the quotient ring R/I is flat over R.

04PS **Lemma 10.108.2.** *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is pure,*
(2) *for every ideal $J \subset R$ we have $J \cap I = IJ$,*
(3) *for every finitely generated ideal $J \subset R$ we have $J \cap I = JI$,*
(4) *for every $x \in R$ we have $(x) \cap I = xI$,*
(5) *for every $x \in I$ we have $x = yx$ for some $y \in I$,*
(6) *for every $x_1, \ldots, x_n \in I$ there exists a $y \in I$ such that $x_i = yx_i$ for all $i = 1, \ldots, n$,*
(7) *for every prime $\mathfrak{p}$ of R we have $IR_\mathfrak{p} = 0$ or $IR_\mathfrak{p} = R_\mathfrak{p}$,*
(8) *$Supp(I) = \mathrm{Spec}(R) \setminus V(I)$,*
(9) *I is the kernel of the map $R \to (1 + I)^{-1}R$,*
(10) *$R/I \cong S^{-1}R$ as R-algebras for some multiplicative subset S of R, and*
(11) *$R/I \cong (1 + I)^{-1}R$ as R-algebras.*

Proof. For any ideal J of R we have the short exact sequence $0 \to J \to R \to R/J \to 0$. Tensoring with R/I we get an exact sequence $J \otimes_R R/I \to R/I \to R/I + J \to 0$ and $J \otimes_R R/I = J/JI$. Thus the equivalence of (1), (2), and (3) follows from Lemma 10.39.5. Moreover, these imply (4).

The implication (4) $\Rightarrow$ (5) is trivial. Assume (5) and let $x_1, \ldots, x_n \in I$. Choose $y_i \in I$ such that $x_i = y_i x_i$. Let $y \in I$ be the element such that $1 - y = \prod_{i=1,\ldots,n}(1 - y_i)$. Then $x_i = yx_i$ for all $i = 1, \ldots, n$. Hence (6) holds, and it follows that (5) $\Leftrightarrow$ (6).

Assume (5). Let $x \in I$. Then $x = yx$ for some $y \in I$. Hence $x(1 - y) = 0$, which shows that x maps to zero in $(1 + I)^{-1}R$. Of course the kernel of the map $R \to (1+I)^{-1}R$ is always contained in I. Hence we see that (5) implies (9). Assume (9). Then for any $x \in I$ we see that $x(1 - y) = 0$ for some $y \in I$. In other words, $x = yx$. We conclude that (5) is equivalent to (9).

Assume (5). Let $\mathfrak{p}$ be a prime of R. If $\mathfrak{p} \notin V(I)$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $\mathfrak{p} \in V(I)$, in other words, if $I \subset \mathfrak{p}$, then $x \in I$ implies $x(1 - y) = 0$ for some $y \in I$, implies x maps to zero in $R_\mathfrak{p}$, i.e., $IR_\mathfrak{p} = 0$. Thus we see that (7) holds.

Assume (7). Then $(R/I)_\mathfrak{p}$ is either 0 or $R_\mathfrak{p}$ for any prime $\mathfrak{p}$ of R. Hence by Lemma 10.39.18 we see that (1) holds. At this point we see that all of (1) – (7) and (9) are equivalent.

As $IR_\mathfrak{p} = I_\mathfrak{p}$ we see that (7) implies (8). Finally, if (8) holds, then this means exactly that $I_\mathfrak{p}$ is the zero module if and only if $\mathfrak{p} \in V(I)$, which is clearly saying that (7) holds. Now (1) – (9) are equivalent.

Assume (1) – (9) hold. Then $R/I \subset (1 + I)^{-1}R$ by (9) and the map $R/I \to (1 + I)^{-1}R$ is also surjective by the description of localizations at primes afforded by (7). Hence (11) holds.

The implication (11) $\Rightarrow$ (10) is trivial. And (10) implies that (1) holds because a localization of R is flat over R, see Lemma 10.39.18. □

04PT **Lemma 10.108.3.** *Let R be a ring. If $I, J \subset R$ are pure ideals, then $V(I) = V(J)$ implies $I = J$.*

Proof. For example, by property (7) of Lemma 10.108.2 we see that $I = \mathrm{Ker}(R \to \prod_{\mathfrak{p} \in V(I)} R_\mathfrak{p})$ can be recovered from the closed subset associated to it. □

04PU **Lemma 10.108.4.** *Let R be a ring. The rule $I \mapsto V(I)$ determines a bijection*

$$\{I \subset R \text{ pure}\} \leftrightarrow \{Z \subset \mathrm{Spec}(R) \text{ closed and closed under generalizations}\}$$

Proof. Let I be a pure ideal. Then since $R \to R/I$ is flat, by going down generalizations lift along the map $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. Hence $V(I)$ is closed under generalizations. This shows that the map is well defined. By Lemma 10.108.3 the map is injective. Suppose that $Z \subset \mathrm{Spec}(R)$ is closed and closed under generalizations. Let $J \subset R$ be the radical ideal such that $Z = V(J)$. Let $I = \{x \in R : x \in xJ\}$. Note that I is an ideal: if $x, y \in I$ then there exist $f, g \in J$ such that $x = xf$ and $y = yg$. Then

$$x + y = (x + y)(f + g - fg)$$

Verification left to the reader. We claim that I is pure and that $V(I) = V(J)$. If the claim is true then the map of the lemma is surjective and the lemma holds.

Note that $I \subset J$, so that $V(J) \subset V(I)$. Let $I \subset \mathfrak{p}$ be a prime. Consider the multiplicative subset $S = (R \setminus \mathfrak{p})(1 + J)$. By definition of I and $I \subset \mathfrak{p}$ we see that $0 \notin S$. Hence we can find a prime $\mathfrak{q}$ of R which is disjoint from S, see Lemmas 10.9.4 and 10.17.5. Hence $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \cap (1 + J) = \emptyset$. This implies that $\mathfrak{q} + J$ is a proper ideal of R. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{q} + J$. Then we get $\mathfrak{m} \in V(J)$ and hence $\mathfrak{q} \in V(J) = Z$ as Z was assumed to be closed under generalization. This in turn implies $\mathfrak{p} \in V(J)$ as $\mathfrak{q} \subset \mathfrak{p}$. Thus we see that $V(I) = V(J)$.

Finally, since $V(I) = V(J)$ (and J radical) we see that $J = \sqrt{I}$. Pick $x \in I$, so that $x = xy$ for some $y \in J$ by definition. Then $x = xy = xy^2 = \ldots = xy^n$. Since $y^n \in I$ for some $n > 0$ we conclude that property (5) of Lemma 10.108.2 holds and we see that I is indeed pure. □

05KK **Lemma 10.108.5.** *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *I is pure and finitely generated,*
(2) *I is generated by an idempotent,*
(3) *I is pure and $V(I)$ is open, and*
(4) *R/I is a projective R-module.*

Proof. If (1) holds, then $I = I \cap I = I^2$ by Lemma 10.108.2. Hence I is generated by an idempotent by Lemma 10.21.5. Thus (1) $\Rightarrow$ (2). If (2) holds, then $I = (e)$ and $R = (1-e) \oplus (e)$ as an R-module hence R/I is flat and I is pure and $V(I) = D(1-e)$ is open. Thus (2) $\Rightarrow$ (1) + (3). Finally, assume (3). Then $V(I)$ is open and closed, hence $V(I) = D(1 - e)$ for some idempotent e of R, see Lemma 10.21.3. The ideal $J = (e)$ is a pure ideal such that $V(J) = V(I)$ hence $I = J$ by Lemma 10.108.3. In this way we see that (3) $\Rightarrow$ (2). By Lemma 10.78.2 we see that (4) is equivalent to the assertion that I is pure and R/I finitely presented. Moreover, R/I is finitely presented if and only if I is finitely generated, see Lemma 10.5.3. Hence (4) is equivalent to (1). □

We can use the above to characterize those rings for which every finite flat module is finitely presented.

052U **Lemma 10.108.6.** *Let R be a ring. The following are equivalent:*

(1) *every $Z \subset \mathrm{Spec}(R)$ which is closed and closed under generalizations is also open, and*
(2) *any finite flat R-module is finite locally free.*

Proof. If any finite flat R-module is finite locally free then the support of R/I where I is a pure ideal is open. Hence the implication (2) $\Rightarrow$ (1) follows from Lemma 10.108.3.

For the converse assume that R satisfies (1). Let M be a finite flat R-module. The support $Z = \mathrm{Supp}(M)$ of M is closed, see Lemma 10.40.5. On the other hand, if $\mathfrak{p} \subset \mathfrak{p}'$, then by Lemma 10.78.5 the module $M_{\mathfrak{p}'}$ is free, and $M_{\mathfrak{p}} = M_{\mathfrak{p}'} \otimes_{R_{\mathfrak{p}'}} R_{\mathfrak{p}}$ Hence $\mathfrak{p}' \in \mathrm{Supp}(M) \Rightarrow \mathfrak{p} \in \mathrm{Supp}(M)$, in other words, the support is closed under generalization. As R satisfies (1) we see that the support of M is open and closed. Suppose that M is generated by r elements $m_1, \ldots, m_r$. The modules $\wedge^i(M)$, $i = 1, \ldots, r$ are finite flat R-modules also, because $\wedge^i(M)_{\mathfrak{p}} = \wedge^i(M_{\mathfrak{p}})$ is free over $R_{\mathfrak{p}}$. Note that $\mathrm{Supp}(\wedge^{i+1}(M)) \subset \mathrm{Supp}(\wedge^i(M))$. Thus we see that there exists a decomposition

$$\mathrm{Spec}(R) = U_0 \amalg U_1 \amalg \ldots \amalg U_r$$

by open and closed subsets such that the support of $\wedge^i(M)$ is $U_r \cup \ldots \cup U_i$ for all $i = 0, \ldots, r$. Let $\mathfrak{p}$ be a prime of R, and say $\mathfrak{p} \in U_i$. Note that $\wedge^i(M) \otimes_R \kappa(\mathfrak{p}) = \wedge^i(M \otimes_R \kappa(\mathfrak{p}))$. Hence, after possibly renumbering $m_1, \ldots, m_r$ we may assume that $m_1, \ldots, m_i$ generate $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's Lemma 10.20.1 we get a surjection

$$R_f^{\oplus i} \longrightarrow M_f, \quad (a_1, \ldots, a_i) \longmapsto \sum a_i m_i$$

for some $f \in R$, $f \notin \mathfrak{p}$. We may also assume that $D(f) \subset U_i$. This means that $\wedge^i(M_f) = \wedge^i(M)_f$ is a flat R_f module whose support is all of $\mathrm{Spec}(R_f)$. By the above it is generated by a single element, namely $m_1 \wedge \ldots \wedge m_i$. Hence $\wedge^i(M)_f \cong R_f/J$ for some pure ideal $J \subset R_f$ with $V(J) = \mathrm{Spec}(R_f)$. Clearly this means that $J = (0)$, see Lemma 10.108.3. Thus $m_1 \wedge \ldots \wedge m_i$ is a basis for $\wedge^i(M_f)$ and it follows that the displayed map is injective as well as surjective. This proves that M is finite locally free as desired. □

10.109. Rings of finite global dimension

00O2 The following lemma is often used to compare different projective resolutions of a given module.

00O3 **Lemma 10.109.1** (Schanuel's lemma)**.** *Let R be a ring. Let M be an R-module. Suppose that*

$$0 \to K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \to 0 \quad \textit{and} \quad 0 \to L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \to 0$$

are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$. More precisely, there exist a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \oplus P_2 & \xrightarrow[(c_1, id)]{} & P_1 \oplus P_2 & \xrightarrow[(p_1, 0)]{} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & P_1 \oplus L & \xrightarrow{(id, c_2)} & P_1 \oplus P_2 & \xrightarrow{(0, p_2)} & M & \longrightarrow & 0 \end{array}$$

whose vertical arrows are isomorphisms.

Proof. Consider the module N defined by the short exact sequence $0 \to N \to P_1 \oplus P_2 \to M \to 0$, where the last map is the sum of the two maps $P_i \to M$. It is easy to see that the projection $N \to P_1$ is surjective with kernel L, and that $N \to P_2$ is surjective with kernel K. Since P_i are projective we have $N \cong K \oplus P_2 \cong L \oplus P_1$. This proves the first statement.

To prove the second statement (and to reprove the first), choose $a : P_1 \to P_2$ and $b : P_2 \to P_1$ such that $p_1 = p_2 \circ a$ and $p_2 = p_1 \circ b$. This is possible because P_1 and P_2 are projective. Then we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \oplus P_2 & \xrightarrow[(c_1,\mathrm{id})]{} & P_1 \oplus P_2 & \xrightarrow[(p_1,0)]{} & M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle T} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & P_1 \oplus P_2 & \xrightarrow[(p_1,p_2)]{} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle S} & & \| & & \\
0 & \longrightarrow & P_1 \oplus L & \xrightarrow{(\mathrm{id},c_2)} & P_1 \oplus P_2 & \xrightarrow{(0,p_2)} & M & \longrightarrow & 0
\end{array}$$

with T and S given by the matrices

$$S = \begin{pmatrix} \mathrm{id} & 0 \\ a & \mathrm{id} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \mathrm{id} & b \\ 0 & \mathrm{id} \end{pmatrix}$$

Then S, T and the maps $N \to P_1 \oplus L$ and $N \to K \oplus P_2$ are isomorphisms as desired. □

00O4 **Definition 10.109.2.** Let R be a ring. Let M be an R-module. We say M has *finite projective dimension* if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the *projective dimension* of M.

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extent the projective dimension is independent of the choice of a projective resolution.

00O5 **Lemma 10.109.3.** *Let R be a ring. Suppose that M is an R-module of projective dimension d. Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d-1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).*

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \mathrm{Ker}(F_0 \to M) \oplus M$, and hence $\mathrm{Ker}(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\mathrm{Ker}(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 10.109.1 we have $P_0 \oplus \mathrm{Ker}(F_0 \to M) \cong F_0 \oplus \mathrm{Ker}(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \mathrm{Ker}(P_0 \to M)$ has the finite projective resolution

$$0 \to P_d \to P_{d-1} \to \ldots \to P_2 \to P_1 \oplus F_0 \to \mathrm{Ker}(P_0 \to M) \oplus F_0 \to 0$$

of length $d-1$. By induction applied to the exact sequence

$$F_e \to F_{e-1} \to \ldots \to F_2 \to P_0 \oplus F_1 \to P_0 \oplus \mathrm{Ker}(F_0 \to M) \to 0$$

of length $e-1$ we conclude $\mathrm{Ker}(F_e \to F_{e-1})$ is projective (if $e \geq 2$) or that $\mathrm{Ker}(F_1 \oplus P_0 \to F_0 \oplus P_0)$ is projective. This implies the lemma. □

0CXC **Lemma 10.109.4.** *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

(1) M *has projective dimension* $\leq d$,
(2) *there exists a resolution* $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ *with* P_i *projective,*
(3) *for some resolution* $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ *with* P_i *projective we have* $\mathrm{Ker}(P_{d-1} \to P_{d-2})$ *is projective if* $d \geq 2$, *or* $\mathrm{Ker}(P_0 \to M)$ *is projective if* $d = 1$, *or* M *is projective if* $d = 0$,
(4) *for any resolution* $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ *with* P_i *projective we have* $\mathrm{Ker}(P_{d-1} \to P_{d-2})$ *is projective if* $d \geq 2$, *or* $\mathrm{Ker}(P_0 \to M)$ *is projective if* $d = 1$, *or* M *is projective if* $d = 0$.

Proof. The equivalence of (1) and (2) is the definition of projective dimension, see Definition 10.109.2. We have (2) $\Rightarrow$ (4) by Lemma 10.109.3. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are immediate. □

0CXD **Lemma 10.109.5.** *Let* R *be a local ring. Let* M *be an* R*-module. Let* $d \geq 0$. *The equivalent conditions (1) – (4) of Lemma 10.109.4 are also equivalent to*

(5) *there exists a resolution* $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ *with* P_i *free.*

Proof. Follows from Lemma 10.109.4 and Theorem 10.85.4. □

0CXE **Lemma 10.109.6.** *Let* R *be a Noetherian ring. Let* M *be a finite* R*-module. Let* $d \geq 0$. *The equivalent conditions (1) – (4) of Lemma 10.109.4 are also equivalent to*

(6) *there exists a resolution* $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ *with* P_i *finite projective.*

Proof. Choose a resolution $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ with F_i finite free (Lemma 10.71.1). By Lemma 10.109.4 we see that $P_d = \mathrm{Ker}(F_{d-1} \to F_{d-2})$ is projective at least if $d \geq 2$. Then P_d is a finite R-module as R is Noetherian and $P_d \subset F_{d-1}$ which is finite free. Whence $0 \to P_d \to F_{d-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ is the desired resolution. □

0CXF **Lemma 10.109.7.** *Let* R *be a local Noetherian ring. Let* M *be a finite* R*-module. Let* $d \geq 0$. *The equivalent conditions (1) – (4) of Lemma 10.109.4, condition (5) of Lemma 10.109.5, and condition (6) of Lemma 10.109.6 are also equivalent to*

(7) *there exists a resolution* $0 \to F_d \to F_{d-1} \to \ldots \to F_0 \to M \to 0$ *with* F_i *finite free.*

Proof. This follows from Lemmas 10.109.4, 10.109.5, and 10.109.6 and because a finite projective module over a local ring is finite free, see Lemma 10.78.2. □

065R **Lemma 10.109.8.** *Let* R *be a ring. Let* M *be an* R*-module. Let* $n \geq 0$. *The following are equivalent*

(1) M *has projective dimension* $\leq n$,
(2) $\mathrm{Ext}^i_R(M, N) = 0$ *for all* R*-modules* N *and all* $i \geq n + 1$, *and*
(3) $\mathrm{Ext}^{n+1}_R(M, N) = 0$ *for all* R*-modules* N.

Proof. Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 10.109.3 we see that $P_e = \mathrm{Ker}(d_e)$ is projective for $e \geq n - 1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex

$\mathrm{Hom}_R(F_\bullet, N)$ is split exact in degrees $\geq n+1$. Whence (2) holds. The implication (2) $\Rightarrow$ (3) is trivial.

Assume (3) holds. If $n = 0$ then M is projective by Lemma 10.77.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By Lemma 10.71.7 and the vanishing of $\mathrm{Ext}^i_R(F, N)$ for all $i > 0$ by part (1) we see that $\mathrm{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that K has projective dimension $\leq n-1$. Then M has projective dimension $\leq n$ as any finite projective resolution of K gives a projective resolution of length one more for M by adding F to the front. □

065S **Lemma 10.109.9.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules.*

(1) *If M has projective dimension $\leq n$ and M'' has projective dimension $\leq n+1$, then M' has projective dimension $\leq n$.*
(2) *If M' and M'' have projective dimension $\leq n$ then M has projective dimension $\leq n$.*
(3) *If M' has projective dimension $\leq n$ and M has projective dimension $\leq n+1$ then M'' has projective dimension $\leq n+1$.*

Proof. Combine the characterization of projective dimension in Lemma 10.109.8 with the long exact sequence of ext groups in Lemma 10.71.7. □

00O6 **Definition 10.109.10.** Let R be a ring. The ring R is said to have *finite global dimension* if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the *global dimension* of R.

The argument in the proof of the following lemma can be found in the paper [**Aus55**] by Auslander.

0D1U **Lemma 10.109.11.** *Let R be a ring. Suppose we have a module $M = \bigcup_{e \in E} M_e$ where the M_e are submodules well-ordered by inclusion. Assume the quotients $M_e / \bigcup_{e' < e} M_{e'}$ have projective dimension $\leq n$. Then M has projective dimension $\leq n$.*

Proof. We will prove this by induction on n.

Base case: $n = 0$. Then $P_e = M_e / \bigcup_{e' < e} M_{e'}$ is projective. Thus we may choose a section $P_e \to M_e$ of the projection $M_e \to P_e$. We claim that the induced map $\psi : \bigoplus_{e \in E} P_e \to M$ is an isomorphism. Namely, if $x = \sum x_e \in \bigoplus P_e$ is nonzero, then we let e_{max} be maximal such that $x_{e_{max}}$ is nonzero and we conclude that $y = \psi(x) = \psi(\sum x_e)$ is nonzero because $y \in M_{e_{max}}$ has nonzero image $x_{e_{max}}$ in $P_{e_{max}}$. On the other hand, let $y \in M$. Then $y \in M_e$ for some e. We show that $y \in \mathrm{Im}(\psi)$ by transfinite induction on e. Let $x_e \in P_e$ be the image of y. Then $y - \psi(x_e) \in \bigcup_{e' < e} M_{e'}$. By induction hypothesis we conclude that $y - \psi(x_e) \in \mathrm{Im}(\psi)$ hence $y \in \mathrm{Im}(\psi)$. Thus the claim is true and ψ is an isomorphism. We conclude that M is projective as a direct sum of projectives, see Lemma 10.77.4.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well-ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e/\bigcup_{e'<e} K_{e'} \to F_e/\bigcup_{e'<e} F_{e'} \to M_e/\bigcup_{e'<e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e/\bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Lemma 10.109.9 we see that the projective dimension of $K_e/\bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. □

065T **Lemma 10.109.12.** *Let R be a ring. The following are equivalent*

(1) *R has finite global dimension $\leq n$,*
(2) *every finite R-module has projective dimension $\leq n$, and*
(3) *every cyclic R-module R/I has projective dimension $\leq n$.*

Proof. It is clear that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Assume (3). Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e \in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e/\bigcup_{e'<e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$ by (3). By Lemma 10.109.11 this means that M has projective dimension $\leq n$. □

00O8 **Lemma 10.109.13.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.*

(1) *If M has projective dimension $\leq n$, then $S^{-1}M$ has projective dimension $\leq n$ over $S^{-1}R$.*
(2) *If R has finite global dimension $\leq n$, then $S^{-1}R$ has finite global dimension $\leq n$.*

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact, see Proposition 10.9.12, and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, see Lemma 10.94.1, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). □

10.110. Regular rings and global dimension

065U We can use the material on rings of finite global dimension to give another characterization of regular local rings.

00O7 **Proposition 10.110.1.** *Let R be a regular local ring of dimension d. Every finite R-module M of depth e has a finite free resolution*

$$0 \to F_{d-e} \to \ldots \to F_0 \to M \to 0.$$

In particular a regular local ring has global dimension $\leq d$.

Proof. The first part holds in view of Lemma 10.106.6 and Lemma 10.104.9. The last part follows from this and Lemma 10.109.12. □

00O9 **Lemma 10.110.2.** *Let R be a Noetherian ring. Then R has finite global dimension if and only if there exists an integer n such that for all maximal ideals $\mathfrak{m}$ of R the ring $R_\mathfrak{m}$ has global dimension $\leq n$.*

Proof. We saw, Lemma 10.109.13 that if R has finite global dimension n, then all the localizations $R_\mathfrak{m}$ have finite global dimension at most n. Conversely, suppose that all the $R_\mathfrak{m}$ have global dimension $\leq n$. Let M be a finite R-module. Let $0 \to K_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ be a resolution with F_i finite free. Then K_n is a finite R-module. According to Lemma 10.109.3 and the assumption all the modules $K_n \otimes_R R_\mathfrak{m}$ are projective. Hence by Lemma 10.78.2 the module K_n is finite projective. □

00OA **Lemma 10.110.3.** *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case the projective dimension of κ is $\geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let $x_1, \ldots, x_n$ be elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Consider the *Koszul complex* on $x_1, \ldots, x_n$. This is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \wedge^{n-2} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R$$

with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{a=1}^{i} (-1)^{a+1} x_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

It is easy to see that this is a complex $K_\bullet(R, x_\bullet)$. Note that the cokernel of the last map of $K_\bullet(R, x_\bullet)$ is κ by Lemma 10.20.1 part (8).

If κ has finite projective dimension d, then we can find a resolution $F_\bullet \to \kappa$ by finite free R-modules of length d (Lemma 10.109.7). By Lemma 10.102.2 we may assume all the maps in the complex $F_\bullet$ have the property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m} F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. By Lemma 10.71.4 we can find a map of complexes $\alpha : K_\bullet(R, x_\bullet) \to F_\bullet$ inducing the identity on κ. We will prove by induction that the maps $\alpha_i : \wedge^i R^n = K_i(R, x_\bullet) \to F_i$ have the property that $\alpha_i \otimes \kappa : \wedge^i \kappa^n \to F_i \otimes \kappa$ are injective. This shows that $F_n \neq 0$ and hence $d \geq n$ as desired.

The result is clear for $i = 0$ because the composition $R \xrightarrow{\alpha_0} F_0 \to \kappa$ is nonzero. Note that F_0 must have rank 1 since otherwise the map $F_1 \to F_0$ whose cokernel is a single copy of κ cannot have image contained in $\mathfrak{m} F_0$.

Next we check the case $i = 1$ as we feel that it is instructive; the reader can skip this as the induction step will deduce the $i = 1$ case from the case $i = 0$. We saw above that $F_0 = R$ and $F_1 \to F_0 = R$ has image $\mathfrak{m}$. We have a commutative diagram

$$\begin{array}{ccccccc} R^n & = & K_1(R, x_\bullet) & \to & K_0(R, x_\bullet) & = & R \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F_1 & \to & F_0 & = & R \end{array}$$

where the rightmost vertical arrow is given by multiplication by a unit. Hence we see that the image of the composition $R^n \to F_1 \to F_0 = R$ is also equal to $\mathfrak{m}$. Thus the map $R^n \otimes \kappa \to F_1 \otimes \kappa$ has to be injective since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = n$.

Let $i \geq 1$ and assume injectivity of $\alpha_j \otimes \kappa$ has been proved for all $j \leq i-1$. Consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^i R^n & = & K_i(R, x_\bullet) & \to & K_{i-1}(R, x_\bullet) & = & \wedge^{i-1} R^n \\ & & \downarrow & & \downarrow & & \\ & & F_i & \to & F_{i-1} & & \end{array}$$

We know that $\wedge^{i-1}\kappa^n \to F_{i-1} \otimes \kappa$ is injective. This proves that $\wedge^{i-1}\kappa^n \otimes_\kappa \mathfrak{m}/\mathfrak{m}^2 \to F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective. Also, by our choice of the complex, F_i maps into $\mathfrak{m}F_{i-1}$, and similarly for the Koszul complex. Hence we get a commutative diagram

$$\begin{array}{ccc} \wedge^i \kappa^n & \to & \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^2 \\ \downarrow & & \downarrow \\ F_i \otimes \kappa & \to & F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

At this point it suffices to verify the map $\wedge^i \kappa^n \to \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective, which can be done by hand. □

00OB **Lemma 10.110.4.** *Let R be a Noetherian local ring. Suppose that the residue field κ has finite projective dimension n over R. In this case* $\dim(R) \geq n$.

Proof. Let $F_\bullet$ be a finite resolution of κ by finite free R-modules (Lemma 10.109.7). By Lemma 10.102.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. Say $F_n \neq 0$ and $F_i = 0$ for $i > n$, so that the projective dimension of κ is n. By Proposition 10.102.9 we see that $\mathrm{depth}_{I(\varphi_n)}(R) \geq n$ since $I(\varphi_n)$ cannot equal R by our choice of the complex. Thus by Lemma 10.72.3 also $\dim(R) \geq n$. □

00OC **Proposition 10.110.5.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The following are equivalent*

(1) *κ has finite projective dimension as an R-module,*
(2) *R has finite global dimension,*
(3) *R is a regular local ring.*

Moreover, in this case the global dimension of R equals $\dim(R) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$.

Proof. We have (3) $\Rightarrow$ (2) by Proposition 10.110.1. The implication (2) $\Rightarrow$ (1) is trivial. Assume (1). By Lemmas 10.110.3 and 10.110.4 we see that $\dim(R) \geq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus R is regular, see Definition 10.60.10 and the discussion preceding it. Assume the equivalent conditions (1) – (3) hold. By Proposition 10.110.1 the global dimension of R is at most $\dim(R)$ and by Lemma 10.110.3 it is at least $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus the stated equality holds. □

0AFS **Lemma 10.110.6.** *A Noetherian local ring R is a regular local ring if and only if it has finite global dimension. In this case $R_\mathfrak{p}$ is a regular local ring for all primes $\mathfrak{p}$.*

Proof. By Propositions 10.110.5 and 10.110.1 we see that a Noetherian local ring is a regular local ring if and only if it has finite global dimension. Furthermore, any localization $R_\mathfrak{p}$ has finite global dimension, see Lemma 10.109.13, and hence is a regular local ring. □

By Lemma 10.110.6 it makes sense to make the following definition, because it does not conflict with the earlier definition of a regular local ring.

00OD **Definition 10.110.7.** A Noetherian ring R is said to be *regular* if all the localizations $R_{\mathfrak{p}}$ at primes are regular local rings.

It is enough to require the local rings at maximal ideals to be regular. Note that this is not the same as asking R to have finite global dimension, even assuming R is Noetherian. This is because there is an example of a regular Noetherian ring which does not have finite global dimension, namely because it does not have finite dimension.

00OE **Lemma 10.110.8.** *Let R be a Noetherian ring. The following are equivalent:*

(1) *R has finite global dimension n,*
(2) *R is a regular ring of dimension n,*
(3) *there exists an integer n such that all the localizations $R_{\mathfrak{m}}$ at maximal ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{m}$, and*
(4) *there exists an integer n such that all the localizations $R_{\mathfrak{p}}$ at prime ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{p}$.*

Proof. This follows from the discussion above. More precisely, it follows by combining Definition 10.110.7 with Lemma 10.110.2 and Proposition 10.110.5. □

00OF **Lemma 10.110.9.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume that $R \to S$ is flat and that S is regular. Then R is regular.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $\kappa = R/\mathfrak{m}$ be the residue field. Let $d = \dim S$. Choose any resolution $F_\bullet \to \kappa$ with each F_i a finite free R-module. Set $K_d = \mathrm{Ker}(F_{d-1} \to F_{d-2})$. By flatness of $R \to S$ the complex $0 \to K_d \otimes_R S \to F_{d-1} \otimes_R S \to \ldots \to F_0 \otimes_R S \to \kappa \otimes_R S \to 0$ is still exact. Because the global dimension of S is d, see Proposition 10.110.1, we see that $K_d \otimes_R S$ is a finite free S-module (see also Lemma 10.109.3). By Lemma 10.78.6 we see that K_d is a finite free R-module. Hence κ has finite projective dimension and R is regular by Proposition 10.110.5. □

10.111. Auslander-Buchsbaum

090U The following result can be found in [**AB57**].

090V **Proposition 10.111.1.** *Let R be a Noetherian local ring. Let M be a nonzero finite R-module which has finite projective dimension $pd_R(M)$. Then we have*

$$depth(R) = pd_R(M) + depth(M)$$

Proof. We prove this by induction on $\mathrm{depth}(M)$. The most interesting case is the case $\mathrm{depth}(M) = 0$. In this case, let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal finite free resolution, so $e = \mathrm{pd}_R(M)$. By Lemma 10.102.2 we may assume all matrix coefficients of the maps in the complex are contained in the maximal ideal of R. Then on the one hand, by Proposition 10.102.9 we see that $\mathrm{depth}(R) \geq e$. On the other hand, breaking the long exact sequence into short

exact sequences

$$\begin{aligned} 0 \to R^{n_e} \to R^{n_{e-1}} \to K_{e-2} \to 0, \\ 0 \to K_{e-2} \to R^{n_{e-2}} \to K_{e-3} \to 0, \\ \ldots, \\ 0 \to K_0 \to R^{n_0} \to M \to 0 \end{aligned}$$

we see, using Lemma 10.72.6, that

$$\begin{aligned} \text{depth}(K_{e-2}) \geq \text{depth}(R) - 1, \\ \text{depth}(K_{e-3}) \geq \text{depth}(R) - 2, \\ \ldots, \\ \text{depth}(K_0) \geq \text{depth}(R) - (e-1), \\ \text{depth}(M) \geq \text{depth}(R) - e \end{aligned}$$

and since $\text{depth}(M) = 0$ we conclude $\text{depth}(R) \leq e$. This finishes the proof of the case $\text{depth}(M) = 0$.

Induction step. If $\text{depth}(M) > 0$, then we pick $x \in \mathfrak{m}$ which is a nonzerodivisor on both M and R. This is possible, because either $\text{pd}_R(M) > 0$ and $\text{depth}(R) > 0$ by the aforementioned Proposition 10.102.9 or $\text{pd}_R(M) = 0$ in which case M is finite free hence also $\text{depth}(R) = \text{depth}(M) > 0$. Thus $\text{depth}(R \oplus M) > 0$ by Lemma 10.72.6 (for example) and we can find an $x \in \mathfrak{m}$ which is a nonzerodivisor on both R and M. Let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal resolution as above. An application of the snake lemma shows that

$$0 \to (R/xR)^{n_e} \to (R/xR)^{n_{e-1}} \to \ldots \to (R/xR)^{n_0} \to M/xM \to 0$$

is a minimal resolution too. Thus $\text{pd}_R(M) = \text{pd}_{R/xR}(M/xM)$. By Lemma 10.72.7 we have $\text{depth}(R/xR) = \text{depth}(R) - 1$ and $\text{depth}(M/xM) = \text{depth}(M) - 1$. Till now depths have all been depths as R modules, but we observe that $\text{depth}_R(M/xM) = \text{depth}_{R/xR}(M/xM)$ and similarly for R/xR. By induction hypothesis we see that the Auslander-Buchsbaum formula holds for M/xM over R/xR. Since the depths of both R/xR and M/xM have decreased by one and the projective dimension has not changed we conclude. □

10.112. Homomorphisms and dimension

00OG This section contains a collection of easy results relating dimensions of rings when there are maps between them.

00OH **Lemma 10.112.1.** *Suppose $R \to S$ is a ring map satisfying either going up, see Definition 10.41.1, or going down see Definition 10.41.1. Assume in addition that $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. Then $\dim(R) \leq \dim(S)$.*

Proof. Assume going up. Take any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e$ of prime ideals in R. By surjectivity we may choose a prime $\mathfrak{q}_0$ mapping to $\mathfrak{p}_0$. By going up we may extend this to a chain of length e of primes $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$. Thus $\dim(S) \geq \dim(R)$. The case of going down is exactly the same. See also Topology, Lemma 5.19.9 for a purely topological version. □

00OI **Lemma 10.112.2.** *Suppose that $R \to S$ is a ring map with the going up property, see Definition 10.41.1. If $\mathfrak{q} \subset S$ is a maximal ideal. Then the inverse image of $\mathfrak{q}$ in R is a maximal ideal too.*

Proof. Trivial. □

00OJ **Lemma 10.112.3.** *Suppose that $R \to S$ is a ring map such that S is integral over R. Then $\dim(R) \geq \dim(S)$, and every closed point of $\mathrm{Spec}(S)$ maps to a closed point of $\mathrm{Spec}(R)$.*

Proof. Immediate from Lemmas 10.36.20 and 10.112.2 and the definitions. □

00OK **Lemma 10.112.4.** *Suppose $R \subset S$ and S integral over R. Then $\dim(R) = \dim(S)$.*

Proof. This is a combination of Lemmas 10.36.22, 10.36.17, 10.112.1, and 10.112.3. □

00OL **Definition 10.112.5.** Suppose that $R \to S$ is a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The *local ring of the fibre at* $\mathfrak{q}$ is the local ring

$$S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S/\mathfrak{p}S)_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_\mathfrak{q}$$

00OM **Lemma 10.112.6.** *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Then*

$$\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

Proof. We use the characterization of dimension of Proposition 10.60.9. Let $x_1, \ldots, x_d$ be elements of $\mathfrak{p}$ generating an ideal of definition of $R_\mathfrak{p}$ with $d = \dim(R_\mathfrak{p})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak{q}$ generating an ideal of definition of $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ with $e = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. It is clear that $S_\mathfrak{q}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition of $S_\mathfrak{q}$. □

00ON **Lemma 10.112.7.** *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Lemma 10.39.19). Then*

$$\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

Proof. By Lemma 10.112.6 we have an inequality $\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. To get equality, choose a chain of primes $\mathfrak{p}S \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_d = \mathfrak{q}$ with $d = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. On the other hand, choose a chain of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{p}$ with $e = \dim(R_\mathfrak{p})$. By the going down theorem we may choose $\mathfrak{q}_{-1} \subset \mathfrak{q}_0$ lying over $\mathfrak{p}_{e-1}$. And then we may choose $\mathfrak{q}_{-2} \subset \mathfrak{q}_{-1}$ lying over $\mathfrak{p}_{e-2}$. Inductively we keep going until we get a chain $\mathfrak{q}_{-e} \subset \ldots \subset \mathfrak{q}_d$ of length $e + d$. □

031E **Lemma 10.112.8.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

(1) *R is regular,*
(2) *$S/\mathfrak{m}_R S$ is regular, and*
(3) *$R \to S$ is flat.*

Then S is regular.

Proof. By Lemma 10.112.7 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak{m}_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak{m}_S$ which generate the maximal ideal of $S/\mathfrak{m}_R S$ with $e = \dim(S/\mathfrak{m}_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e+d = \dim(S)$. □

The lemma below will later be used to show that rings of finite type over a field are Cohen-Macaulay if and only if they are quasi-finite flat over a polynomial ring. It is a partial converse to Lemma 10.128.1.

00R5 **Lemma 10.112.9.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R Cohen-Macaulay. If S is finite flat over R, or if S is flat over R and $\dim(S) \leq \dim(R)$, then S is Cohen-Macaulay and $\dim(R) = \dim(S)$.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}_R$ be a regular sequence of length $d = \dim(R)$. By Lemma 10.68.5 this maps to a regular sequence in S. Hence S is Cohen-Macaulay if $\dim(S) \leq d$. This is true if S is finite flat over R by Lemma 10.112.4. And in the second case we assumed it. □

10.113. The dimension formula

02II Recall the definitions of catenary (Definition 10.105.1) and universally catenary (Definition 10.105.3).

02IJ **Lemma 10.113.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Assume that*

1. *R is Noetherian,*
2. *$R \to S$ is of finite type,*
3. *R, S are domains, and*
4. *$R \subset S$.*

Then we have

$$height(\mathfrak{q}) \leq height(\mathfrak{p}) + trdeg_R(S) - trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$, where S' is a finitely generated R-subalgebra of S. In this case set $\mathfrak{q}' = S' \cap \mathfrak{q}$. The lemma for the ring maps $R \to S'$ and $S' \to S$ implies the lemma for $R \to S$ by additivity of transcendence degree in towers of fields (Fields, Lemma 9.26.5). Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\mathrm{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$$

see Lemma 10.112.7. Let $\mathfrak{r} = \mathfrak{p}S$. Then $\mathrm{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 1$ is equivalent to $\mathfrak{q} = \mathfrak{r}$, and implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$. In the same vein $\mathrm{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 0$ is equivalent to having a strict inclusion $\mathfrak{r} \subset \mathfrak{q}$, which implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak{n}$ with $\mathfrak{n} \neq 0$. In this case we have $\mathrm{trdeg}_R(S) = 0$. Denote $\mathfrak{q}' \subset R[x]$ the prime corresponding to $\mathfrak{q}$. Thus we have

$$S_\mathfrak{q} = (R[x])_{\mathfrak{q}'}/\mathfrak{n}(R[x])_{\mathfrak{q}'}$$

By the previous case we have $\dim((R[x])_{\mathfrak{q}'}) = \dim(R_\mathfrak{p}) + 1 - \mathrm{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$. Since $\mathfrak{n} \neq 0$ we see that the dimension of $S_\mathfrak{q}$ decreases by at least one, see Lemma 10.60.13, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak{n} \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $F[x]$ where F is the fraction field of R. Hence any maximal chain of primes in $S_\mathfrak{q} = R[x]_{\mathfrak{q}'}/\mathfrak{n}R[x]_{\mathfrak{q}'}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak{q}'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak{q}'$. This proves that $\dim(S_\mathfrak{q}) = \dim(R[x]_{\mathfrak{q}'}) - 1$ and this implies equality holds as desired. □

The following lemma says that generically finite maps tend to be quasi-finite in codimension 1.

02MA **Lemma 10.113.2.** *Let $A \to B$ be a ring map. Assume*

(1) *$A \subset B$ is an extension of domains,*
(2) *the induced extension of fraction fields is finite,*
(3) *A is Noetherian, and*
(4) *$A \to B$ is of finite type.*

Let $\mathfrak{p} \subset A$ be a prime of height 1. Then there are at most finitely many primes of B lying over $\mathfrak{p}$ and they all have height 1.

Proof. By the dimension formula (Lemma 10.113.1) for any prime $\mathfrak{q}$ lying over $\mathfrak{p}$ we have

$$\dim(B_\mathfrak{q}) \leq \dim(A_\mathfrak{p}) - \mathrm{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}).$$

As the domain $B_\mathfrak{q}$ has at least 2 prime ideals we see that $\dim(B_\mathfrak{q}) \geq 1$. We conclude that $\dim(B_\mathfrak{q}) = 1$ and that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is algebraic. Hence $\mathfrak{q}$ defines a closed point of its fibre $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, see Lemma 10.35.9. Since $B \otimes_A \kappa(\mathfrak{p})$ is a Noetherian ring the fibre $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ is a Noetherian topological space, see Lemma 10.31.5. A Noetherian topological space consisting of closed points is finite, see for example Topology, Lemma 5.9.2. □

10.114. Dimension of finite type algebras over fields

00OO In this section we compute the dimension of a polynomial ring over a field. We also prove that the dimension of a finite type domain over a field is the dimension of its local rings at maximal ideals. We will establish the connection with the transcendence degree over the ground field in Section 10.116.

00OP **Lemma 10.114.1.** *Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The ideal $\mathfrak{m}$ is generated by n elements. The dimension of $k[x_1, \ldots, x_n]_\mathfrak{m}$ is n. Hence $k[x_1, \ldots, x_n]_\mathfrak{m}$ is a regular local ring of dimension n.*

Proof. By the Hilbert Nullstellensatz (Theorem 10.34.1) we know the residue field $\kappa = \kappa(\mathfrak{m})$ is a finite extension of k. Denote $\alpha_i \in \kappa$ the image of x_i. Denote $\kappa_i = k(\alpha_1, \ldots, \alpha_i) \subset \kappa$, $i = 1, \ldots, n$ and $\kappa_0 = k$. Note that $\kappa_i = k[\alpha_1, \ldots, \alpha_i]$ by field theory. Define inductively elements $f_i \in \mathfrak{m} \cap k[x_1, \ldots, x_i]$ as follows: Let $P_i(T) \in \kappa_{i-1}[T]$ be the monic minimal polynomial of α_i over κ_{i-1}. Let $Q_i(T) \in k[x_1, \ldots, x_{i-1}][T]$ be a monic lift of $P_i(T)$ (of the same degree). Set $f_i = Q_i(x_i)$.

Note that if $d_i = \deg_T(P_i) = \deg_T(Q_i) = \deg_{x_i}(f_i)$ then $d_1 d_2 \ldots d_i = [\kappa_i : k]$ by Fields, Lemmas 9.7.7 and 9.9.2.

We claim that for all $i = 0, 1, \ldots, n$ there is an isomorphism

$$\psi_i : k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \cong \kappa_i.$$

By construction the composition $k[x_1, \ldots, x_i] \to k[x_1, \ldots, x_n] \to \kappa$ is surjective onto κ_i and $f_1, \ldots, f_i$ are in the kernel. This gives a surjective homomorphism. We prove ψ_i is injective by induction. It is clear for $i = 0$. Given the statement for i we prove it for $i + 1$. The ring extension $k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \to k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is generated by 1 element over a field and one irreducible equation. By elementary field theory $k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is a field, and hence ψ_i is injective.

This implies that $\mathfrak{m} = (f_1, \ldots, f_n)$. Moreover, we also conclude that

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_i) \cong \kappa_i[x_{i+1}, \ldots, x_n].$$

Hence $(f_1, \ldots, f_i)$ is a prime ideal. Thus

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \ldots \subset (f_1, \ldots, f_n) = \mathfrak{m}$$

is a chain of primes of length n. The lemma follows. □

00OQ **Proposition 10.114.2.** *A polynomial algebra in n variables over a field is a regular ring. It has global dimension n. All localizations at maximal ideals are regular local rings of dimension n.*

Proof. By Lemma 10.114.1 all localizations $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ at maximal ideals are regular local rings of dimension n. Hence we conclude by Lemma 10.110.8. □

00OR **Lemma 10.114.3.** *Let k be a field. Let $\mathfrak{p} \subset \mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a pair of primes. Any maximal chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length height($\mathfrak{q}$) − height($\mathfrak{p}$).*

Proof. By Proposition 10.114.2 any local ring of $k[x_1, \ldots, x_n]$ is regular. Hence all local rings are Cohen-Macaulay, see Lemma 10.106.3. The local rings at maximal ideals have dimension n hence every maximal chain of primes in $k[x_1, \ldots, x_n]$ has length n, see Lemma 10.104.3. Hence every maximal chain of primes between (0) and $\mathfrak{p}$ has length height($\mathfrak{p}$), see Lemma 10.104.4 for example. Putting these together leads to the assertion of the lemma. □

00OS **Lemma 10.114.4.** *Let k be a field. Let S be a finite type k-algebra which is an integral domain. Then $\dim(S) = \dim(S_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$ of S. In words: every maximal chain of primes has length equal to the dimension of S.*

Proof. Write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Proposition 10.114.2 and Lemma 10.114.3 all the maximal chains of primes in S (which necessarily end with a maximal ideal) have length $n - \text{height}(\mathfrak{p})$. Thus this number is the dimension of S and of $S_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of S. □

Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.10.1.

00OT **Lemma 10.114.5.** *Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal and let $x \in X$ be the corresponding point. The following numbers are equal*

(1) $\dim_x(X)$,

(2) $\max \dim(Z)$ *where the maximum is over those irreducible components* Z *of* X *passing through* x*, and*
(3) $\min \dim(S_{\mathfrak{m}})$ *where the minimum is over maximal ideals* $\mathfrak{m}$ *with* $\mathfrak{p} \subset \mathfrak{m}$.

Proof. Let $X = \bigcup_{i \in I} Z_i$ be the decomposition of X into its irreducible components. There are finitely many of them (see Lemmas 10.31.3 and 10.31.5). Let $I' = \{i \mid x \in Z_i\}$, and let $T = \bigcup_{i \notin I'} Z_i$. Then $U = X \setminus T$ is an open subset of X containing the point x. The number (2) is $\max_{i \in I'} \dim(Z_i)$. For any open $W \subset U$ with $x \in W$ the irreducible components of W are the irreducible sets $W_i = Z_i \cap W$ for $i \in I'$ and x is contained in each of these. Note that each W_i, $i \in I'$ contains a closed point because X is Jacobson, see Section 10.35. Since $W_i \subset Z_i$ we have $\dim(W_i) \leq \dim(Z_i)$. The existence of a closed point implies, via Lemma 10.114.4, that there is a chain of irreducible closed subsets of length equal to $\dim(Z_i)$ in the open W_i. Thus $\dim(W_i) = \dim(Z_i)$ for any $i \in I'$. Hence $\dim(W)$ is equal to the number (2). This proves that (1) = (2).

Let $\mathfrak{m} \supset \mathfrak{p}$ be any maximal ideal containing $\mathfrak{p}$. Let $x_0 \in X$ be the corresponding point. First of all, x_0 is contained in all the irreducible components Z_i, $i \in I'$. Let $\mathfrak{q}_i$ denote the minimal primes of S corresponding to the irreducible components Z_i. For each i such that $x_0 \in Z_i$ (which is equivalent to $\mathfrak{m} \supset \mathfrak{q}_i$) we have a surjection

$$S_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}}/\mathfrak{q}_i S_{\mathfrak{m}} = (S/\mathfrak{q}_i)_{\mathfrak{m}}$$

Moreover, the primes $\mathfrak{q}_i S_{\mathfrak{m}}$ so obtained exhaust the minimal primes of the Noetherian local ring $S_{\mathfrak{m}}$, see Lemma 10.26.3. We conclude, using Lemma 10.114.4, that the dimension of $S_{\mathfrak{m}}$ is the maximum of the dimensions of the Z_i passing through x_0. To finish the proof of the lemma it suffices to show that we can choose x_0 such that $x_0 \in Z_i \Rightarrow i \in I'$. Because S is Jacobson (as we saw above) it is enough to show that $V(\mathfrak{p}) \setminus T$ (with T as above) is nonempty. And this is clear since it contains the point x (i.e. $\mathfrak{p}$). □

00OU **Lemma 10.114.6.** *Let* k *be a field. Let* S *be a finite type* k*-algebra. Let* $X = \operatorname{Spec}(S)$*. Let* $\mathfrak{m} \subset S$ *be a maximal ideal and let* $x \in X$ *be the associated closed point. Then* $\dim_x(X) = \dim(S_{\mathfrak{m}})$.

Proof. This is a special case of Lemma 10.114.5. □

00OV **Lemma 10.114.7.** *Let* k *be a field. Let* S *be a finite type* k *algebra. Assume that* S *is Cohen-Macaulay. Then* $\operatorname{Spec}(S) = \coprod T_d$ *is a finite disjoint union of open and closed subsets* T_d *with* T_d *equidimensional (see Topology, Definition 5.10.5) of dimension* d*. Equivalently,* S *is a product of rings* S_d*,* $d = 0, \ldots, \dim(S)$ *such that every maximal ideal* $\mathfrak{m}$ *of* S_d *has height* d.

Proof. The equivalence of the two statements follows from Lemma 10.24.3. Let $\mathfrak{m} \subset S$ be a maximal ideal. Every maximal chain of primes in $S_{\mathfrak{m}}$ has the same length equal to $\dim(S_{\mathfrak{m}})$, see Lemma 10.104.3. Hence, the dimension of the irreducible components passing through the point corresponding to $\mathfrak{m}$ all have dimension equal to $\dim(S_{\mathfrak{m}})$, see Lemma 10.114.4. Since $\operatorname{Spec}(S)$ is a Jacobson topological space the intersection of any two irreducible components of it contains a closed point if nonempty, see Lemmas 10.35.2 and 10.35.4. Thus we have shown that any two irreducible components that meet have the same dimension. The lemma follows easily from this, and the fact that $\operatorname{Spec}(S)$ has a finite number of irreducible components (see Lemmas 10.31.3 and 10.31.5). □

10.115. Noether normalization

00OW In this section we prove variants of the Noether normalization lemma. The key ingredient we will use is contained in the following two lemmas.

051M **Lemma 10.115.1.** *Let $n \in \mathbf{N}$. Let N be a finite nonempty set of multi-indices $\nu = (\nu_1, \ldots, \nu_n)$. Given $e = (e_1, \ldots, e_n)$ we set $e \cdot \nu = \sum e_i \nu_i$. Then for $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n$ we have: If $\nu, \nu' \in N$ then*

$$(e \cdot \nu = e \cdot \nu') \Leftrightarrow (\nu = \nu')$$

Proof. Say $N = \{\nu_j\}$ with $\nu_j = (\nu_{j1}, \ldots, \nu_{jn})$. Let $A_i = \max_j \nu_{ji} - \min_j \nu_{ji}$. If for each i we have $e_{i-1} > A_i e_i + A_{i+1} e_{i+1} + \ldots + A_n e_n$ then the lemma holds. For suppose that $e \cdot (\nu - \nu') = 0$. Then for $n \geq 2$,

$$e_1(\nu_1 - \nu_1') = \sum\nolimits_{i=2}^{n} e_i(\nu_i' - \nu_i).$$

We may assume that $(\nu_1 - \nu_1') \geq 0$. If $(\nu_1 - \nu_1') > 0$, then

$$e_1(\nu_1 - \nu_1') \geq e_1 > A_2 e_2 + \ldots + A_n e_n \geq \sum\nolimits_{i=2}^{n} e_i |\nu_i' - \nu_i| \geq \sum\nolimits_{i=2}^{n} e_i(\nu_i' - \nu_i).$$

This contradiction implies that $\nu_1' = \nu_1$. By induction, $\nu_i' = \nu_i$ for $2 \leq i \leq n$. □

051N **Lemma 10.115.2.** *Let R be a ring. Let $g \in R[x_1, \ldots, x_n]$ be an element which is nonconstant, i.e., $g \notin R$. For $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n = 1$ the polynomial*

$$g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = a x_n^d + \text{lower order terms in } x_n$$

where $d > 0$ and $a \in R$ is one of the nonzero coefficients of g.

Proof. Write $g = \sum_{\nu \in N} a_\nu x^\nu$ with $a_\nu \in R$ not zero. Here N is a finite set of multi-indices as in Lemma 10.115.1 and $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$. Note that the leading term in

$$(x_1 + x_n^{e_1})^{\nu_1} \ldots (x_{n-1} + x_n^{e_{n-1}})^{\nu_{n-1}} x_n^{\nu_n} \quad \text{is} \quad x_n^{e_1\nu_1 + \ldots + e_{n-1}\nu_{n-1} + \nu_n}.$$

Hence the lemma follows from Lemma 10.115.1 which guarantees that there is exactly one nonzero term $a_\nu x^\nu$ of g which gives rise to the leading term of $g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$, i.e., $a = a_\nu$ for the unique $\nu \in N$ such that $e \cdot \nu$ is maximal. □

00OX **Lemma 10.115.3.** *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some proper ideal I. If $I \neq 0$, then there exist $y_1, \ldots, y_{n-1} \in k[x_1, \ldots, x_n]$ such that S is finite over $k[y_1, \ldots, y_{n-1}]$. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. Pick $f \in I$, $f \neq 0$. It suffices to show the lemma for $k[x_1, \ldots, x_n]/(f)$ since S is a quotient of that ring. We will take $y_i = x_i - x_n^{e_i}$, $i = 1, \ldots, n-1$ for suitable integers e_i. When does this work? It suffices to show that $\overline{x_n} \in k[x_1, \ldots, x_n]/(f)$ is integral over the ring $k[y_1, \ldots, y_{n-1}]$. The equation for $\overline{x_n}$ over this ring is

$$f(y_1 + x_n^{e_1}, \ldots, y_{n-1} + x_n^{e_{n-1}}, x_n) = 0.$$

Hence we are done if we can show there exists integers e_i such that the leading coefficient with respect to x_n of the equation above is a nonzero element of k. This can be achieved for example by choosing $e_1 \gg e_2 \gg \ldots \gg e_{n-1}$, see Lemma 10.115.2. □

00OY **Lemma 10.115.4.** *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq (1)$, there exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that (a) the map $k[y_1, \ldots, y_r] \to S$ is injective where the source is the polynomial ring on $y_1, \ldots, y_r$, and (b) the map $k[y_1, \ldots, y_r] \to S$ is finite. In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. By induction on n, with $n = 0$ being trivial. If $I = 0$, then take $r = n$ and $y_i = x_i$. If $I \neq 0$, then choose $y_1, \ldots, y_{n-1}$ as in Lemma 10.115.3. Let $S' \subset S$ be the subring generated by the images of the y_i. By induction we can choose r and $z_1, \ldots, z_r \in k[y_1, \ldots, y_{n-1}]$ such that (a), (b) hold for $k[z_1, \ldots, z_r] \to S'$. Since $S' \to S$ is injective and finite we see (a), (b) hold for $k[z_1, \ldots, z_r] \to S$. The last assertion follows from Lemma 10.112.4. □

00OZ **Lemma 10.115.5.** *Let k be a field. Let S be a finite type k algebra and denote $X = \mathrm{Spec}(S)$. Let $\mathfrak{q}$ be a prime of S, and let $x \in X$ be the corresponding point. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that $\dim(S_g) = \dim_x(X) =: d$ and such that there exists a finite injective map $k[y_1, \ldots, y_d] \to S_g$.*

Proof. Note that by definition $\dim_x(X)$ is the minimum of the dimensions of S_g for $g \in S$, $g \notin \mathfrak{q}$, i.e., the minimum is attained. Thus the lemma follows from Lemma 10.115.4. □

051P **Lemma 10.115.6.** *Let k be a field. Let $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a prime ideal. Set $r = trdeg_k\, \kappa(\mathfrak{q})$. Then there exists a finite ring map $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ such that $\varphi^{-1}(\mathfrak{q}) = (y_{r+1}, \ldots, y_n)$.*

Proof. By induction on n. The case $n = 0$ is clear. Assume $n > 0$. If $r = n$, then $\mathfrak{q} = (0)$ and the result is clear. Choose a nonzero $f \in \mathfrak{q}$. Of course f is nonconstant. After applying an automorphism of the form

$$k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n], \quad x_n \mapsto x_n, \quad x_i \mapsto x_i + x_n^{e_i} \ (i < n)$$

we may assume that f is monic in x_n over $k[x_1, \ldots, x_n]$, see Lemma 10.115.2. Hence the ring map

$$k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_n], \quad y_n \mapsto f, \quad y_i \mapsto x_i \ (i < n)$$

is finite. Moreover $y_n \in \mathfrak{q} \cap k[y_1, \ldots, y_n]$ by construction. Thus $\mathfrak{q} \cap k[y_1, \ldots, y_n] = \mathfrak{p}k[y_1, \ldots, y_n] + (y_n)$ where $\mathfrak{p} \subset k[y_1, \ldots, y_{n-1}]$ is a prime ideal. Note that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite, and hence $r = \mathrm{trdeg}_k\, \kappa(\mathfrak{p})$. Apply the induction hypothesis to the pair $(k[y_1, \ldots, y_{n-1}], \mathfrak{p})$ and we obtain a finite ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ such that $\mathfrak{p} \cap k[z_1, \ldots, z_{n-1}] = (z_{r+1}, \ldots, z_{n-1})$. We extend the ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ to a ring map $k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n]$ by mapping z_n to y_n. The composition of the ring maps

$$k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$$

solves the problem. □

07NA **Lemma 10.115.7.** *Let $R \to S$ be an injective finite type ring map. Assume R is a domain. Then there exists an integer d and a factorization*

$$R \to R[y_1, \ldots, y_d] \to S' \to S$$

by injective maps such that S' is finite over $R[y_1, \ldots, y_d]$ and such that $S'_f \cong S_f$ for some nonzero $f \in R$.

Proof. Pick $x_1, \ldots, x_n \in S$ which generate S over R. Let K be the fraction field of R and $S_K = S \otimes_R K$. By Lemma 10.115.4 we can find $y_1, \ldots, y_d \in S$ such that $K[y_1, \ldots, y_d] \to S_K$ is a finite injective map. Note that $y_i \in S$ because we may pick the y_j in the $\mathbf{Z}$-algebra generated by $x_1, \ldots, x_n$. As a finite ring map is integral (see Lemma 10.36.3) we can find monic $P_i \in K[y_1, \ldots, y_d][T]$ such that $P_i(x_i) = 0$ in S_K. Let $f \in R$ be a nonzero element such that $fP_i \in R[y_1, \ldots, y_d][T]$ for all i. Then $fP_i(x_i)$ maps to zero in S_K. Hence after replacing f by another nonzero element of R we may also assume $fP_i(x_i)$ is zero in S. Set $x_i' = fx_i$ and let $S' \subset S$ be the R-subalgebra generated by $y_1, \ldots, y_d$ and $x_1', \ldots, x_n'$. Note that x_i' is integral over $R[y_1, \ldots, y_d]$ as we have $Q_i(x_i') = 0$ where $Q_i = f^{\deg_T(P_i)} P_i(T/f)$ which is a monic polynomial in T with coefficients in $R[y_1, \ldots, y_d]$ by our choice of f. Hence $R[y_1, \ldots, y_d] \subset S'$ is finite by Lemma 10.36.5. Since $S' \subset S$ we have $S'_f \subset S_f$ (localization is exact). On the other hand, the elements $x_i = x_i'/f$ in S'_f generate S_f over R_f and hence $S'_f \to S_f$ is surjective. Whence $S'_f \cong S_f$ and we win. □

10.116. Dimension of finite type algebras over fields, reprise

07NB This section is a continuation of Section 10.114. In this section we establish the connection between dimension and transcendence degree over the ground field for finite type domains over a field.

00P0 **Lemma 10.116.1.** *Let k be a field. Let S be a finite type k algebra which is an integral domain. Let K be the field of fractions of S. Let $r = trdeg(K/k)$ be the transcendence degree of K over k. Then $\dim(S) = r$. Moreover, the local ring of S at every maximal ideal has dimension r.*

Proof. We may write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Lemma 10.114.3 all local rings of S at maximal ideals have the same dimension. Apply Lemma 10.115.4. We get a finite injective ring map

$$k[y_1, \ldots, y_d] \to S$$

with $d = \dim(S)$. Clearly, $k(y_1, \ldots, y_d) \subset K$ is a finite extension and we win. □

06RP **Lemma 10.116.2.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset \mathfrak{q}' \subset S$ be distinct prime ideals. Then $trdeg_k\ \kappa(\mathfrak{q}') < trdeg_k\ \kappa(\mathfrak{q})$.*

Proof. By Lemma 10.116.1 we have $\dim V(\mathfrak{q}) = \mathrm{trdeg}_k\ \kappa(\mathfrak{q})$ and similarly for $\mathfrak{q}'$. Hence the result follows as the strict inclusion $V(\mathfrak{q}') \subset V(\mathfrak{q})$ implies a strict inequality of dimensions. □

The following lemma generalizes Lemma 10.114.6.

00P1 **Lemma 10.116.3.** *Let k be a field. Let S be a finite type k algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $x \in X$ be the corresponding point. Then we have*

$$\dim_x(X) = \dim(S_\mathfrak{p}) + trdeg_k\ \kappa(\mathfrak{p}).$$

Proof. By Lemma 10.116.1 we know that $r = \mathrm{trdeg}_k\ \kappa(\mathfrak{p})$ is equal to the dimension of $V(\mathfrak{p})$. Pick any maximal chain of primes $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ starting with $\mathfrak{p}$ in S. This has length r by Lemma 10.114.4. Let $\mathfrak{q}_j$, $j \in J$ be the minimal primes of S which are contained in $\mathfrak{p}$. These correspond $1-1$ to minimal primes in $S_\mathfrak{p}$ via the rule $\mathfrak{q}_j \mapsto \mathfrak{q}_j S_\mathfrak{p}$. By Lemma 10.114.5 we know that $\dim_x(X)$ is equal to the

maximum of the dimensions of the rings $S/\mathfrak{q}_j$. For each j pick a maximal chain of primes $\mathfrak{q}_j \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p}$. Then $\dim(S_\mathfrak{p}) = \max_{j\in J} s(j)$. Now, each chain

$$\mathfrak{q}_i \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$$

is a maximal chain in $S/\mathfrak{q}_j$, and by what was said before we have $\dim_x(X) = \max_{j\in J} r + s(j)$. The lemma follows. □

The following lemma says that the codimension of one finite type Spec in another is the difference of heights.

00P2 **Lemma 10.116.4.** *Let k be a field. Let $S' \to S$ be a surjection of finite type k algebras. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $\mathfrak{p}'$ be the corresponding prime ideal of S'. Let $X = \mathrm{Spec}(S)$, resp. $X' = \mathrm{Spec}(S')$, and let $x \in X$, resp. $x' \in X'$ be the point corresponding to $\mathfrak{p}$, resp. $\mathfrak{p}'$. Then*

$$\dim_{x'} X' - \dim_x X = \mathit{height}(\mathfrak{p}') - \mathit{height}(\mathfrak{p}).$$

Proof. Immediate from Lemma 10.116.3. □

00P3 **Lemma 10.116.5.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Then* $\dim(S) = \dim(K \otimes_k S)$.

Proof. By Lemma 10.115.4 there exists a finite injective map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S)$. Since K is flat over k we also get a finite injective map $K[y_1, \ldots, y_d] \to K \otimes_k S$. The result follows from Lemma 10.112.4. □

00P4 **Lemma 10.116.6.** *Let k be a field. Let S be a finite type k-algebra. Set $X = \mathrm{Spec}(S)$. Let K/k be a field extension. Set $S_K = K \otimes_k S$, and $X_K = \mathrm{Spec}(S_K)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$ and let $\mathfrak{q}_K \subset S_K$ be a prime corresponding to $x_K \in X_K$ lying over $\mathfrak{q}$. Then* $\dim_x X = \dim_{x_K} X_K$.

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $K \otimes_k S = K[x_1, \ldots, x_n]/(K \otimes_k I)$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding primes. Consider the following commutative diagram of Noetherian local rings

$$\begin{array}{ccc} K[x_1, \ldots, x_n]_{\mathfrak{q}'_K} & \longrightarrow & (K \otimes_k S)_{\mathfrak{q}_K} \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_n]_{\mathfrak{q}'} & \longrightarrow & S_\mathfrak{q} \end{array}$$

Both vertical arrows are flat because they are localizations of the flat ring maps $S \to S_K$ and $k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$. Moreover, the vertical arrows have the same fibre rings. Hence, we see from Lemma 10.112.7 that $\mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K)$. Denote $x' \in X' = \mathrm{Spec}(k[x_1, \ldots, x_n])$ and $x'_K \in X'_K = \mathrm{Spec}(K[x_1, \ldots, x_n])$ the points corresponding to $\mathfrak{q}'$ and $\mathfrak{q}'_K$. By Lemma 10.116.4 and what we showed above we have

$$\begin{aligned} n - \dim_x X &= \dim_{x'} X' - \dim_x X \\ &= \mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) \\ &= \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K) \\ &= \dim_{x'_K} X'_K - \dim_{x_K} X_K \\ &= n - \dim_{x_K} X_K \end{aligned}$$

and the lemma follows. □

0CWE **Lemma 10.116.7.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime and let $\mathfrak{q}_K \subset S_K$ be a prime lying over $\mathfrak{q}$. Then*

$$\dim(S_K \otimes_S \kappa(\mathfrak{q}))_{\mathfrak{q}_K} = \dim(S_K)_{\mathfrak{q}_K} - \dim S_{\mathfrak{q}} = trdeg_k\kappa(\mathfrak{q}) - trdeg_K\kappa(\mathfrak{q}_K)$$

Moreover, given $\mathfrak{q}$ we can always choose $\mathfrak{q}_K$ such that the number above is zero.

Proof. Observe that $S_{\mathfrak{q}} \to (S_K)_{\mathfrak{q}_K}$ is a flat local homomorphism of local Noetherian rings with special fibre $(S_K \otimes_S \kappa(\mathfrak{q}))_{\mathfrak{q}_K}$. Hence the first equality by Lemma 10.112.7. The second equality follows from the fact that we have $\dim_x X = \dim_{x_K} X_K$ with notation as in Lemma 10.116.6 and we have $\dim_x X = \dim S_{\mathfrak{q}} + \mathrm{trdeg}_k\kappa(\mathfrak{q})$ by Lemma 10.116.3 and similarly for $\dim_{x_K} X_K$. If we choose $\mathfrak{q}_K$ minimal over $\mathfrak{q}S_K$, then the dimension of the fibre ring will be zero. □

10.117. Dimension of graded algebras over a field

00P5 Here is a basic result.

00P6 **Lemma 10.117.1.** *Let k be a field. Let S be a graded k-algebra generated over k by finitely many elements of degree 1. Assume $S_0 = k$. Let $P(T) \in \mathbf{Q}[T]$ be the polynomial such that $\dim(S_d) = P(d)$ for all $d \gg 0$. See Proposition 10.58.7. Then*

1. *The irrelevant ideal S_+ is a maximal ideal $\mathfrak{m}$.*
2. *Any minimal prime of S is a homogeneous ideal and is contained in $S_+ = \mathfrak{m}$.*
3. *We have $\dim(S) = \deg(P) + 1 = \dim_x \mathrm{Spec}(S)$ (with the convention that $\deg(0) = -1$) where x is the point corresponding to the maximal ideal $S_+ = \mathfrak{m}$.*
4. *The Hilbert function of the local ring $R = S_{\mathfrak{m}}$ is equal to the Hilbert function of S.*

Proof. The first statement is obvious. The second follows from Lemma 10.57.8. By (2) every irreducible component passes through x. Thus we have $\dim(S) = \dim_x \mathrm{Spec}(S) = \dim(S_{\mathfrak{m}})$ by Lemma 10.114.5. Since $\mathfrak{m}^d/\mathfrak{m}^{d+1} \cong \mathfrak{m}^d S_{\mathfrak{m}}/\mathfrak{m}^{d+1}S_{\mathfrak{m}}$ we see that the Hilbert function of the local ring $S_{\mathfrak{m}}$ is equal to the Hilbert function of S, which is (4). We conclude the last equality of (3) by Proposition 10.60.9. □

10.118. Generic flatness

051Q Basically this says that a finite type algebra over a domain becomes flat after inverting a single element of the domain. There are several versions of this result (in increasing order of strength).

051R **Lemma 10.118.1.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is Noetherian,*
2. *R is a domain,*
3. *$R \to S$ is of finite type, and*
4. *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Let K be the fraction field of R. Set $S_K = K \otimes_R S$. This is an algebra of finite type over K. We will argue by induction on $d = \dim(S_K)$ (which is finite for example by Noether normalization, see Section 10.115). Fix $d \geq 0$. Assume we know that the lemma holds in all cases where $\dim(S_K) < d$.

Suppose given $R \to S$ and M as in the lemma with $\dim(S_K) = d$. By Lemma 10.62.1 there exists a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that M_i/M_{i-1} is isomorphic to $S/\mathfrak{q}$ for some prime $\mathfrak{q}$ of S. Note that $\dim((S/\mathfrak{q})_K) \leq \dim(S_K)$. Also, note that an extension of free modules is free (see basic notion 50). Thus we may assume $M = S$ and that S is a domain of finite type over R.

If $R \to S$ has a nontrivial kernel, then take a nonzero $f \in R$ in this kernel. In this case $S_f = 0$ and the lemma holds. (This is really the case $d = -1$ and the start of the induction.) Hence we may assume that $R \to S$ is a finite type extension of Noetherian domains.

Apply Lemma 10.115.7 and replace R by R_f (with f as in the lemma) to get a factorization

$$R \subset R[y_1, \ldots, y_d] \subset S$$

where the second extension is finite. Choose $z_1, \ldots, z_r \in S$ which form a basis for the fraction field of S over the fraction field of $R[y_1, \ldots, y_d]$. This gives a short exact sequence

$$0 \to R[y_1, \ldots, y_d]^{\oplus r} \xrightarrow{(z_1, \ldots, z_r)} S \to N \to 0$$

By construction N is a finite $R[y_1, \ldots, y_d]$-module whose support does not contain the generic point (0) of $\mathrm{Spec}(R[y_1, \ldots, y_d])$. By Lemma 10.40.5 there exists a nonzero $g \in R[y_1, \ldots, y_d]$ such that g annihilates N, so we may view N as a finite module over $S' = R[y_1, \ldots, y_d]/(g)$. Since $\dim(S'_K) < d$ by induction there exists a nonzero $f \in R$ such that N_f is a free R_f-module. Since $(R[y_1, \ldots, y_d])_f \cong R_f[y_1, \ldots, y_d]$ is free also we conclude by the already mentioned fact that an extension of free modules is free. □

051S **Lemma 10.118.2.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

(1) *R is a domain,*
(2) *$R \to S$ is of finite presentation, and*
(3) *M is an S-module of finite presentation.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. For $g \in R[x_1, \ldots, x_n]$ denote $\overline{g}$ its image in S. We may write $M = S^{\oplus t}/\sum Sn_i$ for some $n_i \in S^{\oplus t}$. Write $n_i = (\overline{g}_{i1}, \ldots, \overline{g}_{it})$ for some $g_{ij} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated by all the coefficients of all the elements $g_i, g_{ij} \in R[x_1, \ldots, x_n]$. Define $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Define $M_0 = S_0^{\oplus t}/\sum S_0 n_i$. Then R_0 is a domain of finite type over $\mathbf{Z}$ and hence Noetherian (see Lemma 10.31.1). Moreover via the injection $R_0 \to R$ we have $S \cong R \otimes_{R_0} S_0$ and $M \cong R \otimes_{R_0} M_0$. Applying Lemma 10.118.1 we obtain a nonzero $f \in R_0$ such that $(M_0)_f$ is a free $(R_0)_f$-module. Hence $M_f = R_f \otimes_{(R_0)_f} (M_0)_f$ is a free R_f-module. □

051T **Lemma 10.118.3.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

(1) *R is a domain,*
(2) *$R \to S$ is of finite type, and*
(3) *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that

(a) *M_f and S_f are free as R_f-modules, and*
(b) *S_f is a finitely presented R_f-algebra and M_f is a finitely presented S_f-module.*

Proof. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Assume $S = R[x_1, \ldots, x_n]$. Choose elements $m_1, \ldots, m_t$ which generate M. This gives a short exact sequence

$$0 \to N \to S^{\oplus t} \xrightarrow{(m_1, \ldots, m_t)} M \to 0.$$

Denote K the fraction field of R. Denote $S_K = K \otimes_R S = K[x_1, \ldots, x_n]$, and similarly $N_K = K \otimes_R N$, $M_K = K \otimes_R M$. As $R \to K$ is flat the sequence remains exact after tensoring with K. As $S_K = K[x_1, \ldots, x_n]$ is a Noetherian ring (see Lemma 10.31.1) we can find finitely many elements $n'_1, \ldots, n'_s \in N_K$ which generate it. Choose $n_1, \ldots, n_r \in N$ such that $n'_i = \sum a_{ij} n_j$ for some $a_{ij} \in K$. Set

$$M' = S^{\oplus t} / \sum\nolimits_{i=1,\ldots,r} Sn_i$$

By construction M' is a finitely presented S-module, and there is a surjection $M' \to M$ which induces an isomorphism $M'_K \cong M_K$. We may apply Lemma 10.118.2 to $R \to S$ and M' and we find an $f \in R$ such that M'_f is a free R_f-module. Thus $M'_f \to M_f$ is a surjection of modules over the domain R_f where the source is a free module and which becomes an isomorphism upon tensoring with K. Thus it is injective as $M'_f \subset M'_K$ because M'_f is free and R_f is a domain with fraction field K. Hence $M'_f \to M_f$ is an isomorphism and the result is proved.

For the general case, choose a surjection $R[x_1, \ldots, x_n] \to S$. Think of both S and M as finite modules over $R[x_1, \ldots, x_n]$. By the special case proved above there exists a nonzero $f \in R$ such that both S_f and M_f are free as R_f-modules and finitely presented as $R_f[x_1, \ldots, x_n]$-modules. Clearly this implies that S_f is a finitely presented R_f-algebra and that M_f is a finitely presented S_f-module. □

Let $R \to S$ be a ring map. Let M be an S-module. Consider the following condition on an element $f \in R$:

051U (10.118.3.1)
$$\begin{cases} S_f & \text{is of finite presentation over } R_f \\ M_f & \text{is of finite presentation as } S_f\text{-module} \\ S_f, M_f & \text{are free as } R_f\text{-modules} \end{cases}$$

We define

051V (10.118.3.2)
$$U(R \to S, M) = \bigcup\nolimits_{f \in R \text{ with (10.118.3.1)}} D(f)$$

which is an open subset of $\mathrm{Spec}(R)$.

051W **Lemma 10.118.4.** *Let $R \to S$ be a ring map. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of S-modules. Then*

$$U(R \to S, M_1) \cap U(R \to S, M_3) \subset U(R \to S, M_2).$$

Proof. Let $u \in U(R \to S, M_1) \cap U(R \to S, M_3)$. Choose $f_1, f_3 \in R$ such that $u \in D(f_1)$, $u \in D(f_3)$ and such that (10.118.3.1) holds for f_1 and M_1 and for f_3 and M_3. Then set $f = f_1 f_3$. Then $u \in D(f)$ and (10.118.3.1) holds for f and both M_1 and M_3. An extension of free modules is free, and an extension of finitely presented modules is finitely presented (Lemma 10.5.3). Hence we see that (10.118.3.1) holds for f and M_2. Thus $u \in U(R \to S, M_2)$ and we win. □

051X **Lemma 10.118.5.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $f \in R$. Using the identification* $\mathrm{Spec}(R_f) = D(f)$ *we have $U(R_f \to S_f, M_f) = D(f) \cap U(R \to S, M)$.*

Proof. Suppose that $u \in U(R_f \to S_f, M_f)$. Then there exists an element $g \in R_f$ such that $u \in D(g)$ and such that (10.118.3.1) holds for the pair $((R_f)_g \to (S_f)_g, (M_f)_g)$. Write $g = a/f^n$ for some $a \in R$. Set $h = af$. Then $R_h = (R_f)_g$, $S_h = (S_f)_g$, and $M_h = (M_f)_g$. Moreover $u \in D(h)$. Hence $u \in U(R \to S, M)$. Conversely, suppose that $u \in D(f) \cap U(R \to S, M)$. Then there exists an element $g \in R$ such that $u \in D(g)$ and such that (10.118.3.1) holds for the pair $(R_g \to S_g, M_g)$. Then it is clear that (10.118.3.1) also holds for the pair $(R_{fg} \to S_{fg}, M_{fg}) = ((R_f)_g \to (S_f)_g, (M_f)_g)$. Hence $u \in U(R_f \to S_f, M_f)$ and we win. □

051Y **Lemma 10.118.6.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $U \subset \mathrm{Spec}(R)$ be a dense open. Assume there is a covering $U = \bigcup_{i \in I} D(f_i)$ of opens such that $U(R_{f_i} \to S_{f_i}, M_{f_i})$ is dense in $D(f_i)$ for each $i \in I$. Then $U(R \to S, M)$ is dense in* $\mathrm{Spec}(R)$.

Proof. In view of Lemma 10.118.5 this is a purely topological statement. Namely, by that lemma we see that $U(R \to S, M) \cap D(f_i)$ is dense in $D(f_i)$ for each $i \in I$. By Topology, Lemma 5.21.4 we see that $U(R \to S, M) \cap U$ is dense in U. Since U is dense in $\mathrm{Spec}(R)$ we conclude that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. □

051Z **Lemma 10.118.7.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

(1) *$R \to S$ is of finite type,*
(2) *M is a finite S-module, and*
(3) *R is reduced.*

Then there exists a subset $U \subset \mathrm{Spec}(R)$ such that

(1) *U is open and dense in* $\mathrm{Spec}(R)$,
(2) *for every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that we have*
   (a) *M_f and S_f are free over R_f,*
   (b) *S_f is a finitely presented R_f-algebra, and*
   (c) *M_f is a finitely presented S_f-module.*

Proof. Note that the lemma is equivalent to the statement that the open $U(R \to S, M)$, see Equation (10.118.3.2), is dense in $\mathrm{Spec}(R)$. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Proof of the case $S = R[x_1, \ldots, x_n]$ and M any finite module over S. Note that in this case $S_f = R_f[x_1, \ldots, x_n]$ is free and of finite presentation over R_f, so we do not have to worry about the conditions regarding S, only those that concern M. We will use induction on n.

There exists a finite filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

such that $M_i/M_{i-1} \cong S/J_i$ for some ideal $J_i \subset S$, see Lemma 10.5.4. Since a finite intersection of dense opens is dense open, we see from Lemma 10.118.4 that it suffices to prove the lemma for each of the modules R/J_i. Hence we may assume that $M = S/J$ for some ideal J of $S = R[x_1, \ldots, x_n]$.

Let $I \subset R$ be the ideal generated by the coefficients of elements of J. Let $U_1 = \mathrm{Spec}(R) \setminus V(I)$ and let

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.118.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $I = R$, or (b) $V(I) = \mathrm{Spec}(R)$.

In case (b) we actually have $I = 0$ as R is reduced! Hence $J = 0$ and $M = S$ and the lemma holds in this case.

In case (a) we have to do a little bit more work. Note that every element of I is actually the coefficient of a monomial of an element of J, because the set of coefficients of elements of J forms an ideal (details omitted). Hence we find an element

$$g = \sum\nolimits_{K \in E} a_K x^K \in J$$

where E is a finite set of multi-indices $K = (k_1, \ldots, k_n)$ with at least one coefficient a_{K_0} a unit in R. Actually we can find one which has a coefficient equal to 1 as $1 \in I$ in case (a). Let $m = \#\{K \in E \mid a_K \text{ is not a unit}\}$. Note that $0 \leq m \leq \#E - 1$. We will argue by induction on m.

The case $m = 0$. In this case all the coefficients a_K, $K \in E$ of g are units and $E \neq \emptyset$. If $E = \{K_0\}$ is a singleton and $K_0 = (0, \ldots, 0)$, then g is a unit and $J = S$ so the result holds for sure. (This happens in particular when $n = 0$ and it provides the base case of the induction on n.) If not $E = \{(0, \ldots, 0)\}$, then at least one K is not equal to $(0, \ldots, 0)$, i.e., $g \notin R$. At this point we employ the usual trick of Noether normalization. Namely, we consider

$$G(y_1, \ldots, y_n) = g(y_1 + y_n^{e_1}, y_2 + y_n^{e_2}, \ldots, y_{n-1} + y_n^{e_{n-1}}, y_n)$$

with $0 \ll e_{n-1} \ll e_{n-2} \ll \ldots \ll e_1$. By Lemma 10.115.2 it follows that $G(y_1, \ldots, y_n)$ as a polynomial in y_n looks like

$$a_K y_n^{k_n + \sum_{i=1,\ldots,n-1} e_i k_i} + \text{lower order terms in } y_n$$

As a_K is a unit we conclude that $M = R[x_1, \ldots, x_n]/J$ is finite over $R[y_1, \ldots, y_{n-1}]$. Hence $U(R \to R[x_1, \ldots, x_n], M) = U(R \to R[y_1, \ldots, y_{n-1}], M)$ and we win by induction on n.

The case $m > 0$. Pick a multi-index $K \in E$ such that a_K is not a unit. As before set $U_1 = \mathrm{Spec}(R_{a_K}) = \mathrm{Spec}(R) \setminus V(a_K)$ and set

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.118.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $a_K R = R$, or (b) $V(a_K) = \mathrm{Spec}(R)$. In case (a) the number m drops, as a_K has turned into a unit. In case (b), since R is reduced, we conclude that $a_K = 0$. Hence the set E decreases so the number m drops as well. In both cases we win by induction on m.

At this point we have proven the lemma in case $S = R[x_1, \ldots, x_n]$. Assume that $(R \to S, M)$ is an arbitrary pair satisfying the conditions of the lemma. Choose a surjection $R[x_1, \ldots, x_n] \to S$. Observe that, with the notation introduced in (10.118.3.2), we have

$$U(R \to S, M) = U(R \to R[x_1, \ldots, x_n], S) \cap U(R \to R[x_1, \ldots, x_n], M)$$

Hence as we've just finished proving the right two opens are dense also the open on the left is dense. □

10.119. Around Krull-Akizuki

00P7 One application of Krull-Akizuki is to show that there are plenty of discrete valuation rings. More generally in this section we show how to construct discrete valuation rings dominating Noetherian local rings.

First we show how to dominate a Noetherian local domain by a 1-dimensional Noetherian local domain by blowing up the maximal ideal.

00P8 **Lemma 10.119.1.** *Let R be a local Noetherian domain with fraction field K. Assume R is not a field. Then there exist $R \subset R' \subset K$ with*

(1) *R' local Noetherian of dimension 1,*
(2) *$R \to R'$ a local ring map, i.e., R' dominates R, and*
(3) *$R \to R'$ essentially of finite type.*

Proof. Choose any valuation ring $A \subset K$ dominating R (which exist by Lemma 10.50.2). Denote v the corresponding valuation. Let $x_1, \ldots, x_r$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. We may and do assume that $v(x_r) = \min\{v(x_1), \ldots, v(x_r)\}$. Consider the ring

$$S = R[x_1/x_r, x_2/x_r, \ldots, x_{r-1}/x_r] \subset K.$$

Note that $\mathfrak{m}S = x_rS$ is a principal ideal. Note that $S \subset A$ and that $v(x_r) > 0$, hence we see that $x_rS \neq S$. Choose a minimal prime $\mathfrak{q}$ over x_rS. Then $\text{height}(\mathfrak{q}) = 1$ by Lemma 10.60.11 and $\mathfrak{q}$ lies over $\mathfrak{m}$. Hence we see that $R' = S_{\mathfrak{q}}$ is a solution. □

0BHZ **Lemma 10.119.2** (Kollár)**.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Then exactly one of the following holds:*

(1) *$(R, \mathfrak{m})$ is Artinian,*
(2) *$(R, \mathfrak{m})$ is regular of dimension 1,*
(3) *$\text{depth}(R) \geq 2$, or*
(4) *there exists a finite ring map $R \to R'$ which is not an isomorphism whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$ such that $\mathfrak{m}$ is not an associated prime of R' and $R' \neq 0$.*

This is taken from a forthcoming paper by János Kollár entitled "Variants of normality for Noetherian schemes".

Proof. Observe that $(R, \mathfrak{m})$ is not Artinian if and only if $V(\mathfrak{m}) \subset \text{Spec}(R)$ is nowhere dense. See Proposition 10.60.7. We assume this from now on.

Let $J \subset R$ be the largest ideal killed by a power of $\mathfrak{m}$. If $J \neq 0$ then $R \to R/J$ shows that $(R, \mathfrak{m})$ is as in (4).

Otherwise $J = 0$. In particular $\mathfrak{m}$ is not an associated prime of R and we see that there is a nonzerodivisor $x \in \mathfrak{m}$ by Lemma 10.63.18. If $\mathfrak{m}$ is not an associated prime of R/xR then $\text{depth}(R) \geq 2$ by the same lemma. Thus we are left with the case when there is a $y \in R$, $y \notin xR$ such that $y\mathfrak{m} \subset xR$.

If $y\mathfrak{m} \subset x\mathfrak{m}$ then we can consider the map $\varphi : \mathfrak{m} \to \mathfrak{m}$, $f \mapsto yf/x$ (well defined as x is a nonzerodivisor). By the determinantal trick of Lemma 10.16.2 there exists a monic polynomial P with coefficients in R such that $P(\varphi) = 0$. We conclude that $P(y/x) = 0$ in R_x. Let $R' \subset R_x$ be the ring generated by R and y/x. Then $R \subset R'$ and R'/R is a finite R-module annihilated by a power of $\mathfrak{m}$. Thus R is as in (4).

Otherwise there is a $t \in \mathfrak{m}$ such that $yt = ux$ for some unit u of R. After replacing t by $u^{-1}t$ we get $yt = x$. In particular y is a nonzerodivisor. For any $t' \in \mathfrak{m}$ we have $yt' = xs$ for some $s \in R$. Thus $y(t' - st) = xs - xs = 0$. Since y is not a zero-divisor this implies that $t' = ts$ and so $\mathfrak{m} = (t)$. Thus $(R, \mathfrak{m})$ is regular of dimension 1.

The argument so far shows that every R falls into one of the 4 cases. To finish we have to show that no two of the properties can hold simultaneously for each of the 6 combinations of properties. We'll just show that (2) and (3) each exclude (4); the other cases are left to the reader.

Assume R is regular of dimension 1 and that $R \to R'$ is a finite ring map whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$ and $\mathfrak{m}$ is not an associated prime of R'. Then R' is a finite R-module of depth at least 1 and hence free by Lemma 10.106.6. Since $R \to R'$ is an isomorphism in the generic point, we conclude that R' must be free of rank 1 as an R-module. Then $R' \to \mathrm{End}_R(R') \cong R$ is an inverse to the map $R \to R'$. Thus (4) cannot be true.

Assume R has depth ≥ 2 and that $R \to R'$ is a finite ring map whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$ and $\mathfrak{m}$ is not an associated prime of R'. Then $R \to R'$ is necessarily injective (as the kernel would have both depth 0 and depth ≥ 1) and the cokernel has depth at least 1 (by Lemma 10.72.6 and the fact that R' has depth ≥ 1 by assumption) whence cannot be supported on $\{\mathfrak{m}\}$ unless it is 0 as well. Thus (4) cannot be true. □

00P9 **Lemma 10.119.3.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume R is Noetherian, has dimension 1, and that $\dim(\mathfrak{m}/\mathfrak{m}^2) > 1$. Then there exists a ring map $R \to R'$ such that*

(1) *$R \to R'$ is finite,*
(2) *$R \to R'$ is not an isomorphism,*
(3) *the kernel and cokernel of $R \to R'$ are annihilated by a power of $\mathfrak{m}$, and*
(4) *$\mathfrak{m}$ is not an associated prime of R'.*

Proof. This follows from Lemma 10.119.2 and the fact that R is not Artinian, not regular, and does not have depth ≥ 2 (the last part because the depth does not exceed the dimension by Lemma 10.72.3). □

00PA **Example 10.119.4.** Consider the Noetherian local ring

$$R = k[[x, y]]/(y^2)$$

It has dimension 1 and it is Cohen-Macaulay. An example of an extension as in Lemma 10.119.3 is the extension

$$k[[x, y]]/(y^2) \subset k[[x, z]]/(z^2), \quad y \mapsto xz$$

in other words it is gotten by adjoining y/x to R. The effect of repeating the construction $n > 1$ times is to adjoin the element y/x^n.

00PB **Example 10.119.5.** Let k be a field of characteristic $p > 0$ such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \sum a_i x^i \in k[[x]] \text{ such that } [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}$$

Then A is a discrete valuation ring and its completion is $A^\wedge = k[[x]]$. Note that the induced extension of fraction fields of $A \subset k[[x]]$ is infinite purely inseparable. Choose any $f \in k[[x]]$, $f \notin A$. Let $R = A[f] \subset k[[x]]$. Then R is a Noetherian local domain of dimension 1 whose completion $R^\wedge$ is nonreduced (think!).

00PC **Remark 10.119.6.** Suppose that R is a 1-dimensional semi-local Noetherian domain. If there is a maximal ideal $\mathfrak{m} \subset R$ such that $R_\mathfrak{m}$ is not regular, then we may apply Lemma 10.119.3 to $(R, \mathfrak{m})$ to get a finite ring extension $R \subset R_1$. (For example one can do this so that $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ is the blowup of $\mathrm{Spec}(R)$ in the ideal $\mathfrak{m}$.) Of course R_1 is a 1-dimensional semi-local Noetherian domain with the same fraction field as R. If R_1 is not a regular semi-local ring, then we may repeat the construction to get $R_1 \subset R_2$. Thus we get a sequence

$$R \subset R_1 \subset R_2 \subset R_3 \subset \ldots$$

of finite ring extensions which may stop if R_n is regular for some n. Resolution of singularities would be the claim that eventually R_n is indeed regular. In reality this is not the case. Namely, there exists a characteristic 0 Noetherian local domain A of dimension 1 whose completion is nonreduced, see [**FR70**, Proposition 3.1] or our Examples, Section 110.17. For an example in characteristic $p > 0$ see Example 10.119.5. Since the construction of blowing up commutes with completion it is easy to see the sequence never stabilizes. See [**Ben73**] for a discussion (mostly in positive characteristic). On the other hand, if the completion of R in all of its maximal ideals is reduced, then the procedure stops (insert future reference here).

00PD **Lemma 10.119.7.** *Let A be a ring. The following are equivalent.*

(1) *The ring A is a discrete valuation ring.*
(2) *The ring A is a valuation ring and Noetherian but not a field.*
(3) *The ring A is a regular local ring of dimension 1.*
(4) *The ring A is a Noetherian local domain with maximal ideal $\mathfrak{m}$ generated by a single nonzero element.*
(5) *The ring A is a Noetherian local normal domain of dimension 1.*

In this case if π is a generator of the maximal ideal of A, then every element of A can be uniquely written as $u\pi^n$, where $u \in A$ is a unit.

Proof. The equivalence of (1) and (2) is Lemma 10.50.18. Moreover, in the proof of Lemma 10.50.18 we saw that if A is a discrete valuation ring, then A is a PID, hence (3). Note that a regular local ring is a domain (see Lemma 10.106.2). Using this the equivalence of (3) and (4) follows from dimension theory, see Section 10.60.

Assume (3) and let π be a generator of the maximal ideal $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{A/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ because it is generated by π^n (and it cannot be zero). In particular $\mathfrak{m}^n = (\pi^n)$ and the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the polynomial ring $A/\mathfrak{m}[T]$. For $x \in A \setminus \{0\}$ define $v(x) = \max\{n \mid x \in \mathfrak{m}^n\}$. In other words $x = u\pi^{v(x)}$ with $u \in A^*$. By the remarks above we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A by setting

$v(a/b) = v(a) - v(b)$ (well defined by multiplicativity shown above). Then it is clear that A is the set of elements of K which have valuation ≥ 0. Hence we see that A is a valuation ring by Lemma 10.50.16.

A valuation ring is a normal domain by Lemma 10.50.3. Hence we see that the equivalent conditions (1) – (3) imply (5). Assume (5). Suppose that $\mathfrak{m}$ cannot be generated by 1 element to get a contradiction. Then Lemma 10.119.3 implies there is a finite ring map $A \to A'$ which is an isomorphism after inverting any nonzero element of $\mathfrak{m}$ but not an isomorphism. In particular we may identify A' with a subset of the fraction field of A. Since $A \to A'$ is finite it is integral (see Lemma 10.36.3). Since A is normal we get $A = A'$ a contradiction. □

09DZ **Definition 10.119.8.** Let A be a discrete valuation ring. A *uniformizer* is an element $\pi \in A$ which generates the maximal ideal of A.

By Lemma 10.119.7 any two uniformizers of a discrete valuation ring are associates.

00PE **Lemma 10.119.9.** *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is local Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(R/xR) < \infty$ and*

$$\text{length}_R(M/xM) \leq r \cdot \text{length}_R(R/xR).$$

Proof. If x is a unit then the result is true. Hence we may assume $x \in \mathfrak{m}$ the maximal ideal of R. Since x is not zero and R is a domain we have $\dim(R/xR) = 0$, and hence R/xR has finite length. Consider $M \subset K^{\oplus r}$ as in the lemma. We may assume that the elements of M generate $K^{\oplus r}$ as a K-vector space after replacing $K^{\oplus r}$ by a smaller subspace if necessary.

Suppose first that M is a finite R-module. In that case we can clear denominators and assume $M \subset R^{\oplus r}$. Since M generates $K^{\oplus r}$ as a vectors space we see that $R^{\oplus r}/M$ has finite length. In particular there exists an integer $c \geq 0$ such that $x^c R^{\oplus r} \subset M$. Note that $M \supset xM \supset x^2M \supset \ldots$ is a sequence of modules with successive quotients each isomorphic to M/xM. Hence we see that

$$n\text{length}_R(M/xM) = \text{length}_R(M/x^nM).$$

The same argument for $M = R^{\oplus r}$ shows that

$$n\text{length}_R(R^{\oplus r}/xR^{\oplus r}) = \text{length}_R(R^{\oplus r}/x^nR^{\oplus r}).$$

By our choice of c above we see that x^nM is sandwiched between $x^nR^{\oplus r}$ and $x^{n+c}R^{\oplus r}$. This easily gives that

$$r(n+c)\text{length}_R(R/xR) \geq n\text{length}_R(M/xM) \geq r(n-c)\text{length}_R(R/xR)$$

Hence in the finite case we actually get the result of the lemma with equality.

Suppose now that M is not finite. Suppose that the length of M/xM is $\geq k$ for some natural number k. Then we can find

$$0 \subset N_0 \subset N_1 \subset N_2 \subset \ldots N_k \subset M/xM$$

with $N_i \neq N_{i+1}$ for $i = 0, \ldots k-1$. Choose an element $m_i \in M$ whose congruence class mod xM falls into N_i but not into N_{i-1} for $i = 1, \ldots, k$. Consider the finite R-module $M' = Rm_1 + \ldots + Rm_k \subset M$. Let $N_i' \subset M'/xM'$ be the inverse image of N_i. It is clear that $N_i' \neq N_{i+1}'$ by our choice of m_i. Hence we see that $\text{length}_R(M'/xM') \geq k$. By the finite case we conclude $k \leq r\text{length}_R(R/xR)$ as desired. □

Here is a first application.

031F **Lemma 10.119.10.** *Let $R \to S$ be a homomorphism of domains inducing an injection of fraction fields $K \subset L$. If R is Noetherian local of dimension 1 and $[L : K] < \infty$ then*

(1) *each prime ideal $\mathfrak{n}_i$ of S lying over the maximal ideal $\mathfrak{m}$ of R is maximal,*
(2) *there are finitely many of these, and*
(3) *$[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] < \infty$ for each i.*

Proof. Pick $x \in \mathfrak{m}$ nonzero. Apply Lemma 10.119.9 to the submodule $S \subset L \cong K^{\oplus n}$ where $n = [L : K]$. Thus the ring S/xS has finite length over R. It follows that $S/\mathfrak{m}S$ has finite length over $\kappa(\mathfrak{m})$. In other words, $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S$ is finite (Lemma 10.52.6). Thus $S/\mathfrak{m}S$ is Artinian (Lemma 10.53.2). The structural results on Artinian rings implies parts (1) and (2), see for example Lemma 10.53.6. Part (3) is implied by the finiteness established above. □

00PF **Lemma 10.119.11.** *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(M/xM) < \infty$.*

Proof. Since R has dimension 1 we see that x is contained in finitely many primes $\mathfrak{m}_i$, $i = 1, \ldots, n$, each maximal. Since R is Noetherian we see that R/xR is Artinian and $R/xR = \prod_{i=1,\ldots,n}(R/xR)_{\mathfrak{m}_i}$ by Proposition 10.60.7 and Lemma 10.53.6. Hence M/xM similarly decomposes as the product $M/xM = \prod(M/xM)_{\mathfrak{m}_i}$ of its localizations at the $\mathfrak{m}_i$. By Lemma 10.119.9 applied to $M_{\mathfrak{m}_i}$ over $R_{\mathfrak{m}_i}$ we see each $M_{\mathfrak{m}_i}/xM_{\mathfrak{m}_i} = (M/xM)_{\mathfrak{m}_i}$ has finite length over $R_{\mathfrak{m}_i}$. Thus M/xM has finite length over R as the above implies M/xM has a finite filtration by R-submodules whose successive quotients are isomorphic to the residue fields $\kappa(\mathfrak{m}_i)$. □

00PG **Lemma 10.119.12** (Krull-Akizuki)**.** *Let R be a domain with fraction field K. Let L/K be a finite extension of fields. Assume R is Noetherian and $\dim(R) = 1$. In this case any ring A with $R \subset A \subset L$ is Noetherian.*

Proof. Let $I \subset A$ be a nonzero ideal. By Lemma 10.30.8 we can find a nonzero element $x \in I \cap R$. Then we get $I/xA \subset A/xA$. By Lemma 10.119.11 the R-module A/xA has finite length as an R-module. Hence I/xA has finite length as an R-module. Hence I is finitely generated as an ideal in A. □

00PH **Lemma 10.119.13.** *Let R be a Noetherian local domain with fraction field K. Assume that R is not a field. Let L/K be a finitely generated field extension. Then there exists discrete valuation ring A with fraction field L which dominates R.*

Proof. If L is not finite over K choose a transcendence basis $x_1, \ldots, x_r$ of L over K and replace R by $R[x_1, \ldots, x_r]$ localized at the maximal ideal generated by $\mathfrak{m}_R$ and $x_1, \ldots, x_r$. Thus we may assume $K \subset L$ finite.

By Lemma 10.119.1 we may assume $\dim(R) = 1$.

Let $A \subset L$ be the integral closure of R in L. By Lemma 10.119.12 this is Noetherian. By Lemma 10.36.17 there is a prime ideal $\mathfrak{q} \subset A$ lying over the maximal ideal of R. By Lemma 10.119.7 the ring $A_{\mathfrak{q}}$ is a discrete valuation ring dominating R as desired. □

10.120. Factorization

034O Here are some notions and relations between them that are typically taught in a first year course on algebra at the undergraduate level.

034P **Definition 10.120.1.** Let R be a domain.

(1) Elements $x, y \in R$ are called *associates* if there exists a unit $u \in R^*$ such that $x = uy$.
(2) An element $x \in R$ is called *irreducible* if it is nonzero, not a unit and whenever $x = yz$, $y, z \in R$, then y is either a unit or an associate of x.
(3) An element $x \in R$ is called *prime* if the ideal generated by x is a prime ideal.

034Q **Lemma 10.120.2.** *Let R be a domain. Let $x, y \in R$. Then x, y are associates if and only if $(x) = (y)$.*

Proof. If $x = uy$ for some unit $u \in R$, then $(x) \subset (y)$ and $y = u^{-1}x$ so also $(y) \subset (x)$. Conversely, suppose that $(x) = (y)$. Then $x = fy$ and $y = gx$ for some $f, g \in A$. Then $x = fgx$ and since R is a domain $fg = 1$. Thus x and y are associates. □

034R **Lemma 10.120.3.** *Let R be a domain. Consider the following conditions:*

(1) *The ring R satisfies the ascending chain condition for principal ideals.*
(2) *Every nonzero, nonunit element $a \in R$ has a factorization $a = b_1 \ldots b_k$ with each b_i an irreducible element of R.*

Then (1) implies (2).

Proof. Let x be a nonzero element, not a unit, which does not have a factorization into irreducibles. Set $x_1 = x$. We can write $x = yz$ where neither y nor z is irreducible or a unit. Then either y does not have a factorization into irreducibles, in which case we set $x_2 = y$, or z does not have a factorization into irreducibles, in which case we set $x_2 = z$. Continuing in this fashion we find a sequence

$$\ldots | x_3 | x_2 | x_1$$

of elements of R with x_n/x_{n+1} not a unit. This gives a strictly increasing sequence of principal ideals $(x_1) \subset (x_2) \subset (x_3) \subset \ldots$ thereby finishing the proof. □

034S **Definition 10.120.4.** A *unique factorization domain*, abbreviated *UFD*, is a domain R such that if $x \in R$ is a nonzero, nonunit, then x has a factorization into irreducibles, and if

$$x = a_1 \ldots a_m = b_1 \ldots b_n$$

are factorizations into irreducibles then $n = m$ and there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates.

034T **Lemma 10.120.5.** *Let R be a domain. Assume every nonzero, nonunit factors into irreducibles. Then R is a UFD if and only if every irreducible element is prime.*

Proof. Assume R is a UFD and let $x \in R$ be an irreducible element. Say $ab \in (x)$, i.e., $ab = cx$. Choose factorizations $a = a_1 \ldots a_n$, $b = b_1 \ldots b_m$, and $c = c_1 \ldots c_r$. By uniqueness of the factorization

$$a_1 \ldots a_n b_1 \ldots b_m = c_1 \ldots c_r x$$

we find that x is an associate of one of the elements $a_1, \ldots, b_m$. In other words, either $a \in (x)$ or $b \in (x)$ and we conclude that x is prime.

Assume every irreducible element is prime. We have to prove that factorization into irreducibles is unique up to permutation and taking associates. Say $a_1 \ldots a_m = b_1 \ldots b_n$ with a_i and b_j irreducible. Since a_1 is prime, we see that $b_j \in (a_1)$ for some j. After renumbering we may assume $b_1 \in (a_1)$. Then $b_1 = a_1 u$ and since b_1 is irreducible we see that u is a unit. Hence a_1 and b_1 are associates and $a_2 \ldots a_n = u b_2 \ldots b_m$. By induction on $n + m$ we see that $n = m$ and a_i associate to $b_{\sigma(i)}$ for $i = 2, \ldots, n$ as desired. □

0AFT **Lemma 10.120.6.** *Let R be a Noetherian domain. Then R is a UFD if and only if every height 1 prime ideal is principal.*

Proof. Assume R is a UFD and let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ nonzero and let $x = a_1 \ldots a_n$ be a factorization into irreducibles. Since $\mathfrak{p}$ is prime we see that $a_i \in \mathfrak{p}$ for some i. By Lemma 10.120.5 the ideal (a_i) is prime. Since $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$.

Assume every height 1 prime is principal. Since R is Noetherian every nonzero nonunit element x has a factorization into irreducibles, see Lemma 10.120.3. It suffices to prove that an irreducible element x is prime, see Lemma 10.120.5. Let $(x) \subset \mathfrak{p}$ be a prime minimal over (x). Then $\mathfrak{p}$ has height 1 by Lemma 10.60.11. By assumption $\mathfrak{p} = (y)$. Hence $x = yz$ and z is a unit as x is irreducible. Thus $(x) = (y)$ and we see that x is prime. □

0AFU **Lemma 10.120.7** (Nagata's criterion for factoriality)**.** *Let A be a domain. Let $S \subset A$ be a multiplicative subset generated by prime elements. Let $x \in A$ be irreducible. Then* [**Nag57b**, Lemma 2]

(1) *the image of x in $S^{-1}A$ is irreducible or a unit, and*
(2) *x is prime if and only if the image of x in $S^{-1}A$ is a prime element or a unit in $S^{-1}A$.*

Moreover, then A is a UFD if and only if every nonzero nonunit element of A has a factorization into irreducibles and $S^{-1}A$ is a UFD.

Proof. Say $x = \alpha\beta$ for $\alpha, \beta \in S^{-1}A$. Then $\alpha = a/s$ and $\beta = b/s'$ for $a, b \in A$, $s, s' \in S$. Thus we get $ss'x = ab$. By assumption we can write $ss' = p_1 \ldots p_r$ for some prime elements p_i. For each i the element p_i divides either a or b. Dividing we find a factorization $x = a'b'$ and $a = s''a'$, $b = s'''b'$ for some $s'', s''' \in S$. As x is irreducible, either a' or b' is a unit. Tracing back we find that either α or β is a unit. This proves (1).

Suppose x is prime. Then $A/(x)$ is a domain. Hence $S^{-1}A/xS^{-1}A = S^{-1}(A/(x))$ is a domain or zero. Thus x maps to a prime element or a unit.

Suppose that the image of x in $S^{-1}A$ is a unit. Then $yx = s$ for some $s \in S$ and $y \in A$. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. But if the first case happens for all $i = 1, \ldots, r$, then x is a unit which is a contradiction.

Suppose that the image of x in $S^{-1}A$ is a prime element. Assume $a, b \in A$ and $ab \in (x)$. Then $sa = xy$ or $sb = xy$ for some $s \in S$ and $y \in A$. Say the first case

happens. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. If the first case happens for all $i = 1, \ldots, r$, then $a \in (x)$ as desired. This completes the proof of (2).

The final statement of the lemma follows from (1) and (2) and Lemma 10.120.5. □

0BUD **Lemma 10.120.8.** *A UFD satisfies the ascending chain condition for principal ideals.*

Proof. Consider an ascending chain $(a_1) \subset (a_2) \subset (a_3) \subset \ldots$ of principal ideals in R. Write $a_1 = p_1^{e_1} \ldots p_r^{e_r}$ with p_i prime. Then we see that a_n is an associate of $p_1^{c_1} \ldots p_r^{c_r}$ for some $0 \leq c_i \leq e_i$. Since there are only finitely many possibilities we conclude. □

0BUE **Lemma 10.120.9.** *Let R be a domain. Assume R has the ascending chain condition for principal ideals. Then the same property holds for a polynomial ring over R.*

Proof. Consider an ascending chain $(f_1) \subset (f_2) \subset (f_3) \subset \ldots$ of principal ideals in $R[x]$. Since f_{n+1} divides f_n we see that the degrees decrease in the sequence. Thus f_n has fixed degree $d \geq 0$ for all $n \gg 0$. Let a_n be the leading coefficient of f_n. The condition $f_n \in (f_{n+1})$ implies that a_{n+1} divides a_n for all n. By our assumption on R we see that a_{n+1} and a_n are associates for all n large enough (Lemma 10.120.2). Thus for large n we see that $f_n = uf_{n+1}$ where $u \in R$ (for reasons of degree) is a unit (as a_n and a_{n+1} are associates). □

0BC1 **Lemma 10.120.10.** *A polynomial ring over a UFD is a UFD. In particular, if k is a field, then $k[x_1, \ldots, x_n]$ is a UFD.*

Proof. Let R be a UFD. Then R satisfies the ascending chain condition for principal ideals (Lemma 10.120.8), hence $R[x]$ satisfies the ascending chain condition for principal ideals (Lemma 10.120.9), and hence every element of $R[x]$ has a factorization into irreducibles (Lemma 10.120.3). Let $S \subset R$ be the multiplicative subset generated by prime elements. Since every nonunit of R is a product of prime elements we see that $K = S^{-1}R$ is the fraction field of R. Observe that every prime element of R maps to a prime element of $R[x]$ and that $S^{-1}(R[x]) = S^{-1}R[x] = K[x]$ is a UFD (and even a PID). Thus we may apply Lemma 10.120.7 to conclude. □

0AFV **Lemma 10.120.11.** *A unique factorization domain is normal.*

Proof. Let R be a UFD. Let x be an element of the fraction field of R which is integral over R. Say $x^d - a_1 x^{d-1} - \ldots - a_d = 0$ with $a_i \in R$. We can write $x = up_1^{e_1} \ldots p_r^{e_r}$ with u a unit, $e_i \in \mathbf{Z}$, and $p_1, \ldots, p_r$ irreducible elements which are not associates. To prove the lemma we have to show $e_i \geq 0$. If not, say $e_1 < 0$, then for $N \gg 0$ we get

$$u^d p_2^{de_2+N} \ldots p_r^{de_r+N} = p_1^{-de_1} p_2^N \ldots p_r^N (\textstyle\sum_{i=1,\ldots,d} a_i x^{d-i}) \in (p_1)$$

which contradicts uniqueness of factorization in R. □

034U **Definition 10.120.12.** A *principal ideal domain*, abbreviated *PID*, is a domain R such that every ideal is a principal ideal.

034V **Lemma 10.120.13.** *A principal ideal domain is a unique factorization domain.*

Proof. As a PID is Noetherian this follows from Lemma 10.120.6. □

034W **Definition 10.120.14.** A *Dedekind domain* is a domain R such that every nonzero ideal $I \subset R$ can be written as a product

$$I = \mathfrak{p}_1 \ldots \mathfrak{p}_r$$

of nonzero prime ideals uniquely up to permutation of the $\mathfrak{p}_i$.

0AUQ **Lemma 10.120.15.** *A PID is a Dedekind domain.*

Proof. Let R be a PID. Since every nonzero ideal of R is principal, and R is a UFD (Lemma 10.120.13), this follows from the fact that every irreducible element in R is prime (Lemma 10.120.5) so that factorizations of elements turn into factorizations into primes. □

09ME **Lemma 10.120.16.** *Let A be a ring. Let I and J be nonzero ideals of A such that $IJ = (f)$ for some nonzerodivisor $f \in A$. Then I and J are finitely generated ideals and finitely locally free of rank 1 as A-modules.*

Proof. It suffices to show that I and J are finite locally free A-modules of rank 1, see Lemma 10.78.2. To do this, write $f = \sum_{i=1,\ldots,n} x_i y_i$ with $x_i \in I$ and $y_i \in J$. We can also write $x_i y_i = a_i f$ for some $a_i \in A$. Since f is a nonzerodivisor we see that $\sum a_i = 1$. Thus it suffices to show that each I_{a_i} and J_{a_i} is free of rank 1 over A_{a_i}. After replacing A by A_{a_i} we conclude that $f = xy$ for some $x \in I$ and $y \in J$. Note that both x and y are nonzerodivisors. We claim that $I = (x)$ and $J = (y)$ which finishes the proof. Namely, if $x' \in I$, then $x'y = af = axy$ for some $a \in A$. Hence $x' = ax$ and we win. □

034X **Lemma 10.120.17.** *Let R be a ring. The following are equivalent*

1. *R is a Dedekind domain,*
2. *R is a Noetherian domain and for every nonzero maximal ideal $\mathfrak{m}$ the local ring $R_\mathfrak{m}$ is a discrete valuation ring, and*
3. *R is a Noetherian, normal domain, and $\dim(R) \leq 1$.*

Proof. Assume (1). The argument is nontrivial because we did not assume that R was Noetherian in our definition of a Dedekind domain. Let $\mathfrak{p} \subset R$ be a prime ideal. Observe that $\mathfrak{p} \neq \mathfrak{p}^2$ by uniqueness of the factorizations in the definition. Pick $x \in \mathfrak{p}$ with $x \notin \mathfrak{p}^2$. Let $y \in \mathfrak{p}$ be a second element (for example $y = 0$). Write $(x, y) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$. Since $(x, y) \subset \mathfrak{p}$ at least one of the primes $\mathfrak{p}_i$ is contained in $\mathfrak{p}$. But as $x \notin \mathfrak{p}^2$ there is at most one. Thus exactly one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is contained in $\mathfrak{p}$, say $\mathfrak{p}_1 \subset \mathfrak{p}$. We conclude that $(x, y)R_\mathfrak{p} = \mathfrak{p}_1 R_\mathfrak{p}$ is prime for every choice of y. We claim that $(x)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Namely, pick $y \in \mathfrak{p}$. By the above applied with y^2 we see that $(x, y^2)R_\mathfrak{p}$ is prime. Hence $y \in (x, y^2)R_\mathfrak{p}$, i.e., $y = ax + by^2$ in $R_\mathfrak{p}$. Thus $(1 - by)y = ax \in (x)R_\mathfrak{p}$, i.e., $y \in (x)R_\mathfrak{p}$ as desired.

Writing $(x) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ anew with $\mathfrak{p}_1 \subset \mathfrak{p}$ we conclude that $\mathfrak{p}_1 R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, i.e., $\mathfrak{p}_1 = \mathfrak{p}$. Moreover, $\mathfrak{p}_1 = \mathfrak{p}$ is a finitely generated ideal of R by Lemma 10.120.16. We conclude that R is Noetherian by Lemma 10.28.10. Moreover, it follows that $R_\mathfrak{m}$ is a discrete valuation ring for every prime ideal $\mathfrak{p}$, see Lemma 10.119.7.

The equivalence of (2) and (3) follows from Lemmas 10.37.10 and 10.119.7. Assume (2) and (3) are satisfied. Let $I \subset R$ be an ideal. We will construct a factorization of I. If I is prime, then there is nothing to prove. If not, pick $I \subset \mathfrak{p}$ with $\mathfrak{p} \subset R$

maximal. Let $J = \{x \in R \mid x\mathfrak{p} \subset I\}$. We claim $J\mathfrak{p} = I$. It suffices to check this after localization at the maximal ideals $\mathfrak{m}$ of R (the formation of J commutes with localization and we use Lemma 10.23.1). Then either $\mathfrak{p}R_\mathfrak{m} = R_\mathfrak{m}$ and the result is clear, or $\mathfrak{p}R_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$. In the last case $\mathfrak{p}R_\mathfrak{m} = (\pi)$ and the case where $\mathfrak{p}$ is principal is immediate. By Noetherian induction the ideal J has a factorization and we obtain the desired factorization of I. We omit the proof of uniqueness of the factorization. □

The following is a variant of the Krull-Akizuki lemma.

09IG **Lemma 10.120.18.** *Let A be a Noetherian domain of dimension 1 with fraction field K. Let L/K be a finite extension. Let B be the integral closure of A in L. Then B is a Dedekind domain and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, has finite fibres, and induces finite residue field extensions.*

Proof. By Krull-Akizuki (Lemma 10.119.12) the ring B is Noetherian. By Lemma 10.112.4 $\dim(B) = 1$. Thus B is a Dedekind domain by Lemma 10.120.17. Surjectivity of the map on spectra follows from Lemma 10.36.17. The last two statements follow from Lemma 10.119.10. □

10.121. Orders of vanishing

02MB

02MC **Lemma 10.121.1.** *Let R be a semi-local Noetherian ring of dimension 1. If $a, b \in R$ are nonzerodivisors then*

$$\mathit{length}_R(R/(ab)) = \mathit{length}_R(R/(a)) + \mathit{length}_R(R/(b))$$

and these lengths are finite.

Proof. We saw the finiteness in Lemma 10.119.11. Additivity holds since there is a short exact sequence $0 \to R/(a) \to R/(ab) \to R/(b) \to 0$ where the first map is given by multiplication by b. (Use length is additive, see Lemma 10.52.3.) □

02MD **Definition 10.121.2.** Suppose that K is a field, and $R \subset K$ is a local[10] Noetherian subring of dimension 1 with fraction field K. In this case we define the *order of vanishing along R*

$$\mathrm{ord}_R : K^* \longrightarrow \mathbf{Z}$$

by the rule

$$\mathrm{ord}_R(x) = \mathrm{length}_R(R/(x))$$

if $x \in R$ and we set $\mathrm{ord}_R(x/y) = \mathrm{ord}_R(x) - \mathrm{ord}_R(y)$ for $x, y \in R$ both nonzero.

We can use the order of vanishing to compare lattices in a vector space. Here is the definition.

02ME **Definition 10.121.3.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. A *lattice in V* is a finite R-submodule $M \subset V$ such that $V = K \otimes_R M$.

[10] We could also define this when R is only semi-local but this is probably never really what you want!

The condition $V = K \otimes_R M$ signifies that M contains a basis for the vector space V. We remark that in many places in the literature the notion of a lattice may be defined only in case the ring R is a discrete valuation ring. If R is a discrete valuation ring then any lattice is a free R-module, and this may not be the case in general.

02MF **Lemma 10.121.4.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space.*

(1) *If M is a lattice in V and $M \subset M' \subset V$ is an R-submodule of V containing M then the following are equivalent*
   (a) *M' is a lattice,*
   (b) *$\text{length}_R(M'/M)$ is finite, and*
   (c) *M' is finitely generated.*

(2) *If M is a lattice in V and $M' \subset M$ is an R-submodule of M then M' is a lattice if and only if $\text{length}_R(M/M')$ is finite.*

(3) *If M, M' are lattices in V, then so are $M \cap M'$ and $M + M'$.*

(4) *If $M \subset M' \subset M'' \subset V$ are lattices in V then*
$$\text{length}_R(M''/M) = \text{length}_R(M'/M) + \text{length}_R(M''/M').$$

(5) *If M, M', N, N' are lattices in V and $N \subset M \cap M'$, $M + M' \subset N'$, then we have*
$$\begin{aligned} & \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M') \\ = \ & \text{length}_R(M/N) - \text{length}_R(M'/N) \\ = \ & \text{length}_R(M + M'/M') - \text{length}_R(M + M'/M) \\ = \ & \text{length}_R(N'/M') - \text{length}_R(N'/M) \end{aligned}$$

Proof. Proof of (1). Assume (1)(a). Say $y_1, \ldots, y_m$ generate M'. Then each $y_i = x_i/f_i$ for some $x_i \in M$ and nonzero $f_i \in R$. Hence we see that $f_1 \ldots f_m M' \subset M$. Since R is Noetherian local of dimension 1 we see that $\mathfrak{m}^n \subset (f_1 \ldots f_m)$ for some n (for example combine Lemmas 10.60.13 and Proposition 10.60.7 or combine Lemmas 10.119.9 and 10.52.4). In other words $\mathfrak{m}^n M' \subset M$ for some n Hence $\text{length}(M'/M) < \infty$ by Lemma 10.52.8, in other words (1)(b) holds. Assume (1)(b). Then M'/M is a finite R-module (see Lemma 10.52.2). Hence M' is a finite R-module as an extension of finite R-modules. Hence (1)(c). The implication (1)(c) $\Rightarrow$ (1)(a) follows from the remark following Definition 10.121.3.

Proof of (2). Suppose M is a lattice in V and $M' \subset M$ is an R-submodule. We have seen in (1) that if M' is a lattice, then $\text{length}_R(M/M') < \infty$. Conversely, assume that $\text{length}_R(M/M') < \infty$. Then M' is finitely generated as R is Noetherian and for some n we have $\mathfrak{m}^n M \subset M'$ (Lemma 10.52.4). Hence it follows that M' contains a basis for V, and M' is a lattice.

Proof of (3). Assume M, M' are lattices in V. Since R is Noetherian the submodule $M \cap M'$ of M is finite. As M is a lattice we can find $x_1, \ldots, x_n \in M$ which form a K-basis for V. Because M' is a lattice we can write $x_i = y_i/f_i$ with $y_i \in M'$ and $f_i \in R$. Hence $f_i x_i \in M \cap M'$. Hence $M \cap M'$ is a lattice also. The fact that $M + M'$ is a lattice follows from part (1).

Part (4) follows from additivity of lengths (Lemma 10.52.3) and the exact sequence
$$0 \to M'/M \to M''/M \to M''/M' \to 0$$

Part (5) follows from repeatedly applying part (4). □

02MG **Definition 10.121.5.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let M, M' be two lattices in V. The *distance between M and M'* is the integer

$$d(M, M') = \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M')$$

of Lemma 10.121.4 part (5).

In particular, if $M' \subset M$, then $d(M, M') = \text{length}_R(M/M')$.

02MH **Lemma 10.121.6.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. This distance function has the property that*

$$d(M, M'') = d(M, M') + d(M', M'')$$

whenever given three lattices M, M', M'' of V. In particular we have $d(M, M') = -d(M', M)$.

Proof. Omitted. □

02MI **Lemma 10.121.7.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let $\varphi : V \to V$ be a K-linear isomorphism. For any lattice $M \subset V$ we have*

$$d(M, \varphi(M)) = ord_R(\det(\varphi))$$

Proof. We can see that the integer $d(M, \varphi(M))$ does not depend on the lattice M as follows. Suppose that M' is a second such lattice. Then we see that

$$\begin{aligned} d(M, \varphi(M)) &= d(M, M') + d(M', \varphi(M)) \\ &= d(M, M') + d(\varphi(M'), \varphi(M)) + d(M', \varphi(M')) \end{aligned}$$

Since φ is an isomorphism we see that $d(\varphi(M'), \varphi(M)) = d(M', M) = -d(M, M')$, and hence $d(M, \varphi(M)) = d(M', \varphi(M'))$. Moreover, both sides of the equation (of the lemma) are additive in φ, i.e.,

$$\text{ord}_R(\det(\varphi \circ \psi)) = \text{ord}_R(\det(\varphi)) + \text{ord}_R(\det(\psi))$$

and also

$$\begin{aligned} d(M, \varphi(\psi((M))) &= d(M, \psi(M)) + d(\psi(M), \varphi(\psi(M))) \\ &= d(M, \psi(M)) + d(M, \varphi(M)) \end{aligned}$$

by the independence shown above. Hence it suffices to prove the lemma for generators of $\text{GL}(V)$. Choose an isomorphism $K^{\oplus n} \cong V$. Then $\text{GL}(V) = \text{GL}_n(K)$ is generated by elementary matrices E. The result is clear for E equal to the identity matrix. If $E = E_{ij}(\lambda)$ with $i \neq j$, $\lambda \in K$, $\lambda \neq 0$, for example

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then with respect to a different basis we get $E_{12}(1)$. The result is clear for $E = E_{12}(1)$ by taking as lattice $R^{\oplus n} \subset K^{\oplus n}$. Finally, if $E = E_i(a)$, with $a \in K^*$ for

example

$$E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then $E_1(a)(R^{\oplus b}) = aR \oplus R^{\oplus n-1}$ and it is clear that $d(R^{\oplus n}, aR \oplus R^{\oplus n-1}) = \mathrm{ord}_R(a)$ as desired. □

02MJ **Lemma 10.121.8.** *Let $A \to B$ be a ring map. Assume*

(1) *A is a Noetherian local domain of dimension 1,*
(2) *$A \subset B$ is a finite extension of domains.*

Let L/K be the corresponding finite extension of fraction fields. Let $y \in L^*$ and $x = Nm_{L/K}(y)$. In this situation B is semi-local. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$ord_A(x) = \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(y)$$

where ord is defined as in Definition 10.121.2.

Proof. The ring B is semi-local by Lemma 10.113.2. Write $y = b/b'$ for some $b, b' \in B$. By the additivity of ord and multiplicativity of Nm it suffices to prove the lemma for $y = b$ or $y = b'$. In other words we may assume $y \in B$. In this case the right hand side of the formula is

$$\sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] \mathrm{length}_{B_{\mathfrak{m}_i}}((B/yB)_{\mathfrak{m}_i})$$

By Lemma 10.52.12 this is equal to $\mathrm{length}_A(B/yB)$. By Lemma 10.121.7 we have

$$\mathrm{length}_A(B/yB) = d(B, yB) = \mathrm{ord}_A(\det_K(L \xrightarrow{y} L)).$$

Since $x = \mathrm{Nm}_{L/K}(y) = \det_K(L \xrightarrow{y} L)$ by definition the lemma is proved. □

10.122. Quasi-finite maps

02MK Consider a ring map $R \to S$ of finite type. A map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is quasi-finite at a point if that point is isolated in its fibre. This means that the fibre is zero dimensional at that point. In this section we study the basic properties of this important but technical notion. More advanced material can be found in the next section.

00PJ **Lemma 10.122.1.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:*

(1) *$\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$,*
(2) *$S_{\mathfrak{q}}$ is finite over k,*
(3) *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) = \{\mathfrak{q}\}$,*
(4) *$\dim_{\mathfrak{q}} \mathrm{Spec}(S) = 0$,*
(5) *$\mathfrak{q}$ is a closed point of $\mathrm{Spec}(S)$ and $\dim(S_{\mathfrak{q}}) = 0$, and*
(6) *the field extension $\kappa(\mathfrak{q})/k$ is finite and $\dim(S_{\mathfrak{q}}) = 0$.*

In this case $S = S_{\mathfrak{q}} \times S'$ for some finite type k-algebra S'. Also, the element g as in (3) has the property $S_{\mathfrak{q}} = S_g$.

Proof. Suppose $\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$, i.e., $\{\mathfrak{q}\}$ is open in $\mathrm{Spec}(S)$. Because $\mathrm{Spec}(S)$ is a Jacobson space (see Lemmas 10.35.2 and 10.35.4) we see that $\mathfrak{q}$ is a closed point. Hence $\{\mathfrak{q}\}$ is open and closed in $\mathrm{Spec}(S)$. By Lemmas 10.21.3 and 10.24.3 we may write $S = S_1 \times S_2$ with $\mathfrak{q}$ corresponding to the only point $\mathrm{Spec}(S_1)$. Hence $S_1 = S_\mathfrak{q}$ is a zero dimensional ring of finite type over k. Hence it is finite over k for example by Lemma 10.115.4. We have proved (1) implies (2).

Suppose $S_\mathfrak{q}$ is finite over k. Then $S_\mathfrak{q}$ is Artinian local, see Lemma 10.53.2. So $\mathrm{Spec}(S_\mathfrak{q}) = \{\mathfrak{q}S_\mathfrak{q}\}$ by Lemma 10.53.6. Consider the exact sequence $0 \to K \to S \to S_\mathfrak{q} \to Q \to 0$. It is clear that $K_\mathfrak{q} = Q_\mathfrak{q} = 0$. Also, K is a finite S-module as S is Noetherian and Q is a finite S-module since $S_\mathfrak{q}$ is finite over k. Hence there exists $g \in S$, $g \notin \mathfrak{q}$ such that $K_g = Q_g = 0$. Thus $S_\mathfrak{q} = S_g$ and $D(g) = \{\mathfrak{q}\}$. We have proved that (2) implies (3).

Suppose $D(g) = \{\mathfrak{q}\}$. Since $D(g)$ is open by construction of the topology on $\mathrm{Spec}(S)$ we see that $\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$. We have proved that (3) implies (1). In other words (1), (2) and (3) are equivalent.

Assume $\dim_\mathfrak{q} \mathrm{Spec}(S) = 0$. This means that there is some open neighbourhood of $\mathfrak{q}$ in $\mathrm{Spec}(S)$ which has dimension zero. Then there is an open neighbourhood of the form $D(g)$ which has dimension zero. Since S_g is Noetherian we conclude that S_g is Artinian and $D(g) = \mathrm{Spec}(S_g)$ is a finite discrete set, see Proposition 10.60.7. Thus $\mathfrak{q}$ is an isolated point of $D(g)$ and, by the equivalence of (1) and (2) above applied to $\mathfrak{q}S_g \subset S_g$, we see that $S_\mathfrak{q} = (S_g)_{\mathfrak{q}S_g}$ is finite over k. Hence (4) implies (2). It is clear that (1) implies (4). Thus (1) – (4) are all equivalent.

Lemma 10.114.6 gives the implication (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (6) follows from Lemma 10.116.3. The implication (6) $\Rightarrow$ (5) follows from Lemma 10.35.9. At this point we know (1) – (6) are equivalent.

The two statements at the end of the lemma we saw during the course of the proof of the equivalence of (1), (2) and (3) above. □

00PK **Lemma 10.122.2.** *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $F = \mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ be the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Remark 10.18.5. Denote $\overline{\mathfrak{q}} \in F$ the point corresponding to $\mathfrak{q}$. The following are equivalent*

(1) *$\overline{\mathfrak{q}}$ is an isolated point of F,*
(2) *$S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is finite over $\kappa(\mathfrak{p})$,*
(3) *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the only prime of $D(g)$ mapping to $\mathfrak{p}$ is $\mathfrak{q}$,*
(4) *$\dim_{\overline{\mathfrak{q}}}(F) = 0$,*
(5) *$\overline{\mathfrak{q}}$ is a closed point of F and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$, and*
(6) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$.*

Proof. Note that $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_{\overline{\mathfrak{q}}}$. Moreover $S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. The conditions correspond exactly to the conditions of Lemma 10.122.1 for the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ and the prime $\overline{\mathfrak{q}}$, hence they are equivalent. □

00PL **Definition 10.122.3.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime.

(1) If the equivalent conditions of Lemma 10.122.2 are satisfied then we say $R \to S$ is *quasi-finite at* $\mathfrak{q}$.

(2) We say a ring map $A \to B$ is *quasi-finite* if it is of finite type and quasi-finite at all primes of B.

0H8X **Lemma 10.122.4.** *Let $R \to S$ be a finite type ring map and $\mathfrak{p}$ be a prime ideal of R. Then the following are equivalent:*

(1) *$R \to S$ is quasi-finite at all primes of S lying over $\mathfrak{p}$,*
(2) *$S \otimes_R \kappa(\mathfrak{p})$ is a finite $\kappa(\mathfrak{p})$-algebra, and*
(3) *$\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ is a finite set.*

Proof. Condition (1) says the topology on $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ is discrete (as every point is open). Hence the equivalence of (1), (2), (3) follows from Lemma 10.61.3. □

00PM **Lemma 10.122.5.** *Let $R \to S$ be a finite type ring map. Then $R \to S$ is quasi-finite if and only if for all primes $\mathfrak{p} \subset R$ the ring $S \otimes_R \kappa(\mathfrak{p})$ is finite over $\kappa(\mathfrak{p})$.*

Proof. Follows immediately from the more general Lemma 10.122.4. □

077H **Lemma 10.122.6.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $f \in R$, $f \notin \mathfrak{p}$ and $g \in S$, $g \notin \mathfrak{q}$. Then $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $R_f \to S_{fg}$ is quasi-finite at $\mathfrak{q}S_{fg}$.*

Proof. The fibre of $\mathrm{Spec}(S_{fg}) \to \mathrm{Spec}(R_f)$ is homeomorphic to an open subset of the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Hence the lemma follows from part (1) of the equivalent conditions of Lemma 10.122.2. □

00PN **Lemma 10.122.7.** *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

be a commutative diagram of rings with primes as indicated. Assume $R \to S$ of finite type, and $S \otimes_R R' \to S'$ surjective. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $R' \to S'$ is quasi-finite at $\mathfrak{q}'$.

Proof. Write $S \otimes_R \kappa(\mathfrak{p}) = S_1 \times S_2$ with S_1 finite over $\kappa(\mathfrak{p})$ and such that $\mathfrak{q}$ corresponds to a point of S_1 as in Lemma 10.122.1. This product decomposition induces a corresponding product decomposition for any $S \otimes_R \kappa(\mathfrak{p})$-algebra. In particular, we obtain $S' \otimes_{R'} \kappa(\mathfrak{p}') = S'_1 \times S'_2$. Because $S \otimes_R R' \to S'$ is surjective the canonical map $(S \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S' \otimes_{R'} \kappa(\mathfrak{p}')$ is surjective and hence $S_i \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S'_i$ is surjective. It follows that S'_1 is finite over $\kappa(\mathfrak{p}')$. The map $S' \otimes_{R'} \kappa(\mathfrak{p}') \to \kappa(\mathfrak{q}')$ factors through S'_1 (i.e. it annihilates the factor S'_2) because the map $S \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ factors through S_1 (i.e. it annihilates the factor S_2). Thus $\mathfrak{q}'$ corresponds to a point of $\mathrm{Spec}(S'_1)$ in the disjoint union decomposition of the fibre: $\mathrm{Spec}(S' \otimes_{R'} \kappa(\mathfrak{p}')) = \mathrm{Spec}(S'_1) \amalg \mathrm{Spec}(S'_2)$, see Lemma 10.21.2. Since S'_1 is finite over a field, it is Artinian ring, and hence $\mathrm{Spec}(S'_1)$ is a finite discrete set. (See Proposition 10.60.7.) We conclude $\mathfrak{q}'$ is isolated in its fibre as desired. □

00PO **Lemma 10.122.8.** *A composition of quasi-finite ring maps is quasi-finite.*

Proof. Suppose $A \to B$ and $B \to C$ are quasi-finite ring maps. By Lemma 10.6.2 we see that $A \to C$ is of finite type. Let $\mathfrak{r} \subset C$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. Since $A \to B$ and $B \to C$ are quasi-finite at $\mathfrak{q}$ and $\mathfrak{r}$ respectively, then there exist $b \in B$ and $c \in C$ such that $\mathfrak{q}$ is the only prime of $D(b)$ which maps to $\mathfrak{p}$ and similarly $\mathfrak{r}$ is the only prime of $D(c)$ which maps to $\mathfrak{q}$. If $c' \in C$ is the image of $b \in B$, then $\mathfrak{r}$ is the only prime of $D(cc')$ which maps to $\mathfrak{p}$. Therefore $A \to C$ is quasi-finite at $\mathfrak{r}$. □

00PP **Lemma 10.122.9.** *Let $R \to S$ be a ring map of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$.*

(1) *The set $\{\mathfrak{q}' \mid R' \to S' \text{ quasi-finite at } \mathfrak{q}'\}$ is the inverse image of the corresponding set of $\operatorname{Spec}(S)$ under the canonical map $\operatorname{Spec}(S') \to \operatorname{Spec}(S)$.*
(2) *If $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is surjective, then $R \to S$ is quasi-finite if and only if $R' \to S'$ is quasi-finite.*
(3) *Any base change of a quasi-finite ring map is quasi-finite.*

Proof. Let $\mathfrak{p}' \subset R'$ be a prime lying over $\mathfrak{p} \subset R$. Then the fibre ring $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is the base change of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ by the field extension $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$. Hence the first assertion follows from the invariance of dimension under field extension (Lemma 10.116.6) and Lemma 10.122.1. The stability of quasi-finite maps under base change follows from this and the stability of finite type property under base change. The second assertion follows since the assumption implies that given a prime $\mathfrak{q} \subset S$ we can find a prime $\mathfrak{q}' \subset S'$ lying over it. □

0C6H **Lemma 10.122.10.** *Let $A \to B$ and $B \to C$ be ring homomorphisms such that $A \to C$ is of finite type. Let $\mathfrak{r}$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then $B \to C$ is quasi-finite at $\mathfrak{r}$.*

Proof. Observe that $B \to C$ is of finite type (Lemma 10.6.2) so that the statement makes sense. Let us use characterization (3) of Lemma 10.122.2. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then there exists some $c \in C$ such that

$$\{\mathfrak{r}' \subset C \text{ lying over } \mathfrak{p}\} \cap D(c) = \{\mathfrak{r}\}.$$

Since the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{q}$ form a subset of the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{p}$ we conclude $B \to C$ is quasi-finite at $\mathfrak{r}$. □

02ML **Lemma 10.122.11.** *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{p} \subset R$ be a minimal prime. Assume that there are at most finitely many primes of S lying over $\mathfrak{p}$. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the ring map $R_g \to S_g$ is finite.*

Proof. Let $x_1, \ldots, x_n$ be generators of S over R. By Lemma 10.122.4 the assumption means that $S \otimes_R \kappa(\mathfrak{p}) = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is a finite $\kappa(\mathfrak{p})$-algebra. Thus we may find monic polynomials $P_i \in R_\mathfrak{p}[X]$ such that $P_i(x_i)$ maps to zero in $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. Since $\mathfrak{p}$ is a minimal prime, $\mathfrak{p}R_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.25.1. Hence $\mathfrak{p}S_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.32.3. Thus there exist $e_i \geq 1$ such that $P(x_i)^{e_i} = 0$ in $S_\mathfrak{p}$. Let $g_1 \in R$, $g_1 \notin \mathfrak{p}$ be an element such that P_i has coefficients in $R[1/g_1]$ for all i. Next, let $g_2 \in R$, $g_2 \notin \mathfrak{p}$ be an element such that $P(x_i)^{e_i} = 0$ in $S_{g_1 g_2}$. Setting $g = g_1 g_2$ we win. □

10.123. Zariski's Main Theorem

00PI In this section our aim is to prove the algebraic version of Zariski's Main theorem. This theorem will be the basis of many further developments in the theory of schemes and morphisms of schemes later in the Stacks project.

Let $R \to S$ be a ring map of finite type. Our goal in this section is to show that the set of points of $\mathrm{Spec}(S)$ where the map is quasi-finite is *open* (Theorem 10.123.12). In fact, it will turn out that there exists a finite ring map $R \to S'$ such that in some sense the quasi-finite locus of S/R is open in $\mathrm{Spec}(S')$ (but we will not prove this in the algebra chapter since we do not develop the language of schemes here – for the case where $R \to S$ is quasi-finite see Lemma 10.123.14). These statements are somewhat tricky to prove and we do it by a long list of lemmas concerning integral and finite extensions of rings. This material may be found in [**Ray70**], and [**Pes66**]. We also found notes by Thierry Coquand helpful.

00PQ **Lemma 10.123.1.** *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Then $\varphi(a_n)t$ is integral over R.*

Proof. Namely, multiply the equation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$ with $\varphi(a_n)^{n-1}$ and write it as $\varphi(a_0a_n^{n-1}) + \varphi(a_1a_n^{n-2})(\varphi(a_n)t) + \ldots + (\varphi(a_n)t)^n = 0$. □

The following lemma is in some sense the key lemma in this section.

00PT **Lemma 10.123.2.** *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume that (a) t is integral over $R[x]$, and (b) there exists a monic $p \in R[x]$ such that $t\varphi(p) \in \mathrm{Im}(\varphi)$. Then there exists a $q \in R[x]$ such that $t - \varphi(q)$ is integral over R.*

Proof. Write $t\varphi(p) = \varphi(r)$ for some $r \in R[x]$. Using euclidean division, write $r = qp + r'$ with $q, r' \in R[x]$ and $\deg(r') < \deg(p)$. We may replace t by $t - \varphi(q)$ which is still integral over $R[x]$, so that we obtain $t\varphi(p) = \varphi(r')$. In the ring S_t we may write this as $\varphi(p) - (1/t)\varphi(r') = 0$. This implies that $\varphi(x)$ gives an element of the localization S_t which is integral over $\varphi(R)[1/t] \subset S_t$. On the other hand, t is integral over the subring $\varphi(R)[\varphi(x)] \subset S$. Combined we conclude that t is integral over the subring $\varphi(R)[1/t] \subset S_t$, see Lemma 10.36.6. In other words there exists an equation of the form

$$t^d + \sum\nolimits_{i<d} \left(\sum\nolimits_{j=0,\ldots,n_i} \varphi(r_{i,j})/t^j \right) t^i = 0$$

in S_t with $r_{i,j} \in R$. This means that $t^{d+N} + \sum_{i<d} \sum_{j=0,\ldots,n_i} \varphi(r_{i,j}) t^{i+N-j} = 0$ in S for some N large enough. In other words t is integral over R. □

00PV **Lemma 10.123.3.** *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume t is integral over $R[x]$. Let $p \in R[x]$, $p = a_0 + a_1x + \ldots + a_kx^k$ such that $t\varphi(p) \in \mathrm{Im}(\varphi)$. Then there exists a $q \in R[x]$ and $n \geq 0$ such that $\varphi(a_k)^n t - \varphi(q)$ is integral over R.*

Proof. Let R' and S' be the localization of R and S at the element a_k. Let $\varphi' : R'[x] \to S'$ be the localization of φ. Let $t' \in S'$ be the image of t. Set $p' = p/a_k \in R'[x]$. Then $t'\varphi'(p') \in \mathrm{Im}(\varphi')$ since $t\varphi(p) \in \mathrm{Im}(\varphi)$. As p' is monic, by Lemma 10.123.2 there exists a $q' \in R'[x]$ such that $t' - \varphi'(q')$ is integral over R'. We may choose an $n \geq 0$ and an element $q \in R[x]$ such that $a_k^n q'$ is the image of q. Then $\varphi(a_k)^n t - \varphi(q)$ is an element of S whose image in S' is integral over R'. By

Lemma 10.36.11 there exists an $m \geq 0$ such that $\varphi(a_k)^m(\varphi(a_k)^n t - \varphi(q))$ is integral over R. Thus $\varphi(a_k)^{m+n}t - \varphi(a_k^m q)$ is integral over R as desired. □

00PW **Situation 10.123.4.** Let R be a ring. Let $\varphi : R[x] \to S$ be finite. Let

$$J = \{g \in S \mid gS \subset \mathrm{Im}(\varphi)\}$$

be the "conductor ideal" of φ. Assume $\varphi(R) \subset S$ integrally closed in S.

00PX **Lemma 10.123.5.** *In Situation 10.123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1x + \ldots + a_kx^k) \in J$. Then there exists an $m \geq 0$ such that $u\varphi(a_k)^m \in J$.*

Proof. Assume that S is generated by $t_1, \ldots, t_n$ as an $R[x]$-module. In this case $J = \{g \in S \mid gt_i \in \mathrm{Im}(\varphi) \text{ for all } i\}$. Note that each element ut_i is integral over $R[x]$, see Lemma 10.36.3. We have $\varphi(a_0+a_1x+\ldots+a_kx^k)ut_i \in \mathrm{Im}(\varphi)$. By Lemma 10.123.3, for each i there exists an integer n_i and an element $q_i \in R[x]$ such that $\varphi(a_k^{n_i})ut_i - \varphi(q_i)$ is integral over R. By assumption this element is in $\varphi(R)$ and hence $\varphi(a_k^{n_i})ut_i \in \mathrm{Im}(\varphi)$. It follows that $m = \max\{n_1, \ldots, n_n\}$ works. □

00PY **Lemma 10.123.6.** *In Situation 10.123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1x + \ldots + a_kx^k) \in \sqrt{J}$. Then $u\varphi(a_i) \in \sqrt{J}$ for all i.*

Proof. Under the assumptions of the lemma we have $u^n\varphi(a_0+a_1x+\ldots+a_kx^k)^n \in J$ for some $n \geq 1$. By Lemma 10.123.5 we deduce $u^n\varphi(a_k^{nm}) \in J$ for some $m \geq 1$. Thus $u\varphi(a_k) \in \sqrt{J}$, and so $u\varphi(a_0 + a_1x + \ldots + a_kx^k) - u\varphi(a_kx^k) = u\varphi(a_0 + a_1x + \ldots + a_{k-1}x^{k-1}) \in \sqrt{J}$. We win by induction on k. □

This lemma suggests the following definition.

00PZ **Definition 10.123.7.** Given an inclusion of rings $R \subset S$ and an element $x \in S$ we say that x is *strongly transcendental over R* if whenever $u(a_0+a_1x+\ldots+a_kx^k) = 0$ with $u \in S$ and $a_i \in R$, then we have $ua_i = 0$ for all i.

Note that if S is a domain then this is the same as saying that x as an element of the fraction field of S is transcendental over the fraction field of R.

00Q0 **Lemma 10.123.8.** *Suppose $R \subset S$ is an inclusion of reduced rings and suppose that $x \in S$ is strongly transcendental over R. Let $\mathfrak{q} \subset S$ be a minimal prime and let $\mathfrak{p} = R \cap \mathfrak{q}$. Then the image of x in $S/\mathfrak{q}$ is strongly transcendental over the subring $R/\mathfrak{p}$.*

Proof. Suppose $u(a_0 + a_1x + \ldots + a_kx^k) \in \mathfrak{q}$. By Lemma 10.25.1 the local ring $S_\mathfrak{q}$ is a field, and hence $u(a_0 + a_1x + \ldots + a_kx^k)$ is zero in $S_\mathfrak{q}$. Thus $uu'(a_0 + a_1x + \ldots + a_kx^k) = 0$ for some $u' \in S$, $u' \notin \mathfrak{q}$. Since x is strongly transcendental over R we get $uu'a_i = 0$ for all i. This in turn implies that $ua_i \in \mathfrak{q}$. □

00Q1 **Lemma 10.123.9.** *Suppose $R \subset S$ is an inclusion of domains and let $x \in S$. Assume x is (strongly) transcendental over R and that S is finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. As a first case, assume that R is normal, see Definition 10.37.11. By Lemma 10.37.14 we see that $R[x]$ is normal. Take a prime $\mathfrak{q} \subset S$, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite. This would be the case if $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $\mathfrak{r} = R[x] \cap \mathfrak{q}$. Then since $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q})$ we see that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r})$ is finite too. Thus the inclusion $\mathfrak{r} \supset \mathfrak{p}R[x]$ is strict. By going

down for $R[x] \subset S$, see Proposition 10.38.7, we find a prime $\mathfrak{q}' \subset \mathfrak{q}$, lying over the prime $\mathfrak{p}R[x]$. Hence the fibre $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ contains a point not equal to $\mathfrak{q}$, namely $\mathfrak{q}'$, whose closure contains $\mathfrak{q}$ and hence $\mathfrak{q}$ is not isolated in its fibre.

If R is not normal, let $R \subset R' \subset K$ be the integral closure R' of R in its field of fractions K. Let $S \subset S' \subset L$ be the subring S' of the field of fractions L of S generated by R' and S. Note that by construction the map $S \otimes_R R' \to S'$ is surjective. This implies that $R'[x] \subset S'$ is finite. Also, the map $S \subset S'$ induces a surjection on Spec, see Lemma 10.36.17. We conclude by Lemma 10.122.7 and the normal case we just discussed. □

00Q2 **Lemma 10.123.10.** *Suppose $R \subset S$ is an inclusion of reduced rings. Assume $x \in S$ be strongly transcendental over R, and S finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. Let $\mathfrak{q} \subset S$ be any prime. Choose a minimal prime $\mathfrak{q}' \subset \mathfrak{q}$. According to Lemmas 10.123.8 and 10.123.9 the extension $R/(R \cap \mathfrak{q}') \subset S/\mathfrak{q}'$ is not quasi-finite at the prime corresponding to $\mathfrak{q}$. By Lemma 10.122.7 the extension $R \to S$ is not quasi-finite at $\mathfrak{q}$. □

00Q8 **Lemma 10.123.11.** *Let R be a ring. Let $S = R[x]/I$. Let $\mathfrak{q} \subset S$ be a prime. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $S' \subset S$ be the integral closure of R in S. Then there exists an element $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.*

Proof. Let $\mathfrak{p}$ be the image of $\mathfrak{q}$ in $\mathrm{Spec}(R)$. There exists an $f \in I$, $f = a_n x^n + \ldots + a_0$ such that $a_i \notin \mathfrak{p}$ for some i. Namely, otherwise the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ would be $\kappa(\mathfrak{p})[x]$ and the map would not be quasi-finite at any prime lying over $\mathfrak{p}$. We conclude there exists a relation $b_m x^m + \ldots + b_0 = 0$ with $b_j \in S'$, $j = 0, \ldots, m$ and $b_j \notin \mathfrak{q} \cap S'$ for some j. We prove the lemma by induction on m. The base case is $m = 0$ is vacuous (because the statements $b_0 = 0$ and $b_0 \notin \mathfrak{q}$ are contradictory).

The case $b_m \notin \mathfrak{q}$. In this case x is integral over S'_{b_m}, in fact $b_m x \in S'$ by Lemma 10.123.1. Hence the injective map $S'_{b_m} \to S_{b_m}$ is also surjective, i.e., an isomorphism as desired.

The case $b_m \in \mathfrak{q}$. In this case we have $b_m x \in S'$ by Lemma 10.123.1. Set $b'_{m-1} = b_m x + b_{m-1}$. Then

$$b'_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0 = 0$$

Since b'_{m-1} is congruent to b_{m-1} modulo $S' \cap \mathfrak{q}$ we see that it is still the case that one of $b'_{m-1}, b_{m-2}, \ldots, b_0$ is not in $S' \cap \mathfrak{q}$. Thus we win by induction on m. □

00Q9 **Theorem 10.123.12** (Zariski's Main Theorem)**.** *Let R be a ring. Let $R \to S$ be a finite type R-algebra. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q} \subset S$ be a prime of S. If $R \to S$ is quasi-finite at $\mathfrak{q}$ then there exists a $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.*

Proof. There exist finitely many elements $x_1, \ldots, x_n \in S$ such that S is finite over the R-sub algebra generated by $x_1, \ldots, x_n$. (For example generators of S over R.) We prove the proposition by induction on the minimal such number n.

The case $n = 0$ is trivial, because in this case $S' = S$, see Lemma 10.36.3.

The case $n = 1$. We may replace R by its integral closure in S (Lemma 10.122.10 guarantees that $R \to S$ is still quasi-finite at $\mathfrak{q}$). Thus we may assume $R \subset S$ is

integrally closed in S, in other words $R = S'$. Consider the map $\varphi : R[x] \to S$, $x \mapsto x_1$. (We will see that φ is not injective below.) By assumption φ is finite. Hence we are in Situation 10.123.4. Let $J \subset S$ be the "conductor ideal" defined in Situation 10.123.4. Consider the diagram

$$\begin{array}{ccccccc} R[x] & \longrightarrow & S & \longrightarrow & S/\sqrt{J} & \longleftarrow & R/(R \cap \sqrt{J})[x] \\ & \nwarrow & \uparrow & & \uparrow & \nearrow & \\ & & R & \longrightarrow & R/(R \cap \sqrt{J}) & & \end{array}$$

According to Lemma 10.123.6 the image of x in the quotient $S/\sqrt{J}$ is strongly transcendental over $R/(R \cap \sqrt{J})$. Hence by Lemma 10.123.10 the ring map $R/(R \cap \sqrt{J}) \to S/\sqrt{J}$ is not quasi-finite at any prime of $S/\sqrt{J}$. By Lemma 10.122.7 we deduce that $\mathfrak{q}$ does not lie in $V(J) \subset \mathrm{Spec}(S)$. Thus there exists an element $s \in J$, $s \notin \mathfrak{q}$. By definition of J we may write $s = \varphi(f)$ for some polynomial $f \in R[x]$. Let $I = \mathrm{Ker}(\varphi : R[x] \to S)$. Since $\varphi(f) \in J$ we get $(R[x]/I)_f \cong S_{\varphi(f)}$. Also $s \notin \mathfrak{q}$ means that $f \notin \varphi^{-1}(\mathfrak{q})$. Thus $\varphi^{-1}(\mathfrak{q})/I$ is a prime of $R[x]/I$ at which $R \to R[x]/I$ is quasi-finite, see Lemma 10.122.6. Note that R is integrally closed in $R[x]/I$ since R is integrally closed in S. By Lemma 10.123.11 there exists an element $h \in R$, $h \notin R \cap \mathfrak{q}$ such that $R_h \cong (R[x]/I)_h$. Thus $(R[x]/I)_{fh} = S_{\varphi(fh)}$ is isomorphic to a principal localization $R_{h'}$ of R for some $h' \in R$, $h' \notin \mathfrak{q}$.

The case $n > 1$. Consider the subring $R' \subset S$ which is the integral closure of $R[x_1, \ldots, x_{n-1}]$ in S. By Lemma 10.122.10 the extension S/R' is quasi-finite at $\mathfrak{q}$. Also, note that S is finite over $R'[x_n]$. By the case $n = 1$ above, there exists a $g' \in R'$, $g' \notin \mathfrak{q}$ such that $(R')_{g'} \cong S_{g'}$. At this point we cannot apply induction to $R \to R'$ since R' may not be finite type over R. Since S is finitely generated over R we deduce in particular that $(R')_{g'}$ is finitely generated over R. Say the elements g', and $y_1/(g')^{n_1}, \ldots, y_N/(g')^{n_N}$ with $y_i \in R'$ generate $(R')_{g'}$ over R. Let R'' be the R-sub algebra of R' generated by $x_1, \ldots, x_{n-1}, y_1, \ldots, y_N, g'$. This has the property $(R'')_{g'} \cong S_{g'}$. Surjectivity because of how we chose y_i, injectivity because $R'' \subset R'$, and localization is exact. Note that R'' is finite over $R[x_1, \ldots, x_{n-1}]$ because of our choice of R', see Lemma 10.36.4. Let $\mathfrak{q}'' = R'' \cap \mathfrak{q}$. Since $(R'')_{\mathfrak{q}''} = S_{\mathfrak{q}}$ we see that $R \to R''$ is quasi-finite at $\mathfrak{q}''$, see Lemma 10.122.2. We apply our induction hypothesis to $R \to R''$, $\mathfrak{q}''$ and $x_1, \ldots, x_{n-1} \in R''$ and we find a subring $R''' \subset R''$ which is integral over R and an element $g'' \in R'''$, $g'' \notin \mathfrak{q}''$ such that $(R''')_{g''} \cong (R'')_{g''}$. Write the image of g' in $(R'')_{g''}$ as $g'''/(g'')^n$ for some $g''' \in R'''$. Set $g = g''g''' \in R'''$. Then it is clear that $g \notin \mathfrak{q}$ and $(R''')_g \cong S_g$. Since by construction we have $R''' \subset S'$ we also have $S'_g \cong S_g$ as desired. □

00QA **Lemma 10.123.13.** *Let $R \to S$ be a finite type ring map. The set of points $\mathfrak{q}$ of $\mathrm{Spec}(S)$ at which S/R is quasi-finite is open in $\mathrm{Spec}(S)$.*

Proof. Let $\mathfrak{q} \subset S$ be a point at which the ring map is quasi-finite. By Theorem 10.123.12 there exists an integral ring map $R \to S'$, $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$. Since S and hence S_g are of finite type over R we may find finitely many elements $y_1, \ldots, y_N$ of S' such that $S''_g \cong S_g$ where $S'' \subset S'$ is the sub R-algebra generated by $g, y_1, \ldots, y_N$. Since S'' is finite over R (see Lemma 10.36.4) we see that S'' is quasi-finite over R (see Lemma 10.122.5). It is easy to see that this implies that S''_g is quasi-finite over R, for example because the property

of being quasi-finite at a prime depends only on the local ring at the prime. Thus we see that S_g is quasi-finite over R. By the same token this implies that $R \to S$ is quasi-finite at every prime of S which lies in $D(g)$. □

00QB **Lemma 10.123.14.** *Let $R \to S$ be a finite type ring map. Suppose that S is quasi-finite over R. Let $S' \subset S$ be the integral closure of R in S. Then*

(1) *$\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is a homeomorphism onto an open subset,*
(2) *if $g \in S'$ and $D(g)$ is contained in the image of the map, then $S'_g \cong S_g$, and*
(3) *there exists a finite R-algebra $S'' \subset S'$ such that (1) and (2) hold for the ring map $S'' \to S$.*

Proof. Because S/R is quasi-finite we may apply Theorem 10.123.12 to each point $\mathfrak{q}$ of $\mathrm{Spec}(S)$. Since $\mathrm{Spec}(S)$ is quasi-compact, see Lemma 10.17.8, we may choose a finite number of $g_i \in S'$, $i = 1, \ldots, n$ such that $S'_{g_i} = S_{g_i}$, and such that $g_1, \ldots, g_n$ generate the unit ideal in S (in other words the standard opens of $\mathrm{Spec}(S)$ associated to $g_1, \ldots, g_n$ cover all of $\mathrm{Spec}(S)$).

Suppose that $D(g) \subset \mathrm{Spec}(S')$ is contained in the image. Then $D(g) \subset \bigcup D(g_i)$. In other words, $g_1, \ldots, g_n$ generate the unit ideal of S'_g. Note that $S'_{gg_i} \cong S_{gg_i}$ by our choice of g_i. Hence $S'_g \cong S_g$ by Lemma 10.23.2.

We construct a finite algebra $S'' \subset S'$ as in (3). To do this note that each $S'_{g_i} \cong S_{g_i}$ is a finite type R-algebra. For each i pick some elements $y_{ij} \in S'$ such that each S'_{g_i} is generated as R-algebra by $1/g_i$ and the elements y_{ij}. Then set S'' equal to the sub R-algebra of S' generated by all g_i and all the y_{ij}. Details omitted. □

10.124. Applications of Zariski's Main Theorem

03GB Here is an immediate application characterizing the finite maps of 1-dimensional semi-local rings among the quasi-finite ones as those where equality always holds in the formula of Lemma 10.121.8.

02MM **Lemma 10.124.1.** *Let $A \subset B$ be an extension of domains. Assume*

(1) *A is a local Noetherian ring of dimension 1,*
(2) *$A \to B$ is of finite type, and*
(3) *the induced extension L/K of fraction fields is finite.*

Then B is semi-local. Let $x \in \mathfrak{m}_A$, $x \neq 0$. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$[L : K] ord_A(x) \geq \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(x)$$

where ord is defined as in Definition 10.121.2. We have equality if and only if $A \to B$ is finite.

Proof. The ring B is semi-local by Lemma 10.113.2. Let B' be the integral closure of A in B. By Lemma 10.123.14 we can find a finite A-subalgebra $C \subset B'$ such that on setting $\mathfrak{n}_i = C \cap \mathfrak{m}_i$ we have $C_{\mathfrak{n}_i} \cong B_{\mathfrak{m}_i}$ and the primes $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise distinct. The ring C is semi-local by Lemma 10.113.2. Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the other maximal ideals of C (the "missing points"). By Lemma 10.121.8 we have

$$\mathrm{ord}_A(x^{[L:K]}) = \sum\nolimits_i [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)] \mathrm{ord}_{C_{\mathfrak{n}_i}}(x) + \sum\nolimits_j [\kappa(\mathfrak{p}_j) : \kappa(\mathfrak{m}_A)] \mathrm{ord}_{C_{\mathfrak{p}_j}}(x)$$

hence the inequality follows. In case of equality we conclude that $m = 0$ (no "missing points"). Hence $C \subset B$ is an inclusion of semi-local rings inducing a bijection on maximal ideals and an isomorphism on all localizations at maximal ideals. So if $b \in B$, then $I = \{x \in C \mid xb \in C\}$ is an ideal of C which is not contained in any of the maximal ideals of C, and hence $I = C$, hence $b \in C$. Thus $B = C$ and B is finite over A. □

Here is a more standard application of Zariski's main theorem to the structure of local homomorphisms of local rings.

052V **Lemma 10.124.2.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume*

(1) *$R \to S$ is essentially of finite type,*
(2) *$\kappa(\mathfrak{m}_R) \subset \kappa(\mathfrak{m}_S)$ is finite, and*
(3) *$\dim(S/\mathfrak{m}_R S) = 0$.*

Then S is the localization of a finite R-algebra.

Proof. Let S' be a finite type R-algebra such that $S = S'_{\mathfrak{q}'}$ for some prime $\mathfrak{q}'$ of S'. By Definition 10.122.3 we see that $R \to S'$ is quasi-finite at $\mathfrak{q}'$. After replacing S' by $S'_{g'}$ for some $g' \in S'$, $g' \notin \mathfrak{q}'$ we may assume that $R \to S'$ is quasi-finite, see Lemma 10.123.13. Then by Lemma 10.123.14 there exists a finite R-algebra S'' and elements $g' \in S'$, $g' \notin \mathfrak{q}'$ and $g'' \in S''$ such that $S'_{g'} \cong S''_{g''}$ as R-algebras. This proves the lemma. □

07NC **Lemma 10.124.3.** *Let $R \to S$ be a ring map, $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$ in R. If*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type, and*
(3) *$R \to S$ is quasi-finite at $\mathfrak{q}$,*

then $R^\wedge_\mathfrak{p} \otimes_R S = S^\wedge_\mathfrak{q} \times B$ for some $R^\wedge_\mathfrak{p}$-algebra B.

Proof. There exists a finite R-algebra $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}' = S' \cap \mathfrak{q}$ such that $S'_g = S_g$ and in particular $S'_{\mathfrak{q}'} = S_\mathfrak{q}$, see Lemma 10.123.14. We have

$$R^\wedge_\mathfrak{p} \otimes_R S' = (S'_{\mathfrak{q}'})^\wedge \times B'$$

by Lemma 10.97.8. Observe that under this product decomposition g maps to a pair (u, b') with $u \in (S'_{\mathfrak{q}'})^\wedge$ a unit because $g \notin \mathfrak{q}'$. The product decomposition for $R^\wedge_\mathfrak{p} \otimes_R S'$ induces a product decomposition

$$R^\wedge_\mathfrak{p} \otimes_R S = A \times B$$

Since $S'_g = S_g$ we also have $(R^\wedge_\mathfrak{p} \otimes_R S')_g = (R^\wedge_\mathfrak{p} \otimes_R S)_g$ and since $g \mapsto (u, b')$ where u is a unit we see that $(S'_{\mathfrak{q}'})^\wedge = A$. Since the isomorphism $S'_{\mathfrak{q}'} = S_\mathfrak{q}$ determines an isomorphism on completions this also tells us that $A = S^\wedge_\mathfrak{q}$. This finishes the proof, except that we should perform the sanity check that the induced map $\phi : R^\wedge_\mathfrak{p} \otimes_R S \to A = S^\wedge_\mathfrak{q}$ is the natural one. For elements of the form $x \otimes 1$ with $x \in R^\wedge_\mathfrak{p}$ this is clear as the natural map $R^\wedge_\mathfrak{p} \to S^\wedge_\mathfrak{q}$ factors through $(S'_{\mathfrak{q}'})^\wedge$. For elements of the form $1 \otimes y$ with $y \in S$ we can argue that for some $n \geq 1$ the element $g^n y$ is the image of some $y' \in S'$. Thus $\phi(1 \otimes g^n y)$ is the image of y' by the composition $S' \to (S'_{\mathfrak{q}'})^\wedge \to S^\wedge_\mathfrak{q}$ which is equal to the image of $g^n y$ by the map $S \to S^\wedge_\mathfrak{q}$. Since g maps to a unit this also implies that $\phi(1 \otimes y)$ has the correct value, i.e., the image of y by $S \to S^\wedge_\mathfrak{q}$. □

10.125. Dimension of fibres

00QC We study the behaviour of dimensions of fibres, using Zariski's main theorem. Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.10.1.

00QD **Definition 10.125.1.** Suppose that $R \to S$ is of finite type, and let $\mathfrak{q} \subset S$ be a prime lying over a prime $\mathfrak{p}$ of R. We define the *relative dimension of S/R at $\mathfrak{q}$*, denoted $\dim_{\mathfrak{q}}(S/R)$, to be the dimension of $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ at the point corresponding to $\mathfrak{q}$. We let $\dim(S/R)$ be the supremum of $\dim_{\mathfrak{q}}(S/R)$ over all $\mathfrak{q}$. This is called the *relative dimension of S/R*.

In particular, $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $\dim_{\mathfrak{q}}(S/R) = 0$. The following lemma is more or less a reformulation of Zariski's Main Theorem.

00QE **Lemma 10.125.2.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime. Suppose that $\dim_{\mathfrak{q}}(S/R) = n$. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$.*

Proof. The ring $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{S}$ corresponding to $\mathfrak{q}$. By definition of the dimension of a topological space at a point there exists an open $U \subset \mathrm{Spec}(\overline{S})$ with $\overline{q} \in U$ and $\dim(U) = n$. Since the topology on $\mathrm{Spec}(\overline{S})$ is induced from the topology on $\mathrm{Spec}(S)$ (see Remark 10.18.5), we can find a $g \in S$, $g \notin \mathfrak{q}$ with image $\overline{g} \in \overline{S}$ such that $D(\overline{g}) \subset U$. Thus after replacing S by S_g we see that $\dim(\overline{S}) = n$.

Next, choose generators $x_1, \ldots, x_N$ for S as an R-algebra. By Lemma 10.115.4 there exist elements $y_1, \ldots, y_n$ in the $\mathbf{Z}$-subalgebra of S generated by $x_1, \ldots, x_N$ such that the map $R[t_1, \ldots, t_n] \to S$, $t_i \mapsto y_i$ has the property that $\kappa(\mathfrak{p})[t_1 \ldots, t_n] \to \overline{S}$ is finite. In particular, S is quasi-finite over $R[t_1, \ldots, t_n]$ at $\mathfrak{q}$. Hence, by Lemma 10.123.13 we may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ such that $R[t_1, \ldots, t_n] \to S$ is quasi-finite. □

0520 **Lemma 10.125.3.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

(1) *$R \to S$ is of finite type,*
(2) *$\dim_{\mathfrak{q}}(S/R) = n$, and*
(3) *$trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = r$.*

Then there exist $f \in R$, $f \notin \mathfrak{p}$, $g \in S$, $g \notin \mathfrak{q}$ and a quasi-finite ring map

$$\varphi : R_f[x_1, \ldots, x_n] \longrightarrow S_g$$

such that $\varphi^{-1}(\mathfrak{q}S_g) = (\mathfrak{p}, x_{r+1}, \ldots, x_n)R_f[x_{r+1}, \ldots, x_n]$

Proof. After replacing S by a principal localization we may assume there exists a quasi-finite ring map $\varphi : R[t_1, \ldots, t_n] \to S$, see Lemma 10.125.2. Set $\mathfrak{q}' = \varphi^{-1}(\mathfrak{q})$. Let $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the prime corresponding to $\mathfrak{q}'$. By Lemma 10.115.6 there exists a finite ring map $\kappa(\mathfrak{p})[x_1, \ldots, x_n] \to \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ such that the inverse image of $\overline{\mathfrak{q}}'$ is $(x_{r+1}, \ldots, x_n)$. Let $\overline{h}_i \in \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the image of x_i. We can find an element $f \in R$, $f \notin \mathfrak{p}$ and $h_i \in R_f[t_1, \ldots, t_n]$ which map to $\overline{h}_i$ in $\kappa(\mathfrak{p})[t_1, \ldots, t_n]$. Then the ring map

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n]$$

becomes finite after tensoring with $\kappa(\mathfrak{p})$. In particular, $R_f[t_1, \ldots, t_n]$ is quasi-finite over $R_f[x_1, \ldots, x_n]$ at the prime $\mathfrak{q}'R_f[t_1, \ldots, t_n]$. Hence, by Lemma 10.123.13 there exists a $g \in R_f[t_1, \ldots, t_n]$, $g \notin \mathfrak{q}'R_f[t_1, \ldots, t_n]$ such that $R_f[x_1, \ldots, x_n] \to R_f[t_1, \ldots, t_n, 1/g]$ is quasi-finite. Thus we see that the composition

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n, 1/g] \longrightarrow S_{\varphi(g)}$$

is quasi-finite and we win. □

00QF **Lemma 10.125.4.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p})$.*

Proof. If $R_\mathfrak{p}$ is Noetherian (and hence $S_\mathfrak{q}$ Noetherian since it is essentially of finite type over $R_\mathfrak{p}$) then this follows immediately from Lemma 10.112.6 and the definitions. In the general case, let S' be the integral closure of $R_\mathfrak{p}$ in $S_\mathfrak{p}$. By Zariski's Main Theorem 10.123.12 we have $S_\mathfrak{q} = S'_{\mathfrak{q}'}$ for some $\mathfrak{q}' \subset S'$ lying over $\mathfrak{q}$. By Lemma 10.112.3 we have $\dim(S') \leq \dim(R_\mathfrak{p})$ and hence a fortiori $\dim(S_\mathfrak{q}) = \dim(S'_{\mathfrak{q}'}) \leq \dim(R_\mathfrak{p})$. □

00QG **Lemma 10.125.5.** *Let k be a field. Let S be a finite type k-algebra. Suppose there is a quasi-finite k-algebra map $k[t_1, \ldots, t_n] \subset S$. Then $\dim(S) \leq n$.*

Proof. By Lemma 10.114.1 the dimension of any local ring of $k[t_1, \ldots, t_n]$ is at most n. Thus the result follows from Lemma 10.125.4. □

00QH **Lemma 10.125.6.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime. Suppose that $\dim_\mathfrak{q}(S/R) = n$. There exists an open neighbourhood V of $\mathfrak{q}$ in $\mathrm{Spec}(S)$ such that $\dim_{\mathfrak{q}'}(S/R) \leq n$ for all $\mathfrak{q}' \in V$.*

Proof. By Lemma 10.125.2 we see that we may assume that S is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$. Considering the fibres, we reduce to Lemma 10.125.5. □

In other words, the lemma says that the set of points where the fibre has dimension $\leq n$ is open in $\mathrm{Spec}(S)$. The next lemma says that formation of this open commutes with base change. If the ring map is of finite presentation then this set is quasi-compact open (see below).

00QI **Lemma 10.125.7.** *Let $R \to S$ be a finite type ring map. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$ and denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the associated map on spectra. Let $n \geq 0$. The inverse image $f^{-1}(\{\mathfrak{q} \in \mathrm{Spec}(S) \mid \dim_\mathfrak{q}(S/R) \leq n\})$ is equal to $\{\mathfrak{q}' \in \mathrm{Spec}(S') \mid \dim_{\mathfrak{q}'}(S'/R') \leq n\}$.*

Proof. The condition is formulated in terms of dimensions of fibre rings which are of finite type over a field. Combined with Lemma 10.116.6 this yields the lemma. □

00QJ **Lemma 10.125.8.** *Let $R \to S$ be a ring homomorphism of finite presentation. Let $n \geq 0$. The set*

$$V_n = \{\mathfrak{q} \in \mathrm{Spec}(S) \mid \dim_\mathfrak{q}(S/R) \leq n\}$$

is a quasi-compact open subset of $\mathrm{Spec}(S)$.

Proof. It is open by Lemma 10.125.6. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a presentation of S. Let R_0 be the $\mathbf{Z}$-subalgebra of R generated by the coefficients of the polynomials f_i. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then $S = R \otimes_{R_0} S_0$. By Lemma 10.125.7 V_n is the inverse image of an open $V_{0,n}$ under the quasi-compact continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$. Since S_0 is Noetherian we see that $V_{0,n}$ is quasi-compact. □

00QK **Lemma 10.125.9.** *Let R be a valuation ring with residue field k and field of fractions K. Let S be a domain containing R such that S is of finite type over R. If $S \otimes_R k$ is not the zero ring then*

$$\dim(S \otimes_R k) = \dim(S \otimes_R K)$$

In fact, $\mathrm{Spec}(S \otimes_R k)$ *is equidimensional.*

Proof. It suffices to show that $\dim_{\mathfrak{q}}(S/k)$ is equal to $\dim(S \otimes_R K)$ for every prime $\mathfrak{q}$ of S containing $\mathfrak{m}_R S$. Pick such a prime. By Lemma 10.125.6 the inequality $\dim_{\mathfrak{q}}(S/k) \geq \dim(S \otimes_R K)$ holds. Set $n = \dim_{\mathfrak{q}}(S/k)$. By Lemma 10.125.2 after replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ there exists a quasi-finite ring map $R[t_1, \ldots, t_n] \to S$. If $\dim(S \otimes_R K) < n$, then $K[t_1, \ldots, t_n] \to S \otimes_R K$ has a nonzero kernel. Say $f = \sum a_I t_1^{i_1} \ldots t_n^{i_n}$. After dividing f by a nonzero coefficient of f with minimal valuation, we may assume $f \in R[t_1, \ldots, t_n]$ and some a_I does not map to zero in k. Hence the ring map $k[t_1, \ldots, t_n] \to S \otimes_R k$ has a nonzero kernel which implies that $\dim(S \otimes_R k) < n$. Contradiction. □

10.126. Algebras and modules of finite presentation

05N4 In this section we discuss some standard results where the key feature is that the assumption involves a finite type or finite presentation assumption.

00QP **Lemma 10.126.1.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite type if and only if $R' \to S'$ is of finite type.*

Proof. It is clear that if $R \to S$ is of finite type then $R' \to S'$ is of finite type. Assume that $R' \to S'$ is of finite type. Say $y_1, \ldots, y_m$ generate S' over R'. Write $y_j = \sum_i a_{ij} \otimes x_{ji}$ for some $a_{ij} \in R'$ and $x_{ji} \in S$. Let $A \subset S$ be the R-subalgebra generated by the x_{ij}. By flatness we have $A' := R' \otimes_R A \subset S'$, and by construction $y_j \in A'$. Hence $A' = S'$. By faithful flatness $A = S$. □

00QQ **Lemma 10.126.2.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite presentation if and only if $R' \to S'$ is of finite presentation.*

Proof. It is clear that if $R \to S$ is of finite presentation then $R' \to S'$ is of finite presentation. Assume that $R' \to S'$ is of finite presentation. By Lemma 10.126.1 we see that $R \to S$ is of finite type. Write $S = R[x_1, \ldots, x_n]/I$. By flatness $S' = R'[x_1, \ldots, x_n]/R' \otimes I$. Say $g_1, \ldots, g_m$ generate $R' \otimes I$ over $R'[x_1, \ldots, x_n]$. Write $g_j = \sum_i a_{ij} \otimes f_{ji}$ for some $a_{ij} \in R'$ and $f_{ji} \in I$. Let $J \subset I$ be the ideal generated by the f_{ij}. By flatness we have $R' \otimes_R J \subset R' \otimes_R I$, and both are ideals over $R'[x_1, \ldots, x_n]$. By construction $g_j \in R' \otimes_R J$. Hence $R' \otimes_R J = R' \otimes_R I$. By faithful flatness $J = I$. □

05N5 **Lemma 10.126.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \subset R$ be a multiplicative subset. Set $R' = S^{-1}(R/I) = S^{-1}R/S^{-1}I$.*

(1) *For any finite R'-module M' there exists a finite R-module M such that $S^{-1}(M/IM) \cong M'$.*
(2) *For any finitely presented R'-module M' there exists a finitely presented R-module M such that $S^{-1}(M/IM) \cong M'$.*

Proof. Proof of (1). Choose a short exact sequence $0 \to K' \to (R')^{\oplus n} \to M' \to 0$. Let $K \subset R^{\oplus n}$ be the inverse image of K' under the map $R^{\oplus n} \to (R')^{\oplus n}$. Then $M = R^{\oplus n}/K$ works.

Proof of (2). Choose a presentation $(R')^{\oplus m} \to (R')^{\oplus n} \to M' \to 0$. Suppose that the first map is given by the matrix $A' = (a'_{ij})$ and the second map is determined by generators $x'_i \in M'$, $i = 1, \ldots, n$. As $R' = S^{-1}(R/I)$ we can choose $s \in S$ and a matrix $A = (a_{ij})$ with coefficients in R such that $a'_{ij} = a_{ij}/s \bmod S^{-1}I$. Let M be the finitely presented R-module with presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ where the first map is given by the matrix A and the second map is determined by generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $M \to M'$, $x_i \mapsto x'_i$ induces an isomorphism $S^{-1}(M/IM) \cong M'$. □

05N6 **Lemma 10.126.4.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module.*

(1) *If $S^{-1}M$ is a finite $S^{-1}R$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*
(2) *If $S^{-1}M$ is a finitely presented $S^{-1}R$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let M' be the R-submodule of M generated by $x_1, \ldots, x_n$.

Proof of (2). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let $K = \mathrm{Ker}(R^{\oplus n} \to M)$ where the map is given by the rule $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. By Lemma 10.5.3 we see that $S^{-1}K$ is a finite $S^{-1}R$-module. By (1) we can find a finite submodule $K' \subset K$ with $S^{-1}K' = S^{-1}K$. Take $M' = \mathrm{Coker}(K' \to R^{\oplus n})$. □

05GJ **Lemma 10.126.5.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let M be an R-module.*

(1) *If $M_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $M'_\mathfrak{p} \to M_\mathfrak{p}$.*
(2) *If $M_\mathfrak{p}$ is a finitely presented $R_\mathfrak{p}$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $M'_\mathfrak{p} \to M_\mathfrak{p}$.*

Proof. This is a special case of Lemma 10.126.4 □

00QR **Lemma 10.126.6.** *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Assume*

(1) *S is of finite presentation over R,*
(2) *φ induces an isomorphism $R_\mathfrak{p} \cong S_\mathfrak{q}$.*

Then there exist $f \in R$, $f \notin \mathfrak{p}$ and an R_f-algebra C such that $S_f \cong R_f \times C$ as R_f-algebras.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Let $a_i \in R_\mathfrak{p}$ be an element mapping to the image of x_i in $S_\mathfrak{q}$. Write $a_i = b_i/f$ for some $f \in R$, $f \notin \mathfrak{p}$. After replacing R by R_f and x_i by $x_i - a_i$ we may assume that $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ such that x_i maps to zero in $S_\mathfrak{q}$. Then if c_j denotes the constant term of g_j we conclude that c_j maps to zero in $R_\mathfrak{p}$. After another replacement of R we may assume that the constant coefficients c_j of the g_j are zero. Thus we obtain an R-algebra map $S \to R$, $x_i \mapsto 0$ whose kernel is the ideal $(x_1, \ldots, x_n)$.

We have the isomorphisms $R_\mathfrak{p} \to S_\mathfrak{q} \to R_\mathfrak{p}$ and $S \to R$ sends x_i to zero. Thus we must have $S_\mathfrak{q} = R_\mathfrak{p}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$ and a fortiori $S_\mathfrak{q} = S_\mathfrak{p}/(x_1, \ldots, x_n)S_\mathfrak{p}$. This means that the finitely generated ideal $(x_1, \ldots, x_n)S_\mathfrak{p}$ is pure in $S_\mathfrak{p}$, see Definition 10.108.1. Hence $(x_1, \ldots, x_n)S_\mathfrak{p}$ is generated by an idempotent e in $S_\mathfrak{p}$ by Lemma 10.108.5. After replacing $R \to S$ by $R_f \to S_f$ for some $f \in R$, $f \notin \mathfrak{p}$ we can find an idempotent $e' \in S$ mapping to e. Then $e'S$ and $(x_1, \ldots, x_n)S$ are finitely generated ideals which become equal in $S_\mathfrak{p}$. Hence after replacing $R \to S$ by $R_f \to S_f$ for some $f \in R$, $f \notin \mathfrak{p}$ we may assume $e'S = (x_1, \ldots, x_n)S$. Setting $C = e'S$ finishes the proof. □

00QS **Lemma 10.126.7.** *Let R be a ring. Let S, S' be of finite presentation over R. Let $\mathfrak{q} \subset S$ and $\mathfrak{q}' \subset S'$ be primes. If $S_\mathfrak{q} \cong S'_{\mathfrak{q}'}$ as R-algebras, then there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S_g \cong S'_{g'}$ as R-algebras.*

Proof. Let $\psi : S_\mathfrak{q} \to S'_{\mathfrak{q}'}$ be the isomorphism of the hypothesis of the lemma. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $S' = R[y_1, \ldots, y_m]/J$. For each $i = 1, \ldots, n$ choose a fraction h_i/g_i with $h_i, g_i \in R[y_1, \ldots, y_m]$ and $g_i \bmod J$ not in $\mathfrak{q}'$ which represents the image of x_i under ψ. After replacing S' by $S'_{g_1 \ldots g_n}$ and $R[y_1, \ldots, y_m, y_{m+1}]$ (mapping y_{m+1} to $1/(g_1 \ldots g_n)$) we may assume that $\psi(x_i)$ is the image of some $h_i \in R[y_1, \ldots, y_m]$. Consider the elements $f_j(h_1, \ldots, h_n) \in R[y_1, \ldots, y_m]$. Since ψ kills each f_j we see that there exists a $g \in R[y_1, \ldots, y_m]$, $g \bmod J \notin \mathfrak{q}'$ such that $gf_j(h_1, \ldots, h_n) \in J$ for each $j = 1, \ldots, r$. After replacing S' by S'_g and $R[y_1, \ldots, y_m, y_{m+1}]$ as before we may assume that $f_j(h_1, \ldots, h_n) \in J$. Thus we obtain a ring map $S \to S'$, $x_i \mapsto h_i$ which induces ψ on local rings. By Lemma 10.6.2 the map $S \to S'$ is of finite presentation. By Lemma 10.126.6 we may assume that $S' = S \times C$. Thus localizing S' at the idempotent corresponding to the factor C we obtain the result. □

0G8U **Lemma 10.126.8.** *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let S be an R-algebra such that $R/I \to S/IS$ is of finite type. Then $R \to S$ is of finite type.*

Proof. Choose $s_1, \ldots, s_n \in S$ whose images in S/IS generate S/IS as an algebra over R/I. By Lemma 10.20.1 part (11) we see that the R-algebra map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto s_i$ is surjective and we conclude. □

07RD **Lemma 10.126.9.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $S \to S'$ be an R-algebra map such that $S \to S'/IS'$ is surjective and such that S' is of finite type over R. Then $S \to S'$ is surjective.*

Proof. Write $S' = R[x_1, \ldots, x_m]/K$ for some ideal K. By assumption there exist $g_j = x_j + \sum \delta_{j,J} x^J \in R[x_1, \ldots, x_n]$ with $\delta_{j,J} \in I$ and with $g_j \bmod K \in \mathrm{Im}(S \to S')$.

Hence it suffices to show that $g_1, \ldots, g_m$ generate $R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be a finitely generated $\mathbf{Z}$-subalgebra of R containing at least the $\delta_{j,J}$. Then $R_0 \cap I$ is a nilpotent ideal (by Lemma 10.32.5). It follows that $R_0[x_1, \ldots, x_n]$ is generated by $g_1, \ldots, g_m$ (because $x_j \mapsto g_j$ defines an automorphism of $R_0[x_1, \ldots, x_m]$; details omitted). Since R is the union of the subrings R_0 we win. □

087P **Lemma 10.126.10.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Let $IS \subset \mathfrak{q} \subset S$ be a prime ideal. Assume that*

(1) *$S \to S'$ is surjective,*
(2) *$S_\mathfrak{q}/IS_\mathfrak{q} \to S'_\mathfrak{q}/IS'_\mathfrak{q}$ is an isomorphism,*
(3) *S is of finite type over R,*
(4) *S' of finite presentation over R, and*
(5) *$S'_\mathfrak{q}$ is flat over R.*

Then $S_g \to S'_g$ is an isomorphism for some $g \in S$, $g \notin \mathfrak{q}$.

Proof. Let $J = \mathrm{Ker}(S \to S')$. By Lemma 10.6.2 J is a finitely generated ideal. Since $S'_\mathfrak{q}$ is flat over R we see that $J_\mathfrak{q}/IJ_\mathfrak{q} \subset S_\mathfrak{q}/IS_\mathfrak{q}$ (apply Lemma 10.39.12 to $0 \to J \to S \to S' \to 0$). By assumption (2) we see that $J_\mathfrak{q}/IJ_\mathfrak{q}$ is zero. By Nakayama's lemma (Lemma 10.20.1) we see that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $J_g = 0$. Hence $S_g \cong S'_g$ as desired. □

07RE **Lemma 10.126.11.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Assume that*

(1) *I is locally nilpotent,*
(2) *$S/IS \to S'/IS'$ is an isomorphism,*
(3) *S is of finite type over R,*
(4) *S' of finite presentation over R, and*
(5) *S' is flat over R.*

Then $S \to S'$ is an isomorphism.

Proof. By Lemma 10.126.9 the map $S \to S'$ is surjective. As I is locally nilpotent, so are the ideals IS and IS' (Lemma 10.32.3). Hence every prime ideal $\mathfrak{q}$ of S contains IS and (trivially) $S_\mathfrak{q}/IS_\mathfrak{q} \cong S'_\mathfrak{q}/IS'_\mathfrak{q}$. Thus Lemma 10.126.10 applies and we see that $S_\mathfrak{q} \to S'_\mathfrak{q}$ is an isomorphism for every prime $\mathfrak{q} \subset S$. It follows that $S \to S'$ is injective for example by Lemma 10.23.1. □

10.127. Colimits and maps of finite presentation

00QL In this section we prove some preliminary lemmas which will eventually help us prove result using absolute Noetherian reduction. In Categories, Section 4.19 we discuss filtered colimits in general. Here is an example of this very general notion.

0BUF **Lemma 10.127.1.** *Let $R \to A$ be a ring map. Consider the category $\mathcal{I}$ of all diagrams of R-algebra maps $A' \to A$ with A' finitely presented over R. Then $\mathcal{I}$ is filtered, and the colimit of the A' over $\mathcal{I}$ is isomorphic to A.*

Proof. The category[11] $\mathcal{I}$ is nonempty as $R \to R$ is an object of it. Consider a pair of objects $A' \to A$, $A'' \to A$ of $\mathcal{I}$. Then $A' \otimes_R A'' \to A$ is in $\mathcal{I}$ (use Lemmas 10.6.2 and 10.14.2). The ring maps $A' \to A' \otimes_R A''$ and $A'' \to A' \otimes_R A''$ define arrows in $\mathcal{I}$

[11] To avoid set theoretical difficulties we consider only $A' \to A$ such that A' is a quotient of $R[x_1, x_2, x_3, \ldots]$.

thereby proving the second defining property of a filtered category, see Categories, Definition 4.19.1. Finally, suppose that we have two morphisms $\sigma, \tau : A' \to A''$ in $\mathcal{I}$. If $x_1, \ldots, x_r \in A'$ are generators of A' as an R-algebra, then we can consider $A''' = A''/(\sigma(x_i) - \tau(x_i))$. This is a finitely presented R-algebra and the given R-algebra map $A'' \to A$ factors through the surjection $\nu : A'' \to A'''$. Thus ν is a morphism in $\mathcal{I}$ equalizing σ and τ as desired.

The fact that our index category is cofiltered means that we may compute the value of $B = \operatorname{colim}_{A' \to A} A'$ in the category of sets (some details omitted; compare with the discussion in Categories, Section 4.19). To see that $B \to A$ is surjective, for every $a \in A$ we can use $R[x] \to A$, $x \mapsto a$ to see that a is in the image of $B \to A$. Conversely, if $b \in B$ is mapped to zero in A, then we can find $A' \to A$ in $\mathcal{I}$ and $a' \in A'$ which maps to b. Then $A'/(a') \to A$ is in $\mathcal{I}$ as well and the map $A' \to B$ factors as $A' \to A'/(a') \to B$ which shows that $b = 0$ as desired. □

Often it is easier to think about colimits over preordered sets. Let $(\Lambda, \geq)$ a preordered set. A system of rings over Λ is given by a ring R_λ for every $\lambda \in \Lambda$, and a morphism $R_\lambda \to R_\mu$ whenever $\lambda \leq \mu$. These morphisms have to satisfy the rule that $R_\lambda \to R_\mu \to R_\nu$ is equal to the map $R_\lambda \to R_\nu$ for all $\lambda \leq \mu \leq \nu$. See Categories, Section 4.21. We will often assume that $(I, \leq)$ is *directed*, which means that Λ is nonempty and given $\lambda, \mu \in \Lambda$ there exists a $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Recall that the colimit $\operatorname{colim}_\lambda R_\lambda$ is sometimes called a "direct limit" in this case (but we will not use this terminology).

Note that Categories, Lemma 4.21.5 tells us that colimits over filtered index categories are the same thing as colimits over directed sets.

00QN **Lemma 10.127.2.** *Let $R \to A$ be a ring map. There exists a directed system A_λ of R-algebras of finite presentation such that $A = \operatorname{colim}_\lambda A_\lambda$. If A is of finite type over R we may arrange it so that all the transition maps in the system of A_λ are surjective.*

Proof. The first proof is that this follows from Lemma 10.127.1 and Categories, Lemma 4.21.5.

Second proof. Compare with the proof of Lemma 10.11.3. Consider any finite subset $S \subset A$, and any finite collection of polynomial relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in A$ and each $e \in E$ consists of a polynomial $f_e \in R[X_s; s \in S]$ such that $f_e(x_s) = 0$. Let $A_{S,E} = R[X_s; s \in S]/(f_e; e \in E)$ which is a finitely presented R-algebra. There are canonical maps $A_{S,E} \to A$. If $S \subset S'$ and if the elements of E correspond, via the map $R[X_s; s \in S] \to R[X_s; s \in S']$, to a subset of E', then there is an obvious map $A_{S,E} \to A_{S',E'}$ commuting with the maps to A. Thus, setting Λ equal the set of pairs (S, E) with ordering by inclusion as above, we get a directed partially ordered set. It is clear that the colimit of this directed system is A.

For the last statement, suppose $A = R[x_1, \ldots, x_n]/I$. In this case, consider the subset $\Lambda' \subset \Lambda$ consisting of those systems (S, E) above with $S = \{x_1, \ldots, x_n\}$. It is easy to see that still $A = \operatorname{colim}_{\lambda' \in \Lambda'} A_{\lambda'}$. Moreover, the transition maps are clearly surjective. □

It turns out that we can characterize ring maps of finite presentation as follows. This in some sense says that the algebras of finite presentation are the "compact" objects in the category of R-algebras.

00QO **Lemma 10.127.3.** *Let $\varphi : R \to S$ be a ring map. The following are equivalent*

(1) *φ is of finite presentation,*
(2) *for every directed system A_λ of R-algebras the map*
$$\operatorname{colim}_\lambda \operatorname{Hom}_R(S, A_\lambda) \longrightarrow \operatorname{Hom}_R(S, \operatorname{colim}_\lambda A_\lambda)$$
is bijective, and
(3) *for every directed system A_λ of R-algebras the map*
$$\operatorname{colim}_\lambda \operatorname{Hom}_R(S, A_\lambda) \longrightarrow \operatorname{Hom}_R(S, \operatorname{colim}_\lambda A_\lambda)$$
is surjective.

Proof. Assume (1) and write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $A = \operatorname{colim} A_\lambda$. Observe that an R-algebra homomorphism $S \to A$ or $S \to A_\lambda$ is determined by the images of $x_1, \ldots, x_n$. Hence it is clear that $\operatorname{colim}_\lambda \operatorname{Hom}_R(S, A_\lambda) \to \operatorname{Hom}_R(S, A)$ is injective. To see that it is surjective, let $\chi : S \to A$ be an R-algebra homomorphism. Then each x_i maps to some element in the image of some A_{λ_i}. We may pick $\mu \geq \lambda_i$, $i = 1, \ldots, n$ and assume $\chi(x_i)$ is the image of $y_i \in A_\mu$ for $i = 1, \ldots, n$. Consider $z_j = f_j(y_1, \ldots, y_n) \in A_\mu$. Since χ is a homomorphism the image of z_j in $A = \operatorname{colim}_\lambda A_\lambda$ is zero. Hence there exists a $\mu_j \geq \mu$ such that z_j maps to zero in A_{μ_j}. Pick $\nu \geq \mu_j$, $j = 1, \ldots, m$. Then the images of $z_1, \ldots, z_m$ are zero in A_ν. This exactly means that the y_i map to elements $y_i' \in A_\nu$ which satisfy the relations $f_j(y_1', \ldots, y_n') = 0$. Thus we obtain a ring map $S \to A_\nu$. This shows that (1) implies (2).

It is clear that (2) implies (3). Assume (3). By Lemma 10.127.2 we may write $S = \operatorname{colim}_\lambda S_\lambda$ with S_λ of finite presentation over R. Then the identity map factors as
$$S \to S_\lambda \to S$$
for some λ. This implies that S is finitely presented over S_λ by Lemma 10.6.2 part (4) applied to $S \to S_\lambda \to S$. Applying part (2) of the same lemma to $R \to S_\lambda \to S$ we conclude that S is of finite presentation over R. □

Using the basic material above we can give a criterion of when an algebra A is a filtered colimit of given type of algebra as follows.

07C3 **Lemma 10.127.4.** *Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent*

(1) *Λ is a filtered colimit of elements of $\mathcal{E}$, and*
(2) *for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a filtered diagram such that $\Lambda = \operatorname{colim}_i A_i$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then we get a factorization $A \to A_i \to \Lambda$ by applying Lemma 10.127.3. Thus (1) implies (2).

Consider the category $\mathcal{I}$ of Lemma 10.127.1. By Categories, Lemma 4.19.3 the full subcategory $\mathcal{J}$ consisting of those $A \to \Lambda$ with $A \in \mathcal{E}$ is cofinal in $\mathcal{I}$ and is a filtered category. Then Λ is also the colimit over $\mathcal{J}$ by Categories, Lemma 4.17.2. □

But more is true. Namely, given $R = \operatorname{colim}_\lambda R_\lambda$ we see that the category of finitely presented R-modules is equivalent to the limit of the category of finitely presented R_λ-modules. Similarly for the categories of finitely presented R-algebras.

05LI **Lemma 10.127.5.** *Let A be a ring and let M, N be A-modules. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.*

1. *If M is a finite A-module, and $u, u' : M \to N$ are A-module maps such that $u \otimes 1 = u' \otimes 1 : M \otimes_A R \to N \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.*
2. *If N is a finite A-module and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.*
3. *If N is a finitely presented A-module, and $v : N \otimes_A R \to M \otimes_A R$ is an R-module map, then there exists an i and an R_i-module map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ such that $v = v_i \otimes 1$.*
4. *If M is a finite A-module, N is a finitely presented A-module, and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in M$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $M \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in N$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $N \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.

To prove (3) let $y_1, \ldots, y_m \in N$ be generators. Let $K = \operatorname{Ker}(A^{\oplus m} \to N)$ where the map is given by the rule $(a_1, \ldots, a_m) \mapsto \sum a_j x_j$. Let $k_1, \ldots, k_t$ be generators for K. Say $k_s = (k_{s1}, \ldots, k_{sm})$. Since $M \otimes_A R = \operatorname{colim}_i M \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $M \otimes_A R$ equal $v(y_j \otimes 1)$. We want to use the z_j to define the map $v_i : N \otimes_A R_i \to M \otimes_A R_i$. Since $K \otimes_A R_i \to R_i^{\oplus m} \to N \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = \sum_j k_{sj} z_j$ is zero in $M \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $M \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that M is finite, and that N is finitely presented. Let $v : N \otimes_A R \to M \otimes_A R$ be an inverse to $u \otimes 1$. Apply part (3) to get a map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ for some i. Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{M \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{N \otimes_R R_i}$. □

05N7 **Lemma 10.127.6.** *Suppose that $R = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-modules is the colimit of the categories of finitely presented R_λ-modules. More precisely*

1. *Given a finitely presented R-module M there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-module M_λ such that $M \cong M_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and an R-module map $\varphi : M_\lambda \otimes_{R_\lambda} R \to N_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-module map $\varphi_\mu : M_\lambda \otimes_{R_\lambda} R_\mu \to N_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*

(3) *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and R-module maps $\varphi_\lambda, \psi_\lambda : M_\lambda \to N_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Suppose that the first map is given by the matrix $A = (a_{ij})$. We can choose a $\lambda \in \Lambda$ and a matrix $A_\lambda = (a_{\lambda,ij})$ with coefficients in R_λ which maps to A in R. Then we simply let M_λ be the R_λ-module with presentation $R_\lambda^{\oplus m} \to R_\lambda^{\oplus n} \to M_\lambda \to 0$ where the first arrow is given by A_λ.

Parts (2) and (3) follow from Lemma 10.127.5. □

05N8 **Lemma 10.127.7.** *Let A be a ring and let B, C be A-algebras. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.*

(1) *If B is a finite type A-algebra, and $u, u' : B \to C$ are A-algebra maps such that $u \otimes 1 = u' \otimes 1 : B \otimes_A R \to C \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.*
(2) *If C is a finite type A-algebra and $u : B \to C$ is an A-algebra map such that $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.*
(3) *If C is of finite presentation over A and $v : C \otimes_A R \to B \otimes_A R$ is an R-algebra map, then there exists an i and an R_i-algebra map $v_i : C \otimes_A R_i \to B \otimes_A R_i$ such that $v = v_i \otimes 1$.*
(4) *If B is a finite type A-algebra, C is a finitely presented A-algebra, and $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in B$ be generators. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $B \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in C$ be generators. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $C \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.

To prove (3) let $c_1, \ldots, c_m \in C$ be generators. Let $K = \operatorname{Ker}(A[x_1, \ldots, x_m] \to N)$ where the map is given by the rule $x_j \mapsto \sum c_j$. Let $f_1, \ldots, f_t$ be generators for K as an ideal in $A[x_1, \ldots, x_m]$. We think of $f_j = f_j(x_1, \ldots, x_m)$ as a polynomial. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $B \otimes_A R$ equal $v(c_j \otimes 1)$. We want to use the z_j to define a map $v_i : C \otimes_A R_i \to B \otimes_A R_i$. Since $K \otimes_A R_i \to R_i[x_1, \ldots, x_m] \to C \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = f_j(z_1, \ldots, z_m)$ is zero in $B \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $B \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that B is a finite type A-algebra, and that C is a finitely presented A-algebra. Let $v : B \otimes_A R \to C \otimes_A R$ be an inverse to $u \otimes 1$. Let $v_i : C \otimes_A R_i \to B \otimes_A R_i$ be as in part (3). Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{B \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{C \otimes_R R_i}$. □

05N9 **Lemma 10.127.8.** *Suppose that $R = \operatorname{colim}_{\lambda\in\Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-algebras is the colimit of the categories of finitely presented R_λ-algebras. More precisely*

(1) *Given a finitely presented R-algebra A there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-algebra A_λ such that $A \cong A_\lambda \otimes_{R_\lambda} R$.*
(2) *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and an R-algebra map $\varphi : A_\lambda \otimes_{R_\lambda} R \to B_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-algebra map $\varphi_\mu : A_\lambda \otimes_{R_\lambda} R_\mu \to B_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
(3) *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and R_λ-algebra maps $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We can choose a $\lambda \in \Lambda$ and elements $f_{\lambda,j} \in R_\lambda[x_1, \ldots, x_n]$ mapping to $f_j \in R[x_1, \ldots, x_n]$. Then we simply let $A_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_{\lambda,1}, \ldots, f_{\lambda,m})$.

Parts (2) and (3) follow from Lemma 10.127.7. □

00QT **Lemma 10.127.9.** *Suppose $R \to S$ is a local homomorphism of local rings. There exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let

$$\Lambda = \{(A, B) \mid A \subset R, B \subset S, \#A < \infty, \#B < \infty, \varphi(A) \subset B\}.$$

As partial ordering we take the inclusion relation. For each $\lambda = (A, B) \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by b, $b \in B$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{array}{ccccccc} B & \longrightarrow & S'_\lambda & \longrightarrow & S_\lambda & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & R'_\lambda & \longrightarrow & R_\lambda & \longrightarrow & R \end{array}.$$

The transition maps are clear. We leave the proofs of the other assertions to the reader. □

00QU **Lemma 10.127.10.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let $x_1, \ldots, x_n \in S$ be elements such that S is a localization of the sub R-algebra of S generated by $x_1, \ldots, x_n$. In other words, S is a quotient of a localization of the polynomial ring $R[x_1, \ldots, x_n]$.

Let $\Lambda = \{A \subset R \mid \#A < \infty\}$ be the set of finite subsets of R. As partial ordering we take the inclusion relation. For each $\lambda = A \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by $\varphi(a)$, $a \in A$ and the elements $x_1, \ldots, x_n$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{array}{ccccccc} \varphi(A) \amalg \{x_i\} & \longrightarrow & S'_\lambda & \longrightarrow & S_\lambda & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & R'_\lambda & \longrightarrow & R_\lambda & \longrightarrow & R \end{array}$$

It is clear that if $A \subset B$ corresponds to $\lambda \leq \mu$ in Λ, then there are canonical maps $R_\lambda \to R_\mu$, and $S_\lambda \to S_\mu$ and we obtain a system over the directed set Λ.

The assertion that $R = \operatorname{colim} R_\lambda$ is clear because all the maps $R_\lambda \to R$ are injective and any element of R eventually is in the image. The same argument works for $S = \operatorname{colim} S_\lambda$. Assertions (2), (3) are true by construction. The final assertion holds because clearly the maps $S'_\lambda \otimes_{R'_\lambda} R'_\mu \to S'_\mu$ are surjective. □

00QV **Lemma 10.127.11.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphism $R_\lambda \to S_\lambda$ of local rings such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*

Proof. By assumption we may choose an isomorphism $\Phi : (R[x_1, \ldots, x_n]/I)_{\mathfrak{q}} \to S$ where $I \subset R[x_1, \ldots, x_n]$ is a finitely generated ideal, and $\mathfrak{q} \subset R[x_1, \ldots, x_n]/I$ is a prime. (Note that $R \cap \mathfrak{q}$ is equal to the maximal ideal $\mathfrak{m}$ of R.) We also choose generators $f_1, \ldots, f_m \in I$ for the ideal I. Write R in any way as a colimit $R = \operatorname{colim} R_\lambda$ over a directed set $(\Lambda, \leq)$, with each R_λ local and essentially of finite type over $\mathbf{Z}$. There exists some $\lambda_0 \in \Lambda$ such that f_j is the image of some $f_{j,\lambda_0} \in R_{\lambda_0}[x_1, \ldots, x_n]$. For all $\lambda \geq \lambda_0$ denote $f_{j,\lambda} \in R_\lambda[x_1, \ldots, x_n]$ the image of f_{j,λ_0}. Thus we obtain a system of ring maps

$$R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}) \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to S$$

Set $\mathfrak{q}_\lambda$ the inverse image of $\mathfrak{q}$. Set $S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$. We leave it to the reader to see that this works. □

00QW **Remark 10.127.12.** Suppose that $R \to S$ is a local homomorphism of local rings, which is essentially of finite presentation. Take any system $(\Lambda, \leq)$, $R_\lambda \to S_\lambda$ with the properties listed in Lemma 10.127.10. What may happen is that this is the

"wrong" system, namely, it may happen that property (4) of Lemma 10.127.11 is not satisfied. Here is an example. Let $k = \mathbf{F}_2$. Consider the ring

$$R = \text{localization of } k[z, y_1, y_2, \ldots]/(y_i^2 - zy_{i+1}) \text{ at } (z, y_1, y_2, \ldots)$$

Set $S = R/zR$. As system take $\Lambda = \mathbf{N}$ and

$$R_n = \text{localization of } k[z, y_1, \ldots, y_n]/(\{y_i^2 - zy_{i+1}\}_{i \leq n-1}) \text{ at } (z, y_1, \ldots, y_n)$$

and $S_n = R_n/(z, y_n^2)$. All the maps $S_n \otimes_{R_n} R_{n+1} \to S_{n+1}$ are not localizations (i.e., isomorphisms in this case) since $1 \otimes y_{n+1}^2$ maps to zero. If we take instead $S'_n = R_n/zR_n$ then the maps $S'_n \otimes_{R_n} R_{n+1} \to S'_{n+1}$ are isomorphisms. The moral of this remark is that we do have to be a little careful in choosing the systems.

00QX **Lemma 10.127.13.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings together with S_λ-modules M_λ, such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*
(4) *Each M_λ is finite over S_λ.*
(5) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*
(6) *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

Proof. As in the proof of Lemma 10.127.11 we may first write $R = \operatorname{colim} R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \operatorname{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$$

Choose a presentation

$$S^{\oplus s} \to S^{\oplus t} \to M \to 0$$

of M over S. Let $A \in \operatorname{Mat}(t \times s, S)$ be the matrix of the presentation. For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ we can find a matrix $A_{\lambda_2} \in \operatorname{Mat}(t \times s, S_{\lambda_2})$ which maps to A. For all $\lambda \geq \lambda_2$ we let $M_\lambda = \operatorname{Coker}(S_\lambda^{\oplus s} \xrightarrow{A_\lambda} S_\lambda^{\oplus t})$. We leave it to the reader to see that this works. □

00QY **Lemma 10.127.14.** *Suppose $R \to S$ is a ring map. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*

Proof. This is the non-local version of Lemma 10.127.9. Proof is similar and left to the reader. □

0BTG **Lemma 10.127.15.** *Suppose $R \to S$ is a ring map. Assume that S is integral over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*

(3) *Each S_λ is finite over R_λ.*

Proof. Consider the set Λ of pairs (E, F) where $E \subset R$ is a finite subset, $F \subset S$ is a finite subset, and every element $f \in F$ is the root of a monic $P(X) \in R[X]$ whose coefficients are in E. Say $(E, F) \leq (E', F')$ if $E \subset E'$ and $F \subset F'$. Given $\lambda = (E, F) \in \Lambda$ set $R_\lambda \subset R$ equal to the $\mathbf{Z}$-subalgebra of R generated by E and $S_\lambda \subset S$ equal to the $\mathbf{Z}$-subalgebra generated by F and the image of E in S. It is clear that $R = \operatorname{colim} R_\lambda$. We have $S = \operatorname{colim} S_\lambda$ as every element of S is integral over S. The ring maps $R_\lambda \to S_\lambda$ are finite by Lemma 10.36.5 and the fact that S_λ is generated over R_λ by the elements of F which are integral over R_λ by our condition on the pairs (E, F). The lemma follows. □

00QZ **Lemma 10.127.16.** *Suppose $R \to S$ is a ring map. Assume that S is of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. This is the non-local version of Lemma 10.127.10. Proof is similar and left to the reader. □

00R0 **Lemma 10.127.17.** *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*

Proof. This is the non-local version of Lemma 10.127.11. Proof is similar and left to the reader. □

00R1 **Lemma 10.127.18.** *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ together with S_λ-modules M_λ, such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *Each M_λ is finite over S_λ.*
(5) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*
(6) *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

In particular, for every $\lambda \in \Lambda$ we have

$$M = M_\lambda \otimes_{S_\lambda} S = M_\lambda \otimes_{R_\lambda} R.$$

Proof. This is the non-local version of Lemma 10.127.13. Proof is similar and left to the reader. □

10.128. More flatness criteria

00R3 The following lemma is often used in algebraic geometry to show that a finite morphism from a normal surface to a smooth surface is flat. It is a partial converse to Lemma 10.112.9 because an injective finite local ring map certainly satisfies condition (3).

00R4 **Lemma 10.128.1.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume*

(1) *R is regular,*
(2) *S Cohen-Macaulay,*
(3) $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$.

Then $R \to S$ is flat.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial, because then R is a field. Assume $\dim(R) > 0$. By (3) this implies that $\dim(S) > 0$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of S. Note that $\mathfrak{q}_i \not\supset \mathfrak{m}_R S$ since

$$\dim(S/\mathfrak{q}_i) = \dim(S) > \dim(S/\mathfrak{m}_R S)$$

the first equality by Lemma 10.104.3 and the inequality by (3). Thus $\mathfrak{p}_i = R \cap \mathfrak{q}_i$ is not equal to $\mathfrak{m}_R$. Pick $x \in \mathfrak{m}_R$, $x \notin \mathfrak{m}_R^2$, and $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Hence we see that x is not contained in any of the minimal primes of S. Hence x is a nonzerodivisor on S by (2), see Lemma 10.104.2 and S/xS is Cohen-Macaulay with $\dim(S/xS) = \dim(S) - 1$. By (1) and Lemma 10.106.3 the ring R/xR is regular with $\dim(R/xR) = \dim(R) - 1$. By induction we see that $R/xR \to S/xS$ is flat. Hence we conclude by Lemma 10.99.10 and the remark following it. □

07DY **Lemma 10.128.2.** *Let $R \to S$ be a homomorphism of Noetherian local rings. Assume that R is a regular local ring and that a regular system of parameters maps to a regular sequence in S. Then $R \to S$ is flat.*

Proof. Suppose that $x_1, \ldots, x_d$ are a system of parameters of R which map to a regular sequence in S. Note that $S/(x_1, \ldots, x_d)S$ is flat over $R/(x_1, \ldots, x_d)$ as the latter is a field. Then x_d is a nonzerodivisor in $S/(x_1, \ldots, x_{d-1})S$ hence $S/(x_1, \ldots, x_{d-1})S$ is flat over $R/(x_1, \ldots, x_{d-1})$ by the local criterion of flatness (see Lemma 10.99.10 and remarks following). Then x_{d-1} is a nonzerodivisor in $S/(x_1, \ldots, x_{d-2})S$ hence $S/(x_1, \ldots, x_{d-2})S$ is flat over $R/(x_1, \ldots, x_{d-2})$ by the local criterion of flatness (see Lemma 10.99.10 and remarks following). Continue till one reaches the conclusion that S is flat over R. □

The following lemma is the key to proving that results for finitely presented modules over finitely presented rings over a base ring follow from the corresponding results for finite modules in the Noetherian case.

00R6 **Lemma 10.128.3.** *Let $R \to S$, M, Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.127.13. Assume that M is flat over R. Then for some $\lambda \in \Lambda$ the module M_λ is flat over R_λ.*

Proof. Pick some $\lambda \in \Lambda$ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) = \mathrm{Ker}(\mathfrak{m}_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let

$\xi_1, \ldots, \xi_n$ be generators. Because M is flat over R we have that $0 = \mathrm{Ker}(\mathfrak{m}_\lambda R \otimes_R M \to M)$. Since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathfrak{m}_\lambda R_{\lambda'} \otimes_{R_{\lambda'}} M_{\lambda'}$. Hence we see that

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) \longrightarrow \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/\mathfrak{m}_\lambda R_{\lambda'})$$

is zero. Note that $M_\lambda \otimes_{R_\lambda} R_\lambda/\mathfrak{m}_\lambda$ is flat over $R_\lambda/\mathfrak{m}_\lambda$ because this last ring is a field. Hence we may apply Lemma 10.99.14 to get that $M_{\lambda'}$ is flat over $R_{\lambda'}$. □

Using the lemma above we can start to reprove the results of Section 10.99 in the non-Noetherian case.

046Y **Lemma 10.128.4.** *Suppose that $R \to S$ is a local homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of S-modules. Assume*

(1) *S is essentially of finite presentation over R,*
(2) *M, N are finitely presented over S,*
(3) *N is flat over R, and*
(4) *$\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective.*

Then u is injective, and $N/u(M)$ is flat over R.

Proof. By Lemma 10.127.13 and its proof we can find a system $R_\lambda \to S_\lambda$ of local ring maps together with maps of S_λ-modules $u_\lambda : M_\lambda \to N_\lambda$ satisfying the conclusions (1) – (6) for both N and M of that lemma and such that the colimit of the maps u_λ is u. By Lemma 10.128.3 we may assume that N_λ is flat over R_λ for all sufficiently large λ. Denote $\mathfrak{m}_\lambda \subset R_\lambda$ the maximal ideal and $\kappa_\lambda = R_\lambda/\mathfrak{m}_\lambda$, resp. $\kappa = R/\mathfrak{m}$ the residue fields.

Consider the map

$$\Psi_\lambda : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa \longrightarrow M/\mathfrak{m}M.$$

Since $S_\lambda/\mathfrak{m}_\lambda S_\lambda$ is essentially of finite type over the field κ_λ we see that the tensor product $S_\lambda/\mathfrak{m}_\lambda S_\lambda \otimes_{\kappa_\lambda} \kappa$ is essentially of finite type over κ. Hence it is a Noetherian ring and we conclude the kernel of Ψ_λ is finitely generated. Since $M/\mathfrak{m}M$ is the colimit of the system $M_\lambda/\mathfrak{m}_\lambda M_\lambda$ and κ is the colimit of the fields κ_λ there exists a $\lambda' > \lambda$ such that the kernel of Ψ_λ is generated by the kernel of

$$\Psi_{\lambda,\lambda'} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'} \longrightarrow M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

By construction there exists a multiplicative subset $W \subset S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ such that $S_{\lambda'} = W^{-1}(S_\lambda \otimes_{R_\lambda} R_{\lambda'})$ and

$$W^{-1}(M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'}) = M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

Now suppose that x is an element of the kernel of

$$\Psi_{\lambda'} : M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa \longrightarrow M/\mathfrak{m}M.$$

Then for some $w \in W$ we have $wx \in M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes \kappa$. Hence $wx \in \mathrm{Ker}(\Psi_\lambda)$. Hence wx is a linear combination of elements in the kernel of $\Psi_{\lambda,\lambda'}$. Hence $wx = 0$ in $M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa$, hence $x = 0$ because w is invertible in $S_{\lambda'}$. We conclude that the kernel of $\Psi_{\lambda'}$ is zero for all sufficiently large λ'!

By the result of the preceding paragraph we may assume that the kernel of Ψ_λ is zero for all λ sufficiently large, which implies that the map $M_\lambda/\mathfrak{m}_\lambda M_\lambda \to M/\mathfrak{m}M$ is injective. Combined with $\overline{u}$ being injective this formally implies that also $\overline{u_\lambda} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \to N_\lambda/\mathfrak{m}_\lambda N_\lambda$ is injective. By Lemma 10.99.1 we conclude that (for all

sufficiently large λ) the map u_λ is injective and that $N_\lambda/u_\lambda(M_\lambda)$ is flat over R_λ. The lemma follows. □

046Z **Lemma 10.128.5.** *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *S is essentially of finite presentation over R,*
(2) *S is flat over R, and*
(3) *$f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$.*

Then S/fS is flat over R, and f is a nonzerodivisor in S.

Proof. Follows directly from Lemma 10.128.4. □

0470 **Lemma 10.128.6.** *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *$R \to S$ is essentially of finite presentation,*
(2) *$R \to S$ is flat, and*
(3) *$f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$.*

Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 10.128.5. □

Here is the version of the local criterion of flatness for the case of local ring maps which are locally of finite presentation.

0471 **Lemma 10.128.7.** *Let $R \to S$ be a local homomorphism of local rings. Let $I \neq R$ be an ideal in R. Let M be an S-module. Assume*

(1) *S is essentially of finite presentation over R,*
(2) *M is of finite presentation over S,*
(3) *$\mathit{Tor}_1^R(M, R/I) = 0$, and*
(4) *M/IM is flat over R/I.*

Then M is flat over R.

Proof. Let Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.127.13. Denote $I_\lambda \subset R_\lambda$ the inverse image of I. In this case the system $R/I \to S/IS$, M/IM, $R_\lambda \to S_\lambda/I_\lambda S_\lambda$, and $M_\lambda/I_\lambda M_\lambda$ satisfies the conclusions of Lemma 10.127.13 as well. Hence by Lemma 10.128.3 we may assume (after shrinking the index set Λ) that $M_\lambda/I_\lambda M_\lambda$ is flat for all λ. Pick some λ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) = \mathrm{Ker}(I_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because $\mathrm{Tor}_1^R(M, R/I) = 0$ and since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in

$\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$. The composition of the maps

$$\begin{array}{l} R_{\lambda'} \otimes_{R_\lambda} \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) \\ \qquad\qquad \Big\downarrow \text{surjective by Lemma 10.99.12} \\ \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_{\lambda'}/I_\lambda R_{\lambda'}) \\ \qquad\qquad \Big\downarrow \text{surjective up to localization by Lemma 10.99.13} \\ \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_\lambda R_{\lambda'}) \\ \qquad\qquad \Big\downarrow \text{surjective by Lemma 10.99.12} \\ \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'}). \end{array}$$

is surjective up to a localization by the reasons indicated. The localization is necessary since $M_{\lambda'}$ is not equal to $M_\lambda \otimes_{R_\lambda} R_{\lambda'}$. Namely, it is equal to $M_\lambda \otimes_{S_\lambda} S_{\lambda'}$ and $S_{\lambda'}$ is the localization of $S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ whence the statement up to a localization (or tensoring with $S_{\lambda'}$). Note that Lemma 10.99.12 applies to the first and third arrows because $M_\lambda/I_\lambda M_\lambda$ is flat over R_λ/I_λ and because $M_{\lambda'}/I_\lambda M_{\lambda'}$ is flat over $R_{\lambda'}/I_\lambda R_{\lambda'}$ as it is a base change of the flat module $M_\lambda/I_\lambda M_\lambda$. The composition maps the generators ξ_i to zero as we explained above. We finally conclude that $\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$ is zero. This implies that $M_{\lambda'}$ is flat over $R_{\lambda'}$ by Lemma 10.99.10. □

Please compare the lemma below to Lemma 10.99.15 (the case of Noetherian local rings) and Lemma 10.101.8 (the case of a nilpotent ideal in the base).

00R7 **Lemma 10.128.8** (Critère de platitude par fibres)**.** *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

(1) *The ring maps $R \to S$ and $R \to S'$ are essentially of finite presentation.*
(2) *The module M is of finite presentation over S'.*
(3) *The module M is not zero.*
(4) *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
(5) *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. As in the proof of Lemma 10.127.11 we may first write $R = \mathrm{colim}\, R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Denote $\mathfrak{p}_\lambda$ the maximal ideal of R_λ. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \mathrm{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{u,\lambda}))_{\mathfrak{q}_\lambda}$$

For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ there exist $g_{j,\lambda_2} \in R_{\lambda_2}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with images $\overline{g}_{j,\lambda_2} \in S_{\lambda_2}[y_1, \ldots, y_m]$ such that

$$S' = \mathrm{colim}_{\lambda \geq \lambda_2} S'_\lambda, \text{ with } S'_\lambda = (S_\lambda[y_1, \ldots, y_m]/(\overline{g}_{1,\lambda}, \ldots, \overline{g}_{v,\lambda}))_{\overline{\mathfrak{q}}'_\lambda}$$

Note that this also implies that

$$S'_\lambda = (R_\lambda[x_1, \ldots, x_n, y_1, \ldots, y_m]/(g_{1,\lambda}, \ldots, g_{v,\lambda}))_{\mathfrak{q}'_\lambda}$$

Choose a presentation

$$(S')^{\oplus s} \to (S')^{\oplus t} \to M \to 0$$

of M over S'. Let $A \in \mathrm{Mat}(t \times s, S')$ be the matrix of the presentation. For some $\lambda_3 \in \Lambda$, $\lambda_3 \geq \lambda_2$ we can find a matrix $A_{\lambda_3} \in \mathrm{Mat}(t \times s, S_{\lambda_3})$ which maps to A. For all $\lambda \geq \lambda_3$ we let $M_\lambda = \mathrm{Coker}((S'_\lambda)^{\oplus s} \xrightarrow{A_\lambda} (S'_\lambda)^{\oplus t})$.

With these choices, we have for each $\lambda_3 \leq \lambda \leq \mu$ that $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is a localization, $S'_\lambda \otimes_{S_\lambda} S_\mu \to S'_\mu$ is a localization, and the map $M_\lambda \otimes_{S'_\lambda} S'_\mu \to M_\mu$ is an isomorphism. This also implies that $S'_\lambda \otimes_{R_\lambda} R_\mu \to S'_\mu$ is a localization. Thus, since M is flat over R we see by Lemma 10.128.3 that for all λ big enough the module M_λ is flat over R_λ. Moreover, note that $\mathfrak{m} = \operatorname{colim} \mathfrak{p}_\lambda$, $S/\mathfrak{m}S = \operatorname{colim} S_\lambda/\mathfrak{p}_\lambda S_\lambda$, $S'/\mathfrak{m}S' = \operatorname{colim} S'_\lambda/\mathfrak{p}_\lambda S'_\lambda$, and $M/\mathfrak{m}M = \operatorname{colim} M_\lambda/\mathfrak{p}_\lambda M_\lambda$. Also, for each $\lambda_3 \leq \lambda \leq \mu$ we see (from the properties listed above) that

$$S'_\lambda/\mathfrak{p}_\lambda S'_\lambda \otimes_{S_\lambda/\mathfrak{p}_\lambda S_\lambda} S_\mu/\mathfrak{p}_\mu S_\mu \longrightarrow S'_\mu/\mathfrak{p}_\mu S'_\mu$$

is a localization, and the map

$$M_\lambda/\mathfrak{p}_\lambda M_\lambda \otimes_{S'_\lambda/\mathfrak{p}_\lambda S'_\lambda} S'_\mu/\mathfrak{p}_\mu S'_\mu \longrightarrow M_\mu/\mathfrak{p}_\mu M_\mu$$

is an isomorphism. Hence the system $(S_\lambda/\mathfrak{p}_\lambda S_\lambda \to S'_\lambda/\mathfrak{p}_\lambda S'_\lambda, M_\lambda/\mathfrak{p}_\lambda M_\lambda)$ is a system as in Lemma 10.127.13 as well. We may apply Lemma 10.128.3 again because $M/\mathfrak{m}M$ is assumed flat over $S/\mathfrak{m}S$ and we see that $M_\lambda/\mathfrak{p}_\lambda M_\lambda$ is flat over $S_\lambda/\mathfrak{p}_\lambda S_\lambda$ for all λ big enough. Thus for λ big enough the data $R_\lambda \to S_\lambda \to S'_\lambda, M_\lambda$ satisfies the hypotheses of Lemma 10.99.15. Pick such a λ. Then $S = S_\lambda \otimes_{R_\lambda} R$ is flat over R, and $M = M_\lambda \otimes_{S_\lambda} S$ is flat over S (since the base change of a flat module is flat). □

The following is an easy consequence of the "critère de platitude par fibres" Lemma 10.128.8. For more results of this kind see More on Flatness, Section 38.1.

05UV **Lemma 10.128.9.** *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

(1) *$R \to S'$ is essentially of finite presentation,*
(2) *$R \to S$ is essentially of finite type,*
(3) *M is of finite presentation over S',*
(4) *M is not zero,*
(5) *$M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module, and*
(6) *M is a flat R-module.*

Then S is essentially of finite presentation and flat over R and M is a flat S-module.

Proof. As S is essentially of finite presentation over R we can write $S = C_{\overline{\mathfrak{q}}}$ for some finite type R-algebra C. Write $C = R[x_1, \ldots, x_n]/I$. Denote $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\overline{\mathfrak{q}}$. Then we see that $S = B/J$ where $B = R[x_1, \ldots, x_n]_{\mathfrak{q}}$ is essentially of finite presentation over R and $J = IB$. We can find $f_1, \ldots, f_k \in J$ such that the images $\overline{f}_i \in B/\mathfrak{m}B$ generate the image $\overline{J}$ of J in the Noetherian ring $B/\mathfrak{m}B$. Hence there exist finitely generated ideals $J' \subset J$ such that $B/J' \to B/J$ induces an isomorphism

$$(B/J') \otimes_R R/\mathfrak{m} \longrightarrow B/J \otimes_R R/\mathfrak{m} = S/\mathfrak{m}S.$$

For any J' as above we see that Lemma 10.128.8 applies to the ring maps

$$R \longrightarrow B/J' \longrightarrow S'$$

and the module M. Hence we conclude that B/J' is flat over R for any choice J' as above. Now, if $J' \subset J' \subset J$ are two finitely generated ideals as above, then we conclude that $B/J' \to B/J''$ is a surjective map between flat R-algebras which are essentially of finite presentation which is an isomorphism modulo $\mathfrak{m}$. Hence Lemma 10.128.4 implies that $B/J' = B/J''$, i.e., $J' = J''$. Clearly this means that J is finitely generated, i.e., S is essentially of finite presentation over R. Thus we may apply Lemma 10.128.8 to $R \to S \to S'$ and we win. □

0CEL **Lemma 10.128.10** (Critère de platitude par fibres: locally nilpotent case)**.** *Let*

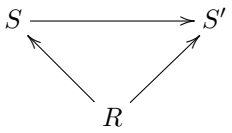

be a commutative diagram in the category of rings. Let $I \subset R$ be a locally nilpotent ideal and M an S'-module. Assume

(1) *$R \to S$ is of finite type,*
(2) *$R \to S'$ is of finite presentation,*
(3) *M is a finitely presented S'-module,*
(4) *M/IM is flat as a S/IS-module, and*
(5) *M is flat as an R-module.*

Then M is a flat S-module and $S_{\mathfrak{q}}$ is flat and essentially of finite presentation over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. If $M \otimes_S \kappa(\mathfrak{q})$ is nonzero, then $S' \otimes_S \kappa(\mathfrak{q})$ is nonzero and hence there exists a prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{q}$ (Lemma 10.18.6). Let $\mathfrak{p} \subset R$ be the image of $\mathfrak{q}$ in $\mathrm{Spec}(R)$. Then $I \subset \mathfrak{p}$ as I is locally nilpotent hence $M/\mathfrak{p}M$ is flat over $S/\mathfrak{p}S$. Hence we may apply Lemma 10.128.9 to $R_{\mathfrak{p}} \to S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ and $M_{\mathfrak{q}'}$. We conclude that $M_{\mathfrak{q}'}$ is flat over S and $S_{\mathfrak{q}}$ is flat and essentially of finite presentation over R. Since $\mathfrak{q}'$ was an arbitrary prime of S' we also see that M is flat over S (Lemma 10.39.18). □

10.129. Openness of the flat locus

00R8 We use Lemma 10.128.3 to reduce to the Noetherian case. The Noetherian case is handled using the characterization of exact complexes given in Section 10.102.

00R9 **Lemma 10.129.1.** *Let k be a field. Let S be a finite type k-algebra. Let $f_1, \ldots, f_i$ be elements of S. Assume that S is Cohen-Macaulay and equidimensional of dimension d, and that $\dim V(f_1, \ldots, f_i) \leq d - i$. Then equality holds and $f_1, \ldots, f_i$ forms a regular sequence in $S_{\mathfrak{q}}$ for every prime $\mathfrak{q}$ of $V(f_1, \ldots, f_i)$.*

Proof. If S is Cohen-Macaulay and equidimensional of dimension d, then we have $\dim(S_{\mathfrak{m}}) = d$ for all maximal ideals $\mathfrak{m}$ of S, see Lemma 10.114.7. By Proposition 10.103.4 we see that for all maximal ideals $\mathfrak{m} \in V(f_1, \ldots, f_i)$ the sequence is a regular sequence in $S_{\mathfrak{m}}$ and the local ring $S_{\mathfrak{m}}/(f_1, \ldots, f_i)$ is Cohen-Macaulay of dimension $d - i$. This actually means that $S/(f_1, \ldots, f_i)$ is Cohen-Macaulay and equidimensional of dimension $d - i$. □

00RA **Lemma 10.129.2.** *Let $R \to S$ be a finite type ring map. Let d be an integer such that all fibres $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay and equidimensional of dimension d. Let $f_1, \ldots, f_i$ be elements of S. The set*

$$\{\mathfrak{q} \in V(f_1, \ldots, f_i) \mid f_1, \ldots, f_i \text{ are a regular sequence in } S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \text{ where } \mathfrak{p} = R \cap \mathfrak{q}\}$$

is open in $V(f_1, \ldots, f_i)$.

Proof. Write $\overline{S} = S/(f_1, \ldots, f_i)$. Suppose $\mathfrak{q}$ is an element of the set defined in the lemma, and $\mathfrak{p}$ is the corresponding prime of R. We will use relative dimension as defined in Definition 10.125.1. First, note that $d = \dim_\mathfrak{q}(S/R) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$ by Lemma 10.116.3. Since $f_1, \ldots, f_i$ form a regular sequence in the Noetherian local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ Lemma 10.60.13 tells us that $\dim(\overline{S}_\mathfrak{q}/\mathfrak{p}\overline{S}_\mathfrak{q}) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) - i$. We conclude that $\dim_\mathfrak{q}(\overline{S}/R) = \dim(\overline{S}_\mathfrak{q}/\mathfrak{p}\overline{S}_\mathfrak{q}) + \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = d - i$ by Lemma 10.116.3. By Lemma 10.125.6 we have $\dim_{\mathfrak{q}'}(\overline{S}/R) \leq d - i$ for all $\mathfrak{q}' \in V(f_1, \ldots, f_i) = \operatorname{Spec}(\overline{S})$ in a neighbourhood of $\mathfrak{q}$. Thus after replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ we may assume that the inequality holds for all $\mathfrak{q}'$. The result follows from Lemma 10.129.1. □

00RB **Lemma 10.129.3.** *Let $R \to S$ be a ring map. Consider a finite homological complex of finite free S-modules:*

$$F_\bullet : 0 \to S^{n_e} \xrightarrow{\varphi_e} S^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} S^{n_i} \xrightarrow{\varphi_i} S^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} S^{n_0}$$

For every prime $\mathfrak{q}$ of S consider the complex $\overline{F}_{\bullet,\mathfrak{q}} = F_{\bullet,\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is inverse image of $\mathfrak{q}$ in R. Assume R is Noetherian and there exists an integer d such that $R \to S$ is finite type, flat with fibres $S \otimes_R \kappa(\mathfrak{p})$ Cohen-Macaulay of dimension d. The set

$$\{\mathfrak{q} \in \operatorname{Spec}(S) \mid \overline{F}_{\bullet,\mathfrak{q}} \text{ is exact}\}$$

is open in $\operatorname{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set defined in the lemma. We are going to use Proposition 10.102.9 to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g)$ is contained in the set defined in the lemma. In other words, we are going to show that after replacing S by S_g, the set of the lemma is all of $\operatorname{Spec}(S)$. Thus during the proof we will, finitely often, replace S by such a localization. Recall that Proposition 10.102.9 characterizes exactness of complexes in terms of ranks of the maps φ_i and the ideals $I(\varphi_i)$, in case the ring is local. We first address the rank condition. Set $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i} n_e$. Note that $r_i + r_{i+1} = n_i$ and note that r_i is the expected rank of φ_i (in the exact case).

By Lemma 10.99.5 we see that if $\overline{F}_{\bullet,\mathfrak{q}}$ is exact, then the localization $F_{\bullet,\mathfrak{q}}$ is exact. In particular the complex $F_\bullet$ becomes exact after localizing by an element $g \in S$, $g \notin \mathfrak{q}$. In this case Proposition 10.102.9 applied to all localizations of S at prime ideals implies that all $(r_i + 1) \times (r_i + 1)$-minors of φ_i are zero. Thus we see that the rank of φ_i is at most r_i.

Let $I_i \subset S$ denote the ideal generated by the $r_i \times r_i$-minors of the matrix of φ_i. By Proposition 10.102.9 the complex $\overline{F}_{\bullet,\mathfrak{q}}$ is exact if and only if for every $1 \leq i \leq e$ we have either $(I_i)_\mathfrak{q} = S_\mathfrak{q}$ or $(I_i)_\mathfrak{q}$ contains a $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$-regular sequence of length i. Namely, by our choice of r_i above and by the bound on the ranks of the φ_i this is the only way the conditions of Proposition 10.102.9 can be satisfied.

If $(I_i)_\mathfrak{q} = S_\mathfrak{q}$, then after localizing S at some element $g \notin \mathfrak{q}$ we may assume that $I_i = S$. Clearly, this is an open condition.

If $(I_i)_\mathfrak{q} \neq S_\mathfrak{q}$, then we have a sequence $f_1, \ldots, f_i \in (I_i)_\mathfrak{q}$ which form a regular sequence in $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. Note that for any prime $\mathfrak{q}' \subset S$ such that $(f_1, \ldots, f_i) \notin \mathfrak{q}'$ we have $(I_i)_{\mathfrak{q}'} = S_{\mathfrak{q}'}$. Thus the result follows from Lemma 10.129.2. □

00RC **Theorem 10.129.4.** *Let R be a ring. Let $R \to S$ be a ring map of finite presentation. Let M be a finitely presented S-module. The set*

$$\{\mathfrak{q} \in \mathrm{Spec}(S) \mid M_\mathfrak{q} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q} \in \mathrm{Spec}(S)$ be a prime. Let $\mathfrak{p} \subset R$ be the inverse image of $\mathfrak{q}$ in R. Note that $M_\mathfrak{q}$ is flat over R if and only if it is flat over $R_\mathfrak{p}$. Let us assume that $M_\mathfrak{q}$ is flat over R. We claim that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that M_g is flat over R.

We first reduce to the case where R and S are of finite type over $\mathbf{Z}$. Choose a directed set Λ and a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in Lemma 10.127.18. Set $\mathfrak{p}_\lambda$ equal to the inverse image of $\mathfrak{p}$ in R_λ. Set $\mathfrak{q}_\lambda$ equal to the inverse image of $\mathfrak{q}$ in S_λ. Then the system

$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$

is a system as in Lemma 10.127.13. Hence by Lemma 10.128.3 we see that for some λ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$. Suppose we can prove our claim for the system $(R_\lambda \to S_\lambda, M_\lambda, \mathfrak{q}_\lambda)$. In other words, suppose that we can find a $g \in S_\lambda$, $g \notin \mathfrak{q}_\lambda$ such that $(M_\lambda)_g$ is flat over R_λ. By Lemma 10.127.18 we have $M = M_\lambda \otimes_{R_\lambda} R$ and hence also $M_g = (M_\lambda)_g \otimes_{R_\lambda} R$. Thus by Lemma 10.39.7 we deduce the claim for the system $(R \to S, M, \mathfrak{q})$.

At this point we may assume that R and S are of finite type over $\mathbf{Z}$. We may write S as a quotient of a polynomial ring $R[x_1, \ldots, x_n]$. Of course, we may replace S by $R[x_1, \ldots, x_n]$ and assume that S is a polynomial ring over R. In particular we see that $R \to S$ is flat and all fibres rings $S \otimes_R \kappa(\mathfrak{p})$ have global dimension n.

Choose a resolution $F_\bullet$ of M over S with each F_i finite free, see Lemma 10.71.1. Let $K_n = \mathrm{Ker}(F_{n-1} \to F_{n-2})$. Note that $(K_n)_\mathfrak{q}$ is flat over R, since each F_i is flat over R and by assumption on M, see Lemma 10.39.13. In addition, the sequence

$$0 \to K_n/\mathfrak{p}K_n \to F_{n-1}/\mathfrak{p}F_{n-1} \to \ldots \to F_0/\mathfrak{p}F_0 \to M/\mathfrak{p}M \to 0$$

is exact upon localizing at $\mathfrak{q}$, because of vanishing of $\mathrm{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{q})$. Since the global dimension of $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is n we conclude that $K_n/\mathfrak{p}K_n$ localized at $\mathfrak{q}$ is a finite free module over $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. By Lemma 10.99.4 $(K_n)_\mathfrak{q}$ is free over $S_\mathfrak{q}$. In particular, there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(K_n)_g$ is finite free over S_g.

By Lemma 10.129.3 there exists a further localization S_g such that the complex

$$0 \to K_n \to F_{n-1} \to \ldots \to F_0$$

is exact on *all fibres* of $R \to S$. By Lemma 10.99.5 this implies that the cokernel of $F_1 \to F_0$ is flat. This proves the theorem in the Noetherian case. □

10.130. Openness of Cohen-Macaulay loci

00RD In this section we characterize the Cohen-Macaulay property of finite type algebras in terms of flatness. We then use this to prove the set of points where such an algebra is Cohen-Macaulay is open.

00RE **Lemma 10.130.1.** *Let S be a finite type algebra over a field k. Let $\varphi : k[y_1, \ldots, y_d] \to S$ be a quasi-finite ring map. As subsets of $\mathrm{Spec}(S)$ we have*

$$\{\mathfrak{q} \mid S_\mathfrak{q} \text{ flat over } k[y_1, \ldots, y_d]\} = \{\mathfrak{q} \mid S_\mathfrak{q} \text{ CM and } \dim_\mathfrak{q}(S/k) = d\}$$

For notation see Definition 10.125.1.

Proof. Let $\mathfrak{q} \subset S$ be a prime. Denote $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. Note that always $\dim(S_\mathfrak{q}) \leq \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$ by Lemma 10.125.4 for example. Moreover, the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite and hence $\mathrm{trdeg}_k(\kappa(\mathfrak{p})) = \mathrm{trdeg}_k(\kappa(\mathfrak{q}))$.

Let $\mathfrak{q}$ be an element of the left hand side. Then Lemma 10.112.9 applies and we conclude that $S_\mathfrak{q}$ is Cohen-Macaulay and $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Combined with the equality of transcendence degrees above and Lemma 10.116.3 this implies that $\dim_\mathfrak{q}(S/k) = d$. Hence $\mathfrak{q}$ is an element of the right hand side.

Let $\mathfrak{q}$ be an element of the right hand side. By the equality of transcendence degrees above, the assumption that $\dim_\mathfrak{q}(S/k) = d$ and Lemma 10.116.3 we conclude that $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Hence Lemma 10.128.1 applies and we see that $\mathfrak{q}$ is an element of the left hand side. □

00RF **Lemma 10.130.2.** *Let S be a finite type algebra over a field k. The set of primes $\mathfrak{q}$ such that $S_\mathfrak{q}$ is Cohen-Macaulay is open in S.*

This lemma is a special case of Lemma 10.130.4 below, so you can skip straight to the proof of that lemma if you like.

Proof. Let $\mathfrak{q} \subset S$ be a prime such that $S_\mathfrak{q}$ is Cohen-Macaulay. We have to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the ring S_g is Cohen-Macaulay. For any $g \in S$, $g \notin \mathfrak{q}$ we may replace S by S_g and $\mathfrak{q}$ by $\mathfrak{q}S_g$. Combining this with Lemmas 10.115.5 and 10.116.3 we may assume that there exists a finite injective ring map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S_\mathfrak{q}) + \mathrm{trdeg}_k(\kappa(\mathfrak{q}))$. Set $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. By construction we see that $\mathfrak{q}$ is an element of the right hand side of the displayed equality of Lemma 10.130.1. Hence it is also an element of the left hand side.

By Theorem 10.129.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring S_g is flat over $k[y_1, \ldots, y_d]$. Hence by the equality of Lemma 10.130.1 again we conclude that all local rings of S_g are Cohen-Macaulay as desired. □

00RG **Lemma 10.130.3.** *Let k be a field. Let S be a finite type k algebra. The set of Cohen-Macaulay primes forms a dense open $U \subset \mathrm{Spec}(S)$.*

Proof. The set is open by Lemma 10.130.2. It contains all minimal primes $\mathfrak{q} \subset S$ since the local ring at a minimal prime $S_\mathfrak{q}$ has dimension zero and hence is Cohen-Macaulay. □

00RH **Lemma 10.130.4.** *Let R be a ring. Let $R \to S$ be of finite presentation and flat. For any $d \geq 0$ the set*

$$\left\{\begin{array}{c}\mathfrak{q} \in \mathrm{Spec}(S) \text{ such that setting } \mathfrak{p} = R \cap \mathfrak{q} \text{ the fibre ring} \\ S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \text{ is Cohen-Macaulay and } \dim_\mathfrak{q}(S/R) = d\end{array}\right\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set indicated, with $\mathfrak{p}$ the corresponding prime of R. We have to find a $g \in S$, $g \notin \mathfrak{q}$ such that all fibre rings of $R \to S_g$ are Cohen-Macaulay. During the course of the proof we may (finitely many times) replace S by S_g for a $g \in S$, $g \notin \mathfrak{q}$. Thus by Lemma 10.125.2 we may assume there is a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$ with $d = \dim_{\mathfrak{q}}(S/R)$. Let $\mathfrak{q}' = R[t_1, \ldots, t_d] \cap \mathfrak{q}$. By Lemma 10.130.1 we see that the ring map

$$R[t_1, \ldots, t_d]_{\mathfrak{q}'}/\mathfrak{p}R[t_1, \ldots, t_d]_{\mathfrak{q}'} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$$

is flat. Hence by the critère de platitude par fibres Lemma 10.128.8 we see that $R[t_1, \ldots, t_d]_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is flat. Hence by Theorem 10.129.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring map $R[t_1, \ldots, t_d] \to S_g$ is flat. Replacing S by S_g we see that for every prime $\mathfrak{r} \subset S$, setting $\mathfrak{r}' = R[t_1, \ldots, t_d] \cap \mathfrak{r}$ and $\mathfrak{p}' = R \cap \mathfrak{r}$ the local ring map $R[t_1, \ldots, t_d]_{\mathfrak{r}'} \to S_{\mathfrak{r}}$ is flat. Hence also the base change

$$R[t_1, \ldots, t_d]_{\mathfrak{r}'}/\mathfrak{p}'R[t_1, \ldots, t_d]_{\mathfrak{r}'} \longrightarrow S_{\mathfrak{r}}/\mathfrak{p}'S_{\mathfrak{r}}$$

is flat. Hence by Lemma 10.130.1 applied with $k = \kappa(\mathfrak{p}')$ we see $\mathfrak{r}$ is in the set of the lemma as desired. □

00RI **Lemma 10.130.5.** *Let R be a ring. Let $R \to S$ be flat of finite presentation. The set of primes $\mathfrak{q}$ such that the fibre ring $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$, with $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay is open and dense in every fibre of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof. The set, call it W, is open by Lemma 10.130.4. It is dense in the fibres because the intersection of W with a fibre is the corresponding set of the fibre to which Lemma 10.130.3 applies. □

00RJ **Lemma 10.130.6.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_{\mathfrak{q}}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.*

Proof. During the course of the proof we may (finitely many times) replace S by S_g for any $g \in S$, $g \notin \mathfrak{q}$. Hence using Lemma 10.115.5 we may assume that $\dim(S) = \dim_{\mathfrak{q}}(S/k) =: d$ and find a finite injective map $k[x_1, \ldots, x_d] \to S$. Note that this also induces a finite injective map $K[x_1, \ldots, x_d] \to S_K$ by base change. By Lemma 10.116.6 we have $\dim_{\mathfrak{q}_K}(S_K/K) = d$. Set $\mathfrak{p} = k[x_1, \ldots, x_d] \cap \mathfrak{q}$ and $\mathfrak{p}_K = K[x_1, \ldots, x_d] \cap \mathfrak{q}_K$. Consider the following commutative diagram of Noetherian local rings

$$\begin{array}{ccc} S_{\mathfrak{q}} & \longrightarrow & (S_K)_{\mathfrak{q}_K} \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_d]_{\mathfrak{p}} & \longrightarrow & K[x_1, \ldots, x_d]_{\mathfrak{p}_K} \end{array}$$

By Lemma 10.130.1 we have to show that the left vertical arrow is flat if and only if the right vertical arrow is flat. Because the bottom arrow is flat this equivalence holds by Lemma 10.100.1. □

00RK **Lemma 10.130.7.** *Let R be a ring. Let $R \to S$ be of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$. Denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the map associated to the ring map $S \to S'$. Set W equal to the set of primes $\mathfrak{q}$ such that*

the fibre ring $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$, $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay, and let W' denote the analogue for S'/R'. Then $W' = f^{-1}(W)$.

Proof. Trivial from Lemma 10.130.6 and the definitions. □

00RL **Lemma 10.130.8.** *Let R be a ring. Let $R \to S$ be a ring map which is (a) flat, (b) of finite presentation, (c) has Cohen-Macaulay fibres. Then we can write $S = S_0 \times \ldots \times S_n$ as a product of R-algebras S_d such that each S_d satisfies (a), (b), (c) and has all fibres equidimensional of dimension d.*

Proof. For each integer d denote $W_d \subset \mathrm{Spec}(S)$ the set defined in Lemma 10.130.4. Clearly we have $\mathrm{Spec}(S) = \coprod W_d$, and each W_d is open by the lemma we just quoted. Hence the result follows from Lemma 10.24.3. □

10.131. Differentials

00RM In this section we define the module of differentials of a ring map.

00RN **Definition 10.131.1.** Let $\varphi : R \to S$ be a ring map and let M be an S-module. A *derivation*, or more precisely an *R-derivation* into M is a map $D : S \to M$ which is additive, annihilates elements of $\varphi(R)$, and satisfies the *Leibniz rule*: $D(ab) = aD(b) + bD(a)$.

Note that $D(ra) = rD(a)$ if $r \in R$ and $a \in S$. An equivalent definition is that an R-derivation is an R-linear map $D : S \to M$ which satisfies the Leibniz rule. The set of all R-derivations forms an S-module: Given two R-derivations D, D' the sum $D + D' : S \to M$, $a \mapsto D(a) + D'(a)$ is an R-derivation, and given an R-derivation D and an element $c \in S$ the scalar multiple $cD : S \to M$, $a \mapsto cD(a)$ is an R-derivation. We denote this S-module

$$\mathrm{Der}_R(S, M).$$

Also, if $\alpha : M \to N$ is an S-module map, then the composition $\alpha \circ D$ is an R-derivation into N. In this way the assignment $M \mapsto \mathrm{Der}_R(S, M)$ is a covariant functor.

Consider the following map of free S-modules

$$\bigoplus\nolimits_{(a,b) \in S^2} S[(a,b)] \oplus \bigoplus\nolimits_{(f,g) \in S^2} S[(f,g)] \oplus \bigoplus\nolimits_{r \in R} S[r] \longrightarrow \bigoplus\nolimits_{a \in S} S[a]$$

defined by the rules

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

with obvious notation. Let $\Omega_{S/R}$ be the cokernel of this map. There is a map $\mathrm{d} : S \to \Omega_{S/R}$ which maps a to the class $\mathrm{d}a$ of $[a]$ in the cokernel. This is an R-derivation by the relations imposed on $\Omega_{S/R}$, in other words

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b, \quad \mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f, \quad \mathrm{d}\varphi(r) = 0$$

where $a, b, f, g \in S$ and $r \in R$.

07BK **Definition 10.131.2.** The pair $(\Omega_{S/R}, \mathrm{d})$ is called the *module of Kähler differentials* or the *module of differentials* of S over R.

00RO **Lemma 10.131.3.** *The module of differentials of S over R has the following universal property. The map*

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \longrightarrow Der_R(S, M), \quad \alpha \longmapsto \alpha \circ d$$

is an isomorphism of functors.

Proof. By definition an R-derivation is a rule which associates to each $a \in S$ an element $D(a) \in M$. Thus D gives rise to a map $[D] : \bigoplus S[a] \to M$. However, the conditions of being an R-derivation exactly mean that $[D]$ annihilates the image of the map in the displayed presentation of $\Omega_{S/R}$ above. □

00RP **Lemma 10.131.4.** *Suppose that $R \to S$ is surjective. Then $\Omega_{S/R} = 0$.*

Proof. You can see this either because all R-derivations clearly have to be zero, or because the map in the presentation of $\Omega_{S/R}$ is surjective. □

Suppose that

00RQ (10.131.4.1)
$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ R & \xrightarrow{\psi} & R' \end{array}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{array}{ccc} \Omega_{S/R} & \longrightarrow & \Omega_{S'/R'} \\ {\scriptstyle \mathrm{d}}\uparrow & & \uparrow{\scriptstyle \mathrm{d}} \\ S & \xrightarrow{\varphi} & S' \end{array}$$

To construct the map just use the obvious map between the presentations for $\Omega_{S/R}$ and $\Omega_{S'/R'}$. Namely,

0H2F (10.131.4.2)
$$\begin{array}{ccc} \bigoplus S'[(a',b')] \oplus \bigoplus S'[(f',g')] \oplus \bigoplus S'[r'] & \longrightarrow & \bigoplus S'[a'] \\ \Big\uparrow {\scriptstyle \begin{array}{c}[(a,b)] \mapsto [(\varphi(a),\varphi(b))] \\ [(f,g)] \mapsto [(\varphi(f),\varphi(g))] \\ [r] \mapsto [\psi(r)]\end{array}} & & \Big\uparrow {\scriptstyle [a] \mapsto [\varphi(a)]} \\ \bigoplus S[(a,b)] \oplus \bigoplus S[(f,g)] \oplus \bigoplus S[r] & \longrightarrow & \bigoplus S[a] \end{array}$$

The result is simply that $f\mathrm{d}g \in \Omega_{S/R}$ is mapped to $\varphi(f)\mathrm{d}\varphi(g)$.

031G **Lemma 10.131.5.** *Let I be a directed set. Let $(R_i \to S_i, \varphi_{ii'})$ be a system of ring maps over I, see Categories, Section 4.21. Then we have*

$$\Omega_{S/R} = \mathrm{colim}_i \, \Omega_{S_i/R_i}.$$

where $R \to S = \mathrm{colim}(R_i \to S_i)$.

Proof. This is clear from the defining presentation of $\Omega_{S/R}$ and the functoriality of this described above. □

00RR **Lemma 10.131.6.** *In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-module by the elements da, where $a \in S$ is such that $\varphi(a) \in \beta(R')$. (This includes in particular the elements $d(i)$, $i \in I$.)*

First proof. Consider the map of presentations (10.131.4.2). Clearly the right vertical map of free modules is surjective. Thus the map is surjective. Suppose that some element η of $\Omega_{S/R}$ maps to zero in $\Omega_{S'/R'}$. Write η as the image of $\sum s_i[a_i]$ for some $s_i, a_i \in S$. Then we see that $\sum \varphi(s_i)[\varphi(a_i)]$ is the image of an element

$$\theta = \sum s'_j[a'_j, b'_j] + \sum s'_k[f'_k, g'_k] + \sum s'_l[r'_l]$$

in the upper left corner of the diagram. Since φ is surjective, the terms $s'_j[a'_j, b'_j]$ and $s'_k[f'_k, g'_k]$ are in the image of elements in the lower right corner. Thus, modifying η and θ by subtracting the images of these elements, we may assume $\theta = \sum s'_l[r'_l]$. In other words, we see $\sum \varphi(s_i)[\varphi(a_i)]$ is of the form $\sum s'_l[\beta(r'_l)]$. Next, we may assume that we have some $a' \in S'$ such that $a' = \varphi(a_i)$ for all i and $a' = \beta(r'_l)$ for all l. This is clear from the direct sum decomposition of the upper right corner of the diagram. Choose $a \in S$ with $\varphi(a) = a'$. Then we can write $a_i = a + x_i$ for some $x_i \in I$. Thus we may assume that all a_i are equal to a by using the relations that are allowed. But then we may assume our element is of the form $s[a]$. We still know that $\varphi(s)[a'] = \sum \varphi(s'_l)[\beta(r'_l)]$. Hence either $\varphi(s) = 0$ and we're done, or $a' = \varphi(a)$ is in the image of β and we're done as well. □

Second proof. We will use the universal property of modules of differentials given in Lemma 10.131.3 without further mention.

In (10.131.4.1) let $R'' = S \times_{S'} R'$. Then we have following diagram:

$$\begin{array}{ccccc} S & \longrightarrow & S & \longrightarrow & S' \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R'' & \longrightarrow & R' \end{array}$$

Let M be an S-module. It follows immediately from the definitions that an R-derivation $D : S \to M$ is an R''-derivation if and only if it annihilates the elements in the image of $R'' \to S$. The universal property translates this into the statement that the natural map $\Omega_{S/R} \to \Omega_{S/R''}$ is surjective with kernel generated as an S-module by the image of R''.

From the previous paragraph we see that it suffices to show that $\Omega_{S/R} \to \Omega_{S'/R'}$ is an isomorphism when $S \to S'$ is surjective and $R = S \times_{S'} R'$. Let M' be an S'-module. Observe that any R'-derivation $D' : S' \to M'$ gives an R-derivation by precomposing with $S \to S'$. Conversely, suppose M is an S-module and $D : S \to M$ is an R-derivation. If $i \in I$, then there exist an $a \in R$ with $\alpha(a) = i$ (as $R = S \times_{S'} R'$). It follows that $D(i) = 0$ and hence $0 = D(is) = iD(s)$ for all $s \in S$. Thus the image of D is contained in the submodule $M' \subset M$ of elements annihilated by I and moreover the induced map $S \to M'$ factors through an R'-derivation $S' \to M'$. It is an exercise to use the universal property to see that this means $\Omega_{S/R} \to \Omega_{S'/R'}$ is an isomorphism; details omitted. □

00RS **Lemma 10.131.7.** *Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules.

Proof. We get a diagram (10.131.4.1) by putting $R = A$, $S = C$, $R' = B$, and $S' = C$. By Lemma 10.131.6 the map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective, and the kernel is generated by the elements $\mathrm{d}(c)$, where $c \in C$ is in the image of $B \to C$. The lemma follows. □

00RT **Lemma 10.131.8.** *Let $\varphi : A \to B$ be a ring map.*

(1) *If $S \subset A$ is a multiplicative subset mapping to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.*
(2) *If $S \subset B$ is a multiplicative subset then $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$.*

Proof. To show the equality of (1) it is enough to show that any A-derivation $D : B \to M$ annihilates the elements $\varphi(s)^{-1}$. This is clear from the Leibniz rule applied to $1 = \varphi(s)\varphi(s)^{-1}$. To show (2) note that there is an obvious map $S^{-1}\Omega_{B/A} \to \Omega_{S^{-1}B/A}$. To show it is an isomorphism it is enough to show that there is a A-derivation d' of $S^{-1}B$ into $S^{-1}\Omega_{B/A}$. To define it we simply set $\mathrm{d}'(b/s) = (1/s)\mathrm{d}b - (1/s^2)b\mathrm{d}s$. Details omitted. □

00RU **Lemma 10.131.9.** *In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Then there is a canonical exact sequence of S'-modules*

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

The leftmost map is characterized by the rule that $f \in I$ maps to $df \otimes 1$.

Proof. The middle term is $\Omega_{S/R} \otimes_S S/I$. For $f \in I$ denote $\overline{f}$ the image of f in I/I^2. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if $f_1, f_2 \in I$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $S' = S/I$-linear.

The map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is the canonical S'-linear map associated to the S-linear map $\Omega_{S/R} \to \Omega_{S'/R}$. It is surjective because $\Omega_{S/R} \to \Omega_{S'/R}$ is surjective by Lemma 10.131.6.

The composite of the two maps is zero because $\mathrm{d}f$ maps to zero in $\Omega_{S'/R}$ for $f \in I$. Note that exactness just says that the kernel of $\Omega_{S/R} \to \Omega_{S'/R}$ is generated as an S-submodule by the submodule $I\Omega_{S/R}$ together with the elements $\mathrm{d}f$, with $f \in I$. We know by Lemma 10.131.6 that this kernel is generated by the elements $\mathrm{d}(a)$ where $\varphi(a) = \beta(r)$ for some $r \in R$. But then $a = \alpha(r) + a - \alpha(r)$, so $\mathrm{d}(a) = \mathrm{d}(a - \alpha(r))$. And $a - \alpha(r) \in I$ since $\varphi(a - \alpha(r)) = \varphi(a) - \varphi(\alpha(r)) = \beta(r) - \beta(r) = 0$. We conclude the elements $\mathrm{d}f$ with $f \in I$ already generate the kernel as an S-module, as desired. □

02HP **Lemma 10.131.10.** *In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Moreover, assume that there exists an*

R-algebra map $S' \to S$ which is a right inverse to $S \to S'$. Then the exact sequence of S'-modules of Lemma 10.131.9 turns into a short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

which is even a split short exact sequence.

Proof. Let $\beta : S' \to S$ be the right inverse to the surjection $\alpha : S \to S'$, so $S = I \oplus \beta(S')$. Clearly we can use $\beta : \Omega_{S'/R} \to \Omega_{S/R}$, to get a right inverse to the map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$. On the other hand, consider the map

$$D : S \longrightarrow I/I^2, \quad x \longmapsto x - \beta(\alpha(x))$$

It is easy to show that D is an R-derivation (omitted). Moreover $xD(s) = 0$ if $x \in I, s \in S$. Hence, by the universal property D induces a map $\tau : \Omega_{S/R} \otimes_S S' \to I/I^2$. We omit the verification that it is a left inverse to $\mathrm{d} : I/I^2 \to \Omega_{S/R} \otimes_S S'$. Hence we win. □

02HQ **Lemma 10.131.11.** *Let $R \to S$ be a ring map. Let $I \subset S$ be an ideal. Let $n \geq 1$ be an integer. Set $S' = S/I^{n+1}$. The map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism*

$$\Omega_{S/R} \otimes_S S/I^n \longrightarrow \Omega_{S'/R} \otimes_{S'} S/I^n.$$

Proof. This follows from Lemma 10.131.9 and the fact that $\mathrm{d}(I^{n+1}) \subset I^n \Omega_{S/R}$ by the Leibniz rule for d. □

00RV **Lemma 10.131.12.** *Suppose that we have ring maps $R \to R'$ and $R \to S$. Set $S' = S \otimes_R R'$, so that we obtain a diagram (10.131.4.1). Then the canonical map defined above induces an isomorphism $\Omega_{S/R} \otimes_R R' = \Omega_{S'/R'}$.*

Proof. Let $\mathrm{d}' : S' = S \otimes_R R' \to \Omega_{S/R} \otimes_R R'$ denote the map $\mathrm{d}'(\sum a_i \otimes x_i) = \sum \mathrm{d}(a_i) \otimes x_i$. It exists because the map $S \times R' \to \Omega_{S/R} \otimes_R R'$, $(a, x) \mapsto \mathrm{d}a \otimes_R x$ is R-bilinear. This is an R'-derivation, as can be verified by a simple computation. We will show that $(\Omega_{S/R} \otimes_R R', \mathrm{d}')$ satisfies the universal property. Let $D : S' \to M'$ be an R'-derivation into an S'-module. The composition $S \to S' \to M'$ is an R-derivation, hence we get an S-linear map $\varphi_D : \Omega_{S/R} \to M'$. We may tensor this with R' and get the map $\varphi'_D : \Omega_{S/R} \otimes_R R' \to M'$, $\varphi'_D(\eta \otimes x) = x\varphi_D(\eta)$. It is clear that $D = \varphi'_D \circ \mathrm{d}'$. □

The multiplication map $S \otimes_R S \to S$ is the R-algebra map which maps $a \otimes b$ to ab in S. It is also an S-algebra map, if we think of $S \otimes_R S$ as an S-algebra via either of the maps $S \to S \otimes_R S$.

00RW **Lemma 10.131.13.** *Let $R \to S$ be a ring map. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules $\Omega_{S/R} \to J/J^2$, $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$.*

First proof. Apply Lemma 10.131.10 to the commutative diagram

$$\begin{array}{ccc} S \otimes_R S & \longrightarrow & S \\ \uparrow & & \uparrow \\ S & \longrightarrow & S \end{array}$$

where the left vertical arrow is $a \mapsto a \otimes 1$. We get the exact sequence $0 \to J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S \to \Omega_{S/S} \to 0$. By Lemma 10.131.4 the term $\Omega_{S/S}$ is 0, and

we obtain an isomorphism between the other two terms. We have $\Omega_{S\otimes_R S/S} = \Omega_{S/R}\otimes_S (S\otimes_R S)$ by Lemma 10.131.12 as $S \to S\otimes_R S$ is the base change of $R \to S$ and hence

$$\Omega_{S\otimes_R S/S}\otimes_{S\otimes_R S} S = \Omega_{S/R}\otimes_S (S\otimes_R S)\otimes_{S\otimes_R S} S = \Omega_{S/R}$$

We omit the verification that the map is given by the rule of the lemma. □

Second proof. First we show that the rule $a\mathrm{d}b \mapsto a\otimes b - ab\otimes 1$ is well defined. In order to do this we have to show that $\mathrm{d}r$ and $a\mathrm{d}b + b\mathrm{d}a - d(ab)$ map to zero. The first because $r\otimes 1 - 1\otimes r = 0$ by definition of the tensor product. The second because

$$(a\otimes b - ab\otimes 1) + (b\otimes a - ba\otimes 1) - (1\otimes ab - ab\otimes 1) = (a\otimes 1 - 1\otimes a)(1\otimes b - b\otimes 1)$$

is in J^2.

We construct a map in the other direction. We may think of $S \to S\otimes_R S$, $a \mapsto a\otimes 1$ as the base change of $R \to S$. Hence we have $\Omega_{S\otimes_R S/S} = \Omega_{S/R}\otimes_S (S\otimes_R S)$, by Lemma 10.131.12. At this point the sequence of Lemma 10.131.9 gives a map

$$J/J^2 \to \Omega_{S\otimes_R S/S}\otimes_{S\otimes_R S} S = (\Omega_{S/R}\otimes_S (S\otimes_R S))\otimes_{S\otimes_R S} S = \Omega_{S/R}.$$

We leave it to the reader to see it is the inverse of the map above. □

00RX **Lemma 10.131.14.** *If $S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R}$ is a finite free S-module with basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$.*

Proof. We first show that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate $\Omega_{S/R}$ as an S-module. To prove this we show that $\mathrm{d}g$ can be expressed as a sum $\sum g_i \mathrm{d}x_i$ for any $g \in R[x_1, \ldots, x_n]$. We do this by induction on the (total) degree of g. It is clear if the degree of g is 0, because then $\mathrm{d}g = 0$. If the degree of g is > 0, then we may write g as $c + \sum g_i x_i$ with $c \in R$ and $\deg(g_i) < \deg(g)$. By the Leibniz rule we have $\mathrm{d}g = \sum g_i \mathrm{d}x_i + \sum x_i \mathrm{d}g_i$, and hence we win by induction.

Consider the R-derivation $\partial/\partial x_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$. (We leave it to the reader to define this; the defining property being that $\partial/\partial x_i(x_j) = \delta_{ij}$.) By the universal property this corresponds to an S-module map $l_i : \Omega_{S/R} \to R[x_1, \ldots, x_n]$ which maps $\mathrm{d}x_i$ to 1 and $\mathrm{d}x_j$ to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. □

00RY **Lemma 10.131.15.** *Suppose $R \to S$ is of finite presentation. Then $\Omega_{S/R}$ is a finitely presented S-module.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Write $I = (f_1, \ldots, f_m)$. According to Lemma 10.131.9 there is an exact sequence of S-modules

$$I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R}\otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

The result follows from the fact that I/I^2 is a finite S-module (generated by the images of the f_i), and that the middle term is finite free by Lemma 10.131.14. □

00RZ **Lemma 10.131.16.** *Suppose $R \to S$ is of finite type. Then $\Omega_{S/R}$ is finitely generated S-module.*

Proof. This is very similar to, but easier than the proof of Lemma 10.131.15. □

10.132. The de Rham complex

0FKF Let $A \to B$ be a ring map. Denote $\mathrm{d} : B \to \Omega_{B/A}$ the module of differentials with its universal A-derivation constructed in Section 10.131. Let $\Omega^i_{B/A} = \wedge^i_B(\Omega_{B/A})$ for $i \geq 0$ be the ith exterior power as in Section 10.13. The *de Rham complex of B over A* is the complex

$$\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \ldots$$

with A-linear differentials constructed and described below.

The map $\mathrm{d} : \Omega^0_{B/A} \to \Omega^1_{B/A}$ is the universal derivation $\mathrm{d} : B \to \Omega_{B/A}$. Observe that this is indeed A-linear.

For $p \geq 1$ we claim there is a unique A-linear map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ such that

0FKG (10.132.0.1) $$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

Recall that $\Omega_{B/A}$ is generated as a B-module by the elements $\mathrm{d}b$. Thus $\Omega^p_{B/A}$ is generated as an A-module by the elements $b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and it follows that the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ if it exists is unique.

Construction of $\mathrm{d} : \Omega^1_{B/A} \to \Omega^2_{B/A}$. By Definition 10.131.2 the elements $\mathrm{d}b$ freely generate $\Omega_{B/A}$ as a B-module subject to the relations $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}(b' + b'') = \mathrm{d}b' + \mathrm{d}b''$ and $\mathrm{d}(b'b'') = b'\mathrm{d}b'' + b''\mathrm{d}b'$ for $b', b'' \in B$. Hence to show that the rule

$$\sum b'_i \mathrm{d}b_i \longmapsto \sum \mathrm{d}b'_i \wedge \mathrm{d}b_i$$

is well defined we have to show that the elements

$$b\mathrm{d}a, \quad \text{and} \quad b\mathrm{d}(b' + b'') - b\mathrm{d}b' - b\mathrm{d}b'' \quad \text{and} \quad b\mathrm{d}(b'b'') - bb'\mathrm{d}b'' - bb''\mathrm{d}b'$$

for $a \in A$ and $b, b', b'' \in B$ are mapped to zero. This is clear by direct computation using the Leibniz rule for d.

Observe that the composition $\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A}$ is zero as $\mathrm{d}(\mathrm{d}(b)) = \mathrm{d}(1\mathrm{d}b) = \mathrm{d}(1) \wedge \mathrm{d}(b) = 0 \wedge \mathrm{d}b = 0$. Here $\mathrm{d}(1) = 0$ as $1 \in B$ is in the image of $A \to B$. We will use this below.

Construction of $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ for $p \geq 2$. We will show the A-linear map

$$\gamma : \Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \longrightarrow \Omega^{p+1}_{B/A}$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1} \omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

factors over the natural surjection $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ to give the desired map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$. According to Lemma 10.13.4 the kernel of $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ is generated as an A-module by the elements $\omega_1 \otimes \ldots \otimes \omega_p$ with $\omega_i = \omega_j$ for some $i \neq j$ and $\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p - \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$ for some $f \in B$. A direct computation shows the first type of element is mapped to 0 by γ, in other words, γ is alternating. To finish we have to show that

$$\gamma(\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p) = \gamma(\omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p)$$

for $f \in B$. By A-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = b\mathrm{d}b'$ and $\omega_2 = c\mathrm{d}c'$ for $b, b', c, c' \in B$. Thus we need to show that

$$\begin{aligned} &\mathrm{d}(fb) \wedge \mathrm{d}b' \wedge c\mathrm{d}c' - fb\mathrm{d}b' \wedge \mathrm{d}c \wedge \mathrm{d}c' \\ &= \mathrm{d}b \wedge \mathrm{d}b' \wedge fc\mathrm{d}c' - b\mathrm{d}b' \wedge \mathrm{d}(fc) \wedge \mathrm{d}c' \end{aligned}$$

in other words that

$$(c\mathrm{d}(fb) + fb\mathrm{d}c - fc\mathrm{d}b - b\mathrm{d}(fc)) \wedge \mathrm{d}b' \wedge \mathrm{d}c' = 0.$$

This follows from the Leibniz rule. Observe that the value of γ on the element $b_0\mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$ is $\mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and hence (10.132.0.1) will be satisfied for the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ so obtained.

Finally, since $\Omega^p_{B/A}$ is additively generated by the elements $b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and since $\mathrm{d}(b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p) = \mathrm{d}b_0 \wedge \ldots \wedge \mathrm{d}b_p$ we see in exactly the same manner that the composition $\Omega^p_{B/A} \to \Omega^{p+1}_{B/A} \to \Omega^{p+2}_{B/A}$ is zero for $p \geq 1$. Thus the de Rham complex is indeed a complex.

Given just a ring R we set $\Omega_R = \Omega_{R/\mathbf{Z}}$. This is sometimes called the absolute module of differentials of R; this makes sense: if Ω_R is the module of differentials where we only assume the Leibniz rule and not the vanishing of $\mathrm{d}1$, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_R. In this case the *absolute de Rham complex of* R is the corresponding complex

$$\Omega^0_R \to \Omega^1_R \to \Omega^2_R \to \ldots$$

where we set $\Omega^i_R = \Omega^i_{R/\mathbf{Z}}$ and so on.

Suppose we have a commutative diagram of rings

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

There is a natural map of de Rham complexes

$$\Omega^\bullet_{B/A} \longrightarrow \Omega^\bullet_{B'/A'}$$

Namely, in degree 0 this is the map $B \to B'$, in degree 1 this is the map $\Omega_{B/A} \to \Omega_{B'/A'}$ constructed in Section 10.131, and for $p \geq 2$ it is the induced map $\Omega^p_{B/A} = \wedge^p_B(\Omega_{B/A}) \to \wedge^p_{B'}(\Omega_{B'/A'}) = \Omega^p_{B'/A'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (10.132.0.1).

0H8Y **Lemma 10.132.1.** *Suppose that we have ring maps $A \to A'$ and $A \to B$. Set $B' = B \otimes_A A'$, so that we obtain a diagram as above. Then the canonical map defined above induces an isomorphism $\Omega^\bullet_{B/A} \otimes_A A' = \Omega^\bullet_{B'/A'}$ of complexes.*

Proof. This follows from Lemma 10.131.12 and the fact that taking exterior powers commutes with base change. □

07HY **Lemma 10.132.2.** *Let $A \to B$ be a ring map. Let $\pi : \Omega_{B/A} \to \Omega$ be a surjective B-module map. Denote $d : B \to \Omega$ the composition of π with the universal derivation $d_{B/A} : B \to \Omega_{B/A}$. Set $\Omega^i = \wedge^i_B(\Omega)$. Assume that the kernel of π is generated, as*

a B-module, by elements $\omega \in \Omega_{B/A}$ such that $d_{B/A}(\omega) \in \Omega^2_{B/A}$ maps to zero in Ω^2. Then there is a de Rham complex

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$$

whose differential is defined by the rule

$$d : \Omega^p \to \Omega^{p+1}, \quad d(f_0 df_1 \wedge \ldots \wedge df_p) = df_0 \wedge df_1 \wedge \ldots \wedge df_p$$

Proof. We will show that there exists a commutative diagram

$$\begin{array}{ccccccc} \Omega^0_{B/A} & \xrightarrow{\mathrm{d}_{B/A}} & \Omega^1_{B/A} & \xrightarrow{\mathrm{d}_{B/A}} & \Omega^2_{B/A} & \xrightarrow{\mathrm{d}_{B/A}} & \ldots \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \wedge^2\pi} & & \\ \Omega^0 & \xrightarrow{\mathrm{d}} & \Omega^1 & \xrightarrow{\mathrm{d}} & \Omega^2 & \xrightarrow{\mathrm{d}} & \ldots \end{array}$$

the description of the map d will follow from the construction of the differentials $\mathrm{d}_{B/A} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ of the de Rham complex of B over A given above. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that $\mathrm{d}_{B/A}$ maps the kernel of π into the kernel of $\wedge^2\pi$. We are given that any element of the kernel of π is of the form $\sum b_i\omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2\pi(\mathrm{d}_{B/A}(\omega_i)) = 0$. By the Leibniz rule for $\mathrm{d}_{B/A}$ we have $\mathrm{d}_{B/A}(\sum b_i\omega_i) = \sum b_i \mathrm{d}_{B/A}(\omega_i) + \sum \mathrm{d}_{B/A}(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2\pi$.

For $i > 1$ we note that $\wedge^i\pi$ is surjective with kernel the image of $\mathrm{Ker}(\pi) \wedge \Omega^{i-1}_{B/A} \to \Omega^i_{B/A}$. For $\omega_1 \in \mathrm{Ker}(\pi)$ and $\omega_2 \in \Omega^{i-1}_{B/A}$ we have

$$\mathrm{d}_{B/A}(\omega_1 \wedge \omega_2) = \mathrm{d}_{B/A}(\omega_1) \wedge \omega_2 - \omega_1 \wedge \mathrm{d}_{B/A}(\omega_2)$$

which is in the kernel of $\wedge^{i+1}\pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. □

10.133. Finite order differential operators

09CH In this section we introduce differential operators of finite order.

09CI **Definition 10.133.1.** Let $R \to S$ be a ring map. Let M, N be S-modules. Let $k \geq 0$ be an integer. We inductively define a *differential operator $D : M \to N$ of order k* to be an R-linear map such that for all $g \in S$ the map $m \mapsto D(gm) - gD(m)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an S-linear map.

If $D : M \to N$ is a differential operator of order k, then for all $g \in S$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(M, N) = \mathrm{Diff}^k_{S/R}(M, N)$$

is an S-module. We have

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

09CJ **Lemma 10.133.2.** *Let $R \to S$ be a ring map. Let L, M, N be S-modules. If $D : L \to M$ and $D' : M \to N$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let $g \in S$. Then the map which sends $x \in L$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

09CK **Lemma 10.133.3.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $k \geq 0$. There exists an S-module $P^k_{S/R}(M)$ and a canonical isomorphism*

$$\mathit{Diff}^k_{S/R}(M, N) = \mathrm{Hom}_S(P^k_{S/R}(M), N)$$

functorial in the S-module N.

Via the Yoneda lemma this tells us there exists a "universal" differential operator $D_{univ} : M \to P^k_{S/R}(M)$ of order k such that for every differential operator $D : M \to N$ of order k there is a unique S-linear map $\alpha : P^k_{S/R}(M) \to N$ with $D = D_{univ} \circ \alpha$. The pair $(P^k_{S/R}(M), D_{univ})$ is unique up to unique isomorphism.

Proof. The existence of $P^k_{S/R}(M)$ follows from general category theoretic arguments (insert future reference here), but we will also give a construction. Set $F = \bigoplus_{m \in M} S[m]$ where $[m]$ is a symbol indicating the basis element in the summand corresponding to m. Given any differential operator $D : M \to N$ we obtain an S-linear map $L_D : F \to N$ sending $[m]$ to $D(m)$. If D has order 0, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where $g_0 \in S$ and $m, m' \in M$. If D has order 1, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where $f \in R$, $g_0, g_1 \in S$, and $m \in M$. If D has order k, then L_D annihilates the elements $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the elements

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : F \to N$ is an S-linear map annihilating all the elements listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $P^k_{S/R}(M)$ is the quotient of F by the submodule generated by these elements. □

09CL **Definition 10.133.4.** Let $R \to S$ be a ring map. Let M be an S-module. The module $P^k_{S/R}(M)$ constructed in Lemma 10.133.3 is called the *module of principal parts of order k* of M.

Note that the inclusions

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to P^2_{S/R}(M) \to P^1_{S/R}(M) \to P^0_{S/R}(M) = M$$

09CM **Example 10.133.5.** Let $R \to S$ be a ring map and let N be an S-module. Observe that $\text{Diff}^1(S, N) = \text{Der}_R(S, N) \oplus N$. Namely, if $D : S \to N$ is a differential operator of order 1 then $\sigma_D : S \to N$ defined by $\sigma_D(g) := D(g) - gD(1)$ is an R-derivation and $D = \sigma_D + \lambda_{D(1)}$ where $\lambda_x : S \to N$ is the linear map sending g to gx. It follows that $P^1_{S/R} = \Omega_{S/R} \oplus S$ by the universal property of $\Omega_{S/R}$.

09CN **Lemma 10.133.6.** *Let $R \to S$ be a ring map. Let M be an S-module. There is a canonical short exact sequence*

$$0 \to \Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

functorial in M called the sequence of principal parts.

Proof. The map $P^1_{S/R}(M) \to M$ is given above. Let N be an S-module and let $D : M \to N$ be a differential operator of order 1. For $m \in M$ the map

$$g \longmapsto D(gm) - gD(m)$$

is an R-derivation $S \to N$ by the axioms for differential operators of order 1. Thus it corresponds to a linear map $D_m : \Omega_{S/R} \to N$ determined by the rule $adb \mapsto aD(bm) - abD(m)$ (see Lemma 10.131.3). The map

$$\Omega_{S/R} \times M \longrightarrow N, \quad (\eta, m) \longmapsto D_m(\eta)$$

is S-bilinear (details omitted) and hence determines an S-linear map

$$\sigma_D : \Omega_{S/R} \otimes_S M \to N$$

In this way we obtain a map $\text{Diff}^1(M, N) \to \text{Hom}_S(\Omega_{S/R} \otimes_S M, N)$, $D \mapsto \sigma_D$ functorial in N. By the Yoneda lemma this corresponds a map $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$. It is immediate from the construction that this map is functorial in M. The sequence

$$\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

is exact because for every module N the sequence

$$0 \to \text{Hom}_S(M, N) \to \text{Diff}^1(M, N) \to \text{Hom}_S(\Omega_{S/R} \otimes_S M, N)$$

is exact by inspection.

To see that $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$ is injective we argue as follows. Choose an exact sequence

$$0 \to M' \to F \to M \to 0$$

with F a free S-module. This induces an exact sequence

$$0 \to \text{Diff}^1(M, N) \to \text{Diff}^1(F, N) \to \text{Diff}^1(M', N)$$

for all N. This proves that in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{S/R} \otimes_S M' & \longrightarrow & P^1_{S/R}(M') & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{S/R} \otimes_S F & \longrightarrow & P^1_{S/R}(F) & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{S/R} \otimes_S M & \longrightarrow & P^1_{S/R}(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the middle column is exact. The left column is exact by right exactness of $\Omega_{S/R} \otimes_S -$. By the snake lemma (see Section 10.4) it suffices to prove exactness on the left for the free module F. Using that $P^1_{S/R}(-)$ commutes with direct sums we reduce to the case $M = S$. This case is a consequence of the discussion in Example 10.133.5. □

09CP **Remark 10.133.7.** Suppose given a commutative diagram of rings

$$\begin{array}{ccc}
B & \longrightarrow & B' \\
\uparrow & & \uparrow \\
A & \longrightarrow & A'
\end{array}$$

a B-module M, a B'-module M', and a B-linear map $M \to M'$. Then we get a compatible system of module maps

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & P^2_{B'/A'}(M') & \longrightarrow & P^1_{B'/A'}(M') & \longrightarrow & P^0_{B'/A'}(M') \\
 & & \uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & P^2_{B/A}(M) & \longrightarrow & P^1_{B/A}(M) & \longrightarrow & P^0_{B/A}(M)
\end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $P^k_{B/A}(M)$ in terms of generators and relations in the proof of Lemma 10.133.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 10.133.6.

0H8Z **Lemma 10.133.8.** *Suppose that we have ring maps $A \to A'$ and $A \to B$ and a B-module M. Set $B' = B \otimes_A A'$ and view $M' = M \otimes_A A'$ as a B'-module. The map of Remark 10.133.7 induces an isomorphism $P^k_{B/A}(M) \otimes_A A' = P^k_{B'/A'}(M')$.*

Proof. Let $D_{univ} : M \to P^k_{B/A}(M)$ be the universal differential operator. The induced A'-linear map $D_{univ} \otimes 1 : M' = M \otimes_A A' \to P^k_{B/A}(M) \otimes_A A'$ is easily verified to be an order k differential operator for $M'/B'/A'$. We will show that $D_{univ} \otimes 1$ satisfies the universal property. Let $D' : M' \to N'$ be a differential operator of order k into a B'-module N'. Then the composition $D : M \to M' \to N'$ is a differential operator into N' viewed as a B-module. Hence there is a B-linear

map $\gamma : P^k_{B/A}(M) \to N'$ such that $D = \gamma \circ D_{univ}$. The reader checks easily that the induced B'-linear map $\gamma' : P^k_{B/A}(M) \otimes_A A' \to N'$ satisfies $\gamma' \circ (D_{univ} \otimes 1) = D'$. We omit the proof that γ' is unique. □

0H90 **Lemma 10.133.9.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules*

$$P^k_{S/R}(M) \longrightarrow (S \otimes_R M)/J^{k+1}(S \otimes_R M)$$

where $s \in S$ acts on the target via multiplication by $s \otimes 1$ and such that the universal differential operators of order k to the map given by $m \mapsto$ class of $1 \otimes m$.

Proof. Consider the map $T : M \to S \otimes M$, $m \mapsto 1 \otimes m$ Since T is R-linear and since

$$g_0 g_1 \ldots g_k T(m) - \sum g_0 \ldots \hat{g}_i \ldots g_k T(g_i m) + \ldots + (-1)^{k+1} T(g_0 \ldots g_k m) = \prod_{i=0,\ldots,k} (g_i \otimes 1 - 1 \otimes g_i) 1 \otimes m$$

for $m \in M$ and $g_0, \ldots, g_k \in S$, we conclude that the rule $m \mapsto$ class of $1 \otimes m$ in the statement of the lemma is a differential operator of order k (see proof of Lemma 10.133.3). By the universal property of $P^k_{S/R}(M)$ we obtain the arrow in the statement of the lemma. On the other hand, if $D : M \to N$ is a differential operator of order k, then we can consider the S-linear map $L : S \otimes_R M \to N$, $g \otimes m \mapsto gD(m)$. The reader checks, by a computation similar to the one above and using that J is generated by elements of the form $g \otimes 1 - 1 \otimes g$, that L annihilates $J^{k+1}(S \otimes_R M)$. In particular, if we apply this to the universal differential operator $D_{univ} : M \to P^k_{S/R}(M)$, then we obtain an S-linear map $(S \otimes_R M)/J^{k+1}(S \otimes_R M) \to P^k_{S/R}(M)$. We leave it to the reader to see that this map is the inverse to the map in the statement of the lemma. □

0G34 **Lemma 10.133.10.** *Let $A \to B$ be a ring map. The differentials $d : \Omega^i_{B/A} \to \Omega^{i+1}_{B/A}$ are differential operators of order 1.*

Proof. Given $b \in B$ we have to show that $\mathrm{d} \circ b - b \circ \mathrm{d}$ is a linear operator. Thus we have to show that

$$\mathrm{d} \circ b \circ b' - b \circ \mathrm{d} \circ b' - b' \circ \mathrm{d} \circ b + b' \circ b \circ \mathrm{d} = 0$$

To see this it suffices to check this on additive generators for $\Omega^i_{B/A}$. Thus it suffices to show that

$$\mathrm{d}(bb'b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_i) - b\mathrm{d}(b'b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_i) - b'\mathrm{d}(bb_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_i) + bb'\mathrm{d}(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_i)$$

is zero. This is a pleasant calculation using the Leibniz rule which is left to the reader. □

0G35 **Lemma 10.133.11.** *Let $A \to B$ be a ring map. Let $g_i \in B$, $i \in I$ be a set of generators for B as an A-algebra. Let M, N be B-modules. Let $D : M \to N$ be an A-linear map. In order to show that D is a differential operator of order k it suffices to show that $D \circ g_i - g_i \circ D$ is a differential operator of order $k-1$ for $i \in I$.*

Proof. Namely, we claim that the set of elements $g \in B$ such that $D \circ g - g \circ D$ is a differential operator of order $k-1$ is an A-subalgebra of B. This follows from the relations

$$D \circ (g + g') - (g + g') \circ D = (D \circ g - g \circ D) + (D \circ g' - g' \circ D)$$

and

$$D \circ gg' - gg' \circ D = (D \circ g - g \circ D) \circ g' + g \circ (D \circ g' - g' \circ D)$$

Strictly speaking, to conclude for products we also use Lemma 10.133.2. □

0G36 **Lemma 10.133.12.** *Let $A \to B$ be a ring map. Let M, N be B-modules. Let $S \subset B$ be a multiplicative subset. Any differential operator $D : M \to N$ of order k extends uniquely to a differential operator $E : S^{-1}M \to S^{-1}N$ of order k.*

Proof. By induction on k. If $k = 0$, then D is B-linear and hence we get the extension by the functoriality of localization. Given $b \in B$ the operator $L_b : m \mapsto D(bm) - bD(m)$ has order $k-1$. Hence it has a unique extension to a differential operator $E_b : S^{-1}M \to S^{-1}N$ of order $k-1$ by induction. Moreover, a computation shows that $L_{b'b} = L_{b'} \circ b + b' \circ L_b$ hence by uniqueness we obtain $E_{b'b} = E_{b'} \circ b + b' \circ E_b$. Similarly, we obtain $E_{b'} \circ b - b \circ E_{b'} = E_b \circ b' - b' \circ E_b$. Now for $m \in M$ and $g \in S$ we set

$$E(m/g) = (1/g)(D(m) - E_g(m/g))$$

To show that this is well defined it suffices to show that for $g' \in S$ if we use the representative $g'm/g'g$ we get the same result. We compute

$$\begin{aligned}(1/g'g)(D(g'm) - E_{g'g}(g'm/gg')) &= (1/gg')(g'D(m) + E_{g'}(m) - E_{g'g}(g'm/gg')) \\ &= (1/g'g)(g'D(m) - g'E_g(m/g))\end{aligned}$$

which is the same as before. It is clear that E is R-linear as D and E_g are R-linear. Taking $g = 1$ and using that $E_1 = 0$ we see that E extends D. By Lemma 10.133.11 it now suffices to show that $E \circ b - b \circ E$ for $b \in B$ and $E \circ 1/g' - 1/g' \circ E$ for $g' \in S$ are differential operators of order $k-1$ in order to show that E is a differential operator of order k. For the first, choose an element m/g in $S^{-1}M$ and observe that

$$\begin{aligned}E(bm/g) - bE(m/g) &= (1/g)(D(bm) - bD(m) - E_g(bm/g) + bE_g(m/g)) \\ &= (1/g)(L_b(m) - E_b(m) + gE_b(m/g)) \\ &= E_b(m/g)\end{aligned}$$

which is a differential operator of order $k-1$. Finally, we have

$$\begin{aligned}E(m/g'g) - (1/g')E(m/g) &= (1/g'g)(D(m) - E_{g'g}(m/g'g)) - (1/g'g)(D(m) - E_g(m/g)) \\ &= -(1/g')E_{g'}(m/g'g)\end{aligned}$$

which also is a differential operator of order $k-1$ as the composition of linear maps (multiplication by $1/g'$ and signs) and $E_{g'}$. We omit the proof of uniqueness. □

0G37 **Lemma 10.133.13.** *Let $R \to A$ and $R \to B$ be ring maps. Let M and M' be A-modules. Let $D : M \to M'$ be a differential operator of order k with respect to $R \to A$. Let N be any B-module. Then the map*

$$D \otimes id_N : M \otimes_R N \to M' \otimes_R N$$

is a differential operator of order k with respect to $B \to A \otimes_R B$.

Proof. It is clear that $D' = D \otimes \mathrm{id}_N$ is B-linear. By Lemma 10.133.11 it suffices to show that

$$D' \circ a \otimes 1 - a \otimes 1 \circ D' = (D \circ a - a \circ D) \otimes \mathrm{id}_N$$

is a differential operator of order $k-1$ which follows by induction on k. □

10.134. The naive cotangent complex

00S0 Let $R \to S$ be a ring map. Denote $R[S]$ the polynomial ring whose variables are the elements $s \in S$. Let's denote $[s] \in R[S]$ the variable corresponding to $s \in S$. Thus $R[S]$ is a free R-module on the basis elements $[s_1] \ldots [s_n]$ where $s_1, \ldots, s_n$ ranges over all unordered sequences of elements of S. There is a canonical surjection

07BL (10.134.0.1)
$$R[S] \longrightarrow S, \quad [s] \longmapsto s$$

whose kernel we denote $I \subset R[S]$. It is a simple observation that I is generated by the elements $[s+s'] - [s] - [s']$, $[s][s'] - [ss']$ and $[r] - r$. According to Lemma 10.131.9 there is a canonical map

07BM (10.134.0.2)
$$I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S$$

whose cokernel is canonically isomorphic to $\Omega_{S/R}$. Observe that the S-module $\Omega_{R[S]/R} \otimes_{R[S]} S$ is free on the generators $\mathrm{d}[s]$.

07BN **Definition 10.134.1.** Let $R \to S$ be a ring map. The *naive cotangent complex* $NL_{S/R}$ is the chain complex (10.134.0.2)

$$NL_{S/R} = \left(I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S\right)$$

with I/I^2 placed in (homological) degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ placed in degree 0. We will denote $H_1(L_{S/R}) = H_1(NL_{S/R})$[12] the homology in degree 1.

Before we continue let us say a few words about the actual cotangent complex (Cotangent, Section 92.3). Given a ring map $R \to S$ there exists a canonical simplicial R-algebra $P_\bullet$ whose terms are polynomial algebras and which comes equipped with a canonical homotopy equivalence

$$P_\bullet \longrightarrow S$$

The cotangent complex $L_{S/R}$ of S over R is defined as the chain complex associated to the simplicial module

$$\Omega_{P_\bullet/R} \otimes_{P_\bullet} S$$

The naive cotangent complex as defined above is canonically isomorphic to the truncation $\tau_{\leq 1} L_{S/R}$ (see Homology, Section 12.15 and Cotangent, Section 92.11). In particular, it is indeed the case that $H_1(NL_{S/R}) = H_1(L_{S/R})$ so our definition is compatible with the one using the cotangent complex. Moreover, $H_0(L_{S/R}) = H_0(NL_{S/R}) = \Omega_{S/R}$ as we've seen above.

Let $R \to S$ be a ring map. A *presentation of S over R* is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra (on a set of variables). Often, when S is of finite type over R we will indicate this by saying: "Let $R[x_1, \ldots, x_n] \to S$ be a presentation of S/R", or "Let $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ be a presentation of S/R" if we want to indicate that I is the kernel of the presentation. Note that the map $R[S] \to S$ used to define the naive cotangent complex is an example of a presentation.

[12] This module is sometimes denoted $\Gamma_{S/R}$ in the literature.

Note that for every presentation α we obtain a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S.$$

Here the term I/I^2 is placed in degree 1 and the term $\Omega_{P/R} \otimes S$ is placed in degree 0. The class of $f \in I$ in I/I^2 is mapped to $\mathrm{d}f \otimes 1$ in $\Omega_{P/R} \otimes S$. The cokernel of this complex is canonically $\Omega_{S/R}$, see Lemma 10.131.9. We call the complex $NL(\alpha)$ the *naive cotangent complex associated to the presentation* $\alpha : P \to S$ *of* S/R. Note that if $P = R[S]$ with its canonical surjection onto S, then we recover $NL_{S/R}$. If $P = R[x_1, \ldots, x_n]$ then will sometimes use the notation $I/I^2 \to \bigoplus_{i=1,\ldots,n} S\mathrm{d}x_i$ to denote this complex.

Suppose we are given a commutative diagram

06RQ (10.134.1.1)
$$\begin{array}{ccc} S & \xrightarrow[\phi]{} & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

of rings. Let $\alpha : P \to S$ be a presentation of S over R and let $\alpha' : P' \to S'$ be a presentation of S' over R'. A *morphism of presentations from* $\alpha : P \to S$ *to* $\alpha' : P' \to S'$ is defined to be an R-algebra map

$$\varphi : P \to P'$$

such that $\phi \circ \alpha = \alpha' \circ \varphi$. Note that in this case $\varphi(I) \subset I'$, where $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. Thus φ induces a map of S-modules $I/I^2 \to I'/(I')^2$ and by functoriality of differentials also an S-module map $\Omega_{P/R} \otimes S \to \Omega_{P'/R'} \otimes S'$. These maps are compatible with the differentials of $NL(\alpha)$ and $NL(\alpha')$ and we obtain a map of naive cotangent complexes

$$NL(\alpha) \longrightarrow NL(\alpha').$$

It is often convenient to consider the induced map $NL(\alpha) \otimes_S S' \to NL(\alpha')$.

In the special case that $P = R[S]$ and $P' = R'[S']$ the map $\phi : S \to S'$ induces a canonical ring map $\varphi : P \to P'$ by the rule $[s] \mapsto [\phi(s)]$. Hence the construction above determines canonical(!) maps of chain complexes

$$NL_{S/R} \longrightarrow NL_{S'/R'}, \quad \text{and} \quad NL_{S/R} \otimes_S S' \longrightarrow NL_{S'/R'}$$

associated to the diagram (10.134.1.1). Note that this construction is compatible with composition: given a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow[\phi]{} & S' & \xrightarrow[\phi']{} & S'' \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & R'' \end{array}$$

we see that the composition of

$$NL_{S/R} \longrightarrow NL_{S'/R'} \longrightarrow NL_{S''/R''}$$

is the map $NL_{S/R} \to NL_{S''/R''}$ given by the outer square.

It turns out that $NL(\alpha)$ is homotopy equivalent to $NL_{S/R}$ and that the maps constructed above are well defined up to homotopy (homotopies of maps of complexes are discussed in Homology, Section 12.13 but we also spell out the exact meaning of the statements in the lemma below in its proof).

00S1 **Lemma 10.134.2.** *Suppose given a diagram (10.134.1.1). Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.*

(1) *There exists a morphism of presentations from α to α'.*
(2) *Any two morphisms of presentations induce homotopic morphisms of complexes $NL(\alpha) \to NL(\alpha')$.*
(3) *The construction is compatible with compositions of morphisms of presentations (see proof for exact statement).*
(4) *If $R \to R'$ and $S \to S'$ are isomorphisms, then for any map φ of presentations from α to α' the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.*

In particular, comparing α to the canonical presentation (10.134.0.1) we conclude there is a quasi-isomorphism $NL(\alpha) \to NL_{S/R}$ well defined up to homotopy and compatible with all functorialities (up to homotopy).

Proof. Since P is a polynomial algebra over R we can write $P = R[x_a, a \in A]$ for some set A. As α' is surjective, we can choose for every $a \in A$ an element $f_a \in P'$ such that $\alpha'(f_a) = \phi(\alpha(x_a))$. Let $\varphi : P = R[x_a, a \in A] \to P'$ be the unique R-algebra map such that $\varphi(x_a) = f_a$. This gives the morphism in (1).

Let φ and φ' morphisms of presentations from α to α'. Let $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. We have to construct the diagonal map h in the diagram

$$\begin{array}{ccc} I/I^2 & \xrightarrow{\mathrm{d}} & \Omega_{P/R} \otimes_P S \\ {\scriptstyle \varphi_1}\downarrow \; \downarrow{\scriptstyle \varphi_1'} & \swarrow{\scriptstyle h} & {\scriptstyle \varphi_0}\downarrow \; \downarrow{\scriptstyle \varphi_0'} \\ I'/(I')^2 & \xrightarrow{\mathrm{d}} & \Omega_{P'/R'} \otimes_{P'} S' \end{array}$$

where the vertical maps are induced by φ, φ' such that

$$\varphi_1 - \varphi_1' = h \circ \mathrm{d} \quad \text{and} \quad \varphi_0 - \varphi_0' = \mathrm{d} \circ h$$

Consider the map $\varphi - \varphi' : P \to P'$. Since both φ and φ' are compatible with α and α' we obtain $\varphi - \varphi' : P \to I'$. This implies that $\varphi, \varphi' : P \to P'$ induce the same P-module structure on $I'/(I')^2$, since $\varphi(p)i' - \varphi'(p)i' = (\varphi - \varphi')(p)i' \in (I')^2$. Also $\varphi - \varphi'$ is R-linear and

$$(\varphi - \varphi')(fg) = \varphi(f)(\varphi - \varphi')(g) + (\varphi - \varphi')(f)\varphi'(g)$$

Hence the induced map $D : P \to I'/(I')^2$ is a R-derivation. Thus we obtain a canonical map $h : \Omega_{P/R} \otimes_P S \to I'/(I')^2$ such that $D = h \circ \mathrm{d}$. A calculation (omitted) shows that h is the desired homotopy.

Suppose that we have a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow[\phi]{} & S' & \xrightarrow[\phi']{} & S'' \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & R'' \end{array}$$

and that

(1) $\alpha : P \to S$,
(2) $\alpha' : P' \to S'$, and
(3) $\alpha'' : P'' \to S''$

are presentations. Suppose that

(1) $\varphi : P \to P'$ is a morphism of presentations from α to α' and
(2) $\varphi' : P' \to P''$ is a morphism of presentations from α' to α''.

Then it is immediate that $\varphi' \circ \varphi : P \to P''$ is a morphism of presentations from α to α'' and that the induced map $NL(\alpha) \to NL(\alpha'')$ of naive cotangent complexes is the composition of the maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha'')$ induced by φ and φ'.

In the simple case of complexes with 2 terms a quasi-isomorphism is just a map that induces an isomorphism on both the cokernel and the kernel of the maps between the terms. Note that homotopic maps of 2 term complexes (as explained above) define the same maps on kernel and cokernel. Hence if φ is a map from a presentation α of S over R to itself, then the induced map $NL(\alpha) \to NL(\alpha)$ is a quasi-isomorphism being homotopic to the identity by part (2). To prove (4) in full generality, consider a morphism φ' from α' to α which exists by (1). The compositions $NL(\alpha) \to NL(\alpha') \to NL(\alpha)$ and $NL(\alpha') \to NL(\alpha) \to NL(\alpha')$ are homotopic to the identity maps by (3), hence these maps are homotopy equivalences by definition. It follows formally that both maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha)$ are quasi-isomorphisms. Some details omitted. □

08Q1 **Lemma 10.134.3.** *Let $A \to B$ be a polynomial algebra. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(0 \to \Omega_{B/A})$ with $\Omega_{B/A}$ in degree 0.*

Proof. Follows from Lemma 10.134.2 and the fact that $\mathrm{id}_B : B \to B$ is a presentation of B over A with zero kernel. □

The following lemma is part of the motivation for introducing the naive cotangent complex. The cotangent complex extends this to a genuine long exact cohomology sequence. If $B \to C$ is a local complete intersection, then one can extend the sequence with a zero on the left, see More on Algebra, Lemma 15.33.6.

00S2 **Lemma 10.134.4** (Jacobi-Zariski sequence)**.** *Let $A \to B \to C$ be ring maps. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_t, t \in T] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A[x_s]/A} \otimes C & \longrightarrow & \Omega_{A[x_s,y_t]/A} \otimes C & \longrightarrow & \Omega_{B[y_t]/B} \otimes C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \end{array}$$

with exact rows. We get the following exact sequence of homology groups

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules extending the sequence of Lemma 10.131.7. If $Tor_1^B(\Omega_{B/A}, C) = 0$ and $Tor_2^B(\Omega_{B/A}, C) = 0$, then $H_1(NL_{B/A} \otimes_B C) = H_1(L_{B/A}) \otimes_B C$.

Proof. The precise definition of the maps is omitted. The exactness of the top row follows as the $\mathrm{d}x_s$, $\mathrm{d}y_t$ form a basis for the middle module. The map γ factors

$$A[x_s, y_t] \to B[y_t] \to C$$

with surjective first arrow and second arrow equal to β. Thus we see that $K \to J$ is surjective. Moreover, the kernel of the first displayed arrow is $IA[x_s, y_t]$. Hence $I/I^2 \otimes C$ surjects onto the kernel of $K/K^2 \to J/J^2$. Finally, we can use Lemma 10.134.2 to identify the terms as homology groups of the naive cotangent complexes.

The final assertion is a statement in homological algebra. Recall that $NL_{B/A} = (N^{-1} \to N^0)$ is a two term complex of B-modules with N^0 free and cohomology modules $H^0 = \Omega_{B/A}$ and $H^{-1} = H_1(L_{B/A})$. Write $M \subset N^0$ for the image of the differential. If $\mathrm{Tor}_1^B(H^0, C) = 0$, then we have an exact sequence

$$0 \to M \otimes_B C \to N^0 \otimes_B C \to H^0 \otimes_B C \to 0$$

Since N^0 is free, we also see that $\mathrm{Tor}_2^B(H^0, C) = \mathrm{Tor}_1^B(M, C)$. Hence if $\mathrm{Tor}_2^B(H^0, C) = 0$ then we also have an exact sequence

$$0 \to H^{-1} \otimes_B C \to N^{-1} \otimes_B C \to M \otimes_B C \to 0$$

Putting everything together we see that if $\mathrm{Tor}_1^B(H^0, C) = 0$ and $\mathrm{Tor}_2^B(H^0, C) = 0$, then $H^{-1} \otimes_B C$ is the kernel of $N^{-1} \otimes_B C \to N^0 \otimes_B C$ as desired. □

07VC **Remark 10.134.5.** Let $A \to B$ and $\phi : B \to C$ be ring maps. Then the composition $NL_{B/A} \to NL_{C/A} \to NL_{C/B}$ is homotopy equivalent to zero. Namely, this composition is the functoriality of the naive cotangent complex for the square

$$\begin{array}{ccc} B & \xrightarrow{\phi} & C \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

Write $J = \mathrm{Ker}(B[C] \to C)$. An explicit homotopy is given by the map $\Omega_{A[B]/A} \otimes_A B \to J/J^2$ which maps the basis element $\mathrm{d}[b]$ to the class of $[\phi(b)] - b$ in J/J^2.

07BP **Lemma 10.134.6.** *Let $A \to B$ be a surjective ring map with kernel I. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(I/I^2 \to 0)$ with I/I^2 in degree 1. In particular $H_1(L_{B/A}) = I/I^2$.*

Proof. Follows from Lemma 10.134.2 and the fact that $A \to B$ is a presentation of B over A. □

065V **Lemma 10.134.7.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is). Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is exact.

Proof. Follows from Lemma 10.134.4 and the description of the naive cotangent complexes $NL_{C/B}$ and $NL_{C/A}$ in Lemma 10.134.6. □

00S4 **Lemma 10.134.8** (Flat base change)**.** *Let $R \to S$ be a ring map. Let $\alpha : P \to S$ be a presentation. Let $R \to R'$ be a flat ring map. Let $\alpha' : P \otimes_R R' \to S' = S \otimes_R R'$ be the induced presentation. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. In particular, the canonical map*

$$NL_{S/R} \otimes_S S' \longrightarrow NL_{S \otimes_R R'/R'}$$

is a homotopy equivalence if $R \to R'$ is flat.

Proof. This is true because $\mathrm{Ker}(\alpha') = R' \otimes_R \mathrm{Ker}(\alpha)$ since $R \to R'$ is flat. □

07BQ **Lemma 10.134.9.** *Let $R_i \to S_i$ be a system of ring maps over the directed set I. Set $R = \mathrm{colim}\, R_i$ and $S = \mathrm{colim}\, S_i$. Then $NL_{S/R} = \mathrm{colim}\, NL_{S_i/R_i}$.*

Proof. Recall that $NL_{S/R}$ is the complex $I/I^2 \to \bigoplus_{s \in S} S\mathrm{d}[s]$ where $I \subset R[S]$ is the kernel of the canonical presentation $R[S] \to S$. Now it is clear that $R[S] = \mathrm{colim}\, R_i[S_i]$ and similarly that $I = \mathrm{colim}\, I_i$ where $I_i = \mathrm{Ker}(R_i[S_i] \to S_i)$. Hence the lemma is clear. □

07BR **Lemma 10.134.10.** *If $S \subset A$ is a multiplicative subset of A, then $NL_{S^{-1}A/A}$ is homotopy equivalent to the zero complex.*

Proof. Since $A \to S^{-1}A$ is flat we see that $NL_{S^{-1}A/A} \otimes_A S^{-1}A \to NL_{S^{-1}A/S^{-1}A}$ is a homotopy equivalence by flat base change (Lemma 10.134.8). Since the source of the arrow is isomorphic to $NL_{S^{-1}A/A}$ and the target of the arrow is zero (by Lemma 10.134.6) we win. □

07BS **Lemma 10.134.11.** *Let $S \subset A$ is a multiplicative subset of A. Let $S^{-1}A \to B$ be a ring map. Then $NL_{B/A} \to NL_{B/S^{-1}A}$ is a homotopy equivalence.*

Proof. Choose a presentation $\alpha : P \to B$ of B over A. Then $\beta : S^{-1}P \to B$ is a presentation of B over $S^{-1}A$. A direct computation shows that we have $NL(\alpha) = NL(\beta)$ which proves the lemma as the naive cotangent complex is well defined up to homotopy by Lemma 10.134.2. □

08JZ **Lemma 10.134.12.** *Let $A \to B$ be a ring map. Let $g \in B$. Suppose $\alpha : P \to B$ is a presentation with kernel I. Then a presentation of B_g over A is the map*

$$\beta : P[x] \longrightarrow B_g$$

extending α and sending x to $1/g$. The kernel J of β is generated by I and the element $fx - 1$ where $f \in P$ is an element mapped to $g \in B$ by α. In this situation we have

1. $J/J^2 = (I/I^2)_g \oplus B_g(fx - 1)$,
2. $\Omega_{P[x]/A} \otimes_{P[x]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g\, dx$,
3. $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$

Hence the canonical map $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof. Since $P[x]/(I, fx-1) = B[x]/(gx-1) = B_g$ we get the statement about I and $fx - 1$ generating J. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{P/A} \otimes B_g & \longrightarrow & \Omega_{P[x]/A} \otimes B_g & \longrightarrow & \Omega_{B[x]/B} \otimes B_g \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & (I/I^2)_g & \longrightarrow & J/J^2 & \longrightarrow & (gx-1)/(gx-1)^2 \longrightarrow 0 \end{array}$$

with exact rows of Lemma 10.134.4. The B_g-module $\Omega_{B[x]/B} \otimes B_g$ is free of rank 1 on $\mathrm{d}x$. The element $\mathrm{d}x$ in the B_g-module $\Omega_{P[x]/A} \otimes B_g$ provides a splitting for the top row. The element $gx - 1 \in (gx-1)/(gx-1)^2$ is mapped to $g\mathrm{d}x$ in $\Omega_{B[x]/B} \otimes B_g$ and hence $(gx-1)/(gx-1)^2$ is free of rank 1 over B_g. (This can also be seen by arguing that $gx - 1$ is a nonzerodivisor in $B[x]$ because it is a polynomial with invertible constant term and any nonzerodivisor gives a quasi-regular sequence of length 1 by Lemma 10.69.2.)

Let us prove $(I/I^2)_g \to J/J^2$ injective. Consider the P-algebra map

$$\pi : P[x] \to (P/I^2)_f = P_f/I_f^2$$

sending x to $1/f$. Since J is generated by I and $fx-1$ we see that $\pi(J) \subset (I/I^2)_f = (I/I^2)_g$. Since this is an ideal of square zero we see that $\pi(J^2) = 0$. If $a \in I$ maps to an element of J^2 in J, then $\pi(a) = 0$, which implies that a maps to zero in I_f/I_f^2. This proves the desired injectivity.

Thus we have a short exact sequence of two term complexes

$$0 \to NL(\alpha) \otimes_B B_g \to NL(\beta) \to (B_g \xrightarrow{g} B_g) \to 0$$

Such a short exact sequence can always be split in the category of complexes. In our particular case we can take as splittings

$$J/J^2 = (I/I^2)_g \oplus B_g(fx-1) \quad \text{and} \quad \Omega_{P[x]/A} \otimes B_g = \Omega_{P/A} \otimes B_g \oplus B_g(g^{-2}\mathrm{d}f + \mathrm{d}x)$$

This works because $\mathrm{d}(fx-1) = x\mathrm{d}f + f\mathrm{d}x = g(g^{-2}\mathrm{d}f + \mathrm{d}x)$ in $\Omega_{P[x]/A} \otimes B_g$. □

00S7 **Lemma 10.134.13.** *Let $A \to B$ be a ring map. Let $S \subset B$ be a multiplicative subset. The canonical map $NL_{B/A} \otimes_B S^{-1}B \to NL_{S^{-1}B/A}$ is a quasi-isomorphism.*

Proof. We have $S^{-1}B = \mathrm{colim}_{g \in S} B_g$ where we think of S as a directed set (ordering by divisibility), see Lemma 10.9.9. By Lemma 10.134.12 each of the maps $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ are quasi-isomorphisms. The lemma follows from Lemma 10.134.9. □

00S3 **Lemma 10.134.14.** *Let R be a ring. Let $A_1 \to A_0$, and $B_1 \to B_0$ be two term complexes. Suppose that there exist morphisms of complexes $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. Then $A_1 \oplus B_0 \cong B_1 \oplus A_0$ as R-modules.*

Proof. Choose a map $h : A_0 \to A_1$ such that

$$\mathrm{id}_{A_1} - \psi_1 \circ \varphi_1 = h \circ d_A \text{ and } \mathrm{id}_{A_0} - \psi_0 \circ \varphi_0 = d_A \circ h.$$

Similarly, choose a map $h' : B_0 \to B_1$ such that

$$\mathrm{id}_{B_1} - \varphi_1 \circ \psi_1 = h' \circ d_B \text{ and } \mathrm{id}_{B_0} - \varphi_0 \circ \psi_0 = d_B \circ h'.$$

A trivial computation shows that

$$\begin{pmatrix} \mathrm{id}_{A_1} & -h' \circ \psi_1 + h \circ \psi_0 \\ 0 & \mathrm{id}_{B_0} \end{pmatrix} = \begin{pmatrix} \psi_1 & h \\ -d_B & \varphi_0 \end{pmatrix} \begin{pmatrix} \varphi_1 & -h' \\ d_A & \psi_0 \end{pmatrix}$$

This shows that both matrices on the right hand side are invertible and proves the lemma. □

00S5 **Lemma 10.134.15.** *Let $R \to S$ be a ring map of finite type. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S$ we have*

$$I/I^2 \oplus S^{\oplus m} \cong J/J^2 \oplus S^{\oplus n}$$

as S-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. See Lemmas 10.134.2 and 10.134.14. □

00S6 **Lemma 10.134.16.** *Let $R \to S$ be a ring map of finite type. Let $g \in S$. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S_g$ we have*

$$(I/I^2)_g \oplus S_g^{\oplus m} \cong J/J^2 \oplus S_g^{\oplus n}$$

as S_g-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. Let $\beta' : R[x_1, \ldots, x_n, x] \to S_g$ be the presentation of Lemma 10.134.12 constructed starting with α. Then we know that $NL(\alpha) \otimes_S S_g$ is homotopy equivalent to $NL(\beta')$. We know that $NL(\beta)$ and $NL(\beta')$ are homotopy equivalent by Lemma 10.134.2. We conclude that $NL(\alpha) \otimes_S S_g$ is homotopy equivalent to $NL(\beta)$. Finally, we apply Lemma 10.134.15. □

10.135. Local complete intersections

00S8 The property of being a local complete intersection is an intrinsic property of a Noetherian local ring. This will be discussed in Divided Power Algebra, Section 23.8. However, for the moment we just define this property for finite type algebras over a field.

00S9 **Definition 10.135.1.** Let k be a field. Let S be a finite type k-algebra.

(1) We say that S is a *global complete intersection over k* if there exists a presentation $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim(S) = n - c$.
(2) We say that S is a *local complete intersection over k* if there exists a covering $\mathrm{Spec}(S) = \bigcup D(g_i)$ such that each of the rings S_{g_i} is a global complete intersection over k.

We will also use the convention that the zero ring is a global complete intersection over k.

Suppose S is a global complete intersection $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Definition 10.135.1. For a maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ we have $\dim(k[x_1, \ldots, x_n]_{\mathfrak{m}}) = n$ (Lemma 10.114.1). If $(f_1, \ldots, f_c) \subset \mathfrak{m}$, then we conclude that $\dim(S_{\mathfrak{m}}) \geq n - c$ by Lemma 10.60.13. Since $\dim(S) = n - c$ by Definition 10.135.1 we conclude that $\dim(S_{\mathfrak{m}}) = n - c$ for all maximal ideals of S and that $\mathrm{Spec}(S)$ is equidimensional (Topology, Definition 5.10.5) of dimension $n - c$, see Lemma 10.114.5. We will often use this without further mention.

00SA **Lemma 10.135.2.** *Let k be a field. Let S be a finite type k-algebra. Let $g \in S$.*

(1) *If S is a global complete intersection so is S_g.*
(2) *If S is a local complete intersection so is S_g.*

Proof. The second statement follows immediately from the first. Proof of the first statement. If S_g is the zero ring, then it is true. Assume S_g is nonzero. Write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $n - c = \dim(S)$ as in Definition 10.135.1. By the remarks following the definition $\dim(S_g) = n - c$. Let $g' \in k[x_1, \ldots, x_n]$ be an element whose residue class corresponds to g. Then $S_g = k[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}g' - 1)$ as desired. □

00SB **Lemma 10.135.3.** *Let k be a field. Let S be a finite type k-algebra. If S is a local complete intersection, then S is a Cohen-Macaulay ring.*

Proof. Choose a maximal prime $\mathfrak{m}$ of S. We have to show that $S_{\mathfrak{m}}$ is Cohen-Macaulay. By assumption we may assume $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim(S) = n - c$. Let $\mathfrak{m}' \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to

$\mathfrak{m}$. According to Proposition 10.114.2 the local ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ is regular local of dimension n. In particular it is Cohen-Macaulay by Lemma 10.106.3. By Lemma 10.60.13 applied c times the local ring $S_{\mathfrak{m}} = k[x_1, \ldots, x_n]_{\mathfrak{m}'}/(f_1, \ldots, f_c)$ has dimension $\geq n - c$. By assumption $\dim(S_{\mathfrak{m}}) \leq n - c$. Thus we get equality. This implies that $f_1, \ldots, f_c$ is a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ and that $S_{\mathfrak{m}}$ is Cohen-Macaulay, see Proposition 10.103.4. □

The following is the technical key to the rest of the material in this section. An important feature of this lemma is that we may choose any presentation for the ring S, but that condition (1) does not depend on this choice.

00SC **Lemma 10.135.4.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. Choose any presentation $S = k[x_1, \ldots, x_n]/I$. Let $\mathfrak{q}'$ be the prime of $k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. Set $c = \text{height}(\mathfrak{q}') - \text{height}(\mathfrak{q})$, in other words $\dim_{\mathfrak{q}}(S) = n - c$ (see Lemma 10.116.4). The following are equivalent*

1. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*
2. *The ideal $I_{\mathfrak{q}'} \subset k[x_1, \ldots, x_n]_{\mathfrak{q}'}$ can be generated by c elements.*
3. *The conormal module $(I/I^2)_{\mathfrak{q}}$ can be generated by c elements over $S_{\mathfrak{q}}$.*
4. *The conormal module $(I/I^2)_{\mathfrak{q}}$ is a free $S_{\mathfrak{q}}$-module of rank c.*
5. *The ideal $I_{\mathfrak{q}'}$ can be generated by a regular sequence in the regular local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.*

In this case any c elements of $I_{\mathfrak{q}'}$ which generate $I_{\mathfrak{q}'}/\mathfrak{q}'I_{\mathfrak{q}'}$ form a regular sequence in the local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.

Proof. Set $R = k[x_1, \ldots, x_n]_{\mathfrak{q}'}$. This is a Cohen-Macaulay local ring of dimension $\text{height}(\mathfrak{q}')$, see for example Lemma 10.135.3. Moreover, $\overline{R} = R/IR = R/I_{\mathfrak{q}'} = S_{\mathfrak{q}}$ is a quotient of dimension $\text{height}(\mathfrak{q})$. Let $f_1, \ldots, f_c \in I_{\mathfrak{q}'}$ be elements which generate $(I/I^2)_{\mathfrak{q}}$. By Lemma 10.20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{q}'}$. Since the dimensions work out, we conclude by Proposition 10.103.4 that $f_1, \ldots, f_c$ is a regular sequence in R. By Lemma 10.69.2 we see that $(I/I^2)_{\mathfrak{q}}$ is free. These arguments show that (2), (3), (4) are equivalent and that they imply the last statement of the lemma, and therefore they imply (5).

If (5) holds, say $I_{\mathfrak{q}'}$ is generated by a regular sequence of length e, then $\text{height}(\mathfrak{q}) = \dim(S_{\mathfrak{q}}) = \dim(k[x_1, \ldots, x_n]_{\mathfrak{q}'}) - e = \text{height}(\mathfrak{q}') - e$ by dimension theory, see Section 10.60. We conclude that $e = c$. Thus (5) implies (2).

We continue with the notation introduced in the first paragraph. For each f_i we may find $d_i \in k[x_1, \ldots, x_n]$, $d_i \notin \mathfrak{q}'$ such that $f_i' = d_i f_i \in k[x_1, \ldots, x_n]$. Then it is still true that $I_{\mathfrak{q}'} = (f_1', \ldots, f_c')R$. Hence there exists a $g' \in k[x_1, \ldots, x_n]$, $g' \notin \mathfrak{q}'$ such that $I_{g'} = (f_1', \ldots, f_c')$. Moreover, pick $g'' \in k[x_1, \ldots, x_n]$, $g'' \notin \mathfrak{q}'$ such that $\dim(S_{g''}) = \dim_{\mathfrak{q}} \text{Spec}(S)$. By Lemma 10.116.4 this dimension is equal to $n - c$. Finally, set g equal to the image of $g'g''$ in S. Then we see that

$$S_g \cong k[x_1, \ldots, x_n, x_{n+1}]/(f_1', \ldots, f_c', x_{n+1}g'g'' - 1)$$

and by our choice of g'' this ring has dimension $n - c$. Therefore it is a global complete intersection. Thus each of (2), (3), and (4) implies (1).

Assume (1). Let $S_g \cong k[y_1, \ldots, y_m]/(f_1, \ldots, f_t)$ be a presentation of S_g as a global complete intersection. Write $J = (f_1, \ldots, f_t)$. Let $\mathfrak{q}'' \subset k[y_1, \ldots, y_m]$ be the prime corresponding to $\mathfrak{q}S_g$. Note that $t = m - \dim(S_g) = \text{height}(\mathfrak{q}'') - \text{height}(\mathfrak{q})$, see

Lemma 10.116.4 for the last equality. As seen in the proof of Lemma 10.135.3 (and also above) the elements $f_1, \ldots, f_t$ form a regular sequence in the local ring $k[y_1, \ldots, y_m]_{\mathfrak{q}''}$. By Lemma 10.69.2 we see that $(J/J^2)_{\mathfrak{q}}$ is free of rank t. By Lemma 10.134.16 we have

$$J/J^2 \oplus S_g^n \cong (I/I^2)_g \oplus S_g^m$$

Thus $(I/I^2)_{\mathfrak{q}}$ is free of rank $t + n - m = m - \dim(S_g) + n - m = n - \dim(S_g) = \text{height}(\mathfrak{q}') - \text{height}(\mathfrak{q}) = c$. Thus we obtain (4). □

The result of Lemma 10.135.4 suggests the following definition.

00SD **Definition 10.135.5.** Let k be a field. Let S be a local k-algebra essentially of finite type over k. We say S is a *complete intersection (over k)* if there exists a local k-algebra R and elements $f_1, \ldots, f_c \in \mathfrak{m}_R$ such that

(1) R is essentially of finite type over k,
(2) R is a regular local ring,
(3) $f_1, \ldots, f_c$ form a regular sequence in R, and
(4) $S \cong R/(f_1, \ldots, f_c)$ as k-algebras.

By the Cohen structure theorem (see Theorem 10.160.8) any complete Noetherian local ring may be written as the quotient of some regular complete local ring. Hence we may use the definition above to define the notion of a complete intersection ring for any complete Noetherian local ring. We will discuss this in Divided Power Algebra, Section 23.8. In the meantime the following lemma shows that such a definition makes sense.

00SE **Lemma 10.135.6.** *Let $A \to B \to C$ be surjective local ring homomorphisms. Assume A and B are regular local rings. The following are equivalent*

(1) $\text{Ker}(A \to C)$ *is generated by a regular sequence,*
(2) $\text{Ker}(A \to C)$ *is generated by* $\dim(A) - \dim(C)$ *elements,*
(3) $\text{Ker}(B \to C)$ *is generated by a regular sequence, and*
(4) $\text{Ker}(B \to C)$ *is generated by* $\dim(B) - \dim(C)$ *elements.*

Proof. A regular local ring is Cohen-Macaulay, see Lemma 10.106.3. Hence the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), see Proposition 10.103.4. By Lemma 10.106.4 the ideal $\text{Ker}(A \to B)$ can be generated by $\dim(A) - \dim(B)$ elements. Hence we see that (4) implies (2).

It remains to show that (1) implies (4). We do this by induction on $\dim(A) - \dim(B)$. The case $\dim(A) - \dim(B) = 0$ is trivial. Assume $\dim(A) > \dim(B)$. Write $I = \text{Ker}(A \to C)$ and $J = \text{Ker}(A \to B)$. Note that $J \subset I$. Our assumption is that the minimal number of generators of I is $\dim(A) - \dim(C)$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Consider the maps

$$J/\mathfrak{m}J \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$$

By Lemma 10.106.4 and its proof the composition is injective. Take any element $x \in J$ which is not zero in $J/\mathfrak{m}J$. By the above and Nakayama's lemma x is an element of a minimal set of generators of I. Hence we may replace A by A/xA and I by I/xA which decreases both $\dim(A)$ and the minimal number of generators of I by 1. Thus we win. □

00SF **Lemma 10.135.7.** *Let k be a field. Let S be a local k-algebra essentially of finite type over k. The following are equivalent:*

(1) *S is a complete intersection over k,*
(2) *for any surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by a regular sequence,*
(3) *for some surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements,*
(4) *there exists a global complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$, and*
(5) *there exists a local complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$.*

Proof. It is clear that (2) implies (1) and (1) implies (3). It is also clear that (4) implies (5). Let us show that (3) implies (4). Thus we assume there exists a surjection $R \to S$ with R a regular local ring essentially of finite presentation over k such that the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements. We may write $R = (k[x_1, \ldots, x_n]/J)_{\mathfrak{q}}$ for some $J \subset k[x_1, \ldots, x_n]$ and some prime $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ with $J \subset \mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of the map $k[x_1, \ldots, x_n] \to S$ so that $S \cong (k[x_1, \ldots, x_n]/I)_{\mathfrak{q}}$. By assumption $(I/J)_{\mathfrak{q}}$ is generated by $\dim(R) - \dim(S)$ elements. We conclude that $I_{\mathfrak{q}}$ can be generated by $\dim(k[x_1, \ldots, x_n]_{\mathfrak{q}}) - \dim(S)$ elements by Lemma 10.135.6. From Lemma 10.135.4 we see that for some $g \in k[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}$ the algebra $(k[x_1, \ldots, x_n]/I)_g$ is a global complete intersection and S is isomorphic to a local ring of it.

To finish the proof of the lemma we have to show that (5) implies (2). Assume (5) and let $\pi : R \to S$ be a surjection with R a regular local k-algebra essentially of finite type over k. By assumption we have $S = A_{\mathfrak{a}}$ for some local complete intersection A over k. Choose a presentation $R = (k[y_1, \ldots, y_m]/J)_{\mathfrak{q}}$ with $J \subset \mathfrak{q} \subset k[y_1, \ldots, y_m]$. We may and do assume that J is the kernel of the map $k[y_1, \ldots, y_m] \to R$. Let $I \subset k[y_1, \ldots, y_m]$ be the kernel of the map $k[y_1, \ldots, y_m] \to S = A_{\mathfrak{a}}$. Then $J \subset I$ and $(I/J)_{\mathfrak{q}}$ is the kernel of the surjection $\pi : R \to S$. So $S = (k[y_1, \ldots, y_m]/I)_{\mathfrak{q}}$.

By Lemma 10.126.7 we see that there exist $g \in A$, $g \notin \mathfrak{a}$ and $g' \in k[y_1, \ldots, y_m]$, $g' \notin \mathfrak{q}$ such that $A_g \cong (k[y_1, \ldots, y_m]/I)_{g'}$. After replacing A by A_g and $k[y_1, \ldots, y_m]$ by $k[y_1, \ldots, y_{m+1}]$ we may assume that $A \cong k[y_1, \ldots, y_m]/I$. Consider the surjective maps of local rings

$$k[y_1, \ldots, y_m]_{\mathfrak{q}} \to R \to S.$$

We have to show that the kernel of $R \to S$ is generated by a regular sequence. By Lemma 10.135.4 we know that $k[y_1, \ldots, y_m]_{\mathfrak{q}} \to A_{\mathfrak{a}} = S$ has this property (as A is a local complete intersection over k). We win by Lemma 10.135.6. □

00SG **Lemma 10.135.8.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:*

(1) *The local ring $S_{\mathfrak{q}}$ is a complete intersection ring (Definition 10.135.5).*
(2) *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a local complete intersection over k.*
(3) *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*

(4) *For any presentation $S = k[x_1, \ldots, x_n]/I$ with $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$ any of the equivalent conditions (1) – (5) of Lemma 10.135.4 hold.*

Proof. This is a combination of Lemmas 10.135.4 and 10.135.7 and the definitions. □

00SH **Lemma 10.135.9.** *Let k be a field. Let S be a finite type k-algebra. The following are equivalent:*

(1) *The ring S is a local complete intersection over k.*
(2) *All local rings of S are complete intersection rings over k.*
(3) *All localizations of S at maximal ideals are complete intersection rings over k.*

Proof. This follows from Lemma 10.135.8, the fact that $\mathrm{Spec}(S)$ is quasi-compact and the definitions. □

The following lemma says that being a complete intersection is preserved under change of base field (in a strong sense).

00SI **Lemma 10.135.10.** *Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then $S_\mathfrak{q}$ is a complete intersection over k (Definition 10.135.5) if and only if $(S_K)_{\mathfrak{q}_K}$ is a complete intersection over K.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $S_K = K[x_1, \ldots, x_n]/I_K$ where $I_K = K \otimes_k I$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding prime. We will show that the equivalent conditions of Lemma 10.135.4 hold for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ if and only if they hold for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. The lemma will follow from this (see Lemma 10.135.8).

By Lemma 10.116.6 we have $\dim_\mathfrak{q} S = \dim_{\mathfrak{q}_K} S_K$. Hence the integer c occurring in Lemma 10.135.4 is the same for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ as for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. On the other hand we have

$$\begin{aligned} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{q}')} \kappa(\mathfrak{q}'_K) &= I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I \otimes_{k[x_1,\ldots,x_n]} K[x_1, \ldots, x_n] \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= (K \otimes_k I) \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K). \end{aligned}$$

Therefore, $\dim_{\kappa(\mathfrak{q}')} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') = \dim_{\kappa(\mathfrak{q}'_K)} I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K)$. Thus it follows from Nakayama's Lemma 10.20.1 that the minimal number of generators of $I_{\mathfrak{q}'}$ is the same as the minimal number of generators of $(I_K)_{\mathfrak{q}'_K}$. Thus the lemma follows from characterization (2) of Lemma 10.135.4. □

00SJ **Lemma 10.135.11.** *Let $k \to K$ be a field extension. Let S be a finite type k-algebra. Then S is a local complete intersection over k if and only if $S \otimes_k K$ is a local complete intersection over K.*

Proof. This follows from a combination of Lemmas 10.135.9 and 10.135.10. But we also give a different proof here (based on the same principles).

Set $S' = S \otimes_k K$. Let $\alpha : k[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : K[x_1, \ldots, x_n] \to S'$ be the induced presentation with kernel I'.

Suppose that S is a local complete intersection. Pick a prime $\mathfrak{q} \subset S'$. Denote $\mathfrak{q}'$ the corresponding prime of $K[x_1, \ldots, x_n]$, $\mathfrak{p}$ the corresponding prime of S, and $\mathfrak{p}'$ the corresponding prime of $k[x_1, \ldots, x_n]$. Consider the following diagram of Noetherian local rings

$$\begin{array}{ccc} S'_{\mathfrak{q}} & \longleftarrow & K[x_1, \ldots, x_n]_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ S_{\mathfrak{p}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{p}'} \end{array}$$

By Lemma 10.135.4 we know that $S_{\mathfrak{p}}$ is cut out by some regular sequence $f_1, \ldots, f_c$ in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$. Since the right vertical arrow is flat we see that the images of $f_1, \ldots, f_c$ form a regular sequence in $K[x_1, \ldots, x_n]_{\mathfrak{q}'}$. Because tensoring with K over k is an exact functor we have $S'_{\mathfrak{q}} = K[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_c)$. Hence by Lemma 10.135.4 again we see that S' is a local complete intersection in a neighbourhood of $\mathfrak{q}$. Since $\mathfrak{q}$ was arbitrary we see that S' is a local complete intersection over K.

Suppose that S' is a local complete intersection. Pick a maximal ideal $\mathfrak{m}$ of S. Let $\mathfrak{m}'$ denote the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Denote $\kappa = \kappa(\mathfrak{m})$ the residue field. By Remark 10.18.5 the primes of S' lying over $\mathfrak{m}$ correspond to primes in $K \otimes_k \kappa$. By the Hilbert-Nullstellensatz Theorem 10.34.1 we have $[\kappa : k] < \infty$. Hence $K \otimes_k \kappa$ is finite nonzero over K. Hence $K \otimes_k \kappa$ has a finite number > 0 of primes which are all maximal, each of which has a residue field finite over K (see Section 10.53). Hence there are finitely many > 0 prime ideals $\mathfrak{n} \subset S'$ lying over $\mathfrak{m}$, each of which is maximal and has a residue field which is finite over K. Pick one, say $\mathfrak{n} \subset S'$, and let $\mathfrak{n}' \subset K[x_1, \ldots, x_n]$ denote the corresponding prime ideal of $K[x_1, \ldots, x_n]$. Note that since $V(\mathfrak{m}S')$ is finite, we see that $\mathfrak{n}$ is an isolated closed point of it, and we deduce that $\mathfrak{m}S'_{\mathfrak{n}}$ is an ideal of definition of $S'_{\mathfrak{n}}$. This implies that $\dim(S_{\mathfrak{m}}) = \dim(S'_{\mathfrak{n}})$ for example by Lemma 10.112.7. (This can also be seen using Lemma 10.116.6.) Consider the corresponding diagram of Noetherian local rings

$$\begin{array}{ccc} S'_{\mathfrak{n}} & \longleftarrow & K[x_1, \ldots, x_n]_{\mathfrak{n}'} \\ \uparrow & & \uparrow \\ S_{\mathfrak{m}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{m}'} \end{array}$$

According to Lemma 10.134.8 we have $NL(\alpha) \otimes_S S' = NL(\alpha')$, in particular $I'/(I')^2 = I/I^2 \otimes_S S'$. Thus $(I/I^2)_{\mathfrak{m}} \otimes_{S_{\mathfrak{m}}} \kappa$ and $(I'/(I')^2)_{\mathfrak{n}} \otimes_{S'_{\mathfrak{n}}} \kappa(\mathfrak{n})$ have the same dimension. Since $(I'/(I')^2)_{\mathfrak{n}}$ is free of rank $n - \dim S'_{\mathfrak{n}}$ we deduce that $(I/I^2)_{\mathfrak{m}}$ can be generated by $n - \dim S'_{\mathfrak{n}} = n - \dim S_{\mathfrak{m}}$ elements. By Lemma 10.135.4 we see that S is a local complete intersection in a neighbourhood of $\mathfrak{m}$. Since $\mathfrak{m}$ was any maximal ideal we conclude that S is a local complete intersection. $\square$

We end with a lemma which we will later use to prove that given ring maps $T \to A \to B$ where B is syntomic over T, and B is syntomic over A, then A is syntomic over T.

02JP **Lemma 10.135.12.** *Let*

$$\begin{array}{ccc} B & \longleftarrow & S \\ \uparrow & & \uparrow \\ A & \longleftarrow & R \end{array}$$

be a commutative square of local rings. Assume

(1) *R and $\overline{S} = S/\mathfrak{m}_R S$ are regular local rings,*
(2) *$A = R/I$ and $B = S/J$ for some ideals I, J,*
(3) *$J \subset S$ and $\overline{J} = J/\mathfrak{m}_R \cap J \subset \overline{S}$ are generated by regular sequences, and*
(4) *$A \to B$ and $R \to S$ are flat.*

Then I is generated by a regular sequence.

Proof. Set $\overline{B} = B/\mathfrak{m}_R B = B/\mathfrak{m}_A B$ so that $\overline{B} = \overline{S}/\overline{J}$. Let $f_1, \ldots, f_{\overline{c}} \in J$ be elements such that $\overline{f}_1, \ldots, \overline{f}_{\overline{c}} \in \overline{J}$ form a regular sequence generating $\overline{J}$. Note that $\overline{c} = \dim(\overline{S}) - \dim(\overline{B})$, see Lemma 10.135.6. By Lemma 10.99.3 the ring $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R. Hence $S/(f_1, \ldots, f_{\overline{c}}) + IS$ is flat over A. The map $S/(f_1, \ldots, f_{\overline{c}}) + IS \to B$ is therefore a surjection of finite S/IS-modules flat over A which is an isomorphism modulo $\mathfrak{m}_A$, and hence an isomorphism by Lemma 10.99.1. In other words, $J = (f_1, \ldots, f_{\overline{c}}) + IS$.

By Lemma 10.135.6 again the ideal J is generated by a regular sequence of $c = \dim(S) - \dim(B)$ elements. Hence $J/\mathfrak{m}_S J$ is a vector space of dimension c. By the description of J above there exist $g_1, \ldots, g_{c-\overline{c}} \in I$ such that J is generated by $f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}}$ (use Nakayama's Lemma 10.20.1). Consider the ring $A' = R/(g_1, \ldots, g_{c-\overline{c}})$ and the surjection $A' \to A$. We see from the above that $B = S/(f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}})$ is flat over A' (as $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R). Hence $A' \to B$ is injective (as it is faithfully flat, see Lemma 10.39.17). Since this map factors through A we get $A' = A$. Note that $\dim(B) = \dim(A) + \dim(\overline{B})$, and $\dim(S) = \dim(R) + \dim(\overline{S})$, see Lemma 10.112.7. Hence $c - \overline{c} = \dim(R) - \dim(A)$ by elementary algebra. Thus $I = (g_1, \ldots, g_{c-\overline{c}})$ is generated by a regular sequence according to Lemma 10.135.6. □

10.136. Syntomic morphisms

00SK Syntomic ring maps are flat finitely presented ring maps all of whose fibers are local complete intersections. We discuss general local complete intersection ring maps in More on Algebra, Section 15.33.

00SL **Definition 10.136.1.** A ring map $R \to S$ is called *syntomic*, or we say S is a *flat local complete intersection over R* if it is flat, of finite presentation, and if all of its fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are local complete intersections, see Definition 10.135.1.

Clearly, an algebra over a field is syntomic over the field if and only if it is a local complete intersection. Here is a pleasing feature of this definition.

00SM **Lemma 10.136.2.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is syntomic if and only if $R' \to S'$ is syntomic.*

Proof. By Lemma 10.126.2 and Lemma 10.39.8 this holds for the property of being flat and for the property of being of finite presentation. The map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, see Lemma 10.39.16. Thus it suffices to show given primes

$\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$ that $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection if and only if $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is a local complete intersection. Note that $S' \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$. Thus Lemma 10.135.11 applies. □

00SN **Lemma 10.136.3.** *Any base change of a syntomic map is syntomic.*

Proof. This is true for being flat, for being of finite presentation, and for having local complete intersections as fibres by Lemmas 10.39.7, 10.6.2 and 10.135.11. □

00SO **Lemma 10.136.4.** *Let $R \to S$ be a ring map. Suppose we have $g_1, \ldots g_m \in S$ which generate the unit ideal such that each $R \to S_{g_i}$ is syntomic. Then $R \to S$ is syntomic.*

Proof. This is true for being flat and for being of finite presentation by Lemmas 10.39.18 and 10.23.3. The property of having fibre rings which are local complete intersections is local on S by its very definition, see Definition 10.135.1. □

00SP **Definition 10.136.5.** Let $R \to S$ be a ring map. We say that $R \to S$ is a *relative global complete intersection* if there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and every nonempty fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ has dimension $n - c$. We will say "let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection" to indicate this situation.

The following lemma is occasionally useful to find global presentations.

07CF **Lemma 10.136.6.** *Let S be a finitely presented R-algebra which has a presentation $S = R[x_1, \ldots, x_n]/I$ such that I/I^2 is free over S. Then S has a presentation $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the classes of $f_1, \ldots, f_c$.*

Proof. Note that I is a finitely generated ideal by Lemma 10.6.3. Let $f_1, \ldots, f_c \in I$ be elements which map to a basis of I/I^2. By Nakayama's lemma (Lemma 10.20.1) there exists a $g \in 1 + I$ such that

$$g \cdot I \subset (f_1, \ldots, f_c)$$

and $I_g \cong (f_1, \ldots, f_c)_g$. Hence we see that

$$S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)[1/g] \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, gx_{n+1} - 1)$$

as desired. It follows that $f_1, \ldots, f_c, gx_{n+1} - 1$ form a basis for $(f_1, \ldots, f_c, gx_{n+1} - 1)/(f_1, \ldots, f_c, gx_{n+1} - 1)^2$ for example by applying Lemma 10.134.12. □

00SQ **Example 10.136.7.** Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

In other words, this is the unique ring map of polynomial rings as indicated such that the polynomial factorization

$$x^{n+m} + a_1x^{n+m-1} + \ldots + a_{n+m} = (x^n + b_1x^{n-1} + \ldots + b_n)(x^m + c_1x^{m-1} + \ldots + c_m)$$

holds. Note that S is generated by $n+m$ elements over R (namely, b_i, c_j) and that there are $n+m$ equations (namely $a_k = a_k(b_i, c_j)$). In order to show that S is a relative global complete intersection over R it suffices to prove that all fibres have dimension 0.

To prove this, let $R \to k$ be a ring map into a field k. Say a_i maps to $\alpha_i \in k$. Consider the fibre ring $S_k = k \otimes_R S$. Let $k \to K$ be a field extension. A k-algebra map of $S_k \to K$ is the same thing as finding $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m \in K$ such that

$$x^{n+m} + \alpha_1 x^{n+m-1} + \ldots + \alpha_{n+m} = (x^n + \beta_1 x^{n-1} + \ldots + \beta_n)(x^m + \gamma_1 x^{m-1} + \ldots + \gamma_m).$$

Hence we see there are at most finitely many choices of such $n+m$-tuples in K. This proves that all fibres have finitely many closed points (use Hilbert's Nullstellensatz to see they all correspond to solutions in $\overline{k}$ for example) and hence that $R \to S$ is a relative global complete intersection.

Another way to argue this is to show $\mathbf{Z}[a_1, \ldots, a_{n+m}] \to \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$ is actually also a *finite* ring map. Namely, by Lemma 10.38.5 each of b_i, c_j is integral over R, and hence $R \to S$ is finite by Lemma 10.36.4.

00SR **Example 10.136.8.** Consider the ring map

$$\begin{array}{ccc} R = \mathbf{Z}[a_1, \ldots, a_n] & \longrightarrow & S = \mathbf{Z}[\alpha_1, \ldots, \alpha_n] \\ a_1 & \longmapsto & \alpha_1 + \ldots + \alpha_n \\ \ldots & \ldots & \ldots \\ a_n & \longmapsto & \alpha_1 \ldots \alpha_n \end{array}$$

In other words this is the unique ring map of polynomial rings as indicated such that

$$x^n + a_1 x^{n-1} + \ldots + a_n = \prod\nolimits_{i=1}^{n} (x + \alpha_i)$$

holds in $\mathbf{Z}[\alpha_i, x]$. Another way to say this is that a_i maps to the ith elementary symmetric function in $\alpha_1, \ldots, \alpha_n$. By the usual theory of elementary symmetric polynomials (details omitted) the ring S is finite free over R with basis the elements $\alpha_1^{e_1}\alpha_2^{e_2}\ldots\alpha_n^{e_n}$ with $0 \leq e_i \leq n-i$. A fortiori, the fibre rings of $R \to S$ are finite and hence have dimension 0. On the other hand, S is generated by n elements over R subject to n equations. Hence S is a relative global complete intersection over R. Since the rank of S over R is positive, we also see that S is faithfully flat over R.

00SS **Lemma 10.136.9.** *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5)*

1. *For any $R \to R'$ the base change $R' \otimes_R S = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection.*
2. *For any $g \in S$ which is the image of $h \in R[x_1, \ldots, x_n]$ the ring $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$ is a relative global complete intersection.*
3. *If $R \to S$ factors as $R \to R_f \to S$ for some $f \in R$. Then the ring $S = R_f[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R_f.*

Proof. By Lemma 10.116.5 the fibres of a base change have the same dimension as the fibres of the original map. Moreover $R' \otimes_R R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) =$

$R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Thus (1) follows. The proof of (2) is that the localization at one element can be described as $S_g \cong S[x_{n+1}]/(gx_{n+1} - 1)$. Assertion (3) follows from (1) since under the assumptions of (3) we have $R_f \otimes_R S \cong S$. □

00ST **Lemma 10.136.10.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. We will find $h \in R[x_1, \ldots, x_n]$ which maps to $g \in S$ such that*

$$S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$$

is a relative global complete intersection with a presentation as in Definition 10.136.5 in each of the following cases:

1. *Let $I \subset R$ be an ideal. If the fibres of $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ have dimension $n - c$, then we can find (h, g) as above such that g maps to $1 \in S/IS$.*
2. *Let $\mathfrak{p} \subset R$ be a prime. If $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$, then we can find (h, g) as above such that g maps to a unit of $S \otimes_R \kappa(\mathfrak{p})$.*
3. *Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $\dim_{\mathfrak{q}}(S/R) = n - c$, then we can find (h, g) as above such that $g \notin \mathfrak{q}$.*

Proof. Ad (1). By Lemma 10.125.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $V(IS)$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J) \cap V(IS) = \emptyset$ hence we can find a $g \in J$ which maps to $1 \in S/IS$. Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (2). By Lemma 10.125.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J \cdot S \otimes_R \kappa(\mathfrak{p})) = \emptyset$. Hence we can find a $g \in J$ which maps to a unit in $S \otimes_R \kappa(\mathfrak{p})$ (details omitted). Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (3). By Lemma 10.125.6 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that all nonempty fibres of $R \to S_g$ have dimension $\leq n - c$. Let $h \in R[x_1, \ldots, x_n]$ be any element that maps to g. □

The following lemma says we can do absolute Noetherian approximation for relative global complete intersections.

00SU **Lemma 10.136.11.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5). There exist a finite type $\mathbf{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_1, \ldots, x_n]$ and such that*

$$S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a relative global complete intersection.

Proof. Let $R_0 \subset R$ be the $\mathbf{Z}$-algebra of R generated by all the coefficients of the polynomials $f_1, \ldots, f_c$. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Clearly, $S = R \otimes_{R_0} S_0$. Pick a prime $\mathfrak{q} \subset S$ and denote $\mathfrak{p} \subset R$, $\mathfrak{q}_0 \subset S_0$, and $\mathfrak{p}_0 \subset R_0$ the primes it lies over. Because $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$ we also have $\dim(S_0 \otimes_{R_0} \kappa(\mathfrak{p}_0)) = n - c$, see Lemma 10.116.5. By Lemma 10.125.6 there exists a $g \in S_0$, $g \notin \mathfrak{q}_0$ such that all nonempty fibres of $R_0 \to (S_0)_g$ have dimension $\leq n - c$. As $\mathfrak{q}$ was arbitrary and $\mathrm{Spec}(S)$ quasi-compact, we can find finitely many $g_1, \ldots, g_m \in S_0$ such that (a) for $j = 1, \ldots, m$ the nonempty fibres of $R_0 \to (S_0)_{g_j}$ have dimension $\leq n - c$ and (b) the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$ is contained in $D(g_1) \cup \ldots \cup D(g_m)$. In other words, the images of $g_1, \ldots, g_m$ in $S = R \otimes_{R_0} S_0$ generate the unit ideal. After

increasing R_0 we may assume that $g_1, \ldots, g_m$ generate the unit ideal in S_0. By (a) the nonempty fibres of $R_0 \to S_0$ all have dimension $\leq n - c$ and we conclude. □

00SV **Lemma 10.136.12.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5). For every prime $\mathfrak{q}$ of S, let $\mathfrak{q}'$ denote the corresponding prime of $R[x_1, \ldots, x_n]$. Then*

(1) *$f_1, \ldots, f_c$ is a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$,*
(2) *each of the rings $R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i)$ is flat over R, and*
(3) *the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$.*

Proof. Assume R is Noetherian. Let $\mathfrak{p} = R \cap \mathfrak{q}'$. By Lemma 10.135.4 for example we see that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$. Moreover, the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}}$. Since R, and hence $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is Noetherian we see from Lemma 10.99.3 that (1) and (2) hold.

Let R be general. Write $R = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ as the filtered colimit of finite type $\mathbf{Z}$-subalgebras (compare with Section 10.127). We may assume that $f_1, \ldots, f_c \in R_\lambda[x_1, \ldots, x_n]$ for all λ. Let $R_0 \subset R$ be as in Lemma 10.136.11. Then we may assume $R_0 \subset R_\lambda$ for all λ. It follows that $S_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection (as base change of S_0 via $R_0 \to R_\lambda$, see Lemma 10.136.9). Denote $\mathfrak{p}_\lambda$, $\mathfrak{q}_\lambda$, $\mathfrak{q}'_\lambda$ the prime of R_λ, S_λ, $R_\lambda[x_1, \ldots, x_n]$ induced by $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$. With this notation, we have (1) and (2) for each λ. Since

$$R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) = \operatorname{colim} R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i)$$

we deduce flatness in (2) over R from Lemma 10.39.6. Since we have

$$\begin{aligned} & R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) \\ = \operatorname{colim} & \Big(R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i) \Big) \end{aligned}$$

and since filtered colimits are exact (Lemma 10.8.8) we conclude that we have (1).

Proof of (3). Denote N the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ and $e_i \in N$ the image of f_i. By Lemma 10.69.2 and (1) we know that $e_1, \ldots, e_c$ is a basis of $N_{\mathfrak{q}}$ for all primes $\mathfrak{q}$ of S. By Lemma 10.23.1 we conclude that (3) is true. □

00SW **Lemma 10.136.13.** *A relative global complete intersection is syntomic, i.e., flat.*

Proof. Let $R \to S$ be a relative global complete intersection. The fibres are global complete intersections, and S is of finite presentation over R. Thus the only thing to prove is that $R \to S$ is flat. This is true by (2) of Lemma 10.136.12. □

03HS **Lemma 10.136.14.** *Suppose that A is a ring, and $P(x) = x^n + b_1 x^{n-1} + \ldots + b_n \in A[x]$ is a monic polynomial over A. Then there exists a syntomic, finite free, faithfully flat ring extension $A \subset A'$ such that $P(x) = \prod_{i=1,\ldots,n}(x - \beta_i)$ for certain $\beta_i \in A'$.*

Proof. Take $A' = A \otimes_R S$, where R and S are as in Example 10.136.8, where $R \to A$ maps a_i to b_i, and let $\beta_i = -1 \otimes \alpha_i$. Observe that $R \to S$ is faithfully flat and finite free and syntomic by Lemma 10.136.13. These properties are inherited by the base change $A \to A'$; some details omitted. □

00SY **Lemma 10.136.15.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The following are equivalent:*

(1) *There exists an element $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is syntomic.*
(2) *There exists an element $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a relative global complete intersection over R.*
(3) *There exists an element $g \in S$, $g \notin \mathfrak{q}$, such that $R \to S_g$ is of finite presentation, the local ring map $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat, and the local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is a complete intersection ring over $\kappa(\mathfrak{p})$ (see Definition 10.135.5).*

Proof. The implication (1) $\Rightarrow$ (3) is Lemma 10.135.8. The implication (2) $\Rightarrow$ (1) is Lemma 10.136.13. It remains to show that (3) implies (2).

Assume (3). After replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ we may assume S is finitely presented over R. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. Let $\mathfrak{q}' \subset R[x_1, \ldots, x_n]$ be the prime corresponding to $\mathfrak{q}$. Write $\kappa(\mathfrak{p}) = k$. Note that $S \otimes_R k = k[x_1, \ldots, x_n]/\overline{I}$ where $\overline{I} \subset k[x_1, \ldots, x_n]$ is the ideal generated by the image of I. Let $\overline{\mathfrak{q}}' \subset k[x_1, \ldots, x_n]$ be the prime ideal generated by the image of $\mathfrak{q}'$. By Lemma 10.135.8 the equivalent conditions of Lemma 10.135.4 hold for $\overline{I}$ and $\overline{\mathfrak{q}}'$. Say the dimension of $\overline{I}_{\overline{\mathfrak{q}}'}/\overline{\mathfrak{q}}'\overline{I}_{\overline{\mathfrak{q}}'}$ over $\kappa(\overline{\mathfrak{q}}')$ is c. Pick $f_1, \ldots, f_c \in I$ mapping to a basis of this vector space. The images $\overline{f}_j \in \overline{I}$ generate $\overline{I}_{\overline{\mathfrak{q}}'}$ (by Lemma 10.135.4). Set $S' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let J be the kernel of the surjection $S' \to S$. Since S is of finite presentation J is a finitely generated ideal (Lemma 10.6.2). Consider the short exact sequence

$$0 \to J \to S' \to S \to 0$$

As $S_\mathfrak{q}$ is flat over R we see that $J_{\mathfrak{q}'} \otimes_R k \to S'_{\mathfrak{q}'} \otimes_R k$ is injective (Lemma 10.39.12). However, by construction $S'_{\mathfrak{q}'} \otimes_R k$ maps isomorphically to $S_\mathfrak{q} \otimes_R k$. Hence we conclude that $J_{\mathfrak{q}'} \otimes_R k = J_{\mathfrak{q}'}/\mathfrak{p}J_{\mathfrak{q}'} = 0$. By Nakayama's lemma (Lemma 10.20.1) we conclude that there exists a $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}'$ such that $J_g = 0$. In other words $S'_g \cong S_g$. After further localizing we see that S' (and hence S) becomes a relative global complete intersection by Lemma 10.136.10 as desired. □

07BT **Lemma 10.136.16.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/I$ for some finitely generated ideal I. If $g \in S$ is such that S_g is syntomic over R, then $(I/I^2)_g$ is a finite projective S_g-module.*

Proof. By Lemma 10.136.15 there exist finitely many elements $g_1, \ldots, g_m \in S$ which generate the unit ideal in S_g such that each S_{gg_j} is a relative global complete intersection over R. Since it suffices to prove that $(I/I^2)_{gg_j}$ is finite projective, see Lemma 10.78.2, we may assume that S_g is a relative global complete intersection. In this case the result follows from Lemmas 10.134.16 and 10.136.12. □

00SZ **Lemma 10.136.17.** *Let $R \to S$, $S \to S'$ be ring maps.*

(1) *If $R \to S$ and $S \to S'$ are syntomic, then $R \to S'$ is syntomic.*
(2) *If $R \to S$ and $S \to S'$ are relative global complete intersections, then $R \to S'$ is a relative global complete intersection.*

Proof. Proof of (2). Say $R \to S$ and $S \to S'$ are relative global complete intersections and we have presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$ as in Definition 10.136.5. Then

$$S' \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, h'_1, \ldots, h'_d)$$

for some lifts $h'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ of the h_j. Hence it suffices to bound the dimensions of the fibre rings. Thus we may assume $R = k$ is a field. In this case we see that we have a ring, namely S, which is of finite type over k and equidimensional of dimension $n - c$, and a finite type ring map $S \to S'$ all of whose nonempty fibre rings are equidimensional of dimension $m - d$. Then, by Lemma 10.112.6 for example applied to localizations at maximal ideals of S', we see that $\dim(S') \leq n - c + m - d$ as desired.

We will reduce part (1) to part (2). Assume $R \to S$ and $S \to S'$ are syntomic. Let $\mathfrak{q}' \subset S$ be a prime ideal lying over $\mathfrak{q} \subset S$. By Lemma 10.136.15 there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S \to S'_{g'}$ is a relative global complete intersection. Similarly, we find $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is a relative global complete intersection. By Lemma 10.136.9 the ring map $S_g \to S_{gg'}$ is a relative global complete intersection. By part (2) we see that $R \to S_{gg'}$ is a relative global complete intersection and $gg' \notin \mathfrak{q}'$. Since $\mathfrak{q}'$ was arbitrary combining Lemmas 10.136.15 and 10.136.4 we see that $R \to S'$ is syntomic (this also uses that the spectrum of S' is quasi-compact, see Lemma 10.17.8). □

The following lemma will be improved later, see Smoothing Ring Maps, Proposition 16.3.2.

00T0 **Lemma 10.136.18.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a syntomic map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some relative global complete intersection S_i over R.*

Proof. By Lemma 10.136.15 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is a relative global complete intersection over R/I. Hence we may assume that $\overline{S}$ is a relative global complete intersection. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.136.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Note that $S/IS \cong \overline{S}$. By Lemma 10.136.10 we can find $g \in S$ mapping to 1 in $\overline{S}$ such that S_g is a relative global complete intersection over R. Since $\overline{S} \cong S_g/IS_g$ this finishes the proof. □

10.137. Smooth ring maps

00T1 Let us motivate the definition of a smooth ring map by an example. Suppose R is a ring and $S = R[x, y]/(f)$ for some nonzero $f \in R[x, y]$. In this case there is an exact sequence

$$S \to S\mathrm{d}x \oplus S\mathrm{d}y \to \Omega_{S/R} \to 0$$

where the first arrow maps 1 to $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ see Section 10.134. We conclude that $\Omega_{S/R}$ is locally free of rank 1 if the partial derivatives of f generate the unit ideal in S. In this case S is smooth of relative dimension 1 over R. But it can happen that $\Omega_{S/R}$ is locally free of rank 2 namely if both partial derivatives of f are zero. For example if for a prime p we have $p = 0$ in R and $f = x^p + y^p$ then this happens. Here $R \to S$ is a relative global complete intersection of relative dimension 1 which is not smooth. Hence, in order to check that a ring map is smooth it is not sufficient to check whether the module of differentials is free. The correct condition is the following.

00T2 **Definition 10.137.1.** A ring map $R \to S$ is *smooth* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0: this means that $H_1(NL_{S/R}) = 0$ and that $\Omega_{S/R}$ is a finite projective S-module.

Let $R \to S$ be a ring map. By Lemma 10.134.2 for any presentation $\alpha : P \to S$ we have $H_1(NL(\alpha)) = H_1(NL_{S/R})$ and $H_0(NL(\alpha)) = \Omega_{S/R}$. Thus, if $R \to S$ is smooth, then for any surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I the map

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$$

is an injective map whose cokernel is a finite projective S-module. Thus the displayed arrow is a split injection. In other words $\bigoplus_{i=1}^n S\mathrm{d}x_i \cong I/I^2 \oplus \Omega_{S/R}$ as S-modules and I/I^2 is a finite projective S-module as well. Conversely, if $R \to S$ is of finite presentation and we have a surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I such that the displayed arrow is a split injection, then $R \to S$ is smooth.

00T3 **Lemma 10.137.2.** *Let $R \to S$ be a smooth ring map. Any localization S_g is smooth over R. If $f \in R$ maps to an invertible element of S, then $R_f \to S$ is smooth.*

Proof. By Lemma 10.134.13 the naive cotangent complex for S_g over R is the base change of the naive cotangent complex of S over R. The assumption is that the naive cotangent complex of S/R is $\Omega_{S/R}$ and that this is a finite projective S-module. Hence so is its base change. Thus S_g is smooth over R.

The second assertion follows in the same way from Lemma 10.134.11. □

00T4 **Lemma 10.137.3.** *Let $R \to S$ be a smooth ring map. Let $R \to R'$ be any ring map. Then the base change $R' \to S' = R' \otimes_R S$ is smooth.*

Proof. Let $\alpha : R[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : R'[x_1, \ldots, x_n] \to R' \otimes_R S$ be the induced presentation. Let $I' = \mathrm{Ker}(\alpha')$. Since $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ is exact, the sequence $R' \otimes_R I \to R'[x_1, \ldots, x_n] \to R' \otimes_R S \to 0$ is exact. Thus $R' \otimes_R I \to I'$ is surjective. By Definition 10.137.1 there is a short exact sequence

$$0 \to I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

and the S-module $\Omega_{S/R}$ is finite projective. In particular I/I^2 is a direct summand of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$. Consider the commutative diagram

$$\begin{array}{ccc} R' \otimes_R (I/I^2) & \longrightarrow & R' \otimes_R (\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S) \\ \downarrow & & \downarrow \\ I'/(I')^2 & \longrightarrow & \Omega_{R'[x_1,\ldots,x_n]/R'} \otimes_{R'[x_1,\ldots,x_n]} (R' \otimes_R S) \end{array}$$

Since the right vertical map is an isomorphism we see that the left vertical map is injective and surjective by what was said above. Thus we conclude that $NL(\alpha')$ is quasi-isomorphic to $\Omega_{S'/R'} \cong S' \otimes_S \Omega_{S/R}$ placed in degree 0. This module is finite projective since it is the base change of a finite projective module. □

00T5 **Lemma 10.137.4.** *Let k be a field. Let S be a smooth k-algebra. Then S is a local complete intersection.*

Proof. By Lemmas 10.137.3 and 10.135.11 it suffices to prove this when k is algebraically closed. Choose a presentation $\alpha : k[x_1, \ldots, x_n] \to S$ with kernel I. Let $\mathfrak{m}$ be a maximal ideal of S, and let $\mathfrak{m}' \supset I$ be the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. We will show that condition (5) of Lemma 10.135.4 holds (with $\mathfrak{m}$ instead of $\mathfrak{q}$). We may write $\mathfrak{m}' = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, because k is algebraically closed, see Theorem 10.34.1. By our assumption that $k \to S$ is smooth the S-module map $\mathrm{d} : I/I^2 \to \bigoplus_{i=1}^n S\mathrm{d}x_i$ is a split injection. Hence the corresponding map $I/\mathfrak{m}'I \to \bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i$ is injective. Say $\dim_{\kappa(\mathfrak{m}')}(I/\mathfrak{m}'I) = c$ and pick $f_1, \ldots, f_c \in I$ which map to a $\kappa(\mathfrak{m}')$-basis of $I/\mathfrak{m}'I$. By Nakayama's Lemma 10.20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{m}'}$ over $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Consider the commutative diagram

$$\begin{array}{ccccc} I & \longrightarrow & I/I^2 & \longrightarrow & I/\mathfrak{m}'I \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{k[x_1,\ldots,x_n]/k} & \longrightarrow & \bigoplus S\mathrm{d}x_i & \xrightarrow{\mathrm{d}x_i \mapsto x_i - a_i} & \mathfrak{m}'/(\mathfrak{m}')^2 \end{array}$$

(proof commutativity omitted). The middle vertical map is the one defining the naive cotangent complex of α. Note that the right lower horizontal arrow induces an isomorphism $\bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i \to \mathfrak{m}'/(\mathfrak{m}')^2$. Hence our generators $f_1, \ldots, f_c$ of $I_{\mathfrak{m}'}$ map to a collection of elements in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ whose classes in $\mathfrak{m}'/(\mathfrak{m}')^2$ are linearly independent over $\kappa(\mathfrak{m}')$. Therefore they form a regular sequence in the ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ by Lemma 10.106.3. This verifies condition (5) of Lemma 10.135.4 hence S_g is a global complete intersection over k for some $g \in S$, $g \notin \mathfrak{m}$. As this works for any maximal ideal of S we conclude that S is a local complete intersection over k. □

00T6 **Definition 10.137.5.** Let R be a ring. Given integers $n \geq c \geq 0$ and $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ we say

$$R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a *standard smooth algebra over* R if the polynomial

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. We say an R-algebra S is *standard smooth* or that the ring map $R \to S$ is *standard smooth* if there exist $n \geq c \geq 0$ and $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ such that $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a standard smooth algebra over R and S is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as an R-algebra.

00T7 **Lemma 10.137.6.** *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R[x_1, \ldots, x_n]/I$ be a standard smooth algebra. Then*

1. *the ring map $R \to S$ is smooth,*
2. *the S-module $\Omega_{S/R}$ is free on $dx_{c+1}, \ldots, dx_n$,*
3. *the S-module I/I^2 is free on the classes of $f_1, \ldots, f_c$,*
4. *for any $g \in S$ the ring map $R \to S_g$ is standard smooth,*
5. *for any ring map $R \to R'$ the base change $R' \to R' \otimes_R S$ is standard smooth,*

(6) *if $f \in R$ maps to an invertible element in S, then $R_f \to S$ is standard smooth, and*
(7) *the ring S is a relative global complete intersection over R.*

Proof. Consider the naive cotangent complex of the given presentation

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1}^{n} S\mathrm{d}x_i$$

Let us compose this map with the projection onto the first c direct summands of the direct sum. According to the definition of a standard smooth algebra the classes $f_i \bmod (f_1, \ldots, f_c)^2$ map to a basis of $\bigoplus_{i=1}^{c} S\mathrm{d}x_i$. We conclude that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free of rank c with a basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$, and that the homology in degree 0, i.e., $\Omega_{S/R}$, of the naive cotangent complex is a free S-module with basis the images of $\mathrm{d}x_{c+j}$, $j = 1, \ldots, n-c$. In particular, this proves $R \to S$ is smooth.

The proofs of (4) and (6) are omitted. But see the example below and the proof of Lemma 10.136.9.

Let $\varphi : R \to R'$ be any ring map. Denote $S' = R'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_c^\varphi)$ where f^φ is the polynomial obtained from $f \in R[x_1, \ldots, x_n]$ by applying φ to all the coefficients. Then $S' \cong R' \otimes_R S$. Moreover, the determinant of Definition 10.137.5 for S'/R' is equal to g^φ. Its image in S' is therefore the image of g via $R[x_1, \ldots, x_n] \to S \to S'$ and hence invertible. This proves (5).

To prove (7) it suffices to show that $S \otimes_R \kappa(\mathfrak{p})$ has dimension $n - c$ for every prime $\mathfrak{p} \subset R$. By (5) it suffices to prove that any standard smooth algebra $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over a field k has dimension $n - c$. We already know that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a local complete intersection by Lemma 10.137.4. Hence, since I/I^2 is free of rank c we see that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ has dimension $n - c$, by Lemma 10.135.4 for example. □

00T8 **Example 10.137.7.** Let R be a ring. Let $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Let

$$h = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}.$$

Set $S = R[x_1, \ldots, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}h - 1)$. This is an example of a standard smooth algebra, except that the presentation is wrong and the variables should be in the following order: $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$.

00T9 **Lemma 10.137.8.** *A composition of standard smooth ring maps is standard smooth.*

Proof. Suppose that $R \to S$ and $S \to S'$ are standard smooth. We choose presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$. Choose elements $g_j' \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to the g_j. In this way we see $S' = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, g_1', \ldots, g_d')$. To show that S' is standard

smooth it suffices to verify that the determinant

$$\det \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_c/\partial x_1 & \partial g_1/\partial x_1 & \ldots & \partial g_d/\partial x_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \ldots & \partial f_c/\partial x_c & \partial g_1/\partial x_c & \ldots & \partial g_d/\partial x_c \\ 0 & \ldots & 0 & \partial g_1/\partial y_1 & \ldots & \partial g_d/\partial y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \partial g_1/\partial y_d & \ldots & \partial g_d/\partial y_d \end{pmatrix}$$

is invertible in S'. This is clear since it is the product of the two determinants which were assumed to be invertible by hypothesis. □

00TA **Lemma 10.137.9.** *Let $R \to S$ be a smooth ring map. There exists an open covering of $\mathrm{Spec}(S)$ by standard opens $D(g)$ such that each S_g is standard smooth over R. In particular $R \to S$ is syntomic.*

Proof. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_m)$. For every subset $E \subset \{1, \ldots, m\}$ consider the open subset U_E where the classes $f_e, e \in E$ freely generate the finite projective S-module I/I^2, see Lemma 10.79.4. We may cover $\mathrm{Spec}(S)$ by standard opens $D(g)$ each completely contained in one of the opens U_E. For such a g we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \longrightarrow S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we use Lemma 10.134.12 to see that $J/J^2 \cong (I/I^2)_g \oplus S_g$ is free. We may and do replace S by S_g. Then using Lemma 10.136.6 we may assume we have a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_c)$ such that I/I^2 is free on the classes of $f_1, \ldots, f_c$.

Using the presentation α obtained at the end of the previous paragraph, we more or less repeat this argument with the basis elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ of $\Omega_{R[x_1,\ldots,x_n]/R}$. Namely, for any subset $E \subset \{1, \ldots, n\}$ of cardinality c we may consider the open subset U_E of $\mathrm{Spec}(S)$ where the differential of $NL(\alpha)$ composed with the projection

$$S^{\oplus c} \cong I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \longrightarrow \bigoplus\nolimits_{i \in E} S\mathrm{d}x_i$$

is an isomorphism. Again we may find a covering of $\mathrm{Spec}(S)$ by (finitely many) standard opens $D(g)$ such that each $D(g)$ is completely contained in one of the opens U_E. By renumbering, we may assume $E = \{1, \ldots, c\}$. For a g with $D(g) \subset U_E$ we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \to S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we conclude from Lemma 10.134.12 that $J = (f_1, \ldots, f_c, fx_{n+1} - 1)$ where $\alpha(f) = g$ and that the composition

$$J/J^2 \longrightarrow \Omega_{R[x_1,\ldots,x_{n+1}]/R} \otimes_{R[x_1,\ldots,x_{n+1}]} S_g \longrightarrow \bigoplus\nolimits_{i=1}^{c} S_g\mathrm{d}x_i \oplus S_g\mathrm{d}x_{n+1}$$

is an isomorphism. Reordering the coordinates as $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$ we conclude that S_g is standard smooth over R as desired.

This finishes the proof as standard smooth algebras are syntomic (Lemmas 10.137.6 and 10.136.13) and being syntomic over R is local on S (Lemma 10.136.4). □

00TB **Definition 10.137.10.** Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S. We say $R \to S$ is *smooth at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth.

For ring maps of finite presentation we can characterize this as follows.

07BU **Lemma 10.137.11.** *Let $R \to S$ be of finite presentation. Let $\mathfrak{q}$ be a prime of S. The following are equivalent*

(1) *$R \to S$ is smooth at $\mathfrak{q}$,*
(2) *$H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a finite free $S_\mathfrak{q}$-module,*
(3) *$H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a projective $S_\mathfrak{q}$-module, and*
(4) *$H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a flat $S_\mathfrak{q}$-module.*

Proof. We will use without further mention that formation of the naive cotangent complex commutes with localization, see Section 10.134, especially Lemma 10.134.13. Note that $\Omega_{S/R}$ is a finitely presented S-module, see Lemma 10.131.15. Hence (2), (3), and (4) are equivalent by Lemma 10.78.2. It is clear that (1) implies the equivalent conditions (2), (3), and (4). Assume (2) holds. Writing $S_\mathfrak{q}$ as the colimit of principal localizations we see from Lemma 10.127.6 that we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g$ is finite free. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I. We may work with $NL(\alpha)$ instead of $NL_{S/R}$, see Lemma 10.134.2. The surjection

$$\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \to \Omega_{S/R} \to 0$$

has a right inverse after inverting g because $(\Omega_{S/R})_g$ is projective. Hence the image of $\mathrm{d} : (I/I^2)_g \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S_g$ is a direct summand and this map has a right inverse too. We conclude that $H_1(L_{S/R})_g$ is a quotient of $(I/I^2)_g$. In particular $H_1(L_{S/R})_g$ is a finite S_g-module. Thus the vanishing of $H_1(L_{S/R})_\mathfrak{q}$ implies the vanishing of $H_1(L_{S/R})_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}$. Then $R \to S_{gg'}$ is smooth by definition. □

00TC **Lemma 10.137.12.** *Let $R \to S$ be a ring map. Then $R \to S$ is smooth if and only if $R \to S$ is smooth at every prime $\mathfrak{q}$ of S.*

Proof. The direct implication is trivial. Suppose that $R \to S$ is smooth at every prime $\mathfrak{q}$ of S. Since $\mathrm{Spec}(S)$ is quasi-compact, see Lemma 10.17.8, there exists a finite covering $\mathrm{Spec}(S) = \bigcup D(g_i)$ such that each S_{g_i} is smooth. By Lemma 10.23.3 this implies that S is of finite presentation over R. According to Lemma 10.134.13 we see that $NL_{S/R} \otimes_S S_{g_i}$ is quasi-isomorphic to a finite projective S_{g_i}-module placed in degree 0. By Lemma 10.78.2 this implies that $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0. □

00TD **Lemma 10.137.13.** *A composition of smooth ring maps is smooth.*

Proof. You can prove this in many different ways. One way is to use the snake lemma (Lemma 10.4.1), the Jacobi-Zariski sequence (Lemma 10.134.4), combined with the characterization of projective modules as being direct summands of free modules (Lemma 10.77.2). Another proof can be obtained by combining Lemmas 10.137.9, 10.137.8 and 10.137.12. □

0GIF **Lemma 10.137.14.** *Let R be a ring. Let $S = S' \times S''$ be a product of R-algebras. Then S is smooth over R if and only if both S' and S'' are smooth over R.*

Proof. Omitted. Hints: By Lemma 10.137.12 we can check smoothness one prime at a time. Since $\mathrm{Spec}(S)$ is the disjoint union of $\mathrm{Spec}(S')$ and $\mathrm{Spec}(S'')$ by Lemma 10.21.2 we find that smoothness of $R \to S$ at $\mathfrak{q}$ corresponds to either smoothness of $R \to S'$ at the corresponding prime or smoothness of $R \to S''$ at the corresponding prime. □

00TE **Lemma 10.137.15.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Let $\mathfrak{q} \subset S$ be a prime. Then $R \to S$ is smooth at $\mathfrak{q}$ if and only if there exists a subset $I \subset \{1, \ldots, n\}$ of cardinality c such that the polynomial*

$$g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I}.$$

does not map to an element of $\mathfrak{q}$.

Proof. By Lemma 10.136.12 we see that the naive cotangent complex associated to the given presentation of S is the complex

$$\bigoplus\nolimits_{j=1}^{c} S \cdot f_j \longrightarrow \bigoplus\nolimits_{i=1}^{n} S \cdot \mathrm{d}x_i, \quad f_j \longmapsto \sum \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i.$$

The maximal minors of the matrix giving the map are exactly the polynomials g_I.

Assume g_I maps to $g \in S$, with $g \notin \mathfrak{q}$. Then the algebra S_g is smooth over R. Namely, its naive cotangent complex is quasi-isomorphic to the complex above localized at g, see Lemma 10.134.13. And by construction it is quasi-isomorphic to a free rank $n - c$ module in degree 0.

Conversely, suppose that all g_I end up in $\mathfrak{q}$. In this case the complex above tensored with $\kappa(\mathfrak{q})$ does not have maximal rank, and hence there is no localization by an element $g \in S$, $g \notin \mathfrak{q}$ where this map becomes a split injection. By Lemma 10.134.13 again there is no such localization which is smooth over R. □

00TF **Lemma 10.137.16.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

1. *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is of finite presentation,*
2. *the local ring homomorphism $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat,*
3. *the fibre $S \otimes_R \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$ at the prime corresponding to $\mathfrak{q}$.*

Then $R \to S$ is smooth at $\mathfrak{q}$.

Proof. By Lemmas 10.136.15 and 10.137.4 we see that there exists a $g \in S$ such that S_g is a relative global complete intersection. Replacing S by S_g we may assume $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection. For any subset $I \subset \{1, \ldots, n\}$ of cardinality c consider the polynomial $g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c, i \in I}$ of Lemma 10.137.15. Note that the image $\overline{g}_I$ of g_I in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$ is the determinant of the partial derivatives of the images $\overline{f}_j$ of the f_j in the ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$. Thus the lemma follows by applying Lemma 10.137.15 both to $R \to S$ and to $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$. □

Note that the sets U, V in the following lemma are open by definition.

00TG **Lemma 10.137.17.** *Let $R \to S$ be a ring map of finite presentation. Let $R \to R'$ be a flat ring map. Denote $S' = R' \otimes_R S$ the base change. Let $U \subset \mathrm{Spec}(S)$ be the set of primes at which $R \to S$ is smooth. Let $V \subset \mathrm{Spec}(S')$ the set of primes at which $R' \to S'$ is smooth. Then V is the inverse image of U under the map $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$.*

Proof. By Lemma 10.134.8 we see that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This already implies that $f^{-1}(U) \subset V$.

Let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{q} \subset S$. Assume $\mathfrak{q}' \in V$. We have to show that $\mathfrak{q} \in U$. Since $S \to S'$ is flat, we see that $S_\mathfrak{q} \to S'_{\mathfrak{q}'}$ is faithfully flat (Lemma

10.39.17). Thus the vanishing of $H_1(L_{S'/R'})_{\mathfrak{q}'}$ implies the vanishing of $H_1(L_{S/R})_{\mathfrak{q}}$. By Lemma 10.78.6 applied to the $S_{\mathfrak{q}}$-module $(\Omega_{S/R})_{\mathfrak{q}}$ and the map $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ we see that $(\Omega_{S/R})_{\mathfrak{q}}$ is projective. Hence $R \to S$ is smooth at $\mathfrak{q}$ by Lemma 10.137.11. □

02UQ **Lemma 10.137.18.** *Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then S is smooth over k at $\mathfrak{q}$ if and only if S_K is smooth at $\mathfrak{q}_K$ over K.*

Proof. This is a special case of Lemma 10.137.17. □

04B1 **Lemma 10.137.19.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a smooth ring map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some (standard) smooth ring S_i over R.*

Proof. By Lemma 10.137.9 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is standard smooth over R/I. Hence we may assume that $\overline{S}$ is standard smooth over R/I. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.137.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_j}{\partial x_i})_{i,j=1,\ldots,c}$ as in Example 10.137.7. This proves the lemma. □

10.138. Formally smooth maps

00TH In this section we define formally smooth ring maps. It will turn out that a ring map of finite presentation is formally smooth if and only if it is smooth, see Proposition 10.138.13.

00TI **Definition 10.138.1.** Let $R \to S$ be a ring map. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

00TJ **Lemma 10.138.2.** *Let $R \to S$ be a formally smooth ring map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally smooth over R'.*

Proof. Let a solid diagram

$$\begin{array}{ccccc} S & \longrightarrow & R' \otimes_R S & \longrightarrow & A/I \\ \uparrow & & \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & A \end{array}$$

as in Definition 10.138.1 be given. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

031H **Lemma 10.138.3.** *A composition of formally smooth ring maps is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

00TK **Lemma 10.138.4.** *A polynomial ring over R is formally smooth over R.*

Proof. Suppose we have a diagram as in Definition 10.138.1 with $S = R[x_j; j \in J]$. Then there exists a dotted arrow simply by choosing lifts $a_j \in A$ of the elements in A/I to which the elements x_j map to under the top horizontal arrow. □

00TL **Lemma 10.138.5.** *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra map $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.*

Proof. Assume $R \to S$ is formally smooth. Consider the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & P/J \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & P/J^2 \end{array}$$

By assumption the dotted arrow exists. This proves that σ exists.

Conversely, suppose we have a σ as in the lemma. Let a solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Definition 10.138.1 be given. Because P is formally smooth by Lemma 10.138.4, there exists an R-algebra homomorphism $\psi : P \to A$ which lifts the map $P \to S \to A/I$. Clearly $\psi(J) \subset I$ and since $I^2 = 0$ we conclude that $\psi(J^2) = 0$. Hence ψ factors as $\overline{\psi} : P/J^2 \to A$. The desired dotted arrow is the composition $\overline{\psi} \circ \sigma : S \to A$. □

00TM **Remark 10.138.6.** Lemma 10.138.5 holds more generally whenever P is formally smooth over R.

031I **Lemma 10.138.7.** *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if the sequence*

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

of Lemma 10.131.9 is a split exact sequence.

Proof. Assume S is formally smooth over R. By Lemma 10.138.5 this means there exists an R-algebra map $S \to P/J^2$ which is a right inverse to the canonical map $P/J^2 \to S$. By Lemma 10.131.11 we have $\Omega_{P/R} \otimes_P S = \Omega_{(P/J^2)/R} \otimes_{P/J^2} S$. By Lemma 10.131.10 the sequence is split.

Assume the exact sequence of the lemma is split exact. Choose a splitting $\sigma : \Omega_{S/R} \to \Omega_{P/R} \otimes_P S$. For each $\lambda \in S$ choose $x_\lambda \in P$ which maps to λ. Next, for each $\lambda \in S$ choose $f_\lambda \in J$ such that

$$\mathrm{d}f_\lambda = \mathrm{d}x_\lambda - \sigma(\mathrm{d}\lambda)$$

in the middle term of the exact sequence. We claim that $s : \lambda \mapsto x_\lambda - f_\lambda \mod J^2$ is an R-algebra homomorphism $s : S \to P/J^2$. To prove this we will repeatedly

use that if $h \in J$ and $\mathrm{d}h = 0$ in $\Omega_{P/R} \otimes_R S$, then $h \in J^2$. Let $\lambda, \mu \in S$. Then $\sigma(\mathrm{d}\lambda + \mathrm{d}\mu - \mathrm{d}(\lambda + \mu)) = 0$. This implies

$$\mathrm{d}(x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu}) = 0$$

which means that $x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu} \in J^2$, which in turn means that $s(\lambda) + s(\mu) = s(\lambda + \mu)$. Similarly, we have $\sigma(\lambda \mathrm{d}\mu + \mu \mathrm{d}\lambda - \mathrm{d}\lambda\mu) = 0$ which implies that

$$\mu(\mathrm{d}x_\lambda - \mathrm{d}f_\lambda) + \lambda(\mathrm{d}x_\mu - \mathrm{d}f_\mu) - \mathrm{d}x_{\lambda\mu} + \mathrm{d}f_{\lambda\mu} = 0$$

in the middle term of the exact sequence. Moreover we have

$$\mathrm{d}(x_\lambda x_\mu) = x_\lambda \mathrm{d}x_\mu + x_\mu \mathrm{d}x_\lambda = \lambda \mathrm{d}x_\mu + \mu \mathrm{d}x_\lambda$$

in the middle term again. Combined these equations mean that $x_\lambda x_\mu - x_{\lambda\mu} - \mu f_\lambda - \lambda f_\mu + f_{\lambda\mu} \in J^2$, hence $(x_\lambda - f_\lambda)(x_\mu - f_\mu) - (x_{\lambda\mu} - f_{\lambda\mu}) \in J^2$ as $f_\lambda f_\mu \in J^2$, which means that $s(\lambda)s(\mu) = s(\lambda\mu)$. If $\lambda \in R$, then $\mathrm{d}\lambda = 0$ and we see that $\mathrm{d}f_\lambda = \mathrm{d}x_\lambda$, hence $\lambda - x_\lambda + f_\lambda \in J^2$ and hence $s(\lambda) = \lambda$ as desired. At this point we can apply Lemma 10.138.5 to conclude that S/R is formally smooth. □

031J **Proposition 10.138.8.** *Let $R \to S$ be a ring map. Consider a formally smooth R-algebra P and a surjection $P \to S$ with kernel J. The following are equivalent*

1. *S is formally smooth over R,*
2. *for some $P \to S$ as above there exists a section to $P/J^2 \to S$,*
3. *for all $P \to S$ as above there exists a section to $P/J^2 \to S$,*
4. *for some $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact,*
5. *for all $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact, and*
6. *the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0: this means that $H_1(NL_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module.*

Proof. It is clear that (1) implies (3) implies (2), see first part of the proof of Lemma 10.138.5. It is also true that (3) implies (5) implies (4) and that (2) implies (4), see first part of the proof of Lemma 10.138.7. Finally, Lemma 10.138.7 applied to the canonical surjection $R[S] \to S$ (10.134.0.1) shows that (1) implies (6).

Assume (4) and let's prove (6). Consider the sequence of Lemma 10.134.4 associated to the ring maps $R \to P \to S$. By the implication (1) $\Rightarrow$ (6) proved above we see that $NL_{P/R} \otimes_R S$ is quasi-isomorphic to $\Omega_{P/R} \otimes_P S$ placed in degree 0. Hence $H_1(NL_{P/R} \otimes_P S) = 0$. Since $P \to S$ is surjective we see that $NL_{S/P}$ is homotopy equivalent to J/J^2 placed in degree 1 (Lemma 10.134.6). Thus we obtain the exact sequence $0 \to H_1(L_{S/R}) \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$. By assumption we see that $H_1(L_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module. Thus (6) follows.

Finally, let's prove that (6) implies (1). The assumption means that the complex $J/J^2 \to \Omega_{P/R} \otimes S$ where $P = R[S]$ and $P \to S$ is the canonical surjection (10.134.0.1) is quasi-isomorphic to a projective S-module placed in degree 0. Hence Lemma 10.138.7 shows that S is formally smooth over R. □

031K **Lemma 10.138.9.** *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is formally smooth. Then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.131.7 is a split short exact sequence.

Proof. Follows from Proposition 10.138.8 and Lemma 10.134.4. □

06A6 **Lemma 10.138.10.** *Let $A \to B \to C$ be ring maps with $A \to C$ formally smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.131.9 is split exact.

Proof. Follows from Proposition 10.138.8, Lemma 10.134.4, and Lemma 10.131.9. □

06A7 **Lemma 10.138.11.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ formally smooth. Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.134.7 is split exact.

Proof. Since $A \to B$ is formally smooth there exists a ring map $\sigma : B \to A/I^2$ whose composition with $A \to B$ equals the quotient map $A \to A/I^2$. Then σ induces a map $J/J^2 \to I/I^2$ which is inverse to the map $I/I^2 \to J/J^2$. □

031L **Lemma 10.138.12.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Assume*

(1) *$I^2 = 0$,*
(2) *$R \to S$ is flat, and*
(3) *$R/I \to S/IS$ is formally smooth.*

Then $R \to S$ is formally smooth.

Proof. Assume (1), (2) and (3). Let $P = R[\{x_t\}_{t \in T}] \to S$ be a surjection of R-algebras with kernel J. Thus $0 \to J \to P \to S \to 0$ is a short exact sequence of flat R-modules. This implies that $I \otimes_R S = IS$, $I \otimes_R P = IP$ and $I \otimes_R J = IJ$ as well as $J \cap IP = IJ$. We will use throughout the proof that

$$\Omega_{(S/IS)/(R/I)} = \Omega_{S/R} \otimes_S (S/IS) = \Omega_{S/R} \otimes_R R/I = \Omega_{S/R}/I\Omega_{S/R}$$

and similarly for P (see Lemma 10.131.12). By Lemma 10.138.7 the sequence

031M (10.138.12.1) $$0 \to J/(IJ + J^2) \to \Omega_{P/R} \otimes_P S/IS \to \Omega_{S/R} \otimes_S S/IS \to 0$$

is split exact. Of course the middle term is $\bigoplus_{t \in T} S/IS\mathrm{d}x_t$. Choose a splitting $\sigma : \Omega_{P/R} \otimes_P S/IS \to J/(IJ + J^2)$. For each $t \in T$ choose an element $f_t \in J$ which maps to $\sigma(\mathrm{d}x_t)$ in $J/(IJ + J^2)$. This determines a unique S-module map

$$\tilde{\sigma} : \Omega_{P/R} \otimes_R S = \bigoplus S\mathrm{d}x_t \longrightarrow J/J^2$$

with the property that $\tilde{\sigma}(\mathrm{d}x_t) = f_t$. As σ is a section to d the difference

$$\Delta = \mathrm{id}_{J/J^2} - \tilde{\sigma} \circ \mathrm{d}$$

is a self map $J/J^2 \to J/J^2$ whose image is contained in $(IJ + J^2)/J^2$. In particular $\Delta((IJ + J^2)/J^2) = 0$ because $I^2 = 0$. This means that Δ factors as

$$J/J^2 \to J/(IJ + J^2) \xrightarrow{\overline{\Delta}} (IJ + J^2)/J^2 \to J/J^2$$

where $\overline{\Delta}$ is a S/IS-module map. Using again that the sequence (10.138.12.1) is split, we can find a S/IS-module map $\overline{\delta} : \Omega_{P/R} \otimes_P S/IS \to (IJ + J^2)/J^2$ such

that $\overline{\delta} \circ d$ is equal to $\overline{\Delta}$. In the same manner as above the map $\overline{\delta}$ determines an S-module map $\delta : \Omega_{P/R} \otimes_P S \to J/J^2$. After replacing $\tilde{\sigma}$ by $\tilde{\sigma} + \delta$ a simple computation shows that $\Delta = 0$. In other words $\tilde{\sigma}$ is a section of $J/J^2 \to \Omega_{P/R} \otimes_P S$. By Lemma 10.138.7 we conclude that $R \to S$ is formally smooth. □

00TN **Proposition 10.138.13.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is of finite presentation and formally smooth,*
(2) *$R \to S$ is smooth.*

Proof. Follows from Proposition 10.138.8 and Definition 10.137.1. (Note that $\Omega_{S/R}$ is a finitely presented S-module if $R \to S$ is of finite presentation, see Lemma 10.131.15.) □

00TP **Lemma 10.138.14.** *Let $R \to S$ be a smooth ring map. Then there exists a subring $R_0 \subset R$ of finite type over $\mathbf{Z}$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$.*

Proof. We are going to use that smooth is equivalent to finite presentation and formally smooth, see Proposition 10.138.13. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and denote $I = (f_1, \ldots, f_m)$. Choose a right inverse $\sigma : S \to R[x_1, \ldots, x_n]/I^2$ to the projection to S as in Lemma 10.138.5. Choose $h_i \in R[x_1, \ldots, x_n]$ such that $\sigma(x_i \bmod I) = h_i \bmod I^2$. Since $x_i - h_i \in I$, there exist $b_{ij} \in R[x_1, \ldots, x_n]$ such that

$$x_i - h_i = \sum\nolimits_j b_{ij} f_j$$

The fact that σ is an R-algebra homomorphism $R[x_1, \ldots, x_n]/I \to R[x_1, \ldots, x_n]/I^2$ is equivalent to the condition that

$$f_j(h_1, \ldots, h_n) = \sum\nolimits_{j_1 j_2} a_{j_1 j_2} f_{j_1} f_{j_2}$$

for certain $a_{kl} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated over $\mathbf{Z}$ by all the coefficients of the polynomials f_j, h_i, a_{kl}, b_{ij}. Set $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with $I_0 = (f_1, \ldots, f_m)$. Since the second displayed equation holds in $R_0[x_1, \ldots, x_n]$ we can let $\sigma_0 : S_0 \to R_0[x_1, \ldots, x_n]/I_0^2$ be the R_0-algebra map defined by the rule $x_i \mapsto h_i \bmod I_0^2$. Since the first displayed equation holds in $R_0[x_1, \ldots, x_n]$ we see that σ_0 is a right inverse to the projection $R_0[x_1, \ldots, x_n]/I_0^2 \to R_0[x_1, \ldots, x_n]/I_0 = S_0$. Thus by Lemma 10.138.5 the ring S_0 is formally smooth over R_0. □

0CAQ **Lemma 10.138.15.** *Let $A = \operatorname{colim} A_i$ be a filtered colimit of rings. Let $A \to B$ be a smooth ring map. There exists an i and a smooth ring map $A_i \to B_i$ such that $B = B_i \otimes_{A_i} A$.*

Proof. Follows from Lemma 10.138.14 since $R_0 \to A$ will factor through A_i for some i by Lemma 10.127.3. □

06CM **Lemma 10.138.16.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = S \otimes_R R'$. Then $R \to S$ is formally smooth if and only if $R' \to S'$ is formally smooth.*

Proof. If $R \to S$ is formally smooth, then $R' \to S'$ is formally smooth by Lemma 10.138.2. To prove the converse, assume $R' \to S'$ is formally smooth. Note that $N \otimes_R R' = N \otimes_S S'$ for any S-module N. In particular $S \to S'$ is faithfully flat also. Choose a polynomial ring $P = R[\{x_i\}_{i \in I}]$ and a surjection of R-algebras $P \to S$ with kernel J. Note that $P' = P \otimes_R R'$ is a polynomial algebra over R'. Since

$R \to R'$ is flat the kernel J' of the surjection $P' \to S'$ is $J \otimes_R R'$. Hence the split exact sequence (see Lemma 10.138.7)

$$0 \to J'/(J')^2 \to \Omega_{P'/R'} \otimes_{P'} S' \to \Omega_{S'/R'} \to 0$$

is the base change via $S \to S'$ of the corresponding sequence

$$J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

see Lemma 10.131.9. As $S \to S'$ is faithfully flat we conclude two things: (1) this sequence (without $'$) is exact too, and (2) $\Omega_{S/R}$ is a projective S-module. Namely, $\Omega_{S'/R'}$ is projective as a direct summand of the free module $\Omega_{P'/R'} \otimes_{P'} S'$ and $\Omega_{S/R} \otimes_S S' = \Omega_{S'/R'}$ by what we said above. Thus (2) follows by descent of projectivity through faithfully flat ring maps, see Theorem 10.95.6. Hence the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ is exact also and we win by applying Lemma 10.138.7 once more. □

It turns out that smooth ring maps satisfy the following strong lifting property.

07K4 **Lemma 10.138.17.** *Let $R \to S$ be a smooth ring map. Given a commutative solid diagram*

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is a locally nilpotent ideal, a dotted arrow exists which makes the diagram commute.

Proof. By Lemma 10.138.14 we can extend the diagram to a commutative diagram

$$\begin{array}{ccccc} S_0 & \longrightarrow & S & \longrightarrow & A/I \\ \uparrow & & \uparrow & \nearrow & \uparrow \\ R_0 & \longrightarrow & R & \longrightarrow & A \end{array}$$

with $R_0 \to S_0$ smooth, R_0 of finite type over $\mathbf{Z}$, and $S = S_0 \otimes_{R_0} R$. Let $x_1, \ldots, x_n \in S_0$ be generators of S_0 over R_0. Let $a_1, \ldots, a_n$ be elements of A which map to the same elements in A/I as the elements $x_1, \ldots, x_n$. Denote $A_0 \subset A$ the subring generated by the image of R_0 and the elements $a_1, \ldots, a_n$. Set $I_0 = A_0 \cap I$. Then $A_0/I_0 \subset A/I$ and $S_0 \to A/I$ maps into A_0/I_0. Thus it suffices to find the dotted arrow in the diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & A_0/I_0 \\ \uparrow & \nearrow & \uparrow \\ R_0 & \longrightarrow & A_0 \end{array}$$

The ring A_0 is of finite type over $\mathbf{Z}$ by construction. Hence A_0 is Noetherian, whence I_0 is nilpotent, see Lemma 10.32.5. Say $I_0^n = 0$. By Proposition 10.138.13 we can successively lift the R_0-algebra map $S_0 \to A_0/I_0$ to $S_0 \to A_0/I_0^2$, $S_0 \to A_0/I_0^3$, ..., and finally $S_0 \to A_0/I_0^n = A_0$. □

10.139. Smoothness and differentials

05D4 Some results on differentials and smooth ring maps.

04B2 **Lemma 10.139.1.** *Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence*

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.131.7 is exact.

Proof. This follows from the more general Lemma 10.138.9 because a smooth ring map is formally smooth, see Proposition 10.138.13. But it also follows directly from Lemma 10.134.4 since $H_1(L_{C/B}) = 0$ is part of the definition of smoothness of $B \to C$. □

06A8 **Lemma 10.139.2.** *Let $A \to B \to C$ be ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.131.9 is split exact.

Proof. This follows from the more general Lemma 10.138.10 because a smooth ring map is formally smooth, see Proposition 10.138.13. □

06A9 **Lemma 10.139.3.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ smooth. Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.134.7 is exact.

Proof. This follows from the more general Lemma 10.138.11 because a smooth ring map is formally smooth, see Proposition 10.138.13. □

05D5 **Lemma 10.139.4.** *Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = \mathrm{Ker}(\sigma)$. Then*

1. *I/I^2 is a finite locally free R-module, and*
2. *if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-algebras, where $S^\wedge$ is the I-adic completion of S.*

Proof. By Lemma 10.131.10 applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds. If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i.$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n.

Namely, as S is formally smooth over R (by Proposition 10.138.13) we see that in the solid diagram

$$\begin{array}{ccc} S & \dashrightarrow & P/J^n \\ & \searrow^{\sigma_{n-1}} & \downarrow \\ & & P/J^{n-1} \end{array}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\overline{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^{n-1}. By construction the map Ψ_n is surjective and now $\overline{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma. □

10.140. Smooth algebras over fields

00TQ Warning: The following two lemmas do not hold over nonperfect fields in general.

00TR **Lemma 10.140.1.** *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. Then*

$$\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Consider the exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \to \Omega_{\kappa(\mathfrak{m})/k} \to 0$$

of Lemma 10.131.9. We would like to show that the first map is an isomorphism. Since k is algebraically closed the composition $k \to \kappa(\mathfrak{m})$ is an isomorphism by Theorem 10.34.1. So the surjection $S \to \kappa(\mathfrak{m})$ splits as a map of k-algebras, and Lemma 10.131.10 shows that the sequence above is exact on the left. Since $\Omega_{\kappa(\mathfrak{m})/k} = 0$, we win. □

00TS **Lemma 10.140.2.** *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. The following are equivalent:*

1. *The ring $S_{\mathfrak{m}}$ is a regular local ring.*
2. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \leq \dim(S_{\mathfrak{m}})$.*
3. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim(S_{\mathfrak{m}})$.*
4. *There exists a $g \in S$, $g \notin \mathfrak{m}$ such that S_g is smooth over k. In other words S/k is smooth at $\mathfrak{m}$.*

Proof. Note that (1), (2) and (3) are equivalent by Lemma 10.140.1 and Definition 10.110.7.

Assume that S is smooth at $\mathfrak{m}$. By Lemma 10.137.9 we see that S_g is standard smooth over k for a suitable $g \in S$, $g \notin \mathfrak{m}$. Hence by Lemma 10.137.6 we see that $\Omega_{S_g/k}$ is free of rank $\dim(S_g)$. Hence by Lemma 10.140.1 we see that $\dim(S_{\mathfrak{m}}) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ in other words $S_{\mathfrak{m}}$ is regular.

Conversely, suppose that $S_{\mathfrak{m}}$ is regular. Let $d = \dim(S_{\mathfrak{m}}) = \dim \mathfrak{m}/\mathfrak{m}^2$. Choose a presentation $S = k[x_1, \ldots, x_n]/I$ such that x_i maps to an element of $\mathfrak{m}$ for all i. In other words, $\mathfrak{m}'' = (x_1, \ldots, x_n)$ is the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Note that we have a short exact sequence

$$I/\mathfrak{m}''I \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}/(\mathfrak{m})^2 \to 0$$

Pick $c = n - d$ elements $f_1, \ldots, f_c \in I$ such that their images in $\mathfrak{m}''/(\mathfrak{m}'')^2$ span the kernel of the map to $\mathfrak{m}/\mathfrak{m}^2$. This is clearly possible. Denote $J = (f_1, \ldots, f_c)$. So $J \subset I$. Denote $S' = k[x_1, \ldots, x_n]/J$ so there is a surjection $S' \to S$. Denote $\mathfrak{m}' = \mathfrak{m}''S'$ the corresponding maximal ideal of S'. Hence we have

$$\begin{array}{ccccc} k[x_1, \ldots, x_n] & \longrightarrow & S' & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{m}'' & \longrightarrow & \mathfrak{m}' & \longrightarrow & \mathfrak{m} \end{array}$$

By our choice of J the exact sequence

$$J/\mathfrak{m}''J \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}'/(\mathfrak{m}')^2 \to 0$$

shows that $\dim(\mathfrak{m}'/(\mathfrak{m}')^2) = d$. Since $S'_{\mathfrak{m}'}$ surjects onto $S_{\mathfrak{m}}$ we see that $\dim(S_{\mathfrak{m}'}) \geq d$. Hence by the discussion preceding Definition 10.60.10 we conclude that $S'_{\mathfrak{m}'}$ is regular of dimension d as well. Because S' was cut out by $c = n - d$ equations we conclude that there exists a $g' \in S'$, $g' \notin \mathfrak{m}'$ such that $S'_{g'}$ is a global complete intersection over k, see Lemma 10.135.4. Also the map $S'_{\mathfrak{m}'} \to S_{\mathfrak{m}}$ is a surjection of Noetherian local domains of the same dimension and hence an isomorphism. Hence $S' \to S$ is surjective with finitely generated kernel and becomes an isomorphism after localizing at $\mathfrak{m}'$. Thus we can find $g' \in S'$, $g \notin \mathfrak{m}'$ such that $S'_{g'} \to S_{g'}$ is an isomorphism. All in all we conclude that after replacing S by a principal localization we may assume that S is a global complete intersection.

At this point we may write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. Recall that the naive cotangent complex of this algebra is given by

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i$$

see Lemma 10.136.12. By Lemma 10.137.15 in order to show that S is smooth at $\mathfrak{m}$ we have to show that one of the $c \times c$ minors g_I of the matrix "A" giving the map above does not vanish at $\mathfrak{m}$. By Lemma 10.140.1 the matrix A mod $\mathfrak{m}$ has rank c. Thus we win. □

00TT **Lemma 10.140.3.** *Let k be any field. Let S be a finite type k-algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$. The following are equivalent:*

1. *The k-algebra S is smooth at $\mathfrak{q}$ over k.*
2. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) \leq \dim_x X$.*
3. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) = \dim_x X$.*

Moreover, in this case the local ring $S_{\mathfrak{q}}$ is regular.

Proof. If S is smooth at $\mathfrak{q}$ over k, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is standard smooth over k, see Lemma 10.137.9. A standard smooth algebra over k has a module of differentials which is free of rank equal to the dimension, see Lemma 10.137.6 (use that a relative global complete intersection over a field has dimension equal to the number of variables minus the number of equations). Thus we see that (1) implies (3). To finish the proof of the lemma it suffices to show that (2) implies (1) and that it implies that $S_{\mathfrak{q}}$ is regular.

Assume (2). By Nakayama's Lemma 10.20.1 we see that $\Omega_{S/k,\mathfrak{q}}$ can be generated by $\leq \dim_x X$ elements. We may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ such that $\Omega_{S/k}$ is generated by at most $\dim_x X$ elements. Let K/k be an algebraically closed

field extension such that there exists a k-algebra map $\psi : \kappa(\mathfrak{q}) \to K$. Consider $S_K = K \otimes_k S$. Let $\mathfrak{m} \subset S_K$ be the maximal ideal corresponding to the surjection

$$S_K = K \otimes_k S \longrightarrow K \otimes_k \kappa(\mathfrak{q}) \xrightarrow{\mathrm{id}_K \otimes \psi} K.$$

Note that $\mathfrak{m} \cap S = \mathfrak{q}$, in other words $\mathfrak{m}$ lies over $\mathfrak{q}$. By Lemma 10.116.6 the dimension of $X_K = \mathrm{Spec}(S_K)$ at the point corresponding to $\mathfrak{m}$ is $\dim_x X$. By Lemma 10.114.6 this is equal to $\dim((S_K)_\mathfrak{m})$. By Lemma 10.131.12 the module of differentials of S_K over K is the base change of $\Omega_{S/k}$, hence also generated by at most $\dim_x X = \dim((S_K)_\mathfrak{m})$ elements. By Lemma 10.140.2 we see that S_K is smooth at $\mathfrak{m}$ over K. By Lemma 10.137.17 this implies that S is smooth at $\mathfrak{q}$ over k. This proves (1). Moreover, we know by Lemma 10.140.2 that the local ring $(S_K)_\mathfrak{m}$ is regular. Since $S_\mathfrak{q} \to (S_K)_\mathfrak{m}$ is flat we conclude from Lemma 10.110.9 that $S_\mathfrak{q}$ is regular. □

The following lemma can be significantly generalized (in several different ways).

00TU **Lemma 10.140.4.** *Let k be a field. Let R be a Noetherian local ring containing k. Assume that the residue field $\kappa = R/\mathfrak{m}$ is a finitely generated separable extension of k. Then the map*

$$d : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective.

Proof. We may replace R by $R/\mathfrak{m}^2$. Hence we may assume that $\mathfrak{m}^2 = 0$. By assumption we may write $\kappa = k(\overline{x}_1, \ldots, \overline{x}_r, \overline{y})$ where $\overline{x}_1, \ldots, \overline{x}_r$ is a transcendence basis of κ over k and $\overline{y}$ is separable algebraic over $k(\overline{x}_1, \ldots, \overline{x}_r)$. Say its minimal equation is $P(\overline{y}) = 0$ with $P(T) = T^d + \sum_{i<d} a_i T^i$, with $a_i \in k(\overline{x}_1, \ldots, \overline{x}_r)$ and $P'(\overline{y}) \neq 0$. Choose any lifts $x_i \in R$ of the elements $\overline{x}_i \in \kappa$. This gives a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \kappa \\ & \overset{\varphi}{\nwarrow} & \uparrow \\ & & k(\overline{x}_1, \ldots, \overline{x}_r) \end{array}$$

of k-algebras. We want to extend the left upwards arrow φ to a k-algebra map from κ to R. To do this choose any $y \in R$ lifting $\overline{y}$. To see that it defines a k-algebra map defined on $\kappa \cong k(\overline{x}_1, \ldots, \overline{x}_r)[T]/(P)$ all we have to show is that we may choose y such that $P^\varphi(y) = 0$. If not then we compute for $\delta \in \mathfrak{m}$ that

$$P(y + \delta) = P(y) + P'(y)\delta$$

because $\mathfrak{m}^2 = 0$. Since $P'(y)\delta = P'(\overline{y})\delta$ we see that we can adjust our choice as desired. This shows that $R \cong \kappa \oplus \mathfrak{m}$ as k-algebras! Now either a direct computation of $\Omega_{\kappa \oplus \mathfrak{m}/k}$ or an application of Lemma 10.131.10 finishes the proof. □

00TV **Lemma 10.140.5.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. The following are equivalent:*

1. *The algebra S is smooth at $\mathfrak{q}$ over k.*
2. *The ring $S_\mathfrak{q}$ is regular.*

Proof. Denote $R = S_\mathfrak{q}$ and denote its maximal by $\mathfrak{m}$ and its residue field κ. By Lemma 10.140.4 and 10.131.9 we see that there is a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0$$

Note that $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$, see Lemma 10.131.8. Moreover, since κ is separable over k we have $\dim_\kappa \Omega_{\kappa/k} = \mathrm{trdeg}_k(\kappa)$. Hence we get

$$\dim_\kappa \Omega_{R/k} \otimes_R \kappa = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 + \mathrm{trdeg}_k(\kappa) \geq \dim R + \mathrm{trdeg}_k(\kappa) = \dim_{\mathfrak{q}} S$$

(see Lemma 10.116.3 for the last equality) with equality if and only if R is regular. Thus we win by applying Lemma 10.140.3. □

00TW **Lemma 10.140.6.** *Let $R \to S$ be a $\mathbf{Q}$-algebra map. Let $f \in S$ be such that $\Omega_{S/R} = S\mathrm{d}f \oplus C$ for some S-submodule C. Then*

(1) *f is not nilpotent, and*
(2) *if S is a Noetherian local ring, then f is a nonzerodivisor in S.*

Proof. For $a \in S$ write $\mathrm{d}(a) = \theta(a)\mathrm{d}f + c(a)$ for some $\theta(a) \in S$ and $c(a) \in C$. Consider the R-derivation $S \to S$, $a \mapsto \theta(a)$. Note that $\theta(f) = 1$.

If $f^n = 0$ with $n > 1$ minimal, then $0 = \theta(f^n) = nf^{n-1}$ contradicting the minimality of n. We conclude that f is not nilpotent.

Suppose $fa = 0$. If f is a unit then $a = 0$ and we win. Assume f is not a unit. Then $0 = \theta(fa) = f\theta(a) + a$ by the Leibniz rule and hence $a \in (f)$. By induction suppose we have shown $fa = 0 \Rightarrow a \in (f^n)$. Then writing $a = f^n b$ we get $0 = \theta(f^{n+1}b) = (n+1)f^n b + f^{n+1}\theta(b)$. Hence $a = f^n b = -f^{n+1}\theta(b)/(n+1) \in (f^{n+1})$. Since in the Noetherian local ring S we have $\bigcap(f^n) = 0$, see Lemma 10.51.4 we win. □

The following is probably quite useless in applications.

00TX **Lemma 10.140.7.** *Let k be a field of characteristic 0. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. The following are equivalent:*

(1) *The algebra S is smooth at $\mathfrak{q}$ over k.*
(2) *The $S_\mathfrak{q}$-module $\Omega_{S/k,\mathfrak{q}}$ is (finite) free.*
(3) *The ring $S_\mathfrak{q}$ is regular.*

Proof. In characteristic zero any field extension is separable and hence the equivalence of (1) and (3) follows from Lemma 10.140.5. Also (1) implies (2) by definition of smooth algebras. Assume that $\Omega_{S/k,\mathfrak{q}}$ is free over $S_\mathfrak{q}$. We are going to use the notation and observations made in the proof of Lemma 10.140.5. So $R = S_\mathfrak{q}$ with maximal ideal $\mathfrak{m}$ and residue field κ. Our goal is to prove R is regular.

If $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = 0$ and $R \cong \kappa$. Hence R is regular and we win.

If $\mathfrak{m}/\mathfrak{m}^2 \neq 0$, then choose any $f \in \mathfrak{m}$ whose image in $\mathfrak{m}/\mathfrak{m}^2$ is not zero. By Lemma 10.140.4 we see that $\mathrm{d}f$ has nonzero image in $\Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$. By assumption $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$ is finite free and hence by Nakayama's Lemma 10.20.1 we see that $\mathrm{d}f$ generates a direct summand. We apply Lemma 10.140.6 to deduce that f is a nonzerodivisor in R. Furthermore, by Lemma 10.131.9 we get an exact sequence

$$(f)/(f^2) \to \Omega_{R/k} \otimes_R R/fR \to \Omega_{(R/fR)/k} \to 0$$

This implies that $\Omega_{(R/fR)/k}$ is finite free as well. Hence by induction we see that R/fR is a regular local ring. Since $f \in \mathfrak{m}$ was a nonzerodivisor we conclude that R is regular, see Lemma 10.106.7. □

00TY **Example 10.140.8.** Lemma 10.140.7 does not hold in characteristic $p > 0$. The standard examples are the ring maps

$$\mathbf{F}_p \longrightarrow \mathbf{F}_p[x]/(x^p)$$

whose module of differentials is free but is clearly not smooth, and the ring map ($p > 2$)

$$\mathbf{F}_p(t) \to \mathbf{F}_p(t)[x, y]/(x^p + y^2 + \alpha)$$

which is not smooth at the prime $\mathfrak{q} = (y, x^p + \alpha)$ but is regular.

Using the material above we can characterize smoothness at the generic point in terms of field extensions.

07ND **Lemma 10.140.9.** *Let $R \to S$ be an injective finite type ring map with R and S domains. Then $R \to S$ is smooth at $\mathfrak{q} = (0)$ if and only if the induced extension L/K of fraction fields is separable.*

Proof. Assume $R \to S$ is smooth at (0). We may replace S by S_g for some nonzero $g \in S$ and assume that $R \to S$ is smooth. Then $K \to S \otimes_R K$ is smooth (Lemma 10.137.3). Moreover, for any field extension K'/K the ring map $K' \to S \otimes_R K'$ is smooth as well. Hence $S \otimes_R K'$ is a regular ring by Lemma 10.140.3, in particular reduced. It follows that $S \otimes_R K$ is a geometrically reduced over K. Hence L is geometrically reduced over K, see Lemma 10.43.3. Hence L/K is separable by Lemma 10.44.2.

Conversely, assume that L/K is separable. We may assume $R \to S$ is of finite presentation, see Lemma 10.30.1. It suffices to prove that $K \to S \otimes_R K$ is smooth at (0), see Lemma 10.137.17. This follows from Lemma 10.140.5, the fact that a field is a regular ring, and the assumption that L/K is separable. □

10.141. Smooth ring maps in the Noetherian case

02HR

02HS **Definition 10.141.1.** Let $\varphi : B' \to B$ be a ring map. We say φ is a *small extension* if B' and B are local Artinian rings, φ is surjective and $I = \mathrm{Ker}(\varphi)$ has length 1 as a B'-module.

Clearly this means that $I^2 = 0$ and that $I = (x)$ for some $x \in B'$ such that $\mathfrak{m}'x = 0$ where $\mathfrak{m}' \subset B'$ is the maximal ideal.

02HT **Lemma 10.141.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p} \subset R$. Assume R is Noetherian and $R \to S$ of finite type. The following are equivalent:*

1. *$R \to S$ is smooth at $\mathfrak{q}$,*
2. *for every surjection of local R-algebras $(B', \mathfrak{m}') \to (B, \mathfrak{m})$ with $\mathrm{Ker}(B' \to B)$ having square zero and every solid commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B' \end{array}$$

   such that $\mathfrak{q} = S \cap \mathfrak{m}$ there exists a dotted arrow making the diagram commute,
3. *same as in (2) but with $B' \to B$ ranging over small extensions, and*

(4) *same as in (2) but with $B' \to B$ ranging over small extensions such that in addition $S \to B$ induces an isomorphism $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{m})$.*

Proof. Assume (1). This means there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. By Proposition 10.138.13 we know that $R \to S_g$ is formally smooth. Note that given any diagram as in (2) the map $S \to B$ factors automatically through $S_\mathfrak{q}$ and a fortiori through S_g. The formal smoothness of S_g over R gives us a morphism $S_g \to B'$ fitting into a similar diagram with S_g at the upper left corner. Composing with $S \to S_g$ gives the desired arrow. In other words, we have shown that (1) implies (2).

Clearly (2) implies (3) and (3) implies (4).

Assume (4). We are going to show that (1) holds, thereby finishing the proof of the lemma. Choose a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. This is possible as S is of finite type over R and therefore of finite presentation (see Lemma 10.31.4). Set $I = (f_1, \ldots, f_m)$. Consider the naive cotangent complex

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{j=1}^{n} S\mathrm{d}x_j$$

of this presentation (see Section 10.134). It suffices to show that when we localize this complex at $\mathfrak{q}$ then the map becomes a split injection, see Lemma 10.137.11. Denote $S' = R[x_1, \ldots, x_n]/I^2$. By Lemma 10.131.11 we have

$$S \otimes_{S'} \Omega_{S'/R} = S \otimes_{R[x_1,\ldots,x_n]} \Omega_{R[x_1,\ldots,x_n]/R} = \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j.$$

Thus the map

$$\mathrm{d} : I/I^2 \longrightarrow S \otimes_{S'} \Omega_{S'/R}$$

is the same as the map in the naive cotangent complex above. In particular the truth of the assertion we are trying to prove depends only on the three rings $R \to S' \to S$. Let $\mathfrak{q}' \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Since localization commutes with taking modules of differentials (Lemma 10.131.8) we see that it suffices to show that the map

02HU (10.141.2.1) $$\mathrm{d} : I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2 \longrightarrow S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

coming from $R \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is a split injection.

Let $N \in \mathbf{N}$ be an integer. Consider the ring

$$B'_N = S'_{\mathfrak{q}'}/(\mathfrak{q}')^N S'_{\mathfrak{q}'} = (S'/(\mathfrak{q}')^N S')_{\mathfrak{q}'}$$

and its quotient $B_N = B'_N/IB'_N$. Note that $B_N \cong S_\mathfrak{q}/\mathfrak{q}^N S_\mathfrak{q}$. Observe that B'_N is an Artinian local ring since it is the quotient of a local Noetherian ring by a power of its maximal ideal. Consider a filtration of the kernel I_N of $B'_N \to B_N$ by B'_N-submodules

$$0 \subset J_{N,1} \subset J_{N,2} \subset \ldots \subset J_{N,n(N)} = I_N$$

such that each successive quotient $J_{N,i}/J_{N,i-1}$ has length 1. (As B'_N is Artinian such a filtration exists.) This gives a sequence of small extensions

$$B'_N \to B'_N/J_{N,1} \to B'_N/J_{N,2} \to \ldots \to B'_N/J_{N,n(N)} = B'_N/I_N = B_N = S_\mathfrak{q}/\mathfrak{q}^N S_\mathfrak{q}$$

Applying condition (4) successively to these small extensions starting with the map $S \to B_N$ we see there exists a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B_N \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B'_N \end{array}$$

Clearly the ring map $S \to B'_N$ factors as $S \to S_\mathfrak{q} \to B'_N$ where $S_\mathfrak{q} \to B'_N$ is a local homomorphism of local rings. Moreover, since the maximal ideal of B'_N to the Nth power is zero we conclude that $S_\mathfrak{q} \to B'_N$ factors through $S_\mathfrak{q}/(\mathfrak{q})^N S_\mathfrak{q} = B_N$. In other words we have shown that for all $N \in \mathbf{N}$ the surjection of R-algebras $B'_N \to B_N$ has a splitting.

Consider the presentation

$$I_N \to B_N \otimes_{B'_N} \Omega_{B'_N/R} \to \Omega_{B_N/R} \to 0$$

coming from the surjection $B'_N \to B_N$ with kernel I_N (see Lemma 10.131.9). By the above the R-algebra map $B'_N \to B_N$ has a right inverse. Hence by Lemma 10.131.10 we see that the sequence above is split exact! Thus for every N the map

$$I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R}$$

is a split injection. The rest of the proof is gotten by unwinding what this means exactly. Note that

$$I_N = I_{\mathfrak{q}'}/(I_{\mathfrak{q}'}^2 + (\mathfrak{q}')^N \cap I_{\mathfrak{q}'})$$

By Artin-Rees (Lemma 10.51.2) we find a $c \geq 0$ such that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_N = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2$$

for all $N \geq c$ (these tensor product are just a fancy way of dividing by $\mathfrak{q}^{N-c}$). We may of course assume $c \geq 1$. By Lemma 10.131.11 we see that

$$S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

we can further tensor this by $B_N = S_\mathfrak{q}/\mathfrak{q}^N$ to see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}.$$

Since a split injection remains a split injection after tensoring with anything we see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} (10.141.2.1) = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}/\mathfrak{q}^N S_\mathfrak{q}} (I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R})$$

is a split injection for all $N \geq c$. By Lemma 10.74.1 we see that (10.141.2.1) is a split injection. This finishes the proof. □

10.142. Overview of results on smooth ring maps

00TZ Here is a list of results on smooth ring maps that we proved in the preceding sections. For more precise statements and definitions please consult the references given.

(1) A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex of S/R is quasi-isomorphic to a finite projective S-module in degree 0, see Definition 10.137.1.

(2) If S is smooth over R, then $\Omega_{S/R}$ is a finite projective S-module, see discussion following Definition 10.137.1.
(3) The property of being smooth is local on S, see Lemma 10.137.12.
(4) The property of being smooth is stable under base change, see Lemma 10.137.3.
(5) The property of being smooth is stable under composition, see Lemma 10.137.13.
(6) A smooth ring map is syntomic, in particular flat, see Lemma 10.137.9.
(7) A finitely presented, flat ring map with smooth fibre rings is smooth, see Lemma 10.137.16.
(8) A finitely presented ring map $R \to S$ is smooth if and only if it is formally smooth, see Proposition 10.138.13.
(9) If $R \to S$ is a finite type ring map with R Noetherian then to check that $R \to S$ is smooth it suffices to check the lifting property of formal smoothness along small extensions of Artinian local rings, see Lemma 10.141.2.
(10) A smooth ring map $R \to S$ is the base change of a smooth ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, see Lemma 10.138.14.
(11) Formation of the set of points where a ring map is smooth commutes with flat base change, see Lemma 10.137.17.
(12) If S is of finite type over an algebraically closed field k, and $\mathfrak{m} \subset S$ a maximal ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak{m}$,
   (b) $S_\mathfrak{m}$ is a regular local ring,
   (c) $\dim(S_\mathfrak{m}) = \dim_{\kappa(m)} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m})$.

   see Lemma 10.140.2.
(13) If S is of finite type over a field k, and $\mathfrak{q} \subset S$ a prime ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak{q}$,
   (b) $\dim_\mathfrak{q}(S/k) = \dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q})$.

   see Lemma 10.140.3.
(14) If S is smooth over a field, then all its local rings are regular, see Lemma 10.140.3.
(15) If S is of finite type over a field k, $\mathfrak{q} \subset S$ a prime ideal, the field extension $\kappa(\mathfrak{q})/k$ is separable and $S_\mathfrak{q}$ is regular, then S is smooth over k at $\mathfrak{q}$, see Lemma 10.140.5.
(16) If S is of finite type over a field k, if k has characteristic 0, if $\mathfrak{q} \subset S$ a prime ideal, and if $\Omega_{S/k,\mathfrak{q}}$ is free, then S is smooth over k at $\mathfrak{q}$, see Lemma 10.140.7.

Some of these results were proved using the notion of a standard smooth ring map, see Definition 10.137.5. This is the analogue of what a relative global complete intersection map is for the case of syntomic morphisms. It is also the easiest way to make examples.

10.143. Étale ring maps

00U0 An étale ring map is a smooth ring map whose relative dimension is equal to zero. This is the same as the following slightly more direct definition.

00U1 **Definition 10.143.1.** Let $R \to S$ be a ring map. We say $R \to S$ is *étale* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero: this means that $H_1(NL_{S/R}) = 0$ and $\Omega_{S/R} = 0$. Given a prime $\mathfrak{q}$ of S we say that $R \to S$ is *étale at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale.

In particular we see that $\Omega_{S/R} = 0$ if S is étale over R. If $R \to S$ is smooth, then $R \to S$ is étale if and only if $\Omega_{S/R} = 0$. From our results on smooth ring maps we automatically get a whole host of results for étale maps. We summarize these in Lemma 10.143.3 below. But before we do so we prove that *any* étale ring map is standard smooth.

00U9 **Lemma 10.143.2.** *Any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, then there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\det(\partial f_j/\partial x_i)$ is invertible in S.*

Proof. Let $R \to S$ be étale. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. As $R \to S$ is étale we know that

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S\mathrm{d}x_i$$

is an isomorphism, in particular I/I^2 is a free S-module. Thus by Lemma 10.136.6 we may assume (after possibly changing the presentation), that $I = (f_1, \ldots, f_c)$ such that the classes $f_i \bmod I^2$ form a basis of I/I^2. It follows immediately from the fact that the displayed map above is an isomorphism that $c = n$ and that $\det(\partial f_j/\partial x_i)$ is invertible in S. □

00U2 **Lemma 10.143.3.** *Results on étale ring maps.*

(1) *The ring map $R \to R_f$ is étale for any ring R and any $f \in R$.*
(2) *Compositions of étale ring maps are étale.*
(3) *A base change of an étale ring map is étale.*
(4) *The property of being étale is local: Given a ring map $R \to S$ and elements $g_1, \ldots, g_m \in S$ which generate the unit ideal such that $R \to S_{g_j}$ is étale for $j = 1, \ldots, m$ then $R \to S$ is étale.*
(5) *Given $R \to S$ of finite presentation, and a flat ring map $R \to R'$, set $S' = R' \otimes_R S$. The set of primes where $R' \to S'$ is étale is the inverse image via $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ of the set of primes where $R \to S$ is étale.*
(6) *An étale ring map is syntomic, in particular flat.*
(7) *If S is finite type over a field k, then S is étale over k if and only if $\Omega_{S/k} = 0$.*
(8) *Any étale ring map $R \to S$ is the base change of an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$.*
(9) *Let $A = \mathrm{colim}\, A_i$ be a filtered colimit of rings. Let $A \to B$ be an étale ring map. Then there exists an étale ring map $A_i \to B_i$ for some i such that $B \cong A \otimes_{A_i} B_i$.*
(10) *Let A be a ring. Let S be a multiplicative subset of A. Let $S^{-1}A \to B'$ be étale. Then there exists an étale ring map $A \to B$ such that $B' \cong S^{-1}B$.*
(11) *Let A be a ring. Let $B = B' \times B''$ be a product of A-algebras. Then B is étale over A if and only if both B' and B'' are étale over A.*

Proof. In each case we use the corresponding result for smooth ring maps with a small argument added to show that $\Omega_{S/R}$ is zero.

Proof of (1). The ring map $R \to R_f$ is smooth and $\Omega_{R_f/R} = 0$.

Proof of (2). The composition $A \to C$ of smooth maps $A \to B$ and $B \to C$ is smooth, see Lemma 10.137.13. By Lemma 10.131.7 we see that $\Omega_{C/A}$ is zero as both $\Omega_{C/B}$ and $\Omega_{B/A}$ are zero.

Proof of (3). Let $R \to S$ be étale and $R \to R'$ be arbitrary. Then $R' \to S' = R' \otimes_R S$ is smooth, see Lemma 10.137.3. Since $\Omega_{S'/R'} = S' \otimes_S \Omega_{S/R}$ by Lemma 10.131.12 we conclude that $\Omega_{S'/R'} = 0$. Hence $R' \to S'$ is étale.

Proof of (4). Assume the hypotheses of (4). By Lemma 10.137.12 we see that $R \to S$ is smooth. We are also given that $\Omega_{S_{g_i}/R} = (\Omega_{S/R})_{g_i} = 0$ for all i. Then $\Omega_{S/R} = 0$, see Lemma 10.23.2.

Proof of (5). The result for smooth maps is Lemma 10.137.17. In the proof of that lemma we used that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This reduces us to showing that if M is a finitely presented S-module the set of primes $\mathfrak{q}'$ of S' such that $(M \otimes_S S')_{\mathfrak{q}'} = 0$ is the inverse image of the set of primes $\mathfrak{q}$ of S such that $M_{\mathfrak{q}} = 0$. This follows from Lemma 10.40.6.

Proof of (6). Follows directly from the corresponding result for smooth ring maps (Lemma 10.137.9).

Proof of (7). Follows from Lemma 10.140.3 and the definitions.

Proof of (8). Lemma 10.138.14 gives the result for smooth ring maps. The resulting smooth ring map $R_0 \to S_0$ satisfies the hypotheses of Lemma 10.130.8, and hence we may replace S_0 by the factor of relative dimension 0 over R_0.

Proof of (9). Follows from (8) since $R_0 \to A$ will factor through A_i for some i by Lemma 10.127.3.

Proof of (10). Follows from (9), (1), and (2) since $S^{-1}A$ is a filtered colimit of principal localizations of A.

Proof of (11). Use Lemma 10.137.14 to see the result for smoothness and then use that $\Omega_{B/A}$ is zero if and only if both $\Omega_{B'/A}$ and $\Omega_{B''/A}$ are zero. □

Next we work out in more detail what it means to be étale over a field.

00U3 **Lemma 10.143.4.** *Let k be a field. A ring map $k \to S$ is étale if and only if S is isomorphic as a k-algebra to a finite product of finite separable extensions of k.*

Proof. We are going to use without further mention: if $S = S_1 \times \ldots \times S_n$ is a finite product of k-algebras, then S is étale over k if and only if each S_i is étale over k. See Lemma 10.143.3 part (11).

If k'/k is a finite separable field extension then we can write $k' = k(\alpha) \cong k[x]/(f)$. Here f is the minimal polynomial of the element α. Since k' is separable over k we have $\gcd(f, f') = 1$. This implies that $\mathrm{d} : k' \cdot f \to k' \cdot \mathrm{d}x$ is an isomorphism. Hence $k \to k'$ is étale. Thus if S is a finite product of finite separable extensions of k, then S is étale over k.

Conversely, suppose that $k \to S$ is étale. Then S is smooth over k and $\Omega_{S/k} = 0$. By Lemma 10.140.3 we see that $\dim_{\mathfrak{m}} \mathrm{Spec}(S) = 0$ for every maximal ideal $\mathfrak{m}$ of S. Thus $\dim(S) = 0$. By Proposition 10.60.7 we find that S is a finite product of Artinian local rings. By the already used Lemma 10.140.3 these local rings are fields. Hence we may assume $S = k'$ is a field. By the Hilbert Nullstellensatz (Theorem 10.34.1) we see that the extension k'/k is finite. The smoothness of

$k \to k'$ implies by Lemma 10.140.9 that k'/k is a separable extension and the proof is complete. □

00U4 **Lemma 10.143.5.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$ in R. If S/R is étale at $\mathfrak{q}$ then*

(1) *we have $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(2) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite separable.*

Proof. First we may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ and assume that $R \to S$ is étale. Then the lemma follows from Lemma 10.143.4 by unwinding the fact that $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$. □

00U5 **Lemma 10.143.6.** *An étale ring map is quasi-finite.*

Proof. Let $R \to S$ be an étale ring map. By definition $R \to S$ is of finite type. For any prime $\mathfrak{p} \subset R$ the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$ and hence a finite products of fields finite separable over $\kappa(\mathfrak{p})$, in particular finite over $\kappa(\mathfrak{p})$. Thus $R \to S$ is quasi-finite by Lemma 10.122.5. □

00U6 **Lemma 10.143.7.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. If*

(1) *$R \to S$ is of finite presentation,*
(2) *$R_\mathfrak{p} \to S_\mathfrak{q}$ is flat*
(3) *$\mathfrak{p}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(4) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite separable,*

then $R \to S$ is étale at $\mathfrak{q}$.

Proof. Apply Lemma 10.122.2 to find a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathfrak{q}$ is the only prime of S_g lying over $\mathfrak{p}$. We may and do replace S by S_g. Then $S \otimes_R \kappa(\mathfrak{p})$ has a unique prime, hence is a local ring, hence is equal to $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \cong \kappa(\mathfrak{q})$. By Lemma 10.137.16 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. Replace S by S_g again we may assume that $R \to S$ is smooth. By Lemma 10.137.9 we may even assume that $R \to S$ is standard smooth, say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Since $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$ has dimension 0 we conclude that $n = c$, i.e., $R \to S$ is étale. □

Here is a completely new phenomenon.

00U7 **Lemma 10.143.8.** *Let $R \to S$ and $R \to S'$ be étale. Then any R-algebra map $S' \to S$ is étale.*

Proof. First of all we note that $S' \to S$ is of finite presentation by Lemma 10.6.2. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the primes $\mathfrak{q}' \subset S'$ and $\mathfrak{p} \subset R$. By Lemma 10.143.5 the ring map $S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} \to S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is a map of finite separable extensions of $\kappa(\mathfrak{p})$. In particular it is flat. Hence by Lemma 10.128.8 we see that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is flat. Thus $S' \to S$ is flat. Moreover, the above also shows that $\mathfrak{q}'S_\mathfrak{q}$ is the maximal ideal of $S_\mathfrak{q}$ and that the residue field extension of $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite separable. Hence from Lemma 10.143.7 we conclude that $S' \to S$ is étale at $\mathfrak{q}$. Since being étale is local (see Lemma 10.143.3) we win. □

00U8 **Lemma 10.143.9.** *Let $\varphi : R \to S$ be a ring map. If $R \to S$ is surjective, flat and finitely presented then there exist an idempotent $e \in R$ such that $S = R_e$.*

First proof. Let I be the kernel of φ. We have that I is finitely generated by Lemma 10.6.3 since φ is of finite presentation. Moreover, since S is flat over R, tensoring the exact sequence $0 \to I \to R \to S \to 0$ over R with S gives $I/I^2 = 0$. Now we conclude by Lemma 10.21.5. □

Second proof. Since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto a closed subset (see Lemma 10.17.7) and is open (see Proposition 10.41.8) we see that the image is $D(e)$ for some idempotent $e \in R$ (see Lemma 10.21.3). Thus $R_e \to S$ induces a bijection on spectra. Now this map induces an isomorphism on all local rings for example by Lemmas 10.78.5 and 10.20.1. Then it follows that $R_e \to S$ is also injective, for example see Lemma 10.23.1. □

04D1 **Lemma 10.143.10.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be an étale ring map. Then there exists an étale ring map $R \to S$ such that $\overline{S} \cong S/IS$ as R/I-algebras.*

Proof. By Lemma 10.143.2 we can write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ as in Definition 10.137.5 with $\overline{\Delta} = \det(\frac{\partial \overline{f}_i}{\partial x_j})_{i,j=1,\ldots,n}$ invertible in $\overline{S}$. Just take some lifts f_i and set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_n, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,n}$ as in Example 10.137.7. This proves the lemma. □

05YT **Lemma 10.143.11.** *Consider a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows where $B' \to B$ and $A' \to A$ are surjective ring maps whose kernels are ideals of square zero. If $A \to B$ is étale, and $J = I \otimes_A B$, then $A' \to B'$ is étale.

Proof. By Lemma 10.143.10 there exists an étale ring map $A' \to C$ such that $C/IC = B$. Then $A' \to C$ is formally smooth (by Proposition 10.138.13) hence we get an A'-algebra map $\varphi : C \to B'$. Since $A' \to C$ is flat we have $I \otimes_A B = I \otimes_A C/IC = IC$. Hence the assumption that $J = I \otimes_A B$ implies that φ induces an isomorphism $IC \to J$ and an isomorphism $C/IC \to B'/IB'$, whence φ is an isomorphism. □

00UA **Example 10.143.12.** Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

of Example 10.136.7. Write symbolically

$$S = R[b_1, \ldots, c_m]/(\{a_k(b_i, c_j) - a_k\}_{k=1,\ldots,n+m})$$

where for example $a_1(b_i, c_j) = b_1 + c_1$. The matrix of partial derivatives is

$$\begin{pmatrix} 1 & c_1 & \ldots & c_m & 0 & \ldots & \ldots & 0 \\ 0 & 1 & c_1 & \ldots & c_m & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & c_1 & c_2 & \ldots & c_m \\ 1 & b_1 & \ldots & b_{n-1} & b_n & 0 & \ldots & 0 \\ 0 & 1 & b_1 & \ldots & b_{n-1} & b_n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 & b_1 & \ldots & b_n \end{pmatrix}$$

The determinant Δ of this matrix is better known as the *resultant* of the polynomials $g = x^n + b_1x^{n-1} + \ldots + b_n$ and $h = x^m + c_1x^{m-1} + \ldots + c_m$, and the matrix above is known as the *Sylvester matrix* associated to g, h. In a formula $\Delta = \mathrm{Res}_x(g, h)$. The Sylvester matrix is the transpose of the matrix of the linear map

$$\begin{aligned} S[x]_{<m} \oplus S[x]_{<n} &\longrightarrow S[x]_{<n+m} \\ a \oplus b &\longmapsto ag + bh \end{aligned}$$

Let $\mathfrak{q} \subset S$ be any prime. By the above the following are equivalent:

(1) $R \to S$ is étale at $\mathfrak{q}$,
(2) $\Delta = \mathrm{Res}_x(g, h) \notin \mathfrak{q}$,
(3) the images $\overline{g}, \overline{h} \in \kappa(\mathfrak{q})[x]$ of the polynomials g, h are relatively prime in $\kappa(\mathfrak{q})[x]$.

The equivalence of (2) and (3) holds because the image of the Sylvester matrix in $\mathrm{Mat}(n + m, \kappa(\mathfrak{q}))$ has a kernel if and only if the polynomials $\overline{g}, \overline{h}$ have a factor in common. We conclude that the ring map

$$R \longrightarrow S[\frac{1}{\Delta}] = S[\frac{1}{\mathrm{Res}_x(g, h)}]$$

is étale.

00UH **Lemma 10.143.13.** *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $\mathfrak{p}$ be a prime of R. Let $f \bmod \mathfrak{p} = \overline{g}\overline{h}$ be a factorization of the image of f in $\kappa(\mathfrak{p})[x]$. If $\gcd(\overline{g}, \overline{h}) = 1$, then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$, and*
(3) *a factorization $f = gh$ in $R'[x]$*

such that

(1) *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
(2) *$\overline{g} = g \bmod \mathfrak{p}'$, $\overline{h} = h \bmod \mathfrak{p}'$, and*
(3) *the polynomials g, h generate the unit ideal in $R'[x]$.*

Proof. Suppose $\overline{g} = \overline{b}_0x^n + \overline{b}_1x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = \overline{c}_0x^m + \overline{c}_1x^{m-1} + \ldots + \overline{c}_m$ with $\overline{b}_0, \overline{c}_0 \in \kappa(\mathfrak{p})$ nonzero. After localizing R at some element of R not contained in $\mathfrak{p}$ we may assume $\overline{b}_0$ is the image of an invertible element $b_0 \in R$. Replacing $\overline{g}$ by $\overline{g}/b_0$ and $\overline{h}$ by $b_0\overline{h}$ we reduce to the case where $\overline{g}$, $\overline{h}$ are monic (verification omitted). Say $\overline{g} = x^n + \overline{b}_1x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = x^m + \overline{c}_1x^{m-1} + \ldots + \overline{c}_m$. Write $f = x^{n+m} + a_1x^{n+m-1} + \ldots + a_{n+m}$. Consider the fibre product

$$R' = R \otimes_{\mathbf{Z}[a_1, \ldots, a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

where the map $\mathbf{Z}[a_k] \to \mathbf{Z}[b_i, c_j]$ is as in Examples 10.136.7 and 10.143.12. By construction there is an R-algebra map

$$R' = R \otimes_{\mathbf{Z}[a_1,\ldots,a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \longrightarrow \kappa(\mathfrak{p})$$

which maps b_i to $\overline{b}_i$ and c_j to $\overline{c}_j$. Denote $\mathfrak{p}' \subset R'$ the kernel of this map. Since by assumption the polynomials $\overline{g}, \overline{h}$ are relatively prime we see that the element $\Delta = \mathrm{Res}_x(g, h) \in \mathbf{Z}[b_i, c_j]$ (see Example 10.143.12) does not map to zero in $\kappa(\mathfrak{p})$ under the displayed map. We conclude that $R \to R'$ is étale at $\mathfrak{p}'$. In fact a solution to the problem posed in the lemma is the ring map $R \to R'[1/\Delta]$ and the prime $\mathfrak{p}'R'[1/\Delta]$. Because $\mathrm{Res}_x(f, g)$ is invertible in this ring the Sylvester matrix is invertible over $R'[1/\Delta]$ and hence $1 = ag + bh$ for some $a, b \in R'[1/\Delta][x]$ see Example 10.143.12. □

10.144. Local structure of étale ring maps

0G1A Lemma 10.143.2 tells us that it does not really make sense to define a standard étale morphism to be a standard smooth morphism of relative dimension 0. As a model for an étale morphism we take the example given by a finite separable extension k'/k of fields. Namely, we can always find an element $\alpha \in k'$ such that $k' = k(\alpha)$ and such that the minimal polynomial $f(x) \in k[x]$ of α has derivative f' which is relatively prime to f.

00UB **Definition 10.144.1.** Let R be a ring. Let $g, f \in R[x]$. Assume that f is monic and the derivative f' is invertible in the localization $R[x]_g/(f)$. In this case the ring map $R \to R[x]_g/(f)$ is said to be *standard étale*.

In Proposition 10.144.4 we show that every étale ring map is locally standard étale.

00UC **Lemma 10.144.2.** *Let $R \to R[x]_g/(f)$ be standard étale.*

(1) *The ring map $R \to R[x]_g/(f)$ is étale.*
(2) *For any ring map $R \to R'$ the base change $R' \to R'[x]_g/(f)$ of the standard étale ring map $R \to R[x]_g/(f)$ is standard étale.*
(3) *Any principal localization of $R[x]_g/(f)$ is standard étale over R.*
(4) *A composition of standard étale maps is* **not** *standard étale in general.*

Proof. Omitted. Here is an example for (4). The ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2}$ is standard étale. The ring map $\mathbf{F}_{2^2} \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is standard étale. But the ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is not standard étale. □

Standard étale morphisms are a convenient way to produce étale maps. Here is an example.

00UD **Lemma 10.144.3.** *Let R be a ring. Let $\mathfrak{p}$ be a prime of R. Let $L/\kappa(\mathfrak{p})$ be a finite separable field extension. There exists an étale ring map $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$.*

Proof. By the theorem of the primitive element we may write $L = \kappa(\mathfrak{p})[\alpha]$. Let $\overline{f} \in \kappa(\mathfrak{p})[x]$ denote the minimal polynomial for α (in particular this is monic). After replacing α by $c\alpha$ for some $c \in R$, $c \notin \mathfrak{p}$ we may assume all the coefficients of $\overline{f}$ are in the image of $R \to \kappa(\mathfrak{p})$ (verification omitted). Thus we can find a monic polynomial $f \in R[x]$ which maps to $\overline{f}$ in $\kappa(\mathfrak{p})[x]$. Since $\kappa(\mathfrak{p}) \subset L$ is separable, we

see that $\gcd(\overline{f}, \overline{f}') = 1$. Hence there is an element $\gamma \in L$ such that $\overline{f}'(\alpha)\gamma = 1$. Thus we get a R-algebra map

$$\begin{aligned} R[x, 1/f']/(f) &\longrightarrow L \\ x &\longmapsto \alpha \\ 1/f' &\longmapsto \gamma \end{aligned}$$

The left hand side is a standard étale algebra R' over R and the kernel of the ring map gives the desired prime. □

00UE **Proposition 10.144.4.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is étale at $\mathfrak{q}$, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale.*

Proof. The following proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.143.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale. Thus we may assume that S is étale over R.

Step 2. By Lemma 10.143.3 there exists an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that $R = R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.143.6. By Lemma 10.123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that of course S' is not étale over R in general.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is étale at $\mathfrak{q}$ (but no longer necessarily étale at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 10.53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^{n} A_i$$

see Proposition 10.60.7. One of the factors, say A_1, is the local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 10.143.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\overline{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^{n} A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\overline{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to

t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

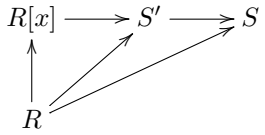

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 10.41.11 (we have going up for $S' \to S$ by Lemma 10.36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak{p} \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$$

The second map is finite and injective. We have $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = \kappa(\mathfrak{q})$, see Lemma 10.143.5. Hence a fortiori $S_\mathfrak{q}/\mathfrak{q}'S_\mathfrak{q} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 10.20.1 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$, the map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_\mathfrak{q}$.

Step 7. By Lemma 10.126.7 there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is étale over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})[x]$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in I \subset R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial

over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1\overline{h}_2^{e_2}\ldots\overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1}\overline{h}_2^{ld_2}\ldots\overline{h}_n^{ld_n} = \overline{h}_1(\overline{h}_2^{e_2}\ldots\overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1-1}\overline{h}_2^{ld_2}\ldots\overline{h}_n^{ld_n}) = \overline{h}_1\overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1\frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.144.2) and $\varphi(g) \notin \mathfrak{q}$. Pick an element $g' \in R[x]/(f)$ such that also $\varphi(g') \notin \mathfrak{q}$ and $S_{\varphi(g')}$ is étale over R (which exists since S is étale over R at $\mathfrak{q}$). Then the ring map $R[x]_{gg'}/(f) \to S_{\varphi(gg')}$ is a surjective map of étale algebras over R. Hence it is étale by Lemma 10.143.8. Hence it is a localization by Lemma 10.143.9. Thus a localization of S at an element not in $\mathfrak{q}$ is isomorphic to a localization of a standard étale algebra over R which is what we wanted to show. □

The following two lemmas say that the étale topology is coarser than the topology generated by Zariski coverings and finite flat morphisms. They should be skipped on a first reading.

00UF **Lemma 10.144.5.** *Let $R \to S$ be a standard étale morphism. There exists a ring map $R \to S'$ with the following properties*

1. *$R \to S'$ is finite, finitely presented, and flat (in other words S' is finite projective as an R-module),*
2. *$\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is surjective,*
3. *for every prime $\mathfrak{q} \subset S$, lying over $\mathfrak{p} \subset R$ and every prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors through a map $\varphi : S \to S'_{g'}$ with $\varphi^{-1}(\mathfrak{q}'S'_{g'}) = \mathfrak{q}$.*

Proof. Let $S = R[x]_g/(f)$ be a presentation of S as in Definition 10.144.1. Write $f = x^n + a_1x^{n-1} + \ldots + a_n$ with $a_i \in R$. By Lemma 10.136.14 there exists a finite free and faithfully flat ring map $R \to S'$ such that $f = \prod(x - \alpha_i)$ for certain $\alpha_i \in S'$. Hence $R \to S'$ satisfies conditions (1), (2). Let $\mathfrak{q} \subset R[x]/(f)$ be a prime ideal with $g \notin \mathfrak{q}$ (i.e., it corresponds to a prime of S). Let $\mathfrak{p} = R \cap \mathfrak{q}$ and let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{p}$. Note that there are n maps of R-algebras

$$\begin{aligned} \varphi_i : R[x]/(f) &\longrightarrow S' \\ x &\longmapsto \alpha_i \end{aligned}$$

To finish the proof we have to show that for some i we have (a) the image of $\varphi_i(g)$ in $\kappa(\mathfrak{q}')$ is not zero, and (b) $\varphi_i^{-1}(\mathfrak{q}') = \mathfrak{q}$. Because then we can just take $g' = \varphi_i(g)$, and $\varphi = \varphi_i$ for that i.

Let $\overline{f}$ denote the image of f in $\kappa(\mathfrak{p})[x]$. Note that as a point of $\mathrm{Spec}(\kappa(\mathfrak{p})[x]/(\overline{f}))$ the prime $\mathfrak{q}$ corresponds to an irreducible factor f_1 of $\overline{f}$. Moreover, $g \notin \mathfrak{q}$ means that f_1 does not divide the image $\overline{g}$ of g in $\kappa(\mathfrak{p})[x]$. Denote $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ the images of $\alpha_1, \ldots, \alpha_n$ in $\kappa(\mathfrak{q}')$. Note that the polynomial $\overline{f}$ splits completely in $\kappa(\mathfrak{q}')[x]$, namely

$$\overline{f} = \prod\nolimits_i (x - \overline{\alpha}_i)$$

Moreover $\varphi_i(g)$ reduces to $\overline{g}(\overline{\alpha}_i)$. It follows we may pick i such that $f_1(\overline{\alpha}_i) = 0$ and $\overline{g}(\overline{\alpha}_i) \neq 0$. For this i properties (a) and (b) hold. Some details omitted. □

00UG **Lemma 10.144.6.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is étale, and*
(2) *$\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Then there exists a ring map $R \to S'$ such that

(1) *$R \to S'$ is finite, finitely presented, and flat (in other words it is finite projective as an R-module),*
(2) *$\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is surjective,*
(3) *for every prime $\mathfrak{q}' \subset S'$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors as $R \to S \to S'_{g'}$.*

Proof. By Proposition 10.144.4 and the quasi-compactness of $\mathrm{Spec}(S)$ (see Lemma 10.17.8) we can find $g_1, \ldots, g_n \in S$ generating the unit ideal of S such that each $R \to S_{g_i}$ is standard étale. If we prove the lemma for the ring map $R \to \prod_{i=1,\ldots,n} S_{g_i}$ then the lemma follows for the ring map $R \to S$. Hence we may assume that $S = \prod_{i=1,\ldots,n} S_i$ is a finite product of standard étale morphisms.

For each i choose a ring map $R \to S'_i$ as in Lemma 10.144.5 adapted to the standard étale morphism $R \to S_i$. Set $S' = S'_1 \otimes_R \ldots \otimes_R S'_n$; we will use the R-algebra maps $S'_i \to S'$ without further mention below. We claim this works. Properties (1) and (2) are immediate. For property (3) suppose that $\mathfrak{q}' \subset S'$ is a prime. Denote $\mathfrak{p}$ its image in $\mathrm{Spec}(R)$. Choose $i \in \{1, \ldots, n\}$ such that $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)$; this is possible by assumption. Set $\mathfrak{q}'_i \subset S'_i$ the image of $\mathfrak{q}'$ in the spectrum of S'_i. By construction of S'_i there exists a $g'_i \in S'_i$ such that $R \to (S'_i)_{g'_i}$ factors as $R \to S_i \to (S'_i)_{g'_i}$. Hence also $R \to S'_{g'_i}$ factors as

$$R \to S_i \to (S'_i)_{g'_i} \to S'_{g'_i}$$

as desired. □

10.145. Étale local structure of quasi-finite ring maps

0G1B The following lemmas say roughly that after an étale extension a quasi-finite ring map becomes finite. To help interpret the results recall that the locus where a finite type ring map is quasi-finite is open (see Lemma 10.123.13) and that formation of this locus commutes with arbitrary base change (see Lemma 10.122.9).

00UI **Lemma 10.145.1.** *Let $R \to S' \to S$ be ring maps. Let $\mathfrak{p} \subset R$ be a prime. Let $g \in S'$ be an element. Assume*

(1) *$R \to S'$ is integral,*
(2) *$R \to S$ is finite type,*
(3) *$S'_g \cong S_g$, and*
(4) *g invertible in $S' \otimes_R \kappa(\mathfrak{p})$.*

Then there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to S_f$ is finite.

Proof. By assumption the image T of $V(g) \subset \mathrm{Spec}(S')$ under the morphism $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ does not contain $\mathfrak{p}$. By Section 10.41 especially, Lemma 10.41.6 we see T is closed. Pick $f \in R$, $f \notin \mathfrak{p}$ such that $T \cap D(f) = \emptyset$. Then we see that g becomes invertible in S'_f. Hence $S'_f \cong S_f$. Thus S_f is both of finite type and integral over R_f, hence finite. □

00UJ **Lemma 10.145.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p} \subset R$. Assume $R \to S$ finite type and quasi-finite at $\mathfrak{q}$. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

1. *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *$R' \to A$ is finite,*
3. *A has exactly one prime $\mathfrak{r}$ lying over $\mathfrak{p}'$,*
4. *$\mathfrak{r}$ lies over $\mathfrak{q}$, and*
5. *B does not have a prime lying over $\mathfrak{q}$ and $\mathfrak{p}'$.*

Proof. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q}' = S' \cap \mathfrak{q}$. By Zariski's Main Theorem 10.123.12 there exists a $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'_g \cong S_g$. Consider the fibre rings $F = S \otimes_R \kappa(\mathfrak{p})$ and $F' = S' \otimes_R \kappa(\mathfrak{p})$. Denote $\overline{\mathfrak{q}}'$ the prime of F' corresponding to $\mathfrak{q}'$. Since F' is integral over $\kappa(\mathfrak{p})$ we see that $\overline{\mathfrak{q}}'$ is a closed point of $\mathrm{Spec}(F')$, see Lemma 10.36.19. Note that $\mathfrak{q}$ defines an isolated closed point $\overline{\mathfrak{q}}$ of $\mathrm{Spec}(F)$ (see Definition 10.122.3). Since $S'_g \cong S_g$ we have $F'_g \cong F_g$, so $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}'$ have isomorphic open neighbourhoods in $\mathrm{Spec}(F)$ and $\mathrm{Spec}(F')$. We conclude the set $\{\overline{\mathfrak{q}}'\} \subset \mathrm{Spec}(F')$ is open. Combined with $\mathfrak{q}'$ being closed (shown above) we conclude that $\overline{\mathfrak{q}}'$ defines an isolated closed point of $\mathrm{Spec}(F')$ as well.

An additional small remark is that under the map $\mathrm{Spec}(F) \to \mathrm{Spec}(F')$ the point $\overline{\mathfrak{q}}$ is the only point mapping to $\overline{\mathfrak{q}}'$. This follows from the discussion above.

By Lemma 10.24.3 we may write $F' = F'_1 \times F'_2$ with $\mathrm{Spec}(F'_1) = \{\overline{\mathfrak{q}}'\}$. Since $F' = S' \otimes_R \kappa(\mathfrak{p})$, there exists an $s' \in S'$ which maps to the element $(r, 0) \in F'_1 \times F'_2 = F'$ for some $r \in R$, $r \notin \mathfrak{p}$. In fact, what we will use about s' is that it is an element of S', not contained in $\mathfrak{q}'$, and contained in any other prime lying over $\mathfrak{p}$.

Let $f(x) \in R[x]$ be a monic polynomial such that $f(s') = 0$. Denote $\overline{f} \in \kappa(\mathfrak{p})[x]$ the image. We can factor it as $\overline{f} = x^e \overline{h}$ where $\overline{h}(0) \neq 0$. After replacing f by xf if necessary, we may assume $e \geq 1$. By Lemma 10.143.13 we can find an étale ring extension $R \to R'$, a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a factorization $f = hi$ in $R'[x]$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $\overline{h} = h \bmod \mathfrak{p}'$, $x^e = i \bmod \mathfrak{p}'$, and we can write $ah + bi = 1$ in $R'[x]$ (for suitable a, b).

Consider the elements $h(s'), i(s') \in R' \otimes_R S'$. By construction we have $h(s')i(s') = f(s') = 0$. On the other hand they generate the unit ideal since $a(s')h(s') + b(s')i(s') = 1$. Thus we see that $R' \otimes_R S'$ is the product of the localizations at these elements:

$$R' \otimes_R S' = (R' \otimes_R S')_{i(s')} \times (R' \otimes_R S')_{h(s')} = S'_1 \times S'_2$$

Moreover this product decomposition is compatible with the product decomposition we found for the fibre ring F'; this comes from our choices of s', i, h which guarantee that $\overline{\mathfrak{q}}'$ is the only prime of F' which does not contain the image of $i(s')$ in F'. Here we use that the fibre ring of $R' \otimes_R S'$ over R' at $\mathfrak{p}'$ is the same as F' due to the fact that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$. It follows that S'_1 has exactly one prime, say $\mathfrak{r}'$, lying over $\mathfrak{p}'$ and that this prime lies over $\mathfrak{q}'$. Hence the element $g \in S'$ maps to an element of S'_1 not contained in $\mathfrak{r}'$.

The base change $R' \otimes_R S$ inherits a similar product decomposition

$$R' \otimes_R S = (R' \otimes_R S)_{i(s')} \times (R' \otimes_R S)_{h(s')} = S_1 \times S_2$$

It follows from the above that S_1 has exactly one prime, say $\mathfrak{r}$, lying over $\mathfrak{p}'$ (consider the fibre ring as above), and that this prime lies over $\mathfrak{q}$.

Now we may apply Lemma 10.145.1 to the ring maps $R' \to S'_1 \to S_1$, the prime $\mathfrak{p}'$ and the element g to see that after replacing R' by a principal localization we can assume that S_1 is finite over R' as desired. □

00UK **Lemma 10.145.3.** *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *we have $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *each A_i is finite over R',*
3. *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$, and*
4. *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As F is of finite type over $\kappa(\mathfrak{p})$ it is Noetherian and hence $\mathrm{Spec}(F)$ has finitely many isolated closed points. If there are no isolated closed points, i.e., no primes $\mathfrak{q}$ of S over $\mathfrak{p}$ such that S/R is quasi-finite at $\mathfrak{q}$, then the lemma holds. If there exists at least one such prime $\mathfrak{q}$, then we may apply Lemma 10.145.2. This gives a diagram

$$\begin{array}{ccccc} S & \longrightarrow & R' \otimes_R S & = & A_1 \times B' \\ \uparrow & & \uparrow & \nearrow & \\ R & \longrightarrow & R' & & \end{array}$$

as in said lemma. Since the residue fields at $\mathfrak{p}$ and $\mathfrak{p}'$ are the same, the fibre rings of S/R and $(A_1 \times B')/R'$ are the same. Hence, by induction on the number of isolated closed points of the fibre we may assume that the lemma holds for $R' \to B'$ and $\mathfrak{p}'$. Thus we get an étale ring map $R' \to R''$, a prime $\mathfrak{p}'' \subset R''$ and a decomposition

$$R'' \otimes_{R'} B' = A_2 \times \ldots \times A_n \times B$$

We omit the verification that the ring map $R \to R''$, the prime $\mathfrak{p}''$ and the resulting decomposition

$$R'' \otimes_R S = (R'' \otimes_{R'} A_1) \times A_2 \times \ldots \times A_n \times B$$

is a solution to the problem posed in the lemma. □

00UL **Lemma 10.145.4.** *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
(3) *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *each A_i is finite over R',*
(2) *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$,*
(3) *the finite field extensions $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p}')$ are purely inseparable, and*
(4) *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. The strategy of the proof is to make two étale ring extensions: first we control the residue fields, then we apply Lemma 10.145.3.

Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As in the proof of Lemma 10.145.3 there are finitely may primes, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of S lying over R at which the ring map $R \to S$ is quasi-finite. Let $\kappa(\mathfrak{p}) \subset L_i \subset \kappa(\mathfrak{q}_i)$ be the subfield such that $\kappa(\mathfrak{p}) \subset L_i$ is separable, and the field extension $\kappa(\mathfrak{q}_i)/L_i$ is purely inseparable. Let $L/\kappa(\mathfrak{p})$ be a finite Galois extension into which L_i embeds for $i = 1, \ldots, n$. By Lemma 10.144.3 we can find an étale ring extension $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$. Thus the fibre ring of $R' \otimes_R S$ at $\mathfrak{p}'$ is isomorphic to $F \otimes_{\kappa(\mathfrak{p})} L$. The primes lying over $\mathfrak{q}_i$ correspond to primes of $\kappa(\mathfrak{q}_i) \otimes_{\kappa(\mathfrak{p})} L$ which is a product of fields purely inseparable over L by our choice of L and elementary field theory. These are also the only primes over $\mathfrak{p}'$ at which $R' \to R' \otimes_R S$ is quasi-finite, by Lemma 10.122.9. Hence after replacing R by R', $\mathfrak{p}$ by $\mathfrak{p}'$, and S by $R' \otimes_R S$ we may assume that for all primes $\mathfrak{q}$ lying over $\mathfrak{p}$ for which S/R is quasi-finite the field extensions $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ are purely inseparable.

Next apply Lemma 10.145.3. The result is what we want since the field extensions do not change under this étale ring extension. □

10.146. Local homomorphisms

053J Some lemmas which don't have a natural section to go into. The first lemma says, loosely speaking, that an étale map of local rings is an isomorphism modulo all powers of a nonunit principal ideal.

0GSD **Lemma 10.146.1.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R and that $\kappa(\mathfrak{m}_R) \to \kappa(\mathfrak{m}_S)$ is an isomorphism. Then there exists an $t \in \mathfrak{m}_R$ such that $R/t^nR \to S/t^nS$ is an isomorphsm for all $n \geq 1$.* [**Lin82**, Lemma on page 321], [**Ces22**, Lemma 4.1.5]

Proof. Write $S = T_{\mathfrak{q}}$ for some étale R-algebra T and prime ideal $\mathfrak{q} \subset T$ lying over $\mathfrak{m}_R$. By Proposition 10.144.4 we may assume $R \to T$ is standard étale. Write $T = R[x]_g/(f)$ as in Definition 10.144.1. By our assumption on residue fields, we may choose $a \in R$ such that x and a have the same image in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{m}_S) = \kappa(\mathfrak{m}_R)$. Then after replacing x by $x - a$ we may assume that $\mathfrak{q}$ is generated by x and $\mathfrak{m}_R$ in T. In particular $t = f(0) \in \mathfrak{m}_R$. We will show that $t = f(0)$ works.

Write $f = x^d + \sum_{i=1,\ldots,d-1} a_i x^i + t$. Since $R \to T$ is standard étale we find that a_1 is a unit in R: the derivative of f is invertible in T in particular is not contained in $\mathfrak{q}$. Let $h = a_1 + a_2 x + \ldots + a_{d-1}x^{d-2} + x^{d-1} \in R[x]$ so that $f = t + xh$ in $R[x]$. We see that $h \notin \mathfrak{q}$ and hence we may replace T by $R[x]_{hg}/(f)$. After this replacement we see that

$$T/tT = (R/tR)[x]_{hg}/(f) = (R/tR)[x]_{hg}/(xh) = (R/tR)[x]_{hg}/(x)$$

is a quotient of R/tR. By Lemma 10.126.9 we conclude that $R/t^nR \to T/t^nT$ is surjective for all $n \geq 1$. On the other hand, we know that the flat local ring map $R/t^nR \to S/t^nS$ factors through $R/t^nR \to T/t^nT$ for all n, hence these maps are also injective (a flat local homomorphism of local rings is faithfully flat and hence injective, see Lemmas 10.39.17 and 10.82.11). As S is the localization of T we see that S/t^nS is the localization of $T/t^nT = R/t^nR$ at a prime lying over the maximal ideal, but this ring is already local and the proof is complete. □

053K **Lemma 10.146.2.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R. Then there exists a finite, finitely presented, faithfully flat ring map $R \to S'$ such that for every maximal ideal $\mathfrak{m}'$ of S' there is a factorization*

$$R \to S \to S'_{\mathfrak{m}'}.$$

of the ring map $R \to S'_{\mathfrak{m}'}$.

Proof. Write $S = T_{\mathfrak{q}}$ for some étale R-algebra T. By Proposition 10.144.4 we may assume T is standard étale. Apply Lemma 10.144.5 to the ring map $R \to T$ to get $R \to S'$. Then in particular for every maximal ideal $\mathfrak{m}'$ of S' we get a factorization $\varphi : T \to S'_{g'}$ for some $g' \notin \mathfrak{m}'$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{m}'S'_{g'})$. Thus φ induces the desired local ring map $S \to S'_{\mathfrak{m}'}$. □

10.147. Integral closure and smooth base change

03GC

03GD **Lemma 10.147.1.** *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $R \to B$ be a ring map. If $h \in B[x]/(f)$ is integral over R, then the element $f'h$ can be written as $f'h = \sum_i b_i x^i$ with $b_i \in B$ integral over R.*

Proof. Say $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ with $r_i \in R$. There exists a finite free ring extension $B \subset B'$ such that $f = (x - \alpha_1)\ldots(x - \alpha_d)$ for some $\alpha_i \in B'$, see Lemma 10.136.14. Note that each α_i is integral over R. We may represent $h = h_0 + h_1 x + \ldots + h_{d-1}x^{d-1}$ with $h_i \in B$. Then it is a universal fact that

$$f'h = \sum\nolimits_{i=1,\ldots,d} h(\alpha_i)(x - \alpha_1)\ldots\widehat{(x - \alpha_i)}\ldots(x - \alpha_d)$$

as elements of $B'[x]/(f)$. You prove this by evaluating both sides at the points α_i over the ring $B_{univ} = \mathbf{Z}[\alpha_i, h_j]$ (some details omitted). By our assumption that h satisfies $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ we see that

$$h(\alpha_i)^e + r_1 h(\alpha_i)^{e-1} + \ldots + r_e = 0$$

in B'. Hence $h(\alpha_i)$ is integral over R. Using the formula above we see that $f'h \equiv \sum_{j=0,\ldots,d-1} b'_j x^j$ in $B'[x]/(f)$ with $b'_j \in B'$ integral over R. However, since $f'h \in B[x]/(f)$ and since $1, x, \ldots, x^{d-1}$ is a B'-basis for $B'[x]/(f)$ we see that $b'_j \in B$ as desired. □

03GE **Lemma 10.147.2.** *Let $R \to S$ be an étale ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. The map $S \otimes_R A \to A'$ is injective because $A \subset B$ and $R \to S$ is flat. We are going to use repeatedly that taking integral closure commutes with localization, see Lemma 10.36.11. Hence we may localize on S, by Lemma 10.23.2 (the criterion for checking whether an S-module map is an isomorphism). Thus we may assume that $S = R[x]_g/(f) = (R[x]/(f))_g$ is standard étale over R, see Proposition 10.144.4. Applying localization one more time we see that A' is $(A'')_g$ where A'' is the integral closure of $R[x]/(f)$ in $B[x]/(f)$. Suppose that $a \in A''$. It suffices to show that a is in $S \otimes_R A$. By Lemma 10.147.1 we see that $f'a = \sum a_i x^i$ with $a_i \in A$. Since f' is invertible in S (by definition of a standard étale ring map) we conclude that $a \in S \otimes_R A$ as desired. □

03GF **Example 10.147.3.** Let p be a prime number. The ring extension

$$R = \mathbf{Z}[1/p] \subset R' = \mathbf{Z}[1/p][x]/(x^{p-1} + \ldots + x + 1)$$

has the following property: For $d < p$ there exist elements $\alpha_0, \ldots, \alpha_{d-1} \in R'$ such that

$$\prod\nolimits_{0 \leq i < j < d} (\alpha_i - \alpha_j)$$

is a unit in R'. Namely, take α_i equal to the class of x^i in R' for $i = 0, \ldots, p-1$. Then we have

$$T^p - 1 = \prod\nolimits_{i=0,\ldots,p-1} (T - \alpha_i)$$

in $R'[T]$. Namely, the ring $\mathbf{Q}[x]/(x^{p-1} + \ldots + x + 1)$ is a field because the cyclotomic polynomial $x^{p-1} + \ldots + x + 1$ is irreducible over $\mathbf{Q}$ and the α_i are pairwise distinct roots of $T^p - 1$, whence the equality. Taking derivatives on both sides and substituting $T = \alpha_i$ we obtain

$$p\alpha_i^{p-1} = (\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_1)$$

and we see this is invertible in R'.

03GG **Lemma 10.147.4.** *Let $R \to S$ be a smooth ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. Arguing as in the proof of Lemma 10.147.2 we may localize on S. Hence we may assume that $R \to S$ is a standard smooth ring map, see Lemma 10.137.9. By definition of a standard smooth ring map we see that S is étale over a polynomial ring $R[x_1, \ldots, x_n]$. Since we have seen the result in the case of an étale ring extension (Lemma 10.147.2) this reduces us to the case where $S = R[x]$. Thus we have to show

$$f = \sum b_i x^i \text{ integral over } R[x] \Leftrightarrow \text{each } b_i \text{ integral over } R.$$

The implication from right to left holds because the set of elements in $B[x]$ integral over $R[x]$ is a ring (Lemma 10.36.7) and contains x.

Suppose that $f \in B[x]$ is integral over $R[x]$, and assume that $f = \sum_{i<d} b_i x^i$ has degree $< d$. Since integral closure and localization commute, it suffices to show there exist distinct primes p, q such that each b_i is integral both over $R[1/p]$ and

over $R[1/q]$. Hence, we can find a finite free ring extension $R \subset R'$ such that R' contains $\alpha_1, \ldots, \alpha_d$ with the property that $\prod_{i<j}(\alpha_i - \alpha_j)$ is a unit in R', see Example 10.147.3. In this case we have the universal equality

$$f = \sum_i f(\alpha_i) \frac{(x-\alpha_1)\ldots\widehat{(x-\alpha_i)}\ldots(x-\alpha_d)}{(\alpha_i-\alpha_1)\ldots\widehat{(\alpha_i-\alpha_i)}\ldots(\alpha_i-\alpha_d)}.$$

OK, and the elements $f(\alpha_i)$ are integral over R' since $(R' \otimes_R B)[x] \to R' \otimes_R B$, $h \mapsto h(\alpha_i)$ is a ring map. Hence we see that the coefficients of f in $(R' \otimes_R B)[x]$ are integral over R'. Since R' is finite over R (hence integral over R) we see that they are integral over R also, as desired. □

0CBF **Lemma 10.147.5.** *Let $R \to S$ and $R \to B$ be ring maps. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. If S is a filtered colimit of smooth R-algebras, then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. This follows from the straightforward fact that taking tensor products and taking integral closures commutes with filtered colimits and Lemma 10.147.4. □

10.148. Formally unramified maps

00UM It turns out to be logically more efficient to define the notion of a formally unramified map before introducing the notion of a formally étale one.

00UN **Definition 10.148.1.** Let $R \to S$ be a ring map. We say S is *formally unramified over R* if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there exists at most one dotted arrow making the diagram commute.

0H91 **Lemma 10.148.2.** *Let $R \to S$ be a formally unramified map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally unramified over R'.*

Proof. Let a solid diagram

$$\begin{array}{ccccc} S & \longrightarrow & R' \otimes_R S & \longrightarrow & A/I \\ \uparrow & & \uparrow & \nearrow\!\nearrow & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & A \end{array}$$

as in Definition 10.148.1 be given. By assumption there exists at most one longer dotted arrow. By the universal property of tensor product we conclude there is at most one shorter dotted arrow. □

00UO **Lemma 10.148.3.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *the module of differentials $\Omega_{S/R}$ is zero.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. Let $A_{univ} = S \otimes_R S/J^2$. Recall that $I_{univ} = J/J^2$ is isomorphic to $\Omega_{S/R}$, see Lemma 10.131.13. Moreover, the two R-algebra maps $\sigma_1, \sigma_2 : S \to A_{univ}$, $\sigma_1(s) = s \otimes 1 \bmod J^2$, and $\sigma_2(s) = 1 \otimes s \bmod J^2$ differ by the universal derivation $\mathrm{d} : S \to \Omega_{S/R} = I_{univ}$.

Assume $R \to S$ formally unramified. Then we see that $\sigma_1 = \sigma_2$. Hence $\mathrm{d}(s) = 0$ for all $s \in S$. Hence $\Omega_{S/R} = 0$.

Assume that $\Omega_{S/R} = 0$. Let $A, I, R \to A, S \to A/I$ be a solid diagram as in Definition 10.148.1. Let $\tau_1, \tau_2 : S \to A$ be two dotted arrows making the diagram commute. Consider the R-algebra map $A_{univ} \to A$ defined by the rule $s_1 \otimes s_2 \mapsto \tau_1(s_1)\tau_2(s_2)$. We omit the verification that this is well defined. Since $A_{univ} \cong S$ as $I_{univ} = \Omega_{S/R} = 0$ we conclude that $\tau_1 = \tau_2$. □

04E8 **Lemma 10.148.4.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *$R \to S_{\mathfrak{q}}$ is formally unramified for all primes $\mathfrak{q}$ of S, and*
(3) *$R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is formally unramified for all primes $\mathfrak{q}$ of S with $\mathfrak{p} = R \cap \mathfrak{q}$.*

Proof. We have seen in Lemma 10.148.3 that (1) is equivalent to $\Omega_{S/R} = 0$. Similarly, by Lemma 10.131.8 we see that (2) and (3) are equivalent to $(\Omega_{S/R})_{\mathfrak{q}} = 0$ for all $\mathfrak{q}$. Hence the equivalence follows from Lemma 10.23.1. □

04E9 **Lemma 10.148.5.** *Let $A \to B$ be a formally unramified ring map.*

(1) *For $S \subset A$ a multiplicative subset, $S^{-1}A \to S^{-1}B$ is formally unramified.*
(2) *For $S \subset B$ a multiplicative subset, $A \to S^{-1}B$ is formally unramified.*

Proof. Follows from Lemma 10.148.4. (You can also deduce it from Lemma 10.148.3 combined with Lemma 10.131.8.) □

07QE **Lemma 10.148.6.** *Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally unramified, then $S = \mathrm{colim}_{i \in I} S_i$ is formally unramified over R*

Proof. Consider a diagram as in Definition 10.148.1. By assumption there exists at most one R-algebra map $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Since every element of S is in the image of one of the maps $S_i \to S$ we see that there is at most one map $S \to A$ fitting into the diagram. □

10.149. Conormal modules and universal thickenings

04EA It turns out that one can define the first infinitesimal neighbourhood not just for a closed immersion of schemes, but already for any formally unramified morphism. This is based on the following algebraic fact.

04EB **Lemma 10.149.1.** *Let $R \to S$ be a formally unramified ring map. There exists a surjection of R-algebras $S' \to S$ whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{a} & A/I \\ \uparrow & & \uparrow \\ R & \xrightarrow{b} & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there is a unique R-algebra map $a' : S' \to A$ such that $S' \to A \to A/I$ is equal to $S' \to S \to A/I$.

Proof. Choose a set of generators $z_i \in S$, $i \in I$ for S as an R-algebra. Let $P = R[\{x_i\}_{i \in I}]$ denote the polynomial ring on generators x_i, $i \in I$. Consider the R-algebra map $P \to S$ which maps x_i to z_i. Let $J = \mathrm{Ker}(P \to S)$. Consider the map

$$\mathrm{d} : J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

see Lemma 10.131.9. This is surjective since $\Omega_{S/R} = 0$ by assumption, see Lemma 10.148.3. Note that $\Omega_{P/R}$ is free on $\mathrm{d}x_i$, and hence the module $\Omega_{P/R} \otimes_P S$ is free over S. Thus we may choose a splitting of the surjection above and write

$$J/J^2 = K \oplus \Omega_{P/R} \otimes_P S$$

Let $J^2 \subset J' \subset J$ be the ideal of P such that J'/J^2 is the second summand in the decomposition above. Set $S' = P/J'$. We obtain a short exact sequence

$$0 \to J/J' \to S' \to S \to 0$$

and we see that $J/J' \cong K$ is a square zero ideal in S'. Hence

$$\begin{array}{ccc} S & \xrightarrow[1]{} & S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S' \end{array}$$

is a diagram as above. In fact we claim that this is an initial object in the category of diagrams. Namely, let $(I \subset A, a, b)$ be an arbitrary diagram. We may choose an R-algebra map $\beta : P \to A$ such that

$$\begin{array}{ccccc} S & \xrightarrow[1]{} & S & \xrightarrow{a} & A/I \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & P & \xrightarrow{\beta} & A \\ & & & \underset{b}{\searrow} & \end{array}$$

is commutative. Now it may not be the case that $\beta(J') = 0$, in other words it may not be true that β factors through $S' = P/J'$. But what is clear is that $\beta(J') \subset I$ and since $\beta(J) \subset I$ and $I^2 = 0$ we have $\beta(J^2) = 0$. Thus the "obstruction" to finding a morphism from $(J/J' \subset S', 1, R \to S')$ to $(I \subset A, a, b)$ is the corresponding S-linear map $\overline{\beta} : J'/J^2 \to I$. The choice in picking β lies in the choice of $\beta(x_i)$. A different choice of β, say β', is gotten by taking $\beta'(x_i) = \beta(x_i) + \delta_i$ with $\delta_i \in I$. In this case, for $g \in J'$, we obtain

$$\beta'(g) = \beta(g) + \sum\nolimits_i \delta_i \beta(\frac{\partial g}{\partial x_i})$$

Since the map $\mathrm{d}|_{J'/J^2} : J'/J^2 \to \Omega_{P/R} \otimes_P S$ given by $g \mapsto \frac{\partial g}{\partial x_i} \mathrm{d}x_i \otimes 1$ is an isomorphism by construction, we see that there is a unique choice of $\delta_i \in I$ such that $\beta'(g) = 0$ for all $g \in J'$. (Namely, δ_i is $-\overline{\beta}(g)$ where $g \in J'/J^2$ is the unique element mapped to $\mathrm{d}x_i \otimes 1$ by $\mathrm{d}|_{J'/J^2}$.) The uniqueness of the solution implies the uniqueness required in the lemma. □

In the situation of Lemma 10.149.1 the R-algebra map $S' \to S$ is unique up to unique isomorphism.

04EC **Definition 10.149.2.** Let $R \to S$ be a formally unramified ring map.

(1) The *universal first order thickening* of S over R is the surjection of R-algebras $S' \to S$ of Lemma 10.149.1.
(2) The *conormal module* of $R \to S$ is the kernel I of the universal first order thickening $S' \to S$, seen as an S-module.

We often denote the conormal module $C_{S/R}$ in this situation.

04ED **Lemma 10.149.3.** *Let $I \subset R$ be an ideal of a ring. The universal first order thickening of R/I over R is the surjection $R/I^2 \to R/I$. The conormal module of R/I over R is $C_{(R/I)/R} = I/I^2$.*

Proof. Omitted. □

04EE **Lemma 10.149.4.** *Let $A \to B$ be a formally unramified ring map. Let $\varphi : B' \to B$ be the universal first order thickening of B over A.*

(1) *Let $S \subset A$ be a multiplicative subset. Then $S^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$.*
(2) *Let $S \subset B$ be a multiplicative subset. Then $S' = \varphi^{-1}(S)$ is a multiplicative subset in B' and $(S')^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over A. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/A}$.*

Note that the lemma makes sense by Lemma 10.148.5.

Proof. With notation and assumptions as in (1). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. Note that $S^{-1}B' \to S^{-1}B$ is a surjection of $S^{-1}A$-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to S^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & S^{-1}B & \longrightarrow & D/I \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & S^{-1}A & \longrightarrow & D \end{array}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S in D is mapped to invertible elements of D, and hence we obtain a compatible map $S^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $S^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above.

With notation and assumptions as in (2). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over A. Note that $(S')^{-1}B' \to S^{-1}B$ is a surjection of A-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to (S')^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & S^{-1}B & \longrightarrow & D/I \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A & \longrightarrow & D \end{array}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S' in D is mapped to invertible elements of D, and hence we obtain a compatible map $(S')^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $(S')^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above. □

04EF **Lemma 10.149.5.** *Let $R \to A \to B$ be ring maps. Assume $A \to B$ formally unramified. Let $B' \to B$ be the universal first order thickening of B over A. Then B' is formally unramified over A, and the canonical map $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ is an isomorphism.*

Proof. We are going to use the construction of B' from the proof of Lemma 10.149.1 although in principle it should be possible to deduce these results formally from the definition. Namely, we choose a presentation $B = P/J$, where $P = A[x_i]$ is a polynomial ring over A. Next, we choose elements $f_i \in J$ such that $\mathrm{d}f_i = \mathrm{d}x_i \otimes 1$ in $\Omega_{P/A} \otimes_P B$. Having made these choices we have $B' = P/J'$ with $J' = (f_i) + J^2$, see proof of Lemma 10.149.1.

Consider the canonical exact sequence

$$J'/(J')^2 \to \Omega_{P/A} \otimes_P B' \to \Omega_{B'/A} \to 0$$

see Lemma 10.131.9. By construction the classes of the $f_i \in J'$ map to elements of the module $\Omega_{P/A} \otimes_P B'$ which generate it modulo J'/J^2 by construction. Since J'/J^2 is a nilpotent ideal, we see that these elements generate the module altogether (by Nakayama's Lemma 10.20.1). This proves that $\Omega_{B'/A} = 0$ and hence that B' is formally unramified over A, see Lemma 10.148.3.

Since P is a polynomial ring over A we have $\Omega_{P/R} = \Omega_{A/R} \otimes_A P \oplus \bigoplus P\mathrm{d}x_i$. We are going to use this decomposition. Consider the following exact sequence

$$J'/(J')^2 \to \Omega_{P/R} \otimes_P B' \to \Omega_{B'/R} \to 0$$

see Lemma 10.131.9. We may tensor this with B and obtain the exact sequence

$$J'/(J')^2 \otimes_{B'} B \to \Omega_{P/R} \otimes_P B \to \Omega_{B'/R} \otimes_{B'} B \to 0$$

If we remember that $J' = (f_i) + J^2$ then we see that the first arrow annihilates the submodule $J^2/(J')^2$. In terms of the direct sum decomposition $\Omega_{P/R} \otimes_P B = \Omega_{A/R} \otimes_A B \oplus \bigoplus B\mathrm{d}x_i$ given we see that the submodule $(f_i)/(J')^2 \otimes_{B'} B$ maps isomorphically onto the summand $\bigoplus B\mathrm{d}x_i$. Hence what is left of this exact sequence is an isomorphism $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ as desired. □

10.150. Formally étale maps

00UP

00UQ **Definition 10.150.1.** Let $R \to S$ be a ring map. We say S is *formally étale over* R if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there exists a unique dotted arrow making the diagram commute.

Clearly a ring map is formally étale if and only if it is both formally smooth and formally unramified.

0H92 **Lemma 10.150.2.** *Let $R \to S$ be a formally étale map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally étale over R'.*

Proof. Combine Lemmas 10.138.2 and 10.148.2. □

00UR **Lemma 10.150.3.** *Let $R \to S$ be a ring map of finite presentation. The following are equivalent:*

(1) *$R \to S$ is formally étale,*
(2) *$R \to S$ is étale.*

Proof. Assume that $R \to S$ is formally étale. Then $R \to S$ is smooth by Proposition 10.138.13. By Lemma 10.148.3 we have $\Omega_{S/R} = 0$. Hence $R \to S$ is étale by definition.

Assume that $R \to S$ is étale. Then $R \to S$ is formally smooth by Proposition 10.138.13. By Lemma 10.148.3 it is formally unramified. Hence $R \to S$ is formally étale. □

031N **Lemma 10.150.4.** *Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally étale, then $S = \operatorname{colim}_{i \in I} S_i$ is formally étale over R*

Proof. Consider a diagram as in Definition 10.150.1. By assumption we get unique R-algebra maps $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Hence these are compatible with the transition maps $\varphi_{ii'}$ and define a lift $S \to A$. This proves existence. The uniqueness is clear by restricting to each S_i. □

04EG **Lemma 10.150.5.** *Let R be a ring. Let $S \subset R$ be any multiplicative subset. Then the ring map $R \to S^{-1}R$ is formally étale.*

Proof. Let $I \subset A$ be an ideal of square zero. What we are saying here is that given a ring map $\varphi : R \to A$ such that $\varphi(f) \mod I$ is invertible for all $f \in S$ we have also that $\varphi(f)$ is invertible in A for all $f \in S$. This is true because A^* is the inverse image of $(A/I)^*$ under the canonical map $A \to A/I$. □

0H1D **Lemma 10.150.6.** *Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal such that $R \to S/J$ is surjective; let $I \subset R$ be the kernel. If $R \to S$ is formally étale, then $R/I^n \to S/J^n$ is an isomorphism for all n and $\bigoplus I^n/I^{n+1} \to \bigoplus J^n/J^{n+1}$ is an isomorphism of graded rings.*

Proof. Using the lifting property inductively we find dotted arrows

$$\begin{array}{ccc} S & \longrightarrow & S/J = R/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R/I^2 \end{array} \qquad \begin{array}{ccc} S & \longrightarrow & R/I^2 \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R/I^3 \end{array} \qquad \begin{array}{ccc} S & \longrightarrow & R/I^3 \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R/I^4 \end{array}$$

The corresponding maps $S/J^n \to R/I^n$ are isomorphisms since the compositions $S/J^n \to R/I^n \to S/J^n$ are (inductively) the identity by the uniqueness in the lifting property of formally étale ring maps. □

0H93 **Lemma 10.150.7.** *Let $R \to S \to S'$ be ring maps. Let J, resp. J' be the kernel of the multiplication map $S \otimes_R S \to S$, resp. $S' \otimes_{R'} S' \to S'$. If $S \to S'$ is formally étale, then the map*

$$S' \otimes_S \left((S \otimes_R S)/J^{k+1}\right) \longrightarrow (S' \otimes_R S')/(J')^{k+1}$$

is an isomorphism for all $k \geq 0$. In particular, the map $S' \otimes_S \Omega_{S/R} \to \Omega_{S'/R}$ is an isomorphism.

Proof. Observe that $S' \otimes_S (S \otimes_R S) = S' \otimes_R S$ and the ideal J generates in $S' \otimes_R S$ the kernel I of the surjection $S' \otimes_R S \to S'$. Whence the left hand side is equal to $S' \otimes_R S/I^{k+1}$. The map $S' \otimes_R S \to S' \otimes_R S'$ is formally étale by Lemma 10.150.2. Thus we conclude that the displayed arrow in the statement of the lemma is an isomorphism by Lemma 10.150.6. The final assertion follows from this and the fact that $\Omega_{S/R} = J/J^2$ and $\Omega_{S'/R} = J'/(J')^2$ by Lemma 10.131.13. □

0H94 **Lemma 10.150.8.** *Let $R \to S \to S'$ be ring maps with $S \to S'$ formally étale (for example étale). Let M be an S-module. Set $M' = S' \otimes_R M$. Then we have*

$$S' \otimes_S P^k_{S/R}(M) = P^k_{S'/R}(M')$$

It follows that for any S-module N and any finite order differential operator $D : M \to N$ there exists a unique extension $D' : M' \to S' \otimes_S N$ of D to a differential operator (of the same or lesser order).

Proof. Let J and J' be as in the statement of Lemma 10.150.7. Then we have

$$\begin{aligned}
S' \otimes_S P^k_{S/R}(M) &= S' \otimes_S \left((S \otimes_R M)/J^{k+1}(S \otimes_R M)\right) \\
&= S' \otimes_S \left((S \otimes_R S)/J^{k+1}\right) \otimes_S M \\
&= \left(S' \otimes_R S'/(J')^{k+1}\right) \otimes_S M \\
&= \left(S' \otimes_R S'/(J')^{k+1}\right) \otimes_{S'} M' \\
&= S' \otimes_R M'/(J')^{k+1}(S' \otimes_R M') \\
&= P^k_{S'/R'}(M')
\end{aligned}$$

The first and the last equalities are from Lemma 10.133.9. The third equality is Lemma 10.150.7. The final assertion holds because if D corresponds to the linear map $\gamma : P^k_{S/R}(M) \to N$, then we can let $D' : M' \to S' \otimes_R N$ correspond to the linear map $1 \otimes \gamma : S' \otimes_R P^k_{S/R}(M) \to S' \otimes_R N$. □

0H95 **Remark 10.150.9.** Let $R \to S \to S'$ be ring maps with $S \to S'$ formally étale (for example étale). Let M_i, $i = 1, 2, 3$ be S-modules and let $D_i : M_i \to M_{i+1}$, $i = 1, 2$ be differential operators of finite order. Then if $D'_i : M'_i \to M'_{i+1}$, $i = 1, 2$ are the extensions of D_i to $M'_i = S' \otimes_S M_i$ as in Lemma 10.150.8, then $D'_2 \circ D'_1$ is the extension of $D_2 \circ D_1$. In particular, if M is an S-module, then $M' = S' \otimes_S M$ is a module over the S-algebra $\mathrm{Diff}_{S/R}(M, M)$.

10.151. Unramified ring maps

00US The definition of a G-unramified ring map is the one from EGA. The definition of an unramified ring map is the one from [**Ray70**].

00UT **Definition 10.151.1.** Let $R \to S$ be a ring map.

(1) We say $R \to S$ is *unramified* if $R \to S$ is of finite type and $\Omega_{S/R} = 0$.

(2) We say $R \to S$ is *G-unramified* if $R \to S$ is of finite presentation and $\Omega_{S/R} = 0$.
(3) Given a prime $\mathfrak{q}$ of S we say that S is *unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified.
(4) Given a prime $\mathfrak{q}$ of S we say that S is *G-unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is G-unramified.

Of course a G-unramified map is unramified.

00UU **Lemma 10.151.2.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is formally unramified and of finite type, and*
(2) *$R \to S$ is unramified.*

Moreover, also the following are equivalent

(1) *$R \to S$ is formally unramified and of finite presentation, and*
(2) *$R \to S$ is G-unramified.*

Proof. Follows from Lemma 10.148.3 and the definitions. □

00UV **Lemma 10.151.3.** *Properties of unramified and G-unramified ring maps.*

(1) *The base change of an unramified ring map is unramified. The base change of a G-unramified ring map is G-unramified.*
(2) *The composition of unramified ring maps is unramified. The composition of G-unramified ring maps is G-unramified.*
(3) *Any principal localization $R \to R_f$ is G-unramified and unramified.*
(4) *If $I \subset R$ is an ideal, then $R \to R/I$ is unramified. If $I \subset R$ is a finitely generated ideal, then $R \to R/I$ is G-unramified.*
(5) *An étale ring map is G-unramified and unramified.*
(6) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $(\Omega_{S/R})_\mathfrak{q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(7) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $\Omega_{S/R} \otimes_S \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(8) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})})_\mathfrak{q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(9) *If $R \to S$ is of finite type (resp. presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}) \otimes_{S \otimes_R \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(10) *If $R \to S$ is a ring map, $g_1, \ldots, g_m \in S$ generate the unit ideal and $R \to S_{g_j}$ is unramified (resp. G-unramified) for $j = 1, \ldots, m$, then $R \to S$ is unramified (resp. G-unramified).*
(11) *If $R \to S$ is a ring map which is unramified (resp. G-unramified) at every prime of S, then $R \to S$ is unramified (resp. G-unramified).*
(12) *If $R \to S$ is G-unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and a G-unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that $S = R \otimes_{R_0} S_0$.*
(13) *If $R \to S$ is unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and an unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$.*

Proof. We prove each point, in order.

Ad (1). Follows from Lemmas 10.131.12 and 10.14.2.

Ad (2). Follows from Lemmas 10.131.7 and 10.14.2.

Ad (3). Follows by direct computation of $\Omega_{R_f/R}$ which we omit.

Ad (4). We have $\Omega_{(R/I)/R} = 0$, see Lemma 10.131.4, and the ring map $R \to R/I$ is of finite type. If I is a finitely generated ideal then $R \to R/I$ is of finite presentation.

Ad (5). See discussion following Definition 10.143.1.

Ad (6). In this case $\Omega_{S/R}$ is a finite S-module (see Lemma 10.131.16) and hence there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g = 0$. By Lemma 10.131.8 this means that $\Omega_{S_g/R} = 0$ and hence $R \to S_g$ is unramified as desired.

Ad (7). Use Nakayama's lemma (Lemma 10.20.1) to see that the condition is equivalent to the condition of (6).

Ad (8) and (9). These are equivalent in the same manner that (6) and (7) are equivalent. Moreover $\Omega_{S\otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})} = \Omega_{S/R} \otimes_S (S \otimes_R \kappa(\mathfrak{p}))$ by Lemma 10.131.12. Hence we see that (9) is equivalent to (7) since the $\kappa(\mathfrak{q})$ vector spaces in both are canonically isomorphic.

Ad (10). Follows from Lemmas 10.23.2 and 10.131.8.

Ad (11). Follows from (6) and (7) and the fact that the spectrum of S is quasi-compact.

Ad (12). Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_j + \sum a_{ijk} g_j \mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij}, a_{ijk} \in R[x_1, \ldots, x_n]$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_i, h_{ij}, a_{ijk}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. This works.

Ad (13). Write $S = R[x_1, \ldots, x_n]/I$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_{ij} + \sum g'_{ik}\mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij} \in R[x_1, \ldots, x_n]$ and $g_{ij}, g'_{ik} \in I$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_{ij}, h_{ij}, g'_{ik}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_{ij}, g'_{ik})$. This works. □

02FL **Lemma 10.151.4.** *Let $R \to S$ be a ring map. If $R \to S$ is unramified, then there exists an idempotent $e \in S \otimes_R S$ such that $S \otimes_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$. By assumption $J/J^2 = 0$, see Lemma 10.131.13. Since S is of finite type over R we see that J is finitely generated, namely by $x_i \otimes 1 - 1 \otimes x_i$, where x_i generate S over R. We win by Lemma 10.21.5. □

00UW **Lemma 10.151.5.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$ in R. If S/R is unramified at $\mathfrak{q}$ then*

(1) *we have $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(2) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite separable.*

Proof. We may first replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ and assume that $R \to S$ is unramified. The base change $S \otimes_R \kappa(\mathfrak{p})$ is unramified over $\kappa(\mathfrak{p})$ by Lemma 10.151.3. By Lemma 10.140.3 it is smooth hence étale over $\kappa(\mathfrak{p})$. Hence we see that $S \otimes_R \kappa(\mathfrak{p}) = (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S$ is a product of finite separable field extensions of $\kappa(\mathfrak{p})$ by Lemma 10.143.4. This implies the lemma. □

02UR **Lemma 10.151.6.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S. If $R \to S$ is unramified at $\mathfrak{q}$ then $R \to S$ is quasi-finite at $\mathfrak{q}$. In particular, an unramified ring map is quasi-finite.*

Proof. An unramified ring map is of finite type. Thus it is clear that the second statement follows from the first. To see the first statement apply the characterization of Lemma 10.122.2 part (2) using Lemma 10.151.5. □

02FM **Lemma 10.151.7.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. If*

(1) *$R \to S$ is of finite type,*
(2) *$\mathfrak{p}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(3) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite separable,*

then $R \to S$ is unramified at $\mathfrak{q}$.

Proof. By Lemma 10.151.3 (8) it suffices to show that $\Omega_{S\otimes_R\kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$ is zero when localized at $\mathfrak{q}$. Hence we may replace S by $S \otimes_R \kappa(\mathfrak{p})$ and R by $\kappa(\mathfrak{p})$. In other words, we may assume that $R = k$ is a field and S is a finite type k-algebra. In this case the hypotheses imply that $S_\mathfrak{q} \cong \kappa(\mathfrak{q})$. Thus $(\Omega_{S/k})_\mathfrak{q} = \Omega_{S_\mathfrak{q}/k} = \Omega_{\kappa(\mathfrak{q})/k}$ is zero as desired (the first equality is Lemma 10.131.8). □

08WD **Lemma 10.151.8.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is étale,*
(2) *$R \to S$ is flat and G-unramified, and*
(3) *$R \to S$ is flat, unramified, and of finite presentation.*

Proof. Parts (2) and (3) are equivalent by definition. The implication (1) $\Rightarrow$ (3) follows from the fact that étale ring maps are of finite presentation, Lemma 10.143.3 (flatness of étale maps), and Lemma 10.151.3 (étale maps are unramified). Conversely, the characterization of étale ring maps in Lemma 10.143.7 and the structure of unramified ring maps in Lemma 10.151.5 shows that (3) implies (1). (This uses that $R \to S$ is étale if $R \to S$ is étale at every prime $\mathfrak{q} \subset S$, see Lemma 10.143.3.) □

0G1C **Lemma 10.151.9.** *Let k be a field. Let*

$$\varphi : k[x_1, \ldots, x_n] \to A, \quad x_i \longmapsto a_i$$

be a finite type ring map. Then φ is étale if and only if we have the following two conditions: (a) the local rings of A at maximal ideals have dimension n, and (b) the elements $d(a_1), \ldots, d(a_n)$ generate $\Omega_{A/k}$ as an A-module.

Proof. Assume (a) and (b). Condition (b) implies that $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence φ is unramified. Thus it suffices to prove that φ is flat, see Lemma 10.151.8. Let $\mathfrak{m} \subset A$ be a maximal ideal. Set $X = \mathrm{Spec}(A)$ and denote $x \in X$ the closed point corresponding to $\mathfrak{m}$. Then $\dim(A_\mathfrak{m})$ is $\dim_x X$, see Lemma 10.114.6. Thus by Lemma 10.140.3 we see that if (a) and (b) hold, then $A_\mathfrak{m}$ is a regular local ring for

every maximal ideal $\mathfrak{m}$. Then $k[x_1, \ldots, x_n]_{\varphi^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ is flat by Lemma 10.128.1 (and the fact that a regular local ring is CM, see Lemma 10.106.3). Thus φ is flat by Lemma 10.39.18.

Assume φ is étale. Then $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence (b) holds. On the other hand, étale ring maps are flat (Lemma 10.143.3) and quasi-finite (Lemma 10.143.6). Hence for every maximal ideal $\mathfrak{m}$ of A we my apply Lemma 10.112.7 to $k[x_1, \ldots, x_n]_{\varphi^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ to see that $\dim(A_{\mathfrak{m}}) = n$ and hence (a) holds. □

10.152. Local structure of unramified ring maps

0G1D An unramified morphism is locally (in a suitable sense) the composition of a closed immersion and an étale morphism. The algebraic underpinnings of this fact are discussed in this section.

0395 **Proposition 10.152.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is unramified at $\mathfrak{q}$, then there exist*

(1) *a $g \in S$, $g \notin \mathfrak{q}$,*
(2) *a standard étale ring map $R \to S'$, and*
(3) *a surjective R-algebra map $S' \to S_g$.*

Proof. This proof is the "same" as the proof of Proposition 10.144.4. The proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.151.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified. Thus we may assume that S is unramified over R.

Step 2. By Lemma 10.151.3 there exists an unramified ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.151.6. By Lemma 10.123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that S' may not be unramified over R.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is unramified at $\mathfrak{q}$ (but no longer necessarily unramified at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 10.53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^{n} A_i$$

see Proposition 10.60.7. One of the factors, say A_1, is the local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 10.151.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ (so even if the field extension is trivial we do

not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\overline{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^{n} A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\overline{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{array}{ccccc} R[x] & \longrightarrow & S' & \longrightarrow & S \\ \uparrow & \nearrow & & \nearrow & \\ R & & & & \end{array}$$

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_{\mathfrak{q}} = S_{\mathfrak{q}'}$, see Lemma 10.41.11 (we have going up for $S' \to S$ by Lemma 10.36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_{\mathfrak{p}} \to S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$$

The second map is finite and injective. We have $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} = \kappa(\mathfrak{q})$, see Lemma 10.151.5. Hence a fortiori $S_{\mathfrak{q}}/\mathfrak{q}'S_{\mathfrak{q}} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 10.20.1 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$, the map $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_{\mathfrak{q}}$.

Step 7. By Lemma 10.126.7 there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is unramified over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1}\overline{h}_2^{d_2}\ldots\overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l\deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1\overline{h}_2^{e_2}\ldots\overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1}\overline{h}_2^{ld_2}\ldots\overline{h}_n^{ld_n} = \overline{h}_1(\overline{h}_2^{e_2}\ldots\overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1-1}\overline{h}_2^{ld_2}\ldots\overline{h}_n^{ld_n}) = \overline{h}_1\overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1\frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.144.2) and $\varphi(g) \notin \mathfrak{q}$. Thus the map $(R[x]/(f))_g \to S_{\varphi(g)}$ is the desired surjection. □

00UX **Lemma 10.152.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Assume that $R \to S$ is of finite type and unramified at $\mathfrak{q}$. Then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
(3) *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

(1) *$R' \to A$ is surjective, and*
(2) *$\mathfrak{p}'A$ is a prime of A lying over $\mathfrak{p}'$ and over $\mathfrak{q}$.*

Proof. We may replace $(R \to S, \mathfrak{p}, \mathfrak{q})$ with any base change $(R' \to R' \otimes_R S, \mathfrak{p}', \mathfrak{q}')$ by an étale ring map $R \to R'$ with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a choice of $\mathfrak{q}'$ lying over both $\mathfrak{q}$ and $\mathfrak{p}'$. Note also that given $R \to R'$ and $\mathfrak{p}'$ a suitable $\mathfrak{q}'$ can always be found.

The assumption that $R \to S$ is of finite type means that we may apply Lemma 10.145.4. Thus we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Then clearly $\mathfrak{q} = \mathfrak{r}_i$ for some i, since an unramified morphism is quasi-finite (see Lemma 10.151.6). Say $\mathfrak{q} = \mathfrak{r}_1$. By Lemma 10.151.5 we see that $\kappa(\mathfrak{r}_1)/\kappa(\mathfrak{p})$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_1)_{\mathfrak{r}_1}$ is the maximal ideal. Also, by Lemma 10.41.11 (which applies to $R \to A_1$ because a finite ring map satisfies going up by Lemma 10.36.22) we have $(A_1)_{\mathfrak{r}_1} = (A_1)_{\mathfrak{p}}$. It follows from Nakayama's Lemma

10.20.1 that the map of local rings $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} = (A_1)_{\mathfrak{r}_1}$ is surjective. Since A_1 is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_1)_f$ is surjective. After replacing R by R_f we win. □

00UY **Lemma 10.152.3.** *Let $R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. If $R \to S$ is unramified then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
(3) *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *$R' \to A_i$ is surjective,*
(2) *$\mathfrak{p}' A_i$ is a prime of A_i lying over $\mathfrak{p}'$, and*
(3) *there is no prime of B lying over $\mathfrak{p}'$.*

Proof. We may apply Lemma 10.145.4. Thus, after an étale base change, we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable, and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Since $R \to S$ is quasi-finite (see Lemma 10.151.6) we see there is no prime of B lying over $\mathfrak{p}$. By Lemma 10.151.5 we see that $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p})$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_i)_{\mathfrak{r}_i}$ is the maximal ideal. Also, by Lemma 10.41.11 (which applies to $R \to A_i$ because a finite ring map satisfies going up by Lemma 10.36.22) we have $(A_i)_{\mathfrak{r}_i} = (A_i)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 10.20.1 that the map of local rings $R_{\mathfrak{p}} \to (A_i)_{\mathfrak{p}} = (A_i)_{\mathfrak{r}_i}$ is surjective. Since A_i is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_i)_f$ is surjective. After replacing R by R_f we win. □

10.153. Henselian local rings

04GE In this section we discuss a bit the notion of a henselian local ring. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For $a \in R$ we denote $\overline{a}$ the image of a in κ. For a polynomial $f \in R[T]$ we often denote $\overline{f}$ the image of f in $\kappa[T]$. Given a polynomial $f \in R[T]$ we denote f' the derivative of f with respect to T. Note that $\overline{f}' = \overline{f'}$.

04GF **Definition 10.153.1.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) We say R is *henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f}'(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.
(2) We say R is *strictly henselian* if R is henselian and its residue field is separably algebraically closed.

Note that the condition $\overline{f'}(a_0) \neq 0$ is equivalent to the condition that a_0 is a simple root of the polynomial $\overline{f}$. In fact, it implies that the lift $a \in R$, if it exists, is unique.

06RR **Lemma 10.153.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $f \in R[T]$. Let $a, b \in R$ such that $f(a) = f(b) = 0$, $a = b \bmod \mathfrak{m}$, and $f'(a) \notin \mathfrak{m}$. Then $a = b$.*

Proof. Write $f(x+y) - f(x) = f'(x)y + g(x,y)y^2$ in $R[x,y]$ (this is possible as one sees by expanding $f(x+y)$; details omitted). Then we see that $0 = f(b) - f(a) = f(a+(b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$

is a unit in R. Hence $(b-a)(1+f'(a)^{-1}c(b-a)) = 0$. By assumption $b-a \in \mathfrak{m}$, hence $1+f'(a)^{-1}c(b-a)$ is a unit in R. Hence $b-a = 0$ in R. $\square$

Here is the characterization of henselian local rings.

04GG **Lemma 10.153.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent*

(1) *R is henselian,*
(2) *for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,*
(3) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(4) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(5) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(6) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(7) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a retraction $\tau : S \to R$ of $R \to S$,*
(8) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a unique retraction $\tau : S \to R$ of $R \to S$ such that $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$,*
(9) *any finite R-algebra is a product of local rings,*
(10) *any finite R-algebra is a finite product of local rings,*
(11) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite and $R \to B$ not quasi-finite at any prime lying over $\mathfrak{m}$,*
(12) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite such that each irreducible component of $\operatorname{Spec}(B \otimes_R \kappa)$ has dimension ≥ 1, and*
(13) *any quasi-finite R-algebra S can be written as $S = A \times B$ with $R \to A$ finite such that $B \otimes_R \kappa = 0$.*

Proof. Here is a list of the easier implications:

(1) 2⇒1 because in (2) we consider all polynomials and in (1) only monic ones,
(2) 5⇒3 because in (5) we consider all polynomials and in (3) only monic ones,
(3) 6⇒4 because in (6) we consider all polynomials and in (4) only monic ones,
(4) 4⇒3 is obvious,
(5) 6⇒5 is obvious,
(6) 8⇒7 is obvious,
(7) 10⇒9 is obvious,
(8) 11⇔12 by definition of being quasi-finite at a prime,
(9) 11⇒13 by definition of being quasi-finite,

Proof of 1⇒8. Assume (1). Let $R \to S$ be étale, and let $\mathfrak{q} \subset S$ be a prime ideal such that $\kappa(\mathfrak{q}) \cong \kappa$. By Proposition 10.144.4 we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale. After replacing S by S_g we may assume that

$S = R[t]_g/(f)$ is standard étale (details omitted). Since the prime $\mathfrak{q}$ has residue field κ it corresponds to a root a_0 of $\overline{f}$ which is not a root of $\overline{g}$. By definition of a standard étale algebra this also means that $\overline{f}'(a_0) \neq 0$. Since also f is monic by definition of a standard étale algebra again we may use that R is henselian to conclude that there exists an $a \in R$ with $a_0 = \overline{a}$ such that $f(a) = 0$. This implies that $g(a)$ is a unit of R and we obtain the desired map $\tau : S = R[t]_g/(f) \to R$ by the rule $t \mapsto a$. By construction $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$. By Lemma 10.153.2 the map τ is unique. This proves (8) holds.

Proof of 7⇒8. (This is really unimportant and should be skipped.) Assume (7) holds and assume $R \to S$ is étale. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$. Namely, we can argue that $\bigcap_{i=1}^r \mathfrak{q}_i \not\subset \mathfrak{q}$ since otherwise $\mathfrak{q}_i \subset \mathfrak{q}$ for some i, but this cannot happen as the fiber of an étale morphism is discrete (use Lemma 10.143.4 for example). Apply (7) to the étale ring map $R \to S_g$ and the prime $\mathfrak{q}S_g$. This gives a retraction $\tau_g : S_g \to R$ such that the composition $\tau : S \to S_g \to R$ has the property $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$. Details omitted.

Proof of 8⇒11. Assume (8) and let $R \to S$ be a finite type ring map. Apply Lemma 10.145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A' \times B'$ with A' finite over R' and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a retraction $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then use that

$$S = (S \otimes_R R') \otimes_{R',\tau} R = (A' \times B') \otimes_{R',\tau} R = (A' \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R)$$

which gives a decomposition as in (11).

Proof of 8⇒10. Assume (8) and let $R \to S$ be a finite ring map. Apply Lemma 10.145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A'_1 \times \ldots \times A'_n \times B'$ with A'_i finite over R' having exactly one prime over $\mathfrak{m}'$ and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a retraction $\tau : R' \to R$ with $\mathfrak{m}' = \tau^{-1}(\mathfrak{m})$. Then we obtain

$$\begin{aligned} S &= (S \otimes_R R') \otimes_{R',\tau} R \\ &= (A'_1 \times \ldots \times A'_n \times B') \otimes_{R',\tau} R \\ &= (A'_1 \otimes_{R',\tau} R) \times \ldots \times (A'_1 \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R) \\ &= A_1 \times \ldots \times A_n \times B \end{aligned}$$

The factor B is finite over R but $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{m}$. Hence $B = 0$. The factors A_i are finite R-algebras having exactly one prime lying over $\mathfrak{m}$, hence they are local rings. This proves that S is a finite product of local rings.

Proof of 9⇒10. This holds because if S is finite over the local ring R, then it has at most finitely many maximal ideals. Namely, by going up for $R \to S$ the maximal ideals of S all lie over $\mathfrak{m}$, and $S/\mathfrak{m}S$ is Artinian hence has finitely many primes.

Proof of 10⇒1. Assume (10). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S = R[T]/(f)$ is a finite R-algebra. Applying (10) we get $S = A_1 \times \ldots \times A_r$ is a finite product of local R-algebras. In particular we see that $S/\mathfrak{m}S = \prod A_i/\mathfrak{m}A_i$ is the decomposition of $\kappa[T]/(\overline{f})$ as a product of local rings. This means that one of the factors, say $A_1/\mathfrak{m}A_1$ is the quotient

$\kappa[T]/(\overline{f}) \to \kappa[T]/(T - a_0)$. Since A_1 is a summand of the finite free R-module S it is a finite free R-module itself. As $A_1/\mathfrak{m}A_1$ is a κ-vector space of dimension 1 we see that $A_1 \cong R$ as an R-module. Clearly this means that $R \to A_1$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A_1 \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 13⇒1. Assume (13). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S_1 = R[T]/(f)$ is a finite R-algebra. Let $g \in R[T]$ be any element such that $\overline{g} = \overline{f}/(T - a_0)$. Then $S = (S_1)_g$ is a quasi-finite R-algebra such that $S \otimes_R \kappa \cong \kappa[T]_{\overline{g}}/(\overline{f}) \cong \kappa[T]/(T - a_0) \cong \kappa$. Applying (13) to S we get $S = A \times B$ with A finite over R and $B \otimes_R \kappa = 0$. In particular we see that $\kappa \cong S/\mathfrak{m}S = A/\mathfrak{m}A$. Since A is a summand of the flat R-algebra S we see that it is finite flat, hence free over R. As $A/\mathfrak{m}A$ is a κ-vector space of dimension 1 we see that $A \cong R$ as an R-module. Clearly this means that $R \to A$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 8⇒2. Assume (8). Let $f \in R[T]$ be any polynomial and let $a_0 \in \kappa$ be a simple root. Then the algebra $S = R[T]_{f'}/(f)$ is étale over R. Let $\mathfrak{q} \subset S$ be the prime generated by $\mathfrak{m}$ and $T - b$ where $b \in R$ is any element such that $\overline{b} = a_0$. Apply (8) to S and $\mathfrak{q}$ to get $\tau : S \to R$. Then the image $\tau(T) = a \in R$ works in (2).

At this point we see that (1), (2), (7), (8), (9), (10), (11), (12), (13) are all equivalent. The weakest assertion of (3), (4), (5) and (6) is (3) and the strongest is (6). Hence we still have to prove that (3) implies (1) and (1) implies (6).

Proof of 3⇒1. Assume (3). Let $f \in R[T]$ be monic and let $a_0 \in \kappa$ be a simple root of $\overline{f}$. This gives a factorization $\overline{f} = (T - a_0)h_0$ with $h_0(a_0) \neq 0$, so $\gcd(T - a_0, h_0) = 1$. Apply (3) to get a factorization $f = gh$ with $\overline{g} = T - a_0$ and $\overline{h} = h_0$. Set $S = R[T]/(f)$ which is a finite free R-algebra. We will write g, h also for the images of g and h in S. Then $gS + hS = S$ by Nakayama's Lemma 10.20.1 as the equality holds modulo $\mathfrak{m}$. Since $gh = f = 0$ in S this also implies that $gS \cap hS = 0$. Hence by the Chinese Remainder theorem we obtain $S = S/(g) \times S/(h)$. This implies that $A = S/(g)$ is a summand of a finite free R-module, hence finite free. Moreover, the rank of A is 1 as $A/\mathfrak{m}A = \kappa[T]/(T - a_0)$. Thus the map $R \to A$ is an isomorphism. Setting $a \in R$ equal to the image of T under the maps $R[T] \to S \to A \to R$ gives an element of R with $f(a) = 0$ and $\overline{a} = a_0$.

Proof of 1⇒6. Assume (1) or equivalently all of (1), (2), (7), (8), (9), (10), (11), (12), (13). Let $f \in R[T]$ be a polynomial. Suppose that $\overline{f} = g_0 h_0$ is a factorization with $\gcd(g_0, h_0) = 1$. We may and do assume that g_0 is monic. Consider $S = R[T]/(f)$. Because we have the factorization we see that the coefficients of f generate the unit ideal in R. This implies that S has finite fibres over R, hence is quasi-finite over R. It also implies that S is flat over R by Lemma 10.128.5. Combining (13) and (10) we may write $S = A_1 \times \ldots \times A_n \times B$ where each A_i is local and finite over R, and $B \otimes_R \kappa = 0$. After reordering the factors $A_1, \ldots, A_n$ we may assume that

$$\kappa[T]/(g_0) = A_1/\mathfrak{m}A_1 \times \ldots \times A_r/\mathfrak{m}A_r,\ \kappa[T]/(h_0) = A_{r+1}/\mathfrak{m}A_{r+1} \times \ldots \times A_n/\mathfrak{m}A_n$$

as quotients of $\kappa[T]$. The finite flat R-algebra $A = A_1 \times \ldots \times A_r$ is free as an R-module, see Lemma 10.78.5. Its rank is $\deg_T(g_0)$. Let $g \in R[T]$ be the characteristic polynomial of the R-linear operator $T : A \to A$. Then g is a monic polynomial of

degree $\deg_T(g) = \deg_T(g_0)$ and moreover $\overline{g} = g_0$. By Cayley-Hamilton (Lemma 10.16.1) we see that $g(T_A) = 0$ where T_A indicates the image of T in A. Hence we obtain a well defined surjective map $R[T]/(g) \to A$ which is an isomorphism by Nakayama's Lemma 10.20.1. The map $R[T] \to A$ factors through $R[T]/(f)$ by construction hence we may write $f = gh$ for some h. This finishes the proof. □

04GH **Lemma 10.153.4.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring.*

(1) *If $R \to S$ is a finite ring map then S is a finite product of henselian local rings each finite over R.*
(2) *If $R \to S$ is a finite ring map and S is local, then S is a henselian local ring and $R \to S$ is a (finite) local ring map.*
(3) *If $R \to S$ is a finite type ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ at which $R \to S$ is quasi-finite, then $S_\mathfrak{q}$ is henselian and finite over R.*
(4) *If $R \to S$ is quasi-finite then $S_\mathfrak{q}$ is henselian and finite over R for every prime $\mathfrak{q}$ lying over $\mathfrak{m}$.*

Proof. Part (2) implies part (1) since S as in part (1) is a finite product of its localizations at the primes lying over $\mathfrak{m}$ by Lemma 10.153.3 part (10). Part (2) also follows from Lemma 10.153.3 part (10) since any finite S-algebra is also a finite R-algebra (of course any finite ring map between local rings is local).

Let $R \to S$ and $\mathfrak{q}$ be as in (3). Write $S = A \times B$ with A finite over R and B not quasi-finite over R at any prime lying over $\mathfrak{m}$, see Lemma 10.153.3 part (11). Hence $S_\mathfrak{q}$ is a localization of A at a maximal ideal and we deduce (3) from (1). Part (4) follows from part (3). □

04GJ **Lemma 10.153.5.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. This is a combination of parts (11) and (10) of Lemma 10.153.3. □

06DD **Lemma 10.153.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ finite purely inseparable and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.153.5. The field extension $\kappa(\mathfrak{m}_{A_i})/\kappa$ is finite and κ is separably algebraically closed, hence it is finite purely inseparable. □

04GK **Lemma 10.153.7.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. The category of finite étale ring extensions $R \to S$ is equivalent to the category of finite étale algebras $\kappa \to \overline{S}$ via the functor $S \mapsto S/\mathfrak{m}S$.*

Proof. Denote $\mathcal{C} \to \mathcal{D}$ the functor of categories of the statement. Suppose that $R \to S$ is finite étale. Then we may write

$$S = A_1 \times \ldots \times A_n$$

with A_i local and finite étale over S, use either Lemma 10.153.5 or Lemma 10.153.3 part (10). In particular $A_i/\mathfrak{m}A_i$ is a finite separable field extension of κ, see Lemma 10.143.5. Thus we see that every object of $\mathcal{C}$ and $\mathcal{D}$ decomposes canonically into irreducible pieces which correspond via the given functor. Next, suppose that S_1,

S_2 are finite étale over R such that $\kappa_1 = S_1/\mathfrak{m}S_1$ and $\kappa_2 = S_2/\mathfrak{m}S_2$ are fields (finite separable over κ). Then $S_1 \otimes_R S_2$ is finite étale over R and we may write

$$S_1 \otimes_R S_2 = A_1 \times \ldots \times A_n$$

as before. Then we see that $\mathrm{Hom}_R(S_1, S_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $S_2 \to A_i$ is an isomorphism. To see this use that given any R-algebra map $\varphi : S_1 \to S_2$ the map $\varphi \times 1 : S_1 \otimes_R S_2 \to S_2$ is surjective, and hence is equal to projection onto one of the factors A_i. But in exactly the same way we see that $\mathrm{Hom}_\kappa(\kappa_1, \kappa_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $\kappa_2 \to A_i/\mathfrak{m}A_i$ is an isomorphism. By the discussion above these sets of indices match, and we conclude that our functor is fully faithful. Finally, let κ'/κ be a finite separable field extension. By Lemma 10.144.3 there exists an étale ring map $R \to S$ and a prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ such that $\kappa \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension. By part (1) we may write $S = A_1 \times \ldots \times A_n \times B$. Since $R \to S$ is quasi-finite we see that there exists no prime of B over $\mathfrak{m}$. Hence $S_\mathfrak{q}$ is equal to A_i for some i. Hence $R \to A_i$ is finite étale and produces the given residue field extension. Thus the functor is essentially surjective and we win. □

04GL **Lemma 10.153.8.** *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Let $R \to S$ be an unramified ring map. Then*

$$S = A_1 \times \ldots \times A_n \times B$$

with each $R \to A_i$ surjective and no prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.153.5. Now we see that $R \to A_i$ is finite unramified and A_i local. Hence the maximal ideal of A_i is $\mathfrak{m}A_i$ and its residue field $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ, see Lemma 10.151.5. However, the condition that R is strictly henselian means that κ is separably algebraically closed, so $\kappa = A_i/\mathfrak{m}A_i$. By Nakayama's Lemma 10.20.1 we conclude that $R \to A_i$ is surjective as desired. □

04GM **Lemma 10.153.9.** *Let $(R, \mathfrak{m}, \kappa)$ be a complete local ring, see Definition 10.160.1. Then R is henselian.*

Proof. Let $f \in R[T]$ be monic. Denote $f_n \in R/\mathfrak{m}^{n+1}[T]$ the image. Denote f_n' the derivative of f_n with respect to T. Let $a_0 \in \kappa$ be a simple root of f_0. We lift this to a solution of f over R inductively as follows: Suppose given $a_n \in R/\mathfrak{m}^{n+1}$ such that $a_n \bmod \mathfrak{m} = a_0$ and $f_n(a_n) = 0$. Pick any element $b \in R/\mathfrak{m}^{n+2}$ such that $a_n = b \bmod \mathfrak{m}^{n+1}$. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f_{n+1}'(b)$$

(Newton's method). This makes sense as $f_{n+1}'(b) \in R/\mathfrak{m}^{n+2}$ is invertible by the condition on a_0. Then we compute $f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$ in $R/\mathfrak{m}^{n+2}$. Since the system of elements $a_n \in R/\mathfrak{m}^{n+1}$ so constructed is compatible we get an element $a \in \lim R/\mathfrak{m}^n = R$ (here we use that R is complete). Moreover, $f(a) = 0$ since it maps to zero in each $R/\mathfrak{m}^n$. Finally $\overline{a} = a_0$ and we win. □

06RS **Lemma 10.153.10.** *Let $(R, \mathfrak{m})$ be a local ring of dimension 0. Then R is henselian.*

Proof. Let $R \to S$ be a finite ring map. By Lemma 10.153.3 it suffices to show that S is a product of local rings. By Lemma 10.36.21 S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ which all lie over $\mathfrak{m}$. There are no inclusions among these primes, see

Lemma 10.36.20, hence they are all maximal. Every element of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ is nilpotent by Lemma 10.17.2. It follows S is the product of the localizations of S at the primes $\mathfrak{m}_i$ by Lemma 10.53.5. □

The following lemma will be the key to the uniqueness and functorial properties of henselization and strict henselization.

08HQ **Lemma 10.153.11.** *Let $R \to S$ be a ring map with S henselian local. Given*

(1) *an étale ring map $R \to A$,*
(2) *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*
(3) *a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and f induces the map τ on residue fields.

Proof. Consider $A \otimes_R S$. This is an étale algebra over S, see Lemma 10.143.3. Moreover, the kernel

$$\mathfrak{q}' = \mathrm{Ker}(A \otimes_R S \to \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{m}_S) \xrightarrow{\tau \otimes 1} \kappa(\mathfrak{m}_S))$$

is a prime ideal lying over $\mathfrak{m}_S$ with residue field equal to the residue field of S. Hence by Lemma 10.153.3 there exists a unique retraction $\sigma : A \otimes_R S \to S$ with $\sigma^{-1}(\mathfrak{m}_S) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S \to S$. We omit the verification of the properties of f; the uniqueness of f comes from the uniqueness of σ (details omitted). □

04GX **Lemma 10.153.12.** *Let $\varphi : R \to S$ be a local homomorphism of strictly henselian local rings. Let $P_1, \ldots, P_n \in R[x_1, \ldots, x_n]$ be polynomials such that $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R. Then the map*

$$R^n \longrightarrow S^n, \quad (h_1, \ldots, h_n) \longmapsto (\varphi(h_1), \ldots, \varphi(h_n))$$

induces a bijection between

$$\{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0,\ i = 1, \ldots, n\}$$

and

$$\{(s_1, \ldots, s_n) \in S^n \mid P_i^\varphi(s_1, \ldots, s_n) = 0,\ i = 1, \ldots, n\}$$

where $P_i^\varphi \in S[x_1, \ldots, x_n]$ are the images of the P_i under φ.

Proof. The first solution set is canonically isomorphic to the set

$$\mathrm{Hom}_R(R[x_1, \ldots, x_n]/(P_1, \ldots, P_n), R).$$

As R is henselian the map $R \to R/\mathfrak{m}_R$ induces a bijection between this set and the set of solutions in the residue field $R/\mathfrak{m}_R$, see Lemma 10.153.3. The same is true for S. Now since $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R and $R/\mathfrak{m}_R$ is separably algebraically closed we see that $R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P}_1, \ldots, \overline{P}_n)$ is a finite product of copies of $R/\mathfrak{m}_R$ where $\overline{P}_i$ is the image of P_i in $R/\mathfrak{m}_R[x_1, \ldots, x_n]$. Hence the tensor product

$$R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P}_1, \ldots, \overline{P}_n) \otimes_{R/\mathfrak{m}_R} S/\mathfrak{m}_S = S/\mathfrak{m}_S[x_1, \ldots, x_n]/(\overline{P}_1^\varphi, \ldots, \overline{P}_n^\varphi)$$

is also a finite product of copies of $S/\mathfrak{m}_S$ with the same index set. This proves the lemma. □

05D6 **Lemma 10.153.13.** *Let R be a henselian local ring. Any countably generated Mittag-Leffler module over R is a direct sum of finitely presented R-modules.*

Proof. Let M be a countably generated and Mittag-Leffler R-module. We claim that for any element $x \in M$ there exists a direct sum decomposition $M = N \oplus K$ with $x \in N$, the module N finitely presented, and K Mittag-Leffler.

Suppose the claim is true. Choose generators $x_1, x_2, x_3, \ldots$ of M. By the claim we can inductively find direct sum decompositions

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_n \oplus K_n$$

with N_i finitely presented, $x_1, \ldots, x_n \in N_1 \oplus \ldots \oplus N_n$, and K_n Mittag-Leffler. Repeating ad infinitum we see that $M = \bigoplus N_i$.

We still have to prove the claim. Let $x \in M$. By Lemma 10.92.2 there exists an endomorphism $\alpha : M \to M$ such that α factors through a finitely presented module, and $\alpha(x) = x$. Say α factors as

$$M \xrightarrow{\pi} P \xrightarrow{i} M$$

Set $a = \pi \circ \alpha \circ i : P \to P$, so $i \circ a \circ \pi = \alpha^3$. By Lemma 10.16.2 there exists a monic polynomial $P \in R[T]$ such that $P(a) = 0$. Note that this implies formally that $\alpha^2 P(\alpha) = 0$. Hence we may think of M as a module over $R[T]/(T^2 P)$. Assume that $x \neq 0$. Then $\alpha(x) = x$ implies that $0 = \alpha^2 P(\alpha)x = P(1)x$ hence $P(1) = 0$ in R/I where $I = \{r \in R \mid rx = 0\}$ is the annihilator of x. As $x \neq 0$ we see $I \subset \mathfrak{m}_R$, hence 1 is a root of $\overline{P} = P \bmod \mathfrak{m}_R \in R/\mathfrak{m}_R[T]$. As R is henselian we can find a factorization

$$T^2 P = (T^2 Q_1) Q_2$$

for some $Q_1, Q_2 \in R[T]$ with $Q_2 = (T-1)^e \bmod \mathfrak{m}_R R[T]$ and $Q_1(1) \neq 0 \bmod \mathfrak{m}_R$, see Lemma 10.153.3. Let $N = \mathrm{Im}(\alpha^2 Q_1(\alpha) : M \to M)$ and $K = \mathrm{Im}(Q_2(\alpha) : M \to M)$. As $T^2 Q_1$ and Q_2 generate the unit ideal of $R[T]$ we get a direct sum decomposition $M = N \oplus K$. Moreover, Q_2 acts as zero on N and $T^2 Q_1$ acts as zero on K. Note that N is a quotient of P hence is finitely generated. Also $x \in N$ because $\alpha^2 Q_1(\alpha)x = Q_1(1)x$ and $Q_1(1)$ is a unit in R. By Lemma 10.89.10 the modules N and K are Mittag-Leffler. Finally, the finitely generated module N is finitely presented as a finitely generated Mittag-Leffler module is finitely presented, see Example 10.91.1 part (1). □

10.154. Filtered colimits of étale ring maps

0BSG This section is a precursor to the section on ind-étale ring maps (Pro-étale Cohomology, Section 61.7). The material will also be useful to prove uniqueness properties of the henselization and strict henselization of a local ring.

0BSH **Lemma 10.154.1.** *Let $R \to A$ and $R \to R'$ be ring maps. If A is a filtered colimit of étale ring maps, then so is $R' \to R' \otimes_R A$.*

Proof. This is true because colimits commute with tensor products and étale ring maps are preserved under base change (Lemma 10.143.3). □

0BSI **Lemma 10.154.2.** *Let $A \to B \to C$ be ring maps. If $A \to B$ is a filtered colimit of étale ring maps and $B \to C$ is a filtered colimit of étale ring maps, then $A \to C$ is a filtered colimit of étale ring maps.*

Proof. We will use the criterion of Lemma 10.127.4. Let $A \to P \to C$ be a factorization of $A \to C$ with P of finite presentation over A. Write $B = \operatorname{colim}_{i \in I} B_i$ where I is a directed set and where B_i is an étale A-algebra. Write $C = \operatorname{colim}_{j \in J} C_j$ where J is a directed set and where C_j is an étale B-algebra. We can factor $P \to C$ as $P \to C_j \to C$ for some j by Lemma 10.127.3. By Lemma 10.143.3 we can find an $i \in I$ and an étale ring map $B_i \to C'_j$ such that $C_j = B \otimes_{B_i} C'_j$. Then $C_j = \operatorname{colim}_{i' \geq i} B_{i'} \otimes_{B_i} C'_j$ and again we see that $P \to C_j$ factors as $P \to B_{i'} \otimes_{B_i} C'_j \to C$. As $A \to C' = B_{i'} \otimes_{B_i} C'_j$ is étale as compositions and tensor products of étale ring maps are étale. Hence we have factored $P \to C$ as $P \to C' \to C$ with C' étale over A and the criterion of Lemma 10.127.4 applies. □

0BSJ **Lemma 10.154.3.** *Let R be a ring. Let $A = \operatorname{colim} A_i$ be a filtered colimit of R-algebras such that each A_i is a filtered colimit of étale R-algebras. Then A is a filtered colimit of étale R-algebras.*

Proof. Write $A_i = \operatorname{colim}_{j \in J_i} A_j$ where J_i is a directed set and A_j is an étale R-algebra. For each $i \leq i'$ and $j \in J_i$ there exists an $j' \in J_{i'}$ and an R-algebra map $\varphi_{jj'} : A_j \to A_{j'}$ making the diagram

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i'} \\ \uparrow & & \uparrow \\ A_j & \xrightarrow{\varphi_{jj'}} & A_{j'} \end{array}$$

commute. This is true because $R \to A_j$ is of finite presentation so that Lemma 10.127.3 applies. Let $\mathcal{J}$ be the category with objects $\coprod_{i \in I} J_i$ and morphisms triples $(j, j', \varphi_{jj'})$ as above (and obvious composition law). Then $\mathcal{J}$ is a filtered category and $A = \operatorname{colim}_{\mathcal{J}} A_j$. Details omitted. □

0GIM **Lemma 10.154.4.** *Let I be a directed set. Let $i \mapsto (R_i \to A_i)$ be a system of arrows of rings over I. Set $R = \operatorname{colim} R_i$ and $A = \operatorname{colim} A_i$. If each A_i is a filtered colimit of étale R_i-algebras, then A is a filtered colimit of étale R-algebras.*

Proof. This is true because $A = A \otimes_R R = \operatorname{colim} A_i \otimes_{R_i} R$ and hence we can apply Lemma 10.154.3 because $R \to A_i \otimes_{R_i} R$ is a filtered colimit of étale ring maps by Lemma 10.154.1. □

08HS **Lemma 10.154.5.** *Let R be a ring. Let $A \to B$ be an R-algebra homomorphism. If A and B are filtered colimits of étale R-algebras, then B is a filtered colimit of étale A-algebras.*

Proof. Write $A = \operatorname{colim} A_i$ and $B = \operatorname{colim} B_j$ as filtered colimits with A_i and B_j étale over R. For each i we can find a j such that $A_i \to B$ factors through B_j, see Lemma 10.127.3. The factorization $A_i \to B_j$ is étale by Lemma 10.143.8. Since $A \to A \otimes_{A_i} B_j$ is étale (Lemma 10.143.3) it suffices to prove that $B = \operatorname{colim} A \otimes_{A_i} B_j$ where the colimit is over pairs (i, j) and factorizations $A_i \to B_j \to B$ of $A_i \to B$ (this is a directed system; details omitted). This is clear because colimits commute with tensor products and hence $\operatorname{colim} A \otimes_{A_i} B_j = A \otimes_A B = B$. □

08HR **Lemma 10.154.6.** *Let $R \to S$ be a ring map with S henselian local. Given*

(1) *an R-algebra A which is a filtered colimit of étale R-algebras,*
(2) *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*

(3) *a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and $f \bmod \mathfrak{q} = \tau$.

Proof. Write $A = \operatorname{colim} A_i$ as a filtered colimit of étale R-algebras. Set $\mathfrak{q}_i = A_i \cap \mathfrak{q}$. We obtain $f_i : A_i \to S$ by applying Lemma 10.153.11. Set $f = \operatorname{colim} f_i$. □

08HT **Lemma 10.154.7.** *Let R be a ring. Given a commutative diagram of ring maps*

$$\begin{array}{ccc} S & \longrightarrow & K \\ \uparrow & & \uparrow \\ R & \longrightarrow & S' \end{array}$$

where S, S' are henselian local, S, S' are filtered colimits of étale R-algebras, K is a field and the arrows $S \to K$ and $S' \to K$ identify K with the residue field of both S and S'. Then there exists a unique R-algebra isomorphism $S \to S'$ compatible with the maps to K.

Proof. Follows immediately from Lemma 10.154.6. □

The following lemma is not strictly speaking about colimits of étale ring maps.

04GI **Lemma 10.154.8.** *A filtered colimit of (strictly) henselian local rings along local homomorphisms is (strictly) henselian.*

Proof. Categories, Lemma 4.21.5 says that this is really just a question about a colimit of (strictly) henselian local rings over a directed set. Let $(R_i, \varphi_{ii'})$ be such a system with each $\varphi_{ii'}$ local. Then $R = \operatorname{colim}_i R_i$ is local, and its residue field κ is $\operatorname{colim} \kappa_i$ (argument omitted). It is easy to see that $\operatorname{colim} \kappa_i$ is separably algebraically closed if each κ_i is so; thus it suffices to prove R is henselian if each R_i is henselian. Suppose that $f \in R[T]$ is monic and that $a_0 \in \kappa$ is a simple root of $\overline{f}$. Then for some large enough i there exists an $f_i \in R_i[T]$ mapping to f and an $a_{0,i} \in \kappa_i$ mapping to a_0. Since $\overline{f_i}(a_{0,i}) \in \kappa_i$, resp. $\overline{f_i'}(a_{0,i}) \in \kappa_i$ maps to $0 = \overline{f}(a_0) \in \kappa$, resp. $0 \neq \overline{f'}(a_0) \in \kappa$ we conclude that $a_{0,i}$ is a simple root of $\overline{f_i}$. As R_i is henselian we can find $a_i \in R_i$ such that $f_i(a_i) = 0$ and $a_{0,i} = \overline{a_i}$. Then the image $a \in R$ of a_i is the desired solution. Thus R is henselian. □

10.155. Henselization and strict henselization

0BSK In this section we construct the henselization. We encourage the reader to keep in mind the uniqueness already proved in Lemma 10.154.7 and the functorial behaviour pointed out in Lemma 10.154.6 while reading this material.

04GN **Lemma 10.155.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties*

(1) *R^h is henselian,*
(2) *R^h is a filtered colimit of étale R-algebras,*
(3) *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and*
(4) *$\kappa = R^h/\mathfrak{m}R^h$.*

Proof. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is an étale ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. We set

$$R^h = \operatorname{colim}_{(S,\mathfrak{q})} S.$$

Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{m})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. For any pair $(S, \mathfrak{q})$ the prime ideal $\mathfrak{q}$ is maximal with residue field κ since the composition $\kappa \to S/\mathfrak{q} \to \kappa(\mathfrak{q})$ is an isomorphism. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two objects. Set $S'' = S \otimes_R S'$ and $\mathfrak{q}'' = \mathfrak{q}S'' + \mathfrak{q}'S''$. Then $S''/\mathfrak{q}'' = S/\mathfrak{q} \otimes_R S'/\mathfrak{q}' = \kappa$ by what we said above. Moreover, $R \to S''$ is étale by Lemma 10.143.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

with prime ideal

$$\mathfrak{q}'' = (\mathfrak{q}' \otimes S' + S' \otimes \mathfrak{q}') \otimes S' + (S' \otimes_{\varphi,S,\psi} S') \otimes \mathfrak{q}'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. Moreover, the canonical map $S' \to S''$ (using the right most factor for example) equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that R^h consists of triples $(S, \mathfrak{q}, f)$ with $f \in S$, and two such triples $(S, \mathfrak{q}, f)$, $(S', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists a pair $(S'', \mathfrak{q}'')$ and morphisms of pairs $\varphi : (S, \mathfrak{q}) \to (S'', \mathfrak{q}'')$ and $\varphi' : (S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

Suppose that $x \in R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then $\mathfrak{q} \not\subset \mathfrak{q}_i$ as we have seen above that $\mathfrak{q}$ is maximal. Thus, since $\mathfrak{q}$ is a prime ideal, we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$. Consider the morphism of pairs $(S, \mathfrak{q}) \to (S_g, \mathfrak{q}S_g)$. In this way we see that we may always assume that x is given by a triple $(S, \mathfrak{q}, f)$ where $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, i.e., $\sqrt{\mathfrak{m}S} = \mathfrak{q}$. But since $R \to S$ is étale, we have $\mathfrak{m}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$, see Lemma 10.143.5. Hence we actually get that $\mathfrak{m}S = \mathfrak{q}$.

Suppose that $x \notin \mathfrak{m}R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \notin \mathfrak{m}S$, i.e., $f \notin \mathfrak{q}$. Hence $(S, \mathfrak{q}) \to (S_f, \mathfrak{q}S_f)$ is a morphism of pairs such that the image of f becomes invertible. Hence x is invertible with inverse represented by the triple $(S_f, \mathfrak{q}S_f, 1/f)$. We conclude that R^h is a local ring with maximal ideal $\mathfrak{m}R^h$. The residue field is κ since we can define $R^h/\mathfrak{m}R^h \to \kappa$ by mapping a triple $(S, \mathfrak{q}, f)$ to the residue class of f modulo $\mathfrak{q}$.

We still have to show that R^h is henselian. Namely, suppose that $P \in R^h[T]$ is a monic polynomial and $a_0 \in \kappa$ is a simple root of the reduction $\overline{P} \in \kappa[T]$. Then we can find a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial $Q \in S[T]$. Set $S' = S[T]/(Q)$ and let $\mathfrak{q}' \subset S'$ be the maximal ideal $\mathfrak{q}' = \mathfrak{q}S' + (T - a')S'$ where $a' \in S$ is any element lifting a_0. By construction $S \to S'$ is étale at $\mathfrak{q}'$ and $\kappa = \kappa(\mathfrak{q}')$. Pick $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'' = S'_g$ is étale over S. Then $(S, \mathfrak{q}) \to (S'', \mathfrak{q}'S'')$ is a morphism of pairs. Now that triple $(S'', \mathfrak{q}'S'', \text{class of } T)$ determines an element $a \in R^h$ with the properties $P(a) = 0$, and $\overline{a} = a_0$ as desired. $\square$

04GP **Lemma 10.155.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a commutative diagram*

$$\begin{array}{ccccc} \kappa & \longrightarrow & \kappa & \longrightarrow & \kappa^{sep} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R^h & \longrightarrow & R^{sh} \end{array}$$

with the following properties

(1) *the map $R^h \to R^{sh}$ is local*
(2) *R^{sh} is strictly henselian,*
(3) *R^{sh} is a filtered colimit of étale R-algebras,*
(4) *$\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and*
(5) *$\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.*

Proof. This is proved by exactly the same proof as used for Lemma 10.155.1. The only difference is that, instead of pairs, one uses triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ étale, $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is an embedding of extensions of κ. □

04GQ **Definition 10.155.3.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) The local ring map $R \to R^h$ constructed in Lemma 10.155.1 is called the *henselization* of R.
(2) Given a separable algebraic closure $\kappa \subset \kappa^{sep}$ the local ring map $R \to R^{sh}$ constructed in Lemma 10.155.2 is called the *strict henselization of R with respect to $\kappa \subset \kappa^{sep}$*.
(3) A local ring map $R \to R^{sh}$ is called a *strict henselization* of R if it is isomorphic to one of the local ring maps constructed in Lemma 10.155.2

The maps $R \to R^h \to R^{sh}$ are flat local ring homomorphisms. By Lemma 10.154.7 the R-algebras R^h and R^{sh} are well defined up to unique isomorphism by the conditions that they are henselian local, filtered colimits of étale R-algebras with residue field κ and κ^{sep}. In the rest of this section we mostly just discuss functoriality of the (strict) henselizations. We will discuss more intricate results concerning the relationship between R and its henselization in More on Algebra, Section 15.45.

0BSL **Remark 10.155.4.** We can also construct R^{sh} from R^h. Namely, for any finite separable subextension $\kappa^{sep}/\kappa'/\kappa$ there exists a unique (up to unique isomorphism) finite étale local ring extension $R^h \subset R^h(\kappa')$ whose residue field extension reproduces the given extension, see Lemma 10.153.7. Hence we can set

$$R^{sh} = \bigcup\nolimits_{\kappa \subset \kappa' \subset \kappa^{sep}} R^h(\kappa')$$

The arrows in this system, compatible with the arrows on the level of residue fields, exist by Lemma 10.153.7. This will produce a henselian local ring by Lemma 10.154.8 since each of the rings $R^h(\kappa')$ is henselian by Lemma 10.153.4. By construction the residue field extension induced by $R^h \to R^{sh}$ is the field extension κ^{sep}/κ. Hence R^{sh} so constructed is strictly henselian. By Lemma 10.154.2 the R-algebra R^{sh} is a colimit of étale R-algebras. Hence the uniqueness of Lemma 10.154.7 shows that R^{sh} is the strict henselization.

04GR **Lemma 10.155.5.** *Let $R \to S$ be a local map of local rings. Let $S \to S^h$ be the henselization. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying*

over $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R \cong \kappa(\mathfrak{q})$. Then there exists a unique morphism of rings $f : A \to S^h$ fitting into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & S^h \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$.

Proof. This is a special case of Lemma 10.153.11. □

04GS **Lemma 10.155.6.** *Let $R \to S$ be a local map of local rings. Let $R \to R^h$ and $S \to S^h$ be the henselizations. There exists a unique local ring map $R^h \to S^h$ fitting into the commutative diagram*

$$\begin{array}{ccc} R^h & \xrightarrow{f} & S^h \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

Proof. Follows immediately from Lemma 10.154.6. □

Here is a slightly different construction of the henselization.

04GV **Lemma 10.155.7.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is étale and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This category is filtered and*

$$(R_\mathfrak{p})^h = \operatorname{colim}_{(S,\mathfrak{q})} S = \operatorname{colim}_{(S,\mathfrak{q})} S_\mathfrak{q}$$

canonically.

Proof. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{p})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two pairs. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields $\kappa(\mathfrak{p})$, hence they correspond to maximal ideals of $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$. Set $S'' = S \otimes_R S'$. By the above there exists a unique prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$ whose residue field is $\kappa(\mathfrak{p})$. The ring map $R \to S''$ is étale by Lemma 10.143.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak{p}$ is

$$F'' = (F' \otimes_{\varphi,F,\psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of pairs the map $F' \to \kappa(\mathfrak{p})$ corresponding to $\mathfrak{p}'$ extends to a map $F'' \to \kappa(\mathfrak{p})$ and in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S''$ whose residue field is $\kappa(\mathfrak{p})$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of pairs

$(S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

Recall that in the proof of Lemma 10.155.1 we constructed $(R_\mathfrak{p})^h$ as the corresponding colimit but starting with $R_\mathfrak{p}$ and its maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Now, given any pair $(S, \mathfrak{q})$ for $(R, \mathfrak{p})$ we obtain a pair $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$. Moreover, in this situation

$$S_\mathfrak{p} = \operatorname{colim}_{f \in R, f \notin \mathfrak{p}} S_f.$$

Hence in order to show the equalities of the lemma, it suffices to show that any pair $(S_{loc}, \mathfrak{q}_{loc})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is of the form $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for some pair $(S, \mathfrak{q})$ over $(R, \mathfrak{p})$ (some details omitted). This follows from Lemma 10.143.3. □

08HU **Lemma 10.155.8.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $R \to R^h$ and $S \to S^h$ be the henselizations of $R_\mathfrak{p}$ and $S_\mathfrak{q}$. The local ring map $R^h \to S^h$ of Lemma 10.155.6 identifies S^h with the henselization of $R^h \otimes_R S$ at the unique prime lying over $\mathfrak{m}^h$ and $\mathfrak{q}$.*

Proof. By Lemma 10.155.7 we see that R^h, resp. S^h are filtered colimits of étale R, resp. S-algebras. Hence we see that $R^h \otimes_R S$ is a filtered colimit of étale S-algebras A_i (Lemma 10.143.3). By Lemma 10.154.5 we see that S^h is a filtered colimit of étale $R^h \otimes_R S$-algebras. Since moreover S^h is a henselian local ring with residue field equal to $\kappa(\mathfrak{q})$, the statement follows from the uniqueness result of Lemma 10.154.7. □

04GT **Lemma 10.155.9.** *Let $\varphi : R \to S$ be a local map of local rings. Let $S/\mathfrak{m}_S \subset \kappa^{sep}$ be a separable algebraic closure. Let $S \to S^{sh}$ be the strict henselization of S with respect to $S/\mathfrak{m}_S \subset \kappa^{sep}$. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$. Given any commutative diagram*

$$\begin{array}{ccc} \kappa(\mathfrak{q}) & \xrightarrow{\phi} & \kappa^{sep} \\ \uparrow & & \uparrow \\ R/\mathfrak{m}_R & \xrightarrow{\varphi} & S/\mathfrak{m}_S \end{array}$$

there exists a unique morphism of rings $f : A \to S^{sh}$ fitting into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & S^{sh} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$ and the induced map $\kappa(\mathfrak{q}) \to \kappa^{sep}$ is the given one.

Proof. This is a special case of Lemma 10.153.11. □

04GU **Lemma 10.155.10.** *Let $R \to S$ be a local map of local rings. Choose separable algebraic closures $R/\mathfrak{m}_R \subset \kappa_1^{sep}$ and $S/\mathfrak{m}_S \subset \kappa_2^{sep}$. Let $R \to R^{sh}$ and $S \to S^{sh}$ be the corresponding strict henselizations. Given any commutative diagram*

$$\begin{array}{ccc} \kappa_1^{sep} & \xrightarrow{\phi} & \kappa_2^{sep} \\ \uparrow & & \uparrow \\ R/\mathfrak{m}_R & \xrightarrow{\varphi} & S/\mathfrak{m}_S \end{array}$$

There exists a unique local ring map $R^{sh} \to S^{sh}$ fitting into the commutative diagram

$$\begin{array}{ccc} R^{sh} & \xrightarrow{f} & S^{sh} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

and inducing ϕ on the residue fields of R^{sh} and S^{sh}.

Proof. Follows immediately from Lemma 10.154.6. □

04GW **Lemma 10.155.11.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\kappa(\mathfrak{p}) \subset \kappa^{sep}$ be a separable algebraic closure. Consider the category of triples $(S, \mathfrak{q}, \phi)$ where $R \to S$ is étale, $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$, and $\phi : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is a $\kappa(\mathfrak{p})$-algebra map. This category is filtered and*

$$(R_{\mathfrak{p}})^{sh} = \operatorname{colim}_{(S,\mathfrak{q},\phi)} S = \operatorname{colim}_{(S,\mathfrak{q},\phi)} S_{\mathfrak{q}}$$

canonically.

Proof. A morphism of triples $(S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$ and such that $\phi' \circ \varphi = \phi$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the triple $(R, \mathfrak{p}, \kappa(\mathfrak{p}) \subset \kappa^{sep})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$ are two triples. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields finite separable over $\kappa(\mathfrak{p})$ and ϕ, resp. ϕ' correspond to maps into κ^{sep}. Hence this data corresponds to $\kappa(\mathfrak{p})$-algebra maps

$$\phi : S \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}, \quad \phi' : S' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}.$$

Set $S'' = S \otimes_R S'$. Combining the maps the above we get a unique $\kappa(\mathfrak{p})$-algebra map

$$\phi'' = \phi \otimes \phi' : S'' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}$$

whose kernel corresponds to a prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$, and whose residue field maps via ϕ'' to the compositum of $\phi(\kappa(\mathfrak{q}))$ and $\phi'(\kappa(\mathfrak{q}'))$ in κ^{sep}. The ring map $R \to S''$ is étale by Lemma 10.143.3. Hence $(S'', \mathfrak{q}'', \phi'')$ is a triple dominating both $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S''' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S''' is étale over R. The fibre ring of S''' over $\mathfrak{p}$ is

$$F''' = (F' \otimes_{\varphi,F,\psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of triples the map $\phi' : F' \to \kappa^{sep}$ extends to a map $\phi'' : F''' \to \kappa^{sep}$ which in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S'''$. The canonical map $S' \to S'''$ (using the right most factor for example) is a morphism of triples $(S', \mathfrak{q}', \phi') \to (S''', \mathfrak{q}'', \phi'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

We still have to show that the colimit R_{colim} of the system is equal to the strict henselization of $R_{\mathfrak{p}}$ with respect to κ^{sep}. To see this note that the system of triples $(S, \mathfrak{q}, \phi)$ contains as a subsystem the pairs $(S, \mathfrak{q})$ of Lemma 10.155.7. Hence R_{colim} contains $R^h_{\mathfrak{p}}$ by the result of that lemma. Moreover, it is clear that $R^h_{\mathfrak{p}} \subset R_{colim}$ is a directed colimit of étale ring extensions. It follows that R_{colim} is henselian by Lemmas 10.153.4 and 10.154.8. Finally, by Lemma 10.144.3 we see that the residue field of R_{colim} is equal to κ^{sep}. Hence we conclude that R_{colim} is strictly henselian and hence equals the strict henselization of $R_{\mathfrak{p}}$ as desired. Some details omitted. □

08HV **Lemma 10.155.12.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Choose separable algebraic closures $\kappa(\mathfrak{p}) \subset \kappa_1^{sep}$ and $\kappa(\mathfrak{q}) \subset \kappa_2^{sep}$. Let R^{sh} and S^{sh} be the corresponding strict henselizations of $R_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$. Given any commutative diagram*

$$\begin{array}{ccc} \kappa_1^{sep} & \xrightarrow[\phi]{} & \kappa_2^{sep} \\ \uparrow & & \uparrow \\ \kappa(\mathfrak{p}) & \xrightarrow{\varphi} & \kappa(\mathfrak{q}) \end{array}$$

The local ring map $R^{sh} \to S^{sh}$ of Lemma 10.155.10 identifies S^{sh} with the strict henselization of $R^{sh} \otimes_R S$ at a prime lying over $\mathfrak{q}$ and the maximal ideal $\mathfrak{m}^{sh} \subset R^{sh}$.

Proof. The proof is identical to the proof of Lemma 10.155.8 except that it uses Lemma 10.155.11 instead of Lemma 10.155.7. □

0C2Z **Lemma 10.155.13.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$ such that $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism. Choose a separable algebraic closure κ^{sep} of $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Then*

$$(S_{\mathfrak{q}})^{sh} = (S_{\mathfrak{q}})^h \otimes_{(R_{\mathfrak{p}})^h} (R_{\mathfrak{p}})^{sh}$$

Proof. This follows from the alternative construction of the strict henselization of a local ring in Remark 10.155.4 and the fact that the residue fields are equal. Some details omitted. □

10.156. Henselization and quasi-finite ring maps

0GIN In this section we prove some results concerning the functorial maps between (strict) henselizations for quasi-finite ring maps.

05WP **Lemma 10.156.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{array}{ccc} R^h_{\mathfrak{p}} & \longrightarrow & S^h_{\mathfrak{q}} \\ \uparrow & & \uparrow \\ R_{\mathfrak{p}} & \longrightarrow & S_{\mathfrak{q}} \end{array}$$

of Lemma 10.155.6 identifies $S^h_{\mathfrak{q}}$ with the localization of $R^h_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ at the prime generated by $\mathfrak{q}$. Moreover, the ring map $R^h_{\mathfrak{p}} \to S^h_{\mathfrak{q}}$ is finite.

Proof. Note that $R^h_\mathfrak{p} \otimes_R S$ is quasi-finite over $R^h_\mathfrak{p}$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 10.122.7. Hence the localization S' of $R^h_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is henselian and finite over $R^h_\mathfrak{p}$, see Lemma 10.153.4. As a localization S' is a filtered colimit of étale $R^h_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$-algebras. By Lemma 10.155.8 we see that $S^h_\mathfrak{q}$ is the henselization of $R^h_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$. Thus $S' = S^h_\mathfrak{q}$ by the uniqueness result of Lemma 10.154.7. □

05WQ **Lemma 10.156.2.** *Let R be a local ring with henselization R^h. Let $I \subset \mathfrak{m}_R$. Then R^h/IR^h is the henselization of R/I.*

Proof. This is a special case of Lemma 10.156.1. □

05WR **Lemma 10.156.3.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $\kappa_2^{sep}/\kappa(\mathfrak{q})$ be a separable algebraic closure and denote $\kappa_1^{sep} \subset \kappa_2^{sep}$ the subfield of elements separable algebraic over $\kappa(\mathfrak{p})$ (Fields, Lemma 9.14.6). The commutative diagram*

$$\begin{array}{ccc} R^{sh}_\mathfrak{p} & \longrightarrow & S^{sh}_\mathfrak{q} \\ \uparrow & & \uparrow \\ R_\mathfrak{p} & \longrightarrow & S_\mathfrak{q} \end{array}$$

of Lemma 10.155.10 identifies $S^{sh}_\mathfrak{q}$ with the localization of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ at the prime ideal which is the kernel of the map

$$R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \longrightarrow \kappa_1^{sep} \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \longrightarrow \kappa_2^{sep}$$

Moreover, the ring map $R^{sh}_\mathfrak{p} \to S^{sh}_\mathfrak{q}$ is a finite local homomorphism of local rings whose residue field extension is the extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is both finite and purely inseparable.

Proof. Since $R \to S$ is quasi-finite at $\mathfrak{q}$ we see that the extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite, see Definition 10.122.3 and Lemma 10.122.2. Hence κ_1^{sep} is a separable algebraic closure of $\kappa(\mathfrak{p})$ (small detail omitted). In particular Lemma 10.155.10 does really apply. Next, the compositum of $\kappa(\mathfrak{q})$ and κ_1^{sep} in κ_2^{sep} is separably algebraically closed and hence equal to κ_2^{sep}. We conclude that $\kappa_2^{sep}/\kappa_1^{sep}$ is finite. By construction the extension $\kappa_2^{sep}/\kappa_1^{sep}$ is purely inseparable. The ring map $R^{sh}_\mathfrak{p} \to S^{sh}_\mathfrak{q}$ is indeed local and induces the residue field extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is indeed finite purely inseparable.

Note that $R^{sh}_\mathfrak{p} \otimes_R S$ is quasi-finite over $R^{sh}_\mathfrak{p}$ at the prime ideal $\mathfrak{q}'$ given in the statement of the lemma, see Lemma 10.122.7. Hence the localization S' of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ at $\mathfrak{q}'$ is henselian and finite over $R^{sh}_\mathfrak{p}$, see Lemma 10.153.4. Note that the residue field of S' is κ_2^{sep} as the map $\kappa_1^{sep} \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \to \kappa_2^{sep}$ is surjective by the discussion in the previous paragraph. Furthermore, as a localization S' is a filtered colimit of étale $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$-algebras. By Lemma 10.155.12 we see that $S^{sh}_\mathfrak{q}$ is the strict henselization of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ at $\mathfrak{q}'$. Thus $S' = S^{sh}_\mathfrak{q}$ by the uniqueness result of Lemma 10.154.7. □

05WS **Lemma 10.156.4.** *Let R be a local ring with strict henselization R^{sh}. Let $I \subset \mathfrak{m}_R$. Then R^{sh}/IR^{sh} is a strict henselization of R/I.*

Proof. This is a special case of Lemma 10.156.3. □

092Y **Lemma 10.156.5.** *Let $A \to B$ and $A \to C$ be local homomorphisms of local rings. If $A \to C$ is integral and either $\kappa(\mathfrak{m}_C)/\kappa(\mathfrak{m}_A)$ or $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ is purely inseparable, then $D = B \otimes_A C$ is a local ring and $B \to D$ and $C \to D$ are local.*

Proof. Any maximal ideal of D lies over the maximal ideal of B by going up for the integral ring map $B \to D$ (Lemma 10.36.22). Now $D/\mathfrak{m}_B D = \kappa(\mathfrak{m}_B) \otimes_A C = \kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} C/\mathfrak{m}_A C$. The spectrum of $C/\mathfrak{m}_A C$ consists of a single point, namely $\mathfrak{m}_C$. Thus the spectrum of $D/\mathfrak{m}_B D$ is the same as the spectrum of $\kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_C)$ which is a single point by our assumption that either $\kappa(\mathfrak{m}_C)/\kappa(\mathfrak{m}_A)$ or $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ is purely inseparable. This proves that D is local and that the ring maps $B \to D$ and $C \to D$ are local. □

0GIP **Lemma 10.156.6.** *Let $A \to B$ and $A \to C$ be ring maps. Let κ be a separably algebraically closed field and let $B \otimes_A C \to \kappa$ be a ring homomorphism. Denote*

$$\begin{array}{ccc} B^{sh} & \longrightarrow & (B \otimes_A C)^{sh} \\ \uparrow & & \uparrow \\ A^{sh} & \longrightarrow & C^{sh} \end{array}$$

the corresponding maps of strict henselizations (see proof). If

(1) *$A \to B$ is quasi-finite at the prime $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, or*
(2) *B is a filtered colimit of quasi-finite A-algebras, or*
(3) *$B_{\mathfrak{p}_B}$ is a filtered colimit of quasi-finite algebras over $A_{\mathfrak{p}_A}$, or*
(4) *B is integral over A,*

then $B^{sh} \otimes_{A^{sh}} C^{sh} \to (B \otimes_A C)^{sh}$ is an isomorphism.

Proof. Write $D = B \otimes_A C$. Denote $\mathfrak{p}_A = \mathrm{Ker}(A \to \kappa)$ and similarly for $\mathfrak{p}_B$, $\mathfrak{p}_C$, and $\mathfrak{p}_D$. Denote $\kappa_A \subset \kappa$ the separable algebraic closure of $\kappa(\mathfrak{p}_A)$ in κ and similarly for κ_B, κ_C, and κ_D. Denote A^{sh} the strict henselization of $A_{\mathfrak{p}_A}$ constructed using the separable algebraic closure $\kappa_A/\kappa(\mathfrak{p}_A)$. Similarly for B^{sh}, C^{sh}, and D^{sh}. We obtain the commutative diagram of the lemma from the functoriality of Lemma 10.155.10.

Consider the map

$$c : B^{sh} \otimes_{A^{sh}} C^{sh} \to D^{sh} = (B \otimes_A C)^{sh}$$

we obtain from the commutative diagram. If $A \to B$ is quasi-finite at $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, then the ring map $C \to D$ is quasi-finite at $\mathfrak{p}_D$ by Lemma 10.122.7. Hence by Lemma 10.156.3 (and Lemma 10.36.13) the ring map c is a homomorphism of finite C^{sh}-algebras and

$$B^{sh} = (B \otimes_A A^{sh})_{\mathfrak{q}} \quad \text{and} \quad D^{sh} = (D \otimes_C C^{sh})_{\mathfrak{r}} = (B \otimes_A C^{sh})_{\mathfrak{r}}$$

for some primes $\mathfrak{q}$ and $\mathfrak{r}$. Since

$$B^{sh} \otimes_{A^{sh}} C^{sh} = (B \otimes_A A^{sh})_{\mathfrak{q}} \otimes_{A^{sh}} C^{sh} = \text{a localization of } B \otimes_A C^{sh}$$

we conclude that source and target of c are both localizations of $B \otimes_A C^{sh}$ (compatibly with the map). Hence it suffices to show that $B^{sh} \otimes_{A^{sh}} C^{sh}$ is local (small detail omitted). This follows from Lemma 10.156.5 and the fact that $A^{sh} \to B^{sh}$ is finite with purely inseparable residue field extension by the already used Lemma 10.156.3. This proves case (1) of the lemma.

In case (2) write $B = \operatorname{colim} B_i$ as a filtered colimit of quasi-finite A-algebras. We correspondingly get $D = \operatorname{colim} D_i$ with $D_i = B_i \otimes_A C$. Observe that $B^{sh} = \operatorname{colim} B_i^{sh}$. Namely, the ring $\operatorname{colim} B_i^{sh}$ is a strictly henselian local ring by Lemma 10.154.8. Also $\operatorname{colim} B_i^{sh}$ is a filtered colimit of étale B-algebras by Lemma 10.154.4. Finally, the residue field of $\operatorname{colim} B_i^{sh}$ is a separable algebraic closure of $\kappa(\mathfrak{p}_B)$ (details omitted). Hence we conclude that $B^{sh} = \operatorname{colim} B_i^{sh}$, see discussion following Definition 10.155.3. Similarly, we have $D^{sh} = \operatorname{colim} D_i^{sh}$. Then we conclude by case (1) because

$$D^{sh} = \operatorname{colim} D_i^{sh} = \operatorname{colim} B_i^{sh} \otimes_{A^{sh}} C^{sh} = B^{sh} \otimes_{A^{sh}} C^{sh}$$

since filtered colimit commute with tensor products.

Case (3). We may replace A, B, C by their localizations at $\mathfrak{p}_A$, $\mathfrak{p}_B$, and $\mathfrak{p}_C$. Thus (3) follows from (2).

Since an integral ring map is a filtered colimit of finite ring maps, we see that (4) follows from (2) as well. □

10.157. Serre's criterion for normality

031O We introduce the following properties of Noetherian rings.

031P **Definition 10.157.1.** Let R be a Noetherian ring. Let $k \geq 0$ be an integer.

(1) We say R has property (R_k) if for every prime $\mathfrak{p}$ of height $\leq k$ the local ring $R_\mathfrak{p}$ is regular. We also say that R is *regular in codimension* $\leq k$.
(2) We say R has property (S_k) if for every prime $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ has depth at least $\min\{k, \dim(R_\mathfrak{p})\}$.
(3) Let M be a finite R-module. We say M has property (S_k) if for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ has depth at least $\min\{k, \dim(\operatorname{Supp}(M_\mathfrak{p}))\}$.

Any Noetherian ring has property (S_0) and so does any finite module over it. Our convention that the depth of the zero module is ∞ (see Section 10.72) and the dimension of the empty set is $-\infty$ (see Topology, Section 5.10) guarantees that the zero module has property (S_k) for all k.

031Q **Lemma 10.157.2.** *Let R be a Noetherian ring. Let M be a finite R-module. The following are equivalent:*

(1) *M has no embedded associated prime, and*
(2) *M has property (S_1).*

Proof. Let $\mathfrak{p}$ be an embedded associated prime of M. Then there exists another associated prime $\mathfrak{q}$ of M such that $\mathfrak{p} \supset \mathfrak{q}$. In particular this implies that $\dim(\operatorname{Supp}(M_\mathfrak{p})) \geq 1$ (since $\mathfrak{q}$ is in the support as well). On the other hand $\mathfrak{p}R_\mathfrak{p}$ is associated to $M_\mathfrak{p}$ (Lemma 10.63.15) and hence $\operatorname{depth}(M_\mathfrak{p}) = 0$ (see Lemma 10.63.18). In other words (S_1) does not hold. Conversely, if (S_1) does not hold then there exists a prime $\mathfrak{p}$ such that $\dim(\operatorname{Supp}(M_\mathfrak{p})) \geq 1$ and $\operatorname{depth}(M_\mathfrak{p}) = 0$. Since $\operatorname{depth}(M_\mathfrak{p}) = 0$, we see that $\mathfrak{p} \in \operatorname{Ass}(M)$ by the two Lemmas 10.63.15 and 10.63.18. Since $\dim(\operatorname{Supp}(M_\mathfrak{p})) \geq 1$, there is a prime $\mathfrak{q} \in \operatorname{Supp}(M)$ with $\mathfrak{q} \subset \mathfrak{p}$, $\mathfrak{q} \neq \mathfrak{p}$. We can take such a $\mathfrak{q}$ that is minimal in $\operatorname{Supp}(M)$. Then by Proposition 10.63.6 we have $\mathfrak{q} \in \operatorname{Ass}(M)$ and hence $\mathfrak{p}$ is an embedded associated prime. □

031R **Lemma 10.157.3.** *Let R be a Noetherian ring. The following are equivalent:*

(1) *R is reduced, and*

(2) *R has properties (R_0) and (S_1).*

Proof. Suppose that R is reduced. Then $R_\mathfrak{p}$ is a field for every minimal prime $\mathfrak{p}$ of R, according to Lemma 10.25.1. Hence we have (R_0). Let $\mathfrak{p}$ be a prime of height ≥ 1. Then $A = R_\mathfrak{p}$ is a reduced local ring of dimension ≥ 1. Hence its maximal ideal $\mathfrak{m}$ is not an associated prime since this would mean there exists an $x \in \mathfrak{m}$ with annihilator $\mathfrak{m}$ so $x^2 = 0$. Hence the depth of $A = R_\mathfrak{p}$ is at least one, by Lemma 10.63.9. This shows that (S_1) holds.

Conversely, assume that R satisfies (R_0) and (S_1). If $\mathfrak{p}$ is a minimal prime of R, then $R_\mathfrak{p}$ is a field by (R_0), and hence is reduced. If $\mathfrak{p}$ is not minimal, then we see that $R_\mathfrak{p}$ has depth ≥ 1 by (S_1) and we conclude there exists an element $t \in \mathfrak{p}R_\mathfrak{p}$ such that $R_\mathfrak{p} \to R_\mathfrak{p}[1/t]$ is injective. Now $R_\mathfrak{p}[1/t]$ is contained in the product of its localizations at prime ideals, see Lemma 10.23.1. This implies that $R_\mathfrak{p}$ is a subring of a product of localizations of R at $\mathfrak{p} \supset \mathfrak{q}$ with $t \notin \mathfrak{q}$. Since these primes have smaller height by induction on the height we conclude that R is reduced. □

031S **Lemma 10.157.4** (Serre's criterion for normality)**.** *Let R be a Noetherian ring. The following are equivalent:* [**DG67**, IV, Theorem 5.8.6]

(1) *R is a normal ring, and*
(2) *R has properties (R_1) and (S_2).*

Proof. Proof of (1) $\Rightarrow$ (2). Assume R is normal, i.e., all localizations $R_\mathfrak{p}$ at primes are normal domains. In particular we see that R has (R_0) and (S_1) by Lemma 10.157.3. Hence it suffices to show that a local Noetherian normal domain R of dimension d has depth $\geq \min(2, d)$ and is regular if $d = 1$. The assertion if $d = 1$ follows from Lemma 10.119.7.

Let R be a local Noetherian normal domain with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$. Apply Lemma 10.119.2 to R. It is clear that R does not fall into cases (1) or (2) of the lemma. Let $R \to R'$ as in (4) of the lemma. Since R is a domain we have $R \subset R'$. Since $\mathfrak{m}$ is not an associated prime of R' there exists an $x \in \mathfrak{m}$ which is a nonzerodivisor on R'. Then $R_x = R'_x$ so R and R' are domains with the same fraction field. But finiteness of $R \subset R'$ implies every element of R' is integral over R (Lemma 10.36.3) and we conclude that $R = R'$ as R is normal. This means (4) does not happen. Thus we get the remaining possibility (3), i.e., $\text{depth}(R) \geq 2$ as desired.

Proof of (2) $\Rightarrow$ (1). Assume R satisfies (R_1) and (S_2). By Lemma 10.157.3 we conclude that R is reduced. Hence it suffices to show that if R is a reduced local Noetherian ring of dimension d satisfying (S_2) and (R_1) then R is a normal domain. If $d = 0$, the result is clear. If $d = 1$, then the result follows from Lemma 10.119.7.

Let R be a reduced local Noetherian ring with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$ which satisfies (R_1) and (S_2). By Lemma 10.37.16 it suffices to show that R is integrally closed in its total ring of fractions $Q(R)$. Pick $x \in Q(R)$ which is integral over R. Then $R' = R[x]$ is a finite ring extension of R (Lemma 10.36.5). Because $\dim(R_\mathfrak{p}) < d$ for every nonmaximal prime $\mathfrak{p} \subset R$ we have $R_\mathfrak{p} = R'_\mathfrak{p}$ by induction. Hence the support of R'/R is $\{\mathfrak{m}\}$. It follows that R'/R is annihilated by a power of $\mathfrak{m}$ (Lemma 10.62.4). By Lemma 10.119.2 this contradicts the assumption that the depth of R is $\geq 2 = \min(2, d)$ and the proof is complete. □

0567 **Lemma 10.157.5.** *A regular ring is normal.*

Proof. Let R be a regular ring. By Lemma 10.157.4 it suffices to prove that R is (R_1) and (S_2). As a regular local ring is Cohen-Macaulay, see Lemma 10.106.3, it is clear that R is (S_2). Property (R_1) is immediate. □

031T **Lemma 10.157.6.** *Let R be a Noetherian normal domain with fraction field K. Then*

(1) *for any nonzero $a \in R$ the quotient R/aR has no embedded primes, and all its associated primes have height 1*
(2) $$R = \bigcap\nolimits_{height(\mathfrak{p})=1} R_{\mathfrak{p}}$$
(3) *For any nonzero $x \in K$ the quotient $R/(R \cap xR)$ has no embedded primes, and all its associates primes have height 1.*

Proof. By Lemma 10.157.4 we see that R has (S_2). Hence for any nonzero element $a \in R$ we see that R/aR has (S_1) (use Lemma 10.72.6 for example) Hence R/aR has no embedded primes (Lemma 10.157.2). We conclude the associated primes of R/aR are exactly the minimal primes $\mathfrak{p}$ over (a), which have height 1 as a is not zero (Lemma 10.60.11). This proves (1).

Thus, given $b \in R$ we have $b \in aR$ if and only if $b \in aR_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ over (a) (see Lemma 10.63.19). These primes all have height 1 as seen above so $b/a \in R$ if and only if $b/a \in R_{\mathfrak{p}}$ for all height 1 primes. Hence (2) holds.

For (3) write $x = a/b$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes over (ab). These all have height 1 by the above. Then we see that $R \cap xR = \bigcap_{i=1,\ldots,r}(R \cap xR_{\mathfrak{p}_i})$ by part (2) of the lemma. Hence $R/(R \cap xR)$ is a submodule of $\bigoplus R/(R \cap xR_{\mathfrak{p}_i})$. As $R_{\mathfrak{p}_i}$ is a discrete valuation ring (by property (R_1) for the Noetherian normal domain R, see Lemma 10.157.4) we have $xR_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i} R_{\mathfrak{p}_i}$ for some $e_i \in \mathbf{Z}$. Hence the direct sum is equal to $\bigoplus_{e_i > 0} R/\mathfrak{p}_i^{(e_i)}$, see Definition 10.64.1. By Lemma 10.64.2 the only associated prime of the module $R/\mathfrak{p}^{(n)}$ is $\mathfrak{p}$. Hence the set of associate primes of $R/(R \cap xR)$ is a subset of $\{\mathfrak{p}_i\}$ and there are no inclusion relations among them. This proves (3). □

10.158. Formal smoothness of fields

031U In this section we show that field extensions are formally smooth if and only if they are separable. However, we first prove finitely generated field extensions are separable algebraic if and only if they are formally unramified.

090W **Lemma 10.158.1.** *Let K/k be a finitely generated field extension. The following are equivalent*

(1) *K is a finite separable field extension of k,*
(2) *$\Omega_{K/k} = 0$,*
(3) *K is formally unramified over k,*
(4) *K is unramified over k,*
(5) *K is formally étale over k,*
(6) *K is étale over k.*

Proof. The equivalence of (2) and (3) is Lemma 10.148.3. By Lemma 10.143.4 we see that (1) is equivalent to (6). Property (6) implies (5) and (4) which both in turn imply (3) (Lemmas 10.150.3, 10.151.3, and 10.151.2). Thus it suffices to show

that (2) implies (1). Choose a finitely generated k-subalgebra $A \subset K$ such that K is the fraction field of the domain A. Set $S = A \setminus \{0\}$. Since $0 = \Omega_{K/k} = S^{-1}\Omega_{A/k}$ (Lemma 10.131.8) and since $\Omega_{A/k}$ is finitely generated (Lemma 10.131.16), we can replace A by a localization A_f to reduce to the case that $\Omega_{A/k} = 0$ (details omitted). Then A is unramified over k, hence K/k is finite separable for example by Lemma 10.151.5 applied with $\mathfrak{q} = (0)$. □

031W **Lemma 10.158.2.** *Let k be a perfect field of characteristic $p > 0$. Let K/k be an extension. Let $a \in K$. Then $da = 0$ in $\Omega_{K/k}$ if and only if a is a pth power.*

Proof. By Lemma 10.131.5 we see that there exists a subfield $k \subset L \subset K$ such that L/k is a finitely generated field extension and such that $\mathrm{d}a$ is zero in $\Omega_{L/k}$. Hence we may assume that K is a finitely generated field extension of k.

Choose a transcendence basis $x_1, \ldots, x_r \in K$ such that K is finite separable over $k(x_1, \ldots, x_r)$. This is possible by the definitions, see Definitions 10.45.1 and 10.42.1. We remark that the result holds for the purely transcendental subfield $k(x_1, \ldots, x_r) \subset K$. Namely,

$$\Omega_{k(x_1,\ldots,x_r)/k} = \bigoplus\nolimits_{i=1}^{r} k(x_1, \ldots, x_r)\mathrm{d}x_i$$

and any rational function all of whose partial derivatives are zero is a pth power. Moreover, we also have

$$\Omega_{K/k} = \bigoplus\nolimits_{i=1}^{r} K\mathrm{d}x_i$$

since $k(x_1, \ldots, x_r) \subset K$ is finite separable (computation omitted). Suppose $a \in K$ is an element such that $\mathrm{d}a = 0$ in the module of differentials. By our choice of x_i we see that the minimal polynomial $P(T) \in k(x_1, \ldots, x_r)[T]$ of a is separable. Write

$$P(T) = T^d + \sum\nolimits_{i=1}^{d} a_i T^{d-i}$$

and hence

$$0 = \mathrm{d}P(a) = \sum\nolimits_{i=1}^{d} a^{d-i}\mathrm{d}a_i$$

in $\Omega_{K/k}$. By the description of $\Omega_{K/k}$ above and the fact that P was the minimal polynomial of a, we see that this implies $\mathrm{d}a_i = 0$. Hence $a_i = b_i^p$ for each i. Therefore by Fields, Lemma 9.28.2 we see that a is a pth power. □

07DZ **Lemma 10.158.3.** *Let k be a field of characteristic $p > 0$. Let $a_1, \ldots, a_n \in k$ be elements such that $da_1, \ldots, da_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Then the field extension $k(a_1^{1/p}, \ldots, a_n^{1/p})$ has degree p^n over k.*

Proof. By induction on n. If $n = 1$ the result is Lemma 10.158.2. For the induction step, suppose that $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$ has degree p^{n-1} over k. We have to show that a_n does not map to a pth power in $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. If it does then we can write

$$\begin{aligned} a_n &= \left(\sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I a_1^{i_1/p} \ldots a_{n-1}^{i_{n-1}/p}\right)^p \\ &= \sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I^p a_1^{i_1} \ldots a_{n-1}^{i_{n-1}} \end{aligned}$$

Applying d we see that $\mathrm{d}a_n$ is linearly dependent on $\mathrm{d}a_i$, $i < n$. This is a contradiction. □

031X **Lemma 10.158.4.** *Let k be a field of characteristic $p > 0$. The following are equivalent:*

(1) *the field extension K/k is separable (see Definition 10.42.1), and*
(2) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective.*

Proof. Write K as a directed colimit $K = \operatorname{colim}_i K_i$ of finitely generated field extensions K_i/k. By definition K is separable if and only if each K_i is separable over k, and by Lemma 10.131.5 we see that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective if and only if each $K_i \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K_i/\mathbf{F}_p}$ is injective. Hence we may assume that K/k is a finitely generated field extension.

Assume K/k is a finitely generated field extension which is separable. Choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 10.42.3. In this case there exists an irreducible polynomial $G(X_1, \ldots, X_{r+1}) \in k[X_1, \ldots, X_{r+1}]$ such that $G(x_1, \ldots, x_{r+1}) = 0$ and such that $\partial G/\partial X_{r+1}$ is not identically zero. Moreover K is the field of fractions of the domain. $S = K[X_1, \ldots, X_{r+1}]/(G)$. Write

$$G = \sum a_I X^I, \quad X^I = X_1^{i_1} \ldots X_{r+1}^{i_{r+1}}.$$

Using the presentation of S above we see that

$$\Omega_{S/\mathbf{F}_p} = \frac{S \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} S\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Since $\Omega_{K/\mathbf{F}_p}$ is the localization of the S-module $\Omega_{S/\mathbf{F}_p}$ (see Lemma 10.131.8) we conclude that

$$\Omega_{K/\mathbf{F}_p} = \frac{K \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} K\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Now, since the polynomial $\partial G/\partial X_{r+1}$ is not identically zero we conclude that the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{S/\mathbf{F}_p}$ is injective as desired.

Assume K/k is a finitely generated field extension and that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective. (This part of the proof is the same as the argument proving Lemma 10.44.2.) Let $x_1, \ldots, x_r$ be a transcendence basis of K over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$ is minimal. If K is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in K$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) = \sum a_{I,i} X^I T^i \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in K. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset K$. We may assume that for some pair (I_0, i_0) the coefficient $a_{I_0,i_0} = 1$. We claim that $\mathrm{d}G/\mathrm{d}X_i$ is not identically zero for at least one i. Namely, if this is not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$. Then this means that

$$\sum\nolimits_{(I,i) \neq (I_0,i_0)} x^I \alpha^i \mathrm{d}a_{I,i}$$

is zero in $\Omega_{K/\mathbf{F}_p}$. Note that there is no k-linear relation among the elements

$$\{x^I \alpha^i \mid a_{I,i} \neq 0 \text{ and } (I,i) \neq (I_0, i_0)\}$$

of K. Hence the assumption that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective implies that $\mathrm{d}a_{I,i} = 0$ in $\Omega_{k/\mathbf{F}_p}$ for all (I, i). By Lemma 10.158.2 we see that each $a_{I,i}$ is a pth power, which implies that G is a pth power contradicting the irreducibility of G. Thus, after renumbering, we may assume that $\mathrm{d}G/\mathrm{d}X_1$ is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of K over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset K$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$, which is a contradiction. □

031Y **Lemma 10.158.5.** *Let K/k be an extension of fields. If K is formally smooth over k, then K is a separable extension of k.*

Proof. Assume K is formally smooth over k. By Lemma 10.138.9 we see that $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective. Hence K is separable over k by Lemma 10.158.4. □

031Z **Lemma 10.158.6.** *Let K/k be an extension of fields. Then K is formally smooth over k if and only if $H_1(L_{K/k}) = 0$.*

Proof. This follows from Proposition 10.138.8 and the fact that a vector spaces is free (hence projective). □

0320 **Lemma 10.158.7.** *Let K/k be an extension of fields.*

(1) *If K is purely transcendental over k, then K is formally smooth over k.*
(2) *If K is separable algebraic over k, then K is formally smooth over k.*
(3) *If K is separable over k, then K is formally smooth over k.*

Proof. For (1) write $K = k(x_j; j \in J)$. Suppose that A is a k-algebra, and $I \subset A$ is an ideal of square zero. Let $\varphi : K \to A/I$ be a k-algebra map. Let $a_j \in A$ be an element such that $a_j \mod I = \varphi(x_j)$. Then it is easy to see that there is a unique k-algebra map $K \to A$ which maps x_j to a_j and which reduces to $\varphi \bmod I$. Hence $k \subset K$ is formally smooth.

In case (2) we see that $k \subset K$ is a colimit of étale ring extensions. An étale ring map is formally étale (Lemma 10.150.3). Hence this case follows from Lemma 10.150.4 and the trivial observation that a formally étale ring map is formally smooth.

In case (3), write $K = \operatorname{colim} K_i$ as the filtered colimit of its finitely generated sub k-extensions. By Definition 10.42.1 each K_i is separable algebraic over a purely transcendental extension of k. Hence K_i/k is formally smooth by cases (1) and (2) and Lemma 10.138.3. Thus $H_1(L_{K_i/k}) = 0$ by Lemma 10.158.6. Hence $H_1(L_{K/k}) = 0$ by Lemma 10.134.9. Hence K/k is formally smooth by Lemma 10.158.6 again. □

0321 **Lemma 10.158.8.** *Let k be a field.*

(1) *If the characteristic of k is zero, then any extension field of k is formally smooth over k.*
(2) *If the characteristic of k is $p > 0$, then K/k is formally smooth if and only if it is a separable field extension.*

Proof. Combine Lemmas 10.158.5 and 10.158.7. □

Here we put together all the different characterizations of separable field extensions.

0322 **Proposition 10.158.9.** *Let K/k be a field extension. If the characteristic of k is zero then*

(1) K *is separable over* k,
(2) K *is geometrically reduced over* k,
(3) K *is formally smooth over* k,
(4) $H_1(L_{K/k}) = 0$, *and*
(5) *the map* $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ *is injective.*

If the characteristic of k *is* $p > 0$, *then the following are equivalent:*

(1) K *is separable over* k,
(2) *the ring* $K \otimes_k k^{1/p}$ *is reduced,*
(3) K *is geometrically reduced over* k,
(4) *the map* $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ *is injective,*
(5) $H_1(L_{K/k}) = 0$, *and*
(6) K *is formally smooth over* k.

Proof. This is a combination of Lemmas 10.44.2, 10.158.8 10.158.5, and 10.158.4. □

Here is yet another characterization of finitely generated separable field extensions.

037X **Lemma 10.158.10.** *Let* K/k *be a finitely generated field extension. Then* K *is separable over* k *if and only if* K *is the localization of a smooth* k*-algebra.*

Proof. Choose a finite type k-algebra R which is a domain whose fraction field is K. Lemma 10.140.9 says that $k \to R$ is smooth at (0) if and only if K/k is separable. This proves the lemma. □

07BV **Lemma 10.158.11.** *Let* K/k *be a field extension. Then* K *is a filtered colimit of global complete intersection algebras over* k. *If* K/k *is separable, then* K *is a filtered colimit of smooth algebras over* k.

Proof. Suppose that $E \subset K$ is a finite subset. It suffices to show that there exists a k subalgebra $A \subset K$ which contains E and which is a global complete intersection (resp. smooth) over k. The separable/smooth case follows from Lemma 10.158.10. In general let $L \subset K$ be the subfield generated by E. Pick a transcendence basis $x_1, \ldots, x_d \in L$ over k. The extension $L/k(x_1, \ldots, x_d)$ is finite. Say $L = k(x_1, \ldots, x_d)[y_1, \ldots, y_r]$. Pick inductively polynomials $P_i \in k(x_1, \ldots, x_d)[Y_1, \ldots, Y_r]$ such that $P_i = P_i(Y_1, \ldots, Y_i)$ is monic in Y_i over $k(x_1, \ldots, x_d)[Y_1, \ldots, Y_{i-1}]$ and maps to the minimum polynomial of y_i in $k(x_1, \ldots, x_d)[y_1, \ldots, y_{i-1}][Y_i]$. Then it is clear that $P_1, \ldots, P_r$ is a regular sequence in $k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]$ and that $L = k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$. If $h \in k[x_1, \ldots, x_d]$ is a polynomial such that $P_i \in k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$, then we see that $P_1, \ldots, P_r$ is a regular sequence in $k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$ and $A = k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$ is a global complete intersection. After adjusting our choice of h we may assume $E \subset A$ and we win. □

10.159. Constructing flat ring maps

03C2 The following lemma is occasionally useful.

03C3 **Lemma 10.159.1.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Let* K/k *be a field extension. There exists a local ring* $(R', \mathfrak{m}', k')$, *a flat local ring map* $R \to R'$ *such that* $\mathfrak{m}' = \mathfrak{m}R'$ *and such that* k' *is isomorphic to* K *as an extension of* k.

Proof. Suppose that $k' = k(\alpha)$ is a monogenic extension of k. Then k' is the residue field of a flat local extension $R \subset R'$ as in the lemma. Namely, if α is transcendental over k, then we let R' be the localization of $R[x]$ at the prime $\mathfrak{m}R[x]$. If α is algebraic with minimal polynomial $T^d + \sum \overline{\lambda}_i T^{d-i}$, then we let $R' = R[T]/(T^d + \sum \lambda_i T^{d-i})$.

Consider the collection of triples $(k', R \to R', \phi)$, where $k \subset k' \subset K$ is a subfield, $R \to R'$ is a local ring map as in the lemma, and $\phi : R' \to k'$ induces an isomorphism $R'/\mathfrak{m}R' \cong k'$ of k-extensions. These form a "big" category $\mathcal{C}$ with morphisms $(k_1, R_1, \phi_1) \to (k_2, R_2, \phi_2)$ given by ring maps $\psi : R_1 \to R_2$ such that

$$\begin{array}{ccccc} R_1 & \xrightarrow[\phi_1]{} & k_1 & \longrightarrow & K \\ {\scriptstyle\psi}\downarrow & & & & \| \\ R_2 & \xrightarrow{\phi_2} & k_2 & \longrightarrow & K \end{array}$$

commutes. This implies that $k_1 \subset k_2$.

Suppose that I is a directed set, and $((R_i, k_i, \phi_i), \psi_{ii'})$ is a system over I, see Categories, Section 4.21. In this case we can consider

$$R' = \operatorname{colim}_{i \in I} R_i$$

This is a local ring with maximal ideal $\mathfrak{m}R'$, and residue field $k' = \bigcup_{i \in I} k_i$. Moreover, the ring map $R \to R'$ is flat as it is a colimit of flat maps (and tensor products commute with directed colimits). Hence we see that (R', k', ϕ') is an "upper bound" for the system.

An almost trivial application of Zorn's Lemma would finish the proof if $\mathcal{C}$ was a set, but it isn't. (Actually, you can make this work by finding a reasonable bound on the cardinals of the local rings occurring.) To get around this problem we choose a well ordering on K. For $x \in K$ we let $K(x)$ be the subfield of K generated by all elements of K which are $\leq x$. By transfinite recursion on $x \in K$ we will produce ring maps $R \subset R(x)$ as in the lemma with residue field extension $K(x)/k$. Moreover, by construction we will have that $R(x)$ will contain $R(y)$ for all $y \leq x$. Namely, if x has a predecessor x', then $K(x) = K(x')[x]$ and hence we can let $R(x') \subset R(x)$ be the local ring extension constructed in the first paragraph of the proof. If x does not have a predecessor, then we first set $R'(x) = \operatorname{colim}_{x' < x} R(x')$ as in the third paragraph of the proof. The residue field of $R'(x)$ is $K'(x) = \bigcup_{x' < x} K(x')$. Since $K(x) = K'(x)[x]$ we see that we can use the construction of the first paragraph of the proof to produce $R'(x) \subset R(x)$. This finishes the proof of the lemma. □

09E0 **Lemma 10.159.2.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If $k \subset K$ is a separable algebraic extension, then there exists a directed set I and a system of finite étale extensions $R \subset R_i$, $i \in I$ of local rings such that $R' = \operatorname{colim} R_i$ has residue field K (as extension of k).*

Proof. Let $R \subset R'$ be the extension constructed in the proof of Lemma 10.159.1. By construction $R' = \operatorname{colim}_{\alpha \in A} R_\alpha$ where A is a well-ordered set and the transition maps $R_\alpha \to R_{\alpha+1}$ are finite étale and $R_\alpha = \operatorname{colim}_{\beta < \alpha} R_\beta$ if α is not a successor. We will prove the result by transfinite induction.

Suppose the result holds for R_α, i.e., $R_\alpha = \operatorname{colim} R_i$ with R_i finite étale over R. Since $R_\alpha \to R_{\alpha+1}$ is finite étale there exists an i and a finite étale extension

$R_i \to R_{i,1}$ such that $R_{\alpha+1} = R_\alpha \otimes_{R_i} R_{i,1}$. Thus $R_{\alpha+1} = \mathrm{colim}_{i' \geq i} R_{i'} \otimes_{R_i} R_{i,1}$ and the result holds for $\alpha + 1$. Suppose α is not a successor and the result holds for R_β for all $\beta < \alpha$. Since every finite subset $E \subset R_\alpha$ is contained in R_β for some $\beta < \alpha$ and we see that E is contained in a finite étale subextension by assumption. Thus the result holds for R_α. □

07NE **Lemma 10.159.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. Then there exists a finite free ring map $R \to S$ such that $\mathfrak{q} = \mathfrak{p}S$ is prime and $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is isomorphic to the given extension $L/\kappa(\mathfrak{p})$.*

Proof. By induction of the degree of $\kappa(\mathfrak{p}) \subset L$. If the degree is 1, then we take $R = S$. In general, if there exists a sub extension $\kappa(\mathfrak{p}) \subset L' \subset L$ then we win by induction on the degree (by first constructing $R \subset S'$ corresponding to $L'/\kappa(\mathfrak{p})$ and then construction $S' \subset S$ corresponding to L/L'). Thus we may assume that $L \supset \kappa(\mathfrak{p})$ is generated by a single element $\alpha \in L$. Let $X^d + \sum_{i<d} a_i X^i$ be the minimal polynomial of α over $\kappa(\mathfrak{p})$, so $a_i \in \kappa(\mathfrak{p})$. We may write a_i as the image of f_i/g for some $f_i, g \in R$ and $g \notin \mathfrak{p}$. After replacing α by $g\alpha$ (and correspondingly replacing a_i by $g^{d-i}a_i$) we may assume that a_i is the image of some $f_i \in R$. Then we simply take $S = R[x]/(x^d + \sum f_i x^i)$. □

0GIL **Lemma 10.159.4.** *Let A be a ring. Let $\kappa = \max(|A|, \aleph_0)$. Then every flat A-algebra B is the filtered colimit of its flat A-subalgebras $B' \subset B$ of cardinality $|B'| \leq \kappa$. (Observe that B' is faithfully flat over A if B is faithfully flat over A.)*

Proof. If B has cardinality $\leq \kappa$ then this is true. Let $E \subset B$ be an A-subalgebra with $|E| \leq \kappa$. We will show that E is contained in a flat A-subalgebra B' with $|B'| \leq \kappa$. The lemma follows because (a) every finite subset of B is contained in an A-subalgebra of cardinality at most κ and (b) every pair of A-subalgebras of B of cardinality at most κ is contained in an A-subalgebra of cardinality at most κ. Details omitted.

We will inductively construct a sequence of A-subalgebras

$$E = E_0 \subset E_1 \subset E_2 \subset \ldots$$

each having cardinality $\leq \kappa$ and we will show that $B' = \bigcup E_k$ is flat over A to finish the proof.

The construction is as follows. Set $E_0 = E$. Given E_k for $k \geq 0$ we consider the set S_k of relations between elements of E_k with coefficients in A. Thus an element $s \in S_k$ is given by an integer $n \geq 1$ and $a_1, \ldots, a_n \in A$, and $e_1, \ldots, e_n \in E_k$ such that $\sum a_i e_i = 0$ in E_k. The flatness of $A \to B$ implies by Lemma 10.39.11 that for every $s = (n, a_1, \ldots, a_n, e_1, \ldots, e_n) \in S_k$ we may choose

$$(m_s, b_{s,1}, \ldots, b_{s,m_s}, a_{s,11}, \ldots, a_{s,nm_s})$$

where $m_s \geq 0$ is an integer, $b_{s,j} \in B$, $a_{s,ij} \in A$, and

$$e_i = \sum\nolimits_j a_{s,ij} b_{s,j}, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i a_i a_{s,ij}, \forall j.$$

Given these choicse, we let $E_{k+1} \subset B$ be the A-subalgebra generated by

(1) E_k and
(2) the elements $b_{s,1}, \ldots, b_{s,m_s}$ for every $s \in S_k$.

Some set theory (omitted) shows that E_{k+1} has at most cardinality κ (this uses that we inductively know $|E_k| \leq \kappa$ and consequently the cardinality of S_k is also at most κ).

To show that $B' = \bigcup E_k$ is flat over A we consider a relation $\sum_{i=1,\ldots,n} a_i b'_i = 0$ in B' with coefficients in A. Choose k large enough so that $b'_i \in E_k$ for $i = 1, \ldots, n$. Then $(n, a_1, \ldots, a_n, b'_1, \ldots, b'_n) \in S_k$ and hence we see that the relation is trivial in E_{k+1} and a fortiori in B'. Thus $A \to B'$ is flat by Lemma 10.39.11. □

10.160. The Cohen structure theorem

0323 Here is a fundamental notion in commutative algebra.

0324 **Definition 10.160.1.** Let $(R, \mathfrak{m})$ be a local ring. We say R is a *complete local ring* if the canonical map

$$R \longrightarrow \lim_n R/\mathfrak{m}^n$$

to the completion of R with respect to $\mathfrak{m}$ is an isomorphism[13].

Note that an Artinian local ring R is a complete local ring because $\mathfrak{m}_R^n = 0$ for some $n > 0$. In this section we mostly focus on Noetherian complete local rings.

0325 **Lemma 10.160.2.** *Let R be a Noetherian complete local ring. Any quotient of R is also a Noetherian complete local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.*

Proof. The ring S is Noetherian by Lemma 10.31.1. As an R-module S is complete by Lemma 10.97.1. Hence S is the product of the completions at its maximal ideals by Lemma 10.97.8. □

032B **Lemma 10.160.3.** *Let $(R, \mathfrak{m})$ be a complete local ring. If $\mathfrak{m}$ is a finitely generated ideal then R is Noetherian.*

Proof. See Lemma 10.97.5. □

0326 **Definition 10.160.4.** Let $(R, \mathfrak{m})$ be a complete local ring. A subring $\Lambda \subset R$ is called a *coefficient ring* if the following conditions hold:

(1) Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak{m}$,
(2) the residue field of Λ maps isomorphically to the residue field of R, and
(3) $\Lambda \cap \mathfrak{m} = p\Lambda$, where p is the characteristic of the residue field of R.

Let us make some remarks on this definition. We split the discussion into the following cases:

(1) The local ring R contains a field. This happens if either $\mathbf{Q} \subset R$, or $pR = 0$ where p is the characteristic of $R/\mathfrak{m}$. In this case a coefficient ring Λ is a field contained in R which maps isomorphically to $R/\mathfrak{m}$.
(2) The characteristic of $R/\mathfrak{m}$ is $p > 0$ but no power of p is zero in R. In this case Λ is a complete discrete valuation ring with uniformizer p and residue field $R/\mathfrak{m}$.

[13] This includes the condition that $\bigcap \mathfrak{m}^n = (0)$; in some texts this may be indicated by saying that R is complete and separated. Warning: It can happen that the completion $\lim_n R/\mathfrak{m}^n$ of a local ring is non-complete, see Examples, Lemma 110.7.1. This does not happen when $\mathfrak{m}$ is finitely generated, see Lemma 10.96.3 in which case the completion is Noetherian, see Lemma 10.97.5.

(3) The characteristic of $R/\mathfrak{m}$ is $p > 0$, and for some $n > 1$ we have $p^{n-1} \neq 0$, $p^n = 0$ in R. In this case Λ is an Artinian local ring whose maximal ideal is generated by p and which has residue field $R/\mathfrak{m}$.

The complete discrete valuation rings with uniformizer p above play a special role and we baptize them as follows.

0327 **Definition 10.160.5.** A *Cohen ring* is a complete discrete valuation ring with uniformizer p a prime number.

0328 **Lemma 10.160.6.** *Let p be a prime number. Let k be a field of characteristic p. There exists a Cohen ring Λ with $\Lambda/p\Lambda \cong k$.*

Proof. First note that the p-adic integers $\mathbf{Z}_p$ form a Cohen ring for $\mathbf{F}_p$. Let k be an arbitrary field of characteristic p. Let $\mathbf{Z}_p \to R$ be a flat local ring map such that $\mathfrak{m}_R = pR$ and $R/pR = k$, see Lemma 10.159.1. By Lemma 10.97.5 the completion $\Lambda = R^\wedge$ is Noetherian. It is a complete Noetherian local ring with maximal ideal (p) as $\Lambda/p\Lambda = R/pR$ is a field (use Lemma 10.96.3). Since $\mathbf{Z}_p \to R \to \Lambda$ is flat (by Lemma 10.97.2) we see that p is a nonzerodivisor in Λ. Hence Λ has dimension ≥ 1 (Lemma 10.60.13) and we conclude that Λ is regular of dimension 1, i.e., a discrete valuation ring by Lemma 10.119.7. We conclude Λ is a Cohen ring for k. □

0329 **Lemma 10.160.7.** *Let $p > 0$ be a prime. Let Λ be a Cohen ring with residue field of characteristic p. For every $n \geq 1$ the ring map*

$$\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$$

is formally smooth.

Proof. If $n = 1$, this follows from Proposition 10.158.9. For general n we argue by induction on n. Namely, if $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth, then we can apply Lemma 10.138.12 to the ring map $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \Lambda/p^{n+1}\Lambda$ and the ideal $I = (p^n) \subset \mathbf{Z}/p^{n+1}\mathbf{Z}$. □

032A **Theorem 10.160.8** (Cohen structure theorem)**.** *Let $(R, \mathfrak{m})$ be a complete local ring.*

(1) *R has a coefficient ring (see Definition 10.160.4),*
(2) *if $\mathfrak{m}$ is a finitely generated ideal, then R is isomorphic to a quotient*

$$\Lambda[[x_1, \ldots, x_n]]/I$$

where Λ is either a field or a Cohen ring.

Proof. Let us prove a coefficient ring exists. First we prove this in case the characteristic of the residue field κ is zero. Namely, in this case we will prove by induction on $n > 0$ that there exists a section

$$\varphi_n : \kappa \longrightarrow R/\mathfrak{m}^n$$

to the canonical map $R/\mathfrak{m}^n \to \kappa = R/\mathfrak{m}$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The field extension $\kappa/\mathbf{Q}$ is formally smooth by Proposition 10.158.9. Hence we can find the dotted arrow in the following diagram

$$\begin{array}{ccc} R/\mathfrak{m}^{n-1} & \longleftarrow & R/\mathfrak{m}^n \\ {\scriptstyle \varphi_{n-1}}\uparrow & \nearrow & \uparrow \\ \kappa & \longleftarrow & \mathbf{Q} \end{array}$$

This proves the induction step. Putting these maps together

$$\lim_n \ \varphi_n : \kappa \longrightarrow R = \lim_n \ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

Next, we prove the existence of a coefficient ring in the case where the characteristic of the residue field κ is $p > 0$. Namely, choose a Cohen ring Λ with $\kappa = \Lambda/p\Lambda$, see Lemma 10.160.6. In this case we will prove by induction on $n > 0$ that there exists a map

$$\varphi_n : \Lambda/p^n\Lambda \longrightarrow R/\mathfrak{m}^n$$

whose composition with the reduction map $R/\mathfrak{m}^n \to \kappa$ produces the given isomorphism $\Lambda/p\Lambda = \kappa$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The ring map $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth by Lemma 10.160.7. Hence we can find the dotted arrow in the following diagram

$$\begin{array}{ccc} R/\mathfrak{m}^{n-1} & \longleftarrow & R/\mathfrak{m}^n \\ {\scriptstyle\varphi_{n-1}}\uparrow & \nearrow & \uparrow \\ \Lambda/p^n\Lambda & \longleftarrow & \mathbf{Z}/p^n\mathbf{Z} \end{array}$$

This proves the induction step. Putting these maps together

$$\lim_n \ \varphi_n : \Lambda = \lim_n \ \Lambda/p^n\Lambda \longrightarrow R = \lim_n \ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

The final statement of the theorem follows readily. Namely, if $y_1, \ldots, y_n$ are generators of the ideal $\mathfrak{m}$, then we can use the map $\Lambda \to R$ just constructed to get a map

$$\Lambda[[x_1, \ldots, x_n]] \longrightarrow R, \quad x_i \longmapsto y_i.$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete this map is surjective by Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. □

032C **Remark 10.160.9.** If k is a field then the power series ring $k[[X_1, \ldots, X_d]]$ is a Noetherian complete local regular ring of dimension d. If Λ is a Cohen ring then $\Lambda[[X_1, \ldots, X_d]]$ is a complete local Noetherian regular ring of dimension $d+1$. Hence the Cohen structure theorem implies that any Noetherian complete local ring is a quotient of a regular local ring. In particular we see that a Noetherian complete local ring is universally catenary, see Lemma 10.105.9 and Lemma 10.106.3.

0C0S **Lemma 10.160.10.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring. Assume R is regular.*

(1) *If R contains either $\mathbf{F}_p$ or $\mathbf{Q}$, then R is isomorphic to a power series ring over its residue field.*
(2) *If k is a field and $k \to R$ is a ring map inducing an isomorphism $k \to R/\mathfrak{m}$, then R is isomorphic as a k-algebra to a power series ring over k.*

Proof. In case (1), by the Cohen structure theorem (Theorem 10.160.8) there exists a coefficient ring which must be a field mapping isomorphically to the residue field. Thus it suffices to prove (2). In case (2) we pick $f_1, \ldots, f_d \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$ and we consider the continuous k-algebra map $k[[x_1, \ldots, x_d]] \to R$ sending x_i to f_i. As both source and target are $(x_1, \ldots, x_d)$-adically complete, this

map is surjective by Lemma 10.96.1. On the other hand, it has to be injective because otherwise the dimension of R would be $< d$ by Lemma 10.60.13. □

032D **Lemma 10.160.11.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then there exists a $R_0 \subset R$ with the following properties*

(1) *R_0 is a regular complete local ring,*
(2) *$R_0 \subset R$ is finite and induces an isomorphism on residue fields,*
(3) *R_0 is either isomorphic to $k[[X_1, \ldots, X_d]]$ where k is a field or $\Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.*

Proof. Let Λ be a coefficient ring of R. Since R is a domain we see that either Λ is a field or Λ is a Cohen ring.

Case I: $\Lambda = k$ is a field. Let $d = \dim(R)$. Choose $x_1, \ldots, x_d \in \mathfrak{m}$ which generate an ideal of definition $I \subset R$. (See Section 10.60.) By Lemma 10.96.9 we see that R is I-adically complete as well. Consider the map $R_0 = k[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.112.3), and the lemma is proved.

Case II: Λ is a Cohen ring. Let $d + 1 = \dim(R)$. Let $p > 0$ be the characteristic of the residue field k. As R is a domain we see that p is a nonzerodivisor in R. Hence $\dim(R/pR) = d$, see Lemma 10.60.13. Choose $x_1, \ldots, x_d \in R$ which generate an ideal of definition in R/pR. Then $I = (p, x_1, \ldots, x_d)$ is an ideal of definition of R. By Lemma 10.96.9 we see that R is I-adically complete as well. Consider the map $R_0 = \Lambda[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (p, X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.112.3), and the lemma is proved. □

10.161. Japanese rings

0BI1 In this section we begin to discuss finiteness of integral closure.

032F **Definition 10.161.1.** Let R be a domain with field of fractions K. [**DG67**, Chapter 0, Definition 23.1.1]

(1) We say R is *N-1* if the integral closure of R in K is a finite R-module.
(2) We say R is *N-2* or *Japanese* if for any finite extension L/K of fields the integral closure of R in L is finite over R.

The main interest in these notions is for Noetherian rings, but here is a non-Noetherian example.

0350 **Example 10.161.2.** Let k be a field. The domain $R = k[x_1, x_2, x_3, \ldots]$ is N-2, but not Noetherian. The reason is the following. Suppose that $R \subset L$ and the field L is a finite extension of the fraction field of R. Then there exists an integer n such that L comes from a finite extension $L_0/k(x_1, \ldots, x_n)$ by adjoining the (transcendental) elements x_{n+1}, x_{n+2}, etc. Let S_0 be the integral closure of $k[x_1, \ldots, x_n]$ in L_0. By Proposition 10.162.16 below it is true that S_0 is finite over $k[x_1, \ldots, x_n]$. Moreover, the integral closure of R in L is $S = S_0[x_{n+1}, x_{n+2}, \ldots]$ (use Lemma 10.37.8) and hence finite over R. The same argument works for $R = \mathbf{Z}[x_1, x_2, x_3, \ldots]$.

032G **Lemma 10.161.3.** *Let R be a domain. If R is N-1 then so is any localization of R. Same for N-2.*

Proof. These statements hold because taking integral closure commutes with localization, see Lemma 10.36.11. □

032H **Lemma 10.161.4.** *Let R be a domain. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If each domain R_{f_i} is N-1 then so is R. Same for N-2.*

Proof. Assume R_{f_i} is N-2 (or N-1). Let L be a finite extension of the fraction field of R (equal to the fraction field in the N-1 case). Let S be the integral closure of R in L. By Lemma 10.36.11 we see that S_{f_i} is the integral closure of R_{f_i} in L. Hence S_{f_i} is finite over R_{f_i} by assumption. Thus S is finite over R by Lemma 10.23.2. □

032I **Lemma 10.161.5.** *Let R be a domain. Let $R \subset S$ be a quasi-finite extension of domains (for example finite). Assume R is N-2 and Noetherian. Then S is N-2.*

Proof. Let L/K be the induced extension of fraction fields. Note that this is a finite field extension (for example by Lemma 10.122.2 (2) applied to the fibre $S \otimes_R K$, and the definition of a quasi-finite ring map). Let S' be the integral closure of R in S. Then S' is contained in the integral closure of R in L which is finite over R by assumption. As R is Noetherian this implies S' is finite over R. By Lemma 10.123.14 there exist elements $g_1, \ldots, g_n \in S'$ such that $S'_{g_i} \cong S_{g_i}$ and such that $g_1, \ldots, g_n$ generate the unit ideal in S. Hence it suffices to show that S' is N-2 by Lemmas 10.161.3 and 10.161.4. Thus we have reduced to the case where S is finite over R.

Assume $R \subset S$ with hypotheses as in the lemma and moreover that S is finite over R. Let M be a finite field extension of the fraction field of S. Then M is also a finite field extension of K and we conclude that the integral closure T of R in M is finite over R. By Lemma 10.36.16 we see that T is also the integral closure of S in M and we win by Lemma 10.36.15. □

032J **Lemma 10.161.6.** *Let R be a Noetherian domain. If $R[z, z^{-1}]$ is N-1, then so is R.*

Proof. Let R' be the integral closure of R in its field of fractions K. Let S' be the integral closure of $R[z, z^{-1}]$ in its field of fractions. Clearly $R' \subset S'$. Since $K[z, z^{-1}]$ is a normal domain we see that $S' \subset K[z, z^{-1}]$. Suppose that $f_1, \ldots, f_n \in S'$ generate S' as $R[z, z^{-1}]$-module. Say $f_i = \sum a_{ij} z^j$ (finite sum), with $a_{ij} \in K$. For any $x \in R'$ we can write

$$x = \sum h_i f_i$$

with $h_i \in R[z, z^{-1}]$. Thus we see that R' is contained in the finite R-submodule $\sum Ra_{ij} \subset K$. Since R is Noetherian we conclude that R' is a finite R-module. □

032K **Lemma 10.161.7.** *Let R be a Noetherian domain, and let $R \subset S$ be a finite extension of domains. If S is N-1, then so is R. If S is N-2, then so is R.*

Proof. Omitted. (Hint: Integral closures of R in extension fields are contained in integral closures of S in extension fields.) □

032L **Lemma 10.161.8.** *Let R be a Noetherian normal domain with fraction field K. Let L/K be a finite separable field extension. Then the integral closure of R in L is finite over R.*

Proof. Consider the trace pairing (Fields, Definition 9.20.6)

$$L \times L \longrightarrow K, \quad (x, y) \longmapsto \langle x, y \rangle := \text{Trace}_{L/K}(xy).$$

Since L/K is separable this is nondegenerate (Fields, Lemma 9.20.7). Moreover, if $x \in L$ is integral over R, then $\text{Trace}_{L/K}(x)$ is in R. This is true because the minimal polynomial of x over K has coefficients in R (Lemma 10.38.6) and because $\text{Trace}_{L/K}(x)$ is an integer multiple of one of these coefficients (Fields, Lemma 9.20.3). Pick $x_1, \ldots, x_n \in L$ which are integral over R and which form a K-basis of L. Then the integral closure $S \subset L$ is contained in the R-module

$$M = \{y \in L \mid \langle x_i, y \rangle \in R, \ i = 1, \ldots, n\}$$

By linear algebra we see that $M \cong R^{\oplus n}$ as an R-module. Hence $S \subset R^{\oplus n}$ is a finitely generated R-module as R is Noetherian. □

03B7 **Example 10.161.9.** Lemma 10.161.8 does not work if the ring is not Noetherian. For example consider the action of $G = \{+1, -1\}$ on $A = \mathbf{C}[x_1, x_2, x_3, \ldots]$ where -1 acts by mapping x_i to $-x_i$. The invariant ring $R = A^G$ is the $\mathbf{C}$-algebra generated by all $x_i x_j$. Hence $R \subset A$ is not finite. But R is a normal domain with fraction field $K = L^G$ the G-invariants in the fraction field L of A. And clearly A is the integral closure of R in L.

The following lemma can sometimes be used as a substitute for Lemma 10.161.8 in case of purely inseparable extensions.

0AE0 **Lemma 10.161.10.** *Let R be a Noetherian normal domain with fraction field K of characteristic $p > 0$. Let $a \in K$ be an element such that there exists a derivation $D : R \to R$ with $D(a) \neq 0$. Then the integral closure of R in $L = K[x]/(x^p - a)$ is finite over R.*

Proof. After replacing x by fx and a by $f^p a$ for some $f \in R$ we may assume $a \in R$. Hence also $D(a) \in R$. We will show by induction on $i \leq p - 1$ that if

$$y = a_0 + a_1 x + \ldots + a_i x^i, \quad a_j \in K$$

is integral over R, then $D(a)^i a_j \in R$. Thus the integral closure is contained in the finite R-module with basis $D(a)^{-p+1} x^j$, $j = 0, \ldots, p - 1$. Since R is Noetherian this proves the lemma.

If $i = 0$, then $y = a_0$ is integral over R if and only if $a_0 \in R$ and the statement is true. Suppose the statement holds for some $i < p - 1$ and suppose that

$$y = a_0 + a_1 x + \ldots + a_{i+1} x^{i+1}, \quad a_j \in K$$

is integral over R. Then

$$y^p = a_0^p + a_1^p a + \ldots + a_{i+1}^p a^{i+1}$$

is an element of R (as it is in K and integral over R). Applying D we obtain

$$(a_1^p + 2a_2^p a + \ldots + (i+1) a_{i+1}^p a^i) D(a)$$

is in R. Hence it follows that

$$D(a) a_1 + 2D(a) a_2 x + \ldots + (i+1) D(a) a_{i+1} x^i$$

is integral over R. By induction we find $D(a)^{i+1} a_j \in R$ for $j = 1, \ldots, i + 1$. (Here we use that $1, \ldots, i + 1$ are invertible.) Hence $D(a)^{i+1} a_0$ is also in R because it is

the difference of $D(a)^{i+1}y$ and $\sum_{j>0} D(a)^{i+1}a_j x^j$ which are integral over R (since x is integral over R as $a \in R$). □

032M **Lemma 10.161.11.** *A Noetherian domain whose fraction field has characteristic zero is N-1 if and only if it is N-2 (i.e., Japanese).*

Proof. This is clear from Lemma 10.161.8 since every field extension in characteristic zero is separable. □

032N **Lemma 10.161.12.** *Let R be a Noetherian domain with fraction field K of characteristic $p > 0$. Then R is N-2 if and only if for every finite purely inseparable extension L/K the integral closure of R in L is finite over R.*

Proof. Assume the integral closure of R in every finite purely inseparable field extension of K is finite. Let L/K be any finite extension. We have to show the integral closure of R in L is finite over R. Choose a finite normal field extension M/K containing L. As R is Noetherian it suffices to show that the integral closure of R in M is finite over R. By Fields, Lemma 9.27.3 there exists a subextension $M/M_{insep}/K$ such that M_{insep}/K is purely inseparable, and M/M_{insep} is separable. By assumption the integral closure R' of R in M_{insep} is finite over R. By Lemma 10.161.8 the integral closure R'' of R' in M is finite over R'. Then R'' is finite over R by Lemma 10.7.3. Since R'' is also the integral closure of R in M (see Lemma 10.36.16) we win. □

032O **Lemma 10.161.13.** *Let R be a Noetherian domain. If R is N-1 then $R[x]$ is N-1. If R is N-2 then $R[x]$ is N-2.*

Proof. Assume R is N-1. Let R' be the integral closure of R which is finite over R. Hence also $R'[x]$ is finite over $R[x]$. The ring $R'[x]$ is normal (see Lemma 10.37.8), hence N-1. This proves the first assertion.

For the second assertion, by Lemma 10.161.7 it suffices to show that $R'[x]$ is N-2. In other words we may and do assume that R is a normal N-2 domain. In characteristic zero we are done by Lemma 10.161.11. In characteristic $p > 0$ we have to show that the integral closure of $R[x]$ is finite in any finite purely inseparable extension of $L/K(x)$ where K is the fraction field of R. There exists a finite purely inseparable field extension L'/K and $q = p^e$ such that $L \subset L'(x^{1/q})$; some details omitted. As $R[x]$ is Noetherian it suffices to show that the integral closure of $R[x]$ in $L'(x^{1/q})$ is finite over $R[x]$. And this integral closure is equal to $R'[x^{1/q}]$ with $R \subset R' \subset L'$ the integral closure of R in L'. Since R is N-2 we see that R' is finite over R and hence $R'[x^{1/q}]$ is finite over $R[x]$. □

0332 **Lemma 10.161.14.** *Let R be a Noetherian domain. If there exists a nonzero $f \in R$ such that R_f is normal then*

$$U = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_\mathfrak{p} \text{ is normal}\}$$

is open in $\mathrm{Spec}(R)$.

Proof. It is clear that the standard open $D(f)$ is contained in U. By Serre's criterion Lemma 10.157.4 we see that $\mathfrak{p} \notin U$ implies that for some $\mathfrak{q} \subset \mathfrak{p}$ we have either

(1) Case I: $\mathrm{depth}(R_\mathfrak{q}) < 2$ and $\dim(R_\mathfrak{q}) \geq 2$, and
(2) Case II: $R_\mathfrak{q}$ is not regular and $\dim(R_\mathfrak{q}) = 1$.

This in particular also means that $R_\mathfrak{q}$ is not normal, and hence $f \in \mathfrak{q}$. In case I we see that $\text{depth}(R_\mathfrak{q}) = \text{depth}(R_\mathfrak{q}/fR_\mathfrak{q}) + 1$. Hence such a prime $\mathfrak{q}$ is the same thing as an embedded associated prime of R/fR. In case II $\mathfrak{q}$ is an associated prime of R/fR of height 1. Thus there is a finite set E of such primes $\mathfrak{q}$ (see Lemma 10.63.5) and

$$\text{Spec}(R) \setminus U = \bigcup\nolimits_{\mathfrak{q} \in E} V(\mathfrak{q})$$

as desired. □

0333 **Lemma 10.161.15.** *Let R be a Noetherian domain. Then R is N-1 if and only if the following two conditions hold*

(1) *there exists a nonzero $f \in R$ such that R_f is normal, and*
(2) *for every maximal ideal $\mathfrak{m} \subset R$ the local ring $R_\mathfrak{m}$ is N-1.*

Proof. First assume R is N-1. Let R' be the integral closure of R in its field of fractions K. By assumption we can find $x_1, \ldots, x_n$ in R' which generate R' as an R-module. Since $R' \subset K$ we can find $f_i \in R$ nonzero such that $f_i x_i \in R$. Then $R_f \cong R'_f$ where $f = f_1 \ldots f_n$. Hence R_f is normal and we have (1). Part (2) follows from Lemma 10.161.3.

Assume (1) and (2). Let K be the fraction field of R. Suppose that $R \subset R' \subset K$ is a finite extension of R contained in K. Note that $R_f = R'_f$ since R_f is already normal. Hence by Lemma 10.161.14 the set of primes $\mathfrak{p}' \in \text{Spec}(R')$ with $R'_{\mathfrak{p}'}$ non-normal is closed in $\text{Spec}(R')$. Since $\text{Spec}(R') \to \text{Spec}(R)$ is closed the image of this set is closed in $\text{Spec}(R)$. For such a ring R' denote $Z_{R'} \subset \text{Spec}(R)$ this image.

Pick a maximal ideal $\mathfrak{m} \subset R$. Let $R_\mathfrak{m} \subset R'_\mathfrak{m}$ be the integral closure of the local ring in K. By assumption this is a finite ring extension. By Lemma 10.36.11 we can find finitely many elements $x_1, \ldots, x_n \in K$ integral over R such that $R'_\mathfrak{m}$ is generated by $x_1, \ldots, x_n$ over $R_\mathfrak{m}$. Let $R' = R[x_1, \ldots, x_n] \subset K$. With this choice it is clear that $\mathfrak{m} \notin Z_{R'}$.

As $\text{Spec}(R)$ is quasi-compact, the above shows that we can find a finite collection $R \subset R'_i \subset K$ such that $\bigcap Z_{R'_i} = \emptyset$. Let R' be the subring of K generated by all of these. It is finite over R. Also $Z_{R'} = \emptyset$. Namely, every prime $\mathfrak{p}'$ lies over a prime $\mathfrak{p}'_i$ such that $(R'_i)_{\mathfrak{p}'_i}$ is normal. This implies that $R'_{\mathfrak{p}'} = (R'_i)_{\mathfrak{p}'_i}$ is normal too. Hence R' is normal, in other words R' is the integral closure of R in K. □

032P **Lemma 10.161.16** (Tate)**.** *Let R be a ring. Let $x \in R$. Assume* [**DG67**, Theorem 23.1.3]

(1) *R is a normal Noetherian domain,*
(2) *R/xR is a domain and N-2,*
(3) *$R \cong \lim_n R/x^n R$ is complete with respect to x.*

Then R is N-2.

Proof. We may assume $x \neq 0$ since otherwise the lemma is trivial. Let K be the fraction field of R. If the characteristic of K is zero the lemma follows from (1), see Lemma 10.161.11. Hence we may assume that the characteristic of K is $p > 0$, and we may apply Lemma 10.161.12. Thus given L/K a finite purely inseparable field extension we have to show that the integral closure S of R in L is finite over R.

Let q be a power of p such that $L^q \subset K$. By enlarging L if necessary we may assume there exists an element $y \in L$ such that $y^q = x$. Since $R \to S$ induces a

homeomorphism of spectra (see Lemma 10.46.7) there is a unique prime ideal $\mathfrak{q} \subset S$ lying over the prime ideal $\mathfrak{p} = xR$. It is clear that

$$\mathfrak{q} = \{f \in S \mid f^q \in \mathfrak{p}\} = yS$$

since $y^q = x$. Observe that $R_\mathfrak{p}$ is a discrete valuation ring by Lemma 10.119.7. Then $S_\mathfrak{q}$ is Noetherian by Krull-Akizuki (Lemma 10.119.12). Whereupon we conclude $S_\mathfrak{q}$ is a discrete valuation ring by Lemma 10.119.7 once again. By Lemma 10.119.10 we see that $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a finite field extension. Hence the integral closure $S' \subset \kappa(\mathfrak{q})$ of R/xR is finite over R/xR by assumption (2). Since $S/yS \subset S'$ this implies that S/yS is finite over R. Note that S/y^nS has a finite filtration whose subquotients are the modules $y^iS/y^{i+1}S \cong S/yS$. Hence we see that each S/y^nS is finite over R. In particular S/xS is finite over R. Also, it is clear that $\bigcap x^nS = (0)$ since an element in the intersection has qth power contained in $\bigcap x^nR = (0)$ (Lemma 10.51.4). Thus we may apply Lemma 10.96.12 to conclude that S is finite over R, and we win. □

032Q **Lemma 10.161.17.** *Let R be a ring. If R is Noetherian, a domain, and N-2, then so is $R[[x]]$.*

Proof. Observe that $R[[x]]$ is Noetherian by Lemma 10.31.2. Let $R' \supset R$ be the integral closure of R in its fraction field. Because R is N-2 this is finite over R. Hence $R'[[x]]$ is finite over $R[[x]]$. By Lemma 10.37.9 we see that $R'[[x]]$ is a normal domain. Apply Lemma 10.161.16 to the element $x \in R'[[x]]$ to see that $R'[[x]]$ is N-2. Then Lemma 10.161.7 shows that $R[[x]]$ is N-2. □

10.162. Nagata rings

032E Here is the definition.

032R **Definition 10.162.1.** Let R be a ring.

(1) We say R is *universally Japanese* if for any finite type ring map $R \to S$ with S a domain we have that S is N-2 (i.e., Japanese).
(2) We say that R is a *Nagata ring* if R is Noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is N-2.

It is clear that a Noetherian universally Japanese ring is a Nagata ring. It is our goal to show that a Nagata ring is universally Japanese. This is not obvious at all, and requires some work. But first, here is a useful lemma.

03GH **Lemma 10.162.2.** *Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure A of R in S is finite over R.*

Proof. As S is essentially of finite type over R it is Noetherian and has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, see Lemma 10.31.6. Since S is reduced we have $S \subset \prod S_{\mathfrak{q}_i}$ and each $S_{\mathfrak{q}_i} = K_i$ is a field, see Lemmas 10.25.4 and 10.25.1. It suffices to show that the integral closure A_i' of R in each K_i is finite over R. This is true because R is Noetherian and $A \subset \prod A_i'$. Let $\mathfrak{p}_i \subset R$ be the prime of R corresponding to $\mathfrak{q}_i$. As S is essentially of finite type over R we see that $K_i = S_{\mathfrak{q}_i} = \kappa(\mathfrak{q}_i)$ is a finitely generated field extension of $\kappa(\mathfrak{p}_i)$. Hence the algebraic closure L_i of $\kappa(\mathfrak{p}_i)$ in K_i is finite over $\kappa(\mathfrak{p}_i)$, see Fields, Lemma 9.26.11. It is clear that A_i' is the integral closure of $R/\mathfrak{p}_i$ in L_i, and hence we win by definition of a Nagata ring. □

0351 **Lemma 10.162.3.** *Let R be a ring. To check that R is universally Japanese it suffices to show: If $R \to S$ is of finite type, and S a domain then S is N-1.*

Proof. Namely, assume the condition of the lemma. Let $R \to S$ be a finite type ring map with S a domain. Let L be a finite extension of the fraction field of S. Then there exists a finite ring extension $S \subset S' \subset L$ such that L is the fraction field of S'. By assumption S' is N-1, and hence the integral closure S'' of S' in L is finite over S'. Thus S'' is finite over S (Lemma 10.7.3) and S'' is the integral closure of S in L (Lemma 10.36.16). We conclude that R is universally Japanese. □

032S **Lemma 10.162.4.** *If R is universally Japanese then any algebra essentially of finite type over R is universally Japanese.*

Proof. The case of an algebra of finite type over R is immediate from the definition. The general case follows on applying Lemma 10.161.3. □

032T **Lemma 10.162.5.** *Let R be a Nagata ring. If $R \to S$ is a quasi-finite ring map (for example finite) then S is a Nagata ring also.*

Proof. First note that S is Noetherian as R is Noetherian and a quasi-finite ring map is of finite type. Let $\mathfrak{q} \subset S$ be a prime ideal, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R/\mathfrak{p} \subset S/\mathfrak{q}$ is quasi-finite and hence we conclude that $S/\mathfrak{q}$ is N-2 by Lemma 10.161.5 as desired. □

032U **Lemma 10.162.6.** *A localization of a Nagata ring is a Nagata ring.*

Proof. Clear from Lemma 10.161.3. □

032V **Lemma 10.162.7.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each R_{f_i} is universally Japanese then so is R.*
(2) *If each R_{f_i} is Nagata then so is R.*

Proof. Let $\varphi : R \to S$ be a finite type ring map so that S is a domain. Then $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in S. Hence if each $S_{f_i} = S_{\varphi(f_i)}$ is N-1 then so is S, see Lemma 10.161.4. This proves (1).

If each R_{f_i} is Nagata, then each R_{f_i} is Noetherian and hence R is Noetherian, see Lemma 10.23.2. And if $\mathfrak{p} \subset R$ is a prime, then we see each $R_{f_i}/\mathfrak{p}R_{f_i} = (R/\mathfrak{p})_{f_i}$ is N-2 and hence we conclude $R/\mathfrak{p}$ is N-2 by Lemma 10.161.4. This proves (2). □

032W **Lemma 10.162.8.** *A Noetherian complete local ring is a Nagata ring.*

Proof. Let R be a complete local Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime. Then $R/\mathfrak{p}$ is also a complete local Noetherian ring, see Lemma 10.160.2. Hence it suffices to show that a Noetherian complete local domain R is N-2. By Lemmas 10.161.5 and 10.160.11 we reduce to the case $R = k[[X_1, \ldots, X_d]]$ where k is a field or $R = \Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

In the case $k[[X_1, \ldots, X_d]]$ we reduce to the statement that a field is N-2 by Lemma 10.161.17. This is clear. In the case $\Lambda[[X_1, \ldots, X_d]]$ we reduce to the statement that a Cohen ring Λ is N-2. Applying Lemma 10.161.16 once more with $x = p \in \Lambda$ we reduce yet again to the case of a field. Thus we win. □

032X **Definition 10.162.9.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is *analytically unramified* if its completion $R^\wedge = \lim_n R/\mathfrak{m}^n$ is reduced. A prime ideal $\mathfrak{p} \subset R$ is said to be *analytically unramified* if $R/\mathfrak{p}$ is analytically unramified.

At this point we know the following are true for any Noetherian local ring R: The map $R \to R^\wedge$ is a faithfully flat local ring homomorphism (Lemma 10.97.3). The completion $R^\wedge$ is Noetherian (Lemma 10.97.5) and complete (Lemma 10.97.4). Hence the completion $R^\wedge$ is a Nagata ring (Lemma 10.162.8). Moreover, we have seen in Section 10.160 that $R^\wedge$ is a quotient of a regular local ring (Theorem 10.160.8), and hence universally catenary (Remark 10.160.9).

032Y **Lemma 10.162.10.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(1) *If R is analytically unramified, then R is reduced.*
(2) *If R is analytically unramified, then each minimal prime of R is analytically unramified.*
(3) *If R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified, then R is analytically unramified.*
(4) *If R is analytically unramified, then the integral closure of R in its total ring of fractions $Q(R)$ is finite over R.*
(5) *If R is a domain and analytically unramified, then R is N-1.*

Proof. In this proof we will use the remarks immediately following Definition 10.162.9. As $R \to R^\wedge$ is a faithfully flat local ring homomorphism it is injective and (1) follows.

Let $\mathfrak{q}$ be a minimal prime of R, and assume R is analytically unramified. Then $\mathfrak{q}$ is an associated prime of R (see Proposition 10.63.6). Hence there exists an $f \in R$ such that $\{x \in R \mid fx = 0\} = \mathfrak{q}$. Note that $(R/\mathfrak{q})^\wedge = R^\wedge/\mathfrak{q}^\wedge$, and that $\{x \in R^\wedge \mid fx = 0\} = \mathfrak{q}^\wedge$, because completion is exact (Lemma 10.97.2). If $x \in R^\wedge$ is such that $x^2 \in \mathfrak{q}^\wedge$, then $fx^2 = 0$ hence $(fx)^2 = 0$ hence $fx = 0$ hence $x \in \mathfrak{q}^\wedge$. Thus $\mathfrak{q}$ is analytically unramified and (2) holds.

Assume R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified. Then $R \to R/\mathfrak{q}_1 \times \ldots \times R/\mathfrak{q}_t$ is injective. Since completion is exact (see Lemma 10.97.2) we see that $R^\wedge \subset (R/\mathfrak{q}_1)^\wedge \times \ldots \times (R/\mathfrak{q}_t)^\wedge$. Hence (3) is clear.

Assume R is analytically unramified. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $R^\wedge$. Then we see that

$$Q(R^\wedge) = R^\wedge_{\mathfrak{p}_1} \times \ldots \times R^\wedge_{\mathfrak{p}_s}$$

with each $R^\wedge_{\mathfrak{p}_i}$ a field as $R^\wedge$ is reduced (see Lemma 10.25.4). Hence the integral closure S of $R^\wedge$ in $Q(R^\wedge)$ is equal to $S = S_1 \times \ldots \times S_s$ with S_i the integral closure of $R^\wedge/\mathfrak{p}_i$ in its fraction field. By Lemma 10.162.8 the ring S_i is finite over $R^\wedge/\mathfrak{p}_i$. Thus S is finite over $R^\wedge$. Denote R' the integral closure of R in $Q(R)$. As $R \to R^\wedge$ is flat we see that $R' \otimes_R R^\wedge \subset Q(R) \otimes_R R^\wedge \subset Q(R^\wedge)$. Moreover $R' \otimes_R R^\wedge$ is integral over $R^\wedge$ (Lemma 10.36.13). Hence $R' \otimes_R R^\wedge \subset S$ is a $R^\wedge$-submodule. As $R^\wedge$ is Noetherian it is a finite $R^\wedge$-module. Thus we may find $f_1, \ldots, f_n \in R'$ such that $R' \otimes_R R^\wedge$ is generated by the elements $f_i \otimes 1$ as a $R^\wedge$-module. By faithful flatness we see that R' is generated by $f_1, \ldots, f_n$ as an R-module. This proves (4).

Part (5) is a special case of part (4). □

032Z **Lemma 10.162.11.** *Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Assume*

(1) *$R_\mathfrak{p}$ is a discrete valuation ring, and*
(2) *$\mathfrak{p}$ is analytically unramified.*

Then for any associated prime $\mathfrak{q}$ of $R^\wedge/\mathfrak{p}R^\wedge$ the local ring $(R^\wedge)_\mathfrak{q}$ is a discrete valuation ring.

Proof. Assumption (2) says that $R^\wedge/\mathfrak{p}R^\wedge$ is a reduced ring. Hence an associated prime $\mathfrak{q} \subset R^\wedge$ of $R^\wedge/\mathfrak{p}R^\wedge$ is the same thing as a minimal prime over $\mathfrak{p}R^\wedge$. In particular we see that the maximal ideal of $(R^\wedge)_\mathfrak{q}$ is $\mathfrak{p}(R^\wedge)_\mathfrak{q}$. Choose $x \in R$ such that $xR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. By the above we see that $x \in (R^\wedge)_\mathfrak{q}$ generates the maximal ideal. As $R \to R^\wedge$ is faithfully flat we see that x is a nonzerodivisor in $(R^\wedge)_\mathfrak{q}$. Hence we win. □

0330 **Lemma 10.162.12.** *Let $(R, \mathfrak{m})$ be a Noetherian local domain. Let $x \in \mathfrak{m}$. Assume*

(1) *$x \neq 0$,*
(2) *R/xR has no embedded primes, and*
(3) *for each associated prime $\mathfrak{p} \subset R$ of R/xR we have*
   (a) *the local ring $R_\mathfrak{p}$ is regular, and*
   (b) *$\mathfrak{p}$ is analytically unramified.*

Then R is analytically unramified.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the associated primes of the R-module R/xR. Since R/xR has no embedded primes we see that each $\mathfrak{p}_i$ has height 1, and is a minimal prime over (x). For each i, let $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{is_i}$ be the associated primes of the $R^\wedge$-module $R^\wedge/\mathfrak{p}_iR^\wedge$. By Lemma 10.162.11 we see that $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular. By Lemma 10.65.3 we see that

$$\mathrm{Ass}_{R^\wedge}(R^\wedge/xR^\wedge) = \bigcup\nolimits_{\mathfrak{p}\in\mathrm{Ass}_R(R/xR)} \mathrm{Ass}_{R^\wedge}(R^\wedge/\mathfrak{p}R^\wedge) = \{\mathfrak{q}_{ij}\}.$$

Let $y \in R^\wedge$ with $y^2 = 0$. As $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular, and hence a domain (Lemma 10.106.2) we see that y maps to zero in $(R^\wedge)_{\mathfrak{q}_{ij}}$. Hence y maps to zero in $R^\wedge/xR^\wedge$ by Lemma 10.63.19. Hence $y = xy'$. Since x is a nonzerodivisor (as $R \to R^\wedge$ is flat) we see that $(y')^2 = 0$. Hence we conclude that $y \in \bigcap x^nR^\wedge = (0)$ (Lemma 10.51.4). □

0331 **Lemma 10.162.13.** *Let $(R, \mathfrak{m})$ be a local ring. If R is Noetherian, a domain, and Nagata, then R is analytically unramified.*

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial. Hence we assume $\dim(R) = d$ and that the lemma holds for all Noetherian Nagata domains of dimension $< d$.

Let $R \subset S$ be the integral closure of R in the field of fractions of R. By assumption S is a finite R-module. By Lemma 10.162.5 we see that S is Nagata. By Lemma 10.112.4 we see $\dim(R) = \dim(S)$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of S. Each of these lies over the maximal ideal $\mathfrak{m}$ of R. Moreover

$$(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_t)^n \subset \mathfrak{m}S$$

for sufficiently large n as $S/\mathfrak{m}S$ is Artinian. By Lemma 10.97.2 $R^\wedge \to S^\wedge$ is an injective map, and by the Chinese Remainder Lemma 10.15.4 combined with Lemma 10.96.9 we have $S^\wedge = \prod S_i^\wedge$ where $S_i^\wedge$ is the completion of S with respect to the maximal ideal $\mathfrak{m}_i$. Hence it suffices to show that $S_{\mathfrak{m}_i}$ is analytically unramified. In other words, we have reduced to the case where R is a Noetherian normal Nagata domain.

Assume R is a Noetherian, normal, local Nagata domain. Pick a nonzero $x \in \mathfrak{m}$ in the maximal ideal. We are going to apply Lemma 10.162.12. We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is clear. We have that R/xR has no embedded primes by Lemma 10.157.6. Thus property (2) holds. The same lemma also tells us each associated prime $\mathfrak{p}$ of R/xR has height 1. Hence $R_\mathfrak{p}$ is a 1-dimensional normal domain hence regular (Lemma 10.119.7). Thus (3)(a) holds. Finally (3)(b) holds by induction hypothesis, since $R/\mathfrak{p}$ is Nagata (by Lemma 10.162.5 or directly from the definition). Thus we conclude R is analytically unramified. □

0BI2 **Lemma 10.162.14.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent*

(1) *R is Nagata,*
(2) *for $R \to S$ finite with S a domain and $\mathfrak{m}' \subset S$ maximal the local ring $S_{\mathfrak{m}'}$ is analytically unramified,*
(3) *for $(R, \mathfrak{m}) \to (S, \mathfrak{m}')$ finite local homomorphism with S a domain, then S is analytically unramified.*

Proof. Assume R is Nagata and let $R \to S$ and $\mathfrak{m}' \subset S$ be as in (2). Then S is Nagata by Lemma 10.162.5. Hence the local ring $S_{\mathfrak{m}'}$ is Nagata (Lemma 10.162.6). Thus it is analytically unramified by Lemma 10.162.13. It is clear that (2) implies (3).

Assume (3) holds. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. To prove (1) we have to show that the integral closure of $R/\mathfrak{p}$ is finite over $R/\mathfrak{p}$. Choose $x_1, \ldots, x_n \in L$ which generate L over $\kappa(\mathfrak{p})$. For each i let $P_i(T) = T^{d_i} + a_{i,1}T^{d_i-1} + \ldots + a_{i,d_i}$ be the minimal polynomial for x_i over $\kappa(\mathfrak{p})$. After replacing x_i by $f_i x_i$ for a suitable $f_i \in R$, $f_i \notin \mathfrak{p}$ we may assume $a_{i,j} \in R/\mathfrak{p}$. In fact, after further multiplying by elements of $\mathfrak{m}$, we may assume $a_{i,j} \in \mathfrak{m}/\mathfrak{p} \subset R/\mathfrak{p}$ for all i, j. Having done this let $S = R/\mathfrak{p}[x_1, \ldots, x_n] \subset L$. Then S is finite over R, a domain, and $S/\mathfrak{m}S$ is a quotient of $R/\mathfrak{m}[T_1, \ldots, T_n]/(T_1^{d_1}, \ldots, T_n^{d_n})$. Hence S is local. By (3) S is analytically unramified and by Lemma 10.162.10 we find that its integral closure S' in L is finite over S. Since S' is also the integral closure of $R/\mathfrak{p}$ in L we win. □

The following proposition says in particular that an algebra of finite type over a Nagata ring is a Nagata ring.

0334 **Proposition 10.162.15** (Nagata)**.** *Let R be a ring. The following are equivalent:*

(1) *R is a Nagata ring,*
(2) *any finite type R-algebra is Nagata, and*
(3) *R is universally Japanese and Noetherian.*

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (2) holds). Let R be a Nagata ring. We will show that any finitely generated R-algebra S is Nagata. This will prove the proposition.

Step 1. There exists a sequence of ring maps $R = R_0 \to R_1 \to R_2 \to \ldots \to R_n = S$ such that each $R_i \to R_{i+1}$ is generated by a single element. Hence by induction it suffices to prove S is Nagata if $S \cong R[x]/I$.

Step 2. Let $\mathfrak{q} \subset S$ be a prime of S, and let $\mathfrak{p} \subset R$ be the corresponding prime of R. We have to show that $S/\mathfrak{q}$ is N-2. Hence we have reduced to proving the following:

(*) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 3: Let R be a Nagata domain and let $R \subset S$ be a monogenic extension of domains. Suppose the induced extension of fraction fields of R and S is purely transcendental. In this case $S = R[x]$. By Lemma 10.161.13 we see that S is N-2. Hence we have reduced to proving the following: (**) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains inducing a finite extension of fraction fields then S is N-2.

Step 4. Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains inducing a finite extension of fraction fields L/K. Choose an element $x \in S$ which generates S as an R-algebra. Let M/L be a finite extension of fields. Let R' be the integral closure of R in M. Then the integral closure S' of S in M is equal to the integral closure of $R'[x]$ in M. Also the fraction field of R' is M and $R \subset R'$ is finite (by the Nagata property of R). This implies that R' is a Nagata ring (Lemma 10.162.5). To show that S' is finite over S is the same as showing that S' is finite over $R'[x]$. Replace R by R' and S by $R'[x]$ to reduce to the following statement: (***) Given a normal Nagata domain R with fraction field K, and $x \in K$, the ring $S \subset K$ generated by R and x is N-1.

Step 5. Let R be a normal Nagata domain with fraction field K. Let $x = b/a \in K$. We have to show that the ring $S \subset K$ generated by R and x is N-1. Note that $S_a \cong R_a$ is normal. Hence by Lemma 10.161.15 it suffices to show that $S_{\mathfrak{m}}$ is N-1 for every maximal ideal $\mathfrak{m}$ of S.

With assumptions as in the preceding paragraph, pick such a maximal ideal and set $\mathfrak{n} = R \cap \mathfrak{m}$. The residue field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{n})$ is finite (Theorem 10.34.1) and generated by the image of x. After replacing R by R_a for some $a \in R$, $a \notin \mathfrak{n}$ and S by S_a, may assume there exists a monic polynomial $f(X) = X^d + \sum_{i=1,\ldots,d} a_i X^{d-i}$ in $R[x]$ with $f(x) \in \mathfrak{m}$ (details omitted). Let K''/K be a finite extension of fields such that the polynomial $f(X)$ splits completely in $K''[X]$. Let R' be the integral closure of R in K''. Let $S' \subset K''$ be the subring generated by R' and x. As R is Nagata we see R' is finite over R and Nagata (Lemma 10.162.5). Moreover, S' is finite over S. If for every maximal ideal $\mathfrak{m}'$ of S' the local ring $S'_{\mathfrak{m}'}$ is N-1, then $S'_{\mathfrak{m}}$ is N-1 by Lemma 10.161.15, which in turn implies that $S_{\mathfrak{m}}$ is N-1 by Lemma 10.161.7. After replacing R by R' and S by S', and $\mathfrak{m}$ by any of the maximal ideals $\mathfrak{m}'$ lying over $\mathfrak{m}$ we reach the situation where the polynomial f above split completely: $f(X) = \prod_{i=1,\ldots,d}(X - a_i)$ with $a_i \in R$. Since $f(x) \in \mathfrak{m}$ we see that $x - a_i \in \mathfrak{m}$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak{m}$.

To recapitulate: R is a normal Nagata domain with fraction field K, $x \in K$ and S is the subring of K generated by x and R, finally $\mathfrak{m} \subset S$ is a maximal ideal with $x \in \mathfrak{m}$. We have to show $S_{\mathfrak{m}}$ is N-1.

We will show that Lemma 10.162.12 applies to the local ring $S_{\mathfrak{m}}$ and the element x. This will imply that $S_{\mathfrak{m}}$ is analytically unramified, whereupon we see that it is N-1 by Lemma 10.162.10.

We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is trivial. Let $I = \mathrm{Ker}(R[X] \to S)$ where $X \mapsto x$. We claim that I is generated by all linear forms $aX - b$ such that $ax = b$ in K. Clearly all these linear forms are in I. If

$g = a_dX^d + \ldots a_1X + a_0 \in I$, then we see that a_dx is integral over R (Lemma 10.123.1) and hence $b := a_dx \in R$ as R is normal. Then $g - (a_dX - b)X^{d-1} \in I$ and we win by induction on the degree. As a consequence we see that

$$S/xS = R[X]/(X, I) = R/J$$

where

$$J = \{b \in R \mid ax = b \text{ for some } a \in R\} = xR \cap R$$

By Lemma 10.157.6 we see that $S/xS = R/J$ has no embedded primes as an R-module, hence as an R/J-module, hence as an S/xS-module, hence as an S-module. This proves property (2). Take such an associated prime $\mathfrak{q} \subset S$ with the property $\mathfrak{q} \subset \mathfrak{m}$ (so that it is an associated prime of $S_\mathfrak{m}/xS_\mathfrak{m}$ – it does not matter for the arguments). Then $\mathfrak{q}$ is minimal over xS and hence has height 1. By the sequence of equalities above we see that $\mathfrak{p} = R \cap \mathfrak{q}$ is an associated prime of R/J, and so has height 1 (see Lemma 10.157.6). Thus $R_\mathfrak{p}$ is a discrete valuation ring and therefore $R_\mathfrak{p} \subset S_\mathfrak{q}$ is an equality. This shows that $S_\mathfrak{q}$ is regular. This proves property (3)(a). Finally, $(S/\mathfrak{q})_\mathfrak{m}$ is a localization of $S/\mathfrak{q}$, which is a quotient of $S/xS = R/J$. Hence $(S/\mathfrak{q})_\mathfrak{m}$ is a localization of a quotient of the Nagata ring R, hence Nagata (Lemmas 10.162.5 and 10.162.6) and hence analytically unramified (Lemma 10.162.13). This shows (3)(b) holds and we are done. □

0335 **Proposition 10.162.16.** *The following types of rings are Nagata and in particular universally Japanese:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. The Noetherian complete local ring case is Lemma 10.162.8. In the other cases you just check if $R/\mathfrak{p}$ is N-2 for every prime ideal $\mathfrak{p}$ of the ring. This is clear whenever $R/\mathfrak{p}$ is a field, i.e., $\mathfrak{p}$ is maximal. Hence for the Dedekind ring case we only need to check it when $\mathfrak{p} = (0)$. But since we assume the fraction field has characteristic zero Lemma 10.161.11 kicks in. □

09E1 **Example 10.162.17.** A discrete valuation ring is Nagata if and only if it is N-2 (because the quotient by the maximal ideal is a field and hence N-2). The discrete valuation ring A of Example 10.119.5 is not Nagata, i.e., it is not N-2. Namely, the finite extension $A \subset R = A[f]$ is not N-1. To see this say $f = \sum a_ix^i$. For every $n \geq 1$ set $g_n = \sum_{i<n} a_ix^i \in A$. Then $h_n = (f - g_n)/x^n$ is an element of the fraction field of R and $h_n^p \in k^p[[x]] \subset A$. Hence the integral closure R' of R contains $h_1, h_2, h_3, \ldots$. Now, if R' were finite over R and hence A, then $f = x^nh_n + g_n$ would be contained in the submodule $A + x^nR'$ for all n. By Artin-Rees this would imply $f \in A$ (Lemma 10.51.4), a contradiction.

09E2 **Lemma 10.162.18.** *Let $(A, \mathfrak{m})$ be a Noetherian local domain which is Nagata and has fraction field of characteristic p. If $a \in A$ has a pth root in $A^\wedge$, then a has a pth root in A.*

Proof. Let $\alpha \in A^\wedge$ be a pth root of a. To get a contradiction, assume a does not have a pth root in A. Then a does not have a pth root in the fraction field K of A. Namely, if $\beta = b/c$ with $b, c \in A$ and $c \neq 0$ satisfies $\beta^p = a$, then $\alpha = b/c$ in

$A^\wedge$ which implies that c divides b in A by faithful flatness of $A \to A^\wedge$ and hence $\beta \in A$, contradiction. Thus $B = A[x]/(x^p - a)$ is a domain because $K[x]/(x^p - a)$ is a field. However, the completion of B isn't reduced by the assumed existence of α. This contradicts our earlier results, as B is a Nagata ring (Proposition 10.162.15) and hence analytically unramified by Lemma 10.162.13. □

10.163. Ascending properties

0336 In this section we start proving some algebraic facts concerning the "ascent" of properties of rings. To do this for depth of rings one uses the following result on ascending depth of modules, see [**DG67**, IV, Proposition 6.3.1].

0338 **Lemma 10.163.1.** *We have*

$$depth(M \otimes_R N) = depth(M) + depth(N/\mathfrak{m}_R N)$$

where $R \to S$ is a local homomorphism of local Noetherian rings, M is a finite R-module, and N is a finite S-module flat over R.

[**DG67**, IV, Proposition 6.3.1]

Proof. In the statement and in the proof below, we take the depth of M as an R-module, the depth of $M \otimes_R N$ as an S-module, and the depth of $N/\mathfrak{m}_R N$ as an $S/\mathfrak{m}_R S$-module. Denote n the right hand side. First assume that n is zero. Then both $\text{depth}(M) = 0$ and $\text{depth}(N/\mathfrak{m}_R N) = 0$. This means there is a $z \in M$ whose annihilator is $\mathfrak{m}_R$ and a $\overline{y} \in N/\mathfrak{m}_R N$ whose annihilator is $\mathfrak{m}_S/\mathfrak{m}_R S$. Let $y \in N$ be a lift of $\overline{y}$. Since N is flat over R the map $z : R/\mathfrak{m}_R \to M$ produces an injective map $N/\mathfrak{m}_R N \to M \otimes_R N$. Hence the annihilator of $z \otimes y$ is $\mathfrak{m}_S$. Thus $\text{depth}(M \otimes_R N) = 0$ as well.

Assume $n > 0$. If $\text{depth}(N/\mathfrak{m}_R N) > 0$, then we may choose $f \in \mathfrak{m}_S$ mapping to $\overline{f} \in S/\mathfrak{m}_R S$ which is a nonzerodivisor on $N/\mathfrak{m}_R N$. Then $\text{depth}(N/\mathfrak{m}_R N) = \text{depth}(N/(f, \mathfrak{m}_R)N) + 1$ by Lemma 10.72.7. According to Lemma 10.99.1 the element $f \in S$ is a nonzerodivisor on N and N/fN is flat over R. Hence by induction on n we have

$$\text{depth}(M \otimes_R N/fN) = \text{depth}(M) + \text{depth}(N/(f, \mathfrak{m}_R)N).$$

Because N/fN is flat over R the sequence

$$0 \to M \otimes_R N \to M \otimes_R N \to M \otimes_R N/fN \to 0$$

is exact where the first map is multiplication by f (Lemma 10.39.12). Hence by Lemma 10.72.7 we find that $\text{depth}(M \otimes_R N) = \text{depth}(M \otimes_R N/fN) + 1$ and we conclude that equality holds in the formula of the lemma.

If $n > 0$, but $\text{depth}(N/\mathfrak{m}_R N) = 0$, then we can choose $f \in \mathfrak{m}_R$ which is a nonzerodivisor on M. As N is flat over R it is also the case that f is a nonzerodivisor on $M \otimes_R N$. By induction on n again we have

$$\text{depth}(M/fM \otimes_R N) = \text{depth}(M/fM) + \text{depth}(N/\mathfrak{m}_R N).$$

In this case $\text{depth}(M \otimes_R N) = \text{depth}(M/fM \otimes_R N) + 1$ and $\text{depth}(M) = \text{depth}(M/fM) + 1$ by Lemma 10.72.7 and we conclude that equality holds in the formula of the lemma. □

0337 **Lemma 10.163.2.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Then*

$$depth(S) = depth(R) + depth(S/\mathfrak{m}_R S).$$

Proof. This is a special case of Lemma 10.163.1. □

045J **Lemma 10.163.3.** *Let $R \to S$ be a flat local homomorphism of local Noetherian rings. Then the following are equivalent*

(1) *S is Cohen-Macaulay, and*
(2) *R and $S/\mathfrak{m}_R S$ are Cohen-Macaulay.*

Proof. Follows from the definitions and Lemmas 10.163.2 and 10.112.7. □

0339 **Lemma 10.163.4.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *S is Noetherian,*
(3) *φ is flat,*
(4) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are (S_k), and*
(5) *R has property (S_k).*

Then S has property (S_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. By Lemma 10.163.2 we have

$$\operatorname{depth}(S_\mathfrak{q}) = \operatorname{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \operatorname{depth}(R_\mathfrak{p}).$$

On the other hand, we have

$$\dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \dim(S_\mathfrak{q})$$

by Lemma 10.112.6. (Actually equality holds, by Lemma 10.112.7 but strictly speaking we do not need this.) Finally, as the fibre rings of the map are assumed (S_k) we see that $\operatorname{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}))$. Thus the lemma follows by the following string of inequalities

$$\begin{aligned}
\operatorname{depth}(S_\mathfrak{q}) &= \operatorname{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \operatorname{depth}(R_\mathfrak{p}) \\
&\geq \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})) + \min(k, \dim(R_\mathfrak{p})) \\
&= \min(2k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + k, k + \dim(R_\mathfrak{p}), \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \dim(R_\mathfrak{p})) \\
&\geq \min(k, \dim(S_\mathfrak{q}))
\end{aligned}$$

as desired. □

033A **Lemma 10.163.5.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *S is Noetherian*
(3) *φ is flat,*
(4) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have property (R_k), and*
(5) *R has property (R_k).*

Then S has property (R_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. Assume that $\dim(S_\mathfrak{q}) \leq k$. Since $\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$ by Lemma 10.112.7 we see that $\dim(R_\mathfrak{p}) \leq k$ and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \leq k$. Hence $R_\mathfrak{p}$ and $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ are regular by assumption. It follows that $S_\mathfrak{q}$ is regular by Lemma 10.112.8. □

0C21 **Lemma 10.163.6.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *S is Noetherian*
(3) *φ is flat,*

(4) *the fibre rings* $S \otimes_R \kappa(\mathfrak{p})$ *are reduced,*
(5) R *is reduced.*

Then S *is reduced.*

Proof. For Noetherian rings reduced is the same as having properties (S_1) and (R_0), see Lemma 10.157.3. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 10.163.4 and 10.163.5 and we see that S is (S_1) and (R_0), in other words reduced by Lemma 10.157.3 again. □

033B **Lemma 10.163.7.** *Let* $\varphi : R \to S$ *be a ring map. Assume*

(1) φ *is smooth,*
(2) R *is reduced.*

Then S *is reduced.*

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.142). In particular, the fibres are reduced. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 10.163.6.

In the general case we may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.142. Now, if $x \in S$ is an element with $x^2 = 0$, then we can enlarge R_0 and assume that x comes from an element $x_0 \in S_0$. After enlarging R_0 once more we may assume that $x_0^2 = 0$ in S_0. However, since $R_0 \subset R$ is reduced we see that S_0 is reduced and hence $x_0 = 0$ as desired. □

0C22 **Lemma 10.163.8.** *Let* $\varphi : R \to S$ *be a ring map. Assume*

(1) R *is Noetherian,*
(2) S *is Noetherian,*
(3) φ *is flat,*
(4) *the fibre rings* $S \otimes_R \kappa(\mathfrak{p})$ *are normal, and*
(5) R *is normal.*

Then S *is normal.*

Proof. For a Noetherian ring being normal is the same as having properties (S_2) and (R_1), see Lemma 10.157.4. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 10.163.4 and 10.163.5 and we see that S is (S_2) and (R_1), in other words normal by Lemma 10.157.4 again. □

033C **Lemma 10.163.9.** *Let* $\varphi : R \to S$ *be a ring map. Assume*

(1) φ *is smooth,*
(2) R *is normal.*

Then S *is normal.*

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.142). In particular, the fibres are normal. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 10.163.8.

The general case. First note that R is reduced and hence S is reduced by Lemma 10.163.7. Let $\mathfrak{q}$ be a prime of S and let $\mathfrak{p}$ be the corresponding prime of R. Note that $R_\mathfrak{p}$ is a normal domain. We have to show that $S_\mathfrak{q}$ is a normal domain. To do this we may replace R by $R_\mathfrak{p}$ and S by $S_\mathfrak{p}$. Hence we may assume that R is a normal domain.

Assume $R \to S$ smooth, and R a normal domain. We may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.142. As R_0 is a Nagata domain (see Proposition 10.162.16) we see that its integral closure R_0' is finite over R_0. Moreover, as R is a normal domain it is clear that $R_0' \subset R$. Hence we may replace R_0 by R_0' and S_0 by $R_0' \otimes_{R_0} S_0$ and assume that R_0 is a normal Noetherian domain. By the first paragraph of the proof we conclude that S_0 is a normal ring (it need not be a domain of course). In this way we see that $R = \bigcup R_\lambda$ is the union of normal Noetherian domains and correspondingly $S = \operatorname{colim} R_\lambda \otimes_{R_0} S_0$ is the colimit of normal rings. This implies that S is a normal ring. Some details omitted. □

07NF **Lemma 10.163.10.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is a regular ring.*

Then S is regular.

Proof. This follows from Lemma 10.163.5 applied for all (R_k) using Lemma 10.140.3 to see that the hypotheses are satisfied. □

10.164. Descending properties

033D In this section we start proving some algebraic facts concerning the "descent" of properties of rings. It turns out that it is often "easier" to descend properties than it is to ascend them. In other words, the assumption on the ring map $R \to S$ are often weaker than the assumptions in the corresponding lemma of the preceding section. However, we warn the reader that the results on descent are often useless unless the corresponding ascent can also be shown! Here is a typical result which illustrates this phenomenon.

033E **Lemma 10.164.1.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian.*

Then R is Noetherian.

Proof. Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be a growing sequence of ideals of R. By assumption we have $I_n S = I_{n+1} S = I_{n+2} S = \ldots$ for some n. By faithful flatness, extending and contracting gives the same ideal, meaning that $I = R \cap IS$ for each ideal I in R (Lemma 10.82.11). So $I_n = I_{n+1} = I_{n+2} = \ldots$ as desired. □

033F **Lemma 10.164.2.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is reduced.*

Then R is reduced.

Proof. This is clear as $R \to S$ is injective, by Lemma 10.82.11. □

033G **Lemma 10.164.3.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a normal ring.*

Then R is a normal ring.

Proof. Since S is reduced it follows that R is reduced. Let $\mathfrak{p}$ be a prime of R. We have to show that $R_\mathfrak{p}$ is a normal domain. Since $S_\mathfrak{p}$ is faithfully over $R_\mathfrak{p}$ too we may assume that R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then we see that $R \to S_\mathfrak{q}$ is faithfully flat (Lemma 10.39.17). Hence we may assume S is local as well. In particular S is a normal domain. Since $R \to S$ is faithfully flat and S is a normal domain we see that R is a domain. Next, suppose that a/b is integral over R with $a, b \in R$. Then $a/b \in S$ as S is normal. Hence $a \in bS$. This means that $a : R \to R/bR$ becomes the zero map after base change to S. By faithful flatness we see that $a \in bR$, so $a/b \in R$. Hence R is normal. □

07NG **Lemma 10.164.4.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a regular ring.*

Then R is a regular ring.

Proof. We see that R is Noetherian by Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. Then Lemma 10.110.9 applies to $R_\mathfrak{p} \to S_\mathfrak{q}$ and we conclude that $R_\mathfrak{p}$ is regular. Since $\mathfrak{p}$ was arbitrary we see R is regular. □

0352 **Lemma 10.164.5.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian and has property (S_k).*

Then R is Noetherian and has property (S_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime ideal. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.112.7) and $\text{depth}(A) = \text{depth}(B)$ (Lemma 10.163.2). Hence since B has (S_k) we see that A has (S_k). □

0353 **Lemma 10.164.6.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian and has property (R_k).*

Then R is Noetherian and has property (R_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime ideal and assume $\dim(R_\mathfrak{p}) \leq k$. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.112.7). As S has (R_k) we conclude that B is a regular local ring. By Lemma 10.110.9 we conclude that A is regular. □

0354 **Lemma 10.164.7.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is smooth and surjective on spectra, and*
(2) *S is a Nagata ring.*

Then R is a Nagata ring.

Proof. Recall that a Nagata ring is the same thing as a Noetherian universally Japanese ring (Proposition 10.162.15). We have already seen that R is Noetherian in Lemma 10.164.1. Let $R \to A$ be a finite type ring map into a domain. According to Lemma 10.162.3 it suffices to check that A is N-1. It is clear that $B = A \otimes_R S$ is a finite type S-algebra and hence Nagata (Proposition 10.162.15). Since $A \to B$ is smooth (Lemma 10.137.3) we see that B is reduced (Lemma 10.163.7). Since B is Noetherian it has only a finite number of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ (see Lemma 10.31.6). As $A \to B$ is flat each of these lies over $(0) \subset A$ (by going down, see Lemma 10.39.19) The total ring of fractions $Q(B)$ is the product of the $L_i = \kappa(\mathfrak{q}_i)$ (Lemmas 10.25.4 and 10.25.1). Moreover, the integral closure B' of B in $Q(B)$ is the product of the integral closures B'_i of the $B/\mathfrak{q}_i$ in the factors L_i (compare with Lemma 10.37.16). Since B is universally Japanese the ring extensions $B/\mathfrak{q}_i \subset B'_i$ are finite and we conclude that $B' = \prod B'_i$ is finite over B. Since $A \to B$ is flat we see that any nonzerodivisor on A maps to a nonzerodivisor on B. The corresponding map

$$Q(A) \otimes_A B = (A \setminus \{0\})^{-1}A \otimes_A B = (A \setminus \{0\})^{-1}B \to Q(B)$$

is injective (we used Lemma 10.12.15). Via this map A' maps into B'. This induces a map

$$A' \otimes_A B \longrightarrow B'$$

which is injective (by the above and the flatness of $A \to B$). Since B' is a finite B-module and B is Noetherian we see that $A' \otimes_A B$ is a finite B-module. Hence there exist finitely many elements $x_i \in A'$ such that the elements $x_i \otimes 1$ generate $A' \otimes_A B$ as a B-module. Finally, by faithful flatness of $A \to B$ we conclude that the x_i also generated A' as an A-module, and we win. □

0355 **Remark 10.164.8.** The property of being "universally catenary" does not descend; not even along étale ring maps. In Examples, Section 110.19 there is a construction of a finite ring map $A \to B$ with A local Noetherian and not universally catenary, B semi-local with two maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ with $B_\mathfrak{m}$ and $B_\mathfrak{n}$ regular of dimension 2 and 1 respectively, and the same residue fields as that of A. Moreover, $\mathfrak{m}_A$ generates the maximal ideal in both $B_\mathfrak{m}$ and $B_\mathfrak{n}$ (so $A \to B$ is unramified as well as finite). By Lemma 10.152.3 there exists a local étale ring map $A \to A'$ such that $B \otimes_A A' = B_1 \times B_2$ decomposes with $A' \to B_i$ surjective. This shows that A' has two minimal primes $\mathfrak{q}_i$ with $A'/\mathfrak{q}_i \cong B_i$. Since B_i is regular local (since it is étale over either $B_\mathfrak{m}$ or $B_\mathfrak{n}$) we conclude that A' is universally catenary.

10.165. Geometrically normal algebras

037Y In this section we put some applications of ascent and descent of properties of rings.

037Z **Lemma 10.165.1.** *Let k be a field. Let A be a k-algebra. The following properties of A are equivalent:*

(1) *$k' \otimes_k A$ is a normal ring for every field extension k'/k,*
(2) *$k' \otimes_k A$ is a normal ring for every finitely generated field extension k'/k,*
(3) *$k' \otimes_k A$ is a normal ring for every finite purely inseparable extension k'/k,*
(4) *$k^{perf} \otimes_k A$ is a normal ring.*

Here normal ring is defined in Definition 10.37.11.

Proof. It is clear that (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4).

If k'/k is a finite purely inseparable extension, then there is an embedding $k' \to k^{perf}$ of k-extensions. The ring map $k' \otimes_k A \to k^{perf} \otimes_k A$ is faithfully flat, hence $k' \otimes_k A$ is normal if $k^{perf} \otimes_k A$ is normal by Lemma 10.164.3. In this way we see that (4) $\Rightarrow$ (3).

Assume (2) and let k'/k be any field extension. Then we can write $k' = \operatorname{colim}_i k_i$ as a directed colimit of finitely generated field extensions. Hence we see that $k' \otimes_k A = \operatorname{colim}_i k_i \otimes_k A$ is a directed colimit of normal rings. Thus we see that $k' \otimes_k A$ is a normal ring by Lemma 10.37.17. Hence (1) holds.

Assume (3) and let K/k be a finitely generated field extension. By Lemma 10.45.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a normal ring because we assumed (3). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a normal ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.137.3) and ascent of normality along smooth maps (Lemma 10.163.9). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a normal ring as it is a localization of a normal ring (Lemma 10.37.13). Step 4. Finally $K \otimes_k A$ is a normal ring by descent of normality along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.164.3). This proves the lemma. □

0380 **Definition 10.165.2.** Let k be a field. A k-algebra R is called *geometrically normal* over k if the equivalent conditions of Lemma 10.165.1 hold.

06DE **Lemma 10.165.3.** *Let k be a field. A localization of a geometrically normal k-algebra is geometrically normal.*

Proof. This is clear as being a normal ring is checked at the localizations at prime ideals. □

0C30 **Lemma 10.165.4.** *Let k be a field. Let K/k be a separable field extension. Then K is geometrically normal over k.*

Proof. This is true because $k^{perf} \otimes_k K$ is a field. Namely, it is reduced by Lemma 10.43.6. By Lemma 10.45.4 (or by Definition 10.45.5) the field extension k^{perf}/k is purely inseparable. Hence by Lemma 10.46.10 the ring $k^{perf} \otimes_k K$ has a unique prime ideal. A reduced ring with a unique prime ideal is a field. □

06DF **Lemma 10.165.5.** *Let k be a field. Let A, B be k-algebras. Assume A is geometrically normal over k and B is a normal ring. Then $A \otimes_k B$ is a normal ring.*

Proof. Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. Denote $\mathfrak{p}$, resp. $\mathfrak{q}$ the corresponding prime of A, resp. B. Then $(A \otimes_k B)_\mathfrak{r}$ is a localization of $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$. Hence it suffices to prove the result for the ring $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$, see Lemma 10.37.13 and Lemma 10.165.3. Thus we may assume A and B are domains.

Assume that A and B are domains with fractions fields K and L. Note that B is the filtered colimit of its finite type normal k-sub algebras (as k is a Nagata ring, see

Proposition 10.162.16, and hence the integral closure of a finite type k-sub algebra is still a finite type k-sub algebra by Proposition 10.162.15). By Lemma 10.37.17 we reduce to the case that B is of finite type over k.

Assume that A and B are domains with fractions fields K and L and B of finite type over k. In this case the ring $K \otimes_k B$ is of finite type over K, hence Noetherian (Lemma 10.31.1). In particular $K \otimes_k B$ has finitely many minimal primes (Lemma 10.31.6). Since $A \to A \otimes_k B$ is flat, this implies that $A \otimes_k B$ has finitely many minimal primes (by going down for flat ring maps – Lemma 10.39.19 – these primes all lie over $(0) \subset A$). Thus it suffices to prove that $A \otimes_k B$ is integrally closed in its total ring of fractions (Lemma 10.37.16).

We claim that $K \otimes_k B$ and $A \otimes_k L$ are both normal rings. If this is true then any element x of $Q(A \otimes_k B)$ which is integral over $A \otimes_k B$ is (by Lemma 10.37.12) contained in $K \otimes_k B \cap A \otimes_k L = A \otimes_k B$ and we're done. Since $A \otimes_K L$ is a normal ring by assumption, it suffices to prove that $K \otimes_k B$ is normal.

As A is geometrically normal over k we see K is geometrically normal over k (Lemma 10.165.3) hence K is geometrically reduced over k. Hence $K = \bigcup K_i$ is the union of finitely generated field extensions of k which are geometrically reduced (Lemma 10.43.2). Each K_i is the localization of a smooth k-algebra (Lemma 10.158.10). So $K_i \otimes_k B$ is the localization of a smooth B-algebra hence normal (Lemma 10.163.9). Thus $K \otimes_k B$ is a normal ring (Lemma 10.37.17) and we win. □

0C31 **Lemma 10.165.6.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically normal over k if and only if it is geometrically normal over k'.*

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 9.28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically normal over k', then A is geometrically normal over k.

Assume A is geometrically normal over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 10.43.8, we see that $K \otimes_{k'} A$ is a localization of a normal ring, hence normal. □

10.166. Geometrically regular algebras

045K Let k be a field. Let A be a Noetherian k-algebra. Let K/k be a finitely generated field extension. Then the ring $K \otimes_k A$ is Noetherian as well, see Lemma 10.31.8. Thus the following lemma makes sense.

0381 **Lemma 10.166.1.** *Let k be a field. Let A be a k-algebra. Assume A is Noetherian. The following properties of A are equivalent:*

(1) *$k' \otimes_k A$ is regular for every finitely generated field extension k'/k, and*
(2) *$k' \otimes_k A$ is regular for every finite purely inseparable extension k'/k.*

Here regular ring is as in Definition 10.110.7.

Proof. The lemma makes sense by the remarks preceding the lemma. It is clear that (1) ⇒ (2).

Assume (2) and let K/k be a finitely generated field extension. By Lemma 10.45.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a regular ring because we assumed (2). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a regular ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.137.3) and ascent of regularity along smooth maps (Lemma 10.163.10). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a regular ring as it is a localization of a regular ring (immediate from the definition). Step 4. Finally $K \otimes_k A$ is a regular ring by descent of regularity along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.164.4). This proves the lemma. □

0382 **Definition 10.166.2.** Let k be a field. Let R be a Noetherian k-algebra. The k-algebra R is called *geometrically regular* over k if the equivalent conditions of Lemma 10.166.1 hold.

It is clear from the definition that $K \otimes_k R$ is a geometrically regular algebra over K for any finitely generated field extension K of k. We will see later (More on Algebra, Proposition 15.35.1) that it suffices to check $R \otimes_k k'$ is regular whenever $k \subset k' \subset k^{1/p}$ (finite).

07NH **Lemma 10.166.3.** *Let k be a field. Let $A \to B$ be a faithfully flat k-algebra map. If B is geometrically regular over k, so is A.*

Proof. Assume B is geometrically regular over k. Let k'/k be a finite, purely inseparable extension. Then $A \otimes_k k' \to B \otimes_k k'$ is faithfully flat as a base change of $A \to B$ (by Lemmas 10.30.3 and 10.39.7) and $B \otimes_k k'$ is regular by our assumption on B over k. Then $A \otimes_k k'$ is regular by Lemma 10.164.4. □

07QF **Lemma 10.166.4.** *Let k be a field. Let $A \to B$ be a smooth ring map of k-algebras. If A is geometrically regular over k, then B is geometrically regular over k.*

Proof. Let k'/k be a finitely generated field extension. Then $A \otimes_k k' \to B \otimes_k k'$ is a smooth ring map (Lemma 10.137.3) and $A \otimes_k k'$ is regular. Hence $B \otimes_k k'$ is regular by Lemma 10.163.10. □

07QG **Lemma 10.166.5.** *Let k be a field. Let A be an algebra over k. Let $k = \operatorname{colim} k_i$ be a directed colimit of subfields. If A is geometrically regular over each k_i, then A is geometrically regular over k.*

Proof. Let k'/k be a finite purely inseparable field extension. We can get k' by adjoining finitely many variables to k and imposing finitely many polynomial relations. Hence we see that there exists an i and a finite purely inseparable field extension k'_i/k_i such that $k_i = k \otimes_{k_i} k'_i$. Thus $A \otimes_k k' = A \otimes_{k_i} k'_i$ and the lemma is clear. □

07QH **Lemma 10.166.6.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically regular over k if and only if it is geometrically regular over k'.*

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 9.28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically regular over k', then A is geometrically regular over k.

Assume A is geometrically regular over k. Since k' is the filtered colimit of finite extensions of k we may assume by Lemma 10.166.5 that k'/k is finite separable. Consider the ring maps

$$k' \to A \otimes_k k' \to A.$$

Note that $A \otimes_k k'$ is geometrically regular over k' as a base change of A to k'. Note that $A \otimes_k k' \to A$ is the base change of $k' \otimes_k k' \to k'$ by the map $k' \to A$. Since k'/k is an étale extension of rings, we see that $k' \otimes_k k' \to k'$ is étale (Lemma 10.143.3). Hence A is geometrically regular over k' by Lemma 10.166.4. □

10.167. Geometrically Cohen-Macaulay algebras

045L This section is a bit of a misnomer, since Cohen-Macaulay algebras are automatically geometrically Cohen-Macaulay. Namely, see Lemma 10.130.6 and Lemma 10.167.2 below.

045M **Lemma 10.167.1.** *Let k be a field and let K/k and L/k be two field extensions such that one of them is a field extension of finite type. Then $K \otimes_k L$ is a Noetherian Cohen-Macaulay ring.*

Proof. The ring $K \otimes_k L$ is Noetherian by Lemma 10.31.8. Say K is a finite extension of the purely transcendental extension $k(t_1, \ldots, t_r)$. Then $k(t_1, \ldots, t_r) \otimes_k L \to K \otimes_k L$ is a finite free ring map. By Lemma 10.112.9 it suffices to show that $k(t_1, \ldots, t_r) \otimes_k L$ is Cohen-Macaulay. This is clear because it is a localization of the polynomial ring $L[t_1, \ldots, t_r]$. (See for example Lemma 10.104.7 for the fact that a polynomial ring is Cohen-Macaulay.) □

045N **Lemma 10.167.2.** *Let k be a field. Let S be a Noetherian k-algebra. Let K/k be a finitely generated field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_\mathfrak{q}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.*

Proof. By Lemma 10.31.8 the ring S_K is Noetherian. Hence $S_\mathfrak{q} \to (S_K)_{\mathfrak{q}_K}$ is a flat local homomorphism of Noetherian local rings. Note that the fibre

$$(S_K)_{\mathfrak{q}_K}/\mathfrak{q}(S_K)_{\mathfrak{q}_K} \cong (\kappa(\mathfrak{q}) \otimes_k K)_{\mathfrak{q}'}$$

is the localization of the Cohen-Macaulay (Lemma 10.167.1) ring $\kappa(\mathfrak{q}) \otimes_k K$ at a suitable prime ideal $\mathfrak{q}'$. Hence the lemma follows from Lemma 10.163.3. □

10.168. Colimits and maps of finite presentation, II

07RF This section is a continuation of Section 10.127.

We start with an application of the openness of flatness. It says that we can approximate flat modules by flat modules which is useful.

02JO **Lemma 10.168.1.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume that*

(1) $R \to S$ *is of finite presentation,*
(2) M *is a finitely presented* S*-module, and*
(3) M *is flat over* R*.*

In this case we have the following:

(1) *There exists a finite type* $\mathbf{Z}$*-algebra* R_0 *and a finite type ring map* $R_0 \to S_0$ *and a finite* S_0*-module* M_0 *such that* M_0 *is flat over* R_0*, together with a ring maps* $R_0 \to R$ *and* $S_0 \to S$ *and an* S_0*-module map* $M_0 \to M$ *such that* $S \cong R \otimes_{R_0} S_0$ *and* $M = S \otimes_{S_0} M_0$*.*
(2) *If* $R = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ *is written as a directed colimit, then there exists a* λ *and a ring map* $R_\lambda \to S_\lambda$ *of finite presentation, and an* S_λ*-module* M_λ *of finite presentation such that* M_λ *is flat over* R_λ *and such that* $S = R \otimes_{R_\lambda} S_\lambda$ *and* $M = S \otimes_{S_\lambda} M_\lambda$*.*
(3) *If*
$$(R \to S, M) = \operatorname{colim}_{\lambda \in \Lambda} (R_\lambda \to S_\lambda, M_\lambda)$$
is written as a directed colimit such that
   (a) $R_\mu \otimes_{R_\lambda} S_\lambda \to S_\mu$ *and* $S_\mu \otimes_{S_\lambda} M_\lambda \to M_\mu$ *are isomorphisms for* $\mu \geq \lambda$*,*
   (b) $R_\lambda \to S_\lambda$ *is of finite presentation,*
   (c) M_λ *is a finitely presented* S_λ*-module,*
then for all sufficiently large λ *the module* M_λ *is flat over* R_λ*.*

Proof. We first write $(R \to S, M)$ as the directed colimit of a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in as in Lemma 10.127.18. Let $\mathfrak{q} \subset S$ be a prime. Let $\mathfrak{p} \subset R$, $\mathfrak{q}_\lambda \subset S_\lambda$, and $\mathfrak{p}_\lambda \subset R_\lambda$ the corresponding primes. As seen in the proof of Theorem 10.129.4
$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$
is a system as in Lemma 10.127.13, and hence by Lemma 10.128.3 we see that for some $\lambda_\mathfrak{q} \in \Lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$.

By Theorem 10.129.4 we get an open subset $U_\lambda \subset \operatorname{Spec}(S_\lambda)$ such that M_λ flat over R_λ at all the primes of U_λ. Denote $V_\lambda \subset \operatorname{Spec}(S)$ the inverse image of U_λ under the map $\operatorname{Spec}(S) \to \operatorname{Spec}(S_\lambda)$. The argument above shows that for every $\mathfrak{q} \in \operatorname{Spec}(S)$ there exists a $\lambda_\mathfrak{q}$ such that $\mathfrak{q} \in V_\lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$. Since $\operatorname{Spec}(S)$ is quasi-compact we see this implies there exists a single $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} = \operatorname{Spec}(S)$.

The complement $\operatorname{Spec}(S_{\lambda_0}) \setminus U_{\lambda_0}$ is $V(I)$ for some ideal $I \subset S_{\lambda_0}$. As $V_{\lambda_0} = \operatorname{Spec}(S)$ we see that $IS = S$. Choose $f_1, \ldots, f_r \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum f_i s_i = 1$. Since $\operatorname{colim} S_\lambda = S$, after increasing λ_0 we may assume there exist $s_{i,\lambda_0} \in S_{\lambda_0}$ such that $\sum f_i s_{i,\lambda_0} = 1$. Hence for this λ_0 we have $U_{\lambda_0} = \operatorname{Spec}(S_{\lambda_0})$. This proves (1).

Proof of (2). Let $(R_0 \to S_0, M_0)$ be as in (1) and suppose that $R = \operatorname{colim} R_\lambda$. Since R_0 is a finite type $\mathbf{Z}$ algebra, there exists a λ and a map $R_0 \to R_\lambda$ such that $R_0 \to R_\lambda \to R$ is the given map $R_0 \to R$ (see Lemma 10.127.3). Then, part (2) follows by taking $S_\lambda = R_\lambda \otimes_{R_0} S_0$ and $M_\lambda = S_\lambda \otimes_{S_0} M_0$.

Finally, we come to the proof of (3). Let $(R_\lambda \to S_\lambda, M_\lambda)$ be as in (3). Choose $(R_0 \to S_0, M_0)$ and $R_0 \to R$ as in (1). As in the proof of (2), there exists a λ_0 and a ring map $R_0 \to R_{\lambda_0}$ such that $R_0 \to R_{\lambda_0} \to R$ is the given map $R_0 \to R$. Since S_0 is of finite presentation over R_0 and since $S = \operatorname{colim} S_\lambda$ we see that for some $\lambda_1 \geq \lambda_0$ we get an R_0-algebra map $S_0 \to S_{\lambda_1}$ such that the composition

$S_0 \to S_{\lambda_1} \to S$ is the given map $S_0 \to S$ (see Lemma 10.127.3). For all $\lambda \geq \lambda_1$ this gives maps

$$\Psi_\lambda : R_\lambda \otimes_{R_0} S_0 \longrightarrow R_\lambda \otimes_{R_{\lambda_1}} S_{\lambda_1} \cong S_\lambda$$

the last isomorphism by assumption. By construction $\operatorname{colim}_\lambda \Psi_\lambda$ is an isomorphism. Hence Ψ_λ is an isomorphism for all λ large enough by Lemma 10.127.8. In the same vein, there exists a $\lambda_2 \geq \lambda_1$ and an S_0-module map $M_0 \to M_{\lambda_2}$ such that $M_0 \to M_{\lambda_2} \to M$ is the given map $M_0 \to M$ (see Lemma 10.127.5). For $\lambda \geq \lambda_2$ there is an induced map

$$S_\lambda \otimes_{S_0} M_0 \longrightarrow S_\lambda \otimes_{S_{\lambda_2}} M_{\lambda_2} \cong M_\lambda$$

and for λ large enough this map is an isomorphism by Lemma 10.127.6. This implies (3) because M_0 is flat over R_0. □

034Y **Lemma 10.168.2.** *Let $R \to A \to B$ be ring maps. Assume $A \to B$ faithfully flat of finite presentation. Then there exists a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & A_0 & \longrightarrow & B_0 \\ \| & & \downarrow & & \downarrow \\ R & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$.

Proof. We first prove the lemma with R replaced $\mathbf{Z}$. By Lemma 10.168.1 there exists a diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \uparrow & & \uparrow \\ B_0 & \longrightarrow & B \end{array}$$

where A_0 is of finite type over $\mathbf{Z}$, B_0 is flat of finite presentation over A_0 such that $B = A \otimes_{A_0} B_0$. As $A_0 \to B_0$ is flat of finite presentation we see that the image of $\operatorname{Spec}(B_0) \to \operatorname{Spec}(A_0)$ is open, see Proposition 10.41.8. Hence the complement of the image is $V(I_0)$ for some ideal $I_0 \subset A_0$. As $A \to B$ is faithfully flat the map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective, see Lemma 10.39.16. Now we use that the base change of the image is the image of the base change. Hence $I_0 A = A$. Pick a relation $\sum f_i r_i = 1$, with $r_i \in A$, $f_i \in I_0$. Then after enlarging A_0 to contain the elements r_i (and correspondingly enlarging B_0) we see that $A_0 \to B_0$ is surjective on spectra also, i.e., faithfully flat.

Thus the lemma holds in case $R = \mathbf{Z}$. In the general case, take the solution $A_0' \to B_0'$ just obtained and set $A_0 = A_0' \otimes_{\mathbf{Z}} R$, $B_0 = B_0' \otimes_{\mathbf{Z}} R$. □

07RG **Lemma 10.168.3.** *Let $A = \operatorname{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is finite,*
(2) *C_0 is of finite type over B_0.*

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is finite.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick monic polynomials $P_j \in A \otimes_{A_0} B_0[T]$ such that $P_j(1 \otimes x_j) = 0$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $P_{j,i} \in A_i \otimes_{A_0} B_0[T]$ mapping to P_j. Since $\otimes$ commutes with colimits we see that $P_{j,i}(1 \otimes x_j)$ is zero in $A_i \otimes_{A_0} C_0$ after possibly increasing i. Then this i works. □

07RH **Lemma 10.168.4.** *Let $A = \operatorname{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is surjective,*
(2) *C_0 is of finite type over B_0.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick $b_j \in A \otimes_{A_0} B_0$ mapping to $1 \otimes x_j$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $b_{j,i} \in A_i \otimes_{A_0} B_0$ mapping to b_j. After increasing i we may assume that $b_{j,i}$ maps to $1 \otimes x_j$ in $A_i \otimes_{A_0} C_0$ for all $j = 1, \ldots, m$. Then this i works. □

0C4F **Lemma 10.168.5.** *Let $A = \operatorname{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is unramified,*
(2) *C_0 is of finite type over B_0.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is unramified.

Proof. Set $B_i = A_i \otimes_{A_0} B_0$, $C_i = A_i \otimes_{A_0} C_0$, $B = A \otimes_{A_0} B_0$, and $C = A \otimes_{A_0} C_0$. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Then $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$ generate Ω_{C_0/B_0} over C_0 and their images generate Ω_{C_i/B_i} over C_i (Lemmas 10.131.14 and 10.131.9). Observe that $0 = \Omega_{C/B} = \operatorname{colim} \Omega_{C_i/B_i}$ (Lemma 10.131.5). Thus there is an i such that $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$ map to zero and hence $\Omega_{C_i/B_i} = 0$ as desired. □

0C32 **Lemma 10.168.6.** *Let $A = \operatorname{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is an isomorphism,*
(2) *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is an isomorphism.

Proof. By Lemma 10.168.4 there exists an i such that $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective. Since the map is of finite presentation the kernel is a finitely generated ideal. Let $g_1, \ldots, g_r \in A_i \otimes_{A_0} B_0$ generate the kernel. Then we may pick $i' \geq i$ such that g_j map to zero in $A_{i'} \otimes_{A_0} B_0$. Then $A_{i'} \otimes_{A_0} B_0 \to A_{i'} \otimes_{A_0} C_0$ is an isomorphism. □

07RI **Lemma 10.168.7.** *Let $A = \operatorname{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is étale,*
(2) *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is étale.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$.

Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C\mathrm{d}x_k$$

is an isomorphism. Thus for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\mathrm{d}\xi_{k,i} = \mathrm{d}x_k$ in $\bigoplus C_i\mathrm{d}x_k$. Moreover, on increasing i if necessary, we see that $\sum(\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

0C0B **Lemma 10.168.8.** *Let $A = \mathrm{colim}_{i\in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is smooth,*
(2) *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is smooth.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A\otimes_{A_0} B_0$ and $C = A\otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C\mathrm{d}x_k$$

is a split injection. Let $\xi_k \in (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2$ be elements such that $\sum(\partial f_j/\partial x_k)\xi_k = f_j \bmod (f_1, \ldots, f_m)^2$. Then for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\sum(\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

0C33 **Lemma 10.168.9.** *Let $A = \mathrm{colim}_{i\in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection),*
(2) *C_0 is of finite presentation over B_0.*

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection).

Proof. Assume $A\otimes_{A_0} B_0 \to A\otimes_{A_0} C_0$ is a relative global complete intersection. By Lemma 10.136.11 there exists a finite type $\mathbf{Z}$-algebra R, a ring map $R \to A\otimes_{A_0} B_0$, a relative global complete intersection $R \to S$, and an isomorphism

$$(A \otimes_{A_0} B_0) \otimes_R S \longrightarrow A \otimes_{A_0} C_0$$

Because R is of finite type (and hence finite presentation) over $\mathbf{Z}$, there exists an i and a map $R \to A_i \otimes_{A_0} B_0$ lifting the map $R \to A \otimes_{A_0} B_0$, see Lemma 10.127.3. Using the same lemma, there exists an $i' \geq i$ such that $(A_i\otimes_{A_0} B_0)\otimes_R S \to A\otimes_{A_0} C_0$ comes from a map $(A_i \otimes_{A_0} B_0) \otimes_R S \to A_{i'} \otimes_{A_0} C_0$. Thus we may assume, after replacing i by i', that the displayed map comes from an $A_i \otimes_{A_0} B_0$-algebra map

$$(A_i \otimes_{A_0} B_0) \otimes_R S \longrightarrow A_i \otimes_{A_0} C_0$$

By Lemma 10.168.6 after increasing i this map is an isomorphism. This finishes the proof in this case because the base change of a relative global complete intersection is a relative global complete intersection by Lemma 10.136.9.

Assume $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic. Then there exist elements $g_1, \ldots, g_m$ in $A \otimes_{A_0} C_0$ generating the unit ideal such that $A \otimes_{A_0} B_0 \to (A \otimes_{A_0} C_0)_{g_j}$ is a relative global complete intersection, see Lemma 10.136.15. We can find an i and elements $g_{i,j} \in A_i \otimes_{A_0} C_0$ mapping to g_j. After increasing i we may assume $g_{i,1}, \ldots, g_{i,m}$ generate the unit ideal of $A_i \otimes_{A_0} C_0$. The result of the previous paragraph implies that, after increasing i, we may assume the maps $A_i \otimes_{A_0} B_0 \to (A_i \otimes_{A_0} C_0)_{g_{i,j}}$ are relative global complete intersections. Then $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic by Lemma 10.136.4 (and the already used Lemma 10.136.15). □

The following lemma is an application of the results above which doesn't seem to fit well anywhere else.

034Z **Lemma 10.168.10.** *Let $R \to S$ be a faithfully flat ring map of finite presentation. Then there exists a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

where $R \to S'$ is quasi-finite, faithfully flat and of finite presentation.

Proof. As a first step we reduce this lemma to the case where R is of finite type over $\mathbf{Z}$. By Lemma 10.168.2 there exists a diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

where R_0 is of finite type over $\mathbf{Z}$, and S_0 is faithfully flat of finite presentation over R_0 such that $S = R \otimes_{R_0} S_0$. If we prove the lemma for the ring map $R_0 \to S_0$, then the lemma follows for $R \to S$ by base change, as the base change of a quasi-finite ring map is quasi-finite, see Lemma 10.122.9. (Of course we also use that base changes of flat maps are flat and base changes of maps of finite presentation are of finite presentation.)

Assume $R \to S$ is a faithfully flat ring map of finite presentation and that R is Noetherian (which we may assume by the preceding paragraph). Let $W \subset \mathrm{Spec}(S)$ be the open set of Lemma 10.130.4. As $R \to S$ is faithfully flat the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Lemma 10.39.16. By Lemma 10.130.5 the map $W \to \mathrm{Spec}(R)$ is also surjective. Hence by replacing S with a product $S_{g_1} \times \ldots \times S_{g_m}$ we may assume $W = \mathrm{Spec}(S)$; here we use that $\mathrm{Spec}(R)$ is quasi-compact (Lemma 10.17.8), and that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open (Proposition 10.41.8). Suppose that $\mathfrak{p} \subset R$ is a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a maximal ideal of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. The Noetherian local ring $\overline{S}_\mathfrak{q} = S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is Cohen-Macaulay, say of dimension d. We may choose $f_1, \ldots, f_d$ in the maximal ideal of $S_\mathfrak{q}$ which map to a regular sequence in $\overline{S}_\mathfrak{q}$. Choose a common denominator

$g \in S$, $g \notin \mathfrak{q}$ of $f_1, \ldots, f_d$, and consider the R-algebra

$$S' = S_g/(f_1, \ldots, f_d).$$

By construction there is a prime ideal $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ and corresponding to $\mathfrak{q}$ (via $S_g \to S'_g$). Also by construction the ring map $R \to S'$ is quasi-finite at $\mathfrak{q}$ as the local ring

$$S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} = S_{\mathfrak{q}}/(f_1, \ldots, f_d) + \mathfrak{p}S_{\mathfrak{q}} = \overline{S}_{\mathfrak{q}}/(\overline{f}_1, \ldots, \overline{f}_d)$$

has dimension zero, see Lemma 10.122.2. Also by construction $R \to S'$ is of finite presentation. Finally, by Lemma 10.99.3 the local ring map $R_{\mathfrak{p}} \to S'_{\mathfrak{q}'}$ is flat (this is where we use that R is Noetherian). Hence, by openness of flatness (Theorem 10.129.4), and openness of quasi-finiteness (Lemma 10.123.13) we may after replacing g by gg' for a suitable $g' \in S$, $g' \notin \mathfrak{q}$ assume that $R \to S'$ is flat and quasi-finite. The image $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is open and contains $\mathfrak{p}$. In other words we have shown a ring S' as in the statement of the lemma exists (except possibly the faithfulness part) whose image contains any given prime. Using one more time the quasi-compactness of $\mathrm{Spec}(R)$ we see that a finite product of such rings does the job. □

10.169. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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CHAPTER 11

Brauer groups

073W

11.1. Introduction

073X A reference is the lectures by Serre in the Seminaire Cartan, see [**Ser55a**]. Serre in turn refers to [**Deu68**] and [**ANT44**]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn's theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

11.2. Noncommutative algebras

073Y Let k be a field. In this chapter an *algebra* A over k is a possibly noncommutative ring A together with a ring map $k \to A$ such that k maps into the center of A and such that 1 maps to an identity element of A. An *A-module* is a right A-module such that the identity of A acts as the identity.

073Z **Definition 11.2.1.** Let A be a k-algebra. We say A is *finite* if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

0740 **Definition 11.2.2.** A *skew field* is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, in which every nonzero element has a multiplicative inverse.

A skew field is a k-algebra for some k (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn's lemma.

0741 **Definition 11.2.3.** Let A be a k-algebra. We say an A-module M is *simple* if it is nonzero and the only A-submodules are 0 and M. We say A is *simple* if the only two-sided ideals of A are 0 and A.

0742 **Definition 11.2.4.** A k-algebra A is *central* if the center of A is the image of $k \to A$.

0743 **Definition 11.2.5.** Given a k-algebra A we denote A^{op} the k-algebra we get by reversing the order of multiplication in A. This is called the *opposite algebra*.

11.3. Wedderburn's theorem

0744 The following cute argument can be found in a paper of Rieffel, see [**Rie65**]. The proof could not be simpler (quote from Carl Faith's review).

0745 **Lemma 11.3.1.** *Let A be a possibly noncommutative ring with 1 which contains no nontrivial two-sided ideal. Let M be a nonzero right ideal in A, and view M as a right A-module. Then A coincides with the bicommutant of M.*

Proof. Let $A' = \mathrm{End}_A(M)$, so M is a left A'-module. Set $A'' = \mathrm{End}_{A'}(M)$ (the bicommutant of M). We view M as a right A''-module[1]. Let $R : A \to A''$ be the natural homomorphism such that $mR(a) = ma$. Then R is injective, since $R(1) = \mathrm{id}_M$ and A contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in A''. Namely, $R(m)a'' = R(ma'')$ for $a'' \in A''$ and m in M, because *left* multiplication of M by any element n of M represents an element of A', and so $(nm)a'' = n(ma'')$ for all n in M. Finally, the product ideal AM is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in A''. But $R(A)$ contains the identity element of A'', and so $R(A) = A''$. □

0746 **Lemma 11.3.2.** *Let A be a k-algebra. If A is finite, then*

(1) *A has a simple module,*
(2) *any nonzero module contains a simple submodule,*
(3) *a simple module over A has finite dimension over k, and*
(4) *if M is a simple A-module, then $End_A(M)$ is a skew field.*

Proof. Of course (1) follows from (2) since A is a nonzero A-module. For (2), any submodule of minimal (finite) dimension as a k-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If M is simple, then $mA \subset M$ is a sub-module, hence we see (3). Any nonzero element of $\mathrm{End}_A(M)$ is an isomorphism, hence (4) holds. □

0747 **Theorem 11.3.3.** *Let A be a simple finite k-algebra. Then A is a matrix algebra over a finite k-algebra K which is a skew field.*

Proof. We may choose a simple submodule $M \subset A$ and then the k-algebra $K = \mathrm{End}_A(M)$ is a skew field, see Lemma 11.3.2. By Lemma 11.3.1 we see that $A = \mathrm{End}_K(M)$. Since K is a skew field and M is finitely generated (since $\dim_k(M) < \infty$) we see that M is finite free as a left K-module. It follows immediately that $A \cong \mathrm{Mat}(n \times n, K^{op})$. □

11.4. Lemmas on algebras

0748 Let A be a k-algebra. Let $B \subset A$ be a subalgebra. The *centralizer of B in A* is the subalgebra

$$C = \{y \in A \mid xy = yx \text{ for all } x \in B\}.$$

It is a k-algebra.

0749 **Lemma 11.4.1.** *Let A, A' be k-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers C, C'. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.*

Proof. Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C' \subset C''$. Conversely, every element of C'' commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C'$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$. □

074A **Lemma 11.4.2.** *Let A be a finite simple k-algebra. Then the center k' of A is a finite field extension of k.*

[1]This means that given $a'' \in A''$ and $m \in M$ we have a product $ma'' \in M$. In particular, the multiplication in A'' is the opposite of what you'd get if you wrote elements of A'' as endomorphisms acting on the left.

Proof. Write $A = \mathrm{Mat}(n \times n, K)$ for some skew field K finite over k, see Theorem 11.3.3. By Lemma 11.4.1 the center of A is $k \otimes_k k'$ where $k' \subset K$ is the center of K. Since the center of a skew field is a field, we win. □

074B **Lemma 11.4.3.** *Let V be a k vector space. Let K be a central k-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided K-sub vector space. Then W is generated as a left K-vector space by $W \cap (V \otimes 1)$.*

Proof. Let $V' \subset V$ be the k-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have

$$W/(V' \otimes_k K) \subset (V/V') \otimes_k K.$$

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/(V' \otimes_k K)$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of V'. Hence we may replace V by V/V' and W by $W/(V' \otimes_k K)$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if W is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $w = \sum_{i=1,\ldots,n} v_i \otimes k_i$ with n minimal. We may right multiply with k_1^{-1} and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$

$$cw - wc = \sum\nolimits_{i=2,\ldots,n} v_i \otimes (ck_i - k_i c) \in W$$

and hence $ck_i - k_i c = 0$ by minimality of n. This implies that k_i is in the center of K which is k by assumption. Hence $w = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of n. □

074C **Lemma 11.4.4.** *Let A be a k-algebra. Let K be a central k-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if A is simple, then so is $A \otimes_k K$.*

Proof. Set $J = \{a \in A \mid a \otimes 1 \in I\}$. This is a two-sided ideal of A. And $I = J \otimes_k K$ by Lemma 11.4.3. □

074D **Lemma 11.4.5.** *Let R be a possibly noncommutative ring. Let $n \geq 1$ be an integer. Let $R_n = Mat(n \times n, R)$.*

1. *The functors $M \mapsto M^{\oplus n}$ and $N \mapsto Ne_{11}$ define quasi-inverse equivalences of categories $Mod_R \leftrightarrow Mod_{R_n}$.*
2. *A two-sided ideal of R_n is of the form IR_n for some two-sided ideal I of R.*
3. *The center of R_n is equal to the center of R.*

Proof. Part (1) proves itself. If $J \subset R_n$ is a two-sided ideal, then $J = \bigoplus e_{ii} J e_{jj}$ and all of the summands $e_{ii} J e_{jj}$ are equal to each other and are a two-sided ideal I of R. This proves (2). Part (3) is clear. □

074E **Lemma 11.4.6.** *Let A be a finite simple k-algebra.*

1. *There exists exactly one simple A-module M up to isomorphism.*
2. *Any finite A-module is a direct sum of copies of a simple module.*
3. *Two finite A-modules are isomorphic if and only if they have the same dimension over k.*
4. *If $A = Mat(n \times n, K)$ with K a finite skew field extension of k, then $M = K^{\oplus n}$ is a simple A-module and $End_A(M) = K^{op}$.*

(5) *If M is a simple A-module, then $L = End_A(M)$ is a skew field finite over k acting on the left on M, we have $A = End_L(M)$, and the centers of A and L agree. Also $[A : k][L : k] = \dim_k(M)^2$.*
(6) *For a finite A-module N the algebra $B = End_A(N)$ is a matrix algebra over the skew field L of (5). Moreover $End_B(N) = A$.*

Proof. By Theorem 11.3.3 we can write $A = \mathrm{Mat}(n \times n, K)$ for some finite skew field extension K of k. By Lemma 11.4.5 the category of modules over A is equivalent to the category of modules over K. Thus (1), (2), and (3) hold because every module over K is free. Part (4) holds because the equivalence transforms the K-module K to $M = K^{\oplus n}$. Using $M = K^{\oplus n}$ in (5) we see that $L = K^{op}$. The statement about the center of $L = K^{op}$ follows from Lemma 11.4.5. The statement about $\mathrm{End}_L(M)$ follows from the explicit form of M. The formula of dimensions is clear. Part (6) follows as N is isomorphic to a direct sum of copies of a simple module. □

074F **Lemma 11.4.7.** *Let A, A' be two simple k-algebras one of which is finite and central over k. Then $A \otimes_k A'$ is simple.*

Proof. Suppose that A' is finite and central over k. Write $A' = \mathrm{Mat}(n \times n, K')$, see Theorem 11.3.3. Then the center of K' is k and we conclude that $A \otimes_k K'$ is simple by Lemma 11.4.4. Hence $A \otimes_k A' = \mathrm{Mat}(n \times n, A \otimes_k K')$ is simple by Lemma 11.4.5. □

074G **Lemma 11.4.8.** *The tensor product of finite central simple algebras over k is finite, central, and simple.*

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

074H **Lemma 11.4.9.** *Let A be a finite central simple algebra over k. Let k'/k be a field extension. Then $A' = A \otimes_k k'$ is a finite central simple algebra over k'.*

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

074I **Lemma 11.4.10.** *Let A be a finite central simple algebra over k. Then $A \otimes_k A^{op} \cong Mat(n \times n, k)$ where $n = [A : k]$.*

Proof. By Lemma 11.4.8 the algebra $A \otimes_k A^{op}$ is simple. Hence the map

$$A \otimes_k A^{op} \longrightarrow \mathrm{End}_k(A), \quad a \otimes a' \longmapsto (x \mapsto axa')$$

is injective. Since both sides of the arrow have the same dimension we win. □

11.5. The Brauer group of a field

074J Let k be a field. Consider two finite central simple algebras A and B over k. We say A and B are *similar* if there exist $n, m > 0$ such that $\mathrm{Mat}(n \times n, A) \cong \mathrm{Mat}(m \times m, B)$ as k-algebras.

074K **Lemma 11.5.1.** *Similarity.*

(1) *Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over k.*
(2) *Every similarity class contains a unique (up to isomorphism) finite central skew field extension of k.*

(3) *If $A = Mat(n \times n, K)$ and $B = Mat(m \times m, K')$ for some finite central skew fields K, K' over k then A and B are similar if and only if $K \cong K'$ as k-algebras.*

Proof. Note that by Wedderburn's theorem (Theorem 11.3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if $A = \mathrm{Mat}(n \times n, K) \cong \mathrm{Mat}(m \times m, K') = B$ then $K \cong K'$. To see this note that for a simple module M of A we have $\mathrm{End}_A(M) = K^{op}$, see Lemma 11.4.6. Hence $A \cong B$ implies $K^{op} \cong (K')^{op}$ and we win. □

Given two finite central simple k-algebras A, B the tensor product $A \otimes_k B$ is another, see Lemma 11.4.8. Moreover if A is similar to A', then $A \otimes_k B$ is similar to $A' \otimes_k B$ because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 11.4.10 shows that $A \otimes_k A^{op}$ is isomorphic to a matrix algebra, i.e., that $A \otimes_k A^{op}$ is in the similarity class of k. Thus we obtain an abelian group.

074L **Definition 11.5.2.** Let k be a field. The *Brauer group* of k is the abelian group of similarity classes of finite central simple k-algebras defined above. Notation $\mathrm{Br}(k)$.

For any map of fields $k \to k'$ we obtain a group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(k'), \quad A \longmapsto A \otimes_k k'$$

see Lemma 11.4.9. In other words, $\mathrm{Br}(-)$ is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension $k \subset K$ is trivial.

074M **Lemma 11.5.3.** *The Brauer group of an algebraically closed field is zero.*

Proof. Let $k \subset K$ be a finite central skew field extension. For any element $x \in K$ the subring $k[x] \subset K$ is a commutative finite integral k-sub algebra, hence a field, see Algebra, Lemma 10.36.19. Since k is algebraically closed we conclude that $k[x] = k$. Since x was arbitrary we conclude $k = K$. □

074N **Lemma 11.5.4.** *Let A be a finite central simple algebra over a field k. Then $[A : k]$ is a square.*

Proof. This is true because $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ by Lemma 11.5.3. □

11.6. Skolem-Noether

074P

074Q **Theorem 11.6.1.** *Let A be a finite central simple k-algebra. Let B be a simple k-algebra. Let $f, g : B \to A$ be two k-algebra homomorphisms. Then there exists an invertible element $x \in A$ such that $f(b) = xg(b)x^{-1}$ for all $b \in B$.*

Proof. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M, see Lemmas 11.3.2 and 11.4.6. Then M has two $B \otimes_k L^{op}$-module structures defined by $m \cdot_1 (b \otimes l) = lmf(b)$ and $m \cdot_2 (b \otimes l) = lmg(b)$. The k-algebra $B \otimes_k L^{op}$ is simple by Lemma 11.4.7. Since B is simple, the existence of a k-algebra homomorphism $B \to A$ implies that B is finite. Thus $B \otimes_k L^{op}$ is finite simple and we conclude the two $B \otimes_k L^{op}$-module structures

on M are isomorphic by Lemma 11.4.6. Hence we find $\varphi : M \to M$ intertwining these operations. In particular φ is in the commutant of L which implies that φ is multiplication by some $x \in A$, see Lemma 11.4.6. Working out the definitions we see that x is a solution to our problem. □

074R **Lemma 11.6.2.** *Let A be a finite central simple k-algebra. Any automorphism of A is inner. In particular, any automorphism of $Mat(n \times n, k)$ is inner.*

Proof. The Skolem-Noether theorem (Theorem 11.6.1) applies. □

11.7. The centralizer theorem

074S

074T **Theorem 11.7.1.** *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. Then*

(1) *the centralizer C of B in A is simple,*
(2) *$[A : k] = [B : k][C : k]$, and*
(3) *the centralizer of C in A is B.*

Proof. Throughout this proof we use the results of Lemma 11.4.6 freely. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M and $A = \mathrm{End}_L(M)$. Then M is a right $B \otimes_k L^{op}$-module and $C = \mathrm{End}_{B \otimes_k L^{op}}(M)$. Since the algebra $B \otimes_k L^{op}$ is simple by Lemma 11.4.7 we see that C is simple (by Lemma 11.4.6 again).

Write $B \otimes_k L^{op} = \mathrm{Mat}(m \times m, K)$ for some skew field K finite over k. Then $C = \mathrm{Mat}(n \times n, K^{op})$ if M is isomorphic to a direct sum of n copies of the simple $B \otimes_k L^{op}$-module $K^{\oplus m}$ (the lemma again). Thus we have $\dim_k(M) = nm[K : k]$, $[B : k][L : k] = m^2[K : k]$, $[C : k] = n^2[K : k]$, and $[A : k][L : k] = \dim_k(M)^2$ (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to $C \subset A$ shows that $[B : k] = [C' : k]$ where C' is the centralizer of C in A (and the obvious fact that $B \subset C'$). □

074U **Lemma 11.7.2.** *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. If B is a central k-algebra, then $A = B \otimes_k C$ where C is the (central simple) centralizer of B in A.*

Proof. We have $\dim_k(A) = \dim_k(B \otimes_k C)$ by Theorem 11.7.1. By Lemma 11.4.7 the tensor product is simple. Hence the natural map $B \otimes_k C \to A$ is injective hence an isomorphism. □

074V **Lemma 11.7.3.** *Let A be a finite central simple algebra over k. If $K \subset A$ is a subfield, then the following are equivalent*

(1) *$[A : k] = [K : k]^2$,*
(2) *K is its own centralizer, and*
(3) *K is a maximal commutative subring.*

Proof. Theorem 11.7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

074W **Lemma 11.7.4.** *Let A be a finite central skew field over k. Then every maximal subfield $K \subset A$ satisfies $[A : k] = [K : k]^2$.*

Proof. Special case of Lemma 11.7.3. □

11.8. Splitting fields

074X

074Y **Definition 11.8.1.** Let A be a finite central simple k-algebra. We say a field extension k'/k *splits* A, or k' is a *splitting field* for A if $A \otimes_k k'$ is a matrix algebra over k'.

Another way to say this is that the class of A maps to zero under the map $\mathrm{Br}(k) \to \mathrm{Br}(k')$.

074Z **Theorem 11.8.2.** *Let A be a finite central simple k-algebra. Let k'/k be a finite field extension. The following are equivalent*

(1) *k' splits A, and*
(2) *there exists a finite central simple algebra B similar to A such that $k' \subset B$ and $[B : k] = [k' : k]^2$.*

Proof. Assume (2). It suffices to show that $B \otimes_k k'$ is a matrix algebra. We know that $B \otimes_k B^{op} \cong \mathrm{End}_k(B)$. Since k' is the centralizer of k' in B^{op} by Lemma 11.7.3 we see that $B \otimes_k k'$ is the centralizer of $k \otimes k'$ in $B \otimes_k B^{op} = \mathrm{End}_k(B)$. Of course this centralizer is just $\mathrm{End}_{k'}(B)$ where we view B as a k' vector space via the embedding $k' \to B$. Thus the result.

Assume (1). This means that we have an isomorphism $A \otimes_k k' \cong \mathrm{End}_{k'}(V)$ for some k'-vector space V. Let B be the commutant of A in $\mathrm{End}_k(V)$. Note that k' sits in B. By Lemma 11.7.2 the classes of A and B add up to zero in $\mathrm{Br}(k)$. From the dimension formula in Theorem 11.7.1 we see that

$$[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_{k'}(V)^2 = [k' : k]^2[A : k].$$

Hence $[B : k] = [k' : k]^2$. Thus we have proved the result for the opposite to the Brauer class of A. However, k' splits the Brauer class of A if and only if it splits the Brauer class of the opposite algebra, so we win anyway. □

0750 **Lemma 11.8.3.** *A maximal subfield of a finite central skew field K over k is a splitting field for K.*

Proof. Combine Lemma 11.7.4 with Theorem 11.8.2. □

0751 **Lemma 11.8.4.** *Consider a finite central skew field K over k. Let $d^2 = [K : k]$. For any finite splitting field k' for K the degree $[k' : k]$ is divisible by d.*

Proof. By Theorem 11.8.2 there exists a finite central simple algebra B in the Brauer class of K such that $[B : k] = [k' : k]^2$. By Lemma 11.5.1 we see that $B = \mathrm{Mat}(n \times n, K)$ for some n. Then $[k' : k]^2 = n^2 d^2$ whence the result. □

0752 **Proposition 11.8.5.** *Consider a finite central skew field K over k. There exists a maximal subfield $k \subset k' \subset K$ which is separable over k. In particular, every Brauer class has a finite separable spitting field.*

Proof. Since every Brauer class is represented by a finite central skew field over k, we see that the second statement follows from the first by Lemma 11.8.3.

To prove the first statement, suppose that we are given a separable subfield $k' \subset K$. Then the centralizer K' of k' in K has center k', and the problem reduces to finding a maximal subfield of K' separable over k'. Thus it suffices to prove, if $k \neq K$, that we can find an element $x \in K$, $x \notin k$ which is separable over k. This statement is

clear in characteristic zero. Hence we may assume that k has characteristic $p > 0$. If the ground field k is finite then, the result is clear as well (because extensions of finite fields are always separable). Thus we may assume that k is an infinite field of positive characteristic.

To get a contradiction assume no element of K is separable over k. By the discussion in Fields, Section 9.28 this means the minimal polynomial of any $x \in K$ is of the form $T^q - a$ where q is a power of p and $a \in k$. Since it is clear that every element of K has a minimal polynomial of degree $\leq \dim_k(K)$ we conclude that there exists a fixed p-power q such that $x^q \in k$ for all $x \in K$.

Consider the map

$$(-)^q : K \longrightarrow K$$

and write it out in terms of a k-basis $\{a_1, \ldots, a_n\}$ of K with $a_1 = 1$. So

$$(\sum x_i a_i)^q = \sum f_i(x_1, \ldots, x_n) a_i.$$

Since multiplication on K is k-bilinear we see that each f_i is a polynomial in $x_1, \ldots, x_n$ (details omitted). The choice of q above and the fact that k is infinite shows that f_i is identically zero for $i \geq 2$. Hence we see that it remains zero on extending k to its algebraic closure $\overline{k}$. But the algebra $K \otimes_k \overline{k}$ is a matrix algebra (for example by Lemmas 11.4.9 and 11.5.3), which implies there are some elements whose qth power is not central (e.g., e_{11}). This is the desired contradiction. □

The results above allow us to characterize finite central simple algebras as follows.

0753 **Lemma 11.8.6.** *Let k be a field. For a k-algebra A the following are equivalent*

(1) *A is finite central simple k-algebra,*
(2) *A is a finite dimensional k-vector space, k is the center of A, and A has no nontrivial two-sided ideal,*
(3) *there exists $d \geq 1$ such that $A \otimes_k \overline{k} \cong Mat(d \times d, \overline{k})$,*
(4) *there exists $d \geq 1$ such that $A \otimes_k k^{sep} \cong Mat(d \times d, k^{sep})$,*
(5) *there exist $d \geq 1$ and a finite Galois extension k'/k such that $A \otimes_k k' \cong Mat(d \times d, k')$,*
(6) *there exist $n \geq 1$ and a finite central skew field K over k such that $A \cong Mat(n \times n, K)$.*

The integer d is called the degree of A.

Proof. The equivalence of (1) and (2) is a consequence of the definitions, see Section 11.2. Assume (1). By Proposition 11.8.5 there exists a separable splitting field $k \subset k'$ for A. Of course, then a Galois closure of k'/k is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Assume (3). Then $A \otimes_k \overline{k}$ is a finite central simple $\overline{k}$-algebra for example by Lemma 11.4.5. This trivially implies that A is a finite central simple k-algebra. Finally, the equivalence of (1) and (6) is Wedderburn's theorem, see Theorem 11.3.3. □

11.9. Other chapters

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CHAPTER 12

Homological Algebra

00ZU

12.1. Introduction

00ZV Basic homological algebra will be explained in this document. We add as needed in the other parts, since there is clearly an infinite amount of this stuff around. A reference is [**ML63**].

12.2. Basic notions

00ZW The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Nothing yet.

12.3. Preadditive and additive categories

09SE Here is the definition of a preadditive category.

00ZY **Definition 12.3.1.** A category $\mathcal{A}$ is called *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called *additive* if and only if $F : \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

00ZZ **Lemma 12.3.2.** *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent*

(1) *x is an initial object,*
(2) *x is a final object, and*
(3) *$id_x = 0$ in $\mathrm{Mor}_{\mathcal{A}}(x, x)$.*

Furthermore, if such an object 0 exists, then a morphism $\alpha : y \to z$ factors through 0 if and only if $\alpha = 0$.

Proof. First assume that x is either (1) initial or (2) final. In both cases, it follows that $\mathrm{Mor}(x, x)$ is a trivial abelian group containing id_x, thus $\mathrm{id}_x = 0$ in $\mathrm{Mor}(x, x)$, which shows that each of (1) and (2) implies (3).

Now assume that $\mathrm{id}_x = 0$ in $\mathrm{Mor}(x, x)$. Let y be an arbitrary object of $\mathcal{A}$ and let $f \in \mathrm{Mor}(x, y)$. Denote $C : \mathrm{Mor}(x, x) \times \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, y)$ the composition map. Then $f = C(0, f)$ and since C is bilinear we have $C(0, f) = 0$. Thus $f = 0$. Hence x is initial in $\mathcal{A}$. A similar argument for $f \in \mathrm{Mor}(y, x)$ can be used to show that x is also final. Thus (3) implies both (1) and (2). □

0100 **Definition 12.3.3.** In a preadditive category $\mathcal{A}$, we call an object that is both final and initial (as in Lemma 12.3.2) a *zero object* and denote it 0.

0101 **Lemma 12.3.4.** *Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case also $x \amalg y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

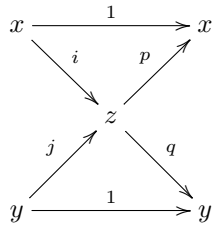

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which shows that $z = x \amalg y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \amalg y$ instead of a product. □

0102 **Definition 12.3.5.** Given a pair of objects x, y in a preadditive category $\mathcal{A}$, the *direct sum* $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 12.3.4 above.

0103 **Remark 12.3.6.** Note that the proof of Lemma 12.3.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

0105 **Lemma 12.3.7.** *Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 12.3.6. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums.

To see that F transforms zero to zero, use the characterization (3) of the zero object in Lemma 12.3.2. □

0104 **Definition 12.3.8.** A category $\mathcal{A}$ is called *additive* if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

0106 **Definition 12.3.9.** Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A *kernel* of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.
(2) If the kernel of f exists, then we denote this $\mathrm{Ker}(f) \to x$.
(3) A *cokernel* of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \mathrm{Coker}(f)$.
(5) If a kernel of f exists, then a *coimage of* f is a cokernel for the morphism $\mathrm{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \mathrm{Coim}(f)$.
(7) If a cokernel of f exists, then the *image of* f is a kernel of the morphism $y \to \mathrm{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\mathrm{Im}(f) \to y$.

In the above definition, we have spoken of "the kernel" and "the cokernel", tacitly using their uniqueness up to unique isomorphism. This follows from the Yoneda lemma (Categories, Section 4.3) because the kernel of $f : x \to y$ represents the functor sending an object z to the set $\mathrm{Ker}(\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y))$. The case of cokernels is dual.

We first relate the direct sum to kernels as follows.

09QG **Lemma 12.3.10.** *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma 12.3.4 be a direct sum in $\mathcal{C}$. Then $i : x \to x \oplus y$ is a kernel of $q : x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f : z' \to x \oplus y$ be a morphism such that $q \circ f = 0$. We have to show that there exists a unique morphism $g : z' \to x$ such that $f = i \circ g$. Since $i \circ p + j \circ q$ is the identity on $x \oplus y$ we see that

$$f = (i \circ p + j \circ q) \circ f = i \circ p \circ f$$

and hence $g = p \circ f$ works. Uniqueness holds because $p \circ i$ is the identity on x. The proof of the second statement is dual. □

0E43 **Lemma 12.3.11.** *Let $\mathcal{C}$ be a preadditive category. Let $f : x \to y$ be a morphism in $\mathcal{C}$.*

(1) *If a kernel of f exists, then this kernel is a monomorphism.*
(2) *If a cokernel of f exists, then this cokernel is an epimorphism.*
(3) *If a kernel and coimage of f exist, then the coimage is an epimorphism.*
(4) *If a cokernel and image of f exist, then the image is a monomorphism.*

Proof. Part (1) follows easily from the uniqueness required in the definition of a kernel. The proof of (2) is dual. Part (3) follows from (2), since the coimage is a cokernel. Similarly, (4) follows from (1). □

0107 **Lemma 12.3.12.** *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to y$.*

Proof. There is a canonical morphism $\mathrm{Coim}(f) \to y$ because $\mathrm{Ker}(f) \to x \to y$ is zero. The composition $\mathrm{Coim}(f) \to y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \mathrm{Coker}(f)$ which is zero (the uniqueness follows from Lemma 12.3.11 (3)). Hence $\mathrm{Coim}(f) \to y$ factors uniquely through $\mathrm{Im}(f) \to y$, which gives us the desired map. □

0108 **Example 12.3.13.** Let k be a field. Consider the category of filtered vector spaces over k. (See Definition 12.19.1.) Consider the filtered vector spaces (V, F) and (W, F) with $V = W = k$ and

$$F^iV = \begin{cases} V & \text{if} \quad i < 0 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \text{ and } F^iW = \begin{cases} W & \text{if} \quad i \leq 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

The map $f : V \to W$ corresponding to id_k on the underlying vector spaces has trivial kernel and cokernel but is not an isomorphism. Note also that $\mathrm{Coim}(f) = V$ and $\mathrm{Im}(f) = W$. This means that the category of filtered vector spaces over k is not abelian.

12.4. Karoubian categories

09SF Skip this section on a first reading.

09SG **Definition 12.4.1.** Let $\mathcal{C}$ be a preadditive category. We say $\mathcal{C}$ is *Karoubian* if every idempotent endomorphism of an object of $\mathcal{C}$ has a kernel.

The dual notion would be that every idempotent endomorphism of an object has a cokernel. However, in view of the (dual of the) following lemma that would be an equivalent notion.

09SH **Lemma 12.4.2.** *Let $\mathcal{C}$ be a preadditive category. The following are equivalent*

(1) *$\mathcal{C}$ is Karoubian,*
(2) *every idempotent endomorphism of an object of $\mathcal{C}$ has a cokernel, and*
(3) *given an idempotent endomorphism $p : z \to z$ of $\mathcal{C}$ there exists a direct sum decomposition $z = x \oplus y$ such that p corresponds to the projection onto y.*

Proof. Assume (1) and let $p : z \to z$ be as in (3). Let $x = \mathrm{Ker}(p)$ and $y = \mathrm{Ker}(1 - p)$. There are maps $x \to z$ and $y \to z$. Since $(1 - p)p = 0$ we see that $p : z \to z$ factors through y, hence we obtain a morphism $z \to y$. Similarly we obtain a morphism $z \to x$. We omit the verification that these four morphisms induce an isomorphism $x = y \oplus z$ as in Remark 12.3.6. Thus (1) $\Rightarrow$ (3). The implication (2) $\Rightarrow$ (3) is dual. Finally, condition (3) implies (1) and (2) by Lemma 12.3.10. □

05QV **Lemma 12.4.3.** *Let $\mathcal{D}$ be a preadditive category.*

(1) *If $\mathcal{D}$ has countable products and kernels of maps which have a right inverse, then $\mathcal{D}$ is Karoubian.*
(2) *If $\mathcal{D}$ has countable coproducts and cokernels of maps which have a left inverse, then $\mathcal{D}$ is Karoubian.*

Proof. Let X be an object of $\mathcal{D}$ and let $e : X \to X$ be an idempotent. The functor

$$W \longmapsto \mathrm{Ker}(\mathrm{Mor}_{\mathcal{D}}(W, X) \xrightarrow{e} \mathrm{Mor}_{\mathcal{D}}(W, X))$$

is representable if and only if e has a kernel. Note that for any abelian group A and idempotent endomorphism $e : A \to A$ we have

$$\mathrm{Ker}(e : A \to A) = \mathrm{Ker}(\Phi : \prod\nolimits_{n\in\mathbf{N}} A \to \prod\nolimits_{n\in\mathbf{N}} A)$$

where

$$\Phi(a_1, a_2, a_3, \ldots) = (ea_1 + (1-e)a_2, ea_2 + (1-e)a_3, \ldots)$$

Moreover, Φ has the right inverse

$$\Psi(a_1, a_2, a_3, \ldots) = (a_1, (1-e)a_1 + ea_2, (1-e)a_2 + ea_3, \ldots).$$

Hence (1) holds. The proof of (2) is dual (using the dual definition of a Karoubian category, namely condition (2) of Lemma 12.4.2). □

12.5. Abelian categories

00ZX An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom (that is sometimes forgotten) is that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 12.3.12 is always an isomorphism. Example 12.3.13 shows that it is necessary.

0109 **Definition 12.5.1.** A category $\mathcal{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

010A **Lemma 12.5.2.** *Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.*

Proof. The first statement is straightforward. To see that $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, recall that additivity can be characterized by the existence of a zero object and direct sums, which are both preserved when passing to the opposite category. Finally, to see that $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian, observes that kernels, cokernels, images and coimages in $\mathcal{A}^{opp}$ correspond to cokernels, kernels, coimages and images in $\mathcal{A}$, respectively. □

010B **Definition 12.5.3.** Let $f : x \to y$ be a morphism in an abelian category.

(1) We say f is *injective* if $\mathrm{Ker}(f) = 0$.
(2) We say f is *surjective* if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a *subobject* of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a *quotient* of x.

010C **Lemma 12.5.4.** *Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$. Then*

(1) *f is injective if and only if f is a monomorphism,*
(2) *f is surjective if and only if f is an epimorphism, and*
(3) *f is an isomorphism if and only if f is injective and surjective.*

Proof. Proof of (1). Recall that $\mathrm{Ker}(f)$ is an object representing the functor sending z to $\mathrm{Ker}(\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y))$, see Definition 12.3.9. Thus $\mathrm{Ker}(f)$ is 0 if and only if $\mathrm{Mor}_{\mathcal{A}}(z, x) \to \mathrm{Mor}_{\mathcal{A}}(z, y)$ is injective for all z if and only if f is a monomorphism. The proof of (2) is similar. For the proof of (3) note that an isomorphism is both a monomorphism and epimorphism, which by (1), (2) proves f is injective and surjective. If f is both injective and surjective, then $x = \mathrm{Coim}(f)$ and $y = \mathrm{Im}(f)$ whence f is an isomorphism. □

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$y/x = \text{Coker}(x \to y).$$

010D **Lemma 12.5.5.** *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 4.18.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. □

05PJ **Example 12.5.6.** Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \text{Ker}((a, -b) : x \oplus z \to y)$.
(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \text{Coker}((a, -b) : y \to x \oplus z)$.

010E **Definition 12.5.7.** Let $\mathcal{A}$ be an additive category. Consider a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots \quad \text{or} \quad x_1 \to x_2 \to \ldots \to x_n$$

in $\mathcal{A}$. We say such a sequence is a *complex* if the composition of any two consecutive (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a complex of the first type above is *exact at y* if $\text{Im}(x \to y) = \text{Ker}(y \to z)$ and we say a complex of the second kind is *exact at x_i* where $1 < i < n$ if $\text{Im}(x_{i-1} \to x_i) = \text{Ker}(x_i \to x_{i+1})$. We a sequence as above is *exact* or is an *exact sequence* or is an *exact complex* if it is a complex and exact at every object (in the first case) or exact at x_i for all $1 < i < n$ (in the second case). There are variants of these notions for sequences of the form

$$\ldots \to x_{-3} \to x_{-2} \to x_{-1} \quad \text{and} \quad x_1 \to x_2 \to x_3 \to \ldots$$

A *short exact sequence* is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we assume the reader knows what it means for a sequence of abelian groups to be exact.

05AA **Lemma 12.5.8.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if*
$$0 \to \text{Hom}_{\mathcal{A}}(M_3, N) \to \text{Hom}_{\mathcal{A}}(M_2, N) \to \text{Hom}_{\mathcal{A}}(M_1, N)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and
(2) *$0 \to M_1 \to M_2 \to M_3$ is exact if and only if*
$$0 \to \text{Hom}_{\mathcal{A}}(N, M_1) \to \text{Hom}_{\mathcal{A}}(N, M_2) \to \text{Hom}_{\mathcal{A}}(N, M_3)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 10.10.1. □

010F **Definition 12.5.9.** Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is *split* if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

010G **Lemma 12.5.10.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.*

(1) *Given a morphism $s : C \to B$ right inverse to $B \to C$, there exists a unique $\pi : B \to A$ such that (s, π) splits the short exact sequence as in Definition 12.5.9.*
(2) *Given a morphism $\pi : B \to A$ left inverse to $A \to B$, there exists a unique $s : C \to B$ such that (s, π) splits the short exact sequence as in Definition 12.5.9.*

Proof. Omitted. □

08N2 **Lemma 12.5.11.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *The diagram is cartesian if and only if*

$$0 \to w \xrightarrow{(g,f)} x \oplus y \xrightarrow{(k,-h)} z$$

is exact.
(2) *The diagram is cocartesian if and only if*

$$w \xrightarrow{(g,-f)} x \oplus y \xrightarrow{(k,h)} z \to 0$$

is exact.

Proof. Let $u = (g, f) : w \to x \oplus y$ and $v = (k, -h) : x \oplus y \to z$. Let $p : x \oplus y \to x$ and $q : x \oplus y \to y$ be the canonical projections. Let $i : \mathrm{Ker}(v) \to x \oplus y$ be the canonical injection. By Example 12.5.6, the diagram is cartesian if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $f \circ r = q \circ i$ and $g \circ r = p \circ i$. The sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z$ is exact if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $u \circ r = i$. But given $r : \mathrm{Ker}(v) \to w$, we have $f \circ r = q \circ i$ and $g \circ r = p \circ i$ if and only if $q \circ u \circ r = f \circ r = q \circ i$ and $p \circ u \circ r = g \circ r = p \circ i$, hence if and only if $u \circ r = i$. This proves (1), and then (2) follows by duality. □

08N3 **Lemma 12.5.12.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *If the diagram is cartesian, then the morphism $\mathrm{Ker}(f) \to \mathrm{Ker}(k)$ induced by g is an isomorphism.*
(2) *If the diagram is cocartesian, then the morphism $\mathrm{Coker}(f) \to \mathrm{Coker}(k)$ induced by h is an isomorphism.*

Proof. Suppose the diagram is cartesian. Let $e : \mathrm{Ker}(f) \to \mathrm{Ker}(k)$ be induced by g. Let $i : \mathrm{Ker}(f) \to w$ and $j : \mathrm{Ker}(k) \to x$ be the canonical injections. There exists $t : \mathrm{Ker}(k) \to w$ with $f \circ t = 0$ and $g \circ t = j$. Hence, there exists $u : \mathrm{Ker}(k) \to \mathrm{Ker}(f)$ with $i \circ u = t$. It follows $g \circ i \circ u \circ e = g \circ t \circ e = j \circ e = g \circ i$ and $f \circ i \circ u \circ e = 0 = f \circ i$, hence $i \circ u \circ e = i$. Since i is a monomorphism this implies $u \circ e = \mathrm{id}_{\mathrm{Ker}(f)}$. Furthermore, we have $j \circ e \circ u = g \circ i \circ u = g \circ t = j$. Since j is a monomorphism this implies $e \circ u = \mathrm{id}_{\mathrm{Ker}(k)}$. This proves (1). Now, (2) follows by duality. □

08N4 **Lemma 12.5.13.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *If the diagram is cartesian and k is an epimorphism, then the diagram is cocartesian and f is an epimorphism.*
(2) *If the diagram is cocartesian and g is a monomorphism, then the diagram is cartesian and h is a monomorphism.*

Proof. Suppose the diagram is cartesian and k is an epimorphism. Let $u = (g, f) : w \to x \oplus y$ and let $v = (k, -h) : x \oplus y \to z$. As k is an epimorphism, v is an epimorphism, too. Therefore and by Lemma 12.5.11, the sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z \to 0$ is exact. Thus, the diagram is cocartesian by Lemma 12.5.11. Finally, f is an epimorphism by Lemma 12.5.12 and Lemma 12.5.4. This proves (1), and (2) follows by duality. □

05PK **Lemma 12.5.14.** *Let $\mathcal{A}$ be an abelian category.*

(1) *If $x \to y$ is surjective, then for every $z \to y$ the projection $x \times_y z \to z$ is surjective.*
(2) *If $x \to y$ is injective, then for every $x \to z$ the morphism $z \to z \amalg_x y$ is injective.*

Proof. Immediately from Lemma 12.5.4 and Lemma 12.5.13. □

08N5 **Lemma 12.5.15.** *Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ and $g : y \to z$ be morphisms with $g \circ f = 0$. Then, the following statements are equivalent:*

(1) *The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ is exact.*
(2) *For every $h : w \to y$ with $g \circ h = 0$ there exist an object v, an epimorphism $k : v \to w$ and a morphism $l : v \to x$ with $h \circ k = f \circ l$.*

Proof. Let $i : \mathrm{Ker}(g) \to y$ be the canonical injection. Let $p : x \to \mathrm{Coim}(f)$ be the canonical projection. Let $j : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ be the canonical injection.

Suppose (1) holds. Let $h : w \to y$ with $g \circ h = 0$. There exists $c : w \to \mathrm{Ker}(g)$ with $i \circ c = h$. Let $v = x \times_{\mathrm{Ker}(g)} w$ with canonical projections $k : v \to w$ and $l : v \to x$, so that $c \circ k = j \circ p \circ l$. Then, $h \circ k = i \circ c \circ k = i \circ j \circ p \circ l = f \circ l$. As $j \circ p$ is an epimorphism by hypothesis, k is an epimorphism by Lemma 12.5.13. This implies (2).

Suppose (2) holds. Then, $g \circ i = 0$. So, there are an object w, an epimorphism $k : w \to \mathrm{Ker}(g)$ and a morphism $l : w \to x$ with $f \circ l = i \circ k$. It follows $i \circ j \circ p \circ l =$

$f \circ l = i \circ k$. Since i is a monomorphism we see that $j \circ p \circ l = k$ is an epimorphism. So, j is an epimorphisms and thus an isomorphism. This implies (1). □

08N6 **Lemma 12.5.16.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ u & \xrightarrow{k} & v & \xrightarrow{l} & w \end{array}$$

be a commutative diagram.

(1) *If the first row is exact and k is a monomorphism, then the induced sequence $\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ is exact.*
(2) *If the second row is exact and g is an epimorphism, then the induced sequence $\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$ is exact.*

Proof. Suppose the first row is exact and k is a monomorphism. Let $a : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $b : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ be the induced morphisms. Let $h : \mathrm{Ker}(\alpha) \to x$, $i : \mathrm{Ker}(\beta) \to y$ and $j : \mathrm{Ker}(\gamma) \to z$ be the canonical injections. As j is a monomorphism we have $b \circ a = 0$. Let $c : s \to \mathrm{Ker}(\beta)$ with $b \circ c = 0$. Then, $g \circ i \circ c = j \circ b \circ c = 0$. By Lemma 12.5.15 there are an object t, an epimorphism $d : t \to s$ and a morphism $e : t \to x$ with $i \circ c \circ d = f \circ e$. Then, $k \circ \alpha \circ e = \beta \circ f \circ e = \beta \circ i \circ c \circ d = 0$. As k is a monomorphism we get $\alpha \circ e = 0$. So, there exists $m : t \to \mathrm{Ker}(\alpha)$ with $h \circ m = e$. It follows $i \circ a \circ m = f \circ h \circ m = f \circ e = i \circ c \circ d$. As i is a monomorphism we get $a \circ m = c \circ d$. Thus, Lemma 12.5.15 implies (1), and then (2) follows by duality. □

010H **Lemma 12.5.17.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccc} & & x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & \\ 0 & \longrightarrow & u & \xrightarrow{k} & v & \xrightarrow{l} & w & \end{array}$$

be a commutative diagram with exact rows.

(1) *There exists a unique morphism $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ such that the diagram*

$$\begin{array}{ccccc} y & \xleftarrow{\pi'} & y \times_z \mathrm{Ker}(\gamma) & \xrightarrow{\pi} & \mathrm{Ker}(\gamma) \\ {\scriptstyle \beta}\downarrow & & & & \downarrow{\scriptstyle \delta} \\ v & \xrightarrow{\iota'} & \mathrm{Coker}(\alpha) \amalg_u v & \xleftarrow{\iota} & \mathrm{Coker}(\alpha) \end{array}$$

commutes, where π and π' are the canonical projections and ι and ι' are the canonical coprojections.

(2) *The induced sequence*

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma)$$

is exact. If f is injective then so is f', and if l is surjective then so is l'.

Proof. As π is an epimorphism and ι is a monomorphism by Lemma 12.5.13, uniqueness of δ is clear. Let $p = y \times_z \mathrm{Ker}(\gamma)$ and $q = \mathrm{Coker}(\alpha) \amalg_u v$. Let $h : \mathrm{Ker}(\beta) \to y$, $i : \mathrm{Ker}(\gamma) \to z$ and $j : \mathrm{Ker}(\pi) \to p$ be the canonical injections. Let $\pi'' : u \to \mathrm{Coker}(\alpha)$ be the canonical projection. Keeping in mind Lemma 12.5.13 we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\pi) & \xrightarrow{j} & p & \xrightarrow{\pi} & \mathrm{Ker}(\gamma) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle i} & & \\
 & & x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & u & \xrightarrow{k} & v & \xrightarrow{l} & w & & \\
 & & \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \iota'} & & & & \\
0 & \longrightarrow & \mathrm{Coker}(\alpha) & \xrightarrow{\iota} & q & & & &
\end{array}$$

As $l \circ \beta \circ \pi' = \gamma \circ i \circ \pi = 0$ and as the third row of the diagram above is exact, there is an $a : p \to u$ with $k \circ a = \beta \circ \pi'$. As the upper right quadrangle of the diagram above is cartesian, Lemma 12.5.12 yields an epimorphism $b : x \to \mathrm{Ker}(\pi)$ with $\pi' \circ j \circ b = f$. It follows $k \circ a \circ j \circ b = \beta \circ \pi' \circ j \circ b = \beta \circ f = k \circ \alpha$. As k is a monomorphism this implies $a \circ j \circ b = \alpha$. It follows $\pi'' \circ a \circ j \circ b = \pi'' \circ \alpha = 0$. As b is an epimorphism this implies $\pi'' \circ a \circ j = 0$. Therefore, as the top row of the diagram above is exact, there exists $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ with $\delta \circ \pi = \pi'' \circ a$. It follows $\iota \circ \delta \circ \pi = \iota \circ \pi'' \circ a = \iota' \circ k \circ a = \iota' \circ \beta \circ \pi'$ as desired.

As the upper right quadrangle in the diagram above is cartesian there is a $c : \mathrm{Ker}(\beta) \to p$ with $\pi' \circ c = h$ and $\pi \circ c = g'$. It follows $\iota \circ \delta \circ g' = \iota \circ \delta \circ \pi \circ c = \iota' \circ \beta \circ \pi' \circ c = \iota' \circ \beta \circ h = 0$. As ι is a monomorphism this implies $\delta \circ g' = 0$.

Next, let $d : r \to \mathrm{Ker}(\gamma)$ with $\delta \circ d = 0$. Applying Lemma 12.5.15 to the exact sequence $p \xrightarrow{\pi} \mathrm{Ker}(\gamma) \to 0$ and d yields an object s, an epimorphism $m : s \to r$ and a morphism $n : s \to p$ with $\pi \circ n = d \circ m$. As $\pi'' \circ a \circ n = \delta \circ d \circ m = 0$, applying Lemma 12.5.15 to the exact sequence $x \xrightarrow{\alpha} u \xrightarrow{p} \mathrm{Coker}(\alpha)$ and $a \circ n$ yields an object t, an epimorphism $\varepsilon : t \to s$ and a morphism $\zeta : t \to x$ with $a \circ n \circ \varepsilon = \alpha \circ \zeta$. It holds $\beta \circ \pi' \circ n \circ \varepsilon = k \circ \alpha \circ \zeta = \beta \circ f \circ \zeta$. Let $\eta = \pi' \circ n \circ \varepsilon - f \circ \zeta : t \to y$. Then, $\beta \circ \eta = 0$. It follows that there is a $\vartheta : t \to \mathrm{Ker}(\beta)$ with $\eta = h \circ \vartheta$. It holds $i \circ g' \circ \vartheta = g \circ h \circ \vartheta = g \circ \pi' \circ n \circ \varepsilon - g \circ f \circ \zeta = i \circ \pi \circ n \circ \varepsilon = i \circ d \circ m \circ \varepsilon$. As i is a monomorphism we get $g' \circ \vartheta = d \circ m \circ \varepsilon$. Thus, as $m \circ \varepsilon$ is an epimorphism, Lemma 12.5.15 implies that $\mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha)$ is exact. Then, the claim follows by Lemma 12.5.16 and duality. $\square$

08N7 **Lemma 12.5.18.** *Let $\mathcal{A}$ be an abelian category. Let*

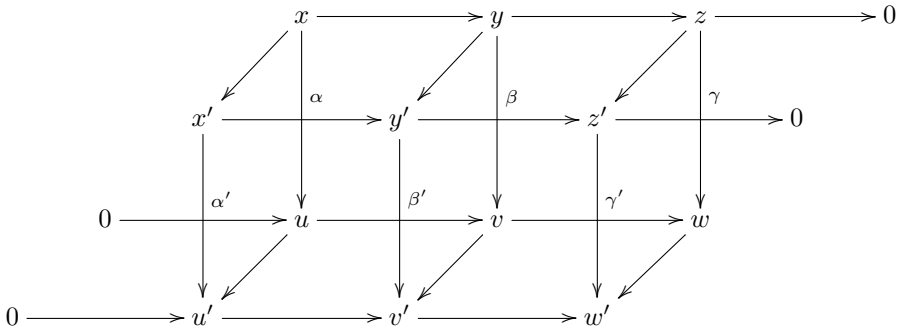

be a commutative diagram with exact rows. Then, the induced diagram

$$\begin{array}{ccccccccc}
\mathrm{Ker}(\alpha) & \longrightarrow & \mathrm{Ker}(\beta) & \longrightarrow & \mathrm{Ker}(\gamma) & \xrightarrow{\delta} & \mathrm{Coker}(\alpha) & \longrightarrow & \mathrm{Coker}(\beta) \longrightarrow \mathrm{Coker}(\gamma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \quad\quad \downarrow \\
\mathrm{Ker}(\alpha') & \longrightarrow & \mathrm{Ker}(\beta') & \longrightarrow & \mathrm{Ker}(\gamma') & \xrightarrow{\delta'} & \mathrm{Coker}(\alpha') & \longrightarrow & \mathrm{Coker}(\beta') \longrightarrow \mathrm{Coker}(\gamma')
\end{array}$$

commutes.

Proof. Omitted. □

05QA **Lemma 12.5.19.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccc}
w & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
w' & \longrightarrow & x' & \longrightarrow & y' & \longrightarrow & z'
\end{array}$$

be a commutative diagram with exact rows.

(1) *If α, γ are surjective and δ is injective, then β is surjective.*
(2) *If β, δ are injective and α is surjective, then γ is injective.*

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 12.5.17 to

$$\begin{array}{ccccccc}
 & & \mathrm{Ker}(y \to z) & \longrightarrow & y & \longrightarrow & z \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ker}(y' \to z') & \longrightarrow & y' & \longrightarrow & z'
\end{array}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 12.5.17 to

$$\begin{array}{ccccccc}
w & \longrightarrow & x & \longrightarrow & \mathrm{Ker}(y \to z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & w' & \longrightarrow & x' & \longrightarrow & \mathrm{Ker}(y' \to z')
\end{array}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

05QB **Lemma 12.5.20.** *Let $\mathcal{A}$ be an abelian category. Let* [**ES52**, Lemma 4.5 page 16]

$$\begin{array}{ccccccccc} v & \longrightarrow & w & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ v' & \longrightarrow & w' & \longrightarrow & x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 12.5.19. □

12.6. Extensions

010I

010J **Definition 12.6.1.** Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. An *extension E of B by A* is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

A *morphism of extensions* between two extensions $0 \to A \to E \to B \to 0$ and $0 \to A \to F \to B \to 0$ means a morphism $f : E \to F$ in $\mathcal{A}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutative. Thus, the extensions of B by A form a category.

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, although they are definitively part of the structure of an extension.

010K **Definition 12.6.2.** Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the Ext-*group*.

This definition works, because by our conventions $\mathrm{Ob}(\mathcal{A})$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of "big" abelian categories listed in Categories, Remark 4.2.2 one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-, -)$ into a functor

$$\mathcal{A} \times \mathcal{A}^{opp} \longrightarrow \mathit{Sets}, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(B, A)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$. By Lemma 12.5.12 we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension E' is called the *pullback of E via $B' \to B$*.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \amalg_A E$. By Lemma 12.5.13 we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension E' is called the *pushout of E via $A \to A'$*.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E \times_B B') \times_{B'} B'' = E \times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \amalg_A (E \times_B B') \cong (A' \amalg_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \amalg_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We push out by the sum map $A \oplus A \to A$ and we pull back by the diagonal map $B \to B \oplus B$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & B \oplus B & \longrightarrow & 0 \\ & & {\scriptstyle \Sigma}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B \oplus B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & {\scriptstyle \Delta}\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E_1 + E_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension $E_1 + E_2$ is called the *Baer sum* of the given extensions.

010L **Lemma 12.6.3.** *The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $\mathrm{Ext}_{\mathcal{A}}(B, A)$ which is functorial in both variables.*

Proof. Omitted. □

05E2 **Lemma 12.6.4.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.*

(1) *There is a canonical six term exact sequence of abelian groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(M_3, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(M_2, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(M_1, N) \\ & & & \searrow & & & \\ & & \mathrm{Ext}_{\mathcal{A}}(M_3, N) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(M_2, N) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(M_1, N) \end{array}$$

for all objects N of $\mathcal{A}$, and

(2) *there is a canonical six term exact sequence of abelian groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_3) \\ & & & \searrow & & & \\ & & \mathrm{Ext}_{\mathcal{A}}(N, M_1) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(N, M_2) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}(N, M_3) \end{array}$$

for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: The boundary maps are defined using either the pushout or pullback of the given short exact sequence. □

12.7. Additive functors

010M First a completely silly lemma characterizing additive functors between additive categories.

0DLP **Lemma 12.7.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. The following are equivalent*

(1) *F is additive,*
(2) *$F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism for all $A, B \in \mathcal{A}$, and*
(3) *$F(A \oplus B) \to F(A) \oplus F(B)$ is an isomorphism for all $A, B \in \mathcal{A}$.*

Proof. Additive functors commute with direct sums by Lemma 12.3.7 hence (1) implies (2) and (3). On the other hand (2) and (3) are equivalent because the composition $F(A) \oplus F(B) \to F(A \oplus B) \to F(A) \oplus F(B)$ is the identity map. Assume (2) and (3) hold. Let $f, g : A \to B$ be maps. Then $f + g$ is equal to the composition

$$A \to A \oplus A \xrightarrow{\mathrm{diag}(f,g)} B \oplus B \to B$$

Apply the functor F and consider the following diagram

$$\begin{array}{ccccccc} F(A) & \longrightarrow & F(A \oplus A) & \xrightarrow[F(\mathrm{diag}(f,g))]{} & F(B \oplus B) & \longrightarrow & F(B) \\ & \searrow & \uparrow & & \downarrow & \nearrow & \\ & & F(A) \oplus F(A) & \xrightarrow{\mathrm{diag}(F(f),F(g))} & F(B) \oplus F(B) & & \end{array}$$

We claim this is commutative. For the middle square we can verify it separately for each of the four induced maps $F(A) \to F(B)$ where it follows from the fact that F is a functor (in other words this square commutes even if F does not satisfy any properties beyond being a functor). For the triangle on the left, we use that $F(A \oplus A) \to F(A) \oplus F(A)$ is an isomorphism to see that it suffice to check after composition with this map and this check is trivial. Dually for the other triangle. Thus going around the bottom is equal to $F(f + g)$ and we conclude. □

Recall that we defined, in Categories, Definition 4.23.1 the notion of a "right exact", "left exact" and "exact" functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

010N **Lemma 12.7.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.*

(1) *If F is either left or right exact, then it is additive.*
(2) *F is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.*
(3) *F is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*
(4) *F is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. If F is left exact, i.e., F commutes with finite limits, then F sends products to products, hence F preserved direct sums, hence F is additive by Lemma 12.7.1. On the other hand, suppose that for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact. Let A, B be two objects. Then we have a short exact sequence

$$0 \to A \to A \oplus B \to B \to 0$$

see for example Lemma 12.3.10. By assumption, the lower row in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(A) & \longrightarrow & F(A) \oplus F(B) & \longrightarrow & F(B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(A) & \longrightarrow & F(A \oplus B) & \longrightarrow & F(B) & & \end{array}$$

is exact. Hence by the snake lemma (Lemma 12.5.17) we conclude that $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism. Hence F is additive in this case as well. Thus for the rest of the proof we may assume F is additive.

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \mathrm{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\mathrm{Ker}(f)) = \mathrm{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C \to 0$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. Also $0 \to \mathrm{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\mathrm{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\mathrm{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

010O **Lemma 12.7.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism*

$$\mathrm{Ext}_{\mathcal{A}}(B, A) \longrightarrow \mathrm{Ext}_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

03A3 **Lemma 12.7.4.** *Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that*

(1) *$\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,*
(2) *$\mathcal{B}$ is abelian and b is left exact, and*
(3) *$ba \cong id_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 12.5.5. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 4.24.6. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \mathrm{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \mathrm{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \mathrm{id}$ we see that every morphism of $\mathcal{A}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\mathrm{Ker}(\psi) = b\,\mathrm{Ker}(a\psi), \quad \text{and} \quad \mathrm{Coker}(\psi) = b\,\mathrm{Coker}(a\psi).$$

Now we still have to show that $\mathrm{Coim}(\psi) = \mathrm{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 12.5.5). Hence we see by Categories, Lemma 4.24.6 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}(\mathrm{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\,\mathrm{Coker}(a(\mathrm{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\,\mathrm{Coker}(\mathrm{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\,\mathrm{Coim}(a\psi) && \text{by definition} \\
&= b\,\mathrm{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\,\mathrm{Ker}(aA_2 \to \mathrm{Coker}(a\psi)) && \text{by definition} \\
&= \mathrm{Ker}(baA_2 \to b\,\mathrm{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \mathrm{Ker}(A_2 \to b\,\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{A}} \\
&= \mathrm{Ker}(A_2 \to \mathrm{Coker}(\psi)) && \text{by formula above} \\
&= \mathrm{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. □

12.8. Localization

05QC In this section we note how Gabriel-Zisman localization interacts with the additive structure on a category.

05QD **Lemma 12.8.1.** *Let $\mathcal{C}$ be a preadditive category. Let S be a left or right multiplicative system. There exists a unique preadditive structure on $S^{-1}\mathcal{C}$ such that the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. We will prove this in the case S is a left multiplicative system. The case where S is a right multiplicative system is dual. Suppose that X, Y are objects of $\mathcal{C}$ and that $\alpha, \beta : X \to Y$ are morphisms in $S^{-1}\mathcal{C}$. According to Categories, Lemma 4.27.5 we may represent these by pairs $s^{-1}f, s^{-1}g$ with common denominator s. Suppose that we have a pre-additive structure on $S^{-1}\mathcal{C}$ and that Q is additive. Then

$$\begin{aligned} s^{-1}f + s^{-1}g &= Q(s)^{-1} \circ Q(f) + Q(s)^{-1} \circ Q(g) \\ &= Q(s)^{-1} \circ (Q(f) + Q(g)) \\ &= Q(s)^{-1} \circ Q(f + g) \\ &= s^{-1}(f + g). \end{aligned}$$

Hence, the pre-additive structure on $S^{-1}\mathcal{C}$ is unique if it exists. Correspondingly, we define $\alpha + \beta$ to be the equivalence class of $s^{-1}(f + g)$ and in the rest of the proof we show that this is well defined and that composition is bilinear. Once this is done it is clear that Q is an additive functor.

Let us show construction above is well defined. An abstract way of saying this is that filtered colimits of abelian groups agree with filtered colimits of sets and to use Categories, Equation (4.27.7.1). We can work this out in a bit more detail as follows. Say $s : Y \to Y_1$ and $f, g : X \to Y_1$. Suppose we have a second representation of α, β as $(s')^{-1}f', (s')^{-1}g'$ with $s' : Y \to Y_2$ and $f', g' : X \to Y_2$. By Categories, Remark 4.27.7 we can find a morphism $s_3 : Y \to Y_3$ and morphisms $a_1 : Y_1 \to Y_3$, $a_2 : Y_2 \to Y_3$ such that $a_1 \circ s = s_3 = a_2 \circ s'$ and also $a_1 \circ f = a_2 \circ f'$ and $a_1 \circ g = a_2 \circ g'$. Hence we see that $s^{-1}(f + g)$ is equivalent to

$$\begin{aligned} s_3^{-1}(a_1 \circ (f + g)) &= s_3^{-1}(a_1 \circ f + a_1 \circ g) \\ &= s_3^{-1}(a_2 \circ f' + a_2 \circ g') \\ &= s_3^{-1}(a_2 \circ (f' + g')) \end{aligned}$$

which is equivalent to $(s')^{-1}(f' + g')$.

Fix $s : Y \to Y'$ and $f, g : X \to Y'$ with $\alpha = s^{-1}f$ and $\beta = s^{-1}g$ as morphisms $X \to Y$ in $S^{-1}\mathcal{C}$. To show that composition is bilinear first consider the case of a morphism $\gamma : Y \to Z$ in $S^{-1}\mathcal{C}$. Say $\gamma = t^{-1}h$ for some $h : Y \to Z'$ and $t : Z \to Z'$ in S. Using LMS2 we choose morphisms $a : Y' \to Z''$ and $t' : Z' \to Z''$ in S such

that $a \circ s = t' \circ h$. Picture

$$\begin{array}{ccccc} & & & & Z \\ & & & & \downarrow t \\ & & Y & \xrightarrow{h} & Z' \\ & & \downarrow s & & \downarrow t' \\ X & \xrightarrow{f,g} & Y' & \xrightarrow{a} & Z'' \end{array}$$

Then $\gamma \circ \alpha = (t' \circ t)^{-1}(a \circ f)$ and $\gamma \circ \beta = (t' \circ t)^{-1}(a \circ g)$. Hence we see that $\gamma \circ (\alpha + \beta)$ is represented by $(t' \circ t)^{-1}(a \circ (f + g)) = (t' \circ t)^{-1}(a \circ f + a \circ g)$ which represents $\gamma \circ \alpha + \gamma \circ \beta$.

Finally, assume that $\delta : W \to X$ is another morphism of $S^{-1}\mathcal{C}$. Say $\delta = r^{-1}i$ for some $i : W \to X'$ and $r : X \to X'$ in S. We claim that we can find a morphism $s' : Y' \to Y''$ in S and morphisms $a'', b'' : X' \to Y''$ such that the following diagram commutes

$$\begin{array}{ccccc} & & & & Y \\ & & & & \downarrow s \\ & & X & \xrightarrow{f,g,f+g} & Y' \\ & & \downarrow r & & \downarrow s' \\ W & \xrightarrow{i} & X' & \xrightarrow{a'',b'',a''+b''} & Y'' \end{array}$$

Namely, using LMS2 we can first choose $s_1 : Y' \to Y_1$, $s_2 : Y' \to Y_2$ in S and $a : X' \to Y_1$, $b : X' \to Y_2$ such that $a \circ r = s_1 \circ f$ and $b \circ r = s_2 \circ f$. Then using that the category Y'/S is filtered (see Categories, Remark 4.27.7), we can find a $s' : Y' \to Y''$ and morphisms $a' : Y_1 \to Y''$, $b' : Y_2 \to Y''$ such that $s' = a' \circ s_1$ and $s' = b' \circ s_2$. Setting $a'' = a' \circ a$ and $b'' = b' \circ b$ works. At this point we see that the compositions $\alpha \circ \delta$ and $\beta \circ \delta$ are represented by $(s' \circ s)^{-1}(a'' \circ i)$ and $(s' \circ s)^{-1}(b'' \circ i)$. Hence $\alpha \circ \delta + \beta \circ \delta$ is represented by $(s' \circ s)^{-1}(a'' \circ i + b'' \circ i) = (s' \circ s)^{-1}((a'' + b'') \circ i)$ which by the diagram again is a representative of $(\alpha + \beta) \circ \delta$. □

05QE **Lemma 12.8.2.** *Let $\mathcal{C}$ be an additive category. Let S be a left or right multiplicative system. Then $S^{-1}\mathcal{C}$ is an additive category and the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. By Lemma 12.8.1 we see that $S^{-1}\mathcal{C}$ is preadditive and that Q is additive. By Lemma 12.3.7 we conclude that $S^{-1}\mathcal{C}$ has a zero object and direct sums. □

The following lemma describes the "kernel" of the localization functor in case we invert a multiplicative system.

05QF **Lemma 12.8.3.** *Let $\mathcal{C}$ be an additive category. Let S be a multiplicative system. Let X be an object of $\mathcal{C}$. The following are equivalent*

1. *$Q(X) = 0$ in $S^{-1}\mathcal{C}$,*
2. *there exists $Y \in \mathrm{Ob}(\mathcal{C})$ such that $0 : X \to Y$ is an element of S, and*
3. *there exists $Z \in \mathrm{Ob}(\mathcal{C})$ such that $0 : Z \to X$ is an element of S.*

Proof. If (2) holds we see that $0 = Q(0) : Q(X) \to Q(Y)$ is an isomorphism. In the additive category $S^{-1}\mathcal{C}$ this implies that $Q(X) = 0$. Hence (2) $\Rightarrow$ (1). Similarly,

(3) $\Rightarrow$ (1). Suppose that $Q(X) = 0$. This implies that the morphism $f : 0 \to X$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Hence by Categories, Lemma 4.27.21 there exists a morphism $g : Z \to 0$ such that $fg \in S$. This proves (1) $\Rightarrow$ (3). Similarly, (1) $\Rightarrow$ (2). □

05QG **Lemma 12.8.4.** *Let $\mathcal{A}$ be an abelian category.*

(1) *If S is a left multiplicative system, then the category $S^{-1}\mathcal{A}$ has cokernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*
(2) *If S is a right multiplicative system, then the category $S^{-1}\mathcal{A}$ has kernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*
(3) *If S is a multiplicative system, then the category $S^{-1}\mathcal{A}$ is abelian and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.*

Proof. Assume S is a left multiplicative system. Let $a : X \to Y$ be a morphism of $S^{-1}\mathcal{A}$. Then $a = s^{-1}f$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Since $Q(s)$ is an isomorphism we see that the existence of $\mathrm{Coker}(a : X \to Y)$ is equivalent to the existence of $\mathrm{Coker}(Q(f) : X \to Y')$. Since $\mathrm{Coker}(Q(f))$ is the coequalizer of 0 and $Q(f)$ we see that $\mathrm{Coker}(Q(f))$ is represented by $Q(\mathrm{Coker}(f))$ by Categories, Lemma 4.27.9. This proves (1).

Part (2) is dual to part (1).

If S is a multiplicative system, then S is both a left and a right multiplicative system. Thus we see that $S^{-1}\mathcal{A}$ has kernels and cokernels and Q commutes with kernels and cokernels. To finish the proof of (3) we have to show that Coim = Im in $S^{-1}\mathcal{A}$. Again using that any arrow in $S^{-1}\mathcal{A}$ is isomorphic to an arrow $Q(f)$ we see that the result follows from the result for $\mathcal{A}$. □

12.9. Jordan-Hölder

0FCD The Jordan-Hölder lemma is Lemma 12.9.7. First we state some definitions.

0FCE **Definition 12.9.1.** Let $\mathcal{A}$ be an abelian category. An object A of $\mathcal{A}$ is said to be *simple* if it is nonzero and the only subobjects of A are 0 and A.

0FCF **Definition 12.9.2.** Let $\mathcal{A}$ be an abelian category.

(1) We say an object A of $\mathcal{A}$ is *Artinian* if and only if it satisfies the descending chain condition for subobjects.
(2) We say $\mathcal{A}$ is *Artinian* if every object of $\mathcal{A}$ is Artinian.

0FCG **Definition 12.9.3.** Let $\mathcal{A}$ be an abelian category.

(1) We say an object A of $\mathcal{A}$ is *Noetherian* if and only if it satisfies the ascending chain condition for subobjects.
(2) We say $\mathcal{A}$ is *Noetherian* if every object of $\mathcal{A}$ is Noetherian.

0FCH **Lemma 12.9.4.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of $\mathcal{A}$. Then A_2 is Artinian if and only if A_1 and A_3 are Artinian.*

Proof. Omitted. □

0FCI **Lemma 12.9.5.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of $\mathcal{A}$. Then A_2 is Noetherian if and only if A_1 and A_3 are Noetherian.*

Proof. Omitted. □

0FCJ **Lemma 12.9.6.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$. The following are equivalent*

(1) *A is Artinian and Noetherian, and*
(2) *there exists a filtration $0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A$ by subobjects such that A_i/A_{i-1} is simple for $i = 1, \ldots, n$.*

Proof. Assume (1). If A is zero, then (2) holds. If A is not zero, then there exists a smallest nonzero object $A_1 \subset A$ by the Artinian property. Of course A_1 is simple. If $A_1 = A$, then we are done. If not, then we can find $A_1 \subset A_2 \subset A$ minimal with $A_2 \neq A_1$. Then A_2/A_1 is simple. Continuing in this way, we can find a sequence $0 \subset A_1 \subset A_2 \subset \ldots$ of subobjects of A such that A_i/A_{i-1} is simple. Since A is Noetherian, we conclude that the process stops. Hence (2) follows.

Assume (2). We will prove (1) by induction on n. If $n = 1$, then A is simple and clearly Noetherian and Artinian. If the result holds for $n-1$, then we use the short exact sequence $0 \to A_{n-1} \to A_n \to A_n/A_{n-1} \to 0$ and Lemmas 12.9.4 and 12.9.5 to conclude for n. □

0FCK **Lemma 12.9.7** (Jordan-Hölder)**.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ satisfying the equivalent conditions of Lemma 12.9.6. Given two filtrations*

$$0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A \quad \text{and} \quad 0 \subset B_1 \subset B_2 \subset \ldots \subset B_m = A$$

with $S_i = A_i/A_{i-1}$ and $T_j = B_j/B_{j-1}$ simple objects we have $n = m$ and there exists a permutation σ of $\{1, \ldots, n\}$ such that $S_i \cong T_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$.

Proof. Let j be the smallest index such that $A_1 \subset B_j$. Then the map $S_1 = A_1 \to B_j/B_{j-1} = T_j$ is an isomorphism. Moreover, the object $A/A_1 = A_n/A_1 = B_m/A_1$ has the two filtrations

$$0 \subset A_2/A_1 \subset A_3/A_1 \subset \ldots \subset A_n/A_1$$

and

$$0 \subset (B_1 + A_1)/A_1 \subset \ldots \subset (B_{j-1} + A_1)/A_1 = B_j/A_1 \subset B_{j+1}/A_1 \subset \ldots \subset B_m/A_1$$

We conclude by induction. □

12.10. Serre subcategories

02MN In [**Ser53**, Chapter I, Section 1] a notion of a "class" of abelian groups is defined. This notion has been extended to abelian categories by many authors (in slightly different ways). We will use the following variant which is virtually identical to Serre's original definition.

02MO **Definition 12.10.1.** Let $\mathcal{A}$ be an abelian category. [**Ser53**, Condition (I) on page 259]

(1) A *Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence[1]

$$A \to B \to C$$

with $A, C \in \mathrm{Ob}(\mathcal{C})$, then also $B \in \mathrm{Ob}(\mathcal{C})$.

[1] By Definition 12.5.7 this means $\mathrm{Im}(A \to B) = \mathrm{Ker}(B \to C)$.

(2) A *weak Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
with A_0, A_1, A_3, A_4 in $\mathcal{C}$, then also A_2 in $\mathcal{C}$.

In some references the second notion is called a "thick" subcategory and in other references the first notion is called a "thick" subcategory. However, it seems that the notion of a Serre subcategory is universally accepted to be the one defined above. Note that in both cases the category $\mathcal{C}$ is abelian and that the inclusion functor $\mathcal{C} \to \mathcal{A}$ is a fully faithful exact functor. Let's characterize these types of subcategories in more detail.

02MP **Lemma 12.10.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *any subobject or quotient of an object of $\mathcal{C}$ is an object of $\mathcal{C}$,*
(4) *if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.*

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

0754 **Lemma 12.10.3.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a weak Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *kernels and cokernels in $\mathcal{A}$ of morphisms between objects of $\mathcal{C}$ are in $\mathcal{C}$,*
(4) *if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.*

Moreover, a weak Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

02MQ **Lemma 12.10.4.** *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.*

Proof. Omitted. □

02MR **Definition 12.10.5.** Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the *kernel of the functor F*, and is sometimes denoted $\mathrm{Ker}(F)$.

Any Serre subcategory of an abelian category is the kernel of an exact functor. In Examples, Section 110.77 we discuss this for Serre's original example of torsion groups.

02MS **Lemma 12.10.6.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*
$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset \mathrm{Ker}(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. Consider the set of arrows of $\mathcal{A}$ defined by the following formula

$$S = \{f \in \mathrm{Arrows}(\mathcal{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{C})\}.$$

We claim that S is a multiplicative system. To prove this we have to check MS1, MS2, MS3, see Categories, Definition 4.27.1.

It is clear that identities are elements of S. Suppose that $f : A \to B$ and $g : B \to C$ are elements of S. There are exact sequences

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g)$$
$$\mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0$$

Hence it follows that $gf \in S$. This proves MS1. (In fact, a similar argument will show that S is a saturated multiplicative system, see Categories, Definition 4.27.20.)

Consider a solid diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle t}\downarrow & & \vdots\,{\scriptstyle s} \\ C & \overset{f}{\dashrightarrow} & C \amalg_A B \end{array}$$

with $t \in S$. Set $W = C \amalg_A B = \mathrm{Coker}((t, -g) : A \to C \oplus B)$. Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism. Hence s is an element of S. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. This means that $(f - g) \circ s = 0$. In turn this means that $I = \mathrm{Im}(f - g) \subset C$ is a quotient of $\mathrm{Coker}(s)$ hence an object of $\mathcal{C}$. Thus $t : C \to C' = C/I$ is an element of S such that $t \circ (f - g) = 0$, i.e., such that $t \circ f = t \circ g$. This proves LMS3 and the proof of RMS3 is dual.

Having proved that S is a multiplicative system we set $\mathcal{A}/\mathcal{C} = S^{-1}\mathcal{A}$, and we set F equal to the localization functor Q. By Lemma 12.8.4 the category $\mathcal{A}/\mathcal{C}$ is abelian and F is exact. If X is in the kernel of $F = Q$, then by Lemma 12.8.3 we see that $0 : X \to Z$ is an element of S and hence X is an object of $\mathcal{C}$, i.e., the kernel of F is $\mathcal{C}$. Finally, if G is as in the statement of the lemma, then G turns every element of S into an isomorphism. Hence we obtain the functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ from the universal property of localization, see Categories, Lemma 4.27.8. We still have to show the functor H is exact. To do this it suffices to show that H commutes with taking kernels and cokernels, see Lemma 12.7.2. Let $A \to B$ be a morphism in $\mathcal{A}/\mathcal{C}$. We may represent $A \to B$ as fs^{-1} where $s : A' \to A$ is in S and $f : A' \to B$ an arbitrary morphism of $\mathcal{A}$. Since $F = Q$ maps s to an isomorphism in the quotient category $\mathcal{A}/\mathcal{C}$, it suffices to show that H commutes with taking kernels and cokernels of morphisms $f : A \to B$ of $\mathcal{A}$. But here we have $H(f) = G(f)$ and the result follows from the fact that G is exact. □

06XK **Lemma 12.10.7.** *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory contained in the kernel of F. Then*

$\mathcal{C} = \mathrm{Ker}(F)$ if and only if the induced functor $\overline{F} : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ (Lemma 12.10.6) is faithful.

Proof. We will use the results of Lemma 12.10.6 without further mention. The "only if" direction is true because the kernel of $\overline{F}$ is zero by construction. Namely, if $f : X \to Y$ is a morphism in $\mathcal{A}/\mathcal{C}$ such that $\overline{F}(f) = 0$, then $\overline{F}(\mathrm{Im}(f)) = \mathrm{Im}(\overline{F}(f)) = 0$, hence $\mathrm{Im}(f) = 0$ by the assumption on the kernel of F. Thus $f = 0$.

For the "if" direction, let X be an object of $\mathcal{A}$ such that $F(X) = 0$. Then $\overline{F}(\mathrm{id}_X) = \mathrm{id}_{\overline{F}(X)} = 0$, thus $\mathrm{id}_X = 0$ in $\mathcal{A}/\mathcal{C}$ by faithfulness of $\overline{F}$. Hence $X = 0$ in $\mathcal{A}/\mathcal{C}$, that is $X \in \mathrm{Ob}(\mathcal{C})$. □

12.11. K-groups

02MT A tiny bit about K_0 of an abelian category.

02MU **Definition 12.11.1.** Let $\mathcal{A}$ be an abelian category. We denote $K_0(\mathcal{A})$ the *zeroth K-group of $\mathcal{A}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects of $\mathcal{A}$ and for every short exact sequence $0 \to A \to B \to C \to 0$ impose the relation $[B] - [A] - [C] = 0$.

Another way to say this is that there is a presentation

$$\bigoplus_{A \to B \to C \text{ ses}} \mathbf{Z}[A \to B \to C] \longrightarrow \bigoplus_{A \in \mathrm{Ob}(\mathcal{A})} \mathbf{Z}[A] \longrightarrow K_0(\mathcal{A}) \longrightarrow 0$$

with $[A \to B \to C] \mapsto [B] - [A] - [C]$ of $K_0(\mathcal{A})$. The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to the relation $[0] = 0$ in $K_0(\mathcal{A})$. There are no set-theoretical issues as all of our categories are "small" if not mentioned otherwise. Some examples of K-groups for categories of modules over rings where computed in Algebra, Section 10.55.

02MV **Lemma 12.11.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F induces a homomorphism of K-groups $K_0(F) : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by simply setting $K_0(F)([A]) = [F(A)]$.*

Proof. Proves itself. □

Suppose we are given an object M of an abelian category $\mathcal{A}$ and a complex of the form

02MW (12.11.2.1) $$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

In this situation we define

$$H^0(M, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi), \quad \text{and} \quad H^1(M, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

02MX **Lemma 12.11.3.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory and set $\mathcal{B} = \mathcal{A}/\mathcal{C}$.*

(1) *The exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence*

$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$$

of K-groups, and

(2) *the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is equal to the collection of elements of the form*
$$[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$$
where (M, φ, ψ) is a complex as in (12.11.2.1) with the property that it becomes exact in $\mathcal{B}$; in other words that $H^0(M, \varphi, \psi)$ and $H^1(M, \varphi, \psi)$ are objects of $\mathcal{C}$.

Proof. Proof of (1). It is clear that $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is surjective and that the composition $K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is zero. Let $x \in K_0(\mathcal{A})$ be an element mapping to zero in $K_0(\mathcal{B})$. We can write $x = [A] - [A']$ with A, A' in $\mathcal{A}$ (fun exercise). Denote B, B' the corresponding objects of $\mathcal{B}$. The fact that x maps to zero in $K_0(\mathcal{B})$ means that there exists a finite set $I = I^+ \amalg I^-$, for each $i \in I$ a short exact sequence
$$0 \to B_i \to B_i' \to B_i'' \to 0$$
in $\mathcal{B}$ such that we have
$$[B] - [B'] = \sum\nolimits_{i \in I^+} ([B_i'] - [B_i] - [B_i'']) - \sum\nolimits_{i \in I^-} ([B_i'] - [B_i] - [B_i''])$$
in the free abelian group on isomorphism classes of objects of $\mathcal{B}$. We can rewrite this as
$$[B] + \sum\nolimits_{i \in I^+} ([B_i] + [B_i'']) + \sum\nolimits_{i \in I^-} [B_i'] = [B'] + \sum\nolimits_{i \in I^-} ([B_i] + [B_i'']) + \sum\nolimits_{i \in I^+} [B_i'].$$
Since the right and left hand side should contain the same isomorphism classes of objects of $\mathcal{B}$ counted with multiplicity, this means there should be a bijection
$$\tau : \{B\} \amalg \{B_i, B_i''; i \in I^+\} \amalg \{B_i'; i \in I^-\} \longrightarrow \{B'\} \amalg \{B_i, B_i''; i \in I^-\} \amalg \{B_i'; i \in I^+\}$$
such that N and $\tau(N)$ are isomorphic in $\mathcal{B}$. The proof of Lemmas 12.10.6 and 12.8.4 show that we choose for $i \in I$ a short exact sequence
$$0 \to A_i \to A_i' \to A_i'' \to 0$$
in $\mathcal{A}$ such that B_i, B_i', B_i'' are isomorphic to the images of A_i, A_i', A_i'' in $\mathcal{B}$. This implies that the corresponding bijection
$$\tau : \{A\} \amalg \{A_i, A_i''; i \in I^+\} \amalg \{A_i'; i \in I^-\} \longrightarrow \{A'\} \amalg \{A_i, A_i''; i \in I^-\} \amalg \{A_i'; i \in I^+\}$$
satisfies the property that M and $\tau(M)$ are objects of $\mathcal{A}$ which become isomorphic in $\mathcal{B}$. This means $[M] - [\tau(M)]$ is in the image of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$. Namely, the isomorphism in $\mathcal{B}$ is given by a diagram $M \leftarrow M' \to \tau(M)$ in $\mathcal{A}$ where both $M' \to M$ and $M' \to \tau(M)$ have kernel and cokernel in $\mathcal{C}$. Working backwards we conclude that $x = [A] - [A']$ is in the image of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ and the proof of part (1) is complete.

Proof of (2). The proof is similar to the proof of (1) but slightly more bookkeeping is involved. First we remark that any class of the type $[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$ is zero in $K_0(\mathcal{A})$ by the following calculation
$$\begin{aligned} 0 &= [M] - [M] \\ &= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\ &= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\ &= [H^1(M, \varphi, \psi)] - [H^0(M, \varphi, \psi)] \end{aligned}$$
as desired. Hence it suffices to show that any element in the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is of this form.

Any element x in $K_0(\mathcal{C})$ can be represented as the difference $x = [P] - [Q]$ of two objects of $\mathcal{C}$ (fun exercise). Suppose that this element maps to zero in $K_0(\mathcal{A})$. This means that there exist

(1) a finite set $I = I^+ \amalg I^-$,
(2) for $i \in I$ a short exact sequence $0 \to A_i \to B_i \to C_i \to 0$ in $\mathcal{A}$

such that

$$[P] - [Q] = \sum\nolimits_{i \in I^+} ([B_i] - [A_i] - [C_i]) - \sum\nolimits_{i \in I^-} ([B_i] - [A_i] - [C_i])$$

in the free abelian group on the objects of $\mathcal{A}$. We can rewrite this as

$$[P] + \sum\nolimits_{i \in I^+} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^-} [B_i] = [Q] + \sum\nolimits_{i \in I^-} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^+} [B_i].$$

Since the right and left hand side should contain the same objects of $\mathcal{A}$ counted with multiplicity, this means there should be a bijection τ between the terms which occur above. Set

$$T^+ = \{p\} \amalg \{a, c\} \times I^+ \amalg \{b\} \times I^-$$

and

$$T^- = \{q\} \amalg \{a, c\} \times I^- \amalg \{b\} \times I^+.$$

Set $T = T^+ \amalg T^- = \{p, q\} \amalg \{a, b, c\} \times I$. For $t \in T$ define

$$O(t) = \begin{cases} P & \text{if} \quad t = p \\ Q & \text{if} \quad t = q \\ A_i & \text{if} \quad t = (a, i) \\ B_i & \text{if} \quad t = (b, i) \\ C_i & \text{if} \quad t = (c, i) \end{cases}$$

Hence we can view $\tau : T^+ \to T^-$ as a bijection such that $O(t) = O(\tau(t))$ for all $t \in T^+$. Let $t_0^- = \tau(p)$ and let $t_0^+ \in T^+$ be the unique element such that $\tau(t_0^+) = q$. Consider the object

$$M^+ = \bigoplus\nolimits_{t \in T^+} O(t)$$

By using τ we see that it is equal to the object

$$M^- = \bigoplus\nolimits_{t \in T^-} O(t)$$

Consider the map

$$\varphi : M^+ \longrightarrow M^-$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^+$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^- and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^-$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^-. The map is zero on the summands corresponding to p and (c, i), $i \in I^+$. Similarly, consider the map

$$\psi : M^- \longrightarrow M^+$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^-$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^+ and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^+$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^+. The map is zero on the summands corresponding to q and (c, i), $i \in I^-$.

Note that the kernel of φ is equal to the direct sum of the summand P and the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands

$O((b,i)) = B_i$, $i \in I^-$. The image of ψ is equal to the direct sum of the summands $O((c,i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b,i)) = B_i$, $i \in I^-$. In other words we see that

$$P \cong \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

In exactly the same way we see that

$$Q \cong \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

Since as we remarked above the existence of the bijection τ shows that $M^+ = M^-$ we see that the lemma follows. □

12.12. Cohomological delta-functors

010P

010Q **Definition 12.12.1.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *cohomological δ-functor* or simply a *δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

(1) a collection $F^n : \mathcal{A} \to \mathcal{B}$, $n \geq 0$ of additive functors, and
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a collection $\delta_{A\to B\to C} : F^n(C) \to F^{n+1}(A)$, $n \geq 0$ of morphisms of $\mathcal{B}$.

These data are assumed to satisfy the following axioms

(1) for every short exact sequence as above the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^0(A) & \longrightarrow & F^0(B) & \longrightarrow & F^0(C) \\ & & & \overset{\delta_{A\to B\to C}}{\swarrow} & & & \\ & & F^1(A) & \longrightarrow & F^1(B) & \longrightarrow & F^1(C) \\ & & & \overset{\delta_{A\to B\to C}}{\swarrow} & & & \\ & & F^2(A) & \longrightarrow & F^2(B) & \longrightarrow & \ldots \end{array}$$

is exact, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow[\delta_{A\to B\to C}]{} & F^{n+1}(A) \\ \downarrow & & \downarrow \\ F^n(C') & \xrightarrow{\delta_{A'\to B'\to C'}} & F^{n+1}(A') \end{array}$$

are commutative.

Note that this in particular implies that F^0 is left exact.

010R **Definition 12.12.2.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let (F^n, δ_F) and (G^n, δ_G) be δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism of δ-functors from F to G* is a collection of transformation of functors $t^n : F^n \to G^n$, $n \geq 0$ such that for every short exact

sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow{\delta_{F,A\to B\to C}} & F^{n+1}(A) \\ {\scriptstyle t^n}\downarrow & & \downarrow{\scriptstyle t^{n+1}} \\ G^n(C) & \xrightarrow{\delta_{G,A\to B\to C}} & G^{n+1}(A) \end{array}$$

are commutative.

010S **Definition 12.12.3.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. We say F is a *universal δ-functor* if and only if for every δ-functor $G = (G^n, \delta_G)$ and any morphism of functors $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$.

010T **Lemma 12.12.4.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Suppose that for every $n > 0$ and any $A \in \mathrm{Ob}(\mathcal{A})$ there exists an injective morphism $u : A \to B$ (depending on A and n) such that $F^n(u) : F^n(A) \to F^n(B)$ is zero. Then F is a universal δ-functor.*

Proof. Let $G = (G^n, \delta_G)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and let $t : F^0 \to G^0$ be a morphism of functors. We have to show there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$. We construct t^n by induction on n. For $n = 0$ we set $t^0 = t$. Suppose we have already constructed a unique sequence of transformation of functors t^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in \mathrm{Ob}(\mathcal{A})$. By assumption we may choose an embedding $u : A \to B$ such that $F^{n+1}(u) = 0$. Let $C = B/u(A)$. The long exact cohomology sequence for the short exact sequence $0 \to A \to B \to C \to 0$ and the δ-functor F gives that $F^{n+1}(A) = \mathrm{Coker}(F^n(B) \to F^n(C))$ by our choice of u. Since we have already defined t^n we can set

$$t^{n+1}_A : F^{n+1}(A) \to G^{n+1}(A)$$

equal to the unique map such that

$$\begin{array}{ccc} \mathrm{Coker}(F^n(B) \to F^n(C)) & \xrightarrow{t^n} & \mathrm{Coker}(G^n(B) \to G^n(C)) \\ {\scriptstyle \delta_{F,A\to B\to C}}\downarrow & & \downarrow{\scriptstyle \delta_{G,A\to B\to C}} \\ F^{n+1}(A) & \xrightarrow{t^{n+1}_A} & G^{n+1}(A) \end{array}$$

commutes. This is clearly uniquely determined by the requirements imposed. We omit the verification that this defines a transformation of functors. □

010U **Lemma 12.12.5.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a universal δ-functor (F^n, δ_F) from $\mathcal{A}$ to $\mathcal{B}$ with $F^0 = F$, then it is determined up to unique isomorphism of δ-functors.*

Proof. Immediate from the definitions. □

12.13. Complexes

010V Of course the notions of a chain complex and a cochain complex are dual and you only have to read one of the two parts of this section. So pick the one you like. (Actually, this doesn't quite work right since the conventions on numbering things are not adapted to an easy transition between chain and cochain complexes.)

A *chain complex* $A_\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A_i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all i. A *morphism of chain complexes* $f : A_\bullet \to B_\bullet$ is given by a family of morphisms $f_i : A_i \to B_i$ such that all the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow[d_i]{} & A_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ B_i & \xrightarrow{d_i} & B_{i-1} \end{array}$$

commute. The *category of chain complexes of* $\mathcal{A}$ is denoted $\mathrm{Ch}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to 0 \to \ldots$$

is denoted $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, a chain complex $A_\bullet$ belongs to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ if and only if $A_i = 0$ for all $i < 0$.

Given an additive category $\mathcal{A}$ we identify $\mathcal{A}$ with the full subcategory of $\mathrm{Ch}(\mathcal{A})$ consisting of chain complexes zero except in degree 0 by the functor

$$\mathcal{A} \longrightarrow \mathrm{Ch}(\mathcal{A}), \quad A \longmapsto (\ldots \to 0 \to A \to 0 \to \ldots)$$

By abuse of notation we often denote the object on the right hand side simply A. If we want to stress that we are viewing A as a chain complex we may sometimes use the notation $A[0]$, see Section 12.14.

A *homotopy* h between a pair of morphisms of chain complexes $f, g : A_\bullet \to B_\bullet$ is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that we have

$$f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

for all i. Two morphisms $f, g : A_\bullet \to B_\bullet$ are said to be *homotopic* if a homotopy between f and g exists. Clearly, the notions of chain complex, morphism of chain complexes, and homotopies between morphisms of chain complexes make sense even in a preadditive category.

010W **Lemma 12.13.1.** *Let $\mathcal{A}$ be an additive category. Let $f, g : B_\bullet \to C_\bullet$ be morphisms of chain complexes. Suppose given morphisms of chain complexes $a : A_\bullet \to B_\bullet$, and $c : C_\bullet \to D_\bullet$. If $\{h_i : B_i \to C_{i+1}\}$ defines a homotopy between f and g, then $\{c_{i+1} \circ h_i \circ a_i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of chain complexes with maps up to homotopy. We'll return to this later.

010X **Definition 12.13.2.** Let $\mathcal{A}$ be an additive category. We say a morphism $a : A_\bullet \to B_\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B_\bullet \to A_\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A_\bullet$ and $B_\bullet$, then we say that $A_\bullet$ and $B_\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

010Y **Lemma 12.13.3.** *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of chain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of chain complexes*

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

is exact at $B_\bullet$ if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *homology group* of a chain complex $A_\bullet$ in an abelian category is defined by the following formula

$$H_i(A_\bullet) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes of $\mathcal{A}$ then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because clearly $f_i(\mathrm{Ker}(d_i : A_i \to A_{i-1})) \subset \mathrm{Ker}(d_i : B_i \to B_{i-1})$, and similarly for $\mathrm{Im}(d_{i+1})$. Thus we obtain a functor

$$H_i : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

010Z **Definition 12.13.4.** Let $\mathcal{A}$ be an abelian category.

(1) A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is called a *quasi-isomorphism* if the induced map $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A chain complex $A_\bullet$ is called *acyclic* if all of its homology objects $H_i(A_\bullet)$ are zero.

0110 **Lemma 12.13.5.** *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.*
(2) *If the map $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

0111 **Lemma 12.13.6.** *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{array}{ccccc} \ldots & & \ldots & & \ldots \\ & \searrow & & \searrow & \\ H_i(A_\bullet) & \longrightarrow & H_i(B_\bullet) & \longrightarrow & H_i(C_\bullet) \\ & \searrow & & \searrow & \\ H_{i-1}(A_\bullet) & \longrightarrow & H_{i-1}(B_\bullet) & \longrightarrow & H_{i-1}(C_\bullet) \\ & \searrow & & \searrow & \\ \ldots & & \ldots & & \ldots \end{array}$$

Proof. Omitted. The maps come from the Snake Lemma 12.5.17 applied to the diagrams

$$\begin{array}{ccccccc} & & A_i/\operatorname{Im}(d_{A,i+1}) & \longrightarrow & B_i/\operatorname{Im}(d_{B,i+1}) & \longrightarrow & C_i/\operatorname{Im}(d_{C,i+1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d_{A,i}} & & \downarrow{\scriptstyle d_{B,i}} & & \downarrow{\scriptstyle d_{C,i}} \\ 0 & \longrightarrow & \operatorname{Ker}(d_{A,i-1}) & \longrightarrow & \operatorname{Ker}(d_{B,i-1}) & \longrightarrow & \operatorname{Ker}(d_{C,i-1}) \end{array}$$

□

A *cochain complex* $A^\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A^i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A *morphism of cochain complexes* $f : A^\bullet \to B^\bullet$ is given by a family of morphisms $f^i : A^i \to B^i$ such that all the diagrams

$$\begin{array}{ccc} A^i & \xrightarrow[d^i]{} & A^{i+1} \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ B^i & \xrightarrow{d^i} & B^{i+1} \end{array}$$

commute. The *category of cochain complexes of* $\mathcal{A}$ is denoted $\mathrm{CoCh}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \ldots$$

is denoted $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$. In other words, a cochain complex $A^\bullet$ belongs to the subcategory $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$ if and only if $A^i = 0$ for all $i < 0$.

Given an additive category $\mathcal{A}$ we identify $\mathcal{A}$ with the full subcategory of $\mathrm{CoCh}(\mathcal{A})$ consisting of cochain complexes zero except in degree 0 by the functor

$$\mathcal{A} \longrightarrow \mathrm{CoCh}(\mathcal{A}), \quad A \longmapsto (\ldots \to 0 \to A \to 0 \to \ldots)$$

By abuse of notation we often denote the object on the right hand side simply A. If we want to stress that we are viewing A as a cochain complex we may sometimes use the notation $A[0]$, see Section 12.14.

A *homotopy* h between a pair of morphisms of cochain complexes $f, g : A^\bullet \to B^\bullet$ is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that we have

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

for all i. Two morphisms $f, g : A^\bullet \to B^\bullet$ are said to be *homotopic* if a homotopy between f and g exists. Clearly, the notions of cochain complex, morphism of cochain complexes, and homotopies between morphisms of cochain complexes make sense even in a preadditive category.

0112 **Lemma 12.13.7.** *Let $\mathcal{A}$ be an additive category. Let $f, g : B^\bullet \to C^\bullet$ be morphisms of cochain complexes. Suppose given morphisms of cochain complexes $a : A^\bullet \to B^\bullet$, and $c : C^\bullet \to D^\bullet$. If $\{h^i : B^i \to C^{i-1}\}$ defines a homotopy between f and g, then $\{c^{i-1} \circ h^i \circ a^i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of cochain complexes with maps up to homotopy. We'll return to this later.

0113 **Definition 12.13.8.** Let $\mathcal{A}$ be an additive category. We say a morphism $a : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B^\bullet \to A^\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A^\bullet$ and $B^\bullet$, then we say that $A^\bullet$ and $B^\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

0114 **Lemma 12.13.9.** *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of cochain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is injective if and only if each $f^n : A^n \to B^n$ is injective.*
(3) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is surjective if and only if each $f^n : A^n \to B^n$ is surjective.*
(4) *A sequence of cochain complexes*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$$

is exact at $B^\bullet$ if and only if each sequence

$$A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i$$

is exact at B^i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *cohomology group* of a cochain complex $A^\bullet$ is defined by the following formula

$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}).$$

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes of $\mathcal{A}$ then we get an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because clearly $f^i(\mathrm{Ker}(d^i : A^i \to A^{i+1})) \subset \mathrm{Ker}(d^i : B^i \to B^{i+1})$, and similarly for $\mathrm{Im}(d^{i-1})$. Thus we obtain a functor

$$H^i : \mathrm{CoCh}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

0115 **Definition 12.13.10.** Let $\mathcal{A}$ be an abelian category.

(1) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ of $\mathcal{A}$ is called a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A cochain complex $A^\bullet$ is called *acyclic* if all of its cohomology objects $H^i(A^\bullet)$ are zero.

0116 **Lemma 12.13.11.** *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A^\bullet \to B^\bullet$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are equal.*
(2) *If $f : A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

0117 **Lemma 12.13.12.** *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence of cochain complexes of $\mathcal{A}$. Then there is a long exact cohomology sequence

$$\begin{array}{ccccc} \ldots & & \ldots & & \ldots \\ & \searrow & & \searrow & \\ H^i(A^\bullet) & \longrightarrow & H^i(B^\bullet) & \longrightarrow & H^i(C^\bullet) \\ & \searrow & & \searrow & \\ H^{i+1}(A^\bullet) & \longrightarrow & H^{i+1}(B^\bullet) & \longrightarrow & H^{i+1}(C^\bullet) \\ & \searrow & & \searrow & \\ \ldots & & \ldots & & \ldots \end{array}$$

The construction produces long exact cohomology sequences which are functorial in the short exact sequence and compatible with shifts as in Definition 12.14.8.

Proof. For the horizontal maps $H^i(A^\bullet) \to H^i(B^\bullet)$ and $H^i(B^\bullet) \to H^i(C^\bullet)$ we use the fact that H^i is a functor, see above. For the "boundary map" $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ we use the map δ of the Snake Lemma 12.5.17 applied to the diagram

$$\begin{array}{ccccccc} & & A^i/\operatorname{Im}(d_A^{i-1}) & \longrightarrow & B^i/\operatorname{Im}(d_B^{i-1}) & \longrightarrow & C^i/\operatorname{Im}(d_C^{i-1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d_A^i} & & \downarrow{\scriptstyle d_B^i} & & \downarrow{\scriptstyle d_C^i} \\ 0 & \longrightarrow & \operatorname{Ker}(d_A^{i+1}) & \longrightarrow & \operatorname{Ker}(d_B^{i+1}) & \longrightarrow & \operatorname{Ker}(d_C^{i+1}) \end{array}$$

This works as the kernel of the right vertical map is equal to $H^i(C^\bullet)$ and the cokernel of the left vertical map is $H^{i+1}(A^\bullet)$. The exactness of the long sequence is the exactnesss in part (2) of Lemma 12.5.17. The functoriality is Lemma 12.5.18. Compatibility with shifts is immediate from the definitions. □

12.14. Homotopy and the shift functor

0119 It is an annoying feature that signs and indices have to be part of any discussion of homological algebra[2].

[2]Please let us know if you notice sign errors or if you have improvements to our conventions.

011A **Definition 12.14.1.** Let $\mathcal{A}$ be an additive category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted chain complex* $A[k]_\bullet$ as follows:

(1) we set $A[k]_n = A_{n+k}$, and
(2) we set $d_{A[k],n} : A[k]_n \to A[k]_{n-1}$ equal to $d_{A[k],n} = (-1)^k d_{A,n+k}$.

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]_\bullet \to B[k]_\bullet$ be the morphism of chain complexes with $f[k]_n = f_{k+n}$.

Of course this means we have functors $[k] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors), such that $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Recall that we view $\mathcal{A}$ as a full subcategory of $\mathrm{Ch}(\mathcal{A})$, see Section 12.13. Thus for any object A of $\mathcal{A}$ the notation $A[k]$ refers to the unique chain complex zero in all degrees except having A in degree $-k$.

011B **Definition 12.14.2.** Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we identify $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$ via the identification $A_{i+k} = A[k]_i$.

This identification is functorial in $A_\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H_{i-k}(A[k]_\bullet)$, which are further compatible with the identifications $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes, $a, b : A_\bullet \to B_\bullet$ are morphisms of chain complexes, and $\{h_i : A_i \to B_{i+1}\}$ is a homotopy between a and b. Recall that this means that $a_i - b_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. What if $a = b$? Then we obtain the formula $0 = d_{i+1} \circ h_i + h_{i-1} \circ d_i$, in other words, $-d_{i+1} \circ h_i = h_{i-1} \circ d_i$. By definition above this means the collection $\{h_i\}$ above defines a morphism of chain complexes

$$A_\bullet \longrightarrow B[1]_\bullet.$$

Such a thing is the same as a morphism $A[-1]_\bullet \to B_\bullet$ by our remarks above. This proves the following lemma.

011C **Lemma 12.14.3.** *Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes. Given any morphism of chain complexes $a : A_\bullet \to B_\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$.*

Proof. See above. □

011D **Lemma 12.14.4.** *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence of complexes. Suppose that $\{s_n : C_n \to B_n\}$ is a family of morphisms which split the short exact sequences $0 \to A_n \to B_n \to C_n \to 0$. Let $\pi_n : B_n \to A_n$ be the associated projections, see Lemma 12.5.10. Then the family of morphisms

$$\pi_{n-1} \circ d_{B,n} \circ s_n : C_n \to A_{n-1}$$

define a morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$.

Proof. Denote $i : A_\bullet \to B_\bullet$ and $q : B_\bullet \to C_\bullet$ the maps of complexes in the short exact sequence. Then $i_{n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}$. Hence $i_{n-2} \circ d_{A,n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n-1} \circ (d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}) = -d_{B,n-1} \circ s_{n-1} \circ d_{C,n}$ as desired. □

011E **Lemma 12.14.5.** *Notation and assumptions as in Lemma 12.14.4 above. The morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$ induces the maps*

$$H_i(\delta(s)) : H_i(C_\bullet) \longrightarrow H_i(A[-1]_\bullet) = H_{i-1}(A_\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of chain complexes by Lemma 12.13.6.

Proof. Omitted. □

011F **Lemma 12.14.6.** *Notation and assumptions as in Lemma 12.14.4 above. Suppose $\{s'_n : C_n \to B_n\}$ is a second choice of splittings. Write $s'_n = s_n + i_n \circ h_n$ for some unique morphisms $h_n : C_n \to A_n$. The family of maps $\{h_n : C_n \to A[-1]_{n+1}\}$ is a homotopy between the associated morphisms $\delta(s), \delta(s') : C_\bullet \to A[-1]_\bullet$.*

Proof. Omitted. □

011G **Definition 12.14.7.** Let $\mathcal{A}$ be an additive category. Let $A^\bullet$ be a cochain complex with boundary maps $d^n_A : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted cochain complex* $A[k]^\bullet$ as follows:

(1) we set $A[k]^n = A^{n+k}$, and
(2) we set $d^n_{A[k]} : A[k]^n \to A[k]^{n+1}$ equal to $d^n_{A[k]} = (-1)^k d^{n+k}_A$.

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of cochain complexes with $f[k]^n = f^{k+n}$.

Of course this means we have functors $[k] : \mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors) and such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Recall that we view $\mathcal{A}$ as a full subcategory of $\mathrm{CoCh}(\mathcal{A})$, see Section 12.13. Thus for any object A of $\mathcal{A}$ the notation $A[k]$ refers to the unique cochain complex zero in all degrees except having A in degree $-k$.

011H **Definition 12.14.8.** Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex with boundary maps $d^n_A : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we identify $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$ via the identification $A^{i+k} = A[k]^i$.

This identification is functorial in $A^\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H^{i-k}(A[k]^\bullet)$, which are further compatible with the identifications $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes, $a, b : A^\bullet \to B^\bullet$ are morphisms of cochain complexes, and $\{h^i : A^i \to B^{i-1}\}$ is a homotopy between a and b. Recall that this means that $a^i - b^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. What if $a = b$? Then we obtain the formula $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, in other words, $-d^{i-1} \circ h^i = h^{i+1} \circ d^i$. By definition above this means the collection $\{h^i\}$ above defines a morphism of cochain complexes

$$A^\bullet \longrightarrow B[-1]^\bullet.$$

Such a thing is the same as a morphism $A[1]^\bullet \to B^\bullet$ by our remarks above. This proves the following lemma.

011I **Lemma 12.14.9.** *Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes. Given any morphism of cochain complexes $a : A^\bullet \to B^\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$.*

Proof. See above. □

011J **Lemma 12.14.10.** *Let $\mathcal{A}$ be an additive category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a complex (!) of complexes. Suppose that we are given splittings $B^n = A^n \oplus C^n$ compatible with the maps in the displayed sequence. Let $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ be the corresponding maps. Then the family of morphisms

$$\pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$$

define a morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$.

Proof. Denote $i : A^\bullet \to B^\bullet$ and $q : B^\bullet \to C^\bullet$ the maps of complexes in the short exact sequence. Then $i^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^n \circ s^n - s^{n+1} \circ d_C^n$. Hence $i^{n+2} \circ d_A^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^{n+1} \circ (d_B^n \circ s^n - s^{n+1} \circ d_C^n) = -d_B^{n+1} \circ s^{n+1} \circ d_C^n$ as desired. □

011K **Lemma 12.14.11.** *Notation and assumptions as in Lemma 12.14.10 above. Assume in addition that $\mathcal{A}$ is abelian. The morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$ induces the maps*

$$H^i(\delta) : H^i(C^\bullet) \longrightarrow H^i(A[1]^\bullet) = H^{i+1}(A^\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of cochain complexes by Lemma 12.13.12.

Proof. Omitted. □

011L **Lemma 12.14.12.** *Notation and assumptions as in Lemma 12.14.10. Let $\alpha : A^\bullet \to B^\bullet$, $\beta : B^\bullet \to C^\bullet$ be the given morphisms of complexes. Suppose $(s')^n : C^n \to B^n$ and $(\pi')^n : B^n \to A^n$ is a second choice of splittings. Write $(s')^n = s^n + \alpha^n \circ h^n$ and $(\pi')^n = \pi^n + g^n \circ \beta^n$ for some unique morphisms $h^n : C^n \to A^n$ and $g^n : C^n \to A^n$. Then*

1. *$g^n = -h^n$, and*
2. *the family of maps $\{g^n : C^n \to A[1]^{n-1}\}$ is a homotopy between $\delta, \delta' : C^\bullet \to A[1]^\bullet$, more precisely $(\delta')^n = \delta^n + g^{n+1} \circ d_C^n + d_{A[1]}^{n-1} \circ g^n$.*

Proof. As $(s')^n$ and $(\pi')^n$ are splittings we have $(\pi')^n \circ (s')^n = 0$. Hence

$$0 = (\pi^n + g^n \circ \beta^n) \circ (s^n + \alpha^n \circ h^n) = g^n \circ \beta^n \circ s^n + \pi^n \circ \alpha^n \circ h^n = g^n + h^n$$

which proves (1). We compute $(\delta')^n$ as follows

$$(\pi^{n+1} + g^{n+1} \circ \beta^{n+1}) \circ d_B^n \circ (s^n + \alpha^n \circ h^n) = \delta^n + g^{n+1} \circ d_C^n + d_A^n \circ h^n$$

Since $h^n = -g^n$ and since $d_{A[1]}^{n-1} = -d_A^n$ we conclude that (2) holds. □

12.15. Truncation of complexes

0118 Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex. There are several ways to *truncate* the complex $A_\bullet$.

(1) The *"stupid" truncation* $\sigma_{\leq n}$ is the subcomplex $\sigma_{\leq n}A_\bullet$ defined by the rule $(\sigma_{\leq n}A_\bullet)_i = 0$ if $i > n$ and $(\sigma_{\leq n}A_\bullet)_i = A_i$ if $i \leq n$. In a picture

$$\begin{array}{ccccccccc}
\sigma_{\leq n}A_\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots
\end{array}$$

Note the property $\sigma_{\leq n}A_\bullet/\sigma_{\leq n-1}A_\bullet = A_n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\geq n}$ is the quotient complex $\sigma_{\geq n}A_\bullet$ defined by the rule $(\sigma_{\geq n}A_\bullet)_i = A_i$ if $i \geq n$ and $(\sigma_{\geq n}A_\bullet)_i = 0$ if $i < n$. In a picture

$$\begin{array}{ccccccccc}
A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\sigma_{\geq n}A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

The map of complexes $\sigma_{\geq n}A_\bullet \to \sigma_{\geq n+1}A_\bullet$ is surjective with kernel $A_n[-n]$.

(3) The *canonical truncation* $\tau_{\geq n}A_\bullet$ is defined by the picture

$$\begin{array}{ccccccccc}
\tau_{\geq n}A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & \mathrm{Ker}(d_n) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots
\end{array}$$

Note that these complexes have the property that

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \geq n \\ 0 & \text{if} \quad i < n \end{cases}$$

(4) The *canonical truncation* $\tau_{\leq n}A_\bullet$ is defined by the picture

$$\begin{array}{ccccccccc}
A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\tau_{\leq n}A_\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d_{n+1}) & \longrightarrow & A_{n-1} & \longrightarrow & \ldots
\end{array}$$

Note that these complexes have the property that

$$H_i(\tau_{\leq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex. There are four ways to truncate the complex $A^\bullet$.

(1) The *"stupid" truncation* $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i < n$ and $(\sigma_{\geq n}A^\bullet)^i = A^i$ if $i \geq n$. In a picture

$$\begin{array}{ccccccccc}
\sigma_{\geq n}A^\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots
\end{array}$$

Note the property $\sigma_{\geq n}A^\bullet/\sigma_{\geq n+1}A^\bullet = A^n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\leq n}$ is the quotient complex $\sigma_{\leq n}A^\bullet$ defined by the rule $(\sigma_{\leq n}A^\bullet)^i = 0$ if $i > n$ and $(\sigma_{\leq n}A^\bullet)^i = A^i$ if $i \leq n$. In a picture

$$\begin{array}{ccccccccc}
A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\sigma_{\leq n}A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

The map of complexes $\sigma_{\leq n}A^\bullet \to \sigma_{\leq n-1}A^\bullet$ is surjective with kernel $A^n[-n]$.

(3) The *canonical truncation* $\tau_{\leq n}A^\bullet$ is defined by the picture

$$\begin{array}{ccccccccc}
\tau_{\leq n}A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & \mathrm{Ker}(d^n) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots
\end{array}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} H^i(A^\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

(4) The *canonical truncation* $\tau_{\geq n}A^\bullet$ is defined by the picture

$$\begin{array}{ccccccccc}
A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\tau_{\geq n}A^\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d^{n-1}) & \longrightarrow & A^{n+1} & \longrightarrow & \ldots
\end{array}$$

Note that these complexes have the property that

$$H^i(\tau_{\geq n}A^\bullet) = \begin{cases} 0 & \text{if} \quad i < n \\ H^i(A^\bullet) & \text{if} \quad i \geq n \end{cases}$$

12.16. Graded objects

09MF We make the following definition.

0125 **Definition 12.16.1.** Let $\mathcal{A}$ be an additive category. The *category of graded objects of* $\mathcal{A}$, denoted $\mathrm{Gr}(\mathcal{A})$, is the category with

(1) objects $A = (A^i)$ are families of objects A^i, $i \in \mathbf{Z}$ of objects of $\mathcal{A}$, and
(2) morphisms $f : A = (A^i) \to B = (B^i)$ are families of morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$.

If $\mathcal{A}$ has countable direct sums, then we can associate to an object $A = (A^i)$ of $\mathrm{Gr}(\mathcal{A})$ the object

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} A^i$$

and set $k^i A = A^i$. In this case $\mathrm{Gr}(\mathcal{A})$ is equivalent to the category of pairs (A, k) consisting of an object A of $\mathcal{A}$ and a direct sum decomposition

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} k^i A$$

by direct summands indexed by $\mathbf{Z}$ and a morphism $(A, k) \to (B, k)$ of such objects is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(k^i A) \subset k^i B$ for all $i \in \mathbf{Z}$. Whenever our additive category $\mathcal{A}$ has countable direct sums we will use this equivalence without further mention.

However, with our definitions an additive or abelian category does not necessarily have all (countable) direct sums. In this case our definition still makes sense. For example, if $\mathcal{A} = \mathrm{Vect}_k$ is the category of finite dimensional vector spaces over a field k, then $\mathrm{Gr}(\mathrm{Vect}_k)$ is the category of vector spaces with a given gradation all of whose graded pieces are finite dimensional, and not the category of finite dimensional vector spaces with a given gradation.

0126 **Lemma 12.16.2.** *Let $\mathcal{A}$ be an abelian category. The category of graded objects $Gr(\mathcal{A})$ is abelian.*

Proof. Let $f : A = (A^i) \to B = (B^i)$ be a morphism of graded objects of $\mathcal{A}$ given by morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$. Then we have $\mathrm{Ker}(f) = (\mathrm{Ker}(f^i))$ and $\mathrm{Coker}(f) = (\mathrm{Coker}(f^i))$ in the category $\mathrm{Gr}(\mathcal{A})$. Since we have $\mathrm{Im} = \mathrm{Coim}$ in $\mathcal{A}$ we see the same thing holds in $\mathrm{Gr}(\mathcal{A})$. □

0AMH **Remark 12.16.3** (Warning)**.** There are abelian categories $\mathcal{A}$ having countable direct sums but where countable direct sums are not exact. An example is the opposite of the category of abelian sheaves on $\mathbf{R}$. Namely, the category of abelian sheaves on $\mathbf{R}$ has countable products, but countable products are not exact. For such a category the functor $\mathrm{Gr}(\mathcal{A}) \to \mathcal{A}$, $(A^i) \mapsto \bigoplus A^i$ described above is not exact. It is still true that $\mathrm{Gr}(\mathcal{A})$ is equivalent to the category of graded objects (A, k) of $\mathcal{A}$, but the kernel in the category of graded objects of a map $\varphi : (A, k) \to (B, k)$ is not equal to $\mathrm{Ker}(\varphi)$ endowed with a direct sum decomposition, but rather it is the direct sum of the kernels of the maps $k^i A \to k^i B$.

09MG **Definition 12.16.4.** Let $\mathcal{A}$ be an additive category. If $A = (A^i)$ is a graded object, then the kth *shift* $A[k]$ is the graded object with $A[k]^i = A^{k+i}$.

If A and B are graded objects of $\mathcal{A}$, then we have

09MH (12.16.4.1) $$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A, B[k]) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A[-k], B)$$

and an element of this group is sometimes called a map of graded objects *homogeneous of degree k*.

Given any set G we can define G-graded objects of $\mathcal{A}$ as the category whose objects are $A = (A^g)_{g \in G}$ families of objects parametrized by elements of G. Morphisms $f : A \to B$ are defined as families of maps $f^g : A^g \to B^g$ where g runs over the elements of G. If G is an abelian group, then we can (unambiguously) define shift functors $[g]$ on the category of G-graded objects by the rule $(A[g])^{g_0} = A^{g+g_0}$. A particular case of this type of construction is when $G = \mathbf{Z} \times \mathbf{Z}$. In this case the

objects of the category are called *bigraded* objects of $\mathcal{A}$. The (p, q) component of a bigraded object A is usually denoted $A^{p,q}$. For $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ we write $A[a, b]$ in stead of $A[(a, b)]$. A morphism $A \to A[a, b]$ is sometimes called a *map of bidegree* (a, b).

12.17. Additive monoidal categories

0FN9 Some material about the interaction between a monoidal structure and an additive structure on a category.

0FNA **Definition 12.17.1.** An *additive monoidal category* is an additive category $\mathcal{A}$ endowed with a monoidal structure $\otimes, \phi$ (Categories, Definition 4.43.1) such that $\otimes$ is an additive functor in each variable.

0FFT **Lemma 12.17.2.** *Let $\mathcal{A}$ be an additive monoidal category. If Y_i, $i = 1, 2$ are left duals of X_i, $i = 1, 2$, then $Y_1 \oplus Y_2$ is a left dual of $X_1 \oplus X_2$.*

Proof. Follows from uniqueness of adjoints and Categories, Remark 4.43.7. □

0FFU **Lemma 12.17.3.** *In a Karoubian additive monoidal category every summand of an object which has a left dual has a left dual.*

Proof. We will use Categories, Lemma 4.43.6 without further mention. Let X be an object which has a left dual Y. We have

$$\mathrm{Hom}(X, X) = \mathrm{Hom}(\mathbf{1}, X \otimes Y) = \mathrm{Hom}(Y, Y)$$

If $a : X \to X$ corresponds to $b : Y \to Y$ then b is the unique endomorphism of Y such that precomposing by a on

$$\mathrm{Hom}(Z' \otimes X, Z) = \mathrm{Hom}(Z', Z \otimes Y)$$

is the same as postcomposing by $1 \otimes b$. Hence the bijection $\mathrm{Hom}(X, X) \to \mathrm{Hom}(Y, Y)$, $a \mapsto b$ is an isomorphism of the opposite of the algebra $\mathrm{Hom}(X, X)$ with the algebra $\mathrm{Hom}(Y, Y)$. In particular, if $X = X_1 \oplus X_2$, then the corresponding projectors e_1, e_2 are mapped to idempotents in $\mathrm{Hom}(Y, Y)$. If $Y = Y_1 \oplus Y_2$ is the corresponding direct sum decomposition of Y (Section 12.4) then we see that under the bijection $\mathrm{Hom}(Z' \otimes X, Z) = \mathrm{Hom}(Z', Z \otimes Y)$ we have $\mathrm{Hom}(Z' \otimes X_i, Z) = \mathrm{Hom}(Z', Z \otimes Y_i)$ functorially as subgroups for $i = 1, 2$. It follows that Y_i is the left dual of X_i by the discussion in Categories, Remark 4.43.7. □

0FFX **Example 12.17.4.** Let F be a field. Let $\mathcal{C}$ be the category of graded F-vector spaces. Given graded vector spaces V and W we let $V \otimes W$ denote the graded F-vector space whose degree n part is

$$(V \otimes W)^n = \bigoplus\nolimits_{n=p+q} V^p \otimes_F W^q$$

Given a third graded vector space U as associativity constraint $\phi : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ we use the "usual" isomorphisms

$$U^p \otimes_F (V^q \otimes_F W^r) \to (U^p \otimes_F V^q) \otimes_F W^r$$

of vectors spaces. As unit we use the graded F-vector space $\mathbf{1}$ which has F in degree 0 and is zero in other degrees. There are two commutativity constraints on $\mathcal{C}$ which turn $\mathcal{C}$ into a symmetric monoidal category: one involves the intervention of signs and the other does not. We will usually use the one that does. To be explicit, if V

and W are graded F-vector spaces we will use the isomorphism $\psi : V \otimes W \to W \otimes V$ which in degree n uses

$$V^p \otimes_F W^q \to W^q \otimes_F V^p, \quad v \otimes w \mapsto (-1)^{pq} w \otimes v$$

We omit the verification that this works.

0FFV **Lemma 12.17.5.** *Let F be a field. Let $\mathcal{C}$ be the category of graded F-vector spaces viewed as a monoidal category as in Example 12.17.4. If V in $\mathcal{C}$ has a left dual W, then $\sum_n \dim_F V^n < \infty$ and the map ϵ defines nondegenerate pairings $W^{-n} \times V^n \to F$.*

Proof. As unit we take By Categories, Definition 4.43.5 we have maps

$$\eta : \mathbf{1} \to V \otimes W \quad \epsilon : W \otimes V \to \mathbf{1}$$

Since $\mathbf{1} = F$ placed in degree 0, we may think of ϵ as a sequence of pairings $W^{-n} \times V^n \to F$ as in the statement of the lemma. Choose bases $\{e_{n,i}\}_{i \in I_n}$ for V^n for all n. Write

$$\eta(1) = \sum e_{n,i} \otimes w_{-n,i}$$

for some elements $w_{-n,i} \in W^{-n}$ almost all of which are zero! The condition that $(\epsilon \otimes 1) \circ (1 \otimes \eta)$ is the identity on W means that

$$\sum\nolimits_{n,i} \epsilon(w, e_{n,i}) w_{-n,i} = w$$

Thus we see that W is generated as a graded vector space by the finitely many nonzero vectors $w_{-n,i}$. The condition that $(1 \otimes \epsilon) \circ (\eta \otimes 1)$ is the identity of V means that

$$\sum\nolimits_{n,i} e_{n,i}\ \epsilon(w_{-n,i}, v) = v$$

In particular, setting $v = e_{n,i}$ we conclude that $\epsilon(w_{-n,i}, e_{n,i'}) = \delta_{ii'}$. Thus we find that the statement of the lemma holds and that $\{w_{-n,i}\}_{i \in I_n}$ is the dual basis for W^{-n} to the chosen basis for V^n. □

12.18. Double complexes and associated total complexes

0FNB We discuss double complexes and associated total complexes.

012Y **Definition 12.18.1.** Let $\mathcal{A}$ be an additive category. A *double complex* in $\mathcal{A}$ is given by a system $(\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q \in \mathbf{Z}})$, where each $A^{p,q}$ is an object of $\mathcal{A}$ and $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ are morphisms of $\mathcal{A}$ such that the following rules hold:

(1) $d_1^{p+1,q} \circ d_1^{p,q} = 0$
(2) $d_2^{p,q+1} \circ d_2^{p,q} = 0$
(3) $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$

for all $p, q \in \mathbf{Z}$.

This is just the cochain version of the definition. It says that each $A^{p,\bullet}$ is a cochain complex and that each $d_1^{p,\bullet}$ is a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$ such that

$d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ as morphisms of complexes. In other words a double complex can be seen as a complex of complexes. So in the diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p,q+1} & \xrightarrow{d_1^{p,q+1}} & A^{p+1,q+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d_2^{p,q}} & & \uparrow{\scriptstyle d_2^{p+1,q}} & & \\
\cdots & \longrightarrow & A^{p,q} & \xrightarrow{d_1^{p,q}} & A^{p+1,q} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

any square commutes. Warning: In the literature one encounters a different definition where a "bicomplex" or a "double complex" has the property that the squares in the diagram anti-commute.

0A5J **Example 12.18.2.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be additive categories. Suppose that

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$

is a functor which is bilinear on morphisms, see Categories, Definition 4.2.20 for the definition of $\mathcal{A} \times \mathcal{B}$. Given complexes $X^\bullet$ of $\mathcal{A}$ and $Y^\bullet$ of $\mathcal{B}$ we obtain a double complex

$$K^{\bullet,\bullet} = X^\bullet \otimes Y^\bullet$$

in $\mathcal{C}$. Here the first differential $K^{p,q} \to K^{p+1,q}$ is the morphism $X^p \otimes Y^q \to X^{p+1} \otimes Y^q$ induced by the morphism $X^p \to X^{p+1}$ and the identity on Y^q. Similarly for the second differential.

012Z **Definition 12.18.3.** Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex. The *associated simple complex*, denoted $sA^\bullet$, also often called the *associated total complex*, denoted $\mathrm{Tot}(A^{\bullet,\bullet})$, is given by

$$sA^n = \mathrm{Tot}^n(A^{\bullet,\bullet}) = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d_{sA^\bullet}^n = d_{\mathrm{Tot}(A^{\bullet,\bullet})}^n = \sum\nolimits_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

If countable direct sums exist in $\mathcal{A}$ or if for each n at most finitely many $A^{p,n-p}$ are nonzero, then $\mathrm{Tot}(A^{\bullet,\bullet})$ exists. Note that the definition is *not* symmetric in the indices (p, q).

08BI **Remark 12.18.4.** Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet,\bullet}$ be a triple complex. The associated total complex is the complex with terms

$$\mathrm{Tot}^n(A^{\bullet,\bullet,\bullet}) = \bigoplus\nolimits_{p+q+r=n} A^{p,q,r}$$

and differential

$$d_{\mathrm{Tot}(A^{\bullet,\bullet,\bullet})}^n = \sum\nolimits_{p+q+r=n} d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r}$$

With this definition a simple calculation shows that the associated total complex is equal to

$$\mathrm{Tot}(A^{\bullet,\bullet,\bullet}) = \mathrm{Tot}(\mathrm{Tot}_{12}(A^{\bullet,\bullet,\bullet})) = \mathrm{Tot}(\mathrm{Tot}_{23}(A^{\bullet,\bullet,\bullet}))$$

In other words, we can either first combine the first two of the variables and then combine sum of those with the last, or we can first combine the last two variables and then combine the first with the sum of the last two.

0FLG **Remark 12.18.5.** Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex with differentials $d_1^{p,q}$ and $d_2^{p,q}$. Denote $A^{\bullet,\bullet}[a,b]$ the double complex with

$$(A^{\bullet,\bullet}[a,b])^{p,q} = A^{p+a,q+b}$$

and differentials

$$d^{p,q}_{A^{\bullet,\bullet}[a,b],1} = (-1)^a d_1^{p+a,q+b} \quad \text{and} \quad d^{p,q}_{A^{\bullet,\bullet}[a,b],2} = (-1)^b d_2^{p+a,q+b}$$

In this situation there is a well defined isomorphism

$$\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[a+b] \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet}[a,b])$$

which in degree n is given by the map

$$\begin{array}{ccc} (\mathrm{Tot}(A^{\bullet,\bullet})[a+b])^n & = & \bigoplus_{p+q=n+a+b} A^{p,q} \\ & & \Big\downarrow {\scriptstyle \epsilon(p,q,a,b)\mathrm{id}_{A^{p,q}}} \\ \mathrm{Tot}(A^{\bullet,\bullet}[a,b])^n & = & \bigoplus_{p'+q'=n} A^{p'+a,q'+b} \end{array}$$

for some sign $\epsilon(p,q,a,b)$. Of course the summand $A^{p,q}$ maps to the summand $A^{p'+a,q'+b}$ when $p = p' + a$ and $q = q' + b$. To figure out the conditions on these signs observe that on the source we have

$$d|_{A^{p,q}} = (-1)^{a+b}\left(d_1^{p,q} + (-1)^p d_2^{p,q}\right)$$

whereas on the target we have

$$d|_{A^{p'+a,q'+b}} = (-1)^a d_1^{p'+a,q'+b} + (-1)^{p'}(-1)^b d_2^{p'+a,q'+b}$$

Thus our constraints are that

$$(-1)^a\epsilon(p,q,a,b) = \epsilon(p+1,q,a,b)(-1)^{a+b} \Leftrightarrow \epsilon(p+1,q,a,b) = (-1)^b\epsilon(p,q,a,b)$$

and

$$(-1)^{p'+b}\epsilon(p,q,a,b) = \epsilon(p,q+1,a,b)(-1)^{a+b+p} \Leftrightarrow \epsilon(p,q,a,b) = \epsilon(p,q+1,a,b)$$

Thus we choose $\epsilon(p,q,a,b) = (-1)^{pb}$.

0G6A **Remark 12.18.6.** Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\mathrm{DoubleComp}(\mathcal{A})$ denote the category of double complexes. We can consider an object $A^{\bullet,\bullet}$ of $\mathrm{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{\bullet,-1} \to A^{\bullet,0} \to A^{\bullet,1} \to \ldots$$

For the variant where we switch the role of the indices, see Remark 12.18.7. In this remark we show that taking the associated total complex is compatible with all the structures on complexes we have studied in the chapter so far.

First, observe that the shift functor on double complexes viewed as complexes of complexes in the manner given above is the functor $[0,1]$ defined in Remark 12.18.5. By Remark 12.18.5 the functor

$$\mathrm{Tot} : \mathrm{DoubleComp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$$

is compatible with shift functors, in the sense that we have a functorial isomorphism $\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[1] \to \mathrm{Tot}(A^{\bullet,\bullet}[0,1])$.

Second, if

$$f, g : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$$

are homotopic when f and g are viewed as morphisms of complexes of complexes in the manner given above, then

$$\mathrm{Tot}(f), \mathrm{Tot}(g) : \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet})$$

are homotopic maps of complexes. Indeed, let $h = (h^q)$ be a homotopy between f and g. If we denote $h^{p,q} : A^{p,q} \to B^{p,q-1}$ the component in degree p of h^q, then this means that

$$f^{p,q} - g^{p,q} = d_2^{p,q-1} \circ h^{p,q} + h^{p,q+1} \circ d_2^{p,q}$$

The fact that $h^q : A^{\bullet,q} \to B^{\bullet,q-1}$ is a map of complexes means that

$$d_1^{p,q-1} \circ h^{p,q} = h^{p+1,q} \circ d_1^{p,q}$$

Let us define $h' = ((h')^n)$ the homotopy given by the maps $(h')^n : \mathrm{Tot}^n(A^{\bullet,\bullet}) \to \mathrm{Tot}^{n-1}(B^{\bullet,\bullet})$ using $(-1)^p h^{p,q}$ on the summand $A^{p,q}$ for $p + q = n$. Then we see that

$$d_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ h' + h' \circ d_{\mathrm{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{p,q}$ is equal to

$$d_1^{p,q-1} \circ (-1)^p h^{p,q} + (-1)^p d_2^{p,q-1} \circ (-1)^p h^{p,q} + (-1)^{p+1} h^{p+1,q} \circ d_1^{p,q} + (-1)^p h^{p,q+1} \circ (-1)^p d_2^{p,q}$$

which evaluates to $f^{p,q} - g^{p,q}$ by the equations given above. This proves the second compatibility.

Third, suppose that in the paragraph above we have $f = g$. Then the assignment $h \rightsquigarrow h'$ above is compatible with the identification of Lemma 12.14.9. More precisely, if we view h as a morphism of complexes of complexes $A^{\bullet,\bullet} \to B^{\bullet,\bullet}[0,-1]$ via this lemma then

$$\mathrm{Tot}(A^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(h)} \mathrm{Tot}(B^{\bullet,\bullet}[0,-1]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(B^{\bullet,\bullet})[-1]$$

is equal to h' viewed as a morphism of complexes via the lemma. Here γ is the identification of Remark 12.18.5. The verification of this third point is immediate.

Fourth, let

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to C^{\bullet,\bullet} \to 0$$

be a complex of double complexes and suppose we are given splittings $s^q : C^{\bullet,q} \to B^{\bullet,q}$ and $\pi^q : B^{\bullet,q} \to A^{\bullet,q}$ of this as in Lemma 12.14.10 when we view double complexes as complexes of complexes in the manner given above. This on the one hand produces a map

$$\delta : C^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}[0,1]$$

by the procedure in Lemma 12.14.10. On the other hand taking Tot we obtain a complex

$$0 \to \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet}) \to \mathrm{Tot}(C^{\bullet,\bullet}) \to 0$$

which is termwise split (see below) and hence comes with a morphism

$$\delta' : \mathrm{Tot}(C^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

well defined up to homotopy by Lemmas 12.14.10 and 12.14.12. Claim: these maps agree in the sense that

$$\mathrm{Tot}(C^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(\delta)} \mathrm{Tot}(A^{\bullet,\bullet}[0,1]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

is equal to δ' where γ is as in Remark 12.18.5. To see this denote $s^{p,q} : C^{p,q} \to B^{\bullet,q}$ and $\pi^{p,q} : B^{p,q} \to A^{p,q}$ the components of s^q and π^q. As splittings $(s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^n(B^{\bullet,\bullet})$ and $(\pi')^n : \mathrm{Tot}^n(B^{\bullet,\bullet}) \to \mathrm{Tot}^n(A^{\bullet,\bullet})$ we use the maps whose components are $s^{p,q}$ and $\pi^{p,q}$ for $p + q = n$. We recall that

$$(\delta')^n = (\pi')^{n+1} \circ d^n_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ (s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^{n+1}(A^{\bullet,\bullet})$$

The restriction of this to the summand $C^{p,q}$ is equal to

$$\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q} + \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q} = \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q}$$

The equality holds because s^q is a morphism of complexes (with d_1 as differential) and because $\pi^{p+1,q} \circ s^{p+1,q} = 0$ as s and π correspond to a direct sum decomposition of B in every bidegree. On the other hand, for δ we have

$$\delta^q = \pi^q \circ d_2 \circ s^q : C^{\bullet,q} \to A^{\bullet,q+1}$$

whose restriction to the summand $C^{p,q}$ is equal to $\pi^{p,q+1} \circ d_2^{p,q} \circ s^{p,q}$. The difference in signs is exactly canceled out by the sign of $(-1)^p$ in the isomorphism γ and the fourth claim is proven.

0G6B **Remark 12.18.7.** Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\mathrm{DoubleComp}(\mathcal{A})$ denote the category of double complexes. We can consider an object $A^{\bullet,\bullet}$ of $\mathrm{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{-1,\bullet} \to A^{0,\bullet} \to A^{1,\bullet} \to \ldots$$

For the variant where we switch the role of the indices, see Remark 12.18.6. In this remark we show that taking the associated total complex is compatible with all the structures on complexes we have studied in the chapter so far.

First, observe that the shift functor on double complexes viewed as complexes of complexes in the manner given above is the functor $[1, 0]$ defined in Remark 12.18.5. By Remark 12.18.5 the functor

$$\mathrm{Tot} : \mathrm{DoubleComp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$$

is compatible with shift functors, in the sense that we have a functorial isomorphism $\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[1] \to \mathrm{Tot}(A^{\bullet,\bullet}[1, 0])$.

Second, if

$$f, g : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$$

are homotopic when f and g are viewed as morphisms of complexes of complexes in the manner given above, then

$$\mathrm{Tot}(f), \mathrm{Tot}(g) : \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet})$$

are homotopic maps of complexes. Indeed, let $h = (h^p)$ be a homotopy between f and g. If we denote $h^{p,q} : A^{p,q} \to B^{p-1,q}$ the component in degree p of h^q, then this means that

$$f^{p,q} - g^{p,q} = d_1^{p-1,q} \circ h^{p,q} + h^{p+1,q} \circ d_1^{p,q}$$

The fact that $h^p : A^{p,\bullet} \to B^{p-1,\bullet}$ is a map of complexes means that

$$d_2^{p-1,q} \circ h^{p,q} = h^{p,q+1} \circ d_2^{p,q}$$

Let us define $h' = ((h')^n)$ the homotopy given by the maps $(h')^n : \mathrm{Tot}^n(A^{\bullet,\bullet}) \to \mathrm{Tot}^{n-1}(B^{\bullet,\bullet})$ using $h^{p,q}$ on the summand $A^{p,q}$ for $p + q = n$. Then we see that

$$d_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ h' + h' \circ d_{\mathrm{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{p,q}$ is equal to

$$d_1^{p-1,q} \circ h^{p,q} + (-1)^{p-1} d_2^{p-1,q} \circ h^{p,q} + h^{p+1,q} \circ d_1^{p,q} + h^{p,q+1} \circ (-1)^p d_2^{p,q}$$

which evaluates to $f^{p,q} - g^{p,q}$ by the equations given above. This proves the second compatibility.

Third, suppose that in the paragraph above we have $f = g$. Then the assignment $h \rightsquigarrow h'$ above is compatible with the identification of Lemma 12.14.9. More precisely, if we view h as a morphism of complexes of complexes $A^{\bullet,\bullet} \to B^{\bullet,\bullet}[-1, 0]$ via this lemma then

$$\mathrm{Tot}(A^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(h)} \mathrm{Tot}(B^{\bullet,\bullet}[-1, 0]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(B^{\bullet,\bullet})[-1]$$

is equal to h' viewed as a morphism of complexes via the lemma. Here γ is the identification of Remark 12.18.5. The verification of this third point is immediate.

Fourth, let

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to C^{\bullet,\bullet} \to 0$$

be a complex of double complexes and suppose we are given splittings $s^p : C^{p,\bullet} \to B^{p,\bullet}$ and $\pi^p : B^{p,\bullet} \to A^{p,\bullet}$ of this as in Lemma 12.14.10 when we view double complexes as complexes of complexes in the manner given above. This on the one hand produces a map

$$\delta : C^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}[0, 1]$$

by the procedure in Lemma 12.14.10. On the other hand taking Tot we obtain a complex

$$0 \to \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet}) \to \mathrm{Tot}(C^{\bullet,\bullet}) \to 0$$

which is termwise split (see below) and hence comes with a morphism

$$\delta' : \mathrm{Tot}(C^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

well defined up to homotopy by Lemmas 12.14.10 and 12.14.12. Claim: these maps agree in the sense that

$$\mathrm{Tot}(C^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(\delta)} \mathrm{Tot}(A^{\bullet,\bullet}[1, 0]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

is equal to δ' where γ is as in Remark 12.18.5. To see this denote $s^{p,q} : C^{p,q} \to B^{\bullet,q}$ and $\pi^{p,q} : B^{p,q} \to A^{p,q}$ the components of s^q and π^q. As splittings $(s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^n(B^{\bullet,\bullet})$ and $(\pi')^n : \mathrm{Tot}^n(B^{\bullet,\bullet}) \to \mathrm{Tot}^n(A^{\bullet,\bullet})$ we use the maps whose components are $s^{p,q}$ and $\pi^{p,q}$ for $p + q = n$. We recall that

$$(\delta')^n = (\pi')^{n+1} \circ d^n_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ (s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^{n+1}(A^{\bullet,\bullet})$$

The restriction of this to the summand $C^{p,q}$ is equal to

$$\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q} + \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q} = \pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q}$$

The equality holds because s^p is a morphism of complexes (with d_2 as differential) and because $\pi^{p,q+1} \circ s^{p,q+1} = 0$ as s and π correspond to a direct sum decomposition of B in every bidegree. On the other hand, for δ we have

$$\delta^p = \pi^p \circ d_1 \circ s^p : C^{p,\bullet} \to A^{p+1,\bullet}$$

whose restriction to the summand $C^{p,q}$ is equal to $\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q}$. Thus we get the same as before which matches with the fact that the isomorphism $\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[1] \to \mathrm{Tot}(A^{\bullet,\bullet}[1, 0])$ is defined without the intervention of signs.

12.19. Filtrations

0120 A nice reference for this material is [**Del71**, Section 1]. (Note that our conventions regarding abelian categories are different.)

0121 **Definition 12.19.1.** Let $\mathcal{A}$ be an abelian category.

(1) A *decreasing filtration* F on an object A is a family $(F^nA)_{n\in\mathbf{Z}}$ of subobjects of A such that
$$A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0$$
(2) A *filtered object of* $\mathcal{A}$ is pair (A, F) consisting of an object A of $\mathcal{A}$ and a decreasing filtration F on A.
(3) A *morphism* $(A, F) \to (B, F)$ *of filtered objects* is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(F^iA) \subset F^iB$ for all $i \in \mathbf{Z}$.
(4) The category of filtered objects is denoted $\mathrm{Fil}(\mathcal{A})$.
(5) Given a filtered object (A, F) and a subobject $X \subset A$ the *induced filtration* on X is the filtration with $F^nX = X \cap F^nA$.
(6) Given a filtered object (A, F) and a surjection $\pi : A \to Y$ the *quotient filtration* is the filtration with $F^nY = \pi(F^nA)$.
(7) A filtration F on an object A is said to be *finite* if there exist n, m such that $F^nA = A$ and $F^mA = 0$.
(8) Given a filtered object (A, F) we say $\bigcap F^iA$ exists if there exists a biggest subobject of A contained in all F^iA. We say $\bigcup F^iA$ exists if there exists a smallest subobject of A containing all F^iA.
(9) The filtration on a filtered object (A, F) is said to be *separated* if $\bigcap F^iA = 0$ and *exhaustive* if $\bigcup F^iA = A$.

By abuse of notation we say that a morphism $f : (A, F) \to (B, F)$ of filtered objects is *injective* if $f : A \to B$ is injective in the abelian category $\mathcal{A}$. Similarly we say f is *surjective* if $f : A \to B$ is surjective in the category $\mathcal{A}$. Being injective (resp. surjective) is equivalent to being a monomorphism (resp. epimorphism) in $\mathrm{Fil}(\mathcal{A})$. By Lemma 12.19.2 this is also equivalent to having zero kernel (resp. cokernel).

0122 **Lemma 12.19.2.** *Let $\mathcal{A}$ be an abelian category. The category of filtered objects $Fil(\mathcal{A})$ has the following properties:*

(1) *It is an additive category.*
(2) *It has a zero object.*
(3) *It has kernels and cokernels, images and coimages.*
(4) *In general it is not an abelian category.*

Proof. It is clear that $\mathrm{Fil}(\mathcal{A})$ is additive with direct sum given by $(A, F) \oplus (B, F) = (A \oplus B, F)$ where $F^p(A \oplus B) = F^pA \oplus F^pB$. The kernel of a morphism $f : (A, F) \to (B, F)$ of filtered objects is the injection $\mathrm{Ker}(f) \subset A$ where $\mathrm{Ker}(f)$ is endowed with the induced filtration. The cokernel of a morphism $f : A \to B$ of filtered objects is the surjection $B \to \mathrm{Coker}(f)$ where $\mathrm{Coker}(f)$ is endowed with the quotient filtration. Since all kernels and cokernels exist, so do all coimages and images. See Example 12.3.13 for the last statement. □

0123 **Definition 12.19.3.** Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects of $\mathcal{A}$ is said to be *strict* if $f(F^iA) = f(A) \cap F^iB$ for all $i \in \mathbf{Z}$.

This also equivalent to requiring that $f^{-1}(F^iB) = F^iA + \mathrm{Ker}(f)$ for all $i \in \mathbf{Z}$. We characterize strict morphisms as follows.

05SI **Lemma 12.19.4.** *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 12.3.12 is an isomorphism.*

Proof. Note that $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism of objects of $\mathcal{A}$, and that part (2) signifies that it is an isomorphism of filtered objects. By the description of kernels and cokernels in the proof of Lemma 12.19.2 we see that the filtration on $\mathrm{Coim}(f)$ is the quotient filtration coming from $A \to \mathrm{Coim}(f)$. Similarly, the filtration on $\mathrm{Im}(f)$ is the induced filtration coming from the injection $\mathrm{Im}(f) \to B$. The definition of strict is exactly that the quotient filtration is the induced filtration. □

05SK **Lemma 12.19.5.** *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a strict monomorphism of filtered objects. Let $g : A \to C$ be a morphism of filtered objects. Then $f \oplus g : A \to B \oplus C$ is a strict monomorphism.*

Proof. Clear from the definitions. □

05SL **Lemma 12.19.6.** *Let $\mathcal{A}$ be an abelian category. Let $f : B \to A$ be a strict epimorphism of filtered objects. Let $g : C \to A$ be a morphism of filtered objects. Then $f \oplus g : B \oplus C \to A$ is a strict epimorphism.*

Proof. Clear from the definitions. □

0124 **Lemma 12.19.7.** *Let $\mathcal{A}$ be an abelian category. Let (A, F), (B, F) be filtered objects. Let $u : A \to B$ be a morphism of filtered objects. If u is injective then u is strict if and only if the filtration on A is the induced filtration. If u is surjective then u is strict if and only if the filtration on B is the quotient filtration.*

Proof. This is immediate from the definition. □

05SJ **Lemma 12.19.8.** *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$, $g : B \to C$ be strict morphisms of filtered objects.*

(1) *In general the composition $g \circ f$ is not strict.*
(2) *If g is injective, then $g \circ f$ is strict.*
(3) *If f is surjective, then $g \circ f$ is strict.*

Proof. Let B a vector space over a field k with basis e_1, e_2, with the filtration $F^nB = B$ for $n < 0$, with $F^0B = ke_1$, and $F^nB = 0$ for $n > 0$. Now take $A = k(e_1 + e_2)$ and $C = B/ke_2$ with filtrations induced by B, i.e., such that $A \to B$ and $B \to C$ are strict (Lemma 12.19.7). Then $F^n(A) = A$ for $n < 0$ and $F^n(A) = 0$ for $n \geq 0$. Also $F^n(C) = C$ for $n \leq 0$ and $F^n(C) = 0$ for $n > 0$. So the (nonzero) composition $A \to C$ is not strict.

Assume g is injective. Then

$$\begin{aligned} g(f(F^pA)) &= g(f(A) \cap F^pB) \\ &= g(f(A)) \cap g(F^p(B)) \\ &= (g \circ f)(A) \cap (g(B) \cap F^pC) \\ &= (g \circ f)(A) \cap F^pC. \end{aligned}$$

The first equality as f is strict, the second because g is injective, the third because g is strict, and the fourth because $(g \circ f)(A) \subset g(B)$.

Assume f is surjective. Then

$$\begin{aligned}(g \circ f)^{-1}(F^i C) &= f^{-1}(F^i B + \mathrm{Ker}(g)) \\ &= f^{-1}(F^i B) + f^{-1}(\mathrm{Ker}(g)) \\ &= F^i A + \mathrm{Ker}(f) + \mathrm{Ker}(g \circ f) \\ &= F^i A + \mathrm{Ker}(g \circ f)\end{aligned}$$

The first equality because g is strict, the second because f is surjective, the third because f is strict, and the last because $\mathrm{Ker}(f) \subset \mathrm{Ker}(g \circ f)$. □

The following lemma says that subobjects of a filtered object have a well defined filtration independent of a choice of writing the object as a cokernel.

0129 **Lemma 12.19.9.** *Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. Let $X \subset Y \subset A$ be subobjects of A. On the object*

$$Y/X = \mathrm{Ker}(A/X \to A/Y)$$

the quotient filtration coming from the induced filtration on Y and the induced filtration coming from the quotient filtration on A/X agree. Any of the morphisms $X \to Y$, $X \to A$, $Y \to A$, $Y \to A/X$, $Y \to Y/X$, $Y/X \to A/X$ are strict (with induced/quotient filtrations).

Proof. The quotient filtration Y/X is given by $F^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX$ because $F^pY = Y \cap F^pA$ and $F^pX = X \cap F^pA$. The induced filtration from the injection $Y/X \to A/X$ is given by

$$\begin{aligned}F^p(Y/X) &= Y/X \cap F^p(A/X) \\ &= Y/X \cap (F^pA + X)/X \\ &= (Y \cap F^pA)/(X \cap F^pA) \\ &= F^pY/F^pX.\end{aligned}$$

Hence the first statement of the lemma. The proof of the other cases is similar. □

05SM **Lemma 12.19.10.** *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : A \to B$ and $g : A \to C$ be morphisms. Then there exists a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{f'} & C \amalg_A B \end{array}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. Set $C \amalg_A B$ equal to $\mathrm{Coker}((g, -f) : A \to C \oplus B)$ in $\mathrm{Fil}(\mathcal{A})$. This cokernel exists, by Lemma 12.19.2. It is a pushout, see Example 12.5.6. Note that $F^p(C \amalg_A B)$ is the image of $F^pC \oplus F^pB$. Hence

$$(f')^{-1}(F^p(C \amalg_A B)) = g(f^{-1}(F^pB)) + F^pC$$

Whence the last statement. □

05SN **Lemma 12.19.11.** *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : B \to A$ and $g : C \to A$ be morphisms. Then there exists a fibre product*

$$\begin{array}{ccc} B \times_A C & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & A \end{array}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. This lemma is dual to Lemma 12.19.10. □

Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. We denote $\mathrm{gr}^p_F(A) = \mathrm{gr}^p(A)$ the object $F^pA/F^{p+1}A$ of $\mathcal{A}$. This defines an additive functor

$$\mathrm{gr}^p : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad (A, F) \longmapsto \mathrm{gr}^p(A).$$

Recall that we have defined the category $\mathrm{Gr}(\mathcal{A})$ of graded objects of $\mathcal{A}$ in Section 12.16. For (A, F) in $\mathrm{Fil}(\mathcal{A})$ we may set

$$\mathrm{gr}(A) = \text{the graded object of } \mathcal{A} \text{ whose } p\text{th graded piece is } \mathrm{gr}^p(A)$$

and if $\mathcal{A}$ has countable direct sums, then we simply have

$$\mathrm{gr}(A) = \bigoplus \mathrm{gr}^p(A)$$

This defines an additive functor

$$\mathrm{gr} : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A}), \quad (A, F) \longmapsto \mathrm{gr}(A).$$

05SP **Lemma 12.19.12.** *Let $\mathcal{A}$ be an abelian category.*

(1) *Let A be a filtered object and $X \subset A$. Then for each p the sequence*

$$0 \to gr^p(X) \to gr^p(A) \to gr^p(A/X) \to 0$$

is exact (with induced filtration on X and quotient filtration on A/X).

(2) *Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. Then for each p the sequences*

$$0 \to gr^p(\mathrm{Ker}(f)) \to gr^p(A) \to gr^p(\mathrm{Coim}(f)) \to 0$$

and

$$0 \to gr^p(\mathrm{Im}(f)) \to gr^p(B) \to gr^p(\mathrm{Coker}(f)) \to 0$$

are exact.

Proof. We have $F^{p+1}X = X \cap F^{p+1}A$, hence map $\mathrm{gr}^p(X) \to \mathrm{gr}^p(A)$ is injective. Dually the map $\mathrm{gr}^p(A) \to \mathrm{gr}^p(A/X)$ is surjective. The kernel of $F^pA/F^{p+1}A \to A/X + F^{p+1}A$ is clearly $F^{p+1}A + X \cap F^pA/F^{p+1}A = F^pX/F^{p+1}X$ hence exactness in the middle. The two short exact sequence of (2) are special cases of the short exact sequence of (1). □

0127 **Lemma 12.19.13.** *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of finite filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism,*
(3) *$gr(\mathrm{Coim}(f)) \to gr(\mathrm{Im}(f))$ is an isomorphism,*
(4) *the sequence $gr(\mathrm{Ker}(f)) \to gr(A) \to gr(B)$ is exact,*
(5) *the sequence $gr(A) \to gr(B) \to gr(\mathrm{Coker}(f))$ is exact, and*

(6) *the sequence*

$$0 \to gr(\mathrm{Ker}(f)) \to gr(A) \to gr(B) \to gr(\mathrm{Coker}(f)) \to 0$$

is exact.

Proof. The equivalence of (1) and (2) is Lemma 12.19.4. By Lemma 12.19.12 we see that (4), (5), (6) imply (3) and that (3) implies (4), (5), (6). Hence it suffices to show that (3) implies (2). Thus we have to show that if $f : A \to B$ is an injective and surjective map of finite filtered objects which induces and isomorphism $\mathrm{gr}(A) \to \mathrm{gr}(B)$, then f induces an isomorphism of filtered objects. In other words, we have to show that $f(F^pA) = F^pB$ for all p. As the filtrations are finite we may prove this by descending induction on p. Suppose that $f(F^{p+1}A) = F^{p+1}B$. Then commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{p+1}A & \longrightarrow & F^pA & \longrightarrow & \mathrm{gr}^p(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{gr}^p(f)} & & \\ 0 & \longrightarrow & F^{p+1}B & \longrightarrow & F^pB & \longrightarrow & \mathrm{gr}^p(B) & \longrightarrow & 0 \end{array}$$

and the five lemma imply that $f(F^pA) = F^pB$. □

0128 **Lemma 12.19.14.** *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume $\alpha : A \to B$ and $\beta : B \to C$ are strict morphisms of filtered objects. Then $gr(\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)) = \mathrm{Ker}(gr(\beta))/\mathrm{Im}(gr(\alpha)))$.*

Proof. This follows formally from Lemma 12.19.12 and the fact that $\mathrm{Coim}(\alpha) \cong \mathrm{Im}(\alpha)$ and $\mathrm{Coim}(\beta) \cong \mathrm{Im}(\beta)$ by Lemma 12.19.4. □

05QH **Lemma 12.19.15.** *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume A, B, C have finite filtrations and that $gr(A) \to gr(B) \to gr(C)$ is exact. Then*

(1) *for each $p \in \mathbf{Z}$ the sequence $gr^p(A) \to gr^p(B) \to gr^p(C)$ is exact,*
(2) *for each $p \in \mathbf{Z}$ the sequence $F^p(A) \to F^p(B) \to F^p(C)$ is exact,*
(3) *for each $p \in \mathbf{Z}$ the sequence $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact,*
(4) *the maps $A \to B$ and $B \to C$ are strict, and*
(5) *$A \to B \to C$ is exact (as a sequence in $\mathcal{A}$).*

Proof. Part (1) is immediate from the definitions. We will prove (3) by induction on the length of the filtrations. If each of A, B, C has only one nonzero graded part, then (3) holds as $\mathrm{gr}(A) = A$, etc. Let n be the largest integer such that at least one of F^nA, F^nB, F^nC is nonzero. Set $A' = A/F^nA$, $B' = B/F^nB$, $C' = C/F^nC$ with induced filtrations. Note that $\mathrm{gr}(A) = F^nA \oplus \mathrm{gr}(A')$ and similarly for B and C. The induction hypothesis applies to $A' \to B' \to C'$, which implies that $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact for $p \geq n$. To conclude the same for

$p = n+1$, i.e., to prove that $A \to B \to C$ is exact we use the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^nA & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^nB & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^nC & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences of objects of $\mathcal{A}$. The proof of (2) is dual. Of course (5) follows from (2).

To prove (4) denote $f : A \to B$ and $g : B \to C$ the given morphisms. We know that $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C))$ by (2) and $f(A) = \mathrm{Ker}(g)$ by (5). Hence $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C)) = \mathrm{Ker}(g) \cap F^p(B) = f(A) \cap F^p(B)$ which proves that f is strict. The proof that g is strict is dual to this. □

12.20. Spectral sequences

011M A nice discussion of spectral sequences may be found in [**Eis95**]. See also [**McC01**], [**Lan02**], etc.

011N **Definition 12.20.1.** Let $\mathcal{A}$ be an abelian category.

(1) A *spectral sequence in* $\mathcal{A}$ is given by a system $(E_r, d_r)_{r \geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$ and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ for $r \geq 1$.

(2) A *morphism of spectral sequences* $f : (E_r, d_r)_{r \geq 1} \to (E'_r, d'_r)_{r \geq 1}$ is given by a family of morphisms $f_r : E_r \to E'_r$ such that $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} is the morphism induced by f_r via the identifications $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \mathrm{Ker}(d'_r)/\mathrm{Im}(d'_r)$.

We will sometimes loosen this definition somewhat and allow E_{r+1} to be an object with a given isomorphism $E_{r+1} \to \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. In addition we sometimes have a system $(E_r, d_r)_{r \geq r_0}$ for some $r_0 \in \mathbf{Z}$ satisfying the properties of the definition above for indices $\geq r_0$. We will also call this a spectral sequence since by a simple renumbering it falls under the definition anyway. In fact, the cases $r_0 = 0$ and $r_0 = -1$ can be found in the literature.

Given a spectral sequence $(E_r, d_r)_{r \geq 1}$ we define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

by the following simple procedure. Set $B_2 = \mathrm{Im}(d_1)$ and $Z_2 = \mathrm{Ker}(d_1)$. Then it is clear that $d_2 : Z_2/B_2 \to Z_2/B_2$. Hence we can define B_3 as the unique subobject of E_1 containing B_2 such that B_3/B_2 is the image of d_2. Similarly we can define Z_3 as the unique subobject of E_1 containing B_2 such that Z_3/B_2 is the kernel of d_2. And so on and so forth. In particular we have

$$E_r = Z_r/B_r$$

for all $r \geq 1$. In case the spectral sequence starts at $r = r_0$ then we can similarly construct B_i, Z_i as subobjects in E_{r_0}. In fact, in the literature one sometimes finds the notation

$$0 = B_r(E_r) \subset B_{r+1}(E_r) \subset B_{r+2}(E_r) \subset \ldots \subset Z_{r+2}(E_r) \subset Z_{r+1}(E_r) \subset Z_r(E_r) = E_r$$

to denote the filtration described above but starting with E_r.

011O **Definition 12.20.2.** Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence.

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ of E_1 exist then we define the *limit*[3] of the spectral sequence to be the object $E_\infty = Z_\infty/B_\infty$.
(2) We say that the spectral sequence *degenerates at* E_r if the differentials $d_r, d_{r+1}, \ldots$ are all zero.

Note that if the spectral sequence degenerates at E_r, then we have $E_r = E_{r+1} = \ldots = E_\infty$ (and the limit exists of course). Also, almost any abelian category we will encounter has countable sums and intersections.

0AMI **Remark 12.20.3** (Variant)**.** It is often the case that the terms of a spectral sequence have additional structure, for example a grading or a bigrading. To accommodate this (and to get around certain technical issues) we introduce the following notion. Let $\mathcal{A}$ be an abelian category. Let $(T_r)_{r\geq 1}$ be a sequence of *translation* or *shift* functors, i.e., $T_r : \mathcal{A} \to \mathcal{A}$ is an isomorphism of categories. In this setting a *spectral sequence* is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to T_r E_r$ is a morphism such that $T_r d_r \circ d_r = 0$ so that

$$\ldots \longrightarrow T_r^{-1} E_r \xrightarrow{T_r^{-1} d_r} E_r \xrightarrow{d_r} T_r E_r \xrightarrow{T_r d_r} T_r^2 E_r \longrightarrow \ldots$$

is a complex and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(T_r^{-1} d_r)$ for $r \geq 1$. It is clear what a *morphism of spectral sequences* means in this setting. In this setting we can still define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

and Z_∞ and B_∞ (if they exist) as above.

12.21. Spectral sequences: exact couples

011P

011Q **Definition 12.21.1.** Let $\mathcal{A}$ be an abelian category.

(1) An *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and α, f, g are morphisms as in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ & {\scriptstyle f}\nwarrow \quad \swarrow{\scriptstyle g} & \\ & E & \end{array}$$

with the property that the kernel of each arrow is the image of its predecessor. So $\mathrm{Ker}(\alpha) = \mathrm{Im}(f)$, $\mathrm{Ker}(f) = \mathrm{Im}(g)$, and $\mathrm{Ker}(g) = \mathrm{Im}(\alpha)$.
(2) A *morphism of exact couples* $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ is given by morphisms $t_A : A \to A'$ and $t_E : E \to E'$ such that $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$, and $g' \circ t_A = t_E \circ g$.

011R **Lemma 12.21.2.** *Let (A, E, α, f, g) be an exact couple in an abelian category $\mathcal{A}$. Set*

(1) *$d = g \circ f : E \to E$ so that $d \circ d = 0$,*

[3] This notation is not universally accepted. In some references an additional pair of subobjects Z_∞ and B_∞ of E_1 such that $0 = B_1 \subset B_2 \subset \ldots \subset B_\infty \subset Z_\infty \subset \ldots \subset Z_2 \subset Z_1 = E_1$ is part of the data comprising a spectral sequence!

(2) $E' = \mathrm{Ker}(d)/\mathrm{Im}(d)$,
(3) $A' = \mathrm{Im}(\alpha)$,
(4) $\alpha' : A' \to A'$ *induced by* α,
(5) $f' : E' \to A'$ *induced by* f,
(6) $g' : A' \to E'$ *induced by "*$g \circ \alpha^{-1}$*"*.

Then we have

(1) $\mathrm{Ker}(d) = f^{-1}(\mathrm{Ker}(g)) = f^{-1}(\mathrm{Im}(\alpha))$,
(2) $\mathrm{Im}(d) = g(\mathrm{Im}(f)) = g(\mathrm{Ker}(\alpha))$,
(3) $(A', E', \alpha', f', g')$ *is an exact couple.*

Proof. Omitted. □

Hence it is clear that given an exact couple (A, E, α, f, g) we get a spectral sequence by setting $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

011S **Definition 12.21.3.** Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. The *spectral sequence associated to the exact couple* is the spectral sequence $(E_r, d_r)_{r \geq 1}$ with $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

011T **Lemma 12.21.4.** *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Let $(E_r, d_r)_{r \geq 1}$ be the spectral sequence associated to the exact couple. In this case we have*

$$0 = B_1 \subset \ldots \subset B_{r+1} = g(\mathrm{Ker}(\alpha^r)) \subset \ldots \subset Z_{r+1} = f^{-1}(\mathrm{Im}(\alpha^r)) \subset \ldots \subset Z_1 = E$$

and the map $d_{r+1} : E_{r+1} \to E_{r+1}$ is described by the following rule: For any (test) object T of $\mathcal{A}$ and any elements $x : T \to Z_{r+1}$ and $y : T \to A$ such that $f \circ x = \alpha^r \circ y$ we have

$$d_{r+1} \circ \overline{x} = \overline{g \circ y}$$

where $\overline{x} : T \to E_{r+1}$ is the induced morphism.

Proof. Omitted. □

Note that in the situation of the lemma we obviously have

$$B_\infty = g\left(\bigcup\nolimits_r \mathrm{Ker}(\alpha^r)\right) \subset Z_\infty = f^{-1}\left(\bigcap\nolimits_r \mathrm{Im}(\alpha^r)\right)$$

provided $\bigcup \mathrm{Ker}(\alpha^r)$ and $\bigcap \mathrm{Im}(\alpha^r)$ exist. This produces as limit $E_\infty = Z_\infty / B_\infty$, see Definition 12.20.2.

0AMJ **Remark 12.21.5** (Variant)**.** Let $\mathcal{A}$ be an abelian category. Let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. We will indicate the n-fold compositions by $S^n A$ and $T^n A$ for $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbf{Z}$. In this situation an *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and $\alpha : A \to T^{-1}A$, $f : E \to A$, $g : A \to SE$ are morphisms such that

$$TE \xrightarrow{Tf} TA \xrightarrow{T\alpha} A \xrightarrow{g} SE \xrightarrow{Sf} SA$$

is an exact complex. Let's visualize this as follows

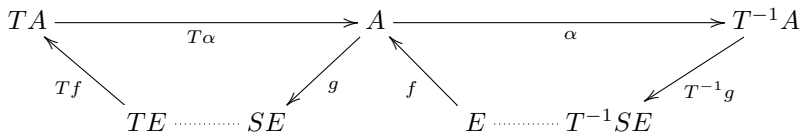

We set $d = g \circ f : E \to SE$. Then $d \circ S^{-1}d = g \circ f \circ S^{-1}g \circ S^{-1}f = 0$ because $f \circ S^{-1}g = 0$. Set $E' = \mathrm{Ker}(d)/\mathrm{Im}(S^{-1}d)$. Set $A' = \mathrm{Im}(T\alpha)$. Let $\alpha' : A' \to T^{-1}A'$ induced by α. Let $f' : E' \to A'$ be induced by f which works because $f(\mathrm{Ker}(d)) \subset \mathrm{Ker}(g) = \mathrm{Im}(T\alpha)$. Finally, let $g' : A' \to TSE'$ induced by "$Tg \circ (T\alpha)^{-1}$"[4].

In exactly the same way as above we find

(1) $\mathrm{Ker}(d) = f^{-1}(\mathrm{Ker}(g)) = f^{-1}(\mathrm{Im}(T\alpha))$,
(2) $\mathrm{Im}(d) = g(\mathrm{Im}(f)) = g(\mathrm{Ker}(\alpha))$,
(3) $(A', E', \alpha', f', g')$ is an exact couple for the shift functors TS and T.

We obtain a spectral sequence (as in Remark 12.20.3) with $E_1 = E$, $E_2 = E'$, etc, with $d_r : E_r \to T^{r-1}SE_r$ for all $r \geq 1$. Lemma 12.21.4 tells us that

$$SB_{r+1} = g(\mathrm{Ker}(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha \circ \alpha))$$

and

$$Z_{r+1} = f^{-1}(\mathrm{Im}(T\alpha \circ T^2\alpha \circ \ldots \circ T^r\alpha))$$

in this situation. The description of the map d_{r+1} is similar to that given in the lemma. (It may be easier to use these explicit descriptions to prove one gets a spectral sequence from such an exact couple.)

12.22. Spectral sequences: differential objects

011U

011V **Definition 12.22.1.** Let $\mathcal{A}$ be an abelian category. A *differential object* of $\mathcal{A}$ is a pair (A, d) consisting of an object A of $\mathcal{A}$ endowed with a selfmap d such that $d \circ d = 0$. A *morphism of differential objects* $(A, d) \to (B, d)$ is given by a morphism $\alpha : A \to B$ such that $d \circ \alpha = \alpha \circ d$.

011W **Lemma 12.22.2.** *Let $\mathcal{A}$ be an abelian category. The category of differential objects of $\mathcal{A}$ is abelian.*

Proof. Omitted. □

011X **Definition 12.22.3.** For a differential object (A, d) we denote

$$H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(d)$$

its *homology*.

011Y **Lemma 12.22.4.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ be a short exact sequence of differential objects. Then we get an exact homology sequence*

$$\ldots \to H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to \ldots$$

Proof. Apply Lemma 12.13.12 to the short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0
\end{array}$$

where the vertical arrows are d. □

[4]This works because $TSE' = \mathrm{Ker}(TSd)/\mathrm{Im}(Td)$ and $Tg(\mathrm{Ker}(T\alpha)) = Tg(\mathrm{Im}(Tf)) = \mathrm{Im}(T(d))$ and $TS(d)(\mathrm{Im}(Tg)) = \mathrm{Im}(TSg \circ TSf \circ Tg) = 0$.

We come to an important example of a spectral sequence. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : (A, d) \to (A, d)$ be an endomorphism of this differential object. If we assume α injective, then we get a short exact sequence

$$0 \to (A, d) \to (A, d) \to (A/\alpha A, d) \to 0$$

of differential objects. By the Lemma 12.22.4 we get an exact couple

$$\begin{array}{ccc} H(A,d) & \xrightarrow{\overline{\alpha}} & H(A,d) \\ & {\scriptstyle f}\nwarrow \quad \swarrow{\scriptstyle g} & \\ & H(A/\alpha A, d) & \end{array}$$

where g is the canonical map and f is the map defined in the snake lemma. Thus we get an associated spectral sequence! Since in this case we have $E_1 = H(A/\alpha A, d)$ we see that it makes sense to define $E_0 = A/\alpha A$ and $d_0 = d$. In other words, we start the spectral sequence with $r = 0$. According to our conventions in Section 12.20 we define a sequence of subobjects

$$0 = B_0 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_0 = E_0$$

with the property that $E_r = Z_r/B_r$. Namely we have for $r \geq 1$ that

(1) B_r is the image of $(\alpha^{r-1})^{-1}(dA)$ under the natural map $A \to A/\alpha A$,
(2) Z_r is the image of $d^{-1}(\alpha^r A)$ under the natural map $A \to A/\alpha A$, and
(3) $d_r : E_r \to E_r$ is given as follows: given an element $z \in Z_r$ choose an element $y \in A$ such that $d(z) = \alpha^r(y)$. Then $d_r(z + B_r + \alpha A) = y + B_r + \alpha A$.

Warning: It is not necessarily the case that $\alpha A \subset (\alpha^{r-1})^{-1}(dA)$, nor $\alpha A \subset d^{-1}(\alpha^r A)$. It is true that $(\alpha^{r-1})^{-1}(dA) \subset d^{-1}(\alpha^r A)$. We have

$$E_r = \frac{d^{-1}(\alpha^r A) + \alpha A}{(\alpha^{r-1})^{-1}(dA) + \alpha A}.$$

It is not hard to verify directly that (1) – (3) give a spectral sequence.

011Z **Definition 12.22.5.** Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : A \to A$ be an injective selfmap of A which commutes with d. The *spectral sequence associated to* (A, d, α) is the spectral sequence $(E_r, d_r)_{r \geq 0}$ described above.

0AMK **Remark 12.22.6** (Variant)**.** Let $\mathcal{A}$ be an abelian category and let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. Assume that $TS = ST$ as functors. Consider pairs (A, d) consisting of an object A of $\mathcal{A}$ and a morphism $d : A \to SA$ such that $d \circ S^{-1}d = 0$. The category of these objects is abelian. We define $H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(S^{-1}d)$ and we observe that $H(SA, Sd) = SH(A, d)$ (canonical isomorphism). Given a short exact sequence

$$0 \to (A, d) \to (B, d) \to (C, d) \to 0$$

we obtain a long exact homology sequence

$$\ldots \to S^{-1}H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to SH(A, d) \to \ldots$$

(note the shifts in the boundary maps). Since $ST = TS$ the functor T defines a shift functor on pairs by setting $T(A, d) = (TA, Td)$. Next, let $\alpha : (A, d) \to T^{-1}(A, d)$

be injective with cokernel (Q, d). Then we get an exact couple as in Remark 12.21.5 with shift functors TS and T given by

$$(H(A, d), S^{-1}H(Q, d), \overline{\alpha}, f, g)$$

where $\overline{\alpha} : H(A, d) \to T^{-1}H(A, d)$ is induced by α, the map $f : S^{-1}H(Q, d) \to H(A, d)$ is the boundary map and $g : H(A, d) \to TH(Q, d) = TS(S^{-1}H(Q, d))$ is induced by the quotient map $A \to TQ$. Thus we get a spectral sequence as above with $E_1 = S^{-1}H(Q, d)$ and differentials $d_r : E_r \to T^r SE_r$. As above we set $E_0 = S^{-1}Q$ and $d_0 : E_0 \to SE_0$ given by $S^{-1}d : S^{-1}Q \to Q$. If according to our conventions we define $B_r \subset Z_r \subset E_0$, then we have for $r \geq 1$ that

(1) SB_r is the image of

$$(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha)^{-1} \operatorname{Im}(T^{-r}S^{-1}d)$$

under the natural map $T^{-1}A \to Q$,

(2) Z_r is the image of

$$(S^{-1}T^{-1}d)^{-1} \operatorname{Im}(\alpha \circ \ldots \circ T^{r-1}\alpha)$$

under the natural map $S^{-1}T^{-1}A \to S^{-1}Q$.

The differentials can be described as follows: if $x \in Z_r$, then pick $x' \in S^{-1}T^{-1}A$ mapping to x. Then $S^{-1}T^{-1}d(x')$ is $(\alpha \circ \ldots \circ T^{r-1}\alpha)(y)$ for some $y \in T^{r-1}A$. Then $d_r(x) \in T^r SE_r$ is represented by the class of the image of y in $T^r SE_0 = T^r Q$ modulo $T^r SB_r$.

12.23. Spectral sequences: filtered differential objects

012A We can build a spectral sequence starting with a filtered differential object.

012B **Definition 12.23.1.** Let $\mathcal{A}$ be an abelian category. A *filtered differential object* (K, F, d) is a filtered object (K, F) of $\mathcal{A}$ endowed with an endomorphism $d : (K, F) \to (K, F)$ whose square is zero: $d \circ d = 0$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 12.16.3). Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Note that each $F^n K$ is a differential object by itself. Consider the object $A = \bigoplus F^n K$ and endow it with a differential d by using d on each summand. Then (A, d) is a differential object of $\mathcal{A}$ which comes equipped with a grading. Consider the map

$$\alpha : A \to A$$

which is given by the inclusions $F^n K \to F^{n-1}K$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)$. Hence, by Definition 12.22.5 we get a spectral sequence. We will call this *the spectral sequence associated to the filtered differential object* (K, F, d).

Let us figure out the terms of this spectral sequence. First, note that $A/\alpha A = \operatorname{gr}(K)$ endowed with its differential $d = \operatorname{gr}(d)$. Hence we see that

$$E_0 = \operatorname{gr}(K), \quad d_0 = \operatorname{gr}(d).$$

Hence the homology of the graded differential object $\operatorname{gr}(K)$ is the next term:

$$E_1 = H(\operatorname{gr}(K), \operatorname{gr}(d)).$$

In addition we see that E_0 is a graded object of $\mathcal{A}$ and that d_0 is compatible with the grading. Hence clearly E_1 is a graded object as well. But it turns out that the differential d_1 does not preserve this grading; instead it shifts the degree by 1.

To work this out precisely, we define

$$Z_r^p = \frac{F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K}{F^{p+1}K}$$

and

$$B_r^p = \frac{F^pK \cap d(F^{p-r+1}K) + F^{p+1}K}{F^{p+1}K}.$$

This notation, although quite natural, seems to be different from the notation in most places in the literature. Perhaps it does not matter, since the literature does not seem to have a consistent choice of notation either. With these choices we see that $B_r \subset E_0$, resp. $Z_r \subset E_0$ (as defined in Section 12.22) is equal to $\bigoplus_p B_r^p$, resp. $\bigoplus_p Z_r^p$. Hence if we define

$$E_r^p = Z_r^p/B_r^p$$

for $r \geq 0$ and $p \in \mathbf{Z}$, then we have $E_r = \bigoplus_p E_r^p$. We can define a differential $d_r^p : E_r^p \to E_r^{p+r}$ by the rule

$$z + F^{p+1}K \longmapsto dz + F^{p+r+1}K$$

where $z \in F^pK \cap d^{-1}(F^{p+r}K)$.

012C **Lemma 12.23.2.** *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ in $Gr(\mathcal{A})$ associated to (K, F, d) such that $d_r : E_r \to E_r[r]$ for all r and such that the graded pieces E_r^p and maps $d_r^p : E_r^p \to E_r^{p+r}$ are as given above. Furthermore, $E_0^p = gr^pK$, $d_0^p = gr^p(d)$, and $E_1^p = H(gr^pK, gr^p(d))$.*

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the object $A = (F^{p+1}K)$ of $\mathrm{Gr}(\mathcal{A})$, i.e., we put $F^{p+1}K$ in degree p (the funny shift in numbering to get numbering correct later on). We endow it with a differential d by using d on each component. Then (A, d) is a differential object of $\mathrm{Gr}(\mathcal{A})$. Consider the map

$$\alpha : A \to A[-1]$$

which is given in degree p by the inclusions $F^{p+1}A \to F^pA$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)[-1]$. Hence, we can apply Remark 12.22.6 with $S = \mathrm{id}$ and $T = [1]$. The corresponding spectral sequence $(E_r, d_r)_{r\geq 0}$ in $\mathrm{Gr}(\mathcal{A})$ is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^p)$ is an object of $\mathrm{Gr}(\mathcal{A})$. Then, since $T^rS = [r]$ we have $d_r : E_r \to E_r[r]$ which means that $d_r^p : E_r^p \to E_r^{p+r}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have $E_0 = \mathrm{Coker}(\alpha : A \to A[-1])$ and by our choice of numbering above this gives $E_0^p = \mathrm{gr}^pK$. The first differential is given by $d_0^p = \mathrm{gr}^p d : E_0^p \to E_0^p$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 12.22.6 translates into a straightforward manner into the formulae for Z_r^p and B_r^p given above. □

012D **Lemma 12.23.3.** *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to (K, F, d) has*

$$d_1^p : E_1^p = H(gr^p K) \longrightarrow H(gr^{p+1} K) = E_1^{p+1}$$

equal to the boundary map in homology associated to the short exact sequence of differential objects

$$0 \to gr^{p+1} K \to F^p K/F^{p+2} K \to gr^p K \to 0.$$

Proof. This is clear from the formula for the differential d_1^p given just above Lemma 12.23.2. □

012E **Definition 12.23.4.** Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The *induced filtration* on $H(K, d)$ is the filtration defined by $F^p H(K, d) = \mathrm{Im}(H(F^p K, d) \to H(K, d))$.

Writing out what this means we see that

$$F^p H(K, d) = \frac{\mathrm{Ker}(d) \cap F^p K + \mathrm{Im}(d)}{\mathrm{Im}(d)}$$

and hence we see that

$$\mathrm{gr}^p H(K) = \frac{\mathrm{Ker}(d) \cap F^p K + \mathrm{Im}(d)}{\mathrm{Ker}(d) \cap F^{p+1} K + \mathrm{Im}(d)} = \frac{\mathrm{Ker}(d) \cap F^p K}{\mathrm{Ker}(d) \cap F^{p+1} K + \mathrm{Im}(d) \cap F^p K}$$

012F **Lemma 12.23.5.** *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. If Z_∞^p and B_∞^p exist (see proof), then*

1. *the limit E_∞ exists and is graded having $E_\infty^p = Z_\infty^p/B_\infty^p$ in degree p, and*
2. *the associated graded $gr(H(K))$ of the cohomology of K is a graded subquotient of the graded limit object E_∞.*

Proof. The objects Z_∞, B_∞, and the limit $E_\infty = Z_\infty/B_\infty$ of Definition 12.20.2 are objects of $\mathrm{Gr}(\mathcal{A})$ by our construction of the spectral sequence in the proof of Lemma 12.23.2. Since Z_r^p, resp. B_r^p is the pth graded part of Z_r, resp. B_r, if we assume that

$$Z_\infty^p = \bigcap\nolimits_r Z_r^p = \frac{\bigcap_r (F^p K \cap d^{-1}(F^{p+r} K) + F^{p+1} K)}{F^{p+1} K}$$

and

$$B_\infty^p = \bigcup\nolimits_r B_r^p = \frac{\bigcup_r (F^p K \cap d(F^{p-r+1} K) + F^{p+1} K)}{F^{p+1} K}.$$

exist, then Z_∞ and B_∞ exist with degree p parts Z_∞^p and B_∞^p (follows from an elementary argument about unions and intersections of graded subobjects). Thus

$$E_\infty^p = \frac{\bigcap_r (F^p K \cap d^{-1}(F^{p+r} K) + F^{p+1} K)}{\bigcup_r (F^p K \cap d(F^{p-r+1} K) + F^{p+1} K)}.$$

where the top and bottom exist. We have

012G (12.23.5.1) $$\mathrm{Ker}(d) \cap F^p K + F^{p+1} K \subset \bigcap\nolimits_r \left(F^p K \cap d^{-1}(F^{p+r} K) + F^{p+1} K\right)$$

and

012H (12.23.5.2) $$\bigcup\nolimits_r \left(F^p K \cap d(F^{p-r+1} K) + F^{p+1} K\right) \subset \mathrm{Im}(d) \cap F^p K + F^{p+1} K.$$

Thus a subquotient of E_∞^p is

$$\frac{\mathrm{Ker}(d) \cap F^pK + F^{p+1}K}{\mathrm{Im}(d) \cap F^pK + F^{p+1}K} = \frac{\mathrm{Ker}(d) \cap F^pK}{\mathrm{Im}(d) \cap F^pK + \mathrm{Ker}(d) \cap F^{p+1}K}$$

Comparing with the formula given for $\mathrm{gr}^p H(K)$ in the discussion following Definition 12.23.4 we conclude. □

012I **Definition 12.23.6.** Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. We say the spectral sequence associated to (K, F, d)

(1) *weakly converges to* $H(K)$ if $\mathrm{gr}H(K) = E_\infty$ via Lemma 12.23.5,
(2) *abuts to* $H(K)$ if it weakly converges to $H(K)$ and we have $\bigcap F^pH(K) = 0$ and $\bigcup F^pH(K) = H(K)$,

Unfortunately, it seems hard to find a consistent terminology for these notions in the literature.

012J **Lemma 12.23.7.** *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated spectral sequence*

(1) *weakly converges to $H(K)$ if and only if for every $p \in \mathbf{Z}$ we have equality in equations (12.23.5.2) and (12.23.5.1),*
(2) *abuts to $H(K)$ if and only if it weakly converges to $H(K)$ and $\bigcap_p(\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)) = \mathrm{Im}(d)$ and $\bigcup_p(\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)) = \mathrm{Ker}(d)$.*

Proof. Immediate from the discussions above. □

12.24. Spectral sequences: filtered complexes

012K

012L **Definition 12.24.1.** Let $\mathcal{A}$ be an abelian category. A *filtered complex $K^\bullet$ of $\mathcal{A}$* is a complex of $\mathrm{Fil}(\mathcal{A})$ (see Definition 12.19.1).

We will denote the filtration on the objects by F. Thus F^pK^n denotes the pth step in the filtration of the nth term of the complex. Note that each $F^pK^\bullet$ is a complex of $\mathcal{A}$. Hence we could also have defined a filtered complex as a filtered object in the (abelian) category of complexes of $\mathcal{A}$. In particular $\mathrm{gr}K^\bullet$ is a graded object of the category of complexes of $\mathcal{A}$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 12.16.3). Let us denote d the differential of K. Forgetting the grading we can think of $\bigoplus K^n$ as a filtered differential object of $\mathcal{A}$. Hence according to Section 12.23 we obtain a spectral sequence $(E_r, d_r)_{r\geq 0}$. In this section we work out the terms of this spectral sequence, and we endow the terms of this spectral sequence with additional structure coming from the grading of K.

First we point out that $E_0^p = \mathrm{gr}^pK^\bullet$ is a complex and hence is graded. Thus E_0 is bigraded in a natural way. It is customary to use the bigrading

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q}, \quad E_0^{p,q} = \mathrm{gr}^pK^{p+q}$$

The idea is that $p+q$ should be thought of as the *total degree* of the (co)homology classes. Also, p is called the *filtration degree*, and q is called the *complementary degree*. The differential d_0 is compatible with this bigrading in the following way

$$d_0 = \bigoplus d_0^{p,q}, \quad d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}.$$

Namely, d_0^p is just the differential on the complex $\mathrm{gr}^p K^\bullet$ (which occurs as $\mathrm{gr}^p E_0$ just shifted a bit).

To go further we identify the objects B_r^p and Z_r^p introduced in Section 12.23 as graded objects and we work out the corresponding decompositions of the differentials. We do this in a completely straightforward manner, but again we warn the reader that our notation is not the same as notation found elsewhere. We define

$$Z_r^{p,q} = \frac{F^p K^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^p K^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and of course $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. With these definitions it is completely clear that $Z_r^p = \bigoplus_q Z_r^{p,q}$, $B_r^p = \bigoplus_q B_r^{p,q}$, and $E_r^p = \bigoplus_q E_r^{p,q}$. Moreover, we have

$$0 \subset \ldots \subset B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset E_0^{p,q}$$

Also, the map d_r^p decomposes as the direct sum of the maps

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad z + F^{p+1}K^{p+q} \mapsto dz + F^{p+r+1}K^{p+q+1}$$

where $z \in F^p K^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})$.

012M **Lemma 12.24.2.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ in the category of bigraded objects of $\mathcal{A}$ associated to $(K^\bullet, F)$ such that d_r has bidegree $(r, -r+1)$ and such that E_r has bigraded pieces $E_r^{p,q}$ and maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ as given above. Furthermore, we have $E_0^{p,q} = gr^p(K^{p+q})$, $d_0^{p,q} = gr^p(d^{p+q})$, and $E_1^{p,q} = H^{p+q}(gr^p(K^\bullet))$.*

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the bigraded object $A = (F^{p+1}K^{p+1+q})$ of $\mathcal{A}$, i.e., we put $F^{p+1}K^{p+1+q}$ in degree (p,q) (the funny shift in numbering to get numbering correct later on). We endow it with a differential $d : A \to A[0,1]$ by using d on each component. Then (A, d) is a differential bigraded object. Consider the map

$$\alpha : A \to A[-1, 1]$$

which is given in degree (p,q) by the inclusion $F^{p+1}K^{p+1+q} \to F^pK^{p+1+q}$. This is an injective morphism of differential objects $\alpha : (A,d) \to (A,d)[-1,1]$. Hence, we can apply Remark 12.22.6 with $S = [0,1]$ and $T = [1,-1]$. The corresponding spectral sequence $(E_r, d_r)_{r\geq 0}$ of bigraded objects is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^{p,q})$. Then, since $T^r S = [r, -r+1]$ we have $d_r : E_r \to E_r[r, -r+1]$ which means that $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have

$$E_0 = \mathrm{Coker}(\alpha : A \to A[-1,1])[0,-1] = \mathrm{Coker}(\alpha[0,-1] : A[0,-1] \to A[-1,0])$$

and by our choice of numbering above this gives

$$E_0^{p,q} = \mathrm{Coker}(F^{p+1}K^{p+q} \to F^pK^{p+q}) = \mathrm{gr}^pK^{p+q}$$

The first differential is given by $d_0^{p,q} = \mathrm{gr}^p d^{p+q} : E_0^{p,q} \to E_0^{p,q+1}$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 12.22.6 translates into a straightforward manner into the formulae for $Z_r^{p,q}$ and $B_r^{p,q}$ given above. □

012N **Lemma 12.24.3.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$.*

(1) *The map*

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(gr^p(K^\bullet)) \longrightarrow E_1^{p+1,q} = H^{p+q+1}(gr^{p+1}(K^\bullet))$$

is equal to the boundary map in cohomology associated to the short exact sequence of complexes

$$0 \to gr^{p+1}(K^\bullet) \to F^pK^\bullet/F^{p+2}K^\bullet \to gr^p(K^\bullet) \to 0.$$

(2) *Assume that $d(F^pK) \subset F^{p+1}K$ for all $p \in \mathbf{Z}$. Then d induces the zero differential on $gr^p(K^\bullet)$ and hence $E_1^{p,q} = gr^p(K^\bullet)^{p+q}$. Furthermore, in this case*

$$d_1^{p,q} : E_1^{p,q} = gr^p(K^\bullet)^{p+q} \longrightarrow E_1^{p+1,q} = gr^{p+1}(K^\bullet)^{p+q+1}$$

is the morphism induced by d.

Proof. This is clear from the formula given for the differential $d_1^{p,q}$ just above Lemma 12.24.2. □

012O **Lemma 12.24.4.** *Let $\mathcal{A}$ be an abelian category. Let $\alpha : (K^\bullet, F) \to (L^\bullet, F)$ be a morphism of filtered complexes of $\mathcal{A}$. Let $(E_r(K), d_r)_{r\geq 0}$, resp. $(E_r(L), d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$, resp. $(L^\bullet, F)$. The morphism α induces a canonical morphism of spectral sequences $\{\alpha_r : E_r(K) \to E_r(L)\}_{r\geq 0}$ compatible with the bigradings.*

Proof. Obvious from the explicit representation of the terms of the spectral sequences. □

012P **Definition 12.24.5.** Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The *induced filtration* on $H^n(K^\bullet)$ is the filtration defined by $F^pH^n(K^\bullet) = \mathrm{Im}(H^n(F^pK^\bullet) \to H^n(K^\bullet))$.

Writing out what this means we see that

012R (12.24.5.1)
$$F^pH^n(K^\bullet, d) = \frac{\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n}{\mathrm{Im}(d) \cap K^n}$$

and hence we see that

0BDT (12.24.5.2)
$$\mathrm{gr}^pH^n(K^\bullet) = \frac{\mathrm{Ker}(d) \cap F^pK^n}{\mathrm{Ker}(d) \cap F^{p+1}K^n + \mathrm{Im}(d) \cap F^pK^n}$$

(one intermediate step omitted).

012Q **Lemma 12.24.6.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. If $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ exist (see proof), then*

(1) *the limit E_∞ exists and is a bigraded object having $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$ in bidegree (p, q),*

(2) *the pth graded part $gr^pH^n(K^\bullet)$ of the nth cohomology object of $K^\bullet$ is a subquotient of $E_\infty^{p,n-p}$.*

Proof. The objects Z_∞, B_∞, and the limit $E_\infty = Z_\infty/B_\infty$ of Definition 12.20.2 are bigraded objects of $\mathcal{A}$ by our construction of the spectral sequence in Lemma 12.24.2. Since $Z_r^{p,q}$, resp. $B_r^{p,q}$ is the bidegree (p,q) part of Z_r, resp. B_r, if we assume that

$$Z_\infty^{p,q} = \bigcap\nolimits_r Z_r^{p,q} = \bigcap\nolimits_r \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_\infty^{p,q} = \bigcup\nolimits_r B_r^{p,q} = \bigcup\nolimits_r \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

exist, then Z_∞ and B_∞ exist with bidegree (p,q) parts $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ (follows from an elementary argument about unions and intersections of bigraded objects). Thus

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q})}{\bigcup_r (F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q})}.$$

where the top and the bottom exist. With $n = p+q$ we have

(12.24.6.1)

012S $$\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n \subset \bigcap\nolimits_r \left(F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n\right)$$

and

(12.24.6.2)

012T $$\bigcup\nolimits_r \left(F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n\right) \subset \mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n.$$

Thus a subquotient of $E_\infty^{p,q}$ is

$$\frac{\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n}{\mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n} = \frac{\mathrm{Ker}(d) \cap F^pK^n}{\mathrm{Im}(d) \cap F^pK^n + \mathrm{Ker}(d) \cap F^{p+1}K^n}$$

Comparing with (12.24.5.2) we conclude. □

0BDU **Definition 12.24.7.** Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r \geq r_0}$ be a spectral sequence of bigraded objects of $\mathcal{A}$ with d_r of bidegree $(r, -r+1)$. We say such a spectral sequence is

(1) *regular* if for all $p, q \in \mathbf{Z}$ there is a $b = b(p,q)$ such that the maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are zero for $r \geq b$,
(2) *coregular* if for all $p, q \in \mathbf{Z}$ there is a $b = b(p,q)$ such that the maps $d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}$ are zero for $r \geq b$,
(3) *bounded* if for all n there are only a finite number of nonzero $E_{r_0}^{p,n-p}$,
(4) *bounded below* if for all n there is a $b = b(n)$ such that $E_{r_0}^{p,n-p} = 0$ for $p \geq b$.
(5) *bounded above* if for all n there is a $b = b(n)$ such that $E_{r_0}^{p,n-p} = 0$ for $p \leq b$.

Bounded below means that if we look at $E_r^{p,q}$ on the line $p+q = n$ (whose slope is -1) we obtain zeros as (p,q) moves down and to the right. As mentioned above there is no consistent terminology regarding these notions in the literature.

0BDV **Lemma 12.24.8.** *In the situation of Definition 12.24.7. Let $Z_r^{p,q}, B_r^{p,q} \subset E_{r_0}^{p,q}$ be the (p,q)-graded parts of Z_r, B_r defined as in Section 12.20.*

(1) *The spectral sequence is regular if and only if for all p, q there exists an $r = r(p, q)$ such that $Z_r^{p,q} = Z_{r+1}^{p,q} = \ldots$*
(2) *The spectral sequence is coregular if and only if for all p, q there exists an $r = r(p, q)$ such that $B_r^{p,q} = B_{r+1}^{p,q} = \ldots$*
(3) *The spectral sequence is bounded if and only if it is both bounded below and bounded above.*
(4) *If the spectral sequence is bounded below, then it is regular.*
(5) *If the spectral sequence is bounded above, then it is coregular.*

Proof. Omitted. Hint: If $E_r^{p,q} = 0$, then we have $E_{r'}^{p,q} = 0$ for all $r' \geq r$. □

012U **Definition 12.24.9.** Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. We say the spectral sequence associated to $(K^\bullet, F)$

(1) *weakly converges to $H^*(K^\bullet)$* if $\mathrm{gr}^p H^n(K^\bullet) = E_\infty^{p,n-p}$ via Lemma 12.24.6 for all $p, n \in \mathbf{Z}$,
(2) *abuts to $H^*(K^\bullet)$* if it weakly converges to $H^*(K^\bullet)$ and $\bigcap_p F^p H^n(K^\bullet) = 0$ and $\bigcup_p F^p H^n(K^\bullet) = H^n(K^\bullet)$ for all n,
(3) *converges to $H^*(K^\bullet)$* if it is regular, abuts to $H^*(K^\bullet)$, and $H^n(K^\bullet) = \lim_p H^n(K^\bullet)/F^p H^n(K^\bullet)$.

Weak convergence, abutment, or convergence is symbolized by the notation $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. As mentioned above there is no consistent terminology regarding these notions in the literature.

012V **Lemma 12.24.10.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated spectral sequence*

(1) *weakly converges to $H^*(K^\bullet)$ if and only if for every $p, q \in \mathbf{Z}$ we have equality in equations (12.24.6.2) and (12.24.6.1),*
(2) *abuts to $H^*(K)$ if and only if it weakly converges to $H^*(K^\bullet)$ and we have $\bigcap_p(\mathrm{Ker}(d) \cap F^p K^n + \mathrm{Im}(d) \cap K^n) = \mathrm{Im}(d) \cap K^n$ and $\bigcup_p(\mathrm{Ker}(d) \cap F^p K^n + \mathrm{Im}(d) \cap K^n) = \mathrm{Ker}(d) \cap K^n$.*

Proof. Immediate from the discussions above. □

012W **Lemma 12.24.11.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 12.19.1). Then*

(1) *the spectral sequence associated to $(K^\bullet, F)$ is bounded,*
(2) *the filtration on each $H^n(K^\bullet)$ is finite,*
(3) *the spectral sequence associated to $(K^\bullet, F)$ converges to $H^*(K^\bullet)$,*
(4) *if $\mathcal{C} \subset \mathcal{A}$ is a weak Serre subcategory and for some r we have $E_r^{p,q} \in \mathcal{C}$ for all $p, q \in \mathbf{Z}$, then $H^n(K^\bullet)$ is in $\mathcal{C}$.*

Proof. Part (1) follows as $E_0^{p,n-p} = \mathrm{gr}^p K^n$. Part (2) is clear from Equation (12.24.5.1). We will use Lemma 12.24.10 to prove that the spectral sequence weakly converges. Fix $p, n \in \mathbf{Z}$. The right hand side of (12.24.6.1) is equal to $F^p K^n \cap \mathrm{Ker}(d) + F^{p+1} K^n$ because $F^{p+r} K^n = 0$ for $r \gg 0$. Thus (12.24.6.1) is an equality. The left hand side of (12.24.6.2) is equal to $F^p K^n \cap \mathrm{Im}(d) + F^{p+1} K^n$ because $F^{p-r+1} K^{n-1} = K^{n-1}$ for $r \gg 0$. Thus (12.24.6.2) is an equality. Since the filtration on $H^n(K^\bullet)$ is finite by (2) we see that we have abutment. To prove we have convergence we have to show the spectral sequence is regular which follows as it is bounded (Lemma 12.24.8) and we have to show that $H^n(K^\bullet) =$

$\lim_p H^n(K^\bullet)/F^pH^n(K^\bullet)$ which follows from the fact that the filtration on $H^*(K^\bullet)$ is finite proved in part (2).

Proof of (4). Assume that for some $r \geq 0$ we have $E_r^{p,q} \in \mathcal{C}$ for some weak Serre subcategory $\mathcal{C}$ of $\mathcal{A}$. Then $E_{r+1}^{p,q}$ is in $\mathcal{C}$ as well, see Lemma 12.10.3. By boundedness proved above (which implies that the spectral sequence is both regular and coregular, see Lemma 12.24.8) we can find an $r' \geq r$ such that $E_\infty^{p,q} = E_{r'}^{p,q}$ for all p, q with $p + q = n$. Thus $H^n(K^\bullet)$ is an object of $\mathcal{A}$ which has a finite filtration whose graded pieces are in $\mathcal{C}$. This implies that $H^n(K^\bullet)$ is in $\mathcal{C}$ by Lemma 12.10.3. □

0BDW **Lemma 12.24.12.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 12.19.1) and that for some r we have only a finite number of nonzero $E_r^{p,q}$. Then only a finite number of $H^n(K^\bullet)$ are nonzero and we have*

$$\sum (-1)^n [H^n(K^\bullet)] = \sum (-1)^{p+q} [E_r^{p,q}]$$

in $K_0(\mathcal{A}')$ where $\mathcal{A}'$ is the smallest weak Serre subcategory of $\mathcal{A}$ containing the objects $E_r^{p,q}$.

Proof. Denote E_r^{even} and E_r^{odd} the even and odd part of E_r defined as the direct sum of the (p, q) components with $p + q$ even and odd. The differential d_r defines maps $\varphi : E_r^{even} \to E_r^{odd}$ and $\psi : E_r^{odd} \to E_r^{even}$ whose compositions either way give zero. Then we see that

$$\begin{aligned} [E_r^{even}] - [E_r^{odd}] &= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\ &= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\ &= [E_{r+1}^{even}] - [E_{r+1}^{odd}] \end{aligned}$$

Note that all the intervening objects are in the smallest Serre subcategory containing the objects $E_r^{p,q}$. Continuing in this manner we see that we can increase r at will. Since there are only a finite number of pairs (p, q) for which $E_r^{p,q}$ is nonzero, a property which is inherited by $E_{r+1}, E_{r+2}, \ldots$, we see that we may assume that $d_r = 0$. At this stage we see that $H^n(K^\bullet)$ has a finite filtration (Lemma 12.24.11) whose graded pieces are exactly the $E_r^{p,n-p}$ and the result is clear. □

The following lemma is more a kind of sanity check for our definitions. Surely, if we have a filtered complex such that for every n we have

$$H^n(F^pK^\bullet) = 0 \text{ for } p \gg 0 \quad \text{and} \quad H^n(F^pK^\bullet) = H^n(K^\bullet) \text{ for } p \ll 0,$$

then the corresponding spectral sequence should converge?

0BK5 **Lemma 12.24.13.** *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume*

1. *for every n there exist $p_0(n)$ such that $H^n(F^pK^\bullet) = 0$ for $p \geq p_0(n)$,*
2. *for every n there exist $p_1(n)$ such that $H^n(F^pK^\bullet) \to H^n(K^\bullet)$ is an isomorphism for $p \leq p_1(n)$.*

Then

1. *the spectral sequence associated to $(K^\bullet, F)$ is bounded,*
2. *the filtration on each $H^n(K^\bullet)$ is finite,*
3. *the spectral sequence associated to $(K^\bullet, F)$ converges to $H^*(K^\bullet)$.*

Proof. Fix n. Using the long exact cohomology sequence associated to the short exact sequence of complexes

$$0 \to F^{p+1}K^\bullet \to F^pK^\bullet \to \mathrm{gr}^pK^\bullet \to 0$$

we find that $E_1^{p,n-p} = 0$ for $p \geq \max(p_0(n), p_0(n+1))$ and $p < \min(p_1(n), p_1(n+1))$. Hence the spectral sequence is bounded (Definition 12.24.7). This proves (1).

It is clear from the assumptions and Definition 12.24.5 that the filtration on $H^n(K^\bullet)$ is finite. This proves (2).

Next we prove that the spectral sequence weakly converges to $H^*(K^\bullet)$ using Lemma 12.24.10. Let us show that we have equality in (12.24.6.1). Namely, for $p + r > p_0(n+1)$ the map

$$d : F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) \to F^{p+r}K^{n+1}$$

ends up in the image of $d : F^{p+r}K^n \to F^{p+r}K^{n+1}$ because the complex $F^{p+r}K^\bullet$ is exact in degree $n+1$. We conclude that $F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) = d(F^{p+r}K^n) + \mathrm{Ker}(d) \cap F^pK^n$. Hence for such r we have

$$\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n = F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n$$

which proves the desired equality. To show that we have equality in (12.24.6.2) we use that for $p - r + 1 < p_1(n-1)$ we have

$$d(F^{p-r+1}K^{n-1}) = \mathrm{Im}(d) \cap F^{p-r+1}K^n$$

because the map $F^{p-r+1}K^\bullet \to K^\bullet$ induces an isomorphism on cohomology in degree $n-1$. This shows that we have

$$F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n = \mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n$$

for such r which proves the desired equality.

To see that the spectral sequence abuts to $H^*(K^\bullet)$ using Lemma 12.24.10 we have to show that $\bigcap_p(\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n) = \mathrm{Im}(d) \cap K^n$ and $\bigcup_p(\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n) = \mathrm{Ker}(d) \cap K^n$. For $p \geq p_0(n)$ we have $\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n = \mathrm{Im}(d) \cap K^n$ and for $p \leq p_1(n)$ we have $\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n = \mathrm{Ker}(d) \cap K^n$. Combining weak convergence, abutment, and boundedness we see that (2) and (3) are true. □

12.25. Spectral sequences: double complexes

012X Let $K^{\bullet,\bullet}$ be a double complex, see Section 12.18. It is customary to denote $H_I^p(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_1^{p,q})/\mathrm{Im}(d_1^{p-1,q})$ (varying q) and differential induced by d_2. Then $H_{II}^q(H_I^p(K^{\bullet,\bullet}))$ denotes its cohomology in degree q. It is also customary to denote $H_{II}^q(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_2^{p,q})/\mathrm{Im}(d_2^{p,q-1})$ (varying p) and differential induced by d_1. Then $H_I^p(H_{II}^q(K^{\bullet,\bullet}))$ denotes its cohomology in degree p. It will turn out that these cohomology groups show up as the terms in the spectral sequence for a filtration on the associated total complex or simple complex, see Definition 12.18.3.

There are two natural filtrations on the total complex $\mathrm{Tot}(K^{\bullet,\bullet})$ associated to the double complex $K^{\bullet,\bullet}$. Namely, we define

$$F_I^p(\mathrm{Tot}^n(K^{\bullet,\bullet})) = \bigoplus\nolimits_{i+j=n,\ i\geq p} K^{i,j} \quad \text{and} \quad F_{II}^p(\mathrm{Tot}^n(K^{\bullet,\bullet})) = \bigoplus\nolimits_{i+j=n,\ j\geq p} K^{i,j}.$$

It is immediately verified that $(\mathrm{Tot}(K^{\bullet,\bullet}), F_I)$ and $(\mathrm{Tot}(K^{\bullet,\bullet}), F_{II})$ are filtered complexes. By Section 12.24 we obtain two spectral sequences. It is customary to denote $({}'E_r, {}'d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_I and to denote $({}''E_r, {}''d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_{II}. Here is a description of these spectral sequences.

0130 **Lemma 12.25.1.** *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. The spectral sequences associated to $K^{\bullet,\bullet}$ have the following terms:*

(1) ${}'E_0^{p,q} = K^{p,q}$ *with* ${}'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$,
(2) ${}''E_0^{p,q} = K^{q,p}$ *with* ${}''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$,
(3) ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ *with* ${}'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
(4) ${}''E_1^{p,q} = H^q(K^{\bullet,p})$ *with* ${}''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,
(5) ${}'E_2^{p,q} = H_I^p(H_{II}^q(K^{\bullet,\bullet}))$,
(6) ${}''E_2^{p,q} = H_{II}^p(H_I^q(K^{\bullet,\bullet}))$.

Proof. Omitted. □

These spectral sequences define two filtrations on $H^n(\mathrm{Tot}(K^{\bullet,\bullet}))$. We will denote these F_I and F_{II}.

0131 **Definition 12.25.2.** Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. We say the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ *weakly converges to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$, *abuts to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$, or *converges to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$ if Definition 12.24.9 applies. Similarly we say the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ *weakly converges to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$, *abuts to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$, or *converges to* $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$ if Definition 12.24.9 applies.

As mentioned above there is no consistent terminology regarding these notions in the literature. In the situation of the definition, we have weak convergence of the first spectral sequence if for all n

$$\mathrm{gr}_{F_I}(H^n(\mathrm{Tot}(K^{\bullet,\bullet}))) = \oplus_{p+q=n} {}'E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.24.6. Similarly the second spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ weakly converges if for all n

$$\mathrm{gr}_{F_{II}}(H^n(\mathrm{Tot}(K^{\bullet,\bullet}))) = \oplus_{p+q=n} {}''E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.24.6.

0132 **Lemma 12.25.3.** *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. Assume that for every $n \in \mathbf{Z}$ there are only finitely many nonzero $K^{p,q}$ with $p+q = n$. Then*

(1) *the two spectral sequences associated to $K^{\bullet,\bullet}$ are bounded,*
(2) *the filtrations F_I, F_{II} on each $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$ are finite,*
(3) *the spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ converge to $H^*(\mathit{Tot}(K^{\bullet,\bullet}))$,*
(4) *if $\mathcal{C} \subset \mathcal{A}$ is a weak Serre subcategory and for some r we have ${}'E_r^{p,q} \in \mathcal{C}$ for all $p, q \in \mathbf{Z}$, then $H^n(\mathit{Tot}(K^{\bullet,\bullet}))$ is in $\mathcal{C}$. Similarly for $({}''E_r, {}''d_r)_{r\geq 0}$.*

Proof. Follows immediately from Lemma 12.24.11. □

Here is our first application of spectral sequences.

0133 **Lemma 12.25.4.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex. Let $A^{\bullet,\bullet}$ be a double complex. Let $\alpha^p : K^p \to A^{p,0}$ be morphisms. Assume that*

(1) *For every $n \in \mathbf{Z}$ there are only finitely many nonzero $A^{p,q}$ with $p+q=n$.*
(2) *We have $A^{p,q} = 0$ if $q < 0$.*
(3) *The morphisms α^p give rise to a morphism of complexes $\alpha : K^\bullet \to A^{\bullet,0}$.*
(4) *The complex $A^{p,\bullet}$ is exact in all degrees $q \neq 0$ and the morphism $K^p \to A^{p,0}$ induces an isomorphism $K^p \to \mathrm{Ker}(d_2^{p,0})$.*

Then α induces a quasi-isomorphism

$$K^\bullet \longrightarrow Tot(A^{\bullet,\bullet})$$

of complexes. Moreover, there is a variant of this lemma involving the second variable q instead of p.

Proof. The map is simply the map given by the morphisms $K^n \to A^{n,0} \to \mathrm{Tot}^n(A^{\bullet,\bullet})$, which are easily seen to define a morphism of complexes. Consider the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ associated to the double complex $A^{\bullet,\bullet}$. By Lemma 12.25.3 this spectral sequence converges and the induced filtration on $H^n(\mathrm{Tot}(A^{\bullet,\bullet}))$ is finite for each n. By Lemma 12.25.1 and assumption (4) we have ${}'E_1^{p,q} = 0$ unless $q = 0$ and ${}'E_1^{p,0} = K^p$ with differential ${}'d_1^{p,0}$ identified with d_K^p. Hence ${}'E_2^{p,0} = H^p(K^\bullet)$ and zero otherwise. This clearly implies ${}'d_2^{p,q} = {}'d_3^{p,q} = \ldots = 0$ for degree reasons. Hence we conclude that $H^n(\mathrm{Tot}(A^{\bullet,\bullet})) = H^n(K^\bullet)$. We omit the verification that this identification is given by the morphism of complexes $K^\bullet \to \mathrm{Tot}(A^{\bullet,\bullet})$ introduced above. □

0FKH **Lemma 12.25.5.** *Let $\mathcal{A}$ be an abelian category. Let $M^\bullet$ be a complex of $\mathcal{A}$. Let*

$$a : M^\bullet[0] \longrightarrow \left(A^{0,\bullet} \to A^{1,\bullet} \to A^{2,\bullet} \to \ldots\right)$$

be a homotopy equivalence in the category of complexes of complexes of $\mathcal{A}$. Then the map $\alpha : M^\bullet \to Tot(A^{\bullet,\bullet})$ induced by $M^\bullet \to A^{0,\bullet}$ is a homotopy equivalence.

Proof. The statement makes sense as a complex of complexes is the same thing as a double complex. The assumption means there is a map

$$b : \left(A^{0,\bullet} \to A^{1,\bullet} \to A^{2,\bullet} \to \ldots\right) \longrightarrow M^\bullet[0]$$

such that $a \circ b$ and $b \circ a$ are homotopic to the identity in the category of complexes of complexes. This means that $b \circ a$ is the identity of $M^\bullet[0]$ (because there is only one term in degree 0). Also, observe that b is given by a map $b^0 : A^{0,\bullet} \to M^\bullet$ and zero in all other degrees. Thus b induces a map $\beta : \mathrm{Tot}(A^{\bullet,\bullet}) \to M^\bullet$ and $\beta \circ \alpha$ is the identity on $M^\bullet$. Finally, we have to show that the map $\alpha \circ \beta$ is homotopic to the identity. For this we choose maps of complexes $h^n : A^{n,\bullet} \to A^{n-1,\bullet}$ such that $a \circ b - \mathrm{id} = d_1 \circ h + h \circ d_1$ which exist by assumption. Here $d_1 : A^{n,\bullet} \to A^{n+1,\bullet}$ are the differentials of the complex of complexes. We will also denote d_2 the differentials of the complexes $A^{n,\bullet}$ for all n. Let $h^{n,m} : A^{n,m} \to A^{n-1,m}$ be the components of h^n. Then we can consider

$$h' : \mathrm{Tot}(A^{\bullet,\bullet})^k = \bigoplus\nolimits_{n+m=k} A^{n,m} \to \bigoplus\nolimits_{n+m=k-1} A^{n,m} = \mathrm{Tot}(A^{\bullet,\bullet})^{k-1}$$

given by $h^{n,m}$ on the summand $A^{n,m}$. Then we compute that the map

$$d_{\mathrm{Tot}(A^{\bullet,\bullet})} \circ h' + h' \circ d_{\mathrm{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{n,m}$ is equal to

$$d_1^{n-1,m} \circ h^{n,m} + (-1)^{n-1} d_2^{n-1,m} \circ h^{n,m} + h^{n+1,m} \circ d_1^{n,m} + h^{n,m+1} \circ (-1)^n d_2^{n,m}$$

Since h^n is a map of complexes, the terms $(-1)^{n-1}d_2^{n-1,m} \circ h^{n,m}$ and $h^{n,m+1} \circ (-1)^n d_2^{n,m}$ cancel. The other two terms give $(\alpha \circ \beta)|_{A^{n,m}} - \mathrm{id}_{A^{n,m}}$ because $a \circ b - \mathrm{id} = d_1 \circ h + h \circ d_1$. This finishes the proof. □

12.26. Double complexes of abelian groups

0E1P In this section we put some results on double complexes of abelian groups for which do not (yet) have the analogues results for general abelian categories. Please be careful not to use these lemmas except when the underlying abelian category is the category of abelian groups or some such (e.g., the category of modules over a ring). Some of the arguments will be difficult to follow without drawing "zig-zags" on a napkin – compare with the proof of Algebra, Lemma 10.75.3.

0E1Q **Lemma 12.26.1.** *Let $M^\bullet$ be a complex of abelian groups. Let*

$$0 \to M^\bullet \to A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$$

be an exact complex of complexes of abelian groups. Set $A^{p,q} = A_p^q$ to obtain a double complex. Then the map $M^\bullet \to Tot(A^{\bullet,\bullet})$ induced by $M^\bullet \to A_0^\bullet$ is a quasi-isomorphism.

Proof. If there exists a $t \in \mathbf{Z}$ such that $A_0^q = 0$ for $q < t$, then this follows immediately from Lemma 12.25.4 (with p and q swapped as in the final statement of that lemma). OK, but for every $t \in \mathbf{Z}$ we have a complex

$$0 \to \sigma_{\geq t} M^\bullet \to \sigma_{\geq t} A_0^\bullet \to \sigma_{\geq t} A_1^\bullet \to \sigma_{\geq t} A_2^\bullet \to \ldots$$

of stupid truncations. Denote $A(t)^{\bullet,\bullet}$ the corresponding double complex. Every element ξ of $H^n(\mathrm{Tot}(A^{\bullet,\bullet}))$ is the image of an element of $H^n(\mathrm{Tot}(A(t)^{\bullet,\bullet}))$ for some t (look at explicit representatives of cohomology classes). Hence ξ is in the image of $H^n(\sigma_{\geq t} M^\bullet)$. Thus the map $H^n(M^\bullet) \to H^n(\mathrm{Tot}(A^{\bullet,\bullet}))$ is surjective. It is injective because for all t the map $H^n(\sigma_{\geq t} M^\bullet) \to H^n(\mathrm{Tot}(A(t)^{\bullet,\bullet}))$ is injective and similar arguments. □

09IZ **Lemma 12.26.2.** *Let $M^\bullet$ be a complex of abelian groups. Let*

$$\ldots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$\ldots \to \mathrm{Ker}(d^p_{A_2^\bullet}) \to \mathrm{Ker}(d^p_{A_1^\bullet}) \to \mathrm{Ker}(d^p_{A_0^\bullet}) \to \mathrm{Ker}(d^p_{M^\bullet}) \to 0$$

are exact as well. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Then $Tot(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A_0^\bullet \to M^\bullet$ is a quasi-isomorphism.

Proof. Using the short exact sequences $0 \to \mathrm{Ker}(d^p_{A_n^\bullet}) \to A_n^p \to \mathrm{Im}(d^p_{A_n^\bullet}) \to 0$ and the assumptions we see that

$$\ldots \to \mathrm{Im}(d^p_{A_2^\bullet}) \to \mathrm{Im}(d^p_{A_1^\bullet}) \to \mathrm{Im}(d^p_{A_0^\bullet}) \to \mathrm{Im}(d^p_{M^\bullet}) \to 0$$

is exact for all $p \in \mathbf{Z}$. Repeating with the exact sequences $0 \to \mathrm{Im}(d^{p-1}_{A_n^\bullet}) \to \mathrm{Ker}(d^p_{A_n^\bullet}) \to H^p(A_n^\bullet) \to 0$ we find that

$$\ldots \to H^p(A_2^\bullet) \to H^p(A_1^\bullet) \to H^p(A_0^\bullet) \to H^p(M^\bullet) \to 0$$

is exact for all $p \in \mathbf{Z}$.

Write $T^\bullet = \mathrm{Tot}(A^{\bullet,\bullet})$. We will show that $H^0(T^\bullet) \to H^0(M^\bullet)$ is an isomorphism. The same argument works for other degrees. Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent an element $\xi \in H^0(T^\bullet)$. Write $x = \sum_{i=n,\ldots,0} x_i$ with $x_i \in A^i_i$. Assume $n > 0$. Then x_n is in the kernel of $d^n_{A^\bullet_n}$ and maps to zero in $H^n(A^\bullet_{n-1})$ because it maps to an element which is the boundary of x_{n-1} up to sign. By the first paragraph of the proof, we find that $x_n \bmod \mathrm{Im}(d^{n-1}_{A^\bullet_n})$ is in the image of $H^n(A^\bullet_{n+1}) \to H^n(A^\bullet_n)$. Thus we can modify x by a boundary and reach the situation where x_n is a boundary. Modifying x once more we see that we may assume $x_n = 0$. By induction we see that every cohomology class ξ is represented by a cocycle $x = x_0$. Finally, the condition on exactness of kernels tells us two such cocycles x_0 and x_0' are cohomologous if and only if their image in $H^0(M^\bullet)$ are the same. □

09J0 **Lemma 12.26.3.** *Let $M^\bullet$ be a complex of abelian groups. Let*

$$0 \to M^\bullet \to A^\bullet_0 \to A^\bullet_1 \to A^\bullet_2 \to \ldots$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$0 \to \mathrm{Coker}(d^p_{M^\bullet}) \to \mathrm{Coker}(d^p_{A^\bullet_0}) \to \mathrm{Coker}(d^p_{A^\bullet_1}) \to \mathrm{Coker}(d^p_{A^\bullet_2}) \to \ldots$$

are exact as well. Set $A^{p,q} = A^q_p$ to obtain a double complex. Let $Tot_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $M^\bullet \to Tot_\pi(A^{\bullet,\bullet})$ induced by $M^\bullet \to A^\bullet_0$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \mathrm{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A^q_p \quad \text{with} \quad \mathrm{d}^n_{T^\bullet} = \prod\nolimits_{n=p+q} (f^q_p + (-1)^p d^q_{A^\bullet_p})$$

where $f^\bullet_p : A^\bullet_p \to A^\bullet_{p+1}$ are the maps of complexes in the lemma.

We will show that $H^0(M^\bullet) \to H^0(T^\bullet)$ is an isomorphism. The same argument works for other degrees. Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent $\xi \in H^0(T^\bullet)$. Write $x = (x_i)$ with $x_i \in A^{-i}_i$. Note that x_0 maps to zero in $\mathrm{Coker}(A^{-1}_1 \to A^0_1)$. Hence we see that $x_0 = m_0 + d^{-1}_{A^\bullet_0}(y)$ for some $m_0 \in M^0$ and $y \in A^{-1}_0$. Then $d_{M^\bullet}(m_0) = 0$ because $\mathrm{d}_{A^\bullet_0}(x_0) = 0$ as $\mathrm{d}_{T^\bullet}(x) = 0$. Thus, replacing ξ by something in the image of $H^0(M^\bullet) \to H^0(T^\bullet)$ we may assume that x_0 is in $\mathrm{Im}(d^{-1}_{A^\bullet_0})$.

Assume $x_0 \in \mathrm{Im}(d^{-1}_{A^\bullet_0})$. We claim that in this case $\xi = 0$. To prove this we find, by induction on n elements $y_0, y_1, \ldots, y_n$ with $y_i \in A^{-i-1}_i$ such that $x_0 = \mathrm{d}^{-1}_{A^\bullet_0}(y_0)$ and $x_j = f^{-j}_{j-1}(y_{j-1}) + (-1)^j d^{-j-1}_{A^\bullet_j}(y_j)$ for $j = 1, \ldots, n$. This is clear for $n = 0$. Proof of induction step: suppose we have found $y_0, \ldots, y_{n-1}$. Then $w_n = x_n - f^{-n}_{n-1}(y_{n-1})$ is in the kernel of $d^{-n}_{A^\bullet_n}$ and maps to zero in $H^{-n}(A^\bullet_{n+1})$ (because it maps to an element which is the boundary of x_{n+1} up to sign). Exactly as in the proof of Lemma 12.26.2 the assumptions of the lemma imply that

$$0 \to H^p(M^\bullet) \to H^p(A^\bullet_0) \to H^p(A^\bullet_1) \to H^p(A^\bullet_2) \to \ldots$$

is exact for all $p \in \mathbf{Z}$. Thus after changing y_{n-1} by an element in $\mathrm{Ker}(d^{n-1}_{A^\bullet_{n-1}})$ we may assume that w_n maps to zero in $H^{-n}(A^\bullet_n)$. This means we can find y_n as desired. Observe that this procedure does not change $y_0, \ldots, y_{n-2}$. Hence continuing ad infinitum we find an element $y = (y_i)$ in T^{n-1} with $d_{T^\bullet}(y) = \xi$. This shows that $H^0(M^\bullet) \to H^0(T^\bullet)$ is surjective.

Suppose that $m_0 \in \mathrm{Ker}(d^0_{M^\bullet})$ maps to zero in $H^0(T^\bullet)$. Say it maps to the differential applied to $y = (y_i) \in T^{-1}$. Then $y_0 \in A_0^{-1}$ maps to zero in $\mathrm{Coker}(d^{-2}_{A_1^\bullet})$. By assumption this means that $y_0 \bmod \mathrm{Im}(d^{-2}_{A_0^\bullet})$ is the image of some $z \in M^{-1}$. It follows that $m_0 = d^{-1}_{M^\bullet}(z)$. This proves injectivity and the proof is complete. □

0E1R **Lemma 12.26.4.** *Let $M^\bullet$ be a complex of abelian groups. Let*

$$\ldots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Let $Tot_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $Tot_\pi(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A_0^\bullet \to M^\bullet$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \mathrm{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A^q_{-p} \quad \text{with} \quad \mathrm{d}^n_{T^\bullet} = \prod\nolimits_{n=p+q} (f^q_{-p} + (-1)^p d^q_{A^\bullet_{-p}})$$

where $f^\bullet_p : A^\bullet_p \to A^\bullet_{p-1}$ are the maps of complexes in the lemma. We will show that $T^\bullet$ is acyclic when $M^\bullet$ is the zero complex. This will suffice by the following trick. Set $B^\bullet_n = A^\bullet_{n+1}$ and $B^\bullet_0 = M^\bullet$. Then we have an exact sequence

$$\ldots \to B_2^\bullet \to B_1^\bullet \to B_0^\bullet \to 0 \to 0$$

as in the lemma. Let $S^\bullet = \mathrm{Tot}_\pi(B^{\bullet,\bullet})$. Then there is an obvious short exact sequence of complexes

$$0 \to M^\bullet \to S^\bullet \to T^\bullet[1] \to 0$$

and we conclude by the long exact cohomology sequence. Some details omitted.

Assume $M^\bullet = 0$. We will show $H^0(T^\bullet) = 0$. The same argument works for other degrees. Let $x = (x_n) \in \mathrm{Ker}(d_{T^\bullet})$ map to $\xi \in H^0(T^\bullet)$ with $x_n \in A^{-n,n} = A^n_n$. Since $M^0 = 0$ we find that $x_0 = f^0_1(y_0)$ for some $y_0 \in A^0_1$. Then $x_1 - d^0_{A^\bullet_1}(y_0) = f^1_2(y_1)$ because it is mapped to zero by f^1_1 as x is a cocycle. for some $y_1 \in A^1_2$. Continuing, using induction, we find $y = (y_i) \in T^{-1}$ with $d_{T^\bullet}(y) = x$ as desired. □

12.27. Injectives

0134

0135 **Definition 12.27.1.** Let $\mathcal{A}$ be an abelian category. An object $J \in \mathrm{Ob}(\mathcal{A})$ is called *injective* if for every injection $A \hookrightarrow B$ and every morphism $A \to J$ there exists a morphism $B \to J$ making the following diagram commute

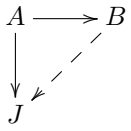

Here is the obligatory characterization of injective objects.

0136 **Lemma 12.27.2.** *Let $\mathcal{A}$ be an abelian category. Let I be an object of $\mathcal{A}$. The following are equivalent:*

(1) *The object I is injective.*
(2) *The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(B, I)$ is exact.*

(3) *Any short exact sequence*

$$0 \to I \to A \to B \to 0$$

in $\mathcal{A}$ is split.

(4) *We have $\mathrm{Ext}_{\mathcal{A}}(B, I) = 0$ for all $B \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Omitted. □

0137 **Lemma 12.27.3.** *Let $\mathcal{A}$ be an abelian category. Suppose I_ω, $\omega \in \Omega$ is a set of injective objects of $\mathcal{A}$. If $\prod_{\omega\in\Omega} I_\omega$ exists then it is injective.*

Proof. Omitted. □

0138 **Definition 12.27.4.** Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough injectives* if every object A has an injective morphism $A \to J$ into an injective object J.

0139 **Definition 12.27.5.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial injective embeddings* if there exists a functor

$$J : \mathcal{A} \longrightarrow \mathrm{Arrows}(\mathcal{A})$$

such that

(1) $s \circ J = \mathrm{id}_{\mathcal{A}}$,
(2) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the morphism $J(A)$ is injective, and
(3) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the object $t(J(A))$ is an injective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (A \to J(A))$.

12.28. Projectives

013A

013B **Definition 12.28.1.** Let $\mathcal{A}$ be an abelian category. An object $P \in \mathrm{Ob}(\mathcal{A})$ is called *projective* if for every surjection $A \to B$ and every morphism $P \to B$ there exists a morphism $P \to A$ making the following diagram commute

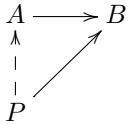

Here is the obligatory characterization of projective objects.

013C **Lemma 12.28.2.** *Let $\mathcal{A}$ be an abelian category. Let P be an object of $\mathcal{A}$. The following are equivalent:*

(1) *The object P is projective.*
(2) *The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(P, B)$ is exact.*
(3) *Any short exact sequence*

$$0 \to A \to B \to P \to 0$$

in $\mathcal{A}$ is split.

(4) *We have $\mathrm{Ext}_{\mathcal{A}}(P, A) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Omitted. □

013D **Lemma 12.28.3.** *Let $\mathcal{A}$ be an abelian category. Suppose P_ω, $\omega \in \Omega$ is a set of projective objects of $\mathcal{A}$. If $\coprod_{\omega\in\Omega} P_\omega$ exists then it is projective.*

Proof. Omitted. □

013E **Definition 12.28.4.** Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough projectives* if every object A has an surjective morphism $P \to A$ from an projective object P onto it.

013F **Definition 12.28.5.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial projective surjections* if there exists a functor

$$P : \mathcal{A} \longrightarrow \text{Arrows}(\mathcal{A})$$

such that

(1) $t \circ P = \text{id}_{\mathcal{A}}$,
(2) for any object $A \in \text{Ob}(\mathcal{A})$ the morphism $P(A)$ is surjective, and
(3) for any object $A \in \text{Ob}(\mathcal{A})$ the object $s(P(A))$ is an projective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (P(A) \to A)$.

12.29. Injectives and adjoint functors

015Y Here are some lemmas on adjoint functors and their relationship with injectives. See also Lemma 12.7.4.

015Z **Lemma 12.29.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors with u right adjoint to v. Consider the following conditions:*

(a) *v transforms injective maps into injective maps,*
(b) *v is exact, and*
(c) *u transforms injectives into injectives.*

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). If $\mathcal{A}$ has enough injectives, then all three conditions are equivalent.

Proof. Observe that v is right exact as a left adjoint (Categories, Lemma 4.24.6). Combined with Lemma 12.7.2 this explains why (a) $\Leftrightarrow$ (b).

Assume (a). Let I be an injective object of $\mathcal{A}$. Let $\varphi : N \to M$ be an injective map in $\mathcal{B}$ and let $\alpha : N \to uI$ be a morphism. By adjointness we get a morphism $\alpha : vN \to I$ and by assumption $v\varphi : vN \to vM$ is injective. Hence as I is an injective object we get a morphism $\beta : vM \to I$ extending α. By adjointness again this corresponds to a morphism $\beta : M \to uI$ extending α. Hence (c) is true.

Assume $\mathcal{A}$ has enough injectives and (c) holds. Let $f : B \to B'$ be a monomorphism in $\mathcal{B}$, and let $A = \text{Ker}(v(f))$. Choose a monomorphism $g : A \to I$ with I injective (it exists by assumption). Then g extends to $g' : v(B) \to I$, whence by adjunction a morphism $B \to u(I)$. Since $u(I)$ is injective, this morphism extends to $h : B' \to u(I)$, hence by adjunction a morphism $k : v(B') \to I$ extending g'. But then k "extends" g, which forces $A = 0$ since g was a monomorphism. Thus (a) is true. □

03B8 **Remark 12.29.2.** Let $R \to S$ be a ring map. Let $u : \text{Mod}_S \to \text{Mod}_R$ be $u(N) = N_R$ and let $v : \text{Mod}_R \to \text{Mod}_S$ be $v(M) = M \otimes_R S$. Then u is right adjoint to v, and u is exact and v is right exact. But conditions (a), (b), (c) of Lemma 12.29.1 do not hold in general. For example, if $R = \mathbf{Z}$ and $S = \mathbf{Z}/p\mathbf{Z}$, then the injective S-module $\mathbf{Z}/p\mathbf{Z}$ is not an injective $\mathbf{Z}$-module. In fact, the lemma shows all injective S-modules are injective as R-modules if and only if $R \to S$ is a flat ring map.

0160 **Lemma 12.29.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives, and*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \mathrm{Ob}(\mathcal{B})$.*

Then $\mathcal{B}$ has enough injectives.

Proof. Pick $B \in \mathrm{Ob}(\mathcal{B})$. Pick an injection $vB \to I$ for I an injective object of $\mathcal{A}$. According to Lemma 12.29.1 and the assumptions the corresponding map $B \to uI$ is the injection of B into an injective object. □

03B9 **Remark 12.29.4.** Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as In Lemma 12.29.3. In the presence of conditions (1) and (2) condition (4) is equivalent to v being faithful. Moreover, condition (4) is needed. An example is to consider the case where the functors u and v are both the zero functor.

0161 **Lemma 12.29.5.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives,*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \mathrm{Ob}(\mathcal{B})$, and*
(5) *$\mathcal{A}$ has functorial injective embeddings.*

Then $\mathcal{B}$ has functorial injective embeddings.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding on $\mathcal{A}$. Then $B \mapsto (B \to uJ(vB))$ is a functorial injective embedding on $\mathcal{B}$. Compare with the proof of Lemma 12.29.3. □

0793 **Lemma 12.29.6.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{B})$ such that*

(1) *every object of $\mathcal{B}$ is a quotient of an element of $\mathcal{P}$, and*
(2) *for every $P \in \mathcal{P}$ there exists an object Q of $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(Q, A) = \mathrm{Hom}_{\mathcal{B}}(P, u(A))$ functorially in A,*

then there exists a left adjoint v of u.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the object Q of $\mathcal{A}$ corresponding to P is defined up to unique isomorphism by the formula $\mathrm{Hom}_{\mathcal{A}}(Q, A) = \mathrm{Hom}_{\mathcal{B}}(P, u(A))$. Let us write $Q = v(P)$. Denote $i_P : P \to u(v(P))$ the map corresponding to $\mathrm{id}_{v(P)}$ in $\mathrm{Hom}_{\mathcal{A}}(v(P), v(P))$. Functoriality in (2) implies that the bijection is given by

$$\mathrm{Hom}_{\mathcal{A}}(v(P), A) \to \mathrm{Hom}_{\mathcal{B}}(P, u(A)), \quad \varphi \mapsto u(\varphi) \circ i_P$$

For any pair of elements $P_1, P_2 \in \mathcal{P}$ there is a canonical map

$$\mathrm{Hom}_{\mathcal{B}}(P_2, P_1) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), v(P_1)), \quad \varphi \mapsto v(\varphi)$$

which is characterized by the rule $u(v(\varphi)) \circ i_{P_2} = i_{P_1} \circ \varphi$ in $\mathrm{Hom}_{\mathcal{B}}(P_2, u(v(P_1)))$. Note that $\varphi \mapsto v(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $P \mapsto v(P)$ is a functor from the full subcategory of $\mathcal{B}$ whose objects are the elements of $\mathcal{P}$.

Given an arbitrary object B of $\mathcal{B}$ choose an exact sequence

$$P_2 \to P_1 \to B \to 0$$

which is possible by assumption (1). Define $v(B)$ to be the object of $\mathcal{A}$ fitting into the exact sequence

$$v(P_2) \to v(P_1) \to v(B) \to 0$$

Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(v(B), A) &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(v(P_1), A) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), A)) \\ &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{B}}(P_1, u(A)) \to \mathrm{Hom}_{\mathcal{B}}(P_2, u(A))) \\ &= \mathrm{Hom}_{\mathcal{B}}(B, u(A)) \end{aligned}$$

Hence we see that we may take $\mathcal{P} = \mathrm{Ob}(\mathcal{B})$, i.e., we see that v is everywhere defined. □

12.30. Essentially constant systems

0A2D In this section we discuss essentially constant systems with values in additive categories.

0A2E **Lemma 12.30.1.** *Let $\mathcal{I}$ be a category, let $\mathcal{A}$ be a pre-additive Karoubian category, and let $M : \mathcal{I} \to \mathcal{A}$ be a diagram.*

(1) *Assume $\mathcal{I}$ is filtered. The following are equivalent*
   (a) *M is essentially constant,*
   (b) *$X = \mathrm{colim}\, M$ exists and there exists a cofinal filtered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that $X_{i'}$ maps isomorphically to X and $Z_{i'}$ to zero in $M_{i''}$ for some $i' \to i''$ in $\mathcal{I}'$.*

(2) *Assume $\mathcal{I}$ is cofiltered. The following are equivalent*
   (a) *M is essentially constant,*
   (b) *$X = \lim M$ exists and there exists an initial cofiltered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that X maps isomorphically to $X_{i'}$ and $M_{i''} \to Z_{i'}$ is zero for some $i'' \to i'$ in $\mathcal{I}'$.*

Proof. Assume (1)(a), i.e., $\mathcal{I}$ is filtered and M is essentially constant. Let $X = \mathrm{colim}\, M_i$. Choose i and $X \to M_i$ as in Categories, Definition 4.22.1. Let $\mathcal{I}'$ be the full subcategory consisting of objects which are the target of a morphism with source i. Suppose $i' \in \mathrm{Ob}(\mathcal{I}')$ and choose a morphism $i \to i'$. Then $X \to M_i \to M_{i'}$ composed with $M_{i'} \to X$ is the identity on X. As $\mathcal{A}$ is Karoubian, we find a direct summand decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$, where $Z_{i'} = \mathrm{Ker}(M_{i'} \to X)$ and $X_{i'}$ maps isomorphically to X. Pick $i \to k$ and $i' \to k$ such that $M_{i'} \to X \to M_i \to M_k$ equals $M_{i'} \to M_k$ as in Categories, Definition 4.22.1. Then we see that $M_{i'} \to M_k$ annihilates $Z_{i'}$. Thus (1)(b) holds.

Assume (1)(b), i.e., $\mathcal{I}$ is filtered and we have $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ as stated in the lemma. To see that M is essentially constant we can replace $\mathcal{I}$ by $\mathcal{I}'$, see Categories, Lemma 4.22.11. Pick any $i \in \mathrm{Ob}(\mathcal{I})$ and denote $X \to M_i$ the inverse of the isomorphism $X_i \to X$ followed by the inclusion map $X_i \to M_i$. If j is a second object, then choose $j \to k$ such that $Z_j \to M_k$ is zero. Since $\mathcal{I}$ is filtered we may also assume there is a morphism

$i \to k$ (after possibly increasing k). Then $M_j \to X \to M_i \to M_k$ and $M_j \to M_k$ both annihilate Z_j. Thus after postcomposing by a morphism $M_k \to M_l$ which annihilates the summand Z_k, we find that $M_j \to X \to M_i \to M_l$ and $M_j \to M_l$ are equal, i.e., M is essentially constant.

The proof of (2) is dual. □

0A2F **Lemma 12.30.2.** *Let $\mathcal{I}$ be a category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent*

(1) $\operatorname{colim}_{\mathcal{I}} F \oplus G$ *exists, and*
(2) $\operatorname{colim}_{\mathcal{I}} F$ *and* $\operatorname{colim}_{\mathcal{I}} G$ *exist.*

In this case $\operatorname{colim}_{\mathcal{I}} F \oplus G = \operatorname{colim}_{\mathcal{I}} F \oplus \operatorname{colim}_{\mathcal{I}} G$.

Proof. Assume (1) holds. Set $W = \operatorname{colim}_{\mathcal{I}} F \oplus G$. Note that the projection onto F defines natural transformation $F \oplus G \to F \oplus G$ which is idempotent. Hence we obtain an idempotent endomorphism $W \to W$ by Categories, Lemma 4.14.8. Since $\mathcal{A}$ is Karoubian we get a corresponding direct sum decomposition $W = X \oplus Y$, see Lemma 12.4.2. A straightforward argument (omitted) shows that $X = \operatorname{colim}_{\mathcal{I}} F$ and $Y = \operatorname{colim}_{\mathcal{I}} G$. Thus (2) holds. We omit the proof that (2) implies (1). □

0A2G **Lemma 12.30.3.** *Let $\mathcal{I}$ be a filtered category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent*

(1) *$F \oplus G : \mathcal{I} \to \mathcal{A}$ is essentially constant, and*
(2) *F and G are essentially constant.*

Proof. Assume (1) holds. In particular $W = \operatorname{colim}_{\mathcal{I}} F \oplus G$ exists and hence by Lemma 12.30.2 we have $W = X \oplus Y$ with $X = \operatorname{colim}_{\mathcal{I}} F$ and $Y = \operatorname{colim}_{\mathcal{I}} G$. A straightforward argument (omitted) using for example the characterization of Categories, Lemma 4.22.9 shows that F is essentially constant with value X and G is essentially constant with value Y. Thus (2) holds. The proof that (2) implies (1) is omitted. □

12.31. Inverse systems

02MY Let $\mathcal{C}$ be a category. In Categories, Section 4.21 we defined the notion of an inverse system over a preordered set (with values in the category $\mathcal{C}$). If the preordered set is $\mathbf{N} = \{1, 2, 3, \ldots\}$ with the usual ordering such an inverse system over $\mathbf{N}$ is often simply called an *inverse system*. It consists quite simply of a pair $(M_i, f_{ii'})$ where each M_i, $i \in \mathbf{N}$ is an object of $\mathcal{C}$, and for each $i > i'$, $i, i' \in \mathbf{N}$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that moreover $f_{i'i''} \circ f_{ii'} = f_{ii''}$ whenever this makes sense. It is clear that in fact it suffices to give the morphisms $M_2 \to M_1$, $M_3 \to M_2$, and so on. Hence an inverse system is frequently pictured as follows

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

Moreover, we often omit the transition maps φ_i from the notation and we simply say "let (M_i) be an inverse system".

The collection of all inverse systems with values in $\mathcal{C}$ forms a category with the obvious notion of morphism.

02MZ **Lemma 12.31.1.** *Let $\mathcal{C}$ be a category.*

(1) *If $\mathcal{C}$ is an additive category, then the category of inverse systems with values in $\mathcal{C}$ is an additive category.*

(2) *If $\mathcal{C}$ is an abelian category, then the category of inverse systems with values in $\mathcal{C}$ is an abelian category. A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to N_i$ is exact.*

Proof. Omitted. □

The limit (see Categories, Section 4.21) of such an inverse system is denoted $\lim M_i$, or $\lim_i M_i$. If $\mathcal{C}$ is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$\lim_i M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

see Categories, Section 4.15. However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

02N0 **Definition 12.31.2.** Let $\mathcal{C}$ be an abelian category. We say the inverse system (A_i) satisfies the *Mittag-Leffler condition*, or for short is *ML*, if for every i there exists a $c = c(i) \geq i$ such that

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_c \to A_i)$$

for all $k \geq c$.

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, modules over a ring, etc. It is shown in a paper by A. Neeman (see [**Nee02**]) that this condition is not strong enough in an abelian category having AB4* (having exact products).

02N1 **Lemma 12.31.3.** *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) *In any case the sequence*

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i$$

is exact.

(2) *If (B_i) is ML, then also (C_i) is ML.*

(3) *If (A_i) is ML, then*

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to 0$$

is exact.

Proof. Nice exercise. See Algebra, Lemma 10.87.1 for part (3). □

070B **Lemma 12.31.4.** *Let*

$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$

be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence

$$\lim_i B_i \to \lim_i C_i \to \lim_i D_i$$

is exact.

Proof. Let $Z_i = \mathrm{Ker}(C_i \to D_i)$ and $I_i = \mathrm{Im}(A_i \to B_i)$. Then $\lim Z_i = \mathrm{Ker}(\lim C_i \to \lim D_i)$ and we get a short exact sequence of systems

$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$

Moreover, by Lemma 12.31.3 we see that (I_i) has (ML), thus another application of Lemma 12.31.3 shows that $\lim B_i \to \lim Z_i$ is surjective which proves the lemma. □

The following characterization of essentially constant inverse systems shows in particular that they have ML.

070C **Lemma 12.31.5.** *Let $\mathcal{A}$ be an abelian category. Let (A_i) be an inverse system in $\mathcal{A}$ with limit $A = \lim A_i$. Then (A_i) is essentially constant (see Categories, Definition 4.22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions, (b) for all j there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant. Then there exists an i and a morphism $A_i \to A$ such that $A \to A_i \to A$ is the identity and for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$ (the last map comes from $A = \lim A_i$). Hence setting $Z_j = \mathrm{Ker}(A_j \to A)$ for all $j \geq i$ works. Proof of the converse is omitted. □

We will improve on the following lemma in More on Algebra, Lemma 15.87.13.

070D **Lemma 12.31.6.** *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be an exact sequence of inverse systems of abelian groups. If (C_i) is essentially constant, then (A_i) has ML if and only if (B_i) has ML.

Proof. After renumbering we may assume that $C_i = C \oplus Z_i$ compatible with transition maps and that for all i there exists an $i' \geq i$ such that $Z_{i'} \to Z_i$ is zero, see Lemma 12.31.5.

First, assume $C = 0$, i.e., we have $C_i = Z_i$. In this case choose $1 = n_1 < n_2 < n_3 < \ldots$ such that $Z_{n_{i+1}} \to Z_{n_i}$ is zero. Then $B_{n_{i+1}} \to B_{n_i}$ factors through $A_{n_i} \subset B_{n_i}$. It follows that for $j \geq i + 1$ we have

$$\mathrm{Im}(A_{n_j} \to A_{n_i}) \subset \mathrm{Im}(B_{n_j} \to B_{n_i}) \subset \mathrm{Im}(A_{n_{j-1}} \to A_{n_i})$$

as subsets of A_{n_i}. Thus the images $\mathrm{Im}(A_{n_j} \to A_{n_i})$ stabilize for $j \geq i + 1$ if and only if the same is true for the images $\mathrm{Im}(B_{n_j} \to B_{n_i})$. The equivalence follows from this (small detail omitted).

If $C \neq 0$, denote $B_i' \subset B_i$ the inverse image of C by the map $B_i \to C \oplus Z_i$. Then by the previous paragraph we see that (B_i') has ML if and only if (B_i) has ML. Thus we may replace (B_i) by (B_i'). In this case we have exact sequences $0 \to A_i \to B_i \to C \to 0$ for all i. It follows that $0 \to \mathrm{Im}(A_j \to A_i) \to \mathrm{Im}(B_j \to B_i) \to C \to 0$ is short exact for all $j \geq i$. Hence the images $\mathrm{Im}(A_j \to A_i)$ stabilize for $j \geq i$ if and only if the same is true for $\mathrm{Im}(B_j \to B_i)$ as desired. □

The "correct" version of the following lemma is More on Algebra, Lemma 15.87.3.

070E **Lemma 12.31.7.** *Let*

$$(A_i^{-2} \to A_i^{-1} \to A_i^0 \to A_i^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_i^{-1}), (H_i^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If (A_i^{-2}) and (A_i^{-1}) are ML and (H_i^{-1}) is essentially constant, then $H^0 = \lim H_i^0$.

Proof. Let $Z_i^j = \mathrm{Ker}(A_i^j \to A_i^{j+1})$ and $I_i^j = \mathrm{Im}(A_i^{j-1} \to A_i^j)$. Note that $\lim Z_i^0 = \mathrm{Ker}(\lim A_i^0 \to \lim A_i^1)$ as taking kernels commutes with limits. The systems (I_i^{-1}) and (I_i^0) have ML as quotients of the systems (A_i^{-2}) and (A_i^{-1}), see Lemma 12.31.3. Thus an exact sequence

$$0 \to (I_i^{-1}) \to (Z_i^{-1}) \to (H_i^{-1}) \to 0$$

of inverse systems where (I_i^{-1}) has ML and where (H_i^{-1}) is essentially constant by assumption. Hence (Z_i^{-1}) has ML by Lemma 12.31.6. The exact sequence

$$0 \to (Z_i^{-1}) \to (A_i^{-1}) \to (I_i^0) \to 0$$

and an application of Lemma 12.31.3 shows that $\lim A_i^{-1} \to \lim I_i^0$ is surjective. Finally, the exact sequence

$$0 \to (I_i^0) \to (Z_i^0) \to (H_i^0) \to 0$$

and Lemma 12.31.3 show that $\lim I_i^0 \to \lim Z_i^0 \to \lim H_i^0 \to 0$ is exact. Putting everything together we win. □

Sometimes we need a version of the lemma above where we take limits over big ordinals.

0AAT **Lemma 12.31.8.** *Let α be an ordinal. Let $K_\beta^\bullet$, $\beta < \alpha$ be an inverse system of complexes of abelian groups over α. If for all $\beta < \alpha$ the complex $K_\beta^\bullet$ is acyclic and the map*

$$K_\beta^n \longrightarrow \lim_{\gamma<\beta} K_\gamma^n$$

is surjective, then the complex $\lim_{\beta<\alpha} K_\beta^\bullet$ is acyclic.

Proof. By transfinite induction we prove this holds for every ordinal α and every system as in the lemma. In particular, whilst proving the result for α we may assume the complexes $\lim_{\gamma<\beta} K_\gamma^n$ are acyclic.

Let $x \in \lim_{\beta<\alpha} K_\beta^0$ with $\mathrm{d}(x) = 0$. We will find a $y \in \lim_{\beta<\alpha} K_\beta^{-1}$ with $\mathrm{d}(y) = x$. Write $x = (x_\beta)$ where $x_\beta \in K_\beta^0$ is the image of x for $\beta < \alpha$. We will construct $y = (y_\beta)$ by transfinite recursion.

For $\beta = 0$ let $y_0 \in K_0^{-1}$ be any element with $\mathrm{d}(y_0) = x_0$.

For $\beta = \gamma + 1$ a successor, we have to find an element y_β which maps both to y_γ by the transition map $f : K_\beta^\bullet \to K_\gamma^\bullet$ and to x_β under the differential. As a first approximation we choose y'_β with $\mathrm{d}(y'_\beta) = x_\beta$. Then the difference $y_\gamma - f(y'_\beta)$ is in the kernel of the differential, hence equal to $\mathrm{d}(z_\gamma)$ for some $z_\gamma \in K_\gamma^{-2}$. By assumption, the map $f^{-2} : K_\beta^{-2} \to K_\gamma^{-2}$ is surjective. Hence we write $z_\gamma = f(z_\beta)$ and change y'_β into $y_\beta = y'_\beta + \mathrm{d}(z_\beta)$ which works.

If β is a limit ordinal, then we have the element $(y_\gamma)_{\gamma<\beta}$ in $\lim_{\gamma<\beta} K_\gamma^{-1}$ whose differential is the image of x_β. Thus we can argue in exactly the same manner as

above using the termwise surjective map of complexes $f : K^\bullet_\beta \to \lim_{\gamma<\beta} K^\bullet_\gamma$ and the fact (see first paragraph of proof) that we may assume $\lim_{\gamma<\beta} K^\bullet_\gamma$ is acyclic by induction. □

12.32. Exactness of products

060J

060K **Lemma 12.32.1.** *Let I be a set. For $i \in I$ let $L_i \to M_i \to N_i$ be a complex of abelian groups. Let $H_i = \mathrm{Ker}(M_i \to N_i)/\mathrm{Im}(L_i \to M_i)$ be the cohomology. Then*

$$\prod L_i \to \prod M_i \to \prod N_i$$

is a complex of abelian groups with homology $\prod H_i$.

Proof. Omitted. □

12.33. Other chapters

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Algebraic Spaces

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CHAPTER 13

Derived Categories

05QI

13.1. Introduction

05QJ We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [**Ver96**].

13.2. Triangulated categories

0143 Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [**Ver96**], [**KS06**], and [**Nee01**].

13.3. The definition of a triangulated category

05QK In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

0144 **Definition 13.3.1.** Let $\mathcal{D}$ be an additive category. Let $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ be an additive functor which is an auto-equivalence of $\mathcal{D}$.

(1) A *triangle* is a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphisms of $\mathcal{D}$.
(2) A *morphism of triangles* $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

In the setting of Definition 13.3.1, we write $[0] = \mathrm{id}$, for $n > 0$ we denote $[n]$ the n-fold composition of $[1]$, we choose a quasi-inverse $[-1]$ of $[1]$, and we set $[-n]$ equal to the n-fold composition of $[-1]$. Then $\{[n]\}_{n \in \mathbf{Z}}$ is a collection of additive auto-equivalences of $\mathcal{D}$ indexed by $n \in \mathbf{Z}$ such that we are given isomorphisms of functors $[n] \circ [m] \cong [n + m]$.

Here is the definition of a triangulated category as given in Verdier's thesis.

0145 **Definition 13.3.2.** A *triangulated category* consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

(1) $\mathcal{D}$ is an additive category,
(2) $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ is an additive auto-equivalence and $[n]$ for $n \in \mathbf{Z}$ is as discussed above, and
(3) $\mathcal{T}$ is a set of triangles (Definition 13.3.1) called the *distinguished triangles*

subject to the following conditions

TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathcal{D}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots & & \downarrow{\scriptstyle a[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

whose rows are distinguished triangles and which satisfies $b \circ f = f' \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of $\mathcal{D}$, and morphisms $f : X \to Y$, $g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
   (a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
   (b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
   (c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\,], \mathcal{T})$ a *pre-triangulated category* if TR1, TR2 and TR3 hold.[1]

The explanation of TR4 is that if you think of Q_1 as Y/X, Q_2 as Z/X and Q_3 as Z/Y, then TR4(a) expresses the isomorphism $(Z/X)/(Y/X) \cong Z/Y$ and TR4(b) and TR4(c) express that we can compare the triangles $X \to Y \to Q_1 \to X[1]$ etc with morphisms of triangles. For a more precise reformulation of this idea see the proof of Lemma 13.10.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence

(13.3.2.1)

05QL $$\ldots \to Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \to \ldots$$

each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category $\mathcal{D}$ without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

[1] We use $[\,]$ as an abbreviation for the family $\{[n]\}_{n \in \mathbf{Z}}$.

014V **Definition 13.3.3.** Let $\mathcal{D}$, $\mathcal{D}'$ be pre-triangulated categories. An *exact functor*, or a *triangulated functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \to F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of $\mathcal{D}'$.

An exact functor is additive, see Lemma 13.4.17. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \to (F', \xi')$ in this 2-category is simply a transformation of functors $a : F \to F'$ which is compatible with ξ and ξ', i.e.,

$$\begin{array}{ccc} F \circ [1] & \xrightarrow{\xi} & [1] \circ F \\ {\scriptstyle a \star 1}\downarrow & & \downarrow{\scriptstyle 1 \star a} \\ F' \circ [1] & \xrightarrow{\xi'} & [1] \circ F' \end{array}$$

commutes.

05QM **Definition 13.3.4.** Let $(\mathcal{D}, [\,], \mathcal{T})$ be a pre-triangulated category. A *pre-triangulated subcategory*[2] is a pair $(\mathcal{D}', \mathcal{T}')$ such that

(1) $\mathcal{D}'$ is an additive subcategory of $\mathcal{D}$ which is preserved under $[1]$ and such that $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence,
(2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$ and $f, g, h \in \mathrm{Arrows}(\mathcal{D}')$, and
(3) $(\mathcal{D}', [\,], \mathcal{T}')$ is a pre-triangulated category.

If $\mathcal{D}$ is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a *triangulated subcategory* if it is a pre-triangulated subcategory and $(\mathcal{D}', [\,], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is an exact functor with $\xi_X : X[1] \to X[1]$ given by the identity on $X[1]$.

We will see in Lemma 13.4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \to Z$ is zero. Thus the sequence (13.3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

0147 **Definition 13.3.5.** Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called *homological* if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact in the abelian category $\mathcal{A}$. An additive functor $H : \mathcal{D}^{opp} \to \mathcal{A}$ is called *cohomological* if the corresponding functor $\mathcal{D} \to \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence
(13.3.5.1)

0148 $$H^{-1}(Z) \xrightarrow{H(h[-1])} H^0(X) \xrightarrow{H(f)} H^0(Y) \xrightarrow{H(g)} H^0(Z) \xrightarrow{H(h)} H^1(X)$$

[2]This definition may be nonstandard. If $\mathcal{D}'$ is a full subcategory then $\mathcal{T}'$ is the intersection of the set of triangles in $\mathcal{D}'$ with $\mathcal{T}$, see Lemma 13.4.16. In this case we drop $\mathcal{T}'$ from the notation.

This will be called the *long exact sequence* associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

0150 **Definition 13.3.6.** Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. A *δ-functor from $\mathcal{A}$ to $\mathcal{D}$* is given by a functor $G : \mathcal{A} \to \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

a morphism $\delta = \delta_{A\to B\to C} : G(C) \to G(A)[1]$ such that

(1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A\to B\to C})$ is a distinguished triangle of $\mathcal{D}$ for any short exact sequence as above, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences the diagram

$$\begin{array}{ccc} G(C) & \xrightarrow[\delta_{A\to B\to C}]{} & G(A)[1] \\ \downarrow & & \downarrow \\ G(C') & \xrightarrow{\delta_{A'\to B'\to C'}} & G(A')[1] \end{array}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A\to B\to C})$ the *image of the short exact sequence under the given δ-functor*.

Note how a δ-functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \to \mathcal{D}$ is a δ-functor, but we will often do so anyway.

13.4. Elementary results on triangulated categories

05QN Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

0146 **Lemma 13.4.1.** *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle 1[1]} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. □

0149 **Lemma 13.4.2.** *Let $\mathcal{D}$ be a pre-triangulated category. For any object W of $\mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$ is homological, and the functor $\mathrm{Hom}_{\mathcal{D}}(-, W)$ is cohomological.*

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h). We have already seen that $g \circ f = 0$, see Lemma 13.4.1. Suppose $a : W \to Y$ is a morphism such that $g \circ a = 0$. Then we get a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{1} & W & \longrightarrow & 0 & \longrightarrow & W[1] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b[1]} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. □

014A **Lemma 13.4.3.** *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = \mathrm{Hom}_{\mathcal{D}}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} H_W(Z[-1]) & \longrightarrow & H_W(X) & \longrightarrow & H_W(Y) & \longrightarrow & H_W(Z) & \longrightarrow & H_W(X[1]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_W(Z'[-1]) & \longrightarrow & H_W(X') & \longrightarrow & H_W(Y') & \longrightarrow & H_W(Z') & \longrightarrow & H_W(X'[1]) \end{array}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 4.3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). □

09WA **Remark 13.4.4.** Let $\mathcal{D}$ be an additive category with translation functors $[n]$ as in Definition 13.3.1. Let us call a triangle (X, Y, Z, f, g, h) *special*[3] if for every object W of $\mathcal{D}$ the long sequence of abelian groups

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to \mathrm{Hom}_{\mathcal{D}}(W, X[1]) \to \ldots$$

is exact. The proof of Lemma 13.4.3 shows that if

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

is a morphism of special triangles and if two among a, b, c are isomorphisms so is the third. There is a dual statement for *co-special* triangles, i.e., triangles which turn into long exact sequences on applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, W)$. Thus distinguished triangles are special and co-special, but in general there are many more (co-)special triangles, than there are distinguished triangles.

05QP **Lemma 13.4.5.** *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \to (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

[3] This is nonstandard notation.

Proof. Picture

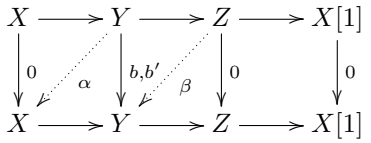

Applying Lemma 13.4.2 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 13.4.1. □

05QQ **Lemma 13.4.6.** *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X[1] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a[1]} \\ Z & \xrightarrow{h} & X[1] \end{array}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \to Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h).

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endomorphism of (X, Y, Z, f, g, h). Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 13.4.5 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. □

014B **Lemma 13.4.7.** *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism*

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 13.4.3. □

0FWZ **Lemma 13.4.8.** *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If one of the following conditions holds

(1) $\mathrm{Hom}(Y, X') = 0$,
(2) $\mathrm{Hom}(Z, Y') = 0$,
(3) $\mathrm{Hom}(X, X') = \mathrm{Hom}(Z, X') = 0$,
(4) $\mathrm{Hom}(Z, X') = \mathrm{Hom}(Z, Z') = 0$, *or*
(5) $\mathrm{Hom}(X[1], Z') = \mathrm{Hom}(Z, X') = 0$

then b is the unique morphism from $Y \to Y'$ such that (a, b, c) is a morphism of triangles.

Proof. If we have a second morphism of triangles (a, b', c) then $(0, b - b', 0)$ is a morphism of triangles. Hence we have to show: the only morphism $b : Y \to Y'$ such that $X \to Y \to Y'$ and $Y \to Y' \to Z'$ are zero is 0. We will use Lemma 13.4.2 without further mention. In particular, condition (3) implies (1). Given condition (1) if the composition $g' \circ b : Y \to Y' \to Z'$ is zero, then b lifts to a morphism $Y \to X'$ which has to be zero. This proves (1).

The proof of (2) and (4) are dual to this argument.

Assume (5). Consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b} & \swarrow{\scriptstyle \epsilon} & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

We may choose ϵ such that $b = \epsilon \circ g$. Then $g' \circ \epsilon \circ g = 0$ which implies that $g' \circ \epsilon = \delta \circ h$ for some $\delta \in \mathrm{Hom}(X[1], Z')$. Since $\mathrm{Hom}(X[1], Z') = 0$ we conclude that $g' \circ \epsilon = 0$. Hence $\epsilon = f' \circ \gamma$ for some $\gamma \in \mathrm{Hom}(Z, X')$. Since $\mathrm{Hom}(Z, X') = 0$ we conclude that $\epsilon = 0$ and hence $b = 0$ as desired. □

05QR **Lemma 13.4.9.** *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f is an isomorphism,*
(2) *$(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and*
(3) *for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.*

Proof. By TR1 the triangle $(X, X, 0, 1, 0, 0)$ is distinguished. Let (X, Y, Z, f, g, h) be a distinguished triangle. By TR3 there is a map of distinguished triangles $(1, f, 0) : (X, X, 0) \to (X, Y, Z)$. If f is an isomorphism, then $(1, f, 0)$ is an isomorphism of triangles by Lemma 13.4.3 and $Z = 0$. Conversely, if $Z = 0$, then $(1, f, 0)$ is an isomorphism of triangles as well, hence f is an isomorphism. □

05QS **Lemma 13.4.10.** *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent*

(1) *$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,*
(2) *both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.*

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{array}{c} (X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \\ \downarrow{\scriptstyle (1,1,c)} \\ (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h''). \end{array}$$

In the terminology of Remark 13.4.4 this is a map of special triangles (because a direct sum of special triangles is special) and we conclude that c is an isomorphism. Thus (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. First observe that (X, Y, Z, f, g, h) is a special triangle (terminology from Remark 13.4.4) as a direct summand of the distinguished hence special triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus Y', Z \oplus$

$Z', f \oplus f', g \oplus g', h \oplus h') \to (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

By Remark 13.4.4 we find that c is an isomorphism and we conclude that (2) holds. □

05QT **Lemma 13.4.11.** *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.*

(1) *If $h = 0$, then there exists a right inverse $s : Z \to Y$ to g.*
(2) *For any right inverse $s : Z \to Y$ of g the map $f \oplus s : X \oplus Z \to Y$ is an isomorphism.*
(3) *For any objects X', Z' of $\mathcal{D}$ the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.*

Proof. To see (1) use that $\mathrm{Hom}_{\mathcal{D}}(Z, Y) \to \mathrm{Hom}_{\mathcal{D}}(Z, Z) \to \mathrm{Hom}_{\mathcal{D}}(Z, X[1])$ is exact by Lemma 13.4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to 0$$

is split exact (split by $s : Z \to Y$). Hence by Yoneda's lemma we see that $X \oplus Z \to Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 13.4.10. □

05QU **Lemma 13.4.12.** *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f has a kernel,*
(2) *f has a cokernel,*
(3) *f is the isomorphic to a composition $K \oplus Z \to Z \to Z \oplus Q$ of a projection and coprojection for some objects K, Z, Q of $\mathcal{D}$.*

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \to Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) $\Rightarrow$ (1), (2). Next we prove (1) $\Rightarrow$ (3). Suppose first that $f : X \to Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h). By Lemma 13.4.1 the composition $f \circ h[-1] = 0$. As f is a monomorphism we see that $h[-1] = 0$ and hence $h = 0$. Then Lemma 13.4.11 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K. As $K \to X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \to Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) $\Rightarrow$ (3) is dual to this. □

0CRG **Lemma 13.4.13.** *Let $\mathcal{D}$ be a pre-triangulated category. Let I be a set.*

(1) *Let X_i, $i \in I$ be a family of objects of $\mathcal{D}$.*
   (a) *If $\prod X_i$ exists, then $(\prod X_i)[1] = \prod X_i[1]$.*
   (b) *If $\bigoplus X_i$ exists, then $(\bigoplus X_i)[1] = \bigoplus X_i[1]$.*
(2) *Let $X_i \to Y_i \to Z_i \to X_i[1]$ be a family of distinguished triangles of $\mathcal{D}$.*
   (a) *If $\prod X_i$, $\prod Y_i$, $\prod Z_i$ exist, then $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is a distinguished triangle.*
   (b) *If $\bigoplus X_i$, $\bigoplus Y_i$, $\bigoplus Z_i$ exist, then $\bigoplus X_i \to \bigoplus Y_i \to \bigoplus Z_i \to \bigoplus X_i[1]$ is a distinguished triangle.*

Proof. Part (1) is true because [1] is an autoequivalence of $\mathcal{D}$ and because direct sums and products are defined in terms of the category structure. Let us prove (2)(a). Choose a distinguished triangle $\prod X_i \to \prod Y_i \to Z \to \prod X_i[1]$. For each

j we can use TR3 to choose a morphism $p_j : Z \to Z_j$ fitting into a morphism of distinguished triangles with the projection maps $\prod X_i \to X_j$ and $\prod Y_i \to Y_j$. Using the definition of products we obtain a map $\prod p_i : Z \to \prod Z_i$ fitting into a morphism of triangles from the distinguished triangle to the triangle made out of the products. Observe that the "product" triangle $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is special in the terminology of Remark 13.4.4 because products of exact sequences of abelian groups are exact. Hence Remark 13.4.4 shows that the morphism of triangles is an isomorphism and we conclude by TR1. The proof of (2)(b) is dual. □

05QW **Lemma 13.4.14.** *Let $\mathcal{D}$ be a pre-triangulated category. If $\mathcal{D}$ has countable products, then $\mathcal{D}$ is Karoubian. If $\mathcal{D}$ has countable coproducts, then $\mathcal{D}$ is Karoubian.*

Proof. Assume $\mathcal{D}$ has countable products. By Homology, Lemma 12.4.3 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 13.4.12. The second statement is dual to the first. □

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

014C **Lemma 13.4.15.** *Let $\mathcal{D}$ be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ there exist*

(1) *isomorphisms $i : X' \to X$, $j : Y' \to Y$ and $k : Z' \to Z$, and then setting $f' = j^{-1}fi : X' \to Y'$ and $g' = k^{-1}gj : Y' \to Z'$ there exist*
(2) *distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.*

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 13.4.7. □

05QX **Lemma 13.4.16.** *Let $\mathcal{D}$ be a pre-triangulated category. Assume that $\mathcal{D}'$ is an additive full subcategory of $\mathcal{D}$. The following are equivalent*

(1) *there exists a set of triangles $\mathcal{T}'$ such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of $\mathcal{D}$,*
(2) *$\mathcal{D}'$ is preserved under $[1]$ and $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence and given any morphism $f : X \to Y$ in $\mathcal{D}'$ there exists a distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{D}$ such that Z is isomorphic to an object of $\mathcal{D}'$.*

In this case $\mathcal{T}'$ as in (1) is the set of distinguished triangles (X, Y, Z, f, g, h) of $\mathcal{D}$ such that $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$. Finally, if $\mathcal{D}$ is a triangulated category, then (1) and (2) are also equivalent to

(3) *$\mathcal{D}'$ is a triangulated subcategory.*

Proof. Omitted. □

05QY **Lemma 13.4.17.** *An exact functor of pre-triangulated categories is additive.*

Proof. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of $\mathcal{D}$ the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in $\mathcal{D}'$. This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 13.4.1. Hence $F(0)$ is the zero object of $\mathcal{D}'$. This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it factors through the zero object). Next, using that $(X, X \oplus Y, Y, (1,0), (0,1), 0)$ is a distinguished triangle by Lemma 13.4.11 part (3), we see that $(F(X), F(X \oplus Y), F(Y), F(1,0), F(0,1), 0)$ is one too. This implies that the map $F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism by Lemma 13.4.11 part (2). To finish we apply Homology, Lemma 12.7.1. □

05SQ **Lemma 13.4.18.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in $\mathcal{D}'$.*

Proof. The "only if" part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in $\mathcal{D}'$. Pick a distinguished triangle (X, Y, Z', f, g', h') in $\mathcal{D}$. By Lemma 13.4.7 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \to Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z'). Hence (X, Y, Z) is distinguished by TR1. □

014Y **Lemma 13.4.19.** *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.*

Proof. Omitted. □

05QZ **Lemma 13.4.20.** *Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor.*

(1) *Let $\mathcal{D}'$ be a pre-triangulated category. Let $F : \mathcal{D}' \to \mathcal{D}$ be an exact functor. Then the composition $H \circ F$ is a homological functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{A}'$ be an exact functor. Then $G \circ H$ is a homological functor as well.*

Proof. Omitted. □

0151 **Lemma 13.4.21.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor.*

(1) *Let $\mathcal{D}'$ be a triangulated category. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ-functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Let $H : \mathcal{A}' \to \mathcal{A}$ be an exact functor. Then $G \circ H$ is a δ-functor as well.*

Proof. Omitted. □

05SR **Lemma 13.4.22.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor. Let $H : \mathcal{D} \to \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in $\mathcal{A}$. Then the collection*

$$\{H^n \circ G, H^n(\delta_{A \to B \to C})\}_{n \geq 0}$$

is a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.12.1.

Proof. The notation signifies the following. If $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$$

is a morphism in $\mathcal{D}$ such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 13.3.6. Then $H^n(\delta) : H^n(G(C)) \to H^n(G(A)[1]) = H^{n+1}(G(A))$ is clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (13.3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \to H^0(G(A)) \to H^0(G(B)) \to H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \to \ldots$$

in $\mathcal{B}$ as desired. □

The proof of the following result uses TR4.

05R0 **Proposition 13.4.23.** *Let $\mathcal{D}$ be a triangulated category. Any commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2] \end{array}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying [1].

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), and (X, Y', A). Note that the morphism $X \to Y'$ is both equal to the composition $X \to Y \to Y'$ and equal to the composition $X \to X' \to Y'$. Hence, we can find morphisms

(1) $a : Z \to A$ and $b : A \to Y''$, and
(2) $a' : X'' \to A$ and $b' : A \to Z'$

as in TR4. Denote $c : Y'' \to Z[1]$ the composition $Y'' \to Y[1] \to Z[1]$ and denote $c' : Z' \to X''[1]$ the composition $Z' \to X'[1] \to X''[1]$. The conclusion of our application TR4 are that

(1) (Z, A, Y'', a, b, c), (X'', A, Z', a', b', c') are distinguished triangles,
(2) $(X, Y, Z) \to (X, Y', A)$, $(X, Y', A) \to (Y, Y', Y'')$, $(X, X', X'') \to (X, Y', A)$, $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \to (X, Y', A)$ and $(X, Y', A) \to (Y, Y', Y'')$. are morphisms of triangles we see the first of the diagrams

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X'' & \xrightarrow{b\circ a'} & Y'' \\ \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] \end{array} \quad \text{and} \quad \begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow{\scriptstyle b'\circ a} & & \downarrow \\ Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

is commutative. The second is commutative too using that $(X, Y, Z) \to (X, Y', A)$ and $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \to Y''$.

Next we apply TR4 one more time to the morphisms $X'' \to A \to Y''$ and the triangles (X'', A, Z', a', b', c'), (X'', Y'', Z''), and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \to Z''$ and $b'' : Z'' \to Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b'\circ a, a'', b'')$. Moreover, $(X'', A, Z') \to (X'', Y'', Z'')$ and $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \to Z'$ factors as $Y' \to A \to Z'$ because $(X, Y', A) \to (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \to Y''$ factors as $Y' \to A \to Y''$ because $(X, Y', A) \to (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \to Z[1]$ is the composition $Y'' \to Y[1] \to Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \to X''[1]$ is the composition $Z' \to X'[1] \to X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. □

13.5. Localization of triangulated categories

05R1 In order to construct the derived category starting from the homotopy category of complexes, we will use a localization process.

05R2 **Definition 13.5.1.** Let $\mathcal{D}$ be a pre-triangulated category. We say a multiplicative system S is *compatible with the triangulated structure* if the following two conditions hold:

MS5 For a morphism f of $\mathcal{D}$ we have $f \in S \Leftrightarrow f[1] \in S$[4].

[4]See Remark 13.5.3.

MS6 Given a solid commutative square

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \vdots & & \downarrow{\scriptstyle s[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

05R3 **Lemma 13.5.2.** *Let $\mathcal{D}$ be a pre-triangulated category. Let $S \subset Arrows(\mathcal{D})$.*

(1) *If S contains all identities and MS6 holds (Definition 13.5.1), then every isomorphism of $\mathcal{D}$ is in S.*
(2) *If MS1, MS5 (Categories, Definition 4.27.1) and MS6 hold, then MS2 holds.*

Proof. Assume S contains all identities and MS6 holds. Let $f : X \to Y$ be an isomorphism of $\mathcal{D}$. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow[1]{} & X & \longrightarrow & 0[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \vdots & & \downarrow{\scriptstyle 1[1]} \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 0[1] \end{array}$$

The rows are distinguished triangles by Lemma 13.4.9. By MS6 we see that the dotted arrow exists and is in S, so f is in S.

Assume MS1, MS5, MS6. Suppose that $f : X \to Y$ is a morphism of $\mathcal{D}$ and $t : X \to X'$ an element of S. Choose a distinguished triangle (X, Y, Z, f, g, h). Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle t} & & \vdots{\scriptstyle s'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle t[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'[1] \end{array}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. □

0H30 **Remark 13.5.3.** In the presence of MS1 and MS6, condition MS5 is equivalent to asking $s[n] \in S$ for all $s \in S$ and $n \in \mathbf{Z}$. For example, suppose MS5 holds, we have $s \in S$, and we want to show $s[-1] \in S$. This isn't immediate because $s[-1][1]$ is not equal to s, only isomorphic to s as an arrow of $\mathcal{D}$. Still, this does imply that $s[-1][1] = f \circ s \circ g$ for isomorphisms f, g. By Lemma 13.5.2 (1) we find $f, g \in S$, hence $s[-1][1] \in S$ by MS1, hence $s[-1] \in S$ by MS5. We leave a complete proof to the reader as an exercise.

05R4 **Lemma 13.5.4.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let*

$$S = \{f \in Arrows(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 4.27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.27.1 and 4.27.20 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 13.4.3. By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, d, h) using TR1. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ d = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[d]{} & Q & \longrightarrow & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle j} & & \\ & & & & W & & \end{array}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 13.4.9. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ d = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. □

05R5 **Lemma 13.5.5.** *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let*

$$S = \{f \in \mathit{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 4.27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.27.1 and 4.27.20 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (13.3.5.1). By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, u, v) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[u]{} & Q & \xrightarrow[v]{} & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle j} & & \\ & & & & W & & \end{array}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (13.3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ u = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. □

05R6 **Proposition 13.5.6.** *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that $[1] \circ Q = Q \circ [1]$ and the localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ is exact. Moreover, if $\mathcal{D}$ is a triangulated category, so is $S^{-1}\mathcal{D}$.*

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 12.8.2. It follows from MS5 that there is a unique additive auto-equivalence $[1] : S^{-1}\mathcal{D} \to S^{-1}\mathcal{D}$ such that $Q \circ [1] = [1] \circ Q$ (equality of functors); we omit the details. We say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q.

Proof of TR1. The only thing to prove here is that if $a : Q(X) \to Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguished triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Choose a distinguished triangle (X, Y', Z, f, g, h) in $\mathcal{D}$. Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of $\mathcal{D}$ and a commutative diagram

$$\begin{array}{ccc} Q(X) & \xrightarrow[Q(f)]{} & Q(Y) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Q(X') & \xrightarrow{Q(f')} & Q(Y') \end{array}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 4.27.10 to find a morphism $f'' : X'' \to Y''$ in $\mathcal{D}$ and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow[k]{} & X'' & \xleftarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{l} & Y'' & \xleftarrow{t} & Y' \end{array}$$

in $\mathcal{D}$ with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for $\mathcal{D}$ and MS6 to find a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle g[1]} \\
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle s[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{array}{c}
(Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \\
\downarrow{\scriptstyle (a,b,c)} \\
(Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h'))
\end{array}$$

This proves the first statement of the lemma. If $\mathcal{D}$ is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 13.4.15 to the following statement: Given composable morphisms $a : Q(X) \to Q(Y)$ and $b : Q(Y) \to Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \to Y$ in $\mathcal{D}$. (More precisely, write $a = s^{-1}f$ with $s : Y \to Y'$ in S and $f : X \to Y'$. Then replace Y by Y'.) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \to Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) in $\mathcal{D}$ (by TR1), and morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$. □

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

05R7 **Lemma 13.5.7.** *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ be the localization functor, see Proposition 13.5.6.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \to \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 4.27.8) is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \to \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 4.27.8) is an exact functor too.*

Proof. This lemma proves itself. Details omitted. □

0GSL **Lemma 13.5.8.** *Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{D}' \subset \mathcal{D}$ be a full, pre-triangulated subcategory. Let S be a saturated multiplicative system of $\mathcal{D}$ compatible with the triangulated structure. Assume that for each X in $\mathcal{D}$ there exists an $s : X' \to X$ in S such that X' is an object of $\mathcal{D}'$. Then $S' = S \cap Arrows(\mathcal{D}')$ is a saturated multiplicative system compatible with the triangulated structure and the functor*

$$(S')^{-1}\mathcal{D}' \longrightarrow S^{-1}\mathcal{D}$$

is an equivalence of pre-triangulated categories.

Proof. Consider the quotient functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ of Proposition 13.5.6. Since S is saturated we have that a morphism $f : X \to Y$ is in S if and only if $Q(f)$ is invertible, see Categories, Lemma 4.27.21. Thus S' is the collection of arrows which are turned into isomorphisms by the composition $\mathcal{D}' \to \mathcal{D} \to S^{-1}\mathcal{D}$. Hence S' is is a saturated multiplicative system compatible with the triangulated structure by Lemma 13.5.4. By Lemma 13.5.7 we obtain the exact functor $(S')^{-1}\mathcal{D}' \to S^{-1}\mathcal{D}$ of pre-triangulated categories. By assumption this functor is essentially surjective. Let X', Y' be objects of $\mathcal{D}'$. By Categories, Remark 4.27.15 we have

$$\mathrm{Mor}_{S^{-1}\mathcal{D}}(X', Y') = \mathrm{colim}_{s : X \to X' \text{ in } S} \mathrm{Mor}_{\mathcal{D}}(X, Y')$$

Our assumption implies that for any $s : X \to X'$ in S we can find a morphism $s' : X'' \to X$ in S with X'' in $\mathcal{D}'$. Then $s \circ s' : X'' \to X'$ is in S'. Hence the colimit above is equal to

$$\mathrm{colim}_{s'' : X'' \to X' \text{ in } S'} \mathrm{Mor}_{\mathcal{D}'}(X'', Y') = \mathrm{Mor}_{(S')^{-1}\mathcal{D}'}(X', Y')$$

This proves our functor is also fully faithful and the proof is complete. □

The following lemma describes the kernel (see Definition 13.6.5) of the localization functor.

05R8 **Lemma 13.5.9.** *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of $\mathcal{D}$. The following are equivalent*

1. *$Q(Z) = 0$ in $S^{-1}\mathcal{D}$,*
2. *there exists $Z' \in \mathrm{Ob}(\mathcal{D})$ such that $0 : Z \to Z'$ is an element of S,*
3. *there exists $Z' \in \mathrm{Ob}(\mathcal{D})$ such that $0 : Z' \to Z$ is an element of S, and*
4. *there exists an object Z' and a distinguished triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.*

If S is saturated, then these are also equivalent to

5. *the morphism $0 \to Z$ is an element of S,*
6. *the morphism $Z \to 0$ is an element of S,*
7. *there exists a distinguished triangle (X, Y, Z, f, g, h) such that $f \in S$.*

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 12.8.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguished triangle (see Lemma 13.4.11) with "$0 \in S$". By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) – (4) are all equivalent.

Next, assume that S is saturated. Note that each of (5), (6), (7) implies one of the equivalent conditions (1) – (4). Suppose that $Q(Z) = 0$. Then $0 \to Z$ is a morphism of $\mathcal{D}$ which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories,

Lemma 4.27.21 the fact that S is saturated implies that $0 \to Z$ is in S. Hence (1) $\Rightarrow$ (5). Dually (1) $\Rightarrow$ (6). Finally, if $0 \to Z$ is in S, then the triangle $(0, Z, Z, 0, \mathrm{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (7). $\square$

05R9 **Lemma 13.5.10.** *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a saturated multiplicative system in $\mathcal{D}$ that is compatible with the triangulated structure. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{D}$. Consider the category of morphisms of triangles*

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h') \mid s, s', s'' \in S\}$$

Then $\mathcal{I}$ is a filtered category and the functors $\mathcal{I} \to X/S$, $\mathcal{I} \to Y/S$, and $\mathcal{I} \to Z/S$ are cofinal.

Proof. We strongly suggest the reader skip the proof of this lemma and instead work it out on a napkin.

The first remark is that using rotation of distinguished triangles (TR2) gives an equivalence of categories between $\mathcal{I}$ and the corresponding category for the distinguished triangle $(Y, Z, X[1], g, h, -f[1])$. Using this we see for example that if we prove the functor $\mathcal{I} \to X/S$ is cofinal, then the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Note that if $s : X \to X'$ is a morphism of S, then using MS2 we can find $s' : Y \to Y'$ and $f' : X' \to Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of $\mathcal{I}$. Hence the functor $\mathcal{I} \to X/S$ is surjective on objects. Using rotation as above this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Suppose given objects $s_1 : X \to X_1$ and $s_2 : X \to X_2$ in X/S and a morphism $a : X_1 \to X_2$ in X/S. Since S is saturated, we see that $a \in S$, see Categories, Lemma 4.27.21. By the argument of the previous paragraph we can complete $s_1 : X \to X_1$ to an object $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ in $\mathcal{I}$. Then we can repeat and find $(a, b, c) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ with $a, b, c \in S$ completing the given $a : X_1 \to X_2$. But then (a, b, c) is a morphism in $\mathcal{I}$. In this way we conclude that the functor $\mathcal{I} \to X/S$ is also surjective on arrows. Using rotation as above, this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

The category $\mathcal{I}$ is nonempty as the identity provides an object. This proves the condition (1) of the definition of a filtered category, see Categories, Definition 4.19.1.

We check condition (2) of Categories, Definition 4.19.1 for the category $\mathcal{I}$. Suppose given objects $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$. We want to find an object of $\mathcal{I}$ which is the target of an arrow from both $(X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(X_2, Y_2, Z_2, f_2, g_2, h_2)$. By Categories, Remark 4.27.7 the categories X/S, Y/S, Z/S are filtered. Thus we can find $X \to X_3$ in X/S and morphisms $s : X_2 \to X_3$ and $a : X_1 \to X_3$. By the above we can find a morphism $(s, s', s'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ with $s', s'' \in S$. After replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) we may assume that there exists a morphism $a : X_1 \to X_2$ in X/S. Repeating the argument for Y and Z (by rotating as above) we may assume there is a morphism $a : X_1 \to X_2$ in X/S, $b : Y_1 \to Y_2$ in Y/S, and $c : Z_1 \to Z_2$ in Z/S. However,

these morphisms do not necessarily give rise to a morphism of distinguished triangles. On the other hand, the necessary diagrams do commute in $S^{-1}\mathcal{D}$. Hence we see (for example) that there exists a morphism $s_2' : Y_2 \to Y_3$ in S such that $s_2' \circ f_2 \circ a = s_2' \circ b \circ f_1$. Another replacement of (X_2, Y_2, Z_2) as above then gets us to the situation where $f_2 \circ a = b \circ f_1$. Rotating and applying the same argument two more times we see that we may assume (a, b, c) is a morphism of triangles. This proves condition (2).

Next we check condition (3) of Categories, Definition 4.19.1. Suppose $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ are objects of $\mathcal{I}$, and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \to X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s_3', s_3'') : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $s_3, s_3', s_3'' \in S$. Thus $(s_3 \circ s_2, s_3' \circ s_2', s_3'' \circ s_2'') : (X, Y, Z) \to (X_3, Y_3, Z_3)$ is also an object of $\mathcal{I}$ and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s_3', s_3'') we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check that $\mathcal{I} \to X/S$ is cofinal, see Categories, Definition 4.17.1. The first condition is true as the functor is surjective. Suppose that we have an object $s : X \to X'$ in X/S and two objects $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$ as well as morphisms $t_1 : X' \to X_1$ and $t_2 : X' \to X_2$ in X/S. By property (2) of $\mathcal{I}$ proved above we can find morphisms $(s_3, s_3', s_3'') : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ and $(s_4, s_4', s_4'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ in $\mathcal{I}$. We would be done if the compositions $X' \to X_1 \to X_3$ and $X' \to X_2 \to X_3$ were equal (see displayed equation in Categories, Definition 4.17.1). If not, then, because X/S is filtered, we can choose a morphism $X_3 \to X_4$ in X/S such that the compositions $X' \to X_1 \to X_3 \to X_4$ and $X' \to X_2 \to X_3 \to X_4$ are equal. Then we finally complete $X_3 \to X_4$ to a morphism $(X_3, Y_3, Z_3) \to (X_4, Y_4, Z_4)$ in $\mathcal{I}$ and compose with that morphism to see that the result is true. □

13.6. Quotients of triangulated categories

05RA Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the "quotient". The construction uses a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 12.10. Before we do the actual construction we briefly discuss kernels of exact functors.

05RB **Definition 13.6.1.** Let $\mathcal{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.

A saturated triangulated subcategory is sometimes called a *thick triangulated subcategory*. In some references, this is only used for strictly full triangulated subcategories (and sometimes the definition is written such that it implies strictness). There is another notion, that of an *épaisse triangulated subcategory*. The definition

is that given a commutative diagram

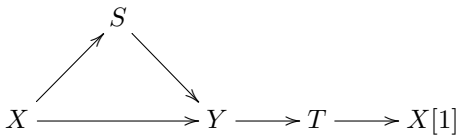

where the second line is a distinguished triangle and S and T are isomorphic to objects of $\mathcal{D}'$, then also X and Y are isomorphic to objects of $\mathcal{D}'$. It turns out that this is equivalent to being saturated (this is elementary and can be found in [**Ric89a**]) and the notion of a saturated category is easier to work with.

05RC **Lemma 13.6.2.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}'') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid F(X) = 0\}$$

Then $\mathcal{D}''$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}''$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}''$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}''$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. □

05RD **Lemma 13.6.3.** *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \in \mathbf{Z}\}$$

Then $\mathcal{D}'$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}'$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}'$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all n, then also $H(Z[n]) = 0$ for all n by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}'$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

05RE **Lemma 13.6.4.** *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}_H^+, \mathcal{D}_H^-, \mathcal{D}_H^b$ be the full subcategory of $\mathcal{D}$ with objects*

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}_H^+) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \ll 0\} \\ \mathrm{Ob}(\mathcal{D}_H^-) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \gg 0\} \\ \mathrm{Ob}(\mathcal{D}_H^b) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for $\mathcal{D}_H^+$. It is clear that it is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}_H^+$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

05RF **Definition 13.6.5.** Let $\mathcal{D}$ be a (pre-)triangulated category.

(1) Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. The *kernel of F* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.2.
(2) Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor. The *kernel of H* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.3.

These are sometimes denoted $\mathrm{Ker}(F)$ or $\mathrm{Ker}(H)$.

The proof of the following lemma uses TR4.

05RG **Lemma 13.6.6.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set*

05RH (13.6.6.1) $$S = \left\{ \begin{array}{c} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \end{array} \right\}$$

Then S is a multiplicative system compatible with the triangulated structure on $\mathcal{D}$. In this situation the following are equivalent

(1) *S is a saturated multiplicative system,*
(2) *$\mathcal{D}'$ is a saturated triangulated subcategory.*

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of $\mathcal{D}$ and because 0 is an object of $\mathcal{D}'$. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms contained in S. Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3). By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathcal{D}'$. By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathcal{D}'$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathcal{D}'$. Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \to Y$ be a morphism and let $t : Z \to X$ be an element of S such that $a \circ t = 0$. To prove LMS3 it suffices to find an $s \in S$ such that $s \circ a = 0$, compare with the proof of Lemma 13.5.4. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there

exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, s, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[g]{} & Q & \longrightarrow & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle s} & & \\ & & & & W & & \end{array}$$

Since $t \in S$ we see that Q is isomorphic to an object of $\mathcal{D}'$. Hence $s \in S$. Finally, $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Lemma 13.4.1. We conclude that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and $\mathcal{D}'$ are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ X' & \longrightarrow & Y' \end{array}$$

with $s, s' \in S$. By Proposition 13.4.23 we can extend this to a nine square diagram. As s, s' are elements of S we see that X'', Y'' are isomorphic to objects of $\mathcal{D}'$. Since $\mathcal{D}'$ is a full triangulated subcategory we see that Z'' is also isomorphic to an object of $\mathcal{D}'$. Whence the morphism $Z \to Z'$ is an element of S. This proves MS6.

MS2 is a formal consequence of MS1, MS5, and MS6, see Lemma 13.5.2. This finishes the proof of the first assertion of the lemma.

Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of $\mathcal{D}'$. Consider the morphism $f : 0 \to X$. The composition $0 \to X \to X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \to 0 \to X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, see Lemma 13.4.11. Hence $0 \to X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of $\mathcal{D}'$ as desired.

Finally, assume $\mathcal{D}'$ is a saturated triangulated subcategory. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of $\mathcal{D}$ such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.

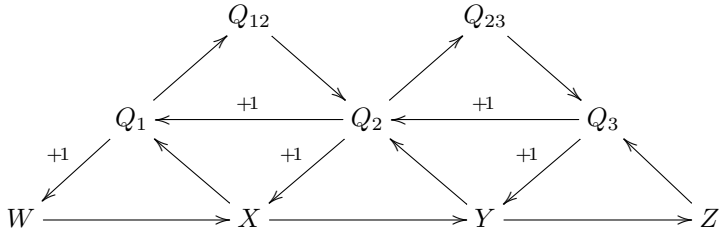

First choose distinguished triangles (W, X, Q_1), (X, Y, Q_2), (Y, Z, Q_3) (W, Y, Q_{12}), and (X, Z, Q_{23}). Denote $s : Q_2 \to Q_1[1]$ the composition $Q_2 \to X[1] \to Q_1[1]$. Denote $t : Q_3 \to Q_2[1]$ the composition $Q_3 \to Y[1] \to Q_2[1]$. By TR4 applied to the composition $W \to X \to Y$ and the composition $X \to Y \to Z$ there exist distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t. The objects Q_{12} and Q_{23} are isomorphic to objects of $\mathcal{D}'$ as $W \to Y$ and $X \to Z$ are assumed in S. Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \to Q_2[1] \to X[2]$ is zero, see Lemma 13.4.1. Hence $Q_3[1] \oplus Q_1[2]$ is isomorphic to an object of $\mathcal{D}'$, see Lemma 13.4.11. By assumption on $\mathcal{D}'$ we conclude that Q_3 and Q_1 are isomorphic to objects of $\mathcal{D}'$. Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of $\mathcal{D}'$. Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) □

05RI **Definition 13.6.7.** Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the *quotient category* $\mathcal{D}/\mathcal{B}$ by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of $\mathcal{D}$ associated to $\mathcal{B}$ via Lemma 13.6.6. The localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is called the *quotient functor* in this case.

Note that the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 13.5.6. The universal property of this construction is the following.

05RJ **Lemma 13.6.8.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ be the quotient functor.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $\mathcal{B} \subset \mathrm{Ker}(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $\mathcal{B} \subset \mathrm{Ker}(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \to \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.*

Proof. This lemma follows from Lemma 13.5.7. Namely, if $f : X \to Y$ is a morphism of $\mathcal{D}$ such that for some distinguished triangle (X, Y, Z, f, g, h) the object Z is isomorphic to an object of $\mathcal{B}$, then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. □

The kernel of the quotient functor can be described as follows.

05RK **Lemma 13.6.9.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of $\mathcal{D}$ whose objects are*

$$\mathrm{Ob}(\mathrm{Ker}(Q)) = \left\{ \begin{array}{l} Z \in \mathrm{Ob}(\mathcal{D}) \text{ such that there exists a } Z' \in \mathrm{Ob}(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

Proof. First note that the kernel is automatically a strictly full triangulated subcategory containing summands of any of its objects, see Lemma 13.6.2. The description of its objects follows from the definitions and Lemma 13.5.9 part (4). □

Let $\mathcal{D}$ be a triangulated category. At this point we have constructions which induce order preserving maps between

(1) the partially ordered set of multiplicative systems S in $\mathcal{D}$ compatible with the triangulated structure, and
(2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (13.6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{array}{c} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

Note that it is not the case that these operations are mutually inverse.

05RL **Lemma 13.6.10.** *Let $\mathcal{D}$ be a triangulated category. The operations described above have the following properties*

(1) *$S(\mathcal{B}(S))$ is the "saturation" of S, i.e., it is the smallest saturated multiplicative system in $\mathcal{D}$ containing S, and*
(2) *$\mathcal{B}(S(\mathcal{B}))$ is the "saturation" of $\mathcal{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathcal{D}$ compatible with the triangulated structure on $\mathcal{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathcal{D}$.

Proof. First, let's start with a full triangulated subcategory $\mathcal{B}$. Then $\mathcal{B}(S(\mathcal{B})) = \mathrm{Ker}(Q : \mathcal{D} \to \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 13.6.9.

Next, suppose that S is multiplicative system in $\mathcal{D}$ compatible with the triangulation on $\mathcal{D}$. Then $\mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$. Hence (using Lemma 13.4.9 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) &= \left\{ \begin{array}{c} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished} \\ \text{triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \end{array} \right\}. \\ &= \{f \in \mathrm{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism}\} \\ &= \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 4.27.21. The final statement of that lemma finishes the proof. □

05RM **Lemma 13.6.11.** *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$, see Definition 13.3.5. The subcategory $\mathrm{Ker}(H)$ of $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ whose corresponding saturated multiplicative system (see Lemma 13.6.10) is the set*

$$S = \{f \in \mathrm{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathrm{Ker}(H)$.

Proof. The category $\mathrm{Ker}(H)$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ by Lemma 13.6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 13.5.5. Recall that the multiplicative system corresponding to $\mathrm{Ker}(H)$ is the set

$$\left\{ \begin{array}{c} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \end{array} \right\}.$$

By the long exact cohomology sequence, see (13.3.5.1), it is clear that f is an element of this set if and only if f is an element of S. Finally, the factorization of H through Q is a consequence of Lemma 13.6.8. □

13.7. Adjoints for exact functors

0A8C Results on adjoint functors between triangulated categories.

0A8D **Lemma 13.7.1.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If F admits a right adjoint $G : \mathcal{D}' \to \mathcal{D}$, then G is also an exact functor.*

Proof. Let X be an object of $\mathcal{D}$ and A an object of $\mathcal{D}'$. Since F is an exact functor we see that

$$\begin{aligned}\mathrm{Mor}_{\mathcal{D}}(X, G(A[1])) &= \mathrm{Mor}_{\mathcal{D}'}(F(X), A[1]) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X)[-1], A) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X[-1]), A) \\ &= \mathrm{Mor}_{\mathcal{D}}(X[-1], G(A)) \\ &= \mathrm{Mor}_{\mathcal{D}}(X, G(A)[1])\end{aligned}$$

By Yoneda's lemma (Categories, Lemma 4.3.5) we obtain a canonical isomorphism $G(A)[1] = G(A[1])$. Let $A \to B \to C \to A[1]$ be a distinguished triangle in $\mathcal{D}'$. Choose a distinguished triangle

$$G(A) \to G(B) \to X \to G(A)[1]$$

in $\mathcal{D}$. Then $F(G(A)) \to F(G(B)) \to F(X) \to F(G(A))[1]$ is a distinguished triangle in $\mathcal{D}'$. By TR3 we can choose a morphism of distinguished triangles

$$\begin{array}{ccccccc} F(G(A)) & \longrightarrow & F(G(B)) & \longrightarrow & F(X) & \longrightarrow & F(G(A))[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

Since G is the adjoint the new morphism determines a morphism $X \to G(C)$ such that the diagram

$$\begin{array}{ccccccc} G(A) & \longrightarrow & G(B) & \longrightarrow & X & \longrightarrow & G(A)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G(A) & \longrightarrow & G(B) & \longrightarrow & G(C) & \longrightarrow & G(A)[1] \end{array}$$

commutes. Applying the homological functor $\mathrm{Hom}_{\mathcal{D}'}(W, -)$ for an object W of $\mathcal{D}'$ we deduce from the 5 lemma that

$$\mathrm{Hom}_{\mathcal{D}'}(W, X) \to \mathrm{Hom}_{\mathcal{D}'}(W, G(C))$$

is a bijection and using the Yoneda lemma once more we conclude that $X \to G(C)$ is an isomorphism. Hence we conclude that $G(A) \to G(B) \to G(C) \to G(A)[1]$ is a distinguished triangle which is what we wanted to show. □

09J1 **Lemma 13.7.2.** *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be functors. Assume that*

(1) *F and G are exact functors,*
(2) *F is fully faithful,*

(3) *G is a right adjoint to F, and*
(4) *the kernel of G is zero.*

Then F is an equivalence of categories.

Proof. Since F is fully faithful the adjunction map $\mathrm{id} \to G \circ F$ is an isomorphism (Categories, Lemma 4.24.4). Let X be an object of $\mathcal{D}'$. Choose a distinguished triangle

$$F(G(X)) \to X \to Y \to F(G(X))[1]$$

in $\mathcal{D}'$. Applying G and using that $G(F(G(X))) = G(X)$ we find a distinguished triangle

$$G(X) \to G(X) \to G(Y) \to G(X)[1]$$

Hence $G(Y) = 0$. Thus $Y = 0$. Thus $F(G(X)) \to X$ is an isomorphism. □

13.8. The homotopy category

05RN Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

013H **Definition 13.8.1.** Let $\mathcal{A}$ be an additive category.

(1) We set $\mathrm{Comp}(\mathcal{A}) = \mathrm{CoCh}(\mathcal{A})$ be the *category of (cochain) complexes*.
(2) A complex $K^\bullet$ is said to be *bounded below* if $K^n = 0$ for all $n \ll 0$.
(3) A complex $K^\bullet$ is said to be *bounded above* if $K^n = 0$ for all $n \gg 0$.
(4) A complex $K^\bullet$ is said to be *bounded* if $K^n = 0$ for all $|n| \gg 0$.
(5) We let $\mathrm{Comp}^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A})$, resp. $\mathrm{Comp}^b(\mathcal{A})$ be the full subcategory of $\mathrm{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
(6) We let $K(\mathcal{A})$ be the category with the same objects as $\mathrm{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 12.13.7).
(7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of $\mathcal{A}$.

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

13.9. Cones and termwise split sequences

014D Let $\mathcal{A}$ be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 12.14.7, give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}$.

014E **Definition 13.9.1.** Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. The *cone* of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \to C(f)^\bullet$ and $p : C(f)^\bullet \to K^\bullet[1]$ induced by the obvious maps $L^n \to C(f)^n \to K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

014F **Lemma 13.9.2.** *Suppose that*

$$\begin{array}{ccc} K_1^\bullet & \xrightarrow{f_1} & L_1^\bullet \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K_2^\bullet & \xrightarrow{f_2} & L_2^\bullet \end{array}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \to (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \to L_2^{n-1}$ be a family of morphisms such that $b \circ f_1 - f_2 \circ a = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : L_1^n \oplus K_1^{n+1} \to L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

Note that the morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ constructed in the proof of Lemma 13.9.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

08RI **Lemma 13.9.3.** *Suppose that $f : K^\bullet \to L^\bullet$ and $g : L^\bullet \to M^\bullet$ are morphisms of complexes such that $g \circ f$ is homotopic to zero. Then*

(1) *g factors through a morphism $C(f)^\bullet \to M^\bullet$, and*
(2) *f factors through a morphism $K^\bullet \to C(g)^\bullet[-1]$.*

Proof. The assumptions say that the diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{f} & L^\bullet \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & M^\bullet \end{array}$$

commutes up to homotopy. Since the cone on $0 \to M^\bullet$ is $M^\bullet$ the map $C(f)^\bullet \to C(0 \to M^\bullet) = M^\bullet$ of Lemma 13.9.2 is the map in (1). The cone on $K^\bullet \to 0$ is $K^\bullet[1]$ and applying Lemma 13.9.2 gives a map $K^\bullet[1] \to C(g)^\bullet$. Applying $[-1]$ we obtain the map in (2). □

Note that the morphisms $C(f)^\bullet \to M^\bullet$ and $K^\bullet \to C(g)^\bullet[-1]$ constructed in the proof of Lemma 13.9.3 in general depend on the chosen homotopy.

014G **Definition 13.9.4.** Let $\mathcal{A}$ be an additive category. A *termwise split injection* $\alpha : A^\bullet \to B^\bullet$ is a morphism of complexes such that each $A^n \to B^n$ is isomorphic to the inclusion of a direct summand. A *termwise split surjection* $\beta : B^\bullet \to C^\bullet$ is a morphism of complexes such that each $B^n \to C^n$ is isomorphic to the projection onto a direct summand.

014H **Lemma 13.9.5.** *Let $\mathcal{A}$ be an additive category. Let*

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ C^\bullet & \xrightarrow{g} & D^\bullet \end{array}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a termwise split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a termwise split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \to D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \to A^n$ are morphisms splitting the morphisms f^n. Take $b' = b - dh\pi - h\pi d$. Suppose $s^n : D^n \to C^n$ are morphisms splitting the morphisms $g^n : C^n \to D^n$. Take $a' = a + dsh + shd$. Computations omitted. □

The following lemma can be used to replace a morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

013N **Lemma 13.9.6.** *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$K^\bullet \xrightarrow{\tilde{\alpha}} \tilde{L}^\bullet \xrightarrow{\pi} L^\bullet, \quad \alpha : K^\bullet \to L^\bullet$$

such that

(1) *$\tilde{\alpha}$ is a termwise split injection (see Definition 13.9.4),*
(2) *there is a map of complexes $s : L^\bullet \to \tilde{L}^\bullet$ such that $\pi \circ s = id_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{L}^\bullet$.

Proof. We set

$$\tilde{L}^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d_{\tilde{L}}^n = \begin{pmatrix} d_L^n & 0 & 0 \\ 0 & d_K^n & \mathrm{id}_{K^{n+1}} \\ 0 & 0 & -d_K^{n+1} \end{pmatrix}$$

In other words, $\tilde{L}^\bullet = L^\bullet \oplus C(1_{K^\bullet})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha \\ \mathrm{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = \begin{pmatrix} \mathrm{id}_{L^n} & 0 & 0 \end{pmatrix}$$

so that clearly $\pi \circ \tilde{\alpha} = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \mathrm{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \to \tilde{L}^{n-1}$ be the map which maps the summand K^n of $\tilde{L}^n$ via the identity morphism to the summand K^n of $\tilde{L}^{n-1}$. Then it is a trivial matter (see computations in remark below) to prove that

$$\mathrm{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

013O **Remark 13.9.7.** To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

0642 **Lemma 13.9.8.** *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$K^\bullet \xrightarrow{i} \tilde{K}^\bullet \xrightarrow{\tilde{\alpha}} L^\bullet, \quad K^\bullet \xrightarrow{\alpha} L^\bullet$$

such that

(1) *$\tilde{\alpha}$ is a termwise split surjection (see Definition 13.9.4),*
(2) *there is a map of complexes $s : \tilde{K}^\bullet \to K^\bullet$ such that $s \circ i = id_{K^\bullet}$ and such that $i \circ s$ is homotopic to $id_{\tilde{K}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{K}^\bullet$.

Proof. Dual to Lemma 13.9.6. Take

$$\tilde{K}^n = K^n \oplus L^{n-1} \oplus L^n$$

and we define

$$d^n_{\tilde{K}} = \begin{pmatrix} d^n_K & 0 & 0 \\ 0 & -d^{n-1}_L & \mathrm{id}_{L^n} \\ 0 & 0 & d^n_L \end{pmatrix}$$

in other words $\tilde{K}^\bullet = K^\bullet \oplus C(1_{L^\bullet[-1]})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & \mathrm{id}_{L^n} \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \mathrm{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \mathrm{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \to \tilde{K}^{n-1}$ be the map which maps the summand L^{n-1} of $\tilde{K}^n$ via the identity morphism to the summand L^{n-1} of $\tilde{K}^{n-1}$. Then it is a trivial matter to prove that

$$\mathrm{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

014I **Definition 13.9.9.** Let $\mathcal{A}$ be an additive category. A *termwise split exact sequence of complexes of* $\mathcal{A}$ is a complex of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n. We often write $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 12.14.10 we get an associated morphism of complexes

$$\delta : C^\bullet \longrightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d_B^n \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

This will be the *triangle associated to the termwise split sequence of complexes.*

05SS **Lemma 13.9.10.** *Let $\mathcal{A}$ be an additive category. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be termwise split exact sequences as in Definition 13.9.9. Let $(\pi')^n$, $(s')^n$ be a second collection of splittings. Denote $\delta' : C^\bullet \longrightarrow A^\bullet[1]$ the morphism associated to this second set of splittings. Then*

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \longrightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$.

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 12.14.12. □

014J **Remark 13.9.11.** Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \to A_2^\bullet$, $b : B_1^\bullet \to B_2^\bullet$, and $c : C_1^\bullet \to C_2^\bullet$ are morphisms of complexes such that

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array}$$

commutes in $K(\mathcal{A})$. In general, there does **not** exist a morphism $b' : B_1^\bullet \to B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes. Namely, consider Examples, Equation (110.64.0.1). If we could replace the middle map there by a homotopic one such that the diagram commutes, then we would have additivity of traces which we do not.

086L **Lemma 13.9.12.** *Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2, 3$ be termwise split exact sequences of complexes. Let $b : B_1^\bullet \to B_2^\bullet$ and $b' : B_2^\bullet \to B_3^\bullet$ be morphisms of complexes such that*

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ {\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array} \quad \text{and} \quad \begin{array}{ccccc} A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\ A_3^\bullet & \longrightarrow & B_3^\bullet & \longrightarrow & C_3^\bullet \end{array}$$

commute in $K(\mathcal{A})$. Then $b' \circ b = 0$ in $K(\mathcal{A})$.

Proof. By Lemma 13.9.5 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b^n) \subset \mathrm{Im}(A_2^n \to B_2^n)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(A_2^n \to B_2^n)$. Then $b' \circ b = 0$ as a map of complexes. □

014K **Lemma 13.9.13.** *Let $\mathcal{A}$ be an additive category. Let $f_1 : K_1^\bullet \to L_1^\bullet$ and $f_2 : K_2^\bullet \to L_2^\bullet$ be morphisms of complexes. Let*

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \longrightarrow (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2^\bullet \to K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \to L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \to C(f_1)^\bullet$ be the morphism from Lemma 13.9.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{array}{ccccc} L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \\ {\scriptstyle 1}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle 1}\downarrow \\ L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. Thus we see that $(c-1)^2 = 0$ in $K(\mathcal{A})$ by Lemma 13.9.12. Hence c is an isomorphism in $K(\mathcal{A})$ with inverse $2 - c$. □

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

014L **Lemma 13.9.14.** *Let $\mathcal{A}$ be an additive category.*

(1) *Given a termwise split sequence of complexes $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ there exists a homotopy equivalence $C(\alpha)^\bullet \to C^\bullet$ such that the diagram*

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C(\alpha)^\bullet & \xrightarrow[-p]{} & A^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \xrightarrow{\delta} & A^\bullet[1] \end{array}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

(2) *Given a morphism of complexes $f : K^\bullet \to L^\bullet$ there exists an isomorphism of triangles*

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & \tilde{L}^\bullet & \longrightarrow & M^\bullet & \xrightarrow[\delta]{} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & C(f)^\bullet & \xrightarrow{-p} & K^\bullet[1] \end{array}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \to \tilde{L}^\bullet \to M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \to C^n$ via the projection onto B^n followed by β^n. This defines a morphism of complexes because the compositions $A^{n+1} \to B^{n+1} \to C^{n+1}$ are zero. To get a homotopy inverse we take $C^\bullet \to C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n. This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \to A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $C^\bullet \to C(\alpha)^\bullet \to C^\bullet$ is the identity. The composition $C(\alpha)^\bullet \to C^\bullet \to C(\alpha)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \to C(\alpha)^{n-1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \to B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n \\ -\delta^n \end{pmatrix} \begin{pmatrix} \beta^n & 0 \end{pmatrix} = \begin{pmatrix} d & \alpha^n \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \to A^\bullet[1]$ (see Definition 13.9.1) and $C(\alpha)^\bullet \to C^\bullet \to A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \to C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \to A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \to \tilde{L}^\bullet$, $s : L^\bullet \to \tilde{L}^\bullet$ and $\pi : \tilde{L}^\bullet \to L^\bullet$ be as in Lemma 13.9.6. By Lemmas 13.9.2 and 13.9.13 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace $L^\bullet$ by $\tilde{L}^\bullet$ and f by $\tilde{f}$. In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). □

014M **Lemma 13.9.15.** *Let $\mathcal{A}$ be an additive category. Let $A_1^\bullet \to A_2^\bullet \to \ldots \to A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram*

$$\begin{array}{ccccccc} A_1^\bullet & \longrightarrow & A_2^\bullet & \longrightarrow & \ldots & \longrightarrow & A_n^\bullet \\ \uparrow & & \uparrow & & & & \uparrow \\ B_1^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & \ldots & \longrightarrow & B_n^\bullet \end{array}$$

such that each morphism $B_i^\bullet \to B_{i+1}^\bullet$ is a termwise split injection and each $B_i^\bullet \to A_i^\bullet$ is a homotopy equivalence. Moreover, if all $A_i^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the $B_i^\bullet$.

Proof. The case $n = 1$ is without content. Lemma 13.9.6 is the case $n = 2$. Suppose we have constructed the diagram except for $B_n^\bullet$. Apply Lemma 13.9.6 to the composition $B_{n-1}^\bullet \to A_{n-1}^\bullet \to A_n^\bullet$. The result is a factorization $B_{n-1}^\bullet \to B_n^\bullet \to A_n^\bullet$ as desired. □

014N **Lemma 13.9.16.** *Let $\mathcal{A}$ be an additive category. Let $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.*

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \to A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. □

014O **Lemma 13.9.17.** *Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence*

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

13.10. Distinguished triangles in the homotopy category

014P Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

014Q **Definition 13.10.1.** Let $\mathcal{A}$ be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a *distinguished triangle of $K(\mathcal{A})$* if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 13.9.9. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 13.9.14 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Proposition 13.10.3. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

014R **Lemma 13.10.2.** *Let $\mathcal{A}$ be an additive category. Suppose that $\alpha : A^\bullet \to B^\bullet$ and $\beta : B^\bullet \to C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1^n : B^n \to A^n$, and $\pi_3^n : C^n \to B^n$ are the splittings. Then also $A^\bullet \to C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write $Q_1^\bullet$, $Q_2^\bullet$ and $Q_3^\bullet$ for the "quotient" complexes. In other words, $Q_1^n = \mathrm{Ker}(\pi_1^n)$, $Q_3^n = \mathrm{Ker}(\pi_3^n)$ and $Q_2^n = \mathrm{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C^n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n. Hence

$Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & Q_1^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^\bullet & \to & C^\bullet & \to & Q_2^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & C^\bullet & \to & Q_3^\bullet & \to & 0 \end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \to B^\bullet \to Q_1^\bullet \to A^\bullet[1]) \longrightarrow (A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1])$$

and

$$(A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1]) \longrightarrow (B^\bullet \to C^\bullet \to Q_3^\bullet \to B^\bullet[1]).$$

Note that the splittings $Q_3^n \to C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \to Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to 0$ upon composing with $C^n \to Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \to Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \to B^\bullet[1] \to Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

014S **Proposition 13.10.3.** *Let $\mathcal{A}$ be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \to A^\bullet \to A^\bullet \to 0 \to 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 13.9.14.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \to B^\bullet \to C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 13.9.16 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 13.9.14. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 13.9.14 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 13.9.17 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ$

f. By Lemma 13.9.14 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 13.9.2 to the commutative diagram given by f, f', a, b.

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 13.4.15. Let $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 13.9.15 we may assume that $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 13.10.2. □

05RP **Remark 13.10.4.** Let $\mathcal{A}$ be an additive category. Exactly the same proof as the proof of Proposition 13.10.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphism between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

05RQ **Lemma 13.10.5.** *Let $\mathcal{A}$ be an additive category. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.*

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 13.4.16 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \to L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 13.9.6 and then one can directly take the associated distinguished triangle.) □

014X **Lemma 13.10.6.** *Let $\mathcal{A}$, $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The induced functors*

$$\begin{aligned} F : K(\mathcal{A}) &\longrightarrow K(\mathcal{B}) \\ F : K^+(\mathcal{A}) &\longrightarrow K^+(\mathcal{B}) \\ F : K^-(\mathcal{A}) &\longrightarrow K^-(\mathcal{B}) \\ F : K^b(\mathcal{A}) &\longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \to B^\bullet \to C^\bullet$ is a termwise split sequence of complexes of $\mathcal{A}$ with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \to A^\bullet[1]$, see Definition 13.9.9. Then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \to F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. □

0G6C **Lemma 13.10.7.** *Let $\mathcal{A}$ be an additive category. Let $(A^\bullet, B^\bullet, C^\bullet, a, b, c)$ be a distinguished triangle in $K(\mathcal{A})$. Then there exists an isomorphic distinguished triangle $(A^\bullet, (B')^\bullet, C^\bullet, a', b', c)$ such that $0 \to A^n \to (B')^n \to C^n \to 0$ is a split short exact sequence for all n.*

Proof. We will use that $K(\mathcal{A})$ is a triangulated category by Proposition 13.10.3. Let $W^\bullet$ be the cone on $c : C^\bullet \to A^\bullet[1]$ with its maps $i : A^\bullet[1] \to W^\bullet$ and $p : W^\bullet \to C^\bullet[1]$. Then $(C^\bullet, A^\bullet[1], W^\bullet, c, i, -p)$ is a distinguished triangle by Lemma 13.9.14. Rotating backwards twice we see that $(A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$

is a distinguished triangle. By TR3 there is a morphism of distinguished triangles $(\mathrm{id}, \beta, \mathrm{id}) : (A^\bullet, B^\bullet, C^\bullet, a, b, c) \to (A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$ which must be an isomorphism by Lemma 13.4.3. This finishes the proof because $0 \to A^\bullet \to W^\bullet[-1] \to C^\bullet \to 0$ is a termwise split short exact sequence of complexes by the very construction of cones in Section 13.9. □

0G6D **Remark 13.10.8.** Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\mathrm{DoubleComp}(\mathcal{A})$ denote the category of double complexes in $\mathcal{A}$, see Homology, Section 12.18. We can use this category to construct two triangulated categories.

(1) We can consider an object $A^{\bullet,\bullet}$ of $\mathrm{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows
$$\ldots \to A^{\bullet,-1} \to A^{\bullet,0} \to A^{\bullet,1} \to \ldots$$
and take the homotopy category $K_{first}(\mathrm{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 13.10.3. By Homology, Remark 12.18.6 the functor
$$\mathrm{Tot} : K_{first}(\mathrm{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$
is an exact functor of triangulated categories.

(2) We can consider an object $A^{\bullet,\bullet}$ of $\mathrm{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows
$$\ldots \to A^{-1,\bullet} \to A^{0,\bullet} \to A^{1,\bullet} \to \ldots$$
and take the homotopy category $K_{second}(\mathrm{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 13.10.3. By Homology, Remark 12.18.7 the functor
$$\mathrm{Tot} : K_{second}(\mathrm{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$
is an exact functor of triangulated categories.

0G6E **Remark 13.10.9.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be additive categories and assume $\mathcal{C}$ has countable direct sums. Suppose that
$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$
is a functor which is bilinear on morphisms. This determines a functor
$$\mathrm{Comp}(\mathcal{A}) \times \mathrm{Comp}(\mathcal{B}) \longrightarrow \mathrm{DoubleComp}(\mathcal{C}), \quad (X^\bullet, Y^\bullet) \longmapsto X^\bullet \otimes Y^\bullet$$
See Homology, Example 12.18.2.

(1) For a fixed object $X^\bullet$ of $\mathrm{Comp}(\mathcal{A})$ the functor
$$K(\mathcal{B}) \longrightarrow K(\mathcal{C}), \quad Y^\bullet \longmapsto \mathrm{Tot}(X^\bullet \otimes Y^\bullet)$$
is an exact functor of triangulated categories.

(2) For a fixed object $Y^\bullet$ of $\mathrm{Comp}(\mathcal{B})$ the functor
$$K(\mathcal{A}) \longrightarrow K(\mathcal{C}), \quad X^\bullet \longmapsto \mathrm{Tot}(X^\bullet \otimes Y^\bullet)$$
is an exact functor of triangulated categories.

This follows from Remark 13.10.8 since the functors $\mathrm{Comp}(\mathcal{A}) \to \mathrm{DoubleComp}(\mathcal{C})$, $Y^\bullet \mapsto X^\bullet \otimes Y^\bullet$ and $\mathrm{Comp}(\mathcal{B}) \to \mathrm{DoubleComp}(\mathcal{C})$, $X^\bullet \mapsto X^\bullet \otimes Y^\bullet$ are immediately seen to be compatible with homotopies and termwise split short exact sequences and hence induce exact functors of triangulated categories
$$K(\mathcal{B}) \to K_{first}(\mathrm{DoubleComp}(\mathcal{C})) \quad \text{and} \quad K(\mathcal{A}) \to K_{second}(\mathrm{DoubleComp}(\mathcal{C}))$$

Observe that for the first of the two the isomorphism

$$\mathrm{Tot}(X^\bullet \otimes Y^\bullet[1]) \cong \mathrm{Tot}(X^\bullet \otimes Y^\bullet)[1]$$

involves signs (this goes back to the signs chosen in Homology, Remark 12.18.5).

13.11. Derived categories

05RR In this section we construct the derived category of an abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 12.13.11. Moreover, in Homology, Definition 12.14.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

05RS **Lemma 13.11.1.** *Let $\mathcal{A}$ be an abelian category. The functor*

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 12.13.12. □

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence
(13.11.1.1)

05ST $$\ldots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \ldots$$

see (13.3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 13.9.9), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 12.13.12 and for agreement of sequences, see Homology, Lemma 12.14.11.

Recall that a complex $K^\bullet$ is *acyclic* if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if and only if $H^i(f)$ is an isomorphism for all i. See Homology, Definition 12.13.10.

05RT **Lemma 13.11.2.** *Let $\mathcal{A}$ be an abelian category. The full subcategory $Ac(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(\mathcal{A})$ is the set $Qis(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \to Qis(\mathcal{A})^{-1}K(\mathcal{A})$ is $Ac(\mathcal{A})$ and the functor H^0 factors through Q.*

Proof. We know that H^0 is a homological functor by Lemma 13.11.1. Thus this lemma is a special case of Lemma 13.6.11. □

05RU **Definition 13.11.3.** Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Ac}(\mathcal{A})$ and $\mathrm{Qis}(\mathcal{A})$ be as in Lemma 13.11.2. The *derived category of* $\mathcal{A}$ is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\mathrm{Ac}(\mathcal{A}) = \mathrm{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \to \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 13.6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{aligned}
\mathrm{Ob}(D^+(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\
\mathrm{Ob}(D^-(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\
\mathrm{Ob}(D^b(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\}
\end{aligned}$$

The category $D^b(\mathcal{A})$ is called the *bounded derived category* of $\mathcal{A}$.

If $K^\bullet$ and $L^\bullet$ are complexes of $\mathcal{A}$ then we sometimes say "$K^\bullet$ is *quasi-isomorphic* to $L^\bullet$" to indicate that $K^\bullet$ and $L^\bullet$ are isomorphic objects of $D(\mathcal{A})$.

09PA **Remark 13.11.4.** In this chapter, we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$, it isn't clear that the derived category $D(\mathcal{A})$ exists, because it isn't clear that morphisms in the derived category are sets. In fact, in general they aren't, see Examples, Lemma 110.62.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Injectives, Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Lemma 13.31.2 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

05RV **Lemma 13.11.5.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex.*

(1) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$, then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$, then there exists a quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^\bullet$ bounded above.*
(3) *If $H^n(K^\bullet) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq b}K^\bullet$, $L^\bullet = \tau_{\geq a}K^\bullet$, and $N^\bullet = \tau_{\leq b}L^\bullet = \tau_{\geq a}M^\bullet$. See Homology, Section 12.15 for the truncation functors. □

To state the following lemma denote $\mathrm{Ac}^+(\mathcal{A})$, $\mathrm{Ac}^-(\mathcal{A})$, resp. $\mathrm{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Ac}(\mathcal{A})$. Denote $\mathrm{Qis}^+(\mathcal{A})$, $\mathrm{Qis}^-(\mathcal{A})$, resp. $\mathrm{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Qis}(\mathcal{A})$.

05RW **Lemma 13.11.6.** *Let $\mathcal{A}$ be an abelian category. The subcategories $Ac^+(\mathcal{A})$, $Ac^-(\mathcal{A})$, resp. $Ac^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $Qis^+(\mathcal{A})$, $Qis^-(\mathcal{A})$, resp. $Qis^b(\mathcal{A})$.*

(1) *The kernel of the functor $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is $Ac^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^+(\mathcal{A})/Ac^+(\mathcal{A}) = Qis^+(\mathcal{A})^{-1}K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$

(2) *The kernel of the functor $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ is $Ac^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^-(\mathcal{A})/Ac^-(\mathcal{A}) = Qis^-(\mathcal{A})^{-1}K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$

(3) *The kernel of the functor $K^b(\mathcal{A}) \to D^b(\mathcal{A})$ is $Ac^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^b(\mathcal{A})/Ac^b(\mathcal{A}) = Qis^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 13.6.11 by considering the restriction of the homological functor H^0. The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 13.11.5. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that $K^\bullet, L^\bullet$ are bounded below complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 13.11.5 we can choose a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. Hence the functor is "full". Finally, suppose that the pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ such that $s' \circ f = 0$. Using Lemma 13.11.5 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \to (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$ which implies that $s^{-1}f$ is zero in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\mathrm{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \to \mathrm{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. □

13.12. The canonical delta-functor

014Z The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ-functor from an abelian category into a triangulated category, see Definition 13.3.6.

Consider the functor $\mathrm{Comp}(\mathcal{A}) \to K(\mathcal{A})$. This functor is **not** a δ-functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \to A \to B \to C \to 0$ of objects of $\mathcal{A}$. Since $\mathrm{Hom}_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\mathrm{Comp}(\mathcal{A}) \to D(\mathcal{A})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

of complexes in the abelian category $\mathcal{A}$. Consider the cone $C(a)^\bullet$ of the morphism a. We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \to C^n$ via the projection to B^n followed by b^n. Hence a morphism of complexes

$$q : C(a)^\bullet \longrightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 13.9.1. Note that, as $a^\bullet$ is injective in each degree, the kernel of q is identified with the cone of $\mathrm{id}_{A^\bullet}$ which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 13.9.14 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

0152 **Lemma 13.12.1.** *Let $\mathcal{A}$ be an abelian category. The functor $Comp(\mathcal{A}) \to D(\mathcal{A})$ defined has the natural structure of a δ-functor, with*

$$\delta_{A^\bullet \to B^\bullet \to C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors $Comp^+(\mathcal{A}) \to D^+(\mathcal{A})$, $Comp^-(\mathcal{A}) \to D^-(\mathcal{A})$, and $Comp^b(\mathcal{A}) \to D^b(\mathcal{A})$ into δ-functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\bullet & \xrightarrow[a]{} & B^\bullet & \xrightarrow[b]{} & C^\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & (A')^\bullet & \xrightarrow{a'} & (B')^\bullet & \xrightarrow{b'} & (C')^\bullet & \longrightarrow & 0
\end{array}$$

we get the desired commutative diagram of Definition 13.3.6 (2). By Lemma 13.9.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \to C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. $\square$

0153 **Lemma 13.12.2.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D^\bullet & \longrightarrow & E^\bullet & \longrightarrow & F^\bullet & \longrightarrow & 0 \end{array}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ-functor of Lemma 13.12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \to D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. □

0154 **Lemma 13.12.3.** *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ-functor of Lemma 13.12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 13.9.14. □

08J5 **Remark 13.12.4.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$. Let $a \in \mathbf{Z}$. We claim there is a canonical distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{A})$. Here we have used the canonical truncation functors τ from Homology, Section 12.15. Namely, we first take the distinguished triangle associated by our δ-functor (Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \tau_{\leq a}K^\bullet \to K^\bullet \to K^\bullet/\tau_{\leq a}K^\bullet \to 0$$

Next, we use that the map $K^\bullet \to \tau_{\geq a+1}K^\bullet$ factors through a quasi-isomorphism $K^\bullet/\tau_{\leq a}K^\bullet \to \tau_{\geq a+1}K^\bullet$ by the description of cohomology groups in Homology, Section 12.15. In a similar way we obtain canonical distinguished triangles

$$\tau_{\leq a}K^\bullet \to \tau_{\leq a+1}K^\bullet \to H^{a+1}(K^\bullet)[-a-1] \to (\tau_{\leq a}K^\bullet)[1]$$

and

$$H^a(K^\bullet)[-a] \to \tau_{\geq a}K^\bullet \to \tau_{\geq a+1}K^\bullet \to H^a(K^\bullet)[-a+1]$$

08Q2 **Lemma 13.12.5.** *Let $\mathcal{A}$ be an abelian category. Let*

$$K_0^\bullet \to K_1^\bullet \to \ldots \to K_n^\bullet$$

be maps of complexes such that

(1) *$H^i(K_0^\bullet) = 0$ for $i > 0$,*
(2) *$H^{-j}(K_j^\bullet) \to H^{-j}(K_{j+1}^\bullet)$ is zero.*

Then the composition $K_0^\bullet \to K_n^\bullet$ factors through $\tau_{\leq -n}K_n^\bullet \to K_n^\bullet$ in $D(\mathcal{A})$. Dually, given maps of complexes

$$K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_0^\bullet$$

such that

(1) $H^i(K_0^\bullet) = 0$ *for* $i < 0$,
(2) $H^j(K_{j+1}^\bullet) \to H^j(K_j^\bullet)$ *is zero,*

then the composition $K_n^\bullet \to K_0^\bullet$ factors through $K_n^\bullet \to \tau_{\geq n}K_n^\bullet$ in $D(\mathcal{A})$.

Proof. The case $n = 1$. Since $\tau_{\leq 0}K_0^\bullet = K_0^\bullet$ in $D(\mathcal{A})$ we can replace $K_0^\bullet$ by $\tau_{\leq 0}K_0^\bullet$ and $K_1^\bullet$ by $\tau_{\leq 0}K_1^\bullet$. Consider the distinguished triangle

$$\tau_{\leq -1}K_1^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0] \to (\tau_{\leq -1}K_1^\bullet)[1]$$

(Remark 13.12.4). The composition $K_0^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0]$ is zero as it is equal to $K_0^\bullet \to H^0(K_0^\bullet)[0] \to H^0(K_1^\bullet)[0]$ which is zero by assumption. The fact that $\mathrm{Hom}_{D(\mathcal{A})}(K_0^\bullet, -)$ is a homological functor (Lemma 13.4.2), allows us to find the desired factorization. For $n = 2$ we get a factorization $K_0^\bullet \to \tau_{\leq -1}K_1^\bullet$ by the case $n = 1$ and we can apply the case $n = 1$ to the map of complexes $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -1}K_2^\bullet$ to get a factorization $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -2}K_2^\bullet$. The general case is proved in exactly the same manner. □

13.13. Filtered derived categories

05RX A reference for this section is [**Ill72**, I, Chapter V]. Let $\mathcal{A}$ be an abelian category. In this section we will define the filtered derived category $DF(\mathcal{A})$ of $\mathcal{A}$. In short, we will define it as the derived category of the exact category of objects of $\mathcal{A}$ endowed with a finite filtration. (Thus our construction is a special case of a more general construction of the derived category of an exact category, see for example [**Büh10**], [**Kel90**].) Illusie's filtered derived category is the full subcategory of ours consisting of those objects whose filtration is finite. (In our category the filtration is still finite in each degree, but may not be uniformly bounded.) The rationale for our choice is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 13.21.3, see also Remark 13.21.4.

We will use the notation regarding filtered objects introduced in Homology, Section 12.19. The category of filtered objects of $\mathcal{A}$ is denoted $\mathrm{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

05RY **Definition 13.13.1.** Let $\mathcal{A}$ be an abelian category. The *category of finite filtered objects of* $\mathcal{A}$ is the category of filtered objects (A, F) of $\mathcal{A}$ whose filtration F is finite. We denote it $\mathrm{Fil}^f(\mathcal{A})$.

Thus $\mathrm{Fil}^f(\mathcal{A})$ is a full subcategory of $\mathrm{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$. There is a functor

$$\mathrm{gr} = \bigoplus\nolimits_{p \in \mathbf{Z}} \mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$$

where $\mathrm{Gr}(\mathcal{A})$ is the category of graded objects of $\mathcal{A}$, see Homology, Definition 12.16.1. Finally, there is a functor

$$(\text{forget } F) : \mathrm{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of $\mathcal{A}$. The category $\mathrm{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 12.3.13.

Because the functors gr^p, gr, (forget F) are additive they induce exact functors of triangulated categories

$$\mathrm{gr}^p, (\text{forget } F) : K(\mathrm{Fil}^f(\mathcal{A})) \to K(\mathcal{A}) \quad \text{and} \quad \mathrm{gr} : K(\mathrm{Fil}^f(\mathcal{A})) \to K(\mathrm{Gr}(\mathcal{A}))$$

by Lemma 13.10.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

05RZ **Definition 13.13.2.** Let $\mathcal{A}$ be an abelian category.

(1) Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of $K(\mathrm{Fil}^f(\mathcal{A}))$. We say that α is a *filtered quasi-isomorphism* if the morphism $\mathrm{gr}(\alpha)$ is a quasi-isomorphism.
(2) Let $K^\bullet$ be an object of $K(\mathrm{Fil}^f(\mathcal{A}))$. We say that $K^\bullet$ is *filtered acyclic* if the complex $\mathrm{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism if and only if each $\mathrm{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex $K^\bullet$ is filtered acyclic if and only if each $\mathrm{gr}^p(K^\bullet)$ is acyclic.

05S0 **Lemma 13.13.3.** *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow Gr(\mathcal{A})$, $K^\bullet \longmapsto H^0(gr(K^\bullet))$ is homological.*
(2) *The functor $K(Fil^f(\mathcal{A})) \to \mathcal{A}$, $K^\bullet \longmapsto H^0(gr^p(K^\bullet))$ is homological.*
(3) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0((forget\ F)K^\bullet)$ is homological.*

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is homological, see Lemma 13.11.1 and the fact that the functors $\mathrm{gr}, \mathrm{gr}^p, (\text{forget } F)$ are exact functors of triangulated categories. See Lemma 13.4.20. □

05S1 **Lemma 13.13.4.** *Let $\mathcal{A}$ be an abelian category. The full subcategory $FAc(\mathcal{A})$ of $K(Fil^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(Fil^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(Fil^f(\mathcal{A}))$ is the set $FQis(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Fil^f(\mathcal{A})) \longrightarrow FQis(\mathcal{A})^{-1}K(Fil^f(\mathcal{A}))$$

is $FAc(\mathcal{A})$ and the functor $H^0 \circ gr$ factors through Q.

Proof. We know that $H^0 \circ \mathrm{gr}$ is a homological functor by Lemma 13.13.3. Thus this lemma is a special case of Lemma 13.6.11. □

05S2 **Definition 13.13.5.** Let $\mathcal{A}$ be an abelian category. Let $\mathrm{FAc}(\mathcal{A})$ and $\mathrm{FQis}(\mathcal{A})$ be as in Lemma 13.13.4. The *filtered derived category of $\mathcal{A}$* is the triangulated category

$$DF(\mathcal{A}) = K(\mathrm{Fil}^f(\mathcal{A}))/\mathrm{FAc}(\mathcal{A}) = \mathrm{FQis}(\mathcal{A})^{-1}K(\mathrm{Fil}^f(\mathcal{A})).$$

05S3 **Lemma 13.13.6.** *The functors $gr^p, gr, (forget\ F)$ induce canonical exact functors*

$$gr^p, (forget\ F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

and

$$gr : DF(\mathcal{A}) \longrightarrow D(Gr(\mathcal{A}))$$

which commute with the localization functors.

Proof. This follows from the universal property of localization, see Lemma 13.5.7, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors $\mathrm{gr}^p, \mathrm{gr}, (\text{forget } F)$. This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \to L^\bullet$ be a filtered quasi-isomorphism in $K(\mathrm{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f. Then $M^\bullet$ is filtered acyclic, see Lemma 13.13.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 12.19.15. □

05S4 **Definition 13.13.7.** Let $\mathcal{A}$ be an abelian category. The *bounded filtered derived category* $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\mathrm{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

05S5 **Lemma 13.13.8.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet \in K(Fil^f(\mathcal{A}))$.*

(1) *If $H^n(gr(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^n = 0$ for all $n < a$.*
(2) *If $H^n(gr(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^n = 0$ for all $n > b$.*
(3) *If $H^n(gr(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are filtered quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Suppose that $H^n(\mathrm{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 12.19.15 the sequence

$$K^{a-2} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of $\mathcal{A}$ and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\mathrm{Coim}(d^{a-1}) = \mathrm{Im}(d^{a-1})$ in $\mathrm{Fil}^f(\mathcal{A})$ and the map $\mathrm{gr}(\mathrm{Im}(d^{a-1})) \to \mathrm{gr}(K^a)$ is injective with image equal to the image of $\mathrm{gr}(K^{a-1}) \to \mathrm{gr}(K^a)$, see Homology, Lemma 12.19.13. This means that the map $K^\bullet \to \tau_{\geq a} K^\bullet$ into the truncation

$$\tau_{\geq a} K^\bullet = (\ldots \to 0 \to K^a / \mathrm{Im}(d^{a-1}) \to K^{a+1} \to \ldots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). □

To state the following lemma denote $\mathrm{FAc}^+(\mathcal{A})$, $\mathrm{FAc}^-(\mathcal{A})$, resp. $\mathrm{FAc}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FAc}(\mathcal{A})$. Denote $\mathrm{FQis}^+(\mathcal{A})$, $\mathrm{FQis}^-(\mathcal{A})$, resp. $\mathrm{FQis}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FQis}(\mathcal{A})$.

05S6 **Lemma 13.13.9.** *Let $\mathcal{A}$ be an abelian category. The subcategories $FAc^+(\mathcal{A})$, $FAc^-(\mathcal{A})$, resp. $FAc^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of*

$K^+(Fil^f\mathcal{A})$, $K^-(Fil^f\mathcal{A})$, resp. $K^b(Fil^f\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $FQis^+(\mathcal{A})$, $FQis^-(\mathcal{A})$, resp. $FQis^b(\mathcal{A})$.

(1) *The kernel of the functor $K^+(Fil^f\mathcal{A}) \to DF^+(\mathcal{A})$ is $FAc^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^+(Fil^f\mathcal{A})/FAc^+(\mathcal{A}) = FQis^+(\mathcal{A})^{-1}K^+(Fil^f\mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$

(2) *The kernel of the functor $K^-(Fil^f\mathcal{A}) \to DF^-(\mathcal{A})$ is $FAc^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^-(Fil^f\mathcal{A})/FAc^-(\mathcal{A}) = FQis^-(\mathcal{A})^{-1}K^-(Fil^f\mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$

(3) *The kernel of the functor $K^b(Fil^f\mathcal{A}) \to DF^b(\mathcal{A})$ is $FAc^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^b(Fil^f\mathcal{A})/FAc^b(\mathcal{A}) = FQis^b(\mathcal{A})^{-1}K^b(Fil^f\mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 13.13.8, by exactly the same arguments as used in the proof of Lemma 13.11.6. □

13.14. Derived functors in general

05S7 A reference for this section is Deligne's exposé XVII in [**AGV71**]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the "correct" extension of the exact functor to the localized category.

05S8 **Situation 13.14.1.** Here $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in $\mathcal{D}$ compatible with the structure of triangulated category on $\mathcal{D}$.

Let $X \in \mathrm{Ob}(\mathcal{D})$. Recall from Categories, Remark 4.27.7 the filtered category X/S of arrows $s : X \to X'$ in S with source X. Dually, in Categories, Remark 4.27.15 we defined the cofiltered category S/X of arrows $s : X' \to X$ in S with target X.

05S9 **Definition 13.14.2.** Assumptions and notation as in Situation 13.14.1. Let $X \in \mathrm{Ob}(\mathcal{D})$.

(1) we say the *right derived functor RF is defined at X* if the ind-object
$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \to X') \longmapsto F(X')$$
is essentially constant[5]; in this case the value Y in $\mathcal{D}'$ is called the *value of RF at X*.

(2) we say the *left derived functor LF is defined at X* if the pro-object
$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \to X) \longmapsto F(X')$$
is essentially constant; in this case the value Y in $\mathcal{D}'$ is called the *value of LF at X*.

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of $\mathcal{D}$ consisting of objects where RF is defined is a triangulated subcategory, and RF will define a functor on this subcategory which transforms morphisms of S into isomorphisms.

[5]For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 4.22.

05SA **Lemma 13.14.3.** *Assumptions and notation as in Situation 13.14.1. Let $f : X \to Y$ be a morphism of $\mathcal{D}$.*

(1) *If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \to RF(Y)$ between the values such that for any commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with $s, s' \in S$ the diagram

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(X') & \longrightarrow & RF(X) \\ \downarrow & & \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(Y') & \longrightarrow & RF(Y) \end{array}$$

commutes.

(2) *If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \to LF(Y)$ between the values such that for any commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow[s]{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{s'} & Y \end{array}$$

with s, s' in S the diagram

$$\begin{array}{ccccc} LF(X) & \longrightarrow & F(X') & \longrightarrow & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ LF(Y) & \longrightarrow & F(Y') & \longrightarrow & F(Y) \end{array}$$

commutes.

Proof. Part (1) holds if we only assume that the colimits

$$RF(X) = \text{colim}_{s:X\to X'} F(X') \quad \text{and} \quad RF(Y) = \text{colim}_{s':Y\to Y'} F(Y')$$

exist. Namely, to give a morphism $RF(X) \to RF(Y)$ between the colimits is the same thing as giving for each $s : X \to X'$ in $\text{Ob}(X/S)$ a morphism $F(X') \to RF(Y)$ compatible with morphisms in the category X/S. To get the morphism we choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with s, s' in S as is possible by MS2 and we set $F(X') \to RF(Y)$ equal to the composition $F(X') \to F(Y') \to RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. □

05SB **Lemma 13.14.4.** *Assumptions and notation as in Situation 13.14.1. Let $s : X \to Y$ be an element of S.*

(1) *RF is defined at X if and only if it is defined at Y. In this case the map $RF(s) : RF(X) \to RF(Y)$ between values is an isomorphism.*
(2) *LF is defined at X if and only if it is defined at Y. In this case the map $LF(s) : LF(X) \to LF(Y)$ between values is an isomorphism.*

Proof. Omitted. □

05SU **Lemma 13.14.5.** *Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.*
(2) *LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \to LF(X[n])$ between values.*

Proof. Omitted. □

05SC **Lemma 13.14.6.** *Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If RF is defined at two out of three of X, Y, Z, then it is defined at the third. Moreover, in this case*

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in $\mathcal{D}'$. Similarly for LF.

Proof. Say RF is defined at X, Y with values A, B. Let $RF(f) : A \to B$ be the induced morphism, see Lemma 13.14.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in $\mathcal{D}'$. We claim that C is a value of RF at Z.

To see this pick $s : X \to X'$ in S such that there exists a morphism $\alpha : A \to F(X')$ as in Categories, Definition 4.22.1. We may choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \to Y''$ in S) assume that there exists a morphism $\beta : B \to F(Y')$ as in Categories, Definition 4.22.1. Picture

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & F(X') & \longrightarrow & A \\ {\scriptstyle RF(f)}\downarrow & & \downarrow{\scriptstyle F(f')} & & \downarrow{\scriptstyle RF(f)} \\ B & \xrightarrow{\beta} & F(Y') & \longrightarrow & B \end{array}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s' : Y' \to Y}$ is essentially constant means that there exists a $s'' : Y \to Y''$ in S and a morphism $h : Y' \to Y''$ such that $s'' = h \circ s'$ and such that $F(h)$ equal to $F(Y') \to B \to F(Y') \to F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 13.14.4 it suffices to prove that $RF(Z')$ is defined and that the arrow $\gamma : C \to F(Z')$ induces an isomorphism $C \to RF(Z')$. Namely, then we will get an isomorphism

$$(A, B, C, RF(f), b, c) \longrightarrow (RF(X'), RF(Y'), RF(Z'), RF(f'), RF(g'), RF(h'))$$

of triangles and by TR1 we conclude that the target is a distinguished triangle. Consider the category $\mathcal{I}$ of Lemma 13.5.10 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \to (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over $\mathcal{I}$ because $\mathcal{I} \to Z'/S$ is cofinal, see Categories, Lemma 4.22.11 (cofinality is proven in Lemma 13.5.10). For any object W in $\mathcal{D}'$ we consider the diagram

$$\begin{array}{ccc}
\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(X'')) & \longleftarrow & \operatorname{Mor}_{\mathcal{D}'}(W, A) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(Y'')) & \longleftarrow & \operatorname{Mor}_{\mathcal{D}'}(W, B) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(Z'')) & \longleftarrow & \operatorname{Mor}_{\mathcal{D}'}(W, C) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(X''[1])) & \longleftarrow & \operatorname{Mor}_{\mathcal{D}'}(W, A[1]) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(Y''[1])) & \longleftarrow & \operatorname{Mor}_{\mathcal{D}'}(W, B[1])
\end{array}$$

where the horizontal arrows are given by composing with (α, β, γ). Since filtered colimits are exact (Algebra, Lemma 10.8.8) the left column is an exact sequence. Thus the 5 lemma (Homology, Lemma 12.5.20) tells us the map

$$\operatorname{colim}_{\mathcal{I}} \operatorname{Mor}_{\mathcal{D}'}(W, F(Z'')) \longrightarrow \operatorname{Mor}_{\mathcal{D}'}(W, C)$$

is bijective. We conclude that $F(Z'')$ is essentially constant over $\mathcal{I}$ with value C by part (4) of Categories, Lemma 4.22.9. □

05SD **Lemma 13.14.7.** *Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$.*

(1) *If RF is defined at X and Y, then RF is defined at $X \oplus Y$.*
(2) *If $\mathcal{D}'$ is Karoubian and RF is defined at $X \oplus Y$, then RF is defined at both X and Y.*

In either case we have $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF.

Proof. If RF is defined at X and Y, then the distinguished triangle $X \to X \oplus Y \to Y \to X[1]$ (Lemma 13.4.11) and Lemma 13.14.6 shows that RF is defined at $X \oplus Y$ and that we have a distinguished triangle $RF(X) \to RF(X \oplus Y) \to$

$RF(Y) \to RF(X)[1]$. Applying Lemma 13.4.11 to this once more we find that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. This proves (1) and the final assertion.

Conversely, assume that RF is defined at $X \oplus Y$ and that $\mathcal{D}'$ is Karoubian. Since S is a saturated system S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$, see Categories, Lemma 4.27.21. Thus for any $s : X \to X'$ and $s' : Y \to Y'$ in S the morphism $s \oplus s' : X \oplus Y \to X' \oplus Y'$ is an element of S. In this way we obtain a functor

$$X/S \times Y/S \longrightarrow (X \oplus Y)/S$$

Recall that the categories $X/S, Y/S, (X \oplus Y)/S$ are filtered (Categories, Remark 4.27.7). By Categories, Lemma 4.22.12 $X/S \times Y/S$ is filtered and $F|_{X/S} : X/S \to \mathcal{D}'$ (resp. $G|_{Y/S} : Y/S \to \mathcal{D}'$) is essentially constant if and only if $F|_{X/S} \circ \mathrm{pr}_1 : X/S \times Y/S \to \mathcal{D}'$ (resp. $G|_{Y/S} \circ \mathrm{pr}_2 : X/S \times Y/S \to \mathcal{D}'$) is essentially constant. Below we will show that the displayed functor is cofinal, hence by Categories, Lemma 4.22.11, we see that $F|_{(X\oplus Y)/S}$ is essentially constant implies that $F|_{X/S} \circ \mathrm{pr}_1 \oplus F|_{Y/S} \circ \mathrm{pr}_2 : X/S \times Y/S \to \mathcal{D}'$ is essentially constant. By Homology, Lemma 12.30.3 (and this is where we use that $\mathcal{D}'$ is Karoubian) we see that $F|_{X/S} \circ \mathrm{pr}_1 \oplus F|_{Y/S} \circ \mathrm{pr}_2$ being essentially constant implies $F|_{X/S} \circ \mathrm{pr}_1$ and $F|_{Y/S} \circ \mathrm{pr}_2$ are essentially constant proving that RF is defined at X and Y.

Proof that the displayed functor is cofinal. To do this pick any $t : X \oplus Y \to Z$ in S. Using MS2 we can find morphisms $Z \to X'$, $Z \to Y'$ and $s : X \to X'$, $s' : Y \to Y'$ in S such that

$$\begin{array}{ccccc} X & \longleftarrow & X \oplus Y & \longrightarrow & Y \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle s'} \\ X' & \longleftarrow & Z & \longrightarrow & Y' \end{array}$$

commutes. This proves there is a map $Z \to X' \oplus Y'$ in $(X \oplus Y)/S$, i.e., we get part (1) of Categories, Definition 4.17.1. To prove part (2) it suffices to prove that given $t : X \oplus Y \to Z$ and morphisms $s_i \oplus s_i' : Z \to X_i' \oplus Y_i'$, $i = 1, 2$ in $(X \oplus Y)/S$ we can find morphisms $a : X_1' \to X'$, $b : X_2' \to X'$, $c : Y_1' \to Y'$, $d : Y_2' \to Y'$ in S such that $a \circ s_1 = b \circ s_2$ and $c \circ s_1' = d \circ s_2'$. To do this we first choose any X' and Y' and maps a, b, c, d in S; this is possible as X/S and Y/S are filtered. Then the two maps $a \circ s_1, b \circ s_2 : Z \to X'$ become equal in $S^{-1}\mathcal{D}$. Hence we can find a morphism $X' \to X''$ in S equalizing them. Similarly we find $Y' \to Y''$ in S equalizing $c \circ s_1'$ and $d \circ s_2'$. Replacing X' by X'' and Y' by Y'' we get $a \circ s_1 = b \circ s_2$ and $c \circ s_1' = d \circ s_2'$.

The proof of the corresponding statements for LF are dual. □

05SE **Proposition 13.14.8.** *Assumptions and notation as in Situation 13.14.1.*

(1) *The full subcategory $\mathcal{E}$ of $\mathcal{D}$ consisting of objects at which RF is defined is a strictly full triangulated subcategory of $\mathcal{D}$.*
(2) *We obtain an exact functor $RF : \mathcal{E} \longrightarrow \mathcal{D}'$ of triangulated categories.*
(3) *Elements of S with either source or target in $\mathcal{E}$ are morphisms of $\mathcal{E}$.*
(4) *Any element of $S_\mathcal{E} = Arrows(\mathcal{E}) \cap S$ is mapped to an isomorphism by RF.*
(5) *The set $S_\mathcal{E}$ is a saturated multiplicative system in $\mathcal{E}$ compatible with the triangulated structure.*
(6) *The functor $S_\mathcal{E}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is a fully faithful exact functor of triangulated categories.*

(7) *We obtain an exact functor*

$$RF : S_{\mathcal{E}}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$

(8) *If $\mathcal{D}'$ is Karoubian, then $\mathcal{E}$ is a saturated triangulated subcategory of $\mathcal{D}$.*

A similar result holds for LF.

Proof. Since S is saturated it contains all isomorphisms (see remark following Categories, Definition 4.27.20). Hence (1) follows from Lemmas 13.14.4, 13.14.6, and 13.14.5. We get (2) from Lemmas 13.14.3, 13.14.5, and 13.14.6. We get (3) from Lemma 13.14.4. Part (4) follows from Lemma 13.14.4. Part (5) follows from the definitions and part (3). The fully faithfulness in (6) follows from (3) and the definitions. The fact that $S_{\mathcal{E}}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is exact follows from the fact that a triangle in $S_{\mathcal{E}}^{-1}\mathcal{E}$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{E}$, see proof of Proposition 13.5.6. The factorization of $RF : \mathcal{E} \to \mathcal{D}'$ through an exact functor $S_{\mathcal{E}}^{-1}\mathcal{E} \to \mathcal{D}'$ follows from Lemma 13.5.7. Finally, part (8) follows from Lemma 13.14.7. □

Proposition 13.14.8 tells us that RF lives on a maximal strictly full triangulated subcategory of $S^{-1}\mathcal{D}$ and is an exact functor on this triangulated category. Picture:

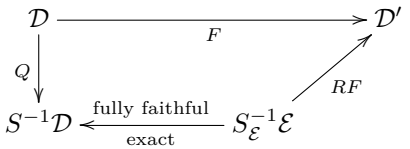

05SV **Definition 13.14.9.** In Situation 13.14.1. We say F is *right derivable*, or that RF *everywhere defined* if RF is defined at every object of $\mathcal{D}$. We say F is *left derivable*, or that LF *everywhere defined* if LF is defined at every object of $\mathcal{D}$.

In this case we obtain a right (resp. left) derived functor

05SW (13.14.9.1) $$RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'),$$

see Proposition 13.14.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F. In fact, it might happen that the canonical map $F(X) \to RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right derivable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category $\mathcal{D}$. This warrants a definition.

05SX **Definition 13.14.10.** In Situation 13.14.1.

(1) An object X of $\mathcal{D}$ *computes* RF if RF is defined at X and the canonical map $F(X) \to RF(X)$ is an isomorphism.
(2) An object X of $\mathcal{D}$ *computes* LF if LF is defined at X and the canonical map $LF(X) \to F(X)$ is an isomorphism.

05SY **Lemma 13.14.11.** *Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *X computes RF if and only if $X[n]$ computes RF.*
(2) *X computes LF if and only if $X[n]$ computes LF.*

Proof. Omitted. □

05SZ **Lemma 13.14.12.** *Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If X, Y compute RF then so does Z. Similar for LF.*

Proof. By Lemma 13.14.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{array}{c} (F(X), F(Y), F(Z), F(f), F(g), F(h)) \\ \downarrow \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h)) \end{array}$$

Two out of three maps are isomorphisms, hence so is the third. □

05T0 **Lemma 13.14.13.** *Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$. If $X \oplus Y$ computes RF, then X and Y compute RF. Similarly for LF.*

Proof. If $X \oplus Y$ computes RF, then $RF(X \oplus Y) = F(X) \oplus F(Y)$. In the proof of Lemma 13.14.7 we have seen that the functor $X/S \times Y/S \to (X \oplus Y)/S$, $(s, s') \mapsto s \oplus s'$ is cofinal. Thus by Categories, Lemma 4.22.11 and by characterization (4) of Categories, Lemma 4.22.9 we know that for any object W in $\mathcal{D}'$ the map

$$\mathrm{Hom}_{\mathcal{D}'}(F(X \oplus Y), W) \longrightarrow \mathrm{colim}_{s:X\to X', s':Y\to Y'} \mathrm{Hom}_{\mathcal{D}'}(F(X' \oplus Y'), W)$$

is bijective. Since this arrow is clearly compatible with direct sum decompositions on both sides, we conclude that the map

$$\mathrm{Hom}_{\mathcal{D}'}(F(X), W) \longrightarrow \mathrm{colim}_{s:X\to X'} \mathrm{Hom}_{\mathcal{D}'}(F(X'), W)$$

is bijective (minor detail omitted). Thus by Categories, Lemma 4.22.9 we conclude RF is defined at X with value $F(X)$. Similarly for Y. □

05T1 **Lemma 13.14.14.** *Assumptions and notation as in Situation 13.14.1.*

(1) *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X \to X'$ in S such that X' computes RF, then RF is everywhere defined.*
(2) *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X' \to X$ in S such that X' computes LF, then LF is everywhere defined.*

Proof. This is clear from the definitions. □

06XN **Lemma 13.14.15.** *Assumptions and notation as in Situation 13.14.1. If there exists a subset $\mathcal{I} \subset \mathrm{Ob}(\mathcal{D})$ such that*

(1) *for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X \to X'$ in S with $X' \in \mathcal{I}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF. Dually, if there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ such that

(1) *for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X' \to X$ in S with $X' \in \mathcal{P}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{P}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then LF is everywhere defined and every $X \in \mathcal{P}$ computes LF.

Proof. Let X be an object of $\mathcal{D}$. Assumption (1) implies that the arrows $s : X \to X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S. Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \to \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \to X'$ in S with $X' \in \mathcal{I}$. □

05T2 **Lemma 13.14.16.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated categories. Let S, resp. S' be a saturated multiplicative system in $\mathcal{A}$, resp. $\mathcal{B}$ compatible with the triangulated structure. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be exact functors. Denote $F' : \mathcal{A} \to (S')^{-1}\mathcal{B}$ the composition of F with the localization functor.*

1. *If RF', RG, $R(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : R(G \circ F) \longrightarrow RG \circ RF'$.*
2. *If LF', LG, $L(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : LG \circ LF' \to L(G \circ F)$.*

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \to (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \to \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \to \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \to S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \to (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of $\mathcal{B}$ there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \to RG \circ Q_B$. Let X be an object of $\mathcal{A}$. We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= \operatorname{colim}_{s:X \to X' \in S} G(F(X')) \\ &\xrightarrow{t'} \operatorname{colim}_{s:X \to X' \in S} RG(Q_B(F(X'))) \\ &= \operatorname{colim}_{s:X \to X' \in S} RG(F'(X')) \\ &= RG(\operatorname{colim}_{s:X \to X' \in S} F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 4.22.8 and its proof. We omit the proof this defines a transformation of functors. The case of left derived functors is similar. □

13.15. Derived functors on derived categories

05T3 In practice derived functors come about most often when given an additive functor between abelian categories.

05T4 **Situation 13.15.1.** Here $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \to K(\mathcal{B}), \quad K^+(\mathcal{A}) \to K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \to K^-(\mathcal{B}).$$

See Lemma 13.10.6. We also denote F the composition $K(\mathcal{A}) \to D(\mathcal{B})$, $K^+(\mathcal{A}) \to D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \to D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

1. The right derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}(\mathcal{A})$.

(2) The right derived functor of $F : K^+(\mathcal{A}) \to D^+(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}^+(\mathcal{A})$.
(3) The left derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}(\mathcal{A})$.
(4) The left derived functor of $F : K^-(\mathcal{A}) \to D^-(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}^-(\mathcal{A})$.

Each of these cases is an example of Situation 13.14.1.

Some of the ambiguity that may arise is alleviated by the following.

05T5 **Lemma 13.15.2.** *In Situation 13.15.1.*

(1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(2) *Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(4) *Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \to X'$ in $K(\mathcal{A})$. By Lemma 13.11.5 there exists quasi-isomorphism $X' \to X''$ with X'' bounded below. Hence we see that $X/\mathrm{Qis}^+(\mathcal{A})$ is cofinal in $X/\mathrm{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). □

Given an object A of an abelian category $\mathcal{A}$ we get a complex

$$A[0] = (\ldots \to 0 \to A \to 0 \to \ldots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \to K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

0157 **Definition 13.15.3.** In Situation 13.15.1.

(1) The *right derived functors of F* are the partial functors RF associated to cases (1) and (2) of Situation 13.15.1.
(2) The *left derived functors of F* are the partial functors LF associated to cases (3) and (4) of Situation 13.15.1.
(3) An object A of $\mathcal{A}$ is said to be *right acyclic for F*, or *acyclic for RF* if $A[0]$ computes RF.
(4) An object A of $\mathcal{A}$ is said to be *left acyclic for F*, or *acyclic for LF* if $A[0]$ computes LF.

The following few lemmas give some criteria for the existence of enough acyclics.

05T7 **Lemma 13.15.4.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $a \in \mathbf{Z}$.*

(1) *Given $K^\bullet$ with $K^n = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ and $P^n \to K^n$ surjective for all n and $P^n = 0$ for $n > a$.*
(2) *Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ for all n and $P^n = 0$ for $n > a$.*

Proof. Proof of part (1). Consider the following induction hypothesis IH_n: There are $P^j \in \mathcal{P}$, $j \geq n$, with $P^j = 0$ for $j > a$, maps $d^j : P^j \to P^{j+1}$ for $j \geq n$, and surjective maps $\alpha^j : P^j \to K^j$ for $j \geq n$ such that the diagram

$$\begin{array}{ccccccccc} & & & & P^n & \longrightarrow & P^{n+1} & \longrightarrow & P^{n+2} & \longrightarrow & \ldots \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} & \longrightarrow & \ldots \end{array}$$

is commutative, such that $d^{j+1} \circ d^j = 0$ for $j \geq n$, such that α induces isomorphisms $\mathrm{Ker}(d^j)/\mathrm{Im}(d^{j-1}) \to H^j(K^\bullet)$ for $j > n$, and such that $\alpha : \mathrm{Ker}(d^n) \to \mathrm{Ker}(d^n_K)$ is surjective. Then we choose a surjection

$$P^{n-1} \longrightarrow K^{n-1} \times_{K^n} \mathrm{Ker}(d^n) = K^{n-1} \times_{\mathrm{Ker}(d^n_K)} \mathrm{Ker}(d^n)$$

with P^{n-1} in $\mathcal{P}$. This allows us to extend the diagram above to

$$\begin{array}{ccccccccc} & & P^{n-1} & \longrightarrow & P^n & \longrightarrow & P^{n+1} & \longrightarrow & P^{n+2} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} & \longrightarrow & \ldots \end{array}$$

The reader easily checks that IH_{n-1} holds with this choice.

We finish the proof of (1) as follows. First we note that IH_n is true for $n = a + 1$ since we can just take $P^j = 0$ for $j > a$. Hence we see that proceeding by descending induction we produce a complex $P^\bullet$ with $P^n = 0$ for $n > a$ consisting of objects from $\mathcal{P}$, and a termwise surjective quasi-isomorphism $\alpha : P^\bullet \to K^\bullet$ as desired.

Proof of part (2). The assumption implies that the morphism $\tau_{\leq a}K^\bullet \to K^\bullet$ (Homology, Section 12.15) is a quasi-isomorphism. Apply part (1) to find $P^\bullet \to \tau_{\leq a}K^\bullet$. The composition $P^\bullet \to K^\bullet$ is the desired quasi-isomorphism. □

05T6 **Lemma 13.15.5.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $a \in \mathbf{Z}$.*

(1) *Given $K^\bullet$ with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $K^n \to I^n$ injective and $I^n \in \mathcal{I}$ for all n and $I^n = 0$ for $n < a$,*
(2) *Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^n \in \mathcal{I}$ and $I^n = 0$ for $n < a$.*

Proof. This lemma is dual to Lemma 13.15.4. □

05T8 **Lemma 13.15.6.** *In Situation 13.15.1. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:*

(1) *every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$,*
(2) *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{I}$ is acyclic for RF.

Proof. Pick $A \in \mathcal{I}$. Let $A[0] \to K^\bullet$ be a quasi-isomorphism with $K^\bullet$ bounded below. Then we can find a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and each $I^n \in \mathcal{I}$, see Lemma 13.15.5[6]. Hence we see that these resolutions are cofinal in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$ we have $F(A)[0] \to F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ and $\mathrm{Im}(d^{-1})$ are elements of $\mathcal{I}$ using property (2) and the exact sequences

$$0 \to \mathrm{Ker}(d^n) \to I^n \to \mathrm{Im}(d^n) \to 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \to F(I^{n_0}) \to F(I^{n_0+1}) \to \ldots \to F(I^{-1}) \to F(\mathrm{Im}(d^{-1})) \to 0$$

is exact. The exact sequence $0 \to \mathrm{Im}(d^{-1}) \to I^0 \to I^0/\mathrm{Im}(d^{-1}) \to 0$ implies that $I^0/\mathrm{Im}(d^{-1})$ is an element of $\mathcal{I}$. The exact sequence $0 \to A \to I^0/\mathrm{Im}(d^{-1}) \to \mathrm{Im}(d^0) \to 0$ then implies that $\mathrm{Im}(d^0) = \mathrm{Ker}(d^1)$ is an elements of $\mathcal{I}$ and from then on one continues as before to show that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ is an element of $\mathcal{I}$ for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \to F(I^\bullet)$ is an isomorphism as desired. □

05T9 **Lemma 13.15.7.** *In Situation 13.15.1. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:*

(1) *every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
(2) *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $Q, R \in \mathcal{P}$, then $P \in \mathcal{P}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{P}$ is acyclic for LF.

Proof. Dual to the proof of Lemma 13.15.6. □

13.16. Higher derived functors

05TB The following simple lemma shows that right derived functors "move to the right".

05TC **Lemma 13.16.1.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $K^\bullet$ be a complex of $\mathcal{A}$ and $a \in \mathbf{Z}$.*

(1) *If $H^i(K^\bullet) = 0$ for all $i < a$ and RF is defined at $K^\bullet$, then $H^i(RF(K^\bullet)) = 0$ for all $i < a$.*
(2) *If RF is defined at $K^\bullet$ and $\tau_{\leq a}K^\bullet$, then $H^i(RF(\tau_{\leq a}K^\bullet)) = H^i(RF(K^\bullet))$ for all $i \leq a$.*

Proof. Assume $K^\bullet$ satisfies the assumptions of (1). Let $s : K^\bullet \to L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \to \tau_{\geq a}L^\bullet$ is a quasi-isomorphism by our assumption on $K^\bullet$. Hence in the category $K^\bullet/\mathrm{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \to L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. From Categories, Lemma 4.22.11 we deduce that RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s. This means that $\mathrm{id} : RF(K^\bullet) \to RF(K^\bullet)$ factors through $F(L^\bullet)$ for some complex $L^\bullet$ with $L^n = 0$ for $n < a$. It follows that $H^i(RF(K^\bullet)) = 0$ for $i < a$.

[6] By (1) we see that $\mathcal{I}$ is nonempty. Pick P in $\mathcal{I}$. Then the short exact sequence $0 \to P \to P \to 0 \to 0$ and assumption (2) shows that 0 is in $\mathcal{I}$. Thus the lemma applies.

To prove (2) we use the distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

of Remark 13.12.4 to conclude via Lemma 13.14.6 that RF is defined at $\tau_{\geq a+1}K^\bullet$ as well and that we have a distinguished triangle

$$RF(\tau_{\leq a}K^\bullet) \to RF(K^\bullet) \to RF(\tau_{\geq a+1}K^\bullet) \to RF(\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{B})$. By part (1) we see that $RF(\tau_{\geq a+1}K^\bullet)$ has vanishing cohomology in degrees $< a+1$. The long exact cohomology sequence of this distinguished triangle then shows what we want. □

015A **Definition 13.16.2.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The *ith right derived functor R^iF of F* is the functor

$$R^iF = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

05TD **Lemma 13.16.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *We have $R^iF = 0$ for $i < 0$,*
(2) *R^0F is left exact,*
(3) *the map $F \to R^0F$ is an isomorphism if and only if F is left exact.*

Proof. Let A be an object of $\mathcal{A}$. By Lemma 13.16.1 we have $H^i(RF(A[0]) = 0$ for $i < 0$. This proves (1).

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$. By Lemma 13.12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $D^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \to R^0F(A) \to R^0F(B) \to R^0F(C)$ is exact. Hence R^0F is left exact. Of course this also proves that if $F \to R^0F$ is an isomorphism, then F is left exact.

Assume F is left exact. Recall that $RF(A[0])$ is the value of the essentially constant system $F(K^\bullet)$ for $s : A[0] \to K^\bullet$ quasi-isomorphisms. It follows that $R^0F(A)$ is the value of the essentially constant system $H^0(F(K^\bullet))$ for $s : A[0] \to K^\bullet$ quasi-isomorphisms, see Categories, Lemma 4.22.8. But if $s : A[0] \to K^\bullet$ is a quasi-isomorphism, then $A[0] \to \tau_{\geq 0}K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \to K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. It follows from Categories, Lemma 4.22.11 that we may restrict to such s. Moreover, for such an s the sequence

$$0 \to A \to K^0 \to K^1$$

is exact. Since F is left exact we see that $0 \to F(A) \to F(K^0) \to F(K^1)$ is exact as well. It follows that $F(A) \to H^0(F(K^\bullet))$ is an isomorphism and the system is actually constant with value $F(A)$. We conclude $R^0F = F$ as desired. □

015C **Lemma 13.16.4.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let A be an object of $\mathcal{A}$.*

(1) *A is right acyclic for F if and only if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$,*

(2) *if F is left exact, then A is right acyclic for F if and only if $R^iF(A) = 0$ for all $i > 0$.*

Proof. If A is right acyclic for F, then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$ then $F(A[0]) \to RF(A[0])$ is a quasi-isomorphism by Lemma 13.16.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0F$, see Lemma 13.16.3. □

015D **Lemma 13.16.5.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$.*

(1) *If A and C are right acyclic for F then so is B.*
(2) *If A and B are right acyclic for F then so is C.*
(3) *If B and C are right acyclic for F and $F(B) \to F(C)$ is surjective then A is right acyclic for F.*

In each of the three cases

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence of $\mathcal{B}$.

Proof. By Lemma 13.12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $D^+(\mathcal{A})$. As RF is an exact functor and since $R^iF = 0$ for $i < 0$ and $R^0F = F$ (Lemma 13.16.3) we obtain an exact cohomology sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \ldots$$

in the abelian category $\mathcal{B}$. Thus the lemma follows from the characterization of acyclic objects in Lemma 13.16.4. □

05TE **Lemma 13.16.6.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *The functors R^iF, $i \geq 0$ come equipped with a canonical structure of a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.12.1.*
(2) *If every object of $\mathcal{A}$ is a subobject of a right acyclic object for F, then $\{R^iF, \delta\}_{i\geq 0}$ is a universal δ-functor, see Homology, Definition 12.12.3.*

Proof. The functor $\mathcal{A} \to \mathrm{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\mathrm{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.12.1. The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \to \mathcal{B}$ is a homological functor, see Definition 13.11.3. Hence we get the structure of a δ-functor from Lemma 13.4.22 and Lemma 13.4.21. Part (2) follows from Homology, Lemma 12.12.4 and the description of acyclics in Lemma 13.16.4. □

015E **Lemma 13.16.7** (Leray's acyclicity lemma)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $A^\bullet$ be a bounded below complex of right F-acyclic objects such that RF is defined at $A^\bullet$[7]. The canonical map*

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., $A^\bullet$ computes RF.

[7]For example this holds if $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.

Proof. Let $A^\bullet$ be a bounded complex of right F-acyclic objects. We claim that RF is defined at $A^\bullet$ and that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$. Namely, it holds for complexes with at most one nonzero right F-acyclic object by Definition 13.15.3. Next, suppose that $A^n = 0$ for $n \notin [a, b]$. Using the "stupid" truncations we obtain a termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} A^\bullet \to A^\bullet \to \sigma_{\leq a} A^\bullet \to 0$$

see Homology, Section 12.15. Thus a distinguished triangle $(\sigma_{\geq a+1} A^\bullet, A^\bullet, \sigma_{\leq a} A^\bullet)$. By induction hypothesis RF is defined for the two outer complexes and these complexes compute RF. Then the same is true for the middle one by Lemma 13.14.12.

Suppose that $A^\bullet$ is a bounded below complex of acyclic objects such that RF is defined at $A^\bullet$. To show that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A^\bullet)) \to H^i(RF(A^\bullet))$ is an isomorphism for all i. Pick i. Consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+2} A^\bullet \to A^\bullet \to \sigma_{\leq i+1} A^\bullet \to 0.$$

Note that this induces a termwise split short exact sequence

$$0 \to \sigma_{\geq i+2} F(A^\bullet) \to F(A^\bullet) \to \sigma_{\leq i+1} F(A^\bullet) \to 0.$$

Hence we get distinguished triangles

$$(\sigma_{\geq i+2} A^\bullet, A^\bullet, \sigma_{\leq i+1} A^\bullet) \quad \text{and} \quad (\sigma_{\geq i+2} F(A^\bullet), F(A^\bullet), \sigma_{\leq i+1} F(A^\bullet))$$

Since RF is defined at $A^\bullet$ (by assumption) and at $\sigma_{\leq i+1} A^\bullet$ (by the first paragraph) we see that RF is defined at $\sigma_{\geq i+2} A^\bullet$ and we get a distinguished triangle

$$(RF(\sigma_{\geq i+2} A^\bullet), RF(A^\bullet), RF(\sigma_{\leq i+1} A^\bullet))$$

See Lemma 13.14.6. Using these distinguished triangles we obtain a map of exact sequences

$$\begin{array}{ccccccc}
H^i(\sigma_{\geq i+2} F(A^\bullet)) & \longrightarrow & H^i(F(A^\bullet)) & \longrightarrow & H^i(\sigma_{\leq i+1} F(A^\bullet)) & \longrightarrow & H^{i+1}(\sigma_{\geq i+2} F(A^\bullet)) \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow \\
H^i(RF(\sigma_{\geq i+2} A^\bullet)) & \longrightarrow & H^i(RF(A^\bullet)) & \longrightarrow & H^i(RF(\sigma_{\leq i+1} A^\bullet)) & \longrightarrow & H^{i+1}(RF(\sigma_{\geq i+2} A^\bullet))
\end{array}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof it suffices to show that $H^i(RF(\sigma_{\geq i+2} A^\bullet)) = 0$ and $H^{i+1}(RF(\sigma_{\geq i+2} A^\bullet)) = 0$. This follows immediately from Lemma 13.16.1. □

05TA **Proposition 13.16.8.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.*

(1) *If every object of $\mathcal{A}$ injects into an object acyclic for RF, then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

see (13.14.9.1). Moreover, any bounded below complex $A^\bullet$ whose terms are acyclic for RF computes RF.

(2) *If every object of $\mathcal{A}$ is quotient of an object acyclic for LF, then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor*

$$LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$$

see (13.14.9.1). Moreover, any bounded above complex $A^\bullet$ whose terms are acyclic for LF computes LF.

Proof. Assume every object of $\mathcal{A}$ injects into an object acyclic for RF. Let $\mathcal{I}$ be the set of objects acyclic for RF. Let $K^\bullet$ be a bounded below complex in $\mathcal{A}$. By Lemma 13.15.5 there exists a quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$. Hence in order to prove (1) it suffices to show that $F(I^\bullet) \to F((I')^\bullet)$ is a quasi-isomorphism when $s : I^\bullet \to (I')^\bullet$ is a quasi-isomorphism of bounded below complexes of objects from $\mathcal{I}$, see Lemma 13.14.15. Note that the cone $C(s)^\bullet$ is an acyclic bounded below complex all of whose terms are in $\mathcal{I}$. Hence it suffices to show: given an acyclic bounded below complex $I^\bullet$ all of whose terms are in $\mathcal{I}$ the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Setting $J^n = \mathrm{Im}(d^n)$ we break $I^\bullet$ into short exact sequences $0 \to J^n \to I^{n+1} \to J^{n+1} \to 0$ for $n \geq n_0$. These sequences induce distinguished triangles (J^n, I^{n+1}, J^{n+1}) in $D^+(\mathcal{A})$ by Lemma 13.12.1. For each $k \in \mathbf{Z}$ denote H_k the assertion: For all $n \leq k$ the object J^n is in $\mathcal{I}$. Then H_k holds trivially for $k < n_0$. If H_n holds, then Lemma 13.14.12 shows that J^{n+1} is in $\mathcal{I}$ and we have H_{n+1}. By Proposition 13.14.8 we have a distinguished triangle $(RF(J^n), RF(I^{n+1}), RF(J^{n+1}))$. Since J^n, I^{n+1}, J^{n+1} are in $\mathcal{I}$ the long exact cohomology sequence (13.11.1.1) associated to this distinguished triangle collapses to an exact sequence

$$0 \to F(J^n) \to F(I^{n+1}) \to F(J^{n+1}) \to 0$$

This in turn proves that $F(I^\bullet)$ is exact.

The proof in the case of LF is dual. □

015F **Lemma 13.16.9.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then*

1. *every object of $\mathcal{A}$ is right acyclic for F,*
2. *$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined,*
3. *$RF : D(\mathcal{A}) \to D(\mathcal{B})$ is everywhere defined,*
4. *every complex computes RF, in other words, the canonical map $F(K^\bullet) \to RF(K^\bullet)$ is an isomorphism for all complexes, and*
5. *$R^iF = 0$ for $i \neq 0$.*

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. □

13.17. Triangulated subcategories of the derived category

06UP Let $\mathcal{A}$ be an abelian category. In this section we look at certain strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$.

Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Definition 12.10.1 and Lemma 12.10.3. We let $D_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$\mathrm{Ob}(D_{\mathcal{B}}(\mathcal{A})) = \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D^+_{\mathcal{B}}(\mathcal{A}) = D^+(\mathcal{A}) \cap D_{\mathcal{B}}(\mathcal{A})$ and similarly for the other bounded versions.

06UQ **Lemma 13.17.1.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_{\mathcal{B}}(\mathcal{A})$ is a strictly full saturated triangulated subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_{\mathcal{B}}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_{\mathcal{B}}(\mathcal{A})$, then both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of objects of $\mathcal{B}$ as $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$. Hence both X and Y are in $D_{\mathcal{B}}(\mathcal{A})$. By Lemma 13.4.16 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_{\mathcal{B}}(\mathcal{A})$ then Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. The long exact cohomology sequence (13.11.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 12.10.1) show that $H^n(Z)$ is an object of $\mathcal{B}$ for all n. Thus Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. □

We continue to assume that $\mathcal{B}$ is a weak Serre subcategory of the abelian category $\mathcal{A}$. Then $\mathcal{B}$ is an abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact. Hence we obtain a derived functor $D(\mathcal{B}) \to D(\mathcal{A})$, see Lemma 13.16.9. Clearly the functor $D(\mathcal{B}) \to D(\mathcal{A})$ factors through a canonical exact functor

06UR (13.17.1.1)
$$D(\mathcal{B}) \longrightarrow D_{\mathcal{B}}(\mathcal{A})$$

After all a complex made from objects of $\mathcal{B}$ certainly gives rise to an object of $D_{\mathcal{B}}(\mathcal{A})$ and as distinguished triangles in $D_{\mathcal{B}}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_{\mathcal{B}}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \to D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \to D^+_{\mathcal{B}}(\mathcal{A})$, $D^-(\mathcal{B}) \to D^-_{\mathcal{B}}(\mathcal{A})$, and $D^b(\mathcal{B}) \to D^b_{\mathcal{B}}(\mathcal{A})$ for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. In this case we have the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, see Homology, Lemma 12.10.6. In this case $D_{\mathcal{B}}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 13.6.8. Similarly for the bounded versions.

06XL **Lemma 13.17.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category $\mathcal{A}/\mathcal{B}$ in the proof of Homology, Lemma 12.10.6. Let $(X^\bullet, d^\bullet)$ be a complex of $\mathcal{A}/\mathcal{B}$. This means that X^i is an object of $\mathcal{A}$ and $d^i : X^i \to X^{i+1}$ is a morphism in $\mathcal{A}/\mathcal{B}$ such that $d^i \circ d^{i-1} = 0$ in $\mathcal{A}/\mathcal{B}$.

For $i \geq 0$ we may write $d^i = (s^i, f^i)$ where $s^i : Y^i \to X^i$ is a morphism of $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ (equivalently s^i becomes an isomorphism in the quotient category) and $f^i : Y^i \to X^{i+1}$ is a morphism of $\mathcal{A}$. By induction we will construct a commutative diagram

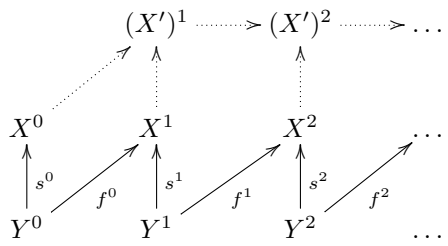

where the vertical arrows $X^i \to (X')^i$ become isomorphisms in the quotient category. Namely, we first let $(X')^1 = \mathrm{Coker}(Y^0 \to X^0 \oplus X^1)$ (or rather the pushout

of the diagram with arrows s^0 and f^0) which gives the first commutative diagram. Next, we take $(X')^2 = \mathrm{Coker}(Y^1 \to (X')^1 \oplus X^2)$. And so on. Setting additionally $(X')^n = X^n$ for $n \leq 0$ we see that the map $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $X^n \to X^{n+1}$ in $\mathcal{A}$ for $n \geq 0$.

Dually, for $i < 0$ we may write $d^i = (g^i, t^{i+1})$ where $t^{i+1} : X^{i+1} \to Z^{i+1}$ is an isomorphism in the quotient category and $g^i : X^i \to Z^{i+1}$ is a morphism. By induction we will construct a commutative diagram

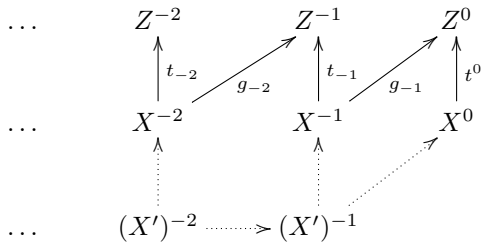

where the vertical arrows $(X')^i \to X^i$ become isomorphisms in the quotient category. Namely, we take $(X')^{-1} = X^{-1} \times_{Z^0} X^0$. Then we take $(X')^{-2} = X^{-2} \times_{Z^{-1}} (X')^{-1}$. And so on. Setting additionally $(X')^n = X^n$ for $n \geq 0$ we see that the map $((X')^\bullet, (d')^\bullet) \to (X^\bullet, d^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $d^n : X^n \to X^{n+1}$ in $\mathcal{A}$ for all $n \in \mathbf{Z}$.

In this case we know the compositions $d^n \circ d^{n-1}$ are zero in $\mathcal{A}/\mathcal{B}$. If for $n > 0$ we replace X^n by

$$(X')^n = X^n / \sum\nolimits_{0<k\leq n} \mathrm{Im}(\mathrm{Im}(X^{k-2} \to X^k) \to X^n)$$

then the compositions $d^n \circ d^{n-1}$ are zero for $n \geq 0$. (Similarly to the second paragraph above we obtain an isomorphism of complexes $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$.) Finally, for $n < 0$ we replace X^n by

$$(X')^n = \bigcap\nolimits_{n\leq k<0} (X^n \to X^k)^{-1} \mathrm{Ker}(X^k \to X^{k+2})$$

and we argue in the same manner to get a complex in $\mathcal{A}$ whose image in $\mathcal{A}/\mathcal{B}$ is isomorphic to the given one. □

06XM **Lemma 13.17.3.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ such that $vu \cong id$. Then*

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 13.17.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \to X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \mathrm{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X

is an element of the multiplicative system used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in \mathrm{Ob}(\mathcal{A})$) the map

$$\mathrm{Hom}_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). □

For certain Serre subcategories $\mathcal{B} \subset \mathcal{A}$ we can prove that the functor $D(\mathcal{B}) \to D_{\mathcal{B}}(\mathcal{A})$ is fully faithful.

0FCL **Lemma 13.17.4.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Assume that for every surjection $X \to Y$ with $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{B})$ there exists $X' \subset X$, $X' \in \mathrm{Ob}(\mathcal{B})$ which surjects onto Y. Then the functor $D^-(\mathcal{B}) \to D^-_{\mathcal{B}}(\mathcal{A})$ of (13.17.1.1) is an equivalence.*

Proof. Let $X^\bullet$ be a bounded above complex of $\mathcal{A}$ such that $H^i(X^\bullet) \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Moreover, suppose we are given $B^i \subset X^i$, $B^i \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Claim: there exists a subcomplex $Y^\bullet \subset X^\bullet$ such that

(1) $Y^\bullet \to X^\bullet$ is a quasi-isomorphism,
(2) $Y^i \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$, and
(3) $B^i \subset Y^i$ for all $i \in \mathbf{Z}$.

To prove the claim, using the assumption of the lemma we first choose $C^i \subset \mathrm{Ker}(d^i : X^i \to X^{i+1})$, $C^i \in \mathrm{Ob}(\mathcal{B})$ surjecting onto $H^i(X^\bullet)$. Setting $D^i = C^i + d^{i-1}(B^{i-1}) + B^i$ we find a subcomplex $D^\bullet$ satisfying (2) and (3) such that $H^i(D^\bullet) \to H^i(X^\bullet)$ is surjective for all $i \in \mathbf{Z}$. For any choice of $E^i \subset X^i$ with $E^i \in \mathrm{Ob}(\mathcal{B})$ and $d^i(E^i) \subset D^{i+1} + E^{i+1}$ we see that setting $Y^i = D^i + E^i$ gives a subcomplex whose terms are in $\mathcal{B}$ and whose cohomology surjects onto the cohomology of $X^\bullet$. Clearly, if $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$ then we see that the map on cohomology is also injective. For $n \gg 0$ we can take E^n equal to 0. By descending induction we can choose E^i for all i with the desired property. Namely, given $E^{i+1}, E^{i+2}, \ldots$ we choose $E^i \subset X^i$ such that $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$. This is possible by our assumption in the lemma combined with the fact that $(D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$ is in $\mathcal{B}$ as $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.

The claim above implies the lemma. Essential surjectivity is immediate from the claim. Let us prove faithfulness. Namely, suppose we have a morphism $f : U^\bullet \to V^\bullet$ of bounded above complexes of $\mathcal{B}$ whose image in $D(\mathcal{A})$ is zero. Then there exists a quasi-isomorphism $s : V^\bullet \to X^\bullet$ into a bounded above complex of $\mathcal{A}$ such that $s \circ f$ is homotopic to zero. Choose a homotopy $h^i : U^i \to X^{i-1}$ between 0 and $s \circ f$. Apply the claim with $B^i = h^{i+1}(U^{i+1}) + s^i(V^i)$. The resulting map $s' : V^\bullet \to Y^\bullet$ is a quasi-isomorphism as well and $s' \circ f$ is homotopic to zero as is clear from the fact that h^i factors through Y^{i-1}. This proves faithfulness. Fully faithfulness is proved in the exact same manner. □

13.18. Injective resolutions

013G In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

013I **Definition 13.18.1.** Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An *injective resolution of A* is a complex $I^\bullet$ together with a map $A \to I^0$ such that:

(1) We have $I^n = 0$ for $n < 0$.
(2) Each I^n is an injective object of $\mathcal{A}$.

(3) The map $A \to I^0$ is an isomorphism onto $\mathrm{Ker}(d^0)$.
(4) We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \to I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\ldots \to 0 \to A \to I^0 \to I^1 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *injective resolution of* $K^\bullet$ is a complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ of complexes such that

(1) We have $I^n = 0$ for $n \ll 0$, i.e., $I^\bullet$ is bounded below.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $\alpha : K^\bullet \to I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \to I^\bullet$ gives rise to a diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1} & \longrightarrow & \ldots
\end{array}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of $\mathcal{A}$ is almost the same thing as an injective resolution of the complex $A[0]$.

013J **Lemma 13.18.2.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*

Proof. Omitted. For the second statement use $L^\bullet = \tau_{\geq n} K^\bullet$ for some $n \ll 0$. See Homology, Section 12.15 for the definition of the truncation $\tau_{\geq n}$. □

013K **Lemma 13.18.3.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.*

(1) *Any object of $\mathcal{A}$ has an injective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then $K^\bullet$ has an injective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \to I^n$ is injective.*

Proof. Proof of (1). First choose an injection $A \to I^0$ of A into an injective object of $\mathcal{A}$. Next, choose an injection $I_0/A \to I^1$ into an injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose an injection $I^1/\mathrm{Im}(d^0) \to I^2$ into an injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. By Lemma 13.18.2 part (2) follows from part (3). Part (3) is a special case of Lemma 13.15.5. □

013R **Lemma 13.18.4.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $I^\bullet$ be bounded below and consisting of injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Note that $\alpha^j = 0$ for $j \ll 0$ as $I^\bullet$ is bounded below. In particular, we can find an n such that there exist $h^j : K^j \to I^{j-1}$ for $j \leq n$ such that $\alpha^j = d^{j-1} \circ h^j + h^{j+1} \circ d^j$ for $j < n$. We will show

that there exists a morphism $h^{n+1} : K^{n+1} \to I^n$ such that $\alpha^n = d^{n-1} \circ h^n + h^{n+1} \circ d^n$. Note that

$$\begin{aligned}(\alpha^n - d^{n-1} \circ h^{n-1}) \circ d^{n-1} &= \alpha^{n-1} \circ d^{n-1} - d^{n-1} \circ h^{n-1} \circ d^{n-1} \\ &= d^{n-1} \circ \alpha^{n-1} - d^{n-1} \circ h^{n-1} \circ d^{n-1} \\ &= d^{n-1} \circ (d^{n-2} \circ h^{n-1} + h^n \circ d^{n-1}) - d^{n-1} \circ h^{n-1} \circ d^{n-1} \\ &= 0\end{aligned}$$

Since $K^\bullet$ is acyclic we have $d^{n-1}(K^{n-1}) = \mathrm{Ker}(K^n \to K^{n+1})$. Hence we can think of $\alpha^n - d^{n-1} \circ h^{n-1}$ as a map into I^n defined on the subobject $\mathrm{Im}(K^n \to K^{n+1})$ of K^{n+1}. By injectivity of the object I^n we can extend this to a map $h^{n+1} : K^{n+1} \to I^n$. With this choice the reader checks that we indeed have $\alpha^n = d^{n-1} \circ h^n + h^{n+1} \circ d^n$.

By induction on n we conclude we can find $h = (h^j)_{j \in \mathbf{Z}}$ which forms a homotopy between α and 0 as desired. □

05TF **Remark 13.18.5.** Let $\mathcal{A}$ be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 13.18.4 to obtain proofs of some of the lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 13.9.14) and $C(\alpha)^\bullet$ is an acyclic complex (Lemma 13.11.2). Next, let $I^\bullet$ be a bounded below complex of injective objects. Then

$$\begin{array}{ccccc}\mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \\ & & & \swarrow & \\ \mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) & & & & \end{array}$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.18.4 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

013P **Lemma 13.18.6.** *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc}K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta} & \\ I^\bullet & & \end{array}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is injective in every degree then we can find a β which makes the diagram commute.*

Proof. The "correct" proof of part (1) is explained in Remark 13.18.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.9.6. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \to I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \to L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n-1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all $j \leq n-1$. Consider the commutative solid diagram

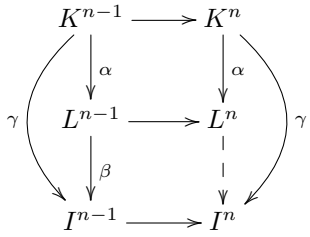

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

013Q (13.18.6.1) $$\alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1})$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \to I^n$ and, using the injectivity of I^n, we can extend this to a morphism from all of L^n into I^n. After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since $I^\bullet$ is bounded below – in this way starting the induction).

It remains to prove the equality (13.18.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of $\mathcal{A}$ and a morphism $x : T \to L^n$ whose image is contained in the subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \to K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \to H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \to H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. □

013S **Lemma 13.18.7.** *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

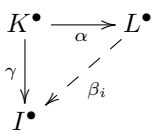

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 13.18.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.9.6. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \to L^n$ is the inclusion of a direct summand for all n. Thus we get a short exact sequence of complexes

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

which is termwise split and such that $M^\bullet$ is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \to I^n$ and $\gamma^n : K^n \to I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \to I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \to I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1}) \circ d$$

is a morphism of complexes $L^\bullet \to I^\bullet$ which is identically zero on the subcomplex $K^\bullet$. Hence it factors as $L^\bullet \to M^\bullet \to I^\bullet$. Thus the result of the lemma follows from Lemma 13.18.4. □

05TG **Lemma 13.18.8.** *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By Lemma 13.18.6 we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then Lemma 13.18.7 shows that β is homotopic to zero also, i.e., $b = 0$. □

013T **Lemma 13.18.9.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $\mathrm{Comp}^+(\mathcal{A})$ there exists a commutative diagram in $\mathrm{Comp}^+(\mathcal{A})$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & I_3^\bullet & \longrightarrow & 0 \end{array}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. Additionally,

(1) *given any injective resolution $A^\bullet \to I^\bullet$ we may assume $I_1^\bullet = I^\bullet$,*
(2) *if $A^n = B^n = C^n = 0$ for $n < 0$, then we may assume $I_j^n = 0$ for $n < 0$,*
(3) *we can combine (1) and (2) if also $I^n = 0$ for $n < 0$, and*
(4) *add more here.*

Proof. Step 1. Choose an injective resolution $A^\bullet \to I^\bullet$ (see Lemma 13.18.3) or use the given one. Recall that $\mathrm{Comp}^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 12.13.9. Hence we may form the pushout along the map $A^\bullet \to I^\bullet$ to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & E^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \end{array}$$

Because of the 5-lemma and the last assertion of Homology, Lemma 12.13.12 the map $B^\bullet \to E^\bullet$ is a quasi-isomorphism. Note that the lower short exact sequence is termwise split, see Homology, Lemma 12.27.2. Hence it suffices to prove the lemma when $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \to A^\bullet[1]$ the morphism as in Homology, Lemma 12.14.10. Choose injective resolutions $f_1 : A^\bullet \to I_1^\bullet$ and $f_3 : C^\bullet \to I_3^\bullet$. (If $A^\bullet$ is a complex of injectives, then use $I_1^\bullet = A^\bullet$. If $A^n = C^n = 0$ for $n < 0$, then pick these such that $I_1^n = I_3^n = 0$ for $n < 0$, see Lemma 13.18.3.) We may assume f_3 is injective in every degree. By Lemma 13.18.6 we may find a morphism $\delta' : I_3^\bullet \to I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \to I_2^n$ to be given as the sum of the maps $A^n \to I_1^n$ and $C^n \to I_3^n$. Everything is clear. □

13.19. Projective resolutions

0643 This section is dual to Section 13.18. We give definitions and state results, but we do not reprove the lemmas.

0644 **Definition 13.19.1.** Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. A *projective resolution of* A is a complex $P^\bullet$ together with a map $P^0 \to A$ such that:

(1) We have $P^n = 0$ for $n > 0$.
(2) Each P^n is a projective object of $\mathcal{A}$.
(3) The map $P^0 \to A$ induces an isomorphism $\mathrm{Coker}(d^{-1}) \to A$.
(4) We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \to A[0]$ is a quasi-isomorphism. In other words the complex

$$\ldots \to P^{-1} \to P^0 \to A \to 0 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. A *projective resolution of* $K^\bullet$ is a complex $P^\bullet$ together with a map $\alpha : P^\bullet \to K^\bullet$ of complexes such that

(1) We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
(2) Each P^n is a projective object of $\mathcal{A}$.
(3) The map $\alpha : P^\bullet \to K^\bullet$ is a quasi-isomorphism.

0645 **Lemma 13.19.2.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.*
(2) *If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \to K^\bullet$ with $L^\bullet$ bounded above.*

Proof. Dual to Lemma 13.18.2. □

0646 **Lemma 13.19.3.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives.*

(1) *Any object of $\mathcal{A}$ has a projective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then $K^\bullet$ has a projective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \to K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \to K^n$ is surjective.*

Proof. Dual to Lemma 13.18.3. □

0647 **Lemma 13.19.4.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $P^\bullet$ be bounded above and consisting of projective objects. Any morphism $P^\bullet \to K^\bullet$ is homotopic to zero.*

Proof. Dual to Lemma 13.18.4. □

0648 **Remark 13.19.5.** Let $\mathcal{A}$ be an abelian category. Suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Let $P^\bullet$ be a bounded above complex of projectives. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 13.18.5.

0649 **Lemma 13.19.6.** *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xleftarrow{\alpha} & L^\bullet \\ \uparrow & \swarrow_{\beta} & \nearrow \\ P^\bullet & & \end{array}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is surjective in every degree then we can find a β which makes the diagram commute.*

Proof. Dual to Lemma 13.18.6. □

064A **Lemma 13.19.7.** *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xleftarrow{\alpha} & L^\bullet \\ \uparrow & \swarrow_{\beta_i} & \nearrow \\ P^\bullet & & \end{array}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 13.18.7. □

064B **Lemma 13.19.8.** *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$ be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(P^\bullet, L^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 13.18.8. □

064C **Lemma 13.19.9.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $Comp^+(\mathcal{A})$ there exists a commutative diagram in $Comp^+(\mathcal{A})$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & P_3^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \end{array}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \to C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 13.18.9. □

064D **Lemma 13.19.10.** *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$, $K^\bullet$ be complexes. Let $n \in \mathbf{Z}$. Assume that*

(1) *$P^\bullet$ is a bounded complex consisting of projective objects,*
(2) *$P^i = 0$ for $i < n$, and*
(3) *$H^i(K^\bullet) = 0$ for $i \geq n$.*

Then $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 13.19.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, \tau_{\geq n}K^\bullet, f, g, h)$$

by Remark 13.12.4. Hence, by Lemma 13.4.2 it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1}K^\bullet) = 0$ and $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n}K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 13.19.4. □

064E **Lemma 13.19.11.** *Let $\mathcal{A}$ be an abelian category. Let $\beta : P^\bullet \to L^\bullet$ and $\alpha : E^\bullet \to L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume*

(1) *$P^\bullet$ is a bounded complex of projectives and $P^i = 0$ for $i < n$,*
(2) *$H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.*

Then there exists a map of complexes $\gamma : P^\bullet \to E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \to C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 13.19.10. Hence we can lift β to $E^\bullet$ by Lemma 13.4.2. □

13.20. Right derived functors and injective resolutions

0156 At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

05TH **Lemma 13.20.1.** *Let $\mathcal{A}$ be an abelian category. Let $I \in \mathrm{Ob}(\mathcal{A})$ be an injective object. Let $I^\bullet$ be a bounded below complex of injectives in $\mathcal{A}$.*

(1) *$I^\bullet$ computes RF relative to $Qis^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into any triangulated category $\mathcal{D}$.*
(2) *I is right acyclic for any additive functor $F : \mathcal{A} \to \mathcal{B}$ into any abelian category $\mathcal{B}$.*

Proof. Part (2) is a direct consequences of part (1) and Definition 13.15.3. To prove (1) let $\alpha : I^\bullet \to K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 13.18.6 we see that α has a left inverse. Hence the category $I^\bullet/\mathrm{Qis}^+(\mathcal{A})$ is essentially constant with value $\mathrm{id} : I^\bullet \to I^\bullet$. Thus also the ind-object

$$I^\bullet/\mathrm{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \to K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 13.14.2 and 13.14.10. □

05TI **Lemma 13.20.2.** *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) *For any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ the right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$

is everywhere defined.

(2) *For any additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ the right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

is everywhere defined.

Proof. Combine Lemma 13.20.1 and Proposition 13.16.8 for the second assertion. To see the first assertion combine Lemma 13.18.3, Lemma 13.20.1, Lemma 13.14.14, and Equation (13.14.9.1). □

0159 **Lemma 13.20.3.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.*

(1) *The functor RF is an exact functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(2) *The functor RF induces an exact functor $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *The functor RF induces a δ-functor $Comp^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(4) *The functor RF induces a δ-functor $\mathcal{A} \to D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.20.2 RF is everywhere defined. Here are some references:

(1) The derived functor is exact: This boils down to Lemma 13.14.6.
(2) This is true because $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.
(3) This is true because $\mathrm{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.12.1.
(4) This is true because $\mathcal{A} \to \mathrm{Comp}^+(\mathcal{A})$ is exact and precomposing a δ-functor by an exact functor gives a δ-functor.

□

015B **Lemma 13.20.4.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.*

(1) *For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of complexes in $Comp^+(\mathcal{A})$ there is an associated long exact sequence*

$$\ldots \to H^i(RF(A^\bullet)) \to H^i(RF(B^\bullet)) \to H^i(RF(C^\bullet)) \to H^{i+1}(RF(A^\bullet)) \to \ldots$$

(2) *The functors $R^iF : \mathcal{A} \to \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \to \mathcal{B}$.*
(3) *We have $R^iF(I) = 0$ for $i > 0$ and I injective.*

(4) *The sequence (R^iF, δ) forms a universal δ-functor (see Homology, Definition 12.12.3) from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.20.2 RF is everywhere defined. Here are some references:

(1) This follows from Lemma 13.20.3 part (3) combined with the long exact cohomology sequence (13.11.1.1) for $D^+(\mathcal{B})$.
(2) This is Lemma 13.16.3.
(3) This is the fact that injective objects are acyclic.
(4) This is Lemma 13.16.6.

□

13.21. Cartan-Eilenberg resolutions

015G This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

015H **Definition 13.21.1.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a bounded below complex. A *Cartan-Eilenberg resolution* of $K^\bullet$ is given by a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$ with the following properties:

(1) There exists a $i \ll 0$ such that $I^{p,q} = 0$ for all $p < i$ and all q.
(2) We have $I^{p,q} = 0$ if $q < 0$.
(3) The complex $I^{p,\bullet}$ is an injective resolution of K^p.
(4) The complex $\mathrm{Ker}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Ker}(d_K^p)$.
(5) The complex $\mathrm{Im}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Im}(d_K^p)$.
(6) The complex $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.

015I **Lemma 13.21.2.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $K^\bullet$ be a bounded below complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.*

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose $K^\bullet$ into short exact sequences as follows: Set $Z^p = \mathrm{Ker}(d^p)$, $B^p = \mathrm{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{aligned}
0 \to Z^n \to K^n \to B^{n+1} \to 0 \\
0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0 \\
0 \to Z^{n+1} \to K^{n+1} \to B^{n+2} \to 0 \\
0 \to B^{n+2} \to Z^{n+2} \to H^{n+2} \to 0 \\
\ldots
\end{aligned}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

(1) Choose an injective resolution $Z^n \to J_Z^{n,\bullet}$ with $J_Z^{n,m} = 0$ for $m < 0$, see Lemma 13.18.3.
(2) Using Lemma 13.18.9 choose injective resolutions $K^n \to I^{n,\bullet}$, $B^{n+1} \to J_B^{n+1,\bullet}$ with $I^{n,m} = 0$ for $m < 0$ and $J_B^{n+1,m} = 0$ for $m < 0$, and an exact sequence of complexes $0 \to J_Z^{n,\bullet} \to I^{n,\bullet} \to J_B^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to Z^n \to K^n \to B^{n+1} \to 0$.
(3) Using Lemma 13.18.9 choose injective resolutions $Z^{n+1} \to J_Z^{n+1,\bullet}$, $H^{n+1} \to J_H^{n+1,\bullet}$ with $J_Z^{n+1,m} = 0$ for $m < 0$ and $J_H^{n+1,m} = 0$ for $m < 0$ and an exact sequence of complexes $0 \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to J_H^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$.
(4) Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to I^{p+1,\bullet}$ everything is clear. □

015J **Lemma 13.21.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $K^\bullet$. The spectral sequences $('E_r, 'd_r)_{r\geq 0}$ and $(''E_r, ''d_r)_{r\geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations*

$$'E_1^{p,q} = R^qF(K^p) \quad \text{and} \quad ''E_2^{p,q} = R^pF(H^q(K^\bullet))$$

Moreover, these spectral sequences are bounded, converge to $H^*(RF(K^\bullet))$, and the associated induced filtrations on $H^n(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:

(1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
(2) As $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H_I^p(I^{\bullet,\bullet}))$.
(3) By Homology, Lemma 12.25.4 the total complex $\mathrm{Tot}(I^{\bullet,\bullet})$ is an injective resolution of $K^\bullet$. Hence RF is defined at $K^\bullet$ with value $F(\mathrm{Tot}(I^{\bullet,\bullet}))$.

Consider the two spectral sequences associated to the double complex $L^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 12.25.1. These are both bounded, converge to $H^*(\mathrm{Tot}(L^{\bullet,\bullet}))$, and induce finite filtrations on $H^n(\mathrm{Tot}(L^{\bullet,\bullet}))$, see Homology, Lemma 12.25.3. Since $\mathrm{Tot}(L^{\bullet,\bullet}) = \mathrm{Tot}(F(I^{\bullet,\bullet})) = F(\mathrm{Tot}(I^{\bullet,\bullet}))$ computes $RF(K^\bullet)$ we find the final assertion of the lemma holds true.

Computation of the first spectral sequence. We have $'E_1^{p,q} = H^q(L^{p,\bullet})$ in other words

$$'E_1^{p,q} = H^q(F(I^{p,\bullet})) = R^qF(K^p)$$

as desired. Observe for later use that the maps $'d_1^{p,q} : 'E_1^{p,q} \to 'E_1^{p+1,q}$ are the maps $R^qF(K^p) \to R^qF(K^{p+1})$ induced by $K^p \to K^{p+1}$ and the fact that R^qF is a functor.

Computation of the second spectral sequence. We have $''E_1^{p,q} = H^q(L^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of $\mathcal{A}$. Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F. Hence $''E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H_I^q(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H_I^p(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. □

015K **Remark 13.21.4.** The spectral sequences of Lemma 13.21.3 are functorial in the complex $K^\bullet$. This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of $\mathcal{A}$. The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 13.26.15.

13.22. Composition of right derived functors

015L Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \to D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$t : R(G \circ F) \longrightarrow RG \circ RF$$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$, see Lemma 13.14.16. This transformation need not always be an isomorphism.

015M **Lemma 13.22.1.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume $\mathcal{A}$, $\mathcal{B}$ have enough injectives. The following are equivalent*

(1) *$F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$, and*
(2) *the canonical map*

$$t : R(G \circ F) \longrightarrow RG \circ RF.$$

is isomorphism of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

Proof. If (2) holds, then (1) follows by evaluating the isomorphism t on $RF(I) = F(I)$. Conversely, assume (1) holds. Let $A^\bullet$ be a bounded below complex of $\mathcal{A}$. Choose an injective resolution $A^\bullet \to I^\bullet$. The map t is given (see proof of Lemma 13.14.16) by the maps

$$R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet))) \to RG(F(I^\bullet)) = RG(RF(A^\bullet))$$

where the arrow is an isomorphism by Lemma 13.16.7. □

015N **Lemma 13.22.2** (Grothendieck spectral sequence)**.** *With assumptions as in Lemma 13.22.1 and assuming the equivalent conditions (1) and (2) hold. Let X be an object of $D^+(\mathcal{A})$. There exists a spectral sequence $(E_r, d_r)_{r \geq 0}$ consisting of bigraded objects E_r of $\mathcal{C}$ with d_r of bidegree $(r, -r+1)$ and with*

$$E_2^{p,q} = R^pG(H^q(RF(X)))$$

Moreover, this spectral sequence is bounded, converges to $H^*(R(G \circ F)(X))$, and induces a finite filtration on each $H^n(R(G \circ F)(X))$.

For an object A of $\mathcal{A}$ we get $E_2^{p,q} = R^pG(R^qF(A))$ converging to $R^{p+q}(G \circ F)(A)$.

Proof. We may represent X by a bounded below complex $A^\bullet$. Choose an injective resolution $A^\bullet \to I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \to I^{\bullet,\bullet}$ using Lemma 13.21.2. Apply the second spectral sequence of Lemma 13.21.3. □

13.23. Resolution functors

013U Let $\mathcal{A}$ be an abelian category with enough injectives. Denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 13.23.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

013V **Proposition 13.23.1.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of $\mathcal{A}$. The functor*

$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 13.18.3. Fully faithfulness is a consequence of Lemma 13.18.8. □

Proposition 13.23.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category $\mathcal{A}$ is a "big" category, i.e., has a class of objects.

013W **Definition 13.23.2.** Let $\mathcal{A}$ be an abelian category with enough injectives. A *resolution functor*[8] for $\mathcal{A}$ is given by the following data:

(1) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
(2) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$.

05TJ **Lemma 13.23.3.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram*

$$\begin{array}{ccc} K^+(\mathcal{A}) & \xrightarrow[j]{} & K^+(\mathcal{I}) \\ & \searrow \quad \swarrow & \\ & D^+(\mathcal{A}) & \end{array}$$

where $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injective objects.

Proof. For every morphism $\alpha : K^\bullet \to L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \to j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{array}{ccc} K^\bullet & \xrightarrow[\alpha]{} & L^\bullet \\ {\scriptstyle i_{K^\bullet}}\downarrow & & \downarrow{\scriptstyle i_{L^\bullet}} \\ j(K^\bullet) & \xrightarrow{j(\alpha)} & j(L^\bullet) \end{array}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 13.18.6 and 13.18.7 or the equivalent Lemma 13.18.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. □

013X **Lemma 13.23.4.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.*

Proof. Consider the set of all objects $K^\bullet$ of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 13.18.3 every object has an injective resolution. By the axiom of choice we can choose for each $K^\bullet$ an injective resolution $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$. □

[8] This is likely nonstandard terminology.

014W **Lemma 13.23.5.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is exact.*

Proof. Denote $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$ the canonical maps of Definition 13.23.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \to j(K^\bullet)[1]$. Consider the diagram

$$\begin{array}{ccc} K^\bullet[1] & = & K^\bullet[1] \\ \downarrow{\scriptstyle i_{K^\bullet[1]}} & & \downarrow{\scriptstyle i_{K^\bullet}[1]} \\ j(K^\bullet[1]) & \xrightarrow{\xi_{K^\bullet}} & j(K^\bullet)[1] \end{array}$$

By Lemmas 13.18.6 and 13.18.7 there exists a unique dotted arrow $\xi_{K^\bullet}$ in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 13.18.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$. Note that we have a commutative diagram

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \xrightarrow{h} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j(K^\bullet) & \xrightarrow{j(f)} & j(L^\bullet) & \xrightarrow{j(g)} & j(M^\bullet) & \xrightarrow{\xi_{K^\bullet} \circ j(h)} & j(K^\bullet)[1] \end{array}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M. Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 13.4.18 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. □

05TK **Lemma 13.23.6.** *Let $\mathcal{A}$ be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \to K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$.*

Proof. The functor Q is a localization by Lemma 13.11.6. To prove the existence of j' it suffices to show that any element of $\mathrm{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j, see Lemma 13.5.7. Consider the commutative square in proof of Lemma 13.23.5. In this square, if α is a quasi-isomorphism, then $i_{L^\bullet} \circ \alpha = j(\alpha) \circ i_{K^\bullet}$ is a quasi-isomorphism too, hence so is $j(\alpha)$. Thus by Proposition 13.23.1 the morphism $j(\alpha)$ is an isomorphism in $K^+(\mathcal{I})$. We omit the verification that j' is quasi-inverse to $K^+(\mathcal{I}) \to D^+(\mathcal{A})$. □

013Y **Remark 13.23.7.** Suppose that $\mathcal{A}$ is a "big" abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i' : K^\bullet \to J^\bullet$ of a bounded below complex $K^\bullet$ into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

13.24. Functorial injective embeddings and resolution functors

0140 In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \to K^+(\mathcal{I})$ in case the category $\mathcal{A}$ has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if $\mathcal{A}$ is a "big" abelian category. See Remark 13.24.3 below.

Let $\mathcal{A}$ be an abelian category. As before denote $\mathcal{I}$ the additive full subcategory of $\mathcal{A}$ consisting of injective objects. Consider the category $\text{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \to I^\bullet$ where $K^\bullet$ is a bounded below complex of $\mathcal{A}$, $I^\bullet$ is a bounded below complex of injectives of $\mathcal{A}$ and α is a quasi-isomorphism. In other words, α is an injective resolution and $K^\bullet$ is bounded below. There is an obvious functor

$$s : \text{InjRes}(\mathcal{A}) \longrightarrow \text{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \text{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto I^\bullet$.

0141 **Lemma 13.24.1.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has functorial injective embeddings, see Homology, Definition 12.27.5.*

(1) *There exists a functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$.*
(2) *For any functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$ we obtain a resolution functor, see Definition 13.23.2.*

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding, see Homology, Definition 12.27.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \to A \to 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \text{Ker}(J(A) \to J(0))$. Note that the functorial morphism $A \to J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \text{Ker}(J(A) \to J(0))$ if needed.

Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet,\bullet}$ of injectives, together with a map $\alpha : K^\bullet \to I^{\bullet,0}$ such that α induces a quasi-isomorphism of $K^\bullet$ with the associated total complex of $I^{\bullet,\bullet}$. First we set $I^{p,q} = 0$ whenever $q < 0$. Next, we set $I^{p,0} = J(K^p)$ and $\alpha^p : K^p \to I^{p,0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet,0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p,0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p,1} = J(\text{Coker}(K^p \to I^{p,0}))$. Again by functoriality we see that $I^{\bullet,1}$ is a complex. And again we get that $I^{p,1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\text{Ker}(I^{p,0} \to I^{p,1})$. As our third step we take $I^{p,2} = J(\text{Coker}(I^{p,0} \to I^{p,1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 12.25.4 to get that the map

$$\alpha : K^\bullet \longrightarrow \text{Tot}(I^{\bullet,\bullet})$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ inj = \text{id}$. For every object $K^\bullet$ of $\text{Comp}^+(\mathcal{A})$ we can write

$$inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$$

This provides us with a resolution functor as in Definition 13.23.2. □

05TL **Remark 13.24.2.** Suppose inj is a functor such that $s \circ inj = \text{id}$ as in part (2) of Lemma 13.24.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \to L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category $\text{InjRes}(\mathcal{A})$. It induces a commutative diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle i_K}\downarrow & & \downarrow{\scriptstyle i_L} \\ j(K)^\bullet & \xrightarrow{inj(\alpha)} & j(L)^\bullet \end{array}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 13.23.3 we see that the functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotopy} \in \text{Hom}_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{array}{ccc} \text{Comp}^+(\mathcal{A}) & \xrightarrow[t\circ inj]{} & K^+(\mathcal{I}) \\ & \searrow \quad \nearrow_{j} & \\ & K^+(\mathcal{A}) & \end{array}$$

is commutative.

0142 **Remark 13.24.3.** Let $Mod(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (or more generally on a ringed site). We will see later that $Mod(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 19.8.4. Note that the proof of Lemma 13.23.4 does not apply to $Mod(\mathcal{O}_X)$. But the proof of Lemma 13.24.1 does apply to $Mod(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where $\mathcal{I}$ is the additive category of injective $\mathcal{O}_X$-modules. This argument also works in the following cases:

(1) The category Mod_R of R-modules over a ring R.
(2) The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on a site endowed with a presheaf of rings.
(3) The category $Mod(\mathcal{O})$ of sheaves of $\mathcal{O}$-modules on a ringed site.
(4) Add more here as needed.

13.25. Right derived functors via resolution functors

05TM The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet))$ if we are given a resolution functor j.

05TN **Lemma 13.25.1.** *Let $\mathcal{A}$ be an abelian category with enough injectives Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 13.23.2. The right derived functor RF of F fits into the following 2-commutative diagram*

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{j'} & K^+(\mathcal{I}) \\ & \searrow^{RF} \quad \swarrow_{F} & \\ & D^+(\mathcal{B}) & \end{array}$$

where j' is the functor from Lemma 13.23.6.

Proof. By Lemma 13.20.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. □

0158 **Remark 13.25.2.** In the situation of Lemma 13.25.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \to K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

13.26. Filtered derived category and injective resolutions

015O Let $\mathcal{A}$ be an abelian category. In this section we will show that if $\mathcal{A}$ has enough injectives, then so does the category $\mathrm{Fil}^f(\mathcal{A})$ in some sense. One can use this observation to compute in the filtered derived category of $\mathcal{A}$.

The category $\mathrm{Fil}^f(\mathcal{A})$ is an example of an exact category, see Injectives, Remark 19.9.6. A special role is played by the strict morphisms, see Homology, Definition 12.19.3, i.e., the morphisms f such that $\mathrm{Coim}(f) = \mathrm{Im}(f)$. We will say that a complex $A \to B \to C$ in $\mathrm{Fil}^f(\mathcal{A})$ is *exact* if the sequence $\mathrm{gr}(A) \to \mathrm{gr}(B) \to \mathrm{gr}(C)$ is exact in $\mathcal{A}$. This implies that $A \to B$ and $B \to C$ are strict morphisms, see Homology, Lemma 12.19.15.

015P **Definition 13.26.1.** Let $\mathcal{A}$ be an abelian category. We say an object I of $\mathrm{Fil}^f(\mathcal{A})$ is *filtered injective* if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

05TP **Lemma 13.26.2.** *Let $\mathcal{A}$ be an abelian category. An object I of $Fil^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^pI = \bigoplus_{n \geq p} I_n$.*

Proof. Follows from the fact that any injection $J \to M$ of $\mathcal{A}$ is split if J is an injective object. Details omitted. □

05TQ **Lemma 13.26.3.** *Let $\mathcal{A}$ be an abelian category. Any strict monomorphism $u : I \to A$ of $Fil^f(\mathcal{A})$ where I is a filtered injective object is a split injection.*

Proof. Let p be the largest integer such that $F^pI \neq 0$. In particular $\mathrm{gr}^p(I) = F^pI$. Let I' be the object of $\mathrm{Fil}^f(\mathcal{A})$ whose underlying object of $\mathcal{A}$ is F^pI and with filtration given by $F^nI' = 0$ for $n > p$ and $F^nI' = I' = F^pI$ for $n \leq p$. Note that $I' \to I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^pI \to A/F^{p+1}(A)$ is injective, see Homology, Lemma 12.19.13. Choose a splitting $s : A/F^{p+1}A \to F^pI$ in $\mathcal{A}$. The induced morphism $s' : A \to I'$ is a strict morphism of filtered objects splitting the composition $I' \to I \to A$. Hence we can write $A = I' \oplus \mathrm{Ker}(s')$ and $I = I' \oplus \mathrm{Ker}(s'|_I)$. Note that $\mathrm{Ker}(s'|_I) \to \mathrm{Ker}(s')$ is a strict monomorphism and that $\mathrm{Ker}(s'|_I)$ is a filtered injective object. By induction on the length of the filtration on I the map $\mathrm{Ker}(s'|_I) \to \mathrm{Ker}(s')$ is a split injection. Thus we win. □

05TR **Lemma 13.26.4.** *Let $\mathcal{A}$ be an abelian category. Let $u : A \to B$ be a strict monomorphism of $Fil^f(\mathcal{A})$ and $f : A \to I$ a morphism from A into a filtered injective object in $Fil^f(\mathcal{A})$. Then there exists a morphism $g : B \to I$ such that $f = g \circ u$.*

Proof. The pushout $f' : I \to I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 12.19.10. Hence the result follows formally from Lemma 13.26.3. □

05TS **Lemma 13.26.5.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $Fil^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where I is a filtered injective object.*

Proof. Pick $a \leq b$ such that $\mathrm{gr}^p(A) = 0$ unless $p \in \{a, a+1, \ldots, b\}$. For each $n \in \{a, a+1, \ldots, b\}$ choose an injection $u_n : A/F^{n+1}A \to I_n$ with I_n an injective object. Set $I = \bigoplus_{a \leq n \leq b} I_n$ with filtration $F^pI = \bigoplus_{n \geq p} I_n$ and set $u : A \to I$ equal to the direct sum of the maps u_n. □

05TT **Lemma 13.26.6.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $Fil^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^\bullet$ where $I^\bullet$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

Proof. First choose a strict monomorphism $u_0 : A \to I^0$ of A into a filtered injective object, see Lemma 13.26.5. Next, choose a strict monomorphism $u_1 : \mathrm{Coker}(u_0) \to I^1$ into a filtered injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose a strict monomorphism $u_2 : \mathrm{Coker}(u_1) \to I^2$ into a filtered injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. This works because each of the sequences

$$0 \to \mathrm{Coker}(u_n) \to I^{n+1} \to \mathrm{Coker}(u_{n+1}) \to 0$$

is short exact, i.e., induces a short exact sequence on applying gr. To see this use Homology, Lemma 12.19.13. □

05TU **Lemma 13.26.7.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism of $Fil^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $B[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{array}{ccc} A[0] & \longrightarrow & B[0] \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $b : B \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 12.19.15. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow[a]{} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \ldots \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ B & \xrightarrow[b]{} & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \ldots \end{array}$$

Because $A \to I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 13.26.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\mathrm{Coker}(a) = \mathrm{Coim}(d_I^0) = \mathrm{Im}(d_I^0) \longrightarrow J^1.$$

As $\mathrm{Im}(d_I^0) \to I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \to J^1$ by Lemma 13.26.4 again. And so on. □

05TV **Lemma 13.26.8.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $Fil^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[0] & \longrightarrow & B[0] & \longrightarrow & C[0] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & M^\bullet & \longrightarrow & J^\bullet & \longrightarrow & 0 \end{array}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $c : C \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 13.13.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{array}{ccccccccc} B & \longrightarrow & C & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \ldots \\ {\scriptstyle\alpha}\uparrow & \overset{\beta}{\searrow}{\scriptstyle b} & & \overset{c}{\searrow}{\scriptstyle \overline{b}} & \downarrow{\scriptstyle \delta^0} & & \downarrow{\scriptstyle \delta^1} & & \\ A & \overset{a}{\longrightarrow} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \ldots \end{array}$$

As $A \to B$ is a strict monomorphism, we can find a morphism $b : B \to I^0$ such that $b \circ \alpha = a$, see Lemma 13.26.4. As A is the kernel of the strict morphism $I^0 \to I^1$ and $\beta = \mathrm{Coker}(\alpha)$ we obtain a unique morphism $\overline{b} : C \to I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find $\delta^0 : J^0 \to I^1$, see Lemma 13.26.4. Because $B \to C$ is a strict epimorphism and because $B \to I^0 \to I^1 \to I^2$ is zero, we see that $C \to I^1 \to I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \mathrm{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\mathrm{Coker}(c) = \mathrm{Coim}(d_J^0) = \mathrm{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\mathrm{Im}(d_J^0) \to J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \to I^2$, see Lemma 13.26.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1}\delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \to M^\bullet$ is given by $b \oplus c \circ \beta : M \to I^0 \oplus J^0$. □

05TW **Lemma 13.26.9.** *Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^\bullet \in K^+(Fil^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below, each I^n a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.*

Proof. After replacing $K^\bullet$ by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{array}{l} 0 \to \mathrm{Ker}(d_K^0) \to K^0 \to \mathrm{Coim}(d_K^0) \to 0 \\ 0 \to \mathrm{Ker}(d_K^1) \to K^1 \to \mathrm{Coim}(d_K^1) \to 0 \\ 0 \to \mathrm{Ker}(d_K^2) \to K^2 \to \mathrm{Coim}(d_K^2) \to 0 \\ \ldots \end{array}$$

of the exact category $\mathrm{Fil}^f(\mathcal{A})$ and the maps $u_i : \mathrm{Coim}(d_K^i) \to \mathrm{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{aligned} \mathrm{Ker}(d_K^i)[0] &\to I_{ker,i}^\bullet \\ \mathrm{Coim}(d_K^i)[0] &\to I_{coim,i}^\bullet \end{aligned}$$

with $I_{ker,i}^n, I_{coim,i}^n$ filtered injective and zero for $n < 0$, see Lemma 13.26.6. By Lemma 13.26.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I_{coim,i}^\bullet \to I_{ker,i+1}^\bullet$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(d_K^i)[0] & \longrightarrow & K^i[0] & \longrightarrow & \mathrm{Coim}(d_K^i)[0] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{ker,i}^\bullet & \xrightarrow{\alpha_i} & I_i^\bullet & \xrightarrow{\beta_i} & I_{coim,i}^\bullet & \longrightarrow & 0 \end{array}$$

with the lower sequence a termwise split exact sequence, see Lemma 13.26.8. For $i \geq 0$ set $d_i : I_i^\bullet \to I_{i+1}^\bullet$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{array}{ccccccc} I_0^\bullet & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & \ldots \\ \uparrow & & \uparrow & & \uparrow & & \\ K^0[0] & \longrightarrow & K^1[0] & \longrightarrow & K^2[0] & \longrightarrow & \ldots \end{array}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no nonzero objects in degree < 0. Thus we obtain a double complex by setting $I^{a,b} = I_a^b$ and using

$$d_1^{a,b} : I^{a,b} = I_a^b \to I_{a+1}^b = I^{a+1,b}$$

the map d_a^b and using for

$$d_2^{a,b} : I^{a,b} = I_a^b \to I_a^{b+1} = I^{a,b+1}$$

the map $d_{I_a}^b$. Denote $\mathrm{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 12.18.3. Observe that the maps $K^n[0] \to I_n^\bullet$ come from maps $K^n \to I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \mathrm{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\mathrm{gr}(-)$ is an additive functor, we see that $\mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet})) = \mathrm{Tot}(\mathrm{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 12.25.4 to conclude that $\mathrm{gr}(K^\bullet) \to \mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. □

05TX **Lemma 13.26.10.** *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet, I^\bullet \in K(Fil^f(\mathcal{A}))$. Assume $K^\bullet$ is filtered acyclic and $I^\bullet$ bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero:* $\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d_K^{n-1} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is filtered acyclic we see that d_K^{n-1} and d_K^n are strict and that

$$0 \to \mathrm{Im}(d_K^{n-1}) \to K^n \to \mathrm{Im}(d_K^n) \to 0$$

is an exact sequence of the exact category $\mathrm{Fil}^f(\mathcal{A})$, see Homology, Lemma 12.19.15. Hence we can think of α^n as a map into I^n defined on $\mathrm{Im}(d_K^n)$. Using that $\mathrm{Im}(d_K^n) \to K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \to I^n$ as desired, see Lemma 13.26.4. □

05TY **Lemma 13.26.11.** *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet \in K(Fil^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.*

(1) *Let $\alpha : K^\bullet \to L^\bullet$ in $K(Fil^f(\mathcal{A}))$ be a filtered quasi-isomorphism. Then the map*

$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$

is bijective.

(2) *Let $L^\bullet \in K(Fil^f(\mathcal{A}))$. Then*

$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) = \mathrm{Hom}_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathrm{Fil}^f(\mathcal{A}))$ (Lemma 13.9.14) and $C(\alpha)^\bullet$ is a filtered acyclic complex (Lemma 13.13.4). Then

$$\mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$

$$\mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet)$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.26.10 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. □

015Q **Lemma 13.26.12.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{I}^f \subset Fil^f(\mathcal{A})$ denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor*

$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams

$$\begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ {\scriptstyle gr^p}\downarrow & & \downarrow{\scriptstyle gr^p} \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array} \qquad \begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ \downarrow{\scriptstyle forget\ F} & & \downarrow{\scriptstyle forget\ F} \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array}$$

are commutative, where $\mathcal{I} \subset \mathcal{A}$ is the strictly full additive subcategory whose objects are the injective objects.

Proof. The functor $K^+(\mathcal{I}^f) \to DF^+(\mathcal{A})$ is essentially surjective by Lemma 13.26.9. It is fully faithful by Lemma 13.26.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. □

015R **Remark 13.26.13.** We can invert the arrow of the lemma only if $\mathcal{A}$ is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category $\mathcal{A}$ with enough injectives, such as $Mod(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i\in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 3.12.1. Thus we may use the lemma above for $\mathcal{A}'$. This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^pT(A) = T(F^pA)$ which makes sense as T is left exact. (Warning: It may not be the case that $\mathrm{gr}(T(A)) = T(\mathrm{gr}(A))$.) This induces functors of triangulated categories

05TZ (13.26.13.1) $$T : K^+(\mathrm{Fil}^f(\mathcal{A})) \longrightarrow K^+(\mathrm{Fil}^f(\mathcal{B}))$$

The filtered right derived functor of T is the right derived functor of Definition 13.14.2 for this exact functor composed with the exact functor $K^+(\mathrm{Fil}^f(\mathcal{B})) \to DF^+(\mathcal{B})$ and the multiplicative set $\mathrm{FQis}^+(\mathcal{A})$. Assume $\mathcal{A}$ has enough injectives. At this point we can redo the discussion of Section 13.20 to define the *filtered right derived functors*

015S (13.26.13.2) $$RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B})$$

of our functor T.

However, instead we will proceed as in Section 13.25, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \to K^+(\mathcal{I}^f)$ of the equivalence of Lemma 13.26.12. By Lemma 13.4.18 we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

015T (13.26.13.3) $$I \cong \bigoplus\nolimits_{p\in\mathbf{Z}} I_p, \quad \text{with} \quad F^pI = \bigoplus\nolimits_{q\geq p} I_q$$

by Lemma 13.26.2. Hence if T is any additive functor $T : \mathcal{A} \to \mathcal{B}$ then we get an additive functor

05U0 (13.26.13.4) $$T_{ext} : \mathcal{I}^f \to \mathrm{Fil}^f(\mathcal{B})$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^pT_{ext}(I) = \bigoplus_{q \geq p} T(I_q)$. Note that we have the property $\mathrm{gr}(T_{ext}(I)) = T(\mathrm{gr}(I))$ by construction. Hence we obtain a functor

05U1 (13.26.13.5) $$T_{ext} : K^+(\mathcal{I}^f) \to K^+(\mathrm{Fil}^f(\mathcal{B}))$$

which commutes with gr. Then we define (13.26.13.2) by the composition

05U2 (13.26.13.6) $$RT = T_{ext} \circ j'.$$

Since $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 13.20.1, the commutation of T with gr, and the commutative diagrams of Lemma 13.26.12 imply that

015U (13.26.13.7) $$\mathrm{gr}^p \circ RT \cong RT \circ \mathrm{gr}^p$$

and

015V (13.26.13.8) $$(\text{forget } F) \circ RT \cong RT \circ (\text{forget } F)$$

as functors $DF^+(\mathcal{A}) \to D^+(\mathcal{B})$.

The filtered derived functor RT (13.26.13.2) induces functors

$$RT : \mathrm{Fil}^f(\mathcal{A}) \to DF^+(\mathcal{B}),$$
$$RT : \mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A})) \to DF^+(\mathcal{B}),$$
$$RT : KF^+(\mathcal{A}) \to DF^+(\mathcal{B}).$$

Note that since $\mathrm{Fil}^f(\mathcal{A})$, and $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ-functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ-functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

015W **Lemma 13.26.14.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume $\mathcal{A}$ has enough injectives. Let $(K^\bullet, F)$ be an object of $Comp^+(Fil^f(\mathcal{A}))$. There exists a spectral sequence $(E_r, d_r)_{r \geq 0}$ consisting of bigraded objects E_r of $\mathcal{B}$ and d_r of bidegree $(r, -r+1)$ and with*

$$E_1^{p,q} = R^{p+q}T(gr^p(K^\bullet))$$

Moreover, this spectral sequence is bounded, converges to $R^*T(K^\bullet)$, and induces a finite filtration on each $R^nT(K^\bullet)$. The construction of this spectral sequence is functorial in the object $K^\bullet$ of $Comp^+(Fil^f(\mathcal{A}))$ and the terms (E_r, d_r) for $r \geq 1$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.26.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (13.26.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to this as a filtered complex in $\mathcal{B}$, see Homology, Section 12.24. By Homology, Lemma 12.24.2 we have $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(T(I^\bullet)))$. By Equation (13.26.13.3) we have $E_1^{p,q} = H^{p+q}(T(\mathrm{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\mathrm{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 12.24.11 to conclude that the spectral sequence is bounded, converges to $H^n(T(I^\bullet)) = R^nT(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \to L^\bullet$ is a morphism of $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \to J^\bullet$ with $J^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.26.9. By our results above, for example Lemma 13.26.11, there exists a diagram

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \to T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 12.24.4. The last statement follows from this. □

015X **Remark 13.26.15.** As promised in Remark 13.21.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, assume $\mathcal{A}$ has enough injectives, and let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. We give an alternative construction of the spectral sequences ${}'E$ and ${}''E$ of Lemma 13.21.3.

First spectral sequence. Consider the "stupid" filtration on $K^\bullet$ obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 12.15. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 12.24.3. Note that $\mathrm{gr}^p(K^\bullet) = K^p[-p]$ with this filtration. According to Lemma 13.26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(K^p[-p]) = R^qT(K^p)$$

as in the spectral sequence ${}'E_r$. Observe moreover that the differentials $E_1^{p,q} \to E_1^{p+1,q}$ agree with the differentials in ${}'E_1$, see Homology, Lemma 12.24.3 part (2) and the description of ${}'d_1$ in the proof of Lemma 13.21.3.

Second spectral sequence. Consider the filtration on the complex $K^\bullet$ obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 12.15. The minus sign is necessary to get a decreasing filtration. Note that $\mathrm{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[p]$ with this filtration. According to Lemma 13.26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = {}''E_2^{i,j}$$

with $i = 2p+q$ and $j = -p$. (This looks unnatural, but note that we could just have well developed the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \to I^{\bullet,\bullet}$ the induced morphism $K^\bullet \to \mathrm{Tot}(I^{\bullet,\bullet})$ into the associated total complex will be a filtered injective resolution for either filtration using suitable filtrations on $\mathrm{Tot}(I^{\bullet,\bullet})$. This can be used to match up the spectral sequences exactly.

13.27. Ext groups

06XP In this section we start describing the Ext groups of objects of an abelian category. First we have the following very general definition.

06XQ **Definition 13.27.1.** Let $\mathcal{A}$ be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The *ith extension group* of X by Y is the group

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in \mathrm{Ob}(\mathcal{A})$ we set $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Ext}^i_{\mathcal{A}}(A[0], B[0])$.

Since $\mathrm{Hom}_{D(\mathcal{A})}(X, -)$, resp. $\mathrm{Hom}_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 13.4.2, we see that a distinguished triangle (Y, Y', Y''), resp. (X, X', X'') leads to a long exact sequence

$$\ldots \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y') \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y'') \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X, Y) \to \ldots$$

respectively

$$\ldots \to \mathrm{Ext}^i_{\mathcal{A}}(X'', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X'', Y) \to \ldots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the Ext groups by Hom groups in these categories provided X, Y are contained in them.

In case the category $\mathcal{A}$ has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 13.18.2 and 13.19.2.

06XR **Lemma 13.27.2.** *Let $\mathcal{A}$ be an abelian category. Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$.*

(1) *Let $Y^\bullet \to I^\bullet$ be an injective resolution (Definition 13.18.1). Then*
$$\mathrm{Ext}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$
(2) *Let $P^\bullet \to X^\bullet$ be a projective resolution (Definition 13.19.1). Then*
$$\mathrm{Ext}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 13.18.8 and Lemma 13.19.8. □

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

06XS **Lemma 13.27.3.** *Let $\mathcal{A}$ be an abelian category.*

(1) *Let X, Y be objects of $D(\mathcal{A})$. Given $a, b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > a$ and $H^j(Y) = 0$ for $j < b$, we have $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ for $n < b - a$ and*
$$\mathrm{Ext}^{b-a}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(H^a(X), H^b(Y))$$
(2) *Let $A, B \in \mathrm{Ob}(\mathcal{A})$. For $i < 0$ we have $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$. We have $\mathrm{Ext}^0_{\mathcal{A}}(B, A) = \mathrm{Hom}_{\mathcal{A}}(B, A)$.*

Proof. Choose complexes $X^\bullet$ and $Y^\bullet$ representing X and Y. Since $Y^\bullet \to \tau_{\geq b} Y^\bullet$ is a quasi-isomorphism, we may assume that $Y^j = 0$ for $j < b$. Let $L^\bullet \to X^\bullet$ be any quasi-isomorphism. Then $\tau_{\leq a} L^\bullet \to X^\bullet$ is a quasi-isomorphism. Hence a morphism $X \to Y[n]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \to X^\bullet$ is a quasi-isomorphism, $f : L^\bullet \to Y^\bullet[n]$ a morphism, and $L^i = 0$ for $i < a$. Note that

f maps L^i to Y^{i+n}. Thus $f = 0$ if $n < b - a$ because always either L^i or Y^{i+n} is zero. If $n = b - a$, then f corresponds exactly to a morphism $H^a(X) \to H^b(Y)$. Part (2) is a special case of (1). □

Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of objects of $\mathcal{A}$. Then $0 \to A[0] \to A'[0] \to A''[0] \to 0$ leads to a distinguished triangle in $D(\mathcal{A})$ (see Lemma 13.12.1) hence a long exact sequence of Ext groups

$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A') \to \mathrm{Ext}^0_{\mathcal{A}}(B, A'') \to \mathrm{Ext}^1_{\mathcal{A}}(B, A) \to \ldots$$

Similarly, given a short exact sequence $0 \to B \to B' \to B'' \to 0$ we obtain a long exact sequence of Ext groups

$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B'', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^1_{\mathcal{A}}(B'', A) \to \ldots$$

We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [**Yon60**].

06XT **Definition 13.27.4.** Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. A degree i *Yoneda extension* of B by A is an exact sequence

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. We say two Yoneda extensions E and E' of the same degree are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & Z_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \longrightarrow & Z''_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z''_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \longrightarrow & Z'_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z'_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 13.27.5 below.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a Yoneda extension $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ we define an associated element $\delta(E) \in \mathrm{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \to A[i]$ where s is the quasi-isomorphism

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

06XU **Lemma 13.27.5.** *Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A. Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.*

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \to B[0]$ and map $f : L^\bullet \to A[i]$. After replacing $L^\bullet$ by $\tau_{\leq 0}L^\bullet$ we may assume that $L^j = 0$ for $j > 0$. Picture

$$\begin{array}{ccccccccc} L^{-i-1} & \longrightarrow & L^{-i} & \longrightarrow & \ldots & \longrightarrow & L^0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & & & & & \\ & & A & & & & & & \end{array}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ.

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \ldots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex $L^\bullet$ and quasi-isomorphisms

$$t : L^\bullet \to (\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots)$$

and

$$t' : L^\bullet \to (\ldots \to 0 \to A \to Z'_{i-1} \to \ldots \to Z'_0 \to 0 \to \ldots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 4.27. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists "morphisms" of degree i Yoneda extensions $E'' \to E$ and $E'' \to E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. □

06XV **Lemma 13.27.6.** *Let $\mathcal{A}$ be an abelian category. Let A, B be objects of $\mathcal{A}$. Then $\mathrm{Ext}^1_{\mathcal{A}}(B, A)$ is the group $\mathrm{Ext}_{\mathcal{A}}(B, A)$ constructed in Homology, Definition 12.6.2.*

Proof. This is the case $i = 1$ of Lemma 13.27.5. □

Given an abelian category $\mathcal{A}$ and objects X, Y, Z of $D(\mathcal{A})$, there are bilinear and associative composition laws

$$\mathrm{Ext}^j_{\mathcal{A}}(Y, Z) \times \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{A}}(X, Z), \quad (\eta, \xi) \longmapsto \eta \circ \xi$$

Namely, if $\xi : X \to Y[i]$ and $\eta : Y \to Z[j]$, then we define $\eta \circ \xi$ as the composition of ξ by $\eta[i] : Y[i] \to Z[i+j]$. If A, B, C are objects of $\mathcal{A}$, then this composition law

$$\mathrm{Ext}^j_{\mathcal{A}}(B, C) \times \mathrm{Ext}^i_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{A}}(A, C)$$

can be described in terms of Yoneda extensions as follows: the composition of

$$0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

and

$$0 \to B \to Z'_{j-1} \to Z'_{j-2} \to \ldots \to Z'_0 \to C \to 0$$

is the Yoneda extension

$$0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to Z'_{j-1} \to Z'_{j-2} \to \ldots \to Z'_0 \to C \to 0$$

0GSM **Lemma 13.27.7.** *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to Z \to B \to 0$ and $0 \to B \to Z' \to C \to 0$ be short exact sequences in $\mathcal{A}$. Denote $[Z] \in \mathrm{Ext}^1(B, A)$ and $[Z'] \in \mathrm{Ext}^1(C, B)$ their classes. Then $[Z] \circ [Z'] \in \mathrm{Ext}^2_{\mathcal{A}}(C, A)$ is 0 if and only if there exists a commutative diagram*

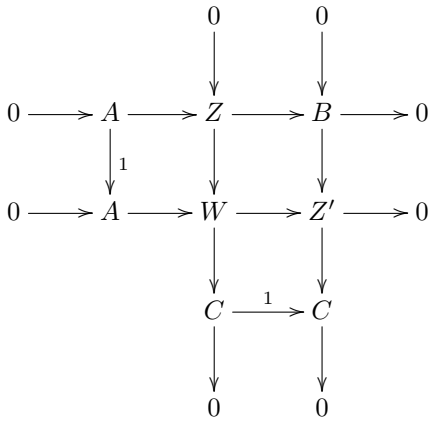

with exact rows and columns in $\mathcal{A}$.

Proof. Omitted. Hints: You can argue this using the result of Lemma 13.27.5 and working out what it means for a 2-extension class to be zero. Or you can use that if $[Z] \circ [Z'] \in \mathrm{Ext}^2_{\mathcal{A}}(C, A)$ is zero, then by the long exact cohomology sequence of Ext the element $[Z] \in \mathrm{Ext}^1(B, A)$ is the image of some element in $\mathrm{Ext}^1(Z', A)$. □

0EWW **Lemma 13.27.8.** *Let $\mathcal{A}$ be an abelian category and let $p \geq 0$. If $\mathrm{Ext}^p_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$, then $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq p$ and any pair of objects A, B of $\mathcal{A}$.*

Proof. For $i > p$ write any class ξ as $\delta(E)$ where E is a Yoneda extension

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

This is possible by Lemma 13.27.5. Set $C = \mathrm{Ker}(Z_{p-1} \to Z_{p-2}) = \mathrm{Im}(Z_p \to Z_{p-1})$. Then $\delta(E)$ is the composition of $\delta(E')$ and $\delta(E'')$ where

$$E' : 0 \to C \to Z_{p-1} \to \ldots \to Z_0 \to B \to 0$$

and

$$E'' : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_p \to C \to 0$$

Since $\delta(E') \in \mathrm{Ext}^p_{\mathcal{A}}(B, C) = 0$ we conclude. □

0GM4 **Lemma 13.27.9.** *Let $\mathcal{A}$ be an abelian category. Let K be an object of $D^b(\mathcal{A})$ such that $\mathrm{Ext}^p_{\mathcal{A}}(H^i(K), H^j(K)) = 0$ for all $p \geq 2$ and $i > j$. Then K is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.*

Proof. Choose a, b such that $H^i(K) = 0$ for $i \notin [a, b]$. We will prove the lemma by induction on $b - a$. If $b - a \leq 0$, then the result is clear. If $b - a > 0$, then we look at the distinguished triangle of truncations

$$\tau_{\leq b-1}K \to K \to H^b(K)[-b] \to (\tau_{\leq b-1}K)[1]$$

see Remark 13.12.4. By Lemma 13.4.11 if the last arrow is zero, then $K \cong \tau_{\leq b-1}K \oplus H^b(K)[-b]$ and we win by induction. Again using induction we see that

$$\mathrm{Hom}_{D(\mathcal{A})}(H^b(K)[-b], (\tau_{\leq b-1}K)[1]) = \bigoplus\nolimits_{i<b} \mathrm{Ext}^{b-i+1}_{\mathcal{A}}(H^b(K), H^i(K))$$

By assumption the direct sum is zero and the proof is complete. □

0EWX **Lemma 13.27.10.** *Let $\mathcal{A}$ be an abelian category. Assume $\mathrm{Ext}^2_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$. Then any object K of $D^b(\mathcal{A})$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.*

Proof. The assumption implies that $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq 2$ and any pair of objects A, B of $\mathcal{A}$ by Lemma 13.27.8. Hence this lemma is a special case of Lemma 13.27.9. □

13.28. K-groups

0FCM A tiny bit about K_0 of a triangulated category.

0FCN **Definition 13.28.1.** Let $\mathcal{D}$ be a triangulated category. We denote $K_0(\mathcal{D})$ the *zeroth K-group of $\mathcal{D}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{D}$ and for every distinguished triangle $X \to Y \to Z$ impose the relation $[Y] - [X] - [Z] = 0$.

Observe that this implies that $[X[n]] = (-1)^n[X]$ because we have the distinguished triangle $(X, 0, X[1], 0, 0, -\mathrm{id}[1])$.

0FCP **Lemma 13.28.2.** *Let $\mathcal{A}$ be an abelian category. Then there is a canonical identification $K_0(D^b(\mathcal{A})) = K_0(\mathcal{A})$ of zeroth K-groups.*

Proof. Given an object A of $\mathcal{A}$ denote $A[0]$ the object A viewed as a complex sitting in degree 0. If $0 \to A \to A' \to A'' \to 0$ is a short exact sequence, then we get a distinguished triangle $A[0] \to A'[0] \to A''[0] \to A[1]$, see Section 13.12. This shows that we obtain a map $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ by sending $[A]$ to $[A[0]]$ with apologies for the horrendous notation.

On the other hand, given an object X of $D^b(\mathcal{A})$ we can consider the element

$$c(X) = \sum (-1)^i [H^i(X)] \in K_0(\mathcal{A})$$

Given a distinguished triangle $X \to Y \to Z$ the long exact sequence of cohomology (13.11.1.1) and the relations in $K_0(\mathcal{A})$ show that $c(Y) = c(X) + c(Z)$. Thus c factors through a map $c : K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$.

We want to show that the two maps above are mutually inverse. It is clear that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$ is the identity. Suppose that $X^\bullet$ is a bounded complex of $\mathcal{A}$. The existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n}X^\bullet \to \sigma_{\geq n-1}X^\bullet \to X^{n-1}[-n+1] \to (\sigma_{\geq n}X^\bullet)[1]$$

and induction show that

$$[X^\bullet] = \sum (-1)^i [X^i[0]]$$

in $K_0(D^b(\mathcal{A}))$ (with again apologies for the notation). It follows that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ is surjective which finishes the proof. □

0FCQ **Lemma 13.28.3.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Then F induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{D}')$.*

Proof. Omitted. □

0FCR **Lemma 13.28.4.** *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category to an abelian category. Assume that for any X in $\mathcal{D}$ only a finite number of the objects $H(X[i])$ are nonzero in $\mathcal{A}$. Then H induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{A})$ sending $[X]$ to $\sum(-1)^i[H(X[i])]$.*

Proof. Omitted. □

0FCS **Lemma 13.28.5.** *Let $\mathcal{B}$ be a weak Serre subcategory of the abelian category $\mathcal{A}$. There is a canonical isomorphism*

$$K_0(\mathcal{B}) \longrightarrow K_0(D^b_{\mathcal{B}}(\mathcal{A})), \quad [B] \longmapsto [B[0]]$$

The inverse sends the class $[X]$ of X to the element $\sum(-1)^i[H^i(X)]$.

Proof. We omit the verification that the rule for the inverse gives a well defined map $K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$. It is immediate that the composition $K_0(\mathcal{B}) \to K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$ is the identity. On the other hand, using the distinguished triangles of Remark 13.12.4 and an induction argument the reader may show that the displayed arrow in the statement of the lemma is surjective (details omitted). The lemma follows. □

0FCT **Lemma 13.28.6.** *Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be triangulated categories. Let*

$$\otimes : \mathcal{D} \times \mathcal{D}' \longrightarrow \mathcal{D}''$$

be a functor such that for fixed X in $\mathcal{D}$ the functor $X \otimes - : \mathcal{D}' \to \mathcal{D}''$ is an exact functor and for fixed X' in $\mathcal{D}'$ the functor $- \otimes X' : \mathcal{D} \to \mathcal{D}''$ is an exact functor. Then $\otimes$ induces a bilinear map $K_0(\mathcal{D}) \times K_0(\mathcal{D}') \to K_0(\mathcal{D}'')$ which sends $([X], [X'])$ to $[X \otimes X']$.

Proof. Omitted. □

13.29. Unbounded complexes

06XW A reference for the material in this section is [**Spa88**]. The following lemma is useful to find "good" left resolutions of unbounded complexes.

06XX **Lemma 13.29.1.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}K^\bullet & \longrightarrow & \tau_{\leq 2}K^\bullet & \longrightarrow & \ldots \end{array}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms and termwise surjective,*
(2) *$P_n^\bullet$ is a bounded above complex with terms in $\mathcal{P}$,*
(3) *the arrows $P_n^\bullet \to P_{n+1}^\bullet$ are termwise split injections and each cokernel P_{n+1}^i/P_n^i is an element of $\mathcal{P}$.*

Proof. We are going to use that the homotopy category $K(\mathcal{A})$ is a triangulated category, see Proposition 13.10.3. By Lemma 13.15.4 we can find a termwise surjective map of complexes $P_1^\bullet \to \tau_{\leq 1}K^\bullet$ which is a quasi-isomorphism such that the terms of $P_1^\bullet$ are in $\mathcal{P}$. By induction it suffices, given $P_1^\bullet, \ldots, P_n^\bullet$ to construct $P_{n+1}^\bullet$ and the maps $P_n^\bullet \to P_{n+1}^\bullet$ and $P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$.

Choose a distinguished triangle $P_n^\bullet \to \tau_{\leq n+1}K^\bullet \to C^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. Applying Lemma 13.15.4 we choose a map of complexes $Q^\bullet \to C^\bullet$ which is a quasi-isomorphism such that the terms of $Q^\bullet$ are in $\mathcal{P}$. By the axioms of triangulated categories we may fit the composition $Q^\bullet \to C^\bullet \to P_n^\bullet[1]$ into a distinguished triangle $P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. By Lemma 13.10.7 we may and do assume $0 \to P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to 0$ is a termwise split short exact sequence. This implies that the terms of $P_{n+1}^\bullet$ are in $\mathcal{P}$ and that $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection whose cokernels are in $\mathcal{P}$. By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{array}{ccccccc} P_n^\bullet & \longrightarrow & P_{n+1}^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & P_n^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_n^\bullet & \longrightarrow & \tau_{\leq n+1}K^\bullet & \longrightarrow & C^\bullet & \longrightarrow & P_n^\bullet[1] \end{array}$$

in the triangulated category $K(\mathcal{A})$. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \to P_n^\bullet$ and $Q^\bullet \to C^\bullet$ are.

At this point we have all the properties we want, except we don't know that the map $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ is termwise surjective. Since we have the commutative diagram

$$\begin{array}{ccc} P_n^\bullet & \longrightarrow & P_{n+1}^\bullet \\ \downarrow & & \downarrow \\ \tau_{\leq n}K^\bullet & \longrightarrow & \tau_{\leq n+1}K^\bullet \end{array}$$

of complexes, by induction hypothesis we see that f is surjective on terms in all degrees except possibly n and $n+1$. Choose an object $P \in \mathcal{P}$ and a surjection $q : P \to K^n$. Consider the map

$$g : P^\bullet = (\ldots \to 0 \to P \xrightarrow{1} P \to 0 \to \ldots) \longrightarrow \tau_{\leq n+1}K^\bullet$$

with first copy of P in degree n and maps given by q in degree n and $d_K \circ q$ in degree $n+1$. This is a surjection in degree n and the cokernel in degree $n+1$ is $H^{n+1}(\tau_{\leq n+1}K^\bullet)$; to see this recall that $\tau_{\leq n+1}K^\bullet$ has $\mathrm{Ker}(d_K^{n+1})$ in degree $n+1$. However, since f is a quasi-isomorphism we know that $H^{n+1}(f)$ is surjective. Hence after replacing $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ by $f \oplus g : P_{n+1}^\bullet \oplus P^\bullet \to \tau_{\leq n+1}K^\bullet$ we win. $\square$

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

0794 **Proposition 13.29.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume*

(1) *$\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
(2) *for any bounded above acyclic complex $P^\bullet$ of $\mathcal{A}$ with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,*
(3) *$\mathcal{A}$ and $\mathcal{B}$ have colimits of systems over $\mathbf{N}$,*
(4) *colimits over $\mathbf{N}$ are exact in both $\mathcal{A}$ and $\mathcal{B}$, and*
(5) *F commutes with colimits over $\mathbf{N}$.*

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 13.15.4 for any bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and $P^n \in \mathcal{P}$ for all n. Suppose that $s : P^\bullet \to (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of $\mathcal{P}$. Then $F(P^\bullet) \to F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of $\mathcal{P}$ computes LF, see Lemma 13.14.15.

Next, let $K^\bullet$ be an arbitrary complex of $\mathcal{A}$. Choose a diagram

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}K^\bullet & \longrightarrow & \tau_{\leq 2}K^\bullet & \longrightarrow & \ldots \end{array}$$

as in Lemma 13.29.1. Note that the map $\operatorname{colim} P_n^\bullet \to K^\bullet$ is a quasi-isomorphism because colimits over $\mathbf{N}$ in $\mathcal{A}$ are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(\operatorname{colim} P_n^\bullet) = \operatorname{colim} F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $\operatorname{colim} P_n^\bullet$ computes LF. To see this, by Lemma 13.14.15, it suffices to prove the following claim. Suppose that

$$\operatorname{colim} Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = \operatorname{colim} P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each $P_n^\bullet$, $Q_n^\bullet$ is a bounded above complex whose terms are in $\mathcal{P}$ and the maps $P_n^\bullet \to \tau_{\leq n}P^\bullet$ and $Q_n^\bullet \to \tau_{\leq n}Q^\bullet$ are quasi-isomorphisms. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes $P_n^\bullet$ and $Q_n^\bullet$. However, for each n we know that the solid arrows in the diagram

$$\begin{array}{ccccc} & & R^\bullet & & \\ & & \vdots & & \\ P_n^\bullet & \leftarrow\cdots & L^\bullet & \cdots\rightarrow & Q_n^\bullet \\ \downarrow & & & & \downarrow \\ \tau_{\leq n}P^\bullet & & \xrightarrow{\tau_{\leq n}\alpha} & & \tau_{\leq n}Q^\bullet \end{array}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 13.11.2) we can find a quasi-isomorphism $L^\bullet \to P_n^\bullet$ and map of complexes $L^\bullet \to Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n}L^\bullet \to L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof)

we can find a bounded above complex $R^\bullet$ whose terms are in $\mathcal{P}$ and a quasi-isomorphism $R^\bullet \to L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \to F(P_n^\bullet)$ and $F(R^\bullet) \to F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain an isomorphism $H^i(F(P_n^\bullet)) \to H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{array}{ccc} H^i(F(P_n^\bullet)) & \longrightarrow & H^i(F(Q_n^\bullet)) \\ \downarrow & & \downarrow \\ H^i(F(P^\bullet)) & \longrightarrow & H^i(F(Q^\bullet)) \end{array}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(\operatorname{colim} P_n^\bullet)) = H^i(\operatorname{colim} F(P_n^\bullet)) = \operatorname{colim} H^i(F(P_n^\bullet))$$

and similarly for $Q^\bullet$ we conclude that $H^i(\alpha) : H^i(F(P^\bullet) \to H^i(F(Q^\bullet)$ is an isomorphism and the claim follows. □

070F **Lemma 13.29.3.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{I}$ contains 0, is closed under (finite) products, and every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}K^\bullet & \longrightarrow & \tau_{\geq -1}K^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & I_2^\bullet & \longrightarrow & I_1^\bullet \end{array}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms and termwise injective,*
(2) *$I_n^\bullet$ is a bounded below complex with terms in $\mathcal{I}$,*
(3) *the arrows $I_{n+1}^\bullet \to I_n^\bullet$ are termwise split surjections and $\mathrm{Ker}(I_{n+1}^i \to I_n^i)$ is an element of $\mathcal{I}$.*

Proof. This lemma is dual to Lemma 13.29.1. □

13.30. Deriving adjoints

0FNC Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be exact functors of triangulated categories. Let S, resp. S' be a multiplicative system for $\mathcal{D}$, resp. $\mathcal{D}'$ compatible with the triangulated structure. Denote $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ and $Q' : \mathcal{D}' \to (S')^{-1}\mathcal{D}'$ the localization functors. In this situation, by abuse of notation, one often denotes RF the partially defined right derived functor corresponding to $Q' \circ F : \mathcal{D} \to (S')^{-1}\mathcal{D}'$ and the multiplicative system S. Similarly one denotes LG the partially defined left derived functor corresponding to $Q \circ G : \mathcal{D}' \to S^{-1}\mathcal{D}$ and the multiplicative system S'. Picture

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{D}' \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\ S^{-1}\mathcal{D} & \overset{RF}{\dashrightarrow} & (S')^{-1}\mathcal{D}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{D}' & \xrightarrow{G} & \mathcal{D} \\ {\scriptstyle Q'}\downarrow & & \downarrow{\scriptstyle Q} \\ (S')^{-1}\mathcal{D}' & \overset{LG}{\dashrightarrow} & S^{-1}\mathcal{D} \end{array}$$

0FND **Lemma 13.30.1.** *In the situation above assume F is right adjoint to G. Let $K \in \mathrm{Ob}(\mathcal{D})$ and $M \in \mathrm{Ob}(\mathcal{D}')$. If RF is defined at K and LG is defined at M, then there is a canonical isomorphism*

$$\mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) = \mathrm{Hom}_{S^{-1}\mathcal{D}}(LG(M), K)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $S^{-1}\mathcal{D}$ and $(S')^{-1}\mathcal{D}'$ where RF and LG are defined.

Proof. Since RF is defined at K, we see that the rule which assigns to an $s : K \to I$ in S the object $F(I)$ is essentially constant as an ind-object of $(S')^{-1}\mathcal{D}'$ with value $RF(K)$. Similarly, the rule which assigns to a $t : P \to M$ in S' the object $G(P)$ is essentially constant as a pro-object of $S^{-1}\mathcal{D}$ with value $LG(M)$. Thus we have

$$\begin{aligned}
\mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) &= \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, F(I)) \\
&= \mathrm{colim}_{s:K\to I} \mathrm{colim}_{t:P\to M} \mathrm{Hom}_{\mathcal{D}'}(P, F(I)) \\
&= \mathrm{colim}_{t:P\to M} \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{\mathcal{D}'}(P, F(I)) \\
&= \mathrm{colim}_{t:P\to M} \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{\mathcal{D}}(G(P), I) \\
&= \mathrm{colim}_{t:P\to M} \mathrm{Hom}_{S^{-1}\mathcal{D}}(G(P), K) \\
&= \mathrm{Hom}_{S^{-1}\mathcal{D}}(LG(M), K)
\end{aligned}$$

The first equality holds by Categories, Lemma 4.22.9. The second equality holds by the definition of morphisms in $D(\mathcal{B})$, see Categories, Remark 4.27.15. The third equality holds by Categories, Lemma 4.14.10. The fourth equality holds because F and G are adjoint. The fifth equality holds by definition of morphism in $D(\mathcal{A})$, see Categories, Remark 4.27.7. The sixth equality holds by Categories, Lemma 4.22.10. We omit the proof of functoriality. □

0DVC **Lemma 13.30.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. Let $K^\bullet$ be a complex of $\mathcal{A}$ and let $M^\bullet$ be a complex of $\mathcal{B}$. If RF is defined at $K^\bullet$ and LG is defined at $M^\bullet$, then there is a canonical isomorphism*

$$\mathrm{Hom}_{D(\mathcal{B})}(M^\bullet, RF(K^\bullet)) = \mathrm{Hom}_{D(\mathcal{A})}(LG(M^\bullet), K^\bullet)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $D(\mathcal{A})$ and $D(\mathcal{B})$ where RF and LG are defined.

Proof. This is a special case of the very general Lemma 13.30.1. □

The following lemma is an example of why it is easier to work with unbounded derived categories. Namely, without having the unbounded derived functors, the lemma could not even be stated.

09T5 **Lemma 13.30.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. If the derived functors $RF : D(\mathcal{A}) \to D(\mathcal{B})$ and $LG : D(\mathcal{B}) \to D(\mathcal{A})$ exist, then RF is a right adjoint to LG.*

Proof. Immediate from Lemma 13.30.2. □

13.31. K-injective complexes

070G The following types of complexes can be used to compute right derived functors on the unbounded derived category.

070H **Definition 13.31.1.** Let $\mathcal{A}$ be an abelian category. A complex $I^\bullet$ is *K-injective* if for every acyclic complex $M^\bullet$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

070I **Lemma 13.31.2.** *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be a complex. The following are equivalent*

(1) *$I^\bullet$ is K-injective,*
(2) *for every quasi-isomorphism $M^\bullet \to N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$
is bijective, and
(3) *for every complex $N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$
is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $\mathrm{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \to I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \to I^\bullet$ where $s : M^\bullet \to N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \to I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \to I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If $M^\bullet$ is acyclic then $M^\bullet$ is isomorphic to the zero complex in $D(\mathcal{A})$ hence $\mathrm{Hom}_{D(\mathcal{A})}(M^\bullet, I^\bullet) = 0$, whence $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. □

090X **Lemma 13.31.3.** *Let $\mathcal{A}$ be an abelian category. Let (K, L, M, f, g, h) be a distinguished triangle of $K(\mathcal{A})$. If two out of K, L, M are K-injective complexes, then the third is too.*

Proof. Follows from the definition, Lemma 13.4.2, and the fact that $K(\mathcal{A})$ is a triangulated category (Proposition 13.10.3). □

070J **Lemma 13.31.4.** *Let $\mathcal{A}$ be an abelian category. A bounded below complex of injectives is K-injective.*

Proof. Follows from Lemmas 13.31.2 and 13.18.8. □

0BK6 **Lemma 13.31.5.** *Let $\mathcal{A}$ be an abelian category. Let T be a set and for each $t \in T$ let $I_t^\bullet$ be a K-injective complex. If $I^n = \prod_t I_t^n$ exists for all n, then $I^\bullet$ is a K-injective complex. Moreover, $I^\bullet$ represents the product of the objects $I_t^\bullet$ in $D(\mathcal{A})$.*

Proof. Let $K^\bullet$ be an complex. Observe that the complex
$$C : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b+1})$$
has cohomology $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$ in the middle. Similarly, the complex
$$C_t : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b+1})$$

computes $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet)$. Next, observe that we have

$$C = \prod\nolimits_{t\in T} C_t$$

as complexes of abelian groups by our choice of I. Taking products is an exact functor on the category of abelian groups. Hence if $K^\bullet$ is acyclic, then $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet) = 0$, hence C_t is acyclic, hence C is acyclic, hence we get $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) = 0$. Thus we find that $I^\bullet$ is K-injective. Having said this, we can use Lemma 13.31.2 to conclude that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \prod\nolimits_{t\in T} \mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I_t^\bullet)$$

and indeed $I^\bullet$ represents the product in the derived category. □

070Y **Lemma 13.31.6.** *Let $\mathcal{A}$ be an abelian category. Let $F : K(\mathcal{A}) \to \mathcal{D}'$ be an exact functor of triangulated categories. Then RF is defined at every complex in $K(\mathcal{A})$ which is quasi-isomorphic to a K-injective complex. In fact, every K-injective complex computes RF.*

Proof. By Lemma 13.14.4 it suffices to show that RF is defined at a K-injective complex, i.e., it suffices to show a K-injective complex $I^\bullet$ computes RF. Consider a quasi-isomorphism $I^\bullet \to N^\bullet$. By Lemma 13.31.2 it has a left inverse. Thus $I^\bullet \to I^\bullet$ is a final object of $I^\bullet/\mathrm{Qis}(\mathcal{A})$ and we win. □

070K **Lemma 13.31.7.** *Let $\mathcal{A}$ be an abelian category. Assume every complex has a quasi-isomorphism towards a K-injective complex. Then any exact functor $F : K(\mathcal{A}) \to \mathcal{D}'$ of triangulated categories has a right derived functor*

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = F(I^\bullet)$ for K-injective complexes $I^\bullet$.

Proof. To see this we apply Lemma 13.14.15 with $\mathcal{I}$ the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \to J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \to F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \to J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 13.31.2. □

The following lemma can be generalized to limits over bigger ordinals.

070L **Lemma 13.31.8.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\ldots \to I_3^\bullet \to I_2^\bullet \to I_1^\bullet$$

be an inverse system of complexes. Assume

(1) *each $I_n^\bullet$ is K-injective,*
(2) *each map $I_{n+1}^m \to I_n^m$ is a split surjection,*
(3) *the limits $I^m = \lim I_n^m$ exist.*

Then the complex $I^\bullet$ is K-injective.

Proof. We urge the reader to skip the proof of this lemma. Let $M^\bullet$ be an acyclic complex. Let us abbreviate $H_n(a,b) = \mathrm{Hom}_{\mathcal{A}}(M^a, I_n^b)$. With this notation $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m-2) \to \prod_m \lim_n H_n(m, m-1) \to \prod_m \lim_n H_n(m, m) \to \prod_m \lim_n H_n(m, m+1)$$

in the third spot from the left. We may exchange the order of $\prod$ and lim and each of the complexes

$$\prod_m H_n(m, m-2) \to \prod_m H_n(m, m-1) \to \prod_m H_n(m, m) \to \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\ldots \to \prod_m H_3(m, m-2) \to \prod_m H_2(m, m-2) \to \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 12.31.4. □

It appears that a combination of Lemmas 13.29.3, 13.31.4, and 13.31.8 produces "enough K-injectives" for any abelian category with enough injectives and countable products. Actually, this may not work! See Lemma 13.34.5 for an explanation.

08BJ **Lemma 13.31.9.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v, and*
(2) *v is exact.*

Then u transforms K-injective complexes into K-injective complexes.

Proof. Let $I^\bullet$ be a K-injective complex of $\mathcal{A}$. Let $M^\bullet$ be a acyclic complex of $\mathcal{B}$. As v is exact we see that $v(M^\bullet)$ is an acyclic complex. By adjointness we get

$$0 = \mathrm{Hom}_{K(\mathcal{A})}(v(M^\bullet), I^\bullet) = \mathrm{Hom}_{K(\mathcal{B})}(M^\bullet, u(I^\bullet))$$

hence the lemma follows. □

13.32. Bounded cohomological dimension

07K5 There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

07K6 **Lemma 13.32.1.** *Let $\mathcal{A}$ be an abelian category. Let $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots, \infty\}$ be a function. Assume that*

(1) *every object of $\mathcal{A}$ is a subobject of an object A with $d(A) = 0$,*
(2) *$d(A \oplus B) \leq \max\{d(A), d(B)\}$ for $A, B \in \mathcal{A}$, and*
(3) *if $0 \to A \to B \to C \to 0$ is short exact, then $d(C) \leq \max\{d(A) - 1, d(B)\}$.*

Let $K^\bullet$ be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$. Then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 13.15.5 we can find a quasi-isomorphism $\sigma_{\geq 0}K^\bullet \to M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then $K^\bullet$ is quasi-isomorphic to the complex

$$\ldots \to K^{-2} \to K^{-1} \to M^0 \to M^1 \to \ldots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \to -\infty$ as $n \to -\infty$ is not violated by this replacement.

We are going to improve $K^\bullet$ by an (infinite) sequence of elementary replacements. An *elementary replacement* is the following. Choose an index n such that $d(K^n) >$

0. Choose an injection $K^n \to M$ where $d(M) = 0$. Set $M' = \text{Coker}(K^n \to M \oplus K^{n+1})$. Consider the map of complexes

$$\begin{array}{ccccccccc} K^\bullet : & & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (K')^\bullet : & & K^{n-1} & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & K^{n+2} \end{array}$$

It is clear that $K^\bullet \to (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and

$$d((K')^{n+1}) \leq \max\{d(K^n) - 1, d(M \oplus K^{n+1})\} \leq \max\{d(K^n) - 1, d(K^{n+1})\}$$

and the other values are unchanged.

To finish the proof we carefully choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m} K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \text{ and } d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$ and the fact that $d(K^n) = 0$ for $n >> 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I, or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. $\square$

07K7 **Lemma 13.32.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume*

(1) *every object of $\mathcal{A}$ is a subobject of an object which is right acyclic for F,*
(2) *there exists an integer $n \geq 0$ such that $R^n F = 0$,*

Then

(1) *$RF : D(\mathcal{A}) \to D(\mathcal{B})$ exists,*
(2) *any complex consisting of right acyclic objects for F computes RF,*
(3) *any complex is the source of a quasi-isomorphism into a complex consisting of right acyclic objects for F,*
(4) *for $E \in D(\mathcal{A})$*
   (a) *$H^i(RF(\tau_{\leq a} E) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,*
   (b) *$H^i(RF(E)) \to H^i(RF(\tau_{\geq b-n+1} E))$ is an isomorphism for $i \geq b$,*
   (c) *if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + n - 1]$.*

Proof. Note that the first assumption implies that $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, see Proposition 13.16.8. Let A be an object of $\mathcal{A}$. Choose an injection $A \to A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^n F(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^m F = 0$ for all $m \geq n$.

We are going to use Lemma 13.32.1 with the function $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots\}$ given by $d(A) = \max\{0\} \cup \{i \mid R^iF(A) \neq 0\}$. The first assumption of Lemma 13.32.1 is our assumption (1). The second assumption of Lemma 13.32.1 follows from the fact that $RF(A \oplus B) = RF(A) \oplus RF(B)$. The third assumption of Lemma 13.32.1 follows from the long exact cohomology sequence. Hence for every complex $K^\bullet$ there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ into a complex of objects right acyclic for F. This proves statement (3).

We claim that if $L^\bullet \to M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F, then $F(L^\bullet) \to F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we get statements (1) and (2) of the lemma by Lemma 13.14.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1}L^\bullet \to \sigma_{\geq i-n-1}M^\bullet \to Q^\bullet,$$

i.e., let $Q^\bullet$ be the cone of the first map. Note that $Q^\bullet$ is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i - n - 1$ or $j = i - n - 2$. We may apply RF to $Q^\bullet$. Using the second spectral sequence of Lemma 13.21.3 and the assumed vanishing of cohomology (2) we conclude that $H^j(RF(Q^\bullet))$ is zero except possibly for $j \in \{i-n-2, \ldots, i-1\}$. Hence we see that $RF(\sigma_{\geq i-n-1}L^\bullet) \to RF(\sigma_{\geq i-n-1}M^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 13.16.8 we know that $RF(\sigma_{\geq i-n-1}L^\bullet) = \sigma_{\geq i-n-1}F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1}M^\bullet) = \sigma_{\geq i-n-1}F(M^\bullet)$. We conclude that $F(L^\bullet) \to F(M^\bullet)$ is an isomorphism in degree i as desired.

Part (4)(a) follows from Lemma 13.16.1.

For part (4)(b) let E be represented by the complex $L^\bullet$ of objects right acyclic for F. By part (2) $RF(E)$ is represented by the complex $F(L^\bullet)$ and $RF(\sigma_{\geq c}L^\bullet)$ is represented by $\sigma_{\geq c}F(L^\bullet)$. Consider the distinguished triangle

$$H^{b-n}(L^\bullet)[n-b] \to \tau_{\geq b-n}L^\bullet \to \tau_{\geq b-n+1}L^\bullet$$

of Remark 13.12.4. The vanishing established above gives that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\tau_{\geq b-n+1}L^\bullet))$ for $i \geq b$. Consider the short exact sequence of complexes

$$0 \to \mathrm{Im}(L^{b-n-1} \to L^{b-n})[n-b] \to \sigma_{\geq b-n}L^\bullet \to \tau_{\geq b-n}L^\bullet \to 0$$

Using the distinguished triangle associated to this (see Section 13.12) and the vanishing as before we conclude that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\sigma_{\geq b-n}L^\bullet))$ for $i \geq b$. Since the map $RF(\sigma_{\geq b-n}L^\bullet) \to RF(L^\bullet)$ is represented by $\sigma_{\geq b-n}F(L^\bullet) \to F(L^\bullet)$ we conclude that this in turn agrees with $H^i(RF(L^\bullet))$ for $i \geq b$ as desired.

Proof of (4)(c). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from part (4)(a). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + n$ from part (4)(b). □

07K8 **Lemma 13.32.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. If*

(1) *every object of $\mathcal{A}$ is a quotient of an object which is left acyclic for F,*
(2) *there exists an integer $n \geq 0$ such that $L^nF = 0$,*

Then

(1) *$LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists,*
(2) *any complex consisting of left acyclic objects for F computes LF,*

(3) *any complex is the target of a quasi-isomorphism from a complex consisting of left acyclic objects for F,*
(4) *for $E \in D(\mathcal{A})$*
   (a) *$H^i(LF(\tau_{\leq a+n-1}E) \to H^i(LF(E))$ is an isomorphism for $i \leq a$,*
   (b) *$H^i(LF(E)) \to H^i(LF(\tau_{\geq b}E))$ is an isomorphism for $i \geq b$,*
   (c) *if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(LF(E)) = 0$ for $i \notin [a - n + 1, b]$.*

Proof. This is dual to Lemma 13.32.2. □

13.33. Derived colimits

0A5K In a triangulated category there is a notion of derived colimit.

090Z **Definition 13.33.1.** Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We say an object K is a *derived colimit*, or a *homotopy colimit* of the system (K_n) if the direct sum $\bigoplus K_n$ exists and there is a distinguished triangle

$$\bigoplus K_n \to \bigoplus K_n \to K \to \bigoplus K_n[1]$$

where the map $\bigoplus K_n \to \bigoplus K_n$ is given by $1 - f_n$ in degree n. If this is the case, then we sometimes indicate this by the notation $K = \text{hocolim} K_n$.

By TR3 a derived colimit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived colimit of K_n exists as soon as $\bigoplus K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example of a triangulated category where all homotopy colimits exist.

The nonuniqueness makes it hard to pin down the derived colimit. In More on Algebra, Lemma 15.87.5 the reader finds an exact sequence

$$0 \to R^1 \lim \text{Hom}(K_n, L[-1]) \to \text{Hom}(\text{hocolim} K_n, L) \to \lim \text{Hom}(K_n, L) \to 0$$

describing the Homs out of a homotopy colimit in terms of the usual Homs.

0CRH **Remark 13.33.2.** Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We may think of a derived colimit as an object K of $\mathcal{D}$ endowed with morphisms $i_n : K_n \to K$ such that $i_{n+1} \circ f_n = i_n$ and such that there exists a morphism $c : K \to \bigoplus K_n[1]$ with the property that

$$\bigoplus K_n \xrightarrow{1-f_n} \bigoplus K_n \xrightarrow{i_n} K \xrightarrow{c} \bigoplus K_n[1]$$

is a distinguished triangle. If (K', i'_n, c') is a second derived colimit, then there exists an isomorphism $\varphi : K \to K'$ such that $\varphi \circ i_n = i'_n$ and $c' \circ \varphi = c$. The existence of φ is TR3 and the fact that φ is an isomorphism is Lemma 13.4.3.

0CRI **Remark 13.33.3.** Let $\mathcal{D}$ be a triangulated category. Let $(a_n) : (K_n, f_n) \to (L_n, g_n)$ be a morphism of systems of objects of $\mathcal{D}$. Let (K, i_n, c) be a derived colimit of the first system and let (L, j_n, d) be a derived colimit of the second system with notation as in Remark 13.33.2. Then there exists a morphism $a : K \to L$ such that $a \circ i_n = j_n$ and $d \circ a = (a_n[1]) \circ c$. This follows from TR3 applied to the defining distinguished triangles.

0CRJ **Lemma 13.33.4.** *Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. Let $n_1 < n_2 < n_3 < \ldots$ be a sequence of integers. Assume $\bigoplus K_n$ and*

$\bigoplus K_{n_i}$ *exist. Then there exists an isomorphism* $hocolim K_{n_i} \to hocolim K_n$ *such that*

$$\begin{array}{ccc} K_{n_i} & \longrightarrow & hocolim K_{n_i} \\ {\scriptstyle id}\downarrow & & \downarrow \\ K_{n_i} & \longrightarrow & hocolim K_n \end{array}$$

commutes for all i.

Proof. Let $g_i : K_{n_i} \to K_{n_{i+1}}$ be the composition $f_{n_{i+1}-1} \circ \ldots \circ f_{n_i}$. We construct commutative diagrams

$$\begin{array}{ccc} \bigoplus_i K_{n_i} & \xrightarrow{1-g_i} & \bigoplus_i K_{n_i} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \bigoplus_n K_n & \xrightarrow{1-f_n} & \bigoplus_n K_n \end{array} \quad \text{and} \quad \begin{array}{ccc} \bigoplus_n K_n & \xrightarrow{1-f_n} & \bigoplus_n K_n \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ \bigoplus_i K_{n_i} & \xrightarrow{1-g_i} & \bigoplus_i K_{n_i} \end{array}$$

as follows. Let $a_i = a|_{K_{n_i}}$ be the inclusion of K_{n_i} into the direct sum. In other words, a is the natural inclusion. Let $b_i = b|_{K_{n_i}}$ be the map

$$K_{n_i} \xrightarrow{1,\ f_{n_i},\ f_{n_i+1} \circ f_{n_i},\ \ldots,\ f_{n_{i+1}-2} \circ \ldots \circ f_{n_i}} K_{n_i} \oplus K_{n_i+1} \oplus \ldots \oplus K_{n_{i+1}-1}$$

If $n_{i-1} < j \leq n_i$, then we let $c_j = c|_{K_j}$ be the map

$$K_j \xrightarrow{f_{n_i-1} \circ \ldots \circ f_j} K_{n_i}$$

We let $d_j = d|_{K_j}$ be zero if $j \neq n_i$ for any i and we let d_{n_i} be the natural inclusion of K_{n_i} into the direct sum. In other words, d is the natural projection. By TR3 these diagrams define morphisms

$$\varphi : \text{hocolim} K_{n_i} \to \text{hocolim} K_n \quad \text{and} \quad \psi : \text{hocolim} K_n \to \text{hocolim} K_{n_i}$$

Since $c \circ a$ and $d \circ b$ are the identity maps we see that $\varphi \circ \psi$ is an isomorphism by Lemma 13.4.3. The other way around we get the morphisms $a \circ c$ and $b \circ d$. Consider the morphism $h = (h_j) : \bigoplus K_n \to \bigoplus K_n$ given by the rule: for $n_{i-1} < j < n_i$ we set

$$h_j : K_j \xrightarrow{1,\ f_j,\ f_{j+1} \circ f_j,\ \ldots,\ f_{n_i-1} \circ \ldots \circ f_j} K_j \oplus \ldots \oplus K_{n_i}$$

Then the reader verifies that $(1 - f) \circ h = \text{id} - a \circ c$ and $h \circ (1 - f) = \text{id} - b \circ d$. This means that $\text{id} - \psi \circ \varphi$ has square zero by Lemma 13.4.5 (small argument omitted). In other words, $\psi \circ \varphi$ differs from the identity by a nilpotent endomorphism, hence is an isomorphism. Thus φ and ψ are isomorphisms as desired. □

0A5L **Lemma 13.33.5.** *Let* $\mathcal{A}$ *be an abelian category. If* $\mathcal{A}$ *has exact countable direct sums, then* $D(\mathcal{A})$ *has countable direct sums. In fact given a collection of complexes* $K_i^\bullet$ *indexed by a countable index set* I *the termwise direct sum* $\bigoplus K_i^\bullet$ *is the direct sum of* $K_i^\bullet$ *in* $D(\mathcal{A})$.

Proof. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : K_i^\bullet \to L^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : M_i^\bullet \to K_i^\bullet$ of complexes and maps of complexes $f_i : M_i^\bullet \to L^\bullet$ such that $\alpha_i = f_i s_i^{-1}$. By assumption the map of complexes

$$s : \bigoplus M_i^\bullet \longrightarrow \bigoplus K_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \bigoplus f_i$ we see that $\alpha = fs^{-1}$ is a map in $D(\mathcal{A})$ whose composition with the coprojection $K_i^\bullet \to \bigoplus K_i^\bullet$ is α_i. We omit the verification that α is unique. □

093W **Lemma 13.33.6.** *Let $\mathcal{A}$ be an abelian category. Assume colimits over $\mathbf{N}$ exist and are exact. Then countable direct sums exists and are exact. Moreover, if (A_n, f_n) is a system over $\mathbf{N}$, then there is a short exact sequence*

$$0 \to \bigoplus A_n \to \bigoplus A_n \to \operatorname{colim} A_n \to 0$$

where the first map in degree n is given by $1 - f_n$.

Proof. The first statement follows from $\bigoplus A_n = \operatorname{colim}(A_1 \oplus \ldots \oplus A_n)$. For the second, note that for each n we have the short exact sequence

$$0 \to A_1 \oplus \ldots \oplus A_{n-1} \to A_1 \oplus \ldots \oplus A_n \to A_n \to 0$$

where the first map is given by the maps $1 - f_i$ and the second map is the sum of the transition maps. Take the colimit to get the sequence of the lemma. □

0949 **Lemma 13.33.7.** *Let $\mathcal{A}$ be an abelian category. Let $L_n^\bullet$ be a system of complexes of $\mathcal{A}$. Assume colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$. Then the termwise colimit $L^\bullet = \operatorname{colim} L_n^\bullet$ is a homotopy colimit of the system in $D(\mathcal{A})$.*

Proof. We have an exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

by Lemma 13.33.6. The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 13.33.5. Thus the result follows from the definition of derived colimits in Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 13.12.1). □

0CRK **Lemma 13.33.8.** *Let $\mathcal{D}$ be a triangulated category having countable direct sums. Let $\mathcal{A}$ be an abelian category with exact colimits over $\mathbf{N}$. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor commuting with countable direct sums. Then $H(\operatorname{hocolim} K_n) = \operatorname{colim} H(K_n)$ for any system of objects of $\mathcal{D}$.*

Proof. Write $K = \operatorname{hocolim} K_n$. Apply H to the defining distinguished triangle to get

$$\bigoplus H(K_n) \to \bigoplus H(K_n) \to H(K) \to \bigoplus H(K_n[1]) \to \bigoplus H(K_n[1])$$

where the first map is given by $1 - H(f_n)$ and the last map is given by $1 - H(f_n[1])$. Apply Lemma 13.33.6 to see that this proves the lemma. □

The following lemma tells us that taking maps out of a compact object (to be defined later) commutes with derived colimits.

094A **Lemma 13.33.9.** *Let $\mathcal{D}$ be a triangulated category with countable direct sums. Let $K \in \mathcal{D}$ be an object such that for every countable set of objects $E_n \in \mathcal{D}$ the canonical map*

$$\bigoplus \operatorname{Hom}_\mathcal{D}(K, E_n) \longrightarrow \operatorname{Hom}_\mathcal{D}(K, \bigoplus E_n)$$

is a bijection. Then, given any system L_n of $\mathcal{D}$ over $\mathbf{N}$ whose derived colimit $L = \operatorname{hocolim} L_n$ exists we have that

$$\operatorname{colim} \operatorname{Hom}_\mathcal{D}(K, L_n) \longrightarrow \operatorname{Hom}_\mathcal{D}(K, L)$$

is a bijection.

Proof. Consider the defining distinguished triangle

$$\bigoplus L_n \to \bigoplus L_n \to L \to \bigoplus L_n[1]$$

Apply the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(K, -)$ (see Lemma 13.4.2). By elementary considerations concerning colimits of abelian groups we get the result. □

13.34. Derived limits

08TB In a triangulated category there is a notion of derived limit.

08TC **Definition 13.34.1.** Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be an inverse system of objects of $\mathcal{D}$. We say an object K is a *derived limit*, or a *homotopy limit* of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example of a triangulated category where all derived limits exist.

The nonuniqueness makes it hard to pin down the derived limit. In More on Algebra, Lemma 15.87.4 the reader finds an exact sequence

$$0 \to R^1 \lim \mathrm{Hom}(L, K_n[-1]) \to \mathrm{Hom}(L, R\lim K_n) \to \lim \mathrm{Hom}(L, K_n) \to 0$$

describing the Homs into a derived limit in terms of the usual Homs.

07KC **Lemma 13.34.2.** *Let $\mathcal{A}$ be an abelian category with exact countable products. Then*

1. *$D(\mathcal{A})$ has countable products,*
2. *countable products $\prod K_i$ in $D(\mathcal{A})$ are obtained by taking termwise products of any complexes representing the K_i, and*
3. *$H^p(\prod K_i) = \prod H^p(K_i)$.*

Proof. Let $K_i^\bullet$ be a complex representing K_i in $D(\mathcal{A})$. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : L^\bullet \to K_i^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : K_i^\bullet \to M_i^\bullet$ of complexes and maps of complexes $f_i : L^\bullet \to M_i^\bullet$ such that $\alpha_i = s_i^{-1} f_i$. By assumption the map of complexes

$$s : \prod K_i^\bullet \longrightarrow \prod M_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \prod f_i$ we see that $\alpha = s^{-1} f$ is a map in $D(\mathcal{A})$ whose composition with the projection $\prod K_i^\bullet \to K_i^\bullet$ is α_i. We omit the verification that α is unique. □

The duals of Lemmas 13.33.6, 13.33.7, and 13.33.9 should be stated here and proved. However, we do not know any applications of these lemmas for now.

0BK7 **Lemma 13.34.3.** *Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let (K_n) be an inverse system of $D^+(\mathcal{A})$. Then $R\lim K_n$ exists.*

Proof. It suffices to show that $\prod K_n$ exists in $D(\mathcal{A})$. For every n we can represent K_n by a bounded below complex $I_n^\bullet$ of injectives (Lemma 13.18.3). Then $\prod K_n$ is represented by $\prod I_n^\bullet$, see Lemma 13.31.5. □

0H72 **Remark 13.34.4.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$. Then $\tau_{\geq -n}K^\bullet$ is an inverse system of complexes and which in particular determines an inverse system in $D(\mathcal{A})$. Let us assume that $R\lim \tau_{\geq -n}K^\bullet$ exists. Then the canonical maps $c_n : K^\bullet \to \tau_{\geq -n}K^\bullet$ are compatible with the transition maps of our inverse system. By the defining distinguished triangle of Definition 13.34.1 and Lemma 13.4.2 we conclude there exists a morphism

$$c : K^\bullet \longrightarrow R\lim \tau_{\geq -n}K^\bullet$$

in $D(\mathcal{A})$ such that the composition of c with the projection $R\lim \tau_{\geq -n}K^\bullet \to \tau_{\geq -m}K^\bullet$ is equal to c_m. Now the morphism c may not be unique, but we claim that whether or not c is an isomorphism is independent of the choice of c (and of our choice of the homotopy limit). Namely, for $i \in \mathbf{Z}$ and for $m > -i$ the composition

$$H^i(K^\bullet) \xrightarrow{H^i(c)} H^i(R\lim \tau_{\geq -n}K^\bullet) \to H^i(\tau_{\geq -m}K^\bullet) = H^i(K^\bullet)$$

is the identity. Hence $H^i(c)$ is an isomorphism if and only if the second map is an isomorphism. This is independent of c and also independent of the choice of the homotopy limit (as any two choices are isomorphic).

070M **Lemma 13.34.5.** *Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let $K^\bullet$ be a complex. Let $I_n^\bullet$ be the inverse system of bounded below complexes of injectives produced by Lemma 13.29.3. Then $I^\bullet = \lim I_n^\bullet$ exists, is K-injective, represents $R\lim \tau_{\geq -n}K^\bullet$ in $D(\mathcal{A})$, and the following are equivalent*

1. *the map $K^\bullet \to I^\bullet$ (see proof) is a quasi-isomorphism,*
2. *the map $K^\bullet \to R\lim \tau_{\geq -n}K^\bullet$ of Remark 13.34.4 is an isomorphism in $D(\mathcal{A})$.*

Proof. The statement of the lemma makes sense as $R\lim \tau_{\geq -n}K^\bullet$ exists by Lemma 13.34.3. Each complex $I_n^\bullet$ is K-injective by Lemma 13.31.4. Choose direct sum decompositions $I_{n+1}^p = C_{n+1}^p \oplus I_n^p$ for all $n \geq 1$. Set $C_1^p = I_1^p$. The complex $I^\bullet = \lim I_n^\bullet$ exists because we can take $I^p = \prod_{n \geq 1} C_n^p$. Fix $p \in \mathbf{Z}$. We claim there is a split short exact sequence

$$0 \to I^p \to \prod I_n^p \to \prod I_n^p \to 0$$

of objects of $\mathcal{A}$. Here the first map is given by the projection maps $I^p \to I_n^p$ and the second map by $(x_n) \mapsto (x_n - f_{n+1}^p(x_{n+1}))$ where $f_n^p : I_n^p \to I_{n-1}^p$ are the transition maps. The splitting comes from the map $\prod I_n^p \to \prod C_n^p = I^p$. We obtain a termwise split short exact sequence of complexes

$$0 \to I^\bullet \to \prod I_n^\bullet \to \prod I_n^\bullet \to 0$$

Hence a corresponding distinguished triangle in $K(\mathcal{A})$ and $D(\mathcal{A})$. By Lemma 13.31.5 the products are K-injective and represent the corresponding products in $D(\mathcal{A})$. It follows that $I^\bullet$ represents $R\lim I_n^\bullet$ (Definition 13.34.1). Since $R\lim I_n^\bullet \cong R\lim \tau_{\geq -n}K^\bullet$ as derived limits are defined on the level of the derived category, we see that $I^\bullet$ represents $R\lim \tau_{\geq -n}K^\bullet$. Moreover, the complex $I^\bullet$ is K-injective by Lemma 13.31.3. By the commutative diagram of Lemma 13.29.3 and since

$K^i = (\tau_{\geq -n}K^\bullet)^i$ for $n \gg 0$ we see that we get a unique map $\gamma : K^\bullet \to I^\bullet$ such that the diagrams

$$\begin{array}{ccc} K^\bullet & \longrightarrow & \tau_{\geq -n}K^\bullet \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ I^\bullet & \longrightarrow & I^\bullet_n \end{array}$$

commute. It follows that γ is a map of complexes which represents the map $c : K^\bullet \to R\lim \tau_{\geq -n}K^\bullet$ of Remark 13.34.4 in $D(\mathcal{A})$. In other words, the diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow[c]{} & R\lim \tau_{\geq -n}K^\bullet \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\cong} \\ I^\bullet & \xrightarrow{\cong} & R\lim I^\bullet_n \end{array}$$

is commutative in $D(\mathcal{A})$. The lemma follows. □

090Y **Lemma 13.34.6.** *Let $\mathcal{A}$ be an abelian category having enough injectives and exact countable products. Then for every complex there is a quasi-isomorphism to a K-injective complex.*

Proof. By Lemma 13.34.5 it suffices to show that $K \to R\lim \tau_{\geq -n}K$ is an isomorphism for all K in $D(\mathcal{A})$. Consider the defining distinguished triangle

$$R\lim \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to (R\lim \tau_{\geq -n}K)[1]$$

By Lemma 13.34.2 we have

$$H^p(\prod \tau_{\geq -n}K) = \prod\nolimits_{p \geq -n} H^p(K)$$

It follows in a straightforward manner from the long exact cohomology sequence of the displayed distinguished triangle that $H^p(R\lim \tau_{\geq -n}K) = H^p(K)$. □

13.35. Operations on full subcategories

0FX0 Let $\mathcal{T}$ be a triangulated category. We will identify full subcategories of $\mathcal{T}$ with subsets of $\mathrm{Ob}(\mathcal{T})$. Given full subcategories $\mathcal{A}, \mathcal{B}, \ldots$ we let

(1) $\mathcal{A}[a, b]$ for $-\infty \leq a \leq b \leq \infty$ be the full subcategory of $\mathcal{T}$ consisting of all objects $A[-i]$ with $i \in [a, b] \cap \mathbf{Z}$ with $A \in \mathrm{Ob}(\mathcal{A})$ (note the minus sign!),
(2) $smd(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to direct summands of objects of $\mathcal{A}$,
(3) $add(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to finite direct sums of objects of $\mathcal{A}$,
(4) $\mathcal{A} \star \mathcal{B}$ be the full subcategory of $\mathcal{T}$ consisting of all objects X of $\mathcal{T}$ which fit into a distinguished triangle $A \to X \to B$ with $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$,
(5) $\mathcal{A}^{\star n} = \mathcal{A} \star \ldots \star \mathcal{A}$ with $n \geq 1$ factors (we will see $\star$ is associative below),
(6) $smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$ with $n \geq 1$ factors.

If E is an object of $\mathcal{T}$, then we think of E sometimes also as the full subcategory of $\mathcal{T}$ whose single object is E. Then we can consider things like $add(E[-1, 2])$ and so on and so forth. We warn the reader that this notation is not universally accepted.

0FX1 **Lemma 13.35.1.** *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we have $(\mathcal{A} \star \mathcal{B}) \star \mathcal{C} = \mathcal{A} \star (\mathcal{B} \star \mathcal{C})$.*

Proof. If we have distinguished triangles $A \to X \to B$ and $X \to Y \to C$ then by Axiom TR4 we have distinguished triangles $A \to Y \to Z$ and $B \to Z \to C$. □

0FX2 **Lemma 13.35.2.** *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ we have $smd(\mathcal{A}) \star smd(\mathcal{B}) \subset smd(\mathcal{A} \star \mathcal{B})$ and $smd(smd(\mathcal{A}) \star smd(\mathcal{B})) = smd(\mathcal{A} \star \mathcal{B})$.*

Proof. Suppose we have a distinguished triangle $A_1 \to X \to B_1$ where $A_1 \oplus A_2 \in \mathrm{Ob}(\mathcal{A})$ and $B_1 \oplus B_2 \in \mathrm{Ob}(\mathcal{B})$. Then we obtain a distinguished triangle $A_1 \oplus A_2 \to A_2 \oplus X \oplus B_2 \to B_1 \oplus B_2$ which proves that X is in $smd(\mathcal{A} \star \mathcal{B})$. This proves the inclusion. The equality follows trivially from this. □

0FX3 **Lemma 13.35.3.** *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ the full subcategories $add(\mathcal{A}) \star add(\mathcal{B})$ and $smd(add(\mathcal{A}))$ are closed under direct sums.*

Proof. Namely, if $A \to X \to B$ and $A' \to X' \to B'$ are distinguished triangles and $A, A' \in add(\mathcal{A})$ and $B, B' \in add(\mathcal{B})$ then $A \oplus A' \to X \oplus X' \to B \oplus B'$ is a distinguished triangle with $A \oplus A' \in add(\mathcal{A})$ and $B \oplus B' \in add(\mathcal{B})$. The result for $smd(add(\mathcal{A}))$ is trivial. □

0FX4 **Lemma 13.35.4.** *Let $\mathcal{T}$ be a triangulated category. Given a full subcategory $\mathcal{A}$ for $n \geq 1$ the subcategory*

$$\mathcal{C}_n = smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$$

defined above is a strictly full subcategory of $\mathcal{T}$ closed under direct sums and direct summands and $\mathcal{C}_{n+m} = smd(\mathcal{C}_n \star \mathcal{C}_m)$ for all $n, m \geq 1$.

Proof. Immediate from Lemmas 13.35.1, 13.35.2, and 13.35.3. □

0FX5 **Remark 13.35.5.** Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor of triangulated categories. Given a full subcategory $\mathcal{A}$ of $\mathcal{T}$ we denote $F(\mathcal{A})$ the full subcategory of $\mathcal{T}'$ whose objects consists of all objects $F(A)$ with $A \in \mathrm{Ob}(\mathcal{A})$. We have

$$\begin{aligned} F(\mathcal{A}[a,b]) &= F(\mathcal{A})[a,b] \\ F(smd(\mathcal{A})) &\subset smd(F(\mathcal{A})), \\ F(add(\mathcal{A})) &\subset add(F(\mathcal{A})), \\ F(\mathcal{A} \star \mathcal{B}) &\subset F(\mathcal{A}) \star F(\mathcal{B}), \\ F(\mathcal{A}^{\star n}) &\subset F(\mathcal{A})^{\star n}. \end{aligned}$$

We omit the trivial verifications.

0FX6 **Remark 13.35.6.** Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots$ and $\mathcal{B}$ of $\mathcal{T}$ we have

$$\begin{aligned} \left(\bigcup \mathcal{A}_i\right)[a,b] &= \bigcup \mathcal{A}_i[a,b] \\ smd\left(\bigcup \mathcal{A}_i\right) &= \bigcup smd(\mathcal{A}_i), \\ add\left(\bigcup \mathcal{A}_i\right) &= \bigcup add(\mathcal{A}_i), \\ \left(\bigcup \mathcal{A}_i\right) \star \mathcal{B} &= \bigcup \mathcal{A}_i \star \mathcal{B}, \end{aligned}$$

$$\mathcal{B} \star \left(\bigcup \mathcal{A}_i\right) = \bigcup \mathcal{B} \star \mathcal{A}_i,$$
$$\left(\bigcup \mathcal{A}_i\right)^{\star n} = \bigcup \mathcal{A}_i^{\star n}.$$

We omit the trivial verifications.

0FX7 **Lemma 13.35.7.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D} = D(\mathcal{A})$. Let $\mathcal{E} \subset \mathrm{Ob}(\mathcal{A})$ be a subset which we view as a subset of $\mathrm{Ob}(\mathcal{D})$ also. Let K be an object of $\mathcal{D}$.*

(1) *Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a, b]$ and $H^i(K) \in \mathcal{E}$ if $i \in [a, b]$. Then K is in $smd(add(\mathcal{E}[a, b])^{\star(b-a+1)})$.*
(2) *Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a, b]$ and $H^i(K) \in smd(add(\mathcal{E}))$ if $i \in [a, b]$. Then K is in $smd(add(\mathcal{E}[a, b])^{\star(b-a+1)})$.*
(3) *Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a, b]$ and $K^i \in \mathcal{E}$ for $i \in [a, b]$. Then K is in $smd(add(\mathcal{E}[a, b])^{\star(b-a+1)})$.*
(4) *Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a, b]$ and $K^i \in smd(add(\mathcal{E}))$ for $i \in [a, b]$. Then K is in $smd(add(\mathcal{E}[a, b])^{\star(b-a+1)})$.*

Proof. We will use Lemma 13.35.4 without further mention. We will prove (2) which trivially implies (1). We use induction on $b - a$. If $b - a = 0$, then K is isomorphic to $H^i(K)[-a]$ in $\mathcal{D}$ and the result is immediate. If $b - a > 0$, then we consider the distinguished triangle

$$\tau_{\leq b-1} K^\bullet \to K^\bullet \to K^b[-b]$$

and we conclude by induction on $b - a$. We omit the proof of (3) and (4). □

0FX8 **Lemma 13.35.8.** *Let $\mathcal{T}$ be a triangulated category. Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor to an abelian category $\mathcal{A}$. Let $a \leq b$ and $\mathcal{E} \subset \mathrm{Ob}(\mathcal{T})$ be a subset such that $H^i(E) = 0$ for $E \in \mathcal{E}$ and $i \notin [a, b]$. Then for $X \in smd(add(\mathcal{E}[-m, m])^{\star n})$ we have $H^i(X) = 0$ for $i \notin [-m + na, m + nb]$.*

Proof. Omitted. Pleasant exercise in the definitions. □

13.36. Generators of triangulated categories

09SI In this section we briefly introduce a few of the different notions of a generator for a triangulated category. Our terminology is taken from [**BV03**] (except that we use "saturated" for what they call "épaisse", see Definition 13.6.1, and our definition of $add(\mathcal{A})$ is different).

Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Denote $\langle E \rangle_1$ the strictly full subcategory of $\mathcal{D}$ consisting of objects in $\mathcal{D}$ isomorphic to direct summands of finite direct sums

$$\bigoplus\nolimits_{i=1,\ldots,r} E[n_i]$$

of shifts of E. It is clear that in the notation of Section 13.35 we have

$$\langle E \rangle_1 = smd(add(E[-\infty, \infty]))$$

For $n > 1$ let $\langle E \rangle_n$ denote the full subcategory of $\mathcal{D}$ consisting of objects of $\mathcal{D}$ isomorphic to direct summands of objects X which fit into a distinguished triangle

$$A \to X \to B \to A[1]$$

where A is an object of $\langle E\rangle_1$ and B an object of $\langle E\rangle_{n-1}$. In the notation of Section 13.35 we have

$$\langle E\rangle_n = smd(\langle E\rangle_1 \star \langle E\rangle_{n-1})$$

Each of the categories $\langle E\rangle_n$ is a strictly full additive (by Lemma 13.35.3) subcategory of $\mathcal{D}$ preserved under shifts and under taking summands. But, $\langle E\rangle_n$ is not necessarily closed under "taking cones" or "extensions", hence not necessarily a triangulated subcategory. This will be true for the subcategory

$$\langle E\rangle = \bigcup\nolimits_n \langle E\rangle_n$$

as will be shown in the lemmas below.

0FX9 **Lemma 13.36.1.** *Let $\mathcal{T}$ be a triangulated category. Let E be an object of $\mathcal{T}$. For $n \geq 1$ we have*

$$\langle E\rangle_n = smd(\langle E\rangle_1 \star \ldots \star \langle E\rangle_1) = smd(\langle E\rangle_1^{\star n}) = \bigcup\nolimits_{m\geq 1} smd(add(E[-m,m])^{\star n})$$

For $n, n' \geq 1$ we have $\langle E\rangle_{n+n'} = smd(\langle E\rangle_n \star \langle E\rangle_{n'})$.

Proof. The left equality in the displayed formula follows from Lemmas 13.35.1 and 13.35.2 and induction. The middle equality is a matter of notation. Since $\langle E\rangle_1 = smd(add(E[-\infty,\infty])])$ and since $E[-\infty,\infty] = \bigcup_{m\geq 1} E[-m,m]$ we see from Remark 13.35.6 and Lemma 13.35.2 that we get the equality on the right. Then the final statement follows from the remark and the corresponding statement of Lemma 13.35.4. □

0ATG **Lemma 13.36.2.** *Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. The subcategory*

$$\langle E\rangle = \bigcup\nolimits_n \langle E\rangle_n = \bigcup\nolimits_{n,m\geq 1} smd(add(E[-m,m])^{\star n})$$

is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ and it is the smallest such subcategory of $\mathcal{D}$ containing the object E.

Proof. The equality on the right follows from Lemma 13.36.1. It is clear that $\langle E\rangle = \bigcup\langle E\rangle_n$ contains E, is preserved under shifts, direct sums, direct summands. If $A \in \langle E\rangle_a$ and $B \in \langle E\rangle_b$ and if $A \to X \to B \to A[1]$ is a distinguished triangle, then $X \in \langle E\rangle_{a+b}$ by Lemma 13.36.1. Hence $\bigcup\langle E\rangle_n$ is also preserved under extensions and it follows that it is a triangulated subcategory.

Finally, let $\mathcal{D}' \subset \mathcal{D}$ be a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E. Then $\mathcal{D}'[-\infty,\infty] \subset \mathcal{D}'$, $add(\mathcal{D}) \subset \mathcal{D}'$, $smd(\mathcal{D}') \subset \mathcal{D}'$, and $\mathcal{D}' \star \mathcal{D}' \subset \mathcal{D}'$. In other words, all the operations we used to construct $\langle E\rangle$ out of E preserve $\mathcal{D}'$. Hence $\langle E\rangle \subset \mathcal{D}'$ and this finishes the proof. □

09SJ **Definition 13.36.3.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$.

(1) We say E is a *classical generator* of $\mathcal{D}$ if the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E is equal to $\mathcal{D}$, in other words, if $\langle E\rangle = \mathcal{D}$.
(2) We say E is a *strong generator* of $\mathcal{D}$ if $\langle E\rangle_n = \mathcal{D}$ for some $n \geq 1$.
(3) We say E is a *weak generator* or a *generator* of $\mathcal{D}$ if for any nonzero object K of $\mathcal{D}$ there exists an integer n and a nonzero map $E \to K[n]$.

This definition can be generalized to the case of a family of objects.

09SK **Lemma 13.36.4.** *Let $\mathcal{D}$ be a triangulated category. Let E, K be objects of $\mathcal{D}$. The following are equivalent*

(1) $\mathrm{Hom}(E, K[i]) = 0$ *for all* $i \in \mathbf{Z}$,
(2) $\mathrm{Hom}(E', K) = 0$ *for all* $E' \in \langle E \rangle$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Conversely, assume (1). Then $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_1$. Arguing by induction on n and using Lemma 13.4.2 we see that $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_n$. □

09SL **Lemma 13.36.5.** *Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a generator.*

Proof. Assume E is a classical generator. Let K be an object of $\mathcal{D}$ such that $\mathrm{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$. By Lemma 13.36.4 $\mathrm{Hom}(E', K) = 0$ for all E' in $\langle E \rangle$. However, since $\mathcal{D} = \langle E \rangle$ we conclude that $\mathrm{id}_K = 0$, i.e., $K = 0$. □

0FXA **Lemma 13.36.6.** *Let $\mathcal{D}$ be a triangulated category which has a strong generator. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a strong generator.*

Proof. Let E' be an object of $\mathcal{D}$ such that $\mathcal{D} = \langle E' \rangle_n$. Since $\mathcal{D} = \langle E \rangle$ we see that $E' \in \langle E \rangle_m$ for some $m \geq 1$ by Lemma 13.36.2. Then $\langle E' \rangle_1 \subset \langle E \rangle_m$ hence

$$\mathcal{D} = \langle E' \rangle_n = smd(\langle E' \rangle_1 \star \ldots \star \langle E' \rangle_1) \subset smd(\langle E \rangle_m \star \ldots \star \langle E \rangle_m) = \langle E \rangle_{nm}$$

as desired. Here we used Lemma 13.36.1. □

0ATH **Remark 13.36.7.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Let T be a property of objects of $\mathcal{D}$. Suppose that

(1) if $K_i \in \mathcal{D}$, $i = 1, \ldots, r$ with $T(K_i)$ for $i = 1, \ldots, r$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
(3) if $T(K \oplus L)$ then $T(K)$ and $T(L)$, and
(4) $T(E[n])$ holds for all n.

Then T holds for all objects of $\langle E \rangle$.

13.37. Compact objects

09SM Here is the definition.

07LS **Definition 13.37.1.** Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A *compact object* of $\mathcal{D}$ is an object K such that the map

$$\bigoplus\nolimits_{i \in I} \mathrm{Hom}_{\mathcal{D}}(K, E_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(K, \bigoplus\nolimits_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in \mathrm{Ob}(\mathcal{D})$ parametrized by $i \in I$.

This notion turns out to be very useful in algebraic geometry. It is an intrinsic condition on objects that forces the objects to be, well, compact.

09QH **Lemma 13.37.2.** *Let $\mathcal{D}$ be a (pre-)triangulated category with direct sums. Then the compact objects of $\mathcal{D}$ form the objects of a Karoubian, saturated, strictly full, (pre-)triangulated subcategory $\mathcal{D}_c$ of $\mathcal{D}$.*

Proof. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$ with X and Y compact. Then it follows from Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) that Z is a compact object too. It is clear that if $X \oplus Y$ is compact, then X, Y are compact objects too. Hence $\mathcal{D}_c$ is a saturated triangulated subcategory. Since $\mathcal{D}$ is Karoubian by Lemma 13.4.14 we conclude that the same is true for $\mathcal{D}_c$. □

09SN **Lemma 13.37.3.** *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E_i, $i \in I$ be a family of compact objects of $\mathcal{D}$ such that $\bigoplus E_i$ generates $\mathcal{D}$. Then every object X of $\mathcal{D}$ can be written as*

$$X = \text{hocolim} X_n$$

where X_1 is a direct sum of shifts of the E_i and each transition morphism fits into a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ where Y_n is a direct sum of shifts of the E_i.

Proof. Set $X_1 = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$ and $\varphi : E_i[m] \to X$. Then X_1 comes equipped with a canonical morphism $X_1 \to X$. Given $X_n \to X$ we set $Y_n = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$, and $\varphi : E_i[m] \to X_n$ is a morphism such that $E_i[m] \to X_n \to X$ is zero. Choose a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ and let $X_{n+1} \to X$ be any morphism such that $X_n \to X_{n+1} \to X$ is the given one; such a morphism exists by our choice of Y_n. We obtain a morphism $\text{hocolim} X_n \to X$ by the construction of our maps $X_n \to X$. Choose a distinguished triangle

$$C \to \text{hocolim} X_n \to X \to C[1]$$

Let $E_i[m] \to C$ be a morphism. Since E_i is compact, the composition $E_i[m] \to C \to \text{hocolim} X_n$ factors through X_n for some n, say by $E_i[m] \to X_n$. Then the construction of Y_n shows that the composition $E_i[m] \to X_n \to X_{n+1}$ is zero. In other words, the composition $E_i[m] \to C \to \text{hocolim} X_n$ is zero. This means that our morphism $E_i[m] \to C$ comes from a morphism $E_i[m] \to X[-1]$. The construction of X_1 then shows that such morphism lifts to $\text{hocolim} X_n$ and we conclude that our morphism $E_i[m] \to C$ is zero. The assumption that $\bigoplus E_i$ generates $\mathcal{D}$ implies that C is zero and the proof is done. □

09SP **Lemma 13.37.4.** *With assumptions and notation as in Lemma 13.37.3. If C is a compact object and $C \to X_n$ is a morphism, then there is a factorization $C \to E \to X_n$ where E is an object of $\langle E_{i_1} \oplus \ldots \oplus E_{i_t} \rangle$ for some $i_1, \ldots, i_t \in I$.*

Proof. We prove this by induction on n. The base case $n = 1$ is clear. If $n > 1$ consider the composition $C \to X_n \to Y_{n-1}[1]$. This can be factored through some $E'[1] \to Y_{n-1}[1]$ where E' is a finite direct sum of shifts of the E_i. Let $I' \subset I$ be the finite set of indices that occur in this direct sum. Thus we obtain

$$\begin{array}{ccccccc} E' & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & E'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X_n & \longrightarrow & Y_{n-1}[1] \end{array}$$

By induction the morphism $C' \to X_{n-1}$ factors through $E'' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I''} E_i \rangle$ for some finite subset $I'' \subset I$. Choose a distinguished triangle

$$E' \to E'' \to E \to E'[1]$$

then E is an object of $\langle \bigoplus_{i \in I' \cup I''} E_i \rangle$. By construction and the axioms of a triangulated category we can choose morphisms $C \to E$ and a morphism $E \to X_n$ fitting into morphisms of triangles $(E', C', C) \to (E', E'', E)$ and $(E', E'', E) \to (Y_{n-1}, X_{n-1}, X_n)$. The composition $C \to E \to X_n$ may not equal the given morphism $C \to X_n$, but the compositions into Y_{n-1} are equal. Let $C \to X_{n-1}$ be a morphism that lifts the difference. By induction assumption we can factor this through a morphism $E''' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I'''} E_i \rangle$ for some finite subset $I' \subset I$. Thus we see that we get a solution on considering $E \oplus E''' \to X_n$ because $E \oplus E'''$ is an object of $\langle \bigoplus_{i \in I' \cup I'' \cup I'''} E_i \rangle$. □

09SQ **Definition 13.37.5.** Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. We say $\mathcal{D}$ is *compactly generated* if there exists a set E_i, $i \in I$ of compact objects such that $\bigoplus E_i$ generates $\mathcal{D}$.

The following proposition clarifies the relationship between classical generators and weak generators.

09SR **Proposition 13.37.6.** *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E be a compact object of $\mathcal{D}$. The following are equivalent*

(1) *E is a classical generator for $\mathcal{D}_c$ and $\mathcal{D}$ is compactly generated, and*
(2) *E is a generator for $\mathcal{D}$.*

Proof. If E is a classical generator for $\mathcal{D}_c$, then $\mathcal{D}_c = \langle E \rangle$. It follows formally from the assumption that $\mathcal{D}$ is compactly generated and Lemma 13.36.4 that E is a generator for $\mathcal{D}$.

The converse is more interesting. Assume that E is a generator for $\mathcal{D}$. Let X be a compact object of $\mathcal{D}$. Apply Lemma 13.37.3 with $I = \{1\}$ and $E_1 = E$ to write

$$X = \text{hocolim} X_n$$

as in the lemma. Since X is compact we find that $X \to \text{hocolim} X_n$ factors through X_n for some n (Lemma 13.33.9). Thus X is a direct summand of X_n. By Lemma 13.37.4 we see that X is an object of $\langle E \rangle$ and the lemma is proven. □

13.38. Brown representability

0A8E A reference for the material in this section is [**Nee96**].

0A8F **Lemma 13.38.1.** *Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $H : \mathcal{D} \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.* [**Nee96**, Theorem 3.1].

Proof. Let E_i, $i \in I$ be a set of compact objects such that $\bigoplus_{i \in I} E_i$ generates $\mathcal{D}$. We may and do assume that the set of objects $\{E_i\}$ is preserved under shifts. Consider pairs (i, a) where $i \in I$ and $a \in H(E_i)$ and set

$$X_1 = \bigoplus\nolimits_{(i,a)} E_i$$

Since $H(X_1) = \prod_{(i,a)} H(E_i)$ we see that $(a)_{(i,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \text{Hom}_{\mathcal{D}}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 4.3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \mathrm{Hom}_{\mathcal{D}}(-, X_n)$. Namely, we apply the procedure above to the functor $\mathrm{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(i,k),\ k \in \mathrm{Ker}(H_n(E_i) \to H(E_i))} E_i$$

and a transformation $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \to \mathrm{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

We claim that $X = \mathrm{hocolim} X_n$ represents H. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ such that

$$\mathrm{Hom}_{\mathcal{D}}(-, X_1) \to \mathrm{Hom}_{\mathcal{D}}(-, X_2) \to \mathrm{Hom}_{\mathcal{D}}(-, X_3) \to \ldots \to \mathrm{Hom}_{\mathcal{D}}(-, X) \to H$$

commutes. For each i the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \to H(E_i)$ is surjective, by construction of X_1. On the other hand, by construction of $X_n \to X_{n+1}$ the kernel of $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \to H(E_i)$ is killed by the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \to \mathrm{Hom}_{\mathcal{D}}(E_i, X_{n+1})$. Since

$$\mathrm{Hom}_{\mathcal{D}}(E_i, X) = \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(E_i, X_n)$$

by Lemma 13.33.9 we see that $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \to H(E_i)$ is injective.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 13.6.3). Moreover, as both H and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\{E_i\}$ is preserved under shifts, we see that E_i is an object of $\mathcal{D}'$ for all i. It follows from Lemma 13.37.3 that $\mathcal{D}' = \mathcal{D}$ and the proof is complete. □

0A8G **Proposition 13.38.2.** *Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.* [**Nee96**, Theorem 4.1].

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 13.38.1 we find an object X of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(W, X) = \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of

the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Lemma 13.7.1. □

13.39. Brown representability, bis

0GYF In this section we explain a version of Brown representability for triangulated categories which have a suitable set of generators; for other versions, please see [**Fra01**], [**Nee01**], and [**Kra02**].

0GYG **Lemma 13.39.1.** *Let $\mathcal{D}$ be a triangulated category with direct sums. Suppose given a set $\mathcal{E}$ of objects of $\mathcal{D}$ such that*

(1) *if X is a nonzero object of $\mathcal{D}$, then there exists an $E \in \mathcal{E}$ and a nonzero map $E \to X$, and*
(2) *given objects X_n, $n \in \mathbf{N}$ of $\mathcal{D}$, $E \in \mathcal{E}$, and $\alpha : E \to \bigoplus X_n$, there exist $E_n \in \mathcal{E}$ and $\beta_n : E_n \to X_n$ and a morphism $\gamma : E \to \bigoplus E_n$ such that $\alpha = (\bigoplus \beta_n) \circ \gamma$.*

Let $H : \mathcal{D} \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.

Weak version of [**Kra02**, Theorem A]

Proof. This proof is very similar to the proof of Lemma 13.38.1. We may replace $\mathcal{E}$ by $\bigcup_{i \in \mathbf{Z}} \mathcal{E}[i]$ and assume that $\mathcal{E}$ is preserved by shifts. Consider pairs (E, a) where $E \in \mathcal{E}$ and $a \in H(E)$ and set

$$X_1 = \bigoplus\nolimits_{(E,a)} E$$

Since $H(X_1) = \prod_{(E,a)} H(E)$ we see that $(a)_{(E,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \mathrm{Hom}_{\mathcal{D}}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 4.3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \mathrm{Hom}_{\mathcal{D}}(-, X_n)$. Namely, we apply the procedure above to the functor $\mathrm{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(E,k),\ k \in \mathrm{Ker}(H_n(E) \to H(E))} E$$

and a transformation $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \to \mathrm{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

Set $X = \mathrm{hocolim} X_n$. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ such that

$$\mathrm{Hom}_{\mathcal{D}}(-, X_1) \to \mathrm{Hom}_{\mathcal{D}}(-, X_2) \to \mathrm{Hom}_{\mathcal{D}}(-, X_3) \to \ldots \to \mathrm{Hom}_{\mathcal{D}}(-, X) \to H$$

commutes. We claim that $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H(-)$ is an isomorphism.

Let $E \in \mathcal{E}$. Let us show that

$$\mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n) \to \mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n)$$

is injective. Namely, let $\alpha : E \to \bigoplus X_n$. Then by assumption (2) we obtain a factorization $\alpha = (\bigoplus \beta_n) \circ \gamma$. Since $E_n \to X_n \to X_{n+1}$ is zero by construction, we see that the composition $\bigoplus E_n \to \bigoplus X_n \to \bigoplus X_n$ is equal to $\bigoplus \beta_n$. Hence also the composition $E \to \bigoplus X_n \to \bigoplus X_n$ is equal to α. This proves the stated injectivity and hence also

$$\mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n[1]) \to \mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n[1])$$

is injective. It follows that we have an exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n) \to \mathrm{Hom}_{\mathcal{D}}(E, \bigoplus X_n) \to \mathrm{Hom}_{\mathcal{D}}(E, X) \to 0$$

for all $E \in \mathcal{E}$.

Let $E \in \mathcal{E}$ and let $f : E \to X$ be a morphism. By the previous paragraph, we may choose $\alpha : E \to \bigoplus X_n$ lifting f. Then by assumption (2) we obtain a factorization $\alpha = (\bigoplus \beta_n) \circ \gamma$. For each n there is a morphism $\delta_n : E_n \to X_1$ such that δ_n and β_n map to the same element of $H(E_n)$. Then the compositions

$$E_n \to X_n \to X_{n+1} \quad \text{and} \quad E_n \to X_1 \to X_{n+1}$$

are equal by construction of $X_n \to X_{n+1}$. It follows that

$$\bigoplus E_n \to \bigoplus X_n \to X \quad \text{and} \quad \bigoplus E_n \to \bigoplus X_1 \to X$$

are the same too. Observing that $\bigoplus X_1 \to X$ factors as $\bigoplus X_1 \to X_1 \to X$, we conclude that

$$\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to \mathrm{Hom}_{\mathcal{D}}(E, X)$$

is surjective. Since by construction the map $\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to H(E)$ is surjective and by construction the kernel of this map is annihilated by $\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to \mathrm{Hom}_{\mathcal{D}}(E, X)$ we conclude that $\mathrm{Hom}_{\mathcal{D}}(E, X) \to H(E)$ is a bijection for all $E \in \mathcal{E}$.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 13.6.3). Moreover, as both H and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\mathcal{E}$ is preserved by shifts, we conclude that $\mathcal{E} \subset \mathrm{Ob}(\mathcal{D}')$ by the result of the previous paragraph. To finish the proof we have to show that $\mathcal{D}' = \mathcal{D}$.

Let Y be an object of $\mathcal{D}$ and set $H(-) = \mathrm{Hom}_{\mathcal{D}}(-, Y)$. Then H is a cohomological functor which transforms direct sums into products. By the construction in the first part of the proof we obtain a morphism $\mathrm{colim}\, X_n = X \to Y$ such that $\mathrm{Hom}_{\mathcal{D}}(E, X) \to \mathrm{Hom}_{\mathcal{D}}(E, Y)$ is bijective for all $E \in \mathcal{E}$. Then assumption (1) tells us that $X \to Y$ is an isomorphism! On the other hand, by construction $X_1, X_2, \ldots$ are in $\mathcal{D}'$ and so is X. Thus $Y \in \mathcal{D}'$ and the proof is complete. $\square$

0GYH **Proposition 13.39.2.** *Let $\mathcal{D}$ be a triangulated category with direct sums. Assume there exists a set $\mathcal{E}$ of objects of $\mathcal{D}$ satisfying conditions (1) and (2) of Lemma 13.39.1. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.*

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 13.39.1 we find an object X of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(W, X) = \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Lemma 13.7.1. □

13.40. Admissible subcategories

0CQP A reference for this section is [**BK89**, Section 1].

0FXB **Definition 13.40.1.** Let $\mathcal{D}$ be an additive category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory. The *right orthogonal* $\mathcal{A}^\perp$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\mathrm{Hom}(A, X) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$. The *left orthogonal* ${}^\perp\mathcal{A}$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\mathrm{Hom}(X, A) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$.

0CQQ **Lemma 13.40.2.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle*

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) *Z is in $\mathcal{A}^\perp$, and*
(2) *$\mathrm{Hom}(A, X) = \mathrm{Hom}(A, Y)$ for all $A \in \mathrm{Ob}(\mathcal{A})$.*

Proof. By Lemma 13.4.2 the functor $\mathrm{Hom}(A, -)$ is homological and hence we get a long exact sequence as in (13.3.5.1). Assume (1) and let $A \in \mathrm{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\mathrm{Hom}(A[1], Z) \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z)$$

Since $A[1] \in \mathrm{Ob}(\mathcal{A})$ we see that the first and last groups are zero. Thus we get (2). Assume (2) and let $A \in \mathrm{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z) \to \mathrm{Hom}(A[-1], X) \to \mathrm{Hom}(A[-1], Y)$$

and we conclude that $\mathrm{Hom}(A, Z) = 0$ as desired. □

0H0M **Lemma 13.40.3.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle*

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) *X is in ${}^\perp\mathcal{B}$, and*
(2) *$\mathrm{Hom}(Y, B) = \mathrm{Hom}(Z, B)$ for all $B \in \mathrm{Ob}(\mathcal{B})$.*

Proof. Dual to Lemma 13.40.2. □

0FXC **Lemma 13.40.4.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Then both the right orthogonal $\mathcal{A}^\perp$ and the left orthogonal ${}^\perp\mathcal{A}$ of $\mathcal{A}$ are strictly full, saturated[9], triangulated subcagories of $\mathcal{D}$.*

Proof. It is immediate from the definitions that the orthogonals are preserved under taking shifts, direct sums, and direct summands. Consider a distinguished triangle

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. By Lemma 13.4.16 it suffices to show that if X and Y are in $\mathcal{A}^\perp$, then Z is in $\mathcal{A}^\perp$. This is immediate from Lemma 13.40.2. □

0CQR **Lemma 13.40.5.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with A in $\mathcal{A}$ and B in $\mathcal{A}^\perp$.*

(1) *If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.*
(2) *If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.*

Proof. Let $X_1 \to X_2 \to X_3 \to X_1[1]$ be a distinguished triangle and assume P holds for X_1 and X_2. Choose distinguished triangles

$$A_1 \to X_1 \to B_1 \to A_1[1] \quad \text{and} \quad A_2 \to X_2 \to B_2 \to A_2[1]$$

as in condition P. Since $\mathrm{Hom}(A_1, A_2) = \mathrm{Hom}(A_1, X_2)$ by Lemma 13.40.2 there is a unique morphism $A_1 \to A_2$ such that the diagram

$$\begin{array}{ccc} A_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & X_2 \end{array}$$

commutes. Choose an extension of this to a diagram

$$\begin{array}{ccccccc} A_1 & \longrightarrow & X_1 & \longrightarrow & Q_1 & \longrightarrow & A_1[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_2 & \longrightarrow & X_2 & \longrightarrow & Q_2 & \longrightarrow & A_2[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_3 & \longrightarrow & X_3 & \longrightarrow & Q_3 & \longrightarrow & A_3[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_1[1] & \longrightarrow & X_1[1] & \longrightarrow & Q_1[1] & \longrightarrow & A_1[2] \end{array}$$

as in Proposition 13.4.23. By TR3 we see that $Q_1 \cong B_1$ and $Q_2 \cong B_2$ and hence $Q_1, Q_2 \in \mathrm{Ob}(\mathcal{A}^\perp)$. As $Q_1 \to Q_2 \to Q_3 \to Q_1[1]$ is a distinguished triangle we see that $Q_3 \in \mathrm{Ob}(\mathcal{A}^\perp)$ by Lemma 13.40.4. Since $\mathcal{A}$ is a full triangulated subcategory, we see that A_3 is isomorphic to an object of $\mathcal{A}$. Thus X_3 satisfies P. The other cases of (1) follow from this case by translation. Part (2) is a special case of (1) via Lemma 13.4.11. □

[9] Definition 13.6.1.

0H0N **Lemma 13.40.6.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with B in $\mathcal{B}$ and A in ${}^\perp\mathcal{B}$.*

(1) *If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.*
(2) *If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.*

Proof. Dual to Lemma 13.40.5. □

0CQS **Lemma 13.40.7.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent*

(1) *the inclusion functor $\mathcal{A} \to \mathcal{D}$ has a right adjoint, and*
(2) *for every X in $\mathcal{D}$ there exists a distinguished triangle*

$$A \to X \to B \to A[1]$$

in $\mathcal{D}$ with $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{A}^\perp)$.

If this holds, then $\mathcal{A}$ is saturated (Definition 13.6.1) and if $\mathcal{A}$ is strictly full in $\mathcal{D}$, then $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$.

Proof. Assume (1) and denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint. Let $X \in \mathrm{Ob}(\mathcal{D})$. Set $A = v(X)$. We may extend the adjunction mapping $A \to X$ to a distinguished triangle $A \to X \to B \to A[1]$. Since

$$\mathrm{Hom}_{\mathcal{A}}(A', A) = \mathrm{Hom}_{\mathcal{A}}(A', v(X)) = \mathrm{Hom}_{\mathcal{D}}(A', X)$$

for $A' \in \mathrm{Ob}(\mathcal{A})$, we conclude that $B \in \mathrm{Ob}(\mathcal{A}^\perp)$ by Lemma 13.40.2.

Assume (2). We will construct the adjoint v explicitly. Let $X \in \mathrm{Ob}(\mathcal{D})$. Choose $A \to X \to B \to A[1]$ as in (2). Set $v(X) = A$. Let $f : X \to Y$ be a morphism in $\mathcal{D}$. Choose $A' \to Y \to B' \to A'[1]$ as in (2). Since $\mathrm{Hom}(A, A') = \mathrm{Hom}(A, Y)$ by Lemma 13.40.2 there is a unique morphism $f' : A \to A'$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \longrightarrow & Y \end{array}$$

commutes. Hence we can set $v(f) = f'$ to get a functor. To see that v is adjoint to the inclusion morphism use Lemma 13.40.2 again.

Proof of the final statement. In order to prove that $\mathcal{A}$ is saturated we may replace $\mathcal{A}$ by the strictly full subcategory having the same isomorphism classes as $\mathcal{A}$; details omitted. Assume $\mathcal{A}$ is strictly full. If we show that $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$, then $\mathcal{A}$ will be saturated by Lemma 13.40.4. Since the incusion $\mathcal{A} \subset {}^\perp(\mathcal{A}^\perp)$ is clear it suffices to prove the other inclusion. Let X be an object of ${}^\perp(\mathcal{A}^\perp)$. Choose a distinguished triangle $A \to X \to B \to A[1]$ as in (2). As $\mathrm{Hom}(X, B) = 0$ by assumption we see that $A \cong X \oplus B[-1]$ by Lemma 13.4.11. Since $\mathrm{Hom}(A, B[-1]) = 0$ as $B \in \mathcal{A}^\perp$ this implies $B[-1] = 0$ and $A \cong X$ as desired. □

0CQT **Lemma 13.40.8.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent*

(1) *the inclusion functor $\mathcal{B} \to \mathcal{D}$ has a left adjoint, and*

(2) *for every X in $\mathcal{D}$ there exists a distinguished triangle*

$$A \to X \to B \to A[1]$$

in $\mathcal{D}$ with $B \in \mathrm{Ob}(\mathcal{B})$ and $A \in \mathrm{Ob}({}^\perp\mathcal{B})$.

If this holds, then $\mathcal{B}$ is saturated (Definition 13.6.1) and if $\mathcal{B}$ is strictly full in $\mathcal{D}$, then $\mathcal{B} = ({}^\perp\mathcal{B})^\perp$.

Proof. Dual to Lemma 13.40.7. □

0FXD **Definition 13.40.9.** Let $\mathcal{D}$ be a triangulated category. A *right admissible* subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 13.40.7. A *left admissible* subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 13.40.8. A *two-sided admissible* subcategory is one which is both right and left admissible.

Let $\mathcal{A}$ be a right admissible subcategory of the triangulated category $\mathcal{D}$. Then we observe that for $X \in \mathcal{D}$ the distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathcal{A}$ and $B \in \mathcal{A}^\perp$ is canonical in the following sense: for any other distinguished triangle $A' \to X \to B' \to A'[1]$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{A}^\perp$ there is an isomorphism $(\alpha, \mathrm{id}_X, \beta) : (A, X, B) \to (A', X, B')$ of triangles. The following proposition summarizes what was said above.

0H0P **Proposition 13.40.10.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$ be subcategories. The following are equivalent*

(1) *$\mathcal{A}$ is right admissible and $\mathcal{B} = \mathcal{A}^\perp$,*
(2) *$\mathcal{B}$ is left admissible and $\mathcal{A} = {}^\perp\mathcal{B}$,*
(3) *$\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and for every X in $\mathcal{D}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

If this is true, then $\mathcal{A} \to \mathcal{D}/\mathcal{B}$ and $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ are equivalences of triangulated categories, the right adjoint to the inclusion functor $\mathcal{A} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{B} \to \mathcal{A}$, and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$.

Proof. The equivalence between (1), (2), and (3) follows in a straightforward manner from Lemmas 13.40.7 and 13.40.8 (small detail omitted). Denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint of the inclusion functor $i : \mathcal{A} \to \mathcal{D}$. It is immediate that $\mathrm{Ker}(v) = \mathcal{A}^\perp = \mathcal{B}$. Thus v factors over a functor $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ by the universal property of the quotient. Since $v \circ i = \mathrm{id}_\mathcal{A}$ by Categories, Lemma 4.24.4 we see that $\overline{v}$ is a left quasi-inverse to $\overline{i} : \mathcal{A} \to \mathcal{D}/\mathcal{B}$. We claim also the composition $\overline{i} \circ \overline{v}$ is isomorphic to $\mathrm{id}_{\mathcal{D}/\mathcal{B}}$. Namely, suppose we have X fitting into a distinguished triangle $A \to X \to B \to A[1]$ as in (3). Then $v(X) = A$ as was seen in the proof of Lemma 13.40.7. Viewing X as an object of $\mathcal{D}/\mathcal{B}$ we have $\overline{i}(\overline{v}(X)) = A$ and there is a functorial isomorphism $\overline{i}(\overline{v}(X)) = A \to X$ in $\mathcal{D}/\mathcal{B}$. Thus we find that indeed $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ is an equivalence. To show that $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ is an equivalence and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$ is dual to what we just said. □

13.41. Postnikov systems

0D7Y A reference for this section is [**Orl97**]. Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. In this section we consider the problem of constructing a "totalization" of this complex.

0D7Z **Definition 13.41.1.** Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. A *Postnikov system* is defined inductively as follows.

(1) If $n = 0$, then it is an isomorphism $Y_0 \to X_0$.
(2) If $n = 1$, then it is a choice of an isomorphism $Y_0 \to X_0$ and a choice of a distinguished triangle
$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$
where $X_1 \to Y_0$ composed with $Y_0 \to X_0$ is the given morphism $X_1 \to X_0$.
(3) If $n > 1$, then it is a choice of a Postnikov system for $X_{n-1} \to \ldots \to X_0$ and a choice of a distinguished triangle
$$Y_n \to X_n \to Y_{n-1} \to Y_n[1]$$
where the morphism $X_n \to Y_{n-1}$ composed with $Y_{n-1} \to X_{n-1}$ is the given morphism $X_n \to X_{n-1}$.

Given a morphism

0D80 (13.41.1.1)
$$\begin{array}{ccccccc} X_n & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_0 \\ \downarrow & & \downarrow & & & & \downarrow \\ X'_n & \longrightarrow & X'_{n-1} & \longrightarrow & \ldots & \longrightarrow & X'_0 \end{array}$$

between complexes of the same length in $\mathcal{D}$ there is an obvious notion of a *morphism of Postnikov systems*.

Here is a key example.

0D8Z **Example 13.41.2.** Let $\mathcal{A}$ be an abelian category. Let $\ldots \to A_2 \to A_1 \to A_0$ be a chain complex in $\mathcal{A}$. Then we can consider the objects

$$X_n = A_n \quad \text{and} \quad Y_n = (A_n \to A_{n-1} \to \ldots \to A_0)[-n]$$

of $D(\mathcal{A})$. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system as in Definition 13.41.1 for $\ldots \to X_2 \to X_1 \to X_0$. Here we are using the obvious extension of Postnikov systems for an infinite complex of $D(\mathcal{A})$. Finally, if colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$ then

$$\text{hocolim} Y_n[n] = (\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots)$$

in $D(\mathcal{A})$. This follows immediately from Lemma 13.33.7.

Given a complex $X_n \to X_{n-1} \to \ldots \to X_0$ and a Postnikov system as in Definition 13.41.1 we can consider the maps

$$Y_0 \to Y_1[1] \to \ldots \to Y_n[n]$$

These maps fit together in certain distinguished triangles and fit with the given maps between the X_i. Here is a picture for $n = 3$:

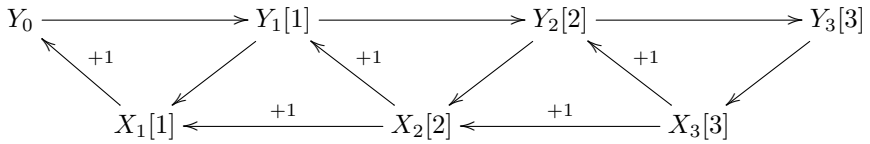

We encourage the reader to think of $Y_n[n]$ as obtained from $X_0, X_1[1], \ldots, X_n[n]$; for example if the maps $X_i \to X_{i-1}$ are zero, then we can take $Y_n[n] = \bigoplus_{i=0,\ldots,n} X_i[i]$. Postnikov systems do not always exist. Here is a simple lemma for low n.

0D81 **Lemma 13.41.3.** *Let $\mathcal{D}$ be a triangulated category. Consider Postnikov systems for complexes of length n.*

(1) *For $n = 0$ Postnikov systems always exist and any morphism (13.41.1.1) of complexes extends to a unique morphism of Postnikov systems.*
(2) *For $n = 1$ Postnikov systems always exist and any morphism (13.41.1.1) of complexes extends to a (nonunique) morphism of Postnikov systems.*
(3) *For $n = 2$ Postnikov systems always exist but morphisms (13.41.1.1) of complexes in general do not extend to morphisms of Postnikov systems.*
(4) *For $n > 2$ Postnikov systems do not always exist.*

Proof. The case $n = 0$ is immediate as isomorphisms are invertible. The case $n = 1$ follows immediately from TR1 (existence of triangles) and TR3 (extending morphisms to triangles). For the case $n = 2$ we argue as follows. Set $Y_0 = X_0$. By the case $n = 1$ we can choose a Postnikov system

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

Since the composition $X_2 \to X_1 \to X_0$ is zero, we can factor $X_2 \to X_1$ (nonuniquely) as $X_2 \to Y_1 \to X_1$ by Lemma 13.4.2. Then we simply fit the morphism $X_2 \to Y_1$ into a distinguished triangle

$$Y_2 \to X_2 \to Y_1 \to Y_2[1]$$

to get the Postnikov system for $n = 2$. For $n > 2$ we cannot argue similarly, as we do not know whether the composition $X_n \to X_{n-1} \to Y_{n-1}$ is zero in $\mathcal{D}$. □

0D82 **Lemma 13.41.4.** *Let $\mathcal{D}$ be a triangulated category. Given a map (13.41.1.1) consider the condition*

0DW1 (13.41.4.1) $$\mathrm{Hom}(X_i[i-j-1], X'_j) = 0 \text{ for } i > j+1$$

Then

(1) *If we have a Postnikov system for $X'_n \to X'_{n-1} \to \ldots \to X'_0$ then property (13.41.4.1) implies that*
$$\mathrm{Hom}(X_i[i-j-1], Y'_j) = 0 \text{ for } i > j+1$$
(2) *If we are given Postnikov systems for both complexes and we have (13.41.4.1), then the map extends to a (nonunique) map of Postnikov systems.*

Proof. We first prove (1) by induction on j. For the base case $j = 0$ there is nothing to prove as $Y'_0 \to X'_0$ is an isomorphism. Say the result holds for $j - 1$. We consider the distinguished triangle

$$Y'_j \to X'_j \to Y'_{j-1} \to Y'_j[1]$$

The long exact sequence of Lemma 13.4.2 gives an exact sequence

$$\mathrm{Hom}(X_i[i-j-1], Y'_{j-1}[-1]) \to \mathrm{Hom}(X_i[i-j-1], Y'_j) \to \mathrm{Hom}(X_i[i-j-1], X'_j)$$

From the induction hypothesis and (13.41.4.1) we conclude the outer groups are zero and we win.

Proof of (2). For $n = 1$ the existence of morphisms has been established in Lemma 13.41.3. For $n > 1$ by induction, we may assume given the map of Postnikov systems of length $n-1$. The problem is that we do not know whether the diagram

$$\begin{array}{ccc} X_n & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ X'_n & \longrightarrow & Y'_{n-1} \end{array}$$

is commutative. Denote $\alpha : X_n \to Y'_{n-1}$ the difference. Then we do know that the composition of α with $Y'_{n-1} \to X'_{n-1}$ is zero (because of what it means to be a map of Postnikov systems of length $n-1$). By the distinguished triangle $Y'_{n-1} \to X'_{n-1} \to Y'_{n-2} \to Y'_{n-1}[1]$, this means that α is the composition of $Y'_{n-2}[-1] \to Y'_{n-1}$ with a map $\alpha' : X_n \to Y'_{n-2}[-1]$. Then (13.41.4.1) guarantees α' is zero by part (1) of the lemma. Thus α is zero. To finish the proof of existence, the commutativity guarantees we can choose the dotted arrow fitting into the diagram

$$\begin{array}{ccccccc} Y_{n-1}[-1] & \longrightarrow & Y_n & \longrightarrow & X_n & \longrightarrow & Y_{n-1} \\ \downarrow & & \vdots & & \downarrow & & \downarrow \\ Y'_{n-1}[-1] & \longrightarrow & Y'_n & \longrightarrow & X'_n & \longrightarrow & Y'_{n-1} \end{array}$$

by TR3. □

0FXE **Lemma 13.41.5.** *Let $\mathcal{D}$ be a triangulated category. Given a map (13.41.1.1) assume we are given Postnikov systems for both complexes. If*

(1) $\mathrm{Hom}(X_i[i], Y'_n[n]) = 0$ *for* $i = 1, \ldots, n$, *or*
(2) $\mathrm{Hom}(Y_n[n], X'_{n-i}[n-i]) = 0$ *for* $i = 1, \ldots, n$, *or*
(3) $\mathrm{Hom}(X_{j-i}[-i+1], X'_j) = 0$ *and* $\mathrm{Hom}(X_j, X'_{j-i}[-i]) = 0$ *for* $j \geq i > 0$,

then there exists at most one morphism between these Postnikov systems.

Proof. Proof of (1). Look at the following diagram

$$\begin{array}{ccccccccc} Y_0 & \longrightarrow & Y_1[1] & \longrightarrow & Y_2[2] & \longrightarrow & \ldots & \longrightarrow & Y_n[n] \\ \downarrow & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ Y'_n[n] & & & & & & & & \end{array}$$

The arrows are the composition of the morphism $Y_n[n] \to Y'_n[n]$ and the morphism $Y_i[i] \to Y_n[n]$. The arrow $Y_0 \to Y'_n[n]$ is determined as it is the composition $Y_0 = X_0 \to X'_0 = Y'_0 \to Y'_n[n]$. Since we have the distinguished triangle $Y_0 \to Y_1[1] \to X_1[1]$ we see that $\mathrm{Hom}(X_1[1], Y'_n[n]) = 0$ guarantees that the second vertical arrow is unique. Since we have the distinguished triangle $Y_1[1] \to Y_2[2] \to X_2[2]$ we see that $\mathrm{Hom}(X_2[2], Y'_n[n]) = 0$ guarantees that the third vertical arrow is unique. And so on.

Proof of (2). The composition $Y_n[n] \to Y'_n[n] \to X_n[n]$ is the same as the composition $Y_n[n] \to X_n[n] \to X'_n[n]$ and hence is unique. Then using the distinguished triangle $Y'_{n-1}[n-1] \to Y'_n[n] \to X'_n[n]$ we see that it suffices to show $\mathrm{Hom}(Y_n[n], Y'_{n-1}[n-1]) = 0$. Using the distinguished triangles

$$Y'_{n-i-1}[n-i-1] \to Y'_{n-i}[n-i] \to X'_{n-i}[n-i]$$

we get this vanishing from our assumption. Small details omitted.

Proof of (3). Looking at the proof of Lemma 13.41.4 and arguing by induction on n it suffices to show that the dotted arrow in the morphism of triangles

$$\begin{array}{ccccccc} Y_{n-1}[-1] & \longrightarrow & Y_n & \longrightarrow & X_n & \longrightarrow & Y_{n-1} \\ \downarrow & & \vdots & & \downarrow & & \downarrow \\ Y'_{n-1}[-1] & \longrightarrow & Y'_n & \longrightarrow & X'_n & \longrightarrow & Y'_{n-1} \end{array}$$

is unique. By Lemma 13.4.8 part (5) it suffices to show that $\mathrm{Hom}(Y_{n-1}, X'_n) = 0$ and $\mathrm{Hom}(X_n, Y'_{n-1}[-1]) = 0$. To prove the first vanishing we use the distinguished triangles $Y_{n-i-1}[-i] \to Y_{n-i}[-(i-1)] \to X_{n-i}[-(i-1)]$ for $i > 0$ and induction on i to see that the assumed vanishing of $\mathrm{Hom}(X_{n-i}[-i+1], X'_n)$ is enough. For the second we similarly use the distinguished triangles $Y'_{n-i-1}[-i-1] \to Y'_{n-i}[-i] \to X'_{n-i}[-i]$ to see that the assumed vanishing of $\mathrm{Hom}(X_n, X'_{n-i}[-i])$ is enough as well. □

0D83 **Lemma 13.41.6.** *Let $\mathcal{D}$ be a triangulated category. Let $X_n \to X_{n-1} \to \ldots \to X_0$ be a complex in $\mathcal{D}$. If*

$$\mathrm{Hom}(X_i[i-j-2], X_j) = 0 \text{ for } i > j+2$$

then there exists a Postnikov system. If we have

$$\mathrm{Hom}(X_i[i-j-1], X_j) = 0 \text{ for } i > j+1$$

then any two Postnikov systems are isomorphic.

Proof. We argue by induction on n. The cases $n = 0, 1, 2$ follow from Lemma 13.41.3. Assume $n > 2$. Suppose given a Postnikov system for the complex $X_{n-1} \to X_{n-2} \to \ldots \to X_0$. The only obstruction to extending this to a Postnikov system of length n is that we have to find a morphism $X_n \to Y_{n-1}$ such that the composition $X_n \to Y_{n-1} \to X_{n-1}$ is equal to the given map $X_n \to X_{n-1}$. Considering the distinguished triangle

$$Y_{n-1} \to X_{n-1} \to Y_{n-2} \to Y_{n-1}[1]$$

and the associated long exact sequence coming from this and the functor $\mathrm{Hom}(X_n, -)$ (see Lemma 13.4.2) we find that it suffices to show that the composition $X_n \to X_{n-1} \to Y_{n-2}$ is zero. Since we know that $X_n \to X_{n-1} \to X_{n-2}$ is zero we can apply the distinguished triangle

$$Y_{n-2} \to X_{n-2} \to Y_{n-3} \to Y_{n-2}[1]$$

to see that it suffices if $\mathrm{Hom}(X_n, Y_{n-3}[-1]) = 0$. Arguing exactly as in the proof of Lemma 13.41.4 part (1) the reader easily sees this follows from the condition stated in the lemma.

The statement on isomorphisms follows from the existence of a map between the Postnikov systems extending the identity on the complex proven in Lemma 13.41.4 part (2) and Lemma 13.4.3 to show all the maps are isomorphisms. □

13.42. Essentially constant systems

0G38 Some preliminary lemmas on essentially constant systems in triangulated categories.

0G39 **Lemma 13.42.1.** *Let $\mathcal{D}$ be a triangulated category. Let (A_i) be an inverse system in $\mathcal{D}$. Then (A_i) is essentially constant (see Categories, Definition 4.22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions and identity on A, (b) for all $j \geq i$ there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant with value A. Then $A = \lim A_i$ and there exists an i and a morphism $A_i \to A$ such that (1) the composition $A \to A_i \to A$ is the identity on A and (2) for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$. From (1) we conclude that for $j \geq i$ the maps $A \to A_j$ and $A_j \to A_i \to A$ compose to the identity on A. It follows that $A_j \to A$ has a kernel Z_j and that the map $A \oplus Z_j \to A_j$ is an isomorphism, see Lemmas 13.4.12 and 13.4.11. These direct sum decompositions clearly satisfy (a). From (2) we conclude that for all j there is a $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero, so (b) holds. Proof of the converse is omitted. □

0G3A **Lemma 13.42.2.** *Let $\mathcal{D}$ be a triangulated category. Let*

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles in $\mathcal{D}$. If (A_n) and (C_n) are essentially constant, then (B_n) is essentially constant and their values fit into a distinguished triangle $A \to B \to C \to A[1]$ such that for some $n \geq 1$ there is a map

$$\begin{array}{ccccccc} A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & A_n[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

of distinguished triangles which induces an isomorphism $\lim_{n' \geq n} A_{n'} \to A$ and similarly for B and C.

Proof. After renumbering we may assume that $A_n = A \oplus A'_n$ and $C_n = C \oplus C'_n$ for inverse systems (A'_n) and (C'_n) which are essentially zero, see Lemma 13.42.1. In particular, the morphism

$$C \oplus C'_n \to (A \oplus A'_n)[1]$$

maps the summand C into the summand $A[1]$ for all n by a map $\delta : C \to A[1]$ which is independent of n. Choose a distinguished triangle

$$A \to B \to C \xrightarrow{\delta} A[1]$$

Next, choose a morphism of distinguished triangles

$$(A_1 \to B_1 \to C_1 \to A_1[1]) \to (A \to B \to C \to A[1])$$

which is possible by TR3. For any object D of $\mathcal{D}$ this induces a commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(C, D) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(B, D) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A, D) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(C_n, D) & \longrightarrow & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(B_n, D) & \longrightarrow & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(A_n, D) & \longrightarrow & \ldots \end{array}$$

The left and right vertical arrows are isomorphisms and so are the ones to the left and right of those. Thus by the 5-lemma we conclude that the middle arrow is an isomorphism. It follows that (B_n) is isomorphic to the constant inverse system with value B by the discussion in Categories, Remark 4.22.7. Since this is equivalent to (B_n) being essentially constant with value B by Categories, Remark 4.22.5 the proof is complete. □

0G3B **Lemma 13.42.3.** *Let $\mathcal{A}$ be an abelian category. Let A_n be an inverse system of objects of $D(\mathcal{A})$. Assume*

(1) *there exist integers $a \leq b$ such that $H^i(A_n) = 0$ for $i \notin [a, b]$, and*
(2) *the inverse systems $H^i(A_n)$ of $\mathcal{A}$ are essentially constant for all $i \in \mathbf{Z}$.*

Then A_n is an essentially constant system of $D(\mathcal{A})$ whose value A satisfies that $H^i(A)$ is the value of the constant system $H^i(A_n)$ for each $i \in \mathbf{Z}$.

Proof. By Remark 13.12.4 we obtain an inverse system of distinguished triangles

$$\tau_{\leq a} A_n \to A_n \to \tau_{\geq a+1} A_n \to (\tau_{\leq a} A_n)[1]$$

Of course we have $\tau_{\leq a} A_n = H^a(A_n)[-a]$ in $D(\mathcal{A})$. Thus by assumption these form an essentially constant system. By induction on $b - a$ we find that the inverse system $\tau_{\geq a+1} A_n$ is essentially constant, say with value A'. By Lemma 13.42.2 we find that A_n is an essentially constant system. We omit the proof of the statement on cohomologies (hint: use the final part of Lemma 13.42.2). □

0G3C **Lemma 13.42.4.** *Let $\mathcal{D}$ be a triangulated category. Let*

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles. If the system C_n is pro-zero (essentially constant with value 0), then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. For any object X of $\mathcal{D}$ consider the exact sequence

$$\mathrm{colim}\, \mathrm{Hom}(C_n, X) \to \mathrm{colim}\, \mathrm{Hom}(B_n, X) \to \mathrm{colim}\, \mathrm{Hom}(A_n, X) \to \mathrm{colim}\, \mathrm{Hom}(C_n[-1], X) \to$$

Exactness follows from Lemma 13.4.2 combined with Algebra, Lemma 10.8.8. By assumption the first and last term are zero. Hence the map $\mathrm{colim}\, \mathrm{Hom}(B_n, X) \to \mathrm{colim}\, \mathrm{Hom}(A_n, X)$ is an isomorphism for all X. The lemma follows from this and Categories, Remark 4.22.7. □

0G3D **Lemma 13.42.5.** *Let $\mathcal{A}$ be an abelian category.*

$$A_n \to B_n$$

be an inverse system of maps of $D(\mathcal{A})$. Assume

(1) *there exist integers $a \leq b$ such that $H^i(A_n) = 0$ and $H^i(B_n) = 0$ for $i \notin [a, b]$, and*

(2) *the inverse system of maps $H^i(A_n) \to H^i(B_n)$ of $\mathcal{A}$ define an isomorphism of pro-objects of $\mathcal{A}$ for all $i \in \mathbf{Z}$.*

Then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. We can inductively extend the maps $A_n \to B_n$ to an inverse system of distinguished triangles $A_n \to B_n \to C_n \to A_n[1]$ by axiom TR3. By Lemma 13.42.4 it suffices to prove that C_n is pro-zero. By Lemma 13.42.3 it suffices to show that $H^p(C_n)$ is pro-zero for each p. This follows from assumption (2) and the long exact sequences

$$H^p(A_n) \xrightarrow{\alpha_n} H^p(B_n) \xrightarrow{\beta_n} H^p(C_n) \xrightarrow{\delta_n} H^{p+1}(A_n) \xrightarrow{\epsilon_n} H^{p+1}(B_n)$$

Namely, for every n we can find an $m > n$ such that $\mathrm{Im}(\beta_m)$ maps to zero in $H^p(C_n)$ because we may choose m such that $H^p(B_m) \to H^p(B_n)$ factors through $\alpha_n : H^p(A_n) \to H^p(B_n)$. For a similar reason we may then choose $k > m$ such that $\mathrm{Im}(\delta_k)$ maps to zero in $H^{p+1}(A_m)$. Then $H^p(C_k) \to H^p(C_n)$ is zero because $H^p(C_k) \to H^p(C_m)$ maps into $\mathrm{Ker}(\delta_m)$ and $H^p(C_m) \to H^p(C_n)$ annihilates $\mathrm{Ker}(\delta_m) = \mathrm{Im}(\beta_m)$. □

13.43. Other chapters

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CHAPTER 14

Simplicial Methods

0162

14.1. Introduction

0163 This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [**GJ99**]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [**Qui67**] or the paper on Étale Homotopy by Artin and Mazur, see [**AM69**].

14.2. The category of finite ordered sets

0164 The category Δ is the category with

(1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
(2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$ between the corresponding sets.

Here *nondecreasing* for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the "big" category of nonempty finite totally ordered sets and nondecreasing maps. There are exactly $n + 1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n+1)(n+2)/2$ morphisms $[1] \to [n]$ and there are exactly $n+2$ morphisms $[n] \to [1]$. And so on and so forth.

0165 **Definition 14.2.1.** For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta^n_j : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma^n_j : [n+1] \to [n]$ the surjective order preserving map with $(\sigma^n_j)^{-1}(\{j\}) = \{j, j+1\}$.

0166 **Lemma 14.2.2.** *Any morphism in Δ can be written as a composition of the morphisms δ^n_j and σ^n_j.*

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta^m_j \circ \psi$ for some morphism $\psi : [n] \to [m-1]$. If $\varphi(j) = \varphi(j+1)$ then we can write φ as $\psi \circ \sigma^{n-1}_j$ for some morphism $\psi : [n-1] \to [m]$. The result follows because each replacement as above lowers $n+m$ and hence at some point φ is both injective and surjective, hence an identity morphism. □

0167 **Lemma 14.2.3.** *The morphisms δ^n_j and σ^n_j satisfy the following relations.*

(1) *If* $0 \leq i < j \leq n+1$, *then* $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. *In other words the diagram*

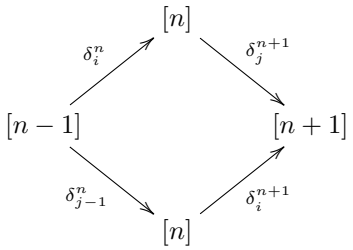

commutes.

(2) *If* $0 \leq i < j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. *In other words the diagram*

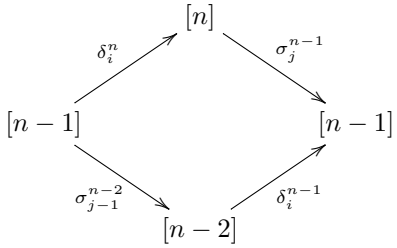

commutes.

(3) *If* $0 \leq j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_j^n = id_{[n-1]}$ *and* $\sigma_j^{n-1} \circ \delta_{j+1}^n = id_{[n-1]}$. *In other words the diagram*

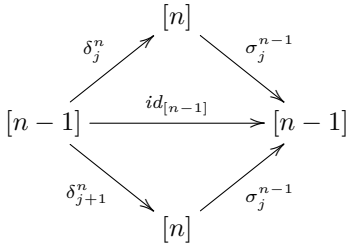

commutes.

(4) *If* $0 < j+1 < i \leq n$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. *In other words the diagram*

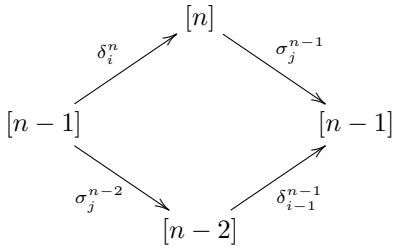

commutes.

(5) *If* $0 \leq i \leq j \leq n-1$, *then* $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$. *In other words the diagram*

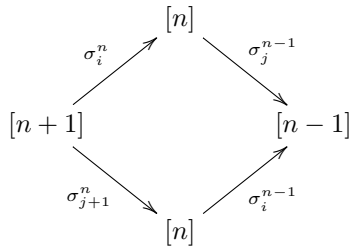

commutes.

Proof. Omitted. □

0168 **Lemma 14.2.4.** *The category* Δ *is the universal category with objects* $[n]$, $n \geq 0$ *and morphisms* δ_j^n *and* σ_j^n *such that (a) every morphism is a composition of these morphisms, (b) the relations listed in Lemma 14.2.3 are satisfied, and (c) any relation among the morphisms is a consequence of those relations.*

Proof. Omitted. □

14.3. Simplicial objects

0169

016A **Definition 14.3.1.** Let $\mathcal{C}$ be a category.

(1) A *simplicial object* U *of* $\mathcal{C}$ is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *simplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *simplicial abelian group*.
(4) A *morphism of simplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of simplicial objects of* $\mathcal{C}$ is denoted $\mathrm{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

016B **Lemma 14.3.2.** *Let* $\mathcal{C}$ *be a category.*

(1) *Given a simplicial object* U *in* $\mathcal{C}$ *we obtain a sequence of objects* $U_n = U([n])$ *endowed with the morphisms* $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ *and* $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$. *These morphisms satisfy the opposites of the relations displayed in Lemma 14.2.3, namely*
   (a) *If* $0 \leq i < j \leq n+1$, *then* $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$.
   (b) *If* $0 \leq i < j \leq n-1$, *then* $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$.

(c) *If* $0 \leq j \leq n-1$, *then* $id = d^n_j \circ s^{n-1}_j = d^n_{j+1} \circ s^{n-1}_j$.
(d) *If* $0 < j+1 < i \leq n$, *then* $d^n_i \circ s^{n-1}_j = s^{n-2}_j \circ d^{n-1}_{i-1}$.
(e) *If* $0 \leq i \leq j \leq n-1$, *then* $s^n_i \circ s^{n-1}_j = s^n_{j+1} \circ s^{n-1}_i$.

(2) *Conversely, given a sequence of objects* U_n *and morphisms* d^n_j, s^n_j *satisfying (1)(a) – (e) there exists a unique simplicial object* U *in* $\mathcal{C}$ *such that* $U_n = U([n])$, $d^n_j = U(\delta^n_j)$, *and* $s^n_j = U(\sigma^n_j)$.
(3) *A morphism between simplicial objects* U *and* U' *is given by a family of morphisms* $U_n \to U'_n$ *commuting with the morphisms* d^n_j *and* s^n_j.

Proof. This follows from Lemma 14.2.4. □

016C **Remark 14.3.3.** By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d^n_i, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d^n_i and s^n_i may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j+1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s^0_0 = U(\sigma^0_0) : U_0 \to U_1$ and two morphisms $d^1_0 = U(\delta^1_0)$, and $d^1_1 = U(\delta^1_1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s^1_0 = U(\sigma^1_0)$, $s^1_1 = U(\sigma^1_1)$ which are morphisms $U_1 \to U_2$. Three morphisms $d^2_0 = U(\delta^2_0)$, $d^2_1 = U(\delta^2_1)$, $d^2_2 = U(\delta^2_2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$U_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow}\; U_0$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

016D **Example 14.3.4.** The simplest example is the *constant* simplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

016E **Example 14.3.5.** Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 14.3.5 above is an important example. It tells us that it is a good idea to think of the maps $d^n_j : U_n \to U_{n-1}$ as projection maps (forgetting the jth component), and to think of the maps $s^n_j : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

016F **Lemma 14.3.6.** *Let* $\mathcal{C}$ *be a category. Let* U *be a simplicial object of* $\mathcal{C}$. *Each of the morphisms* $s^n_i : U_n \to U_{n+1}$ *has a left inverse. In particular* s^n_i *is a monomorphism.*

Proof. This is true because $d^{n+1}_i \circ s^n_i = \mathrm{id}_{U_n}$. □

14.4. Simplicial objects as presheaves

016G Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 7.2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

016H (14.4.0.1) $$\mathrm{Mor}(U, U') = \mathrm{Mor}_{PSh(\Delta)}(U, U').$$

Some of the material below could be replaced by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

14.5. Cosimplicial objects

016I A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

016J **Definition 14.5.1.** Let $\mathcal{C}$ be a category.

(1) A *cosimplicial object U of $\mathcal{C}$* is a covariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *cosimplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *cosimplicial abelian group*.
(4) A *morphism of cosimplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of cosimplicial objects of $\mathcal{C}$* is denoted $\mathrm{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

016K **Lemma 14.5.2.** *Let $\mathcal{C}$ be a category.*

(1) *Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta^n_j = U(\delta^n_j) : U_{n-1} \to U_n$ and $\sigma^n_j = U(\sigma^n_j) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 14.2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms δ^n_j, σ^n_j satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta^n_j = U(\delta^n_j)$, and $\sigma^n_j = U(\sigma^n_j)$.*
(3) *A morphism between cosimplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ^n_j and σ^n_j.*

Proof. This follows from Lemma 14.2.4. □

016L **Remark 14.5.3.** By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ^n_i, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ^n_i and σ^n_i may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.
(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.
(3) We have $\mathrm{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.
(4) If $i > j+1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.
(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma_0^0 = U(\sigma_0^0) : U_1 \to U_0$ and two morphisms $\delta_0^1 = U(\delta_0^1)$, and $\delta_1^1 = U(\delta_1^1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma_0^1 = U(\sigma_0^1)$, $\sigma_1^1 = U(\sigma_1^1)$ which are morphisms $U_2 \to U_1$. Three morphisms $\delta_0^2 = U(\delta_0^2)$, $\delta_1^2 = U(\delta_1^2)$, $\delta_2^2 = U(\delta_2^2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:

$$U_0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_2$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

016M **Example 14.5.4.** The simplest example is the *constant* cosimplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

016N **Example 14.5.5.** Suppose that $X \to Y$ is a morphism of C such that all the pushouts $Y \amalg_X Y \amalg_X \ldots \amalg_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold pushout, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on "coordinates". In other words, the map $U_1 = Y \amalg_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \amalg_X Y$ are the two coprojections.

0B13 **Example 14.5.6.** For every $n \geq 0$ we denote $C[n]$ the cosimplicial set

$$\Delta \longrightarrow \mathit{Sets}, \quad [k] \longmapsto \mathrm{Mor}_\Delta([n], [k])$$

This example is dual to Example 14.11.2.

016O **Lemma 14.5.7.** *Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta_i^n : U_{n-1} \to U_n$ has a left inverse. In particular δ_i^n is a monomorphism.*

Proof. This is true because $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{U_n}$ for $j < n$. □

14.6. Products of simplicial objects

016P Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

016Q **Definition 14.6.1.** Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the simplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and

(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

016R **Lemma 14.6.2.** *If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.7. Fibre products of simplicial objects

016S Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

016T **Definition 14.7.1.** Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the simplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

016U **Lemma 14.7.2.** *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.8. Pushouts of simplicial objects

016V Of course we should define the pushout of simplicial objects as the pushout in the category of simplicial objects. This may lead to the potentially confusing situation where the pushouts exist but are not as described below. To avoid this we define the pushout directly as follows.

016W **Definition 14.8.1.** Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the pushouts $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The *pushout of V and W over U* is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the pushout of the presheaves V and W over the presheaf U on Δ.

016X **Lemma 14.8.2.** *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have*

$$\mathrm{Mor}(V \amalg_U W, T) = \mathrm{Mor}(V, T) \times_{\mathrm{Mor}(U,T)} \mathrm{Mor}(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.9. Products of cosimplicial objects

016Y Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

016Z **Definition 14.9.1.** Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

0170 **Lemma 14.9.2.** *If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.10. Fibre products of cosimplicial objects

0171 Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

0172 **Definition 14.10.1.** Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

0173 **Lemma 14.10.2.** *If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.11. Simplicial sets

0174 Let U be a simplicial set. It is a good idea to think of U_0 as the 0-*simplices*, the set U_1 as the 1-*simplices*, the set U_2 as the 2-*simplices*, and so on.

We think of the maps $s_j^n : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d_j^n : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s_j^n and d_j^n geometrically.

0175 **Definition 14.11.1.** Let U be a simplicial set. We say x is an *n-simplex of U* to signify that x is an element of U_n. We say that y is the jth *face of x* to signify that $d_j^n x = y$. We say that z is the jth *degeneracy of x* if $z = s_j^n x$. A simplex is called *degenerate* if it is the degeneracy of another simplex.

Here are a few fundamental examples.

0176 **Example 14.11.2.** For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\Delta^{opp} \longrightarrow \mathit{Sets}, \quad [k] \longmapsto \mathrm{Mor}_\Delta([k], [n])$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\mathrm{id}_{[n]}$.

0177 **Lemma 14.11.3.** *Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection*

$$\mathrm{Mor}(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. □

0178 **Example 14.11.4.** Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 4.2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 4.35, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 4.38.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (14.4.0.1) and Lemma 14.11.3 above we get the formula

$$\mathrm{Mor}_{PSh(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

0179 **Lemma 14.11.5.** *Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.*

Proof. Suppose $n > a+b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$

such that $\alpha(i) = \alpha(i+1)$ and $\beta(i) = \beta(i+1)$. It follows immediately that (u, v) is in the image of s_i^{n-1}. □

14.12. Truncated simplicial objects and skeleton functors

017Z Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

0180 **Definition 14.12.1.** An *n-truncated simplicial object of $\mathcal{C}$* is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A *morphism of n-truncated simplicial objects* is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\mathrm{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a *skeleton functor*

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 14.21.6 to avoid possible confusion with other functors in the literature.

14.13. Products with simplicial sets

017A Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of $\mathcal{C}$ the set

(14.13.0.1)

017B $$\left\{(f_{n,u} : V_n \to W_n)_{n\geq 0, u\in U_n} \text{ such that } \begin{array}{c} \forall \varphi : [m] \to [n] \\ f_{m,U(\varphi)(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u} \end{array}\right\}$$

If this functor is of the form $\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

017C **Definition 14.13.1.** Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the *product $U \times V$ of U and V* to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod\nolimits_{u\in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod\nolimits_{u\in U_n} V_n \longrightarrow \coprod\nolimits_{u'\in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the *product $U \times V$ exists.*

017D **Lemma 14.13.2.** *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto \mathrm{Mor}_{Simp(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (14.13.0.1).*

Proof. Omitted. □

017E **Lemma 14.13.3.** *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.*

(1) *The rule $(U, V) \mapsto U \times V$ defines a functor $FSSets \times Simp(\mathcal{C}) \to Simp(\mathcal{C})$.*
(2) *For every U, V as above there is a canonical map of simplicial objects*

$$U \times V \longrightarrow V$$

defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. □

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \amalg \ldots \amalg X$ exist then according to the definition above the product

$$X \times \Delta[k]$$

exists, where we think of X as the corresponding constant simplicial object.

017F **Lemma 14.13.4.** *With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, V_k).$$

which maps γ to the restriction of the morphism γ_k to the component corresponding to $id_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have

$$\mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n(X \times \Delta[k]), W) = \mathrm{Mor}_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha : X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\gamma_\alpha} & V_n \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xrightarrow{\gamma_{\alpha\circ\varphi}} & V_m \end{array}$$

commute. Taking $\alpha = \mathrm{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi) \circ \gamma_{\mathrm{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha) \circ f$.

The truncated case is similar, and left to the reader. □

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that

$$\mathrm{Mor}_{\mathcal{C}}(X, V_0) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, V)$$

where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14.14. Hom from simplicial sets into cosimplicial objects

07K9 Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_{\mathcal{C}}(U, V)_n = \mathrm{Mor}_{\mathcal{C}}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_{\mathcal{C}}(U, V)_m \to \mathrm{Hom}_{\mathcal{C}}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

019V **Definition 14.14.1.** Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy
$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_{\mathcal{C}}(X, V_n))$$
and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.13.3 in the case of products of simplicial sets with simplicial objects.

14.15. Hom from cosimplicial sets into simplicial objects

0B14 Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$, and let V be a simplicial object of $\mathcal{C}$. Then we get a simplicial set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_{\mathcal{C}}(U, V)_n = \mathrm{Mor}_{\mathcal{C}}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_{\mathcal{C}}(U, V)_n \to \mathrm{Hom}_{\mathcal{C}}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

0B15 **Definition 14.15.1.** Let $\mathcal{C}$ be a category with finite products. Let V be a simplicial object of $\mathcal{C}$. Let U be a cosimplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the simplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy
$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_{\mathcal{C}}(X, V_n))$$
and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_n \to \mathrm{Hom}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.13.3 in the case of products of simplicial sets with simplicial objects.

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$U_n = \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X$$

and maps given $\varphi : [m] \to [n]$

$$U(\varphi) : \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X \longrightarrow \prod\nolimits_{\alpha' \in \mathrm{Mor}([k],[m])} X, \quad (f_\alpha)_\alpha \longmapsto (f_{\varphi \circ \alpha'})_{\alpha'}$$

In terms of "coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi\circ\alpha'})_{\alpha'}$. We claim this object is equal to $\mathrm{Hom}(C[k], X)$ where we think of X as the constant simplicial object X and where $C[k]$ is the cosimplicial set from Example 14.5.6.

017M **Lemma 14.15.2.** *With X, k and U as above.*

(1) *For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(V, U) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(V_k, X).$$

which maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $id_{[k]}$.

(2) *Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have*

$$\mathrm{Mor}_{Simp_k(\mathcal{C})}(W, sk_k U) = \mathrm{Mor}_{\mathcal{C}}(W_k, X).$$

(3) *The object U constructed above is an incarnation of $\mathrm{Hom}(C[k], X)$ where $C[k]$ is the cosimplicial set from Example 14.5.6.*

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\alpha : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{array}{ccc} X & \xleftarrow{\gamma_{\varphi\circ\alpha'}} & V_n \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xleftarrow{\gamma_{\alpha'}} & V_m \end{array}$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \mathrm{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\mathrm{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

Part (3) is immediate from the construction of U and the fact that $C[k]_n = \mathrm{Mor}([k], [n])$ which are the index sets used in the construction of U_n. □

14.16. Internal Hom

017G Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor

$$\mathrm{Simp}(\mathcal{C})^{opp} \longrightarrow Sets, \quad W \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times V, U)$$

is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the *internal hom of V into U exists*. Moreover, in this case, given X in $\mathcal{C}$, we would have

$$\begin{aligned}\mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(V, U)_n) &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V, U)) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\ &= \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(\Delta[n] \times V, U)_0)\end{aligned}$$

provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. The first and last equalities follow from Lemma 14.13.4.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects

$$\mathcal{H}om(\Delta[n] \times V, U)_0$$

because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects "$\mathrm{Hom}(\Delta[n], U)$".

14.17. Hom from simplicial sets into simplicial objects

017H Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

017I **Definition 14.17.1.** Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $\mathrm{Hom}(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $\mathrm{Hom}(U, V)$ need not exist. Also, by the discussion in Section 14.16 we expect that if it does exist, then $\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $\mathrm{Hom}(U, V)$ is not an internal hom.

017J **Lemma 14.17.2.** *Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned}\mathcal{C}^{opp} &\longrightarrow \mathit{Sets} \\ X &\longmapsto \mathrm{Mor}_{\mathit{Simp}(\mathcal{C})}(X \times U, V)\end{aligned}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f_u} & V_n \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xrightarrow{f_{U(\varphi)(u)}} & V_m \end{array}$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

(1) The set of objects of $\mathcal{U}$ is $\coprod_{n\geq 0} U_n$,
(2) a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
(3) for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
(4) for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$\lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. □

017K **Lemma 14.17.3.** *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned} \mathcal{C}^{opp} &\longrightarrow Sets \\ X &\longmapsto \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V) \end{aligned}$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 14.17.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 4.17.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0\leq n\leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of Categories, Definition 4.17.3.

Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pigeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma_i^{k-1}$ and $\varphi' = \psi' \circ \sigma_i^{k-1}$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s_i^{k-1} : U_{k-1} \to U_k$ is injective, see Lemma 14.3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with $u, u' \in \mathcal{U}_{\leq m_0}$, there exists a commutative diagram

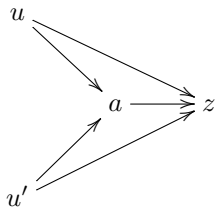

with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that

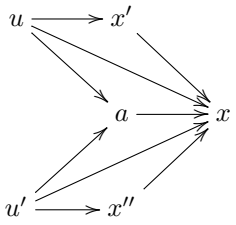

is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of Categories, Definition 4.17.3. □

017L **Lemma 14.17.4.** *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then* $\mathrm{Hom}(U, V)$ *exists, moreover we have the expected equalities*

$$\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $\mathrm{Hom}(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 14.17.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 14.11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$\mathrm{Hom}(U, V)(\varphi) : \mathrm{Hom}(U, V)_n \longrightarrow \mathrm{Hom}(U, V)_m.$$

Clearly this defines a contravariant functor $\mathrm{Hom}(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $\mathrm{Hom}(U, V)$ satisfies the desired universal property

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to \mathrm{Hom}(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to \mathrm{Hom}(U, V)_n$ corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} W_n \times U \times \Delta[m] & \xrightarrow{W(\varphi) \times \mathrm{id} \times \mathrm{id}} & W_m \times U \times \Delta[m] \\ \downarrow{\scriptstyle \mathrm{id} \times \mathrm{id} \times \varphi} & & \downarrow{\scriptstyle f_m} \\ W_n \times U \times \Delta[n] & \xrightarrow{f_n} & V \end{array}$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f'_n : W_n \times U_n \to V_n$

as $f'_n = (f_n)_{n,\mathrm{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \mathrm{id}_{[n]}$. To see that the collection (f'_n) defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f'_n = f'_m \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\mathrm{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$\begin{array}{ccc} W_n \times U_n \times (\Delta[n])_n & \xrightarrow[(f_n)_n]{} & V_n \\ {\scriptstyle \mathrm{id} \times U(\varphi) \times \varphi}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ W_n \times U_m \times (\Delta[n])_m & \xrightarrow{(f_n)_m} & V_m \end{array}$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\mathrm{id}}$. Altogether we obtain $V(\varphi) \circ (f_n)_{n,\mathrm{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to \mathrm{Hom}(U, V)$ as follows. By Lemma 14.13.4 the morphisms $\mathrm{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$W_n \times \Delta[n] \times U \xrightarrow{c_n} W \times U \xrightarrow{f'} V.$$

By construction this corresponds to a unique morphism $f_n : W_n \to \mathrm{Hom}(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. □

017N **Lemma 14.17.5.** *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then*

$$\mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T) = \mathrm{Hom}(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 14.17.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the object representing the functor that associates to X the first set of the following sequence of functorial equalities

$$\begin{aligned} &\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T)) \\ &= \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T)) \times_{\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(U,T))} \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(W, T)) \\ &= \mathrm{Mor}(X \times \Delta[n] \times V, T) \times_{\mathrm{Mor}(X \times \Delta[n] \times U, T)} \mathrm{Mor}(X \times \Delta[n] \times W, T) \\ &= \mathrm{Mor}(X \times \Delta[n] \times (V \amalg_U W), T)) \end{aligned}$$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X \times \Delta[n] \times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $\mathrm{Hom}(V \amalg_U W, T)_n$. □

14.18. Splitting simplicial objects

017O A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abuse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

017P **Definition 14.18.1.** Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is *split* if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

017Q (14.18.1.1)
$$\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) \longrightarrow U_n$$

is an isomorphism for all $n \geq 0$. If U is an r-truncated simplicial object of $\mathcal{C}$ then we say U is *split* if there exist subobjects $N(U_m)$ of U_m, $r \geq m \geq 0$ with the property that (14.18.1.1) is an isomorphism for $r \geq n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \amalg N(U_1)$. Second we have

$$U_2 = U_0 \amalg N(U_1) \amalg N(U_1) \amalg N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

017R **Lemma 14.18.2.** *Let U be a simplicial set. Then U has a unique splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.*

Proof. From the definition it follows immediately, that if there is a splitting then $N(U_m)$ has to be the set of nondegenerate simplices. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \mathrm{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. □

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 14.18.2 is not functorial. Here is a case where it is functorial.

017S **Lemma 14.18.3.** *Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegenerate simplex of V and (b) the restriction of f to a map from the set of nondegenerate simplices of U to the set of nondegenerate simplices of V is injective. Then f_n is injective for all n. Same holds with "injective" replaced by "surjective" or "bijective".*

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 14.18.2, in other words that we get commutative diagrams

$$\begin{array}{ccc} \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(V_m) & \longrightarrow & V_n. \end{array}$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). □

017T **Lemma 14.18.4.** *Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \bigcup\nolimits_{\varphi:[m]\to[i],\ i\leq n} \mathrm{Im}(U(\varphi))$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n + 1$ and higher are degenerate. □

017U **Lemma 14.18.5.** *Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma guarantees that (14.18.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 14.3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in \mathrm{Im}(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_1(x_1) = 0$. Continue like this to get

$$\begin{aligned} x &= x_0 + s_0(z_0), \\ x_0 &= x_1 + s_1(z_1), \\ x_1 &= x_2 + s_2(z_2), \\ \ldots &\ \ldots\ \ldots \\ x_{n-1} &= x_n + s_n(z_n) \end{aligned}$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$

and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \leq n$ assuming the result for $i-1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \leq j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 14.3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_n$$

with $x_n \in N(U_{n+1})$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=0}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

with the property that

$$\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=j+1}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_n + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum\nolimits_{0 \leq i \leq j \leq n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n - k + 1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma_{i_k}^{n-k} \ldots \sigma_{i_2}^{n-1} \sigma_{i_1}^{n}$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n - k + 1$. □

017V **Lemma 14.18.6.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $\mathrm{Mor}_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 14.18.5. Moreover,

$$N(\mathrm{Mor}_{\mathcal{A}}(A, U_m)) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d_i^m) = \mathrm{Mor}_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (14.18.1.1) becomes an isomorphism after applying the functor $\mathrm{Mor}_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. □

017W **Lemma 14.18.7.** *Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with "injective" replaced with "surjective", or "isomorphism".*

Proof. This is clear from Lemma 14.18.6 and the definition of a splitting. □

017X **Lemma 14.18.8.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 14.18.6 above. Then $d_m^m(N(U_m)) \subset N(U_{m-1})$.*

Proof. For $j = 0, \ldots, m-2$ we have $d_j^{m-1} d_m^m = d_{m-1}^{m-1} d_j^m$ by the relations in Remark 14.3.3. The result follows. □

017Y **Lemma 14.18.9.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \sum\nolimits_{\varphi:[m]\to[i],\ i\leq n} \mathrm{Im}(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, $N(U'_m) = 0$ for all $m > n$.

Proof. Pick m, $i \leq n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\mathrm{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U'_m) = 0$. By definition of $N(U_m)$ and $N(U'_m)$ we have $N(U'_m) = U'_m \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 14.18.6, it suffices to show that U'_m is contained in the sum

$$\sum\nolimits_{\varphi:[m]\to[m'] \text{ surjective},\ m'<m} \mathrm{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum\nolimits_{\varphi:[m]\to[m'] \text{ surjective},\ m'<m} \mathrm{Im}(U(\varphi)).$$

Clearly U'_m is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \leq n$ occurring in the definition of U'_m may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. □

14.19. Coskeleton functors

0AMA Let $\mathcal{C}$ be a category. The *coskeleton functor* (if it exists) is a functor

$$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{C}) \longrightarrow \mathrm{Simp}(\mathcal{C})$$

which is right adjoint to the skeleton functor. In a formula

0181 (14.19.0.1) $$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$$

Given a n-truncated simplicial object V we say that *$cosk_n V$ exists* if there exists a $\text{cosk}_n V \in \text{Ob}(\text{Simp}(\mathcal{C}))$ and a morphism $\text{sk}_n\text{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto \text{Mor}_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 4.3.

0182 **Example 14.19.1.** Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\text{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 14.3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$\begin{array}{ccc} V_n & \longrightarrow & X^{n+1} \\ {\scriptstyle V([0]\to[n],0\mapsto i)}\downarrow & & \downarrow{\scriptstyle \text{pr}_i} \\ V_0 & \longrightarrow & X \end{array}$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (14.19.0.1) holds.

Recall the category $\Delta/[n]$, see Example 14.11.4. We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. In other words we have the following commutative diagram of categories and functors

$$\begin{array}{ccc} (\Delta/[n])_{\leq m} & \longrightarrow & \Delta/[n] \\ \downarrow & & \downarrow \\ \Delta_{\leq m} & \longrightarrow & \Delta \end{array}$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])^{opp}_{\leq m} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{array}{rcl} ([k] \to [n]) & \longmapsto & U_k \\ \psi : ([k'] \to [n]) \to ([k] \to [n]) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{array}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\overline{\varphi} : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ \overline{\varphi}$ is equal to the functor $U(n)$.

0183 **Lemma 14.19.2.** *If the category $\mathcal{C}$ has finite limits, then $cosk_m$ functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $cosk_m U$ is described by the formula*

$$(cosk_m U)_n = \lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $cosk_m U(\varphi)$ comes from the identification $U(n') \circ \overline{\varphi} = U(n)$ above via Categories, Lemma 4.14.9.

Proof. During the proof of this lemma we denote $\mathrm{cosk}_m U$ the simplicial object with $(\mathrm{cosk}_m U)_n$ equal to $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$. We will conclude at the end of the proof that it does satisfy the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to \mathrm{cosk}_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (\mathrm{cosk}_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$\begin{array}{ccc} V_n & \xrightarrow[\gamma(\alpha)]{} & U_k \\ {\scriptstyle V(\varphi)}\uparrow & & \uparrow{\scriptstyle U(\psi)} \\ V_{n'} & \xrightarrow{\gamma(\alpha')} & U_{k'} \end{array}$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k = k'$, $\varphi = \alpha'$, and $\alpha = \psi = \mathrm{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\mathrm{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $\mathrm{sk}_m V \to U$.

Conversely, given a morphism $\gamma : \mathrm{sk}_m V \to U$, we obtain a family of morphisms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to \mathrm{cosk}_m U$. □

0184 **Lemma 14.19.3.** *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$ exists and is canonically isomorphic to U_n.*

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. □

0185 **Lemma 14.19.4.** *Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $sk_n cosk_n U \to U$ is an isomorphism.*

Proof. Combine Lemmas 14.19.2 and 14.19.3. □

Let us describe a particular instance of the coskeleton functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\mathrm{Simp}_{n'}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $\mathrm{sk}_n : \mathrm{Simp}_{n+1}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta^{n+1}_{i,j} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta^{n+1}_{i,j} = \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1}$, see Lemma 14.2.3. This motivates the following lemma.

0186 **Lemma 14.19.5.** *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha:[k]\to[n+1],k\leq n} \mid f_{\alpha\circ\psi} = U(\psi)\circ f_\alpha \ \forall\ \psi : [k'] \to [k], \alpha : [k] \to [n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta_0^{n+1}}, \ldots, f_{\delta_{n+1}^{n+1}}).$$

This satisfies the conditions of the lemma because

$$d_{j-1}^n \circ f_{\delta_i^{n+1}} = f_{\delta_i^{n+1}\circ\delta_{j-1}^n} = f_{\delta_j^{n+1}\circ\delta_i^n} = d_i^n \circ f_{\delta_j^{n+1}}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha : [k] \to [n+1]$ be given. Since $k \leq n$ the map α is not surjective. Hence we can write $\alpha = \delta_i^{n+1} \circ \psi$ for some $0 \leq i \leq n+1$ and some $\psi : [k] \to [n]$. We have no choice but to define

$$f_\alpha = U(\psi) \circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i, ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j, ϕ) is another pair. Then $i \neq j$ and we may assume $i < j$. Since both i, j are not in the image of α we may actually write $\alpha = \delta_{i,j}^{n+1} \circ \xi$ and then we see that $\psi = \delta_{j-1}^n \circ \xi$ and $\phi = \delta_i^n \circ \xi$. Thus

$$\begin{aligned} U(\psi) \circ f_i &= U(\delta_{j-1}^n \circ \xi) \circ f_i \\ &= U(\xi) \circ d_{j-1}^n \circ f_i \\ &= U(\xi) \circ d_i^n \circ f_j \\ &= U(\delta_i^n \circ \xi) \circ f_j \\ &= U(\phi) \circ f_j \end{aligned}$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi : [k'] \to [k]$, $\alpha : [k] \to [n+1]$ with $k, k' \leq n$. Set $\alpha' = \alpha \circ \psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta_i^{n+1} \circ \phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta_i^{n+1} \circ \phi \circ \psi$. By construction above we then have

$$U(\psi) \circ f_\alpha = U(\psi) \circ U(\phi) \circ f_i = U(\phi \circ \psi) \circ f_i = f_{\alpha\circ\psi} = f_{\alpha'}$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. □

0187 **Lemma 14.19.6.** *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d_{j-1}^n \circ f_i = d_i^n \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde{U}$, with $sk_n\tilde{U} = U$ and $\tilde{U}_{n+1} = U_{n+1}$ such that the following adjointness holds

$$\mathrm{Mor}_{Simp_{n+1}(\mathcal{C})}(V, \tilde{U}) = \mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n V, U)$$

Proof. By Lemma 14.19.3 there are identifications

$$U_i = \lim_{(\Delta/[i])^{opp}_{\leq n}} U(i)$$

for $0 \leq i \leq n$. By Lemma 14.19.5 we have

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n).$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n+1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 14.19.2. This defines an $(n+1)$-truncated simplicial object $\tilde{U}$ with $\mathrm{sk}_n \tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : \mathrm{sk}_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d_i^{n+1} : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d_{j-1}^n \circ f_i = d_i^n \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphism $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. □

0188 **Remark 14.19.7.** Let U, and U_{n+1} be as in Lemma 14.19.6. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d_i^{n+1} : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s_j^n : U_n \to U_{n+1}$ are given by

$$\begin{aligned} f \quad \longmapsto \quad & (s_{j-1}^{n-1} \circ d_0^{n-1} \circ f, \\ & s_{j-1}^{n-1} \circ d_1^{n-1} \circ f, \\ & \ldots \\ & s_{j-1}^{n-1} \circ d_{j-1}^{n-1} \circ f, \\ & f, \\ & f, \\ & s_j^{n-1} \circ d_{j+1}^{n-1} \circ f, \\ & s_j^{n-1} \circ d_{j+2}^{n-1} \circ f, \\ & \ldots \\ & s_j^{n-1} \circ d_n^{n-1} \circ f) \end{aligned}$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 14.19.6. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$ there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

0189 **Remark 14.19.8.** The construction of Lemma 14.19.6 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n+1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n+1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n+1)$-simplex. In other words, the ith face of f_j is the $(j-1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face

and degeneracy maps for $\tilde{U}$. If V is an $(n+1)$-truncated simplicial set, then its $(n+1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $\mathrm{Mor}(\mathrm{sk}_n V, U) \to \mathrm{Mor}(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $\mathrm{Mor}(T, U)$. We may apply the construction to this, and take its set of $(n+1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 14.19.6.

018A **Remark 14.19.9.** *Inductive construction of coskeleta.* Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 14.19.6.

Since the construction of Lemma 14.19.6 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\mathrm{cosk}_m U$ to be the simplicial object with $(\mathrm{cosk}_m U)_n = U^n_n = U^{n+1}_n = \ldots$. And it follows formally from Lemma 14.19.6 that U^n satisfies the formula

$$\mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(V, U^n) = \mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(\mathrm{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(V, \mathrm{cosk}_m U) = \mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(\mathrm{sk}_m V, U)$$

with $\mathrm{cosk}_m U$ chosen as above.

018B **Lemma 14.19.10.** *Let $\mathcal{C}$ be a category which has finite limits.*

(1) *For every n the functor $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint $cosk_n$.*
(2) *For every $n' \geq n$ the functor $sk_n : Simp_{n'}(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint, namely $sk_{n'} cosk_n$.*
(3) *For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $cosk_m sk_m cosk_n U = cosk_n U$.*
(4) *If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to cosk_n sk_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to cosk_m sk_m U$ is an isomorphism for all $m \geq n$.*

Proof. The existence in (1) follows from Lemma 14.19.2 above. Parts (2) and (3) follow from the discussion in Remark 14.19.9. After this (4) is obvious. □

09VS **Remark 14.19.11.** We do not need all finite limits in order to be able to define the coskeleton functors. Here are some remarks

(1) We have seen in Example 14.19.1 that if $\mathcal{C}$ has products of pairs of objects then cosk_0 exists.
(2) For $k > 0$ the functor cosk_k exists if $\mathcal{C}$ has finite connected limits.

This is clear from the inductive procedure of constructing coskeleta (Remarks 14.19.8 and 14.19.9) but it also follows from the fact that the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.19.2 are connected. Observe that we

do not need the categories for $n \leq k$ by Lemma 14.19.3 or Lemma 14.19.4. (As k gets higher the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ are more and more connected in a topological sense.)

018C **Lemma 14.19.12.** *Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then*

$$cosk_n(U \times V) = cosk_n U \times cosk_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$\begin{aligned} \mathrm{Mor}(W, \mathrm{cosk}_n(U \times V)) &= \mathrm{Mor}(\mathrm{sk}_n W, U \times V) \\ &= \mathrm{Mor}(\mathrm{sk}_n W, U) \times \mathrm{Mor}(\mathrm{sk}_n W, V) \\ &= \mathrm{Mor}(W, \mathrm{cosk}_n U) \times \mathrm{Mor}(W, \mathrm{cosk}_n V) \\ &= \mathrm{Mor}(W, \mathrm{cosk}_n U \times \mathrm{cosk}_n V) \end{aligned}$$

The lemma follows. □

018D **Lemma 14.19.13.** *Assume $\mathcal{C}$ has fibre products. Let $U \to V$ and $W \to V$ be morphisms of n-truncated simplicial objects of the category $\mathcal{C}$. Then*

$$cosk_n(U \times_V W) = cosk_n U \times_{cosk_n V} cosk_n W$$

whenever the left and right hand side exist.

Proof. Omitted, but very similar to the proof of Lemma 14.19.12 above. □

08NJ **Lemma 14.19.14.** *Let $\mathcal{C}$ be a category with finite limits. Let $X \in \mathrm{Ob}(\mathcal{C})$. The functor $\mathcal{C}/X \to \mathcal{C}$ commutes with the coskeleton functors $cosk_k$ for $k \geq 1$.*

Proof. The statement means that if U is a simplicial object of $\mathcal{C}/X$ which we can think of as a simplicial object of $\mathcal{C}$ with a morphism towards the constant simplicial object X, then $\mathrm{cosk}_k U$ computed in $\mathcal{C}/X$ is the same as computed in $\mathcal{C}$. This follows for example from Categories, Lemma 4.16.2 because the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.19.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 14.19.3 or Lemma 14.19.4. □

018E **Lemma 14.19.15.** *The canonical map $\Delta[n] \to cosk_1 sk_1 \Delta[n]$ is an isomorphism.*

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote $\epsilon^i_j : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon^i_j(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon^i_j(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent

to the condition that $F(\epsilon^i_j(u)) \leq F(\epsilon^i_{j'}(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon^i_j, \epsilon^i_{j'} : [0] \to [i]$$

both factor through the map $\epsilon^i_{j,j'} : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$\mathrm{Mor}(U, \Delta[n]) = \mathrm{Mor}(\mathrm{sk}_1 U, \mathrm{sk}_1 \Delta[n])$$

as desired. □

14.20. Augmentations

018F

018G **Definition 14.20.1.** Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$* is a morphism from U into the constant simplicial object X.

018H **Lemma 14.20.2.** *Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d^1_0 = \epsilon_0 \circ d^1_1$.*

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. □

018I **Lemma 14.20.3.** *Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 14.3.5. For any simplicial object V of $\mathcal{C}$ we have*

$$\begin{aligned} \mathrm{Mor}_{Simp(\mathcal{C})}(V, U) &= \mathrm{Mor}_{Simp_1(\mathcal{C})}(sk_1 V, sk_1 U) \\ &= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1\} \end{aligned}$$

In particular we have $U = cosk_1 sk_1 U$.

Proof. Suppose that $g : \mathrm{sk}_1 V \to \mathrm{sk}_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

$$\begin{array}{ccccc} & & V_1 & \xrightarrow[d^1_1]{d^1_0} & V_0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\ Y \times_X Y & \xrightarrow[pr_0]{pr_1} & Y & \longrightarrow & X \end{array}$$

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done

as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram

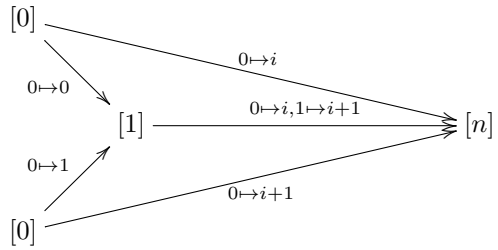

Hence we get $(g_{n,0}, \ldots, g_{n,n}) : V_n \to Y \times_X \ldots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. □

018J **Remark 14.20.4.** Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 14.20.2 and 14.20.3 we obtain a canonical morphism $V \to U$.

14.21. Left adjoints to the skeleton functors

018K In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$\mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(U, \mathrm{sk}_m V) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(i_{m!}U, V).$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 14.12. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 4.2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : ([n]/\Delta)^{opp}_{\leq m} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{aligned} ([n] \to [k]) &\longmapsto U_k \\ \psi : ([n] \to [k']) \to ([n] \to [k]) &\longmapsto U(\psi) : U_k \to U_{k'} \end{aligned}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\underline{\varphi} : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ \underline{\varphi}$ is equal to the functor $U(n')$.

018L **Lemma 14.21.1.** *Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula*

$$(i_{m!}U)_n = \mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ \underline{\varphi} = U(n')$ above via Categories, Lemma 4.14.8.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \mathrm{sk}_m V$ be given. A morphism

$$\mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Conversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \mathrm{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\mathrm{id}_{[i]}} \mathrm{colim}_{([i]/\Delta)^{opp}_{\leq m}} U(i) \xrightarrow{\gamma'_i} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. □

018M **Lemma 14.21.2.** *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit*

$$\mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\mathrm{id} : [n] \to [n]$. □

018N **Lemma 14.21.3.** *Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to sk_m i_{m!}U$ is an isomorphism.*

Proof. Combine Lemmas 14.21.1 and 14.21.2. □

018O **Lemma 14.21.4.** *If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.*

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 14.21.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V'_i = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 14.18.4. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

018P **Lemma 14.21.5.** *Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_n U \to U$ identifies $i_{n!}sk_n U$ with the simplicial set $U' \subset U$ defined in Lemma 14.18.4.*

Proof. By Lemma 14.21.4 the only nondegenerate simplices of $i_{n!}\mathrm{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 14.18.3 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. □

018Q **Remark 14.21.6.** In some texts the composite functor

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \mathrm{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense for simplicial sets, because then Lemma 14.21.5 says that $i_{m!}\mathrm{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{\mathrm{cosk}_m} \mathrm{Simp}(\mathcal{C})$$

by cosk_m.

018R **Lemma 14.21.7.** *Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) *$V_i = U_i$ for $i < n$,*
(2) *$V_n = U_n \cup \{x\}$,*
(3) *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$\begin{array}{ccc} \Delta[n] & \longrightarrow & V \\ \uparrow & & \uparrow \\ i_{(n-1)!}sk_{n-1}\Delta[n] & \longrightarrow & U \end{array}$$

is a pushout diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s_i^{j-1}(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ is surjective. Note that a nondegenerate simplex of $U \amalg_{\partial\Delta[n]} \Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 14.18.3. □

018S **Lemma 14.21.8.** *Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets*

$$U = W^0 \subset W^1 \subset W^2 \subset \ldots W^r = V$$

such that Lemma 14.21.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 14.21.7. Moreover the number of nondegenerate simplices of V which are

not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. □

018T **Lemma 14.21.9.** *Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.*

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V'_i = V_i$ for $i \leq m$ and $N(V'_i) = 0$ for $i > m$, see Lemma 14.18.9. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

018U **Lemma 14.21.10.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_nU \to U$ identifies $i_{n!}sk_nU$ with the simplicial subobject $U' \subset U$ defined in Lemma 14.18.9.*

Proof. By Lemma 14.21.9 we have $N(i_{n!}\mathrm{sk}_n U)_i = 0$ for $i > n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$, see Lemma 14.21.3. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ induces injective maps $N(i_{n!}\mathrm{sk}_n U)_i \to N(U)_i$ for all i. Hence Lemma 14.18.7 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. □

Here is another way to think about the coskeleton functor using the material above.

018V **Lemma 14.21.11.** *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case*

$$(cosk_n sk_n V)_{n+1} = \mathrm{Hom}(i_{n!}sk_n\Delta[n+1], V)_0.$$

Proof. By Lemma 14.13.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$\begin{aligned} \mathrm{Mor}(X \times \Delta[n+1], \mathrm{cosk}_n\mathrm{sk}_n V) &= \mathrm{Mor}(X \times \mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n V) \\ &= \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n\Delta[n+1], V). \end{aligned}$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\mathrm{sk}_n(X \times \Delta[n+1]) = X \times \mathrm{sk}_n\Delta[n+1]$ and that $i_{n!}(X \times \mathrm{sk}_n\Delta[n+1]) = X \times i_{n!}\mathrm{sk}_n\Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $\mathrm{Mor}_{\mathcal{C}}(X, W)$ the (possibly truncated) simplicial set $[n] \mapsto \mathrm{Mor}_{\mathcal{C}}(X, W_n)$. From the definitions it follows that $\mathrm{Mor}(U \times X, W) = \mathrm{Mor}(U, \mathrm{Mor}_{\mathcal{C}}(X, W))$ for any (possibly truncated) simplicial set U. Hence

$$\begin{aligned} \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n\Delta[n+1], W) &= \mathrm{Mor}(i_{n!}\mathrm{sk}_n\Delta[n+1], \mathrm{Mor}_{\mathcal{C}}(X, W)) \\ &= \mathrm{Mor}(\mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n \mathrm{Mor}_{\mathcal{C}}(X, W)) \\ &= \mathrm{Mor}(X \times \mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n W) \\ &= \mathrm{Mor}(i_{n!}(X \times \mathrm{sk}_n\Delta[n+1]), W). \end{aligned}$$

This proves the second equality used, and ends the proof of the lemma. □

14.22. Simplicial objects in abelian categories

018Y Recall that an abelian category is defined in Homology, Section 12.5.

018Z **Lemma 14.22.1.** *Let $\mathcal{A}$ be an abelian category.*

(1) *The categories Simp($\mathcal{A}$) and CoSimp($\mathcal{A}$) are abelian.*

(2) *A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of (co)simplicial objects*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 14.6.2 and 14.9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. □

For an object A of $\mathcal{A}$ and an integer k consider the k-truncated simplicial object U with

(1) $U_i = 0$ for $i < k$,
(2) $U_k = A$,
(3) all morphisms $U(\varphi)$ equal to zero, except $U(\mathrm{id}_{[k]}) = \mathrm{id}_A$.

Since $\mathcal{A}$ has both finite limits and finite colimits we see that both $\mathrm{cosk}_k U$ and $i_{k!}U$ exist. We will describe both of these and the canonical map $i_{k!}U \to \mathrm{cosk}_k U$.

0190 **Lemma 14.22.2.** *With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.*

(1) *Given a k-truncated simplicial object V we have*

$$\mathrm{Mor}(U, V) = \{f : A \to V_k \mid d_i^k \circ f = 0,\ i = 0, \ldots, k\}$$

and

$$\mathrm{Mor}(V, U) = \{f : V_k \to A \mid f \circ s_i^{k-1} = 0,\ i = 0, \ldots, k-1\}.$$

(2) *The object $i_{k!}U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.*
(3) *For any $\varphi : [m] \to [n]$ the map $i_{k!}U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.*
(4) *The object $cosk_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.*
(5) *For any $\varphi : [m] \to [n]$ the map $cosk_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.*
(6) *The canonical map $c : i_{k!}U \to cosk_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.*
(7) *The canonical map $c : i_{k!}U \to cosk_k U$ is injective.*

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!}U$. This will prove (2), (3) at the

same time. We have to show that given a morphism $f : U \to \mathrm{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d_i^k for all i. For any surjective $\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow{\tilde{f}_n} & V_n \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ \bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{\tilde{f}_m} & V_m \end{array}$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left upper corner separately. It is clear for the summands corresponding to α where $\alpha \circ \varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta_i^k \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d_i^k \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\mathrm{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \mathrm{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s_i^{k-1} for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\tilde{f}_n} & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\ {\scriptstyle V(\varphi)}\downarrow & & \downarrow{\scriptstyle (5)} \\ V_m & \xrightarrow{\tilde{f}_m} & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A \end{array}$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi \circ \beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\varphi \circ \beta')$. Since $\varphi \circ \beta'$ is not injective we can write it as $\psi \circ \sigma_i^{k-1}$, and we deduce that $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ s_i^{k-1} \circ V(\psi) = 0$ see above.

The composition $i_{k!}U \to \mathrm{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!}U)_k = (\mathrm{cosk}_k U)_k$. Hence it suffices to check that the proposed rule defines a morphism of simplicial objects. To see this we have to show that for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow[(6)]{} & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (5)} \\ \bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{(6)} & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A \end{array}$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \mathrm{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions and seeing they are equal. This comes down to the following equality

$$\#\left\{\beta \mid \beta = \varphi \circ \beta' \text{ and } \alpha \circ \beta = \mathrm{id}_{[k]}\right\} = \#\left\{\alpha' \mid \alpha' = \alpha \circ \varphi \text{ and } \alpha' \circ \beta' = \mathrm{id}_{[k]}\right\}$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 14.18.7, 14.19.4, 14.21.3 and 14.21.9. □

0191 **Definition 14.22.3.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The *Eilenberg-Maclane object* $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 14.22.2 above.

0192 **Lemma 14.22.4.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules*

(1) *$E_n = \bigoplus_\alpha A$, where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.*
(2) *Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $\mathrm{Im}(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.*

Then there exists a short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A, k)_n \to E_n$ resp. $E_n \to K(A, k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. □

0193 **Lemma 14.22.5.** *Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have*

$$V = \mathrm{colim}_n \, i_{n!} sk_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\mathrm{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 14.21.10 for the transition maps. □

14.23. Simplicial objects and chain complexes

0194 Let $\mathcal{A}$ be an abelian category. See Homology, Section 12.13 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The *associated chain complex $s(U)$ of U*, sometimes called the *Moore complex*, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum\nolimits_{i=0}^{n} (-1)^i d_i^n.$$

This is a complex because, by the relations listed in Remark 14.3.3, we have

$$\begin{aligned} d_n \circ d_{n+1} &= (\sum\nolimits_{i=0}^{n} (-1)^i d_i^n) \circ (\sum\nolimits_{j=0}^{n+1} (-1)^j d_j^{n+1}) \\ &= \sum\nolimits_{0 \le i < j \le n+1} (-1)^{i+j} d_{j-1}^n \circ d_i^{n+1} + \sum\nolimits_{n \ge i \ge j \ge 0} (-1)^{i+j} d_i^n \circ d_j^{n+1} \\ &= 0. \end{aligned}$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\ge 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

0195 **Lemma 14.23.1.** *The functor s is exact.*

Proof. Clear from Lemma 14.22.1. □

0196 **Lemma 14.23.2.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 14.22.4. Then the complex $s(E)$ is acyclic.*

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occurring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\mathrm{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum\nolimits_{i=0}^{n+1} (-1)^i x_{\alpha' \circ \delta_i^{n+1}}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum\nolimits_{i=0}^{n} (-1)^i x_{(\alpha \circ \delta_i^n)'}$$

It is easy to see that $\alpha' \circ \delta_i^{n+1} = (\alpha \circ \delta_i^n)'$ for $i = 0, \ldots, n$. It is also easy to see that $\alpha' \circ \delta_{n+1}^{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \mathrm{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by

the system of maps $h'_n = (-1)^{n+1}h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 12.13.5. □

0197 **Lemma 14.23.3.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A,k))) = A$ if $i = k$ and 0 else.*

Proof. First, let us prove this if $k = 0$. In this case we have $K(A,0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A,0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^{n}(-1)^i\mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A,0))$ looks like this

$$\ldots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A,k) \to E \to K(A,k+1) \to 0$$

from Lemma 14.22.4. By Lemma 14.22.1 the associated sequence of chain complexes is exact. By Lemma 14.23.2 we see that $s(E)$ is acyclic. Hence the result for $k+1$ follows from the long exact sequence of homology, see Homology, Lemma 12.13.6. □

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 14.18.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1}\mathrm{Ker}(d_i^m)$. We define the *normalized chain complex* $N(U)$ to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 14.18.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

0198 **Lemma 14.23.4.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.*

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum(-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. □

0199 **Lemma 14.23.5.** *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A,k))_i = A$ if $i = k$ and 0 else.*

Proof. It is clear that $N(K(A,k))_i = 0$ when $i < k$ because $K(A,k)_i = 0$ in that case. It is clear that $N(K(A,k))_k = A$ since $K(A,k)_{k-1} = 0$ and $K(A,k)_k = A$. For $i > k$ we have $N(K(A,k))_i = 0$ by Lemma 14.21.9 and the definition of $K(A,k)$, see Definition 14.22.3. □

019A **Lemma 14.23.6.** *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,*

$$s(U) = N(U) \oplus D(U)$$

for some complex $D(U)$. The construction $U \mapsto D(U)$ is functorial.

Proof. Define $D(U)_n$ to be the image of

$$\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective},\ m<n} N(U_m) \xrightarrow{\bigoplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 14.18.6 and Definition 14.18.1. We show that $D(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi\circ\delta_i^n)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta_i^n)$ map $N(U_m)$ into $D(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta_i^n : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta_i^n)$ is contained in $D(U)_{n-1}$ by definition. If $\varphi \circ \delta_i^n : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta_i^n = \delta_j^m \circ \psi \circ \delta_i^n$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta_i^n) = U(\psi \circ \delta_i^n) \circ d_j^m$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d_m^m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi \circ \delta_i^n)(N(U_m)) \subset U(\psi \circ \delta_i^n)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $D(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma_j^{n-1}$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s_j^{n-1}$. Thus the composition is given by the sum

$$\sum (-1)^i d_i^n \circ s_j^{n-1}$$

Recall from Remark 14.3.3 that $d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1} = \mathrm{id}$ and these drop out because the corresponding terms have opposite signs. The map $d_n^n \circ s_j^{n-1}$, if $j < n-1$, is equal to $s_j^{n-2} \circ d_{n-1}^{n-1}$. Since d_{n-1}^{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d_n^n(s_j^{n-1}(N(U_{n-1}))$ is contained in $s_j^{n-2}(N(U_{n-2}))$ which is contained in $D(U_{n-1})$ by definition. For all other combinations of (i, j) we have either $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$ (if $i < j$), or $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. □

0FKI **Remark 14.23.7.** In the situation of Lemma 14.23.6 the subcomplex $D(U) \subset s(U)$ can also be defined as the subcomplex with terms

$$D(U)_n = \mathrm{Im}\left(\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective},\ m<n} U_m \xrightarrow{\bigoplus U(\varphi)} U_n\right)$$

Namely, since U_m is the direct sum of the subobject $N(U_m)$ and the images of $N(U_k)$ for surjections $[m] \to [k]$ with $k < m$ this is clearly the same as the definition of $D(U)_n$ given in the proof of Lemma 14.23.6. Thus we see that if U is a simplicial abelian group, then elements of $D(U)_n$ are exactly the sums of degenerate n-simplices.

019B **Lemma 14.23.8.** *The functor N is exact.*

Proof. By Lemma 14.23.1 and the functorial decomposition of Lemma 14.23.6. □

019C **Lemma 14.23.9.** *Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism. In other words, the complex $D(V)$ of Lemma 14.23.6 is acyclic.*

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$, by Lemmas 14.23.3 and 14.23.5. Consider the hypothesis $IH_{n,m}$: for all V such that $V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism $H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case $N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$. By Lemma 14.22.2 and Definition 14.22.3 we have $K(V_m, m) = i_{m!}\mathrm{sk}_m V$. By Lemma 14.21.10 the natural morphism

$$K(V_m, m) = i_{m!}\mathrm{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a morphism of short exact sequence of associated complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(K(V_m, m)) & \longrightarrow & N(V) & \longrightarrow & N(W) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & s(K(V_m, m)) & \longrightarrow & s(V) & \longrightarrow & s(W) & \longrightarrow & 0 \end{array}$$

see Lemmas 14.23.1 and 14.23.8. Hence we deduce the result for V from the result on the ends. □

14.24. Dold-Kan

019D In this section we prove the Dold-Kan theorem relating simplicial objects in an abelian category with chain complexes.

019E **Lemma 14.24.1.** *Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects isomorphisms, injections and surjections.*

Proof. The faithfulness is immediate from the canonical splitting of Lemma 14.18.6. The statement on reflecting injections, surjections, and isomorphisms follows from Lemma 14.18.7. □

019F **Lemma 14.24.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. Suppose that*

(1) *the functors S and N are exact,*
(2) *there is an isomorphism $g : N \circ S \to id_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,*
(3) *N is faithful, and*
(4) *S is essentially surjective.*

Then S and N are quasi-inverse equivalences of categories.

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired isomorphism $S \circ N \to \mathrm{id}_{\mathcal{A}}$. □

019G **Theorem 14.24.3.** *Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 14.22.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \Big\{ \alpha : [n] \to \{0, 1, 2, \ldots\} \mid \mathrm{Im}(\alpha) = [k] \text{ for some } k \Big\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\mathrm{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

(1) $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of "α" as a basis vector for the summand corresponding to it,
(2) given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows
   (a) $\alpha \circ \varphi \notin I_m$ then we set it equal to zero,
   (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,
   (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and
   (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

Let us show that $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$. To do this, assume we have maps $\varphi : [m] \to [n]$ and $\psi : [n] \to [p]$. We will show that $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi) = S(A_\bullet)(\psi \circ \varphi)$. Choose $\beta \in I_p$ and set $\alpha = \beta \circ \psi$ and $\gamma = \alpha \circ \varphi$ viewed as maps $\alpha : [n] \to \{0, 1, 2, \ldots\}$ and $\gamma : [m] \to \{0, 1, 2, \ldots\}$. Picture

$$\begin{array}{ccccc} [m] & \xrightarrow[\varphi]{} & [n] & \xrightarrow[\psi]{} & [p] \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathrm{Im}(\gamma) & \longrightarrow & \mathrm{Im}(\alpha) & \longrightarrow & [k(\beta)] \end{array}$$

We will show that the restriction of the maps $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi)$ and $S(A_\bullet)(\psi \circ \varphi)$. to the summand $A_{k(\beta)} \cdot \beta$ agree. There are several cases to consider

(1) Say $\alpha \notin I_n$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] = \mathrm{Im}(\gamma) \subset \mathrm{Im}(\alpha) \subset [k(\beta)]$ and the subset $\mathrm{Im}(\alpha)$ of $[k(\beta)]$ has a gap so $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.
(2) Say $\alpha \in I_n$ and $k(\alpha) < k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] \subset [k(\alpha)] \subset [k(\beta)]$ and it follows that $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.
(3) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta)$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is the identity map from $A_{k(\beta)} \cdot \beta$ to $A_{k(\alpha)} \cdot \alpha$. In this case because $\mathrm{Im}(\alpha) = [k(\beta)]$ the rule describing the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is exactly the same as the rule describing the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ and hence agreement holds.
(4) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\alpha)} \cdot \alpha$. Subcases
   (a) If $\gamma \notin I_m$, then both the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ are zero and we get agreement.
   (b) If $\gamma \in I_m$ but $k(\gamma) < k(\alpha) - 1$, then again both restrictions are zero and we get agreement.
   (c) If $\gamma \in I_m$ and $k(\gamma) = k(\alpha)$ then $\mathrm{Im}(\gamma) = \mathrm{Im}(\alpha)$. In this case the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\gamma)} \cdot \gamma$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is the identity map $A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$. Hence agreement holds.
   (d) Finally, if $\gamma \in I_m$ and $k(\gamma) = k(\alpha) - 1$ then the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is given by $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ as a map $A_{k(\alpha)} \cdot \alpha \to A_{k(\beta)} \cdot \beta$. Since $A_\bullet$ is a complex we see that the composition $A_{k(\beta)} \cdot \beta \to A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$ is zero which matches what we get for the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ because $k(\gamma) = k(\beta) - 2 < k(\beta) - 1$.

Thus $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$.

Let us construct an isomorphism $A_\bullet \to N(S(A_\bullet))$ functorial in $A_\bullet$. Recall that

$$S(A_\bullet) = N(S(A_\bullet)) \oplus D(S(A_\bullet))$$

as chain complexes by Lemma 14.23.6. On the other hand it follows from Remark 14.23.7 and the construction of $S(A_\bullet)$ that

$$D(S(A_\bullet))_n = \bigoplus\nolimits_{\alpha \in I_n,\ k(\alpha) < n} A_{k(\alpha)} \cdot \alpha \subset \bigoplus\nolimits_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$$

However, if $\alpha \in I_n$ then we have $k(\alpha) \geq n \Leftrightarrow \alpha = \mathrm{id}_{[n]} : [n] \to [n]$. Thus the summand $A_n \cdot \mathrm{id}_{[n]}$ of $S(A_\bullet)_n$ is a complement to the summand $D(S(A_\bullet))_n$. All the maps $d_i^n : S(A_\bullet)_n \to S(A_\bullet)_n$ restrict to zero on the summand $A_n \cdot \mathrm{id}_{[n]}$ except for d_n^n which produces $(-1)^n d_{A,n}$ from $A_n \cdot \mathrm{id}_{[n]}$ to $A_{n-1} \cdot \mathrm{id}_{[n-1]}$. We conclude that $A_n \cdot \mathrm{id}_{[n]}$ must be equal to the summand $N(S(A_\bullet))_n$ and moreover the restriction of the differential $d_n = \sum (-1)^i d_i^n : S(A_\bullet)_n \to S(A_\bullet)_{n-1}$ to the summand $A_n \cdot \mathrm{id}_{[n]}$ gives what we want!

Finally, we have to show that $S \circ N$ is isomorphic to the identity functor. Let U be a simplicial object of $\mathcal{A}$. Then we can define an obvious map

$$S(N(U))_n = \bigoplus\nolimits_{\alpha \in I_n} N(U)_{k(\alpha)} \cdot \alpha \longrightarrow U_n$$

by using $U(\alpha) : N(U)_{k(\alpha)} \to U_n$ on the summand corresponding to α. By Definition 14.18.1 this is an isomorphism. To finish the proof we have to show that this is compatible with the maps in the simplicial objects. Thus let $\varphi : [m] \to [n]$ and let $\alpha \in I_n$. Set $\beta = \alpha \circ \varphi$. Picture

$$\begin{array}{ccc} [m] & \xrightarrow{\varphi} & [n] \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathrm{Im}(\beta) & \longrightarrow & [k(\alpha)] \end{array}$$

There are several cases to consider

(1) Say $\beta \notin I_m$. Then there exists an index $0 \leq j < k(\alpha)$ with $j \notin \mathrm{Im}(\alpha \circ \varphi)$ and hence we can choose a factorization $\alpha \circ \varphi = \delta^{k(\alpha)}_j \circ \psi$ for some $\psi : [m] \to [k(\alpha)-1]$. It follows that $U(\varphi)$ is zero on the image of the summand $N(U)_{k(\alpha)} \cdot \alpha$ because $U(\varphi) \circ U(\alpha) = U(\alpha \circ \varphi) = U(\psi) \circ d^{k(\alpha)}_j$ is zero on $N(U)_{k(\alpha)}$ by construction of N. This matches our rule for $S(N(U))$ given above.

(2) Say $\beta \in I_m$ and $k(\beta) < k(\alpha) - 1$. Here we argue exactly as in case (1) with $j = k(\alpha) - 1$.

(3) Say $\beta \in I_m$ and $k(\beta) = k(\alpha)$. Here the summand $N(U)_{k(\alpha)} \cdot \alpha$ is mapped by the identity to the summand $N(U)_{k(\beta)} \cdot \beta$. This is the same as the effect of $U(\varphi)$ since in this case $U(\varphi) \circ U(\alpha) = U(\beta)$.

(4) Say $\beta \in I_m$ and $k(\beta) = k(\alpha)-1$. Here we use the differential $(-1)^{k(\alpha)} d_{N(U),k(\alpha)}$ to map the summand $N(U)_{k(\alpha)} \cdot \alpha$ to the summand $N(U)_{k(\beta)} \cdot \beta$. On the other hand, since $\mathrm{Im}(\beta) = [k(\beta)]$ in this case we get $\alpha \circ \varphi = \delta^{k(\alpha)}_{k(\alpha)} \circ \beta$. Thus we see that $U(\varphi)$ composed with the restriction of $U(\alpha)$ to $N(U)_{k(\alpha)}$ is equal to $U(\beta)$ precomposed with $d^{k(\alpha)}_{k(\alpha)}$ restricted to $N(U)_{k(\alpha)}$. Since $d_{N(U),k(\alpha)} = \sum (-1)^i d^{k(\alpha)}_i$ and since $d^{k(\alpha)}_i$ restricts to zero on $N(U)_{k(\alpha)}$ for $i < k(\alpha)$ we see that equality holds.

This finishes the proof of the theorem. □

14.25. Dold-Kan for cosimplicial objects

019H Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 12.5.2 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\mathrm{CoSimp}(\mathcal{A}) = \mathrm{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 14.24.3) implies that $\mathrm{CoSimp}(\mathcal{A})$ is equivalent to the category $\mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\mathrm{CoCh}_{\geq 0}(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \mathrm{CoSimp}(\mathcal{A}) \longrightarrow \mathrm{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the *cochain complex $s(U)$ associated to a cosimplicial object U*. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1 - \delta_1^1} U_1 \xrightarrow{\delta_0^2 - \delta_1^2 + \delta_2^2} U_2 \longrightarrow \ldots$$

This is sometimes also called the *Moore complex* associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \text{Coker}(\textstyle\bigoplus_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1}\delta_{n+1}^{n+1}$, i.e., by fitting the morphism $(-1)^{n+1}\delta_{n+1}^{n+1}$ into a commutative diagram

$$\begin{array}{ccc} U_n & \xrightarrow[(-1)^{n+1}\delta_{n+1}^{n+1}]{} & U_{n+1} \\ \downarrow & & \downarrow \\ Q(U)^n & \xrightarrow{d_n} & Q(U)^{n+1}. \end{array}$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ_i^n maps into the kernel of the right vertical arrow for $i = 0, \ldots, n-1$. (This is dual to Lemma 14.18.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \ldots$$

This is called the *normalized cochain complex associated to U*. The dual to the Dold-Kan Theorem 14.24.3 is the following.

019I **Lemma 14.25.1.** *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $s : CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$ is exact.*
(2) *The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.*
(3) *There exists a functorial direct sum decomposition $s(U) = D(U) \oplus Q(U)$ in $CoCh_{\geq 0}(\mathcal{A})$.*
(4) *The functor Q is exact.*
(5) *The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.*
(6) *The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$.*

Proof. Omitted. But the results are the exact dual statements to Lemmas 14.23.1, 14.23.4, 14.23.6, 14.23.8, 14.23.9, and Theorem 14.24.3. □

14.26. Homotopies

019J Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 14.13.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

019K **Definition 14.26.1.** Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.

(1) We say a morphism

$$h : U \times \Delta[1] \longrightarrow V$$

is a *homotopy from a to b* if $a = h \circ e_0$ and $b = h \circ e_1$.

(2) We say the morphisms a and b are *homotopic* or are *in the same homotopy class* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}.

The relation "there is a homotopy from a to b" is in general not transitive or symmetric; we will see it is reflexive in Example 14.26.3. Of course, "being homotopic" is an equivalence relation on the set $\mathrm{Mor}(U, V)$ and it is the equivalence relation generated by the relation "there is a homotopy from a to b" . It turns out we can define homotopies between pairs of maps of simplicial objects in any category. We will do this in Remark 14.26.4 after we work out in some detail what it means to have a morphism $h : U \times \Delta[1] \to V$.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \ldots, n+1$.

019L **Lemma 14.26.2.** *In the situation above, we have the following relations:*

(1) *We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.*
(2) *We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.*
(3) *We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.*
(4) *We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.*
(5) *We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.*

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphism h which is a homotopy from a to b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 14.2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d_j^n and s_j^n. □

07KA **Example 14.26.3.** Suppose in the situation above $a = b$. Then there is a *trivial* homotopy from a to b, namely the one with $h_{n,i} = a_n = b_n$.

019M **Remark 14.26.4.** Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{C}$. A *homotopy from a to b* is given by morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 14.26.2. As in Definition 14.26.1 we say the morphisms a and b are *homotopic* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the original notion whenever both categories have finite coproducts.

08RJ **Remark 14.26.5.** Let $\mathcal{C}$ be any category. Suppose two morphisms $a, a' : U \to V$ of simplicial objects are homotopic. Then for any morphism $b : V \to W$ the two maps $b \circ a, b \circ a' : U \to W$ are homotopic. Similarly, for any morphism $c : X \to U$ the two maps $a \circ c, a' \circ c : X \to V$ are homotopic. In fact the maps $b \circ a \circ c, b \circ a' \circ c : X \to W$ are homotopic. Namely, if the maps $h_{n,i} : U_n \to V_n$ define a homotopy from a to a' then the maps $b \circ h_{n,i} \circ c$ define a homotopy from $b \circ a \circ c$ to $b \circ a' \circ c$. In this way we see that we obtain a new category $\mathrm{hSimp}(\mathcal{C})$ with the same objects as $\mathrm{Simp}(\mathcal{C})$ but whose morphisms are homotopy classes of morphisms of $\mathrm{Simp}(\mathcal{C})$. Thus there is a canonical functor

$$\mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{hSimp}(\mathcal{C})$$

which is essentially surjective and surjective on sets of morphisms.

019N **Definition 14.26.6.** Let U and V be two simplicial objects of a category $\mathcal{C}$. We say a morphism $a : U \to V$ is a *homotopy equivalence* if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. We say U and V are *homotopy equivalent* if there exists a homotopy equivalence $a : U \to V$.

08Q3 **Example 14.26.7.** The simplicial set $\Delta[m]$ is homotopy equivalent to $\Delta[0]$. Namely, consider the unique morphism $f : \Delta[m] \to \Delta[0]$ and the morphism $g : \Delta[0] \to \Delta[m]$ given by the inclusion of the last 0-simplex of $\Delta[m]$. We have $f \circ g = \mathrm{id}$. We will give a homotopy $h : \Delta[m] \times \Delta[1] \to \Delta[m]$ from $\mathrm{id}_{\Delta[m]}$ to $g \circ f$. Namely h is given by the maps

$$\mathrm{Mor}_\Delta([n], [m]) \times \mathrm{Mor}_\Delta([n], [1]) \to \mathrm{Mor}_\Delta([n], [m])$$

which send (φ, α) to

$$k \mapsto \begin{cases} \varphi(k) & \text{if} \quad \alpha(k) = 0 \\ m & \text{if} \quad \alpha(k) = 1 \end{cases}$$

Note that this only works because we took g to be the inclusion of the last 0-simplex. If we took g to be the inclusion of the first 0-simplex we could find a homotopy from $g \circ f$ to $\mathrm{id}_{\Delta[m]}$. This is an illustration of the asymmetry inherent in homotopies in the category of simplicial sets.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

[1] In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 14.26.2.

019O **Lemma 14.26.8.** *Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 14.13.3.*

(1) *We have $\pi \circ e_1 = \pi \circ e_0 = id_U$, and*
(2) *The morphisms $id_{U \times \Delta[1]}$, and $e_0 \circ \pi$ are homotopic.*
(3) *The morphisms $id_{U \times \Delta[1]}$, and $e_1 \circ \pi$ are homotopic.*

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 14.26, we have $\beta = \beta_1$ if $\beta_2 = \alpha^n_0$ and $\beta = \alpha^n_{n+1}$ if $\beta_2 = \alpha^n_{n+1}$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 14.13.3 is a homotopy from $\mathrm{id}_{U \times \Delta[1]}$ to $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). □

019P **Lemma 14.26.9.** *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 14.3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \ldots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.*

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$\begin{aligned} Y \times_X \ldots \times_X Y \times \mathrm{Mor}([n], [1]) &\to Y \times_X \ldots \times_X Y \\ (y_0, \ldots, y_n) \times \alpha &\mapsto (g^{\alpha(0)}(y_0), \ldots, g^{\alpha(n)}(y_n)) \end{aligned}$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satisfy the relations of Lemma 14.26.2. Hence they define the desired homotopy. See also Remark 14.26.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. □

08Q4 **Lemma 14.26.10.** *Let $\mathcal{C}$ be a category. Let T be a set. For $t \in T$ let X_t, Y_t be simplicial objects of $\mathcal{C}$. Assume $X = \prod_{t \in T} X_t$ and $Y = \prod_{t \in T} Y_t$ exist.*

(1) *If X_t and Y_t are homotopy equivalent for all $t \in T$ and T is finite, then X and Y are homotopy equivalent.*

For $t \in T$ let $a_t, b_t : X_t \to Y_t$ be morphisms. Set $a = \prod a_t : X \to Y$ and $b = \prod b_t : X \to Y$.

(2) *If there exists a homotopy from a_t to b_t for all $t \in T$, then there exists a homotopy from a to b.*
(3) *If T is finite and $a_t, b_t : X_t \to Y_t$ for $t \in T$ are homotopic, then a and b are homotopic.*

Proof. If $h_t = (h_{t,n,i})$ is a homotopy from a_t to b_t (see Remark 14.26.4), then $h = (\prod_t h_{t,n,i})$ is a homotopy from $\prod a_t$ to $\prod b_t$. This proves (2).

Proof of (3). Choose $t \in T$. There exists an integer $n \geq 0$ and a chain $a_t = a_{t,0}, a_{t,1}, \ldots, a_{t,n} = b_t$ such that for every $1 \leq i \leq n$ either there is a homotopy from $a_{t,i-1}$ to $a_{t,i}$ or there is a homotopy from $a_{t,i}$ to $a_{t,i-1}$. If $n = 0$, then we pick another t. (We're done if $a_t = b_t$ for all $t \in T$.) So assume $n > 0$. By Example 14.26.3 there are is a homotopy from $b_{t'}$ to $b_{t'}$ for all $t' \in T \setminus \{t\}$. Thus by (2) there is a homotopy from $a_{t,n-1} \times \prod_{t'} b_{t'}$ to b or there is a homotopy from b to $a_{t,n-1} \times \prod_{t'} b_{t'}$. In this way we can decrease n by 1. This proves (3).

Part (1) follows from part (3) and the definitions. □

14.27. Homotopies in abelian categories

019Q Let $\mathcal{A}$ be an additive category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. Let us prove the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$ are homotopic in the sense of Homology, Section 12.13. Using the notation introduced in Section 14.26 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

019R (14.27.0.1) $$s(h)_n = \sum\nolimits_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n.$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$\begin{aligned}
d_{n+1} \circ s(h)_n &= \sum\nolimits_{j=0}^{n+1} \sum\nolimits_{i=0}^{n} (-1)^{j+i+1} d_j^{n+1} \circ h_{n+1,i+1} \circ s_i^n \\
&= \sum\nolimits_{1 \leq i+1 \leq j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ d_j^{n+1} \circ s_i^n \\
&\quad + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j+1} h_{n,i} \circ d_j^{n+1} \circ s_i^n \\
&= \sum\nolimits_{1 \leq i+1 < j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ s_i^{n-1} \circ d_{j-1}^n \\
&\quad + \sum\nolimits_{1 \leq i+1 = j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \\
&\quad + \sum\nolimits_{n \geq i = j \geq 0} (-1)^{i+j+1} h_{n,i} \\
&\quad + \sum\nolimits_{n \geq i > j \geq 0} (-1)^{i+j+1} h_{n,i} \circ s_{i-1}^{n-1} \circ d_j^n
\end{aligned}$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s_i^{n-1} \circ d_j^n.$$

Hence we obtain

$$\begin{aligned}
d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\
&= h_{n,n+1} - h_{n,0} \\
&= a_n - b_n
\end{aligned}$$

as desired.

019S **Lemma 14.27.1.** *Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of chain complexes. If $\mathcal{A}$ is abelian, then also $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.*

Proof. We may choose a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ either there is a homotopy from a_i to a_{i-1} or there is a homotopy from a_{i-1} to a_i. The calculation above shows that in this case either $s(a_i)$ is homotopic to $s(a_{i-1})$ as a map of chain complexes or $s(a_{i-1})$ is homotopic to $s(a_i)$ as a map of chain complexes. Of course, these things are equivalent and moreover being homotopic is an equivalence relation on the set of maps of chain complexes, see Homology, Section 12.13. This proves that $s(a)$ and $s(b)$ are homotopic as maps of chain complexes.

Next, we turn to $N(a)$ and $N(b)$. It follows from Lemma 14.23.6 that $N(a)$, $N(b)$ are compositions

$$N(U) \to s(U) \to s(V) \to N(V)$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 12.13.1. □

019T **Lemma 14.27.2.** *Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $N(a) : N(U) \to N(V)$ is a homotopy equivalence of chain complexes.*

Proof. Omitted. See Lemma 14.27.1 above. □

14.28. Homotopies and cosimplicial objects

019U Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $\mathrm{Hom}(\Delta[1], V)$, see Section 14.14. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$.

019W **Definition 14.28.1.** Let $\mathcal{C}$ be a category having finite products. Let U and V be two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms of cosimplicial objects of $\mathcal{C}$.

(1) We say a morphism
$$h : U \longrightarrow \mathrm{Hom}(\Delta[1], V)$$
such that $a = e_0 \circ h$ and $b = e_1 \circ h$ is a *homotopy from a to b*.
(2) We say a and b are *homotopic* or are *in the same homotopy class* if there exists a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_i to a_{i-1} or there exists a homotopy from a_{i-1} to a_i.

This is dual to the notion we introduced for simplicial objects in Section 14.26. To explain this, consider a homotopy $h : U \to \mathrm{Hom}(\Delta[1], V)$ from a to b as in the definition. Recall that $\Delta[1]_n$ is a finite set. The degree n component of h is a morphism

$$h_n = (h_{n,\alpha}) : U \longrightarrow \mathrm{Hom}(\Delta[1], V)_n = \prod\nolimits_{\alpha \in \Delta[1]_n} V_n$$

The morphisms $h_{n,\alpha} : U_n \to V_n$ of $\mathcal{C}$ have the property that for every morphism $f : [n] \to [m]$ of Δ we have

07KB (14.28.1.1) $$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f}$$

Moreover, the condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n]\to[1]}$ where $0 : [n] \to [1]$ is the constant map with value 0. Similarly, the condition that $b = e_1 \circ h$ means that $b_n = h_{n,1:[n]\to[1]}$ where $1 : [n] \to [1]$ is the constant map with value 1. Conversely, given a family of morphisms $\{h_{n,\alpha}\}$ such that (14.28.1.1) holds for all morphisms f of Δ and such that $a_n = h_{n,0:[n]\to[1]}$ and $b_n = h_{n,1:[n]\to[1]}$ for all $n \geq 0$, then we obtain a homotopy h from a to b by setting $h = \prod_{\alpha \in \Delta[1]_n} h_{n,\alpha}$.

0FKJ **Remark 14.28.2.** Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. A *homotopy from a to b* is given by morphisms $h_{n,\alpha} : U_n \to V_n$, for $n \geq 0$, $\alpha \in \Delta[1]_n$ satisfying (14.28.1.1) for all morphisms f of Δ and such that $a_n = h_{n,0:[n]\to[1]}$ and $b_n = h_{n,1:[n]\to[1]}$ for all $n \geq 0$. As in Definition 14.28.1 we say the morphisms a and b are *homotopic* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. This new notion is the same as the old one in case finite products exist. We deduce in particular that functors preserve the original notion whenever both categories have finite products.

019X **Lemma 14.28.3.** *Let $\mathcal{C}$ be a category. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent*

(1) *There exists a homotopy $h = \{h_{n,\alpha}\}$ from a to b as in Remark 14.28.2.*
(2) *There exists a homotopy $h = \{h_{n,i}\}$ from a' to b' as in Remark 14.26.4.*

Thus a is homotopic to b as in Remark 14.28.2 if and only if a' is homotopic to b' as in Remark 14.26.4.

Proof. In case $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts and we may use Definitions 14.28.1 and 14.26.1 instead of Remarks 14.28.2 and 14.26.4. In this case $h : U \to \mathrm{Hom}(\Delta[1], V)$ is the same as a morphism $h' : \mathrm{Hom}(\Delta[1], V)' \to U'$. Since products and coproducts get switched too, it is immediate that $(\mathrm{Hom}(\Delta[1], V))' = V' \times \Delta[1]$. Moreover, the "primed" version of the morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$ are the morphisms $e_0, e_1 : V' \to \Delta[1] \times V$. Thus $e_0 \circ h = a$ translates into $h' \circ e_0 = a'$ and similarly $e_1 \circ h = b$ translates into $h' \circ e_1 = b'$. This proves the lemma in this case.

In the general case, one needs to translate the relations given by (14.28.1.1) into the relations given in Lemma 14.26.2. We omit the details.

The final assertion is formal from the equivalence of (1) and (2). □

019Y **Lemma 14.28.4.** *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be categories. With terminology as in Remarks 14.28.2 and 14.26.4.*

(1) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.*
(2) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.*
(3) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.*
(4) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.*

Proof. By Lemma 14.28.3 above, we can turn F into a covariant functor between a pair of categories, and we have to show that the functor preserves homotopic pairs of maps. This is explained in Remark 14.26.4. □

019Z **Lemma 14.28.5.** *Let $f : X \to Y$ be a morphism of a category $\mathcal{C}$ with pushouts. Assume there is a morphism $s : Y \to X$ with $s \circ f = id_X$. Consider the cosimplicial object U constructed in Example 14.5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $f \circ s$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.*

Proof. This lemma is dual to Lemma 14.26.9. Hence this lemma follows on applying Lemma 14.28.3. □

01A0 **Lemma 14.28.6.** *Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of cochain complexes. If in addition $\mathcal{A}$ is abelian, then $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.*

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 14.28.5 the maps a' and b' are homotopic. By Lemma 14.27.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows in the same manner using that $Q(U)' = N(U')$. □

0FKK **Lemma 14.28.7.** *Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of cosimplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $Q(a) : Q(U) \to Q(V)$ is a homotopy equivalence of chain complexes.*

Proof. Omitted. See Lemma 14.28.6 above. □

14.29. More homotopies in abelian categories

01A1 Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 14.27.1. We first develop some material on homotopies between morphisms of chain complexes.

01A2 **Lemma 14.29.1.** *Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor*

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $\mathrm{Mor}_{Ch(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A,n}$ by the matrix

$$d_{\diamond A,n} = \begin{pmatrix} d_{A,n} & 0 & \mathrm{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\mathrm{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A,n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps $\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$\begin{aligned} f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\ &= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\ &= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z)) \end{aligned}$$

which are the same by definition of a homotopy. □

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 12.14.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

01A3 **Lemma 14.29.2.** *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of

complexes, see Homology, Lemma 12.14.4. If $\delta(s)$ factors through the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A \oplus A & \longrightarrow & \diamond A & \longrightarrow & A[-1] & \longrightarrow & 0 \end{array}$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 12.14.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1, -1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian groups. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$ and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$\begin{aligned} (p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) &= (p_n, q_n, f_n \circ r_n)(\\ &\qquad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\ &= (f_n(x_3), -f_n(x_3), f_n(d_n(x_3))) \end{aligned}$$

by definition of f_n. And

$$\begin{aligned} d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) &= d_n(0, 0, f_n(x_3)) \\ &= (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3))) \end{aligned}$$

The result follows as f is a morphism of complexes. □

01A4 **Lemma 14.29.3.** *Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then there exists a homotopy from a to b as in Definition 14.26.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 14.27.*

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 14.29.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (14.27.0.1) and Lemma 14.27.1 with $h = \mathrm{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $s : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus

it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 14.26.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\mathrm{id}_{U \times \Delta[1]}$. By Lemma 14.27.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 14.29.2 above we get a corresponding morphism

$$\Phi : \diamond s(U) \to s(U \times \Delta[1])$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$h_{n-1} = \sum\nolimits_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus\nolimits_j U_n \cdot \alpha_j^n.$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) : U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n = \bigoplus_{\alpha:[n]\to[1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^n U_n \cdot \alpha_i^n$ with α_i^n as in Section 14.26. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U \oplus U & \longrightarrow & U \times \Delta[1] & \longrightarrow & W & \longrightarrow & 0 \\ & & & \searrow^{(1,1)} & \downarrow \pi & & & & \\ & & & & U & & & & \end{array}$$

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U \times \Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1, 1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1, 1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 14.29.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 12.14.4 and its proof, to do this we have compute

$$d_{s(U \times \delta[1]),n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W),n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U \times \delta[1]),n} = \sum\nolimits_{i=0}^n (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U \times \delta[1]),n}(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U \times \delta[1]),n} \circ \sigma_n)(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha_0^n \circ \delta_i^n}$$

To compute $d_{s(W),n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta_i^n = \alpha_0^{n-1}, \alpha_n^{n-1}$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W),n})(x_\alpha) = \sum_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n \neq \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}}\right)$$

Clearly the difference of the two terms is the sum

$$\sum_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n = \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}}\right)$$

Of course, if $\alpha \circ \delta_i^n = \alpha_0^{n-1}$ then the term drops out. Recall that $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha_j^n \circ \delta_i^n = \alpha_n^{n-1}$ is if $j = n$ and $i = n$. Thus we actually get 0 unless $j = n$ and in that case we get $(-1)^n (d_n^n(x))_{\alpha_n^{n-1}} - (-1)^n (d_n^n(x))_{\alpha_0^{n-1}}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha_n^n$ and on that summand it is equal to $((-1)^n d_n^n, -(-1)^n d_n^n)$. (Namely, the first summand of the two corresponds to the factor with α_n^{n-1} because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \Phi & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & s(U \times \Delta[1]) & \longrightarrow & s(W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \Psi & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0
\end{array}$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d_0^n, -(-1)^n d_0^n) \circ (-1)^n s_n^n = (1, -1)$ as desired. □

14.30. Trivial Kan fibrations

08NK Recall that for $n \geq 0$ the simplicial set $\Delta[n]$ is given by the rule $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$, see Example 14.11.2. Recall that $\Delta[n]$ has a unique nondegenerate n-simplex and all nondegenerate simplices are faces of this n-simplex. In fact, the nondegenerate simplices of $\Delta[n]$ correspond exactly to injective morphisms $[k] \to [n]$, which we may identify with subsets of $[n]$. Moreover, recall that $\mathrm{Mor}(\Delta[n], X) = X_n$ for any simplicial set X (Lemma 14.11.3). We set

$$\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$$

and we call it the *boundary* of $\Delta[n]$. From Lemma 14.21.5 we see that $\partial\Delta[n] \subset \Delta[n]$ is the simplicial subset having the same nondegenerate simplices in degrees $\leq n-1$ but not containing the nondegenerate n-simplex.

08NL **Definition 14.30.1.** A map $X \to Y$ of simplicial sets is called a *trivial Kan fibration* if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and any commutative solid diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

a dotted arrow exists making the diagram commute.

A trivial Kan fibration satisfies a very general lifting property.

08NM **Lemma 14.30.2.** *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. For any solid commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{b} & X \\ \downarrow & \nearrow & \downarrow \\ W & \xrightarrow{a} & Y \end{array}$$

of simplicial sets with $Z \to W$ (termwise) injective a dotted arrow exists making the diagram commute.

Proof. Suppose that $Z \neq W$. Let n be the smallest integer such that $Z_n \neq W_n$. Let $x \in W_n$, $x \notin Z_n$. Denote $Z' \subset W$ the simplicial subset containing Z, x, and all degeneracies of x. Let $\varphi : \Delta[n] \to Z'$ be the morphism corresponding to x (Lemma 14.11.3). Then $\varphi|_{\partial\Delta[n]}$ maps into Z as all the nondegenerate simplices of $\partial\Delta[n]$ end up in Z. By assumption we can extend $b \circ \varphi|_{\partial\Delta[n]}$ to $\beta : \Delta[n] \to X$. By Lemma 14.21.7 the simplicial set Z' is the pushout of $\Delta[n]$ and Z along $\partial\Delta[n]$. Hence b and β define a morphism $b' : Z' \to X$. In other words, we have extended the morphism b to a bigger simplicial subset of Z.

The proof is finished by an application of Zorn's lemma (omitted). □

08NN **Lemma 14.30.3.** *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. □

08NP **Lemma 14.30.4.** *The composition of two trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. □

08NQ **Lemma 14.30.5.** *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of trivial Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U^t_n$. Then $U \to U^0$ is a trivial Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. □

08NR **Lemma 14.30.6.** *Let $X_i \to Y_i$ be a set of trivial Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a trivial Kan fibration.*

Proof. Omitted. □

08Q5 **Lemma 14.30.7.** *A filtered colimit of trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. Hint: See description of filtered colimits of sets in Categories, Section 4.19. □

08NS **Lemma 14.30.8.** *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.*

Proof. By Lemma 14.30.2 we can choose an right inverse $g : Y \to X$ to f. Consider the diagram

$$\begin{array}{ccc} \partial\Delta[1] \times X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[1] \times X & \longrightarrow & Y \end{array}$$

Here the top horizontal arrow is given by id_X and $g \circ f$ where we use that $(\partial\Delta[1] \times X)_n = X_n \amalg X_n$ for all $n \geq 0$. The bottom horizontal arrow is given by the map $\Delta[1] \to \Delta[0]$ and $f : X \to Y$. The diagram commutes as $f \circ g \circ f = f$. By Lemma 14.30.2 we can fill in the dotted arrow and we win. □

14.31. Kan fibrations

08NT Let n, k be integers with $0 \leq k \leq n$ and $1 \leq n$. Let $\sigma_0, \ldots, \sigma_n$ be the $n+1$ faces of the unique nondegenerate n-simplex σ of $\Delta[n]$, i.e., $\sigma_i = d_i\sigma$. We let

$$\Lambda_k[n] \subset \Delta[n]$$

be the *kth horn* of the n-simplex $\Delta[n]$. It is the simplicial subset of $\Delta[n]$ generated by $\sigma_0, \ldots, \hat{\sigma}_k, \ldots, \sigma_n$. In other words, the image of the displayed inclusion contains all the nondegenerate simplices of $\Delta[n]$ except for σ and σ_k.

08NU **Definition 14.31.1.** A map $X \to Y$ of simplicial sets is called a *Kan fibration* if for all k, n with $1 \leq n$, $0 \leq k \leq n$ and any commutative solid diagram

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

a dotted arrow exists making the diagram commute. A *Kan complex* is a simplicial set X such that $X \to *$ is a Kan fibration, where $*$ is the constant simplicial set on a singleton.

Note that $\Lambda_k[n]$ is always nonempty. Thus a morphism from the empty simplicial set to any simplicial set is always a Kan fibration. It follows from Lemma 14.30.2 that a trivial Kan fibration is a Kan fibration.

08NV **Lemma 14.31.2.** *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. □

08NW **Lemma 14.31.3.** *The composition of two Kan fibrations is a Kan fibration.*

Proof. Omitted. □

08NX **Lemma 14.31.4.** *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U^t_n$. Then $U \to U^0$ is a Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. □

08NY **Lemma 14.31.5.** *Let $X_i \to Y_i$ be a set of Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a Kan fibration.*

Proof. Omitted. □

The following lemma is due to J.C. Moore, see [**Moo55**].

08NZ **Lemma 14.31.6.** *Let X be a simplicial group. Then X is a Kan complex.*

Proof. The following proof is basically just a translation into English of the proof in the reference mentioned above. Using the terminology as explained in the introduction to this section, suppose $f : \Lambda_k[n] \to X$ is a morphism from a horn. Set $x_i = f(\sigma_i) \in X_{n-1}$ for $i = 0, \ldots, \hat{k}, \ldots, n$. This means that for $i < j$ we have $d_i x_j = d_{j-1} x_i$ whenever $i, j \neq k$. We have to find an $x \in X_n$ such that $x_i = d_i x$ for $i = 0, \ldots, \hat{k}, \ldots, n$.

We first prove there exists a $u \in X_n$ such that $d_i u = x_i$ for $i < k$. This is trivial for $k = 0$. If $k > 0$, one defines by induction an element $u^r \in X_n$ such that $d_i u^r = x_i$ for $0 \leq i \leq r$. Start with $u^0 = s_0 x_0$. If $r < k - 1$, we set

$$y^r = s_{r+1}((d_{r+1}u^r)^{-1} x_{r+1}), \quad u^{r+1} = u^r y^r.$$

An easy calculation shows that $d_i y^r = 1$ (unit element of the group X_{n-1}) for $i \leq r$ and $d_{r+1} y^r = (d_{r+1} u^r)^{-1} x_{r+1}$. It follows that $d_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally, take $u = u^{k-1}$ to get u as promised.

Next we prove, by induction on the integer r, $0 \leq r \leq n - k$, there exists a $x^r \in X_n$ such that

$$d_i x^r = x_i \quad \text{for } i < k \text{ and } i > n - r.$$

Start with $x^0 = u$ for $r = 0$. Having defined x^r for $r \leq n - k - 1$ we set

$$z^r = s_{n-r-1}((d_{n-r} x^r)^{-1} x_{n-r}), \quad x^{r+1} = x^r z^r$$

A simple calculation, using the given relations, shows that $d_i z^r = 1$ for $i < k$ and $i > n - r$ and that $d_{n-r}(z^r) = (d_{n-r} x^r)^{-1} x_{n-r}$. It follows that $d_i x^{r+1} = x_i$ for $i < k$ and $i > n - r - 1$. Finally, we take $x = x^{n-k}$ which finishes the proof. □

08P0 **Lemma 14.31.7.** *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective. Then f is a Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

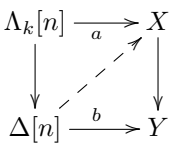

as in Definition 14.31.1. The map a corresponds to $x_0, \ldots, \hat{x}_k, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$ for $i \neq k$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ for $i \neq k$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$ for $i \neq k$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. In this case the statement become Lemma 14.31.6. □

08P1 **Lemma 14.31.8.** *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective and induces a quasi-isomorphism on associated chain complexes. Then f is a trivial Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{a} & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \xrightarrow{b} & Y \end{array}$$

as in Definition 14.30.1. The map a corresponds to $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. This works, because by Homology, Lemma 12.13.6 the homology of the chain complex associated to $\mathrm{Ker}(f)$ is zero and hence $\mathrm{Ker}(f) \to 0$ induces a quasi-isomorphism on associated chain complexes.

Since X is a Kan complex (Lemma 14.31.6) we can find $x \in X_n$ with $d_i x = x_i$ for $i = 0, \ldots, n-1$. After replacing x_i by $x_i - d_i x$ for $i = 0, \ldots, n$ we may assume that $x_0 = x_1 = \ldots = x_{n-1} = 0$. In this case we see that $d_i x_n = 0$ for $i = 0, \ldots, n-1$. Thus $x_n \in N(X)_{n-1}$ and lies in the kernel of the differential $N(X)_{n-1} \to N(X)_{n-2}$. Here $N(X)$ is the normalized chain complex associated to X, see Section 14.23. Since $N(X)$ is quasi-isomorphic to $s(X)$ (Lemma 14.23.9) and thus acyclic we find $x \in N(X_n)$ whose differential is x_n. This x answers the question posed by the lemma and we are done. □

08P2 **Lemma 14.31.9.** *Let $f : X \to Y$ be a map of simplicial abelian groups. If f is a homotopy equivalence of simplicial sets, then f induces a quasi-isomorphism of associated chain complexes.*

Proof. In this proof we will write $H_n(Z) = H_n(s(Z)) = H_n(N(Z))$ when Z is a simplicial abelian group, with s and N as in Section 14.23. Let $\mathbf{Z}[X]$ denote the free abelian group on X viewed as a simplicial set and similarly for $\mathbf{Z}[Y]$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}[X] & \xrightarrow{g} & \mathbf{Z}[Y] \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

of simplicial abelian groups. Since taking the free abelian group on a set is a functor, we see that the horizontal arrow is a homotopy equivalence of simplicial abelian groups, see Lemma 14.28.4. By Lemma 14.27.2 we see that $H_n(g) : H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective for all $n \geq 0$.

Let $\xi \in H_n(Y)$. By definition of $N(Y)$ we can represent ξ by an element $y \in N(Y_n)$ whose boundary is zero. This means $y \in Y_n$ with $d_0^n(y) = \ldots = d_{n-1}^n(y) = 0$ because $y \in N(Y_n)$ and $d_n^n(y) = 0$ because the boundary of y is zero. Denote $0_n \in Y_n$ the zero element. Then we see that

$$\tilde{y} = [y] - [0_n] \in (\mathbf{Z}[Y])_n$$

is an element with $d_0^n(\tilde{y}) = \ldots = d_{n-1}^n(\tilde{y}) = 0$ and $d_n^n(\tilde{y}) = 0$. Thus $\tilde{y}$ is in $N(\mathbf{Z}[Y])_n$ has boundary 0, i.e., $\tilde{y}$ determines a class $\tilde{\xi} \in H_n(\mathbf{Z}[Y])$ mapping to ξ. Because $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we can lift $\tilde{\xi}$ to a class in $H_n(\mathbf{Z}[X])$. Looking at the commutative diagram above we see that ξ is in the image of $H_n(X) \to H_n(Y)$.

Let $\xi \in H_n(X)$ be an element mapping to zero in $H_n(Y)$. Exactly as in the previous parapgraph we can represent ξ by an element $x \in N(X_n)$ whose boundary is zero, i.e., $d_0^n(x) = \ldots = d_{n-1}^n(x) = d_n^n(x) = 0$. In particular, we see that $[x] - [0_n]$ is an element of $N(\mathbf{Z}[X])_n$ whose boundary is zero, whence defines a lift $\tilde{\xi} \in H_n(\mathbf{Z}[x])$ of ξ. The fact that ξ maps to zero in $H_n(Y)$ means there exists a $y \in N(Y_{n+1})$ whose boundary is $f_n(x)$. This means $d_0^{n+1}(y) = \ldots = d_n^{n+1}(y) = 0$ and $d_{n+1}^{n+1}(y) = f(x)$. However, this means exactly that $z = [y] - [0_{n+1}]$ is in $N(\mathbf{Z}[y])_{n+1}$ and

$$g([x] - [0_n]) = [f(x)] - [0_n] = \text{boundary of } z$$

This proves that $\tilde{\xi}$ maps to zero in $H_n(\mathbf{Z}[y])$. As $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we conclude $\tilde{\xi} = 0$ and hence $\xi = 0$. □

14.32. A homotopy equivalence

01A5 Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\begin{array}{ccc} \mathrm{cosk}_0(A) & = & \Big(\ldots A \times A \times A \rightrightarrows\rightrightarrows A \times A \rightrightarrows A \Big) \\ & & \downarrow \qquad\quad \downarrow \qquad \downarrow \\ \mathrm{cosk}_0(B) & = & \Big(\ldots B \times B \times B \rightrightarrows\rightrightarrows B \times B \rightrightarrows B \Big) \end{array}$$

See Example 14.19.1. The case $n = 0$ of the following lemma says that this map of simplicial sets is a trivial Kan fibration if f is surjective.

01A6 **Lemma 14.32.1.** *Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The map $f_i : V_i \to U_i$ is a bijection for $i < n$.*
(2) *The map $f_n : V_n \to U_n$ is a surjection.*
(3) *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
(4) *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f is a trivial Kan fibration.

Proof. Consider a solid diagram

$$\begin{array}{ccc} \partial\Delta[k] & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \Delta[k] & \longrightarrow & U \end{array}$$

as in Definition 14.30.1. Let $x \in U_k$ be the k-simplex corresponding to the lower horizontal arrow. If $k \leq n$ then the dotted arrow is the one corresponding to a lift $y \in V_k$ of x; the diagram will commute as the other nondegenerate simplices of $\Delta[k]$ are in degrees $< k$ where f is an isomorphism. If $k > n$, then by conditions (3) and (4) we have (using adjointness of skeleton and coskeleton functors)

$$\mathrm{Mor}(\Delta[k], U) = \mathrm{Mor}(\mathrm{sk}_n\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\mathrm{sk}_n\partial\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\partial\Delta[k], U)$$

and similarly for V because $\mathrm{sk}_n\Delta[k] = \mathrm{sk}_n\partial\Delta[k]$ for $k > n$. Thus we obtain a unique dotted arrow fitting into the diagram in this case also. □

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \mathrm{cosk}_0(A) \to \mathrm{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f^0 is homotopic to f^1. In fact, there is a homotopy $h : \mathrm{cosk}_0(A) \times \Delta[1] \to \mathrm{cosk}_0(A)$ from f^0 to f^1 with components

$$\begin{aligned} h_m : A \times \ldots \times A \times \mathrm{Mor}_\Delta([m], [1]) &\longrightarrow B \times \ldots \times B, \\ (a_0, \ldots, a_m, \alpha) &\longmapsto (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m)) \end{aligned}$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

01A9 **Lemma 14.32.2.** *Let $f^0, f^1 : V \to U$ be maps of simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The maps $f_i^j : V_i \to U_i$, $j = 0, 1$ are equal for $i < n$.*
(2) *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
(3) *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f^0 is homotopic to f^1.

First proof. Let W be the n-truncated simplicial set with $W_i = U_i$ for $i < n$ and $W_n = U_n/\sim$ where $\sim$ is the equivalence relation generated by $f^0(y) \sim f^1(y)$ for $y \in V_n$. This makes sense as the morphisms $U(\varphi) : U_n \to U_i$ corresponding to $\varphi : [i] \to [n]$ for $i < n$ factor through the quotient map $U_n \to W_n$ because f^0 and f^1 are morphisms of simplicial sets and equal in degrees $< n$. Next, we upgrade W to a simplicial set by taking $\mathrm{cosk}_n W$. By Lemma 14.32.1 the morphism $g : U \to W$ is a trivial Kan fibration. Observe that $g \circ f^0 = g \circ f^1$ by construction and denote this morphism $f : V \to W$. Consider the diagram

$$\begin{array}{ccc} \partial\Delta[1] \times V & \xrightarrow[f^0, f^1]{} & U \\ \downarrow & \nearrow_{f} & \downarrow \\ \Delta[1] \times V & \longrightarrow & W \end{array}$$

By Lemma 14.30.2 the dotted arrow exists and the proof is done. □

Second proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to \text{cosk}_1\text{sk}_1\Delta[1]$ is an isomorphism, see Lemma 14.19.15. Thus we see that $\Delta[1] \to \text{cosk}_n\text{sk}_n\Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 14.19.10. And hence $V \times \Delta[1] \to \text{cosk}_n\text{sk}_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 14.19.12. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : \text{sk}_n V \times \text{sk}_n\Delta[1] \to \text{sk}_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times \text{Mor}_\Delta([k], [1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

(1) If $\text{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(2) If $\text{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(3) If $\text{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$\begin{array}{ccc} V_l \times \text{Mor}([l], [1]) & \longrightarrow & U_l \\ \downarrow & & \downarrow \\ V_k \times \text{Mor}([k], [1]) & \longrightarrow & U_k \end{array}$$

commutes. Pick $v \in V_l$ and $\alpha : [l] \to [1]$. The commutativity means that

$$h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v, \alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\text{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases above down even further to the cases (A) $\text{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\text{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \text{id}_{[n]}$, which is a trivial case. □

01AB **Lemma 14.32.3.** *Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices*

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \text{ factors}).$$

see Example 14.3.5. If f is surjective, the morphism $U \to B$ where B indicates the constant simplicial set with value B is a trivial Kan fibration.

Proof. Observe that U fits into a cartesian square

$$\begin{array}{ccc} U & \longrightarrow & \text{cosk}_0(A) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \text{cosk}_0(B) \end{array}$$

Since the right vertical arrow is a trivial Kan fibration by Lemma 14.32.1, so is the left by Lemma 14.30.3. □

14.33. Preparation for standard resolutions

0G5L The material in this section can be found in [**God73**, Appendix 1]

0G5M **Example 14.33.1.** Let $Y : \mathcal{C} \to \mathcal{C}$ be a functor from a category to itself and suppose given transformations of functors

$$d : Y \longrightarrow \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad s : Y \longrightarrow Y \circ Y$$

Using these transformations we can construct something that looks like a simplicial object. Namely, for $n \geq 0$ we define

$$X_n = Y \circ \ldots \circ Y \quad (n+1 \text{ compositions})$$

Observe that $X_{n+m+1} = X_n \circ X_m$ for $n, m \geq 0$. Next, for $n \geq 0$ and $0 \leq j \leq n$ we define using notation as in Categories, Section 4.28

$$d^n_j = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} : X_n \to X_{n-1} \quad \text{and} \quad s^n_j = 1_{X_{j-1}} \star s \star 1_{X_{n-j-1}} : X_n \to X_{n+1}$$

So d^n_j, resp. s^n_j is the natural transformation using d, resp. s on the jth Y (counted from the left) in the composition defining X_n.

0G5N **Lemma 14.33.2.** *In Example 14.33.1 if*

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad \textit{and} \quad (s \star 1) \circ s = (1 \star s) \circ s$$

then $X = (X_n, d^n_j, s^n_j)$ is a simplicial object in the category of endofunctors of $\mathcal{C}$ and $d : X_0 = Y \to id_{\mathcal{C}}$ defines an augmentation.

Proof. To see that we obtain a simplicial object we have to check that the relations (1)(a) – (e) of Lemma 14.3.2 are satisfied. We will use the short hand notation

$$1_a = 1_{X_{a-1}} = 1_Y \star \ldots \star 1_Y \quad (a \text{ factors})$$

for $a \geq 0$. With this notation we have

$$d^n_j = 1_j \star d \star 1_{n-j} \quad \text{and} \quad s^n_j = 1_j \star s \star 1_{n-j}$$

We are repeatedly going to use the rule that for transformations of funtors a, a', b, b' we have $(a' \circ a) \star (b' \circ b) = (a' \star b') \circ (a \star b)$ provided that the $\star$ and $\circ$ compositions in this formula make sense, see Categories, Lemma 4.28.2.

Condition (1)(a) always holds (no conditions needed on d and s). Namely, let $0 \leq i < j \leq n+1$. We have to show that $d^n_i \circ d^{n+1}_j = d^n_{j-1} \circ d^{n+1}_i$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star d \star 1_{n+1-j}) = (1_{j-1} \star d \star 1_{n+1-j}) \circ (1_i \star d \star 1_{n+1-i})$$

We can rewrite the left hand side as

$$\begin{aligned} & (1_i \star d \star 1_{j-i-1} \star 1_{n+1-j}) \circ (1_i \star 1_1 \star 1_{j-i-1} \star d \star 1_{n+1-j}) \\ & = 1_i \star ((d \star 1_{j-i-1}) \circ (1_1 \star 1_{j-i-1} \star d)) \star 1_{n+1-j} \\ & = 1_i \star d \star 1_{j-i-1} \star d \star 1_{n+1-j} \end{aligned}$$

The second equality is true because $d \circ 1_1 = d$ and $1_{j-i} \circ (1_{j-i-1} \star d) = 1_{j-i-1} \star d$. A similar computation gives the same result for the right hand side.

We check condition (1)(b). Let $0 \leq i < j \leq n-1$. We have to show that $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j-1} \star s \star 1_{n-1-j}) \circ (1_i \star d \star 1_{n-1-i})$$

By the same kind of calculus as in case (1)(a) both sides simplify to $1_i \star d \star 1_{j-i-1} \star s \star 1_{n-j-1}$.

We check condition (1)(c). Let $0 \leq j \leq n-1$. We have to show $\mathrm{id} = d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1}$, i.e.,

$$1_n = (1_j \star d \star 1_{n-j}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star d \star 1_{n-j-1}) \circ (1_j \star s \star 1_{n-1-j})$$

This is easily seen to be implied by the first assumption of the lemma.

We check condition (1)(d). Let $0 < j+1 < i \leq n$. We have to show $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_j \star s \star 1_{n-2-j}) \circ (1_{i-1} \star d \star 1_{n-i})$$

By the same kind of calculus as in case (1)(a) both sides simplify to $1_j \star s \star 1_{i-j-2} \star d \star 1_{n-i}$.

We check condition (1)(e). Let $0 \leq i \leq j \leq n-1$. We have to show that $s_i^n \circ s_j^{n-1} = s_{j+1}^n \circ s_i^{n-1}$, i.e.,

$$(1_i \star s \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star s \star 1_{n-1-j}) \circ (1_i \star s \star 1_{n-1-i})$$

By the same kind of calculus as in case (1)(a) this reduces to

$$(s \star 1_{j-i+1}) \circ (1_{j-i} \star s) = (1_{j-i+1} \star s) \circ (s \star 1_{j-i})$$

If $j = i$ this is exactly one of the two assumptions of the lemma. For $j > i$ left and right hand side both reduce to the equality $s \star 1_{j-i-1} \star s$ by calculations similar to those we did in case (1)(a).

Finally, in order to show that d defines an augmentation we have to show that $d \circ (1_1 \star d) = d \circ (d \star 1_1)$ which is true because both sides are equal to $d \star d$. □

0G5P **Example 14.33.3.** Let $\mathcal{C}$, Y, d, s be as in Example 14.33.1 satisfying the equations of Lemma 14.33.2. Given functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ we obtain a simplicial object $G \circ X \circ F$ in the category of functors from $\mathcal{A}$ to $\mathcal{B}$ which comes with an augmentation to $G \circ F$.

0G5Q **Lemma 14.33.4.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 14.33.3. Given a transformation of functors $h_0 : G \circ F \to G \circ Y \circ F$ such that*

$$1_{G \circ F} = (1_G \star d \star 1_F) \circ h_0$$

Then there is a morphism $h : G \circ F \to G \circ X \circ F$ of simplicial objects such that $\epsilon \circ h = id$ where $\epsilon : G \circ X \circ F \to G \circ F$ is the augmentation.

Proof. Denote $u_n : Y = X_0 \to X_n$ the map of the simplicial object X corresponding to the unique morphism $[n] \to [0]$ in Δ. Set $h_n : G \circ F \to G \circ X_n \circ F$ equal to $(1_G \star u_n \star 1_F) \circ h_0$.

For any simplicial object $X = (X_n)$ in any category $u = (u_n) : X_0 \to X$ is a morphism from the constant simplicial object on X_0 to X. Hence h is a morphism of simplicial objects because it is the composition of $1_G \star u \star 1_F$ and h_0.

Let us check that $\epsilon \circ h = \mathrm{id}$. We compute

$$\epsilon_n \circ (1_G \star u_n \star 1_F) \circ h_0 = \epsilon_0 \circ h_0 = \mathrm{id}$$

The first equality because ϵ is a morphism of simplicial objects and the second equality because $\epsilon_0 = (1_G \star d \star 1_F)$ and we can apply the assumption in the statement of the lemma. □

0G5R **Lemma 14.33.5.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 14.33.3. Let $F' : \mathcal{A} \to \mathcal{C}$ and $G' : \mathcal{C} \to \mathcal{B}$ be two functors. Let $(a_n) : G \circ X \to G' \circ X$ be a morphism of simplicial objects compatible via augmentations with $a : G \to G'$. Let $(b_n) : X \circ F \to X \circ F'$ be a morphism of simplicial objects compatible via augmentations with $b : F \to F'$. Then the two maps*

$$a \star (b_n), (a_n) \star b : G \circ X \circ F \to G' \circ X \circ F'$$

are homotopic.

Proof. To show the morphisms are homotopic we construct morphisms

$$h_{n,i} : G \circ X_n \circ F \to G' \circ X_n \circ F'$$

for $n \geq 0$ and $0 \leq i \leq n+1$ satisfying the relations described in Lemma 14.26.2. See also Remark 14.26.4. To satisfy condition (1) of Lemma 14.26.2 we are forced to set $h_{n,0} = a \star b_n$ and $h_{n,n+1} = a_n \star b$. Thus a logical choice is

$$h_{n,i} = a_{i-1} \star b_{n-i}$$

for $1 \leq i \leq n$. Setting $a = a_{-1}$ and $b = b_{-1}$ we see the displayed formular holds for $0 \leq i \leq n+1$.

Recall that

$$d^n_j = 1_G \star 1_j \star d \star 1_{n-j} \star 1_F$$

on $G \circ X \circ F$ where we use the notation $1_a = 1_{Y \circ \ldots \circ Y}$ introduced in the proof of Lemma 14.33.2. We are going to use below that we can rewrite this as

$$\begin{aligned} d^n_j &= d^j_j \star 1_{n-j} = d^{j+1}_j \star 1_{n-j} = \ldots = d^{n-1}_j \star 1_1 \\ &= 1_j \star d^{n-j}_0 = 1_{j-1} \star d^{n-j+1}_1 = \ldots = 1_1 \star d^{n-1}_{j-1} \end{aligned}$$

Of course we have the analogous formulae for d^n_j on $G' \circ X \circ F'$.

We check condition (2) of Lemma 14.26.2. Let $i > j$. We have to show

$$d^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-2} \star b_{n-i}) \circ d^n_j$$

Since $i - 1 \geq j$ we can use one of the possible descriptions of d^n_j to rewrite the left hand side as

$$(d^{i-1}_j \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (d^{i-1}_j \circ a_{i-1}) \star b_{n-i} = (a_{i-2} \circ d^{i-1}_j) \star b_{n-i}$$

Similarly the right hand side becomes

$$(a_{i-2} \star b_{n-i}) \circ (d^{i-1}_j \star 1_{n-i+1}) = (a_{i-2} \circ d^{i-1}_j) \star b_{n-i}$$

Thus we obtain the same result and (2) is checked.

We check condition (3) of Lemma 14.26.2. Let $i \leq j$. We have to show

$$d^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n-1-i}) \circ d^n_j$$

Since $j \geq i$ we may rewrite the left hand side as

$$(1_i \star d^{n-i}_{j-i}) \circ (a_{i-1} \star b_{n-i}) = a_{i-1} \star (b_{n-1-i} \circ d^{n-i}_{j-i})$$

A similar manipulation shows this agrees with the right hand side.

Recall that

$$s^n_j = 1_G \star 1_j \star s \star 1_{n-j} \star 1_F$$

on $G \circ X \circ F$. We are going to use below that we can rewrite this as

$$\begin{aligned} s^n_j &= s^j_j \star 1_{n-j} = s^{j+1}_j \star 1_{n-j-1} = \ldots = s^{n-1}_j \star 1_1 \\ &= 1_j \star s^{n-j}_0 = 1_{j-1} \star s^{n-j+1}_1 = \ldots = 1_1 \star s^{n-1}_{j-1} \end{aligned}$$

Of course we have the analogous formulae for s^n_j on $G' \circ X \circ F'$.

We check condition (4) of Lemma 14.26.2. Let $i > j$. We have to show

$$s^n_j \circ (a_{i-1} \star b_{n-i}) = (a_i \star b_{n-i}) \circ s^n_j$$

Since $i - 1 \geq j$ we can rewrite the left hand side as

$$(s^{i-1}_j \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (s^{i-1}_j \circ a_{i-1}) \star b_{n-i} = (a_i \circ s^{i-1}_j) \star b_{n-i}$$

Similarly the right hand side becomes

$$(a_i \star b_{n-i}) \circ (s^{i-1}_j \star 1_{n-i+1}) = (a_i \circ s^{i-1}_j) \star b_{n-i}$$

as desired.

We check condition (5) of Lemma 14.26.2. Let $i \leq j$. We have to show

$$s^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n+1-i}) \circ s^n_j$$

This equality holds because both sides evaluate to $a_{i-1} \star (s^{n-i}_{j-i} \circ b_{n-i}) = a_{i-1} \star (b_{n+1-i} \circ s^{n-i}_{j-i})$ by exactly the same arguments as above. □

0G5S **Lemma 14.33.6.** *Let $\mathcal{C}$, Y, d, s be as in Example 14.33.1 satisfying the equations of Lemma 14.33.2. Let $f : id_{\mathcal{C}} \to id_{\mathcal{C}}$ be an endomorphism of the identity functor. Then $f \star 1_X, 1_X \star f : X \to X$ are maps of simplicial objects compatible with f via the augmentation $\epsilon : X \to id_{\mathcal{C}}$. Moreover, $f \star 1_X$ and $1_X \star f$ are homotopic.*

Proof. The map $f \star 1_X$ is the map with components

$$X_n = \mathrm{id}_{\mathcal{C}} \circ X_n \xrightarrow{f \star 1_{X_n}} \mathrm{id}_{\mathcal{C}} \circ X_n = X_n$$

For a transformation $a : F \to G$ of endofunctors of $\mathcal{C}$ we have $a \circ (f \star 1_F) = f \star a = (f \star 1_G) \circ a$. Thus $f \star 1_X$ is indeed a morphism of simplicial objects. Similarly for $1_X \star f$.

To show the morphisms are homotopic we construct morphisms $h_{n,i} : X_n \to X_n$ for $n \geq 0$ and $0 \leq i \leq n + 1$ satisfying the relations described in Lemma 14.26.2. See also Remark 14.26.4. It turns out we can take

$$h_{n,i} = 1_i \star f \star 1_{n+1-i}$$

where 1_i is the identity transformation on $Y \circ \ldots \circ Y$ as in the proof of Lemma 14.33.2. We have $h_{n,0} = f \star 1_{X_n}$ and $h_{n,n+1} = 1_{X_n} \star f$ which checks the first condition. In checking the other conditions we use the comments made in the proof of Lemma 14.33.5 about the maps d^n_j and s^n_j.

We check condition (2) of Lemma 14.26.2. Let $i > j$. We have to show

$$d^n_j \circ (1_i \star f \star 1_{n+1-i}) = (1_{i-1} \star f \star 1_{n+1-i}) \circ d^n_j$$

Since $i-1 \geq j$ we can use one of the possible descriptions of d^n_j to rewrite the left hand side as

$$(d^{i-1}_j \star 1_{n-i+1}) \circ (1_i \star f \star 1_{n+1-i}) = d^{i-1}_j \star f \star 1_{n+1-i}$$

Similarly the right hand side becomes

$$(1_{i-1} \star f \star 1_{n+1-i}) \circ (d^{i-1}_j \star 1_{n-i+1}) = d^{i-1}_j \star f \star 1_{n+1-i}$$

Thus we obtain the same result and (2) is checked.

The conditions (3), (4), and (5) of Lemma 14.26.2 are checked in exactly the same manner using the strategy of the proof of Lemma 14.33.5. We omit the details[2]. □

14.34. Standard resolutions

08N8 Some of the material in this section can be found in [**God73**, Appendix 1] and [**Ill72**, I 1.5].

08N9 **Situation 14.34.1.** Let $\mathcal{A}$, $\mathcal{S}$ be categories and let $V : \mathcal{A} \to \mathcal{S}$ be a functor with a left adjoint $U : \mathcal{S} \to \mathcal{A}$.

In this very general situation we will construct a simplicial object X in the category of functors from $\mathcal{A}$ to $\mathcal{A}$. We suggest looking at the examples presented later on before reading the text of this section.

For the construction we will use the horizontal composition as defined in Categories, Section 4.28. The definition of the adjunction morphisms[3]

$$d : U \circ V \to \mathrm{id}_{\mathcal{A}} \quad \text{(counit)} \quad \text{and} \quad \eta : \mathrm{id}_{\mathcal{S}} \to V \circ U \quad \text{(unit)}$$

in Categories, Section 4.24 shows that the compositions

08NB (14.34.1.1) $$V \xrightarrow{\eta \star 1_V} V \circ U \circ V \xrightarrow{1_V \star d} V \quad \text{and} \quad U \xrightarrow{1_U \star \eta} U \circ V \circ U \xrightarrow{d \star 1_U} U$$

are the identity morphisms. Here to define the morphism $\eta \star 1_V$ we silently identify V with $\mathrm{id}_{\mathcal{S}} \circ V$ and 1_V stands for $\mathrm{id}_V : V \to V$. We will use this notation and these relations repeatedly in what follows. For $n \geq 0$ we set

$$X_n = (U \circ V)^{\circ(n+1)} = U \circ V \circ U \circ \ldots \circ U \circ V$$

In other words, X_n is the $(n+1)$-fold composition of $U \circ V$ with itself. We also set $X_{-1} = \mathrm{id}_{\mathcal{A}}$. We have $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geq -1$. We will endow this sequence of functors with the structure of a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ by constructing the morphisms of functors

$$d^n_j : X_n \to X_{n-1}, \quad s^n_j : X_n \to X_{n+1}$$

satisfying the relations displayed in Lemma 14.2.3. Namely, we set

$$d^n_j = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} \quad \text{and} \quad s^n_j = 1_{X_{j-1} \circ U} \star \eta \star 1_{V \circ X_{n-j-1}}$$

Finally, write $\epsilon_0 = d : X_0 \to X_{-1}$.

08NC **Lemma 14.34.2.** *In Situation 14.34.1 the system $X = (X_n, d^n_j, s^n_j)$ is a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = \mathrm{id}_{\mathcal{A}}$.*

[2] When f is invertible it suffices to prove that $(a_n) = 1_X$ and $(b_n) = f^{-1} \star 1_X \star f$ are homotopic. But this follows from Lemma 14.33.5 because in this case $a = b = 1_{\mathrm{id}_{\mathcal{C}}}$.

[3] We can't use ϵ for the counit of the adjunction because we want to use ϵ for the augmentation of our simplicial object.

Proof. Consider $Y = U \circ V : \mathcal{A} \to \mathcal{A}$. We already have the transformation $d : Y = U \circ V \to \mathrm{id}_{\mathcal{A}}$. Let us denote

$$s = 1_U \star \eta \star 1_V : Y = U \circ \mathrm{id}_{\mathcal{S}} \circ V \longrightarrow U \circ V \circ U \circ V = Y \circ Y$$

This places us in the situation of Example 14.33.1. It is immediate from the formulas that the X, d_i^n, s_i^n constructed above and the X, s_i^n, s_i^n constructed from Y, d, s in Example 14.33.1 agree. Thus, according to Lemma 14.33.2 it suffices to prove that

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad \text{and} \quad (s \star 1) \circ s = (1 \star s) \circ s$$

The first equal sign translates into the equality

$$1_U \star 1_V = (d \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V)$$

which holds if we have $1_U = (d \star 1_U) \circ (1_U \star \eta)$ which in turn holds by (14.34.1.1). Similarly for the second equal sign. For the last equation we need to prove

$$(1_U \star \eta \star 1_V \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V) = (1_U \star 1_V \star 1_U \star \eta \star 1_V) \circ (1_U \star \eta \star 1_V)$$

For this it suffices to prove $(\eta \star 1_V \star 1_U) \circ \eta = (1_V \star 1_U \star \eta) \circ \eta$ which is true because both sides are the same as $\eta \star \eta$. □

Before reading the proof of the following lemma, we advise the reader to look at the example discussed in Example 14.34.8 in order to understand the purpose of the lemma.

08ND **Lemma 14.34.3.** *In Situation 14.34.1 the maps*

$$1_V \star \epsilon : V \circ X \to V, \quad \textit{and} \quad \epsilon \star 1_U : X \circ U \to U$$

are homotopy equivalences.

Proof. As in the proof of Lemma 14.34.2 we set $Y = U \circ V$ so that we are in the situation of Example 14.33.1.

Proof of the first homotopy equivalence. By Lemma 14.33.4 to construct a map $h : V \to V \circ X$ right inverse to $1_V \star \epsilon$ it suffices to construct a map $h_0 : V \to V \circ Y = V \circ U \circ V$ such that $1_V = (1_V \star d) \circ h_0$. Of course we take $h_0 = \eta \star 1_V$ and the equality holds by (14.34.1.1). To finish the proof we need to show the two maps

$$(1_V \star \epsilon) \circ h, 1_V \star \mathrm{id}_X : V \circ X \longrightarrow V \circ X$$

are homotopic. This follows immediately from Lemma 14.33.5 (with $G = G' = V$ and $F = F' = \mathrm{id}_{\mathcal{S}}$).

The proof of the second homotopy equivalence. By Lemma 14.33.4 to construct a map $h : U \to X \circ U$ right inverse to $\epsilon \star 1_U$ it suffices to construct a map $h_0 : U \to Y \circ U = U \circ V \circ U$ such that $1_U = (d \star 1_U) \circ h_0$. Of course we take $h_0 = 1_U \star \eta$ and the equality holds by (14.34.1.1). To finish the proof we need to show the two maps

$$(\epsilon \star 1_U) \circ h, \mathrm{id}_X \star 1_U : X \circ U \longrightarrow X \circ U$$

are homotopic. This follows immediately from Lemma 14.33.5 (with $G = G' = \mathrm{id}_{\mathcal{A}}$ and $F = F' = U$). □

0G5T **Example 14.34.4.** Let R be a ring. As an example of the above we can take $i : \mathrm{Mod}_R \to \mathit{Sets}$ to be the forgetful functor and $F : \mathit{Sets} \to \mathrm{Mod}_R$ to be the functor that associates to a set E the free R-module $R[E]$ on E. For an R-module M the simplicial R-module $X(M)$ will have the following shape

$$X(M) = \Big(\ldots \; R[R[R[M]]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R[R[M]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R[M] \Big)$$

which comes with an augmentation towards M. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 14.30.8, 14.31.9, and 14.31.8 this is equivalent to asking M to be the only nonzero cohomology group of the chain complex associated to the simplicial module $X(M)$.

08NA **Example 14.34.5.** Let A be a ring. Let Alg_A be the category of commutative A-algebras. As an example of the above we can take $i : \mathit{Alg}_A \to \mathit{Sets}$ to be the forgetful functor and $F : \mathit{Sets} \to \mathit{Alg}_A$ to be the functor that associates to a set E the polynomial algebra $A[E]$ on E over A. (We apologize for the overlap in notation between this example and Example 14.34.4.) For an A-algebra B the simplicial A-algebra $X(B)$ will have the following shape

$$X(B) = \Big(\ldots \; A[A[A[B]]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; A[A[B]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; A[B] \Big)$$

which comes with an augmentation towards B. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 14.30.8, 14.31.9, and 14.31.8 this is equivalent to asking B to be the only nonzero cohomology group of the chain complex of A-modules associated to $X(B)$ viewed as a simplicial A-module.

0G5U **Example 14.34.6.** In Example 14.34.4 we have $X_n(M) = R[R[\ldots[M]\ldots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum\nolimits_i r_i \Big[\sum\nolimits_j r_{ij}[m_{ij}] \Big]$$

of $X_1(M)$. The maps $d_0, d_1 : R[R[M]] \to R[M]$ are given by

$$d_0(\xi) = \sum\nolimits_{i,j} r_i r_{ij}[m_{ij}] \quad \text{and} \quad d_1(\xi) = \sum\nolimits_i r_i \Big[\sum\nolimits_j r_{ij} m_{ij} \Big].$$

The maps $s_0, s_1 : R[R[M]] \to R[R[R[M]]]$ are given by

$$s_0(\xi) = \sum\nolimits_i r_i \Big[\Big[\sum\nolimits_j r_{ij}[m_{ij}] \Big] \Big] \quad \text{and} \quad s_1(\xi) = \sum\nolimits_i r_i \Big[\sum\nolimits_j r_{ij}[[m_{ij}]] \Big].$$

09CB **Example 14.34.7.** In Example 14.34.5 we have $X_n(B) = A[A[\ldots[B]\ldots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum\nolimits_i a_i [x_{i,1}] \ldots [x_{i,m_i}] \in A[A[B]] = X_1(B)$$

where for each i, j we can write

$$x_{i,j} = \sum a_{i,j,k} [b_{i,j,k,1}] \ldots [b_{i,j,k,n_{i,j,k}}] \in A[B]$$

Obviously this is horrendous! To ease the notation, to see what the A-algebra maps $d_0, d_1 : A[A[B]] \to A[B]$ are doing it suffices to see what happens to the variables $[x]$ where

$$x = \sum a_k [b_{k,1}] \ldots [b_{k,n_k}] \in A[B]$$

is a general element. For these we get

$$d_0([x]) = x = \sum a_k [b_{k,1}] \ldots [b_{k,n_k}] \quad \text{and} \quad d_1([x]) = \Big[\sum a_k b_{k,1} \ldots b_{k,n_k} \Big]$$

The maps $s_0, s_1 : A[A[B]] \to A[A[A[B]]]$ are given by

$$s_0([x]) = \Big[\Big[\sum a_k[b_{k,1}] \ldots [b_{k,n_k}]\Big]\Big] \quad \text{and} \quad s_1([x]) = \Big[\sum a_k[[b_{k,1}]] \ldots [[b_{k,n_k}]]\Big]$$

08NE **Example 14.34.8.** Going back to the example discussed in Example 14.34.5 our Lemma 14.34.3 signifies that for any ring map $A \to B$ the map of simplicial rings

$$\begin{array}{ccccc} A[A[A[B]]] & \rightleftarrows & A[A[B]] & \rightleftarrows & A[B] \\ \downarrow & & \downarrow & & \downarrow \\ B & \rightleftarrows & B & \rightleftarrows & B \end{array}$$

is a homotopy equivalence on underlying simplicial sets. Moreover, the inverse map constructed in Lemma 14.34.3 is in degree n given by

$$b \longmapsto [\ldots[b]\ldots]$$

with obvious notation. In the other direction the lemma tells us that for every set E there is a homotopy equivalence

$$\begin{array}{ccccc} A[A[A[A[E]]]] & \rightleftarrows & A[A[A[E]]] & \rightleftarrows & A[A[E]] \\ \downarrow & & \downarrow & & \downarrow \\ A[E] & \rightleftarrows & A[E] & \rightleftarrows & A[E] \end{array}$$

of rings. The inverse map constructed in the lemma is in degree n given by the ring map

$$\sum a_{e_1,\ldots,e_p}[e_1][e_2]\ldots[e_p] \longmapsto \sum a_{e_1,\ldots,e_p}[\ldots[e_1]\ldots][\ldots[e_2]\ldots]\ldots[\ldots[e_p]\ldots]$$

(with obvious notation).

14.35. Other chapters

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(7) Sites and Sheaves
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CHAPTER 15

More on Algebra

05E3

15.1. Introduction

05E4 In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 10.1. A reference is [**Mat70a**].

15.2. Advice for the reader

0910 More than in the chapter on commutative algebra, each of the sections in this chapter stands on its own. Starting with Section 15.56 we freely use the (unbounded) derived category of modules over rings and all the machinery that comes with it.

15.3. Stably free modules

0BC2 Here is what seems to be the generally accepted definition.

0BC3 **Definition 15.3.1.** Let R be a ring.

(1) Two modules M, N over R are said to be *stably isomorphic* if there exist $n, m \geq 0$ such that $M \oplus R^{\oplus m} \cong N \oplus R^{\oplus n}$ as R-modules.
(2) A module M is *stably free* if it is stably isomorphic to a free module.

Observe that a stably free module is projective.

0BC4 **Lemma 15.3.2.** *Let R be a ring. Let $0 \to P' \to P \to P'' \to 0$ be a short exact sequence of finite projective R-modules. If 2 out of 3 of these modules are stably free, then so is the third.*

Proof. Since the modules are projective, the sequence is split. Thus we can choose an isomorphism $P = P' \oplus P''$. If $P' \oplus R^{\oplus n}$ and $P'' \oplus R^{\oplus m}$ are free, then we see that $P \oplus R^{\oplus n+m}$ is free. Suppose that P' and P are stably free, say $P \oplus R^{\oplus n}$ is free and $P' \oplus R^{\oplus m}$ is free. Then

$$P'' \oplus (P' \oplus R^{\oplus m}) \oplus R^{\oplus n} = (P'' \oplus P') \oplus R^{\oplus m} \oplus R^{\oplus n} = (P \oplus R^{\oplus n}) \oplus R^{\oplus m}$$

is free. Thus P'' is stably free. By symmetry we get the last of the three cases. □

0BC5 **Lemma 15.3.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). For every finite stably free R/I-module E there exists a finite stably free R-module M such that $M/IM \cong E$.*

Proof. Choose a n and m and an isomorphism $E \oplus (R/I)^{\oplus n} \cong (R/I)^{\oplus m}$. Choose R-linear maps $\varphi : R^{\oplus m} \to R^{\oplus n}$ and $\psi : R^{\oplus n} \to R^{\oplus m}$ lifting the projection $(R/I)^{\oplus m} \to (R/I)^{\oplus n}$ and injection $(R/I)^{\oplus n} \to (R/I)^{\oplus m}$. Then $\varphi \circ \psi : R^{\oplus n} \to R^{\oplus n}$ reduces to the identity modulo I. Thus the determinant of this map is invertible by our assumption on I. Hence $P = \mathrm{Ker}(\varphi)$ is stably free and lifts E. □

0D48 **Lemma 15.3.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). Let M be a finite flat R-module such that M/IM is a projective R/I-module. Then M is a finite projective R-module.*

Proof. By Algebra, Lemma 10.78.5 we see that $M_\mathfrak{p}$ is finite free for all prime ideals $\mathfrak{p} \subset R$. By Algebra, Lemma 10.78.2 it suffices to show that the function $\rho_M : \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant on $\mathrm{Spec}(R)$. Because M/IM is finite projective, this is true on $V(I) \subset \mathrm{Spec}(R)$. Since every closed point of $\mathrm{Spec}(R)$ is in $V(I)$ and since $\rho_M(\mathfrak{p}) = \rho_M(\mathfrak{q})$ whenever $\mathfrak{p} \subset \mathfrak{q} \subset R$ are prime ideals, we conclude by an elementary argument on topological spaces which we omit. □

The lift of Lemma 15.3.3 is unique up to isomorphism by the following lemma.

0BC6 **Lemma 15.3.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). If P and P' are finite projective R-modules, then*

(1) *if $\varphi : P \to P'$ is an R-module map inducing an isomorphism $\overline{\varphi} : P/IP \to P'/IP'$, then φ is an isomorphism,*
(2) *if $P/IP \cong P'/IP'$, then $P \cong P'$.*

Proof. Proof of (1). As P' is projective as an R-module we may choose a lift $\psi : P' \to P$ of the map $P' \to P'/IP' \xrightarrow{\overline{\varphi}^{-1}} P/IP$. By Nakayama's lemma (Algebra, Lemma 10.20.1) $\psi \circ \varphi$ and $\varphi \circ \psi$ are surjective. Hence these maps are isomorphisms (Algebra, Lemma 10.16.4). Thus φ is an isomorphism.

Proof of (2). Choose an isomorphism $P/IP \cong P'/IP'$. Since P is projective we can choose a lift $\varphi : P \to P'$ of the map $P \to P/IP \to P'/IP'$. Then φ is an isomorphism by (1). □

15.4. A comment on the Artin-Rees property

07VD Some of this material is taken from [**CdJ02**]. A general discussion with additional references can be found in [**EH05**, Section 1].

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Given a homomorphism $f : M \to N$ of finite A-modules there exists a $c \geq 0$ such that

$$f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$, see Algebra, Lemma 10.51.3. In this situation we will say *c works for f in the Artin-Rees lemma.*

07VE **Lemma 15.4.1.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal contained in the Jacobson radical of A. Let*

$$S : L \xrightarrow{f} M \xrightarrow{g} N \quad \text{and} \quad S' : L \xrightarrow{f'} M \xrightarrow{g'} N$$

be two complexes of finite A-modules as shown. Assume that

(1) *c works in the Artin-Rees lemma for f and g,*
(2) *the complex S is exact, and*
(3) *$f' = f \bmod I^{c+1} M$ and $g' = g \bmod I^{c+1} N$.*

Then c works in the Artin-Rees lemma for g' and the complex S' is exact.

Proof. We first show that $g'(M) \cap I^n N \subset g'(I^{n-c}M)$ for $n \geq c$. Let a be an element of M such that $g'(a) \in I^n N$. We want to adjust a by an element of $f'(L)$, i.e, without changing $g'(a)$, so that $a \in I^{n-c}M$. Assume that $a \in I^r M$, where $r < n - c$. Then

$$g(a) = g'(a) + (g - g')(a) \in I^n N + I^{r+c+1}N = I^{r+c+1}N.$$

By Artin-Rees for g we have $g(a) \in g(I^{r+1}M)$. Say $g(a) = g(a_1)$ with $a_1 \in I^{r+1}M$. Since the sequence S is exact, $a - a_1 \in f(L)$. Accordingly, we write $a = f(b) + a_1$ for some $b \in L$. Then $f(b) = a - a_1 \in I^r M$. Artin-Rees for f shows that if $r \geq c$, we may replace b by an element of $I^{r-c}L$. Then in all cases, $a = f'(b) + a_2$, where $a_2 = (f - f')(b) + a_1 \in I^{r+1}M$. (Namely, either $c \geq r$ and $(f - f')(b) \in I^{r+1}M$ by assumption, or $c < r$ and $b \in I^{r-c}$, whence again $(f - f')(b) \in I^{c+1}I^{r-c}M = I^{r+1}M$.) So we can adjust a by the element $f'(b) \in f'(L)$ to increase r by 1.

In fact, the argument above shows that $(g')^{-1}(I^n N) \subset f'(L) + I^{n-c}M$ for all $n \geq c$. Hence S' is exact because

$$(g')^{-1}(0) = (g')^{-1}(\bigcap I^n N) \subset \bigcap f'(L) + I^{n-c}M = f'(L)$$

as I is contained in the Jacobson radical of A, see Algebra, Lemma 10.51.5. □

Given an ideal $I \subset A$ of a ring A and an A-module M we set

$$\mathrm{Gr}_I(M) = \bigoplus I^n M / I^{n+1} M.$$

We think of this as a graded $\mathrm{Gr}_I(A)$-module.

07VF **Lemma 15.4.2.** *Assumptions as in Lemma 15.4.1. Let $Q = \mathrm{Coker}(g)$ and $Q' = \mathrm{Coker}(g')$. Then $Gr_I(Q) \cong Gr_I(Q')$ as graded $Gr_I(A)$-modules.*

Proof. In degree n we have $\mathrm{Gr}_I(Q)_n = I^n N/(I^{n+1}N + g(M) \cap I^n N)$ and similarly for Q'. We claim that

$$g(M) \cap I^n N \subset I^{n+1}N + g'(M) \cap I^n N.$$

By symmetry (the proof of the claim will only use that c works for g which also holds for g' by the lemma) this will imply that

$$I^{n+1}N + g(M) \cap I^n N = I^{n+1}N + g'(M) \cap I^n N$$

whence $\mathrm{Gr}_I(Q)_n$ and $\mathrm{Gr}_I(Q')_n$ agree as subquotients of N, implying the lemma. Observe that the claim is clear for $n \leq c$ as $f = f' \bmod I^{c+1}N$. If $n > c$, then suppose $b \in g(M) \cap I^n N$. Write $b = g(a)$ for $a \in I^{n-c}M$. Set $b' = g'(a)$. We have $b - b' = (g - g')(a) \in I^{n+1}N$ as desired. □

07VG **Lemma 15.4.3.** *Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal. Let $f : M \to N$ be a homomorphism of finite A-modules. Assume that c works for f in the Artin-Rees lemma. Then c works for $f \otimes 1 : M \otimes_A B \to N \otimes_A B$ in the Artin-Rees lemma for the ideal IB.*

Proof. Note that

$$(f \otimes 1)(M) \cap I^n N \otimes_A B = (f \otimes 1)\left((f \otimes 1)^{-1}(I^n N \otimes_A B)\right)$$

On the other hand,

$$\begin{aligned}(f \otimes 1)^{-1}(I^n N \otimes_A B) &= \mathrm{Ker}(M \otimes_A B \to N \otimes_A B/(I^n N \otimes_A B)) \\ &= \mathrm{Ker}(M \otimes_A B \to (N/I^n N) \otimes_A B)\end{aligned}$$

As $A \to B$ is flat taking kernels and cokernels commutes with tensoring with B, whence this is equal to $f^{-1}(I^nN) \otimes_A B$. By assumption $f^{-1}(I^nN)$ is contained in $\mathrm{Ker}(f) + I^{n-c}M$. Thus the lemma holds. □

15.5. Fibre products of rings, I

08KG Fibre products of rings have to do with pushouts of schemes. Some cases of pushouts of schemes are discussed in More on Morphisms, Section 37.14.

00IT **Lemma 15.5.1.** *Let R be a ring. Let $A \to B$ and $C \to B$ be R-algebra maps. Assume*

(1) *R is Noetherian,*
(2) *A, B, C are of finite type over R,*
(3) *$A \to B$ is surjective, and*
(4) *B is finite over C.*

Then $A \times_B C$ is of finite type over R.

Proof. Set $D = A \times_B C$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \leftarrow & B & \leftarrow & A & \leftarrow & I & \leftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & C & \leftarrow & D & \leftarrow & I & \leftarrow & 0 \end{array}$$

with exact rows. Choose $y_1, \ldots, y_n \in B$ which are generators for B as a C-module. Choose $x_i \in A$ mapping to y_i. Then $1, x_1, \ldots, x_n$ are generators for A as a D-module. The map $D \to A \times C$ is injective, and the ring $A \times C$ is finite as a D-module (because it is the direct sum of the finite D-modules A and C). Hence the lemma follows from the Artin-Tate lemma (Algebra, Lemma 10.51.7). □

08NI **Lemma 15.5.2.** *Let R be a Noetherian ring. Let I be a finite set. Suppose given a cartesian diagram*

$$\begin{array}{ccc} \prod B_i & \xleftarrow{\prod \varphi_i} & \prod A_i \\ {\scriptstyle \prod \psi_i}\uparrow & & \uparrow \\ Q & \leftarrow & P \end{array}$$

with ψ_i and φ_i surjective, and Q, A_i, B_i of finite type over R. Then P is of finite type over R.

Proof. Follows from Lemma 15.5.1 and induction on the size of I. Namely, let $I = I' \amalg \{i_0\}$. Let P' be the ring defined by the diagram of the lemma using I'. Then P' is of finite type by induction hypothesis. Finally, P sits in a fibre product diagram

$$\begin{array}{ccc} B_{i_0} & \leftarrow & A_{i_0} \\ \uparrow & & \uparrow \\ P' & \leftarrow & P \end{array}$$

to which the lemma applies. □

01Z8 **Lemma 15.5.3.** *Suppose given a cartesian diagram of rings*

$$\begin{array}{ccc} R & \xleftarrow{t} & R' \\ \uparrow{\scriptstyle s} & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

i.e., $B' = B \times_R R'$. If $h \in B'$ corresponds to $g \in B$ and $f \in R'$ such that $s(g) = t(f)$, then the diagram

$$\begin{array}{ccc} R_{s(g)} = R_{t(f)} & \xleftarrow{t} & (R')_f \\ \uparrow{\scriptstyle s} & & \uparrow \\ B_g & \longleftarrow & (B')_h \end{array}$$

is cartesian too.

Proof. The equality $B' = B \times_R R'$ tells us that

$$0 \to B' \to B \oplus R' \xrightarrow{s,-t} R$$

is an exact sequence of B'-modules. We have $B_g = B_h$, $R'_f = R'_h$, and $R_{s(g)} = R_{t(f)} = R_h$ as B'-modules. By exactness of localization (Algebra, Proposition 10.9.12) we find that

$$0 \to B'_h \to B_g \oplus R'_f \xrightarrow{s,-t} R_{s(g)} = R_{t(f)}$$

is an exact sequence. This proves the lemma. □

Consider a commutative diagram of rings

$$\begin{array}{ccc} R & \longleftarrow & R' \\ \uparrow & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

Consider the functor (where the fibre product of categories is as constructed in Categories, Example 4.31.3)

0D2E (15.5.3.1) $$\mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B \times_{\mathrm{Mod}_R} \mathrm{Mod}_{R'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} R', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B R = L' \otimes_{B'} R' \otimes_{R'} R$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an R'-module and $\varphi : N \otimes_B R \to M' \otimes_{R'} R$ is an isomorphism.

0D2F **Lemma 15.5.4.** *Given a commutative diagram of rings*

$$\begin{array}{ccc} R & \longleftarrow & R' \\ \uparrow & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

the functor (15.5.3.1) has a right adjoint, namely the functor

$$F : (N, M', \varphi) \longmapsto N \times_\varphi M'$$

(see proof for elucidation).

Proof. Given an object (N, M', φ) of the category $\mathrm{Mod}_B \times_{\mathrm{Mod}_R} \mathrm{Mod}_{R'}$ we set

$$N \times_\varphi M' = \{(n, m') \in N \times M' \mid \varphi(n \otimes 1) = m' \otimes 1 \text{ in } M' \otimes_{R'} R\}$$

viewed as a B'-module. The adjointness statement is that for a B'-module L' and a triple (N, M', φ) we have

$$\mathrm{Hom}_{B'}(L', N \times_\varphi M') = \mathrm{Hom}_B(L' \otimes_{B'} B, N) \times_{\mathrm{Hom}_R(L' \otimes_{B'} R, M' \otimes_{R'} R)} \mathrm{Hom}_{R'}(L' \otimes_{B'} R', M')$$

By Algebra, Lemma 10.14.3 the right hand side is equal to

$$\mathrm{Hom}_{B'}(L', N) \times_{\mathrm{Hom}_{B'}(L', M' \otimes_{R'} R)} \mathrm{Hom}_{B'}(L', M')$$

Thus it is clear that for a pair (g, f') of elements of this fibre product we get an B'-linear map $L' \to N \times_\varphi M'$, $l' \mapsto (g(l'), f'(l'))$. Conversely, given a B' linear map $g' : L' \to N \times_\varphi M'$ we can set g equal to the composition $L' \to N \times_\varphi M' \to N$ and f' equal to the composition $L' \to N \times_\varphi M' \to M'$. These constructions are mutually inverse to each other and define the desired isomorphism. □

15.6. Fibre products of rings, II

0D2G In this section we discuss fibre products in the following situation.

08KH **Situation 15.6.1.** In the following we will consider ring maps

$$B \longrightarrow A \longleftarrow A'$$

where we assume $A' \to A$ is surjective with kernel I. In this situation we set $B' = B \times_A A'$ to obtain a cartesian square

$$\begin{array}{ccc} A & \longleftarrow & A' \\ \uparrow & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

0B7J **Lemma 15.6.2.** *In Situation 15.6.1 we have*

$$\mathrm{Spec}(B') = \mathrm{Spec}(B) \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$$

as topological spaces.

Proof. Since $B' = B \times_A A'$ we obtain a commutative square of spectra, which induces a continuous map

$$can : \mathrm{Spec}(B) \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \longrightarrow \mathrm{Spec}(B')$$

as the source is a pushout in the category of topological spaces (which exists by Topology, Section 5.29).

To show the map *can* is surjective, let $\mathfrak{q}' \subset B'$ be a prime ideal. If $I \subset \mathfrak{q}'$ (here and below we take the liberty of considering I as an ideal of B' as well as an ideal of A'), then $\mathfrak{q}'$ corresponds to a prime ideal of B and is in the image. If not, then pick $h \in I$, $h \notin \mathfrak{q}'$. In this case $B_h = A_h = 0$ and the ring map $B'_h \to A'_h$ is an isomorphism, see Lemma 15.5.3. Thus we see that $\mathfrak{q}'$ corresponds to a unique prime ideal $\mathfrak{p}' \subset A'$ which does not contain I.

Since $B' \to B$ is surjective, we see that *can* is injective on the summand $\mathrm{Spec}(B)$. We have seen above that $\mathrm{Spec}(A') \to \mathrm{Spec}(B')$ is injective on the complement of $V(I) \subset \mathrm{Spec}(A')$. Since $V(I) \subset \mathrm{Spec}(A')$ is exactly the image of $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ a trivial set theoretic argument shows that *can* is injective.

To finish the proof we have to show that *can* is open. To do this, observe that an open of the pushout is of the form $V \amalg U'$ where $V \subset \mathrm{Spec}(B)$ and $U' \subset \mathrm{Spec}(A')$ are opens whose inverse images in $\mathrm{Spec}(A)$ agree. Let $v \in V$. We can find a $g \in B$ such that $v \in D(g) \subset V$. Let $f \in A$ be the image. Pick $f' \in A'$ mapping to f. Then $D(f') \cap U' \cap V(I) = D(f') \cap V(I)$. Hence $V(I) \cap D(f')$ and $D(f') \cap (U')^c$ are disjoint closed subsets of $D(f') = \mathrm{Spec}(A'_{f'})$. Write $(U')^c = V(J)$ for some ideal $J \subset A'$. Since $A'_{f'} \to A'_{f'}/IA'_{f'} \times A'_{f'}/JA'_{f'}$ is surjective by the disjointness just shown, we can find an $a'' \in A'_{f'}$ mapping to 1 in $A'_{f'}/IA'_{f'}$ and mapping to zero in $A'_{f'}/JA'_{f'}$. Clearing denominators, we find an element $a' \in J$ mapping to f^n in A. Then $D(a'f') \subset U'$. Let $h' = (g^{n+1}, a'f') \in B'$. Since $B'_{h'} = B_{g^{n+1}} \times_{A_{f^{n+1}}} A'_{a'f'}$ by a previously cited lemma, we see that $D(h')$ pulls back to an open neighbourhood of v in the pushout, i.e., the image of $V \amalg U'$ contains an open neighbourhood of the image of v. We omit the (easier) proof that the same thing is true for $u' \in U'$ with $u' \notin V(I)$. □

0E1S **Lemma 15.6.3.** *In Situation 15.6.1 if $B \to A$ is integral, then $B' \to A'$ is integral.*

Proof. Let $a' \in A'$ with image $a \in A$. Let $x^d + b_1 x^{d-1} + \ldots + b_d$ be a monical polynomial with coefficients in B satisfied by a. Choose $b'_i \in B'$ mapping to $b_i \in B$ (possible). Then $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d$ is in the kernel of $A' \to A$. Since $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$ we can modify our choice of b'_d to get $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d = 0$ as desired. □

In Situation 15.6.1 we'd like to understand B'-modules in terms of modules over A', A, and B. In order to do this we consider the functor (where the fibre product of categories as constructed in Categories, Example 4.31.3)

08KI (15.6.3.1) $$\mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} A', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B A = L' \otimes_{B'} A' \otimes_{A'} A$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an A'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism. However, it is often more convenient think of φ as a B-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

07RU **Lemma 15.6.4.** *In Situation 15.6.1 the functor (15.6.3.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (15.6.3.1) is the identity functor on $Mod_B \times_{Mod_A} Mod_{A'}$. In other words, setting $N' = N \times_{\varphi,M} M'$ we have $N' \otimes_{B'} B = N$ and $N' \otimes_{B'} A' = M'$.

Proof. The adjointness statement follows from the more general Lemma 15.5.4. To prove the final assertion, recall that $B' = B \times_A A'$ and $N' = N \times_{\varphi,M} M'$ and extend these equalities to

$$\begin{array}{ccccc} A & \leftarrow & A' & \leftarrow & I \\ \uparrow & & \uparrow & & \uparrow \\ B & \leftarrow & B' & \leftarrow & J \end{array} \quad \text{and} \quad \begin{array}{ccccc} M & \leftarrow & M' & \leftarrow & K \\ \uparrow{\scriptstyle \varphi} & & \uparrow & & \uparrow \\ N & \leftarrow & N' & \leftarrow & L \end{array}$$

where I, J, K, L are the kernels of the horizontal maps of the original diagrams. We present the proof as a sequence of observations:

(1) $K = IM'$ (see statement lemma),
(2) $B' \to B$ is surjective with kernel J and $J \to I$ is bijective,
(3) $N' \to N$ is surjective with kernel L and $L \to K$ is bijective,
(4) $JN' \subset L$,
(5) $\mathrm{Im}(N \to M)$ generates M as an A-module (because $N \otimes_B A = M$),
(6) $\mathrm{Im}(N' \to M')$ generates M' as an A'-module (because it holds modulo K and L maps isomorphically to K),
(7) $JN' = L$ (because $L \cong K = IM'$ is generated by images of elements xn' with $x \in I$ and $n' \in N'$ by the previous statement),
(8) $N' \otimes_{B'} B = N$ (because $N = N'/L$, $B = B'/J$, and the previous statement),
(9) there is a map $\gamma : N' \otimes_{B'} A' \to M'$,
(10) γ is surjective (see above),
(11) the kernel of the composition $N' \otimes_{B'} A' \to M' \to M$ is generated by elements $l \otimes 1$ and $n' \otimes x$ with $l \in K$, $n' \in N'$, $x \in I$ (because $M = N \otimes_B A$ by assumption and because $N' \to N$ and $A' \to A$ are surjective with kernels L and I),
(12) any element of $N' \otimes_{B'} A'$ in the submodule generated by the elements $l \otimes 1$ and $n' \otimes x$ with $l \in L$, $n' \in N'$, $x \in I$ can be written as $l \otimes 1$ for some $l \in L$ (because J maps isomorphically to I we see that $n' \otimes x = n'x \otimes 1$ in $N' \otimes_{B'} A'$; similarly $xn' \otimes a' = n' \otimes xa' = n'(xa') \otimes 1$ in $N' \otimes_{B'} A'$ when $n' \in N'$, $x \in J$ and $a' \in A'$; since we have seen that $JN' = L$ this proves the assertion),
(13) the kernel of γ is zero (because by (10) and (11) any element of the kernel is of the form $l \otimes 1$ with $l \in L$ which is mapped to $l \in K \subset M'$ by γ).

This finishes the proof. □

08IG **Lemma 15.6.5.** *In the situation of Lemma 15.6.4 for a B'-module L' the adjunction map*

$$L' \longrightarrow (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

is surjective but in general not injective.

Proof. As in the proof of Lemma 15.6.4 let $J \subset B'$ be the kernel of the map $B' \to B$. Then $L' \otimes_{B'} B = L'/JL'$. Hence to prove surjectivity it suffices to show that elements of the form $(0, z)$ of the fibre product are in the image of the map of the lemma. The kernel of the map $L' \otimes_{B'} A' \to L' \otimes_{B'} A$ is the image of $L' \otimes_{B'} I \to L' \otimes_{B'} A'$. Since the map $J \to I$ induced by $B' \to A'$ is an isomorphism the composition

$$L' \otimes_{B'} J \to L' \to (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

induces a surjection of $L' \otimes_{B'} J$ onto the set of elements of the form $(0, z)$. To see the map is not injective in general we present a simple example. Namely, take a field k, set $B' = k[x, y]/(xy)$, $A' = B'/(x)$, $B = B'/(y)$, $A = B'/(x, y)$ and $L' = B'/(x - y)$. In that case the class of x in L' is nonzero but is mapped to zero under the displayed arrow. □

08KJ **Lemma 15.6.6.** *In Situation 15.6.1 let $(N_1, M_1', \varphi_1) \to (N_2, M_2', \varphi_2)$ be a morphism of $Mod_B \times_{Mod_A} Mod_{A'}$ with $N_1 \to N_2$ and $M_1' \to M_2'$ surjective. Then*

$$N_1 \times_{\varphi_1, M_1} M_1' \to N_2 \times_{\varphi_2, M_2} M_2'$$

where $M_1 = M_1'/IM_1'$ and $M_2 = M_2'/IM_2'$ is surjective.

Proof. Pick $(x_2, y_2) \in N_2 \times_{\varphi_2, M_2} M_2'$. Choose $x_1 \in N_1$ mapping to x_2. Since $M_1' \to M_1$ is surjective we can find $y_1 \in M_1'$ mapping to $\varphi_1(x_1)$. Then (x_1, y_1) maps to (x_2, y_2') in $N_2 \times_{\varphi_2, M_2} M_2'$. Thus it suffices to show that elements of the form $(0, y_2)$ are in the image of the map. Here we see that $y_2 \in IM_2'$. Write $y_2 = \sum t_i y_{2,i}$ with $t_i \in I$. Choose $y_{1,i} \in M_1'$ mapping to $y_{2,i}$. Then $y_1 = \sum t_i y_{1,i} \in IM_1'$ and the element $(0, y_1)$ does the job. □

0D2H **Lemma 15.6.7.** *Let $A, A', B, B', I, M, M', N, \varphi$ be as in Lemma 15.6.4. If N finite over B and M' finite over A', then $N' = N \times_{\varphi, M} M'$ is finite over B'.*

Proof. We will use the results of Lemma 15.6.4 without further mention. Choose generators $y_1, \ldots, y_r$ of N over B and generators $x_1, \ldots, x_s$ of M' over A'. Using that $N = N' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in N'$ mapping to $y_1, \ldots, y_r$ in N. Using that $M' = N' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in N'$ such that $x_i = \sum v_j \otimes a_{ij}'$ for some $a_{ij}' \in A'$. In particular we see that the images $\overline{v}_j \in M'$ of the v_j generate M' over A'. We claim that $u_1, \ldots, u_r, v_1, \ldots, v_t$ generate N' as a B'-module. Namely, pick $\xi \in N'$. We first choose $b_1', \ldots, b_r' \in B'$ such that ξ and $\sum b_i' u_i$ map to the same element of N. This is possible because $B' \to B$ is surjective and $y_1, \ldots, y_r$ generate N over B. The difference $\xi - \sum b_i' u_i$ is of the form $(0, \theta)$ for some θ in IM'. Say θ is $\sum t_j \overline{v}_j$ with $t_j \in I$. As $J = \mathrm{Ker}(B' \to B)$ maps isomorphically to I we can choose $s_j \in J \subset B'$ mapping to t_j. Because $N' = N \times_{\varphi, M} M'$ it follows that $\xi = \sum b_i' u_i + \sum s_j v_j$ as desired. □

0D2I **Lemma 15.6.8.** *With A, A', B, B', I as in Situation 15.6.1.*

1. *Let (N, M', φ) be an object of $Mod_B \times_{Mod_A} Mod_{A'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi, M} M'$ is flat over B'.*
2. *If L' is a flat B'-module, then $L' = (L \otimes_{B'} B) \times_{(L \otimes_{B'} A)} (L \otimes_{B'} A')$.*
3. *The category of flat B'-modules is equivalent to the full subcategory of $Mod_B \times_{Mod_A} Mod_{A'}$ consisting of triples (N, M', φ) with N flat over B and M' flat over A'.*

Proof. In the proof we will use Lemma 15.6.4 without further mention.

Proof of (1). Set $J = \mathrm{Ker}(B' \to B)$. This is an ideal of B' mapping isomorphically to $I = \mathrm{Ker}(A' \to A)$. Let $\mathfrak{b}' \subset B'$ be an ideal. We have to show that $\mathfrak{b}' \otimes_{B'} N' \to N'$ is injective, see Algebra, Lemma 10.39.5. We know that

$$\mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_{B'} N' = \mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_B N \to N$$

is injective as N is flat over B. As $\mathfrak{b}' \cap J \to \mathfrak{b}' \to \mathfrak{b}'/(\mathfrak{b}' \cap J) \to 0$ is exact, we conclude that it suffices to show that $(\mathfrak{b}' \cap J) \otimes_{B'} N' \to N'$ is injective. Thus we may assume that $\mathfrak{b}' \subset J$. Next, since $J \to I$ is an isomorphism we have

$$J \otimes_{B'} N' = I \otimes_{A'} A' \otimes_{B'} N' = I \otimes_{A'} M'$$

which maps injectively into M' as M' is a flat A'-module. Hence $J \otimes_{B'} N' \to N'$ is injective and we conclude that $\mathrm{Tor}_1^{B'}(B'/J, N') = 0$, see Algebra, Remark 10.75.9. Thus we may apply Algebra, Lemma 10.99.8 to N' over B' and the ideal J. Going back to our ideal $\mathfrak{b}' \subset J$, let $\mathfrak{b}' \subset \mathfrak{b}'' \subset J$ be the smallest ideal whose image in I is an A'-submodule of I. In other words, we have $\mathfrak{b}'' = A'\mathfrak{b}'$ if we view $J = I$ as A'-module. Then $\mathfrak{b}''/\mathfrak{b}'$ is killed by J and we get a short exact sequence

$$0 \to \mathfrak{b}' \otimes_{B'} N' \to \mathfrak{b}'' \otimes_{B'} N' \to \mathfrak{b}''/\mathfrak{b}' \otimes_{B'} N' \to 0$$

by the vanishing of $\mathrm{Tor}_1^{B'}(\mathfrak{b}''/\mathfrak{b}', N')$ we get from the application of the lemma. Thus we may replace $\mathfrak{b}'$ by $\mathfrak{b}''$. In particular we may assume $\mathfrak{b}'$ is an A'-module and maps to an ideal of A'. Then

$$\mathfrak{b}' \otimes_{B'} N' = \mathfrak{b}' \otimes_{A'} A' \otimes_{B'} N' = \mathfrak{b}' \otimes_{A'} M'$$

This tensor product maps injectively into M' by our assumption that M' is flat over A'. We conclude that $\mathfrak{b}' \otimes_{B'} N' \to N' \to M'$ is injective and hence the first map is injective as desired.

Proof of (2). This follows by tensoring the short exact sequence $0 \to B' \to B \oplus A' \to A \to 0$ with L' over B'.

Proof of (3). Immediate consequence of (1) and (2). □

0D2J **Lemma 15.6.9.** *Let A, A', B, B', I be as in Situation 15.6.1. The category of finite projective B'-modules is equivalent to the full subcategory of $Mod_B \times_{Mod_A} Mod_{A'}$ consisting of triples (N, M', φ) with N finite projective over B and M' finite projective over A'.*

Proof. Recall that a module is finite projective if and only if it is finitely presented and flat, see Algebra, Lemma 10.78.2. Using Lemmas 15.6.8 and 15.6.7 we reduce to showing that $N' = N \times_{\varphi,M} M'$ is a B'-module of finite presentation if N finite projective over B and M' finite projective over A'.

By Lemma 15.6.7 the module N' is finite over B'. Choose a surjection $(B')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $B^{\oplus n} \to N$, $(A')^{\oplus n} \to M'$, and $A^{\oplus n} \to M$ with kernels K_B, $K_{A'}$, and K_A. There is a canonical map

$$K' \longrightarrow K_B \times_{K_A} K_{A'}$$

On the other hand, since $N' = N \times_{\varphi,M} M'$ and $B' = B \times_A A'$ there is also a canonical map $K_B \times_{K_A} K_{A'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 10.5.3 the modules K_B and $K_{A'}$ are finite. We conclude from Lemma 15.6.7 that K' is a finite B'-module provided that $K_B \to K_A$ and $K_{A'} \to K_A$ induce isomorphisms $K_B \otimes_B A = K_A = K_{A'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_B \to B^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{A'} \to (A')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 10.39.12. □

15.7. Fibre products of rings, III

0D2K In this section we discuss fibre products in the following situation.

08KK **Situation 15.7.1.** Let A, A', B, B', I be as in Situation 15.6.1. Let $B' \to D'$ be a ring map. Set $D = D' \otimes_{B'} B$, $C' = D' \otimes_{B'} A'$, and $C = D' \otimes_{B'} A$. This leads to a

big commutative diagram

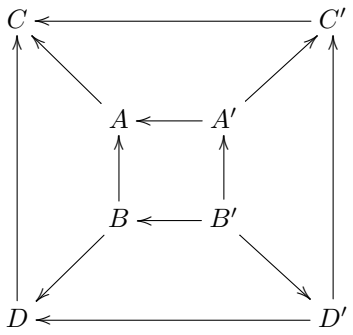

of rings. Observe that we do **not** assume that the map $D' \to D \times_C C'$ is an isomorphism[1]. In this situation we have the functor

08KL (15.7.1.1) $$\mathrm{Mod}_{D'} \longrightarrow \mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'}, \quad L' \longmapsto (L' \otimes_{D'} D, L' \otimes_{D'} C', can)$$

analogous to (15.6.3.1). Note that $L' \otimes_{D'} D = L \otimes_{D'} (D' \otimes_{B'} B) = L \otimes_{B'} B$ and similarly $L' \otimes_{D'} C' = L \otimes_{D'} (D' \otimes_{B'} A') = L \otimes_{B'} A'$ hence the diagram

$$\begin{array}{ccc} \mathrm{Mod}_{D'} & \longrightarrow & \mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{B'} & \longrightarrow & \mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'} \end{array}$$

is commutative. In the following we will write (N, M', φ) for an object of $\mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'}$, i.e., N is a D-module, M' is an C'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism of C-modules. However, it is often more convenient think of φ as a D-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

08KM **Lemma 15.7.2.** *In Situation 15.7.1 the functor (15.7.1.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (15.7.1.1) is the identity functor on $Mod_D \times_{Mod_C} Mod_{C'}$. In other words, setting $N' = N \times_{\varphi,M} M'$ we have $N' \otimes_{D'} D = N$ and $N' \otimes_{D'} C' = M'$.

Proof. The adjointness statement follows from the more general Lemma 15.5.4. The final assertion follows from the corresponding assertion of Lemma 15.6.4 because $N' \otimes_{D'} D = N' \otimes_{D'} D' \otimes_{B'} B = N' \otimes_{B'} B$ and $N' \otimes_{D'} C' = N' \otimes_{D'} D' \otimes_{B'} A' = N' \otimes_{B'} A'$. □

08KN **Lemma 15.7.3.** *In Situation 15.7.1 the map $JD' \to IC'$ is surjective where $J = \mathrm{Ker}(B' \to B)$.*

Proof. Since $C' = D' \otimes_{B'} A'$ we have that IC' is the image of $D' \otimes_{B'} I = C' \otimes_{A'} I \to C'$. As the ring map $B' \to A'$ induces an isomorphism $J \to I$ the lemma follows. □

08IH **Lemma 15.7.4.** *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.7.2. If N finite over D and M' finite over C', then $N' = N \times_{\varphi,M} M'$ is finite over D'.*

[1] But $D' \to D \times_C C'$ is surjective by Lemma 15.6.5.

Proof. Recall that $D' \to D \times_C C'$ is surjective by Lemma 15.6.5. Observe that $N' = N \times_{\varphi,M} M'$ is a module over $D \times_C C'$. We can apply Lemma 15.6.7 to the data $C, C', D, D', IC', M', M, N, \varphi$ to see that $N' = N \times_{\varphi,M} M'$ is finite over $D \times_C C'$. Thus it is finite over D'. □

07RW **Lemma 15.7.5.** *With $A, A', B, B', C, C', D, D', I$ as in Situation 15.7.1.*

(1) *Let (N, M', φ) be an object of $Mod_D \times_{Mod_C} Mod_{C'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi,M} M'$ is flat over B'.*
(2) *If L' is a D'-module flat over B', then $L' = (L \otimes_{D'} D) \times_{(L \otimes_{D'} C)} (L \otimes_{D'} C')$.*
(3) *The category of D'-modules flat over B' is equivalent to the categories of objects (N, M', φ) of $Mod_D \times_{Mod_C} Mod_{C'}$ with N flat over B and M' flat over A'.*

Proof. Part (1) follows from part (1) of Lemma 15.6.8.

Part (2) follows from part (2) of Lemma 15.6.8 using that $L' \otimes_{D'} D = L' \otimes_{B'} B$, $L' \otimes_{D'} C' = L' \otimes_{B'} A'$, and $L' \otimes_{D'} C = L' \otimes_{B'} A$, see discussion in Situation 15.7.1.

Part (3) is an immediate consequence of (1) and (2). □

The following lemma is a good deal more interesting than its counter part in the absolute case (Lemma 15.6.9), although the proof is essentially the same.

08KP **Lemma 15.7.6.** *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.7.2. If*

(1) *N is finitely presented over D and flat over B,*
(2) *M' finitely presented over C' and flat over A', and*
(3) *the ring map $B' \to D'$ factors as $B' \to D'' \to D'$ with $B' \to D''$ flat and $D'' \to D'$ of finite presentation,*

then $N' = N \times_M M'$ is finitely presented over D'.

Proof. Choose a surjection $D''' = D''[x_1, \ldots, x_n] \to D'$ with finitely generated kernel J. By Algebra, Lemma 10.36.23 it suffices to show that N' is finitely presented as a D'''-module. Moreover, $D''' \otimes_{B'} B \to D' \otimes_{B'} B = D$ and $D''' \otimes_{B'} A' \to D' \otimes_{B'} A' = C'$ are surjections whose kernels are generated by the image of J, hence N is a finitely presented $D''' \otimes_{B'} B$-module and M' is a finitely presented $D''' \otimes_{B'} A'$-module by Algebra, Lemma 10.36.23 again. Thus we may replace D' by D''' and D by $D''' \otimes_{B'} B$, etc. Since D''' is flat over B', it follows that we may assume that $B' \to D'$ is flat.

Assume $B' \to D'$ is flat. By Lemma 15.7.4 the module N' is finite over D'. Choose a surjection $(D')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $D^{\oplus n} \to N$, $(C')^{\oplus n} \to M'$, and $C^{\oplus n} \to M$ with kernels K_D, $K_{C'}$, and K_C. There is a canonical map

$$K' \longrightarrow K_D \times_{K_C} K_{C'}$$

On the other hand, since $N' = N \times_M M'$ and $D' = D \times_C C'$ (by Lemma 15.6.8; applied to the flat B'-module D') there is also a canonical map $K_D \times_{K_C} K_{C'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 10.5.3 the modules K_D and $K_{C'}$ are finite. We conclude from Lemma 15.7.4 that K' is a finite D'-module provided that $K_D \to K_C$ and $K_{C'} \to$

K_C induce isomorphisms $K_D \otimes_B A = K_C = K_{C'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_D \to D^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{C'} \to (C')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 10.39.12. □

08KQ **Lemma 15.7.7.** *Let A, A', B, B', I be as in Situation 15.6.1. Let (D, C', φ) be a system consisting of an B-algebra D, a A'-algebra C' and an isomorphism $D \otimes_B A \to C'/IC' = C$. Set $D' = D \times_C C'$ (as in Lemma 15.6.4). Then*

1. *$B' \to D'$ is finite type if and only if $B \to D$ and $A' \to C'$ are finite type,*
2. *$B' \to D'$ is flat if and only if $B \to D$ and $A' \to C'$ are flat,*
3. *$B' \to D'$ is flat and of finite presentation if and only if $B \to D$ and $A' \to C'$ are flat and of finite presentation,*
4. *$B' \to D'$ is smooth if and only if $B \to D$ and $A' \to C'$ are smooth,*
5. *$B' \to D'$ is étale if and only if $B \to D$ and $A' \to C'$ are étale.*

Moreover, if D' is a flat B'-algebra, then $D' \to (D' \otimes_{B'} B) \times_{(D' \otimes_{B'} A)} (D' \otimes_{B'} A')$ is an isomorphism. In this way the category of flat B'-algebras is equivalent to the categories of systems (D, C', φ) as above with D flat over B and C' flat over A'.

Proof. The implication "⇒" follows from Algebra, Lemmas 10.14.2, 10.39.7, 10.137.3, and 10.143.3 because we have $D' \otimes_{B'} B = D$ and $D' \otimes_{B'} A' = C'$ by Lemma 15.6.4. Thus it suffices to prove the implications in the other direction.

Ad (1). Assume D of finite type over B and C' of finite type over A'. We will use the results of Lemma 15.6.4 without further mention. Choose generators $x_1, \ldots, x_r$ of D over B and generators $y_1, \ldots, y_s$ of C' over A'. Using that $D = D' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in D'$ mapping to $x_1, \ldots, x_r$ in D. Using that $C' = D' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in D'$ such that $y_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular, the images of v_j in C' generate C' as an A'-algebra. Set $N = r + t$ and consider the cube of rings

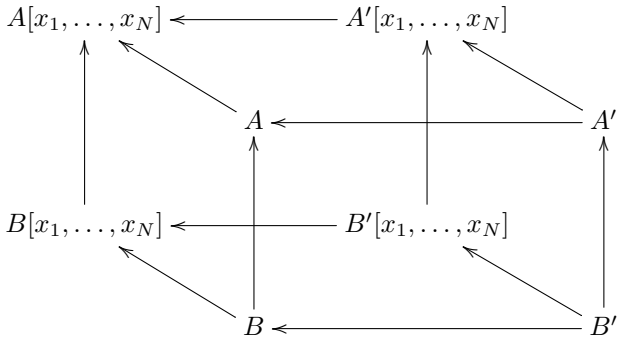

Observe that the back square is cartesian as well. Consider the ring map

$$B'[x_1, \ldots, x_N] \to D', \quad x_i \mapsto u_i \quad \text{and} \quad x_{r+j} \mapsto v_j.$$

Then we see that the induced maps $B[x_1, \ldots, x_N] \to D$ and $A'[x_1, \ldots, x_N] \to C'$ are surjective, in particular finite. We conclude from Lemma 15.7.4 that $B'[x_1, \ldots, x_N] \to D'$ is finite, which implies that D' is of finite type over B' for example by Algebra, Lemma 10.6.2.

Ad (2). The implication "⇐" follows from Lemma 15.7.5. Moreover, the final statement follows from the final statement of Lemma 15.7.5.

Ad (3). Assume $B \to D$ and $A' \to C'$ are flat and of finite presentation. The flatness of $B' \to D'$ we've seen in (2). We know $B' \to D'$ is of finite type by (1). Choose a surjection $B'[x_1, \ldots, x_N] \to D'$. By Algebra, Lemma 10.6.3 the ring D is of finite presentation as a $B[x_1, \ldots, x_N]$-module and the ring C' is of finite presentation as a $A'[x_1, \ldots, x_N]$-module. By Lemma 15.7.6 we see that D' is of finite presentation as a $B'[x_1, \ldots, x_N]$-module, i.e., $B' \to D'$ is of finite presentation.

Ad (4). Assume $B \to D$ and $A' \to C'$ smooth. By (3) we see that $B' \to D'$ is flat and of finite presentation. By Algebra, Lemma 10.137.16 it suffices to check that $D' \otimes_{B'} k$ is smooth for any field k over B'. If the composition $J \to B' \to k$ is zero, then $B' \to k$ factors as $B' \to B \to k$ and we see that

$$D' \otimes_{B'} k = D' \otimes_{B'} B \otimes_B k = D \otimes_B k$$

is smooth as $B \to D$ is smooth. If the composition $J \to B' \to k$ is nonzero, then there exists an $h \in J$ which does not map to zero in k. Then $B' \to k$ factors as $B' \to B'_h \to k$. Observe that h maps to zero in B, hence $B_h = 0$. Thus by Lemma 15.5.3 we have $B'_h = A'_h$ and we get

$$D' \otimes_{B'} k = D' \otimes_{B'} B'_h \otimes_{B'_h} k = C'_h \otimes_{A'_h} k$$

is smooth as $A' \to C'$ is smooth.

Ad (5). Assume $B \to D$ and $A' \to C'$ are étale. By (4) we see that $B' \to D'$ is smooth. As we can read off whether or not a smooth map is étale from the dimension of fibres we see that (5) holds (argue as in the proof of (4) to identify fibres – some details omitted). □

08KR **Remark 15.7.8.** In Situation 15.7.1. Assume $B' \to D'$ is of finite presentation and suppose we are given a D'-module L'. We claim there is a bijective correspondence between

(1) surjections of D'-modules $L' \to Q'$ with Q' of finite presentation over D' and flat over B', and
(2) pairs of surjections of modules $(L' \otimes_{D'} D \to Q_1, L' \otimes_{D'} C' \to Q_2)$ with
   (a) Q_1 of finite presentation over D and flat over B,
   (b) Q_2 of finite presentation over C' and flat over A',
   (c) $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ as quotients of $L' \otimes_{D'} C$.

The correspondence between these is given by $Q \mapsto (Q_1, Q_2)$ with $Q_1 = Q \otimes_{D'} D$ and $Q_2 = Q \otimes_{D'} C'$. And for the converse we use $Q = Q_1 \times_{Q_{12}} Q_2$ where Q_{12} the common quotient $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ of $L' \otimes_{D'} C$. As quotient map we use

$$L' \longrightarrow (L' \otimes_{D'} D) \times_{(L' \otimes_{D'} C)} (L' \otimes_{D'} C') \longrightarrow Q_1 \times_{Q_{12}} Q_2 = Q$$

where the first arrow is surjective by Lemma 15.6.5 and the second by Lemma 15.6.6. The claim follows by Lemmas 15.7.5 and 15.7.6.

15.8. Fitting ideals

07Z6 The Fitting ideals of a finite module are the ideals determined by the construction of Lemma 15.8.2.

07Z7 **Lemma 15.8.1.** *Let R be a ring. Let A be an $n \times m$ matrix with coefficients in R. Let $I_r(A)$ be the ideal generated by the $r \times r$-minors of A with the convention that $I_0(A) = R$ and $I_r(A) = 0$ if $r > \min(n, m)$. Then*

(1) $I_0(A) \supset I_1(A) \supset I_2(A) \supset \ldots$,

(2) *if B is an $(n+n') \times m$ matrix, and A is the first n rows of B, then $I_{r+n'}(B) \subset I_r(A)$,*
(3) *if C is an $n \times n$ matrix then $I_r(CA) \subset I_r(A)$.*
(4) *If A is a block matrix*
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
then $I_r(A) = \sum_{r_1+r_2=r} I_{r_1}(A_1)I_{r_2}(A_2)$.
(5) *Add more here.*

Proof. Omitted. (Hint: Use that a determinant can be computed by expanding along a column or a row.) □

07Z8 **Lemma 15.8.2.** *Let R be a ring. Let M be a finite R-module. Choose a presentation*
$$\bigoplus\nolimits_{j\in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$
of M. Let $A = (a_{ij})_{i=1,\ldots,n, j\in J}$ be the matrix of the map $\bigoplus_{j\in J} R \to R^{\oplus n}$. The ideal $Fit_k(M)$ generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation.

Proof. Let $K \subset R^{\oplus n}$ be the kernel of the surjection $R^{\oplus n} \to M$. Pick $z_1, \ldots, z_{n-k} \in K$ and write $z_j = (z_{1j}, \ldots, z_{nj})$. Another description of the ideal $\mathrm{Fit}_k(M)$ is that it is the ideal generated by the $(n-k) \times (n-k)$ minors of all the matrices (z_{ij}) we obtain in this way.

Suppose we change the surjection into the surjection $R^{\oplus n+n'} \to M$ with kernel K' where we use the original map on the first n standard basis elements of $R^{\oplus n+n'}$ and 0 on the last n' basis vectors. Then the corresponding ideals are the same. Namely, if $z_1, \ldots, z_{n-k} \in K$ as above, let $z'_j = (z_{1j}, \ldots, z_{nj}, 0, \ldots, 0) \in K'$ for $j = 1, \ldots, n-k$ and $z'_{n+j'} = (0, \ldots, 0, 1, 0, \ldots, 0) \in K'$. Then we see that the ideal of $(n-k) \times (n-k)$ minors of (z_{ij}) agrees with the ideal of $(n+n'-k) \times (n+n'-k)$ minors of (z'_{ij}). This gives one of the inclusions. Conversely, given $z'_1, \ldots, z'_{n+n'-k}$ in K' we can project these to $R^{\oplus n}$ to get $z_1, \ldots, z_{n+n'-k}$ in K. By Lemma 15.8.1 we see that the ideal generated by the $(n+n'-k) \times (n+n'-k)$ minors of (z'_{ij}) is contained in the ideal generated by the $(n-k) \times (n-k)$ minors of (z_{ij}). This gives the other inclusion.

Let $R^{\oplus m} \to M$ be another surjection with kernel L. By Schanuel's lemma (Algebra, Lemma 10.109.1) and the results of the previous paragraph, we may assume $m = n$ and that there is an isomorphism $R^{\oplus n} \to R^{\oplus m}$ commuting with the surjections to M. Let $C = (c_{li})$ be the (invertible) matrix of this map (it is a square matrix as $n = m$). Then given $z'_1, \ldots, z'_{n-k} \in L$ as above we can find $z_1, \ldots, z_{n-k} \in K$ with $z'_1 = Cz_1, \ldots, z'_{n-k} = Cz_{n-k}$. By Lemma 15.8.1 we get one of the inclusions. By symmetry we get the other. □

07Z9 **Definition 15.8.3.** Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. The *kth Fitting ideal* of M is the ideal $\mathrm{Fit}_k(M)$ constructed in Lemma 15.8.2. Set $\mathrm{Fit}_{-1}(M) = 0$.

Since the Fitting ideals are the ideals of minors of a big matrix (numbered in reverse ordering from the ordering in Lemma 15.8.1) we see that
$$0 = \mathrm{Fit}_{-1}(M) \subset \mathrm{Fit}_0(M) \subset \mathrm{Fit}_1(M) \subset \ldots \subset \mathrm{Fit}_t(M) = R$$

for some $t \gg 0$. Here are some basic properties of Fitting ideals.

07ZA **Lemma 15.8.4.** *Let R be a ring. Let M be a finite R-module.*

(1) *If M can be generated by n elements, then $Fit_n(M) = R$.*
(2) *Given a second finite R-module M' we have $Fit_0(M \oplus M') = Fit_0(M)Fit_0(M')$ and more generally*
$$Fit_l(M \oplus M') = \sum\nolimits_{k+k'=l} Fit_k(M)Fit_{k'}(M')$$
(3) *If $R \to R'$ is a ring map, then $Fit_k(M \otimes_R R')$ is the ideal of R' generated by the image of $Fit_k(M)$.*
(4) *If M is of finite presentation, then $Fit_k(M)$ is a finitely generated ideal.*
(5) *If $M \to M'$ is a surjection, then $Fit_k(M) \subset Fit_k(M')$.*
(6) *We have $Fit_0(M) \subset Ann_R(M)$.*
(7) *We have $V(Fit_0(M)) = Supp(M)$.*
(8) *If I is an ideal of R, then $Fit_0(R/I) = I$.*
(9) *Add more here.*

Proof. Part (1) follows from the fact that $I_0(A) = R$ in Lemma 15.8.1.

Part (2) follows form the corresponding statement in Lemma 15.8.1.

Part (3) follows from the fact that $\otimes_R R'$ is right exact, so the base change of a presentation of M is a presentation of $M \otimes_R R'$.

Proof of (4). Let $R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$ be a presentation. Then $\mathrm{Fit}_k(M)$ is the ideal generated by the $(n-k) \times (n-k)$ minors of the matrix A.

Part (5) is immediate from the definition.

Proof of (6). Choose a presentation of M with matrix A as in Lemma 15.8.2. Let $J' \subset J$ be a subset of cardinality n. It suffices to show that $f = \det(a_{ij})_{i=1,\ldots,n,j\in J'}$ annihilates M. This is clear because the cokernel of
$$R^{\oplus n} \xrightarrow{A'=(a_{ij})_{i=1,\ldots,n,j\in J'}} R^{\oplus n} \to M \to 0$$
is killed by f as there is a matrix B with $A'B = f1_{n\times n}$.

Proof of (7). Choose a presentation of M with matrix A as in Lemma 15.8.2. By Nakayama's lemma (Algebra, Lemma 10.20.1) we have
$$M_{\mathfrak{p}} \neq 0 \Leftrightarrow M \otimes_R \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \mathrm{rank}(\text{image } A \text{ in } \kappa(\mathfrak{p})) < n$$
Clearly $\mathrm{Fit}_0(M)$ exactly cuts out the set of primes with this property. □

07ZB **Example 15.8.5.** Let R be a ring. The Fitting ideals of the finite free module $M = R^{\oplus n}$ are $\mathrm{Fit}_k(M) = 0$ for $k < n$ and $\mathrm{Fit}_k(M) = R$ for $k \geq n$.

0H73 **Example 15.8.6.** Let R be a ring and let $I, J \subset R$ be ideals. Then $\mathrm{Fit}_0(R/I) = I$, $\mathrm{Fit}_0(R/I \oplus R/J) = IJ$, $\mathrm{Fit}_1(R/I \oplus R/J) = I + J$.

07ZC **Lemma 15.8.7.** *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. Let $\mathfrak{p} \subset R$ be a prime ideal. The following are equivalent*

(1) *$Fit_k(M) \not\subset \mathfrak{p}$,*
(2) *$\dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}) \leq k$,*
(3) *$M_{\mathfrak{p}}$ can be generated by k elements over $R_{\mathfrak{p}}$, and*
(4) *M_f can be generated by k elements over R_f for some $f \in R$, $f \notin \mathfrak{p}$.*

Proof. By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that M_f can be generated by k elements over R_f for some $f \in R$, $f \notin \mathfrak{p}$ if $M \otimes_R \kappa(\mathfrak{p})$ can be generated by k elements. Hence (2), (3), and (4) are equivalent. Using Lemma 15.8.4 part (3) this reduces the problem to the case where R is a field and $\mathfrak{p} = (0)$. In this case the result follows from Example 15.8.5. □

07ZD **Lemma 15.8.8.** *Let R be a ring. Let M be a finite R-module. Let $r \geq 0$. The following are equivalent*

(1) *M is finite locally free of rank r (Algebra, Definition 10.78.1),*
(2) *$Fit_{r-1}(M) = 0$ and $Fit_r(M) = R$, and*
(3) *$Fit_k(M) = 0$ for $k < r$ and $Fit_k(M) = R$ for $k \geq r$.*

Proof. It is immediate that (2) is equivalent to (3) because the Fitting ideals form an increasing sequence of ideals. Since the formation of $\mathrm{Fit}_k(M)$ commutes with base change (Lemma 15.8.4) we see that (1) implies (2) by Example 15.8.5 and glueing results (Algebra, Section 10.23). Conversely, assume (2). By Lemma 15.8.7 we may assume that M is generated by r elements. Thus a presentation $\bigoplus_{j\in J} R \to R^{\oplus r} \to M \to 0$. But now the assumption that $\mathrm{Fit}_{r-1}(M) = 0$ implies that all entries of the matrix of the map $\bigoplus_{j\in J} R \to R^{\oplus r}$ are zero. Thus M is free. □

080Z **Lemma 15.8.9.** *Let R be a local ring. Let M be a finite R-module. Let $k \geq 0$. Assume that $Fit_k(M) = (f)$ for some $f \in R$. Let M' be the quotient of M by $\{x \in M \mid fx = 0\}$. Then M' can be generated by k elements.*

Proof. Choose generators $x_1, \ldots, x_n \in M$ corresponding to the surjection $R^{\oplus n} \to M$. Since R is local if a set of elements $E \subset (f)$ generates (f), then some $e \in E$ generates (f), see Algebra, Lemma 10.20.1. Hence we may pick $z_1, \ldots, z_{n-k}$ in the kernel of $R^{\oplus n} \to M$ such that some $(n-k)\times(n-k)$ minor of the $n \times (n-k)$ matrix $A = (z_{ij})$ generates (f). After renumbering the x_i we may assume the first minor $\det(z_{ij})_{1\leq i,j\leq n-k}$ generates (f), i.e., $\det(z_{ij})_{1\leq i,j\leq n-k} = uf$ for some unit $u \in R$. Every other minor is a multiple of f. By Algebra, Lemma 10.15.6 there exists a $n - k \times n - k$ matrix B such that

$$AB = f \begin{pmatrix} u1_{n-k\times n-k} \\ C \end{pmatrix}$$

for some matrix C with coefficients in R. This implies that for every $i \leq n - k$ the element $y_i = ux_i + \sum_j c_{ji}x_j$ is annihilated by f. Since $M/\sum Ry_i$ is generated by the images of $x_{n-k+1}, \ldots, x_n$ we win. □

0F7M **Lemma 15.8.10.** *Let R be a ring. Let M be a finitely presented R-module. Let $k \geq 0$. Assume that $Fit_k(M) = (f)$ for some nonzerodivisor $f \in R$ and $Fit_{k-1}(M) = 0$. Then*

(1) *M has projective dimension ≤ 1,*
(2) *$M' = \mathrm{Ker}(f : M \to M)$ is the f-power torsion submodule of M,*
(3) *M' has projective dimension ≤ 1,*
(4) *M/M' is finite locally free of rank k, and*
(5) *$M \cong M/M' \oplus M'$.*

Proof. Choose a presentation

$$R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$$

for some matrix A with coefficients in R.

We first prove the lemma when R is local. Set $M' = \{x \in M \mid fx = 0\}$ as in the statement. By Lemma 15.8.9 we can choose $x_1, \ldots, x_k \in M$ which generate M/M'. Then $x_1, \ldots, x_k$ generate $M_f = (M/M')_f$. Hence, if there is a relation $\sum a_i x_i = 0$ in M, then we see that $a_1, \ldots, a_k$ map to zero in R_f since otherwise $\mathrm{Fit}_{k-1}(M)R_f = \mathrm{Fit}_{k-1}(M_f)$ would be nonzero. Since f is a nonzerodivisor, we conclude $a_1 = \ldots = a_k = 0$. Thus $M \cong R^{\oplus k} \oplus M'$. After a change of basis in our presentation above, we may assume the first $n - k$ basis vectors of $R^{\oplus n}$ map into the summand M' of M and the last k-basis vectors of $R^{\oplus n}$ map to basis elements of the summand $R^{\oplus k}$ of M. Having done so, the last k rows of the matrix A vanish. In this way we see that, replacing M by M', k by 0, n by $n - k$, and A by the submatrix where we delete the last k rows, we reduce to the case discussed in the next paragraph.

Assume R is local, $k = 0$, and M annihilated by f. Now the 0th Fitting ideal of M is (f) and is generated by the $n \times n$ minors of the matrix A of size $n \times m$. (This in particular implies $m \geq n$.) Since R is local, some $n \times n$ minor of A is uf for a unit $u \in R$. After renumbering we may assume this minor is the first one. Moreover, we know all other $n \times n$ minors of A are divisible by f. Write $A = (A_1 A_2)$ in block form where A_1 is an $n \times n$ matrix and A_2 is an $n \times (m - n)$ matrix. By Algebra, Lemma 10.15.6 applied to the transpose of A (!) we find there exists an $n \times n$ matrix B such that

$$BA = B(A_1 A_2) = f \begin{pmatrix} u1_{n\times n} & C \end{pmatrix}$$

for some $n \times (m - n)$ matrix C with coefficients in R. Then we first conclude $BA_1 = fu1_{n\times n}$. Thus

$$BA_2 = fC = u^{-1}fuC = u^{-1}BA_1C$$

Since the determinant of B is a nonzerodivisor we conclude that $A_2 = u^{-1}A_1C$. Therefore the image of A is equal to the image of A_1 which is isomorphic to $R^{\oplus n}$ because the determinant of A_1 is a nonzerodivisor. Hence M has projective dimension ≤ 1.

We return to the case of a general ring R. By the local case we see that M/M' is a finite locally free module of rank k, see Algebra, Lemma 10.78.2. Hence the extension $0 \to M' \to M \to M/M' \to 0$ splits. It follows that M' is a finitely presented module. Choose a short exact sequence $0 \to K \to R^{\oplus a} \to M' \to 0$. Then K is a finite R-module, see Algebra, Lemma 10.5.3. By the local case we see that $K_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus a}$ for all primes. Hence by Algebra, Lemma 10.78.2 again we see that K is finite locally free of rank a. It follows that M' has projective dimension ≤ 1 and the lemma is proved. □

15.9. Lifting

07LW In this section we collection some lemmas concerning lifting statements of the following kind: If A is a ring and $I \subset A$ is an ideal, and $\overline{\xi}$ is some kind of structure over A/I, then we can lift $\overline{\xi}$ to a similar kind of structure ξ over A or over some étale extension of A. Here are some types of structure for which we have already proved some results:

(1) idempotents, see Algebra, Lemmas 10.32.6 and 10.32.7,
(2) projective modules, see Algebra, Lemmas 10.77.5 and 10.77.6,

(3) finite stably free modules, see Lemma 15.3.3,
(4) basis elements, see Algebra, Lemmas 10.101.1 and 10.101.3,
(5) ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma 10.138.4, Proposition 10.138.13, and Lemma 10.138.17,
(6) syntomic ring maps, see Algebra, Lemma 10.136.18,
(7) smooth ring maps, see Algebra, Lemma 10.137.19,
(8) étale ring maps, see Algebra, Lemma 10.143.10,
(9) factoring polynomials, see Algebra, Lemma 10.143.13, and
(10) Algebra, Section 10.153 discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section 16.3 in particular Smoothing Ring Maps, Proposition 16.3.2.

Let A be a ring and let $I \subset A$ be an ideal. Let $\overline{\xi}$ be some kind of structure over A/I. In the following lemmas we look for étale ring maps $A \to A'$ which induce isomorphisms $A/I \to A'/IA'$ and objects ξ' over A' lifting $\overline{\xi}$. A general remark is that given étale ring maps $A \to A' \to A''$ such that $A/I \cong A'/IA'$ and $A'/IA' \cong A''/IA''$ the composition $A \to A''$ is also étale (Algebra, Lemma 10.143.3) and also satisfies $A/I \cong A''/IA''$. We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

07LX **Lemma 15.9.1.** *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{u} \in A/I$ be an invertible element. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an invertible element $u' \in A'$ lifting $\overline{u}$.*

Proof. Choose any lift $f \in A$ of $\overline{u}$ and set $A' = A_f$ and u the image of f in A'. □

07LY **Lemma 15.9.2.** *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{e} \in A/I$ be an idempotent. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in A'$ lifting $\overline{e}$.*

Proof. Choose any lift $x \in A$ of $\overline{e}$. Set

$$A' = A[t]/(t^2 - t)\left[\frac{1}{t - 1 + x}\right].$$

The ring map $A \to A'$ is étale because $(2t-1)\mathrm{d}t = 0$ and $(2t-1)(2t-1) = 1$ which is invertible. We have $A'/IA' = A/I[t]/(t^2-t)[\frac{1}{t-1+\overline{e}}] \cong A/I$ the last map sending t to $\overline{e}$ which works as $\overline{e}$ is a root of $t^2 - t$. This also shows that setting e' equal to the class of t in A' works. □

07LZ **Lemma 15.9.3.** *Let A be a ring, let $I \subset A$ be an ideal. Let $\mathrm{Spec}(A/I) = \coprod_{j \in J} \overline{U}_j$ be a finite disjoint open covering. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite disjoint open covering $\mathrm{Spec}(A') = \coprod_{j \in J} U'_j$ lifting the given covering.*

Proof. This follows from Lemma 15.9.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 10.21.3. □

07M0 **Lemma 15.9.4.** *Let $A \to B$ be a ring map and $J \subset B$ an ideal. If $A \to B$ is étale at every prime of $V(J)$, then there exists a $g \in B$ mapping to an invertible element of B/J such that $A' = B_g$ is étale over A.*

Proof. The set of points of $\mathrm{Spec}(B)$ where $A \to B$ is not étale is a closed subset of $\mathrm{Spec}(B)$, see Algebra, Definition 10.143.1. Write this as $V(J')$ for some ideal $J' \subset B$. Then $V(J') \cap V(J) = \emptyset$ hence $J + J' = B$ by Algebra, Lemma 10.17.2. Write $1 = f + g$ with $f \in J$ and $g \in J'$. Then g works. □

Next we have three lemmas saying we can lift factorizations of polynomials.

0ALH **Lemma 15.9.5.** *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ such that $\overline{g}$ and $\overline{h}$ are monic and generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ with g', h' monic lifting the given factorization over A/I.*

Proof. We will deduce this from results on the universal factorization proved earlier; however, we encourage the reader to find their own proof not using this trick. Say $\deg(\overline{g}) = n$ and $\deg(\overline{h}) = m$ so that $\deg(f) = n + m$. Write $f = x^{n+m} + \sum \alpha_i x^{n+m-i}$ for some $\alpha_1, \ldots, \alpha_{n+m} \in A$. Consider the ring map

$$R = \mathbf{Z}[a_1, \ldots, a_{n+m}] \longrightarrow S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

of Algebra, Example 10.143.12. Let $R \to A$ be the ring map which sends a_i to α_i. Set

$$B = A \otimes_R S$$

By construction the image f_B of f in $B[x]$ factors, say $f_B = g_B h_B$ with $g_B = x^n + \sum (1 \otimes b_i) x^{n-i}$ and similarly for h_B. Write $\overline{g} = x^n + \sum \overline{\beta}_i x^{n-i}$ and $\overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i}$. The A-algebra map

$$B \longrightarrow A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i$$

maps g_B and h_B to $\overline{g}$ and $\overline{h}$ in $A/I[x]$. The displayed map is surjective; denote $J \subset B$ its kernel. From the discussion in Algebra, Example 10.143.12 it is clear that $A \to B$ is etale at all points of $V(J) \subset \mathrm{Spec}(B)$. Choose $g \in B$ as in Lemma 15.9.4 and consider the A-algebra B_g. Since g maps to a unit in $B/J = A/I$ we obtain also a map $B_g/IB_g \to A/I$ of A/I-algebras. Since $A/I \to B_g/IB_g$ is étale, also $B_g/IB_g \to A/I$ is étale (Algebra, Lemma 10.143.8). Hence there exists an idempotent $e \in B_g/IB_g$ such that $A/I = (B_g/IB_g)_e$ (Algebra, Lemma 10.143.9). Choose a lift $h \in B_g$ of e. Then $A \to A' = (B_g)_h$ with factorization given by the image of the factorization $f_B = g_B h_B$ in A' is a solution to the problem posed by the lemma. □

07M1 **Lemma 15.9.6.** *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume*

1. *the leading coefficient of $\overline{g}$ is an invertible element of A/I, and*
2. *$\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$.*

Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Applying Lemma 15.9.1 we may assume that the leading coefficient of $\overline{g}$ is the reduction of an invertible element $u \in A$. Then we may replace $\overline{g}$ by $\overline{u}^{-1}\overline{g}$ and $\overline{h}$ by $\overline{u}\overline{h}$. Thus we may assume that $\overline{g}$ is monic. Since f is monic we conclude that $\overline{h}$ is monic too. In this case the result follows from Lemma 15.9.5. □

07M2 **Example 15.9.7.** This example shows that one cannot drop the condition on the leading coefficient in Lemma 15.9.6. Namely, let $A = \mathbf{Z}$, $I = 4\mathbf{Z}$, $f = 1$ and $\bar{g} = \bar{h} = 2x^2 + 2x + 1$ in $A/I[x]$. If $A \to A'$ is étale and $A'/IA' = A/I$, then the 2-adic completion of A' is isomorphic to $\mathbf{Z}_2$. Now any polynomial g' in $\mathbf{Z}_2[x]$ congruent to $2x^2 + 2x + 1$ mod 4 will not have a polynomial multiplicative inverse in $\mathbf{Z}_2[x]$, whence the conclusion of the lemma does not hold.

07M3 **Lemma 15.9.8.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal of R and let $J \subset S$ be an ideal of S. If the closure of the image of $V(J)$ in $\mathrm{Spec}(R)$ is disjoint from $V(I)$, then there exists an element $f \in R$ which maps to 1 in R/I and to an element of J in S.*

Proof. Let $I' \subset R$ be an ideal such that $V(I')$ is the closure of the image of $V(J)$. Then $V(I) \cap V(I') = \emptyset$ by assumption and hence $I + I' = R$ by Algebra, Lemma 10.17.2. Write $1 = g + f$ with $g \in I$ and $f \in I'$. We have $V(f') \supset V(J)$ where f' is the image of f in S. Hence $(f')^n \in J$ for some n, see Algebra, Lemma 10.17.2. Replacing f by f^n we win. □

09XG **Lemma 15.9.9.** *Let I be an ideal of a ring A. Let $A \to B$ be an integral ring map. Let $b \in B$ map to an idempotent in B/IB. Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = x^d(x-1)^d$ for some $d \geq 1$.*

Proof. Observe that $z = b^2 - b$ is an element of IB. By Algebra, Lemma 10.38.4 there exist a monic polynomial $g(x) = x^d + \sum a_j x^j$ of degree d with $a_j \in I$ such that $g(z) = 0$ in B. Hence $f(x) = g(x^2 - x) \in A[x]$ is a monic polynomial such that $f(x) \equiv x^d(x-1)^d \bmod I$ and such that $f(b) = 0$ in B. □

07M4 **Lemma 15.9.10.** *Let A be a ring, let $I \subset A$ be an ideal. Let $A \to B$ be an integral ring map. Let $\overline{e} \in B/IB$ be an idempotent. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in B \otimes_A A'$ lifting $\overline{e}$.*

Proof. Choose an element $y \in B$ lifting $\overline{e}$. Choose $f \in A[x]$ as in Lemma 15.9.9 for y. By Lemma 15.9.6 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and such that $f = gh$ in $A[x]$ with $g(x) = x^d \bmod IA'$ and $h(x) = (x-1)^d \bmod IA'$. After replacing A by A' we may assume that the factorization is defined over A. In that case we see that $b_1 = g(y) \in B$ is a lift of $\overline{e}^d = \overline{e}$ and $b_2 = h(y) \in B$ is a lift of $(\overline{e} - 1)^d = (-1)^d(1 - \overline{e})^d = (-1)^d(1 - \overline{e})$ and moreover $b_1 b_2 = 0$. Thus $(b_1, b_2)B/IB = B/IB$ and $V(b_1, b_2) \subset \mathrm{Spec}(B)$ is disjoint from $V(IB)$. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed (see Algebra, Lemmas 10.36.22 and 10.41.6) we can find an $a \in A$ which maps to an invertible element of A/I whose image in B lies in (b_1, b_2), see Lemma 15.9.8. After replacing A by the localization A_a we get that $(b_1, b_2) = B$. Then $\mathrm{Spec}(B) = D(b_1) \amalg D(b_2)$; disjoint union because $b_1 b_2 = 0$ and covers $\mathrm{Spec}(B)$ because $(b_1, b_2) = B$. Let $e \in B$ be the idempotent corresponding to the open and closed subset $D(b_1)$, see Algebra, Lemma 10.21.3. Since b_1 is a lift of $\overline{e}$ and b_2 is a lift of $\pm(1 - \overline{e})$ we conclude that e is a lift of $\overline{e}$ by the uniqueness statement in Algebra, Lemma 10.21.3. □

07M5 **Lemma 15.9.11.** *Let A be a ring, let $I \subset A$ be an ideal. Let $\overline{P}$ be a finite projective A/I-module. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A'-module P' lifting $\overline{P}$.*

Proof. We can choose an integer n and a direct sum decomposition $(A/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Choose a lift $\varphi : A^{\oplus n} \to A^{\oplus n}$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$. Let $f \in A[x]$ be the characteristic polynomial of φ. Set $B = A[x]/(f)$. By Cayley-Hamilton (Algebra, Lemma 10.16.1) there is a map $B \to \mathrm{End}_A(A^{\oplus n})$ mapping x to φ. For every prime $\mathfrak{p} \supset I$ the image of f in $\kappa(\mathfrak{p})$ is $(x-1)^r x^{n-r}$ where r is the dimension of $\overline{P} \otimes_{A/I} \kappa(\mathfrak{p})$. Hence $(x-1)^n x^n$ maps to zero in $B \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \supset I$. Thus $x(1-x)$ is contained in every prime ideal of B/IB. Hence $x^N(1-x)^N$ is contained in IB for some $N \geq 1$. It follows that $x^N + (1-x)^N$ is a unit in B/IB and that

$$\overline{e} = \text{image of } \frac{x^N}{x^N + (1-x)^N} \text{ in } B/IB$$

is an idempotent as both assertions hold in $\mathbf{Z}[x]/(x^N(x-1)^N)$. The image of $\overline{e}$ in $\mathrm{End}_{A/I}((A/I)^{\oplus n})$ is

$$\frac{\overline{p}^N}{\overline{p}^N + (1-\overline{p})^N} = \overline{p}$$

as $\overline{p}$ is an idempotent. After replacing A by an étale extension A' as in the lemma, we may assume there exists an idempotent $e \in B$ which maps to $\overline{e}$ in B/IB, see Lemma 15.9.10. Then the image of e under the map

$$B = A[x]/(f) \longrightarrow \mathrm{End}_A(A^{\oplus n}).$$

is an idempotent element p which lifts $\overline{p}$. Setting $P = \mathrm{Im}(p)$ we win. □

07EV **Lemma 15.9.12.** *Let A be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of A-modules. Consider the A-algebra $C = Sym^*_A(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then*

$$NL(\alpha) = (K \otimes_A C \to \bigoplus\nolimits_{j=1,\ldots,m} C\, dy_j)$$

(see Algebra, Section 10.134) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \mathrm{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that α is a homomorphism of graded algebras. We will prove that in degree d we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \mathrm{Ker}(\mathrm{Sym}^d_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(M)) = \mathrm{Im}(K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(A^{\oplus m})),$$

see Algebra, Lemma 10.13.2. It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^d_A(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^{d-1}_A(A^{\oplus m})$ is $\mathrm{Sym}^{d-1}_A(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\begin{aligned} \mathrm{Coker}(K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\otimes m})) &= K \otimes_A \mathrm{Sym}^{d-1}_A(M) \\ &= K \otimes_A C_{d-1} \end{aligned}$$

as desired. □

07M6 **Lemma 15.9.13.** *Let A be a ring. Let M be an A-module. Then $C = Sym^*_A(M)$ is smooth over A if and only if M is a finite projective A-module.*

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of C. Then $J = \mathrm{Ker}(\sigma)$ is the part of degree > 0 and we see that $J/J^2 = M$ as an A-module. Hence if $A \to C$ is smooth then M is a finite projective A-module by Algebra, Lemma 10.139.4.

Conversely, assume that M is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel K. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as M is projective. In particular we see that K is a finite A-module and hence C is of finite presentation over A as C is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus Ax_i$. The computation of Lemma 15.9.12 shows that $NL_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $NL_{C/A}$ is quasi-isomorphic to $C \otimes_A M$ placed in degree 0 which means that C is smooth over A by Algebra, Definition 10.137.1. □

07M7 **Lemma 15.9.14.** *Let A be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram*

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow & \\ A & \longrightarrow & A/I \end{array}$$

where B is a smooth A-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an A-algebra map $B \to A'$ lifting the ring map $B \to A/I$.

Proof. Let $J \subset B$ be the kernel of $B \to A/I$ so that $B/J = A/I$. By Algebra, Lemma 10.139.3 the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is split exact. Thus $\overline{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a finite projective A/I-module. Choose an integer n and a direct sum decomposition $A/I^{\oplus n} = \overline{P} \oplus \overline{K}$. By Lemma 15.9.11 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A-module K which lifts $\overline{K}$. We may and do replace A by A'. Set $B' = B \otimes_A \mathrm{Sym}^*_A(K)$. Since $A \to \mathrm{Sym}^*_A(K)$ is smooth by Lemma 15.9.13 we see that $B \to B'$ is smooth which in turn implies that $A \to B'$ is smooth (see Algebra, Lemmas 10.137.3 and 10.137.12). Moreover the section $\mathrm{Sym}^*_A(K) \to A$ determines a section $B' \to B$ and we let $B' \to A/I$ be the composition $B' \to B \to A/I$. Let $J' \subset B'$ be the kernel of $B' \to A/I$. We have $JB' \subset J'$ and $B \otimes_A K \subset J'$. These maps combine to give an isomorphism

$$(A/I)^{\oplus n} \cong J/J^2 \oplus \overline{K} \longrightarrow J'/((J')^2 + IB')$$

Thus, after replacing B by B' we may assume that $J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a free A/I-module of rank n.

In this case, choose $f_1, \ldots, f_n \in J$ which map to a basis of $J/(J^2 + IB)$. Consider the finitely presented A-algebra $C = B/(f_1, \ldots, f_n)$. Note that we have an exact sequence

$$0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

see Algebra, Lemma 10.134.4 (note that $H_1(L_{B/A}) = 0$ and that $\Omega_{B/A}$ is finite projective, in particular flat so the Tor group vanishes). For any prime $\mathfrak{q} \supset J$ of B the module $\Omega_{B/A,\mathfrak{q}}$ is free of rank n because $\Omega_{B/A}$ is finite projective and because

$\Omega_{B/A} \otimes_B B/J$ is free of rank n (see Algebra, Lemma 10.78.2). By our choice of $f_1, \ldots, f_n$ the map

$$\big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2\big)_{\mathfrak{q}} \to \Omega_{B/A,\mathfrak{q}}$$

is surjective modulo J. Hence we see that this map of modules over the local ring $C_{\mathfrak{q}}$ has to be an isomorphism (this is because by Nakayama's Algebra, Lemma 10.20.1 the map is surjective and then for example by Algebra, Lemma 10.16.4 because $((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2)_{\mathfrak{q}}$ is generated by n elements the map is injective). Thus $H_1(L_{C/A})_{\mathfrak{q}} = 0$ and $\Omega_{C/A,\mathfrak{q}} = 0$. By Algebra, Lemma 10.137.11 we see that $A \to C$ is smooth at the prime $\overline{\mathfrak{q}}$ of C corresponding to $\mathfrak{q}$. Since $\Omega_{C/A,\mathfrak{q}} = 0$ it is actually étale at $\overline{\mathfrak{q}}$. Thus $A \to C$ is étale at all primes of C containing JC. By Lemma 15.9.4 we can find an $f \in C$ mapping to an invertible element of C/JC such that $A \to C_f$ is étale. By our choice of f it is still true that $C_f/JC_f = A/I$. The map $C_f/IC_f \to A/I$ is surjective and étale by Algebra, Lemma 10.143.8. Hence A/I is isomorphic to the localization of C_f/IC_f at some element $g \in C$, see Algebra, Lemma 10.143.9. Set $A' = C_{fg}$ to conclude the proof. □

15.10. Zariski pairs

0ELX In this section and the next a *pair* is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$.

0ELY **Definition 15.10.1.** A *Zariski pair* is a pair (A, I) such that I is contained in the Jacobson radical of A.

09XF **Lemma 15.10.2.** *Let (A, I) be a Zariski pair. Then the map from idempotents of A to idempotents of A/I is injective.*

Proof. An idempotent of a local ring is either 0 or 1. Thus an idempotent is determined by the set of maximal ideals where it vanishes, by Algebra, Lemma 10.23.1. □

0ELZ **Lemma 15.10.3.** *Let (A, I) be a Zariski pair. Let $A \to B$ be a flat, integral, finitely presented ring map such that $A/I \to B/IB$ is an isomorphism. Then $A \to B$ is an isomorphism.*

Proof. The ring map $A \to B$ is finite by Algebra, Lemma 10.36.5. Hence B is finitely presented as an A-module by Algebra, Lemma 10.36.23. Hence B is a finite locally free A-module by Algebra, Lemma 10.78.2. Since the module B has rank 1 along $V(I)$ (see rank function described in Algebra, Lemma 10.78.2), and as (A, I) is a Zariski pair, we conclude that the rank is 1 everywhere. It follows that $A \to B$ is an isomorphism: it is a pleasant exercise to show that a ring map $R \to S$ such that S is a locally free R-module of rank 1 is an isomorphism (hint: look at local rings). □

0EM0 **Lemma 15.10.4.** *Let (A, I) be a Zariski pair. Let $A \to B$ be a finite ring map. Assume*

(1) *$B/IB = B_1 \times B_2$ is a product of A/I-algebras*
(2) *$A/I \to B_1/IB_1$ is surjective,*
(3) *$b \in B$ maps to $(1, 0)$ in the product.*

Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = (x-1)x^d$ for some $d \geq 1$.

Proof. By Lemma 15.9.10 we can find an étale ring map $A \to A'$ inducing an isomorphism $A/I \to A'/IA'$ such that $B' = B \otimes_A A'$ contains an idempotent e' lifting the image of b in B'/IB'. Consider the corresponding A'-algebra decomposition

$$B' = B'_1 \times B'_2$$

which is compatible with the one given in the lemma upon reduction modulo I. The map $A' \to B'_1$ is surjective modulo IA'. By Nakayama's lemma (Algebra, Lemma 10.20.1) we can find $i \in IA'$ such that after replacing A' by A'_{1+i} the map $A' \to B'_1$ is surjective. Observe that the image $b'_1 \in B'_1$ of b satisfies $b'_1 - 1 \in IB'_1$. Thus we may pick $a' \in IA'$ mapping to $b'_1 - 1$. On the other hand, the image $b'_2 \in B'_2$ of b is in IB'_2. By Algebra, Lemma 10.38.4 there exist a monic polynomial $g(x) = x^d + \sum a'_j x^j$ of degree d with $a'_j \in IA'$ such that $g(b'_2) = 0$ in B'_2. Thus the image $b' = (b'_1, b'_2) \in B'$ of b is a root of the polynomial $(x - 1 - a')g(x)$. We conclude that

$$(b'-1)(b')^d \in \sum\nolimits_{j=0,\ldots,d} IA' \cdot (b')^j$$

We claim that this implies

$$(b-1)b^d \in \sum\nolimits_{j=0,\ldots,d} I \cdot b^j$$

in B. For this it is enough to see that the ring map $A \to A'$ is faithfully flat, because the condition is that the image of $(b-1)b^d$ is zero in $B/\sum_{j=0,\ldots,d} Ib^j$ (use Algebra, Lemma 10.82.11). The map $A \to A'$ flat because it is étale (Algebra, Lemma 10.143.3). On the other hand, the induced map on spectra is open (see Algebra, Proposition 10.41.8 and use previous lemma referenced) and the image contains $V(I)$. Since I is contained in the Jacobson radical of A we conclude. □

0GED **Lemma 15.10.5.** *Let (A, I) be a Zariski pair with A Noetherian. Let $f \in I$. Then A_f is a Jacobson ring.*

Proof. We will use the criterion of Algebra, Lemma 10.61.4. Let $\mathfrak{p} \subset A$ be a prime ideal such that $\mathfrak{p}_f = \mathfrak{p}A_f$ is prime and not maximal. We have to show that $A_f/\mathfrak{p}_f = (A/\mathfrak{p})_f$ has infinitely many prime ideals. After replacing A by $A/\mathfrak{p}$ we may assume A is a domain, $\dim A_f > 0$, and our goal is to show that $\mathrm{Spec}(A_f)$ is infinite. Since $\dim A_f > 0$ we can find a nonzero prime ideal $\mathfrak{q} \subset A$ not containing f. Choose a maximal ideal $\mathfrak{m} \subset A$ containing $\mathfrak{q}$. Since (A, I) is a Zariski pair, we see $I \subset \mathfrak{m}$. Hence $\mathfrak{m} \neq \mathfrak{q}$ and $\dim(A_\mathfrak{m}) > 1$. Hence $\mathrm{Spec}((A_\mathfrak{m})_f) \subset \mathrm{Spec}(A_f)$ is infinite by Algebra, Lemma 10.61.1 and we win. □

15.11. Henselian pairs

09XD Some of the results of Section 15.9 may be viewed as results about henselian pairs. In this section a *pair* is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$. As in Section 15.9 given an object ξ over A we denote $\bar{\xi}$ the "base change" of ξ to an object over A/I (provided this makes sense).

09XE **Definition 15.11.1.** A *henselian pair* is a pair (A, I) satisfying

(1) I is contained in the Jacobson radical of A, and

(2) for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with g, h monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$.

Observe that if A is a local ring and $I = \mathfrak{m}$ is the maximal ideal, then (A, I) is a henselian pair if and only if A is a henselian local ring, see Algebra, Lemma 10.153.3. In Lemma 15.11.6 we give a number of equivalent characterizations of henselian pairs (and we will add more as time goes on).

0ALI **Lemma 15.11.2.** *Let (A, I) be a pair with I locally nilpotent. Then the functor $B \mapsto B/IB$ induces an equivalence between the category of étale algebras over A and the category of étale algebras over A/I. Moreover, the pair is henselian.*

Proof. Essential surjectivity holds by Algebra, Lemma 10.143.10. If B, B' are étale over A and $B/IB \to B'/IB'$ is a morphism of A/I-algebras, then we can lift this by Algebra, Lemma 10.138.17. Finally, suppose that $f, g : B \to B'$ are two A-algebra maps with $f \bmod I = g \bmod I$. Choose an idempotent $e \in B \otimes_A B$ generating the kernel of the multiplication map $B \otimes_A B \to B$, see Algebra, Lemmas 10.151.4 and 10.151.3 (to see that étale is unramified). Then $(f \otimes g)(e) \in IB'$. Since IB' is locally nilpotent (Algebra, Lemma 10.32.3) this implies $(f \otimes g)(e) = 0$ by Algebra, Lemma 10.32.6. Thus $f = g$.

It is clear that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 15.9.5 there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ such that the factorization lifts to a factorization into monic polynomials over A'. By the above we have $A = A'$ and the factorization is over A. □

0CT7 **Lemma 15.11.3.** *Let $A = \lim A_n$ where (A_n) is an inverse system of rings whose transition maps are surjective and have locally nilpotent kernels. Then (A, I_n) is a henselian pair, where $I_n = \mathrm{Ker}(A \to A_n)$.*

Proof. Fix n. Let $a \in A$ be an element which maps to 1 in A_n. By Algebra, Lemma 10.32.4 we see that a maps to a unit in A_m for all $m \geq n$. Hence a is a unit in A. Thus by Algebra, Lemma 10.19.1 the ideal I_n is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_n h_n$ be a factorization of $\overline{f} = f \bmod I_n$ with $g_n, h_n \in A_n[T]$ monic generating the unit ideal in $A_n[T]$. By Lemma 15.11.2 we can successively lift this factorization to $f \bmod I_m = g_m h_m$ with g_m, h_m monic in $A_m[T]$ for all $m \geq n$. At each step we have to verify that our lifts g_m, h_m generate the unit ideal in $A_n[T]$; this follows from the corresponding fact for g_n, h_n and the fact that $\mathrm{Spec}(A_n[T]) = \mathrm{Spec}(A_m[T])$ because the kernel of $A_m \to A_n$ is locally nilpotent. As $A = \lim A_m$ this finishes the proof. □

0ALJ **Lemma 15.11.4.** *Let (A, I) be a pair. If A is I-adically complete, then the pair is henselian.*

Proof. By Algebra, Lemma 10.96.6 the ideal I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 15.11.2 we can successively lift this factorization to $f \bmod I^n = g_n h_n$ with g_n, h_n monic in $A/I^n[T]$ for all $n \geq 1$. As $A = \lim A/I^n$ this finishes the proof. □

09XH **Lemma 15.11.5.** *Let (A, I) be a pair. Let $A \to B$ be a finite type ring map such that $B/IB = C_1 \times C_2$ with $A/I \to C_1$ finite. Let B' be the integral closure of A in B. Then we can write $B'/IB' = C_1 \times C_2'$ such that the map $B'/IB' \to B/IB$ preserves product decompositions and there exists a $g \in B'$ mapping to $(1, 0)$ in $C_1 \times C_2'$ with $B'_g \to B_g$ an isomorphism.*

Proof. Observe that $A \to B$ is quasi-finite at every prime of the closed subset $T = \mathrm{Spec}(C_1) \subset \mathrm{Spec}(B)$ (this follows by looking at fibre rings, see Algebra, Definition 10.122.3). Consider the diagram of topological spaces

$$\begin{array}{ccc} \mathrm{Spec}(B) & \xrightarrow{\phi} & \mathrm{Spec}(B') \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi'} & \\ & \mathrm{Spec}(A) & \end{array}$$

By Algebra, Theorem 10.123.12 for every $\mathfrak{p} \in T$ there is a $h_\mathfrak{p} \in B'$, $h_\mathfrak{p} \notin \mathfrak{p}$ such that $B'_h \to B_h$ is an isomorphism. The union $U = \bigcup D(h_\mathfrak{p})$ gives an open $U \subset \mathrm{Spec}(B')$ such that $\phi^{-1}(U) \to U$ is a homeomorphism and $T \subset \phi^{-1}(U)$. Since T is open in $\psi^{-1}(V(I))$ we conclude that $\phi(T)$ is open in $U \cap (\psi')^{-1}(V(I))$. Thus $\phi(T)$ is open in $(\psi')^{-1}(V(I))$. On the other hand, since C_1 is finite over A/I it is finite over B'. Hence $\phi(T)$ is a closed subset of $\mathrm{Spec}(B')$ by Algebra, Lemmas 10.41.6 and 10.36.22. We conclude that $\mathrm{Spec}(B'/IB') \supset \phi(T)$ is open and closed. By Algebra, Lemma 10.24.3 we get a corresponding product decomposition $B'/IB' = C_1' \times C_2'$. The map $B'/IB' \to B/IB$ maps C_1' into C_1 and C_2' into C_2 as one sees by looking at what happens on spectra (hint: the inverse image of $\phi(T)$ is exactly T; some details omitted). Pick a $g \in B'$ mapping to $(1, 0)$ in $C_1' \times C_2'$ such that $D(g) \subset U$; this is possible because $\mathrm{Spec}(C_1')$ and $\mathrm{Spec}(C_2')$ are disjoint and closed in $\mathrm{Spec}(B')$ and $\mathrm{Spec}(C_1')$ is contained in U. Then $B'_g \to B_g$ defines a homeomorphism on spectra and an isomorphism on local rings (by our choice of U above). Hence it is an isomorphism, as follows for example from Algebra, Lemma 10.23.1. Finally, it follows that $C_1' = C_1$ and the proof is complete. □

09XI **Lemma 15.11.6.** *Let (A, I) be a pair. The following are equivalent* [**Ray70**, Chapter XI] and [**Gab92**, Proposition 1]

1. *(A, I) is a henselian pair,*
2. *given an étale ring map $A \to A'$ and an A-algebra map $\sigma : A' \to A/I$, there exists an A-algebra map $A' \to A$ lifting σ,*
3. *for any finite A-algebra B the map $B \to B/IB$ induces a bijection on idempotents,*
4. *for any integral A-algebra B the map $B \to B/IB$ induces a bijection on idempotents, and*
5. *(Gabber) I is contained in the Jacobson radical of A and every monic polynomial $f(T) \in A[T]$ of the form*
$$f(T) = T^n(T - 1) + a_nT^n + \ldots + a_1T + a_0$$
with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ has a root $\alpha \in 1 + I$.

Moreover, in part (5) the root is unique.

Proof. Assume (2) holds. Then I is contained in the Jacobson radical of A, since otherwise there would be a nonunit $f \in A$ congruent to 1 modulo I and the map $A \to A_f$ would contradict (2). Hence $IB \subset B$ is contained in the Jacobson radical of

B for B integral over A because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed by Algebra, Lemmas 10.41.6 and 10.36.22. Thus the map from idempotents of B to idempotents of B/IB is injective by Lemma 15.10.2. On the other hand, since (2) holds, every idempotent of B/IB lifts to an idempotent of B by Lemma 15.9.10. In this way we see that (2) implies (4).

The implication (4) $\Rightarrow$ (3) is trivial.

Assume (3). Let $\mathfrak{m}$ be a maximal ideal and consider the finite map $A \to B = A/(I \cap \mathfrak{m})$. The condition that $B \to B/IB$ induces a bijection on idempotents implies that $I \subset \mathfrak{m}$ (if not, then $B = A/I \times A/\mathfrak{m}$ and $B/IB = A/I$). Thus we see that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be monic and suppose given a factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. Set $B = A[T]/(f)$. Let $\overline{e}$ be the idempotent of B/IB corresponding to the decomposition

$$B/IB = A/I[T]/(g_0) \times A/I[T]/(h_0)$$

of A-algebras. Let $e \in B$ be an idempotent lifting $\overline{e}$ which exists as we assumed (3). This gives a product decomposition

$$B = eB \times (1-e)B$$

Note that B is free of rank $\deg(f)$ as an A-module. Hence eB and $(1-e)B$ are finite locally free A-modules. However, since eB and $(1-e)B$ have constant rank $\deg(g_0)$ and $\deg(h_0)$ over A/I we find that the same is true over $\mathrm{Spec}(A)$. We conclude that

$$\begin{aligned} f &= \mathrm{CharPol}_A(T : B \to B) \\ &= \mathrm{CharPol}_A(T : eB \to eB)\mathrm{CharPol}_A(T : (1-e)B \to (1-e)B) \end{aligned}$$

is a factorization into monic polynomials reducing to the given factorization modulo I. Here $\mathrm{CharPol}_A$ denotes the characteristic polynomial of an endomorphism of a finite locally free module over A. If the module is free the $\mathrm{CharPol}_A$ is defined as the characteristic polynomial of the corresponding matrix and in general one uses Algebra, Lemma 10.24.2 to glue. Details omitted. Thus (3) implies (1).

Assume (1). Let f be as in (5). The factorization of $f \bmod I$ as T^n times $T-1$ lifts to a factorization $f = gh$ with g and h monic by Definition 15.11.1. Then h has to have degree 1 and we see that f has a root reducing to 1 modulo I. Finally, I is contained in the Jacobson radical by the definition of a henselian pair. Thus (1) implies (5).

Before we give the proof of the last step, let us show that the root α in (5), if it exists, is unique. Namely, due to the explicit shape of $f(T)$, we have $f'(\alpha) \in 1 + I$ where f' is the derivative of f with respect to T. An elementary argument shows that

$$f(T) = f(\alpha + T - \alpha) = f(\alpha) + f'(\alpha) \cdot (T - \alpha) \bmod (T-\alpha)^2 A[T]$$

This shows that any other root $\alpha' \in 1 + I$ of $f(T)$ satisfies $0 = f(\alpha') - f(\alpha) = (\alpha' - \alpha)(1 + i)$ for some $i \in I$, so that, since $1 + i$ is a unit in A, we have $\alpha = \alpha'$.

Assume (5). We will show that (2) holds, in other words, that for every étale map $A \to A'$, every section $\sigma : A' \to A/I$ modulo I lifts to a section $A' \to A$. Since $A \to A'$ is étale, the section σ determines a decomposition

0EM1 (15.11.6.1) $$A'/IA' \cong A/I \times C$$

of A/I-algebras. Namely, the surjective ring map $A'/IA' \to A/I$ is étale by Algebra, Lemma 10.143.8 and then we get the desired idempotent by Algebra, Lemma 10.143.9. We will show that this decomposition lifts to a decomposition

0EM2 (15.11.6.2) $$A' \cong A'_1 \times A'_2$$

of A-algebras with A'_1 integral over A. Then $A \to A'_1$ is integral and étale and $A/I \to A'_1/IA'_1$ is an isomorphism, thus $A \to A'_1$ is an isomorphism by Lemma 15.10.3 (here we also use that an étale ring map is flat and of finite presentation, see Algebra, Lemma 10.143.3).

Let B' be the integral closure of A in A'. By Lemma 15.11.5 we may decompose

0EM3 (15.11.6.3) $$B'/IB' \cong A/I \times C'$$

as A/I-algebras compatibly with (15.11.6.1) and we may find $b \in B'$ that lifts $(1, 0)$ such that $B'_b \to A'_b$ is an isomorphism. If the decomposition (15.11.6.3) lifts to a decomposition

0EM4 (15.11.6.4) $$B' \cong B'_1 \times B'_2$$

of A-algebras, then the induced decomposition $A' = A'_1 \times A'_2$ will give the desired (15.11.6.2): indeed, since b is a unit in B'_1 (details omitted), we will have $B'_1 \cong A'_1$, so that A'_1 will be integral over A.

Choose a finite A-subalgebra $B'' \subset B'$ containing b (observe that any finitely generated A-subalgebra of B' is finite over A). After enlarging B'' we may assume b maps to an idempotent in B''/IB'' producing

0EM5 (15.11.6.5) $$B''/IB'' \cong C''_1 \times C''_2$$

Since $B'_b \cong A'_b$ we see that B'_b is of finite type over A. Say B'_b is generated by $b_1/b^n, \ldots, b_t/b^n$ over A and enlarge B'' so that $b_1, \ldots, b_t \in B''$. Then $B''_b \to B'_b$ is surjective as well as injective, hence an isomorphism. In particular, we see that $C''_1 = A/I$! Therefore $A/I \to C''_1$ is an isomorphism, in particular surjective. By Lemma 15.10.4 we can find an $f(T) \in A[T]$ of the form

$$f(T) = T^n(T - 1) + a_nT^n + \ldots + a_1T + a_0$$

with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ such that $f(b) = 0$. In particular, we find that B' is a $A[T]/(f)$-algebra. By (5) we deduce there is a root $a \in 1 + I$ of f. This produces a product decomposition $A[T]/(f) = A[T]/(T - a) \times D$ compatible with the splitting (15.11.6.3) of B'/IB'. The induced splitting of B' is then a desired (15.11.6.4). □

09XJ **Lemma 15.11.7.** *Let A be a ring. Let $I, J \subset A$ be ideals with $V(I) = V(J)$. Then (A, I) is henselian if and only if (A, J) is henselian.*

Proof. For any integral ring map $A \to B$ we see that $V(IB) = V(JB)$. Hence idempotents of B/IB and B/JB are in bijective correspondence (Algebra, Lemma 10.21.3). It follows that $B \to B/IB$ induces a bijection on sets of idempotents if and only if $B \to B/JB$ induces a bijection on sets of idempotents. Thus we conclude by Lemma 15.11.6. □

09XK **Lemma 15.11.8.** *Let (A, I) be a henselian pair and let $A \to B$ be an integral ring map. Then (B, IB) is a henselian pair.*

Proof. Immediate from the fourth characterization of henselian pairs in Lemma 15.11.6 and the fact that the composition of integral ring maps is integral. □

0DYD **Lemma 15.11.9.** *Let $I \subset J \subset A$ be ideals of a ring A. The following are equivalent*

(1) *(A, I) and $(A/I, J/I)$ are henselian pairs, and*
(2) *(A, J) is an henselian pair.*

Proof. Assume (1). Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 15.11.6 we find that both arrows induce bijections on idempotents. Hence so does the composition. Whence (A, J) is a henselian pair by Lemma 15.11.6.

Conversely, assume (2) holds. Then $(A/I, J/I)$ is a henselian pair by Lemma 15.11.8. Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 15.11.6 we find that the composition and the second arrow induce bijections on idempotents. Hence so does the first arrow. It follows that (A, I) is a henselian pair (by the lemma again). □

0G1R **Lemma 15.11.10.** *Let A be a ring and let (A, I) and (A, I') be henselian pairs. Then $(A, I + I')$ is an henselian pair.*

Proof. By Lemma 15.11.8 the pair $(A/I, (I' + I)/I)$ is henselian. Thus we get the conclusion from Lemma 15.11.9. □

0ATD **Lemma 15.11.11.** *Let J be a set and let $\{(A_j, I_j)\}_{j\in J}$ be a collection of pairs. Then $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian if and only if so is each (A_j, I_j).*

Proof. For every $j \in J$, the projection $\prod_{j\in J} A_j \to A_j$ is an integral ring map, so Lemma 15.11.8 proves that each (A_j, I_j) is Henselian if $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian.

Conversely, suppose that each (A_j, I_j) is a Henselian pair. Then every $1 + x$ with $x \in \prod_{j\in J} I_j$ is a unit in $\prod_{j\in J} A_j$ because it is so componentwise by Algebra, Lemma 10.19.1 and Definition 15.11.1. Thus, by Algebra, Lemma 10.19.1 again, $\prod_{j\in J} I_j$ is contained in the Jacobson radical of $\prod_{j\in J} A_j$. Continuing to work componentwise, it likewise follows that for every monic $f \in (\prod_{j\in J} A_j)[T]$ and every factorization $\overline{f} = g_0 h_0$ with monic $g_0, h_0 \in (\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T] = (\prod_{j\in J} A_j/I_j)[T]$ that generate the unit ideal in $(\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T]$, there exists a factorization $f = gh$ in $(\prod_{j\in J} A_j)[T]$ with g, h monic and reducing to g_0, h_0. In conclusion, according to Definition 15.11.1 $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is a Henselian pair. □

0EM6 **Lemma 15.11.12.** *The property of being Henselian is preserved under limits of pairs. More precisely, let J be a preordered set and let (A_j, I_j) be an inverse system of henselian pairs over J. Then $A = \lim A_j$ equipped with the ideal $I = \lim I_j$ is a henselian pair (A, I).*

Proof. By Categories, Lemma 4.14.11, we only need to consider products and equalizers. For products, the claim follows from Lemma 15.11.11. Thus, consider

an equalizer diagram

$$(A, I) \longrightarrow (A', I') \xrightarrow[\psi]{\varphi} (A'', I'')$$

in which the pairs (A', I') and (A'', I'') are henselian. To check that the pair (A, I) is also henselian, we will use the Gabber's criterion in Lemma 15.11.6. Every element of $1 + I$ is a unit in A because, due to the uniqueness of the inverses of units, this may be checked in (A', I'). Thus I is contained in the Jacobson radical of A, see Algebra, Lemma 10.19.1. Thus, let

$$f(T) = T^{N-1}(T - 1) + a_{N-1}T^{N-1} + \cdots + a_1T + a_0$$

be a polynomial in $A[T]$ with $a_{N-1}, \ldots, a_0 \in I$ and $N \geq 1$. The image of $f(T)$ in $A'[T]$ has a unique root $\alpha' \in 1 + I'$ and likewise for the further image in $A''[T]$. Thus, due to the uniqueness, $\varphi(\alpha') = \psi(\alpha')$, to the effect that α' defines a root of $f(T)$ in $1 + I$, as desired. □

0FWT **Lemma 15.11.13.** *The property of being Henselian is preserved under filtered colimits of pairs. More precisely, let J be a directed set and let (A_j, I_j) be a system of henselian pairs over J. Then $A = \operatorname{colim} A_j$ equipped with the ideal $I = \operatorname{colim} I_j$ is a henselian pair (A, I).*

Proof. If $u \in 1 + I$ then for some $j \in J$ we see that u is the image of some $u_j \in 1 + I_j$. Then u_j is invertible in A_j by Algebra, Lemma 10.19.1 and the assumption that I_j is contained in the Jacobson radical of A_j. Hence u is invertible in A. Thus I is contained in the Jacobson radical of A (by the lemma).

Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0h_0$ be a factorization with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. Write $1 = g_0g'_0 + h_0h'_0$ for some $g'_0, h'_0 \in A/I[T]$. Since $A = \operatorname{colim} A_j$ and $A/I = \operatorname{colim} A_j/I_j$ are filtered colimits we can find a $j \in J$ and $f_j \in A_j$ and a factorization $\overline{f}_j = g_{j,0}h_{j,0}$ with $g_{j,0}, h_{j,0} \in A_j/I_j[T]$ monic and $1 = g_{j,0}g'_{j,0} + h_{j,0}h'_{j,0}$ for some $g'_{j,0}, h'_{j,0} \in A_j/I_j[T]$ with $f_j, g_{j,0}, h_{j,0}, g'_{j,0}, h'_{j,0}$ mapping to f, g_0, h_0, g'_0, h'_0. Since (A_j, I_j) is a henselian pair, we can lift $\overline{f}_j = g_{j,0}h_{j,0}$ to a factorization over A_j and taking the image in A we obtain a corresponding factorization in A. Hence (A, I) is henselian. □

0FWU **Example 15.11.14** (Moret-Bailly)**.** Lemma 15.11.13 is wrong if the colimit isn't filtered. For example, if we take the coproduct of the henselian pairs $(\mathbf{Z}_p, (p))$ and $(\mathbf{Z}_p, (p))$, then we obtain (A, pA) with $A = \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Z}_p$. This isn't a henselian pair: $A/pA = \mathbf{F}_p$ hence if (A, pA) where henselian, then A would have to be local. However, $\operatorname{Spec}(A)$ is disconnected; for example for odd primes p we have the nontrivial idempotent

$$(1/2 \otimes 1)\left(1 \otimes 1 - (1 + p)^{-1}u \otimes u\right)$$

where $u \in \mathbf{Z}_p$ is a square root of $1 + p$. Some details omitted.

0G1S **Lemma 15.11.15.** *Let A be a ring. There exists a largest ideal $I \subset A$ such that (A, I) is a henselian pair.*

Proof. Combine Lemmas 15.11.9, 15.11.10, and 15.11.13. □

09Y6 **Lemma 15.11.16.** *Let (A, I) be a henselian pair. Let $\mathfrak{p} \subset A$ be a prime ideal. Then $V(\mathfrak{p} + I)$ is connected.*

Proof. By Lemma 15.11.8 we see that $(A/\mathfrak{p}, I + \mathfrak{p}/\mathfrak{p})$ is a henselian pair. Thus it suffices to prove: If (A, I) is a henselian pair and A is a domain, then $\mathrm{Spec}(A/I) = V(I)$ is connected. If not, then A/I has a nontrivial idempotent by Algebra, Lemma 10.21.4. By Lemma 15.11.6 this would imply A has a nontrivial idempotent. This is a contradiction. □

15.12. Henselization of pairs

0EM7 We continue the discussion started in Section 15.11.

0A02 **Lemma 15.12.1.** *The inclusion functor*

$$\textit{category of henselian pairs} \longrightarrow \textit{category of pairs}$$

has a left adjoint $(A, I) \mapsto (A^h, I^h)$.

Proof. Let (A, I) be a pair. Consider the category $\mathcal{C}$ consisting of étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism. We will show that the category $\mathcal{C}$ is directed and that $A^h = \mathrm{colim}_{B \in \mathcal{C}} B$ with ideal $I^h = IA^h$ gives the desired adjoint.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 4.19.1). It is nonempty because $\mathrm{id} : A \to A$ is an object. If B and B' are two objects of $\mathcal{C}$, then $B'' = B \otimes_A B'$ is an object of $\mathcal{C}$ (use Algebra, Lemma 10.143.3) and there are morphisms $B \to B''$ and $B' \to B''$. Suppose that $f, g : B \to B'$ are two maps between objects of $\mathcal{C}$. Then a coequalizer is

$$(B' \otimes_{f,B,g} B') \otimes_{(B' \otimes_A B')} B'$$

which is étale over A by Algebra, Lemmas 10.143.3 and 10.143.8. Thus the category $\mathcal{C}$ is directed.

Since $B/IB = A/I$ for all objects B of $\mathcal{C}$ we see that $A^h/I^h = A^h/IA^h = \mathrm{colim}\, B/IB = \mathrm{colim}\, A/I = A/I$.

Next, we show that $A^h = \mathrm{colim}_{B \in \mathcal{C}} B$ with $I^h = IA^h$ is a henselian pair. To do this we will verify condition (2) of Lemma 15.11.6. Namely, suppose given an étale ring map $A^h \to A'$ and A^h-algebra map $\sigma : A' \to A^h/I^h$. Then there exists a $B \in \mathcal{C}$ and an étale ring map $B \to B'$ such that $A' = B' \otimes_B A^h$. See Algebra, Lemma 10.143.3. Since $A^h/I^h = A/I$, the map σ induces an A-algebra map $s : B' \to A/I$. Then $B'/IB' = A/I \times C$ as A/I-algebra, where C is the kernel of the map $B'/IB' \to A/I$ induced by s. Let $g \in B'$ map to $(1, 0) \in A/I \times C$. Then $B \to B'_g$ is étale and $A/I \to B'_g/IB'_g$ is an isomorphism, i.e., B'_g is an object of $\mathcal{C}$. Thus we obtain a canonical map $B'_g \to A^h$ such that

$$\begin{array}{ccc} B'_g & \longrightarrow & A^h \\ \uparrow & \nearrow & \\ B & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} B' & \longrightarrow & B'_g & \longrightarrow & A^h \\ & \searrow^{s} & & & \downarrow \\ & & & & A/I \end{array}$$

commute. This induces a map $A' = B' \otimes_B A^h \to A^h$ compatible with σ as desired.

Let $(A, I) \to (A', I')$ be a morphism of pairs with (A', I') henselian. We will show there is a unique factorization $A \to A^h \to A'$ which will finish the proof. Namely, for each $A \to B$ in $\mathcal{C}$ the ring map $A' \to B' = A' \otimes_A B$ is étale and induces an

isomorphism $A'/I' \to B'/I'B'$. Hence there is a section $\sigma_B : B' \to A'$ by Lemma 15.11.6. Given a morphism $B_1 \to B_2$ in $\mathcal{C}$ we claim the diagram

$$\begin{array}{ccc} B_1' & \longrightarrow & B_2' \\ & {\scriptstyle \sigma_{B_1}} \searrow \quad \swarrow {\scriptstyle \sigma_{B_2}} & \\ & A' & \end{array}$$

commutes. This follows once we prove that for every B in $\mathcal{C}$ the section σ_B is the unique A'-algebra map $B' \to A'$. We have $B' \otimes_{A'} B' = B' \times R$ for some ring R, see Algebra, Lemma 10.151.4. In our case $R/I'R = 0$ as $B'/I'B' = A'/I'$. Thus given two A'-algebra maps $\sigma_B, \sigma'_B : B' \to A'$ then $e = (\sigma_B \otimes \sigma'_B)(0,1) \in A'$ is an idempotent contained in I'. We conclude that $e = 0$ by Lemma 15.10.2. Hence $\sigma_B = \sigma'_B$ as desired. Using the commutativity we obtain

$$A^h = \operatorname{colim}_{B \in \mathcal{C}} B \to \operatorname{colim}_{B \in \mathcal{C}} A' \otimes_A B \xrightarrow{\operatorname{colim} \sigma_B} A'$$

as desired. The uniqueness of the maps σ_B also guarantees that this map is unique. Hence $(A, I) \mapsto (A^h, I^h)$ is the desired adjoint. □

0AGU **Lemma 15.12.2.** *Let (A, I) be a pair. Let (A^h, I^h) be as in Lemma 15.12.1. Then $A \to A^h$ is flat, $I^h = IA^h$ and $A/I^n \to A^h/I^nA^h$ is an isomorphism for all n.*

Proof. In the proof of Lemma 15.12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism and we have seen that $I^h = IA^h$. As an étale ring map is flat (Algebra, Lemma 10.143.3) we conclude that $A \to A^h$ is flat by Algebra, Lemma 10.39.3. Since each $A \to B$ is flat we find that the maps $A/I^n \to B/I^nB$ are isomorphisms as well (for example by Algebra, Lemma 10.101.3). Taking the colimit we find that $A/I^n = A^h/I^nA^h$ as desired. □

0A03 **Lemma 15.12.3.** *The functor of Lemma 15.12.1 associates to a local ring $(A, \mathfrak{m})$ its henselization.*

Proof. Let $(A^h, \mathfrak{m}^h)$ be the henselization of the pair $(A, \mathfrak{m})$ constructed in Lemma 15.12.1. Then $\mathfrak{m}^h = \mathfrak{m}A^h$ is a maximal ideal by Lemma 15.12.2 and since it is contained in the Jacobson radical, we conclude A^h is local with maximal ideal $\mathfrak{m}^h$. Having said this there are two ways to finish the proof.

First proof: observe that the construction in the proof of Algebra, Lemma 10.155.1 as a colimit is the same as the colimit used to construct A^h in Lemma 15.12.1. Second proof: Both the henselization $A \to S$ and $A \to A^h$ of Lemma 15.12.1 are local ring homomorphisms, both S and A^h are filtered colimits of étale A-algebras, both S and A^h are henselian local rings, and both S and A^h have residue fields equal to $\kappa(\mathfrak{m})$ (by Lemma 15.12.2 for the second case). Hence they are canonically isomorphic by Algebra, Lemma 10.154.7. □

0AGV **Lemma 15.12.4.** *Let (A, I) be a pair with A Noetherian. Let (A^h, I^h) be as in Lemma 15.12.1. Then the map of I-adic completions*

$$A^\wedge \to (A^h)^\wedge$$

is an isomorphism. Moreover, A^h is Noetherian, the maps $A \to A^h \to A^\wedge$ are flat, and $A^h \to A^\wedge$ is faithfully flat.

Proof. The first statement is an immediate consequence of Lemma 15.12.2 and in fact holds without assuming A is Noetherian. In the proof of Lemma 15.12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. For each such $A \to B$ the induced map $A^\wedge \to B^\wedge$ is an isomorphism (see proof of Lemma 15.12.2). By Algebra, Lemma 10.97.2 the ring map $B \to A^\wedge = B^\wedge = (A^h)^\wedge$ is flat for each B. Thus $A^h \to A^\wedge = (A^h)^\wedge$ is flat by Algebra, Lemma 10.39.6. Since $I^h = IA^h$ is contained in the Jacobson radical of A^h and since $A^h \to A^\wedge$ induces an isomorphism $A^h/I^h \to A/I$ we see that $A^h \to A^\wedge$ is faithfully flat by Algebra, Lemma 10.39.15. By Algebra, Lemma 10.97.6 the ring $A^\wedge$ is Noetherian. Hence we conclude that A^h is Noetherian by Algebra, Lemma 10.164.1. □

0A04 **Lemma 15.12.5.** *Let $(A, I) = \operatorname{colim}(A_i, I_i)$ be a filtered colimit of pairs. The functor of Lemma 15.12.1 gives $A^h = \operatorname{colim} A_i^h$ and $I^h = \operatorname{colim} I_i^h$.*

This lemma is false for non-filtered colimits, see Example 15.11.14.

Proof. By Categories, Lemma 4.24.5 we see that (A^h, I^h) is the colimit of the system (A_i^h, I_i^h) in the category of henselian pairs. Thus for a henselian pair (B, J) we have

$$\operatorname{Mor}((A^h, I^h), (B, J)) = \lim \operatorname{Mor}((A_i^h, I_i^h), (B, J)) = \operatorname{Mor}(\operatorname{colim}(A_i^h, I_i^h), (B, J))$$

Here the colimit is in the category of pairs. Since the colimit is filtered we obtain $\operatorname{colim}(A_i^h, I_i^h) = (\operatorname{colim} A_i^h, \operatorname{colim} I_i^h)$ in the category of pairs; details omitted. Again using the colimit is filtered, this is a henselian pair (Lemma 15.11.13). Hence by the Yoneda lemma we find $(A^h, I^h) = (\operatorname{colim} A_i^h, \operatorname{colim} I_i^h)$. □

0F0L **Lemma 15.12.6.** *Let A be a ring with ideals I and J. If $V(I) = V(J)$ then the functor of Lemma 15.12.1 produces the same ring for the pair (A, I) as for the pair (A, J).*

Proof. Let (A', IA') be the pair produced by Lemma 15.12.1 starting with the pair (A, I), see Lemma 15.12.2. Let (A'', JA'') be the pair produced by Lemma 15.12.1 starting with the pair (A, J). By Lemma 15.11.7 we see that (A', JA') is a henselian pair and (A'', IA'') is a henselian pair. By the universal property of the construction we obtain unique A-algebra maps $A'' \to A'$ and $A' \to A''$. The uniqueness shows that these are mutually inverse. □

0DYE **Lemma 15.12.7.** *Let $(A, I) \to (B, J)$ be a map of pairs such that $V(J) = V(IB)$. Let $(A^h, I^h) \to (B^h, J^h)$ be the induced map on henselizations (Lemma 15.12.1). If $A \to B$ is integral, then the induced map $A^h \otimes_A B \to B^h$ is an isomorphism.*

Proof. By Lemma 15.12.6 we may assume $J = IB$. By Lemma 15.11.8 the pair $(A^h \otimes_A B, I^h(A^h \otimes_A B))$ is henselian. By the universal property of (B^h, IB^h) we obtain a map $B^h \to A^h \otimes_A B$. We omit the proof that this map is the inverse of the map in the lemma. □

0H7Q **Lemma 15.12.8.** *Let $I_1, I_2, \ldots, I_n \subset A$ be ideals in a ring A. Suppose $I_i + I_j = A$ for $i \neq j$, $1 \leq i, j \leq n$. Denote A^h the henselization of the pair $(A, I_1 \cap \ldots \cap I_n)$ and A_i^h the henselization of the pair (A, I_i). Then the natural morphism*

$$A^h \to \prod\nolimits_{i=1,\ldots,n} A_i^h$$

is an isomorphism.

Proof. The map comes from the functoriality of the construction. If $n = 1$ the result is immediate. If $n > 1$, observe that $(I_1 \cap \ldots \cap I_{n-1}) + I_n = A$ by our condition on the ideals. Thus if we prove the lemma for $n = 2$, then the lemma follows for all n by induction.

The case $n = 2$. Choose $f_i \in I_i$ such that $f_1 + f_2 = 1$. Recall that A^h is constructed as the colimit of A-algebras B such that $A \to B$ is étale and $A/I_1 \cap I_2 \to B/(I_1 \cap I_2)B$ is an isomorphism. See proof of Lemma 15.12.1. Denote $Hens$ the category of such A-algebras B. Similarly, denote $Hens_i$, $i = 1, 2$ the category of étale A-algebras B such that $A/I_i \to B/I_iB$ is an isomorphism.

Consider the subcategory $Hens_1(f_2) \subset Hens_1$ consisting of those B such that f_2 is invertible. This is a cofinal subcategory since if B is in $Hens_1$, then B_{f_2} is in $Hens_1$ as well (hint: $B_{f_2}/I_1B_{f_2} = (B/I_1B)_{f_2} = B/I_1B$ because f_2 is a unit modulo I_1). Similarly, the subcategory $Hens_2(f_1) \subset Hens_2$ is cofinal. Hence to compute A_1^h and A_2^h we may work with this smaller categories, see Categories, Lemma 4.17.2.

Consider the functor

$$Hens_1(f_2) \times Hens_2(f_1) \to Hens, \quad (B_1, B_2) \mapsto B_1 \times B_2$$

This functor makes sense as

$$\begin{aligned}(B_1 \times B_2)/(I_1 \cap I_2)(B_1 \times B_2) &= B_1/(I_1 \cap I_2)B_1 \times B_2/(I_1 \cap I_2)B_2 \\ &= B_1/I_1B_1 \times B_2/I_2B_2 \\ &= A/I_1 \times A/I_2 \\ &= A/(I_1 \cap I_2)\end{aligned}$$

The second equality holds because $(I_1 \cap I_2)B_1 = I_1B_1 \cap I_2B_1 = I_1B_1$ as $f_2 \in I_2$ is invertible in B_1 (we have also used Algebra, Lemma 10.39.2). The last equality holds by Algebra, Lemma 10.15.4. To finish the proof, it suffices to prove that the displayed functor is cofinal (see lemma quoted above). The first condition of Categories, Definition 4.17.1 holds because an object B of $Hens$ maps to $B_{f_2} \times B_{f_1}$ which is in the image of the functor. The second one always holds because if we have $B \to B_1 \times B_2$ and $B \to B_1' \times B_2'$ with $B_1, B_1' \in Hens_1(f_2)$ and $B_2, B_2' \in Hens_2(f_1)$, then we can set $B_1'' = B_1 \otimes_B B_1'$ which is in $Hens_1(f_2)$ and similarly $B_2'' = B_2 \otimes_B B_2''$ in $Hens_2(f_1)$ to find a map $B \to B_1'' \times B_2''$ fitting into the commutative diagram

$$\begin{array}{ccccc} & & B & & \\ & \swarrow & \downarrow & \searrow & \\ B_1 \times B_2 & \longrightarrow & B_1'' \times B_2'' & \longleftarrow & B_1' \times B_2' \end{array}$$

This finishes the proof. □

15.13. Lifting and henselian pairs

0D49 In this section we mostly combine results from Sections 15.9 and 15.11.

0D4A **Lemma 15.13.1.** *Let (R, I) be a henselian pair. The map*

$$P \longrightarrow P/IP$$

induces a bijection between the sets of isomorphism classes of finite projective R-modules and finite projective R/I-modules. In particular, any finite projective R/I-module is isomorphic to P/IP for some finite projective R-module P.

Proof. We first prove the final statement. Let $\overline{P}$ be a finite projective R/I-module. We can find a finite projective module P' over some R' étale over R with $R/I = R'/IR'$ such that P'/IP' is isomorphic to $\overline{P}$, see Lemma 15.9.11. Then, since (R, I) is a henselian pair, the étale ring map $R \to R'$ has a section $\tau : R' \to R$ (Lemma 15.11.6). Setting $P = P' \otimes_{R',\tau} R$ we conclude that P/IP is isomorphic to $\overline{P}$. Of course, this tells us that the map in the statement of the lemma is surjective.

Injectivity. Suppose that P_1 and P_2 are finite projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$ as R/I-modules. Since P_1 is projective, we can find an R-module map $u : P_1 \to P_2$ lifting the given isomorphism. Then u is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1). We similarly find a surjection $v : P_2 \to P_1$. By Algebra, Lemma 10.16.4 the map $v \circ u$ is an isomorphism and we conclude u is an isomorphism. □

09ZL **Lemma 15.13.2.** *Let (A, I) be a henselian pair. The functor $B \to B/IB$ determines an equivalence between finite étale A-algebras and finite étale A/I-algebras.*

Proof. Let B, B' be two A-algebras finite étale over A. Then $B' \to B'' = B \otimes_A B'$ is finite étale as well (Algebra, Lemmas 10.143.3 and 10.36.13). Now we have 1-to-1 correspondences between

(1) A-algebra maps $B \to B'$,
(2) sections of $B' \to B''$, and
(3) idempotents e of B'' such that $B' \to B'' \to eB''$ is an isomorphism.

The bijection between (2) and (3) sends $\sigma : B'' \to B'$ to e such that $(1 - e)$ is the idempotent that generates the kernel of σ which exists by Algebra, Lemmas 10.143.8 and 10.143.9. There is a similar correspondence between A/I-algebra maps $B/IB \to B'/IB'$ and idempotents $\overline{e}$ of B''/IB'' such that $B'/IB' \to B''/IB'' \to \overline{e}(B''/IB'')$ is an isomorphism. However every idempotent $\overline{e}$ of B''/IB'' lifts uniquely to an idempotent e of B'' (Lemma 15.11.6). Moreover, if $B'/IB' \to \overline{e}(B''/IB'')$ is an isomorphism, then $B' \to eB''$ is an isomorphism too by Nakayama's lemma (Algebra, Lemma 10.20.1). In this way we see that the functor is fully faithful.

Essential surjectivity. Let $A/I \to C$ be a finite étale map. By Algebra, Lemma 10.143.10 there exists an étale map $A \to B$ such that $B/IB \cong C$. Let B' be the integral closure of A in B. By Lemma 15.11.5 we have $B'/IB' = C \times C'$ for some ring C' and $B'_g \cong B_g$ for some $g \in B'$ mapping to $(1, 0) \in C \times C'$. Since idempotents lift (Lemma 15.11.6) we get $B' = B'_1 \times B'_2$ with $C = B'_1/IB'_1$ and $C' = B'_2/IB'_2$. The image of g in B'_1 is invertible. Then $B_g = B'_g = B'_1 \times (B_2)_g$ and this implies that $A \to B'_1$ is étale. We conclude that B'_1 is finite étale over A (integral étale implies finite étale by Algebra, Lemma 10.36.5 for example) and the proof is done. □

0H74 **Lemma 15.13.3.** *Let R be a ring and S a smooth R-algebra. Assume that A is an R-algebra and (A, I) is a henselian pair. Then any R-algebra map $S \to A/I$ can be lifted to an R-algebra map $S \to A$.*

Proof. Let $\tau : S \to A/I$ be an R-algebra map. Observe that $S \otimes_R A$ is a smooth A-algebra by Algebra, Lemma 10.137.3. Thus by Lemma 15.9.14 we can lift the induced map $S \otimes_R A \to A/I$ to an A-algebra homorphism $S \otimes_R A \to A'$ where $A \to A'$ is étale and induces an isomorphism $A/I \to A'/IA'$. Since (A, I) is henselian there is an A-algebra map $A' \to A$, see Lemma 15.11.6. The composition $S \to S \otimes_R A \to A' \to A$ is the desired lift. □

0D4B **Lemma 15.13.4.** *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given A_n-modules M_n and A_{n+1}-module maps $M_{n+1} \to M_n$. Assume*

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *M_1 is a finite projective A_1-module,*
(3) *M_n is a finite flat A_n-module, and*
(4) *the maps induce isomorphisms $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$.*

Then $M = \lim M_n$ is a finite projective A-module and $M \otimes_A A_n \to M_n$ is an isomorphism for all n.

Proof. By Lemma 15.11.3 the pair $(A, \mathrm{Ker}(A \to A_1))$ is henselian. By Lemma 15.13.1 we can choose a finite projective A-module P and an isomorphism $P \otimes_A A_1 \to M_1$. Since P is projective, we can successively lift the A-module map $P \to M_1$ to A-module maps $P \to M_2$, $P \to M_3$, and so on. Thus we obtain a map

$$P \longrightarrow M$$

Since P is finite projective, we can write $A^{\oplus m} = P \oplus Q$ for some $m \geq 0$ and A-module Q. Since $A = \lim A_n$ we conclude that $P = \lim P \otimes_A A_n$. Hence, in order to show that the displayed A-module map is an isomorphism, it suffices to show that the maps $P \otimes_A A_n \to M_n$ are isomorphisms. From Lemma 15.3.4 we see that M_n is a finite projective module. By Lemma 15.3.5 the maps $P \otimes_A A_n \to M_n$ are isomorphisms. □

15.14. Absolute integral closure

0DCK Here is our definition.

0DCL **Definition 15.14.1.** A ring A is *absolutely integrally closed* if every monic $f \in A[T]$ is a product of linear factors.

Be careful: it may be possible to write f as a product of linear factors in many different ways.

0DCM **Lemma 15.14.2.** *Let A be a ring. The following are equivalent*

(1) *A is absolutely integrally closed, and*
(2) *any monic $f \in A[T]$ has a root in A.*

Proof. Omitted. □

0DCN **Lemma 15.14.3.** *Let A be absolutely integrally closed.*

(1) *Any quotient ring A/I of A is absolutely integrally closed.*
(2) *Any localization $S^{-1}A$ is absolutely integrally closed.*

Proof. Omitted. □

0DCP **Lemma 15.14.4.** *Let A be a ring. Let $S \subset A$ be a multiplicative subset consisting of nonzerodivisors. If $S^{-1}A$ is absolutely integrally closed and $A \subset S^{-1}A$ is integrally closed in $S^{-1}A$, then A is absolutely integrally closed.*

Proof. Omitted. □

0DCQ **Lemma 15.14.5.** *Let A be a normal domain. Then A is absolutely integrally closed if and only if its fraction field is algebraically closed.*

Proof. Observe that a field is algebraically closed if and only if it is absolutely integrally closed as a ring. Hence the lemma follows from Lemmas 15.14.3 and 15.14.4. □

0DCR **Lemma 15.14.6.** *For any ring A there exists an extension $A \subset B$ such that*

(1) *B is a filtered colimit of finite free A-algebras,*
(2) *B is free as an A-module, and*
(3) *B is absolutely integrally closed.*

Proof. Let I be the set of monic polynomials over A. For $i \in I$ denote x_i a variable and P_i the corresponding monic polynomial in the variable x_i. Then we set

$$F(A) = A[x_i; i \in I]/(P_i; i \in I)$$

As the notation suggests F is a functor from the category of rings to itself. Note that $A \subset F(A)$, that $F(A)$ is free as an A-module, and that $F(A)$ is a filtered colimit of finite free A-algebras. Then we take

$$B = \operatorname{colim} F^n(A)$$

where the transition maps are the inclusions $F^n(A) \subset F(F^n(A)) = F^{n+1}(A)$. Any monic polynomial with coefficients in B actually has coefficients in $F^n(A)$ for some n and hence has a solution in $F^{n+1}(A)$ by construction. This implies that B is absolutely integrally closed by Lemma 15.14.2. We omit the proof of the other properties. □

0DCS **Lemma 15.14.7.** *Let A be absolutely integrally closed. Let $\mathfrak{p} \subset A$ be a prime. Then the local ring $A_\mathfrak{p}$ is strictly henselian.*

Proof. By Lemma 15.14.3 we may assume A is a local ring and $\mathfrak{p}$ is its maximal ideal. The residue field is algebraically closed by Lemma 15.14.3. Every monic polynomial decomposes completely into linear factors hence Algebra, Definition 10.153.1 applies directly. □

0DCT **Lemma 15.14.8.** *Let A be absolutely integrally closed. Let $I \subset A$ be an ideal. Then (A, I) is a henselian pair if (and only if) the following conditions hold*

(1) *I is contained in the Jacobson radical of A,*
(2) *$A \to A/I$ induces a bijection on idempotents.*

Proof. Let $f \in A[T]$ be a monic polynomial and let $f \bmod I = g_0 h_0$ be a factorization over A/I with g_0, h_0 monic such that g_0 and h_0 generate the unit ideal of $A/I[T]$. This means that

$$A/I[T]/(f) = A/I[T]/(g_0) \times A/I[T]/(h_0)$$

Denote $e \in A/I[T]/(f)$ the element correspoing to the idempotent $(1, 0)$ in the ring on the right. Write $f = (T - a_1) \ldots (T - a_d)$ with $a_i \in A$. For each $i \in \{1, \ldots, d\}$ we obtain an A-algebra map $\varphi_i : A[T]/(f) \to A$, $T \mapsto a_i$ which induces a similar A/I-algebra map $\overline{\varphi}_i : A/I[T]/(f) \to A/I$. Denote $e_i = \overline{\varphi}_i(e) \in A/I$. These are idempotents. By our assumption (2) we can lift e_i to an idempotent in A. This means we can write $A = \prod A_j$ as a finite product of rings such that in A_j/IA_j each e_i is either 0 or 1. Some details omitted. Observe that A_j is absolutely integrally closed as a factor ring of A. It suffices to lift the factorization of f over A_j/IA_j to A_j. This reduces us to the situation discussed in the next paragraph.

Assume $e_i = 1$ for $i = 1, \ldots, r$ and $e_i = 0$ for $i = r+1, \ldots, d$. From $(g_0, h_0) = A/I[T]$ we have that there are $k_0, l_0 \in A/I[T]$ such that $g_0k_0 + h_0l_0 = 1$. We see that $e = h_0l_0$ and $e_i = h_0(a_i)l_0(a_i)$. We conclude that $h_0(a_i)$ is a unit for $i = 1, \ldots, r$. Since $f(a_i) = 0$ we find $0 = h_0(a_i)g_0(a_i)$ and we conclude that $g_0(a_i) = 0$ for $i = 1, \ldots, r$. Thus $(T - a_1)$ divides g_0 in $A/I[T]$, say $g_0 = (T - a_1)g_0'$. Set $f' = (T - a_2) \ldots (T - a_d)$ and $h_0' = h_0$. By induction on d we can lift the factorization $f' \bmod I = g_0'h_0'$ to a factorization of $f' = g'h'$ over over A which gives the factorization $f = (T - a_1)g'h'$ lifting the factorization $f \bmod I = g_0h_0$ as desired. □

15.15. Auto-associated rings

05GL Some of this material is in [**Laz69**].

05GM **Definition 15.15.1.** A ring R is said to be *auto-associated* if R is local and its maximal ideal $\mathfrak{m}$ is weakly associated to R.

05GN **Lemma 15.15.2.** *An auto-associated ring R has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.*

Proof. By assumption there exists a nonzero element $x \in R$ such that for every $f \in \mathfrak{m}$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then x is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all i, see Algebra, Lemma 10.24.4. As $y \in \mathrm{Ann}_R(I)$ we win. □

05GP **Lemma 15.15.3.** *Let R be a ring having property (P) of Lemma 15.15.2. Let $u : N \to M$ be a homomorphism of projective R-modules. Then u is universally injective if and only if u is injective.*

Proof. Assume u is injective. Our goal is to show u is universally injective. First we choose a module Q such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case N is free. In this case N is a directed colimit of finite free R-modules. Thus we reduce to the case that N is a finite free R-module, say $N = R^{\oplus n}$. We prove the lemma by induction on n. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with M projective. Choose an R-module Q such that $M \oplus Q$ is free. After replacing u by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that M is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case u is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As u is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as R has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective R-module. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^{\oplus n} & \longrightarrow & R^{\oplus n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{(a_{11},\ldots,a_{1m})} & R^{\oplus m} & \longrightarrow & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) & \longrightarrow & 0 \end{array}$$

with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right vertical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

05GQ **Lemma 15.15.4.** *Let R be a ring. The following are equivalent*

(1) *R has property (P) of Lemma 15.15.2,*
(2) *any injective map of projective R-modules is universally injective,*
(3) *if $u : N \to M$ is injective and N, M are finite projective R-modules then $\mathrm{Coker}(u)$ is a finite projective R-module,*
(4) *if $N \subset M$ and N, M are finite projective as R-modules, then N is a direct summand of M, and*
(5) *any injective map $R \to R^{\oplus n}$ is a split injection.*

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 15.15.3. It is clear that (3) and (4) are equivalent. We have (2) $\Rightarrow$ (3), (4) by Algebra, Lemma 10.82.4. Part (5) is a special case of (4). Assume (5). Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of R. As $I \neq R$ we see that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\mathrm{Ann}_R(I) \neq 0$. Thus (1) holds. □

05GR **Example 15.15.5.** If the equivalent conditions of Lemma 15.15.4 hold, then it is not always the case that every injective map of free R-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$. This is an auto-associated ring. Consider the map of free R-modules

$$u : \bigoplus\nolimits_{i \geq 1} Re_i \longrightarrow \bigoplus\nolimits_{i \geq 1} Rf_i, \quad e_i \longmapsto f_i - x_i f_{i+1}.$$

For any integer n the restriction of u to $\bigoplus_{i=1,\ldots,n} Re_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are R-linearly independent. Hence u is injective and hence universally injective by the lemma. Since $u \otimes \mathrm{id}_k$ is bijective we see that if u were a split injection then u would be surjective. But u is not surjective because the inverse image of f_1 would be the element

$$\sum\nolimits_{i \geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that $\mathrm{Coker}(u)$ is a flat (because u is universally injective), countably generated R-module which is not projective (as u is not split), hence not Mittag-Leffler (see Algebra, Lemma 10.93.1).

The following lemma is a special case of Algebra, Proposition 10.102.9 in case the local ring is Noetherian.

00MX **Lemma 15.15.6.** *Let $(R, \mathfrak{m})$ be a local ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules. The following are equivalent*

(1) *φ is injective,*
(2) *the rank of φ is m and the annihilator of $I(\varphi)$ in R is zero.*

If R is Noetherian these are also equivalent to

(3) *the rank of φ is m and either $I(\varphi) = R$ or it contains a nonzerodivisor.*

Here the rank of φ and $I(\varphi)$ are defined as in Algebra, Definition 10.102.5.

Proof. If any matrix coefficient of φ is not in $\mathfrak{m}$, then we apply Algebra, Lemma 10.102.2 to write φ as the sum of $1 : R \to R$ and a map $\varphi' : R^{m-1} \to R^{n-1}$. It is easy to see that the lemma for φ' implies the lemma for φ. Thus we may assume from the outset that all the matrix coefficients of φ are in $\mathfrak{m}$.

Suppose φ is injective. We may assume $m > 0$. Let $\mathfrak{q} \in \text{WeakAss}(R)$ so that $R_\mathfrak{q}$ is an auto-associated ring. Then φ induces an injective map $R_\mathfrak{q}^m \to R_\mathfrak{q}^n$ which is universally injective by Lemmas 15.15.2 and 15.15.3. Thus $\varphi : \kappa(\mathfrak{q})^m \to \kappa(\mathfrak{q})^n$ is injective. Hence the rank of φ mod $\mathfrak{q}$ is m and $I(\varphi \otimes \kappa(\mathfrak{q}))$ is not the zero ideal. Since m is the maximum rank φ can have, we conclude that φ has rank m as well (ranks of matrices can only drop after base change). Hence $I(\varphi) \cdot \kappa(\mathfrak{q}) = I(\varphi \otimes \kappa(\mathfrak{q}))$ is not zero. Thus $I(\varphi)$ is not contained in $\mathfrak{q}$. Thus none of the weakly associated primes of R are weakly associated primes of the R-module $\text{Ann}_R I(\varphi)$. Thus $\text{Ann}_R I(\varphi)$ has no weakly associated primes, see Algebra, Lemma 10.66.4. It follows from Algebra, Lemma 10.66.5 that $\text{Ann}_R I(\varphi)$ is zero.

Conversely, assume (2). The rank being m implies $n \geq m$. Write $I(\varphi) = (f_1, \ldots, f_r)$ which is possible as $I(\varphi)$ is finitely generated. By Algebra, Lemma 10.15.5 we can find maps $\psi_i : R^n \to R^m$ such that $\psi \circ \varphi = f_i \text{id}_{R^m}$. Thus $\varphi(x) = 0$ implies $f_i x = 0$ for $i = 1, \ldots, r$. This implies $x = 0$ and hence φ is injective.

For the equivalence of (1) and (3) in the Noetherian local case we refer to Algebra, Proposition 10.102.9. If the ring R is Noetherian but not local, then the reader can deduce it from the local case; details omitted. Another option is to redo the argument above using associated primes, using that there are finitely many of these, using prime avoidance, and using the characterization of nonzerodivisors as elements of a Noetherian ring not contained in any associated prime. □

0EWY **Lemma 15.15.7.** *Let R be a ring. Suppose that $\varphi : R^n \to R^n$ be an injective map of finite free modules of the same rank. Then $\text{Hom}_R(\text{Coker}(\varphi), R) = 0$.*

Proof. Let $\varphi^t : R^n \to R^n$ be the transpose of φ. The lemma claims that φ^t is injective. With notation as in Lemma 15.15.6 we see that the rank of φ^t is n and that $I(\varphi) = I(\varphi^t)$. Thus we conclude by the equivalence of (1) and (2) of the lemma. □

15.16. Flattening stratification

0521 Let $R \to S$ be a ring map and let M be an S-module. For any R-algebra R' we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ *flattens* M if the module M' is flat over R'. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten M. In particular we would like to know if there exists a *universal flattening* $R \to R_{univ}$ *of* M, i.e., a ring map $R \to R_{univ}$ which flattens M and has the property that any ring map $R \to R'$ which flattens M factors through $R \to R_{univ}$. It turns out that such a universal solution usually does not exist.

We will discuss *universal flattenings* and *flattening stratifications* in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 38.21. If the universal flattening $R \to R_{univ}$ exists then the morphism of schemes $\text{Spec}(R_{univ}) \to \text{Spec}(R)$ is the universal flattening of the quasi-coherent module $\widetilde{M}$ on $\text{Spec}(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

0522 **Lemma 15.16.1.** *Let R be a ring. Let M be an R-module. Let I_1, I_2 be ideals of R. If M/I_1M is flat over R/I_1 and M/I_2M is flat over R/I_2, then $M/(I_1 \cap I_2)M$ is flat over $R/(I_1 \cap I_2)$.*

Proof. By replacing R with $R/(I_1 \cap I_2)$ and M by $M/(I_1 \cap I_2)M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that M is flat over R we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 10.39.5. By flatness of M/I_1M over R/I_1 the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1M \longrightarrow M/I_1M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$\begin{array}{ccccccc} (J \cap I_1) \otimes_R M & \longrightarrow & J \otimes_R M & \longrightarrow & J/(J \cap I_1) \otimes_R M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & = & M & \longrightarrow & M/I_1M & & \end{array}$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2M$. By flatness of M/I_2M over R/I_2 the map $J' \otimes_{R/I_2} M/I_2M \to M/I_2M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. □

15.17. Flattening over an Artinian ring

05LJ A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flatting stratification exists when the base scheme is the spectrum of an Artinian local ring.

0524 **Lemma 15.17.1.** *Let R be an Artinian ring. Let M be an R-module. Then there exists a smallest ideal $I \subset R$ such that M/IM is flat over R/I.*

Proof. This follows directly from Lemma 15.16.1 and the Artinian property. □

This ideal has the following universal property.

0525 **Lemma 15.17.2.** *Let R be an Artinian ring. Let M be an R-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that M/IM is flat over R/I. Then I has the following universal property: For every ring map $\varphi : R \to R'$ we have*

$$R' \otimes_R M \text{ is flat over } R' \Leftrightarrow \text{we have } \varphi(I) = 0.$$

Proof. Note that I exists by Lemma 15.17.1. The implication $\Rightarrow$ follows from Algebra, Lemma 10.39.7. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over R'. Let $J = \mathrm{Ker}(\varphi)$. By Algebra, Lemma 10.101.7 and as $R' \otimes_R M = R' \otimes_{R/J} M/JM$ is flat over R' we conclude that M/JM is flat over R/J. Hence $I \subset J$ as desired. □

15.18. Flattening over a closed subset of the base

05LK Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. In the following we will consider the following condition

052W (15.18.0.1) $$\forall \mathfrak{q} \in V(IS) \subset \mathrm{Spec}(S) : M_{\mathfrak{q}} \text{ is flat over } R.$$

Geometrically, this means that M is flat over R along the inverse image of $V(I)$ in $\mathrm{Spec}(S)$. If R and S are Noetherian rings and M is a finite S-module, then (15.18.0.1) is equivalent to the condition that M/I^nM is flat over R/I^n for all $n \geq 1$, see Algebra, Lemma 10.99.11.

052X **Lemma 15.18.1.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If (15.18.0.1) holds for $(R \to S, I, M)$, then (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.*

Proof. Assume (15.18.0.1) holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset \mathfrak{q}'$ be a prime of $S \otimes_R R'$. Let $\mathfrak{q} \subset S$ be the corresponding prime of S. Then $IS \subset \mathfrak{q}$. Note that $(M \otimes_R R')_{\mathfrak{q}'}$ is a localization of the base change $M_{\mathfrak{q}} \otimes_R R'$. Hence $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over R' as a localization of a flat module, see Algebra, Lemmas 10.39.7 and 10.39.18. □

05LL **Lemma 15.18.2.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that*

(1) *the map $V(I') \to V(I)$ induced by $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, and*
(2) *$R'_{\mathfrak{p}'}$ is flat over R for all primes $\mathfrak{p}' \in V(I')$.*

If (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then (15.18.0.1) holds for $(R \to S, I, M)$.

Proof. Assume (15.18.0.1) holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset \mathfrak{q} \subset S$. Let $I \subset \mathfrak{p} \subset R$ be the corresponding prime of R. By assumption there exists a prime $\mathfrak{p}' \in V(I')$ of R' lying over $\mathfrak{p}$ and $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ is flat. Choose a prime $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$ which corresponds to a prime $\mathfrak{q}' \subset S \otimes_R R'$ which lies over $\mathfrak{q}$ and over $\mathfrak{p}'$. Note that $(S \otimes_R R')_{\mathfrak{q}'}$ is a localization of $S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}$. By assumption the module $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$. Hence Algebra, Lemma 10.100.1 implies that $M_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ which is what we wanted to prove. □

05LM **Lemma 15.18.3.** *Let $R \to S$ be a ring map of finite presentation. Let M be an S-module of finite presentation. Let $R' = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \mathrm{colim}_\lambda I_\lambda$. If (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists a $\lambda \in \Lambda$ such that (15.18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.*

Proof. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.14.2. By Algebra, Theorem 10.129.4 the set

$$U' = \{\mathfrak{q}' \in \mathrm{Spec}(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $\mathrm{Spec}(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S') \subset \bigcup D(g'_j)$ means that $I'S' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as $1 = \sum z_s h_s + \sum f_j g'_j$ for some $z_s \in I'$, $h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes lying over the ideal I_λ. □

15.19. Flattening over a closed subsets of source and base

05LN In this section we slightly generalize the discussion in Section 15.18. We strongly suggest the reader first read and understand that section.

05LP **Situation 15.19.1.** Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal. Let M be an S-module.

In this situation, given an R-algebra R' and an ideal $I' \subset R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

05LQ (15.19.1.1) $$\forall \mathfrak{q}' \in V(I'S' + JS') \subset \mathrm{Spec}(S') : M'_{\mathfrak{q}'} \text{ is flat over } R'.$$

Geometrically, this means that M' is flat over R' along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (15.19.1.1) only depends on the pair (R', I') where R' is viewed as an R-algebra.

05LR **Lemma 15.19.2.** *In Situation 15.19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. If (15.19.1.1) holds for (R', I'), then (15.19.1.1) holds for (R'', I'').*

Proof. Assume (15.19.1.1) holds for (R', I'). Let $I''S'' + JS'' \subset \mathfrak{q}''$ be a prime of S''. Let $\mathfrak{q}' \subset S'$ be the corresponding prime of S'. Then both $I'S' \subset \mathfrak{q}'$ and $JS' \subset \mathfrak{q}'$ because the corresponding conditions hold for $\mathfrak{q}''$. Note that $(M'')_{\mathfrak{q}''}$ is a localization of the base change $M'_{\mathfrak{q}'} \otimes_R R''$. Hence $(M'')_{\mathfrak{q}''}$ is flat over R'' as a localization of a flat module, see Algebra, Lemmas 10.39.7 and 10.39.18. □

05LS **Lemma 15.19.3.** *In Situation 15.19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. Assume*

1. *the map $V(I'') \to V(I')$ induced by $\mathrm{Spec}(R'') \to \mathrm{Spec}(R')$ is surjective, and*
2. *$R''_{\mathfrak{p}''}$ is flat over R' for all primes $\mathfrak{p}'' \in V(I'')$.*

If (15.19.1.1) holds for (R'', I''), then (15.19.1.1) holds for (R', I').

Proof. Assume (15.19.1.1) holds for (R'', I''). Pick a prime $I'S' + JS' \subset \mathfrak{q}' \subset S'$. Let $I' \subset \mathfrak{p}' \subset R'$ be the corresponding prime of R'. By assumption there exists a prime $\mathfrak{p}'' \in V(I'')$ of R'' lying over $\mathfrak{p}'$ and $R'_{\mathfrak{p}'} \to R''_{\mathfrak{p}''}$ is flat. Choose a prime $\overline{\mathfrak{q}}'' \subset \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{p}')} \kappa(\mathfrak{p}'')$. This corresponds to a prime $\mathfrak{q}'' \subset S'' = S' \otimes_{R'} R''$ which lies over $\mathfrak{q}'$ and over $\mathfrak{p}''$. In particular we see that $I''S'' \subset \mathfrak{q}''$ and that $JS'' \subset \mathfrak{q}''$. Note that $(S' \otimes_{R'} R'')_{\mathfrak{q}''}$ is a localization of $S'_{\mathfrak{q}'} \otimes_{R'_{\mathfrak{p}'}} R''_{\mathfrak{p}''}$. By assumption the module $(M' \otimes_{R'} R'')_{\mathfrak{q}''}$ is flat over $R''_{\mathfrak{p}''}$. Hence Algebra, Lemma 10.100.1 implies that $M'_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$ which is what we wanted to prove. □

05LT **Lemma 15.19.4.** *In Situation 15.19.1 assume $R \to S$ is essentially of finite presentation and M is an S-module of finite presentation. Let $R' = \operatorname{colim}_{\lambda\in\Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \operatorname{colim}_\lambda I_\lambda$. If (15.19.1.1) holds for (R', I'), then there exists a $\lambda \in \Lambda$ such that (15.19.1.1) holds for (R_λ, I_λ).*

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.14.2. By Algebra, Theorem 10.129.4 the set

$$U' = \{\mathfrak{q}' \in \operatorname{Spec}(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $\operatorname{Spec}(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S'+JS') \subset \bigcup D(g'_j)$ means that $I'S' + JS' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as

$$1 = \sum y_t k_t + \sum z_s h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that S is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented C-module N, see Algebra, Lemma 10.126.3. At this point we introduce C_λ, C', N_λ, N'. Then in the discussion above we obtain an open $U' \subset \operatorname{Spec}(C')$ over which N' is flat over R'. The assumption that (15.19.1.1) is true means that $V(I'S'+JS')$ maps into U', because for a prime $\mathfrak{q}' \subset S'$, corresponding to a prime $\mathfrak{r}' \subset C'$ we have $M'_{\mathfrak{q}'} = N'_{\mathfrak{r}'}$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before. □

05LU **Lemma 15.19.5.** *In Situation 15.19.1. Let $I \subset R$ be an ideal. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *M is a finite S-module, and*
4. *for each $n \geq 1$ and any prime $\mathfrak{q} \in V(J + IS)$ the module $(M/I^nM)_\mathfrak{q}$ is flat over R/I^n.*

Then (15.19.1.1) holds for (R, I), i.e., for every prime $\mathfrak{q} \in V(J+IS)$ the localization $M_\mathfrak{q}$ is flat over R.

Proof. Let $\mathfrak{q} \in V(J + IS)$. Then Algebra, Lemma 10.99.11 applied to $R \to S_\mathfrak{q}$ and $M_\mathfrak{q}$ implies that $M_\mathfrak{q}$ is flat over R. □

15.20. Flattening over a Noetherian complete local ring

05LV The following three lemmas give a completely algebraic proof of the existence of the "local" flattening stratification when the base is a complete local Noetherian ring R and the given module is finite over a finite type R-algebra S.

0526 **Lemma 15.20.1.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

(1) *$(R, \mathfrak{m})$ is a complete local Noetherian ring,*
(2) *S is a Noetherian ring, and*
(3) *M is finite over S.*

Then there exists an ideal $I \subset \mathfrak{m}$ such that

(1) *$(M/IM)_\mathfrak{q}$ is flat over R/I for all primes $\mathfrak{q}$ of S/IS lying over $\mathfrak{m}$, and*
(2) *if $J \subset R$ is an ideal such that $(M/JM)_\mathfrak{q}$ is flat over R/J for all primes $\mathfrak{q}$ lying over $\mathfrak{m}$, then $I \subset J$.*

In other words, I is the smallest ideal of R such that (15.18.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

Proof. Let $J \subset R$ be an ideal. Apply Algebra, Lemma 10.99.11 to the module M/JM over the ring R/J. Then we see that $(M/JM)_\mathfrak{q}$ is flat over R/J for all primes $\mathfrak{q}$ of S/JS if and only if $M/(J+\mathfrak{m}^n)M$ is flat over $R/(J+\mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 15.17.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that M/I_nM is flat over R/I_n. It is clear that $I_{n+1} + \mathfrak{m}^n$ is contains I_n and applying Lemma 15.16.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = \lim_n R/\mathfrak{m}^n$ we see that $I = \lim_n I_n/\mathfrak{m}^n$ is an ideal in R such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that I verifies (1) and (2). Some details omitted. □

0527 **Lemma 15.20.2.** *With notation $R \to S$, M, and I and assumptions as in Lemma 15.20.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is Noetherian. Then the following are equivalent*

(1) *condition (15.18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
(2) *$\varphi(I) = 0$.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring R' (see Algebra, Lemma 10.51.4) this is equivalent to the condition that $\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates I for each n. Now assumption (1) for φ implies assumption (1) for φ_n by Lemma 15.18.1. This reduces us to the case where R' is Artinian local.

Assume R' Artinian. Let $J = \mathrm{Ker}(\varphi)$. We have to show that $I \subset J$. By the construction of I in Lemma 15.20.1 it suffices to show that $(M/JM)_\mathfrak{q}$ is flat over R/J for every prime $\mathfrak{q}$ of S/JS lying over $\mathfrak{m}$. As R' is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over R'. As R' is Artinian and $R/J \to R'$ is a local injective ring map, it follows that R/J is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over R' implies that M/JM is flat over R/J by Algebra, Lemma 10.101.7. This concludes the proof. □

0528 **Lemma 15.20.3.** *With notation $R \to S$, M, and I and assumptions as in Lemma 15.20.1. In addition assume that $R \to S$ is of finite type. Then for any local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ the following are equivalent*

(1) *condition (15.18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
(2) *$\varphi(I) = 0$.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). As R is Noetherian we see that $R \to S$ is of finite presentation and M is an S-module of finite presentation. Write $R' = \operatorname{colim}_\lambda R_\lambda$ as a directed colimit of local R-subalgebras $R_\lambda \subset R'$, with maximal ideals $\mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}'$ such that each R_λ is essentially of finite type over R. By Lemma 15.18.3 we see that condition (15.18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda, M \otimes_R R_\lambda)$ for some λ. Hence Lemma 15.20.2 applies to the ring map $R \to R_\lambda$ and we see that I maps to zero in R_λ, a fortiori it maps to zero in R'. □

15.21. Descent of flatness along integral maps

052Y First a few simple lemmas.

052Z **Lemma 15.21.1.** *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. Let $\alpha \in R$ be such that $P(\alpha) = 0$. Then $P(T) = (T - \alpha)Q(T)$ for some monic polynomial $Q(T) \in R[T]$.*

Proof. By induction on the degree of P. If $\deg(P) = 1$, then $P(T) = T - \alpha$ and the result is true. If $\deg(P) > 1$, then we can write $P(T) = (T - \alpha)Q(T) + r$ for some polynomial $Q \in R[T]$ of degree $< \deg(P)$ and some $r \in R$ by long division. By assumption $0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r$ and we conclude that $r = 0$ as desired. □

0530 **Lemma 15.21.2.** *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. There exists a finite free ring map $R \to R'$ such that $P(T) = (T - \alpha)Q(T)$ for some $\alpha \in R'$ and some monic polynomial $Q(T) \in R'[T]$.*

Proof. Write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$. Set $R' = R[x]/(x^d + a_1 x^{d-1} + \ldots + a_0)$. Set α equal to the congruence class of x. Then it is clear that $P(\alpha) = 0$. Thus we win by Lemma 15.21.1. □

0531 **Lemma 15.21.3.** *Let $R \to S$ be a finite ring map. There exists a finite free ring extension $R \subset R'$ such that $S \otimes_R R'$ is a quotient of a ring of the form*

$$R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$.

Proof. Let $x_1, \ldots, x_n \in S$ be generators of S over R. For each i we can choose a monic polynomial $P_i(T) \in R[T]$ such that $P_i(x_i) = 0$ in S, see Algebra, Lemma 10.36.3. Say $\deg(P_i) = d_i$. By Lemma 15.21.2 (applied $\sum d_i$ times) there exists a finite free ring extension $R \subset R'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{j=1,\ldots,d_i}(T - \alpha_{ij})$$

for certain $\alpha_{ik} \in R'$. Let $R'[T_1, \ldots, T_n] \to S \otimes_R R'$ be the R'-algebra map which maps T_i to $x_i \otimes 1$. As this maps $P_i(T_i)$ to zero, this induces the desired surjection. □

0532 **Lemma 15.21.4.** *Let R be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume J contains elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $\underline{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map*

$$\Phi_{\underline{k}} : R[T_1, \ldots, T_n] \to R, \quad T_i \longmapsto \alpha_{ik_i}$$

Set $J_{\underline{k}} = \Phi_{\underline{k}}(J)$. Then the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is equal to $V(\bigcap J_{\underline{k}})$.

Proof. This lemma proves itself. Hint: $V(\bigcap J_{\underline{k}}) = \bigcup V(J_{\underline{k}})$. □

The following result is due to Ferrand, see [**Fer69**].

0533 **Lemma 15.21.5.** *Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let M be an R-module. If $M \otimes_R S$ is a flat S-module, then M is a flat R-module.*

Proof. Let M be an R-module such that $M \otimes_R S$ is flat over S. By Algebra, Lemma 10.39.8 in order to prove that M is flat we may replace R by any faithfully flat ring extension. By Lemma 15.21.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 15.21.4. Note that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Algebra, Lemma 10.36.17. Thus, using Lemma 15.21.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = \mathrm{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{array}{ccc} S & \longrightarrow & R/J_{\underline{k}} \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

from which we conclude that $M/J_{\underline{k}}M$ is flat over $R/J_{\underline{k}}$. By Lemma 15.16.1 we see that M/IM is flat over R/I. Finally, applying Algebra, Lemma 10.101.5 we conclude that M is flat over R. □

0534 **Lemma 15.21.6.** *Let $R \to S$ be an injective integral ring map. Let M be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over S, then M is flat over R.*

Proof. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Let's say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij,I} x^I$ with $f_{ij,I} \in R$ (multi-index notation). Consider diagrams

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_\lambda & \longrightarrow & S_\lambda \end{array}$$

where R_λ is a finitely generated $\mathbf{Z}$-subalgebra of R containing all $f_{ij,I}$ and S_λ is a finite R_λ-subalgebra of S. Let M_λ be the finite $R_\lambda[x_1, \ldots, x_n]$-module defined by

a presentation as above, using the same matrix T but now viewed as a matrix over $R_\lambda[x_1, \ldots, x_n]$. Note that S is the directed colimit of the S_λ (details omitted). By Algebra, Lemma 10.168.1 we see that for some λ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over S_λ. By Lemma 15.21.5 we conclude that M_λ is flat over R_λ. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 10.39.7. □

0H7R **Lemma 15.21.7.** *Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let P be an R-module. If $P \otimes_R S$ is a projective S-module, then P is a projective R-module.*

Proof. Let P be an R-module such that $P \otimes_R S$ is projective over S. By Algebra, Theorem 10.95.6 in order to prove that P is projective we may replace R by any faithfully flat ring extension. By Lemma 15.21.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 15.21.4. Note that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Algebra, Lemma 10.36.17. Thus, using Lemma 15.21.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = \mathrm{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{array}{ccc} S & \longrightarrow & R/J_{\underline{k}} \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

from which we conclude that $P/J_{\underline{k}}P$ is projective over $R/J_{\underline{k}}$. By Algebra, Lemma 10.77.8 we see that P/IP is projective over R/I. Since P is flat over R by Lemma 15.21.5, we may apply Algebra, Lemma 10.77.7 to conclude that P is projective over R. □

15.22. Torsion free modules

0549 In this section we discuss torsion free modules and the relationship with flatness (especially over dimension 1 rings).

0536 **Definition 15.22.1.** Let R be a domain. Let M be an R-module.

(1) We say an element $x \in M$ is *torsion* if there exists a nonzero $f \in R$ such that $fx = 0$.
(2) We say M is *torsion free* if the only torsion element of M is 0.
(3) We say M is a *torsion module* if every element of M is torsion.

Let R be a domain and let $S = R \setminus \{0\}$ be the multiplicative set of nonzero elements of R. Then an R-module M is torsion free if and only if $M \to S^{-1}M$ is injective. In other words, if and only if the map $M \to M \otimes_R K$ is injective where $K = S^{-1}R$ is the fraction field of R.

0537 **Lemma 15.22.2.** *Let R be a domain. Let M be an R-module. The set of torsion elements of M forms a submodule $M_{tors} \subset M$. The quotient module M/M_{tors} is torsion free.*

Proof. Omitted. □

0AUR **Lemma 15.22.3.** *Let R be a domain. Let M be a torsion free R-module. For any multiplicative set $S \subset R$ the module $S^{-1}M$ is a torsion free $S^{-1}R$-module.*

Proof. Omitted. □

0AXM **Lemma 15.22.4.** *Let $R \to R'$ be a flat homomorphism of domains. If M is a torsion free R-module, then $M \otimes_R R'$ is a torsion free R'-module.*

Proof. If M is torsion free, then $M \subset M \otimes_R K$ is injective where K is the fraction field of R. Since R' is flat over R we see that $M \otimes_R R' \to (M \otimes_R K) \otimes_R R'$ is injective. Since $M \otimes_R K$ is isomorphic to a direct sum of copies of K, it suffices to see that $K \otimes_R R'$ is torsion free. This is true because it is a localization of R'. □

0AUS **Lemma 15.22.5.** *Let R be a domain. Let $0 \to M \to M' \to M'' \to 0$ be a short exact sequence of R-modules. If M and M'' are torsion free, then M' is torsion free.*

Proof. Omitted. □

0AUT **Lemma 15.22.6.** *Let R be a domain. Let M be an R-module. Then M is torsion free if and only if $M_\mathfrak{m}$ is a torsion free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Omitted. Hint: Use Lemma 15.22.3 and Algebra, Lemma 10.23.1. □

0AUU **Lemma 15.22.7.** *Let R be a domain. Let M be a finite R-module. Then M is torsion free if and only if M is a submodule of a finite free module.*

Proof. If M is a submodule of $R^{\oplus n}$, then M is torsion free. For the converse, assume M is torsion free. Let K be the fraction field of R. Then $M \otimes_R K$ is a finite dimensional K-vector space. Choose a basis $e_1, \ldots, e_r$ for this vector space. Let $x_1, \ldots, x_n$ be generators of M. Write $x_i = \sum (a_{ij}/b_{ij}) e_j$ for some $a_{ij}, b_{ij} \in R$ with $b_{ij} \neq 0$. Set $b = \prod_{i,j} b_{ij}$. Since M is torsion free the map $M \to M \otimes_R K$ is injective and the image is contained in $R^{\oplus r} = Re_1/b \oplus \ldots \oplus Re_r/b$. □

0AUV **Lemma 15.22.8.** *Let R be a Noetherian domain. Let M be a nonzero finite R-module. The following are equivalent*

(1) *M is torsion free,*
(2) *M is a submodule of a finite free module,*
(3) *(0) is the only associated prime of M,*
(4) *(0) is in the support of M and M has property (S_1), and*
(5) *(0) is in the support of M and M has no embedded associated prime.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 15.22.7. We have seen the equivalence of (4) and (5) in Algebra, Lemma 10.157.2. The equivalence between (3) and (5) is immediate from the definition. A localization of a torsion free module is torsion free (Lemma 15.22.3), hence it is clear that a M has no associated primes different from (0). Thus (1) implies (5). Conversely, assume (5). If M has torsion, then there exists an embedding $R/I \subset M$ for some nonzero ideal I of R. Hence M has an associated prime different from (0) (see Algebra, Lemmas 10.63.3 and 10.63.7). This is an embedded associated prime which contradicts the assumption. □

0538 **Lemma 15.22.9.** *Let R be a domain. Any flat R-module is torsion free.*

Proof. If $x \in R$ is nonzero, then $x : R \to R$ is injective, and hence if M is flat over R, then $x : M \to M$ is injective. Thus if M is flat over R, then M is torsion free. □

0539 **Lemma 15.22.10.** *Let A be a valuation ring. An A-module M is flat over A if and only if M is torsion free.*

Proof. The implication "flat ⇒ torsion free" is Lemma 15.22.9. For the converse, assume M is torsion free. By the equational criterion of flatness (see Algebra, Lemma 10.39.11) we have to show that every relation in M is trivial. To do this assume that $\sum_{i=1,\ldots,n} a_i x_i = 0$ with $x_i \in M$ and $a_i \in A$. After renumbering we may assume that $v(a_1) \leq v(a_i)$ for all i. Hence we can write $a_i = a_i' a_1$ for some $a_i' \in A$. Note that $a_1' = 1$. As M is torsion free we see that $x_1 = -\sum_{i \geq 2} a_i' x_i$. Thus, if we choose $y_i = x_i$, $i = 2, \ldots, n$ then

$$x_1 = \sum\nolimits_{j \geq 2} -a_j' y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a_j') + a_j \cdot 1(j \geq 2)$$

shows that the relation was trivial (to be explicit the elements a_{ij} are defined by setting $a_{11} = 0$, $a_{1j} = -a_j'$ for $j > 1$, and $a_{ij} = \delta_{ij}$ for $i, j \geq 2$). □

0AUW **Lemma 15.22.11.** *Let A be a Dedekind domain (for example a discrete valuation ring or more generally a PID).*

(1) *An A-module is flat if and only if it is torsion free.*
(2) *A finite torsion free A-module is finite locally free.*
(3) *A finite torsion free A-module is finite free if A is a PID.*

Proof. (For the parenthetical remark in the statement of the lemma, see Algebra, Lemma 10.120.15.) Proof of (1). By Lemma 15.22.6 and Algebra, Lemma 10.39.18 it suffices to check the statement over $A_{\mathfrak{m}}$ for $\mathfrak{m} \subset A$ maximal. Since $A_{\mathfrak{m}}$ is a discrete valuation ring (Algebra, Lemma 10.120.17) we win by Lemma 15.22.10.

Proof of (2). Follows from Algebra, Lemma 10.78.2 and (1).

Proof of (3). Let A be a PID and let M be a finite torsion free module. By Lemma 15.22.7 we see that $M \subset A^{\oplus n}$ for some n. We argue that M is free by induction on n. The case $n = 1$ expresses exactly the fact that A is a PID. If $n > 1$ let $M' \subset A^{\oplus n-1}$ be the image of the projection onto the last $n-1$ summands of $A^{\oplus n}$. Then we obtain a short exact sequence $0 \to I \to M \to M' \to 0$ where I is the intersection of M with the first summand A of $A^{\oplus n}$. By induction we see that M is an extension of finite free A-modules, whence finite free. □

0AUX **Lemma 15.22.12.** *Let R be a domain. Let M, N be R-modules. If N is torsion free, so is* $\mathrm{Hom}_R(M, N)$.

Proof. Choose a surjection $\bigoplus_{i \in I} R \to M$. Then $\mathrm{Hom}_R(M, N) \subset \prod_{i \in I} N$. □

15.23. Reflexive modules

0AUY Here is our definition.

0AUZ **Definition 15.23.1.** Let R be a domain. We say an R-module M is *reflexive* if the natural map

$$j : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$$

which sends $m \in M$ to the map sending $\varphi \in \mathrm{Hom}_R(M, R)$ to $\varphi(m) \in R$ is an isomorphism.

We can make this definition for more general rings, but already the definition above has drawbacks. It would be wise to restrict to Noetherian domains and finite torsion free modules and (perhaps) impose some regularity conditions on R (e.g., R is normal).

0AV0 **Lemma 15.23.2.** *Let R be a domain and let M be an R-module.*

(1) *If M is reflexive, then M is torsion free.*
(2) *If M is finite, then the kernel and cokernel of $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ are torsion modules.*
(3) *If M is finite, then j is injective if and only if M is torsion free.*

Proof. To see (1) use that the double dual of M is torsion free by Lemma 15.22.12. Assume that M is finite. Choose generators $x_1, \ldots, x_n$ of M over R. Let K be the fraction field of R. After renumbering we may assume that for some $0 \leq r \leq n$ the elements $x_1, \ldots, x_r$ map to a basis of $M \otimes_R K$ over K. For $r < i \leq n$ write $x_i \otimes 1 = \sum_{1 \leq j \leq r} x_j \otimes k_{ij}$ with $k_{ij} \in K$. Choose $c \in R$ such that $a_{ij} = ck_{ij} \in R$ and such that $cx_i = \sum_{1 \leq j \leq r} a_{ij} x_j$ in M (small detail omitted). Then we get maps

$$R^{\oplus r} \xrightarrow{\alpha} M \xrightarrow{\beta} R^{\oplus r} \xrightarrow{\alpha} M$$

with both $\alpha \circ \beta$ and $\beta \circ \alpha$ given by multiplication with c. Here α uses $x_1, \ldots, x_r$ and β sends x_i to c times the ith basis vector for $1 \leq i \leq r$ and to the vector $(a_{i1}, \ldots, a_{ir})$ for $r < i \leq n$. Consider the commutative diagram

$$\begin{array}{ccccccc}
R^{\oplus r} & \longrightarrow & M & \longrightarrow & R^{\oplus r} & \longrightarrow & M \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} \\
R^{\oplus r} & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) & \longrightarrow & R^{\oplus r} & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)
\end{array}$$

Of course the compositions of consecutive arrows on the bottom row are equal to multiplication by c. The diagram shows that the kernel and cokernel of j are annihilated by j which proves (2). For (3) note that if M is torsion free, then β is injective which implies that j is injective. The converse is clear because we've seen already that the double dual of M is torsion free. □

0B36 **Lemma 15.23.3.** *Let R be a discrete valuation ring and let M be a finite R-module. Then the map $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is surjective.*

Proof. Let $M_{tors} \subset M$ be the torsion submodule. Then we have $\mathrm{Hom}_R(M, R) = \mathrm{Hom}_R(M/M_{tors}, R)$ (holds over any domain). Hence we may assume that M is torsion free. Then M is free by Lemma 15.22.11 and the lemma is clear. □

0AV1 **Lemma 15.23.4.** *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent:*

(1) *M is reflexive,*
(2) *$M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module for all primes $\mathfrak{p} \subset R$, and*
(3) *$M_\mathfrak{m}$ is a reflexive $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. The localization of $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ at a prime $\mathfrak{p}$ is the corresponding map for the module $M_\mathfrak{p}$ over the Noetherian local domain $R_\mathfrak{p}$. See Algebra, Lemma 10.10.2. Thus the lemma holds by Algebra, Lemma 10.23.1. □

0EB8 **Lemma 15.23.5.** *Let R be a Noetherian domain. Let $0 \to M \to M' \to M''$ an exact sequence of finite R-modules. If M' is reflexive and M'' is torsion free, then M is reflexive.*

Proof. We will use without further mention that $\mathrm{Hom}_R(N, N')$ is a finite R-module for any finite R-modules N and N', see Algebra, Lemma 10.71.9. We take duals to get a sequence

$$\mathrm{Hom}_R(M, R) \leftarrow \mathrm{Hom}_R(M', R) \leftarrow \mathrm{Hom}_R(M'', R)$$

Dualizing again we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R) & \xrightarrow[j]{} & \mathrm{Hom}_R(\mathrm{Hom}_R(M',R),R) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M'',R),R) \\ \uparrow & & \uparrow & & \uparrow \\ M & \longrightarrow & M' & \longrightarrow & M'' \end{array}$$

We do not know the top row is exact. But, by assumption the middle vertical arrow is an isomorphism and the right vertical arrow is injective (Lemma 15.23.2). We claim j is injective. Assuming the claim a diagram chase shows that the left vertical arrow is an isomorphism, i.e., M is reflexive.

Proof of the claim. Consider the exact sequence $\mathrm{Hom}_R(M', R) \to \mathrm{Hom}_R(M, R) \to Q \to 0$ defining Q. One applies Algebra, Lemma 10.10.2 to obtain

$$\mathrm{Hom}_K(M' \otimes_R K, K) \to \mathrm{Hom}_K(M \otimes_R K, K) \to Q \otimes_R K \to 0$$

But $M \otimes_R K \to M' \otimes_R K$ is an injective map of vector spaces, hence split injective, so $Q \otimes_R K = 0$, that is, Q is torsion. Then one gets the exact sequence

$$0 \to \mathrm{Hom}_R(Q, R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M', R), R)$$

and $\mathrm{Hom}_R(Q, R) = 0$ because Q is torsion. □

0AV2 **Lemma 15.23.6.** *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent*

1. *M is reflexive,*
2. *there exists a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free.*

Proof. Observe that a finite free module is reflexive. By Lemma 15.23.5 we see that (2) implies (1). Assume M is reflexive. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to \mathrm{Hom}_R(M, R) \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to R^{\oplus n} \to N \to 0$$

with $N = \mathrm{Im}(R^{\oplus n} \to R^{\oplus m})$ a torsion free module. As $M = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ we get an exact sequence as in (2). □

0EB9 **Lemma 15.23.7.** *Let $R \to R'$ be a flat homomorphism of Noetherian domains. If M is a finite reflexive R-module, then $M \otimes_R R'$ is a finite reflexive R'-module.*

Proof. Choose a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free, see Lemma 15.23.6. Since $R \to R'$ is flat we obtain a short exact sequence $0 \to M \otimes_R R' \to F \otimes_R R' \to N \otimes_R R' \to 0$ with $F \otimes_R R'$ finite free and $N \otimes_R R'$ torsion free (Lemma 15.22.4). Thus $M \otimes_R R'$ is reflexive by Lemma 15.23.6. □

0AV3 **Lemma 15.23.8.** *Let R be a Noetherian domain. Let M be a finite R-module. Let N be a finite reflexive R-module. Then $\mathrm{Hom}_R(M, N)$ is reflexive.*

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Then we obtain

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$ torsion free. We conclude by Lemma 15.23.5. □

0AV4 **Definition 15.23.9.** Let R be a Noetherian domain. Let M be a finite R-module. The module $M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is called the *reflexive hull* of M.

This makes sense because the reflexive hull is reflexive by Lemma 15.23.8. The assignment $M \mapsto M^{**}$ is a functor. If $\varphi : M \to N$ is an R-module map into a reflexive R-module N, then φ factors $M \to M^{**} \to N$ through the reflexive hull of M. Another way to say this is that taking the reflexive hull is the left adjoint to the inclusion functor

$$\text{finite reflexive modules} \subset \text{finite modules}$$

over a Noetherian domain R.

0AV5 **Lemma 15.23.10.** *Let R be a Noetherian local ring. Let M, N be finite R-modules.*

(1) *If N has depth ≥ 1, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 1.*
(2) *If N has depth ≥ 2, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 2.*

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$. A submodule of a module with depth ≥ 1 has depth ≥ 1; this follows immediately from the definition. Thus part (1) is clear. For (2) note that here the assumption and the previous remark implies N' has depth ≥ 1. The module $N^{\oplus n}$ has depth ≥ 2. From Algebra, Lemma 10.72.6 we conclude $\mathrm{Hom}_R(M, N)$ has depth ≥ 2. □

0AV6 **Lemma 15.23.11.** *Let R be a Noetherian ring. Let M, N be finite R-modules.*

(1) *If N has property (S_1), then $\mathrm{Hom}_R(M, N)$ has property (S_1).*
(2) *If N has property (S_2), then $\mathrm{Hom}_R(M, N)$ has property (S_2).*
(3) *If R is a domain, N is torsion free and (S_2), then $\mathrm{Hom}_R(M, N)$ is torsion free and has property (S_2).*

Proof. Since localizing at primes commutes with taking Hom_R for finite R-modules (Algebra, Lemma 10.71.9) parts (1) and (2) follow immediately from Lemma 15.23.10. Part (3) follows from (2) and Lemma 15.22.12. □

0AV7 **Lemma 15.23.12.** *Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume that for every prime $\mathfrak{p}$ of R at least one of the following happens*

(1) *$M_\mathfrak{p} \to N_\mathfrak{p}$ is injective, or*
(2) *$\mathfrak{p} \notin Ass(M)$.*

Then φ is injective.

Proof. Let $\mathfrak{p}$ be an associated prime of $\mathrm{Ker}(\varphi)$. Then there exists an element $x \in M_\mathfrak{p}$ which is in the kernel of $M_\mathfrak{p} \to N_\mathfrak{p}$ and whose annihilator is $\mathfrak{p}R_\mathfrak{p}$ (Algebra, Lemma 10.63.15). This is impossible in both cases. Hence $\mathrm{Ass}(\mathrm{Ker}(\varphi)) = \emptyset$ and we conclude $\mathrm{Ker}(\varphi) = 0$ by Algebra, Lemma 10.63.7. □

0AV8 **Lemma 15.23.13.** *Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite and that for every prime $\mathfrak{p}$ of R one of the following happens*

(1) *$M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism, or*
(2) *$\mathrm{depth}(M_\mathfrak{p}) \geq 2$ and $\mathfrak{p} \notin \mathrm{Ass}(N)$.*

Then φ is an isomorphism.

Proof. By Lemma 15.23.12 we see that φ is injective. Let $N' \subset N$ be an finitely generated R-module containing the image of M. Then $\mathrm{Ass}(N_\mathfrak{p}) = \emptyset$ implies $\mathrm{Ass}(N'_\mathfrak{p}) = \emptyset$. Hence the assumptions of the lemma hold for $M \to N'$. In order to prove that φ is an isomorphism, it suffices to prove the same thing for every such $N' \subset N$. Thus we may assume N is a finite R-module. In this case, $\mathfrak{p} \notin \mathrm{Ass}(N) \Rightarrow \mathrm{depth}(N_\mathfrak{p}) \geq 1$, see Algebra, Lemma 10.63.18. Consider the short exact sequence

$$0 \to M \to N \to Q \to 0$$

defining Q. Looking at the conditions we see that either $Q_\mathfrak{p} = 0$ in case (1) or $\mathrm{depth}(Q_\mathfrak{p}) \geq 1$ in case (2) by Algebra, Lemma 10.72.6. This implies that Q does not have any associated primes, hence $Q = 0$ by Algebra, Lemma 10.63.7. □

0AV9 **Lemma 15.23.14.** *Let R be a Noetherian domain. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite, N is torsion free, and that for every prime $\mathfrak{p}$ of R one of the following happens*

(1) *$M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism, or*
(2) *$\mathrm{depth}(M_\mathfrak{p}) \geq 2$.*

Then φ is an isomorphism.

Proof. This is a special case of Lemma 15.23.13. □

0AVA **Lemma 15.23.15.** *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent*

(1) *M is reflexive,*
(2) *for every prime $\mathfrak{p}$ of R one of the following happens*
   (a) *$M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module, or*
   (b) *$\mathrm{depth}(M_\mathfrak{p}) \geq 2$.*

Proof. If (1) is true, then $M_\mathfrak{p}$ is a reflexive module for all primes of $\mathfrak{p}$ by Lemma 15.23.4. Thus (1) $\Rightarrow$ (2). Assume (2). Set $N = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ so that

$$N_\mathfrak{p} = \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}), R_\mathfrak{p})$$

for every prime $\mathfrak{p}$ of R. See Algebra, Lemma 10.10.2. We apply Lemma 15.23.14 to the map $j : M \to N$. This is allowed because M is finite and N is torsion free by Lemma 15.22.12. In case (2)(a) the map $M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism and in case (2)(b) we have $\mathrm{depth}(M_\mathfrak{p}) \geq 2$. □

0EBA **Lemma 15.23.16.** *Let R be a Noetherian domain. Let M be a finite reflexive R-module. Let $\mathfrak{p} \subset R$ be a prime ideal.*

(1) *If $\text{depth}(R_\mathfrak{p}) \geq 2$, then $\text{depth}(M_\mathfrak{p}) \geq 2$.*
(2) *If R is (S_2), then M is (S_2).*

Proof. Since formation of reflexive hull $\text{Hom}_R(\text{Hom}_R(M, R), R)$ commutes with localization (Algebra, Lemma 10.10.2) part (1) follows from Lemma 15.23.10. Part (2) is immediate from Lemma 15.23.11. □

0EBB **Example 15.23.17.** The results above and below suggest reflexivity is related to the (S_2) condition; here is an example to prevent too optimistic conjectures. Let k be a field. Let R be the k-subalgebra of $k[x, y]$ generated by $1, y, x^2, xy, x^3$. Then R is not (S_2). So R as an R-module is an example of a reflexive R-module which is not (S_2). Let $M = k[x, y]$ viewed as an R-module. Then M is a reflexive R-module because

$$\text{Hom}_R(M, R) = \mathfrak{m} = (y, x^2, xy, x^3) \quad \text{and} \quad \text{Hom}_R(\mathfrak{m}, R) = M$$

and M is (S_2) as an R-module (computations omitted). Thus R is a Noetherian domain possessing a reflexive (S_2) module but R is not (S_2) itself.

0AVB **Lemma 15.23.18.** *Let R be a Noetherian normal domain with fraction field K. Let M be a finite R-module. The following are equivalent*

(1) *M is reflexive,*
(2) *M is torsion free and has property (S_2),*
(3) *M is torsion free and $M = \bigcap_{height(\mathfrak{p})=1} M_\mathfrak{p}$ where the intersection happens in $M_K = M \otimes_R K$.*

Proof. By Algebra, Lemma 10.157.4 we see that R satisfies (R_1) and (S_2).

Assume (1). Then M is torsion free by Lemma 15.23.2 and satisfies (S_2) by Lemma 15.23.16. Thus (2) holds.

Assume (2). By definition $M' = \bigcap_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}$ is the kernel of the map

$$M_K \longrightarrow \bigoplus\nolimits_{\text{height}(\mathfrak{p})=1} M_K/M_\mathfrak{p} \subset \prod\nolimits_{\text{height}(\mathfrak{p})=1} M_K/M_\mathfrak{p}$$

Observe that our map indeed factors through the direct sum as indicated since given $a/b \in K$ there are at most finitely many height 1 primes $\mathfrak{p}$ with $b \in \mathfrak{p}$. Let $\mathfrak{p}_0$ be a prime of height 1. Then $(M_K/M_\mathfrak{p})_{\mathfrak{p}_0} = 0$ unless $\mathfrak{p} = \mathfrak{p}_0$ in which case we get $(M_K/M_\mathfrak{p})_{\mathfrak{p}_0} = M_K/M_{\mathfrak{p}_0}$. Thus by exactness of localization and the fact that localization commutes with direct sums, we see that $M'_{\mathfrak{p}_0} = M_{\mathfrak{p}_0}$. Since M has depth ≥ 2 at primes of height > 1, we see that $M \to M'$ is an isomorphism by Lemma 15.23.14. Hence (3) holds.

Assume (3). Let $\mathfrak{p}$ be a prime of height 1. Then $R_\mathfrak{p}$ is a discrete valuation ring by (R_1). By Lemma 15.22.11 we see that $M_\mathfrak{p}$ is finite free, in particular reflexive. Hence the map $M \to M^{**}$ induces an isomorphism at all the primes $\mathfrak{p}$ of height 1. Thus the condition $M = \bigcap_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}$ implies that $M = M^{**}$ and (1) holds. □

0AVC **Lemma 15.23.19.** *Let R be a Noetherian normal domain. Let M be a finite R-module. Then the reflexive hull of M is the intersection*

$$M^{**} = \bigcap\nolimits_{height(\mathfrak{p})=1} M_\mathfrak{p}/(M_\mathfrak{p})_{tors} = \bigcap\nolimits_{height(\mathfrak{p})=1} (M/M_{tors})_\mathfrak{p}$$

taken in $M \otimes_R K$.

Proof. Let $\mathfrak{p}$ be a prime of height 1. The kernel of $M_\mathfrak{p} \to M \otimes_R K$ is the torsion submodule $(M_\mathfrak{p})_{tors}$ of $M_\mathfrak{p}$. Moreover, we have $(M/M_{tors})_\mathfrak{p} = M_\mathfrak{p}/(M_\mathfrak{p})_{tors}$ and this is a finite free module over the discrete valuation ring $R_\mathfrak{p}$ (Lemma 15.22.11). Then $M_\mathfrak{p}/(M_\mathfrak{p})_{tors} \to (M_\mathfrak{p})^{**} = (M^{**})_\mathfrak{p}$ is an isomorphism, hence the lemma is a consequence of Lemma 15.23.18. □

0BM4 **Lemma 15.23.20.** *Let A be a Noetherian normal domain with fraction field K. Let L be a finite extension of K. If the integral closure B of A in L is finite over A, then B is reflexive as an A-module.*

Proof. It suffices to show that $B = \bigcap B_\mathfrak{p}$ where the intersection is over height 1 primes $\mathfrak{p} \subset A$, see Lemma 15.23.18. Let $b \in \bigcap B_\mathfrak{p}$. Let $x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of b over K. We want to show $a_i \in A$. By Algebra, Lemma 10.38.6 we see that $a_i \in A_\mathfrak{p}$ for all i and all height one primes $\mathfrak{p}$. Hence we get what we want from Algebra, Lemma 10.157.6 (or the lemma already cited as A is a reflexive module over itself). □

15.24. Content ideals

0AS9 The definition may not be what you expect.

0ASA **Definition 15.24.1.** Let A be a ring. Let M be a flat A-module. Let $x \in M$. If the set of ideals I in A such that $x \in IM$ has a smallest element, we call it the *content ideal of x*.

Note that since M is flat over A, for a pair of ideals I, I' of A we have $IM \cap I'M = (I \cap I')M$ as can be seen by tensoring the exact sequence $0 \to I \cap I' \to I \oplus I' \to I + I' \to 0$ by M.

0ASB **Lemma 15.24.2.** *Let A be a ring. Let M be a flat A-module. Let $x \in M$. The content ideal of x, if it exists, is finitely generated.*

Proof. Say $x \in IM$. Then we can write $x = \sum_{i=1,\ldots,n} f_i x_i$ with $f_i \in I$ and $x_i \in M$. Hence $x \in I'M$ with $I' = (f_1, \ldots, f_n)$. □

0ASC **Lemma 15.24.3.** *Let $(A, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be a map of flat A-modules such that $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective. If $x \in M$ has content ideal I, then $u(x)$ has content ideal I as well.*

Proof. It is clear that $u(x) \in IN$. If $u(x) \in I'N$, then $u(x) \in (I' \cap I)N$, see discussion following Definition 15.24.1. Hence it suffices to show: if $x \in I'N$ and $I' \subset I$, $I' \neq I$, then $u(x) \notin I'N$. Since I/I' is a nonzero finite A-module (Lemma 15.24.2) there is a nonzero map $\chi : I/I' \to A/\mathfrak{m}$ of A-modules by Nakayama's lemma (Algebra, Lemma 10.20.1). Since I is the content ideal of x we see that $x \notin I''M$ where $I'' = \mathrm{Ker}(\chi)$. Hence x is not in the kernel of the map

$$IM = I \otimes_A M \xrightarrow{\chi \otimes 1} A/\mathfrak{m} \otimes M \cong M/\mathfrak{m}M$$

Applying our hypothesis on $\overline{u}$ we conclude that $u(x)$ does not map to zero under the map

$$IN = I \otimes_A N \xrightarrow{\chi \otimes 1} A/\mathfrak{m} \otimes N \cong N/\mathfrak{m}N$$

and we conclude. □

0ASD **Lemma 15.24.4.** *Let A be a ring. Let M be a flat Mittag-Leffler module. Then every element of M has a content ideal.*

Proof. This is a special case of Algebra, Lemma 10.91.2. □

15.25. Flatness and finiteness conditions

054A In this section we discuss some implications of the type "flat + finite type ⇒ finite presentation". We will revisit this result in the chapter on flatness, see More on Flatness, Section 38.1. A first result of this type was proved in Algebra, Lemma 10.108.6.

053A **Lemma 15.25.1.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be an S-module. Assume*

(1) *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
(2) *M is a finite S-module,*
(3) *M flat over R, and*
(4) *for every prime $\mathfrak{p}$ of R the module $M_{\mathfrak{p}}$ is of finite presentation over $S_{\mathfrak{p}}$.*

Then M is of finite presentation over S.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of M as an S-module. Let $\mathfrak{q}$ be a prime ideal of S lying over a prime $\mathfrak{p}$ of R. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum Sk_j \subset K$ then $K'_{\mathfrak{p}} = K_{\mathfrak{p}}$ and $K'_{\mathfrak{p}_j} = K_{\mathfrak{p}_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_{\mathfrak{q}} = M_{\mathfrak{q}}$. By openness of flatness, see Algebra, Theorem 10.129.4 we conclude that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that M'_g is flat over R. Thus $M'_g \to M_g$ is a surjective map of flat R-modules. Consider the commutative diagram

$$\begin{array}{ccc} M'_g & \longrightarrow & M_g \\ \downarrow & & \downarrow \\ \prod (M'_g)_{\mathfrak{p}_j} & \longrightarrow & \prod (M_g)_{\mathfrak{p}_j} \end{array}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{\mathfrak{p}_j}$ is injective and M'_g is flat over R. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that M_g is of finite presentation over S_g. We conclude by applying Algebra, Lemma 10.23.2. □

053B **Lemma 15.25.2.** *Let $R \to S$ be a ring homomorphism. Assume*

(1) *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
(2) *$R \to S$ is of finite type,*
(3) *S flat over R, and*
(4) *for every prime $\mathfrak{p}$ of R the ring $S_{\mathfrak{p}}$ is of finite presentation over $R_{\mathfrak{p}}$.*

Then S is of finite presentation over R.

Proof. By assumption S is a quotient of a polynomial ring over R. Thus the result follows directly from Lemma 15.25.1. □

053C **Lemma 15.25.3.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over R, i.e., $\deg(x_i) > 0$ but not necessarily equal to 1. Let M be a graded S-module. Assume*

(1) *R is a local ring,*
(2) *M is a finite S-module, and*
(3) *M is flat over R.*

Then M is finitely presented as an S-module.

Proof. Let $M = \bigoplus M_d$ be the grading on M. Pick homogeneous generators $m_1, \ldots, m_r \in M$ of M. Say $\deg(m_i) = d_i \in \mathbf{Z}$. This gives us a presentation

$$0 \to K \to \bigoplus\nolimits_{i=1,\ldots,r} S(-d_i) \to M \to 0$$

which in each degree d leads to the short exact sequence

$$0 \to K_d \to \bigoplus\nolimits_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.$$

By assumption each M_d is a finite flat R-module. By Algebra, Lemma 10.78.5 this implies each M_d is a finite free R-module. Hence we see each K_d is a finite R-module. Also each K_d is flat over R by Algebra, Lemma 10.39.13. Hence we conclude that each K_d is finite free by Algebra, Lemma 10.78.5 again.

Let $\mathfrak{m}$ be the maximal ideal of R. By the flatness of M over R the short exact sequences above remain short exact after tensoring with $\kappa = \kappa(\mathfrak{m})$. As the ring $S \otimes_R \kappa$ is Noetherian we see that there exist homogeneous elements $k_1, \ldots, k_t \in K$ such that the images $\overline{k}_j$ generate $K \otimes_R \kappa$ over $S \otimes_R \kappa$. Say $\deg(k_j) = e_j$. Thus for any d the map

$$\bigoplus\nolimits_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d$$

becomes surjective after tensoring with κ. By Nakayama's lemma (Algebra, Lemma 10.20.1) this implies the map is surjective over R. Hence K is generated by $k_1, \ldots, k_t$ over S and we win. □

053D **Lemma 15.25.4.** *Let R be a ring. Let $S = \bigoplus_{n \geq 0} S_n$ be a graded R-algebra. Let $M = \bigoplus_{d \in \mathbf{Z}} M_d$ be a graded S-module. Assume S is finitely generated as an R-algebra, assume S_0 is a finite R-algebra, and assume there exist finitely many primes $\mathfrak{p}_j$, $i = 1, \ldots, m$ such that $R \to \prod R_{\mathfrak{p}_j}$ is injective.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module and finite as an S-module, then M is finitely presented as an S-module.*

Proof. As S is finitely generated as an R-algebra, it is finitely generated as an S_0 algebra, say by homogeneous elements $t_1, \ldots, t_n \in S$ of degrees $d_1, \ldots, d_n > 0$. Set $P = R[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i$. The ring map $P \to S$, $x_i \to t_i$ is finite as S_0 is a finite R-module. To prove (1) it suffices to prove that S is a finitely presented P-module. To prove (2) it suffices to prove that M is a finitely presented P-module. Thus it suffices to prove that if $S = P$ is a graded polynomial ring and M is a finite S-module flat over R, then M is finitely presented as an S-module. By Lemma 15.25.3 we see $M_{\mathfrak{p}}$ is a finitely presented $S_{\mathfrak{p}}$-module for every prime $\mathfrak{p}$ of R. Thus the result follows from Lemma 15.25.1. □

05GS **Remark 15.25.5.** Let R be a ring. When does R satisfy the condition mentioned in Lemmas 15.25.1, 15.25.2, and 15.25.4? This holds if

(1) R is local,
(2) R is Noetherian,
(3) R is a domain,
(4) R is a reduced ring with finitely many minimal primes, or
(5) R has finitely many weakly associated primes, see Algebra, Lemma 10.66.17.

Thus these lemmas hold in all cases listed above.

The following lemma will be improved on in More on Flatness, Proposition 38.13.10.

053E **Lemma 15.25.6.** *Let A be a valuation ring. Let $A \to B$ be a ring map of finite type. Let M be a finite B-module.* [**Nag66**, Theorem 3]

(1) *If B is flat over A, then B is a finitely presented A-algebra.*
(2) *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. We are going to use that an A-module is flat if and only if it is torsion free, see Lemma 15.22.10. By Algebra, Lemma 10.57.10 we can find a graded A-algebra S with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded S-module N such that $B \cong S_{(f)}$ and $M \cong N_{(f)}$. If M is torsion free, then we can take N torsion free by replacing it by N/N_{tors}, see Lemma 15.22.2. Similarly, if B is torsion free, then we can take S torsion free by replacing it by S/S_{tors}. Hence in case (1), we may apply Lemma 15.25.4 to see that S is a finitely presented A-algebra, which implies that $B = S_{(f)}$ is a finitely presented A-algebra. To see (2) we may first replace S by a graded polynomial ring, and then we may apply Lemma 15.25.3 to conclude. □

0GSE **Lemma 15.25.7.** *Let A be a valuation ring. Let $A \to B$ be a local homomorphism which is essentially of finite type. Let M be a finite B-module.*

(1) *If B is flat over A, then B is essentially of finite presentation over A.*
(2) *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak{q}$. In case (1) we consider $M = B$ as a finite module over $P_\mathfrak{q}$ and in case (2) we consider M as a finite module over $P_\mathfrak{q}$. In both cases, we have to show that this is a finitely presented $P_\mathfrak{q}$-module, see Algebra, Lemma 10.6.4 for case (2).

Choose a presentation $0 \to K \to P_\mathfrak{q}^{\oplus r} \to M \to 0$ which is possible because M is finite over $P_\mathfrak{q}$. Let $L = P^{\oplus r} \cap K$. Then $K = L_\mathfrak{q}$, see Algebra, Lemma 10.9.15. Then $N = P^{\oplus r}/L$ is a submodule of M and hence flat by Lemma 15.22.10. Since also N is a finite P-module, we see that N is finitely presented as a P-module by Lemma 15.25.6. Since localization is exact (Algebra, Proposition 10.9.12) we see that $N_\mathfrak{q} = M$ and we conclude. □

15.26. Blowing up and flatness

0535 In this section we begin our discussion of results of the form: "After a blowup the strict transform becomes flat". More results of this type may be found in Divisors, Section 31.35 and More on Flatness, Section 38.30.

053H **Definition 15.26.1.** Let R be a ring. Let $I \subset R$ be an ideal and $a \in I$. Let $R[\frac{I}{a}]$ be the affine blowup algebra, see Algebra, Definition 10.70.1. Let M be an R-module. The *strict transform of M along $R \to R[\frac{I}{a}]$* is the $R[\frac{I}{a}]$-module

$$M' = \left(M \otimes_R R[\tfrac{I}{a}]\right)/a\text{-power torsion}$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

053I **Lemma 15.26.2.** *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let S be a finite type R-algebra. Let M be a finite S-module. For every valuation ring $A \subset K$ dominating R there exists an ideal $I \subset \mathfrak{m}$ and a nonzero element $a \in I$ such that*

(1) *I is finitely generated,*
(2) *A has center on $R[\frac{I}{a}]$,*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero, and*
(4) *the strict transform $S_{I,a}$ of S along $R \to R[\frac{I}{a}]$ is flat and of finite presentation over R, and the strict transform $M_{I,a}$ of M along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented over $S_{I,a}$.*

Proof. Write $S = R[x_1, \ldots, x_n]/J$ and denote $N = S \oplus M$ viewed as a module over $P = R[x_1, \ldots, x_n]$. If we can prove the lemma in case S is a polynomial algebra over R, then we can find I, a satisfying (1), (2), (3) such that the strict transform $N_{I,a}$ of N along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented as a module over the strict transform $P_{I,a}]$ of P. Since $P_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$ (small detail omitted) we find that the summand $S_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $R[\frac{I}{a}][x_1, \ldots, x_n]$. Hence $S_{I,a}$ is finitely presented as an $R[\frac{I}{a}]$-algebra. Moreover, the summand $M_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $P_{I,a}$ hence also finitely presented as a module over $S_{I,a}$, see Algebra, Lemma 10.6.4. This reduces us to the case discussed in the next paragraph.

Assume $S = R[x_1, \ldots, x_n]$. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let M_A be the quotient of $M \otimes_R A$ by its torsion submodule, see Lemma 15.22.2. Then M_A is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma 15.22.10 we see that M_A is flat over A. By Lemma 15.25.6 we see that M_A is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an S_A-module. For any choice of $a \in I \subset \mathfrak{m}$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of M along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 10.70.12 we have $A = \operatorname{colim}_{I,a} R[\frac{I}{a}]$. This implies that $S_A = \operatorname{colim} S_{I,a}$ and

$$\operatorname{colim} M \otimes_R R[\tfrac{I}{a}] = M \otimes_R A$$

Choose I, a and lifts $k_1, \ldots, k_t \in S_{I,a}^{\oplus r}$. Since M_A is the quotient of $M \otimes_R A$ by torsion, we see that the images of $k_1, \ldots, k_t$ in $M \otimes_R A$ are annihilated by a nonzero element $\alpha \in A$. After replacing I, a by a different pair (recall that the colimit is filtered), we may assume $\alpha = x/a^n$ for some $x \in I^n$ nonzero. Then we find that $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R A$. Hence after replacing I, a by a different pair we may assume $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R R[\frac{I}{a}]$ for some nonzero $x \in R$. Then finally replacing I, a by xI, xa we find that we may assume $k_1, \ldots, k_t$ map to a-power torsion elements of $M \otimes_R R[\frac{I}{a}]$. For any such pair (I, a) we set

$$M'_{I,a} = S_{I,a}^{\oplus r} / \sum S_{I,a} k_j.$$

Since $M_A = S_A^{\oplus r} / \sum S_A k_j$ we see that $M_A = \operatorname{colim}_{I,a} M'_{I,a}$. At this point we finally apply Algebra, Lemma 10.168.1 (3) to conclude that $M'_{I,a}$ is flat for some pair (I, a)

as above. This lemma does not apply a priori to the system of strict transforms

$$M_{I,a} = (M \otimes_R R[\tfrac{I}{a}])/a\text{-power torsion}$$

as the transition maps may not satisfy the assumptions of the lemma. But now, since flatness implies torsion free (Lemma 15.22.9) and since $M_{I,a}$ is the quotient of $M'_{I,a}$ (because we arranged it so the elements $k_1, \ldots, k_t$ map to zero in $M_{I,a}$) by the a-power torsion submodule we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. □

0CZM **Lemma 15.26.3.** *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = Fit_k(M)$. For every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

has $Fit_k(M') = R'$.

Proof. First observe that $\mathrm{Fit}_k(M \otimes_R R') = IR' = aR'$. The first equality by Lemma 15.8.4 part (3) and the second equality by Algebra, Lemma 10.70.2. From Lemma 15.8.9 and exactness of localization we see that $M'_{\mathfrak{p}'}$ can be generated by $\leq k$ elements for every prime $\mathfrak{p}'$ of R'. Then $\mathrm{Fit}_k(M') = R'$ for example by Lemma 15.8.7. □

0CZN **Lemma 15.26.4.** *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = Fit_k(M)$. Assume that $M_{\mathfrak{p}}$ is free of rank k for every $\mathfrak{p} \notin V(I)$. Then for every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank k.

Proof. By Lemma 15.26.3 we have $\mathrm{Fit}_k(M') = R'$. By Lemma 15.8.8 it suffices to show that $\mathrm{Fit}_{k-1}(M') = 0$. Recall that $R' \subset R'_a = R_a$, see Algebra, Lemma 10.70.2. Hence it suffices to prove that $\mathrm{Fit}_{k-1}(M')$ maps to zero in $R'_a = R_a$. Since clearly $(M')_a = M_a$ this reduces us to showing that $\mathrm{Fit}_{k-1}(M_a) = 0$ because formation of Fitting ideals commutes with base change according to Lemma 15.8.4 part (3). This is true by our assumption that M_a is finite locally free of rank k (see Algebra, Lemma 10.78.2) and the already cited Lemma 15.8.8. □

0BBJ **Lemma 15.26.5.** *Let R be a ring. Let M be a finite R-module. Let $f \in R$ be an element such that M_f is finite locally free of rank r. Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank r.

Proof. Choose a surjection $R^{\oplus n} \to M$. Choose a finite submodule $K \subset \mathrm{Ker}(R^{\oplus n} \to M)$ such that $R^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible because M_f is of finite presentation for example by Algebra, Lemma 10.78.2. Set $M_1 = R^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 10.70.7. But a is a nonzerodivisor on M'_1, whence the displayed

map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented R-module.

Assume M is a finitely presented R-module. Then $J = \mathrm{Fit}_r(M) \subset S$ is a finitely generated ideal. We claim that $I = fJ$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{p}$ is not an element of $V(J)$ by Lemma 15.8.7. Thus $\mathfrak{p} \notin V(fJ) = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fb$ for some $b \in J$. By Algebra, Lemmas 10.70.2 and 10.70.8 we see that $JR' = bR'$ and b is a nonzerodivisor in R'. Let $\mathfrak{p}' \subset R' = R[\frac{I}{a}]$ be a prime ideal. Then $JR'_{\mathfrak{p}'}$ is generated by b. It follows from Lemma 15.8.9 that $M'_{\mathfrak{p}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in R'$, $g \notin \mathfrak{p}'$ such that the corresponding map $(R')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \mathrm{Fit}_{k-1}(M')$. Note that $J'R'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see Lemma 15.8.8. Since $R'_a = R_f$ (Algebra, Lemma 10.70.7) and $M'_a = M_f$ is free of rank r we see that $J'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. □

15.27. Completion and flatness

06LD In this section we discuss when the completion of a "big" flat module is flat.

05BC **Lemma 15.27.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. There is a canonical map*

$$\left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow \prod\nolimits_{\alpha \in A} R$$

from the I-adic completion of the direct sum into the product which is universally injective.

Proof. By definition an element x of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \bmod I^n$. As $R = R^\wedge$ we see that for any α there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \bmod I^n$. Note that for each n there are only finitely many α such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_\alpha R$ such that for each n there are only finitely many α such that $y_\alpha \bmod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite "convergent sums" $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each n there are only finitely many y_α which are nonzero modulo I^n. The displayed map maps this element to the element to (y_α) in the product. In particular the map is injective.

Let Q be a finite R-module. We have to show that the map

$$Q \otimes_R \left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow Q \otimes_R \left(\prod\nolimits_{\alpha \in A} R\right)$$

is injective, see Algebra, Theorem 10.82.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for Q. By Artin-Rees (Algebra, Lemma 10.51.2) there exists a constant c such that $\mathrm{Im}(R^{\oplus k} \to$

$R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \mathrm{Im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{array}{ccccccc} \bigoplus_{l=1}^k \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & \bigoplus_{j=1}^m \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & Q \otimes_R \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{l=1}^k \left(\prod_{\alpha\in A} R\right) & \longrightarrow & \bigoplus_{j=1}^m \left(\prod_{\alpha\in A} R\right) & \longrightarrow & Q \otimes_R \left(\prod_{\alpha\in A} R\right) & \longrightarrow & 0 \end{array}$$

with exact rows. Pick an element $\sum_j \sum_\alpha y_{j,\alpha}$ of $\bigoplus_{j=1,\ldots,m} \left(\bigoplus_{\alpha\in A} R\right)^\wedge$. If this element maps to zero in the module $Q \otimes_R \left(\prod_{\alpha\in A} R\right)$, then we see in particular that $\sum_j q_j \otimes y_{j,\alpha} = 0$ in Q for each α. Thus we can find an element $(z_{1,\alpha}, \ldots, z_{k,\alpha}) \in \bigoplus_{l=1,\ldots,k} R$ which maps to $(y_{1,\alpha}, \ldots, y_{m,\alpha}) \in \bigoplus_{j=1,\ldots,m} R$. Moreover, if $y_{j,\alpha} \in I^{N_\alpha}$ for $j = 1, \ldots, m$, then we may assume that $z_{l,\alpha} \in I^{N_\alpha - c}$ for $l = 1, \ldots, k$. Hence the sum $\sum_l \sum_\alpha z_{l,\alpha}$ is "convergent" and defines an element of $\bigoplus_{l=1,\ldots,k} \left(\bigoplus_{\alpha\in A} R\right)^\wedge$ which maps to the element $\sum_j \sum_\alpha y_{j,\alpha}$ we started out with. Thus the right vertical arrow is injective and we win. □

The following lemma can also be deduced from Lemma 15.27.4 below.

06LE **Lemma 15.27.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian. The completion $(\bigoplus_{\alpha\in A} R)^\wedge$ is a flat R-module.*

Proof. Denote $R^\wedge$ the completion of R with respect to I. As $R \to R^\wedge$ is flat by Algebra, Lemma 10.97.2 it suffices to prove that $(\bigoplus_{\alpha\in A} R)^\wedge$ is a flat $R^\wedge$-module (use Algebra, Lemma 10.39.4). Since

$$(\bigoplus\nolimits_{\alpha\in A} R)^\wedge = (\bigoplus\nolimits_{\alpha\in A} R^\wedge)^\wedge$$

we may replace R by $R^\wedge$ and assume that R is complete with respect to I (see Algebra, Lemma 10.97.4). In this case Lemma 15.27.1 tells us the map $(\bigoplus_{\alpha\in A} R)^\wedge \to \prod_{\alpha\in A} R$ is universally injective. Thus, by Algebra, Lemma 10.82.7 it suffices to show that $\prod_{\alpha\in A} R$ is flat. By Algebra, Proposition 10.90.6 (and Algebra, Lemma 10.90.5) we see that $\prod_{\alpha\in A} R$ is flat. □

0911 **Lemma 15.27.3.** *Let A be a Noetherian ring. Let I be an ideal of A. Let M be a finite A-module. For every $p > 0$ there exists a $c > 0$ such that $Tor_p^A(M, A/I^n) \to Tor_p^A(M, A/I^{n-c})$ is zero for all $n \geq c$.*

This is [**Qui**, Lemma 9.9]; note that the author forgot the word "strict" in the statement although it was clearly intended.

Proof. Proof for $p = 1$. Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. Then $\mathrm{Tor}_1^A(M, A/I^n) = K \cap (I^n)^{\oplus t}/I^n K$. By Artin-Rees (Algebra, Lemma 10.51.2) there is a constant $c \geq 0$ such that $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ for $n \geq c$. Thus the result for $p = 1$. For $p > 1$ we have $\mathrm{Tor}_p^A(M, A/I^n) = \mathrm{Tor}_{p-1}^A(K, A/I^n)$. Thus the lemma follows by induction. □

0912 **Lemma 15.27.4.** *Let A be a Noetherian ring. Let I be an ideal of A. Let (M_n) be an inverse system of A-modules such that*

(1) *M_n is a flat A/I^n-module,*
(2) *$M_{n+1} \to M_n$ is surjective.*

Then $M = \lim M_n$ is a flat A-module and $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q.

Proof. We first show that $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q. Choose a resolution $F_2 \to F_1 \to F_0 \to Q \to 0$ by finite free A-modules F_i. Then

$$F_2 \otimes_A M_n \to F_1 \otimes_A M_n \to F_0 \otimes_A M_n$$

is a chain complex whose homology in degree 0 is $Q \otimes_A M_n$ and whose homology in degree 1 is

$$\mathrm{Tor}_1^A(Q, M_n) = \mathrm{Tor}_1^A(Q, A/I^n) \otimes_{A/I^n} M_n$$

as M_n is flat over A/I^n. By Lemma 15.27.3 we see that this system is essentially constant (with value 0). It follows from Homology, Lemma 12.31.7 that $\lim Q \otimes_A M_n = \mathrm{Coker}(\lim F_1 \otimes_A M_n \to \lim F_0 \otimes_A M_n)$. Since F_i is finite free this equals $\mathrm{Coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M$.

Next, let $Q \to Q'$ be an injective map of finite A-modules. We have to show that $Q \otimes_A M \to Q' \otimes_A M$ is injective (Algebra, Lemma 10.39.5). By the above we see

$$\mathrm{Ker}(Q \otimes_A M \to Q' \otimes_A M) = \mathrm{Ker}(\lim Q \otimes_A M_n \to \lim Q' \otimes_A M_n).$$

For each n we have an exact sequence

$$\mathrm{Tor}_1^A(Q', M_n) \to \mathrm{Tor}_1^A(Q'', M_n) \to Q \otimes_A M_n \to Q' \otimes_A M_n$$

where $Q'' = \mathrm{Coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor's are essentially constant with value 0. It follows from Homology, Lemma 12.31.7 that the inverse limit of the right most maps is injective. □

0AGW **Lemma 15.27.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *I is finitely generated,*
(2) *R/I is Noetherian,*
(3) *M/IM is flat over R/I,*
(4) *$Tor_1^R(M, R/I) = 0$.*

Then the I-adic completion $R^\wedge$ is a Noetherian ring and $M^\wedge$ is flat over $R^\wedge$.

Proof. By Algebra, Lemma 10.99.8 the modules M/I^nM are flat over R/I^n for all n. By Algebra, Lemma 10.96.3 we have (a) $R^\wedge$ and $M^\wedge$ are I-adically complete and (b) $R/I^n = R^\wedge/I^nR^\wedge$ for all n. By Algebra, Lemma 10.97.5 the ring $R^\wedge$ is Noetherian. Applying Lemma 15.27.4 we conclude that $M^\wedge = \lim M/I^nM$ is flat as an $R^\wedge$-module. □

15.28. The Koszul complex

0621 We define the Koszul complex as follows.

0622 **Definition 15.28.1.** Let R be a ring. Let $\varphi : E \to R$ be an R-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the commutative differential graded algebra defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree n in $K_\bullet(\varphi)$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that E is a finite free module, say $E = R^{\oplus n}$. In this case the map φ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

0623 **Definition 15.28.2.** Let R be a ring and let $f_1, \ldots, f_r \in R$. The *Koszul complex on $f_1, \ldots, f_r$* is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^{\oplus r} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_r)$, $K_\bullet(R, f_1, \ldots, f_r)$, or $K_\bullet(R, f_\bullet)$.

Of course, if E is finite locally free, then $K_\bullet(\varphi)$ is locally on $\mathrm{Spec}(R)$ isomorphic to a Koszul complex $K_\bullet(f_1, \ldots, f_r)$. This complex has many interesting formal properties.

0624 **Lemma 15.28.3.** *Let $\varphi : E \to R$ and $\varphi' : E' \to R$ be R-module maps. Let $\psi : E \to E'$ be an R-module map such that $\varphi' \circ \psi = \varphi$. Then ψ induces a homomorphism of differential graded algebras $K_\bullet(\varphi) \to K_\bullet(\varphi')$.*

Proof. This is immediate from the definitions. □

0625 **Lemma 15.28.4.** *Let $f_1, \ldots, f_r \in R$ be a sequence. Let (x_{ij}) be an invertible $r \times r$-matrix with coefficients in R. Then the complexes $K_\bullet(f_\bullet)$ and*

$$K_\bullet(\sum x_{1j} f_j, \sum x_{2j} f_j, \ldots, \sum x_{rj} f_j)$$

are isomorphic.

Proof. Set $g_i = \sum x_{ij} f_j$. The matrix (x_{ji}) gives an isomorphism $x : R^{\oplus r} \to R^{\oplus r}$ such that $(g_1, \ldots, g_r) = (f_1, \ldots, f_r) \circ x$. Hence this follows from the functoriality of the Koszul complex described in Lemma 15.28.3. □

0626 **Lemma 15.28.5.** *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \varphi(e)$ in R. Then*

$$f = de + ed$$

as endomorphisms of $K_\bullet(\varphi)$.

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

0663 **Lemma 15.28.6.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be a sequence. Multiplication by f_i on $K_\bullet(f_\bullet)$ is homotopic to zero, and in particular the cohomology modules $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_r)$.*

Proof. Special case of Lemma 15.28.5. □

In Derived Categories, Section 13.9 we defined the cone of a morphism of cochain complexes. The cone $C(f)_\bullet$ of a morphism of chain complexes $f : A_\bullet \to B_\bullet$ is the complex $C(f)_\bullet$ given by $C(f)_n = B_n \oplus A_{n-1}$ and differential

0627 (15.28.6.1) $$d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : B_\bullet \to C(f)_\bullet$ and $p : C(f)_\bullet \to A_\bullet[-1]$ induced by the obvious maps $B_n \to C(f)_n \to A_{n-1}$.

0628 **Lemma 15.28.7.** *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f \in R$. Set $E' = E \oplus R$ and define $\varphi' : E' \to R$ by φ on E and multiplication by f on R. The complex $K_\bullet(\varphi')$ is isomorphic to the cone of the map of complexes*

$$f : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi).$$

Proof. Denote $e_0 \in E'$ the element $1 \in R \subset R \oplus E$. By our definition of the cone above we see that

$$C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi) = \wedge^n(E) \oplus \wedge^{n-1}(E) = \wedge^n(E')$$

where in the last $=$ we map $(0, e_1 \wedge \ldots \wedge e_{n-1})$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}$ in $\wedge^n(E')$. A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as $\varphi'|_E = \varphi$ and $d_{C(f)}$ restricted to the first summand is just $d_{K_\bullet(\varphi)}$. On the other hand, if $e_1 \wedge \ldots \wedge e_{n-1}$ is in the second summand, then

$$d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = fe_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})$$

and on the other hand

$$\begin{aligned}
& d_{K_\bullet(\varphi')}(0, e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}) \\
& = \sum\nolimits_{i=0,\ldots,n-1} (-1)^i \varphi'(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
& = fe_1 \wedge \ldots \wedge e_{n-1} + \sum\nolimits_{i=1,\ldots,n-1} (-1)^i \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
& = fe_1 \wedge \ldots \wedge e_{n-1} - e_0 \left(\sum\nolimits_{i=1,\ldots,n-1} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \right)
\end{aligned}$$

which is the image of the result of the previous computation. □

0629 **Lemma 15.28.8.** *Let R be a ring. Let $f_1, \ldots, f_r$ be a sequence of elements of R. The complex $K_\bullet(f_1, \ldots, f_r)$ is isomorphic to the cone of the map of complexes*

$$f_r : K_\bullet(f_1, \ldots, f_{r-1}) \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}).$$

Proof. Special case of Lemma 15.28.7. □

062A **Lemma 15.28.9.** *Let R be a ring. Let $A_\bullet$ be a complex of R-modules. Let $f, g \in R$. Let $C(f)_\bullet$ be the cone of $f : A_\bullet \to A_\bullet$. Define similarly $C(g)_\bullet$ and $C(fg)_\bullet$. Then $C(fg)_\bullet$ is homotopy equivalent to the cone of a map*

$$C(f)_\bullet[1] \longrightarrow C(g)_\bullet$$

Proof. We first prove this if $A_\bullet$ is the complex consisting of R placed in degree 0. In this case the complex $C(f)_\bullet$ is the complex

$$\ldots \to 0 \to R \xrightarrow{f} R \to 0 \to \ldots$$

with R placed in (homological) degrees 1 and 0. The map of complexes we use is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
0 & \longrightarrow & R & \xrightarrow{g} & R & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The cone of this is the chain complex consisting of $R^{\oplus 2}$ placed in degrees 1 and 0 and differential (15.28.6.1)

$$\begin{pmatrix} g & 1 \\ 0 & -f \end{pmatrix} : R^{\oplus 2} \longrightarrow R^{\oplus 2}$$

To see this chain complex is homotopic to $C(fg)_\bullet$, i.e., to $R \xrightarrow{fg} R$, consider the maps of complexes

$$\begin{array}{ccc} R & \xrightarrow{fg} & R \\ {\scriptstyle (1,-g)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ R^{\oplus 2} & \longrightarrow & R^{\oplus 2} \end{array} \qquad \begin{array}{ccc} R^{\oplus 2} & \longrightarrow & R^{\oplus 2} \\ {\scriptstyle (1,0)}\downarrow & & \downarrow{\scriptstyle (f,1)} \\ R & \xrightarrow{fg} & R \end{array}$$

with obvious notation. The composition of these two maps in one direction is the identity on $C(fg)_\bullet$, but in the other direction it isn't the identity. We omit writing out the required homotopy.

To see the result holds in general, we use that we have a functor $K_\bullet \mapsto \mathrm{Tot}(A_\bullet \otimes_R K_\bullet)$ on the category of complexes which is compatible with homotopies and cones. Then we write $C(f)_\bullet$ and $C(g)_\bullet$ as the total complex of the double complexes

$$(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet$$

and in this way we deduce the result from the special case discussed above. Some details omitted. □

062B **Lemma 15.28.10.** *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f, g \in R$. Set $E' = E \oplus R$ and define $\varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R$ by φ on E and multiplication by f, g, fg on R. The complex $K_\bullet(\varphi'_{fg})$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(\varphi'_f)[1] \longrightarrow K_\bullet(\varphi'_g).$$

Proof. By Lemma 15.28.7 the complex $K_\bullet(\varphi'_f)$ is isomorphic to the cone of multiplication by f on $K_\bullet(\varphi)$ and similarly for the other two cases. Hence the lemma follows from Lemma 15.28.9. □

062C **Lemma 15.28.11.** *Let R be a ring. Let $f_1, \ldots, f_{r-1}$ be a sequence of elements of R. Let $f, g \in R$. The complex $K_\bullet(f_1, \ldots, f_{r-1}, fg)$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(f_1, \ldots, f_{r-1}, f)[1] \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}, g)$$

Proof. Special case of Lemma 15.28.10. □

0664 **Lemma 15.28.12.** *Let R be a ring. Let $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ be elements of R. Then there is an isomorphism of Koszul complexes*

$$K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s) = \mathit{Tot}(K_\bullet(R, f_1, \ldots, f_r) \otimes_R K_\bullet(R, g_1, \ldots, g_s)).$$

Proof. Omitted. Hint: If $K_\bullet(R, f_1, \ldots, f_r)$ is generated as a differential graded algebra by $x_1, \ldots, x_r$ with $\mathrm{d}(x_i) = f_i$ and $K_\bullet(R, g_1, \ldots, g_s)$ is generated as a differential graded algebra by $y_1, \ldots, y_s$ with $\mathrm{d}(y_j) = g_j$, then we can think of $K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s)$ as the differential graded algebra generated by the sequence of elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ with $\mathrm{d}(x_i) = f_i$ and $\mathrm{d}(y_j) = g_j$. □

15.29. The extended alternating Čech complex

0G6F Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex of R is the cochain complex

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}$$

where R is in degree 0, the term $\bigoplus_{i_0} R_{f_{i_0}}$ is in degre 1, and so on. The maps are defined as follows

(1) The map $R \to \bigoplus_{i_0} R_{f_{i_0}}$ is given by the canonical maps $R \to R_{f_{i_0}}$.
(2) Given $1 \leq i_0 < \ldots < i_{p+1} \leq r$ and $0 \leq j \leq p+1$ we have the canonical localization map
$$R_{f_{i_0}\ldots\hat{f}_{i_j}\ldots f_{i_{p+1}}} \to R_{f_{i_0}\ldots f_{i_{p+1}}}$$
(3) The differentials use the canonical maps of (2) with sign $(-1)^j$.

If M is any R-module, the extended alternating Čech complex of M is the similarly constructed cochain complex

$$M \to \bigoplus\nolimits_{i_0} M_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1\ldots f_r}$$

where M is in degree 0 as before.

0G6G **Lemma 15.29.1.** *The extended alternating Čech complexes defined above are complexes of R-modules.*

Proof. Omitted. □

0G6H **Lemma 15.29.2.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. The extended alternating Čech complex of M is the tensor product over R of M with the extended alternating Čech complex of R.*

Proof. Omitted. □

0G6I **Lemma 15.29.3.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let $R \to S$ be a ring map, denote $g_1, \ldots, g_r \in S$ the images of $f_1, \ldots, f_r$, and set $N = M \otimes_R S$. The extended alternating Čech complex constructed using S, $g_1, \ldots, g_r$, and N is the tensor product of the extended alternating Čech complex of M with S over R.*

Proof. Omitted. □

0G6J **Lemma 15.29.4.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. If there exists an $i \in \{1, \ldots, r\}$ such that f_i is a unit, then the extended alternating Čech complex of M is homotopy equivalent to 0.*

Proof. We will use the following notation: a cochain x of degree $p+1$ in the extended alternating Čech complex of M is $x = (x_{i_0\ldots i_p})$ where $x_{i_0\ldots i_p}$ is in $M_{f_{i_0}\ldots f_{i_p}}$. With this notation we have

$$d(x)_{i_0\ldots i_{p+1}} = \sum\nolimits_j (-1)^j x_{i_0\ldots\hat{i}_j\ldots i_{p+1}}$$

As homotopy we use the maps

$$h : \text{cochains of degree } p+2 \to \text{cochains of degree } p+1$$

given by the rule

$$h(x)_{i_0\ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(x)_{i_0\ldots i_p} = (-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p} \text{ if not}$$

Here j is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since f_i is a unit we have the equality

$$M_{f_{i_0}\ldots f_{i_p}} = M_{f_{i_0}\ldots f_{i_j} f_i f_{i_{j+1}}\ldots f_{i_p}}$$

which we can use to make sense of thinking of $(-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p}$ as an element of $M_{f_{i_0}\ldots f_{i_p}}$. We will show by a computation that $dh + hd$ equals the negative of the identity map which finishes the proof. To do this fix x a cochain of degree $p+1$ and let $1 \leq i_0 < \ldots < i_p \leq r$.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(x))_{i_0\ldots i_p} = 0$. On the other hand we have

$$d(h(x))_{i_0\ldots i_p} = \sum (-1)^j h(x)_{i_0\ldots \hat{i}_j \ldots i_p} = (-1)^t h(x)_{i_0\ldots \hat{i}\ldots i_p} = (-1)^t (-1)^{t-1} x_{i_0\ldots i_p}$$

Thus $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired.

Case II: $i \notin \{i_0, \ldots, i_p\}$. Let j be such that $i_j < i < i_{j+1}$. Then we see that

$$\begin{aligned} h(d(x))_{i_0\ldots i_p} &= (-1)^j d(x)_{i_0\ldots i_j i i_{j+1}\ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j+j'} x_{i_0\ldots \hat{i}_{j'}\ldots i_j i i_{j+1}\ldots i_p} - x_{i_0\ldots i_p} \\ &\quad + \sum\nolimits_{j' > j} (-1)^{j+j'+1} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'}\ldots i_p} \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(h(x))_{i_0\ldots i_p} &= \sum\nolimits_{j'} (-1)^{j'} h(x)_{i_0\ldots \hat{i}_{j'}\ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j'+j-1} x_{i_0\ldots \hat{i}_{j'}\ldots i_j i i_{j+1}\ldots i_p} \\ &\quad + \sum\nolimits_{j' > j} (-1)^{j'+j} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'}\ldots i_p} \end{aligned}$$

Adding these up we obtain $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired. □

0G6K **Lemma 15.29.5.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let H^q be the qth cohomology module of the extended alternation Čech complex of M. Then*

1. *$H^q = 0$ if $q \notin [0, r]$,*
2. *for $x \in H^i$ there exists an $n \geq 1$ such that $f_i^n x = 0$ for $i = 1, \ldots, r$,*
3. *the support of H^q is contained in $V(f_1, \ldots, f_r)$,*
4. *if there is an $f \in (f_1, \ldots, f_r)$ which acts invertibly on M, then $H^q = 0$.*

Proof. Part (1) follows from the fact that the extended alternating Čech complex is zero in degrees < 0 and $> r$. To prove (2) it suffices to show that for each i there exists an $n \geq 1$ such that $f_i^n x = 0$. To see this it suffices to show that $(H^q)_{f_i} = 0$. Since localization is exact, $(H^q)_{f_i}$ is the qth cohomology module of the localization of the extended alternating complex of M at f_i. By Lemma 15.29.3 this localization is the extended alternating Čech complex of M_{f_i} over R_{f_i} with respect to the images of $f_1, \ldots, f_r$ in R_{f_i}. Thus we reduce to showing that H^q is zero if f_i is invertible, which follows from Lemma 15.29.4. Part (3) follows from the observation that $(H^q)_{f_i} = 0$ for all i that we just proved. To see part (4) note

that in this case f acts invertibly on H^q and H^q is supported on $V(f)$ by (3). This forces H^q to be zero (small detail omitted). □

0913 **Lemma 15.29.6.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex*

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K(R, f_1^n, \ldots, f_r^n)$; see proof for a precise statement.

Proof. We urge the reader to prove this for themselves. Denote $K(R, f_1^n, \ldots, f_r^n)$ the Koszul complex of Definition 15.28.2 viewed as a cochain complex sitting in degrees $0, \ldots, r$. Thus we have

$$K(R, f_1^n, \ldots, f_r^n) : 0 \to \wedge^r(R^{\oplus r}) \to \wedge^{r-1}(R^{\oplus r}) \to \ldots \to R^{\oplus r} \to R \to 0$$

with the term $\wedge^r(R^{\oplus r})$ sitting in degree 0. Let $e_1^n, \ldots, e_r^n$ be the standard basis of $R^{\oplus r}$. Then the elements $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ for $1 \leq j_1 < \ldots < j_{r-p} \leq r$ form a basis for the term in degree p of the Koszul complex. Further, observe that

$$d(e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n) = \sum (-1)^{a+1} f_{j_a}^n e_{j_1}^n \wedge \ldots \wedge \hat{e}_{j_a}^n \wedge \ldots \wedge e_{j_{r-p}}^n$$

by our construction of the Koszul complex in Section 15.28. The transition maps of our system

$$K(R, f_1^n, \ldots, f_r^n) \to K(R, f_1^{n+1}, \ldots, f_r^{n+1})$$

are given by the rule

$$e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n \longmapsto f_{i_0} \ldots f_{i_{p-1}} e_{j_1}^{n+1} \wedge \ldots \wedge e_{j_{r-p}}^{n+1}$$

where the indices $1 \leq i_0 < \ldots < i_{p-1} \leq r$ are such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to $f_1 \ldots f_r$ in degree r.

Denote $K^p(R, f_1^n, \ldots, f_r^n)$ the term of degree p in the Koszul complex. Observe that for any $f \in R$ we have

$$R_f = \operatorname{colim}(R \xrightarrow{f} R \xrightarrow{f} R \to \ldots)$$

Hence we see that in degree p we obtain

$$\operatorname{colim} K^p(R, f_1^n, \ldots f_r^n) = \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

Here the element $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ of the Koszul complex above maps in the colimit to the element $(f_{i_0} \ldots f_{i_{p-1}})^{-n}$ in the summand $R_{f_{i_0} \ldots f_{i_{p-1}}}$ where the indices are chosen such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}) = \sum\nolimits_{i \notin \{i_0, \ldots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0} \ldots f_{i_t} f_i f_{i_{t+1}} \ldots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree p by the map

$$\bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}} \longrightarrow \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}} \longmapsto (-1)^{i_0 + \ldots + i_{p-1} + p} \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from $\operatorname{colim} K(R, f_1^n, \ldots, f_r^n)$ to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out. □

15.30. Koszul regular sequences

062D Please take a look at Algebra, Sections 10.68, 10.69, and 10.72 before looking at this one.

062E **Definition 15.30.1.** Let R be a ring. Let $r \geq 0$ and let $f_1, \ldots, f_r \in R$ be a sequence of elements. Let M be an R-module. The sequence $f_1, \ldots, f_r$ is called

(1) *M-Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$ for all $i \neq 0$,
(2) *M-H_1-regular* if $H_1(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$,
(3) *Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for all $i \neq 0$, and
(4) *H_1-regular* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$.

We will see in Lemmas 15.30.2, 15.30.3, and 15.30.6 that for elements $f_1, \ldots, f_r$ of a ring R we have the following implications

$$\begin{aligned} f_1, \ldots, f_r \text{ is a regular sequence} &\Rightarrow f_1, \ldots, f_r \text{ is a Koszul-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is an } H_1\text{-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is a quasi-regular sequence.} \end{aligned}$$

In general none of these implications can be reversed, but if R is a Noetherian local ring and $f_1, \ldots, f_r \in \mathfrak{m}_R$, then the four conditions are all equivalent (Lemma 15.30.7). If $f = f_1 \in R$ is a length 1 sequence and f is not a unit of R then it is clear that the following are all equivalent

(1) f is a regular sequence of length one,
(2) f is a Koszul-regular sequence of length one, and
(3) f is a H_1-regular sequence of length one.

It is also clear that these imply that f is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

$$R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)$$

and let f be the image of x in R. Then f is a zerodivisor, but $\bigoplus_{n\geq 0}(f^n)/(f^{n+1}) \cong k[x]$ is a polynomial ring.

062F **Lemma 15.30.2.** *An M-regular sequence is M-Koszul-regular. A regular sequence is Koszul-regular.*

Proof. Let R be a ring and let M be an R-module. It is immediate that an M-regular sequence of length 1 is M-Koszul-regular. Let $f_1, \ldots, f_r$ be an M-regular sequence. Then f_1 is a nonzerodivisor on M. Hence

$$0 \to K_\bullet(f_2, \ldots, f_r) \otimes M \xrightarrow{f_1} K_\bullet(f_2, \ldots, f_r) \otimes M \to K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \to 0$$

is a short exact sequence of complexes where $\overline{f}_i$ is the image of f_i in $R/(f_1)$. By Lemma 15.28.8 the complex $K_\bullet(R, f_1, \ldots, f_r)$ is isomorphic to the cone of multiplication by f_1 on $K_\bullet(f_2, \ldots, f_r)$. Thus $K_\bullet(R, f_1, \ldots, f_r) \otimes M$ is isomorphic to the cone on the first map. Hence $K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M$ is quasi-isomorphic to $K_\bullet(f_1, \ldots, f_r) \otimes M$. As $\overline{f}_2, \ldots, \overline{f}_r$ is an M/f_1M-regular sequence in $R/(f_1)$ the result follows from the case $r = 1$ and induction. □

0CEM **Lemma 15.30.3.** *A M-Koszul-regular sequence is M-H_1-regular. A Koszul-regular sequence is H_1-regular.*

Proof. This is immediate from the definition. □

062G **Lemma 15.30.4.** *Let $f_1, \ldots, f_{r-1} \in R$ be a sequence and $f, g \in R$. Let M be an R-module.*

(1) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-H_1-regular then $f_1, \ldots, f_{r-1}, fg$ is M-H_1-regular too.*
(2) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-Koszul-regular then $f_1, \ldots, f_{r-1}, fg$ is M-Koszul-regular too.*

Proof. By Lemma 15.28.11 we have exact sequences

$$H_i(K_\bullet(f_1, \ldots, f_{r-1}, f) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, fg) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, g) \otimes M)$$

for all i. □

062H **Lemma 15.30.5.** *Let $\varphi : R \to S$ be a flat ring map. Let $f_1, \ldots, f_r \in R$. Let M be an R-module and set $N = M \otimes_R S$.*

(1) *If $f_1, \ldots, f_r$ in R is an M-H_1-regular sequence, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-H_1-regular sequence in S.*
(2) *If $f_1, \ldots, f_r$ is an M-Koszul-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-Koszul-regular sequence in S.*

Proof. This is true because $K_\bullet(f_1, \ldots, f_r) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r))$ and therefore $(K_\bullet(f_1, \ldots, f_r) \otimes_R M) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r)) \otimes_S N$. □

062I **Lemma 15.30.6.** *An M-H_1-regular sequence is M-quasi-regular.*

Proof. Let R be a ring and let M be an R-module. Let $f_1, \ldots, f_r$ be an M-H_1-regular sequence. Denote $J = (f_1, \ldots, f_r)$. The assumption means that we have an exact sequence

$$\wedge^2(R^r) \otimes M \to R^{\oplus r} \otimes M \to JM \to 0$$

where the first arrow is given by $e_i \wedge e_j \otimes m \mapsto (f_i e_j - f_j e_i) \otimes m$. Tensoring the sequence with R/J we see that

$$JM/J^2M = (R/J)^{\oplus r} \otimes_R M = (M/JM)^{\oplus r}$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I f_1^{i_1} \ldots f_r^{i_r} \in J^{n+1}M$$

then $m_I \in JM$ for all I. In the next paragraph, we prove $m_I \in JM$ for $I = (0, \ldots, 0, n)$ and in the last paragraph we deduce the general case from this special case.

Let $I = (0, \ldots, 0, n)$. Let ξ be as above. We can write $\xi = m_1 f_1 + \ldots + m_{r-1} f_{r-1} + m_I f_r^n$. As we have assumed $\xi \in J^{n+1}M$, we can also write $\xi = \sum_{1 \leq i \leq j \leq r-1} m_{ij} f_i f_j + \sum_{1 \leq i \leq r-1} m_i' f_i f_r^n + m'' f_r^{n+1}$. Then we see that

$$\begin{gathered}(m_1 - m_{11}f_1 - m_1' f_r^n) f_1 + \\ (m_2 - m_{12}f_1 - m_{22}f_2 - m_2' f_r^n) f_2 + \\ \ldots + \\ (m_{r-1} - m_{1r-1}f_1 - \ldots - m_{r-1r-1}f_{r-1} - m_{r-1}' f_r^n) f_{r-1} + \\ (m_I - m'' f_r) f_r^n = 0\end{gathered}$$

Since $f_1, \ldots, f_{r-1}, f_r^n$ is M-H_1-regular by Lemma 15.30.4 we see that $m_I - m'' f_r$ is in the submodule $f_1 M + \ldots + f_{r-1} M + f_r^n M$. Thus $m_I \in f_1 M + \ldots + f_r M$.

Let $S = R[x_1, x_2, \ldots, x_r, 1/x_r]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_r$ is an M-H_1-regular sequence in S, see Lemma 15.30.5. By Lemma 15.28.4 we see that

$$g_1 = f_1 - \frac{x_1}{x_r} f_r, \ \ldots, \ g_{r-1} = f_{r-1} - \frac{x_{r-1}}{x_r} f_r, \ g_r = \frac{1}{x_r} f_r$$

is an M-H_1-regular sequence in S. Finally, note that our element ξ can be rewritten

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I (g_1 + x_1 g_r)^{i_1} \ldots (g_{r-1} + x_{r-1} g_r)^{i_{r-1}} (x_r g_r)^{i_r}$$

and the coefficient of g_r^n in this expression is

$$\sum m_I x_1^{i_1} \ldots x_r^{i_r}$$

By the case discussed in the previous paragraph this sum is in $J(M \otimes_R S)$. Since the monomials $x_1^{i_1} \ldots x_r^{i_r}$ form part of an R-basis of S over R we conclude that $m_I \in J$ for all I as desired. □

For nonzero finite modules over Noetherian local rings all of the types of regular sequences introduced so far are equivalent.

09CC **Lemma 15.30.7.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_r \in \mathfrak{m}$. The following are equivalent*

(1) *$f_1, \ldots, f_r$ is an M-regular sequence,*
(2) *$f_1, \ldots, f_r$ is a M-Koszul-regular sequence,*
(3) *$f_1, \ldots, f_r$ is an M-H_1-regular sequence,*
(4) *$f_1, \ldots, f_r$ is an M-quasi-regular sequence.*

In particular the sequence $f_1, \ldots, f_r$ is a regular sequence in R if and only if it is a Koszul regular sequence, if and only if it is a H_1-regular sequence, if and only if it is a quasi-regular sequence.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 15.30.2. The implication (2) $\Rightarrow$ (3) is Lemma 15.30.3. The implication (3) $\Rightarrow$ (4) is Lemma 15.30.6. The implication (4) $\Rightarrow$ (1) is Algebra, Lemma 10.69.6. □

0665 **Lemma 15.30.8.** *Let A be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in A whose image in A/I is H_1-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.*

Proof. Consider the exact sequence of complexes

$$0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0$$

Since the complex on the right has $H_1 = 0$ by assumption we see that

$$\mathrm{Coker}(I^{\oplus m} \to I) \longrightarrow \mathrm{Coker}(A^{\oplus m} \to A)$$

is injective. This is equivalent to the assertion of the lemma. □

0666 **Lemma 15.30.9.** *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Then $I \cap J^2 = IJ$.*

Proof. To prove this choose $g_1, \ldots, g_m \in J$ whose images in A/I form a H_1-regular sequence which generates J/I. In particular $J = I + (g_1, \ldots, g_m)$. Suppose that $x \in I \cap J^2$. Because $x \in J^2$ can write

$$x = \sum a_{ij} g_i g_j + \sum a_j g_j + a$$

with $a_{ij} \in A$, $a_j \in I$ and $a \in I^2$. Then $\sum a_{ij} g_i g_j \in I \cap (g_1, \ldots, g_m)$ hence by Lemma 15.30.8 we see that $\sum a_{ij} g_i g_j \in I(g_1, \ldots, g_m)$. Thus $x \in IJ$ as desired. □

0667 **Lemma 15.30.10.** *Let A be a ring. Let I be an ideal generated by a quasi-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A.*

Proof. We claim that $g_1, \ldots, g_m$ forms an H_1-regular sequence in A/I^d for every d. By induction assume that this holds in A/I^{d-1}. We have a short exact sequence of complexes

$$0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0$$

Since $f_1, \ldots, f_n$ is quasi-regular we see that the first complex is a direct sum of copies of $K_\bullet(A/I, g_1, \ldots, g_m)$ hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence $g_1, \ldots, g_m$ forms a quasi-regular sequence in A/I^d for every $d \geq 1$, see Lemma 15.30.6. Now we are ready to prove that $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A. Namely, set $J = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and suppose that (with multinomial notation)

$$\sum\nolimits_{|N|+|M|=d} a_{N,M} f^N g^M \in J^{d+1}$$

for some $a_{N,M} \in A$. We have to show that $a_{N,M} \in J$ for all N, M. Let $e \in \{0, 1, \ldots, d\}$. Then

$$\sum\nolimits_{|N|=d-e,\ |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1}$$

Because $g_1, \ldots, g_m$ is a quasi-regular sequence in A/I^{d-e+1} we deduce

$$\sum\nolimits_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1}$$

for each M with $|M| = e$. By Lemma 15.30.8 applied to I^{d-e}/I^{d-e+1} in the ring A/I^{d-e+1} this implies $\sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m)$. Since $f_1, \ldots, f_n$ is quasi-regular in A this implies that $a_{N,M} \in J$ for each N, M with $|N| = d - e$ and $|M| = e$. This proves the lemma. □

0668 **Lemma 15.30.11.** *Let A be a ring. Let I be an ideal generated by an H_1-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an H_1-regular sequence in A.*

Proof. We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^2(A^{\oplus n+m}) & \longrightarrow & A^{\oplus n+m} & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \wedge^2(A/I^{\oplus m}) & \longrightarrow & A/I^{\oplus m} & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in A. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I we see that $(\overline{a}_{n+1}, \ldots, \overline{a}_{n+m})$ is the image of some element $\overline{\alpha}$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\overline{\alpha}$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and subtract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements

$$(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$$

in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero. □

068L **Lemma 15.30.12.** *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an H_1-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an H_1-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1,\ldots,m} \overline{a}_j \overline{g}_j$ in A/I is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$\sum\nolimits_{j=1,\ldots,m} a_j g_j + \sum\nolimits_{i=1,\ldots,n} b_i f_i = 0$$

in A. By assumption this relation in A is a linear combination of trivial relations. Taking the image in A/I we obtain what we want. □

0669 **Lemma 15.30.13.** *Let A be a ring. Let I be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in A.*

Proof. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A/I. Then

$$\begin{aligned} K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) &= \mathrm{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \\ &\cong A/(f_i, g_j) \end{aligned}$$

The first equality by Lemma 15.28.12. The first quasi-isomorphism $\cong$ by (the dual of) Homology, Lemma 12.25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. □

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 110.15.1.

068M **Lemma 15.30.14.** *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in A, then $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A. Then

$$\begin{aligned} A/(f_i, g_j) &\cong K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) \\ &= \mathrm{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \end{aligned}$$

The first quasi-isomorphism $\cong$ by assumption. The first equality by Lemma 15.28.12. The second quasi-isomorphism by (the dual of) Homology, Lemma 12.25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. Hence we win. □

066A **Lemma 15.30.15.** *Let R be a ring. Let I be an ideal generated by $f_1, \ldots, f_r \in R$.*

1. *If I can be generated by a quasi-regular sequence of length r, then $f_1, \ldots, f_r$ is a quasi-regular sequence.*
2. *If I can be generated by an H_1-regular sequence of length r, then $f_1, \ldots, f_r$ is an H_1-regular sequence.*
3. *If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.*

Proof. If I can be generated by a quasi-regular sequence of length r, then I/I^2 is free of rank r over R/I. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f}_i$ form a basis of I/I^2 over R/I. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of I/I^2, is that the maps $\mathrm{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that I can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod I. The matrix a_{ij} gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet(R, f_1, \ldots, f_r) \to K_\bullet(R, g_1, \ldots, g_r)$, see Lemma 15.28.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet(R, f_1, \ldots, f_r)$ and $K_\bullet(R, g_1, \ldots, g_r)$ are annihilated by I, see Lemma 15.28.6, we see that α is a quasi-isomorphism.

Now assume that $g_1, \ldots, g_r$ is a H_1-regular sequence generating I. Then $g_1, \ldots, g_r$ is a quasi-regular sequence by Lemma 15.30.6. By the previous paragraph we conclude that $f_1, \ldots, f_r$ is a H_1-regular sequence. Similarly for Koszul-regular sequences. □

068P **Lemma 15.30.16.** *Let R be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_i t_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzerodivisor.*

This is a particular case of [**McC57**, Corollary]

Proof. If one of the a_i is a unit this is just the statement that any element of the form $t_1 + a_2t_2 + \ldots + a_nt_n$ is a nonzerodivisor in the polynomial ring over R.

Case I: R is Noetherian. Let $\mathfrak{q}_j$, $j = 1, \ldots, m$ be the associated primes of R. We have to show that each of the maps

$$\sum a_i t_i : \mathrm{Sym}^d(R^{\oplus n}) \longrightarrow \mathrm{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\mathrm{Sym}^d(R^{\oplus n})$ is a free R-module its associated primes are $\mathfrak{q}_j$, $j = 1, \ldots, m$. For each j there exists an $i = i(j)$ such that $a_i \notin \mathfrak{q}_j$ because there exists an $x \in R$ with $\mathfrak{q}_j x = 0$ but $a_i x \neq 0$ for some i by assumption. Hence a_i is a unit in $R_{\mathfrak{q}_j}$ and the map is injective after localizing at $\mathfrak{q}_j$. Thus the map is injective, see Algebra, Lemma 10.63.19.

Case II: R general. We can write R as the union of Noetherian rings R_λ with $a_1, \ldots, a_n \in R_\lambda$. For each R_λ the result holds, hence the result holds for R. □

068Q **Lemma 15.30.17.** *Let R be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in R such that $(f_1, \ldots, f_n) \neq R$. Consider the faithfully flat, smooth ring map*

$$R \longrightarrow S = R[\{t_{ij}\}_{i \leq j}, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$$

For $1 \leq i \leq n$ set

$$g_i = \sum\nolimits_{i \leq j} t_{ij} f_j \in S.$$

Then $g_1, \ldots, g_n$ is a regular sequence in S and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$.

Proof. The equality of ideals is obvious as the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is invertible in S. Because $f_1, \ldots, f_n$ is a Koszul-regular sequence we see that the kernel of $R \to R^{\oplus n}$, $x \mapsto (xf_1, \ldots, xf_n)$ is zero (as it computes the $n$the Koszul homology of R w.r.t. $f_1, \ldots, f_n$). Hence by Lemma 15.30.16 we see that $g_1 = f_1t_{11} + \ldots + f_nt_{1n}$ is a nonzerodivisor in $S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}^{-1}]$. We see that $g_1, f_2, \ldots, f_n$ is a Koszul-sequence in S' by Lemma 15.30.5 and 15.30.15. We conclude that $\overline{f}_2, \ldots, \overline{f}_n$ is a Koszul-regular sequence in $S'/(g_1)$ by Lemma 15.30.14. Hence by induction on n we see that the images $\overline{g}_2, \ldots, \overline{g}_n$ of $g_2, \ldots, g_n$ in $S'/(g_1)[\{t_{ij}\}_{2 \leq i \leq j}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$ form a regular sequence. This in turn means that $g_1, \ldots, g_n$ forms a regular sequence in S. □

15.31. More on Koszul regular sequences

0CEN We continue the discussion from Section 15.30.

0G6L **Lemma 15.31.1.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be an Koszul-regular sequence. Then the extended alternating Čech complex $R \to \bigoplus_{i_0} R_{f_{i_0}} \to \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$ from Section 15.29 only has cohomology in degree r.*

Proof. By Lemma 15.30.4 and induction the sequence $f_1, \ldots, f_{r-1}, f_r^n$ is Koszul regular for all $n \geq 1$. By Lemma 15.28.4 any permutation of a Koszul regular sequence is a Koszul regular sequence. Hence we see that we may replace any (or all) f_i by its nth power and still have a Koszul regular sequence. Thus $K_\bullet(R, f_1^n, \ldots, f_r^n)$

has nonzero cohomology only in homological degree 0. This implies what we want by Lemma 15.29.6. □

0BIQ **Lemma 15.31.2.** *Let $a, a_2, \ldots, a_r$ be an H_1-regular sequence in a ring R (for example a Koszul regular sequence or a regular sequence, see Lemmas 15.30.2 and 15.30.3). With $I = (a, a_2, \ldots, a_r)$ the blowup algebra $R' = R[\frac{I}{a}]$ is isomorphic to $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$.*

Proof. By Algebra, Lemma 10.70.6 it suffices to show that R'' is a-torsion free.

We claim $a, ay_2 - a_2, \ldots, ay_n - a_r$ is a H_1-regular sequence in $R[y_2, \ldots, y_r]$. Namely, the map

$$(a, ay_2 - a_2, \ldots, ay_n - a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to define the Koszul complex on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is isomorphic to the map

$$(a, a_2, \ldots, a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to the define the Koszul complex on $a, a_2, \ldots, a_r$ via the isomorphism

$$R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]^{\oplus r}$$

sending $(b_1, \ldots, b_r)$ to $(b_1 - b_2y_2 \ldots - b_ry_r, -b_2, \ldots, -b_r)$. By Lemma 15.28.3 these Koszul complexes are isomorphic. By Lemma 15.30.5 applied to the flat ring map $R \to R[y_2, \ldots, y_r]$ we conclude our claim is true. By Lemma 15.28.8 we see that the Koszul complex K on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is the cone on $a : L \to L$ where L is the Koszul complex on $ay_2 - a_2, \ldots, ay_n - a_r$. Since $H_1(K) = 0$ by the claim, we conclude that $a : H_0(L) \to H_0(L)$ is injective, in other words that $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$ has no nonzero a-torsion elements as desired. □

063Q **Lemma 15.31.3.** *Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in B such that $B/(f_1, \ldots, f_r)$ is A-flat. Let $A \to A'$ be a ring map. Then the canonical map*

$$H_1(K_\bullet(B, f_1, \ldots, f_r)) \otimes_A A' \longrightarrow H_1(K_\bullet(B', f'_1, \ldots, f'_r))$$

is surjective. Here $B' = B \otimes_A A'$ and $f'_i \in B'$ is the image of f_i.

Proof. The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of A-modules with B/J flat over A and cohomology group $H_1 = H_1(K_\bullet(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with A' we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at B' and B'/J'. In order to compute its cohomology group $H'_1 = H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ at $(B')^{\oplus r}$ we split the first sequence above into the exact sequences $0 \to J \to B \to B/J \to 0$, $0 \to K \to B^{\oplus r} \to J \to 0$, and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring over A with A' we obtain the exact sequences

$$0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0$$
$$K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0$$
$$\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0$$

where the first one is exact as B/J is flat over A, see Algebra, Lemma 10.39.12. We conclude that $J' = J \otimes_A A' \subset B'$ and that $K \otimes_A A' \to \mathrm{Ker}((B')^{\oplus r} \to B')$ is surjective. Thus

$$\begin{aligned} H_1 \otimes_A A' &= \mathrm{Coker}\left(\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A'\right) \\ &\to \mathrm{Coker}\left(\wedge^2((B')^{\oplus r}) \to \mathrm{Ker}((B')^{\oplus r} \to B')\right) = H_1' \end{aligned}$$

is surjective too. □

0CEP **Lemma 15.31.4.** *Let $A \to B$ and $A \to A'$ be ring maps. Set $B' = B \otimes_A A'$. Let $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A*

1. *If $f_1, \ldots, f_r$ is a quasi-regular sequence, then the image in B' is a quasi-regular sequence.*
2. *If $f_1, \ldots, f_r$ is a H_1-regular sequence, then the image in B' is a H_1-regular sequence.*

Proof. Assume $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 10.39.13 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 10.39.12. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Assume $f_1, \ldots, f_r$ is H_1-regular. By Lemma 15.31.3 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanishing of $H_1(K_\bullet(B', f_1', \ldots, f_r'))$ and we win. □

0CEQ **Lemma 15.31.5.** *Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f_1', \ldots, f_r' \in B'$. Assume*

1. *$A' \to B'$ is flat and of finite presentation,*
2. *I is locally nilpotent,*
3. *the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,*
4. *$B/(f_1, \ldots, f_r)$ is flat over A.*

Then $B'/(f_1', \ldots, f_r')$ is flat over A'.

Proof. Set $C' = B'/(f_1', \ldots, f_r')$. We have to show $A' \to C'$ is flat. Let $\mathfrak{r}' \subset C'$ be a prime ideal lying over $\mathfrak{p}' \subset A'$. We let $\mathfrak{q}' \subset B'$ be the inverse image of $\mathfrak{r}'$. By Algebra, Lemma 10.39.18 it suffices to show that $A'_{\mathfrak{p}'} \to C'_{\mathfrak{q}'}$ is flat. Algebra, Lemma 10.128.6 tells us it suffices to show that $f_1', \ldots, f_r'$ map to a regular sequence in

$$B'_{\mathfrak{q}'}/\mathfrak{p}'B'_{\mathfrak{q}'} = B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = (B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$$

with obvious notation. What we know is that $f_1, \ldots, f_r$ is a quasi-regular sequence in B and that $B/(f_1, \ldots, f_r)$ is flat over A. By Lemma 15.31.4 the images $\overline{f}_1, \ldots, \overline{f}_r$ of $f_1', \ldots, f_r'$ in $B \otimes_A \kappa(\mathfrak{p})$ form a quasi-regular sequence. Since $(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$ is a Noetherian local ring, we conclude by Lemma 15.30.7. □

0CER **Lemma 15.31.6.** *Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f_1', \ldots, f_r' \in B'$. Assume*

1. *$A' \to B'$ is flat and of finite presentation (for example smooth),*
2. *I is locally nilpotent,*
3. *the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,*
4. *$B/(f_1, \ldots, f_r)$ is smooth over A.*

Then $B'/(f'_1, \ldots, f'_r)$ is smooth over A'.

Proof. Set $C' = B'/(f'_1, \ldots, f'_r)$ and $C = B/(f_1, \ldots, f_r)$. Then $A' \to C'$ is of finite presentation. By Lemma 15.31.5 we see that $A' \to C'$ is flat. The fibre rings of $A' \to C'$ are equal to the fibre rings of $A \to C$ and hence smooth by assumption (4). It follows that $A' \to C'$ is smooth by Algebra, Lemma 10.137.16. □

15.32. Regular ideals

07CU We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section 31.20. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

07CV **Definition 15.32.1.** Let R be a ring and let $I \subset R$ be an ideal.

(1) We say I is a *regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(2) We say I is a *Koszul-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a Koszul-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(3) We say I is a *H_1-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and an H_1-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(4) We say I is a *quasi-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a quasi-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.

It is clear that given $I \subset R$ we have the implications

$$\begin{aligned} I \text{ is a regular ideal} &\Rightarrow I \text{ is a Koszul-regular ideal} \\ &\Rightarrow I \text{ is a } H_1\text{-regular ideal} \\ &\Rightarrow I \text{ is a quasi-regular ideal} \end{aligned}$$

see Lemmas 15.30.2, 15.30.3, and 15.30.6. Such an ideal is always finitely generated.

07CW **Lemma 15.32.2.** *A quasi-regular ideal is finitely generated.*

Proof. Let $I \subset R$ be a quasi-regular ideal. Since $V(I)$ is quasi-compact, there exist $g_1, \ldots, g_m \in R$ such that $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$ and such that I_{g_j} is generated by a quasi-regular sequence $g_{j1}, \ldots, g_{jr_j} \in R_{g_j}$. Write $g_{ji} = g'_{ji}/g_j^{e_{ij}}$ for some $g'_{ij} \in I$. Write $1 + x = \sum g_j h_j$ for some $x \in I$ which is possible as $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$. Note that $\mathrm{Spec}(R) = D(g_1) \cup \ldots \cup D(g_m) \bigcup D(x)$ Then I is generated by the elements g'_{ij} and x as these generate on each of the pieces of the cover, see Algebra, Lemma 10.23.2. □

08RK **Lemma 15.32.3.** *Let $I \subset R$ be a quasi-regular ideal of a ring. Then I/I^2 is a finite projective R/I-module.*

Proof. This follows from Algebra, Lemma 10.78.2 and the definitions. □

We prove flat descent for Koszul-regular, H_1-regular, quasi-regular ideals.

068N **Lemma 15.32.4.** *Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in B, then I is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in A.*

Proof. We fix the prime $\mathfrak{p} \supset I$ throughout the proof. Assume IB is quasi-regular. By Lemma 15.32.2 IB is a finite module, hence I is a finite A-module by Algebra, Lemma 10.83.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$

As IB is quasi-regular, the B/IB-module $IB/(IB)^2$ is finite locally free. Hence I/I^2 is finite projective, see Algebra, Proposition 10.83.3. In particular, after replacing A by A_f for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that I/I^2 is free of rank r. Pick $f_1, \ldots, f_r \in I$ which give a basis of I/I^2. By Nakayama's lemma (see Algebra, Lemma 10.20.1) we see that, after another replacement $A \rightsquigarrow A_f$ as above, I is generated by $f_1, \ldots, f_r$.

Proof of the "quasi-regular" case. Above we have seen that I/I^2 is free on the r-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\mathrm{Sym}^d(I/I^2) \to I^d/I^{d+1}$ are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\mathrm{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on B by assumption. Details omitted.

Proof of the "H_1-regular" and "Koszul-regular" case. Consider the sequence of elements $f_1, \ldots, f_r$ generating I we constructed above. By Lemma 15.30.15 we see that $f_1, \ldots, f_r$ map to a H_1-regular or Koszul-regular sequence in B_g for any $g \in B$ such that IB is generated by an H_1-regular or Koszul-regular sequence. Hence $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing H_1 or H_i, $i > 0$. Since the homology of $K_\bullet(B, f_1, \ldots, f_r) = K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by IB (see Lemma 15.28.6) and since $V(IB) \subset \bigcup_{g \text{ as above}} D(g)$ we conclude that $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_\bullet(A, f_1, \ldots, f_r)$. □

07CX **Lemma 15.32.5.** *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a H_1-regular ideal. Then $I \cap J^2 = IJ$.*

Proof. Follows immediately from Lemma 15.30.9 by localizing. □

15.33. Local complete intersection maps

07CY We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

07CZ **Lemma 15.33.1.** *Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel I is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel J is a Koszul-regular ideal.*

Proof. Choose $f_j \in A[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in A[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} A[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & A[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ A[y_1, \ldots, y_m] & \longrightarrow & B \end{array}$$

Note that the kernel K of $A[x_i, y_j] \to B$ is equal to $K = (I, y_j - f_j) = (J, x_i - f_i)$. In particular, as I is finitely generated by Lemma 15.32.2 we see that $J = K/(x_i - f_i)$ is finitely generated too.

Pick a prime $\mathfrak{q} \subset B$. Since $I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n}$ (Algebra, Lemma 10.134.15) we see that

$$\dim J/J^2 \otimes_B \kappa(\mathfrak{q}) + n = \dim I/I^2 \otimes_B \kappa(\mathfrak{q}) + m.$$

Pick $p_1, \ldots, p_t \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q}) = I \otimes_{A[x_i]} \kappa(\mathfrak{q})$. Pick $q_1, \ldots, q_s \in J$ which map to a basis of $J/J^2 \otimes \kappa(\mathfrak{q}) = J \otimes_{A[y_j]} \kappa(\mathfrak{q})$. So $s+n = t+m$. By Nakayama's lemma there exist $h \in A[x_i]$ and $h' \in A[y_j]$ both mapping to a nonzero element of $\kappa(\mathfrak{q})$ such that $I_h = (p_1, \ldots, p_t)$ in $A[x_i, 1/h]$ and $J_{h'} = (q_1, \ldots, q_s)$ in $A[y_j, 1/h']$. As I is Koszul-regular we may also assume that I_h is generated by a Koszul regular sequence. This sequence must necessarily have length $t = \dim I/I^2 \otimes_B \kappa(\mathfrak{q})$, hence we see that $p_1, \ldots, p_t$ is a Koszul-regular sequence by Lemma 15.30.15. As also $y_1 - f_1, \ldots, y_m - f_m$ is a regular sequence we conclude

$$y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t$$

is a Koszul-regular sequence in $A[x_i, y_j, 1/h]$ (see Lemma 15.30.13). This sequence generates the ideal K_h. Hence the ideal $K_{hh'}$ is generated by a Koszul-regular sequence of length $m + t = n + s$. But it is also generated by the sequence

$$x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s$$

of the same length which is thus a Koszul-regular sequence by Lemma 15.30.15. Finally, by Lemma 15.30.14 we conclude that the images of $q_1, \ldots, q_s$ in

$$A[x_i, y_j, 1/hh'] / (x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'']$$

form a Koszul-regular sequence generating $J_{h''}$. Since h'' is the image of hh' it doesn't map to zero in $\kappa(\mathfrak{q})$ and we win. □

This lemma allows us to make the following definition.

07D0 **Definition 15.33.2.** A ring map $A \to B$ is called a *local complete intersection* if it is of finite type and for some (equivalently any) presentation $B = A[x_1, \ldots, x_n]/I$ the ideal I is Koszul-regular.

This notion is local.

07D1 **Lemma 15.33.3.** *Let $R \to S$ be a ring map. Let $g_1, \ldots, g_m \in S$ generate the unit ideal. If each $R \to S_{g_j}$ is a local complete intersection so is $R \to S$.*

Proof. Let $S = R[x_1, \ldots, x_n]/I$ be a presentation. Pick $h_j \in R[x_1, \ldots, x_n]$ mapping to g_j in S. Then $R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1)$ is a presentation of S_{g_j}. Hence $I_j = (I, x_{n+1}h_j - 1)$ is a Koszul-regular ideal in $R[x_1, \ldots, x_n, x_{n+1}]$. Pick a prime $I \subset \mathfrak{q} \subset R[x_1, \ldots, x_n]$. Then $h_j \notin \mathfrak{q}$ for some j and $\mathfrak{q}_j = (\mathfrak{q}, x_{n+1}h_j - 1)$ is a prime ideal of $V(I_j)$ lying over $\mathfrak{q}$. Pick $f_1, \ldots, f_r \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q})$. Then $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a sequence of elements of I_j which map to a basis of $I_j \otimes \kappa(\mathfrak{q}_j)$. By Nakayama's lemma there exists an $h \in R[x_1, \ldots, x_n, x_{n+1}]$ such that $(I_j)_h$ is generated by $x_{n+1}h_j - 1, f_1, \ldots, f_r$. We may also assume that $(I_j)_h$ is generated by a Koszul regular sequence of some length e. Looking at the dimension of $I_j \otimes \kappa(\mathfrak{q}_j)$ we see that $e = r + 1$. Hence by Lemma 15.30.15 we see that $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a Koszul-regular sequence generating $(I_j)_h$ for some $h \in R[x_1, \ldots, x_n, x_{n+1}]$, $h \notin \mathfrak{q}_j$. By Lemma 15.30.14 we see that $I_{h'}$ is generated by a Koszul-regular sequence for some $h' \in R[x_1, \ldots, x_n]$, $h' \notin \mathfrak{q}$ as desired. □

07D2 **Lemma 15.33.4.** *Let R be a ring. If $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection, then $f_1, \ldots, f_c$ is a Koszul regular sequence.*

Proof. Recall that the homology groups $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_c)$. Hence it suffices to show that $H_i(K_\bullet(f_\bullet))_\mathfrak{q}$ is zero for all primes $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_c)$. This follows from Algebra, Lemma 10.136.12 and the fact that a regular sequence is Koszul regular (Lemma 15.30.2). □

07D3 **Lemma 15.33.5.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is syntomic (Algebra, Definition 10.136.1), and*
(2) *$R \to S$ is flat and a local complete intersection.*

Proof. Assume (1). Then $R \to S$ is flat by definition. By Algebra, Lemma 10.136.15 and Lemma 15.33.3 we see that it suffices to show a relative global complete intersection is a local complete intersection homomorphism which is Lemma 15.33.4.

Assume (2). A local complete intersection is of finite presentation because a Koszul-regular ideal is finitely generated. Let $R \to k$ be a map to a field. It suffices to show that $S' = S \otimes_R k$ is a local complete intersection over k, see Algebra, Definition 10.135.1. Choose a prime $\mathfrak{q}' \subset S'$. Write $S = R[x_1, \ldots, x_n]/I$. Then $S' = k[x_1, \ldots, x_n]/I'$ where $I' \subset k[x_1, \ldots, x_n]$ is the image of I. Let $\mathfrak{p}' \subset k[x_1, \ldots, x_n]$, $\mathfrak{q} \subset S$, and $\mathfrak{p} \subset R[x_1, \ldots, x_n]$ be the corresponding primes. By Definition 15.32.1 exists an $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak{p}$ and $f_1, \ldots, f_r \in R[x_1, \ldots, x_n]_g$ which form a Koszul-regular sequence generating I_g. Since S and hence S_g is flat over R we see that the images $f'_1, \ldots, f'_r$ in $k[x_1, \ldots, x_n]_g$ form a H_1-regular sequence generating I'_g, see Lemma 15.31.4. Thus $f'_1, \ldots, f'_r$ map to a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$ generating $I'_{\mathfrak{p}'}$ by Lemma 15.30.7. Applying Algebra, Lemma 10.135.4 we conclude $S'_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}'$ is a global complete intersection over k as desired. □

For a local complete intersection $R \to S$ we have $H_n(L_{S/R}) = 0$ for $n \geq 2$. Since we haven't (yet) defined the full cotangent complex we can't state and prove this, but we can deduce one of the consequences.

07D4 **Lemma 15.33.6.** *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is a local complete intersection homomorphism. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_1, \ldots, y_m] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A[x_s]/A} \otimes C & \longrightarrow & \Omega_{A[x_s,y_t]/A} \otimes C & \longrightarrow & \Omega_{B[y_t]/B} \otimes C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \end{array}$$

with exact rows. In particular, the six term exact sequence of Algebra, Lemma 10.134.4 can be completed with a zero on the left, i.e., the sequence

$$0 \to H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact.

Proof. The only thing to prove is the injectivity of the map $I/I^2 \otimes C \to K/K^2$. By assumption the ideal J is Koszul-regular. Hence we have $IA[x_s, y_j] \cap K^2 = IK$ by Lemma 15.32.5. This means that the kernel of $K/K^2 \to J/J^2$ is isomorphic to $IA[x_s, y_j]/IK$. Since $I/I^2 \otimes_A C = IA[x_s, y_j]/IK$ by right exactness of tensor product, this provides us with the desired injectivity of $I/I^2 \otimes_A C \to K/K^2$. □

07D5 **Lemma 15.33.7.** *Let $A \to B \to C$ be ring maps. If $B \to C$ is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma 15.33.6 remains valid.*

Proof. Follows from Lemma 15.33.6 and Algebra, Lemma 10.134.9. □

0D08 **Lemma 15.33.8.** *Let $A \to B$ be a local homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced map on henselizations, resp. strict henselizations (Algebra, Lemma 10.155.6, resp. Lemma 10.155.10). Then $NL_{B/A} \otimes_B B^h \to NL_{B^h/A^h}$ and $NL_{B/A} \otimes_B B^{sh} \to NL_{B^{sh}/A^{sh}}$ induce isomorphisms on cohomology groups.*

Proof. Since A^h is a filtered colimit of étale algebras over A we see that $NL_{A^h/A}$ is an acyclic complex by Algebra, Lemma 10.134.9 and Algebra, Definition 10.143.1. The same is true for B^h/B. Using the Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) for $A \to A^h \to B^h$ we find that $NL_{B^h/A} \to NL_{B^h/A^h}$ induces isomorphisms on cohomology groups. Moreover, an étale ring map is a local complete intersection as it is even a global complete intersection, see Algebra, Lemma 10.143.2. By Lemma 15.33.7 we get a six term exact Jacobi-Zariski sequence associated to $A \to B \to B^h$ which proves that $NL_{B/A} \otimes_B B^h \to NL_{B^h/A}$ induces isomorphisms on cohomology groups. This finishes the proof in the case of the map on henselizations. The case of strict henselization is proved in exactly the same manner. □

15.34. Cartier's equality and geometric regularity

07E0 A reference for this section and the next is [**Mat70a**, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 10.134.

07E1 **Lemma 15.34.1** (Cartier equality)**.** *Let K/k be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $trdeg_k(K) = \dim_K \Omega_{K/k} - \dim_K H_1(L_{K/k})$.*

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus\nolimits_{j=1,\ldots,c} K \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} K\mathrm{d}x_i$$

by Algebra, Lemmas 10.134.2 and 10.134.13. The transcendence degree of K over k is the dimension of A (by Algebra, Lemma 10.116.1) which is $n - c$ and we win. □

07E2 **Lemma 15.34.2.** *Let $M/L/K$ be field extensions. Then the Jacobi-Zariski sequence*

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 15.33.7 with Algebra, Lemma 10.158.11. □

07E3 **Lemma 15.34.3.** *Given a commutative diagram of fields*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

with k'/k and K'/K finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'} \quad \text{and} \quad \beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Coker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = \mathit{trdeg}_k(k') - \mathit{trdeg}_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k'} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/K}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

By Lemma 15.34.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$. The lemma follows. □

15.35. Geometric regularity

07E4 Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) is a canonical exact sequence

$$H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ by Algebra, Lemma 10.134.6. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring A is geometrically regular over k. We will link this to the notion of formal smoothness in Section 15.40.

07E5 **Proposition 15.35.1.** *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The following are equivalent*

1. *A is geometrically regular over k,*
2. *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
3. *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
4. *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. Proof of (3) ⇒ (1). Assume (3). Let k'/k be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $\mathfrak{m}'$. Set $K' = A'/\mathfrak{m}'$. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(L_{K/k}) \otimes K' & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \otimes K' & \longrightarrow & \Omega_{A/k} \otimes_A K' & \longrightarrow & \Omega_{K/k} \otimes K' \longrightarrow 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \alpha} \\ & & H_1(L_{K'/k'}) & \longrightarrow & \mathfrak{m}'/(\mathfrak{m}')^2 & \longrightarrow & \Omega_{A'/k'} \otimes_{A'} K' & \longrightarrow & \Omega_{K'/k'} \longrightarrow 0 \end{array}$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 10.131.12). Thus α is surjective. By Lemma 15.34.3 we have

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim \mathfrak{m}'/(\mathfrak{m}')^2 \leq \dim \mathfrak{m}/\mathfrak{m}^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 10.112.6. Hence A' is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{array}{ccccc} \mathbf{F}_p & \longrightarrow & A & \longrightarrow & K \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{F}_p & \longrightarrow & k & \longrightarrow & K \end{array}$$

to get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \Omega_{A/\mathbf{F}_p} \otimes_A K & \longrightarrow & \Omega_{K/\mathbf{F}_p} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H_1(L_{K/k}) & \longrightarrow & \Omega_{k/\mathbf{F}_p} \otimes_k K & \longrightarrow & \Omega_{K/\mathbf{F}_p} & \longrightarrow & \Omega_{K/k} \longrightarrow 0 \end{array}$$

with exact rows. We have used that $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ and that $H_1(L_{K/\mathbf{F}_p}) = 0$ as $K/\mathbf{F}_p$ is separable, see Algebra, Proposition 10.158.9. Thus it is clear that the kernels of $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ and $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ have the same dimension.

Proof of (2) $\Rightarrow$ (4) following Faltings, see [**Fal78a**]. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Consider the field extension $k' = k(a_1^{1/p}, \ldots, a_n^{1/p})$. By Algebra, Lemma 10.158.3 we see that $k' = k[x_1, \ldots, x_n]/(x_1^p - a_1, \ldots, x_n^p - a_n)$. In particular we see that the naive cotangent complex of k'/k is homotopic to the complex $\bigoplus_{j=1,\ldots,n} k' \to \bigoplus_{i=1,\ldots,n} k'$ with the zero differential as $\mathrm{d}(x_j^p - a_j) = 0$ in $\Omega_{k[x_1,\ldots,x_n]/k}$. Set $A' = A \otimes_k k'$ and $K' = A'/\mathfrak{m}'$ as above. By Algebra, Lemma 10.134.8 we see that $NL_{A'/A}$ is homotopy equivalent to the complex $\bigoplus_{j=1,\ldots,n} A' \to \bigoplus_{i=1,\ldots,n} A'$ with the zero differential, i.e., $H_1(L_{A'/A})$ and $\Omega_{A'/A}$ are free of rank n. The Jacobi-Zariski sequence for $\mathbf{F}_p \to A \to A'$ is

$$H_1(L_{A'/A}) \to \Omega_{A/\mathbf{F}_p} \otimes_A A' \to \Omega_{A'/\mathbf{F}_p} \to \Omega_{A'/A} \to 0$$

Using the presentation $A[x_1, \ldots, x_n] \to A'$ with kernel $(x_j^p - a_j)$ we see, unwinding the maps in Algebra, Lemma 10.134.4, that the jth basis vector of $H_1(L_{A'/A})$ maps to $\mathrm{d}a_j \otimes 1$ in $\Omega_{A/\mathbf{F}_p} \otimes A'$. As $\Omega_{A'/A}$ is free (hence flat) we get on tensoring with K' an exact sequence

$$K'^{\oplus n} \to \Omega_{A/\mathbf{F}_p} \otimes_A K' \xrightarrow{\beta} \Omega_{A'/\mathbf{F}_p} \otimes_{A'} K' \to K'^{\oplus n} \to 0$$

We conclude that the elements $\mathrm{d}a_j \otimes 1$ generate $\mathrm{Ker}(\beta)$ and we have to show that are linearly independent, i.e., we have to show $\dim(\mathrm{Ker}(\beta)) = n$. Consider the

following big diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}'/(\mathfrak{m}')^2 & \longrightarrow & \Omega_{A'/\mathbf{F}_p} \otimes K' & \longrightarrow & \Omega_{K'/\mathbf{F}_p} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \otimes K' & \longrightarrow & \Omega_{A/\mathbf{F}_p} \otimes K' & \longrightarrow & \Omega_{K/\mathbf{F}_p} \otimes K' & \longrightarrow & 0 \end{array}$$

By Lemma 15.34.1 and the Jacobi-Zariski sequence for $\mathbf{F}_p \to K \to K'$ we see that the kernel and cokernel of γ have the same finite dimension. By assumption A' is regular (and of the same dimension as A, see above) hence the kernel and cokernel of α have the same dimension. It follows that the kernel and cokernel of β have the same dimension which is what we wanted to show.

The implication (1) $\Rightarrow$ (2) is trivial. This finishes the proof of the proposition. □

07E6 **Lemma 15.35.2.** *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. Assume A is geometrically regular over k. Let $K/F/k$ be a finitely generated subextension. Let $\varphi : k[y_1, \ldots, y_m] \to A$ be a k-algebra map such that y_i maps to an element of F in K and such that $dy_1, \ldots, dy_m$ map to a basis of $\Omega_{F/k}$. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then*

$$k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$$

is flat and $A/\mathfrak{p}A$ is regular.

Proof. Set $A_0 = k[y_1, \ldots, y_m]_{\mathfrak{p}}$ with maximal ideal $\mathfrak{m}_0$ and residue field K_0. Note that $\Omega_{A_0/k}$ is free of rank m and $\Omega_{A_0/k} \otimes K_0 \to \Omega_{K_0/k}$ is an isomorphism. It is clear that A_0 is geometrically regular over k. Hence $H_1(L_{K_0/k}) \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is an isomorphism, see Proposition 15.35.1. Now consider

$$\begin{array}{ccc} H_1(L_{K_0/k}) \otimes K & \longrightarrow & \mathfrak{m}_0/\mathfrak{m}_0^2 \otimes K \\ \downarrow & & \downarrow \\ H_1(L_{K/k}) & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

Since the left vertical arrow is injective by Lemma 15.34.2 and the lower horizontal by Proposition 15.35.1 we conclude that the right vertical one is too. Hence a regular system of parameters in A_0 maps to part of a regular system of parameters in A. We win by Algebra, Lemmas 10.128.2 and 10.106.3. □

15.36. Topological rings and modules

07E7 Let's quickly discuss some properties of topological abelian groups. An abelian group M is a *topological abelian group* if M is endowed with a topology such that addition $M \times M \to M$, $(x, y) \mapsto x + y$ and inverse $M \to M$, $x \mapsto -x$ are continuous. A *homomorphism of topological abelian groups* is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom Im = Coim doesn't hold). If $N \subset M$ is a subgroup, then we think of N and M/N as topological groups also, namely using the induced topology on N and the quotient topology on M/N (i.e., such that $M \to M/N$ is submersive). Note that if $N \subset M$ is an open subgroup, then the topology on M/N is discrete.

We say the topology on M is *linear* if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let I be a directed set and let G_i be an inverse system of (discrete) abelian groups over I. Then

$$G = \lim_{i \in I} G_i$$

with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by $\mathrm{Ker}(G \to G_i)$. Conversely, let M be a linearly topologized abelian group. Choose any fundamental system of open subgroups $U_i \subset M$, $i \in I$ (i.e., the U_i form a fundamental system of open neighbourhoods and each U_i is a subgroup of M). Setting $i \geq i' \Leftrightarrow U_i \subset U_{i'}$ we see that I is a directed set. We obtain a homomorphism of linearly topologized abelian groups

$$c : M \longrightarrow \lim_{i \in I} M/U_i.$$

It is clear that M is *separated* (as a topological space) if and only if c is injective. We say that M is *complete* if c is an isomorphism[2]. We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups $\{U_i\}_{i \in I}$ chosen above. In fact the topological abelian group $M^\wedge = \lim_{i \in I} M/U_i$ is independent of this choice and is sometimes called the *completion* of M. Any $G = \lim G_i$ as above is complete, in particular, the completion $M^\wedge$ is always complete.

07E8 **Definition 15.36.1** (Topological rings)**.** Let R be a ring and let M be an R-module. [**GD60**, Chapter 0, Sections 7.1 and 7.2]

(1) We say R is a *topological ring* if R is endowed with a topology such that both addition and multiplication are continuous as maps $R \times R \to R$ where $R \times R$ has the product topology. In this case we say M is a *topological module* if M is endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous.
(2) A *homomorphism of topological modules* is just a continuous R-module map. A *homomorphism of topological rings* is a ring homomorphism which is continuous for the given topologies.
(3) We say M is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of submodules. We say R is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of ideals.
(4) If R is linearly topologized, we say that $I \subset R$ is an *ideal of definition* if I is open and if every neighbourhood of 0 contains I^n for some n.
(5) If R is linearly topologized, we say that R is *pre-admissible* if R has an ideal of definition.
(6) If R is linearly topologized, we say that R is *admissible* if it is pre-admissible and complete[3].
(7) If R is linearly topologized, we say that R is *pre-adic* if there exists an ideal of definition I such that $\{I^n\}_{n \geq 0}$ forms a fundamental system of neighbourhoods of 0.

[2]We include being separated as part of being complete as we'd like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.

[3]By our conventions this includes separated.

(8) If R is linearly topologized, we say that R is *adic* if R is pre-adic and complete.

Note that a (pre)adic topological ring is the same thing as a (pre)admissible topological ring which has an ideal of definition I such that I^n is open for all $n \geq 1$.

Let R be a ring and let M be an R-module. Let $I \subset R$ be an ideal. Then we can consider the linear topology on R which has $\{I^n\}_{n\geq 0}$ as a fundamental system of neighbourhoods of 0. This topology is called the I-adic topology; R is a pre-adic topological ring in the I-adic topology[4]. Moreover, the linear topology on M which has $\{I^n M\}_{n\geq 0}$ as a fundamental system of open neighbourhoods of 0 turns M into a topological R-module. This is called the *I-adic topology* on M. We see that M is I-adically complete (as defined in Algebra, Definition 10.96.2) if and only if M is complete in the I-adic topology[5]. In particular, we see that R is I-adically complete if and only if R is an adic topological ring in the I-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

07E9 **Lemma 15.36.2.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ and $J \subset S$ be ideals and endow R with the I-adic topology and S with the J-adic topology. Then φ is a homomorphism of topological rings if and only if $\varphi(I^n) \subset J$ for some $n \geq 1$.*

Proof. Omitted. □

0CQU **Lemma 15.36.3** (Baire category theorem)**.** *Let M be a topological abelian group. Assume M is linearly topologized, complete, and has a countable fundamental system of neighbourhoods of 0. If $U_n \subset M$, $n \geq 1$ are open dense subsets, then $\bigcap_{n\geq 1} U_n$ is dense.*

Proof. Let U_n be as in the statement of the lemma. After replacing U_n by $U_1 \cap \ldots \cap U_n$, we may assume that $U_1 \supset U_2 \supset \ldots$. Let M_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that $M_{n+1} \subset M_n$. Pick $x \in M$. We will show that for every $k \geq 1$ there exists a $y \in \bigcap_{n\geq 1} U_n$ with $x - y \in M_k$.

To construct y we argue as follows. First, we pick a $y_1 \in U_1$ with $y_1 \in x + M_k$. This is possible because U_1 is dense and $x + M_k$ is open. Then we pick a $k_1 > k$ such that $y_1 + M_{k_1} \subset U_1$. This is possible because U_1 is open. Next, we pick a $y_2 \in U_2$ with $y_2 \in y_1 + M_{k_1}$. This is possible because U_2 is dense and $y_2 + M_{k_1}$ is open. Then we pick a $k_2 > k_1$ such that $y_2 + M_{k_2} \subset U_2$. This is possible because U_2 is open.

Continuing in this fashion we get a converging sequence y_i of elements of M with limit y. By construction $x - y \in M_k$. Since

$$y - y_i = (y_{i+1} - y_i) + (y_{i+2} - y_{i+1}) + \ldots$$

is in M_{k_i} we see that $y \in y_i + M_{k_i} \subset U_i$ for all i as desired. □

0CQV **Lemma 15.36.4.** *With same assumptions as Lemma 15.36.3 if $M = \bigcup_{n\geq 1} N_n$ for some closed subgroups N_n, then N_n is open for some n.*

[4]Thus the I-adic topology is sometimes called the I-pre-adic topology.

[5]It may happen that the I-adic completion $M^\wedge$ is not I-adically complete, even though $M^\wedge$ is always complete with respect to the limit topology. If I is finitely generated then the I-adic topology and the limit topology on $M^\wedge$ agree, see Algebra, Lemma 10.96.3 and its proof.

Proof. If not, then $U_n = M \setminus N_n$ is dense for all n and we get a contradiction with Lemma 15.36.3. □

0CQW **Lemma 15.36.5** (Open mapping lemma)**.** *Let $u : N \to M$ be a continuous map of topologized abelian groups. Assume that M separated and that N is complete, linearly topologized, and has a countable fundamental system of neighbourhoods of 0. Then exactly one of the following holds*

(1) *u is open, or*
(2) *for some open subgroup $N' \subset N$ the image $u(N')$ is nowhere dense in M.*

Proof. Let N_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that N_n is a subgroup and that $N_{n+1} \subset N_n$. If (2) does not hold, then the closure M_n of $u(N_n)$ is an open subgroup for $n = 1, 2, 3, \ldots$. Since u is continuous, we see that M_n, $n \in \mathbf{N}$ must be a fundamental system of open neighbourhoods of 0 in M. Also, since M_n is the closure of $u(N_n)$ we see that

$$u(N_n) + M_{n+1} = M_n$$

for all $n \geq 1$. Pick $x_1 \in M_1$. Then we can inductively choose $y_i \in N_i$ and $x_{i+1} \in M_{i+1}$ such that

$$u(y_i) + x_{i+1} = x_i$$

The element $y = y_1 + y_2 + y_3 + \ldots$ of N exists because N is complete. Whereupon we see that $x_1 = u(y)$ because M is separated. Thus $M_1 = u(N_1)$. In exactly the same way the reader shows that $M_i = u(N_i)$ for all $i \geq 2$ and we see that u is open. □

15.37. Formally smooth maps of topological rings

07EA There is a version of formal smoothness which applies to homomorphisms of topological rings.

07EB **Definition 15.37.1.** Let $R \to S$ be a homomorphism of topological rings with R and S linearly topologized. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

of homomorphisms of topological rings where A is a discrete ring and $J \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ and we endow R with the $\mathfrak{m}$-adic topology and S with the $\mathfrak{n}$-adic topology. Continuity of $\varphi : R \to S$ holds if and only if $\varphi(\mathfrak{m}^m) \subset \mathfrak{n}$ for some $m \geq 1$, see Lemma 15.36.2. It turns out that in this case only the topology on S is relevant.

07EC **Lemma 15.37.2.** *Let $\varphi : R \to S$ be a ring map.*

(1) *If $R \to S$ is formally smooth in the sense of Algebra, Definition 10.138.1, then $R \to S$ is formally smooth for any linear topology on R and any pre-adic topology on S such that $R \to S$ is continuous.*

(2) *Let $\mathfrak{n} \subset S$ and $\mathfrak{m} \subset R$ ideals such that φ is continuous for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Then the following are equivalent*
   (a) *φ is formally smooth for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S, and*
   (b) *φ is formally smooth for the discrete topology on R and the $\mathfrak{n}$-adic topology on S.*

Proof. Assume $R \to S$ is formally smooth in the sense of Algebra, Definition 10.138.1. If S has a pre-adic topology, then there exists an ideal $\mathfrak{n} \subset S$ such that S has the $\mathfrak{n}$-adic topology. Suppose given a solid commutative diagram as in Definition 15.37.1. Continuity of $S \to A/J$ means that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$, see Lemma 15.36.2. We obtain a ring map $\psi : S \to A$ from the assumed formal smoothness of S over R. Then $\psi(\mathfrak{n}^k) \subset J$ hence $\psi(\mathfrak{n}^{2k}) = 0$ as $J^2 = 0$. Hence ψ is continuous by Lemma 15.36.2. This proves (1).

The proof of (2)(b) $\Rightarrow$ (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 15.37.1 where we use the discrete topology on R. Since φ is continuous we see that $\varphi(\mathfrak{m}^n) \subset \mathfrak{n}$ for some $n \geq 1$. As $S \to A/J$ is continuous we see that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$. Hence $\mathfrak{m}^{nk}$ maps into J under the map $R \to A$. Thus $\mathfrak{m}^{2nk}$ maps to zero in A and we see that $R \to A$ is continuous in the $\mathfrak{m}$-adic topology. Thus (2)(a) gives a dotted arrow as desired. □

07NI **Definition 15.37.3.** Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset S$ be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 15.37.2 hold, then we say $R \to S$ is *formally smooth for the $\mathfrak{n}$-adic topology*.

This property is inherited by the completions.

07ED **Lemma 15.37.4.** *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings endowed with finitely generated ideals. Endow R and S with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $R \to S$ be a homomorphism of topological rings. The following are equivalent*

(1) *$R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology,*
(2) *$R \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology,*
(3) *$R^\wedge \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology.*

Here $R^\wedge$ and $S^\wedge$ are the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completions of R and S.

Proof. The assumption that $\mathfrak{m}$ is finitely generated implies that $R^\wedge$ is $\mathfrak{m}R^\wedge$-adically complete, that $\mathfrak{m}R^\wedge = \mathfrak{m}^\wedge$ and that $R^\wedge/\mathfrak{m}^n R^\wedge = R/\mathfrak{m}^n$, see Algebra, Lemma 10.96.3 and its proof. Similarly for $(S, \mathfrak{n})$. Thus it is clear that diagrams as in Definition 15.37.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. □

The advantage of working with adic rings is that one gets a stronger lifting property.

07NJ **Lemma 15.37.5.** *Let $R \to S$ be a ring map. Let $\mathfrak{n}$ be an ideal of S. Assume that $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology. Consider a solid commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ & \overset{\psi}{\searrow} & \uparrow \\ R & \longrightarrow & A \end{array}$$

of homomorphisms of topological rings where A is adic and A/J is the quotient (as topological ring) of A by a closed ideal $J \subset A$ such that J^t is contained in an ideal of definition of A for some $t \geq 1$. Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.

Proof. Let $I \subset A$ be an ideal of definition so that $I \supset J^t$ for some n. Then $A = \lim A/I^n$ and $A/J = \lim A/J + I^n$ because J is assumed closed. Consider the following diagram of discrete R algebras $A_{n,m} = A/J^n + I^m$:

$$\begin{array}{ccccc} A/J^3 + I^3 & \longrightarrow & A/J^2 + I^3 & \longrightarrow & A/J + I^3 \\ \downarrow & & \downarrow & & \downarrow \\ A/J^3 + I^2 & \longrightarrow & A/J^2 + I^2 & \longrightarrow & A/J + I^2 \\ \downarrow & & \downarrow & & \downarrow \\ A/J^3 + I & \longrightarrow & A/J^2 + I & \longrightarrow & A/J + I \end{array}$$

Note that each of the commutative squares defines a surjection

$$A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$$

of R-algebras whose kernel has square zero. We will inductively construct R-algebra maps $\varphi_{n,m} : S \to A_{n,m}$. Namely, we have the maps $\varphi_{1,m} = \psi \bmod J + I^m$. Note that each of these maps is continuous as ψ is. We can inductively choose the maps $\varphi_{n,1}$ by starting with our choice of $\varphi_{1,1}$ and lifting up, using the formal smoothness of S over R, along the right column of the diagram above. We construct the remaining maps $\varphi_{n,m}$ by induction on $n + m$. Namely, we choose $\varphi_{n+1,m+1}$ by lifting the pair $(\varphi_{n+1,m}, \varphi_{n,m+1})$ along the displayed surjection above (again using the formal smoothness of S over R). In this way all of the maps $\varphi_{n,m}$ are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \bmod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = \lim \varphi_n$ we obtain a map $S \to A = \lim A/I^n$. The composition into A/J agrees with ψ as we have seen that $A/J = \lim A/J + I^n$. Finally we show that φ is continuous. Namely, we know that $\psi(\mathfrak{n}^r) \subset J + I/J$ for some $r \geq 1$ by our assumption that ψ is a morphism of topological rings, see Lemma 15.36.2. Hence $\varphi(\mathfrak{n}^r) \subset J + I$ hence $\varphi(\mathfrak{n}^{rt}) \subset I$ as desired. □

07EE **Lemma 15.37.6.** *Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset \mathfrak{n}' \subset S$ be ideals. If $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology, then $R \to S$ is formally smooth for the $\mathfrak{n}'$-adic topology.*

Proof. Omitted. □

07EF **Lemma 15.37.7.** *A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

07EG **Lemma 15.37.8.** *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $\mathfrak{n}$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $\mathfrak{n}' = \mathfrak{n}S'$-adic topology.*

Proof. Let a solid diagram

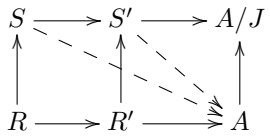

as in Definition 15.37.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 10.138.16. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

07EH **Lemma 15.37.9.** *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $\mathfrak{n}' = \mathfrak{n}S$. If*

(1) *the map $R \to R'$ embeds R as a direct summand of R' as an R-module, and*
(2) *$R' \to S'$ is formally smooth for the $\mathfrak{n}'$-adic topology,*

then $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology.

Proof. Let a solid diagram

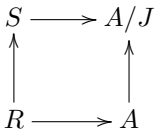

as in Definition 15.37.1 be given. Set $A' = A \otimes_R R'$ and $J' = \mathrm{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

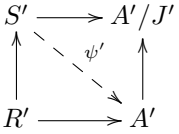

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as R-modules. (Warning: C isn't an ideal in R'.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \mathrm{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.$$

Then $J' = J \oplus J''$ as A-modules. The image of the composition $\psi : S \to A'$ of ψ' with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the A-submodule J'' is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for. □

15.38. Formally smooth maps of local rings

0DYF In the case of a local homomorphism of local rings one can limit the diagrams for which the lifting property has to be checked. Please compare with Algebra, Lemma 10.141.2.

0DYG **Lemma 15.38.1.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. The following are equivalent*

(1) *$R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology,*
(2) *for every solid commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

of local homomorphisms of local rings where $J \subset A$ is an ideal of square zero, $\mathfrak{m}_A^n = 0$ for some $n > 0$, and $S \to A/J$ induces an isomorphism on residue fields, a dotted arrow exists which makes the diagram commute.

If S is Noetherian these conditions are also equivalent to

(3) *same as in (2) but only for diagrams where in addition $A \to A/J$ is a small extension (Algebra, Definition 10.141.1).*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definitions. Consider a diagram

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Definition 15.37.1 for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Pick $m > 0$ with $\mathfrak{n}^m(A/J) = 0$ (possible by continuity of maps in diagram). Consider the subring A' of A which is the inverse image of the image of S in A/J. Set $J' = J$ viewed as an ideal in A'. Then J' is an ideal of square zero in A' and A'/J' is a quotient of $S/\mathfrak{n}^m$. Hence A' is local and $\mathfrak{m}_{A'}^{2m} = 0$. Thus we get a diagram

$$\begin{array}{ccc} S & \longrightarrow & A'/J' \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A' \end{array}$$

as in (2). If we can construct the dotted arrow in this diagram, then we obtain the dotted arrow in the original one by composing with $A' \to A$. In this way we see that (2) implies (1).

Assume S Noetherian. The implication (1) $\Rightarrow$ (3) is immediate. Assume (3) and suppose a diagram as in (2) is given. Then $\mathfrak{m}_A^n J = 0$ for some $n > 0$. Considering the maps

$$A \to A/\mathfrak{m}_A^{n-1}J \to \ldots \to A/\mathfrak{m}J \to A/J$$

we see that it suffices to produce the lifting if $\mathfrak{m}_A J = 0$. Assume $\mathfrak{m}_A J = 0$ and let $A' \subset A$ be the ring constructed above. Then A'/J' is Artinian as a quotient of the Artinian local ring $S/\mathfrak{n}^m$. Thus it suffices to show that given property (3) we can find the dotted arrow in diagrams as in (2) with A/J Artinian and $\mathfrak{m}_A J = 0$. Let κ be the common residue field of A, A/J, and S. By (3), if $J_0 \subset J$ is an ideal with $\dim_\kappa(J/J_0) = 1$, then we can produce a dotted arrow $S \to A/J_0$. Taking the product we obtain

$$S \longrightarrow \prod\nolimits_{J_0 \text{ as above}} A/J_0$$

Clearly the image of this arrow is contained in the sub R-algebra A' of elements which map into the small diagonal $A/J \subset \prod_{J_0} A/J$. Let $J' \subset A'$ be the elements mapping to zero in A/J. Then J' is an ideal of square zero and as κ-vector space equal to

$$J' = \prod\nolimits_{J_0 \text{ as above}} J/J_0$$

Thus the map $J \to J'$ is injective. By the theory of vector spaces we can choose a splitting $J' = J \oplus M$. It follows that

$$A' = A \oplus M$$

as an R-algebra. Hence the map $S \to A'$ can be composed with the projection $A' \to A$ to give the desired dotted arrow thereby finishing the proof of the lemma. □

The following lemma will be improved on in Section 15.40.

07EI **Lemma 15.38.2.** *Let k be a field and let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology, then A is a regular local ring.*

Proof. Let $k_0 \subset k$ be the prime field. Then k_0 is perfect, hence k/k_0 is separable, hence formally smooth by Algebra, Lemma 10.158.7. By Lemmas 15.37.2 and 15.37.7 we see that $k_0 \to A$ is formally smooth for the $\mathfrak{m}$-adic topology on A. Hence we may assume $k = \mathbf{Q}$ or $k = \mathbf{F}_p$.

By Algebra, Lemmas 10.97.3 and 10.110.9 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 15.37.4 we may replace A by $A^\wedge$. Thus we may assume that A is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 10.160.8) there exist a map $K \to A$. As k is the prime field we see that $K \to A$ is a k-algebra map.

Let $x_1, \ldots, x_n \in \mathfrak{m}$ be elements whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that

$$A/\mathfrak{m}^2 \cong K[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/\mathfrak{m}_T^2 \cong K[X_1, \ldots, X_n]/(X_i X_j).$$

Let $A/\mathfrak{m}^2 \to T/m_T^2$ be the local K-algebra isomorphism given by mapping the class of x_i to the class of X_i. Denote $f_1 : A \to T/\mathfrak{m}_T^2$ the composition of this isomorphism with the quotient map $A \to A/\mathfrak{m}^2$. The assumption that $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology means we can lift f_1 to a map $f_2 : A \to T/\mathfrak{m}_T^3$, then to a map $f_3 : A \to T/\mathfrak{m}_T^4$, and so on, for all $n \geq 1$. Warning: the maps f_n are continuous k-algebra maps and may not be K-algebra maps. We get an induced map $f : A \to T = \lim T/\mathfrak{m}_T^n$ of local k-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ hence each f_n is surjective and we conclude f is surjective as A is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence A is regular by definition. (It also follows that f is an isomorphism.) □

0C34 **Lemma 15.38.3.** *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable, then there exists a k-algebra map $\kappa \to A$ such that $\kappa \to A \to \kappa$ is id_κ.*

Proof. By Algebra, Proposition 10.158.9 the extension κ/k is formally smooth. By Lemma 15.37.2 $k \to \kappa$ is formally smooth in the sense of Definition 15.37.1. Then we get $\kappa \to A$ from Lemma 15.37.5. □

0C35 **Lemma 15.38.4.** *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable and A regular, then there exists an isomorphism of $A \cong \kappa[[t_1, \ldots, t_d]]$ as k-algebras.*

Proof. Choose $\kappa \to A$ as in Lemma 15.38.3 and apply Algebra, Lemma 10.160.10. □

The following result will be improved on in Section 15.40

07EJ **Lemma 15.38.5.** *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a regular local k-algebra such that K/k is separable. Then $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. It suffices to prove that the completion of A is formally smooth over k, see Lemma 15.37.4. Hence we may assume that A is a complete local regular k-algebra with residue field K separable over k. By Lemma 15.38.4 we see that $A = K[[x_1, \ldots, x_n]]$.

The power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over k. Namely, K is formally smooth over k and $K[x_1, \ldots, x_n]$ is formally smooth over K as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over k by Algebra, Lemma 10.138.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.37.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.37.4. □

07VH **Lemma 15.38.6.** *Let $A \to B$ be a finite type ring map with A Noetherian. Let $\mathfrak{q} \subset B$ be a prime ideal lying over $\mathfrak{p} \subset A$. The following are equivalent*

(1) *$A \to B$ is smooth at $\mathfrak{q}$, and*
(2) *$A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Algebra, Lemma 10.141.2. Conversely, if $A \to B$ is smooth at $\mathfrak{q}$, then $A \to B_g$ is smooth for some $g \in B$, $g \notin \mathfrak{q}$. Then $A \to B_g$ is formally smooth by Algebra, Proposition 10.138.13. Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth as localization preserves formal smoothness (for example by the criterion of Algebra, Proposition 10.138.8 and the fact that the cotangent complex behaves well with respect to localization, see Algebra, Lemmas 10.134.11 and 10.134.13). Finally, Lemma 15.37.2 implies that $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology. □

15.39. Some results on power series rings

07NK Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

07NL **Lemma 15.39.1.** *Let K be a field of characteristic 0 and $A = K[[x_1, \ldots, x_n]]$. Let L be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$.*

(1) *$\mathbf{Q} \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
(2) *$\mathbf{F}_p \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.*
(3) *$\mathbf{Z} \to C$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. By the universal property of power series rings it suffices to prove:

(1) $\mathbf{Q} \to K$ is formally smooth.
(2) $\mathbf{F}_p \to L$ is formally smooth.

(3) $\mathbf{Z} \to \Lambda$ is formally smooth in the $\mathfrak{m}$-adic topology.

The first two are Algebra, Proposition 10.158.9. The third follows from Algebra, Lemma 10.160.7 since for any test diagram as in Definition 15.37.1 some power of p will be zero in A/J and hence some power of p will be zero in A. □

07NM **Lemma 15.39.2.** *Let K be a field and $A = K[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring and let $B = \Lambda[[x_1, \ldots, x_n]]$.*

(1) *If $y_1, \ldots, y_n \in A$ is a regular system of parameters then $K[[y_1, \ldots, y_n]] \to A$ is an isomorphism.*
(2) *If $z_1, \ldots, z_r \in A$ form part of a regular system of parameters for A, then $r \leq n$ and $A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$.*
(3) *If $p, y_1, \ldots, y_n \in B$ is a regular system of parameters then $\Lambda[[y_1, \ldots, y_n]] \to B$ is an isomorphism.*
(4) *If $p, z_1, \ldots, z_r \in B$ form part of a regular system of parameters for B, then $r \leq n$ and $B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]]$.*

Proof. Proof of (1). Set $A' = K[[y_1, \ldots, y_n]]$. It is clear that the map $A' \to A$ induces an isomorphism $A'/\mathfrak{m}_{A'}^n \to A/\mathfrak{m}_A^n$ for all $n \geq 1$. Since A and A' are both complete we deduce that $A' \to A$ is an isomorphism. Proof of (2). Extend $z_1, \ldots, z_r$ to a regular system of parameters $z_1, \ldots, z_r, y_1, \ldots, y_{n-r}$ of A. Consider the map $A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A$. This is an isomorphism by (1). Hence (2) follows as it is clear that $A'/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$. The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). □

07NN **Lemma 15.39.3.** *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

with the following properties:

(1) *the horizontal arrows are surjective,*
(2) *if the characteristic of $A/\mathfrak{m}_A$ is zero, then S and R are power series rings over fields,*
(3) *if the characteristic of $A/\mathfrak{m}_A$ is $p > 0$, then S and R are power series rings over Cohen rings, and*
(4) *$R \to S$ maps a regular system of parameters of R to part of a regular system of parameters of S.*

In particular $R \to S$ is flat (see Algebra, Lemma 10.128.2) with regular fibre $S/\mathfrak{m}_R S$ (see Algebra, Lemma 10.106.3).

Proof. Use the Cohen structure theorem (Algebra, Theorem 10.160.8) to choose a surjection $S \to B$ as in the statement of the lemma where we choose S to be a power series over a Cohen ring if the residue characteristic is $p > 0$ and a power series over a field else. Let $J \subset S$ be the kernel of $S \to B$. Next, choose a surjection $R = \Lambda[[x_1, \ldots, x_n]] \to A$ where we choose Λ to be a Cohen ring if the residue characteristic of A is $p > 0$ and Λ equal to the residue field of A otherwise. We lift the composition $\Lambda[[x_1, \ldots, x_n]] \to A \to B$ to a map $\varphi : R \to S$. This is possible because $\Lambda[[x_1, \ldots, x_n]]$ is formally smooth over $\mathbf{Z}$ in the $\mathfrak{m}$-adic topology

(see Lemma 15.39.1) by an application of Lemma 15.37.5. Finally, we replace φ by the map $\varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]]$ with $\varphi'|_\Lambda = \varphi|_\Lambda$ and $\varphi'(x_i) = \varphi(x_i) + y_i$. We also replace $S \to B$ by the map $S' \to B$ which maps y_i to zero. After this replacement it is clear that a regular system of parameters of R maps to part of a regular sequence in S' and we win. □

There should be an elementary proof of the following lemma.

09Q8 **Lemma 15.39.4.** *Let $S \to R$ and $S' \to R$ be surjective maps of complete Noetherian local rings. Then $S \times_R S'$ is a complete Noetherian local ring.*

Proof. Let k be the residue field of R. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.160.5) with residue field k (Algebra, Lemma 10.160.6). If the characteristic of k is 0 we set $\Lambda = k$. Choose a surjection $\Lambda[[x_1, \ldots, x_n]] \to R$ (as in the Cohen structure theorem, see Algebra, Theorem 10.160.8) and lift this to maps $\Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi : \Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi' : \Lambda[[x_1, \ldots, x_n]] \to S'$ using Lemmas 15.39.1 and 15.37.5. Next, choose $f_1, \ldots, f_m \in S$ generating the kernel of $S \to R$ and $f'_1, \ldots, f'_{m'} \in S'$ generating the kernel of $S' \to R$. Then the map

$$\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_{m'}]] \longrightarrow S \times_R S,$$

which sends x_i to $(\varphi(x_i), \varphi'(x_i))$ and y_j to $(f_j, 0)$ and $z_{j'}$ to $(0, f'_j)$ is surjective. Thus $S \times_R S'$ is a quotient of a complete local ring, whence complete. □

15.40. Geometric regularity and formal smoothness

07EK In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the $\mathfrak{m}$-adic topology between Noetherian local rings.

07EL **Theorem 15.40.1.** *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If the characteristic of k is zero then the following are equivalent*

(1) *A is a regular local ring, and*
(2) *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

If the characteristic of k is $p > 0$ then the following are equivalent

(1) *A is geometrically regular over k,*
(2) *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
(3) *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
(4) *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
(5) *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. If the characteristic of k is zero, then the equivalence of (1) and (2) follows from Lemmas 15.38.2 and 15.38.5.

If the characteristic of k is $p > 0$, then it follows from Proposition 15.35.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 15.37.8 we see that $k' \to A' = A \otimes_k k'$ is formally smooth for the $\mathfrak{m}' = \mathfrak{m}A'$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then A' is a regular local ring by Lemma 15.38.2. Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/J \\ {\scriptstyle i}\uparrow & \overset{\bar\psi}{\nearrow} & \uparrow{\scriptstyle \pi} \\ k & \xrightarrow[\varphi]{} & B \end{array}$$

as in Definition 15.37.1. As $J^2 = 0$ we see that J has a canonical B/J module structure and via $\bar\psi$ an A-module structure. As $\bar\psi$ is continuous for the $\mathfrak{m}$-adic topology we see that $\mathfrak{m}^n J = 0$ for some n. Hence we can filter J by B/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}$. Considering the sequence of ring maps $B \to B/J_1 \to B/J_2 \to \ldots \to B/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}$. By Lemma 15.38.5 we see that $\mathbf{F}_p \to A$ is formally smooth in the $\mathfrak{m}$-adic topology. Hence we can find a ring map $\psi : A \to B$ such that $\pi \circ \psi = \bar\psi$. Then $\psi \circ i, \varphi : k \to B$ are two maps whose compositions with π are equal. Hence $D = \psi \circ i - \varphi : k \to J$ is a derivation. By Algebra, Lemma 10.131.3 we can write $D = \xi \circ \mathrm{d}$ for some k-linear map $\xi : \Omega_{k/\mathbf{F}_p} \to J$. Using the K-vector space structure on J we extend ξ to a K-linear map $\xi' : \Omega_{k/\mathbf{F}_p} \otimes_k K \to J$. Using (5) we can find a K-linear map $\xi'' : \Omega_{A/\mathbf{F}_p} \otimes_A K$ whose restriction to $\Omega_{k/\mathbf{F}_p} \otimes_k K$ is ξ'. Write

$$D' : A \xrightarrow{\mathrm{d}} \Omega_{A/\mathbf{F}_p} \to \Omega_{A/\mathbf{F}_p} \otimes_A K \xrightarrow{\xi''} J.$$

Finally, set $\psi' = \psi - D' : A \to B$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar\psi$ and such that $\psi' \circ i = \varphi$ as desired. □

07EM **Example 15.40.2.** Let k be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a pth power. A standard example of a geometrically regular local k-algebra whose residue field is purely inseparable over k is the ring

$$A = k[x, y]_{(x, y^p - a)}/(y^p - a - x)$$

Namely, A is a localization of a smooth algebra over k hence $k \to A$ is formally smooth, hence $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology. A closely related example is the following. Let $k = \mathbf{F}_p(s)$ and $K = \mathbf{F}_p(t)^{perf}$. We claim the ring map

$$k \longrightarrow A = K[[x]], \quad s \longmapsto t + x$$

is formally smooth for the (x)-adic topology on A. Namely, $\Omega_{k/\mathbf{F}_p}$ is 1-dimensional with basis $\mathrm{d}s$. It maps to the element $\mathrm{d}x + \mathrm{d}t = \mathrm{d}x$ in $\Omega_{A/\mathbf{F}_p}$. We leave it to the reader to show that $\Omega_{A/\mathbf{F}_p}$ is free on $\mathrm{d}x$ as an A-module. Hence we see that condition (5) of Theorem 15.40.1 holds and we conclude that $k \to A$ is formally smooth in the (x)-adic topology.

07NP **Lemma 15.40.3.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then $A \to B$ is flat.*

Proof. We may assume that A and B a Noetherian complete local rings by Lemma 15.37.4 and Algebra, Lemma 10.97.6 (this also uses Algebra, Lemma 10.39.9 and

10.97.3 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.39.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because B is formally smooth over A we see that the A-algebra map

$$S/IS \longrightarrow B$$

has a section, see Lemma 15.37.5. Hence B is a direct summand of the flat A-module S/IS (by base change of flatness, see Algebra, Lemma 10.39.7), whence flat. □

0DYH **Lemma 15.40.4.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Let K be the residue field of B. Then the Jacobi-Zariski sequence for $A \to B \to K$ gives an exact sequence*

$$0 \to H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \Omega_{B/A} \otimes_B K \to \Omega_{K/A} \to 0$$

Proof. Observe that $\mathfrak{m}_B/\mathfrak{m}_B^2 = H_1(NL_{K/B})$ by Algebra, Lemma 10.134.6. By Algebra, Lemma 10.134.4 it remains to show injectivity of $H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2$. With k the residue field of A, the Jacobi-Zariski sequence for $A \to k \to K$ gives $\Omega_{K/A} = \Omega_{K/k}$ and an exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to H_1(NL_{K/A}) \to H_1(NL_{K/k}) \to 0$$

Set $\overline{B} = B \otimes_A k$. Since $\overline{B}$ is regular the ideal $\mathfrak{m}_{\overline{B}}$ is generated by a regular sequence. Applying Lemmas 15.30.9 and 15.30.7 to $\mathfrak{m}_A B \subset \mathfrak{m}_B$ we find $\mathfrak{m}_A B/(\mathfrak{m}_A B \cap \mathfrak{m}_B^2) = \mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B$ which is equal to $\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K$ as $A \to B$ is flat by Lemma 15.40.3. Thus we obtain a short exact sequence

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \to 0$$

Functoriality of the Jacobi-Zariski sequences shows that we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K & \longrightarrow & H_1(NL_{K/A}) & \longrightarrow & H_1(NL_{K/k}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 & \longrightarrow & 0 \end{array}$$

The left vertical arrow is injective by Theorem 15.40.1 as $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology by Lemma 15.37.8. This finishes the proof by the snake lemma. □

07NQ **Proposition 15.40.5.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

(1) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*

(2) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
(3) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. The equivalence of (1) and (2) follows from Theorem 15.40.1.

Assume (3). By Lemma 15.40.3 we see that $A \to B$ is flat. By Lemma 15.37.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 15.37.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 10.97.3. Hence we may replace A and B by their respective completions and assume that A and B are Noetherian complete local rings. In this case choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.39.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of R with $t_1 = p$ in case the characteristic of k is $p > 0$. Set $\overline{S} = S \otimes_R k$. Consider the short exact sequence

$$0 \to J \to S \to B \to 0$$

As $\overline{B}$ and $\overline{S}$ are regular, the kernel of $\overline{S} \to \overline{B}$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.106.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.39.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective, an isomorphism on special fibres, and source and target are flat over A (for example use Algebra, Lemma 10.99.1 or use that tensoring the short exact sequence $0 \to I \to S \otimes_R A \to B \to 0$ over A with k we find $I \otimes_A k = 0$ hence $I = 0$ by Nakayama). Thus by Lemma 15.37.8 it suffices to show that $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/\mathfrak{m}_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for S. If the characteristic of k is zero, choose a coefficient field $K \subset S$ and if the characteristic of k is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field K. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case)

or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 15.39.2. From now on we think of S as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 15.40.1. Choose a solid diagram

$$\begin{array}{ccc} S & \longrightarrow & N/J \\ {\scriptstyle i}\uparrow & \nearrow_{\bar\psi} & \uparrow{\scriptstyle \pi} \\ R & \xrightarrow{\varphi} & N \end{array}$$

as in Definition 15.37.1. As $J^2 = 0$ we see that J has a canonical N/J module structure and via $\bar\psi$ a S-module structure. As $\bar\psi$ is continuous for the $\mathfrak{m}_S$-adic topology we see that $\mathfrak{m}_S^n J = 0$ for some n. Hence we can filter J by N/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}_S$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}_S$. As $\mathbf{Q} \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $\mathfrak{m}_S$-adic topology (see Lemma 15.39.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \bar\psi$. Since S is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of ψ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic p case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with π are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 10.131.3 we can write $D = \xi \circ \mathrm{d}$ for some R-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ (by construction) and $\mathfrak{m}_R \Omega_{R/\mathbf{Z}}$ (as J is annihilated by $\mathfrak{m}_R$). Hence ξ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi'} J$$

where ξ' is k-linear. Using the K-vector space structure on J we extend ξ' to a K-linear map

$$\xi'' : \Omega_{k/\mathbf{Z}} \otimes_k K \longrightarrow J.$$

Using that $\overline{S}/k$ is formally smooth we see that

$$\Omega_{k/\mathbf{Z}} \otimes_k K \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K$$

is injective by Theorem 15.40.1 (this is true also in the characteristic zero case as it is even true that $\Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective in characteristic zero, see Algebra, Proposition 10.158.9). Hence we can find a K-linear map $\xi''' : \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \to J$ whose restriction to $\Omega_{k/\mathbf{Z}} \otimes_k K$ is ξ''. Write

$$D' : S \xrightarrow{\mathrm{d}} \Omega_{S/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \xrightarrow{\xi'''} J.$$

Finally, set $\psi' = \psi - D' : S \to N$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar\psi$ and such that $\psi' \circ i = \varphi$ as desired. □

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

07NR **Lemma 15.40.6.** *Let A be a Noetherian complete local ring with residue field k. Let B be a Noetherian complete local k-algebra. Assume $k \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists a Noetherian complete local ring C and a local homomorphism $A \to C$ which is formally smooth in the $\mathfrak{m}_C$-adic topology such that $C \otimes_A k \cong B$.*

Proof. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.39.3. Let $t_1, \ldots, t_d$ be a regular system of parameters for R with $t_1 = p$ in case the characteristic of k is $p > 0$. As B and $\overline{S} = S \otimes_R k$ are regular we see that $\mathrm{Ker}(\overline{S} \to B)$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.106.4. Lift these elements to $x_1, \ldots, x_r \in S$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.39.3 with moreover $\overline{S} = B$. In this case $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by Proposition 15.40.5. Hence the base change $C = S \otimes_R A$ is formally smooth over A in the $\mathfrak{m}_C$-adic topology by Lemma 15.37.8. □

07NS **Remark 15.40.7.** The assertion of Lemma 15.40.6 is quite strong. Namely, suppose that we have a diagram

$$\begin{array}{ccc} & & B \\ & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of local homomorphisms of Noetherian complete local rings where $A \to A'$ induces an isomorphism of residue fields $k = A/\mathfrak{m}_A = A'/\mathfrak{m}_{A'}$ and with $B \otimes_{A'} k$ formally smooth over k. Then we can extend this to a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of local homomorphisms of Noetherian complete local rings where $A \to C$ is formally smooth in the $\mathfrak{m}_C$-adic topology and where $C \otimes_A k \cong B \otimes_{A'} k$. Namely, pick $A \to C$ as in Lemma 15.40.6 lifting $B \otimes_{A'} k$ over k. By formal smoothness we can find the arrow $C \to B$, see Lemma 15.37.5. Denote $C \otimes_A^\wedge A'$ the completion of $C \otimes_A A'$ with respect to the ideal $C \otimes_A \mathfrak{m}_{A'}$. Note that $C \otimes_A^\wedge A'$ is a Noetherian

complete local ring (see Algebra, Lemma 10.97.5) which is flat over A' (see Algebra, Lemma 10.99.11). We have moreover

(1) $C \otimes^\wedge_A A' \to B$ is surjective,
(2) if $A \to A'$ is surjective, then $C \to B$ is surjective,
(3) if $A \to A'$ is finite, then $C \to B$ is finite, and
(4) if $A' \to B$ is flat, then $C \otimes^\wedge_A A' \cong B$.

Namely, by Nakayama's lemma for nilpotent ideals (see Algebra, Lemma 10.20.1) we see that $C \otimes_A k \cong B \otimes_{A'} k$ implies that $C \otimes_A A'/\mathfrak{m}^n_{A'} \to B/\mathfrak{m}^n_{A'}B$ is surjective for all n. This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 10.99.1.

15.41. Regular ring maps

07BY Let k be a field. Recall that a Noetherian k-algebra A is said to be *geometrically regular* over k if and only if $A \otimes_k k'$ is regular for all finite purely inseparable extensions k' of k, see Algebra, Definition 10.166.2. Moreover, if this is the case then $A \otimes_k k'$ is regular for every finitely generated field extension k'/k, see Algebra, Lemma 10.166.1. We use this notion in the following definition.

07BZ **Definition 15.41.1.** A ring map $R \to \Lambda$ is *regular* if it is flat and for every prime $\mathfrak{p} \subset R$ the fibre ring

$$\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

If $R \to \Lambda$ is a ring map with Λ Noetherian, then the fibre rings are always Noetherian.

07C0 **Lemma 15.41.2** (Regular is a local property)**.** *Let $R \to \Lambda$ be a ring map with Λ Noetherian. The following are equivalent*

(1) *$R \to \Lambda$ is regular,*
(2) *$R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is regular for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$, and*
(3) *$R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ is regular for all maximal ideals $\mathfrak{m}' \subset \Lambda$ lying over $\mathfrak{m}$ in R.*

Proof. This is true because a Noetherian ring is regular if and only if all the local rings are regular local rings, see Algebra, Definition 10.110.7 and a ring map is flat if and only if all the induced maps of local rings are flat, see Algebra, Lemma 10.39.18. □

07C1 **Lemma 15.41.3** (Regular maps and base change)**.** *Let $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.*

Proof. Flatness is preserved under any base change, see Algebra, Lemma 10.39.7. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is finitely generated as R' is of finite type over R. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 10.31.8 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 10.166.1. □

07QI **Lemma 15.41.4** (Composition of regular maps)**.** *Let $A \to B$ and $B \to C$ be regular ring maps. If the fibre rings of $A \to C$ are Noetherian, then $A \to C$ is regular.*

Proof. Let $\mathfrak{p} \subset A$ be a prime. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. We have to show that $C \otimes_A k$ is regular. By Lemma 15.41.3 we may assume that $A = k$ and we reduce to proving that C is regular. The assumption is that B is regular and that $B \to C$ is flat with regular fibres. Then C is regular by Algebra, Lemma 10.112.8. Some details omitted. □

07EP **Lemma 15.41.5.** *Let R be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth R-algebras. Set $\Lambda = \operatorname{colim} A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak{p})$ are Noetherian for all $\mathfrak{p} \subset R$, then $R \to \Lambda$ is regular.*

Proof. Note that Λ is flat over R by Algebra, Lemmas 10.39.3 and 10.137.9. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} k = \Lambda \otimes_R k = \operatorname{colim} A_i \otimes_R k$$

is a colimit of smooth k-algebras, see Algebra, Lemma 10.137.3. Since each local ring of a smooth k-algebra is regular by Algebra, Lemma 10.140.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 10.106.8. This proves the lemma. □

Let's see when a field extension defines a regular ring map.

07EQ **Lemma 15.41.6.** *Let K/k be a field extension. Then $k \to K$ is a regular ring map if and only if K is a separable field extension of k.*

Proof. If $k \to K$ is regular, then K is geometrically reduced over k, hence K is separable over k by Algebra, Proposition 10.158.9. Conversely, if K/k is separable, then K is a colimit of smooth k-algebras, see Algebra, Lemma 10.158.11 hence is regular by Lemma 15.41.5. □

07NT **Lemma 15.41.7.** *Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.*

Proof. By Algebra, Lemma 10.39.10 we see that $A \to B$ is flat. Let $\mathfrak{p} \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak{p}) \to C \otimes_A \kappa(\mathfrak{p})$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular by Algebra, Lemma 10.166.3. □

15.42. Ascending properties along regular ring maps

07QJ This section is the analogue of Algebra, Section 10.163 but where the ring map $R \to S$ is regular.

07QK **Lemma 15.42.1.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and reduced.*

Then S is reduced.

Proof. For Noetherian rings being reduced is the same as having properties (S_1) and (R_0), see Algebra, Lemma 10.157.3. Hence we may apply Algebra, Lemmas 10.163.4 and 10.163.5. □

0BFK **Lemma 15.42.2.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and normal.*

Then S is normal.

Proof. For Noetherian rings being normal is the same as having properties (S_2) and (R_1), see Algebra, Lemma 10.157.4. Hence we may apply Algebra, Lemmas 10.163.4 and 10.163.5. □

0H7S **Lemma 15.42.3.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and regular.*

Then S is regular.

Proof. For Noetherian rings being regular is the same as having properties (R_k) for all k. Hence we may apply Algebra, Lemma 10.163.5. □

0H7T **Lemma 15.42.4.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and Cohen-Macaulay.*

Then S is Cohen-Macaulay.

Proof. For Noetherian rings being Cohen-Macaulay is the same as having properties (S_k) for all k. Hence we may apply Algebra, Lemma 10.163.4. □

15.43. Permanence of properties under completion

07NU Given a Noetherian local ring $(A, \mathfrak{m})$ we denote $A^\wedge$ the completion of A with respect to $\mathfrak{m}$. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring with maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}A^\wedge$ and that $A \to A^\wedge$ is faithfully flat. See Algebra, Lemmas 10.97.6, 10.97.4, and 10.97.3.

07NV **Lemma 15.43.1.** *Let A be a Noetherian local ring. Then $\dim(A) = \dim(A^\wedge)$.*

Proof. By Algebra, Lemma 10.97.4 the map $A \to A^\wedge$ induces isomorphisms $A/\mathfrak{m}^n = A^\wedge/(\mathfrak{m}^\wedge)^n$ for $n \geq 1$. By Algebra, Lemma 10.52.12 this implies that

$$\text{length}_A(A/\mathfrak{m}^n) = \text{length}_{A^\wedge}(A^\wedge/(\mathfrak{m}^\wedge)^n)$$

for all $n \geq 1$. Thus $d(A) = d(A^\wedge)$ and we conclude by Algebra, Proposition 10.60.9. An alternative proof is to use Algebra, Lemma 10.112.7. □

07NW **Lemma 15.43.2.** *Let A be a Noetherian local ring. Then $\mathit{depth}(A) = \mathit{depth}(A^\wedge)$.*

Proof. See Algebra, Lemma 10.163.2. □

07NX **Lemma 15.43.3.** *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if $A^\wedge$ is so.*

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants are preserved under completion (Lemmas 15.43.1 and 15.43.2) the claim follows. □

07NY **Lemma 15.43.4.** *Let A be a Noetherian local ring. Then A is regular if and only if $A^\wedge$ is so.*

Proof. If $A^\wedge$ is regular, then A is regular by Algebra, Lemma 10.110.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(A^\wedge)$ (Lemma 15.43.1). On the other hand, $\mathfrak{m}A^\wedge$ is the maximal ideal of $A^\wedge$ and hence $\mathfrak{m}_{A^\wedge}$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 10.112.8.) □

0AP1 **Lemma 15.43.5.** *Let A be a Noetherian local ring. Then A is a discrete valuation ring if and only if $A^\wedge$ is so.*

Proof. This follows from Lemmas 15.43.1 and 15.43.4 and Algebra, Lemma 10.119.7. □

07NZ **Lemma 15.43.6.** *Let A be a Noetherian local ring.*

(1) *If $A^\wedge$ is reduced, then so is A.*
(2) *In general A reduced does not imply $A^\wedge$ is reduced.*
(3) *If A is Nagata, then A is reduced if and only if $A^\wedge$ is reduced.*

Proof. As $A \to A^\wedge$ is faithfully flat we have (1) by Algebra, Lemma 10.164.2. For (2) see Algebra, Example 10.119.5 (there are also examples in characteristic zero, see Algebra, Remark 10.119.6). For (3) see Algebra, Lemmas 10.162.13 and 10.162.10. □

0FIZ **Lemma 15.43.7.** *Let A be a Noetherian local ring. If $A^\wedge$ is normal, then so is A.*

Proof. As $A \to A^\wedge$ is faithfully flat this follows from Algebra, Lemma 10.164.3. □

0C4G **Lemma 15.43.8.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the induced map of completions $A^\wedge \to B^\wedge$ is flat if and only if $A \to B$ is flat.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A^\wedge & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

The vertical arrows are faithfully flat. Assume that $A^\wedge \to B^\wedge$ is flat. Then $A \to B^\wedge$ is flat. Hence B is flat over A by Algebra, Lemma 10.39.9.

Assume that $A \to B$ is flat. Then $A \to B^\wedge$ is flat. Hence $B^\wedge/\mathfrak{m}_A^n B^\wedge$ is flat over $A/\mathfrak{m}_A^n$ for all $n \geq 1$. Note that $\mathfrak{m}_A^n A^\wedge$ is the nth power of the maximal ideal $\mathfrak{m}_A^\wedge$ of $A^\wedge$ and $A/\mathfrak{m}_A^n = A^\wedge/(\mathfrak{m}_A^\wedge)^n$. Thus we see that $B^\wedge$ is flat over $A^\wedge$ by applying Algebra, Lemma 10.99.11 (with $R = A^\wedge$, $I = \mathfrak{m}_A^\wedge$, $S = B^\wedge$, $M = S$). □

0AGX **Lemma 15.43.9.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings such that $\mathfrak{m}_A B = \mathfrak{m}_B$ and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$. Then $A \to B$ induces an isomorphism $A^\wedge \to B^\wedge$ of completions.*

Proof. By Algebra, Lemma 10.97.7 we see that $B^\wedge$ is the $\mathfrak{m}_A$-adic completion of B and that $A^\wedge \to B^\wedge$ is finite. Since $A \to B$ is flat we have $\mathrm{Tor}_1^A(B, \kappa(\mathfrak{m}_A)) = 0$. Hence we see that $B^\wedge$ is flat over $A^\wedge$ by Lemma 15.27.5. Thus $B^\wedge$ is a free $A^\wedge$-module by Algebra, Lemma 10.78.5. Since $A^\wedge \to B^\wedge$ induces an isomorphism

$\kappa(\mathfrak{m}_A) = A^\wedge/\mathfrak{m}_A A^\wedge \to B^\wedge/\mathfrak{m}_A B^\wedge = B^\wedge/\mathfrak{m}_B B^\wedge = \kappa(\mathfrak{m}_B)$ by our assumptions (and Algebra, Lemma 10.96.3), we see that $B^\wedge$ is free of rank 1. Thus $A^\wedge \to B^\wedge$ is an isomorphism. □

15.44. Permanence of properties under étale maps

0AGY In this section we consider an étale ring map $\varphi : A \to B$ and we study which properties of A are inherited by B and which properties of the local ring of B at $\mathfrak{q}$ are inherited by the local ring of A at $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Basically, this section reviews and collects earlier results and does not add any new material.

We will use without further mention that an étale ring map is flat (Algebra, Lemma 10.143.3) and that a flat local homomorphism of local rings is faithfully flat (Algebra, Lemma 10.39.17).

0AGZ **Lemma 15.44.1.** *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p}$ is Noetherian if and only if $B_\mathfrak{q}$ is Noetherian.*

Proof. Since $A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat we see that $B_\mathfrak{q}$ Noetherian implies that $A_\mathfrak{p}$ is Noetherian, see Algebra, Lemma 10.164.1. Conversely, if $A_\mathfrak{p}$ is Noetherian, then $B_\mathfrak{q}$ is Noetherian as it is a localization of a finite type $A_\mathfrak{p}$-algebra. □

07QP **Lemma 15.44.2.** *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$.*

Proof. Namely, because $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat we have going down, and hence the inequality $\dim(A_\mathfrak{p}) \leq \dim(B_\mathfrak{q})$, see Algebra, Lemma 10.112.1. On the other hand, suppose that $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of primes in $B_\mathfrak{q}$. Then the corresponding sequence of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (with $\mathfrak{p}_i = \mathfrak{q}_i \cap A_\mathfrak{p}$) is chain also (i.e., no equalities in the sequence) as an étale ring map is quasi-finite (see Algebra, Lemma 10.143.6) and a quasi-finite ring map induces a map of spectra with discrete fibres (by definition). This means that $\dim(A_\mathfrak{p}) \geq \dim(B_\mathfrak{q})$ as desired. □

0AH0 **Lemma 15.44.3.** *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p}$ is regular if and only if $B_\mathfrak{q}$ is regular.*

Proof. By Lemma 15.44.1 we may assume both $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are Noetherian in order to prove the equivalence. Let $x_1, \ldots, x_t \in \mathfrak{p}A_\mathfrak{p}$ be a minimal set of generators. As $A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat we see that the images $y_1, \ldots, y_t$ in $B_\mathfrak{q}$ form a minimal system of generators for $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ (Algebra, Lemma 10.143.5). Regularity of $A_\mathfrak{p}$ by definition means $t = \dim(A_\mathfrak{p})$ and similarly for $B_\mathfrak{q}$. Hence the lemma follows from the equality $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$ of Lemma 15.44.2. □

0AP2 **Lemma 15.44.4.** *If $A \to B$ is an étale ring map and A is a Dedekind domain, then B is a finite product of Dedekind domains. In particular, the localizations $B_\mathfrak{q}$ for $\mathfrak{q} \subset B$ maximal are discrete valuation rings.*

Proof. The statement on the local rings follows from Lemmas 15.44.2 and 15.44.3 and Algebra, Lemma 10.119.7. It follows that B is a Noetherian normal ring of dimension 1. By Algebra, Lemma 10.37.16 we conclude that B is a finite product of normal domains of dimension 1. These are Dedekind domains by Algebra, Lemma 10.120.17. □

15.45. Permanence of properties under henselization

07QL Given a local ring R we denote R^h, resp. R^{sh} the henselization, resp. strict henselization of R, see Algebra, Definition 10.155.3. Many of the properties of R are reflected in R^h and R^{sh} as we will show in this section.

07QM **Lemma 15.45.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then we have the following*

(1) *$R \to R^h \to R^{sh}$ are faithfully flat ring maps,*
(2) *$\mathfrak{m}R^h = \mathfrak{m}^h$ and $\mathfrak{m}R^{sh} = \mathfrak{m}^h R^{sh} = \mathfrak{m}^{sh}$,*
(3) *$R/\mathfrak{m}^n = R^h/\mathfrak{m}^n R^h$ for all n,*
(4) *there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak{m}^n R^{sh}$ is a free $R/\mathfrak{m}^n$-module on x_i mod $\mathfrak{m}^n R^{sh}$.*

Proof. By construction R^h is a colimit of étale R-algebras, see Algebra, Lemma 10.155.1. Since étale ring maps are flat (Algebra, Lemma 10.143.3) we see that R^h is flat over R by Algebra, Lemma 10.39.3. As a flat local ring homomorphism is faithfully flat (Algebra, Lemma 10.39.17) we see that $R \to R^h$ is faithfully flat. The ring map $R^h \to R^{sh}$ is a colimit of finite étale ring maps, see proof of Algebra, Lemma 10.155.2. Hence the same arguments as above show that $R^h \to R^{sh}$ is faithfully flat.

Part (2) follows from Algebra, Lemmas 10.155.1 and 10.155.2. Part (3) follows from Algebra, Lemma 10.101.1 because $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism and $R/\mathfrak{m}^n \to R^h/\mathfrak{m}^n R^h$ is flat as a base change of the flat ring map $R \to R^h$ (Algebra, Lemma 10.39.7). Let κ^{sep} be the residue field of R^{sh} (it is a separable algebraic closure of κ). Choose $x_i \in R^{sh}$ mapping to a basis of κ^{sep} as a κ-vector space. Then (4) follows from Algebra, Lemma 10.101.1 in exactly the same way as above. □

07QN **Lemma 15.45.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then*

(1) *$R \to R^h$, $R^h \to R^{sh}$, and $R \to R^{sh}$ are formally étale,*
(2) *$R \to R^h$, $R^h \to R^{sh}$, resp. $R \to R^{sh}$ are formally smooth in the $\mathfrak{m}^h$, $\mathfrak{m}^{sh}$, resp. $\mathfrak{m}^{sh}$-topology.*

Proof. Part (1) follows from the fact that R^h and R^{sh} are directed colimits of étale algebras (by construction), that étale algebras are formally étale (Algebra, Lemma 10.150.3), and that colimits of formally étale algebras are formally étale (Algebra, Lemma 10.150.4). Part (2) follows from the fact that a formally étale ring map is formally smooth and Lemma 15.37.2. □

06LJ **Lemma 15.45.3.** *Let R be a local ring. The following are equivalent* [**DG67**, IV, Theorem 18.6.6 and Proposition 18.8.8]

(1) *R is Noetherian,*
(2) *R^h is Noetherian, and*
(3) *R^{sh} is Noetherian.*

In this case we have

(a) *$(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian complete local rings,*
(b) *$R^\wedge \to (R^h)^\wedge$ is an isomorphism,*
(c) *$R^h \to (R^h)^\wedge$ and $R^{sh} \to (R^{sh})^\wedge$ are flat,*
(d) *$R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology,*
(e) *$(R^\wedge)^{sh} = R^\wedge \otimes_{R^h} R^{sh}$, and*
(f) *$((R^\wedge)^{sh})^\wedge = (R^{sh})^\wedge$.*

Proof. Since $R \to R^h \to R^{sh}$ are faithfully flat (Lemma 15.45.1), we see that R^h or R^{sh} being Noetherian implies that R is Noetherian, see Algebra, Lemma 10.164.1. In the rest of the proof we assume R is Noetherian.

As $\mathfrak{m} \subset R$ is finitely generated it follows that $\mathfrak{m}^h = \mathfrak{m}R^h$ and $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$ are finitely generated, see Lemma 15.45.1. Hence $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian by Algebra, Lemma 10.160.3. This proves (a).

Note that (b) is immediate from Lemma 15.45.1. In particular we see that $(R^h)^\wedge$ is flat over R, see Algebra, Lemma 10.97.3.

Next, we show that $R^h \to (R^h)^\wedge$ is flat. Write $R^h = \operatorname{colim}_i R_i$ as a directed colimit of localizations of étale R-algebras. By Algebra, Lemma 10.39.6 if $(R^h)^\wedge$ is flat over each R_i, then $R^h \to (R^h)^\wedge$ is flat. Note that $R^h = R_i^h$ (by construction). Hence $R_i^\wedge = (R^h)^\wedge$ by part (b) is flat over R_i as desired. To finish the proof of (c) we show that $R^{sh} \to (R^{sh})^\wedge$ is flat. To do this, by a limit argument as above, it suffices to show that $(R^{sh})^\wedge$ is flat over R. Note that it follows from Lemma 15.45.1 that $(R^{sh})^\wedge$ is the completion of a free R-module. By Lemma 15.27.2 we see this is flat over R as desired. This finishes the proof of (c).

At this point we know (c) is true and that $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian. It follows from Algebra, Lemma 10.164.1 that R^h and R^{sh} are Noetherian.

Part (d) follows from Lemma 15.45.2 and Lemma 15.37.4.

Part (e) follows from Algebra, Lemma 10.155.13 and the fact that $R^\wedge$ is henselian by Algebra, Lemma 10.153.9.

Proof of (f). Using (e) there is a map $R^{sh} \to (R^\wedge)^{sh}$ which induces a map $(R^{sh})^\wedge \to ((R^\wedge)^{sh})^\wedge$ upon completion. Using (e) there is a map $R^\wedge \to (R^{sh})^\wedge$. Since $(R^{sh})^\wedge$ is strictly henselian (see above) this map induces a map $(R^\wedge)^{sh} \to (R^{sh})^\wedge$ by Algebra, Lemma 10.155.10. Completing we obtain a map $((R^\wedge)^{sh})^\wedge \to (R^{sh})^\wedge$. We omit the verification that these two maps are mutually inverse. □

06DH **Lemma 15.45.4.** *Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.*

Proof. The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.164.2. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 10.163.7. □

0ASE **Lemma 15.45.5.** *Let R be a local ring. Let $nil(R)$ denote the ideal of nilpotent elements of R. Then $nil(R)R^h = nil(R^h)$ and $nil(R)R^{sh} = nil(R^{sh})$.*

Proof. Note that $nil(R)$ is the biggest ideal consisting of nilpotent elements such that the quotient $R/nil(R)$ is reduced. Note that $nil(R)R^h$ consists of nilpotent elements by Algebra, Lemma 10.32.3. Also, note that $R^h/nil(R)R^h$ is the henselization of $R/nil(R)$ by Algebra, Lemma 10.156.2. Hence $R^h/nil(R)R^h$ is reduced by Lemma 15.45.4. We conclude that $nil(R)R^h = nil(R^h)$ as desired. Similarly for the strict henselization but using Algebra, Lemma 10.156.4. □

06DI **Lemma 15.45.6.** *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.*

Proof. A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition 10.37.11). The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.164.3. Conversely, assume R is normal. Since R^h and R^{sh} are filtered colimits of étale hence smooth R-algebras, the result follows from Algebra, Lemmas 10.163.9 and 10.37.17. □

06LK **Lemma 15.45.7.** *Given any local ring R we have $\dim(R) = \dim(R^h) = \dim(R^{sh})$.*

Proof. Since $R \to R^{sh}$ is faithfully flat (Lemma 15.45.1) we see that $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Lemma 10.112.1. For the converse, we write $R^{sh} = \operatorname{colim} R_i$ as a directed colimit of local rings R_i each of which is a localization of an étale R-algebra. Now if $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of prime ideals in R^{sh}, then for some sufficiently large i the sequence

$$R_i \cap \mathfrak{q}_0 \subset R_i \cap \mathfrak{q}_1 \subset \ldots \subset R_i \cap \mathfrak{q}_n$$

is a chain of primes in R_i. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Lemma 15.44.2 we have $\dim(R_i) = \dim(R)$ for each i and we win. □

06LL **Lemma 15.45.8.** *Given a Noetherian local ring R we have $\mathit{depth}(R) = \mathit{depth}(R^h) = \mathit{depth}(R^{sh})$.*

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian. Hence the lemma follows from Algebra, Lemma 10.163.2. □

06LM **Lemma 15.45.9.** *Let R be a Noetherian local ring. The following are equivalent: R is Cohen-Macaulay, the henselization R^h of R is Cohen-Macaulay, and the strict henselization R^{sh} of R is Cohen-Macaulay.*

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Since we have $\operatorname{depth}(R) = \operatorname{depth}(R^h) = \operatorname{depth}(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 15.45.8 and 15.45.7 we conclude. □

06LN **Lemma 15.45.10.** *Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.*

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Let $\mathfrak{m}$ be the maximal ideal of R. Let $x_1, \ldots, x_t \in \mathfrak{m}$ be a minimal system of generators of $\mathfrak{m}$, i.e., such that the images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis over $\kappa = R/\mathfrak{m}$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in R^h, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in R^{sh} are a minimal system of generators for $\mathfrak{m}^h = \mathfrak{m}R^h$, resp. $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$. Regularity of R by definition means $t = \dim(R)$ and similarly for R^h and R^{sh}. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 15.45.7 □

0AP3 **Lemma 15.45.11.** *Let R be a Noetherian local ring. Then R is a discrete valuation ring if and only if R^h is a discrete valuation ring if and only if R^{sh} is a discrete valuation ring.*

Proof. This follows from Lemmas 15.45.7 and 15.45.10 and Algebra, Lemma 10.119.7. □

0AH1 **Lemma 15.45.12.** *Let A be a ring. Let B be a filtered colimit of étale A-algebras. Let $\mathfrak{p}$ be a prime of A. If B is Noetherian, then there are finitely many primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ lying over $\mathfrak{p}$, we have $B \otimes_A \kappa(\mathfrak{p}) = \prod \kappa(\mathfrak{q}_i)$, and each of the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ is separable algebraic.*

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ with $A \to B_i$ étale. Then on the one hand $B \otimes_A \kappa(\mathfrak{p}) = \operatorname{colim} B_i \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras, and on the other hand it is Noetherian. An étale $\kappa(\mathfrak{p})$-algebra is a finite product of finite separable field extensions (Algebra, Lemma 10.143.4). Hence there are no nontrivial specializations between the primes (which are all maximal and minimal primes) of the algebras $B_i \otimes_A \kappa(\mathfrak{p})$ and hence there are no nontrivial specializations between the primes of $B \otimes_A \kappa(\mathfrak{p})$. Thus $B \otimes_A \kappa(\mathfrak{p})$ is reduced and has finitely many primes which all minimal. Thus it is a finite product of fields (use Algebra, Lemma 10.25.4 or Algebra, Proposition 10.60.7). Each of these fields is a colimit of finite separable extensions and hence the final statement of the lemma follows. □

07QQ **Lemma 15.45.13.** *Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Then*

$$R^h \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,t} \kappa(\mathfrak{q}_i) \quad \textit{resp.} \quad R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$ are the prime of R^h, resp. R^{sh} lying over $\mathfrak{p}$. Moreover, the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ resp. $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p})$ are separable algebraic.

Proof. This can be deduced from the more general Lemma 15.45.12 using that the henselization and strict henselization are Noetherian (as we've seen above). But we also give a direct proof as follows.

We will use without further mention the results of Lemmas 15.45.1 and 15.45.3. Note that $R^h/\mathfrak{p}R^h$, resp. $R^{sh}/\mathfrak{p}R^{sh}$ is the henselization, resp. strict henselization of $R/\mathfrak{p}$, see Algebra, Lemma 10.156.2 resp. Algebra, Lemma 10.156.4. Hence we may replace R by $R/\mathfrak{p}$ and assume that R is a Noetherian local domain and that $\mathfrak{p} = (0)$. Since R^h, resp. R^{sh} is Noetherian, it has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$. Since $R \to R^h$, resp. $R \to R^{sh}$ is flat these are exactly the primes lying over $\mathfrak{p} = (0)$ (by going down). Finally, as R is a domain, we see that R^h, resp. R^{sh} is reduced, see Lemma 15.45.4. Thus we see that $R^h \otimes_R \kappa(\mathfrak{p})$ resp. $R^{sh} \otimes_R \kappa(\mathfrak{p})$ is a reduced Noetherian ring with finitely many primes, all of which are minimal (and hence maximal). Thus these rings are Artinian and are products of their localizations at maximal ideals, each necessarily a field (see Algebra, Proposition 10.60.7 and Algebra, Lemma 10.25.1).

The final statement follows from the fact that $R \to R^h$, resp. $R \to R^{sh}$ is a colimit of étale ring maps and hence the induced residue field extensions are colimits of finite separable extensions, see Algebra, Lemma 10.143.5. □

15.46. Field extensions, revisited

07P0 In this section we study some peculiarities of field extensions in characteristic $p > 0$.

07P1 **Definition 15.46.1.** Let p be a prime number. Let $k \to K$ be an extension of fields of characteristic p. Denote kK^p the compositum of k and K^p in K.

(1) A subset $\{x_i\} \subset K$ is called *p-independent over k* if the elements $x^E = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over kK^p.

(2) A subset $\{x_i\}$ of K is called a *p-basis of K over k* if the elements x^E form a basis of K over kK^p.

This is related to the notion of a p-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).

07P2 **Lemma 15.46.2.** *Let K/k be a field extension. Assume k has characteristic $p > 0$. Let $\{x_i\}$ be a subset of K. The following are equivalent*

(1) *the elements $\{x_i\}$ are p-independent over k, and*
(2) *the elements dx_i are K-linearly independent in $\Omega_{K/k}$.*

Any p-independent collection can be extended to a p-basis of K over k. In particular, the field K has a p-basis over k. Moreover, the following are equivalent:

(a) *$\{x_i\}$ is a p-basis of K over k, and*
(b) *dx_i is a basis of the K-vector space $\Omega_{K/k}$.*

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a k-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any k-derivation of K annihilates kK^p. Applying θ_i to the given relation we obtain new relations

$$\sum\nolimits_{E, e_i > 0} e_i a_E x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \neq 0$, then we get a contradiction. Hence (1) holds.

If $\{x_i\}$ is a p-basis for K over k, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $\mathrm{d}x_i$ forms a basis for $\Omega_{K/k}$ over K. Thus (a) implies (b).

Let $\{x_i\}$ be a p-independent subset of K over k. An application of Zorn's lemma shows that we can enlarge this to a maximal p-independent subset of K over k. We claim that any maximal p-independent subset $\{x_i\}$ of K is a p-basis of K over k. The claim will imply that (1) implies (2) and establish the existence of p-bases. To prove the claim let L be the subfield of K generated by kK^p and the x_i. We have to show that $L = K$. If $x \in K$ but $x \notin L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is p-independent over k, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal p-independent subset of K over k. Hence by the claim above it is a p-basis. □

07P3 **Lemma 15.46.3.** *Let K/k be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

(1) *$k \subset K_\alpha$ for all $\alpha \in A$,*
(2) *$k = \bigcap_{\alpha \in A} K_\alpha$,*
(3) *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then for $n \geq 1$ and $V \subset K^{\oplus n}$ a K-vector space we have $V \cap k^{\oplus n} \neq 0$ if and only if $V \cap K_\alpha^{\oplus n} \neq 0$ for all $\alpha \in A$.

Proof. By induction on n. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\oplus n-1}$. Assume that $V \subset K^{\oplus n}$ has nonzero intersection with $K_\alpha^{\oplus n}$ for all $\alpha \in A$. If $V \cap 0 \oplus k^{\oplus n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an α such that $V \cap 0 \oplus K_\alpha^{\oplus n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha^{\oplus n}$ be a nonzero

element. By our choice of α we see that x_1 is not zero. Replace v by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x'_1, \ldots, x'_n) \in V \cap K_\alpha$, then $v' - x'_1 v = 0$ by our choice of α. Hence we see that $V \cap K_\alpha^{\oplus n} = K_\alpha v$. If we choose some α' such that $K_{\alpha'} \subset K_\alpha$, then we see that necessarily $v \in V \cap K_{\alpha'}^{\oplus n}$ (by the same arguments applied to α'). Hence

$$x_2, \ldots, x_n \in \bigcap\nolimits_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}$$

which equals k by (2) and (3). □

07P4 **Lemma 15.46.4.** *Let K be a field of characteristic p. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

(1) *$K^p \subset K_\alpha$ for all $\alpha \in A$,*
(2) *$K^p = \bigcap_{\alpha \in A} K_\alpha$,*
(3) *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then

(1) *the intersection of the kernels of the maps $\Omega_{K/\mathbf{F}_p} \to \Omega_{K/K_\alpha}$ is zero,*
(2) *for any finite extension L/K we have $L^p = \bigcap_{\alpha \in A} L^p K_\alpha$.*

Proof. Proof of (1). Choose a p-basis $\{x_i\}$ for K over $\mathbf{F}_p$. Suppose that $\eta = \sum_{i \in I'} y_i \mathrm{d}x_i$ maps to zero in Ω_{K/K_α} for every $\alpha \in A$. Here the index set I' is finite. By Lemma 15.46.2 this means that for every α there exists a relation

$$\sum\nolimits_E a_{E,\alpha} x^E = 0, \quad a_{E,\alpha} \in K_\alpha$$

where E runs over multi-indices $E = (e_i)_{i \in I'}$ with $0 \leq e_i < p$. On the other hand, Lemma 15.46.2 guarantees there is no such relation $\sum a_E x^E = 0$ with $a_E \in K^p$. This is a contradiction by Lemma 15.46.3.

Proof of (2). Suppose that we have a tower $L/M/K$ of finite extensions of fields. Set $M_\alpha = M^p K_\alpha$ and $L_\alpha = L^p K_\alpha = L^p M_\alpha$. Then we can first prove that $M^p = \bigcap_{\alpha \in A} M_\alpha$, and after that prove that $L^p = \bigcap_{\alpha \in A} L_\alpha$. Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume that $L = K(\theta)$ is separable over K. Then L is generated by θ^p over K, hence we may assume that $\theta \in L^p$. In this case we see that

$$L^p = K^p \oplus K^p\theta \oplus \ldots K^p\theta^{d-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha\theta \oplus \ldots K_\alpha\theta^{d-1}$$

where $d = [L : K]$. Thus the conclusion is clear in this case. The other case is where $L = K(\theta)$ with $\theta^p = t \in K$, $t \notin K^p$. In this case we have

$$L^p = K^p \oplus K^p t \oplus \ldots K^p t^{p-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots K_\alpha t^{p-1}$$

Again the result is clear. □

07P5 **Lemma 15.46.5.** *Let k be a field of characteristic $p > 0$. Let $\{x_i\}_{i \in I}$ be a p-basis for k. Let $n, m \geq 0$. Let K be the fraction field of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. Let J be a finite subset of I. Consider the subfield subfield $k/k_J/k^p$ generated by k^p and x_i with $i \in I \setminus J$. The fraction fields K_J of*

$$A_J = k_J[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$$

form a family of subfields of K as in Lemma 15.46.4. Moreover, each of the ring extensions $A_J \subset A$ is finite.

Proof. Since k/k_J is finite, the ring extension $k_J[[x_1^p, \ldots, x_d^p]] \subset k[[x_1, \ldots, x_d]]$ is finite by Algebra, Lemma 10.97.7. This implies that $A_J \to A$ is finite.

Let us check properties (1), (2), (3) of Lemma 15.46.4. Proof of (1). For $a \in A$ we see that $a^p \in A_J$. Hence $K^p \subset K_J$. Proof of (2). Suppose that $f/g^p \in K$, $f, g \in A$, $g \neq 0$ is contained in K_J for every choice of J. Fix J for the moment. Since $f/g^p \in K_J$ we can write $f/g^p = a/b^p$ with $a \in A_J$ and $b \in A$ nonzero. Hence $b^p f \in A_J$. For any A_J-derivation $D : A \to A$ we see that $0 = D(b^p f) = b^p D(f)$ hence $D(f) = 0$ as A is a domain. Taking $D = \partial_{x_i}$ and $D = \partial_{y_j}$ we conclude that $f \in k[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_d^p]$. Applying a k_J-derivation $\theta : k \to k$ we similarly conclude that all coefficients of f are in k_J, i.e., $f \in A_J$. Since it is clear that $A^p = \bigcap_J A_J$ where J ranges over all subfields as in the lemma we conclude $f \in A^p$ as desired. Proof of (3). This is clear because $K_{J \cup J'} \subset K_J \cap K_{J'}$. □

15.47. The singular locus

07P6 Let R be a Noetherian ring. The *regular locus* $\mathrm{Reg}(X)$ of $X = \mathrm{Spec}(R)$ is the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a regular local ring. The *singular locus* $\mathrm{Sing}(X)$ of $X = \mathrm{Spec}(R)$ is the complement $X \setminus \mathrm{Reg}(X)$, i.e., the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not a regular local ring. By the discussion preceding Algebra, Definition 10.110.7 we see that $\mathrm{Reg}(X)$ is stable under generalization. In this section we study conditions that guarantee that $\mathrm{Reg}(X)$ is open.

07P7 **Definition 15.47.1.** Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. [**Mat70a**, (32.B)]

(1) We say R is *J-0* if $\mathrm{Reg}(X)$ contains a nonempty open.
(2) We say R is *J-1* if $\mathrm{Reg}(X)$ is open.
(3) We say R is *J-2* if any finite type R-algebra is J-1.

The ring $\mathbf{Q}[x]/(x^2)$ does not satisfy J-0, but it does satisfy J-1. On the other hand, J-1 implies J-0 for Noetherian domains and more generally nonzero reduced Noetherian rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

07P8 **Lemma 15.47.2.** *Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. The ring R is J-1 if and only if $V(\mathfrak{p}) \cap Reg(X)$ contains a nonempty open subset of $V(\mathfrak{p})$ for all $\mathfrak{p} \in Reg(X)$.*

Proof. This follows from Topology, Lemma 5.16.5 and the fact that $\mathrm{Reg}(X)$ is stable under generalization by Algebra, Lemma 10.110.6. □

07P9 **Lemma 15.47.3.** *Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. Assume that for all primes $\mathfrak{p} \subset R$ the ring $R/\mathfrak{p}$ is J-0. Then R is J-1.*

Proof. We will show that the criterion of Lemma 15.47.2 applies. Let $\mathfrak{p} \in \mathrm{Reg}(X)$ be a prime of height r. Pick $f_1, \ldots, f_r \in \mathfrak{p}$ which map to generators of $\mathfrak{p}R_\mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{Reg}(X)$ we see that $f_1, \ldots, f_r$ maps to a regular sequence in $R_\mathfrak{p}$, see Algebra, Lemma 10.106.3. Thus by Algebra, Lemma 10.68.6 we see that after replacing R by R_g for some $g \in R$, $g \notin \mathfrak{p}$ the sequence $f_1, \ldots, f_r$ is a regular sequence in R. After another replacement we may also assume $f_1, \ldots, f_r$ generate $\mathfrak{p}$. Next, let $\mathfrak{p} \subset \mathfrak{q}$ be a prime ideal such that $(R/\mathfrak{p})_\mathfrak{q}$ is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of $V(\mathfrak{p})$ consisting of such primes, hence it suffices to prove $R_\mathfrak{q}$ is regular. Note that $f_1, \ldots, f_r$ is a regular sequence in

$R_{\mathfrak{q}}$ such that $R_{\mathfrak{q}}/(f_1, \ldots, f_r)R_{\mathfrak{q}}$ is regular. Hence $R_{\mathfrak{q}}$ is regular by Algebra, Lemma 10.106.7. □

07PA **Lemma 15.47.4.** *Let $R \to S$ be a ring map. Assume that*

(1) *R is a Noetherian domain,*
(2) *$R \to S$ is injective and of finite type, and*
(3) *S is a domain and J-0.*

Then R is J-0.

Proof. After replacing S by S_g for some nonzero $g \in S$ we may assume that S is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma 10.118.1. Then R is regular by Algebra, Lemma 10.164.4. □

07PB **Lemma 15.47.5.** *Let $R \to S$ be a ring map. Assume that*

(1) *R is a Noetherian domain and J-0,*
(2) *$R \to S$ is injective and of finite type, and*
(3) *S is a domain, and*
(4) *the induced extension of fraction fields is separable.*

Then S is J-0.

Proof. We may replace R by a principal localization and assume R is a regular ring. By Algebra, Lemma 10.140.9 the ring map $R \to S$ is smooth at (0). Hence after replacing S by a principal localization we may assume that S is smooth over R. Then S is regular too, see Algebra, Lemma 10.163.10. □

07PC **Lemma 15.47.6.** *Let R be a Noetherian ring. The following are equivalent*

(1) *R is J-2,*
(2) *every finite type R-algebra which is a domain is J-0,*
(3) *every finite R-algebra is J-1,*
(4) *for every prime $\mathfrak{p}$ and every finite purely inseparable extension $L/\kappa(\mathfrak{p})$ there exists a finite R-algebra R' which is a domain, which is J-0, and whose field of fractions is L.*

Proof. It is clear that we have the implications (1) ⇒ (2) and (2) ⇒ (4). Recall that a domain which is J-1 is J-0. Hence we also have the implications (1) ⇒ (3) and (3) ⇒ (4).

Let $R \to S$ be a finite type ring map and let's try to show S is J-1. By Lemma 15.47.3 it suffices to prove that $S/\mathfrak{q}$ is J-0 for every prime $\mathfrak{q}$ of S. In this way we see (2) ⇒ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with S a domain. Let $\mathfrak{p} = \mathrm{Ker}(R \to S)$. Let K be the fraction field of S. There exists a diagram of fields

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ \kappa(\mathfrak{p}) & \longrightarrow & L \end{array}$$

where the horizontal arrows are finite purely inseparable field extensions and where K'/L is separable, see Algebra, Lemma 10.42.4. Choose $R' \subset L$ as in (4) and let S' be the image of the map $S \otimes_R R' \to K'$. Then S' is a domain whose fraction

field is K', hence S' is J-0 by Lemma 15.47.5 and our choice of R'. Then we apply Lemma 15.47.4 to see that S is J-0 as desired. □

15.48. Regularity and derivations

07PD Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that D *extends to* S if there exists a derivation $D' : S \to S$ such that

$$\begin{array}{ccc} S & \xrightarrow[D']{} & S \\ \uparrow & & \uparrow \\ R & \xrightarrow{D} & R \end{array}$$

is commutative.

07PE **Lemma 15.48.1.** *Let R be a ring. Let $D : R \to R$ be a derivation.*

(1) *For any ideal $I \subset R$ the derivation D extends canonically to a derivation $D^\wedge : R^\wedge \to R^\wedge$ on the I-adic completion.*
(2) *For any multiplicative subset $S \subset R$ the derivation D extends uniquely to the localization $S^{-1}R$ of R.*

If $R \subset R'$ is a finite type extension of rings such that $R_g \cong R'_g$ for some $g \in R$ which is a nonzerodivisor in R', then $g^N D$ extends to R' for some $N \geq 0$.

Proof. Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence D induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend D to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of R' over R. Choose an N such that $g^N x_i \in R$. Consider $g^{N+1}D$. By (2) this extends to R_g. Moreover, by the Leibniz rule and our construction of the extension above we have

$$g^{N+1}D(x_i) = g^{N+1}D(g^{-N}g^N x_i) = -Ng^N x_i D(g) + gD(g^N x_i)$$

and both terms are in R. This implies that

$$g^{N+1}D(x_1^{e_1} \ldots x_n^{e_n}) = \sum e_i x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} g^{N+1} D(x_i)$$

is an element of R'. Hence every element of R' (which can be written as a sum of monomials in the x_i with coefficients in R) is mapped to an element of R' by $g^{N+1}D$ and we win. □

07PF **Lemma 15.48.2.** *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/(f)$. Then $R/(f)$ is regular.*

Proof. It suffices to prove this when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case it suffices to prove that $f \notin \mathfrak{m}^2$, see Algebra, Lemma 10.106.3. However, if $f \in \mathfrak{m}^2$ then $D(f) \in \mathfrak{m}$ by the Leibniz rule, a contradiction. □

0GEE **Lemma 15.48.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring. Let $m \geq 1$. Let $f_1, \ldots, f_m \in \mathfrak{m}$. Assume there exist derivations $D_1, \ldots, D_m : R \to R$ such that $\det_{1 \leq i,j \leq m}(D_i(f_j))$ is a unit of R. Then $R/(f_1, \ldots, f_m)$ is regular and $f_1, \ldots, f_m$ is a regular sequence.*

Proof. It suffices to prove that $f_1, \ldots, f_m$ are κ-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, see Algebra, Lemma 10.106.3. However, if there is a nontrivial linear relation the we get $\sum a_i f_i \in \mathfrak{m}^2$ for some $a_i \in R$ but not all $a_i \in \mathfrak{m}$. Observe that $D_i(\mathfrak{m}^2) \subset \mathfrak{m}$ and $D_i(a_j f_j) \equiv a_j D_i(f_j) \bmod \mathfrak{m}$ by the Leibniz rule for derivations. Hence this would imply

$$\sum a_j D_i(f_j) \in \mathfrak{m}$$

which would contradict the assumption on the determinant. □

07PG **Lemma 15.48.4.** *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of R. Then $R[z]/(z^n - f)$ is regular for any integer $n \geq 1$. More generally, $R[z]/(p(z) - f)$ is regular for any $p \in \mathbf{Z}[z]$.*

Proof. By Algebra, Lemma 10.163.10 we see that $R[z]$ is a regular ring. Apply Lemma 15.48.2 to the extension of D to $R[z]$ which maps z to zero. This works because D annihilates any polynomial with integer coefficients and sends f to a unit. □

07PH **Lemma 15.48.5.** *Let p be a prime number. Let B be a domain with $p = 0$ in B. Let $f \in B$ be an element which is not a pth power in the fraction field of B. If B is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.*

Proof. Let R be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace R by its image in B, hence we may assume R is a domain of characteristic $p > 0$ (as well as Noetherian complete local). By Algebra, Lemma 10.160.11 we can write R as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field k and integer n. Hence we may replace R by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma 10.115.7 to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with B' finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^{pN} f \in B'$ for some large integer N. It is clear that f' is not a pth power in the fraction field of B'. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma 15.48.1 guarantees that $D = g^M D'$ extends to B for some $M > 0$. Then $D(f) = g^M D'(f) = g^M D'(g^{-pN} f') = g^{M-pN} D'(f')$ is nonzero. Thus it suffices to prove the lemma in case B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Assume B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. Denote L the fraction field of B. Note that $\mathrm{d}f$ is not zero in $\Omega_{L/\mathbf{F}_p}$, see Algebra, Lemma 10.158.2. We apply Lemma 15.46.5 to find a subfield $k' \subset k$ of finite index such that with $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$ the element $\mathrm{d}f$ does not map to zero in $\Omega_{L/K'}$ where K' is the fraction field of A'. Thus we can choose a K'-derivation $D' : L \to L$ with $D'(f) \neq 0$. Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate B as an A'-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_t D'$ we win. □

07PI **Lemma 15.48.6.** *Let A be a Noetherian complete local domain. Then A is J-0.*

Proof. By Algebra, Lemma 10.160.11 we can find a regular subring $A_0 \subset A$ with A finite over A_0. The induced extension K/K_0 of fraction fields is finite. If K/K_0 is separable, then we are done by Lemma 15.47.5. If not, then A_0 and A have characteristic $p > 0$. For any subextension $K/M/K_0$ there exists a finite subextension

$A_0 \subset B \subset A$ whose fraction field is M. Hence, arguing by induction on $[K : K_0]$ we may assume there exists $A_0 \subset B \subset A$ such that B is J-0 and K/M has no nontrivial subextensions. In this case, if K/M is separable, then we see that A is J-0 by Lemma 15.47.5. If not, then $K = M[z]/(z^p - b_1/b_2)$ for some $b_1, b_2 \in B$ with $b_2 \neq 0$ and b_1/b_2 not a pth power in M. Choose $a \in A$ nonzero such that $az \in A$. After replacing z by $b_2 a^p z$ we obtain $K = M[z]/(z^p - b)$ with $z \in A$ and $b \in B$ not a pth power in M. By Lemma 15.48.5 we can find a derivation $D : B \to B$ with $D(b) \neq 0$. Applying Lemma 15.48.4 we see that $A_{\mathfrak{p}}$ is regular for any prime $\mathfrak{p}$ of A lying over a regular prime of B and not containing $D(b)$. As B is J-0 we conclude A is too. □

07PJ **Proposition 15.48.7.** *The following types of rings are J-2:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Noetherian local rings of dimension* 1*,*
(5) *Nagata rings of dimension* 1*,*
(6) *Dedekind domains with fraction field of characteristic zero,*
(7) *finite type ring extensions of any of the above.*

Proof. For cases (1), (3), (5), and (6) this is proved by checking condition (4) of Lemma 15.47.6. We will only do this in case R is a Nagata ring of dimension 1. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite purely inseparable extension. If $\mathfrak{p} \subset R$ is a maximal ideal, then $R \to L$ is finite and L is a regular ring and we've checked the condition. If $\mathfrak{p} \subset R$ is a minimal prime, then the Nagata condition insures that the integral closure $R' \subset L$ of R in L is finite over R. Then R' is a normal domain of dimension 1 (Algebra, Lemma 10.112.3) hence regular (Algebra, Lemma 10.157.4) and we've checked the condition in this case as well.

For case (2), we will use condition (3) of Lemma 15.47.6. Let R be a Noetherian complete local ring. Note that if $R \to R'$ is finite, then R' is a product of Noetherian complete local rings, see Algebra, Lemma 10.160.2. Thus by Lemma 15.47.3 it suffices to prove that a Noetherian complete local ring which is a domain is J-0, which is Lemma 15.48.6.

For case (4), we also use condition (3) of Lemma 15.47.6. Namely, if R is a local Noetherian ring of dimension 1 and $R \to R'$ is finite, then $\mathrm{Spec}(R')$ is finite. Since the regular locus is stable under generalization, we see that R' is J-1. □

15.49. Formal smoothness and regularity

07PK The title of this section refers to André's Theorem 15.49.3.

07PL **Lemma 15.49.1.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $D : A \to A$ be a derivation. Assume that B is complete and $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists an extension $D' : B \to B$ of D.*

Proof. Denote $B[\epsilon] = B[x]/(x^2)$ the ring of dual numbers over B. Consider the ring map $\psi : A \to B[\epsilon]$, $a \mapsto a + \epsilon D(a)$. Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{1} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\psi} & B[\epsilon] \end{array}$$

By Lemma 15.37.5 and the assumption of formal smoothness of B/A we find a map $\varphi : B \to B[\epsilon]$ fitting into the diagram. Write $\varphi(b) = b + \epsilon D'(b)$. Then $D' : B \to B$ is the desired extension. □

07PM **Proposition 15.49.2.** *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

1. *$A \to B$ is regular,*
2. *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
3. *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
4. *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 15.40.5. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let $\mathfrak{p}$ be a prime of A. We will show that $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. By Lemma 15.37.8 we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Thus we may assume that A is a domain and that $\mathfrak{p} = (0)$.

Choose $A_0 \subset A$ as in Algebra, Lemma 10.160.11. We will use all the properties stated in that lemma without further mention. As $A_0 \to A$ induces an isomorphism on residue fields, and as $B/\mathfrak{m}_A B$ is geometrically regular over $A/\mathfrak{m}_A$ we can find a diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_0 & \longrightarrow & A \end{array}$$

with $A_0 \to C$ formally smooth in the $\mathfrak{m}_C$-adic topology such that $B = C \otimes_{A_0} A$, see Remark 15.40.7. (Completion in the tensor product is not needed as $A_0 \to A$ is finite, see Algebra, Lemma 10.97.1.) Hence it suffices to show that $C \otimes_{A_0} K_0$ is a geometrically regular algebra over the fraction field K_0 of A_0.

The upshot of the preceding paragraph is that we may assume that $A = k[[x_1, \ldots, x_n]]$ where k is a field or $A = \Lambda[[x_1, \ldots, x_n]]$ where Λ is a Cohen ring. In this case B is a regular ring, see Algebra, Lemma 10.112.8. Hence $B \otimes_A K$ is a regular ring too (where K is the fraction field of A) and we win if the characteristic of K is zero.

Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic $p > 0$. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we've seen above. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let A' be a finite

A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of A'' is L. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 15.48.5 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $\mathfrak{m}$-adic topology by Lemma 15.37.9 we can use Lemma 15.49.1 to extend D to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 15.48.4 and we win. □

0H7U **Theorem 15.49.3** (André)**.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. Assume $A \to A^\wedge$ is regular*[6]*. The following are equivalent*

(1) *$A \to B$ is regular,*
(2) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
(3) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
(4) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

The implication (4) ⇒ (1) is the main result of [**And74b**].

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 15.40.5. It is clear that (1) implies (2). Thus we assume (4) holds and we prove (1).

By Lemma 15.37.4 we see that $A^\wedge \to B^\wedge$ is formally smooth. By Proposition 15.49.2 we see that $A^\wedge \to B^\wedge$ is regular. By assumption $A \to A^\wedge$ is regular, hence $A \to B^\wedge$ is regular by Lemma 15.41.4. Since $B \to B^\wedge$ is faithfully flat, we conclude that $A \to B$ is regular by Lemma 15.41.7. □

15.50. G-rings

07GG Let A be a Noetherian local ring. In Section 15.43 we have seen that some but not all properties of A are reflected in the completion $A^\wedge$ of A. To study this further we introduce some terminology. For a prime $\mathfrak{q}$ of A the fibre ring

$$A^\wedge \otimes_A \kappa(\mathfrak{q}) = (A^\wedge)_\mathfrak{q}/\mathfrak{q}(A^\wedge)_\mathfrak{q} = (A/\mathfrak{q})^\wedge \otimes_{A/\mathfrak{q}} \kappa(\mathfrak{q})$$

is called a *formal fibre* of A. We think of the formal fibre as an algebra over $\kappa(\mathfrak{q})$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

07GH **Definition 15.50.1.** A ring R is called a *G-ring* if R is Noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_\mathfrak{p} \to (R_\mathfrak{p})^\wedge$ is regular.

By the discussion above we see that R is a G-ring if and only if every local ring $R_\mathfrak{p}$ has geometrically regular formal fibres. Note that if $\mathbf{Q} \subset R$, then it suffices to check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

07PN **Lemma 15.50.2.** *Let R be a Noetherian ring. Then R is a G-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the algebra*

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$.

[6] For example, if A is a G-ring or (quasi-)excellent, see Sections 15.50 and 15.52.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

07PP **Lemma 15.50.3.** *Let $R \to R'$ be a finite type map of Noetherian rings and let*

$$\begin{array}{ccccc} \mathfrak{q}' & \longrightarrow & \mathfrak{p}' & \longrightarrow & R' \\ \downarrow & & \downarrow & & \uparrow \\ \mathfrak{q} & \longrightarrow & \mathfrak{p} & \longrightarrow & R \end{array}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$.

1. *If the formal fibre $R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$, then the formal fibre $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is geometrically regular over $\kappa(\mathfrak{q}')$.*
2. *If the formal fibres of $R_\mathfrak{p}$ are geometrically regular, then the formal fibres of $R'_{\mathfrak{p}'}$ are geometrically regular.*
3. *If $R \to R'$ is quasi-finite and R is a G-ring, then R' is a G-ring.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume $R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. By Algebra, Lemma 10.124.3 we see that

$$R^\wedge_\mathfrak{p} \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R^\wedge_\mathfrak{p}$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R^\wedge_\mathfrak{p} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma 10.166.1. □

07PQ **Lemma 15.50.4.** *Let R be a Noetherian ring. Then R is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of S are regular rings.*

Proof. Assume that for any finite free ring map $R \to S$ the ring S has regular formal fibres. Let $\mathfrak{q} \subset \mathfrak{p} \subset R$ be primes and let $\kappa(\mathfrak{q}) \subset L$ be a finite purely inseparable extension. To show that R is a G-ring it suffices to show that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $\mathfrak{q}' = \mathfrak{q}R'$ is a prime and such that $\kappa(\mathfrak{q}')$ is isomorphic to L over $\kappa(\mathfrak{q})$, see Algebra, Lemma 10.159.3. By Algebra, Lemma 10.97.8 we have

$$R^\wedge_\mathfrak{p} \otimes_R R' = \prod (R'_{\mathfrak{p}'_i})^\wedge$$

where $\mathfrak{p}'_i$ are the primes of R' lying over $\mathfrak{p}$. Thus we have

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L = R^\wedge_\mathfrak{p} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = \prod (R'_{\mathfrak{p}'_i})^\wedge \otimes_{R'_{\mathfrak{p}'_i}} \kappa(\mathfrak{q}')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus R is a G-ring. The converse follows from Lemma 15.50.3. □

07PR **Lemma 15.50.5.** *Let k be a field of characteristic p. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote K the fraction field of A. Let $\mathfrak{p} \subset A$ be a prime. Then $A^\wedge_\mathfrak{p} \otimes_A K$ is geometrically regular over K.*

Proof. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A^\wedge_\mathfrak{p} \otimes L$ is regular. The base case is $L = K$: as A is regular, $A^\wedge_\mathfrak{p}$ is regular (Lemma 15.43.4), hence the localization $A^\wedge_\mathfrak{p} \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let B be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of C is L. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for every element of B or C some power of it lies in A. Hence there is a unique prime $\mathfrak{r} \subset B$, resp. $\mathfrak{q} \subset C$ lying over $\mathfrak{p}$. Note that

$$A^\wedge_\mathfrak{p} \otimes_A M = B^\wedge_\mathfrak{r} \otimes_B M$$

see Algebra, Lemma 10.97.8. By induction we know that this ring is regular. In the same manner we have

$$A^\wedge_\mathfrak{p} \otimes_A L = C^\wedge_\mathfrak{r} \otimes_C L = B^\wedge_\mathfrak{r} \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B^\wedge_\mathfrak{r}[z]/(z^p - f)$. By Lemma 15.48.5 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in M! By Lemma 15.48.1 D extends to a derivation of $B_\mathfrak{r}$, $B^\wedge_\mathfrak{r}$ and $B^\wedge_\mathfrak{r} \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B^\wedge_\mathfrak{r} \otimes_B M$ which maps f to g and g is a unit of the ring $B^\wedge_\mathfrak{r} \otimes_B M$. Hence $A^\wedge_\mathfrak{p} \otimes_A L$ is regular by Lemma 15.48.4 and we win. □

07PS **Proposition 15.50.6.** *A Noetherian complete local ring is a G-ring.*

Proof. Let A be a Noetherian complete local ring. By Lemma 15.50.2 it suffices to check that $B = A/\mathfrak{q}$ has geometrically regular formal fibres over the minimal prime (0) of B. Thus we may assume that A is a domain and it suffices to check the condition for the formal fibres over the minimal prime (0) of A. Let K be the fraction field of A.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that A is finite over A_0, see Algebra, Lemma 10.160.11. Moreover, we may assume that A_0 is a power series ring over a field or a Cohen ring. By Lemma 15.50.3 we see that it suffices to prove the result for A_0.

Assume that A is a power series ring over a field or a Cohen ring. Since A is regular the localizations $A_\mathfrak{p}$ are regular (see Algebra, Definition 10.110.7 and the discussion preceding it). Hence the completions $A^\wedge_\mathfrak{p}$ are regular, see Lemma 15.43.4. Hence the fibre $A^\wedge_\mathfrak{p} \otimes_A K$ is, as a localization of $A^\wedge_\mathfrak{p}$, also regular. Thus we are done if the characteristic of K is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 15.50.5. □

07PT **Lemma 15.50.7.** *Let R be a Noetherian ring. Then R is a G-ring if and only if $R_\mathfrak{m}$ has geometrically regular formal fibres for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is faithfully flat we

can choose a prime $\mathfrak{p}'$ in $R^\wedge_\mathfrak{m}$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{array}{ccccc} R^\wedge_\mathfrak{m} & \longrightarrow & (R^\wedge_\mathfrak{m})_{\mathfrak{p}'} & \longrightarrow & (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'} \\ \uparrow & & \uparrow & & \uparrow \\ R_\mathfrak{m} & \longrightarrow & R_\mathfrak{p} & \longrightarrow & R^\wedge_\mathfrak{p} \end{array}$$

By assumption the ring map $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is regular. By Proposition 15.50.6 $(R^\wedge_\mathfrak{m})_{\mathfrak{p}'} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular. The localization $R^\wedge_\mathfrak{m} \to (R^\wedge_\mathfrak{m})_{\mathfrak{p}'}$ is regular. Hence $R_\mathfrak{m} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular by Lemma 15.41.4. Since it factors through the localization $R_\mathfrak{p}$, also the ring map $R_\mathfrak{p} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular. Thus we may apply Lemma 15.41.7 to see that $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$ is regular. □

07QR **Lemma 15.50.8.** *Let R be a Noetherian local ring which is a G-ring. Then the henselization R^h and the strict henselization R^{sh} are G-rings.*

Proof. We will use the criterion of Lemma 15.50.7. Let $\mathfrak{q} \subset R^h$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{q}$. Set $\mathfrak{q}_1 = \mathfrak{q}$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of R^h lying over $\mathfrak{p}$, so that $R^h \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.13. Using that $(R^h)^\wedge = R^\wedge$ (Lemma 15.45.3) we see

$$\prod\nolimits_{i=1,\ldots,t} (R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i) = (R^h)^\wedge \otimes_{R^h} (R^h \otimes_R \kappa(\mathfrak{p})) = R^\wedge \otimes_R \kappa(\mathfrak{p})$$

Hence $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{q}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.166.6 that $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{q}_i)$.

Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$ and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$, see Lemma 15.45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,s} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 15.45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 15.49.2. We conclude that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i)$ is regular over $\kappa(\mathfrak{p})$ by Lemma 15.41.4 as $R^\wedge \otimes_R \kappa(\mathfrak{p})$ is regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{r}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.166.6 that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i)$ is geometrically regular over $\kappa(\mathfrak{r}_i)$. □

07PU **Lemma 15.50.9.** *Let p be a prime number. Let A be a Noetherian complete local domain with fraction field K of characteristic p. Let $\mathfrak{q} \subset A[x]$ be a maximal ideal lying over the maximal ideal of A and let $(0) \neq \mathfrak{r} \subset \mathfrak{q}$ be a prime lying over $(0) \subset A$. Then $A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(\mathfrak{r})$.*

Proof. Note that $K \subset \kappa(\mathfrak{r})$ is finite. Hence, given a finite purely inseparable extension $L/\kappa(\mathfrak{r})$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(\mathfrak{r}) \otimes_A B$ surjects onto L. Namely, you take $B \subset L$ a finite A-subalgebra whose field of fractions is L. Denote $\mathfrak{r}' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_A B \to \kappa(\mathfrak{r}) \otimes_A B \to L$ so that $\kappa(\mathfrak{r}') = L$. Then

$$A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} L = A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\mathfrak{r}') = \prod B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the primes of $B[x]$ lying over $\mathfrak{q}$, see Algebra, Lemma 10.97.8. Thus we see that it suffices to prove the rings $B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$ are regular. This reduces us to showing that $A[x]^\wedge_{\mathfrak{q}} \otimes_{A[x]} \kappa(\mathfrak{r})$ is regular in the special case that $K = \kappa(\mathfrak{r})$.

Assume $K = \kappa(\mathfrak{r})$. In this case we see that $\mathfrak{r}K[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]^\wedge_{\mathfrak{q}} \otimes_{A[x]} \kappa(\mathfrak{r}) = (A[x]^\wedge_{\mathfrak{q}} \otimes_A K)/(x - f)$$

The derivation $D = \mathrm{d}/\mathrm{d}x$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover D extends to $A[x]^\wedge_{\mathfrak{q}} \otimes_A K$ by Lemma 15.48.1. As $A \to A[x]^\wedge_{\mathfrak{q}}$ is formally smooth (see Lemmas 15.37.2 and 15.37.4) the ring $A[x]^\wedge_{\mathfrak{q}} \otimes_A K$ is regular by Proposition 15.49.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 15.48.2. □

07PV **Proposition 15.50.10.** *Let R be a G-ring. If $R \to S$ is essentially of finite type then S is a G-ring.*

Proof. Since being a G-ring is a property of the local rings it is clear that a localization of a G-ring is a G-ring. Conversely, if every localization at a prime is a G-ring, then the ring is a G-ring. Thus it suffices to show that $S_{\mathfrak{q}}$ is a G-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 15.50.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a G-ring. By induction on n it suffices to prove that $R[x]$ is a G-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 15.50.7 it suffices to show that

$$R[x]_{\mathfrak{q}} \longrightarrow R[x]^\wedge_{\mathfrak{q}}$$

is regular. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_{\mathfrak{p}}$. Hence we may assume that R is a Noetherian local G-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{array}{ccc} R[x]^\wedge_{\mathfrak{q}} & \longrightarrow & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ \uparrow & & \uparrow \\ R[x]_{\mathfrak{q}} & \longrightarrow & R^\wedge[x]_{\mathfrak{q}'} \end{array}$$

Since R is a G-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_{\mathfrak{q}} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 15.41.7. Hence we see that we may assume R is a Noetherian complete local ring and $\mathfrak{q}$ a prime lying over the maximal ideal of R.

Let R be a Noetherian complete local ring and let $\mathfrak{q} \subset R[x]$ be a maximal ideal lying over the maximal ideal of R. Let $\mathfrak{r} \subset \mathfrak{q}$ be a prime ideal. We want to show that $R[x]^\wedge_{\mathfrak{q}} \otimes_{R[x]} \kappa(\mathfrak{r})$ is a geometrically regular algebra over $\kappa(\mathfrak{r})$. Set $\mathfrak{p} = R \cap \mathfrak{r}$. Then we can replace R by $R/\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{r}$ by their images in $R/\mathfrak{p}[x]$, see Lemma 15.50.2. Hence we may assume that R is a domain and that $\mathfrak{r} \cap R = (0)$.

By Algebra, Lemma 10.160.11 we can find $R_0 \subset R$ which is regular and such that R is finite over R_0. Applying Lemma 15.50.3 we see that it suffices to prove

$R[x]^\wedge_{\mathfrak{q}} \otimes_{R[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(r)$ when, in addition to the above, R is a regular complete local ring.

Now R is a regular complete local ring, we have $\mathfrak{q} \subset \mathfrak{r} \subset R[x]$, we have $(0) = R \cap \mathfrak{r}$ and $\mathfrak{q}$ is a maximal ideal lying over the maximal ideal of R. Since R is regular the ring $R[x]$ is regular (Algebra, Lemma 10.163.10). Hence the localization $R[x]_{\mathfrak{q}}$ is regular. Hence the completions $R[x]^\wedge_{\mathfrak{q}}$ are regular, see Lemma 15.43.4. Hence the fibre $R[x]^\wedge_{\mathfrak{q}} \otimes_{R[x]} \kappa(\mathfrak{r})$ is, as a localization of $R[x]^\wedge_{\mathfrak{q}}$, also regular. Thus we are done if the characteristic of the fraction field of R is 0.

If the characteristic of R is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

(1) The case $\mathfrak{r} = (0)$. The result is a direct consequence of Lemma 15.50.5.
(2) The case $\mathfrak{r} \neq (0)$. This is Lemma 15.50.9.

□

07PW **Remark 15.50.11.** Let R be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the I-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [**Nis81**]. A generalization and, in some sense, clarification of this example can be found in the last section of [**Dum00**].

07PX **Proposition 15.50.12.** *The following types of rings are G-rings:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, $\mathbf{Z}$ and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 15.50.6. The statement on finite type overrings is Proposition 15.50.10. □

0A41 **Lemma 15.50.13.** *Let $(A, \mathfrak{m})$ be a henselian local ring. Then A is a filtered colimit of a system of henselian local G-rings with local transition maps.*

Proof. Write $A = \operatorname{colim} A_i$ as a filtered colimit of finite type $\mathbf{Z}$-algebras. Let $\mathfrak{p}_i$ be the prime ideal of A_i lying under $\mathfrak{m}$. We may replace A_i by the localization of A_i at $\mathfrak{p}_i$. Then A_i is a Noetherian local G-ring (Proposition 15.50.12). By Lemma 15.12.5 we see that $A = \operatorname{colim} A_i^h$. By Lemma 15.50.8 the rings A_i^h are G-rings. □

0AH2 **Lemma 15.50.14.** *Let A be a G-ring. Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then $A \to A^\wedge$ is regular.* [**Mat70a**, Theorem 79]

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 10.97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 10.97.6. Thus it suffices to check the third condition of Lemma 15.41.2. Let $\mathfrak{m}' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak{m} \subset A$. By Algebra, Lemma 10.96.6 we have $IA^\wedge \subset \mathfrak{m}'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak{m}$, $\mathfrak{m}/I = \mathfrak{m}'/IA^\wedge$, and $A/\mathfrak{m} = A^\wedge/\mathfrak{m}'$. Since $A^\wedge/\mathfrak{m}'$ is a field, we conclude that $\mathfrak{m}$ is a maximal ideal as well. Then $A_{\mathfrak{m}} \to A^\wedge_{\mathfrak{m}'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak{m}A^\wedge_{\mathfrak{m}'} = \mathfrak{m}'A^\wedge_{\mathfrak{m}'}$.

Thus it induces an isomorphism on complete local rings, see Lemma 15.43.9. Let $(A_{\mathfrak{m}})^\wedge$ be the completion of $A_{\mathfrak{m}}$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak{m}'} \to ((A^\wedge)_{\mathfrak{m}'})^\wedge = (A_{\mathfrak{m}})^\wedge$$

is faithfully flat (Algebra, Lemma 10.97.3). Thus we can apply Lemma 15.41.7 to the ring maps

$$A_{\mathfrak{m}} \to (A^\wedge)_{\mathfrak{m}'} \to (A_{\mathfrak{m}})^\wedge$$

to conclude because $A_{\mathfrak{m}} \to (A_{\mathfrak{m}})^\wedge$ is regular as A is a G-ring. □

0AH3 **Lemma 15.50.15.** *Let A be a G-ring. Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 15.12.1. Then A^h is a G-ring.* [**Gre76**, Theorem 5.3 i)]

Proof. Let $\mathfrak{m}^h \subset A^h$ be a maximal ideal. We have to show that the map from $A^h_{\mathfrak{m}^h}$ to its completion has geometrically regular fibres, see Lemma 15.50.7. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}^h$ in A. Note that $I^h \subset \mathfrak{m}^h$ and hence $I \subset \mathfrak{m}$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 15.12.4. Then the local homomorphism of Noetherian local rings

$$A_{\mathfrak{m}} \to A^h_{\mathfrak{m}^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 15.43.9 (details omitted). Let $\mathfrak{q}^h$ be a prime of $A^h_{\mathfrak{m}^h}$ lying over $\mathfrak{q} \subset A_{\mathfrak{m}}$. Set $\mathfrak{q}_1 = \mathfrak{q}^h$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of A^h lying over $\mathfrak{q}$, so that $A^h \otimes_A \kappa(\mathfrak{q}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.12. Using that $(A^h)^\wedge_{\mathfrak{m}^h} = (A_{\mathfrak{m}})^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}_i) = (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} (A^h_{\mathfrak{m}^h} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})) = (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$$

Hence, as one of the components, the ring

$$(A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

is geometrically regular over $\kappa(\mathfrak{q})$ by assumption on A. Since $\kappa(\mathfrak{q}^h)$ is separable algebraic over $\kappa(\mathfrak{q})$ it follows from Algebra, Lemma 10.166.6 that

$$(A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

is geometrically regular over $\kappa(\mathfrak{q}^h)$ as desired. □

15.51. Properties of formal fibres

0BIR In this section we redo some of the arguments of Section 15.50 for to be able to talk intelligently about properties of the formal fibres of Noetherian rings.

Let P be a property of ring maps $k \to R$ where k is a field and R is Noetherian. We say P holds for the fibres of a ring homomorphism $A \to B$ with B Noetherian if P holds for $\kappa(\mathfrak{q}) \to B \otimes_A \kappa(\mathfrak{q})$ for all primes $\mathfrak{q}$ of A. In the following we will use the following assertions

(A) $P(k \to R) \Rightarrow P(k' \to R \otimes_k k')$ for finitely generated field extensions k'/k,
(B) $P(k \to R_{\mathfrak{p}})$, $\forall \mathfrak{p} \in \mathrm{Spec}(R) \Leftrightarrow P(k \to R)$,
(C) given flat maps $A \to B \to C$ of Noetherian rings, if the fibres of $A \to B$ have P and $B \to C$ is regular, then the fibres of $A \to C$ have P,
(D) given flat maps $A \to B \to C$ of Noetherian rings if the fibres of $A \to C$ have P and $B \to C$ is faithfully flat, then the fibres of $A \to B$ have P,

(E) given $k \to k' \to R$ with R Noetherian if k'/k is separable algebraic and $P(k \to R)$, then $P(k' \to R)$, and
(F) add more here.

Given a Noetherian local ring A we say "*the formal fibres of A have P*" if P holds for the fibres of $A \to A^\wedge$. We say that R *is a P-ring* if R is Noetherian and for all primes $\mathfrak{p}$ of R the formal fibres of $R_\mathfrak{p}$ have P.

0BIS **Lemma 15.51.1.** *Let R be a Noetherian ring. Let P be a property as above. Then R is a P-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the $\kappa(\mathfrak{q})$-algebra*

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

has property P.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

0BK8 **Lemma 15.51.2.** *Let $R \to \Lambda$ be a homomorphism of Noetherian rings. Assume P has property (B). The following are equivalent*

(1) *the fibres of $R \to \Lambda$ have P,*
(2) *the fibres of $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ have P for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$, and*
(3) *the fibres of $R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ have P for all maximal ideals $\mathfrak{m}' \subset \Lambda$ lying over $\mathfrak{m}$ in R.*

Proof. Let $\mathfrak{p} \subset R$ be a prime. Then the fibre over $\mathfrak{p}$ is the ring $\Lambda \otimes_R \kappa(\mathfrak{p})$ whose spectrum maps bijectively onto the subset of $\mathrm{Spec}(\Lambda)$ consisting of primes $\mathfrak{q}$ lying over $\mathfrak{p}$, see Algebra, Remark 10.18.5. For such a prime $\mathfrak{q}$ choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m}'$ and set $\mathfrak{m} = R \cap \mathfrak{m}'$. Then $\mathfrak{p} \subset \mathfrak{m}$ and we have

$$(\Lambda \otimes_R \kappa(\mathfrak{p}))_\mathfrak{q} \cong (\Lambda_{\mathfrak{m}'} \otimes_{R_\mathfrak{m}} \kappa(\mathfrak{p}))_\mathfrak{q}$$

as $\kappa(\mathfrak{q})$-algebras. Thus (1), (2), and (3) are equivalent because by (B) we can check property P on local rings. □

0BIT **Lemma 15.51.3.** *Let $R \to R'$ be a finite type map of Noetherian rings and let*

$$\begin{array}{ccccc} \mathfrak{q}' & \longrightarrow & \mathfrak{p}' & \longrightarrow & R' \\ \downarrow & & \downarrow & & \uparrow \\ \mathfrak{q} & \longrightarrow & \mathfrak{p} & \longrightarrow & R \end{array}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$. Assume P satisfies (A) and (B).

(1) *If $\kappa(\mathfrak{q}) \to R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ has P, then $\kappa(\mathfrak{q}') \to R'_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ has P.*
(2) *If the formal fibres of $R_\mathfrak{p}$ have P, then the formal fibres of $R'_{\mathfrak{p}'}$ have P.*
(3) *If $R \to R'$ is quasi-finite and R is a P-ring, then R' is a P-ring.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume P holds for $\kappa(\mathfrak{q}) \to R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$. By Algebra, Lemma 10.124.3 we see that

$$R^\wedge_\mathfrak{p} \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R'^{\wedge}_{\mathfrak{p}}$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^{\wedge}$ is a factor of a base change of the map $R_{\mathfrak{p}} \to R^{\wedge}_{\mathfrak{p}}$. It follows that $(R'_{\mathfrak{p}'})^{\wedge} \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R^{\wedge}_{\mathfrak{p}} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R^{\wedge}_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows from the assumptions on P. □

0BIU **Lemma 15.51.4.** *Let R be a Noetherian ring. Assume P satisfies (C) and (D). Then R is a P-ring if and only if the formal fibres of $R_{\mathfrak{m}}$ have P for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume the formal fibres of $R_{\mathfrak{m}}$ have P for all maximal ideals $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_{\mathfrak{m}} \to R^{\wedge}_{\mathfrak{m}}$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R^{\wedge}_{\mathfrak{m}}$ lying over $\mathfrak{p}R_{\mathfrak{m}}$. Consider the commutative diagram

$$\begin{array}{ccccc} R^{\wedge}_{\mathfrak{m}} & \longrightarrow & (R^{\wedge}_{\mathfrak{m}})_{\mathfrak{p}'} & \longrightarrow & (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'} \\ \uparrow & & \uparrow & & \uparrow \\ R_{\mathfrak{m}} & \longrightarrow & R_{\mathfrak{p}} & \longrightarrow & R^{\wedge}_{\mathfrak{p}} \end{array}$$

By assumption the fibres of the ring map $R_{\mathfrak{m}} \to R^{\wedge}_{\mathfrak{m}}$ have P. By Proposition 15.50.6 $(R^{\wedge}_{\mathfrak{m}})_{\mathfrak{p}'} \to (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'}$ is regular. The localization $R^{\wedge}_{\mathfrak{m}} \to (R^{\wedge}_{\mathfrak{m}})_{\mathfrak{p}'}$ is regular. Hence $R^{\wedge}_{\mathfrak{m}} \to (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'}$ is regular by Lemma 15.41.4. Hence the fibres of $R_{\mathfrak{m}} \to (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'}$ have P by (C). Since $R_{\mathfrak{m}} \to (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'}$ factors through the localization $R_{\mathfrak{p}}$, also the fibres of $R_{\mathfrak{p}} \to (R^{\wedge}_{\mathfrak{m}})^{\wedge}_{\mathfrak{p}'}$ have P. Thus we may apply (D) to see that the fibres of $R_{\mathfrak{p}} \to R^{\wedge}_{\mathfrak{p}}$ have P. □

0BIV **Proposition 15.51.5.** *Let R be a P-ring where P satisfies (A), (B), (C), and (D). If $R \to S$ is essentially of finite type then S is a P-ring.*

Proof. Since being a P-ring is a property of the local rings it is clear that a localization of a P-ring is a P-ring. Conversely, if every localization at a prime is a P-ring, then the ring is a P-ring. Thus it suffices to show that $S_{\mathfrak{q}}$ is a P-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 15.51.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a P-ring. By induction on n it suffices to prove that $R[x]$ is a P-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 15.51.4 it suffices to show that the fibres of

$$R[x]_{\mathfrak{q}} \longrightarrow R[x]^{\wedge}_{\mathfrak{q}}$$

have P. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_{\mathfrak{p}}$. Hence we may assume that R is a Noetherian local P-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^{\wedge}[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{array}{ccc} R[x]^{\wedge}_{\mathfrak{q}} & \longrightarrow & (R^{\wedge}[x]_{\mathfrak{q}'})^{\wedge} \\ \uparrow & & \uparrow \\ R[x]_{\mathfrak{q}} & \longrightarrow & R^{\wedge}[x]_{\mathfrak{q}'} \end{array}$$

Since R is a P-ring the fibres of $R[x] \to R^{\wedge}[x]$ have P because they are base changes of the fibres of $R \to R^{\wedge}$ by a finitely generated field extension so (A) applies. Hence the fibres of the lower horizontal arrow have P for example by Lemma 15.51.2. The right vertical arrow is regular because $R^{\wedge}$ is a G-ring (Propositions 15.50.6 and

15.50.10). It follows that the fibres of the composition $R[x]_{\mathfrak{q}} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ have P by (C). Hence the fibres of the left vertical arrow have P by (D) and the proof is complete. □

0BK9 **Lemma 15.51.6.** *Let A be a P-ring where P satisfies (B) and (D). Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then the fibres of $A \to A^\wedge$ have P.*

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 10.97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 10.97.6. Thus it suffices to check the third condition of Lemma 15.51.2. Let $\mathfrak{m}' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak{m} \subset A$. By Algebra, Lemma 10.96.6 we have $IA^\wedge \subset \mathfrak{m}'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak{m}$, $\mathfrak{m}/I = \mathfrak{m}'/IA^\wedge$, and $A/\mathfrak{m} = A^\wedge/\mathfrak{m}'$. Since $A^\wedge/\mathfrak{m}'$ is a field, we conclude that $\mathfrak{m}$ is a maximal ideal as well. Then $A_{\mathfrak{m}} \to A^\wedge_{\mathfrak{m}'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak{m}A^\wedge_{\mathfrak{m}'} = \mathfrak{m}'A^\wedge_{\mathfrak{m}'}$. Thus it induces an isomorphism on complete local rings, see Lemma 15.43.9. Let $(A_{\mathfrak{m}})^\wedge$ be the completion of $A_{\mathfrak{m}}$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak{m}'} \to ((A^\wedge)_{\mathfrak{m}'})^\wedge = (A_{\mathfrak{m}})^\wedge$$

is faithfully flat (Algebra, Lemma 10.97.3). Thus we can apply (D) to the ring maps

$$A_{\mathfrak{m}} \to (A^\wedge)_{\mathfrak{m}'} \to (A_{\mathfrak{m}})^\wedge$$

to conclude because the fibres of $A_{\mathfrak{m}} \to (A_{\mathfrak{m}})^\wedge$ have P as A is a P-ring. □

0BKA **Lemma 15.51.7.** *Let A be a P-ring where P satisfies (B), (C), (D), and (E). Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 15.12.1. Then A^h is a P-ring.*

Proof. Let $\mathfrak{m}^h \subset A^h$ be a maximal ideal. We have to show that the fibres of $A^h_{\mathfrak{m}^h} \to (A^h_{\mathfrak{m}^h})^\wedge$ have P, see Lemma 15.51.4. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}^h$ in A. Note that $I^h \subset \mathfrak{m}^h$ and hence $I \subset \mathfrak{m}$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 15.12.4. Then the local homomorphism of Noetherian local rings

$$A_{\mathfrak{m}} \to A^h_{\mathfrak{m}^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 15.43.9 (details omitted). Let $\mathfrak{q}^h$ be a prime of $A^h_{\mathfrak{m}^h}$ lying over $\mathfrak{q} \subset A_{\mathfrak{m}}$. Set $\mathfrak{q}_1 = \mathfrak{q}^h$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of A^h lying over $\mathfrak{q}$, so that $A^h \otimes_A \kappa(\mathfrak{q}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.12. Using that $(A^h)^\wedge_{\mathfrak{m}^h} = (A_{\mathfrak{m}})^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}_i) = (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} (A^h_{\mathfrak{m}^h} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})) = (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$$

Hence, looking at local rings and using (B), we see that

$$\kappa(\mathfrak{q}) \longrightarrow (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

has P as $\kappa(\mathfrak{q}) \to (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$ does by assumption on A. Since $\kappa(\mathfrak{q}^h)/\kappa(\mathfrak{q})$ is separable algebraic, by (E) we find that $\kappa(\mathfrak{q}^h) \to (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$ has P as desired. □

0C36 **Lemma 15.51.8.** *Let R be a Noetherian local ring which is a P-ring where P satisfies (B), (C), (D), and (E). Then the henselization R^h and the strict henselization R^{sh} are P-rings.*

Proof. We have seen this for the henselization in Lemma 15.51.7. To prove it for the strict henselization, it suffices to show that the formal fibres of R^{sh} have P, see Lemma 15.51.4. Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$ and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$, see Lemma 15.45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,t} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 15.45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 15.49.2. We conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$ by (C) and our assumption on R. Using property (B), using the decomposition above, and looking at local rings we conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. Since $\kappa(\mathfrak{r})/\kappa(\mathfrak{p})$ is separable algebraic, it follows from (E) that P holds for $\kappa(\mathfrak{r}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. □

0BIW **Lemma 15.51.9.** *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically reduced over k".*

Proof. Part (A) follows from the definition of geometrically reduced algebras (Algebra, Definition 10.43.1). Part (B) follows too: a ring is reduced if and only if all local rings are reduced. Part (C). This follows from Lemma 15.42.1. Part (D). This follows from Algebra, Lemma 10.164.2. Part (E). This follows from Algebra, Lemma 10.43.9. □

0BIX **Lemma 15.51.10.** *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically normal over k".*

Proof. Part (A) follows from the definition of geometrically normal algebras (Algebra, Definition 10.165.2). Part (B) follows too: a ring is normal if and only if all of its local rings are normal. Part (C). This follows from Lemma 15.42.2. Part (D). This follows from Algebra, Lemma 10.164.3. Part (E). This follows from Algebra, Lemma 10.165.6. □

0BIY **Lemma 15.51.11.** *Fix $n \geq 1$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R has (S_n)".*

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Let k'/k be a finitely generated field extension. Then the fibres of the ring map $R \to R \otimes_k k'$ are Cohen-Macaulay by Algebra, Lemma 10.167.1. Hence we may apply Algebra, Lemma 10.163.4 to the ring map $R \to R \otimes_k k'$ to see that if R has (S_n) so does $R \otimes_k k'$. This proves (A). Part (B) follows too: a Noetherian rings has (S_n) if and only if all of its local rings have (S_n). Part (C). This follows from Algebra, Lemma 10.163.4 as the fibres of a regular homomorphism are regular and in particular Cohen-Macaulay. Part (D). This follows from Algebra, Lemma 10.164.5. Part (E). This is immediate as the condition does not refer to the ground field. □

0BJ9 **Lemma 15.51.12.** *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is Cohen-Macaulay".*

Proof. Follows immediately from Lemma 15.51.11 and the fact that a Noetherian ring is Cohen-Macaulay if and only if it satisfies conditions (S_n) for all n. □

0BIZ **Lemma 15.51.13.** *Fix $n \geq 0$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"$R \otimes_k k'$ has (R_n) for all finite extensions k'/k".*

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Assume $P(k \to R)$ is true. Let K/k be a finitely generated field extension. By Algebra, Lemma 10.45.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Algebra, Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $R \otimes_k k'$ satisfies (S_n) because we assumed P for $k \to R$. Step 2: $R \otimes_k k' \to R \otimes_k k' \otimes_{k'} B$ is a smooth ring map (Algebra, Lemma 10.137.3) and we conclude $R \otimes_k k' \otimes_{k'} B$ satisfies (S_n) by Algebra, Lemma 10.163.5 (and using Algebra, Lemma 10.140.3 to see that the hypotheses are satisfied). Step 3. $R \otimes_k k' \otimes_{k'} K' = R \otimes_k K'$ satisfies (R_n) as it is a localization of a ring having (R_n). Step 4. Finally $R \otimes_k K$ satisfies (R_n) by descent of (R_n) along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Algebra, Lemma 10.164.6). This proves (A). Part (B) follows too: a Noetherian ring has (R_n) if and only if all of its local rings have (R_n). Part (C). This follows from Algebra, Lemma 10.163.5 as the fibres of a regular homomorphism are regular (small detail omitted). Part (D). This follows from Algebra, Lemma 10.164.6 (small detail omitted).

Part (E). Let l/k be a separable algebraic extension of fields and let $l \to R$ be a ring map with R Noetherian. Assume that $k \to R$ has P. We have to show that $l \to R$ has P. Let l'/l be a finite extension. First observe that there exists a finite subextension $l/m/k$ and a finite extension m'/m such that $l' = l \otimes_m m'$. Then $R \otimes_l l' = R \otimes_m m'$. Hence it suffices to prove that $m \to R$ has property P, i.e., we may assume that l/k is finite. If l/k is finite, then l'/k is finite and we see that

$$l' \otimes_l R = (l' \otimes_k R) \otimes_{l \otimes_k l} l$$

is a localization (by Algebra, Lemma 10.43.8) of the Noetherian ring $l' \otimes_k R$ which has property (R_n) by assumption P for $k \to R$. This proves that $l' \otimes_l R$ has property (R_n) as desired. □

15.52. Excellent rings

07QS In this section we discuss Grothendieck's notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 15.50.1, Definition 15.47.1, and Algebra, Definition 10.105.3.

07QT **Definition 15.52.1.** Let R be a ring.

(1) We say R is *quasi-excellent* if R is Noetherian, a G-ring, and J-2.

(2) We say R is *excellent* if R is quasi-excellent and universally catenary.

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function. We will see later (Section 15.110) that to be universally catenary can be formulated as a condition on the maps $R_{\mathfrak{m}} \to R^\wedge_{\mathfrak{m}}$ for maximal ideals $\mathfrak{m}$ of R.

07QU **Lemma 15.52.2.** *Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.*

Proof. For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 15.50.10. We omit the proof that localization preserves (quasi-)excellency. □

07QW **Proposition 15.52.3.** *The following types of rings are excellent:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. See Propositions 15.50.12 and 15.48.7 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 10.105.9. In particular fields, Dedekind rings, and more generally regular rings are universally catenary. Via the Cohen structure theorem we see that complete local rings are universally catenary, see Algebra, Remark 10.160.9. □

The material developed above has some consequences for Nagata rings.

0BJ0 **Lemma 15.52.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent*

(1) *A is Nagata, and*
(2) *the formal fibres of A are geometrically reduced.*

Proof. Assume (2). By Algebra, Lemma 10.162.14 we have to show that if $A \to B$ is finite, B is a domain, and $\mathfrak{m}' \subset B$ is a maximal ideal, then $B_{\mathfrak{m}'}$ is analytically unramified. Combining Lemmas 15.51.9 and 15.51.4 and Proposition 15.51.5 we see that the formal fibres of $B_{\mathfrak{m}'}$ are geometrically reduced. In particular $B^\wedge_{\mathfrak{m}'} \otimes_B L$ is reduced where L is the fraction field of B. It follows that $B^\wedge_{\mathfrak{m}'}$ is reduced, i.e., $B_{\mathfrak{m}'}$ is analytically unramified.

Assume (1). Let $\mathfrak{q} \subset A$ be a prime ideal and let $K/\kappa(\mathfrak{q})$ be a finite extension. We have to show that $A^\wedge \otimes_A K$ is reduced. Let $A/\mathfrak{q} \subset B \subset K$ be a local subring finite over A whose fraction field is K. To construct B choose $x_1, \ldots, x_n \in K$ which generate K over $\kappa(\mathfrak{q})$ and which satisfy monic polynomials $P_i(T) = T^{d_i} + a_{i,1}T^{d_i-1} + \ldots + a_{i,d_i} = 0$ with $a_{i,j} \in \mathfrak{m}$. Then let B be the A-subalgebra of K generated by $x_1, \ldots, x_n$. (For more details see the proof of Algebra, Lemma 10.162.14.) Then

$$A^\wedge \otimes_A K = (A^\wedge \otimes_A B)_{\mathfrak{q}} = B^\wedge_{\mathfrak{q}}$$

Since $B^\wedge$ is reduced by Algebra, Lemma 10.162.14 the proof is complete. □

07QV **Lemma 15.52.5.** *A quasi-excellent ring is Nagata.*

Proof. Let R be quasi-excellent. Using that a finite type algebra over R is quasi-excellent (Lemma 15.52.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 10.162.3. Applying Algebra, Lemma 10.161.15 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain R is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 15.42.1. In other words, R is analytically unramified. Hence R is N-1 by Algebra, Lemma 10.162.10. □

0C23 **Lemma 15.52.6.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is normal and the formal fibres of A are normal (for example if A is excellent or quasi-excellent), then $A^\wedge$ is normal.*

Proof. Follows immediately from Algebra, Lemma 10.163.8. □

15.53. Abelian categories of modules

0AZ5 Let R be a ring. The category Mod_R of R-modules is an abelian category. Here are some examples of subcategories of Mod_R which are abelian (we use the terminology introduced in Homology, Definition 12.10.1 as well as Homology, Lemmas 12.10.2 and 12.10.3):

(1) The category of coherent R-modules is a weak Serre subcategory of Mod_R. This follows from Algebra, Lemma 10.90.3.
(2) Let $S \subset R$ be a multiplicative subset. The full subcategory consisting of R-modules M such that multiplication by $s \in S$ is an isomorphism on M is a Serre subcategory of Mod_R. This follows from Algebra, Lemma 10.9.5.
(3) Let $I \subset R$ be a finitely generated ideal. The full subcategory of I-power torsion modules is a Serre subcategory of Mod_R. See Lemma 15.89.5.
(4) In some texts a *torsion module* is defined as a module M such that for all $x \in M$ there exists a nonzerodivisor $f \in R$ such that $fx = 0$. The full subcategory of torsion modules is a Serre subcategory of Mod_R.
(5) If R is not Noetherian, then the category Mod_R^{fg} of finitely generated R-modules is **not** abelian. Namely, if $I \subset R$ is a non-finitely generated ideal, then the map $R \to R/I$ does not have a kernel in Mod_R^{fg}.
(6) If R is Noetherian, then coherent R-modules agree with finitely generated (i.e., finite) R-modules, see Algebra, Lemmas 10.90.5, 10.90.4, and 10.31.4. Hence Mod_R^{fg} is abelian by (1) above, but in fact,in this case the category Mod_R^{fg} is a (strong) Serre subcategory of Mod_R.

15.54. Injective abelian groups

01D6 In this section we show the category of abelian groups has enough injectives. Recall that an abelian group M is *divisible* if and only if for every $x \in M$ and every $n \in \mathbf{N}$ there exists a $y \in M$ such that $ny = x$.

01D7 **Lemma 15.54.1.** *An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.*

Proof. Suppose that J is not divisible. Then there exists an $x \in J$ and $n \in \mathbf{N}$ such that there is no $y \in J$ with $ny = x$. Then the morphism $\mathbf{Z} \to J$, $m \mapsto mx$ does not extend to $\frac{1}{n}\mathbf{Z} \supset \mathbf{Z}$. Hence J is not injective.

Let $A \subset B$ be abelian groups. Assume that J is a divisible abelian group. Let $\varphi : A \to J$ be a morphism. Consider the set of homomorphisms $\varphi' : A' \to J$ with $A \subset A' \subset B$ and $\varphi'|_A = \varphi$. Define $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A' \supset A''$ and $\varphi'|_{A''} = \varphi''$. If $(A_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} A_i \to J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (A', φ') is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \to J$, then we can find $\varphi' : A' \to J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbf{N}$ such that $nx \in A$, then $A \oplus \mathbf{Z} \cong A + \mathbf{Z}x$. Hence we can extend φ to $A' = A + \mathbf{Z}x$ by using φ on A and mapping x to zero for example. If there does exist an $n \in \mathbf{N}$ such that $nx \in A$, then let n be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$. Define a morphism $\tilde{\varphi} : A \oplus \mathbf{Z} \to J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of z the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbf{Z} \to B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbf{Z}x$, and this extends the morphism φ. □

We can use this lemma to show that every abelian group can be embedded in an injective abelian group. But this is a special case of the result of the following section.

15.55. Injective modules

01D8 Some lemmas on injective modules.

0AVD **Definition 15.55.1.** Let R be a ring. An R-module J is *injective* if and only if the functor $\mathrm{Hom}_R(-, J) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(-, M)$ is left exact for any R-module M, see Algebra, Lemma 10.10.1. Hence the condition for J to be injective really signifies that given an injection of R-modules $M \to M'$ the map $\mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M, J)$ is surjective.

Before we reformulate this in terms of *Ext*-modules we discuss the relationship between $\mathrm{Ext}^1_R(M, N)$ and extensions as in Homology, Section 12.6.

0AUL **Lemma 15.55.2.** *Let R be a ring. Let $\mathcal{A}$ be the abelian category of R-modules. There is a canonical isomorphism $\mathrm{Ext}_{\mathcal{A}}(M, N) = \mathrm{Ext}^1_R(M, N)$ compatible with the long exact sequences of Algebra, Lemmas 10.71.6 and 10.71.7 and the 6-term exact sequences of Homology, Lemma 12.6.4.*

Proof. Omitted. □

0AVE **Lemma 15.55.3.** *Let R be a ring. Let J be an R-module. The following are equivalent*

(1) *J is injective,*
(2) *$\mathrm{Ext}^1_R(M, J) = 0$ for every R-module M.*

Proof. Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence of R-modules. Consider the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, J) \to \mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M'', J) \\ \to \mathrm{Ext}^1_R(M, J) \to \mathrm{Ext}^1_R(M', J) \to \mathrm{Ext}^1_R(M'', J) \to \ldots \end{aligned}$$

of Algebra, Lemma 10.71.7. Thus we see that (2) implies (1). Conversely, if J is injective then the Ext-group is zero by Homology, Lemma 12.27.2 and Lemma 15.55.2. □

0AVF **Lemma 15.55.4.** *Let R be a ring. Let J be an R-module. The following are equivalent*

(1) *J is injective,*
(2) *$\mathrm{Ext}^1_R(R/I, J) = 0$ for every ideal $I \subset R$, and*
(3) *for an ideal $I \subset R$ and module map $I \to J$ there exists an extension $R \to J$.*

Proof. If $I \subset R$ is an ideal, then the short exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence

$$\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J) \to \mathrm{Ext}^1_R(R/I, J) \to 0$$

by Algebra, Lemma 10.71.7 and the fact that $\mathrm{Ext}^1_R(R, J) = 0$ as R is projective (Algebra, Lemma 10.77.2). Thus (2) and (3) are equivalent. In this proof we will show that (1) $\Leftrightarrow$ (3) which is known as Baer's criterion.

Assume (1). Given a module map $I \to J$ as in (3) we find the extension $R \to J$ because the map $\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J)$ is surjective by definition.

Assume (3). Let $M \subset N$ be an inclusion of R-modules. Let $\varphi : M \to J$ be a homomorphism. We will show that φ extends to N which finishes the proof of the lemma. Consider the set of homomorphisms $\varphi' : M' \to J$ with $M \subset M' \subset N$ and $\varphi'|_M = \varphi$. Define $(M', \varphi') \geq (M'', \varphi'')$ if and only if $M' \supset M''$ and $\varphi'|_{M''} = \varphi''$. If $(M_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} M_i \to J$ defined by $a \in M_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (M', φ') is maximal then $M' = N$. In other words, it suffices to show, given any subgroup $M \subset N$, $M \neq N$ and any $\varphi : M \to J$, then we can find $\varphi' : M' \to J$ with $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in N$, $x \notin M$. Let $I = \{f \in R \mid fx \in M\}$. This is an ideal of R. Define a homomorphism $\psi : I \to J$ by $f \mapsto \varphi(fx)$. Extend to a map $\tilde{\psi} : R \to J$ which is possible by assumption (3). By our choice of I the kernel of $M \oplus R \to J$, $(y, f) \mapsto \varphi(y) + \tilde{\psi}(f)$ contains the kernel of the map $M \oplus R \to N$, $(y, f) \mapsto y + fx$. Hence this homomorphism factors through the image $M' = M + Rx$ and this extends the given homomorphism as desired. □

In the rest of this section we prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This follows from Lemma 15.54.1.

01D9 **Definition 15.55.5.** Let R be a ring.

(1) For any R-module M over R we denote $M^\vee = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. We think of $M \mapsto M^\vee$ as a contravariant functor from the category of R-modules to itself.
(2) For any R-module M we denote

$$F(M) = \bigoplus\nolimits_{m \in M} R[m]$$

the *free module* with basis given by the elements $[m]$ with $m \in M$. We let $F(M) \to M$, $\sum f_i[m_i] \mapsto \sum f_i m_i$ be the natural surjection of R-modules. We think of $M \mapsto (F(M) \to M)$ as a functor from the category of R-modules to the category of arrows in R-modules.

01DA **Lemma 15.55.6.** *Let R be a ring. The functor $M \mapsto M^\vee$ is exact.*

Proof. This because $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group by Lemma 15.54.1. □

There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = \mathrm{Hom}(M^\vee, \mathbf{Q}/\mathbf{Z})$ be the map $\varphi \mapsto \varphi(x)$.

01DB **Lemma 15.55.7.** *For any R-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.*

Proof. You can check this using that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbf{Q}/\mathbf{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbf{Q}/\mathbf{Z}$ and then $ev(x)(\varphi) = \varphi(x) \neq 0$. □

The canonical surjection $F(M) \to M$ of R-modules turns into a canonical injection, see above, of R-modules

$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$

Set $J(M) = (F(M^\vee))^\vee$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

01DC **Lemma 15.55.8.** *Let R be a ring. For every R-module M the R-module $J(M)$ is injective.*

Proof. Note that $J(M) \cong \prod_{\varphi \in M^\vee} R^\vee$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that

$$\mathrm{Hom}_R(N, R^\vee) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and the fact that $(-)^\vee$ is an exact functor by Lemma 15.55.6. □

01DD **Lemma 15.55.9.** *Let R be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.*

Proof. Follows from the above. □

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ J(M) & \longrightarrow & J(N) \end{array}$$

This is the kind of construction we would like to have in general. In Homology, Section 12.27 we introduced terminology to express this. Namely, we say this means that the category of R-modules has functorial injective embeddings.

15.56. Derived categories of modules

0914 In this section we put some generalities concerning the derived category of modules over a ring.

Let A be a ring. The category of A-modules is denoted Mod_A. We will use the symbol $K(A)$ to denote the homotopy category of complexes of A-modules, i.e., we set $K(A) = K(\mathrm{Mod}_A)$ as a category, see Derived Categories, Section 13.8. The bounded versions are $K^+(A)$, $K^-(A)$, and $K^b(A)$. We view $K(A)$ as a triangulated category as in Derived Categories, Section 13.10. The *derived category* of A, denoted $D(A)$, is the category obtained from $K(A)$ by inverting quasi-isomorphisms, i.e., we set $D(A) = D(\mathrm{Mod}_A)$, see Derived Categories, Section 13.11[7]. The bounded versions are $D^+(A)$, $D^-(A)$, and $D^b(A)$.

Let A be a ring. The category of A-modules has products and products are exact. The category of A-modules has enough injectives by Lemma 15.55.9. Hence every complex of A-modules is quasi-isomorphic to a K-injective complex (Derived Categories, Lemma 13.34.6). It follows that $D(A)$ has countable products (Derived Categories, Lemma 13.34.2) and in fact arbitrary products (Injectives, Lemma 19.13.4). This implies that every inverse system of objects of $D(A)$ has a derived limit (well defined up to isomorphism), see Derived Categories, Section 13.34.

0915 **Lemma 15.56.1.** *Let $R \to S$ be a flat ring map. If $I^\bullet$ is a K-injective complex of S-modules, then $I^\bullet$ is K-injective as a complex of R-modules.*

Proof. This is true because $\mathrm{Hom}_{K(R)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(S)}(M^\bullet \otimes_R S, I^\bullet)$ by Algebra, Lemma 10.14.3 and the fact that tensoring with S is exact. □

0916 **Lemma 15.56.2.** *Let $R \to S$ be an epimorphism of rings. Let $I^\bullet$ be a complex of S-modules. If $I^\bullet$ is K-injective as a complex of R-modules, then $I^\bullet$ is a K-injective complex of S-modules.*

Proof. This is true because $\mathrm{Hom}_{K(R)}(N^\bullet, I^\bullet) = \mathrm{Hom}_{K(S)}(N^\bullet, I^\bullet)$ for any complex of S-modules $N^\bullet$, see Algebra, Lemma 10.107.14. □

0917 **Lemma 15.56.3.** *Let $A \to B$ be a ring map. If $I^\bullet$ is a K-injective complex of A-modules, then $\mathrm{Hom}_A(B, I^\bullet)$ is a K-injective complex of B-modules.*

Proof. This is true because $\mathrm{Hom}_{K(B)}(N^\bullet, \mathrm{Hom}_A(B, I^\bullet)) = \mathrm{Hom}_{K(A)}(N^\bullet, I^\bullet)$ by Algebra, Lemma 10.14.4. □

15.57. Computing Tor

064F Let R be a ring. We denote $D(R)$ the derived category of the abelian category Mod_R of R-modules. Note that Mod_R has enough projectives as every free R-module is projective. Thus we can define the left derived functors of any additive functor from Mod_R to any abelian category.

This applies in particular to the functor $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ whose left derived functors are the Tor functors $\mathrm{Tor}_i^R(-, M)$, see Algebra, Section 10.75. There is also a total left derived functor

064G (15.57.0.1) $$- \otimes^{\mathbf{L}}_R M : D^-(R) \longrightarrow D^-(R)$$

[7] See also Injectives, Remark 19.13.3.

which is denoted $- \otimes^{\mathbf{L}}_R M$. Its satellites are the Tor modules, i.e., we have

$$H^{-p}(N \otimes^{\mathbf{L}}_R M) = \mathrm{Tor}^R_p(N, M).$$

A special situation occurs when we consider the tensor product with an R-algebra A. In this case we think of $- \otimes_R A$ as a functor from Mod_R to Mod_A. Hence the total right derived functor

064H (15.57.0.2)
$$- \otimes^{\mathbf{L}}_R A : D^-(R) \longrightarrow D^-(A)$$

which is denoted $- \otimes^{\mathbf{L}}_R A$. Its satellites are the tor groups, i.e., we have

$$H^{-p}(N \otimes^{\mathbf{L}}_R A) = \mathrm{Tor}^R_p(N, A).$$

In particular these Tor groups naturally have the structure of A-modules.

We will generalize the material in this section to unbounded complexes in the next few sections.

15.58. Tensor products of complexes

0GWN Let R be a ring. The category $\mathrm{Comp}(R)$ of complexes of R-modules has a symmetric monoidal structure. Namely, suppose that we have two complexes of R-modules $L^\bullet$ and $M^\bullet$. Using Homology, Example 12.18.2 and Homology, Definition 12.18.3 we obtain a third complex of R-modules, namely

$$\mathrm{Tot}(L^\bullet \otimes_R M^\bullet)$$

Clearly this construction is functorial in both $L^\bullet$ and $M^\bullet$. The associativity constraint will be the canonical isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \longrightarrow \mathrm{Tot}(K^\bullet \otimes_R \mathrm{Tot}(L^\bullet \otimes_R M^\bullet))$$

constructed in Homology, Remark 12.18.4 from the triple complex $K^\bullet \otimes_R L^\bullet \otimes_R M^\bullet$. The commutativity constraint is the canonical isomorphism

$$\mathrm{Tot}(L^\bullet \otimes_R M^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R L^\bullet)$$

which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. To see that it is a map of complexes we compute for $x \in L^p$ and $y \in M^q$ that

$$\mathrm{d}(x \otimes y) = \mathrm{d}_L(x) \otimes y + (-1)^p x \otimes \mathrm{d}_M(y)$$

Our rule says the right hand side is mapped to

$$(-1)^{(p+1)q} y \otimes \mathrm{d}_L(x) + (-1)^{p+p(q+1)} \mathrm{d}_M(y) \otimes x$$

On the other hand, we see that

$$\mathrm{d}((-1)^{pq} y \otimes x) = (-1)^{pq} \mathrm{d}_M(y) \otimes x + (-1)^{pq+q} y \otimes \mathrm{d}_L(x)$$

These two expressions agree by inspection as desired.

0FNI **Lemma 15.58.1.** *Let R be a ring. The category $\mathrm{Comp}(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto \mathrm{Tot}(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.*

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having R in degree 0 and zero in other degrees with obvious isomorphisms $\mathrm{Tot}(\mathbf{1} \otimes_R M^\bullet) = M^\bullet$ and $\mathrm{Tot}(K^\bullet \otimes_R \mathbf{1}) = K^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. □

064I **Lemma 15.58.2.** *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopic maps of complexes. Then α and β induce homotopic maps*

$$Tot(\alpha \otimes id_P), Tot(\beta \otimes id_P) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto Tot(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

Proof. Say $\alpha = \beta + dh + hd$ for some homotopy h defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus\nolimits_{a+b=n} h^a \otimes \mathrm{id}_{P^b} : \bigoplus\nolimits_{a+b=n} L^a \otimes_R P^b \longrightarrow \bigoplus\nolimits_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\mathrm{Tot}(\alpha \otimes \mathrm{id}_P) = \mathrm{Tot}(\beta \otimes \mathrm{id}_P) + dH + Hd$$

as maps $\mathrm{Tot}(L^\bullet \otimes_R P^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R P^\bullet)$. □

0GWP **Lemma 15.58.3.** *Let R be a ring. The homotopy category $K(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto Tot(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.*

Proof. This follows from Lemmas 15.58.1 and 15.58.2. Details omitted. □

064J **Lemma 15.58.4.** *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. The functors*

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(P^\bullet \otimes_R L^\bullet)$$

and

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R P^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. □

15.59. Derived tensor product

06XY We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions.

06XZ **Definition 15.59.1.** Let R be a ring. A complex $K^\bullet$ is called *K-flat* if for every acyclic complex $M^\bullet$ the total complex $\mathrm{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

06Y0 **Lemma 15.59.2.** *Let R be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor*

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 15.58.4 and the fact that quasi-isomorphisms in $K(R)$ are characterized by having acyclic cones. □

06Y1 **Lemma 15.59.3.** *Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of R-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of R'-modules.*

Proof. Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of R'-modules. □

0795 **Lemma 15.59.4.** *Let R be a ring. If $K^\bullet$, $L^\bullet$ are K-flat complexes of R-modules, then $Tot(K^\bullet \otimes_R L^\bullet)$ is a K-flat complex of R-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(M^\bullet \otimes_R \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)$$

and the definition. □

06Y2 **Lemma 15.59.5.** *Let R be a ring. Let $(K_1^\bullet, K_2^\bullet, K_3^\bullet)$ be a distinguished triangle in $K(R)$. If two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 15.58.4 and the fact that in a distinguished triangle in $K(R)$ if two out of three are acyclic, so is the third. □

0BYH **Lemma 15.59.6.** *Let R be a ring. Let $0 \to K_1^\bullet \to K_2^\bullet \to K_3^\bullet \to 0$ be a short exact sequence of complexes. If K_3^n is flat for all $n \in \mathbf{Z}$ and two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Let $L^\bullet$ be a complex of R-modules. Then

$$0 \to \mathrm{Tot}(L^\bullet \otimes_R K_1^\bullet) \to \mathrm{Tot}(L^\bullet \otimes_R K_2^\bullet) \to \mathrm{Tot}(L^\bullet \otimes_R K_3^\bullet) \to 0$$

is a short exact sequence of complexes. Namely, for each n, m the sequence of modules $0 \to L^n \otimes_R K_1^m \to L^n \otimes_R K_2^m \to L^n \otimes_R K_3^m \to 0$ is exact by Algebra, Lemma 10.39.12 and the sequence of complexes is a direct sum of these. Thus the lemma follows from this and the fact that in a short exact sequence of complexes if two out of three are acyclic, so is the third. □

064K **Lemma 15.59.7.** *Let R be a ring. Let $P^\bullet$ be a bounded above complex of flat R-modules. Then $P^\bullet$ is K-flat.*

Proof. Let $L^\bullet$ be an acyclic complex of R-modules. Let $\xi \in H^n(\mathrm{Tot}(L^\bullet \otimes_R P^\bullet))$. We have to show that $\xi = 0$. Since $\mathrm{Tot}^n(L^\bullet \otimes_R P^\bullet)$ is a direct sum with terms $L^a \otimes_R P^b$ we see that ξ comes from an element in $H^n(\mathrm{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet))$ for some $m \in \mathbf{Z}$. Since $\tau_{\leq m} L^\bullet$ is also acyclic we may replace $L^\bullet$ by $\tau_{\leq m} L^\bullet$. Hence we may assume that $L^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.25.3 has

$${}'E_1^{p,q} = H^p(L^\bullet \otimes_R P^q)$$

which is zero as P^q is flat and $L^\bullet$ acyclic. Hence $H^*(\mathrm{Tot}(L^\bullet \otimes_R P^\bullet)) = 0$. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

06Y3 **Lemma 15.59.8.** *Let R be a ring. Let $K_1^\bullet \to K_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i K_i^\bullet$ is K-flat. More generally any filtered colimit of K-flat complexes is K-flat.*

Proof. Because we are taking termwise colimits we have

$$\mathrm{colim}_i \mathrm{Tot}(M^\bullet \otimes_R K_i^\bullet) = \mathrm{Tot}(M^\bullet \otimes_R \mathrm{colim}_i K_i^\bullet)$$

by Algebra, Lemma 10.12.9. Hence the lemma follows from the fact that filtered colimits are exact, see Algebra, Lemma 10.8.8. □

0E8F **Lemma 15.59.9.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. If $K^\bullet \otimes_R M$ is acyclic for all finitely presented R-modules M, then $K^\bullet$ is K-flat.*

Proof. We will use repeatedly that tensor product commute with colimits (Algebra, Lemma 10.12.9). Thus we see that $K^\bullet \otimes_R M$ is acyclic for any R-module M, because any R-module is a filtered colimit of finitely presented R-modules M, see Algebra, Lemma 10.11.3. Let $M^\bullet$ be an acyclic complex of R-modules. We have to show that $\mathrm{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic. Since $M^\bullet = \mathrm{colim}\, \tau_{\leq n} M^\bullet$ (termwise colimit) we have

$$\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) = \mathrm{colim}\, \mathrm{Tot}(\tau_{\leq n} M^\bullet \otimes_R K^\bullet)$$

with truncations as in Homology, Section 12.15. As filtered colimits are exact (Algebra, Lemma 10.8.8) we may replace $M^\bullet$ by $\tau_{\leq n} M^\bullet$ and assume that $M^\bullet$ is bounded above. In the bounded above case, we can write $M^\bullet = \mathrm{colim}\, \sigma_{\geq -n} M^\bullet$ where the complexes $\sigma_{\geq -n} M^\bullet$ are bounded but possibly no longer acyclic. Arguing as above we reduce to the case where $M^\bullet$ is a bounded complex. Finally, for a bounded complex $M^a \to \ldots \to M^b$ we can argue by induction on the length $b - a$ of the complex. The case $b - a = 1$ we have seen above. For $b - a > 1$ we consider the split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} M^\bullet \to M^\bullet \to M^a[-a] \to 0$$

and we apply Lemma 15.58.4 to do the induction step. Some details omitted. □

06Y4 **Lemma 15.59.10.** *Let R be a ring. For any complex $M^\bullet$ there exists a K-flat complex $K^\bullet$ whose terms are flat R-modules and a quasi-isomorphism $K^\bullet \to M^\bullet$ which is termwise surjective.*

Proof. Let $\mathcal{P} \subset \mathrm{Ob}(\mathrm{Mod}_R)$ be the class of flat R-modules. By Derived Categories, Lemma 13.29.1 there exists a system $K_1^\bullet \to K_2^\bullet \to \ldots$ and a diagram

$$\begin{array}{ccccc} K_1^\bullet & \longrightarrow & K_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1} M^\bullet & \longrightarrow & \tau_{\leq 2} M^\bullet & \longrightarrow & \ldots \end{array}$$

with the properties (1), (2), (3) listed in that lemma. These properties imply each complex $K_i^\bullet$ is a bounded above complex of flat modules. Hence $K_i^\bullet$ is K-flat by Lemma 15.59.7. The induced map $\mathrm{colim}_i K_i^\bullet \to M^\bullet$ is a quasi-isomorphism and termwise surjective by construction. The complex $\mathrm{colim}_i K_i^\bullet$ is K-flat by Lemma 15.59.8. The terms $\mathrm{colim}\, K_i^n$ are flat because filtered colimits of flat modules are flat, see Algebra, Lemma 10.39.3. □

09PB **Remark 15.59.11.** In fact, we can do better than Lemma 15.59.10. Namely, we can find a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a complex of R-modules endowed with a filtration

$$0 = F_{-1} P^\bullet \subset F_0 P^\bullet \subset F_1 P^\bullet \subset \ldots \subset P^\bullet$$

by subcomplexes such that

(1) $P^\bullet = \bigcup F_p P^\bullet$,
(2) the inclusions $F_i P^\bullet \to F_{i+1} P^\bullet$ are termwise split injections,
(3) the quotients $F_{i+1} P^\bullet / F_i P^\bullet$ are isomorphic to direct sums of shifts $R[k]$ (as complexes, so differentials are zero).

This will be shown in Differential Graded Algebra, Lemma 22.20.4 (or you can argue as in the proof of Lemma 15.64.2). Moreover, given such a complex we obtain a distinguished triangle

$$\bigoplus F_i P^\bullet \to \bigoplus F_i P^\bullet \to M^\bullet \to \bigoplus F_i P^\bullet[1]$$

in $D(R)$. Using this we can sometimes reduce statements about general complexes to statements about $R[k]$ (this of course only works if the statement is preserved under taking direct sums). More precisely, let T be a property of objects of $D(R)$. Suppose that

(1) if $K_i \in D(R)$, $i \in I$ is a family of objects with $T(K_i)$ for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
(3) $T(R[k])$ holds for all k.

Then T holds for all objects of $D(R)$.

064L **Lemma 15.59.12.** *Let R be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of R-modules. For every complex $L^\bullet$ of R-modules the induced map*

$$Tot(id_L \otimes \alpha) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 15.59.10. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(K^\bullet \otimes_R P^\bullet) & \longrightarrow & \mathrm{Tot}(K^\bullet \otimes_R Q^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(L^\bullet \otimes_R P^\bullet) & \longrightarrow & \mathrm{Tot}(L^\bullet \otimes_R Q^\bullet) \end{array}$$

The result follows as by Lemma 15.59.2 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 15.59.10. By Lemmas 15.58.2 and 15.58.4 we obtain an exact functor of triangulated categories

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \mathrm{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 15.59.2 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(R)$ at quasi-isomorphism. By Lemma 15.59.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

064M **Definition 15.59.13.** Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_R M^\bullet : D(R) \longrightarrow D(R)$$

is the exact functor of triangulated categories described above.

This functor extends the functor (15.57.0.1). It is clear from our explicit constructions that there is an isomorphism (involving a choice of signs, see below)

$$M^\bullet \otimes^{\mathbf{L}}_R L^\bullet \cong L^\bullet \otimes^{\mathbf{L}}_R M^\bullet$$

whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes^{\mathbf{L}}_R L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

0BYI **Lemma 15.59.14.** *Let R be a ring. Let $K^\bullet, L^\bullet$ be complexes of R-modules. There is a canonical isomorphism*

$$K^\bullet \otimes^{\mathbf{L}}_R L^\bullet \longrightarrow L^\bullet \otimes^{\mathbf{L}}_R K^\bullet$$

functorial in both complexes which uses a sign of $(-1)^{pq}$ for the map $K^p \otimes_R L^q \to L^q \otimes_R K^p$ (see proof for explanation).

Proof. We may and do replace the complexes by K-flat complexes $K^\bullet$ and $L^\bullet$ and then we use the commutativity constraint discussed in Section 15.58. □

0BYJ **Lemma 15.59.15.** *Let R be a ring. Let $K^\bullet, L^\bullet, M^\bullet$ be complexes of R-modules. There is a canonical isomorphism*

$$(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet) \otimes^{\mathbf{L}}_R M^\bullet = K^\bullet \otimes^{\mathbf{L}}_R (L^\bullet \otimes^{\mathbf{L}}_R M^\bullet)$$

functorial in all three complexes.

Proof. Replace the complexes by K-flat complexes and use the associativity constraint in Section 15.58. □

0G6M **Lemma 15.59.16.** *Let R be a ring. Let $a : K^\bullet \to L^\bullet$ be a map of complexes of R-modules. If $K^\bullet$ is K-flat, then there exist a complex $N^\bullet$ and maps of complexes $b : K^\bullet \to N^\bullet$ and $c : N^\bullet \to L^\bullet$ such that*

(1) *$N^\bullet$ is K-flat,*
(2) *c is a quasi-isomorphism,*
(3) *a is homotopic to $c \circ b$.*

If the terms of $K^\bullet$ are flat, then we may choose $N^\bullet$, b, and c such that the same is true for $N^\bullet$.

Proof. We will use that the homotopy category $K(R)$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $K^\bullet \to L^\bullet \to C^\bullet \to K^\bullet[1]$. Choose a quasi-isomorphism $M^\bullet \to C^\bullet$ with $M^\bullet$ K-flat with flat terms, see Lemma 15.59.10. By the axioms of triangulated categories, we may fit the composition $M^\bullet \to C^\bullet \to K^\bullet[1]$ into a distinguished triangle $K^\bullet \to N^\bullet \to M^\bullet \to K^\bullet[1]$. By Lemma 15.59.5 we see that $N^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $N^\bullet \to L^\bullet$ fitting into the following morphism of distinguished triangles

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & N^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & C^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $N^\bullet \to L^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(R)$ and use Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $N^\bullet$ such that $N^n \cong M^n \oplus K^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $K^\bullet$ are flat. □

15.60. Derived change of rings

06Y5 Let $R \to A$ be a ring map. Let $N^\bullet$ be a complex of A-modules. We can also use K-flat resolutions to define a functor

$$- \otimes^{\mathbf{L}}_R N^\bullet : D(R) \to D(A)$$

as the left derived functor of the functor $K(R) \to K(A)$, $M^\bullet \mapsto \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$. In particular, taking $N^\bullet = A[0]$ we obtain a derived base change functor

$$- \otimes^{\mathbf{L}}_R A : D(R) \to D(A)$$

extending the functor (15.57.0.2). Namely, for every complex of R-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set

$$M^\bullet \otimes^{\mathbf{L}}_R N^\bullet = \mathrm{Tot}(K^\bullet \otimes_R N^\bullet).$$

You can use Lemmas 15.59.10 and 15.59.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

06Y6 **Lemma 15.60.1.** *The construction above is independent of choices and defines an exact functor of triangulated categories $- \otimes^{\mathbf{L}}_R N^\bullet : D(R) \to D(A)$. There is a functorial isomorphism*

$$E^\bullet \otimes^{\mathbf{L}}_R N^\bullet = (E^\bullet \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A N^\bullet$$

for $E^\bullet$ in $D(R)$.

Proof. To prove the existence of the derived functor $- \otimes^{\mathbf{L}}_R N^\bullet$ we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$. To prove the stated properties of F use Lemmas 15.58.2 and 15.58.4. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(R)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 15.59.10 and (2) follows from Lemma 15.59.12. Thus we obtain a derived functor

$$LF : D(R) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(A)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 guarantees that $LF(K^\bullet) = F(K^\bullet) = \mathrm{Tot}(K^\bullet \otimes_R N^\bullet)$ when $K^\bullet$ is K-flat, i.e., LF is indeed computed in the way described above. Moreover, by Lemma 15.59.3 the complex $K^\bullet \otimes_R A$ is a K-flat complex of A-modules. Hence

$$(K^\bullet \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A N^\bullet = \mathrm{Tot}((K^\bullet \otimes_R A) \otimes_A N^\bullet) = \mathrm{Tot}(K^\bullet \otimes_A N^\bullet) = K^\bullet \otimes^{\mathbf{L}}_A N^\bullet$$

which proves the final statement of the lemma. □

0BYK **Lemma 15.60.2.** *Let $R \to A$ be a ring map. Let $f : L^\bullet \to N^\bullet$ be a map of complexes of A-modules. Then f induces a transformation of functors*

$$1 \otimes f : - \otimes^{\mathbf{L}}_A L^\bullet \longrightarrow - \otimes^{\mathbf{L}}_A N^\bullet$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Since the functors are computing by evaluating on K-flat complexes $K^\bullet$ we can simply use the functoriality

$$\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R N^\bullet)$$

to define the transformation. The last statement follows from Lemma 15.59.2. □

0GMT **Lemma 15.60.3.** *Let $R \to A$ be a ring map. The functor $D(R) \to D(A)$, $E \mapsto E \otimes^{\mathbf{L}}_R A$ of Lemma 15.60.1 is left adjoint to the restriction functor $D(A) \to D(R)$.*

Proof. This follows from Derived Categories, Lemma 13.30.1 and the fact that $- \otimes_R A$ and restriction are adjoint by Algebra, Lemma 10.14.3. □

08YT **Remark 15.60.4** (Warning)**.** Let $R \to A$ be a ring map, and let N and N' be A-modules. Denote N_R and N'_R the restriction of N and N' to R-modules, see Algebra, Section 10.14. In this situation, the objects $N_R \otimes^{\mathbf{L}}_R N'$ and $N \otimes^{\mathbf{L}}_R N'_R$ of $D(A)$ are in general not isomorphic! In other words, one has to pay careful attention as to which of the two sides is being used to provide the A-module structure.

For a specific example, set $R = k[x, y]$, $A = R/(xy)$, $N = R/(x)$ and $N' = A = R/(xy)$. The resolution $0 \to R \xrightarrow{xy} R \to N'_R \to 0$ shows that $N \otimes^{\mathbf{L}}_R N'_R = N[1] \oplus N$ in $D(A)$. The resolution $0 \to R \xrightarrow{x} R \to N_R \to 0$ shows that $N_R \otimes^{\mathbf{L}}_R N'$ is represented by the complex $A \xrightarrow{x} A$. To see these two complexes are not isomorphic, one can show that the second complex is not isomorphic in $D(A)$ to the direct sum of its cohomology groups, or one can show that the first complex is not a perfect object of $D(A)$ whereas the second one is. Some details omitted.

08YU **Lemma 15.60.5.** *Let $A \to B \to C$ be ring maps. Let $N^\bullet$ be a complex of B-modules and $K^\bullet$ a complex of C-modules. The compositions of the functors*

$$D(A) \xrightarrow{-\otimes^{\mathbf{L}}_A N^\bullet} D(B) \xrightarrow{-\otimes^{\mathbf{L}}_B K^\bullet} D(C)$$

is the functor $- \otimes^{\mathbf{L}}_A (N^\bullet \otimes^{\mathbf{L}}_B K^\bullet) : D(A) \to D(C)$. If M, N, K are modules over A, B, C, then we have

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K = M \otimes^{\mathbf{L}}_A (N \otimes^{\mathbf{L}}_B K) = (M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B K)$$

in $D(C)$. We also have a canonical isomorphism

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K \longrightarrow (M \otimes^{\mathbf{L}}_A K) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C)$$

using signs. Similar results holds for complexes.

Proof. Choose a K-flat complex $P^\bullet$ of B-modules and a quasi-isomorphism $P^\bullet \to N^\bullet$ (Lemma 15.59.10). Let $M^\bullet$ be a K-flat complex of A-modules representing an arbitrary object of $D(A)$. Then we see that

$$(M^\bullet \otimes^{\mathbf{L}}_A P^\bullet) \otimes^{\mathbf{L}}_B K^\bullet \longrightarrow (M^\bullet \otimes^{\mathbf{L}}_A N^\bullet) \otimes^{\mathbf{L}}_B K^\bullet$$

is an isomorphism by Lemma 15.60.2 applied to the material inside the brackets. By Lemmas 15.59.3 and 15.59.4 the complex

$$\mathrm{Tot}(M^\bullet \otimes_A P^\bullet) = \mathrm{Tot}((M^\bullet \otimes_R A) \otimes_A P^\bullet$$

is K-flat as a complex of B-modules and it represents the derived tensor product in $D(B)$ by construction. Hence we see that $(M^\bullet \otimes^{\mathbf{L}}_A P^\bullet) \otimes^{\mathbf{L}}_B K^\bullet$ is represented by the complex

$$\mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_A P^\bullet) \otimes_B K^\bullet) = \mathrm{Tot}(M^\bullet \otimes_A \mathrm{Tot}(P^\bullet \otimes_B K^\bullet))$$

of C-modules. Equality by Homology, Remark 12.18.4. Going back the way we came we see that this is equal to

$$M^\bullet \otimes^{\mathbf{L}}_A (P^\bullet \otimes^{\mathbf{L}}_B K^\bullet) \longleftarrow M^\bullet \otimes^{\mathbf{L}}_A (N^\bullet \otimes^{\mathbf{L}}_B K^\bullet)$$

The arrow is an isomorphism by definition of the functor $- \otimes^{\mathbf{L}}_B K^\bullet$. All of these constructions are functorial in the complex $M^\bullet$ and hence we obtain our isomorphism of functors.

By the above we have the first equality in

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K = M \otimes^{\mathbf{L}}_A (N \otimes^{\mathbf{L}}_B K) = (M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B K)$$

The second equality follows from the final statement of Lemma 15.60.1. The same thing allows us to write $N \otimes^{\mathbf{L}}_B K = (N \otimes^{\mathbf{L}}_B C) \otimes^{\mathbf{L}}_C K$ and substituting we get

$$\begin{aligned}
(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K &= (M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C ((N \otimes^{\mathbf{L}}_B C) \otimes^{\mathbf{L}}_C K) \\
&= (M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C (K \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C)) \\
&= ((M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C K) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C)) \\
&= (M \otimes^{\mathbf{L}}_C K) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C)
\end{aligned}$$

by Lemmas 15.59.14 and 15.59.15 as well as the previously mentioned lemma. □

15.61. Tor independence

065Y Consider a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

of rings. Given an object K of $D(A)$ we can consider its derived base change $K \otimes^{\mathbf{L}}_A A'$ to an object of $D(A')$. Or we can take the restriction of K to an object of $D(R)$ and consider the derived base change of this to an object of $D(R')$, denoted $K \otimes^{\mathbf{L}}_R R'$. We claim there is a functorial comparison map

065Z (15.61.0.1) $$K \otimes^{\mathbf{L}}_R R' \longrightarrow K \otimes^{\mathbf{L}}_A A'$$

in $D(R')$. To construct this comparison map choose a K-flat complex $K^\bullet$ of A-modules representing K. Next, choose a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a K-flat complex of R-modules. The map above is the map

$$K \otimes^{\mathbf{L}}_R R' = E^\bullet \otimes_R R' \longrightarrow K^\bullet \otimes_A A' = K \otimes^{\mathbf{L}}_A A'$$

In general there is no chance that this map is an isomorphism.

However, we often encounter the situation where the diagram above is a "base change" diagram of rings, i.e., $A' = A \otimes_R R'$. In this situation, for any A-module M we have $M \otimes_A A' = M \otimes_R R'$. Thus $- \otimes_R R'$ is equal to $- \otimes_A A'$ as a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{A'}$. In general this equality **does not extend** to derived tensor products. In other words, the comparison map is not an isomorphism. A simple example is to take $R = k[x]$, $A = R' = A' = k[x]/(x) = k$ and $K^\bullet = A[0]$. Clearly, a necessary condition is that $\mathrm{Tor}^R_p(A, R') = 0$ for all $p > 0$.

0660 **Definition 15.61.1.** Let R be a ring. Let A, B be R-algebras. We say A and B are *Tor independent over* R if $\mathrm{Tor}^R_p(A, B) = 0$ for all $p > 0$.

0661 **Lemma 15.61.2.** *The comparison map (15.61.0.1) is an isomorphism if $A' = A \otimes_R R'$ and A and R' are Tor independent over R.*

Proof. To prove this we choose a free resolution $F^\bullet \to R'$ of R' as an R-module. Because A and R' are Tor independent over R we see that $F^\bullet \otimes_R A$ is a free A-module resolution of A' over A. By our general construction of the derived tensor product above we see that

$$K^\bullet \otimes_A A' \cong \mathrm{Tot}(K^\bullet \otimes_A (F^\bullet \otimes_R A)) = \mathrm{Tot}(K^\bullet \otimes_R F^\bullet) \cong \mathrm{Tot}(E^\bullet \otimes_R F^\bullet) \cong E^\bullet \otimes_R R'$$

as desired. □

08HW **Lemma 15.61.3.** *Consider a commutative diagram of rings*

$$\begin{array}{ccccc} A' & \longleftarrow & R' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & R & \longrightarrow & B \end{array}$$

Assume that R' is flat over R and A' is flat over $A \otimes_R R'$ and B' is flat over $R' \otimes_R B$. Then

$$Tor_i^R(A, B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B') = Tor_i^{R'}(A', B')$$

Proof. By Algebra, Section 10.76 there are canonical maps

$$\mathrm{Tor}_i^R(A, B) \longrightarrow \mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') \longrightarrow \mathrm{Tor}_i^{R'}(A', B')$$

These induce a map from left to right in the formula of the lemma.

Take a free resolution $F_\bullet \to A$ of A as an R-module. Then we see that $F_\bullet \otimes_R R'$ is a resolution of $A \otimes_R R'$. Hence $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R')$ is computed by $F_\bullet \otimes_R B \otimes_R R'$. By our assumption that R' is flat over R, this computes $\mathrm{Tor}_i^R(A, B) \otimes_R R'$. Thus $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') = \mathrm{Tor}_i^R(A, B) \otimes_R R'$ (uses only flatness of R' over R).

By Lazard's theorem (Algebra, Theorem 10.81.4) we can write A', resp. B' as a filtered colimit of finite free $A \otimes_R R'$, resp. $B \otimes_R R'$-modules. Say $A' = \mathrm{colim}\, M_i$ and $B' = \mathrm{colim}\, N_j$. The result above gives

$$\mathrm{Tor}_i^{R'}(M_i, N_j) = \mathrm{Tor}_i^R(A, B) \otimes_{A \otimes_R B} (M_i \otimes_{R'} N_j)$$

as one can see by writing everything out in terms of bases. Taking the colimit we get the result of the lemma. □

0FXF **Lemma 15.61.4.** *Let $R \to A$ and $R \to B$ be ring maps. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$ and $B' = B \otimes_R R'$. If A and B are tor independent over R and $R \to R'$ is flat, then A' and B' are tor independent over R'.*

Proof. Follows immediately from Lemma 15.61.3 and Definition 15.61.1. □

0DJD **Lemma 15.61.5.** *Assumptions as in Lemma 15.61.3. For $M \in D(A)$ there are canonical isomorphisms*

$$H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B') = H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')$$

of $A' \otimes_{R'} B'$-modules.

Proof. Let us elucidate the two sides of the equation. On the left hand side we have the composition of the functors $D(A) \to D(A') \to D(R') \to D(B')$ with the functor $H^i : D(B') \to \text{Mod}_{B'}$. Since there is a map from A' to the endomorphisms of the object $(M \otimes^{\mathbf{L}}_A A') \otimes^{\mathbf{L}}_{R'} B'$ in $D(B')$, we see that the left hand side is indeed an $A' \otimes_{R'} B'$-module. By the same arguments we see that $H^i(M \otimes^{\mathbf{L}}_R B)$ has an $A \otimes_R B$-module structure.

We first prove the result in case $B' = R' \otimes_R B$. In this case we choose a resolution $F^\bullet \to B$ by free R-modules. We also choose a K-flat complex $M^\bullet$ of A-modules representing M. Then the left hand side is represented by

$$\begin{aligned} H^i(\text{Tot}((M^\bullet \otimes_A A') \otimes_{R'} (R' \otimes_R F^\bullet))) &= H^i(\text{Tot}(M^\bullet \otimes_A A' \otimes_R F^\bullet)) \\ &= H^i(\text{Tot}(M^\bullet \otimes_R F^\bullet) \otimes_A A') \\ &= H^i(M \otimes^{\mathbf{L}}_R B) \otimes_A A' \end{aligned}$$

The final equality because $A \to A'$ is flat. The final module is the desired module because $A' \otimes_{R'} B' = A' \otimes_R B$ since we've assumed $B' = R' \otimes_R B$ in this paragraph.

General case. Suppose that $B' \to B''$ is a flat ring map. Then it is easy to see that

$$H^i((M \otimes^{\mathbf{L}}_A A') \otimes^{\mathbf{L}}_{R'} B'') = H^i((M \otimes^{\mathbf{L}}_A A') \otimes^{\mathbf{L}}_{R'} B') \otimes_{B'} B''$$

and

$$H^i(M \otimes^{\mathbf{L}}_R B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B'') = \big(H^i(M \otimes^{\mathbf{L}}_R B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')\big) \otimes_{B'} B''$$

Thus the result for B' implies the result for B''. Since we've proven the result for $R' \otimes_R B$ in the previous paragraph, this implies the result in general. □

08HX **Lemma 15.61.6.** *Let R be a ring. Let A, B be R-algebras. The following are equivalent*

1. *A and B are Tor independent over R,*
2. *for every pair of primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ lying over the same prime $\mathfrak{r} \subset R$ the rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are Tor independent over $R_\mathfrak{r}$, and*
3. *For every prime $\mathfrak{s}$ of $A \otimes_R B$ the module*

$$Tor_i^R(A, B)_\mathfrak{s} = Tor_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

(where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$) is zero.

Proof. Let $\mathfrak{s}$ be a prime of $A \otimes_R B$ as in (3). The equality

$$\text{Tor}_i^R(A, B)_\mathfrak{s} = \text{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$ follows from Lemma 15.61.3. Hence (2) implies (3). Since we can test the vanishing of modules by localizing at primes (Algebra, Lemma 10.23.1) we conclude that (3) implies (1). For (1) $\Rightarrow$ (2) we use that

$$\text{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q}) = \text{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A_\mathfrak{p} \otimes_{R_\mathfrak{r}} B_\mathfrak{q})$$

again by Lemma 15.61.3. □

15.62. Spectral sequences for Tor

061Y In this section we collect various spectral sequences that come up when considering the Tor functors.

061Z **Example 15.62.1.** Let R be a ring. Let $K_\bullet$ be a chain complex of R-modules with $K_n = 0$ for $n \ll 0$. Let M be an R-module. Choose a resolution $P_\bullet \to M$ of M by free R-modules. We obtain a double chain complex $K_\bullet \otimes_R P_\bullet$. Applying the material in Homology, Section 12.25 (especially Homology, Lemma 12.25.3) translated into the language of chain complexes we find two spectral sequences converging to $H_*(K_\bullet \otimes^{\mathbf{L}}_R M)$. Namely, on the one hand a spectral sequence with E_2-page

$$(E_2)_{i,j} = \operatorname{Tor}^R_j(H_i(K_\bullet), M) \Rightarrow H_{i+j}(K_\bullet \otimes^{\mathbf{L}}_R M)$$

and differential d_2 given by maps $\operatorname{Tor}^R_j(H_i(K_\bullet), M) \to \operatorname{Tor}^R_{j-2}(H_{i+1}(K_\bullet), M)$. Another spectral sequence with E_1-page

$$(E_1)_{i,j} = \operatorname{Tor}^R_j(K_i, M) \Rightarrow H_{i+j}(K_\bullet \otimes^{\mathbf{L}}_R M)$$

with differential d_1 given by maps $\operatorname{Tor}^R_j(K_i, M) \to \operatorname{Tor}^R_j(K_{i-1}, M)$ induced by $K_i \to K_{i-1}$.

068F **Example 15.62.2.** Let $R \to S$ be a ring map. Let M be an R-module and let N be an S-module. Then there is a spectral sequence

$$\operatorname{Tor}^S_n(\operatorname{Tor}^R_m(M, S), N) \Rightarrow \operatorname{Tor}^R_{n+m}(M, N).$$

To construct it choose a R-free resolution $P^\bullet$ of M. Then we have

$$M \otimes^{\mathbf{L}}_R N = P^\bullet \otimes_R N = (P^\bullet \otimes_R S) \otimes_S N$$

and then apply the first spectral sequence of Example 15.62.1.

0620 **Example 15.62.3.** Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B' = B \otimes_A A' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

and B-modules M, N. Set $M' = M \otimes_A A' = M \otimes_B B'$ and $N' = N \otimes_A A' = N \otimes_B B'$. *Assume that $A \to B$ is flat and that M and N are A-flat.* Then there is a spectral sequence

$$\operatorname{Tor}^A_i(\operatorname{Tor}^B_j(M, N), A') \Rightarrow \operatorname{Tor}^{B'}_{i+j}(M', N')$$

The reason is as follows. Choose free resolution $F_\bullet \to M$ as a B-module. As B and M are A-flat we see that $F_\bullet \otimes_A A'$ is a free B'-resolution of M'. Hence we see that the groups $\operatorname{Tor}^{B'}_n(M', N')$ are computed by the complex

$$(F_\bullet \otimes_A A') \otimes_{B'} N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes^{\mathbf{L}}_A A'$$

the last equality because $F_\bullet \otimes_B N$ is a complex of flat A-modules as N is flat over A. Hence we obtain the spectral sequence by applying the spectral sequence of Example 15.62.1.

0662 **Example 15.62.4.** Let $K^\bullet, L^\bullet$ be objects of $D^-(R)$. Then there is a spectral sequence with

$$E_2^{p,q} = H^p(K^\bullet \otimes^{\mathbf{L}}_R H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$$

and another spectral sequence with

$$E_2^{p,q} = H^p(H^q(K^\bullet) \otimes^{\mathbf{L}}_R L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$$

Both spectral sequences have $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$. After replacing $K^\bullet$ and $L^\bullet$ by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of $\mathrm{Tot}(K^\bullet \otimes L^\bullet)$ discussed in Homology, Section 12.25.

15.63. Products and Tor

068G The simplest example of the product maps comes from the following situation. Suppose that $K^\bullet, L^\bullet \in D(R)$. Then there are maps

068H (15.63.0.1)
$$H^i(K^\bullet) \otimes_R H^j(L^\bullet) \longrightarrow H^{i+j}(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$$

Namely, to define these maps we may assume that one of $K^\bullet, L^\bullet$ is a K-flat complex of R-modules (for example a bounded above complex of free or projective R-modules). In that case $K^\bullet \otimes^{\mathbf{L}}_R L^\bullet$ is represented by the complex $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$, see Section 15.59 (or Section 15.57). Next, suppose that $\xi \in H^i(K^\bullet)$ and $\zeta \in H^j(L^\bullet)$. Choose $k \in \mathrm{Ker}(K^i \to K^{i+1})$ and $l \in \mathrm{Ker}(L^j \to L^{j+1})$ representing ξ and ζ. Then we set

$$\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)).$$

This make sense because the formula (see Homology, Definition 12.18.3) for the differential d on the total complex shows that $k \otimes l$ is a cocycle. Moreover, if $k' = d_K(k'')$ for some $k'' \in K^{i-1}$, then $k' \otimes l = \mathrm{d}(k'' \otimes l)$ because l is a cocycle. Similarly, altering the choice of l representing ζ does not change the class of $k \otimes l$. It is equally clear that $\cup$ is bilinear, and hence to a general element of $H^i(K^\bullet) \otimes_R H^j(L^\bullet)$ we assign

$$\sum \xi_i \otimes \zeta_i \longmapsto \sum \xi_i \cup \zeta_i$$

in $H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$.

Let $R \to A$ be a ring map. Let $K^\bullet, L^\bullet \in D(R)$. Then we have a canonical identification

068I (15.63.0.2)
$$(K^\bullet \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A (L^\bullet \otimes^{\mathbf{L}}_R A) = (K^\bullet \otimes^{\mathbf{L}}_R L^\bullet) \otimes^{\mathbf{L}}_R A$$

in $D(A)$. It is constructed as follows. First, choose K-flat resolutions $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ over R. Then the left hand side is represented by the complex $\mathrm{Tot}((P^\bullet \otimes_R A) \otimes_A (Q^\bullet \otimes_R A))$ and the right hand side by the complex $\mathrm{Tot}(P^\bullet \otimes_R Q^\bullet) \otimes_R A$. These complexes are canonically isomorphic. Thus the construction above induces products

$$\mathrm{Tor}^R_n(K^\bullet, A) \otimes_A \mathrm{Tor}^R_m(L^\bullet, A) \longrightarrow \mathrm{Tor}^R_{n+m}(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet, A)$$

which are occasionally useful.

Let M, N be R-modules. Using the general construction above, the canonical map $M \otimes^{\mathbf{L}}_R N \to M \otimes_R N$ and functoriality of Tor we obtain canonical maps

068J (15.63.0.3)
$$\mathrm{Tor}^R_n(M, A) \otimes_A \mathrm{Tor}^R_m(N, A) \longrightarrow \mathrm{Tor}^R_{n+m}(M \otimes_R N, A)$$

Here is a direct construction using projective resolutions. First, choose projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N$$

over R. We have $H_0(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N$ by right exactness of $\otimes_R$. Hence Derived Categories, Lemmas 13.19.6 and 13.19.7 guarantee the existence and uniqueness of a map of complexes $\mu : \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet$ such that $H_0(\mu) = \mathrm{id}_{M \otimes_R N}$. This induces a canonical map

$$\begin{aligned}(M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A) &= \mathrm{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A)) \\ &= \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A \\ &\to T_\bullet \otimes_R A \\ &= (M \otimes_R N) \otimes_R^{\mathbf{L}} A\end{aligned}$$

in $D(A)$. Hence the products (15.63.0.3) above are constructed using (15.63.0.1) over A to construct

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \to H^{-n-m}((M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A))$$

and then composing by the displayed map above to end up in $\mathrm{Tor}_{n+m}^R(M \otimes_R N, A)$.

An interesting special case of the above occurs when $M = N = B$ where B is an R-algebra. In this case we obtain maps

$$\mathrm{Tor}_n^R(B, A) \otimes_A \mathrm{Tor}_m^R(B, A) \longrightarrow \mathrm{Tor}_{n+m}^R(B \otimes_R B, A) \longrightarrow \mathrm{Tor}_{n+m}^R(B, A)$$

the second arrow being induced by the multiplication map $B \otimes_R B \to B$ via functoriality for Tor. In other words we obtain an A-algebra structure on $\mathrm{Tor}_\star^R(B, A)$. This algebra structure has many intriguing properties (associativity, graded commutative, B-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

068K **Lemma 15.63.1.** *Let R be a ring. Let A, B, C be R-algebras and let $B \to C$ be an R-algebra map. Then the induced map*

$$Tor_\star^R(B, A) \longrightarrow Tor_\star^R(C, A)$$

is an A-algebra homomorphism.

Proof. Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. □

15.64. Künneth spectral sequence

0H7V Let R be a ring. Let $K^\bullet$ and $L^\bullet$ be filtered complexes of R-modules (see Homology, Definition 12.24.1; note that our filtrations are decreasing). Then the complex

$$T^\bullet = \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$$

also has a decreasing filtration defined by the formula

$$F^n T^\bullet = \mathrm{Im}\left(\bigoplus\nolimits_{i+j=n} \mathrm{Tot}(F^i K^\bullet \otimes_R F^j L^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)\right)$$

Under some assumptions on our filtered complexes, we will determine the spectral sequence that arises from this by the construction in Homology, Section 12.24.

Assume that each K^n, $F^i K^n$, $\mathrm{gr}^i K^n$, L^m, $F^j L^m$, and $\mathrm{gr}^j L^m$ is a flat R-module. In this case the modules $F^i K^n \otimes_R L^m$, $K^n \otimes_R F^j L^m$, and $F^i K^n \otimes_R F^j L^m$ are

submodules of $K^n \otimes_R L^m$. Similarly, the module $\mathrm{gr}^i K^n \otimes_R \mathrm{gr}^j L^m$ is a submodule of $K^n/F^{i+1}K^n \otimes_R L^m/F^{j+1}L^m$. Consider the map

$$F^n T^n \longrightarrow \bigoplus\nolimits_{i+j=n} \mathrm{Tot}(K^\bullet/F^{i+1}K^\bullet \otimes_R L^\bullet/F^{j+1}L^\bullet)$$

We leave it to the reader to show that we do indeed end up in the direct sum and not the direct product, due to our definition of $F^n T^\bullet$. For $a+b=n$, the restriction of the displayed arrow to the subcomplex $\mathrm{Tot}(F^a K^\bullet \otimes_R F^b L^\bullet)$ of $F^n T^\bullet$ maps into the summand with $i=a$ and $j=b$. Moreover, by our flatness assumptions, the image is isomorphic to $\mathrm{Tot}(\mathrm{gr}^i K^\bullet \otimes_R \mathrm{gr}^j L^\bullet)$ and the kernel of $\mathrm{Tot}(F^i K^\bullet \otimes_R F^j L^\bullet) \to \mathrm{Tot}(\mathrm{gr}^i K^\bullet \otimes_R \mathrm{gr}^j L^\bullet)$ is $\mathrm{Tot}(F^{i+1}K^\bullet \otimes_R F^j L^\bullet) + \mathrm{Tot}(F^i K^\bullet \otimes_R F^{j+1} L^\bullet)$. It follows that we have a short exact sequence of complexes

$$0 \to F^{n+1}T^\bullet \to F^n T^\bullet \to \bigoplus\nolimits_{i+j=n} \mathrm{Tot}(\mathrm{gr}^i K^\bullet \otimes_R \mathrm{gr}^j L^\bullet) \to 0$$

under our assumptions. In other words, this tells us that $\mathrm{gr}^n T^\bullet$ is a direct sum of the complexes $\mathrm{Tot}(\mathrm{gr}^i K^\bullet \otimes_R \mathrm{gr}^j L^\bullet)$ for $i+j=n$.

Assume in addition that the complexes of R-modules $K^\bullet$, $F^i K^\bullet$, $\mathrm{gr}^i K^\bullet$, $L^\bullet$, $F^j L^\bullet$, and $\mathrm{gr}^j L^\bullet$ are K-flat. In this case we conclude that the spectral sequence of Homology, Section 12.24 associated to the filtered complex $T^\bullet$ has terms

$$E_1^{p,q} = \bigoplus\nolimits_{i+j=p} H^{p+q}(\mathrm{gr}^i K^\bullet \otimes^{\mathbf{L}}_R \mathrm{gr}^j L^\bullet)$$

with differentials induced from the short exact sequences

$$0 \to \mathrm{gr}^{n+1} T^\bullet \to F^n T^\bullet/F^{n+2}T^\bullet \to \mathrm{gr}^n T^\bullet \to 0$$

as explained in Homology, Lemma 12.24.3. In particular, the reader can show this means that the summand $H^{p+q}(\mathrm{gr}^i K^\bullet \otimes^{\mathbf{L}}_R \mathrm{gr}^j L^\bullet)$ of $E_1^{p,q}$ maps into the sum

$$H^{p+q+1}(\mathrm{gr}^{i+1} K^\bullet \otimes^{\mathbf{L}}_R \mathrm{gr}^j L^\bullet) \oplus H^{p+q+1}(\mathrm{gr}^i K^\bullet \otimes^{\mathbf{L}}_R \mathrm{gr}^{j+1} L^\bullet)$$

inside $E_1^{p+1,q}$.

0H7W **Lemma 15.64.1.** *Under the assumptions above, if in addition*

(1) *the filtration on $K^\bullet$ is finite and the filtration on $L^\bullet$ is finite, or*
(2) *more generally the following are true*
   (a) *$F^i K^\bullet$ is acyclic for $i \gg 0$,*
   (b) *$F^i K^\bullet \to K^\bullet$ is a quasi-isomorphism for $i \ll 0$,*
   (c) *$F^j L^\bullet$ is acyclic for $j \gg 0$, and*
   (d) *$F^j L^\bullet \to L^\bullet$ is a quasi-isomorphism for $j \ll 0$.*

Then the spectral sequence is bounded, the associated filtration on each $H^n(T^\bullet) = H^n(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$ is finite and we have convergence

$$E_1^{p,q} = \bigoplus\nolimits_{i+j=p} H^{p+q}(gr^i K^\bullet \otimes^{\mathbf{L}}_R gr^j L^\bullet) \Rightarrow H^n(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$$

Proof. In case (1) the filtration on $T^\bullet$ is finite and the lemma follows immediately from Homology, Lemma 12.24.11. In case (2) choose $a < b$ such that $F^i K^\bullet$ and $F^j L^\bullet$ are acyclic for $i, j > b$ and $F^i K^\bullet \to K^\bullet$ and $F^j L^\bullet \to L^\bullet$ are quasi-isomorphisms for $i, j < a$. We claim that in this case the complex $F^n T^\bullet$ is acyclic for $n > 2b$ and that $F^n T^\bullet \to T^\bullet$ is a quasi-isomorphism for $n < 2a - 1$. Since the claim shows that Homology, Lemma 12.24.13 applies we see that our lemma is true.

Proof of the claim. Given n and integers $i_1 \leq i_2$ consider

$$S^{n,i_1,i_2} = \mathrm{Im}\left(\bigoplus\nolimits_{i_1 \leq i \leq i_2} \mathrm{Tot}(F^iK^\bullet \otimes_R F^{n-i}L^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)\right)$$

viewed as a subcomplex of $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$. Observe that $F^nT^\bullet = \mathrm{colim}\, S^{n,i_1,i_2}$ is a filtered colimit. Thus to show the claim it suffices to show that S^{n,i_1,i_2} is acyclic for $n > 2b$ and that $S^{n,i_1,i_2} \to \mathrm{Tot}(K^\bullet \otimes L^\bullet)$ is a quasi-isomorphism for $n < 2a - 1$ and a cofinal set of choices of pairs i_1, i_2.

If $i_1 = i_2 = i$ and $n > 2b$, then

$$S^{n,i,i} = \mathrm{Tot}(F^iK^\bullet \otimes_R F^{n-i}L^\bullet)$$

is acyclic: either $i > b$ and this complex is acyclic by our assumption that $F^{n-i}L^\bullet$ is K-flat, or $i \leq b$ and then $n - i \geq n - b > b$ and the same holds. Using our flatness assumptions the reader shows that there is a short exact sequence

$$0 \to S^{n+1,i_2+1,i_2+1} \to S^{n,i_1,i_2} \oplus S^{n,i_2+1,i_2+1} \to S^{n,i_1,i_2+1} \to 0$$

of complexes. Thus we see by induction on $i_2 - i_1$ that all of the complexes S^{n,i_1,i_2} for $n > 2b$ are acyclic.

Assume $n < 2a - 1$. Observe that

$$S^{n,a-1,a-1} = \mathrm{Tot}(F^{a-1}K^\bullet \otimes_R F^{n-a+1}L^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$$

is a quasi-isomorphism because both $a - 1 < a$ and $n - a + 1 < a$ and all our complexes are K-flat hence both sides compute the same object of $D(R)$. Observe that for $i_2 \geq a - 1$ and $n < 2a - 1$ the map

$$S^{n+1,i_2+1,i_2+1} \to S^{n,i_2+1,i_2+1}$$

is a quasi-isomorphism because left and right hand side are quasi-isomorphic to $\mathrm{Tot}(F^{i_2+1}K^\bullet \otimes L^\bullet)$. Thus using the short exact sequence of complexes in the previous paragraph we find that $S^{n,a-1,a-1} \to S^{n,a-1,t}$ is a quasi-isomorphism for all $t \geq a - 1$. On the other hand, there are similarly short exact sequences

$$0 \to S^{n+1,i_1,i_1} \to S^{n,i_1-1,i_1-1} \oplus S^{n,i_1,i_2} \to S^{n,i_1-1,i_2} \to 0$$

of complexes. Similarly to the above, we note that for $i_1 \leq a - 1$ the map

$$S^{n+1,i_1,i_1} \to S^{n,i_1-1,i_1-1}$$

is a quasi-isomorphism because left and right hand side are quasi-isomorphic to $\mathrm{Tot}(K^\bullet \otimes F^{n-i_1+1}L^\bullet)$. Thus we conclude that $S^{n,s,t} \to S^{n,a-1,t}$ is a quasi-isomorphism for all $s \leq a - 1$. Combined we find that $S^{n,s,t}$ for $s \leq a - 1$ and $t \geq a - 1$ maps quasi-isomorphically to $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$. This finishes the proof of the lemma. □

0H7X **Lemma 15.64.2.** *Let R be a ring. Let $K^\bullet$ be a filtered complex. There exists a map $f : P^\bullet \to K^\bullet$ of filtered complexes such that*

1. *each P^n, F^iP^n, gr^iP^n is a free R-module,*
2. *the complexes of R-modules $P^\bullet$, $F^iP^\bullet$, and $gr^iP^\bullet$ are K-flat,*
3. *f induces quasi-isomorphisms $P^\bullet \to K^\bullet$, $F^iP^\bullet \to F^iK^\bullet$, and $gr^iP^\bullet \to gr^iK^\bullet$.*

Proof. Let us say a filtered complex $L^\bullet$ is *basic* if each L^n, F^iL^n, gr^iL^n is a free R-module and if all differentials are zero.

There exists a basic filtered complex $P_0^\bullet$ and a map $f_0 : P_0^\bullet \to K^\bullet$ of filtered complexes such that f_0 and F^if_0, $i \in \mathbf{Z}$ are surjective on cohomology. To see this set

$$P_0^n = \bigoplus\nolimits_{z \in \mathrm{Ker}(d_K^n)} R\xi_z \oplus \bigoplus\nolimits_{j \in \mathbf{Z}} \bigoplus\nolimits_{z \in \mathrm{Ker}(d_{F^jK}^n)} R\xi_z$$

with zero differentials and the map f_0 defined by $f_0(\xi_z) = z$. As for the filtration, we set

$$F^iP_0^n = \bigoplus\nolimits_{j \geq i \in \mathbf{Z}} \bigoplus\nolimits_{z \in \mathrm{Ker}(d_{F^jK}^n)} R\xi_z$$

We leave it to the reader to check that this gives $f_0 : P_0^\bullet \to K^\bullet$ as claimed.

By induction on $m \geq 0$ we are going to construct embeddings

$$P_0^\bullet \subset \ldots P_m^\bullet \subset P_{m+1}^\bullet$$

of filtered complexes and maps $f_m : P_m^\bullet \to K^\bullet$ with the following properties

(1) the filtered complex $P_{m+1}^\bullet/P_m^\bullet$ is basic,
(2) the kernel of $H^n(f_m)$ and $H^n(F^if_m)$ maps to zero in $H^n(P_{m+1}^\bullet)$ and $H^n(F^iP_{m+1}^\bullet)$,
(3) the map $f_{m+1} : P_{m+1}^\bullet \to K^\bullet$ extends the map f_m.

To to this, set

$$\Omega_{n,m} = \mathrm{Ker}\left(\mathrm{Ker}(d_{P_m}^n) \to H^n(K^\bullet)\right), \text{ resp. } \Omega_{n,i,m} = \mathrm{Ker}\left(\mathrm{Ker}(d_{F^iP_m}^n) \to H^n(F^iK^\bullet)\right),$$

Note that $\Omega_{n,m}$ surjects onto the kernel of $H^n(f_m)$ and that $\Omega_{n,i,m}$ surjects onto the kernel of $H^n(F^if_n)$. For each $z \in \Omega_{n,m}$, resp. $z \in \Omega_{n,i,m}$ we choose a $y_z \in K^{n-1}$, resp. $y_z \in F^iK^{n-1}$ such that $d_K(y_z) = f_m(z)$. Then we set

$$P_{m+1}^n = P_m^n \oplus \bigoplus\nolimits_{z \in \Omega_{n+1,m}} R\eta_z \oplus \bigoplus\nolimits_{j \in \mathbf{Z}} \bigoplus\nolimits_{z \in \Omega_{n+1,j,m}} R\eta_z$$

We set $d_{P_m}(\eta_z) = z$ and we set $f_{m+1}(\eta_z) = y_z$. Finally, we set

$$F^iP_{m+1}^n = F^iP_m^n \oplus \bigoplus\nolimits_{j \geq i} \bigoplus\nolimits_{z \in \Omega_{n+1,j,m}} R\eta_z$$

We leave it to the reader to check that this gives $P_m^\bullet \subset P_{m+1}^\bullet$ and $f_{m+1} : P_{m+1}^\bullet \to K^\bullet$ as claimed.

At this point we simply take

$$P^\bullet = \bigcup P_m^\bullet$$

as a filtered complex with map $f : P^\bullet \to K^\bullet$ given by $\bigcup f_m$.

Part (1) of the statement of the lemma holds because P^n as a filtered module is isomorphic to the direct sum of P_0^n and P_{m+1}^n/P_m^n for $m \geq 0$. Small detail omitted.

Part (3) of the statement. Observe that $H^n(P^\bullet) = \mathrm{colim}\, H^n(P_m^\bullet)$. The map f_0 is surjective on cohomology; whence the same holds for each f_m. For each m by construction the embedding $P_m^\bullet \subset P_{m+1}^\bullet$ kills the kernel of $H^n(f_m)$. Combining these facts the reader easily concludes that $H^n(f)$ is an isomorphism. Similarly for $H^n(F^if_n)$. Then also $H^n(\mathrm{gr}^if)$ must be an isomorphism because of the short exact sequence $0 \to F^{i+1} \to F^i \to \mathrm{gr}^i \to 0$ (of functors on the category of filtered complexes, say). Small detail omitted.

Part (2) of the statement. To see that $P^\bullet$ is K-flat, by Lemma 15.59.8, it suffices to show that $P^\bullet_m$ is K-flat. By Lemma 15.59.6 and induction it suffices to note that a complex with zero differentials and free terms is K-flat. The same argument works to show that $F^iP^\bullet$ is K-flat for all $i \in \mathbf{Z}$. Finally, we see that $\mathrm{gr}^iP^\bullet$ is K-flat by another application of Lemma 15.59.6. □

0H7Y **Proposition 15.64.3.** *Let R be a ring. Let $K^\bullet$ and $L^\bullet$ be filtered complexes of R-modules. Then there exists a filtered complex $T^\bullet$ representing $K^\bullet \otimes^{\mathbf{L}}_R L^\bullet$ in $D(R)$ such that the associated spectral sequence has E_1-page*

$$E_1^{p,q} = \bigoplus\nolimits_{i+j=p} H^{p+q}(gr^iK^\bullet \otimes^{\mathbf{L}}_R gr^jL^\bullet)$$

If

(1) *$F^iK^\bullet$ is acyclic for $i \gg 0$,*
(2) *$F^iK^\bullet \to K^\bullet$ is a quasi-isomorphism for $i \ll 0$,*
(3) *$F^jL^\bullet$ is acyclic for $j \gg 0$, and*
(4) *$F^jL^\bullet \to L^\bullet$ is a quasi-isomorphism for $j \ll 0$.*

then the spectral sequence is bounded, the associated filtration on each $H^n(K^\bullet \otimes^{\mathbf{L}}_R L^\bullet)$ is finite and the spectral sequence convergences.

Proof. Choose $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ as in Lemma 15.64.2. Then we use the spectral sequence for the filtered complex

$$T^\bullet = \mathrm{Tot}(P^\bullet \otimes_R Q^\bullet)$$

described in the text of this section and in Lemma 15.64.1. □

0H7Z **Lemma 15.64.4** (Künneth Spectral Sequence)**.** *Let R be a ring. Let K and L be objects of $D^b(R)$. There exists a bigraded bounded spectral sequence $\{E_r\}_{r\geq 2}$ with*

$$E_2^{p,q} = \bigoplus\nolimits_{i+j=q} Tor^R_{-p}(H^i(K), H^j(L))$$

and d_r of bidegree $(r, -r+1)$ converging to $H^{p+q}(K \otimes^{\mathbf{L}}_R L)$.

Proof. Let $K^\bullet$ be a complex of R-modules representing K and let $L^\bullet$ be a complex of R-modules representing L. Set $F^iK^\bullet = \tau_{\leq -i}K^\bullet$ and similarly for L, see Homology, Section 12.15. Apply Proposition 15.64.3 noting that $\mathrm{gr}^iK^\bullet = H^{-i}(K)[i]$ to get a bounded spectral sequence $\{(E')_r\}_{r\geq 1}$ with

$$(E')_1^{p,q} = \bigoplus\nolimits_{i+j=p} \mathrm{Tor}^R_{-2p-q}(H^{-i}(K), H^{-j}(L))$$

converging to the cohomology of $K \otimes^{\mathbf{L}}_R L$. By construction this spectral sequence has differentials $d_r^{p,q} : (E')_r^{p,q} \to (E')_r^{p+r,q-r+1}$ for $r \geq 1$. To get the spectral sequence of the lemma we set for $r \geq 2$

$$E_r^{p,q} = (E')_{r-1}^{-q,p+2q}$$

We leave it to the reader to show that this works. □

0H80 **Example 15.64.5.** If $R = \mathbf{Z}$ or more generally if R is a Dedekind domain, then $\mathrm{Tor}^R_i(M,N) = 0$ for $i \notin \{0,1\}$ for all R-modules M and N. Hence the spectral sequence of Lemma 15.64.4 degenerates at the E_2 page and we get short exact sequences

$$0 \to \bigoplus_{i+j=n} H^i(K) \otimes_R H^j(L) \to H^n(K \otimes^{\mathbf{L}}_R L) \to \bigoplus_{i+j=n+1} \mathrm{Tor}^R_1(H^i(K), H^j(L)) \to 0$$

for all $n \in \mathbf{Z}$.

15.65. Pseudo-coherent modules, I

064N Suppose that R is a ring. Recall that an R-module M is of finite type if there exists a surjection $R^{\oplus a} \to M$ and of finite presentation if there exists a presentation $R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$. Similarly, we can consider those R-modules for which there exists a length n resolution

064P (15.65.0.1) $$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$

by finite free R-modules. A module is called *pseudo-coherent* if we can find such a resolution for every n. Here is the formal definition.

064Q **Definition 15.65.1.** Let R be a ring. Denote $D(R)$ its derived category. Let $m \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ is *m-pseudo-coherent* if there exists a bounded complex $E^\bullet$ of finite free R-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.
(2) An object $K^\bullet$ of $D(R)$ is *pseudo-coherent* if it is quasi-isomorphic to a bounded above complex of finite free R-modules.
(3) An R-module M is called *m-pseudo-coherent* if $M[0]$ is an m-pseudo-coherent object of $D(R)$.
(4) An R-module M is called *pseudo-coherent*[8] if $M[0]$ is a pseudo-coherent object of $D(R)$.

As usual we apply this terminology also to complexes of R-modules. Since any morphism $E^\bullet \to K^\bullet$ in $D(R)$ is represented by an actual map of complexes, see Derived Categories, Lemma 13.19.8, there is no ambiguity. It turns out that $K^\bullet$ is pseudo-coherent if and only if $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.65.5. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma 15.65.17. Let us first relate this to the informal discussion above.

064R **Lemma 15.65.2.** *Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent and $L^\bullet$ is m-pseudo-coherent then $M^\bullet$ is m-pseudo-coherent.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent, then $L^\bullet$ is m-pseudo-coherent.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent and $M^\bullet$ is m-pseudo-coherent, then $K^\bullet$ is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m+1$ and surjective for $i = m+1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1} P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m+1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 13.19.11 we can find a map $\gamma : P^\bullet \to E^\bullet$ such that the

[8] This clashes with what is meant by a pseudo-coherent module in [**Bou61**].

diagram

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow{\scriptstyle \beta} \\ P^\bullet & \xrightarrow{\gamma} & E^\bullet \end{array}$$

is commutative in $D(R)$. The cone $C(\gamma)^\bullet$ is a bounded complex of finite free R-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\gamma)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma)^\bullet \to M^\bullet$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence $M^\bullet$ is m-pseudo-coherent.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

064S **Lemma 15.65.3.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

(1) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type R-module.*
(2) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m+1$, then $H^{m+1}(K^\bullet)$ is a finitely presented R-module.*

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It is clear that it suffices to prove the result for $E^\bullet$. Let n be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As E^n is finite projective we see that $E^{n-1} = E' \oplus E^n$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has E' in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \mathrm{Coker}(E^m \to E^{m+1})$ which is of finite presentation. □

064T **Lemma 15.65.4.** *Let R be a ring. Let M be an R-module. Then*

(1) *M is 0-pseudo-coherent if and only if M is a finite R-module,*
(2) *M is (-1)-pseudo-coherent if and only if M is a finitely presented R-module,*
(3) *M is $(-d)$-pseudo-coherent if and only if there exists a resolution*

$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$

of length d, and
(4) *M is pseudo-coherent if and only if there exists an infinite resolution*

$$\ldots \to R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$$

by finite free R-modules.

Proof. If M is of finite type (resp. of finite presentation), then M is 0-pseudo-coherent (resp. (-1)-pseudo-coherent) as follows from the discussion preceding Definition 15.65.1. Conversely, if M is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 15.65.3. If M is (-1)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^{\oplus a} \to M$ and denote $K = \mathrm{Ker}(R^{\oplus a} \to M)$. By Lemma 15.65.2 we see that K is 0-pseudo-coherent, hence of finite type, whence M is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). □

064U **Lemma 15.65.5.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is pseudo-coherent,*
(2) *$K^\bullet$ is m-pseudo-coherent for every $m \in \mathbf{Z}$, and*
(3) *$K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective R-modules.*

If (1), (2), and (3) hold and $H^i(K^\bullet) = 0$ for $i > b$, then we can find a quasi-isomorphism $F^\bullet \to K^\bullet$ with F^i finite free R-modules and $F^i = 0$ for $i > b$.

Proof. We see that (1) $\Rightarrow$ (3) as a finite free module is a finite projective R-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective R-modules. Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with F^{n_0} a finite free R-module, and inductively

$$F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1}$$

for $n \leq n_0$ with F^{n_0} a finite free R-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on C^{n-1}. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free R-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m} E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is m-pseudo-coherent. Hence (1) implies (2).

Assume (2). Since $K^\bullet$ is 0-pseudo-coherent we see in particular that $K^\bullet$ is bounded above. Let b be an integer such that $H^i(K^\bullet) = 0$ for $i > b$. By descending induction on $n \in \mathbf{Z}$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > b$. Picture

$$\begin{array}{ccccccc} & & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \ldots \end{array}$$

The base case is $n = b + 1$ where we can take $F^i = 0$ for all i. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology

$$0 \to H^{n-1}(K^\bullet) \to H^{n-1}(C^\bullet) \to H^n(F^\bullet) \to H^n(K^\bullet) \to H^n(C^\bullet) \to \ldots$$

shows that $H^i(C^\bullet) = 0$ for $i \geq n$. By Lemma 15.65.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent. By Lemma 15.65.3 we see that $H^{n-1}(C^\bullet)$ is a finite R-module. In particular, we see that the kernel of $H^n(F^\bullet) \to H^n(K^\bullet)$ is a finite R-module. Choose a finite free R-module F^{n-1} and a map $F^{n-1} \to \mathrm{Ker}(F^n \to F^{n-1})$ such that F^{n-1} surjects onto the kernel of $H^n(F^\bullet) \to H^n(K^\bullet)$. We extend our of complexes to

$$\begin{array}{ccccccccc} F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots \longrightarrow K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \ldots \end{array}$$

Note that α^{n-1} exists because the image of $F^{n-1} \to F^n \to K^n$ is in the image of $K^{n-1} \to K^n$ by construction. Dnote again $C^\bullet$ the cone of this extended map of complexes. At this point we see that we get an exact sequence

$$H^{n-1}(F^\bullet) \to H^{n-1}(K^\bullet) \to H^{n-1}(C^\bullet) \to H^n(F^\bullet) \cong H^n(K^\bullet) \to H^n(C^\bullet) \to \ldots$$

In other words, we see that the cokernel of $H^{n-1}(F^\bullet) \to H^{n-1}(K^\bullet)$ is a finite R-module, say generated by the classes of $\xi_1, \ldots, \xi_r \in \mathrm{Ker}(K^{n-1} \to K^n)$. Then we replace F^{n-1} by $F^{n-1} \oplus R^{\oplus r}$ where the basis elements in the free summand map to zero in F^n and the to ξ_i in K^{n-1}. This finishes the proof of the induction step. □

064V **Lemma 15.65.6.** *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent then the third is also pseudo-coherent.*

Proof. Combine Lemmas 15.65.2 and 15.65.5. □

064W **Lemma 15.65.7.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

1. *If $H^i(K^\bullet) = 0$ for all $i \geq m$, then $K^\bullet$ is m-pseudo-coherent.*
2. *If $H^i(K^\bullet) = 0$ for $i > m$ and $H^m(K^\bullet)$ is a finite R-module, then $K^\bullet$ is m-pseudo-coherent.*
3. *If $H^i(K^\bullet) = 0$ for $i > m+1$, the module $H^{m+1}(K^\bullet)$ is of finite presentation, and $H^m(K^\bullet)$ is of finite type, then $K^\bullet$ is m-pseudo-coherent.*

Proof. It suffices to prove (3). Set $M = H^{m+1}(K^\bullet)$. Note that $\tau_{\geq m+1}K^\bullet$ is quasi-isomorphic to $M[-m-1]$. By Lemma 15.65.4 we see that $M[-m-1]$ is m-pseudo-coherent. Since we have the distinguished triangle

$$(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)$$

(Derived Categories, Remark 13.12.4) by Lemma 15.65.2 it suffices to prove that $\tau_{\leq m}K^\bullet$ is pseudo-coherent. By assumption $H^m(\tau_{\leq m}K^\bullet)$ is a finite type R-module. Hence we can find a finite free R-module E and a map $E \to \mathrm{Ker}(d_K^m)$ such that the composition $E \to \mathrm{Ker}(d_K^m) \to H^m(\tau_{\leq m}K^\bullet)$ is surjective. Then $E[-m] \to \tau_{\leq m}K^\bullet$ witnesses the fact that $\tau_{\leq m}K^\bullet$ is m-pseudo-coherent. □

064X **Lemma 15.65.8.** *Let R be a ring. Let $m \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) so are $K^\bullet$ and $L^\bullet$.*

Proof. In this proof we drop the superscript $^\bullet$. Assume that $K \oplus L$ is m-pseudo-coherent. It is clear that $K, L \in D^-(R)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 15.65.2 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. By Lemma 15.65.7 we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n], L[n-1], \ldots, L$ are m-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 15.65.5. □

064Y **Lemma 15.65.9.** *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of R-modules such that K^i is $(m-i)$-pseudo-coherent for all i. Then $K^\bullet$ is m-pseudo-coherent. In particular, if $K^\bullet$ is a bounded above complex of pseudo-coherent R-modules, then $K^\bullet$ is pseudo-coherent.*

Proof. We may replace $K^\bullet$ by $\sigma_{\geq m-1}K^\bullet$ (for example) and hence assume that $K^\bullet$ is bounded. Then the complex $K^\bullet$ is m-pseudo-coherent as each $K^i[-i]$ is m-pseudo-coherent by induction on the length of the complex: use Lemma 15.65.2 and the stupid truncations. For the final statement, it suffices to prove that $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.65.5. This follows from the first part. □

066B **Lemma 15.65.10.** *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(R)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. Assume $K^\bullet$ is an object of $D^-(R)$ such that each $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent. Let n be the largest integer such that $H^n(K^\bullet)$ is nonzero. We will prove the lemma by induction on n. If $n < m$, then $K^\bullet$ is m-pseudo-coherent by Lemma 15.65.7. If $n \geq m$, then we have the distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, H^n(K^\bullet)[-n])$$

(Derived Categories, Remark 13.12.4) Since $H^n(K^\bullet)[-n]$ is m-pseudo-coherent by assumption, we can use Lemma 15.65.2 to see that it suffices to prove that $\tau_{\leq n-1}K^\bullet$ is m-pseudo-coherent. By induction on n we win. (The pseudo-coherent case follows from this and Lemma 15.65.5.) □

064Z **Lemma 15.65.11.** *Let $A \to B$ be a ring map. Assume that B is pseudo-coherent as an A-module. Let $K^\bullet$ be a complex of B-modules. The following are equivalent*

1. *$K^\bullet$ is m-pseudo-coherent as a complex of B-modules, and*
2. *$K^\bullet$ is m-pseudo-coherent as a complex of A-modules.*

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free B-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the distinguished triangle $(E^\bullet, K^\bullet, C(\alpha)^\bullet)$. By Lemma 15.65.7 $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of A-modules. Hence it suffices to prove that $E^\bullet$ is pseudo-coherent as a complex of A-modules, which follows from Lemma 15.65.9. The pseudo-coherent case of (1) $\Rightarrow$ (2) follows from this and Lemma 15.65.5.

Assume (2). Let n be the largest integer such that $H^n(K^\bullet) \neq 0$. We will prove that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules by induction on $n - m$. The

case $n < m$ follows from Lemma 15.65.7. Choose a bounded complex of finite free A-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the induced map of complexes

$$\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.$$

Note that $C(\alpha\otimes 1)^\bullet$ is acyclic in degrees $\geq n$ as $H^n(E) \to H^n(E^\bullet \otimes_A B) \to H^n(K^\bullet)$ is surjective by construction and since $H^i(E^\bullet \otimes_A B) = 0$ for $i > n$ by the spectral sequence of Example 15.62.4. On the other hand, $C(\alpha\otimes 1)^\bullet$ is m-pseudo-coherent as a complex of A-modules because both $K^\bullet$ and $E^\bullet \otimes_A B$ (see Lemma 15.65.9) are so, see Lemma 15.65.2. Hence by induction we see that $C(\alpha\otimes 1)^\bullet$ is m-pseudo-coherent as a complex of B-modules. Finally another application of Lemma 15.65.2 shows that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules (as clearly $E^\bullet \otimes_A B$ is pseudo-coherent as a complex of B-modules). The pseudo-coherent case of (2) $\Rightarrow$ (1) follows from this and Lemma 15.65.5. □

0650 **Lemma 15.65.12.** *Let $A \to B$ be a ring map. Let $K^\bullet$ be an m-pseudo-coherent (resp. pseudo-coherent) complex of A-modules. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) complex of B-modules.*

Proof. First we note that the statement of the lemma makes sense as $K^\bullet$ is bounded above and hence $K^\bullet \otimes^{\mathbf{L}}_A B$ is defined by Equation (15.57.0.2). Having said this, choose a bounded complex $E^\bullet$ of finite free A-modules and $\alpha : E^\bullet \to K^\bullet$ with $H^i(\alpha)$ an isomorphism for $i > m$ and surjective for $i = m$. Then the cone $C(\alpha)^\bullet$ is acyclic in degrees $\geq m$. Since $- \otimes^{\mathbf{L}}_A B$ is an exact functor we get a distinguished triangle

$$(E^\bullet \otimes^{\mathbf{L}}_A B, K^\bullet \otimes^{\mathbf{L}}_A B, C(\alpha)^\bullet \otimes^{\mathbf{L}}_A B)$$

of complexes of B-modules. By the dual to Derived Categories, Lemma 13.16.1 we see that $H^i(C(\alpha)^\bullet \otimes^{\mathbf{L}}_A B) = 0$ for $i \geq m$. Since $E^\bullet$ is a complex of projective R-modules we see that $E^\bullet \otimes^{\mathbf{L}}_A B = E^\bullet \otimes_A B$ and hence

$$E^\bullet \otimes_A B \longrightarrow K^\bullet \otimes^{\mathbf{L}}_A B$$

is a morphism of complexes of B-modules that witnesses the fact that $K^\bullet \otimes^{\mathbf{L}}_A B$ is m-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of m-pseudo-coherent complexes via Lemma 15.65.5. □

066C **Lemma 15.65.13.** *Let $A \to B$ be a flat ring map. Let M be an m-pseudo-coherent (resp. pseudo-coherent) A-module. Then $M \otimes_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) B-module.*

Proof. Immediate consequence of Lemma 15.65.12 and the fact that $M \otimes^{\mathbf{L}}_A B = M \otimes_A B$ because B is flat over A. □

The following lemma also follows from the stronger Lemma 15.65.15.

066D **Lemma 15.65.14.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

Assume $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent for $i = 1, \ldots, r$. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet \otimes_R R_{f_i})$ is nonzero for some i. This implies in particular that $H^i(K^\bullet) = 0$ for $i > n$ (and that $H^n(K^\bullet) \neq 0$) see Algebra, Lemma 10.23.2. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 15.65.7. If $n \geq m$, then $H^n(K^\bullet)_{f_i}$ is a finite R_{f_i}-module for each i, see Lemma 15.65.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.23.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complexes $C(\alpha)^\bullet \otimes_R R_{f_i}$ are m-pseudo-coherent for each i, see Lemma 15.65.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.65.2 once more we conclude. □

068R **Lemma 15.65.15.** *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R'$ is an exact functor and that therefore

$$H^i(K^\bullet) \otimes_R R' = H^i(K^\bullet \otimes_R R').$$

Assume $K^\bullet \otimes_R R'$ is m-pseudo-coherent. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet)$ is nonzero; then n is also the largest integer such that $H^n(K^\bullet \otimes_R R')$ is nonzero. We will prove the lemma by induction on $n-m$. If $n < m$, then the lemma is true by Lemma 15.65.7. If $n \geq m$, then $H^n(K^\bullet) \otimes_R R'$ is a finite R'-module, see Lemma 15.65.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.83.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $C(\alpha)^\bullet \otimes_R R'$ is m-pseudo-coherent, see Lemma 15.65.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.65.2 once more we conclude. □

0DJE **Lemma 15.65.16.** *Let R be a ring. Let K, L be objects of $D(R)$.*

1. *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_R L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
2. *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_R L$ is pseudo-coherent.*

Proof. Proof of (1). We may assume there exist bounded complexes $K^\bullet$ and $L^\bullet$ of finite free R-modules and maps $\alpha : K^\bullet \to K$ and $\beta : L^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to K \otimes^{\mathbf{L}}_R L$$

induces isomorphisms on cohomology in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted). Part (2) follows from part (1) and Lemma 15.65.5. □

066E **Lemma 15.65.17.** *Let R be a Noetherian ring. Then*

(1) *A complex of R-modules $K^\bullet$ is m-pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for $i \geq m$.*
(2) *A complex of R-modules $K^\bullet$ is pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for all i.*
(3) *An R-module is pseudo-coherent if and only if it is finite.*

Proof. In Algebra, Lemma 10.71.1 we have seen that any finite R-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 15.65.4. Hence (3) holds. Suppose that $K^\bullet$ is an m-pseudo-coherent complex. Then there exists a bounded complex of finite free R-modules $E^\bullet$ such that $H^i(K^\bullet)$ is isomorphic to $H^i(E^\bullet)$ for $i > m$ and such that $H^m(K^\bullet)$ is a quotient of $H^m(E^\bullet)$. Thus it is clear that each $H^i(K^\bullet)$, $i \geq m$ is a finite module. The converse implication in (1) follows from Lemma 15.65.10 and part (3). Part (2) follows from (1) and Lemma 15.65.5. □

0EWZ **Lemma 15.65.18.** *Let R be a coherent ring (Algebra, Definition 10.90.1). Let $K \in D^-(R)$. The following are equivalent*

(1) *K is m-pseudo-coherent,*
(2) *$H^m(K)$ is a finite R-module and $H^i(K)$ is coherent for $i > m$, and*
(3) *$H^m(K)$ is a finite R-module and $H^i(K)$ is finitely presented for $i > m$.*

Thus K is pseudo-coherent if and only if $H^i(K)$ is a coherent module for all i.

Proof. Recall that an R-module M is coherent if and only if it is of finite presentation (Algebra, Lemma 10.90.4). This explains the equivalence of (2) and (3). If so and if we choose an exact sequence $0 \to N \to R^{\oplus m} \to M \to 0$, then N is coherent by Algebra, Lemma 10.90.3. Thus in this case, repeating this procedure with N we find a resolution

$$\ldots \to R^{\oplus n} \to R^{\oplus m} \to M \to 0$$

by finite free R-modules. In other words, M is pseudo-coherent. The equivalence of (1) and (2) follows from this and Lemmas 15.65.10 and 15.65.4. The final assertion follows from the equivalence of (1) and (2) combined with Lemma 15.65.5. □

15.66. Pseudo-coherent modules, II

0G8V We continue the discussion started in Section 15.65.

0G8W **Lemma 15.66.1.** *Let R be a ring. Let $M = \operatorname{colim} M_i$ be a filtered colimit of R-modules. Let $K \in D(R)$ be m-pseudo-coherent. Then $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective.*

Proof. By definition we can find a distinguished triangle

$$E \to K \to L \to E[1]$$

in $D(R)$ such that E is represented by a bounded complex of finite free R-modules and such that $H^i(L) = 0$ for $i \geq m$. Then $\operatorname{Ext}^n_R(L, N) = 0$ for any R-module N and $n \leq -m$, see Derived Categories, Lemma 13.27.3. By the long exact sequence of Ext associated to the distinguished triangle we see that $\operatorname{Ext}^n_R(K, N) \to \operatorname{Ext}^n_R(E, N)$ is an isomorphism for $n < -m$ and injective for $n = -m$. Thus it suffices to prove that $M \mapsto \operatorname{Ext}^n_R(E, M)$ commutes with filtered colimits when E can be represented by a bounded complex of finite free R-modules $E^\bullet$. The modules $\operatorname{Ext}^n_R(E, M)$ are computed by the complex $\operatorname{Hom}_R(E^\bullet, M)$, see Derived Categories, Lemma 13.19.8. The functor $M \mapsto \operatorname{Hom}_R(E^p, M)$ commutes with filtered colimits as E^p is finite free.

Thus $\mathrm{Hom}_R(E^\bullet, M) = \mathrm{colim}\, \mathrm{Hom}_R(E^\bullet, M_i)$ as complexes. Since filtered colimits are exact (Algebra, Lemma 10.8.8) we conclude. □

0G8X **Lemma 15.66.2.** *Let R be a ring. Let $K \in D^-(R)$. Let $m \in \mathbf{Z}$. Then K is m-pseudo-coherent if and only if for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules we have $\mathrm{colim}\, \mathrm{Ext}^n_R(K, M_i) = \mathrm{Ext}^n_R(K, M)$ for $n < -m$ and $\mathrm{colim}\, \mathrm{Ext}^{-m}_R(K, M_i) \to \mathrm{Ext}^{-m}_R(K, M)$ is injective.*

Proof. One implication was shown in Lemma 15.66.1. Assume for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules we have $\mathrm{colim}\, \mathrm{Ext}^n_R(K, M_i) = \mathrm{Ext}^n_R(K, M)$ for $n < -m$ and $\mathrm{colim}\, \mathrm{Ext}^{-m}_R(K, M_i) \to \mathrm{Ext}^{-m}_R(K, M)$ is injective. We will show K is m-pseudo-coherent.

Let t be the maximal integer such that $H^t(K)$ is nonzero. We will use induction on t. If $t < m$, then K is m-pseudo-coherent by Lemma 15.65.7. If $t \geq m$, then since $\mathrm{Hom}_R(H^t(K), M) = \mathrm{Ext}^{-t}_R(K, M)$ we conclude that $\mathrm{colim}\, \mathrm{Hom}_R(H^t(K), M_i) \to \mathrm{Hom}_R(H^t(K), M)$ is injective for any filtered colimit $M = \mathrm{colim}\, M_i$. This implies that $H^t(K)$ is a finite R-module by Algebra, Lemma 10.11.1. Choose a finite free R-module F and a surjection $F \to H^t(K)$. We can lift this to a morphism $F[-t] \to K$ in $D(R)$ and choose a distinguished triangle

$$F[-t] \to K \to L \to F[-t+1]$$

in $D(R)$. Then $H^i(L) = 0$ for $i \geq t$. Moreover, the long exact sequence of Ext associated to this distinguished triangle shows that L inherits the assumption we made on K by a small argument we omit. By induction on t we conclude that L is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 15.65.2. □

087Q **Lemma 15.66.3.** *Let R be a ring. Let L, M, N be R-modules.*

(1) *If M is finitely presented and L is flat, then the canonical map $\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$ is an isomorphism.*
(2) *If M is $(-m)$-pseudo-coherent and L is flat, then the canonical map $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ is an isomorphism for $i < m$.*

Proof. Choose a resolution $F_\bullet \to M$ whose terms are free R-modules, see Algebra, Lemma 10.71.1. The complex $\mathrm{Hom}_R(F_\bullet, N)$ computes $\mathrm{Ext}^i_R(M, N)$ and the complex $\mathrm{Hom}_R(F_\bullet, N \otimes_R L)$ computes $\mathrm{Ext}^i_R(M, N \otimes_R L)$. There always is a map of cochain complexes

$$\mathrm{Hom}_R(F_\bullet, N) \otimes_R L \longrightarrow \mathrm{Hom}_R(F_\bullet, N \otimes_R L)$$

which induces canonical maps $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ for all $i \geq 0$ (canonical for example in the sense that these maps do not depend on the choice of the resolution $F_\bullet$). If L is flat, then the complex $\mathrm{Hom}_R(F_\bullet, N) \otimes_R L$ computes $\mathrm{Ext}^i_R(M, N) \otimes_R L$ since taking cohomology commutes with tensoring by L.

Having said all of the above, if M is $(-m)$-pseudo-coherent, then we may choose $F_\bullet$ such that F_i is finite free for $i = 0, \ldots, m$. Then the map of cochain complexes displayed above is an isomorphism in degrees $\leq m$ and hence an isomorphism on cohomology groups in degrees $< m$. This proves (2). If M is finitely presented, then M is (-1)-pseudo-coherent by Lemma 15.65.4 and we get the result because $\mathrm{Hom} = \mathrm{Ext}^0$. □

087R **Lemma 15.66.4.** *Let $R \to R'$ be a flat ring map. Let M, N be R-modules.*

(1) *If M is a finitely presented R-module, then* $\mathrm{Hom}_R(M, N) \otimes_R R' = \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$.
(2) *If M is $(-m)$-pseudo-coherent, then* $\mathrm{Ext}^i_R(M, N) \otimes_R R' = \mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$ *for $i < m$.*

In particular if R is Noetherian and M is a finite module this holds for all i.

Proof. By Algebra, Lemma 10.73.1 we have $\mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') = \mathrm{Ext}^i_R(M, N \otimes_R R')$. Combined with Lemma 15.66.3 we conclude (1) and (2) holds. The final statement follows from this and Lemma 15.65.17. □

0CYB **Lemma 15.66.5.** *Let R be a ring. Let $K \in D^-(R)$. The following are equivalent:*

(1) *K is pseudo-coherent,*
(2) *for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map*
$$\alpha : K \otimes^{\mathbf{L}}_R \left(\prod\nolimits_\alpha Q_\alpha \right) \longrightarrow \prod\nolimits_\alpha (K \otimes^{\mathbf{L}}_R Q_\alpha)$$
is an isomorphism in $D(R)$,
(3) *for every R-module Q and every set A, the canonical map*
$$\beta : K \otimes^{\mathbf{L}}_R Q^A \longrightarrow (K \otimes^{\mathbf{L}}_R Q)^A$$
is an isomorphism in $D(R)$, and
(4) *for every set A, the canonical map*
$$\gamma : K \otimes^{\mathbf{L}}_R R^A \longrightarrow K^A$$
is an isomorphism in $D(R)$.

Given $m \in \mathbf{Z}$ the following are equivalent

(a) *K is m-pseudo-coherent,*
(b) *for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, with α as above $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$,*
(c) *for every R-module Q and every set A, with β as above $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$,*
(d) *for every set A, with γ as above $H^i(\gamma)$ is an isomorphism for $i > m$ and surjective for $i = m$.*

Proof. If K is pseudo-coherent, then K can be represented by a bounded above complex of finite free R-modules. Then the derived tensor products are computed by tensoring with this complex. Also, products in $D(R)$ are given by taking products of any choices of representative complexes. Hence (1) implies (2), (3), (4) by the corresponding fact for modules, see Algebra, Proposition 10.89.3.

In the same way (using the tensor product is right exact) the reader shows that (a) implies (b), (c), and (d).

Assume (4) holds. To show that K is pseudo-coherent it suffices to show that K is m-pseudo-coherent for all m (Lemma 15.65.5). Hence to finish then proof it suffices to prove that (d) implies (a).

Assume (d). Let i be the largest integer such that $H^i(K)$ is nonzero. If $i < m$, then we are done. If not, then from (d) and the description of products in $D(R)$ given above we find that $H^i(K) \otimes_R R^A \to H^i(K)^A$ is surjective. Hence $H^i(K)$ is a finitely generated R-module by Algebra, Proposition 10.89.2. Thus we may choose a complex L consisting of a single finite free module sitting in degree i and a map of

complexes $L \to K$ such that $H^i(L) \to H^i(K)$ is surjective. In particular L satisfies (1), (2), (3), and (4). Choose a distinguished triangle

$$L \to K \to M \to L[1]$$

Then we see that $H^j(M) = 0$ for $j \geq i$. On the other hand, M still has property (d) by a small argument which we omit. By induction on i we find that M is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 15.65.2. □

0G8Y **Lemma 15.66.6.** *Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let $i \in \mathbf{Z}$. There exists a finitely presented R-module M and a map $K \to M[-i]$ in $D(R)$ which induces an injection $H^i(K) \to M$.*

Proof. By Definition 15.65.1 we may represent K by a complex $P^\bullet$ of finite free R-modules. Set $M = \mathrm{Coker}(P^{i-1} \to P^i)$. □

0A7D **Lemma 15.66.7.** *Let A be a Noetherian ring. Let $K \in D(A)$ be pseudo-coherent, i.e., $K \in D^-(A)$ with finite cohomology modules. Let $\mathfrak{m}$ be a maximal ideal of A. If $H^i(K)/\mathfrak{m}H^i(K) \neq 0$, then there exists a finite A-module E annihilated by a power of $\mathfrak{m}$ and a map $K \to E[-i]$ which is nonzero on $H^i(K)$.*

Proof. (The equivalent formulation of pseudo-coherence in the statement of the lemma is Lemma 15.65.17.) Choose $K \to M[-i]$ as in Lemma 15.66.6. By Artin-Rees (Algebra, Lemma 10.51.2) we can find an n such that $H^i(K) \cap \mathfrak{m}^n M \subset \mathfrak{m}H^i(K)$. Take $E = M/\mathfrak{m}^n M$. □

15.67. Tor dimension

0651 Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

0652 **Definition 15.67.1.** Let R be a ring. Denote $D(R)$ its derived category. Let $a, b \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ has *tor-amplitude in* $[a, b]$ if $H^i(K^\bullet \otimes^{\mathbf{L}}_R M) = 0$ for all R-modules M and all $i \notin [a, b]$.
(2) An object $K^\bullet$ of $D(R)$ has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) An R-module M has *tor dimension* $\leq d$ if $M[0]$ as an object of $D(R)$ has tor-amplitude in $[-d, 0]$.
(4) An R-module M has *finite tor dimension* if $M[0]$ as an object of $D(R)$ has finite tor dimension.

We observe that if $K^\bullet$ has finite tor dimension, then $K^\bullet \in D^b(R)$.

0653 **Lemma 15.67.2.** *Let R be a ring. Let $K^\bullet$ be a bounded above complex of flat R-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d_K^{a-1})$ is a flat R-module.*

Proof. As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes^{\mathbf{L}}_R M$. Hence for every R-module M the sequence

$$K^{a-2} \otimes_R M \to K^{a-1} \otimes_R M \to K^a \otimes_R M$$

is exact in the middle. Since $K^{a-2} \to K^{a-1} \to K^a \to \mathrm{Coker}(d_K^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^R(\mathrm{Coker}(d_K^{a-1}), M) = 0$ for all R-modules M. This means that $\mathrm{Coker}(d_K^{a-1})$ is flat, see Algebra, Lemma 10.75.8. □

0654 **Lemma 15.67.3.** *Let R be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *$K^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *$K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat R-modules with $E^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $K^\bullet \otimes^{\mathbf{L}}_R M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution with $K^i = 0$ for $i > b$. See Derived Categories, Lemma 13.19.3. Set $E^\bullet = \tau_{\geq a} K^\bullet$. Everything is clear except that E^a is flat which follows immediately from Lemma 15.67.2 and the definitions. □

0BYL **Lemma 15.67.4.** *Let R be a ring. Let $a \in \mathbf{Z}$ and let K be an object of $D(R)$. The following are equivalent*

(1) *K has tor-amplitude in $[a, \infty]$, and*
(2) *K is quasi-isomorphic to a K-flat complex $E^\bullet$ whose terms are flat R-modules with $E^i = 0$ for $i \notin [a, \infty]$.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $K^\bullet$ with flat terms representing K, see Lemma 15.59.10. For any R-module M the cohomology of

$$K^{n-1} \otimes_R M \to K^n \otimes_R M \to K^{n+1} \otimes_R M$$

computes $H^n(K \otimes^{\mathbf{L}}_R M)$. This is always zero for $n < a$. Hence if we apply Lemma 15.67.2 to the complex $\ldots \to K^{a-1} \to K^a \to K^{a+1}$ we conclude that $N = \mathrm{Coker}(K^{a-1} \to K^a)$ is a flat R-module. We set

$$E^\bullet = \tau_{\geq a} K^\bullet = (\ldots \to 0 \to N \to K^{a+1} \to \ldots)$$

The kernel $L^\bullet$ of $K^\bullet \to E^\bullet$ is the complex

$$L^\bullet = (\ldots \to K^{a-1} \to I \to 0 \to \ldots)$$

where $I \subset K^a$ is the image of $K^{a-1} \to K^a$. Since we have the short exact sequence $0 \to I \to K^a \to N \to 0$ we see that I is a flat R-module. Thus $L^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 15.59.7. It follows that $E^\bullet$ is K-flat by Lemma 15.59.6. □

0655 **Lemma 15.67.5.** *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbf{Z}$.*

(1) *If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.*
(3) *If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_R M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

066F **Lemma 15.67.6.** *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

(1) *M has tor dimension $\leq d$, and*

(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with F_i a flat R-module.

In particular an R-module has tor dimension 0 if and only if it is a flat R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to M. Hence the Tor dimension of M is at most d by Lemma 15.67.3. Conversely, assume (1). Let $P^\bullet \to M$ be a projective resolution of M. By Lemma 15.67.2 we see that $\tau_{\geq -d}P^\bullet$ is a flat resolution of M of length d, i.e., (2) holds. □

066G **Lemma 15.67.7.** *Let R be a ring. Let $a, b \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.*

Proof. Clear from the fact that the Tor functors are additive. □

066H **Lemma 15.67.8.** *Let R be a ring. Let $K^\bullet$ be a bounded complex of R-modules such that K^i has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of R-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.67.5 and the stupid truncations. □

066I **Lemma 15.67.9.** *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^b(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.67.5 and the canonical truncations. □

0B66 **Lemma 15.67.10.** *Let $A \to B$ be a ring map. Let $K^\bullet$ and $L^\bullet$ be complexes of B-modules. Let $a, b, c, d \in \mathbf{Z}$. If*

(1) *$K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$,*
(2) *$L^\bullet$ as a complex of A-modules has tor amplitude in $[c, d]$,*

then $K^\bullet \otimes^{\mathbf{L}}_B L^\bullet$ as a complex of A-modules has tor amplitude in $[a+c, b+d]$.

Proof. We may assume that $K^\bullet$ is a complex of flat B-modules with $K^i = 0$ for $i \notin [a, b]$, see Lemma 15.67.3. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$(K^\bullet \otimes^{\mathbf{L}}_B L^\bullet) \otimes^{\mathbf{L}}_A M = \mathrm{Tot}(\mathrm{Tot}(K^\bullet \otimes_B L^\bullet) \otimes_A F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_B \mathrm{Tot}(L^\bullet \otimes_A F^\bullet))$$

see Homology, Remark 12.18.4 for the second equality. By assumption (2) the complex $\mathrm{Tot}(L^\bullet \otimes_A F^\bullet)$ has nonzero cohomology only in degrees $[c, d]$. Hence the spectral sequence of Homology, Lemma 12.25.1 for the double complex $K^\bullet \otimes_B \mathrm{Tot}(L^\bullet \otimes_A F^\bullet)$ proves that $(K^\bullet \otimes^{\mathbf{L}}_B L^\bullet) \otimes^{\mathbf{L}}_A M$ has nonzero cohomology only in degrees $[a+c, b+d]$. □

066J **Lemma 15.67.11.** *Let $A \to B$ be a ring map. Assume that B is flat as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a, b]$.*

Proof. This is a special case of Lemma 15.67.10, but can also be seen directly as follows. We have $K^\bullet \otimes^{\mathbf{L}}_A M = K^\bullet \otimes^{\mathbf{L}}_B (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of B-modules is a flat resolution of $K^\bullet$ as a complex of A-modules and can be used to compute $K^\bullet \otimes^{\mathbf{L}}_A M$. □

066K **Lemma 15.67.12.** *Let $A \to B$ be a ring map. Assume that B has tor dimension $\leq d$ as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a - d, b]$.*

Proof. This is a special case of Lemma 15.67.10, but can also be seen directly as follows. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes^{\mathbf{L}}_A M = \mathrm{Tot}(K^\bullet \otimes_A F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes^{\mathbf{L}}_B (M \otimes^{\mathbf{L}}_A B).$$

By our assumption on B as an A-module we see that $M \otimes^{\mathbf{L}}_A B$ has cohomology only in degrees $-d, -d+1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 15.62.4 that $K^\bullet \otimes^{\mathbf{L}}_B (M \otimes^{\mathbf{L}}_A B)$ has cohomology only in degrees $[-d + a, b]$ as desired. □

066L **Lemma 15.67.13.** *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ as a complex of B-modules has tor amplitude in $[a, b]$.*

Proof. By Lemma 15.67.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat A-modules with $E^i = 0$ for $i \notin [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes^{\mathbf{L}}_A B$ by construction and each $E^i \otimes_A B$ is a flat B-module by Algebra, Lemma 10.39.7. Hence we conclude by Lemma 15.67.3. □

066M **Lemma 15.67.14.** *Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let M be an A-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a B-module of tor dimension $\leq d$.*

Proof. Immediate consequence of Lemma 15.67.13 and the fact that $M \otimes^{\mathbf{L}}_A B = M \otimes_A B$ because B is flat over A. □

0B67 **Lemma 15.67.15.** *Let $A \to B$ be a ring map. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules,*
2. *$K^\bullet_{\mathfrak{q}}$ has tor amplitude in $[a, b]$ as a complex of $A_{\mathfrak{p}}$-modules for every prime $\mathfrak{q} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{q}$,*
3. *$K^\bullet_{\mathfrak{m}}$ has tor amplitude in $[a, b]$ as a complex of $A_{\mathfrak{p}}$-modules for every maximal ideal $\mathfrak{m} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{m}$.*

Proof. Assume (3) and let M be an A-module. Then $H^i = H^i(K^\bullet \otimes^{\mathbf{L}}_A M)$ is a B-module and $(H^i)_{\mathfrak{m}} = H^i(K^\bullet_{\mathfrak{m}} \otimes^{\mathbf{L}}_{A_{\mathfrak{p}}} M_{\mathfrak{p}})$. Hence $H^i = 0$ for $i \notin [a, b]$ by Algebra, Lemma 10.23.1. Thus (3) $\Rightarrow$ (1). We omit the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). □

066N **Lemma 15.67.16.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. This follows immediately from Lemma 15.67.15 but can also be seen directly as follows. Note that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

and similarly for every R-module M we have

$$H^i(K^\bullet \otimes^{\mathbf{L}}_R M)_{f_i} = H^i(K^\bullet \otimes^{\mathbf{L}}_R M) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i} \otimes^{\mathbf{L}}_{R_{f_i}} M_{f_i}).$$

Hence the result follows from the fact that an R-module N is zero if and only if N_{f_i} is zero for each i, see Algebra, Lemma 10.23.2. □

068S **Lemma 15.67.17.** *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Let M be an R-module. Since $R \to R'$ is flat we see that

$$(M \otimes^{\mathbf{L}}_R K^\bullet) \otimes_R R' = ((M \otimes_R R') \otimes^{\mathbf{L}}_{R'} (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with R'. Hence $\mathrm{Tor}^R_i(M, K^\bullet) \otimes_R R' = \mathrm{Tor}^{R'}_i(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\mathrm{Tor}^{R'}_i(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \notin [a, b]$ implies the same thing for $\mathrm{Tor}^R_i(M, K^\bullet)$. □

0DJF **Lemma 15.67.18.** *Given ring maps $R \to A \to B$ with $A \to B$ faithfully flat and $K \in D(A)$ the tor amplitude of K over R is the same as the tor amplitude of $K \otimes^{\mathbf{L}}_A B$ over R.*

Proof. This is true because for an R-module M we have $H^i(K \otimes^{\mathbf{L}}_R M) \otimes_A B = H^i((K \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_R M)$ for all i. Namely, represent K by a complex $K^\bullet$ of A-modules and choose a free resolution $F^\bullet \to M$. Then we have the equality

$$\mathrm{Tot}(K^\bullet \otimes_A B \otimes_R F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_R F^\bullet) \otimes_A B$$

The cohomology groups of the left hand side are $H^i((K \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_R M)$ and on the right hand side we obtain $H^i(K \otimes^{\mathbf{L}}_R M) \otimes_A B$. □

066P **Lemma 15.67.19.** *Let R be a ring of finite global dimension d. Then*

1. *every module has tor dimension $\leq d$,*
2. *a complex of R-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and*
3. *a complex of R-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.*

Proof. The assumption on R means that every module has a finite projective resolution of length at most d, in particular every module has tor dimension $\leq d$. The second statement follows from Lemma 15.67.9 and the definitions. The third statement is a rephrasing of the second. □

0H75 **Lemma 15.67.20.** *Let $R' \to R$ be a surjective ring map whose kernel is a nilpotent ideal. Let $K' \in D(R')$ and set $K = K' \otimes^{\mathbf{L}}_{R'} R$. Let $a, b \in \mathbf{Z}$. Then K has tor amplitude in $[a, b]$ if and only if K' does.*

Proof. One direction follows from Lemma 15.67.13. For the other, assume K has tor amplitude in $[a, b]$ and let M' be an R'-module. We have to show that $K' \otimes^{\mathbf{L}}_{R'} M'$ has nonzero cohomology only for degrees contained in the interval $[a, b]$.

Let $I = \mathrm{Ker}(R' \to R)$. Then $I^n = 0$ for some n. If $IM' = 0$, then we can view M' as an R-module and argue as follows

$$K' \otimes^{\mathbf{L}}_{R'} M' = K' \otimes^{\mathbf{L}}_{R'} (R \otimes^{\mathbf{L}}_{R} M') = (K' \otimes^{\mathbf{L}}_{R'} R) \otimes^{\mathbf{L}}_{R} M' = K \otimes^{\mathbf{L}}_{R} M'$$

which has nonvanishing cohomology only in the interval $[a, b]$ by assumption on K. If $I^{t+1}M' = 0$, then we consider the short exact sequence

$$0 \to IM' \to M' \to M'/IM' \to 0$$

By induction on t we have that both $K' \otimes^{\mathbf{L}}_{R'} IM'$ and $K' \otimes^{\mathbf{L}}_{R'} M'/IM'$ have nonzero cohomology only for degrees in the interval $[a, b]$. Then the distinguished triangle

$$K' \otimes^{\mathbf{L}}_{R'} IM' \to K' \otimes^{\mathbf{L}}_{R'} M' \to K' \otimes^{\mathbf{L}}_{R'} M'/IM' \to (K' \otimes^{\mathbf{L}}_{R'} IM')[1]$$

proves the same is true for $K' \otimes^{\mathbf{L}}_{R'} M'$ as desired. □

15.68. Spectral sequences for Ext

0AVG In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of objects L, K of the derived category $D(R)$ of a ring R we denote

$$\mathrm{Ext}^n_R(L, K) = \mathrm{Hom}_{D(R)}(L, K[n])$$

according to our general conventions in Derived Categories, Section 13.27.

For M an R-module and $K \in D^+(R)$ there is a spectral sequence

0AVH (15.68.0.1) $$E_2^{i,j} = \mathrm{Ext}^i_R(M, H^j(K)) \Rightarrow \mathrm{Ext}^{i+j}_R(M, K)$$

and if K is represented by the bounded below complex $K^\bullet$ of R-modules there is a spectral sequence

0AVI (15.68.0.2) $$E_1^{i,j} = \mathrm{Ext}^j_R(M, K^i) \Rightarrow \mathrm{Ext}^{i+j}_R(M, K)$$

These spectral sequences come from applying Derived Categories, Lemma 13.21.3 to the functor $\mathrm{Hom}_R(M, -)$.

15.69. Projective dimension

0A5M We defined the projective dimension of a module in Algebra, Definition 10.109.2.

0A5N **Definition 15.69.1.** Let R be a ring. Let K be an object of $D(R)$. We say K has *finite projective dimension* if K can be represented by a bounded complex of projective modules. We say K has *projective-amplitude in* $[a, b]$ if K is quasi-isomorphic to a complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$.

Clearly, K has finite projective dimension if and only if K has projective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has finite projective dimension, then K is bounded. Here is a lemma to detect such objects of $D(R)$.

0A5P **Lemma 15.69.2.** *Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *K has projective-amplitude in $[a, b]$,*
2. *$\mathrm{Ext}^i_R(K, N) = 0$ for all R-modules N and all $i \notin [-b, -a]$,*

(3) $H^n(K) = 0$ *for* $n > b$ *and* $\mathrm{Ext}^i_R(K, N) = 0$ *for all* R*-modules* N *and all* $i > -a$*, and*
(4) $H^n(K) = 0$ *for* $n \notin [a-1, b]$ *and* $\mathrm{Ext}^{-a+1}_R(K, N) = 0$ *for all* R*-modules* N*.*

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \mathrm{Hom}_R(P^b, N) \to \ldots \to \mathrm{Hom}_R(P^a, N) \to 0 \to \ldots$$

and we obtain (2).

Assume (2) holds. Choose an injection $H^n(K) \to I$ where I is an injective R-module. Since $\mathrm{Hom}_R(-, I)$ is an exact functor, we see that $\mathrm{Ext}^{-n}(K, I) = \mathrm{Hom}_R(H^n(K), I)$. We conclude in particular that $H^n(K)$ is zero for $n > b$. Thus (2) implies (3).

By the same argument as in (2) implies (3) gives that (3) implies (4).

Assume (4). The same argument as in (2) implies (3) shows that $H^{a-1}(K) = 0$, i.e., we have $H^i(K) = 0$ unless $i \in [a, b]$. In particular, K is bounded above and we can choose a a complex $P^\bullet$ representing K with P^i projective (for example free) for all $i \in \mathbf{Z}$ and $P^i = 0$ for $i > b$. See Derived Categories, Lemma 13.15.4. Let $Q = \mathrm{Coker}(P^{a-1} \to P^a)$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to Q \to P^{a+1} \to \ldots \to P^b \to 0 \to \ldots$$

as $H^i(K) = 0$ for $i < a$. Denote $K' = (P^{a+1} \to \ldots \to P^b)$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$K' \to K \to Q[-a] \to K'[1]$$

in $D(R)$. Thus for every R-module N an exact sequence

$$\mathrm{Ext}^{-a}(K', N) \to \mathrm{Ext}^1(Q, N) \to \mathrm{Ext}^{1-a}(K, N)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus Q is a projective R-module by Algebra, Lemma 10.77.2. Hence (1) holds and the proof is complete. □

0A5Q **Example 15.69.3.** Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R \to \ldots$$

but M does not have finite projective dimension as defined in Algebra, Definition 10.109.2. This explains why we consider bounded (in both directions) complexes of projective modules in our definition of finite projective dimension of objects of $D(R)$.

15.70. Injective dimension

0A5R This section is the dual of the section on projective dimension.

0A5S **Definition 15.70.1.** Let R be a ring. Let K be an object of $D(R)$. We say K has *finite injective dimension* if K can be represented by a finite complex of injective R-modules. We say K has *injective-amplitude in* $[a, b]$ if K is isomorphic to a complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

with I^i an injective R-module for all $i \in \mathbf{Z}$.

Clearly, K has bounded injective dimension if and only if K has injective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has bounded injective dimension, then K is bounded. Here is the obligatory lemma.

0A5T **Lemma 15.70.2.** *Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *K has injective-amplitude in $[a, b]$,*
2. *$\mathrm{Ext}^i_R(N, K) = 0$ for all R-modules N and all $i \notin [a, b]$,*
3. *$\mathrm{Ext}^i(R/I, K) = 0$ for all ideals $I \subset R$ and all $i \notin [a, b]$.*

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

where I^i is an injective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \mathrm{Hom}_R(N, I^a) \to \ldots \to \mathrm{Hom}_R(N, I^b) \to 0 \to \ldots$$

and we obtain (2). It is clear that (2) implies (3).

Assume (3) holds. Choose a nonzero map $R \to H^n(K)$. Since $\mathrm{Hom}_R(R, -)$ is an exact functor, we see that $\mathrm{Ext}^n_R(R, K) = \mathrm{Hom}_R(R, H^n(K)) = H^n(K)$. We conclude that $H^n(K)$ is zero for $n \notin [a, b]$. In particular, K is bounded below and we can choose a quasi-isomorphism

$$K \to I^\bullet$$

with I^i injective for all $i \in \mathbf{Z}$ and $I^i = 0$ for $i < a$. See Derived Categories, Lemma 13.15.5. Let $J = \mathrm{Ker}(I^b \to I^{b+1})$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to I^a \to \ldots \to I^{b-1} \to J \to 0 \to \ldots$$

Denote $K' = (I^a \to \ldots \to I^{b-1})$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$J[-b] \to K \to K' \to J[1 - b]$$

in $D(R)$. Thus for every ideal $I \subset R$ an exact sequence

$$\mathrm{Ext}^b(R/I, K') \to \mathrm{Ext}^1(R/I, J) \to \mathrm{Ext}^{1+b}(R/I, K)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus J is an injective R-module by Lemma 15.55.4. $\square$

0EX0 **Example 15.70.3.** Let R be a Dedekind domain. Then every nonzero ideal I is a finite projective module, see Lemma 15.22.11. Thus R/I has projective dimension 1. Hence every R-module M has injective dimension ≤ 1 by Lemma 15.70.2. Thus $\mathrm{Ext}^i_R(M, N) = 0$ for $i \geq 2$ and any pair of R-modules M, N. It follows that any object K in $D^b(R)$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$, see Derived Categories, Lemma 13.27.10.

0A5U **Example 15.70.4.** Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$\ldots \to R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R$$

and R is an injective R-module. However one usually does not consider M to have finite injective dimension in this situation. This explains why we consider bounded (in both directions) complexes of injective modules in our definition of bounded injective dimension of objects of $D(R)$.

0A5V **Lemma 15.70.5.** *Let R be a ring. Let $K \in D(R)$.*

(1) *If K is in $D^b(R)$ and $H^i(K)$ has finite injective dimension for all i, then K has finite injective dimension.*
(2) *If $K^\bullet$ represents K, is a bounded complex of R-modules, and K^i has finite injective dimension for all i, then K has finite injective dimension.*

Proof. Omitted. Hint: Apply the spectral sequences of Derived Categories, Lemma 13.21.3 to the functor $F = \mathrm{Hom}_R(N, -)$ to get a computation of $\mathrm{Ext}^i_A(N, K)$ and use the criterion of Lemma 15.70.2. □

0DW2 **Lemma 15.70.6.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent*

(1) *K has finite injective dimension, and*
(2) *there exists a b such that $\mathrm{Ext}^i_R(R/J, K) = 0$ for $i > b$ and any ideal $J \supset I$.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). Say $H^i(K) = 0$ for $i < a$. Then $\mathrm{Ext}^i(M, K) = 0$ for $i < a$ and all R-modules M. Thus it suffices to show that $\mathrm{Ext}^i(M, K) = 0$ for $i > b$ any finite R-module M, see Lemma 15.70.2. By Algebra, Lemma 10.62.1 the module M has a finite filtration whose successive quotients are of the form $R/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. If $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence and $\mathrm{Ext}^i(M_j, K) = 0$ for $i > b$ and $j = 1, 2$, then $\mathrm{Ext}^i(M, K) = 0$ for $i > b$. Thus we may assume $M = R/\mathfrak{p}$. If $I \subset \mathfrak{p}$, then the vanishing follows from the assumption. If not, then choose $f \in I$, $f \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{f} R/\mathfrak{p} \to R/(\mathfrak{p}, f) \to 0$$

The R-module $R/(\mathfrak{p}, f)$ has a filtration whose successive quotients are $R/\mathfrak{q}$ with $(\mathfrak{p}, f) \subset \mathfrak{q}$. Thus by Noetherian induction and the argument above we may assume the vanishing holds for $R/(\mathfrak{p}, f)$. On the other hand, the modules $E^i = \mathrm{Ext}^i(R/\mathfrak{p}, K)$ are finite by our assumption on K (bounded below with finite cohomology modules), the spectral sequence (15.68.0.1), and Algebra, Lemma 10.71.9. Thus E^i for $i > b$ is a finite R-module such that $E^i/fE^i = 0$. We conclude by Nakayama's lemma (Algebra, Lemma 10.20.1) that E^i is zero. □

0AVJ **Lemma 15.70.7.** *Let $(R, \mathfrak{m}, \kappa)$ be a local Noetherian ring. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent*

(1) *K has finite injective dimension, and*
(2) *$\mathrm{Ext}^i_R(\kappa, K) = 0$ for $i \gg 0$.*

Proof. This is a special case of Lemma 15.70.6. □

15.71. Modules which are close to being projective

0G8Z There seem to be many different of definitions in the literature of "almost projective modules". In this section we discuss just one of the many possibilities.

0G90 **Lemma 15.71.1.** *Let R be a ring. Let M, N be R-modules.*

(1) *Given an R-module map $\varphi : M \to N$ the following are equivalent: (a) φ factors through a projective R-module, and (b) φ factors through a free R-module.*
(2) *The set of $\varphi : M \to N$ satisfying the equivalent conditions of (1) is an R-submodule of $\mathrm{Hom}_R(M, N)$.*
(3) *Given maps $\psi : M' \to M$ and $\xi : N \to N'$, if $\varphi : M \to N$ satisfies the equivalent conditions of (1), then $\xi \circ \varphi \circ \psi : M' \to N'$ does too.*

Proof. The equivalence of (1)(a) and (1)(b) follows from Algebra, Lemma 10.77.2. If $\varphi : M \to N$ and $\varphi' : M \to N$ factor through the modules P and P' then $\varphi + \varphi'$ factors through $P \oplus P'$ and $\lambda\varphi$ factors through P for all $\lambda \in R$. This proves (2). If $\varphi : M \to N$ factors through the module P and ψ and ξ are as in (3), then $\xi \circ \varphi \circ \psi$ factors through P. This proves (3). □

0G91 **Lemma 15.71.2.** *Let R be a ring. Let $\varphi : M \to N$ be an R-module map. If φ factors through a projective module and M is a finite R-module, then φ factors through a finite projective module.*

Proof. By Lemma 15.71.1 we can factor $\varphi = \tau \circ \sigma$ where the target of σ is $\bigoplus_{i \in I} R$ for some set I. Choose generators $x_1, \ldots, x_n$ for M. Write $\sigma(x_j) = (a_{ji})_{i \in I}$. For each j only a finite number of a_{ij} are nonzero. Hence σ has image contained in a finite free R-module and we conclude. □

Let R be a ring. Observe that an R-module is projective if and only if the identity on R factors through a projective module.

0G92 **Lemma 15.71.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. The following conditions are equivalent*

(1) *for every $a \in I$ the map $a : M \to M$ factors through a projective R-module,*
(2) *for every $a \in I$ the map $a : M \to M$ factors through a free R-module, and*
(3) *$\mathrm{Ext}^1_R(M, N)$ is annihilated by I for every R-module N.*

Proof. The equivalence of (1) and (2) follows from Lemma 15.71.1. If (1) holds, then (3) holds because $\mathrm{Ext}^1_R(P, N)$ for any N and any projective module P. Conversely, assume (3) holds. Choose a short exact sequence $0 \to N \to P \to M \to 0$ with P projective (or even free). By assumption the corresponding element of $\mathrm{Ext}^1_R(M, N)$ is annihilated by I. Hence for every $a \in I$ the map $a : M \to M$ can be factored through the surjection $P \to M$ and we conclude (1) holds. □

In order to comfortably talk about modules satisfying the equivalent conditions of Lemma 15.71.3 we give the property a name.

0G93 **Definition 15.71.4.** Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-projective*[9] if the equivalent conditions of Lemma 15.71.3 hold.

Modules annihilated by I are I-projective.

0G94 **Lemma 15.71.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M is annihilated by I, then M is I-projective.*

Proof. Immediate from the definition and the fact that the zero module is projective. □

0G95 **Lemma 15.71.6.** *Let R be a ring. Let $I \subset R$ be an ideal. Let*

$$0 \to K \to P \to M \to 0$$

be a short exact sequence of R-modules. If M is I-projective and P is projective, then K is I-projective.

Proof. The element $\mathrm{id}_K \in \mathrm{Hom}_R(K, K)$ maps to the class of the given extension in $\mathrm{Ext}^1_R(M, K)$. Since by assumption this class is annihilated by any $a \in I$ we see that $a : K \to K$ factors through $K \to P$ and we conclude. □

0G96 **Lemma 15.71.7.** *Let R be a ring. Let $I \subset R$ be an ideal. If M is a finite, I-projective R-module, then $M^\vee = \mathrm{Hom}_R(M, R)$ is I-projective.*

Proof. Assume M is finite and I-projective. Choose a short exact sequence $0 \to K \to R^{\oplus r} \to M \to 0$. This produces an injection $M^\vee \to R^{\oplus r} = (R^{\oplus r})^\vee$. Since the extension class in $\mathrm{Ext}^1_R(M, K)$ corresponding to the short exact sequence is annihilated by I, we see that for any $a \in I$ we can find a map $M \to R^{\oplus r}$ such that the composition with the given map $R^{\oplus r} \to M$ is equal to $a : M \to M$. Taking duals we find that $a : M^\vee \to M^\vee$ factors through the map $M^\vee \to R^{\oplus r}$ given above and we conclude. □

15.72. Hom complexes

0A8H Let R be a ring. Let $L^\bullet$ and $M^\bullet$ be two complexes of R-modules. We construct a complex $\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)$. Namely, for each n we set

$$\mathrm{Hom}^n(L^\bullet, M^\bullet) = \prod\nolimits_{n=p+q} \mathrm{Hom}_R(L^{-q}, M^p)$$

It is a good idea to think of Hom^n as the R-module of all R-linear maps from $L^\bullet$ to $M^\bullet$ (viewed as graded modules) which are homogeneous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

for $f \in \mathrm{Hom}^n(L^\bullet, M^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. See Section 15.73 for sign rules. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A5X (15.72.0.1) $$H^n(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}_{K(R)}(L^\bullet, M^\bullet[n])$$

for all $n \in \mathbf{Z}$.

[9] This is nonstandard notation.

0A5Y **Lemma 15.72.1.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical isomorphism*

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathit{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

of complexes of R-modules.

Proof. Let α be an element of degree n on the left hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}^p(L^\bullet, M^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r))$$

If we make the identifications

0A5Z (15.72.1.1) $$\mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r)) = \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

then by our sign rules we get

$$\begin{aligned} \mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_K \\ &= \mathrm{d}_M \circ \alpha^{r,s,q} - (-1)^{r+s} \alpha^{r,s,q} \circ \mathrm{d}_L - (-1)^{r+s+q} \alpha^{r,s,q} \circ \mathrm{d}_K \end{aligned}$$

On the other hand, if β is an element of degree n of the right hand side, then

$$\beta = (\beta^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

and by our sign rule (Homology, Definition 12.18.3) we get

$$\begin{aligned} \mathrm{d}(\beta^{r,s,q}) &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^n \beta^{r,s,q} \circ \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes L^\bullet)} \\ &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^{r+s+q} \left(\beta^{r,s,q} \circ \mathrm{d}_K + (-1)^{-q} \beta^{r,s,q} \circ \mathrm{d}_L\right) \end{aligned}$$

Thus we see that the map induced by the identifications (15.72.1.1) indeed is a morphism of complexes. □

0GWQ **Remark 15.72.2.** Let R be a ring. The category $\mathrm{Comp}(R)$ of complexes of R-modules is a symmetric monoidal category with tensor product given by $\mathrm{Tot}(- \otimes_R -)$, see Lemma 15.58.1. Given $L^\bullet$ and $M^\bullet$ in $\mathrm{Comp}(R)$ an element $f \in \mathrm{Hom}^0(L^\bullet, M^\bullet)$ defines a map of complexes $f : L^\bullet \to M^\bullet$ if and only if $\mathrm{d}(f) = 0$. Hence Lemma 15.72.1 also tells us that

$$\mathrm{Mor}_{\mathrm{Comp}(R)}(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Mor}_{\mathrm{Comp}(R)}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

functorially in $K^\bullet, L^\bullet, M^\bullet$ in $\mathrm{Comp}(R)$. This means that $\mathrm{Hom}^\bullet(-,-)$ is an internal hom for the symmetric monoidal category $\mathrm{Comp}(R)$ as discussed in Categories, Remark 4.43.12.

0A8I **Lemma 15.72.3.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$\mathit{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R \mathrm{Hom}^\bullet(K^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, M^\bullet)$$

of complexes of R-modules.

Proof. Via the discussion in Remark 15.72.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

An element α of degree n of the left hand side is

$$\alpha = (\alpha^{p,q}) \in \bigoplus\nolimits_{p+q=n} \mathrm{Hom}^p(L^\bullet, M^\bullet) \otimes_R \mathrm{Hom}^q(K^\bullet, L^\bullet)$$

The element $\alpha^{p,q}$ is a finite sum $\alpha^{p,q} = \sum \beta_i^p \otimes \gamma_i^q$ with

$$\beta_i^p = (\beta_i^{r,s}) \in \prod\nolimits_{r+s=p} \mathrm{Hom}_R(L^{-s}, M^r)$$

and

$$\gamma_i^q = (\gamma_i^{u,v}) \in \prod\nolimits_{u+v=q} \mathrm{Hom}_R(K^{-v}, L^u)$$

The map is given by sending α to $\delta = (\delta^{r,v})$ with

$$\delta^{r,v} = \sum\nolimits_{i,s} \beta_i^{r,s} \circ \gamma_i^{-s,v} \in \mathrm{Hom}_R(K^{-v}, M^r)$$

For given $r + v = n$ this sum is finite as there are only finitely many nonzero $\alpha^{p,q}$, hence only finitely many nonzero β_i^p and γ_i^q. By our sign rules we have

$$\begin{aligned}
\mathrm{d}(\alpha^{p,q}) &= \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q}) + (-1)^p \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)}(\alpha^{p,q}) \\
&= \sum \left(\mathrm{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^p \beta_i^p \circ \mathrm{d}_L \circ \gamma_i^q \right) \\
&\quad + (-1)^p \sum \left(\beta_i^p \circ \mathrm{d}_L \circ \gamma_i^q - (-1)^q \beta_i^p \circ \gamma_i^q \circ \mathrm{d}_K \right) \\
&= \sum \left(\mathrm{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^n \beta_i^p \circ \gamma_i^q \circ \mathrm{d}_K \right)
\end{aligned}$$

It follows that the rules $\alpha \mapsto \delta$ is compatible with differentials and the lemma is proved. □

0BYM **Lemma 15.72.4.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$Tot(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, Tot(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 15.72.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(M^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(M^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in M^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned}
\mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M \\
&= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M
\end{aligned}$$

On the other hand, if β is an element of degree n of the left hand side, then

$$\beta = (\beta^{p,q}) \in \bigoplus\nolimits_{p+q=n} K^p \otimes_R \mathrm{Hom}^q(M^\bullet, L^\bullet)$$

and we can write $\beta^{p,q} = \sum \gamma_i^p \otimes \delta_i^q$ with $\gamma_i^p \in K^p$ and

$$\delta_i^q = (\delta_i^{r,s}) \in \prod\nolimits_{r+s=q} \mathrm{Hom}_R(M^{-s}, L^r)$$

By our sign rules we have

$$\begin{aligned}\mathrm{d}(\beta^{p,q}) &= \mathrm{d}_K(\beta^{p,q}) + (-1)^p \mathrm{d}_{\mathrm{Hom}^\bullet(M^\bullet, L^\bullet)}(\beta^{p,q}) \\ &= \sum \mathrm{d}_K(\gamma_i^p) \otimes \delta_i^q + (-1)^p \sum \gamma_i^p \otimes (\mathrm{d}_L \circ \delta_i^q - (-1)^q \delta_i^q \circ \mathrm{d}_M)\end{aligned}$$

We send the element β to α with

$$\alpha^{r,s,q} = c^{r,s,q}(\sum \gamma_i^r \otimes \delta_i^{s,q})$$

where $c^{r,s,q} : K^r \otimes_R \mathrm{Hom}_R(M^{-q}, L^s) \to \mathrm{Hom}_R(M^{-q}, K^r \otimes_R L^s)$ is the canonical map. For a given β and r there are only finitely many nonzero γ_i^r hence only finitely many nonzero $\alpha^{r,s,q}$ are nonzero (for a given r). Thus this family of maps satisfies the conditions above and the map is well defined. Comparing signs we see that this is compatible with differentials. □

0A62 **Lemma 15.72.5.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet$ of R-modules there is a canonical morphism*

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathit{Tot}(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in both complexes.

Proof. Via the discussion in Remark 15.72.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(L^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(L^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in L^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned}\mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L \\ &= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L\end{aligned}$$

Now an element $\beta \in K^n$ we send to α with $\alpha^{n,-q,q} = \beta \otimes \mathrm{id}_{L^{-q}}$ and $\alpha^{r,s,q} = 0$ if $r \neq n$. This is indeed an element as above, as for fixed q there is only one nonzero $\alpha^{r,s,q}$. The description of the differential shows this is compatible with differentials. □

0A60 **Lemma 15.72.6.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$\mathit{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 15.72.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Consider an element β of degree n of the right hand side. Then

$$\beta = (\beta^{p,s}) \in \prod\nolimits_{p+s=n} \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

Our sign rules tell us that

$$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M \circ \beta^{p,s} - (-1)^n \beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)}$$

We can describe the last term as follows

$$(\beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)})(f) = \beta^{p,s}(\mathrm{d}_L \circ f - (-1)^{s+1} f \circ \mathrm{d}_K)$$

if $f \in \mathrm{Hom}^{-s-1}(K^\bullet, L^\bullet)$. We conclude that in some unspecified sense $\mathrm{d}(\beta^{p,s})$ is a sum of three terms with signs as follows

0FNE (15.72.6.1) $$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M(\beta^{p,s}) - (-1)^n \mathrm{d}_L(\beta^{p,s}) + (-1)^{p+1} \mathrm{d}_K(\beta^{p,s})$$

Next, we consider an element α of degree n of the left hand side. We can write it like so

$$\alpha = (\alpha^{t,r}) \in \bigoplus\nolimits_{t+r=n} \mathrm{Hom}^t(L^\bullet, M^\bullet) \otimes K^r$$

Each $\alpha^{t,r}$ maps to an element

$$\alpha^{t,r} \mapsto (\alpha^{p,q,r}) \in \prod\nolimits_{p+q=t} \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r$$

Our sign rules tell us that

$$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r})$$

where if we further write $\alpha^{p,q,r} = \sum g_i^{p,q} \otimes k_i^r$ then we have

$$\mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) = \sum (\mathrm{d}_M \circ g_i^{p,q}) \otimes k_i^r - (-1)^{p+q} \sum (g_i^{p,q} \circ \mathrm{d}_L) \otimes k_i^r$$

We conclude that in some unspecified sense $\mathrm{d}(\alpha^{p,q,r})$ is a sum of three terms with signs as follows

0FNF (15.72.6.2) $$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_M(\alpha^{p,q,r}) - (-1)^{p+q} \mathrm{d}_L(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r})$$

To define our map we will use the canonical maps

$$c_{p,q,r} : \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p)$$

which sends $\varphi \otimes k$ to the map $\psi \mapsto \varphi(\psi(k))$. This is functorial in all three variables. With $s = q + r$ there is an inclusion

$$\mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p) \subset \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

coming from the projection $\mathrm{Hom}^{-s}(K^\bullet, L^\bullet) \to \mathrm{Hom}_R(K^r, L^{-q})$. Since $\alpha^{p,q,r}$ is nonzero only for a finite number of r we see that for a given s there is only a finite number of q, r with $q + r = s$. Thus we can send α to the element β with

$$\beta^{p,s} = \sum\nolimits_{q+r=s} \epsilon_{p,q,r} c_{p,q,r}(\alpha^{p,q,r})$$

where where the sum uses the inclusions given above and where $\epsilon_{p,q,r} \in \{\pm 1\}$. Comparing signs in the equations (15.72.6.1) and (15.72.6.2) we see that

(1) $\epsilon_{p,q,r} = \epsilon_{p+1,q,r}$
(2) $-(-1)^n \epsilon_{p,q,r} = -(-1)^{p+q} \epsilon_{p,q-1,r}$ or equivalently $\epsilon_{p,q,r} = (-1)^r \epsilon_{p,q-1,r}$
(3) $(-1)^{p+1} \epsilon_{p,q,r} = (-1)^{p+q} \epsilon_{p,q,r+1}$ or equivalently $(-1)^{q+1} \epsilon_{p,q,r} = \epsilon_{p,q,r+1}$.

A good solution is to take

$$\epsilon_{p,r,s} = (-1)^{r+qr}$$

The choice of this sign is explained in the remark following the proof. □

0A61 **Remark 15.72.7.** Let us explain why the sign used in the direct construction in the proof of Lemma 15.72.6 agrees with the sign we get from the construction using the discussion in Remark 15.72.2 and Categories, Remark 4.43.12. Denote $- \otimes - = \mathrm{Tot}(- \otimes_R -)$ and $hom(-,-) = \mathrm{Hom}^\bullet(-,-)$. The construction using monoidal category language tells us to use the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \longrightarrow hom(hom(K^\bullet, L^\bullet), M^\bullet)$$

in $\mathrm{Comp}(R)$ corresponding to the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \otimes hom(K^\bullet, L^\bullet) \longrightarrow M^\bullet$$

gotten by swapping the order of the last two tensor products and then using the evaluation maps $hom(K^\bullet, L^\bullet) \otimes K^\bullet \to L^\bullet$ and $hom(L^\bullet, K^\bullet) \otimes L^\bullet \to M^\bullet$. Only in swapping does a sign intervene. Namely, in the isomorphism

$$K^\bullet \otimes hom(K^\bullet, L^\bullet) \to hom(K^\bullet, L^\bullet) \otimes K^\bullet$$

there is a sign $(-1)^{r(q+r')}$ on $K^r \otimes_R \mathrm{Hom}_R(K^{-r'}, L^q)$, see Section 15.73 item (9). The reader can convince themselves that, because of the correspondence we are using to describe maps into an internal hom, this sign only matters if $r = r'$ and in this case we obtain $(-1)^{r(q+r)} = (-1)^{r+qr}$ as in the direct proof.

15.73. Sign rules

0FNG In this section we review the sign rules used so far and we discuss some of their ramifications. It also seems appropriate to discuss these issues in the setting of the category of complexes of modules over a ring, as most interesting phenomena already occur in this case. We sincerely hope the reader will not need to use the more esoteric aspects of this section.

For the rest of this section, we fix a ring R and we denote $M^\bullet$ a complex of R-modules with differentials $d_M^n : M^n \to M^{n+1}$.

(1) The kth shifted complex $M^\bullet[k]$ has terms $(M^\bullet[k])^n = M^{n+k}$ and differentials $d^n_{M[k]} = (-1)^k d_M^{n+k}$, see Homology, Definition 12.14.7.
(2) Given a map $f : M^\bullet \to N^\bullet$ of complexes, we define $f[k] : M^\bullet[k] \to N^\bullet[k]$ without the intervention of signs, see Homology, Definition 12.14.7.
(3) We identify $H^n(M^\bullet[k])$ with $H^{n+k}(M^\bullet)$ without the intervention of signs, see Homology, Definition 12.14.8.
(4) The boundary map of a short exact sequence of complexes is defined as in the snake lemma without the intervention of signs, see Homology, Lemma 12.13.12.
(5) The distinguished triangle associated to a termwise split short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of complexes is given by

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$$

where $M^n \to K^{n+1}$ is the map $\pi^{n+1} \circ d_L^n \circ s^n$ if s and π are compatible termwise splittings. In other words, without the intervention of signs. See Derived Categories, Definitions 13.10.1 and 13.9.9.
(6) The total complex $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ has differential d satisfying the Leibniz rule $d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y)$. See Homology, Example 12.18.2 and Homology, Definition 12.18.3.

0FNH (7) There is a canonical isomorphism

$$\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)[a+b] \to \mathrm{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])$$

which uses the sign $(-1)^{pb}$ on the summand $M^p \otimes_R N^q$, see Homology, Remark 12.18.5. It is often more convenient to consider the corresponding shifted map $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet) \to \mathrm{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])[-a-b]$.

(8) There is a canonical isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R \mathrm{Tot}(L^\bullet \otimes_R M^\bullet))$$

defined without the intervention of signs. See Section 15.58.

0GWR (9) There is a canonical isomorphism

$$\mathrm{Tot}(L^\bullet \otimes_R M^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R L^\bullet)$$

which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. See Section 15.58.

Before we get into a discussion of the sign conventions regarding Hom-complexes, we construct the dual of a complex with respect to the conventions above.

0FNJ **Lemma 15.73.1.** *Let R be a ring. Let M be an R-module. Let N, η, ϵ be a left dual of M in the monoidal category of R-modules, see Categories, Definition 4.43.5. Then*

(1) *M and N are finite projective R-modules,*
(2) *the map $e : \mathrm{Hom}_R(M, R) \to N$, $\lambda \mapsto (\lambda \otimes 1)(\eta)$ is an isomorphism,*
(3) *we have $\epsilon(n, m) = e^{-1}(n)(m)$ for $n \in N$ and $m \in M$.*

Proof. The assumptions mean that

$$M \xrightarrow{\eta\otimes 1} M \otimes_R N \otimes_R M \xrightarrow{1\otimes\epsilon} M \quad \text{and} \quad N \xrightarrow{1\otimes\eta} N \otimes_R M \otimes_R N \xrightarrow{\epsilon\otimes 1} N$$

are the identity map. We can choose a finite free module F, an R-module map $F \to M$, and a lift $\tilde{\eta} : R \to F \otimes_R N$ of η. We obtain a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\eta\otimes 1} & M \otimes_R N \otimes_R M & \xrightarrow{1\otimes\epsilon} & M \\ & \searrow^{\tilde{\eta}\otimes 1} & \uparrow & & \uparrow \\ & & F \otimes_R N \otimes_R M & \xrightarrow{1\otimes\epsilon} & F \end{array}$$

This shows that the identity on M factors through a finite free module and hence M is finite projective. By symmetry we see that N is finite projective. This proves part (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. □

0FNK **Lemma 15.73.2.** *Let R be a ring. Let $M^\bullet$ be a complex of R-modules. Let $N^\bullet, \eta, \epsilon$ be a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules. Then*

(1) *$M^\bullet$ and $N^\bullet$ are bounded,*
(2) *M^n and N^n are finite projective R-modules,*
(3) *writing $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ and $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ then $(N^{-n}, \eta_n, \epsilon_n)$ is the left dual of M^n as in Lemma 15.73.1,*
(4) *the differential $d_N^n : N^n \to N^{n+1}$ is equal to $-(-1)^n$ times the map*

$$N^n = \mathrm{Hom}_R(M^{-n}, R) \xrightarrow{d_M^{-n-1}} \mathrm{Hom}_R(M^{-n-1}, R) = N^{n+1}$$

where the equality signs are the identifications from Lemma 15.73.1 part (2).

Conversely, given a bounded complex $M^\bullet$ of finite projective R-modules, setting $N^n = \mathrm{Hom}_R(M^{-n}, R)$ with differentials as above, setting $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ given by evaluation, and setting $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ mapping 1 to id_{M_n} we obtain a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules.

Proof. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{M^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{N^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{M^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{N^{-n}}$ which proves (3). By Lemma 15.73.1 we have (2). Since the sum $\eta = \sum \eta_n$ is finite, we get (1). Since $\eta = \sum \eta_n$ is a map of complexes $R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ we see that

$$(d_M^{-n-1} \otimes 1) \circ \eta_{-n-1} + (-1)^n (1 \otimes d_N^{-n}) \circ \eta_{-n} = 0$$

by our choice of signs for the differential on $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$. Unwinding definitions, this proves (4). To see the final statement of the lemma one reads the above backwards. □

We will use the description of the left dual of a complex in Lemma 15.73.2 as a motivation for our sign rule on the Hom-complex. Namelly, we choose the signs such that (11) holds. We continue with the discussion of various sign rules as above

(10) Given complexes $K^\bullet$, $M^\bullet$ we let $\mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$ be the complex with terms

$$\mathrm{Hom}^n(M^\bullet, K^\bullet) = \prod\nolimits_{n=p+q} \mathrm{Hom}_R(M^{-q}, K^p)$$

and differential given by the rule

$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$

0FNL (11) The choice above is such that if $M^\bullet$ has a left dual $N^\bullet$ as in Lemma 15.73.2, then we have a canonical isomorphism

$$\mathrm{Tot}(K^\bullet \otimes_R N^\bullet) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs sending the summand $K^p \otimes_R N^q$ to the summand $\mathrm{Hom}_R(M^{-q}, K^p)$ via $N^q = \mathrm{Hom}_R(M^{-q}, R)$ and the canonical map $K^p \otimes_R \mathrm{Hom}_R(M^{-q}, R) \to \mathrm{Hom}_R(M^{-q}, K^p)$.

(12) There is a composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, K^\bullet) \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs, see Lemma 15.72.3.

(13) There is a canonical isomorphism

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

defined without the intervention of signs, see Lemma 15.72.1.

(14) There is a canonical map

$$\mathrm{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 15.72.4.

(15) There is a canonical map

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 15.72.5.

(16) By Lemma 15.72.6 is a canonical map

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$

which uses a sign $(-1)^{r+qr}$ on the module $\mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r$ whose reason is explained in Remark 15.72.7.

0FNM (17) Taking $L^\bullet = M^\bullet$ and using $R \to \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$ the map from the previous item becomes the evaluation map

$$ev : K^\bullet \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, M^\bullet), M^\bullet)$$

It sends $x \in K^n$ to the map which sends $f \in \mathrm{Hom}^m(K^\bullet, M^\bullet)$ to $(-1)^{nm} f(x)$.

0FNN (18) There is a canonical identification

$$\mathrm{Hom}^\bullet(M^\bullet, K^\bullet)[a-b] \to \mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])$$

which uses signs. It is defined as the map whose corresponding shifted map

$$\mathrm{Hom}^\bullet(M^\bullet, K^\bullet) \to \mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a]$$

uses the sign $(-1)^{nb}$ on the module $\mathrm{Hom}_R(M^{-q}, K^p)$ with $p + q = n$. Namely, if $f \in \mathrm{Hom}^n(M^\bullet, K^\bullet)$ then

$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$

on the source, whereas on the target f lies in $(\mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a])^n = \mathrm{Hom}^{n+b-a}(M^\bullet[b], K^\bullet[a])$ and hence we get

$$\begin{aligned} d(f) &= (-1)^{b-a}\left(d_{K[a]} \circ f - (-1)^{n+b-a} f \circ d_{M[b]}\right) \\ &= (-1)^{b-a}\left((-1)^a d_K \circ f - (-1)^{n+b-a} f \circ (-1)^b d_M\right) \\ &= (-1)^b d_K \circ f - (-1)^{n+b} f \circ d_M \end{aligned}$$

and one sees that the chosen sign of $(-1)^{nb}$ in degree n produces a map of complexes for these differentials.

15.74. Derived hom

0A5W Let R be a ring. The derived hom we will define in this section is a functor

$$D(R)^{opp} \times D(R) \longrightarrow D(R), \quad (K, L) \longmapsto R\,\mathrm{Hom}_R(K, L)$$

This is an internal hom in the derived category of R-modules in the sense that it is characterized by the formula

0A63 (15.74.0.1) $$\mathrm{Hom}_{D(R)}(K, R\,\mathrm{Hom}_R(L, M)) = \mathrm{Hom}_{D(R)}(K \otimes^{\mathbf{L}}_R L, M)$$

for objects K, L, M of $D(R)$. Note that this formula characterizes the objects up to unique isomorphism by the Yoneda lemma. A construction can be given as follows. Choose a K-injective complex $I^\bullet$ of R-modules representing M, choose a complex $L^\bullet$ representing L, and set

$$R\,\mathrm{Hom}_R(L, M) = \mathrm{Hom}^\bullet(L^\bullet, I^\bullet)$$

with notation as in Section 15.72. A generalization of this construction is discussed in Differential Graded Algebra, Section 22.31. From (15.72.0.1) and Derived Categories, Lemma 13.31.2 that we have

0A64 (15.74.0.2) $$H^n(R\mathrm{Hom}_R(L, M)) = \mathrm{Hom}_{D(R)}(L, M[n])$$

for all $n \in \mathbf{Z}$. In particular, the object $R\mathrm{Hom}_R(L, M)$ of $D(R)$ is well defined, i.e., independent of the choice of the K-injective complex $I^\bullet$.

0A65 **Lemma 15.74.1.** *Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical isomorphism*

$$R\mathrm{Hom}_R(K, R\mathrm{Hom}_R(L, M)) = R\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R L, M)$$

in $D(R)$ functorial in K, L, M which recovers (15.74.0.1) by taking H^0.

Proof. Choose a K-injective complex $I^\bullet$ representing M and a K-flat complex of R-modules $L^\bullet$ representing L. For any complex of R-modules $K^\bullet$ we have

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, I^\bullet)) = \mathrm{Hom}^\bullet(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), I^\bullet)$$

by Lemma 15.72.1. The lemma follows by the definition of $R\mathrm{Hom}$ and because $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$ represents the derived tensor product. □

0A66 **Lemma 15.74.2.** *Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $L^\bullet$ be a complex of R-modules. Then $R\mathrm{Hom}_R(P^\bullet, L^\bullet)$ is represented by the complex $\mathrm{Hom}^\bullet(P^\bullet, L^\bullet)$.*

Proof. By (15.72.0.1) and Derived Categories, Lemma 13.19.8 the cohomology groups of the complex are "correct". Hence if we choose a quasi-isomorphism $L^\bullet \to I^\bullet$ with $I^\bullet$ a K-injective complex of R-modules then the induced map

$$\mathrm{Hom}^\bullet(P^\bullet, L^\bullet) \longrightarrow \mathrm{Hom}^\bullet(P^\bullet, I^\bullet)$$

is a quasi-isomorphism. As the right hand side is our definition of $R\mathrm{Hom}_R(P^\bullet, L^\bullet)$ we win. □

0A67 **Lemma 15.74.3.** *Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical morphism*

$$R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R K \longrightarrow R\mathrm{Hom}_R(R\mathrm{Hom}_R(K, L), M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a K-flat complex $K^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 15.72.6. We omit the proof that this is functorial in all three objects of $D(R)$. □

0A8J **Lemma 15.74.4.** *Let R be a ring. Given K, L, M in $D(R)$ there is a canonical morphism*

$$R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R R\mathrm{Hom}_R(K, L) \longrightarrow R\mathrm{Hom}_R(K, M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and any complex of R-modules $K^\bullet$ representing K. By Lemma 15.72.3 there is a map of complexes

$$\mathrm{Tot}\left(\mathrm{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \mathrm{Hom}^\bullet(K^\bullet, J^\bullet)\right) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, I^\bullet)$$

The complexes of R-modules $\mathrm{Hom}^\bullet(J^\bullet, I^\bullet)$, $\mathrm{Hom}^\bullet(K^\bullet, J^\bullet)$, and $\mathrm{Hom}^\bullet(K^\bullet, I^\bullet)$ represent $R\mathrm{Hom}_R(L, M)$, $R\mathrm{Hom}_R(K, L)$, and $R\mathrm{Hom}_R(K, M)$. If we choose a K-flat complex $H^\bullet$ and a quasi-isomorphism $H^\bullet \to \mathrm{Hom}^\bullet(K^\bullet, J^\bullet)$, then there is a map

$$\mathrm{Tot}\left(\mathrm{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R H^\bullet\right) \longrightarrow \mathrm{Tot}\left(\mathrm{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \mathrm{Hom}^\bullet(K^\bullet, J^\bullet)\right)$$

whose source represents $R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R R\mathrm{Hom}_R(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. □

0BYN **Lemma 15.74.5.** *Let R be a ring. Given complexes K, L, M in $D(R)$ there is a canonical morphism*

$$K \otimes^{\mathbf{L}}_R R\mathrm{Hom}_R(M, L) \longrightarrow R\mathrm{Hom}_R(M, K \otimes^{\mathbf{L}}_R L)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-flat complex $K^\bullet$ representing K, and a K-injective complex $I^\bullet$ representing L, and choose any complex $M^\bullet$ representing M. Choose a quasi-isomorphism $\mathrm{Tot}(K^\bullet \otimes_R I^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\left(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, I^\bullet)\right) \to \mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R I^\bullet)) \to \mathrm{Hom}^\bullet(M^\bullet, J^\bullet)$$

where the first map is the map from Lemma 15.72.4. □

0A6B **Lemma 15.74.6.** *Let R be a ring. Given complexes K, L in $D(R)$ there is a canonical morphism*

$$K \longrightarrow R\mathrm{Hom}_R(L, K \otimes^{\mathbf{L}}_R L)$$

in $D(R)$ functorial in both K and L.

Proof. This is a special case of Lemma 15.74.5 but we will also prove it directly. Choose a K-flat complex $K^\bullet$ representing K and any complex $L^\bullet$ representing L. Choose a quasi-isomorphism $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$K^\bullet \to \mathrm{Hom}^\bullet(L^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)) \to \mathrm{Hom}^\bullet(L^\bullet, J^\bullet)$$

where the first map is the map from Lemma 15.72.5. □

15.75. Perfect complexes

0656 A perfect complex is a pseudo-coherent complex of finite tor dimension. We will not use this as the definition, but define perfect complexes over a ring directly as follows.

0657 **Definition 15.75.1.** Let R be a ring. Denote $D(R)$ the derived category of the abelian category of R-modules.

(1) An object K of $D(R)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.
(2) An R-module M is *perfect* if $M[0]$ is a perfect object in $D(R)$.

For example, over a Noetherian ring a finite module is perfect if and only if it has finite projective dimension, see Lemma 15.75.3 and Algebra, Definition 10.109.2.

0658 **Lemma 15.75.2.** *Let $K^\bullet$ be an object of $D(R)$. The following are equivalent*

(1) *$K^\bullet$ is perfect, and*
(2) *$K^\bullet$ is pseudo-coherent and has finite tor dimension.*

If (1) and (2) hold and $K^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of finite projective R-modules with $E^i = 0$ for $i \notin [a, b]$.

Proof. It is clear that (1) implies (2), see Lemmas 15.65.5 and 15.67.3. Assume (2) holds and that $K^\bullet$ has tor-amplitude in $[a, b]$. In particular, $H^i(K^\bullet) = 0$ for $i > b$. Choose a complex $F^\bullet$ of finite free R-modules with $F^i = 0$ for $i > b$ and a quasi-isomorphism $F^\bullet \to K^\bullet$ (Lemma 15.65.5). Set $E^\bullet = \tau_{\geq a} F^\bullet$. Note that E^i is finite free except E^a which is a finitely presented R-module. By Lemma 15.67.2 E^a is flat. Hence by Algebra, Lemma 10.78.2 we see that E^a is finite projective. □

066Q **Lemma 15.75.3.** *Let M be a module over a ring R. The following are equivalent*

(1) *M is a perfect module, and*
(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with each F_i a finite projective R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence M is perfect. Conversely, assume (1). By Lemmas 15.75.2 and 15.65.4 we can find resolution $E^\bullet \to M$ with E^{-i} a finite free R-module. By Lemma 15.67.2 we see that $F_d = \mathrm{Coker}(E^{d-1} \to E^d)$ is flat for some d sufficiently large. By Algebra, Lemma 10.78.2 we see that F_d is finite projective. Hence

$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$

is the desired resolution. □

066R **Lemma 15.75.4.** *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.*

Proof. Combine Lemmas 15.75.2, 15.65.6, and 15.67.5. □

066S **Lemma 15.75.5.** *Let R be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.*

Proof. Follows from Lemmas 15.75.2, 15.65.8, and 15.67.7. □

066T **Lemma 15.75.6.** *Let R be a ring. Let $K^\bullet$ be a bounded complex of perfect R-modules. Then $K^\bullet$ is a perfect complex.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.75.4 and the stupid truncations. □

066U **Lemma 15.75.7.** *Let R be a ring. If $K^\bullet \in D^b(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.75.4 and the canonical truncations. □

066V **Lemma 15.75.8.** *Let $A \to B$ be a ring map. Assume that B is perfect as an A-module. Let $K^\bullet$ be a perfect complex of B-modules. Then $K^\bullet$ is perfect as a complex of A-modules.*

Proof. Using Lemma 15.75.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.67.12 and Lemma 15.65.11 for those results. □

066W **Lemma 15.75.9.** *Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of A-modules. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ is a perfect complex of B-modules.*

Proof. Using Lemma 15.75.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.67.13 and Lemma 15.65.12 for those results. □

066X **Lemma 15.75.10.** *Let $A \to B$ be a flat ring map. Let M be a perfect A-module. Then $M \otimes_A B$ is a perfect B-module.*

Proof. By Lemma 15.75.3 the assumption implies that M has a finite resolution $F_\bullet$ by finite projective R-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over B. Hence $M \otimes_A B$ is perfect. □

0GM0 **Lemma 15.75.11.** *Let R be a ring. If K and L are perfect objects of $D(R)$, then $K \otimes^{\mathbf{L}}_R L$ is a perfect object too.*

Proof. We can prove this using the definition as follows. We may represent K, resp. L by a bounded complex $K^\bullet$, resp. $L^\bullet$ of finite projective R-modules. Then $K \otimes^{\mathbf{L}}_R L$ is represented by the bounded complex $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$. The terms of this complex are direct sums of the modules $M^a \otimes_R L^b$. Since M^a and L^b are direct summands of finite free R-modules, so is $M^a \otimes_R L^b$. Hence we conclude the terms of the complex $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$ are finite projective.

Another proof can be given using the characterization of perfect complexes in Lemma 15.75.2 and the corresponding lemmas for pseudo-coherent complexes (Lemma 15.65.16) and for tor amplitude (Lemma 15.67.10 used with $A = B = R$). □

066Y **Lemma 15.75.12.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 15.75.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.67.16 and Lemma 15.65.14 for those results. □

068T **Lemma 15.75.13.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 15.75.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.67.17 and Lemma 15.65.15 for those results. □

066Z **Lemma 15.75.14.** *Let R be a regular ring. Then*

(1) *an R-module is perfect if and only if it is a finite R-module, and*
(2) *a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.*

Proof. Any perfect R-module is finite by definition. Conversely, let M be a finite R-module. Choose a resolution

$$\ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

with F_i finite free R-modules (Algebra, Lemma 10.71.1). Set $M_i = \mathrm{Ker}(d_i)$. Denote $U_i \subset \mathrm{Spec}(R)$ the set of primes $\mathfrak{p}$ such that $M_{i,\mathfrak{p}}$ is free; U_i is open by Algebra, Lemma 10.79.3. We have a exact sequence $0 \to M_{i+1} \to F_{i+1} \to M_i \to 0$. If $\mathfrak{p} \in U_i$, then $0 \to M_{i+1,\mathfrak{p}} \to F_{i+1,\mathfrak{p}} \to M_{i,\mathfrak{p}} \to 0$ splits. Thus $M_{i+1,\mathfrak{p}}$ is finite projective, hence free (Algebra, Lemma 10.78.2). This shows that $U_i \subset U_{i+1}$. We claim that $\mathrm{Spec}(R) = \bigcup U_i$. Namely, for every prime ideal $\mathfrak{p}$ the regular local ring $R_\mathfrak{p}$ has finite global dimension by Algebra, Proposition 10.110.1. It follows that $M_{i,\mathfrak{p}}$ is finite projective (hence free) for $i \gg 0$ for example by Algebra, Lemma 10.109.3. Since the spectrum of R is Noetherian (Algebra, Lemma 10.31.5) we conclude that $U_n = \mathrm{Spec}(R)$ for some n. Then M_n is a projective R-module by Algebra, Lemma 10.78.2. Thus

$$0 \to M_n \to F_n \to \ldots \to F_1 \to M \to 0$$

is a bounded resolution by finite projective modules and hence M is perfect. This proves part (1).

Let $K^\bullet$ be a complex of R-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective R-modules so certainly each $H^i(K^\bullet)$ is a finite R-module (as R is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module. Then by (1) each $H^i(K^\bullet)$ is a perfect R-module, whence $K^\bullet$ is perfect by Lemma 15.75.7 □

07VI **Lemma 15.75.15.** *Let A be a ring. Let $K \in D(A)$ be perfect. Then $K^\vee = R\mathrm{Hom}_A(K, A)$ is a perfect complex and $K \cong (K^\vee)^\vee$. There are functorial isomorphisms*

$$L \otimes^{\mathbf{L}}_A K^\vee = R\mathrm{Hom}_A(K, L) \quad \textit{and} \quad H^0(L \otimes^{\mathbf{L}}_A K^\vee) = \mathrm{Ext}^0_A(K, L)$$

for $L \in D(A)$.

Proof. We can represent K by a bounded complex $K^\bullet$ of finite projective A-modules. By Lemma 15.74.2 the object $K^\vee$ is represented by the complex $E^\bullet = \mathrm{Hom}^\bullet(K^\bullet, A)$. Note that $E^n = \mathrm{Hom}_A(K^{-n}, A)$ is finite projective as well. Thus $E^\bullet$ is a bounded complex of finite projective modules and we conclude $K^\vee$ is perfect. There is a canonical map

$$K^\bullet = \mathrm{Tot}(\mathrm{Hom}^\bullet(A, A) \otimes_A K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, A), A)$$

which up to sign uses the evaluation map in each degree, see Lemma 15.72.6. (For sign rules see Section 15.73.) Thus this map defines a canonical isomorphism $(K^\vee)^\vee \cong K$ as the double dual of a finite projective module is itself.

The second equality follows from the first by Lemma 15.74.1 and Derived Categories, Lemma 13.19.8 as well as the definition of Ext groups, see Derived Categories, Section 13.27. Let $L^\bullet$ be a complex of A-modules representing L. By Section 15.73 item (11) there is a canonical isomorphism

$$\mathrm{Tot}(L^\bullet \otimes_A E^\bullet) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, L^\bullet)$$

of complexes of A-modules (this uses Lemma 15.73.2 to see that $E^\bullet$ is the left dual of $K^\bullet$ in the category of complexes). This proves the first displayed equality and the proof is complete. □

0H81 **Remark 15.75.16.** Let A be a ring. Let $K \in D(A)$ be perfect. If K has tor-amplitude in $[a, b]$ then its dual $K^\vee$ has tor-amplitude in $[-b, -a]$. This follows from the last statement in Lemma 15.75.2 and the description of $K^\vee$ in the proof of Lemma 15.75.15.

0BKB **Lemma 15.75.17.** *Let A be a ring. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(A)$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(A)$ we have*

$$R\operatorname{Hom}_A(K, E) = R\lim E \otimes^{\mathbf{L}}_A K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 15.75.15 we have $R\lim E\otimes^{\mathbf{L}}_A K_n^\vee = R\lim R\operatorname{Hom}_A(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\operatorname{Hom}_A(K_n, E) \to \prod R\operatorname{Hom}_A(K_n, E) \to \prod R\operatorname{Hom}_A(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\operatorname{Hom}_A(K_n, E) = R\operatorname{Hom}_A(\bigoplus K_n, E)$. This is a formal consequence of (15.74.0.1) and the fact that derived tensor product commutes with direct sums. □

0BC7 **Lemma 15.75.18.** *Let $R = \operatorname{colim}_{i\in I} R_i$ be a filtered colimit of rings.*

1. *Given a perfect K in $D(R)$ there exists an $i \in I$ and a perfect K_i in $D(R_i)$ such that $K \cong K_i \otimes^{\mathbf{L}}_{R_i} R$ in $D(R)$.*
2. *Given $0 \in I$ and $K_0, L_0 \in D(R_0)$ with K_0 perfect, we have*

$$\operatorname{Hom}_{D(R)}(K_0 \otimes^{\mathbf{L}}_{R_0} R, L_0 \otimes^{\mathbf{L}}_{R_0} R) = \operatorname{colim}_{i\geq 0} \operatorname{Hom}_{D(R_i)}(K_0 \otimes^{\mathbf{L}}_{R_0} R_i, L_0 \otimes^{\mathbf{L}}_{R_0} R_i)$$

In other words, the triangulated category of perfect complexes over R is the colimit of the triangulated categories of perfect complexes over R_i.

Proof. We will use the results of Algebra, Lemmas 10.127.5 and 10.127.6 without further mention. These lemmas in particular say that the category of finitely presented R-modules is the colimit of the categories of finitely presented R_i-modules. Since finite projective modules can be characterized as summands of finite free modules (Algebra, Lemma 10.78.2) we see that the same is true for the category of finite projective modules. This proves (1) by our definition of perfect objects of $D(R)$.

To prove (2) we may represent K_0 by a bounded complex $K_0^\bullet$ of finite projective R_0-modules. We may represent L_0 by a K-flat complex $L_0^\bullet$ (Lemma 15.59.10). Then we have

$$\operatorname{Hom}_{D(R)}(K_0 \otimes^{\mathbf{L}}_{R_0} R, L_0 \otimes^{\mathbf{L}}_{R_0} R) = \operatorname{Hom}_{K(R)}(K_0^\bullet \otimes_{R_0} R, L_0^\bullet \otimes_{R_0} R)$$

by Derived Categories, Lemma 13.19.8. Similarly for the Hom with R replaced by R_i. Since in the right hand side only a finite number of terms are involved, since

$$\operatorname{Hom}_R(K_0^p \otimes_{R_0} R, L_0^q \otimes_{R_0} R) = \operatorname{colim}_{i\geq 0} \operatorname{Hom}_{R_i}(K_0^p \otimes_{R_0} R_i, L_0^q \otimes_{R_0} R_i)$$

by the lemmas cited at the beginning of the proof, and since filtered colimits are exact (Algebra, Lemma 10.8.8) we conclude that (2) holds as well. □

15.76. Lifting complexes

0BC8 Let R be a ring. Let $I \subset R$ be an ideal. The lifting problem we will consider is the following. Suppose given an object K of $D(R)$ and a complex $E^\bullet$ of R/I-modules such that $E^\bullet$ represents $K \otimes^{\mathbf{L}}_R R/I$ in $D(R)$. Question: Does there exist a complex of R-modules $P^\bullet$ lifting $E^\bullet$ representing K in $D(R)$? In general the answer to this question is no, but in good cases something can be done. We first discuss lifting acyclic complexes.

0BC9 **Lemma 15.76.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Assume*

(1) *each $P \in \mathcal{P}$ is a projective R-module,*
(2) *if $P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$, then $P_2 \in \mathcal{P}$, and*
(3) *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective.*

Then given any bounded above acyclic complex $E^\bullet$ whose terms are of the form P/IP for $P \in \mathcal{P}$ there exists a bounded above acyclic complex $P^\bullet$ whose terms are in $\mathcal{P}$ lifting $E^\bullet$.

Proof. Say $E^i = 0$ for $i > b$. Assume given n and a morphism of complexes

$$\begin{array}{ccccccccccc} & & & & P^n & \longrightarrow & P^{n+1} & \longrightarrow \ldots \longrightarrow & P^b & \longrightarrow & 0 \longrightarrow \ldots \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & E^{n-1} & \longrightarrow & E^n & \longrightarrow & E^{n+1} & \longrightarrow \ldots \longrightarrow & E^b & \longrightarrow & 0 \longrightarrow \ldots \end{array}$$

with $P^i \in \mathcal{P}$, with $P^n \to P^{n+1} \to \ldots \to P^b$ acyclic in degrees $\geq n+1$, and with vertical maps inducing isomorphisms $P^i/IP^i \to E^i$. In this situation one can inductively choose isomorphisms $P^i = Z^i \oplus Z^{i+1}$ such that the maps $P^i \to P^{i+1}$ are given by $Z^i \oplus Z^{i+1} \to Z^{i+1} \to Z^{i+1} \oplus Z^{i+2}$. By property (2) and arguing inductively we see that $Z^i \in \mathcal{P}$. Choose $P^{n-1} \in \mathcal{P}$ and an isomorphism $P^{n-1}/IP^{n-1} \to E^{n-1}$. Since P^{n-1} is projective and since $Z^n/IZ^n = \mathrm{Im}(E^{n-1} \to E^n)$, we can lift the map $P^{n-1} \to E^{n-1} \to E^n$ to a map $P^{n-1} \to Z^n$. By property (3) the map $P^{n-1} \to Z^n$ is surjective. Thus we obtain an extension of the diagram by adding P^{n-1} and the maps just constructed to the left of P^n. Since a diagram of the desired form exists for $n > b$ we conclude by induction on n. □

0BCA **Lemma 15.76.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Let $K \in D(R)$ and let $E^\bullet$ be a complex of R/I-modules representing $K \otimes^{\mathbf{L}}_R R/I$. Assume*

(1) *each $P \in \mathcal{P}$ is a projective R-module,*
(2) *$P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$ if and only if $P_1, P_2 \in \mathcal{P}$,*
(3) *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective,*
(4) *$E^\bullet$ is bounded above and E^i is of the form P/IP for $P \in \mathcal{P}$, and*
(5) *K can be represented by a bounded above complex whose terms are in $\mathcal{P}$.*

Then there exists a bounded above complex $P^\bullet$ whose terms are in $\mathcal{P}$ with $P^\bullet/IP^\bullet$ isomorphic to $E^\bullet$ and representing K in $D(R)$.

Proof. By assumption (5) we can represent K by a bounded above complex $K^\bullet$ whose terms are in $\mathcal{P}$. Then $K \otimes^{\mathbf{L}}_R R/I$ is represented by $K^\bullet/IK^\bullet$. Since $E^\bullet$ is a bounded above complex of projective R/I-modules by (4), we can choose a quasi-isomorphism $\delta : E^\bullet \to K^\bullet/IK^\bullet$ (Derived Categories, Lemma 13.19.8). Let $C^\bullet$ be

cone on δ (Derived Categories, Definition 13.9.1). The module C^i is the direct sum $K^i/IK^i \oplus E^{i+1}$ hence is of the form P/IP for some $P \in \mathcal{P}$ as (2) says in particular that $\mathcal{P}$ is preserved under taking sums. Since $C^\bullet$ is acyclic, we can apply Lemma 15.76.1 and find a acyclic lift $A^\bullet$ of $C^\bullet$. The complex $A^\bullet$ is bounded above and has terms in $\mathcal{P}$. In

$$\begin{array}{ccccc} K^\bullet & \dashrightarrow & A^\bullet & & \\ \downarrow & & \downarrow & & \\ K^\bullet/IK^\bullet & \longrightarrow & C^\bullet & \longrightarrow & E^\bullet[1] \end{array}$$

we can find the dotted arrow making the diagram commute by Derived Categories, Lemma 13.19.6. We will show below that it follows from (1), (2), (3) that $K^i \to A^i$ is the inclusion of a direct summand for every i. By property (2) we see that $P^i = \mathrm{Coker}(K^i \to A^i)$ is in $\mathcal{P}$. Thus we can take $P^\bullet = \mathrm{Coker}(K^\bullet \to A^\bullet)[-1]$ to conclude.

To finish the proof we have to show the following: Let $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ and $P_1/IP_1 \to P_2/IP_2$ is split injective with cokernel of the form P_3/IP_3 for some $P_3 \in \mathcal{P}$, then f is split injective. Write $E_i = P_i/IP_i$. Then $E_2 = E_1 \oplus E_3$. Since P_2 is projective we can choose a map $g : P_2 \to P_3$ lifting the map $E_2 \to E_3$. By condition (3) the map g is surjective, hence split as P_3 is projective. Set $P_1' = \mathrm{Ker}(g)$ and choose a splitting $P_2 = P_1' \oplus P_3$. Then $P_1' \in \mathcal{P}$ by (2). We do not know that $g \circ f = 0$, but we can consider the map

$$P_1 \xrightarrow{f} P_2 \xrightarrow{projection} P_1'$$

The composition modulo I is an isomorphism. Since P_1' is projective we can split $P_1 = T \oplus P_1'$. If $T = 0$, then we are done, because then $P_2 \to P_1'$ is a splitting of f. We see that $T \in \mathcal{P}$ by (2). Calculating modulo I we see that $T/IT = 0$. Since $0 \in \mathcal{P}$ (as the summand of any P in $\mathcal{P}$) we see the map $0 \to T$ is surjective and we conclude that $T = 0$ as desired. □

09AR **Lemma 15.76.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

(1) *$E^\bullet$ is a bounded above complex of projective R/I-modules,*
(2) *$K \otimes^{\mathbf{L}}_R R/I$ is represented by $E^\bullet$ in $D(R/I)$, and*
(3) *I is a nilpotent ideal.*

Then there exists a bounded above complex $P^\bullet$ of projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$.

Proof. We apply Lemma 15.76.2 using the class $\mathcal{P}$ of all projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift projective R/I-modules to projective R-modules, see Algebra, Lemma 10.77.5. To see that (5) holds it suffices to show that K is in $D^-(R)$. Since we are given that $K \otimes^{\mathbf{L}}_R R/I$ is in $D^-(R/I)$ because $E^\bullet$ is bounded above, this follows from Lemma 15.67.20. □

0H76 **Lemma 15.76.4.** *Let $R' \to R$ be a surjective ring map whose kernel is a nilpotent ideal. Let $K' \in D(R')$ and set $K = K' \otimes^{\mathbf{L}}_{R'} R$. Then K is pseudo-coherent if and only if K' is pseudo-coherent.*

Proof. One direction follows from Lemma 15.65.12. For the other direction, assume K is pseudo-coherent. Then by Lemma 15.65.5 we can represent K by a bounded above complex $E^\bullet$ of finite free R-modules. By Lemma 15.76.3 we can represent K' by a bounded above complex $P^\bullet$ of projective R'-modules such that $P^n \otimes_{R'} R = E^n$. By Nakayama's lemma we see that P^n is finite free and we conclude that K' is pseudo-coherent as well. □

0BCB **Lemma 15.76.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

(1) *$E^\bullet$ is a bounded above complex of finite stably free R/I-modules,*
(2) *$K \otimes^{\mathbf{L}}_R R/I$ is represented by $E^\bullet$ in $D(R/I)$,*
(3) *$K^\bullet$ is pseudo-coherent, and*
(4) *every element of $1 + I$ is invertible.*

Then there exists a bounded above complex $P^\bullet$ of finite stably free R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$. Moreover, if E^i is free, then P^i is free.

Proof. We apply Lemma 15.76.2 using the class $\mathcal{P}$ of all finite stably free R-modules. Property (1) of the lemma is immediate. Property (2) follows from Lemma 15.3.2. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift finite stably free R/I-modules to finite stably free R-modules, see Lemma 15.3.3. Part (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. The final assertion of the lemma follows from Lemma 15.3.5. □

0BCC **Lemma 15.76.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $K \in D(R)$ be pseudo-coherent. Set $d_i = \dim_\kappa H^i(K \otimes^{\mathbf{L}}_R \kappa)$. Then $d_i < \infty$ and for some $b \in \mathbf{Z}$ we have $d_i = 0$ for $i > b$. Then there exists a complex*

$$\ldots \to R^{\oplus d_{b-2}} \to R^{\oplus d_{b-1}} \to R^{\oplus d_b} \to 0 \to \ldots$$

representing K in $D(R)$. Moreover, this complex is unique up to isomorphism(!).

Proof. Observe that $K \otimes^{\mathbf{L}}_R \kappa$ is pseudo-coherent as an object of $D(\kappa)$, see Lemma 15.65.12. Hence the cohomology spaces are finite dimensional and vanish above some cutoff. Every object of $D(\kappa)$ is isomorphic in $D(\kappa)$ to a complex $E^\bullet$ with zero differentials. In particular $E^i \cong \kappa^{\oplus d_i}$ is finite free. Applying Lemma 15.76.5 we obtain the existence.

If we have two complexes $F^\bullet$ and $G^\bullet$ with F^i and G^i free of rank d_i representing K. Then we may choose a map of complexes $\beta : F^\bullet \to G^\bullet$ representing the isomorphism $F^\bullet \cong K \cong G^\bullet$, see Derived Categories, Lemma 13.19.8. The induced map of complexes $\beta \otimes 1 : F^\bullet \otimes^{\mathbf{L}}_R \kappa \to G^\bullet \otimes^{\mathbf{L}}_R \kappa$ must be an isomorphism of complexes as the differentials in $F^\bullet \otimes^{\mathbf{L}}_R \kappa$ and $G^\bullet \otimes^{\mathbf{L}}_R \kappa$ are zero. Thus $\beta^i : F^i \to G^i$ is a map of finite free R-modules whose reduction modulo $\mathfrak{m}$ is an isomorphism. Hence β^i is an isomorphism and we win. □

0BCD **Lemma 15.76.7.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime. Let $K \in D(R)$ be perfect. Set $d_i = \dim_{\kappa(\mathfrak{p})} H^i(K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}))$. Then $d_i < \infty$ and only a finite number are nonzero. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ and a complex*

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes^{\mathbf{L}}_R R_f$ in $D(R_f)$.

Proof. Observe that $K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ is perfect as an object of $D(\kappa(\mathfrak{p}))$, see Lemma 15.75.9. Hence only a finite number of d_i are nonzero and they are all finite. Applying Lemma 15.76.6 we get a complex representing K having the desired shape over the local ring $R_\mathfrak{p}$. We have $R_\mathfrak{p} = \operatorname{colim} R_f$ for $f \in R$, $f \notin \mathfrak{p}$ (Algebra, Lemma 10.9.9). We conclude by Lemma 15.75.18. Some details omitted. □

0F9V **Lemma 15.76.8.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime. Let $M^\bullet$ and $N^\bullet$ be bounded complexes of finite projective R-modules representing the same object of $D(R)$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that there is an isomorphism (!) of complexes*

$$M^\bullet_f \oplus P^\bullet \cong N^\bullet_f \oplus Q^\bullet$$

where $P^\bullet$ and $Q^\bullet$ are finite direct sums of trivial complexes, i.e., complexes of the form the form $\ldots \to 0 \to R_f \xrightarrow{1} R_f \to 0 \to \ldots$ (placed in arbitrary degrees).

Proof. If we have an isomorphism of the type described over the localization $R_\mathfrak{p}$, then using that $R_\mathfrak{p} = \operatorname{colim} R_f$ (Algebra, Lemma 10.9.9) we can descend the isomorphism to an isomorphism over R_f for some f. Thus we may assume R is local and $\mathfrak{p}$ is the maximal ideal. In this case the result follows from the uniqueness of a "minimal" complex representing a perfect object, see Lemma 15.76.6, and the fact that any complex is a direct sum of a trivial complex and a minimal one (Algebra, Lemma 10.102.2). □

0BCE **Lemma 15.76.9.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

(1) *$E^\bullet$ is a bounded above complex of finite projective R/I-modules,*
(2) *$K \otimes^{\mathbf{L}}_R R/I$ is represented by $E^\bullet$ in $D(R/I)$,*
(3) *K is pseudo-coherent, and*
(4) *(R, I) is a henselian pair.*

Then there exists a bounded above complex $P^\bullet$ of finite projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$. Moreover, if E^i is free, then P^i is free.

Proof. We apply Lemma 15.76.2 using the class $\mathcal{P}$ of all finite projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift finite projective R/I-modules to finite projective R-modules, see Lemma 15.13.1. Property (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. Thus Lemma 15.76.2 applies and we find $P^\bullet$ as desired. The final assertion of the lemma follows from Lemma 15.3.5. □

15.77. Splitting complexes

0BCF In this section we discuss conditions which imply an object of the derived category of a ring is a direct sum of its truncations. Our method is to use the following lemma (under suitable hypotheses) to split the canonical distinguished triangles

$$\tau_{\leq i}K \to K \to \tau_{\geq i+1}K \to (\tau_{\leq i}K)[1]$$

in $D(R)$, see Derived Categories, Remark 13.12.4.

0BCG **Lemma 15.77.1.** *Let R be a ring. Let K and L be objects of $D(R)$. Assume L has projective-amplitude in $[a, b]$, for example if L is perfect of tor-amplitude in $[a, b]$.*

(1) *If $H^i(K) = 0$ for $i \geq a$, then $\mathrm{Hom}_{D(R)}(L, K) = 0$.*
(2) *If $H^i(K) = 0$ for $i \geq a + 1$, then given any distinguished triangle $K \to M \to L \to K[1]$ there is an isomorphism $M \cong K \oplus L$ in $D(R)$ compatible with the maps in the distinguished triangle.*
(3) *If $H^i(K) = 0$ for $i \geq a$, then the isomorphism in (2) exists and is unique.*

Proof. The assumption that L has projective-amplitude in $[a, b]$ means we can represent L by a complex $L^\bullet$ of projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Definition 15.69.1. If L is perfect of tor-amplitude in $[a, b]$, then we can represent L by a complex $L^\bullet$ of finite projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Lemma 15.75.2. If $H^i(K) = 0$ for $i \geq a$, then K is quasi-isomorphic to $\tau_{\leq a-1}K$. Hence we can represent K by a complex $K^\bullet$ of R-modules with $K^i = 0$ for $i \geq a$. Then we obtain

$$\mathrm{Hom}_{D(R)}(L, K) = \mathrm{Hom}_{K(R)}(L^\bullet, K^\bullet) = 0$$

by Derived Categories, Lemma 13.19.8. This proves (1). Under the hypotheses of (2) we see that $\mathrm{Hom}_{D(R)}(L, K[1]) = 0$ by (1), hence the distinguished triangle is split by Derived Categories, Lemma 13.4.11. The uniqueness of (3) follows from (1). □

0A1U **Lemma 15.77.2.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the map*

$$H^i(K^\bullet) \otimes_R \kappa(\mathfrak{p}) \longrightarrow H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}))$$

is surjective. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$ and a canonical isomorphism

$$K^\bullet \otimes_R R_f \cong \tau_{\leq i}(K^\bullet \otimes_R R_f) \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$.

Proof. In this proof all tensor products are over R and we write $\kappa = \kappa(\mathfrak{p})$. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \ldots$$

Let $0 \subset V \subset W \subset K^i \otimes \kappa$ be defined by the formulas

$$V = \mathrm{Im}\left(K^{i-1} \otimes \kappa \to K^i \otimes \kappa\right) \quad \text{and} \quad W = \mathrm{Ker}\left(K^i \otimes \kappa \to K^{i+1} \otimes \kappa\right)$$

Set $\dim(V) = r$, $\dim(W/V) = s$, and $\dim(K^i \otimes \kappa/W) = t$. We can pick $x_1, \ldots, x_r \in K^{i-1}$ which map by d^{i-1} to a basis of V. By our assumption we can pick $y_1, \ldots, y_s \in \mathrm{Ker}(d^i)$ mapping to a basis of W/V. Finally, choose $z_1, \ldots, z_t \in K^i$ mapping to a basis of $K^i \otimes \kappa/W$. Then we see that the elements $d^i(z_1), \ldots, d^i(z_t) \in K^{i+1}$ are linearly independent in $K^{i+1} \otimes_R \kappa$. By Algebra, Lemma 10.79.4 we may after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{p}$ assume that

(1) $d^i(x_a), y_b, z_c$ is an R-basis of K^i,
(2) $d^i(z_1), \ldots, d^i(z_t)$ are R-linearly independent in K^{i+1}, and
(3) the quotient $E^{i+1} = K^{i+1}/\sum Rd^i(z_c)$ is finite projective.

Since d^i annihilates $d^{i-1}(x_a)$ and y_b, we deduce from condition (2) that $E^{i+1} = \mathrm{Coker}(d^i : K^i \to K^{i+1})$. Thus we see that

$$\tau_{\geq i+1}K^\bullet = (\ldots \to 0 \to E^{i+1} \to K^{i+2} \to \ldots)$$

is a bounded complex of finite projective modules sitting in degrees $[i+1, b]$ for some b. Thus $\tau_{\geq i+1}K^\bullet$ is perfect of amplitude $[i+1, b]$. Since $\tau_{\leq i}K^\bullet$ has no cohomology in degrees $> i$, we may apply Lemma 15.77.1 to the distinguished triangle

$$\tau_{\leq i}K^\bullet \to K^\bullet \to \tau_{\geq i+1}K^\bullet \to (\tau_{\leq i}K^\bullet)[1]$$

(Derived Categories, Remark 13.12.4) to conclude. □

0A1V **Lemma 15.77.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the maps*

$$H^i(K^\bullet) \otimes_R \kappa(\mathfrak{p}) \longrightarrow H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) \quad \text{and} \quad H^{i-1}(K^\bullet) \otimes_R \kappa(\mathfrak{p}) \longrightarrow H^{i-1}(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}))$$

are surjective. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that

1. *$\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$,*
2. *$H^i(K^\bullet)_f$ is a finite free R_f-module, and*
3. *there is a canonical direct sum decomposition*

$$K^\bullet \otimes_R R_f \cong \tau_{\leq i-1}(K^\bullet \otimes_R R_f) \oplus H^i(K^\bullet)_f[-i] \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$.

Proof. We get (1) from Lemma 15.77.2 as well as a splitting $K^\bullet \otimes_R R_f = \tau_{\leq i}K^\bullet \otimes_R R_f \oplus \tau_{\geq i+1}K^\bullet \otimes_R R_f$ in $D(R_f)$. Applying Lemma 15.77.2 once more to $\tau_{\leq i}K^\bullet \otimes_R R_f$ we obtain (after suitably choosing f) a splitting $\tau_{\leq i}K^\bullet \otimes_R R_f = \tau_{\leq i-1}K^\bullet \otimes_R R_f \oplus H^i(K^\bullet)_f$ in $D(R_f)$ as well as the conclusion that $H^i(K)_f$ is a flat perfect module, i.e., finite projective. □

068U **Lemma 15.77.4.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $i \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules such that $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ and a canonical direct sum decomposition*

$$K^\bullet \otimes_R R_f = \tau_{\geq i+1}(K^\bullet \otimes_R R_f) \oplus \tau_{\leq i-1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$ with $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ a perfect complex with tor-amplitude in $[i+1, \infty]$.

Proof. This is an often used special case of Lemma 15.77.2. A direct proof is as follows. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$

Let A be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let B be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that A mod $\mathfrak{p}$ has rank r and that B mod $\mathfrak{p}$ has rank $m - r$. In other words, there is some $r \times r$ minor a of A which is not in $\mathfrak{p}$ and there is some $(m-r) \times (m-r)$-minor b of B which is not in $\mathfrak{p}$. Set $f = ab$. Then after inverting f we can find direct sum decompositions $K^{i-1} = R^{\oplus l-r} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus l-r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of K^i and such that the module map $K^i \to K^{i+1}$ kills

of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus m-r}$ onto the corresponding summand of K^{i+1}. Thus $K^\bullet$ becomes quasi-isomorphic to

$$\ldots \to K^{i-2} \to R^{\oplus l-r} \to 0 \to R^{\oplus n-m+r} \to K^{i+2} \to \ldots$$

and everything is clear. □

0G97 **Lemma 15.77.5.** *Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume that for any injective R-module map $M \to M'$ the map $\mathrm{Ext}^{-a}_R(K, M) \to \mathrm{Ext}^{-a}_R(K, M')$ is injective. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.*

Proof. Consider the distinguished triangle

$$\tau_{\leq a}K \to K \to \tau_{\geq a+1}K \to (\tau_{\leq a}K)[1]$$

in $D(R)$, see Derived Categories, Remark 13.12.4. Observe that $\mathrm{Ext}^{-a}_R(\tau_{\leq a}K, M) = \mathrm{Hom}_R(H^a(K), M)$ and $\mathrm{Ext}^{-a-1}_R(\tau_{\leq a}K, M) = 0$, see Derived Categories, Lemma 13.27.3. Thus the long exact sequence of Ext gives an exact sequence

$$0 \to \mathrm{Ext}^{-a}_R(\tau_{\geq a+1}K, M) \to \mathrm{Ext}^{-a}_R(K, M) \to \mathrm{Hom}_R(H^a(K), M)$$

functorial in the R-module M. Now if I is an injective R-module, then $\mathrm{Ext}^{-a}_R(\tau_{\geq a+1}K, I) = 0$ for example by Derived Categories, Lemma 13.27.2. Since every module injects into an injective module, we conclude that $\mathrm{Ext}^{-a}_R(\tau_{\geq a+1}K, M) = 0$ for every R-module M. By Lemma 15.69.2 we conclude that $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b (this is where we use that K is bounded above). We obtain the splitting by Lemma 15.77.1. □

0G98 **Lemma 15.77.6.** *Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume $\mathrm{Ext}^{-a}_R(K, M) = 0$ for any R-module M. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a-1}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.*

Proof. By Lemma 15.77.5 we have a direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b. Clearly, we must have $H^a(K) = 0$ and we conclude that $\tau_{\leq a}K = \tau_{\leq a-1}K$ in $D(R)$. □

15.78. Recognizing perfect complexes

0G99 Some lemmas that allow us to prove certain complexes are perfect.

0BYP **Lemma 15.78.1.** *Let R be a ring and let $\mathfrak{p} \subset R$ be a prime. Let K be pseudo-coherent and bounded below. Set $d_i = \dim_{\kappa(\mathfrak{p})} H^i(K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}))$. If there exists an $a \in \mathbf{Z}$ such that $d_i = 0$ for $i < a$, then there exists an $f \in R$, $f \notin \mathfrak{p}$ and a complex*

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes^{\mathbf{L}}_R R_f$ in $D(R_f)$. In particular $K \otimes^{\mathbf{L}}_R R_f$ is perfect.

Proof. After decreasing a we may assume that also $H^i(K^\bullet) = 0$ for $i < a$. By Lemma 15.77.4 after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{p}$ we can write $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect. Since $H^i(K^\bullet) = 0$ for $i < a$ we see that $\tau_{\leq a-1}K^\bullet = 0$ in $D(R)$. Hence $K^\bullet$ is perfect. Then we can conclude using Lemma 15.76.7. □

068V **Lemma 15.78.2.** *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is perfect with tor amplitude in $[a, b]$,*
(2) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \notin [a, b]$, and*
(3) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \notin [a, b]$.*

Proof. We omit the proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (3). Let $i \in \mathbf{Z}$ with $i \notin [a, b]$. By Lemma 15.77.4 we see that the assumption implies that $H^i(K^\bullet)_\mathfrak{m} = 0$ for all maximal ideals of R. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 10.23.1. Moreover, Lemma 15.77.4 now also implies that for every maximal ideal $\mathfrak{m}$ there exists an element $f \in R$, $f \notin \mathfrak{m}$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by appealing to Lemmas 15.75.12 and 15.67.16. □

068W **Lemma 15.78.3.** *Let R be a ring. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Consider the following conditions*

(1) *$K^\bullet$ is perfect,*
(2) *for every prime ideal $\mathfrak{p}$ the complex $K^\bullet \otimes_R R_\mathfrak{p}$ is perfect,*
(3) *for every maximal ideal $\mathfrak{m}$ the complex $K^\bullet \otimes_R R_\mathfrak{m}$ is perfect,*
(4) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \ll 0$,*
(5) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \ll 0$.*

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$$

If $K^\bullet$ is bounded below, then all conditions are equivalent.

Proof. By Lemma 15.75.9 we see that (1) implies (2). It is immediate that (2) $\Rightarrow$ (3). Since every prime $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$, we can apply Lemma 15.75.9 to the map $R_\mathfrak{m} \to R_\mathfrak{p}$ to see that (3) implies (2). Applying Lemma 15.75.9 to the residue maps $R_\mathfrak{p} \to \kappa(\mathfrak{p})$ and $R_\mathfrak{m} \to \kappa(\mathfrak{m})$ we see that (2) implies (4) and (3) implies (5).

Assume R is local with maximal ideal $\mathfrak{m}$ and residue field κ. We will show that if $H^i(K^\bullet \otimes^{\mathbf{L}} \kappa) = 0$ for $i < a$ for some a, then K is perfect. This will show that (4) implies (2) and (5) implies (3) whence the first part of the lemma. First we apply Lemma 15.77.4 with $i = a - 1$ to see that $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect of tor-amplitude contained in $[a, \infty]$. To finish we need to show that $\tau_{\leq a-1}K$ is zero, i.e., that its cohomology groups are zero. If not let i be the largest index such that $M = H^i(\tau_{\leq a-1}K)$ is not zero. Then M is a finite R-module because $\tau_{\leq a-1}K^\bullet$ is pseudo-coherent (Lemmas 15.65.3 and 15.65.8). Thus by Nakayama's lemma (Algebra, Lemma 10.20.1) we find that $M \otimes_R \kappa$ is nonzero. This implies that

$$H^i((\tau_{\leq a-1}K^\bullet) \otimes^{\mathbf{L}}_R \kappa) = H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa)$$

is nonzero which is a contradiction.

Assume the equivalent conditions (2) – (5) hold and that $K^\bullet$ is bounded below. Say $H^i(K^\bullet) = 0$ for $i < a$. Pick a maximal ideal $\mathfrak{m}$ of R. It suffices to show there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $K^\bullet \otimes^{\mathbf{L}}_R R_f$ is perfect (Lemma 15.75.12 and Algebra, Lemma 10.17.8). This follows from Lemma 15.78.1. □

0G9A **Lemma 15.78.4.** *Let R be a ring. Let K be a pseudo-coherent object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *K has projective-amplitude in $[a, b]$,*
(2) *K is perfect of tor-amplitude in $[a, b]$,*
(3) *$\mathrm{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i \notin [-b, -a]$,*
(4) *$H^n(K) = 0$ for $n > b$ and $\mathrm{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i > -a$, and*
(5) *$H^n(K) = 0$ for $n \notin [a - 1, b]$ and $\mathrm{Ext}^{-a+1}_R(K, N) = 0$ for all finitely presented R-modules N.*

Proof. From the final statement of Lemma 15.75.2 we see that (2) implies (1). If (1) holds, then K can be represented by a complex of projective modules P^i with $P^i = 0$ for $i \notin [a, b]$. Since projective modules are flat (as summands of free modules), we see that K has tor-amplitude in $[a, b]$, see Lemma 15.67.3. Thus by Lemma 15.75.2 we see that (2) holds.

In conditions (3), (4), (5) the assumed vanishing of ext groups $\mathrm{Ext}^i_R(K, M)$ for M of finite presentation is equivalent to the vanishing for all R-modules M by Lemma 15.66.1 and Algebra, Lemma 10.11.3. Thus the equivalence of (1), (3), (4), and (5) follows from Lemma 15.69.2. □

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

068X **Lemma 15.78.5.** *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

(1) *the ring map $A \to B$ is flat,*
(2) *for every prime $\mathfrak{p} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{p})$ has finite global dimension $\leq d$,*
(3) *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
(4) *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. In particular, $K^\bullet$ is flat as a complex of A-modules and $K^\bullet \otimes_A M = K^\bullet \otimes^{\mathbf{L}}_A M$ for any A-module M. For every prime $\mathfrak{p}$ of A the complex

$$K^\bullet \otimes_A \kappa(\mathfrak{p})$$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(\mathfrak{p})$ with vanishing H^i except for $i \in [a, b]$. As $B \otimes_A \kappa(\mathfrak{p})$ has global dimension d we see from Lemma 15.67.19 that $K^\bullet \otimes_A \kappa(\mathfrak{p})$ has tor amplitude in $[a - d, b]$. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$. Since $K^\bullet \otimes_A \kappa(\mathfrak{p})$ is a bounded above complex of free $B \otimes_A \kappa(\mathfrak{p})$-modules we see that

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_B \kappa(\mathfrak{q}) &= K^\bullet \otimes_B \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes^{\mathbf{L}}_{B \otimes_A \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \end{aligned}$$

Hence the arguments above imply that $H^i(K^\bullet \otimes^{\mathbf{L}}_B \kappa(\mathfrak{q})) = 0$ for $i \notin [a - d, b]$. We conclude by Lemma 15.78.2. □

The following lemma is a local version of Lemma 15.78.5. It can be used to find perfect complexes over regular local rings.

09PC **Lemma 15.78.6.** *Let $A \to B$ be a local ring homomorphism. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

(1) *the ring map $A \to B$ is flat,*
(2) *the ring $B/\mathfrak{m}_A B$ is regular of dimension d,*
(3) *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
(4) *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules, in fact it suffices if $H^i(K^\bullet \otimes^{\mathbf{L}}_A \kappa(\mathfrak{m}_A))$ is nonzero only for $i \in [a, b]$.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. By (3) we may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. We compute

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_B \kappa(\mathfrak{m}_B) &= K^\bullet \otimes_B \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes^{\mathbf{L}}_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \end{aligned}$$

The first equality because $K^\bullet$ is a bounded above complex of flat B-modules. The second equality follows from basic properties of the tensor product. The third equality holds because $K^\bullet \otimes_A \kappa(\mathfrak{m}_A) = K^\bullet/\mathfrak{m}_A K^\bullet$ is a bounded above complex of flat $B/\mathfrak{m}_A B$-modules. Since $K^\bullet$ is a bounded above complex of flat A-modules by (1), the cohomology modules H^i of the complex $K^\bullet \otimes_A \kappa(\mathfrak{m}_A)$ are nonzero only for $i \in [a, b]$ by assumption (4). Thus the spectral sequence of Example 15.62.1 and the fact that $B/\mathfrak{m}_A B$ has finite global dimension d (by (2) and Algebra, Proposition 10.110.1) shows that $H^j(K^\bullet \otimes^{\mathbf{L}}_B \kappa(\mathfrak{m}_B))$ is zero for $j \notin [a - d, b]$. This finishes the proof by Lemma 15.78.2. □

15.79. Characterizing perfect complexes

07LQ In this section we prove that the perfect complexes are exactly the compact objects of the derived category of a ring. First we show the following.

0ATI **Lemma 15.79.1.** *Let R be a ring. The full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects is the smallest strictly full, saturated, triangulated subcategory containing $R = R[0]$. In other words $D_{perf}(R) = \langle R \rangle$. In particular, R is a classical generator for $D_{perf}(R)$.*

Proof. To see what the statement means, please look at Derived Categories, Definitions 13.6.1 and 13.36.3. It was shown in Lemmas 15.75.4 and 15.75.5 that $D_{perf}(R) \subset D(R)$ is a strictly full, saturated, triangulated subcategory of $D(R)$. Of course $R \in D_{perf}(R)$.

Recall that $\langle R \rangle = \bigcup \langle R \rangle_n$. To finish the proof we will show that if $M \in D_{perf}(R)$ is represented by

$$\ldots \to 0 \to M^a \to M^{a+1} \to \ldots \to M^b \to 0 \to \ldots$$

with M^i finite projective, then $M \in \langle R \rangle_{b-a+1}$. The proof is by induction on $b - a$. By definition $\langle R \rangle_1$ contains any finite projective R-module placed in any degree; this deals with the base case $b - a = 0$ of the induction. In general, we consider the distinguished triangle

$$M_b[-b] \to M^\bullet \to \sigma_{\leq b-1} M^\bullet \to M_b[-b+1]$$

By induction the truncated complex $\sigma_{\leq b-1} M^\bullet$ is in $\langle R \rangle_{b-a}$ and $M_b[-b]$ is in $\langle R \rangle_1$. Hence $M^\bullet \in \langle R \rangle_{b-a+1}$ by definition. □

Let R be a ring. Recall that $D(R)$ has direct sums which are given simply by taking direct sums of complexes, see Derived Categories, Lemma 13.33.5. We will use this in the lemmas of this section without further mention.

07LR **Lemma 15.79.2.** *Let R be a ring. Let $K \in D(R)$ be an object such that for every countable set of objects $E_n \in D(R)$ the canonical map*

$$\bigoplus \mathrm{Hom}_{D(R)}(K, E_n) \longrightarrow \mathrm{Hom}_{D(R)}(K, \bigoplus E_n)$$

is a bijection. Then, given any system $L_n^\bullet$ of complexes over $\mathbf{N}$ we have that

$$\mathrm{colim}\, \mathrm{Hom}_{D(R)}(K, L_n^\bullet) \longrightarrow \mathrm{Hom}_{D(R)}(K, L^\bullet)$$

is a bijection, where $L^\bullet$ is the termwise colimit, i.e., $L^m = \mathrm{colim}\, L_n^m$ for all $m \in \mathbf{Z}$.

Proof. Consider the short exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

where the first map is given by $1 - t_n$ in degree n where $t_n : L_n^\bullet \to L_{n+1}^\bullet$ is the transition map. By Derived Categories, Lemma 13.12.1 this is a distinguished triangle in $D(R)$. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$, see Derived Categories, Lemma 13.4.2. Thus a long exact cohomology sequence

$$\begin{array}{ccccc} \ldots & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet[-1]) & & \\ & \searrow & & & \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet) \\ & \searrow & & & \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet[1]) & \longrightarrow & \ldots & & \end{array}$$

Since we have assumed that $\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet)$ is equal to $\bigoplus \mathrm{Hom}_{D(R)}(K, L_n^\bullet)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $\mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show. □

The following proposition, characterizing perfect complexes as the compact objects (Derived Categories, Definition 13.37.1) of the derived category, shows up in various places. See for example [**Ric89b**, proof of Proposition 6.3] (this treats the bounded case), [**TT90**, Theorem 2.4.3] (the statement doesn't match exactly), and [**BN93**, Proposition 6.4] (watch out for horrendous notational conventions).

07LT **Proposition 15.79.3.** *Let R be a ring. For an object K of $D(R)$ the following are equivalent*

1. *K is perfect, and*
2. *K is a compact object of $D(R)$.*

Proof. Assume K is perfect, i.e., K is quasi-isomorphic to a bounded complex $P^\bullet$ of finite projective modules, see Definition 15.75.1. If E_i is represented by the complex $E_i^\bullet$, then $\bigoplus E_i$ is represented by the complex whose degree n term is $\bigoplus E_i^n$. On the other hand, as P^n is projective for all n we have $\mathrm{Hom}_{D(R)}(P^\bullet, K^\bullet) =$

$\mathrm{Hom}_{K(R)}(P^\bullet, K^\bullet)$ for every complex of R-modules $K^\bullet$, see Derived Categories, Lemma 13.19.8. Thus $\mathrm{Hom}_{D(R)}(P^\bullet, E^\bullet)$ is the cohomology of the complex

$$\prod \mathrm{Hom}_R(P^n, E^{n-1}) \to \prod \mathrm{Hom}_R(P^n, E^n) \to \prod \mathrm{Hom}_R(P^n, E^{n+1}).$$

Since $P^\bullet$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each P^n is a finite R-module we see that $\mathrm{Hom}_R(P^n, \bigoplus_i E_i^m) = \bigoplus_i \mathrm{Hom}_R(P^n, E_i^m)$ for all n, m. Combining these remarks we see that the map of Derived Categories, Definition 13.37.1 is a bijection.

Conversely, assume K is compact. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

Since $K \in D^-(R)$ and since every R-module is a quotient of a free module, we may represent K by a bounded above complex $K^\bullet$ of free R-modules, see Derived Categories, Lemma 13.15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.15. Hence by Lemma 15.79.2 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free R-modules. Say $L^i = 0$ for $i \notin [a, b]$. Fix a, b from now on. Let c be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with L^i is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda \in \Lambda} R$. Since L^{c-1} is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} R)[-c]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree c. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 13.12.1). Since $\mathrm{Hom}_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where L can be represented by a finite complex of free R-modules. In particular we see that L is perfect. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.14 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1-e)$ in $D(R)$. Hence we finally conclude that K is perfect by Lemma 15.75.5. □

07LU **Lemma 15.79.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes^{\mathbf{L}}_R R/I$ is perfect in $D(R/I)$, and*
(2) *I is a nilpotent ideal.*

Then K is perfect in $D(R)$.

Proof. Choose a finite complex $\overline{P}^\bullet$ of finite projective R/I-modules representing $K \otimes^{\mathbf{L}}_R R/I$, see Definition 15.75.1. By Lemma 15.76.3 there exists a complex $P^\bullet$ of projective R-modules representing K such that $\overline{P}^\bullet = P^\bullet/IP^\bullet$. It follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $P^\bullet$ is a finite complex of finite projective R-modules. □

09AS **Lemma 15.79.5.** *Let R be a ring. Let $I, J \subset R$ be ideals. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes^{\mathbf{L}}_R R/I$ is perfect in $D(R/I)$, and*
(2) *$K \otimes^{\mathbf{L}}_R R/J$ is perfect in $D(R/J)$.*

Then $K \otimes^{\mathbf{L}}_R R/IJ$ is perfect in $D(R/IJ)$.

Proof. It is clear that we may assume replace R by R/IJ and K by $K \otimes^{\mathbf{L}}_R R/IJ$. Then $R \to R/(I \cap J)$ is a surjection whose kernel has square zero. Hence by Lemma 15.79.4 it suffices to prove that $K \otimes^{\mathbf{L}}_R R/(I \cap J)$ is perfect. Thus we may assume that $I \cap J = 0$.

We prove the lemma in case $I \cap J = 0$. First, we may represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 15.59.10. Then we see that we have a short exact sequence of complexes

$$0 \to K^\bullet \to K^\bullet/IK^\bullet \oplus K^\bullet/JK^\bullet \to K^\bullet/(I+J)K^\bullet \to 0$$

Note that $K^\bullet/IK^\bullet$ represents $K \otimes^{\mathbf{L}}_R R/I$ by construction of the derived tensor product. Similarly for $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$. Note that $K^\bullet/(I+J)K^\bullet$ is a perfect complex of $R/(I+J)$-modules, see Lemma 15.75.9. Hence the complexes $K^\bullet/IK^\bullet$, and $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$ have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 15.75.2). We conclude that $K \in D^b(R)$ by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we assume $K^\bullet$ is a bounded above complex of free R-modules (see Derived Categories, Lemma 13.15.4).

We will now show that K is perfect using the criterion of Proposition 15.79.3. Thus we let $E_j \in D(R)$ be a family of objects parametrized by a set J. We choose complexes $E_j^\bullet$ with flat terms representing E_j, see for example Lemma 15.59.10. It is clear that

$$0 \to E_j^\bullet \to E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to E_j^\bullet/(I+J)E_j^\bullet \to 0$$

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

$$0 \to \bigoplus E_j^\bullet \to \bigoplus E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to \bigoplus E_j^\bullet/(I+J)E_j^\bullet \to 0$$

(Note that $- \otimes_R R/I$ commutes with direct sums.) This short exact sequence determines a distinguished triangle in $D(R)$, see Derived Categories, Lemma 13.12.1. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$ (see Derived Categories, Lemma 13.4.2) to get a commutative diagram

$$\begin{array}{ccc}
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J))[-1] & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))[-1] \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J)[-1] & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet) \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J) \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J)) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))
\end{array}$$

with exact columns. It is clear that, for any complex $E^\bullet$ of R-modules we have

$$\begin{aligned}
\mathrm{Hom}_{D(R)}(K^\bullet, E^\bullet/I) &= \mathrm{Hom}_{K(R)}(K^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{K(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{D(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I)
\end{aligned}$$

and similarly for when dividing by J or $I+J$, see Derived Categories, Lemma 13.19.8. Derived Categories. Thus all the horizontal arrows, except for possibly the middle one, are isomorphisms as the complexes $K^\bullet/IK^\bullet$, $K^\bullet/JK^\bullet$, $K^\bullet/(I+J)K^\bullet$ are perfect complexes of R/I, R/J, $R/(I+J)$-modules, see Proposition 15.79.3. It follows from the 5-lemma (Homology, Lemma 12.5.20) that the middle map is an isomorphism and the lemma follows by Proposition 15.79.3. □

15.80. Strong generators and regular rings

0FXG Let R be a ring. Denote $D(R)_c$ the saturated full triangulated subcategory of $D(R)$. We already know that

$$\langle R \rangle = D_{perf}(R) = D(R)_c$$

See Lemma 15.79.1 and Proposition 15.79.3. It turns out that if R is regular, then R is a strong generator (Derived Categories, Definition 13.36.3).

0FXH **Lemma 15.80.1.** *Let R be a ring. Let $n \geq 1$. Let $K \in \langle R \rangle_n$ with notation as in Derived Categories, Section 13.36. Consider maps* [**Kel65**]

$$K \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$. If $H^i(f_j) = 0$ for all i, j, then $f_n \circ \ldots \circ f_1 = 0$.

Proof. If $n = 1$, then K is a direct summand in $D(R)$ of a bounded complex $P^\bullet$ whose terms are finite free R-modules and whose differentials are zero. Thus it suffices to show any morphism $f : P^\bullet \to K_1$ in $D(R)$ with $H^i(f) = 0$ for all i is zero. Since $P^\bullet$ is a finite direct sum $P^\bullet = \bigoplus R[m_j]$ it suffices to show any morphism $g : R[m] \to K_1$ with $H^{-m}(g) = 0$ in $D(R)$ is zero. This follows from the fact that $\mathrm{Hom}_{D(R)}(R[-m], K) = H^m(K)$.

For $n > 1$ we proceed by induction on n. Namely, we know that K is a summand in $D(R)$ of an object P which sits in a distinguished triangle

$$P' \xrightarrow{i} P \xrightarrow{p} P'' \to P'[1]$$

with $P' \in \langle R \rangle_1$ and $P'' \in \langle R \rangle_{n-1}$. As above we may replace K by P and assume that we have

$$P \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$ with f_j zero on cohomology. By the case $n = 1$ the composition $f_1 \circ i$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can find a morphism $h : P'' \to K_1$ such that $f_1 = h \circ p$. Observe that $f_2 \circ h$ is zero on cohomology. Hence by induction we find that $f_n \circ \ldots \circ f_2 \circ h = 0$ which implies $f_n \circ \ldots \circ f_1 = f_n \circ \ldots \circ f_2 \circ h \circ p = 0$ as desired. □

0FXI **Lemma 15.80.2.** *Let R be a Noetherian ring. If R is a strong generator for $D_{perf}(R)$, then R is regular of finite dimension.*

Proof. Assume $D_{perf}(R) = \langle R \rangle_n$ for some $n \geq 1$. For any finite R-module M we can choose a complex

$$P = (P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^1} \ldots \xrightarrow{d^{-1}} P^0)$$

of finite free R-modules with $H^i(P) = 0$ for $i = -n, \ldots, -1$ and $M \cong \mathrm{Coker}(d^{-1})$. Note that P is in $D_{perf}(R)$. For any R-module N we can compute $\mathrm{Ext}^n_R(M, N)$ the finite free resolution P of M, see Algebra, Section 10.71 and compare with Derived Categories, Section 13.27. In particular, the sequence above defines an element

$$\xi \in \mathrm{Ext}^n_R(\mathrm{Coker}(d^{-1}), \mathrm{Coker}(d^{-n-1})) = \mathrm{Ext}^n_R(M, \mathrm{Coker}(d^{-n-1}))$$

and for any element $\overline{\xi}$ in $\mathrm{Ext}^n_R(M, N)$ there is a R-module map $\varphi : \mathrm{Coker}(d^{-n-1}) \to N$ such that φ maps ξ to $\overline{\xi}$. For $j = 1, \ldots, n-1$ consider the complexes

$$K_j = (\mathrm{Coker}(d^{-n-1}) \to P^{-n+1} \to \ldots \to P^{-j})$$

with $\mathrm{Coker}(d^{-n-1})$ in degree $-n$ and P^t in degree t. We also set $K_n = \mathrm{Coker}(d^{-n-1})[n]$. Then we have maps

$$P \to K_1 \to K_2 \to \ldots \to K_n$$

which induce vanishing maps on cohomology. By Lemma 15.80.1 since $P \in D_{perf}(R) = \langle R \rangle_n$ we find that the composition of this maps is zero in $D(R)$. Since $\mathrm{Hom}_{D(R)}(P, K_n) = \mathrm{Hom}_{K(R)}(P, K_n)$ by Derived Categories, Lemma 13.19.8 we conclude $\xi = 0$. Hence $\mathrm{Ext}^n_R(M, N) = 0$ for all R-modules N, see discussion above. It follows that M has projective dimension $\leq n - 1$ by Algebra, Lemma 10.109.8. Since this holds for all finite R-modules M we conclude that R has finite global dimension, see Algebra, Lemma 10.109.12. We finally conclude by Algebra, Lemma 10.110.8. □

0FXJ **Lemma 15.80.3.** *Let R be a Noetherian regular ring of dimension $d < \infty$. Let $K, L \in D^-(R)$. Assume there exists an k such that $H^i(K) = 0$ for $i \leq k$ and $H^i(L) = 0$ for $i \geq k - d + 1$. Then $\mathrm{Hom}_{D(R)}(K, L) = 0$.*

Proof. Let $K^\bullet$ be a bounded above complex representing K, say $K^i = 0$ for $i \geq n+1$. After replacing $K^\bullet$ by $\tau_{\geq k+1}K^\bullet$ we may assume $K^i = 0$ for $i \leq k$. Then we may use the distinguished triangle

$$K^n[-n] \to K^\bullet \to \sigma_{\leq n-1}K^\bullet$$

to see it suffices to prove the lemma for $K^n[-n]$ and $\sigma_{\leq n-1}K^\bullet$. By induction on n, we conclude that it suffices to prove the lemma in case K is represented by the complex $M[-m]$ for some R-module M and some $m \geq k+1$. Since R has global dimension d by Algebra, Lemma 10.110.8 we see that M has a projective resolution $0 \to P_d \to \ldots \to P_0 \to M \to 0$. Then the complex $P^\bullet$ having P_i in degree $m - i$ is a bounded complex of projectives representing $M[-m]$. On the other hand, we can choose a complex $L^\bullet$ representing L with $L^i = 0$ for $i \geq k - d + 1$. Hence any map of complexes $P^\bullet \to L^\bullet$ is zero. This implies the lemma by Derived Categories, Lemma 13.19.8. □

0FXK **Lemma 15.80.4.** *Let R be a Noetherian regular ring of dimension $1 \leq d < \infty$. Let $K \in D(R)$ be perfect and let $k \in \mathbf{Z}$ such that $H^i(K) = 0$ for $i = k-d+2, \ldots, k$ (empty condition if $d = 1$). Then $K = \tau_{\leq k-d+1}K \oplus \tau_{\geq k+1}K$.*

Proof. The vanishing of cohomology shows that we have a distinguished triangle

$$\tau_{\leq k-d+1}K \to K \to \tau_{\geq k+1}K \to (\tau_{\leq k-d+1}K)[1]$$

By Derived Categories, Lemma 13.4.11 it suffices to show that the third arrow is zero. Thus it suffices to show that $\mathrm{Hom}_{D(R)}(\tau_{\geq k+1}K, (\tau_{\leq k-d+1}K)[1]) = 0$ which follows from Lemma 15.80.3. □

0FXL **Lemma 15.80.5.** *Let R be a Noetherian regular ring of finite dimension. Then R is a strong generator for the full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects.*

Proof. We will use that an object K of $D(R)$ is perfect if and only if K is bounded and has finite cohomology modules, see Lemma 15.75.14. Strong generators of triangulated categories are defined in Derived Categories, Definition 13.36.3. Let $d = \dim(R)$.

Let $K \in D_{perf}(R)$. We will show $K \in \langle R \rangle_{d+1}$. By Algebra, Lemma 10.110.8 every finite R-module has projective dimension $\leq d$. We will show by induction on $0 \leq i \leq d$ that if $H^n(K)$ has projective dimension $\leq i$ for all $n \in \mathbf{Z}$, then K is in $\langle R \rangle_{i+1}$.

Base case $i = 0$. In this case $H^n(K)$ is a finite R-module of projective dimension 0. In other words, each cohomology is a projective R-module. Thus $\mathrm{Ext}^i_R(H^n(K), H^m(K)) = 0$ for all $i > 0$ and $m, n \in \mathbf{Z}$. By Derived Categories, Lemma 13.27.9 we find that K is isomorphic to the direct sum of the shifts of its cohomology modules. Since each cohomology module is a finite projective R-module, it is a direct summand of a direct sum of copies of R. Hence by definition we see that K is contained in $\langle R \rangle_1$.

Induction step. Assume the claim holds for $i < d$ and let $K \in D_{perf}(R)$ have the property that $H^n(K)$ has projective dimension $\leq i+1$ for all $n \in \mathbf{Z}$. Choose $a \leq b$ such that $H^n(K)$ is zero for $n \notin [a, b]$. For each $n \in [a, b]$ choose a surjection $F^n \to H^n(K)$ where F^n is a finite free R-module. Since F^n is projective, we can lift $F^n \to H^n(K)$ to a map $F^n[-n] \to K$ in $D(R)$ (small detail omitted). Thus

we obtain a morphism $\bigoplus_{a\leq n\leq b} F^n[-n] \to K$ which is surjective on cohomology modules. Choose a distinguished triangle

$$K' \to \bigoplus\nolimits_{a\leq n\leq b} F^n[-n] \to K \to K'[1]$$

in $D(R)$. Of course, the object K' is bounded and has finite cohomology modules. The long exact sequence of cohomology breaks into short exact sequences

$$0 \to H^n(K') \to F^n \to H^n(K) \to 0$$

by the choices we made. By Algebra, Lemma 10.109.9 we see that the projective dimension of $H^n(K')$ is $\leq \max(0, i)$. Thus $K' \in \langle R\rangle_{i+1}$. By definition this means that K is in $\langle R\rangle_{i+1+1}$ as desired. □

0FXM **Proposition 15.80.6.** *Let R be a Noetherian ring. The following are equivalent*

(1) *R is regular of finite dimension,*
(2) *$D_{perf}(R)$ has a strong generator, and*
(3) *R is a strong generator for $D_{perf}(R)$.*

Proof. This is a formal consequence of Lemmas 15.79.1, 15.80.2, and 15.80.5 as well as Derived Categories, Lemma 13.36.6. □

15.81. Relatively finitely presented modules

0659 Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a finite B-module. In this case it is **not true** that

$$M \text{ of finite presentation over } B \Leftrightarrow M \text{ of finite presentation over } A$$

A counter example is $R = k[x_1, x_2, x_3, \ldots]$, $A = R$, $B = R/(x_i)$, and $M = B$. To "fix" this we introduce a relative notion of finite presentation.

05GY **Lemma 15.81.1.** *Let $R \to A$ be a ring map of finite type. Let M be an A-module. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module,*
(2) *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module, and*
(3) *for any surjection $A' \to A$ where A' is a finitely presented R-algebra, the module M is finitely presented as A'-module.*

In this case M is a finitely presented A-module.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas 10.6.4 and 10.36.23. The equivalence of (2) and (3) follows by choosing a presentation $A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and using Algebra, Lemma 10.36.23 to show that

M is finitely presented as A'-module if and only if M is finitely presented as a $R[x_1, \ldots, x_n]$-module. □

05GZ **Definition 15.81.2.** Let $R \to A$ be a finite type ring map. Let M be an A-module. We say M is an A-module *finitely presented relative to* R if the equivalent conditions of Lemma 15.81.1 hold.

Note that if $R \to A$ is of finite presentation, then M is an A-module finitely presented relative to R if and only if M is a finitely presented A-module. It is equally clear that A as an A-module is finitely presented relative to R if and only if A is of finite presentation over R. If R is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

05H0 **Lemma 15.81.3.** *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a B-module. Then M is an A-module finitely presented relative to R if and only if M is a B-module finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n] & \longrightarrow & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma 10.36.23 and the definition. □

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over R. It is also stable under localization, stable under base change, and "glues" well.

065A **Lemma 15.81.4.** *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and M an A-module. If M of finite presentation relative to R_f, then M_g is an A_g-module of finite presentation relative to R.*

Proof. Choose a presentation $R_f[x_1, \ldots, x_n] \to A$. We write $R_f = R[x_0]/(fx_0 - 1)$. Consider the presentation $R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g$ which extends the given map, maps x_0 to the image of $1/f$, and maps x_{n+1} to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to g (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^{\oplus s} \to R_f[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to h_{ij}. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus t} \to M_g \to 0$$

is a presentation of M_f. Here the $t \times (s + 2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h'_{ij}, a second $t \times t$ block which is $(x_0 f -)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. □

065B **Lemma 15.81.5.** *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. For any ring map $R \to R'$ the $A \otimes_R R'$-module*

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^{\oplus s} \to R'[x_1, \ldots, x_n]^{\oplus t} \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. □

0670 **Lemma 15.81.6.** *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. Let $A \to A'$ be a ring map of finite presentation. The A'-module $M \otimes_A A'$ is finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g_i' \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to g_i. Say

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus s+tl} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus t} \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + lt)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h_{ij}, followed by l blocks of size $t \times t$ which are $g_i' I_t$. □

065C **Lemma 15.81.7.** *Let $R \to A \to B$ be finite type ring maps. Let M be a B-module. If M is finitely presented relative to A and A is of finite presentation over R, then M is finitely presented relative to R.*

Proof. Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^{\oplus s} \to A[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

given by a matrix (h_{ij}). Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_u).$$

Choose $h_{ij}' \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to h_{ij}. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

where the $t \times (s + tu)$-matrix is given by a first $t \times s$ block consisting of h_{ij}' followed by u blocks of size $t \times t$ given by $g_i I_t$, $i = 1, \ldots, u$. □

065D **Lemma 15.81.8.** *Let $R \to A$ be a finite type ring map. Let M be an A-module. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

(1) *each M_{f_i} is finitely presented relative to R, and*
(2) *M is finitely presented relative to R.*

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.81.4. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

By Lemma 15.81.1 we see that M_{f_i} is a finitely presented A_{f_i}-module, hence by Algebra, Lemma 10.23.2 we see that M is a finitely presented A-module. Hence M is a finite P-module (with P as above). Choose a surjection $P^{\oplus t} \to M$. We have to show that the kernel K of this map is a finite P-module. Since P_{y_i} surjects onto A_{f_i} we see by Lemma 15.81.1 and Algebra, Lemma 10.5.3 that the localization K_{y_i} is a finitely generated P_{y_i}-module. Choose elements $k_{i,j} \in K$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ such that the images of $k_{i,j}$ in K_{y_i} generate. Set $K' \subset K$ equal to the P-module generated by the elements $k_{i,j}$. Then K/K' is a module whose localization at y_i is zero for all i. Since $(y_1, \ldots, y_r) = P$ we see that $K/K' = 0$ as desired. □

0671 **Lemma 15.81.9.** *Let $R \to A$ be a finite type ring map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules.*

(1) *If M', M'' are finitely presented relative to R, then so is M.*
(2) *If M' is a finite type A-module and M is finitely presented relative to R, then M'' is finitely presented relative to R.*

Proof. Follows immediately from Algebra, Lemma 10.5.3. □

0672 **Lemma 15.81.10.** *Let $R \to A$ be a finite type ring map. Let M, M' be A-modules. If $M \oplus M'$ is finitely presented relative to R, then so are M and M'.*

Proof. Omitted. □

15.82. Relatively pseudo-coherent modules

065E This section is the analogue of Section 15.81 for pseudo-coherence.

065F **Lemma 15.82.1.** *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then*

$$K^\bullet \otimes^{\mathbf{L}}_{R[x]} R \cong K^\bullet \oplus K^\bullet[1]$$

in $D(R)$.

Proof. Choose a K-flat resolution $P^\bullet \to K^\bullet$ over R such that P^n is a flat R-module for all n, see Lemma 15.59.10. Then $P^\bullet \otimes_R R[x]$ is a K-flat complex of $R[x]$-modules whose terms are flat $R[x]$-modules, see Lemma 15.59.3 and Algebra, Lemma 10.39.7. In particular $x : P^n \otimes_R R[x] \to P^n \otimes_R R[x]$ is injective with cokernel isomorphic to P^n. Thus

$$P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]$$

is a double complex of $R[x]$-modules whose associated total complex is quasi-isomorphic to $P^\bullet$ and hence $K^\bullet$. Moreover, this associated total complex is a K-flat complex of $R[x]$-modules for example by Lemma 15.59.4 or by Lemma 15.59.5. Hence

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x]} R &\cong \mathrm{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]) \otimes_{R[x]} R = \mathrm{Tot}(P^\bullet \xrightarrow{0} P^\bullet) \\ &= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1] \end{aligned}$$

as desired. □

065G **Lemma 15.82.2.** *Let R be a ring and $K^\bullet$ a complex of R-modules. Let $m \in \mathbf{Z}$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then $K^\bullet$ is m-pseudo-coherent as a complex of R-modules if and only if $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules.*

Proof. This is a special case of Lemma 15.65.11. We also prove it in another way as follows.

Note that $0 \to R[x] \to R[x] \to R \to 0$ is exact. Hence R is pseudo-coherent as an $R[x]$-module. Thus one implication of the lemma follows from Lemma 15.65.11. To prove the other implication, assume that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules. By Lemma 15.65.12 we see that $K^\bullet \otimes^{\mathbf{L}}_{R[x]} R$ is m-pseudo-coherent as a complex of R-modules. By Lemma 15.82.1 we see that $K^\bullet \oplus K^\bullet[1]$ is m-pseudo-coherent as a complex of R-modules. Finally, we conclude that $K^\bullet$ is m-pseudo-coherent as a complex of R-modules from Lemma 15.65.8. □

065H **Lemma 15.82.3.** *Let $R \to A$ be a ring map of finite type. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. The following are equivalent*

1. *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules,*
2. *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules.*

In particular the same equivalence holds for pseudo-coherence.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

After a change of coordinates the ring homomorphism $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n]$ is isomorphic to the ring homomorphism which maps each y_i to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the number of variables this lemma follows from Lemma 15.82.2. The pseudo-coherent case follows from this and Lemma 15.65.5. □

065I **Definition 15.82.4.** Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let M be an A-module. Let $m \in \mathbf{Z}$.

1. We say $K^\bullet$ is *m-pseudo-coherent relative to R* if the equivalent conditions of Lemma 15.82.3 hold.
2. We say $K^\bullet$ is *pseudo-coherent relative to R* if $K^\bullet$ is m-pseudo-coherent relative to R for all $m \in \mathbf{Z}$.
3. We say M is *m-pseudo-coherent relative to R* if $M[0]$ is m-pseudo-coherent relative to R.
4. We say M is *pseudo-coherent relative to R* if $M[0]$ is pseudo-coherent relative to R.

Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 15.65.5. This definition has the following pleasing property.

0673 **Lemma 15.82.5.** *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R if and only if $K^\bullet$ seen as a complex of A-modules is m-pseudo-coherent (pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{array}{ccc} & & R[x_1, \ldots, x_n, y_1, \ldots, y_m] \\ & & \downarrow \\ R[x_1, \ldots, x_n] & \longrightarrow & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

The top horizontal arrow and the top right vertical arrow satisfy the assumptions of Lemma 15.65.11. Hence $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. □

0674 **Lemma 15.82.6.** *Let R be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $L^\bullet$ is m-pseudo-coherent relative to R then $M^\bullet$ is m-pseudo-coherent relative to R.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent relative to R, then $L^\bullet$ is m-pseudo-coherent relative to R.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $M^\bullet$ is m-pseudo-coherent relative to R, then $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R.*

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to R, the so is the third.

Proof. Follows immediately from Lemma 15.65.2 and the definitions. □

0675 **Lemma 15.82.7.** *Let $R \to A$ be a finite type ring map. Let M be an A-module. Then*

(1) *M is 0-pseudo-coherent relative to R if and only if M is a finite type A-module,*
(2) *M is (-1)-pseudo-coherent relative to R if and only if M is a finitely presented relative to R,*
(3) *M is $(-d)$-pseudo-coherent relative to R if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution*

$$R[x_1, \ldots, x_n]^{\oplus a_d} \to R[x_1, \ldots, x_n]^{\oplus a_{d-1}} \to \ldots \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

of length d, and

(4) *M is pseudo-coherent relative to R if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution*

$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

by finite free $R[x_1, \ldots, x_n]$-modules.

Proof. Follows immediately from Lemma 15.65.4 and the definitions. □

0676 **Lemma 15.82.8.** *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R so are $K^\bullet$ and $L^\bullet$.*

Proof. Immediate from Lemma 15.65.8 and the definitions. □

0677 **Lemma 15.82.9.** *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of A-modules such that K^i is $(m-i)$-pseudo-coherent relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent relative to R. In particular, if $K^\bullet$ is a bounded above complex of A-modules pseudo-coherent relative to R, then $K^\bullet$ is pseudo-coherent relative to R.*

Proof. Immediate from Lemma 15.65.9 and the definitions. □

0678 **Lemma 15.82.10.** *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 15.65.10 and the definitions. □

0679 **Lemma 15.82.11.** *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of A-modules. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R_f, then $K^\bullet \otimes_A A_g$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. First we show that $K^\bullet$ is m-pseudo-coherent relative to R. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 15.65.11 we see that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 15.65.12 we see that

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x_0, x_1, \ldots, x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] &= K^\bullet \otimes_{R[x_0, x_1, \ldots, x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] \\ &= K^\bullet \otimes_A A_f \end{aligned}$$

is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$-modules. write

$$R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$

As $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 15.65.11) that $K^\bullet \otimes_A A_g$ is m-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. □

067A **Lemma 15.82.12.** *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules which is m-pseudo-coherent (resp. pseudo-coherent) relative to R. Let $R \to R'$ be a ring map such that A and R' are Tor independent over R. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes^{\mathbf{L}}_A A'$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R'.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes^{\mathbf{L}}_A A' = K^\bullet \otimes^{\mathbf{L}}_R R' = K^\bullet \otimes^{\mathbf{L}}_{R[x_1,\ldots,x_n]} R'[x_1, \ldots, x_n]$$

by Lemma 15.61.2 applied twice. Hence we win by Lemma 15.65.12. □

067B **Lemma 15.82.13.** *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules. Assume B as a B-module is pseudo-coherent relative to A. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $K^\bullet \otimes^{\mathbf{L}}_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots y_m] \to B$. Choose a resolution $E^\bullet \to B$ of B by a complex of finite free $A[y_1, \ldots, y_n]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat A-modules. Then

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_A B &= \mathrm{Tot}(K^\bullet \otimes_A B[0]) \\ &= \mathrm{Tot}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_{A[y_1,\ldots,y_m]} B[0]) \\ &\cong \mathrm{Tot}\left((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_{A[y_1,\ldots,y_m]} E^\bullet\right) \\ &= \mathrm{Tot}(K^\bullet \otimes_A E^\bullet) \end{aligned}$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 15.59.7. Thus we have to show that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ is m-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots y_m]$-modules. Note that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^i[-i]$. Note that for $i \ll 0$ the complexes $K^\bullet \otimes_A E^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent (over any ring). Hence, applying Lemma 15.82.6 and induction, it suffices to show that $K^\bullet \otimes_A E^i[-i]$ is pseudo-coherent relative to R for all i. Note that $E^i = 0$ for $i > 0$. Since also E^i is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is m-pseudo-coherent relative to R which follows from Lemma 15.82.12 for instance. □

067C **Lemma 15.82.14.** *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let M be an A-module. Assume B is flat over A and B as a B-module is pseudo-coherent relative to A. If M is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $M \otimes_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 15.82.13. □

067D **Lemma 15.82.15.** *Let R be a ring. Let $A \to B$ be a map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Assume A is pseudo-coherent relative to R. Then the following are equivalent*

(1) *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to A, and*
(2) *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption A is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 15.65.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 15.65.11 and the definitions. □

067E **Lemma 15.82.16.** *Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

(1) *each $K^\bullet \otimes_A A_{f_i}$ is m-pseudo-coherent relative to R, and*
(2) *$K^\bullet$ is m-pseudo-coherent relative to R.*

The same equivalence holds for pseudo-coherence relative to R.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.82.11. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

Note that P is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 15.65.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an m-pseudo-coherent complex of $P[1/y_i]$-modules for each i. Thus by Lemma 15.65.14 we see that $K^\bullet$ is pseudo-coherent as a complex of P-modules, and Lemma 15.65.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. □

067F **Lemma 15.82.17.** *Let R be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then*

(1) *A complex of A-modules $K^\bullet$ is m-pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for $i \geq m$.*
(2) *A complex of A-modules $K^\bullet$ is pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for all i.*
(3) *An A-module is pseudo-coherent relative to R if and only if it is finite.*

Proof. Immediate consequence of Lemma 15.65.17 and the definitions. □

15.83. Pseudo-coherent and perfect ring maps

067G We can define these types of ring maps as follows.

067H **Definition 15.83.1.** Let $A \to B$ be a ring map.

(1) We say $A \to B$ is a *pseudo-coherent ring map* if it is of finite type and B, as a B-module, is pseudo-coherent relative to A.
(2) We say $A \to B$ is a *perfect ring map* if it is a pseudo-coherent ring map such that B as an A-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 15.82.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for

the definition of a perfect ring map is Lemma 15.75.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

068Y **Lemma 15.83.2.** *A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.*

Proof. If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an A-module. Hence Lemma 15.78.5 implies that B is perfect as a $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 15.75.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent. Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat A-modules and hence B has finite tor dimension as an A-module. □

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 37.60 and 37.61 for the corresponding discussion concerning morphisms of schemes.

067I **Lemma 15.83.3.** *A finite type ring map of Noetherian rings is pseudo-coherent.*

Proof. See Lemma 15.82.17. □

067J **Lemma 15.83.4.** *A ring map which is flat and of finite presentation is perfect.*

Proof. Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that B has finite tor dimension. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 15.82.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that B_0 and A are tor independent over A_0, hence we may use Lemma 15.82.12 to conclude that $A \to B$ is pseudo-coherent. □

067K **Lemma 15.83.5.** *Let $A \to B$ be a finite type ring map with A a regular ring of finite dimension. Then $A \to B$ is perfect.*

Proof. By Algebra, Lemma 10.110.8 the assumption on A means that A has finite global dimension. Hence every module has finite tor dimension, see Lemma 15.67.19, in particular B does. By Lemma 15.83.3 the map is pseudo-coherent. □

07EN **Lemma 15.83.6.** *A local complete intersection homomorphism is perfect.*

Proof. Let $A \to B$ be a local complete intersection homomorphism. By Definition 15.33.2 this means that $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 15.83.2 and 15.75.3 it suffices to show that I is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 15.75.12 this is a local question. Hence we may assume that I is generated by a Koszul-regular sequence (by Definition 15.32.1). Of course this means that I has a finite free resolution and we win. □

0DHQ **Lemma 15.83.7.** *Let $R \to A$ be a pseudo-coherent ring map. Let $K \in D(A)$. The following are equivalent*

(1) *K is m-pseudo-coherent (resp. pseudo-coherent) relative to R, and*
(2) *K is m-pseudo-coherent (resp. pseudo-coherent) in $D(A)$.*

Proof. Reformulation of a special case of Lemma 15.82.15. □

0E1T **Lemma 15.83.8.** *Let $R \to B \to A$ be ring maps with $\varphi : B \to A$ surjective and $R \to B$ and $R \to A$ flat and of finite presentation. For $K \in D(A)$ denote $\varphi_* K \in D(B)$ the restriction. The following are equivalent*

(1) *K is pseudo-coherent,*
(2) *K is pseudo-coherent relative to R,*
(3) *K is pseudo-coherent relative to A,*
(4) *$\varphi_* K$ is pseudo-coherent,*
(5) *$\varphi_* K$ is pseudo-coherent relative to R.*

Similar holds for m-pseudo-coherence.

Proof. Observe that $R \to A$ and $R \to B$ are perfect ring maps (Lemma 15.83.4) hence a fortiori pseudo-coherent ring maps. Thus (1) $\Leftrightarrow$ (2) and (4) $\Leftrightarrow$ (5) by Lemma 15.83.7.

Using that A is pseudo-coherent relative to R we use Lemma 15.82.15 to see that (2) $\Leftrightarrow$ (3). However, since $A \to B$ is surjective, we see directly from Definition 15.82.4 that (3) is equivalent with (4). □

15.84. Relatively perfect modules

0DHR This section is the analogue of Section 15.82 for perfect objects of the derived category. we only define this notion in a limited generality as we are not sure what the correct definition is in general. See Derived Categories of Schemes, Remark 36.35.14 for a discussion.

0DHS **Definition 15.84.1.** Let $R \to A$ be a flat ring map of finite presentation. An object K of $D(A)$ is *R-perfect* or *perfect relative to R* if K is pseudo-coherent (Definition 15.65.1) and has finite tor dimension over R (Definition 15.67.1).

By Lemma 15.83.8 it would have been the same thing to ask K to be pseudo-coherent relative to R. Here are some obligatory lemmas.

0DHT **Lemma 15.84.2.** *Let $R \to A$ be a flat ring map of finite presentation. The R-perfect objects of $D(A)$ form a saturated[10] triangulated strictly full subcategory.*

Proof. This follows from Lemmas 15.65.2, 15.65.8, 15.67.5, and 15.67.7. □

0DHU **Lemma 15.84.3.** *Let $R \to A$ be a flat ring map of finite presentation. A perfect object of $D(A)$ is R-perfect. If $K, M \in D(A)$ then $K \otimes^{\mathbf{L}}_A M$ is R-perfect if K is perfect and M is R-perfect.*

Proof. The first statement follows from the second by taking $M = A$. The second statement follows from Lemmas 15.75.2, 15.67.10, and 15.65.16. □

0DHV **Lemma 15.84.4.** *Let $R \to A$ be a flat ring map of finite presentation. Let $K \in D(A)$. The following are equivalent*

(1) *K is R-perfect, and*
(2) *K is isomorphic to a finite complex of R-flat, finitely presented A-modules.*

[10] Derived Categories, Definition 13.6.1.

Proof. To prove (2) implies (1) it suffices by Lemma 15.84.2 to show that an R-flat, finitely presented A-module M defines an R-perfect object of $D(A)$. Since M has finite tor dimension over R, it suffices to show that M is pseudo-coherent. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$-algebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ and a finite A_0-module M_0 flat over R_0 such that $A = A_0 \otimes_{R_0} R$ and $M = M_0 \otimes_{R_0} R$. By Lemma 15.65.17 we see that M_0 is pseudo-coherent A_0-module. Choose a resolution $P_0^\bullet \to M_0$ by finite free A_0-modules P_0^n. Since A_0 is flat over R_0, this is a flat resolution. Since M_0 is flat over R_0 we find that $P^\bullet = P_0^\bullet \otimes_{R_0} R$ still resolves $M = M_0 \otimes_{R_0} R$. (You can use Lemma 15.61.2 to see this.) Hence $P^\bullet$ is a finite free resolution of M over A and we conclude that M is pseudo-coherent.

Assume (1). We can represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Assume that K viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. By Lemma 15.67.2 we see that $\tau_{\geq a} P^\bullet$ is a complex of R-flat, finitely presented A-modules representing K. □

0DHW **Lemma 15.84.5.** *Let $R \to A$ be a flat ring map of finite presentation. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$. If $K \in D(A)$ is R-perfect, then $K \otimes^{\mathbf{L}}_A A'$ is R'-perfect.*

Proof. By Lemma 15.65.12 we see that $K \otimes^{\mathbf{L}}_A A'$ is pseudo-coherent. By Lemma 15.61.2 we see that $K \otimes^{\mathbf{L}}_A A'$ is equal to $K \otimes^{\mathbf{L}}_R R'$ in $D(R')$. Then we can apply Lemma 15.67.13 to see that $K \otimes^{\mathbf{L}}_R R'$ in $D(R')$ has finite tor dimension. □

0E1U **Lemma 15.84.6.** *Let $R \to A$ be a flat ring map. Let $K, L \in D(A)$ with K pseudo-coherent and L finite tor dimension over R. We may choose*

(1) *a bounded above complex $P^\bullet$ of finite free A-modules representing K, and*
(2) *a bounded complex of R-flat A-modules $F^\bullet$ representing L.*

Given these choices we have

(a) *$E^\bullet = \mathrm{Hom}^\bullet(P^\bullet, F^\bullet)$ is a bounded below complex of R-flat A-modules representing $R\mathrm{Hom}_A(K, L)$,*
(b) *for any ring map $R \to R'$ with $A' = A \otimes_R R'$ the complex $E^\bullet \otimes_R R'$ represents $R\mathrm{Hom}_{A'}(K \otimes^{\mathbf{L}}_A A', L \otimes^{\mathbf{L}}_A A')$.*

If in addition $R \to A$ is of finite presentation and L is R-perfect, then we may choose F^p to be finitely presented A-modules and consequently E^n will be finitely presented A-modules as well.

Proof. The existence of $P^\bullet$ is the definition of a pseudo-coherent complex. We first represent L by a bounded above complex $F^\bullet$ of free A-modules (this is possible because bounded tor dimension in particular implies bounded). Next, say L viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. Then, after replacing $F^\bullet$ by $\tau_{\geq a} F^\bullet$, we get a complex as in (2). This follows from Lemma 15.67.2.

Proof of (a). Since $F^\bullet$ is bounded an since $P^\bullet$ is bounded above, we see that $E^n = 0$ for $n \ll 0$ and that E^n is a finite (!) direct sum

$$E^n = \bigoplus\nolimits_{p+q=n} \mathrm{Hom}_A(P^{-q}, F^p)$$

and since P^{-q} is finite free, this is indeed an R-flat A-module. The fact that $E^\bullet$ represents $R\mathrm{Hom}_A(K, L)$ follows from Lemma 15.74.2.

Proof of (b). Let $R \to R'$ be a ring map and $A' = A \otimes_R R'$. By Lemma 15.61.2 the object $L \otimes^{\mathbf{L}}_A A'$ is represented by $F^\bullet \otimes_R R'$ viewed as a complex of A'-modules (by flatness of F^p over R). Similarly for $P^\bullet \otimes_R R'$. As above $R\operatorname{Hom}_{A'}(K \otimes^{\mathbf{L}}_A A', L \otimes^{\mathbf{L}}_A A')$ is represented by

$$\operatorname{Hom}^\bullet(P^\bullet \otimes_R R', F^\bullet \otimes_R R') = E^\bullet \otimes_R R'$$

The equality holds by looking at the terms of the complex individually and using that $\operatorname{Hom}_{A'}(P^{-q} \otimes_R R', F^p \otimes_R R') = \operatorname{Hom}_A(P^{-q}, F^p) \otimes_R R'$. □

0DHX **Lemma 15.84.7.** *Let $R = \operatorname{colim}_{i \in I} R_i$ be a filtered colimit of rings. Let $0 \in I$ and $R_0 \to A_0$ be a flat ring map of finite presentation. For $i \geq 0$ set $A_i = R_i \otimes_{R_0} A_0$ and set $A = R \otimes_{R_0} A_0$.*

(1) *Given an R-perfect K in $D(A)$ there exists an $i \in I$ and an R_i-perfect K_i in $D(A_i)$ such that $K \cong K_i \otimes^{\mathbf{L}}_{A_i} A$ in $D(A)$.*
(2) *Given $K_0, L_0 \in D(A_0)$ with K_0 pseudo-coherent and L_0 finite tor dimension over R_0, then we have*

$$\operatorname{Hom}_{D(A)}(K_0 \otimes^{\mathbf{L}}_{A_0} A, L_0 \otimes^{\mathbf{L}}_{A_0} A) = \operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(A_i)}(K_0 \otimes^{\mathbf{L}}_{A_0} A_i, L_0 \otimes^{\mathbf{L}}_{A_0} A_i)$$

In particular, the triangulated category of R-perfect complexes over A is the colimit of the triangulated categories of R_i-perfect complexes over A_i.

Proof. By Algebra, Lemma 10.127.6 the category of finitely presented A-modules is the colimit of the categories of finitely presented A_i-modules. Given this, Algebra, Lemma 10.168.1 tells us that category of R-flat, finitely presented A-modules is the colimit of the categories of R_i-flat, finitely presented A_i-modules. Thus the characterization in Lemma 15.84.4 proves that (1) is true.

To prove (2) we choose $P_0^\bullet$ representing K_0 and $F_0^\bullet$ representing L_0 as in Lemma 15.84.6. Then $E_0^\bullet = \operatorname{Hom}^\bullet(P_0^\bullet, F_0^\bullet)$ satisfies

$$H^0(E_0^\bullet \otimes_{R_0} R_i) = \operatorname{Hom}_{D(A_i)}(K_0 \otimes^{\mathbf{L}}_{A_0} A_i, L_0 \otimes^{\mathbf{L}}_{A_0} A_i)$$

and

$$H^0(E_0^\bullet \otimes_{R_0} R) = \operatorname{Hom}_{D(A)}(K_0 \otimes^{\mathbf{L}}_{A_0} A, L_0 \otimes^{\mathbf{L}}_{A_0} A)$$

by the lemma. Thus the result because tensor product commutes with colimits and filtered colimits are exact (Algebra, Lemma 10.8.8). □

0DJG **Lemma 15.84.8.** *Let $R' \to A'$ be a flat ring map of finite presentation. Let $R' \to R$ be a surjective ring map whose kernel is a nilpotent ideal. Set $A = A' \otimes_{R'} R$. Let $K' \in D(A')$ and set $K = K' \otimes^{\mathbf{L}}_{A'} A$ in $D(A)$. If K is R-perfect, then K' is R'-perfect.*

Proof. Observe that $A' \to A$ has nilpotent kernel and that by flatness of $R' \to A'$ we have $K = K' \otimes^{\mathbf{L}}_{R'} R$ (see Section 15.61). Hence the lemma follows by combining Lemmas 15.76.4 and 15.67.20. □

0DJH **Lemma 15.84.9.** *Let R be a ring. Let $A = R[x_1, \ldots, x_d]/I$ be flat and of finite presentation over R. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(A)$ be pseudo-coherent. Let $a, b \in \mathbf{Z}$. If $H^i(K_{\mathfrak{q}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}))$ is nonzero only for $i \in [a, b]$, then $K_{\mathfrak{q}}$ has tor amplitude in $[a - d, b]$ over R.*

Proof. By Lemma 15.83.8 K is pseudo-coherent as a complex of $R[x_1, \ldots, x_d]$-modules. Therefore we may assume $A = R[x_1, \ldots, x_d]$. Applying Lemma 15.78.6 to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$ with tor amplitude in $[a - d, b]$. Since $R_\mathfrak{p} \to A_\mathfrak{q}$ is flat, we conclude by Lemma 15.67.11. □

0GHJ **Lemma 15.84.10.** *Let $R \to A$ be a ring map which is flat and of finite presentation. Let $K \in D(A)$ be pseudo-coherent. The following are equivalent*

(1) *K is R-perfect, and*
(2) *K is bounded below and for every prime ideal $\mathfrak{p} \subset R$ the object $K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ is bounded below.*

Proof. Observe that (1) implies (2) as an R-perfect complex has bounded tor dimension as a complex of R-modules by definition. Let us prove the other implication.

Write $A = R[x_1, \ldots, x_d]/I$. Denote L in $D(R[x_1, \ldots, x_d])$ the restriction of K. By Lemma 15.83.8 we see that L is pseudo-coherent. Since L and K have the same image in $D(R)$ we see that L is R-perfect if and only if K is R-perfect. Also $L \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ and $K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ are the same objects of $D(\kappa(\mathfrak{p}))$. This reduces us to the case $A = R[x_1, \ldots, x_d]$.

Say $A = R[x_1, \ldots, x_d]$ and K satisfies (2). Let $\mathfrak{q} \subset A$ be a prime lying over a prime $\mathfrak{p} \subset R$. By Lemma 15.78.6 applied to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$. Since K is bounded below, we see that K is perfect in $D(A)$ by Lemma 15.78.3. This implies that K is R-perfect by Lemma 15.84.3 and the proof is complete. □

15.85. Two term complexes

0G9B In this section we prove some results on two term complexes of modules which will help us understand conditions on the naive cotangent complex.

0G9C **Lemma 15.85.1.** *Let R be a ring. Let $K \in D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. The following are equivalent*

(1) *$H^{-1}(K) = 0$ and $H^0(K)$ is a projective module and*
(2) *$\mathrm{Ext}^1_R(K, M) = 0$ for every R-module M.*

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

(3) *$\mathrm{Ext}^1_R(K, M) = 0$ for every finite R-module M.*

Proof. The equivalence of (1) and (2) follows from Lemma 15.69.2. If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K is pseudo-coherent, see Lemma 15.65.17. Thus the equivalence of (1) and (3) follows from Lemma 15.78.4. □

0G9D **Remark 15.85.2.** The following two statements follow from Lemma 15.85.1, Algebra, Definition 10.137.1, and Algebra, Proposition 10.138.8.

(1) A ring map $A \to B$ is smooth if and only if $A \to B$ is of finite presentation and $\mathrm{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.
(2) A ring map $A \to B$ is formally smooth if and only if $\mathrm{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.

0G9E **Lemma 15.85.3.** *Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Then*

(1) *K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a free module, and*
(2) *if R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a finite free module and K^{-1} finite.*

Proof. Proof of (1). Suppose K is given by the complex of modules $M^\bullet$. We may first replace $M^\bullet$ by $\tau_{\leq 0}M^\bullet$. Thus we may assume $M^i = 0$ for $i > 0$, Next, we may choose a free resolution $P^\bullet \to M^\bullet$ with $P^i = 0$ for $i > 0$, see Derived Categories, Lemma 13.15.4. Finally, we can set $K^\bullet = \tau_{\geq -1}P^\bullet$.

Proof of (2). Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 15.65.5 we can choose a quasi-isomorphism $F^\bullet \to M^\bullet$ with $F^i = 0$ for $i > 0$ and F^i finite free. Then we can set $K^\bullet = \tau_{\geq -1}F^\bullet$. □

Maps in the derived category out of the naive cotangent complex $NL_{B/A}$ or $NL(\alpha)$ (see Algebra, Section 10.134) are easy to understand by the result of the following lemma.

0ALN **Lemma 15.85.4.** *Let R be a ring. Let $M^\bullet$ be a complex of modules over R with $M^i = 0$ for $i > 0$ and M^0 a projective R-module. Let $K^\bullet$ be a second complex.*

(1) *Assume $K^i = 0$ for $i \leq -2$. Then $\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(M^\bullet, K^\bullet)$.*
(2) *Assume $K^i = 0$ for $i \notin [-1, 0]$ and K^0 a projective R-module. Then for a map of complexes $a^\bullet : M^\bullet \to K^\bullet$, the following are equivalent*
   (a) *$a^\bullet$ induces the zero map $\mathrm{Ext}^1_R(K^\bullet, N) \to \mathrm{Ext}^1_R(M^\bullet, N)$ for all R-modules N, and*
   (b) *there is a map $h^0 : M^0 \to K^{-1}$ such that $a^{-1} + h^0 \circ d^{-1}_K = 0$.*
(3) *Assume $K^i = 0$ for $i \leq -3$. Let $\alpha \in \mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet)$. If the composition of α with $K^\bullet \to K^{-2}[2]$ comes from an R-module map $a : M^{-2} \to K^{-2}$ with $a \circ d^{-3}_M = 0$, then α can be represented by a map of complexes $a^\bullet : M^\bullet \to K^\bullet$ with $a^{-2} = a$.*
(4) *In (2) for any second map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ representing α with $a = (a')^{-2}$ there exist $h^i : M^i \to K^{i-1}$ for $i = 0, -1$ such that*

$$h^{-1} \circ d^{-2}_M = 0, \quad (a')^{-1} = a^{-1} + d^{-2}_K \circ h^{-1} + h^0 \circ d^{-1}_M, \quad (a')^0 = a^0 + d^{-1}_K \circ h^0$$

Proof. Set $F^0 = M^0$. Choose a free R-module F^{-1} and a surjection $F^{-1} \to M^{-1}$. Choose a free R-module F^{-2} and a surjection $F^{-2} \to M^{-2} \times_{M^{-1}} F^{-1}$. Continuing in this way we obtain a quasi-isomorphism $p^\bullet : F^\bullet \to M^\bullet$ which is termwise surjective and with F^i projective for all i.

Proof of (1). By Derived Categories, Lemma 13.19.8 we have

$$\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(F^\bullet, K^\bullet)$$

If $K^i = 0$ for $i \leq -2$, then any morphism of complexes $F^\bullet \to K^\bullet$ factors through $p^\bullet$. Similarly, any homotopy $\{h^i : F^i \to K^{i-1}\}$ factors through $p^\bullet$. Thus (1) holds.

Proof of (2). If (2)(b) holds, then $a^\bullet$ is homotopic to a map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ which is zero in degree -1. On the other hand, let $N \to I^\bullet$ be an injective resolution. We have

$$\mathrm{Ext}^1_R(K^\bullet, N) = \mathrm{Hom}_{D(R)}(K^\bullet, I^\bullet[1]) = \mathrm{Hom}_{K(R)}(K^\bullet, I^\bullet[1])$$

by Derived Categories, Lemma 13.18.8. Let $b^\bullet : K^\bullet \to I^\bullet[1]$ be a map of complexes. Since $K^1 = 0$ the map $b^0 : K^0 \to I^1$ maps into the kernel of $I^1 \to I^2$ which is the image of $I^0 \to I^1$. Since K^0 is projective we can lift b^0 to a map $h : K^0 \to I^0$. Thus we see that $b^\bullet$ is homotopic to a map of complexes $(b')^\bullet$ with $(b')^0 = 0$. Since $K^i = 0$ for $i \notin [-1, 0]$ it follows that $(b')^\bullet \circ (a')^\bullet = 0$ as a map of complexes. Hence the map $\mathrm{Ext}^1_R(K^\bullet, N) \to \mathrm{Ext}^1_R(M^\bullet, N)$ is zero. In this way we see that (2)(b) implies (2)(a). Conversely, assume (2)(a). We see that the canonical element in $\mathrm{Ext}^1_R(K^\bullet, K^{-1})$ maps to zero in $\mathrm{Ext}^1_R(M^\bullet, K^{-1})$. Using (1) we see immediately that we get a map h^0 as in (2)(b).

Proof of (3). Choose $b^\bullet : F^\bullet \to K^\bullet$ representing α. The composition of α with $K^\bullet \to K^{-2}[2]$ is represented by $b^{-2} : F^{-2} \to K^{-2}$. As this is homotopic to $a \circ p^{-2} : F^{-2} \to M^{-2} \to K^{-2}$, there is a map $h : F^{-1} \to K^{-2}$ such that $b^{-2} = a \circ p^{-2} + h \circ d_F^{-2}$. Adjusting $b^\bullet$ by h viewed as a homotopy from $F^\bullet$ to $K^\bullet$, we find that $b^{-2} = a \circ p^{-2}$. Hence b^{-2} factors through p^{-2}. Since $F^0 = M^0$ the kernel of p^{-2} surjects onto the kernel of p^{-1} (for example because the kernel of $p^\bullet$ is an acyclic complex or by a diagram chase). Hence b^{-1} necessarily factors through p^{-1} as well and we see that (3) holds for these factorizations and $a^0 = b^0$.

Proof of (4) is omitted. Hint: There is a homotopy between $a^\bullet \circ p^\bullet$ and $(a')^\bullet \circ p^\bullet$ and we argue as before that this homotopy factors through $p^\bullet$. □

Let $A \to B$ be a finitely presented ring map. Given an ideal $I \subset B$ we can consider the condition

(*) $\mathrm{Ext}^1_B(NL_{B/A}, N)$ is annihilated by I for all B-modules N.

This condition is one possible precise mathematical formulation of the notion "the singular locus of $A \to B$ is scheme theoretically contained in $V(I)$". Please compare with Remark 15.85.2 and the following lemmas.

0G9F **Lemma 15.85.5.** *Let R be a ring and let $I \subset R$ be an ideal. Let $K \in D(R)$. Assume $H^i(K) = 0$ for $i \notin \{-1, 0\}$. The following are equivalent*

1. *$\mathrm{Ext}^1_R(K, N)$ is annihilated by I for all R-modules N,*
2. *K can be represented by a complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$,*
3. *whenever K is represented by a two term complex $K^{-1} \to K^0$ with K^0 projective, then for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.*

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

4. *$\mathrm{Ext}^1_R(K, N)$ is annihilated by I for every finite R-module N,*
5. *K can be represented by a complex $K^{-1} \to K^0$ with K^0 finite free and K^{-1} finite such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.*

Proof. Assume (1) and let $K^{-1} \to K^0$ be a two term complex representing K with K^0 projective. We will use the description of maps in $D(R)$ out of $K^\bullet$ given in Lemma 15.85.4 without further mention. Choosing $N = K^{-1}$ consider the element ξ of $\mathrm{Ext}^1_R(K, N)$ given by $\mathrm{id}_{K^{-1}} : K^{-1} \to K^{-1}$. Since is annihilated by $a \in I$ we

see that we get the dotted arrow fitting into the following commutative diagram

$$\begin{array}{ccc} K^{-1} & \xrightarrow{d_K^{-1}} & K^0 \\ {\scriptstyle a}\downarrow & \swarrow {\scriptstyle h} & \\ K^{-1} & & \end{array}$$

This proves that (3) holds. Part (3) implies (2) in view of Lemma 15.85.3 part (1). Assume $K^\bullet$ is as in (2) and N is an arbitrary R-module. Any element ξ of $\mathrm{Ext}^1_R(K, N)$ is given as the class of a map $\varphi : K^{-1} \to N$. Then for $a \in I$ by assumption we may choose a map h as in the diagram above and we see that $a\varphi = \varphi \circ a = \varphi \circ h \circ \mathrm{d}_K^{-1}$ which proves that $a\xi$ is zero in $\mathrm{Ext}^1_R(K, N)$. Thus (1), (2), and (3) are equivalent.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. Part (3) implies (5) in view of Lemma 15.85.3 part (2). It is clear that (5) implies (2). Trivially (1) implies (4). Thus to finish the proof it suffices to show that (4) implies any of the other conditions. Let $K^{-1} \to K^0$ be a complex representing K with K^0 finite free and K^{-1} finite as in Lemma 15.85.3 part (2). The argument given in the proof of (2) $\Rightarrow$ (1) shows that if $\mathrm{Ext}^1_R(K, K^{-1})$ is annihilated by I, then (1) holds. In this way we see that (4) implies (1) and the proof is complete. □

0G9G **Lemma 15.85.6.** *Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $K^{-1} \to K^0$ be a two term complex of R-modules representing K such that K^0 is a flat R-module (for example projective or free). Let $R \to R'$ be a ring map. Then the complex $K^\bullet \otimes_R R'$ represents $\tau_{\geq -1}(K \otimes^{\mathbf{L}}_R R')$.*

Proof. We have a distinguished triangle

$$K^0 \to K^\bullet \to K^{-1}[1] \to K^0[1]$$

in $D(R)$. This determines a map of distinguished triangles

$$\begin{array}{ccccccc} K^0 \otimes^{\mathbf{L}}_R R' & \longrightarrow & K^\bullet \otimes^{\mathbf{L}}_R R' & \longrightarrow & K^{-1} \otimes^{\mathbf{L}}_R R'[1] & \longrightarrow & K^0 \otimes^{\mathbf{L}}_R R'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^0 \otimes_R R' & \longrightarrow & K^\bullet \otimes_R R' & \longrightarrow & K^{-1} \otimes_R R'[1] & \longrightarrow & K^0 \otimes_R R'[1] \end{array}$$

The left and right vertical arrows are isomorphisms as K^0 is flat. Since $K^{-1} \otimes^{\mathbf{L}}_R R' \to K^{-1} \otimes_R R'$ is an isomorphism on cohomology in degree 0 we conclude. □

0G9H **Lemma 15.85.7.** *Let I be an ideal of a ring R. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $R \to R'$ be a ring map. If K satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.85.5 with respect to (R, I), then $\tau_{\geq -1}(K \otimes^{\mathbf{L}}_R R')$ satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.85.5 with respect to (R', IR')*

Proof. We may assume K is represented by a two term complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ is equal to $h_a \circ d_K^{-1}$ for some map $h_a : K^0 \to K^{-1}$. By Lemma 15.85.6 we see that $\tau_{\geq -1}(K \otimes^{\mathbf{L}}_R R')$ is represented by $K^\bullet \otimes_R R'$. Then of course for every $a \in I$ we see that $a \otimes 1 : K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ is equal to $(h_a \otimes 1) \circ (d_K^{-1} \otimes 1)$. Since the collection of maps $K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ which factor through $\mathrm{d}_K^{-1} \otimes 1$ forms an R'-module we conclude. □

0G9I **Lemma 15.85.8.** *Let R be a ring. Let $\alpha : K \to K'$ be a morphism of $D(R)$. Assume*

(1) *$H^i(K) = H^i(K') = 0$ for $i \notin \{-1, 0\}$*
(2) *$H^0(\alpha)$ is an isomorphism and $H^{-1}(\alpha)$ is surjective.*

For any $f \in R$ if $f : K \to K$ is 0, then $f : K' \to K'$ is 0.

Proof. Set $M = \mathrm{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished triangle

$$M[1] \to K \to K' \to M[2]$$

Since $K \to K' \xrightarrow{f} K'$ is zero by our assumption, we see that $f : K' \to K'$ factors over a map $M[2] \to K'$. However $\mathrm{Hom}(M[2], K') = 0$ for example by Derived Categories, Lemma 13.27.3. □

0G9J **Lemma 15.85.9.** *Let I be an ideal of a ring R. Let $\alpha : K \to K'$ be a morphism of $D(R)$. Assume*

(1) *$H^i(K) = H^i(K') = 0$ for $i \notin \{-1, 0\}$*
(2) *$H^0(\alpha)$ is an isomorphism and $H^{-1}(\alpha)$ is surjective.*

If K satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.85.5, then K' does too.

Proof. Set $M = \mathrm{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished triangle

$$M[1] \to K \to K' \to M[2]$$

For any R-module N this determines an exact sequence

$$\mathrm{Ext}^0_R(M[1], N) \to \mathrm{Ext}^1_R(K', N) \to \mathrm{Ext}^1_R(K, N)$$

Since $\mathrm{Ext}^0_R(M[1], N) = \mathrm{Ext}^{-1}_R(M, N) = 0$ we see that $\mathrm{Ext}^1_R(K', N)$ is a submodule of $\mathrm{Ext}^1_R(K, N)$. Hence if $\mathrm{Ext}^1_R(K, N)$ is annihilated by I so is $\mathrm{Ext}^1_R(K', N)$. □

0G9K **Lemma 15.85.10.** *Let R be ring and let $I \subset R$ be an ideal. Let $K \in D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. The following are equivalent*

(1) *there exists a $c \geq 0$ such that the equivalent conditions (1), (2), (3) of Lemma 15.85.5 hold for K and the ideal I^c,*
(2) *there exists a $c \geq 0$ such that (a) I^c annihilates $H^{-1}(K)$ and (b) $H^0(K)$ is an I^c-projective module (see Section 15.71).*

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

(3) *there exists a $c \geq 0$ such that the equivalent conditions (4), (5) of Lemma 15.85.5 hold for K and the ideal I^c,*
(4) *$H^{-1}(K)$ is I-power torsion and there exist $f_1, \ldots, f_s \in R$ with $V(f_1, \ldots, f_s) \subset V(I)$ such that the localizations $H^0(K)_{f_i}$ are projective R_{f_i}-modules,*
(5) *$H^{-1}(K)$ is I-power torsion and there exist $f_1, \ldots, f_s \in I$ with $V(f_1, \ldots, f_s) = V(I)$ such that the localizations $H^0(K)_{f_i}$ are projective R_{f_i}-modules, and*
(6) *$H^{-1}(K)$ is I-power torsion and for any $f_1, \ldots, f_s \in I$ with $V(f_1, \ldots, f_s) = V(I)$ the localizations $H^0(K)_{f_i}$ are projective R_{f_i}-modules.*

Proof. The distinguished triangle $H^{-1}(K)[1] \to K \to H^0(K)[0] \to H^{-1}(K)[2]$ determines an exact sequence

$$0 \to \mathrm{Ext}^1_R(H^0(K), N) \to \mathrm{Ext}^1_R(K, N) \to \mathrm{Hom}_R(H^{-1}(K), N) \to \mathrm{Ext}^2_R(H^0(K), N)$$

Thus (2) implies that I^{2c} annihilates $\mathrm{Ext}^1_R(K, N)$ for every R-module N. Assuming (1) we immediately see that $H^0(K)$ is I^c-projective. On the other hand, we may choose an injective map $H^{-1}(K) \to N$ for some injective R-module N. Then this map is the image of an element of $\mathrm{Ext}^1_R(K, N)$ by the vanishing of the Ext^2 in the sequence and we conclude $H^{-1}(K)$ is annihilated by I^c.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 15.85.5 we see that (3) is equivalent to (1) and (2). Also, if (3) holds then for $f \in I$ the multiplication by f on $H^0(K)$ factors through a projective module, which implies that $H^0(K)_f$ is a summand of a projective R_f-module and hence itself a projective R_f-module. Thus the equivalent conditions (1), (2), and (3) imply (6). Of course (6) implies (5) and (5) trivially implies (4).

Assume (4). Since $H^{-1}(K)$ is a finite R-module and I-power torsion we see that I^{c_1} annihilates $H^{-1}(K)$ for some $c_1 \geq 0$. Choose a short exact sequence

$$0 \to M \to R^{\oplus r} \to H^0(K) \to 0$$

which determines an element $\xi \in \mathrm{Ext}^1_R(H^0(K), M)$. For any $f \in I$ we have $\mathrm{Ext}^1_R(H^0(K), M)_f = \mathrm{Ext}^1_{R_f}(H^0(K)_f, M_f)$ by Lemma 15.66.4. Hence if $H^0(K)_f$ is projective, then a power of f annihilates ξ. We conclude that ξ is annihilated by $(f_1, \ldots, f_s)^{c_2}$ for some $c_2 \geq 0$. Since $V(f_1, \ldots, f_s) \subset V(I)$ we have $\sqrt{I} \subset (f_1, \ldots, f_s)$ (Algebra, Lemma 10.17.2). Since R is Noetherian we find $I^{c_3} \subset (f_1, \ldots, f_s)$ for some $c_3 \geq 0$ (Algebra, Lemma 10.32.5). Hence I^{c2c3} annihilates ξ. This in turn says that $H^0(K)$ is $I^{c_2c_3}$-projective (as multiplication by $a \in I$ which annihilate ξ factor through $R^{\oplus r}$). Hence taking $c = \max(c_1, c_2c_3)$ we see that (2) holds. □

0AJT **Lemma 15.85.11.** *Let R be a ring. Let $K_j \in D(R)$, $j = 1, 2, 3$ with $H^i(K_j) = 0$ for $i \notin \{-1, 0\}$. Let $\varphi : K_1 \to K_2$ and $\psi : K_2 \to K_3$ be maps in $D(R)$. If $H^0(\varphi) = 0$ and $H^{-1}(\psi) = 0$, then $\varphi \circ \psi = 0$.*

Proof. Apply Derived Categories, Lemma 13.12.5 to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2}K_2 = 0$. □

0G9L **Lemma 15.85.12.** *Let R be a ring. Let $K \in D(R)$ be given by a two term complex of the form $R^{\oplus n} \to R^{\oplus n}$. Denote $A \in Mat(n \times n, R)$ the matrix of the differential. Then $\det(a) : K \to K$ is zero in $D(R)$.*

Proof. Omitted. Good exercise. □

15.86. The naive cotangent complex

0FUX In this section we continue the discussion started in Algebra, Section 10.134. We begin with a discussion of base change. The first lemma shows that taking the naive tensor product of the naive cotangent complex with a ring extension isn't quite as naive as one might think.

0FUY **Lemma 15.86.1.** *Let $R \to S$ and $S \to S'$ be ring maps. The canonical map $NL_{S/R} \otimes^{\mathbf{L}}_S S' \to NL_{S/R} \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL_{S/R} \otimes^{\mathbf{L}}_S S') \to NL_{S/R} \otimes_S S'$ in $D(S')$. Similarly, given a presentation α of S over R the canonical map $NL(\alpha) \otimes^{\mathbf{L}}_S S' \to NL(\alpha) \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL(\alpha) \otimes^{\mathbf{L}}_S S') \to NL(\alpha) \otimes_S S'$ in $D(S')$.*

Proof. Special case of Lemma 15.85.6. □

0FUZ **Lemma 15.86.2.** *Let $R \to S$ and $R \to R'$ be ring maps. Let $\alpha : P \to S$ be a presentation of S over R. Then $\alpha' : P \otimes_R R' \to S \otimes_R R'$ is a presentation of $S' = S \otimes_R R'$ over R'. The canonical map*

$$NL(\alpha) \otimes_S S' \to NL(\alpha')$$

is an isomorphism on H^0 and surjective on H^{-1}. In particular, the canonical map

$$NL_{S/R} \otimes_S S' \to NL_{S'/R'}$$

is an isomorphism on H^0 and surjective on H^{-1}.

Proof. Denote $I = \mathrm{Ker}(P \to S)$. Denote $P' = P \otimes_R R'$ and $I' = \mathrm{Ker}(P' \to S')$. Suppose P is a polynomial algebra on x_j for $j \in J$. The map displayed in the lemma becomes

$$\begin{array}{ccc} \bigoplus_{j \in J} S' \mathrm{d}x_j & \longrightarrow & \bigoplus_{j \in J} S' \mathrm{d}x_j \\ \uparrow & & \uparrow \\ I/I^2 \otimes_S S' & \longrightarrow & I'/(I')^2 \end{array}$$

where the left column is $NL(\alpha) \otimes_S S'$ and the right column is $NL(\alpha')$. By right exactness of tensor product we see that $I \otimes_R R' \to I'$ is surjective. Hence the bottom arrow is a surjection. This proves the first statement of the lemma. The statement for $NL_{S/R} \otimes_S S' \to NL_{S'/R'}$ follows as these complexes are homotopic to $NL(\alpha) \otimes_S S'$ and $NL(\alpha')$. □

0FJU **Lemma 15.86.3.** *Consider a cocartesian diagram of rings*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

If B is flat over A, then the canonical map $NL_{B/A} \otimes_B B' \to NL_{B'/A'}$ is a quasi-isomorphism. If in addition $NL_{B/A}$ has tor-amplitude in $[-1, 0]$ then $NL_{B/A} \otimes^{\mathbf{L}}_B B' \to NL_{B'/A'}$ is a quasi-isomorphism too.

Proof. Choose a presentation $\alpha : P \to B$ as in Algebra, Section 10.134. Let $I = \mathrm{Ker}(\alpha)$. Set $P' = P \otimes_A A'$ and denote $\alpha' : P' \to B'$ the corresponding presentation of B' over A'. As B is flat over A we see that $I' = \mathrm{Ker}(\alpha')$ is equal to $I \otimes_A A'$. Hence

$$I'/(I')^2 = \mathrm{Coker}(I^2 \otimes_A A' \to I \otimes_A A') = I/I^2 \otimes_A A' = I/I^2 \otimes_B B'$$

We have $\Omega_{P'/A'} = \Omega_{P/A} \otimes_A A'$ because both sides have the same basis. It follows that $\Omega_{P'/A'} \otimes_{P'} B' = \Omega_{P/A} \otimes_P B \otimes_B B'$. This proves that $NL(\alpha) \otimes_B B' \to NL(\alpha')$ is an isomorphism of complexes and hence the first statement holds.

We have

$$NL(\alpha) = I/I^2 \longrightarrow \Omega_{P/A} \otimes_P B$$

as a complex of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 15.67.2. If this holds, then $NL(\alpha) \otimes^{\mathbf{L}}_B B' = NL(\alpha) \otimes_B B'$ and we get the second statement. □

0FV0 **Lemma 15.86.4.** *Let $A \to B$ be a local complete intersection as in Definition 15.33.2. Then $NL_{B/A}$ is a perfect object of $D(B)$ with tor amplitude in $[-1, 0]$.*

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B\mathrm{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 15.67.2. By definition I is a Koszul regular ideal and hence a quasi-regular ideal, see Section 15.32. Thus I/I^2 is a finite projective B-module (Lemma 15.32.3) and we conclude both that $NL_{B/A}$ is perfect and that it has tor amplitude in $[-1, 0]$. □

0FV1 **Lemma 15.86.5.** *Consider a cocartesian diagram of rings*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

If $A \to B$ and $A' \to B'$ are local complete intersections as in Definition 15.33.2, then the kernel of $H^{-1}(NL_{B/A} \otimes_B B') \to H^{-1}(NL_{B'/A'})$ is a finite projective B'-module.

Proof. By Lemma 15.86.4 the complexes $NL_{B/A}$ and $NL_{B'/A'}$ are perfect of tor-amplitude in $[-1, 0]$. Combining Lemmas 15.86.1, 15.75.9, and 15.67.13 we have $NL_{B/A} \otimes_B B' = NL_{B/A} \otimes^{\mathbf{L}}_B B'$ and this complex is also perfect of tor-amplitude in $[-1, 0]$. Choose a distinguished triangle

$$C \to NL_{B/A} \otimes_B B' \to NL_{B'/A'} \to C[1]$$

in $D(B')$. By Lemmas 15.75.4 and 15.67.5 we conclude that C is perfect with tor-amplitude in $[-1, 1]$. By Lemma 15.86.2 the complex C has only one nonzero cohomology module, namely the module of the lemma sitting in degree -1. This module is of finite presentation (Lemma 15.65.4) and flat (Lemma 15.67.6). Hence it is finite projective by Algebra, Lemma 10.78.2. □

15.87. Rlim of abelian groups

07KV We briefly discuss $R\lim$ on abelian groups. In this section we will denote $Ab(\mathbf{N})$ the abelian category of inverse systems of abelian groups. The notation is compatible with the notation for sheaves of abelian groups on a site, as an inverse system of abelian groups is the same thing as a sheaf of groups on the category $\mathbf{N}$ (with a unique morphism $i \to j$ if $i \leq j$), see Remark 15.87.6. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for sheaves of abelian groups on sites.

07KW **Lemma 15.87.1.** *The functor $\lim : Ab(\mathbf{N}) \to Ab$ has a right derived functor*

08U4 (15.87.1.1) $$R\lim : D(Ab(\mathbf{N})) \longrightarrow D(Ab)$$

As usual we set $R^p \lim(K) = H^p(R\lim(K))$. Moreover, we have

(1) *for any (A_n) in $Ab(\mathbf{N})$ we have $R^p \lim A_n = 0$ for $p > 1$,*

(2) *the object* $R\lim A_n$ *of* $D(Ab)$ *is represented by the complex*

$$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

sitting in degrees 0 *and* 1,
(3) *if* (A_n) *is ML, then* $R^1\lim A_n = 0$*, i.e.,* (A_n) *is right acyclic for* lim,
(4) *every* $K^\bullet \in D(Ab(\mathbf{N}))$ *is quasi-isomorphic to a complex whose terms are right acyclic for* lim*, and*
(5) *if each* $K^p = (K^p_n)$ *is right acyclic for* lim*, i.e., of* $R^1\lim_n K^p_n = 0$*, then* $R\lim K$ *is represented by the complex whose term in degree* p *is* $\lim_n K^p_n$*.*

Proof. Let (A_n) be an arbitrary inverse system. Let (B_n) be the inverse system with

$$B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$$

and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n)$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system (Homology, Section 12.31). It follows from Derived Categories, Lemma 13.15.6 using Homology, Lemma 12.31.3 that every ML system is right acyclic for lim, i.e., (3) holds. This already implies that RF is defined on $D^+(Ab(\mathbf{N}))$, see Derived Categories, Proposition 13.16.8. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n > 1$ and $C_1 = 0$ with transition maps given by projections as well. Then there is a short exact sequence of inverse systems $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since (C_n) is ML as well, we conclude that (2) holds (by proposition reference above) which also implies (1). Finally, this implies by Derived Categories, Lemma 13.32.2 that $R\lim$ is in fact defined on all of $D(Ab(\mathbf{N}))$. In fact, the proof of Derived Categories, Lemma 13.32.2 proceeds by proving assertions (4) and (5). □

0H31 **Lemma 15.87.2.** *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups. Then there is an associated 6 *term exact sequence* $0 \to \lim A_i \to \lim B_i \to \lim C_i \to R^1\lim A_i \to R^1\lim B_i \to R^1\lim C_i \to 0$*.*

Proof. Follows from the vanishing in Lemma 15.87.1. □

Here is the "correct" formulation of Homology, Lemma 12.31.7.

0918 **Lemma 15.87.3.** *Let*

$$(A^{-2}_n \to A^{-1}_n \to A^0_n \to A^1_n)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ *its limit. Denote* (H^{-1}_n)*,* (H^0_n) *the inverse systems of cohomologies, and denote* H^{-1}*,* H^0 *the cohomologies of* $A^{-2} \to A^{-1} \to A^0 \to A^1$*. If*

(1) (A^{-2}_n) *and* (A^{-1}_n) *have vanishing* $R^1\lim$*,*
(2) (H^{-1}_n) *has vanishing* $R^1\lim$*,*

then $H^0 = \lim H^0_n$*.*

Proof. Let $K \in D(Ab(\mathbf{N}))$ be the object represented by the system of complexes whose nth constituent is the complex $A^{-2}_n \to A^{-1}_n \to A^0_n \to A^1_n$. We will compute

$H^0(R\lim K)$ using both spectral sequences[11] of Derived Categories, Lemma 13.21.3. The first has E_1-page

$$\begin{matrix} 0 & 0 & R^1\lim A_n^0 & R^1\lim A_n^1 \\ A^{-2} & A^{-1} & A^0 & A^1 \end{matrix}$$

with horizontal differentials and all higher differentials are zero. The second has E_2 page

$$\begin{matrix} R^1\lim H_n^{-2} & 0 & R^1\lim H_n^0 & R^1\lim H_n^1 \\ \lim H_n^{-2} & \lim H_n^{-1} & \lim H_n^0 & \lim H_n^1 \end{matrix}$$

and degenerates at this point. The result follows. □

0919 **Lemma 15.87.4.** *Let $\mathcal{D}$ be a triangulated category. Let (K_n) be an inverse system of objects of $\mathcal{D}$. Let K be a derived limit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence*

$$0 \to R^1\lim \mathrm{Hom}_{\mathcal{D}}(L, K_n[-1]) \to \mathrm{Hom}_{\mathcal{D}}(L, K) \to \lim \mathrm{Hom}_{\mathcal{D}}(L, K_n) \to 0$$

Proof. This follows from Derived Categories, Definition 13.34.1 and Lemma 13.4.2, and the description of lim and $R^1\lim$ in Lemma 15.87.1 above. □

0CQX **Lemma 15.87.5.** *Let $\mathcal{D}$ be a triangulated category. Let (K_n) be a system of objects of $\mathcal{D}$. Let K be a derived colimit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence*

$$0 \to R^1\lim \mathrm{Hom}_{\mathcal{D}}(K_n, L[-1]) \to \mathrm{Hom}_{\mathcal{D}}(K, L) \to \lim \mathrm{Hom}_{\mathcal{D}}(K_n, L) \to 0$$

Proof. This follows from Derived Categories, Definition 13.33.1 and Lemma 13.4.2, and the description of lim and $R^1\lim$ in Lemma 15.87.1 above. □

091A **Remark 15.87.6** (Rlim as cohomology)**.** Consider the category $\mathbf{N}$ whose objects are natural numbers and whose morphisms are unique arrows $i \to j$ if $j \geq i$. Endow $\mathbf{N}$ with the chaotic topology (Sites, Example 7.6.6) so that a sheaf $\mathcal{F}$ is the same thing as an inverse system

$$\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \mathcal{F}_3 \leftarrow \ldots$$

of sets over $\mathbf{N}$. Note that $\Gamma(\mathbf{N}, \mathcal{F}) = \lim \mathcal{F}_n$. For an inverse system of abelian groups $\mathcal{F}_n$ we have

$$R^p\lim \mathcal{F}_n = H^p(\mathbf{N}, \mathcal{F})$$

because both sides are the higher right derived functors of $\mathcal{F} \mapsto \lim \mathcal{F}_n = H^0(\mathbf{N}, \mathcal{F})$. Thus the existence of $R\lim$ also follows from the general material in Cohomology on Sites, Sections 21.2 and 21.19.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(Ab)$, see Derived Categories, Lemma 13.34.2.

[11] To use these spectral sequences we have to show that $Ab(\mathbf{N})$ has enough injectives. A inverse system (I_n) of abelian groups is injective if and only if each I_n is an injective abelian group and the transition maps are split surjections. Every system embeds in one of these. Details omitted.

07KX **Lemma 15.87.7.** *Let $K = (K_n^\bullet)$ be an object of $D(Ab(\mathbf{N}))$. There exists a canonical distinguished triangle*

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(Ab)$. In other words, $R\lim K$ is a derived limit of the inverse system $(K_n^\bullet)$ of $D(Ab)$, see Derived Categories, Definition 13.34.1.

Proof. Suppose that for each p the inverse system (K_n^p) is right acyclic for lim. By Lemma 15.87.1 this gives a short exact sequence

$$0 \to \lim\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to 0$$

for each p. Since the complex consisting of $\lim_n K_n^p$ computes $R\lim K$ by Lemma 15.87.1 we see that the lemma holds in this case.

Next, assume $K = (K_n^\bullet)$ is general. By Lemma 15.87.1 there is a quasi-isomorphism $K \to L$ in $D(Ab(\mathbf{N}))$ such that (L_n^p) is acyclic for each p. Then $\prod K_n^\bullet$ is quasi-isomorphic to $\prod L_n^\bullet$ as products are exact in Ab, whence the result for L (proved above) implies the result for K. □

07KY **Lemma 15.87.8.** *With notation as in Lemma 15.87.7 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim\nolimits_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim\nolimits_n H^p(K_n^\bullet) \to 0$$

Proof. The long exact sequence of the distinguished triangle is

$$\ldots \to H^p(R\lim K) \to \prod\nolimits_n H^p(K_n^\bullet) \to \prod\nolimits_n H^p(K_n^\bullet) \to H^{p+1}(R\lim K) \to \ldots$$

The map in the middle has kernel $\lim_n H^p(K_n^\bullet)$ by its explicit description given in the lemma. The cokernel of this map is $R^1 \lim_n H^p(K_n^\bullet)$ by Lemma 15.87.1. □

Warning. An object of $D(Ab(\mathbf{N}))$ is a complex of inverse systems of abelian groups. You can also think of this as an inverse system $(K_n^\bullet)$ of complexes. However, this is **not** the same thing as an inverse system of objects of $D(Ab)$; the following lemma and remark explain the difference.

0CQ9 **Lemma 15.87.9.** *Let (K_n) be an inverse system of objects of $D(Ab)$. Then there exists an object $M = (M_n^\bullet)$ of $D(Ab(\mathbf{N}))$ and isomorphisms $M_n^\bullet \to K_n$ in $D(Ab)$ such that the diagrams*

$$\begin{array}{ccc} M_{n+1}^\bullet & \longrightarrow & M_n^\bullet \\ \downarrow & & \downarrow \\ K_{n+1} & \longrightarrow & K_n \end{array}$$

commute in $D(Ab)$.

Proof. Namely, let $M_1^\bullet$ be a complex of abelian groups representing K_1. Suppose we have constructed $M_e^\bullet \to M_{e-1}^\bullet \to \ldots \to M_1^\bullet$ and maps $\psi_i : M_i^\bullet \to K_i$ such that the diagrams in the statement of the lemma commute for all $n < e$. Then we consider the diagram

$$\begin{array}{ccc} & & M_n^\bullet \\ & & \downarrow{\scriptstyle \psi_n} \\ K_{n+1} & \longrightarrow & K_n \end{array}$$

in $D(Ab)$. By the definition of morphisms in $D(Ab)$ we can find a complex $M^\bullet_{n+1}$ of abelian groups, an isomorphism $M^\bullet_{n+1} \to K_{n+1}$ in $D(Ab)$, and a morphism of complexes $M^\bullet_{n+1} \to M^\bullet_n$ representing the composition

$$K_{n+1} \to K_n \xrightarrow{\psi_n^{-1}} M^\bullet_n$$

in $D(Ab)$. Thus the lemma holds by induction. □

08U5 **Remark 15.87.10.** Let (K_n) be an inverse system of objects of $D(Ab)$. Let $K = R\lim K_n$ be a derived limit of this system (see Derived Categories, Section 13.34). Such a derived limit exists because $D(Ab)$ has countable products (Derived Categories, Lemma 13.34.2). By Lemma 15.87.9 we can also lift (K_n) to an object M of $D(Ab(\mathbf{N}))$. Then $K \cong R\lim M$ where $R\lim$ is the functor (15.87.1.1) because $R\lim M$ is also a derived limit of the system (K_n) by Lemma 15.87.7. Thus, although there may be many isomorphism classes of lifts M of the system (K_n), the isomorphism type of $R\lim M$ is independent of the choice because it is isomorphic to the derived limit $K = R\lim K_n$ of the system. Thus we may apply results on $R\lim$ proved in this section to derived limits. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1\lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 15.87.8 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 15.87.8 to the (defining) distinguished triangle $K \to \prod K_n \to \prod K_n \to K[1]$.

091B **Lemma 15.87.11.** *Let $E \to D$ be a morphism of $D(Ab(\mathbf{N}))$. Let (E_n), resp. (D_n) be the system of objects of $D(Ab)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then $R\lim E \to R\lim D$ is an isomorphism in $D(Ab)$.*

Proof. The assumption in particular implies that the pro-objects $H^p(E_n)$ and $H^p(D_n)$ are isomorphic. By the short exact sequences of Lemma 15.87.8 it suffices to show that given a map $(A_n) \to (B_n)$ of inverse systems of abelian groupsc which induces an isomorphism of pro-objects, then $\lim A_n \cong \lim B_n$ and $R^1\lim A_n \cong R^1\lim B_n$.

The assumption implies there are $1 \leq m_1 < m_2 < m_3 < \ldots$ and maps $\varphi_n : B_{m_n} \to A_n$ such that $(\varphi_n) : (B_{m_n}) \to (A_n)$ is a map of systems which is inverse to the given map $\psi = (\psi_n) : (A_n) \to (B_n)$ as a morphism of pro-objects. What this means is that (after possibly replacing m_n by larger integers) we may assume that the compositions $A_{m_n} \to B_{m_n} \to A_n$ and $B_{m_n} \to A_n \to B_n$ are equal to the transition maps of the inverse systems. Now, if $(b_n) \in \lim B_n$ we can set $a_n = \varphi_{m_n}(b_{m_n})$. This defines an inverse $\lim B_n \to \lim A_n$ (computation omitted). Let us use the cokernel of the map

$$\prod B_n \longrightarrow \prod B_n$$

as an avatar of $R^1\lim B_n$ (Lemma 15.87.1). Any element in this cokernel can be represented by an element (b_i) with $b_i = 0$ if $i \neq m_n$ for some n (computation omitted). We can define a map $R^1\lim B_n \to R^1\lim A_n$ by mapping the class of such a special element (b_n) to the class of $(\varphi_n(b_{m_n}))$. We omit the verification this map is inverse to the map $R^1\lim A_n \to R^1\lim B_n$. □

0CQA **Lemma 15.87.12** (Emmanouil)**.** *Let (A_n) be an inverse system of abelian groups. The following are equivalent* Taken from [**Emm96**].

(1) *(A_n) is Mittag-Leffler,*
(2) *$R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}}(A_n)$.*

Proof. Set $B = \bigoplus_{i \in \mathbf{N}}(A_n)$ and hence $B = (B_n)$ with $B_n = \bigoplus_{i \in \mathbf{N}} A_n$. If (A_n) is ML, then B is ML and hence $R^1 \lim A_n = 0$ and $R^1 \lim B_n = 0$ by Lemma 15.87.1.

Conversely, assume (A_n) is not ML. Then we can pick an m and a sequence of integers $m < m_1 < m_2 < \ldots$ and elements $x_i \in A_{m_i}$ whose image $y_i \in A_m$ is not in the image of $A_{m_i+1} \to A_m$. We will use the elements x_i and y_i to show that $R^1 \lim B_n \neq 0$ in two ways. This will finish the proof of the lemma.

First proof. Set $C = (C_n)$ with $C_n = \prod_{i \in \mathbf{N}} A_n$. There is a canonical injective map $B_n \to C_n$ with cokernel Q_n. Set $Q = (Q_n)$. We may and do think of elements q_n of Q_n as sequences of elements $q_n = (q_{n,1}, q_{n,2}, \ldots)$ with $q_{n,i} \in A_n$ modulo sequences whose tail is zero (in other words, we identify sequences which differ in finitely many places). We have a short exact sequence of inverse systems

$$0 \to (B_n) \to (C_n) \to (Q_n) \to 0$$

Consider the element $q_n \in Q_n$ given by

$$q_{n,i} = \begin{cases} \text{image of } x_i & \text{if} \quad m_i \geq n \\ 0 & \text{else} \end{cases}$$

Then it is clear that q_{n+1} maps to q_n. Hence we obtain $q = (q_n) \in \lim Q_n$. On the other hand, we claim that q is not in the image of $\lim C_n \to \lim Q_n$. Namely, say that $c = (c_n)$ maps to q. Then we can write $c_n = (c_{n,i})$ and since $c_{n',i} \mapsto c_{n,i}$ for $n' \geq n$, we see that $c_{n,i} \in \mathrm{Im}(C_{n'} \to C_n)$ for all $n, i, n' \geq n$. In particular, the image of $c_{m,i}$ in A_m is in $\mathrm{Im}(A_{m_i+1} \to A_m)$ whence cannot be equal to y_i. Thus c_m and $q_m = (y_1, y_2, y_3, \ldots)$ differ in infinitely many spots, which is a contradiction. Considering the long exact cohomology sequence

$$0 \to \lim B_n \to \lim C_n \to \lim Q_n \to R^1 \lim B_n$$

we conclude that the last group is nonzero as desired.

Second proof. For $n' \geq n$ we denote $A_{n,n'} = \mathrm{Im}(A_{n'} \to A_n)$. Then we have $y_i \in A_m$, $y_i \notin A_{m,m_i+1}$. Let $\xi = (\xi_n) \in \prod B_n$ be the element with $\xi_n = 0$ unless $n = m_i$ and $\xi_{m_i} = (0, \ldots, 0, x_i, 0, \ldots)$ with x_i placed in the ith summand. We claim that ξ is not in the image of the map $\prod B_n \to \prod B_n$ of Lemma 15.87.1. This shows that $R^1 \lim B_n$ is nonzero and finishes the proof. Namely, suppose that ξ is the image of $\eta = (z_1, z_2, \ldots)$ with $z_n = \sum z_{n,i} \in \bigoplus_i A_n$. Observe that $x_i = z_{m_i,i} \bmod A_{m_i,m_i+1}$. Then $z_{m_i-1,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_{m_i-1}$, and so on, and we conclude that $z_{m,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_m$. We conclude that $z_{m,i}$ is congruent to y_i modulo A_{m,m_i+1}. In particular $z_{m,i} \neq 0$. This is impossible as $\sum z_{m,i} \in \bigoplus_i A_m$ hence only a finite number of $z_{m,i}$ can be nonzero. □

0CQB **Lemma 15.87.13.** *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups. If (A_i) and (C_i) are ML, then so is (B_i).

Proof. This follows from Lemma 15.87.12, the fact that taking infinite direct sums is exact, and the long exact sequence of cohomology associated to $R\lim$. □

091C **Lemma 15.87.14.** *Let (A_n) be an inverse system of abelian groups. The following are equivalent*

(1) *(A_n) is zero as a pro-object,*
(2) *$\lim A_n = 0$ and $R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}} (A_n)$.*

Proof. It follows from Lemma 15.87.11 that (1) implies (2). Assume (2). Then (A_n) is ML by Lemma 15.87.12. For $m \geq n$ let $A_{n,m} = \mathrm{Im}(A_m \to A_n)$ so that $A_n = A_{n,n} \supset A_{n,n+1} \supset \ldots$. Note that (A_n) is zero as a pro-object if and only if for every n there is an $m \geq n$ such that $A_{n,m} = 0$. Note that (A_n) is ML if and only if for every n there is an $m_n \geq n$ such that $A_{n,m} = A_{n,m+1} = \ldots$. In the ML case it is clear that $\lim A_n = 0$ implies that $A_{n,m_n} = 0$ because the maps $A_{n+1,m_{n+1}} \to A_{n,m}$ are surjective. This finishes the proof. □

15.88. Rlim of modules

0CQC We briefly discuss $R\lim$ on modules. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for modules on ringed sites.

Let (A_n) be an inverse system of rings. We will denote $Mod(\mathbf{N}, (A_n))$ the category of inverse systems (M_n) of abelian groups such that each M_n is given the structure of a A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps. This is an abelian category. Set $A = \lim A_n$. Given an object (M_n) of $Mod(\mathbf{N}, (A_n))$ the limit $\lim M_n$ is an A-module.

091D **Lemma 15.88.1.** *In the situation above. The functor $\lim : Mod(\mathbf{N}, (A_n)) \to Mod_A$ has a right derived functor*

$$R\lim : D(Mod(\mathbf{N}, (A_n))) \longrightarrow D(A)$$

As usual we set $R^p \lim(K) = H^p(R\lim(K))$. Moreover, we have

(1) *for any (M_n) in $Mod(\mathbf{N}, (A_n))$ we have $R^p \lim M_n = 0$ for $p > 1$,*
(2) *the object $R\lim M_n$ of $D(Mod_A)$ is represented by the complex*
$$\prod M_n \to \prod M_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$
sitting in degrees 0 and 1,
(3) *if (M_n) is ML, then $R^1 \lim M_n = 0$, i.e., (M_n) is right acyclic for $\lim$,*
(4) *every $K^\bullet \in D(Mod(\mathbf{N}, (A_n)))$ is quasi-isomorphic to a complex whose terms are right acyclic for $\lim$, and*
(5) *if each $K^p = (K^p_n)$ is right acyclic for $\lim$, i.e., of $R^1 \lim_n K^p_n = 0$, then $R\lim K$ is represented by the complex whose term in degree p is $\lim_n K^p_n$.*

Proof. The proof of this is word for word the same as the proof of Lemma 15.87.1. □

091E **Remark 15.88.2.** This remark is a continuation of Remark 15.87.6. A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings (A_n). A sheaf of modules over (A_n) is exactly the same thing as an object of the category $Mod(\mathbf{N}, (A_n))$ defined above. The derived functor $R\lim$ of Lemma 15.88.1 is simply $R\Gamma(\mathbf{N}, -)$ from the derived category of modules to the derived category of modules over the global sections of

the structure sheaf. It is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 21.12.4.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(A)$, see Derived Categories, Lemma 13.34.2.

0CQD **Lemma 15.88.3.** *Let $K = (K_n^\bullet)$ be an object of $D(Mod(\mathbf{N}, (A_n)))$. There exists a canonical distinguished triangle*

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(A)$. In other words, $R\lim K$ is a derived limit of the inverse system $(K_n^\bullet)$ of $D(A)$, see Derived Categories, Definition 13.34.1.

Proof. The proof is exactly the same as the proof of Lemma 15.87.7 using Lemma 15.88.1 in stead of Lemma 15.87.1. □

0CQE **Lemma 15.88.4.** *With notation as in Lemma 15.88.3 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim\nolimits_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim\nolimits_n H^p(K_n^\bullet) \to 0$$

of A-modules.

Proof. The proof is exactly the same as the proof of Lemma 15.87.8 using Lemma 15.88.1 in stead of Lemma 15.87.1. □

Warning. As in the case of abelian groups an object $M = (M_n^\bullet)$ of $D(Mod(\mathbf{N}, (A_n)))$ is an inverse system of complexes of modules, which is **not** the same thing as an inverse system of objects in the derived categories. In the following lemma we show how an inverse system of objects in derived categories always lifts to an object of $D(Mod(\mathbf{N}, (A_n)))$.

091I **Lemma 15.88.5.** *Let (A_n) be an inverse system of rings. Suppose that we are given*

(1) *for every n an object K_n of $D(A_n)$, and*
(2) *for every n a map $\varphi_n : K_{n+1} \to K_n$ of $D(A_{n+1})$ where we think of K_n as an object of $D(A_{n+1})$ by restriction via $A_{n+1} \to A_n$.*

There exists an object $M = (M_n^\bullet) \in D(Mod(\mathbf{N}, (A_n)))$ and isomorphisms $\psi_n : M_n^\bullet \to K_n$ in $D(A_n)$ such that the diagrams

$$\begin{array}{ccc} M_{n+1}^\bullet & \longrightarrow & M_n^\bullet \\ {\scriptstyle \psi_{n+1}}\downarrow & & \downarrow{\scriptstyle \psi_n} \\ K_{n+1} & \xrightarrow{\varphi_n} & K_n \end{array}$$

commute in $D(A_{n+1})$.

Proof. We write out the proof in detail. For an A_n-module T we write $T_{A_{n+1}}$ for the same module viewd as an A_{n+1}-module. Suppose that $K_n^\bullet$ is a complex of A_n-modules representing K_n. Then $K_{n,A_{n+1}}^\bullet$ is the same complex, but viewed as a complex of A_{n+1}-modules. By the construction of the derived category, the map ψ_n can be given as

$$\psi_n = \tau_n \circ \sigma_n^{-1}$$

where $\sigma_n : L^\bullet_{n+1} \to K^\bullet_{n+1}$ is a quasi-isomorphism of complexes of A_{n+1}-modules and $\tau_n : L^\bullet_{n+1} \to K^\bullet_{n,A_{n+1}}$ is a map of complexes of A_{n+1}-modules.

Now we construct the complexes $M^\bullet_n$ by induction. As base case we let $M^\bullet_1 = K^\bullet_1$. Suppose we have already constructed $M^\bullet_e \to M^\bullet_{e-1} \to \ldots \to M^\bullet_1$ and maps of complexes $\psi_i : M^\bullet_i \to K^\bullet_i$ such that the diagrams

$$\begin{array}{ccccc} M^\bullet_{n+1} & & \longrightarrow & & M^\bullet_{n,A_{n+1}} \\ {\scriptstyle \psi_{n+1}}\downarrow & & & & \downarrow{\scriptstyle \psi_{n,A_{n+1}}} \\ K^\bullet_{n+1} & \xleftarrow{\sigma_n} & L^\bullet_{n+1} & \xrightarrow{\tau_n} & K^\bullet_{n,A_{n+1}} \end{array}$$

above commute in $D(A_{n+1})$ for all $n < e$. Then we consider the diagram

$$\begin{array}{ccccc} & & & & M^\bullet_{e,A_{e+1}} \\ & & & & \downarrow{\scriptstyle \psi_{e,A_{e+1}}} \\ K^\bullet_{e+1} & \xleftarrow{\sigma_e} & L^\bullet_{e+1} & \xrightarrow{\tau_e} & K^\bullet_{e,A_{e+1}} \end{array}$$

in $D(A_{e+1})$. Because ψ_e is a quasi-isomorphism, we see that $\psi_{e,A_{e+1}}$ is a quasi-isomorphism too. By the definition of morphisms in $D(A_{e+1})$ we can find a quasi-isomorphism $\psi_{e+1} : M^\bullet_{e+1} \to K^\bullet_{e+1}$ of complexes of A_{e+1}-modules such that there exists a morphism of complexes $M^\bullet_{e+1} \to M^\bullet_{e,A_{e+1}}$ of A_{e+1}-modules representing the composition $\psi^{-1}_{e,A_{e+1}} \circ \tau_e \circ \sigma_e^{-1}$ in $D(A_{e+1})$. Thus the lemma holds by induction. □

07KZ **Remark 15.88.6.** With assumptions as in Lemma 15.88.5. A priori there are many isomorphism classes of objects M of $D(Mod(\mathbf{N}, (A_n)))$ which give rise to the system (K_n, φ_n) of the lemma. For each such M we can consider the complex $R\lim M \in D(A)$ where $A = \lim A_n$. By Lemma 15.88.3 we see that $R\lim M$ is a derived limit of the inverse system (K_n) of $D(A)$. Hence we see that the isomorphism class of $R\lim M$ in $D(A)$ is independent of the choices made in constructing M. In particular, we may apply results on $R\lim$ proved in this section to derived limits of inverse systems in $D(A)$. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 15.88.3 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 15.88.3 to the (defining) distinguished triangle $R\lim K_n \to \prod K_n \to \prod K_n \to (R\lim K_n)[1]$ of the derived limit.

091F **Lemma 15.88.7.** *Let (A_n) be an inverse system of rings. Every $K \in D(Mod(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(M^\bullet_n)$ such that all the transition maps $M^\bullet_{n+1} \to M^\bullet_n$ are surjective.*

Proof. Let K be represented by the system $(K^\bullet_n)$. Set $M^\bullet_1 = K^\bullet_1$. Suppose we have constructed surjective maps of complexes $M^\bullet_n \to M^\bullet_{n-1} \to \ldots \to M^\bullet_1$ and homotopy

equivalences $\psi_e : K_e^\bullet \to M_e^\bullet$ such that the diagrams

$$\begin{array}{ccc} K_{e+1}^\bullet & \longrightarrow & K_e^\bullet \\ \downarrow & & \downarrow \\ M_{e+1}^\bullet & \longrightarrow & M_e^\bullet \end{array}$$

commute for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc} K_{n+1}^\bullet & \longrightarrow & K_n^\bullet \\ & & \downarrow \\ & & M_n^\bullet \end{array}$$

By Derived Categories, Lemma 13.9.8 we can factor the composition $K_{n+1}^\bullet \to M_n^\bullet$ as $K_{n+1}^\bullet \to M_{n+1}^\bullet \to M_n^\bullet$ such that the first arrow is a homotopy equivalence and the second a termwise split surjection. The lemma follows from this and induction. □

091G **Lemma 15.88.8.** *Let (A_n) be an inverse system of rings. Every $K \in D(Mod(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(K_n^\bullet)$ such that each $K_n^\bullet$ is K-flat.*

Proof. First use Lemma 15.88.7 to represent K by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^\bullet \to M_n^\bullet$ are surjective. Next, let $K_1^\bullet \to M_1^\bullet$ be a quasi-isomorphism with $K_1^\bullet$ a K-flat complex of A_1-modules (Lemma 15.59.10). Suppose we have constructed $K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_1^\bullet$ and maps of complexes $\psi_e : K_e^\bullet \to M_e^\bullet$ such that

$$\begin{array}{ccc} K_{e+1}^\bullet & \longrightarrow & K_e^\bullet \\ \downarrow & & \downarrow \\ M_{e+1}^\bullet & \longrightarrow & M_e^\bullet \end{array}$$

commutes for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc} C^\bullet & \dashrightarrow & K_n^\bullet \\ \vdots & & \downarrow{\scriptstyle \psi_n} \\ M_{n+1}^\bullet & \xrightarrow{\varphi_n} & M_n^\bullet \end{array}$$

in $D(A_{n+1})$. As $M_{n+1}^\bullet \to M_n^\bullet$ is termwise surjective, the complex $C^\bullet$ fitting into the left upper corner with terms

$$C^p = M_{n+1}^p \times_{M_n^p} K_n^p$$

is quasi-isomorphic to $M_{n+1}^\bullet$ (details omitted). Choose a quasi-isomorphism $K_{n+1}^\bullet \to C^\bullet$ with $K_{n+1}^\bullet$ K-flat. Thus the lemma holds by induction. □

091H **Lemma 15.88.9.** *Let (A_n) be an inverse system of rings. Given $K, L \in D(Mod(\mathbf{N}, (A_n)))$ there is a canonical derived tensor product $K \otimes^{\mathbf{L}} L$ in $D(\mathbf{N}, (A_n))$ compatible with the maps to $D(A_n)$. The construction is symmetric in K and L and an exact functor of triangulated categories in each variable.*

Proof. Choose a representative $(K_n^\bullet)$ for K such that each $K_n^\bullet$ is a K-flat complex (Lemma 15.88.8). Then you can define $K \otimes^{\mathbf{L}} L$ as the object represented by the system of complexes

$$(\mathrm{Tot}(K_n^\bullet \otimes_{A_n} L_n^\bullet))$$

for any choice of representative $(L_n^\bullet)$ for L. This is well defined in both variables by Lemmas 15.59.2 and 15.59.12. Compatibility with the map to $D(A_n)$ is clear. Exactness follows exactly as in Lemma 15.58.4. □

091J **Remark 15.88.10.** Let A be a ring. Let (E_n) be an inverse system of objects of $D(A)$. We've seen above that a derived limit $R\lim E_n$ exists. Thus for every object K of $D(A)$ also the derived limit $R\lim(K \otimes^{\mathbf{L}}_A E_n)$ exists. It turns out that we can construct these derived limits functorially in K and obtain an exact functor

$$R\lim(- \otimes^{\mathbf{L}}_A E_n) : D(A) \longrightarrow D(A)$$

of triangulated categories. Namely, we first lift (E_n) to an object E of $D(\mathbf{N}, A)$, see Lemma 15.88.5. (The functor will depend on the choice of this lift.) Next, observe that there is a "diagonal" or "constant" functor

$$\Delta : D(A) \longrightarrow D(\mathbf{N}, A)$$

mapping the complex $K^\bullet$ to the constant inverse system of complexes with value $K^\bullet$. Then we simply define

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) = R\lim(\Delta(K) \otimes^{\mathbf{L}} E)$$

where on the right hand side we use the functor $R\lim$ of Lemma 15.88.1 and the functor $- \otimes^{\mathbf{L}} -$ of Lemma 15.88.9.

091K **Lemma 15.88.11.** *Let A be a ring. Let $E \to D \to F \to E[1]$ be a distinguished triangle of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n), resp. (F_n) be the system of objects of $D(A)$ associated to E, resp. D, resp. F. Then for every $K \in D(A)$ there is a canonical distinguished triangle*

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) \to R\lim(K \otimes^{\mathbf{L}}_A D_n) \to R\lim(K \otimes^{\mathbf{L}}_A F_n) \to R\lim(K \otimes^{\mathbf{L}}_A E_n)[1]$$

in $D(A)$ with notation as in Remark 15.88.10.

Proof. This is clear from the construction in Remark 15.88.10 and the fact that $\Delta : D(A) \to D(\mathbf{N}, A)$, $- \otimes^{\mathbf{L}} -$, and $R\lim$ are exact functors of triangulated categories. □

091L **Lemma 15.88.12.** *Let A be a ring. Let $E \to D$ be a morphism of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n) be the system of objects of $D(A)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then for every $K \in D(A)$ the corresponding map*

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) \longrightarrow R\lim(K \otimes^{\mathbf{L}}_A D_n)$$

in $D(A)$ is an isomorphism (notation as in Remark 15.88.10).

Proof. Follows from the definitions and Lemma 15.87.11. □

15.89. Torsion modules

0ALX In this section "torsion modules" will refer to modules supported on a given closed subset $V(I)$ of an affine scheme $\mathrm{Spec}(R)$. This is different, but analogous to, the notion of a torsion module over a domain (Definition 15.22.1).

05E6 **Definition 15.89.1.** Let R be a ring. Let M be an R-module.

(1) Let $I \subset R$ be an ideal. We say M is an *I-power torsion module* if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
(2) Let $f \in R$. We say M is *an f-power torsion module* if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an f-power torsion module is the same thing as an I-power torsion module for $I = (f)$. We will use the notation

$$M[I^n] = \{m \in M \mid I^n m = 0\}$$

and

$$M[I^\infty] = \bigcup M[I^n]$$

for an R-module M. Thus M is I-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

05E8 **Lemma 15.89.2.** *Let R be a ring. Let I be an ideal of R. Let M be an I-power torsion module. Then M admits a resolution*

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable.

Proof. There is a canonical surjection

$$\oplus_{m \in M} R/I^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I-power torsion module. Proceeding inductively, we construct the desired resolution of M. □

05EA **Lemma 15.89.3.** *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If I is finitely generated then the following are equivalent*

(1) *$M[I] = 0$,*
(2) *$M[I^n] = 0$ for all $n \geq 1$, and*
(3) *if $I = (f_1, \ldots, f_t)$, then the map $M \to \bigoplus M_{f_i}$ is injective.*

Proof. This follows from Algebra, Lemma 10.24.4. □

05EB **Lemma 15.89.4.** *Let R be a ring. Let I be a finitely generated ideal of R.*

(1) *For any R-module M we have $(M/M[I^\infty])[I] = 0$.*
(2) *An extension of I-power torsion modules is I-power torsion.*

Proof. Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_i m \in M[I^\infty]$, i.e., $I^{n_i} f_i m = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence m maps to zero in $(M/M[I^\infty])$ which proves the first statement of the lemma.

For the second, suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules with M' and M'' both I-power torsion modules. Then $M[I^\infty] \supset M'$

and hence $M/M[I^\infty]$ is a quotient of M'' and therefore I-power torsion. Combined with the first statement and Lemma 15.89.3 this implies that it is zero. □

0A6K **Lemma 15.89.5.** *Let I be a finitely generated ideal of a ring R. The I-power torsion modules form a Serre subcategory of the abelian category Mod_R, see Homology, Definition 12.10.1.*

Proof. It is clear that a submodule and a quotient module of an I-power torsion module is I-power torsion. Moreover, the extension of two I-power torsion modules is I-power torsion by Lemma 15.89.4. Hence the statement of the lemma by Homology, Lemma 12.10.2. □

0953 **Lemma 15.89.6.** *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The subcategory I^∞-torsion $\subset Mod_R$ depends only on the closed subset $Z = V(I) \subset \mathrm{Spec}(R)$. In fact, an R-module M is I-power torsion if and only if its support is contained in Z.*

Proof. Let M be an R-module. Let $x \in M$. If $x \in M[I^\infty]$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$. Conversely, if x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, \ldots, f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^{\sum n_i}]$. Thus $M[I^\infty]$ is the kernel of $M \to \prod_{\mathfrak{p} \notin Z} M_\mathfrak{p}$. The second statement of the lemma follows and it implies the first. □

The next two lemmas should probably go somewhere else.

0H82 **Lemma 15.89.7.** *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$ such that $H^i(K \otimes^{\mathbf{L}}_R R/I) = 0$ for $i > 0$. Then*

1. *$H^i(K \otimes^{\mathbf{L}}_R R/I^n) = 0$ for all $n \geq 1$ and $i > 0$,*
2. *$H^i(K \otimes^{\mathbf{L}}_R N) = 0$ for any I-power torsion R-module N and $i > 0$, and*
3. *for any $M \in D^b(R)$ whose cohomology modules $H^i(M)$ are I-power torsion and 0 for $i > 0$ we have $H^i(K \otimes^{\mathbf{L}}_R M) = 0$ for $i > 0$.*

Proof. Proof of (2). We can write $N = \bigcup N[I^n]$. We have $K \otimes^{\mathbf{L}}_R N = \mathrm{hocolim}_n K \otimes^{\mathbf{L}}_R N[I^n]$ as tensor products commute with colimits (details omitted; hint: represent K by a K-flat complex and compute directly). Hence we may assume N is annihilated by I^n. Consider the R-algebra $R' = R/I^n \oplus N$ where N is an ideal of square zero. It suffices to show that the object $K' = K \otimes^{\mathbf{L}}_R R'$ of $D(R')$ has vanishing cohomology in positive degrees. We have a surjection $R' \to R/I$ of R-algebras whose kernel J is nilpotent (any product of n elements in the kernel is zero). We have

$$K \otimes^{\mathbf{L}}_R R/I = (K \otimes^{\mathbf{L}}_R R') \otimes^{\mathbf{L}}_{R'} R/I = K' \otimes^{\mathbf{L}}_{R'} R/I$$

By assumption the complex $K \otimes^{\mathbf{L}}_R R/I$ has tor-amplitude in $[-\infty, 0]$. Thus the conclusion by Lemma 15.67.20.

Part (1) follows trivially from part (2). Part (3) follows from part (2), induction on the number of nonzero cohomology modules of M, and the distinguished triangles of truncation from Derived Categories, Remark 13.12.4. Details omitted. □

0G1T **Lemma 15.89.8.** *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$ such that $K \otimes^{\mathbf{L}}_R R/I = 0$ in $D(R)$. Then*

1. *$K \otimes^{\mathbf{L}}_R R/I^n = 0$ for all $n \geq 1$,*
2. *$K \otimes^{\mathbf{L}}_R N = 0$ for any I-power torsion R-module N,*

(3) $K \otimes^{\mathbf{L}}_R M = 0$ *for any* $M \in D^b(R)$ *whose cohomology modules are* I*-power torsion.*

Proof. Consequence of Lemma 15.89.7 (some details omitted). □

0BNK **Lemma 15.89.9.** *Let* $R \to R'$ *be a ring map. Let* $I \subset R$ *be an ideal such that* $R/I^n \to R'/I^nR'$ *is an isomorphism for* $n > 0$*. For any* I*-power torsion* R*-module* M *the map* $M \to M \otimes_R R'$ *is an isomorphism. For example, if* I *is finitely generated and* $R^\wedge$ *is the completion of* R *with respect to* I*, then we have* $M \cong M \otimes_R R^\wedge$*.* Slight generalization of [**BL95**, Lemme 1].

Proof. If M is annihilated by I^n, then

$$M \otimes_R R' \cong M \otimes_{R/I^n} R'/I^nR' \cong M \otimes_{R/I^n} R/I^n \cong M.$$

If M is I-power torsion, then $M = \bigcup M[I^n]$. Since tensor products commute with direct limits (Algebra, Lemma 10.12.9), we obtain the desired isomorphism. The last statement is a special case of the first statement by Algebra, Lemma 10.96.3. □

15.90. Formal glueing of module categories

05E5 Fix a Noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap. We first do this using only commutative algebra (this section) and later we explain this in the setting of algebraic spaces (Pushouts of Spaces, Section 81.10).

Here are some references treating some of the material in this section: [**Art70**, Section 2], [**FR70**, Appendix], [**BL95**], [**MB96**], and [**dJ95**, Section 4.6].

05E7 **Lemma 15.90.1.** *Let* $\varphi : R \to S$ *be a ring map. Let* $I \subset R$ *be an ideal. The following are equivalent*

(1) φ *is flat and* $R/I \to S/IS$ *is faithfully flat,*
(2) φ *is flat, and the map* $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ *is surjective.*
(3) φ *is flat, and the base change functor* $M \mapsto M \otimes_R S$ *is faithful on modules annihilated by* I*, and*
(4) φ *is flat, and the base change functor* $M \mapsto M \otimes_R S$ *is faithful on* I*-power torsion modules.*

Proof. If $R \to S$ is flat, then $R/I^n \to S/I^nS$ is flat for every n, see Algebra, Lemma 10.39.7. Hence (1) and (2) are equivalent by Algebra, Lemma 10.39.16. The equivalence of (1) with (3) follows by identifying I-torsion R-modules with R/I-modules, using that

$$M \otimes_R S = M \otimes_{R/I} S/IS$$

for R-modules M annihilated by I, and Algebra, Lemma 10.39.14. The implication (4) $\Rightarrow$ (3) is immediate. Assume (3). We have seen above that $R/I^n \to S/I^nS$ is flat, and by assumption it induces a surjection on spectra, as $\mathrm{Spec}(R/I^n) = \mathrm{Spec}(R/I)$ and similarly for S. Hence the base change functor is faithful on modules annihilated by I^n. Since any I-power torsion module M is the union $M = \bigcup M_n$ where M_n is annihilated by I^n we see that the base change functor is faithful on the category of all I-power torsion modules (as tensor product commutes with colimits). □

05E9 **Lemma 15.90.2.** *Assume $(\varphi : R \to S, I)$ satisfies the equivalent conditions of Lemma 15.90.1. The following are equivalent*

(1) *for any I-power torsion module M, the natural map $M \to M \otimes_R S$ is an isomorphism, and*
(2) *$R/I \to S/IS$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). First assume that M is annihilated by I. In this case, M is an R/I-module. Hence, we have an isomorphism

$$M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M$$

proving the claim. Next we prove by induction that $M \to M \otimes_R S$ is an isomorphism for any module M is annihilated by I^n. Assume the induction hypothesis holds for n and assume M is annihilated by I^{n+1}. Then we have a short exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0$$

and as $R \to S$ is flat this gives rise to a short exact sequence

$$0 \to I^n M \otimes_R S \to M \otimes_R S \to M/I^n M \otimes_R S \to 0$$

Using that the canonical map is an isomorphism for $M' = I^n M$ and $M'' = M/I^n M$ (by induction hypothesis) we conclude the same thing is true for M. Finally, suppose that M is a general I-power torsion module. Then $M = \bigcup M_n$ where M_n is annihilated by I^n and we conclude using that tensor products commute with colimits. □

05EC **Lemma 15.90.3.** *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Then*

(1) *for any R-module M the map $M \to M \otimes_R S$ induces an isomorphism $M[I^\infty] \to (M \otimes_R S)[(IS)^\infty]$ of I-power torsion submodules,*
(2) *the natural map*

$$\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism if either M or N is I-power torsion, and
(3) *the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between I-power torsion modules and IS-power torsion modules.*

Proof. Note that the equivalent conditions of both Lemma 15.90.1 and Lemma 15.90.2 are satisfied. We will use these without further mention. We first prove (1). Let M be any R-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 15.89.4 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \longrightarrow \bigoplus\nolimits_{i=1,\ldots t} M', \quad x \longmapsto (f_1^n x, \ldots, f_t^n x)$$

is injective. As S is flat over R also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1,\ldots t} M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[I^n] = 0$ as desired.

Next we prove (2). If N is I-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma 10.14.3. If M is I-power torsion, then the image of any map $M \to N$ factors through $N[I^\infty]$ and the image of any

map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace N by $N[I^\infty]$ and the result follows from the case where N is I-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any IS-power torsion S-module N, the natural map $N \otimes_R S \to N$ is an isomorphism. □

091M **Lemma 15.90.4.** *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. For any $f_1, \ldots, f_r \in R$ such that $V(f_1, \ldots, f_r) = V(I)$*

(1) *the map of Koszul complexes $K(R, f_1, \ldots, f_r) \to K(S, f_1, \ldots, f_r)$ is a quasi-isomorphism, and*
(2) *The map of extended alternating Čech complexes*

$$\begin{array}{ccccccc} R & \to & \prod_{i_0} R_{f_{i_0}} & \to & \prod_{i_0<i_1} R_{f_{i_0}f_{i_1}} & \to \ldots \to & R_{f_1\ldots f_r} \\ & & & & \downarrow & & \\ S & \to & \prod_{i_0} S_{f_{i_0}} & \to & \prod_{i_0<i_1} S_{f_{i_0}f_{i_1}} & \to \ldots \to & S_{f_1\ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. In both cases we have a complex $K_\bullet$ of R modules and we want to show that $K_\bullet \to K_\bullet \otimes_R S$ is a quasi-isomorphism. By Lemma 15.90.2 and the flatness of $R \to S$ this will hold as soon as all homology groups of K are I-power torsion. This is true for the Koszul complex by Lemma 15.28.6 and for the extended alternating Čech complex by Lemma 15.29.5. □

05ED **Lemma 15.90.5.** *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. Let M be the R-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_i e_j - f_j e_i = 0$. There exists a short exact sequence*

$$0 \to K \to M \to I \to 0$$

such that K is annihilated by I.

Proof. This is just a truncation of the Koszul complex. The map $M \to I$ is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_i e_i$ is in the kernel of $M \to I$, i.e., $\sum a_i f_i = 0$, then $f_j m = \sum f_j a_i e_i = (\sum f_i a_i) e_j = 0$. □

05EE **Lemma 15.90.6.** *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N set*

$$H_1(N, f_\bullet) = \frac{\{(x_1, \ldots, x_n) \in N^{\oplus n} \mid f_i x_j = f_j x_i\}}{\{f_1 x, \ldots, f_n x) \mid x \in N\}}$$

For any R-module N there exists a canonical short exact sequence

$$0 \to \mathrm{Ext}_R(R/I, N) \to H_1(N, f_\bullet) \to \mathrm{Hom}_R(K, N)$$

where K is as in Lemma 15.90.5.

Proof. The notation above indicates the Ext-groups in Mod_R as defined in Homology, Section 12.6. These are denoted $\mathrm{Ext}_R(M, N)$. Using the long exact sequence

of Homology, Lemma 12.6.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\mathrm{Ext}_R(R, N) = 0$ we see that

$$\mathrm{Ext}_R(R/I, N) = \mathrm{Coker}(N \longrightarrow \mathrm{Hom}(I, N))$$

Using the short exact sequence of Lemma 15.90.5 we see that we get a complex

$$N \to \mathrm{Hom}(M, N) \to \mathrm{Hom}_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\mathrm{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. □

05EF **Lemma 15.90.7.** *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 15.90.6 is annihilated by I.*

Proof. Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i(x_1, \ldots, x_n) = (f_i x_i, \ldots, f_i x_n)$. In other words f_i annihilates $H_1(N, f_\bullet)$. □

We can improve on the full faithfulness of Lemma 15.90.3 by showing that Ext-groups whose source is I-power torsion are insensitive to passing to S as well. See Dualizing Complexes, Lemma 47.9.8 for a derived version of the following lemma.

05EG **Lemma 15.90.8.** *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules. Assume M is I-power torsion. Given an short exact sequence*

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N \otimes_R S & \longrightarrow & \tilde{E} & \longrightarrow & M \otimes_R S & \longrightarrow & 0 \end{array}$$

with exact rows.

Proof. As M is I-power torsion we see that $M \otimes_R S = M$, see Lemma 15.90.2. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\mathrm{Ext}_R(M, N) \longrightarrow \mathrm{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma 12.7.3. The claim of the lemma is that this map is surjective when M is I-power torsion. In fact we will show that it is an isomorphism. By Lemma 15.89.2 we can find a surjection $M' \to M$ with M' a direct sum of modules of the form R/I^n. Using the long exact sequence of Homology, Lemma 12.6.4 and using Lemma 15.90.3 we see that it suffices to prove the lemma for M'. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some n.

Let $f_1, \ldots, f_t$ be generators for I^n. By Lemma 15.90.6 we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_R(R/I^n, N) & \longrightarrow & H_1(N, f_\bullet) & \longrightarrow & \mathrm{Hom}_R(K, N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}_S(S/I^nS, N \otimes S) & \longrightarrow & H_1(N \otimes S, f_\bullet) & \longrightarrow & \mathrm{Hom}_S(K \otimes S, N \otimes S) \end{array}$$

with exact rows where K is as in Lemma 15.90.5. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since K is annihilated by I^n we see that $\mathrm{Hom}_R(K, N) = \mathrm{Hom}_S(K \otimes_R S, N \otimes_R S)$ by Lemma 15.90.3. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by I^n, see Lemma 15.90.7 we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma 15.90.2. □

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any R-module M we can define a complex

05EJ (15.90.8.1) $$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and

$$\beta(m', m_1, \ldots, m_t) = ((m'/1 - m_1 \otimes 1, \ldots, m'/1 - m_t \otimes 1), (m_1 - m_2, \ldots, m_{t-1} - m_t).$$

We would like to know when this complex is exact.

05EK **Lemma 15.90.9.** *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let M be an R-module. Then the complex (15.90.8.1) is exact.*

Proof. First proof. Denote $\check{\mathcal{C}}_R \to \check{\mathcal{C}}_S$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.90.4. Since these complexes are bounded with flat terms, we see that $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ is a quasi-isomorphism too (Lemmas 15.59.7 and 15.59.12). Now the complex (15.90.8.1) is a truncation of the cone of the map $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ and we win.

Second computational proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^\infty]$, see Lemma 15.89.3. Pick n such that $I^n m = 0$ and consider the map $\varphi : R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma 15.90.3) hence $m = 0$. In this way we see that α is injective.

Let $(m', m'_1, \ldots, m'_t) \in \mathrm{Ker}(\beta)$. Write $m'_i = m_i/f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging n assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_j^n m_i - f_i^n m_j = 0$ in M. In particular we see that $(m_1, \ldots, m_t)$ defines an element ξ of $H_1(M, (f_1^n, \ldots, f_t^n))$. Since $H_1(M, (f_1^n, \ldots, f_t^n))$ is annihilated by I^{tn+1} (see Lemma 15.90.7) and since $R \to S$ is flat we see that

$$H_1(M, (f_1^n, \ldots, f_t^n)) = H_1(M, (f_1^n, \ldots, f_t^n)) \otimes_R S = H_1(M \otimes_R S, (f_1^n, \ldots, f_t^n))$$

by Lemma 15.90.2 The existence of m' implies that ξ maps to zero in the last group, i.e., the element ξ is zero. Thus there exists an $m \in M$ such that $m_i = f_i^n m$. Then $(m', m'_1, \ldots, m'_t) - \alpha(m) = (m'', 0, \ldots, 0)$ for some $m'' \in (M \otimes_R S)[(IS)^\infty]$. By Lemma 15.90.3 we conclude that $m'' \in M[I^\infty]$ and we win. □

05EL **Remark 15.90.10.** In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose

(1) objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where M' is an S-module, M_i is an R_{f_i}-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms such that
   (a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_i}$, and
   (b) $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ as maps $(M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ (cocycle condition).

(2) morphisms $(M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ are given by maps $\varphi' : M' \to N'$ and $\varphi_i : M_i \to N_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\text{Can} : \text{Mod}_R \longrightarrow \text{Glue}(R \to S, f_1, \ldots, f_t), \quad M \longmapsto (M \otimes_R S, M_{f_i}, \text{can}_i, \text{can}_{ij})$$

where $\text{can}_i : (M \otimes_R S)_{f_i} \to M_{f_i} \otimes_R S$ and $\text{can}_{ij} : (M_{f_i})_{f_j} \to (M_{f_j})_{f_i}$ are the canonical isomorphisms. For any object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(\mathbf{M}) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(\mathbf{M}) \to M' \times \prod M_i \to \prod M'_{f_i} \times \prod (M_i)_{f_j}$$

similar to (15.90.8.1). We think of $H^0(\mathbf{M})$ as an R-module. Thus we also get a functor

$$H^0 : \text{Glue}(R \to S, f_1, \ldots, f_t) \longrightarrow \text{Mod}_R$$

Our next goal is to show that the functors Can and H^0 are sometimes quasi-inverse to each other.

0H77 **Lemma 15.90.11.** *In Remark 15.90.10 the functor $H^0 : Glue(R \to S, f_1, \ldots, f_t) \to Mod_R$ is a right adjoint to the functor $Can : Mod_R \to Glue(R \to S, f_1, \ldots, f_t)$.*

Proof. Let $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ be an object of $\text{Glue}(R \to S, f_1, \ldots, f_t)$. For any R-module N there is a map

$$\text{Hom}_{\text{Glue}(R \to S, f_1, \ldots, f_t)}(\text{Can}(N), \mathbf{M}) \to \text{Hom}_R(N, H^0(\mathbf{M}))$$

sending ψ to $H^0(\psi)$ composed with the obvious map $N \to H^0(\text{Can}(N))$. By construction the displayed map is an isomorphism for $N = R$ (even if $R \to H^0(\text{Can}(R))$ is not an isomorphism in general). The category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ has direct sums and cokernels. The functor Can commutes with direct sums and cokernels. From these observations we find the displayed map is bijective by writing N as a cokernel of a map between free R-modules. We omit the details. □

05EM **Lemma 15.90.12.** *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor H^0 is a left quasi-inverse to the functor Can of Remark 15.90.10.*

Proof. This is a reformulation of Lemma 15.90.9. □

05EN **Lemma 15.90.13.** *Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then $Glue(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor Can is exact and commutes with arbitrary colimits.*

Proof. Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\text{Ker}(\varphi'), \text{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\text{Coker}(\varphi'), \text{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking

kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows from the R-flatness of R_{f_i} and S, while commuting with colimits follows as tensor products commute with colimits. □

05EP **Lemma 15.90.14.** *Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_t) = R$. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_t)$. By Algebra, Lemma 10.24.5 there exists a unique module M and isomorphisms $M_{f_i} \to M_i$ which recover the glueing data α_{ij}. Then both M' and $M \otimes_R S$ are S-modules which recover the modules $M_i \otimes_R S$ upon localizing at f_i. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $\mathbf{M}$ is in the essential image of Can. Combined with Lemma 15.90.12 the lemma follows. □

05EQ **Lemma 15.90.15.** *Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_t)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors*

$$\begin{array}{ccccc} Mod_R & \xrightarrow[Can]{} & Glue(R \to S, f_1, \ldots, f_t) & \xrightarrow[H^0]{} & Mod_R \\ \downarrow{\scriptstyle -\otimes_R R'} & & \downarrow{\scriptstyle -\otimes_R R'} & & \downarrow{\scriptstyle -\otimes_R R'} \\ Mod_{R'} & \xrightarrow{Can} & Glue(R' \to S', f_1, \ldots, f_t) & \xrightarrow{H^0} & Mod_{R'} \end{array}$$

Proof. Omitted. □

05ER **Proposition 15.90.16.** *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. We have already seen that $H^0 \circ \text{Can}$ is isomorphic to the identity functor, see Lemma 15.90.12. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_t)$. We get a natural morphism

$$\Psi : (H^0(\mathbf{M}) \otimes_R S, H^0(\mathbf{M})_{f_i}, \text{can}_i, \text{can}_{ij}) \longrightarrow (M', M_i, \alpha_i, \alpha_{ij}).$$

Namely, by definition $H^0(\mathbf{M})$ comes equipped with compatible R-module maps $H^0(\mathbf{M}) \to M'$ and $H^0(\mathbf{M}) \to M_i$. We have to show that this map is an isomorphism.

Pick an index i and set $R' = R_{f_i}$. Combining Lemmas 15.90.15 and 15.90.14 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of Ψ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that H^0 is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of Ψ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$0 \to H^0(\mathbf{M}) \otimes_R S \to M' \to Q \to 0$$

and that $M_i = H^0(\mathbf{M})_{f_i}$. Note that we may think of Q as an R-module which is I-power torsion so that $Q = Q \otimes_R S$. By Lemma 15.90.8 we see that there exists a

commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{M}) & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathbf{M}) \otimes_R S & \longrightarrow & M' & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

with exact rows. This clearly determines an isomorphism $\mathrm{Can}(E) \to (M', M_i, \alpha_i, \alpha_{ij})$ in the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ and we win. (Of course, a posteriori we have $Q = 0$.) □

0ALK **Lemma 15.90.17.** *Let $\varphi : R \to S$ be a flat ring map and let $I \subset R$ be a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism.*

(1) *Given an R-module N, an S-module M' and an S-module map $\varphi : M' \to N \otimes_R S$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $M \otimes_R S = M'$ compatible with φ and ψ.*
(2) *Given an R-module M, an S-module N' and an S-module map $\varphi : M \otimes_R S \to N'$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $N \otimes_R S = N'$ compatible with φ and ψ.*

In both cases we have $\mathrm{Ker}(\varphi) \cong \mathrm{Ker}(\psi)$ *and* $\mathrm{Coker}(\varphi) \cong \mathrm{Coker}(\psi)$.

Proof. Proof of (1). Say $I = (f_1, \ldots, f_t)$. It is clear that the localization φ_{f_i} is an isomorphism. Thus we see that $(M', N_{f_i}, \varphi_{f_i}, can_{ij})$ is an object of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 15.90.10. By Proposition 15.90.16 we conclude that there exists an R-module M such that $M' = M \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms φ_{f_i} and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) = (M', N_{f_i}, \varphi_{f_i}, can_{ij}) \to (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition 15.90.16). The composition of the base change of $M \to N$ with the isomorphism $M' \cong M \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

Proof of (2). This is just the dual of the argument above. Namely, the localization φ_{f_i} is an isomorphism. Thus we see that $(N', M_{f_i}, \varphi_{f_i}^{-1}, can_{ij})$ is an object of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 15.90.10. By Proposition 15.90.16 we conclude that there exists an R-module N such that $N' = N \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms $\varphi_{f_i}^{-1}$ and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) \to (N', M_{f_i}, \varphi_{f_i}, can_{ij}) = (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition 15.90.16). The composition of the base change of $M \to N$ with the isomorphism $N' \cong N \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

The final statement follows for example from Lemma 15.90.3. □

Next, we specialize Proposition 15.90.16 to get something more usable. Namely, if $I = (f)$ is a principal ideal then the objects of $\mathrm{Glue}(R \to S, f)$ are simply triples (M', M_1, α_1) and there is *no* cocycle condition to check!

05ES **Theorem 15.90.18.** *Let R be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor*

$$Mod_R \longrightarrow Mod_S \times_{Mod_{S_f}} Mod_{R_f}, \quad M \longmapsto (M \otimes_R S, M_f, can)$$

is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples (M', M_1, α_1) where M' is an S-module, M_1 is a R_f-module, and $\alpha_1 : M'_f \to M_1 \otimes_R S$ is a S_f-isomorphism, see Categories, Example 4.31.3. Hence this theorem is a special case of Proposition 15.90.16. □

A useful special case of Theorem 15.90.18 is when R is Noetherian, and S is a completion of R at an element f. The completion $R \to S$ is flat, and the functor $M \mapsto M \otimes_R S$ can be identified with the f-adic completion functor when M is finitely generated. To state this more precisely, let Mod_R^{fg} denote the category of finitely generated R-modules.

05ET **Proposition 15.90.19.** *Let R be a Noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the f-adic completion of R. Then the functor $M \mapsto (M^\wedge, M_f, can)$ defines an equivalence*

$$Mod_R^{fg} \longrightarrow Mod_{R^\wedge}^{fg} \times_{Mod_{(R^\wedge)_f}^{fg}} Mod_{R_f}^{fg}$$

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 10.97.2. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 10.97.1 the completion of a finite R-module M is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occurring in Theorem 15.90.18. In particular it is fully faithful. Let (M_1, M_2, ψ) be an object of the right hand side. By Theorem 15.90.18 there exists an R-module M such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 10.23.2 that M is finitely generated, i.e., $M \in \mathrm{Mod}_R^{fg}$. This proves the proposition. □

05EU **Remark 15.90.20.** The equivalences of Proposition 15.90.16, Theorem 15.90.18, and Proposition 15.90.19 preserve properties of modules. For example if M corresponds to $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ then M is finite, or finitely presented, or flat, or projective over R if and only if M' and M_i have the corresponding property over S and R_{f_i}. This follows from the fact that $R \to S \times \prod R_{f_i}$ is faithfully flat and descend and ascent of these properties along faithfully flat maps, see Algebra, Lemma 10.83.2 and Theorem 10.95.6. These functors also preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\mathrm{Mod}_R, \otimes)$, such as the category of algebras.

05EV **Remark 15.90.21.** Given a differential manifold X with a compact closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 15.90.16, Theorem 15.90.18, and Proposition 15.90.19 allow us to work with formal neighbourhoods instead.

15.91. The Beauville-Laszlo theorem

0BNI Let R be a ring and let f be an element of R. Denote $R^\wedge = \lim R/f^nR$ the f-adic completion of R. In this section we discuss and slightly generalize a theorem of Beauville and Laszlo, see [**BL95**]. The theorem asserts that under suitable conditions, a module over R can be constructed by "glueing together" modules over $R^\wedge$ and R_f along an isomorphism between the base extensions to $(R^\wedge)_f$.

In [**BL95**] it is assumed that f is a nonzerodivisor on both R and M. In fact, one only needs to assume that

$$R[f^\infty] \longrightarrow R^\wedge[f^\infty]$$

is bijective and that

$$M[f^\infty] \longrightarrow M \otimes_R R^\wedge$$

is injective. This optimization was partly inspired by an alternate approach to glueing introduced in [**KL15**, §1.3] for use in the theory of nonarchimedean analytic spaces.

In fact, we will establish the Beauville-Laszlo theorem in the more general setting of a ring map

$$R \longrightarrow R'$$

which induces isomorphisms $R/f^nR \to R'/f^nR'$ for every $n > 0$ and an isomorphism $R[f^\infty] \to R'[f^\infty]$. This is better suited for globalizing and does not formally follow from the case when R' is the completion of R because, for instance, the condition that $R[f^\infty] \to R'[f^\infty]$ is a bijection does not imply that $R[f^\infty] \to R^\wedge[f^\infty]$ is a bijection.

The theorem of Beauville and Laszlo as proved in this section can be viewed as a non-flat version of Theorem 15.90.18 and in the case where $R' = R^\wedge$ can be viewed as a non-Noetherian version of Proposition 15.90.19. For a comparison with flat descent, please see Remark 15.91.5.

One can establish even stronger results (without imposing restrictions on M for example) but for this one must work at the level of derived categories. See [**Bha16**, §5] for more details.

0BNJ **Lemma 15.91.1.** *Let R be a ring and let $f \in R$. For every positive integer n the map $R/f^nR \to R^\wedge/f^nR^\wedge$ is an isomorphism.*

Proof. This is a special case of Algebra, Lemma 10.96.3. □

We will use the notation introduced in Section 15.89. Thus for an R-module M, we denote $M[f^n]$ the submodule of M annihilated by f^n and we put

$$M[f^\infty] = \bigcup\nolimits_{n=1}^{\infty} M[f^n] = \mathrm{Ker}(M \to M_f).$$

If $M = M[f^\infty]$, we say that M is an f-power torsion module.

0BNL **Lemma 15.91.2.** *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The R-module $R' \oplus R_f$ is faithful: for every nonzero R-module M, the module $M \otimes_R (R' \oplus R_f)$ is also nonzero. For example, if M is nonzero, then $M \otimes_R (R^\wedge \oplus R_f)$ is nonzero.*

However, the map $M \to M \otimes_R (R' \oplus R_f)$ need not be injective; see Example 15.91.9.

Proof. If $M \neq 0$ but $M \otimes_R R_f = 0$, then M is f-power torsion. By Lemma 15.89.9 we find that $M \otimes_R R' \cong M \neq 0$. The last statement is a special case of the first statement by Lemma 15.91.1. □

0BNM **Lemma 15.91.3.** *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces an isomorphism $R/fR \to R'/fR'$. The map $\mathrm{Spec}(R') \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective. For example, the map $\mathrm{Spec}(R^\wedge) \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Recall that $\mathrm{Spec}(R) = V(f) \amalg D(f)$ where $V(f) = \mathrm{Spec}(R/fR)$ and $D(f) = \mathrm{Spec}(R_f)$, see Algebra, Section 10.17 and especially Lemmas 10.17.7 and 10.17.6. Thus the lemma follows as the map $R \to R/fR$ factors through R'. The last statement is a special case of the first statement by Lemma 15.91.1. □

0BNN **Lemma 15.91.4.** *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. An R-module M is finitely generated if and only if the $(R' \oplus R_f)$-module $M \otimes_R (R' \oplus R_f)$ is finitely generated. For example, if $M \otimes_R (R^\wedge \oplus R_f)$ is finitely generated as a module over $R^\wedge \oplus R_f$, then M is a finitely generated R-module.*

Slight generalization of [**BL95**, Lemme 2(a)].

Proof. The 'only if' is clear, so we assume that $M \otimes_R (R' \oplus R_f)$ is finitely generated. In this case, by writing each generator as a sum of simple tensors, $M \otimes_R (R' \oplus R_f)$ admits a finite generating set consisting of elements of M. That is, there exists a morphism from a finite free R-module to M whose cokernel is killed by tensoring with $R' \oplus R_f$; we may thus deduce M is finite generated by applying Lemma 15.91.2 to this cokernel. The last statement is a special case of the first statement by Lemma 15.91.1. □

0BNP **Remark 15.91.5.** While $R \to R_f$ is always flat, $R \to R^\wedge$ is typically not flat unless R is Noetherian (see Algebra, Lemma 10.97.2 and the discussion in Examples, Section 110.12). Consequently, we cannot in general apply faithfully flat descent as discussed in Descent, Section 35.3 to the morphism $R \to R^\wedge \oplus R_f$. Moreover, even in the Noetherian case, the usual definition of a descent datum for this morphism refers to the ring $R^\wedge \otimes_R R^\wedge$, which we will avoid considering in this section.

Glueing pairs. Let $R \to R'$ be a ring map that induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. Consider the sequence

0F1Q (15.91.5.1) $$0 \to R \to R' \oplus R_f \to R'_f \to 0,$$

in which the map on the right is the difference between the two canonical homomorphisms. If this sequence is exact, then we say that $(R \to R', f)$ is a *glueing pair*. We will say that (R, f) is a *glueing pair* if $(R \to R^\wedge, f)$ is a glueing pair; this makes sense by Lemma 15.91.1. Thus (R, f) is a glueing pair if and only if the sequence

0BNQ (15.91.5.2) $$0 \to R \to R^\wedge \oplus R_f \to (R^\wedge)_f \to 0,$$

is exact.

0BNR **Lemma 15.91.6.** *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (15.91.5.1) is*

(1) *exact on the right,*
(2) *exact on the left if and only if $R[f^\infty] \to R'[f^\infty]$ is injective, and*

(3) *exact in the middle if and only if* $R[f^\infty] \to R'[f^\infty]$ *is surjective.*

In particular, $(R \to R', f)$ *is a glueing pair if and only if* $R[f^\infty] \to R'[f^\infty]$ *is bijective. For example,* (R, f) *is a glueing pair if and only if* $R[f^\infty] \to R^\wedge[f^\infty]$ *is bijective.*

Proof. Let $x \in R'_f$. Write $x = x'/f^n$ with $x' \in R'$. Write $x' = x'' + f^n y$ with $x'' \in R$ and $y \in R'$. Then we see that $(y, -x''/f^n)$ maps to x. Thus (1) holds.

Part (2) follows from the fact that $\mathrm{Ker}(R \to R_f) = R[f^\infty]$.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in R'[f^\infty]$ are in the image of the first arrow. This implies that $R[f^\infty] \to R'[f^\infty]$ is surjective. Conversely, assume that $R[f^\infty] \to R'[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in R$. Then we see that $f^n x - y'$ is annihilated by some power of f in R'. By assumption we can write $f^n x - y' = z$ for some $z \in R[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $R \to R/f^n R$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in R$. Then $x - y'''$ is in $R'[f^\infty]$. Thus $x - y''' = z' \in R[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

The last statement of the lemma is a special case of the penultimate statement by Lemma 15.91.1. □

0BNS **Remark 15.91.7.** Suppose that f is a nonzerodivisor. Then Algebra, Lemma 10.96.4 shows that f is a nonzerodivisor in $R^\wedge$. Hence (R, f) is a glueing pair.

0BNT **Remark 15.91.8.** If $R \to R^\wedge$ is flat, then for each positive integer n tensoring the sequence $0 \to R[f^n] \to R \to R$ with $R^\wedge$ gives the sequence $0 \to R[f^n] \otimes_R R^\wedge \to R^\wedge \to R^\wedge$. Combined with Lemma 15.89.9 we conclude that $R[f^n] \to R^\wedge[f^n]$ is an isomorphism. Thus (R, f) is a glueing pair. This holds in particular if R is Noetherian, see Algebra, Lemma 10.97.2.

0BNU **Example 15.91.9.** Let k be a field and put

$$R = k[f, T_1, T_2, \ldots]/(fT_1, fT_2 - T_1, fT_3 - T_2, \ldots).$$

Then (R, f) is not a glueing pair because the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not injective as the image of T_1 is f-divisible in $R^\wedge$. For

$$R = k[f, T_1, T_2, \ldots]/(fT_1, f^2T_2, \ldots),$$

the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not surjective as the element $T_1 + fT_2 + f^2T_3 + \ldots$ is not in the image. In particular, by Remark 15.91.8, these are both examples where $R \to R^\wedge$ is not flat.

Glueable modules. Let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. For any R-module M, we may tensor (15.91.5.1) with M to obtain a sequence

0F1R (15.91.9.1) $$0 \to M \to (M \otimes_R R') \oplus (M \otimes_R R_f) \to M \otimes_R R'_f \to 0$$

Observe that $M \otimes_R R_f = M_f$ and that $M \otimes_R R'_f = (M \otimes_R R')_f$. If this sequence is exact, we say that M is *glueable for* $(R \to R', f)$. If R is a ring and $f \in R$, then we say an R-module is *glueable* if M is glueable for $(R \to R^\wedge, f)$. Thus M is glueable if and only if the sequence

0BNV (15.91.9.2) $$0 \to M \to (M \otimes_R R^\wedge) \oplus (M \otimes_R R_f) \to M \otimes_R (R^\wedge)_f \to 0$$

is exact.

0BNW **Lemma 15.91.10.** *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (15.91.9.1) is*

(1) *exact on the right,*
(2) *exact on the left if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective, and*
(3) *exact in the middle if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective.*

Thus M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is bijective. If $(R \to R', f)$ is a glueing pair, then M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective. For example, if (R, f) is a glueing pair, then M is glueable if and only if $M[f^\infty] \to (M \otimes_R R^\wedge)[f^\infty]$ is injective.

Proof. We will use the results of Lemma 15.91.6 without further mention. The functor $M \otimes_R -$ is right exact (Algebra, Lemma 10.12.10) hence we get (1).

The kernel of $M \to M \otimes_R R_f = M_f$ is $M[f^\infty]$. Thus (2) follows.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in (M \otimes_R R')[f^\infty]$ are in the image of the first arrow. This implies that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Conversely, assume that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in M$. Then we see that $f^nx - y'$ is annihilated by some power of f in $M \otimes_R R'$. By assumption we can write $f^nx - y' = z$ for some $z \in M[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $M \to M/f^nM$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in M$. Then $x - y'''$ is in $(M \otimes_R R')[f^\infty]$. Thus $x - y''' = z' \in M[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

If $(R \to R', f)$ is a glueing pair, then (15.91.9.1) is exact in the middle for any M by Algebra, Lemma 10.12.10. This gives the penultimate statement of the lemma. The final statement of the lemma follows from this and the fact that (R, f) is a glueing pair if and only if $(R \to R^\wedge, f)$ is a glueing pair. □

0BNX **Remark 15.91.11.** Let $(R \to R', f)$ be a glueing pair and let M be an R-module. Here are some observations which can be used to determine whether M is glueable for $(R \to R', f)$.

(1) By Lemma 15.91.10 we see that M is glueable for $(R \to R^\wedge, f)$ if and only if $M[f^\infty] \to M \otimes_R R^\wedge$ is injective. This holds if $M[f] \to M^\wedge$ is injective, i.e., when $M[f] \cap \bigcap_{n=1}^\infty f^nM = 0$.
(2) If $\mathrm{Tor}_1^R(M, R'_f) = 0$, then M is glueable for $(R \to R', f)$ (use Algebra, Lemma 10.75.2). This is equivalent to saying that $\mathrm{Tor}_1^R(M, R')$ is f-power torsion. In particular, any flat R-module is glueable for $(R \to R', f)$.
(3) If $R \to R'$ is flat, then $\mathrm{Tor}_1^R(M, R') = 0$ for every R-module so every R-module is glueable for $(R \to R', f)$. This holds in particular when R is Noetherian and $R' = R^\wedge$, see Algebra, Lemma 10.97.2

0BNY **Example 15.91.12** (Non glueable module)**.** Let R be the ring of germs at 0 of C^∞ functions on $\mathbf{R}$. Let $f \in R$ be the function $f(x) = x$. Then f is a nonzerodivisor in R, so (R, f) is a glueing pair and $R^\wedge \cong \mathbf{R}[[x]]$. Let $\varphi \in R$ be the function $\varphi(x) = \exp(-1/x^2)$. Then φ has zero Taylor series, so $\varphi \in \mathrm{Ker}(R \to R^\wedge)$. Since $\varphi(x) \neq 0$ for $x \neq 0$, we see that φ is a nonzerodivisor in R. The function φ/f also [**BL95**, §4, Remarques]

has zero Taylor series, so its image in $M = R/\varphi R$ is a nonzero element of $M[f]$ which maps to zero in $M \otimes_R R^\wedge = R^\wedge/\varphi R^\wedge = R^\wedge$. Hence M is not glueable.

We next make some calculations of Tor groups.

0BNZ **Lemma 15.91.13.** *Let $(R \to R', f)$ be a glueing pair. Then $Tor_1^R(R', f^n R) = 0$ for each $n > 0$.*

Proof. From the exact sequence $0 \to R[f^n] \to R \to f^n R \to 0$ we see that it suffices to check that $R[f^n] \otimes_R R' \to R'$ is injective. By Lemma 15.89.9 we have $R[f^n] \otimes_R R' = R[f^n]$ and by Lemma 15.91.6 we see that $R[f^n] \to R'$ is injective as $(R \to R', f)$ is a glueing pair. □

0BP0 **Lemma 15.91.14.** *Let $(R \to R', f)$ be a glueing pair. Then $Tor_1^R(R', R/R[f^\infty]) = 0$.*

Proof. We have $R/R[f^\infty] = \operatorname{colim} R/R[f^n] = \operatorname{colim} f^n R$. As formation of Tor groups commutes with filtered colimits (Algebra, Lemma 10.76.2) we may apply Lemma 15.91.13. □

0BP1 **Lemma 15.91.15.** *Let $(R \to R', f)$ be a glueing pair. For every R-module M, we have $Tor_1^R(R', \operatorname{Coker}(M \to M_f)) = 0$.*

Slight generalization of [**BL95**, Lemme 3(a)]

Proof. Set $\overline{M} = M/M[f^\infty]$. Then $\operatorname{Coker}(M \to M_f) \cong \operatorname{Coker}(\overline{M} \to \overline{M}_f)$ hence we may and do assume that f is a nonzerodivisor on M. In this case $M \subset M_f$ and $M_f/M = \operatorname{colim} M/f^n M$ where the transition maps are given by multiplication by f. Since formation of Tor groups commutes with colimits (Algebra, Lemma 10.76.2) it suffices to show that $\operatorname{Tor}_1^R(R', M/f^n M) = 0$.

We first treat the case $M = R/R[f^\infty]$. By Lemma 15.91.6 we have $M \otimes_R R' = R'/R'[f^\infty]$. From the short exact sequence $0 \to M \to M \to M/f^n M \to 0$ we obtain the exact sequence

$$\begin{array}{ccccc} \operatorname{Tor}_1^R(R', R/R[f^\infty]) & \longrightarrow & \operatorname{Tor}_1^R(R', M/f^n M) & \longrightarrow & R'/R'[f^\infty] \\ & & & \overset{f^n}{\searrow} & \\ R'/R'[f^\infty] & \longrightarrow & (R'/R'[f^\infty])/(f^n(R'/R'[f^\infty])) & \longrightarrow & 0 \end{array}$$

by Algebra, Lemma 10.75.2. Here the diagonal arrow is injective. Since the first group $\operatorname{Tor}_1^R(R', R/R[f^\infty])$ is zero by Lemma 15.91.14, we deduce that $\operatorname{Tor}_1^R(R', M/f^n M) = 0$ as desired.

To treat the general case, choose a surjection $F \to M$ with F a free $R/R[f^\infty]$-module, and form an exact sequence

$$0 \to N \to F/f^n F \to M/f^n M \to 0.$$

By Lemma 15.89.9 this sequence remains unchanged, and hence exact, upon tensoring with R'. Since $\operatorname{Tor}_1^R(R', F/f^n F) = 0$ by the previous paragraph, we deduce that $\operatorname{Tor}_1^R(R', M/f^n M) = 0$ as desired. □

Let $(R \to R', f)$ be a glueing pair. This means that $R/f^n R \to R'/f^n R'$ is an isomorphism for $n > 0$ and the sequence

$$0 \to R \to R' \oplus R_f \to R'_f \to 0$$

is exact. Consider the category $\mathrm{Glue}(R \to R', f)$ introduced in Remark 15.90.10. We will call an object (M', M_1, α_1) of $\mathrm{Glue}(R \to R', f)$ a *glueing datum*. It consists of an R'-module M', an R_f-module M_1, and an isomorphism $\alpha_1 : (M')_f \to M_1 \otimes_R R'$. There is an obvious functor

$$\mathrm{Can} : \mathrm{Mod}_R \longrightarrow \mathrm{Glue}(R \to R', f), \quad M \longmapsto (M \otimes_R R', M_f, \mathrm{can}),$$

and there is a functor

$$H^0 : \mathrm{Glue}(R \to R', f) \longrightarrow \mathrm{Mod}_R, \quad (M', M_1, \alpha_1) \longmapsto \mathrm{Ker}(M' \oplus M_1 \to (M')_f)$$

in the reverse direction, see Remark 15.90.10 for the precise definition.

0BP2 **Theorem 15.91.16.** *Let $(R \to R', f)$ be a glueing pair. The functor $Can : Mod_R \longrightarrow Glue(R \to R', f)$ determines an equivalence of the category of R-modules glueable for $(R \to R', f)$ and the category $Glue(R \to R', f)$ of glueing data.* Slight generalization of the main theorem of [**BL95**].

Proof. Let (M', M_1, α_1) be a glueing datum. We will show that $M = H^0((M', M_1, \alpha_1))$ is a glueable module for $(R \to R', f)$ and that $(M', M_1, \alpha_1) \cong \mathrm{Can}(M)$.

We first check that the map $\mathrm{d} : M' \oplus M_1 \to (M')_f$ used in the definition of the functor H^0 is surjective. Observe that $(x, y) \in M' \oplus M_1$ maps to $\mathrm{d}(x, y) = x/1 - \alpha_1^{-1}(y \otimes 1)$ in $(M')_f$. If $z \in (M')_f$, then we can write $\alpha_1(z) = \sum y_i \otimes g_i$ with $g_i \in R'$ and $y_i \in M_1$. Write $\alpha_1^{-1}(y_i \otimes 1) = y_i'/f^n$ for some $y_i' \in M'$ and $n \geq 0$ (we can pick the same n for all i). Write $g_i = a_i + f^n b_i$ with $a_i \in R$ and $b_i \in R'$. Then with $y = \sum a_i y_i \in M_1$ and $x = \sum b_i y_i' \in M'$ we have $\mathrm{d}(x, -y) = z$ as desired.

Since $M = H^0((M', M_1, \alpha_1)) = \mathrm{Ker}(\mathrm{d})$ we obtain an exact sequence of R-modules

0BP3 (15.91.16.1) $$0 \to M \to M' \oplus M_1 \to (M')_f \to 0.$$

We will prove that the maps $M \to M'$ and $M \to M_1$ induce isomorphisms $M \otimes_R R' \to M'$ and $M \otimes_R R_f \to M_1$. This will imply that M is glueable for $(R \to R', f)$ and $\mathrm{Can}(M) \cong (M', M_1, \alpha_1)$ as desired.

Since f is a nonzerodivisor on M_1, we have $M[f^\infty] \cong M'[f^\infty]$. This yields an exact sequence

0BP4 (15.91.16.2) $$0 \to M/M[f^\infty] \to M_1 \to (M')_f/M' \to 0.$$

Since $R \to R_f$ is flat, we may tensor this exact sequence with R_f to deduce that $M \otimes_R R_f = (M/M[f^\infty]) \otimes_R R_f \to M_1$ is an isomorphism.

By Lemma 15.91.15 we have $\mathrm{Tor}_1^R(R', \mathrm{Coker}(M' \to (M')_f)) = 0$. The sequence (15.91.16.2) thus remains exact upon tensoring over R with R'. Using α_1 and Lemma 15.89.9 the resulting exact sequence can be written as

0BP5 (15.91.16.3) $$0 \to (M/M[f^\infty]) \otimes_R R' \to (M')_f \to (M')_f/M' \to 0$$

This yields an isomorphism $(M/M[f^\infty]) \otimes_R R' \cong M'/M'[f^\infty]$. This implies that in the diagram

$$\begin{array}{ccccccccc} & & M[f^\infty] \otimes_R R' & \longrightarrow & M \otimes_R R' & \longrightarrow & (M/M[f^\infty]) \otimes_R R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'[f^\infty] & \longrightarrow & M' & \longrightarrow & M'/M'[f^\infty] & \longrightarrow & 0, \end{array}$$

the third vertical arrow is an isomorphism. Since the rows are exact and the first vertical arrow is an isomorphism by Lemma 15.89.9 and $M[f^\infty] = M'[f^\infty]$, the five lemma implies that $M \otimes_R R' \to M'$ is an isomorphism.

The above shows that Can is essentially surjective and that the functor H^0 maps into the category of glueable modules. Due to the exactness of (15.91.9.1) for glueable modules we have $H^0 \circ \text{Can} = \text{id}$ on the category of glueable modules. This implies Can is fully faithful by Lemma 15.90.11 combined with Categories, Lemma 4.24.4. This finishes the proof. □

0BP9 **Remark 15.91.17.** Let $(R \to R', f)$ be a glueing pair. Let M be an R-module that is not necessarily glueable for $(R \to R', f)$. Setting $M' = M \otimes_R R'$ and $M_1 = M_f$ we obtain the glueing datum $\text{Can}(M) = (M', M_1, \text{can})$. Then $\tilde{M} = H^0(M', M_1, \text{can})$ is an R-module that is glueable for $(R \to R', f)$ and the canonical map $M \to \tilde{M}$ gives isomorphisms $M \otimes_R R' \to \tilde{M} \otimes_R R'$ and $M_f \to \tilde{M}_f$, see Theorem 15.91.16. From the exactness of the sequences

$$M \to (M \otimes_R R') \oplus M_f \to M \otimes_R (R')_f \to 0$$

and

$$0 \to \tilde{M} \to (\tilde{M} \otimes_R R') \oplus \tilde{M}_f \to \tilde{M} \otimes_R (R')_f \to 0$$

we conclude that the map $M \to \tilde{M}$ is surjective.

Recall that flat R-modules over a glueing pair $(R \to R', f)$ are glueable (Remark 15.91.11). Hence the following lemma shows that Theorem 15.91.16 determines an equivalence between the category of flat R-modules and the category of glueing data (M', M_1, α_1) where M' and M_1 are flat over R' and R_f.

0BP7 **Lemma 15.91.18.** *Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is flat over R if and only if $M \otimes_R R'$ is flat over R' and M_f is flat over R_f.*

Proof. One direction of the lemma follows from Algebra, Lemma 10.39.7. For the other direction, assume $M \otimes_R R'$ is flat over R' and M_f is flat over R_f. Let $\tilde{M}$ be as in Remark 15.91.17. If $\tilde{M}$ is flat over R, then applying Algebra, Lemma 10.39.12 to the short exact sequence $0 \to \text{Ker}(M \to \tilde{M}) \to M \to \tilde{M} \to 0$ we find that $\text{Ker}(M \to \tilde{M}) \otimes_R (R' \oplus R_f)$ is zero. Hence $M = \tilde{M}$ by Lemma 15.91.2 and we conclude. In other words, we may replace M by $\tilde{M}$ and assume M is glueable for $(R \to R', f)$. Let N be a second R-module. It suffices to prove that $\text{Tor}_1^R(M, N) = 0$, see Algebra, Lemma 10.75.8.

The long exact sequence of Tors associated to the short exact sequence $0 \to R \to R' \oplus R_f \to (R')_f \to 0$ and N gives an exact sequence

$$0 \to \text{Tor}_1^R(R', N) \to \text{Tor}_1^R((R')_f, N)$$

and isomorphisms $\text{Tor}_i^R(R', N) = \text{Tor}_i^R((R')_f, N)$ for $i \geq 2$. Since $\text{Tor}_i^R((R')_f, N) = \text{Tor}_i^R(R', N)_f$ we conclude that f is a nonzerodivisor on $\text{Tor}_1^R(R', N)$ and invertible on $\text{Tor}_i^R(R', N)$ for $i \geq 2$. Since $M \otimes_R R'$ is flat over R' we have

$$\text{Tor}_i^R(M \otimes_R R', N) = (M \otimes_R R') \otimes_{R'} \text{Tor}_i^R(R', N)$$

by the spectral sequence of Example 15.62.2. Writing $M \otimes_R R'$ as a filtered colimit of finite free R'-modules (Algebra, Theorem 10.81.4) we conclude that f is a nonzerodivisor on $\text{Tor}_1^R(M \otimes_R R', N)$ and invertible on $\text{Tor}_i^R(M \otimes_R R', N)$. Next,

we consider the exact sequence $0 \to M \to M \otimes_R R' \oplus M_f \to M \otimes_R (R')_f \to 0$ coming from the fact that M is glueable and the associated long exact sequence of Tor. The relevant part is

$$\begin{array}{ccccc} \mathrm{Tor}_1^R(M,N) & \longrightarrow & \mathrm{Tor}_1^R(M \otimes_R R', N) & \longrightarrow & \mathrm{Tor}_1^R(M \otimes_R (R')_f, N) \\ & \searrow & & & \\ & & \mathrm{Tor}_2^R(M \otimes_R R', N) & \longrightarrow & \mathrm{Tor}_2^R(M \otimes_R (R')_f, N) \end{array}$$

We conclude that $\mathrm{Tor}_1^R(M,N) = 0$ by our remarks above on the action on f on $\mathrm{Tor}_i^R(M \otimes_R R', N)$. □

Observe that we have seen the result of the following lemma for "finitely generated" in Lemma 15.91.4.

0BP6 **Lemma 15.91.19.** *Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is a finite projective R-module if and only if $M \otimes_R R'$ is finite projective over R' and M_f is finite projective over R_f.*

Proof. Assume that $M \otimes_R R'$ is a finite projective module over R' and that M_f is a finite projective module over R_f. Our task is to prove that M is finite projective over R. We will use Algebra, Lemma 10.78.2 without further mention. By Lemma 15.91.18 we see that M is flat. By Lemma 15.91.4 we see that M is finite. Choose a short exact sequence $0 \to K \to R^{\oplus n} \to M \to 0$. Since a finite projective module is of finite presentation and since the sequence remains exact after tensoring with R' (by Algebra, Lemma 10.39.12) and R_f, we conclude that $K \otimes_R R'$ and K_f are finite modules. Using the lemma above we conclude that K is finitely generated. Hence M is finitely presented and hence finite projective. □

0BP8 **Remark 15.91.20.** In [**BL95**] it is assumed that f is a nonzerodivisor in R and $R' = R^\wedge$, which gives a glueing pair by Lemma 15.91.6. Even in this setting Theorem 15.91.16 says something new: the results of [**BL95**] only apply to modules on which f is a nonzerodivisor (and hence glueable in our sense, see Lemma 15.91.10). Lemma 15.91.19 also provides a slight extension of the results of [**BL95**]: not only can we allow M to have nonzero f-power torsion, we do not even require it to be glueable.

15.92. Derived Completion

091N Some references for the material in this section are [**DG02**], [**GM92**], [**PSY14a**], [**Lur11**] (especially Chapter 4). Our exposition follows [**BS13**]. The analogue (or "dual") of this section for torsion modules is Dualizing Complexes, Section 47.9. The relationship between the derived category of complexes with torsion cohomology and derived complete complexes can be found in Dualizing Complexes, Section 47.12.

Let $K \in D(A)$. Let $f \in A$. We denote $T(K, f)$ a derived limit of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(A)$.

091P **Lemma 15.92.1.** *Let A be a ring. Let $f \in A$. Let $K \in D(A)$. The following are equivalent*

(1) $\mathrm{Ext}^n_A(A_f, K) = 0$ *for all* n,
(2) $\mathrm{Hom}_{D(A)}(E, K) = 0$ *for all* E *in* $D(A_f)$,
(3) $T(K, f) = 0$,
(4) *for every* $p \in \mathbf{Z}$ *we have* $T(H^p(K), f) = 0$,
(5) *for every* $p \in \mathbf{Z}$ *we have* $\mathrm{Hom}_A(A_f, H^p(K)) = 0$ *and* $\mathrm{Ext}^1_A(A_f, H^p(K)) = 0$,
(6) $R\mathrm{Hom}_A(A_f, K) = 0$,
(7) *the map* $\prod_{n\geq 0} K \to \prod_{n\geq 0} K$, $(x_0, x_1, \ldots) \mapsto (x_0 - fx_1, x_1 - fx_2, \ldots)$ *is an isomorphism in* $D(A)$, *and*
(8) *add more here.*

Proof. It is clear that (2) implies (1) and that (1) is equivalent to (6). Assume (1). Let $I^\bullet$ be a K-injective complex of A-modules representing K. Condition (1) signifies that $\mathrm{Hom}_A(A_f, I^\bullet)$ is acyclic. Let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\mathrm{Hom}_{D(A)}(E, K) = \mathrm{Hom}_{K(A)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(A_f)}(M^\bullet, \mathrm{Hom}_A(A_f, I^\bullet))$$

by Algebra, Lemma 10.14.4. As $\mathrm{Hom}_A(A_f, I^\bullet)$ is a K-injective complex of A_f-modules by Lemma 15.56.3 the fact that it is acyclic implies that it is homotopy equivalent to zero (Derived Categories, Lemma 13.31.2). Thus we get (2).

A free resolution of the A-module A_f is given by

$$0 \to \bigoplus\nolimits_{n\in\mathbf{N}} A \to \bigoplus\nolimits_{n\in\mathbf{N}} A \to A_f \to 0$$

where the first map sends the $(a_0, a_1, a_2, \ldots)$ to $(a_0, a_1 - fa_0, a_2 - fa_1, \ldots)$ and the second map sends $(a_0, a_1, a_2, \ldots)$ to $a_0 + a_1/f + a_2/f^2 + \ldots$. Applying $\mathrm{Hom}_A(-, I^\bullet)$ we get

$$0 \to \mathrm{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

Since $\prod I^\bullet$ represents $\prod_{n\geq 0} K$ this proves the equivalence of (1) and (7). On the other hand, by construction of derived limits in Derived Categories, Section 13.34 the displayed exact sequence shows the object $T(K, f)$ is a representative of $R\mathrm{Hom}_A(A_f, K)$ in $D(A)$. Thus the equivalence of (1) and (3).

There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_A(A_f, H^q(K)) \Rightarrow \mathrm{Ext}^{p+q}_A(A_f, K)$$

See Equation (15.68.0.1)[12]. This spectral sequence degenerates at E_2 because A_f has a length 1 resolution by projective A-modules (see above) hence the E_2-page has only 2 nonzero columns. Thus we obtain short exact sequences

$$0 \to \mathrm{Ext}^1_A(A_f, H^{p-1}(K)) \to \mathrm{Ext}^p_A(A_f, K) \to \mathrm{Hom}_A(A_f, H^p(K)) \to 0$$

This proves (4) and (5) are equivalent to (1). □

[12] If K is not bounded below, we can't use the given reference. In that case, we chooses a complex $K^\bullet$ representing K and a free two-term resolution $0 \to F^{-1} \to F^0 \to A_f \to 0$ (see above). Then we get a double complex with terms $\mathrm{Hom}_A(F^{-q}, K^p)$ whose totalization computes $R\mathrm{Hom}(A_f, K)$. The second spectral sequence associated to this double complex (Homology, Section 12.25) gives us what we want. Details omitted.

091Q **Lemma 15.92.2.** *Let A be a ring. Let $K \in D(A)$. The set I of $f \in A$ such that $T(K, f) = 0$ is a radical ideal of A.*

Proof. We will use the results of Lemma 15.92.1 without further mention. If $f \in I$, and $g \in A$, then A_{gf} is an A_f-module hence $\mathrm{Ext}^n_A(A_{gf}, K) = 0$ for all n, hence $gf \in I$. Suppose $f, g \in I$. Then there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{gf(f+g)} \to 0$$

because f, g generate the unit ideal in A_{f+g}. This follows from Algebra, Lemma 10.24.2 and the easy fact that the last arrow is surjective. From the long exact sequence of Ext and the vanishing of $\mathrm{Ext}^n_A(A_{f(f+g)}, K)$, $\mathrm{Ext}^n_A(A_{g(f+g)}, K)$, and $\mathrm{Ext}^n_A(A_{gf(f+g)}, K)$ for all n we deduce the vanishing of $\mathrm{Ext}^n_A(A_{f+g}, K)$ for all n. Finally, if $f^n \in I$ for some $n > 0$, then $f \in I$ because $T(K, f) = T(K, f^n)$ or because $A_f \cong A_{f^n}$. □

091R **Lemma 15.92.3.** *Let A be a ring. Let $I \subset A$ be an ideal. Let M be an A-module.*

(1) *If M is I-adically complete, then $T(M, f) = 0$ for all $f \in I$.*
(2) *Conversely, if $T(M, f) = 0$ for all $f \in I$ and I is finitely generated, then $M \to \lim M/I^n M$ is surjective.*

Proof. Proof of (1). Assume M is I-adically complete. By Lemma 15.92.1 it suffices to prove $\mathrm{Ext}^1_A(A_f, M) = 0$ and $\mathrm{Hom}_A(A_f, M) = 0$. Since $M = \lim M/I^n M$ and since $\mathrm{Hom}_A(A_f, M/I^n M) = 0$ it follows that $\mathrm{Hom}_A(A_f, M) = 0$. Suppose we have an extension

$$0 \to M \to E \to A_f \to 0$$

For $n \geq 0$ pick $e_n \in E$ mapping to $1/f^n$. Set $\delta_n = fe_{n+1} - e_n \in M$ for $n \geq 0$. Replace e_n by

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots$$

The infinite sum exists as M is complete with respect to I and $f \in I$. A simple calculation shows that $fe'_{n+1} = e'_n$. Thus we get a splitting of the extension by mapping $1/f^n$ to e'_n.

Proof of (2). Assume that $I = (f_1, \ldots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \ldots, r$. By Algebra, Lemma 10.96.7 we may assume $I = (f)$ and $T(M, f) = 0$. Let $x_n \in M$ for $n \geq 0$. Consider the extension

$$0 \to M \to E \to A_f \to 0$$

given by

$$E = M \oplus \bigoplus Ae_n \Big/ \langle x_n - fe_{n+1} + e_n \rangle$$

mapping e_n to $1/f^n$ in A_f (see above). By assumption and Lemma 15.92.1 this extension is split, hence we obtain an element $x + e_0$ which generates a copy of A_f in E. Then

$$x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2 e_2 = \ldots$$

Since $M/f^n M = E/f^n E$ by the snake lemma, we see that $x = x_0 + fx_1 + \ldots + f^{n-1}x_{n-1}$ modulo $f^n M$. In other words, the map $M \to \lim M/f^n M$ is surjective as desired. □

Motivated by the results above we make the following definition.

091S **Definition 15.92.4.** Let A be a ring. Let $K \in D(A)$. Let $I \subset A$ be an ideal. We say K is *derived complete with respect to* I if for every $f \in I$ we have $T(K, f) = 0$. If M is an A-module, then we say M is *derived complete with respect to* I if $M[0] \in D(A)$ is derived complete with respect to I.

The full subcategory $D_{comp}(A) = D_{comp}(A, I) \subset D(A)$ consisting of derived complete objects is a strictly full, saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. By Lemma 15.92.2 the subcategory $D_{comp}(A, I)$ depends only on the radical $\sqrt{I}$ of I, in other words it depends only on the closed subset $Z = V(I)$ of $\mathrm{Spec}(A)$. The subcategory $D_{comp}(A, I)$ is preserved under products and homotopy limits in $D(A)$. But it is not preserved under countable direct sums in general. We will often simply say M is a derived complete module if the choice of the ideal I is clear from the context.

091T **Proposition 15.92.5.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Let M be an A-module. The following are equivalent*

(1) *M is I-adically complete, and*
(2) *M is derived complete with respect to I and $\bigcap I^n M = 0$.*

Proof. This is clear from the results of Lemma 15.92.3. □

The next lemma shows that the category $\mathcal{C}$ of derived complete modules is abelian. It turns out that $\mathcal{C}$ is not a Grothendieck abelian category, see Examples, Section 110.11.

091U **Lemma 15.92.6.** *Let I be an ideal of a ring A.*

(1) *The derived complete A-modules form a weak Serre subcategory $\mathcal{C}$ of Mod_A.*
(2) *$D_{\mathcal{C}}(A) \subset D(A)$ is the full subcategory of derived complete objects.*

Proof. Part (2) is immediate from Lemma 15.92.1 and the definitions. For part (1), suppose that $M \to N$ is a map of derived complete modules. Denote $K = (M \to N)$ the corresponding object of $D(A)$. Pick $f \in I$. Then $\mathrm{Ext}^n_A(A_f, K)$ is zero for all n because $\mathrm{Ext}^n_A(A_f, M)$ and $\mathrm{Ext}^n_A(A_f, N)$ are zero for all n. Hence K is derived complete. By (2) we see that $\mathrm{Ker}(M \to N)$ and $\mathrm{Coker}(M \to N)$ are objects of $\mathcal{C}$. Finally, suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules and M_1, M_3 are derived complete. Then it follows from the long exact sequence of Ext's that M_2 is derived complete. Thus $\mathcal{C}$ is a weak Serre subcategory by Homology, Lemma 12.10.3. □

We will generalize the following lemma in Lemma 15.92.20.

09B9 **Lemma 15.92.7.** *Let I be a finitely generated ideal of a ring A. Let M be a derived complete A-module. If $M/IM = 0$, then $M = 0$.*

Proof. Assume that M/IM is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $N = M/(f_1, \ldots, f_i)M$ is nonzero. If i does not exist, then $M = 0$ which is what we want to show. Then N is derived complete as a cokernel of a map between derived complete modules, see Lemma 15.92.6. By our choice of i we have that $f_{i+1} : N \to N$ is surjective. Hence

$$\lim(\ldots \to N \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N)$$

is nonzero, contradicting the derived completeness of N. □

If the ring is I-adically complete, then one obtains an ample supply of derived complete complexes.

0A05 **Lemma 15.92.8.** *Let A be a ring and $I \subset A$ an ideal. If A is derived complete (eg. I-adically complete) then any pseudo-coherent object of $D(A)$ is derived complete.*

Proof. (Lemma 15.92.3 explains the parenthetical statement of the lemma.) Let K be a pseudo-coherent object of $D(A)$. By definition this means K is represented by a bounded above complex $K^\bullet$ of finite free A-modules. Since A is derived complete it follows that $H^n(K)$ is derived complete for all n, by part (1) of Lemma 15.92.6. This in turn implies that K is derived complete by part (2) of the same lemma. □

0A6C **Lemma 15.92.9.** *Let A be a ring. Let $f, g \in A$. Then for $K \in D(A)$ we have $R\mathrm{Hom}_A(A_f, R\mathrm{Hom}_A(A_g, K)) = R\mathrm{Hom}_A(A_{fg}, K)$.*

Proof. This follows from Lemma 15.74.1. □

091V **Lemma 15.92.10.** *Let I be a finitely generated ideal of a ring A. The inclusion functor $D_{comp}(A, I) \to D(A)$ has a left adjoint, i.e., given any object K of $D(A)$ there exists a map $K \to K^\wedge$ of K into a derived complete object of $D(A)$ such that the map*

$$\mathrm{Hom}_{D(A)}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(A)}(K, E)$$

is bijective whenever E is a derived complete object of $D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have

$$K^\wedge = R\mathrm{Hom}\left((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}), K\right)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \longrightarrow A$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete[13].

Let $f \in A$. By Lemma 15.92.9 the object $R\mathrm{Hom}_A(A_f, K^\wedge)$ is equal to

$$R\mathrm{Hom}\left((A_f \to \prod\nolimits_{i_0} A_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{ff_{i_0}f_{i_1}} \to \ldots \to A_{ff_1\ldots f_r}), K\right)$$

If $f \in I$, then $f_1, \ldots, f_r$ generate the unit ideal in A_f, hence the extended alternating Čech complex

$$A_f \to \prod\nolimits_{i_0} A_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{ff_{i_0}f_{i_1}} \to \ldots \to A_{ff_1\ldots f_r}$$

[13]Namely, if E is derived complete and $a : K \to E$ is a map, then the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{a} & E \\ \downarrow & & \downarrow \cong \\ K^\wedge & \xrightarrow{a^\wedge} & E^\wedge \end{array}$$

shows that any map into E factors through $K^\wedge$. Choose a distinguished triangle $K \to K^\wedge \to C$ and apply $^\wedge$. We find that $C^\wedge = 0$. By the above this means that $\mathrm{Hom}(C, E) = 0$ for E derived complete and this proves that $\mathrm{Hom}(K, E) = \mathrm{Hom}(K^\wedge, E)$ as desired.

is zero in $D(A)$ by Lemma 15.29.5. (In fact, if $f = f_i$ for some i, then this complex is homotopic to zero by Lemma 15.29.4; this is the only case we need.) Hence $R\mathrm{Hom}_A(A_f, K^\wedge) = 0$ and we conclude that $K^\wedge$ is derived complete by Lemma 15.92.1.

Conversely, if K is derived complete, then $R\mathrm{Hom}_A(A_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(A)$. □

0G3E **Remark 15.92.11.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. The left adjoint to the inclusion functor $D_{comp}(A, I) \to D(A)$ which exists by Lemma 15.92.10 is called the *derived completion*. To indicate this we will say "let $K^\wedge$ be the derived completion of K". Please keep in mind that the unit of the adjunction is a functorial map $K \to K^\wedge$.

0A6D **Lemma 15.92.12.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then the derived completion of $K^\bullet$ is zero.*

Proof. Indeed, in this case the $R\mathrm{Hom}_A(K, L)$ is zero for any derived complete complex L, see Lemma 15.92.1. Hence $K^\wedge$ is zero by the universal property in Lemma 15.92.10. □

0A6E **Lemma 15.92.13.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K, L \in D(A)$. Then*

$$R\mathrm{Hom}_A(K, L)^\wedge = R\mathrm{Hom}_A(K, L^\wedge) = R\mathrm{Hom}_A(K^\wedge, L^\wedge)$$

Proof. By Lemma 15.92.10 we know that derived completion is given by $R\mathrm{Hom}_A(C, -)$ for some $C \in D(A)$. Then

$$\begin{aligned} R\mathrm{Hom}_A(C, R\mathrm{Hom}_A(K, L)) &= R\mathrm{Hom}_A(C \otimes^{\mathbf{L}}_A K, L) \\ &= R\mathrm{Hom}_A(K, R\mathrm{Hom}_A(C, L)) \end{aligned}$$

by Lemma 15.74.1. This proves the first equation. The map $K \to K^\wedge$ induces a map

$$R\mathrm{Hom}_A(K^\wedge, L^\wedge) \to R\mathrm{Hom}_A(K, L^\wedge)$$

which is an isomorphism in $D(A)$ by definition of the derived completion as the left adjoint to the inclusion functor. □

091W **Lemma 15.92.14.** *Let A be a ring and let $I \subset A$ be an ideal. Let (K_n) be an inverse system of objects of $D(A)$ such that for all $f \in I$ and n there exists an $e = e(n, f)$ such that f^e is zero on K_n. Then for $K \in D(A)$ the object $K' = R\lim(K \otimes^{\mathbf{L}}_A K_n)$ is derived complete with respect to I.*

Proof. Since the category of derived complete objects is preserved under $R\lim$ it suffices to show that each $K \otimes^{\mathbf{L}}_A K_n$ is derived complete. By assumption for all $f \in I$ there is an e such that f^e is zero on $K \otimes^{\mathbf{L}}_A K_n$. Of course this implies that $T(K \otimes^{\mathbf{L}}_A K_n, f) = 0$ and we win. □

0BKC **Situation 15.92.15.** Let A be a ring. Let $I = (f_1, \ldots, f_r) \subset A$. Let $K^\bullet_n = K_\bullet(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a cochain complex in degrees $-r, -r+1, \ldots, 0$. Using the functoriality of Lemma 15.28.3 we obtain an inverse system

$$\ldots \to K^\bullet_3 \to K^\bullet_2 \to K^\bullet_1$$

compatible with the inverse system $H^0(K_n^\bullet) = A/(f_1^n, \ldots, f_r^n)$ and compatible with the maps $A \to K_n^\bullet$.

A key feature of the discussion below will use that for $m > n$ the map

$$K_m^{-p} = \wedge^p(A^{\oplus r}) \to \wedge^p(A^{\oplus r}) = K_n^{-p}$$

is given by multiplication by $f_{i_1}^{m-n} \ldots f_{i_p}^{m-n}$ on the basis element $e_{i_1} \wedge \ldots \wedge e_{i_p}$.

091Y **Lemma 15.92.16.** *In Situation 15.92.15. For $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is derived complete with respect to I.*

Proof. This is a special case of Lemma 15.92.14 because f_i^n acts by an endomorphism of $K_n^\bullet$ which is homotopic to zero by Lemma 15.28.6. □

091Z **Lemma 15.92.17.** *In Situation 15.92.15. Let $K \in D(A)$. The following are equivalent*

(1) *K is derived complete with respect to I, and*
(2) *the canonical map $K \to R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is an isomorphism of $D(A)$.*

Proof. If (2) holds, then K is derived complete with respect to I by Lemma 15.92.16. Conversely, assume that K is derived complete with respect to I. Consider the filtrations

$$K_n^\bullet \supset \sigma_{\geq -r+1}K_n^\bullet \supset \sigma_{\geq -r+2}K_n^\bullet \supset \ldots \supset \sigma_{\geq -1}K_n^\bullet \supset \sigma_{\geq 0}K_n^\bullet = A$$

by stupid truncations (Homology, Section 12.15). Because the construction $R\lim(K \otimes E)$ is exact in the second variable (Lemma 15.88.11) we see that it suffices to show

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p}K_n^\bullet/\sigma_{\geq p+1}K_n^\bullet)\right) = 0$$

for $p < 0$. The explicit description of the Koszul complexes above shows that

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p}K_n^\bullet/\sigma_{\geq p+1}K_n^\bullet)\right) = \bigoplus\nolimits_{i_1,\ldots,i_{-p}} T(K, f_{i_1} \ldots f_{i_{-p}})$$

which is zero for $p < 0$ by assumption on K. □

0920 **Lemma 15.92.18.** *In Situation 15.92.15. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.92.10.*

First proof. The assignment $K \rightsquigarrow K'$ is a functor and K' is derived complete with respect to I by Lemma 15.92.16. By a formal argument (omitted) we see that it suffices to show $K \to K'$ is an isomorphism if K is derived complete with respect to I. This is Lemma 15.92.17. □

Second proof. Denote $K \mapsto K^\wedge$ the adjoint constructed in Lemma 15.92.10. By that lemma we have

$$K^\wedge = R\mathrm{Hom}\left((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}), K\right)$$

In Lemma 15.29.6 we have seen that the extended alternating Čech complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(A, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free A-modules with dual

(as in Lemma 15.75.15) $R\mathrm{Hom}_A(K^n, A) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). Thus it suffices to show that

$$R\mathrm{Hom}_A(\mathrm{hocolim} K^n, K) = R\lim(K \otimes^{\mathbf{L}}_A K_n)$$

This follows from Lemma 15.75.17. □

0H83 **Lemma 15.92.19.** *Let $I = (f_1, \ldots, f_r)$ be a finitely generated ideal of a ring A. Let K be a derived complete object of $D(A)$. The following are equivalent*

(1) *$H^i(K) = 0$ for $i > 0$,*
(2) *$H^i(K \otimes^{\mathbf{L}}_A A/I) = 0$ for $i > 0$,*
(3) *$H^i(K \otimes^{\mathbf{L}}_A K_1^\bullet) = 0$ for $i > 0$ where $K_1^\bullet$ is as in Situation 15.92.15.*

Proof. The implication (1) ⇒ (2) is always true. The implication (2) ⇒ (3) follows from Lemma 15.89.7. Assume (3). For $s = 0, \ldots, r$ consider the complex

$$K(s) = K \otimes^{\mathbf{L}}_A K_\bullet(A, f_1, \ldots, f_s)$$

where $K_\bullet(A, f_1, \ldots, f_s)$ is the Koszul complex placed in cohomological degrees $-s, \ldots, 0$. We have a distinguished triangles

$$K(s-1) \xrightarrow{f_s} K(s-1) \to K(s) \to K(s-1)[1]$$

by Lemma 15.28.8. Since $K(0) = K$ is derived complete, it follows by ascending induction that each $K(s)$ is a derived complete object of $D(A)$. By descending induction we'll show that $K(s)$ has no nonzero cohomology in degrees > 0. Namely, this holds for $s = r$ by our assumption (3). If it holds for $K(s)$ and $s > 0$, then $f_s : H^i(K(s-1)) \to H^i(K(s-1))$ is surjective for $i > 0$ and we conclude that $H^i(K(s-1))$ vanishes from Lemmas 15.92.6 and 15.92.7. □

0G1U **Lemma 15.92.20.** *Let I be a finitely generated ideal of a ring A. Let K be a derived complete object of $D(A)$. If $K \otimes^{\mathbf{L}}_A A/I = 0$, then $K = 0$.*

Proof. This is a special case of Lemma 15.92.19 but we'll also give a direct proof. Choose generators $f_1, \ldots, f_r$ of I. Denote K_n the Koszul complex on $f_1^n, \ldots, f_r^n$ over A. Recall that K_n is bounded and that the cohomology modules of K_n are annihilated by $f_1^n, \ldots, f_r^n$ and hence by I^{nr}. By Lemma 15.89.8 we see that $K \otimes^{\mathbf{L}}_A K_n = 0$. Since K is derived complete by Lemma 15.92.18 we have $K = R\lim K \otimes^{\mathbf{L}}_A K_n = 0$ as desired. □

A related result is [**DG02**, Proposition 6.5]. The derived Nakayama lemma can for example be found in Bhatt's 3rd lecture on Prismatic cohomology at Columbia University in Fall 2018 as Section 2 property (2). Leonid Positselski proposed a proof in `https://mathoverflow.net/a/331501`. However, we follow the proof suggested by Anonymous in the comments.

As an application of the relationship with the Koszul complex we obtain that derived completion has finite cohomological dimension.

0AAJ **Lemma 15.92.21.** *Let A be a ring and let $I \subset A$ be an ideal which can be generated by r elements. Then derived completion has finite cohomological dimension:*

(1) *Let $K \to L$ be a morphism in $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$.*
(2) *Let $K \to L$ be a morphism of $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \leq -1$ and injective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \leq -r - 1$ and injective for $i = -r$.*

Proof. Say I is generated by $f_1, \ldots, f_r$. For any $K \in D(A)$ by Lemma 15.92.18 we have $K^\wedge = R\lim K \otimes^{\mathbf{L}}_A K_n$ where K_n is the Koszul complex on $f_1^n, \ldots, f_r^n$ and hence we obtain a short exact sequence

$$0 \to R^1 \lim H^{i-1}(K \otimes^{\mathbf{L}}_A K_n) \to H^i(K^\wedge) \to \lim H^i(K \otimes^{\mathbf{L}}_A K_n) \to 0$$

by Lemma 15.88.4.

Proof of (1). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i \geq 0$. Since K_n is sitting in degrees ≤ 0 we see that $H^i(C \otimes^{\mathbf{L}}_A K_n) = 0$ for $i \geq 0$ and that $H^{-1}(C \otimes^{\mathbf{L}}_A K_n) = H^{-1}(C) \otimes_A A/(f_1^n, \ldots, f_r^n)$ is a system with surjective transition maps. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i \geq 0$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (1).

Proof of (2). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i < 0$. Since K_n is sitting in degrees $-r, \ldots, 0$ we see that $H^i(C \otimes^{\mathbf{L}}_A K_n) = 0$ for $i < -r$. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i < r$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (2). □

0BKD **Lemma 15.92.22.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a filtered complex of A-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded derived complete A-modules with d_r of bidegree $(r, -r+1)$ and with*

$$E_1^{p,q} = H^{p+q}((gr^p K^\bullet)^\wedge)$$

If the filtration on each K^n is finite, then the spectral sequence is bounded and converges to $H^*((K^\bullet)^\wedge)$.

Proof. By Lemma 15.92.10 we know that derived completion is given by $R\mathrm{Hom}_A(C, -)$ for some $C \in D^b(A)$. By Lemmas 15.92.21 and 15.69.2 we see that C has finite projective dimension. Thus we may choose a bounded complex of projective modules $P^\bullet$ representing C. Then

$$M^\bullet = \mathrm{Hom}^\bullet(P^\bullet, K^\bullet)$$

is a complex of A-modules representing $(K^\bullet)^\wedge$. It comes with a filtration given by $F^p M^\bullet = \mathrm{Hom}^\bullet(P^\bullet, F^p K^\bullet)$. We see that $F^p M^\bullet$ represents $(F^p K^\bullet)^\wedge$ and hence $\mathrm{gr}^p M^\bullet$ represents $(\mathrm{gr} K^\bullet)^\wedge$. Thus we find our spectral sequence by taking the spectral sequence of the filtered complex $M^\bullet$, see Homology, Section 12.24. If the filtration on each K^n is finite, then the filtration on each M^n is finite because $P^\bullet$ is a bounded complex. Hence the final statement follows from Homology, Lemma 12.24.11. □

0BKE **Example 15.92.23.** Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules. We can apply Lemma 15.92.22 with $F^p K^\bullet = \tau_{\leq -p} K^\bullet$. Then we get a bounded spectral sequence

$$E_1^{p,q} = H^{p+q}(H^{-p}(K^\bullet)^\wedge[p]) = H^{2p+q}(H^{-p}(K^\bullet)^\wedge)$$

converging to $H^{p+q}((K^\bullet)^\wedge)$. After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(A)$ there is a bounded spectral sequence $(E'_r, d'_r)_{r \geq 2}$ of bigraded derived complete modules with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = H^i(H^j(K)^\wedge)$$

and converging to $H^{i+j}(K^\wedge)$.

0924 **Lemma 15.92.24.** *Let $A \to B$ be a ring map. Let $I \subset A$ be an ideal. The inverse image of $D_{comp}(A, I)$ under the restriction functor $D(B) \to D(A)$ is $D_{comp}(B, IB)$.*

Proof. Using Lemma 15.92.2 we see that $L \in D(B)$ is in $D_{comp}(B, IB)$ if and only if $T(L, f)$ is zero for every local section $f \in I$. Observe that the cohomology of $T(L, f)$ is computed in the category of abelian groups, so it doesn't matter whether we think of f as an element of A or take the image of f in B. The lemma follows immediately from this and the definition of derived complete objects. □

0925 **Lemma 15.92.25.** *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. If $A \to B$ is flat and $A/I \cong B/IB$, then the restriction functor $D(B) \to D(A)$ induces an equivalence $D_{comp}(B, IB) \to D_{comp}(A, I)$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote $\check{\mathcal{C}}^\bullet_A \to \check{\mathcal{C}}^\bullet_B$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.90.4. Let $K \in D_{comp}(A, I)$. Let $I^\bullet$ be a K-injective complex of A-modules representing K. Since $\mathrm{Ext}^n_A(A_f, K)$ and $\mathrm{Ext}^n_A(B_f, K)$ are zero for all $f \in I$ and $n \in \mathbf{Z}$ (Lemma 15.92.1) we conclude that $\check{\mathcal{C}}^\bullet_A \to A$ and $\check{\mathcal{C}}^\bullet_B \to B$ induce quasi-isomorphisms

$$I^\bullet = \mathrm{Hom}_A(A, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}^\bullet_A, I^\bullet))$$

and

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}^\bullet_B, I^\bullet))$$

Some details omitted. Since $\check{\mathcal{C}}^\bullet_A \to \check{\mathcal{C}}^\bullet_B$ is a quasi-isomorphism and $I^\bullet$ is K-injective we conclude that $\mathrm{Hom}_A(B, I^\bullet) \to I^\bullet$ is a quasi-isomorphism. As the complex $\mathrm{Hom}_A(B, I^\bullet)$ is a complex of B-modules we conclude that K is in the image of the restriction map, i.e., the functor is essentially surjective

In fact, the argument shows that $F : D_{comp}(A, I) \to D_{comp}(B, IB)$, $K \mapsto \mathrm{Hom}_A(B, I^\bullet)$ is a left inverse to restriction. Finally, suppose that $L \in D_{comp}(B, IB)$. Represent L by a K-injective complex $J^\bullet$ of B-modules. Then $J^\bullet$ is also K-injective as a complex of A-modules (Lemma 15.56.1) hence $F(\text{restriction of } L) = \mathrm{Hom}_A(B, J^\bullet)$. There is a map $J^\bullet \to \mathrm{Hom}_A(B, J^\bullet)$ of complexes of B-modules, whose composition with $\mathrm{Hom}_A(B, J^\bullet) \to J^\bullet$ is the identity. We conclude that F is also a right inverse to restriction and the proof is finished. □

15.93. The category of derived complete modules

0GLN Let A be a ring and let I be an ideal. Denote $\mathcal{C}$ the category of derived complete modules, see Definition 15.92.4. In this section we discuss some properties of this category. In Examples, Section 110.11 we show that $\mathcal{C}$ isn't a Grothendieck abelian category in general.

By Lemma 15.92.6 the category $\mathcal{C}$ is abelian and the inclusion functor $\mathcal{C} \to \mathrm{Mod}_A$ is exact.

Since $D_{comp}(A) \subset D(A)$ is closed under products (see discussion following Definition 15.92.4) and since products in $D(A)$ are computed on the level of complexes, we see that $\mathcal{C}$ has products which agree with products in Mod_A. Thus $\mathcal{C}$ in fact has arbitrary limits and the inclusion functor $\mathcal{C} \to \mathrm{Mod}_A$ commutes with them, see Categories, Lemma 4.14.11.

Assume I is finitely generated. Let $^\wedge : D(A) \to D(A)$ denote the derived completion functor of Lemma 15.92.10. Let us show the functor

$$\mathrm{Mod}_A \longrightarrow \mathcal{C}, \quad M \longmapsto H^0(M^\wedge)$$

is a left adjoint to the inclusion functor $\mathcal{C} \to \mathrm{Mod}_A$. Note that $H^i(M^\wedge) = 0$ for $i > 0$ for example by Lemma 15.92.21. Hence, if N is a derived complete A-module, then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(H^0(M^\wedge), N) &= \mathrm{Hom}_{D_{comp}(A)}(M^\wedge, N) \\ &= \mathrm{Hom}_{D(A)}(M, N) \\ &= \mathrm{Hom}_A(M, N) \end{aligned}$$

as desired.

Let T be a preordered set and let $t \mapsto M_t$ be a system of derived complete A-modules, i.e., a system over T in $\mathcal{C}$, see Categories, Section 4.21. Denote $\mathrm{colim}_{t\in T} M_t$ the colimit of the system in Mod_A. It follows formally from the above that

$$H^0((\mathrm{colim}_{t\in T} M_t)^\wedge)$$

is the colimit of the system in $\mathcal{C}$. In this way we see that $\mathcal{C}$ has all colimits. In general the inclusion functor $\mathcal{C} \to \mathrm{Mod}_A$ will not commute with colimits, see Examples, Section 110.11.

0GLP **Lemma 15.93.1.** *Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian, has arbitrary limits, and the inclusion functor $F : \mathcal{C} \to Mod_A$ is exact and commutes with limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary colimits and F has a left adjoint*

Proof. This summarizes the discussion above. □

15.94. Derived completion for a principal ideal

0BKF In this section we discuss what happens with derived completion when the ideal is generated by a single element.

091X **Lemma 15.94.1.** *Let A be a ring. Let $f \in A$. If there exists an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = A[f^{c+2}] = \ldots$ (for example if A is Noetherian), then for all $n \geq 1$ there exist maps*

$$(A \xrightarrow{f^n} A) \longrightarrow A/(f^n), \quad \text{and} \quad A/(f^{n+c}) \longrightarrow (A \xrightarrow{f^n} A)$$

in $D(A)$ inducing an isomorphism of the pro-objects $\{A/(f^n)\}$ and $\{(f^n : A \to A)\}$ in $D(A)$.

Proof. The first displayed arrow is obvious. We can define the second arrow of the lemma by the diagram

$$\begin{array}{ccc} A/A[f^c] & \xrightarrow[f^{n+c}]{} & A \\ {\scriptstyle f^c}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f^n} & A \end{array}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $A/f^{n+c}A$. We omit the calculation of compositions needed to show the statement on pro objects. □

0923 **Lemma 15.94.2.** *Let A be a ring and $f \in A$. Set $I = (f)$. In this situation we have the naive derived completion $K \mapsto K' = R\lim(K \otimes^{\mathbf{L}}_A A/f^nA)$ and the derived completion*

$$K \mapsto K^\wedge = R\lim(K \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A))$$

of Lemma 15.92.18. The natural transformation of functors $K^\wedge \to K'$ is an isomorphism if and only if the f-power torsion of A is bounded.

Proof. If the f-power torsion is bounded, then the pro-objects $\{(f^n : A \to A)\}$ and $\{A/f^nA\}$ are isomorphic by Lemma 15.94.1. Hence the functors are isomorphic by Lemma 15.87.11. Conversely, we see from Lemma 15.88.11 that the condition is exactly that

$$R\lim(K \otimes^{\mathbf{L}}_A A[f^n])$$

is zero for all $K \in D(A)$. Here the maps of the system $(A[f^n])$ are given by multiplication by f. Taking $K = A$ and $K = \bigoplus_{i\in\mathbf{N}} A$ we see from Lemma 15.87.14 this implies $(A[f^n])$ is zero as a pro-object, i.e., $f^{n-1}A[f^n] = 0$ for some n, i.e., $A[f^{n-1}] = A[f^n]$, i.e., the f-power torsion is bounded. □

09AT **Example 15.94.3.** Let A be a ring. Let $f \in A$ be a nonzerodivisor. An example to keep in mind is $A = \mathbf{Z}_p$ and $f = p$. Let M be an A-module. Claim: M is derived complete with respect to f if and only if there exists a short exact sequence

$$0 \to K \to L \to M \to 0$$

where K, L are f-adically complete modules whose f-torsion is zero. Namely, if there is a such a short exact sequence, then

$$M \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A) = (K/f^nK \to L/f^nL)$$

because f is a nonzerodivisor on K and L and we conclude that $R\lim(M \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A))$ is quasi-isomorphic to $K \to L$, i.e., M. This shows that M is derived complete by Lemma 15.92.17. Conversely, suppose that M is derived complete. Choose a surjection $F \to M$ where F is a free A-module. Since f is a nonzerodivisor on F the derived completion of F is $L = \lim F/f^nF$. Note that L is f-torsion free: if (x_n) with $x_n \in F$ represents an element ξ of L and $f\xi = 0$, then $x_n = x_{n+1} + f^n z_n$ and $fx_n = f^n y_n$ for some $z_n, y_n \in F$. Then $f^n y_n = fx_n = fx_{n+1} + f^{n+1}z_n = f^{n+1}y_{n+1} + f^{n+1}z_n$ and since f is a nonzerodivisor on F we see that $y_n \in fF$ which implies that $x_n \in f^nF$, i.e., $\xi = 0$. Since L is the derived completion, the universal property gives a map $L \to M$ factoring $F \to M$. Let $K = \mathrm{Ker}(L \to M)$ be the kernel. Again K is f-torsion free, hence the derived completion of K is $\lim K/f^nK$. On the other hand, both M and L are derived complete, hence K is too by Lemma 15.92.6. It follows that $K = \lim K/f^nK$ and the claim is proved.

0G3F **Example 15.94.4.** Let p be a prime number. Consider the map $\mathbf{Z}_p[x] \to \mathbf{Z}_p[y]$ of polynomial algebras sending x to py. Consider the cokernel $M = \mathrm{Coker}(\mathbf{Z}_p[x]^\wedge \to \mathbf{Z}_p[y]^\wedge)$ of the induced map on (ordinary) p-adic completions. Then M is a derived complete $\mathbf{Z}_p$-module by Proposition 15.92.5 and Lemma 15.92.6; see also discussion in Example 15.94.3. However, M is not p-adically complete as $1 + py + p^2y^2 + \ldots$ maps to a nonzero element of M which is contained in $\bigcap p^nM$.

0BKG **Example 15.94.5.** Let A be a ring and let $f \in A$. Denote $K \mapsto K^\wedge$ the derived completion with respect to (f). Let M be an A-module. Using that

$$M^\wedge = R\lim(M \xrightarrow{f^n} M)$$

by Lemma 15.92.18 and using Lemma 15.88.4 we obtain

$$H^{-1}(M^\wedge) = \lim M[f^n] = T_f(M)$$

the *f-adic Tate module of M*. Here the maps $M[f^n] \to M[f^{n-1}]$ are given by multiplication by f. Then there is a short exact sequence

$$0 \to R^1 \lim M[f^n] \to H^0(M^\wedge) \to \lim M/f^nM \to 0$$

describing $H^0(M^\wedge)$. We have $H^1(M^\wedge) = R^1 \lim M/f^nM = 0$ as the transition maps are surjective (Lemma 15.88.1). All the other cohomologies of $M^\wedge$ are zero for trivial reasons. Finally, for $K \in D(A)$ and $p \in \mathbf{Z}$ there is a short exact sequence

$$0 \to H^0(H^p(K)^\wedge) \to H^p(K^\wedge) \to T_f(H^{p+1}(K)) \to 0$$

This follows from the spectral sequence of Example 15.92.23 because it degenerates at E_2 (as only $i = -1, 0$ give nonzero terms); the next lemma gives more information.

0H32 **Lemma 15.94.6.** *Let A be a ring and let $f \in A$. Let K be an object of $D(A)$. Denote $K_n = K \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A)$. For all $p \in \mathbf{Z}$ there is a commutative diagram*

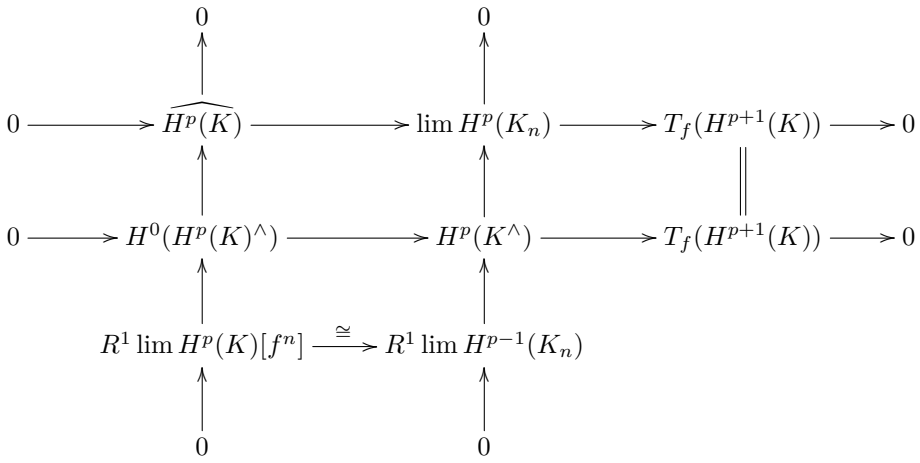

with exact rows and columns where $\widehat{H^p(K)} = \lim H^p(K)/f^nH^p(K)$ is the usual f-adic completion. The left vertical short exact sequence and the middle horizontal short exact sequence are taken from Example 15.94.5 The middle vertical short exact sequence is the one from Lemma 15.88.4.

Proof. To construct the top horizontal short exact sequence, observe that we have the following inverse system short exact sequences

$$0 \to H^p(K)/f^nH^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$$

coming from the construction of K_n as a shift of the cone on $f^n : K \to K$. Taking the inverse limit of these we obtain the top horizontal short exact sequence, see Homology, Lemma 12.31.3.

Let us prove that we have a commutative diagram as in the lemma. We consider the map $L = \tau_{\leq p}K \to K$. Setting $L_n = L \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A)$ we obtain a map $(L_n) \to (K_n)$ of inverse systems which induces a map of short exact sequences

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\lim H^p(L_n) & \longrightarrow & \lim H^p(K_n) \\
\uparrow & & \uparrow \\
H^p(L^\wedge) & \longrightarrow & H^p(K^\wedge) \\
\uparrow & & \uparrow \\
R^1 \lim H^{p-1}(L_n) & \longrightarrow & R^1 \lim H^{p-1}(K_n) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Since $H^i(L) = 0$ for $i > p$ and $H^p(L) = H^p(K)$, a computation using the references in the statement of the lemma shows that $H^p(L^\wedge) = H^0(H^p(K)^\wedge)$ and that $H^p(L_n) = H^p(K)/f^n H^p(K)$. On the other hand, we have $H^{p-1}(L_n) = H^{p-1}(K_n)$ and hence we see that we get the isomorphism as indicated in the statement of the lemma since we already know the kernel of $H^0(H^p(K)^\wedge) \to \widehat{H^p(K)}$ is equal to $R^1 \lim H^p(K)[f^n]$. We omit the verification that the rightmost square in the diagram commutes if we define the top row by the construction in the first paragraph of the proof. □

0H33 **Remark 15.94.7.** With notation as in Lemma 15.94.6 we also see that the inverse system $H^p(K_n)$ has ML if and only if the inverse system $H^{p+1}(K)[f^n]$ has ML. This follows from the inverse system of short exact sequences $0 \to H^p(K)/f^n H^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$ (see proof of the lemma) combined with Homology, Lemma 12.31.3 and Lemma 15.87.13.

0CQY **Lemma 15.94.8** (Bhatt)**.** *Let I be a finitely generated ideal in a ring A. Let M be a derived complete A-module. If M is an I-power torsion module, then $I^n M = 0$ for some n.*

Proof. Say $I = (f_1, \ldots, f_r)$. It suffices to show that for each i there is an n_i such that $f_i^{n_i} M = 0$. Hence we may assume that $I = (f)$ is a principal ideal. Let $B = \mathbf{Z}[x] \to A$ be the ring map sending x to f. By Lemma 15.92.24 we see that M is derived complete as a B-module with respect to the ideal (x). After replacing A by B, we may assume that f is a nonzerodivisor in A.

Assume $I = (f)$ with $f \in A$ a nonzerodivisor. According to Example 15.94.3 there exists a short exact sequence

$$0 \to K \xrightarrow{u} L \to M \to 0$$

where K and L are I-adically complete A-modules whose f-torsion is zero[14]. Consider K and L as topological modules with the I-adic topology. Then u is continuous. Let

$$L_n = \{x \in L \mid f^n x \in u(K)\}$$

Since M is f-power torsion we see that $L = \bigcup L_n$. Let N_n be the closure of L_n in L. By Lemma 15.36.4 we see that N_n is open in L for some n. Fix such an n. Since $f^{n+m} : L \to L$ is a continuous open map, and since $f^{n+m}L_n \subset u(f^m K)$ we conclude that the closure of $u(f^m K)$ is open for all $m \geq 1$. Thus by Lemma 15.36.5 we conclude that u is open. Hence $f^t L \subset \mathrm{Im}(u)$ for some t and we conclude that f^t annihilates M as desired. □

0G3G **Lemma 15.94.9.** *Let $f \in A$ be an element of a ring. Set $J = \bigcap f^n A$. Let M be an A-module derived complete with respect to f. Then $JM' = 0$ where $M' = \mathrm{Ker}(M \to \lim M/f^n M)$. In particular, if A is derived complete then J is an ideal of square zero.*

Proof. Take $x \in M'$ and $g \in J$. For every $n \geq 1$ we may write $x = f^n x_n$. Since g is in $f^n A$ we see that the element $y_n = g x_n$ in M' is independent of the choice of x_n. In particular, we may take $x_n = f x_{n+1}$ and we find that $y_n = f y_{n+1}$. Thus we obtain a map $A_f \to M$ sending $1/f^n$ to y_n. This map has to be zero as M is derived complete (Lemma 15.92.1) and hence $y_n = 0$ for all n. Since $gx = gfx_1 = fy_1$ this completes the proof. □

0G3H **Lemma 15.94.10.** *Let A be a ring derived complete with respect to an ideal I. Then (A, I) is a henselian pair.*

Proof. Let $f \in I$. By Lemma 15.11.15 it suffices to show that (A, fA) is a henselian pair. Observe that A is derived complete with respect to fA (follows immediately from Definition 15.92.4). By Lemma 15.92.3 the map from A to the f-adic completion A' of A is surjective. By Lemma 15.11.4 the pair (A', fA') is henselian. Thus it suffices to show that $(A, \bigcap f^n A)$ is a henselian pair, see Lemma 15.11.9. This follows from Lemmas 15.94.9 and 15.11.2. □

0G3I **Lemma 15.94.11.** *Let A be a ring derived complete with respect to an ideal I. Set $J = \bigcap I^n$. If I can be generated by r elements then $J^N = 0$ where $N = 2^r$.*

Proof. When $r = 1$ this is Lemma 15.94.9. Say $I = (f_1, \ldots, f_r)$ with $r > 1$. By Lemma 15.92.6 the ring $A_t = A/f_r^t A$ is derived complete with respect to I and hence a fortiori derived complete with respect to $I_t = (f_1, \ldots, f_{r-1})A_t$. Observe that $A \to A_t$ sends J into $J_t = \bigcap I_t^n$. By induction $J_t^{N/2} = 0$ with $N = 2^r$. The ideal $\bigcap \mathrm{Ker}(A \to A_t) = \bigcap f_r^t A$ has square zero by the case $r = 1$. This finishes the proof. □

0G5V **Lemma 15.94.12.** *Let A be a reduced ring derived complete with respect to a finitely generated ideal I. Then A is I-adically complete.*

Proof. Follows from Lemma 15.94.11 and Proposition 15.92.5. □

[14]For the proof it is enough to show that there exists a sequence $K \xrightarrow{u} L \to M \to 0$ where K and L are I-adically complete A-modules. This can be shown by choosing a presentation $F_1 \to F_0 \to M \to 0$ with F_i free and then setting K and L equal to the f-adic completions of F_1 and F_0. Namely, as f is a nonzerodivisor these completions will be the derived completions and the sequence will remain exact.

15.95. Derived completion for Noetherian rings

0BKH Let A be a ring and let $I \subset A$ be an ideal. For any $K \in D(A)$ we can consider the derived limit

$$K' = R\lim(K \otimes^{\mathbf{L}}_A A/I^n)$$

This is a functor in K, see Remark 15.88.10. The system of maps $A \to A/I^n$ induces a map $K \to K'$ and K' is derived complete with respect to I (Lemma 15.92.14). This "naive" derived completion construction does not agree with the adjoint of Lemma 15.92.10 in general. For example, if $A = \mathbf{Z}_p \oplus \mathbf{Q}_p/\mathbf{Z}_p$ with the second summand an ideal of square zero, $K = A[0]$, and $I = (p)$, then the naive derived completion gives $\mathbf{Z}_p[0]$, but the construction of Lemma 15.92.10 gives $K^\wedge \cong \mathbf{Z}_p[1] \oplus \mathbf{Z}_p[0]$ (computation omitted). Lemma 15.94.2 characterizes when the two functors agree in the case I is generated by a single element.

The main goal of this section is the show that the naive derived completion is equal to derived completion if A is Noetherian.

0921 **Lemma 15.95.1.** *In Situation 15.92.15. If A is Noetherian, then the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic*[15].

Proof. We have an inverse system of distinguished triangles

$$\tau_{\leq -1}K_n^\bullet \to K_n^\bullet \to A/(f_1^m, \ldots, f_r^m) \to (\tau_{\leq -1}K_n^\bullet)[1]$$

See Derived Categories, Remark 13.12.4. By Derived Categories, Lemma 13.42.4 it suffices to show that the inverse system $\tau_{\leq -1}K_n^\bullet$ is pro-zero. Recall that $K_n^\bullet$ has nonzero terms only in degrees i with $-r \leq i \leq 0$. Thus by Derived Categories, Lemma 13.42.3 it suffices to show that $H^p(K_n^\bullet)$ is pro-zero for $p \leq -1$. In other words, for every $n \in \mathbf{N}$ we have to show there exists an $m \geq n$ such that $H^p(K_m^\bullet) \to H^p(K_n^\bullet)$ is zero. Since A is Noetherian, we see that

$$H^p(K_n^\bullet) = \frac{\mathrm{Ker}(K_n^p \to K_n^{p+1})}{\mathrm{Im}(K_n^{p-1} \to K_n^p)}$$

is a finite A-module. Moreover, the map $K_m^p \to K_n^p$ is given by a diagonal matrix whose entries are in the ideal $(f_1^{m-n}, \ldots, f_r^{m-n})$ as $p < 0$. Note that $H^p(K_n^\bullet)$ is annihilated by $J = (f_1^n, \ldots, f_r^n)$, see Lemma 15.28.6. Now $(f_1^{m-n}, \ldots, f_r^{m-n}) \subset J^t$ for $m - n \geq tn$. Thus by Algebra, Lemma 10.51.2 (Artin-Rees) applied to the ideal J and the module $M = K_n^p$ with submodule $N = \mathrm{Ker}(K_n^p \to K_n^{p+1})$ for m large enough the image of $K_m^p \to K_n^p$ intersected with $\mathrm{Ker}(K_n^p \to K_n^{p+1})$ is contained in $J\,\mathrm{Ker}(K_n^p \to K_n^{p+1})$. For such m we get the zero map. □

0922 **Proposition 15.95.2.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes^{\mathbf{L}}_A A/I^n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.92.10.*

Proof. Say $(f_1, \ldots, f_r) = I$ and let $K_n^\bullet$ be the Koszul complex with respect to $f_1^n, \ldots, f_r^n$. By Lemma 15.92.18 it suffices to prove that

$$R\lim(K \otimes^{\mathbf{L}}_A K_n^\bullet) = R\lim(K \otimes^{\mathbf{L}}_A A/(f_1^n, \ldots, f_r^n)) = R\lim(K \otimes^{\mathbf{L}}_A A/I^n).$$

[15] In particular, for every n there exists an $m \geq n$ such that $K_m^\bullet \to K_n^\bullet$ factors through the map $K_m^\bullet \to A/(f_1^m, \ldots, f_r^m)$.

By Lemma 15.95.1 the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic. It is clear that the pro-objects $\{A/(f_1^n, \ldots, f_r^n)\}$ and $\{A/I^n\}$ are isomorphic. Thus the map from left to right is an isomorphism by Lemma 15.88.12. □

0EET **Lemma 15.95.3.** *Let I be an ideal of a Noetherian ring A. Let M be an A-module with derived completion $M^\wedge$. Then there are short exact sequences*

$$0 \to R^1 \lim Tor_{i+1}^A(M, A/I^n) \to H^{-i}(M^\wedge) \to \lim Tor_i^A(M, A/I^n) \to 0$$

A similar result holds for $M \in D^-(A)$.

Proof. Immediate consequence of Proposition 15.95.2 and Lemma 15.88.4. □

As an application of the proposition above we identify the derived completion in the Noetherian case for pseudo-coherent complexes.

0A06 **Lemma 15.95.4.** *Let A be a Noetherian ring and $I \subset A$ an ideal. Let K be an object of $D(A)$ such that $H^n(K)$ a finite A-module for all $n \in \mathbf{Z}$. Then the cohomology modules $H^n(K^\wedge)$ of the derived completion are the I-adic completions of the cohomology modules $H^n(K)$.*

Proof. The complex $\tau_{\leq m}K$ is pseudo-coherent for all m by Lemma 15.65.17. Thus $\tau_{\leq m}K$ is represented by a bounded above complex $P^\bullet$ of finite free A-modules. Then $\tau_{\leq m}K \otimes_A^{\mathbf{L}} A/I^n = P^\bullet/I^nP^\bullet$. Hence $(\tau_{\leq m}K)^\wedge = R\lim P^\bullet/I^nP^\bullet$ (Proposition 15.95.2) and since the $R\lim$ is just given by termwise lim (Lemma 15.88.1) and since I-adic completion is an exact functor on finite A-modules (Algebra, Lemma 10.97.2) we conclude the result holds for $\tau_{\leq m}K$. Hence the result holds for K as derived completion has finite cohomological dimension, see Lemma 15.92.21. □

09BA **Lemma 15.95.5.** *Let I be an ideal of a Noetherian ring A. Let M be a derived complete A-module. If M/IM is a finite A/I-module, then $M = \lim M/I^nM$ and M is a finite $A^\wedge$-module.*

Proof. Assume M/IM is finite. Pick $x_1, \ldots, x_t \in M$ which map to generators of M/IM. We obtain a map $A^{\oplus t} \to M$ mapping the ith basis vector to x_i. By Proposition 15.95.2 the derived completion of A is $A^\wedge = \lim A/I^n$. As M is derived complete, we see that our map factors through a map $q : (A^\wedge)^{\oplus t} \to M$. The module $\mathrm{Coker}(q)$ is zero by Lemma 15.92.7. Thus M is a finite $A^\wedge$-module. Since $A^\wedge$ is Noetherian and complete with respect to $IA^\wedge$, it follows that M is I-adically complete (use Algebra, Lemmas 10.97.5, 10.96.11, and 10.51.2). □

0EEU **Lemma 15.95.6.** *Let I be an ideal in a Noetherian ring A.*

(1) *If M is a finite A-module and N is a flat A-module, then the derived I-adic completion of $M \otimes_A N$ is the usual I-adic completion of $M \otimes_A N$.*
(2) *If M is a finite A-module and $f \in A$, then the derived I-adic completion of M_f is the usual I-adic completion of M_f.*

Proof. For an A-module M denote $M^\wedge$ the derived completion and $\lim M/I^nM$ the usual completion. Assume M is finite. The system $\mathrm{Tor}_i^A(M, A/I^n)$ is pro-zero for $i > 0$, see Lemma 15.27.3. Since $\mathrm{Tor}_i^A(M \otimes_A N, A/I^n) = \mathrm{Tor}_i^A(M, A/I^n) \otimes_A N$ as N is flat, the same is true for the system $\mathrm{Tor}_i^A(M \otimes_A N, A/I^n)$. By Lemma 15.95.3 we conclude $R\lim(M \otimes_A N) \otimes_A^{\mathbf{L}} A/I^n$ only has cohomology in degree 0 given by the usual completion $\lim M \otimes_A N/I^n(M \otimes_A N)$. This proves (1). Part (2) follows from (1) and the fact that $M_f = M \otimes_A A_f$. □

0EEV **Lemma 15.95.7.** *Let I be an ideal in a Noetherian ring A. Let $^\wedge$ denote derived completion with respect to I. Let $K \in D^-(A)$.*

(1) *If M is a finite A-module, then $(K \otimes^{\mathbf{L}}_A M)^\wedge = K^\wedge \otimes^{\mathbf{L}}_A M$.*
(2) *If $L \in D(A)$ is pseudo-coherent, then $(K \otimes^{\mathbf{L}}_A L)^\wedge = K^\wedge \otimes^{\mathbf{L}}_A L$.*

Proof. Let L be as in (2). We may represent K by a bounded above complex $P^\bullet$ of free A-modules. We may represent L by a bounded above complex $F^\bullet$ of finite free A-modules. Since $\mathrm{Tot}(P^\bullet \otimes_A F^\bullet)$ represents $K \otimes^{\mathbf{L}}_A L$ we see that $(K \otimes^{\mathbf{L}}_A L)^\wedge$ is represented by

$$\mathrm{Tot}((P^\bullet)^\wedge \otimes_A F^\bullet)$$

where $(P^\bullet)^\wedge$ is the complex whose terms are the usual = derived completions $(P^n)^\wedge$, see for example Proposition 15.95.2 and Lemma 15.95.6. This proves (2). Part (1) is a special case of (2). □

15.96. An operator introduced by Berthelot and Ogus

0F7N In this section we discuss a construction introduced in [**BO78**, Section 8] and generalized in [**BMS18**, Section 6]. We urge the reader to look at the original papers discussing this notion.

Let A be a ring and let $f \in A$ be a nonzerodivisor. If M is a A-module then by Lemma 15.89.3 following are equivalent

(1) f is a nonzerodivisor on M,
(2) $M[f] = 0$,
(3) $M[f^n] = 0$ for all $n \geq 1$, and
(4) the map $M \to M_f$ is injective.

If these equivalent conditions hold, then (in this section) we will say *M is f-torsion free*. If so, then we denote $f^iM \subset M_f$ the submodule consisting of elements of the form f^ix with $x \in M$. Of course f^iM is isomorphic to M as an A-module. Let $M^\bullet$ be a complex of f-torsion free A-modules with differentials $d^i : M^i \to M^{i+1}$. In this case we define $\eta_f M^\bullet$ to be the complex with terms

$$(\eta_f M)^i = \{x \in f^iM^i \mid d^i(x) \in f^{i+1}M^{i+1}\}$$

and differential induced by d^i. Observe that $\eta_f M^\bullet$ is another complex of f-torsion free A-modules. If $a^\bullet : M^\bullet \to N^\bullet$ is a map of complexes of f-torsion free A-modules, then we obtain a map of complexes

$$\eta_f a^\bullet : \eta_f M^\bullet \longrightarrow \eta_f N^\bullet$$

induced by the maps $f^iM^i \to f^iN^i$. The reader checks that we obtain an endofunctor on the category of complexes of f-torsion free A-modules. If $a^\bullet, b^\bullet : M^\bullet \to N^\bullet$ are two maps of complexes of f-torsion free A-modules and $h = \{h^i : M^i \to N^{i-1}\}$ is a homotopy between $a^\bullet$ and $b^\bullet$, then we define $\eta_f h$ to be the family of maps $(\eta_f h)^i : (\eta_f M)^i \to (\eta_f N)^{i-1}$ which sends x to $h^i(x)$; this makes sense as $x \in f^iM^i$ implies $h^i(x) \in f^iN^{i-1}$ which is certainly contained in $(\eta_f N)^{i-1}$. The reader checks that $\eta_f h$ is a homotopy between $\eta_f a^\bullet$ and $\eta_f b^\bullet$. All in all we see that we obtain a functor

$$\eta_f : K(f\text{-torsion free } A\text{-modules}) \longrightarrow K(f\text{-torsion free } A\text{-modules})$$

on the homotopy category (Derived Categories, Section 13.8) of the additive category of f-torsion free A-modules. There is no sense in which η_f is an exact functor of triangulated categories, see Example 15.96.1.

0GSN **Example 15.96.1.** Let A be a ring. Let $f \in A$ be a nonzerodivisor. Consider the functor $\eta_f : K(f\text{-torsion free } A\text{-modules}) \to K(f\text{-torsion free } A\text{-modules})$. Let $M^\bullet$ be a complex of f-torsion free A-modules. Multiplication by f defines an isomorphism $\eta_f(M^\bullet[1]) \to (\eta_f M^\bullet)[1]$, so in this sense η_f is compatible with shifts. However, consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & A & \xrightarrow{1} & A & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{-1} & A \end{array}$$

Think of each column as a complex of f-torsion free A-modules with the module on top in degree 1 and the module under it in degree 0. Then this diagram provides us with a distinguished triangle in $K(f\text{-torsion free } A\text{-modules})$ with triangulated structure as given in Derived Categories, Section 13.10. Namely the third complex is the cone of the map between the first two complexes. However, applying η_f to each column we obtain

$$\begin{array}{ccccccc} fA & \xrightarrow{f} & fA & \xrightarrow{1} & fA & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{-1} & A \end{array}$$

However, the third complex is acyclic and even homotopic to zero. Hence if this were a distinguished triangle, then the first arrow would have to be an isomorphism in the homotopy category, which is not true unless f is a unit.

0F7P **Lemma 15.96.2.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical isomorphism*

$$f^i : H^i(M^\bullet)/H^i(M^\bullet)[f] \longrightarrow H^i(\eta_f M^\bullet)$$

given by multiplication by f^i.

Proof. Observe that $\mathrm{Ker}(d^i : (\eta_f M)^i \to (\eta_f M)^{i+1})$ is equal to $\mathrm{Ker}(d^i : f^i M^i \to f^i M^{i+1}) = f^i \mathrm{Ker}(d^i : M^i \to M^{i+1})$. This we get a surjection $f^i : H^i(M^\bullet) \to H^i(\eta_f M^\bullet)$ by sending the class of $z \in \mathrm{Ker}(d^i : M^i \to M^{i+1})$ to the class of $f^i z$. If we obtain the zero class in $H^i(\eta_f M^\bullet)$ then we see that $f^i z = d^{i-1}(f^{i-1} y)$ for some $y \in M^{i-1}$. Since f is a nonzerodivisor on all the modules involved, this means $fz = d^{i-1}(y)$ which exactly means that the class of z is f-torsion as desired. □

0F7Q **Lemma 15.96.3.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. If $M^\bullet \to N^\bullet$ is a quasi-isomorphism of complexes of f-torsion free A-modules, then the induced map $\eta_f M^\bullet \to \eta_f N^\bullet$ is a quasi-isomorphism too.*

Proof. This is true because the isomorphisms of Lemma 15.96.2 are compatible with maps of complexes. □

0F7R **Lemma 15.96.4.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. There is an additive functor[16] $L\eta_f : D(A) \to D(A)$ such that if $M \in D(A)$ is represented by a complex $M^\bullet$ of f-torsion free A-modules, then $L\eta_f M = \eta_f M^\bullet$ and similarly for morphisms.*

[16]Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles. See Example 15.96.1.

Proof. Denote $\mathcal{T} \subset \mathrm{Mod}_A$ the full subcategory of f-torsion free A-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(\mathrm{Mod}_A) = K(A)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(A)$. Let $S \subset \mathrm{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 13.5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(A)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $M^\bullet$ of A-modules, there exists a quasi-isomorphism $K^\bullet \to M^\bullet$ with $K^\bullet$ a complex of f-torsion free modules. By Lemma 15.59.10 we can find a quasi-isomorphism $K^\bullet \to M^\bullet$ such that the complex $K^\bullet$ is K-flat (we won't use this) and consists of flat A-modules K^i. In particular, f is a nonzerodivisor on K^i for all i as desired.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion at the start of this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(A) \to D(A)$$

which according to Lemma 15.96.3 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(A)$ by Categories, Lemma 4.27.8. $\square$

0F7S **Remark 15.96.5.** Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every i set $\overline{M}^i = M^i/fM^i$. Denote $B^i \subset Z^i \subset \overline{M}^i$ the boundaries and cocycles for the differentials on the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$. We claim that there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^{i+1} & \longrightarrow & B^{i+1} \oplus B^i & \longrightarrow & B^i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle s,s'} & & \downarrow & & \\
0 & \longrightarrow & B^{i+1} & \xrightarrow{s} & (\eta_f M)^i/f(\eta_f M)^i & \xrightarrow{t} & Z^i & \longrightarrow & 0
\end{array}$$

with exact rows. Here are the constructions of the maps

(1) If $x \in (\eta_f M)^i$ then $x = f^i x'$ with $d^i(x') = 0$ in $\overline{M}^{i+1}$. Hence we can define the map t by sending x to the class of x'.
(2) If $y \in M^{i+1}$ has class $\overline{y}$ in $B^{i+1} \subset \overline{M}^{i+1}$ then we can write $y = fy' + d^i(x)$ for $y' \in M^{i+1}$ and $x \in M^i$. Hence we can define the map s sending $\overline{y}$ to the class of $f^{i+1}x$ in $(\eta_f M)^i/f(\eta_f M)^i$; we omit the verification that this is well defined.
(3) If $x \in M^i$ has class $\overline{x}$ in $B^i \subset \overline{M}^i$ then we can write $x = fx' + d^{i-1}(z)$ for $x' \in M^i$ and $z \in M^{i-1}$. We define the map s' by sending $\overline{x}$ to the class of $f^i d^{i-1}(z)$ in $(\eta_f M)^i/f(\eta_f M)^i$. This is well defined because if $fx' + d^{i-1}(z) = 0$, then $f^i x'$ is in $(\eta_f M)^i$ and consequently $f^i d^{i-1}(z)$ is in $f(\eta_f M)^i$.

We omit the verification that the lower row in the displayed diagram is a short exact sequence of modules. It is immediately clear from these constructions that

we have commutative diagrams

$$\begin{array}{ccc} B^{i+1} \oplus B^i & \longrightarrow & B^{i+2} \oplus B^{i+1} \\ \downarrow{\scriptstyle s,s'} & & \downarrow{\scriptstyle s,s'} \\ (\eta_f M)^i/f(\eta_f M)^i & \longrightarrow & (\eta_f M)^{i+1}/f(\eta_f M)^{i+1} \end{array}$$

where the upper horizontal arrow is given by the identification of the summands B^{i+1} in source and target. In other words, we have found an acyclic subcomplex of $\eta_f M^\bullet/f(\eta_f M^\bullet) = \eta_f M^\bullet \otimes_A A/fA$ and the quotient by this subcomplex is a complex whose terms Z^i/B^i are the cohomology modules of the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$.

To explain the phenomenon observed in Remark 15.96.5 in a more canonical manner, we are going to construct the Bockstein operators. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every $i \in \mathbf{Z}$ there is a commutative diagram (with tensor products over A)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^\bullet \otimes f^{i+1}A & \longrightarrow & M^\bullet \otimes f^i A & \longrightarrow & M^\bullet \otimes f^i A/f^{i+1}A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M^\bullet \otimes f^{i+1}A/f^{i+2}A & \longrightarrow & M^\bullet \otimes f^i A/f^{i+2}A & \longrightarrow & M^\bullet \otimes f^i A/f^{i+1}A & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences of complexes. Of course these short exact sequences for different i are all isomorphic to each other by suitably multiplying with powers of f. The long exact sequence of cohomology of the bottom sequence in particular determines the *Bockstein operator*

$$\beta = \beta^i : H^i(M^\bullet \otimes f^i A/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A)$$

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

0GSP (15.96.5.1)
$$\begin{array}{ccc} H^i(M^\bullet \otimes f^i A/f^{i+1}A) & \xrightarrow{\delta} & H^{i+1}(M^\bullet \otimes f^{i+1}A) \\ & \searrow^{\beta} & \downarrow \\ & & H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \end{array}$$

of the Bockstein operator where δ is the boundary operator coming from the top row in the commutative diagram above. Let us show that we obtain a complex

0GSQ (15.96.5.2)
$$H^\bullet(M^\bullet/f) = \left[\begin{array}{c} \cdots \\ \downarrow \\ H^{i-1}(M^\bullet \otimes f^{i-1}A/f^i A) \\ \downarrow \beta \\ H^i(M^\bullet \otimes f^i A/f^{i+1}A) \\ \downarrow \beta \\ H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \\ \downarrow \\ \cdots \end{array}\right]$$

i.e., that $\beta \circ \beta = 0$[17]. Namely, using the factorization (15.96.5.1) we see that it suffices to show that

$$H^{i+1}(M^\bullet \otimes f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \xrightarrow{\beta^{i+1}} H^{i+2}(M^\bullet \otimes f^{i+2}A/f^{i+3}A)$$

is zero. This is true because the kernel of β^{i+1} consists of the cohomology classes which can be lifted to $H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+3}A)$ and those in the image of the first map certainly can!

0F7T **Lemma 15.96.6.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical map of complexes*

$$\eta_f M^\bullet \otimes_A A/fA \longrightarrow H^\bullet(M^\bullet/f)$$

which is a quasi-isomorphism where the right hand side is the complex (15.96.5.2).

Proof. Let $x \in (\eta_f M)^i$. Then $x = f^i x' \in f^i M$ and $d^i(x) = f^{i+1}y \in f^{i+1}M^{i+1}$. Thus d^i maps $x' \otimes f^i$ to zero in $M^{i+1} \otimes f^i A/f^{i+1}A$. All tensor products are over A in this proof. Hence we may map x to the class of $x' \otimes f^i$ in $H^i(M^\bullet \otimes f^i A/f^{i+1}A)$. It is clear that this rule defines a map

$$(\eta_f M)^i \otimes A/fA \longrightarrow H^i(M^\bullet \otimes f^i A/f^{i+1}A)$$

of A/fA-modules. Observe that in the situation above, we may view $x' \otimes f^i$ as an element of $M^i \otimes f^i A/f^{i+2}A$ with differential $d^i(x' \otimes f^i) = y \otimes f^{i+1}$. By the construction of β above we find that $\beta(x' \otimes f^i) = y \otimes f^{i+1}$ and we conclude that our maps are compatible with differentials, i.e., we have a map of complexes.

To finish the proof, we observe that the construction given in the previous paragraph agrees with the maps $(\eta_f M)^i \otimes A/fA \to Z^i/B^i$ discussed in Remark 15.96.5. Since we have seen that the kernel of these maps is an acyclic subcomplex of $\eta_f M^\bullet \otimes A/fA$, the lemma is proved. □

0F7Y **Lemma 15.96.7.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For $i \in \mathbf{Z}$ the following are equivalent*

(1) $\mathrm{Ker}(d^i \bmod f^2)$ *surjects onto* $\mathrm{Ker}(d^i \bmod f)$,
(2) $\beta : H^i(M^\bullet \otimes_A f^i A/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes_A f^{i+1}A/f^{i+2}A)$ *is zero.*

These equivalent conditions are implied by the condition $H^{i+1}(M^\bullet)[f] = 0$.

Proof. The equivalence of (1) and (2) follows from the definition of β as the boundary map on cohomology of a short exact sequence of complexes isomorphic to the short exact sequence of complexes $0 \to fM^\bullet/f^2M^\bullet \to M^\bullet/f^2M^\bullet \to M^\bullet/fM^\bullet \to 0$. If $\beta \neq 0$, then $H^{i+1}(M^\bullet)[f] \neq 0$ because of the factorization (15.96.5.1). □

[17] An alternative is to argue that β occurs as the differential for the spectral sequence for the complex $(M^\bullet)_f$ filtered by the subcomplexes $f^i M^\bullet$. Yet another argument, which proves something stronger, is to first consider the case $M^\bullet = A$. Here the short exact sequences $0 \to f^{i+1}A/f^{i+2}A \to f^iA/f^{i+2}A \to f^iA/f^{i+1}A \to 0$ define maps $\beta^i : f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1]$ in $D(A)$. Then one computes (arguing similarly to the text) that the composition $f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1] \to f^{i+2}A/f^{i+3}A[2]$ is zero in $D(A)$. Since $M^\bullet \otimes f^iA/f^{i+1}A = M^\bullet \otimes^{\mathbf{L}} f^iA/f^{i+1}A$ by our assumption on $M^\bullet$ having f-torsion free terms, we conclude the compostion

$$(M^\bullet \otimes f^iA/f^{i+1}A) \to (M^\bullet \otimes f^{i+1}A/f^{i+2}A)[1] \to (M^\bullet \otimes f^{i+2}A/f^{i+3}A)[2]$$

in $D(A)$ is zero as well.

0F7Z **Lemma 15.96.8.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. If $\mathrm{Ker}(d^i \bmod f^2)$ surjects onto $\mathrm{Ker}(d^i \bmod f)$, then the canonical map*

$$(1, d^i) : (\eta_f M)^i/f(\eta_f M)^i \longrightarrow f^i M^i/f^{i+1}M^i \oplus f^{i+1}M^{i+1}/f^{i+2}M^{i+1}$$

identifies the left hand side with a direct sum of submodules of the right hand side.

Proof. With notation as in Remark 15.96.5 we define a map $t^{-1} : Z^i \to (\eta_f M)^i/f(\eta_f M)^i$. Namely, for $x \in M^i$ with $d^i(x) = f^2 y$ we send the class of x in Z^i to the class of $f^i x$ in $(\eta_f M)^i/f(\eta_f M)^i$. We omit the verification that this is well defined; the assumption of the lemma exactly signifies that the domain of this operation is all of Z^i. Then $t \circ t^{-1} = \mathrm{id}_{Z^i}$. Hence t^{-1} defines a splitting of the short exact sequence in Remark 15.96.5 and the resulting direct sum decomposition

$$(\eta_f M)^i/f(\eta_f M)^i = Z^i \oplus B^{i+1}$$

is compatible with the map displayed in the lemma. □

0F7U **Lemma 15.96.9.** *Let A be a ring and let $f, g \in A$ be nonzerodivisors. Let $M^\bullet$ be a complex of A-modules such that fg is a nonzerodivisor on all M^i. Then $\eta_f \eta_g M^\bullet = \eta_{fg} M^\bullet$.*

Proof. The statement means that in degree i we obtain the same submodule of the localization $M^i_{fg} = (M^i_g)_f$. We omit the details. □

0GSR **Lemma 15.96.10.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $A \to B$ be a flat ring map and let $g \in B$ the image of f. Let $M^\bullet$ be a complex of f-torsion free A-modules. Then g is a nonzerodivisor, $M^\bullet \otimes_A B$ is a complex of g-torsion free modules, and $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.*

Proof. Omitted. □

15.97. Perfect complexes and the eta operator

0F7V In this section we do some algebra to prepare for our version of Macpherson's graph construction, see More on Flatness, Section 38.44. We will use the η_f operator introduced in Section 15.96.

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. For each i let r_i be the rank of M^i and set

$$I_i(M^\bullet, f) = \text{ideal generated by the } r_i \times r_i\text{-minors of } (f, d^i) : M^i \to M^i \oplus M^{i+1}$$

Observe that $f^{r_i} \in I_i(M^\bullet, f)$.

0GSS **Lemma 15.97.1.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object of $D(A)$. Then*

$$f^m I_i(M^\bullet, f) = f^n I_i(N^\bullet, f)$$

as ideals of A for integers $n, m \geq 0$ such that

$$m + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(M^j) = n + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(N^j)$$

Proof. It suffices to prove the equality after localization at every prime ideal of A. Thus by Lemma 15.76.8 and an induction argument we omit we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j+1$. If $j \neq i-1, i, i+1$ then we clearly have equality $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$ and we have the desired equality. If $j = i+1$ then the maps

$$(f, d^i) : M^i \to M^i \oplus M^{i+1} \quad \text{and} \quad (f, d^i, 0) : M^i \to M^i \oplus M^{i+1} \oplus A$$

have the same nonzero minors hence in this case we also have $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$. If $j = i$, then $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i+1) \times (r_i+1)$-minors of

$$(f \oplus f, d^i \oplus 1) : (M^i \oplus A) \to (M^i \oplus A) \oplus (M^{i+1} \oplus A)$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \\ 1 \end{pmatrix}$$

With notation as in Lemma 15.8.1 we have $I_0(T_2) = A$, $I_1(T_2) = A$, $I_p(T_2) = 0$ for $p \geq 2$ and hence $I_{r_i+1}(T) = I_{r_i+1}(T_1) + I_{r_i}(T_1) = I_{r_i}(T_1)$ which means that $I_i(M^\bullet, f) = I_i(N^\bullet, f)$. We also have $m = n$ so this finishes the case $j = i$. Finally, say $j = i-1$. Then we see that $m = n+1$, thus we have to show that $f I_i(M^\bullet, f) = I_i(N^\bullet, f)$. In this case $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i+1) \times (r_i+1)$-minors of

$$(f \oplus f, d^i) : (M^i \oplus A) \to (M^i \oplus A) \oplus M^{i+1}$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \end{pmatrix}$$

Arguing as above we find that indeed $f I_i(M^\bullet, f) = I_i(N^\bullet, f)$. □

0GST **Lemma 15.97.2.** *Let $f \in A$ be a nonzerodivisor of a ring A. Let $u \in A$ be a unit. Let $M^\bullet$ be a bounded complex of finite free A-modules. Then $I_i(M^\bullet, f) = I_i(M^\bullet, uf)$.*

Proof. Omitted. □

0GSU **Lemma 15.97.3.** *Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B. Then $I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$.*

Proof. The minors of $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ map to the corresponding minors of $(g, d^i) : M^i \otimes_A B \to M^i \otimes_A B \oplus M^{i+1} \otimes_A B$. □

0GSV **Lemma 15.97.4.** *Let A be a ring, let $\mathfrak{p} \subset A$ be a prime ideal, and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. If $H^i(M^\bullet)_\mathfrak{p}$ is free for all i, then $I_i(M^\bullet, f)_\mathfrak{p}$ is a principal ideal and in fact generated by a power of f for all i.*

Proof. We may assume A is local with maximal ideal $\mathfrak{p}$ by Lemma 15.97.3. We may also replace $M^\bullet$ with a quasi-isomorphic complex by Lemma 15.97.1. By our assumption on the freeness of cohomology modules we see that $M^\bullet$ is quasi-isomorphic to the complex whose term in degree i is $H^i(M^\bullet)$ with vanishing differentials, see for example Derived Categories, Lemma 13.27.9. In other words, we may assume the differentials in the complex $M^\bullet$ are all zero. In this case it is clear that $I_i(M^\bullet, f) = (f^{r_i})$ is principal. □

0F7W **Lemma 15.97.5.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal. Then $(\eta_f M)^i$ is locally free of rank r_i and the map $(1, d^i) : (\eta_f M)^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is the inclusion of a direct summand.*

Proof. Choose a generator g for $I_i(M^\bullet, f)$. Since $f^{r_i} \in I_i(M^\bullet, f)$ we see that g divides a power of f. In particular g is a nonzerodivisor in A. The $r_i \times r_i$-minors of the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ generate the ideal $I_i(M^\bullet, f)$ and the $(r_i + 1) \times (r_i + 1)$-minors of (f, d^i) are zero: we may check this after localizing at f where the rank of the map is equal to r_i. Consider the surjection

$$M^i \oplus M^{i+1} \longrightarrow Q = \mathrm{Coker}(f, d^i)/g\text{-torsion}$$

By Lemma 15.8.10 the module Q is finite locally free of rank r_{i+1}. Hence Q is f-torsion free and we conclude the cokernel of (f, d^i) modulo f-power torsion is Q as well.

Consider the complex of finite free A-modules

$$0 \to f^{i+1} M^i \xrightarrow{1, d^i} f^i M^i \oplus f^{i+1} M^{i+1} \xrightarrow{d^i, -1} f^i M^{i+1} \to 0$$

which becomes split exact after localizing at f. The map $(1, d^i) : f^{i+1} M^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is isomorphic to the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ we studied above. Hence the image

$$Q' = \mathrm{Im}(f^i M^i \oplus f^{i+1} M^{i+1} \xrightarrow{d^i, -1} f^i M^{i+1})$$

is isomorphic to Q in particular projective. On the other hand, by construction of η_f in Section 15.96 the image of the injective map $(1, d^i) : (\eta_f M)^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is the kernel of $(d^i, -1)$. We conclude that we obtain an isomorphism $(\eta_f M)^i \oplus Q' = f^i M^i \oplus f^{i+1} M^{i+1}$ and we see that indeed $\eta_f M^i$ is finite locally free of rank r_i and that $(1, d^i)$ is the inclusion of a direct summand. □

0F7X **Lemma 15.97.6.** *Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B and $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then there is a canonical isomorphism $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.*

Proof. Set $N^i = M^i \otimes_A B$. Observe that $f^i M^i \otimes_A B = g^i N^i$ as submodules of $(N^i)_g$. The maps

$$(\eta_f M)^i \otimes_A B \to g^i N^i \otimes g^{i+1} N^{i+1} \quad \text{and} \quad (\eta_g N)^i \to g^i N^i \otimes g^{i+1} N^{i+1}$$

are inclusions of direct summands by Lemma 15.97.5. Since their images agree after localizing at g we conclude. □

0F80 **Lemma 15.97.7.** *Let A be a ring. Let M, N_1, N_2 be finite projective A-modules. Let $s : M \to N_1 \oplus N_2$ be a split injection. There exists a finitely generated ideal $J \subset A$ with the following property: a ring map $A \to B$ factors through A/J if and only if $s \otimes id_B$ identifies $M \otimes_A B$ with a direct sum of submodules of $N_1 \otimes_A B \oplus N_2 \otimes_A B$.*

Proof. Choose a splitting $\pi : N_1 \oplus N_2 \to M$ of s. Denote $q_i : N_1 \oplus N_2 \to N_1 \oplus N_2$ the projector onto N_i. Set $p_i = \pi \circ q_i \circ s$. Observe that $p_1 + p_2 = \mathrm{id}_M$. We claim M is a direct sum of submodules of $N_1 \oplus N_2$ if and only if p_1 and p_2 are orthogonal projectors. Thus J is the smallest ideal of A such that $p_1 \circ p_1 - p_1$, $p_2 \circ p_2 - p_2$, $p_1 \circ p_2$, and $p_2 \circ p_1$ are contained in $J \otimes_A \mathrm{End}_A(M)$. Some details omitted. □

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume the ideals $I_i(M^\bullet, f)$ are principal for all $i \in \mathbf{Z}$. Then the maps

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^i M^i / f^{i+1} M^i \oplus f^{i+1} M^{i+1} / f^{i+2} M^{i+1}$$

are split injections by Lemma 15.97.5. Denote $J_i(M^\bullet, f) \subset A/fA$ the finitely generated ideal of Lemma 15.97.7 corresponding to the split injection $(1, d^i)$ displayed above.

0GSW **Lemma 15.97.8.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object in $D(A)$. Assume $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then $J_i(M^\bullet, f) = J_i(N^\bullet, f)$ as ideals in A/fA.*

Proof. Observe that the fact that $I_i(M^\bullet, f)$ is a principal ideal implies that $I_i(M^\bullet, f)$ is a principal ideal by Lemma 15.97.1 and hence the statement makes sense. As in the proof of Lemma 15.97.1 we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j + 1$. Since η_f is compatible with direct sums, we see that the map

$$(1, d^i) : (\eta_f N)^i / f(\eta_f N)^i \longrightarrow f^i N^i / f^{i+1} N^i \oplus f^{i+1} N^{i+1} / f^{i+2} N^{i+1}$$

is the direct sum of the corresponding map for $M^\bullet$ and for $Q^\bullet$. By the universal property defining the ideals in question, we conclude that $J_i(N^\bullet, f) = J_i(M^\bullet, f) + J_i(Q^\bullet, f)$. Hence it suffices to show that $J_i(Q^\bullet, f) = 0$ for all i. This is a computation that we omit. □

0F81 **Lemma 15.97.9.** *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Consider the ideal $J(M^\bullet, f) = \sum_i J_i(M^\bullet, f)$ of A/fA. Consider the set of prime ideals*

$$\begin{aligned} E &= \{f \in \mathfrak{p} \subset A \mid \mathrm{Ker}(d^i \bmod f^2)_\mathfrak{p} \text{ surjects onto } \mathrm{Ker}(d^i \bmod f)_\mathfrak{p} \text{ for all } i \in \mathbf{Z}\} \\ &= \{f \in \mathfrak{p} \subset A \mid \text{the localizations } \beta_\mathfrak{p} \text{ of the Bockstein operators are zero}\} \end{aligned}$$

Then we have

(1) *$J(M^\bullet, f)$ is finitely generated,*
(2) *$A/fA \to C = (A/fA)/J(M^\bullet, f)$ is surjective of finite presentation,*

(3) $J(M^\bullet, f)_\mathfrak{p} = 0$ *for* $\mathfrak{p} \in E$,
(4) *if* $f \in \mathfrak{p}$ *and* $H^i(M^\bullet)_\mathfrak{p}$ *is free for all* $i \in \mathbf{Z}$, *then* $\mathfrak{p} \in E$, *and*
(5) *the cohomology modules of* $\eta_f M^\bullet \otimes_A C$ *are finite locally free* C*-modules.*

Proof. The equality in the definition of E follows from Lemma 15.96.7 and in addition the final statement of that lemma implies part (4).

Part (1) is true because the ideals $J_i(M^\bullet, f)$ are finitely generated and because $M^\bullet$ is bounded and hence $J_i(M^\bullet, f)$ is zero for almost all i. Part (2) is just a reformulation of part (1).

Proof of (3). By Lemma 15.97.5 we find that $(\eta_f M)^i$ is finite locally free of rank r_i for all i. Consider the map

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^i M^i / f^{i+1} M^i \oplus f^{i+1} M^{i+1} / f^{i+2} M^{i+1}$$

Pick $\mathfrak{p} \in E$. By Lemma 15.96.8 and the local freeness of the modules $(\eta_f M)^i$ we may write

$$\big((\eta_f M)^i / f(\eta_f M)^i\big)_\mathfrak{p} = (A/fA)_\mathfrak{p}^{\oplus m_i} \oplus (A/fA)_\mathfrak{p}^{\oplus n_i}$$

compatible with the arrow $(1, d^i)$ above. By the universal property of the ideal $J_i(M^\bullet, f)$ we conclude that $J_i(M^\bullet, f)_\mathfrak{p} = 0$. Hence $I_\mathfrak{p} = fA_\mathfrak{p}$ for $\mathfrak{p} \in E$.

Proof of (5). Observe that the differential on $\eta_f M^\bullet$ fits into a commutative diagram

$$\begin{array}{ccc} (\eta_f M)^i & \longrightarrow & f^i M^i \oplus f^{i+1} M^{i+1} \\ \downarrow & & \downarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ (\eta_f M)^{i+1} & \longrightarrow & f^{i+1} M^i \oplus f^{i+2} M^{i+2} \end{array}$$

By construction, after tensoring with C, the modules on the left are direct sums of direct summands of the summands on the right. Picture

$$\begin{array}{ccccc} (\eta_f M)^i \otimes_A C & = & K^i \oplus L^i & \longrightarrow & f^i M^i \otimes_A C \oplus f^{i+1} M^{i+1} \otimes_A C \\ \downarrow & & \downarrow & & \downarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ (\eta_f M)^{i+1} \otimes_A C & = & K^{i+1} \oplus L^{i+1} & \longrightarrow & f^{i+1} M^i \otimes_A C \oplus f^{i+2} M^{i+2} \otimes_A C \end{array}$$

where the horizontal arrows are compatible with direct sum decompositions as well as inclusions of direct summands. It follows that the differential identifies L^i with a direct summand of K^{i+1} and we conclude that the cohomology of $\eta_f M^\bullet \otimes_A C$ in degree i is the module K^{i+1}/L^i which is finite projective as desired. □

15.98. Taking limits of complexes

09B6 In this section we discuss what happens when we have a "formal deformation" of a complex and we take its limit. We will consider two cases

(1) we have a limit $A = \lim A_n$ of an inverse system of rings whose transition maps are surjective with locally nilpotent kernels and objects $K_n \in D(A_n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^\mathbf{L}_{A_{n+1}} A_n$, or
(2) we have a ring A, an ideal I, and objects $K_n \in D(A/I^n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^\mathbf{L}_{A/I^{n+1}} A/I^n$.

Under additional hypotheses we can show that $K = R\lim K_n$ reproduces the system in the sense that $K_n = K \otimes^\mathbf{L}_A A_n$ or $K_n = K \otimes^\mathbf{L}_A A/I^n$.

0CQF **Lemma 15.98.1.** *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume*

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *either all K_n are pseudo-coherent or there exists an integer n_0 such that K_{n_0} is pseudo-coherent and the kernels of $A_{n+1} \to A_n$ are nilpotent ideals for $n \geq n_0$,*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n \to K_n$.*

Then $K = R\lim K_n$ is a pseudo-coherent object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n.

Proof. By assumption we can find a bounded above complex of finite free A_1-modules $P_1^\bullet$ representing K_1, see Definition 15.65.1. By Lemma 15.76.4 we conclude that K_n is pseudo-coherent for all n. Then by Lemma 15.76.5 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of finite free A_n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet \otimes_{A_n} A_{n-1} \to P_{n-1}^\bullet$. Thus $K = R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.88.1 and Remark 15.88.6. Since P_n^i is a finite free A_n-module for each n and $A = \lim A_n$ we see that P^i is finite free of the same rank as P_1^i for each i. This means that K is pseudo-coherent. It also follows that $K \otimes^{\mathbf{L}}_A A_n$ is represented by $P^\bullet \otimes_A A_n = P_n^\bullet$ which proves the final assertion. □

09AV **Lemma 15.98.2.** *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

(1) *A is I-adically complete,*
(2) *K_1 is pseudo-coherent, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a pseudo-coherent, derived complete object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n, see Lemma 15.98.1. Finally, K is derived complete by Lemma 15.92.14. □

0CQG **Lemma 15.98.3.** *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume* [**Bha16**, Lemma 4.2]

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *either all K_n are perfect or there exists an integer n_0 such that K_{n_0} is perfect and the kernels $A_{n+1} \to A_n$ are nilpotent for $n \geq n_0$, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n \to K_n$.*

Then $K = R\lim K_n$ is a perfect object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n by Lemma 15.98.1. Thus it suffices to show that $H^i(K \otimes^{\mathbf{L}}_A \kappa) = 0$ for $i \ll 0$ and every surjective map $A \to \kappa$ whose kernel is a maximal ideal $\mathfrak{m}$, see Lemma 15.78.3. Any element of A which maps to a unit in A_1 is a unit in A by Algebra, Lemma 10.32.4 and hence $\mathrm{Ker}(A \to A_1)$ is contained in the Jacobson

radical of A by Algebra, Lemma 10.19.1. Hence $A \to \kappa$ factors as $A \to A_1 \to \kappa$. Hence

$$K \otimes^{\mathbf{L}}_A \kappa = K \otimes^{\mathbf{L}}_A A_1 \otimes^{\mathbf{L}}_{A_1} \kappa = K_1 \otimes^{\mathbf{L}}_{A_1} \kappa$$

and we get what we want as K_1 has finite tor dimension by Lemma 15.75.2. □

09AW **Lemma 15.98.4.** *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

(1) *A is I-adically complete,*
(2) *K_1 is a perfect object, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a perfect, derived complete object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. Combine Lemmas 15.98.3 and 15.98.2 (to get derived completeness). □

We do not know if the following lemma holds for unbounded complexes.

09AU **Lemma 15.98.5.** *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. If*

(1) *A is Noetherian,*
(2) *K_1 is bounded above, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$,*

then $K = R\lim K_n$ is a derived complete object of $D^-(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. The object K of $D(A)$ is derived complete by Lemma 15.92.14.

Suppose that $H^i(K_1) = 0$ for $i > b$. Then we can find a complex of free A/I-modules $P_1^\bullet$ representing K_1 with $P_1^i = 0$ for $i > b$. By Lemma 15.76.3 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of free A/I^n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet/I^{n-1}P_n^\bullet \to P_{n-1}^\bullet$.

Thus we have arrived at the situation where $R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.88.1 and Remark 15.88.6. The complexes $P_n^\bullet$ are uniformly bounded above complexes of flat A/I^n-modules and the transition maps are termwise surjective. Then $P^\bullet$ is a bounded above complex of flat A-modules by Lemma 15.27.4. It follows that $K \otimes^{\mathbf{L}}_A A/I^t$ is represented by $P^\bullet \otimes_A A/I^t$. We have $P^\bullet \otimes_A A/I^t = \lim P_n^\bullet \otimes_A A/I^t$ termwise by Lemma 15.27.4. The transition maps $P_{n+1}^\bullet \otimes_A A/I^t \to P_n^\bullet \otimes_A A/I^t$ are isomorphisms for $n \geq t$ by our choice of $P_n^\bullet$, hence we have $\lim P_n^\bullet \otimes_A A/I^t = P_t^\bullet \otimes_A A/I^t = P_t^\bullet$. Since $P_t^\bullet$ represents K_t, we see that $K \otimes^{\mathbf{L}}_A A/I^t \to K_t$ is an isomorphism. □

Here is a different type of result.

0EGS **Lemma 15.98.6** (Kollár-Kovács)**.** *Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$. Set $K_n = K \otimes^{\mathbf{L}}_A A/I^n$. Assume for all $i \in \mathbf{Z}$ we have* Email from Kovacs of 23/02/2018.

(1) *$H^i(K)$ is a finite A-module, and*
(2) *the system $H^i(K_n)$ satisfies Mittag-Leffler.*

Then $\lim H^i(K)/I^nH^i(K)$ is equal to $\lim H^i(K_n)$ for all $i \in \mathbf{Z}$.

Proof. Recall that $K^\wedge = R\lim K_n$ is the derived completion of K, see Proposition 15.95.2. By Lemma 15.95.4 we have $H^i(K^\wedge) = \lim H^i(K)/I^n H^i(K)$. By Lemma 15.88.4 we get short exact sequences

$$0 \to R^1 \lim H^{i-1}(K_n) \to H^i(K^\wedge) \to \lim H^i(K_n) \to 0$$

The Mittag-Leffler condition guarantees that the left terms are zero (Lemma 15.88.1) and we conclude the lemma is true. □

15.99. Some evaluation maps

0ATJ In this section we prove that certain canonical maps of $R\mathrm{Hom}$'s are isomorphisms for suitable types of complexes.

0A68 **Lemma 15.99.1.** *Let R be a ring. Let K, L, M be objects of $D(R)$. the map*

$$R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R K \longrightarrow R\mathrm{Hom}_R(R\mathrm{Hom}_R(K, L), M)$$

of Lemma 15.74.3 is an isomorphism in the following two cases

(1) *K perfect, or*
(2) *K is pseudo-coherent, $L \in D^+(R)$, and M finite injective dimension.*

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a bounded above complex of finite projective modules $K^\bullet$ representing K. Consider the map of complexes

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 15.72.6. Note that

$$\Big(\prod\nolimits_{p+r=t} \mathrm{Hom}_R(J^{-r}, I^p)\Big) \otimes_R K^s = \prod\nolimits_{p+r=t} \mathrm{Hom}_R(J^{-r}, I^p) \otimes_R K^s$$

because K^s is finite projective. The map is given by the maps

$$c_{p,r,s} : \mathrm{Hom}_R(J^{-r}, I^p) \otimes_R K^s \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^s, J^{-r}), I^p)$$

which are isomorphisms as K^s is finite projective. For every element $\alpha = (\alpha^{p,r,s})$ of degree n of the left hand side, there are only finitely many values of s such that $\alpha^{p,r,s}$ is nonzero (for some p, r with $n = p+r+s$). Hence our map is an isomorphism if the same vanishing condition is forced on the elements $\beta = (\beta^{p,r,s})$ of the right hand side. If $K^\bullet$ is a bounded complex of finite projective modules, this is clear. On the other hand, if we can choose $I^\bullet$ bounded and $J^\bullet$ bounded below, then $\beta^{p,r,s}$ is zero for p outside a fixed range, for $s \gg 0$, and for $r \gg 0$. Hence among solutions of $n = p + r + s$ with $\beta^{p,r,s}$ nonzero only a finite number of s values occur. □

0A69 **Lemma 15.99.2.** *Let R be a ring. Let K, L, M be objects of $D(R)$. the map*

$$R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R K \longrightarrow R\mathrm{Hom}_R(R\mathrm{Hom}_R(K, L), M)$$

of Lemma 15.74.3 is an isomorphism if the following three conditions are satisfied

(1) *L, M have finite injective dimension,*
(2) *$R\mathrm{Hom}_R(L, M)$ has finite tor dimension,*
(3) *for every $n \in \mathbf{Z}$ the truncation $\tau_{\leq n}K$ is pseudo-coherent*

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 13.12.4. By assumption (3) and Lemma 15.99.1 the map is an isomorphism for $\tau_{\leq n}K$. Hence it suffices to show that both

$$R\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R \tau_{\geq n+1}K \quad \text{and} \quad R\mathrm{Hom}_R(R\mathrm{Hom}_R(\tau_{\geq n+1}K, L), M)$$

have vanishing cohomology in degrees $\leq n - c$ for some c. This follows immediately from assumptions (2) and (1). □

0ATK **Lemma 15.99.3.** *Let R be a ring. Let K, L, M be objects of $D(R)$. The map*

$$K \otimes^{\mathbf{L}}_R R\mathrm{Hom}_R(M, L) \longrightarrow R\mathrm{Hom}_R(M, K \otimes^{\mathbf{L}}_R L)$$

of Lemma 15.74.5 is an isomorphism in the following cases

(1) *M perfect, or*
(2) *K is perfect, or*
(3) *M is pseudo-coherent, $L \in D^+(R)$, and K has tor amplitude in $[a, \infty]$.*

Proof. Proof in case M is perfect. Note that both sides of the arrow transform distinguished triangles in M into distinguished triangles and commute with direct sums. Hence it suffices to check it holds when $M = R[n]$, see Derived Categories, Remark 13.36.7 and Lemma 15.79.1. In this case the result is obvious.

Proof in case K is perfect. Same argument as in the previous case.

Proof in case (3). We may represent K and L by bounded below complexes of R-modules $K^\bullet$ and $L^\bullet$. We may assume that $K^\bullet$ is a K-flat complex consisting of flat R-modules, see Lemma 15.67.4. We may represent M by a bounded above complex $M^\bullet$ of finite free R-modules, see Definition 15.65.1. Then the object on the LHS is represented by

$$\mathrm{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet))$$

and the object on the RHS by

$$\mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

This uses Lemma 15.74.2. Both complexes have in degree n the module

$$\bigoplus\nolimits_{p+q+r=n} K^p \otimes \mathrm{Hom}_R(M^{-r}, L^q) = \bigoplus\nolimits_{p+q+r=n} \mathrm{Hom}_R(M^{-r}, K^p \otimes_R L^q)$$

because M^{-r} is finite free (as well these are finite direct sums). The map defined in Lemma 15.74.5 comes from the map of complexes defined in Lemma 15.72.4 which uses the canonical isomorphisms between these modules. □

0BYQ **Lemma 15.99.4.** *Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $K^\bullet$ be a K-flat complex of R-modules. If $P^\bullet$ is a perfect object of $D(R)$, then $\mathrm{Hom}^\bullet(P^\bullet, K^\bullet)$ is K-flat and represents $R\mathrm{Hom}_R(P^\bullet, K^\bullet)$.*

Proof. The last statement is Lemma 15.74.2. Since $P^\bullet$ represents a perfect object, there exists a finite complex of finite projective R-modules $F^\bullet$ such that $P^\bullet$ and $F^\bullet$ are isomorphic in $D(R)$, see Definition 15.75.1. Then $P^\bullet$ and $F^\bullet$ are homotopy equivalent, see Derived Categories, Lemma 13.19.8. Then $\mathrm{Hom}^\bullet(P^\bullet, K^\bullet)$ and

$\mathrm{Hom}^\bullet(F^\bullet, K^\bullet)$ are homotopy equivalent. Hence the first is K-flat if and only if the second is (follows from Definition 15.59.1 and Lemma 15.58.2). It is clear that

$$\mathrm{Hom}^\bullet(F^\bullet, K^\bullet) = \mathrm{Tot}(E^\bullet \otimes_R K^\bullet)$$

where $E^\bullet$ is the dual complex to $F^\bullet$ with terms $E^n = \mathrm{Hom}_R(F^{-n}, R)$, see Lemma 15.75.15 and its proof. Since $E^\bullet$ is a bounded complex of projectives we find that it is K-flat by Lemma 15.59.7. Then we conclude by Lemma 15.59.4. □

15.100. Base change for derived hom

0E1V We have already seen some material discussing this in Lemma 15.66.4 and in Algebra, Section 10.73.

0E1W **Lemma 15.100.1.** *Let $R \to R'$ be a ring map. For $K \in D(R)$ and $M \in D(R')$ there is a canonical isomorphism*

$$R\mathrm{Hom}_R(K, M) = R\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R R', M)$$

Proof. Choose a K-injective complex of R'-modules $J^\bullet$ representing M. Choose a quasi-isomorphism $J^\bullet \to I^\bullet$ where $I^\bullet$ is a K-injective complex of R-modules. Choose a K-flat complex $K^\bullet$ of R-modules representing K. Consider the map

$$\mathrm{Hom}^\bullet(K^\bullet \otimes_R R', J^\bullet) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, I^\bullet)$$

The map on degree n terms is given by the map

$$\prod\nolimits_{n=p+q} \mathrm{Hom}_{R'}(K^{-q} \otimes_R R', J^p) \longrightarrow \prod\nolimits_{n=p+q} \mathrm{Hom}_R(K^{-q}, I^p)$$

coming from precomposing by $K^{-q} \to K^{-q} \otimes_R R'$ and postcomposing by $J^p \to I^p$. To finish the proof it suffices to show that we get isomorphisms on cohomology groups:

$$\mathrm{Hom}_{D(R)}(K, M) = \mathrm{Hom}_{D(R')}(K \otimes^{\mathbf{L}}_R R', M)$$

which is true because base change $- \otimes^{\mathbf{L}}_R R' : D(R) \to D(R')$ is left adjoint to the restriction functor $D(R') \to D(R)$ by Lemma 15.60.3. □

Let $R \to R'$ be a ring map. There is a base change map

0E1X (15.100.1.1) $$R\mathrm{Hom}_R(K, M) \otimes^{\mathbf{L}}_R R' \longrightarrow R\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R R', M \otimes^{\mathbf{L}}_R R')$$

in $D(R')$ functorial in $K, M \in D(R)$. Namely, by adjointness of $- \otimes^{\mathbf{L}}_R R' : D(R) \to D(R')$ and the restriction functor $D(R') \to D(R)$, this is the same thing as a map

$$R\mathrm{Hom}_R(K, M) \longrightarrow R\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R R', M \otimes^{\mathbf{L}}_R R') = R\mathrm{Hom}_R(K, M \otimes^{\mathbf{L}}_R R')$$

(equality by Lemma 15.100.1) for which we can use the canonical map $M \to M \otimes^{\mathbf{L}}_R R'$ (unit of the adjunction).

0A6A **Lemma 15.100.2.** *Let $R \to R'$ be a ring map. Let $K, M \in D(R)$. The map (15.100.1.1)*

$$R\mathrm{Hom}_R(K, M) \otimes^{\mathbf{L}}_R R' \longrightarrow R\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R R', M \otimes^{\mathbf{L}}_R R')$$

is an isomorphism in $D(R')$ in the following cases

1. *K is perfect,*
2. *R' is perfect as an R-module,*
3. *$R \to R'$ is flat, K is pseudo-coherent, and $M \in D^+(R)$, or*
4. *R' has finite tor dimension as an R-module, K is pseudo-coherent, and $M \in D^+(R)$*

Proof. We may check the map is an isomorphism after applying the restriction functor $D(R') \to D(R)$. After applying this functor our map becomes the map

$$R\operatorname{Hom}_R(K, L) \otimes^{\mathbf{L}}_R R' \longrightarrow R\operatorname{Hom}_R(K, L \otimes^{\mathbf{L}}_R R')$$

of Lemma 15.74.5. See discussion above the lemma to match the left and right hand sides; in particular, this uses Lemma 15.100.1. Thus we conclude by Lemma 15.99.3. □

15.101. Systems of modules

0EGT Let I be an ideal of a Noetherian ring A. In this section we add to our knowledge of the relationship between finite modules over A and systems of finite A/I^n-modules.

0EGU **Lemma 15.101.1.** *Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \operatorname{Ker}(\beta)/\operatorname{Im}(\alpha)$. For $n \geq 0$ let*

$$K/I^nK \xrightarrow{\alpha_n} L/I^nL \xrightarrow{\beta_n} M/I^nM$$

be the induced complex. Set $H_n = \operatorname{Ker}(\beta_n)/\operatorname{Im}(\alpha_n)$. Then there are canonical A-module maps giving a commutative diagram

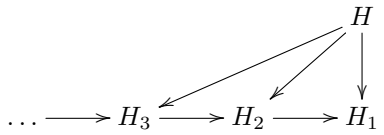

Moreover, there exists a $c > 0$ and canonical A-module maps $H_n \to H/I^{n-c}H$ for $n \geq c$ such that the compositions

$$H/I^nH \to H_n \to H/I^{n-c}H \quad \text{and} \quad H_n \to H/I^{n-c}H \to H_{n-c}$$

are the canonical ones. Moreover, we have

1. *(H_n) and (H/I^nH) are isomorphic as pro-objects of Mod_A,*
2. *$\lim H_n = \lim H/I^nH$,*
3. *the inverse system (H_n) is Mittag-Leffler,*
4. *the image of $H_{n+c} \to H_n$ is equal to the image of $H \to H_n$,*
5. *the composition $I^cH_n \to H_n \to H/I^{n-c}H \to H_n/I^{n-c}H_n$ is the inclusion $I^cH_n \to H_n$ followed by the quotient map $H_n \to H_n/I^{n-c}H_n$, and*
6. *the kernel and cokernel of $H/I^nH \to H_n$ is annihilated by I^c.*

Proof. Observe that $H_n = \beta^{-1}(I^nM)/\operatorname{Im}(\alpha) + I^nL$. For $n \geq 2$ we have $\beta^{-1}(I^nM) \subset \beta^{-1}(I^{n-1}M)$ and $\operatorname{Im}(\alpha) + I^nL \subset \operatorname{Im}(\alpha) + I^{n-1}L$. Thus we obtain our canonical map $H_n \to H_{n-1}$. Similarly, we have $\operatorname{Ker}(\beta) \subset \beta^{-1}(I^nM)$ and $\operatorname{Im}(\alpha) \subset \operatorname{Im}(\alpha) + I^nL$ which produces the canonical map $H \to H_n$. We omit the verification that the diagram commutes.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\beta^{-1}(I^nM) \subset \operatorname{Ker}(\beta) + I^{n-c_1}L$ for $n \geq c_1$ and $\operatorname{Ker}(\beta) \cap I^nL \subset I^{n-c_2}\operatorname{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 10.51.3 and 10.51.2. Set $c = c_1 + c_2$.

Let $n \geq c$. We define $\psi_n : H_n \to H/I^{n-c}H$ as follows. Say $x \in H_n$. Choose $y \in \beta^{-1}(I^nM)$ representing x. Write $y = z + w$ with $z \in \operatorname{Ker}(\beta)$ and $w \in I^{n-c_1}L$ (this is possible by our choice of c_1). We set $\psi_n(x)$ equal to the class of z in $H/I^{n-c}H$. To see this is well defined, suppose we have a second set of choices

y', z', w' as above for x with obvious notation. Then $y' - y \in \mathrm{Im}(\alpha) + I^n L$, say $y' - y = \alpha(v) + u$ with $v \in K$ and $u \in I^n L$. Thus

$$y' = z' + w' = \alpha(v) + u + z + w \Rightarrow z' = z + \alpha(v) + u + w - w'$$

Since $\beta(z' - z - \alpha(v)) = 0$ we find that $u + w - w' \in \mathrm{Ker}(\beta) \cap I^{n-c_1} L$ which is contained in $I^{n-c_1-c_2} \mathrm{Ker}(\beta) = I^{n-c} \mathrm{Ker}(\beta)$ by our choice of c_2. Thus z' and z have the same image in $H/I^{n-c}H$ as desired.

The composition $H/I^n H \to H_n \to H/I^{n-c}H$ is the canonical map because if $z \in \mathrm{Ker}(\beta)$ represents an element x in $H/I^n H = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha) + I^n \mathrm{Ker}(\beta)$ then it is clear from the above that x maps to the class of z in $H/I^{n-c}H$ under the maps constructed above.

Let us consider the composition $H_n \to H/I^{n-c}H \to H_{n-c}$. Given x, y, z, w as in the construction of ψ_n above, we see that x is mapped to the cass of z in H_{n-c}. On the other hand, the canonical map $H_n \to H_{n-c}$ from the first paragraph of the proof sends x to the class of y. Thus we have to show that $y - z \in \mathrm{Im}(\alpha) + I^{n-c} L$ which is the case because $y - z = w \in I^{n-c_1} L \subset I^{n-c} L$.

Statements (1) – (4) are formal consequences of what we just proved. Namely, (1) follows from the existence of the maps and the definition of morphisms of pro-objects in Categories, Remark 4.22.5. Part (2) holds because isomorphic pro-objects have isomorphic limits. Part (3) is immediate from part (4). Part (4) follows from the factorization $H_{n+c} \to H/I^n H \to H_n$ of the canonical map $H_{n+c} \to H_n$.

Proof of part (5). Let $x \in I^c H_n$. Write $x = \sum f_i x_i$ with $x_i \in H_n$ and $f_i \in I^c$. Choose y_i, z_i, w_i as in the construction of ψ_n for x_i. Then for the computation of ψ_n of x we may choose $y = \sum f_i y_i$, $z = \sum f_i z_i$ and $w = \sum f_i w_i$ and we see that $\psi_n(x)$ is given by the class of z. The image of this in $H_n/I^{n-c}H_n$ is equal to the class of y as $w = \sum f_i w_i$ is in $I^n L$. This proves (5).

Proof of part (6). Let $y \in \mathrm{Ker}(\beta)$ whose class is x in H. If x maps to zero in H_n, then $y \in I^n L + \mathrm{Im}(\alpha)$. Hence $y - \alpha(v) \in \mathrm{Ker}(\beta) \cap I^n L$ for some $v \in K$. Then $y - \alpha(v) \in I^{n-c_2} \mathrm{Ker}(\beta)$ and hence the class of y in $H/I^n H$ is annihilated by I^{c_2}. Finally, let $x \in H_n$ be the class of $y \in \beta^{-1}(I^n M)$. Then we write $y = z + w$ with $z \in \mathrm{Ker}(\beta)$ and $w \in I^{n-c_1} L$ as above. Clearly, if $f \in I^{c_1}$ then fx is the class of $fy + fw \equiv fy$ modulo $\mathrm{Im}(\alpha) + I^n L$ and hence fx is the image of the class of fy in H as desired. □

0EGV **Lemma 15.101.2.** *Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$ be pseudo-coherent. Set $K_n = K \otimes^{\mathbf{L}}_A A/I^n$. Then for all $i \in \mathbf{Z}$ the system $H^i(K_n)$ satisfies Mittag-Leffler and $\lim H^i(K)/I^n H^i(K)$ is equal to $\lim H^i(K_n)$.* Email from Kovacs of 23/02/2018.

Proof. We may represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Then K_n is represented by $P^\bullet/I^n P^\bullet$. Hence the Mittag-Leffler property by Lemma 15.101.1. The final statement follows then from Lemma 15.98.6. □

0G9M **Lemma 15.101.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps*

$$M^\bullet \otimes^{\mathbf{L}}_A A/I^n \longrightarrow M^\bullet/I^n M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Let $K_n \in D(A)$ be the object represented by the Koszul complex on $f_1^n, \ldots, f_r^n$. Recall that we have maps $K_n \to A/I^n$ which induce a pro-isomorphism of inverse systems, see Lemma 15.95.1. Hence it suffices to show that

$$M^\bullet \otimes^{\mathbf{L}}_A K_n \longrightarrow M^\bullet/I^n M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Since K_n is represented by a complex of finite free A-modules sitting in degrees $-r, \ldots, 0$ there exist $a, b \in \mathbf{Z}$ such that the source and target of the displayed arrow have vanishing cohomology in degrees outside $[a, b]$ for all n. Thus we may apply Derived Categories, Lemma 13.42.5 and we find that it suffices to show that the maps

$$H^i(M^\bullet \otimes^{\mathbf{L}}_A A/I^n) \to H^i(M^\bullet/I^n M^\bullet)$$

define isomorphisms of pro-systems of A-modules for any $i \in \mathbf{Z}$. To see this choose a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a bounded above complex of finite free A-modules. The arrows above are given by the maps

$$H^i(P^\bullet/I^n P^\bullet) \to H^i(M^\bullet/I^n M^\bullet)$$

These define an isomorphism of pro-systems by Lemma 15.101.1. Namely, the lemma shows both are isomorphic to the pro-system $H^i/I^n H^i$ with $H^i = H^i(M^\bullet) = H^i(P^\bullet)$. □

09BB **Lemma 15.101.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^n M$ and $N_n = N/I^n N$. Then*

1. *the systems $(\mathrm{Hom}_A(M_n, N_n))$ and $(Isom_A(M_n, N_n))$ are Mittag-Leffler,*
2. *there exists a $c \geq 0$ such that the kernels and cokernels of*
$$\mathrm{Hom}_A(M, N)/I^n \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M_n, N_n)$$
are killed by I^c for all n,
3. *we have $\lim \mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N)^\wedge = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$*
4. *$\lim Isom_A(M_n, N_n) = Isom_{A^\wedge}(M^\wedge, N^\wedge)$.*

Here $^\wedge$ denotes usual I-adic completion.

Proof. Note that $\mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N_n)$. Choose a presentation

$$A^{\oplus t} \to A^{\oplus s} \to M \to 0$$

Applying the right exact functor $\mathrm{Hom}_A(-, N)$ we obtain a complex

$$0 \xrightarrow{\alpha} N^{\oplus s} \xrightarrow{\beta} N^{\oplus t}$$

whose cohomology in the middle is $\mathrm{Hom}_A(M, N)$ and such that for $n \geq 0$ the cohomology of

$$0 \xrightarrow{\alpha_n} N_n^{\oplus s} \xrightarrow{\beta_n} N_n^{\oplus t}$$

is $\mathrm{Hom}_A(M_n, N_n)$. Let $c \geq 0$ be as in Lemma 15.101.1 for this A, I, α, and β. By part (3) of the lemma we deduce the Mittag-Leffler property for $(\mathrm{Hom}_A(M_n, N_n))$. The kernel and cokernel of the maps $\mathrm{Hom}_A(M, N)/I^n \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M_n, N_n)$ are killed by I^c by [art part (6) of the lemma. We find that $\lim \mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N)^\wedge$ by part (2) of the lemma. The equality

$$\mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge) = \lim \mathrm{Hom}_A(M_n, N_n)$$

follows formally from the fact that $M^\wedge = \lim M_n$ and $M_n = M^\wedge/I^n M^\wedge$ and the corresponding facts for N, see Algebra, Lemma 10.97.4.

The result for isomorphisms follows from the case of homomorphisms applied to both $(\mathrm{Hom}(M_n, N_n))$ and $(\mathrm{Hom}(N_n, M_n))$ and the following fact: for $n > m > 0$, if we have maps $\alpha : M_n \to N_n$ and $\beta : N_n \to M_n$ which induce an isomorphisms $M_m \to N_m$ and $N_m \to M_m$, then α and β are isomorphisms. Namely, then $\alpha \circ \beta$ is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1) hence $\alpha \circ \beta$ is an isomorphism by Algebra, Lemma 10.16.4. □

09BC **Lemma 15.101.5.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. If $M_n \cong N_n$ for all n, then $M^\wedge \cong N^\wedge$ as $A^\wedge$-modules.*

Proof. By Lemma 15.101.4 the system $(\mathrm{Isom}_A(M_n, N_n))$ is Mittag-Leffler. By assumption each of the sets $\mathrm{Isom}_A(M_n, N_n)$ is nonempty. Hence $\lim \mathrm{Isom}_A(M_n, N_n)$ is nonempty. Since $\lim \mathrm{Isom}_A(M_n, N_n) = \mathrm{Isom}_{A^\wedge}(M^\wedge, N^\wedge)$ we obtain an isomorphism. □

0EGW **Remark 15.101.6.** Let I be an ideal of a Noetherian ring A. Set $A_n = A/I^n$ for $n \geq 1$. Consider the following category:

(1) An object is a sequence $\{E_n\}_{n\geq 1}$ where E_n is a finite A_n-module.
(2) A morphism $\{E_n\} \to \{E'_n\}$ is given by maps

$$\varphi_n : I^c E_n \longrightarrow E'_n/E'_n[I^c] \quad \text{for } n \geq c$$

where $E'_n[I^c]$ is the torsion submodule (Section 15.89) up to equivalence: we say (c, φ_n) is the same as $(c+1, \overline{\varphi}_n)$ where $\overline{\varphi}_n : I^{c+1}E_n \longrightarrow E'_n/E'_n[I^{c+1}]$ is the induced map.

Composition of $(c, \varphi_n) : \{E_n\} \to \{E'_n\}$ and $(c', \varphi'_n) : \{E'_n\} \to \{E''_n\}$ is defined by the obvious compositions

$$I^{c+c'}E_n \to I^{c'}E'_n/E'_n[I^c] \to E''_n/E''_n[I^{c+c'}]$$

for $n \geq c + c'$. We omit the verification that this is a category.

0EGX **Lemma 15.101.7.** *A morphism (c, φ_n) of the category of Remark 15.101.6 is an isomorphism if and only if there exists a $c' \geq 0$ such that $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all $n \gg 0$.*

Proof. We may and do assume $c' \geq c$ and that the $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all n. For $n \geq c'$ and $x \in I^{c'}E'_n$ we can choose $y \in I^cE_n$ with $x = \varphi_n(y) \bmod E'_n[I^c]$ as $\mathrm{Coker}(\varphi_n)$ is annihilated by $I^{c'}$. Set $\psi_n(x)$ equal to the class of y in $E_n/E_n[I^{c'}]$. For a different choice $y' \in I^cE_n$ with $x = \varphi_n(y') \bmod E'_n[I^c]$ the difference $y - y'$ maps to zero in $E'_n/E'_n[I^c]$ and hence is annihilated by $I^{c'}$ in I^cE_n. Thus the maps $\psi_n : I^{c'}E'_n \to E_n/E_n[I^{c'}]$ are well defined. We omit the verification that (c', ψ_n) is the inverse of (c, φ_n) in the category. □

0EGY **Lemma 15.101.8.** *Let I be an ideal of the Noetherian ring A. Let M and N be finite A-modules. Write $A_n = A/I^n$, $M_n = M/I^nM$, and $N_n = N/I^nN$. For every $i \geq 0$ the objects*

$$\{\mathrm{Ext}^i_A(M, N)/I^n \mathrm{Ext}^i_A(M, N)\}_{n\geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(M_n, N_n)\}_{n\geq 1}$$

are isomorphic in the category $\mathcal{C}$ of Remark 15.101.6.

Email correspondence between Janos Kollar, Sandor Kovacs, and Johan de Jong of 23/02/2018.

Proof. Choose a short exact sequence

$$0 \to K \to A^{\oplus r} \to M \to 0$$

and set $K_n = K/I^nK$. For $n \geq 1$ define $K(n) = \mathrm{Ker}(A_n^{\oplus r} \to M_n)$ so that we have exact sequences

$$0 \to K(n) \to A_n^{\oplus r} \to M_n \to 0$$

and surjections $K_n \to K(n)$. In fact, by Lemma 15.101.1 there is a $c \geq 0$ and maps $K(n) \to K_n/I^{n-c}K_n$ which are "almost inverse". Since $I^{n-c}K_n \subset K_n[I^c]$ these maps which witness the fact that the systems $\{K(n)\}_{n\geq 1}$ and $\{K_n\}_{n\geq 1}$ are isomorphic in $\mathcal{C}$.

We claim the systems

$$\{\mathrm{Ext}^i_{A_n}(K(n), N_n)\}_{n\geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(K_n, N_n)\}_{n\geq 1}$$

are isomorphic in the category $\mathcal{C}$. Namely, the surjective maps $K_n \to K(n)$ have kernels annihilated by I^c and therefore determine maps

$$\mathrm{Ext}^i_{A_n}(K(n), N_n) \to \mathrm{Ext}^i_{A_n}(K_n, N_n)$$

whose kernel and cokernel are annihilated by I^c. Hence the claim by Lemma 15.101.7.

For $i \geq 2$ we have isomorphisms

$$\mathrm{Ext}^{i-1}_A(K, N) = \mathrm{Ext}^i_A(M, N) \quad \text{and} \quad \mathrm{Ext}^{i-1}_{A_n}(K(n), N_n) = \mathrm{Ext}^i_{A_n}(M_n, N_n)$$

In this way we see that it suffices to prove the lemma for $i = 0, 1$.

For $i = 0, 1$ we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(M, N) & \longrightarrow & N^{\oplus r} & \xrightarrow{\varphi} & \mathrm{Hom}(K, N) & \longrightarrow & \mathrm{Ext}^1(M, N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & \mathrm{Hom}(K_n, N_n) & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(M_n, N_n) & \longrightarrow & N_n^{\oplus r} & \longrightarrow & \mathrm{Hom}(K(n), N_n) & \longrightarrow & \mathrm{Ext}^1(M_n, N_n) \longrightarrow 0
\end{array}$$

By Lemma 15.101.4 we see that the kernel and cokernel of $\mathrm{Hom}(M, N)/I^n \mathrm{Hom}(M, N) \to \mathrm{Hom}(M_n, N_n)$ and $\mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and are I^c-torsion for some $c \geq 0$ independent of n. Above we have seen the cokernel of the injective maps $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$ are annihilated by I^c after possibly increasing c. For such a c we obtain maps $\delta_n : I^c \mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K(n), N_n)$ fitting into the diagram (precise formulation omitted). The kernel and cokernel of δ_n are annihilated by I^c after possibly increasing c since we know that the same thing is true for $\mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$. Then we can use commutativity of the solid diagram

$$\begin{array}{ccccccc}
\varphi^{-1}(I^c \mathrm{Hom}(K, N)) & \xrightarrow{\varphi} & I^c \mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) & \longrightarrow & I^c \mathrm{Ext}^1(M, N)/I^n \mathrm{Ext}^1(M, N) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \delta_n} & & \vdots & & \\
N_n^{\oplus r} & \longrightarrow & \mathrm{Hom}(K(n), N_n) & \longrightarrow & \mathrm{Ext}^1(M_n, N_n) & \longrightarrow & 0
\end{array}$$

to define the dotted arrow. A straightforward diagram chase (omitted) shows that the kernel and cokernel of the dotted arrow are annihilated buy I^c after possibly increasing c one final time. □

0EGZ **Remark 15.101.9.** The awkwardness in the statement of Lemma 15.101.8 is partly due to the fact that there are no obvious maps between the modules $\mathrm{Ext}^i_{A_n}(M_n, N_n)$ for varying n. What we may conclude from the lemma is that there exists a $c \geq 0$ such that for $m \gg n \gg 0$ there are (canonical) maps

$$I^c \mathrm{Ext}^i_{A_n}(M_m, N_m)/I^n \mathrm{Ext}^i_{A_n}(M_m, N_m) \to \mathrm{Ext}^i_{A_n}(M_n, N_n)/\mathrm{Ext}^i_{A_n}(M_n, N_n)[I^c]$$

whose kernel and cokernel are annihilated by I^c. This is the (weak) sense in which we get a system of modules.

0EH2 **Example 15.101.10.** Let k be a field. Let $A = k[[x, y]]/(xy)$. By abuse of notation we denote x and y the images of x and y in A. Let $I = (x)$. Let $M = A/(y)$. There is a free resolution

$$\ldots \to A \xrightarrow{y} A \xrightarrow{x} A \xrightarrow{y} A \to M \to 0$$

We conclude that

$$\mathrm{Ext}^2_A(M, N) = N[y]/xN$$

where $N[y] = \mathrm{Ker}(y : N \to N)$. We denote $A_n = A/I^n$, $M_n = M/I^n M$, and $N_n = N/I^n N$. For each n we have a free resolution

$$\ldots \to A_n^{\oplus 2} \xrightarrow{y, x^{n-1}} A_n \xrightarrow{x} A_n \xrightarrow{y} A_n \to M_n \to 0$$

We conclude that

$$\mathrm{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n$$

where $N_n[y] = \mathrm{Ker}(y : N_n \to N_n)$ and $N[x^{n-1}] = \mathrm{Ker}(x^{n-1} : N_n \to N_n)$. Take $N = A/(y)$. Then we see that

$$\mathrm{Ext}^2_A(M, N) = N[y]/xN = N/xN \cong k$$

but

$$\mathrm{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n = N_n[x^{n-1}]/xN_n = 0$$

for all r because $N_n = k[x]/(x^n)$ and the sequence

$$N_n \xrightarrow{x} N_n \xrightarrow{x^{n-1}} N_n$$

is exact. Thus ignoring some kind of I-power torsion is necessary to get a result as in Lemma 15.101.8.

0EH0 **Lemma 15.101.11.** *Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $I \subset A$ be an ideal. Let M, N be A-modules. Set $B_n = B/I^n B$, $M_n = M/I^n M$, $N_n = N/I^n N$. If M is flat over A, then we have*

$$\lim \mathrm{Ext}^i_B(M, N)/I^n \mathrm{Ext}^i_B(M, N) = \lim \mathrm{Ext}^i_{B_n}(M_n, N_n)$$

for all $i \in \mathbf{Z}$.

Email correspondence between Janos Kollar, Sandor Kovacs, and Johan de Jong of 23/02/2018.

Proof. Choose a resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

by finite free B-modues P_i. Set $P_{i,n} = P_i/I^nP_i$. Since M and B are flat over A, the sequence

$$\ldots \to P_{2,n} \to P_{1,n} \to P_{0,n} \to M_n \to 0$$

is exact. We see that on the one hand the complex

$$\mathrm{Hom}_B(P_0, N) \to \mathrm{Hom}_B(P_1, N) \to \mathrm{Hom}_B(P_2, N) \to \ldots$$

computes the modules $\mathrm{Ext}^i_B(M, N)$ and on the other hand the complex

$$\mathrm{Hom}_{B_n}(P_{0,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{1,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{2,n}, N_n) \to \ldots$$

computes the modules $\mathrm{Ext}^i_{B_n}(M_n, N_n)$. Since

$$\mathrm{Hom}_{B_n}(P_{i,n}, N_n) = \mathrm{Hom}_B(P_i, N)/I^n \mathrm{Hom}_B(P_i, N)$$

we obtain the result from Lemma 15.101.1 part (2). □

15.102. Systems of modules, bis

0G3J Let I be an ideal of a Noetherian ring A. In Section 15.101 we considered what happens when considering systems of the form M/I^nM for finite A-modules M. In this section we consider the systems I^nM instead.

0G3K **Lemma 15.102.1.** *Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$. For $n \geq 0$ let*

$$I^nK \xrightarrow{\alpha_n} I^nL \xrightarrow{\beta_n} I^nM$$

be the induced complex. Set $H_n = \mathrm{Ker}(\beta_n)/\mathrm{Im}(\alpha_n)$. Then there are canonical A-module maps

$$\ldots \to H_3 \to H_2 \to H_1 \to H$$

There exists a $c > 0$ such that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$ and there is a canonical A-module map $I^nH \to H_{n-c}$ such that the compositions

$$I^nH \to H_{n-c} \to I^{n-2c}H \quad \text{and} \quad H_n \to I^{n-c}H \to H_{n-2c}$$

are the canonical ones. In particular, the inverse systems (H_n) and (I^nH) are isomorphic as pro-objects of Mod_A.

Proof. We have $H_n = \mathrm{Ker}(\beta) \cap I^nL/\alpha(I^nK)$. Since $\mathrm{Ker}(\beta) \cap I^nL \subset \mathrm{Ker}(\beta) \cap I^{n-1}L$ and $\alpha(I^nK) \subset \alpha(I^{n-1}K)$ we get the maps $H_n \to H_{n-1}$. Similarly for the map $H_1 \to H$.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\mathrm{Im}(\alpha) \cap I^nL \subset \alpha(I^{n-c_1}K)$ for $n \geq c_1$ and $\mathrm{Ker}(\beta) \cap I^nL \subset I^{n-c_2}\mathrm{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 10.51.3 and 10.51.2. Set $c = c_1 + c_2$.

It follows immediately from our choice of $c \geq c_2$ that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$.

Let $n \geq c$. We define $\psi_n : I^nH \to H_{n-c}$ as follows. Say $x \in I^nH$. Choose $y \in I^n \mathrm{Ker}(\beta)$ representing x. We set $\psi_n(x)$ equal to the class of y in H_{n-c}. To see this is well defined, suppose we have a second choice y' as above for x. Then $y' - y \in \mathrm{Im}(\alpha)$. By our choice of $c \geq c_1$ we conclude that $y' - y \in \alpha(I^{n-c}K)$ which implies that y and y' represent the same element of H_{n-c}. Thus ψ_n is well defined.

The statements on the compositions $I^nH \to H_{n-c} \to I^{n-2c}H$ and $H_n \to I^{n-c}H \to H_{n-2c}$ follow immediately from our definitions. $\square$

0G3L **Lemma 15.102.2.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite. For each $p > 0$ there exists a $c \geq 0$ such that for $n \geq c$ the map $\mathrm{Ext}^p_A(M, N) \to \mathrm{Ext}^p_A(I^nM, N)$ factors through $\mathrm{Ext}^p_A(I^nM, I^{n-c}N) \to \mathrm{Ext}^p_A(I^nM, N)$.*

Proof. For $p = 0$, if $\varphi : M \to N$ is an A-linear map, then $\varphi(\sum f_i m_i) = \sum f_i \varphi(m_i)$ for $f_i \in A$ and $m_i \in M$. Hence φ induces a map $I^nM \to I^nN$ for all n and the result is true with $c = 0$.

Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. For each n we pick a short exact sequence $0 \to L_n \to A^{\oplus s_n} \to I^nM \to 0$. It is clear that we can construct a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_n & \longrightarrow & A^{\oplus s_n} & \longrightarrow & I^nM & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & A^{\oplus t} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

such that $A^{\oplus s_n} \to A^{\oplus t}$ has image in $(I^n)^{\oplus t}$. By Artin-Rees (Algebra, Lemma 10.51.2) there exists a $c \geq 0$ such that $L_n \to K$ factors through $I^{n-c}K$ if $n \geq c$.

For $p = 1$ our choices above induce a solid commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_A(A^{\oplus s_n}, N) & \longrightarrow & \mathrm{Hom}_A(L_n, N) & \longrightarrow & \mathrm{Ext}^1_A(I^nM, N) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{Hom}_A((I^n)^{\oplus t}, I^{n-c}N) & \longrightarrow & \mathrm{Hom}_A(K \cap (I^n)^{\oplus t}, I^{n-c}N) & \longrightarrow & \mathrm{Ext}^1_A(I^nM, I^{n-c}N) & & \\ \uparrow & & \uparrow & & \vdots & & \\ \mathrm{Hom}_A(A^{\oplus t}, N) & \longrightarrow & \mathrm{Hom}_A(K, N) & \longrightarrow & \mathrm{Ext}^1_A(M, N) & \longrightarrow & 0 \end{array}$$

whose horizontal arrows are exact. The lower middle vertical arrow arises because $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ and hence any A-linear map $K \to N$ induces an A-linear map $(I^n)^{\oplus t} \to I^{n-c}N$ by the argument of the first paragraph. Thus we obtain the dotted arrow as desired.

For $p > 1$ we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}^{p-1}_A(I^{n-c}K, N) & \longrightarrow & \mathrm{Ext}^{p-1}_A(L_n, N) & \longrightarrow & \mathrm{Ext}^p_A(I^nM, N) \\ \uparrow & & & & \uparrow \\ \mathrm{Ext}^{p-1}_A(K, N) & & \longrightarrow & & \mathrm{Ext}^p_A(M, N) \end{array}$$

whose bottom horizontal arrow is an isomorphism. By induction on p the left vertical map factors through $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N)$ for some $c' \geq 0$ and all $n \geq c + c'$. Using the composition $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N) \to \mathrm{Ext}^{p-1}_A(L_n, I^{n-c-c'}N) \to \mathrm{Ext}^p_A(I^nM, I^{n-c-c'}N)$ we obtain the desired factorization (for $n \geq c + c'$ and with c replaced by $c + c'$). $\square$

0927 **Lemma 15.102.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite and N annihilated by a power of I. For each $p > 0$ there exists an n such that the map $\mathrm{Ext}^p_A(M, N) \to \mathrm{Ext}^p_A(I^n M, N)$ is zero.*

Proof. Immediate consequence of Lemma 15.102.2 and the fact that $I^m N = 0$ for some $m > 0$. □

0DYI **Lemma 15.102.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent and let M be a finite A-module. For each $p \in \mathbf{Z}$ there exists an c such that the image of $\mathrm{Ext}^p_A(K, I^n M) \to \mathrm{Ext}^p_A(K, M)$ is contained in $I^{n-c} \mathrm{Ext}^p_A(K, M)$ for $n \geq c$.*

Proof. Choose a bounded above complex $P^\bullet$ of finite free A-modules representing K. Then $\mathrm{Ext}^p_A(K, M)$ is the cohomology of

$$\mathrm{Hom}_A(F^{-p+1}, M) \xrightarrow{a} \mathrm{Hom}_A(F^{-p}, M) \xrightarrow{b} \mathrm{Hom}_A(F^{-p-1}, M)$$

and $\mathrm{Ext}^p_A(K, I^n M)$ is computed by replacing these finite A-modules by I^n times themselves. Thus the result by Lemma 15.102.1 (and much more is true). □

In Situation 15.92.15 we define complexes $I^\bullet_n$ such that we have distinguished triangles

$$I^\bullet_n \to A \to K^\bullet_n \to I^\bullet_n[1]$$

in the triangulated category $K(A)$ of complexes of A-modules up to homotopy. Namely, we set $I^\bullet_n = \sigma_{\leq -1} K^\bullet_n[-1]$. We have termwise split short exact sequences of complexes

$$0 \to A \to K^\bullet_n \to I^\bullet_n[1] \to 0$$

defining distinguished triangles by definition of the triangulated structure on $K(A)$. Their rotations determine the desired distinguished triangles above. Note that $I^0_n = A^{\oplus r} \to A$ is given by multiplication by f_i^n on the ith factor. Hence $I^\bullet_n \to A$ factors as

$$I^\bullet_n \to (f_1^n, \ldots, f_r^n) \to A$$

In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet_n) \to H^0(I^\bullet_n) \to (f_1^n, \ldots, f_r^n) \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet_n) = H^{i-1}(K^\bullet_n)$. The maps $K^\bullet_{n+1} \to K^\bullet_n$ induce maps $I^\bullet_{n+1} \to I^\bullet_n$ and we obtain a commutative diagram

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & I^\bullet_3 & \longrightarrow & I^\bullet_2 & \longrightarrow & I^\bullet_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & (f_1^3, \ldots, f_r^3) & \longrightarrow & (f_1^2, \ldots, f_r^2) & \longrightarrow & (f_1, \ldots, f_r)
\end{array}$$

in $K(A)$.

0G3M **Lemma 15.102.5.** *In Situation 15.92.15 assume A is Noetherian. With notation as above, the inverse system (I^n) is pro-isomorphic in $D(A)$ to the inverse system $(I^\bullet_n)$.*

Proof. It is elementary to show that the inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. Consider the inverse system of distinguished triangles

$$I^\bullet_n \to (f_1^n, \ldots, f_r^n) \to C^\bullet_n \to I^\bullet_n[1]$$

where $C_n^\bullet$ is the cone of the first arrow. By Derived Categories, Lemma 13.42.4 it suffices to show that the inverse system $C_n^\bullet$ is pro-zero. The complex $I_n^\bullet$ has nonzero terms only in degrees i with $-r+1 \leq i \leq 0$ hence $C_n^\bullet$ is bounded similarly. Thus by Derived Categories, Lemma 13.42.3 it suffices to show that $H^p(C_n^\bullet)$ is pro-zero. By the discussion above we have $H^p(C_n^\bullet) = H^p(K_n^\bullet)$ for $p \leq -1$ and $H^p(C_n^\bullet) = 0$ for $p \geq 0$. The fact that the inverse systems $H^p(K_n^\bullet)$ are pro-zero was shown in the proof of Lemma 15.95.1 (and this is where the assumption that A is Noetherian is used). □

0G3N **Lemma 15.102.6.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps*

$$I^n \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow I^n M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Choose generators $f_1, \ldots, f_r \in I$ of I. The inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. With notation as in Lemma 15.102.5 we find that it suffices to prove the inverse system of maps

$$I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow (f_1^n, \ldots, f_r^n)M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Say we have $a \leq b$ such that $M^i = 0$ if $i \notin [a, b]$. Then source and target of the arrows above have cohomology only in degrees $[-r + a, b]$. Thus it suffices to show that for any $p \in \mathbf{Z}$ the inverse system of maps

$$H^p(I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p((f_1^n, \ldots, f_r^n)M^\bullet)$$

defines an isomorphism of pro-objects of A-modules, see Derived Categories, Lemma 13.42.5. Using the pro-isomorphism between $I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet$ and $I^n \otimes_A^{\mathbf{L}} M^\bullet$ and the pro-isomorphism between $(f_1^n, \ldots, f_r^n)M^\bullet$ and $I^n M^\bullet$ this is equivalent to showing that the inverse system of maps

$$H^p(I^n \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

defines an isomorphism of pro-objects of A-modules Choose a bounded above complex of finite free A-modules $P^\bullet$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. Then it suffices to show that the inverse system of maps

$$H^p(I^n P^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

is a pro-isomorphism. This follows from Lemma 15.102.1 as $H^p(P^\bullet) = H^p(M^\bullet)$. □

0928 **Lemma 15.102.7.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. There exists an integer $n > 0$ such that $I^n M \to M$ factors through the map $I \otimes_A^{\mathbf{L}} M \to M$ in $D(A)$.*

Proof. This follows from Lemma 15.102.6. It can also been seen directly as follows. Consider the distinguished triangle

$$I \otimes_A^{\mathbf{L}} M \to M \to A/I \otimes_A^{\mathbf{L}} M \to I \otimes_A^{\mathbf{L}} M[1]$$

By the axioms of a triangulated category it suffices to prove that $I^n M \to A/I \otimes_A^{\mathbf{L}} M$ is zero in $D(A)$ for some n. Choose generators $f_1, \ldots, f_r$ of I and let $K = K_\bullet(A, f_1, \ldots, f_r)$ be the Koszul complex and consider the factorization $A \to K \to A/I$ of the quotient map. Then we see that it suffices to show that $I^n M \to K \otimes_A M$

is zero in $D(A)$ for some $n > 0$. Suppose that we have found an $n > 0$ such that $I^n M \to K \otimes_A M$ factors through $\tau_{\geq t}(K \otimes_A M)$ in $D(A)$. Then the obstruction to factoring through $\tau_{\geq t+1}(K \otimes_A M)$ is an element in $\mathrm{Ext}^t(I^n M, H_t(K \otimes_A M))$. The finite A-module $H_t(K \otimes_A M)$ is annihilated by I. Then by Lemma 15.102.3 we can after increasing n assume this obstruction element is zero. Repeating this a finite number of times we find n such that $I^n M \to K \otimes_A M$ factors through $0 = \tau_{\geq r+1}(K \otimes_A M)$ in $D(A)$ and we win. □

15.103. Miscellany

0926 Some results which do not fit anywhere else.

0DYJ **Lemma 15.103.1.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent. Let $a \in \mathbf{Z}$. Assume that for every finite A-module M the modules $\mathrm{Ext}^i_A(K, M)$ are I-power torsion for $i \geq a$. Then for $i \geq a$ and M finite the system $\mathrm{Ext}^i_A(K, M/I^n M)$ is essentially constant with value*

$$\mathrm{Ext}^i_A(K, M) = \lim \mathrm{Ext}^i_A(K, M/I^n M)$$

Proof. Let M be a finite A-module. Since K is pseudo-coherent we see that $\mathrm{Ext}^i_A(K, M)$ is a finite A-module. Thus for $i \geq a$ it is annihilated by I^t for some $t \geq 0$. By Lemma 15.102.4 we see that the image of $\mathrm{Ext}^i_A(K, I^n M) \to \mathrm{Ext}^i_A(K, M)$ is zero for some $n > 0$. The short exact sequence $0 \to I^n M \to M \to M/I^n M \to 0$ gives a long exact sequence

$$\mathrm{Ext}^i_A(K, I^n M) \to \mathrm{Ext}^i_A(K, M) \to \mathrm{Ext}^i_A(K, M/I^n M) \to \mathrm{Ext}^{i+1}_A(K, I^n M)$$

The systems $\mathrm{Ext}^i_A(K, I^n M)$ and $\mathrm{Ext}^{i+1}_A(K, I^n M)$ are essentially constant with value 0 by what we just said (applied to the finite A-modules $I^m M$). A diagram chase shows $\mathrm{Ext}^i_A(K, M/I^n M)$ is essentially constant with value $\mathrm{Ext}^i_A(K, M)$. □

0FXN **Lemma 15.103.2.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let N be an A-module annihilated by I. There exists an integer $n > 0$ such that $Tor^A_p(I^n M, N) \to Tor^A_p(M, N)$ is zero for all $p \geq 0$.*

Proof. By Lemma 15.102.7 we can factor $I^n M \to M$ as $I^n M \to M \otimes^{\mathbf{L}}_A I \to M$. We claim the composition

$$I^n M \otimes^{\mathbf{L}}_A N \to (M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N \to M \otimes^{\mathbf{L}}_A N$$

is zero. Namely, the diagram

$$\begin{array}{ccc} (M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N & \longrightarrow & M \otimes^{\mathbf{L}}_A (I \otimes^{\mathbf{L}}_A N) \\ & \searrow \quad \swarrow & \\ & M \otimes^{\mathbf{L}}_A N & \end{array}$$

commutes (details omitted) and the map $I \otimes^{\mathbf{L}}_A N \to N$ is zero as N is annihilated by I. □

0D2L **Lemma 15.103.3.** *Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let (M_n) be an inverse system of R-modules. Then $R\lim K \otimes^{\mathbf{L}}_R M_n = K \otimes^{\mathbf{L}}_R R\lim M_n$.*

Proof. Consider the defining distinguished triangle

$$R\lim M_n \to \prod M_n \to \prod M_n \to R\lim M_n[1]$$

and apply Lemma 15.66.5. □

0929 **Lemma 15.103.4.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal and let E be a nonzero finite module over R/I. If R/I has finite projective dimension and E has finite projective dimension over R/I, then E has finite projective dimension over R and*

$$pd_R(E) = pd_R(R/I) + pd_{R/I}(E)$$

Proof. We will use that, for a finite module, having finite projective dimension over R, resp. R/I is the same as being a perfect module, see discussion following Definition 15.75.1. We see that E has finite projective dimension over R by Lemma 15.75.7. Thus we can apply Auslander-Buchsbaum (Algebra, Proposition 10.111.1) to see that

$$\mathrm{pd}_R(E) + \mathrm{depth}(E) = \mathrm{depth}(R), \quad \mathrm{pd}_{R/I}(E) + \mathrm{depth}(E) = \mathrm{depth}(R/I),$$

and

$$\mathrm{pd}_R(R/I) + \mathrm{depth}(R/I) = \mathrm{depth}(R)$$

Note that in the first equation we take the depth of E as an R-module and in the second as an R/I-module. However these depths are the same (this is trivial but also follows from Algebra, Lemma 10.72.11). This concludes the proof. □

0GYI **Lemma 15.103.5.** *Let $A \to B$ be a ring map. There exists a cardinal $\kappa = \kappa(A \to B)$ with the following property: Let $M^\bullet$, resp. $N^\bullet$ be a complex of A-modules, resp. B-modules. Let $a : M^\bullet \to N^\bullet$ be a map of complexes of A-modules which induces an isomorphism $M^\bullet \otimes^{\mathbf{L}}_A B \to N^\bullet$ in $D(B)$. Let $M_1^\bullet \subset M^\bullet$, resp. $N_1^\bullet \subset N^\bullet$ be a subcomplex of A-modules, resp. B-modules such that $a(M_1^\bullet) \subset N_1^\bullet$. Then there exist subcomplexes*

$$M_1^\bullet \subset M_2^\bullet \subset M^\bullet \quad \text{and} \quad N_1^\bullet \subset N_2^\bullet \subset N^\bullet$$

such that $a(M_2^\bullet) \subset N_2^\bullet$ with the following properties:

1. *$\mathrm{Ker}(H^i(M_1^\bullet \otimes^{\mathbf{L}}_A B) \to H^i(N_1^\bullet))$ maps to zero in $H^i(M_2^\bullet \otimes^{\mathbf{L}}_A B)$,*
2. *$\mathrm{Im}(H^i(N_1^\bullet) \to H^i(N_2^\bullet))$ is contained in $\mathrm{Im}(H^i(M_2^\bullet \otimes^{\mathbf{L}}_A B) \to H^2(N_2^\bullet))$,*
3. *$|\bigcup M_2^i \cup \bigcup N_2^i| \leq \max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|)$.*

Proof. Let $\kappa = \max(|A|, |B|, \aleph_0)$. Set $|M^\bullet| = |\bigcup M^i|$ and similarly for other complexes. With this notation we have

$$\max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|) = \max(\kappa, |M_1^\bullet|, |M_2^\bullet|)$$

for the quantity used in the statement of the lemma. We are going to use this and other observations coming from arithmetic of cardinals without further mention.

First, let us show that there are plenty of "small" subcomplexes. For every pair of collections $E = \{E^i\}$ and $F = \{F^i\}$ of finite subsets $E^i \subset M^i$, $i \in \mathbf{Z}$ and $F^i \subset N^i$, $i \in \mathbf{Z}$ we can let

$$M_1^\bullet \subset M_1(E,F)^\bullet \subset M^\bullet \quad \text{and} \quad N_1^\bullet \subset N_1(E,F)^\bullet \subset N^\bullet$$

be the smallest subcomplexes of A and B-modules such that $a(M_1(E,F)^\bullet) \subset N_1(E,F)^\bullet$ and such that $E^i \subset M_1(E,F)^i$ and $F^i \subset M_2(E,F)^i$. Then it is easy to see that

$$|M_1(E,F)^\bullet| \leq \max(\kappa, |M_1^\bullet|) \quad \text{and} \quad |M_2(E,F)^\bullet| \leq \max(\kappa, |M_2^\bullet|)$$

Details omitted. It is clear that we have

$$M^\bullet = \operatorname{colim}_{(E,F)} M_1(E,F)^\bullet \quad \text{and} \quad N^\bullet = \operatorname{colim}_{(E,F)} N_1(E,F)^\bullet$$

and the colimits are (termwise) filtered colimits.

There exists a resolution $\ldots \to F^{-1} \to F^0 \to B$ by free A-modules F_i with $|F_i| \leq \kappa$ (details omitted). The cohomology modules of $M_1^\bullet \otimes_A^{\mathbf{L}} B$ are computed by $\operatorname{Tot}(M_1^\bullet \otimes_A F^\bullet)$. It follows that $|H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)| \leq \max(\kappa, |M_1^\bullet|)$.

Let $i \in \mathbf{Z}$ and let $\xi \in H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)$ be an element which maps to zero in $H^i(N_1^\bullet)$. Then ξ maps to zero in $H^i(N^\bullet)$ and hence ξ maps to zero in $H^i(M^\bullet \otimes_A^{\mathbf{L}} B)$. Since derived tensor product commutes with filtered colimits, we can find finite collections E_ξ and F_ξ as above such that ξ maps to zero in $H^i(M_1(E_\xi, F_\xi)^\bullet \otimes_A^{\mathbf{L}} B)$.

Let $i \in \mathbf{Z}$ and let $\eta \in H^i(N_1^\bullet)$. Then the image of η in $H^i(N^\bullet)$ is in the image of $H^i(M^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N^\bullet)$. Hence as before, we can find finite collections E_η and F_η as above such that η maps to an element of $H^i(N_1(E_\eta, F_\eta)$ which is in the image of the map $H^i(M_1(E_\eta, F_\eta)^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N_1(E_\eta, F_\eta)$.

Now we simply define

$$M_2^\bullet = \sum\nolimits_\xi M_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta M_1(E_\eta, F_\eta)^\bullet$$

where the sum is over ξ and η as in the previous two paragraphs and the sum is taken inside $M^\bullet$. Similarly we set

$$N_2^\bullet = \sum\nolimits_\xi N_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta N_1(E_\eta, F_\eta)^\bullet$$

where the sum is taken inside $N^\bullet$. By construction we will have properties (1) and (2) with these choices. The bound (3) also follows as the set of ξ and η has cardinality at most $\max(\kappa, |M_1^\bullet|, |N_1^\bullet|)$. □

0H84 **Lemma 15.103.6.** *Let R be a ring and $f \in R$. Let $M \in D(R)$ and let C be the cone of $f : M \to M$. If $H^i(M)_f = 0$ for $i < 0$ and $H^i(C) = 0$ for $i < -1$, then $H^i(M) = 0$ for $i < 0$.*

Proof. Denote $M_f = M \otimes_R^{\mathbf{L}} R_f$ and choose a distinguished triangle $F \to M \to M_f$. By assumption $H^i(M_f) = H^i(M)_f = 0$ for $i < 0$. Thus it suffices to show that $H^i(F) = 0$ for $i < 0$. Note that $H^i(F)$ is f-power torsion for all $i \in \mathbf{Z}$. On the other hand, since $f : M_f \to M_f$ is an isomorphism, we see that C is isomorphic to the cone of $f : F \to F$ (use Derived Categories, Proposition 13.4.23). Now, if $H^i(F) \neq 0$, then the kernel of $f : H^i(F) \to H^i(F)$ is nonzero, which implies that $H^{i-1}(C)$ is nonzero. Our assumption implies this cannot happen if $i - 1 < -1$ which finishes the proof. □

0H85 **Lemma 15.103.7.** *Let R be a ring and $f \in R$. Let $M \in D(R)$. Assume*

1. *$H^i(M) = 0$ for $i > 0$,*
2. *$M \otimes_R^{\mathbf{L}} R_f$ is isomorphic to a flat R_f-module placed in degree 0,*
3. *$M \otimes_R^{\mathbf{L}} R/fR$ is isomorphic to a flat R/fR-module placed in degree 0.*

Then M is isomorphic to a flat R-module placed in degree 0.

Proof. Let K be an R-module. It suffices to show that $N = M \otimes^{\mathbf{L}}_R K$ is isomorphic to an R-module placed in degree 0, see Section 15.67. By assumption (1) we see that N only has nonvanishing cohomology in degrees ≤ 0. Thus, by Lemma 15.103.6, it suffices to show that $H^i(N)_f = 0$ for $i < 0$ and that $H^i(C) = 0$ for $i < -1$ where C is the cone of $f : N \to N$. For the first, we note that

$$H^i(N)_f = H^i(N \otimes^{\mathbf{L}}_R R_f) = H^i(M \otimes^{\mathbf{L}}_R K \otimes^{\mathbf{L}}_R R_f) = H^i((M \otimes^{\mathbf{L}}_R R_f) \otimes^{\mathbf{L}}_{R_f} (K \otimes^{\mathbf{L}}_R R_f))$$

By our assumption (2) this is zero, except if $i = 0$ and then one gets $H^0(M)_f \otimes_R K$. For the second, let C' be the cone of $f : K \to K$. Then

$$C = M \otimes^{\mathbf{L}}_R C'$$

Now C' only has nonzero cohomology in degrees 0 and -1 equal to K/fK and $K[f]$; in other words, there is a distinguished triangle

$$(K[f])[1] \to C' \to K/fK$$

On the other hand, for any R-module K' annihilated by f we have

$$M \otimes^{\mathbf{L}}_R K' = M \otimes^{\mathbf{L}}_R R/fR \otimes^{\mathbf{L}}_{R/fR} K'$$

By our assumption (3) this is equal to the module $H^0(M \otimes^{\mathbf{L}}_R R/fR) \otimes_{R/fR} K'$ placed in degree 0. Combining the above we conclude that C only has nonzero cohomology in degrees 0 and -1 and the proof is complete. □

15.104. Tricks with double complexes

0EYW This section continues the discussion in Homology, Section 12.26.

0H0Q **Lemma 15.104.1.** *Let $A^\bullet_0 \to A^\bullet_1 \to A^\bullet_2 \to \ldots$ be a complex of complexes of abelian groups. Assume $H^{-p}(A^\bullet_p) = 0$ for all $p \geq 0$. Set $A^{p,q} = A^q_p$ and view $A^{\bullet,\bullet}$ as a double complex. Then $H^0(Tot_\pi(A^{\bullet,\bullet})) = 0$.*

Proof. Denote $f_p : A^\bullet_p \to A^\bullet_{p+1}$ the given maps of complexes. Recall that the differential on $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ is given by

$$\prod\nolimits_{p+q=n} (f^q_p + (-1)^p \mathrm{d}^q_{A^\bullet_p})$$

on elements in degree n. Let $\xi \in H^0(\mathrm{Tot}_\pi(A^{\bullet,\bullet}))$ be a cohomology class. We will show ξ is zero. Represent ξ as the class of an cocycle $x = (x_p) \in \prod A^{p,-p}$. Since $\mathrm{d}(x) = 0$ we find that $\mathrm{d}_{A^\bullet_0}(x_0) = 0$. Since $H^0(A^\bullet_0) = 0$ there exists a $y_{-1} \in A^{0,-1}$ with $\mathrm{d}_{A^\bullet_0}(y_{-1}) = x_0$. Then we see that $\mathrm{d}_{A^\bullet_1}(x_1 + f_0(y_{-1})) = 0$. Since $H^{-1}(A^\bullet_1) = 0$ we can find a $y_{-2} \in A^{1,-2}$ such that $-\mathrm{d}_{A^\bullet_1}(y_{-2}) = x_1 + f_0(y_{-1})$. By induction we can find $y_{-p-1} \in A^{p,-p-1}$ such that

$$(-1)^p \mathrm{d}_{A^\bullet_p}(y_{-p-1}) = x_p + f_{p-1}(y_{-p})$$

This implies that $\mathrm{d}(y) = x$ where $y = (y_{-p-1})$. □

0EYX **Lemma 15.104.2.** *Let*

$$(A^\bullet_0 \to A^\bullet_1 \to A^\bullet_2 \to \ldots) \longrightarrow (B^\bullet_0 \to B^\bullet_1 \to B^\bullet_2 \to \ldots)$$

be a map between two complexes of complexes of abelian groups. Set $A^{p,q} = A_p^q$, $B^{p,q} = B_p^q$ to obtain double complexes. Let $Tot_\pi(A^{\bullet,\bullet})$ and $Tot_\pi(B^{\bullet,\bullet})$ be the product total complexes associated to the double complexes. If each $A_p^\bullet \to B_p^\bullet$ is a quasi-isomorphism, then $Tot_\pi(A^{\bullet,\bullet}) \to Tot_\pi(B^{\bullet,\bullet})$ is a quasi-isomorphism.

Proof. Recall that $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ in degree n is given by $\prod_{p+q=n} A^{p,q} = \prod_{p+1=n} A_p^q$. Let $C_p^\bullet$ be the cone on the map $A_p^\bullet \to B_p^\bullet$, see Derived Categories, Section 13.9. By the functoriality of the cone construction we obtain a complex of complexes

$$C_0^\bullet \to C_1^\bullet \to C_2^\bullet \to \ldots$$

Then we see $\mathrm{Tot}_\pi(C^{\bullet,\bullet})$ in degree n is given by

$$\prod_{p+q=n} C^{p,q} = \prod_{p+q=n} C_p^q = \prod_{p+q=n} (B_p^q \oplus A_p^{q+1}) = \prod_{p+q=n} B_p^q \oplus \prod_{p+q=n} A_p^{q+1}$$

We conclude that $\mathrm{Tot}_\pi(C^{\bullet,\bullet})$ is the cone of the map $\mathrm{Tot}_\pi(A^{\bullet,\bullet}) \to \mathrm{Tot}_\pi(B^{\bullet,\bullet})$ (We omit the verification that the differentials agree.) Thus it suffices to show $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ is acyclic if each $A_p^\bullet$ is acyclic. This follows from Lemma 15.104.1. □

15.105. Weakly étale ring maps

092A Most of the results in this section are from the paper [**Oli83**] by Olivier. See also the related paper [**Fer67a**].

092B **Definition 15.105.1.** A ring A is called *absolutely flat* if every A-module is flat over A. A ring map $A \to B$ is *weakly étale* or *absolutely flat* if both $A \to B$ and $B \otimes_A B \to B$ are flat.

Absolutely flat rings are sometimes called von Neumann regular rings (often in the setting of noncommutative rings). A localization is a weakly étale ring map. An étale ring map is weakly étale. Here is a simple, yet key property.

092C **Lemma 15.105.2.** *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Let N be a B-module. If N is flat as an A-module, then N is flat as a B-module.*

Proof. Assume N is a flat as an A-module. Then the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_{B \otimes_A B}, \quad N' \mapsto N \otimes_A N'$$

is exact. As $B \otimes_A B \to B$ is flat we conclude that the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_B, \quad N' \mapsto (N \otimes_A N') \otimes_{B \otimes_A B} B = N \otimes_B N'$$

is exact, hence N is flat over B. □

092D **Definition 15.105.3.** Let A be a ring. Let $d \geq 0$ be an integer. We say that A has *weak dimension* $\leq d$ if every A-module has tor dimension $\leq d$.

092E **Lemma 15.105.4.** *Let $A \to B$ be a weakly étale ring map. If A has weak dimension at most d, then so does B.*

Proof. Let N be a B-module. If $d = 0$, then N is flat as an A-module, hence flat as a B-module by Lemma 15.105.2. Assume $d > 0$. Choose a resolution $F_\bullet \to N$ by free B-modules. Our assumption implies that $K = \mathrm{Im}(F_d \to F_{d-1})$ is A-flat, see Lemma 15.67.2. Hence it is B-flat by Lemma 15.105.2. Thus $0 \to K \to F_{d-1} \to \ldots \to F_0 \to N \to 0$ is a flat resolution of length d and we see that N has tor dimension at most d. □

092F **Lemma 15.105.5.** *Let A be a ring. The following are equivalent*

(1) *A has weak dimension ≤ 0,*
(2) *A is absolutely flat,*
(3) *A is reduced and every prime is maximal, and*
(4) *every local ring of A is a field.*

Proof. The equivalence of (1) and (2) is immediate.

Proof of (1) $\Rightarrow$ (3). Assume A is absolutely flat. Then every ideal of A is pure, see Algebra, Definition 10.108.1. Hence every finitely generated ideal is generated by an idempotent by Algebra, Lemma 10.108.5. If $f \in A$, then $(f) = (e)$ for some idempotent $e \in A$ and $D(f) = D(e)$ is open and closed (Algebra, Lemma 10.21.1). This implies every prime ideal of A is maximal by Algebra, Lemma 10.26.5. Moreover, if f is nilpotent, then $e = 0$ hence $f = 0$. Thus A is reduced.

Proof of (3) $\Rightarrow$ (4). If A is reduced and every prime is maximal, then every local ring is a field by Algebra, Lemma 10.25.1.

Proof of (4) $\Rightarrow$ (2). If every local ring of A is a field, then every A-module is flat by Algebra, Lemma 10.39.18. □

092G **Lemma 15.105.6.** *A product of fields is an absolutely flat ring.*

Proof. Let K_i be a family of fields. If $f = (f_i) \in \prod K_i$, then the ideal generated by f is the same as the ideal generated by the idempotent $e = (e_i)$ with $e_i = 0, 1$ according to whether f_i is 0 or not. Thus $D(f) = D(e)$ is open and closed and we conclude by Lemma 15.105.5 and Algebra, Lemma 10.26.5. □

092H **Lemma 15.105.7.** *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

(1) *If $B \otimes_A B \to B$ is flat, then $B' \otimes_{A'} B' \to B'$ is flat.*
(2) *If $A \to B$ is weakly étale, then $A' \to B'$ is weakly étale.*

Proof. Assume $B \otimes_A B \to B$ is flat. The ring map $B' \otimes_{A'} B' \to B'$ is the base change of $B \otimes_A B \to B$ by $A \to A'$. Hence it is flat by Algebra, Lemma 10.39.7. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat ring map is flat. □

092I **Lemma 15.105.8.** *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.*

(1) *If A is an absolutely flat ring, then so is B.*
(2) *If A is reduced and $A \to B$ is weakly étale, then B is reduced.*

Proof. Part (1) follows immediately from Lemma 15.105.2 and the definitions. If A is reduced, then there exists an injection $A \to A' = \prod_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}}$ of A into an absolutely flat ring (Algebra, Lemma 10.25.2 and Lemma 15.105.6). If $A \to B$ is flat, then the induced map $B \to B' = B \otimes_A A'$ is injective too. By Lemma 15.105.7 the ring map $A' \to B'$ is weakly étale. By part (1) we see that B' is absolutely flat. By Lemma 15.105.5 the ring B' is reduced. Hence B is reduced. □

092J **Lemma 15.105.9.** *Let $A \to B$ and $B \to C$ be ring maps.*

(1) *If $B \otimes_A B \to B$ and $C \otimes_B C \to C$ are flat, then $C \otimes_A C \to C$ is flat.*
(2) *If $A \to B$ and $B \to C$ are weakly étale, then $A \to C$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$C \otimes_A C \longrightarrow C \otimes_B C \longrightarrow C$$

of the multiplication map, the fact that

$$C \otimes_B C = (C \otimes_A C) \otimes_{B \otimes_A B} B,$$

the fact that a base change of a flat map is flat, and the fact that the composition of flat ring maps is flat. See Algebra, Lemmas 10.39.7 and 10.39.4. Part (2) follows from (1) and the fact (just used) that the composition of flat ring maps is flat. □

092K **Lemma 15.105.10.** *Let $A \to B \to C$ be ring maps.*

(1) *If $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat, then $B \otimes_A B \to B$ is flat.*
(2) *If $B \to C$ is faithfully flat and $A \to C$ is weakly étale, then $A \to B$ is weakly étale.*

Proof. Assume $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat. Consider the commutative diagram

$$\begin{array}{ccc} C \otimes_A C & \longrightarrow & C \\ \uparrow & & \uparrow \\ B \otimes_A B & \longrightarrow & B \end{array}$$

The vertical arrows are flat, the top horizontal arrow is flat. Hence C is flat as a $B \otimes_A B$-module. The map $B \to C$ is faithfully flat and $C = B \otimes_B C$. Hence B is flat as a $B \otimes_A B$-module by Algebra, Lemma 10.39.9. This proves (1). Part (2) follows from (1) and the fact that $A \to B$ is flat if $A \to C$ is flat and $B \to C$ is faithfully flat (Algebra, Lemma 10.39.9). □

092L **Lemma 15.105.11.** *Let A be a ring. Let $B \to C$ be an A-algebra map of weakly étale A-algebras. Then $B \to C$ is weakly étale.*

Proof. The ring map $B \to C$ is flat by Lemma 15.105.2. The ring map $C \otimes_A C \to C \otimes_B C$ is surjective, hence an epimorphism. Thus Lemma 15.105.2 implies, that since C is flat over $C \otimes_A C$ also C is flat over $C \otimes_B C$. □

092M **Lemma 15.105.12.** *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Then $\Omega_{B/A} = 0$, i.e., B is formally unramified over A.*

Proof. Let $I \subset B \otimes_A B$ be the kernel of the flat surjective map $B \otimes_A B \to B$. Then I is a pure ideal (Algebra, Definition 10.108.1), so $I^2 = I$ (Algebra, Lemma 10.108.2). Since $\Omega_{B/A} = I/I^2$ (Algebra, Lemma 10.131.13) we obtain the vanishing. This means B is formally unramified over A by Algebra, Lemma 10.148.3. □

0CKP **Lemma 15.105.13.** *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.*

(1) *If $A \to B$ is of finite type, then $A \to B$ is unramified.*
(2) *If $A \to B$ is of finite presentation and flat, then $A \to B$ is étale.*

In particular a weakly étale ring map of finite presentation is étale.

Proof. Part (1) follows from Lemma 15.105.12 and Algebra, Definition 10.151.1. Part (2) follows from part (1) and Algebra, Lemma 10.151.8. □

092N **Lemma 15.105.14.** *Let $A \to B$ be a ring map. Then $A \to B$ is weakly étale in each of the following cases*

(1) $B = S^{-1}A$ *is a localization of* A,
(2) $A \to B$ *is étale,*
(3) B *is a filtered colimit of weakly étale* A*-algebras.*

Proof. An étale ring map is flat and the map $B \otimes_A B \to B$ is also étale as a map between étale A-algebras (Algebra, Lemma 10.143.8). This proves (2).

Let B_i be a directed system of weakly étale A-algebras. Then $B = \operatorname{colim} B_i$ is flat over A by Algebra, Lemma 10.39.3. Note that the transition maps $B_i \to B_{i'}$ are flat by Lemma 15.105.11. Hence B is flat over B_i for each i, and we see that B is flat over $B_i \otimes_A B_i$ by Algebra, Lemma 10.39.4. Thus B is flat over $B \otimes_A B = \operatorname{colim} B_i \otimes_A B_i$ by Algebra, Lemma 10.39.6.

Part (1) can be proved directly, but also follows by combining (2) and (3). □

092P **Lemma 15.105.15.** *Let L/K be an extension of fields. If $L \otimes_K L \to L$ is flat, then L is an algebraic separable extension of K.*

Proof. By Lemma 15.105.10 we see that any subfield $K \subset L' \subset L$ the map $L' \otimes_K L' \to L'$ is flat. Thus we may assume L is a finitely generated field extension of K. In this case the fact that L/K is formally unramified (Lemma 15.105.12) implies that L/K is finite separable, see Algebra, Lemma 10.158.1. □

092Q **Lemma 15.105.16.** *Let B be an algebra over a field K. The following are equivalent*

(1) $B \otimes_K B \to B$ *is flat,*
(2) $K \to B$ *is weakly étale, and*
(3) B *is a filtered colimit of étale* K*-algebras.*

Moreover, every finitely generated K-subalgebra of B is étale over K.

Proof. Parts (1) and (2) are equivalent because every K-algebra is flat over K. Part (3) implies (1) and (2) by Lemma 15.105.14

Assume (1) and (2) hold. We will prove (3) and the finite statement of the lemma. A field is absolutely flat ring, hence B is a absolutely flat ring by Lemma 15.105.8. Hence B is reduced and every local ring is a field, see Lemma 15.105.5.

Let $\mathfrak{q} \subset B$ be a prime. The ring map $B \to B_\mathfrak{q}$ is weakly étale, hence $B_\mathfrak{q}$ is weakly étale over K (Lemma 15.105.9). Thus $B_\mathfrak{q}$ is a separable algebraic extension of K by Lemma 15.105.15.

Let $K \subset A \subset B$ be a finitely generated K-sub algebra. We will show that A is étale over K which will finish the proof of the lemma. Then every minimal prime $\mathfrak{p} \subset A$ is the image of a prime $\mathfrak{q}$ of B, see Algebra, Lemma 10.30.5. Thus $\kappa(\mathfrak{p})$ as a subfield of $B_\mathfrak{q} = \kappa(\mathfrak{q})$ is separable algebraic over K. Hence every generic point of $\operatorname{Spec}(A)$ is closed (Algebra, Lemma 10.35.9). Thus $\dim(A) = 0$. Then A is the product of its local rings, e.g., by Algebra, Proposition 10.60.7. Moreover, since A is reduced, all local rings are equal to their residue fields which are finite separable over K. This means that A is étale over K by Algebra, Lemma 10.143.4 and finishes the proof. □

092R **Lemma 15.105.17.** *Let $A \to B$ be a ring map. If $A \to B$ is weakly étale, then $A \to B$ induces separable algebraic residue field extensions.*

Proof. Let $\mathfrak{p}$ be a prime of A. Then $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is weakly étale by Lemma 15.105.7. Hence $B \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras by Lemma 15.105.16. Hence for $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ the extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a filtered colimit of finite separable extensions by Algebra, Lemma 10.143.4. □

092S **Lemma 15.105.18.** *Let A be a ring. The following are equivalent*

(1) *A has weak dimension ≤ 1,*
(2) *every ideal of A is flat,*
(3) *every finitely generated ideal of A is flat,*
(4) *every submodule of a flat A-module is flat, and*
(5) *every local ring of A is a valuation ring.*

Proof. If A has weak dimension ≤ 1, then the resolution $0 \to I \to A \to A/I \to 0$ shows that every ideal I is flat by Lemma 15.67.2. Hence (1) $\Rightarrow$ (2).

Assume (4). Let M be an A-module. Choose a surjection $F \to M$ where F is a free A-module. Then $\mathrm{Ker}(F \to M)$ is flat by assumption, and we see that M has tor dimension ≤ 1 by Lemma 15.67.6. Hence (4) $\Rightarrow$ (1).

Every ideal is the union of the finitely generated ideals contained in it. Hence (3) implies (2) by Algebra, Lemma 10.39.3. Thus (3) $\Leftrightarrow$ (2).

Assume (2). Suppose that $N \subset M$ with M a flat A-module. We will prove that N is flat. We can write $M = \mathrm{colim}\, M_i$ with each M_i finite free, see Algebra, Theorem 10.81.4. Setting $N_i \subset M_i$ the inverse image of N we see that $N = \mathrm{colim}\, N_i$. By Algebra, Lemma 10.39.3. it suffices to prove N_i is flat and we reduce to the case $M = R^{\oplus n}$. In this case the module N has a finite filtration by the submodules $R^{\oplus j} \cap N$ whose subquotients are ideals. By (2) these ideals are flat and hence N is flat by Algebra, Lemma 10.39.13. Thus (2) $\Rightarrow$ (4).

Assume A satisfies (1) and let $\mathfrak{p} \subset A$ be a prime ideal. By Lemmas 15.105.14 and 15.105.4 we see that $A_\mathfrak{p}$ satisfies (1). We will show A is a valuation ring if A is a local ring satisfying (3). Let $f \in \mathfrak{m}$ be a nonzero element. Then (f) is a flat nonzero module generated by one element. Hence it is a free A-module by Algebra, Lemma 10.78.5. It follows that f is a nonzerodivisor and A is a domain. If $I \subset A$ is a finitely generated ideal, then we similarly see that I is a finite free A-module, hence (by considering the rank) free of rank 1 and I is a principal ideal. Thus A is a valuation ring by Algebra, Lemma 10.50.15. Thus (1) $\Rightarrow$ (5).

Assume (5). Let $I \subset A$ be a finitely generated ideal. Then $I_\mathfrak{p} \subset A_\mathfrak{p}$ is a finitely generated ideal in a valuation ring, hence principal (Algebra, Lemma 10.50.15), hence flat. Thus I is flat by Algebra, Lemma 10.39.18. Thus (5) $\Rightarrow$ (3). This finishes the proof of the lemma. □

092T **Lemma 15.105.19.** *Let J be a set. For each $j \in J$ let A_j be a valuation ring with fraction field K_j. Set $A = \prod A_j$ and $K = \prod K_j$. Then A has weak dimension at most 1 and $A \to K$ is a localization.*

Proof. Let $I \subset A$ be a finitely generated ideal. By Lemma 15.105.18 it suffices to show that I is a flat A-module. Let $I_j \subset A_j$ be the image of I. Observe that $I_j = I \otimes_A A_j$, hence $I \to \prod I_j$ is surjective by Algebra, Proposition 10.89.2. Thus $I = \prod I_j$. Since A_j is a valuation ring, the ideal I_j is generated by a single element (Algebra, Lemma 10.50.15). Say $I_j = (f_j)$. Then I is generated by the element $f = (f_j)$. Let $e \in A$ be the idempotent which has a 0 or 1 in A_j depending on

whether f_j is 0 or not. Then $f = ge$ for some nonzerodivisor $g \in A$: take $g = (g_j)$ with $g_j = 1$ if $f_j = 0$ and $g_j = f_j$ else. Thus $I \cong (e)$ as a module. We conclude I is flat as (e) is a direct summand of A. The final statement is true because $K = S^{-1}A$ where $S = \prod(A_j \setminus \{0\})$. □

092U **Lemma 15.105.20.** *Let A be a normal domain with fraction field K. There exists a cartesian diagram*

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ V & \longrightarrow & L \end{array}$$

of rings where V has weak dimension at most 1 and $V \to L$ is a flat, injective, epimorphism of rings.

Proof. For every $x \in K$, $x \notin A$ pick $V_x \subset K$ as in Algebra, Lemma 10.50.11. Set $V = \prod_{x \in K \setminus A} V_x$ and $L = \prod_{x \in K \setminus A} K$. The ring V has weak dimension at most 1 by Lemma 15.105.19 which also shows that $V \to L$ is a localization. A localization is flat and an epimorphism, see Algebra, Lemmas 10.39.18 and 10.107.5. □

092V **Lemma 15.105.21.** *Let A be a ring of weak dimension at most 1. If $A \to B$ is a flat, injective, epimorphism of rings, then A is integrally closed in B.*

Proof. Let $x \in B$ be integral over A. Let $A' = A[x] \subset B$. Then A' is a finite ring extension of A by Algebra, Lemma 10.36.5. To show $A = A'$ it suffices to show $A \to A'$ is an epimorphism by Algebra, Lemma 10.107.6. Note that A' is flat over A by assumption on A and the fact that B is flat over A (Lemma 15.105.18). Hence the composition

$$A' \otimes_A A' \to B \otimes_A A' \to B \otimes_A B \to B$$

is injective, i.e., $A' \otimes_A A' \cong A'$ and the lemma is proved. □

092W **Lemma 15.105.22.** *Let A be a normal domain with fraction field K. Let $A \to B$ be weakly étale. Then B is integrally closed in $B \otimes_A K$.*

Proof. Choose a diagram as in Lemma 15.105.20. As $A \to B$ is flat, the base change gives a cartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_A K \\ \downarrow & & \downarrow \\ B \otimes_A V & \longrightarrow & B \otimes_A L \end{array}$$

of rings. Note that $V \to B \otimes_A V$ is weakly étale (Lemma 15.105.7), hence $B \otimes_A V$ has weak dimension at most 1 by Lemma 15.105.4. Note that $B \otimes_A V \to B \otimes_A L$ is a flat, injective, epimorphism of rings as a flat base change of such (Algebra, Lemmas 10.39.7 and 10.107.3). By Lemma 15.105.21 we see that $B \otimes_A V$ is integrally closed in $B \otimes_A L$. It follows from the cartesian property of the diagram that B is integrally closed in $B \otimes_A K$. □

092X **Lemma 15.105.23.** *Let $A \to B$ be a ring homomorphism. Assume*

(1) *A is a henselian local ring,*
(2) *$A \to B$ is integral,*
(3) *B is a domain.*

Then B is a henselian local ring and $A \to B$ is a local homomorphism. If A is strictly henselian, then B is a strictly henselian local ring and the extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ of residue fields is purely inseparable.

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ of finite A-sub algebras. If we prove the results for each B_i, then the result follows for B. See Algebra, Lemma 10.154.8. If $A \to B$ is finite, then B is a product of local henselian rings by Algebra, Lemma 10.153.4. Since B is a domain we see that B is a local ring. The maximal ideal of B lies over the maximal ideal of A by going up for $A \to B$ (Algebra, Lemma 10.36.22). If A is strictly henselian, then the field extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ being algebraic, has to be purely inseparable. Of course, then $\kappa(\mathfrak{m}_B)$ is separably algebraically closed and B is strictly henselian. □

092Z **Theorem 15.105.24** (Olivier)**.** *Let $A \to B$ be a local homomorphism of local rings. If A is strictly henselian and $A \to B$ is weakly étale, then $A = B$.*

Proof. We will show that for all $\mathfrak{p} \subset A$ there is a unique prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This implies that $B \otimes_A B \to B$ is bijective on spectra as well as surjective and flat. Hence it is an isomorphism for example by the description of pure ideals in Algebra, Lemma 10.108.4. Hence $A \to B$ is a faithfully flat epimorphism of rings. We get $A = B$ by Algebra, Lemma 10.107.7.

Note that the fibre ring $B \otimes_A \kappa(\mathfrak{p})$ is a colimit of étale extensions of $\kappa(\mathfrak{p})$ by Lemmas 15.105.7 and 15.105.16. Hence, if there exists more than one prime lying over $\mathfrak{p}$ or if $\kappa(\mathfrak{p}) \neq \kappa(\mathfrak{q})$ for some $\mathfrak{q}$, then $B \otimes_A L$ has a nontrivial idempotent for some (separable) algebraic field extension $L/\kappa(\mathfrak{p})$.

Let $L/\kappa(\mathfrak{p})$ be an algebraic field extension. Let $A' \subset L$ be the integral closure of $A/\mathfrak{p}$ in L. By Lemma 15.105.23 we see that A' is a strictly henselian local ring whose residue field is a purely inseparable extension of the residue field of A. Thus $B \otimes_A A'$ is a local ring by Algebra, Lemma 10.156.5. On the other hand, $B \otimes_A A'$ is integrally closed in $B \otimes_A L$ by Lemma 15.105.22. Since $B \otimes_A A'$ is local, it follows that the ring $B \otimes_A L$ does not have nontrivial idempotents which is what we wanted to prove. □

15.106. Weakly étale algebras over fields

0CKQ If K is a field, then an algebra B is weakly étale over K if and only if it is a filtered colimit of étale K-algebras. This is Lemma 15.105.16.

0CKR **Lemma 15.106.1.** *Let K be a field. If B is weakly étale over K, then*

1. *B is reduced,*
2. *B is integral over K,*
3. *any finitely generated K-subalgebra of B is a finite product of finite separable extensions of K,*
4. *B is a field if and only if B does not have nontrivial idempotents and in this case it is a separable algebraic extension of K,*
5. *any sub or quotient K-algebra of B is weakly étale over K,*
6. *if B' is weakly étale over K, then $B \otimes_K B'$ is weakly étale over K.*

Proof. Part (1) follows from Lemma 15.105.8 but of course it follows from part (3) as well. Part (3) follows from Lemma 15.105.16 and the fact that étale K-algebras are finite products of finite separable extensions of K, see Algebra, Lemma 10.143.4.

Part (3) implies (2). Part (4) follows from (3) as a product of fields is a field if and only if it has no nontrivial idempotents.

If $S \subset B$ is a subalgebra, then it is the filtered colimit of its finitely generated subalgebras which are all étale over K by the above and hence S is weakly étale over K by Lemma 15.105.16. If $B \to Q$ is a quotient algebra, then Q is the filtered colimit of K-algebra quotients of finite products $\prod_{i \in I} L_i$ of finite separable extensions L_i/K. Such a quotient is of the form $\prod_{i \in J} L_i$ for some subset $J \subset I$ and hence the result holds for quotients by the same reasoning.

The statement on tensor products follows in a similar manner or by combining Lemmas 15.105.7 and 15.105.9. □

0CKS **Lemma 15.106.2.** *Let K be a field. Let A be a K-algebra. There exists a maximal weakly étale K-subalgebra $B_{max} \subset A$.*

Proof. Let $B_1, B_2 \subset A$ be weakly étale K-subalgebras. Then $B_1 \otimes_K B_2$ is weakly étale over K and so is the image of $B_1 \otimes_K B_2 \to A$ (Lemma 15.106.1). Thus the collection $\mathcal{B}$ of weakly étale K-subalgebras $B \subset A$ is directed and the colimit $B_{max} = \operatorname{colim}_{B \in \mathcal{B}} B$ is a weakly étale K-algebra by Lemma 15.105.14. Hence the image of $B_{max} \to A$ is weakly étale over K (previous lemma cited). It follows that this image is in $\mathcal{B}$ and hence $\mathcal{B}$ has a maximal element (and the image is the same as B_{max}). □

0CKT **Lemma 15.106.3.** *Let K be a field. For a K-algebra A denote $B_{max}(A)$ the maximal weakly étale K-subalgebra of A as in Lemma 15.106.2. Then*

1. *any K-algebra map $A' \to A$ induces a K-algebra map $B_{max}(A') \to B_{max}(A)$,*
2. *if $A' \subset A$, then $B_{max}(A') = B_{max}(A) \cap A'$,*
3. *if $A = \operatorname{colim} A_i$ is a filtered colimit, then $B_{max}(A) = \operatorname{colim} B_{max}(A_i)$,*
4. *the map $B_{max}(A) \to B_{max}(A_{red})$ is an isomorphism,*
5. *$B_{max}(A_1 \times \ldots \times A_n) = B_{max}(A_1) \times \ldots \times B_{max}(A_n)$,*
6. *if A has no nontrivial idempotents, then $B_{max}(A)$ is a field and a separable algebraic extension of K,*
7. *add more here.*

Proof. Proof of (1). This is true because the image of $B_{max}(A') \to A$ is weakly étale over K by Lemma 15.106.1.

Proof of (2). By (1) we have $B_{max}(A') \subset B_{max}(A)$. Conversely, $B_{max}(A) \cap A'$ is a weakly étale K-algebra by Lemma 15.106.1 and hence contained in $B_{max}(A')$.

Proof of (3). By (1) there is a map $\operatorname{colim} B_{max}(A_i) \to A$ which is injective because the system is filtered and $B_{max}(A_i) \subset A_i$. The colimit $\operatorname{colim} B_{max}(A_i)$ is weakly étale over K by Lemma 15.105.14. Hence we get an injective map $\operatorname{colim} B_{max}(A_i) \to B_{max}(A)$. Suppose that $a \in B_{max}(A)$. Then a generates a finitely presented K-subalgebra $B \subset B_{max}(A)$. By Algebra, Lemma 10.127.3 there is an i and a K-algebra map $f : B \to A_i$ lifting the given map $B \to A$. Since B is weakly étale by Lemma 15.106.1, we see that $f(B) \subset B_{max}(A_i)$ and we conclude that a is in the image of $\operatorname{colim} B_{max}(A_i) \to B_{max}(A)$.

Proof of (4). Write $B_{max}(A_{red}) = \operatorname{colim} B_i$ as a filtered colimit of étale K-algebras (Lemma 15.105.16). By Algebra, Lemma 10.138.17 for each i there is a K-algebra map $f_i : B_i \to A$ lifting the given map $B_i \to A_{red}$. It follows that the canonical map

$B_{max}(A_{red}) \to B_{max}(A)$ is surjective. The kernel consists of nilpotent elements and hence is zero as $B_{max}(A_{red})$ is reduced (Lemma 15.106.1).

Proof of (5). Omitted.

Proof of (6). Follows from Lemma 15.106.1 part (4). □

0CKU **Lemma 15.106.4.** *Let L/K be an extension of fields. Let A be a K-algebra. Let $B \subset A$ be the maximal weakly étale K-subalgebra of A as in Lemma 15.106.2. Then $B \otimes_K L$ is the maximal weakly étale L-subalgebra of $A \otimes_K L$.*

Proof. For an algebra A over K we write $B_{max}(A/K)$ for the maximal weakly étale K-subalgebra of A. Similarly we write $B_{max}(A'/L)$ for the maximal weakly étale L-subalgebra of A' if A' is an L-algebra. Since $B_{max}(A/K) \otimes_K L$ is weakly étale over L (Lemma 15.105.7) and since $B_{max}(A/K) \otimes_K L \subset A \otimes_K L$ we obtain a canonical injective map

$$B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$$

The lemma states that this map is an isomorphism.

To prove the lemma for L and our K-algebra A, it suffices to prove the lemma for any field extension L' of L. Namely, we have the factorization

$$B_{max}(A/K) \otimes_K L' \to B_{max}((A \otimes_K L)/L) \otimes_L L' \to B_{max}((A \otimes_K L')/L')$$

hence the composition cannot be surjective without $B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$ being surjective. Thus we may assume L is algebraically closed.

Reduction to finite type K-algebra. We may write A is the filtered colimit of its finite type K-subalgebras. Using Lemma 15.106.3 we see that it suffices to prove the lemma for finite type K-algebras.

Assume A is a finite type K-algebra. Since the kernel of $A \to A_{red}$ is nilpotent, the same is true for $A \otimes_K L \to A_{red} \otimes_K L$. Then

$$B_{max}((A \otimes_K L)/L) \to B_{max}((A_{red} \otimes_K L)/L)$$

is injective because the kernel is nilpotent and the weakly étale L-algebra $B_{max}((A \otimes_K L)/L)$ is reduced (Lemma 15.106.1). Since $B_{max}(A/K) = B_{max}(A_{red}/K)$ by Lemma 15.106.3 we conclude that it suffices to prove the lemma for A_{red}.

Assume A is a reduced finite type K-algebra. Let $Q = Q(A)$ be the total quotient ring of A. Then $A \subset Q$ and $A \otimes_K L \subset Q \otimes_A L$ and hence

$$B_{max}(A/K) = A \cap B_{max}(Q/K)$$

and

$$B_{max}((A \otimes_K L)/L) = (A \otimes_K L) \cap B_{max}((Q \otimes_K L)/L)$$

by Lemma 15.106.3. Since $- \otimes_K L$ is an exact functor, it follows that if we prove the result for Q, then the result follows for A. Since Q is a finite product of fields (Algebra, Lemmas 10.25.4, 10.25.1, 10.31.6, and 10.31.1) and since B_{max} commutes with products (Lemma 15.106.3) it suffices to prove the lemma when A is a field.

Assume A is a field. We reduce to A being finitely generated over K by the argument in the third paragraph of the proof. (In fact the way we reduced to the case of a field produces a finitely generated field extension of K.)

Assume A is a finitely generated field extension of K. Then $K' = B_{max}(A/K)$ is a field separable algebraic over K by Lemma 15.106.3 part (6). Hence K' is a

finite separable field extension of K and A is geometrically irreducible over K' by Algebra, Lemma 10.47.13. Since L is algebraically closed and K'/K finite separable we see that

$$K' \otimes_K L \to \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} L, \quad \alpha \otimes \beta \mapsto (\sigma(\alpha)\beta)_\sigma$$

is an isomorphism (Fields, Lemma 9.13.4). We conclude

$$A \otimes_K L = A \otimes_{K'} (K' \otimes_K L) = \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} A \otimes_{K', \sigma} L$$

Since A is geometrically irreducible over K' we see that $A \otimes_{K', \sigma} L$ has a unique minimal prime. Since L is algebraically closed it follows that $B_{max}((A \otimes_{K', \sigma} L)/L) = L$ because this L-algebra is a field algebraic over L by Lemma 15.106.3 part (6). It follows that the maximal weakly étale $K' \otimes_K L$-subalgebra of $A \otimes_K L$ is $K' \otimes_K L$ because we can decompose these subalgebras into products as above. Hence the inclusion $K' \otimes_K L \subset B_{max}((A \otimes_K L)/L)$ is an equality: the ring map $K' \otimes_K L \to B_{max}((A \otimes_K L)/L)$ is weakly étale by Lemma 15.105.11. □

15.107. Local irreducibility

06DT The following definition seems to be the generally accepted one. To parse it, observe that if $A \subset B$ is an integral extension of local domains, then $A \to B$ is a local ring homomorphism by going up (Algebra, Lemma 10.36.22).

0BPZ **Definition 15.107.1.** Let A be a local ring. We say A is *unibranch* if the reduction A_{red} is a domain and if the integral closure A' of A_{red} in its field of fractions is local. We say A is *geometrically unibranch* if A is unibranch and moreover the residue field of A' is purely inseparable over the residue field of A. [**GD67**, Chapter 0 (23.2.1)]

Let A be a local ring. Here is an equivalent formulation

(1) A is unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring, and
(2) A is geometrically unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.

A local ring which is normal is geometrically unibranch (follows from Definition 15.107.1 and Algebra, Definition 10.37.11). Lemmas 15.107.3 and 15.107.5 suggest that being (geometrically) unibranch is a reasonable property to look at.

0C24 **Lemma 15.107.2.** *Let A be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Let A^h be the henselization of A. Consider the maps*

$$\mathrm{Spec}(A') \leftarrow \mathrm{Spec}((A')^h) \to \mathrm{Spec}(A^h)$$

where $(A')^h = A' \otimes_A A^h$. Then

(1) *the left arrow is bijective on maximal ideals,*
(2) *the right arrow is bijective on minimal primes,*
(3) *every minimal prime of $(A')^h$ is contained in a unique maximal ideal and every maximal ideal contains exactly one minimal prime.*

Proof. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^h = A^h/IA^h$ by (Algebra, Lemma 10.156.2). The spectra of A, A^h, A', and $(A')^h$ are the same as the spectra of A/I, A^h/IA^h, A', and $(A')^h = A' \otimes_{A/I} A^h/IA^h$. Thus we may

replace A by $A_{red} = A/I$ and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). As A' is integral over A we see that $(A')^h$ is integral over A^h. By going up (Algebra, Lemma 10.36.22) every maximal ideal of A', resp. $(A')^h$ lies over the maximal ideal $\mathfrak{m}$, resp. $\mathfrak{m}^h$ of A, resp. A^h. Thus (1) follows from the isomorphism

$$(A')^h \otimes_{A^h} \kappa^h = A' \otimes_A A^h \otimes_{A^h} \kappa^h = A' \otimes_A \kappa$$

because the residue field extension κ^h/κ induced by $A \to A^h$ is trivial. We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 10.36.19 and 10.26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 10.37.16. Since A^h is a filtered colimit of étale A-algebras, $(A')^h$ is filtered colimit of étale A'-algebras hence $(A')^h$ is a normal ring by Algebra, Lemmas 10.163.9 and 10.37.17. Thus every local ring of $(A')^h$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 15.11.8 applied to $A^h \to (A')^h$ we see that $((A')^h, \mathfrak{m}(A')^h)$ is a henselian pair. If $\mathfrak{q} \subset (A')^h$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak{q})$ with $V(\mathfrak{m}(A')^h)$ is connected by Lemma 15.11.16 Since $V(\mathfrak{m}(A')^h) = \mathrm{Spec}((A')^h \otimes \kappa^h)$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak{q}$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^h$ is flat by going down (Algebra, Lemma 10.39.19) every minimal prime of $(A')^h$ lies over a minimal prime of A'. Conversely, any prime of $(A')^h$ lying over a minimal prime of A' is minimal because $(A')^h$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^h$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^h are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^h \otimes_A Q(A) = A^h \otimes_A A' \otimes_A Q(A) = A^h \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^h$ and the spectrum of the ring on the right is the is the set of minimal primes of A^h. This finishes the proof. □

0BQ0 **Lemma 15.107.3.** *Let A be a local ring. Let A^h be the henselization of A. The following are equivalent* [**GD67**, Chapter IV Proposition 18.6.12]

(1) *A is unibranch, and*
(2) *A^h has a unique minimal prime.*

Proof. This follows from Lemma 15.107.2 but we will also give a direct proof. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Recall that the residue field $\kappa = A/\mathfrak{m}$ is the same as the residue field of A^h.

Assume (2). Let $\mathfrak{p}^h$ be the unique minimal prime of A^h. The flatness of $A \to A^h$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^h$ is the unique minimal prime of A (by going down, see Algebra, Lemma 10.39.19). Also, since $A^h/\mathfrak{p}A^h = (A/\mathfrak{p})^h$ (see Algebra, Lemma 10.156.2) is reduced by Lemma 15.45.4 we see that $\mathfrak{p}^h = \mathfrak{p}A^h$. Let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. We have to show that A' is local. Since

$A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak{m}$ (by going up for integral ring maps, see Algebra, Lemma 10.36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$. Choose elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak{m}_i$ and $f_i \notin \mathfrak{m}_{3-i}$. We find a finite subalgebra $B = A/\mathfrak{p}[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak{m}_i$, $i = 1, 2$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^h$ the inclusions

$$A^h/\mathfrak{p}^h \subset B \otimes_A A^h \subset \kappa(\mathfrak{p}) \otimes_A A^h \subset \kappa(\mathfrak{p}^h)$$

the last inclusion because $\kappa(\mathfrak{p}) \otimes_A A^h = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} A^h/\mathfrak{p}^h$ is a localization of the domain $A^h/\mathfrak{p}^h$. Note that $B \otimes_A \kappa$ has at least two maximal ideals because $B/\mathfrak{m}B$ has two maximal ideals. Hence, as A^h is henselian we see that $B \otimes_A A^h$ is a product of ≥ 2 local rings, see Algebra, Lemma 10.153.5. But we've just seen that $B \otimes_A A^h$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak{p} \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. Let $A \to B$ be a local map of local rings inducing an isomorphism of residue fields which is a localization of an étale A-algebra. In particular $\mathfrak{m}_B$ is the unique prime containing $\mathfrak{m}B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local implies that B' is local (Algebra, Lemma 10.156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 10.163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak{p}B \subset B \otimes_A \kappa(\mathfrak{p}) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak{p}B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak{p}$). Since A^h is a filtered colimit of the local rings B it follows that A^h has a unique minimal prime. Namely, if $fg = 0$ in A^h for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. □

0C25 **Lemma 15.107.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Choose an algebraic closure $\overline{\kappa}$ of κ and denote $\kappa^{sep} \subset \overline{\kappa}$ the separable algebraic closure of κ. Let A^{sh} be the strict henselization of A with respect to κ^{sep}. Consider the maps*

$$\mathrm{Spec}(A') \xleftarrow{c} \mathrm{Spec}((A')^{sh}) \xrightarrow{e} \mathrm{Spec}(A^{sh})$$

where $(A')^{sh} = A' \otimes_A A^{sh}$. Then

1. *for $\mathfrak{m}' \subset A'$ maximal the residue field κ' is algebraic over κ and the fibre of c over $\mathfrak{m}'$ can be canonically identified with $\mathrm{Hom}_\kappa(\kappa', \overline{\kappa})$,*
2. *the right arrow is bijective on minimal primes,*
3. *every minimal prime of $(A')^{sh}$ is contained in a unique maximal ideal and every maximal ideal contains a unique minimal prime.*

Proof. The proof is almost exactly the same as for Lemma 15.107.2. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^{sh} = A^{sh}/IA^{sh}$ by (Algebra, Lemma 10.156.2). The spectra of A, A^{sh}, A', and $(A')^h$ are the same as the spectra of A/I, A^{sh}/IA^{sh}, A', and $(A')^{sh} = A' \otimes_{A/I} A^{sh}/IA^{sh}$. Thus we may replace A by

$A_{red} = A/I$ and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). The field extension κ'/κ is algebraic because A' is integral over A. Since A' is integral over A, we see that $(A')^{sh}$ is integral over A^{sh}. By going up (Algebra, Lemma 10.36.22) every maximal ideal of A', resp. $(A')^{sh}$ lies over the maximal ideal $\mathfrak{m}$, resp. $\mathfrak{m}^{sh}$ of A, resp. A^h. We have

$$(A')^{sh} \otimes_{A^{sh}} \kappa^{sep} = A' \otimes_A A^h \otimes_{A^h} \kappa^{sep} = (A' \otimes_A \kappa) \otimes_\kappa \kappa^{sep}$$

because the residue field of A^{sh} is κ^{sep}. Thus the fibre of c over $\mathfrak{m}'$ is the spectrum of $\kappa' \otimes_\kappa \kappa^{sep}$. We conclude (1) is true because there is a bijection

$$\mathrm{Hom}_\kappa(\kappa', \overline{\kappa}) \to \mathrm{Spec}(\kappa' \otimes_\kappa \kappa^{sep}), \quad \sigma \mapsto \mathrm{Ker}(\sigma \otimes 1 : \kappa' \otimes_\kappa \kappa^{sep} \to \overline{\kappa})$$

We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 10.36.19 and 10.26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 10.37.16. Since A^{sh} is a filtered colimit of étale A-algebras, $(A')^{sh}$ is filtered colimit of étale A'-algebras hence $(A')^{sh}$ is a normal ring by Algebra, Lemmas 10.163.9 and 10.37.17. Thus every local ring of $(A')^{sh}$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 15.11.8 applied to $A^{sh} \to (A')^{sh}$ to see that $((A')^{sh}, \mathfrak{m}(A')^{sh})$ is a henselian pair. If $\mathfrak{q} \subset (A')^{sh}$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak{q})$ with $V(\mathfrak{m}(A')^{sh})$ is connected by Lemma 15.11.16 Since $V(\mathfrak{m}(A')^{sh}) = \mathrm{Spec}((A')^{sh} \otimes \kappa^{sh})$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak{q}$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^{sh}$ is flat by going down (Algebra, Lemma 10.39.19) every minimal prime of $(A')^{sh}$ lies over a minimal prime of A'. Conversely, any prime of $(A')^{sh}$ lying over a minimal prime of A' is minimal because $(A')^{sh}$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^{sh}$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^{sh} are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^{sh} \otimes_A Q(A) = A^{sh} \otimes_A A' \otimes_A Q(A) = A^{sh} \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^{sh}$ and the spectrum of the ring on the right is the is the set of minimal primes of A^{sh}. This finishes the proof. □

06DM **Lemma 15.107.5.** *Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent*

(1) *A is geometrically unibranch, and*
(2) *A^{sh} has a unique minimal prime.*

[**Art66**, Lemma 2.2] and [**GD67**, Chapter IV Proposition 18.8.15]

Proof. This follows from Lemma 15.107.4 but we will also give a direct proof; this direct proof is almost exactly the same as the direct proof of Lemma 15.107.3.

Denote $\mathfrak{m}$ the maximal ideal of the ring A. Denote κ, κ^{sh} the residue field of A, A^{sh}.

Assume (2). Let $\mathfrak{p}^{sh}$ be the unique minimal prime of A^{sh}. The flatness of $A \to A^{sh}$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^{sh}$ is the unique minimal prime of A (by going down, see Algebra, Lemma 10.39.19). Also, since $A^{sh}/\mathfrak{p}A^{sh} = (A/\mathfrak{p})^{sh}$ (see Algebra, Lemma 10.156.4) is reduced by Lemma 15.45.4 we see that $\mathfrak{p}^{sh} = \mathfrak{p}A^{sh}$. Let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. We have to show that A' is local and that its residue field is purely inseparable over κ. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak{m}$ (by going up for integral ring maps, see Algebra, Lemma 10.36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$. Choosing elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak{m}_i$, $f_i \notin \mathfrak{m}_{3-i}$ we find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak{m}_i$, $i = 1, 2$. If A' is local with maximal ideal $\mathfrak{m}'$, but $A/\mathfrak{m} \subset A'/\mathfrak{m}'$ is not purely inseparable, then we can find $f \in A'$ whose image in $A'/\mathfrak{m}'$ generates a finite, not purely inseparable extension of $A/\mathfrak{m}$ and we find a finite local subalgebra $B = A[f] \subset A'$ whose residue field is not a purely inseparable extension of $A/\mathfrak{m}$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^{sh}$ the inclusions

$$A^{sh}/\mathfrak{p}^{sh} \subset B \otimes_A A^{sh} \subset \kappa(\mathfrak{p}) \otimes_A A^{sh} \subset \kappa(\mathfrak{p}^{sh})$$

the last inclusion because $\kappa(\mathfrak{p}) \otimes_A A^{sh} = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} A^{sh}/\mathfrak{p}^{sh}$ is a localization of the domain $A^{sh}/\mathfrak{p}^{sh}$. Note that $B \otimes_A \kappa^{sh}$ has at least two maximal ideals because $B/\mathfrak{m}B$ either has two maximal ideals or one whose residue field is not purely inseparable over κ, and because κ^{sh} is separably algebraically closed. Hence, as A^{sh} is strictly henselian we see that $B \otimes_A A^{sh}$ is a product of ≥ 2 local rings, see Algebra, Lemma 10.153.6. But we've just seen that $B \otimes_A A^{sh}$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak{p} \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. Let $A \to B$ be a local map of local rings which is a localization of an étale A-algebra. In particular $\mathfrak{m}_B$ is the unique prime containing $\mathfrak{m}_A B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local with purely inseparable residue field extension implies that B' is local (Algebra, Lemma 10.156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 10.163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak{p}B \subset B \otimes_A \kappa(\mathfrak{p}) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak{p}B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak{p}$). Since A^{sh} is a filtered colimit of the local rings B it follows that A^{sh} has a unique minimal prime. Namely, if $fg = 0$ in A^{sh} for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. □

0C26 **Definition 15.107.6.** Let A be a local ring with henselization A^h and strict henselization A^{sh}. The *number of branches of* A is the number of minimal primes of A^h if finite and ∞ otherwise. The *number of geometric branches of* A is the number of minimal primes of A^{sh} if finite and ∞ otherwise.

We spell out the relationship with Definition 15.107.1.

0C37 **Lemma 15.107.7.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring.*

(1) *If A has infinitely many minimal prime ideals, then the number of (geometric) branches of A is ∞.*
(2) *The number of branches of A is 1 if and only if A is unibranch.*
(3) *The number of geometric branches of A is 1 if and only if A is geometrically unibranch.*

Assume A has finitely many minimal primes and let A' be the integral closure of A in the total ring of fractions of A_{red}. Then

(4) *the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A',*
(5) *to get the number of geometric branches of A we have to count each maximal ideal $\mathfrak{m}'$ of A' with multiplicity given by the separable degree of $\kappa(\mathfrak{m}')/\kappa$.*

Proof. This lemma follows immediately from the definitions, Lemma 15.107.2, Lemma 15.107.4, and Fields, Lemma 9.14.8. □

0DQ1 **Lemma 15.107.8.** *Let $A \to B$ be a local homomorphism of local rings which is the localization of a smooth ring map.*

(1) *The number of geometric branches of A is equal to the number of geometric branches of B.*
(2) *If $A \to B$ induces a purely inseparable extension of residue fields, then the number of branches of A is the number of branches of B.*

Proof. We will use that smooth ring maps are flat (Algebra, Lemma 10.137.9), that localizations are flat (Algebra, Lemma 10.39.18), that compositions of flat ring maps are flat (Algebra, Lemma 10.39.4), that base change of a flat ring map is flat (Algebra, Lemma 10.39.7), that flat local homomorphisms are faithfully flat (Algebra, Lemma 10.39.17), that (strict) henselization is flat (Lemma 15.45.1), and Going down for flat ring maps (Algebra, Lemma 10.39.19).

Proof of (2). Let A^h, B^h be the henselizations of A, B. Then B^h is the henselization of $A^h \otimes_A B$ at the unique maximal ideal lying over $\mathfrak{m}_B$, see Algebra, Lemma 10.155.8. Thus we may and do assume A is henselian. Since $A \to B \to B^h$ is flat, every minimal prime of B^h lies over a minimal prime of A and since $A \to B^h$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^h; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak{p} \subset A$, there is at most one minimal prime of B^h lying over $\mathfrak{p}$. After replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a domain; the A is still henselian by Algebra, Lemma 10.156.2. By Lemma 15.107.3 we see that the integral closure A' of A in its field of fractions is a local domain. Of course A' is a normal domain. By Algebra, Lemma 10.163.9 we see that $A' \otimes_A B^h$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^h$ use that B^h is a colimit of étale B-algebras and use Algebra, Lemma 10.37.17). By Algebra, Lemma 10.156.5 we see that $A' \otimes_A B^h$ is local (this is where we use the assumption on the residue fields of A and B). Hence $A' \otimes_A B^h$ is a local normal ring, hence a local domain. Since $B^h \subset A' \otimes_A B^h$ by flatness of $A \to B^h$ we conclude that B^h is a domain as desired.

Proof of (1). Let A^{sh}, B^{sh} be strict henselizations of A, B. Then B^{sh} is a strict henselization of $A^h \otimes_A B$ at a maximal ideal lying over $\mathfrak{m}_B$ and $\mathfrak{m}_{A^h}$, see Algebra, Lemma 10.155.12. Thus we may and do assume A is strictly henselian. Since $A \to B \to B^{sh}$ is flat, every minimal prime of B^{sh} lies over a minimal prime of A and since $A \to B^{sh}$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^{sh}; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak{p} \subset A$, there is at most one minimal prime of B^{sh} lying over $\mathfrak{p}$. After replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a domain; then A is still strictly henselian by Algebra, Lemma 10.156.4. By Lemma 15.107.5 we see that the integral closure A' of A in its field of fractions is a local domain whose residue field is a purely inseparable extension of the residue field of A. Of course A' is a normal domain. By Algebra, Lemma 10.163.9 we see that $A' \otimes_A B^{sh}$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^{sh}$ use that B^{sh} is a colimit of étale B-algebras and use Algebra, Lemma 10.37.17). By Algebra, Lemma 10.156.5 we see that $A' \otimes_A B^{sh}$ is local (since $A \subset A'$ induces a purely inseparable residue field extension). Hence $A' \otimes_A B^{sh}$ is a local normal ring, hence a local domain. Since $B^{sh} \subset A' \otimes_A B^{sh}$ by flatness of $A \to B^{sh}$ we conclude that B^{sh} is a domain as desired. □

15.108. Miscellaneous on branches

0GS4 Some results related to branches of local rings as defined in Section 15.107.

0GS5 **Lemma 15.108.1.** *Let A and B be domains and let $A \to B$ be a ring map. Assume $A \to B$ has additionally at least one of the following properties*

(1) *it is the localization of an étale ring map,*
(2) *it is flat and the localization of an unramified ring map,*
(3) *it is flat and the localization of a quasi-finite ring map,*
(4) *it is flat and the localization of an integral ring map,*
(5) *it is flat and there are no nontrivial specializations between points of fibres of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$,
(6) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *maps the generic point to the generic point and there are no nontrivial specializations between points of fibres, or*
(7) *exactly one point of* $\mathrm{Spec}(B)$ *is mapped to the generic point of* $\mathrm{Spec}(A)$.

Then $A \cap J$ is nonzero for every nonzero ideal J of B.

Proof. Proof in case (7). Let K, resp. L be the fraction field of A, resp. B. By Algebra, Lemma 10.30.7 we see that the unique point of $\mathrm{Spec}(B)$ which maps to the generic point $(0) \in \mathrm{Spec}(A)$ is $(0) \in \mathrm{Spec}(B)$. We conclude that $B \otimes_A K$ is a ring with a unique prime ideal whose residue field is L (in fact it is equal to L but we do not need this). Choose $b \in J$ nonzero. Then b maps to a unit of L. Hence b maps to a unit of $B \otimes_A K$ (Algebra, Lemma 10.19.2). Since $B \otimes_A K = \mathrm{colim}_{f \in A \setminus \{0\}} B_f$ we see that b maps to a unit of B_f for some $f \in A$ nonzero. This means that $bb' = f^n$ for some $b' \in B$ and $n \geq 1$. Thus $f^n \in A \cap J$ as desired.

In the rest of the proof, we show that each of the other assumptions imply (7). Under assumptions (1) – (5), the ring map $A \to B$ is flat and hence $A \to B$ is injective (since flat local homomorphisms are faithfully flat by Algebra, Lemma 10.39.17). Hence the generic point of $\mathrm{Spec}(B)$ maps to the generic point of $\mathrm{Spec}(A)$. Now, if there are no nontrivial specializations between points of fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then of course this generic point of $\mathrm{Spec}(B)$ has to be the unique point

mapping to the generic point of $\mathrm{Spec}(A)$. So (6) implies (7). Finally, to finish we show that in cases (1) – (5) there are no nontrivial specializations between the points of fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Namely, see Algebra, Lemma 10.36.20 for the integral case, Algebra, Definition 10.122.3 for the quasi-finite case, and use that unramified and étale ring maps are quasi-finite (Algebra, Lemmas 10.151.6 and 10.143.6). □

0GSC **Lemma 15.108.2.** *Let $A \to B$ be a ring map. Let $\mathfrak{q} \subset B$ be a prime ideal lying over the prime $\mathfrak{p} \subset A$. Assume*

(1) *A is a domain,*
(2) *$A_\mathfrak{p}$ is geometrically unibranch,*
(3) *$A \to B$ is unramified at $\mathfrak{q}$, and*
(4) *$A_\mathfrak{p} \to B_\mathfrak{q}$ is injective.*

Then there exists a $g \in B$, $g \notin \mathfrak{q}$ such that B_g is étale over A.

Proof. By Algebra, Proposition 10.152.1 after replacing B by a principal localization, we can find a standard étale ring map $A \to B'$ and a surjection $B' \to B$. Denote $\mathfrak{q}' \subset B'$ the inverse image of $\mathfrak{q}$. We will show that $B' \to B$ is injective after possibly replacing B' by a principal localization.

In this paragraph we reduce to the case that B' is a domain. Since A is a domain, the ring B' is reduced, see Algebra, Lemma 15.42.1. Let K be the fraction field of A. Then $B' \otimes_A K$ is étale over a field, hence is a finite product of fields, see Algebra, Lemma 10.143.4. Since $A \to B'$ is étale (hence flat) the minimal primes of B' are lie over $(0) \subset A$ (by going down for flat ring maps). We conclude that B' has finitely many minimal primes, say $\mathfrak{r}_1, \ldots, \mathfrak{r}_r \subset B'$. Since $A_\mathfrak{p}$ is geometrically unibranch and $A \to B'$ étale, the ring $B'_{\mathfrak{q}'}$ is a domain, see Lemmas 15.107.8 and 15.107.7. Hence $\mathfrak{q}' \supset \mathfrak{r}_i$ for exactly one $i = i_0$. Choose $g' \in B'$, $g' \notin \mathfrak{r}_{i_0}$ but $g' \in \mathfrak{r}_i$ for $i \neq i_0$, see Algebra, Lemma 10.15.2. After replacing B' and B by $B'_{g'}$ and $B_{g'}$ we obtain that B' is a domain.

Assume B' is a domain, in particular $B' \subset B'_{\mathfrak{q}'}$. If $B' \to B$ is not injective, then $J = \mathrm{Ker}(B'_{\mathfrak{q}'} \to B_\mathfrak{q})$ is nonzero. By Lemma 15.108.1 applied to $A_\mathfrak{p} \to B'_{\mathfrak{q}'}$ we find a nonzero element $a \in A_\mathfrak{p}$ mapping to zero in $B_\mathfrak{q}$ contradicting assumption (4). This finishes the proof. □

0GS6 **Lemma 15.108.3.** *Let $(A, \mathfrak{m})$ be a geometrically unibranch local domain. Let $A \to B$ be an injective local homomorphism of local rings, which is essentially of finite type. If $\mathfrak{m}B$ is the maximal ideal of B and the induced extension of residue fields is separable, then $A \to B$ is the localization of an étale ring map.*

Generalization of [**Gro71**, Expose I, Theorem 9.5 part (ii)]

Proof. We may write $B = C_\mathfrak{q}$ where $A \to C$ is a finite type ring map and $\mathfrak{q} \subset C$ is a prime ideal lying over $\mathfrak{m}$. By Algebra, Lemma 10.151.7 the ring map $A \to C$ is unramified at $\mathfrak{q}$. By Algebra, Proposition 10.152.1 after replacing C by a principal localization, we can find a standard étale ring map $A \to C'$ and a surjection $C' \to C$. Denote $\mathfrak{q}' \subset C'$ the inverse image of $\mathfrak{q}$ and set $B' = C'_{\mathfrak{q}'}$. Then $B' \to B$ is surjective. It suffices to show that $B' \to B$ is also injective.

Since A is a domain, the rings C' and B' are reduced, see Algebra, Lemma 15.42.1. Since A is geometrically unibranch, the ring B' is a domain, see by Lemmas 15.107.8 and 15.107.7. If $B' \to B$ is not injective, then $A \cap \mathrm{Ker}(B' \to B)$ is nonzero by Lemma 15.108.1 which contradicts the assumption that $A \to B$ is injective. □

06DU **Lemma 15.108.4.** *Let k be an algebraically closed field. Let A, B be strictly henselian local k-algebras with residue field equal to k. Let C be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Then the minimal primes of C correspond 1-to-1 to pairs of minimal primes of A and B.*

Proof. First note that a minimal prime $\mathfrak{r}$ of C maps to a minimal prime $\mathfrak{p}$ in A and to a minimal prime $\mathfrak{q}$ of B because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 10.39.19). Hence it suffices to show that the strict henselization of $(A/\mathfrak{p} \otimes_k B/\mathfrak{q})_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ has a unique minimal prime ideal. By Algebra, Lemma 10.156.4 the rings $A/\mathfrak{p}$, $B/\mathfrak{q}$ are strictly henselian. Hence we may assume that A and B are strictly henselian local domains and our goal is to show that C has a unique minimal prime. By Lemma 15.107.5 the integral closure A' of A in its fraction field is a normal local domain with residue field k. Similarly for the integral closure B' of B into its fraction field. By Algebra, Lemma 10.165.5 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds – Algebra, Lemma 10.36.22) and that $\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$$

by Algebra, Lemma 10.41.11. It follows that

$$(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B} \longrightarrow R$$

is integral. We conclude that R is the integral closure of $(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ in its fraction field, and by Lemma 15.107.5 once again we conclude that C has a unique prime ideal. □

15.109. Branches of the completion

0C27 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Consider the maps $A \to A^h \to A^\wedge$. In general the map $A^h \to A^\wedge$ need not induce a bijection on minimal primes, see Examples, Section 110.20. In other words, the number of branches of A (as defined in Definition 15.107.6) may be different from the number of branches of $A^\wedge$. However, under some conditions the number of branches is the same, for example if the dimension of A is 1.

0C28 **Lemma 15.109.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *The map $A^h \to A^\wedge$ defines a surjective map from minimal primes of $A^\wedge$ to minimal primes of A^h.*
2. *The number of branches of A is at most the number of branches of $A^\wedge$.*
3. *The number of geometric branches of A is at most the number of geometric branches of $A^\wedge$.*

Proof. By Lemma 15.45.3 the map $A^h \to A^\wedge$ is flat and injective. Combining going down (Algebra, Lemma 10.39.19) and Algebra, Lemma 10.30.5 we see that part (1) holds. Part (2) follows from this, Definition 15.107.6, and the fact that $A^\wedge$ is henselian (Algebra, Lemma 10.153.9). By Lemma 15.45.3 we have $(A^\wedge)^{sh} = A^{sh} \otimes_{A^h} A^\wedge$. Thus we can repeat the arguments above using the flat injective map $A^{sh} \to (A^\wedge)^{sh}$ to prove (3). □

0C29 **Lemma 15.109.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The number of branches of A is the same as the number of branches of $A^\wedge$ if and only if $\sqrt{\mathfrak{q}A^\wedge}$ is prime for every minimal prime $\mathfrak{q} \subset A^h$ of the henselization.*

Proof. Follows from Lemma 15.109.1 and the fact that there are only a finite number of branches for both A and $A^\wedge$ by Algebra, Lemma 10.31.6 and the fact that A^h and $A^\wedge$ are Noetherian (Lemma 15.45.3). □

A simple glueing lemma.

0C2A **Lemma 15.109.3.** *Let A be a ring and let I be a finitely generated ideal. Let $A \to C$ be a ring map such that for all $f \in I$ the ring map $A_f \to C_f$ is localization at an idempotent. Then there exists a surjection $A \to C'$ such that $A_f \to (C \times C')_f$ is an isomorphism for all $f \in I$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Write

$$C_{f_i} = (A_{f_i})_{e_i}$$

for some idempotent $e_i \in A_{f_i}$. Write $e_i = a_i/f_i^n$ for some $a_i \in A$ and $n \geq 0$; we may use the same n for all $i = 1, \ldots, r$. After replacing a_i by $f_i^m a_i$ and n by $n + m$ for a suitable $m \gg 0$, we may assume $a_i^2 = f_i^n a_i$ for all i. Since e_i maps to 1 in $C_{f_i f_j} = (A_{f_i f_j})_{e_j} = A_{f_i f_j a_j}$ we see that

$$(f_i f_j a_j)^N (f_j^n a_i - f_i^n a_j) = 0$$

for some N (we can pick the same N for all pairs i, j). Using $a_j^2 = f_j^n a_j$ this gives

$$f_i^{N+n} f_j^{N+nN} a_j = f_i^N f_j^{N+n} a_i a_j^N$$

After increasing n to $n + N + nN$ and replacing a_i by $f_i^{N+nN} a_i$ we see that $f_i^n a_j$ is in the ideal of a_i for all pairs i, j. Let $C' = A/(a_1, \ldots, a_r)$. Then

$$C'_{f_i} = A_{f_i}/(a_i) = A_{f_i}/(e_i)$$

because a_j is in the ideal generated by a_i after inverting f_i. Since for an idempotent e of a ring B we have $B = B_e \times B/(e)$ we see that the conclusion of the lemma holds for f equal to one of $f_1, \ldots, f_r$. Using glueing of functions, in the form of Algebra, Lemma 10.23.2, we conclude that the result holds for all $f \in I$. Namely, for $f \in I$ the elements $f_1, \ldots, f_r$ generate the unit ideal in A_f so $A_f \to (C \times C')_f$ is an isomorphism if and only if this is the case after localizing at $f_1, \ldots, f_r$. □

Lemma 15.109.4 can be used to construct finite type extensions from given finite type extensions of the formal completion. We will generalize this lemma in Algebraization of Formal Spaces, Lemma 88.10.3.

0ALR **Lemma 15.109.4.** *Let A be a Noetherian ring and I an ideal. Let B be a finite type A-algebra. Let $B^\wedge \to C$ be a surjective ring map with kernel J where $B^\wedge$ is the I-adic completion. If J/J^2 is annihilated by I^c for some $c \geq 0$, then C is isomorphic to the completion of a finite type A-algebra.*

Proof. Let $f \in I$. Since $B^\wedge$ is Noetherian (Algebra, Lemma 10.97.6), we see that J is a finitely generated ideal. Hence we conclude from Algebra, Lemma 10.21.5 that

$$C_f = ((B^\wedge)_f)_e$$

for some idempotent $e \in (B^\wedge)_f$. By Lemma 15.109.3 we can find a surjection $B^\wedge \to C'$ such that $B^\wedge \to C \times C'$ becomes an isomorphism after inverting any $f \in I$. Observe that $C \times C'$ is a finite $B^\wedge$-algebra.

Choose generators $f_1, \ldots, f_r \in I$. Denote $\alpha_i : (C \times C')_{f_i} \to B_{f_i} \otimes_B B^\wedge$ the inverse of the isomorphism of $(B^\wedge)_{f_i}$-algebras we obtained above. Denote $\alpha_{ij} : (B_{f_i})_{f_j} \to (B_{f_j})_{f_i}$ the obvious B-algebra isomorphism. Consider the object

$$(C \times C', B_{f_i}, \alpha_i, \alpha_{ij})$$

of the category $\mathrm{Glue}(B \to B^\wedge, f_1, \ldots, f_r)$ introduced in Remark 15.90.10. We omit the verification of conditions (1)(a) and (1)(b). Since $B \to B^\wedge$ is a flat map (Algebra, Lemma 10.97.2) inducing an isomorphism $B/IB \to B^\wedge/IB^\wedge$ we may apply Proposition 15.90.16 and Remark 15.90.20. We conclude that $C \times C'$ is isomorphic to $D \otimes_B B^\wedge$ for some finite B-algebra D. Then $D/ID \cong C/IC \times C'/IC'$. Let $\overline{e} \in D/ID$ be the idempotent corresponding to the factor C/IC. By Lemma 15.9.10 there exists an étale ring map $B \to B'$ which induces an isomorphism $B/IB \to B'/IB'$ such that $D' = D \otimes_B B'$ contains an idempotent e lifting $\overline{e}$. Since $C \times C'$ is I-adically complete the pair $(C \times C', IC \times IC')$ is henselian (Lemma 15.11.4). Thus we can factor the map $B \to C \times C'$ through B' (see Lemma 15.11.6). Doing so we may replace B by B' and D by D' (because the I-adic completetions are unaltered by this). Then we find that $D = D_e \times D_{1-e} = D/(1-e) \times D/(e)$ is a product of finite type A-algebras and the completion of the first part is C and the completion of the second part is C'. □

0C2B **Lemma 15.109.5.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with henselization A^h. Let $\mathfrak{q} \subset A^\wedge$ be a minimal prime with $\dim(A^\wedge/\mathfrak{q}) = 1$. Then there exists a minimal prime $\mathfrak{q}^h$ of A^h such that $\mathfrak{q} = \sqrt{\mathfrak{q}^h A^\wedge}$.*

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 15.45.3). We will apply Lemma 15.109.4 to $A^\wedge \to A^\wedge/J$ where $J = \mathrm{Ker}(A^\wedge \to (A^\wedge)_\mathfrak{q})$. Since $\dim((A^\wedge)_\mathfrak{q}) = 0$ we see that $\mathfrak{q}^n \subset J$ for some n. Hence J/J^2 is annihilated by $\mathfrak{q}^n$. On the other hand $(J/J^2)_\mathfrak{q} = 0$ because $J_\mathfrak{q} = 0$. Hence $\mathfrak{m}$ is the only associated prime of J/J^2 and we find that a power of $\mathfrak{m}$ annihilates J/J^2. Thus the lemma applies and we find that $A^\wedge/J = C^\wedge$ for some finite type A-algebra C.

Then $C/\mathfrak{m}C = A/\mathfrak{m}$ because $A^\wedge/J$ has the same property. Hence $\mathfrak{m}_C = \mathfrak{m}C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak{m}_C$ (Algebra, Lemma 10.151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 10.152.1. However, since the residue field extension of $A \to C_{\mathfrak{m}_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 10.97.2) and $I = J \cap A$ (by flatness of $A \to A^\wedge$). Since $\mathfrak{q}^n \subset J \subset \mathfrak{q}$ we see that $\mathfrak{p} = \mathfrak{q} \cap A$ satisfies $\mathfrak{p}^n \subset I \subset \mathfrak{p}$. Then $\sqrt{\mathfrak{p}A^\wedge} = \mathfrak{q}$ and the proof is complete. □

0C2C **Lemma 15.109.6.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The punctured spectrum of $A^\wedge$ is disconnected if and only if the punctured spectrum of A^h is disconnected.*

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 15.45.3).

Since $A \to A^\wedge$ is faithfully flat (see reference just given) the map from the punctured spectrum of $A^\wedge$ to the punctured spectrum of A is surjective (see Algebra, Lemma 10.39.16). Hence if the punctured spectrum of A is disconnected, then the same is true for $A^\wedge$.

Assume the punctured spectrum of $A^\wedge$ is disconnected. This means that

$$\mathrm{Spec}(A^\wedge) \setminus \{\mathfrak{m}^\wedge\} = Z \amalg Z'$$

with Z and Z' closed. Let $\overline{Z}, \overline{Z}' \subset \mathrm{Spec}(A^\wedge)$ be the closures. Say $\overline{Z} = V(J)$, $\overline{Z}' = V(J')$ for some ideals $J, J' \subset A^\wedge$. Then $V(J+J') = \{\mathfrak{m}^\wedge\}$ and $V(JJ') = \mathrm{Spec}(A^\wedge)$. The first equality means that $\mathfrak{m}^\wedge = \sqrt{J + J'}$ which implies $(\mathfrak{m}^\wedge)^e \subset J + J'$ for some $e \geq 1$. The second equality implies every element of JJ' is nilpotent hence $(JJ')^n = 0$ for some $n \geq 1$. Combined this means that J^n/J^{2n} is annihilated by J^n and $(J')^n$ and hence by $(\mathfrak{m}^\wedge)^{2en}$. Thus we may apply Lemma 15.109.4 to see that there is a finite type A-algebra C and an isomorphism $A^\wedge/J^n = C^\wedge$.

The rest of the proof is exactly the same as the second part of the proof of Lemma 15.109.5; of course that lemma is a special case of this one! We have $C/\mathfrak{m}C = A/\mathfrak{m}$ because $A^\wedge/J^n$ has the same property. Hence $\mathfrak{m}_C = \mathfrak{m}C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak{m}_C$ (Algebra, Lemma 10.151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 10.152.1. However, since the residue field extension of $A \to C_{\mathfrak{m}_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J^n = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 10.97.2) and $I = J^n \cap A$ (by flatness of $A \to A^\wedge$). By symmetry $I' = (J')^n \cap A$ satisfies $(J')^n = I'A^\wedge$. Then $\mathfrak{m}^e \subset I + I'$ and $II' = 0$ and we conclude that $V(I)$ and $V(I')$ are closed subschemes which give the desired disjoint union decomposition of the punctured spectrum of A. □

0C2D **Lemma 15.109.7.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Then the number of (geometric) branches of A and $A^\wedge$ is the same.*

Proof. To see this for the number of branches, combine Lemmas 15.109.1, 15.109.2, and 15.109.5 and use that the dimension of $A^\wedge$ is one, see Lemma 15.43.1. To see this is true for the number of geometric branches we use the result for branches, the fact that the dimension does not change under strict henselization (Lemma 15.45.7), and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 15.45.3. □

0C2E **Lemma 15.109.8.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If the formal fibres of A are geometrically normal (for example if A is excellent or quasi-excellent), then A is Nagata and the number of (geometric) branches of A and $A^\wedge$ is the same.* [**Bed13**, Theorem 2.3]

Proof. Since a normal ring is reduced, we see that A is Nagata by Lemma 15.52.4. In the rest of the proof we will use Lemma 15.51.10, Proposition 15.51.5, and Lemma 15.51.4. This tells us that A is a P-ring where $P(k \to R) =$"R is geometrically normal over k" and the same is true for any (essentially of) finite type A-algebra.

Let $\mathfrak{q} \subset A$ be a minimal prime. Then $A^\wedge/\mathfrak{q}A^\wedge = (A/\mathfrak{q})^\wedge$ and $A^h/\mathfrak{q}A^h = (A/\mathfrak{q})^h$ (Algebra, Lemma 10.156.2). Hence the number of branches of A is the sum of the number of branches of the rings $A/\mathfrak{q}$ and similarly for $A^\wedge$. In this way we reduce to the case that A is a domain.

Assume A is a domain. Let A' be the integral closure of A in the fraction field K of A. Since A is Nagata, we see that $A \to A'$ is finite. Recall that the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A' (Lemma 15.107.2). Also, recall that

$$(A')^\wedge = A' \otimes_A A^\wedge = \prod\nolimits_{\mathfrak{m}' \subset A'} (A'_{\mathfrak{m}'})^\wedge$$

by Algebra, Lemma 10.97.8. Because $A'_{\mathfrak{m}'}$ is a local ring whose formal fibres are geometrically normal, we see that $(A'_{\mathfrak{m}'})^\wedge$ is normal (Lemma 15.52.6). Hence the minimal primes of $A' \otimes_A A^\wedge$ are in 1-to-1 correspondence with the factors in the decomposition above. By flatness of $A \to A^\wedge$ we have

$$A^\wedge \subset A' \otimes_A A^\wedge \subset K \otimes_A A^\wedge$$

Since the left and the right ring have the same set of minimal primes, the same is true for the ring in the middle (small detail omitted) and this finishes the proof.

To see this is true for the number of geometric branches we use the result for branches, the fact that the formal fibres of A^{sh} are geometrically normal (Lemmas 15.51.10 and 15.51.8) and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 15.45.3. $\square$

15.110. Formally catenary rings

0AW1 In this section we prove a theorem of Ratliff [**Rat71**] that a Noetherian local ring is universally catenary if and only if it is formally catenary.

0AW2 **Definition 15.110.1.** A Noetherian local ring A is *formally catenary* if for every minimal prime $\mathfrak{p} \subset A$ the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is equidimensional.

Let A be a Noetherian local ring which is formally catenary. By Ratliff's result (Proposition 15.110.5) we see that any quotient of A is also formally catenary (because the class of universally catenary rings is stable under quotients). We conclude that the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is equidimensional for every prime ideal $\mathfrak{p}$ of A.

0AW3 **Lemma 15.110.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which is not formally catenary. Then A is not universally catenary.*

Proof. By assumption there exists a minimal prime $\mathfrak{p} \subset A$ such that the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is not equidimensional. After replacing A by $A/\mathfrak{p}$ we may assume that A is a domain and that the spectrum of $A^\wedge$ is not equidimensional. Let $\mathfrak{q}$ be a minimal prime of $A^\wedge$ such that $d = \dim(A^\wedge/\mathfrak{q})$ is minimal and hence $0 < d < \dim(A)$. We prove the lemma by induction on d.

The case $d = 1$. In this case $\dim(A^\wedge_{\mathfrak{q}}) = 0$. Hence $A^\wedge_{\mathfrak{q}}$ is Artinian local and we see that for some $n > 0$ the ideal $J = \mathfrak{q}^n$ maps to zero in $A^\wedge_{\mathfrak{q}}$. It follows that $\mathfrak{m}$ is the only associated prime of J/J^2, whence $\mathfrak{m}^m$ annihilates J/J^2 for some $m > 0$. Thus we can use Lemma 15.109.4 to find $A \to B$ of finite type such that $B^\wedge \cong A^\wedge/J$. It follows that $\mathfrak{m}_B = \sqrt{\mathfrak{m}B}$ is a maximal ideal with the same residue field as $\mathfrak{m}$ and $B^\wedge$ is the $\mathfrak{m}_B$-adic completion (Algebra, Lemma 10.97.7). Then

$$\dim(B_{\mathfrak{m}_B}) = \dim(B^\wedge) = 1 = d.$$

Since we have the factorization $A \to B \to A^\wedge/J$ the inverse image of $\mathfrak{q}/J$ is a prime $\mathfrak{q}' \subset \mathfrak{m}_B$ lying over (0) in A. Thus, if A were universally catenary, the dimension

formula (Algebra, Lemma 10.113.1) would give

$$\begin{aligned}\dim(B_{\mathfrak{m}_B}) &\geq \dim((B/\mathfrak{q}')_{\mathfrak{m}_B}) \\ &= \dim(A) + \mathrm{trdeg}_A(B/\mathfrak{q}') - \mathrm{trdeg}_{\kappa(\mathfrak{m})}(\kappa(\mathfrak{m}_B)) \\ &= \dim(A) + \mathrm{trdeg}_A(B/\mathfrak{q}')\end{aligned}$$

This contradiction finishes the argument in case $d = 1$.

Assume $d > 1$. Let $Z \subset \mathrm{Spec}(A^\wedge)$ be the union of the irreducible components distinct from $V(\mathfrak{q})$. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_m \subset A^\wedge$ be the prime ideals corresponding to irreducible components of $V(\mathfrak{q}) \cap Z$ of dimension > 0. Choose $f \in \mathfrak{m}$, $f \notin A \cap \mathfrak{r}_j$ using prime avoidance (Algebra, Lemma 10.15.2). Then $\dim(A/fA) = \dim(A) - 1$ and there is some irreducible component of $V(\mathfrak{q}, f)$ of dimension $d-1$. Thus A/fA is not formally catenary and the invariant d has decreased. By induction A/fA is not universally catenary, hence A is not universally catenary. □

0AW4 **Lemma 15.110.3.** *Let $A \to B$ be a flat local ring map of local Noetherian rings. Assume B is catenary and is $\mathrm{Spec}(B)$ equidimensional. Then*

(1) *$\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional for all $\mathfrak{p} \subset A$ and*
(2) *A is catenary and $\mathrm{Spec}(A)$ is equidimensional.*

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $\mathfrak{q} \subset B$ be a prime minimal over $\mathfrak{p}B$. Then $\mathfrak{q} \cap A = \mathfrak{p}$ by going down for $A \to B$ (Algebra, Lemma 10.39.19). Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is a flat local ring map with special fibre of dimension 0 and hence

$$\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q}) = \dim(B) - \dim(B/\mathfrak{q})$$

(Algebra, Lemma 10.112.7). The second equality because $\mathrm{Spec}(B)$ is equidimensional and B is catenary. Thus $\dim(B/\mathfrak{q})$ is independent of the choice of $\mathfrak{q}$ and we conclude that $\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional of dimension $\dim(B) - \dim(A_\mathfrak{p})$. On the other hand, we have $\dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim(B/\mathfrak{m}_A B)$ and $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ by flatness (see lemma cited above) and we get

$$\dim(A_\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p})$$

for all $\mathfrak{p}$ in A. Applying this to all minimal primes in A we see that A is equidimensional. If $\mathfrak{p} \subset \mathfrak{p}'$ is a strict inclusion with no primes in between, then we may apply the above to the prime $\mathfrak{p}'/\mathfrak{p}$ in $A/\mathfrak{p}$ because $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat and $\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional, to get

$$1 = \dim((A/\mathfrak{p})_{\mathfrak{p}'}) = \dim(A/\mathfrak{p}) - \dim(A/\mathfrak{p}')$$

Thus $\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function and we conclude that A is catenary. □

0AW5 **Lemma 15.110.4.** *Let A be a formally catenary Noetherian local ring. Then A is universally catenary.*

Proof. We may replace A by $A/\mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime of A, see Algebra, Lemma 10.105.8. Thus we may assume that the spectrum of $A^\wedge$ is equidimensional. It suffices to show that every local ring essentially of finite type over A is catenary (see for example Algebra, Lemma 10.105.6). Hence it suffices to show that $A[x_1, \ldots, x_n]_\mathfrak{m}$ is catenary where $\mathfrak{m} \subset A[x_1, \ldots, x_n]$ is a maximal ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 10.54.5 (and Algebra, Lemmas 10.105.7 and 10.105.4). Let $\mathfrak{m}' \subset A^\wedge[x_1, \ldots, x_n]$ be the unique maximal ideal lying over $\mathfrak{m}$. Then

$$A[x_1, \ldots, x_n]_\mathfrak{m} \to A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$$

is local and flat (Algebra, Lemma 10.97.2). Hence it suffices to show that the ring on the right hand side catenary with equidimensional spectrum, see Lemma 15.110.3. It is catenary because complete local rings are universally catenary (Algebra, Remark 10.160.9). Pick any minimal prime $\mathfrak{q}$ of $A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Then $\mathfrak{q} = \mathfrak{p}A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$ for some minimal prime $\mathfrak{p}$ of $A^\wedge$ (small detail omitted). Hence

$$\dim(A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{p}) + n = \dim(A^\wedge) + n$$

the first equality by Algebra, Lemma 10.112.7 and the second because the spectrum of $A^\wedge$ is equidimensional. This finishes the proof. □

0AW6 **Proposition 15.110.5** (Ratliff)**.** *A Noetherian local ring is universally catenary if and only if it is formally catenary.* [**Rat71**]

Proof. Combine Lemmas 15.110.2 and 15.110.4. □

0C2F **Lemma 15.110.6.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with geometrically normal formal fibres. Then* [**HRW04**, Corollary 2.3]

(1) *A^h is universally catenary, and*
(2) *if A is unibranch (for example normal), then A is universally catenary.*

Proof. By Lemma 15.109.8 the number of branches of A and $A^\wedge$ are the same, hence Lemma 15.109.2 applies. Then for any minimal prime $\mathfrak{q} \subset A^h$ we see that $A^\wedge/\mathfrak{q}A^\wedge$ has a unique minimal prime. Thus A^h is formally catenary (by definition) and hence universally catenary by Proposition 15.110.5. If A is unibranch, then A^h has a unique minimal prime, hence $A^\wedge$ has a unique minimal prime, hence A is formally catenary and we conclude in the same way. □

15.111. Group actions and integral closure

0BRE This section is in some sense a continuation of Algebra, Section 10.38. More material of a similar kind can be found in Fundamental Groups, Section 58.12

0BRF **Lemma 15.111.1.** *Let $\varphi : A \to B$ be a surjection of rings. Let G be a finite group of order n acting on $\varphi : A \to B$. If $b \in B^G$, then there exists a monic polynomial $P \in A^G[T]$ which maps to $(T - b)^n$ in $B^G[T]$.*

Proof. Choose $a \in A$ lifting b and set $P = \prod_{\sigma \in G}(T - \sigma(a))$. □

09EG **Lemma 15.111.2.** *Let R be a ring. Let G be a finite group acting on R. Let $I \subset R$ be an ideal such that $\sigma(I) \subset I$ for all $\sigma \in G$. Then $R^G/I^G \subset (R/I)^G$ is an integral extension of rings which induces a homeomorphism on spectra and purely inseparable extensions of residue fields.*

Proof. Since $I^G = R^G \cap I$ it is clear that the map is injective. Lemma 15.111.1 shows that Algebra, Lemma 10.46.11 applies. □

0H34 **Lemma 15.111.3.** *Let G be a finite group of order n acting on a ring R. Let $J \subset R^G$ be an ideal. For $x \in JR$ we have $\prod_{\sigma \in G}(T - \sigma(x)) = T^n + a_1T^{n-1} + \ldots + a_n$ with $a_i \in J$.*

Proof. Observe that the polynomial is indeed monic and has coefficients in R^G. We can write $x = f_1b_1 + \ldots + f_mb_m$ with $f_j \in J$ and $b_j \in R$. Thus, arguing by

induction on m, we may assume that $x = y - fb$ with $f \in J$, $b \in R$, and $y \in JR$ such that the result holds for y. Then we see that

$$\prod\nolimits_{\sigma\in G}(T - \sigma(x)) = \prod\nolimits_{\sigma\in G}(T - \sigma(y) + f\sigma(b)) = \prod\nolimits_{\sigma\in G}(T - \sigma(y)) + \sum_{i=1,\ldots,n} f^i a_i$$

where we have

$$a_i = \sum\nolimits_{S\subset G,\ |S|=i} \prod\nolimits_{\sigma\in S} \sigma(b) \prod\nolimits_{\sigma\notin S}(T - \sigma(y))$$

A computation we omit shows that $a_i \in R^G$ (hint: the given expression is symmetric). Thus the polynomial of the statement of the lemma for x is congruent modulo J to the polynomial for y and this proves the induction step. □

0H35 **Lemma 15.111.4.** *Let R be a ring. Let G be a finite group of order n acting on R. Let $J \subset R^G$ be an ideal. Then $R^G/J \to (R/JR)^G$ is ring map such that*

1. *for $b \in (R/JR)^G$ there is a monic polynomial $P \in R^G/J[T]$ whose image in $(R/JR)^G[T]$ is $(T - b)^n$,*
2. *for $a \in \mathrm{Ker}(R^G/J \to (R/JR)^G)$ we have $(T - a)^n = T^n$ in $R^G/J[T]$.*

In particular, $R^G/J \to (R/JR)^G$ is an integral ring map which induces homeomorphisms on spectra and purely inseparable extensions of residue fields.

Proof. Part (1) follow from Lemma 15.111.1 with $I = JR$. If a is as in part (2), then a is the image of $x \in R^G \cap JR$. Hence $(T - x)^n = \prod_{\sigma\in G}(T - \sigma(x))$ is congruent to T^n modulo J by Lemma 15.111.3. This proves part (2). To see the final statement we may apply Algebra, Lemma 10.46.11. □

0H36 **Remark 15.111.5.** In Lemma 15.111.4 we see that the map $R^G/J \to (R/JR)^G$ is an isomorphism if n is invertible in R.

0BRG **Lemma 15.111.6.** *Let R be a ring. Let G be a finite group of order n acting on R. Let A be an R^G-algebra.*

1. *for $b \in (A \otimes_{R^G} R)^G$ there exists a monic polynomial $P \in A[T]$ whose image in $(A \otimes_{R^G} R)^G[T]$ is $(T - b)^n$,*
2. *for $a \in \mathrm{Ker}(A \to (A \otimes_{R^G} R)^G)$ we have $(T - a)^n = T^n$ in $A[T]$.*

Proof. Choose a surjection $E \to A$ where E is a polynomial algebra over R^G. Then $(E \otimes_{R^G} R)^G = E$ because E is free as an R^G-module. Denote $J = \mathrm{Ker}(E \to A)$. Since tensor product is right exact we see that $A \otimes_{R^G} R$ is the quotient of $E \otimes_{R^G} R$ by the ideal generated by J. In this way we see that our lemma is a special case of Lemma 15.111.4. □

0BRH **Lemma 15.111.7.** *Let R be a ring. Let G be a finite group acting on R. Let $R^G \to A$ be a ring map. The map*

$$A \to (A \otimes_{R^G} R)^G$$

is an isomorphism if $R^G \to A$ is flat. In general the map is integral, induces a homeomorphism on spectra, and induces purely inseparable residue field extensions.

Proof. To see the first statement consider the exact sequence $0 \to R^G \to R \to \bigoplus_{\sigma\in G} R$ where the second map sends x to $(\sigma(x) - x)_{\sigma\in G}$. Tensoring with A the sequence remains exact if $R^G \to A$ is flat. Thus A is the G-invariants in $(A \otimes_{R^G} R)^G$.

The second statement follows from Lemma 15.111.6 and Algebra, Lemma 10.46.11. □

0BRI **Lemma 15.111.8.** *Let G be a finite group acting on a ring R. For any two primes $\mathfrak{q}, \mathfrak{q}' \subset R$ lying over the same prime in R^G there exists a $\sigma \in G$ with $\sigma(\mathfrak{q}) = \mathfrak{q}'$.*

Proof. The extension $R^G \subset R$ is integral because every $x \in R$ is a root of the monic polynomial $\prod_{\sigma \in G}(T - \sigma(x))$ in $R^G[T]$. Thus there are no inclusion relations among the primes lying over a given prime $\mathfrak{p}$ (Algebra, Lemma 10.36.20). If the lemma is wrong, then we can choose $x \in \mathfrak{q}'$, $x \notin \sigma(\mathfrak{q})$ for all $\sigma \in G$. See Algebra, Lemma 10.15.2. Then $y = \prod_{\sigma \in G} \sigma(x)$ is in R^G and in $\mathfrak{p} = R^G \cap \mathfrak{q}'$. On the other hand, $x \notin \sigma(\mathfrak{q})$ for all σ means $\sigma(x) \notin \mathfrak{q}$ for all σ. Hence $y \notin \mathfrak{q}$ as $\mathfrak{q}$ is a prime ideal. This is impossible as $y \in \mathfrak{p} \subset \mathfrak{q}$. □

0BRJ **Lemma 15.111.9.** *Let G be a finite group acting on a ring R. Let $\mathfrak{q} \subset R$ be a prime lying over $\mathfrak{p} \subset R^G$. Then $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is an algebraic normal extension and the map*

$$D = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\} \longrightarrow Aut(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$$

is surjective[18].

Proof. With $A = (R^G)_{\mathfrak{p}}$ and $B = A \otimes_{R^G} R$ we see that $A = B^G$ as localization is flat, see Lemma 15.111.7. Observe that $\mathfrak{p}A$ and $\mathfrak{q}B$ are prime ideals, D is the stabilizer of $\mathfrak{q}B$, and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}A)$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}B)$. Thus we may replace R by B and assume that $\mathfrak{p}$ is a maximal ideal. Since $R^G \subset R$ is an integral ring extension, we find that the maximal ideals of R are exactly the primes lying over $\mathfrak{p}$ (follows from Algebra, Lemmas 10.36.20 and 10.36.22). By Lemma 15.111.8 there are finitely many of them $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m$ and they form a single orbit for G. By the Chinese remainder theorem (Algebra, Lemma 10.15.4) the map $R \to \prod_{j=1,\ldots,m} R/\mathfrak{q}_j$ is surjective.

First we prove that the extension is normal. Pick an element $\alpha \in \kappa(\mathfrak{q})$. We have to show that the minimal polynomial P of α over $\kappa(\mathfrak{p})$ splits completely. By the above we can choose $a \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_m$ mapping to α in $\kappa(\mathfrak{q})$. Consider the polynomial $Q = \prod_{\sigma \in G}(T - \sigma(a))$ in $R^G[T]$. The image of Q in $R[T]$ splits completely into linear factors, hence the same is true for its image in $\kappa(\mathfrak{q})[T]$. Since P divides the image of Q in $\kappa(\mathfrak{p})[T]$ we conclude that P splits completely into linear factors over $\kappa(\mathfrak{q})$ as desired.

Since $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is normal we may assume $\kappa(\mathfrak{q}) = \kappa_1 \otimes_{\kappa(\mathfrak{p})} \kappa_2$ with $\kappa_1/\kappa(\mathfrak{p})$ purely inseparable and $\kappa_2/\kappa(\mathfrak{p})$ Galois, see Fields, Lemma 9.27.3. Pick $\alpha \in \kappa_2$ which generates κ_2 over $\kappa(\mathfrak{p})$ if it is finite and a subfield of degree $> |G|$ if it is infinite (to get a contradiction). This is possible by Fields, Lemma 9.19.1. Pick a, P, and Q as in the previous paragraph. If $\alpha' \in \kappa_2$ is a Galois conjugate of α over $\kappa(\mathfrak{p})$, then the fact that P divides the image of P in $\kappa(\mathfrak{p})[T]$ shows there exists a $\sigma \in G$ such that $\sigma(a)$ maps to α'. By our choice of a (vanishing at other maximal ideals) this implies $\sigma \in D$ and that the image of σ in $\mathrm{Aut}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$ maps α to α'. Hence the surjectivity or the desired absurdity in case α has degree $> |G|$ over $\kappa(\mathfrak{p})$. □

0BRK **Lemma 15.111.10.** *Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = Gal(L/K)$ and let B be the integral closure of A in L.*

(1) *For any two primes $\mathfrak{q}, \mathfrak{q}' \subset B$ lying over the same prime in A there exists a $\sigma \in G$ with $\sigma(\mathfrak{q}) = \mathfrak{q}'$.*

[18] Recall that we use the notation Gal only in the case of Galois extensions.

(2) *Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Then $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is an algebraic normal extension and the map*

$$D = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\} \longrightarrow Aut(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$$

is surjective.

Proof. Proof of (1). Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \mathrm{Gal}(M/K)$ with $\sigma(\mathfrak{q} \cap M) = \mathfrak{q}' \cap M$. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. Observe that (K, id_K) is such a pair as $A = K \cap B$ since A is a normal domain. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ). If $M \neq L$, then we can find $M \subset M' \subset L$ with M'/M nontrivial and finite and M'/K Galois (Fields, Lemma 9.16.5). Choose $\sigma' \in \mathrm{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 9.22.2). Then the primes $\sigma'(\mathfrak{q} \cap M')$ and $\mathfrak{q}' \cap M'$ restrict to the same prime of $B \cap M$. Since $B \cap M = (B \cap M')^{\mathrm{Gal}(M'/M)}$ we can use Lemma 15.111.8 to find $\tau \in \mathrm{Gal}(M'/M)$ with $\tau(\sigma'(\mathfrak{q} \cap M')) = \mathfrak{q}' \cap M'$. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ).

Part (2) is proved in exactly the same manner as part (1). We write out the details. Pick $\overline{\sigma} \in \mathrm{Aut}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$. Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \mathrm{Gal}(M/K)$ with $\sigma(\mathfrak{q} \cap M) = \mathfrak{q} \cap M$ and

$$\begin{array}{ccc} \kappa(\mathfrak{q} \cap M) & \longrightarrow & \kappa(\mathfrak{q}) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \overline{\sigma}} \\ \kappa(\mathfrak{q} \cap M) & \longrightarrow & \kappa(\mathfrak{q}) \end{array}$$

commutes. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. As above (K, id_K) is such a pair. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ). If $M \neq L$, then we can find $M \subset M' \subset L$ with M'/M finite and M'/K Galois (Fields, Lemma 9.16.5). Choose $\sigma' \in \mathrm{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 9.22.2). Then the primes $\sigma'(\mathfrak{q} \cap M')$ and $\mathfrak{q} \cap M'$ restrict to the same prime of $B \cap M$. Adjusting the choice of σ' as in the first paragraph, we may assume that $\sigma'(\mathfrak{q} \cap M') = \mathfrak{q} \cap M'$. Then σ' and $\overline{\sigma}$ define maps $\kappa(\mathfrak{q} \cap M') \to \kappa(\mathfrak{q})$ which agree on $\kappa(\mathfrak{q} \cap M)$. Since $B \cap M = (B \cap M')^{\mathrm{Gal}(M'/M)}$ we can use Lemma 15.111.9 to find $\tau \in \mathrm{Gal}(M'/M)$ with $\tau(\mathfrak{q} \cap M') = \mathfrak{q} \cap M'$ such that $\tau \circ \sigma$ and $\overline{\sigma}$ induce the same map on $\kappa(\mathfrak{q} \cap M')$. There is a small detail here in that the lemma first guarantees that $\kappa(\mathfrak{q} \cap M')/\kappa(\mathfrak{q} \cap M)$ is normal, which then tells us that the difference between the maps is an automorphism of this extension (Fields, Lemma 9.15.10), to which we can apply the lemma to get τ. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ). □

0BSX **Lemma 15.111.11.** *Let A be a normal domain with fraction field K. Let $M/L/K$ be a tower of (possibly infinite) Galois extensions of K. Let $H = Gal(M/K)$ and $G = Gal(L/K)$ and let C and B be the integral closure of A in M and L. Let $\mathfrak{r} \subset C$ and $\mathfrak{q} = B \cap \mathfrak{r}$. Set $D_\mathfrak{r} = \{\tau \in H \mid \tau(\mathfrak{r}) = \mathfrak{r}\}$ and $I_\mathfrak{r} = \{\tau \in D_\mathfrak{r} \mid \tau \bmod \mathfrak{r} = id_{\kappa(\mathfrak{r})}\}$ and similarly for $D_\mathfrak{q}$ and $I_\mathfrak{q}$. Under the map $H \to G$ the induced maps $D_\mathfrak{r} \to D_\mathfrak{q}$ and $I_\mathfrak{r} \to I_\mathfrak{q}$ are surjective.*

Proof. Let $\sigma \in D_\mathfrak{q}$. Pick $\tau \in H$ mapping to σ. This is possible by Fields, Lemma 9.22.2. Then $\tau(\mathfrak{r})$ and $\mathfrak{r}$ both lie over $\mathfrak{q}$. Hence by Lemma 15.111.10 there exists a

$\sigma' \in \mathrm{Gal}(M/L)$ with $\sigma'(\tau(\mathfrak{r})) = \mathfrak{r}$. Hence $\sigma'\tau \in D_\mathfrak{r}$ maps to σ. The case of inertia groups is proved in exactly the same way using surjectivity onto automorphism groups. □

15.112. Extensions of discrete valuation rings

0EXQ In this section and the next few we use the following definitions.

09E4 **Definition 15.112.1.** We say that $A \to B$ or $A \subset B$ is an *extension of discrete valuation rings* if A and B are discrete valuation rings and $A \to B$ is injective and local. In particular, if π_A and π_B are uniformizers of A and B, then $\pi_A = u\pi_B^e$ for some $e \geq 1$ and unit u of B. The integer e does not depend on the choice of the uniformizers as it is also the unique integer ≥ 1 such that

$$\mathfrak{m}_A B = \mathfrak{m}_B^e$$

The integer e is called the *ramification index* of B over A. We say that B is *weakly unramified* over A if $e = 1$. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or *residue degree* of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

09E5 **Lemma 15.112.2.** *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite and we have $ef \leq [L : K]$.*

Proof. Finiteness of the residue field extension is Algebra, Lemma 10.119.10. The inequality follows from Algebra, Lemmas 10.119.9 and 10.52.12. □

0BRL **Lemma 15.112.3.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings. Then the ramification indices of B/A and C/B multiply to give the ramification index of C/A. In a formula $e_{C/A} = e_{B/A}e_{C/B}$. Similarly for the residual degrees in case they are finite.*

Proof. This is immediate from the definitions and Fields, Lemma 9.7.7. □

09E6 **Lemma 15.112.4.** *Let $A \subset B$ be an extension of discrete valuation rings inducing the field extension $K \subset L$. If the characteristic of K is $p > 0$ and L is purely inseparable over K, then the ramification index e is a power of p.*

Proof. Write $\pi_A = u\pi_B^e$ for some $u \in B^*$. On the other hand, we have $\pi_B^q \in K$ for some p-power q. Write $\pi_B^q = v\pi_A^k$ for some $v \in A^*$ and $k \in \mathbf{Z}$. Then $\pi_A^q = u^q\pi_B^{qe} = u^q v^e \pi_A^{ke}$. Taking valuations in B we conclude that $ke = q$. □

In the following lemma we discuss what it means for an extension $A \subset B$ of discrete valuation rings to be "unramified", i.e., have ramification index 1 and separable (possibly nonalgebraic) extension of residue fields. However, we cannot use the term "unramified" itself because there already exists a notion of an unramified ring map, see Algebra, Section 10.151.

09E7 **Lemma 15.112.5.** *Let $A \subset B$ be an extension of discrete valuation rings. The following are equivalent*

(1) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology, and*
(2) *$A \to B$ is weakly unramified and κ_B/κ_A is a separable field extension.*

Proof. This follows from Proposition 15.40.5 and Algebra, Proposition 10.158.9. □

09E8 **Remark 15.112.6.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable field extension. Let $B \subset L$ be the integral closure of A in L. Picture:

$$\begin{array}{ccc} B & \longrightarrow & L \\ \uparrow & & \uparrow \\ A & \longrightarrow & K \end{array}$$

By Algebra, Lemma 10.161.8 the ring extension $A \subset B$ is finite, hence B is Noetherian. By Algebra, Lemma 10.112.4 the dimension of B is 1, hence B is a Dedekind domain, see Algebra, Lemma 10.120.17. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak{m}_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak{m}_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L : K] = \sum\nolimits_{i=1,\ldots,n} e_i f_i$$

by Algebra, Lemma 10.121.8 applied to a uniformizer in A. We observe that $n = 1$ if A is henselian (by Algebra, Lemma 10.153.4), e.g. if A is complete.

09E9 **Definition 15.112.7.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable extension. With B and $\mathfrak{m}_i$, $i = 1, \ldots, n$ as in Remark 15.112.6 we say the extension L/K is

(1) *unramified with respect to* A if $e_i = 1$ and the extension $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable for all i,
(2) *tamely ramified with respect to* A if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$, the field extensions $\kappa(\mathfrak{m}_i)/\kappa_A$ are separable, and the ramification indices e_i are prime to p, and
(3) *totally ramified with respect to* A if $n = 1$ and the residue field extension $\kappa(\mathfrak{m}_1)/\kappa_A$ is trivial.

If the discrete valuation ring A is clear from context, then we sometimes say L/K is unramified, totally ramified, or tamely ramified for short.

For unramified extensions we have the following basic lemma.

0EXR **Lemma 15.112.8.** *Let A be a discrete valuation ring with fraction field K.*

(1) *If $M/L/K$ are finite separable extensions and M is unramified with respect to A, then L is unramified with respect to A.*
(2) *If L/K is a finite separable extension which is unramified with respect to A, then there exists a Galois extension M/K containing L which is unramified with respect to A.*
(3) *If L_1/K, L_2/K are finite separable extensions which are unramified with respect to A, then there exists a a finite separable extension L/K which is unramified with respect to A containing L_1 and L_2.*

Proof. We will use the results of the discussion in Remark 15.112.6 without further mention.

Proof of (1). Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak{m} \subset B$ there is a maximal ideal $\mathfrak{m}' \subset C$ lying over $\mathfrak{m}$. By the multiplicativity of ramification indices (Lemma 15.112.3) and the assumption, we conclude that the ramification index of $B_{\mathfrak{m}}$ over A is 1. Since $\kappa(\mathfrak{m}')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak{m})/\kappa_A$.

Proof of (2). Let M be the normal closure of L over K, see Fields, Definition 9.16.4. Then M/K is Galois by Fields, Lemma 9.21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 9.16.6. Let B be the integral closure of A in L as in Remark 15.112.6. The condition that L is unramified with respect to A exactly means that $A \to B$ is an étale ring map, see Algebra, Lemma 10.143.7. By permanence properties of étale ring maps we see that

$$B \otimes_A \ldots \otimes_A B$$

is étale over A, see Algebra, Lemma 10.143.3. Hence the displayed ring is a product of Dedekind domains, see Lemma 15.44.4. We conclude that M is the fraction field of a Dedekind domain finite étale over A. This means that M is unramified with respect to A as desired.

Proof of (3). Let $B_i \subset L_i$ be the integral closure of A. Argue in the same manner as above to show that $B_1 \otimes_A B_2$ is finite étale over A. Details omitted. □

0EXS **Lemma 15.112.9.** *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is unramified with respect to A and M/L is unramified with respect to $B_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of B, then M/K is unramified with respect to A.*

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak{m}'$ of C lies over a maximal ideal $\mathfrak{m}$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 15.112.3) and the fact that we have the tower $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})/\kappa_A$ of finite extensions of fields. □

15.113. Galois extensions and ramification

09E3 In the case of Galois extensions, we can elaborate on the discussion in Section 15.112.

09EA **Lemma 15.113.1.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Then G acts on the ring B of Remark 15.112.6 and acts transitively on the set of maximal ideals of B.*

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Hence this lemma is a special case of Lemma 15.111.8. □

09EB **Lemma 15.113.2.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Then there are $e \geq 1$ and $f \geq 1$ such that $e_i = e$ and $f_i = f$ for all i (notation as in Remark 15.112.6). In particular $[L : K] = nef$.*

Proof. Immediate consequence of Lemma 15.113.1 and the definitions. □

09EC **Definition 15.113.3.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m} \subset B$ be a maximal ideal.

(1) The *decomposition group of* $\mathfrak{m}$ is the subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$.
(2) The *inertia group of* $\mathfrak{m}$ is the kernel I of the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$.

Note that the field $\kappa(\mathfrak{m})$ may be inseparable over κ_A. In particular the field extension $\kappa(\mathfrak{m})/\kappa_A$ need not be Galois. If κ_A is perfect, then it is.

09ED **Lemma 15.113.4.** *Let A be a discrete valuation ring with fraction field K and residue field κ. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Then*

(1) *the field extension $\kappa(\mathfrak{m})/\kappa$ is normal, and*
(2) *$D \to Aut(\kappa(\mathfrak{m})/\kappa)$ is surjective.*

If for some (equivalently all) maximal ideal(s) $\mathfrak{m} \subset B$ the field extension $\kappa(\mathfrak{m})/\kappa$ is separable, then

(3) *$\kappa(\mathfrak{m})/\kappa$ is Galois, and*
(4) *$D \to Gal(\kappa(\mathfrak{m})/\kappa)$ is surjective.*

Here $D \subset G$ is the decomposition group of $\mathfrak{m}$.

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Thus parts (1) and (2) follow from Lemma 15.111.9. The "equivalently all" part of the lemma follows from Lemma 15.113.1. Assume $\kappa(\mathfrak{m})/\kappa$ is separable. Then parts (3) and (4) follow immediately from (1) and (2). □

09EE **Lemma 15.113.5.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m} \subset B$ be a maximal ideal. The inertia group I of $\mathfrak{m}$ sits in a canonical exact sequence*

$$1 \to P \to I \to I_t \to 1$$

such that

(1) *$P = \{\sigma \in D \mid \sigma|_{B/\mathfrak{m}^2} = id_{B/\mathfrak{m}^2}\}$ where D is the decomposition group,*
(2) *P is a normal subgroup of D,*
(3) *P is a p-group if the characteristic of κ_A is $p > 0$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
(4) *I_t is cyclic of order the prime to p part of the integer e, and*
(5) *there is a canonical isomorphism $\theta : I_t \to \mu_e(\kappa(\mathfrak{m}))$.*

Here e is the integer of Lemma 15.113.2.

Proof. Recall that $|G| = [L : K] = nef$, see Lemma 15.113.2. Since G acts transitively on the set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ of maximal ideals of B (Lemma 15.113.1) and since D is the stabilizer of an element we see that $|D| = ef$. By Lemma 15.113.4 we have

$$ef = |D| = |I| \cdot |\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)|$$

where κ is the residue field of A. As $\kappa(\mathfrak{m})$ is normal over κ the order of $\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)$ differs from f by a power of p (see Fields, Lemma 9.15.9 and discussion following Fields, Definition 9.14.7). Hence the prime to p part of $|I|$ is equal to the prime to p part of e.

Set $C = B_{\mathfrak{m}}$. Then I acts on C over A and trivially on the residue field of C. Let $\pi_A \in A$ and $\pi_C \in C$ be uniformizers. Write $\pi_A = u\pi_C^e$ for some unit u in C. For $\sigma \in I$ write $\sigma(\pi_C) = \theta_\sigma \pi_C$ for some unit θ_σ in C. Then we have

$$\pi_A = \sigma(\pi_A) = \sigma(u)(\theta_\sigma \pi_C)^e = \sigma(u)\theta_\sigma^e \pi_C^e = \frac{\sigma(u)}{u}\theta_\sigma^e \pi_A$$

Since $\sigma(u) \equiv u \bmod \mathfrak{m}_C$ as $\sigma \in I$ we see that the image $\overline{\theta}_\sigma$ of θ_σ in $\kappa_C = \kappa(\mathfrak{m})$ is an eth root of unity. We obtain a map

0BU3 (15.113.5.1)
$$\theta : I \longrightarrow \mu_e(\kappa(\mathfrak{m})), \quad \sigma \mapsto \overline{\theta}_\sigma$$

We claim that θ is a homomorphism of groups and independent of the choice of uniformizer π_C. Namely, if τ is a second element of I, then $\tau(\sigma(\pi_C)) = \tau(\theta_\sigma \pi_C) = \tau(\theta_\sigma)\theta_\tau \pi_C$, hence $\theta_{\tau\sigma} = \tau(\theta_\sigma)\theta_\tau$ and since $\tau \in I$ we conclude that $\overline{\theta}_{\tau\sigma} = \overline{\theta}_\sigma \overline{\theta}_\tau$. If π'_C is a second uniformizer, then we see that $\pi'_C = w\pi_C$ for some unit w of C and $\sigma(\pi'_C) = w^{-1}\sigma(w)\theta_\sigma \pi'_C$, hence $\theta'_\sigma = w^{-1}\sigma(w)\theta_\sigma$, hence θ'_σ and θ_σ map to the same element of the residue field as before.

Since $\kappa(\mathfrak{m})$ has characteristic p, the group $\mu_e(\kappa(\mathfrak{m}))$ is cyclic of order at most the prime to p part of e (see Fields, Section 9.17).

Let $P = \mathrm{Ker}(\theta)$. The elements of P are exactly the elements of D acting trivially on $C/\pi_C^2 C \cong B/\mathfrak{m}^2$. Thus (a) is true. This implies (b) as P is the kernel of the map $D \to \mathrm{Aut}(B/\mathfrak{m}^2)$. If we can prove (c), then parts (d) and (e) will follow as I_t will be isomorphic to $\mu_e(\kappa(\mathfrak{m}))$ as the arguments above show that $|I_t| \geq |\mu_e(\kappa(\mathfrak{m}))|$.

Thus it suffices to prove that the kernel P of θ is a p-group. Let σ be a nontrivial element of the kernel. Then $\sigma - \mathrm{id}$ sends $\mathfrak{m}_C^i$ into $\mathfrak{m}_C^{i+1}$ for all i. Let m be the order of σ. Pick $c \in C$ such that $\sigma(c) \neq c$. Then $\sigma(c) - c \in \mathfrak{m}_C^i$, $\sigma(c) - c \notin \mathfrak{m}_C^{i+1}$ for some i and we have

$$\begin{aligned} 0 &= \sigma^m(c) - c \\ &= \sigma^m(c) - \sigma^{m-1}(c) + \ldots + \sigma(c) - c \\ &= \sum\nolimits_{j=0,\ldots,m-1} \sigma^j(\sigma(c) - c) \\ &\equiv m(\sigma(c) - c) \bmod \mathfrak{m}_C^{i+1} \end{aligned}$$

It follows that $p|m$ (or $m = 0$ if $p = 1$). Thus every element of the kernel of θ has order divisible by p, i.e., $\mathrm{Ker}(\theta)$ is a p-group. □

0BU4 **Definition 15.113.6.** With assumptions and notation as in Lemma 15.113.5.

(1) The *wild inertia group of* $\mathfrak{m}$ is the subgroup P.
(2) The *tame inertia group of* $\mathfrak{m}$ is the quotient $I \to I_t$.

We denote $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ the surjective map (15.113.5.1) whose kernel is P and which induces the isomorphism $I_t \to \mu_e(\kappa(\mathfrak{m}))$.

0BU5 **Lemma 15.113.7.** *With assumptions and notation as in Lemma 15.113.5. The inertia character $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ satisfies the following property*

$$\theta(\tau\sigma\tau^{-1}) = \tau(\theta(\sigma))$$

for $\tau \in D$ and $\sigma \in I$.

Proof. The formula makes sense as I is a normal subgroup of D and as τ acts on $\kappa(\mathfrak{m})$ via the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m}))$ discussed in Lemma 15.113.4 for example. Recall the construction of θ. Choose a uniformizer π of $B_\mathfrak{m}$ and for $\sigma \in I$ write $\sigma(\pi) = \theta_\sigma \pi$. Then $\theta(\sigma)$ is the image $\overline{\theta}_\sigma$ of θ_σ in the residue field. For any $\tau \in D$ we can write $\tau(\pi) = \theta_\tau \pi$ for some unit θ_τ. Then $\theta_{\tau^{-1}} = \tau^{-1}(\theta_\tau^{-1})$. We compute

$$\begin{aligned}
\theta_{\tau\sigma\tau^{-1}} &= \tau(\sigma(\tau^{-1}(\pi)))/\pi \\
&= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})\pi))/\pi \\
&= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1}))\theta_\sigma\pi)/\pi \\
&= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\tau(\theta_\sigma)\theta_\tau
\end{aligned}$$

However, since σ acts trivially modulo π we see that the product $\tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\theta_\tau$ maps to 1 in the residue field. This proves the lemma. □

We will generalize the following lemma in Fundamental Groups, Lemma 58.12.5.

09EH **Lemma 15.113.8.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak{m} \subset B$ be a maximal ideal of the integral closure of A in L. Let $I \subset G$ be the inertia group of $\mathfrak{m}$. Then B^I is the integral closure of A in L^I and $A \to (B^I)_{B^I \cap \mathfrak{m}}$ is étale.*

Proof. Write $B' = B^I$. It follows from the definitions that $B' = B^I$ is the integral closure of A in L^I. Write $\mathfrak{m}' = B^I \cap \mathfrak{m} = B' \cap \mathfrak{m} \subset B'$. By Lemma 15.111.8 the maximal ideal $\mathfrak{m}$ is the unique prime ideal of B lying over $\mathfrak{m}'$. As I acts trivially on $\kappa(\mathfrak{m})$ we see from Lemma 15.111.2 that the extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{m}')$ is purely inseparable (perhaps an easier alternative is to apply the result of Lemma 15.111.9). Since D/I acts faithfully on $\kappa(\mathfrak{m}')$, we conclude that D/I acts faithfully on $\kappa(\mathfrak{m})$. Of course the elements of the residue field κ of A are fixed by this action. By Galois theory we see that $[\kappa(\mathfrak{m}') : \kappa] \geq |D/I|$, see Fields, Lemma 9.21.6.

Let π be the uniformizer of A. Since $\mathrm{Norm}_{L/K}(\pi) = \pi^{[L:K]}$ we see from Algebra, Lemma 10.121.8 that

$$|G| = [L : K] = [L : K] \, \mathrm{ord}_A(\pi) = |G/D| \, [\kappa(\mathfrak{m}) : \kappa] \, \mathrm{ord}_{B_\mathfrak{m}}(\pi)$$

as there are $n = |G/D|$ maximal ideals of B which are all conjugate under G, see Remark 15.112.6 and Lemma 15.113.1. Applying the same reasoning to the finite extension the finite extension L/L^I of degree $|I|$ we find

$$|I| \, \mathrm{ord}_{B'_{\mathfrak{m}'}}(\pi) = [\kappa(\mathfrak{m}) : \kappa(\mathfrak{m}')] \, \mathrm{ord}_{B_\mathfrak{m}}(\pi)$$

We conclude that

$$\mathrm{ord}_{B'_{\mathfrak{m}'}}(\pi) = \frac{|D/I|}{[\kappa(\mathfrak{m}') : \kappa]}$$

Since the left hand side is a positive integer and since the right hand side is ≤ 1 by the above, we conclude that we have equality, $\mathrm{ord}_{B'_{\mathfrak{m}'}}(\pi) = 1$ and $\kappa(\mathfrak{m}')/\kappa$ has degree $|D/I|$. Thus $\pi B'_{\mathfrak{m}'} = \mathfrak{m}' B'_\mathfrak{m}$ and $\kappa(\mathfrak{m}')$ is Galois over κ with Galois group D/I, in particular separable, see Fields, Lemma 9.21.2. By Algebra, Lemma 10.143.7 we find that $A \to B'_{\mathfrak{m}'}$ is étale as desired. □

0BU6 **Remark 15.113.9.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak{m} \subset B$ be a maximal ideal of the integral closure of A in L. Let

$$P \subset I \subset D \subset G$$

be the wild inertia, inertia, decomposition group of $\mathfrak{m}$. Consider the diagram

$$\begin{array}{ccccccccc} \mathfrak{m} & \longrightarrow & \mathfrak{m}^P & \longrightarrow & \mathfrak{m}^I & \longrightarrow & \mathfrak{m}^D & \longrightarrow & A \cap \mathfrak{m} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B & \longleftarrow & B^P & \longleftarrow & B^I & \longleftarrow & B^D & \longleftarrow & A \end{array}$$

Observe that B^P, B^I, B^D are the integral closures of A in the fields L^P, L^I, L^D. Thus we also see that B^P is the integral closure of B^I in L^P and so on. Observe that $\mathfrak{m}^P = \mathfrak{m} \cap B^P$, $\mathfrak{m}^I = \mathfrak{m} \cap B^I$, and $\mathfrak{m}^D = \mathfrak{m} \cap B^D$. Hence the top line of the diagram corresponds to the images of $\mathfrak{m} \in \mathrm{Spec}(B)$ under the induced maps of spectra. Having said all of this we have the following

(1) the extension L^I/L^D is Galois with group D/I,
(2) the extension L^P/L^I is Galois with group $I_t = I/P$,
(3) the extension L^P/L^D is Galois with group D/P,
(4) $\mathfrak{m}^I$ is the unique prime of B^I lying over $\mathfrak{m}^D$,
(5) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^I$,
(6) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^P$,
(7) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^D$,
(8) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^I$,
(9) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^D$,
(10) $A \to B^D_{\mathfrak{m}^D}$ is étale and induces a trivial residue field extension,
(11) $B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ is étale and induces a Galois extension of residue fields with Galois group D/I,
(12) $A \to B^I_{\mathfrak{m}^I}$ is étale,
(13) $B^I_{\mathfrak{m}^I} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a trivial residue field extension,
(14) $B^D_{\mathfrak{m}^D} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension,
(15) $A \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension.

Statements (1), (2), and (3) are immediate from Galois theory (Fields, Section 9.21) and Lemma 15.113.5. Statements (4) – (9) are clear from Lemma 15.113.1. Part (12) is Lemma 15.113.8. Since we have the factorization $A \to B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ we obtain the étaleness in (10) and (11) as a consequence. The residue field extension in (10) must be trivial because it is separable and D/I maps onto $\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ as shown in Lemma 15.113.4. The same argument provides the proof of the statement on residue fields in (11). To see (13), (14), and (15) it suffices to prove (13). By the above, the extension L^P/L^I is Galois with a cyclic Galois group of order prime to p, the prime $\mathfrak{m}^P$ is the unique prime lying over $\mathfrak{m}^I$ and the action of I/P on the residue field is trivial. Thus we can apply Lemma 15.113.5 to this extension and the discrete valuation ring $B^I_{\mathfrak{m}^I}$ to see that (13) holds.

0BU7 **Lemma 15.113.10.** *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be a tower with M/K and L/K finite Galois. Let C, B be the integral closure of A in M, L. Let $\mathfrak{m}' \subset C$ be a maximal ideal and set $\mathfrak{m} = \mathfrak{m}' \cap B$. Let*

$$P \subset I \subset D \subset Gal(L/K) \quad \text{and} \quad P' \subset I' \subset D' \subset Gal(M/K)$$

be the wild inertia, inertia, decomposition group of $\mathfrak{m}$ and $\mathfrak{m}'$. Then the canonical surjection $Gal(M/K) \to Gal(L/K)$ induces surjections $P' \to P$, $I' \to I$, and

$D' \to D$. Moreover these fit into commutative diagrams

$$\begin{array}{ccc} D' & \longrightarrow & Aut(\kappa(\mathfrak{m}')/\kappa_A) \\ \downarrow & & \downarrow \\ D & \longrightarrow & Aut(\kappa(\mathfrak{m})/\kappa_A) \end{array} \quad \text{and} \quad \begin{array}{ccc} I' & \xrightarrow[\theta']{} & \mu_{e'}(\kappa(\mathfrak{m}')) \\ \downarrow & & \downarrow{\scriptstyle (-)^{e'/e}} \\ I & \xrightarrow{\theta} & \mu_e(\kappa(\mathfrak{m})) \end{array}$$

where e' and e are the ramification indices of $A \to C_{\mathfrak{m}'}$ and $A \to B_{\mathfrak{m}}$.

Proof. The fact that under the map $\mathrm{Gal}(M/K) \to \mathrm{Gal}(L/K)$ the groups P', I', D' map into P, I, D is immediate from the definitions of these groups. The commutativity of the first diagram is clear (observe that since $\kappa(\mathfrak{m})/\kappa_A$ is normal every automorphism of $\kappa(\mathfrak{m}')$ over κ_A indeed induces an automorphism of $\kappa(\mathfrak{m})$ over κ_A and hence we obtain the right vertical arrow in the first diagram, see Lemma 15.113.4 and Fields, Lemma 9.15.7).

The maps $I' \to I$ and $D' \to D$ are surjective by Lemma 15.111.11. The surjectivity of $P' \to P$ follows as P' and P are p-Sylow subgroups of I' and I.

To see the commutativity of the second diagram we choose a uniformizer π' of $C_{\mathfrak{m}'}$ and a uniformizer π of $B_{\mathfrak{m}}$. Then $\pi = c'(\pi')^{e'/e}$ for some unit c' of $C_{\mathfrak{m}'}$. For $\sigma' \in I'$ the image $\sigma \in I$ is simply the restriction of σ' to L. Write $\sigma'(\pi') = c\pi'$ for a unit $c \in C_{\mathfrak{m}'}$ and write $\sigma(\pi) = b\pi$ for a unit b of $B_{\mathfrak{m}}$. Then $\sigma'(\pi) = b\pi$ and we obtain

$$b\pi = \sigma'(\pi) = \sigma'(c'(\pi')^{e'/e}) = \sigma'(c')c^{e'/e}(\pi')^{e'/e} = \frac{\sigma'(c')}{c'}c^{e'/e}\pi$$

As $\sigma' \in I'$ we see that b and $c^{e'/e}$ have the same image in the residue field which proves what we want. □

0BU8 **Remark 15.113.11.** In order to use the inertia character $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ for infinite Galois extensions, it is convenient to scale it. Let $A, K, L, B, \mathfrak{m}, G, P, I, D, e, \theta$ be as in Lemma 15.113.5 and Definition 15.113.6. Then $e = q|I_t|$ with q is a power of the characteristic p of $\kappa(\mathfrak{m})$ if positive or 1 if zero. Note that $\mu_e(\kappa(\mathfrak{m})) = \mu_{|I_t|}(\kappa(\mathfrak{m}))$ because the characteristic of $\kappa(\mathfrak{m})$ is p. Consider the map

$$\theta_{can} = q\theta : I \longrightarrow \mu_{|I_t|}(\kappa(\mathfrak{m}))$$

This map induces an isomorphism $\theta_{can} : I_t \to \mu_{|I_t|}(\kappa(\mathfrak{m}))$. We have $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$ and $\sigma \in I$ by Lemma 15.113.7. Finally, if M/L is an extension such that M/K is Galois and $\mathfrak{m}'$ is a prime of the integral closure of A in M lying over $\mathfrak{m}$, then we get the commutative diagram

$$\begin{array}{ccc} I' & \xrightarrow[\theta'_{can}]{} & \mu_{|I'_t|}(\kappa(\mathfrak{m}')) \\ \downarrow & & \downarrow{\scriptstyle (-)^{|I'_t|/|I_t|}} \\ I & \xrightarrow{\theta_{can}} & \mu_{|I_t|}(\kappa(\mathfrak{m})) \end{array}$$

by Lemma 15.113.10.

15.114. Krasner's lemma

0BU9 Here is Krasner's lemma in the case of discretely valued fields.

09EI **Lemma 15.114.1** (Krasner's lemma)**.** *Let A be a complete local domain of dimension 1. Let $P(t) \in A[t]$ be a polynomial with coefficients in A. Let $\alpha \in A$ be a root of P but not a root of the derivative $P' = dP/dt$. For every $c \geq 0$ there exists an integer n such that for any $Q \in A[t]$ whose coefficients are in $\mathfrak{m}_A^n$ the polynomial $P + Q$ has a root $\beta \in A$ with $\beta - \alpha \in \mathfrak{m}_A^c$.*

Proof. Choose a nonzero $\pi \in \mathfrak{m}$. Since the dimension of A is 1 we have $\mathfrak{m} = \sqrt{(\pi)}$. By assumption we may write $P'(\alpha)^{-1} = \pi^{-m}a$ for some $m \geq 0$ and $a \in A$. We may and do assume that $c \geq m + 1$. Pick n such that $\mathfrak{m}_A^n \subset (\pi^{c+m})$. Pick any Q as in the statement. For later use we observe that we can write

$$P(x + y) = P(x) + P'(x)y + R(x, y)y^2$$

for some $R(x, y) \in A[x, y]$. We will show by induction that we can find a sequence $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ such that

(1) $\alpha_k \equiv \alpha \bmod \pi^c$,
(2) $\alpha_{k+1} - \alpha_k \in (\pi^k)$, and
(3) $(P + Q)(\alpha_k) \in (\pi^{m+k})$.

Setting $\beta = \lim \alpha_k$ will finish the proof.

Base case. Since the coefficients of Q are in (π^{c+m}) we have $(P + Q)(\alpha) \in (\pi^{c+m})$. Hence $\alpha_m = \alpha$ works. This choice guarantees that $\alpha_k \equiv \alpha \bmod \pi^c$ for all $k \geq m$.

Induction step. Given α_k we write $\alpha_{k+1} = \alpha_k + \delta$ for some $\delta \in (\pi^k)$. Then we have

$$(P + Q)(\alpha_{k+1}) = P(\alpha_k + \delta) + Q(\alpha_k + \delta)$$

Because the coefficients of Q are in (π^{c+m}) we see that $Q(\alpha_k + \delta) \equiv Q(\alpha_k) \bmod \pi^{c+m+k}$. On the other hand we have

$$P(\alpha_k + \delta) = P(\alpha_k) + P'(\alpha_k)\delta + R(\alpha_k, \delta)\delta^2$$

Note that $P'(\alpha_k) \equiv P'(\alpha) \bmod (\pi^{m+1})$ as $\alpha_k \equiv \alpha \bmod \pi^{m+1}$. Hence we obtain

$$P(\alpha_k + \delta) \equiv P(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Recombining the two terms we see that

$$(P + Q)(\alpha_{k+1}) \equiv (P + Q)(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Thus a solution is to take $\delta = -P'(\alpha)^{-1}(P + Q)(\alpha_k) = -\pi^{-m}a(P + Q)(\alpha_k)$ which is contained in (π^k) by induction assumption. □

09EJ **Lemma 15.114.2.** *Let A be a discrete valuation ring with field of fractions K. Let $A^\wedge$ be the completion of A with fraction field $K^\wedge$. If $M/K^\wedge$ is a finite separable extension, then there exists a finite separable extension L/K such that $M = K^\wedge \otimes_K L$.*

Proof. Note that $A^\wedge$ is a discrete valuation ring too (by Lemmas 15.43.4 and 15.43.1). In particular $A^\wedge$ is a domain. The proof will work more generally for Noetherian local rings A such that $A^\wedge$ is a local domain of dimension 1.

Let $\theta \in M$ be an element that generates M over $K^\wedge$. (Theorem of the primitive element.) Let $P(t) \in K^\wedge[t]$ be the minimal polynomial of θ over $K^\wedge$. Let $\pi \in \mathfrak{m}_A$ be a nonzero element. After replacing θ by $\pi^n\theta$ we may assume that the coefficients of $P(t)$ are in $A^\wedge$. Let $B = A^\wedge[\theta] = A^\wedge[t]/(P(t))$. Note that B is a complete local domain of dimension 1 because it is finite over A and contained in M. Since M is separable over K the element θ is not a root of the derivative of P. For any integer

n we can find a monic polynomial $P_1 \in A[t]$ such that $P - P_1$ has coefficients in $\pi^n A^\wedge[t]$. By Krasner's lemma (Lemma 15.114.1) we see that P_1 has a root β in B for n sufficiently large. Moreover, we may assume (if n is chosen large enough) that $\theta - \beta \in \pi B$. Consider the map $\Phi : A^\wedge[t]/(P_1) \to B$ of $A^\wedge$-algebras which maps t to β. Since $B = \pi B + \sum_{i<\deg(P)} A^\wedge \theta^i$, the map Φ is surjective by Nakayama's lemma. As $\deg(P_1) = \deg(P)$ it follows that Φ is an isomorphism. We conclude that the ring extension $L = K[t]/(P_1(t))$ satisfies $K^\wedge \otimes_K L \cong M$. This implies that L is a field and the proof is complete. □

09EK **Definition 15.114.3.** Let A be a discrete valuation ring. We say A has *mixed characteristic* if the characteristic of the residue field of A is $p > 0$ and the characteristic of the fraction field of A is 0. In this case we obtain an extension of discrete valuation rings $\mathbf{Z}_{(p)} \subset A$ and the *absolute ramification index* of A is the ramification index of this extension.

15.115. Abhyankar's lemma and tame ramification

0EXT In this section we prove what we think is the most general version of Abhyankar's lemma for discrete valuation rings. After doing so, we apply this to prove some results about tamely ramified extensions of the fraction field of a discrete valuation ring.

09EM **Remark 15.115.1.** Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a finite extension of fields. Let $A_1 \subset K_1$ be the integral closure of A in K_1. On the other hand, let $L_1 = (L \otimes_K K_1)_{red}$. Then L_1 is a nonempty finite product of finite field extensions of L. Let B_1 be the integral closure of B in L_1. We obtain compatible commutative diagrams

$$\begin{array}{ccc} L & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & K_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_1 \end{array}$$

In this situation we have the following

(1) By Algebra, Lemma 10.120.18 the ring A_1 is a Dedekind domain and B_1 is a finite product of Dedekind domains.
(2) Note that $L \otimes_K K_1 = (B \otimes_A A_1)_\pi$ where $\pi \in A$ is a uniformizer and that π is a nonzerodivisor on $B \otimes_A A_1$. Thus the ring map $B \otimes_A A_1 \to B_1$ is integral with kernel consisting of nilpotent elements. Hence $\mathrm{Spec}(B_1) \to \mathrm{Spec}(B \otimes_A A_1)$ is surjective on spectra (Algebra, Lemma 10.36.17). The map $\mathrm{Spec}(B \otimes_A A_1) \to \mathrm{Spec}(A_1)$ is surjective as $A_1/\mathfrak{m}_A A_1 \to B/\mathfrak{m}_A B \otimes_{\kappa_A} A_1/\mathfrak{m}_A A_1$ is an injective ring map with $A_1/\mathfrak{m}_A A_1$ Artinian. We conclude that $\mathrm{Spec}(B_1) \to \mathrm{Spec}(A_1)$ is surjective.
(3) Let $\mathfrak{m}_i$, $i = 1, \ldots n$ with $n \geq 1$ be the maximal ideals of A_1. For each $i = 1, \ldots, n$ let $\mathfrak{m}_{ij}$, $j = 1, \ldots, m_i$ with $m_i \geq 1$ be the maximal ideals of B_1 lying over $\mathfrak{m}_i$. We obtain diagrams

$$\begin{array}{ccc} B & \longrightarrow & (B_1)_{\mathfrak{m}_{ij}} \\ \uparrow & & \uparrow \\ A & \longrightarrow & (A_1)_{\mathfrak{m}_i} \end{array}$$

of extensions of discrete valuation rings.
(4) If A is henselian (for example complete), then A_1 is a discrete valuation ring, i.e., $n = 1$. Namely, A_1 is a union of finite extensions of A which are domains, hence local by Algebra, Lemma 10.153.4.
(5) If B is henselian (for example complete), then B_1 is a product of discrete valuation rings, i.e., $m_i = 1$ for $i = 1, \ldots, n$.
(6) If $K \subset K_1$ is purely inseparable, then A_1 and B_1 are both discrete valuation rings, i.e., $n = 1$ and $m_1 = 1$. This is true because for every $b \in B_1$ a p-power power of b is in B, hence B_1 can only have one maximal ideal.
(7) If $K \subset K_1$ is finite separable, then $L_1 = L \otimes_K K_1$ and is a finite product of finite separable extensions too. Hence $A \subset A_1$ and $B \subset B_1$ are finite by Algebra, Lemma 10.161.8.
(8) If A is Nagata, then $A \subset A_1$ is finite.
(9) If B is Nagata, then $B \subset B_1$ is finite.

09EV **Lemma 15.115.2.** *Let A be a discrete valuation ring with uniformizer π. Let $n \geq 2$. Then $K_1 = K[\pi^{1/n}]$ is a degree n extension of K and the integral closure A_1 of A in K_1 is the ring $A[\pi^{1/n}]$ which is a discrete valuation ring with ramification index n over A.*

Proof. This lemma proves itself. □

09EQ **Lemma 15.115.3.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then for any finite extension K_1/K we have $L_1 = L \otimes_K K_1$, $B_1 = B \otimes_A A_1$, and each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ (see Remark 15.115.1) is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.*

Proof. We will use the equivalence of Lemma 15.112.5 without further mention. Let $\pi \in A$ and $\pi_i \in (A_1)_{\mathfrak{m}_i}$ be uniformizers. As $\kappa_A \subset \kappa_B$ is separable, the ring

$$(B \otimes_A (A_1)_{\mathfrak{m}_i})/\pi_i(B \otimes_A (A_1)_{\mathfrak{m}_i}) = B/\pi B \otimes_{A/\pi A} (A_1)_{\mathfrak{m}_i}/\pi_i(A_1)_{\mathfrak{m}_i}$$

is a product of fields each separable over $\kappa_{\mathfrak{m}_i}$. Hence the element π_i in $B \otimes_A (A_1)_{\mathfrak{m}_i}$ is a nonzerodivisor and the quotient by this element is a product of fields. It follows that $B \otimes_A A_1$ is a Dedekind domain in particular reduced. Thus $B \otimes_A A_1 \subset B_1$ is an equality. □

The following lemma is our version of Abhyankar's lemma for discrete valuation rings. Observe that κ_B/κ_A is not assumed to be an algebraic extension of fields.

0BRM **Lemma 15.115.4** (Abhyankar's lemma)**.** *Let $A \subset B$ be an extension of discrete valuation rings. Assume that either the residue characteristic of A is 0 or it is p, the ramification index e is prime to p, and κ_B/κ_A is a separable field extension. Let K_1/K be a finite extension. Using the notation of Remark 15.115.1 assume e divides the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$ for some i. Then $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ is formally smooth in the $\mathfrak{m}_{ij}$-adic topology for all $j = 1, \ldots, m_i$.*

Proof. Let $\pi \in A$ be a uniformizer. Let π_1 be a uniformizer of $(A_1)_{\mathfrak{m}_i}$. Write $\pi = u\pi_1^{e_1}$ with u a unit of $(A_1)_{\mathfrak{m}_i}$ and e_1 the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$.

Claim: we may assume that u is an eth power in K_1. Namely, let K_2 be an extension of K_1 obtained by adjoining a root of $x^e = u$; thus K_2 is a factor of $K_1[x]/(x^e - u)$. Then K_2/K_1 is a finite separable extension (by our assumption on e) and hence

$A_1 \subset A_2$ is finite. Since $(A_1)_{\mathfrak{m}_i} \to (A_1)_{\mathfrak{m}_i}[x]/(x^e - u)$ is finite étale (as e is prime to the residue characteristic and u a unit) we conclude that $(A_2)_{\mathfrak{m}_i}$ is a factor of a finite étale extension of $(A_1)_{\mathfrak{m}_i}$ hence finite étale over $(A_1)_{\mathfrak{m}_i}$ itself. The same reasoning shows that $B_1 \subset B_2$ induces finite étale extensions $(B_1)_{\mathfrak{m}_{ij}} \subset (B_2)_{\mathfrak{m}_{ij}}$. Pick a maximal ideal $\mathfrak{m}'_{ij} \subset B_2$ lying over $\mathfrak{m}_{ij} \subset B_1$ (of course there may be more than one) and consider

$$\begin{array}{ccc} (B_1)_{\mathfrak{m}_{ij}} & \longrightarrow & (B_2)_{\mathfrak{m}'_{ij}} \\ \uparrow & & \uparrow \\ (A_1)_{\mathfrak{m}_i} & \longrightarrow & (A_2)_{\mathfrak{m}'_i} \end{array}$$

where $\mathfrak{m}'_i \subset A_2$ is the image. Now the horizontal arrows have ramification index 1 and induce finite separable residue field extensions. Thus, using the equivalence of Lemma 15.112.5, we see that it suffices to show that the right vertical arrow is formally smooth in the $\mathfrak{m}'_{ij}$-adic topology. Since u has a eth root in K_2 we obtain the claim.

Assume u has an eth root in K_1. Since $e|e_1$ and since u has a eth root in K_1 we see that $\pi = \theta^e$ for some $\theta \in K_1$. Let $K'_1 = K[\theta] \subset K_1$ be the subfield generated by θ. By Lemma 15.115.2 the integral closure A'_1 of A in $K[\theta]$ is the discrete valuation ring $A'_1 = A[\theta]$ which has ramification index e over A. If we can prove the lemma for the extension K'_1/K, then we conclude by Lemma 15.115.3 applied to the diagram

$$\begin{array}{ccc} (B'_1)_{B'_1 \cap \mathfrak{m}_{ij}} & \longrightarrow & (B_1)_{\mathfrak{m}_{ij}} \\ \uparrow & & \uparrow \\ A'_1 & \longrightarrow & (A_1)_{\mathfrak{m}_i} \end{array}$$

for all $j = 1, \ldots, m_i$. This reduces us to the case discussed in the next paragraph.

Assume $K_1 = K[\pi^{1/e}]$ and set $\theta = \pi^{1/e}$. Let π_B be a uniformizer for B and write $\pi = w\pi_B^e$ for some unit w of B. Then we see that $L_1 = L \otimes_K K_1$ is obtained by adjoining θ/π_B which is an eth root of the unit w. Thus $B \subset B_1$ is finite étale. Thus for any maximal ideal $\mathfrak{m} \subset B_1$ consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow[1]{} & (B_1)_{\mathfrak{m}} \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e_{\mathfrak{m}}} \\ A & \xrightarrow{e} & A_1 \end{array}$$

Here the numbers along the arrows are the ramification indices. By multiplicativity of ramification indices (Lemma 15.112.3) we conclude $e_{\mathfrak{m}} = 1$. Looking at the residue field extensions we find that $\kappa(\mathfrak{m})$ is a finite separable extension of κ_B which is separable over κ_A. Therefore $\kappa(\mathfrak{m})$ is separable over κ_A which is equal to the residue field of A_1 and we win by Lemma 15.112.5. □

0EXU **Lemma 15.115.5.** *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is tamely ramified with respect to A and M/L is tamely ramified with respect to $B_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of B, then M/K is tamely ramified with respect to A.*

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak{m}'$ of C lies over a maximal ideal $\mathfrak{m}$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 15.112.3) and the fact that we have the tower $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})/\kappa_A$ of finite extensions of fields. □

0EXV **Lemma 15.115.6.** *Let A be a discrete valuation ring with fraction field K. If $M/L/K$ are finite separable extensions and M is tamely ramified with respect to A, then L is tamely ramified with respect to A.*

Proof. We will use the results of the discussion in Remark 15.112.6 without further mention. Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak{m} \subset B$ there is a maximal ideal $\mathfrak{m}' \subset C$ lying over $\mathfrak{m}$. By the multiplicativity of ramification indices (Lemma 15.112.3) and the assumption, we conclude that the ramification index of $B_{\mathfrak{m}}$ over A is prime to the residue characteristic. Since $\kappa(\mathfrak{m}')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak{m})/\kappa_A$. □

0EXW **Lemma 15.115.7.** *Let A be a discrete valuation ring with fraction field K. Let $\pi \in A$ be a uniformizer. Let L/K be a finite separable extension. The following are equivalent*

1. *L is tamely ramified with respect to A,*
2. *there exists an $e \geq 1$ invertible in κ_A and an extension $L'/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ such that L is contained in L', and*
3. *there exists an $e_0 \geq 1$ invertible in κ_A such that for every $d \geq 1$ invertible in κ_A (2) holds with $e = de_0$.*

Proof. Observe that A' is a discrete valuation ring with fraction field K', see Lemma 15.115.2. Of course the ramification index of A' over A is e. Thus if (2) holds, then L' is tamely ramified with respect to A by Lemma 15.115.5. Hence L is tamely ramified with respect to A by Lemma 15.115.6.

The implication (3) $\Rightarrow$ (2) is immediate.

Assume that (1) holds. Let B be the integral closure of A in L and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be its maximal ideals. Denote e_i the ramification index of $A \to B_{\mathfrak{m}_i}$. Let e_0 be the least common multiple of $e_1, \ldots, e_r$. This is invertible in κ_A by our assumption (1). Let $e = de_0$ as in (3). Set $A' = A[\pi^{1/e}]$. Then $A \to A'$ is an extension of discrete valuation rings with fraction field $K' = K[\pi^{1/e}]$, see Lemma 15.115.2. Choose a product decomposition

$$L \otimes_K K' = \prod L'_j$$

where L'_j are fields. Let B'_j be the integral closure of A in L'_j. Let $\mathfrak{m}_{ijk}$ be the maximal ideals of B'_j lying over $\mathfrak{m}_i$. Observe that $(B'_j)_{\mathfrak{m}_i}$ is the integral closure of $B_{\mathfrak{m}_i}$ in L'_j. By Abhyankar's lemma (Lemma 15.115.4) applied to $A \subset B_{\mathfrak{m}_i}$ and the extension K'/K we see that $A' \to (B'_j)_{\mathfrak{m}_{ijk}}$ is formally smooth in the $\mathfrak{m}_{ijk}$-adic topology. This implies that the ramification index is 1 and that the residue field extension is separable (Lemma 15.112.5). In this way we see that L'_j is unramified with respect to A'. This finishes the proof: we take $L' = L'_j$ for some j. □

0EXX **Lemma 15.115.8.** *Let A be a discrete valuation ring with fraction field K.*

(1) *If L/K is a finite separable extension which is tamely ramified with respect to A, then there exists a Galois extension M/K containing L which is tamely ramified with respect to A.*
(2) *If L_1/K, L_2/K are finite separable extensions which are tamely ramified with respect to A, then there exists a finite separable extension L/K which is tamely ramified with respect to A containing L_1 and L_2.*

Proof. Proof of (2). Choose a uniformizer $\pi \in A$. We can choose an integer e invertible in κ_A and extensions $L_i'/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ with L_i'/L_i as extensions of K, see Lemma 15.115.7. By Lemma 15.112.8 we can find an extension L'/K' which is unramified with respect to A' such that L_i'/K is isomorphic to a subextension of L'/K' for $i = 1, 2$. This finishes the proof of (3) as L'/K is tamely ramified (use same lemma as above).

Proof of (1). We may first replace L by a larger extension and assume that L is an extension of $K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ where e is invertible in κ_A, see Lemma 15.115.7. Let M be the normal closure of L over K, see Fields, Definition 9.16.4. Then M/K is Galois by Fields, Lemma 9.21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 9.16.6. Let B be the integral closure of A in L as in Remark 15.112.6. The condition that L is unramified with respect to $A' = A[\pi^{1/e}]$ exactly means that $A' \to B$ is an étale ring map, see Algebra, Lemma 10.143.7. Claim:

$$K' \otimes_K \ldots \otimes_K K' = \prod K_i'$$

is a product of field extensions K_i'/K tamely ramified with respect to A. Then if A_i' is the integral closure of A in K_i' we see that

$$\prod A_i' \otimes_{(A' \otimes_A \ldots \otimes_A A')} (B \otimes_A \ldots \otimes_A B)$$

is finite étale over $\prod A_i'$ and hence a product of Dedekind domains (Lemma 15.44.4). We conclude that M is the fraction field of one of these Dedekind domains which is finite étale over A_i' for some i. It follows that M/K_i' is unramified with respect to every maximal ideal of A_i' and hence M/K is tamely ramified by Lemma 15.115.5.

It remains the prove the claim. For this we write $A' = A[x]/(x^e - \pi)$ and we see that

$$A' \otimes_A \ldots \otimes_A A' = A'[x_1, \ldots, x_r]/(x_1^e - \pi, \ldots, x_r^e - \pi)$$

The normalization of this ring certainly contains the elements $y_i = x_i/x_1$ for $i = 2, \ldots, r$ subject to the relations $y_i^e - 1 = 0$ and we obtain

$$A[x_1, y_2, \ldots, y_r]/(x_1^e - \pi, y_2^e - 1, \ldots, y_r^e - 1) = A'[y_2, \ldots, y_r]/(y_2^e - 1, \ldots, y_r^e - 1)$$

This ring is finite étale over A' because e is invertible in A'. Hence it is a product of Dedekind domains each unramified over A' as desired (see references given above in case of confusion). □

0EXY **Lemma 15.115.9.** *Let $A \subset B$ be an extension of discrete valuation rings. Denote L/K the corresponding extension of fraction fields. Let K'/K be a finite separable extension. Then*

$$K' \otimes_K L = \prod L_i'$$

is a finite product of fields and the following is true

(1) *If K' is unramified with respect to A, then each L'_i is unramified with respect to B.*
(2) *If K' is tamely ramified with respect to A, then each L'_i is tamely ramified with respect to B.*

Proof. The algebra $K' \otimes_K L$ is a finite product of fields as it is a finite étale algebra over L. Let A' be the integral closure of A in K'.

In case (1) the ring map $A \to A'$ is finite étale. Hence $B' = B \otimes_A A'$ is finite étale over B and is a finite product of Dedekind domains (Lemma 15.44.4). Hence B' is the integral closure of B in $K' \otimes_K L$. It follows immediately that each L'_i is unramified with respect to B.

Choose a uniformizer $\pi \in A$. To prove (2) we may replace K' by a larger extension tame ramified with respect to A (details omitted; hint: use Lemma 15.115.6). Thus by Lemma 15.115.7 we may assume there exists some $e \geq 1$ invertible in κ_A such that K' contains $K[\pi^{1/e}]$ and such that K' is unramified with respect to $A[\pi^{1/e}]$. Choose a product decomposition

$$K[\pi^{1/e}] \otimes_K L = \prod L_{e,j}$$

For every i there exists a j_i such that $L'_i/L_{e,j_i}$ is a finite separable extension. Let $B_{e,j}$ be the integral closure of B in $L_{e,j}$. By (1) applied to $K'/K[\pi^{1/e}]$ and $A[\pi^{1/e}] \subset (B_{e,j_i})_{\mathfrak{m}}$ we see that L'_i is unramified with respect to $(B_{e,j_i})_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset B_{e,j_i}$. Hence the proof will be complete if we can show that $L_{e,j}$ is tamely ramified with respect to B, see Lemma 15.115.5.

Choose a uniformizer θ in B. Write $\pi = u\theta^t$ where u is a unit of B and $t \geq 1$. Then we have

$$A[\pi^{1/e}] \otimes_A B = B[x]/(x^e - u\theta^t) \subset B[y,z]/(y^{e'} - \theta, z^e - u)$$

where $e' = e/\gcd(e,t)$. The map sends x to $zy^{t/\gcd(e,t)}$. Since the right hand side is a product of Dedekind domains each tamely ramified over B the proof is complete (details omitted). □

15.116. Eliminating ramification

09EL In this section we discuss a result of Helmut Epp, see [**Epp73**]. We strongly encourage the reader to read the original. Our approach is slightly different as we try to handle the mixed and equicharacteristic cases by the same method. For related results, see also [**Pon98**], [**Pon99**], [**Kuh03**], and [**ZK99**].

Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. The goal in this section is to find a finite extension K_1/K such that with

$$\begin{array}{ccc} L & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & K_1 \end{array} \quad \text{and} \quad \begin{array}{ccccc} B & \longrightarrow & B_1 & \longrightarrow & (B_1)_{\mathfrak{m}_{ij}} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A_1 & \longrightarrow & (A_1)_{\mathfrak{m}_i} \end{array}$$

as in Remark 15.115.1 the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ are all weakly unramified or even formally smooth in the relevant adic topologies. The simplest (but nontrivial) example of this is Abhyankar's lemma, see Lemma 15.115.4.

09EN **Definition 15.116.1.** Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$.

(1) We say a finite field extension K_1/K is a *weak solution for* $A \subset B$ if all the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.115.1 are weakly unramified.
(2) We say a finite field extension K_1/K is a *solution for* $A \subset B$ if each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.115.1 is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.

We say a solution K_1/K is a *separable solution* if K_1/K is separable.

In general (weak) solutions do not exist; there is an example in [**Epp73**]. Under a mild hypothesis on the residue field extension, we will prove the existence of weak solutions in Theorem 15.116.18 following [**Epp73**]. In the next section, we will deduce the existence of solutions and sometimes separable solutions in geometrically meaningful cases, see Proposition 15.117.8 and Lemma 15.117.9. However, the following example shows that in general one needs inseparable extensions to get even a weak solution.

09EP **Example 15.116.2.** Let k be a perfect field of characteristic $p > 0$. Let $A = k[[x]]$ and $K = k((x))$. Let $B = A[x^{1/p}]$. Any weak solution K_1/K for $A \to B$ is inseparable (and any finite inseparable extension of K is a solution). We omit the proof.

Solutions are stable under further extensions, see Lemma 15.117.1. This may not be true for weak solutions. Weak solutions are in some sense stable under totally ramified extensions, see Lemma 15.116.3.

09ER **Lemma 15.116.3.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is weakly unramified. Then for any finite separable extension K_1/K totally ramified with respect to A we have that $L_1 = L \otimes_K K_1$ is a field, A_1 and $B_1 = B \otimes_A A_1$ are discrete valuation rings, and the extension $A_1 \subset B_1$ (see Remark 15.115.1) is weakly unramified.*

Proof. Let $\pi \in A$ and $\pi_1 \in A_1$ be uniformizers. As K_1/K is totally ramified with respect to A we have $\pi_1^e = u_1\pi$ for some unit u_1 in A_1. Hence A_1 is generated by π_1 over A and the minimal polynomial $P(t)$ of π_1 over K has the form

$$P(t) = t^e + a_{e-1}t^{e-1} + \ldots + a_0$$

with $a_i \in (\pi)$ and $a_0 = u\pi$ for some unit u of A. Note that $e = [K_1 : K]$ as well. Since $A \to B$ is weakly unramified we see that π is a uniformizer of B and hence $B_1 = B[t]/(P(t))$ is a discrete valuation ring with uniformizer the class of t. Thus the lemma is clear. □

09ES **Lemma 15.116.4.** *Let $A \to B \to C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Let K_1/K be a finite extension.*

(1) *If K_1 is a (weak) solution for $A \to C$, then K_1 is a (weak) solution for $A \to B$.*
(2) *If K_1 is a (weak) solution for $A \to B$ and $L_1 = (L \otimes_K K_1)_{red}$ is a product of fields which are (weak) solutions for $B \to C$, then K_1 is a (weak) solution for $A \to C$.*

Proof. Let $L_1 = (L \otimes_K K_1)_{red}$ and $M_1 = (M \otimes_K K_1)_{red}$ and let $B_1 \subset L_1$ and $C_1 \subset M_1$ be the integral closure of B and C. Note that $M_1 = (M \otimes_L L_1)_{red}$ and

that L_1 is a (nonempty) finite product of finite extensions of L. Hence the ring map $B_1 \to C_1$ is a finite product of ring maps of the form discussed in Remark 15.115.1. In particular, the map $\mathrm{Spec}(C_1) \to \mathrm{Spec}(B_1)$ is surjective. Choose a maximal ideal $\mathfrak{m} \subset C_1$ and consider the extensions of discrete valuation rings

$$(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}} \to (C_1)_{\mathfrak{m}}$$

If the composition is weakly unramified, so is the map $(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}}$. If the residue field extension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{\mathfrak{m}}$ is separable, so is the subextension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{B_1 \cap \mathfrak{m}}$. Taking into account Lemma 15.112.5 this proves (1). A similar argument works for (2). □

09ET **Lemma 15.116.5.** *Let $A \to B$ be an extension of discrete valuation rings. There exists a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of extensions of discrete valuation rings such that

(1) *the extensions K'/K and L'/L of fraction fields are separable algebraic,*
(2) *the residue fields of A' and B' are separable algebraic closures of the residue fields of A and B, and*
(3) *if a solution, weak solution, or separable solution exists for $A' \to B'$, then a solution, weak solution, or separable solution exists for $A \to B$.*

Proof. By Algebra, Lemma 10.159.2 there exists an extension $A \subset A'$ which is a filtered colimit of finite étale extensions such that the residue field of A' is a separable algebraic closure of the residue field of A. Then $A \subset A'$ is an extension of discrete valuation rings such that the induced extension K'/K of fraction fields is separable algebraic.

Let $B \subset B'$ be a strict henselization of B. Then $B \subset B'$ is an extension of discrete valuation rings whose fraction field extension is separable algebraic. By Algebra, Lemma 10.155.9 there exists a commutative diagram as in the statement of the lemma. Parts (1) and (2) of the lemma are clear.

Let K'_1/K' be a (weak) solution for $A' \to B'$. Since A' is a colimit, we can find a finite étale extension $A \subset A'_1$ and a finite extension K_1 of the fraction field F of A'_1 such that $K'_1 = K' \otimes_F K_1$. As $A \subset A'_1$ is finite étale and B' strictly henselian, it follows that $B' \otimes_A A'_1$ is a finite product of rings isomorphic to B'. Hence

$$L' \otimes_K K_1 = L' \otimes_K F \otimes_F K_1$$

is a finite product of rings isomorphic to $L' \otimes_{K'} K'_1$. Thus we see that K_1/K is a (weak) solution for $A \to B'$. Hence it is also a (weak) solution for $A \to B$ by Lemma 15.116.4. □

09EU **Lemma 15.116.6.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a normal extension. Say $G = Aut(K_1/K)$. Then G acts on the rings K_1, L_1, A_1 and B_1 of Remark 15.115.1 and acts transitively on the set of maximal ideals of B_1.*

Proof. Everything is clear apart from the last assertion. If there are two or more orbits of the action, then we can find an element $b \in B_1$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a G-invariant element of $B_1 \subset L_1 = (L \otimes_K K_1)_{red}$ which is in some maximal ideals of B_1 but not in all maximal ideals of B_1. Lifting it to an element of $L \otimes_K K_1$ and raising to a high power we obtain a G-invariant element b'' of $L \otimes_K K_1$ mapping to $(b')^N$ for some $N > 0$; in fact, we only need to do this in case the characteristic is $p > 0$ and in this case raising to a suitably large p-power q defines a canonical map $(L \otimes_K K_1)_{red} \to L \otimes_K K_1$. Since $K = (K_1)^G$ we conclude that $b'' \in L$. Since b'' maps to an element of B_1 we see that $b'' \in B$ (as B is normal). Then on the one hand it must be true that $b'' \in \mathfrak{m}_B$ as b' is in some maximal ideal of B_1 and on the other hand it must be true that $b'' \notin \mathfrak{m}_B$ as b' is not in all maximal ideals of B_1. This contradiction finishes the proof of the lemma. □

09EW **Lemma 15.116.7.** *Let A be a discrete valuation ring with uniformizer π. If the residue characteristic of A is $p > 0$, then for every $n > 1$ and p-power q there exists a degree q separable extension L/K totally ramified with respect to A such that the integral closure B of A in L has ramification index q and a uniformizer π_B such that $\pi_B^q = \pi + \pi^n b$ and $\pi_B^q = \pi + (\pi_B)^{nq} b'$ for some $b, b' \in B$.*

Proof. If the characteristic of K is zero, then we can take the extension given by $\pi_B^q = \pi$, see Lemma 15.115.2. If the characteristic of K is $p > 0$, then we can take the extension of K given by $z^q - \pi^n z = \pi^{1-q}$. Namely, then we see that $y^q - \pi^{n+q-1} y = \pi$ where $y = \pi z$. Taking $\pi_B = y$ we obtain the desired result. □

09EX **Lemma 15.116.8.** *Let A be a discrete valuation ring. Assume the reside field κ_A has characteristic $p > 0$ and that $a \in A$ is an element whose residue class in κ_A is not a pth power. Then a is not a pth power in K and the integral closure of A in $K[a^{1/p}]$ is the ring $A[a^{1/p}]$ which is a discrete valuation ring weakly unramified over A.*

Proof. This lemma proves itself. □

09EY **Lemma 15.116.9.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Let $\pi \in A$ be a uniformizer. Assume*

1. *B is a Nagata ring,*
2. *$A \subset B$ is weakly unramified,*
3. *M is a degree p purely inseparable extension of L.*

Then either

1. *$A \to C$ is weakly unramified, or*
2. *$C = B[\pi^{1/p}]$, or*
3. *there exists a degree p separable extension K_1/K totally ramified with respect to A such that $L_1 = L \otimes_K K_1$ and $M_1 = M \otimes_K K_1$ are fields and the maps of integral closures $A_1 \to B_1 \to C_1$ are weakly unramified extensions of discrete valuation rings.*

Proof. Let e be the ramification index of C over B. If $e = 1$, then we are done. If not, then $e = p$ by Lemmas 15.112.2 and 15.112.4. This in turn implies that the residue fields of B and C agree. Choose a uniformizer π_C of C. Write $\pi_C^p = u\pi$ for some unit u of C. Since $\pi_C^p \in L$, we see that $u \in B^*$. Also $M = L[\pi_C]$.

Suppose there exists an integer $m \geq 0$ such that

$$u = \sum\nolimits_{0 \leq i < m} b_i^p \pi^i + b\pi^m$$

with $b_i \in B$ and with $b \in B$ an element whose image in κ_B is not a pth power. Choose an extension K_1/K as in Lemma 15.116.7 with $n = m + 2$ and denote π' the uniformizer of the integral closure A_1 of A in K_1 such that $\pi = (\pi')^p + (\pi')^{np}a$ for some $a \in A_1$. Let B_1 be the integral closure of B in $L \otimes_K K_1$. Observe that $A_1 \to B_1$ is weakly unramified by Lemma 15.116.3. In B_1 we have

$$u\pi = \left(\sum\nolimits_{0 \leq i < m} b_i (\pi')^{i+1}\right)^p + b(\pi')^{(m+1)p} + (\pi')^{np} b_1$$

for some $b_1 \in B_1$ (computation omitted). We conclude that M_1 is obtained from L_1 by adjoining a pth root of

$$b + (\pi')^{n-m-1} b_1$$

Since the residue field of B_1 equals the residue field of B we see from Lemma 15.116.8 that M_1/L_1 has degree p and the integral closure C_1 of B_1 is weakly unramified over B_1. Thus we conclude in this case.

If there does not exist an integer m as in the preceding paragraph, then u is a pth power in the π-adic completion of B_1. Since B is Nagata, this means that u is a pth power in B_1 by Algebra, Lemma 10.162.18. Whence the second case of the statement of the lemma holds. □

09EZ **Lemma 15.116.10.** *Let A be a local ring annihilated by a prime p whose maximal ideal is nilpotent. There exists a ring map $\sigma : \kappa_A \to A$ which is a section to the residue map $A \to \kappa_A$. If $A \to A'$ is a local homomorphism of local rings, then we can choose a similar ring map $\sigma' : \kappa_{A'} \to A'$ compatible with σ provided that the extension $\kappa_{A'}/\kappa_A$ is separable.*

Proof. Separable extensions are formally smooth by Algebra, Proposition 10.158.9. Thus the existence of σ follows from the fact that $\mathbf{F}_p \to \kappa_A$ is separable. Similarly for the existence of σ' compatible with σ. □

09F0 **Lemma 15.116.11.** *Let A be a discrete valuation ring with fraction field K of characteristic $p > 0$. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $z^p - z = \xi$. Then L/K is Galois and one of the following happens*

(1) *$L = K$,*
(2) *L/K is unramified with respect to A of degree p,*
(3) *L/K is totally ramified with respect to A with ramification index p, and*
(4) *the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

(A) *If $\xi \in A$, then we are in case (1) or (2).*
(B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
(C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. The extension is Galois of order dividing p by the discussion in Fields, Section 9.25. It immediately follows from the discussion in Section 15.113 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(x^p - x - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) = a$$

Since $A[y]/(y^p - \pi^{n-k}y - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. □

09F1 **Lemma 15.116.12.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

(1) *$A \subset B$ weakly unramified,*
(2) *the characteristic of K is p,*
(3) *M is a degree p Galois extension of L, and*
(4) *$\kappa_A = \bigcap_{n\geq 1} \kappa_B^{p^n}$.*

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is a weak solution for $A \to C$.

Proof. Since the characteristic of L is p we know that M is an Artin-Schreier extension of L (Fields, Lemma 9.25.1). Thus we may pick $z \in M$, $z \notin L$ such that $\xi = z^p - z \in L$. Choose $n \geq 0$ such that $\pi^n\xi \in B$. We pick z such that n is minimal. If $n = 0$, then M/L is unramified with respect to B (Lemma 15.116.11) and we are done. Thus we have $n > 0$.

Assumption (4) implies that κ_A is perfect. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^n A$ and $\overline{\sigma} : \kappa_B \to B/\pi^n B$ as in Lemma 15.116.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$[19]. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,1} \sigma(\lambda_i)\pi^{-i} + b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, 1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, 1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, 1\}$ we have $\sigma(\lambda_i) = a_i + \pi^n b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of σ. Thus $\xi = \pi^{-n}a + b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). We compute

$$(z - \pi^{-n/p}a_0)^p - (z - \pi^{-n/p}a_0) = \pi^{-(n-1)}(a_1 + \pi^{n-1-n/p}a_0) + b'$$

[19] If B is complete, then we can choose σ to be a ring map. If A is also complete and σ is a ring map, then σ maps κ_A into A.

for some $b' \in B$. This would contradict the minimality of n. Thus p does not divide n. Consider the degree p extension K_1 of K given by $w^p - w = \pi^{-n}a$. By Lemma 15.116.11 this extension is Galois and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.116.3). By Lemma 15.116.11 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, and totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z - w$ and

$$(z-w)^p - (z-w) = z^p - z - (w^p - w) = b$$

with $b \in B$ (see above). Thus by Lemma 15.116.11 once more the extension M_1/L_1 is unramified with respect to B_1 and we conclude that K_1 is a weak solution for $A \to C$. From now on we assume $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we change our choice of z into

$$z' = z - (\sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^p)\pi^{-pj} + \ldots + \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j})$$

Then ξ changes into $\xi' = (z')^p - (z')$ as follows

$$\xi' = \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} + \text{something in } B$$

Writing $\xi' = \sum_{i=n,\ldots,1} \sigma(\lambda_i')\pi^{-i} + b'$ as before we find that $\lambda_i' = \lambda_i$ for $i \neq j, j'$ and $\lambda_j' = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume no such pair j, j' exists. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Choose a separable extension K_1/K totally ramified with respect to A of degree p^r such that the corresponding discrete valuation ring $A_1 \subset K_1$ has uniformizer π' with $(\pi')^{p^r} = \pi + \pi^{n+1}a$ for some $a \in A_1$ (Lemma 15.116.7). Observe that $L_1 = L \otimes_K K_1$ is a field and that L_1/L is totally ramified with respect to B (Lemma 15.116.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^n, i.e., modulo $(\pi')^{np^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i \in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

As in the previous paragraph we change our choice of z into

$$\begin{aligned} z' = z & \\ & - \sum\nolimits_{i \in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}}(\pi')^{-ip^{r-1}} \right) \\ & - \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}}(\pi')^{-jp^{r-1}} \right) \end{aligned}$$

to obtain

$$(z')^p - z' = \sum\nolimits_{i\in I} x_i(\pi')^{-i} + \sum\nolimits_{j\in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + b_1'$$

for some $b_1' \in B_1$. Since there is a unique j such that jp^{r-r_j} is maximal and since jp^{r-r_j} is bigger than $i \in I$ and divisible by p, we see that M_1/L_1 falls into case (C) of Lemma 15.116.11. This finishes the proof. □

09F2 **Lemma 15.116.13.** *Let A be a ring which contains a primitive pth root of unity ζ. Set $w = 1 - \zeta$. Then*

$$P(z) = \frac{(1+wz)^p - 1}{w^p} = z^p - z + \sum\nolimits_{0<i<p} a_i z^i$$

is an element of $A[z]$ and in fact $a_i \in (w)$. Moreover, we have

$$P(z_1 + z_2 + wz_1z_2) = P(z_1) + P(z_2) + w^pP(z_1)P(z_2)$$

in the polynomial ring $A[z_1, z_2]$.

Proof. It suffices to prove this when

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{p-1} + \ldots + x + 1)$$

is the ring of integers of the cyclotomic field. The polynomial identity $t^p - 1 = (t-1)(t-\zeta)\ldots(t-\zeta^{p-1})$ (which is proved by looking at the roots on both sides) shows that $t^{p-1} + \ldots + t + 1 = (t-\zeta)\ldots(t-\zeta^{p-1})$. Substituting $t = 1$ we obtain $p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$. The maximal ideal $(p, w) = (w)$ is the unique prime ideal of A lying over p (as fields of characteristic p do not have nontrivial pth roots of 1). It follows that $p = uw^{p-1}$ for some unit u. This implies that

$$a_i = \frac{1}{p}\binom{p}{i} uw^{i-1}$$

for $p > i > 1$ and $-1 + a_1 = pw/w^p = u$. Since $P(-1) = 0$ we see that $0 = (-1)^p - u$ modulo (w). Hence $a_1 \in (w)$ and the proof if the first part is done. The second part follows from a direct computation we omit. □

09F3 **Lemma 15.116.14.** *Let A be a discrete valuation ring of mixed characteristic $(0, p)$ which contains a primitive pth root of 1. Let $P(t) \in A[t]$ be the polynomial of Lemma 15.116.13. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $P(z) = \xi$. Then L/K is Galois and one of the following happens*

(1) *$L = K$,*
(2) *L/K is unramified with respect to A of degree p,*
(3) *L/K is totally ramified with respect to A with ramification index p, and*
(4) *the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

(A) *If $\xi \in A$, then we are in case (1) or (2).*
(B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
(C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. Adjoining a root of $P(z) = \xi$ is the same thing as adjoining a root of $y^p = w^p(1 + \xi)$. Since K contains a primitive pth root of 1 the extension is Galois of order dividing p by the discussion in Fields, Section 9.24. It immediately follows from the discussion in Section 15.113 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(P(x) - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) + \sum a_i \pi^{n-ik}(\pi^k z)^i = a$$

Since $A[y]/(y^p - \pi^{n-k}y - \sum a_i \pi^{n-ik} y^i - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. □

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.116.13. Let L be a finite extension of K. We say L/K is a *degree p extension of finite level* if L is a degree p extension of K obtained by adjoining a root of the equation $P(z) = \xi$ where $\xi \in K$ is an element with $w^p\xi \in \mathfrak{m}_A$.

This definition is relevant to the discussion in this section due to the following straightforward lemma.

09F4 **Lemma 15.116.15.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume that*

(1) *A has mixed characteristic $(0, p)$,*
(2) *$A \subset B$ is weakly unramified,*
(3) *B contains a primitive pth root of 1, and*
(4) *M/L is Galois of degree p.*

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is either a weak solution for $A \to C$ or is such that M_1/L_1 is a degree p extension of finite level.

Proof. Let $\pi \in A$ be a uniformizer. By Kummer theory (Fields, Lemma 9.24.1) M is obtained from L by adjoining the root of $y^p = b$ for some $b \in L$.

If $\text{ord}_B(b)$ is prime to p, then we choose a degree p separable extension K_1/K totally ramified with respect to A (for example using Lemma 15.116.7). Let A_1 be the integral closure of A in K_1. By Lemma 15.116.3 the integral closure B_1 of B in $L_1 = L \otimes_K K_1$ is a discrete valuation ring weakly unramified over A_1. If K_1/K is not a weak solution for $A \to C$, then the integral closure C_1 of C in $M_1 = M \otimes_K K_1$ is a discrete valuation ring and $B_1 \to C_1$ has ramification index p. In this case, the field M_1 is obtained from L_1 by adjoining the pth root of b with $\text{ord}_{B_1}(b)$ divisible by p. Replacing A by A_1, etc we may assume that $b = \pi^n u$ where $u \in B$ is a unit and n is divisible by p. Of course, in this case the extension M is obtained from L by adjoining the pth root of a unit.

Suppose M is obtained from L by adjoining the root of $y^p = u$ for some unit u of B. If the residue class of u in κ_B is not a pth power, then $B \subset C$ is weakly unramified (Lemma 15.116.8) and we are done. Otherwise, we can replace our choice of y by y/v where v^p and u have the same image in κ_B. After such a replacement we have

$$y^p = 1 + \pi b$$

for some $b \in B$. Then we see that $P(z) = \pi b/w^p$ where $z = (y-1)/w$. Thus we see that the extension is a degree p extension of finite level with $\xi = \pi b/w^p$. $\square$

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.116.13. Let L be a degree p extension of K of finite level. Choose $z \in L$ generating L over K with $\xi = P(z) \in K$. Choose a uniformizer π for A and write $w = u\pi^{e_1}$ for some integer $e_1 = \text{ord}_A(w)$ and unit $u \in A$. Finally, pick $n \geq 0$ such that

$$\pi^n \xi \in A$$

The *level* of L/K is the smallest value of the quantity n/e_1 taking over all z generating L/K with $\xi = P(z) \in K$.

We make a couple of remarks. Since the extension is of finite level we know that we can choose z such that $n < pe_1$. Thus the level is a rational number contained in $[0, p)$. If the level is zero then L/K is unramified with respect to A by Lemma 15.116.14. Our next goal is to lower the level.

09F5 **Lemma 15.116.16.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

(1) *A has mixed characteristic $(0, p)$,*
(2) *$A \subset B$ weakly unramified,*
(3) *B contains a primitive pth root of 1,*
(4) *M/L is a degree p extension of finite level $l > 0$,*
(5) *$\kappa_A = \bigcap_{n \geq 1} \kappa_B^{p^n}$.*

Then there exists a finite separable extension K_1 of K totally ramified with respect to A such that either K_1 is a weak solution for $A \to C$, or the extension M_1/L_1 is a degree p extension of finite level $\leq \max(0, l-1, 2l-p)$.

Proof. Let $\pi \in A$ be a uniformizer. Let $w \in B$ and $P \in B[t]$ be as in Lemma 15.116.13 (for B). Set $e_1 = \text{ord}_B(w)$, so that w and π^{e_1} are associates in B. Pick $z \in M$ generating M over L with $\xi = P(z) \in K$ and n such that $\pi^n \xi \in B$ as in the definition of the level of M over L, i.e., $l = n/e_1$.

The proof of this lemma is completely similar to the proof of Lemma 15.116.12. To explain what is going on, observe that

09F6 (15.116.16.1) $$P(z) \equiv z^p - z \bmod \pi^{-n+e_1} B$$

for any $z \in L$ such that $\pi^{-n} P(z) \in B$ (use that z has valuation at worst $-n/p$ and the shape of the polynomial P). Moreover, we have

09F7 (15.116.16.2) $$\xi_1 + \xi_2 + w^p \xi_1 \xi_2 \equiv \xi_1 + \xi_2 \bmod \pi^{-2n+pe_1} B$$

for $\xi_1, \xi_2 \in \pi^{-n} B$. Finally, observe that $n - e_1 = (l-1)/e_1$ and $-2n + pe_1 = -(2l-p)e_1$. Write $m = n - e_1 \max(0, l-1, 2l-p)$. The above shows that doing calculations in $\pi^{-n}B/\pi^{-n+m}B$ the polynomial P behaves exactly as the polynomial $z^p - z$. This explains why the lemma is true but we also give the details below.

Assumption (4) implies that κ_A is perfect. Observe that $m \leq e_1$ and hence A/π^m is annihilated by w and hence p. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^m A$ and $\overline{\sigma} : \kappa_B \to B/\pi^m B$ as in Lemma 15.116.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i)\pi^{-i} + \pi^{-n+m)}b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, n-m+1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, n-m+1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, n-m+1\}$ we have $\sigma(\lambda_i) = a_i + \pi^{n-m}b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of $\overline{\sigma}$. Thus $\xi = \pi^{-n}a + \pi^{-n+m}b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). Set $z_1 = -\pi^{-n/p}a_0$. Note that $P(z_1) \in \pi^{-n}B$ and that $z + z_1 + wzz_1$ is an element generating M over L (note that $wz_1 \neq -1$ as $n < pe_1$). Moreover, by Lemma 15.116.13 we have

$$P(z + z_1 + wzz_1) = P(z) + P(z_1) + w^p P(z)P(z_1) \in K$$

and by equations (15.116.16.1) and (15.116.16.2) we have

$$P(z) + P(z_1) + w^p P(z)P(z_1) \equiv \xi + z_1^p - z_1 \bmod \pi^{-n+m}B$$

for some $b' \in B$. This contradict the minimality of n! Thus p does not divide n. Consider the degree p extension K_1 of K given by $P(y) = -\pi^{-n}a$. By Lemma 15.116.14 this extension is separable and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.116.3). By Lemma 15.116.14 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z + y + wzy$ and we see that $P(z + y + wzy) \in L_1$ and

$$\begin{aligned} P(z + y + wzy) &= P(z) + P(y) + w^p P(z)P(y) \\ &\equiv \xi - \pi^{-n}a \bmod \pi^{-n+m}B_1 \\ &\equiv 0 \bmod \pi^{-n+m}B_1 \end{aligned}$$

in exactly the same manner as above. By our choice of m this means exactly that M_1/L_1 has level at most $\max(0, l-1, 2l-p)$. From now on we assume that $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we set

$$z_1 = -\sigma(\lambda_j)\pi^{-j} - \sigma(\lambda_j^p)\pi^{-pj} - \ldots - \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j}$$

and we change z into $z' = z + z_1 + wzz_1$. Observe that $z' \in M$ generates M over L and that we have $\xi' = P(z') = P(z) + P(z_1) + wP(z)P(z_1) \in L$ with

$$\xi' \equiv \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} \bmod \pi^{-n+m}B$$

by using equations (15.116.16.1) and (15.116.16.2) as above. Writing

$$\xi' = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i')\pi^{-i} + \pi^{-n+m}b'$$

as before we find that $\lambda_i' = \lambda_i$ for $i \neq j, j'$ and $\lambda_j' = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume there is no pair j, j' of J such that j'/j is a power of p. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Let K_1/K be the extension of degree p^r, totally ramified with respect to A, defined by $(\pi')^{p^r} = \pi$. Observe that π' is the uniformizer of the corresponding discrete valuation ring $A_1 \subset K_1$. Observe that $L_1 = L \otimes_K K_1$ is a field and L_1/L is totally ramified with respect to B (Lemma 15.116.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i\in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^m, i.e., modulo $(\pi')^{mp^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i\in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

Similar to our choice in the previous paragraph we set

$$\begin{aligned} z_1 - \sum\nolimits_{i\in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}}(\pi')^{-ip^{r-1}}\right) \\ - \sum\nolimits_{j\in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}}(\pi')^{-jp^{r-1}}\right) \end{aligned}$$

and we change our choice of z into $z' = z + z_1 + wzz_1$. Then z' generates M_1 over L_1 and $\xi' = P(z') = P(z) + P(z_1) + w^pP(z)P(z_1) \in L_1$ and a calculation shows that

$$\xi' \equiv \sum\nolimits_{i\in I} x_i(\pi')^{-i} + \sum\nolimits_{j\in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + (\pi')^{(-n+m)p^r}b_1'$$

for some $b_1' \in B_1$. There is a unique j such that jp^{r-r_j} is maximal and jp^{r-r_j} is bigger than $i \in I$. If $jp^{r-r_j} \leq (n-m)p^r$ then the level of the extension M_1/L_1 is less than $\max(0, l-1, 2l-p)$. If not, then, as p divides jp^{r-r_j}, we see that M_1/L_1 falls into case (C) of Lemma 15.116.14. This finishes the proof. □

09F8 **Lemma 15.116.17.** *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Assume*

(1) *the residue field k of A is algebraically closed of characteristic $p > 0$,*
(2) *A and B are complete,*
(3) *$A \to B$ is weakly unramified,*
(4) *M is a finite extension of L,*
(5) *$k = \bigcap_{n\geq 1} \kappa_B^{p^n}$*

Then there exists a finite extension K_1/K which is a weak solution for $A \to C$.

Proof. Let M' be any finite extension of L and consider the integral closure C' of B in M'. Then C' is finite over B as B is Nagata by Algebra, Lemma 10.162.8. Moreover, C' is a discrete valuation ring, see discussion in Remark 15.115.1. Moreover C' is complete as a B-module, hence complete as a discrete valuation ring, see Algebra, Section 10.96. It follows in particular that C is the integral closure of B in M (by definition of valuation rings as maximal for the relation of domination).

Let $M \subset M'$ be a finite extension and let $C' \subset M'$ be the integral closure of B as above. By Lemma 15.116.4 it suffices to prove the result for $A \to B \to C'$. Hence we may assume that M/L is normal, see Fields, Lemma 9.16.3.

If M/L is normal, we can find a chain of finite extensions

$$L = L^0 \subset L^1 \subset L^2 \subset \ldots \subset L^r = M$$

such that each extension L^{j+1}/L^j is either:

(a) purely inseparable of degree p,
(b) totally ramified with respect to B^j and Galois of degree p,
(c) totally ramified with respect to B^j and Galois cyclic of order prime to p,
(d) Galois and unramified with respect to B^j.

Here B^j is the integral closure of B in L^j. Namely, since M/L is normal we can write it as a compositum of a Galois extension and a purely inseparable extension (Fields, Lemma 9.27.3). For the purely inseparable extension the existence of the filtration is clear. In the Galois case, note that G is "the" decomposition group and let $I \subset G$ be the inertia group. Then on the one hand I is solvable by Lemma 15.113.5 and on the other hand the extension M^I/L is unramified with respect to B by Lemma 15.113.8. This proves we have a filtration as stated.

We are going to argue by induction on the integer r. Suppose that we can find a finite extension K_1/K which is a weak solution for $A \to B^1$ where B^1 is the integral closure of B in L^1. Let K_1' be the normal closure of K_1/K (Fields, Lemma 9.16.3). Since A is complete and the residue field of A is algebraically closed we see that K_1'/K_1 is separable and totally ramified with respect to A_1 (some details omitted). Hence K_1'/K is a weak solution for $A \to B^1$ as well by Lemma 15.116.3. In other words, we may and do assume that K_1 is a normal extension of K. Having done so we consider the sequence

$$L_1^0 = (L^0 \otimes_K K_1)_{red} \subset L_1^1 = (L^1 \otimes_K K_1)_{red} \subset \ldots \subset L_1^r = (L^r \otimes_K K_1)_{red}$$

and the corresponding integral closures B_1^i. Note that $C_1 = B_1^r$ is a product of discrete valuation rings which are transitively permuted by $G = \mathrm{Aut}(K_1/K)$ by Lemma 15.116.6. In particular all the extensions of discrete valuation rings $A_1 \to (C_1)_{\mathfrak{m}}$ are isomorphic and a weak solution for one will be a weak solution for all of them. We can apply the induction hypothesis to the sequence

$$A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}} \to (B_1^2)_{B_1^2 \cap \mathfrak{m}} \to \ldots \to (B_1^r)_{B_1^r \cap \mathfrak{m}} = (C_1)_{\mathfrak{m}}$$

to get a weak solution K_2/K_1 for $A_1 \to (C_1)_{\mathfrak{m}}$. The extension K_2/K will then be a weak solution for $A \to C$ by what we said before. Note that the induction hypothesis applies: the ring map $A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}}$ is weakly unramified by our choice of K_1 and the sequence of fraction field extensions each still have one of the properties (a), (b), (c), or (d) listed above. Moreover, observe that for any finite extension $\kappa_B \subset \kappa$ we still have $k = \bigcap \kappa^{p^n}$.

Thus everything boils down to finding a weak solution for $A \subset C$ when the field extension M/L satisfies one of the properties (a), (b), (c), or (d).

Case (d). This case is trivial as here $B \to C$ is unramified already.

Case (c). Say M/L is cyclic of order n prime to p. Because M/L is totally ramified with respect to B we see that the ramification index of $B \subset C$ is n and hence the ramification index of $A \subset C$ is n as well. Choose a uniformizer $\pi \in A$ and set $K_1 = K[\pi^{1/n}]$. Then K_1/K is a solution for $A \subset C$ by Abhyankar's lemma (Lemma 15.115.4).

Case (b). We divide this case into the mixed characteristic case and the equicharacteristic case. In the equicharacteristic case this is Lemma 15.116.12. In the mixed characteristic case, we first replace K by a finite extension to get to the situation where M/L is a degree p extension of finite level using Lemma 15.116.15. Then the level is a rational number $l \in [0, p)$, see discussion preceding Lemma 15.116.16. If the level is 0, then $B \to C$ is weakly unramified and we're done. If not, then we can replacing the field K by a finite extension to obtain a new situation with level $l' \leq \max(0, l-1, 2l-p)$ by Lemma 15.116.16. If $l = p - \epsilon$ for $\epsilon < 1$ then we see that $l' \leq p - 2\epsilon$. Hence after a finite number of replacements we obtain a case with level $\leq p - 1$. Then after at most $p - 1$ more such replacements we reach the situation where the level is zero.

Case (a) is Lemma 15.116.9. This is the only case where we possibly need a purely inseparable extension of K, namely, in case (2) of the statement of the lemma we win by adjoining a pth power of the element π. This finishes the proof of the lemma. □

At this point we have collected all the lemmas we need to prove the main result of this section.

09F9 **Theorem 15.116.18** (Epp)**.** *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the characteristic of κ_A is $p > 0$, assume that every element of*

$$\bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

is separable algebraic over κ_A. Then there exists a finite extension K_1/K which is a weak solution for $A \to B$ as defined in Definition 15.116.1.

Proof. If the characteristic of κ_A is zero or if the residue characteristic is p, the ramification index is prime to p, and the residue field extension is separable, then this follows from Abhyankar's lemma (Lemma 15.115.4). Namely, suppose the ramification index is e. Choose a uniformizer $\pi \in A$. Let K_1/K be the extension obtained by adjoining an eth root of π. By Lemma 15.115.2 we see that the integral closure A_1 of A in K_1 is a discrete valuation ring with ramification index over A. Thus $A_1 \to (B_1)_{\mathfrak{m}}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_1 by Lemma 15.115.4 and a fortiori these are weakly unramified extensions of discrete valuation rings.

From now on we let p be a prime number and we assume that κ_A has characteristic p. We first apply Lemma 15.116.5 to reduce to the case that A and B have separably

closed residue fields. Since κ_A and κ_B are replaced by their separable algebraic closures by this procedure we see that we obtain

$$\kappa_A \supset \bigcap\nolimits_{n\geq 1} \kappa_B^{p^n}$$

from the condition of the theorem.

Let $\pi \in A$ be a uniformizer. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B. We have a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & A^\wedge \end{array}$$

of extensions of discrete valuation rings. Let $K^\wedge$ be the fraction field of $A^\wedge$. Suppose that we can find a finite extension $M/K^\wedge$ which is (a) a weak solution for $A^\wedge \to B^\wedge$ and (b) a compositum of a separable extension and an extension obtained by adjoining a p-power root of π. Then by Lemma 15.114.2 we can find a finite extension K_1/K such that $K^\wedge \otimes_K K_1 = M$. Let A_1, resp. $A_1^\wedge$ be the integral closure of A, resp. $A^\wedge$ in K_1, resp. M. Since $A \to A^\wedge$ is formally smooth in the $\mathfrak{m}^\wedge$-adic topology (Lemma 15.112.5) we see that $A_1 \to A_1^\wedge$ is formally smooth in the $\mathfrak{m}_1^\wedge$-adic topology (Lemma 15.115.3 and A_1 and $A_1^\wedge$ are discrete valuation rings by discussion in Remark 15.115.1). We conclude from Lemma 15.116.4 part (2) that K_1/K is a weak solution for $A \to B^\wedge$. Applying Lemma 15.116.4 part (1) we see that K_1/K is a weak solution for $A \to B$.

Thus we may assume A and B are complete discrete valuation rings with separably closed residue fields of characteristic p and with $\kappa_A \supset \bigcap_{n\geq 1} \kappa_B^{p^n}$. We are also given a uniformizer $\pi \in A$ and we have to find a weak solution for $A \to B$ which is a compositum of a separable extension and a field obtained by taking p-power roots of π. Note that the second condition is automatic if A has mixed characteristic.

Set $k = \bigcap_{n\geq 1} \kappa_B^{p^n}$. Observe that k is an algebraically closed field of characteristic p. If A has mixed characteristic let Λ be a Cohen ring for k and in the equicharacteristic case set $\Lambda = k[[t]]$. We can choose a ring map $\Lambda \to A$ which maps t to π in the equicharacteristic case. In the equicharacteristic case this follows from the Cohen structure theorem (Algebra, Theorem 10.160.8) and in the mixed characteristic case this follows as $\mathbf{Z}_p \to \Lambda$ is formally smooth in the adic topology (Lemmas 15.112.5 and 15.37.5). Applying Lemma 15.116.4 we see that it suffices to prove the existence of a weak solution for $\Lambda \to B$ which in the equicharacteristic p case is a compositum of a separable extension and a field obtained by taking p-power roots of t. However, since $\Lambda = k[[t]]$ in the equicharacteristic case and any extension of $k((t))$ is such a compositum, we can now drop this requirement!

Thus we arrive at the situation where A and B are complete, the residue field k of A is algebraically closed of characteristic $p > 0$, we have $k = \bigcap \kappa_B^{p^n}$, and in the mixed characteristic case p is a uniformizer of A (i.e., A is a Cohen ring for k). If A has mixed characteristic choose a Cohen ring Λ for κ_B and in the equicharacteristic case set $\Lambda = \kappa_B[[t]]$. Arguing as above we may choose a ring map $A \to \Lambda$ lifting $k \to \kappa_B$ and mapping a uniformizer to a uniformizer. Since $k \subset \kappa_B$ is separable the ring map $A \to \Lambda$ is formally smooth in the adic topology (Lemma 15.112.5). Hence we can find a ring map $\Lambda \to B$ such that the composition $A \to \Lambda \to B$ is the

given ring map $A \to B$ (see Lemma 15.37.5). Since Λ and B are complete discrete valuation rings with the same residue field, B is finite over Λ (Algebra, Lemma 10.96.12). This reduces us to the special case discussed in Lemma 15.116.17. □

15.117. Eliminating ramification, II

0GLQ In this section we use the results of Section 15.116 to obtain (separable) solutions in some cases.

0GLR **Lemma 15.117.1.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If K_1/K is a solution for $A \subset B$, then for any finite extension K_2/K_1 the extension K_2/K is a solution for $A \subset B$.*

Proof. This follows from Lemma 15.115.3. Details omitted. □

0GLS **Lemma 15.117.2.** *Let $A \subset B$ be an extension of discrete valuation rings. If B is Nagata and the extension L/K of fraction fields is separable, then A is Nagata.*

Proof. A discrete valuation ring is Nagata if and only if it is N-2. Let K_1/K be a finite purely inseparable field extension. We have to show that the integral closure A_1 of A in K_1 is finite over A, see Algebra, Lemma 10.161.12. Since L/K is separable and K_1/K is purely inseparable, the algebra $L \otimes_K K_1$ is a field (by Algebra, Lemmas 10.43.6 and 10.46.10). Let B_1 be the integral closure of B in $L \otimes_K K_1$. Since B is Nagata, B_1 is finite over B. Since $B \otimes_A A_1 \subset B_1$ and B is Noetherian, we see that $B \otimes_A A_1$ is finite over B. As $A \to B$ is faithfully flat, this implies A_1 is finite over A, see Algebra, Lemma 10.83.2. □

0GLT **Lemma 15.117.3.** *Let $A' \subset A$ be an extension of rings. Let $f \in A'$. Assume that (a) A is finite over A', (b) f is a nonzerodivisor on A, and (c) $A'_f = A_f$. Then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$ the following is true: given a ring B', a nonzerodivisor $g \in B'$, and an isomorphism $\varphi' : A'/f^nA' \to B'/g^nB'$ with $\varphi'(f) \equiv g$, there is a finite extension $B' \subset B$ and an isomorphism $\varphi : A/fA \to B/gB$ compatible with φ'.*

Proof. Since A is finite over A' and since $A'_f = A_f$ we can Cchoose $t > 0$ such that $f^tA \subset A'$. Set $n_0 = 2t$. Given n, B', g, φ' as in the statement of the lemma, denote $N \subset B'$ the set of elements $b \in B'$ such that $b \bmod g^nB' \in \varphi'(f^tA)$. Set $B = g^{-t}N$. As $f^tA' \subset f^tA$ and φ' sends f to g we have $g^tB' \subset N$, hence $B' \subset B$. Since $f^tA \cdot f^tA \subset f^t \cdot f^tA$ and φ' sends f to g, we see that $N \cdot N \subset g^tN$. Hence we obtain a multiplication on B extending the multiplication of B'. We have an isomorphism of A'/f^nA'-modules

$$A/f^tA' \xrightarrow{f^t} f^tA/f^nA' \xrightarrow{\varphi'} g^tB/g^nB' \xrightarrow{g^{-t}} B/g^tB'$$

where the module structures on the right are defined using φ'. Since A/f^tA' is a finite A'-module, we conclude that B/g^tB' is a finite B'-module and hence we see that $B' \to B$ is finite. Finally, we leave it to the reader to see that the displayed isomorphism of modules sends fA into gB and induces an isomorphism of rings $\varphi : A/fA \to B/gB$ compatible with φ' (it even induces an isomorphism $A/f^tA \to B/g^tB$ but we don't need this). □

0GLU **Remark 15.117.4.** The construction in Lemma 15.117.3 satisfies the following "functoriality". Suppose we have a commutative diagram

$$\begin{array}{ccc} A_2' & \longrightarrow & A_2 \\ \uparrow & & \uparrow \\ A_1' & \longrightarrow & A_1 \end{array}$$

with injective horizontal arrows. Suppose given an element $f \in A_1'$ such that $(A_1' \subset A_1, f)$ and $(A_2' \subset A_2, f)$ satisfy properties (a), (b), (c) of Lemma 15.117.3. Let $n_{0,1}$ and $n_{0,2}$ be the integers found in the lemma for these two situations. Finally, let $B_1' \to B_2'$ be a ring map, let $g \in B_1'$ be a nonzerodivisor on B_1 and B_2, let $n \geq \max(n_{0,1}, n_{0,2})$, and let a commutative diagram

$$\begin{array}{ccc} A_2'/f^nA_2' & \xrightarrow[\varphi_2']{} & B_2'/g^nB_2' \\ \uparrow & & \uparrow \\ A_1'/f^nA_1' & \xrightarrow{\varphi_1'} & B_2'/g^nB_2' \end{array}$$

be given whose horizontal arrows are isomorphisms and where $\varphi_1'(f) \equiv g$. Then we obtain commutative diagrams

$$\begin{array}{ccc} B_2' & \longrightarrow & B_2 \\ \uparrow & & \uparrow \\ B_1' & \longrightarrow & B_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} A_2/fA_2 & \xrightarrow[\varphi_2]{} & B_2/gB_2 \\ \uparrow & & \uparrow \\ A_1/fA_1 & \xrightarrow{\varphi_1} & B_2/gB_2 \end{array}$$

where $(B_1' \subset B_1, \varphi_1)$ and $(B_2' \subset B_2, \varphi_2)$ are constructed as in the proof of Lemma 15.117.3. We omit the detailed verification.

0GLV **Lemma 15.117.5.** *Let p be a prime number. Let $A \subset B$ be an extension of discrete valuation rings with fraction field extension L/K. Let $K_2/K_1/K$ be a tower of finite field extensions. Assume*

1. *K has characteristic p,*
2. *L/K is separable,*
3. *B is Nagata,*
4. *K_2 is a solution for $A \subset B$,*
5. *K_2/K_1 is purely inseparable of degree p.*

Then there exists a separable extension K_3/K_1 which is a solution for $A \subset B$.

Proof. Let us use notation as in Remark 15.115.1; we will use all the observations made there. Since L/K is separable, the algebra $L_1 = L \otimes_K K_1$ is reduced (Algebra, Lemma 10.43.6). Since B is Nagata, the ring extension $B \subset B_1$ is finite where B_1 is the integral closure of B in L_1 and B_1 is a Nagata ring. Similarly, the ring A is Nagata by Lemma 15.117.2 hence $A \subset A_1$ is finite and A_1 is a Nagata ring too. Moreover, the same assertions are true for K_2, i.e., $L_2 = L \otimes_K K_2$ is reduced, the ring extensions $A_1 \subset A_2$ and $B_1 \subset B_2$ are finite where A_2, resp. B_2 is the integral closure of A, resp. B in K_2, resp. L_2.

Let $\pi \in A$ be a uniformizer. Observe that π is a nonzerodivisor on K_1, K_2, A_1, A_2, L_1, L_2, B_1, and B_2 and we have $K_1 = (A_1)_\pi$, $K_2 = (A_2)_\pi$, $L_1 = (B_1)_\pi$, and $L_2 = (B_2)_\pi$. We may write $K_2 = K_1(\alpha)$ where $\alpha^p = a_1 \in K_1$, see Fields, Lemma 9.14.5.

After multiplying α by a power of π we may and do assume $a_1 \in A_1$. For the rest of the proof it is convenient to write $K_2 = K_1[x]/(x^p - a_1)$ and $L_2 = L_1[x]/(x^p - a_1)$. Consider the extensions of rings

$$A_2' = A_1[x]/(x^p - a_1) \subset A_2 \quad \text{and} \quad B_2' = B_1[x]/(x^p - a_1) \subset B_2$$

We may apply Lemma 15.117.3 to $A_2' \subset A_2$ and $f = \pi^2$ and to $B_2' \subset B_2$ and $f = \pi^2$. Choose an integer n large enough which works for both of these.

Consider the algebras

$$K_3 = K_1[x]/(x^p - \pi^{2n}x - a_1) \quad \text{and} \quad L_3 = L_1[x]/(x^p - \pi^{2n}x - a_1)$$

Observe that K_3/K_1 and L_3/L_1 are finite étale algebra extensions of degree p. Consider the subrings

$$A_3' = A_1[x]/(x^p - \pi^n x - a_1) \quad \text{and} \quad B_3' = B_1[x]/(x^p - \pi^n x - a_1)$$

of $K_3 = (A_2')_\pi$ and $L_3 = (B_3')_\pi$. We are going to construct a commutative diagram

$$\begin{array}{ccc} B_2'/\pi^{2n}B_2' & \xrightarrow{\psi'} & B_3'/\pi^{2n}B_3' \\ \uparrow & & \uparrow \\ A_2'/\pi^{2n}A_2' & \xrightarrow{\varphi'} & A_3'/\pi^{2n}A_3' \end{array}$$

Namely, φ' is the unique A_1-algebra isomorphism sending the class of x to the class of x. Similarly, ψ' is the unique B_1-algebra isomorphism sending the class of x to the class of x. By our choice of n we obtain, via Lemma 15.117.3 and Remark 15.117.4 finite ring extensions $A_3' \subset A_3$ and $B_3' \subset B_3$ such that $A_3' \to B_3'$ extends to a ring map $A_3 \to B_3$ and a commutative diagram

$$\begin{array}{ccc} B_2/\pi^2 B_2 & \xrightarrow{\psi} & B_3/\pi^2 B_3 \\ \uparrow & & \uparrow \\ A_2/\pi^2 A_2 & \xrightarrow{\varphi} & A_3/\pi^2 A_3 \end{array}$$

with all the properties asserted in the references mentioned above (in particular φ and ψ are isomorphisms).

With all of this data in hand, we can finish the proof. Namely, we first observe that A_3 and B_3 are finite products of Dedekind domains with π contained in all of the maximal ideals. Namely, if $\mathfrak{p} \subset A_3$ is a maximal ideal, then $\pi \in \mathfrak{p}$ as $A \to A_3$ is finite. Then $\mathfrak{p}/\pi^2 A_3$ corresponds via φ to a maximal ideal in $A_2/\pi^2 A_2$ which is principal as A_2 is a finite product of Dedekind domains. We conclude that $\mathfrak{p}/\pi^2 A_3$ is principal and hence by Nakayama we see that $\mathfrak{p}(A_3)_\mathfrak{p}$ is principal. The same argument works for B_3. We conclude that A_3 is the integral closure of A in K_3 and that B_3 is the integral closure of B in L_3. Let $\mathfrak{q} \subset B_3$ be a maximal ideal lying over $\mathfrak{p} \subset A_3$. To finish the proof we have to show that $(A_3)_\mathfrak{p} \to (B_3)_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology. By the criterion of Lemma 15.112.5 it suffices to show that $\mathfrak{p}(B_3)_\mathfrak{q} = \mathfrak{q}(B_3)_\mathfrak{q}$ and that the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is separable. This is true because we may check both assertions by looking at the ring map $A_3/\pi^2 A_3 \to B_3/\pi^2 B_3$ and this is isomorphic to the ring map $A_2/\pi^2 A_2 \to B_2/\pi^2 B_2$ where the corresponding statement holds by our assumption that K_2 is a solution for $A \subset B$. Some details omitted. $\square$

0BRN **Lemma 15.117.6.** *Let $A \subset B$ be an extension of discrete valuation rings. Assume*

(1) *the extension L/K of fraction fields is separable,*
(2) *B is Nagata, and*
(3) *there exists a solution for $A \subset B$.*

Then there exists a separable solution for $A \subset B$.

Proof. The lemma is trivial if the characteristic of K is zero; thus we may and do assume that the characteristic of K is $p > 0$.

Let K_2/K be a solution for $A \to B$. We will use induction on the inseparable degree $[K_2 : K]_i$ (Fields, Definition 9.14.7) of K_2/K. If $[K_2 : K]_i = 1$, then K_2 is separable over K and we are done. If not, then there exists a subfield $K_2/K_1/K$ such that K_2/K_1 is purely inseparable of degree p (Fields, Lemmas 9.14.6 and 9.14.5). By Lemma 15.117.5 there exists a separable extension K_3/K_1 which is a solution for $A \subset B$. Then $[K_3 : K]_i = [K_1 : K]_i = [K_2 : K]_i/p$ (Fields, Lemma 9.14.9) is smaller and we conclude by induction. □

09IH **Lemma 15.117.7.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume B is essentially of finite type over A. Let K'/K be an algebraic extension of fields such that the integral closure A' of A in K' is Noetherian. Then the integral closure B' of B in $L' = (L \otimes_K K')_{red}$ is Noetherian as well. Moreover, the map $\mathrm{Spec}(B') \to \mathrm{Spec}(A')$ is surjective and the corresponding residue field extensions are finitely generated field extensions.*

Proof. Let $A \to C$ be a finite type ring map such that B is a localization of C at a prime $\mathfrak{p}$. Then $C' = C \otimes_A A'$ is a finite type A'-algebra, in particular Noetherian. Since $A \to A'$ is integral, so is $C \to C'$. Thus $B = C_{\mathfrak{p}} \subset C'_{\mathfrak{p}}$ is integral too. It follows that the dimension of $C'_{\mathfrak{p}}$ is 1 (Algebra, Lemma 10.112.4). Of course $C'_{\mathfrak{p}}$ is Noetherian. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the minimal primes of $C'_{\mathfrak{p}}$. Let B'_i be the integral closure of $B = C_{\mathfrak{p}}$, or equivalently by the above of $C'_{\mathfrak{p}}$ in the field of fractions of $C'_{\mathfrak{p}'}/\mathfrak{q}_i$. It follows from Krull-Akizuki (Algebra, Lemma 10.119.12 applied to the finitely many localizations of $C'_{\mathfrak{p}}$ at its maximal ideals) that each B'_i is Noetherian. Moreover the residue field extensions in $C'_{\mathfrak{p}} \to B'_i$ are finite by Algebra, Lemma 10.119.10. Finally, we observe that $B' = \prod B'_i$ is the integral closure of B in $L' = (L \otimes_K K')_{red}$. □

09II **Proposition 15.117.8.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If B is essentially of finite type over A, then there exists a finite extension K_1/K which is a solution for $A \to B$ as defined in Definition 15.116.1.*

See [**dJ96**, Lemma 2.13] for a special case.

Proof. Observe that a weak solution is a solution if the residue field of A is perfect, see Lemma 15.112.5. Thus the proposition follows immediately from Theorem 15.116.18 if the residue characteristic of A is 0 (and in fact we do not need the assumption that $A \to B$ is essentially of finite type). If the residue characteristic of A is $p > 0$ we will also deduce it from Epp's theorem.

Let $x_i \in A$, $i \in I$ be a set of elements mapping to a p-base of the residue field κ of A. Set

$$A' = \bigcup\nolimits_{n \geq 1} A[t_{i,n}]/(t_{i,n}^{p^n} - x_i)$$

where the transition maps send $t_{i,n+1}$ to $t_{i,n}^p$. Observe that A' is a filtered colimit of weakly unramified finite extensions of discrete valuation rings over A. Thus A' is a discrete valuation ring and $A \to A'$ is weakly unramified. By construction the residue field $\kappa' = A'/\mathfrak{m}_A A'$ is the perfection of κ.

Let K' be the fraction field of A'. We may apply Lemma 15.117.7 to the extension K'/K. Thus B' is a finite product of Dedekind domains. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B'. Using Epp's theorem (Theorem 15.116.18) we find a weak solution K_i'/K' for each of the extensions $A' \subset B'_{\mathfrak{m}_i}$. Since the residue field of A' is perfect, these are actually solutions. Let K_1'/K' be a finite extension which contains each K_i'. Then K_1'/K' is still a solution for each $A' \subset B'_{\mathfrak{m}_i}$ by Lemma 15.117.1.

Let A_1' be the integral closure of A in K_1'. Note that A_1' is a Dedekind domain by the discussion in Remark 15.115.1 applied to $K' \subset K_1'$. Thus Lemma 15.117.7 applies to K_1'/K. Therefore the integral closure B_1' of B in $L_1' = (L \otimes_K K_1')_{red}$ is a Dedekind domain and because K_1'/K' is a solution for each $A' \subset B'_{\mathfrak{m}_i}$ we see that $(A_1')_{A_1' \cap \mathfrak{m}} \to (B_1')_{\mathfrak{m}}$ is formally smooth in the $\mathfrak{m}$-adic topology for each maximal ideal $\mathfrak{m} \subset B_1'$.

By construction, the field K_1' is a filtered colimit of finite extensions of K. Say $K_1' = \operatorname{colim}_{i \in I} K_i$. For each i let A_i, resp. B_i be the integral closure of A, resp. B in K_i, resp. $L_i = (L \otimes_K K_i)_{red}$. Then it is clear that

$$A_1' = \operatorname{colim} A_i \quad \text{and} \quad B_1' = \operatorname{colim} B_i$$

Since the ring maps $A_i \to A_1'$ and $B_i \to B_1'$ are injective integral ring maps and since A_1' and B_1' have finite spectra, we see that for all i large enough the ring maps $A_i \to A_1'$ and $B_i \to B_1'$ are bijective on spectra. Once this is true, for all i large enough the maps $A_i \to A_1'$ and $B_i \to B_1'$ will be weakly unramified (once the uniformizer is in the image). It follows from multiplicativity of ramification indices that $A_i \to B_i$ induces weakly unramified maps on all localizations at maximal ideals of B_i for such i. Increasing i a bit more we see that

$$B_i \otimes_{A_i} A_1' \longrightarrow B_1'$$

induces surjective maps on residue fields (because the residue fields of B_1' are finitely generated over those of A_1' by Lemma 15.117.7). Picture of residue fields at maximal ideals lying under a chosen maximal ideal of B_1':

$$\begin{array}{ccccccc} \kappa_{B_i} & \longrightarrow & \kappa_{B_{i'}} & \longrightarrow & \ldots & & \kappa_{B_1'} \\ \uparrow & & \uparrow & & & & \uparrow \\ \kappa_{A_i} & \longrightarrow & \kappa_{A_{i'}} & \longrightarrow & \ldots & & \kappa_{A_1'} \end{array}$$

Thus κ_{B_i} is a finitely generated extension of κ_{A_i} such that the compositum of κ_{B_i} and $\kappa_{A_1'}$ in $\kappa_{B_1'}$ is separable over $\kappa_{A_1'}$. Then that happens already at a finite stage: for example, say $\kappa_{B_1'}$ is finite separable over $\kappa_{A_1'}(x_1, \ldots, x_n)$, then just increase i such that $x_1, \ldots, x_n$ are in κ_{B_i} and such that all generators satisfy separable polynomial equations over $\kappa_{A_i}(x_1, \ldots, x_n)$. This means that $A_i \to (B_i)_{\mathfrak{m}}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_i and the proof is complete. $\square$

0BRP **Lemma 15.117.9.** *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume*

(1) *B is essentially of finite type over A,*
(2) *either A or B is a Nagata ring, and*
(3) *L/K is separable.*

Then there exists a separable solution for $A \to B$ (Definition 15.116.1).

Proof. Observe that if A is Nagata, then so is B (Algebra, Lemma 10.162.6 and Proposition 10.162.15). Thus the lemma follows on combining Proposition 15.117.8 and Lemma 15.117.6. □

15.118. Picard groups of rings

0AFW We first define invertible modules as follows.

0B8H **Definition 15.118.1.** Let R be a ring. An R-module M is *invertible* if the functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R, \quad N \longmapsto M \otimes_R N$$

is an equivalence of categories. An invertible R-module is said to be *trivial* if it is isomorphic to R as an R-module.

0B8I **Lemma 15.118.2.** *Let R be a ring. Let M be an R-module. Equivalent are*

(1) *M is finite locally free module of rank 1,*
(2) *M is invertible, and*
(3) *there exists an R-module N such that $M \otimes_R N \cong R$.*

Moreover, in this case the module N in (3) is isomorphic to $\text{Hom}_R(M, R)$.

Proof. Assume (1). Consider the module $N = \text{Hom}_R(M, R)$ and the evaluation map $M \otimes_R N = M \otimes_R \text{Hom}_R(M, R) \to R$. If $f \in R$ such that $M_f \cong R_f$, then the evaluation map becomes an isomorphism after localization at f (details omitted). Thus we see the evaluation map is an isomorphism by Algebra, Lemma 10.23.2. Thus (1) $\Rightarrow$ (3).

Assume (3). Then the functor $K \mapsto K \otimes_R N$ is a quasi-inverse to the functor $K \mapsto K \otimes_R M$. Thus (3) $\Rightarrow$ (2). Conversely, if (2) holds, then $K \mapsto K \otimes_R M$ is essentially surjective and we see that (3) holds.

Assume the equivalent conditions (2) and (3) hold. Denote $\psi : M \otimes_R N \to R$ the isomorphism from (3). Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$M \to M \otimes_R M \otimes_R N \to M$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of M. This automorphism factors as

$$M \to R^{\oplus n} \to M$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that M is a direct summand of a finite free R-module. This means that M is finite locally free (Algebra, Lemma 10.78.2). Since the same is true for N by symmetry and since $M \otimes_R N \cong R$, we see that M and N both have to have rank 1. □

The set of isomorphism classes of these modules is often called the *class group* or *Picard group* of R. The group structure is determined by assigning to the isomorphism classes of the invertible modules L and L' the isomorphism class of $L \otimes_R L'$. The inverse of an invertible module L is the module

$$L^{\otimes -1} = \mathrm{Hom}_R(L, R),$$

because as seen in the proof of Lemma 15.118.2 the evaluation map $L \otimes_R L^{\otimes -1} \to R$ is an isomorphism. Let us denote the Picard group of R by $\mathrm{Pic}(R)$.

0BCH **Lemma 15.118.3.** *Let R be a UFD. Then $\mathrm{Pic}(R)$ is trivial.*

Proof. Let L be an invertible R-module. By Lemma 15.118.2 we see that L is a finite locally free R-module. In particular L is torsion free and finite over R. Pick a nonzero element $\varphi \in \mathrm{Hom}_R(L, R)$ of the dual invertible module. Then $I = \varphi(L) \subset R$ is an ideal which is an invertible module. Pick a nonzero $f \in I$ and let

$$f = up_1^{e_1} \ldots p_r^{e_r}$$

be the factorization into prime elements with p_i pairwise distinct. Since L is finite locally free there exist $a_i \in R$, $a_i \notin (p_i)$ such that $I_{a_i} = (g_i)$ for some $g_i \in R_{a_i}$. Then p_i is still a prime element of the UFD R_{a_i} and we can write $g_i = p_i^{c_i} g_i'$ for some $g_i' \in R_{a_i}$ not divisible by p_i. Since $f \in I_{a_i}$ we see that $e_i \geq c_i$. We claim that I is generated by $h = p_1^{c_1} \ldots p_r^{c_r}$ which finishes the proof.

To prove the claim it suffices to show that I_a is generated by h for any $a \in R$ such that I_a is a principal ideal (Algebra, Lemma 10.23.2). Say $I_a = (g)$. Let $J \subset \{1, \ldots, r\}$ be the set of i such that p_i is a nonunit (and hence a prime element) in R_a. Because $f \in I_a = (g)$ we find the prime factorization $g = v \prod_{i \in J} p_j^{b_j}$ with v a unit and $b_j \leq e_j$. For each $j \in J$ we have $I_{aa_j} = gR_{aa_j} = g_j R_{aa_j}$, in other words g and g_j map to associates in R_{aa_j}. By uniqueness of factorization this implies that $b_j = c_j$ and the proof is complete. □

15.119. Determinants

0FJ9 Let R be a ring. Let M be a finite projective R-module. There exists a product decomposition $R = R_0 \times \ldots \times R_t$ such that in the corresponding decomposition $M = M_0 \times \ldots \times M_t$ of M we have that M_i is finite locally free of rank i over R_i. This follows from Algebra, Lemma 10.78.2 (to see that the rank is locally constant) and Algebra, Lemmas 10.21.3 and 10.24.3 (to decompose R into a product). In this situation we define

$$\det(M) = \wedge^0_{R_0}(M_0) \times \ldots \times \wedge^t_{R_t}(M_t)$$

as an R-module. This is a finite locally free module of rank 1 as each term is finite locally free of rank 1. If $\varphi : M \to N$ is an isomorphism of finite projective R-modules, then we obtain a canonical isomorphism

$$\det(\varphi) : \det(M) \longrightarrow \det(N)$$

of locally free modules of rank 1. More generally, if for all primes $\mathfrak{p}$ of R the ranks of the free modules $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are the same, then any R-module homomorphism $\varphi : M \to N$ induces an R-module map $\det(\varphi) : \det(M) \to \det(N)$. Finally, if

$M = N$ then $\det(\varphi) : \det(M) \to \det(M)$ is an endomorphism of an invertible R-module. Since $R = \mathrm{Hom}_R(L, L)$ for an invertible R-module we may and do view $\det(\varphi)$ as an element of R. In this way we obtain the *determinant*

$$\det : \mathrm{Hom}_R(M, M) \longrightarrow R$$

which is a multiplicative map.

0FJA **Remark 15.119.1.** Let R be a ring. Let M be a finite projective R-module. Then we can consider the graded commutative R-algebra exterior algebra $\wedge_R^*(M)$ on M over R. A formula for $\det(M)$ is that $\det(M) \subset \wedge_R^*(M)$ is the annihilator of $M \subset \wedge_R^*(M)$. This is sometimes useful as it does not refer to the decomposition of R into a product. Of course, to prove this satisfies the desired properties one has to either decompose R into a product (as above), or one has to look at the localizations at primes of R.

Next, we consider what happens to the determinant give a short exact sequence of finite projective modules.

0FJB **Lemma 15.119.2.** *Let R be a ring. Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finite projective R-modules. Then there is a canonical isomorphism

$$\gamma : \det(M') \otimes \det(M'') \longrightarrow \det(M)$$

First proof. First proof. Decompose R into a product of rings R_{ij} such that $M' = \prod M'_{ij}$ and $M'' = \prod M''_{ij}$ where M'_{ij} has rank i and M''_{ij} has rank j. Of course then $M = \prod M_{ij}$ and M_{ij} has rank $i + j$. This reduces us to the case where M' and M'' have constant rank say i and j. In this case we have to construct a canonical map

$$\wedge^i(M') \otimes \wedge^j(M'') \longrightarrow \wedge^{i+j}(M)$$

To do this choose $m'_1, \ldots, m'_i$ in M' and $m''_1, \ldots, m''_j$ in M''. Denote $m_1, \ldots, m_i \in M$ the images of $m'_1, \ldots, m'_i$ and denote $m_{i+1}, \ldots, m_{i+j} \in M$ elements mapping to $m''_1, \ldots, m''_j$ in M''. Our rule will be that

$$m'_1 \wedge \ldots \wedge m'_i \otimes m''_1 \wedge \ldots \wedge m''_j \longmapsto m_1 \wedge \ldots \wedge m_{i+j}$$

We omit the detailed proof that this is well defined and an isomorphism. □

Second proof. We will use the description of $\det(M)$, $\det(M')$, and $\det(M'')$ given in Remark 15.119.1. Consider the R-algebra maps $\wedge_R^*(M') \to \wedge_R^*(M)$ and $\wedge_R^*(M) \to \wedge_R^*(M'')$. The first is injective and the second is surjective. Take an element $x' \in \det(M') \subset \wedge_R^*(M')$ and an element $x'' \in \det(M'') \subset \wedge_R^*(M'')$. Choose an element $y'' \in \wedge^*(M)$ mapping to x'' and set

$$\gamma(x' \otimes x'') = x' \wedge y'' \in \det(M) \subset \wedge_R^*(M)$$

The reader verifies easily by looking at localizations at primes that this well defined and an isomorphism. Moreover, this construction gives the same map as the construction given in the first proof. □

0FJC **Lemma 15.119.3.** *Let R be a ring. Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of finite projective R-modules whose vertical arrows are isomorphisms. Then we get a commutative diagram of isomorphisms

$$\begin{array}{ccc} \det(M') \otimes \det(M'') & \xrightarrow{\gamma} & \det(M) \\ {\scriptstyle \det(u)\otimes\det(w)}\downarrow & & \downarrow{\scriptstyle \det(v)} \\ \det(K') \otimes \det(K'') & \xrightarrow{\gamma} & \det(K) \end{array}$$

where the horizontal arrows are the ones constructed in Lemma 15.119.2.

Proof. Omitted. Hint: use the second construction of the maps γ in Lemma 15.119.2. □

0FJD **Lemma 15.119.4.** *Let R be a ring. Let*

$$K \subset L \subset M$$

be R-modules such that K, L/K, and M/L are finite projective R-modules. Then the diagram

$$\begin{array}{ccc} \det(K) \otimes \det(L/K) \otimes \det(M/L) & \longrightarrow & \det(L) \otimes \det(M/L) \\ \downarrow & & \downarrow \\ \det(K) \otimes \det(M/K) & \longrightarrow & \det(M) \end{array}$$

commutes where the maps are those of Lemma 15.119.2.

Proof. Omitted. Hint: after localizing at a prime of R we can assume $K \subset L \subset M$ is isomorphic to $R^{\oplus a} \subset R^{\oplus a+b} \subset R^{\oplus a+b+c}$ and in this case the result is an evident computation. □

0FJE **Lemma 15.119.5.** *Let R be a ring. Let M' and M'' be two finite projective R-modules. Then the diagram*

$$\begin{array}{ccc} \det(M') \otimes \det(M'') & \longrightarrow & \det(M' \oplus M'') \\ {\scriptstyle \epsilon\cdot(\mathit{switch\ tensors})}\downarrow & & \downarrow{\scriptstyle \det(\mathit{switch\ summands})} \\ \det(M'') \otimes \det(M') & \longrightarrow & \det(M'' \oplus M') \end{array}$$

commutes where $\epsilon = \det(-id_{M' \otimes M''}) \in R^*$ and the horizontal arrows are those of Lemma 15.119.2.

Proof. Omitted. □

0FJF **Lemma 15.119.6.** *Let R be a ring. Let M, N be finite projective R-modules. Let $a : M \to N$ and $b : N \to M$ be R-linear maps. Then*

$$\det(id + a \circ b) = \det(id + b \circ a)$$

as elements of R.

Proof. It suffices to prove the assertion after replacing R by a localization at a prime ideal. Thus we may assume R is local and M and N are finite free. In this case we have to prove the equality

$$\det(I_n + AB) = \det(I_m + BA)$$

of usual determinants of matrices where A has size $n \times m$ and B has size $m \times n$. This reduces to the case of the ring $R = \mathbf{Z}[a_{ij}, b_{ji}; 1 \leq i \leq n, 1 \leq j \leq m]$ where a_{ij} and b_{ij} are variables and the entries of the matrices A and B. Taking the fraction field, this reduces to the case of a field of characteristic zero. In characteristic zero there is a universal polynomial expressing the determinant of a matrix of size $\leq N$ in the traces of the powers of said matrix. Hence it suffices to prove

$$\text{Trace}((I_n + AB)^k) = \text{Trace}((I_m + BA)^k)$$

for all $k \geq 1$. Expanding we see that it suffices to prove $\text{Trace}((AB)^k) = \text{Trace}((BA)^k)$ for all $k \geq 0$. For $k = 1$ this is the well known fact that $\text{Trace}(AB) = \text{Trace}(BA)$. For $k > 1$ it follows from this by writing $(AB)^k = A(BA)^{k-1}B$ and $(BA)^k = (BA)^{k-1}AB$. □

Recall that we have defined in Algebra, Section 10.55 a group $K_0(R)$ as the free group on isomorphism classes of finite projective R-modules modulo the relations $[M'] + [M''] = [M' \oplus M'']$.

0AFX **Lemma 15.119.7.** *Let R be a ring. There is a map*

$$\det : K_0(R) \longrightarrow \text{Pic}(R)$$

which maps $[M]$ to the class of the invertible module $\wedge^n(M)$ if M is a finite locally free module of rank n.

Proof. This follows immediately from the constructions above and in particular Lemma 15.119.2 to see that the relations are mapped to 0. □

15.120. Perfect complexes and K-groups

0FJG We quickly show that the zeroth K-group of the derived category of perfect complexes of a ring R is the same as $K_0(R)$ defined in Algebra, Section 10.55.

0AFY **Lemma 15.120.1.** *Let R be a ring. There is a map*

$$c : \textit{perfect complexes over } R \longrightarrow K_0(R)$$

with the following properties

1. *$c(K[n]) = (-1)^n c(K)$ for a perfect complex K,*
2. *if $K \to L \to M \to K[1]$ is a distinguished triangle of perfect complexes, then $c(L) = c(K) + c(M)$,*
3. *if K is represented by a finite complex $M^\bullet$ consisting of finite projective modules, then $c(K) = \sum(-1)^i[M_i]$.*

Proof. Let K be a perfect object of $D(R)$. By definition we can represent K by a finite complex $M^\bullet$ of finite projective R-modules. We define c by setting

$$c(K) = \sum(-1)^n[M^n]$$

in $K_0(R)$. Of course we have to show that this is well defined, but once it is well defined, then (1) and (3) are immediate. For the moment we view the map c as defined on complexes of finite projective R-modules.

Suppose that $L^\bullet \to M^\bullet$ is a surjective map of finite complexes of finite projective R-modules. Let $K^\bullet$ be the kernel. Then we obtain short exact sequences of R-modules

$$0 \to K^n \to L^n \to M^n \to 0$$

which are split because M^n is projective. Hence $K^\bullet$ is also a finite complex of finite projective R-modules and $c(L^\bullet) = c(K^\bullet) + c(M^\bullet)$ in $K_0(R)$.

Suppose given finite complex $M^\bullet$ of finite projective R-modules which is acyclic. Say $M^n = 0$ for $n \notin [a, b]$. Then we can break $M^\bullet$ into short exact sequences

$$\begin{aligned} 0 \to M^a \to M^{a+1} \to N^{a+1} \to 0, \\ 0 \to N^{a+1} \to M^{a+2} \to N^{a+3} \to 0, \\ \ldots \\ 0 \to N^{b-3} \to M^{b-2} \to N^{b-2} \to 0, \\ 0 \to N^{b-2} \to M^{b-1} \to M^b \to 0 \end{aligned}$$

Arguing by descending induction we see that $N^{b-2}, \ldots, N^{a+1}$ are finite projective R-modules, the sequences are split exact, and

$$c(M^\bullet) = \sum (-1)[M^n] = \sum (-1)^n([N^{n-1}] + [N^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows formally from the results of the preceding two paragraphs that c is well defined and satisfies (2). Namely, suppose the finite complexes $M^\bullet$ and $L^\bullet$ of finite projective R-modules represent the same object of $D(R)$. Then we can represent the isomorphism by a map $f : M^\bullet \to L^\bullet$ of complexes, see Derived Categories, Lemma 13.19.8. We obtain a short exact sequence of complexes

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

see Derived Categories, Definition 13.9.1. Since f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 13.12). Hence

$$0 = c(C(f)^\bullet) = c(L^\bullet) + c(K^\bullet[1]) = c(L^\bullet) - c(K^\bullet)$$

as desired. We omit the proof of (2) which is similar. □

The following lemma shows that $K_0(R)$ is equal to $K_0(D_{perf}(R))$.

0FCU **Lemma 15.120.2.** *Let R be a ring. Let $D_{perf}(R)$ be the derived category of perfect objects, see Lemma 15.79.1. The map c of Lemma 15.120.1 gives an isomorphism $K_0(D_{perf}(R)) = K_0(R)$.*

Proof. It follows from the definition of $K_0(D_{perf}(R))$ (Derived Categories, Definition 13.28.1) that c induces a homomorphism $K_0(D_{perf}(R)) \to K_0(R)$.

Given a finite projective module M over R let us denote $M[0]$ the perfect complex over R which has M sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to M \to M' \to M'' \to 0$ of finite projective modules we obtain a distinguished triangle $M[0] \to M'[0] \to M''[0] \to M[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(R) \to K_0(D_{perf}(R))$ by sending $[M]$ to $[M[0]]$ with apologies for the horrendous notation.

It is clear that $K_0(R) \to K_0(D_{perf}(R)) \to K_0(R)$ is the identity. On the other hand, if $M^\bullet$ is a bounded complex of finite projective R-modules, then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n} M^\bullet \to \sigma_{\geq n-1} M^\bullet \to M^{n-1}[-n+1] \to (\sigma_{\geq n} M^\bullet)[1]$$

and induction show that

$$[M^\bullet] = \sum (-1)^i [M^i[0]]$$

in $K_0(D_{perf}(R))$ (with again apologies for the notation). Hence the map $K_0(R) \to K_0(D_{perf}(R))$ is surjective which finishes the proof. □

15.121. Determinants of endomorphisms of finite length modules

0GSX Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Consider the category of pairs (M, φ) consisting of a finite length R-module and an endomorphism $\varphi : M \to M$. This category is abelian and every object is Artinian as well as Noetherian. See Homology, Section 12.9 for definitions.

If (M, φ) is a simple object of this category, then M is annihilated by $\mathfrak{m}$ since otherwise $(\mathfrak{m}M, \varphi|_{\mathfrak{m}M})$ would be a nontrivial subobject. Also $\dim_\kappa(M) = \text{length}_R(M)$ is finite. Thus we may define the determinant and the trace

$$\det_\kappa(\varphi), \quad \text{Trace}_\kappa(\varphi)$$

as elements of κ using linear algebra. Similarly for the characteristic polynomial of φ in this case.

By Homology, Lemma 12.9.6 for an arbitrary object (M, φ) of our category we have a finite filtration

$$0 \subset M_1 \subset \ldots \subset M_n = M$$

by submodules stable under φ such that $(M_i/M_{i-1}, \varphi_i)$ is a simple object of the category where $\varphi_i : M_i/M_{i-1} \to M_i/M_{i-1}$ is the induced map. We define the *determinant* of (M, φ) over κ as

$$\det_\kappa(\varphi) = \prod \det_\kappa(\varphi_i)$$

with $\det_\kappa(\varphi_i)$ as defined in the previous paragraph. We define the *trace* of (M, φ) over κ as

$$\text{Trace}_\kappa(\varphi) = \sum \text{Trace}_\kappa(\varphi_i)$$

with $\text{Trace}_\kappa(\varphi_i)$ as defined in the previous paragraph. We can similarly define the characteristic polynomial of φ over κ as the product of the characteristic polynomials of φ_i as defined in the previous paragraph. By Jordan-Hölder (Homology, Lemma 12.9.7) this is well defined.

0GSY **Lemma 15.121.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $0 \to (M, \varphi) \to (M', \varphi') \to (M'', \varphi'') \to 0$ be a short exact sequence in the category discussed above. Then*

$$\det_\kappa(\varphi') = \det_\kappa(\varphi) \det_\kappa(\varphi''), \quad \textit{Trace}_\kappa(\varphi') = \textit{Trace}_\kappa(\varphi) + \textit{Trace}_\kappa(\varphi'')$$

Also, the characteristic polynomial of φ' over κ is the product of the characteristic polynomials of φ and φ''.

Proof. Left as an exercise. □

0GSZ **Lemma 15.121.2.** *Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a local homomorphism of local rings. Assume that κ'/κ is a finite extension. Let $u \in R'$. Then for any finite length R'-module M' we have*

$$\det_\kappa(u : M' \to M') = Norm_{\kappa'/\kappa}(u \bmod \mathfrak{m}')^m$$

where $m = length_{R'}(M')$.

Proof. Observe that the statement makes sense as $\text{length}_R(M') = \text{length}_{R'}(M')[\kappa' : \kappa]$. If $M' = \kappa'$, then the equality holds by definition of the norm as the determinant of the linear operator given by multiplication by u. In general one reduces to this case by choosing a suitable filtration and using the multiplicativity of Lemma 15.121.1. Some details omitted. □

0GT0 **Lemma 15.121.3.** *Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a flat local homomorphism of local rings such that $m = length_{R'}(R'/\mathfrak{m}R') < \infty$. For any (M, φ) as above, the element $\det_\kappa(\varphi)^m$ maps to $\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R')$ in κ'.*

Proof. The flatness of $R \to R'$ assures us that short exact sequences as in Lemma 15.121.1 base change to short exact sequences over R'. Hence by the multiplicativity of Lemma 15.121.1 we may assume that (M, φ) is a simple object of our category (see introduction to this section). In the simple case M is annihilated by $\mathfrak{m}$. Choose a filtration

$$0 \subset I_1 \subset I_2 \subset \ldots \subset I_{m-1} \subset R'/\mathfrak{m}R'$$

whose successive quotients are isomorphic to κ' as R'-modules. Then we obtain the filtration

$$0 \subset M \otimes_\kappa I_1 \subset M \otimes_\kappa I_2 \subset \ldots \subset M \otimes_\kappa I_{m-1} \subset M \otimes_\kappa R'/\mathfrak{m}R' = M \otimes_R R'$$

whose successive quotients are isomorphic to $M \otimes_\kappa \kappa'$. Also, these submodules are invariant under $\varphi \otimes 1$. By Lemma 15.121.1 we find

$$\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R') = \det_{\kappa'}(\varphi \otimes 1 : M \otimes_\kappa \kappa' \to M \otimes_\kappa \kappa')^m = \det_\kappa(\varphi)^m$$

The last equality holds by the compatibility of determinants of linear maps with field extensions. This proves the lemma. □

15.122. A regular local ring is a UFD

0FJH We prove the result mentioned in the section title.

0AFZ **Lemma 15.122.1.** *Let R be a regular local ring. Let $f \in R$. Then $\text{Pic}(R_f) = 0$.*

Proof. Let L be an invertible R_f-module. In particular L is a finite R_f-module. There exists a finite R-module M such that $M_f \cong L$, see Algebra, Lemma 10.126.3. By Algebra, Proposition 10.110.1 we see that M has a finite free resolution $F_\bullet$ over R. It follows that L is quasi-isomorphic to a finite complex of *free* R_f-modules. Hence by Lemma 15.120.1 we see that $[L_f] = n[R_f]$ in $K_0(R_f)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 15.119.7 we see that L is trivial. □

0AG0 **Lemma 15.122.2.** *A regular local ring is a UFD.*

Proof. Recall that a regular local ring is a domain, see Algebra, Lemma 10.106.2. We will prove the unique factorization property by induction on the dimension of the regular local ring R. If $\dim(R) = 0$, then R is a field and in particular a UFD. Assume $\dim(R) > 0$. Let $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. Then $R/(x)$ is regular by Algebra, Lemma 10.106.3, hence a domain by Algebra, Lemma 10.106.2, hence x is a prime

element. Let $\mathfrak{p} \subset R$ be a height 1 prime. We have to show that $\mathfrak{p}$ is principal, see Algebra, Lemma 10.120.6. We may assume $x \notin \mathfrak{p}$, since if $x \in \mathfrak{p}$, then $\mathfrak{p} = (x)$ and we are done. For every nonmaximal prime $\mathfrak{q} \subset R$ the local ring $R_\mathfrak{q}$ is a regular local ring, see Algebra, Lemma 10.110.6. By induction we see that $\mathfrak{p}R_\mathfrak{q}$ is principal. In particular, the R_x-module $\mathfrak{p}_x = \mathfrak{p}R_x \subset R_x$ is a finitely presented R_x-module whose localization at any prime is free of rank 1. By Algebra, Lemma 10.78.2 we see that $\mathfrak{p}_x$ is an invertible R_x-module. By Lemma 15.122.1 we see that $\mathfrak{p}_x = (y)$ for some $y \in R_x$. We can write $y = x^e f$ for some $f \in \mathfrak{p}$ and $e \in \mathbf{Z}$. Factor $f = a_1 \ldots a_r$ into irreducible elements of R (Algebra, Lemma 10.120.3). Since $\mathfrak{p}$ is prime, we see that $a_i \in \mathfrak{p}$ for some i. Since $\mathfrak{p}_x = (y)$ is prime and $a_i | y$ in R_x, it follows that $\mathfrak{p}_x$ is generated by a_i in R_x, i.e., the image of a_i in R_x is prime. As x is a prime element, we find that a_i is prime in R by Algebra, Lemma 10.120.7. Since $(a_i) \subset \mathfrak{p}$ and $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$ as desired. □

0DLQ **Lemma 15.122.3.** *Let R be a valuation ring with fraction field K and residue field κ. Let $R \to A$ be a homomorphism of rings such that*

(1) *A is local and $R \to A$ is local,*
(2) *A is flat and essentially of finite type over R,*
(3) *$A \otimes_R \kappa$ regular.*

Then $\mathrm{Pic}(A \otimes_R K) = 0$.

Proof. Let L be an invertible $A \otimes_R K$-module. In particular L is a finite module. There exists a finite A-module M such that $M \otimes_R K \cong L$, see Algebra, Lemma 10.126.3. We may assume M is torsion free as an R-module. Thus M is flat as an R-module (Lemma 15.22.10). From Lemma 15.25.6 we deduce that M is of finite presentation as an A-module and A is essentially of finite presentation as an R-algebra. By Lemma 15.84.4 we see that M is perfect relative to R, in particular M is pseudo-coherent as an A-module. By Lemma 15.78.6 we see that M is perfect, hence M has a finite free resolution $F_\bullet$ over A. It follows that L is quasi-isomorphic to a finite complex of *free* $A \otimes_R K$-modules. Hence by Lemma 15.120.1 we see that $[L] = n[A \otimes_R K]$ in $K_0(A \otimes_R K)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 15.119.7 we see that L is trivial. □

15.123. Determinants of complexes

0FJI In Section 15.120 we have seen how to a perfect complex K over a ring R there is associated an isomorphism class of invertible R-modules, i.e., an element of $\mathrm{Pic}(R)$. In fact, analogously to Section 15.119 it turns out there is a functor

$$\det : \left\{\begin{matrix}\text{category of perfect complexes} \\ \text{morphisms are isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category of invertible modules} \\ \text{morphisms are isomorphisms}\end{matrix}\right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(R)$ of R whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\mathrm{gr}L) \to \det(L)$. See [**KM76**] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case of perfect objects L in $D(R)$ of tor-amplitude in $[-1, 0]$. Such an object may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules, see Lemma 15.75.2. In this case we set

$$\det(L^\bullet) = \det(L^0) \otimes_R \det(L^{-1})^{\otimes -1} = \mathrm{Hom}_R(\det(L^{-1}), \det(L^0))$$

Let us say a complex of this form has *rank* 0 if $L^{-1}_\mathfrak{p}$ and $L^0_\mathfrak{p}$ have the same rank for all primes of R. If $L^\bullet$ has rank 0, then we have seen in Section 15.119 that there is a canonical element

$$\delta(L^\bullet) \in \det(L^\bullet)$$

which is simply the determininant of $d : L^{-1} \to L^0$. Note that $\delta(L^\bullet)$ is a trivialization of $\det(L^\bullet)$ if and only if $L^\bullet$ is acyclic.

Consider a map of complexes $a^\bullet : K^\bullet \to L^\bullet$ such that

(1) $a^\bullet$ is a quasi-isomorphism,
(2) $a^n : K^n \to L^n$ is surjective for all n,
(3) K^n, L^n are finite projective R-modules, nonzero only for $n \in \{-1, 0\}$.

In this situation we will construct an isomorphism

$$\det(a^\bullet) : \det(K^\bullet) \longrightarrow \det(L^\bullet)$$

Using the exact sequences $0 \to \mathrm{Ker}(a^i) \to K^i \to L^i \to 0$ we obtain isomorphisms

$$\gamma^i : \det(\mathrm{Ker}(a^i)) \otimes \det(L^i) \to \det(K^i)$$

for $i = -1, 0$ by Lemma 15.119.2. Since $a^\bullet$ is a quasi-isomorphism the complex $\mathrm{Ker}(a^\bullet)$ is acyclic and has rank 0. Hence the canonical element $\delta(\mathrm{Ker}(a^\bullet))$ is a trivialization of the invertible R-module $\det(\mathrm{Ker}(a^\bullet))$, see above. We define $\det(a^\bullet) : \det(K^\bullet) \to \det(L^\bullet)$ as the unique isomorphism such that the diagram

$$\begin{array}{ccc} \det(K^\bullet) & \xrightarrow{\det(a^\bullet)} & \det(L^\bullet) \\ & {\scriptstyle \gamma^0 \otimes (\gamma^{-1})^{\otimes -1}} \nwarrow \quad \swarrow {\scriptstyle \delta(\mathrm{Ker}(a^\bullet))} & \\ & \det(L^\bullet) \otimes \det(\mathrm{Ker}(a^\bullet)) & \end{array}$$

commutes.

0FJJ **Lemma 15.123.1.** *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. If $L^\bullet$ has rank 0, then $\det(a^\bullet)$ maps the canonical element $\delta(K^\bullet)$ to $\delta(L^\bullet)$.*

Proof. Write $M^i = \mathrm{Ker}(a^i)$. Thus we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{-1} & \longrightarrow & K^{-1} & \longrightarrow & L^{-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_K} & & \downarrow{\scriptstyle d_L} & & \\ 0 & \longrightarrow & M^0 & \longrightarrow & K^0 & \longrightarrow & L^0 & \longrightarrow & 0 \end{array}$$

By Lemma 15.119.3 we know that $\det(d_K)$ corresponds to $\det(d_M) \otimes \det(d_L)$ as maps. Unwinding the definitions this gives the required equality. □

0FJK **Lemma 15.123.2.** *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. Let $h : K^0 \to L^{-1}$ be a map such that $b^0 = a^0 + d \circ h$ and $b^{-1} = a^{-1} + h \circ d$ are surjective. Then $\det(a^\bullet) = \det(b^\bullet)$ as maps $\det(K^\bullet) \to \det(L^\bullet)$.*

Proof. Suppose there exists a map $\tilde{h} : K^0 \to K^{-1}$ such that $h = a^{-1} \circ \tilde{h}$ and such that $k^0 = \mathrm{id} + d \circ \tilde{h} : K^0 \to K^0$ and $k^1 = \mathrm{id} + \tilde{h} \circ d : K^{-1} \to K^{-1}$ are isomorphisms. Then we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(b^\bullet) & \longrightarrow & K^\bullet & \xrightarrow[b^\bullet]{} & L^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c^\bullet} & & \downarrow{\scriptstyle k^\bullet} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{Ker}(a^\bullet) & \longrightarrow & K^\bullet & \xrightarrow{a^\bullet} & L^\bullet & \longrightarrow & 0 \end{array}$$

of complexes, where $c^\bullet$ is the induced isomorphism of kernels. Using Lemma 15.119.3 we see that

$$\begin{array}{ccc} \det(\mathrm{Ker}(b^i)) \otimes \det(L^i) & \longrightarrow & \det(K^i) \\ \downarrow{\scriptstyle \det(c^i)\otimes 1} & & \downarrow{\scriptstyle \det(k^i)} \\ \det(\mathrm{Ker}(a^i)) \otimes \det(L^i) & \longrightarrow & \det(K^i) \end{array}$$

commutes. Since $\det(c^\bullet)$ maps the canonical trivialization of $\det(\mathrm{Ker}(a^\bullet))$ to the canonical trivializatio of $\mathrm{Ker}(b^\bullet)$ (Lemma 15.123.1) we see that we conclude if (and only if)

$$\det(k^0) = \det(k^{-1})$$

as elements of R which follows from Lemma 15.119.6.

Suppose there exists a direct summand $U \subset K^{-1}$ such that both $a^{-1}|_U : U \to L^{-1}$ and $b^{-1}|_U : U \to L^{-1}$ are isomorphisms. Define $\tilde{h}$ as the composition of h with the inverse of $a^{-1}|_U$. We claim that $\tilde{h}$ is a map as in the first paragraph of the proof. Namely, we have $h = a^{-1} \circ \tilde{h}$ by construction. To show that $k^{-1} : K^{-1} \to K^{-1}$ is an isomorphism it suffices to show that it is surjective (Algebra, Lemma 10.16.4). Let $u \in U$. We may choose $u' \in U$ such that $b^{-1}(u') = a^{-1}(u)$. Then $u = k^{-1}(u')$. Namely, both u and $k^{-1}(u')$ are in U and $a^{-1}(u) = a^{-1}(k^{-1}(u'))$ by a calculation[20] Since $a^{-1}|_U$ is an isomorphism we get the equality. Thus $U \subset \mathrm{Im}(k^{-1})$. On the other hand, if $x \in \mathrm{Ker}(a^{-1})$ then $x = k^{-1}(x) \bmod U$. Since $K^{-1} = \mathrm{Ker}(a^{-1}) + U$ we conclude k^{-1} is surjective. Finally, we show that $k^0 : K^0 \to K^0$ is surjective. First, since $a^0 \circ k^0 = b^0$ we see that $a^0 \circ k^0$ is surjective. If $x \in \mathrm{Ker}(a^0)$, then $x = d(y)$ for some $y \in \mathrm{Ker}(a^{-1})$. We may write $y = k^{-1}(z)$ for some $z \in K^{-1}$ by the above. Then $x = k^0(d(z))$ and we conclude.

Final step of the proof. It suffices to find U as in the preceding paragraph, but this may not always be possible. However, in order to show equality of two maps of R-modules, it suffices to do so after localization at primes of R. Hence we may assume R is local. Then we get the following problem: suppose

$$\alpha, \beta : R^{\oplus n} \longrightarrow R^{\oplus m}$$

are two surjective R-linear maps. Find a direct summand $U \subset R^{\oplus n}$ such that both $\alpha|_U$ and $\beta|_U$ are isomorphisms. If R is a field, this is possible by linear algebra. In general, one takes a solution over the residue field and lifts this to a solution over the local ring R. Some details omitted. $\square$

[20] $a^{-1}(k^{-1}(u')) = a^{-1}(u') + a^{-1}(\tilde{h}(d(u'))) = a^{-1}(u') + h(d(u')) = b^{-1}(u') = a^{-1}(u)$

0FJL **Lemma 15.123.3.** *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ and $b^\bullet : L^\bullet \to M^\bullet$ be maps of complexes of R-modules satisfying (1), (2), (3) above. Then we have $\det(b^\bullet) \circ \det(a^\bullet) = \det(b^\bullet \circ a^\bullet)$ as maps $\det(M^\bullet) \to \det(K^\bullet)$.*

Proof. Omitted. Hints: Straightforward from Lemmas 15.119.2, 15.119.3, and 15.119.4. □

0FJM **Lemma 15.123.4.** *Let R be a ring. The constructions above determine a functor*

$$\det : \left\{ \begin{array}{l} \text{category of perfect complexes} \\ \text{with tor amplitude in } [-1, 0] \\ \text{morphisms are isomorphisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{array} \right\}$$

Moreover, given a rank 0 perfect object L of $D(R)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \det(L)$ such that for any isomorphism $a : L \to K$ in $D(R)$ we have $\det(a)(\delta(L)) = \delta(K)$.

Proof. By Lemma 15.75.2 every object of the source category may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules. Let us temporarily call a complex of this type good. By Derived Categories, Lemma 13.19.8 morphisms between good complexes in the derived category are homotopy classes of maps of complexes. Thus we may work with good complexes and we can use the determinant $\det(L^\bullet) = \det(L^0) \otimes \det(L^{-1})^{\otimes -1}$ we investigated above.

Let $a^\bullet : L^\bullet \to K^\bullet$ be a morphism of good complexes which is an isomorphism in $D(R)$, i.e., a quasi-isomorphism. We say that

$$\begin{array}{ccc} L^\bullet & \xrightarrow{\quad a^\bullet \quad} & K^\bullet \\ & {\scriptstyle b^\bullet} \nwarrow \quad \nearrow {\scriptstyle c^\bullet} & \\ & M^\bullet & \end{array}$$

is a good diagram if it commutes up to homotopy and $b^\bullet$ and $c^\bullet$ satisfy conditions (1), (2), (3) above. Whenever we have such a diagram it makes sense to define

$$\det(a^\bullet) = \det(c^\bullet) \circ \det(b^\bullet)^{-1}$$

where $\det(c^\bullet)$ and $\det(b^\bullet)$ are the isomorphisms constructed in the text above. We will show that good diagrams always exist and that the resulting map $\det(a^\bullet)$ is independent of the choice of good diagram.

Existence of good diagrams for a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes. Choose a surjection $p : R^{\oplus n} \to K^{-1}$. Then we can consider the new good complex

$$M^\bullet = \ldots \to 0 \to L^{-1} \oplus R^{\oplus n} \xrightarrow{d \oplus 1} L^0 \oplus R^{\oplus n} \to 0 \to \ldots$$

with the projection map $b^\bullet : M^\bullet \to L^\bullet$ and the map $c^\bullet : M^\bullet \to K^\bullet$ using $a^{-1} \oplus p$ in degree -1 and using $a^0 \oplus d \circ p$ in degree 0. The maps $b^\bullet : M^\bullet \to L^\bullet$ and $c^\bullet : M^\bullet \to K^\bullet$ satisfy conditions (1), (2), (3) above and we get a good diagram.

Suppose that we have a good diagram

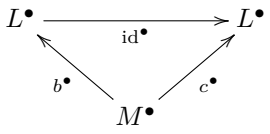

Then by Lemma 15.123.2 we see that $\det(c^\bullet) = \det(b^\bullet)$. Thus we see that $\det(\mathrm{id}^\bullet) = \mathrm{id}$ is independent of the choice of good diagram.

Before we prove independence in general, we think about composition. Suppose we have quasi-isomorphisms $L_1^\bullet \to L_2^\bullet$ and $L_2^\bullet \to L_3^\bullet$ of good complexes and good diagrams

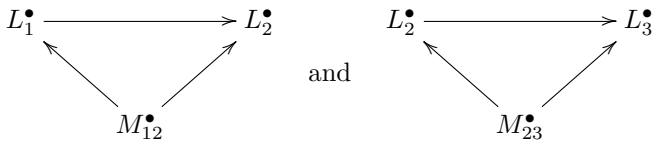

We can extend this to a diagram

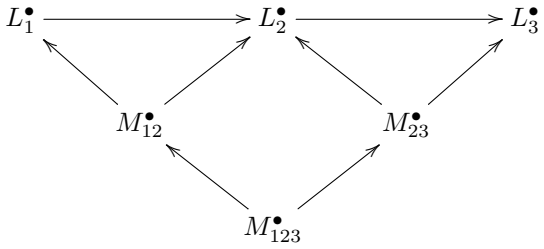

where $M_{123}^\bullet \to M_{12}^\bullet$ and $M_{123}^\bullet \to M_{23}^\bullet$ have properties (1), (2), (3) and the square in the diagram commutes: we can just take $M_{123}^n = M_{12}^n \times_{L_2^n} M_{23}^n$. Then Lemma 15.123.3 shows that

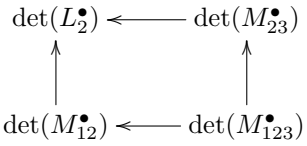

commutes. A diagram chase shows that the composition $\det(L_1^\bullet) \to \det(L_2^\bullet) \to \det(L_3^\bullet)$ of the maps associated to the two good diagrams using $M_{12}^\bullet$ and $M_{23}^\bullet$ is equal to the map associated to the good diagram

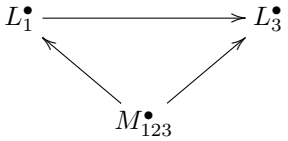

Thus if we can show that these maps are independent of choices, then the composition law is satisfied too and we obtain our functor.

Independence. Let a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes be given. Choose an inverse quasi-isomorphism $b^\bullet : K^\bullet \to L^\bullet$. Setting $L_1^\bullet = L$, $L_2^\bullet = K^\bullet$ and $L_3^\bullet = L^\bullet$ may fix our choice of good diagram for $b^\bullet$ and consider varying good diagrams for $a^\bullet$. Then the result of the previous paragraphs is that no matter what

choices, the composition always equals the identity map on $\det(L^\bullet)$. This clearly proves independence of those choices.

The statement on canonical elements follows immediately from Lemma 15.123.1 and our construction. □

15.124. Extensions of valuation rings

0ASF This section is the analogue of Section 15.112 for general valuation rings.

0ASG **Definition 15.124.1.** We say that $A \to B$ or $A \subset B$ is an *extension of valuation rings* if A and B are valuation rings and $A \to B$ is injective and local. Such an extension induces a commutative diagram

$$\begin{array}{ccc} A \setminus \{0\} & \longrightarrow & B \setminus \{0\} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ \Gamma_A & \longrightarrow & \Gamma_B \end{array}$$

where Γ_A and Γ_B are the value groups. We say that B is *weakly unramified* over A if the lower horizontal arrow is a bijection. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or *residue degree* of the extension $A \subset B$.

Note that $\Gamma_A \to \Gamma_B$ is injective, because the units of A are the inverse of the units of B under the map $A \to B$. Note also, that we do not require the extension of fraction fields to be finite.

0ASH **Lemma 15.124.2.** *Let $A \subset B$ be an extension of valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite, the index of Γ_A in Γ_B is finite, and*

$$[\Gamma_B : \Gamma_A][\kappa_B : \kappa_A] \leq [L : K].$$

Proof. Let $b_1, \ldots, b_n \in B$ be units whose images in κ_B are linearly independent over κ_A. Let $c_1, \ldots, c_m \in B$ be nonzero elements whose images in Γ_B/Γ_A are pairwise distinct. We claim that $b_i c_j$ are K-linearly independent in L. Namely, we claim a sum

$$\sum a_{ij} b_i c_j$$

with $a_{ij} \in K$ not all zero cannot be zero. Choose (i_0, j_0) with $v(a_{i_0 j_0} b_{i_0} c_{j_0})$ minimal. Replace a_{ij} by $a_{ij}/a_{i_0 j_0}$, so that $a_{i_0 j_0} = 1$. Let

$$P = \{(i, j) \mid v(a_{ij} b_i c_j) = v(a_{i_0 j_0} b_{i_0} c_{j_0})\}$$

By our choice of $c_1, \ldots, c_m$ we see that $(i, j) \in P$ implies $j = j_0$. Hence if $(i, j) \in P$, then $v(a_{ij}) = v(a_{i_0 j_0}) = 0$, i.e., a_{ij} is a unit. By our choice of $b_1, \ldots, b_n$ we see that

$$\sum\nolimits_{(i,j) \in P} a_{ij} b_i$$

is a unit in B. Thus the valuation of $\sum_{(i,j) \in P} a_{ij} b_i c_j$ is $v(c_{j_0}) = v(a_{i_0 j_0} b_{i_0} c_{j_0})$. Since the terms with $(i, j) \notin P$ in the first displayed sum have strictly bigger valuation, we conclude that this sum cannot be zero, thereby proving the lemma. □

0H37 **Lemma 15.124.3.** *Let A be a valuation ring with fraction field K of characteristic $p > 0$. Let L/K be a purely inseparable extension. Then the integral closure B of A in L is a valuation ring with fraction field L and $A \subset B$ is an extension of valuation rings.*

Proof. Omitted. Hints: use Algebra, Lemmas 10.50.5 and 10.36.17 for example. □

0ASI **Lemma 15.124.4.** *Let $A \to B$ be a flat local homomorphism of Noetherian local normal domains. Let $f \in A$ and $h \in B$ such that $f = wh^n$ for some $n > 1$ and some unit w of B. Assume that for every height 1 prime $\mathfrak{p} \subset A$ there is a height 1 prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ such that the extension $A_\mathfrak{p} \subset B_\mathfrak{q}$ is weakly unramified. Then $f = ug^n$ for some $g \in A$ and unit u of A.*

Proof. The local rings of A and B at height 1 primes are discrete valuation rings (Algebra, Lemma 10.119.7). Thus the assumption makes sense (via Definition 15.112.1). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the primes of A minimal over f. These have height 1 by Algebra, Lemma 10.60.11. For each i let $\mathfrak{q}_{i,j} \subset B$, $j = 1, \ldots, r_i$ be the height 1 primes of B lying over $\mathfrak{p}_i$. Say we number them so that $A_{\mathfrak{p}_i} \to B_{\mathfrak{q}_{i,1}}$ is weakly unramified. Since f maps to an nth power times a unit in $B_{\mathfrak{q}_{i,1}}$ we see that the valuation v_i of f in $A_{\mathfrak{p}_i}$ is divisible by n. Say $v_i = nw_i$ for some $w_i \geq 0$. Consider the exact sequence

$$0 \to I \to A \to \prod\nolimits_{i=1,\ldots,r} A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i} A_{\mathfrak{p}_i}$$

defining the ideal I. Applying the exact functor $- \otimes_A B$ we obtain an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} (A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i} A_{\mathfrak{p}_i}) \otimes_A B$$

Fix i. We claim that the canonical map

$$(A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i} A_{\mathfrak{p}_i}) \otimes_A B \to \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak{q}_{i,j}}/\mathfrak{q}_{i,j}^{e_{i,j}w_i} B_{\mathfrak{q}_{i,j}}$$

is injective. Here $e_{i,j}$ is the ramification index of $A_{\mathfrak{p}_i} \to B_{\mathfrak{q}_{i,j}}$. The claim asserts that $\mathfrak{p}_i^{w_i} B_{\mathfrak{p}_i}$ is equal to the set of elements b of $B_{\mathfrak{p}_i}$ whose valuation at $\mathfrak{q}_{i,j}$ is $\geq e_{i,j}w_i$. Choose a generator $a \in A_{\mathfrak{p}_i}$ of the principal ideal $\mathfrak{p}_i^{w_i}$. Then the valuation of a at $\mathfrak{q}_{i,j}$ is equal to $e_{i,j}w_i$. Hence, as $B_{\mathfrak{p}_i}$ is a normal domain whose height one primes are the primes $\mathfrak{q}_{i,j}$, $j = 1, \ldots, r_i$, we see that, for b as above, we have $b/a \in B_{\mathfrak{p}_i}$ by Algebra, Lemma 10.157.6. Thus the claim.

The claim combined with the second exact sequence above determines an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak{q}_{i,j}}/\mathfrak{q}_{i,j}^{e_{i,j}w_i} B_{\mathfrak{q}_{i,j}}$$

It follows that $I \otimes_A B$ is the set of elements h' of B which have valuation $\geq e_{i,j}w_i$ at $\mathfrak{q}_{i,j}$. Since $f = wh^n$ in B we see that h has valuation $e_{i,j}w_i$ at $\mathfrak{q}_{i,j}$. Thus $h'/h \in B$ by Algebra, Lemma 10.157.6. It follows that $I \otimes_A B$ is a free B-module of rank 1 (generated by h). Therefore I is a free A-module of rank 1, see Algebra, Lemma 10.78.6. Let $g \in I$ be a generator. Then we see that g and h differ by a unit in B. Working backwards we conclude that the valuation of g in $A_{\mathfrak{p}_i}$ is $w_i = v_i/n$. Hence g^n and f differ by a unit in A (by Algebra, Lemma 10.157.6) as desired. □

0ASJ **Lemma 15.124.5.** *Let A be a valuation ring. Let $A \to B$ be an étale ring map and let $\mathfrak{m} \subset B$ be a prime lying over the maximal ideal of A. Then $A \subset B_{\mathfrak{m}}$ is an extension of valuation rings which is weakly unramified.*

Proof. The ring A has weak dimension ≤ 1 by Lemma 15.105.18. Then B has weak dimension ≤ 1 by Lemmas 15.105.4 and 15.105.14. hence the local ring $B_{\mathfrak{m}}$ is a valuation ring by Lemma 15.105.18. Since the extension $A \subset B_{\mathfrak{m}}$ induces a finite extension of fraction fields, we see that the Γ_A has finite index in the value group of $B_{\mathfrak{m}}$. Thus for every $h \in B_{\mathfrak{m}}$ there exists an $n > 0$, an element $f \in A$, and a unit $w \in B_{\mathfrak{m}}$ such that $f = wh^n$ in $B_{\mathfrak{m}}$. We will show that this implies $f = ug^n$ for some $g \in A$ and unit $u \in A$; this will show that the value groups of A and $B_{\mathfrak{m}}$ agree, as claimed in the lemma.

Write $A = \operatorname{colim} A_i$ as the colimit of its local subrings which are essentially of finite type over $\mathbf{Z}$. Since A is a normal domain (Algebra, Lemma 10.50.3), we may assume that each A_i is normal (here we use that taking normalizations the local rings remain essentially of finite type over $\mathbf{Z}$ by Algebra, Proposition 10.162.16). For some i we can find an étale extension $A_i \to B_i$ such that $B = A \otimes_{A_i} B_i$, see Algebra, Lemma 10.143.3. Let $\mathfrak{m}_i$ be the intersection of B_i with $\mathfrak{m}$. Then we may apply Lemma 15.124.4 to the ring map $A_i \to (B_i)_{\mathfrak{m}_i}$ to conclude. The hypotheses of the lemma are satisfied because:

(1) A_i and $(B_i)_{\mathfrak{m}_i}$ are Noetherian as they are essentially of finite type over $\mathbf{Z}$,
(2) $A_i \to (B_i)_{\mathfrak{m}_i}$ is flat as $A_i \to B_i$ is étale,
(3) B_i is normal as $A_i \to B_i$ is étale, see Algebra, Lemma 10.163.9,
(4) for every height 1 prime of A_i there exists a height 1 prime of $(B_i)_{\mathfrak{m}_i}$ lying over it by Algebra, Lemma 10.113.2 and the fact that $\operatorname{Spec}((B_i)_{\mathfrak{m}_i}) \to \operatorname{Spec}(A_i)$ is surjective,
(5) the induced extensions $(A_i)_{\mathfrak{p}} \to (B_i)_{\mathfrak{q}}$ are unramified for every prime $\mathfrak{q}$ lying over a prime $\mathfrak{p}$ as $A_i \to B_i$ is étale.

This concludes the proof of the lemma. □

0ASK **Lemma 15.124.6.** *Let A be a valuation ring. Let A^h, resp. A^{sh} be its henselization, resp. strict henselization. Then*

$$A \subset A^h \subset A^{sh}$$

are extensions of valuation rings which induce bijections on value groups, i.e., which are weakly unramified.

Proof. Write $A^h = \operatorname{colim}(B_i)_{\mathfrak{q}_i}$ where $A \to B_i$ is étale and $\mathfrak{q}_i \subset B_i$ is a prime ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 10.155.7. Then Lemma 15.124.5 tells us that $(B_i)_{\mathfrak{q}_i}$ is a valuation ring and that the induced map

$$(A \setminus \{0\})/A^* \longrightarrow ((B_i)_{\mathfrak{q}_i} \setminus \{0\})/(B_i)^*_{\mathfrak{q}_i}$$

is bijective. By Algebra, Lemma 10.50.6 we conclude that A^h is a valuation ring. It also follows that $(A \setminus \{0\})/A^* \to (A^h \setminus \{0\})/(A^h)^*$ is bijective. This proves the lemma for the inclusion $A \subset A^h$. To prove it for $A \subset A^{sh}$ we can use exactly the same argument except we replace Algebra, Lemma 10.155.7 by Algebra, Lemma 10.155.11. Since $A^{sh} = (A^h)^{sh}$ we see that this also proves the assertions of the lemma for the inclusion $A^h \subset A^{sh}$. □

15.125. Structure of modules over a PID

0ASL We work a little bit more generally (following the papers [**War69**] and [**War70**] by Warfield) so that the proofs work over valuation rings.

0ASM **Lemma 15.125.1.** *Let P be a module over a ring R. The following are equivalent* [**War69**, Corollary 1]

(1) *P is a direct summand of a direct sum of modules of the form R/fR, for $f \in R$ varying.*
(2) *for every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules such that $fA = A \cap fB$ for all $f \in R$ the map $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is surjective.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence as in (2). To prove that (1) implies (2) it suffices to prove that $\mathrm{Hom}_R(R/fR, B) \to \mathrm{Hom}_R(R/fR, C)$ is surjective for every $f \in R$. Let $\psi : R/fR \to C$ be a map. Say $\psi(1)$ is the image of $b \in B$. Then $fb \in A$. Hence there exists an $a \in A$ such that $fa = fb$. Then $f(b - a) = 0$ hence we get a morphism $\varphi : R/fR \to B$ mapping 1 to $b - a$ which lifts ψ.

Conversely, assume that (2) holds. Let I be the set of pairs (f, φ) where $f \in R$ and $\varphi : R/fR \to P$. For $i \in I$ denote (f_i, φ_i) the corresponding pair. Consider the map

$$B = \bigoplus\nolimits_{i \in I} R/f_iR \longrightarrow P$$

which sends the element r in the summand R/f_iR to $\varphi_i(r)$ in P. Let $A = \mathrm{Ker}(B \to P)$. Then we see that (1) is true if the sequence

$$0 \to A \to B \to P \to 0$$

is an exact sequence as in (2). To see this suppose $f \in R$ and $a \in A$ maps to fb in B. Write $b = (r_i)_{i \in I}$ with almost all $r_i = 0$. Then we see that

$$f \sum \varphi_i(r_i) = 0$$

in P. Hence there is an $i_0 \in I$ such that $f_{i_0} = f$ and $\varphi_{i_0}(1) = \sum \varphi_i(r_i)$. Let $x_{i_0} \in R/f_{i_0}R$ be the class of 1. Then we see that

$$a' = (r_i)_{i \in I} - (0, \ldots, 0, x_{i_0}, 0, \ldots)$$

is an element of A and $fa' = a$ as desired. □

0ASN **Lemma 15.125.2** (Generalized valuation rings)**.** *Let R be a nonzero ring. The following are equivalent* [**War70**]

(1) *For $a, b \in R$ either a divides b or b divides a.*
(2) *Every finitely generated ideal is principal and R is local.*
(3) *The set of ideals of R is linearly ordered by inclusion.*

This holds in particular if R is a valuation ring.

Proof. Assume (2) and let $a, b \in R$. Then $(a, b) = (c)$. If $c = 0$, then $a = b = 0$ and a divides b. Assume $c \neq 0$. Write $c = ua + vb$ and $a = wc$ and $b = zc$. Then $c(1 - uw - vz) = 0$. Since R is local, this implies that $1 - uw - vz \in \mathfrak{m}$. Hence either w or z is a unit, so either a divides b or b divides a. Thus (2) implies (1).

Assume (1). If R has two maximal ideals $\mathfrak{m}_i$ we can choose $a \in \mathfrak{m}_1$ with $a \notin \mathfrak{m}_2$ and $b \in \mathfrak{m}_2$ with $b \notin \mathfrak{m}_1$. Then a does not divide b and b does not divide a. Hence

R has a unique maximal ideal and is local. It follows easily from condition (1) and induction that every finitely generated ideal is principal. Thus (1) implies (2).

It is straightforward to prove that (1) and (3) are equivalent. The final statement is Algebra, Lemma 10.50.4. □

0ASP **Lemma 15.125.3.** *Let R be a ring satisfying the equivalent conditions of Lemma 15.125.2. Then every finitely presented R-module is isomorphic to a finite direct sum of modules of the form R/fR.* [**War70**, Theorem 1]

Proof. Let M be a finitely presented R-module. We will use all the equivalent properties of R from Lemma 15.125.2 without further mention. Denote $\mathfrak{m} \subset R$ the maximal ideal and $\kappa = R/\mathfrak{m}$ the residue field. Let $I \subset R$ be the annihilator of M. Choose a basis $y_1, \ldots, y_n$ of the finite dimensional κ-vector space $M/\mathfrak{m}M$. We will argue by induction on n.

By Nakayama's lemma any collection of elements $x_1, \ldots, x_n \in M$ lifting the elements $y_1, \ldots, y_n$ in $M/\mathfrak{m}M$ generate M, see Algebra, Lemma 10.20.1. This immediately proves the base case $n = 0$ of the induction.

We claim there exists an index i such that for any choice of $x_i \in M$ mapping to y_i the annihilator of x_i is I. Namely, if not, then we can choose $x_1, \ldots, x_n$ such that $I_i = \mathrm{Ann}(x_i) \neq I$ for all i. But as $I \subset I_i$ for all i, ideals being totally ordered implies I_i is strictly bigger than I for $i = 1, \ldots, n$, and by total ordering once more we would see that $\mathrm{Ann}(M) = I_1 \cap \ldots \cap I_n$ is bigger than I which is a contradiction. After renumbering we may assume that y_1 has the property: for any $x_1 \in M$ lifting y_1 the annihilator of x_1 is I.

We set $A = Rx_1 \subset M$. Consider the exact sequence $0 \to A \to M \to M/A \to 0$. Since A is finite, we see that M/A is a finitely presented R-module (Algebra, Lemma 10.5.3) with fewer generators. Hence $M/A \cong \bigoplus_{j=1,\ldots,m} R/f_jR$ by induction. On the other hand, we claim that $A \to M$ satisfies the property: if $f \in R$, then $fA = A \cap fM$. The inclusion $fA \subset A \cap fM$ is trivial. Conversely, if $x \in A \cap fM$, then $x = gx_1 = fy$ for some $g \in R$ and $y \in M$. If f divides g, then $x \in fA$ as desired. If not, then we can write $f = hg$ for some $h \in \mathfrak{m}$. The element $x_1' = x_1 - hy$ has annihilator I by the previous paragraph. Thus $g \in I$ and we see that $x = 0$ as desired. The claim and Lemma 15.125.1 imply the sequence $0 \to A \to M \to M/A \to 0$ is split and we find $M \cong A \oplus \bigoplus_{j=1,\ldots,m} R/f_jR$. Then $A = R/I$ is finitely presented (as a summand of M) and hence I is finitely generated, hence principal. This finishes the proof. □

0ASQ **Lemma 15.125.4.** *Let R be a ring such that every local ring of R at a maximal ideal satisfies the equivalent conditions of Lemma 15.125.2. Then every finitely presented R-module is a summand of a finite direct sum of modules of the form R/fR for f in R varying.* [**War70**, Theorem 3]

Proof. Let M be a finitely presented R-module. We first show that M is a summand of a direct sum of modules of the form R/fR and at the end we argue the direct sum can be taken to be finite. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules such that $fA = A \cap fB$ for all $f \in R$. By Lemma 15.125.1 we have to show that $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is surjective.

It suffices to prove this after localization at maximal ideals $\mathfrak{m}$, see Algebra, Lemma 10.23.1. Note that the localized sequences $0 \to A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m} \to 0$ satisfy the condition that $fA_\mathfrak{m} = A_\mathfrak{m} \cap fB_\mathfrak{m}$ for all $f \in R_\mathfrak{m}$ (because we can write $f = uf'$ with $u \in R_\mathfrak{m}$ a unit and $f' \in R$ and because localization is exact). Since M is finitely presented, we see that

$$\mathrm{Hom}_R(M, B)_\mathfrak{m} = \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) \quad \text{and} \quad \mathrm{Hom}_R(M, C)_\mathfrak{m} = \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m})$$

by Algebra, Lemma 10.10.2. The module $M_\mathfrak{m}$ is a finitely presented $R_\mathfrak{m}$-module. By Lemma 15.125.3 we see that $M_\mathfrak{m}$ is a direct sum of modules of the form $R_\mathfrak{m}/fR_\mathfrak{m}$. Thus we conclude by Lemma 15.125.1 that the map on localizations is surjective.

At this point we know that M is a summand of $\bigoplus_{i \in I} R/f_iR$. Consider the map $M \to \bigoplus_{i \in I} R/f_iR$. Since M is a finite R-module, the image is contained in $\bigoplus_{i \in I'} R/f_iR$ for some finite subset $I' \subset I$. This finishes the proof. □

0ASR **Definition 15.125.5.** Let R be a domain.

(1) We say R is a *Bézout domain* if every finitely generated ideal of R is principal.
(2) We say R is an *elementary divisor domain* if for all $n, m \geq 1$ and every $n \times m$ matrix A, there exist invertible matrices U, V of size $n \times n, m \times m$ such that
$$UAV = \begin{pmatrix} f_1 & 0 & 0 & \ldots \\ 0 & f_2 & 0 & \ldots \\ 0 & 0 & f_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$
with $f_1, \ldots, f_{\min(n,m)} \in R$ and $f_1 | f_2 | \ldots$.

It is apparently still an open question as to whether every Bézout domain R is an elementary divisor domain (or not). This is equivalent to the question of whether every finitely presented module over R is a direct sum of cyclic modules. The converse implication is true.

0ASS **Lemma 15.125.6.** *An elementary divisor domain is Bézout.*

Proof. Let $a, b \in R$ be nonzero. Consider the 1×2 matrix $A = (a\ b)$. Then we see that $u(a\ b)V = (f\ 0)$ with $u \in R$ invertible and $V = (g_{ij})$ an invertible 2×2 matrix. Then $f = uag_{11} + ubg_{21}$ and $(g_{11}, g_{21}) = R$. It follows that $(a, b) = (f)$. An induction argument (omitted) then shows any finitely generated ideal in R is generated by one element. □

0AST **Lemma 15.125.7.** *The localization of a Bézout domain is Bézout. Every local ring of a Bézout domain is a valuation ring. A local domain is Bézout if and only if it is a valuation ring.*

Proof. We omit the proof of the statement on localizations. The final statement is Algebra, Lemma 10.50.15. The second statement follows from the other two. □

0ASU **Lemma 15.125.8.** *Let R be a Bézout domain.*

(1) *Every finite submodule of a free module is finite free.*
(2) *Every finitely presented R-module M is a direct sum of a finite free module and a torsion module M_{tors} which is a summand of a module of the form $\bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_1, \ldots, f_n \in R$ nonzero.*

Proof. Proof of (1). Let $M \subset F$ be a finite submodule of a free module F. Since M is finite, we may assume F is a finite free module (details omitted). Say $F = R^{\oplus n}$. We argue by induction on n. If $n = 1$, then M is a finitely generated ideal, hence principal by our assumption that R is Bézout. If $n > 1$, then we consider the image I of M under the projection $R^{\oplus n} \to R$ onto the last summand. If $I = (0)$, then $M \subset R^{\oplus n-1}$ and we are done by induction. If $I \neq 0$, then $I = (f) \cong R$. Hence $M \cong R \oplus \mathrm{Ker}(M \to I)$ and we are done by induction as well.

Let M be a finitely presented R-module. Since the localizations of R are maximal ideals are valuation rings (Lemma 15.125.7) we may apply Lemma 15.125.4. Thus M is a summand of a module of the form $R^{\oplus r} \oplus \bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_i \neq 0$. Since taking the torsion submodule is a functor we see that M_{tors} is a summand of the module $\bigoplus_{i=1,\ldots,n} R/f_iR$ and M/M_{tors} is a summand of $R^{\oplus r}$. By the first part of the proof we see that M/M_{tors} is finite free. Hence $M \cong M_{tors} \oplus M/M_{tors}$ as desired. □

0ASV **Lemma 15.125.9.** *Let R be a PID. Every finite R-module M is of isomorphic to a module of the form*

$$R^{\oplus r} \oplus \bigoplus\nolimits_{i=1,\ldots,n} R/f_iR$$

for some $r, n \geq 0$ and $f_1, \ldots, f_n \in R$ nonzero.

Proof. A PID is a Noetherian Bézout ring. By Lemma 15.125.8 it suffices to prove the result if M is torsion. Since M is finite, this means that the annihilator of M is nonzero. Say $fM = 0$ for some $f \in R$ nonzero. Then we can think of M as a module over R/fR. Since R/fR is Noetherian of dimension 0 (small detail omitted) we see that $R/fR = \prod R_j$ is a finite product of Artinian local rings R_i (Algebra, Proposition 10.60.7). Each R_i, being a local ring and a quotient of a PID, is a generalized valuation ring in the sense of Lemma 15.125.2 (small detail omitted). Write $M = \prod M_j$ with $M_j = e_jM$ where $e_j \in R/fR$ is the idempotent corresponding to the factor R_j. By Lemma 15.125.3 we see that $M_j = \bigoplus_{i=1,\ldots,n_j} R_j/\overline{f}_{ji}R_j$ for some $\overline{f}_{ji} \in R_j$. Choose lifts $f_{ji} \in R$ and choose $g_{ji} \in R$ with $(g_{ji}) = (f_j, f_{ji})$. Then we conclude that

$$M \cong \bigoplus R/g_{ji}R$$

as an R-module which finishes the proof. □

One can also prove that a PID is a elementary divisor domain (insert future reference here), by proving lemmas similar to the following.

0ASW **Lemma 15.125.10.** *Let R be a Bézout domain. Let $n \geq 1$ and $f_1, \ldots, f_n \in R$ generate the unit ideal. There exists an invertible $n \times n$ matrix in R whose first row is $f_1 \ldots f_n$.*

Proof. This follows from Lemma 15.125.8 but we can also prove it directly as follows. By induction on n. The result holds for $n = 1$. Assume $n > 1$. We may assume $f_1 \neq 0$ after renumbering. Choose $f \in R$ such that $(f) = (f_1, \ldots, f_{n-1})$. Let A be an $(n-1) \times (n-1)$ matrix whose first row is $f_1/f, \ldots, f_{n-1}/f$. Choose $a, b \in R$ such that $af - bf_n = 1$ which is possible because $1 \in (f_1, \ldots, f_n) = (f, f_n)$.

Then a solution is the matrix

$$\begin{pmatrix} f & 0 & \ldots & 0 & f_n \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 \\ b & 0 & \ldots & 0 & a \end{pmatrix} \begin{pmatrix} & & & 0 \\ & A & & \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Observe that the left matrix is invertible because it has determinant 1. □

15.126. Principal radical ideals

0BWR In this section we prove that a catenary Noetherian normal local domain there exists a nontrivial principal radical ideal. This result can be found in [**Art86**].

0BWS **Lemma 15.126.1.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one, and let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Then*

(1) *the function $P : n \mapsto \text{length}_R(R/x^nR)$ satisfies $P(n) \leq nP(1)$ for $n \geq 0$,*
(2) *if x is a nonzerodivisor, then $P(n) = nP(1)$ for $n \geq 0$.*

Proof. Since $\dim(R) = 1$, we have $\dim(R/x^nR) = 0$ and so $\text{length}_R(R/x^nR)$ is finite for each n (Algebra, Lemma 10.62.3). To show the lemma we will induct on n. Since $x^0R = R$, we have that $P(0) = \text{length}_R(R/x^0R) = \text{length}_R 0 = 0$. The statement also holds for $n = 1$. Now let $n \geq 2$ and suppose the statement holds for $n - 1$. The following sequence is exact

$$R/x^{n-1}R \xrightarrow{x} R/x^nR \to R/xR \to 0$$

where x denotes the multiplication by x map. Since length is additive (Algebra, Lemma 10.52.3), we have that $P(n) \leq P(n-1) + P(1)$. By induction $P(n-1) \leq (n-1)P(1)$, whence $P(n) \leq nP(1)$. This proves the induction step.

If x is a nonzerodivisor, then the displayed exact sequence above is exact on the left also. Hence we get $P(n) = P(n-1) + P(1)$ for all $n \geq 1$. □

0BWT **Lemma 15.126.2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Let t be the number of minimal prime ideals of R. Then $t \leq \text{length}_R(R/xR)$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of R. Set $R' = R/\sqrt{0} = R/(\bigcap_{i=1}^t \mathfrak{p}_i)$. We claim it suffices to prove the lemma for R'. Namely, it is clear that R' has t minimal primes too and $\text{length}_{R'}(R'/xR') = \text{length}_R(R'/xR')$ is less than $\text{length}_R(R/xR)$ as there is a surjection $R/xR \to R'/xR'$. Thus we may assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. We conclude that $\text{Supp}(Q) = \{\mathfrak{m}\}$ as $\mathfrak{m}$ is the only prime of R different from the $\mathfrak{p}_i$. It follows that Q has finite length (Algebra,

Lemma 10.62.3). Since $\text{Supp}(Q) = \{\mathfrak{m}\}$ we can pick an $n \gg 0$ such that x^n acts as 0 on Q (Algebra, Lemma 10.62.4). Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x^n} & & \downarrow{\scriptstyle x^n} & & \downarrow{\scriptstyle x^n} & & \\ 0 & \longrightarrow & R & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

where the vertical maps are multiplication by x^n. This is injective on R and on M since x is not contained in any of the $\mathfrak{p}_i$. By the snake lemma (Algebra, Lemma 10.4.1), the following sequence is exact:

$$0 \to Q \to R/x^nR \to M/x^nM \to Q \to 0$$

Hence we find that $\text{length}_R(R/x^nR) = \text{length}_R(M/x^nM)$ for large enough n. Writing $R_i = R/\mathfrak{p}_i$ we see that $\text{length}(M/x^nM) = \sum_{i=1}^t \text{length}_R(R_i/x^nR_i)$. Applying Lemma 15.126.1 and the fact that x is a nonzerodivisor on R and R_i, we conclude that

$$n\text{length}_R(R/xR) = \sum\nolimits_{i=1}^t n\text{length}_{R_i}(R_i/xR_i)$$

Since $\text{length}_{R_i}(R_i/xR_i) \geq 1$ the lemma is proved. □

0BWU **Lemma 15.126.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 1$, let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R, and let $k \in \mathbf{N}$. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak{m}^k$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters.*

Proof. We have $\dim(R/fR) = d-1$ by Algebra, Lemma 10.60.13. Choose a system of parameters $\overline{g}_1, \ldots, \overline{g}_{d-1}$ in R/fR (Algebra, Proposition 10.60.9) and take lifts $g_1, \ldots, g_{d-1}$ in R. It is straightforward to see that $f, g_1, \ldots, g_{d-1}$ is a system of parameters in R. Then $f, g_1^k, \ldots, g_{d-1}^k$ is also a system of parameters and the proof is complete. □

0BWV **Lemma 15.126.4.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension two, and let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R. Then there exist $g \in \mathfrak{m}$ and $N \in \mathbf{N}$ such that*

(a) *f, g form a system of parameters for R.*
(b) *If $h \in \mathfrak{m}^N$, then $f+h, g$ is a system of parameters and $\text{length}_R(R/(f,g)) = \text{length}_R(R/(f+h,g))$.*

Proof. By Lemma 15.126.3 there exists a $g \in \mathfrak{m}$ such that f, g is a system of parameters for R. Then $\mathfrak{m} = \sqrt{(f,g)}$. Thus there exists an n such that $\mathfrak{m}^n \subset (f,g)$, see Algebra, Lemma 10.32.5. We claim that $N = n+1$ works. Namely, let $h \in \mathfrak{m}^N$. By our choice of N we can write $h = af + bg$ with $a, b \in \mathfrak{m}$. Thus

$$(f+h, g) = (f + af + bg, g) = ((1+a)f, g) = (f, g)$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g$ is a system of parameters. □

0AXH **Lemma 15.126.5.** *Let R be a Noetherian local normal domain of dimension 2. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be pairwise distinct primes of height 1. There exists a nonzero element $f \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ such that R/fR is reduced.*

Proof. Let $f \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak{p}$ of R at each height one prime ideal $\mathfrak{p}$ is a discrete valuation ring, see Algebra, Lemma 10.119.7 or Algebra, Lemma 10.157.4. We denote by $\text{ord}_\mathfrak{p}(f)$ the corresponding valuation of f in $R_\mathfrak{p}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak{q}_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^s m_j \mathfrak{q}_j$ as a formal linear combination of height one primes with integer coefficients. Note for later use that each of the primes $\mathfrak{p}_i$ occurs among the primes $\mathfrak{q}_j$. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak{q}_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak{q}_j}$ as $\mathfrak{q}_1, \ldots, \mathfrak{q}_j$ are the associated primes of R/fR, see Algebra, Lemmas 10.63.19 and 10.157.6.

Choose and fix g and N as in Lemma 15.126.4. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 10.60.11 and 10.157.6). For example $t(f) = s$ is the number of primes $\mathfrak{q}_j$ occurring in $\text{div}(f)$. Let $h \in \mathfrak{m}^N$. By Lemma 15.126.2 we have

$$\begin{aligned} t(f+h) &\leq \text{length}_{R/(f+h)}(R/(f+h, g)) \\ &= \text{length}_R(R/(f+h, g)) \\ &= \text{length}_R(R/(f, g)) \end{aligned}$$

see Algebra, Lemma 10.52.5 for the first equality. Therefore we see that $t(f+h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ such that $t(f+h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we see that the number $t(f' + h') \leq t(f + h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we have $t(f + h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N$ such that for each j we have $\text{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\text{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Observe that $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$. Then $\text{ord}_{\mathfrak{q}_j}(f + h) = 1$ for every j by elementary properties of valuations. Thus

$$\text{div}(f + h) = \sum\nolimits_{j=1}^s \mathfrak{q}_j + \sum\nolimits_{k=1}^v e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f + h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 10.15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j$ such that $a_j \notin \mathfrak{q}_{j'}$ for $j' \neq j$ and $a_j \notin \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \text{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Then we take

$$h = \prod\nolimits_{m_j = 1} a_j^2 \times \prod\nolimits_{m_j > 1} a_j$$

Then h clearly satisfies the conditions on valuations imposed above. If $h \notin \mathfrak{m}^N$, then we multiply by an element of $\mathfrak{m}^N$ which is not contained in $\mathfrak{q}_j$ for all j. □

0AXI **Lemma 15.126.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. If $a \in \mathfrak{m}$ is nonzero, then there exists an element $c \in A$ such that A/cA is reduced and such that a divides c^n for some n.*

Proof. Let $\mathrm{div}(a) = \sum_{i=1,\ldots,r} n_i \mathfrak{p}_i$ with notation as in the proof of Lemma 15.126.5. Choose $c \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ with A/cA reduced, see Lemma 15.126.5. For $n \geq \max(n_i)$ we see that $-\mathrm{div}(a) + \mathrm{div}(c^n)$ is an effective divisor (all coefficients nonnegative). Thus $c^n/a \in A$ by Algebra, Lemma 10.157.6. □

In the rest of this section we prove the result in dimension > 2.

0BWW **Lemma 15.126.7.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let $g_1, \ldots, g_d$ be a system of parameters, and let $I = (g_1, \ldots, g_d)$. If $e_I/d!$ is the leading coefficient of the numerical polynomial $n \mapsto \mathrm{length}_R(R/I^{n+1})$, then $e_I \leq \mathrm{length}_R(R/I)$.*

Proof. The function is a numerical polynomial by Algebra, Proposition 10.59.5. It has degree d by Algebra, Proposition 10.60.9. If $d = 0$, then the result is trivial. If $d = 1$, then the result is Lemma 15.126.1. To prove it in general, observe that there is a surjection

$$\bigoplus\nolimits_{i_1,\ldots,i_d \geq 0,\ \sum i_j = n} R/I \longrightarrow I^n/I^{n+1}$$

sending the basis element corresponding to $i_1, \ldots, i_d$ to the class of $g_1^{i_1} \ldots g_d^{i_d}$ in I^n/I^{n+1}. Thus we see that

$$\mathrm{length}_R(R/I^{n+1}) - \mathrm{length}_R(R/I^n) \leq \mathrm{length}_R(R/I)\binom{n+d-1}{d-1}$$

Since $d \geq 2$ the numerical polynomial on the left has degree $d-1$ with leading coefficient $e_I/(d-1)!$. The polynomial on the right has degree $d-1$ and its leading coefficient is $\mathrm{length}_R(R/I)/(d-1)!$. This proves the lemma. □

0BWX **Lemma 15.126.8.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let t be the number of minimal prime ideals of R of dimension d, and let $(g_1, \ldots, g_d)$ be a system of parameters. Then $t \leq \mathrm{length}_R(R/(g_1, \ldots, g_n))$.*

Proof. If $d = 0$ the lemma is trivial. If $d = 1$ the lemma is Lemma 15.126.2. Thus we may assume $d > 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R where the first t have dimension d, and denote $I = (g_1, \ldots, g_n)$. Arguing in exactly the same way as in the proof of Lemma 15.126.2 we can assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod\nolimits_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod\nolimits_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. Therefore no height 0 prime of R is in the support of Q. It follows that the degree of the numerical polynomial $n \mapsto \mathrm{length}_R(Q/I^nQ)$ equals $\dim(\mathrm{Supp}(Q)) < d$, see Algebra, Lemma 10.62.6. By Algebra, Lemma 10.59.10 (which applies as R does not have finite length) the polynomial

$$n \longmapsto \mathrm{length}_R(M/I^nM) - \mathrm{length}_R(R/I^n) - \mathrm{length}_R(Q/I^nQ)$$

has degree $< d$. Since $M = \prod R/\mathfrak{p}_i$ and since $n \to \text{length}_R(R/\mathfrak{p}_i + I^n)$ is a numerical polynomial of degree exactly(!) d for $i = 1, \ldots, t$ (by Algebra, Lemma 10.62.6) we see that the leading coefficient of $n \mapsto \text{length}_R(M/I^nM)$ is at least $t/d!$. Thus we conclude by Lemma 15.126.7. □

0BWY **Lemma 15.126.9.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ and $N \in \mathbf{N}$ such that*

(1) *$f, g_1, \ldots, g_{d-1}$ form a system of parameters for R*
(2) *If $h \in \mathfrak{m}^N$, then $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters and we have $length_R R/(f, g_1, \ldots, g_{d-1}) = length_R R/(f + h, g_1, \ldots, g_{d-1})$.*

Proof. By Lemma 15.126.3 there exist $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters for R. Then $\mathfrak{m} = \sqrt{(f, g_1, \ldots, g_{d-1})}$. Thus there exists an n such that $\mathfrak{m}^n \subset (f, g)$, see Algebra, Lemma 10.32.5. We claim that $N = n + 1$ works. Namely, let $h \in \mathfrak{m}^N$. By our choice of N we can write $h = af + \sum b_i g_i$ with $a, b_i \in \mathfrak{m}$. Thus

$$\begin{aligned}(f + h, g_1, \ldots, g_{d-1}) &= (f + af + \sum b_i g_i, g_1, \ldots, g_{d-1}) \\ &= ((1 + a)f, g_1, \ldots, g_{d-1}) \\ &= (f, g_1, \ldots, g_{d-1})\end{aligned}$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters. □

0BWZ **Proposition 15.126.10.** *Let R be a catenary Noetherian local normal domain. Let $J \subset R$ be a radical ideal. Then there exists a nonzero element $f \in J$ such that R/fR is reduced.*

[**Art86**, Lemma 3.14] has this result without the assumption that the ring is catenary

Proof. The proof is the same as that of Lemma 15.126.5, using Lemma 15.126.8 instead of Lemma 15.126.2 and Lemma 15.126.9 instead of Lemma 15.126.4. We can use Lemma 15.126.8 because R is a catenary domain, so every height one prime ideal of R has dimension $d - 1$, and hence the spectrum of $R/(f + h)$ is equidimensional. For the convenience of the reader we write out the details.

Let $f \in J$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak{p}$ of R at each height one prime ideal $\mathfrak{p}$ is a discrete valuation ring, see Algebra, Lemma 10.119.7 or Algebra, Lemma 10.157.4. We denote by $\text{ord}_\mathfrak{p}(f)$ the corresponding valuation of f in $R_\mathfrak{p}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak{q}_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^s m_j \mathfrak{q}_j$ as a formal linear combination of height one primes with integer coefficients. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak{q}_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak{q}_j}$ as $\mathfrak{q}_1, \ldots, \mathfrak{q}_j$ are the associated primes of R/fR, see Algebra, Lemmas 10.63.19 and 10.157.6.

Choose and fix $g_2, \ldots, g_{d-1}$ and N as in Lemma 15.126.9. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 10.60.11 and 10.157.6). For example $t(f) = s$ is the number of

primes $\mathfrak{q}_j$ occurring in $\mathrm{div}(f)$. Let $h \in \mathfrak{m}^N$. Because R is catenary, for each height one prime $\mathfrak{p}$ of R we have $\dim(R/\mathfrak{p}) = d$. Hence by Lemma 15.126.8 we have

$$\begin{aligned} t(f+h) &\leq \mathrm{length}_{R/(f+h)}(R/(f+h, g_1, \ldots, g_{d-1})) \\ &= \mathrm{length}_R(R/(f+h, g_1, \ldots, g_{d-1})) \\ &= \mathrm{length}_R(R/(f, g_1, \ldots, g_{d-1})) \end{aligned}$$

see Algebra, Lemma 10.52.5 for the first equality. Therefore we see that $t(f+h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap J$ such that $t(f+h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap J$ we see that the number $t(f' + h') \leq t(f+h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap J$ we have $t(f+h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N \cap J$ such that for each j we have $\mathrm{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\mathrm{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Then $\mathrm{ord}_{\mathfrak{q}_j}(f+h) = 1$ for every j by elementary properties of valuations. Thus

$$\mathrm{div}(f+h) = \sum\nolimits_{j=1}^{s} \mathfrak{q}_j + \sum\nolimits_{k=1}^{v} e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f+h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 10.15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j \cap J$ such that $a_j \notin \mathfrak{q}_{j'}$ for $j' \neq j$. Next, we can pick $b_j \in J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ with $b_j \notin \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \mathrm{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Prime avoidance applies because the ideal $J' = J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ is radical, hence R/J' is reduced, hence $(R/J')_{\mathfrak{q}_j}$ is reduced, hence J' contains an element x with $\mathrm{ord}_{\mathfrak{q}_j}(x) = 1$, hence $J' \not\subset \mathfrak{q}_j^{(2)}$. Then the element

$$c = \sum\nolimits_{j=1,\ldots,s} b_j \times \prod\nolimits_{j' \neq j} a_{j'}$$

is an element of J with $\mathrm{ord}_{\mathfrak{q}_j}(c) = 1$ for all $j = 1, \ldots, s$ by elementary properties of valuations. Finally, we let

$$h = c \times \prod\nolimits_{m_j = 1} a_j \times y$$

where $y \in \mathfrak{m}^N$ is an element which is not contained in $\mathfrak{q}_j$ for all j. □

15.127. Invertible objects in the derived category

0FNP We characterize invertible objects in the derived category of a ring.

0FNQ **Lemma 15.127.1.** *Let R be a ring. The derived category $D(R)$ of R is a symmetric monoidal category with tensor product given by derived tensor product and associativity and commutativity constraints as in Section 15.73.*

Proof. Omitted. Hints: The associativity constraint is the isomorphism of Lemma 15.59.15 and the commutativity constraint is the isomorphism of Lemma 15.59.14. Having said this the commutativity of various diagrams follows from the corresponding result for the category of complexes of R-modules, see Section 15.58. □

Thus we know what it means for an object of $D(R)$ to have a (left) dual or to be invertible. Before we can work out what this amounts to we need a simple lemma.

0FNR **Lemma 15.127.2.** *Let R be a ring. Let $F^\bullet$ be a bounded above complex of free R-modules. Given pairs (n_i, f_i), $i = 1, \ldots, N$ with $n_i \in \mathbf{Z}$ and $f_i \in F^{n_i}$ there exists a subcomplex $G^\bullet \subset F^\bullet$ containing all f_i which is bounded and consists of finite free R-modules.*

Proof. By descending induction on $a = \min(n_i; i = 1, \ldots, N)$. If $F^n = 0$ for $n \geq a$, then the result is true with $G^\bullet$ equal to the zero complex. In general, after renumbering we may assume there exists an $1 \leq r \leq N$ such that $n_1 = \ldots = n_r = a$ and $n_i > a$ for $i > r$. Choose a basis $b_j, j \in J$ for F^a. We can choose a finite subset $J' \subset J$ such that $f_i \in \bigoplus_{j \in J'} Rb_j$ for $i = 1, \ldots, r$. Choose a basis $c_k, k \in K$ for F^{a+1}. We can choose a finite subset $K' \subset K$ such that $\mathrm{d}_F^a(b_j) \in \bigoplus_{k \in K'} Rc_k$ for $j \in J'$. Then we can apply the induction hypothesis to find a subcomplex $H^\bullet \subset F^\bullet$ containing $c_k \in F^{a+1}$ for $k \in K'$ and $f_i \in F^{n_i}$ for $i > r$. Take $G^\bullet$ equal to $H^\bullet$ in degrees $> a$ and equal to $\bigoplus_{j \in J'} Rb_j$ in degree a. □

0FNS **Lemma 15.127.3.** *Let R be a ring. Let M be an object of $D(R)$. The following are equivalent*

(1) *M has a left dual in $D(R)$ as in Categories, Definition 4.43.5,*
(2) *M is a perfect object of $D(R)$.*

Moreover, in this case the left dual of M is the object $M^\vee$ of Lemma 15.75.15.

Proof. If M is perfect, then we can represent M by a bounded complex $M^\bullet$ of finite projective R-modules. In this case $M^\bullet$ has a left dual in the category of complexes by Lemma 15.73.2 which is a fortiori a left dual in $D(R)$.

Assume (1). Say N, $\eta : R \to M \otimes^{\mathbf{L}}_R N$, and $\epsilon : M \otimes^{\mathbf{L}}_R N \to R$ is a left dual as in Categories, Definition 4.43.5. Choose a complex $M^\bullet$ representing M. Choose a K-flat complexes $N^\bullet$ with flat terms representing N, see Lemma 15.59.10. Then η is given by a map of complexes

$$\eta : R \longrightarrow \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$$

We can write the image of 1 as a finite sum

$$\eta(1) = \sum\nolimits_n \sum\nolimits_i m_{n,i} \otimes n_{-n,i}$$

with $m_{n,i} \in M^n$ and $n_{-n,i} \in N^{-n}$. Let $K^\bullet \subset M^\bullet$ be the subcomplex generated by all the elements $m_{n,i}$ and $\mathrm{d}(m_{n,i})$. By our choice of $N^\bullet$ we find that $\mathrm{Tot}(K^\bullet \otimes_R N^\bullet) \subset \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ and $\eta(1)$ is in the subcomplex by our choice above. Denote K the object of $D(R)$ represented by $K^\bullet$. Then we see that η factors over a map $\tilde{\eta} : R \longrightarrow K \otimes^{\mathbf{L}}_R N$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_M$ we conclude that the identity on M factors through K by the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\eta \otimes 1} & M \otimes^{\mathbf{L}}_R N \otimes^{\mathbf{L}}_R M & \xrightarrow{1 \otimes \epsilon} & M \\ & \searrow^{\tilde{\eta} \otimes 1} & \uparrow & & \uparrow \\ & & K \otimes^{\mathbf{L}}_R N \otimes^{\mathbf{L}}_R M & \xrightarrow{1 \otimes \epsilon} & K \end{array}$$

Since K is bounded above it follows that $M \in D^-(R)$. Thus we can represent M by a bounded above complex $M^\bullet$ of free R-modules, see for example Derived

Categories, Lemma 13.15.4. Write $\eta(1) = \sum_n \sum_i m_{n,i} \otimes n_{-n,i}$ as before. By Lemma 15.127.2 we can find a subcomplex $K^\bullet \subset M^\bullet$ containing all the elements $m_{n,i}$ which is bounded and consists of finite free R-modules. As above we find that the identity on M factors through K. Since K is perfect we conclude M is perfect too, see Lemma 15.75.5. □

0FNT **Lemma 15.127.4.** *Let R be a ring. Let M be an object of $D(R)$. The following are equivalent*

(1) *M is invertible in $D(R)$, see Categories, Definition 4.43.4, and*
(2) *for every prime ideal $\mathfrak{p} \subset R$ there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $M_f \cong R_f[-n]$ for some $n \in \mathbf{Z}$.*

Moreover, in this case

(a) *M is a perfect object of $D(R)$,*
(b) *$M = \bigoplus H^n(M)[-n]$ in $D(R)$,*
(c) *each $H^n(M)$ is a finite projective R-module,*
(d) *we can write $R = \prod_{a \leq n \leq b} R_n$ such that $H^n(M)$ corresponds to an invertible R_n-module.*

Proof. Assume (2). Consider the object $R\mathrm{Hom}_R(M, R)$ and the composition map

$$R\mathrm{Hom}(M, R) \otimes^{\mathbf{L}}_R M \to R$$

Checking locally we see that this is an isomorphism; we omit the details. Because $D(R)$ is symmetric monoidal we see that M is invertible.

Assume (1). Observe that an invertible object of a monoidal category has a left dual, namely, its inverse. Thus M is perfect by Lemma 15.127.3. Consider a prime ideal $\mathfrak{p} \subset R$ with residue field κ. Then we see that $M \otimes^{\mathbf{L}}_R \kappa$ is an invertible object of $D(\kappa)$. Clearly this implies that $\dim H^i(M \otimes^{\mathbf{L}}_R \kappa)$ is nonzero exactly for one i and equal to 1 in that case. By Lemma 15.76.7 this gives (2).

In the proof above we have seen that (a) holds. Let $U_n \subset \mathrm{Spec}(R)$ be the union of the opens of the form $D(f)$ such that $M_f \cong R_f[-n]$. Clearly, $U_n \cap U_{n'} = \emptyset$ if $n \neq n'$. If M has tor amplitude in $[a, b]$, then $U_n = \emptyset$ if $n \notin [a, b]$. Hence we see that we have a product decomposition $R = \prod_{a \leq n \leq b} R_n$ as in (d) such that U_n corresponds to $\mathrm{Spec}(R_n)$, see Algebra, Lemma 10.24.3. Since $D(R) = \prod_{a \leq n \leq b} D(R_n)$ and similarly for the category of modules parts (b), (c), and (d) follow immediately. □

15.128. Splitting off a free module

0GV7 The arguments in this section are due to Serre, see [**Ser58**].

0GV8 **Situation 15.128.1.** Here R is a ring and M is a finitely presented R-module. Denote $\Omega \subset \mathrm{Spec}(R)$ the set of closed points with the induced topology. For $x \in \Omega$ denote $M(x) = M/xM$ the fibre of M at x. This is a finite dimensional vector space over the residue field $\kappa(x)$ at x. Given $s \in M$ we denote $s(x)$ the image of s in $M(x)$.

0GV9 **Lemma 15.128.2.** *In Situation 15.128.1 let $x \in \Omega$. There exists a canonical short exact sequence*

$$0 \to B(x) \to M(x) \to V(x) \to 0$$

of $\kappa(x)$-vector spaces which the following property: for $s_1, \ldots, s_r \in M$ the following are equivalent

(1) *there exists an $f \in R$, $f \notin x$ such that the map $s_1, \ldots, s_r : R^{\oplus r} \to M$ becomes the inclusion of a direct summand after inverting f, and*
(2) *$s_1(x), \ldots, s_r(x)$ map to linearly independent elements of $V(x)$.*

Proof. Define $B(x) \subset M(x)$ as the perpendicular of the image of the map

$$\mathrm{Hom}_R(M, R) \to \mathrm{Hom}_{\kappa(x)}(M(x), \kappa(x))$$

and set $V(x) = M(x)/B(x)$. Then any R-linear map $\varphi : M \to R$ induces a map $\overline{\varphi} : V(x) \to \kappa(x)$ and conversely any $\kappa(x)$-linear map $\lambda : V(x) \to \kappa(x)$ is equal to $\overline{\varphi}$ for some φ. Let $s_1, \ldots, s_r \in M$.

Suppose $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$. Then we can find $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_R(M, R)$ such that $\varphi_i(s_j)$ maps to δ_{ij}[21] in $\kappa(x)$. Hence the matrix of the composition

$$R^{\oplus r} \xrightarrow{s_1, \ldots, s_r} M \xrightarrow{\varphi_1, \ldots, \varphi_r} R^{\oplus r}$$

has a determinant $f \in R$ which maps to 1 in $\kappa(x)$ Clearly, this implies that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f.

Conversely, suppose that we have an $f \in R$, $f \notin x$ such that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f. Hence we can find R_f-linear maps $\varphi_i : M_f \to R_f$ such that $\varphi_i(s_j) = \delta_{ij} \in R_f$. Since $\mathrm{Hom}_R(M, R)_f = \mathrm{Hom}_{R_f}(M_f, R_f)$ by Algebra, Lemma 10.10.2 we conclude that we can find $n \geq 0$ and $\varphi_i' \in \mathrm{Hom}_R(M, R)$ such that $\varphi_i'(s_j) = f^n \delta_{ij} \in R$. It follows that $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$ as $\overline{\varphi}_i'(s_j) = f^n \delta_{ij}$. □

In Situation 15.128.1 given $s_1, \ldots, s_r \in M$ we denote $Z(s_1, \ldots, s_r) \subset \Omega$ the set of $x \in \Omega$ such that $s_1(x), \ldots, s_r(x)$ map to linearly dependent elements of $V(x)$. By the lemma this is a closed subset of Ω.

0GVA **Lemma 15.128.3.** *In Situation 15.128.1 let $x_1, \ldots, x_n \in \Omega$ be pairwise distinct. Let $v_i \in V(x_i)$. Then there exists an $s \in M$ such that $s(x_i)$ maps to v_i for $i = 1, \ldots, n$.*

Proof. Since x_i is a maximal ideal of R we may use Algebra, Lemma 10.15.4 to see that $M(x_1) \oplus \ldots \oplus M(x_n)$ is a quotient of M. □

0GVB **Proposition 15.128.4.** *In Situation 15.128.1 assume Ω is a Noetherian topological space. Let $s_1, \ldots, s_h \in M$. Let $Z(s_1, \ldots, s_h) \subset F \subset \Omega$ be closed. Let $x_1, \ldots, x_n \in F$ be pairwise distinct. Let $v_i \in V(x_i)$. Let $k \geq 0$ be an integer such that*

$$(*) \quad h + k \leq \dim_{\kappa(x)} V(x) \text{ for all } x \in \Omega$$

Then there exist $s \in M$ and $F' \subset \Omega$ closed such that

(a) *$s(x_i)$ maps to v_i,*
(b) *$Z(s_1, \ldots, s_h, s) \subset F \cup F'$, and*
(c) *every irreducible component of F' has codimension $\geq k$ in Ω.*

[**Ser58**, Theorem 2]

Proof. We note that codimension was defined in Topology, Section 5.11 and that we will use some results on Noetherian topological spaces contained in Topology, Section 5.9.

[21] Kronecker delta.

The proof is by induction on k. If $k = 0$, then we choose $s \in M$ as in Lemma 15.128.3 and we choose $F' = \Omega$.

Assume $k > 0$. By our induction hypothesis we may choose $u \in M$ and $G \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h$, F, $x_1, \ldots, x_n$, $v_1, \ldots, v_n$, and $k - 1$.

Let $G = G_1 \cup \ldots \cup G_m$ be the decomposition of G into its irreducible components. If $G_j \subset F$, then we can remove it from the list. Thus we may assume G_j is not contained in F for $j = 1, \ldots, m$. For $j = 1, \ldots, m$ choose $y_j \in G_j$ with $y_j \notin F$ and $y_j \notin G_{j'}$ for $j' \neq j$. This is possible as there are no inclusions among the irreducible components of G. Choose $w_j \in V(y_j)$ not contained in the span of the images of $s_1(y_j), \ldots, s_h(y_j)$; this is possible because $h + k \leq \dim V(y_j)$ and $k > 0$.

Apply the induction hypothesis to the $h + 1$ sections $s_1, \ldots, s_h, u$, the closed set $F \cup G$, the points $x_1, \ldots, x_n, y_1, \ldots, y_m \in F \cup G$, the elements $0 \in V(x_i)$ and $w_j \in V(y_j)$, and the integer $k - 1$. Note that we have increased h by 1 and decreased k by 1 hence the assumption $(*)$ of the proposition remains valid. This produces $t \in M$ and $H \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h, u$, $F \cup G$, $x_1, \ldots, x_n, y_1, \ldots, y_m$, $0, \ldots, 0, w_1, \ldots, w_m$, and $k - 1$.

Let $H_1, \ldots, H_p \subset H$ be the irreducible components of H which are not contained in $F \cup G$. As before pick $z_l \in H_l$, $z_l \notin F \cup G$ and $z_l \notin H_{l'}$ for $l' \neq l$. Using Algebra, Lemma 10.15.4 we may choose $f \in R$ such that $f(y_j) = 1$, $j = 1, \ldots, m$ and $f(z_l) = 0$, $l = 1, \ldots, p$. Claim: the element $s = u + ft$ works.

First, the value $s(x_i)$ agrees with $u(x_i)$ because $t(x_i) = 0$ and hence we see that $s(x_i)$ maps to v_i. This proves (a). To finish the proof it suffices to show that every irreducible component Z of $Z(s_1, \ldots, s_h, s)$ not contained in F has codimension $\geq k$ in Ω. Namely, then we can set F' equal to the union of these and we get (b) and (c). We can see that irreducible components Z of $Z(s_1, \ldots, s_h, s)$ of codimension $\leq k - 1$ do not exist as follows:

(1) Observe that $Z(s_1, \ldots, s_h, s) \subset Z(s_1, \ldots, s_h, u, t) = F \cup H$ as $s = u + ft$. Hence $Z \subset H$.
(2) The irreducible components of H have codimension $\geq k - 1$. Hence Z is equal to an irreducible component of H as Z has codimension $\leq k - 1$. Hence $Z = H_l$ for some $l \in \{1, \ldots, p\}$ or $Z = G_j$ for some $j \in \{1, \ldots m\}$.
(3) But $Z = G_j$ is impossible as $s_1(y_j), \ldots, s_h(y_j)$ map to linearly independent elements of $V(y_j)$ and $s(y_j) = u(y_j) + f(y_j)t(y_j) = u(y_j) + t(y_j)$ maps to an element of the form
$$\text{linear combination images of } s_i(y_j) + w_j$$
which is linearly independent of the images of $s_1(y_j), \ldots, s_h(y_j)$ in $V(y_j)$ by our choice of w_j.
(4) Also $Z = Z_l$ is impossible. Namely, again $s_1(z_l), \ldots, s_h(z_l)$ map to linearly independent elements of $V(z_l)$ and $s(z_l) = u(z_l) + f(z_l)t(z_l) = u(z_l)$ maps to an element of $V(z_l)$ linearly independent of those as $z_l \notin F \cup G$.

This finishes the proof. □

0GVC **Theorem 15.128.5.** *Let R be a ring whose max spectrum $\Omega \subset \mathrm{Spec}(R)$ is a Noetherian topological space of dimension $d < \infty$. Let M be a finitely presented R-module such that for all $\mathfrak{m} \in \Omega$ the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ has a free direct summand of rank $> d$. Then $M \cong R \oplus M'$.* [**Ser58**, Theorem 1]

Proof. For $\mathfrak{m} \in \Omega$ suppose that $R_{\mathfrak{m}}^{\oplus r}$ is a direct summand of $M_{\mathfrak{m}}$. Then by Algebra, Lemmas 10.9.9 and 10.127.6 we see that $R_f^{\oplus r}$ is a direct summand of M_f for some $f \in R$, $f \notin \mathfrak{m}$. Hence the assumption means that $\dim V(x) > d$ for all $x \in \Omega$ where $V(x)$ is as in Lemma 15.128.2. By Proposition 15.128.4 applied with $F = \emptyset$, $h = 0$ and no s_i, $n = 0$ and no x_i, v_i, and $k = d + 1$ we find an $s \in M$ and $F' \subset \Omega$ such that every irreducible component of F' has codimension $\geq d + 1$ and $Z(s) \subset F'$. Since $d = \dim(\Omega)$ this forces $F' = \emptyset$. Hence $s : R \to M$ is the inclusion of a direct summand at all maximal ideals. It follows that s is universally injective, see Algebra, Lemma 10.82.12. Then s is split injective by Algebra, Lemma 10.82.4. □

15.129. Big projective modules are free

0GVE In this section we discuss one of the results of [**Bas63**]; we suggest the reader look at the original paper. Our argument will use the slightly simplified proof given in the papers [**Aka70**] and [**Hin63**].

0GVF **Lemma 15.129.1** (Eilenberg's lemma)**.** *If $P \oplus Q \cong F$ with F a nonfinitely generated free module, then $P \oplus F \cong F$.* [**Bas63**, Eilenberg's lemma]

Proof.

$$F \cong F \oplus F \oplus \ldots \cong P \oplus Q \oplus P \oplus Q \oplus \ldots \cong P \oplus F \oplus F \oplus \ldots \cong P \oplus F$$

□

0GVG **Lemma 15.129.2.** *Let R be a ring. Let P be a projective module. There exists a free module F such that $P \oplus F$ is free.*

Proof. Since P is projective we see that $F_0 = P \oplus Q$ is a free module for some module Q. Set $F = \bigoplus_{n \geq 1} F_0$. Then $P \oplus F \cong F$ by Lemma 15.129.1. □

0GVH **Lemma 15.129.3.** *Let R be a ring. Let P be a projective module. Let $s \in P$. There exists a finite free module F and a finite free direct summand $K \subset F \oplus P$ with $(0, s) \in K$.*

Proof. By Lemma 15.129.2 we can find a (possibly infinite) free module F such that $F \oplus P$ is free. Then of course $(0, s)$ is contained in a finite free direct summand $K \subset F \oplus P$. In turn K is contained in $F' \oplus P$ where $F' \subset F$ is a finite free direct summand. □

0GVI **Lemma 15.129.4.** *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_{\mathfrak{m}}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$ and $M \subset P$ such that $Rs + M = P$. Then we can find $m \in M$ such that $R(s + m)$ is a free direct summand of P.*

Proof. The statement makes sense as $P_{\mathfrak{m}}$ is free by Algebra, Theorem 10.85.4.

Denote $M' \subset P/JP$ the image of M and $s' \in P/JP$ the image of s. Observe that $R/Js' + M' = P/JP$. Suppose we can find $m' \in M'$ such that $R/J(s' + m')$ is a free direct summand of M'. Choose $\varphi' : P/JP \to R/J$ which gives a splitting, i.e., we have $\varphi'(s' + m') = 1$ in R/J. Then since P is a projective R-module we can find a lift $\varphi : P \to R$ of φ'. Choose $m \in M$ mapping to m'. Then $\varphi(s + m) \in R$ is congruent to 1 modulo J and hence a unit in R (Algebra, Lemma 10.19.1). Whence $R(s + m)$ is a free direct summand of P. This reduces us to the case discussed in the next paragraph.

Assume R is Noetherian. Let $m \in M$ be an element and let $\varphi_1, \ldots, \varphi_n : P \to R$ be R-linear maps. Denote

$$Z(s + m, \varphi_1, \ldots, \varphi_n) \subset \mathrm{Spec}(R)$$

the vanishing locus of $\varphi_1(s + m), \ldots, \varphi_n(s + m) \in R$.

Suppose $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{m} \in Z(s + m, \varphi_1, \ldots, \varphi_n)$. Set $K = M \cap \bigcap \mathrm{Ker}(\varphi_i)$. We claim the image of

$$K/\mathfrak{m}K \to P/\mathfrak{m}P$$

has infinite dimension. Namely, the quotient P/K is a finite R-module as it is isomorphic to a submodule of $P/M \oplus R^{\oplus n}$. Thus we see that the kernel of the displayed arrow is a quotient of $\mathrm{Tor}_1^R(P/K, \kappa(\mathfrak{m}))$ which is finite by Algebra, Lemma 10.75.7. Combined with the fact that $P/\mathfrak{m}P$ has infinite dimension we obtain our claim. Thus we can find a $t \in K$ which maps to an element $\bar{t}$ of the vector space $P/\mathfrak{m}P$ which is linearly independent of the image $\overline{s + m}$ of $s+m$. By linear algebra, we find an R-linear map $\overline{\varphi} : P \to \kappa(\mathfrak{m})$ such that $\overline{\varphi}(\bar{t}) = 1$ and $\overline{\varphi}(\overline{s + m}) = 0$. Since P is projective, we can find an R-linear map $\varphi : P \to R$ lifting $\overline{\varphi}$. Then we see that the vanishing locus $Z(s + m + t, \varphi_1, \ldots, \varphi_n, \varphi)$ is contained in $Z(s + m, \varphi_1, \ldots, \varphi_n)$ but does not contain $\mathfrak{m}$, i.e., it is strictly smaller than $Z(s + m, \varphi_1, \ldots, \varphi_n)$.

Since $\mathrm{Spec}(R)$ is a Noetherian topological space, we see from the arguments above that we may find $m \in M$ and $\varphi_1, \ldots, \varphi_n : P \to R$ such that the closed subset $Z(s + m, \varphi_1, \ldots, \varphi_n)$ does not contain any closed points of $\mathrm{Spec}(R)$. Hence $Z(s + m, \varphi_1, \ldots, \varphi_n) = \emptyset$. Hence we can find $r_1, \ldots, r_n \in R$ such that $\sum r_i \varphi_i(s + m) = 1$. Hence

$$R \xrightarrow{s+m} P \xrightarrow{\sum r_i \varphi_i} R$$

is the desired splitting. □

0GVJ **Lemma 15.129.5.** *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_{\mathfrak{m}}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$. Then we can find a finite stably free direct summand $M \subset P$ such that $s \in M$.*

Proof. By Lemma 15.129.3 we can find a finite free module F and a finite free direct summand $K \subset F \oplus P$ such that $(0, s) \in K$. By induction on the rank of F we reduce to the case discussed in the next paragraph.

Assume there exists a finite stably free direct summand $K \subset R \oplus P$ such that $(0, s) \in K$. Choose a complement K' of K, i.e., such that $R \oplus P = K \oplus K'$. The projection $\pi : R \oplus P \to K'$ is surjectve, hence by Lemma 15.129.4 we find a $p \in P$ such that $\pi(1, p) \in K'$ generates a free direct summand. Accordingly we write $K' = R\pi(1, p) \oplus K''$. We see that

$$R \oplus P = K \oplus K' = K \oplus R\pi(1, p) \oplus K''$$

The projection $\pi' : P \to K''$ is surjective[22] and hence split (as K'' is projective). Thus $\mathrm{Ker}(\pi') \subset P$ is a direct summand containing s. Finally, by construction we

[22]Namely, if $k'' \in K''$ then k'' viewed as an element of K' can be written as $k'' = \lambda\pi(1, 0) + \pi(0, q)$ for some $\lambda \in R$ and $q \in P$. This means $k'' = \lambda\pi(1, p) + \pi(0, q - \lambda p)$. This in turn means that $q - \lambda p$ maps to k'' by the composition $P \to R \oplus P \xrightarrow{\pi} K' \to K''$ since $K' \to K''$ annihilates $\pi(1, p)$.

have an isomorphism

$$R \oplus \mathrm{Ker}(\pi') \cong K \oplus R\pi(1, p)$$

and hence since K is finite and stably free, so is $\mathrm{Ker}(\pi')$. □

0GVK **Theorem 15.129.6.** *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a countably generated projective R-module such that $P_{\mathfrak{m}}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Then P is free.*

Commutative case of [**Bas63**, Theorem 3.1]

Proof. We first prove that P is a countable direct sum of finite stably free modules. Let $x_1, x_2, \ldots$ be a countable set of generators for P. We inductively construct finite stably free direct summands $F_1, F_2, \ldots$ of P such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of P which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$P = F_1 \oplus \ldots \oplus F_n \oplus P'$$

and let $s \in P'$ be the image of x_{n+1}. By Lemma 15.129.5 we can find a finite stably free direct summand $F_{n+1} \subset P'$ containing s. Then $P = \bigoplus_{i=1}^{\infty} F_i$.

Assume that P is an infinite direct sum $P = \bigoplus_{i=1}^{\infty} F_i$ of nonzero finite stably free modules. The stable freeness of the modules F_i will be used in the following manner: the rank of each F_i is constant (and positive). Hence we see that $P_{\mathfrak{m}}$ is free of countably infinite rank for each maximal ideal $\mathfrak{m}$ of R. By Lemma 15.129.4 applied with $s = 0$ and $M = P$, we can find a $t_1 \in P$ such that Rt_1 is a free direct summand of P. Then t_1 is contained in $F_1 \oplus \ldots \oplus F_{n_1}$ for some $n_1 > n_0 = 0$. The same reasoning applied to $\bigoplus_{n>n_1} F_n$ produces an $n_1 < n_2$ and $t_2 \in F_{n_1+1} \oplus \ldots \oplus F_{n_2}$ which generates a free direct summand. Continuing in this fashion we obtain a free direct summand

$$\bigoplus\nolimits_{i\geq 1} t_i : \bigoplus\nolimits_{i\geq 1} R \longrightarrow \bigoplus\nolimits_{i\geq 1} \bigoplus\nolimits_{n_i \geq n > n_{i-1}} F_n = P$$

of infinite rank. Thus we see that $P \cong Q \oplus F$ for some free R-module F of countable rank. Since Q is countably generated it follows that $Q \oplus Q' \cong F$ for some module Q'. Then the Eilenberg swindle (Lemma 15.129.1) implies that $Q \oplus F \cong F$ and P is free. □

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CHAPTER 16

Smoothing Ring Maps

07BW

16.1. Introduction

07BX The main result of this chapter is the following:

> A regular map of Noetherian rings is a filtered colimit of smooth ones.

This theorem is due to Popescu, see [**Pop90**]. A readable exposition of Popescu's proof was given by Richard Swan, see [**Swa98**] who used notes by André and a paper of Ogoma, see [**Ogo94**].

Our exposition follows Swan's, but we first prove an intermediate result which lets us work in a slightly simpler situation. Here is an overview. We first solve the following "lifting problem": A flat infinitesimal deformation of a filtered colimit of smooth algebras is a filtered colimit of smooth algebras. This result essentially says that it suffices to prove the main theorem for maps between reduced Noetherian rings. Next we prove two very clever lemmas called the "lifting lemma" and the "desingularization lemma". We show that these lemmas combined reduce the main theorem to proving a Noetherian, geometrically regular algebra Λ over a field k is a filtered colimit of smooth k-algebras. Next, we discuss the necessary local tricks that go into the Popescu-Ogoma-Swan-André proof. Finally, in the last three sections we give the proof.

We end this introduction with some pointers to references. Let A be a henselian Noetherian local ring. We say A has the *approximation property* if for any $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ the system of equations $f_1 = 0, \ldots, f_m = 0$ has a solution in the completion of A if and only if it has a solution in A. This definition is due to Artin. Artin first proved the approximation property for analytic systems of equations, see [**Art68**]. In [**Art69a**] Artin proved the approximation property for local rings essentially of finite type over an excellent discrete valuation ring. Artin conjectured (page 26 of [**Art69a**]) that every excellent henselian local ring should have the approximation property.

At some point in time it became a conjecture that every regular homomorphism of Noetherian rings is a filtered colimit of smooth algebras (see for example [**Ray72**], [**Pop81**], [**Art82**], [**AD83**]). We're not sure who this conjecture[1] is due to. The relationship with the approximation property is that if $A \to A^\wedge$ is a colimit of smooth algebras with A as above, then the approximation property holds (insert future reference here). Moreover, the main theorem applies to the map $A \to A^\wedge$ if A is an excellent local ring, as one of the conditions of an excellent local ring

[1] The question/conjecture as formulated in [**Art82**], [**AD83**], and [**Pop81**] is stronger and was shown to be equivalent to the original version in [**CP84**].

is that the formal fibres are geometrically regular. Note that excellent local rings were defined by Grothendieck and their definition appeared in print in 1965.

In [**Art82**] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over a field. In [**AR88**] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over an excellent discrete valuation ring. Finally, the main theorem was shown in [**Pop85**], [**Pop86**], [**Pop90**], [**Ogo94**], and [**Swa98**] as discussed above.

Conversely, using some of the results above, in [**Rot90**] it was shown that any Noetherian local ring with the approximation property is excellent.

The paper [**Spi99**] provides an alternative approach to the main theorem, but it seems hard to read (for example [**Spi99**, Lemma 5.2] appears to be an incorrectly reformulated version of [**Elk73**, Lemma 3]). There is also a Bourbaki lecture about this material, see [**Tei95**].

16.2. Singular ideals

07C4 Let $R \to A$ be a ring map. The singular ideal of A over R is the radical ideal in A cutting out the singular locus of the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. Here is a formal definition.

07C5 **Definition 16.2.1.** Let $R \to A$ be a ring map. The *singular ideal of A over R*, denoted $H_{A/R}$ is the unique radical ideal $H_{A/R} \subset A$ with

$$V(H_{A/R}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid R \to A \text{ not smooth at } \mathfrak{q}\}$$

This makes sense because the set of primes where $R \to A$ is smooth is open, see Algebra, Definition 10.137.10. In order to find an explicit set of generators for the singular ideal we first prove the following lemma.

07C6 **Lemma 16.2.2.** *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{q} \subset A$ be a prime ideal. Assume $R \to A$ is smooth at $\mathfrak{q}$. Then there exists an $a \in A$, $a \notin \mathfrak{q}$, an integer c, $0 \leq c \leq \min(n, m)$, subsets $U \subset \{1, \ldots, n\}$, $V \subset \{1, \ldots, m\}$ of cardinality c such that*

$$a = a' \det(\partial f_j / \partial x_i)_{j \in V, i \in U}$$

for some $a' \in A$ and

$$af_\ell \in (f_j, j \in V) + (f_1, \ldots, f_m)^2$$

for all $\ell \in \{1, \ldots, m\}$.

Proof. Set $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is homotopy equivalent to $I/I^2 \to \bigoplus A\mathrm{d}x_i$, see Algebra, Lemma 10.134.2. We will use the formation of the naive cotangent complex commutes with localization, see Algebra, Section 10.134, especially Algebra, Lemma 10.134.13. By Algebra, Definitions 10.137.1 and 10.137.10 we see that $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ is a split injection for some $a \in A$, $a \notin \mathfrak{q}$. After renumbering $x_1, \ldots, x_n$ and $f_1, \ldots, f_m$ we may assume that $f_1, \ldots, f_c$ form a basis for the vector space $I/I^2 \otimes_A \kappa(\mathfrak{q})$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{q})$. Hence after replacing a by aa' for some $a' \in A$, $a' \notin \mathfrak{q}$ we may assume $f_1, \ldots, f_c$ form a basis for $(I/I^2)_a$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $(\Omega_{A/R})_a$. In this situation a^N for some large integer N satisfies the conditions of the lemma (with $U = V = \{1, \ldots, c\}$). □

We will use the notion of a *strictly standard* element in A over R. Our notion is slightly weaker than the one in Swan's paper [**Swa98**]. We also define an *elementary standard* element to be one of the type we found in the lemma above. We compare the different types of elements in Lemma 16.3.7.

07C7 **Definition 16.2.3.** Let $R \to A$ be a ring map of finite presentation. We say an element $a \in A$ is *elementary standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

07C8 (16.2.3.1) $$a = a' \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$$

for some $a' \in A$ and

07C9 (16.2.3.2) $$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m - c$. We say $a \in A$ is *strictly standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

07ER (16.2.3.3) $$a = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} a_I \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I}$$

for some $a_I \in A$ and

07ES (16.2.3.4) $$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m - c$.

The following lemma is useful to find implications of (16.2.3.3).

07ET **Lemma 16.2.4.** *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and write $I = (f_1, \ldots, f_m)$. Let $a \in A$. Then (16.2.3.3) implies there exists an A-linear map $\psi : \bigoplus_{i=1,\ldots,n} A dx_i \to A^{\oplus c}$ such that the composition*

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} I/I^2 \xrightarrow{f \mapsto df} \bigoplus\nolimits_{i=1,\ldots,n} A dx_i \xrightarrow{\psi} A^{\oplus c}$$

is multiplication by a. Conversely, if such a ψ exists, then a^c satisfies (16.2.3.3).

Proof. This is a special case of Algebra, Lemma 10.15.5. □

07CA **Lemma 16.2.5** (Elkik)**.** *Let $R \to A$ be a ring map of finite presentation. The singular ideal $H_{A/R}$ is the radical of the ideal generated by strictly standard elements in A over R and also the radical of the ideal generated by elementary standard elements in A over R.*

Proof. Assume a is strictly standard in A over R. We claim that A_a is smooth over R, which proves that $a \in H_{A/R}$. Namely, let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, c, and $a' \in A$ be as in Definition 16.2.3. Write $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is given by $I/I^2 \to \bigoplus A \mathrm{d}x_i$. Assumption (16.2.3.4) implies that $(I/I^2)_a$ is generated by the classes of $f_1, \ldots, f_c$. Assumption (16.2.3.3) implies that the differential $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ has a left inverse, see Lemma 16.2.4. Hence $R \to A_a$ is smooth by definition and Algebra, Lemma 10.134.13.

Let $H_e, H_s \subset A$ be the radical of the ideal generated by elementary, resp. strictly standard elements of A over R. By definition and what we just proved we have $H_e \subset H_s \subset H_{A/R}$. The inclusion $H_{A/R} \subset H_e$ follows from Lemma 16.2.2. □

07CB **Example 16.2.6.** The set of points where a finitely presented ring map is smooth needn't be a quasi-compact open. For example, let $R = k[x, y_1, y_2, y_3, \ldots]/(xy_i)$ and $A = R/(x)$. Then the smooth locus of $R \to A$ is $\bigcup D(y_i)$ which is not quasi-compact.

07CC **Lemma 16.2.7.** *Let $R \to A$ be a ring map of finite presentation. Let $R \to R'$ be a ring map. If $a \in A$ is elementary, resp. strictly standard in A over R, then $a \otimes 1$ is elementary, resp. strictly standard in $A \otimes_R R'$ over R'.*

Proof. If $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a presentation of A over R, then $A \otimes_R R' = R'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_m)$ is a presentation of $A \otimes_R R'$ over R'. Here f'_j is the image of f_j in $R'[x_1, \ldots, x_n]$. Hence the result follows from the definitions. □

07EU **Lemma 16.2.8.** *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Assume that $H_{A/R}\Lambda = \Lambda$. Then there exists a factorization $A \to B \to \Lambda$ with B smooth over R.*

Proof. Choose $f_1, \ldots, f_r \in H_{A/R}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\sum f_i\lambda_i = 1$ in Λ. Set $B = A[x_1, \ldots, x_r]/(f_1x_1 + \ldots + f_rx_r - 1)$ and define $B \to \Lambda$ by mapping x_i to λ_i. To check that B is smooth over R use that A_{f_i} is smooth over R by definition of $H_{A/R}$ and that B_{f_i} is smooth over A_{f_i}. Details omitted. □

16.3. Presentations of algebras

07CD Some of the results in this section are due to Elkik. Note that the algebra C in the following lemma is a symmetric algebra over A. Moreover, if R is Noetherian, then C is of finite presentation over R.

07CE **Lemma 16.3.1.** *Let R be a ring and let A be a finitely presented R-algebra. There exists finite type R-algebra map $A \to C$ which has a retraction with the following two properties*

1. *for each $a \in A$ such that $R \to A_a$ is a local complete intersection (More on Algebra, Definition 15.33.2) the ring C_a is smooth over A_a and has a presentation $C_a = R[y_1, \ldots, y_m]/J$ such that J/J^2 is free over C_a, and*
2. *for each $a \in A$ such that A_a is smooth over R the module $\Omega_{C_a/R}$ is free over C_a.*

Proof. Choose a presentation $A = R[x_1, \ldots, x_n]/I$ and write $I = (f_1, \ldots, f_m)$. Define the A-module K by the short exact sequence

$$0 \to K \to A^{\oplus m} \to I/I^2 \to 0$$

where the jth basis vector e_j in the middle is mapped to the class of f_j on the right. Set

$$C = \mathrm{Sym}^*_A(I/I^2).$$

The retraction is just the projection onto the degree 0 part of C. We have a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to C$ which maps y_j to the class of f_j in I/I^2. The kernel J of this map is generated by the elements $f_1, \ldots, f_m$ and by elements $\sum h_jy_j$ with $h_j \in R[x_1, \ldots, x_n]$ such that $\sum h_je_j$ defines an element of K. By Algebra, Lemma 10.134.4 applied to $R \to A \to C$ and the presentations above and More on Algebra, Lemma 15.9.12 there is a short exact sequence

07EW (16.3.1.1) $$I/I^2 \otimes_A C \to J/J^2 \to K \otimes_A C \to 0$$

of C-modules. Let $h \in R[x_1, \ldots, x_n]$ be an element with image $a \in A$. We will use as presentations for the localized rings

$$A_a = R[x_0, x_1, \ldots, x_n]/I' \quad \text{and} \quad C_a = R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]/J'$$

where $I' = (hx_0 - 1, I)$ and $J' = (hx_0 - 1, J)$. Hence $I'/(I')^2 = A_a \oplus (I/I^2)_a$ as A_a-modules and $J'/(J')^2 = C_a \oplus (J/J^2)_a$ as C_a-modules. Thus we obtain

07EX (16.3.1.2)
$$C_a \oplus I/I^2 \otimes_A C_a \to C_a \oplus (J/J^2)_a \to K \otimes_A C_a \to 0$$

as the sequence of Algebra, Lemma 10.134.4 corresponding to $R \to A_a \to C_a$ and the presentations above.

Next, assume that $a \in A$ is such that A_a is a local complete intersection over R. Then $(I/I^2)_a$ is finite projective over A_a, see More on Algebra, Lemma 15.32.3. Hence we see $K_a \oplus (I/I^2)_a \cong A_a^{\oplus m}$ is free. In particular K_a is finite projective too. By More on Algebra, Lemma 15.33.6 the sequence (16.3.1.2) is exact on the left. Hence

$$J'/(J')^2 \cong C_a \oplus I/I^2 \otimes_A C_a \oplus K \otimes_A C_a \cong C_a^{\oplus m+1}$$

This proves (1). Finally, suppose that in addition A_a is smooth over R. Then the same presentation shows that $\Omega_{C_a/R}$ is the cokernel of the map

$$J'/(J')^2 \longrightarrow \bigoplus\nolimits_i C_a \mathrm{d}x_i \oplus \bigoplus\nolimits_j C_a \mathrm{d}y_j$$

The summand C_a of $J'/(J')^2$ in the decomposition above corresponds to $hx_0 - 1$ and hence maps isomorphically to the summand $C_a \mathrm{d}x_0$. The summand $I/I^2 \otimes_A C_a$ of $J'/(J')^2$ maps injectively to $\bigoplus_{i=1,\ldots,n} C_a \mathrm{d}x_i$ with quotient $\Omega_{A_a/R} \otimes_{A_a} C_a$. The summand $K \otimes_A C_a$ maps injectively to $\bigoplus_{j \geq 1} C_a \mathrm{d}y_j$ with quotient isomorphic to $I/I^2 \otimes_A C_a$. Thus the cokernel of the last displayed map is the module $I/I^2 \otimes_A C_a \oplus \Omega_{A_a/R} \otimes_{A_a} C_a$. Since $(I/I^2)_a \oplus \Omega_{A_a/R}$ is free (from the definition of smooth ring maps) we see that (2) holds. □

The following proposition was proved for smooth ring maps over henselian pairs by Elkik in [**Elk73**]. For smooth ring maps it can be found in [**Ara01**], where it is also proven that ring maps between smooth algebras can be lifted.

07M8 **Proposition 16.3.2.** *Let $R \to R_0$ be a surjective ring map with kernel I.*

1. *If $R_0 \to A_0$ is a syntomic ring map, then there exists a syntomic ring map $R \to A$ such that $A/IA \cong A_0$.*
2. *If $R_0 \to A_0$ is a smooth ring map, then there exists a smooth ring map $R \to A$ such that $A/IA \cong A_0$.*

Proof. Assume $R_0 \to A_0$ syntomic, in particular a local complete intersection (More on Algebra, Lemma 15.33.5). Choose a presentation $A_0 = R_0[x_1, \ldots, x_n]/J_0$. Set $C_0 = \mathrm{Sym}^*_{A_0}(J_0/J_0^2)$. Note that J_0/J_0^2 is a finite projective A_0-module (Algebra, Lemma 10.136.16). By Lemma 16.3.1 the ring map $A_0 \to C_0$ is smooth and we can find a presentation $C_0 = R_0[y_1, \ldots, y_m]/K_0$ with K_0/K_0^2 free over C_0. By Algebra, Lemma 10.136.6 we can assume $C_0 = R_0[y_1, \ldots, y_m]/(\overline{f}_1, \ldots, \overline{f}_c)$ where $\overline{f}_1, \ldots, \overline{f}_c$ maps to a basis of K_0/K_0^2 over C_0. Choose $f_1, \ldots, f_c \in R[y_1, \ldots, y_c]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$ and set

$$C = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$$

By construction $C_0 = C/IC$. By Algebra, Lemma 10.136.10 we can after replacing C by C_g assume that C is a relative global complete intersection over R. We conclude that there exists a finite projective A_0-module P_0 such that $C_0 = \mathrm{Sym}^*_{A_0}(P_0)$ is isomorphic to C/IC for some syntomic R-algebra C.

Choose an integer n and a direct sum decomposition $A_0^{\oplus n} = P_0 \oplus Q_0$. By More on Algebra, Lemma 15.9.11 we can find an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$ and a finite projective C'-module Q such that Q/IQ is isomorphic to $Q_0 \otimes_{A_0} C/IC$. Then $D = \mathrm{Sym}^*_{C'}(Q)$ is a smooth C'-algebra (see More on Algebra, Lemma 15.9.13). Picture

$$\begin{array}{ccccccc} R & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R/I & \longrightarrow & A_0 & \longrightarrow & C/IC & \xrightarrow{\cong} & C'/IC' \longrightarrow D/ID \end{array}$$

Observe that our choice of Q gives

$$\begin{aligned} D/ID &= \mathrm{Sym}^*_{C/IC}(Q_0 \otimes_{A_0} C/IC) \\ &= \mathrm{Sym}^*_{A_0}(Q_0) \otimes_{A_0} C/IC \\ &= \mathrm{Sym}^*_{A_0}(Q_0) \otimes_{A_0} \mathrm{Sym}^*_{A_0}(P_0) \\ &= \mathrm{Sym}^*_{A_0}(Q_0 \oplus P_0) \\ &= \mathrm{Sym}^*_{A_0}(A_0^{\oplus n}) \\ &= A_0[x_1, \ldots, x_n] \end{aligned}$$

Choose $f_1, \ldots, f_n \in D$ which map to $x_1, \ldots, x_n$ in $D/ID = A_0[x_1, \ldots, x_n]$. Set $A = D/(f_1, \ldots, f_n)$. Note that $A_0 = A/IA$. We claim that $R \to A$ is syntomic in a neighbourhood of $V(IA)$. If the claim is true, then we can find a $f \in A$ mapping to $1 \in A_0$ such that A_f is syntomic over R and the proof of (1) is finished.

Proof of the claim. Observe that $R \to D$ is syntomic as a composition of the syntomic ring map $R \to C$, the étale ring map $C \to C'$ and the smooth ring map $C' \to D$ (Algebra, Lemmas 10.136.17 and 10.137.9). The question is local on $\mathrm{Spec}(D)$, hence we may assume that D is a relative global complete intersection (Algebra, Lemma 10.136.15). Say $D = R[y_1, \ldots, y_m]/(g_1, \ldots, g_s)$. Let $f_1', \ldots, f_n' \in R[y_1, \ldots, y_m]$ be lifts of $f_1, \ldots, f_n$. Then we can apply Algebra, Lemma 10.136.10 to get the claim.

Proof of (2). Since a smooth ring map is syntomic, we can find a syntomic ring map $R \to A$ such that $A_0 = A/IA$. By assumption the fibres of $R \to A$ are smooth over primes in $V(I)$ hence $R \to A$ is smooth in an open neighbourhood of $V(IA)$ (Algebra, Lemma 10.137.16). Thus we can replace A by a localization to obtain the result we want. □

We know that any syntomic ring map $R \to A$ is locally a relative global complete intersection, see Algebra, Lemma 10.136.15. The next lemma says that a vector bundle over $\mathrm{Spec}(A)$ is a relative global complete intersection.

07CG **Lemma 16.3.3.** *Let $R \to A$ be a syntomic ring map. Then there exists a smooth R-algebra map $A \to C$ with a retraction such that C is a global relative complete intersection over R, i.e.,*

$$C \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

flat over R and all fibres of dimension $n - c$.

Proof. Apply Lemma 16.3.1 to get $A \to C$. By Algebra, Lemma 10.136.6 we can write $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with f_i mapping to a basis of J/J^2. The ring map $R \to C$ is syntomic (hence flat) as it is a composition of a syntomic and a smooth ring map. The dimension of the fibres is $n - c$ by Algebra, Lemma 10.135.4 (the fibres are local complete intersections, so the lemma applies). □

07CH **Lemma 16.3.4.** *Let $R \to A$ be a smooth ring map. Then there exists a smooth R-algebra map $A \to B$ with a retraction such that B is standard smooth over R, i.e.,*

$$B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

and $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B.

Proof. Apply Lemma 16.3.3 to get a smooth R-algebra map $A \to C$ with a retraction such that $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. As C is smooth over R we have a short exact sequence

$$0 \to \bigoplus\nolimits_{j=1,\ldots,c} Cf_j \to \bigoplus\nolimits_{i=1,\ldots,n} C\mathrm{d}x_i \to \Omega_{C/R} \to 0$$

Since $\Omega_{C/R}$ is a projective C-module this sequence is split. Choose a left inverse t to the first map. Say $t(\mathrm{d}x_i) = \sum c_{ij} f_j$ so that $\sum_i \frac{\partial f_j}{\partial x_i} c_{i\ell} = \delta_{j\ell}$ (Kronecker delta). Let

$$B' = C[y_1, \ldots, y_c] = R[x_1, \ldots, x_n, y_1, \ldots, y_c]/(f_1, \ldots, f_c)$$

The R-algebra map $C \to B'$ has a retraction given by mapping y_j to zero. We claim that the map

$$R[z_1, \ldots, z_n] \longrightarrow B', \quad z_i \longmapsto x_i - \sum\nolimits_j c_{ij} y_j$$

is étale at every point in the image of $\mathrm{Spec}(C) \to \mathrm{Spec}(B')$. In $\Omega_{B'/R[z_1,\ldots,z_n]}$ we have

$$0 = \mathrm{d}f_j - \sum\nolimits_i \frac{\partial f_j}{\partial x_i} \mathrm{d}z_i \equiv \sum\nolimits_{i,\ell} \frac{\partial f_j}{\partial x_i} c_{i\ell} \mathrm{d}y_\ell \equiv \mathrm{d}y_j \bmod (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$$

Since $0 = \mathrm{d}z_i = \mathrm{d}x_i$ modulo $\sum B' \mathrm{d}y_j + (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$ we conclude that

$$\Omega_{B'/R[z_1,\ldots,z_n]}/(y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]} = 0.$$

As $\Omega_{B'/R[z_1,\ldots,z_n]}$ is a finite B'-module by Nakayama's lemma there exists a $g \in 1 + (y_1, \ldots, y_c)$ that $(\Omega_{B'/R[z_1,\ldots,z_n]})_g = 0$. This proves that $R[z_1, \ldots, z_n] \to B'_g$ is unramified, see Algebra, Definition 10.151.1. For any ring map $R \to k$ where k is a field we obtain an unramified ring map $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ between smooth k-algebras of dimension n. It follows that $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ is flat by Algebra, Lemmas 10.128.1 and 10.140.2. By the critère de platitude par fibre (Algebra, Lemma 10.128.8) we conclude that $R[z_1, \ldots, z_n] \to B'_g$ is flat. Finally, Algebra, Lemma 10.143.7 implies that $R[z_1, \ldots, z_n] \to B'_g$ is étale. Set $B = B'_g$. Note that $C \to B$ is smooth and has a retraction, so also $A \to B$ is smooth and has a retraction. Moreover, $R[z_1, \ldots, z_n] \to B$ is étale. By Algebra, Lemma 10.143.2 we can write

$$B = R[z_1, \ldots, z_n, w_1, \ldots, w_m]/(g_1, \ldots, g_m)$$

with $\det(\partial g_j/\partial w_i)$ invertible in B. This proves the lemma. □

07CI **Lemma 16.3.5.** *Let $R \to \Lambda$ be a ring map. If Λ is a filtered colimit of smooth R-algebras, then Λ is a filtered colimit of standard smooth R-algebras.*

Proof. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. According to Algebra, Lemma 10.127.4 we have to factor this map through a standard smooth algebra, and we know we can factor it as $A \to B \to \Lambda$ with B smooth over R. Choose an R-algebra map $B \to C$ with a retraction $C \to B$ such that C is standard smooth over R, see Lemma 16.3.4. Then the desired factorization is $A \to B \to C \to B \to \Lambda$. □

07EY **Lemma 16.3.6.** *Let $R \to A$ be a standard smooth ring map. Let $E \subset A$ be a finite subset of order $|E| = n$. Then there exists a presentation $A = R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ with $c \geq n$, with $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A, and such that E is the set of congruence classes of $x_1, \ldots, x_n$.*

Proof. Choose a presentation $A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ such that the image of $\det(\partial g_j/\partial y_i)_{i,j=1,\ldots,d}$ is invertible in A. Choose an enumerations $E = \{a_1, \ldots, a_n\}$ and choose $h_i \in R[y_1, \ldots, y_m]$ whose image in A is a_i. Consider the presentation

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(x_1 - h_1, \ldots, x_n - h_n, g_1, \ldots, g_d)$$

and set $c = n + d$. □

07EZ **Lemma 16.3.7.** *Let $R \to A$ be a ring map of finite presentation. Let $a \in A$. Consider the following conditions on a:*

(1) *A_a is smooth over R,*
(2) *A_a is smooth over R and $\Omega_{A_a/R}$ is stably free,*
(3) *A_a is smooth over R and $\Omega_{A_a/R}$ is free,*
(4) *A_a is standard smooth over R,*
(5) *a is strictly standard in A over R,*
(6) *a is elementary standard in A over R.*

Then we have

(a) *(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1),*
(b) *(6) $\Rightarrow$ (5),*
(c) *(6) $\Rightarrow$ (4),*
(d) *(5) $\Rightarrow$ (2),*
(e) *(2) $\Rightarrow$ the elements a^e, $e \geq e_0$ are strictly standard in A over R,*
(f) *(4) $\Rightarrow$ the elements a^e, $e \geq e_0$ are elementary standard in A over R.*

Proof. Part (a) is clear from the definitions and Algebra, Lemma 10.137.6. Part (b) is clear from Definition 16.2.3.

Proof of (c). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.2.3.1) and (16.2.3.2) hold. Choose $h \in R[x_1, \ldots, x_n]$ mapping to a. Then

$$A_a = R[x_0, x_1, \ldots, x_n]/(x_0 h - 1, f_1, \ldots, f_m).$$

Write $J = (x_0 h - 1, f_1, \ldots, f_m)$. By (16.2.3.2) we see that the A_a-module J/J^2 is generated by $x_0 h - 1, f_1, \ldots, f_c$ over A_a. Hence, as in the proof of Algebra, Lemma 10.136.6, we can choose a $g \in 1 + J$ such that

$$A_a = R[x_0, \ldots, x_n, x_{n+1}]/(x_0 h - 1, f_1, \ldots, f_c, g x_{n+1} - 1).$$

At this point (16.2.3.1) implies that $R \to A_a$ is standard smooth (use the coordinates $x_0, x_1, \ldots, x_c, x_{n+1}$ to take derivatives).

Proof of (d). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.2.3.3) and (16.2.3.4) hold. Write $I = (f_1, \ldots, f_m)$. We already know that A_a is smooth over R, see Lemma 16.2.5. By Lemma 16.2.4 we see that $(I/I^2)_a$ is free on $f_1, \ldots, f_c$ and maps isomorphically to a direct summand of $\bigoplus A_a \mathrm{d}x_i$. Since $\Omega_{A_a/R} = (\Omega_{A/R})_a$ is the cokernel of the map $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ we conclude that it is stably free.

Proof of (e). Choose a presentation $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. By assumption we have a short exact sequence

$$0 \to (I/I^2)_a \to \bigoplus\nolimits_{i=1,\ldots,n} A_a \mathrm{d}x_i \to \Omega_{A_a/R} \to 0$$

which is split exact. Hence we see that $(I/I^2)_a \oplus \Omega_{A_a/R}$ is a free A_a-module. Since $\Omega_{A_a/R}$ is stably free we see that $(I/I^2)_a$ is stably free as well. Thus replacing the presentation chosen above by $A = R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}]/J$ with $J = (I, x_{n+1}, \ldots, x_{n+r})$ for some r we get that $(J/J^2)_a$ is (finite) free. Choose $f_1, \ldots, f_c \in J$ which map to a basis of $(J/J^2)_a$. Extend this to a list of generators $f_1, \ldots, f_m \in J$. Consider the presentation $A = R[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_m)$. Then (16.2.3.4) holds for a^e for all sufficiently large e by construction. Moreover, since $(J/J^2)_a \to \bigoplus_{i=1,\ldots,n+r} A_a \mathrm{d}x_i$ is a split injection we can find an A_a-linear left inverse. Writing this left inverse in terms of the basis $f_1, \ldots, f_c$ and clearing denominators we find a linear map $\psi_0 : A^{\oplus n+r} \to A^{\oplus c}$ such that

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} J/J^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n+r} A \mathrm{d}x_i \xrightarrow{\psi_0} A^{\oplus c}$$

is multiplication by a^{e_0} for some $e_0 \geq 1$. By Lemma 16.2.4 we see (16.2.3.3) holds for all a^{ce_0} and hence for a^e for all e with $e \geq ce_0$.

Proof of (f). Choose a presentation $A_a = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in A_a. We may assume that for some $m < n$ the classes of the elements $x_1, \ldots, x_m$ correspond $a_i/1$ where $a_1, \ldots, a_m \in A$ are generators of A over R, see Lemma 16.3.6. After replacing x_i by $a^N x_i$ for $m < i \leq n$ we may assume the class of x_i is $a_i/1 \in A_a$ for some $a_i \in A$. Consider the ring map

$$\Psi : R[x_1, \ldots, x_n] \longrightarrow A, \quad x_i \longmapsto a_i.$$

This is a surjective ring map. By replacing f_j by $a^N f_j$ we may assume that $f_j \in R[x_1, \ldots, x_n]$ and that $\Psi(f_j) = 0$ (since after all $f_j(a_1/1, \ldots, a_n/1) = 0$ in A_a). Let $J = \mathrm{Ker}(\Psi)$. Then $A = R[x_1, \ldots, x_n]/J$ is a presentation and $f_1, \ldots, f_c \in J$ are elements such that $(J/J^2)_a$ is freely generated by $f_1, \ldots, f_c$ and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ maps to an invertible element of A_a. It follows that (16.2.3.1) and (16.2.3.2) hold for a^e and all large enough e as desired. □

16.4. Intermezzo: Néron desingularization

0BJ1 We interrupt the attack on the general case of Popescu's theorem to an easier but already very interesting case, namely, when $R \to \Lambda$ is a homomorphism of discrete valuation rings. This is discussed in [**Art69a**, Section 4].

0BJ2 **Situation 16.4.1.** Here $R \subset \Lambda$ is an extension of discrete valuation rings with ramification index 1 (More on Algebra, Definition 15.112.1). We assume given a factorization

$$R \to A \xrightarrow{\varphi} \Lambda$$

with $R \to A$ flat and of finite type. Let $\mathfrak{q} = \mathrm{Ker}(\varphi)$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_\Lambda)$.

In Situation 16.4.1 let $\pi \in R$ be a uniformizer. Recall that flatness of A over R signifies that π is a nonzerodivisor on A (More on Algebra, Lemma 15.22.10). By our assumption on $R \subset \Lambda$ we see that π maps to a uniformizer of Λ. Since $\pi \in \mathfrak{p}$ we can consider Néron's affine blowup algebra (see Algebra, Section 10.70)

$$\varphi' : A' = A[\tfrac{\mathfrak{p}}{\pi}] \longrightarrow \Lambda$$

which comes endowed with an induced map to Λ sending a/π^n, $a \in \mathfrak{p}^n$ to $\pi^{-n}\varphi(a)$ in Λ. We will denote $\mathfrak{q}' \subset \mathfrak{p}' \subset A'$ the corresponding prime ideals of A'. Observe that the isomorphism class of A' does not depend on our choice of uniformizer. Repeating the construction we obtain a sequence

$$A \to A' \to A'' \to \ldots \to \Lambda$$

0BJ3 **Lemma 16.4.2.** *In Situation 16.4.1 Néron's blowup is functorial in the following sense*

(1) *if $a \in A$, $a \notin \mathfrak{p}$, then Néron's blowup of A_a is A'_a, and*
(2) *if $B \to A$ is a surjection of flat finite type R-algebras with kernel I, then A' is the quotient of B'/IB' by its π-power torsion.*

Proof. Both (1) and (2) are special cases of Algebra, Lemma 10.70.3. In fact, whenever we have $A_1 \to A_2 \to \Lambda$ such that $\mathfrak{p}_1 A_2 = \mathfrak{p}_2$, we have that A'_2 is the quotient of $A'_1 \otimes_{A_1} A_2$ by its π-power torsion. □

0BJ4 **Lemma 16.4.3.** *In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak{p}$ and that $R/\pi R \subset \Lambda/\pi\Lambda$ is a separable field extension. Then $R \to A'$ is smooth at $\mathfrak{p}'$ and there is a short exact sequence*

$$0 \to \Omega_{A/R} \otimes_A A'_{\mathfrak{p}'} \to \Omega_{A'/R,\mathfrak{p}'} \to (A'/\pi A')^{\oplus c}_{\mathfrak{p}'} \to 0$$

where $c = \dim((A/\pi A)_{\mathfrak{p}})$.

Proof. By Lemma 16.4.2 we may replace A by a localization at an element not in $\mathfrak{p}$; we will use this without further mention. Write $\kappa = R/\pi R$. Since smoothness is stable under base change (Algebra, Lemma 10.137.3) we see that $A/\pi A$ is smooth over κ at $\mathfrak{p}$. Hence $(A/\pi A)_{\mathfrak{p}}$ is a regular local ring (Algebra, Lemma 10.140.3). Choose $g_1, \ldots, g_c \in \mathfrak{p}$ which map to a regular system of parameters in $(A/\pi A)_{\mathfrak{p}}$. Then we see that $\mathfrak{p} = (\pi, g_1, \ldots, g_c)$ after possibly replacing A by a localization. Note that $\pi, g_1, \ldots, g_c$ is a regular sequence in $A_{\mathfrak{p}}$ (first π is a nonzerodivisor and then Algebra, Lemma 10.106.3 for the rest of the sequence). After replacing A by a localization we may assume that $\pi, g_1, \ldots, g_c$ is a regular sequence in A (Algebra, Lemma 10.68.6). It follows that

$$A' = A[y_1, \ldots, y_c]/(\pi y_1 - g_1, \ldots, \pi y_c - g_c) = A[y_1, \ldots, y_c]/I$$

by More on Algebra, Lemma 15.31.2. In the following we will use the definition of smoothness using the naive cotangent complex (Algebra, Definition 10.137.1) and

the criterion of Algebra, Lemma 10.137.11 without further mention. The exact sequence of Algebra, Lemma 10.134.4 for $R \to A[y_1, \ldots, y_c] \to A'$ looks like this

$$0 \to H_1(NL_{A'/R}) \to I/I^2 \to \Omega_{A/R} \otimes_A A' \oplus \bigoplus\nolimits_{i=1,\ldots,c} A' \mathrm{d}y_i \to \Omega_{A'/R} \to 0$$

where the class of $\pi y_i - g_i$ in I/I^2 is mapped to $-\mathrm{d}g_i + \pi \mathrm{d}y_i$ in the next term. Here we have used Algebra, Lemma 10.134.6 to compute $NL_{A'/A[y_1,\ldots,y_c]}$ and we have used that $R \to A[y_1, \ldots, y_c]$ is smooth, so $H_1(NL_{A[y_1,\ldots,y_c]/R}) = 0$ and $\Omega_{A[y_1,\ldots,y_c]/R}$ is a finite projective (a fortiori flat) $A[y_1, \ldots, y_c]$-module which is in fact the direct sum of $\Omega_{A/R} \otimes_A A[y_1, \ldots, y_c]$ and a free module with basis $\mathrm{d}y_i$. To finish the proof it suffices to show that $\mathrm{d}g_1, \ldots, \mathrm{d}g_c$ forms part of a basis for the finite free module $\Omega_{A/R,\mathfrak{p}}$. Namely, this will show $(I/I^2)_\mathfrak{p}$ is free on $\pi y_i - g_i$, the localization at $\mathfrak{p}$ of the middle map in the sequence is injective, so $H_1(NL_{A'/R})_\mathfrak{p} = 0$, and that the cokernel $\Omega_{A'/R,\mathfrak{p}}$ is finite free. To do this it suffices to show that the images of $\mathrm{d}g_i$ are $\kappa(\mathfrak{p})$-linearly independent in $\Omega_{A/R,\mathfrak{p}}/\pi = \Omega_{(A/\pi A)/\kappa,\mathfrak{p}}$ (equality by Algebra, Lemma 10.131.12). Since $\kappa \subset \kappa(\mathfrak{p}) \subset \Lambda/\pi\Lambda$ we see that $\kappa(\mathfrak{p})$ is separable over κ (Algebra, Definition 10.42.1). The desired linear independence now follows from Algebra, Lemma 10.140.4. □

0BJ5 **Lemma 16.4.4.** *In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak{q}$ and that we have a surjection of R-algebras $B \to A$ with kernel I. Assume $R \to B$ smooth at $\mathfrak{p}_B = (B \to A)^{-1}\mathfrak{p}$. If the cokernel of*

$$I/I^2 \otimes_A \Lambda \to \Omega_{B/R} \otimes_B \Lambda$$

is a free Λ-module, then $R \to A$ is smooth at $\mathfrak{p}$.

Proof. The cokernel of the map $I/I^2 \to \Omega_{B/R} \otimes_B A$ is $\Omega_{A/R}$, see Algebra, Lemma 10.131.9. Let $d = \dim_\mathfrak{q}(A/R)$ be the relative dimension of $R \to A$ at $\mathfrak{q}$, i.e., the dimension of $\mathrm{Spec}(A[1/\pi])$ at $\mathfrak{q}$. See Algebra, Definition 10.125.1. Then $\Omega_{A/R,\mathfrak{q}}$ is free over $A_\mathfrak{q}$ of rank d (Algebra, Lemma 10.140.3). Thus if the hypothesis of the lemma holds, then $\Omega_{A/R} \otimes_A \Lambda$ is free of rank d. It follows that $\Omega_{A/R} \otimes_A \kappa(\mathfrak{p})$ has dimension d (as it is true upon tensoring with $\Lambda/\pi\Lambda$). Since $R \to A$ is flat and since $\mathfrak{p}$ is a specialization of $\mathfrak{q}$, we see that $\dim_\mathfrak{p}(A/R) \geq d$ by Algebra, Lemma 10.125.6. Then it follows that $R \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemmas 10.137.16 and 10.140.3. □

0BJ6 **Lemma 16.4.5.** *In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak{q}$ and that $R/\pi R \subset \Lambda/\pi\Lambda$ is a separable extension of fields. Then after a finite number of affine Néron blowups the algebra A becomes smooth over R at $\mathfrak{p}$.*

Proof. We choose an R-algebra B and a surjection $B \to A$. Set $\mathfrak{p}_B = (B \to A)^{-1}(\mathfrak{p})$ and denote r the relative dimension of $R \to B$ at $\mathfrak{p}_B$. We choose B such that $R \to B$ is smooth at $\mathfrak{p}_B$. For example we can take B to be a polynomial algebra in r variables over R. Consider the complex

$$I/I^2 \otimes_A \Lambda \longrightarrow \Omega_{B/R} \otimes_B \Lambda$$

of Lemma 16.4.4. By the structure of finite modules over Λ (More on Algebra, Lemma 15.125.9) we see that the cokernel looks like

$$\Lambda^{\oplus d} \oplus \bigoplus\nolimits_{i=1,\ldots,n} \Lambda/\pi^{e_i}\Lambda$$

for some $d \geq 0$, $n \geq 0$, and $e_i \geq 1$. Observe that d is the relative dimension of A/R at $\mathfrak{q}$ (Algebra, Lemma 10.140.3). If the defect $e = \sum_{i=1,\ldots,n} e_i$ is zero, then we are done by Lemma 16.4.4.

Next, we consider what happens when we perform the Néron blowup. Recall that A' is the quotient of B'/IB' by its π-power torsion (Lemma 16.4.2) and that $R \to B'$ is smooth at $\mathfrak{p}_{B'}$ (Lemma 16.4.3). Thus after blowup we have exactly the same setup. Picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Since $I \subset \mathfrak{p}_B$, we see that $I \to I'$ factors through $\pi I'$. Looking at the induced map of complexes we get

$$\begin{array}{ccccc} I'/(I')^2 \otimes_{A'} \Lambda & \longrightarrow & \Omega_{B'/R} \otimes_{B'} \Lambda & = & M' \\ \uparrow & & \uparrow & & \\ I/I^2 \otimes_A \Lambda & \longrightarrow & \Omega_{B/R} \otimes_B \Lambda & = & M \end{array}$$

Then $M \subset M'$ are finite free Λ-modules with quotient M'/M annihilated by π, see Lemma 16.4.3. Let $N \subset M$ and $N' \subset M'$ be the images of the horizontal maps and denote $Q = M/N$ and $Q' = M'/N'$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Then $N \subset N'$ are free Λ-modules of rank $r - d$. Since I maps into $\pi I'$ we see that $N \subset \pi N'$.

Let $K = \Lambda_\pi$ be the fraction field of Λ. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N'_K \cap M' & \longrightarrow & Q'_{tor} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & N_K \cap M & \longrightarrow & Q_{tor} & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences. This shows that the change in defect is given by

$$e - e' = \text{length}(Q_{tor}) - \text{length}(Q'_{tor}) = \text{length}(N'/N) - \text{length}(N'_K \cap M'/N_K \cap M)$$

Since M'/M is annihilated by π, so is $N'_K \cap M'/N_K \cap M$, and its length is at most $\dim_K(N_K)$. Since $N \subset \pi N'$ we get $\text{length}(N'/N) \geq \dim_K(N_K)$, with equality if and only if $N = \pi N'$.

To finish the proof we have to show that N is strictly smaller than $\pi N'$ when A is not smooth at $\mathfrak{p}$; this is the key computation one has to do in Néron's argument. To do this, we consider the exact sequence

$$I/I^2 \otimes_B \kappa(\mathfrak{p}_B) \to \Omega_{B/R} \otimes_B \kappa(\mathfrak{p}_B) \to \Omega_{A/R} \otimes_A \kappa(\mathfrak{p}) \to 0$$

(follows from Algebra, Lemma 10.131.9). Since $R \to A$ is not smooth at $\mathfrak{p}$ we see that the dimension s of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{p})$ is bigger than d. On the other hand the first arrow factors through the injective map

$$\mathfrak{p}B_{\mathfrak{p}}/\mathfrak{p}^2B_{\mathfrak{p}} \to \Omega_{B/R} \otimes_B \kappa(\mathfrak{p}_B)$$

of Algebra, Lemma 10.140.4; note that $\kappa(\mathfrak{p})$ is separable over k by our assumption on $R/\pi R \subset \Lambda/\pi\Lambda$. Hence we conclude that we can find generators $g_1, \ldots, g_t \in I$ such that $g_j \in \mathfrak{p}^2$ for $j > r-s$. Then the images of g_j in A' are in $\pi^2 I'$ for $j > r-s$. Since $r-s < r-d$ we find that at least one of the minimal generators of N becomes divisible by π^2 in N'. Thus we see that e decreases by at least 1 and we win. □

If $R \to \Lambda$ is an extension of discrete valuation rings, then $R \to \Lambda$ is regular if and only if (a) the ramification index is 1, (b) the extension of fraction fields is separable, and (c) $R/\mathfrak{m}_R \subset \Lambda/\mathfrak{m}_\Lambda$ is separable. Thus the following result is a special case of general Néron desingularization in Theorem 16.12.1.

0BJ7 **Lemma 16.4.6.** *Let $R \subset \Lambda$ be an extension of discrete valuation rings which has ramification index 1 and induces a separable extension of residue fields and of fraction fields. Then Λ is a filtered colimit of smooth R-algebras.*

Proof. By Algebra, Lemma 10.127.4 it suffices to show that any $R \to A \to \Lambda$ as in Situation 16.4.1 can be factored as $A \to B \to \Lambda$ with B a smooth R-algebra. After replacing A by its image in Λ we may assume that A is a domain whose fraction field K is a subfield of the fraction field of Λ. In particular, A is separable over the fraction field of R by our assumptions. Then $R \to A$ is smooth at $\mathfrak{q} = (0)$ by Algebra, Lemma 10.140.9. After a finite number of Néron blowups, we may assume $R \to A$ is smooth at $\mathfrak{p}$, see Lemma 16.4.5. Then, after replacing A by a localization at an element $a \in A$, $a \notin \mathfrak{p}$ it becomes smooth over R and the lemma is proved. □

16.5. The lifting problem

07CJ The goal in this section is to prove (Proposition 16.5.3) that the collection of algebras which are filtered colimits of smooth algebras is closed under infinitesimal flat deformations. The proof is elementary and only uses the results on presentations of smooth algebras from Section 16.3.

07CK **Lemma 16.5.1.** *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) *$I^2 = 0$, and*
(2) *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras.*

Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then there exists a factorization

$$A \to B/J \to \Lambda$$

where B is a smooth R-algebra and $J \subset IB$ is a finitely generated ideal.

Proof. Choose a factorization

$$A/IA \to \bar{B} \to \Lambda/I\Lambda$$

with $\bar{B}$ standard smooth over R/I; this is possible by assumption and Lemma 16.3.5. Write

$$\bar{B} = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)$$

and say $\bar{B} \to \Lambda/I\Lambda$ maps t_i to the class of λ_i modulo $I\Lambda$. Choose $g_1, \ldots, g_s \in A[t_1, \ldots, t_r]$ lifting $\bar{g}_1, \ldots, \bar{g}_s$. Write $\varphi(g_i)(\lambda_1, \ldots, \lambda_r) = \sum \epsilon_{ij}\mu_{ij}$ for some $\epsilon_{ij} \in I$ and $\mu_{ij} \in \Lambda$. Define

$$A' = A[t_1, \ldots, t_r, \delta_{i,j}]/(g_i - \sum \epsilon_{ij}\delta_{ij})$$

and consider the map

$$A' \longrightarrow \Lambda, \quad a \longmapsto \varphi(a), \quad t_i \longmapsto \lambda_i, \quad \delta_{ij} \longmapsto \mu_{ij}$$

We have

$$A'/IA' = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)[\delta_{ij}] \cong \bar{B}[\delta_{ij}]$$

This is a standard smooth algebra over R/I as $\bar{B}$ is standard smooth. Choose a presentation $A'/IA' = R/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ with $\det(\partial \bar{f}_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A'/IA'. Choose lifts $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ of $\bar{f}_1, \ldots, \bar{f}_c$. Then

$$B = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} - 1)$$

is smooth over R. Since smooth ring maps are formally smooth (Algebra, Proposition 10.138.13) there exists an R-algebra map $B \to A'$ which is an isomorphism modulo I. Then $B \to A'$ is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1). Thus $A' = B/J$ with $J \subset IB$ finitely generated (see Algebra, Lemma 10.6.3). □

07CL **Lemma 16.5.2.** *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) $I^2 = 0$,
(2) $\Lambda/I\Lambda$ *is a filtered colimit of smooth R/I-algebras, and*
(3) $R \to \Lambda$ *is flat.*

Let $\varphi : B \to \Lambda$ be an R-algebra map with B smooth over R. Let $J \subset IB$ be a finitely generated ideal such that $\varphi(J) = 0$. Then there exists R-algebra maps

$$B \xrightarrow{\alpha} B' \xrightarrow{\beta} \Lambda$$

such that B' is smooth over R, such that $\alpha(J) = 0$ and such that $\beta \circ \alpha = \varphi \bmod I\Lambda$.

Proof. If we can prove the lemma in case $J = (h)$, then we can prove the lemma by induction on the number of generators of J. Namely, suppose that J can be generated by n elements $h_1, \ldots, h_n$ and the lemma holds for all cases where J is generated by $n-1$ elements. Then we apply the case $n = 1$ to produce $B \to B' \to \Lambda$ where the first map kills of h_n. Then we let J' be the ideal of B' generated by the images of $h_1, \ldots, h_{n-1}$ and we apply the case for $n - 1$ to produce $B' \to B'' \to \Lambda$. It is easy to verify that $B \to B'' \to \Lambda$ does the job.

Assume $J = (h)$ and write $h = \sum \epsilon_i b_i$ for some $\epsilon_i \in I$ and $b_i \in B$. Note that $0 = \varphi(h) = \sum \epsilon_i \varphi(b_i)$. As Λ is flat over R, the equational criterion for flatness (Algebra, Lemma 10.39.11) implies that we can find $\lambda_j \in \Lambda$, $j = 1, \ldots, m$ and $a_{ij} \in R$ such that $\varphi(b_i) = \sum_j a_{ij}\lambda_j$ and $\sum_i \epsilon_i a_{ij} = 0$. Set

$$C = B[x_1, \ldots, x_m]/(b_i - \sum a_{ij}x_j)$$

with $C \to \Lambda$ given by φ and $x_j \mapsto \lambda_j$. Choose a factorization

$$C \to B'/J' \to \Lambda$$

as in Lemma 16.5.1. Since B is smooth over R we can lift the map $B \to C \to B'/J'$ to a map $\psi : B \to B'$. We claim that $\psi(h) = 0$. Namely, the fact that ψ agrees with $B \to C \to B'/J'$ mod I implies that

$$\psi(b_i) = \sum a_{ij}\xi_j + \theta_i$$

for some $\xi_i \in B'$ and $\theta_i \in IB'$. Hence we see that

$$\psi(h) = \psi(\sum \epsilon_i b_i) = \sum \epsilon_i a_{ij}\xi_j + \sum \epsilon_i \theta_i = 0$$

because of the relations above and the fact that $I^2 = 0$. □

07CM **Proposition 16.5.3.** *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) *I is nilpotent,*
(2) *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and*
(3) *$R \to \Lambda$ is flat.*

Then Λ is a filtered colimit of smooth R-algebras.

Proof. Since $I^n = 0$ for some n, it follows by induction on n that it suffices to consider the case where $I^2 = 0$. Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Algebra, Lemma 10.127.4. By Lemma 16.5.1 we may assume that $A = B/J$ with B smooth over R and $J \subset IB$ a finitely generated ideal. By Lemma 16.5.2 we can find a (possibly noncommutative) diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & \Lambda & \end{array}$$

of R-algebras which commutes modulo I and such that $\alpha(J) = 0$. The map

$$D : B \longrightarrow I\Lambda, \quad b \longmapsto \varphi(b) - \beta(\alpha(b))$$

is a derivation over R hence we can write it as $D = \xi \circ \mathrm{d}_{B/R}$ for some B-linear map $\xi : \Omega_{B/R} \to I\Lambda$. Since $\Omega_{B/R}$ is a finite projective B-module we can write $\xi = \sum_{i=1,\ldots,n} \epsilon_i \Xi_i$ for some $\epsilon_i \in I$ and B-linear maps $\Xi_i : \Omega_{B/R} \to \Lambda$. (Details omitted. Hint: write $\Omega_{B/R}$ as a direct sum of a finite free module to reduce to the finite free case.) We define

$$B'' = \mathrm{Sym}^*_{B'}\left(\bigoplus\nolimits_{i=1,\ldots,n} \Omega_{B/R} \otimes_{B,\alpha} B'\right)$$

and we define $\beta' : B'' \to \Lambda$ by β on B' and by

$$\beta'|_{i\text{th summand } \Omega_{B/R}\otimes_{B,\alpha}B'} = \Xi_i \otimes \beta$$

and $\alpha' : B \to B''$ by

$$\alpha'(b) = \alpha(b) \oplus \sum \epsilon_i \mathrm{d}_{B/R}(b) \otimes 1 \oplus 0 \oplus \ldots$$

At this point the diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha'} & B'' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & \Lambda & \end{array}$$

does commute. Moreover, it is direct from the definitions that $\alpha'(J) = 0$ as $I^2 = 0$. Hence the desired factorization. □

16.6. The lifting lemma

07CN Here is a fiendishly clever lemma.

07CP **Lemma 16.6.1.** *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Suppose we have R-algebra maps $R/\pi^2R \to \bar{C} \to \Lambda/\pi^2\Lambda$ with $\bar{C}$ of finite presentation. Then there exists an R-algebra homomorphism $D \to \Lambda$ and a commutative diagram*

$$\begin{array}{ccccc} R/\pi^2R & \longrightarrow & \bar{C} & \longrightarrow & \Lambda/\pi^2\Lambda \\ \downarrow & & \downarrow & & \downarrow \\ R/\pi R & \longrightarrow & D/\pi D & \longrightarrow & \Lambda/\pi\Lambda \end{array}$$

with the following properties

(a) *D is of finite presentation,*
(b) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \notin \mathfrak{q}$,*
(c) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \in \mathfrak{q}$ lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth, and*
(d) *$\bar{C}/\pi\bar{C} \to D/\pi D$ is smooth at any prime lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth.*

Proof. We choose a presentation

$$\bar{C} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

We also denote $I = (f_1, \ldots, f_m)$ and $\bar{I}$ the image of I in $R/\pi^2R[x_1, \ldots, x_n]$. Since R is Noetherian, so is $\bar{C}$. Hence the smooth locus of $R/\pi^2R \to \bar{C}$ is quasi-compact, see Topology, Lemma 5.9.2. Applying Lemma 16.2.2 we may choose a finite list of elements $a_1, \ldots, a_r \in R[x_1, \ldots, x_n]$ such that

(1) the union of the open subspaces $\mathrm{Spec}(\bar{C}_{a_k}) \subset \mathrm{Spec}(\bar{C})$ cover the smooth locus of $R/\pi^2R \to \bar{C}$, and
(2) for each $k = 1, \ldots, r$ there exists a finite subset $E_k \subset \{1, \ldots, m\}$ such that $(\bar{I}/\bar{I}^2)_{a_k}$ is freely generated by the classes of f_j, $j \in E_k$.

Set $I_k = (f_j, j \in E_k) \subset I$ and denote $\bar{I}_k$ the image of I_k in $R/\pi^2R[x_1, \ldots, x_n]$. By (2) and Nakayama's lemma we see that $(\bar{I}/\bar{I}_k)_{a_k}$ is annihilated by $1 + b'_k$ for some $b'_k \in \bar{I}_{a_k}$. Suppose b'_k is the image of $b_k/(a_k)^N$ for some $b_k \in I$ and some integer N. After replacing a_k by a_kb_k we get

(3) $(\bar{I}_k)_{a_k} = (\bar{I})_{a_k}$.

Thus, after possibly replacing a_k by a high power, we may write

(4) $a_kf_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j + \pi^2 g_{k,\ell}$

for any $\ell \in \{1, \ldots, m\}$ and some $h^j_{i,\ell}, g_{i,\ell} \in R[x_1, \ldots, x_n]$. If $\ell \in E_k$ we choose $h^j_{k,\ell} = a_k\delta_{\ell,j}$ (Kronecker delta) and $g_{k,\ell} = 0$. Set

$$D = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, p_{k,\ell}).$$

Here $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, m\}$, and

$$p_{k,\ell} = a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j - \pi g_{k,\ell}.$$

Note that for $\ell \in E_k$ we have $p_{k,\ell} = 0$ by our choices above.

The map $R \to D$ is the given one. Say $\bar{C} \to \Lambda/\pi^2\Lambda$ maps x_i to the class of λ_i modulo π^2. For an element $f \in R[x_1, \ldots, x_n]$ we denote $f(\lambda) \in \Lambda$ the result of substituting λ_i for x_i. Then we know that $f_j(\lambda) = \pi^2 \mu_j$ for some $\mu_j \in \Lambda$. Define $D \to \Lambda$ by the rules $x_i \mapsto \lambda_i$ and $z_j \mapsto \pi\mu_j$. This is well defined because

$$\begin{aligned} p_{k,\ell} &\mapsto a_k(\lambda)\pi\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\pi\mu_j - \pi g_{k,\ell}(\lambda) \\ &= \pi \left(a_k(\lambda)\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\mu_j - g_{k,\ell}(\lambda) \right) \end{aligned}$$

Substituting $x_i = \lambda_i$ in (4) above we see that the expression inside the brackets is annihilated by π^2, hence it is annihilated by π as we have assumed $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. The map $\bar{C} \to D/\pi D$ is determined by $x_i \mapsto x_i$ (clearly well defined). Thus we are done if we can prove (b), (c), and (d).

Using (4) we obtain the following key equality

$$\begin{aligned} \pi p_{k,\ell} &= \pi a_k z_\ell - \sum\nolimits_{j \in E_k} \pi h^j_{k,\ell} z_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + a_k f_\ell + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) - \sum\nolimits_{j \in E_k} h^j_{k,\ell} f_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) \end{aligned}$$

The end result is an element of the ideal generated by $f_j - \pi z_j$. In particular, we see that $D[1/\pi]$ is isomorphic to $R[1/\pi][x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j)$ which is isomorphic to $R[1/\pi][x_1, \ldots, x_n]$ hence smooth over R. This proves (b).

For fixed $k \in \{1, \ldots, r\}$ consider the ring

$$D_k = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, j \in E_k, p_{k,\ell})$$

The number of equations is $m = |E_k| + (m - |E_k|)$ as $p_{k,\ell}$ is zero if $\ell \in E_k$. Also, note that

$$\begin{aligned} (D_k/\pi D_k)_{a_k} &= R/\pi R[x_1, \ldots, x_n, 1/a_k, z_1, \ldots, z_m]/(f_j, j \in E_k, p_{k,\ell}) \\ &= (\bar{C}/\pi\bar{C})_{a_k}[z_1, \ldots, z_m]/(a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j) \\ &\cong (\bar{C}/\pi\bar{C})_{a_k}[z_j, j \in E_k] \end{aligned}$$

In particular $(D_k/\pi D_k)_{a_k}$ is smooth over $(\bar{C}/\pi\bar{C})_{a_k}$. By our choice of a_k we have that $(\bar{C}/\pi\bar{C})_{a_k}$ is smooth over $R/\pi R$ of relative dimension $n - |E_k|$, see (2). Hence for a prime $\mathfrak{q}_k \subset D_k$ containing π and lying over $\mathrm{Spec}(\bar{C}_{a_k})$ the fibre ring of $R \to D_k$ is smooth at $\mathfrak{q}_k$ of dimension n. Thus $R \to D_k$ is syntomic at $\mathfrak{q}_k$ by our count of the number of equations above, see Algebra, Lemma 10.136.10. Hence $R \to D_k$ is smooth at $\mathfrak{q}_k$, see Algebra, Lemma 10.137.16.

To finish the proof, let $\mathfrak{q} \subset D$ be a prime containing π lying over a prime where $R/\pi^2 R \to \bar{C}$ is smooth. Then $a_k \notin \mathfrak{q}$ for some k by (1). We will show that the surjection $D_k \to D$ induces an isomorphism on local rings at $\mathfrak{q}$. Since we know that the ring maps $\bar{C}/\pi\bar{C} \to D_k/\pi D_k$ and $R \to D_k$ are smooth at the corresponding

prime $\mathfrak{q}_k$ by the preceding paragraph this will prove (c) and (d) and thus finish the proof.

First, note that for any ℓ the equation $\pi p_{k,\ell} = -a_k(f_\ell - \pi z_\ell) + \sum_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j)$ proved above shows that $f_\ell - \pi z_\ell$ maps to zero in $(D_k)_{a_k}$ and in particular in $(D_k)_{\mathfrak{q}_k}$. The relations (4) imply that $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j$ in I/I^2. Since $(\bar{I}_k/\bar{I}_k^2)_{a_k}$ is free on f_j, $j \in E_k$ we see that

$$a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'}$$

is zero in $\bar{C}_{a_k}$ for every k, k', ℓ and $j \in E_k$. Hence we can find a large integer N such that

$$a_k^N \left(a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'} \right)$$

is in $I_k + \pi^2 R[x_1, \ldots, x_n]$. Computing modulo π we have

$$\begin{aligned} & a_k p_{k',\ell} - a_{k'} p_{k,\ell} + \sum h^{j'}_{k',\ell} p_{k,j'} \\ & = -a_k \sum h^{j'}_{k',\ell} z_{j'} + a_{k'} \sum h^j_{k,\ell} z_j + \sum h^{j'}_{k',\ell} a_k z_{j'} - \sum \sum h^{j'}_{k',\ell} h^j_{k,j'} z_j \\ & = \sum \left(a_{k'} h^j_{k,\ell} - \sum h^{j'}_{k',\ell} h^j_{k,j'} \right) z_j \end{aligned}$$

with Einstein summation convention. Combining with the above we see $a_k^{N+1} p_{k',\ell}$ is contained in the ideal generated by I_k and π in $R[x_1, \ldots, x_n, z_1, \ldots, z_m]$. Thus $p_{k',\ell}$ maps into $\pi(D_k)_{a_k}$. On the other hand, the equation

$$\pi p_{k',\ell} = -a_{k'}(f_\ell - \pi z_\ell) + \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell}(f_{j'} - \pi z_{j'})$$

shows that $\pi p_{k',\ell}$ is zero in $(D_k)_{a_k}$. Since we have assumed that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and since $(D_k)_{\mathfrak{q}_k}$ is smooth hence flat over R we see that $\mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi) = \mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi^2)$. We conclude that $p_{k',\ell}$ maps to zero as well, hence $D_{\mathfrak{q}} = (D_k)_{\mathfrak{q}_k}$ and we win. □

16.7. The desingularization lemma

07CQ Here is another fiendishly clever lemma.

07CR **Lemma 16.7.1.** *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation. Assume*

1. *the image of π is strictly standard in A over R, and*
2. *there exists a section $\rho : A/\pi^4 A \to R/\pi^4 R$ which is compatible with the map to $\Lambda/\pi^4\Lambda$.*

Then we can find R-algebra maps $A \to B \to \Lambda$ with B of finite presentation such that $\mathfrak{a}B \subset H_{B/R}$ where $\mathfrak{a} = Ann_R(Ann_R(\pi^2)/Ann_R(\pi))$.

Proof. Choose a presentation

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and $0 \leq c \leq \min(n, m)$ such that (16.2.3.3) holds for π and such that

07CS (16.7.1.1) $$\pi f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m-c$. Say ρ maps x_i to the class of $r_i \in R$. Then we can replace x_i by $x_i - r_i$. Hence we may assume $\rho(x_i) = 0$ in $R/\pi^4 R$. This implies that $f_j(0) \in \pi^4 R$ and that $A \to \Lambda$ maps x_i to $\pi^4\lambda_i$ for some $\lambda_i \in \Lambda$. Write

$$f_j = f_j(0) + \sum\nolimits_{i=1,\ldots,n} r_{ji} x_i + \text{h.o.t.}$$

This implies that the constant term of $\partial f_j/\partial x_i$ is r_{ji}. Apply ρ to (16.2.3.3) for π and we see that

$$\pi = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i \in I} \bmod \pi^4 R$$

for some $r_I \in R$. Thus we have

$$u\pi = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i \in I}$$

for some $u \in 1 + \pi^3 R$. By Algebra, Lemma 10.15.5 this implies there exists a $n \times c$ matrix (s_{ik}) such that

$$u\pi\delta_{jk} = \sum\nolimits_{i=1,\ldots,n} r_{ji} s_{ik} \quad \text{for all } j, k = 1, \ldots, c$$

(Kronecker delta). We introduce auxiliary variables $v_1, \ldots, v_c, w_1, \ldots, w_n$ and we set

$$h_i = x_i - \pi^2 \sum\nolimits_{j=1,\ldots c} s_{ij} v_j - \pi^3 w_i$$

In the following we will use that

$$R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n) = R[v_1, \ldots, v_c, w_1, \ldots, w_n]$$

without further mention. In $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$ we have

$$\begin{aligned} f_j &= f_j(x_1 - h_1, \ldots, x_n - h_n) \\ &= \pi^2 \sum\nolimits_{k=1}^{c} \left(\sum\nolimits_{i=1}^{n} r_{ji} s_{ik} \right) v_k + \pi^3 \sum\nolimits_{i=1}^{n} r_{ji} w_i \bmod \pi^4 \\ &= \pi^3 v_j + \pi^3 \sum\nolimits_{i=1}^{n} r_{ji} w_i \bmod \pi^4 \end{aligned}$$

for $1 \leq j \leq c$. Hence we can choose elements $g_j \in R[v_1, \ldots, v_c, w_1, \ldots, w_n]$ such that $g_j = v_j + \sum r_{ji} w_i \bmod \pi$ and such that $f_j = \pi^3 g_j$ in the R-algebra $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$. We set

$$B = R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(f_1, \ldots, f_m, h_1, \ldots, h_n, g_1, \ldots, g_c).$$

The map $A \to B$ is clear. We define $B \to \Lambda$ by mapping $x_i \to \pi^4\lambda_i$, $v_i \mapsto 0$, and $w_i \mapsto \pi\lambda_i$. Then it is clear that the elements f_j and h_i are mapped to zero in Λ. Moreover, it is clear that g_i is mapped to an element t of $\pi\Lambda$ such that $\pi^3 t = 0$ (as $f_i = \pi^3 g_i$ modulo the ideal generated by the h's). Hence our assumption that $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$ implies that $t = 0$. Thus we are done if we can prove the statement about smoothness.

Note that $B_\pi \cong A_\pi[v_1, \ldots, v_c]$ because the equations $g_i = 0$ are implied by $f_i = 0$. Hence B_π is smooth over R as A_π is smooth over R by the assumption that π is strictly standard in A over R, see Lemma 16.2.5.

Set $B' = R[v_1, \ldots, v_c, w_1, \ldots, w_n]/(g_1, \ldots, g_c)$. As $g_i = v_i + \sum r_{ji} w_i \bmod \pi$ we see that $B'/\pi B' = R/\pi R[w_1, \ldots, w_n]$. Hence $R \to B'$ is smooth of relative dimension n at every point of $V(\pi)$ by Algebra, Lemmas 10.136.10 and 10.137.16 (the first

lemma shows it is syntomic at those primes, in particular flat, whereupon the second lemma shows it is smooth).

Let $\mathfrak{q} \subset B$ be a prime with $\pi \in \mathfrak{q}$ and for some $r \in \mathfrak{a}$, $r \notin \mathfrak{q}$. Denote $\mathfrak{q}' = B' \cap \mathfrak{q}$. We claim the surjection $B' \to B$ induces an isomorphism of local rings $(B')_{\mathfrak{q}'} \to B_{\mathfrak{q}}$. This will conclude the proof of the lemma. Note that $B_{\mathfrak{q}}$ is the quotient of $(B')_{\mathfrak{q}'}$ by the ideal generated by f_{c+j}, $j = 1, \ldots, m-c$. We observe two things: first the image of f_{c+j} in $(B')_{\mathfrak{q}'}$ is divisible by π^2 and second the image of πf_{c+j} in $(B')_{\mathfrak{q}'}$ can be written as $\sum b_{j_1 j_2} f_{c+j_1} f_{c+j_2}$ by (16.7.1.1). Thus we see that the image of each πf_{c+j} is contained in the ideal generated by the elements $\pi^2 f_{c+j'}$. Hence $\pi f_{c+j} = 0$ in $(B')_{\mathfrak{q}'}$ as this is a Noetherian local ring, see Algebra, Lemma 10.51.4. As $R \to (B')_{\mathfrak{q}'}$ is flat we see that

$$\big(\mathrm{Ann}_R(\pi^2)/\mathrm{Ann}_R(\pi)\big) \otimes_R (B')_{\mathfrak{q}'} = \mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi^2)/\mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi)$$

Because $r \in \mathfrak{a}$ is invertible in $(B')_{\mathfrak{q}'}$ we see that this module is zero. Hence we see that the image of f_{c+j} is zero in $(B')_{\mathfrak{q}'}$ as desired. □

07CT **Lemma 16.7.2.** *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ and $D \to \Lambda$ be R-algebra maps with A and D of finite presentation. Assume*

(1) *π is strictly standard in A over R, and*
(2) *there exists an R-algebra map $A/\pi^4 A \to D/\pi^4 D$ compatible with the maps to $\Lambda/\pi^4\Lambda$.*

Then we can find an R-algebra map $B \to \Lambda$ with B of finite presentation and R-algebra maps $A \to B$ and $D \to B$ compatible with the maps to Λ such that $H_{D/R}B \subset H_{B/D}$ and $H_{D/R}B \subset H_{B/R}$.

Proof. We apply Lemma 16.7.1 to

$$D \longrightarrow A \otimes_R D \longrightarrow \Lambda$$

and the image of π in D. By Lemma 16.2.7 we see that π is strictly standard in $A \otimes_R D$ over D. As our section $\rho : (A \otimes_R D)/\pi^4(A \otimes_R D) \to D/\pi^4 D$ we take the map induced by the map in (2). Thus Lemma 16.7.1 applies and we obtain a factorization $A \otimes_R D \to B \to \Lambda$ with B of finite presentation and $\mathfrak{a}B \subset H_{B/D}$ where

$$\mathfrak{a} = \mathrm{Ann}_D(\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)).$$

For any prime $\mathfrak{q}$ of D such that $D_{\mathfrak{q}}$ is flat over R we have $\mathrm{Ann}_{D_{\mathfrak{q}}}(\pi^2)/\mathrm{Ann}_{D_{\mathfrak{q}}}(\pi) = 0$ because annihilators of elements commutes with flat base change and we assumed $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$. Because D is Noetherian we see that $\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)$ is a finite D-module, hence formation of its annihilator commutes with localization. Thus we see that $\mathfrak{a} \not\subset \mathfrak{q}$. Hence we see that $D \to B$ is smooth at any prime of B lying over $\mathfrak{q}$. Since any prime of D where $R \to D$ is smooth is one where $D_{\mathfrak{q}}$ is flat over R we conclude that $H_{D/R}B \subset H_{B/D}$. The final inclusion $H_{D/R}B \subset H_{B/R}$ follows because compositions of smooth ring maps are smooth (Algebra, Lemma 10.137.13). □

07F0 **Lemma 16.7.3.** *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be*

an R-algebra map with A of finite presentation and assume π is strictly standard in A over R. Let

$$A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$$

be a factorization with $\bar{C}$ of finite presentation. Then we can find a factorization $A \to B \to \Lambda$ with B of finite presentation such that $R_\pi \to B_\pi$ is smooth and such that

$$H_{\bar{C}/(R/\pi^8 R)} \cdot \Lambda/\pi^8\Lambda \subset \sqrt{H_{B/R}\Lambda} \bmod \pi^8\Lambda.$$

Proof. Apply Lemma 16.6.1 to get $R \to D \to \Lambda$ with a factorization $\bar{C}/\pi^4\bar{C} \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$ such that $R \to D$ is smooth at any prime not containing π and at any prime lying over a prime of $\bar{C}/\pi^4\bar{C}$ where $R/\pi^8 R \to \bar{C}$ is smooth. By Lemma 16.7.2 we can find a finitely presented R-algebra B and factorizations $A \to B \to \Lambda$ and $D \to B \to \Lambda$ such that $H_{D/R}B \subset H_{B/R}$. We omit the verification that this is a solution to the problem posed by the lemma. □

16.8. Warmup: reduction to a base field

07F1 In this section we apply the lemmas in the previous sections to prove that it suffices to prove the main result when the base ring is a field, see Lemma 16.8.4.

07F2 **Situation 16.8.1.** Here $R \to \Lambda$ is a regular ring map of Noetherian rings.

Let $R \to \Lambda$ be as in Situation 16.8.1. We say *PT holds for $R \to \Lambda$* if Λ is a filtered colimit of smooth R-algebras.

07F3 **Lemma 16.8.2.** *Let $R_i \to \Lambda_i$, $i = 1, 2$ be as in Situation 16.8.1. If PT holds for $R_i \to \Lambda_i$, $i = 1, 2$, then PT holds for $R_1 \times R_2 \to \Lambda_1 \times \Lambda_2$.*

Proof. Omitted. Hint: A product of filtered colimits is a filtered colimit. □

07F4 **Lemma 16.8.3.** *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Let $S \subset R$ be a multiplicative set. Let $S^{-1}A \to B' \to S^{-1}\Lambda$ be a factorization with B' smooth over $S^{-1}R$. Then we can find a factorization $A \to B \to \Lambda$ such that some $s \in S$ maps to an elementary standard element (Definition 16.2.3) in B over R.*

Proof. We first apply Lemma 16.3.4 to $S^{-1}R \to B'$. Thus we may assume B' is standard smooth over $S^{-1}R$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $x_i \mapsto \lambda_i$ in Λ. We may write $B' = S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ where $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B' and such that $A \to B'$ is given by $x_i \mapsto x_i$, see Lemma 16.3.6. After multiplying x_i, $i > n$ by an element of S and correspondingly modifying the equations f_j we may assume $B' \to S^{-1}\Lambda$ maps x_i to $\lambda_i/1$ for some $\lambda_i \in \Lambda$ for $i > n$. Choose a relation

$$1 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $a_j \in S^{-1}R[x_1, \ldots, x_{n+m}]$. Since each element of S is invertible in B' we may (by clearing denominators) assume that $f_j, a_j \in R[x_1, \ldots, x_{n+m}]$ and that

$$s_0 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $s_0 \in S$. Since g_j maps to zero in $S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ we can find elements $s_j \in S$ such that $s_j g_j = 0$ in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Since f_j

maps to zero in $S^{-1}\Lambda$ we can find $s'_j \in S$ such that $s'_j f_j(\lambda_1, \ldots, \lambda_{n+m}) = 0$ in Λ. Consider the ring

$$B = R[x_1, \ldots, x_{n+m}]/(s'_1 f_1, \ldots, s'_c f_c, g_1, \ldots, g_t)$$

and the factorization $A \to B \to \Lambda$ with $B \to \Lambda$ given by $x_i \mapsto \lambda_i$. We claim that $s = s_0 s_1 \ldots s_t s'_1 \ldots s'_c$ is elementary standard in B over R which finishes the proof. Namely, $s_j g_j \in (f_1, \ldots, f_c)$ and hence $s g_j \in (s'_1 f_1, \ldots, s'_c f_c)$. Finally, we have

$$a_0 \det(\partial s'_j f_j / \partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} (s'_1 \ldots \hat{s'_j} \ldots s'_c) a_j s'_j f_j = s_0 s'_1 \ldots s'_c$$

which divides s as desired. □

07F5 **Lemma 16.8.4.** *If for every Situation 16.8.1 where R is a field PT holds, then PT holds in general.*

Proof. Assume PT holds for any Situation 16.8.1 where R is a field. Let $R \to \Lambda$ be as in Situation 16.8.1 arbitrary. Note that $R/I \to \Lambda/I\Lambda$ is another regular ring map of Noetherian rings, see More on Algebra, Lemma 15.41.3. Consider the set of ideals

$$\mathcal{I} = \{I \subset R \mid R/I \to \Lambda/I\Lambda \text{ does not have PT}\}$$

We have to show that $\mathcal{I}$ is empty. If this set is nonempty, then it contains a maximal element because R is Noetherian. Replacing R by R/I and Λ by Λ/I we obtain a situation where PT holds for $R/I \to \Lambda/I\Lambda$ for any nonzero ideal of R. In particular, we see by applying Proposition 16.5.3 that R is a reduced ring.

Let $A \to \Lambda$ be an R-algebra homomorphism with A of finite presentation. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Algebra, Lemma 10.127.4.

Let $S \subset R$ be the set of nonzerodivisors and consider the total ring of fractions $Q = S^{-1}R$ of R. We know that $Q = K_1 \times \ldots \times K_n$ is a product of fields, see Algebra, Lemmas 10.25.4 and 10.31.6. By Lemma 16.8.2 and our assumption PT holds for the ring map $S^{-1}R \to S^{-1}\Lambda$. Hence we can find a factorization $S^{-1}A \to B' \to S^{-1}\Lambda$ with B' smooth over $S^{-1}R$.

We apply Lemma 16.8.3 and find a factorization $A \to B \to \Lambda$ such that some $\pi \in S$ is elementary standard in B over R. After replacing A by B we may assume that π is elementary standard, hence strictly standard in A. We know that $R/\pi^8 R \to \Lambda/\pi^8\Lambda$ satisfies PT. Hence we can find a factorization $R/\pi^8 R \to A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$ with $R/\pi^8 R \to \bar{C}$ smooth. By Lemma 16.6.1 we can find an R-algebra map $D \to \Lambda$ with D smooth over R and a factorization $R/\pi^4 R \to A/\pi^4 A \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$. By Lemma 16.7.2 we can find $A \to B \to \Lambda$ with B smooth over R which finishes the proof. □

16.9. Local tricks

07F6

07F7 **Situation 16.9.1.** We are given a Noetherian ring R and an R-algebra map $A \to \Lambda$ and a prime $\mathfrak{q} \subset \Lambda$. We assume A is of finite presentation over R. In this situation we denote $\mathfrak{h}_A = \sqrt{H_{A/R}\Lambda}$.

Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. We say $R \to A \to \Lambda \supset \mathfrak{q}$ *can be resolved* if there exists a factorization $A \to B \to \Lambda$ with B of finite presentation and $\mathfrak{h}_A \subset \mathfrak{h}_B \not\subset \mathfrak{q}$. In this case we will call the factorization $A \to B \to \Lambda$ a *resolution of* $R \to A \to \Lambda \supset \mathfrak{q}$.

07F8 **Lemma 16.9.2.** *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $r \geq 1$ and $\pi_1, \ldots, \pi_r \in R$ map to elements of $\mathfrak{q}$. Assume*

(1) *for $i = 1, \ldots, r$ we have*

$$Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i) = Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i^2)$$

and

$$Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i^2)$$

(2) *for $i = 1, \ldots, r$ the element π_i maps to a strictly standard element in A over R.*

Then, if

$$R/(\pi_1^8, \ldots, \pi_r^8)R \to A/(\pi_1^8, \ldots, \pi_r^8)A \to \Lambda/(\pi_1^8, \ldots, \pi_r^8)\Lambda \supset \mathfrak{q}/(\pi_1^8, \ldots, \pi_r^8)\Lambda$$

can be resolved, so can $R \to A \to \Lambda \supset \mathfrak{q}$.

Proof. We are going to prove this by induction on r.

The case $r = 1$. Here the assumption is that there exists a factorization $A/\pi_1^8 \to \bar{C} \to \Lambda/\pi_1^8$ which resolves the situation modulo π_1^8. Conditions (1) and (2) are the assumptions needed to apply Lemma 16.7.3. Thus we can "lift" the resolution $\bar{C}$ to a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

The case $r > 1$. In this case we apply the induction hypothesis for $r - 1$ to the situation $R/\pi_1^8 \to A/\pi_1^8 \to \Lambda/\pi_1^8 \supset \mathfrak{q}/\pi_1^8\Lambda$. Note that property (2) is preserved by Lemma 16.2.7. □

07F9 **Lemma 16.9.3.** *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ and that $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved. Then there exists a factorization $A \to C \to \Lambda$ with C of finite presentation such that $H_{C/R}\Lambda \not\subset \mathfrak{q}$.* [**Swa98**, Lemma 12.2] or [**Pop85**, Lemma 2]

Proof. Let $A_\mathfrak{p} \to C \to \Lambda_\mathfrak{q}$ be a resolution of $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$. By our assumption that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ this means that $H_{C/R_\mathfrak{p}}\Lambda_\mathfrak{q} = \Lambda_\mathfrak{q}$. By Lemma 16.2.8 we may assume that C is smooth over $R_\mathfrak{p}$. By Lemma 16.3.4 we may assume that C is standard smooth over $R_\mathfrak{p}$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $A \to \Lambda$ is given by $x_i \mapsto \lambda_i$. Write $C = R_\mathfrak{p}[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ such that $A \to C$ maps x_i to x_i and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in C, see Lemma 16.3.6. After clearing denominators we may assume $f_1, \ldots, f_c$ are elements of $R[x_1, \ldots, x_{n+m}]$. Of course $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is not invertible in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ but it becomes invertible after inverting some element $s_0 \in R$, $s_0 \notin \mathfrak{p}$. As g_j maps to zero under $R[x_1, \ldots, x_n] \to A \to C$ we can find $s_j \in R$, $s_j \notin \mathfrak{p}$ such that $s_j g_j$ is zero in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Write $f_j = F_j(x_1, \ldots, x_{n+m}, 1)$ for some polynomial $F_j \in R[x_1, \ldots, x_n, X_{n+1}, \ldots, X_{n+m+1}]$ homogeneous in $X_{n+1}, \ldots, X_{n+m+1}$. Pick $\lambda_{n+i} \in \Lambda$, $i = 1, \ldots, m+1$ with $\lambda_{n+m+1} \notin \mathfrak{q}$

such that x_{n+i} maps to $\lambda_{n+i}/\lambda_{n+m+1}$ in $\Lambda_{\mathfrak{q}}$. Then

$$\begin{aligned} F_j(\lambda_1, \ldots, \lambda_{n+m+1}) &= (\lambda_{n+m+1})^{\deg(F_j)} F_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}, 1) \\ &= (\lambda_{n+m+1})^{\deg(F_j)} f_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}) \\ &= 0 \end{aligned}$$

in $\Lambda_{\mathfrak{q}}$. Thus we can find $\lambda_0 \in \Lambda$, $\lambda_0 \notin \mathfrak{q}$ such that $\lambda_0 F_j(\lambda_1, \ldots, \lambda_{n+m+1}) = 0$ in Λ. Now we set B equal to

$$R[x_0, \ldots, x_{n+m+1}]/(g_1, \ldots, g_t, x_0 F_1(x_1, \ldots, x_{n+m+1}), \ldots, x_0 F_c(x_1, \ldots, x_{n+m+1}))$$

which we map to Λ by mapping x_i to λ_i. Let b be the image of $x_0 x_{n+m+1} s_0 s_1 \ldots s_t$ in B. Then B_b is isomorphic to

$$R_{s_0 s_1 \ldots s_t}[x_0, x_1, \ldots, x_{n+m+1}, 1/x_0 x_{n+m+1}]/(f_1, \ldots, f_c)$$

which is smooth over R by construction. Since b does not map to an element of $\mathfrak{q}$, we win. □

07FA **Lemma 16.9.4.** *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume*

(1) *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,*
(2) *$R_{\mathfrak{p}} \to A_{\mathfrak{p}} \to \Lambda_{\mathfrak{q}} \supset \mathfrak{q}\Lambda_{\mathfrak{q}}$ can be resolved, and*
(3) *$\dim(\Lambda_{\mathfrak{q}}) = 0$.*

Then $R \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. By (3) the ring $\Lambda_{\mathfrak{q}}$ is Artinian local hence $\mathfrak{q}\Lambda_{\mathfrak{q}}$ is nilpotent. Thus $(\mathfrak{h}_A)^N \Lambda_{\mathfrak{q}} = 0$ for some $N > 0$. Thus there exists a $\lambda \in \Lambda$, $\lambda \notin \mathfrak{q}$ such that $\lambda(\mathfrak{h}_A)^N = 0$ in Λ. Say $H_{A/R} = (a_1, \ldots, a_r)$ so that $\lambda a_i^N = 0$ in Λ. By Lemma 16.9.3 we can find a factorization $A \to C \to \Lambda$ with C of finite presentation such that $\mathfrak{h}_C \not\subset \mathfrak{q}$. Write $C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Set

$$B = A[x_1, \ldots, x_n, y_1, \ldots, y_r, z, t_{ij}]/(f_j - \sum y_i t_{ij}, z y_i)$$

where t_{ij} is a set of rm variables. Note that there is a map $B \to C[y_i, z]/(y_i z)$ given by setting t_{ij} equal to zero. The map $B \to \Lambda$ is the composition $B \to C[y_i, z]/(y_i z) \to \Lambda$ where $C[y_i, z]/(y_i z) \to \Lambda$ is the given map $C \to \Lambda$, maps z to λ, and maps y_i to the image of a_i^N in Λ.

We claim that B is a solution for $R \to A \to \Lambda \supset \mathfrak{q}$. First note that B_z is isomorphic to $C[y_1, \ldots, y_r, z, z^{-1}]$ and hence is smooth. On the other hand, $B_{y_\ell} \cong A[x_i, y_i, y_\ell^{-1}, t_{ij}, i \neq \ell]$ which is smooth over A. Thus we see that z and $a_\ell y_\ell$ (compositions of smooth maps are smooth) are all elements of $H_{B/R}$. This proves the lemma. □

16.10. Separable residue fields

07FB In this section we explain how to solve a local problem in the case of a separable residue field extension.

07FC **Lemma 16.10.1** (Ogoma)**.** *Let A be a Noetherian ring and let M be a finite A-module. Let $S \subset A$ be a multiplicative set. If $\pi \in A$ and $\mathrm{Ker}(\pi : S^{-1}M \to S^{-1}M) = \mathrm{Ker}(\pi^2 : S^{-1}M \to S^{-1}M)$ then there exists an $s \in S$ such that for any $n > 0$ we have $\mathrm{Ker}(s^n\pi : M \to M) = \mathrm{Ker}((s^n\pi)^2 : M \to M)$.*

Proof. Let $K = \mathrm{Ker}(\pi : M \to M)$ and $K' = \{m \in M \mid \pi^2 m = 0 \text{ in } S^{-1}M\}$ and $Q = K'/K$. Note that $S^{-1}Q = 0$ by assumption. Since A is Noetherian we see that Q is a finite A-module. Hence we can find an $s \in S$ such that s annihilates Q. Then s works. □

07FD **Lemma 16.10.2.** *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $I \subset \mathfrak{q}$ be a prime. Let n, e be positive integers Assume that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset I\Lambda_\mathfrak{q}$ and that $\Lambda_\mathfrak{q}$ is a regular local ring of dimension d. Then there exists an $n > 0$ and $\pi_1, \ldots, \pi_d \in \Lambda$ such that*

(1) $(\pi_1, \ldots, \pi_d)\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$,
(2) $\pi_1^n, \ldots, \pi_d^n \in I$, *and*
(3) *for $i = 1, \ldots, d$ we have*

$$Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i^2).$$

Proof. Set $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. First pick $\pi_1, \ldots, \pi_d$ with (1) which is possible as $\Lambda_\mathfrak{q}$ is regular. By assumption $\pi_i^n \in I\Lambda_\mathfrak{q}$. Thus we can find $s_1, \ldots, s_d \in S$ such that $s_i\pi_i^n \in I$. Replacing π_i by $s_i\pi_i$ we get (2). Note that (1) and (2) are preserved by further multiplying by elements of S. Suppose that (3) holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\pi_1, \ldots, \pi_d$ is a regular sequence in $S^{-1}\Lambda$, see Algebra, Lemma 10.106.3. In particular $\pi_1^e, \ldots, \pi_t^e, \pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda = \Lambda_\mathfrak{q}$ by Algebra, Lemma 10.68.9. Hence we see that

$$\mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i) = \mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i^2).$$

Thus we get (3) for $i = t+1$ after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces a sequence satisfying (1), (2), and (3). □

07FE **Lemma 16.10.3.** *Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1 where*

(1) *k is a field,*
(2) *Λ is Noetherian,*
(3) *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,*
(4) *$\Lambda_\mathfrak{q}$ is a regular local ring, and*
(5) *the field extension $\kappa(\mathfrak{q})/k$ is separable.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. Set $d = \dim \Lambda_\mathfrak{q}$. Set $R = k[x_1, \ldots, x_d]$. Choose $n > 0$ such that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{h}_A\Lambda_\mathfrak{q}$ which is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. Choose generators $a_1, \ldots, a_r$ of $H_{A/R}$. Set

$$B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^n - \sum z_{ij}a_j)$$

Each B_{a_j} is smooth over R it is a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$ and A_{a_j} is smooth over k. Hence B_{x_i} is smooth over R. Let $B \to C$ be the R-algebra map constructed in Lemma 16.3.1 which comes with a R-algebra retraction $C \to B$. In particular a map $C \to \Lambda$ fitting into the diagram above. By construction C_{x_i} is a smooth R-algebra with $\Omega_{C_{x_i}/R}$ free. Hence we can find $c > 0$ such that x_i^c is strictly standard in C/R, see Lemma 16.3.7. Now choose $\pi_1, \ldots, \pi_d \in \Lambda$ as in Lemma 16.10.2 where $n = n$, $e = 8c$, $\mathfrak{q} = \mathfrak{q}$ and $I = \mathfrak{h}_A$. Write $\pi_i^n = \sum \lambda_{ij}a_j$ for some $\pi_{ij} \in \Lambda$. There is a map $B \to \Lambda$ given by $x_i \mapsto \pi_i$ and $z_{ij} \mapsto \lambda_{ij}$. Set

$R = k[x_1, \ldots, x_d]$. Diagram

$$\begin{array}{ccccc} R & \longrightarrow & B & & \\ \uparrow & & \uparrow & \searrow & \\ k & \longrightarrow & A & \longrightarrow & \Lambda \end{array}$$

Now we apply Lemma 16.9.2 to $R \to C \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $x_1^c, \ldots, x_d^c$ of R. Assumption (2) is clear. Assumption (1) holds for R by inspection and for Λ by our choice of $\pi_1, \ldots, \pi_d$. (Note that if $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$, then we have $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^c)$ for all $c > 0$.) Thus it suffices to resolve

$$R/(x_1^e, \ldots, x_d^e) \to C/(x_1^e, \ldots, x_d^e) \to \Lambda/(\pi_1^e, \ldots, \pi_d^e) \supset \mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

for $e = 8c$. By Lemma 16.9.4 it suffices to resolve this after localizing at $\mathfrak{q}$. But since $x_1, \ldots, x_d$ map to a regular sequence in $\Lambda_\mathfrak{q}$ we see that $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat, see Algebra, Lemma 10.128.2. Hence

$$R_\mathfrak{p}/(x_1^e, \ldots, x_d^e) \to \Lambda_\mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

is a flat ring map of Artinian local rings. Moreover, this map induces a separable field extension on residue fields by assumption. Thus this map is a filtered colimit of smooth algebras by Algebra, Lemma 10.158.11 and Proposition 16.5.3. Existence of the desired solution follows from Algebra, Lemma 10.127.4. □

16.11. Inseparable residue fields

07FF In this section we explain how to solve a local problem in the case of an inseparable residue field extension.

07FG **Lemma 16.11.1.** *Let k be a field of characteristic $p > 0$. Let $(\Lambda, \mathfrak{m}, K)$ be an Artinian local k-algebra. Assume that $\dim H_1(L_{K/k}) < \infty$. Then Λ is a filtered colimit of Artinian local k-algebras A with each map $A \to \Lambda$ flat, with $\mathfrak{m}_A\Lambda = \mathfrak{m}$, and with A essentially of finite type over k.*

Proof. Note that the flatness of $A \to \Lambda$ implies that $A \to \Lambda$ is injective, so the lemma really tells us that Λ is a directed union of these types of subrings $A \subset \Lambda$. Let n be the minimal integer such that $\mathfrak{m}^n = 0$. We will prove this lemma by induction on n. The case $n = 1$ is clear as a field extension is a union of finitely generated field extensions.

Pick $\lambda_1, \ldots, \lambda_d \in \mathfrak{m}$ which generate $\mathfrak{m}$. As K is formally smooth over $\mathbf{F}_p$ (see Algebra, Lemma 10.158.7) we can find a ring map $\sigma : K \to \Lambda$ which is a section of the quotient map $\Lambda \to K$. In general σ is **not** a k-algebra map. Given σ we define

$$\Psi_\sigma : K[x_1, \ldots, x_d] \longrightarrow \Lambda$$

using σ on elements of K and mapping x_i to λ_i. Claim: there exists a $\sigma : K \to \Lambda$ and a subfield $k \subset F \subset K$ finitely generated over k such that the image of k in Λ is contained in $\Psi_\sigma(F[x_1, \ldots, x_d])$.

We will prove the claim by induction on the least integer n such that $\mathfrak{m}^n = 0$. It is clear for $n = 1$. If $n > 1$ set $I = \mathfrak{m}^{n-1}$ and $\Lambda' = \Lambda/I$. By induction we may assume given $\sigma' : K \to \Lambda'$ and $k \subset F' \subset K$ finitely generated such that the image of $k \to \Lambda \to \Lambda'$ is contained in $A' = \Psi_{\sigma'}(F'[x_1, \ldots, x_d])$. Denote $\tau' : k \to A'$ the induced map. Choose a lift $\sigma : K \to \Lambda$ of σ' (this is possible by the formal smoothness of $K/\mathbf{F}_p$ we mentioned above). For later reference we note that we can change σ

to $\sigma + D$ for some derivation $D : K \to I$. Set $A = F[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n$. Then Ψ_σ induces a ring map $\Psi_\sigma : A \to \Lambda$. The composition with the quotient map $\Lambda \to \Lambda'$ induces a surjective map $A \to A'$ with nilpotent kernel. Choose a lift $\tau : k \to A$ of τ' (possible as $k/\mathbf{F}_p$ is formally smooth). Thus we obtain two maps $k \to \Lambda$, namely $\Psi_\sigma \circ \tau : k \to \Lambda$ and the given map $i : k \to \Lambda$. These maps agree modulo I, whence the difference is a derivation $\theta = i - \Psi_\sigma \circ \tau : k \to I$. Note that if we change σ into $\sigma + D$ then we change θ into $\theta - D|_k$.

Choose a set of elements $\{y_j\}_{j \in J}$ of k whose differentials $\mathrm{d}y_j$ form a basis of $\Omega_{k/\mathbf{F}_p}$. The Jacobi-Zariski sequence for $\mathbf{F}_p \subset k \subset K$ is

$$0 \to H_1(L_{K/k}) \to \Omega_{k/\mathbf{F}_p} \otimes K \to \Omega_{K/\mathbf{F}_p} \to \Omega_{K/k} \to 0$$

As $\dim H_1(L_{K/k}) < \infty$ we can find a finite subset $J_0 \subset J$ such that the image of the first map is contained in $\bigoplus_{j \in J_0} K\mathrm{d}y_j$. Hence the elements $\mathrm{d}y_j$, $j \in J \setminus J_0$ map to K-linearly independent elements of $\Omega_{K/\mathbf{F}_p}$. Therefore we can choose a $D : K \to I$ such that $\theta - D|_k = \xi \circ \mathrm{d}$ where ξ is a composition

$$\Omega_{k/\mathbf{F}_p} = \bigoplus\nolimits_{j \in J} k\mathrm{d}y_j \longrightarrow \bigoplus\nolimits_{j \in J_0} k\mathrm{d}y_j \longrightarrow I$$

Let $f_j = \xi(\mathrm{d}y_j) \in I$ for $j \in J_0$. Change σ into $\sigma + D$ as above. Then we see that $\theta(a) = \sum_{j \in J_0} a_j f_j$ for $a \in k$ where $\mathrm{d}a = \sum a_j \mathrm{d}y_j$ in $\Omega_{k/\mathbf{F}_p}$. Note that I is generated by the monomials $\lambda^E = \lambda_1^{e_1} \ldots \lambda_d^{e_d}$ of total degree $|E| = \sum e_i = n - 1$ in $\lambda_1, \ldots, \lambda_d$. Write $f_j = \sum_E c_{j,E} \lambda^E$ with $c_{j,E} \in K$. Replace F' by $F = F'(c_{j,E})$. Then the claim holds.

Choose σ and F as in the claim. The kernel of Ψ_σ is generated by finitely many polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_d]$ and we may assume their coefficients are in F after enlarging F by adjoining finitely many elements. In this case it is clear that the map $A = F[x_1, \ldots, x_d]/(g_1, \ldots, g_t) \to K[x_1, \ldots, x_d]/(g_1, \ldots, g_t) = \Lambda$ is flat. By the claim A is a k-subalgebra of Λ. It is clear that Λ is the filtered colimit of these algebras, as K is the filtered union of the subfields F. Finally, these algebras are essentially of finite type over k by Algebra, Lemma 10.54.4. □

07FH **Lemma 16.11.2.** *Let k be a field of characteristic $p > 0$. Let Λ be a Noetherian geometrically regular k-algebra. Let $\mathfrak{q} \subset \Lambda$ be a prime ideal. Let $n \geq 1$ be an integer and let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be a finite subset. Then we can find $m \geq 0$ and $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ with the following properties*

1. *setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $\mathfrak{q}\Lambda_\mathfrak{q} = \mathfrak{p}\Lambda_\mathfrak{q}$ and $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat,*
2. *there is a factorization by homomorphisms of local Artinian rings*
$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
where the first arrow is essentially smooth and the second is flat,
3. *E is contained in D modulo $\mathfrak{q}^n\Lambda_\mathfrak{q}$.*

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Note that $\dim H_1(L_{\kappa(\mathfrak{q})/k}) < \infty$ by More on Algebra, Proposition 15.35.1. Pick $A \subset \bar{\Lambda}$ containing E such that A is local Artinian, essentially of finite type over k, the map $A \to \bar{\Lambda}$ is flat, and $\mathfrak{m}_A$ generates the maximal ideal of $\bar{\Lambda}$, see Lemma 16.11.1. Denote $F = A/\mathfrak{m}_A$ the residue field so that $k \subset F \subset K$. Pick $\lambda_1, \ldots, \lambda_t \in \Lambda$ which map to elements of A in $\bar{\Lambda}$ such that moreover the images of $\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_t$ form a basis of $\Omega_{F/k}$. Consider the map $\varphi' : k[y_1, \ldots, y_t] \to \Lambda$ sending y_j to λ_j. Set $\mathfrak{p}' = (\varphi')^{-1}(\mathfrak{q})$. By More on Algebra,

Lemma 15.35.2 the ring map $k[y_1, \ldots, y_t]_{\mathfrak{p}'} \to \Lambda_{\mathfrak{q}}$ is flat and $\Lambda_{\mathfrak{q}}/\mathfrak{p}'\Lambda_{\mathfrak{q}}$ is regular. Thus we can choose further elements $\lambda_{t+1}, \ldots, \lambda_m \in \Lambda$ which map into $A \subset \bar{\Lambda}$ and which map to a regular system of parameters of $\Lambda_{\mathfrak{q}}/\mathfrak{p}'\Lambda_{\mathfrak{q}}$. We obtain $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ having property (1) such that $k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n k[y_1, \ldots, y_m]_{\mathfrak{p}} \to \bar{\Lambda}$ factors through A. Thus $k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$ is flat by Algebra, Lemma 10.39.9. By construction the residue field extension $F/\kappa(\mathfrak{p})$ is finitely generated and $\Omega_{F/\kappa(\mathfrak{p})} = 0$. Hence it is finite separable by More on Algebra, Lemma 15.34.1. Thus $k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$ is finite by Algebra, Lemma 10.54.4. Finally, we conclude that it is étale by Algebra, Lemma 10.143.7. Since an étale ring map is certainly essentially smooth we win. □

07FI **Lemma 16.11.3.** *Let $\varphi : k[y_1, \ldots, y_m] \to \Lambda$, n, $\mathfrak{q}$, $\mathfrak{p}$ and*

$$k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n \to D \to \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$$

be as in Lemma 16.11.2. Then for any $\lambda \in \Lambda \setminus \mathfrak{q}$ there exists an integer $q > 0$ and a factorization

$$k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n \to D \to D' \to \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$$

such that $D \to D'$ is an essentially smooth map of local Artinian rings, the last arrow is flat, and λ^q is in D'.

Proof. Set $\bar{\Lambda} = \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$. Let $\bar{\lambda}$ be the image of λ in $\bar{\Lambda}$. Let $\alpha \in \kappa(\mathfrak{q})$ be the image of λ in the residue field. Let $k \subset F \subset \kappa(\mathfrak{q})$ be the residue field of D. If α is in F then we can find an $x \in D$ such that $x\bar{\lambda} = 1 \bmod \mathfrak{q}$. Hence $(x\bar{\lambda})^q = 1 \bmod (\mathfrak{q})^q$ if q is divisible by p. Hence $\bar{\lambda}^q$ is in D. If α is transcendental over F, then we can take $D' = (D[\bar{\lambda}])_{\mathfrak{m}}$ equal to the subring generated by D and $\bar{\lambda}$ localized at $\mathfrak{m} = D[\bar{\lambda}] \cap \mathfrak{q}\bar{\Lambda}$. This works because $D[\bar{\lambda}]$ is in fact a polynomial algebra over D in this case. Finally, if $\lambda \bmod \mathfrak{q}$ is algebraic over F, then we can find a p-power q such that α^q is separable algebraic over F, see Fields, Section 9.28. Note that D and $\bar{\Lambda}$ are henselian local rings, see Algebra, Lemma 10.153.10. Let $D \to D'$ be a finite étale extension whose residue field extension is $F(\alpha^q)/F$, see Algebra, Lemma 10.153.7. Since $\bar{\Lambda}$ is henselian and $F(\alpha^q)$ is contained in its residue field we can find a factorization $D' \to \bar{\Lambda}$. By the first part of the argument we see that $\bar{\lambda}^{qq'} \in D'$ for some $q' > 0$. □

07FJ **Lemma 16.11.4.** *Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1 where*

(1) *k is a field of characteristic $p > 0$,*
(2) *Λ is Noetherian and geometrically regular over k,*
(3) *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. The lemma is proven by the following steps in the given order. We will justify each of these steps below.

07FK (1) Pick an integer $N > 0$ such that $\mathfrak{q}^N\Lambda_{\mathfrak{q}} \subset H_{A/k}\Lambda_{\mathfrak{q}}$.
07FL (2) Pick generators $a_1, \ldots, a_t \in A$ of the ideal $H_{A/R}$.
07FM (3) Set $d = \dim(\Lambda_{\mathfrak{q}})$.
07FN (4) Set $B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^{2N} - \sum z_{ij}a_j)$.
07FP (5) Consider B as a $k[x_1, \ldots, x_d]$-algebra and let $B \to C$ be as in Lemma 16.3.1. We also obtain a section $C \to B$.
07FQ (6) Choose $c > 0$ such that each x_i^c is strictly standard in C over $k[x_1, \ldots, x_d]$.

07FR (7) Set $n = N + dc$ and $e = 8c$.

07FS (8) Let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be the images of generators of A as a k-algebra.

07FT (9) Choose an integer m and a k-algebra map $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ and a factorization by local Artinian rings

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

such that the first arrow is essentially smooth, the second is flat, E is contained in D, with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ the map $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat, and $\mathfrak{p}\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$.

07FU (10) Choose $\pi_1, \ldots, \pi_d \in \mathfrak{p}$ which map to a regular system of parameters of $k[y_1, \ldots, y_m]_\mathfrak{p}$.

07FV (11) Let $R = k[y_1, \ldots, y_m, t_1, \ldots, t_m]$ and $\gamma_i = \pi_i t_i$.

07FW (12) If necessary modify the choice of π_i such that for $i = 1, \ldots, d$ we have

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

07FX (13) There exist $\delta_1, \ldots, \delta_d \in \Lambda$, $\delta_i \notin \mathfrak{q}$ and a factorization $D \to D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ with D' local Artinian, $D \to D'$ essentially smooth, the map $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ flat such that, with $\pi_i' = \delta_i\pi_i$, we have for $i = 1, \ldots, d$
   (a) $(\pi_i')^{2N} = \sum a_j\lambda_{ij}$ in Λ where $\lambda_{ij} \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D',
   (b) $\mathrm{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'_i) = \mathrm{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'^2_i)$,
   (c) $\delta_i \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D'.

07FY (14) Define $B \to \Lambda$ by sending x_i to π_i' and z_{ij} to λ_{ij} found above. Define $C \to \Lambda$ by composing the map $B \to \Lambda$ with the retraction $C \to B$.

07FZ (15) Map $R \to \Lambda$ by φ on $k[y_1, \ldots, y_m]$ and by sending t_i to δ_i. Further introduce a map

$$k[x_1, \ldots, x_d] \longrightarrow R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$$

by sending x_i to $\gamma_i = \pi_i t_i$.

07G0 (16) It suffices to resolve

$$R \to C \otimes_{k[x_1, \ldots, x_d]} R \to \Lambda \supset \mathfrak{q}$$

07G1 (17) Set $I = (\gamma_1^e, \ldots, \gamma_d^e) \subset R$.

07G2 (18) It suffices to resolve

$$R/I \to C \otimes_{k[x_1, \ldots, x_d]} R/I \to \Lambda/I\Lambda \supset \mathfrak{q}/I\Lambda$$

07G3 (19) We denote $\mathfrak{r} \subset R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$ the inverse image of $\mathfrak{q}$.

07G4 (20) It suffices to resolve

$$(R/I)_\mathfrak{r} \to C \otimes_{k[x_1, \ldots, x_d]} (R/I)_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$$

07G5 (21) Set $J = (\pi_1^e, \ldots, \pi_d^e)$ in $k[y_1, \ldots, y_m]$.

07G6 (22) It suffices to resolve

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

07G7 (23) It suffices to resolve

$$(R/\mathfrak{p}^nR)_\mathfrak{p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak{p}^nR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

07G8 (24) It suffices to resolve

$$(R/\mathfrak{p}^nR)_\mathfrak{p} \to B \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak{p}^nR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

07G9 (25) The ring $D'[t_1, \ldots, t_d]$ is given the structure of an $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebra by the given map $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D'$ and by sending t_i to t_i. It suffices to find a factorization

$$B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

where the second arrow sends t_i to δ_i and induces the given homomorphism $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$.

07GA (26) Such a factorization exists by our choice of D' above.

We now give the justification for each of the steps, except that we skip justifying the steps which just introduce notation.

Ad (1). This is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A = \sqrt{H_{A/k}\Lambda}$.

Ad (6). Note that A_{a_i} is smooth over k. Hence B_{a_j}, which is isomorphic to a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$, is smooth over $k[x_1, \ldots, x_d]$. Thus B_{x_i} is smooth over $k[x_1, \ldots, x_d]$. By Lemma 16.3.1 we see that C_{x_i} is smooth over $k[x_1, \ldots, x_d]$ with finite free module of differentials. Hence some power of x_i is strictly standard in C over $k[x_1, \ldots, x_n]$ by Lemma 16.3.7.

Ad (9). This follows by applying Lemma 16.11.2.

Ad (10). Since $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat and $\mathfrak{p}\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$ by construction we see that $\dim(k[y_1, \ldots, y_m]_\mathfrak{p}) = d$ by Algebra, Lemma 10.112.7. Thus we can find $\pi_1, \ldots, \pi_d \in \Lambda$ which map to a regular system of parameters in $\Lambda_\mathfrak{q}$.

Ad (12). By Algebra, Lemma 10.106.3 any permutation of the sequence $\pi_1, \ldots, \pi_d$ is a regular sequence in $k[y_1, \ldots, y_m]_\mathfrak{p}$. Hence $\gamma_1 = \pi_1 t_1, \ldots, \gamma_d = \pi_d t_d$ is a regular sequence in $R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d]$, see Algebra, Lemma 10.68.10. Let $S = k[y_1, \ldots, y_m] \setminus \mathfrak{p}$ so that $R_\mathfrak{p} = S^{-1}R$. Note that $\pi_1, \ldots, \pi_d$ and $\gamma_1, \ldots, \gamma_d$ remain regular sequences if we multiply our π_i by elements of S. Suppose that

$$\mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)R}(\gamma_i^2)$$

holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\gamma_1^e, \ldots, \gamma_t^e, \gamma_{t+1}$ is a regular sequence in $S^{-1}R$ by Algebra, Lemma 10.68.9. Hence we see that

$$\mathrm{Ann}_{S^{-1}R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)}(\gamma_i) = \mathrm{Ann}_{S^{-1}R/(\gamma_1^e,\ldots,\gamma_{i-1}^e)}(\gamma_i^2).$$

Thus we get

$$\mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_t^e)R}(\gamma_{t+1}) = \mathrm{Ann}_{R/(\gamma_1^e,\ldots,\gamma_t^e)R}(\gamma_{t+1}^2)$$

after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces the desired sequence.

Ad (13). Let $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Suppose that we have a $t \in \{0, \ldots, d\}$ and $\delta_1, \ldots, \delta_t \in S$ and a factorization $D \to D' \to \bar{\Lambda}$ as in (13) such that (a), (b), (c) hold for $i = 1, \ldots, t$. We have $\pi_{t+1}^N \in H_{A/k}\Lambda_\mathfrak{q}$ as $\mathfrak{q}^N\Lambda_\mathfrak{q} \subset H_{A/k}\Lambda_\mathfrak{q}$ by (1). Hence $\pi_{t+1}^N \in H_{A/k}\bar{\Lambda}$. Hence $\pi_{t+1}^N \in H_{A/k}D'$ as $D' \to \bar{\Lambda}$ is faithfully flat, see Algebra, Lemma 10.82.11. Recall that $H_{A/k} = (a_1, \ldots, a_t)$. Say $\pi_{t+1}^N = \sum a_j d_j$ in D' and choose $c_j \in \Lambda_\mathfrak{q}$ lifting $d_j \in D'$. Then $\pi_{t+1}^N = \sum c_j a_j + \epsilon$ with $\epsilon \in \mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{q}^{n-N}H_{A/k}\Lambda_\mathfrak{q}$. Write $\epsilon = \sum a_j c_j'$ for some $c_j' \in \mathfrak{q}^{n-N}\Lambda_\mathfrak{q}$. Hence $\pi_{t+1}^{2N} = \sum(\pi_{t+1}^N c_j + \pi_{t+1}^N c_j')a_j$. Note that $\pi_{t+1}^N c_j'$ maps to zero in $\bar{\Lambda}$; this trivial but key observation will ensure later that (a) holds. Now we choose $s \in S$ such that there exist $\mu_{t+1j} \in \Lambda$ such that on the one hand $\pi_{t+1}^N c_j + \pi_{t+1}^N c_j' = \mu_{t+1j}/s^{2N}$ in $S^{-1}\Lambda$ and on the other $(s\pi_{t+1})^{2N} = \sum \mu_{t+1j} a_j$ in Λ (minor detail omitted). We

may further replace s by a power and enlarge D' such that s maps to an element of D'. With these choices μ_{t+1j} maps to $s^{2N}d_j$ which is an element of D'. Note that $\pi_1, \ldots, \pi_d$ are a regular sequence of parameters in $S^{-1}\Lambda$ by our choice of φ. Hence $\pi_1, \ldots, \pi_d$ forms a regular sequence in $\Lambda_\mathfrak{q}$ by Algebra, Lemma 10.106.3. It follows that ${\pi'}_1^e, \ldots, {\pi'}_t^e, s\pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda$ by Algebra, Lemma 10.68.9. Thus we get

$$\mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(s\pi_{t+1}) = \mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s\pi_{t+1})^2).$$

Hence we may apply Lemma 16.10.1 to find an $s' \in S$ such that

$$\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s')^q s\pi_{t+1}) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(((s')^q s\pi_{t+1})^2).$$

for any $q > 0$. By Lemma 16.11.3 we can choose q and enlarge D' such that $(s')^q$ maps to an element of D'. Setting $\delta_{t+1} = (s')^q s$ and we conclude that (a), (b), (c) hold for $i = 1, \ldots, t+1$. For (a) note that $\lambda_{t+1j} = (s')^{2Nq}\mu_{t+1j}$ works. By induction on t we win.

Ad (16). By construction the radical of $H_{(C \otimes_{k[x_1,\ldots,x_d]} R)/R}\Lambda$ contains $\mathfrak{h}_A$. Namely, the elements $a_j \in H_{A/k}$ map to elements of $H_{B/k[x_1,\ldots,x_n]}$, hence map to elements of $H_{C/k[x_1,\ldots,x_n]}$, hence $a_j \otimes 1$ map to elements of $H_{C \otimes_{k[x_1,\ldots,x_d]} R/R}$. Moreover, if we have a solution $C \otimes_{k[x_1,\ldots,x_n]} R \to T \to \Lambda$ of

$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$

then $H_{T/R} \subset H_{T/k}$ as R is smooth over k. Hence T will also be a solution for the original situation $k \to A \to \Lambda \supset \mathfrak{q}$.

Ad (18). Follows on applying Lemma 16.9.2 to $R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $\gamma_1^c, \ldots, \gamma_d^c$. We note that since x_i^c are strictly standard in C over $k[x_1, \ldots, x_d]$ the elements γ_i^c are strictly standard in $C \otimes_{k[x_1,\ldots,x_d]} R$ over R by Lemma 16.2.7. The other assumption of Lemma 16.9.2 holds by steps (12) and (13).

Ad (20). Apply Lemma 16.9.4 to the situation in (18). In the rest of the arguments the target ring is local Artinian, hence we are looking for a factorization by a smooth algebra T over the source ring.

Ad (22). Suppose that $C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$ is a solution to

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

Then $C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to T_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$ is a solution to the situation in (20).

Ad (23). Our $n = N + dc$ is large enough so that $\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \subset J_\mathfrak{p}$ and $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset J\Lambda_\mathfrak{q}$. Hence if we have a solution $C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ of (22 then we can take T/JT as the solution for (23).

Ad (24). This is true because we have a section $C \to B$ in the category of R-algebras.

Ad (25). This is true because D' is essentially smooth over the local Artinian ring $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d].$$

Hence $D'[t_1, \ldots, t_d]$ is a filtered colimit of smooth $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebras and $B \otimes_{k[x_1,\ldots,x_d]}$ $(R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ factors through one of these.

Ad (26). The final twist of the proof is that we cannot just use the map $B \to D'$ which maps x_i to the image of π_i' in D' and z_{ij} to the image of λ_{ij} in D' because we need the diagram

$$\begin{array}{ccc} B & \longrightarrow & D'[t_1, \ldots, t_d] \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_d] & \longrightarrow & R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}} \end{array}$$

to commute and we need the composition $B \to D'[t_1, \ldots, t_d] \to \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$ to be the map of (14). This requires us to map x_i to the image of $\pi_i t_i$ in $D'[t_1, \ldots, t_d]$. Hence we map z_{ij} to the image of $\lambda_{ij} t_i^{2N}/\delta_i^{2N}$ in $D'[t_1, \ldots, t_d]$ and everything is clear. □

16.12. The main theorem

07GB In this section we wrap up the discussion.

07GC **Theorem 16.12.1** (Popescu)**.** *Any regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.*

Proof. By Lemma 16.8.4 it suffices to prove this for $k \to \Lambda$ where Λ is Noetherian and geometrically regular over k. Let $k \to A \to \Lambda$ be a factorization with A a finite type k-algebra. It suffices to construct a factorization $A \to B \to \Lambda$ with B of finite type such that $\mathfrak{h}_B = \Lambda$, see Lemma 16.2.8. Hence we may perform Noetherian induction on the ideal $\mathfrak{h}_A$. Pick a prime $\mathfrak{q} \supset \mathfrak{h}_A$ such that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. It now suffices to resolve $k \to A \to \Lambda \supset \mathfrak{q}$ (as defined in the text following Situation 16.9.1). If the characteristic of k is zero, this follows from Lemma 16.10.3. If the characteristic of k is $p > 0$, this follows from Lemma 16.11.4. □

16.13. The approximation property for G-rings

07QX Let R be a Noetherian local ring. In this case R is a G-ring if and only if the ring map $R \to R^\wedge$ is regular, see More on Algebra, Lemma 15.50.7. In this case it is true that the henselization R^h and the strict henselization R^{sh} of R are G-rings, see More on Algebra, Lemma 15.50.8. Moreover, any algebra essentially of finite type over a field, over a complete local ring, over $\mathbf{Z}$, or over a characteristic zero Dedekind ring is a G-ring, see More on Algebra, Proposition 15.50.12. This gives an ample supply of rings to which the result below applies.

Let R be a ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Let S be an R-algebra. In this situation we say a vector $(a_1, \ldots, a_n) \in S^n$ is a *solution in S* if and only if

$$f_j(a_1, \ldots, a_n) = 0 \text{ in } S, \text{ for } j = 1, \ldots, m$$

Of course an important question in algebraic geometry is to see when systems of polynomial equations have solutions. The following theorem tells us that having solutions in the completion of a local Noetherian ring is often enough to show there exist solutions in the henselization of the ring.

07QY **Theorem 16.13.1.** *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution in $R^\wedge$. If R is a henselian G-ring, then for every integer N there exists a solution $(b_1, \ldots, b_n) \in R^n$ in R such that $a_i - b_i \in \mathfrak{m}^N R^\wedge$.*

Proof. Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $\mathfrak{m}^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over R.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

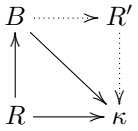

Since the vertical arrow is smooth, More on Algebra, Lemma 15.9.14 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Since R is henselian we see that $R \to R'$ has a section, see Algebra, Lemma 10.153.3. Let $b_i \in R$ be the image of a_i under the ring maps $A \to B \to R' \to R$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R. □

Given a Noetherian local ring $(R, \mathfrak{m})$, an étale ring map $R \to R'$, and a maximal ideal $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$, then we have inclusions

$$R \subset R_{\mathfrak{m}'} \subset R^h \subset R^\wedge,$$

by Algebra, Lemma 10.155.5 and More on Algebra, Lemma 15.45.3.

07QZ **Theorem 16.13.2.** *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution. If R is a G-ring, then for every integer N there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a maximal ideal $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$*
(3) *a solution $(b_1, \ldots, b_n) \in (R')^n$ in R'*

such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$ and $a_i - b_i \in (\mathfrak{m}')^N R^\wedge$.

Proof. We could deduce this theorem from Theorem 16.13.1 using that the henselization R^h is a G-ring by More on Algebra, Lemma 15.50.8 and writing R^h as a directed colimit of étale extension R'. Instead we prove this by redoing the proof of the previous theorem in this case.

Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R' for some étale ring map $R \to R'$ endowed with a maximal ideal $\mathfrak{m}'$ such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $(\mathfrak{m}')^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over some étale ring extension which induces a trivial residue field extension at some prime over $\mathfrak{m}$.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

$$\begin{array}{ccc} B & \dashrightarrow & R' \\ \uparrow & \nearrow & \downarrow \\ R & \longrightarrow & \kappa \end{array}$$

Since the vertical arrow is smooth, More on Algebra, Lemma 15.9.14 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Let $b_i \in R'$ be the image of a_i under the ring maps $A \to B \to R'$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R'. □

0A1W **Example 16.13.3.** Let $(R, \mathfrak{m})$ be a Noetherian local ring with henselization R^h. The map on completions $R^\wedge \to (R^h)^\wedge$ is an isomorphism, see More on Algebra, Lemma 15.45.3. Since also R^h is Noetherian (ibid.) we may think of R^h as a subring of its completion (because the completion is faithfully flat). In this way we see that we may identify R^h with a subring of $R^\wedge$.

Let us try to understand which elements of $R^\wedge$ are in R^h. For simplicity we assume R is a domain with fraction field K. Clearly, every element f of R^h is algebraic over R, in the sense that there exists an equation of the form $a_n f^n + \ldots + a_1 f + a_0 = 0$ for some $a_i \in R$ with $n > 0$ and $a_n \neq 0$.

Conversely, assume that $f \in R^\wedge$, $n \in \mathbf{N}$, and $a_0, \ldots, a_n \in R$ with $a_n \neq 0$ such that $a_n f^n + \ldots + a_1 f + a_0 = 0$. If R is a G-ring, then, for every $N > 0$ there exists an element $g \in R^h$ with $a_n g^n + \ldots + a_1 g + a_0 = 0$ and $f - g \in \mathfrak{m}^N R^\wedge$, see Theorem 16.13.2. We'd like to conclude that $f = g$ when $N \gg 0$. If this is not true, then we find infinitely many roots g of $P(T)$ in R^h. This is impossible because (1) $R^h \subset R^h \otimes_R K$ and (2) $R^h \otimes_R K$ is a finite product of field extensions of K. Namely, $R \to K$ is injective and $R \to R^h$ is flat, hence $R^h \to R^h \otimes_R K$ is injective and (2) follows from More on Algebra, Lemma 15.45.13.

Conclusion: If R is a Noetherian local domain with fraction field K and a G-ring, then $R^h \subset R^\wedge$ is the set of all elements which are algebraic over K.

Here is another variant of the main theorem of this section.

0CAR **Lemma 16.13.4.** *Let R be a Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in ((R_\mathfrak{p})^\wedge)^n$ is a solution. If $R_\mathfrak{p}$ is a G-ring, then for every integer N there exist*

(1) *an étale ring map $R \to R'$,*

(2) *a prime ideal* $\mathfrak{p}' \subset R'$ *lying over* $\mathfrak{p}$
(3) *a solution* $(b_1, \ldots, b_n) \in (R')^n$ *in* R'

such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ *and* $a_i - b_i \in (\mathfrak{p}')^N (R'_{\mathfrak{p}'})^\wedge$.

Proof. By Theorem 16.13.2 we can find a solution $(b'_1, \ldots, b'_n)$ in some ring R'' étale over $R_{\mathfrak{p}}$ which comes with a prime ideal $\mathfrak{p}''$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}'')$ and $a_i - b'_i \in (\mathfrak{p}'')^N (R''_{\mathfrak{p}''})^\wedge$. We can write $R'' = R' \otimes_R R_{\mathfrak{p}}$ for some étale R-algebra R' (see Algebra, Lemma 10.143.3). After replacing R' by a principal localization if necessary we may assume $(b'_1, \ldots, b'_n)$ come from a solution $(b_1, \ldots, b_n)$ in R'. Setting $\mathfrak{p}' = R' \cap \mathfrak{p}''$ we see that $R''_{\mathfrak{p}''} = R'_{\mathfrak{p}'}$ which finishes the proof. □

16.14. Approximation for henselian pairs

0AH4 We can generalize the discussion of Section 16.13 to the case of henselian pairs. Henselian pairs where defined in More on Algebra, Section 15.11.

0AH5 **Lemma 16.14.1.** *Let* (A, I) *be a henselian pair with* A *Noetherian. Let* $A^\wedge$ *be the* I*-adic completion of* A*. Assume at least one of the following conditions holds*

(1) $A \to A^\wedge$ *is a regular ring map,*
(2) A *is a Noetherian G-ring, or*
(3) (A, I) *is the henselization (More on Algebra, Lemma 15.12.1) of a pair* (B, J) *where* B *is a Noetherian G-ring.*

Given $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ *and* $\hat{a}_1, \ldots, \hat{a}_n \in A^\wedge$ *such that* $f_j(\hat{a}_1, \ldots, \hat{a}_n) = 0$ *for* $j = 1, \ldots, m$*, for every* $N \geq 1$ *there exist* $a_1, \ldots, a_n \in A$ *such that* $\hat{a}_i - a_i \in I^N$ *and such that* $f_j(a_1, \ldots, a_n) = 0$ *for* $j = 1, \ldots, m$*.*

Proof. By More on Algebra, Lemma 15.50.15 we see that (3) implies (2). By More on Algebra, Lemma 15.50.14 we see that (2) implies (1). Thus it suffices to prove the lemma in case $A \to A^\wedge$ is a regular ring map.

Let $\hat{a}_1, \ldots, \hat{a}_n$ be as in the statement of the lemma. By Theorem 16.12.1 we can find a factorization $A \to B \to A^\wedge$ with $A \to P$ smooth and $b_1, \ldots, b_n \in B$ with $f_j(b_1, \ldots, b_n) = 0$ in B. Denote $\sigma : B \to A^\wedge \to A/I^N$ the composition. By More on Algebra, Lemma 15.9.14 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I^N \to A'/I^N A'$ and an A-algebra map $\tilde{\sigma} : B \to A'$ lifting σ. Since (A, I) is henselian, there is an A-algebra map $\chi : A' \to A$, see More on Algebra, Lemma 15.11.6. Then setting $a_i = \chi(\tilde{\sigma}(b_i))$ gives a solution. □

16.15. Other chapters

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CHAPTER 17

Sheaves of Modules

01AC

17.1. Introduction

01AD In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [**Ser55b**], [**DG67**] and [**AGV71**].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 18.1), although there we will mostly just duplicate the discussion from this chapter.

17.2. Pathology

01AE A ringed space is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of X, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

17.3. The abelian category of sheaves of modules

01AF Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 6.25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\mathit{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\mathit{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 6.16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $\mathit{Mod}(\mathcal{O}_X)$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$.

On the other hand, we define $\mathrm{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \mathrm{Coker}(\mathcal{F}(U) \to \mathcal{G}(U)) = \mathcal{G}(U)/\varphi(\mathcal{F}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 6.17.2 we see that $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 6.16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

01AG **Lemma 17.3.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition 12.5.1 we have to show that image and coimage agree. By Sheaves, Lemma 6.16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product

$$\prod_{i \in I} \mathcal{F}_i$$

which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 4.14.6.

(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum

$$\bigoplus_{i \in I} \mathcal{F}_i$$

which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 4.14.7. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

01AH **Lemma 17.3.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
(2) *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
(3) *Filtered colimits are exact.*
(4) *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

Proof. As $Mod(\mathcal{O}_X)$ is abelian (Lemma 17.3.1) it has all finite limits and colimits (Homology, Lemma 12.5.5). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas 4.14.11 and 4.14.12. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 17.3.1) this follows from the case of modules which is Algebra, Lemma 10.8.8. We omit the proof of (4). □

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.2 for a description of exactness properties in terms of short exact sequences.

01AJ **Lemma 17.3.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

(1) *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
(2) *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
(3) *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma 6.26.2 and Categories, Section 4.24. Part (3) holds because exactness can be checked on stalks (Lemma 17.3.1) and the description of stalks of the pullback, see Sheaves, Lemma 6.22.1. □

01AK **Lemma 17.3.4.** *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. Follows from the description of stalks given in Sheaves, Lemma 6.31.6. □

01AI **Lemma 17.3.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because

U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j\in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i\in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 17.3.2 above. □

17.4. Sections of sheaves of modules

01AL Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique *map of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X \longrightarrow \mathcal{F},\ f \longmapsto fs$$

associated to s. The notation above signifies that a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that φ is the morphism associated to s.

01AM **Definition 17.4.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i *generate* $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 6.11 where, given a local section s of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote s_x, or even s the image of s in the stalk $\mathcal{F}_x$.

01AN **Lemma 17.4.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ be global sections. The sections s_i generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. Omitted. □

01AO **Lemma 17.4.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. Omitted. □

01AP **Lemma 17.4.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let s_i, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.*

Proof. Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i\in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 6.17.6. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. □

01AQ **Definition 17.4.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 17.4.4 above is the *subsheaf generated by the s_i*.

01AR **Lemma 17.4.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 17.4.4. □

17.5. Supports of modules and sections

01AS

01AT **Definition 17.5.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) The *support of $\mathcal{F}$* is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
(2) We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
(3) Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The *support of s* is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

01AU **Lemma 17.5.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.*

(1) *The support of $s \in \mathcal{F}(U)$ is closed in U.*
(2) *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.*
(3) *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

Proof. This is true because if $s_x = 0$, then s is zero in an open neighbourhood of x by definition of stalks. Similarly for f. Details omitted. □

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbf{R}$. Then the support would be the set of points p_i which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the support of this sheaf is exactly U.

01AV **Lemma 17.5.3.** *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

17.6. Closed immersions and abelian sheaves

01AW Recall that we think of an abelian sheaf on a topological space X as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

01AX **Lemma 17.6.1.** *Let X be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in Z. The functor i^{-1} is a left inverse to i_*.

Proof. Exactness follows from the description of stalks in Sheaves, Lemma 6.32.1 and Lemma 17.3.1. The rest was shown in Sheaves, Lemma 6.32.3. □

Let $\mathcal{F}$ be an abelian sheaf on the topological space X. Given a closed subset Z, there is a canonical abelian subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in Z. Here is the exact statement.

01AY **Remark 17.6.2.** Let X be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. For $U \subset X$ open set, define

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}.$$

Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in Z. By Lemma 17.6.1 we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z. In this way we obtain a left exact functor

$$Ab(X) \longrightarrow Ab(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as abelian sheaf on } Z$$

All of the statements made above follow directly from Lemma 17.5.2.

This seems like a good opportunity to show that the functor i_* has a right adjoint on abelian sheaves.

01AZ **Lemma 17.6.3.** *Let $i : Z \to X$ be the inclusion of a closed subset into the topological space X. The functor $Ab(X) \to Ab(Z)$, $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ of Remark 17.6.2 is a right adjoint to $i_* : Ab(Z) \to Ab(X)$. In particular i_* commutes with arbitrary colimits.*

Proof. We have to show that for any abelian sheaf $\mathcal{F}$ on X and any abelian sheaf $\mathcal{G}$ on Z we have

$$\mathrm{Hom}_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Ab(Z)}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F}))$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. □

01B0 **Remark 17.6.4.** In Sheaves, Remark 6.32.5 we showed that i_* as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor i_* is exact on sheaves of pointed sets, sections with support in Z can be defined for sheaves of pointed sets, and $\mathcal{H}_Z$ makes sense and is a right adjoint to i_*.

17.7. A canonical exact sequence

02US We give this exact sequence its own section.

02UT **Lemma 17.7.1.** *Let X be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on X the adjunction mappings $j_!j^{-1}\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ give a short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on X we obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!j^{-1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_*i^{-1}\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!j^{-1}\mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & i_*i^{-1}\mathcal{G} & \longrightarrow & 0 \end{array}$$

Proof. The functoriality of the short exact sequence is immediate from the naturality of the adjunction mappings. We may check exactness on stalks (Lemma 17.3.1). For a description of the stalks in question see Sheaves, Lemmas 6.31.6 and 6.32.1. □

17.8. Modules locally generated by sections

01B1 Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 6.31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 6.31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

01B2 **Definition 17.8.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

01B3 **Lemma 17.8.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.*

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{array}{ccc} (f^{-1}V, \mathcal{O}_{f^{-1}V}) & \xrightarrow{j'} & (X, \mathcal{O}_X) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (V, \mathcal{O}_V) & \xrightarrow{j} & (Y, \mathcal{O}_Y) \end{array}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 6.26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

17.9. Modules of finite type

01B4

01B5 **Definition 17.9.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

01B6 **Lemma 17.9.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.*

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

01B7 **Lemma 17.9.3.** *Let X be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

Proof. The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). □

01B8 **Lemma 17.9.4.** *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.*

Proof. Choose an open neighbourhood $U \subset X$ of x such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over U. By assumption of surjectivity of φ_x, after shrinking U we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then U works. □

01B9 **Lemma 17.9.5.** *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.*

Proof. This is a special case of Lemma 17.9.4 applied to the morphism $0 \to \mathcal{F}$. □

01BA **Lemma 17.9.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.*

Proof. This is a reformulation of Lemma 17.9.5. □

01BB **Lemma 17.9.7.** *Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be the colimit. Assume (a) I is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and (c) X is quasi-compact. Then there exists an i such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.*

Proof. Let $x \in X$. There exists an open neighbourhood $U \subset X$ of x and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking U to a smaller open neighbourhood of x we may assume that each s_j comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since X is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each j an index i_j and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to U_j. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all i_j. □

01BC **Lemma 17.9.8.** *Let X be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i \in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on X is isomorphic to exactly one of the $\mathcal{F}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r_j}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_j \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_j} / (\mathcal{K}_j|_{U_j \cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_{j'}} / (\mathcal{K}_{j'}|_{U_j \cap U_{j'}})$$

see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). □

17.10. Quasi-coherent modules

01BD In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

01BE **Definition 17.10.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if for every point

$x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

01BF **Lemma 17.10.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Omitted. □

02CF **Remark 17.10.3.** Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 17.10.9.

01BG **Lemma 17.10.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.*

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_Y \longrightarrow \mathcal{G} \longrightarrow 0$$

then upon applying f^* we obtain the exact sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow f^*\mathcal{G} \longrightarrow 0.$$

This implies the lemma. □

This gives plenty of examples of quasi-coherent sheaves.

01BH **Lemma 17.10.5.** *Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:*

(1) *Let $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ be the morphism of ringed spaces with $\pi : X \to \{*\}$ the unique map and with π-map $\pi^\sharp$ the given map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$. Set $\mathcal{F}_1 = \pi^*M$.*
(2) *Choose a presentation $\bigoplus_{j \in J} R \to \bigoplus_{i \in I} R \to M \to 0$. Set*

$$\mathcal{F}_2 = \operatorname{Coker}\left(\bigoplus\nolimits_{j \in J} \mathcal{O}_X \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X\right).$$

Here the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ given by the section $\sum_i \alpha(r_{ij})$ where the r_{ij} are the matrix coefficients of the map in the presentation of M.

(3) *Set $\mathcal{F}_3$ equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.*

This construction has the following properties:

(1) *The resulting sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ is quasi-coherent.*
(2) *The construction gives a functor from the category of R-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.*
(3) *For any $x \in X$ we have $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ functorial in M.*
(4) *Given any $\mathcal{O}_X$-module $\mathcal{G}$ we have*

$$\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$$

where the R-module structure on $\Gamma(X, \mathcal{G})$ comes from the $\Gamma(X, \mathcal{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (3), see Sheaves, Section 6.26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor π^* is right exact, so $\pi^*(\bigoplus_{j\in J} R) \to \pi^*(\bigoplus_{i\in I} R) \to \pi^*M \to 0$ is exact, π^* commutes with arbitrary direct sums, see Lemma 17.3.3, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since π^* has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 6.26.4. Assertion (4) follows from adjointness of π_* and π^*. □

01BI **Definition 17.10.6.** In the situation of Lemma 17.10.5 we say $\mathcal{F}_M$ is the *sheaf associated to the module M and the ring map α*. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say $\mathcal{F}_M$ is the *sheaf associated to the module M*.

01BJ **Lemma 17.10.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. Let $\mathcal{F}_M$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to M. If $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, then $g^*\mathcal{F}_M$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.*

Proof. The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 17.10.5 as π^*M, and the following commutative diagram of ringed spaces

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{\pi} & (\{*\}, \Gamma(Y, \mathcal{O}_Y)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \text{induced by } g^\sharp} \\ (X, \mathcal{O}_X) & \xrightarrow{\pi} & (\{*\}, \Gamma(X, \mathcal{O}_X)) \end{array}$$

(Also use Sheaves, Lemma 6.26.3.) □

01BK **Lemma 17.10.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that x has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood U of x such*

that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module M.

Proof. First we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a "matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of x (note: E may not be open). Let $x \in U \subset E$ be an open neighbourhood of x contained in E. Next, we proceed as in the proof of Lemma 17.3.5. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to j. There exists a finite collection of opens U_{jk}, $k \in K_j$ such that $E \subset \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and f_{jki} in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k \in K_j} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathcal{O}_X$. By Lemma 17.3.2 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module M as the cokernel of the map

$$\bigoplus\nolimits_{j \in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i \in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the (f_{ij}). By construction (2) of Lemma 17.10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. □

01BL **Example 17.10.9.** Let X be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n \in \mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2) \cup (2, \infty)$. Denote f_n the continuous function on X which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let 1_{L_j} be the characteristic function of L_j. We consider the map

$$\bigoplus\nolimits_{j \in \mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i \in \mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{i \in \mathbf{N}} f_j 1_{L_i} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite as f_j is zero in a neighbourhood of 0. Over U_n the image of e_j, for $j > 2n$ is not a finite linear combination $\sum g_{ij} e_{ij}$ with g_{ij} continuous. Thus there is no neighbourhood of $0 \in X$ such that the displayed map is given by a "matrix" as in the proof of Lemma 17.10.8 above.

Note that $\bigoplus_{j \in \mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis e_j and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on X does not come from a morphism of modules. Similarly there should be an example of a ringed space X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces X

and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 17.10.8 shows this cannot happen if N is free).

17.11. Modules of finite presentation

01BM Here is the definition.

01BN **Definition 17.11.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite presentation* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

01BO **Lemma 17.11.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.*

Proof. Immediate from definitions. □

0H96 **Lemma 17.11.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{O}_X$-modules. If $\mathcal{G}$ is finite type, then* $\mathrm{Coker}(\varphi)$ *is finitely presented.*

Proof. Locally on X, we can write $\mathcal{F} = \mathcal{O}_X^{\oplus n}/\mathcal{M}$, where $\mathcal{M} \subset \mathcal{O}_X^{\oplus n}$ is a finite type $\mathcal{O}_X$-submodule. Therefore $\mathrm{Im}(\varphi) = \mathcal{N}/\mathcal{M}$, where $\mathcal{N} \subset \mathcal{O}_X^{\oplus n}$ is a $\mathcal{O}_X$-submodule containing $\mathcal{M}$. The $\mathcal{O}_X$-module $\mathrm{Im}(\varphi)$ is of finite type because $\mathcal{G} \to \mathrm{Im}(\varphi)$ is surjective and $\mathcal{G}$ is finite type. By Lemma 17.9.3, $\mathcal{N}$ is finite type. Thus $\mathrm{Coker}(\varphi) = \mathcal{O}_X^{\oplus n}/\mathcal{N}$ is finitely presented. □

01BP **Lemma 17.11.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation.*

(1) *If $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ is a surjection, then* $\mathrm{Ker}(\psi)$ *is of finite type.*
(2) *If $\theta : \mathcal{G} \to \mathcal{F}$ is surjective with $\mathcal{G}$ of finite type, then* $\mathrm{Ker}(\theta)$ *is of finite type.*

Proof. Proof of (1). Let $x \in X$. Choose an open neighbourhood $U \subset X$ of x such that there exists a presentation

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$

Let e_k be the section generating the kth factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking U to a small neighbourhood of x, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over U. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$.

Similarly, after shrinking U, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map

$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta\circ\chi, 1-\beta\circ\alpha} \mathcal{O}_U^{\oplus r}$$

is a surjection onto the kernel of ψ.

To prove (2) we may locally choose a surjection $\eta : \mathcal{O}_X^{\oplus r} \to \mathcal{G}$. By part (1) we see $\mathrm{Ker}(\theta \circ \eta)$ is of finite type. Since $\mathrm{Ker}(\theta) = \eta(\mathrm{Ker}(\theta \circ \eta))$ we win. □

01BQ **Lemma 17.11.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.*

Proof. Exactly the same as the proof of Lemma 17.10.4 but with finite index sets. □

01BR **Lemma 17.11.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to M is a directed colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. This follows immediately from Lemma 17.10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 10.11.3. □

0B8J **Lemma 17.11.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$.*

Proof. Choose $s_1, \ldots, s_r \in \mathcal{F}_x$ mapping to a basis of $\mathcal{O}_{X,x}^{\oplus r}$ by the isomorphism. Choose an open neighbourhood U of x such that s_i lifts to $s_i \in \mathcal{F}(U)$. After shrinking U we see that the induced map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective (Lemma 17.9.4). By Lemma 17.11.4 we see that $\mathrm{Ker}(\psi)$ is of finite type. Then $\mathrm{Ker}(\psi)_x = 0$ implies that $\mathrm{Ker}(\psi)$ becomes zero after shrinking U once more (Lemma 17.9.5). □

17.12. Coherent modules

01BU A reference for this section is [**Ser55b**].

The category of coherent sheaves on a ringed space X is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $Mod(\mathcal{O}_X)$ no matter what X is. On the other hand, the pullback of a coherent module is "almost never" coherent in the general setting of ringed spaces.

01BV **Definition 17.12.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *coherent $\mathcal{O}_X$-module* if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and
(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $Coh(\mathcal{O}_X)$.

01BW **Lemma 17.12.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood U and sections s_i, $i = 1, \ldots, n$ of $\mathcal{F}$ over U such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood V, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over V we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. □

01BX **Example 17.12.3.** Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 10.90 for more discussion.

01BY **Lemma 17.12.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any finite type subsheaf of a coherent sheaf is coherent.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then $\mathrm{Ker}(\varphi)$ is of finite type.*
(3) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent.*
(4) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.*
(5) *The category $Coh(\mathcal{O}_X)$ is a weak Serre subcategory of $Mod(\mathcal{O}_X)$. In particular, the category of coherent modules is abelian and the inclusion functor $Coh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. Condition (2) of Definition 17.12.1 holds for any subsheaf of a coherent sheaf. Thus we get (1).

Assume the hypotheses of (2). Let us show that $\mathrm{Ker}(\varphi)$ is of finite type. Pick $x \in X$. Choose an open neighbourhood U of x in X such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 17.12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since $\mathcal{G}$ is of finite type so is $\mathrm{Coker}(\varphi)$. Let $U \subset X$ be open and let $\overline{s}_i \in \mathrm{Coker}(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathrm{Coker}(\varphi)$ is of finite type. There exists an open covering of U such that on each open all the sections $\overline{s}_i$ lift to sections s_i of $\mathcal{G}$. Hence we may assume this is the case over U. We may in addition assume there are sections t_j, $j = 1, \ldots, m$ of $\mathrm{Im}(\varphi)$ over U which generate $\mathrm{Im}(\varphi)$ over U. Let $\Phi : \bigoplus_{j=1}^m \mathcal{O}_U \to \mathrm{Im}(\varphi)$ be defined using t_j and $\Psi : \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ using t_j and s_i. Consider the following

commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{j=1}^m \mathcal{O}_U & \longrightarrow & \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \overline{\Psi}} & & \\ 0 & \longrightarrow & \mathrm{Im}(\varphi) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Coker}(\varphi) & \longrightarrow & 0 \end{array}$$

By the snake lemma we get an exact sequence $\mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to 0$. Since $\mathrm{Ker}(\Psi)$ is a finite type module, we see that $\mathrm{Ker}(\overline{\Psi})$ has finite type.

Proof of part (4). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. By part (3) it suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 17.9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections s_i in $\mathcal{F}_3$. Since $\mathcal{F}_1$ is coherent we see that $\mathcal{K}$ is the kernel of a map from a finite type module to a coherent module and hence finite type by (2).

Proof of (5). This follows because (3) and (4) show that Homology, Lemma 12.10.3 applies. □

01BZ **Lemma 17.12.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.*

Proof. Omitted. □

01C0 **Lemma 17.12.6.** *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.*

Proof. Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of φ. By Lemma 17.12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 17.9.5 there exists an open neighbourhood U of x such that $\mathcal{K}|_U = 0$. Then U works. □

17.13. Closed immersions of ringed spaces

01C1 When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion?

Motivated by the example of a closed immersion of normal topological spaces (ringed with the sheaf of continuous functors), or differential manifolds (ringed with the sheaf of differentiable functions), it seems natural to assume at least:

(1) The map i is a closed immersion of topological spaces.

(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.

Already these conditions imply a number of pleasing results: For example we prove that the category of $\mathcal{O}_Z$-modules is equivalent to the category of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ generalizing the result on abelian sheaves of Section 17.6.

However, in the Stacks project we choose the definition that guarantees that if i is a closed immersion and $(X, \mathcal{O}_X)$ is a scheme, then also $(Z, \mathcal{O}_Z)$ is a scheme. Moreover, in this situation we want i_* and i^* to provide an equivalence between the category of quasi-coherent $\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A good way to guarantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. We can interpret this condition as saying "$(Z, \mathcal{O}_Z)$ is locally on $(X, \mathcal{O}_X)$ defined by setting some regular functions f_i, i.e., local sections of $\mathcal{O}_X$, equal to zero". This leads to the following definition.

01C2 **Definition 17.13.1.** A *closed immersion of ringed spaces*[1] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Actually, this definition still does not guarantee that i_* of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

01C3 **Lemma 17.13.2.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on X the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on Z the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the *the same* quasi-coherent sheaf is quasi-coherent. And finally, i_* commutes with arbitrary colimits, see Lemma 17.6.3. Some details omitted. □

01C4 **Lemma 17.13.3.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \notin Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of x and hence finitely generated in a neighbourhood of x. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over V. Write $V = Z \cap U$ for some open $U \subset X$. Note that U is a neighbourhood of x. Clearly the sections s_i give sections s_i of $i_*\mathcal{F}$ over U. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

[1] This is nonstandard notation; see discussion above.

is surjective by inspection of what it does on stalks (here we use that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of x, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over U. Set $V = Z \cap U$. By definition of i_* the sections s_i correspond to sections s_i of $\mathcal{F}$ over V. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks. Hence $\mathcal{F}$ is of finite type. □

08KS **Lemma 17.13.4.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}_X$ the kernel of $i^\sharp$. The functor*

$$i_* : \mathit{Mod}(\mathcal{O}_Z) \longrightarrow \mathit{Mod}(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. We claim that for an $\mathcal{O}_Z$-module $\mathcal{F}$ the canonical map

$$i^*i_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. We check this on stalks. Say $z \in Z$ and $x = i(z)$. We have

$$(i^*i_*\mathcal{F})_z = (i_*\mathcal{F})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z$$

by Sheaves, Lemma 6.26.4, the fact that $\mathcal{O}_{Z,z}$ is a quotient of $\mathcal{O}_{X,x}$, and Sheaves, Lemma 6.32.1. It follows that i_* is fully faithful.

Let $\mathcal{G}$ be a $\mathcal{O}_X$-module with $\mathcal{I}\mathcal{G} = 0$. We will prove the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism. This proves that $\mathcal{G} = i_*\mathcal{F}$ with $\mathcal{F} = i^*\mathcal{G}$ which finishes the proof. We check the displayed map induces an isomorphism on stalks. If $x \in X$, $x \notin i(Z)$, then $\mathcal{G}_x = 0$ because $\mathcal{I}_x = \mathcal{O}_{X,x}$ in this case. As above $(i_*i^*\mathcal{G})_x = 0$ by Sheaves, Lemma 6.32.1. On the other hand, if $x \in Z$, then we obtain the map

$$\mathcal{G}_x \longrightarrow \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,x}$$

by Sheaves, Lemmas 6.26.4 and 6.32.1. This map is an isomorphism because $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/\mathcal{I}_x$ and because $\mathcal{G}_x$ is annihilated by $\mathcal{I}_x$ by assumption. □

0G6N **Remark 17.13.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the *submodule of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\}$$

Observe that $\mathcal{H}_Z(\mathcal{F})(U)$ is a module over $\mathcal{O}_X(U)$, i.e., $\mathcal{H}_Z(\mathcal{F})$ is an $\mathcal{O}_X$-module. By construction $\mathcal{H}_Z(\mathcal{F})$ is the largest $\mathcal{O}_X$-submodule of $\mathcal{F}$ whose support is contained in Z. Applying Lemma 17.13.4 to the morphism of ringed spaces $(Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$ we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on Z. Thus we obtain a functor

$$\mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{Mod}(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. All of the statements made above follow directly from Lemma 17.5.2. Clearly the construction is compatible with the construction in Remark 17.6.2.

0G6P **Lemma 17.13.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor $\mathcal{H}_Z : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X|_Z)$ of Remark 17.13.5 is right adjoint to $i_* : Mod(\mathcal{O}_X|_Z) \to Mod(\mathcal{O}_X)$.*

Proof. We have to show that for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_X|_Z$-module $\mathcal{G}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X|_Z}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F})$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. □

17.14. Locally free sheaves

01C5 Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little careful when defining the rank of a locally free sheaf.

01C6 **Definition 17.14.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say $\mathcal{F}$ is *locally free* if for every point $x \in X$ there exist a set I and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i\in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if we may choose the index sets I to be finite.
(3) We say $\mathcal{F}$ is *finite locally free of rank r* if we may choose the index sets I to have cardinality r.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

01C7 **Lemma 17.14.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.*

Proof. Omitted. □

01C8 **Lemma 17.14.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

Proof. Omitted. □

01C9 **Lemma 17.14.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is X, i.e., all stalks of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function*

$$rank_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any point $x \in X$ the cardinality of any set I such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i\in I} \mathcal{O}_X$ in a neighbourhood of x is $rank_{\mathcal{F}}(x)$.

Proof. Under the assumption of the lemma the cardinality of I can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of x. □

089Q **Lemma 17.14.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $r \geq 0$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of finite locally free $\mathcal{O}_X$-modules of rank r. Then φ is an isomorphism if and only if φ is surjective.*

Proof. Assume φ is surjective. Pick $x \in X$. There exists an open neighbourhood U of x such that both $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic to $\mathcal{O}_U^{\oplus r}$. Pick lifts of the free generators of $\mathcal{G}|_U$ to obtain a map $\psi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi|_U \circ \psi = \mathrm{id}$. Hence we conclude that the map $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ induced by φ is surjective. Since both $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are isomorphic to $\Gamma(U, \mathcal{O}_U)^{\oplus r}$ as an $\Gamma(U, \mathcal{O}_U)$-module we may apply Algebra, Lemma 10.16.4 to see that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is injective. This finishes the proof. □

0BCI **Lemma 17.14.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If all stalks $\mathcal{O}_{X,x}$ are local rings, then any direct summand of a finite locally free $\mathcal{O}_X$-module is finite locally free.*

Proof. Assume $\mathcal{F}$ is a direct summand of the finite locally free $\mathcal{O}_X$-module $\mathcal{H}$. Let $x \in X$ be a point. Then $\mathcal{H}_x$ is a finite free $\mathcal{O}_{X,x}$-module. Because $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$ for some r, see Algebra, Lemma 10.78.2. By Lemma 17.11.7 we see that $\mathcal{F}$ is free of rank r in an open neighbourhood of x. (Note that $\mathcal{F}$ is of finite presentation as a summand of $\mathcal{H}$.) □

17.15. Bilinear maps

0GIG Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be $\mathcal{O}_X$-modules. A *bilinear map* $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ of sheaves of $\mathcal{O}_X$-modules is a map of sheaves of sets as indicated such that for every open $U \subset X$ the induced map

$$\mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{H}(U)$$

is an $\mathcal{O}_X(U)$-bilinear map of modules. Equivalently you can ask certain diagrams of maps of sheaves of sets commute, imitating the usual axioms for bilinear maps of modules. For example, the axiom $f(x + y, z) = f(x, z) + f(y, z)$ is represented by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} \times \mathcal{G} & \xrightarrow[(f \circ \mathrm{pr}_{13}, f \circ \mathrm{pr}_{23})]{} & \mathcal{H} \times \mathcal{H} \\ {\scriptstyle (+ \circ \mathrm{pr}_{12}, \mathrm{pr}_3)} \downarrow & & \downarrow {\scriptstyle +} \\ \mathcal{F} \times \mathcal{G} & \xrightarrow{f} & \mathcal{H} \end{array}$$

Another characterization is this: if $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is a map of sheaves of sets and it induces a bilinar map of modules on stalks for all points of X, then f is a bilinear map of sheaves of modules. This is true as you can test whether local sections are equal by checking on stalks.

Let $\mathrm{Mor}(-, -)$ denote morphisms in the category of sheaves of sets on X. Another characterization of a bilinear map is this: a map of sheaves of sets $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is bilinear if given any sheaf of sets $\mathcal{S}$ the rule

$$\mathrm{Mor}(\mathcal{S}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{S}, \mathcal{G}) \to \mathrm{Mor}(\mathcal{S}, \mathcal{H}), \quad (a, b) \mapsto f \circ (a \times b)$$

is a bilinear map of modules over the ring $\mathrm{Mor}(\mathcal{S}, \mathcal{O}_X)$. We don't usually take this point of view as it is easier to think about sets of local sections and it is clearly equivalent.

Finally, here is yet another way to say the definition: $\mathcal{O}_X$ is a ring object in the category of sheaves of sets and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are module objects over this ring. Then a

bilinear map can be defined for module objects over a ring object in any category. To formulate what is a ring object and what is a module object over a ring object, and what is a bilinear map of such in a category it is pleasant (but not strictly necessary) to assume the category has finite products; and this is true for the category of sheaves of sets.

17.16. Tensor product

01CA We have already briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6.6 and 6.20. Let us generalize this to tensor products of modules.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. We define first the *tensor product presheaf*

$$\mathcal{F} \otimes_{p,\mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the *tensor product sheaf* as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}_X} \mathcal{G})^\#$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules as defined in Section 17.15.

The tensor product of modules M, N over a ring R satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}$, $\mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

01CB **Lemma 17.16.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Omitted. □

05NA **Lemma 17.16.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p,\mathcal{O}_X} \mathcal{G}')^\#$.*

Proof. Omitted. □

01CC **Lemma 17.16.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence*

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks (Lemma 17.3.1), the description of stalks (Lemma 17.16.1) and the corresponding result for tensor products of modules (Algebra, Lemma 10.12.10). □

01CD **Lemma 17.16.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. Omitted. □

05NB **Lemma 17.16.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. Let I be a preordered set and let $\{\mathcal{G}_i\}$ be a system over I. Set $\mathcal{G} = \operatorname{colim}_i \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto \operatorname{colim}_i \mathcal{G}_i(U)$, see Sheaves, Section 6.29. By Lemma 17.16.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \operatorname{colim}_i \mathcal{G}_i(U) = \operatorname{colim}_i \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 10.12.9. Hence the lemma follows from the description of colimits in $Mod(\mathcal{O}_X)$, see Lemma 17.3.2. □

01CE **Lemma 17.16.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(5) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(7) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i\in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j\in J} \mathcal{O}_X) \cong \bigoplus_{(i,j)\in I\times J} \mathcal{O}_X$. The sheaf $\bigoplus_{i\in I} \mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i\in I} \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j\in J} \mathcal{O}_X(U)).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) = \bigoplus_{(i,j)\in I\times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 17.16.3 the complex $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$ (for example by Lemma 17.16.5). Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 17.12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. □

17.17. Flat modules

05NC We can define flat modules exactly as in the case of modules over rings.

05ND **Definition 17.17.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

05NE **Lemma 17.17.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 17.16.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where M is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = (i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F})_x$$

again by Lemma 17.16.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. □

Thus the following definition makes sense.

05NF **Definition 17.17.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at x* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

0H97 **Lemma 17.17.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a flat $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a flat $\mathcal{O}_X$-module.*

Proof. Combine Lemma 17.17.2 with Sheaves, Lemma 6.26.4 and Algebra, Lemma 10.39.7. □

05NG **Lemma 17.17.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.*

Proof. This follows from Lemma 17.16.5, Lemma 17.16.1, Algebra, Lemma 10.8.8, and the fact that we can check exactness at stalks. □

05NH **Lemma 17.17.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.*

Proof. The stalks of $j_{U!}\mathcal{O}_U$ are either zero or equal to $\mathcal{O}_{X,x}$. Apply Lemma 17.17.2. □

05NI **Lemma 17.17.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(2) *Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 17.17.5 and 17.17.6. □

05NJ **Lemma 17.17.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 10.39.12. □

05NK **Lemma 17.17.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

Proof. Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 10.39.13. □

05NL **Lemma 17.17.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact also.

Proof. Follows from Lemma 17.17.8 by splitting the complex into short exact sequences and using Lemma 17.17.9 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 10.39.11).

08BK **Lemma 17.17.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Let $U \subset X$ be open and let*

$$\mathcal{O}_U \xrightarrow{(f_1,\ldots,f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. For every $x \in U$ there exists an open neighbourhood $V \subset U$ of x and a factorization

$$\mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus m} \xrightarrow{(t_1,\ldots,t_m)} \mathcal{F}|_V$$

of $(s_1, \ldots, s_n)|_V$ such that $A \circ (f_1, \ldots, f_n)|_V = 0$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_i \otimes s_i$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists an open neighbourhood $V \subset U$ of x such that $\sum f_i|_V \otimes s_i|_V$ is zero in $\mathcal{I}(V) \otimes_{\mathcal{O}(V)} \mathcal{F}(V)$. Unwinding the definitions using Algebra, Lemma 10.107.10 we find $t_1, \ldots, t_m \in \mathcal{F}(V)$ and $a_{ij} \in \mathcal{O}(V)$ such that $\sum a_{ij} f_i|_V = 0$ and $s_i|_V = \sum a_{ij} t_j$. □

17.18. Duals

0FNU Let $(X, \mathcal{O}_X)$ be a ringed space. The category of $\mathcal{O}_X$-modules endowed with the tensor product constructed in Section 17.16 is a symmetric monoidal category. For an $\mathcal{O}_X$-module $\mathcal{F}$ the following are equivalent

(1) $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}_X$-modules,
(2) $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module, and
(3) $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}_X$-module.

This is proved in Example 17.18.1 and Lemmas 17.18.2 and 17.18.3 of this section.

0FNV **Example 17.18.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is locally a direct summand of a finite free $\mathcal{O}_X$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true. Denote

$$\eta : \mathcal{O}_X \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

the map sending 1 to the section corresponding to $\mathrm{id}_{\mathcal{F}}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{O}_X$$

the evaluation map. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)}$.

0FNW **Lemma 17.18.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}_X$-modules, see Categories, Definition 4.43.5. Then*

(1) *$\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module,*
(2) *the map $e : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,*
(3) *we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.*

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let $x \in X$. We can find an open neighbourhood U of x, a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde{\eta} : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}|_U & \xrightarrow{\eta\otimes 1} & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U & \xrightarrow{1\otimes\epsilon} & \mathcal{F}|_U \\ & \searrow^{\tilde{\eta}\otimes 1} & \uparrow & & \uparrow \\ & & \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U & \xrightarrow{1\otimes\epsilon} & \mathcal{O}_U^{\oplus n} \end{array}$$

This shows that the identity on $\mathcal{F}$ locally on X factors through a finite free module. This proves (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 4.43.7) and the construction of the left dual in Example 17.18.1. □

08BL **Lemma 17.18.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module of finite presentation. Then $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module.*

Proof. After replacing X by the members of an open covering, we may assume there exists a presentation

$$\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Let $x \in X$. By Lemma 17.17.11 we can, after shrinking X to an open neighbourhood of x, assume there exists a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_X^{\oplus r}$. Repeating this argument $r - 1$ more times we obtain a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$ has a section and we win. □

17.19. Constructible sheaves of sets

0CAG Let X be a topological space. Given a set S recall that $\underline{S}$ or $\underline{S}_X$ denotes the constant sheaf with value S, see Sheaves, Definition 6.7.4. Let $U \subset X$ be an open of a topological space X. We will denote j_U the inclusion morphism and we will denote $j_{U!} : Sh(U) \to Sh(X)$ the extension by the empty set described in Sheaves, Section 6.31.

0CAH **Lemma 17.19.1.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on X. There exists a set I and for each $i \in I$ an element $U_i \in \mathcal{B}$ and a finite set S_i such that there exists a surjection $\coprod_{i\in I} j_{U_i!}\underline{S_i} \to \mathcal{F}$.*

Proof. Let S be a singleton set. We will prove the result with $S_i = S$. For every $x \in X$ and element $s \in \mathcal{F}_x$ we can choose a $U(x, s) \in \mathcal{B}$ and $s(x, s) \in \mathcal{F}(U(x, s))$

which maps to s in $\mathcal{F}_x$. By Sheaves, Lemma 6.31.4 the section $s(x, s)$ corresponds to a map of sheaves $j_{U(x,s)!}\underline{S} \to \mathcal{F}$. Then

$$\coprod_{(x,s)} j_{U(x,s)!}\underline{S} \to \mathcal{F}$$

is surjective on stalks and hence surjective. □

0CAI **Lemma 17.19.2.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X and assume that each $U \in \mathcal{B}$ is quasi-compact. Then every sheaf of sets on X is a filtered colimit of sheaves of the form*

0CAJ (17.19.2.1) $$\text{Coequalizer}\Big(\coprod\nolimits_{b=1,\ldots,m} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a=1,\ldots,n} j_{U_a!}\underline{S_a}\Big)$$

with U_a and V_b in $\mathcal{B}$ and S_a and S_b finite sets.

Proof. By Lemma 17.19.1 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source $\mathcal{F}_0$ is a coproduct of sheaves the form $j_{U!}\underline{S}$ with $U \in \mathcal{B}$ and S finite. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{b\in B} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}$$

for some index sets A, B and V_b and U_a in $\mathcal{B}$ and S_a and S_b finite. For every finite subset $B' \subset B$ there is a finite subset $A' \subset A$ such that the coproduct over $b \in B'$ maps into the coproduct over $a \in A'$ via both maps. Namely, we can view the right hand side as a filtered colimit with injective transition maps. Hence taking sections over the quasi-compact opens V_b, $b \in B'$ commutes with this coproduct, see Sheaves, Lemma 6.29.1. Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts. □

0CAK **Lemma 17.19.3.** *Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (17.19.2.1). Then there exists a continuous spectral map $f : X \to Y$ to a finite sober topological space Y and a sheaf of sets $\mathcal{G}$ on Y with finite stalks such that $f^{-1}\mathcal{G} \cong \mathcal{F}$.*

Proof. We can write $X = \lim X_i$ as a directed limit of finite sober spaces, see Topology, Lemma 5.23.14. Of course the transition maps $X_{i'} \to X_i$ are spectral and hence by Topology, Lemma 5.24.5 the maps $p_i : X \to X_i$ are spectral. For some i we can find opens $U_{a,i}$ and $V_{b,i}$ of X_i whose inverse images are U_a and V_b, see Topology, Lemma 5.24.6. The two maps

$$\beta, \gamma : \coprod\nolimits_{b\in B} j_{V_b!}\underline{S_b} \longrightarrow \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}$$

whose coequalizer is $\mathcal{F}$ correspond by adjunction to two families

$$\beta_b, \gamma_b : S_b \longrightarrow \Gamma(V_b, \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}), \quad b \in B$$

of maps of sets. Observe that $p_i^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_a!}\underline{S_a}$ and $(X_{i'} \to X_i)^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_{a,i'}!}\underline{S_a}$. It follows from Sheaves, Lemma 6.29.3 (and using that S_b and B are finite sets) that after increasing i we find maps

$$\beta_{b,i}, \gamma_{b,i} : S_b \longrightarrow \Gamma(V_{b,i}, \coprod\nolimits_{a\in A} j_{U_{a,i}!}\underline{S_a}), \quad b \in B$$

which give rise to the maps β_b and γ_b after pulling back by p_i. These maps correspond in turn to maps of sheaves

$$\beta_i, \gamma_i : \coprod\nolimits_{b\in B} j_{V_{b,i}!}\underline{S_b} \longrightarrow \coprod\nolimits_{a\in A} j_{U_{a,i}!}\underline{S_a}$$

on X_i. Then we can take $Y = X_i$ and

$$\mathcal{G} = \text{Coequalizer}\left(\coprod_{b=1,\ldots,m} j_{V_{b,i}!}\underline{S_b} \longrightarrow \coprod_{a=1,\ldots,n} j_{U_{a,i}!}\underline{S_a} \right)$$

We omit some details. □

0CAL **Lemma 17.19.4.** *Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (17.19.2.1). Then there exist finitely many constructible closed subsets $Z_1, \ldots, Z_n \subset X$ and finite sets S_i such that $\mathcal{F}$ is isomorphic to a subsheaf of $\prod (Z_i \to X)_* \underline{S_i}$.*

Proof. By Lemma 17.19.3 we reduce to the case of a finite sober topological space and a sheaf with finite stalks. In this case $\mathcal{F} \subset \prod_{x \in X} i_{x,*}\mathcal{F}_x$ where $i_x : \overline{\{x\}} \to X$ is the embedding. We omit the proof that $i_{x,*}\mathcal{F}_x$ is a constant sheaf on $\overline{\{x\}}$. □

17.20. Flat morphisms of ringed spaces

02N2 The pointwise definition is motivated by Lemma 17.17.2 and Definition 17.17.3 above.

02N3 **Definition 17.20.1.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say f is *flat at x* if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say f is *flat* if f is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at x is the ring map $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence f is flat at x if and only if $\mathcal{O}_X$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module. And f is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

02N4 **Lemma 17.20.2.** *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. The functor f^* is the composition of the exact functor $f^{-1} : Mod(\mathcal{O}_Y) \to Mod(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$Mod(f^{-1}\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 17.20.1. □

08KT **Definition 17.20.3.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say that $\mathcal{F}$ is *flat over Y at a point $x \in X$* if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
(2) We say that $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at every point x of X.

With this definition we see that $\mathcal{F}$ is flat over Y at x if and only if $\mathcal{F}$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module because $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ by Sheaves, Lemma 6.21.5.

0GMU **Lemma 17.20.4.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module flat over Y. Then the functor*

$$Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. This is true because $f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{F}$, the functor f^{-1} is exact, and $\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_Y$-module. □

17.21. Symmetric and exterior powers

01CF Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *tensor algebra* of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \quad (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where s is a local section of $\mathcal{F}$. This is called the *exterior algebra* of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

01CG **Lemma 17.21.1.** *In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

See Algebra, Section 10.13. Similarly, the sheaf $Sym^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto Sym^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

Proof. Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. □

01CH **Lemma 17.21.2.** *In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $T(\mathcal{F})_x = T(\mathcal{F}_x)$, $Sym(\mathcal{F})_x = Sym(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.*

Proof. Clear from Lemma 17.21.1 above, and Algebra, Lemma 10.13.5. □

01CI **Lemma 17.21.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*T(\mathcal{F}) = T(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.*

Proof. Omitted. □

01CJ **Lemma 17.21.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} Sym^{n-1}(\mathcal{F}_1) \to Sym^n(\mathcal{F}_1) \to Sym^n(\mathcal{F}) \to 0$$

and similarly an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

Proof. See Algebra, Lemma 10.13.2. □

01CK **Lemma 17.21.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}$ is locally generated by sections, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
2. *If $\mathcal{F}$ is of finite type, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
3. *If $\mathcal{F}$ is of finite presentation, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

(4) *If $\mathcal{F}$ is coherent, then for $n > 0$ each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$ is coherent.*
(5) *If $\mathcal{F}$ is quasi-coherent, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
(6) *If $\mathcal{F}$ is locally free, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

Proof. These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 17.16.6.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 10.13.1.

For (3) and (5) we will use Lemma 17.21.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 17.16.6.

To prove (4) we will use Algebra, Lemma 10.13.3. We may localize on X and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i\in I}$ of global sections. The lemma mentioned above combined with Lemma 17.21.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set J is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 17.12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. □

01CL **Lemma 17.21.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*
(1) *If $\mathcal{F}$ is quasi-coherent, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is locally free, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*

Proof. It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods U_i of x on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of x. However, in the proof of Lemma 17.21.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. □

17.22. Internal Hom

01CM Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 6.33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

01CN **Lemma 17.22.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.

Proof. This is the analogue of Algebra, Lemma 10.12.8. The proof is the same, and is omitted. □

01CO **Lemma 17.22.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$

is exact.

(2) *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$

is exact.

Proof. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be as in (1). For every $U \subset X$ open the sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_1|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_2|_U, \mathcal{G}|_U)$$

is exact by Homology, Lemma 12.5.8. This means that taking sections over U of the sequence of sheaves in (1) produces an exact sequence of abelian groups. Hence the sequence in (1) is exact by definition. The proof of (2) is exactly the same. □

0A6F **Lemma 17.22.3.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}_{\mathcal{O}_1}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G}))$$

bifunctorially in $\mathcal{F} \in Mod(\mathcal{O}_2)$ and $\mathcal{G} \in Mod(\mathcal{O}_1)$.

Proof. Omitted. This is the analogue of Algebra, Lemma 10.14.4 and is proved in exactly the same way. □

01CP **Lemma 17.22.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then the canonical map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is injective. If $\mathcal{F}$ is finitely presented, this canonical morphism is an isomorphism.

Proof. The map sends the equivalence class of (U, φ) in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$, where $x \in U \subset X$ is open and $\varphi \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, to the induced map on stalks at x, namely $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Suppose $\mathcal{F}$ is of finite type. Pick a representative (U, φ) of an element σ in the kernel of the map, i.e., $\varphi_x = 0$. Shrinking U if necessary, choose sections $s^1, \ldots, s^n \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since $\varphi_x(s^i_x) = 0$ and we are dealing with a finite number of

sections, we can find an open neighborhood $V \subset U$ of x such that $\varphi_V(s^i|_V) = 0$ for all $i = 1, \ldots, n$. Since $s^i|_V$, $i = 1, \ldots, n$ generate $\mathcal{F}|_V$ this means that $\varphi|_V = 0$. Since (U, φ) is equivalent to $(V, \varphi|_V)$ we conclude $\sigma = 0$ and injectivity of the map follows.

Next, assume $\mathcal{F}$ is finitely presented. By localizing on X we may assume that $\mathcal{F}$ has a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$

By Lemma 17.22.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. □

0C6I **Lemma 17.22.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map*

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

is an isomorphism.

Proof. Note that $f^*\mathcal{F}$ is also finitely presented (Lemma 17.11.5). Let $x \in X$ map to $y \in Y$. Looking at the stalks at x we get an isomorphism by Lemma 17.22.4 and More on Algebra, Lemma 15.66.4 to see that in this case Hom commutes with base change by $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Second proof: use the exact same argument as given in the proof of Lemma 17.22.4. □

01CQ **Lemma 17.22.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent too.*

Proof. The first assertion we saw in the proof of Lemma 17.22.4. And the result for coherent sheaves then follows from Lemma 17.12.4. □

0GMV **Lemma 17.22.7.** *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}_X$-modules. Then the canonical map*

$$\mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. Taking colimits of sheaves of modules commutes with restriction to opens, see Sheaves, Section 6.29. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_{\mathcal{O}_X}(-, -)$ commutes with finite direct sums in either variable and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -)$ is the identity functor. By this and by Lemma 17.22.2 we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $Mod(\mathcal{O}_X)$ also the top row in the following commutative diagram is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) & \longrightarrow & \operatorname{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda & \longrightarrow & \operatorname{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \bigoplus_{i=1,\ldots,n} \mathcal{G} & \longrightarrow & \bigoplus_{j=1,\ldots,m} \mathcal{G} \end{array}$$

Since the right two vertical arrows are isomorphisms we conclude. □

01BS **Lemma 17.22.8.** *Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Assume*

(1) *I is directed,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and*
(3) *X has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$.*

Then we have

$$\operatorname{colim}_i \operatorname{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_X(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \operatorname{colim} \mathcal{F}_i)$ and $\mathcal{H}_i = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}_i)$. Recall that

$$\operatorname{Hom}_X(\mathcal{G}, \mathcal{F}) = \Gamma(X, \mathcal{H}) \quad \text{and} \quad \operatorname{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \Gamma(X, \mathcal{H}_i)$$

by construction. By Lemma 17.22.7 we have $\mathcal{H} = \operatorname{colim} \mathcal{H}_i$. Thus the lemma follows from Sheaves, Lemma 6.29.1. □

01BT **Remark 17.22.9.** In the lemma above some condition beyond the condition that X is quasi-compact is necessary. See Sheaves, Example 6.29.2.

17.23. The annihilator of a sheaf of modules

0H2G Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a canonical map of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

which sends a local section $f \in \mathcal{O}_X(U)$ to the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ given by multiplication by f.

0H2H **Definition 17.23.1.** Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The *annihilator* of $\mathcal{F}$, denoted $\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of the map $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ discussed above.

For each $x \in X$, there is an inclusion of ideals of $\mathcal{O}_{X,x}$:

0H2I (17.23.1.1) $$(\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x \subset \operatorname{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

since after all any section of $\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F})$ will annihilate the stalks of $\mathcal{F}$ at all points at which it is defined. Here is a simple situation in which (17.23.1.1) becomes an equality.

0H2J **Lemma 17.23.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type, then $(Ann_{\mathcal{O}_X}(\mathcal{F}))_x = Ann_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$.*

Proof. By Lemma 17.22.4 the map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{F}_x)$$

is injective. Thus any section f of $\mathcal{O}_X$ over an open neighbourhood U of x which acts as zero on $\mathcal{F}_x$ will act as zero on $\mathcal{F}|_V$ for some $U \supset V \ni x$ open. Hence the inclusion (17.23.1.1) is an equality. □

0H2K **Lemma 17.23.3.** *Let $(X,\mathcal{O}_X)$ be a ringed space, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. If $\mathcal{I} \subset Ann_{\mathcal{O}_X}(\mathcal{F})$, then $\mathcal{F}$ has a natural $\mathcal{O}_X/\mathcal{I}$-module structure which agrees with the usual commutative algebra construction on stalks.*

Proof. Applying the universal property of the cokernel of the inclusion $\mathcal{I} \to \mathcal{O}_X$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \\ \downarrow & \nearrow & \\ \mathcal{O}_X/\mathcal{I} & & \end{array}$$

of $\mathcal{O}_X$-modules. By Lemma 17.22.1 the resulting map $\mathcal{O}_X/\mathcal{I} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ corresponds to a map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$$

which means we have an $\mathcal{O}_X/\mathcal{I}$-module structure on $\mathcal{F}$ compatible with the given $\mathcal{O}_X$-module structure. We omit the verification of the statement on stalks. □

0H2L **Lemma 17.23.4.** *Let $(X,\mathcal{O}_X)$ be a ringed space. If $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, then so is $Ann_{\mathcal{O}_X}(\mathcal{F})$.*

Proof. Since $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ by Lemma 17.12.4 it suffices to show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ is coherent. This follows from Lemma 17.22.6 and the fact that $\mathcal{F}$ is coherent and a fortiori finitely presented (Lemma 17.12.2). □

17.24. Koszul complexes

062J We suggest first reading the section on Koszul complexes in More on Algebra, Section 15.28. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

062K **Definition 17.24.1.** Let X be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the sheaf of commutative differential graded algebras defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

062L **Definition 17.24.2.** Let X be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The *Koszul complex on* $f_1, \ldots, f_n$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on X isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

17.25. Invertible modules

01CR Similarly to the case of modules over rings (More on Algebra, Section 15.118) we have the following definition.

01CS **Definition 17.25.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an equivalence of categories. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 17.25.4 below explains the relationship with locally free modules of rank 1.

0B8K **Lemma 17.25.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Equivalent are*

(1) *$\mathcal{L}$ is invertible, and*
(2) *there exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.*

In this case $\mathcal{L}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module and the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}_X$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$ the given isomorphism. Let $x \in X$. Choose an open neighbourhood U an integer $n \geq 1$ and sections $s_i \in \mathcal{L}(U)$, $t_i \in \mathcal{N}(U)$ such that $\psi(\sum s_i \otimes t_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends s to $\sum s_i \otimes s \otimes t_i$ and the second arrow sends $s \otimes s' \otimes t$ to $\psi(s' \otimes t)s$. We conclude that $s \mapsto \sum \psi(s \otimes t_i)s_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $s \mapsto (\psi(s \otimes t_1), \ldots, \psi(s \otimes t_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i s_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism by checking it induces isomorphisms on stalks. Let $x \in X$. Since we know (by the previous paragraph) that $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module we can use Lemma 17.22.4 to see that it suffices to show that

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism. Since $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x = (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{O}_{X,x}$ (Lemma 17.16.1) the desired result follows from More on Algebra, Lemma 15.118.2. □

0B8L **Lemma 17.25.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_Y$-module is invertible.*

Proof. By Lemma 17.25.2 there exists an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{O}_Y$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong \mathcal{O}_X$ by Lemma 17.16.4. Thus $f^*\mathcal{L}$ is invertible by Lemma 17.25.2. □

0B8M **Lemma 17.25.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any locally free $\mathcal{O}_X$-module of rank 1 is invertible. If all stalks $\mathcal{O}_{X,x}$ are local rings, then the converse holds as well (but in general this is not the case).*

Proof. The parenthetical statement follows by considering a one point space X with sheaf of rings $\mathcal{O}_X$ given by a ring R. Then invertible $\mathcal{O}_X$-modules correspond to invertible R-modules, hence as soon as $\mathrm{Pic}(R)$ is not the trivial group, then we get an example.

Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

Looking over an open covering trivialization $\mathcal{L}$, we see that this map is an isomorphism. Hence $\mathcal{L}$ is invertible by Lemma 17.25.2.

Assume all stalks $\mathcal{O}_{X,x}$ are local rings and $\mathcal{L}$ invertible. In the proof of Lemma 17.25.2 we have seen that $\mathcal{L}_x$ is an invertible $\mathcal{O}_{X,x}$-module for all $x \in X$. Since $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ (More on Algebra, Section 15.118). Since $\mathcal{L}$ is of finite presentation by Lemma 17.25.2 we conclude that $\mathcal{L}$ is locally free of rank 1 by Lemma 17.11.7. □

01CT **Lemma 17.25.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ and the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism.*

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 17.25.2 and its proof. □

01CU **Definition 17.25.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X and $n \in \mathbf{Z}$ we define the nth *tensor power* $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}_X$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ exactly n times.

This makes sense also for negative n as we've defined an invertible $\mathcal{O}_X$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

see Lemma 17.25.5. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

Let $(X, \mathcal{O}_X)$ be a ringed space. We can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 10.56 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

01CV **Definition 17.25.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (although this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces and if $\mathcal{L}$ is invertible on X, then we get an invertible sheaf $f^*\mathcal{L}$ on Y (Lemma 17.25.3) and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilities between the constructions above whose statements we omit.

01CW **Lemma 17.25.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on X is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Recall that any invertible $\mathcal{O}_X$-module is locally a direct summand of a finite free $\mathcal{O}_X$-module, see Lemma 17.25.2. For each open covering $\mathcal{U} : X = \bigcup_{j \in J} U_j$ and map $r : J \to \mathbf{N}$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{F}_j = \mathcal{F}|_{U_j}$ is a direct summand of $\mathcal{O}_{U_j}^{\oplus r(j)}$. The collection of isomorphism classes of $\mathcal{F}_j$ is a set, because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U^{\oplus r})$ is a set. The sheaf $\mathcal{F}$ is gotten by glueing $\mathcal{F}_j$, see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. For each covering there is a set of maps $r : J \to \mathbf{N}$. Hence the collection of all $\mathcal{F}$ forms a set. □

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 17.25.5 says that tensor product defines the structure of an abelian group on this set.

01CX **Definition 17.25.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. The *Picard group* $\mathrm{Pic}(X)$ of X is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

01CY **Lemma 17.25.10.** *Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \text{image } s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 17.25.5. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 10.20.1 that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. □

0H78 **Remark 17.25.11.** Given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 26.2.1) the inverse image $f^{-1}(X_s)$ is equal to Y_{f^*s}, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pullback of s.

17.26. Rank and determinant

0B37 Let $(X, \mathcal{O}_X)$ be a ringed space. Consider the category $\mathit{Vect}(X)$ of finite locally free $\mathcal{O}_X$-modules. This is an exact category (see Injectives, Remark 19.9.6) whose admissible epimorphisms are surjections and whose admissible monomorphisms are kernels of surjections. Moreover, there is a set of isomorphism classes of objects of $\mathit{Vect}(X)$ (proof omitted). Thus we can form the zeroth Grothendieck K-group $K_0(\mathit{Vect}(X))$. Explicitly, in this case $K_0(\mathit{Vect}(X))$ is the abelian group generated by $[\mathcal{E}]$ for $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module, subject to the relations

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$$

whenever there is a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules.

Ranks. Assume all stalks $\mathcal{O}_{X,x}$ are nonzero rings. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$, the *rank* is a locally constant function

$$\text{rank}_{\mathcal{E}} : X \longrightarrow \mathbf{Z}_{\geq 0}, \quad x \longmapsto \text{rank}_{\mathcal{O}_{X,x}} \mathcal{E}_x$$

See Lemma 17.14.4. By definition of locally free modules the function $\text{rank}_{\mathcal{E}}$ is locally constant. If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules, then $\text{rank}_{\mathcal{E}} = \text{rank}_{\mathcal{E}'} + \text{rank}_{\mathcal{E}''}$, Thus the rank defines a homomorphism

$$K_0(\mathit{Vect}(X)) \longrightarrow \text{Map}_{cont}(X, \mathbf{Z}), \quad [\mathcal{E}] \longmapsto \text{rank}_{\mathcal{E}}$$

Determinants. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we obtain a disjoint union decomposition

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

with X_i open and closed, such that $\mathcal{E}$ is finite locally free of rank i on X_i (this is exactly the same as saying the $\text{rank}_{\mathcal{E}}$ is locally constant). In this case we define $\det(\mathcal{E})$ as the invertible sheaf on X which is equal to $\wedge^i(\mathcal{E}|_{X_i})$ on X_i for all $i \geq 0$. Since the decomposition above is disjoint, there are no glueing conditions to check. By Lemma 17.26.1 below this defines a homomorphism

$$\det : K_0(\mathit{Vect}(X)) \longrightarrow \text{Pic}(X), \quad [\mathcal{E}] \longmapsto \det(\mathcal{E})$$

of abelian groups. The elements of $\text{Pic}(X)$ we get in this manner are locally free of rank 1 (see below the lemma for a generalization).

0B38 **Lemma 17.26.1.** *Let X be a ringed space. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$\det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'') \longrightarrow \det(\mathcal{E})$$

of $\mathcal{O}_X$-modules.

Proof. We can decompose X into disjoint open and closed subsets such that both $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank on them. Thus we reduce to the case where $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank, say r' and r''. In this situation we define

$$\wedge^{r'}(\mathcal{E}') \otimes_{\mathcal{O}_X} \wedge^{r''}(\mathcal{E}'') \longrightarrow \wedge^{r'+r''}(\mathcal{E})$$

as follows. Given local sections $s'_1, \ldots, s'_{r'}$ of $\mathcal{E}'$ and local sections $s''_1, \ldots, s''_{r''}$ of $\mathcal{E}''$ we map

$$s'_1 \wedge \ldots \wedge s'_{r'} \otimes s''_1 \wedge \ldots \wedge s''_{r''} \quad \text{to} \quad s'_1 \wedge \ldots \wedge s'_{r'} \wedge \tilde{s}''_1 \wedge \ldots \wedge \tilde{s}''_{r''}$$

where $\tilde{s}''_i$ is a local lift of the section s''_i to a section of $\mathcal{E}$. We omit the details. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Instead of looking at finite locally free $\mathcal{O}_X$-modules we can look at those $\mathcal{O}_X$-modules $\mathcal{F}$ which are locally on X a direct summand of a finite free $\mathcal{O}_X$-module. This is the same thing as asking $\mathcal{F}$ to be a flat $\mathcal{O}_X$-module of finite presentation, see Lemma 17.18.3. If all the stalks $\mathcal{O}_{X,x}$ are local, then such a module $\mathcal{F}$ is finite locally free, see Lemma 17.14.6. In general however this will not be the case; for example X could be a point and $\Gamma(X, \mathcal{O}_X)$ could be the product $A \times B$ of two nonzero rings and $\mathcal{F}$ could correspond to $A \times 0$. Thus for such a module the rank function is undefined. However, it turns out we can still define $\det(\mathcal{F})$ and this will be an invertible $\mathcal{O}_X$-module in the sense of Definition 17.25.1 (not necessarily locally free of rank 1). Our construction will agree with the one above in the case that $\mathcal{F}$ is finite locally free. We urge the reader to skip the rest of this section.

0FJN **Lemma 17.26.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat and finitely presented $\mathcal{O}_X$-module. Denote*

$$\det(\mathcal{F}) \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$$

the annihilator of $\mathcal{F} \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$. Then $\det(\mathcal{F})$ is an invertible $\mathcal{O}_X$-module.

Proof. To prove this we may work locally on X. Hence we may assume $\mathcal{F}$ is a direct summand of a finite free module, see Lemma 17.18.3. Say $\mathcal{F} \oplus \mathcal{G} = \mathcal{O}_X^{\oplus n}$. Set $R = \mathcal{O}_X(X)$. Then we see $\mathcal{F}(X) \oplus \mathcal{G}(X) = R^{\oplus n}$ and correspondingly $\mathcal{F}(U) \oplus \mathcal{G}(U) = \mathcal{O}_X(U)^{\oplus n}$ for all opens $U \subset X$. We conclude that $\mathcal{F} = \mathcal{F}_M$ as in Lemma 17.10.5 with $M = \mathcal{F}(X)$ a finite projective R-module. In other words, we have $\mathcal{F}(U) = M \otimes_R \mathcal{O}_X(U)$. This implies that $\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U))$ for all

open $U \subset X$ with det as in More on Algebra, Section 15.119. By More on Algebra, Remark 15.119.1 we see that

$$\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U)) \subset \wedge^*_{\mathcal{O}_X(U)}(\mathcal{F}(U))$$

is the annihilator of $\mathcal{F}(U)$. We conclude that $\det(\mathcal{F})$ as defined in the statement of the lemma is equal to $\mathcal{F}_{\det(M)}$. Some details omitted; one has to be careful as annihilators cannot be defined as the sheafification of taking annihilators on sections over opens. Thus $\det(\mathcal{F})$ is the pullback of an invertible module and we conclude. □

17.27. Localizing sheaves of rings

01CZ Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 10.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of X.

01D0 **Lemma 17.27.1.** *Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.*

1. *There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.*
2. *For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.*
3. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
4. *The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.*
5. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)^\#_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

Proof. Omitted. □

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of X.

01D1 **Lemma 17.27.2.** *Let X be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have*

$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.20 for notation).

Proof. Omitted. □

17.28. Modules of differentials

08RL In this section we briefly explain how to define the module of relative differentials for a morphism of ringed spaces. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131).

01UN **Definition 17.28.1.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. An $\mathcal{O}_1$-*derivation* or more precisely a φ-*derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 10.131.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(X, \mathcal{O}_2) \longrightarrow \Gamma(X, \mathcal{F})$$

is a $\Gamma(X, \mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

08RM **Lemma 17.28.2.** *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on X, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

08RN
$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & \begin{array}{c}[a+b]-[a]-[b]+ \\ [fg]-g[f]-f[g]+ \\ [\varphi(h)]\end{array} \end{array} \tag{17.28.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (17.28.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

08RP **Definition 17.28.3.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. The *module of differentials* of φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 17.28.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

08TD **Lemma 17.28.4.** *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (17.28.2.1). There is a similar map of presheaves whose value on the open U is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.131.2. On the other hand, the sheaves in (17.28.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

08RQ **Lemma 17.28.5.** *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For $U \subset X$ open there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. Holds because $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1). □

08RR **Lemma 17.28.6.** *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

08TE **Lemma 17.28.7.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $\Omega_{\mathcal{O}_2/\mathcal{O}_1,x} = \Omega_{\mathcal{O}_{2,x}/\mathcal{O}_{1,x}}$.*

Proof. This is a special case of Lemma 17.28.6 for the inclusion map $\{x\} \to X$. An alternative proof is to use Lemma 17.28.4, Sheaves, Lemma 6.17.2, and Algebra, Lemma 10.131.5. □

08RS **Lemma 17.28.8.** *Let X be a topological space. Let*

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi} & \mathcal{O}_2' \\ \uparrow & & \uparrow \\ \mathcal{O}_1 & \longrightarrow & \mathcal{O}_1' \end{array}$$

be a commutative diagram of sheaves of rings on X. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $d : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $d(f) \mapsto d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

08TF **Lemma 17.28.9.** *In Lemma 17.28.8 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $\mathcal{O}_2' = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 17.28.8. To see that the sequence is exact, we can check on stalks (Lemma 17.3.1). By Lemma 17.28.7 this follows from Algebra, Lemma 10.131.9. □

08RT **Definition 17.28.10.** Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *S-derivation* into $\mathcal{F}$ is a $f^{-1}\mathcal{O}_S$-derivation, or more precisely a $f^\sharp$-derivation in the sense of Definition 17.28.1. We denote $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of S-derivations into $\mathcal{F}$.
(2) The *sheaf of differentials $\Omega_{X/S}$ of X over S* is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ endowed with its universal S-derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Here is a particular situation where derivations come up naturally.

01UP **Lemma 17.28.11.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$ of π is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_X \to \mathcal{A}$ of π and any S-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique S-derivation D.

Proof. Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{I}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}
(s + D)(ab) &= s(ab) + D(ab) \\
&= s(a)s(b) + aD(b) + D(a)b \\
&= s(a)s(b) + s(a)D(b) + D(a)s(b) \\
&= (s(a) + D(a))(s(b) + D(b))
\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because D is an S-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

08RU **Lemma 17.28.12.** *Let*

$$\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
{\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\
S' & \xrightarrow{g} & S
\end{array}$$

be a commutative diagram of ringed spaces.

(1) *The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation and we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$.*

(2) *The commutative diagram*

$$\begin{array}{ccc}
f^{-1}\mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} \\
\uparrow & & \uparrow \\
f^{-1}h^{-1}\mathcal{O}_S & \longrightarrow & (h')^{-1}\mathcal{O}_{S'}
\end{array}$$

induces by Lemmas 17.28.6 and 17.28.8 a canonical map $f^{-1}\Omega_{X/S} \to \Omega_{X'/S'}$.

These two maps correspond (via adjointness of f_* and f^* and via $f^*\Omega_{X/S} = f^{-1}\Omega_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ and Sheaves, Lemma 6.20.2) to the same $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}$$

which is uniquely characterized by the property that $f^* d_{X/S}(a)$ maps to $d_{X'/S'}(f^*a)$ for any local section a of $\mathcal{O}_X$.

Proof. Omitted. □

01UW **Lemma 17.28.13.** *Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{g} & X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

be a commutative diagram of ringed spaces. With notation as in Lemma 17.28.12 we have

$$c_{f \circ g} = c_g \circ g^* c_f$$

as maps $(f \circ g)^* \Omega_{X/S} \to \Omega_{X''/S''}$.

Proof. Omitted. □

0H79 **Lemma 17.28.14.** *Let $f : X \to Y$, $g : Y \to S$ be morphisms of ringed spaces Then there is a canonical exact sequence*

$$f^* \Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 17.28.12.

Proof. By taking induced maps in stalks at $x \in X$ and using Lemma 17.28.7 we obtain a sequence

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \Omega_{\mathcal{O}_{Y,f(x)}/\mathcal{O}_{S,g(f(x))}} \to \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,g(f(x))}} \to \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}} \to 0$$

It suffices to see that the maps of the sequence are the same as the ones in Algebra, Lemma 10.131.7. This is because the characterization of the maps in Lemma 17.28.12 and because via the isomorphism of Sheaves, Lemma 6.26.4 we have $(f^*s)_x = s_x \otimes 1$, for any local section s of a sheaf of $\mathcal{O}_Y$-modules. □

17.29. Finite order differential operators

0G3P In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133).

0G3Q **Definition 17.29.1.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k* is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\text{Diff}^k(\mathcal{F}, \mathcal{G}) = \text{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(X, \mathcal{O}_2)$-module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \subset X$ open to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on X. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

0G3R **Lemma 17.29.2.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

0G3S **Lemma 17.29.3.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism*

$$\mathit{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 17.28.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. □

0G3T **Definition 17.29.4.** Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 17.29.3 is called the *module of principal parts of order k* of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F},\mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F},\mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F},\mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

0G3U **Lemma 17.29.5.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.*

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 17.28.4. Perhaps a more pleasing approach is to use the universal property of Lemma 17.29.3 directly to see the equality. We omit the details. □

0G3V **Lemma 17.29.6.** *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence*

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 10.133.6) and Lemmas 17.28.4 and 17.29.5. □

0G3W **Remark 17.29.7.** Let X be a topological space. Suppose given a commutative diagram of sheaves of rings

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

on X, a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ & & \uparrow & & \uparrow & & \uparrow \\ \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 17.29.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 17.29.6.

Next, we extend our definition to morphisms of ringed spaces.

0G3X **Definition 17.29.8.** Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $k \geq 0$ be an integer. A *differential operator of order k on X/S* is a differential operator $D : \mathcal{F} \to \mathcal{G}$ with respect to $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ We denote $\mathrm{Diff}^k_{X/S}(\mathcal{F},\mathcal{G})$ the set of these differential operators.

17.30. The de Rham complex

0FKL The section is the analogue of Algebra, Section 10.132 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Denote $\mathrm{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ the module of differentials with its universal $\mathcal{A}$-derivation constructed in Section 17.28. Let

$$\Omega^i_{\mathcal{B}/\mathcal{A}} = \wedge^i_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}})$$

for $i \geq 0$ be the ith exterior power as in Section 17.21.

0FKM **Definition 17.30.1.** In the situation above, the *de Rham complex of $\mathcal{B}$ over $\mathcal{A}$* is the unique complex

$$\Omega^0_{\mathcal{B}/\mathcal{A}} \to \Omega^1_{\mathcal{B}/\mathcal{A}} \to \Omega^2_{\mathcal{B}/\mathcal{A}} \to \ldots$$

of sheaves of $\mathcal{A}$-modules whose differential in degree 0 is given by $\mathrm{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ and whose differentials in higher degrees have the following property

0FKN (17.30.1.1) $$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

where $b_0, \ldots, b_p \in \mathcal{B}(U)$ are sections over a common open $U \subset X$.

We could construct this complex by repeating the cumbersome arguments given in Algebra, Section 10.132. Instead we recall that $\Omega_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 17.28.4. Thus $\Omega^i_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega^i_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 17.21.1. Therefore we can define the de Rham complex as the sheafification of the rule

$$U \longmapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$$

0FKP **Lemma 17.30.2.** *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega^\bullet_{\mathcal{B}/\mathcal{A}} = \Omega^\bullet_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$ of de Rham complexes.*

Proof. Omitted. Hint: compare with Lemma 17.28.6. □

0G3Y **Lemma 17.30.3.** *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. The differentials $d : \Omega^i_{\mathcal{B}/\mathcal{A}} \to \Omega^{i+1}_{\mathcal{B}/\mathcal{A}}$ are differential operators of order 1.*

Proof. Via our construction of the de Rham complex above as the sheafification of the rule $U \mapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$ this follows from Algebra, Lemma 10.133.10. □

Let X be a topological space. Let

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

be a commutative diagram of sheaves of rings on X. There is a natural map of de Rham complexes

$$\Omega^\bullet_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega^\bullet_{\mathcal{B}'/\mathcal{A}'}$$

Namely, in degree 0 this is the map $\mathcal{B} \to \mathcal{B}'$, in degree 1 this is the map $\Omega_{\mathcal{B}/\mathcal{A}} \to \Omega_{\mathcal{B}'/\mathcal{A}'}$ constructed in Section 17.28, and for $p \geq 2$ it is the induced map $\Omega^p_{\mathcal{B}/\mathcal{A}} =$

$\wedge^p_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}}) \to \wedge^p_{\mathcal{B}'}(\Omega_{\mathcal{B}'/\mathcal{A}'}) = \Omega^p_{\mathcal{B}'/\mathcal{A}'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (17.30.1.1).

0FKQ **Definition 17.30.4.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The *de Rham complex* of f or of X over Y is the complex

$$\Omega^\bullet_{X/Y} = \Omega^\bullet_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$$

Consider a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ S' & \xrightarrow{g} & S \end{array}$$

Then we obtain a canonical map

$$\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$$

of de Rham complexes. Namely, the commutative diagram of sheaves of rings

$$\begin{array}{ccc} f^{-1}\mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} \\ \uparrow & & \uparrow \\ f^{-1}h^{-1}\mathcal{O}_S & \longrightarrow & (h')^{-1}\mathcal{O}_{S'} \end{array}$$

on X' produces a map of complexes (see above)

$$f^{-1}\Omega^\bullet_{X/S} = \Omega^\bullet_{f^{-1}\mathcal{O}_X/f^{-1}h^{-1}\mathcal{O}_S} \longrightarrow \Omega^\bullet_{\mathcal{O}_{X'}/(h')^{-1}\mathcal{O}_{S'}} = \Omega^\bullet_{X'/S'}$$

(using Lemma 17.30.2 for the first equality) and then we can use adjunction.

0G3Z **Lemma 17.30.5.** *Let $f : X \to Y$ be a morphism of ringed spaces. The differentials $d : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ are differential operators of order 1 on X/Y.*

Proof. Immediate from Lemma 17.30.3 and the definition. □

17.31. The naive cotangent complex

08TG This section is the analogue of Algebra, Section 10.134 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. In this section, for any sheaf of sets $\mathcal{E}$ on X we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

08TH (17.31.0.1) $$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 17.28.9 there is a canonical map

08TI (17.31.0.2) $$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

08TJ **Definition 17.31.1.** Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (17.31.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 17.28.8 for modules of differentials. Namely, given a commutative diagram

08TK (17.31.1.1)
$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

of sheaves of rings on X there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes. The map is compatible with compositions in the following sense: given a commutative diagram

$$\begin{array}{ccccc} \mathcal{B} & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{B}'' \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{A}'' \end{array}$$

of sheaves of rings then the composition

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'} \longrightarrow NL_{\mathcal{B}''/\mathcal{A}''}$$

is the map for the outer rectangle.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on X and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. If this map is surjective, i.e., if $\alpha(\mathcal{E})$ generates $\mathcal{B}$ as an $\mathcal{A}$-algebra, then we set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

where $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ is the kernel of the surjection $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Here is the result.

08TL **Lemma 17.31.2.** *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this it suffices to check on stalks (Lemma 17.3.1). If $x \in X$ then the stalk of $NL(\alpha)$ is the complex

$NL(\alpha_x)$ of Algebra, Section 10.134 associated to the presentation $\mathcal{A}_x[\mathcal{E}_x] \to \mathcal{B}_x$ coming from the map $\alpha_x : \mathcal{E}_x \to \mathcal{B}_x$. (Some details omitted; use Lemma 17.28.7 to see compatibility of forming differentials and taking stalks.) We conclude the result holds by Algebra, Lemma 10.134.2. □

08TM **Lemma 17.31.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on Y. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 17.28.6. □

0D09 **Lemma 17.31.4.** *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $NL_{\mathcal{B}/\mathcal{A},x} = NL_{\mathcal{B}_x/\mathcal{A}_x}$.*

Proof. This is a special case of Lemma 17.31.3 for the inclusion map $\{x\} \to X$. □

0E1Y **Lemma 17.31.5.** *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be maps of sheaves of rings. Let C be the cone (Derived Categories, Definition 13.9.1) of the map of complexes $NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$. There is a canonical map*

$$c : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow C[-1]$$

of complexes of $\mathcal{C}$-modules which produces a canonical six term exact sequence

$$\begin{array}{ccccccc} H^0(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) & \longrightarrow & H^0(NL_{\mathcal{C}/\mathcal{A}}) & \longrightarrow & H^0(NL_{\mathcal{C}/\mathcal{B}}) & \longrightarrow & 0 \\ & \searrow & & & & & \\ H^{-1}(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) & \longrightarrow & H^{-1}(NL_{\mathcal{C}/\mathcal{A}}) & \longrightarrow & H^{-1}(NL_{\mathcal{C}/\mathcal{B}}) & & \end{array}$$

of cohomology sheaves.

Proof. To give the map c we have to give a map $c_1 : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}}$ and an explicit homotopy between the composition

$$NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$$

and the zero map, see Derived Categories, Lemma 13.9.3. For c_1 we use the functoriality described above for the obvious diagram. For the homotopy we use the map

$$NL^0_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow NL^{-1}_{\mathcal{C}/\mathcal{B}}, \quad \mathrm{d}[b] \otimes 1 \longmapsto [\varphi(b)] - b[1]$$

where $\varphi : \mathcal{B} \to \mathcal{C}$ is the given map. Please compare with Algebra, Remark 10.134.5. To see the consequence for cohomology sheaves, it suffices to show that $H^0(c)$ is an isomorphism and $H^{-1}(c)$ surjective. To see this we can look at stalks, see Lemma 17.31.4, and then we can use the corresponding result in commutative algebra, see Algebra, Lemma 10.134.4. Some details omitted. □

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

08TN **Definition 17.31.6.** The *naive cotangent complex* $NL_f = NL_{X/Y}$ of a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow[g]{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y \end{array}$$

of ringed spaces, there is a canonical map $c : g^* NL_{X/Y} \to NL_{X'/Y'}$. Namely, it is the map

$$g^* NL_{X/Y} = \mathcal{O}_{X'} \otimes_{g^{-1}\mathcal{O}_X} NL_{g^{-1}\mathcal{O}_X / g^{-1}f^{-1}\mathcal{O}_Y} \longrightarrow NL_{\mathcal{O}_{X'}/(f')^{-1}\mathcal{O}_{Y'}} = NL_{X'/Y'}$$

where the arrow comes from the commutative diagram of sheaves of rings

$$\begin{array}{ccc} g^{-1}\mathcal{O}_X & \xrightarrow[g^\sharp]{} & \mathcal{O}_{X'} \\ {\scriptstyle g^{-1}f^\sharp}\uparrow & & \uparrow{\scriptstyle (f')^\sharp} \\ g^{-1}f^{-1}\mathcal{O}_Y & \xrightarrow{g^{-1}h^\sharp} & (f')^{-1}\mathcal{O}_{Y'} \end{array}$$

as in (17.31.1.1) above. Given a second such diagram

$$\begin{array}{ccc} X'' & \xrightarrow[g']{} & X' \\ \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' \end{array}$$

the composition of $(g')^* c$ and the map $c' : (g')^* NL_{X'/Y'} \to NL_{X''/Y''}$ is the map $(g \circ g')^* NL_{X''/Y''} \to NL_{X/Y}$.

0E1Z **Lemma 17.31.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let C be the cone of the map $NL_{X/Z} \to NL_{X/Y}$ of complexes of $\mathcal{O}_X$-modules. There is a canonical map*

$$f^* NL_{Y/Z} \to C[-1]$$

which produces a canonical six term exact sequence

$$H^0(f^* NL_{Y/Z}) \longrightarrow H^0(NL_{X/Z}) \longrightarrow H^0(NL_{X/Y}) \longrightarrow 0$$

$$H^{-1}(f^* NL_{Y/Z}) \longrightarrow H^{-1}(NL_{X/Z}) \longrightarrow H^{-1}(NL_{X/Y})$$

of cohomology sheaves.

Proof. Consider the maps of sheaves rings

$$(g \circ f)^{-1}\mathcal{O}_Z \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

and apply Lemma 17.31.5. □

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CHAPTER 18

Modules on Sites

18.1. Introduction

03A4

03A5 In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of $\mathcal{O}$-modules, analogous to the material on (pre)sheaves of $\mathcal{O}$-modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 17.1). Basic references are [**Ser55b**], [**DG67**] and [**AGV71**].

18.2. Abelian presheaves

03A6 Let $\mathcal{C}$ be a category. Abelian presheaves were introduced in Sites, Sections 7.2 and 7.7 and discussed a bit more in Sites, Section 7.44. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on $\mathcal{C}$ is denoted $PAb(\mathcal{C})$.

The category $PAb(\mathcal{C})$ is abelian as defined in Homology, Definition 12.5.1. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ the kernel of φ is the abelian presheaf $U \mapsto \mathrm{Ker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \mathrm{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$. Since the category of abelian groups is abelian it follows that $\mathrm{Coim} = \mathrm{Im}$ because this holds over each U. A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if $\mathcal{G}_1(U) \to \mathcal{G}_2(U) \to \mathcal{G}_3(U)$ is an exact sequence of abelian groups for all $U \in \mathrm{Ob}(\mathcal{C})$. We leave the verifications to the reader.

03CL **Lemma 18.2.1.** *Let $\mathcal{C}$ be a category.*

(1) *All limits and colimits exist in $PAb(\mathcal{C})$.*
(2) *All limits and colimits commute with taking sections over objects of $\mathcal{C}$.*

Proof. Let $\mathcal{I} \to PAb(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. We can simply define abelian presheaves L and C by the rules

$$L : U \longmapsto \lim_i \mathcal{F}_i(U)$$

and

$$C : U \longmapsto \mathrm{colim}_i \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves $L \to \mathcal{F}_i$ and $\mathcal{F}_i \to C$, by using the corresponding maps on groups of sections over each U. It is straightforward to check that L and C endowed with these maps are the limit and colimit of the diagram in $PAb(\mathcal{C})$. This proves (1) and (2). Details omitted. □

18.3. Abelian sheaves

03CM Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$ is denoted $Ab(\mathcal{C})$. It is the full subcategory of $PAb(\mathcal{C})$ consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties (α) – (ζ) of Sites, Section 7.44 hold, see Sites, Proposition 7.44.3. In particular the inclusion functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ has a left adjoint, namely the sheafification functor $\mathcal{G} \mapsto \mathcal{G}^\#$.

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

03CN **Lemma 18.3.1.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.*

(1) *The category $Ab(\mathcal{C})$ is an abelian category.*
(2) *The kernel $\mathrm{Ker}(\varphi)$ of φ is the same as the kernel of φ as a morphism of presheaves.*
(3) *The morphism φ is injective (Homology, Definition 12.5.3) if and only if φ is injective as a map of presheaves (Sites, Definition 7.3.1), if and only if φ is injective as a map of sheaves (Sites, Definition 7.11.1).*
(4) *The cokernel $\mathrm{Coker}(\varphi)$ of φ is the sheafification of the cokernel of φ as a morphism of presheaves.*
(5) *The morphism φ is surjective (Homology, Definition 12.5.3) if and only if φ is surjective as a map of sheaves (Sites, Definition 7.11.1).*
(6) *A complex of abelian sheaves*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $s \in \mathcal{G}(U)$ mapping to zero in $\mathcal{H}(U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that each $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof. We claim that Homology, Lemma 12.7.4 applies to the categories $\mathcal{A} = Ab(\mathcal{C})$ and $\mathcal{B} = PAb(\mathcal{C})$, and the functors $a : \mathcal{A} \to \mathcal{B}$ (inclusion), and $b : \mathcal{B} \to \mathcal{A}$ (sheafification). Let us check the assumptions of Homology, Lemma 12.7.4. Assumption (1) is that $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b. The first two statements are clear and adjointness is Sites, Section 7.44 (ϵ). Assumption (2) says that $PAb(\mathcal{C})$ is abelian which we saw in Section 18.2 and that sheafification is left exact, which is Sites, Section 7.44 (ζ). The final assumption is that $ba \cong \mathrm{id}_{\mathcal{A}}$ which is Sites, Section 7.44 (δ). Hence Homology, Lemma 12.7.4 applies and we conclude that $Ab(\mathcal{C})$ is abelian.

In the proof of Homology, Lemma 12.7.4 it is shown that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are equal to the sheafification of the kernel and cokernel of φ as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafification commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if $\mathrm{Im}(\mathcal{F} \to \mathcal{G}) = \mathrm{Ker}(\mathcal{G} \to \mathcal{H})$. □

Another way to say part (6) of the lemma is that a sequence of abelian sheaves

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

is exact if and only if the sheafification of $U \mapsto \mathrm{Im}(\mathcal{F}_1(U) \to \mathcal{F}_2(U))$ is equal to the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$.

03CO **Lemma 18.3.2.** *Let $\mathcal{C}$ be a site.*

(1) *All limits and colimits exist in $Ab(\mathcal{C})$.*
(2) *Limits are the same as the corresponding limits of abelian presheaves over $\mathcal{C}$ (i.e., commute with taking sections over objects of $\mathcal{C}$).*
(3) *Finite direct sums are the same as the corresponding finite direct sums in the category of abelian pre-sheaves over $\mathcal{C}$.*
(4) *A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.*
(5) *Filtered colimits are exact.*

Proof. By Lemma 18.2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let $\mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram as an abelian presheaf. By Sites, Lemma 7.10.1 this is an abelian sheaf. Then it is quite easy to see that $\lim_i \mathcal{F}_i$ is the limit of the diagram in $Ab(\mathcal{C})$. This proves limits exist and (2) holds.

By Categories, Lemma 4.24.5, and because sheafification is left adjoint to the inclusion functor we see that $\mathrm{colim}_i \mathcal{F}$ exists and is the sheafification of the colimit in $PAb(\mathcal{C})$. This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2).

Proof of (5). The statement means that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of abelian sheaves over a directed set I the sequence $0 \to \mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact as well. A formal argument using Homology, Lemma 12.5.8 and the definition of colimits shows that the sequence $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact. Note that $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i$ is the sheafification of the map of presheaf colimits which is injective as each of the maps $\mathcal{F}_i \to \mathcal{G}_i$ is injective. Since sheafification is exact we conclude. □

18.4. Free abelian presheaves

03CP In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set S as[1] $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$. It is characterized by the property

$$\mathrm{Mor}_{Ab}(\mathbf{Z}[S], A) = \mathrm{Mor}_{Sets}(S, A)$$

In other words the construction $S \mapsto \mathbf{Z}[S]$ is a left adjoint to the forgetful functor $Ab \to Sets$.

03A7 **Definition 18.4.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian presheaf* $\mathbf{Z}_\mathcal{G}$ on $\mathcal{G}$ is the abelian presheaf defined by the rule

$$U \longmapsto \mathbf{Z}[\mathcal{G}(U)].$$

In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X = \mathbf{Z}_{h_X}$. In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(U, X)].$$

[1] In other chapters the notation $\mathbf{Z}[S]$ sometimes indicates the polynomial ring over $\mathbf{Z}$ on S.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it is adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Here is the precise statement.

03A8 **Lemma 18.4.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{G}$, $\mathcal{F}$ be a presheaves of sets. Let $\mathcal{A}$ be an abelian presheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\ \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{G}}, \mathcal{A}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. □

03A9 **Lemma 18.4.3.** *Let $\mathcal{C}$ be a category. Let I be a set. For each $i \in I$ let $\mathcal{G}_i$ be a presheaf of sets. Then*

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus\nolimits_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

in $PAb(\mathcal{C})$.

Proof. Omitted. □

18.5. Free abelian sheaves

03CQ Here is the notion of a free abelian sheaf on a sheaf of sets.

03AA **Definition 18.5.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian sheaf* $\mathbf{Z}^\#_{\mathcal{G}}$ on $\mathcal{G}$ is the abelian sheaf $\mathbf{Z}^\#_{\mathcal{G}}$ which is the sheafification of the free abelian presheaf on $\mathcal{G}$. In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}^\#_X$.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it provides an adjoint to the forgetful functor $Ab(\mathcal{C}) \to Sh(\mathcal{C})$. Here is the precise statement.

03AB **Lemma 18.5.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$, $\mathcal{F}$ be a sheaves of sets. Let $\mathcal{A}$ be an abelian sheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(h^\#_U, \mathcal{F}) &= \mathcal{F}(U), \\ \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^\#_{\mathcal{G}}, \mathcal{A}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^\#_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. □

03AC **Lemma 18.5.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. Then $\mathbf{Z}^\#_{\mathcal{G}} = (\mathbf{Z}_{\mathcal{G}^\#})^\#$.*

Proof. Omitted. □

18.6. Ringed sites

04KQ In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

03AD **Definition 18.6.1.** Ringed sites.

(1) A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.

(2) Let $(\mathcal{C}, \mathcal{O})$, $(\mathcal{C}', \mathcal{O}')$ be ringed sites. A *morphism of ringed sites*
$$(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$
is given by a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 7.14.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ be morphisms of ringed sites. Then we define the *composition of morphisms of ringed sites* by the rule
$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$
Here we use composition of morphisms of sites defined in Sites, Definition 7.14.5 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings
$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_* f^\sharp} g_* f_* \mathcal{O}_1 = (g \circ f)_* \mathcal{O}_1$$

18.7. Ringed topoi

01D2 A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

01D3 **Definition 18.7.1.** Ringed topoi.

(1) A *ringed topos* is a pair $(Sh(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed topos.

(2) Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. A *morphism of ringed topoi*
$$(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$
is given by a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ (see Sites, Definition 7.15.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then we define the *composition of morphisms of ringed topoi* by the rule
$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$
Here we use composition of morphisms of topoi defined in Sites, Definition 7.15.1 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings
$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_* f^\sharp} g_* f_* \mathcal{O}_1 = (g \circ f)_* \mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morphism of ringed sites. Here is the precise statement.

03CR **Lemma 18.7.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. There exists a factorization*
$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) & \xrightarrow[(f,f^\sharp)]{} & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \\ \downarrow{\scriptstyle (g,g^\sharp)} & & \downarrow{\scriptstyle (e,e^\sharp)} \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{(h,h^\sharp)} & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \end{array}$$
where

(1) $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ *is an equivalence of topoi induced by a special cocontinuous functor* $\mathcal{C} \to \mathcal{C}'$ *(see Sites, Definition 7.29.2),*
(2) $e : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ *is an equivalence of topoi induced by a special cocontinuous functor* $\mathcal{D} \to \mathcal{D}'$ *(see Sites, Definition 7.29.2),*
(3) $\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_{\mathcal{C}}$ *and* $g^\sharp$ *is the obvious map,*
(4) $\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_{\mathcal{D}}$ *and* $e^\sharp$ *is the obvious map,*
(5) *the sites* $\mathcal{C}'$ *and* $\mathcal{D}'$ *have final objects and fibre products (i.e., all finite limits),*
(6) h *is a morphism of sites induced by a continuous functor* $u : \mathcal{D}' \to \mathcal{C}'$ *which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 7.14.7), and*
(7) *given any set of sheaves* $\mathcal{F}_i$ *(resp.* $\mathcal{G}_j$*) on* $\mathcal{C}$ *(resp.* $\mathcal{D}$*) we may assume each of these is a representable sheaf on* $\mathcal{C}'$ *(resp.* $\mathcal{D}'$*).*

Moreover, if $(f, f^\sharp)$ *is an equivalence of ringed topoi, then we can choose the diagram such that* $\mathcal{C}' = \mathcal{D}'$, $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$ *and* $(h, h^\sharp)$ *is the identity.*

Proof. This follows from Sites, Lemma 7.29.6, and Sites, Remarks 7.29.7 and 7.29.8. You just have to carry along the sheaves of rings. Some details omitted. □

18.8. 2-morphisms of ringed topoi

04IB This is a brief section concerning the notion of a 2-morphism of ringed topoi.

04IC **Definition 18.8.1.** Let $f, g : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be two morphisms of ringed topoi. A *2-morphism from* f *to* g is given by a transformation of functors $t : f_* \to g_*$ such that

$$\begin{array}{ccc} & \mathcal{O}_{\mathcal{D}} & \\ {\scriptstyle f^\sharp}\swarrow & & \searrow{\scriptstyle g^\sharp} \\ f_*\mathcal{O}_{\mathcal{C}} & \xrightarrow{\quad t \quad} & g_*\mathcal{O}_{\mathcal{C}} \end{array}$$

is commutative.

Pictorially we sometimes represent t as follows:

$$(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \underset{g}{\overset{f}{\Downarrow t}} (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$$

As in Sites, Section 7.36 giving a 2-morphism $t : f_* \to g_*$ is equivalent to giving $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathcal{D}} & \xleftarrow{\quad t \quad} & g^{-1}\mathcal{O}_{\mathcal{D}} \\ & {\scriptstyle f^\sharp}\searrow \quad \swarrow{\scriptstyle g^\sharp} & \\ & \mathcal{O}_{\mathcal{C}} & \end{array}$$

is commutative. As in Sites, Section 7.36 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

18.9. Presheaves of modules

03CS Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. At this point we have not yet defined a presheaf of $\mathcal{O}$-modules. Thus we do so right now.

03CT **Definition 18.9.1.** Let $\mathcal{C}$ be a category, and let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets
$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$
such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram
$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle \mathrm{id} \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$
commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $PMod(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on the category $\mathcal{C}$. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition
$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$
We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of $\mathcal{F}$*. We obtain the restriction functor
$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule
$$U \longmapsto (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$
where $U \in \mathrm{Ob}(\mathcal{C})$, with obvious restriction mappings. The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor
$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

03CU **Lemma 18.9.2.** *With $\mathcal{C}$, $\mathcal{O}_1 \to \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*
$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$
In other words, the restriction and change of rings functors defined above are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

18.10. Sheaves of modules

03CV

03CW **Definition 18.10.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 18.9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\mathit{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

18.11. Sheafification of presheaves of modules

03CX

03CY **Lemma 18.11.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$ as a presheaf of rings, see Sites, Section 7.44. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a unique map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}^\# \times \mathcal{F}^\# & \longrightarrow & \mathcal{F}^\# \end{array}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : \mathit{Mod}(\mathcal{O}^\#) \to \mathit{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \mathit{PMod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when $\mathcal{O}$ is already a sheaf of rings. In this case the formula reads

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because $\mathcal{O} = \mathcal{O}^\#$ in this case.

03EI **Lemma 18.11.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ The sheafification functor*

$$\mathit{PMod}(\mathcal{O}) \longrightarrow \mathit{Mod}(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

is exact.

Proof. This is true because it holds for sheafification $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. See the discussion in Section 18.3. □

Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on $\mathcal{C}$. In Section 18.9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

03CZ **Lemma 18.11.3.** *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 18.9.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

0930 **Lemma 18.11.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be an epimorphism of sheaves of rings. Let $\mathcal{G}_1, \mathcal{G}_2$ be $\mathcal{O}'$-modules. Then*

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}_1, \mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}_1, \mathcal{G}_2).$$

In other words, the restriction functor $Mod(\mathcal{O}') \to Mod(\mathcal{O})$ is fully faithful.

Proof. This is the sheaf version of Algebra, Lemma 10.107.14 and is proved in exactly the same way. □

18.12. Morphisms of topoi and sheaves of modules

03D0 All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

03D1 **Lemma 18.12.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ the multiplication map. Recall that f_* (on sheaves of sets) is left exact and hence commutes with products. Hence $f_*\mu$ is a map as indicated. This proves the lemma. □

03D2 **Lemma 18.12.2.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{G} \to \mathcal{G}$ the multiplication map. Recall that f^{-1} (on sheaves of sets) is exact and hence commutes with products. Hence $f^{-1}\mu$ is a map as indicated. This proves the lemma. □

03D3 **Lemma 18.12.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. First we note that we have

$$\mathrm{Mor}_{Ab(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 7.44.3. Suppose that $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond via the formula above. We have to show that α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. For example, suppose α is $f^{-1}\mathcal{O}$-linear, then clearly $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear, and hence (as restriction is a functor) is $\mathcal{O}$-linear. Hence it suffices to prove that the adjunction map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is $\mathcal{O}$-linear. Using that both f_* and f^{-1} commute with products (on sheaves of sets) this comes down to showing that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{G} & \longrightarrow & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & f_*f^{-1}\mathcal{G} \end{array}$$

is commutative. This holds because the adjunction mapping $\mathrm{id}_{Sh(\mathcal{D})} \to f_*f^{-1}$ is a transformation of functors. We omit the proof of the implication β linear $\Rightarrow$ α linear. □

03D4 **Lemma 18.12.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. Note that we have

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 18.11.3. Hence the result follows from Lemma 18.12.3. □

18.13. Morphisms of ringed topoi and modules

03D5 We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

03D6 **Definition 18.13.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_\mathcal{D}$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_\mathcal{D} \to f_*\mathcal{O}_\mathcal{C}$ of the module structure

$$f_*\mathcal{O}_\mathcal{C} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

from Lemma 18.12.1.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by the formula

$$f^*\mathcal{G} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}$$

where the ring map $f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}$ is $f^\sharp$, and where the module structure is given by Lemma 18.12.2.

Thus we have defined functors

$$\begin{aligned} f_* : Mod(\mathcal{O}_\mathcal{C}) &\longrightarrow Mod(\mathcal{O}_\mathcal{D}) \\ f^* : Mod(\mathcal{O}_\mathcal{D}) &\longrightarrow Mod(\mathcal{O}_\mathcal{C}) \end{aligned}$$

The final result on these functors is that they are indeed adjoint as expected.

03D7 **Lemma 18.13.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(f^{-1}\mathcal{O}_\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_\mathcal{D}}) \\ &= \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 18.11.3 and 18.12.3. □

03D8 **Lemma 18.13.3.** *$(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. There are canonical isomorphisms of functors $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. This is clear from the definitions. □

18.14. The abelian category of sheaves of modules

03D9 Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the sum of φ and ψ as morphisms of abelian sheaves. This is clearly again a map of $\mathcal{O}$-modules. It is also clear that composition of maps of $\mathcal{O}$-modules is bilinear with respect to this addition. Thus $Mod(\mathcal{O})$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}$-modules which has constant value $\{0\}$ for all objects U of $\mathcal{C}$. Clearly this is both a final and an initial object of $Mod(\mathcal{O})$. Given a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each object U.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $Mod(\mathcal{O})$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the kernel of φ as a map of abelian sheaves. By Section 18.3 this is the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects U of $\mathcal{C}$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$.

Similarly, we define $\mathrm{Coker}(\varphi)$ as the cokernel of φ as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \mathrm{Coker}(\varphi) \longrightarrow \mathrm{Coker}(\varphi)$$

such that the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ becomes a morphism of $\mathcal{O}$-modules (verification omitted). The map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets, see Section 18.3). To show that $\mathrm{Coker}(\varphi)$ is a cokernel in $Mod(\mathcal{O})$, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}$-modules such that $\beta \circ \varphi$ is zero, then you get for every object U of $\mathcal{C}$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

03DA **Lemma 18.14.1.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. The category $Mod(\mathcal{O})$ is an abelian category. The forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}$-modules, correspond to the corresponding notions for abelian sheaves.*

Proof. Above we have seen that $Mod(\mathcal{O})$ is an additive category, with kernels and cokernels and that $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ preserves kernels and cokernels. By Homology, Definition 12.5.1 we have to show that image and coimage agree. This is clear because it is true in $Ab(\mathcal{C})$. The lemma follows. □

03DB **Lemma 18.14.2.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. All limits and colimits exist in $Mod(\mathcal{O})$ and the forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ commutes with them. Moreover, filtered colimits are exact.*

Proof. The final statement follows from the first as filtered colimits are exact in $Ab(\mathcal{C})$ by Lemma 18.3.2. Let $\mathcal{I} \to Mod(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram in $Ab(\mathcal{C})$. By the description of this limit in Lemma 18.3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times \lim_i \mathcal{F}_i \longrightarrow \lim_i \mathcal{F}_i$$

which turns $\lim_i \mathcal{F}_i$ into a sheaf of $\mathcal{O}$-modules. It is easy to see that this is the limit of the diagram in $Mod(\mathcal{C})$. Let $\operatorname{colim}_i \mathcal{F}_i$ be the colimit of the diagram in $PAb(\mathcal{C})$. By the description of this colimit in the proof of Lemma 18.2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times \operatorname{colim}_i \mathcal{F}_i \longrightarrow \operatorname{colim}_i \mathcal{F}_i$$

which turns $\operatorname{colim}_i \mathcal{F}_i$ into a presheaf of $\mathcal{O}$-modules. Applying sheafification we get a sheaf of $\mathcal{O}$-modules $(\operatorname{colim}_i \mathcal{F}_i)^\#$, see Lemma 18.11.1. It is easy to see that $(\operatorname{colim}_i \mathcal{F}_i)^\#$ is the colimit of the diagram in $Mod(\mathcal{O})$, and by Lemma 18.3.2 forgetting the $\mathcal{O}$-module structure is the colimit in $Ab(\mathcal{C})$. □

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.2 for a description of exactness properties in terms of short exact sequences.

03DC **Lemma 18.14.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.*

(1) *The functor f_* is left exact. In fact it commutes with all limits.*
(2) *The functor f^* is right exact. In fact it commutes with all colimits.*

Proof. This is true because (f^*, f_*) is an adjoint pair of functors, see Lemma 18.13.2. See Categories, Section 4.24. □

05V3 **Lemma 18.14.4.** *Let $\mathcal{C}$ be a site. If $\{p_i\}_{i \in I}$ is a conservative family of points, then we may check exactness of a sequence of abelian sheaves on the stalks at the points p_i, $i \in I$. If $\mathcal{C}$ has enough points, then exactness of a sequence of abelian sheaves may be checked on stalks.*

Proof. This is immediate from Sites, Lemma 7.38.2. □

18.15. Exactness of pushforward

04BC Some technical lemmas concerning exactness properties of pushforward.

04DA **Lemma 18.15.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. The following are equivalent:*

(1) *$f^{-1} f_* \mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Ab(\mathcal{C})$, and*
(2) *$f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ reflects surjections.*

In this case the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is faithful.

Proof. Assume (1). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of abelian sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (1) this implies that a is surjective. This means that (2) holds.

Assume (2). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. By (2) it suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

We omit the proof of the final assertion. □

04DB **Lemma 18.15.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Assume at least one of the following properties holds*

(1) *f_* transforms surjections of sheaves of sets into surjections,*
(2) *f_* transforms surjections of abelian sheaves into surjections,*
(3) *f_* commutes with coequalizers on sheaves of sets,*
(4) *f_* commutes with pushouts on sheaves of sets,*

Then $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Since $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is a right adjoint we already know that it transforms a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of abelian sheaves on $\mathcal{C}$ into an exact sequence

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$$

see Categories, Sections 4.23 and 4.24 and Homology, Section 12.7. Hence it suffices to prove that the map $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 7.41.1 we see that either (3) or (4) formally implies (1), hence in these cases we are done also. □

04BD **Lemma 18.15.3.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume u is almost cocontinuous. Then*

(1) *$f_* : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ is exact.*
(2) *if $f^\sharp : f^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ is given so that f becomes a morphism of ringed sites, then $f_* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact.*

Proof. Part (2) follows from part (1) by Lemma 18.14.2. Part (1) follows from Sites, Lemmas 7.42.6 and 7.41.1. □

18.16. Exactness of lower shriek

04BE Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between sites. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 7.21.1. Recall that $g^{-1} = u^p$, i.e., g^{-1} is given by the simple formula $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$, see Sites, Lemma 7.21.5. We would like to show that $g^{-1} : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ has a left adjoint $g_!$. By Sites, Lemma 7.21.5 the functor $g_!^{Sh} = (u_p\ )^\#$ is a left adjoint on sheaves of sets. Moreover, we know that $g_!^{Sh}\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

where the colimit is over $(\mathcal{I}^u_V)^{opp}$ and is taken in the category of sets. Hence the following definition is natural.

04BF **Definition 18.16.1.** With $u : \mathcal{C} \to \mathcal{D}$ satisfying (a), (b) above. For $\mathcal{F} \in PAb(\mathcal{C})$ we define $g_{p!}\mathcal{F}$ as the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

with colimits over $(\mathcal{I}^u_V)^{opp}$ taken in Ab. For $\mathcal{F} \in PAb(\mathcal{C})$ we set $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$.

The reason for being so explicit with this is that the functors $g_!^{Sh}$ and $g_!$ are different. Whenever we use both we have to be careful to make the distinction clear.

04BG **Lemma 18.16.2.** *The functor $g_{p!}$ is a left adjoint to the functor u^p. The functor $g_!$ is a left adjoint to the functor g^{-1}. In other words the formulas*

$$\operatorname{Mor}_{PAb(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}),$$
$$\operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) = \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})$$

hold bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The second formula follows formally from the first, since if $\mathcal{F}$ and $\mathcal{G}$ are abelian sheaves then

$$\begin{aligned}\operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})\end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let $\mathcal{F}$, $\mathcal{G}$ be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves $\mathcal{F} \to u^p g_{p!}\mathcal{F}$ which on sections over U is the natural map $\mathcal{F}(U) \to \operatorname{colim}_{u(U) \to u(U')} \mathcal{F}(U')$, see Sites, Lemma 7.5.3. Given a map $\alpha : g_{p!}\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p g_{p!}\mathcal{F} \to u^p\mathcal{G}$. which we can precompose by the map $\mathcal{F} \to u^p g_{p!}\mathcal{F}$.

Note that there is a canonical map of abelian presheaves $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ which on sections over V is the natural map $\operatorname{colim}_{V \to u(U)} \mathcal{G}(u(U)) \to \mathcal{G}(V)$. It maps a section $s \in u(U)$ in the summand corresponding to $t : V \to u(U)$ to $t^*s \in \mathcal{G}(V)$. Hence, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $g_{p!}\beta : g_{p!}\mathcal{F} \to g_{p!}u^p\mathcal{G}$ which we can postcompose with the map $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ above. □

04BH **Lemma 18.16.3.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_! : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Compare with Sites, Lemma 7.21.6. Assume (a), (b), and (c). We already know that $g_!$ is right exact as it is a left adjoint, see Categories, Lemma 4.24.6 and Homology, Section 12.7. We have $g_! = (g_{p!}\ )^\#$. We have to show that $g_!$ transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 18.3.2. Thus it suffices to show that $g_{p!}$ transforms injective maps of abelian presheaves into injective maps

of abelian presheaves. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Sites, Section 7.5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 7.5.1 and Algebra, Lemma 10.8.10. □

077I **Lemma 18.16.4.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all abelian sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. The map $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism by Sites, Lemma 7.21.7 and the fact that pullback and pushforward of abelian sheaves agrees with pullback and pushforward on the underlying sheaves of sets.

Pick $U \in \mathrm{Ob}(\mathcal{C})$. We will show that $g^{-1}g_!\mathcal{F}(U) = \mathcal{F}(U)$. First, note that $g^{-1}g_!\mathcal{F}(U) = g_!\mathcal{F}(u(U))$. Hence it suffices to show that $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$. We know that $g_!\mathcal{F}$ is the (abelian) sheaf associated to the presheaf $g_{p!}\mathcal{F}$ which is defined by the rule

$$V \longmapsto \mathrm{colim}_{V \to u(U')} \mathcal{F}(U')$$

with colimit taken in Ab. If $V = u(U)$, then, as u is fully faithful this colimit is over $U \to U'$. Hence we conclude that $g_{p!}\mathcal{F}(u(U) = \mathcal{F}(U)$. Since u is cocontinuous and continuous any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(U_i) \to u(U)\}$ of $\mathcal{D}$ where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. This implies that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ also, since in the colimit defining the value of $(g_{p!}\mathcal{F})^+$ on $u(U)$ we may restrict to the cofinal system of coverings $\{u(U_i) \to u(U)\}$ as above. Hence we see that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

04BI **Remark 18.16.5.** In general the functor $g_!$ cannot be extended to categories of modules in case g is (part of) a morphism of ringed topoi. Namely, given any ring map $A \to B$ the functor $M \mapsto B \otimes_A M$ has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that $A \to B$ is flat). We will see in Section 18.19 below that it is possible to define $j_!$ on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 18.41 below.

08P3 **Lemma 18.16.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$.*

(1) *If u has a left adjoint w, then $g_!$ agrees with $g_!^{Sh}$ on underlying sheaves of sets and $g_!$ is exact.*
(2) *If in addition w is cocontinuous, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. This Lemma is the analogue of Sites, Lemma 7.23.1. From Sites, Lemma 7.19.3 we see that the categories $\mathcal{I}_V^u$ have an initial object. Thus the underlying set of a colimit of a system of abelian groups over $(\mathcal{I}_V^u)^{opp}$ is the colimit of the underlying sets. Whence the agreement of $g_!^{Sh}$ and $g_!$ by our construction of $g_!$ in Definition 18.16.1. The exactness and (2) follow immediately from the corresponding statements of Sites, Lemma 7.23.1. □

18.17. Global types of modules

03DD

03DE **Definition 18.17.1.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules.

(1) We say $\mathcal{F}$ is a *free $\mathcal{O}$-module* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i\in I} \mathcal{O}$.
(2) We say $\mathcal{F}$ is *finite free* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i\in I} \mathcal{O}$ with a finite index set I.
(3) We say $\mathcal{F}$ is *generated by global sections* if there exists a surjection
$$\bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F}$$
from a free $\mathcal{O}$-module onto $\mathcal{F}$.
(4) Given $r \geq 0$ we say $\mathcal{F}$ is *generated by r global sections* if there exists a surjection $\mathcal{O}^{\oplus r} \to \mathcal{F}$.
(5) We say $\mathcal{F}$ is *generated by finitely many global sections* if it is generated by r global sections for some $r \geq 0$.
(6) We say $\mathcal{F}$ has a *global presentation* if there exists an exact sequence
$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules.
(7) We say $\mathcal{F}$ has a *global finite presentation* if there exists an exact sequence
$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules with I and J finite sets.

Note that for any set I the direct sum $\bigoplus_{i\in I} \mathcal{O}$ exists (Lemma 18.14.2) and is the sheafification of the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{O}(U)$. This module is called the *free $\mathcal{O}$-module on the set I*.

03DF **Lemma 18.17.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is free then $f^*\mathcal{F}$ is free.*
(2) *If $\mathcal{F}$ is finite free then $f^*\mathcal{F}$ is finite free.*
(3) *If $\mathcal{F}$ is generated by global sections then $f^*\mathcal{F}$ is generated by global sections.*
(4) *Given $r \geq 0$ if $\mathcal{F}$ is generated by r global sections, then $f^*\mathcal{F}$ is generated by r global sections.*
(5) *If $\mathcal{F}$ is generated by finitely many global sections then $f^*\mathcal{F}$ is generated by finitely many global sections.*
(6) *If $\mathcal{F}$ has a global presentation then $f^*\mathcal{F}$ has a global presentation.*
(7) *If $\mathcal{F}$ has a finite global presentation then $f^*\mathcal{F}$ has a finite global presentation.*

Proof. This is true because f^* commutes with arbitrary colimits (Lemma 18.14.3) and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$. □

18.18. Intrinsic properties of modules

03DG Let $\mathcal{P}$ be a property of sheaves of modules on ringed topoi. We say $\mathcal{P}$ is an *intrinsic property* if we have $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$ whenever $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free $\mathcal{O}$-module on the set I is characterized by the property that

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\bigoplus\nolimits_{i\in I} \mathcal{O}, \mathcal{F}) = \prod\nolimits_{i\in I} \mathrm{Mor}_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable $\mathcal{F}$ in $Mod(\mathcal{O})$. Alternatively, we can also use Lemma 18.17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition 18.17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of $\mathcal{O}$-modules?

The upshot of Lemma 18.7.2 is the following: Suppose you want to define an intrinsic property $\mathcal{P}$ of an $\mathcal{O}$-module on a topos. Then you can proceed as follows:

(1) Given any site $\mathcal{C}$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ and any $\mathcal{O}$-module $\mathcal{F}$ define the corresponding property $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$.
(2) For any pair of sites $\mathcal{C}$, $\mathcal{C}'$, any special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ any $\mathcal{O}$-module $\mathcal{F}$, show that

$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$

where $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is the equivalence of topoi associated to u.

In this case, given any ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ and any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we simply say that $\mathcal{F}$ has property $\mathcal{P}$ if $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ is true. And Lemma 18.7.2 combined with (2) above guarantees that this is well defined.

Moreover, the same Lemma 18.7.2 also guarantees that if in addition

(3) For any morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Sites, Proposition 7.14.7, and any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\mathcal{P}(\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathcal{F}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_\mathcal{C}, f^*\mathcal{F})$$

then it is true that $\mathcal{P}$ is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, locally generated by r sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

18.19. Localization of ringed sites

03DH Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We explain the counterparts of the results in Sites, Section 7.25 in this setting.

Denote $\mathcal{O}_U = j_U^{-1}\mathcal{O}$ the restriction of $\mathcal{O}$ to the site $\mathcal{C}/U$. It is described by the simple rule $\mathcal{O}_U(V/U) = \mathcal{O}(V)$. With this notation the localization morphism j_U becomes a morphism of ringed topoi

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take $j_U^\sharp : j_U^{-1}\mathcal{O} \to \mathcal{O}_U$ the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

04IX **Definition 18.19.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The ringed site $(\mathcal{C}/U, \mathcal{O}_U)$ is called the *localization of the ringed site* $(\mathcal{C}, \mathcal{O})$ *at the object* U.
(2) The morphism of ringed topoi $(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ is called the *localization morphism*.
(3) The functor $j_{U*} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is called the *direct image functor*.
(4) For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^*\mathcal{F}$ is called the *restriction of* $\mathcal{F}$ *to* $\mathcal{C}/U$. We will sometimes denote it by $\mathcal{F}|_{\mathcal{C}/U}$ or even $\mathcal{F}|_U$. It is described by the simple rule $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$.
(5) The left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ of restriction is called *extension by zero*. It exists and is exact by Lemmas 18.19.2 and 18.19.3.

As in the topological case, see Sheaves, Section 6.31, the extension by zero $j_{U!}$ functor is different from extension by the empty set $j_{U!}$ defined on sheaves of sets. Here is the lemma defining extension by zero.

03DI **Lemma 18.19.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The restriction functor $j_U^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_U)$ has a left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$. So*

$$\mathrm{Mor}_{Mod(\mathcal{O}_U)}(\mathcal{G}, j_U^*\mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(j_{U!}\mathcal{G}, \mathcal{F})$$

for $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}))$ and $\mathcal{G} \in \mathrm{Ob}(Mod(\mathcal{O}_U))$. Moreover, the extension by zero $j_{U!}\mathcal{G}$ of $\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious $\mathcal{O}$-module structure.

Proof. The $\mathcal{O}$-module structure on the presheaf is defined as follows. If $f \in \mathcal{O}(V)$ and $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$, then we define $f \cdot s = fs$ where $f \in \mathcal{O}_U(\varphi : V \to U) = \mathcal{O}(V)$ (because $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ to $\mathcal{C}/U$).

Similarly, let $\alpha : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. In this case we can define a map from the presheaf of the lemma into $\mathcal{F}$ by mapping

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ maps to $\alpha(s) \in \mathcal{F}(V)$. It is clear that this is $\mathcal{O}$-linear, and hence induces a morphism of $\mathcal{O}$-modules $\alpha' : j_{U!}\mathcal{G} \to \mathcal{F}$ by the properties of sheafification of modules (Lemma 18.11.1).

Conversely, let $\beta : j_{U!}\mathcal{G} \to \mathcal{F}$ by a map of $\mathcal{O}$-modules. Recall from Sites, Section 7.25 that there exists an extension by the empty set $j_{U!}^{Sh} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ on sheaves of sets which is left adjoint to j_U^{-1}. Moreover, $j_{U!}^{Sh}\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map $j_{U!}^{Sh}\mathcal{G} \to j_{U!}\mathcal{G}$ of sheaves of sets. Hence precomposing β by this map we get a map of sheaves of sets $j_{U!}^{Sh}\mathcal{G} \to \mathcal{F}$ which by adjunction corresponds to a map of sheaves of sets $\beta' : \mathcal{G} \to \mathcal{F}|_U$. We claim that β' is $\mathcal{O}_U$-linear. Namely, suppose that $\varphi : V \to U$ is an object of $\mathcal{C}/U$ and that $s, s' \in \mathcal{G}(\varphi : V \to U)$, and $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \to U)$. Then by the discussion above we see that $\beta'(s + s')$, resp. $\beta'(fs)$ in $\mathcal{F}|_U(\varphi : V \to U)$ correspond to $\beta(s + s')$, resp. $\beta(fs)$ in $\mathcal{F}(V)$. Since β is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$ are mutually inverse. We omit the verifications. □

Note that we have in the situation of Definition 18.19.1 we have

0G1V (18.19.2.1) $$\mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = \mathcal{F}(U)$$

for every $\mathcal{O}$-module $\mathcal{F}$. Namely, the first equality holds by the adjointness of $j_{U!}$ and j_U^* and the second because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = j_U^*\mathcal{F}(U/U) = \mathcal{F}|_U(U/U) = \mathcal{F}(U)$.

03DJ **Lemma 18.19.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is exact.*

Proof. Since $j_{U!}$ is a left adjoint to j_U^* we see that it is right exact (see Categories, Lemma 4.24.6 and Homology, Section 12.7). Hence it suffices to show that if $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of $\mathcal{O}_U$-modules, then $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective. The map on sections of presheaves over an object V (as in Lemma 18.19.2) is the map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 18.11.2 we conclude $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective and we win. □

0E8G **Lemma 18.19.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. A complex of $\mathcal{O}_U$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2 \to j_{U!}\mathcal{G}_3$ is exact as a sequence of $\mathcal{O}$-modules.*

Proof. We already know that $j_{U!}$ is exact, see Lemma 18.19.3. Thus it suffices to show that $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ reflects injections and surjections.

For every $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ we have the unit $\mathcal{G} \to j_U^* j_{U!}\mathcal{G}$ of the adjunction. We claim this map is an injection of sheaves. Namely, looking at the construction of Lemma 18.19.2 we see that this map is the sheafification of the rule sending the object V/U of $\mathcal{C}/U$ to the injective map

$$\mathcal{G}(V/U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

given by the inclusion of the summand corresponding to the structure morphism $V \to U$. Since sheafification is exact the claim follows. Some details omitted.

If $\mathcal{G} \to \mathcal{G}'$ is a map of $\mathcal{O}_U$-modules with $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ injective, then $j_U^* j_{U!}\mathcal{G} \to j_U^* j_{U!}\mathcal{G}'$ is injective (restriction is exact), hence $\mathcal{G} \to j_U^* j_{U!}\mathcal{G}'$ is injective, hence $\mathcal{G} \to \mathcal{G}'$ is injective. We conclude that $j_{U!}$ reflects injections.

Let $a : \mathcal{G} \to \mathcal{G}'$ be a map of $\mathcal{O}_U$-modules such that $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ is surjective. Let $\mathcal{H}$ be the cokernel of a. Then $j_{U!}\mathcal{H} = 0$ as $j_{U!}$ is exact. By the above the map $\mathcal{H} \to j_U^* j_{U!}\mathcal{H}$ is injective. Hence $\mathcal{H} = 0$ as desired. □

04IY **Lemma 18.19.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}/V), \mathcal{O}_V) & \xrightarrow[(j,j^\sharp)]{} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ & {\scriptstyle (j_V, j_V^\sharp)} \searrow \quad \swarrow {\scriptstyle (j_U, j_U^\sharp)} & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

of ringed topoi. Here $(j, j^\sharp)$ is the localization morphism associated to the object V/U of the ringed site $(\mathcal{C}/V, \mathcal{O}_V)$.

Proof. The only thing to check is that $j_V^\sharp = j^\sharp \circ j^{-1}(j_U^\sharp)$, since everything else follows directly from Sites, Lemma 7.25.8 and Sites, Equation (7.25.8.1). We omit the verification of the equality. □

08P4 **Remark 18.19.6.** In the situation of Lemma 18.19.2 the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_U) & \xrightarrow[j_{U!}]{} & Mod(\mathcal{O}_\mathcal{C}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ Ab(\mathcal{C}/U) & \xrightarrow{j_{U!}^{Ab}} & Ab(\mathcal{C}) \end{array}$$

commutes. This is clear from the explicit description of the functor $j_{U!}$ in the lemma.

03EJ **Remark 18.19.7.** Localization and presheaves of modules; see Sites, Remark 7.25.10. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^*, j_{U*} and $j_{U!}$ have not been defined for presheaves of $\mathcal{O}$-modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Sites, Examples 7.6.6). Hence we also obtain a functor

$$j_U^* : PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : PMod(\mathcal{O}_U) \longrightarrow PMod(\mathcal{O})$$

which are right, left adjoint to j_U^*. Inspecting the proof of Lemma 18.19.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ is exact (by Lemma 18.19.3 in the case of the discrete topologies). Moreover, if $\mathcal{C}$ is actually a site, and $\mathcal{O}$ is actually a sheaf of rings, then the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_U) & \xrightarrow[j_{U!}]{} & Mod(\mathcal{O}) \\ {\scriptstyle forget}\downarrow & & \uparrow{\scriptstyle (\ )^\#} \\ PMod(\mathcal{O}_U) & \xrightarrow{j_{U!}} & PMod(\mathcal{O}) \end{array}$$

commutes.

0F6Z **Lemma 18.19.8.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. This is a special case of Lemma 18.16.4 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. □

18.20. Localization of morphisms of ringed sites

04IZ This section is the analogue of Sites, Section 7.28.

04J0 **Lemma 18.20.1.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and set $U = u(V)$. Then there is a canonical map of sheaves of rings $(f')^\sharp$ such that the diagram of Sites, Lemma 7.28.1 is turned into a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow[(j_U, j_U^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ {\scriptstyle (f', (f')^\sharp)}\downarrow & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

Moreover, in this situation we have $f'_* j_U^{-1} = j_V^{-1} f_*$ and $f'_* j_U^* = j_V^* f_*$.

Proof. Just take $(f')^\sharp$ to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1} j_V^{-1}\mathcal{O}' = j_U^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_U^{-1} f^\sharp} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 7.28.1. (Note that $j^{-1} = j^*$ on sheaves of modules if j is a localization morphism, hence the first equality of functors implies the second.) □

04J1 **Lemma 18.20.2.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow[(j_U, j_U^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ {\scriptstyle (f_c, f_c^\sharp)}\downarrow & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

The morphism $(f_c, f_c^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \longrightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 18.20.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 18.19.5. Given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{array}{ccc} U' & \xrightarrow[c']{} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) & \xrightarrow{(j_{U'/U}, j^\sharp_{U'/U})} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ \downarrow{\scriptstyle (f_{c'}, f^\sharp_{c'})} & & \downarrow{\scriptstyle (f_c, f^\sharp_c)} \\ (Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) & \xrightarrow{(j_{V'/V}, j^\sharp_{V'/V})} & (Sh(\mathcal{D}/V), \mathcal{O}'_{V'}) \end{array}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.28.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 18.19.5 and 18.20.1 exactly as in the proof of Sites, Lemma 7.28.3. □

18.21. Localization of ringed topoi

04ID This section is the analogue of Sites, Section 7.30 in the setting of ringed topoi.

04IE **Lemma 18.21.1.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$ be a sheaf. For a sheaf $\mathcal{H}$ on $\mathcal{C}$ denote $\mathcal{H}_\mathcal{F}$ the sheaf $\mathcal{H} \times \mathcal{F}$ seen as an object of the category $Sh(\mathcal{C})/\mathcal{F}$. The pair $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a ringed topos and there is a canonical morphism of ringed topoi*

$$(j_\mathcal{F}, j^\sharp_\mathcal{F}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 18.19 such that

1. *the functor $j_\mathcal{F}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
2. *the functor $j_{\mathcal{F}*}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
3. *the functor $j_{\mathcal{F}!}$ on sheaves of sets is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$,*
4. *the functor $j_{\mathcal{F}!}$ on sheaves of modules associates to the $\mathcal{O}_\mathcal{F}$-module $\varphi : \mathcal{G} \to \mathcal{F}$ the $\mathcal{O}$-module which is the sheafification of the presheaf*

$$V \longmapsto \bigoplus\nolimits_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$

for $V \in \mathrm{Ob}(\mathcal{C})$.

Proof. By Sites, Lemma 7.30.1 we see that $Sh(\mathcal{C})/\mathcal{F}$ is a topos and that (1) and (3) are true. In particular this shows that $j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}$ and shows that $\mathcal{O}_\mathcal{F}$ is a sheaf of rings. Thus we may choose the map $j^\sharp_\mathcal{F}$ to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 7.30.1 shows that we may assume $\mathcal{C}$ is a site with all finite limits and a subcanonical topology and that $\mathcal{F} = h_U$ for some object U of $\mathcal{C}$. Then (4) follows from the description of $j_{\mathcal{F}!}$ in Lemma 18.19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to $j^*_\mathcal{F}$. □

04J2 **Definition 18.21.2.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$.

1. The ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is called the *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$*.
2. The morphism of ringed topoi $(j_\mathcal{F}, j^\sharp_\mathcal{F}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$ of Lemma 18.21.1 is called the *localization morphism*.

We continue the tradition, established in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

04J3 **Lemma 18.21.3.** *With $(Sh(\mathcal{C}), \mathcal{O})$ and $\mathcal{F} \in Sh(\mathcal{C})$ as in Lemma 18.21.1. If $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$ then via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Sites, Lemma 7.25.4 we have*

(1) *canonically $\mathcal{O}_U = \mathcal{O}_\mathcal{F}$, and*
(2) *with these identifications we have $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$.*

Proof. The assertion for underlying topoi is Sites, Lemma 7.30.5. Note that $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ which by Sites, Lemma 7.25.7 corresponds to $\mathcal{O} \times h_U^\#$ under the equivalence of Sites, Lemma 7.25.4. By definition of $\mathcal{O}_\mathcal{F}$ we get (1). What's left is to prove that $j_\mathcal{F}^\sharp = j_U^\sharp$ under this identification. We omit the verification. □

Localization is functorial in the following two ways: We can "relocalize" a localization (see Lemma 18.21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of ringed topoi (see Lemma 18.22.1).

04J4 **Lemma 18.21.4.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) & \xrightarrow[(j,j^\sharp)]{} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \\ & {\scriptstyle (j_\mathcal{G}, j_\mathcal{G}^\sharp)} \searrow \quad \swarrow {\scriptstyle (j_\mathcal{F}, j_\mathcal{F}^\sharp)} & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

where $(j, j^\sharp)$ is the localization morphism of the ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ at the object $\mathcal{G}/\mathcal{F}$.

Proof. All assertions follow from Sites, Lemma 7.30.6 except the assertion that $j_\mathcal{G}^\sharp = j^\sharp \circ j^{-1}(j_\mathcal{F}^\sharp)$. We omit the verification. □

04J5 **Lemma 18.21.5.** *With $(Sh(\mathcal{C}), \mathcal{O})$, $s : \mathcal{G} \to \mathcal{F}$ as in Lemma 18.21.4. If there exist a morphism $f : V \to U$ of $\mathcal{C}$ such that $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and s is induced by f, then the diagrams of Lemma 18.19.5 and Lemma 18.21.4 agree via the identifications $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$ and $(j_\mathcal{G}, j_\mathcal{G}^\sharp) = (j_V, j_V^\sharp)$ of Lemma 18.21.3.*

Proof. All assertions follow from Sites, Lemma 7.30.7 except for the assertion that the two maps $j^\sharp$ agree. This holds since in both cases the map $j^\sharp$ is simply the identity. Some details omitted. □

18.22. Localization of morphisms of ringed topoi

04J6 This section is the analogue of Sites, Section 7.31.

04IF **Lemma 18.22.1.** *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) & \xrightarrow[(j_\mathcal{F}, j_\mathcal{F}^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle (f', (f')^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) & \xrightarrow{(j_\mathcal{G}, j_\mathcal{G}^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

We have $f'_* j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1} f_*$ and $f'_* j_\mathcal{F}^* = j_\mathcal{G}^* f_*$. Moreover, the morphism f' is characterized by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

Proof. By Sites, Lemma 7.31.1 we have the diagram of underlying topoi, the equality $f'_* j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1} f_*$, and the description of $(f')^{-1}$. To define $(f')^\sharp$ we use the map

$$(f')^\sharp : \mathcal{O}'_\mathcal{G} = j_\mathcal{G}^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{G}^{-1} f^\sharp} j_\mathcal{G}^{-1} f_* \mathcal{O} = f'_* j_\mathcal{F}^{-1}\mathcal{O} = f'_* \mathcal{O}_\mathcal{F}$$

or equivalently the map

$$(f')^\sharp : (f')^{-1}\mathcal{O}'_\mathcal{G} = (f')^{-1} j_\mathcal{G}^{-1}\mathcal{O}' = j_\mathcal{F}^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{F}^{-1} f^\sharp} j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}.$$

We omit the verification that these two maps are indeed adjoint to each other. The second construction of $(f')^\sharp$ shows that the diagram commutes in the 2-category of ringed topoi (as the maps $j_\mathcal{F}^\sharp$ and $j_\mathcal{G}^\sharp$ are identities). Finally, the equality $f'_* j_\mathcal{F}^* = j_\mathcal{G}^* f_*$ follows from the equality $f'_* j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1} f_*$ and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 18.21.1. □

04J7 **Lemma 18.22.2.** *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, then the commutative diagrams of Lemma 18.20.1 and Lemma 18.22.1 agree via the identifications of Lemma 18.21.3.

Proof. At the level of morphisms of topoi this is Sites, Lemma 7.31.2. This works also on the level of morphisms of ringed topoi since the formulas defining $(f')^\sharp$ in the proofs of Lemma 18.20.1 and Lemma 18.22.1 agree. □

04J8 **Lemma 18.22.3.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) & \xrightarrow[(j_\mathcal{F}, j_\mathcal{F}^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle (f_c, f_c^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) & \xrightarrow{(j_\mathcal{G}, j_\mathcal{G}^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

The morphism $(f_s, f_s^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 18.22.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 18.21.4. Given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$, and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{s'} & f^{-1}\mathcal{G}' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f^{-1}b} \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) & \xrightarrow{(j_{\mathcal{F}'/\mathcal{F}}, j^\sharp_{\mathcal{F}'/\mathcal{F}})} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \\ {\scriptstyle (f_{s'}, f^\sharp_{s'})}\downarrow & & \downarrow{\scriptstyle (f_s, f^\sharp_s)} \\ (Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) & \xrightarrow{(j_{\mathcal{G}'/\mathcal{G}}, j^\sharp_{\mathcal{G}'/\mathcal{G}})} & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}'}) \end{array}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.31.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 18.21.4 and 18.22.1. □

04J9 **Lemma 18.22.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$, $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be as in Lemma 18.22.3. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h^\#_V$, $\mathcal{F} = h^\#_U$ and s comes from a morphism $c : U \to u(V)$, then the commutative diagrams of Lemma 18.20.2 and Lemma 18.22.3 agree via the identifications of Lemma 18.21.3.*

Proof. This is formal using Lemmas 18.21.5 and 18.22.2. □

18.23. Local types of modules

03DK According to our general strategy explained in Section 18.18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

03DL **Definition 18.23.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We will freely use the notions defined in Definition 18.17.1.

(1) We say $\mathcal{F}$ is *locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a free $\mathcal{O}_{U_i}$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a finite free $\mathcal{O}_{U_i}$-module.
(3) We say $\mathcal{F}$ is *locally generated by sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by global sections.
(4) Given $r \geq 0$ we sat $\mathcal{F}$ is *locally generated by r sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by r global sections.
(5) We say $\mathcal{F}$ is *of finite type* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by finitely many global sections.

(6) We say $\mathcal{F}$ is *quasi-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation.
(7) We say $\mathcal{F}$ is *of finite presentation* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a finite global presentation.
(8) We say $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type, and for every object U of $\mathcal{C}$ and any $s_1, \ldots, s_n \in \mathcal{F}(U)$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $(\mathcal{C}/U, \mathcal{O}_U)$.

03DM **Lemma 18.23.2.** *Any of the properties (1) – (8) of Definition 18.23.1 is intrinsic (see discussion in Section 18.18).*

Proof. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the equivalence of topoi associated to u. Set $\mathcal{O}' = g_*\mathcal{O}$, and let $g^\sharp : \mathcal{O}' \to g_*\mathcal{O}$ be the identity. Finally, set $\mathcal{F}' = g_*\mathcal{F}$. Let $\mathcal{P}_l$ be one of the properties (1) – (7) listed in Definition 18.23.1. (We will discuss the coherent case at the end of the proof.) Let $\mathcal{P}_g$ denote the corresponding property listed in Definition 18.17.1. We have already seen that $\mathcal{P}_g$ is intrinsic. We have to show that $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$ holds if and only if $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$ holds.

Assume that $\mathcal{F}$ has $\mathcal{P}_l$. Let V be an object of $\mathcal{D}$. One of the properties of a special cocontinuous functor is that there exists a covering $\{u(U_i) \to V\}_{i\in I}$ in the site $\mathcal{D}$. By assumption, for each i there exists a covering $\{U_{ij} \to U_i\}_{j\in J_i}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{U_{ij}}$ is $\mathcal{P}_g$. By Sites, Lemma 7.29.3 we have commutative diagrams of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow & & \downarrow \\ (Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

where the vertical arrows are equivalences. Hence we conclude that $\mathcal{F}'|_{u(U_{ij})}$ has property $\mathcal{P}_g$ also. And moreover, $\{u(U_{ij}) \to V\}_{i\in I, j\in J_i}$ is a covering of the site $\mathcal{D}$. Hence $\mathcal{F}'$ has property $\mathcal{P}_l$.

Assume that $\mathcal{F}'$ has $\mathcal{P}_l$. Let U be an object of $\mathcal{C}$. By assumption, there exists a covering $\{V_i \to u(U)\}_{i\in I}$ such that $\mathcal{F}'|_{V_i}$ has property $\mathcal{P}_g$. Because u is cocontinuous we can refine this covering by a family $\{u(U_j) \to u(U)\}_{j\in J}$ where $\{U_j \to U\}_{j\in J}$ is a covering in $\mathcal{C}$. Say the refinement is given by $\alpha : J \to I$ and $u(U_j) \to V_{\alpha(j)}$. Restricting is transitive, i.e., $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$. Hence by Lemma 18.17.2 we see that $\mathcal{F}'|_{u(U_j)}$ has property $\mathcal{P}_g$. Hence the diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow & & \downarrow \\ (Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

where the vertical arrows are equivalences shows that $\mathcal{F}|_{U_j}$ has property $\mathcal{P}_g$ also. Thus $\mathcal{F}$ has property $\mathcal{P}_l$ as desired.

Finally, we prove the lemma in case $\mathcal{P}_l = coherent$[2]. Assume $\mathcal{F}$ is coherent. This implies that $\mathcal{F}$ is of finite type and hence $\mathcal{F}'$ is of finite type also by the first part of the proof. Let V be an object of $\mathcal{D}$ and let $s_1, \ldots, s_n \in \mathcal{F}'(V)$. We have to show that the kernel $\mathcal{K}'$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}'|_V$ is of finite type on $\mathcal{D}/V$. This means we have to show that for any V'/V there exists a covering $\{V'_i \to V'\}$ such that $\mathcal{F}'|_{V'_i}$ is generated by finitely many sections. Replacing V by V' (and restricting the sections s_j to V') we reduce to the case where $V' = V$. Since u is a special cocontinuous functor, there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. Using the isomorphism of topoi $Sh(\mathcal{C}/U_i) = Sh(\mathcal{D}/u(U_i))$ we see that $\mathcal{K}'|_{u(U_i)}$ corresponds to the kernel $\mathcal{K}_i$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$. Since $\mathcal{F}$ is coherent we see that $\mathcal{K}_i$ is of finite type. Hence we conclude (by the first part of the proof again) that $\mathcal{K}|_{u(U_i)}$ is of finite type. Thus there exist coverings $\{V_{il} \to u(U_i)\}$ such that $\mathcal{K}|_{V_{il}}$ is generated by finitely many global sections. Since $\{V_{il} \to V\}$ is a covering of $\mathcal{D}$ we conclude that $\mathcal{K}$ is of finite type as desired.

Assume $\mathcal{F}'$ is coherent. This implies that $\mathcal{F}'$ is of finite type and hence $\mathcal{F}$ is of finite type also by the first part of the proof. Let U be an object of $\mathcal{C}$, and let $s_1, \ldots, s_n \in \mathcal{F}(U)$. We have to show that the kernel $\mathcal{K}$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $\mathcal{C}/U$. Using the isomorphism of topoi $Sh(\mathcal{C}/U) = Sh(\mathcal{D}/u(U))$ we see that $\mathcal{K}|_U$ corresponds to the kernel $\mathcal{K}'$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{u(U)} \to \mathcal{F}'|_{u(U)}$. As $\mathcal{F}'$ is coherent, we see that $\mathcal{K}'$ is of finite type. Hence, by the first part of the proof again, we conclude that $\mathcal{K}$ is of finite type. □

Hence from now on we may refer to the properties of $\mathcal{O}$-modules defined in Definition 18.23.1 without specifying a site.

03DN **Lemma 18.23.3.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that the site $\mathcal{C}$ has a final object X. Then*

1. *The following are equivalent*
   (a) *$\mathcal{F}$ is locally free,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a locally free $\mathcal{O}_{X_i}$-module, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a free $\mathcal{O}_{X_i}$-module.*
2. *The following are equivalent*
   (a) *$\mathcal{F}$ is finite locally free,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite locally free $\mathcal{O}_{X_i}$-module, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite free $\mathcal{O}_{X_i}$-module.*
3. *The following are equivalent*
   (a) *$\mathcal{F}$ is locally generated by sections,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by sections, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by sections.*
4. *Given $r \geq 0$, the following are equivalent*

[2]The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

(a) *$\mathcal{F}$ is locally generated by r sections,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by r sections, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by r sections.*

(5) *The following are equivalent*
(a) *$\mathcal{F}$ is of finite type,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite type, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by finitely many sections.*

(6) *The following are equivalent*
(a) *$\mathcal{F}$ is quasi-coherent,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module which has a global presentation.*

(7) *The following are equivalent*
(a) *$\mathcal{F}$ is of finite presentation,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite presentation, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module has a finite global presentation.*

(8) *The following are equivalent*
(a) *$\mathcal{F}$ is coherent, and*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a coherent $\mathcal{O}_{X_i}$-module.*

Proof. In each case we have (a) $\Rightarrow$ (*b*). In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 7.6.2) and the definition of the local types of modules. Suppose $\{X_i \to X\}$ is a covering. Then for every object U of $\mathcal{C}$ we get an induced covering $\{X_i \times_X U \to U\}$. Moreover, the global property for $\mathcal{F}|_{\mathcal{C}/X_i}$ in part (c) implies the corresponding global property for $\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$ by Lemma 18.17.2, hence the sheaf has property (a) by definition. We omit the proof of (b) $\Rightarrow$ (a) in case (7). □

03DO **Lemma 18.23.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is locally free then $f^*\mathcal{F}$ is locally free.*
(2) *If $\mathcal{F}$ is finite locally free then $f^*\mathcal{F}$ is finite locally free.*
(3) *If $\mathcal{F}$ is locally generated by sections then $f^*\mathcal{F}$ is locally generated by sections.*
(4) *If $\mathcal{F}$ is locally generated by r sections then $f^*\mathcal{F}$ is locally generated by r sections.*
(5) *If $\mathcal{F}$ is of finite type then $f^*\mathcal{F}$ is of finite type.*
(6) *If $\mathcal{F}$ is quasi-coherent then $f^*\mathcal{F}$ is quasi-coherent.*
(7) *If $\mathcal{F}$ is of finite presentation then $f^*\mathcal{F}$ is of finite presentation.*

Proof. According to the discussion in Section 18.18 we need only check preservation under pullback for a morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$

such that f is given by a left exact, continuous functor $u : \mathcal{D} \to \mathcal{C}$ between sites which have all finite limits. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules which has one of the properties (1) – (6) of Definition 18.23.1. We know $\mathcal{D}$ has a final object Y and $X = u(Y)$ is a final object for $\mathcal{C}$. By assumption we have a covering $\{Y_i \to Y\}$ such that $\mathcal{G}|_{\mathcal{D}/Y_i}$ has the corresponding global property. Set $X_i = u(Y_i)$ so that $\{X_i \to X\}$ is a covering in $\mathcal{C}$. We get a commutative diagram of morphisms ringed sites

$$\begin{array}{ccc} (\mathcal{C}/X_i, \mathcal{O}_\mathcal{C}|_{X_i}) & \longrightarrow & (\mathcal{C}, \mathcal{O}_\mathcal{C}) \\ \downarrow & & \downarrow \\ (\mathcal{D}/Y_i, \mathcal{O}_\mathcal{D}|_{Y_i}) & \longrightarrow & (\mathcal{D}, \mathcal{O}_\mathcal{D}) \end{array}$$

by Sites, Lemma 7.28.2. Hence by Lemma 18.17.2 that $f^*\mathcal{G}|_{X_i}$ has the corresponding global property. Hence we conclude that $\mathcal{G}$ has the local property we started out with by Lemma 18.23.3. □

18.24. Basic results on local types of modules

082S Basic lemmas related to the definitions made above.

0H98 **Lemma 18.24.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}$-module. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of $\mathcal{O}$-modules. If $\mathcal{G}$ is finite type, then* $\mathrm{Coker}(\varphi)$ *is finitely presented.*

Proof. Omitted. Hint: See Modules, Lemma 17.11.3. □

082T **Lemma 18.24.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\theta : \mathcal{G} \to \mathcal{F}$ be a surjective $\mathcal{O}$-module map with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ of finite type. Then* $\mathrm{Ker}(\theta)$ *is of finite type.*

Proof. Omitted. Hint: See Modules, Lemma 17.11.4. □

0GZN **Lemma 18.24.3.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 7.6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if for all $U \to V$ in $\mathcal{C}$ the canonical map*

$$\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{F}(U)$$

is an isomorphism.

Proof. Assume $\mathcal{F}$ is quasi-coherent and let $U \to V$ be a morphism of $\mathcal{C}$. Since every covering of V is given by an isomorphism we conclude from Definition 18.23.1 that there exists a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

Since the topology on $\mathcal{C}$ is chaotic, taking sections over any object of $\mathcal{C}$ is exact. We conclude that we obtain a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module and similarly for $\mathcal{F}(U)$. This easily shows that the displayed map in the statement of the lemma is an isomorphism.

Assume the displayed map in the statement of the lemma is an isomorphism for every morphism $U \to V$ in $\mathcal{C}$. Fix V and choose a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module. Then the assumption on $\mathcal{F}$ exactly means that the corresponding sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

is exact and we conclude that $\mathcal{F}$ is quasi-coherent. □

0GZP **Lemma 18.24.4.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 7.6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then the category of quasi-coherent $\mathcal{O}$-modules is a weak Serre subcategory of $Mod(\mathcal{O})$.*

Proof. We will check the definition of a weak Serre subcategory, see Homology, Definition 12.10.1. To do this we will use the characterization of quasi-coherent modules given in Lemma 18.24.3. Consider an exact sequence

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_4$$

in $Mod(\mathcal{O})$ with $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_3$, and $\mathcal{F}_4$ quasi-coherent. Let $U \to V$ be a morphism of $\mathcal{C}$ and consider the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}_0(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) & \longrightarrow & \mathcal{F}_1(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) & \longrightarrow & \mathcal{F}_2(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) & \longrightarrow & \mathcal{F}_3(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{F}_4(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_0(U) & \longrightarrow & \mathcal{F}_1(U) & \longrightarrow & \mathcal{F}_2(U) & \longrightarrow & \mathcal{F}_3(U) \longrightarrow
\end{array}$$

By assumption the vertical arrows with indices 0, 1, 3, 4 are isomorphisms. Since the topology on $\mathcal{C}$ is chaotic taking sections over an object of $\mathcal{C}$ is exact and hence the lower row is exact. Since $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat also the upper row is exact. Thus we conclude that the middle arrow is an isomorphism by the 5 lemma (Homology, Lemma 12.5.20). □

18.25. Closed immersions of ringed topoi

08M2 When do we declare a morphism of ringed topoi $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ to be a closed immersion? By analogy with the discussion in Modules, Section 17.13 it seems natural to assume at least:

(1) The functor i is a closed immersion of topoi (Sites, Definition 7.43.7).
(2) The associated map $\mathcal{O}' \to i_*\mathcal{O}$ is surjective.

These conditions already imply a number of pleasing results which we discuss in this section. However, it seems prudent to not actually define the notion of a closed immersion of ringed topoi as there are many different definitions we could use.

08M3 **Lemma 18.25.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume i is a closed immersion of topoi and $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}'$ the kernel of $i^\sharp$. The functor*

$$i_* : Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}')$$

is exact, fully faithful, with essential image those $\mathcal{O}'$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. By Lemma 18.15.2 and Sites, Lemma 7.43.8 we see that i_* is exact. From the fact that i_* is fully faithful on sheaves of sets, and the fact that $i^\sharp$ is surjective it follows that i_* is fully faithful as a functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$. Namely, suppose that $\alpha : i_*\mathcal{F}_1 \to i_*\mathcal{F}_2$ is an $\mathcal{O}'$-module map. By the fully faithfulness of i_* we obtain a map $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of sets. To prove β is a map of modules we have to show that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{O} \times \mathcal{F}_2 & \longrightarrow & \mathcal{F}_2 \end{array}$$

commutes. It suffices to prove commutativity after applying i_*. Consider

$$\begin{array}{ccccc} \mathcal{O}' \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{F}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}' \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{F}_2 \end{array}$$

We know the outer rectangle commutes. Since $i^\sharp$ is surjective we conclude.

To finish the proof we have to prove the statement on the essential image of i_*. It is clear that $i_*\mathcal{F}$ is annihilated by $\mathcal{I}$ for any $\mathcal{O}$-module $\mathcal{F}$. Conversely, let $\mathcal{G}$ be a $\mathcal{O}'$-module with $\mathcal{I}\mathcal{G} = 0$. By definition of a closed subtopos there exists a subsheaf $\mathcal{U}$ of the final object of $\mathcal{D}$ such that the essential image of i_* on sheaves of sets is the class of sheaves of sets $\mathcal{H}$ such that $\mathcal{H} \times \mathcal{U} \to \mathcal{U}$ is an isomorphism. In particular, $i_*\mathcal{O} \times \mathcal{U} = \mathcal{U}$. This implies that $\mathcal{I} \times \mathcal{U} = \mathcal{O} \times \mathcal{U}$. Hence our module $\mathcal{G}$ satisfies $\mathcal{G} \times \mathcal{U} = \{0\} \times \mathcal{U} = \mathcal{U}$ (because the zero module is isomorphic to the final object of sheaves of sets). Thus there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with $i_*\mathcal{F} = \mathcal{G}$. Since i_* is fully faithful on sheaves of sets, we see that in order to define the addition $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and the multiplication $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ it suffices to use the addition

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

(given to us as $\mathcal{G}$ is a $\mathcal{O}'$-module) and the multiplication

$$i_*\mathcal{O} \times \mathcal{G} \to \mathcal{G}$$

which is given to us as we have the multiplication by $\mathcal{O}'$ which annihilates $\mathcal{I}$ by assumption and $i_*\mathcal{O} = \mathcal{O}'/\mathcal{I}$. By construction $\mathcal{G}$ is isomorphic to the pushforward of the $\mathcal{O}$-module $\mathcal{F}$ so constructed. □

18.26. Tensor product

03EK In Sections 18.9 and 18.11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of $\mathcal{O}$-modules. To be precise, suppose $\mathcal{C}$ is a category, $\mathcal{O}$ is a presheaf of rings, and $\mathcal{F}$, $\mathcal{G}$ are presheaves of $\mathcal{O}$-modules. In this case we define $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}$-modules then we define

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$$

to be the sheaf of $\mathcal{O}$-modules associated to the presheaf $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$.

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

0GMW **Lemma 18.26.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Then $\mathcal{F}^\# \otimes_{\mathcal{O}^\#} \mathcal{G}^\#$ is equal to $(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$.*

Proof. Omitted. Hint: use the characterization of tensor product in terms of bilinear maps below and use the universal property of sheafification. □

Let $\mathcal{C}$ be a site, let $\mathcal{O}$ be a sheaf of rings and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of $\mathcal{O}$-modules. In this case we define

$$\text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$\text{Hom}_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ represents the functor which associates to $\mathcal{H}$ the set of bilinear maps $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 18.18). In fact we have the following.

03EL **Lemma 18.26.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_\mathcal{D}$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. For a sheaf $\mathcal{H}$ of $\mathcal{O}_\mathcal{C}$ modules we have

$$\begin{aligned}
\text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}), \mathcal{H}) &= \text{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{\mathcal{O}_\mathcal{D}}(\mathcal{F} \times \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{f^{-1}\mathcal{O}_\mathcal{D}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{f^{-1}\mathcal{O}_\mathcal{D}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F} \otimes_{f^*\mathcal{O}_\mathcal{D}} f^*\mathcal{G}, \mathcal{H})
\end{aligned}$$

The interesting "=" in this sequence of equalities is the third equality. It follows from the definition and adjointness of f_* and f^{-1} (as discussed in previous sections) in a straightforward manner. □

03L6 **Lemma 18.26.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are finite locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(5) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(7) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is coherent.*

(8) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*

Proof. Omitted. Hint: Compare with Sheaves of Modules, Lemma 17.16.6. □

18.27. Internal Hom

04TT Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\varphi : V \to U$ in $\mathcal{C}$ we define a restriction mapping

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism $j : \mathcal{C}/V \to \mathcal{C}/U$, see Sites, Lemma 7.25.8. Hence this defines a presheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a presheaf of $\mathcal{O}$-modules. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EM **Lemma 18.27.1.** *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules, and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}$-modules.*

Proof. Omitted. Hints: Note first that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^{\#}, \mathcal{G})$, which reduces the question to the case where both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. The result for sheaves of sets is Sites, Lemma 7.26.1. □

0E8H **Lemma 18.27.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules. Then formation of $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ commutes with restriction to U for $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Immediate from the definition. □

0GMX **Remark 18.27.3.** Let $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed sites. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_{\mathcal{D}}$-modules. There is a canonical map

$$f^* \mathcal{H}om_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F}, f^*\mathcal{G})$$

Namely, this map is adjoint to the map

$$\mathcal{H}om_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F}, \mathcal{G}) \longrightarrow f_* \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F}, f^*\mathcal{G})$$

defined as follows. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For sections over $V \in \mathrm{Ob}(\mathcal{D})$ we use the map

$$\begin{aligned} \Gamma(V, \mathcal{H}om_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F}, \mathcal{G})) &= \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{u(V)}}(f^*\mathcal{F}|_{u(V)}, f^*\mathcal{G}|_{u(V)}) \\ &= \Gamma(u(V), \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F}, f^*\mathcal{G})) \\ &= \Gamma(V, f_* \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F}, f^*\mathcal{G})) \end{aligned}$$

where for the arrow we use pullback by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f.

In the situation of Lemma 18.27.1 the "evaluation" morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EN **Lemma 18.27.4.** *Internal hom and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(2) *For any presheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_{\mathcal{O}}(\operatorname{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings.

(3) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(4) *For any sheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_{\mathcal{O}}(\operatorname{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Proof. Let $\mathcal{I} \to PMod(\mathcal{O})$, $i \mapsto \mathcal{G}_i$ be a diagram. Let U be an object of the category $\mathcal{C}$. As j_U^* is both a left and a right adjoint we see that $\lim_i j_U^* \mathcal{G}_i = j_U^* \lim_i \mathcal{G}_i$. Hence we have
$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \lim_i \mathcal{G}_i)(U) &= \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \lim_i \mathcal{G}_i|_U) \\ &= \lim_i \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$
by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have
$$\operatorname{Mor}_{Mod(\mathcal{O})}(\operatorname{colim}_i \mathcal{F}_i, \mathcal{G}) = \operatorname{Mor}_{PMod(\mathcal{O})}(\operatorname{colim}_i^{PSh} \mathcal{F}_i, \mathcal{G})$$
as the colimit $\operatorname{colim}_i \mathcal{F}_i$ is the sheafification of the colimit $\operatorname{colim}_i^{PSh} \mathcal{F}_i$ in $PMod(\mathcal{O})$. Hence (4) follows from (2) (by the remark on limits above again). □

0GMY **Lemma 18.27.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules.*

(1) *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_2, \mathcal{G})$$
is exact.

(2) *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_2)$$
is exact.

Proof. Follows from Lemma 18.27.4 and Homology, Lemma 12.7.2. □

03EO **Lemma 18.27.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular,

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

(2) *Suppose that $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are sheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular,

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

Proof. This is the analogue of Algebra, Lemma 10.12.8. The proof is the same, and is omitted. □

03EP **Lemma 18.27.7.** *Tensor product and colimits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

(2) *Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. This is because tensor product is adjoint to internal hom according to Lemma 18.27.6. See Categories, Lemma 4.24.5. □

0932 **Lemma 18.27.8.** *Let $\mathcal{C}$ be a category, resp. a site Let $\mathcal{O} \to \mathcal{O}'$ be a map of presheaves, resp. sheaves of rings. Then*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}', \mathcal{F}))$$

for any $\mathcal{O}'$-module $\mathcal{G}$ and $\mathcal{O}$-module $\mathcal{F}$.

Proof. This is the analogue of Algebra, Lemma 10.14.4. The proof is the same, and is omitted. □

0E8I **Lemma 18.27.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ and $\mathcal{F}$ in $Mod(\mathcal{O})$ we have $j_{U!}\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} = j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U)$.*

Proof. Let $\mathcal{H}$ be an object of $Mod(\mathcal{O})$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}}(j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U), \mathcal{H}) &= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U, \mathcal{H}|_U) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{H}|_U)) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H})|_U) \\
&= \mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H})) \\
&= \mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}, \mathcal{H})
\end{aligned}$$

The first equality because $j_{U!}$ is a left adjoint to restriction of modules. The second by Lemma 18.27.6. The third by Lemma 18.27.2. The fourth because $j_{U!}$ is a left adjoint to restriction of modules. The fifth by Lemma 18.27.6. The lemma follows from this and the Yoneda lemma. □

0EYY **Remark 18.27.10.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$ and consider the localization morphism $j : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$. See Sites, Definition 7.30.4. We claim that (a) $j_!\mathbf{Z} = \mathbf{Z}^\#_\mathcal{F}$ and (b) $j_!(j^{-1}\mathcal{H}) = j_!\mathbf{Z} \otimes_\mathbf{Z} \mathcal{H}$ for any abelian sheaf $\mathcal{H}$ on $\mathcal{C}$. Let $\mathcal{G}$ be an abelian on $\mathcal{C}$. Part (a) follows from the Yoneda lemma because

$$\mathrm{Hom}(j_!\mathbf{Z}, \mathcal{G}) = \mathrm{Hom}(\mathbf{Z}, j^{-1}\mathcal{G}) = \mathrm{Hom}(\mathbf{Z}^\#_\mathcal{F}, \mathcal{G})$$

where the second equality holds because both sides of the equality evaluate to the set of maps from $\mathcal{F} \to \mathcal{G}$ viewed as an abelian group. For (b) we use the Yoneda lemma and

$$\begin{aligned}
\mathrm{Hom}(j_!(j^{-1}\mathcal{H}), \mathcal{G}) &= \mathrm{Hom}(j^{-1}\mathcal{H}, j^{-1}\mathcal{G}) \\
&= \mathrm{Hom}(\mathbf{Z}, \mathcal{H}om(j^{-1}\mathcal{H}, j^{-1}\mathcal{G})) \\
&= \mathrm{Hom}(\mathbf{Z}, j^{-1}\,\mathcal{H}om(\mathcal{H}, \mathcal{G})) \\
&= \mathrm{Hom}(j_!\mathbf{Z}, \mathcal{H}om(\mathcal{H}, \mathcal{G})) \\
&= \mathrm{Hom}(j_!\mathbf{Z} \otimes_\mathbf{Z} \mathcal{H}, \mathcal{G})
\end{aligned}$$

Here we use adjunction, the fact that taking $\mathcal{H}om$ commutes with localization, and Lemma 18.27.6.

0GMZ **Lemma 18.27.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda\in\Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Then the canonical map*

$$\mathrm{colim}_\lambda \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. It suffices to show the arrow is an isomorphism after restriction to U for all U in $\mathcal{C}$. Both taking colimits of sheaves of modules and taking internal hom commute with restriction to U. See for example Lemmas 18.14.3 and 18.27.2. Fix U. Given a covering $\{U_i \to U\}_{i\in I}$, then it suffices to prove the restriction to each U_i is an isomorphism. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O} \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_\mathcal{O}(-,-)$ commutes with finite direct sums in either variable and $\mathcal{H}om_\mathcal{O}(\mathcal{O}, -)$ is the identity functor. By this and by Lemma 18.27.5 we obtain an exact sequence

$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $Mod(\mathcal{O})$ by Lemma 18.14.2 also the top row in the following commutative diagram is exact

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{colim}_\lambda \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) & \longrightarrow & \mathrm{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda & \longrightarrow & \mathrm{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \bigoplus_{i=1,\ldots,n} \mathcal{G} & \longrightarrow & \bigoplus_{j=1,\ldots,m} \mathcal{G}
\end{array}$$

Since the right two vertical arrows are isomorphisms we conclude. □

0GN0 **Lemma 18.27.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G} = \operatorname{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Then we have*

$$\operatorname{colim}_\lambda \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_\lambda) = \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}).$$

if the hypotheses of Sites, Lemma 7.17.8 part (4) are satisfied for the site $\mathcal{C}$; please see Sites, Remark 7.17.9.

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \operatorname{colim} \mathcal{G}_\lambda)$ and $\mathcal{H}_\lambda = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_\lambda)$. Recall that

$$\operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \Gamma(\mathcal{C}, \mathcal{H}) \quad \text{and} \quad \operatorname{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_\lambda) = \Gamma(\mathcal{C}, \mathcal{H}_\lambda)$$

by construction. By Lemma 18.27.11 we have $\mathcal{H} = \operatorname{colim} \mathcal{H}_\lambda$. Thus the lemma follows from Sites, Lemma 7.17.8. □

18.28. Flat modules

03EQ We can define flat modules exactly as in the case of modules over rings.

03ER **Definition 18.28.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called *flat* if the functor
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$$
is exact.
(2) A map $\mathcal{O} \to \mathcal{O}'$ of presheaves of rings is called *flat* if $\mathcal{O}'$ is flat as a presheaf of $\mathcal{O}$-modules.
(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then we say $\mathcal{F}$ is *flat* if the functor
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$
is exact.
(4) A map $\mathcal{O} \to \mathcal{O}'$ of sheaves of rings on a site is called *flat* if $\mathcal{O}'$ is flat as a sheaf of $\mathcal{O}$-modules.

The notion of a flat module or flat ring map is intrinsic (Section 18.18).

03ES **Lemma 18.28.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If each $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then $\mathcal{F}$ is flat.*

Proof. This is immediate from the definitions. □

03ET **Lemma 18.28.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Omitted. (Hint: Sheafification is exact.) □

0GN1 **Lemma 18.28.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Assume that every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}(U_i)$-module. Then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Let $\mathcal{G} \subset \mathcal{G}'$ be an inclusion of $\mathcal{O}^\#$-modules. We have to show that

$$\mathcal{G} \otimes_{\mathcal{O}^\#} \mathcal{F}^\# \longrightarrow \mathcal{G}' \otimes_{\mathcal{O}^\#} \mathcal{F}^\#$$

is injective. By Lemma 18.26.1 the source of this arrow is the sheafification of the presheaf $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ and similarly for the target. If U is an object of $\mathcal{C}$ such that $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then

$$(\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F})(U) = \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \longrightarrow \mathcal{G}'(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) = (\mathcal{G}' \otimes_{p,\mathcal{O}} \mathcal{F})(U)$$

is injective. Hence we reduce to showing: given a map of presheaves $f : \mathcal{H} \to \mathcal{H}'$ on $\mathcal{C}$ such that every U in $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ with $\mathcal{H}(U_i) \to \mathcal{H}'(U_i)$ injective, then $f^\#$ is injective. This we leave to the reader as an exercise. □

03EU **Lemma 18.28.5.** *Colimits and tensor product.*

(1) *A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.*
(2) *A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.*

Proof. Part (1) follows from Lemma 18.27.7 and Algebra, Lemma 10.8.8 by looking at sections over objects. To see part (2), use Lemma 18.27.7 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted. □

0E8J **Lemma 18.28.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}|_U$ is a flat $\mathcal{O}_U$-module.*

Proof. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Lemma 18.19.3) and $\mathcal{F}$ is flat as an $\mathcal{O}$-modules then we see that the complex made up of the modules

$$j_{U!}(\mathcal{G}_i \otimes_{\mathcal{O}_U} \mathcal{F}|_U) = j_{U!}\mathcal{G}_i \otimes_{\mathcal{O}} \mathcal{F}$$

(Lemma 18.27.9) is exact. We conclude that $\mathcal{G}_1 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_2 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_3 \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is exact by Lemma 18.19.4. □

03EV **Lemma 18.28.7.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Consider the functor $j_U : \mathcal{C}/U \to \mathcal{C}$.*

(1) *The presheaf of $\mathcal{O}$-modules $j_{U!}\mathcal{O}_U$ (see Remark 18.19.7) is flat.*
(2) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). By the discussion in Remark 18.19.7 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{O}(V)$$

which is a flat $\mathcal{O}(V)$-module. Hence (1) follows from Lemma 18.28.2. Then (2) follows as $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$ (the first $j_{U!}$ on sheaves, the second on presheaves) and Lemma 18.28.3. □

03EW **Lemma 18.28.8.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Any presheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(2) *Any presheaf of $\mathcal{O}$-modules is a quotient of a flat presheaf of $\mathcal{O}$-modules.*
(3) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(4) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). For every object U of $\mathcal{C}$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 18.28.5 and 18.28.7 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument. □

03EX **Lemma 18.28.9.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}$ is a flat presheaf of modules, then the sequence*

$$0 \to \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact.

(2) *If $\mathcal{C}$ is a site, $\mathcal{O}$, $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{G}$ are sheaves, and $\mathcal{F}$ is flat as a sheaf of modules, then the sequence*

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Choose a flat presheaf of $\mathcal{O}$-modules $\mathcal{G}'$ which surjects onto $\mathcal{G}$. This is possible by Lemma 18.28.8. Let $\mathcal{G}'' = \mathrm{Ker}(\mathcal{G}' \to \mathcal{G})$. The lemma follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the flat module $\mathcal{G}'$ is exact. The proof in the case of sheaves is exactly the same. □

03EY **Lemma 18.28.10.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds in $Mod(\mathcal{O})$.

Proof. Let $\mathcal{G}^\bullet$ be an arbitrary exact complex of presheaves of $\mathcal{O}$-modules. Assume that $\mathcal{F}_0$ is flat. By Lemma 18.28.9 we see that

$$0 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \to 0$$

is a short exact sequence of complexes of presheaves of $\mathcal{O}$-modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way. □

03EZ **Lemma 18.28.11.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of presheaves of $\mathcal{O}$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}$-modules, then for any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules the complex

$$\ldots \to \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact also. If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds $Mod(\mathcal{O})$.

Proof. Follows from Lemma 18.28.9 by splitting the complex into short exact sequences and using Lemma 18.28.10 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

0G6Q **Lemma 18.28.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{G}$ and $\mathcal{F}$ are flat $\mathcal{O}$-modules, then $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is a flat $\mathcal{O}$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}) \otimes_{\mathcal{O}} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}} (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H})$$

and a composition of exact functors is exact. □

05V4 **Lemma 18.28.13.** *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on a site $\mathcal{C}$. If $\mathcal{G}$ is a flat $\mathcal{O}_1$-module, then $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is a flat $\mathcal{O}_2$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example). □

The following lemma is the analogue of the equational criterion of flatness (Algebra, Lemma 10.39.11).

08FC **Lemma 18.28.14.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}$-module.*
(2) *Let U be an object of $\mathcal{C}$ and let*

$$\mathcal{O}_U \xrightarrow{(f_1,\ldots,f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1},\ldots,t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ (f_1, \ldots, f_n)|_{U_i} = 0$.

(3) *Let U be an object of $\mathcal{C}$ and let*

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1},\ldots,t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1,\ldots,s_n)|_{U_i}$ such that $B_i \circ A|_{U_i} = 0$.

Proof. Assume (1). Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1,\ldots,f_n$. Then $\sum f_j \otimes s_j$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat (Lemma 18.28.6) the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists a covering $\{U_i \to U\}$ such that $\sum f_j|_{U_i} \otimes s_j|_{U_i}$ is zero in $\mathcal{I}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. Unwinding the definitions using Algebra, Lemma 10.107.10 we find $t_{i1},\ldots,t_{il_i} \in \mathcal{F}(U_i)$ and $a_{ijk} \in \mathcal{O}(U_i)$ such that $\sum_j a_{ijk} f_j|_{U_i} = 0$ and $s_j|_{U_i} = \sum_k a_{ijk} t_{ik}$. Thus (2) holds.

Assume (2). Let U, n, m, A and $s_1,\ldots,s_n$ as in (3) be given. Observe that A has m columns. We will prove the assertion of (3) is true by induction on m. For the base case $m = 0$ we can take the covering $\{U \to U\}$ and for the factorization we use $\mathcal{O}_U^n \to \mathcal{O}_U^n \to \mathcal{F}|_U$ where the first map is the identity and the second is $(s_1,\ldots,s_n)$. Assume $m > 0$. Let $(f_1,\ldots,f_n)$ be the last column of A and apply (2). This gives new diagrams

$$\mathcal{O}_{U_i}^{\oplus m} \xrightarrow{B_i \circ A|_{U_i}} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1},\ldots,t_{il_i})} \mathcal{F}|_{U_i}$$

but the last column of $A_i = B_i \circ A|_{U_i}$ is zero. Hence we can apply the induction hypothesis to U_i, l_i, $m-1$, the matrix consisting of the first $m-1$ columns of A_i, and $t_{i1},\ldots,t_{il_i}$ to get coverings $\{U_{ij} \to U_j\}$ and factorizations

$$\mathcal{O}_{U_{ij}}^{\oplus l_i} \xrightarrow{C_{ij}} \mathcal{O}_{U_{ij}}^{\oplus k_{ij}} \xrightarrow{(v_{ij1},\ldots,v_{ijk_{ij}})} \mathcal{F}|_{U_{ij}}$$

of $(t_{i1},\ldots,t_{il_i})|_{U_{ij}}$ such that $C_i \circ B_i|_{U_{ij}} \circ A|_{U_{ij}} = 0$. Then $\{U_{ij} \to U\}$ is a covering and we get the desired factorizations using $B_{ij} = C_i \circ B_i|_{U_{ij}}$ and v_{ija}. In this way we see that (2) implies (3).

Assume (3). Let $\mathcal{G} \to \mathcal{H}$ be an injective homomorphism of $\mathcal{O}$-modules. We have to show that $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ is injective. Let U be an object of $\mathcal{C}$ and let $s \in (\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F})(U)$ be a section which maps to zero in $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$. We have to show that s is zero. Since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is a sheaf, it suffices to find a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that $s|_{U_i}$ is zero for all $i \in I$. Hence we may always replace U by the members of a covering. In particular, since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is the sheafification of $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ we may assume that s is the image of $s' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Arguing similarly for $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ we may assume that s' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Write $\mathcal{F}(U) = \operatorname{colim} M_\alpha$ as a filtered colimit of finitely presented $\mathcal{O}(U)$-modules M_α (Algebra, Lemma 10.11.3). Since tensor product commutes with filtered colimits (Algebra, Lemma 10.12.9) we can choose an α such that s' comes from some $s'' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} M_\alpha$ and such that s'' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} M_\alpha$. Fix α and s''. Choose a presentation

$$\mathcal{O}(U)^{\oplus m} \xrightarrow{A} \mathcal{O}(U)^{\oplus n} \to M_\alpha \to 0$$

We apply (3) to the corresponding complex of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U$$

After replacing U by the members of the covering U_i we find that the map

$$M_\alpha \to \mathcal{F}(U)$$

factors through a free module $\mathcal{O}(U)^{\oplus l}$ for some l. Since $\mathcal{G}(U) \to \mathcal{H}(U)$ is injective we conclude that

$$\mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l} \to \mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l}$$

is injective too. Hence as s'' maps to zero in the module on the right, it also maps to zero in the module on the left, i.e., s is zero as desired. □

08M4 **Lemma 18.28.15.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I}$ is an ideal of square zero. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = \mathcal{F}'/\mathcal{I}\mathcal{F}'$. The following are equivalent*

(1) *$\mathcal{F}'$ is a flat $\mathcal{O}'$-module, and*
(2) *$\mathcal{F}$ is a flat $\mathcal{O}$-module and $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective.*

Proof. If (1) holds, then $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$ is flat over $\mathcal{O}$ by Lemma 18.28.13 and we see the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective by applying $- \otimes_{\mathcal{O}'} \mathcal{F}'$ to the exact sequence $0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$, see Lemma 18.28.9. Assume (2). In the rest of the proof we will use without further mention that $\mathcal{K} \otimes_{\mathcal{O}'} \mathcal{F}' = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}'$-module $\mathcal{K}$ annihilated by $\mathcal{I}$. Let $\alpha : \mathcal{G}' \to \mathcal{H}'$ be an injective map of $\mathcal{O}'$-modules. Let $\mathcal{G} \subset \mathcal{G}'$, resp. $\mathcal{H} \subset \mathcal{H}'$ be the subsheaf of sections annihilated by $\mathcal{I}$. Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}'/\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}'/\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0
\end{array}$$

Note that $\mathcal{G}'/\mathcal{G}$ and $\mathcal{H}'/\mathcal{H}$ are annihilated by $\mathcal{I}$ and that $\mathcal{G}'/\mathcal{G} \to \mathcal{H}'/\mathcal{H}$ is injective. Thus the right vertical arrow is injective as $\mathcal{F}$ is flat over $\mathcal{O}$. The same is true for the left vertical arrow. Hence the middle vertical arrow is injective and $\mathcal{F}'$ is flat. □

0GLY **Lemma 18.28.16.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a flat homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals such that the induced map $\mathcal{O}/\mathcal{I} \to \mathcal{O}'/\mathcal{I}\mathcal{O}'$ is an isomorphism. For any $\mathcal{O}$-module $\mathcal{F}$ annihilated by $\mathcal{I}^n$ for some $n \geq 0$ the map $id \otimes 1 : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism.*

Proof. Omitted. Hint: See More on Algebra, Lemma 15.90.2. □

18.29. Duals

0FNX Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category of $\mathcal{O}$-modules endowed with the tensor product constructed in Section 18.26 is a symmetric monoidal category. For an $\mathcal{O}$-module $\mathcal{F}$ the following are equivalent

(1) $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}$-modules,
(2) for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module, and
(3) $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}$-module.

This is proved in Example 18.29.1 and Lemmas 18.29.2 and 18.29.3 of this section.

0FNY **Example 18.29.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module such that for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true (details omitted). Denote

$$\eta : \mathcal{O} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$$

the map sending 1 to the section corresponding to $\mathrm{id}_{\mathcal{F}}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{O}$$

the evaluation map. Then we see that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})}$.

0FNZ **Lemma 18.29.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a $\mathcal{O}$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}$-modules, see Categories, Definition 4.43.5. Then*

1. *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module,*
2. *the map $e : \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,*
3. *we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.*

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering of U, we can find a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde{\eta} : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}|_U & \xrightarrow{\eta \otimes 1} & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U & \xrightarrow{1 \otimes \epsilon} & \mathcal{F}|_U \\ & \searrow^{\tilde{\eta} \otimes 1} & \uparrow & & \uparrow \\ & & \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U & \xrightarrow{1 \otimes \epsilon} & \mathcal{O}_U^{\oplus n} \end{array}$$

This shows that the identity on $\mathcal{F}|_U$ factors through a finite free $\mathcal{O}_U$-module. This proves (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 4.43.7) and the construction of the left dual in Example 18.29.1. □

08FD **Lemma 18.29.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be locally of finite presentation and flat. Then given an object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module.*

Proof. Choose an object U of $\mathcal{C}$. After replacing U by the members of a covering, we may assume there exists a presentation

$$\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

By Lemma 18.28.14 we may, after replacing U by the members of a covering, assume there exists a factorization

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_1} \to \mathcal{F}|_U$$

such that the composition $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_U^{\oplus n_r} \to \mathcal{F}|_U$ has a section and we win. □

18.30. Towards constructible modules

0933 Recall that a quasi-compact object of a site is roughly an object such that every covering of it can be refined by a finite covering (the actual definition is slightly more involved, see Sites, Section 7.17). It turns out that if every object of a site has a covering by quasi-compact objects, then the modules $j_!\mathcal{O}_U$ with U quasi-compact form a particularly nice set of generators for the category of all modules.

0934 **Lemma 18.30.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then the sequence*

$$\bigoplus j_{U_i \times_U U_j !}\mathcal{O}_{U_i \times_U U_j} \to \bigoplus j_{U_i !}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. For any $\mathcal{O}$-module $\mathcal{F}$ the functor $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{F})$ turns our sequence into the exact sequence $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \times_U U_j)$, see (18.19.2.1). The lemma follows from this and Homology, Lemma 12.5.8. □

0G1W **Lemma 18.30.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be covering of $\mathcal{C}$. If U is quasi-compact, then there exist a finite subset $I' \subset I$ such that the sequence*

$$\bigoplus\nolimits_{i,i' \in I'} j_{U_i \times_U U_{i'} !}\mathcal{O}_{U_i \times_U U_{i'}} \to \bigoplus\nolimits_{i \in I'} j_{U_i !}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. This lemma is immediate from Lemma 18.30.1 if U satisfies condition (3) of Sites, Lemma 7.17.2. We urge the reader to skip the proof in the general case. By definition there exists a covering $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $f_j : V_j \to U_{\alpha(j)}$ (see Sites, Definition 7.8.1) such that the image $I' \subset I$ of α is finite. By Homology, Lemma 12.5.8 it suffices to show that for any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{F})$ turns the sequence of the lemma into an exact sequence. By (18.19.2.1) we obtain the usual sequence

$$0 \to \mathcal{F}(U) \to \prod\nolimits_{i \in I'} \mathcal{F}(U_i) \to \prod\nolimits_{i,i' \in I'} \mathcal{F}(U_i \times_U U_{i'})$$

This is an exact sequence by Sites, Lemma 7.8.6 applied to the family of maps $\{U_i \to U\}_{i \in I'}$ which is refined by the covering $\mathcal{V}$. □

0935 **Lemma 18.30.3.** *Let $\mathcal{C}$ be a site. Let W be a quasi-compact object of $\mathcal{C}$.*

(1) *The functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with coproducts.*

(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The functor $Mod(\mathcal{O}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with direct sums.*

Proof. Proof of (1). Taking sections over W commutes with filtered colimits with injective transition maps by Sites, Lemma 7.17.7. If $\mathcal{F}_i$ is a family of sheaves of sets indexed by a set I. Then $\coprod \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the coproducts $\mathcal{F}_E = \coprod_{i \in E} \mathcal{F}_i$. Since the transition maps are injective we conclude.

Proof of (2). Let $\mathcal{F}_i$ be a family of sheaves of $\mathcal{O}$-modules indexed by a set I. Then $\bigoplus \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the direct sums $\mathcal{F}_E = \bigoplus_{i \in E} \mathcal{F}_i$. A filtered colimit of abelian sheaves can be computed in the category of sheaves of sets. Moreover, for $E \subset E'$ the transition map $\mathcal{F}_E \to \mathcal{F}_{E'}$ is injective (as sheafification is exact and the injectivity is clear on underlying presheaves). Hence it suffices to show the result for a finite index set by Sites, Lemma 7.17.7. The finite case is dealt with in Lemma 18.3.2 (it holds over any object of $\mathcal{C}$). □

0936 **Lemma 18.30.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be a quasi-compact object of $\mathcal{C}$. Then the functor $\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, -)$ commutes with direct sums.*

Proof. This is true because $\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by (18.19.2.1) and because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with direct sums by Lemma 18.30.3. □

In order to state the sharpest possible results in the following we introduce some notation.

0937 **Situation 18.30.5.** Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a set of objects. We consider the following conditions

0938 (1) Every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$.

0939 (2) Every $U \in \mathcal{B}$ is quasi-compact (Sites, Section 7.17).

093A (3) For a covering $\{U_i \to U\}$ with $U_i, U \in \mathcal{B}$ the fibre products $U_i \times_U U_j$ are quasi-compact.

093B **Lemma 18.30.6.** *In Situation 18.30.5 assume (1) holds.*

(1) *Every sheaf of sets is the target of a surjective map whose source is a coproduct $\coprod h^\#_{U_i}$ with U_i in $\mathcal{B}$.*

(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ with U_i in $\mathcal{B}$.*

Proof. Part (1) follows from Sites, Lemmas 7.12.5 and 7.12.4. Part (2) follows from Lemmas 18.28.8 and 18.30.1. □

093C **Lemma 18.30.7.** *In Situation 18.30.5 assume (1) and (2) hold.*

(1) *Every sheaf of sets is a filtered colimit of sheaves of the form*

09Y7 (18.30.7.1) $$\mathit{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h^\#_{V_j} \longrightarrow\!\!\!\!\!\rightarrow \coprod\nolimits_{i=1,\ldots,n} h^\#_{U_i} \right)$$

with U_i and V_j in $\mathcal{B}$.

(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a filtered colimit of sheaves of the form*

093D (18.30.7.2) $$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i} \right)$$

with U_i and V_j in $\mathcal{B}$.

Proof. Proof of (1). By Lemma 18.30.6 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source is a coprod $\mathcal{F}_0$ of sheaves the form $h_U^\#$ with $U \in \mathcal{B}$. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod_{j\in J} h_{V_j}^\# \longrightarrow \coprod_{i\in I} h_{U_i}^\#$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the coproduct over $j \in J'$ maps into the coprod over $i \in I'$ via both maps, see Sites, Lemma 7.17.7. (Details omitted; hint: an infinite coproduct is the filtered colimit of the finite sub-coproducts.) Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts.

Proof of (2). By Lemma 18.30.6 every module is a quotient of a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ with $U \in \mathcal{B}$. Thus every module is a cokernel

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j\in J} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i\in I} j_{U_i!}\mathcal{O}_{U_i}\right)$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the direct sum over $j \in J'$ maps into the direct sum over $i \in I'$, see Lemma 18.30.4. Thus our module is the colimit of the cokernels of these maps between finite direct sums. □

093E **Lemma 18.30.8.** *In Situation 18.30.5 assume (1) and (2) hold. Let $\mathcal{O}$ be a sheaf of rings. Then a cokernel of a map between modules as in (18.30.7.2) is another module as in (18.30.7.2).*

Proof. Let $\mathcal{F} = \mathrm{Coker}(\bigoplus j_{V_j!}\mathcal{O}_{V_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i})$ as in (18.30.7.2). It suffices to show that the cokernel of a map $\varphi : j_{W!}\mathcal{O}_W \to \mathcal{F}$ with $W \in \mathcal{B}$ is another module of the same type. The map φ corresponds to $s \in \mathcal{F}(W)$. Since $\bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{F}$ is surjective, by (1) we may choose a covering $\{W_k \to W\}_{k\in K}$ with $W_k \in \mathcal{B}$ such that $s|_{W_k}$ is the image of some section s_k of $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$). By (2) the object W is quasi-compact. By Lemma 18.30.2 there is a finite subset $K' \subset K$ such that $\bigoplus_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \to j_{W!}\mathcal{O}_W$ is surjective. We conclude that $\mathrm{Coker}(\varphi)$ is equal to

$$\mathrm{Coker}\left(\bigoplus\nolimits_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \oplus \bigoplus j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i}\right)$$

where the map $\bigoplus_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i}$ corresponds to $\sum_{k\in K'} s_k$. This finishes the proof. □

093F **Lemma 18.30.9.** *In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Assume given a map*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, and coverings $\{U_{ik} \to U_i\}_{k\in K_i}$ with $U_{ik} \in \mathcal{B}$. Then there exist finite subsets $K_i' \subset K_i$ and a finite set L of $W_l \in \mathcal{B}$ and a commutative diagram

$$\begin{array}{ccc} \bigoplus_{l\in L} j_{W_l!}\mathcal{O}_{W_l} & \longrightarrow & \bigoplus_{i=1,\ldots,n} \bigoplus_{k\in K_i'} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} & \longrightarrow & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i} \end{array}$$

inducing an isomorphism on cokernels of the horizontal maps.

Proof. Since U_i is quasi-compact, we may choose finite subsets $K_i' \subset K_i$ as in Lemma 18.30.2. Then since $\bigoplus_{i=1,\ldots,n} \bigoplus_{k\in K_i'} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \to \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$ is surjective, we can find coverings $\{V_{jm} \to V_j\}_{m\in M_j}$ with $V_{jm} \in \mathcal{B}$ such that we can find a commutative diagram

$$\begin{array}{ccc} \bigoplus_{j=1,\ldots,m} \bigoplus_{m\in M_j} j_{V_{jm}!}\mathcal{O}_{V_{jm}} & \longrightarrow & \bigoplus_{i=1,\ldots,n} \bigoplus_{k\in K_i'} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} & \longrightarrow & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i} \end{array}$$

Since V_j is quasi-compact, we can choose finite subsets $M_j' \subset M_j$ as in Lemma 18.30.2. Set

$$L = \Big(\coprod\nolimits_{i=1,\ldots,n} K_i' \times K_i'\Big) \coprod \Big(\coprod\nolimits_{j=1,\ldots,m} M_j'\Big)$$

and for $l = (k, k') \in K_i' \times K_i' \subset L$ set $W_l = U_{ik} \times_{U_i} U_{ik'}$ and for $l = m \in M_j' \subset L$ set $W_l = V_{jm}$. Since we have the exact sequences of Lemma 18.30.2 for the families $\{U_{ik} \to U_i\}_{k\in K_i'}$ we conclude that we get a diagram as in the statement of the lemma (details omitted), except that it is not yet clear that $W_l \in \mathcal{B}$. However, since W_l is quasi-compact for all $l \in L$ we do another application of Lemma 18.30.2 and find finite families of maps $\{W_{lt} \to W_l\}_{t\in T_l}$ with $W_{lt} \in \mathcal{B}$ such that $\bigoplus j_{W_{lt}!}\mathcal{O}_{W_{lt}} \to j_{W_l!}\mathcal{O}_{W_l}$ is surjective. Then we replace L by $\coprod_{l\in L} T_l$ and everything is clear. □

093G **Lemma 18.30.10.** *In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Then an extension of modules as in (18.30.7.2) is another module as in (18.30.7.2).*

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_3$ as in (18.30.7.2). Choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F}_1 \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{F}_3 \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Since $\mathcal{F}_2 \to \mathcal{F}_3$ is surjective, we can choose coverings $\{W_{ik} \to W_i\}$ with $W_{ik} \in \mathcal{B}$ such that $A_{W_{ik}} \to \mathcal{F}_3$ lifts to a map $A_{W_{ik}} \to \mathcal{F}_2$. By Lemma 18.30.9 we may replace our collection $\{W_i\}$ by a finite subcollection of the collection $\{W_{ik}\}$ and assume the map $\bigoplus A_{W_i} \to \mathcal{F}_3$ lifts to a map into $\mathcal{F}_2$. Consider the kernel

$$\mathcal{K}_2 = \mathrm{Ker}(\bigoplus A_{U_i} \oplus \bigoplus A_{W_i} \longrightarrow \mathcal{F}_2)$$

By the snake lemma this kernel surjects onto $\mathcal{K}_3 = \mathrm{Ker}(\bigoplus A_{W_i} \to \mathcal{F}_3)$. Thus, arguing as above, after replacing each T_j by a finite family of elements of $\mathcal{B}$ (permissible by Lemma 18.30.2) we may assume there is a map $\bigoplus A_{T_j} \to \mathcal{K}_2$ lifting the given map $\bigoplus A_{T_j} \to \mathcal{K}_3$. Then $\bigoplus A_{V_j} \oplus \bigoplus A_{T_j} \to \mathcal{K}_2$ is surjective which finishes the proof. □

093H **Lemma 18.30.11.** *In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{A} \subset Mod(\mathcal{O})$ be the full subcategory of modules isomorphic to a cokernel as in (18.30.7.2). If the kernel of every map of $\mathcal{O}$-modules of the form*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, is in $\mathcal{A}$, then $\mathcal{A}$ is weak Serre subcategory of $Mod(\mathcal{O})$.

Proof. We will use the criterion of Homology, Lemma 12.10.3. By the results of Lemmas 18.30.8 and 18.30.10 it suffices to see that the kernel of a map $\mathcal{F} \to \mathcal{G}$ between objects of $\mathcal{A}$ is in $\mathcal{A}$. To prove this choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{G} \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Using Lemmas 18.30.1 and 18.30.9 and arguing as in the proof of Lemma 18.30.10 we may assume that the map $\mathcal{F} \to \mathcal{G}$ lifts to a map of presentations

$$\begin{array}{ccccccc} \bigoplus A_{V_j} & \longrightarrow & \bigoplus A_{U_i} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus A_{T_j} & \longrightarrow & \bigoplus A_{W_i} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Then we see that

$$\mathrm{Ker}(\mathcal{F} \to \mathcal{G}) = \mathrm{Coker}\left(\bigoplus A_{V_j} \to \mathrm{Ker}\left(\bigoplus A_{T_j} \oplus \bigoplus A_{U_i} \to \bigoplus A_{W_i}\right)\right)$$

and the lemma follows from the assumption and Lemma 18.30.8. □

18.31. Flat morphisms

04JA

04JB **Definition 18.31.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We say $(f, f^\sharp)$ is *flat* if the ring map $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat. We say a morphism of ringed sites is *flat* if the associated morphism of ringed topoi is flat.

04JC **Lemma 18.31.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of ringed topoi. Then*

$$f^{-1} : Ab(\mathcal{C}') \longrightarrow Ab(\mathcal{C}), \quad \mathcal{F} \longmapsto f^{-1}\mathcal{F}$$

is exact. If $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a flat morphism of ringed topoi then

$$f^* : Mod(\mathcal{O}') \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto f^*\mathcal{F}$$

is exact.

Proof. Given an abelian sheaf $\mathcal{G}$ on $\mathcal{C}'$ the underlying sheaf of sets of $f^{-1}\mathcal{G}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{G}$, see Sites, Section 7.44. Hence the exactness of f^{-1} for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 7.15.1) implies the exactness of f^{-1} as a functor on abelian sheaves.

To see the statement on modules recall that $f^*\mathcal{F}$ is defined as the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\sharp} \mathcal{O}$. Hence f^* is a composition of functors both of which are exact. □

08M5 **Definition 18.31.3.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We say that $\mathcal{F}$ is *flat over* $(Sh(\mathcal{D}), \mathcal{O}')$ if $\mathcal{F}$ is flat as an $f^{-1}\mathcal{O}'$-module.

This is compatible with the notion as defined for morphisms of ringed spaces, see Modules, Definition 17.20.3 and the discussion following.

0GN2 **Lemma 18.31.4.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_\mathcal{D}$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map*

$$f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

of Remark 18.27.3 is an isomorphism.

Proof. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. We have to show that the restriction of the map to $\mathcal{C}/U$ for any $U \in \mathrm{Ob}(\mathcal{C})$ is an isomorphism. We may replace U by the members of a covering of U. Hence by Sites, Lemma 7.14.10 we may assume there exists a morphism $U \to u(V)$ for some $V \in \mathrm{Ob}(\mathcal{C})$. Of course, then we may replace U by $u(V)$. Then since u is continuous, we may replace V by a covering and assume there is a presentation $\mathcal{O}_V^{\oplus m} \to \mathcal{O}_V^{\oplus n} \to \mathcal{F}|_V \to 0$ over $\mathcal{D}/V$. Since formation of $\mathcal{H}om$ commutes with localization (Lemma 18.27.2) we may replace f by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f. Hence we reduce to the case where $\mathcal{F}$ has a global presentation $\mathcal{O}_\mathcal{D}^{\oplus m} \to \mathcal{O}_\mathcal{D}^{\oplus n} \to \mathcal{F} \to 0$. Since f is flat and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$ we obtain a corresponding presentation $\mathcal{O}_\mathcal{C}^{\oplus m} \to \mathcal{O}_\mathcal{C}^{\oplus n} \to f^*\mathcal{F} \to 0$, see Lemma 18.31.2. Using that $\mathcal{H}om$ commutes with finite direct sums in the first variable, using that both $\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_\mathcal{C}, -)$ and $\mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ are the identity functor, and using the functoriality of the construction of Remark 18.27.3 we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) & \longrightarrow & f^*\mathcal{G}^{\oplus n} & \longrightarrow & f^*\mathcal{G}^{\oplus n} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G}) & \longrightarrow & f^*\mathcal{G}^{\oplus n} & \longrightarrow & f^*\mathcal{G}^{\oplus n}
\end{array}$$

where the right two vertical arrows are isomorphisms. By Lemma 18.27.5 the rows are exact. We conclude by the 5 lemma. □

18.32. Invertible modules

0408 Here is the definition.

0409 **Definition 18.32.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.

(1) A finite locally free $\mathcal{O}$-module $\mathcal{F}$ is said to have *rank* r if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ of U such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^{\oplus r}$ as an $\mathcal{O}_{U_i}$-module.

(2) An $\mathcal{O}$-module $\mathcal{L}$ is *invertible* if the functor

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_\mathcal{O} \mathcal{L}$$

is an equivalence.

(3) The sheaf $\mathcal{O}^*$ is the subsheaf of $\mathcal{O}$ defined by the rule

$$U \longmapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$

It is a sheaf of abelian groups with multiplication as the group law.

Lemma 18.40.7 below explains the relationship with locally free modules of rank 1.

0B8N **Lemma 18.32.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}$ be an $\mathcal{O}$-module. The following are equivalent:*

(1) *$\mathcal{L}$ is invertible, and*

(2) *there exists an $\mathcal{O}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_\mathcal{O} \mathcal{N} \cong \mathcal{O}$.*

In this case we have

(a) *$\mathcal{L}$ is a flat $\mathcal{O}$-module of finite presentation,*
(b) *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{L}|_{U_i}$ is a direct summand of a finite free module, and*
(c) *the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.*

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Since $- \otimes_{\mathcal{O}} \mathcal{L}$ is an equivalence, it is exact, and hence $\mathcal{L}$ is flat. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}} \mathcal{N} \to \mathcal{O}$ the given isomorphism. Let U be an object of $\mathcal{C}$. We will show that the restriction $\mathcal{L}$ to the members of a covering of U is a direct summand of a free module, which will certainly imply that $\mathcal{L}$ is of finite presentation. By construction of $\otimes$ we may assume (after replacing U by the members of a covering) that there exists an integer $n \geq 1$ and sections $x_i \in \mathcal{L}(U)$, $y_i \in \mathcal{N}(U)$ such that $\psi(\sum x_i \otimes y_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism. By Lemma 18.27.6 we have

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{N} \otimes_{\mathcal{O}} \mathcal{L}, \mathcal{O}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}))$$

The image of 1 gives a morphism $\mathcal{N} \to \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$. Tensoring with $\mathcal{L}$ we obtain

$$\mathcal{O} = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{N} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$$

This map is the inverse to the evaluation map; computation omitted. □

0B8P **Lemma 18.32.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_{\mathcal{D}}$-module is invertible.*

Proof. By Lemma 18.32.2 there exists an $\mathcal{O}_{\mathcal{D}}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{N} \cong \mathcal{O}_{\mathcal{D}}$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{C}}} f^*\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ by Lemma 18.26.2. Thus $f^*\mathcal{L}$ is invertible by Lemma 18.32.2. □

040A **Lemma 18.32.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}$-module, then so is $\mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$ and the evaluation map $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \to \mathcal{O}$ is an isomorphism.*

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 18.32.2 and its proof. □

040B **Lemma 18.32.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $(\mathcal{C}, \mathcal{O})$ is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Omitted, but see Sheaves of Modules, Lemma 17.25.8. □

Lemma 18.32.5 says that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 18.32.4 says that tensor product defines the structure of an abelian group on this set with inverse of $\mathcal{L}$ given by $\mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.

In fact, given an invertible $\mathcal{O}$-module $\mathcal{L}$ and $n \in \mathbf{Z}$ we define the nth *tensor power* $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}$ exactly n times. This makes sense also for negative n as we've defined an invertible $\mathcal{O}$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O} & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

see Lemma 18.32.4. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

040C **Definition 18.32.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The *Picard group* $\mathrm{Pic}(\mathcal{O})$ of the ringed site is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}$-modules, with addition corresponding to tensor product.

18.33. Modules of differentials

04BJ In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131).

04BK **Definition 18.33.1.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$-*derivation* or more precisely a φ-*derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 18.33.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a $\Gamma(\mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

04BL **Lemma 18.33.2.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $\mathcal{C}$, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

04BM (18.33.2.1)
$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & [a+b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{array}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (18.33.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

04BN **Definition 18.33.3.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The *module of differentials* of the ring map φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 18.33.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 18.18). Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.33.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

08TP **Lemma 18.33.4.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Then $\Omega_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (18.33.2.1). There is a similar map of presheaves whose value on $U \in \mathrm{Ob}(\mathcal{C})$ is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.131.2. On the other hand, the sheaves in (18.33.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

08TQ **Lemma 18.33.5.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.33.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

04BO **Lemma 18.33.6.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For any object U of $\mathcal{C}$ there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. This is a special case of Lemma 18.33.5. □

08TR **Lemma 18.33.7.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi} & \mathcal{O}_2' \\ \uparrow & & \uparrow \\ \mathcal{O}_1 & \longrightarrow & \mathcal{O}_1' \end{array}$$

be a commutative diagram of sheaves of rings on $\mathcal{C}$. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $d : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $d(f)$ maps to $d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

08TS **Lemma 18.33.8.** *In Lemma 18.33.7 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}_2' = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 18.33.7.

To see that the sequence is exact, we argue as follows. Let $\mathcal{O}_2'' \subset \mathcal{O}_2'$ be the presheaf of $\mathcal{O}_1$-algebras whose value on U is the image of $\mathcal{O}_2(U) \to \mathcal{O}_2'(U)$. By Algebra, Lemma 10.131.9 the sequences

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \longrightarrow \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}_2''(U) \longrightarrow \Omega_{\mathcal{O}_2''(U)/\mathcal{O}_1(U)} \longrightarrow 0$$

are exact for all objects U of $\mathcal{C}$. Since sheafification is exact this gives an exact sequence of sheaves of $(\mathcal{O}_2')^\#$-modules. By Lemma 18.33.4 and the fact that $(\mathcal{O}_2'')^\# = \mathcal{O}_2'$ we conclude. □

Here is a particular situation where derivations come up naturally.

04BP **Lemma 18.33.9.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Consider a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{O}_2 \to 0$$

Here $\mathcal{A}$ is a sheaf of $\mathcal{O}_1$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_2$ is a surjection of sheaves of $\mathcal{O}_1$-algebras, and $\mathcal{F} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{F}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{F}$ has a natural structure of an $\mathcal{O}_2$-module. A section $s : \mathcal{O}_2 \to \mathcal{A}$ of π is a $\mathcal{O}_1$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_2 \to \mathcal{F}$ of π and any φ-derivation $D : \mathcal{O}_1 \to \mathcal{F}$ the map

$$s + D : \mathcal{O}_1 \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique φ-derivation D.

Proof. Recall that the $\mathcal{O}_2$-module structure on $\mathcal{F}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_2$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{F}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}(s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b))\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $\mathcal{O}_1$-algebra map because D is an $\mathcal{O}_1$-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

04BQ **Definition 18.33.10.** Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. In this situation

(1) for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules a *Y-derivation* $D : \mathcal{O} \to \mathcal{F}$ is just a $f^\sharp$-derivation, and
(2) the *sheaf of differentials* $\Omega_{X/Y}$ *of X over Y* is the module of differentials of $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, see Definition 18.33.3.

Thus $\Omega_{X/Y}$ comes equipped with a *universal Y-derivation* $\mathrm{d}_{X/Y} : \mathcal{O} \longrightarrow \Omega_{X/Y}$. We sometimes write $\Omega_{X/Y} = \Omega_f$.

Recall that $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ so that this definition makes sense.

04BR **Lemma 18.33.11.** *Let $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$, $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$, $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$, and $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$ be ringed topoi. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of morphisms of ringed topoi. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\mathrm{d}_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*\mathrm{d}_{X/Y}(t)$ maps to $\mathrm{d}_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is clear except for the last assertion. Let us explain the meaning of this. Let $U \in \mathrm{Ob}(\mathcal{C}_X)$ and let $t \in \mathcal{O}_X(U)$. This is what it means for t to be a local section of $\mathcal{O}_X$. Now, we may think of t as a map of sheaves of sets $t : h_U^\# \to \mathcal{O}_X$. Then $f^{-1}t : f^{-1}h_U^\# \to f^{-1}\mathcal{O}_X$. By f^*t we mean the composition

$$f^{-1}h_U^\# \xrightarrow{f^{-1}t} f^{-1}\mathcal{O}_X \xrightarrow{f^\sharp} \mathcal{O}_{X'}, \quad f^{-1}h_U^\# \xrightarrow{f^*t} \mathcal{O}_{X'}$$

Note that $\mathrm{d}_{X/Y}(t) \in \Omega_{X/Y}(U)$. Hence we may think of $\mathrm{d}_{X/Y}(t)$ as a map $\mathrm{d}_{X/Y}(t) : h_U^\# \to \Omega_{X/Y}$. Then $f^{-1}\mathrm{d}_{X/Y}(t) : f^{-1}h_U^\# \to f^{-1}\Omega_{X/Y}$. By $f^*\mathrm{d}_{X/Y}(t)$ we mean the composition

$$f^{-1}h_U^\# \xrightarrow{f^{-1}\mathrm{d}_{X/Y}(t)} f^{-1}\Omega_{X/Y} \xrightarrow{1\otimes\mathrm{id}} f^*\Omega_{X/Y}, \quad f^{-1}h_U^\# \xrightarrow{f^*\mathrm{d}_{X/Y}(t)} f^*\Omega_{X/Y}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*\mathrm{d}_{X/Y}(t)$$

as maps from $f^{-1}h_U^\#$ to $\Omega_{X'/Y'}$. We omit the verification that this property holds for c_f as defined in the lemma. (Hint: The first map $c'_f : \Omega_{X/Y} \to f_*\Omega_{X'/Y'}$ satisfies $c'_f(\mathrm{d}_{X/Y}(t)) = f_*\mathrm{d}_{X'/Y'}(f^\sharp(t))$ as sections of $f_*\Omega_{X'/Y'}$ over U, and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes c_f is that the images of $f^*\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_{X'}$-module $f^*\Omega_{X/Y}$ simply because the local sections $\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_X$-module $\Omega_{X/Y}$. □

18.34. Finite order differential operators

09CQ In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133).

09CR **Definition 18.34.1.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k* is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(\mathcal{F}, \mathcal{G}) = \mathrm{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(\mathcal{C}, \mathcal{O}_2)$-module. We have

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \in \mathrm{Ob}(\mathcal{C})$ to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on the site $\mathcal{C}$. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

09CS **Lemma 18.34.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

09CT **Lemma 18.34.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism*

$$\mathit{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 18.33.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. □

09CU **Definition 18.34.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 18.34.3 is called the *module of principal parts of order k* of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

09CV **Lemma 18.34.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.*

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 18.33.4. Perhaps a more pleasing approach is to use the universal property of Lemma 18.34.3 directly to see the equality. We omit the details. □

09CW **Lemma 18.34.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence*

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 10.133.6) and Lemmas 18.33.4 and 18.34.5. □

09CX **Remark 18.34.7.** Let $\mathcal{C}$ be a site. Suppose given a commutative diagram of sheaves of rings

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ & & \uparrow & & \uparrow & & \uparrow \\ \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 18.34.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 18.34.6.

18.35. The naive cotangent complex

08TT This section is the analogue of Algebra, Section 10.134 and Modules, Section 17.31. We advise the reader to read those sections first.

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. In this section, for any sheaf of sets $\mathcal{E}$ on $\mathcal{C}$ we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$

whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

08TU (18.35.0.1) $$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 18.33.8 there is a canonical map

08TV (18.35.0.2) $$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

08TW **Definition 18.35.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (18.35.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 18.33.7 for modules of differentials. Namely, given a commutative diagram

08TX (18.35.1.1)
$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on $\mathcal{C}$ and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

08TY **Lemma 18.35.2.** *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this we argue as follows. First, observe that $H^0(NL(\alpha_i)) = \Omega_{\mathcal{B}/\mathcal{A}}$ and $H^0(NL(\alpha)) = \Omega_{\mathcal{B}/\mathcal{A}}$ by Lemma 18.33.8 hence the map is an isomorphism on cohomology sheaves in degree 0. Similarly, we claim that $H^{-1}(NL(\alpha_i))$ and $H^{-1}(NL(\alpha))$ are the sheaves associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ where $H_1(L_{-/-})$ is as in Algebra, Definition 10.134.1. If the claim holds, then the proof is finished.

Proof of the claim. Let $\alpha : \mathcal{E} \to \mathcal{B}$ be as above. Let $\mathcal{B}' \subset \mathcal{B}$ be the subpresheaf of $\mathcal{A}$-algebras whose value on U is the image of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Let $\mathcal{I}'$ be the presheaf whose value on U is the kernel of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Then $\mathcal{I}$ is the sheafification of $\mathcal{I}'$ and $\mathcal{B}$ is the sheafification of $\mathcal{B}'$. Similarly, $H^{-1}(NL(\alpha))$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Ker}(\mathcal{I}'(U)/\mathcal{I}'(U)^2 \to \Omega_{\mathcal{A}(U)[\mathcal{E}(U)]/\mathcal{A}(U)} \otimes_{\mathcal{A}(U)[\mathcal{E}(U)]} \mathcal{B}'(U))$$

by Lemma 18.33.4. By Algebra, Lemma 10.134.2 we conclude $H^{-1}(NL(\alpha))$ is the sheaf associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$. Thus we have to show that the maps $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)}) \to H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ induce an isomorphism $\mathcal{H}'_1 \to \mathcal{H}_1$ of sheafifications.

Injectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$. Let $f \in H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ map to zero in $\mathcal{H}_1(U)$. To show: f maps to zero in $\mathcal{H}'_1(U)$. The assumption means there is a covering $\{U_i \to U\}$ such that f maps to zero in $H_1(L_{\mathcal{B}(U_i)/\mathcal{A}(U_i)})$ for all i. Replace U by U_i to get to the point where f maps to zero in $H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. By Algebra, Lemma 10.134.9 we can find a finitely generated subalgebra $\mathcal{B}'(U) \subset B \subset \mathcal{B}(U)$ such that f maps to zero in $H_1(L_{B/\mathcal{A}(U)})$. Since $\mathcal{B} = (\mathcal{B}')^\#$ we can find a covering $\{U_i \to U\}$ such that $B \to \mathcal{B}(U_i)$ factors through $\mathcal{B}'(U_i)$. Hence f maps to zero in $H_1(L_{\mathcal{B}'(U_i)/\mathcal{A}(U_i)})$ as desired.

The surjectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$ is proved in exactly the same way. □

08TZ **Lemma 18.35.3.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{D}$. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 18.33.5. □

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

08U0 **Definition 18.35.4.** Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. The *naive cotangent complex* $NL_f = NL_{X/Y}$ of the given morphism of ringed topoi is $NL_{\mathcal{O}/f^{-1}\mathcal{O}'}$. We sometimes write $NL_{X/Y} = NL_{\mathcal{O}/\mathcal{O}'}$.

18.36. Stalks of modules

04EM We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 7.32.1.1) may not be filtered[3]. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

04EN **Lemma 18.36.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

[3]Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

(1) *We have* $(\mathcal{F}^\#)_p = \mathcal{F}_p$ *for any presheaf of sets on* $\mathcal{C}$.
(2) *The stalk functor* $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ *is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.*
(3) *The stalk functor* $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ *is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.*

Proof. By Sites, Lemma 7.32.5 we have (1). By Sites, Lemmas 7.32.4 we see that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is a left adjoint, and by Sites, Lemma 7.32.5 we see the same thing for $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$. Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 4.24.5). It follows from the definition of a point of a site, see Sites, Definition 7.32.2 that $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. Since sheafification is exact (Sites, Lemma 7.10.15) it follows that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. □

In particular, since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 7.44.3. The result of such an argument is that if $\mathcal{F}$ is a (pre)sheaf of algebraic structures listed in Sites, Proposition 7.44.3 then the stalk $\mathcal{F}_p$ is naturally an algebraic structure of the same kind. Let us explain this in detail when $\mathcal{F}$ is an abelian presheaf. In this case the addition map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ induces a map

$$+ : \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk $\mathcal{F}_p$. In this way we obtain our stalk functor

$$PAb(\mathcal{C}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion $Ab(\mathcal{C}) \subset PAb(\mathcal{C})$.

04EP **Lemma 18.36.2.** *Let* $\mathcal{C}$ *be a site. Let* p *be a point of* $\mathcal{C}$.

(1) *The functor* $Ab(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ *is exact.*
(2) *The stalk functor* $PAb(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ *is exact.*
(3) *For* $\mathcal{F} \in \mathrm{Ob}(PAb(\mathcal{C}))$ *we have* $\mathcal{F}_p = \mathcal{F}_p^\#$.

Proof. This is formal from the results of Lemma 18.36.1 and the construction of the stalk functor above. □

Next, we turn to the case of sheaves of modules. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. (It suffices for the discussion that $\mathcal{O}$ be a presheaf of rings.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let p be a point of $\mathcal{C}$. In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on $\mathcal{O}_p$ and an $\mathcal{O}_p$-module structure on $\mathcal{F}_p$. In this way we obtain a functor

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of $\mathcal{O}$-modules by precomposing with the inclusion $Mod(\mathcal{O}) \subset PMod(\mathcal{O})$.

04EQ **Lemma 18.36.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$.*

(1) *The functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(2) *The stalk functor $PMod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(3) *For $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}))$ we have $\mathcal{F}_p = \mathcal{F}_p^\#$.*

Proof. This is formal from the results of Lemma 18.36.2, the construction of the stalk functor above, and Lemma 18.14.1. □

05V5 **Lemma 18.36.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let p be a point of $\mathcal{C}$ or $Sh(\mathcal{C})$ and set $q = f \circ p$. Then*

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{F}$.

Proof. We have

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C}$$

by definition. Since taking stalks at p (i.e., applying p^{-1}) commutes with $\otimes$ by Lemma 18.26.2 we win by the relation between the stalk of pullbacks at p and stalks at q explained in Sites, Lemma 7.34.2 or Sites, Lemma 7.34.3. □

18.37. Skyscraper sheaves

05V6 Let p be a point of a site $\mathcal{C}$ or a topos $Sh(\mathcal{C})$. In this section we study the exactness properties of the functor which associates to an abelian group A the skyscraper sheaf p_*A. First, recall that $p_* : Sets \to Sh(\mathcal{C})$ has a lot of exactness properties, see Sites, Lemmas 7.32.9 and 7.32.10.

05V7 **Lemma 18.37.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ or of its associated topos.*

(1) *The functor $p_* : Ab \to Ab(\mathcal{C})$, $A \mapsto p_*A$ is exact.*
(2) *There is a functorial direct sum decomposition*

$$p^{-1}p_*A = A \oplus I(A)$$

for $A \in \mathrm{Ob}(Ab)$.

Proof. By Sites, Lemma 7.32.9 there are functorial maps $A \to p^{-1}p_*A \to A$ whose composition equals id_A. Hence a functorial direct sum decomposition as in (2) with $I(A)$ the kernel of the adjunction map $p^{-1}p_*A \to A$. The functor p_* is left exact by Lemma 18.14.3. The functor p_* transforms surjections into surjections by Sites, Lemma 7.32.10. Hence (1) holds. □

To do the same thing for sheaves of modules, suppose given a point p of a ringed topos $(Sh(\mathcal{C}), \mathcal{O})$. Recall that p^{-1} is just the stalk functor. Hence we can think of p as a morphism of ringed topoi

$$(p, \mathrm{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ which equals the stalk functor, and which we discussed in Lemma 18.36.3. In this section we consider the functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$.

05V8 **Lemma 18.37.2.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let p be a point of the topos $Sh(\mathcal{C})$.*

(1) *The functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$, $M \mapsto p_*M$ is exact.*
(2) *The canonical surjection $p^{-1}p_*M \to M$ is $\mathcal{O}_p$-linear.*
(3) *The functorial direct sum decomposition $p^{-1}p_*M = M \oplus I(M)$ of Lemma 18.37.1 is* **not** *$\mathcal{O}_p$-linear in general.*

Proof. Part (1) and surjectivity in (2) follow immediately from the corresponding result for abelian sheaves in Lemma 18.37.1. Since $p^{-1}\mathcal{O} = \mathcal{O}_p$ we have $p^{-1} = p^*$ and hence $p^{-1}p_*M \to M$ is the same as the counit $p^*p_*M \to M$ of the adjunction for modules, whence linear.

Proof of (3). Suppose that G is a group. Consider the topos $G\text{-}Sets = Sh(\mathcal{T}_G)$ and the point $p : Sets \to G\text{-}Sets$. See Sites, Section 7.9 and Example 7.33.7. Here p^{-1} is the functor forgetting about the G-action. And p_* is the right adjoint of the forgetful functor, sending M to $\mathrm{Map}(G, M)$. The maps in the direct sum decomposition are the maps

$$M \to \mathrm{Map}(G, M) \to M$$

where the first sends $m \in M$ to the constant map with value m and where the second map is evaluation at the identity element 1 of G. Next, suppose that R is a ring endowed with an action of G. This determines a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. The category of $\mathcal{O}$-modules is the category of R-modules M endowed with an action of G compatible with the action on R. The R-module structure on $\mathrm{Map}(G, M)$ is given by

$$(rf)(\sigma) = \sigma(r)f(\sigma)$$

for $r \in R$ and $f \in \mathrm{Map}(G, M)$. This is true because it is the unique G-invariant R-module structure compatible with evaluation at 1. The reader observes that in general the image of $M \to \mathrm{Map}(G, M)$ is not an R-submodule (for example take $M = R$ and assume the G-action is nontrivial), which concludes the proof. □

05V9 **Example 18.37.3.** Let G be a group. Consider the site $\mathcal{T}_G$ and its point p, see Sites, Example 7.33.7. Let R be a ring with a G-action which corresponds to a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. Then $\mathcal{O}_p = R$ where we forget the G-action. In this case $p^{-1}p_*M = \mathrm{Map}(G, M)$ and $I(M) = \{f : G \to M \mid f(1_G) = 0\}$ and $M \to \mathrm{Map}(G, M)$ assigns to $m \in M$ the constant function with value m.

18.38. Localization and points

070Z

0710 **Lemma 18.38.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. Let U be an object of $\mathcal{C}$. For $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ we have*

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ lying over p, see Sites, Lemma 7.35.2.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \bigoplus_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 18.19.2. The stalk of $j_{U!}\mathcal{G}$ at p is equal

to the stalk of this presheaf, see Lemma 18.36.3. Let $u : \mathcal{C} \to Sets$ be the functor corresponding to p (see Sites, Section 7.32). Hence we see that

$$(j_{U!}\mathcal{G})_p = \text{colim}_{(V,y)} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple (V, y, φ, s) occurring in this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_{x \in u(U)} \text{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by the description of the points q lying over x in Sites, Lemma 7.35.2. □

0711 **Remark 18.38.2.** Warning: The result of Lemma 18.38.1 has no analogue for $j_{U,*}$.

18.39. Pullbacks of flat modules

05VA The pullback of a flat module along a morphism of ringed topoi is flat. This is a bit tricky to prove.

05VD **Lemma 18.39.1.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Then $f^*\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{C}$-module whenever $\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{D}$-module.* [**AGV71**, Exposé V, Corollary 1.7.1]

Proof. Choose a diagram as in Lemma 18.7.2. Recall that being a flat module is intrinsic (see Section 18.18 and Definition 18.28.1). Hence it suffices to prove the lemma for the morphism $(h, h^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \to (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})$. In other words, we may assume that our sites $\mathcal{C}$ and $\mathcal{D}$ have all finite limits and that f is a morphism of sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ which commutes with finite limits.

Recall that $f^*\mathcal{F} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{F}$ (Definition 18.13.1). By Lemma 18.28.13 it suffices to prove that $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_\mathcal{D}$-module. Combined with the previous paragraph this reduces us to the situation of the next paragraph.

Assume $\mathcal{C}$ and $\mathcal{D}$ are sites which have all finite limits and that $u : \mathcal{D} \to \mathcal{C}$ is a continuous functor which commutes with finite limits. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$ and let $\mathcal{F}$ be a flat $\mathcal{O}$-module. Then u defines a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ (Sites, Proposition 7.14.7). To show: $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}$-module. Let U be an object of $\mathcal{C}$ and let

$$f^{-1}\mathcal{O}|_U \xrightarrow{(f_1, \ldots, f_n)} f^{-1}\mathcal{O}|_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} f^{-1}\mathcal{F}|_U$$

be a complex of $f^{-1}\mathcal{O}|_U$-modules. Our goal is to construct a factorization of $(s_1, \ldots, s_n)$ on the members of a covering of U as in Lemma 18.28.14 part (2). Consider the elements $s_a \in f^{-1}\mathcal{F}(U)$ and $f_a \in f^{-1}\mathcal{O}(U)$. Since $f^{-1}\mathcal{F}$, resp. $f^{-1}\mathcal{O}$ is the sheafification of $u_p\mathcal{F}$ we may, after replacing U by the members of a covering, assume that s_a is the image of an element $s'_a \in u_p\mathcal{F}(U)$ and f_a is the image of an element $f'_a \in u_p\mathcal{O}(U)$. Then after another replacement of U by the members of a covering we may assume that $\sum f'_a s'_a$ is zero in $u_p\mathcal{F}(U)$. Recall that the category $(\mathcal{I}^u_U)^{opp}$ is directed (Sites, Lemma 7.5.2) and that $u_p\mathcal{F}(U) = \text{colim}_{(\mathcal{I}^u_U)^{opp}} \mathcal{F}(V)$ and $u_p\mathcal{O}(U) = \text{colim}_{(\mathcal{I}^u_U)^{opp}} \mathcal{O}(V)$. Hence we may assume there is a pair $(V, \phi) \in \text{Ob}(\mathcal{I}^u_U)$ where V is an object of $\mathcal{D}$ and ϕ is a morphism $\phi : U \to u(V)$ of $\mathcal{D}$ and elements

$s''_a \in \mathcal{F}(V)$ and $f''_a \in \mathcal{O}(V)$ whose images in $u_p\mathcal{F}(U)$ and $u_p\mathcal{O}(U)$ are equal to s'_a and f'_a and such that $\sum f''_a s''_a = 0$ in $\mathcal{F}(V)$. Then we obtain a complex

$$\mathcal{O}|_V \xrightarrow{(f''_1,\ldots,f''_n)} \mathcal{O}|_V^{\oplus n} \xrightarrow{(s''_1,\ldots,s''_n)} \mathcal{F}|_V$$

and we can apply the other direction of Lemma 18.28.14 to see there exists a covering $\{V_i \to V\}$ of $\mathcal{D}$ and for each i a factorization

$$\mathcal{O}|_{V_i}^{\oplus n} \xrightarrow{B''_i} \mathcal{O}|_{V_i}^{\oplus l_i} \xrightarrow{(t''_{i1},\ldots,t''_{il_i})} \mathcal{F}|_{V_i}$$

of $(s''_1,\ldots,s''_n)|_{V_i}$ such that $B_i \circ (f''_1,\ldots,f''_n)|_{V_i} = 0$. Set $U_i = U \times_{\phi,u(V)} u(V_i)$, denote $B_i \in \mathrm{Mat}(l_i \times n, f^{-1}\mathcal{O}(U_i))$ the image of B''_i, and denote $t_{ij} \in f^{-1}\mathcal{F}(U_i)$ the image of t''_{ij}. Then we get a factorization

$$f^{-1}\mathcal{O}|_{U_i}^{\oplus n} \xrightarrow{B_i} f^{-1}\mathcal{O}|_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1},\ldots,t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1,\ldots,s_n)|_{U_i}$ such that $B_i \circ (f_1,\ldots,f_n)|_{U_i} = 0$. This finishes the proof. □

05VB **Lemma 18.39.2.** *Let $(\mathcal{C},\mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module.*

Proof. In Section 18.37 we have seen that we can think of p as a morphism of ringed topoi

$$(p, \mathrm{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

such that the pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ equals the stalk functor. Thus the lemma follows from Lemma 18.39.1. □

05VC **Lemma 18.39.3.** *Let $(\mathcal{C},\mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\{p_i\}_{i\in I}$ be a conservative family of points of $\mathcal{C}$. Then $\mathcal{F}$ is flat if and only if $\mathcal{F}_{p_i}$ is a flat $\mathcal{O}_{p_i}$-module for all $i \in I$.*

Proof. By Lemma 18.39.2 we see one of the implications. For the converse, use that $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$ by Lemma 18.26.2 (as taking stalks at p is given by p^{-1}) and Lemma 18.14.4. □

0G6R **Lemma 18.39.4.** *Let $f : (Sh(\mathcal{C}'),\mathcal{O}') \to (Sh(\mathcal{C}),\mathcal{O})$ be a morphism of ringed topoi. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{H}$ a flat $\mathcal{O}$-module. Then the sequence $0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$ is exact as well.*

Proof. Since f^{-1} is exact we have the short exact sequence $0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$ of $f^{-1}\mathcal{O}$-modules. By Lemma 18.39.1 the $f^{-1}\mathcal{O}$-module $f^{-1}\mathcal{H}$ is flat. By Lemma 18.28.9 this implies that tensoring the sequence over $f^{-1}\mathcal{O}$ with $\mathcal{O}'$ the sequence remains exact. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}} \mathcal{O}'$ and similarly for $\mathcal{G}$ and $\mathcal{H}$ we conclude. □

18.40. Locally ringed topoi

04ER A reference for this section is [**AGV71**, Exposé IV, Exercice 13.9].

04ES **Lemma 18.40.1.** *Let $(\mathcal{C},\mathcal{O})$ be a ringed site. The following are equivalent*

1. *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
2. *For $U \in \mathrm{Ob}(\mathcal{C})$, $n \geq 1$, and $f_1,\ldots,f_n \in \mathcal{O}(U)$ which generate the unit ideal in $\mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j there exists an i such that $f_i|_{U_j}$ is invertible.*

(3) *The map of sheaves of sets*

$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$

which maps (f, a) in the first component to (f, af) and (f, b) in the second component to $(f, b(1-f))$ is surjective.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on n. The first case is $n = 2$ (since $n = 1$ is trivial). In this case we have $a_1 f_1 + a_2 f_2 = 1$ for some $a_1, a_2 \in \mathcal{O}(U)$. By assumption we can find a covering $\{U_j \to U\}$ such that for each j either $a_1 f_1|_{U_j}$ is invertible or $a_2 f_2|_{U_j}$ is invertible. Hence either $f_1|_{U_j}$ is invertible or $f_2|_{U_j}$ is invertible as desired. For $n > 2$ we have $a_1 f_1 + \ldots + a_n f_n = 1$ for some $a_1, \ldots, a_n \in \mathcal{O}(U)$. By the case $n = 2$ we see that we have some covering $\{U_j \to U\}_{j \in J}$ such that for each j either $f_n|_{U_j}$ is invertible or $a_1 f_1 + \ldots + a_{n-1} f_{n-1}|_{U_j}$ is invertible. Say the first case happens for $j \in J_n$. Set $J' = J \setminus J_n$. By induction hypothesis, for each $j \in J'$ we can find a covering $\{U_{jk} \to U_j\}_{k \in K_j}$ such that for each $k \in K_j$ there exists an $i \in \{1, \ldots, n-1\}$ such that $f_i|_{U_{jk}}$ is invertible. By the axioms of a site the family of morphisms $\{U_j \to U\}_{j \in J_n} \cup \{U_{jk} \to U\}_{j \in J', k \in K_j}$ is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let (f, c) be a section of $\mathcal{O} \times \mathcal{O}$ over U. By assumption there exists a covering $\{U_j \to U\}$ such that for each j either f or $1 - f$ restricts to an invertible section. In the first case we can take $a = c|_{U_j}(f|_{U_j})^{-1}$, and in the second case we can take $b = c|_{U_j}(1 - f|_{U_j})^{-1}$. Hence (f, c) is in the image of the map on each of the members. Conversely, assume (3) holds. For any U and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ of U such that the section $(f, 1)|_{U_j}$ is in the image of the map in (3) on sections over U_j. This means precisely that either f or $1 - f$ restricts to an invertible section over U_j, and we see that (1) holds. □

04ET **Lemma 18.40.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the following conditions*

(1) *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1 - f)|_{U_j}$ is invertible.*
(2) *For every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is either the zero ring or a local ring.*

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent.

Proof. Assume (1). Let p be a point of $\mathcal{C}$ given by a functor $u : \mathcal{C} \to Sets$. Let $f_p \in \mathcal{O}_p$. Since $\mathcal{O}_p$ is computed by Sites, Equation (7.32.1.1) we may represent f_p by a triple (U, x, f) where $x \in U(U)$ and $f \in \mathcal{O}(U)$. By assumption there exists a covering $\{U_i \to U\}$ such that for each i either f or $1 - f$ is invertible on U_i. Because u defines a point of the site we see that for some i there exists an $x_i \in u(U_i)$ which maps to $x \in u(U)$. By the discussion surrounding Sites, Equation (7.32.1.1) we see that (U, x, f) and $(U_i, x_i, f|_{U_i})$ define the same element of $\mathcal{O}_p$. Hence we conclude that either f_p or $1 - f_p$ is invertible. Thus $\mathcal{O}_p$ is a ring such that for every element a either a or $1 - a$ is invertible. This means that $\mathcal{O}_p$ is either zero or a local ring, see Algebra, Lemma 10.18.3.

Assume (2) and assume that $\mathcal{C}$ has enough points. Consider the map of sheaves of sets

$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$

of Lemma 18.40.1 part (3). For any local ring R the corresponding map $(R \times R) \amalg (R \times R) \to R \times R$ is surjective, see for example Algebra, Lemma 10.18.3. Since each

$\mathcal{O}_p$ is a local ring or zero the map is surjective on stalks. Hence, by our assumption that $\mathcal{C}$ has enough points it is surjective and we win. □

In Modules, Section 17.2 we pointed out how in a ringed space $(X, \mathcal{O}_X)$ there can be an open subspace over which the structure sheaf is zero. To prevent this we can require the sections 1 and 0 to have different values in every stalk of the space X. In the setting of ringed topoi and ringed sites the condition is that

05D7 (18.40.2.1) $$\emptyset^\# \longrightarrow \text{Equalizer}(0, 1 : * \longrightarrow \mathcal{O})$$

is an isomorphism of sheaves. Here $*$ is the singleton sheaf, resp. $\emptyset^\#$ is the "empty sheaf", i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 7.10.2, resp. Section 7.42. In other words, the condition is that whenever $U \in \text{Ob}(\mathcal{C})$ is not sheaf theoretically empty, then $1, 0 \in \mathcal{O}(U)$ are not equal. Let us state the obligatory lemma.

05D8 **Lemma 18.40.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the statements*

(1) *(18.40.2.1) is an isomorphism, and*
(2) *for every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is not the zero ring.*

We always have (1) $\Rightarrow$ (2) and if $\mathcal{C}$ has enough points then (1) $\Leftrightarrow$ (2).

Proof. Omitted. □

Lemmas 18.40.1, 18.40.2, and 18.40.3 motivate the following definition.

04EU **Definition 18.40.4.** A ringed site $(\mathcal{C}, \mathcal{O})$ is said to be *locally ringed site* if (18.40.2.1) is an isomorphism, and the equivalent properties of Lemma 18.40.1 are satisfied.

In [**AGV71**, Exposé IV, Exercice 13.9] the condition that (18.40.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topos. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

04H7 **Lemma 18.40.5.** *Being a locally ringed site is an intrinsic property. More precisely,*

(1) *if $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site, and*
(2) *if $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi, then $(\mathcal{C}, \mathcal{O})$ is locally ringed if and only if $(\mathcal{C}', \mathcal{O}')$ is locally ringed.*

Proof. It is clear that (2) follows from (1). To prove (1) note that as f^{-1} is exact we have $f^{-1}* = *$, $f^{-1}\emptyset^\# = \emptyset^\#$, and f^{-1} commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus f^{-1} transforms the isomorphism (18.40.2.1) into its analogue for $f^{-1}\mathcal{O}$ and transforms the surjection of Lemma 18.40.1 part (3) into the corresponding surjection for $f^{-1}\mathcal{O}$. □

In fact Lemma 18.40.5 part (2) is the analogue of Schemes, Lemma 26.2.2. It assures us that the following definition makes sense.

04H8 **Definition 18.40.6.** A ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is said to be *locally ringed* if the underlying ringed site $(\mathcal{C}, \mathcal{O})$ is locally ringed.

Here is an example of a consequence of being locally ringed.

0B8Q **Lemma 18.40.7.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Any locally free $\mathcal{O}$-module of rank 1 is invertible. If $(\mathcal{C}, \mathcal{O})$ is locally ringed, then the converse holds as well (but in general this is not the case).*

Proof. Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \longrightarrow \mathcal{O}$$

Given any object U of $\mathcal{C}$ and restricting to the members of a covering trivializing $\mathcal{L}$, we see that this map is an isomorphism (details omitted). Hence $\mathcal{L}$ is invertible by Lemma 18.32.2.

Assume $(Sh(\mathcal{C}), \mathcal{O})$ is locally ringed. Let U be an object of $\mathcal{C}$. In the proof of Lemma 18.32.2 we have seen that there exists a covering $\{U_i \to U\}$ such that $\mathcal{L}|_{\mathcal{C}/U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module. After replacing U by U_i, let $p : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus r}$ be a projector whose image is isomorphic to $\mathcal{L}|_{\mathcal{C}/U}$. Then p corresponds to a matrix

$$P = (p_{ij}) \in \mathrm{Mat}(r \times r, \mathcal{O}(U))$$

which is a projector: $P^2 = P$. Set $A = \mathcal{O}(U)$ so that $P \in \mathrm{Mat}(r \times r, A)$. By Algebra, Lemma 10.78.2 the image of P is a finite locally free module M over A. Hence there are $f_1, \ldots, f_t \in A$ generating the unit ideal, such that M_{f_i} is finite free. By Lemma 18.40.1 after replacing U by the members of an open covering, we may assume that M is free. This means that $\mathcal{L}|_U$ is free (details omitted). Of course, since $\mathcal{L}$ is invertible, this is only possible if the rank of $\mathcal{L}|_U$ is 1 and the proof is complete. □

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

04H9 **Lemma 18.40.8.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the following conditions*

(1) *The diagram of sheaves*

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_\mathcal{D}^*) & \xrightarrow[f^\sharp]{} & \mathcal{O}_\mathcal{C}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_\mathcal{D}) & \xrightarrow{f^\sharp} & \mathcal{O}_\mathcal{C} \end{array}$$

is cartesian.

(2) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the diagram*

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{D},q}^* & \longrightarrow & \mathcal{O}_{\mathcal{C},p}^* \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{D},q} & \longrightarrow & \mathcal{O}_{\mathcal{C},p} \end{array}$$

of sets is cartesian.

We always have (1) ⇒ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent. If $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi then (2) is equivalent to

(3) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the ring map $\mathcal{O}_{\mathcal{D},q} \to \mathcal{O}_{\mathcal{C},p}$ is a local ring map.*

In fact, properties (2), or (3) for a conservative family of points implies (1).

Proof. This lemma proves itself, in other words, it follows by unwinding the definitions. □

04HA **Definition 18.40.9.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Assume $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi. We say that $(f, f^\sharp)$ is a *morphism of locally ringed topoi* if and only if the diagram of sheaves

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_\mathcal{D}^*) & \xrightarrow[f^\sharp]{} & \mathcal{O}_\mathcal{C}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_\mathcal{D}) & \xrightarrow{f^\sharp} & \mathcal{O}_\mathcal{C} \end{array}$$

(see Lemma 18.40.8) is cartesian. If $(f, f^\sharp)$ is a morphism of ringed sites, then we say that it is a *morphism of locally ringed sites* if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

04IG **Lemma 18.40.10.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of locally ringed topoi. Then the composition $(g, g^\sharp) \circ (f, f^\sharp)$ (see Definition 18.7.1) is also a morphism of locally ringed topoi.*

Proof. Omitted. □

04KR **Lemma 18.40.11.** *Let $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be a morphism of topoi. If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, then*

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if $\mathcal{O}$ turns $\mathcal{C}$ into a locally ringed site, then setting $f^\sharp = id$ the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \to (Sh(\mathcal{C}, \mathcal{O})$$

is a morphism of locally ringed topoi.

Proof. Note that the diagram

$$\begin{array}{ccc} \mathcal{O}^* & \longrightarrow & * \\ {\scriptstyle u\mapsto(u,u^{-1})}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathcal{O} \times \mathcal{O} & \xrightarrow{(a,b)\mapsto ab} & \mathcal{O} \end{array}$$

is cartesian. Since f^{-1} is exact we conclude that

$$\begin{array}{ccc} f^{-1}(\mathcal{O}^*) & \longrightarrow & * \\ {\scriptstyle u\mapsto(u,u^{-1})}\downarrow & & \downarrow{\scriptstyle 1} \\ f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} & \xrightarrow{(a,b)\mapsto ab} & f^{-1}\mathcal{O} \end{array}$$

is cartesian which implies the first assertion. For the second, note that $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site by Lemma 18.40.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi. □

04IH **Lemma 18.40.12.** *Localization of locally ringed sites and topoi.*

(1) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let U be an object of $\mathcal{C}$. Then the localization $(\mathcal{C}/U, \mathcal{O}_U)$ is a locally ringed site, and the localization morphism*
$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(2) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then the morphism*
$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \to (Sh(\mathcal{C}/U), \mathcal{O}_U)$$
of Lemma 18.19.5 is a morphism of locally ringed topoi.

(3) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and let $U = u(V)$. Then the morphism*
$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.1 is a morphism of locally ringed sites.

(4) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$, and $c : U \to u(V)$. Then the morphism*
$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.2 is a morphism of locally ringed topoi.

(5) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the localization $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a locally ringed topos and the localization morphism*
$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(6) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $s : \mathcal{G} \to \mathcal{F}$ be a map of sheaves on $\mathcal{C}$. Then the morphism*
$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$$
of Lemma 18.21.4 is a morphism of locally ringed topoi.

(7) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then the morphism*
$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.1 is a morphism of locally ringed topoi.

(8) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be a morphism of sheaves. Then the morphism*
$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.3 is a morphism of locally ringed topoi.

Proof. Part (1) is clear since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}$, so Lemmas 18.40.5 and 18.40.11 apply. Part (2) is clear as the morphism $(j, j^\sharp)$ is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since $(f')^\sharp$ is just the restriction of $f^\sharp$ to the topos $Sh(\mathcal{C})/\mathcal{F}$, see proof of Lemma 18.22.1 (hence the diagram of Definition 18.40.9 for the morphism f' is just the restriction of the corresponding diagram for f, and restriction is an exact functor). Part (4)

follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 18.7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 18.21.3, 18.21.5, 18.22.2, and 18.22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.) □

18.41. Lower shriek for modules

0796 In this section we extend the construction of $g_!$ discussed in Section 18.16 to the case of sheaves of modules.

0797 **Lemma 18.41.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the associated morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. Hence g becomes a morphism of ringed topoi with $g^* = g^{-1}$. In this case there exists a functor*

$$g_! : Mod(\mathcal{O}_\mathcal{C}) \longrightarrow Mod(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. Let U be an object of $\mathcal{C}$. For any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because g^{-1} is described by restriction, see Sites, Lemma 7.21.5. Of course a similar formula holds a direct sum of modules of the form $j_{U!}\mathcal{O}_U$. By Homology, Lemma 12.29.6 and Lemma 18.28.8 we see that $g_!$ exists. □

0798 **Remark 18.41.2.** Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 18.41.1. In general it is **not** the case that the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_\mathcal{C}) & \xrightarrow{g_!} & Mod(\mathcal{O}_\mathcal{D}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ Ab(\mathcal{C}) & \xrightarrow{g_!^{Ab}} & Ab(\mathcal{D}) \end{array}$$

commutes (here $g_!^{Ab}$ is the one from Lemma 18.16.2). There is a transformation of functors

$$g_!^{Ab} \circ forget \longrightarrow forget \circ g_!$$

From the proof of Lemma 18.41.1 we see that this is an isomorphism if and only if $g_!^{Ab} j_{U!}\mathcal{O}_U \to g_! j_{U!}\mathcal{O}_U$ is an isomorphism for all objects U of $\mathcal{C}$. Since we have $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$ this holds if and only if

$$g_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$. Note that for such a U we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{u'} & \mathcal{D}/u(U) \\ {\scriptstyle j_U}\downarrow & & \downarrow{\scriptstyle j_{u(U)}} \\ \mathcal{C} & \xrightarrow{u} & \mathcal{D} \end{array}$$

of cocontinuous functors of sites, see Sites, Lemma 7.28.4 and therefore $g_!^{Ab} j_{U!} = j_{u(U)!}(g')_!^{Ab}$ where $g' : Sh(\mathcal{C}/U) \to Sh(\mathcal{D}/u(U))$ is the morphism of topoi induced by the cocontinuous functor u'. Hence we see that $g_! = g_!^{Ab}$ if the canonical map

0799 (18.41.2.1) $$(g')_!^{Ab}\mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$.

The following two results are of a slightly different nature.

0FN3 **Lemma 18.41.3.** *Assume given a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[(g',(g')^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ {\scriptstyle (f',(f')^\sharp)}\downarrow & & \downarrow{\scriptstyle (f,f^\sharp)} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{(g,g^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Assume

(1) *f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 7.21.1,*
(2) *$v \circ u' = u \circ v'$,*
(3) *v and v' are continuous as well as cocontinuous,*
(4) *for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^{u}_{v(V')}\mathcal{I}$ given by v is cofinal, and*
(5) *$g^{-1}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$.*

Then we have $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 7.28.6. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 18.41.1 we see that the second equality follows by uniqueness of adjoint functors. □

0FN4 **Lemma 18.41.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[(g',(g')^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ {\scriptstyle (f',(f')^\sharp)}\downarrow & & \downarrow{\scriptstyle (f,f^\sharp)} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{(g,g^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi and suppose we have functors

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{v'} & \mathcal{C} \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

such that (with notation as in Sites, Sections 7.14 and 7.21) we have

(1) *u and u' are continuous and give rise to the morphisms f and f',*
(2) *v and v' are cocontinuous giving rise to the morphisms g and g',*
(3) *$u \circ v = v' \circ u'$,*
(4) *v and v' are continuous as well as cocontinuous, and*
(5) *$g^{-1}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$.*

Then $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 7.28.7. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 18.41.1 we see that the second equality follows by uniqueness of adjoint functors. □

18.42. Constant sheaves

093I Let E be a set and let $\mathcal{C}$ be a site. The constant sheaf with value E is denoted $\underline{E}$, see Sites, Example 7.10.13. If E is an abelian group, ring, module, etc, then $\underline{E}$ is a sheaf of abelian groups, rings, modules, etc.

093J **Lemma 18.42.1.** *Let $\mathcal{C}$ be a site. If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves and in fact it is even exact as a sequence of abelian presheaves.*

Proof. Since sheafification is exact it is clear that $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves. Thus $0 \to \underline{A} \to \underline{B} \to \underline{C}$ is an exact sequence of abelian presheaves. To see that $\underline{B} \to \underline{C}$ is surjective, pick a set theoretical section $s : C \to B$. This induces a section $\underline{s} : \underline{C} \to \underline{B}$ of sheaves of sets left inverse to the surjection $\underline{B} \to \underline{C}$. □

093K **Lemma 18.42.2.** *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M and Q be Λ-modules. If Q is a finitely presented Λ-module, then we have $\underline{M \otimes_\Lambda Q}(U) = \underline{M}(U) \otimes_\Lambda Q$ for all $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Choose a presentation $\Lambda^{\oplus m} \to \Lambda^{\oplus n} \to Q \to 0$. This gives an exact sequence $M^{\oplus m} \to M^{\oplus n} \to M \otimes Q \to 0$. By Lemma 18.42.1 we obtain an exact sequence

$$\underline{M}(U)^{\oplus m} \to \underline{M}(U)^{\oplus n} \to \underline{M \otimes Q}(U) \to 0$$

which proves the lemma. (Note that taking sections over U always commutes with finite direct sums, but not arbitrary direct sums.) □

093L **Lemma 18.42.3.** *Let $\mathcal{C}$ be a site. Let Λ be a coherent ring. Let M be a flat Λ-module. For $U \in \mathrm{Ob}(\mathcal{C})$ the module $\underline{M}(U)$ is a flat Λ-module.*

Proof. Let $I \subset \Lambda$ be a finitely generated ideal. By Algebra, Lemma 10.39.5 it suffices to show that $\underline{M}(U) \otimes_\Lambda I \to \underline{M}(U)$ is injective. As Λ is coherent I is finitely presented as a Λ-module. By Lemma 18.42.2 we see that $\underline{M}(U) \otimes I = \underline{M \otimes I}$. Since M is flat the map $M \otimes I \to M$ is injective, whence $\underline{M \otimes I} \to \underline{M}$ is injective. □

093M **Lemma 18.42.4.** *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. The sheaf $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ is a flat $\underline{\Lambda}$-algebra. Moreover we have canonical identifications*

$$\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I} \cdot \underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$$

where $\underline{I}^\wedge = \lim \underline{I/I^n}$.

Proof. To prove $\underline{\Lambda}^\wedge$ is flat, it suffices to show that $\underline{\Lambda}^\wedge(U)$ is flat as a Λ-module for each $U \in \mathrm{Ob}(\mathcal{C})$, see Lemmas 18.28.2 and 18.28.3. By Lemma 18.42.3 we see that

$$\underline{\Lambda}^\wedge(U) = \lim \underline{\Lambda/I^n}(U)$$

is a limit of a system of flat Λ/I^n-modules. By Lemma 18.42.1 we see that the transition maps are surjective. We conclude by More on Algebra, Lemma 15.27.4.

To see the equalities, note that $\underline{\Lambda}(U)/I\underline{\Lambda}(U) = \underline{\Lambda/I}(U)$ by Lemma 18.42.2. It follows that $\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda/I}$. The system of short exact sequences

$$0 \to \underline{I/I^n}(U) \to \underline{\Lambda/I^n}(U) \to \underline{\Lambda/I}(U) \to 0$$

has surjective transition maps, hence gives a short exact sequence

$$0 \to \lim \underline{I/I^n}(U) \to \lim \underline{\Lambda/I^n}(U) \to \lim \underline{\Lambda/I}(U) \to 0$$

see Homology, Lemma 12.31.3. Thus we see that $\underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$. Since

$$I\underline{\Lambda}^\wedge \subset \underline{I} \cdot \underline{\Lambda}^\wedge \subset \underline{I}^\wedge$$

it suffices to show that $I\underline{\Lambda}^\wedge(U) = \underline{I}^\wedge(U)$ for all U. Choose generators $I = (f_1, \ldots, f_r)$. For every n we obtain a short exact sequence

$$0 \to K_n/(I^n)^{\oplus r} \to (\Lambda/I^n)^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} I/I^{n+1} \to 0$$

where $K_n = \{(x_1, \ldots, x_r) \in \Lambda^{\oplus r} \mid \sum x_i f_i \in I^{n+1}\}$. We obtain short exact sequences

$$0 \to \underline{K_n/(I^n)^{\oplus r}}(U) \to \underline{(\Lambda/I^n)^{\oplus r}}(U) \to \underline{I/I^{n+1}}(U) \to 0$$

A calculation shows $K_n = K + (I^n)^{\oplus r}$, hence the transition maps $K_{n+1}/(I^{n+1})^{\oplus r} \to K_n/(I^n)^{\oplus r}$ are surjective. Hence the system of modules on the left hand side has surjective transition maps and a fortiori has ML. Thus we see that $(f_1, \ldots, f_r) : (\underline{\Lambda}^\wedge)^{\oplus r}(U) \to \underline{I}^\wedge(U)$ is surjective by Homology, Lemma 12.31.3 which is what we wanted to show. □

093N **Lemma 18.42.5.** *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M be a Λ-module. Assume $Sh(\mathcal{C})$ is not the empty topos. Then*

(1) *$\underline{M}$ is a finite type sheaf of $\underline{\Lambda}$-modules if and only if M is a finite Λ-module, and*
(2) *$\underline{M}$ is a finitely presented sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module.*

Proof. Proof of (1). If M is generated by $x_1, \ldots, x_r$ then $x_1, \ldots, x_r$ define global sections of $\underline{M}$ which generate it, hence $\underline{M}$ is of finite type. Conversely, assume $\underline{M}$ is of finite type. Let $U \in \mathcal{C}$ be an object which is not sheaf theoretically empty (Sites, Definition 7.42.1). Such an object exists as we assumed $Sh(\mathcal{C})$ is not the empty topos. Then there exists a covering $\{U_i \to U\}$ and finitely many sections $s_{ij} \in \underline{M}(U_i)$ generating $\underline{M}|_{U_i}$. After refining the covering we may assume that s_{ij} come from elements x_{ij} of M. Then x_{ij} define global sections of $\underline{M}$ whose restriction to U generate $\underline{M}$.

Assume there exist elements $x_1, \ldots, x_r$ of M which define global sections of $\underline{M}$ generating $\underline{M}$ as a sheaf of $\underline{\Lambda}$-modules. We will show that $x_1, \ldots, x_r$ generate M as a Λ-module. Let $x \in M$. We can find a covering $\{U_i \to U\}_{i \in I}$ and $f_{i,j} \in \underline{\Lambda}(U_i)$ such that $x|_{U_i} = \sum f_{i,j} x_j|_{U_i}$. After refining the covering we may assume $f_{i,j} \in \Lambda$. Since U is not sheaf theoretically empty, there is at least one $i \in I$ such that U_i is not sheaf theoretically empty. Then the map $M \to \underline{M}(U_i)$ is injective (details omitted). We conclude that $x = \sum f_{i,j} x_j$ in M as desired.

Proof of (2). Assume $\underline{M}$ is a $\underline{\Lambda}$-module of finite presentation. By (1) we see that M is of finite type. Choose generators $x_1, \ldots, x_r$ of M as a Λ-module. This determines a short exact sequence $0 \to K \to \Lambda^{\oplus r} \to M \to 0$ which turns into a short exact sequence

$$0 \to \underline{K} \to \underline{\Lambda}^{\oplus r} \to \underline{M} \to 0$$

by Lemma 18.42.1. By Lemma 18.24.2 we see that $\underline{K}$ is of finite type. Hence K is a finite Λ-module by (1). Thus M is a Λ-module of finite presentation. □

18.43. Locally constant sheaves

093P Here is the general definition.

093Q **Definition 18.43.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc.

(1) We say $\mathcal{F}$ is a *constant sheaf* of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc if it is isomorphic as a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc to a constant sheaf $\underline{E}$ as in Section 18.42.
(2) We say $\mathcal{F}$ is *locally constant* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(3) If $\mathcal{F}$ is a sheaf of sets or groups, then we say $\mathcal{F}$ is *finite locally constant* if the constant values are finite sets or finite groups.

093R **Lemma 18.43.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. If $\mathcal{G}$ is a locally constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc on $\mathcal{D}$, the same is true for $f^{-1}\mathcal{G}$ on $\mathcal{C}$.*

Proof. Omitted. □

093S **Lemma 18.43.3.** *Let $\mathcal{C}$ be a site with a final object X.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $\mathcal{C}$. If $\mathcal{F}$ is of finite type, then there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. Proof omitted. □

093T **Lemma 18.43.4.** *Let $\mathcal{C}$ be a site. Let Λ be a ring. Let M, N be Λ-modules. Let $\mathcal{F}, \mathcal{G}$ be a locally constant sheaves of Λ-modules.*

(1) *If M is of finite presentation, then*

$$\underline{\mathrm{Hom}_\Lambda(M, N)} = \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

(2) *If M and N are both of finite presentation, then*

$$\underline{Isom_\Lambda(M, N)} = Isom_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

(3) *If $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of Λ-modules.*
(4) *If $\mathcal{F}$ and $\mathcal{G}$ are both of finite presentation, then $Isom_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of sets.*

Proof. Proof of (1). Set $E = \mathrm{Hom}_\Lambda(M, N)$. We want to show the canonical map

$$\underline{E} \longrightarrow \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

is an isomorphism. The module M has a presentation $\Lambda^{\oplus s} \to \Lambda^{\oplus t} \to M \to 0$. Then E sits in an exact sequence

$$0 \to E \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus t}, N) \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus s}, N)$$

and we have similarly

$$0 \to \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus t}}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus s}}, \underline{N})$$

This reduces the question to the case where M is a finite free module where the result is clear.

Proof of (3). The question is local on $\mathcal{C}$, hence we may assume $\mathcal{F} = \underline{M}$ and $\mathcal{G} = \underline{N}$ for some Λ-modules M and N. By Lemma 18.42.5 the module M is of finite presentation. Thus the result follows from (1).

Parts (2) and (4) follow from parts (1) and (3) and the fact that *Isom* can be viewed as the subsheaf of sections of $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ which have an inverse in $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{G}, \mathcal{F})$. □

093U **Lemma 18.43.5.** *Let $\mathcal{C}$ be a site.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(\mathcal{C})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(\mathcal{C})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $\mathcal{C}$ is a weak Serre subcategory of $Mod(\mathcal{C}, \Lambda)$.*

Proof. Proof of (1). We may work locally on $\mathcal{C}$. Hence by Lemma 18.43.3 we may assume we are given a finite diagram of finite sets such that our diagram of sheaves is the associated diagram of constant sheaves. Then we just take the limit or colimit in the category of sets and take the associated constant sheaf. Some details omitted.

To prove (2) and (3) we use the criterion of Homology, Lemma 12.10.3. Existence of kernels and cokernels is argued in the same way as above. Of course, the reason for using a Noetherian ring in (3) is to assure us that the kernel of a map of finite Λ-modules is a finite Λ-module. To see that the category is closed under extensions (in the case of sheaves Λ-modules), assume given an extension of sheaves of Λ-modules

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

on $\mathcal{C}$ with $\mathcal{F}$, $\mathcal{G}$ finite type and locally constant. Localizing on $\mathcal{C}$ we may assume $\mathcal{F}$ and $\mathcal{G}$ are constant, i.e., we get

$$0 \to \underline{M} \to \mathcal{E} \to \underline{N} \to 0$$

for some Λ-modules M, N. Choose generators $y_1, \ldots, y_m$ of N, so that we get a short exact sequence $0 \to K \to \Lambda^{\oplus m} \to N \to 0$ of Λ-modules. Localizing further

we may assume y_j lifts to a section s_j of $\mathcal{E}$. Thus we see that $\mathcal{E}$ is a pushout as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{K} & \longrightarrow & \underline{\Lambda^{\oplus m}} & \longrightarrow & \underline{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{N} & \longrightarrow & 0 \end{array}$$

By Lemma 18.43.3 again (and the fact that K is a finite Λ-module as Λ is Noetherian) we see that the map $\underline{K} \to \underline{M}$ is locally constant, hence we conclude. □

093V **Lemma 18.43.6.** *Let $\mathcal{C}$ be a site. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $\mathcal{C}$ is a locally constant sheaf of Λ-modules.*

Proof. Omitted. □

18.44. Localizing sheaves of rings

0EMB Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset, see Algebra, Definition 10.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ for $V \to U$ in $\mathcal{C}$.

0EMC **Lemma 18.44.1.** *In the situation above the map to the sheafification*

$$\mathcal{O} \longrightarrow (\mathcal{S}^{-1}\mathcal{O})^\#$$

is a homomorphism of sheaves of rings with the following universal property: for any homomorphism of sheaves of rings $\mathcal{O} \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{O})^\# \to \mathcal{A}$.

Proof. Omitted. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \mathcal{C}$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \to U$ is a morphism of $\mathcal{C}$. Then $\mathcal{S}^{-1}\mathcal{F}$ is a presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules.

0EMD **Lemma 18.44.2.** *In the situation above the map to the sheafification*

$$\mathcal{F} \longrightarrow (\mathcal{S}^{-1}\mathcal{F})^\#$$

has the following universal property: for any homomorphism of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ such that each local section of $\mathcal{S}$ acts invertibly on $\mathcal{G}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{F})^\# \to \mathcal{G}$. Moreover we have

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O})^\# \otimes_{\mathcal{O}} \mathcal{F}$$

as sheaves of $(\mathcal{S}^{-1}\mathcal{O})^\#$-modules.

Proof. Omitted. □

18.45. Sheaves of pointed sets

0F4H In this section we collect some facts about sheaves of pointed sets which we've previously mentioned only for abelian sheaves.

A pointed set is a pair $(S, 0)$ where S is a set and $0 \in S$ is an element of S. A morphism $(S, 0) \to (S', 0')$ of pointed sets is simply a map of sets $S \to S'$ sending 0 to $0'$. We'll abuse notation and say "let S be a pointed set" to mean S is endowed with a marked element $0 \in S$. A sheaf of pointed sets is the same thing as a sheaf of sets $\mathcal{F}$ endowed with a "marking" $0 : * \to \mathcal{F}$ where $*$ is the final sheaf (Sites, Example 7.10.2).

Given a morphism of sites or of topoi, there are pushforward and pullback functors on the categories of sheaves of pointed sets, see Sites, Section 7.44. These are constructed by taking the pushforward, resp. pullback of the underlying sheaf of sets and suitably marking it (using that the pullback of the final sheaf is the final sheaf).

Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 7.21.1. Then g^{-1} on sheaves of pointed sets has an left adjoint $g_!$ as well. The construction of this functor is entirely analogous to the construction of $g_!$ on abelian sheaves in Section 18.16.

Similarly, if $j : \mathcal{C}/U \to \mathcal{C}$ is as in Section 18.19 then there is a left adjoint $j_!$ to the functor j^{-1} on sheaves of pointed sets

If we ever need these facts and constructions we will precisely state and prove here the corresponding lemmas.

18.46. Other chapters

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CHAPTER 19

Injectives

01D4 **19.1. Introduction**

01D5 In future chapters we will use the existence of injectives and K-injective complexes to do cohomology of sheaves of modules on ringed sites. In this chapter we explain how to produce injectives and K-injective complexes first for modules on sites and later more generally for Grothendieck abelian categories.

We observe that we already know that the category of abelian groups and the category of modules over a ring have enough injectives, see More on Algebra, Sections 15.54 and 15.55

19.2. Baer's argument for modules

05NM There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of More on Algebra, Section 15.55, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [**CE56**] and by Grothendieck in [**Gro57**]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (Section 19.11).

We begin with a few set theoretic remarks. Let $\{B_\beta\}_{\beta\in\alpha}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal α. Assume that $\operatorname{colim}_{\beta\in\alpha} B_\beta$ exists in $\mathcal{C}$. If A is an object of $\mathcal{C}$, then there is a natural map

05NN (19.2.0.1) $$\operatorname{colim}_{\beta\in\alpha} \operatorname{Mor}_{\mathcal{C}}(A, B_\beta) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(A, \operatorname{colim}_{\beta\in\alpha} B_\beta).$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to \operatorname{colim}_{\beta\in\alpha} B_\alpha$. Note that the left colimit is one of sets! In general, (19.2.0.1) is neither injective or surjective.

05NP **Example 19.2.1.** Consider the category of sets. Let $A = \mathbf{N}$ and $B_n = \{1, \ldots, n\}$ be the inductive system indexed by the natural numbers where $B_n \to B_m$ for $n \leq m$ is the obvious map. Then $\operatorname{colim} B_n = \mathbf{N}$, so there is a map $A \to \operatorname{colim} B_n$, which does not factor as $A \to B_m$ for any m. Consequently, $\operatorname{colim} \operatorname{Mor}(A, B_n) \to \operatorname{Mor}(A, \operatorname{colim} B_n)$ is not surjective.

05NQ **Example 19.2.2.** Next we give an example where the map fails to be injective. Let $B_n = \mathbf{N}/\{1, 2, \ldots, n\}$, that is, the quotient set of $\mathbf{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form a system of sets over $\mathbf{N}$. It is easy to see that $\operatorname{colim} B_n = \{*\}$: it is the one-point set. So it follows that $\operatorname{Mor}(A, \operatorname{colim} B_n)$ is a one-element set for every set A. However, $\operatorname{colim} \operatorname{Mor}(A, B_n)$ is **not** a one-element set. Consider the family

of maps $A \to B_n$ which are just the natural projections $\mathbf{N} \to \mathbf{N}/\{1, 2, \ldots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $\operatorname{colim} \operatorname{Mor}(A, B_n)$, but they induce the same map $A \to \operatorname{colim} B_n$.

Nonetheless, if we map out of a finite set then (19.2.0.1) is an isomorphism always.

05NR **Lemma 19.2.3.** *Suppose that, in (19.2.0.1), $\mathcal{C}$ is the category of sets and A is a finite set, then the map is a bijection.*

Proof. Let $f : A \to \operatorname{colim} B_\beta$. The range of f is finite, containing say elements $c_1, \ldots, c_r \in \operatorname{colim} B_\beta$. These all come from some elements in B_β for $\beta \in \alpha$ large by definition of the colimit. Thus we can define $\widetilde{f} : A \to B_\beta$ lifting f at a finite stage. This proves that (19.2.0.1) is surjective. Next, suppose two maps $f : A \to B_\gamma, f' : A \to B_{\gamma'}$ define the same map $A \to \operatorname{colim} B_\beta$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $\beta \geq \gamma, \gamma'$ such that the finitely many elements of A get sent to the same points in B_β under f and f'. This proves that (19.2.0.1) is injective. □

The most interesting case of the lemma is when $\alpha = \omega$, i.e., when the system $\{B_\beta\}$ is a system $\{B_n\}_{n \in \mathbf{N}}$ over the natural numbers as in Examples 19.2.1 and 19.2.2. The essential idea is that A is "small" relative to the long chain of compositions $B_1 \to B_2 \to \ldots$, so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 3.7.1. Next, we generalize this to the category of modules.

05NS **Definition 19.2.4.** Let $\mathcal{C}$ be a category, let $I \subset \operatorname{Arrows}(\mathcal{C})$, and let α be an ordinal. An object A of $\mathcal{C}$ is said to be *α-small with respect to I* if whenever $\{B_\beta\}$ is a system over α with transition maps in I, then the map (19.2.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the collection of injective maps, i.e., the *monomorphisms* in the category of modules over R. In this case, for any system $\{B_\beta\}$ as in the definition each of the maps

$$B_\beta \to \operatorname{colim}_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (19.2.0.1) is an *injection*. We can in fact interpret the B_β's as submodules of the module $B = \operatorname{colim}_{\beta \in \alpha} B_\beta$, and then we have $B = \bigcup_{\beta \in \alpha} B_\beta$. This is not an abuse of notation if we identify B_α with the image in the colimit. We now want to show that modules are always small for "large" ordinals α.

05NT **Proposition 19.2.5.** *Let R be a ring. Let M be an R-module. Let κ the cardinality of the set of submodules of M. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\beta\}$ with injections between them, parametrized by a limit ordinal, any map $M \to \operatorname{colim} B_\beta$ factors through one of the B_β. And this we proved in Lemma 19.2.3.

Now we start the proof in the general case. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \operatorname{colim} B_\beta$ be a map. Consider the subobjects

$\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. □

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall Baer's criterion.

05NU **Lemma 19.2.6** (Baer's criterion)**.** *Let R be a ring. An R-module Q is injective if and only if in every commutative diagram* [**Bae40**, Theorem 1]

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & \nearrow & \\ R & & \end{array}$$

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists.

Proof. This is the equivalence of (1) and (3) in More on Algebra, Lemma 15.55.4; please observe that the proof given there is elementary (and does not use Ext groups or the existence of injectives or projectives in the category of R-modules). □

If M is an R-module, then in general we may have a semi-complete diagram as in Lemma 19.2.6. In it, we can form the *push-out*

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & M \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus_{\mathfrak{a}} M. \end{array}$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to M$ to R *if* we extend M to the larger module $R \oplus_{\mathfrak{a}} M$.

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an R-module M the following (huge) pushout

05NV (19.2.6.1)
$$\begin{array}{ccc} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R & \longrightarrow & \mathbf{M}(M). \end{array}$$

Here the top horizontal arrow maps the element $a \in \mathfrak{a}$ in the summand corresponding to φ to the element $\varphi(a) \in M$. The left vertical arrow maps $a \in \mathfrak{a}$ in the

summand corresponding to φ simply to the element $a \in R$ in the summand corresponding to φ. The fundamental properties of this construction are formulated in the following lemma.

05NW **Lemma 19.2.7.** *Let R be a ring.*

(1) *The construction $M \mapsto (M \to \mathbf{M}(M))$ is functorial in M.*
(2) *The map $M \to \mathbf{M}(M)$ is injective.*
(3) *For any ideal $\mathfrak{a}$ and any R-module map $\varphi : \mathfrak{a} \to M$ there is an R-module map $\varphi' : R \to \mathbf{M}(M)$ such that*

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\varphi} & M \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi'} & \mathbf{M}(M) \end{array}$$

commutes.

Proof. Parts (2) and (3) are immediate from the construction. To see (1), let $\chi : M \to N$ be an R-module map. We claim there exists a canonical commutative diagram

$$\begin{array}{ccccc} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M & & \\ \downarrow & \searrow & & \searrow^{\chi} & \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} \mathfrak{a} & \longrightarrow & N \\ & \searrow & \downarrow & & \\ & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} R & & \end{array}$$

which induces the desired map $\mathbf{M}(M) \to \mathbf{M}(N)$. The middle east-south-east arrow maps the summand $\mathfrak{a}$ corresponding to φ via $\mathrm{id}_{\mathfrak{a}}$ to the summand $\mathfrak{a}$ corresponding to $\psi = \chi \circ \varphi$. Similarly for the lower east-south-east arrow. Details omitted. □

The idea will now be to apply the functor $\mathbf{M}$ a transfinite number of times. We define for each ordinal α a functor $\mathbf{M}_\alpha$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\alpha(N)$. We do this by transfinite recursion. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal α, and suppose $\mathbf{M}_{\alpha'}$ is defined for $\alpha' < \alpha$. If α has an immediate predecessor $\widetilde{\alpha}$, we let

$$\mathbf{M}_\alpha = \mathbf{M} \circ \mathbf{M}_{\widetilde{\alpha}}.$$

If not, i.e., if α is a limit ordinal, we let

$$\mathbf{M}_\alpha(N) = \mathrm{colim}_{\alpha' < \alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the $\mathbf{M}_\alpha(N)$ form an inductive system over ordinals, so this is reasonable.

05NX **Theorem 19.2.8.** *Let κ be the cardinality of the set of ideals in R, and let α be an ordinal whose cofinality is greater than κ. Then $\mathbf{M}_\alpha(N)$ is an injective R-module, and $N \to \mathbf{M}_\alpha(N)$ is a functorial injective embedding.*

Proof. By Baer's criterion Lemma 19.2.6, it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\alpha(N)$ extends to $R \to \mathbf{M}_\alpha(N)$. However, we know since α is a limit ordinal that

$$\mathbf{M}_\alpha(N) = \operatorname{colim}_{\beta<\alpha} \mathbf{M}_\beta(N),$$

so by Proposition 19.2.5, we find that

$$\operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\alpha(N)) = \operatorname{colim}_{\beta<\alpha} \operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\beta(N)).$$

This means in particular that there is some $\beta' < \alpha$ such that f factors through the submodule $\mathbf{M}_{\beta'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\beta'}(N) \to \mathbf{M}_\alpha(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, see Lemma 19.2.7 part (3), we know that the map $\mathfrak{a} \to \mathbf{M}_{\beta'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in $\mathbf{M}_\alpha(N)$ (as $\beta' + 1 < \alpha$ since α is a limit ordinal). In particular, f can be extended to $\mathbf{M}_\alpha(N)$. □

19.3. G-modules

04JE We will see later (Differential Graded Algebra, Section 22.17) that the category of modules over an algebra has functorial injective embeddings. The construction is exactly the same as the construction in More on Algebra, Section 15.55.

04JF **Lemma 19.3.1.** *Let G be a topological group. Let R be a ring. The category $Mod_{R,G}$ of R-G-modules, see Étale Cohomology, Definition 59.57.1, has functorial injective embeddings. In particular this holds for the category of discrete G-modules.*

Proof. By the remark above the lemma the category $\operatorname{Mod}_{R[G]}$ has functorial injective embeddings. Consider the forgetful functor $v : \operatorname{Mod}_{R,G} \to \operatorname{Mod}_{R[G]}$. This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since $v(M) = 0 \Rightarrow M = 0$ we conclude by Homology, Lemma 12.29.5. □

19.4. Abelian sheaves on a space

01DF

01DG **Lemma 19.4.1.** *Let X be a topological space. The category of abelian sheaves on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For an abelian group A we denote $j : A \to J(A)$ the functorial injective embedding constructed in More on Algebra, Section 15.55. Let $\mathcal{F}$ be an abelian sheaf on X. By Sheaves, Example 6.7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x \in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective abelian group the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Mor}_{Ab}(\mathcal{F}_x, I_x) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is injective. Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.27.3. □

19.5. Sheaves of modules on a ringed space

01DH

01DI **Lemma 19.5.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Section 6.25. The category of sheaves of $\mathcal{O}_X$-modules on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For any ring R and any R-module M we denote $j : M \to J_R(M)$ the functorial injective embedding constructed in More on Algebra, Section 15.55. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. By Sheaves, Examples 6.7.5 and 6.15.6 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective $\mathcal{O}_{X,x}$-module the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is an injective $\mathcal{O}_X$-module (see Homology, Lemma 12.29.1 or argue directly). Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.27.3. □

19.6. Abelian presheaves on a category

01DJ Let $\mathcal{C}$ be a category. Recall that this means that $\mathrm{Ob}(\mathcal{C})$ is a set. On the one hand, consider abelian presheaves on $\mathcal{C}$, see Sites, Section 7.2. On the other hand, consider families of abelian groups indexed by elements of $\mathrm{Ob}(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $\mathrm{Ob}(\mathcal{C})$. Let us denote this discrete category simply $\mathrm{Ob}(\mathcal{C})$. There is a natural functor

$$i : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : PAb(\mathcal{C}) \longrightarrow PAb(\mathrm{Ob}(\mathcal{C}))$$

compare Sites, Section 7.5. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $\mathrm{Ob}(\mathcal{C})$ by A.

There are also two functors, namely i_p and ${}_pi$ which assign an abelian presheaf on $\mathcal{C}$ to an abelian presheaf on $\mathrm{Ob}(\mathcal{C})$, see Sites, Sections 7.5 and 7.19. Here we will use $u = {}_pi$ which is defined (in the case at hand) as follows:

$$uA(U) = \prod\nolimits_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g\circ\psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical injective map $B \to uvB$. We leave it to the reader to show that

$$\mathrm{Mor}_{PAb(\mathcal{C})}(B, uA) = \mathrm{Mor}_{PAb(\mathrm{Ob}(\mathcal{C}))}(vB, A).$$

in this simple case; the general case is in Sites, Section 7.5. Thus the pair (u, v) is an example of a pair of adjoint functors, see Categories, Section 4.24.

At this point we can list the following facts about the situation above.

(1) The functors u and v are exact. This follows from the explicit description of these functors given above.
(2) In particular the functor v transforms injective maps into injective maps.
(3) The category $PAb(\mathrm{Ob}(\mathcal{C}))$ has enough injectives.
(4) In fact there is a functorial injective embedding $A \mapsto \big(A \to J(A)\big)$ as in Homology, Definition 12.27.5. Namely, we can take $J(A)$ to be the presheaf $U \mapsto J(A(U))$, where $J(-)$ is the functor constructed in More on Algebra, Section 15.55 for the ring $\mathbf{Z}$.

Putting all of this together gives us the following procedure for embedding objects B of $PAb(\mathcal{C})$ into an injective object: $B \to uJ(vB)$. See Homology, Lemma 12.29.5.

01DK **Proposition 19.6.1.** *For abelian presheaves on a category there is a functorial injective embedding.*

Proof. See discussion above. □

19.7. Abelian Sheaves on a site

01DL Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote $i : Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C})$ the forgetful functor from abelian sheaves to abelian presheaves. Let ${}^\# : PAb(\mathcal{C}) \longrightarrow Ab(\mathcal{C})$ denote the sheafification functor. Recall that ${}^\#$ is a left adjoint to i, that ${}^\#$ is exact, and that $i\mathcal{F}^\# = \mathcal{F}$ for any abelian sheaf $\mathcal{F}$. Finally, let $\mathcal{G} \to J(\mathcal{G})$ denote the canonical embedding into an injective presheaf we found in Section 19.6.

For any sheaf $\mathcal{F}$ in $Ab(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite recursion. We set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i\mathcal{F})^\#$. Sheafification of the canonical map $i\mathcal{F} \to J(i\mathcal{F})$ gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as # is exact. We set $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$. So that there are canonical injective maps $J_\alpha(\mathcal{F}) \to J_{\alpha+1}(\mathcal{F})$. For a limit ordinal β, we define

$$J_\beta(\mathcal{F}) = \operatorname{colim}_{\alpha<\beta} J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals $\alpha < \beta$ we have an injective map $J_\alpha(\mathcal{F}) \to J_\beta(\mathcal{F})$.

01DM **Lemma 19.7.1.** *With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ J_\alpha(\mathcal{F}) & \longrightarrow & J_{\alpha+1}(\mathcal{F}) \end{array}$$

Proof. This is because the map $i\mathcal{G}_1 \to i\mathcal{G}_2$ is injective and hence $i\mathcal{G}_1 \to iJ_\alpha(\mathcal{F})$ extends to $i\mathcal{G}_2 \to J(iJ_\alpha(\mathcal{F}))$ which gives the desired map after applying the sheafification functor. □

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is an injective embedding of $\mathcal{F}$. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

01DN **Lemma 19.7.2.** *Suppose that $\mathcal{G}_i$, $i \in I$ is set of abelian sheaves on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.*

Proof. This reduces to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.

Consider the sets

$$S = \coprod\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} \mathcal{G}(U).$$

and

$$T_\beta = \coprod\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

The transition maps between the sets T_β are injective. If the cofinality of β is large enough, then $T_\beta = \operatorname{colim}_{\alpha<\beta} T_\alpha$, see Sites, Lemma 7.17.10. A morphism $\mathcal{G} \to J_\beta(\mathcal{F})$ factors through $J_\alpha(\mathcal{F})$ if and only if the associated map $S \to T_\beta$ factors through T_α. By Sets, Lemma 3.7.1 if the cofinality of β is bigger than the cardinality of S, then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 3.7.2. □

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \to X} \mathbf{Z}$, see Modules on Sites, Section 18.4. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$, see Modules on Sites, Section 18.5. It can be characterized by the property

05NY (19.7.2.1) $$\operatorname{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X)$$

where the element φ of the left hand side is mapped to $\varphi(1 \cdot \mathrm{id}_X)$ in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

01DO **Lemma 19.7.3.** *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in \mathrm{Ob}(\mathcal{C})$, for any abelian subsheaf $\mathcal{S} \subset \mathbf{Z}_X^\#$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of abelian sheaves. Suppose $\varphi : \mathcal{F} \to \mathcal{J}$ is a morphism. Arguing as in the proof of More on Algebra, Lemma 15.54.1 we see that it suffices to prove that if $\mathcal{F} \neq \mathcal{G}$, then we can find an abelian sheaf $\mathcal{F}'$, $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ such that (a) the inclusion $\mathcal{F} \subset \mathcal{F}'$ is strict, and (b) φ can be extended to $\mathcal{F}'$. To find $\mathcal{F}'$, let X be an object of $\mathcal{C}$ such that the inclusion $\mathcal{F}(X) \subset \mathcal{G}(X)$ is strict. Pick $s \in \mathcal{G}(X)$, $s \notin \mathcal{F}(X)$. Let $\psi : \mathbf{Z}_X^\# \to \mathcal{G}$ be the morphism corresponding to the section s via (19.7.2.1). Set $\mathcal{S} = \psi^{-1}(\mathcal{F})$. By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism $\varphi' : \mathbf{Z}_X^\# \to \mathcal{J}$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to \mathcal{J}$ which agrees with φ on the intersection $\mathcal{F} \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger subsheaf $\mathcal{F}' = \mathcal{F} + \mathrm{Im}(\psi)$. □

01DP **Theorem 19.7.4.** *The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.27.5.*

Proof. Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 19.7.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assignment $\mathcal{F} \mapsto \big(\mathcal{F} \to J_\beta(\mathcal{F})\big)$ is indeed functorial.

The proof of the claim comes from the fact that by Lemma 19.7.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 19.7.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 19.7.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. □

19.8. Modules on a ringed site

01DQ Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. By analogy with More on Algebra, Section 15.55 let us try to prove that there are enough injective $\mathcal{O}$-modules. First of all, we pick an injective embedding

$$\bigoplus\nolimits_{U,\mathcal{I}} j_{U!}\mathcal{O}_U/\mathcal{I} \longrightarrow \mathcal{J}$$

where $\mathcal{J}$ is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects U of $\mathcal{C}$ and over all $\mathcal{O}$-submodules $\mathcal{I} \subset j_{U!}\mathcal{O}_U$. Please see Modules on Sites, Section 18.19 to read about the functors restriction and extension by 0 for the localization functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural $\mathcal{O}$-module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of $\mathcal{O}$-modules. This we can do as follows: For any $\mathcal{O}$-module $\mathcal{F}$ we denote

$$F(\mathcal{F}) = \bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}\mathcal{O}_U.$$

This is a flat sheaf of $\mathcal{O}$-modules which comes equipped with a canonical surjection $F(\mathcal{F}) \to \mathcal{F}$, see Modules on Sites, Lemma 18.28.8. Moreover the construction $\mathcal{F} \mapsto F(\mathcal{F})$ is functorial in $\mathcal{F}$.

01DR **Lemma 19.8.1.** *The functor $\mathcal{F} \mapsto \mathcal{F}^\vee$ is exact.*

Proof. This because $\mathcal{J}$ is an injective abelian sheaf. □

There is a canonical map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ given by evaluation: given $x \in \mathcal{F}(U)$ we let $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$ be the map $\varphi \mapsto \varphi(x)$.

01DS **Lemma 19.8.2.** *For any $\mathcal{O}$-module $\mathcal{F}$ the evaluation map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.*

Proof. You can check this using the definition of $\mathcal{J}$. Namely, if $s \in \mathcal{F}(U)$ is not zero, then let $j_{U!}\mathcal{O}_U \to \mathcal{F}$ be the map of $\mathcal{O}$-modules it corresponds to via adjunction. Let $\mathcal{I}$ be the kernel of this map. There exists a nonzero map $\mathcal{F} \supset j_{U!}\mathcal{O}_U/\mathcal{I} \to \mathcal{J}$ which does not annihilate s. As $\mathcal{J}$ is an injective $\mathcal{O}$-module, this extends to a map $\varphi : \mathcal{F} \to \mathcal{J}$. Then $ev(s)(\varphi) = \varphi(s) \neq 0$ which is what we had to prove. □

The canonical surjection $F(\mathcal{F}) \to \mathcal{F}$ of $\mathcal{O}$-modules turns into a canonical injection, see above, of $\mathcal{O}$-modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$. The composition of ev with this the displayed map gives $\mathcal{F} \to J(\mathcal{F})$ functorially in $\mathcal{F}$.

01DT **Lemma 19.8.3.** *Let $\mathcal{O}$ be a sheaf of rings. For every $\mathcal{O}$-module $\mathcal{F}$ the $\mathcal{O}$-module $J(\mathcal{F})$ is injective.*

Proof. We have to show that the functor $\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F}))$ is exact. Note that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F})) &= \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\ &= \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{J})) \\ &= \mathrm{Hom}(\mathcal{G} \otimes_{\mathcal{O}} F(\mathcal{F}^\vee), \mathcal{J}) \end{aligned}$$

Thus what we want follows from the fact that $F(\mathcal{F}^\vee)$ is flat and $\mathcal{J}$ is injective. □

01DU **Theorem 19.8.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The category of sheaves of $\mathcal{O}$-modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.27.5.*

Proof. From the discussion in this section. □

01DV **Proposition 19.8.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules has functorial injective embeddings.*

Proof. We could prove this along the lines of the discussion in Section 19.6. But instead we argue using the theorem above. Endow $\mathcal{C}$ with the structure of a site by letting the set of coverings of an object U consist of all singletons $\{f : V \to U\}$ where f is an isomorphism. We omit the verification that this defines a site. A

sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies. □

19.9. Embedding abelian categories

05PL In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

05PM **Lemma 19.9.1.** *Let $\mathcal{A}$ be an abelian category. Let*

$$Cov = \{\{f : V \to U\} \mid f \text{ is surjective}\}.$$

Then $(\mathcal{A}, Cov)$ is a site, see Sites, Definition 7.6.2.

Proof. Note that $\mathrm{Ob}(\mathcal{A})$ is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in $\mathcal{A}$ is surjective, see Homology, Lemma 12.5.14. □

Let $\mathcal{A}$ be a pre-additive category. In this case the Yoneda embedding $\mathcal{A} \to PSh(\mathcal{A})$, $X \mapsto h_X$ factors through a functor $\mathcal{A} \to PAb(\mathcal{A})$.

05PN **Lemma 19.9.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} = (\mathcal{A}, Cov)$ be the site defined in Lemma 19.9.1. Then $X \mapsto h_X$ defines a fully faithful, exact functor*

$$\mathcal{A} \longrightarrow Ab(\mathcal{C}).$$

Moreover, the site $\mathcal{C}$ has enough points.

Proof. Suppose that $f : V \to U$ is a surjective morphism of $\mathcal{A}$. Let $K = \mathrm{Ker}(f)$. Recall that $V \times_U V = \mathrm{Ker}((f, -f) : V \oplus V \to U)$, see Homology, Example 12.5.6. In particular there exists an injection $K \oplus K \to V \times_U V$. Let $p, q : V \times_U V \to V$ be the two projection morphisms. Note that $p - q : V \times_U V \to V$ is a morphism such that $f \circ (p - q) = 0$. Hence $p - q$ factors through $K \to V$. Let us denote this morphism by $c : V \times_U V \to K$. And since the composition $K \oplus K \to V \times_U V \to K$ is surjective, we conclude that c is surjective. It follows that

$$V \times_U V \xrightarrow{p-q} V \to U \to 0$$

is an exact sequence of $\mathcal{A}$. Hence for an object X of $\mathcal{A}$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(U, X) \to \mathrm{Hom}_{\mathcal{A}}(V, X) \to \mathrm{Hom}_{\mathcal{A}}(V \times_U V, X)$$

is an exact sequence of abelian groups, see Homology, Lemma 12.5.8. This means that h_X satisfies the sheaf condition on $\mathcal{C}$.

The functor is fully faithful by Categories, Lemma 4.3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 12.5.8. To show that it is right exact, let $X \to Y$ be a surjective morphism of $\mathcal{A}$. Let U be an object of $\mathcal{A}$, and let $s \in h_Y(U) = \mathrm{Mor}_{\mathcal{A}}(U, Y)$ be a section of h_Y over U. By Homology, Lemma 12.5.14 the projection $U \times_Y X \to U$ is surjective. Hence $\{V = U \times_Y X \to U\}$ is a covering of U such that $s|_V$ lifts to a section of h_X. This proves that $h_X \to h_Y$ is a surjection of abelian sheaves, see Sites, Lemma 7.11.2.

The site $\mathcal{C}$ has enough points by Sites, Proposition 7.39.3. □

05PP **Remark 19.9.3.** The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category $\mathcal{A}$ to the category of modules over a ring. Lemma 19.9.2 is not quite as strong. But the result is suitable for the Stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, "diagram chasing" works in the category of abelian sheaves on $\mathcal{C}$, for example by working with sections over objects, or by working on the level of stalks using that $\mathcal{C}$ has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 19.9.2 see Remark 19.9.5.

05PQ **Remark 19.9.4.** If $\mathcal{A}$ is a "big" abelian category, i.e., if $\mathcal{A}$ has a class of objects, then Lemma 19.9.2 does not work. In this case, given any set of objects $E \subset \mathrm{Ob}(\mathcal{A})$ there exists an abelian full subcategory $\mathcal{A}' \subset \mathcal{A}$ such that $\mathrm{Ob}(\mathcal{A}')$ is a set and $E \subset \mathrm{Ob}(\mathcal{A}')$. Then one can apply Lemma 19.9.2 to $\mathcal{A}'$. One can use this to prove that results depending on a diagram chase hold in $\mathcal{A}$.

05PR **Remark 19.9.5.** Let $\mathcal{C}$ be a site. Note that $Ab(\mathcal{C})$ has enough injectives, see Theorem 19.7.4. (In the case that $\mathcal{C}$ has enough points this is straightforward because p_*I is an injective sheaf if I is an injective $\mathbf{Z}$-module and p is a point.) Also, $Ab(\mathcal{C})$ has a cogenerator (details omitted). Hence Lemma 19.9.2 proves that we have a fully faithful, exact embedding $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has a cogenerator and enough injectives. We can apply this to $\mathcal{A}^{opp}$ and we get a fully faithful exact functor $i : \mathcal{A} \to \mathcal{D} = \mathcal{B}^{opp}$ where $\mathcal{D}$ has enough projectives and a generator. Hence $\mathcal{D}$ has a projective generator P. Set $R = \mathrm{Mor}_{\mathcal{D}}(P, P)$. Then

$$\mathcal{A} \longrightarrow \mathrm{Mod}_R, \quad X \longmapsto \mathrm{Hom}_{\mathcal{D}}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 19.9.3 above.

05SF **Remark 19.9.6.** The arguments proving Lemmas 19.9.1 and 19.9.2 work also for *exact categories*, see [**Büh10**, Appendix A] and [**BBD82**, 1.1.4]. We quickly review this here and we add more details if we ever need it in the Stacks project.

Let $\mathcal{A}$ be an additive category. A *kernel-cokernel* pair is a pair (i, p) of morphisms of $\mathcal{A}$ with $i : A \to B$, $p : B \to C$ such that i is the kernel of p and p is the cokernel of i. Given a set $\mathcal{E}$ of kernel-cokernel pairs we say $i : A \to B$ is an *admissible monomorphism* if $(i, p) \in \mathcal{E}$ for some morphism p. Similarly we say a morphism $p : B \to C$ is an *admissible epimorphism* if $(i, p) \in \mathcal{E}$ for some morphism i. The pair $(\mathcal{A}, \mathcal{E})$ is said to be an *exact category* if the following axioms hold

(1) $\mathcal{E}$ is closed under isomorphisms of kernel-cokernel pairs,
(2) for any object A the morphism 1_A is both an admissible epimorphism and an admissible monomorphism,
(3) admissible monomorphisms are stable under composition,
(4) admissible epimorphisms are stable under composition,
(5) the push-out of an admissible monomorphism $i : A \to B$ via any morphism $A \to A'$ exist and the induced morphism $i' : A' \to B'$ is an admissible monomorphism, and
(6) the base change of an admissible epimorphism $p : B \to C$ via any morphism $C' \to C$ exist and the induced morphism $p' : B' \to C'$ is an admissible epimorphism.

Given such a structure let $\mathcal{C} = (\mathcal{A}, \mathrm{Cov})$ where coverings (i.e., elements of Cov) are given by admissible epimorphisms. The axioms listed above immediately imply

that this is a site. Consider the functor

$$F : \mathcal{A} \longrightarrow Ab(\mathcal{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 19.9.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of F is in the essential image of F.

19.10. Grothendieck's AB conditions

079A This and the next few sections are mostly interesting for "big" abelian categories, i.e., those categories listed in Categories, Remark 4.2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [**Gro57**]. He used the following conditions to single out abelian categories with special properties.

079B **Definition 19.10.1.** Let $\mathcal{A}$ be an abelian category. We name some conditions

- AB3 $\mathcal{A}$ has direct sums,
- AB4 $\mathcal{A}$ has AB3 and direct sums are exact,
- AB5 $\mathcal{A}$ has AB3 and filtered colimits are exact.

Here are the dual notions

- AB3* $\mathcal{A}$ has products,
- AB4* $\mathcal{A}$ has AB3* and products are exact,
- AB5* $\mathcal{A}$ has AB3* and cofiltered limits are exact.

We say an object U of $\mathcal{A}$ is a *generator* if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through N. We say $\mathcal{A}$ is a *Grothendieck abelian category* if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 4.14.12. Similarly if $\mathcal{A}$ has AB3* then it has limits, see Categories, Lemma 4.14.11. Exactness of direct sums means the following: given an index set I and short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i \in I$$

in $\mathcal{A}$ then the sequence

$$0 \to \bigoplus\nolimits_{i \in I} A_i \to \bigoplus\nolimits_{i \in I} B_i \to \bigoplus\nolimits_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums exist). Similarly, exactness of filtered colimits means the following: given a directed set I and a system of short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

over I in $\mathcal{A}$ then the sequence

$$0 \to \operatorname{colim}_{i \in I} A_i \to \operatorname{colim}_{i \in I} B_i \to \operatorname{colim}_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4* and AB5*.

19.11. Injectives in Grothendieck categories

05AB The existence of a generator implies that given an object M of a Grothendieck abelian category $\mathcal{A}$ there is a set of subobjects. (This may not be true for a general "big" abelian category.)

0E8N **Lemma 19.11.1.** *Let $\mathcal{A}$ be an abelian category with a generator U and X and object of $\mathcal{A}$. If κ is the cardinality of $\mathrm{Mor}(U, X)$ then*

(1) *There does not exist a strictly increasing (or strictly decreasing) chain of subobjects of X indexed by a cardinal bigger than κ.*
(2) *If α is an ordinal of cofinality $> \kappa$ then any increasing (or decreasing) sequence of subobjects of X indexed by α is eventually constant.*
(3) *The cardinality of the set of subobjects of X is $\leq 2^\kappa$.*

Proof. For (1) assume $\kappa' > \kappa$ is a cardinal and assume X_i, $i \in \kappa'$ is strictly increasing. Then take for each i a $\phi_i \in \mathrm{Mor}(U, X)$ such that ϕ_i factors through X_{i+1} but not through X_i. Then the morphisms ϕ_i are distinct, which contradicts the definition of κ.

Part (2) follows from the definition of cofinality and (1).

Proof of (3). For any subobject $Y \subset X$ define $S_Y \in \mathcal{P}(\mathrm{Mor}(U, X))$ (power set) as $S_Y = \{\phi \in \mathrm{Mor}(U, X) : \phi)$ factors through $Y\}$. Then $Y = Y'$ if and only if $S_Y = S_{Y'}$. Hence the cardinality of the set of subobjects is at most the cardinality of this power set. □

By Lemma 19.11.1 the following definition makes sense.

079C **Definition 19.11.2.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. The *size* $|M|$ of M is the cardinality of the set of subobjects of M.

079D **Lemma 19.11.3.** *Let $\mathcal{A}$ be a Grothendieck abelian category. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{A}$, then $|M'|, |M''| \leq |M|$.*

Proof. Immediate from the definitions. □

079E **Lemma 19.11.4.** *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U.*

(1) *If $|M| \leq \kappa$, then M is the quotient of a direct sum of at most κ copies of U.*
(2) *For every cardinal κ the isomorphism classes of objects M with $|M| \leq \kappa$ form a set.*

Proof. For (1) choose for every proper subobject $M' \subset M$ a morphism $\varphi_{M'} : U \to M$ whose image is not contained in M'. Then $\bigoplus_{M' \subset M} \varphi_{M'} : \bigoplus_{M' \subset M} U \to M$ is surjective. It is clear that (1) implies (2). □

079F **Proposition 19.11.5.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. Let $\kappa = |M|$. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. Please compare with Proposition 19.2.5. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \mathrm{colim}\, B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

Because $\mathcal{A}$ has AB5 we have

$$\operatorname{colim} f^{-1}(B_\beta) = f^{-1}\left(\operatorname{colim} B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. □

079G **Lemma 19.11.6.** *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. An object I of $\mathcal{A}$ is injective if and only if in every commutative diagram*

$$\begin{array}{ccc} M & \longrightarrow & I \\ \downarrow & \nearrow & \\ U & & \end{array}$$

for $M \subset U$ a subobject, the dotted arrow exists.

Proof. Please see Lemma 19.2.6 for the case of modules. Choose an injection $A \subset B$ and a morphism $\varphi : A \to I$. Consider the set S of pairs (A', φ') consisting of subobjects $A \subset A' \subset B$ and a morphism $\varphi' : A' \to I$ extending φ. Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset $T \subset S$. Then

$$A' = \operatorname{colim}_{t \in T} A_t \xrightarrow{\operatorname{colim}_{t\in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set S has a maximal element (A', φ'). We claim that $A' = B$. If not, then choose a morphism $\psi : U \to B$ which does not factor through A'. Set $N = A' \cap \psi(U)$. Set $M = \psi^{-1}(N)$. Then the map

$$M \to N \to A' \xrightarrow{\varphi'} I$$

can be extended to a morphism $\chi : U \to I$. Since $\chi|_{\operatorname{Ker}(\psi)} = 0$ we see that χ factors as

$$U \to \operatorname{Im}(\psi) \xrightarrow{\varphi''} I$$

Since φ' and φ'' agree on $N = A' \cap \operatorname{Im}(\psi)$ we see that combined the define a morphism $A' + \operatorname{Im}(\psi) \to I$ contradicting the assumed maximality of A'. □

079H **Theorem 19.11.7.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has functorial injective embeddings.*

Proof. Please compare with the proof of Theorem 19.2.8. Choose a generator U of $\mathcal{A}$. For an object M we define $\mathbf{M}(M)$ by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{N\subset U} \bigoplus_{\varphi\in \operatorname{Hom}(N,M)} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{N\subset U} \bigoplus_{\varphi\in \operatorname{Hom}(N,M)} U & \longrightarrow & \mathbf{M}(M). \end{array}$$

Note that $M \mapsto \mathbf{M}(M)$ is a functor and that there exist functorial injective maps $M \to \mathbf{M}(M)$. By transfinite induction we define functors $\mathbf{M}_\alpha(M)$ for every ordinal

α. Namely, set $\mathbf{M}_0(M) = M$. Given $\mathbf{M}_\alpha(M)$ set $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$. For a limit ordinal β set

$$\mathbf{M}_\beta(M) = \operatorname{colim}_{\alpha<\beta} \mathbf{M}_\alpha(M).$$

Finally, pick any ordinal α whose cofinality is greater than $|U|$. Such an ordinal exists by Sets, Proposition 3.7.2. We claim that $M \to \mathbf{M}_\alpha(M)$ is the desired functorial injective embedding. Namely, if $N \subset U$ is a subobject and $\varphi : N \to \mathbf{M}_\alpha(M)$ is a morphism, then we see that φ factors through $\mathbf{M}_{\alpha'}(M)$ for some $\alpha' < \alpha$ by Proposition 19.11.5. By construction of $\mathbf{M}(-)$ we see that φ extends to a morphism from U into $\mathbf{M}_{\alpha'+1}(M)$ and hence into $\mathbf{M}_\alpha(M)$. By Lemma 19.11.6 we conclude that $\mathbf{M}_\alpha(M)$ is injective. □

19.12. K-injectives in Grothendieck categories

079I The material in this section is taken from the paper [**Ser03**] authored by Serpé. This paper generalizes some of the results of [**Spa88**] by Spaltenstein to general Grothendieck abelian categories. Our Lemma 19.12.3 is only implicit in the paper by Serpé. Our approach is to mimic Grothendieck's proof of Theorem 19.11.7.

079J **Lemma 19.12.1.** *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. Let c be the function on cardinals defined by $c(\kappa) = |\bigoplus_{\alpha\in\kappa} U|$. If $\pi : M \to N$ is a surjection then there exists a subobject $M' \subset M$ which surjects onto N with $|M'| \leq c(|N|)$.*

Proof. For every proper subobject $N' \subset N$ choose a morphism $\varphi_{N'} : U \to M$ such that $U \to M \to N$ does not factor through N'. Set

$$M' = \operatorname{Im}\Big(\bigoplus\nolimits_{N'\subset N} \varphi_{N'} : \bigoplus\nolimits_{N'\subset N} U \longrightarrow M\Big)$$

Then M' works. □

079K **Lemma 19.12.2.** *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a cardinal κ such that given any acyclic complex $M^\bullet$ we have*

(1) *if $M^\bullet$ is nonzero, there is a nonzero subcomplex $N^\bullet$ which is bounded above, acyclic, and $|N^n| \leq \kappa$,*
(2) *there exists a surjection of complexes*

$$\bigoplus\nolimits_{i\in I} M_i^\bullet \longrightarrow M^\bullet$$

where $M_i^\bullet$ is bounded above, acyclic, and $|M_i^n| \leq \kappa$.

Proof. Choose a generator U of $\mathcal{A}$. Denote c the function of Lemma 19.12.1. Set $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \ldots\}$. Let $n \in \mathbf{Z}$ and let $\psi : U \to M^n$ be a morphism. In order to prove (1) and (2) it suffices to prove there exists a subcomplex $N^\bullet \subset M^\bullet$ which is bounded above, acyclic, and $|N^m| \leq \kappa$, such that ψ factors through N^n. To do this set $N^n = \operatorname{Im}(\psi)$, $N^{n+1} = \operatorname{Im}(U \to M^n \to M^{n+1})$, and $N^m = 0$ for $m \geq n + 2$. Suppose for some $k \leq n$ we have constructed $N^m \subset M^m$ for all $n > m \geq k$ such that

(1) $\mathrm{d}(N^m) \subset N^{m+1}$ for all $m \geq k$,
(2) $\operatorname{Im}(N^{m-1} \to N^m) = \operatorname{Ker}(N^m \to N^{m+1})$ for all $m \geq k + 1$, and
(3) $|N^m| \leq c^{\max\{n-m,0\}}(|U|)$ for all $m \geq k$.

Because $M^\bullet$ is acyclic, we see that the subobject $\mathrm{d}^{-1}(\mathrm{Ker}(N^k \to N^{k+1})) \subset M^{k-1}$ surjects onto $\mathrm{Ker}(N^k \to N^{k+1})$. Thus we can choose $N^{k-1} \subset M^{k-1}$ surjecting onto $\mathrm{Ker}(N^k \to N^{k+1})$ with $|N^{k-1}| \leq c^{n-k+1}(|U|)$ by Lemma 19.12.1. The proof is finished by induction on k. □

079L **Lemma 19.12.3.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let κ be a cardinal as in Lemma 19.12.2. Suppose that $I^\bullet$ is a complex such that*

(1) *each I^j is injective, and*
(2) *for every bounded above acyclic complex $M^\bullet$ such that $|M^n| \leq \kappa$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.*

Then $I^\bullet$ is an K-injective complex.

Proof. Let $M^\bullet$ be an acyclic complex. We are going to construct by induction on the ordinal α an acyclic subcomplex $K^\bullet_\alpha \subset M^\bullet$ as follows. For $\alpha = 0$ we set $K^\bullet_0 = 0$. For $\alpha > 0$ we proceed as follows:

(1) If $\alpha = \beta + 1$ and $K^\bullet_\beta = M^\bullet$ then we choose $K^\bullet_\alpha = K^\bullet_\beta$.
(2) If $\alpha = \beta + 1$ and $K^\bullet_\beta \neq M^\bullet$ then $M^\bullet/K^\bullet_\beta$ is a nonzero acyclic complex. We choose a subcomplex $N^\bullet_\alpha \subset M^\bullet/K^\bullet_\beta$ as in Lemma 19.12.2. Finally, we let $K^\bullet_\alpha \subset M^\bullet$ be the inverse image of $N^\bullet_\alpha$.
(3) If α is a limit ordinal we set $K^\bullet_\beta = \mathrm{colim}\, K^\bullet_\alpha$.

It is clear that $M^\bullet = K^\bullet_\alpha$ for a suitably large ordinal α. We will prove that

$$\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet_\alpha, I^\bullet)$$

is zero by transfinite induction on α. It holds for $\alpha = 0$ since $K^\bullet_0$ is zero. Suppose it holds for β and $\alpha = \beta + 1$. In case (1) of the list above the result is clear. In case (2) there is a short exact sequence of complexes

$$0 \to K^\bullet_\beta \to K^\bullet_\alpha \to N^\bullet_\alpha \to 0$$

Since each component of $I^\bullet$ is injective we see that we obtain an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet_\beta, I^\bullet) \leftarrow \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet_\alpha, I^\bullet) \leftarrow \mathrm{Hom}_{K(\mathcal{A})}(N^\bullet_\alpha, I^\bullet)$$

By induction the term on the left is zero and by assumption on $I^\bullet$ the term on the right is zero. Thus the middle group is zero too. Finally, suppose that α is a limit ordinal. Then we see that

$$\mathrm{Hom}^\bullet(K^\bullet_\alpha, I^\bullet) = \lim\nolimits_{\beta<\alpha} \mathrm{Hom}^\bullet(K^\bullet_\beta, I^\bullet)$$

with notation as in More on Algebra, Section 15.72. These complexes compute morphisms in $K(\mathcal{A})$ by More on Algebra, Equation (15.72.0.1). Note that the transition maps in the system are surjective because I^j is injective for each j. Moreover, for a limit ordinal α we have equality of limit and value (see displayed formula above). Thus we may apply Homology, Lemma 12.31.8 to conclude. □

079M **Lemma 19.12.4.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $(K^\bullet_i)_{i\in I}$ be a set of acyclic complexes. There exists a functor $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ such*

(1) *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
(2) *for every $i \in I$ and $w : K^\bullet_i \to M^\bullet$ the morphism $j_{M^\bullet} \circ w$ is homotopic to zero.*

Proof. For every $i \in I$ choose a (termwise) injective map of complexes $K_i^\bullet \to L_i^\bullet$ which is homotopic to zero with $L_i^\bullet$ quasi-isomorphic to zero. For example, take $L_i^\bullet$ to be the cone on the identity of $K_i^\bullet$. We define $\mathbf{M}^\bullet(M^\bullet)$ by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} K_i^\bullet & \longrightarrow & M^\bullet \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} L_i^\bullet & \longrightarrow & \mathbf{M}^\bullet(M^\bullet). \end{array}$$

Then $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ is a functor. The right vertical arrow defines the functorial injective map $j_{M^\bullet}$. The cokernel of $j_{M^\bullet}$ is isomorphic to the direct sum of the cokernels of the maps $K_i^\bullet \to L_i^\bullet$ hence acyclic. Thus $j_{M^\bullet}$ is a quasi-isomorphism. Part (2) holds by construction. □

079N **Lemma 19.12.5.** *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a functor $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ such*

(1) *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
(2) *for every $n \in \mathbf{Z}$ the map $M^n \to \mathbf{N}^n(M^\bullet)$ factors through a subobject $I^n \subset \mathbf{N}^n(M^\bullet)$ where I^n is an injective object of $\mathcal{A}$.*

Proof. Choose a functorial injective embeddings $i_M : M \to I(M)$, see Theorem 19.11.7. For every complex $M^\bullet$ denote $J^\bullet(M^\bullet)$ the complex with terms $J^n(M^\bullet) = I(M^n) \oplus I(M^{n+1})$ and differential

$$d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a canonical injective map of complexes $u_{M^\bullet} : M^\bullet \to J^\bullet(M^\bullet)$ by mapping M^n to $I(M^n) \oplus I(M^{n+1})$ via the maps $i_{M^n} : M^n \to I(M^n)$ and $i_{M^{n+1}} \circ d : M^n \to M^{n+1} \to I(M^{n+1})$. Hence a short exact sequence of complexes

$$0 \to M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \to 0$$

functorial in $M^\bullet$. Set

$$\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].$$

Note that

$$\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)$$

with differential

$$\begin{pmatrix} -d^{n-1}_{Q^\bullet(M^\bullet)} & -v^n_{M^\bullet} \\ 0 & d^n_{J^\bullet(M)} \end{pmatrix}$$

Hence we see that there is a map of complexes $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ induced by u. It is injective and factors through an injective subobject by construction. The map $j_{M^\bullet}$ is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above. □

079P **Theorem 19.12.6.** *Let $\mathcal{A}$ be a Grothendieck abelian category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ such that $M^n \to I^n$ is injective and I^n is an injective object of $\mathcal{A}$ for all n and $I^\bullet$ is a K-injective complex. Moreover, the construction is functorial in $M^\bullet$.*

Proof. Please compare with the proof of Theorem 19.2.8 and Theorem 19.11.7. Choose a cardinal κ as in Lemmas 19.12.2 and 19.12.3. Choose a set $(K_i^\bullet)_{i\in I}$ of bounded above, acyclic complexes such that every bounded above acyclic complex $K^\bullet$ such that $|K^n| \leq \kappa$ is isomorphic to $K_i^\bullet$ for some $i \in I$. This is possible by Lemma 19.11.4. Denote $\mathbf{M}^\bullet(-)$ the functor constructed in Lemma 19.12.4. Denote $\mathbf{N}^\bullet(-)$ the functor constructed in Lemma 19.12.5. Both of these functors come with injective transformations $\mathrm{id} \to \mathbf{M}$ and $\mathrm{id} \to \mathbf{N}$.

Using transfinite recursion we define a sequence of functors $\mathbf{T}_\alpha(-)$ and corresponding transformations $\mathrm{id} \to \mathbf{T}_\alpha$. Namely we set $\mathbf{T}_0(M^\bullet) = M^\bullet$. If $\mathbf{T}_\alpha$ is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If β is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = \mathrm{colim}_{\alpha<\beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to $M^\bullet$. We claim that $M^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ does the job if the cofinality of α is larger than $\max(\kappa, |U|)$ where U is a generator of $\mathcal{A}$. Namely, it suffices to check conditions (1) and (2) of Lemma 19.12.3.

For (1) we use the criterion of Lemma 19.11.6. Suppose that $M \subset U$ and $\varphi : M \to \mathbf{T}_\alpha^n(M^\bullet)$ is a morphism for some $n \in \mathbf{Z}$. By Proposition 19.11.5 we see that φ factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{N}^\bullet(-)$ we see that φ factors through an injective object of $\mathcal{A}$ which shows that φ lifts to a morphism on U.

For (2) let $w : K^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ be a morphism of complexes where $K^\bullet$ is a bounded above acyclic complex such that $|K^n| \leq \kappa$. Then $K^\bullet \cong K_i^\bullet$ for some $i \in I$. Moreover, by Proposition 19.11.5 once again we see that w factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{M}^\bullet(-)$ we see that w is homotopic to zero. This finishes the proof. □

19.13. Additional remarks on Grothendieck abelian categories

07D6 In this section we put some results on Grothendieck abelian categories which are folklore.

07D7 **Lemma 19.13.1.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A}^{opp} \to Sets$ be a functor. Then F is representable if and only if F commutes with colimits, i.e.,*

$$F(\mathrm{colim}_i N_i) = \lim F(N_i)$$

for any diagram $\mathcal{I} \to \mathcal{A}$, $i \in \mathcal{I}$.

Proof. If F is representable, then it commutes with colimits by definition of colimits.

Assume that F commutes with colimits. Then $F(M \oplus N) = F(M) \times F(N)$ and we can use this to define a group structure on $F(M)$. Hence we get $F : \mathcal{A} \to Ab$ which is additive and right exact, i.e., transforms a short exact sequence $0 \to K \to L \to M \to 0$ into an exact sequence $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$ (compare with Homology, Section 12.7).

Let U be a generator for $\mathcal{A}$. Set $A = \bigoplus_{s\in F(U)} U$. Let $s_{univ} = (s)_{s\in F(U)} \in F(A) = \prod_{s\in F(U)} F(U)$. Let $A' \subset A$ be the largest subobject such that s_{univ} restricts to zero

on A'. This exists because $\mathcal{A}$ is a Grothendieck category and because F commutes with colimits. Because F commutes with colimits there exists a unique element $\overline{s}_{univ} \in F(A/A')$ which maps to s_{univ} in $F(A)$. We claim that A/A' represents F, in other words, the Yoneda map

$$\overline{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let $M \in \mathrm{Ob}(\mathcal{A})$ and $s \in F(M)$. Consider the surjection

$$c_M : A_M = \bigoplus\nolimits_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow M.$$

This gives $F(c_M)(s) = (s_\varphi) \in \prod_\varphi F(U)$. Consider the map

$$\psi : A_M = \bigoplus\nolimits_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow \bigoplus\nolimits_{s \in F(U)} U = A$$

which maps the summand corresponding to φ to the summand corresponding to s_φ by the identity map on U. Then s_{univ} maps to $(s_\varphi)_\varphi$ by construction. in other words the right square in the diagram

$$\begin{array}{ccccc} A' & \longrightarrow & A & \overset{s_{univ}}{\dashrightarrow} & F \\ \uparrow{\scriptstyle ?} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle s} \\ K & \longrightarrow & A_M & \longrightarrow & M \end{array}$$

commutes. Let $K = \mathrm{Ker}(A_M \to M)$. Since s restricts to zero on K we see that $\psi(K) \subset A'$ by definition of A'. Hence there is an induced morphism $M \to A/A'$. This construction gives an inverse to the map $h_{A/A'}(M) \to F(M)$ (details omitted). □

07D8 **Lemma 19.13.2.** *A Grothendieck abelian category has Ab3*.*

Proof. Let M_i, $i \in I$ be a family of objects of $\mathcal{A}$ indexed by a set I. The functor $F = \prod_{i \in I} h_{M_i}$ commutes with colimits. Hence Lemma 19.13.1 applies. □

079Q **Remark 19.13.3.** In the chapter on derived categories we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$ it isn't clear that the derived category $D(\mathcal{A})$ exists because it isn't clear that morphisms in the derived category are sets. In general this isn't true, see Examples, Lemma 110.62.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Derived Categories, Lemma 13.31.2 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

07D9 **Lemma 19.13.4.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then*

(1) *$D(\mathcal{A})$ has both direct sums and products,*
(2) *direct sums are obtained by taking termwise direct sums of any complexes,*
(3) *products are obtained by taking termwise products of K-injective complexes.*

Proof. Let $K_i^\bullet$, $i \in I$ be a family of objects of $D(\mathcal{A})$ indexed by a set I. We claim that the termwise direct sum $\bigoplus_{i\in I} K_i^\bullet$ is a direct sum in $D(\mathcal{A})$. Namely, let $I^\bullet$ be a K-injective complex. Then we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) &= \mathrm{Hom}_{K(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{K(\mathcal{A})}(K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(K_i^\bullet, I^\bullet)\end{aligned}$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 19.12.6. To construct the product, choose a K-injective resolution $K_i^\bullet \to I_i^\bullet$ for each i. Then we claim that $\prod_{i\in I} I_i^\bullet$ is a product in $D(\mathcal{A})$. This follows from Derived Categories, Lemma 13.31.5. □

07DA **Remark 19.13.5.** Let R be a ring. Suppose that M_n, $n \in \mathbf{Z}$ are R-modules. Denote $E_n = M_n[-n] \in D(R)$. We claim that $E = \bigoplus M_n[-n]$ is *both* the direct sum and the product of the objects E_n in $D(R)$. To see that it is the direct sum, take a look at the proof of Lemma 19.13.4. To see that it is the direct product, take injective resolutions $M_n \to I_n^\bullet$. By the proof of Lemma 19.13.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in $D(R)$. Since products in Mod_R are exact, we see that $\prod I_n^\bullet[-n]$ is quasi-isomorphic to E. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with Ab4*.

08U1 **Lemma 19.13.6.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Assume*

(1) *$\mathcal{A}$ is a Grothendieck abelian category,*
(2) *$\mathcal{B}$ has exact countable products, and*
(3) *F commutes with countable products.*

Then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ commutes with derived limits.

Proof. Observe that RF exists as $\mathcal{A}$ has enough K-injectives (Theorem 19.12.6 and Derived Categories, Lemma 13.31.6). The statement means that if $K = R\lim K_n$, then $RF(K) = R\lim RF(K_n)$. See Derived Categories, Definition 13.34.1 for notation. Since RF is an exact functor of triangulated categories it suffices to see that RF commutes with countable products of objects of $D(\mathcal{A})$. In the proof of Lemma 19.13.4 we have seen that products in $D(\mathcal{A})$ are computed by taking products of K-injective complexes and moreover that a product of K-injective complexes is K-injective. Moreover, in Derived Categories, Lemma 13.34.2 we have seen that products in $D(\mathcal{B})$ are computed by taking termwise products. Since RF is computed by applying F to a K-injective representative and since we've assumed F commutes with countable products, the lemma follows. □

The following lemma is some kind of generalization of the existence of Cartan-Eilenberg resolutions (Derived Categories, Section 13.21).

0BKI **Lemma 19.13.7.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. Then there exists a morphism $j : K^\bullet \to J^\bullet$ of filtered complexes of $\mathcal{A}$ such that*

(1) *J^n, F^pJ^n, J^n/F^pJ^n and $F^pJ^n/F^{p'}J^n$ are injective objects of $\mathcal{A}$,*

(2) $J^\bullet$, $F^pJ^\bullet$, $J^\bullet/F^pJ^\bullet$, *and* $F^pJ^\bullet/F^{p'}J^\bullet$ *are K-injective complexes,*
(3) j *induces quasi-isomorphisms* $K^\bullet \to J^\bullet$, $F^pK^\bullet \to F^pJ^\bullet$, $K^\bullet/F^pK^\bullet \to J^\bullet/F^pJ^\bullet$, *and* $F^pK^\bullet/F^{p'}K^\bullet \to F^pJ^\bullet/F^{p'}J^\bullet$.

Proof. By Theorem 19.12.6 we obtain quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i^p : F^pK^\bullet \to I^{p,\bullet}$ as well as commutative diagrams

$$\begin{array}{ccc} K^\bullet & \longleftarrow & F^pK^\bullet \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^p} \\ I^\bullet & \xleftarrow{\alpha^p} & I^{p,\bullet} \end{array} \quad \text{and} \quad \begin{array}{ccc} F^{p'}K^\bullet & \longleftarrow & F^pK^\bullet \\ \downarrow{\scriptstyle i^{p'}} & & \downarrow{\scriptstyle i^p} \\ I^{p',\bullet} & \xleftarrow{\alpha^{pp'}} & I^{p,\bullet} \end{array} \quad \text{for } p' \leq p$$

such that $\alpha^p \circ \alpha^{p'p} = \alpha^{p'}$ and $\alpha^{p'p''} \circ \alpha^{pp'} = \alpha^{pp''}$. The problem is that the maps $\alpha^p : I^{p,\bullet} \to I^\bullet$ need not be injective. For each p we choose an injection $t^p : I^{p,\bullet} \to J^{p,\bullet}$ into an acyclic K-injective complex $J^{p,\bullet}$ whose terms are injective objects of $\mathcal{A}$ (first map to the cone on the identity and then use the theorem). Choose a map of complexes $s^p : I^\bullet \to J^{p,\bullet}$ such that the following diagram commutes

$$\begin{array}{ccc} K^\bullet & \longleftarrow & F^pK^\bullet \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^p} \\ I^\bullet & & I^{p,\bullet} \\ & \searrow^{s^p} & \downarrow{\scriptstyle t^p} \\ & & J^{p,\bullet} \end{array}$$

This is possible: the composition $F^pK^\bullet \to J^{p,\bullet}$ is homotopic to zero because $J^{p,\bullet}$ is acyclic and K-injective (Derived Categories, Lemma 13.31.2). Since the objects $J^{p,n-1}$ are injective and since $F^pK^n \to K^n \to I^n$ are injective morphisms, we can lift the maps $F^pK^n \to J^{p,n-1}$ giving the homotopy to a map $h^n : I^n \to J^{p,n-1}$. Then we set s^p equal to $h \circ \mathrm{d} + \mathrm{d} \circ h$. (Warning: It will not be the case that $t^p = s^p \circ \alpha^p$, so we have to be careful not to use this below.)

Consider

$$J^\bullet = I^\bullet \times \prod\nolimits_p J^{p,\bullet}$$

Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 19.13.4) and since $J^{p,\bullet}$ is isomorphic to 0 in $D(\mathcal{A})$ we see that $J^\bullet \to I^\bullet$ is an isomorphism in $D(\mathcal{A})$. Consider the map

$$j = i \times (s^p \circ i)_{p \in \mathbf{Z}} : K^\bullet \longrightarrow I^\bullet \times \prod\nolimits_p J^{p,\bullet} = J^\bullet$$

By our remarks above this is a quasi-isomorphism. It is also injective. For $p \in \mathbf{Z}$ we let $F^pJ^\bullet \subset J^\bullet$ be

$$\mathrm{Im}\left(\alpha^p \times (t^{p'} \circ \alpha^{pp'})_{p' \leq p} : I^{p,\bullet} \to I^\bullet \times \prod\nolimits_{p' \leq p} J^{p',\bullet}\right) \times \prod\nolimits_{p' > p} J^{p',\bullet}$$

This complex is isomorphic to the complex $I^{p,\bullet} \times \prod_{p' > p} J^{p,\bullet}$ as $\alpha^{pp} = \mathrm{id}$ and t^p is injective. Hence $F^pJ^\bullet$ is quasi-isomorphic to $I^{p,\bullet}$ (argue as above). We have $j(F^pK^\bullet) \subset F^pJ^\bullet$ because of the commutativity of the diagram above. The corresponding map of complexes $F^pK^\bullet \to F^pJ^\bullet$ is a quasi-isomorphism by what we just said. Finally, to see that $F^{p+1}J^\bullet \subset F^pJ^\bullet$ use that $\alpha^{p+1p} \circ \alpha^{pp'} = \alpha^{p+1p'}$

and the commutativity of the first displayed diagram in the first paragraph of the proof.

We claim that $j : K^\bullet \to J^\bullet$ is a solution to the problem posed by the lemma. Namely, F^pJ^n is an injective object of $\mathcal{A}$ because it is isomorphic to $I^{p,n} \times \prod_{p'>p} J^{p',n}$ and products of injectives are injective. Then the injective map $F^pJ^n \to J^n$ splits and hence the quotient J^n/F^pJ^n is injective as well as a direct summand of the injective object J^n. Similarly for $F^pJ^n/F^{p'}J^n$. This in particular means that $0 \to F^pJ^\bullet \to J^\bullet \to J^\bullet/F^pJ^\bullet \to 0$ is a termwise split short exact sequence of complexes, hence defines a distinguished triangle in $K(\mathcal{A})$ by fiat. Since $J^\bullet$ and $F^pJ^\bullet$ are K-injective complexes we see that the same is true for $J^\bullet/F^pJ^\bullet$ by Derived Categories, Lemma 13.31.3. A similar argument shows that $F^pJ^\bullet/F^{p'}J^\bullet$ is K-injective. By construction $j : K^\bullet \to J^\bullet$ and the induced maps $F^pK^\bullet \to F^pJ^\bullet$ are quasi-isomorphisms. Using the long exact cohomology sequences of the complexes in play we find that the same holds for $K^\bullet/F^pK^\bullet \to J^\bullet/F^pJ^\bullet$ and $F^pK^\bullet/F^{p'}K^\bullet \to F^pJ^\bullet/F^{p'}J^\bullet$. □

0G1X **Remark 19.13.8.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. For ease of notation denote K, F^pK, gr^pK the object of $D(\mathcal{A})$ represented by $K^\bullet$, $F^pK^\bullet$, $\mathrm{gr}^pK^\bullet$. Let $M \in D(\mathcal{A})$. Using Lemma 19.13.7 we can construct a spectral sequence $(E_r, d_r)_{r\geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidegree $(r, -r+1)$ and with

$$E_1^{p,q} = \mathrm{Ext}^{p+q}(M, \mathrm{gr}^pK)$$

If for every n we have

$$\mathrm{Ext}^n(M, F^pK) = 0 \text{ for } p \gg 0 \quad \text{and} \quad \mathrm{Ext}^n(M, F^pK) = \mathrm{Ext}^n(M, K) \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$. Namely, choose any complex $M^\bullet$ representing M, choose $j : K^\bullet \to J^\bullet$ as in the lemma, and consider the complex

$$\mathrm{Hom}^\bullet(M^\bullet, J^\bullet)$$

defined exactly as in More on Algebra, Section 15.72. Setting $F^p \mathrm{Hom}^\bullet(M^\bullet, J^\bullet) = \mathrm{Hom}^\bullet(M^\bullet, F^pJ^\bullet)$ we obtain a filtered complex. The spectral sequence of Homology, Section 12.24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 12.24.13.

0G1Y **Remark 19.13.9.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $K^\bullet$ representing K we can use the filtration $F^pK^\bullet = \tau_{\leq -p}K^\bullet$ and the discussion in Remark 19.13.8 to get a spectral sequence with

$$E_1^{p,q} = \mathrm{Ext}^{2p+q}(M, H^{-p}(K))$$

This spectral sequence is independent of the choice of complex $K^\bullet$ representing K. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E'_r, d'_r)_{r\geq 2}$ with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = \mathrm{Ext}^i(M, H^j(K))$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then both E_r and E'_r are bounded and converge to $\mathrm{Ext}^{p+q}(M, K)$. If we use the filtration $F^pK^\bullet = \sigma_{\geq p}K^\bullet$ then we get

$$E_1^{p,q} = \mathrm{Ext}^q(M, K^p)$$

If $M \in D^-(\mathcal{A})$ and $K^\bullet$ is bounded below, then this spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$.

0G1Z **Remark 19.13.10.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K \in D(\mathcal{A})$. Let $M^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. For ease of notation denote M, M/F^pM, gr^pM the object of $D(\mathcal{A})$ represented by $M^\bullet$, $M^\bullet/F^pM^\bullet$, $\mathrm{gr}^pM^\bullet$. Dually to Remark 19.13.8 we can construct a spectral sequence $(E_r, d_r)_{r\geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidegree $(r, -r+1)$ and with

$$E_1^{p,q} = \mathrm{Ext}^{p+q}(\mathrm{gr}^{-p}M, K)$$

If for every n we have

$$\mathrm{Ext}^n(M/F^pM, K) = 0 \text{ for } p \ll 0 \quad \text{and} \quad \mathrm{Ext}^n(M/F^pM, K) = \mathrm{Ext}^n(M, K) \text{ for } p \gg 0$$

then the spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$. Namely, choose a K-injective complex $I^\bullet$ with injective terms representing K, see Theorem 19.12.6. Consider the complex

$$\mathrm{Hom}^\bullet(M^\bullet, I^\bullet)$$

defined exactly as in More on Algebra, Section 15.72. Setting

$$F^p \mathrm{Hom}^\bullet(M^\bullet, I^\bullet) = \mathrm{Hom}^\bullet(M^\bullet/F^{-p+1}M^\bullet, I^\bullet)$$

we obtain a filtered complex (note sign and shift in filtration). The spectral sequence of Homology, Section 12.24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 12.24.13.

0G20 **Remark 19.13.11.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $M^\bullet$ representing M we can use the filtration $F^pM^\bullet = \tau_{\leq -p}M^\bullet$ and the discussion in Remark 19.13.8 to get a spectral sequence with

$$E_1^{p,q} = \mathrm{Ext}^{2p+q}(H^p(M), K)$$

This spectral sequence is independent of the choice of complex $M^\bullet$ representing M. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E'_r, d'_r)_{r\geq 2}$ with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = \mathrm{Ext}^i(H^{-j}(M), K)$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then E_r and E'_r are bounded and converge to $\mathrm{Ext}^{p+q}(M, K)$. If we use the filtration $F^pM^\bullet = \sigma_{\geq p}M^\bullet$ then we get

$$E_1^{p,q} = \mathrm{Ext}^q(M^{-p}, K)$$

If $K \in D^+(\mathcal{A})$ and $M^\bullet$ is bounded above, then this spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$.

0ESJ **Lemma 19.13.12.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Suppose given an object $E \in D(\mathcal{A})$ and an inverse system $\{E^i\}_{i\in\mathbf{Z}}$ of objects of $D(\mathcal{A})$ over $\mathbf{Z}$ together with a compatible system of maps $E^i \to E$. Picture:*

$$\ldots \to E^{i+1} \to E^i \to E^{i-1} \to \ldots \to E$$

Then there exists a filtered complex $K^\bullet$ of $\mathcal{A}$ (Homology, Definition 12.24.1) such that $K^\bullet$ represents E and $F^iK^\bullet$ represents E^i compatibly with the given maps.

Proof. By Theorem 19.12.6 we can choose a K-injective complex $I^\bullet$ representing E all of whose terms I^n are injective objects of $\mathcal{A}$. Choose a complex $G^{0,\bullet}$ representing E^0. Choose a map of complexes $\varphi^0 : G^{0,\bullet} \to I^\bullet$ representing $E^0 \to E$. For $i > 0$ we inductively represent $E^i \to E^{i-1}$ by a map of complexes $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$ and we set $\varphi^i = \delta \circ \varphi^{i-1}$. For $i < 0$ we inductively represent $E^{i+1} \to E^i$ by a termwise injective map of complexes $\delta : G^{i+1,\bullet} \to G^{i,\bullet}$ (for example you can use Derived Categories, Lemma 13.9.6). Claim: we can find a map of complexes $\varphi^i : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$ and fitting into the commutative diagram

$$\begin{array}{ccc} G^{i+1,\bullet} & \xrightarrow{\delta} & G^{i,\bullet} \\ {\scriptstyle \varphi^{i+1}}\downarrow & \swarrow{\scriptstyle \varphi^i} & \\ I^\bullet & & \end{array}$$

Namely, we first choose any map of complexes $\varphi : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$. Then we see that $\varphi \circ \delta$ and φ^{i+1} are homotopic by some homotopy $h^p : G^{i+1,p} \to I^{p-1}$. Since the terms of $I^\bullet$ are injective and since δ is termwise injective, we can lift h^p to $(h')^p : G^{i,p} \to I^{p-1}$. Then we set $\varphi^i = \varphi + h' \circ d + d \circ h'$ and we get what we claimed.

Next, we choose for every i a termwise injective map of complexes $a^i : G^{i,\bullet} \to J^{i,\bullet}$ with $J^{i,\bullet}$ acyclic, K-injective, with $J^{i,p}$ injective objects of $\mathcal{A}$. To do this first map $G^{i,\bullet}$ to the cone on the identity and then apply the theorem cited above. Arguing as above we can find maps of complexes $\delta' : J^{i,\bullet} \to J^{i-1,\bullet}$ such that the diagrams

$$\begin{array}{ccc} G^{i,\bullet} & \xrightarrow{\delta} & G^{i-1,\bullet} \\ {\scriptstyle a^i}\downarrow & & \downarrow{\scriptstyle a^{i-1}} \\ J^{i,\bullet} & \xrightarrow{\delta'} & J^{i-1,\bullet} \end{array}$$

commute. (You could also use the functoriality of cones plus the functoriality in the theorem to get this.) Then we consider the maps

$$\begin{array}{ccccc} G^{i+1,\bullet} \times \prod_{p>i+1} J^{p,\bullet} & \longrightarrow & G^{i,\bullet} \times \prod_{p>i} J^{p,\bullet} & \longrightarrow & G^{i-1,\bullet} \times \prod_{p>i-1} J^{p,\bullet} \\ & \searrow & \downarrow & \swarrow & \\ & & I^\bullet \times \prod_p J^{p,\bullet} & & \end{array}$$

Here the arrows on $J^{p,\bullet}$ are the obvious ones (identity or zero). On the factor $G^{i,\bullet}$ we use $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$, the map $\varphi^i : G^{i,\bullet} \to I^\bullet$, the zero map $0 : G^{i,\bullet} \to J^{p,\bullet}$ for $p > i$, the map $a^i : G^{i,\bullet} \to J^{p,\bullet}$ for $p = i$, and $(\delta')^{i-p} \circ a^i = a^p \circ \delta^{i-p} : G^{i,\bullet} \to J^{p,\bullet}$ for $p < i$. We omit the verification that all the arrows in the diagram are termwise injective. Thus we obtain a filtered complex. Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 19.13.4) and because $J^{p,\bullet}$ is zero in $D(\mathcal{A})$ we conclude this diagram represents the given diagram in the derived category. This finishes the proof. □

0ESK **Lemma 19.13.13.** *In the situation of Lemma 19.13.12 assume we have a second inverse system $\{(E')^i\}_{i\in\mathbf{Z}}$ and a compatible system of maps $(E')^i \to E$. Then there exists a bi-filtered complex $K^\bullet$ of $\mathcal{A}$ such that $K^\bullet$ represents E, $F^iK^\bullet$ represents E^i, and $(F')^iK^\bullet$ represents $(E')^i$ compatibly with the given maps.*

Proof. Using the lemma we can first choose $K^\bullet$ and F. Then we can choose $(K')^\bullet$ and F' which work for $\{(E')^i\}_{i\in\mathbf{Z}}$ and the maps $(E')^i \to E$. Using Lemma 19.13.7 we can assume $K^\bullet$ is a K-injective complex. Then we can choose a map of complexes $(K')^\bullet \to K^\bullet$ corresponding to the given identifications $(K')^\bullet \cong E \cong K^\bullet$. We can additionally choose a termwise injective map $(K')^\bullet \to J^\bullet$ with $J^\bullet$ acyclic and K-injective. (To do this first map $(K')^\bullet$ to the cone on the identity and then apply Theorem 19.12.6.) Then $(K')^\bullet \to K^\bullet \times J^\bullet$ and $K^\bullet \to K^\bullet \times J^\bullet$ are both termwise injective and quasi-isomorphisms (as the product represents E by Lemma 19.13.4). Then we can simply take the images of the filtrations on $K^\bullet$ and $(K')^\bullet$ under these maps to conclude. □

19.14. The Gabriel-Popescu theorem

0F5R In this section we discuss the main theorem of [**PG64**]. The method of proof follows a write-up by Jacob Lurie and another by Akhil Mathew who in turn follow the presentation by Kuhn in [**Kuh94**]. See also [**Tak71**].

Let $\mathcal{A}$ be a Grothendieck abelian category and let U be a generator for $\mathcal{A}$, see Definition 19.10.1. Let $R = \mathrm{Hom}_{\mathcal{A}}(U, U)$. Consider the functor $G : \mathcal{A} \to \mathrm{Mod}_R$ given by

$$G(A) = \mathrm{Hom}_{\mathcal{A}}(U, A)$$

endowed with its canonical right R-module structure.

0F5S **Lemma 19.14.1.** *The functor G above has a left adjoint $F : Mod_R \to \mathcal{A}$.*

Proof. We will give two proofs of this lemma.

The first proof will use the adjoint functor theorem, see Categories, Theorem 4.25.3. Observe that $G : \mathcal{A} \to \mathrm{Mod}_R$ is left exact and sends products to products. Hence G commutes with limits. To check the set theoretical condition in the theorem, suppose that M is an object of Mod_R. Choose a suitably large cardinal κ and denote E a set of objects of $\mathcal{A}$ such that every object A with $|A| \leq \kappa$ is isomorphic to an element of E. This is possible by Lemma 19.11.4. Set $I = \coprod_{A\in E} \mathrm{Hom}_R(M, G(A))$. We think of an element $i \in I$ as a pair (A_i, f_i). Finally, let A be an arbitrary object of $\mathcal{A}$ and $f : M \to G(A)$ arbitrary. We are going to think of elements of $\mathrm{Im}(f) \subset G(A) = \mathrm{Hom}_{\mathcal{A}}(U, A)$ as maps $u : U \to A$. Set

$$A' = \mathrm{Im}(\bigoplus\nolimits_{u\in\mathrm{Im}(f)} U \xrightarrow{u} A)$$

Since G is left exact, we see that $G(A') \subset G(A)$ contains $\mathrm{Im}(f)$ and we get $f' : M \to G(A')$ factoring f. On the other hand, the object A' is the quotient of a direct sum of at most $|M|$ copies of U. Hence if $\kappa = |\bigoplus_{|M|} U|$, then we see that (A', f') is isomorphic to an element (A_i, f_i) of E and we conclude that f factors as $M \xrightarrow{f_i} G(A_i) \to G(A)$ as desired.

The second proof will give a construction of F which will show that "$F(M) = M \otimes_R U$" in some sense. Namely, for any R-module M we can choose a resolution

$$\bigoplus\nolimits_{j\in J} R \to \bigoplus\nolimits_{i\in I} R \to M \to 0$$

Then we define $F(M)$ by the corresponding exact sequence

$$\bigoplus\nolimits_{j\in J} U \to \bigoplus\nolimits_{i\in I} U \to F(M) \to 0$$

This construction is independent of the choice of the resolution and is functorial; we omit the details. For any A in $\mathcal{A}$ we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(F(M), A) \to \prod\nolimits_{i \in I} G(A) \to \prod\nolimits_{j \in J} G(A)$$

which is isomorphic to the sequence

$$0 \to \mathrm{Hom}_R(M, G(A)) \to \mathrm{Hom}_R(\bigoplus\nolimits_{i \in I} R, G(A)) \to \mathrm{Hom}_R(\bigoplus\nolimits_{j \in J} R, G(A))$$

which shows that F is the left adjoint to G. □

0F5T **Lemma 19.14.2.** *Let $f : M \to G(A)$ be an injective map in Mod_R. Then the adjoint map $f' : F(M) \to A$ is injective too.*

Proof. Choose a map $R^{\oplus n} \to M$ and consider the corresponding map $U^{\oplus n} \to F(M)$. Consider a map $v : U \to U^{\oplus n}$ such that the composition $U \to U^{\oplus n} \to F(M) \to A$ is 0. Then this arrow $v : U \to U^{\oplus n}$ is an element v of $R^{\oplus n}$ mapping to zero in $G(A)$. Since f is injective, we conclude that v maps to zero in M which means that $U \to U^{\oplus n} \to F(M)$ is zero by construction of $F(M)$ in the proof of Lemma 19.14.1. Since U is a generator we conclude that

$$\mathrm{Ker}(U^{\oplus n} \to F(M) \to A) = \mathrm{Ker}(U^{\oplus n} \to F(M))$$

To finish the proof we choose a surjection $\bigoplus_{i \in I} R \to M$ and we consider the corresponding surjection

$$\pi : \bigoplus\nolimits_{i \in I} U \longrightarrow F(M)$$

To prove f' is injective it suffices to show that $\mathrm{Ker}(\pi) = \mathrm{Ker}(f' \circ \pi)$ as subobjects of $\bigoplus_{i \in I} U$. However, now we can write $\bigoplus_{i \in I} U$ as the filtered colimit of its subobjects $\bigoplus_{i \in I'} U$ where $I' \subset I$ ranges over the finite subsets. Since filtered colimits are exact by AB5 for $\mathcal{A}$, we see that

$$\mathrm{Ker}(\pi) = \mathrm{colim}_{I' \subset I \text{ finite}} \left(\bigoplus\nolimits_{i \in I'} U \right) \bigcap \mathrm{Ker}(\pi)$$

and

$$\mathrm{Ker}(f' \circ \pi) = \mathrm{colim}_{I' \subset I \text{ finite}} \left(\bigoplus\nolimits_{i \in I'} U \right) \bigcap \mathrm{Ker}(f' \circ \pi)$$

and we get equality because the same is true for each I' by the first displayed equality above. □

0F5U **Theorem 19.14.3.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there exists a (noncommutative) ring R and functors $G : \mathcal{A} \to Mod_R$ and $F : Mod_R \to \mathcal{A}$ such that*

(1) *F is the left adjoint to G,*
(2) *G is fully faithful, and*
(3) *F is exact.*

Moreover, the functors are the ones constructed above.

Proof. We first prove G is fully faithful, or equivalently that $F \circ G \to \mathrm{id}$ is an isomorphism, see Categories, Lemma 4.24.4. First, given an object A the map $F(G(A)) \to A$ is surjective, because every map of $U \to A$ factors through $F(G(A))$ by construction. On the other hand, the map $F(G(A)) \to A$ is the adjoint of the map $\mathrm{id} : G(A) \to G(A)$ and hence injective by Lemma 19.14.2.

The functor F is right exact as it is a left adjoint. Since Mod_R has enough projectives, to show that F is exact, it is enough to show that the first left derived

functor L_1F is zero. To prove $L_1F(M) = 0$ for some R-module M choose an exact sequence $0 \to K \to P \to M \to 0$ of R-modules with P free. It suffices to show $F(K) \to F(P)$ is injective. Now we can write this sequence as a filtered colimit of sequences $0 \to K_i \to P_i \to M_i \to 0$ with P_i a finite free R-module: just write P in this manner and set $K_i = K \cap P_i$ and $M_i = \mathrm{Im}(P_i \to M)$. Because F is a left adjoint it commutes with colimits and because $\mathcal{A}$ is a Grothendieck abelian category, we find that $F(K) \to F(P)$ is injective if each $F(K_i) \to F(P_i)$ is injective. Thus it suffices to check $F(K) \to F(P)$ is injective when $K \subset P = R^{\oplus n}$. Thus $F(K) \to U^{\oplus n}$ is injective by an application of Lemma 19.14.2. □

0F5V **Lemma 19.14.4.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let R, F, G be as in the Gabriel-Popescu theorem (Theorem 19.14.3). Then we obtain derived functors* [**Ser03**, Corollary 4.1]

$$RG : D(\mathcal{A}) \to D(Mod_R) \quad \text{and} \quad F : D(Mod_R) \to D(\mathcal{A})$$

such that F is left adjoint to RG, RG is fully faithful, and $F \circ RG = id$.

Proof. The existence and adjointness of the functors follows from Theorems 19.14.3 and 19.12.6 and Derived Categories, Lemmas 13.31.6, 13.16.9, and 13.30.3. The statement $F \circ RG = \mathrm{id}$ follows because we can compute RG on an object of $D(\mathcal{A})$ by applying G to a suitable representative complex $I^\bullet$ (for example a K-injective one) and then $F(G(I^\bullet)) = I^\bullet$ because $F \circ G = \mathrm{id}$. Fully faithfulness of RG follows from this by Categories, Lemma 4.24.4. □

19.15. Brown representability and Grothendieck abelian categories

0F5W In this section we quickly prove a representability theorem for derived categories of Grothendieck abelian categories. The reader should first read the case of compactly generated triangulated categories in Derived Categories, Section 13.38. After that, instead of reading this section, it makes sense to consult the literature for more general results of this nature, for example see [**Fra01**], [**Nee01**], [**Kra02**], or take a look at Derived Categories, Section 13.39.

0F5X **Lemma 19.15.1.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $H : D(\mathcal{A}) \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.*

Proof. Let R, F, G, RG be as in Lemma 19.14.4 and consider the functor $H \circ F : D(\mathrm{Mod}_R) \to Ab$. Observe that since F is a left adjoint it sends direct sums to direct sums and hence $H \circ F$ transforms direct sums into products. On the other hand, the derived category $D(\mathrm{Mod}_R)$ is generated by a single compact object, namely R. By Derived Categories, Lemma 13.38.1 we see that $H \circ F$ is representable, say by $L \in D(\mathrm{Mod}_R)$. Choose a distinguished triangle

$$M \to L \to RG(F(L)) \to M[1]$$

in $D(\mathrm{Mod}_R)$. Then $F(M) = 0$ because $F \circ RG = \mathrm{id}$. Hence $H(F(M)) = 0$ hence $\mathrm{Hom}(M, L) = 0$. It follows that $L \to RG(F(L))$ is the inclusion of a direct

summand, see Derived Categories, Lemma 13.4.11. For A in $D(\mathcal{A})$ we obtain

$$\begin{aligned} H(A) &= H(F(RG(A)) \\ &= \mathrm{Hom}(RG(A), L) \\ &\to \mathrm{Hom}(RG(A), RG(F(L))) \\ &= \mathrm{Hom}(F(RG(A)), F(L)) \\ &= \mathrm{Hom}(A, F(L)) \end{aligned}$$

where the arrow has a left inverse functorial in A. In other words, we find that H is the direct summand of a representable functor. Since $D(\mathcal{A})$ is Karoubian (Derived Categories, Lemma 13.4.14) we conclude. □

0F5Y **Proposition 19.15.2.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F : D(\mathcal{A}) \to \mathcal{D}$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.*

Proof. For an object Y of $\mathcal{D}$ consider the contravariant functor

$$D(\mathcal{A}) \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 19.15.1 we find an object X of $D(\mathcal{A})$ such that $\mathrm{Hom}_{D(\mathcal{A})}(W, X) = \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Derived Categories, Lemma 13.7.1. □

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CHAPTER 20

Cohomology of Sheaves

01DW

20.1. Introduction

01DX In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites take a look at the chapter Cohomology on Sites, Section 21.1. Basic references are [**God73**] and [**Ive86**].

20.2. Cohomology of sheaves

01DZ Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. We know that the category of abelian sheaves on X has enough injectives, see Injectives, Lemma 19.4.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

0712 (20.2.0.1) $$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of the abelian sheaf* $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal δ-functor from $Ab(X) \to Ab$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

0713 (20.2.0.2) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(X) \to Ab(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on X has enough injectives, see Injectives, Lemma 19.5.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

0714 (20.2.0.3) $$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of* $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(X)}$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

0715 (20.2.0.4) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

20.3. Derived functors

0716 We briefly explain how to get right derived functors using resolution functors. For the unbounded derived functors, please see Section 20.28.

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is abelian, see Modules, Lemma 17.3.1. In this chapter we will write

$$K(\mathcal{O}_X) = K(Mod(\mathcal{O}_X)) \quad \text{and} \quad D(\mathcal{O}_X) = D(Mod(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.8.1 and Definition 13.11.3. By Derived Categories, Remark 13.24.3 there exists a resolution functor

$$j = j_X : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : Mod(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor described in Derived Categories, Section 13.20 and constructed using the resolution functor j_X just described:

05U3 (20.3.0.1) $$RF = F \circ j_X' : D^+(X) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 13.25.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O}_X)$, $\mathrm{Comp}^+(Mod(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the ith right derived functor

05U4 (20.3.0.2) $$R^iF = H^i \circ RF : Mod(\mathcal{O}_X) \longrightarrow \mathcal{B}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\mathrm{Mod}_R)$ and $D(R) = D(\mathrm{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \longrightarrow \mathrm{Mod}_{\mathcal{O}_X(U)}$ which gives rise to

0717 (20.3.0.3) $$R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U))$$

by the discussion above. We set $H^i(U, -) = R^i\Gamma(U, -)$. If $U = X$ we recover (20.2.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ which gives rise to the *derived pushforward*

0718 (20.3.0.4) $$Rf_* : D^+(X) \longrightarrow D^+(Y)$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (20.2.0.4). The two displayed functors above are exact functors of derived categories.

Abuse of notation: When the functor Rf_*, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on X or a complex of sheaves it is understood that $\mathcal{F}$ has been replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(\mathcal{O}_X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of $Mod(\mathcal{O}_X)$ *represents* $\mathcal{F}^\bullet$ *in the derived category* if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase "let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(\mathcal{O}_X)$" does not mean that α is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

20.4. First cohomology and torsors

02FN

02FO **Definition 20.4.1.** Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
(2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

02FP **Lemma 20.4.2.** *Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.*

Proof. Omitted. □

02FQ **Lemma 20.4.3.** *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s' = s$. Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. □

20.5. First cohomology and extensions

0B39

0B3A **Lemma 20.5.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. There is a canonical bijection*

$$\mathrm{Ext}^1_{Mod(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_X \to 0$$

the image of $1 \in \Gamma(X, \mathcal{O}_X)$ in $H^1(X, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(X, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O}_X)$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of a section $\tilde{\xi} \in \Gamma(X, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{Q})$. Now, we just form the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tilde{\xi}} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

see Homology, Section 12.6. □

20.6. First cohomology and invertible sheaves

09NT The Picard group of a ringed space is defined in Modules, Section 17.25.

09NU **Lemma 20.6.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If all stalks $\mathcal{O}_{X,x}$ are local rings, then there is a canonical isomorphism*

$$H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}_X^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}_X^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules, Lemma 17.25.4. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}_X^*$-torsor. Thus we get a map

$$\begin{array}{c} \text{invertible sheaves on } (X, \mathcal{O}_X) \\ \text{up to isomorphism} \end{array} \longrightarrow \begin{array}{c} \mathcal{O}_X^*\text{-torsors} \\ \text{up to isomorphism} \end{array}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 20.4.3 the right hand side is canonically bijective to $H^1(X, \mathcal{O}_X^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 20.4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}_X$ is the neutral element in $\mathrm{Pic}(X)$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}_X^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}_X(U))/\mathcal{O}_X^*(U)$$

where the action of $f \in \mathcal{O}_X^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}_X$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where s'/s is the local section f of $\mathcal{O}_X^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}_X$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}_X$-module whose associated $\mathcal{O}_X^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

20.7. Locality of cohomology

01E0 The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

01E1 **Lemma 20.7.1.** *Let X be a ringed space. Let $U \subset X$ be an open subspace.*

(1) *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
(2) *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.*

Proof. Denote $j : U \to X$ the open immersion. Recall that the functor j^{-1} of restriction to U is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 6.31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 12.29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (2). □

Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical *restriction mapping*

01E2 (20.7.1.1) $$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 20.11.4.

01E3 **Lemma 20.7.2.** *Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since

$\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

01E4 **Lemma 20.7.3.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with restriction mappings as in Equation (20.7.1.1). There is a similar statement for $R^i f_*$ applied to a bounded below complex $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to f_* \mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_* \mathcal{I}^i(V) \to f_* \mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_* \mathcal{I}^{i-1}(V) \to f_* \mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

01E5 **Lemma 20.7.4.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$R^p g_*(\mathcal{F}|_{f^{-1}(V)}) = (R^p f_* \mathcal{F})|_V$$

There is a similar statement for the derived image $Rf_* \mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. First proof. Apply Lemmas 20.7.3 and 20.7.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. □

03BA **Remark 20.7.5.** Here is a different approach to the proofs of Lemmas 20.7.2 and 20.7.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : \mathit{Mod}(\mathcal{O}_X) \to \mathit{PMod}(\mathcal{O}_X)$ be the inclusion functor and let $\#$ be the sheafification functor. Recall that i_X is left exact and $\#$ is exact.

(1) First prove Lemma 20.11.4 below which says that the right derived functors of i_X are given by $R^p i_X \mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^p i_X)_{p \geq 0}$ and $(\underline{H}^p)_{p \geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 12.12.

(2) A restatement of Lemma 20.7.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $\#$ is exact so

$$(\underline{H}^p(\mathcal{F}))^\# = (R^p i_X \mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$

because $\# \circ i_X$ is the identity functor.

(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 20.7.3 says that $R^p f_*\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that $\#$ is exact a that $\# \circ i_Y$ is the identity functor we see that

$$R^p f_*\mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$

as desired.

20.8. Mayer-Vietoris

01E9 Below will construct the Čech-to-cohomology spectral sequence, see Lemma 20.11.5. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

01EA **Lemma 20.8.1.** *Let X be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_!\mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 6.31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 6.31.8. Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). By the description of the stalks of the sheaves $j'_!\mathcal{O}_{U'}$, $j_!\mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{I}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j'_!\mathcal{O}_{U'}, \mathcal{I})$$

is surjective, see Homology, Lemma 12.27.2. Putting everything together we obtain the lemma. □

01EB **Lemma 20.8.2** (Mayer-Vietoris)**.** *Let X be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 20.8.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 12.13.12). We omit the proof of the functoriality of the sequence. □

01EC **Lemma 20.8.3** (Relative Mayer-Vietoris)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U\cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(U\cap f^{-1}(W))\oplus\mathcal{I}^n(V\cap f^{-1}(W)) \to \mathcal{I}^n(U\cap V\cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 20.8.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U\cap V}$ see Lemma 20.7.1. □

20.9. The Čech complex and Čech cohomology

01ED Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering, see Topology, Basic notion (13). As is customary we denote $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0\ldots i_p}$ its value in $\mathcal{F}(U_{i_0\ldots i_p})$. Note that if $s \in \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are *not* working with alternating cochains (these will be defined in Section 20.23). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

01EE (20.9.0.1) $$d(s)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0\ldots\hat{i}_j\ldots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

01EF **Definition 20.9.1.** Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

01EG **Lemma 20.9.2.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf and*

(2) *for every open covering* $\mathcal{U} : U = \bigcup_{i\in I} U_i$ *the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

0G6S **Lemma 20.9.3.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended Čech complex*

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

obtained by putting $\mathcal{F}(U)$ in degree -1 with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0.

Proof. Fix an element $i \in I$ with $U = U_i$. Observe that $U_{i_0\ldots i_p} = U_{i_0\ldots\hat{i}_j\ldots i_p}$ if $i_j = i$. Let us define a homotopy

$$h : \prod\nolimits_{i_0\ldots i_{p+1}} \mathcal{F}(U_{i_0\ldots i_{p+1}}) \longrightarrow \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

by the rule

$$h(s)_{i_0\ldots i_p} = s_{ii_0\ldots i_p}$$

In other words, $h : \prod_{i_0} \mathcal{F}(U_{i_0}) \to \mathcal{F}(U)$ is projection onto the factor $\mathcal{F}(U_i) = \mathcal{F}(U)$ and in general the map h equals the projection onto the factors $\mathcal{F}(U_{ii_1\ldots i_{p+1}}) = \mathcal{F}(U_{i_1\ldots i_{p+1}})$. We compute

$$\begin{aligned}(dh + hd)(s)_{i_0\ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0\ldots\hat{i}_j\ldots i_p} + d(s)_{ii_0\ldots i_p} \\ &= \sum\nolimits_{j=0}^{p} (-1)^j s_{ii_0\ldots\hat{i}_j\ldots i_p} + s_{i_0\ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{ii_0\ldots\hat{i}_j\ldots i_p} \\ &= s_{i_0\ldots i_p}\end{aligned}$$

This proves the identity map is homotopic to zero as desired. □

20.10. Čech cohomology as a functor on presheaves

01EH Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

01EI (20.10.0.1) $$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow \mathrm{Comp}^+(\mathrm{Mod}_{\mathcal{O}_X(U)})$$

see Derived Categories, Definition 13.8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.13.9.

01EJ **Lemma 20.10.1.** *The functor given by Equation (20.10.0.1) is an exact functor (see Homology, Lemma 12.7.2).*

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PMod(\mathcal{O}_X)$ to $\mathrm{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

01EK **Lemma 20.10.2.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 12.12.1).*

Proof. By Lemma 20.10.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 12.13.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 6.31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 6.31.8. In particular we have the following formula

$$\mathrm{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $PMod(\mathcal{O}_X)$, see Homology, Lemma 12.28.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\mathrm{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

01EL **Lemma 20.10.3.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!}\mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!}\mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0 \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$\mathrm{Hom}_{\mathcal{O}_X}((j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence by Categories, Yoneda Lemma 4.3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. □

01EM **Lemma 20.10.4.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 20.10.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & \text{if} \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0}(j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0\ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1} \sum\nolimits_{i\in I_1} (-1)^j s_{i_0\ldots i_{j-1} i i_j \ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\mathrm{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i_{\mathrm{fix}} \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\mathrm{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\mathrm{fix}}) s_{i_1\ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0\ldots i_p} \\ = & \sum_j \sum_{i\in I_1} (-1)^j h(s)_{i_0\ldots i_{j-1} i i_j \ldots i_p} + (i = i_0) d(s)_{i_1\ldots i_p} \\ = & \ s_{i_0\ldots i_p} + \sum_{j\geq 1} \sum_{i\in I_1} (-1)^j (i_0 = i_{\mathrm{fix}}) s_{i_1\ldots i_{j-1} i i_j \ldots i_p} + (i_0 = i_{\mathrm{fix}}) d(s)_{i_1\ldots i_p} \end{aligned}$$

which is equal to $s_{i_0 \ldots i_p}$ as desired. □

01EN **Lemma 20.10.5.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow Mod_{\mathcal{O}_X(U)}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PMod(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 19.8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.20.

Let $\mathcal{I}$ be an injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $PMod(\mathcal{O}_X)$. By Lemma 20.10.3 we have

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 20.10.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 20.10.2) satisfies the assumptions of Homology, Lemma 12.12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PMod(\mathcal{O}_X)$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Consider the associated total complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 12.18.3. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 20.10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

20.11. Čech cohomology and cohomology

01EO

01EP **Lemma 20.11.1.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category $PMod(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 12.29.1). Hence we can apply Lemma 20.10.5 (or its proof) to see the result. □

01EQ **Lemma 20.11.2.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Mod(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Mod(\mathcal{O}_X)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.25.4 to see that α is a quasi-isomorphism. Namely, Lemma 20.11.1 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

0B8R **Lemma 20.11.3.** *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be an open covering. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{H}) \longrightarrow H^1(X, \mathcal{H})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{H})$ via the bijection of Lemma 20.4.3 with the set of isomorphism classes of $\mathcal{H}$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{H}$-torsor whose restriction to U_i is trivial. By Lemma 20.4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \cap U_{i_1}$ there is a unique section $s_{i_0 i_1}$ of $\mathcal{H}$ such that $s_{i_0 i_1} \cdot s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}$. A computation shows that $s_{i_0 i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0 i_1})$. We omit the verification that this map is indeed an inverse. □

01ER **Lemma 20.11.4.** *Let X be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 20.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^p(U) \to \mathcal{I}^{p+1}(U))}{\mathrm{Im}(\mathcal{I}^{p-1}(U) \to \mathcal{I}^p(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

01ES **Lemma 20.11.5.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.22.2) for the functors

$$i : \mathit{Mod}(\mathcal{O}_X) \to \mathit{PMod}(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : \mathit{PMod}(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 20.9.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 20.11.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $\mathit{PMod}(\mathcal{O}_X)$ by Lemma 20.10.5. Putting everything together gives the lemma. □

01ET **Lemma 20.11.6.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0 \ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

Proof. We will use the spectral sequence of Lemma 20.11.5. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

01EU **Lemma 20.11.7.** *Let X be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find $\mathcal{U}$ such that (b) holds it follows from the assumptions of the lemma that we can find $\mathcal{U}$ such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

01EV **Lemma 20.11.8.** *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.11.7 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 20.10.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

01EW **Lemma 20.11.9.** *(Variant of Lemma 20.11.8.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

1. *For every $\mathcal{U} \in$ Cov with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov is a cofinal system of open coverings of U.*
3. *For every $\mathcal{U} \in$ Cov we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

01EX **Lemma 20.11.10.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

(1) *$\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$ and any open covering $\mathcal{V} : V = \bigcup_{j \in J} V_j$ of Y.*
(2) *$H^p(V, f_*\mathcal{I}) = 0$ for all $p > 0$ and every open $V \subset Y$.*

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(V, -)$ (see Derived Categories, Definition 13.15.3) for any $V \subset Y$ open.

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 20.11.1. The second statement follows from the first and Lemma 20.11.8. □

The following lemma implies in particular that $f_* : Ab(X) \to Ab(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

02N5 **Lemma 20.11.11.** *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. In this case the functor f^* transforms injections into injections (Modules, Lemma 17.20.2). Hence the result follows from Homology, Lemma 12.29.1. □

0D0A **Lemma 20.11.12.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an $\mathcal{O}_X$-module. Let $U \subset X$ be open. The canonical map*

$$H^p(U, \prod\nolimits_{i\in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i\in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sheaves, Section 6.29. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 20.7.2, there exists an open covering $\mathcal{U} : U = \bigcup U_j$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 20.11.3 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 20.11.3), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 12.32.1 we conclude that $\check{\xi} = 0$ as desired. □

20.12. Flasque sheaves

09SV Here is the definition.

09SW **Definition 20.12.1.** Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if X is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

09SX **Lemma 20.12.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

Proof. This is a reformulation of Lemma 20.8.1. □

09SY **Lemma 20.12.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open U of X.*

Proof. We will prove this using Derived Categories, Lemma 13.15.6. Since every injective module is flasque we see that we can embed every $\mathcal{O}_X$-module into a flasque module, see Injectives, Lemma 19.4.1. Thus it suffices to show that given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$, $\mathcal{G}$ flasque, then $\mathcal{H}$ is flasque and the sequence remains short exact after taking sections on any open of X. In fact, the second statement implies the first. Thus, let $U \subset X$ be an open subspace. Let $s \in \mathcal{H}(U)$. We will show that we can lift s to a section of $\mathcal{G}$ over U. To do this consider the set T of pairs (V, t) where $V \subset U$ is open and $t \in \mathcal{G}(V)$ is a section mapping to $s|_V$ in $\mathcal{H}$. We put a partial ordering on T by setting $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$. If (V_α, t_α), $\alpha \in A$ is a totally ordered subset of T, then $V = \bigcup V_\alpha$ is open and there is a unique section $t \in \mathcal{G}(V)$ restricting to t_α over V_α by the sheaf condition on $\mathcal{G}$.

Thus by Zorn's lemma there exists a maximal element (V, t) in T. We will show that $V = U$ thereby finishing the proof. Namely, pick any $x \in U$. We can find a small open neighbourhood $W \subset U$ of x and $t' \in \mathcal{G}(W)$ mapping to $s|_W$ in $\mathcal{H}$. Then $t'|_{W\cap V} - t|_{W\cap V}$ maps to zero in $\mathcal{H}$, hence comes from some section $r' \in \mathcal{F}(W \cap V)$. Using that $\mathcal{F}$ is flasque we find a section $r \in \mathcal{F}(W)$ restricting to r' over $W \cap V$. Modifying t' by the image of r we may assume that t and t' restrict to the same section over $W \cap V$. By the sheaf condition of $\mathcal{G}$ we can find a section $\tilde{t}$ of $\mathcal{G}$ over $W \cup V$ restricting to t and t'. By maximality of (V, t) we see that $V \cup W = V$. Thus $x \in V$ and we are done. □

The following lemma does not hold for flasque presheaves.

09SZ **Lemma 20.12.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} : U = \bigcup U_i$ be an open covering. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

Proof. The presheaves $\underline{H}^q(\mathcal{F})$ used in the statement of Lemma 20.11.5 are zero by Lemma 20.12.3. Hence $\check{H}^p(U, \mathcal{F}) = H^p(U, \mathcal{F}) = 0$ by Lemma 20.12.3 again. □

09T0 **Lemma 20.12.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $R^p f_* \mathcal{F} = 0$ for $p > 0$.*

Proof. Immediate from Lemma 20.7.3 and Lemma 20.12.3. □

The following lemma can be proved by an elementary induction argument for finite coverings, compare with the discussion of Čech cohomology in [**Vak**].

0A36 **Lemma 20.12.6.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume the restriction mappings $\mathcal{F}(U) \to \mathcal{F}(U')$ are surjective for U' an arbitrary union of opens of the form $U_{i_0 \ldots i_p}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanishes for $p > 0$.*

Proof. Let Y be the set of nonempty subsets of I. We will use the letters $A, B, C, \ldots$ to denote elements of Y, i.e., nonempty subsets of I. For a finite nonempty subset $J \subset I$ let

$$V_J = \{A \in Y \mid J \subset A\}$$

This means that $V_{\{i\}} = \{A \in Y \mid i \in A\}$ and $V_J = \bigcap_{j\in J} V_{\{j\}}$. Then $V_J \subset V_K$ if and only if $J \supset K$. There is a unique topology on Y such that the collection of subsets V_J is a basis for the topology on Y. Any open is of the form

$$V = \bigcup\nolimits_{t\in T} V_{J_t}$$

for some family of finite subsets J_t. If $J_t \subset J_{t'}$ then we may remove $J_{t'}$ from the family without changing V. Thus we may assume there are no inclusions among the J_t. In this case the minimal elements of V are the sets $A = J_t$. Hence we can read off the family $(J_t)_{t\in T}$ from the open V.

We can completely understand open coverings in Y. First, because the elements $A \in Y$ are nonempty subsets of I we have

$$Y = \bigcup\nolimits_{i\in I} V_{\{i\}}$$

To understand other coverings, let V be as above and let $V_s \subset Y$ be an open corresponding to the family $(J_{s,t})_{t\in T_s}$. Then

$$V = \bigcup\nolimits_{s\in S} V_s$$

if and only if for each $t \in T$ there exists an $s \in S$ and $t_s \in T_s$ such that $J_t = J_{s,t_s}$. Namely, as the family $(J_t)_{t\in T}$ is minimal, the minimal element $A = J_t$ has to be in V_s for some s, hence $A \in V_{J_{t_s}}$ for some $t_s \in T_s$. But since A is also minimal in V_s we conclude that $J_{t_s} = J_t$.

Next we map the set of opens of Y to opens of X. Namely, we send Y to U, we use the rule

$$V_J \mapsto U_J = \bigcap\nolimits_{i\in J} U_i$$

on the opens V_J, and we extend it to arbitrary opens V by the rule

$$V = \bigcup\nolimits_{t\in T} V_{J_t} \mapsto \bigcup\nolimits_{t\in T} U_{J_t}$$

The classification of open coverings of Y given above shows that this rule transforms open coverings into open coverings. Thus we obtain an abelian sheaf $\mathcal{G}$ on Y by setting $\mathcal{G}(Y) = \mathcal{F}(U)$ and for $V = \bigcup_{t\in T} V_{J_t}$ setting

$$\mathcal{G}(V) = \mathcal{F}\left(\bigcup\nolimits_{t\in T} U_{J_t}\right)$$

and using the restriction maps of $\mathcal{F}$.

With these preliminaries out of the way we can prove our lemma as follows. We have an open covering $\mathcal{V} : Y = \bigcup_{i\in I} V_{\{i\}}$ of Y. By construction we have an equality

$$\check{C}^\bullet(\mathcal{V}, \mathcal{G}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

of Čech complexes. Since the sheaf $\mathcal{G}$ is flasque on Y (by our assumption on $\mathcal{F}$ in the statement of the lemma) the vanishing follows from Lemma 20.12.4. □

20.13. The Leray spectral sequence

01EY

01EZ **Lemma 20.13.1.** *Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram*

$$\begin{array}{ccc} D^+(X) & \xrightarrow[R\Gamma(X,-)]{} & D^+(\mathcal{O}_X(X)) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\ D^+(Y) & \xrightarrow{R\Gamma(Y,-)} & D^+(\mathcal{O}_Y(Y)) \end{array}$$

More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram

$$\begin{array}{ccc} D^+(X) & \xrightarrow[R\Gamma(U,-)]{} & D^+(\mathcal{O}_X(U)) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\ D^+(Y) & \xrightarrow{R\Gamma(V,-)} & D^+(\mathcal{O}_Y(V)) \end{array}$$

See also Remark 20.13.2 for more explanation.

Proof. Let $\Gamma_{res} : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_Y(Y)}$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as an $\mathcal{O}_Y(Y)$-module via the map $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let $restriction : \mathrm{Mod}_{\mathcal{O}_X(X)} \to \mathrm{Mod}_{\mathcal{O}_Y(Y)}$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that *restriction* is exact so that

its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 13.16.9. It is clear that

$$\Gamma_{res} = restriction \circ \Gamma(X, -) = \Gamma(Y, -) \circ f_*$$

We claim that Derived Categories, Lemma 13.22.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 20.11.10 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y, -)$-acyclic sheaves on Y. □

01F0 **Remark 20.13.2.** Here is a down-to-earth explanation of the meaning of Lemma 20.13.1. It says that given $f : X \to Y$ and $\mathcal{F} \in Mod(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$$\begin{array}{ccc} R\Gamma(X, \mathcal{F}) & \text{is represented by} & \Gamma(X, \mathcal{I}^\bullet) \\ Rf_*\mathcal{F} & \text{is represented by} & f_*\mathcal{I}^\bullet \\ R\Gamma(Y, Rf_*\mathcal{F}) & \text{is represented by} & \Gamma(Y, f_*\mathcal{I}^\bullet) \end{array}$$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Lemma 20.11.10. Finally, it combines this with the trivial observation that

$$\Gamma(X, \mathcal{I}^\bullet) = \Gamma(Y, f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

01F1 **Lemma 20.13.3.** *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *The cohomology groups $H^i(U, \mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
(2) *Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^i f_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*

There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.

Proof. Consider the morphism of ringed spaces $(X, \mathcal{O}_X) \to (X, \underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seen as an $\underline{\mathbf{Z}}_X$-module, i.e., seen as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 20.13.2 we see that $\Gamma(X, \mathcal{I}^\bullet)$ computes both $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X, \mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 20.7.3. The result follows immediately. □

01F2 **Lemma 20.13.4** (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.22.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where Γ_{res} is as in the proof of Lemma 20.13.1. To see that the assumptions of Derived Categories, Lemma 13.22.2 are satisfied, see the proof of Lemma 20.13.1 or Remark 20.13.2. □

01F3 **Remark 20.13.5.** The Leray spectral sequence, the way we proved it in Lemma 20.13.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example f gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 20.13.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and f are identical with those for $\mathcal{F}$ and f' at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

01F4 **Lemma 20.13.6.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.*
(2) *If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to degenerate at E_2. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

01F5 **Lemma 20.13.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. In this case $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors from $D^+(X) \to D^+(Z)$.*

Proof. We are going to apply Derived Categories, Lemma 13.22.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 6.21.2. It remains to show that $f_*\mathcal{I}$ is g_*-acyclic. This follows from Lemma 20.11.10 and the description of the higher direct images $R^i g_*$ in Lemma 20.7.3. □

01F6 **Lemma 20.13.8** (Relative Leray spectral sequence)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_X$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 20.13.7 and Derived Categories, Lemma 13.22.2. □

20.14. Functoriality of cohomology

01F7

01F8 **Lemma 20.14.1.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}^\bullet$, resp. $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_Y$-modules, resp. $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ be a morphism of complexes. There is a canonical morphism*

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

in $D^+(Y)$. Moreover this construction is functorial in the triple $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

Proof. Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an f-map φ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 6.21.7. Such an f-map gives rise to a morphism of complexes

01F9 (20.14.1.1) $$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 20.14.1, and we apply $R\Gamma(Y, -)$. By Lemma 20.13.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 20.14.2 below. In particular it gives rise to maps on cohomology

01FA (20.14.1.2) $$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}).$$

01FB **Remark 20.14.2.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let φ be an f-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $f_*\mathcal{I}^\bullet \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 13.18.6 there exists a map of complexes β such that the diagram

01FC (20.14.2.1)
$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{I}^\bullet \\ \downarrow & & & & \downarrow \\ \mathcal{J}^\bullet & & \xrightarrow{\beta} & & (\mathcal{J}')^\bullet \end{array}$$

commutes. Applying global section functors we see that we get a diagram

$$\begin{array}{ccc} \Gamma(Y, f_*\mathcal{I}^\bullet) & = & \Gamma(X, \mathcal{I}^\bullet) \\ {\scriptstyle qis}\downarrow & & \\ \Gamma(Y, \mathcal{J}^\bullet) & \xrightarrow{\beta} & \Gamma(Y, (\mathcal{J}')^\bullet) \end{array}$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 20.13.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (20.14.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

20.15. Refinements and Čech cohomology

09UY Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : X = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Choose a map $c : I \to J$ such that $U_i \subset V_{c(i)}$ for all $i \in I$. This induces a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}), \quad (\xi_{j_0 \ldots j_p}) \longmapsto (\xi_{c(i_0) \ldots c(i_p)}|_{U_{i_0 \ldots i_p}})$$

functorial in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Suppose that $c' : I \to J$ is a second map such that $U_i \subset V_{c'(i)}$ for all $i \in I$. Then the corresponding maps γ and γ' are homotopic. Namely, $\gamma - \gamma' = \mathrm{d} \circ h + h \circ \mathrm{d}$ with $h : \check{\mathcal{C}}^{p+1}(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ given by the rule

$$h(\alpha)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{c(i_0) \ldots c(i_a) c'(i_a) \ldots c'(i_p)}$$

We omit the computation showing this works; please see the discussion following (20.25.0.2) for the proof in a more general case. In particular, the map on Čech

cohomology groups is independent of the choice of c. Moreover, it is clear that if $\mathcal{W} : X = \bigcup_{k\in K} W_k$ is a third open covering and $\mathcal{V}$ is a refinement of $\mathcal{W}$, then the composition of the maps

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

associated to maps $I \to J$ and $J \to K$ is the map associated to the composition $I \to K$. In particular, we can define the Čech cohomology groups

$$\check{H}^p(X, \mathcal{F}) = \operatorname{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

where the colimit is over all open coverings of X preordered by refinement.

It turns out that the maps γ defined above are compatible with the map to cohomology, in other words, the composition

$$\check{H}^p(\mathcal{V}, \mathcal{F}) \to \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{Lemma 20.11.2}} H^p(X, \mathcal{F})$$

is the canonical map from the first group to cohomology of Lemma 20.11.2. In the lemma below we will prove this in a slightly more general setting. A consequence is that we obtain a well defined map

09UZ (20.15.0.1) $$\check{H}^p(X, \mathcal{F}) = \operatorname{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

from Čech cohomology to cohomology.

01FD **Lemma 20.15.1.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\varphi : f^*\mathcal{G} \to \mathcal{F}$ be an f-map from an $\mathcal{O}_Y$-module $\mathcal{G}$ to an $\mathcal{O}_X$-module $\mathcal{F}$. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j\in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X = \bigcup_{j\in J} f^{-1}(V_j)$. In this case there exists a commutative diagram*

$$\begin{array}{ccc} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & R\Gamma(X, \mathcal{F}) \\ \gamma \uparrow & & \uparrow \\ \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) & \longrightarrow & R\Gamma(Y, \mathcal{G}) \end{array}$$

in $D^+(\mathcal{O}_X(X))$ with horizontal arrows given by Lemma 20.11.2 and right vertical arrow by (20.14.1.1). In particular we get commutative diagrams of cohomology groups

$$\begin{array}{ccc} \check{H}^p(\mathcal{U}, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) \\ \gamma \uparrow & & \uparrow \\ \check{H}^p(\mathcal{V}, \mathcal{G}) & \longrightarrow & H^p(Y, \mathcal{G}) \end{array}$$

where the right vertical arrow is (20.14.1.2)

Proof. We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree p we define the map by the rule

$$\gamma(s)_{i_0 \ldots i_p} = \varphi(s)_{c(i_0)\ldots c(i_p)}$$

This makes sense because φ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on X and $\mathcal{G} \to J^\bullet$ on Y. According to the proof of Lemma 20.11.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, \mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

As in Remark 20.14.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on Y and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (20.14.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')^{\bullet,\bullet}$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, (\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, f_*\mathcal{I}^q).$$

Note that there is an f-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes

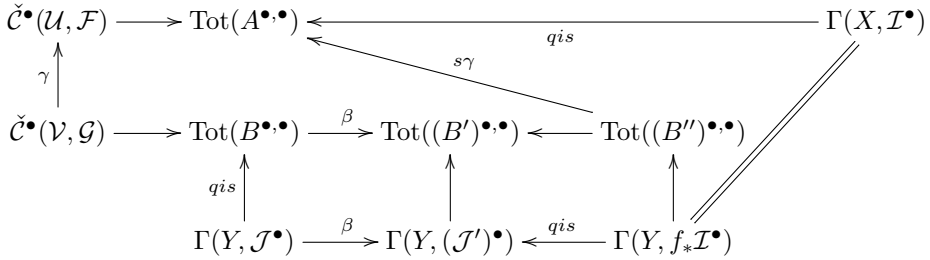

The two horizontal arrows with targets $\mathrm{Tot}(A^{\bullet,\bullet})$ and $\mathrm{Tot}(B^{\bullet,\bullet})$ are the ones explained in Lemma 20.11.2. The left upper shape (a pentagon) is commutative simply because (20.14.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$ to $\Gamma(X, \mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). □

20.16. Cohomology on Hausdorff quasi-compact spaces

09V0 For such a space Čech cohomology agrees with cohomology.

09V1 **Lemma 20.16.1.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. Then the map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ defined in (20.15.0.1) is an isomorphism.*

Proof. Let $\mathcal{U}$ be an open covering of X. By Lemma 20.11.5 there is an exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$$

Thus the map is injective. To show surjectivity it suffices to show that any element of $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ maps to zero after replacing $\mathcal{U}$ by a refinement. This is immediate from the definitions and the fact that $\underline{H}^1(\mathcal{F})$ is a presheaf of abelian groups whose sheafification is zero by locality of cohomology, see Lemma 20.7.2. □

09V2 **Lemma 20.16.2.** *Let X be a Hausdorff and quasi-compact topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Then the map $\check{H}^n(X, \mathcal{F}) \to H^n(X, \mathcal{F})$ defined in (20.15.0.1) is an isomorphism for all n.*

Proof. We already know that $\check{H}^n(X, -) \to H^n(X, -)$ is an isomorphism of functors for $n = 0, 1$, see Lemma 20.16.1. The functors $H^n(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 13.20.4. If we show that $\check{H}^n(X, -)$ forms

a universal δ-functor and that $\check{H}^n(X, -) \to H^n(X, -)$ is compatible with boundary maps, then the map will automatically be an isomorphism by uniqueness of universal δ-functors, see Homology, Lemma 12.12.5.

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of abelian sheaves on X. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. This gives a complex of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

which is in general not exact on the right. The sequence defines the maps

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{U}, \mathcal{G}) \to \check{H}^n(\mathcal{U}, \mathcal{H})$$

but isn't good enough to define a boundary operator $\delta : \check{H}^n(\mathcal{U}, \mathcal{H}) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F})$. Indeed such a thing will not exist in general. However, given an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{H})$ which is the cohomology class of a cocycle $h = (h_{i_0 \ldots i_n})$ we can choose open coverings

$$U_{i_0 \ldots i_n} = \bigcup W_{i_0 \ldots i_n, k}$$

such that $h_{i_0 \ldots i_n}|_{W_{i_0 \ldots i_n, k}}$ lifts to a section of $\mathcal{G}$ over $W_{i_0 \ldots i_n, k}$. By Topology, Lemma 5.13.5 (this is where we use the assumption that X is hausdorff and quasi-compact) we can choose an open covering $\mathcal{V} : X = \bigcup_{j \in J} V_j$ and $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$ (it is a refinement) and such that for all $j_0, \ldots, j_n \in J$ there is a k such that $V_{j_0 \ldots j_n} \subset W_{\alpha(j_0) \ldots \alpha(j_n), k}$. We obtain maps of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{H}) & \longrightarrow & 0
\end{array}$$

In fact, the vertical arrows are the maps of complexes used to define the transition maps between the Čech cohomology groups. Our choice of refinement shows that we may choose

$$g_{j_0 \ldots j_n} \in \mathcal{G}(V_{j_0 \ldots j_n}), \quad g_{j_0 \ldots j_n} \longmapsto h_{\alpha(j_0) \ldots \alpha(j_n)}|_{V_{j_0 \ldots j_n}}$$

The cochain $g = (g_{j_0 \ldots j_n})$ is not a cocycle in general but we know that its Čech boundary $\mathrm{d}(g)$ maps to zero in $\check{\mathcal{C}}^{n+1}(\mathcal{V}, \mathcal{H})$ (by the commutative diagram above and the fact that h is a cocycle). Hence $\mathrm{d}(g)$ is a cocycle in $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$. This allows us to define

$$\delta(\overline{h}) = \text{class of } \mathrm{d}(g) \text{ in } \check{H}^{n+1}(\mathcal{V}, \mathcal{F})$$

Now, given an element $\xi \in \check{H}^n(X, \mathcal{G})$ we choose an open covering $\mathcal{U}$ and an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{G})$ mapping to ξ in the colimit defining Čech cohomology. Then we choose $\mathcal{V}$ and g as above and set $\delta(\xi)$ equal to the image of $\delta(\overline{h})$ in $\check{H}^n(X, \mathcal{F})$. At this point a lot of properties have to be checked, all of which are straightforward. For example, we need to check that our construction is independent of the choice of $\mathcal{U}, \overline{h}, \mathcal{V}, \alpha : J \to I, g$. The class of $\mathrm{d}(g)$ is independent of the choice of the lifts $g_{i_0 \ldots i_n}$ because the difference will be a coboundary. Independence of α holds[1] because a different choice of α determines homotopic vertical maps of complexes

[1]This is an important check because the nonuniqueness of α is the only thing preventing us from taking the colimit of Čech complexes over all open coverings of X to get a short exact sequence of complexes computing Čech cohomology.

in the diagram above, see Section 20.15. For the other choices we use that given a finite collection of open coverings of X we can always find an open covering refining all of them. We also need to check additivity which is shown in the same manner. Finally, we need to check that the maps $\check{H}^n(X, -) \to H^n(X, -)$ are compatible with boundary maps. To do this we choose injective resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_1^\bullet & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_3^\bullet & \longrightarrow & 0 \end{array}$$

as in Derived Categories, Lemma 13.18.9. This will give a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_1^\bullet)) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_2^\bullet)) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_3^\bullet)) & \longrightarrow & 0 \end{array}$$

Here $\mathcal{U}$ is an open covering as above and the vertical maps are those used to define the maps $\check{H}^n(\mathcal{U}, -) \to H^n(X, -)$, see Lemma 20.11.2. The bottom complex is exact as the sequence of complexes of injectives is termwise split exact. Hence the boundary map in cohomology is computed by the usual procedure for this lower exact sequence, see Homology, Lemma 12.13.12. The same will be true after passing to the refinement $\mathcal{V}$ where the boundary map for Čech cohomology was defined. Hence the boundary maps agree because they use the same construction (whenever the first one is defined on an element in Čech cohomology on a given open covering). This finishes our discussion of the construction of the structure of a δ-functor on Čech cohomology and why this structure is compatible with the given δ-functor structure on usual cohomology.

Finally, we may apply Lemma 20.11.1 to see that higher Čech cohomology is trivial on injective sheaves. Hence we see that Čech cohomology is a universal δ-functor by Homology, Lemma 12.12.4. □

09V3 **Lemma 20.16.3.** *Let X be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of Z have disjoint open neighbourhoods in X. For every abelian sheaf $\mathcal{F}$ on X the canonical map*

[**AGV71**, Expose V bis, 4.1.3]

$$\operatorname{colim} H^p(U, \mathcal{F}) \longrightarrow H^p(Z, \mathcal{F}|_Z)$$

where the colimit is over open neighbourhoods U of Z in X is an isomorphism.

Proof. We first prove this for $p = 0$. Injectivity follows from the definition of $\mathcal{F}|_Z$ and holds in general (for any subset of any topological space X). Next, suppose that $s \in H^0(Z, \mathcal{F}|_Z)$. Then we can find opens $U_i \subset X$ such that $Z \subset \bigcup U_i$ and such that $s|_{Z \cap U_i}$ comes from $s_i \in \mathcal{F}(U_i)$. It follows that there exist opens $W_{ij} \subset U_i \cap U_j$ with $W_{ij} \cap Z = U_i \cap U_j \cap Z$ such that $s_i|_{W_{ij}} = s_j|_{W_{ij}}$. Applying Topology, Lemma 5.13.7 we find opens V_i of X such that $V_i \subset U_i$ and such that $V_i \cap V_j \subset W_{ij}$. Hence we see that $s_i|_{V_i}$ glue to a section of $\mathcal{F}$ over the open neighbourhood $\bigcup V_i$ of Z.

To finish the proof, it suffices to show that if $\mathcal{I}$ is an injective abelian sheaf on X, then $H^p(Z, \mathcal{I}|_Z) = 0$ for $p > 0$. This follows using short exact sequences and dimension shifting; details omitted. Thus, suppose $\overline{\xi}$ is an element of $H^p(Z, \mathcal{I}|_Z)$

for some $p > 0$. By Lemma 20.16.2 the element $\overline{\xi}$ comes from $\check{H}^p(\mathcal{V}, \mathcal{I}|_Z)$ for some open covering $\mathcal{V} : Z = \bigcup V_i$ of Z. Say $\overline{\xi}$ is the image of the class of a cocycle $\xi = (\xi_{i_0 \ldots i_p})$ in $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{I}|_Z)$.

Let $\mathcal{I}' \subset \mathcal{I}|_Z$ be the subpresheaf defined by the rule

$$\mathcal{I}'(V) = \{s \in \mathcal{I}|_Z(V) \mid \exists (U, t),\ U \subset X \text{ open},\ t \in \mathcal{I}(U),\ V = Z \cap U,\ s = t|_{Z \cap U}\}$$

Then $\mathcal{I}|_Z$ is the sheafification of $\mathcal{I}'$. Thus for every $(p+1)$-tuple $i_0 \ldots i_p$ we can find an open covering $V_{i_0 \ldots i_p} = \bigcup W_{i_0 \ldots i_p, k}$ such that $\xi_{i_0 \ldots i_p}|_{W_{i_0 \ldots i_p, k}}$ is a section of $\mathcal{I}'$. Applying Topology, Lemma 5.13.5 we may after refining $\mathcal{V}$ assume that each $\xi_{i_0 \ldots i_p}$ is a section of the presheaf $\mathcal{I}'$.

Write $V_i = Z \cap U_i$ for some opens $U_i \subset X$. Since $\mathcal{I}$ is flasque (Lemma 20.12.2) and since $\xi_{i_0 \ldots i_p}$ is a section of $\mathcal{I}'$ for every $(p+1)$-tuple $i_0 \ldots i_p$ we can choose a section $s_{i_0 \ldots i_p} \in \mathcal{I}(U_{i_0 \ldots i_p})$ which restricts to $\xi_{i_0 \ldots i_p}$ on $V_{i_0 \ldots i_p} = Z \cap U_{i_0 \ldots i_p}$. (This appeal to injectives being flasque can be avoided by an additional application of Topology, Lemma 5.13.7.) Let $s = (s_{i_0 \ldots i_p})$ be the corresponding cochain for the open covering $U = \bigcup U_i$. Since $\mathrm{d}(\xi) = 0$ we see that the sections $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}$ restrict to zero on $Z \cap U_{i_0 \ldots i_{p+1}}$. Hence, by the initial remarks of the proof, there exists open subsets $W_{i_0 \ldots i_{p+1}} \subset U_{i_0 \ldots i_{p+1}}$ with $Z \cap W_{i_0 \ldots i_{p+1}} = Z \cap U_{i_0 \ldots i_{p+1}}$ such that $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}|_{W_{i_0 \ldots i_{p+1}}} = 0$. By Topology, Lemma 5.13.7 we can find $U_i' \subset U_i$ such that $Z \subset \bigcup U_i'$ and such that $U'_{i_0 \ldots i_{p+1}} \subset W_{i_0 \ldots i_{p+1}}$. Then $s' = (s'_{i_0 \ldots i_p})$ with $s'_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}|_{U'_{i_0 \ldots i_p}}$ is a cocycle for $\mathcal{I}$ for the open covering $U' = \bigcup U_i'$ of an open neighbourhood of Z. Since $\mathcal{I}$ has trivial higher Čech cohomology groups (Lemma 20.11.1) we conclude that s' is a coboundary. It follows that the image of ξ in the Čech complex for the open covering $Z = \bigcup Z \cap U_i'$ is a coboundary and we are done. □

20.17. The base change map

02N6 We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces. By Modules, Lemma 17.20.2 the functor $g^* : \mathit{Mod}(\mathcal{O}_Y) \to \mathit{Mod}(\mathcal{O}_X)$ is exact. Hence it has a derived functor

$$g^* : D^+(Y) \to D^+(X)$$

which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 13.16.9. Hence as indicated we indicate this functor by g^* rather than Lg^*.

02N7 **Lemma 20.17.1.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base

change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(S')$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 20.11.11 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.18.6 and 13.18.7 the arrow β in the diagram

$$\begin{array}{ccc} (g')_*(g')^*\mathcal{F}^\bullet & \longrightarrow & (g')_*\mathcal{J}^\bullet \\ {\scriptstyle adjunction}\uparrow & & \uparrow{\scriptstyle \beta} \\ \mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

exists and is unique up to homotopy. Pushing down to S we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. □

02N8 **Remark 20.17.2.** The "correct" version of the base change map is the map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet.$$

The construction of this map involves unbounded complexes, see Remark 20.28.3.

20.18. Proper base change in topology

09V4 In this section we prove a very general version of the proper base change theorem in topology. It tells us that the stalks of the higher direct images R^pf_* can be computed on the fibre.

09V5 **Lemma 20.18.1.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $y \in Y$. Assume that*

(1) *f is closed,*
(2) *f is separated, and*
(3) *$f^{-1}(y)$ is quasi-compact.*

Then for E in $D^+(\mathcal{O}_X)$ we have $(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ in $D^+(\mathcal{O}_{Y,y})$.

Proof. The base change map of Lemma 20.17.1 gives a canonical map $(Rf_*E)_y \to R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$. To prove this map is an isomorphism, we represent E by a bounded below complex of injectives $\mathcal{I}^\bullet$. Set $Z = f^{-1}(\{y\})$. The assumptions of Lemma 20.16.3 are satisfied, see Topology, Lemma 5.4.2. Hence the restrictions $\mathcal{I}^n|_Z$ are acyclic for $\Gamma(Z, -)$. Thus $R\Gamma(Z, E|_Z)$ is represented by the complex $\Gamma(Z, \mathcal{I}^\bullet|_Z)$, see Derived Categories, Lemma 13.16.7. In other words, we have to show the map

$$\operatorname{colim}_V \mathcal{I}^\bullet(f^{-1}(V)) \longrightarrow \Gamma(Z, \mathcal{I}^\bullet|_Z)$$

is an isomorphism. Using Lemma 20.16.3 we see that it suffices to show that the collection of open neighbourhoods $f^{-1}(V)$ of $Z = f^{-1}(\{y\})$ is cofinal in the system of all open neighbourhoods. If $f^{-1}(\{y\}) \subset U$ is an open neighbourhood, then as f is closed the set $V = Y \setminus f(X \setminus U)$ is an open neighbourhood of y with $f^{-1}(V) \subset U$. This proves the lemma. □

09V6 **Theorem 20.18.2** (Proper base change)**.** *Consider a cartesian square of topological spaces* [**AGV71**, Expose V bis, 4.1.1]

$$\begin{array}{ccc} X' = Y' \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Assume that f is proper. Let E be an object of $D^+(X)$. Then the base change map

$$g^{-1}Rf_*E \longrightarrow Rf'_*(g')^{-1}E$$

of Lemma 20.17.1 is an isomorphism in $D^+(Y')$.

Proof. Let $y' \in Y'$ be a point with image $y \in Y$. It suffices to show that the base change map induces an isomorphism on stalks at y'. As f is proper it follows that f' is proper, the fibres of f and f' are quasi-compact and f and f' are closed, see Topology, Theorem 5.17.5 and Lemma 5.4.4. Thus we can apply Lemma 20.18.1 twice to see that

$$(Rf'_*(g')^{-1}E)_{y'} = R\Gamma((f')^{-1}(y'), (g')^{-1}E|_{(f')^{-1}(y')})$$

and

$$(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$$

The induced map of fibres $(f')^{-1}(y') \to f^{-1}(y)$ is a homeomorphism of topological spaces and the pull back of $E|_{f^{-1}(y)}$ is $(g')^{-1}E|_{(f')^{-1}(y')}$. The desired result follows. □

0D90 **Lemma 20.18.3** (Proper base change for sheaves of sets)**.** *Consider a cartesian square of topological spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Assume that f is proper. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$ for any sheaf of sets $\mathcal{F}$ on X.

Proof. We argue exactly as in the proof of Theorem 20.18.2 and we find it suffices to show $(f_*\mathcal{F})_y = \Gamma(X_y, \mathcal{F}|_{X_y})$. Then we argue as in Lemma 20.18.1 to reduce this to the $p = 0$ case of Lemma 20.16.3 for sheaves of sets. The first part of the proof of Lemma 20.16.3 works for sheaves of sets and this finishes the proof. Some details omitted. □

20.19. Cohomology and colimits

01FE Let X be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of $\mathcal{O}_X$-modules over the directed set I, see Categories, Section 4.21. Since for each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ we get a canonical map

$$\operatorname{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 6.29.1. See also Modules, Lemma 17.22.8 (in the special case $\mathcal{G} = \mathcal{O}_X$).

01FF **Lemma 20.19.1.** *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

(1) *there exists a basis of quasi-compact open subsets, and*
(2) *the intersection of any two quasi-compact opens is quasi-compact.*

Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map

$$\operatorname{colim}_i H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $q \geq 0$.

Proof. It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $q = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 6.29.1 (combined with Topology, Lemma 5.27.1). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set I is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over I is directed. By Injectives, Lemma 19.5.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 6.28 and 6.29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 10.8.8) we deduce the result. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{array}{ccccccc}
\operatorname{colim}_i H^{q_0}(U, \mathcal{I}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0}(U, \mathcal{Q}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q_0}(U, \operatorname{colim}_i \mathcal{I}_i) & \longrightarrow & H^{q_0}(U, \operatorname{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{q_0+1}(U, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & 0
\end{array}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 10.8.8. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 20.11.9. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of X. This is a basis for the topology on X by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\ldots,m} U_j$ with each of U, U_j quasi-compact open in X. By the result for $q = 0$ we see that for $\mathcal{U} \in \mathrm{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = \operatorname{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0 \ldots j_p}$ are quasi-compact. By Lemma 20.11.1 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1.

Hence by Algebra, Lemma 10.8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 20.11.9 are satisfied and the claim follows. □

0H7A **Lemma 20.19.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of abelian sheaves on X. Set $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$. Let $p \geq 0$ be an integer. Assume the set of opens $V \subset Y$ such that $H^p(f^{-1}(V), \mathcal{F}) = \operatorname{colim} H^p(f^{-1}(V), \mathcal{F}_i)$ is a basis for the topology on Y. Then $R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i$.*

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheafification of the presheaf $\mathcal{G}$ sending V to $H^p(f^{-1}(V), \mathcal{F})$, see Lemma 20.7.3. Similarly, $R^p f_* \mathcal{F}_i$ is the sheafification of the presheaf $\mathcal{G}_i$ sending V to $H^p(f^{-1}(V), \mathcal{F}_i)$. Recall that sheafification is the left adjoint to the inclusion from sheaves to presheaves, see Sheaves, Section 6.17. Hence sheafification commutes with colimits, see Categories, Lemma 4.24.5. Hence it suffices to show that the map of presheaves (with colimit in the category of presheaves)

$$\operatorname{colim} \mathcal{G}_i \longrightarrow \mathcal{G}$$

induces an isomorphism on sheafifications. For this it suffices to show that the presheaves $\mathcal{G}$ and $\operatorname{colim} \mathcal{G}_i$ agree on a basis for the topology of Y. Namely, in this case the stalks of their sheafifications, which can be computed directly from the presheaf values on elements of the basis, agree. The required agreement is exactly the assumption of the lemma. □

Next we formulate the analogy of Sheaves, Lemma 6.29.4 for cohomology. Let X be a spectral space which is written as a cofiltered limit of spectral spaces X_i for a diagram with spectral transition morphisms as in Topology, Lemma 5.24.5. Assume given

(1) an abelian sheaf $\mathcal{F}_i$ on X_i for all $i \in \operatorname{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves (see Sheaves, Definition 6.21.7)

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \operatorname{colim} p_i^{-1} \mathcal{F}_i$ on X.

0A37 **Lemma 20.19.3.** *In the situation discussed above. Let $i \in \operatorname{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be quasi-compact open. Then*

$$\operatorname{colim}_{a : j \to i} H^p(f_a^{-1}(U_i), \mathcal{F}_j) = H^p(p_i^{-1}(U_i), \mathcal{F})$$

for all $p \geq 0$. In particular we have $H^p(X, \mathcal{F}) = \operatorname{colim} H^p(X_i, \mathcal{F}_i)$.

Proof. The case $p = 0$ is Sheaves, Lemma 6.29.4.

In this paragraph we show that we can find a map of systems $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ with $\mathcal{G}_i$ an injective abelian sheaf and γ_i injective. For each i we pick an injection $\mathcal{F}_i \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on X_i. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b : k \to i} f_{b,*} \mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1} \mathcal{F}_i \to \mathcal{F}_k \to \mathcal{I}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1} \mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1} f_{a \circ b,*} \mathcal{I}_k \to f_{b,*} \mathcal{I}_k$ for $b : k \to j$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves.

Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on X_i, see Lemma 20.11.11 and Homology, Lemma 12.27.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 20.19.1 we see that it suffices to prove that $H^p(X, \operatorname{colim} f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every quasi-compact open of X, it suffices to show that the Čech cohomology of $\mathcal{G}$ for any open covering $\mathcal{U}$ of a quasi-compact open of X by finitely many quasi-compact opens is zero, see Lemma 20.11.9. Such a covering is the inverse by p_i of such a covering $\mathcal{U}_i$ on the space X_i for some i by Topology, Lemma 5.24.6. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{a:j\to i} \check{\mathcal{C}}^\bullet(f_a^{-1}(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is a filtered colimit of complexes each of which is acyclic in positive degrees by Lemma 20.11.1. Thus we conclude by Algebra, Lemma 10.8.8. □

20.20. Vanishing on Noetherian topological spaces

02UU The aim is to prove a theorem of Grothendieck namely Proposition 20.20.7. See [**Gro57**].

02UV **Lemma 20.20.1.** *Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on Z we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

Proof. This is true because i_* is exact (see Modules, Lemma 17.6.1), and hence $R^p i_* = 0$ as a functor (Derived Categories, Lemma 13.16.9). Thus we may apply Lemma 20.13.6. □

02UW **Lemma 20.20.2.** *Let X be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group A.*

Proof. Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 6.7.4. Since X is irreducible, any nonempty open U is irreducible and a fortiori connected. Hence for $U \subset X$ nonempty open we have $\underline{A}(U) = A$. We have $\underline{A}(\emptyset) = 0$. Thus $\underline{A}$ is a flasque abelian sheaf on X. The vanishing follows from Lemma 20.12.3. □

0A38 **Lemma 20.20.3.** *Let X be a topological space such that the intersection of any two quasi-compact opens is quasi-compact. Let $\mathcal{F} \subset \underline{\mathbf{Z}}$ be a subsheaf generated by finitely many sections over quasi-compact opens. Then there exists a finite filtration* [**Gro57**, Page 168].

$$(0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

by abelian subsheaves such that for each $0 < i \leq n$ there exists a short exact sequence

$$0 \to j'_!\underline{\mathbf{Z}}_V \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$$

with $j : U \to X$ and $j' : V \to X$ the inclusion of quasi-compact opens into X.

Proof. Say $\mathcal{F}$ is generated by the sections $s_1, \ldots, s_t$ over the quasi-compact opens $U_1, \ldots, U_t$. Since U_i is quasi-compact and s_i a locally constant function to $\mathbf{Z}$ we may assume, after possibly replacing U_i by the parts of a finite decomposition into open and closed subsets, that s_i is a constant section. Say $s_i = n_i$ with $n_i \in \mathbf{Z}$. Of course we can remove (U_i, n_i) from the list if $n_i = 0$. Flipping signs if necessary we may also assume $n_i > 0$. Next, for any subset $I \subset \{1, \ldots, t\}$ we may add $\bigcap_{i\in I} U_i$ and $\gcd(n_i, i \in I)$ to the list. After doing this we see that our list $(U_1, n_1), \ldots, (U_t, n_t)$

satisfies the following property: For $x \in X$ set $I_x = \{i \in \{1, \ldots, t\} \mid x \in U_i\}$. Then $\gcd(n_i, i \in I_x)$ is attained by n_i for some $i \in I_x$.

As our filtration we take $\mathcal{F}_0 = (0)$ and $\mathcal{F}_n$ generated by the sections n_i over U_i for those i such that $n_i \leq n$. It is clear that $\mathcal{F}_n = \mathcal{F}$ for $n \gg 0$. Moreover, the quotient $\mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $U = \bigcup_{n_i \leq n} U_i$ and the kernel of the map $j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $V = \bigcup_{n_i \leq n-1} U_i$. Thus a short exact sequence as in the statement of the lemma. □

02UX **Lemma 20.20.4.** *Let X be a topological space. Let $d \geq 0$ be an integer. Assume*

(1) *X is quasi-compact,*
(2) *the quasi-compact opens form a basis for X, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*
(4) *$H^p(X, j_!\underline{\mathbf{Z}}_U) = 0$ for all $p > d$ and any quasi-compact open $j : U \to X$.*

Then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X.

This is a special case of [**Gro57**, Proposition 3.6.1].

Proof. Let $S = \coprod_{U \subset X} \mathcal{F}(U)$ where U runs over the quasi-compact opens of X. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all s_i (see Modules, Definition 17.4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the directed partially ordered set of finite subsets of S. By Modules, Lemma 17.4.6 it is clear that

$$\operatorname{colim}_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 20.19.1 we have

$$H^p(X, \mathcal{F}) = \operatorname{colim}_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections over quasi-compact opens of X.

Suppose that $\mathcal{F}$ is generated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than n local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ where U is a quasi-compact open of X.

Assume now that we have a short exact sequence

$$0 \to \mathcal{K} \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F} \to 0$$

with U quasi-compact open in X. It suffices to show that $H^q(X, \mathcal{K})$ is zero for $q \geq d+1$. As above we can write $\mathcal{K}$ as the filtered colimit of subsheaves $\mathcal{K}'$ generated by finitely many sections over quasi-compact opens. Then $\mathcal{F}$ is the filtered colimit of the sheaves $j_!\underline{\mathbf{Z}}_U/\mathcal{K}'$. In this way we reduce to the case that $\mathcal{K}$ is generated by finitely many sections over quasi-compact opens. Note that $\mathcal{K}$ is a subsheaf of $\underline{\mathbf{Z}}_X$. Thus by Lemma 20.20.3 there exists a finite filtration of $\mathcal{K}$ whose successive quotients $\mathcal{Q}$ fit into a short exact sequence

$$0 \to j''_!\underline{\mathbf{Z}}_W \to j'_!\underline{\mathbf{Z}}_V \to \mathcal{Q} \to 0$$

with $j'' : W \to X$ and $j' : V \to X$ the inclusions of quasi-compact opens. Hence the vanishing of $H^p(X, \mathcal{Q})$ for $p > d$ follows from our assumption (in the lemma) on the vanishing of the cohomology groups of $j''_!\underline{\mathbf{Z}}_W$ and $j'_!\underline{\mathbf{Z}}_V$. Returning to $\mathcal{K}$ this, via an induction argument using the long exact cohomology sequence, implies the desired vanishing for it as well. □

0BX0 **Example 20.20.5.** Let $X = \mathbf{N}$ endowed with the topology whose opens are $\emptyset$, X, and $U_n = \{i \mid i \leq n\}$ for $n \geq 1$. An abelian sheaf $\mathcal{F}$ on X is the same as an inverse system of abelian groups $A_n = \mathcal{F}(U_n)$ and $\Gamma(X, \mathcal{F}) = \lim A_n$. Since the inverse limit functor is not an exact functor on the category of inverse systems, we see that there is an abelian sheaf with nonzero H^1. Finally, the reader can check that $H^p(X, j_!\mathbf{Z}_U) = 0$, $p \geq 1$ if $j : U = U_n \to X$ is the inclusion. Thus we see that X is an example of a space satisfying conditions (2), (3), and (4) of Lemma 20.20.4 for $d = 0$ but not the conclusion.

02UY **Lemma 20.20.6.** *Let X be an irreducible topological space. Let $\mathcal{H} \subset \underline{\mathbf{Z}}$ be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open $U \subset X$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$ for some $d \in \mathbf{Z}$.*

Proof. Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open V of X (see proof of Lemma 20.20.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}}_U \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}}_{U'} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. □

02UZ **Proposition 20.20.7** (Grothendieck)**.** *Let X be a Noetherian topological space. If $\dim(X) \leq d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X.* [**Gro57**, Theorem 3.6.5].

Proof. We prove this lemma by induction on d. So fix d and assume the lemma holds for all Noetherian topological spaces of dimension $< d$.

Let $\mathcal{F}$ be an abelian sheaf on X. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

see Modules, Lemma 17.7.1. Note that $j_!j^*\mathcal{F}$ is supported on the topological closure Z' of U, i.e., it is of the form $i'_*\mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on Z', where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where X is irreducible. Namely, according to Topology, Lemma 5.9.2 X has finitely many irreducible components. If X has more than one irreducible component, then let $Z \subset X$ be an irreducible component of X and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_*i^*\mathcal{F})$ and $H^p(X, i'_*\mathcal{F}')$ for $p > d$. By Lemma 20.20.1 it suffices to prove $H^p(Z, i^*\mathcal{F})$ and $H^p(Z', \mathcal{F}')$ vanish for $p > d$. Since Z' and Z have fewer irreducible components we indeed reduce to the case of an irreducible X.

If $d = 0$ and X is irreducible, then X is the only nonempty open subset of X. Hence every sheaf is constant and higher cohomology groups vanish (for example by Lemma 20.20.2).

Suppose X is irreducible of dimension $d > 0$. By Lemma 20.20.4 we reduce to the case where $\mathcal{F} = j_!\underline{\mathbf{Z}}_U$ for some open $U \subset X$. In this case we look at the short exact sequence

$$0 \to j_!(\underline{\mathbf{Z}}_U) \to \underline{\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}}_Z \to 0$$

where $Z = X \setminus U$. By Lemma 20.20.2 we have the vanishing of $H^p(X, \underline{\mathbf{Z}}_X)$ for all $p \geq 1$. By induction we have $H^p(X, i_*\underline{\mathbf{Z}}_Z) = H^p(Z, \underline{\mathbf{Z}}_Z) = 0$ for $p \geq d$. Hence we win by the long exact cohomology sequence. □

20.21. Cohomology with support in a closed subset

0A39 This section just discusses the bare minimum – the discussion will be continued in Section 20.34.

Let X be a topological space and let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the subset of sections whose support is contained in Z. The support of a section is defined in Modules, Definition 17.5.1. Modules, Lemma 17.5.2 implies that $\Gamma_Z(X, \mathcal{F})$ is a subgroup of $\Gamma(X, \mathcal{F})$. The same lemma guarantees that the assignment $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is a functor in $\mathcal{F}$. This functor is left exact but not exact in general.

Since the category of abelian sheaves has enough injectives (Injectives, Lemma 19.4.1) we we obtain a right derived functor

$$R\Gamma_Z(X, -) : D^+(X) \longrightarrow D^+(Ab)$$

by Derived Categories, Lemma 13.20.2. The value of $R\Gamma_Z(X, -)$ on an object K is computed by representing K by a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves and taking $\Gamma_Z(X, \mathcal{I}^\bullet)$, see Derived Categories, Lemma 13.20.1. The cohomology groups of an abelian sheaf $\mathcal{F}$ with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on X. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Lemma 20.8.1) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D^+(X)$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D^+(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D^+(X)$.

For an abelian sheaf $\mathcal{F}$ on X we can consider the *subsheaf of sections with support in* Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \cap Z\} = \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$$

Using the equivalence of Modules, Lemma 17.6.1 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z, see Modules, Remark 17.6.2. Thus we obtain a functor

$$Ab(X) \longrightarrow Ab(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as a sheaf on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D^+(X) \longrightarrow D^+(Z)$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$.

Observe that we have $\Gamma_Z(X, \mathcal{F}) = \Gamma(Z, \mathcal{H}_Z(\mathcal{F}))$ for any abelian sheaf $\mathcal{F}$. By Lemma 20.21.1 below the functor $\mathcal{H}_Z$ transforms injective abelian sheaves into sheaves right acyclic for $\Gamma(Z, -)$. Thus by Derived Categories, Lemma 13.22.2 we obtain a convergent Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(K)) \Rightarrow H^{p+q}_Z(X, K)$$

functorial in K in $D^+(X)$.

0A3A **Lemma 20.21.1.** *Let $i : Z \to X$ be the inclusion of a closed subset. Let $\mathcal{I}$ be an injective abelian sheaf on X. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on Z.*

Proof. This follows from Homology, Lemma 12.29.1 as $\mathcal{H}_Z(-)$ is right adjoint to the exact functor i_*. See Modules, Lemmas 17.6.1 and 17.6.3. □

20.22. Cohomology on spectral spaces

0A3C A key result on the cohomology of spectral spaces is Lemma 20.19.3 which loosely speaking says that cohomology commutes with cofiltered limits in the category of spectral spaces as defined in Topology, Definition 5.23.1. This can be applied to give analogues of Lemmas 20.16.3 and 20.18.1 as follows.

0A3D **Lemma 20.22.1.** *Let X be a spectral space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $E \subset X$ be a quasi-compact subset. Let $W \subset X$ be the set of points of X which specialize to a point of E.*

1. *$H^p(W, \mathcal{F}|_W) = \operatorname{colim} H^p(U, \mathcal{F})$ where the colimit is over quasi-compact open neighbourhoods of E,*
2. *$H^p(W \backslash E, \mathcal{F}|_{W\backslash E}) = \operatorname{colim} H^p(U \backslash E, \mathcal{F}|_{U\backslash E})$ if E is a constructible subset.*

Proof. From Topology, Lemma 5.24.7 we see that $W = \lim U$ where the limit is over the quasi-compact opens containing E. Each U is a spectral space by Topology, Lemma 5.23.5. Thus we may apply Lemma 20.19.3 to conclude that (1) holds. The same proof works for part (2) except we use Topology, Lemma 5.24.8. □

0A3E **Lemma 20.22.2.** *Let $f : X \to Y$ be a spectral map of spectral spaces. Let $y \in Y$. Let $E \subset Y$ be the set of points specializing to y. Let $\mathcal{F}$ be an abelian sheaf on X. Then $(R^pf_*\mathcal{F})_y = H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)})$.*

Proof. Observe that $E = \bigcap V$ where V runs over the quasi-compact open neighbourhoods of y in Y. Hence $f^{-1}(E) = \bigcap f^{-1}(V)$. This implies that $f^{-1}(E) = \lim f^{-1}(V)$ as topological spaces. Since f is spectral, each $f^{-1}(V)$ is a spectral space too (Topology, Lemma 5.23.5). We conclude that $f^{-1}(E)$ is a spectral space and that

$$H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)}) = \operatorname{colim} H^p(f^{-1}(V), \mathcal{F})$$

by Lemma 20.19.3. On the other hand, the stalk of $R^pf_*\mathcal{F}$ at y is given by the colimit on the right. □

0A3F **Lemma 20.22.3.** *Let X be a profinite topological space. Then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$ and all abelian sheaves $\mathcal{F}$.*

Proof. Any open covering of X can be refined by a finite disjoint union decomposition with open parts, see Topology, Lemma 5.22.4. Hence if $\mathcal{F} \to \mathcal{G}$ is a surjection of abelian sheaves on X, then $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. In other words, the global sections functor is an exact functor. Therefore its higher derived functors are zero, see Derived Categories, Lemma 13.16.9. □

The following result on cohomological vanishing improves Grothendieck's result (Proposition 20.20.7) and can be found in [**Sch92**].

0A3G **Proposition 20.22.4.** *Let X be a spectral space of Krull dimension d. Let $\mathcal{F}$ be an abelian sheaf on X.* Part (1) is the main theorem of [**Sch92**].

(1) *$H^q(X, \mathcal{F}) = 0$ for $q > d$,*
(2) *$H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for every quasi-compact open $U \subset X$,*
(3) *$H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ and any constructible closed subset $Z \subset X$.*

Proof. We prove this result by induction on d.

If $d = 0$, then X is a profinite space, see Topology, Lemma 5.23.8. Thus (1) holds by Lemma 20.22.3. If $U \subset X$ is quasi-compact open, then U is also closed as a quasi-compact subset of a Hausdorff space. Hence $X = U \amalg (X \setminus U)$ as a topological space and we see that (2) holds. Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are profinite (namely U is quasi-compact because Z is constructible) and since we have (2) and (1) we obtain the desired vanishing of the cohomology groups with support in Z.

Induction step. Assume $d \geq 1$ and assume the proposition is valid for all spectral spaces of dimension $< d$. We first prove part (2) for X. Let U be a quasi-compact open. Let $\xi \in H^d(U, \mathcal{F})$. Set $Z = X \setminus U$. Let $W \subset X$ be the set of points specializing to Z. By Lemma 20.22.1 we have

$$H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z}) = \operatorname{colim}_{Z \subset V} H^d(V \setminus Z, \mathcal{F})$$

where the colimit is over the quasi-compact open neighbourhoods V of Z in X. By Topology, Lemma 5.24.7 we see that $W \setminus Z$ is a spectral space. Since every point of W specializes to a point of Z, we see that $W \setminus Z$ is a spectral space of Krull dimension $< d$. By induction hypothesis we see that the image of ξ in $H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z})$ is zero. By the displayed formula, there exists a $Z \subset V \subset X$ quasi-compact open such that $\xi|_{V \setminus Z} = 0$. Since $V \setminus Z = V \cap U$ we conclude by the Mayer-Vietoris (Lemma 20.8.2) for the covering $X = U \cup V$ that there exists a $\tilde{\xi} \in H^d(X, \mathcal{F})$ which restricts to ξ on U and to zero on V. In other words, part (2) is true.

Proof of part (1) assuming (2). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Set

$$\mathcal{G} = \operatorname{Im}(\mathcal{I}^{d-1} \to \mathcal{I}^d) = \operatorname{Ker}(\mathcal{I}^d \to \mathcal{I}^{d+1})$$

For $U \subset X$ quasi-compact open we have a map of exact sequences as follows

$$\begin{array}{ccccccc}
\mathcal{I}^{d-1}(X) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & H^d(X, \mathcal{F}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}^{d-1}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & H^d(U, \mathcal{F}) & \longrightarrow & 0
\end{array}$$

The sheaf $\mathcal{I}^{d-1}$ is flasque by Lemma 20.12.2 and the fact that $d \geq 1$. By part (2) we see that the right vertical arrow is surjective. We conclude by a diagram chase that the map $\mathcal{G}(X) \to \mathcal{G}(U)$ is surjective. By Lemma 20.12.6 we conclude that $\check{H}^q(\mathcal{U}, \mathcal{G}) = 0$ for $q > 0$ and any finite covering $\mathcal{U} : U = U_1 \cup \ldots \cup U_n$ of a quasi-compact open by quasi-compact opens. Applying Lemma 20.11.9 we find that $H^q(U, \mathcal{G}) = 0$ for all $q > 0$ and all quasi-compact opens U of X. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we conclude that

$$H^q(X, \mathcal{F}) = H^q\left(\Gamma(X, \mathcal{I}^0) \to \ldots \to \Gamma(X, \mathcal{I}^{d-1}) \to \Gamma(X, \mathcal{G})\right)$$

In particular the cohomology group vanishes if $q > d$.

Proof of (3). Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are spectral spaces (Topology, Lemma 5.23.5) of dimension $\leq d$ and since we have (2) and (1) we obtain the desired vanishing. □

20.23. The alternating Čech complex

01FG This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{array}{c} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0 \ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0 \ldots i_n \ldots i_m \ldots i_p} = -s_{i_0 \ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{array} \right\}$$

We omit the verification that the differential d of Equation (20.9.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

01FH **Definition 20.23.1.** Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the *alternating Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on I. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0 \ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (20.9.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

01FI **Definition 20.23.2.** Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume given a total ordering on I. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the *ordered Čech complex* associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \text{sgn}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{cases}$$

Here σ denotes a permutation of $\{0, \ldots, p\}$ and $\text{sgn}(\sigma)$ denotes its sign. The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

01FJ **Lemma 20.23.3.** *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

of complexes.

Proof. Omitted. □

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

01FK **Lemma 20.23.4.** *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism*

$$\pi : \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

of complexes which is a left inverse to the morphism c.

Proof. Omitted. □

01FL **Remark 20.23.5.** This means that if we have two total orderings $<_1$ and $<_2$ on the index set I, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

01FM **Lemma 20.23.6.** *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.*

Proof. For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0 \ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any a, $0 \leq a \leq p$ we denote σ_a the unique permutation of $\{0, \ldots, p\}$ such that $\sigma_a(j) = \sigma(j)$ for $0 \leq j < a$ and such that $\sigma_a(a) < \sigma_a(a+1) < \ldots < \sigma_a(p)$. So if $p = 3$ and σ, τ are given by

$$\begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \sigma & 3 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \tau & 3 & 0 & 2 & 1 \end{array}$$

then we have

$$\begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \sigma_0 & 0 & 1 & 2 & 3 \\ \sigma_1 & 3 & 0 & 1 & 2 \\ \sigma_2 & 3 & 2 & 0 & 1 \\ \sigma_3 & 3 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \tau_0 & 0 & 1 & 2 & 3 \\ \tau_1 & 3 & 0 & 1 & 2 \\ \tau_2 & 3 & 0 & 1 & 2 \\ \tau_3 & 3 & 0 & 2 & 1 \end{array}$$

It is clear that always $\sigma_0 = \text{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

01FN (20.23.6.1) $$h(s)_{i_0 \ldots i_p} = \sum\nolimits_{0 \leq a \leq p} (-1)^a \text{sign}(\sigma_a) s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a+1$ indices and once in the second group of $p-a+1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of σ_a. Hence the sum makes sense since each of the elements $s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$ is defined over the open $U_{i_0 \ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0 \ldots i_p}$ and for $a = p$ we get $(-1)^p \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p} - \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0 \ldots i_p} = \text{sign}(\sigma^{i_0 \ldots i_p}) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}.$$

The claim above implies that κ is a morphism of complexes and that κ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator κ is not the alternating subcomplex. Namely,

the image of κ is the "semi-alternating" complex $\check{\mathcal{C}}^p_{semi\text{-}alt}(\mathcal{U}, \mathcal{F})$ where s is a p-cochain of this complex if and only if

$$s_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0 \ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Čech complex defined by

$$\check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$

It is easy to see that Equation (20.9.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 20.23.4) that the projection map

$$\check{\mathcal{C}}^\bullet_{semi\text{-}alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy

(20.23.6.2)

01FO
$$h(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0 \ldots i_{a-1} i_a i_a i_{a+1} \ldots i_p} & \text{if} \quad i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that

$$(dh + hd)(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ s_{i_0 \ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) The claim clearly shows that the composition

$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as desired. □

0G6T **Lemma 20.23.7.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended alternating Čech complex*

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

obtained by putting $\mathcal{F}(U)$ in degree -1 with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0. Similarly, for any total ordering on I the extended ordered Čech complex

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

is homotopy equivalent to 0.

First proof. Combine Lemmas 20.9.3 and 20.23.6. □

Second proof. Since the alternating and ordered Čech complexes are isomorphic it suffices to prove this for the ordered one. We will use standard notation: a cochain s of degree p in the extended ordered Čech complex has the form $s = (s_{i_0 \ldots i_p})$ where $s_{i_0 \ldots i_p}$ is in $\mathcal{F}(U_{i_0 \ldots i_p})$ and $i_0 < \ldots < i_p$. With this notation we have

$$d(x)_{i_0 \ldots i_{p+1}} = \sum\nolimits_j (-1)^j x_{i_0 \ldots \hat{i}_j \ldots i_p}$$

Fix an index $i \in I$ with $U = U_i$. As homotopy we use the maps

$$h : \text{cochains of degree } p+1 \to \text{cochains of degree } p$$

given by the rule

$$h(s)_{i_0 \ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(s)_{i_0 \ldots i_p} = (-1)^j s_{i_0 \ldots i_j i i_{j+1} \ldots i_p} \text{ if not}$$

Here j is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since $U = U_i$ we have the equality

$$\mathcal{F}(U_{i_0 \ldots i_p}) = \mathcal{F}(U_{i_0 \ldots i_j i i_{j+1} \ldots i_p})$$

which we can use to make sense of thinking of $(-1)^j s_{i_0 \ldots i_j i i_{j+1} \ldots i_p}$ as an element of $\mathcal{F}(U_{i_0 \ldots i_p})$. We will show by a computation that $dh + hd$ equals the negative of the identity map which finishes the proof. To do this fix s a cochain of degree p and let $i_0 < \ldots < i_p$ be elements of I.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(s))_{i_0 \ldots i_p} = 0$. On the other hand we have

$$d(h(s))_{i_0 \ldots i_p} = \sum (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} = (-1)^t h(s)_{i_0 \ldots \hat{i} \ldots i_p} = (-1)^t (-1)^{t-1} s_{i_0 \ldots i_p}$$

Thus $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired.

Case II: $i \notin \{i_0, \ldots, i_p\}$. Let j be such that $i_j < i < i_{j+1}$. Then we see that

$$\begin{aligned} h(d(s))_{i_0 \ldots i_p} &= (-1)^j d(s)_{i_0 \ldots i_j i i_{j+1} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j+j'} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} - s_{i_0 \ldots i_p} \\ &\quad + \sum\nolimits_{j' > j} (-1)^{j+j'+1} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(h(s))_{i_0 \ldots i_p} &= \sum\nolimits_{j'} (-1)^{j'} h(s)_{i_0 \ldots \hat{i}_{j'} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j'+j-1} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} \\ &\quad + \sum\nolimits_{j' > j} (-1)^{j'+j} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

Adding these up we obtain $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired. □

20.24. Alternative view of the Čech complex

02FR In this section we discuss an alternative way to establish the relationship between the Čech complex and cohomology.

02FU **Lemma 20.24.1.** *Let X be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with*

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}$$

and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ *as in Equation (20.9.0.1). Moreover, there exists a canonical map*

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ *is a resolution of* $\mathcal{F}$*.*

Proof. We check

$$0 \to \mathcal{F} \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F}) \to \ldots$$

is exact on stalks. Let $x \in X$ and choose $i_{\text{fix}} \in I$ such that $x \in U_{i_{\text{fix}}}$. Then define

$$h : \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

as follows: If $s \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x$, take a representative

$$\widetilde{s} \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(V \cap U_{i_0} \cap \ldots \cap U_{i_p})$$

defined on some neighborhood V of x, and set

$$h(s)_{i_0 \ldots i_{p-1}} = \widetilde{s}_{i_{\text{fix}} i_0 \ldots i_{p-1}, x}.$$

By the same formula (for $p = 0$) we get a map $\mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathcal{F}_x$. We compute formally as follows:

$$\begin{aligned}
(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\
&= \sum\nolimits_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}$$

This shows h is a homotopy from the identity map of the extended complex

$$0 \to \mathcal{F}_x \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F})_x \to \ldots$$

to zero and we conclude. □

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 20.11.5. Namely, let X, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 20.24.1 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$A^{\bullet,\bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

By construction we have

$$A^{p,q} = \prod\nolimits_{i_0 \ldots i_p} \mathcal{I}^q(U_{i_0 \ldots i_p})$$

Consider the two spectral sequences of Homology, Section 12.25 associated to this double complex, see especially Homology, Lemma 12.25.1. For the spectral sequence $({}'E_r, {}'d_r)_{r \geq 0}$ we get ${}'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ because taking products is exact (Homology, Lemma 12.32.1). For the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$ we get ${}''E_2^{p,q} = 0$ if $p > 0$ and ${}''E_2^{0,q} = H^q(X, \mathcal{F})$. Namely, for fixed q the complex of sheaves $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution (Lemma 20.24.1) of the injective sheaf $\mathcal{I}^q$ by injective sheaves (by Lemmas 20.7.1 and 20.11.11 and Homology, Lemma 12.27.3). Hence the cohomology of $\Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q))$ is zero in positive degrees and equal to $\Gamma(X, \mathcal{I}^q)$ in degree 0. Taking cohomology of the next differential we get our claim about the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$. Whence the result since both spectral sequences converge to the cohomology of the associated total complex of $A^{\bullet,\bullet}$.

02FS **Definition 20.24.2.** Let X be a topological space. An open covering $X = \bigcup_{i \in I} U_i$ is said to be *locally finite* if for every $x \in X$ there exists an open neighbourhood W of x such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

02FT **Remark 20.24.3.** Let $X = \bigcup_{i \in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each i we are given an abelian sheaf $\mathcal{F}_i$ on U_i. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i \in I} (j_i)_* \mathcal{F}_i$. Then for $V \subset X$ open we actually have

$$\Gamma(V, \mathcal{G}) = \prod\nolimits_{i \in I} \mathcal{F}_i(V \cap U_i).$$

In other words we have

$$\bigoplus\nolimits_{i \in I} (j_i)_* \mathcal{F}_i = \prod\nolimits_{i \in I} (j_i)_* \mathcal{F}_i$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 17.3. Thus we conclude that in this case the complex of Lemma 20.24.1 has terms

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}$$

which is sometimes useful.

20.25. Čech cohomology of complexes

01FP In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cup product map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocycles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any topos (insert future reference here).

Let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups on X. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Since the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ (Definition 20.9.1) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. The associated total complex to $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ is the complex with degree n term

$$\mathrm{Tot}^n(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \bigoplus\nolimits_{p+q=n} \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

see Homology, Definition 12.18.3. A typical element in Tot^n will be denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ where $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is $q = n - p$. This notation requires us to be aware of the degree α lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. According to our conventions in Homology, Definition 12.18.3 the differential of an element α of degree n is given by

$$d(\alpha)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}^\bullet$. The expression $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ means the restriction of $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \in \mathcal{F}(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})$ to $U_{i_0 \ldots i_{p+1}}$.

The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

07M9 (20.25.0.1) $$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

08BM (20.25.0.2) $$T_t : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)).$$

defined by the rule

$$T_t(\alpha)_{j_0 \ldots j_p} = \alpha_{t(j_0) \ldots t(j_p)}|_{V_{j_0 \ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0 \ldots j_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an element of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0 \ldots j_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{j_0 \ldots \hat{j_k} \ldots j_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0 \ldots j_p}) + \\
& \sum\nolimits_{a=0}^{p} (-1)^a d(\alpha)_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)} \\
= & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^{k+a} \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(\hat{j_k}) \ldots t'(j_p)} + \\
& \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^{k+a-1} \alpha_{t(j_0) \ldots t(\hat{j_k}) \ldots t(j_a) t'(j_a) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} (-1)^{p+a} d_{\mathcal{F}}(\alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}) + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} (-1)^{a+k} \alpha_{t(j_0) \ldots t(\hat{j_k}) \ldots t(j_a) t'(j_a) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} (-1)^{a+k+1} \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(\hat{j_k}) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} (-1)^{a+p+1} d_{\mathcal{F}}(\alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}) \\
= & \alpha_{t'(j_0) \ldots t'(j_p)} + (-1)^{2p+1} \alpha_{t(j_0) \ldots t(j_p)} \\
= & T_{t'}(\alpha)_{j_0 \ldots j_p} - T_t(\alpha)_{j_0 \ldots j_p}
\end{aligned}$$

We leave it to the reader to verify the cancellations. (Note that the terms having both k and a in the 1st, 2nd and 4th, 5th summands cancel, except the ones where $a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$H^n(T_t) : H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))$$

is independent of the choice of t. We define *Čech hypercohomology* as the colimit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

In the colimit (over all open coverings of X) the following lemma provides a map of Čech hypercohomology into cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

08BN **Lemma 20.25.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. For a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.25.0.1) and (20.25.0.2). There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = H^p(\, Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}^\bullet)))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. The map (20.25.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.25.4 applied to the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 20.11.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a bounded below complex of injectives. Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. To construct the spectral sequence of the lemma, choose a Cartan-Eilenberg resolution $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$, see Derived Categories, Lemma 13.21.2. In this case $\mathcal{F}^\bullet \to \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})$ is an injective resolution and hence

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})))$$

computes $R\Gamma(X, \mathcal{F}^\bullet)$ as we've seen above. By Homology, Remark 12.18.4 we can view this as the total complex associated to the triple complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^{\bullet,\bullet})$ hence, using the same remark we can view it as the total complex associate to the double complex $A^{\bullet,\bullet}$ with terms

$$A^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^{q,m})$$

Since $\mathcal{I}^{q,\bullet}$ is an injective resolution of $\mathcal{F}^q$ we can apply the first spectral sequence associated to $A^{\bullet,\bullet}$ (Homology, Lemma 12.25.1) to get a spectral sequence with

$$E_1^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \underline{H}^m(\mathcal{F}^q))$$

which is the nth term of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^m(\mathcal{F}^\bullet))$. Hence we obtain E_2 terms as described in the lemma. Convergence by Homology, Lemma 12.25.3. □

0FLH **Lemma 20.25.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. If $H^i(U_{i_0 \ldots i_p}, \mathcal{F}^q) = 0$ for all $i > 0$ and all $p, i_0, \ldots, i_p, q$, then the map $Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to R\Gamma(X, \mathcal{F}^\bullet)$ of Lemma 20.25.1 is an isomorphism.*

Proof. Immediate from the spectral sequence of Lemma 20.25.1. □

0FLI **Remark 20.25.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Let b be an integer. We claim there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))[b] & \longrightarrow & R\Gamma(X, \mathcal{F}^\bullet)[b] \\ \downarrow{\scriptstyle \gamma} & & \downarrow \\ \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])) & \longrightarrow & R\Gamma(X, \mathcal{F}^\bullet[b]) \end{array}$$

in the derived category where the map γ is the map on complexes constructed in Homology, Remark 12.18.5. This makes sense because the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])$ is clearly the same as the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)[0, b]$ introduced in Homology, Remark 12.18.5. To check that the diagram commutes, we may choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ as in the proof of Lemma 20.25.1. Chasing diagrams, we see that it suffices to check the diagram commutes when we replace $\mathcal{F}^\bullet$ by $\mathcal{I}^\bullet$. Then we consider the extended diagram

$$\begin{array}{ccccc} \Gamma(X, \mathcal{I}^\bullet)[b] & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))[b] & \longrightarrow & R\Gamma(X, \mathcal{I}^\bullet)[b] \\ \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ \Gamma(X, \mathcal{I}^\bullet[b]) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet[b])) & \longrightarrow & R\Gamma(X, \mathcal{I}^\bullet[b]) \end{array}$$

where the left horizontal arrows are (20.25.0.1). Since in this case the horizontal arrows are isomorphisms in the derived category (see proof of Lemma 20.25.1) it suffices to show that the left square commutes. This is true because the map γ uses the sign 1 on the summands $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^{q+b})$, see formula in Homology, Remark 12.18.5.

Let X be a topological space, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering, and let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups. Consider the map $\tau : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ defined by

$$\tau(\alpha)_{i_0 \ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p \ldots i_0}.$$

Then we have for an element α of degree n that

$$\begin{aligned} & d(\tau(\alpha))_{i_0 \ldots i_{p+1}} \\ & = \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\tau(\alpha)_{i_0 \ldots i_{p+1}}) \\ & = \sum\nolimits_{j=0}^{p+1} (-1)^{j + \frac{p(p+1)}{2}} \alpha_{i_{p+1} \ldots \hat{i}_j \ldots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0}) \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \tau(d(\alpha))_{i_0 \ldots i_{p+1}} \\ & = (-1)^{\frac{(p+1)(p+2)}{2}} d(\alpha)_{i_{p+1} \ldots i_0} \\ & = (-1)^{\frac{(p+1)(p+2)}{2}} \left(\sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1} \ldots \hat{i}_{p+1-j} \ldots i_0} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0}) \right) \end{aligned}$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p + 1 \bmod 2$. In other words τ is an endomorphism of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$.

Note that the diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \\ \downarrow \mathrm{id} & & \downarrow \tau \\ \Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \end{array}$$

commutes. In addition τ is clearly compatible with refinements. This suggests that τ acts as the identity on Čech cohomology (i.e., in the colimit – provided Čech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that τ actually is homotopic to the identity on the total Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$h(\alpha)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} \epsilon_p(a) \alpha_{i_0 \ldots i_a i_p \ldots i_a} \quad \text{with} \quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}$$

for α of degree n. As usual we omit writing $|_{U_{i_0 \ldots i_p}}$. This works because of the following computation, again with α an element of degree n:

$$\begin{aligned} (d(h(\alpha)) + h(d(\alpha)))_{i_0 \ldots i_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0 \ldots \hat{i_k} \ldots i_p} + \\ & (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0 \ldots i_p}) + \\ & \sum\nolimits_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0 \ldots i_a i_p \ldots i_a} \\ = & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^k \epsilon_{p-1}(a) \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\ & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\ & \sum\nolimits_{a=0}^{p} (-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) + \\ & \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} \epsilon_p(a) (-1)^k \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\ & \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} \epsilon_p(a) (-1)^{p+a+1-k} \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\ & \sum\nolimits_{a=0}^{p} \epsilon_p(a) (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) \\ = & \epsilon_p(0) \alpha_{i_p \ldots i_0} + \epsilon_p(p) (-1)^{p+1} \alpha_{i_0 \ldots i_p} \\ = & (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p \ldots i_0} - \alpha_{i_0 \ldots i_p} \end{aligned}$$

The cancellations follow because

$$(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule

07MA (20.25.3.1) $$d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta)$$

when α and β are homogeneous, see Homology, Definition 12.18.3.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\mathrm{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cup product

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 15.63.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

(20.25.3.2)

07MB
$$\mathrm{Tot}\left(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)))$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0 \ldots i_p} = \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_p}.$$

where α has degree n and β has degree m and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n + rp + r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0 \ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [**AGV71**, Exposee XVII] by Deligne. To check (20.25.3.2) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d(\beta)$$

by the definition of the differential on $\mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 12.18.3. We compute first

$$\begin{aligned}
d(\alpha \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F} \otimes \mathcal{G}}((\alpha \cup \beta)_{i_0 \ldots i_{p+1}}) \\
&= \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=0}^{j-1} (-1)^j \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=j+1}^{p+1} (-1)^j \epsilon(n, m, p, r-1) \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} (-1)^{p+1} \epsilon(n, m, p+1, r) d_{\mathcal{F} \otimes \mathcal{G}}(\alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}})
\end{aligned}$$

and note that the summands in the last term equal

$$(-1)^{p+1} \epsilon(n, m, p+1, r) \left(d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} + (-1)^{n-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}})\right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) = n - r$. On the other hand

$$\begin{aligned}
(d(\alpha) \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) d(\alpha)_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \\
&= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=0}^{r} \epsilon(n+1, m, p+1, r) (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) (-1)^r d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}}
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \cup d(\beta))_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r) \alpha_{i_0 \ldots i_r} \otimes d(\beta)_{i_r \ldots i_{p+1}} \\
&= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=r}^{p+1} \epsilon(n, m+1, p+1, r) (-1)^{j-r} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i_j} \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r) (-1)^{p+1-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}})
\end{aligned}$$

The desired equality holds if we have

$$\begin{aligned}
(-1)^{p+1}\epsilon(n,m,p+1,r) &= \epsilon(n+1,m,p+1,r)(-1)^r \\
(-1)^{p+1}\epsilon(n,m,p+1,r)(-1)^{n-r} &= (-1)^n\epsilon(n,m+1,p+1,r)(-1)^{p+1-r} \\
\epsilon(n+1,m,p+1,r)(-1)^r &= (-1)^{1+n}\epsilon(n,m+1,p+1,r-1) \\
(-1)^j\epsilon(n,m,p,r) &= (-1)^n\epsilon(n,m+1,p+1,r)(-1)^{j-r} \\
(-1)^j\epsilon(n,m,p,r-1) &= \epsilon(n+1,m,p+1,r)(-1)^j
\end{aligned}$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n\alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Associativity of the cup product. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet) \otimes_{\mathbf{Z}} \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha\otimes\beta\otimes\gamma) = d(\alpha)\otimes\beta\otimes\gamma + (-1)^{\deg(\alpha)}\alpha\otimes d(\beta)\otimes\gamma + (-1)^{\deg(\alpha)+\deg(\beta)}\alpha\otimes\beta\otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned}
((\alpha \cup \beta) \cup \gamma)_{i_0\ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(a+b,c,p,r)(\alpha\cup\beta)_{i_0\ldots i_r} \otimes \gamma_{i_r\ldots i_p} \\
&= \sum\nolimits_{r=0}^{p}\sum\nolimits_{s=0}^{r} \epsilon(a+b,c,p,r)\epsilon(a,b,r,s)\alpha_{i_0\ldots i_s}\otimes\beta_{i_s\ldots i_r}\otimes\gamma_{i_r\ldots i_p}
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \cup (\beta \cup \gamma)_{i_0\ldots i_p} &= \sum\nolimits_{s=0}^{p} \epsilon(a,b+c,p,s)\alpha_{i_0\ldots i_s} \otimes (\beta\cup\gamma)_{i_s\ldots i_p} \\
&= \sum\nolimits_{s=0}^{p}\sum\nolimits_{r=s}^{p} \epsilon(a,b+c,p,s)\epsilon(b,c,p-s,r-s)\alpha_{i_0\ldots i_s}\otimes\beta_{i_s\ldots i_r}\otimes\gamma_{i_r\ldots i_p}
\end{aligned}$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab}t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s\wedge t) = d(s)\wedge t + (-1)^a s \wedge d(t)$. The composition

$$\mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))\otimes\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathrm{Tot}(\mathcal{F}^\bullet\otimes_{\mathbf{Z}}\mathcal{F}^\bullet))) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the colimit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element α of degree n in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element β of degree m in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$\begin{aligned}
\wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\
&= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) (-1)^{\deg(\alpha_{i_0 \ldots i_r}) \deg(\beta_{i_r \ldots i_p})} \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}
\end{aligned}$$

because $\wedge$ is graded commutative. On the other hand we have

$$\begin{aligned}
\tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \ldots i_p} &= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \tau(\beta)_{i_p \ldots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \ldots i_0} \\
&= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \chi(r) \chi(p-r) \beta_{i_{p-r} \ldots i_p} \wedge \alpha_{i_0 \ldots i_{p-r}} \\
&= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, p-r) \chi(r) \chi(p-r) \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}
\end{aligned}$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that τ acts as the identity on cohomology we have to verify that

$$\epsilon(n, m, p, r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m, n, p, p-r) \chi(r) \chi(p-r)$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$\gamma_i : \mathrm{Tot}(\mathcal{F}_i^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$\begin{array}{ccc}
\mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_1^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow \\
\mathcal{H}^\bullet & \xleftarrow{\gamma_2} & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet)
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \\
\downarrow{\scriptstyle \gamma_2} & & \downarrow \\
\mathcal{H}^\bullet & \xleftarrow{\gamma_3} & \mathrm{Tot}(\mathcal{F}_3^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet)
\end{array}$$

are commutative.

07MC **Lemma 20.25.4.** *In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have*

$$\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1}\gamma_3(a_3 \cup \partial b_1)$$

in $H^{n+m+1}(X, \mathcal{H}^\bullet)$ where ∂ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.

Proof. We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^p$ we use s to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections s of $\mathcal{F}_2^p$ and t of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Suppose that α_3, resp. β_1 is a degree n, resp. m cocycle of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing a_3, resp. b_1. After refining $\mathcal{U}$ if necessary, we can find cochains α_2, resp. β_2 of degree n, resp. m in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to α_3, resp. β_1. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ representing ∂a_3. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing ∂b_1. Thus we may compute

$$\begin{aligned} d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\ &= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\ &= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[2]. □

0B8S **Lemma 20.25.5.** *Let X be a topological space. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I} \subset \mathcal{O}'$ has square zero. Then $M = H^1(X, \mathcal{I})$ is a $R = H^0(X, \mathcal{O})$-module and the boundary map $\partial : R \to M$ associated to the short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

is a derivation (Algebra, Definition 10.131.1).

Proof. The map $\mathcal{O}' \to \mathcal{H}om(\mathcal{I}, \mathcal{I})$ factors through $\mathcal{O}$ as $\mathcal{I} \cdot \mathcal{I} = 0$ by assumption. Hence $\mathcal{I}$ is a sheaf of $\mathcal{O}$-modules and this defines the R-module structure on M. The boundary map is additive hence it suffices to prove the Leibniz rule. Let $f \in R$. Choose an open covering $\mathcal{U} : X = \bigcup U_i$ such that there exist $f_i \in \mathcal{O}'(U_i)$ lifting $f|_{U_i} \in \mathcal{O}(U_i)$. Observe that $f_i - f_j$ is an element of $\mathcal{I}(U_i \cap U_j)$. Then $\partial(f)$

[2]The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the Stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 13.10.

corresponds to the Čech cohomology class of the 1-cocycle α with $\alpha_{i_0i_1} = f_{i_0} - f_{i_1}$. (Observe that by Lemma 20.11.3 the first Čech cohomology group with respect to $\mathcal{U}$ is a submodule of M.) Next, let $g \in R$ be a second element and assume (after possibly refining the open covering) that $g_i \in \mathcal{O}'(U_i)$ lifts $g|_{U_i} \in \mathcal{O}(U_i)$. Then we see that $\partial(g)$ is given by the cocycle β with $\beta_{i_0i_1} = g_{i_0} - g_{i_1}$. Since $f_ig_i \in \mathcal{O}'(U_i)$ lifts $fg|_{U_i}$ we see that $\partial(fg)$ is given by the cocycle γ with

$$\gamma_{i_0i_1} = f_{i_0}g_{i_0} - f_{i_1}g_{i_1} = (f_{i_0} - f_{i_1})g_{i_0} + f_{i_1}(g_{i_0} - g_{i_1}) = \alpha_{i_0i_1}g + f\beta_{i_0i_1}$$

by our definition of the $\mathcal{O}$-module structure on $\mathcal{I}$. This proves the Leibniz rule and the proof is complete. □

20.26. Flat resolutions

06Y7 A reference for the material in this section is [**Spa88**]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 17.17.7 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 13.15.4 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

06Y8 **Lemma 20.26.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functors*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

and

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. □

06Y9 **Definition 20.26.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

is acyclic.

06YA **Lemma 20.26.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 20.26.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

06YB **Lemma 20.26.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}_x^\bullet$ of $\mathcal{O}_{X,x}$-modules is K-flat (More on Algebra, Definition 15.59.1).*

Proof. If $\mathcal{K}_x^\bullet$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 17.3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ and let $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic

complex of $\mathcal{O}_X$-modules. Thus $\mathrm{Tot}(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at x shows that $\mathrm{Tot}(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x^\bullet)$ is acyclic. □

079R **Lemma 20.26.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then Tot$(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. □

079S **Lemma 20.26.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}_X))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 20.26.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. □

0G6U **Lemma 20.26.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}_X$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. By Modules, Lemma 17.17.8 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. □

06YC **Lemma 20.26.8.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. We can check this on stalks, see Lemma 20.26.4. Hence this follows from Sheaves, Lemma 6.26.4 and More on Algebra, Lemma 15.59.3. □

06YD **Lemma 20.26.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.*

Proof. We can check this on stalks, see Lemma 20.26.4. Thus this lemma follows from Modules, Lemma 17.17.2 and More on Algebra, Lemma 15.59.7. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

06YE **Lemma 20.26.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

079T **Lemma 20.26.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules*

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 17.17.7 and Derived Categories, Lemma 13.29.1. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

06YF **Lemma 20.26.12.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.*

Proof. Choose a diagram as in Lemma 20.26.11. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules, Lemma 17.17.6. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 20.26.9. Thus $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 20.26.10. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 20.26.11 shows that $\operatorname{colim} \mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. □

06YG **Lemma 20.26.13.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 20.26.12. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \\ \downarrow & & \downarrow \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \end{array}$$

The result follows as by Lemma 20.26.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 20.26.12. By Lemma 20.26.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 20.26.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 20.26.13 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

06YH **Definition 20.26.14.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Here we use sign rules as given in More on Algebra, Section 15.73. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

08BP **Definition 20.26.15.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\operatorname{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{aligned} &\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0 \\ &\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{G}) \end{aligned}$$

for every $\mathcal{O}_X$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

08BQ **Lemma 20.26.16.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module, and*
(2) *$\mathit{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}_X$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}$ is a quotient of $\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. □

0G6V **Lemma 20.26.17.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}_X$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that*

(1) *$\mathcal{N}^\bullet$ is K-flat,*
(2) *c is a quasi-isomorphism,*
(3) *a is homotopic to $c \circ b$.*

If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, b, and c such that the same is true for $\mathcal{N}^\bullet$.

Proof. We will use that the homotopy category $K(Mod(\mathcal{O}_X))$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 20.26.12. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 20.26.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathcal{K}^\bullet & \longrightarrow & \mathcal{N}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{C}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1] \end{array}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O}_X)$ and use Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. □

20.27. Derived pullback

06YI Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 20.26.8, 20.26.12, and 20.26.13 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

06YJ **Lemma 20.27.1.** *The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 20.26.12 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 20.26.8 applied to the morphism $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 20.26.13

guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. □

0D5S **Lemma 20.27.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.*

Proof. Let E be an object of $D(\mathcal{O}_Z)$. By construction Lg^*E is computed by choosing a K-flat complex $\mathcal{K}^\bullet$ representing E on Z and setting $Lg^*E = g^*\mathcal{K}^\bullet$. By Lemma 20.26.8 we see that $g^*\mathcal{K}^\bullet$ is K-flat on Y. Then Lf^*Lg^*E is given by $f^*g^*\mathcal{K}^\bullet = (g \circ f)^*\mathcal{K}^\bullet$ which also represents $L(g \circ f)^*E$. □

079U **Lemma 20.27.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_Y))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 20.26.5 $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 17.16.4. □

08DE **Lemma 20.27.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}_Y)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

0FP0 **Lemma 20.27.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_Y$-modules. The diagram*

$$\begin{array}{ccc}
Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{M}^\bullet) & \longrightarrow & Lf^*\,Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{M}^\bullet & & f^*\,Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^*\mathcal{M}^\bullet & \longrightarrow & Tot(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet)
\end{array}$$

commutes.

Proof. We will use the existence of K-flat resolutions as in Lemma 20.26.8. If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$\begin{array}{ccc} Lf^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) & \longrightarrow & Lf^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ f^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) & \longrightarrow & f^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{Q}^\bullet) & \longrightarrow & \mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet) \end{array}$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 20.26.5. □

20.28. Cohomology of unbounded complexes

079V Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 17.3 and Lemmas 17.17.6 and 17.17.7. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules all of whose terms are injective $\mathcal{O}_X$-modules and moreover this embedding can be chosen functorial in the complex $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 13.31.7 that

(1) any exact functor $F : K(Mod(\mathcal{O}_X)) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}_X) \to \mathcal{D}$,
(2) for any additive functor $F : Mod(\mathcal{O}_X) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(Mod(\mathcal{O}_X)) \to K(\mathcal{A})$ induced by F and we obtain a right derived functor $RF : D(\mathcal{O}_X) \to D(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(X, -) : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\Gamma(X,\mathcal{O}_X)}$ gives rise to
$$R\Gamma(X, -) : D(\mathcal{O}_X) \to D(\Gamma(X, \mathcal{O}_X))$$
We shall use the notation $H^i(X, K) = H^i(R\Gamma(X, K))$ for cohomology.
(2) For an open $U \subset X$ we consider the functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\Gamma(U,\mathcal{O}_X)}$. This gives rise to
$$R\Gamma(U, -) : D(\mathcal{O}_X) \to D(\Gamma(U, \mathcal{O}_X))$$
We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.
(3) For a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we consider the functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ which gives rise to the total direct image
$$Rf_* : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y)$$
on unbounded derived categories.

079W **Lemma 20.28.1.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor Rf_* defined above and the functor Lf^* defined in Lemma 20.27.1 are adjoint:*

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_X))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_Y))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.30.3. □

0D5T **Lemma 20.28.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_Z)$.*

Proof. By Lemma 20.28.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 20.27.2 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. □

08HY **Remark 20.28.3.** The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is a commutative diagram of ringed spaces. Let K be an object of $D(\mathcal{O}_X)$. Then there exists a canonical base change map

$$Lg^*Rf_*K \longrightarrow R(f')_*L(g')^*K$$

in $D(\mathcal{O}_{S'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*K \to L(g')^*K$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*K \to L(g')^*K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*K \to K$.

0ATL **Remark 20.28.4.** Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{l} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{m} & Z \end{array}$$

of ringed spaces. Then the base change maps of Remark 20.28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* &= Lm^* \circ Rg_* \circ Rf_* \\ &\to Rg'_* \circ Ll^* \circ Rf_* \\ &\to Rg'_* \circ Rf'_* \circ Lk^* \\ &= R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0ATM **Remark 20.28.5.** Consider a commutative diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h'} & Y' & \xrightarrow{h} & Y \end{array}$$

of ringed spaces. Then the base change maps of Remark 20.28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} L(h \circ h')^* \circ Rf_* &= L(h')^* \circ Lh_* \circ Rf_* \\ &\to L(h')^* \circ Rf'_* \circ Lg^* \\ &\to Rf''_* \circ L(g')^* \circ Lg^* \\ &= Rf''_* \circ L(g \circ g')^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0FP1 **Lemma 20.28.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The diagram*

$$\begin{array}{ccc} Lf^*f_*\mathcal{K}^\bullet & \longrightarrow & f^*f_*\mathcal{K}^\bullet \\ \downarrow & & \downarrow \\ Lf^*Rf_*\mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet \end{array}$$

coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to id$ commutes in $D(\mathcal{O}_X)$.

Proof. We will use the existence of K-flat resolutions and K-injective resolutions, see Lemma 20.26.8 and the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_X$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_*\mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is K-flat as a complex of $\mathcal{O}_Y$-modules. We can choose a K-flat complex of $\mathcal{O}_Y$-modules $\mathcal{P}^\bullet$ and a diagram of morphisms of complexes

$$\begin{array}{ccc} \mathcal{P}^\bullet & \longrightarrow & f_*\mathcal{K}^\bullet \\ \downarrow & & \downarrow \\ \mathcal{Q}^\bullet & \longrightarrow & f_*\mathcal{I}^\bullet \end{array}$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can replace $\mathcal{P}^\bullet$ by a K-flat complex. Taking pullbacks we obtain a diagram of morphisms of complexes

$$\begin{array}{ccccc} f^*\mathcal{P}^\bullet & \longrightarrow & f^*f_*\mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ f^*\mathcal{Q}^\bullet & \longrightarrow & f^*f_*\mathcal{I}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. □

0B68 **Remark 20.28.7.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The adjointness of Lf^* and Rf_* allows us to construct a relative cup product

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \longrightarrow Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_Y)$ for all K, L in $D(\mathcal{O}_X)$. Namely, this map is adjoint to a map $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) = Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*L$ (Lemma 20.27.3) with the map $Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*L \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ coming from the counit $Lf^* \circ Rf_* \to \mathrm{id}$.

20.29. Cohomology of filtered complexes

0FLJ Filtered complexes of sheaves frequently come up in a natural fashion when studying cohomology of algebraic varieties, for example the de Rham complex comes with its Hodge filtration. In this sectionwe use the very general Injectives, Lemma 19.13.7 to find construct spectral sequences on cohomology and we relate these to previously constructed spectral sequences.

0BKK **Lemma 20.29.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r\geq 1}$ of bigraded $\Gamma(X, \mathcal{O}_X)$-modules with d_r of bidegree $(r, -r+1)$ and*

$$E_1^{p,q} = H^{p+q}(X, gr^p\mathcal{F}^\bullet)$$

If for every n we have

$$H^n(X, F^p\mathcal{F}^\bullet) = 0 \text{ for } p \gg 0 \quad \text{and} \quad H^n(X, F^p\mathcal{F}^\bullet) = H^n(X, \mathcal{F}^\bullet) \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $H^*(X, \mathcal{F}^\bullet)$.

Proof. (For a proof in case the complex is a bounded below complex of modules with finite filtrations, see the remark below.) Choose an map of filtered complexes $j : \mathcal{F}^\bullet \to \mathcal{J}^\bullet$ as in Injectives, Lemma 19.13.7. The spectral sequence is the spectral sequence of Homology, Section 12.24 associated to the filtered complex

$$\Gamma(X, \mathcal{J}^\bullet) \quad \text{with} \quad F^p\Gamma(X, \mathcal{J}^\bullet) = \Gamma(X, F^p\mathcal{J}^\bullet)$$

Since cohomology is computed by evaluating on K-injective representatives we see that the E_1 page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 12.24.13. □

0BKL **Remark 20.29.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ is bounded from below and for each n the filtration on $\mathcal{F}^n$ is finite, then there is a construction of the spectral sequence in Lemma 20.29.1 avoiding Injectives, Lemma 19.13.7. Namely, by Derived Categories, Lemma 13.26.9 there is a filtered quasi-isomorphism $i : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ of filtered complexes with $\mathcal{I}^\bullet$ bounded below, the filtration on $\mathcal{I}^n$ is finite for all n, and with each $\mathrm{gr}^p\mathcal{I}^n$ an injective $\mathcal{O}_X$-module. Then we take the spectral sequence associated to

$$\Gamma(X, \mathcal{I}^\bullet) \quad \text{with} \quad F^p\Gamma(X, \mathcal{I}^\bullet) = \Gamma(X, F^p\mathcal{I}^\bullet)$$

Since cohomology can be computed by evaluating on bounded below complexes of injectives we see that the E_1 page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 12.24.11. In fact, this is a special case of the spectral sequence in Derived Categories, Lemma 13.26.14.

0BKM **Example 20.29.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 20.29.1 with $F^p\mathcal{F}^\bullet = \tau_{\leq -p}\mathcal{F}^\bullet$. (If $\mathcal{F}^\bullet$ is bounded below we can use Remark 20.29.2.) Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, H^{-p}(\mathcal{F}^\bullet)[p]) = H^{2p+q}(X, H^{-p}(\mathcal{F}^\bullet))$$

After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(\mathcal{O}_X)$ there is a spectral sequence $(E'_r, d'_r)_{r\geq 2}$ of bigraded modules with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = H^i(X, H^j(K))$$

If K is bounded below (for example), then this spectral sequence is bounded and converges to $H^{i+j}(X, K)$. In the bounded below case this spectral sequence is an example of the second spectral sequence of Derived Categories, Lemma 13.21.3 (constructed using Cartan-Eilenberg resolutions).

0FLK **Example 20.29.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 20.29.1 with $F^p\mathcal{F}^\bullet = \sigma_{\geq p}\mathcal{F}^\bullet$. Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathcal{F}^p[-p]) = H^q(X, \mathcal{F}^p)$$

If $\mathcal{F}^\bullet$ is bounded below, then

(1) we can use Remark 20.29.2 to construct this spectral sequence,
(2) the spectral sequence is bounded and converges to $H^{i+j}(X, \mathcal{F}^\bullet)$, and
(3) the spectral sequence is equal to the first spectral sequence of Derived Categories, Lemma 13.21.3 (constructed using Cartan-Eilenberg resolutions).

0FLL **Lemma 20.29.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r\geq 1}$ of bigraded $\mathcal{O}_Y$-modules with d_r of bidegree $(r, -r+1)$ and*

$$E_1^{p,q} = R^{p+q}f_* gr^p\mathcal{F}^\bullet$$

If for every n we have

$$R^n f_* F^p\mathcal{F}^\bullet = 0 \text{ for } p \gg 0 \quad \text{and} \quad R^n f_* F^p\mathcal{F}^\bullet = R^n f_*\mathcal{F}^\bullet \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $Rf_*\mathcal{F}^\bullet$.

Proof. The proof is exactly the same as the proof of Lemma 20.29.1. □

20.30. Godement resolution

0FKR A reference is [**God73**].

Let $(X, \mathcal{O}_X)$ be a ringed space. Denote X_{disc} the discrete topological space with the same points as X. Denote $f : X_{disc} \to X$ the obvious continuous map. Set $\mathcal{O}_{X_{disc}} = f^{-1}\mathcal{O}_X$. Then $f : (X_{disc}, \mathcal{O}_{X_{disc}}) \to (X, \mathcal{O}_X)$ is a flat morphism of ringed spaces. We can apply the *dual* of the material in Simplicial, Section 14.34 to the adjoint pair of functors f^*, f_* on sheaves of modules. Thus we obtain an augmented cosimplicial object

$$\mathrm{id} \longrightarrow f_*f^* \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; f_*f^*f_*f^* \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; f_*f^*f_*f^*f_*f^*$$

in the category of functors from $Mod(\mathcal{O}_X)$ to itself, see Simplicial, Lemma 14.34.2. Moreover, the augmentation

$$f^* \longrightarrow f^*f_*f^* \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; f^*f_*f^*f_*f^* \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; f^*f_*f^*f_*f^*f_*f^*$$

is a homotopy equivalence, see Simplicial, Lemma 14.34.3.

0FKS **Lemma 20.30.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a resolution*

$$0 \to \mathcal{F} \to f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$$

functorial in $\mathcal{F}$ such that each term $f_*f^* \ldots f_*f^*\mathcal{F}$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map

$$\mathcal{F}_x[0] \to \Big((f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F})_x \to \ldots\Big)$$

is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.

Proof. The complex $f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$ is the complex associated to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$ described above, see Simplicial, Section 14.25. The augmentation gives rise to the map $\mathcal{F} \to f_*f^*\mathcal{F}$ as indicated. For any abelian sheaf $\mathcal{H}$ on X_{disc} the pushforward $f_*\mathcal{H}$ is flasque because X_{disc} is a discrete space and the pushforward of a flasque sheaf is flasque. Hence the terms of the complex are flasque $\mathcal{O}_X$-modules.

If $x \in X_{disc} = X$ is a point, then $(f^*\mathcal{G})_x = \mathcal{G}_x$ for any $\mathcal{O}_X$-module $\mathcal{G}$. Hence f^* is an exact functor and a complex of $\mathcal{O}_X$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $f^*\mathcal{G}_1 \to f^*\mathcal{G}_2 \to f^*\mathcal{G}_3$ is exact (see Modules, Lemma 17.3.1). The result mentioned in the introduction to this section proves the pullback by f^* gives a homotopy equivalence from the constant cosimplicial object $f^*\mathcal{F}$ to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$. By Simplicial, Lemma 14.28.7 we obtain that

$$f^*\mathcal{F}[0] \to \Big(f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots\Big)$$

is a homotopy equivalence. This immediately implies the two remaining statements of the lemma. □

0FKT **Lemma 20.30.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ where $\mathcal{G}^\bullet$ be a bounded below complex of flasque $\mathcal{O}_X$-modules and for all $x \in X$ the map $\mathcal{F}^\bullet_x \to \mathcal{G}^\bullet_x$ is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.*

Proof. Let $\mathcal{A}$ be the category of complexes of $\mathcal{O}_X$-modules and let $\mathcal{B}$ be the category of complexes of $\mathcal{O}_X$-modules. Then we can apply the discussion above to the adjoint functors f^* and f_* between $\mathcal{A}$ and $\mathcal{B}$. Arguing exactly as in the proof of Lemma 20.30.1 we get a resolution

$$0 \to \mathcal{F}^\bullet \to f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots$$

in the abelian category $\mathcal{A}$ such that each term of each $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map

$$\mathcal{F}^\bullet_x[0] \to \Big((f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to \ldots\Big)$$

is a homotopy equivalence in the category of complexes of complexes of $\mathcal{O}_{X,x}$-modules. Since a complex of complexes is the same thing as a double complex, we can consider the induced map

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet = \mathrm{Tot}(f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots)$$

Since the complex $\mathcal{F}^\bullet$ is bounded below, the same is true for $\mathcal{G}^\bullet$ and in fact each term of $\mathcal{G}^\bullet$ is a finite direct sum of terms of the complexes $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ and hence is flasque. The final assertion of the lemma now follows from Homology, Lemma 12.25.5. Since this in particular shows that $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism, the proof is complete. □

20.31. Cup product

0FKU Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. The (global) cup product in this setting is a map

$$\mu : R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M) \longrightarrow R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (pt, A)$ as in Remark 20.28.7 via $D(pt, A) = D(A)$. This map in particular defines pairings

$$\cup : H^i(X, K) \times H^j(X, M) \longrightarrow H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

Namely, given $\xi \in H^i(X, K) = H^i(R\Gamma(X, K))$ and $\eta \in H^j(X, M) = H^j(R\Gamma(X, M))$ we can first "tensor" them to get an element $\xi \otimes \eta$ in $H^{i+j}(R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M))$, see More on Algebra, Section 15.63. Then we can apply μ to get the desired element $\xi \cup \eta = \mu(\xi \otimes \eta)$ of $H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$.

Here is another way to think of the cup product of ξ and η. Namely, we can write

$$H^i(R\Gamma(X, K)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[-i], K) \quad \text{and} \quad H^j(R\Gamma(X, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[-j], M)$$

because $\mathrm{Hom}(\mathcal{O}_X, -) = \Gamma(X, -)$. Thus ξ and η are the "same" thing as maps

$$\tilde{\xi} : \mathcal{O}_X[-i] \to K \quad \text{and} \quad \tilde{\eta} : \mathcal{O}_X[-j] \to M$$

Combining this with the functoriality of the derived tensor product we obtain

$$\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X[-j] \xrightarrow{\tilde{\xi} \otimes \tilde{\eta}} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

which by the same token as above is an element of $H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$.

0FP2 **Lemma 20.31.1.** *This construction gives the cup product.*

Proof. With $f : (X, \mathcal{O}_X) \to (pt, A)$ as above we have $Rf_*(-) = R\Gamma(X, -)$ and our map μ is adjoint to the map

$$Lf^*(Rf_*K \otimes^{\mathbf{L}}_A Rf_*M) = Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*M \xrightarrow{\epsilon_K \otimes \epsilon_M} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

where ϵ is the counit of the adjunction between Lf^* and Rf_*. If we think of ξ and η as maps $\xi : A[-i] \to R\Gamma(X, K)$ and $\eta : A[-j] \to R\Gamma(X, M)$, then the tensor $\xi \otimes \eta$

corresponds to the map[3]

$$A[-i-j] = A[-i] \otimes^{\mathbf{L}}_A A[-j] \xrightarrow{\xi\otimes\eta} R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M)$$

By definition the cup product $\xi \cup \eta$ is the map $A[-i-j] \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$ which is adjoint to

$$(\epsilon_K \otimes \epsilon_M) \circ Lf^*(\xi \otimes \eta) = (\epsilon_K \circ Lf^*\xi) \otimes (\epsilon_M \circ Lf^*\eta)$$

However, it is easy to see that $\epsilon_K \circ Lf^*\xi = \tilde{\xi}$ and $\epsilon_M \circ Lf^*\eta = \tilde{\eta}$. We conclude that $\widetilde{\xi \cup \eta} = \tilde{\xi} \otimes \tilde{\eta}$ which means we have the desired agreement. □

0G6W **Remark 20.31.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. Given $\xi \in H^i(X, K)$ we get an associated map

$$\xi = \text{“}\xi \cup -\text{”} : R\Gamma(X, M)[-i] \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

by representing ξ as a map $\xi : A[-i] \to R\Gamma(X, K)$ as in the proof of Lemma 20.31.1 and then using the composition

$$R\Gamma(X, M)[-i] = A[-i] \otimes^{\mathbf{L}}_A R\Gamma(X, M) \xrightarrow{\xi\otimes 1} R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M) \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

where the second arrow is the global cup product μ above. On cohomology this recovers the cup product by ξ as is clear from Lemma 20.31.1 and its proof.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. There is a naive cup product

$$\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \longrightarrow f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

0FP3 **Lemma 20.31.3.** *In the situation above the following diagram commutes*

$$\begin{array}{ccc}
f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet & \longrightarrow & Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\mathcal{M}^\bullet \\
\downarrow & & \downarrow{\scriptstyle \text{Remark 20.28.7}} \\
Tot(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) & & Rf_*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
{\scriptstyle \text{naive cup product}}\downarrow & & \downarrow \\
f_*\,Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) & \longrightarrow & Rf_*\,Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{array}$$

Proof. By the construction in Remark 20.28.7 we see that going around the diagram clockwise the map

$$f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet \longrightarrow Rf_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

[3]There is a sign hidden here, namely, the equality is defined by the composition

$$A[-i-j] \to (A \otimes^{\mathbf{L}}_A A)[-i-j] \to A[-i] \otimes^{\mathbf{L}}_A A[-j]$$

where in the second step we use the identification of More on Algebra, Item (7) which uses a sign in principle. Except, in this case the sign is $+1$ by our convention and even if it wasn't $+1$ it wouldn't matter since we used the same sign in the identification $\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X[-j]$.

is adjoint to the map

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*f_*\mathcal{M}^\bullet \\ &\to Lf^*Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*Rf_*\mathcal{M}^\bullet \\ &\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{M}^\bullet \\ &\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

By Lemma 20.28.6 this is also equal to

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*f_*\mathcal{M}^\bullet \\ &\to f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f^*f_*\mathcal{M}^\bullet \\ &\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{M}^\bullet \\ &\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

Going around anti-clockwise we obtain the map adjoint to the map

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) &\to Lf^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \\ &\to Lf^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to Lf^*Rf_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

By Lemma 20.28.6 this is also equal to

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) &\to Lf^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \\ &\to Lf^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to f^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

Now the proof is finished by a contemplation of the diagram

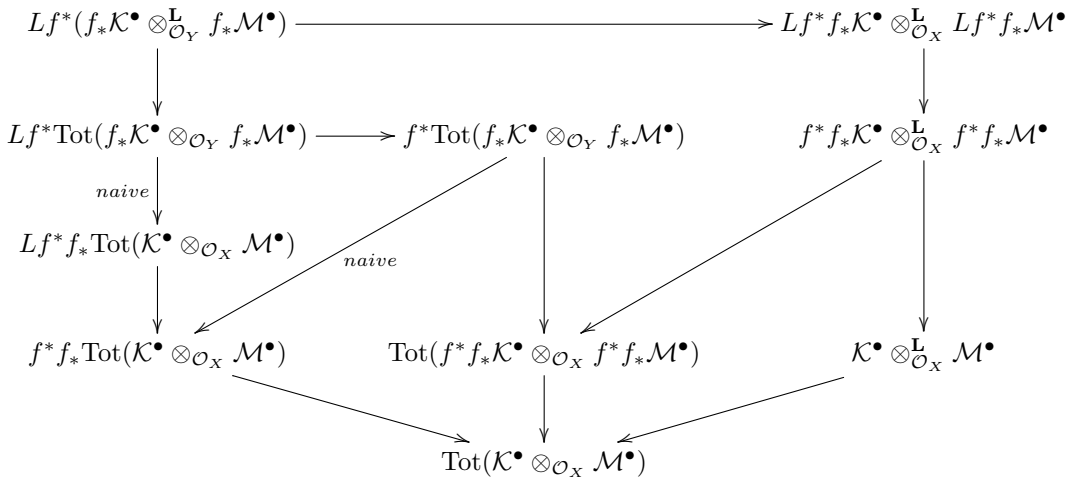

All of the polygons in this diagram commute. The top one commutes by Lemma 20.27.5. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \mathrm{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definition of the

naive cup product (by the adjointness of f^* and f_*). The proof is finished because going around the diagram on the outside are the two maps given above. $\square$

Let $(X, \mathcal{O}_X)$ be a ring space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. Then we have a "naive" cup product

$$\mu' : \mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) \longrightarrow \Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))$$

By Lemma 20.31.3 applied to the morphism $(X, \mathcal{O}_X) \to (pt, A)$ this naive cup product is related to the cup product μ defined in the first paragraph of this section by the following commutative diagram

$$\begin{array}{ccc}
\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A \Gamma(X, \mathcal{M}^\bullet) & \longrightarrow & R\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(X, \mathcal{M}^\bullet) \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) & & R\Gamma(X, \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
\downarrow{\scriptstyle \mu'} & & \downarrow \\
\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) & \longrightarrow & R\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{array}$$

in $D(A)$. On cohomology we obtain the commutative diagram

$$\begin{array}{ccc}
H^i(\Gamma(X, \mathcal{K}^\bullet)) \times H^j(\Gamma(X, \mathcal{M}^\bullet)) & \longrightarrow & H^{i+j}(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) \\
\downarrow & & \uparrow \\
H^i(X, \mathcal{K}^\bullet) \times H^j(X, \mathcal{M}^\bullet) & \xrightarrow{\cup} & H^{i+j}(X, \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{array}$$

relating the naive cup product with the actual cuproduct.

0FKV **Lemma 20.31.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be bounded below complexes of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering Then*

$$\begin{array}{ccc}
Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes^{\mathbf{L}}_A Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) & \longrightarrow & R\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(X, \mathcal{M}^\bullet) \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
Tot(\, Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) & & R\Gamma(X, \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
\downarrow{\scriptstyle (20.25.3.2)} & & \downarrow \\
Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))) & \longrightarrow & R\Gamma(X, \, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{array}$$

where the horizontal arrows are the ones in Lemma 20.25.1 commutes in $D(A)$.

Proof. Choose quasi-isomorphisms of complexes $a : \mathcal{K}^\bullet \to \mathcal{K}_1^\bullet$ and $b : \mathcal{M}^\bullet \to \mathcal{M}_1^\bullet$ as in Lemma 20.30.2. Since the maps a and b on stalks are homotopy equivalences we see that the induced map

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \to \mathrm{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet)$$

is a homotopy equivalence on stalks too (More on Algebra, Lemma 15.58.2) and hence a quasi-isomorphism. Thus the targets

$$R\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) = R\Gamma(X, \mathrm{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet))$$

of the two diagrams are the same in $D(A)$. It follows that it suffices to prove the diagram commutes for $\mathcal{K}$ and $\mathcal{M}$ replaced by $\mathcal{K}_1$ and $\mathcal{M}_1$. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ are bounded below complexes of flasque $\mathcal{O}_X$-modules and consider the diagram relating the cup product with the cup product (20.25.3.2) on Čech complexes. Then we can consider the commutative diagram

$$\begin{array}{ccc}
\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A \Gamma(X, \mathcal{M}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes^{\mathbf{L}}_A \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) & \longrightarrow & \mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) \\
\downarrow & & \downarrow (20.25.3.2) \\
\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)))
\end{array}$$

In this diagram the horizontal arrows are isomorphisms in $D(A)$ because for a bounded below complex of flasque modules such as $\mathcal{K}^\bullet$ we have

$$\Gamma(X, \mathcal{K}^\bullet) = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) = R\Gamma(X, \mathcal{K}^\bullet)$$

in $D(A)$. This follows from Lemma 20.12.3, Derived Categories, Lemma 13.16.7, and Lemma 20.25.2. Hence the commutativity of the diagram of the lemma involving (20.25.3.2) follows from the already proven commutativity of Lemma 20.31.3 where f is the morphism to a point (see discussion following Lemma 20.31.3). □

0FP4 **Lemma 20.31.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 20.28.7 is associative in the sense that the diagram*

$$\begin{array}{ccc}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \\
\downarrow & & \downarrow \\
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)
\end{array}$$

is commutative in $D(\mathcal{O}_Y)$ for all K, L, M in $D(\mathcal{O}_X)$.

Proof. Going around either side we obtain the map adjoint to the obvious map

$$\begin{aligned}
Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M) &= Lf^*(Rf_*K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(Rf_*L) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(Rf_*M) \\
&\to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M
\end{aligned}$$

in $D(\mathcal{O}_X)$. □

0FP5 **Lemma 20.31.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 20.28.7 is commutative in the sense that the diagram*

$$\begin{array}{ccc}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle Rf_*\psi} \\
Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*K & \longrightarrow & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K)
\end{array}$$

is commutative in $D(\mathcal{O}_Y)$ for all K, L in $D(\mathcal{O}_X)$. Here ψ is the commutativity constraint on the derived category (Lemma 20.50.6).

Proof. Omitted. □

0FP6 **Lemma 20.31.7.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. The relative cup product of Remark 20.28.7 is compatible with compositions in the sense that the diagram*

$$\begin{array}{ccccc} R(g\circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g\circ f)_*L & = & & = & Rg_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Rg_*Rf_*L \\ \downarrow & & & & \downarrow \\ R(g\circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) & = & Rg_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) & \leftarrow & Rg_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) \end{array}$$

is commutative in $D(\mathcal{O}_Z)$ for all K, L in $D(\mathcal{O}_X)$.

Proof. This is true because going around the diagram either way we obtain the map adjoint to the map

$$\begin{aligned} & L(g\circ f)^*\left(R(g\circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g\circ f)_*L\right) \\ & = L(g\circ f)^*R(g\circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L(g\circ f)^*R(g\circ f)_*L) \\ & \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \end{aligned}$$

in $D(\mathcal{O}_X)$. To see this one uses that the composition of the counits like so

$$L(g\circ f)^*R(g\circ f)_* = Lf^*Lg^*Rg_*Rf_* \to Lf^*Rf_* \to \mathrm{id}$$

is the counit for $L(g\circ f)^*$ and $R(g\circ f)_*$. See Categories, Lemma 4.24.9. □

0H99 **Lemma 20.31.8.** *Consider a commutative square*

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}) & \xrightarrow{g'} & (X, \mathcal{O}_X) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Y', \mathcal{O}_{Y'}) & \xrightarrow{g} & (Y, \mathcal{O}_Y) \end{array}$$

of ringed spaces. Let K, L in $D(\mathcal{O}_X)$. The relative cup product is compatible with the square in the sense that the diagram

$$\begin{array}{ccc} Lg^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) & \longrightarrow & Lg^*(Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)) \\ \| & & \downarrow \\ Lg^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*Rf_*L & & R(f')_*L(g')^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \\ \downarrow & & \| \\ R(f')_*(L(g')^*K \otimes_{\mathcal{O}_{Y'}} R(f')_*(L(g')^*L & \longrightarrow & R(f')_*(L(g')^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*L) \end{array}$$

is commutative in $D(\mathcal{O}_{Y'})$. The horizontal arrows are given by the relative cup product (Remark 20.28.7) and the vertical arrows are given by the base change map (Remark 20.28.3) and Lemma 20.27.3.

Proof. Omitted. □

20.32. Some properties of K-injective complexes

0D5U Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}_X) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O}_X)$ (see Sheaves, Section 6.31) is an exact functor (Modules, Lemma 17.3.4). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

08BS **Lemma 20.32.1.** *Let X be a ringed space. Let $U \subset X$ be an open subspace. The restriction of a K-injective complex of $\mathcal{O}_X$-modules to U is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows from Derived Categories, Lemma 13.31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. For the construction of $j_!$ see Sheaves, Section 6.31 and for exactness see Modules, Lemma 17.3.4. □

0D5V **Lemma 20.32.2.** *Let X be a ringed space. Let $U \subset X$ be an open subspace. For K in $D(\mathcal{O}_X)$ we have $H^p(U, K) = H^p(U, K|_U)$.*

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules representing K. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(U, \mathcal{I}^\bullet|_U))$$

by construction of cohomology. By Lemma 20.32.1 the complex $\mathcal{I}^\bullet|_U$ is a K-injective complex representing $K|_U$ and the lemma follows. □

0BKJ **Lemma 20.32.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. The sheafification of*

$$U \mapsto H^q(U, K) = H^q(U, K|_U)$$

is the qth cohomology sheaf $H^q(K)$ of K.

Proof. The equality $H^q(U, K) = H^q(U, K|_U)$ holds by Lemma 20.32.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Then

$$H^q(U, K) = \frac{\mathrm{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\mathrm{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \mathrm{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1})/\mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. □

08FE **Lemma 20.32.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Given an open subspace $V \subset Y$, set $U = f^{-1}(V)$ and denote $g : U \to V$ the induced morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Lemma 20.32.1. Since it is clear that $(f_*\mathcal{F})|_V = g_*(\mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X the result follows. □

0D5W **Lemma 20.32.5.** *Let $f : X \to Y$ be a morphism of ringed spaces. Then $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ as functors $D(\mathcal{O}_X) \to D(\Gamma(Y, \mathcal{O}_Y))$. More generally for $V \subset Y$ open and $U = f^{-1}(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.*

Proof. Let Z be the ringed space consisting of a singleton space with $\Gamma(Z, \mathcal{O}_Z) = \Gamma(Y, \mathcal{O}_Y)$. There is a canonical morphism $Y \to Z$ of ringed spaces inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_Z) = D(\Gamma(Y, \mathcal{O}_Y))$. Hence the assertion $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ follows from Lemma 20.28.2 applied to $X \to Y \to Z$.

The second (more general) statement follows from the first statement after applying Lemma 20.32.4. □

0D5X **Lemma 20.32.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let K be in $D(\mathcal{O}_X)$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$$

Proof. The equality $H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 20.32.5. Then the statement on sheafification follows from Lemma 20.32.3. □

0D5Y **Lemma 20.32.7.** *Let X be a ringed space. Let K be an object of $D(\mathcal{O}_X)$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_X)$.*

1. *For any open $U \subset X$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(Ab)$.*
2. *Let $f : X \to Y$ be a morphism of ringed spaces. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Y)$.*

Proof. The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (2) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 20.32.2 and 20.32.6 it suffices to show that the map

$$H^0(X, K) \longrightarrow H^0(X, K_{ab})$$

is an isomorphism. Observe that

$$H^0(X, K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing K. By construction of the derived category as a localization we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K) = \mathrm{colim}_{s:\mathcal{F}^\bullet \to \mathcal{O}_X} \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms s of complexes of $\mathcal{O}_X$-modules. Similarly, we have

$$\mathrm{Hom}_{D(\underline{\mathbf{Z}}_X)}(\underline{\mathbf{Z}}_X, K) = \mathrm{colim}_{s:\mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X} \mathrm{Hom}_{K(\underline{\mathbf{Z}}_X)}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules is cofinal in the system. (This follows from Modules, Lemma 17.17.7 and Derived Categories, Lemma 13.15.4; see discussion in Section

20.26.) Hence we can construct an inverse to the map $H^0(X, K) \longrightarrow H^0(X, K_{ab})$ by representing an element $\xi \in H^0(X, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{O}_X, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 20.26.13 and 20.26.9. We omit the detailed verification that this construction is indeed an inverse. □

08BT **Lemma 20.32.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The restriction functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X)$.*

Proof. This follows formally from the fact that $j_!$ and j^* are adjoint and exact (and hence $Lj_! = j_!$ and $Rj^* = j^*$ exist), see Derived Categories, Lemma 13.30.3. □

0D5Z **Lemma 20.32.9.** *Let $f : X \to Y$ be a flat morphism of ringed spaces. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $f_*\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_Y$-modules.*

Proof. This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_Y)}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Sheaves, Lemma 6.26.2 and the fact that f^* is exact as f is assumed to be flat. □

20.33. Unbounded Mayer-Vietoris

08BR There is a Mayer-Vietoris sequence for unbounded cohomology as well.

08BU **Lemma 20.33.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$j_{U\cap V!}E|_{U\cap V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U\cap V!}E|_{U\cap V}[1]$$

in $D(\mathcal{O}_X)$.

Proof. We have seen in Section 20.32 that the restriction functors and the extension by zero functors are computed by just applying the functors to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U\cap V!}\mathcal{E}^\bullet|_{U\cap V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

To see this sequence is exact one checks on stalks using Sheaves, Lemma 6.31.8 (computation omitted). □

08BV **Lemma 20.33.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. We have seen that $\mathcal{I}^\bullet|U$ is a K-injective complex as well (Lemma 20.32.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \cap V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\cap V,*}\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This sequence is exact because for any $W \subset X$ open and any n the sequence

$$0 \to \mathcal{I}^n(W) \to \mathcal{I}^n(W \cap U) \oplus \mathcal{I}^n(W \cap V) \to \mathcal{I}^n(W \cap U \cap V) \to 0$$

is exact (see proof of Lemma 20.8.2). □

08BW **Lemma 20.33.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\begin{array}{ccccc} \ldots & \longrightarrow & \mathrm{Ext}^{-1}(E_{U\cap V}, F_{U\cap V}) & & \\ & \searrow & & & \\ \mathrm{Hom}(E,F) & \longrightarrow & \mathrm{Hom}(E_U,F_U) \oplus \mathrm{Hom}(E_V,F_V) & \longrightarrow & \mathrm{Hom}(E_{U\cap V}, F_{U\cap V}) \end{array}$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s and* Ext*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 20.33.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_{U!}E|_U, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, F|_U)$$

by Lemma 20.32.8. □

08BX **Lemma 20.33.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$R\Gamma(X,E) \to R\Gamma(U,E) \oplus R\Gamma(V,E) \to R\Gamma(U \cap V, E) \to R\Gamma(X,E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X,E) \to H^n(U,E) \oplus H^0(V,E) \to H^n(U \cap V, E) \to H^{n+1}(X,E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then $R\Gamma(X,E)$ is computed by $\Gamma(X, \mathcal{I}^\bullet)$. Similarly for U, V, and $U \cap V$ by Lemma 20.32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

We have seen this is a short exact sequence in the proof of Lemma 20.8.2. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

08HZ **Lemma 20.33.5.** *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \cap V}) \to Rf_*E[1]$$

This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Lemma 20.32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

This is a short exact sequence of complexes by Lemma 20.8.3 and the fact that $R^1 f_*\mathcal{I} = 0$ for an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

08DF **Lemma 20.33.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $T \subset X$ be a closed subset contained in U.*

(1) *If E is an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T, then $E \to Rj_*(E|_U)$ is an isomorphism.*
(2) *If F is an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T, then $j_!F \to Rj_*F$ is an isomorphism.*

Proof. Let $V = X \setminus T$ and $W = U \cap V$. Note that $X = U \cup V$ is an open covering of X. Denote $j_W : W \to V$ the open immersion. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. By Lemma 20.32.4 we have $(Rj_*E|_U)|_V = Rj_{W,*}(E|_W) = 0$ because $E|_W = 0$ by our assumption. On the other hand, $Rj_*(E|_U)|_U = E|_U$. Thus (1) is clear. Let F be an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T. By Lemma 20.32.4 we have $(Rj_*F)|_V = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. We also have $(j_!F)|_V = j_{W!}(F|_W) = 0$ (the first equality is immediate from the definition of extension by zero). Since both $(Rj_*F)|_U = F$ and $(j_!F)|_U = F$ we see that (2) holds. □

0G6X **Lemma 20.33.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $A = \Gamma(X, \mathcal{O}_X)$. Suppose that $X = U \cup V$ is a union of two open subsets. For objects K and M of $D(\mathcal{O}_X)$*

we have a map of distinguished triangles

$$\begin{array}{ccc}
R\Gamma(X,K) \otimes^{\mathbf{L}}_A R\Gamma(X,M) & \longrightarrow & R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes^{\mathbf{L}}_A (R\Gamma(U,M) \oplus R\Gamma(V,M)) & \longrightarrow & R\Gamma(U, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \oplus R\Gamma(V, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes^{\mathbf{L}}_A R\Gamma(U \cap V,M) & \longrightarrow & R\Gamma(U \cap V, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes^{\mathbf{L}}_A R\Gamma(X,M)[1] & \longrightarrow & R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)[1]
\end{array}$$

where

(1) *the horizontal arrows are given by cup product,*
(2) *on the right hand side we have the distinguished triangle of Lemma 20.33.4 for* $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$*, and*
(3) *on the left hand side we have the exact functor* $R\Gamma(X,K) \otimes^{\mathbf{L}}_A -$ *applied to the distinguished triangle of Lemma 20.33.4 for* M*.*

Proof. Choose a K-flat complex $T^\bullet$ of flat A-modules representing $R\Gamma(X,K)$, see More on Algebra, Lemma 15.59.10. Denote $T^\bullet \otimes_A \mathcal{O}_X$ the pullback of $T^\bullet$ by the morphism of ringed spaces $(X, \mathcal{O}_X) \to (pt, A)$. There is a natural adjunction map $\epsilon : T^\bullet \otimes_A \mathcal{O}_X \to K$ in $D(\mathcal{O}_X)$. Observe that $T^\bullet \otimes_A \mathcal{O}_X$ is a K-flat complex of $\mathcal{O}_X$-modules with flat terms, see Lemma 20.26.8 and Modules, Lemma 17.20.2. By Lemma 20.26.17 we can find a morphism of complexes

$$T^\bullet \otimes_A \mathcal{O}_X \longrightarrow \mathcal{K}^\bullet$$

of $\mathcal{O}_X$-modules representing ϵ such that $\mathcal{K}^\bullet$ is a K-flat complex with flat terms. Namely, by the construction of $D(\mathcal{O}_X)$ we can first represent ϵ by some map of complexes $e : T^\bullet \otimes_A \mathcal{O}_X \to \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules representing ϵ and then we can apply the lemma to e. Choose a K-injective complex $\mathcal{I}^\bullet$ whose terms are injective $\mathcal{O}_X$-modules representing M. Finally, choose a quasi-isomorphism

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet) \longrightarrow \mathcal{J}^\bullet$$

into a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. Observe that source and target of this arrow represent $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ in $D(\mathcal{O}_X)$. At this point, for any open $W \subset X$ we obtain a map of complexes

$$\mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(W)) \to \mathrm{Tot}(\mathcal{K}^\bullet(W) \otimes_A \mathcal{I}^\bullet(W)) \to \mathcal{J}^\bullet(W)$$

of A-modules whose composition represents the map

$$R\Gamma(X,K) \otimes^{\mathbf{L}}_A R\Gamma(W,M) \longrightarrow R\Gamma(W, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(A)$. Clearly, these maps are compatible with restriction mappings. OK, so now we can consider the following commutative(!) diagram of complexes of A-modules

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(X)) & \longrightarrow & \mathcal{J}^\bullet(X) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A (\mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V))) & \longrightarrow & \mathcal{J}^\bullet(U) \oplus \mathcal{J}^\bullet(V) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(U \cap V)) & \longrightarrow & \mathcal{J}^\bullet(U \cap V) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By the proof of Lemma 20.8.2 the columns are exact sequences of complexes of A-modules (this also uses that $\mathrm{Tot}(T^\bullet \otimes_A -)$ transforms short exact sequences of complexes of A-modules into short exact sequences as the terms of $T^\bullet$ are flat A-modules). Since the distinguished triangles of Lemma 20.33.4 are the distinguished triangles associated to these short exact sequences of complexes, the desired result follows from the functoriality of "taking the associated distinguished triangle" discussed in Derived Categories, Section 13.12. □

20.34. Cohomology with support in a closed subset, II

0G6Y We continue the discussion started in Section 20.21.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. In this situation we can consider the functor $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X(X))$ given by $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$. See Modules, Definition 17.5.1 and Modules, Lemma 17.5.2. Using K-injective resolutions, see Section 20.28, we obtain the right derived functor

$$R\Gamma_Z(X, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$$

Given an object K in $D(\mathcal{O}_X)$ we denote $H^q_Z(X, K) = H^q(R\Gamma_Z(X, K))$ the cohomology module with support in Z. We will see later (Lemma 20.34.8) that this agrees with the construction in Section 20.21.

For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the *subsheaf of sections with support in* Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\} = \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$$

As discussed in Modules, Remark 17.13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on Z and we obtain a functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X|_Z)$$

We set $\mathcal{H}^q_Z(K) = H^q(R\mathcal{H}_Z(K))$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

0A3B **Lemma 20.34.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.*

(1) *$R\mathcal{H}_Z : D(\mathcal{O}_X) \to D(\mathcal{O}_X|_Z)$ is right adjoint to $i_* : D(\mathcal{O}_X|_Z) \to D(\mathcal{O}_X)$.*
(2) *For K in $D(\mathcal{O}_X|_Z)$ we have $R\mathcal{H}_Z(i_*K) = K$.*
(3) *Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X|_Z$-modules on Z. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. The functor i_* is exact, so $i_* = Ri_* = Li_*$. Hence part (1) of the lemma follows from Modules, Lemma 17.13.6 and Derived Categories, Lemma 13.30.3. Let K be as in (2). We can represent K by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X|_Z$-modules. By Lemma 20.32.9 the complex $i_*\mathcal{I}^\bullet$, which represents i_*K, is a K-injective complex of $\mathcal{O}_X$-modules. Thus $R\mathcal{H}_Z(i_*K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ which proves (2). Part (3) is a special case of (2). □

Let $(X, \mathcal{O}_X)$ be a ringed space and let $Z \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in Z is a Serre subcategory of the category of all $\mathcal{O}_X$-modules, see Homology, Definition 12.10.1 and Modules, Lemma 17.5.2. We denote $D_Z(\mathcal{O}_X)$ the strictly full saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of complexes whose cohomology sheaves are supported on Z, see Derived Categories, Section 13.17.

0AEF **Lemma 20.34.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.*

(1) *For K in $D(\mathcal{O}_X|_Z)$ we have i_*K in $D_Z(\mathcal{O}_X)$.*
(2) *The functor $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence with quasi-inverse $i^{-1}|_{D_Z(\mathcal{O}_X)} = R\mathcal{H}_Z|_{D_Z(\mathcal{O}_X)}$.*
(3) *The functor $i_* \circ R\mathcal{H}_Z : D(\mathcal{O}_X) \to D_Z(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$.*

Proof. Part (1) is immediate from the definitions. Part (3) is a formal consequence of part (2) and Lemma 20.34.1. In the rest of the proof we prove part (2).

Let us think of i as the morphism of ringed spaces $i : (Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$. Recall that i^* and i_* is an adjoint pair of functors. Since i is a closed immersion, i_* is exact. Since $i^{-1}\mathcal{O}_X = \mathcal{O}_X|_Z$ is the structure sheaf of $(Z, \mathcal{O}_X|_Z)$ we see that $i^* = i^{-1}$ is exact and we see that that $i^*i_* = i^{-1}i_*$ is isomorphic to the identify functor. See Modules, Lemmas 17.3.3 and 17.6.1. Thus $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is fully faithful and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(\mathcal{O}_X)$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence.

To finish the proof it suffices to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(\mathcal{O}_X)$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then Lemma 20.34.1 tells us what we want. □

0G6Z **Lemma 20.34.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}_Z(\mathcal{I}^\bullet)$ is K-injective complex of $\mathcal{O}_X|_Z$-modules.*

Proof. Since $i_* : Mod(\mathcal{O}_X|_Z) \to Mod(\mathcal{O}_X)$ is exact and left adjoint to $\mathcal{H}_Z$ (Modules, Lemma 17.13.6) this follows from Derived Categories, Lemma 13.31.9. □

0G70 **Lemma 20.34.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Then $R\Gamma(Z, -) \circ R\mathcal{H}_Z = R\Gamma_Z(X, -)$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$.*

Proof. Follows from the construction of right derived functors using K-injective resolutions, Lemma 20.34.3, and the fact that $\Gamma_Z(X, -) = \Gamma(Z, -) \circ \mathcal{H}_Z$. □

0G71 **Lemma 20.34.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $U = X \setminus Z$. There is a distinguished triangle*

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(\mathcal{O}_X(X))$ functorial for K in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing K. See Section 20.28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 20.32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$. Hence we find that it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to \Gamma_Z(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}) \to \Gamma(U, \mathcal{I}) \to 0$$

This follows from Lemma 20.8.1 and the definitions. □

0G72 **Lemma 20.34.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Denote $j : U = X \setminus Z \to X$ the inclusion of the complement. There is a distinguished triangle*

$$i_* R\mathcal{H}_Z(K) \to K \to Rj_*(K|_U) \to i_* R\mathcal{H}_Z(K)[1]$$

in $D(\mathcal{O}_X)$ functorial for K in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing K. See Section 20.28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 20.32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$. Hence it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to i_* \mathcal{H}_Z(\mathcal{I}) \to \mathcal{I} \to j_*(\mathcal{I}|_U) \to 0$$

This follows from Lemma 20.8.1 and the definitions. □

0G73 **Lemma 20.34.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let $j : U \to X$ be the inclusion of an open subset with $U \cap Z = \emptyset$. Then $R\mathcal{H}_Z(Rj_*K) = 0$ for all K in $D(\mathcal{O}_U)$.*

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_U$-modules representing K. Then $j_*\mathcal{I}^\bullet$ represents Rj_*K. By Lemma 20.32.9 the complex $j_*\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules. Hence $\mathcal{H}_Z(j_*\mathcal{I}^\bullet)$ represents $R\mathcal{H}_Z(Rj_*K)$. Thus it suffices to show that $\mathcal{H}_Z(j_*\mathcal{G}) = 0$ for any abelian sheaf $\mathcal{G}$ on U. Thus we have to show that a section s of $j_*\mathcal{G}$ over some open W which is supported on $W \cap Z$ is zero. The support condition means that $s|_{W \setminus W \cap Z} = 0$. Since $j_*\mathcal{G}(W) = \mathcal{G}(U \cap W) = j_*\mathcal{G}(W \setminus W \cap Z)$ this implies that s is zero as desired. □

0G74 **Lemma 20.34.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let K be an object of $D(\mathcal{O}_X)$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_X)$.*

(1) *There is a canonical map $R\Gamma_Z(X, K) \to R\Gamma_Z(X, K_{ab})$ which is an isomorphism in $D(Ab)$.*
(2) *There is a canonical map $R\mathcal{H}_Z(K) \to R\mathcal{H}_Z(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Z)$.*

Proof. Proof of (1). The map is constructed as follows. Choose a K-injective complex of $\mathcal{O}_X$-modules $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by

$$\Gamma_Z(X, \mathcal{I}^\bullet) \to \Gamma_Z(X, \mathcal{J}^\bullet)$$

determined by the fact that Γ_Z is a functor on abelian sheaves. An easy check shows that the resulting map combined with the canonical maps of Lemma 20.32.7 fit into a morphism of distinguished triangles

$$\begin{array}{ccccc} R\Gamma_Z(X, K) & \longrightarrow & R\Gamma(X, K) & \longrightarrow & R\Gamma(U, K) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_Z(X, K_{ab}) & \longrightarrow & R\Gamma(X, K_{ab}) & \longrightarrow & R\Gamma(U, K_{ab}) \end{array}$$

of Lemma 20.34.5. Since two of the three arrows are isomorphisms by the lemma cited, we conclude by Derived Categories, Lemma 13.4.3.

The proof of (2) is omitted. Hint: use the same argument with Lemma 20.34.6 for the distinguished triangle. □

0G75 **Remark 20.34.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Given K and M in $D(\mathcal{O}_X)$ there is a canonical map

$$K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M) \longrightarrow R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(\mathcal{O}_X|_Z)$. Here $K|_Z = i^{-1}K$ is the restriction of K to Z viewed as an object of $D(\mathcal{O}_X|_Z)$. By adjointness of i_* and $R\mathcal{H}_Z$ of Lemma 20.34.1 to construct this map it suffices to produce a canonical map

$$i_*\left(K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)\right) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

To construct this map, we choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing M and a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing K. Observe that $\mathcal{K}^\bullet|_Z$ is a K-flat complex of $\mathcal{O}_X|_Z$-modules representing $K|_Z$, see Lemma 20.26.8. Hence we need to produce a map of complexes

$$i_*\mathrm{Tot}\left(\mathcal{K}^\bullet|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^\bullet)\right) \longrightarrow \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)$$

of $\mathcal{O}_X$-modules. For this it suffices to produce maps

$$i_*(\mathcal{K}^a|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^b)) \longrightarrow \mathcal{K}^a \otimes_{\mathcal{O}_X} \mathcal{I}^b$$

Looking at stalks (for example), we see that the left hand side of this formula is equal to $\mathcal{K}^a \otimes_{\mathcal{O}_X} i_*\mathcal{H}_Z(\mathcal{I}^b)$ and we can use the inclusion $\mathcal{H}_Z(\mathcal{I}^b) \to \mathcal{I}^b$ to get our map.

0G76 **Remark 20.34.10.** With notation as in Remark 20.34.9 we obtain a canonical cup product

$$\begin{aligned} H^a(X, K) \times H^b_Z(X, M) &= H^a(X, K) \times H^b(Z, R\mathcal{H}_Z(M)) \\ &\to H^a(Z, K|_Z) \times H^b(Z, R\mathcal{H}_Z(M)) \\ &\to H^{a+b}(Z, K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)) \\ &\to H^{a+b}(Z, R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)) \\ &= H^{a+b}_Z(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \end{aligned}$$

Here the equal signs are given by Lemma 20.34.4, the first arrow is restriction to Z, the second arrow is the cup product (Section 20.31), and the third arrow is the map from Remark 20.34.9.

0G77 **Lemma 20.34.11.** *With notation as in Remark 20.34.9 the diagram*

$$\begin{array}{ccc} H^i(X, K) \times H^j_Z(X, M) & \longrightarrow & H^{i+j}_Z(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \\ \downarrow & & \downarrow \\ H^i(X, K) \times H^j(X, M) & \longrightarrow & H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \end{array}$$

commutes where the top horizontal arrow is the cup product of Remark 20.34.10.

Proof. Omitted. □

0G78 **Remark 20.34.12.** Let $f : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Let $Z \subset X$ be a closed subset and $Z' = f^{-1}(Z)$. Denote $f|_{Z'} : (Z', \mathcal{O}_{X'}|_{Z'}) \to (Z, \mathcal{O}_X|Z)$ be the induced morphism of ringed spaces. For any K in $D(\mathcal{O}_X)$ there is a canonical map

$$L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow R\mathcal{H}_{Z'}(Lf^*K)$$

in $D(\mathcal{O}_{X'}|_{Z'})$. Denote $i : Z \to X$ and $i' : Z' \to X'$ the inclusion maps. By Lemma 20.34.2 part (2) applied to i' it is the same thing to give a map

$$i'_* L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow i'_* R\mathcal{H}_{Z'}(Lf^*K)$$

in $D_{Z'}(\mathcal{O}_{X'})$. The map of functors $Lf^* \circ i_* \to i'_* \circ L(f|_{Z'})^*$ of Remark 20.28.3 is an isomorphism in this case (follows by checking what happens on stalks using that i_* and i'_* are exact and that $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}$ and similarly for Z'). Hence it suffices to construct a the top horizontal arrow in the following diagram

$$\begin{array}{ccc} Lf^* i_* R\mathcal{H}_Z(K) & \longrightarrow & i'_* R\mathcal{H}_{Z'}(Lf^*K) \\ & \searrow \quad \swarrow & \\ & Lf^*K & \end{array}$$

The complex $Lf^* i_* R\mathcal{H}_Z(K)$ is supported on Z'. The south-east arrow comes from the adjunction mapping $i_* R\mathcal{H}_Z(K) \to K$ (Lemma 20.34.1). Since the adjunction mapping $i'_* R\mathcal{H}_{Z'}(Lf^*K) \to Lf^*K$ is universal by Lemma 20.34.2 part (3), we find that the south-east arrow factors uniquely over the south-west arrow and we obtain the desired arrow.

0G79 **Lemma 20.34.13.** *With notation and assumptions as in Remark 20.34.12 the diagram*

$$\begin{array}{ccc} H^p_Z(X, K) & \longrightarrow & H^p_{Z'}(X', Lf^*K) \\ \downarrow & & \downarrow \\ H^p(X, K) & \longrightarrow & H^p(X', Lf^*K) \end{array}$$

commutes. Here the top horizontal arrow comes from the identifications $H^p_Z(X, K) = H^p(Z, R\mathcal{H}_Z(K))$ and $H^p_{Z'}(X', Lf^*K) = H^p(Z', R\mathcal{H}_{Z'}(K'))$, the pullback map $H^p(Z, R\mathcal{H}_Z(K)) \to H^p(Z', L(f|_{Z'})^* R\mathcal{H}_Z(K))$, and the map constructed in Remark 20.34.12.

Proof. Omitted. Hints: Using that $H^p(Z, R\mathcal{H}_Z(K)) = H^p(X, i_* R\mathcal{H}_Z(K))$ and similarly for $R\mathcal{H}_{Z'}(Lf^*K)$ this follows from the functoriality of the pullback maps and the commutative diagram used to define the map of Remark 20.34.12. □

20.35. Inverse systems and cohomology, I

0GYJ Let A be a ring and let $I \subset A$ be an ideal. We prove some results on inverse systems of sheaves of A/I^n-modules.

0GYK **Lemma 20.35.1.** *Let I be an ideal of a ring A. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^{p+1}(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[4]*.*

Proof. Set $N_n = H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$. Then $\bigoplus \mathrm{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & I^{n+m}\mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m} & \longrightarrow & 0 \end{array}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \mathrm{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j \delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_j s_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_j s_j$ in M_n. It follows that

$$\mathrm{Im}(M_{n+1} \to M_{n-c}) = \mathrm{Im}(M_n \to M_{n-c})$$

Namely, the elements $f_j s_j$ map to zero in M_{n-c}. This proves the lemma. □

[4] In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

0GYL **Lemma 20.35.2.** *Let I be an ideal of a ring A. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^{p+1}(X, I^n\mathcal{F}_{m+1}) \to H^{p+1}(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[5].

Proof. The proof is exactly the same as the proof of Lemma 20.35.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(X, I^n\mathcal{F}_{n+1})$ have image contained in N_n.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(X, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I^{n+k}\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{n+k} & \longrightarrow & 0
\end{array}$$

constructed as in the proof of Lemma 20.35.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(X, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in N_{n+k} as desired.

To see the boundary maps δ_n have image contained in N_n we consider the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0
\end{array}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. □

0GYM **Lemma 20.35.3.** *Let I be an ideal of a ring A. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^p(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \mathrm{Ker}(M \to H^p(X, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$. Hence the I-adic topology is finer than the limit topology. For the converse, we will show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$[6]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(X, \mathcal{F}_{n+1})$$

[5] In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

[6] In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^nM$ for all n.

whose image is contained in the image of $H^p(X, I^n\mathcal{F}_{n+1}) \to H^p(X, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(X, I^n\mathcal{F}_{n+1})$$

the inverse image of F^n/F^{n+1}. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(X, I^n\mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^nF^c \subset I^nM$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_i a_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j}h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j}a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_i x_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 15.36.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

0GYN **Lemma 20.35.4.** *Let I be an ideal of a ring A. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. The proof is exactly the same as the proof of Lemma 20.35.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(X, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(X, \mathcal{F}_{n+1})$. In the proof of Lemma 20.35.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(X, I^n\mathcal{F}_{n+1})$ which maps to the image of t in $H^p(X, \mathcal{F}_{n+1})$. We have to show that s is in N_n. Now F^n is the kernel of the map from $M \to H^p(X, \mathcal{F}_n)$ hence for all $m \geq n$ we can map t to an element $t_m \in H^p(X, \mathcal{F}_{m+1})$ which maps to zero in $H^p(X, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n\mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(X, I^n\mathcal{F}_{m+1})$ mapping to t_m. Comparing the sequence above with the one for $m = n$ we see that s_m maps to s up to an element in the image

of $H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n\mathcal{F}_{n+1})$. However, this map factors through the map $H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, I^n\mathcal{F}_{n+1})$ and we see that s is in the image as desired. □

20.36. Inverse systems and cohomology, II

0H38 This section continues the discussion in Section 20.35 in the setting where the ideal is principal.

0H39 **Lemma 20.36.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be inverse system of $\mathcal{O}_X$-modules. Consider the conditions

1. *for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,*
2. *for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$*
3. *there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is f-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$, and*
4. *there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is f-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$.*

Then (4) ⇒ (3) ⇔ (2) ⇔ (1).

Proof. We omit the proof of the equivalence of (1) and (2). We omit the proof that (3) implies (1). Given $\mathcal{F}_n$ as in (1) to prove (3) we set $\mathcal{G} = \operatorname{colim} \mathcal{F}_n$ where the maps $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \ldots$ are as in (1). The map $f : \mathcal{G} \to \mathcal{G}$ is surjective as the image of $\mathcal{F}_{n+1} \subset \mathcal{G}$ is $\mathcal{F}_n \subset \mathcal{G}$ by the short exact sequence (1). Thus $\mathcal{G}$ is an f-divisible $\mathcal{O}_X$-module with $\mathcal{F}_n = \mathcal{G}[f^n]$.

Assume given $\mathcal{F}$ as in (4). The map $\mathcal{F}/f^{n+1}\mathcal{F} \to \mathcal{F}/f^n\mathcal{F}$ is always surjective with kernel the image of the map $\mathcal{F}/f\mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ induced by multiplication with f^n. To verify (2) it suffices to see that the kernel of $f^n : \mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ is $f\mathcal{F}$. To see this it suffices to show that given sections s, t of $\mathcal{F}$ over an open $U \subset X$ with $f^n s = f^{n+1}t$ we have $s = ft$. This is clear because $f : \mathcal{F} \to \mathcal{F}$ is injective as $\mathcal{F}$ is f-torsion free. □

0EHA **Lemma 20.36.2.** *Suppose X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1. Let $p \geq 0$ and set $H^p = \lim H^p(X, \mathcal{F}_n)$. Then $f^c H^p$ is the kernel of $H^p \to H^p(X, \mathcal{F}_c)$ for all $c \geq 1$. Thus the limit topology on H^p is the f-adic topology.*

Proof. Let $c \geq 1$. It is clear that $f^c H^p$ maps to zero in $H^p(X, \mathcal{F}_c)$. If $\xi = (\xi_n) \in H^p$ is small in the limit topology, then $\xi_c = 0$, and hence ξ_n maps to zero in $H^p(X, \mathcal{F}_c)$ for $n \geq c$. Consider the inverse system of short exact sequences

$$0 \to \mathcal{F}_{n-c} \xrightarrow{f^c} \mathcal{F}_n \to \mathcal{F}_c \to 0$$

and the corresponding inverse system of long exact cohomology sequences

$$H^{p-1}(X, \mathcal{F}_c) \to H^p(X, \mathcal{F}_{n-c}) \to H^p(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}_c)$$

Since the term $H^{p-1}(X, \mathcal{F}_c)$ is independent of n we can choose a compatible sequence of elements $\xi'_n \in H^p(X, \mathcal{F}_{n-c})$ lifting ξ_n. Setting $\xi' = (\xi'_n)$ we see that $\xi = f^c\xi'$ as desired. □

0BLB **Lemma 20.36.3.** *Let A be a Noetherian ring complete with respect to a principal ideal (f). Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Assume

(1) *$\Gamma(X, \mathcal{F}_1)$ is a finite A-module,*
(2) *X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.*

Then

$$M = \lim \Gamma(X, \mathcal{F}_n)$$

is a finite A-module, f is a nonzerodivisor on M, and M/fM is the image of M in $\Gamma(X, \mathcal{F}_1)$.

Proof. By Lemma 20.36.2 we have $M/fM \subset H^0(X, \mathcal{F}_1)$. From (1) and the Noetherian property of A we get that M/fM is a finite A-module. Observe that $\bigcap f^n M = 0$ as $f^n M$ maps to zero in $H^0(X, \mathcal{F}_n)$. By Algebra, Lemma 10.96.12 we conclude that M is finite over A. Finally, suppose $s = (s_n) \in M = \lim H^0(X, \mathcal{F}_n)$ satisfies $fs = 0$. Then s_{n+1} is in the kernel of $\mathcal{F}_{n+1} \to \mathcal{F}_n$ by condition (1) of Lemma 20.36.1. Hence $s_n = 0$. Since n was arbitrary, we see $s = 0$. Thus f is a nonzerodivisor on M. □

0BLC **Lemma 20.36.4.** *Let A be a ring. Let $f \in A$. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume

(1) *either $H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and $H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,*
(2) *X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.*

Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 20.35.1. Observe that $f^n : \mathcal{F}_1 \to I^n \mathcal{F}_{n+1}$ is an isomorphism for all $n \geq 0$. Thus it suffices to show that

$$\bigoplus\nolimits_{n \geq 1} H^{p+1}(X, \mathcal{F}_1) \cdot f^{n+1}$$

is a graded $S = \bigoplus_{n \geq 0} A/(f) \cdot f^n$-module satisfying the ascending chain condition. If A is not Noetherian, then $H^{p+1}(X, \mathcal{F}_1)$ has finite length and the result holds. If A is Noetherian, then S is a Noetherian ring and the result holds as the module is finite over S by the assumed finiteness of $H^{p+1}(X, \mathcal{F}_1)$. Some details omitted. □

0DXG **Lemma 20.36.5.** *Let A be a ring. Let $f \in A$. Let X be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume

(1) *either there is an $m \geq 1$ such that the image of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and the intersection of the images of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,*
(2) *X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.*

Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 20.35.2 involving the modules N_n. For $m \geq n$ we have $I^n \mathcal{F}_{m+1} = \mathcal{F}_{m+1-n}$. Thus we see that

$$N_n = \bigcap\nolimits_{m \geq 1} \mathrm{Im}\left(H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)\right)$$

is independent of n and $\bigoplus N_n = \bigoplus N_1 \cdot f^{n+1}$. Thus we conclude exactly as in the proof of Lemma 20.36.4. □

0H3A **Remark 20.36.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be $\mathcal{O}_X$-module. If $\mathcal{F}$ is f-torsion free, then for every $p \geq 0$ we have a short exact sequence of inverse systems

$$0 \to \{H^p(X, \mathcal{F})/f^n H^p(X, \mathcal{F})\} \to \{H^p(X, \mathcal{F}/f^n\mathcal{F})\} \to \{H^{p+1}(X, \mathcal{F})[f^n]\} \to 0$$

Since the first inverse system has the Mittag-Leffler condition (ML) we learn three things from this:

(1) There is a short exact sequence

$$0 \to \widehat{H^p(X, \mathcal{F})} \to \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \to T_f(H^{p+1}(X, \mathcal{F})) \to 0$$

where $\widehat{\ }$ denotes the usual f-adic completion and $T_f(-)$ denotes the f-adic Tate module from More on Algebra, Example 15.94.5.
(2) We have $R^1 \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) = R^1 \lim H^{p+1}(X, \mathcal{F})[f^n]$.
(3) The system $\{H^{p+1}(X, \mathcal{F})[f^n]\}$ is ML if and only if $\{H^p(X, \mathcal{F}/f^n\mathcal{F})\}$ is ML.

See Homology, Lemma 12.31.3 and More on Algebra, Lemmas 15.87.2 and 15.87.13.

20.37. Derived limits

0BKN Let $(X, \mathcal{O}_X)$ be a ringed space. Since the triangulated category $D(\mathcal{O}_X)$ has products (Injectives, Lemma 19.13.4) it follows that $D(\mathcal{O}_X)$ has derived limits, see Derived Categories, Definition 13.34.1. If (K_n) is an inverse system in $D(\mathcal{O}_X)$ then we denote $R\lim K_n$ the derived limit.

0D60 **Lemma 20.37.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For $U \subset X$ open the functor $R\Gamma(U, -)$ commutes with $R\lim$. Moreover, there are short exact sequences*

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

for any inverse system (K_n) in $D(\mathcal{O}_X)$ and any $m \in \mathbf{Z}$.

Proof. The first statement follows from Injectives, Lemma 19.13.6. Then we may apply More on Algebra, Remark 15.87.10 to $R\lim R\Gamma(U, K_n) = R\Gamma(U, R\lim K_n)$ to get the short exact sequences. □

0BKP **Lemma 20.37.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.*

Proof. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O}_X)$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}_Y)$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 19.13.4). However, since Rf_* is a right adjoint by Lemma 20.28.1 this follows formally (see Categories, Lemma 4.24.5). Caution: Note that we cannot apply Categories, Lemma 4.24.5 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O}_X)$. $\square$

0BKQ **Remark 20.37.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Set $K = R\lim K_n$. For each n and m let $\mathcal{H}^m_n = H^m(K_n)$ be the mth cohomology sheaf of K_n and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}^m_n$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}^m_n(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 20.32.3 we see that $\mathcal{H}^m_n$ is the sheafification of $\underline{\mathcal{H}}^m_n$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$\begin{array}{ccc} K & & \underline{\mathcal{H}}^m \longrightarrow \mathcal{H}^m \\ \| & & \downarrow \quad\quad \downarrow \\ R\lim K_n & & \lim \underline{\mathcal{H}}^m_n \longrightarrow \lim \mathcal{H}^m_n \end{array}$$

In general it may not be the case that $\lim \mathcal{H}^m_n$ is the sheafification of $\lim \underline{\mathcal{H}}^m_n$. If $U \subset X$ is an open, then we have short exact sequences

0BKR (20.37.3.1) $$0 \to R^1 \lim \underline{\mathcal{H}}^{m-1}_n(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}^m_n(U) \to 0$$

by Lemma 20.37.1.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on a scheme.

0BKS **Lemma 20.37.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of opens of X. Assume*

1. *every open of X has a covering whose members are elements of $\mathcal{B}$,*
2. *$H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,*
3. *the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1\lim$ for $U \in \mathcal{B}$.*

Then $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.

Proof. Set $K_n = \mathcal{F}_n$ and $K = R\lim \mathcal{F}_n$. Using the notation of Remark 20.37.3 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}^m_n(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}^0_n(U) = \mathcal{F}_n(U)$. From Equation (20.37.3.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. $\square$

0D61 **Lemma 20.37.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Let $x \in X$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(x)$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that for $U \in \mathfrak{U}_x$*

1. *$R^1 \lim H^{m-1}(U, K_n) = 0$, and*
2. *$H^m(U, K_n) \to H^m(U, K_{n(x)})$ is injective for $n \geq n(x)$.*

Then the map on stalks $H^m(R\lim K_n)_x \to H^m(K_{n(x)})_x$ is injective.

Proof. Let γ be an element of $H^m(R\lim K_n)_x$ which maps to zero in $H^m(K_{n(x)})_x$. Since $H^m(R\lim K_n)$ is the sheafification of $U \mapsto H^m(U, R\lim K_n)$ (by Lemma 20.32.3) we can choose $U \in \mathfrak{U}_x$ and an element $\tilde{\gamma} \in H^m(U, R\lim K_n)$ mapping to γ. Then $\tilde{\gamma}$ maps to $\tilde{\gamma}_{n(x)} \in H^m(U, K_{n(x)})$. Using that $H^m(K_{n(x)})$ is the sheafification of $U \mapsto H^m(U, K_{n(x)})$ (by Lemma 20.32.3 again) we see that after shrinking U we may assume that $\tilde{\gamma}_{n(x)} = 0$. For this U we consider the short exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

of Lemma 20.37.1. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(U, K_{n(x)})$. We conclude $\tilde{\gamma} = 0$ and hence $\gamma = 0$ as desired. □

0D62 **Lemma 20.37.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist a function $p(x, -) : \mathbf{Z} \to \mathbf{Z}$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that*

$$H^p(U, H^{m-p}(E)) = 0 \text{ for } U \in \mathfrak{U}_x \text{ and } p > p(x, m)$$

Then the map $E \to R\lim \tau_{\geq -n} E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Set $K_n = \tau_{\geq -n} E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n} E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix m. If $n \geq -m$, then the map $E \to \tau_{\geq -n} E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $x \in X$ there exists an integer $n(x) \geq -m$ such that the map $H^m(K)_x \to H^m(K_{n(x)})_x$ is injective. Namely, then the composition

$$H^m(E)_x \to H^m(K)_x \to H^m(K_{n(x)})_x$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. Set

$$n(x) = 1 + \max\{-m, p(x, m-1) - m, -1 + p(x, m) - m, -2 + p(x, m+1) - m\}.$$

so that in any case $n(x) \geq -m$. Claim: the maps

$$H^{m-1}(U, K_{n+1}) \to H^{m-1}(U, K_n) \quad \text{and} \quad H^m(U, K_{n+1}) \to H^m(U, K_n)$$

are isomorphisms for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. The claim implies conditions (1) and (2) of Lemma 20.37.5 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 13.12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n+2]$$

Looking at the associated long exact cohomology sequence the claim follows if

$$H^{m+n}(U, H^{-n-1}(E)), \quad H^{m+n+1}(U, H^{-n-1}(E)), \quad H^{m+n+2}(U, H^{-n-1}(E))$$

are zero for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. This follows from our choice of $n(x)$ and the assumption in the lemma. □

0D63 **Lemma 20.37.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist an integer $d_x \geq 0$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that* [**Spa88**, Proposition 3.13]

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathfrak{U}_x,\ p > d_x,\ \text{and } q < 0$$

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O}_X)$.

Proof. This follows from Lemma 20.37.6 with $p(x, m) = d_x + \max(0, m)$. □

08U2 **Lemma 20.37.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a set $\mathcal{B}$ of opens of X such that*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$, and*
(2) *$H^p(U, H^{m-p}(E)) = 0$ for $p > p(m)$ and $U \in \mathcal{B}$.*

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Apply Lemma 20.37.6 with $p(x, m) = p(m)$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$. □

0D64 **Lemma 20.37.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist an integer $d \geq 0$ and a basis $\mathcal{B}$ for the topology of X such that*

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathcal{B},\ p > d,\ \text{and } q < 0$$

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Apply Lemma 20.37.7 with $d_x = d$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$. □

The lemmas above can be used to compute cohomology in certain situations.

0BKT **Lemma 20.37.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of X. Assume*

(1) *every open of X has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.*

Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.

Proof. Observe that $K = R\lim \tau_{\geq -n}K$ by Lemma 20.37.9 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (20.37.3.1) we get a short exact sequence

$$0 \to R^1 \lim H^{q-1}(U, \tau_{\geq -n}K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n}K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n}K) = H^0(U, H^q(\tau_{\geq -n}K))$ for all q by using the spectral sequence of Example 20.29.3. The spectral sequence converges because $\tau_{\geq -n}K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n}K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$. The lemma follows. □

Here is another case where we can describe the derived limit.

0BKU **Lemma 20.37.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system of objects of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of X. Assume*

(1) *every open of X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have*
   (a) *$H^p(U, H^q(K_n)) = 0$ for $p > 0$,*
   (b) *the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1 \lim$.*

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. We will use notation as in Remark 20.37.3. Let $U \in \mathcal{B}$. By Lemma 20.37.10 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 15.87.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma. □

20.38. Producing K-injective resolutions

0719 Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $Mod(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 13.29.3 produce a diagram

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet \end{array}$$

in the category of complexes of $\mathcal{O}_X$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.31.4 and 13.31.8 this is a K-injective complex. In general the canonical map

071A (20.38.0.1) $$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

071B **Lemma 20.38.1.** *In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of X. Let $d \in \mathbf{N}$. Assume*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$ and $q < 0$[7].*

Then (20.38.0.1) is a quasi-isomorphism.

Proof. By Derived Categories, Lemma 13.34.5 it suffices to show that the map $\mathcal{F}^\bullet \to R\lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This is Lemma 20.37.9. □

Here is a technical lemma about the cohomology sheaves of the inverse limit of a system of complexes of sheaves. In some sense this lemma is the wrong thing to try to prove as one should take derived limits and not actual inverse limits.

08BY **Lemma 20.38.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$. Assume there exist a set $\mathcal{B}$ of open subsets of X and an integer n_0 such that*

[7] It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 20.37.8.

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$*
   (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler condition),*
   (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler condition), and*
   (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 15.87.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

20.39. Inverse systems and cohomology, III

0H3B This section continues the discussion in Section 20.36 using derived limits.

0H3C **Lemma 20.39.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let E be an object of $D(\mathcal{O}_X)$. Denote*

$$E_n = E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \xrightarrow{f^n} \mathcal{O}_X)$$

and set $E^\wedge = R\lim E_n$. For $p \in \mathbf{Z}$ is a canonical commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \widehat{H^p(X,E)} & \longrightarrow & \lim H^p(X, E_n) & \longrightarrow & T_f(H^{p+1}(X,E)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & H^0(H^p(X,E)^\wedge) & \longrightarrow & H^p(X, E^\wedge) & \longrightarrow & T_f(H^{p+1}(X,E)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & R^1\lim H^p(X,E)[f^n] & \xrightarrow{\cong} & R^1\lim H^{p-1}(X, E_n) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns where $\widehat{H^p(X,E)} = \lim H^p(X,E)/f^n H^p(X,E)$ is the usual f-adic completion, $H^p(X,E)^\wedge$ is the derived f-adic completion, and

$T_f(H^{p+1}(X, E))$ is the f-adic Tate module, see More on Algebra, Example 15.94.5. Finally, we have $H^p(X, E^\wedge) = H^p(R\Gamma(X, E)^\wedge)$.

Proof. Observe that $R\Gamma(X, E^\wedge) = R\lim R\Gamma(X, E_n)$ by Lemma 20.37.2. On the other hand, we have

$$R\Gamma(X, E_n) = R\Gamma(X, E) \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A)$$

(details omitted). We find that $R\Gamma(X, E^\wedge)$ is the derived f-adic completion $R\Gamma(X, E)^\wedge$. Whence the diagram by More on Algebra, Lemma 15.94.6. □

0H3D **Lemma 20.39.2.** *Let $\mathcal{A}$ be an abelian category. Let $f : M \to M$ be a morphism of $\mathcal{A}$. If $M[f^n] = \mathrm{Ker}(f^n : M \to M)$ stabilizes, then the inverse systems*

$$(M \xrightarrow{f^n} M) \quad \text{and} \quad \mathrm{Coker}(f^n : M \to M)$$

are pro-isomorphic in $D(\mathcal{A})$.

Proof. There is clearly a map from the first inverse system to the second. Suppose that $M[f^c] = M[f^{c+1}] = M[f^{c+2}] = \ldots$. Then we can define an arrow of inverse systems in $D(\mathcal{A})$ in the other direction by the diagrams

$$\begin{array}{ccc} M/M[f^c] & \xrightarrow[f^{n+c}]{} & M \\ {\scriptstyle f^c}\downarrow & & \downarrow{\scriptstyle 1} \\ M & \xrightarrow{f^n} & M \end{array}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $\mathrm{Coker}(f^{n+c} : M \to M)$. Some details omitted. □

0H3E **Example 20.39.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there is a c such that $\mathcal{F}[f^c] = \mathcal{F}[f^n]$ for all $n \geq c$. We are going to apply Lemma 20.39.1 with $E = \mathcal{F}$. By Lemma 20.39.2 we see that the inverse system (E_n) is pro-isomorphic to the inverse system $(\mathcal{F}/f^n\mathcal{F})$. We conclude that for $p \in \mathbf{Z}$ we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \widehat{H^p(X, \mathcal{F})} & \longrightarrow & \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) & \longrightarrow & T_f(H^{p+1}(X, \mathcal{F})) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & H^0(H^p(X, \mathcal{F})^\wedge) & \longrightarrow & H^p(R\Gamma(X, \mathcal{F})^\wedge) & \longrightarrow & T_f(H^{p+1}(X, \mathcal{F})) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & R^1\lim H^p(X, \mathcal{F})[f^n] & \xrightarrow{\cong} & R^1\lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns where $\widehat{H^p(X, \mathcal{F})} = \lim H^p(X, \mathcal{F})/f^n H^p(X, \mathcal{F})$ is the usual f-adic completion and $M^\wedge$ denotes derived f-adic completion for M in $D(A)$.

20.40. Čech cohomology of unbounded complexes

08BZ The construction of Section 20.25 isn't the "correct" one for unbounded complexes. The problem is that in the Stacks project we use direct sums in the totalization of a double complex and we would have to replace this by a product. Instead of doing so in this section we assume the covering is finite and we use the alternating Čech complex.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of presheaves of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a **finite** open covering of X. Since the alternating Čech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ (Section 20.23) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)$. In this section we work with the associated total complex. The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

08C0 (20.40.0.1) $$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0\ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0\ldots i_p} = 0$ for $p > 0$.

08C1 **Lemma 20.40.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a finite open covering. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.40.0.1).

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. The map (20.40.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.25.4 applied to the double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemmas 20.11.1 and 20.23.6. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a K-injective complex whose terms are injectives (Injectives, Theorem 19.12.6). Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. □

08C2 **Lemma 20.40.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a finite open covering. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of open subsets of X. Assume*

1. *every open in X has a covering whose members are elements of $\mathcal{B}$,*
2. *we have $U_{i_0\ldots i_p} \in \mathcal{B}$ for all $i_0, \ldots, i_p \in I$,*
3. *for every $U \in \mathcal{B}$ and $p > 0$ we have*
   (a) *$H^p(U, \mathcal{F}^q) = 0$,*
   (b) *$H^p(U, \mathrm{Coker}(\mathcal{F}^{q-1} \to \mathcal{F}^q)) = 0$, and*
   (c) *$H^p(U, H^q(\mathcal{F})) = 0$.*

Then the map

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

of Lemma 20.40.1 is an isomorphism in $D(Ab)$.

Proof. First assume $\mathcal{F}^\bullet$ is bounded below. In this case the map

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

is a quasi-isomorphism by Lemma 20.23.6. Namely, the map of double complexes $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ induces an isomorphism between the first pages of the second spectral sequences associated to these complexes (by Homology, Lemma 12.25.1) and these spectral sequences converge (Homology, Lemma 12.25.3). Thus the conclusion in this case by Lemma 20.25.2 and assumption (3)(a).

In general, by assumption (3)(c) we may choose a resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}^\bullet_n$ as in Lemma 20.38.1. Then the map of the lemma becomes

$$\lim_n \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) = \lim_n \Gamma(X, \mathcal{I}^\bullet_n)$$

Here the arrow is in the derived category, but the equality on the right holds on the level of complexes. Note that (3)(b) shows that $\tau_{\geq -n}\mathcal{F}^\bullet$ is a bounded below complex satisfying the hypothesis of the lemma. Thus the case of bounded below complexes shows each of the maps

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet_n)$$

is a quasi-isomorphism. The cohomologies of the complexes on the left hand side in given degree are eventually constant (as the alternating Čech complex is finite). Hence the same is true on the right hand side. Thus the cohomology of the limit on the right hand side is this constant value by Homology, Lemma 12.31.7 (or the stronger More on Algebra, Lemma 15.87.3) and we win. □

20.41. Hom complexes

0A8K Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}_X$-modules. We construct a complex of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}_X$-modules of all $\mathcal{O}_X$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}_X$-modules) which are homogeneous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_\mathcal{M} \circ f - (-1)^n f \circ \mathrm{d}_\mathcal{L}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A8L
$$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n]) \tag{20.41.0.1}$$

for all $n \in \mathbf{Z}$ and every open $U \subset X$.

0A8M **Lemma 20.41.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.1. □

0A8N **Lemma 20.41.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.3. □

0BYR **Lemma 20.41.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.4. □

0A8Q **Lemma 20.41.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}_X$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.5. □

0A8P **Lemma 20.41.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.6. □

0A8R **Lemma 20.41.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L and M be objects of $D(\mathcal{O}_X)$. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules representing M. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing L. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \subset X$ open.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (20.41.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 20.32.1. □

0A8S **Lemma 20.41.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}_X$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 20.41.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

0A8T **Lemma 20.41.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}_X$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules, then

$$\begin{aligned}
\mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}$$

The first equality by (20.41.0.1). The second equality by Lemma 20.41.1. The third equality by (20.41.0.1). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 20.26.2) and because $\mathcal{I}^\bullet$ is K-injective. □

20.42. Internal hom in the derived category

08DH Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$ such that for every third object K of $D(\mathcal{O}_X)$ there exists a canonical bijection

08DI (20.42.0.1) $$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}_X$-modules constructed in Section 20.41. This is well defined by Lemma 20.41.7. We get a functor

$$D(\mathcal{O}_X)^{opp} \times D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (20.42.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

08DK **Lemma 20.42.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. For every open U we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and in particular $H^0(X, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 20.41.8. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 20.41.6. □

08DJ **Lemma 20.42.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M which recovers (20.42.0.1) by taking $H^0(X, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{K}^\bullet$ be any complex of $\mathcal{O}_X$-modules representing K. Then we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 20.41.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 20.41.8) and that the right hand side represents $R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 20.42.1 we obtain (20.42.0.1). □

08DL **Lemma 20.42.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions to opens, i.e., for every open U we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. This is clear from the construction and Lemma 20.32.1. □

08I0 **Lemma 20.42.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. □

0A8V **Lemma 20.42.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L, M in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and any complex of $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$ representing K. By Lemma 20.41.2 there is a map of complexes

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet\right) \longrightarrow \mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right)$$

whose source represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. □

0BYS **Lemma 20.42.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L, M in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K, and a K-injective complex $\mathcal{I}^\bullet$ representing L, and choose any complex of $\mathcal{O}_X$-modules $\mathcal{M}^\bullet$ representing M. Choose a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\right) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 20.41.3. □

0A8W **Lemma 20.42.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_X)$ functorial in both K and L.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 20.41.4. □

08I1 **Lemma 20.42.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L be an object of $D(\mathcal{O}_X)$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O}_X)$. For M in $D(\mathcal{O}_X)$ there is a canonical map*

08I2 (20.42.8.1) $$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee \longrightarrow R\mathcal{H}om(L, M)$$

which induces a canonical map

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$$

functorial in M in $D(\mathcal{O}_X)$.

Proof. The map (20.42.8.1) is a special case of Lemma 20.42.5 using the identification $M = R\mathcal{H}om(\mathcal{O}_X, M)$. □

0A8U **Lemma 20.42.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 20.41.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O}_X)$. □

0FXP **Remark 20.42.10.** Let $(X, \mathcal{O}_X)$ be a ringed space. For K, K', M, M' in $D(\mathcal{O}_X)$ there is a canonical map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \longrightarrow R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M, K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M')$$

Namely, by (20.42.0.1) is the same thing as a map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M'$$

For this we can first flip the middle two factors (with sign rules as in More on Algebra, Section 15.73) and use the maps

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K' \quad \text{and} \quad R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to M'$$

from Lemma 20.42.5 when thinking of $K = R\mathcal{H}om(\mathcal{O}_X, K)$ and similarly for K', M, and M'.

0B69 **Remark 20.42.11.** Let $f : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there is a canonical map

$$Rf_*R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_*L, Rf_*K)$$

Namely, by (20.42.0.1) this is the same thing as a map $Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \to Rf_*K$. For this we can use the composition

$$Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \to Rf_*(R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \to Rf_*K$$

where the first arrow is the relative cup product (Remark 20.28.7) and the second arrow is Rf_* applied to the canonical map $R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \to K$ coming from Lemma 20.42.5 (with $\mathcal{O}_X$ in one of the spots).

0G7A **Remark 20.42.12.** Let $h : X \to Y$ be a morphism of ringed spaces. Let K, M be objects of $D(\mathcal{O}_Y)$. The diagram

$$\begin{array}{ccc} Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, M) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*K & \longrightarrow & Rf_*\left(R\mathcal{H}om_{\mathcal{O}_X}(K, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K\right) \\ \downarrow & & \downarrow \\ R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, Rf_*M) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*K & \longrightarrow & Rf_*M \end{array}$$

is commutative. Here the left vertical arrow comes from Remark 20.42.11. The top horizontal arrow is Remark 20.28.7. The other two arrows are instances of the map in Lemma 20.42.5 (with one of the entries replaced with $\mathcal{O}_X$ or $\mathcal{O}_Y$).

08I3 **Remark 20.42.13.** Let $h : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_Y)$. We claim there is a canonical map

$$Lh^*R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_X)$. Namely, by (20.42.0.1) proved in Lemma 20.42.2 such a map is the same thing as a map

$$Lh^*R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 20.27.3 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (20.42.0.1).

08I4 **Remark 20.42.14.** Suppose that

$$\begin{array}{ccc} X' & \xrightarrow[h]{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is a commutative diagram of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there exists a canonical base change map

$$Lg^* Rf_* R\mathcal{H}om(K, L) \longrightarrow R(f')_* R\mathcal{H}om(Lh^* K, Lh^* L)$$

in $D(\mathcal{O}_{S'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^* Lg^* Rf_* R\mathcal{H}om(K, L) &= Lh^* Lf^* Rf_* R\mathcal{H}om(K, L) \\ &\to Lh^* R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^* K, Lh^* L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^* Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 20.42.13.

20.43. Ext sheaves

0BQP Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined sheaves of $\mathcal{O}_X$-modules

$$\mathcal{E}xt^n(K, L) = H^n(R\mathcal{H}om(K, L))$$

by taking the nth cohomology sheaf of the object $R\mathcal{H}om(K, L)$ of $D(\mathcal{O}_X)$. We will sometimes write $\mathcal{E}xt^n_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 20.42.1 we see that this $\mathcal{E}xt^n$-sheaf is the sheafification of the rule

$$U \longmapsto \mathrm{Ext}^n_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

By Example 20.29.3 there is always a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(K, L))$$

converging to $\mathrm{Ext}^{p+q}_{D(\mathcal{O}_X)}(K, L)$ in favorable situations (for example if L is bounded below and K is bounded above).

20.44. Global derived hom

0B6A Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\mathrm{Hom}_X(K, L) = R\Gamma(X, R\mathcal{H}om(K, L))$$

in $D(\Gamma(X, \mathcal{O}_X))$. We will sometimes write $R\mathrm{Hom}_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 20.42.1 we have

$$H^0(R\mathrm{Hom}_X(K, L)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L), \quad H^p(R\mathrm{Hom}_X(K, L)) = \mathrm{Ext}^p_{D(\mathcal{O}_X)}(K, L)$$

If $f : Y \to X$ is a morphism of ringed spaces, then there is a canonical map

$$R\mathrm{Hom}_X(K, L) \longrightarrow R\mathrm{Hom}_Y(Lf^* K, Lf^* L)$$

in $D(\Gamma(X, \mathcal{O}_X))$ by taking global sections of the map defined in Remark 20.42.13.

20.45. Glueing complexes

0D65 We can glue complexes! More precisely, in certain circumstances we can glue locally given objects of the derived category to a global object. We first prove some easy cases and then we'll prove the very general [**BBD82**, Theorem 3.2.4] in the setting of topological spaces and open coverings.

08DG **Lemma 20.45.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. Suppose given*

(1) *an object A of $D(\mathcal{O}_U)$,*
(2) *an object B of $D(\mathcal{O}_V)$, and*
(3) *an isomorphism $c : A|_{U\cap V} \to B|_{U\cap V}$.*

Then there exists an object F of $D(\mathcal{O}_X)$ and isomorphisms $f : F|_U \to A$, $g : F|_V \to B$ such that $c = g|_{U\cap V} \circ f^{-1}|_{U\cap V}$. Moreover, given

(1) *an object E of $D(\mathcal{O}_X)$,*
(2) *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
(3) *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*

such that

$$a|_{U\cap V} = b|_{U\cap V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is $a \circ f$ and whose restriction to V is $b \circ g$.

Proof. Denote j_U, j_V, $j_{U\cap V}$ the corresponding open immersions. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V}) \to F[1]$$

where the map $Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the obvious one and where $Rj_{U,*}A \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the composition of $Rj_{U,*}A \to Rj_{U\cap V,*}(A|_{U\cap V})$ with $Rj_{U\cap V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U\cap V,*}(B|_{U\cap V}))|_U \to F|_U[1]$$

Denote $j : U \cap V \to U$. Compatibility of restriction to opens and cohomology shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U\cap V,*}(B|_{U\cap V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U\cap V})$. Hence the second arrow of the last displayed diagram has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism. Similarly, the morphism $F|_V \to B$ is an isomorphism. The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 20.33.3. □

0D66 **Lemma 20.45.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{B}$ be a basis for the topology on Y.*

(1) *Assume K is in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $H^i(f^{-1}(V), K) = 0$ for $i < 0$. Then Rf_*K has vanishing cohomology sheaves in negative degrees, $H^i(f^{-1}(V), K) = 0$ for $i < 0$ for all opens $V \subset Y$, and the rule $V \mapsto H^0(f^{-1}V, K)$ is a sheaf on Y.*
(2) *Assume K, L are in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$. Then $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$ for all opens $V \subset Y$ and the rule $V \mapsto \mathrm{Hom}(K|_{f^{-1}V}, L|_{f^{-1}V})$ is a sheaf on Y.*

Proof. Lemma 20.32.6 tells us $H^i(Rf_*K)$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$. The assumptions in (1) imply that Rf_*K

has vanishing cohomology sheaves in degrees < 0. We conclude that for any open $V \subset Y$ the cohomology group $H^i(V, Rf_*K)$ is zero for $i < 0$ and is equal to $H^0(V, H^0(Rf_*K))$ for $i = 0$. This proves (1).

To prove (2) apply (1) to the complex $R\mathcal{H}om(K, L)$ using Lemma 20.42.1 to do the translation. □

0D67 **Situation 20.45.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. We are given

(1) a collection of opens $\mathcal{B}$ of X,
(2) for $U \in \mathcal{B}$ an object K_U in $D(\mathcal{O}_U)$,
(3) for $V \subset U$ with $V, U \in \mathcal{B}$ an isomorphism $\rho_V^U : K_U|_V \to K_V$ in $D(\mathcal{O}_V)$,

such that whenever we have $W \subset V \subset U$ with U, V, W in $\mathcal{B}$, then $\rho_W^U = \rho_W^V \circ \rho_V^U|_W$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a basis for the topology on X. Another is the case where we have a morphism $f : X \to Y$ of topological spaces and the elements of $\mathcal{B}$ are the inverse images of the elements of a basis for the topology of Y.

In Situation 20.45.3 a *solution* will be a pair (K, ρ_U) where K is an object of $D(\mathcal{O}_X)$ and $\rho_U : K|_U \to K_U$, $U \in \mathcal{B}$ are isomorphisms such that we have $\rho_V^U \circ \rho_U|_V = \rho_V$ for all $V \subset U$, $U, V \in \mathcal{B}$. In certain cases solutions are unique.

0D68 **Lemma 20.45.4.** *In Situation 20.45.3 assume*

(1) $X = \bigcup_{U \in \mathcal{B}} U$ *and for* $U, V \in \mathcal{B}$ *we have* $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,
(2) *for any* $U \in \mathcal{B}$ *we have* $\mathrm{Ext}^i(K_U, K_U) = 0$ *for* $i < 0$.

If a solution (K, ρ_U) *exists, then it is unique up to unique isomorphism and moreover* $\mathrm{Ext}^i(K, K) = 0$ *for* $i < 0$.

Proof. Let (K, ρ_U) and (K', ρ'_U) be a pair of solutions. Let $f : X \to Y$ be the continuous map constructed in Topology, Lemma 5.5.6. Set $\mathcal{O}_Y = f_*\mathcal{O}_X$. Then K, K' and $\mathcal{B}$ are as in Lemma 20.45.2 part (2). Hence we obtain the vanishing of negative exts for K and we see that the rule

$$V \longmapsto \mathrm{Hom}(K|_{f^{-1}V}, K'|_{f^{-1}V})$$

is a sheaf on Y. As both (K, ρ_U) and (K', ρ'_U) are solutions the maps

$$(\rho'_U)^{-1} \circ \rho_U : K|_U \longrightarrow K'|_U$$

over $U = f^{-1}(f(U))$ agree on overlaps. Hence we get a unique global section of the sheaf above which defines the desired isomorphism $K \to K'$ compatible with all structure available. □

0D69 **Remark 20.45.5.** With notation and assumptions as in Lemma 20.45.4. Suppose that $U, V \in \mathcal{B}$. Let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in $U \cap V$. Then

$$(\{K_{U'}\}_{U' \in \mathcal{B}'}, \{\rho_{V'}^{U'}\}_{V' \subset U' \text{ with } U', V' \in \mathcal{B}'})$$

is a system on the ringed space $U \cap V$ satisfying the assumptions of Lemma 20.45.4. Moreover, both $(K_U|_{U \cap V}, \rho_{U'}^U)$ and $(K_V|_{U \cap V}, \rho_{U'}^V)$ are solutions to this system. By the lemma we find a unique isomorphism

$$\rho_{U,V} : K_U|_{U \cap V} \longrightarrow K_V|_{U \cap V}$$

such that for every $U' \subset U \cap V$, $U' \in \mathcal{B}$ the diagram

$$\begin{array}{ccc} K_U|_{U'} & \xrightarrow{\rho_{U,V}|_{U'}} & K_V|_{U'} \\ & {\scriptstyle \rho^U_{U'}} \searrow \quad \swarrow {\scriptstyle \rho^V_{U'}} & \\ & K_{U'} & \end{array}$$

commutes. Pick a third element $W \in \mathcal{B}$. We obtain isomorphisms $\rho_{U,W} : K_U|_{U\cap W} \to K_W|_{U\cap W}$ and $\rho_{V,W} : K_U|_{V\cap W} \to K_W|_{V\cap W}$ satisfying similar properties to those of $\rho_{U,V}$. Finally, we have

$$\rho_{U,W}|_{U\cap V\cap W} = \rho_{V,W}|_{U\cap V\cap W} \circ \rho_{U,V}|_{U\cap V\cap W}$$

This is true by the uniqueness in the lemma because both sides of the equality are the unique isomorphism compatible with the maps $\rho^U_{U''}$ and $\rho^W_{U''}$ for $U'' \subset U\cap V\cap W$, $U'' \in \mathcal{B}$. Some minor details omitted. The collection $(K_U, \rho_{U,V})$ is a descent datum in the derived category for the open covering $\mathcal{U} : X = \bigcup_{U\in\mathcal{B}} U$ of X. In this language we are looking for "effectiveness of the descent datum" when we look for the existence of a solution.

0D6A **Lemma 20.45.6.** *In Situation 20.45.3 assume*

(1) $X = U_1 \cup \ldots \cup U_n$ *with* $U_i \in \mathcal{B}$,
(2) *for* $U, V \in \mathcal{B}$ *we have* $U \cap V = \bigcup_{W\in\mathcal{B}, W\subset U\cap V} W$,
(3) *for any* $U \in \mathcal{B}$ *we have* $\mathrm{Ext}^i(K_U, K_U) = 0$ *for* $i < 0$.

Then a solution exists and is unique up to unique isomorphism.

Proof. Uniqueness was seen in Lemma 20.45.4. We may prove the lemma by induction on n. The case $n = 1$ is immediate.

The case $n = 2$. Consider the isomorphism $\rho_{U_1,U_2} : K_{U_1}|_{U_1\cap U_2} \to K_{U_2}|_{U_1\cap U_2}$ constructed in Remark 20.45.5. By Lemma 20.45.1 we obtain an object K in $D(\mathcal{O}_X)$ and isomorphisms $\rho_{U_1} : K|_{U_1} \to K_{U_1}$ and $\rho_{U_2} : K|_{U_2} \to K_{U_2}$ compatible with ρ_{U_1,U_2}. Take $U \in \mathcal{B}$. We will construct an isomorphism $\rho_U : K|_U \to K_U$ and we will leave it to the reader to verify that (K, ρ_U) is a solution. Consider the set $\mathcal{B}'$ of elements of $\mathcal{B}$ contained in either $U \cap U_1$ or contained in $U \cap U_2$. Then $(K_U, \rho^U_{U'})$ is a solution for the system $(\{K_{U'}\}_{U'\in\mathcal{B}'}, \{\rho^{U'}_{V'}\}_{V'\subset U' \text{ with } U',V'\in\mathcal{B}'})$ on the ringed space U. We claim that $(K|_U, \tau_{U'})$ is another solution where $\tau_{U'}$ for $U' \in \mathcal{B}'$ is chosen as follows: if $U' \subset U_1$ then we take the composition

$$K|_{U'} \xrightarrow{\rho_{U_1}|_{U'}} K_{U_1}|_{U'} \xrightarrow{\rho^{U_1}_{U'}} K_{U'}$$

and if $U' \subset U_2$ then we take the composition

$$K|_{U'} \xrightarrow{\rho_{U_2}|_{U'}} K_{U_2}|_{U'} \xrightarrow{\rho^{U_2}_{U'}} K_{U'}.$$

To verify this is a solution use the property of the map ρ_{U_1,U_2} described in Remark 20.45.5 and the compatibility of ρ_{U_1} and ρ_{U_2} with ρ_{U_1,U_2}. Having said this we apply Lemma 20.45.4 to see that we obtain a unique isomorphism $K|_{U'} \to K_{U'}$ compatible with the maps $\tau_{U'}$ and $\rho^U_{U'}$ for $U' \in \mathcal{B}'$.

The case $n > 2$. Consider the open subspace $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in X'. Then we find a system $(\{K_U\}_{U\in\mathcal{B}'}, \{\rho^U_V\}_{U,V\in\mathcal{B}'})$ on the ringed space X' to which we may apply our induction hypothesis. We find a solution $(K_{X'}, \rho^{X'}_U)$. Then we can consider the collection $\mathcal{B}^* = \mathcal{B} \cup \{X'\}$ of opens

of X and we see that we obtain a system $(\{K_U\}_{U\in\mathcal{B}^*}, \{\rho_V^U\}_{V\subset U \text{ with } U,V\in\mathcal{B}^*})$. Note that this new system also satisfies condition (3) by Lemma 20.45.4 applied to the solution $K_{X'}$. For this system we have $X = X' \cup U_n$. This reduces us to the case $n = 2$ we worked out above. □

0D6B **Lemma 20.45.7.** *Let X be a ringed space. Let E be a well ordered set and let*

$$X = \bigcup\nolimits_{\alpha\in E} W_\alpha$$

be an open covering with $W_\alpha \subset W_{\alpha+1}$ and $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ if α is not a successor. Let K_α be an object of $D(\mathcal{O}_{W_\alpha})$ with $\mathrm{Ext}^i(K_\alpha, K_\alpha) = 0$ for $i < 0$. Assume given isomorphisms $\rho_\beta^\alpha : K_\alpha|_{W_\beta} \to K_\beta$ in $D(\mathcal{O}_{W_\beta})$ for all $\beta < \alpha$ with $\rho_\gamma^\alpha = \rho_\gamma^\beta \circ \rho_\beta^\alpha|_{W_\gamma}$ for $\gamma < \beta < \alpha$. Then there exists an object K in $D(\mathcal{O}_X)$ and isomorphisms $K|_{W_\alpha} \to K_\alpha$ for $\alpha \in E$ compatible with the isomorphisms ρ_β^α.

Proof. In this proof $\alpha, \beta, \gamma, \ldots$ represent elements of E. Choose a K-injective complex $I_\alpha^\bullet$ on W_α representing K_α. For $\beta < \alpha$ denote $j_{\beta,\alpha} : W_\beta \to W_\alpha$ the inclusion morphism. Using transfinite recursion we will construct for all $\beta < \alpha$ a map of complexes

$$\tau_{\beta,\alpha} : (j_{\beta,\alpha})_! I_\beta^\bullet \longrightarrow I_\alpha^\bullet$$

representing the adjoint to the inverse of the isomorphism $\rho_\beta^\alpha : K_\alpha|_{W_\beta} \to K_\beta$. Moreover, we will do this in such that for $\gamma < \beta < \alpha$ we have

$$\tau_{\gamma,\alpha} = \tau_{\beta,\alpha} \circ (j_{\beta,\alpha})_! \tau_{\gamma,\beta}$$

as maps of complexes. Namely, suppose already given $\tau_{\gamma,\beta}$ composing correctly for all $\gamma < \beta < \alpha$. If $\alpha = \alpha' + 1$ is a successor, then we choose any map of complexes

$$(j_{\alpha',\alpha})_! I_{\alpha'}^\bullet \to I_\alpha^\bullet$$

which is adjoint to the inverse of the isomorphism $\rho_{\alpha'}^\alpha : K_\alpha|_{W_{\alpha'}} \to K_{\alpha'}$ (possible because $I_\alpha^\bullet$ is K-injective) and for any $\beta < \alpha'$ we set

$$\tau_{\beta,\alpha} = \tau_{\alpha',\alpha} \circ (j_{\alpha',\alpha})_! \tau_{\beta,\alpha'}$$

If α is not a successor, then we can consider the complex on W_α given by

$$C^\bullet = \mathrm{colim}_{\beta<\alpha} (j_{\beta,\alpha})_! I_\beta^\bullet$$

(termwise colimit) where the transition maps of the sequence are given by the maps $\tau_{\beta',\beta}$ for $\beta' < \beta < \alpha$. We claim that $C^\bullet$ represents K_α. Namely, for $\beta < \alpha$ the restriction of the coprojection $(j_{\beta,\alpha})_! I_\beta^\bullet \to C^\bullet$ gives a map

$$\sigma_\beta : I_\beta^\bullet \longrightarrow C^\bullet|_{W_\beta}$$

which is a quasi-isomorphism: if $x \in W_\beta$ then looking at stalks we get

$$(C^\bullet)_x = \mathrm{colim}_{\beta'<\alpha} \left((j_{\beta',\alpha})_! I_{\beta'}^\bullet\right)_x = \mathrm{colim}_{\beta\le\beta'<\alpha} (I_{\beta'}^\bullet)_x \longleftarrow (I_\beta^\bullet)_x$$

which is a quasi-isomorphism. Here we used that taking stalks commutes with colimits, that filtered colimits are exact, and that the maps $(I_\beta^\bullet)_x \to (I_{\beta'}^\bullet)_x$ are quasi-isomorphisms for $\beta \le \beta' < \alpha$. Hence $(C^\bullet, \sigma_\beta^{-1})$ is a solution to the system $(\{K_\beta\}_{\beta<\alpha}, \{\rho_{\beta'}^\beta\}_{\beta'<\beta<\alpha})$. Since $(K_\alpha, \rho_\beta^\alpha)$ is another solution we obtain a unique isomorphism $\sigma : K_\alpha \to C^\bullet$ in $D(\mathcal{O}_{W_\alpha})$ compatible with all our maps, see Lemma 20.45.6 (this is where we use the vanishing of negative ext groups). Choose a morphism $\tau : C^\bullet \to I_\alpha^\bullet$ of complexes representing σ. Then we set

$$\tau_{\beta,\alpha} = \tau|_{W_\beta} \circ \sigma_\beta$$

to get the desired maps. Finally, we take K to be the object of the derived category represented by the complex

$$K^\bullet = \text{colim}_{\alpha \in E} (W_\alpha \to X)_! I_\alpha^\bullet$$

where the transition maps are given by our carefully constructed maps $\tau_{\beta,\alpha}$ for $\beta < \alpha$. Arguing exactly as above we see that for all α the restriction of the coprojection determines an isomorphism

$$K|_{W_\alpha} \longrightarrow K_\alpha$$

compatible with the given maps ρ_β^α. □

Using transfinite induction we can prove the result in the general case.

0D6C **Theorem 20.45.8** (BBD gluing lemma)**.** *In Situation 20.45.3 assume*

(1) $X = \bigcup_{U \in \mathcal{B}} U$,
(2) *for* $U, V \in \mathcal{B}$ *we have* $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,
(3) *for any* $U \in \mathcal{B}$ *we have* $\text{Ext}^i(K_U, K_U) = 0$ *for* $i < 0$.

Then there exists an object K *of* $D(\mathcal{O}_X)$ *and isomorphisms* $\rho_U : K|_U \to K_U$ *in* $D(\mathcal{O}_U)$ *for* $U \in \mathcal{B}$ *such that* $\rho_V^U \circ \rho_U|_V = \rho_V$ *for all* $V \subset U$ *with* $U, V \in \mathcal{B}$*. The pair* (K, ρ_U) *is unique up to unique isomorphism.*

Special case of [**BBD82**, Theorem 3.2.4] without boundedness assumption.

Proof. A pair (K, ρ_U) is called a solution in the text above. The uniqueness follows from Lemma 20.45.4. If X has a finite covering by elements of $\mathcal{B}$ (for example if X is quasi-compact), then the theorem is a consequence of Lemma 20.45.6. In the general case we argue in exactly the same manner, using transfinite induction and Lemma 20.45.7.

First we use transfinite recursion to choose opens $W_\alpha \subset X$ for any ordinal α. Namely, we set $W_0 = \emptyset$. If $\alpha = \beta + 1$ is a successor, then either $W_\beta = X$ and we set $W_\alpha = X$ or $W_\beta \neq X$ and we set $W_\alpha = W_\beta \cup U_\alpha$ where $U_\alpha \in \mathcal{B}$ is not contained in W_β. If α is a limit ordinal we set $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$. Then for large enough α we have $W_\alpha = X$. Observe that for every α the open W_α is a union of elements of $\mathcal{B}$. Hence if $\mathcal{B}_\alpha = \{U \in \mathcal{B}, U \subset W_\alpha\}$, then

$$S_\alpha = (\{K_U\}_{U \in \mathcal{B}_\alpha}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha})$$

is a system as in Lemma 20.45.4 on the ringed space W_α.

We will show by transfinite induction that for every α the system S_α has a solution. This will prove the theorem as this system is the system given in the theorem for large α.

The case where $\alpha = \beta + 1$ is a successor ordinal. (This case was already treated in the proof of the lemma above but for clarity we repeat the argument.) Recall that $W_\alpha = W_\beta \cup U_\alpha$ for some $U_\alpha \in \mathcal{B}$ in this case. By induction hypothesis we have a solution $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ for the system S_β. Then we can consider the collection $\mathcal{B}_\alpha^* = \mathcal{B}_\alpha \cup \{W_\beta\}$ of opens of W_α and we see that we obtain a system $(\{K_U\}_{U \in \mathcal{B}_\alpha^*}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha^*})$. Note that this new system also satisfies condition (3) by Lemma 20.45.4 applied to the solution K_{W_β}. For this system we have $W_\alpha = W_\beta \cup U_\alpha$. This reduces us to the case handled in Lemma 20.45.6.

The case where α is a limit ordinal. Recall that $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ in this case. For $\beta < \alpha$ let $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ be the solution for S_β. For $\gamma < \beta < \alpha$ the

restriction $K_{W_\beta}|_{W_\gamma}$ endowed with the maps $\rho_U^{W_\beta}$, $U \in \mathcal{B}_\gamma$ is a solution for S_γ. By uniqueness we get unique isomorphisms $\rho_{W_\gamma}^{W_\beta} : K_{W_\beta}|_{W_\gamma} \to K_{W_\gamma}$ compatible with the maps $\rho_U^{W_\beta}$ and $\rho_U^{W_\gamma}$ for $U \in \mathcal{B}_\gamma$. These maps compose in the correct manner, i.e., $\rho_{W_\delta}^{W_\gamma} \circ \rho_{W_\gamma}^{W_\beta}|_{W_\delta} = \rho_{W_\beta}^{W_\delta}$ for $\delta < \gamma < \beta < \alpha$. Thus we may apply Lemma 20.45.7 (note that the vanishing of negative exts is true for K_{W_β} by Lemma 20.45.4 applied to the solution K_{W_β}) to obtain K_{W_α} and isomorphisms

$$\rho_{W_\beta}^{W_\alpha} : K_{W_\alpha}|_{W_\beta} \longrightarrow K_{W_\beta}$$

compatible with the maps $\rho_{W_\gamma}^{W_\beta}$ for $\gamma < \beta < \alpha$.

To show that K_{W_α} is a solution we still need to construct the isomorphisms $\rho_U^{W_\alpha} : K_{W_\alpha}|_U \to K_U$ for $U \in \mathcal{B}_\alpha$ satisfying certain compatibilities. We choose $\rho_U^{W_\alpha}$ to be the unique map such that for any $\beta < \alpha$ and any $V \in \mathcal{B}_\beta$ with $V \subset U$ the diagram

$$\begin{array}{ccc} K_{W_\alpha}|_V & \xrightarrow[\rho_U^{W_\alpha}|_V]{} & K_U|_V \\ {\scriptstyle \rho_{W_\beta}^{W_\alpha}|_V}\downarrow & & \downarrow{\scriptstyle \rho_U^V} \\ K_{W_\beta} & \xrightarrow[\rho_V^{W_\beta}]{} & K_V \end{array}$$

commutes. This makes sense because

$$(\{K_V\}_{V \subset U, V \in \mathcal{B}_\beta \text{ for some } \beta < \alpha}, \{\rho_V^{V'}\}_{V \subset V' \text{ with } V, V' \subset U \text{ and } V, V' \in \mathcal{B}_\beta \text{ for some } \beta < \alpha})$$

is a system as in Lemma 20.45.4 on the ringed space U and because (K_U, ρ_V^U) and $(K_{W_\alpha}|_U, \rho_V^{W_\beta} \circ \rho_{W_\beta}^{W_\alpha}|_V)$ are both solutions for this system. This gives existence and uniqueness. We omit the proof that these maps satisfy the desired compatibilities (it is just bookkeeping). □

20.46. Strictly perfect complexes

08C3 Strictly perfect complexes of modules are used to define the notions of pseudo-coherent and perfect complexes later on. They are defined as follows.

08C4 **Definition 20.46.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}_X$-module for all i.

Warning: Since we do not assume that X is a locally ringed space, it may not be true that a direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

08C5 **Lemma 20.46.2.** *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

09J2 **Lemma 20.46.3.** *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

09U6 **Lemma 20.46.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.*

Proof. The pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

08C6 **Lemma 20.46.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of $\mathcal{O}_X$-modules*

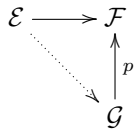

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_X$-module and p surjective, then a dotted arrow making the diagram commute exists locally on X.

Proof. We may assume $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(X, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sheaves, Section 6.16). □

08C7 **Lemma 20.46.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then α is locally on X homotopic to zero.*
(2) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then α is locally on X homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}_X$-module and integer $n \geq a$, then the result follows from Lemma 20.46.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_X)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is locally homotopic to zero, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is locally zero by induction hypothesis. □

08C8 **Lemma 20.46.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of complexes of $\mathcal{O}_X$-modules*

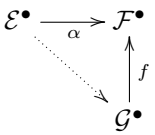

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then a dotted arrow making the diagram commute up to homotopy exists locally on X.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 20.46.6 guarantees there is an open covering $X = \bigcup U_i$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

08C9 **Lemma 20.46.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

(1) *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
(2) *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 20.46.7. We omit the proof of (2). □

08DM **Lemma 20.46.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 20.41.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 20.42. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an open $U \subset X$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{I}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{I}^\bullet|_U)$$

By Lemma 20.46.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

0GM5 **Lemma 20.46.10.** *In the situation of Lemma 20.46.9 if $\mathcal{F}^\bullet$ is K-flat, then $\mathcal{H}^\bullet$ is K-flat.*

Proof. Observe that $\mathcal{H}^\bullet$ is simply the hom complex $\mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ since the boundedness of the strictly prefect complex $\mathcal{E}^\bullet$ insures that the products in the definition of the hom complex turn into direct sums. Let $\mathcal{K}^\bullet$ be an acyclic complex of $\mathcal{O}_X$-modules. Consider the map

$$\gamma : \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet))$$

of Lemma 20.41.3. Since $\mathcal{F}^\bullet$ is K-flat, the complex $\mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet)$ is acyclic, and hence by Lemma 20.46.8 (or Lemma 20.46.9 if you like) the target of γ is acyclic too. Hence to prove the lemma it suffices to show that γ is an isomorphism of complexes. To see this, we may argue by induction on the length of the complex $\mathcal{E}^\bullet$. If the length is ≤ 1 then the $\mathcal{E}^\bullet$ is a direct summand of $\mathcal{O}_X^{\oplus n}[k]$ for some $n \geq 0$ and $k \in \mathbf{Z}$ and in this case the result follows by inspection. If the length is > 1, then we reduce to smaller length by considering the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq a}\mathcal{E}^\bullet \to 0$$

for a suitable $a \in \mathbf{Z}$, see Homology, Section 12.15. Then γ fits into a morphism of termwise split short exact sequences of complexes. By induction γ is an isomorphism for $\sigma_{\geq a+1}\mathcal{E}^\bullet$ and $\sigma_{\leq a}\mathcal{E}^\bullet$ and hence the result for $\mathcal{E}^\bullet$ follows. Some details omitted. □

08I5 **Lemma 20.46.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with*

(1) *$\mathcal{F}^n = 0$ for $n \ll 0$,*
(2) *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
(3) *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}_X$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 20.42.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.4). Hence the construction in Section 20.42 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.25.3)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

20.47. Pseudo-coherent modules

08CA In this section we discuss pseudo-coherent complexes.

08CB **Definition 20.47.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if there exists an open covering $X = \bigcup U_i$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i^\bullet$ is strictly perfect on U_i and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O}_X)$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}_X$-modules.

If X is quasi-compact, then an m-pseudo-coherent object of $D(\mathcal{O}_X)$ is in $D^-(\mathcal{O}_X)$. But this need not be the case if X is not quasi-compact.

08CC **Lemma 20.47.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*

(1) *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
(2) *If E is m-pseudo-coherent, then any complex representing E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let $X = \bigcup U_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) simultaneously. By Lemma 20.46.8 we can after refining the open covering $X = \bigcup U_i$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent. □

09U7 **Lemma 20.47.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules and choose an open covering $Y = \bigcup V_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i}$ as in Definition 20.47.1. Set $U_i = f^{-1}(V_i)$. By Lemma 20.47.2 it suffices to show that $Lf^*\mathcal{E}^\bullet|_{U_i}$ is m-pseudo-coherent. Choose a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i} \to C \to \mathcal{E}_i^\bullet[1]$$

The assumption on α_i means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Denote $f_i : U_i \to V_i$ the restriction of f. Note that $Lf^*\mathcal{E}^\bullet|_{U_i} = Lf_i^*(\mathcal{E}|_{V_i})$. Applying Lf_i^* we obtain the distinguished triangle

$$Lf_i^*\mathcal{E}_i^\bullet \to Lf_i^*\mathcal{E}|_{V_i} \to Lf_i^*C \to Lf_i^*\mathcal{E}_i^\bullet[1]$$

By the construction of Lf_i^* as a left derived functor we see that $H^j(Lf_i^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf_i^*\alpha_i)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha_i)$ is surjective. On the other hand, $Lf_i^*\mathcal{E}_i^\bullet = f_i^*\mathcal{E}_i^\bullet$. is strictly perfect by Lemma 20.46.4. Thus we conclude. □

08CD **Lemma 20.47.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$.*

(1) *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*
(2) *If K and M are m-pseudo-coherent, then L is m-pseudo-coherent.*

(3) *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose an open covering $X = \bigcup U_i$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the open covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 20.46.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{array}{ccc} K|_{U_i} & \longrightarrow & L|_{U_i} \\ \uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \beta_i} \\ \mathcal{K}_i^\bullet & \xrightarrow{\gamma_i} & \mathcal{L}_i^\bullet \end{array}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 20.46.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 20.47.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

09J3 **Lemma 20.47.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$.*

(1) *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
(2) *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is pseudo-coherent.*

Proof. Proof of (1). By replacing X by the members of an open covering we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). We may first replace X by the members of an open covering to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

08CE **Lemma 20.47.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O}_X)$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. After replacing X by the members of an open covering we may assume $K \oplus L \in D^-(\mathcal{O}_X)$, hence $L \in D^-(\mathcal{O}_X)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 20.47.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

09V7 **Lemma 20.47.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 20.47.4 and truncations as in the proof of More on Algebra, Lemma 15.65.9. □

09V8 **Lemma 20.47.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent for all i, then E is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 20.47.4 and truncations as in the proof of More on Algebra, Lemma 15.65.10. □

08DN **Lemma 20.47.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

1. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}_X$-module.*
2. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}_X$-module.*

Proof. Proof of (1). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 20.46.5 we can locally find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

09V9 **Lemma 20.47.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(1) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is 0-pseudo-coherent if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
(2) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is (-1)-pseudo-coherent if and only if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Use Lemma 20.47.9 to prove the implications in one direction and Lemma 20.47.8 for the other. □

20.48. Tor dimension

08CF In this section we take a closer look at resolutions by flat modules.

08CG **Definition 20.48.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) We say E *locally has finite tor dimension* if there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ has finite tor dimension for all i.

An $\mathcal{O}_X$-module $\mathcal{F}$ has *tor dimension* $\leq d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O}_X)$ has tor-amplitude in $[-d, 0]$.

Note that if E as in the definition has finite tor dimension, then E is an object of $D^b(\mathcal{O}_X)$ as can be seen by taking $\mathcal{F} = \mathcal{O}_X$ in the definition above.

08CH **Lemma 20.48.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}_X$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Hence for every $\mathcal{O}_X$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 20.26.16. □

08CI **Lemma 20.48.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}_X$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$, see Section 20.26. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules, Lemma 17.17.9). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}_X$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 20.48.2 and the definitions. □

09U8 **Lemma 20.48.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 20.48.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules, Lemma 17.17.4 the modules $f^*\mathcal{E}^i$ are flat. Thus by Lemma 20.48.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

09U9 **Lemma 20.48.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *for every $x \in X$ the object E_x of $D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$.*

Proof. Taking stalks at x is the same thing as pulling back by the morphism of ringed spaces $(x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 20.48.4. For the converse, note that taking stalks commutes with tensor products (Modules, Lemma 17.16.1). Hence

$$(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x$$

On the other hand, taking stalks is exact, so

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = H^i((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x) = H^i(E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x)$$

and we can check whether $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ is zero by checking whether all of its stalks are zero (Modules, Lemma 17.3.1). Thus (2) implies (1). □

08CJ **Lemma 20.48.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$.*

(1) *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
(2) *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
(3) *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

09J4 **Lemma 20.48.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

08CK **Lemma 20.48.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O}_X)$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

20.49. Perfect complexes

08CL In this section we discuss properties of perfect complexes on ringed spaces.

08CM **Definition 20.49.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if there exists an open covering $X = \bigcup U_i$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ a strictly perfect complex of $\mathcal{O}_{U_i}$-modules. An object E of $D(\mathcal{O}_X)$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.

If X is quasi-compact, then a perfect object of $D(\mathcal{O}_X)$ is in $D^b(\mathcal{O}_X)$. But this need not be the case if X is not quasi-compact.

08CN **Lemma 20.49.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*

(1) *If there exists an open covering $X = \bigcup U_i$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i such that $\mathcal{E}_i^\bullet$ represents $E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
(2) *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 20.47.2. □

0BCJ **Lemma 20.49.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the following are equivalent*

(1) *E is perfect,*
(2) *there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite locally free $\mathcal{O}_{U_i}$-modules, and*
(3) *there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite free $\mathcal{O}_{U_i}$-modules.*

Proof. This follows from Lemma 20.49.2 and the fact that on X every direct summand of a finite free module is finite locally free. See Modules, Lemma 17.14.6. □

08CP **Lemma 20.49.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. After replacing X by the members of an open covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[2-a]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Applying $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 20.26.16). Thus $\mathcal{E}'$ is locally a direct summand of a finite free module by Modules, Lemma 17.18.3. Thus locally the complex

$$\mathcal{E}' \to \mathcal{E}^{a+1} \to \ldots \to \mathcal{E}^b$$

is quasi-isomorphic to E and E is perfect. □

08CQ **Lemma 20.49.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) E *is perfect, and*
(2) E *is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). By definition this means there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 20.47.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 20.48.3. Thus (2) holds.

Assume (2). After replacing X by the members of an open covering we may assume there exist integers $a \leq b$ such that E has tor amplitude in $[a, b]$. Since E is m-pseudo-coherent for all m we conclude using Lemma 20.49.4. □

09UA **Lemma 20.49.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is perfect in $D(\mathcal{O}_Y)$, then Lf^*E is perfect in $D(\mathcal{O}_X)$.*

Proof. This follows from Lemma 20.49.5, 20.48.4, and 20.47.3. (An alternative proof is to copy the proof of Lemma 20.47.3.) □

08CR **Lemma 20.49.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 20.49.5, 20.47.4, and 20.48.6. Second proof (sketch): Say K and L are perfect. After replacing X by the members of an open covering we may assume that K and L are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing X by the members of an open covering we may assume the map $K \to L$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 20.46.8. Then M is isomorphic to the cone of α which is strictly perfect by Lemma 20.46.2. □

09J5 **Lemma 20.49.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If K, L are perfect objects of $D(\mathcal{O}_X)$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$.*

Proof. Follows from Lemmas 20.49.5, 20.47.5, and 20.48.7. □

08CS **Lemma 20.49.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If $K \oplus L$ is a perfect object of $D(\mathcal{O}_X)$, then so are K and L.*

Proof. Follows from Lemmas 20.49.5, 20.47.6, and 20.48.8. □

08DP **Lemma 20.49.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let E be a perfect object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on a closed subset $T \subset U$ with $j(T)$ closed in X. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being a perfect complex is local on X. Thus it suffices to check that Rj_*E is perfect when restricted to U and $V = X \setminus j(T)$. We have $Rj_*E|_U = E$ which is perfect. We have $Rj_*E|_V = 0$ because $E|_{U \setminus T} = 0$. □

0GT1 **Lemma 20.49.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E in $D(\mathcal{O}_X)$ be perfect. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the set*

$$U = \{x \in X \mid H^i(E)_x \text{ is a finite free } \mathcal{O}_{X,x}\text{-module for all } i \in \mathbf{Z}\}$$

is open in X and is the maximal open set $U \subset X$ such that $H^i(E)|_U$ is finite locally free for all $i \in \mathbf{Z}$.

Proof. Note that if $V \subset X$ is some open such that $H^i(E)|_V$ is finite locally free for all $i \in \mathbf{Z}$ then $V \subset U$. Let $x \in U$. We will show that an open neighbourhood of x is contained in U and that $H^i(E)$ is finite locally free on this neighbourhood for all i. This will finish the proof. During the proof we may (finitely many times) replace X by an open neighbourhood of x. Hence we may assume E is represented by a strictly perfect complex $\mathcal{E}^\bullet$. Say $\mathcal{E}^i = 0$ for $i \notin [a, b]$. We will prove the result by induction on $b - a$. The module $H^b(E) = \mathrm{Coker}(d^{b-1} : \mathcal{E}^{b-1} \to \mathcal{E}^b)$ is of finite presentation. Since $H^b(E)_x$ is finite free, we conclude $H^b(E)$ is finite free in an open neighbourhood of x by Modules, Lemma 17.11.7. Thus after replacing X by a (possibly smaller) open neighbourhood we may assume we have a direct sum decomposition $\mathcal{E}^b = \mathrm{Im}(d^{b-1}) \oplus H^b(E)$ and $H^b(E)$ is finite free, see Lemma 20.46.5. Doing the same argument again, we see that we may assume $\mathcal{E}^{b-1} = \mathrm{Ker}(d^{b-1}) \oplus \mathrm{Im}(d^{b-1})$. The complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-2} \to \mathrm{Ker}(d^{b-1})$ is a strictly perfect complex representing a perfect object E' with $H^i(E) = H^i(E')$ for $i \neq b$. Hence we conclude by our induction hypothesis. □

20.50. Duals

0FP7 In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O}_X)$ has a dual if and only if it is perfect (this follows from Example 20.50.7 and Lemma 20.50.8).

0FP8 **Lemma 20.50.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The category of complexes of $\mathcal{O}_X$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a symmetric monoidal category (for sign rules, see More on Algebra, Section 15.73).*

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}_X$ in degree 0 and zero in other degrees with obvious isomorphisms $\mathrm{Tot}(\mathbf{1} \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. □

0FP9 **Example 20.50.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $\mathcal{O}_X$-modules such that each $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module. In other words, there is an open covering $X = \bigcup U_i$ such that $\mathcal{F}^\bullet|_{U_i}$ is a strictly perfect complex. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)$$

as in Section 20.41. Let

$$\eta : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to \mathcal{O}_X$ are as in Modules, Example 17.18.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 15.73.2.

0FPA **Lemma 20.50.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}_X$-modules*

(Categories, Definition 4.43.5) then $\mathcal{F}^\bullet$ is a locally bounded complex whose terms are locally direct summands of finite free $\mathcal{O}_X$-modules and the left dual is as constructed in Example 20.50.2.

Proof. By uniqueness of left duals (Categories, Remark 4.43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to \mathcal{O}_X$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{\mathcal{G}^{-n}}$. Hence we see that $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module by Modules, Lemma 17.18.2. Since the sum $\eta = \sum \eta_n$ is locally finite, we conclude that $\mathcal{F}^\bullet$ is locally bounded. □

0G40 **Lemma 20.50.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L, M \in D(\mathcal{O}_X)$. If K is perfect, then the map*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

of Lemma 20.42.9 is an isomorphism.

Proof. Since the map is globally defined and since formation of the right and left hand side commute with localization (see Lemma 20.42.3), to prove this we may work locally on X. Thus we may assume K is represented by a strictly perfect complex $\mathcal{E}^\bullet$.

If $K_1 \to K_2 \to K_3$ is a distinguished triangle in $D(\mathcal{O}_X)$, then we get distinguished triangles

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_1 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_2 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_3$$

and

$$R\mathcal{H}om(R\mathcal{H}om(K_1, L), M) \to R\mathcal{H}om(R\mathcal{H}om(K_2, L), M) R\mathcal{H}om(R\mathcal{H}om(K_3, L), M)$$

See Section 20.26 and Lemma 20.42.4. The arrow of Lemma 20.42.9 is functorial in K hence we get a morphism between these distinguished triangles. Thus, if the result holds for K_1 and K_3, then the result holds for K_2 by Derived Categories, Lemma 13.4.3.

Combining the remarks above with the distinguished triangles

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq n-1}\mathcal{E}^\bullet$$

of stupid trunctions, we reduce to the case where K consists of a direct summand of a finite free $\mathcal{O}_X$-module placed in some degree. By an obvious compatibility of the problem with direct sums (similar to what was said above) and shifts this reduces us to the case where $K = \mathcal{O}_X^{\oplus n}$ for some integer n. This case is clear. □

08DQ **Lemma 20.50.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms*

$$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee = R\mathcal{H}om(K, M)$$

and

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, M)$$

for M in $D(\mathcal{O}_X)$.

Proof. By Lemma 20.42.9 there is a canonical map

$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$

which is an isomorphism by Lemma 20.50.4. To check the other statements we will use without further mention that formation of internal hom commutes with restriction to opens (Lemma 20.42.3). We may check $K^\vee$ is perfect locally on X. By Lemma 20.42.8 to see the final statement it suffices to check that the map (20.42.8.1)

$$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is local on X as well. Hence it suffices to prove these two statements K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 20.46.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^{-n})^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-n}, \mathcal{O}_X)$ in degree n. Since $\mathcal{E}^{-n}$ is a direct summand of a finite free $\mathcal{O}_X$-module, so is $(\mathcal{E}^{-n})^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. To see that (20.42.8.1) is an isomorphism, represent M by a complex $\mathcal{F}^\bullet$. By Lemma 20.46.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}_X} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (20.42.8.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module and $\mathcal{F}$ is any $\mathcal{O}_X$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

0FPB **Lemma 20.50.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The derived category $D(\mathcal{O}_X)$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 15.73).*

Proof. Omitted. Compare with Lemma 20.50.1. □

0FPC **Example 20.50.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ as in Lemma 20.50.5. Then the map

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O}_X \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee$$

the map sending 1 to the section corresponding to id_K under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow \mathcal{O}_X$$

the evaluation map (to construct it you can use Lemma 20.42.5 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for K as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{K^\vee}$.

0FPD **Lemma 20.50.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let M be an object of $D(\mathcal{O}_X)$. If M has a left dual in the monoidal category $D(\mathcal{O}_X)$ (Categories, Definition 4.43.5) then M is perfect and the left dual is as constructed in Example 20.50.7.*

Proof. Let $x \in X$. It suffices to find an open neighbourhood U of x such that M restricts to a perfect complex over U. Hence during the proof we can (finitely often) replace X by an open neighbourhood of x. Let N, η, ϵ be a left dual.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules representing M. Choose a K-flat complex $\mathcal{N}^\bullet$ representing N whose terms are flat $\mathcal{O}_X$-modules, see Lemma 20.26.12. Consider the map

$$\eta : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

After shrinking X we can find an integer N and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections f_i and g_i of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum\nolimits_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}_X$-modules containing the sections f_i for $i = 1, \ldots, N$. Since $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet) \subset \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that η factors through

$$\tilde{\eta} : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

Denoting K the object of $D(\mathcal{O}_X)$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\eta \otimes 1} & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1 \otimes \epsilon} & M \\ & \searrow^{\tilde{\eta} \otimes 1} & \uparrow & & \uparrow \\ & & K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1 \otimes \epsilon} & K \end{array}$$

Since the composition of the upper row is the identity on M we conclude that M is a direct summand of K in $D(\mathcal{O}_X)$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a}\tau_{\leq b}\mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus M is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex and we conclude we may assume M is in $D^b(\mathcal{O}_X)$. (Recall that the process above involves shrinking X.)

Since M is in $D^b(\mathcal{O}_X)$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.17.7 and Derived Categories, Lemma 13.15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a}\mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume M is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex of flat modules. In particular, M has finite tor amplitude.

Say M has tor amplitude in $[a, b]$. Assuming M is m-pseudo-coherent we are going to show that (after shrinking X) we may assume M is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 20.49.4 and the fact that M is $(b+1)$-pseudo-coherent in any case. After shrinking X we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O}_X)$ such that $H^i(\alpha)$ is an

isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent L by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 13.27.3. Then M is represented by the complex

$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections f_i of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}_X$-submodule generated by the sections f_i for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{E}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & \longrightarrow & \sigma_{\leq m-1}\mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1]
\end{array}$$

where all the morphisms are as indicated above. Denote K the object of $D(\mathcal{O}_X)$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O}_X)$ such that the composition $M \to K \to M$ is the identity on M. We don't know that the diagram

$$\begin{array}{ccccc}
\mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & = & K \\
{\scriptstyle \mathrm{id}}\uparrow & & & & \uparrow{\scriptstyle s} \\
\mathcal{E}^\bullet & \xrightarrow{i} & \mathcal{M}^\bullet & = & M
\end{array}$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 20.46.8 after shrinking X we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of σ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that σ itself is homotopic to zero! This means that we have a commutative solid diagram

$$\begin{array}{ccccccc}
\mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & K & \longrightarrow & \sigma_{\leq m-1}\mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & {\scriptstyle s}\uparrow & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1]
\end{array}$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m-1$ we see that we obtain

$$\begin{array}{ccccc} H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ H^{m-1}(K) & \longrightarrow & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \end{array}$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m-1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}_X^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}_X^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m-1$ and the proof is complete. □

20.51. Miscellany

0GM6 Some results which do not fit anywhere else.

0DJI **Lemma 20.51.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O}_X)$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(\mathcal{O}_X)$ we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 20.50.5 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^\vee = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (20.42.0.1) and the fact that derived tensor product commutes with direct sums. □

0FVB **Lemma 20.51.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K and E be objects of $D(\mathcal{O}_X)$ with E perfect. The diagram*

$$\begin{array}{ccc} H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee) \times H^0(X, E) & \longrightarrow & H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(E, K) \times H^0(X, E) & \longrightarrow & H^0(X, K) \end{array}$$

commutes where the top horizontal arrow is the cup product, the right vertical arrow uses $\epsilon : E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to \mathcal{O}_X$ *(Example 20.50.7), the left vertical arrow uses Lemma 20.50.5, and the bottom horizontal arrow is the obvious one.*

Proof. We will abbreviate $\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X}$ and $\mathcal{O} = \mathcal{O}_X$. We will identify E and K with $R\mathcal{H}om(\mathcal{O}, E)$ and $R\mathcal{H}om(\mathcal{O}, K)$ and we will identify $E^\vee$ with $R\mathcal{H}om(E, \mathcal{O})$.

Let $\xi \in H^0(X, K \otimes E^\vee)$ and $\eta \in H^0(X, E)$. Denote $\tilde{\xi} : \mathcal{O} \to K \otimes E^\vee$ and $\tilde{\eta} : \mathcal{O} \to E$ the corresponding maps in $D(\mathcal{O})$. By Lemma 20.31.1 the cup product $\xi \cup \eta$ corresponds to $\tilde{\xi} \otimes \tilde{\eta} : \mathcal{O} \to K \otimes E^\vee \otimes E$.

We claim the map $\xi' : E \to K$ corresponding to ξ by Lemma 20.50.5 is the composition

$$E = \mathcal{O} \otimes E \xrightarrow{\tilde{\xi} \otimes 1_E} K \otimes E^\vee \otimes E \xrightarrow{1_K \otimes \epsilon} K$$

The construction in Lemma 20.50.5 uses the evaluation map (20.42.8.1) which in turn is constructed using the identification of E with $R\mathcal{H}om(\mathcal{O}, E)$ and the composition $\underline{\circ}$ constructed in Lemma 20.42.5. Hence ξ' is the composition

$$\begin{aligned} E = \mathcal{O} \otimes R\mathcal{H}om(\mathcal{O}, E) &\xrightarrow{\tilde{\xi} \otimes 1} R\mathcal{H}om(\mathcal{O}, K) \otimes R\mathcal{H}om(E, \mathcal{O}) \otimes R\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ} \otimes 1} R\mathcal{H}om(E, K) \otimes R\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ}} R\mathcal{H}om(\mathcal{O}, K) = K \end{aligned}$$

The claim follows immediately from this and the fact that the composition $\underline{\circ}$ constructed in Lemma 20.42.5 is associative (insert future reference here) and the fact that ϵ is defined as the composition $\underline{\circ} : E^\vee \otimes E \to \mathcal{O}$ in Example 20.50.7.

Using the results from the previous two paragraphs, we find the statement of the lemma is that $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes \tilde{\eta})$ is equal to $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes 1_E) \circ (1_{\mathcal{O}} \otimes \tilde{\eta})$ which is immediate. □

0GM7 **Lemma 20.51.3.** *Let* $h : X \to Y$ *be a morphism of ringed spaces. Let* K, M *be objects of* $D(\mathcal{O}_Y)$*. The canonical map*

$$Lh^* R\mathcal{H}om(K, M) \longrightarrow R\mathcal{H}om(Lh^* K, Lh^* M)$$

of Remark 20.42.13 is an isomorphism in the following cases

1. *K is perfect,*
2. *h is flat, K is pseudo-coherent, and M is (locally) bounded below,*
3. *$\mathcal{O}_X$ has finite tor dimension over $h^{-1}\mathcal{O}_Y$, K is pseudo-coherent, and M is (locally) bounded below,*

Proof. Proof of (1). The question is local on Y, hence we may assume that K is represented by a strictly perfect complex $\mathcal{E}^\bullet$, see Section 20.49. Choose a K-flat complex $\mathcal{F}^\bullet$ representing M. Apply Lemma 20.46.9 to see that $R\mathcal{H}om(K, L)$ is represented by the complex $\mathcal{H}^\bullet = \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ with terms $\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$. By the construction of Lh^* in Section 20.27 we see that $Lh^* K$ is represented by the strictly perfect complex $h^*\mathcal{E}^\bullet$ (Lemma 20.46.4). Similarly, the object $Lh^* M$ is represented by the complex $h^*\mathcal{F}^\bullet$. Finally, the object $Lh^* R\mathcal{H}om(K, M)$ is represented by $h^*\mathcal{H}^\bullet$ as $\mathcal{H}^\bullet$ is K-flat by Lemma 20.46.10. Thus to finish the proof it suffices to show that $h^*\mathcal{H}^\bullet = \mathcal{H}om^\bullet(h^*\mathcal{E}^\bullet, h^*\mathcal{F}^\bullet)$. For this it suffices to note that $h^* \mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{H}om(h^*\mathcal{E}, \mathcal{F})$ whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module.

Proof of (2). Since h is flat, we can compute Lh^* by simply using h^* on any complex of $\mathcal{O}_Y$-modules. In particular we have $H^i(Lh^*K) = h^*H^i(K)$ for all $i \in \mathbf{Z}$. Say $H^i(M) = 0$ for $i < a$. Let $K' \to K$ be a morphism of $D(\mathcal{O}_Y)$ which defines an isomorphism $H^i(K') \to H^i(K)$ for all $i \geq b$. Then the corresponding maps

$$R\mathcal{H}om(K, M) \to R\mathcal{H}om(K', M)$$

and

$$R\mathcal{H}om(Lh^*K, Lh^*M) \to R\mathcal{H}om(Lh^*K', Lh^*M)$$

are isomorphisms on cohomology sheaves in degrees $< a - b$ (details omitted). Thus to prove the map in the statement of the lemma induces an isomorphism on cohomology sheaves in degrees $< a-b$ it suffices to prove the result for K' in those degrees. Also, as in the proof of part (1) the question is local on Y. Thus we may assume K is represented by a strictly perfect complex, see Section 20.47. This reduces us to case (1).

Proof of (3). The proof is the same as the proof of (2) except one uses that Lh^* has bounded cohomological dimension to get the desired vanishing. We omit the details. □

0GM8 **Lemma 20.51.4.** *Let X be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Let $x \in X$. The canonical map*

$$R\mathcal{H}om(K, M)_x \longrightarrow R\operatorname{Hom}_{\mathcal{O}_{X,x}}(K_x, M_x)$$

is an isomorphism in the following cases

(1) *K is perfect,*
(2) *K is pseudo-coherent and M is (locally) bounded below.*

Proof. Let $Y = \{x\}$ be the singleton ringed space with structure sheaf given by $\mathcal{O}_{X,x}$. Then apply Lemma 20.51.3 to the flat inclusion morphism $Y \to X$. □

20.52. Invertible objects in the derived category

0FPE We characterize invertible objects in the derived category of a ringed space (both in the case where the stalks of the structure sheaf are local and where not).

0FPF **Lemma 20.52.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. The category of $\mathcal{O}_X$-modules which are summands of finite free $\mathcal{O}_X$-modules is equivalent to the category of finite projective R-modules.*

Proof. Observe that a finite projective R-module is the same thing as a summand of a finite free R-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(X, \mathcal{E})$. The inverse functor is given by the construction of Modules, Lemma 17.10.5. □

0FPG **Lemma 20.52.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let M be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *M is invertible in $D(\mathcal{O}_X)$, see Categories, Definition 4.43.4, and*
(2) *there is a locally finite direct product decomposition*

$$\mathcal{O}_X = \prod\nolimits_{n \in \mathbf{Z}} \mathcal{O}_n$$

and for each n there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules, Definition 17.25.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O}_X)$.

If (1) and (2) hold, then M is a perfect object of $D(\mathcal{O}_X)$. If $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$ these condition are also equivalent to

(3) *there exists an open covering $X = \bigcup U_i$ and for each i an integer n_i such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree n_i.*

Proof. Assume (2). Consider the object $R\mathcal{H}om(M, \mathcal{O}_X)$ and the composition map

$$R\mathcal{H}om(M, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to \mathcal{O}_X$$

To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O}_X = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that

$$R\mathcal{H}om(\mathcal{H}^m, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{H}^n$$

is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}_X$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}_X} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}_X$-modules (Modules, Lemma 17.25.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O}_X)$ is symmetric monoidal, we conclude that M is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}_X}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O}_X = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}_X$ coming from the product decomposition. This shows that in order to prove (2) we may work locally on X.

Choose an object N of $D(\mathcal{O}_X)$ and an isomorphism $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N \cong \mathcal{O}_X$. Let $x \in X$. Then N is a left dual for M in the monoidal category $D(\mathcal{O}_X)$ and we conclude that M is perfect by Lemma 20.50.8. By symmetry we see that N is perfect. After replacing X by an open neighbourhood of x, we may assume M and N are represented by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N$ is represented by $\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$. After another shinking of X we may assume the mutually inverse isomorphisms $\mathcal{O}_X \to M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N$ and $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N \to \mathcal{O}_X$ are given by maps of complexes

$$\alpha : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

See Lemma 20.46.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O}_X)$. After shrinking X we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy θ with components

$$\theta^n : \mathrm{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathrm{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(X, \mathcal{O}_X)$. By Lemma 20.52.1 we find that we obtain

(1) $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective R-modules,
(2) $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective R-modules,
(3) α and β correspond to maps of complexes $a : R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \mathrm{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
(4) θ^n corresponds to a map $h^n : \mathrm{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \mathrm{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
(5) $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 15.127.4 we find a product decomposition $R = \prod_{a \leq n \leq b} R_n$ and invertible R_n-modules H^n such that $M^\bullet \cong \bigoplus_{a \leq n \leq b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each H^n is projective as an R-module. Correspondingly, using Lemma 20.52.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}_X[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O}_X)$. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}_X$ we conclude (2) is true.

If all stalks of $\mathcal{O}_X$ are local, then it is straightforward to prove the equivalence of (2) and (3). We omit the details. □

20.53. Compact objects

09J6 In this section we study compact objects in the derived category of modules on a ringed space. We recall that compact objects are defined in Derived Categories, Definition 13.37.1. On suitable ringed spaces the perfect objects are compact.

0F5Z **Lemma 20.53.1.** *Let X be a ringed space. Let $j : U \to X$ be the inclusion of an open. The $\mathcal{O}_X$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O}_X)$ if there exists an integer d such that*

(1) *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
(2) *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

Proof. Assume (1) and (2). Since $\mathrm{Hom}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by Sheaves, Lemma 6.31.8 we have $\mathrm{Hom}(j_!\mathcal{O}_U, K) = R\Gamma(U, K)$ for K in $D(\mathcal{O}_X)$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second assumption implies any direct sum of injective modules is acyclic for F. Let K_i be a family of objects of $D(\mathcal{O}_X)$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing K_i, see Injectives, Theorem 19.12.6. Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 13.32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 19.13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. □

09J7 **Lemma 20.53.2.** *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

(1) *X is quasi-compact,*
(2) *there exists a basis of quasi-compact open subsets, and*
(3) *the intersection of any two quasi-compact opens is quasi-compact.*

Let K be a perfect object of $D(\mathcal{O}_X)$. Then

(a) *K is a compact object of $D^+(\mathcal{O}_X)$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\mathrm{Hom}(K, M) = \bigoplus_{i \in I} \mathrm{Hom}(K, M_i)$.*
(b) *If X has finite cohomological dimension, i.e., if there exists a d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$, then K is a compact object of $D(\mathcal{O}_X)$.*

Proof. Let $K^\vee$ be the dual of K, see Lemma 20.50.5. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (b). Since $R\Gamma(X, K) = R\,\mathrm{Hom}(\mathcal{O}_X, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 20.19.1 this is a special case of Lemma 20.53.1

Proof of (a). Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}_X$-modules. By Lemma 20.19.1 we see that

$$H^p(X, \bigoplus\nolimits_{i\in I} \mathcal{I}_i) = \bigoplus\nolimits_{i\in I} H^p(X, \mathcal{I}_i) = 0$$

for all p. Now if $M = \bigoplus M_i$ is as in (a), then we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}_X$-modules $\mathcal{I}_i^\bullet$ representing M_i with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 13.18.3. By Injectives, Lemma 19.13.4 we see that the direct sum complex $\bigoplus \mathcal{I}_i^\bullet$ represents M. By Leray acyclicity (Derived Categories, Lemma 13.16.7) we see that

$$R\Gamma(X, M) = \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(X, M_i)$$

as desired. □

20.54. Projection formula

01E6 In this section we collect variants of the projection formula. The most basic version is Lemma 20.54.2. After we state and prove it, we discuss a more general version involving perfect complexes.

01E7 **Lemma 20.54.1.** *Let X be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, see Modules, Definition 17.14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Proof. This is true because under the assumptions of the lemma we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee, \mathcal{I})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 12.27.2. □

01E8 **Lemma 20.54.2.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y, see Modules, Definition 17.14.1. Then there exist isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \longrightarrow R^q f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

for all $q \geq 0$. In fact there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_* \mathcal{F} \longrightarrow Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

in $D^+(Y)$ functorial in $\mathcal{F}$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on X. Note that $f^*\mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 20.54.1. Apply f_* to see that

$$Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on Y. Details omitted. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Without any further assumptions there is a map

0B53 (20.54.2.1) $$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$ and Lemmas 20.27.3 and 20.28.1. A reasonably general version of the projection formula is the following.

0B54 **Lemma 20.54.3.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

in $D(\mathcal{O}_Y)$.

Proof. To check (20.54.2.1) is an isomorphism we may work locally on Y, i.e., we have to find an open covering $\{V_j \to Y\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_Y$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_Y$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_Y$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_Y[n]$ in which case it is trivial. □

Here is a case where the projection formula is true in complete generality.

0B55 **Lemma 20.54.4.** *Let $f : X \to Y$ be a morphism of ringed spaces such that f is a homeomorphism onto a closed subset. Then (20.54.2.1) is an isomorphism always.*

Proof. Since f is a homeomorphism onto a closed subset, the functor f_* is exact (Modules, Lemma 17.6.1). Hence Rf_* is computed by applying f_* to any representative complex. Choose a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_Y$-modules representing K and choose any complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules representing E. Then Lf^*K is represented by $f^*\mathcal{K}^\bullet$ which is a K-flat complex of $\mathcal{O}_X$-modules (Lemma 20.26.8). Thus the right hand side of (20.54.2.1) is represented by

$$f_*\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{K}^\bullet)$$

By the same reasoning we see that the left hand side is represented by

$$\mathrm{Tot}(f_*\mathcal{E}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{K}^\bullet)$$

Since f_* commutes with direct sums (Modules, Lemma 17.6.3) it suffices to show that

$$f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*\mathcal{K}) = f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{K}$$

for any $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{K}$. We will check this by checking on stalks. Let $y \in Y$. If $y \notin f(X)$, then the stalks of both sides are zero. If $y = f(x)$, then we see that we have to show

$$\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y) = \mathcal{E}_x \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y$$

(using Sheaves, Lemma 6.32.1 and Lemma 6.26.4). This equality holds and therefore the lemma has been proved. □

0B6B **Remark 20.54.5.** The map (20.54.2.1) is compatible with the base change map of Remark 20.28.3 in the following sense. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a commutative diagram of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Then the diagram

$$\begin{array}{ccc} Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) & \xrightarrow{p} & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle b} \\ Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K & & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle t} \\ Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K & & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(g')^*Lf^*K) \\ & \searrow^{p} & \downarrow{\scriptstyle c} \\ & & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(f')^*Lg^*K) \end{array}$$

is commutative. Here arrows labeled t are gotten by an application of Lemma 20.27.3, arrows labeled b by an application of Remark 20.28.3, arrows labeled p by an application of (20.54.2.1), and c comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

20.55. An operator introduced by Berthelot and Ogus

0GT2 This section continuous the discussion started in More on Algebra, Section 15.96. We strongly encourage the reader to read that section first.

0GT3 **Lemma 20.55.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. Consider the following two conditions*

1. *for every $x \in X$ there exists an open neighbourhood $U \subset X$ of x and $f \in \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ and $f : \mathcal{O}_U \to \mathcal{O}_U$ is injective, and*
2. *$\mathcal{I}$ is invertible as an $\mathcal{O}_X$-module.*

Then (1) implies (2) and the converse is true if all stalks $\mathcal{O}_{X,x}$ of the structure sheaf are local rings.

Proof. Omitted. Hint: Use Modules, Lemma 17.25.4. □

0GT4 **Situation 20.55.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals satisfying condition (1) of Lemma 20.55.1[8].

0GT5 **Lemma 20.55.3.** *In Situation 20.55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *the subsheaf $\mathcal{F}[\mathcal{I}] \subset \mathcal{F}$ of sections annihilated by $\mathcal{I}$ is zero,*
(2) *the subsheaf $\mathcal{F}[\mathcal{I}^n]$ is zero for all $n \geq 1$,*
(3) *the multiplication map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ is injective,*
(4) *for every open $U \subset X$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ for some $f \in \mathcal{I}(U)$ the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ is injective,*
(5) *for every $x \in X$ and generator f of the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ the element f is a nonzerodivisor on the stalk $\mathcal{F}_x$.*

Proof. Omitted. □

In Situation 20.55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If the equivalent conditions of Lemma 20.55.3 hold, then we will say that $\mathcal{F}$ is *$\mathcal{I}$-torsion free*. If so, then for any $i \in \mathbf{Z}$ we will denote

$$\mathcal{I}^i\mathcal{F} = \mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{F}$$

so that we have inclusions

$$\ldots \subset \mathcal{I}^{i+1}\mathcal{F} \subset \mathcal{I}^i\mathcal{F} \subset \mathcal{I}^{i-1}\mathcal{F} \subset \ldots$$

The modules $\mathcal{I}^i\mathcal{F}$ are locally isomorphic to $\mathcal{F}$ as $\mathcal{O}_X$-modules, but not globally.

Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules with differentials $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$. In this case we define $\eta_\mathcal{I}\mathcal{F}^\bullet$ to be the complex with terms

$$\begin{aligned}(\eta_\mathcal{I}\mathcal{F})^i &= \mathrm{Ker}\left(d^i, -1 : \mathcal{I}^i\mathcal{F}^i \oplus \mathcal{I}^{i+1}\mathcal{F}^{i+1} \to \mathcal{I}^i\mathcal{F}^{i+1}\right) \\ &= \mathrm{Ker}\left(d^i : \mathcal{I}^i\mathcal{F}^i \to \mathcal{I}^i\mathcal{F}^{i+1}/\mathcal{I}^{i+1}\mathcal{F}^{i+1}\right)\end{aligned}$$

and differential induced by d^i. In other words, a local section s of $(\eta_\mathcal{I}\mathcal{F})^i$ is the same thing as a local section s of $\mathcal{I}^i\mathcal{F}^i$ such that its image $d^i(s)$ in $\mathcal{I}^i\mathcal{F}^{i+1}$ is in the subsheaf $\mathcal{I}^{i+1}\mathcal{F}^{i+1}$. Observe that $\eta_\mathcal{I}\mathcal{F}^\bullet$ is another complex of $\mathcal{I}$-torsion free modules.

Let $a^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then we obtain a map of complexes

$$\eta_\mathcal{I}a^\bullet : \eta_\mathcal{I}\mathcal{F}^\bullet \longrightarrow \eta_\mathcal{I}\mathcal{G}^\bullet$$

induced by the maps $\mathcal{I}^i\mathcal{F}^i \to \mathcal{I}^i\mathcal{G}^i$. The reader checks that we obtain an endofunctor on the category of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules.

If $a^\bullet, b^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ are two maps of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules and $h = \{h^i : \mathcal{F}^i \to \mathcal{G}^{i-1}\}$ is a homotopy between $a^\bullet$ and $b^\bullet$, then we define $\eta_\mathcal{I}h$ to be the family of maps $(\eta_\mathcal{I}h)^i : (\eta_\mathcal{I}\mathcal{F})^i \to (\eta_\mathcal{I}\mathcal{G})^{i-1}$ which sends x to $h^i(x)$; this makes sense as x a local section of $\mathcal{I}^i\mathcal{F}^i$ implies $h^i(x)$ is a local section of $\mathcal{I}^i\mathcal{G}^{i-1}$ which is certainly contained in $(\eta_\mathcal{I}\mathcal{G})^{i-1}$. The reader checks that $\eta_\mathcal{I}h$ is a homotopy between $\eta_\mathcal{I}a^\bullet$ and $\eta_\mathcal{I}b^\bullet$. All in all we see that we obtain a functor

$$\eta_f : K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules}) \longrightarrow K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules})$$

[8]The discussion in this section can be generalized to the case where all we require is that $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module as defined in Modules, Section 17.25.

on the homotopy category (Derived Categories, Section 13.8) of the additive category of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is no sense in which $\eta_{\mathcal{I}}$ is an exact functor of triangulated categories; compare with More on Algebra, Example 15.96.1.

0GT6 **Lemma 20.55.4.** *In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. For $x \in X$ choose a generator $f \in \mathcal{I}_x$. Then the stalk $(\eta_{\mathcal{I}}\mathcal{F}^\bullet)_x$ is canonically isomorphic to the complex $\eta_f \mathcal{F}^\bullet_x$ constructed in More on Algebra, Section 15.96.*

Proof. Omitted. □

0F8N **Lemma 20.55.5.** *In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \left(H^i(\mathcal{F}^\bullet)/H^i(\mathcal{F}^\bullet)[\mathcal{I}]\right) \longrightarrow H^i(\eta_{\mathcal{I}}\mathcal{F}^\bullet)$$

of cohomology sheaves.

Proof. We define a map

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} H^i(\mathcal{F}^\bullet) \longrightarrow H^i(\eta_{\mathcal{I}}\mathcal{F}^\bullet)$$

as follows. Let g be a local section of $\mathcal{I}^{\otimes i}$ and let $\overline{s}$ be a local section of $H^i(\mathcal{F}^\bullet)$. Then $\overline{s}$ is (locally) the class of a local section s of $\mathrm{Ker}(d^i : \mathcal{F}^i \to \mathcal{F}^{i+1})$. Then we send $g \otimes \overline{s}$ to the local section gs of $(\eta_{\mathcal{I}}\mathcal{F})^i \subset \mathcal{I}^i\mathcal{F}$. Of course gs is in the kernel of d^i on $\eta_{\mathcal{I}}\mathcal{F}^\bullet$ and hence defines a local section of $H^i(\eta_{\mathcal{I}}\mathcal{F}^\bullet)$. Checking that this is well defined is without problems. We claim that this map factors through an isomorphism as given in the lemma. This we my check on stalks and hence via Lemma 20.55.4 this translates into the result of More on Algebra, Lemma 15.96.2. □

0F8P **Lemma 20.55.6.** *In Situation 20.55.2 let $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then the induced map $\eta_{\mathcal{I}}\mathcal{F}^\bullet \to \eta_{\mathcal{I}}\mathcal{G}^\bullet$ is a quasi-isomorphism too.*

Proof. This is true because the isomorphisms of Lemma 20.55.5 are compatible with maps of complexes. □

0F8Q **Lemma 20.55.7.** *In Situation 20.55.2 there is an additive functor[9] $L\eta_{\mathcal{I}} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ such that if M in $D(\mathcal{O}_X)$ is represented by a complex $\mathcal{F}^\bullet$ of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules, then $L\eta_{\mathcal{I}}M = \eta_{\mathcal{I}}\mathcal{F}^\bullet$. Similarly for morphisms.*

Proof. Denote $\mathcal{T} \subset \mathit{Mod}(\mathcal{O}_X)$ the full subcategory of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(\mathit{Mod}(\mathcal{O}_X)) = K(\mathcal{O}_X)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(\mathcal{O}_X)$. Let $S \subset \mathrm{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 13.5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(\mathcal{O}_X)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules, there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ with $\mathcal{F}^\bullet$ a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. By Lemma 20.26.12

[9] Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles.

we can find a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ such that the complex $\mathcal{F}^\bullet$ is K-flat (we won't use this) and consists of flat $\mathcal{O}_X$-modules $\mathcal{F}^i$. By the third characterization of Lemma 20.55.3 we see that a flat $\mathcal{O}_X$-module is an $\mathcal{I}$-torsion free $\mathcal{O}_X$-module and we are done.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion following Lemma 20.55.3 this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(\mathcal{O}_X) \to D(\mathcal{O}_X)$$

which according to Lemma 20.55.6 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(\mathcal{O}_X)$ by Categories, Lemma 4.27.8. □

In Situation 20.55.2 let us construct the Bockstein operators. First we observe that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{i+1} & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{I}^{i+1}/\mathcal{I}^{i+2} & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+2} & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+1} & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences of $\mathcal{O}_X$-modules. Let M be an object of $D(\mathcal{O}_X)$. Tensoring the above diagram with M gives a morphism

$$\begin{array}{ccccc} M \otimes^{\mathbf{L}} \mathcal{I}^{i+1} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1} \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+2} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1} \end{array}$$

of distinguished triangles. The long exact sequence of cohomology sheaves associated the bottom triangle in particular determines the *Bockstein operator*

$$\beta = \beta^i : H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \longrightarrow H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2})$$

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

0GT7 (20.55.7.1)
$$\begin{array}{ccc} H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) & \xrightarrow{\delta} & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}) \\ & \searrow^{\beta} & \downarrow \\ & & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2}) \end{array}$$

of the Bockstein operator where δ is the boundary operator coming from the top distinguished triangle in the commutative diagram above. We obtain a complex

0GT8 (20.55.7.2)
$$H^\bullet(M/\mathcal{I}) = \begin{bmatrix} \cdots \\ \downarrow \\ H^{i-1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i-1}/\mathcal{I}^i) \\ \downarrow \beta \\ H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \\ \downarrow \beta \\ H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2}) \\ \downarrow \\ \cdots \end{bmatrix}$$

i.e., that $\beta \circ \beta = 0$. Namely, we can check this on stalks and in this case we can deduce it from the corresponding result in algebra shown in More on Algebra, Section 15.96. Alternative proof: the short exact sequences $0 \to \mathcal{I}^{i+1}/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+1} \to 0$ define maps $b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1]$ in $D(\mathcal{O}_X)$ which induce the maps β above by tensoring with M and taking cohomology sheaves. Then one shows that the composition $b^{i+1}[1] \circ b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1] \to (\mathcal{I}^{i+2}/\mathcal{I}^{i+3})[2]$ is zero in $D(\mathcal{O}_X)$ by using the criterion in Derived Categories, Lemma 13.27.7 using that the module $\mathcal{I}^i/\mathcal{I}^{i+3}$ is an extension of $\mathcal{I}^{i+1}/\mathcal{I}^{i+3}$ by $\mathcal{I}^i/\mathcal{I}^{i+1}$.

0GT9 **Lemma 20.55.8.** *In Situation 20.55.2 let M be an object of $D(\mathcal{O}_X)$. There is a canonical isomorphism*

$$L\eta_{\mathcal{I}} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I} \longrightarrow H^\bullet(M/\mathcal{I})$$

in $D(\mathcal{O}_X)$ where the right hand side is the complex (20.55.7.2).

Proof. By the construction of $L\eta_{\mathcal{I}}$ in Lemma 20.55.6 we may assume M is represented by a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$. Then $L\eta_{\mathcal{I}} M$ is represented by the complex $\eta_{\mathcal{I}}\mathcal{F}^\bullet$ which is a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules as well. Thus $L\eta_{\mathcal{I}} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I}$ is represented by the complex $\eta_{\mathcal{I}}\mathcal{F}^\bullet \otimes \mathcal{O}_X/\mathcal{I}$. Similarly, the complex $H^\bullet(M/\mathcal{I})$ has terms $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$.

Let f be a local generator for $\mathcal{I}$. Let s be a local section of $(\eta_{\mathcal{I}}\mathcal{F})^i$. Then we can write $s = f^i s'$ for a local section s' of $\mathcal{F}^i$ and similarly $d^i(s) = f^{i+1}t$ for a local section t of $\mathcal{F}^{i+1}$. Thus d^i maps $f^i s'$ to zero in $\mathcal{F}^{i+1} \otimes \mathcal{I}^i/\mathcal{I}^{i+1}$. Hence we may map s to the class of $f^i s'$ in $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$. This rule defines a map

$$(\eta_{\mathcal{I}}\mathcal{F})^i \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$$

of $\mathcal{O}_X$-modules. A calculation shows that these maps are compatible with differentials (essentially because β sends the class of $f^i s'$ to the class of $f^{i+1}t$), whence a map of complexes representing the arrow in the statement of the lemma.

To finish the proof, we observe that the construction given in the previous paragraph agrees on stalks with the maps constructed in More on Algebra, Lemma 15.96.6 hence we conclude. □

0F9W **Lemma 20.55.9.** *In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\eta_{\mathcal{I}}(\mathcal{F}^\bullet \otimes \mathcal{L}) = (\eta_{\mathcal{I}}\mathcal{F}^\bullet) \otimes \mathcal{L}$.*

Proof. Immediate from the construction. □

0GTA **Lemma 20.55.10.** *In Situation 20.55.2 let M be an object of $D(\mathcal{O}_X)$. Let $x \in X$ with $\mathcal{O}_{X,x}$ nonzero. If $H^i(M)_x$ is finite free over $\mathcal{O}_{X,x}$, then $H^i(L\eta_{\mathcal{I}} M)_x$ is finite free over $\mathcal{O}_{X,x}$ of the same rank.*

Proof. Namely, say $f \in \mathcal{O}_{X,x}$ generates the stalk $\mathcal{I}_x$. Then f is a nonzerodivisor in $\mathcal{O}_{X,x}$ and hence $H^i(M)_x[f] = 0$. Thus by Lemma 20.55.5 we see that $H^i(L\eta_{\mathcal{I}} M)_x$ is isomorphic to $\mathcal{I}_x^i \otimes_{\mathcal{O}_{X,x}} H^i(M)_x$ which is free of the same rank as desired. □

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CHAPTER 21

Cohomology on Sites

01FQ

21.1. Introduction

01FR In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [**AGV71**], [**God73**] and [**Ive86**].

21.2. Cohomology of sheaves

01FT Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 19.7.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

071C (21.2.0.1) $$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U*. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the *global sections* of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

071D (21.2.0.2) $$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(e, \mathcal{F})$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$* as follows

071E (21.2.0.3) $$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 7.15.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

071F (21.2.0.4) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(\mathcal{C}) \to Ab(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 18.6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 19.8.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

071G (21.2.0.5) $$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the *the ith cohomology group of $\mathcal{F}$ over U*. The family of functors $H^i(U, -)$ forms a universal δ-functor $Mod(\mathcal{O}) \to \mathrm{Mod}_{\mathcal{O}(U)}$. Similarly

071H (21.2.0.6) $$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

it the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$*. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Mod(\mathcal{C}) \to \mathrm{Mod}_{\Gamma(\mathcal{C},\mathcal{O})}$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 18.7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

071I (21.2.0.7) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$.

21.3. Derived functors

071J We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.8.1 and Definition 13.11.3. By Derived Categories, Remark 13.24.3 there exists a resolution functor

$$j = j_{(\mathcal{C},\mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : Mod(\mathcal{O}) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 13.20 constructed using the resolution functor j just described:

05U5 (21.3.0.1) $$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 13.25.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O})$, $\mathrm{Comp}^+(Mod(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the $i$the right derived functor

05U6 (21.3.0.2) $$R^i F = H^i \circ RF : Mod(\mathcal{O}) \longrightarrow \mathcal{B}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\mathrm{Mod}_R)$ and $D(R) = D(\mathrm{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}) \longrightarrow \mathrm{Mod}_{\mathcal{O}(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (21.2.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (21.2.0.6). If $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}')$ which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^i f_* \mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (21.2.0.7). The displayed functors above are exact functor of derived categories.

21.4. First cohomology and torsors

03AG

03AH **Definition 21.4.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *pseudo torsor*, or more precisely a *pseudo $\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A *morphism of pseudo $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a pseudo $\mathcal{G}$-torsor such that in addition

(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A *morphism of $\mathcal{G}$-torsors* is simply a morphism of pseudo $\mathcal{G}$-torsors. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

03AI **Lemma 21.4.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$.*

Proof. Omitted. □

03AJ **Lemma 21.4.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in \mathrm{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s' = s$. Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. □

21.5. First cohomology and extensions

03F0

03F1 **Lemma 21.5.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection*

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ in $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tilde{\xi}} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

see Homology, Section 12.6. □

The following lemma will be superseded by the more general Lemma 21.12.4.

03F2 **Lemma 21.5.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $Ab(\mathcal{C}) = Mod(\underline{\mathbf{Z}})$ we may apply Lemma 21.5.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{Mod(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $Mod(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension

$\mathcal{E}_{ab}$, like so:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} & \longrightarrow & \mathcal{E}_{ab} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $Mod(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 18.28.9. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can form the pushout via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. □

21.6. First cohomology and invertible sheaves

040D The Picard group of a ringed site is defined in Modules on Sites, Section 18.32.

040E **Lemma 21.6.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = \mathrm{Pic}(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules on Sites, Lemma 18.40.7. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{array}{c}\text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism}\end{array} \longrightarrow \begin{array}{c}\text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism}\end{array}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 21.4.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 21.4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\mathrm{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g)+(s', g') = (s, g+(s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

21.7. Locality of cohomology

01FU The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

03F3 **Lemma 21.7.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

(1) *If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
(2) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.*

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 18.19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 12.29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). □

The following lemma will be used to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

03YU **Lemma 21.7.2.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 7.21.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in \mathrm{Ob}(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 12.29.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 7.21.8 which is an exact functor. Hence the lemma does indeed apply. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical *restriction mapping*

01FV (21.7.2.1) $$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 21.10.5.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

01FW **Lemma 21.7.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

072W **Lemma 21.7.4.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. □

21.8. The Čech complex and Čech cohomology

03AK Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 7.6.1. Assume that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

03AL (21.8.0.1) $$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0} \times_U \ldots \times_U U_{i_{p+1}}}$$

where the restriction is via the projection map

$$U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \times_U \widehat{U_{i_j}} \times_U \ldots \times_U U_{i_{p+1}}.$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

03AM **Definition 21.8.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

We observe that any covering $\{U_i \to U\}$ of a site $\mathcal{C}$ is a family of morphisms with fixed target to which the definition applies.

03AN **Lemma 21.8.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and*
(2) *for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map*
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
(see Sites, Section 7.10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every covering of $\mathcal{C}$. □

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 7.8. In this case we get a map of Čech complexes

03F4 (21.8.2.1) $$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

which in degree p is given by

$$\varphi(s)_{i_0 \ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0) \ldots \alpha(i_p)}$$

21.9. Čech cohomology as a functor on presheaves

03AO Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

03AP (21.9.0.1) $$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \mathrm{Comp}^+(Ab)$$

see Derived Categories, Definition 13.8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.13.9.

03AQ **Lemma 21.9.1.** *The functor given by Equation (21.9.0.1) is an exact functor (see Homology, Lemma 12.7.2).*

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PAb(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

03AR **Lemma 21.9.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $PAb(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 12.12.1).*

Proof. By Lemma 21.9.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 12.13.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

03AS **Lemma 21.9.3.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Consider the chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of abelian presheaves*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \to \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0} \times_U \ldots \times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PAb(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned} \mathrm{Hom}_{PAb(\mathcal{C})}(\bigoplus_{i_0 \ldots i_p} \mathbf{Z}_{U_{i_0} \times_U \ldots \times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0 \ldots i_p} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U \ldots \times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0 \ldots i_p} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}) \end{aligned}$$

see Modules on Sites, Lemma 18.4.2. □

03AT **Lemma 21.9.4.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of presheaves of Lemma 21.9.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U},\bullet})$ are zero for $i > 0$.*

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{array}{c} \vdots \\ \downarrow \\ \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

For any morphism $\varphi : V \to U$ denote $\mathrm{Mor}_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i \circ \varphi_i = \varphi\}$. We will use a similar notation for $\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c} \vdots \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

Next, we make the remark that we have

$$\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = \mathrm{Mor}_\varphi(V, U_{i_0}) \times \ldots \times \mathrm{Mor}_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \amalg B]$ we see that the complex becomes

$$\begin{array}{c} \vdots \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1}) \times \mathrm{Mor}_\varphi(V, U_{i_2})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \\ \downarrow \\ 0 \end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi (\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. □

03F5 **Lemma 21.9.5.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex*

$$\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U},\bullet}$ is the chain complex of Lemma 21.9.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 21.9.4 we saw that $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U},\bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 18.28.11, which applies since the presheaves $\mathbf{Z}_{\mathcal{U},i}$ are flat, and the proof of Lemma 21.9.4 shows that $H_0(\mathbf{Z}_{\mathcal{U},\bullet})$ is a flat presheaf also. □

03AU **Lemma 21.9.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 19.6.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.20.

Let $\mathcal{I}$ be an injective abelian presheaf. In this case the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact on $PAb(\mathcal{C})$. By Lemma 21.9.3 we have

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 21.9.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 21.9.2) satisfies the assumptions of Homology, Lemma 12.12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PAb(\mathcal{C})$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the total complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$ associated to this double complex, see Homology, Section 12.18. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 21.9.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

21.10. Čech cohomology and cohomology

03AV The relationship between cohomology and Čech cohomology comes from the fact that the Čech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Čech cohomology in the preceding section.

03F6 **Lemma 21.10.1.** *Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $PAb(\mathcal{C})$.*

Proof. Apply Homology, Lemma 12.29.1 to the categories $\mathcal{A} = Ab(\mathcal{C})$, $\mathcal{B} = PAb(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.3 to see that all assumptions of the lemma are satisfied.) □

03AW **Lemma 21.10.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $Ab(\mathcal{C})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. By Lemma 21.10.1 we see that $\mathcal{I}$ is an injective object in $PAb(\mathcal{C})$. Hence we can apply Lemma 21.9.6 (or its proof) to see the vanishing of higher Čech cohomology group. For the zeroth see Lemma 21.8.2. □

03AX **Lemma 21.10.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Ab(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Ab(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the associated total complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 12.18.3. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.25.4 to see that α is a quasi-isomorphism. Namely, Lemma 21.10.2 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

0A6G **Lemma 21.10.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be an abelian sheaf on $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(U, \mathcal{G})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{G})$ via the bijection of Lemma 21.4.3 with the set of isomorphism classes of $\mathcal{G}|_U$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{G}|_U$-torsor on $\mathcal{C}/U$ whose restriction to $\mathcal{C}/U_i$ is trivial. By Lemma 21.4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \times_U U_{i_1}$ there is a unique section $s_{i_0 i_1}$ of $\mathcal{G}$ such that $s_{i_0 i_1} \cdot s_{i_0}|_{U_{i_0}\times_U U_{i_1}} = s_{i_1}|_{U_{i_0}\times_U U_{i_1}}$. An easy computation shows that $s_{i_0 i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0 i_1})$. We omit the verification that this map is indeed an inverse. □

03AY **Lemma 21.10.5.** *Let $\mathcal{C}$ be a site. Consider the functor $i : Ab(\mathcal{C}) \to PAb(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

03AZ **Lemma 21.10.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.22.2) for the functors

$$i : Ab(\mathcal{C}) \to PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \to Ab.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 21.8.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 21.10.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PAb(\mathcal{C})$ by Lemma 21.9.6. Putting everything together gives the lemma. □

03F7 **Lemma 21.10.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. We will use the spectral sequence of Lemma 21.10.6. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

03F8 **Lemma 21.10.8.** *Let $\mathcal{C}$ be a site. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

03F9 **Lemma 21.10.9.** *(Variant of Cohomology, Lemma 20.11.8.) Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that*

(1) *For every $\mathcal{U} \in Cov$, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*
(3) *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \mathrm{Cov}$". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 21.10.2 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \mathrm{Cov}$. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 21.10.8 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \mathrm{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any covering $\mathcal{U} \in \mathrm{Cov}$. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in \mathrm{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in \mathrm{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

21.11. Second cohomology and gerbes

0CJZ Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site all of whose automorphism groups are commutative. In this situation the first and second cohomology groups of the sheaf of automorphisms (Stacks, Lemma 8.11.8) controls the existence of objects.

The following lemma will be made obsolete by a more complete discussion of this relationship we will add in the future.

0CK0 **Lemma 21.11.1.** *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site whose automorphism sheaves are abelian. Let $\mathcal{G}$ be the sheaf of abelian groups constructed in Stacks, Lemma 8.11.8. Let U be an object of $\mathcal{C}$ such that*

(1) *there exists a cofinal system of coverings $\{U_i \to U\}$ of U in $\mathcal{C}$ such that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_i \times_U U_j, \mathcal{G}) = 0$ for all i, j, and*
(2) *$H^2(U, \mathcal{G}) = 0$.*

Then there exists an object of $\mathcal{S}$ lying over U.

Proof. By Stacks, Definition 8.11.1 there exists a covering $\mathcal{U} = \{U_i \to U\}$ and x_i in $\mathcal{S}$ lying over U_i. Write $U_{ij} = U_i \times_U U_j$. By (1) after refining the covering we may assume that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_{ij}, \mathcal{G}) = 0$. Consider the sheaf

$$\mathcal{F}_{ij} = Isom(x_i|_{U_{ij}}, x_j|_{U_{ij}})$$

on $\mathcal{C}/U_{ij}$. Since $\mathcal{G}|_{U_{ij}} = Aut(x_i|_{U_{ij}})$ we see that there is an action

$$\mathcal{G}|_{U_{ij}} \times \mathcal{F}_{ij} \to \mathcal{F}_{ij}$$

by precomposition. It is clear that $\mathcal{F}_{ij}$ is a pseudo $\mathcal{G}|_{U_{ij}}$-torsor and in fact a torsor because any two objects of a gerbe are locally isomorphic. By our choice of the covering and by Lemma 21.4.3 these torsors are trivial (and hence have global sections by Lemma 21.4.2). In other words, we can choose isomorphisms

$$\varphi_{ij} : x_i|_{U_{ij}} \longrightarrow x_j|_{U_{ij}}$$

To find an object x over U we are going to massage our choice of these φ_{ij} to get a descent datum (which is necessarily effective as $p : \mathcal{S} \to \mathcal{C}$ is a stack). Namely, the obstruction to being a descent datum is that the cocycle condition may not hold. Namely, set $U_{ijk} = U_i \times_U U_j \times_U U_k$. Then we can consider

$$g_{ijk} = \varphi_{ik}^{-1}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

which is an automorphism of x_i over U_{ijk}. Thus we may and do consider g_{ijk} as a section of $\mathcal{G}$ over U_{ijk}. A computation (omitted) shows that $(g_{i_0 i_1 i_2})$ is a 2-cocycle in the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{G})$ of $\mathcal{G}$ with respect to the covering $\mathcal{U}$. By the spectral sequence of Lemma 21.10.6 and since $H^1(U_i, \mathcal{G}) = 0$ for all i we see that $\check{H}^2(\mathcal{U}, \mathcal{G}) \to H^2(U, \mathcal{G})$ is injective. Hence $(g_{i_0 i_1 i_2})$ is a coboundary by our assumption that $H^2(U, \mathcal{G}) = 0$. Thus we can find sections $g_{ij} \in \mathcal{G}(U_{ij})$ such that $g_{ik}^{-1}|_{U_{ijk}} g_{jk}|_{U_{ijk}} g_{ij}|_{U_{ijk}} = g_{ijk}$ for all i, j, k. After replacing φ_{ij} by $\varphi_{ij} g_{ij}^{-1}$ we see that φ_{ij} gives a descent datum on the objects x_i over U_i and the proof is complete. □

21.12. Cohomology of modules

03FA Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

03FB **Lemma 21.12.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An injective sheaf of modules is also injective as an object in the category $PMod(\mathcal{O})$.*

Proof. Apply Homology, Lemma 12.29.1 to the categories $\mathcal{A} = Mod(\mathcal{O})$, $\mathcal{B} = PMod(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.11 to see that all assumptions of the lemma are satisfied.) □

06YK **Lemma 21.12.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : Mod(\mathcal{C}) \to PMod(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Mod(\mathcal{O})$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

03FC **Lemma 21.12.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $Mod(\mathcal{O})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. Lemma 21.9.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) &= \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 18.9.2. By Lemma 21.12.1 we see that $\mathcal{I}$ is an injective object in $PMod(\mathcal{O})$. Hence $\mathrm{Hom}_{PMod(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 21.9.5 we see the vanishing of higher Čech cohomology groups. For the zeroth see Lemma 21.8.2. □

03FD **Lemma 21.12.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. By Derived Categories, Lemma 13.20.4 the δ-functor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 12.12.5. Since $Mod(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $Mod(\mathcal{O})$, see Homology, Lemma 12.12.4.

Let $\mathcal{I}$ be an injective object of $Mod(\mathcal{O})$. Apply Lemma 21.10.9 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\text{Cov} = \text{Cov}_\mathcal{C}$. Assumption (3) of that lemma holds by Lemma 21.12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 7.29.5 we see that the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows. □

060L **Lemma 21.12.5.** *Let $\mathcal{C}$ be a site. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \text{Ob}(\mathcal{C})$. The canonical map*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sites, Lemma 7.10.1. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 21.7.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 21.10.4 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 21.10.4), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 12.32.1 we conclude that $\check{\xi} = 0$ as desired. □

093X **Lemma 21.12.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : U' \to U$ be a monomorphism in $\mathcal{C}$. Then for any injective $\mathcal{O}$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : \mathcal{C}/U \to \mathcal{C}$ and $j' : \mathcal{C}/U' \to \mathcal{C}$ be the localization morphisms (Modules on Sites, Section 18.19). Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules

$$\text{Hom}_\mathcal{O}(j_!\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover below we describe a canonical map of $\mathcal{O}$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 12.27.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object V of $\mathcal{C}$ the $\mathcal{O}(V)$-module

$$\bigoplus\nolimits_{\varphi' \in \text{Mor}_\mathcal{C}(V, U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus\nolimits_{\varphi \in \text{Mor}_\mathcal{C}(V, U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 18.19.2. We take the map which maps the summand corresponding to φ' to the summand corresponding to $\varphi = a \circ \varphi'$ by the identity map on $\mathcal{O}(V)$. As a is a monomorphism, this map is injective. As sheafification is exact, the result follows. □

21.13. Totally acyclic sheaves

079X Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we set

$$\mathcal{F}(K) = \mathrm{Mor}_{PSh(\mathcal{C})}(K, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(K^\#, \mathcal{F})$$

The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $Mod(\mathcal{O}) \to Ab$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

Here are some observations:

(1) Since $\mathcal{F}(K) = \mathcal{F}(K^\#)$, we have $H^p(K, \mathcal{F}) = H^p(K^\#, \mathcal{F})$. Allowing K to be a presheaf in the definition above is a purely notational convenience.
(2) Suppose that $K = h_U$ or $K = h_U^\#$ for some object U of $\mathcal{C}$. Then $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 7.12.
(3) If $\mathcal{O} = \mathbf{Z}$ (the constant sheaf), then the cohomology groups are functors $H^p(K, -) : Ab(\mathcal{C}) \to Ab$ since $Mod(\mathcal{O}) = Ab(\mathcal{C})$ in this case.

We can translate some of our already proven results using this language.

079Y **Lemma 21.13.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.*

Proof. We may replace K by its sheafification and assume K is a sheaf. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 7.29.5 we may replace $\mathcal{C}$ by a "larger" site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 21.12.4. □

079Z **Lemma 21.13.2.** *Let $\mathcal{C}$ be a site. Let $K' \to K$ be a map of presheaves of sets on $\mathcal{C}$ whose sheafification is surjective. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p+1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.*

Proof. Since sheafification is exact, we see that $(K'_p)^\#$ is equal to $(K')^\# \times_{K^\#} \ldots \times_{K^\#} (K')^\#$ ($p+1$-factors). Thus we may replace K and K' by their sheafifications and assume $K \to K'$ is a surjective map of sheaves. After replacing $\mathcal{C}$ by a "larger" site as in Sites, Lemma 7.29.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 21.10.6 whose E_1 page is as indicated in the statement of the lemma. □

07A0 **Lemma 21.13.3.** *Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 7.30.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. By Sites, Lemmas 7.30.1 and 7.30.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 21.7.1. □

Keeping in mind Lemma 21.13.1 we see that the following definition is the "correct one" also for sheaves of modules on ringed sites.

072Y **Definition 21.13.4.** Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is *totally acyclic*[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being totally acyclic is an intrinsic property, i.e., preserved under equivalences of topoi. A totally acyclic sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being totally acyclic is strictly stronger. Here is a characterization of totally acyclic sheaves which is sometimes useful.

07A1 **Lemma 21.13.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If*

(1) *$H^p(U, \mathcal{F}) = 0$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *for every surjection $K' \to K$ of sheaves of sets the extended Čech complex*

$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$

is exact,

then $\mathcal{F}$ is totally acyclic (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection

$$K' = \coprod h_{U_i}^\# \longrightarrow K$$

see Sites, Lemma 7.12.5. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is totally acyclic. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $Sh(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with

$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$

covering to $H^{p+q}(K, \mathcal{F})$, see Lemma 21.13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. □

21.14. The Leray spectral sequence

072X The key to proving the existence of the Leray spectral sequence is the following lemma.

072Z **Lemma 21.14.1.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $Mod(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is totally acyclic.*

[1] Although this terminology is is used in [**AGV71**, Vbis, Proposition 1.3.10] this is probably nonstandard notation. In [**AGV71**, V, Definition 4.1] this property is dubbed "flasque", but we cannot use this because it would clash with our definition of flasque sheaves on topological spaces. Please email stacks.project@gmail.com if you have a better suggestion.

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 18.7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by "larger" sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(V)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 21.12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 21.10.9. □

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

0730 **Lemma 21.14.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.*

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 18.31.2. Hence the result follows from Homology, Lemma 12.29.1. □

0731 **Lemma 21.14.3.** *Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A totally acyclic sheaf is right acyclic for the following functors:*

1. *the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,*
2. *the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,*
3. *the functor $\Gamma(\mathcal{C}, -)$ of global sections,*
4. *the functor f_* for any morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.*

Proof. Part (2) is the definition of a totally acyclic sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of totally acyclic sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $Sh(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 18.7.2 that f is given by a morphism of sites. In this case we see that $R^i f_*$, $i > 0$ of a totally acyclic sheaf are zero by the description of higher direct images in Lemma 21.7.4. □

08J6 **Remark 21.14.4.** As a consequence of the results above we find that Derived Categories, Lemma 13.22.1 applies to a number of situations. For example, given a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{F}$. Namely, for an injective $\mathcal{O}_\mathcal{X}$-module $\mathcal{I}$ the $\mathcal{O}_\mathcal{D}$-module $f_*\mathcal{I}$ is totally acyclic by Lemma 21.14.1 and a totally acyclic sheaf is acyclic for $\Gamma(\mathcal{D}, -)$ by Lemma 21.14.3.

0732 **Lemma 21.14.5** (Leray spectral sequence)**.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.22.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 13.22.2 are satisfied, see Lemmas 21.14.1 and 21.14.3. □

0733 **Lemma 21.14.6.** *Let* $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ *be a morphism of ringed topoi. Let* $\mathcal{F}$ *be an* $\mathcal{O}_\mathcal{C}$*-module.*

(1) *If* $R^q f_* \mathcal{F} = 0$ *for* $q > 0$*, then* $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_* \mathcal{F})$ *for all* p.
(2) *If* $H^p(\mathcal{D}, R^q f_* \mathcal{F}) = 0$ *for all* q *and* $p > 0$*, then* $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^q f_* \mathcal{F})$ *for all* q.

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

0734 **Lemma 21.14.7** (Relative Leray spectral sequence)**.** *Let* $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ *and* $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ *be morphisms of ringed topoi. Let* $\mathcal{F}$ *be an* $\mathcal{O}_\mathcal{C}$*-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_* (R^q f_* \mathcal{F})$$

converging to $R^{p+q}(g \circ f)_* \mathcal{F}$*. This spectral sequence is functorial in* $\mathcal{F}$*, and there is a version for bounded below complexes of* $\mathcal{O}_\mathcal{C}$*-modules.*

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 13.22.2 and Lemmas 21.14.1 and 21.14.3. □

21.15. The base change map

0735 In this section we construct the base change map in some cases; the general case is treated in Remark 21.19.3. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 18.31.2 the functor $g^* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{D}) \to D(\mathcal{O}_\mathcal{C})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 13.16.9. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

0736 **Lemma 21.15.1.** *Let*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{array}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^* Rf_* \mathcal{F}^\bullet \longrightarrow R(f')_* (g')^* \mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 21.14.2 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.18.6 and 13.18.7 the arrow β in the diagram

$$\begin{array}{ccc} (g')_*(g')^*\mathcal{F}^\bullet & \longrightarrow & (g')_*\mathcal{J}^\bullet \\ {\scriptstyle adjunction}\uparrow & & \uparrow{\scriptstyle \beta} \\ \mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^* f_* \mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. □

21.16. Cohomology and colimits

0737 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to Mod(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 4.14. For each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

The following lemma is the analogue of Sites, Lemma 7.17.7 for cohomology.

0739 **Lemma 21.16.1.** *Let $\mathcal{C}$ be a site. Let $Cov_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_\mathcal{C}$ be subsets. Assume that*

1. *For every $\mathcal{U} \in Cov$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then the map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to Ab(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in \mathrm{Cov}$ to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satisfies the hypothesis of Sites, Lemma 7.17.7 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $Ab(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 19.7.4, we can find a morphism of

filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 10.8.8), we see the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{array}{ccccccc}
\operatorname{colim}_i H^p(U, \mathcal{I}_i) & \longrightarrow & \operatorname{colim}_i H^p(U, \mathcal{Q}_i) & \longrightarrow & \operatorname{colim}_i H^{p+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^p(U, \operatorname{colim}_i \mathcal{I}_i) & \longrightarrow & H^p(U, \operatorname{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{p+1}(U, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & 0
\end{array}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 10.8.8. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 21.10.9. By the result for $p = 0$ we see that for $\mathcal{U} \in \operatorname{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = \operatorname{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 21.10.2 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 10.8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 21.10.9. are satisfied and the claim follows. □

0GN3 **Lemma 21.16.2.** *Let $\mathcal{C}$ be a site. Let $S \subset \operatorname{Ob}(Sh(\mathcal{C}))$ be a subset. Denote $*$ the final object of $Sh(\mathcal{C})$. Assume*

1. *for some $K \in S$ the map $K \to *$ is surjective,*
2. *given a surjective map of sheaves $\mathcal{F} \to K$ with $K \in S$ there exists a $K' \in S$ and a map $K' \to \mathcal{F}$ such that the composition $K' \to K$ is surjective,*
3. *given $K, K' \in S$ there is a surjection $K'' \to K \times K'$ with $K'' \in S$,*
4. *given $a, b : K \to K'$ with $K, K' \in S$ there exists a surjection $K'' \to \mathit{Equalizer}(a, b)$ with $K'' \in S$, and*
5. *every $K \in S$ is quasi-compact (Sites, Definition 7.17.4).*

Then for all $p \geq 0$ the map

$$\operatorname{colim}_\lambda H^p(\mathcal{C}, \mathcal{F}_\lambda) \longrightarrow H^p(\mathcal{C}, \operatorname{colim}_\lambda \mathcal{F}_\lambda)$$

is an isomorphism for every filtered diagram $\Lambda \to Ab(\mathcal{C})$, $\lambda \mapsto \mathcal{F}_\lambda$.

Proof. We will prove this by induction on p. The base case $p = 0$ follows from Sites, Lemma 7.17.8 part (4). We check the assumptions hold, but we urge the reader to skip this part. Suppose $\mathcal{F} \to *$ is surjective. Choose $K \in S$ and $K \to *$ surjective as in (1). Then $\mathcal{F} \times K \to K$ is surjective. Choose $K' \to \mathcal{F} \times K$ with $K' \in S$ and $K' \to K$ surjective as in (2). Then there is a map $K' \to \mathcal{F}$ and $K' \to *$

is surjective. Hence Sites, Lemma 7.17.8 assumption (4)(a) is satisfied. By Sites, Lemma 7.17.5, assumptions (3) and (5) we see that $K \times K$ is quasi-compact for all $K \in S$. Hence Sites, Lemma 7.17.8 assumption (4)(b) is satisfied. This finishes the proof of the base case.

Induction step. Assume the result holds for H^p for $p \leq p_0$ and for all topoi $Sh(\mathcal{C})$ such that a set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ can be found satisfying (1) – (5). Arguing exactly as in the proof of Lemma 21.16.1 we see that it suffices to show: given a filtered colimit $\mathcal{I} = \mathrm{colim}\, \mathcal{I}_\lambda$ with $\mathcal{I}_\lambda$ injective abelian sheaves, we have $H^{p_0+1}(\mathcal{C}, \mathcal{I}) = 0$. Choose $K \to *$ surjective with $K \in S$ as in (1). Denote K^n the n-fold self product of K. Consider the spectral sequence

$$E_1^{p,q} = H^q(K^{p+1}, \mathcal{I}) \Rightarrow H^{p+q}(*, \mathcal{I}) = H^{p+q}(\mathcal{C}, \mathcal{I})$$

of Lemma 21.13.2. Recall that $H^q(K^{p+1}, \mathcal{F}) = H^q(\mathcal{C}/K^{p+1}, j^{-1}\mathcal{F})$, for any abelian sheaf on $\mathcal{C}$, see Lemma 21.13.3. We have $j^{-1}\mathcal{I} = \mathrm{colim}\, j^{-1}\mathcal{I}_\lambda$ as j^{-1} commutes with colimits. The restrictions $j^{-1}\mathcal{I}_\lambda$ are injective abelian sheaves on $\mathcal{C}/K^{p+1}$ by Lemma 21.7.1. Below we will show that the induction hypothesis applies to $\mathcal{C}/K^{p+1}$ and hence we see that $H^q(K^{p+1}, \mathcal{I}) = \mathrm{colim}\, H^q(K^{p+1}, \mathcal{I}_\lambda) = 0$ for $q < p_0 + 1$ (vanishing as $\mathcal{I}_\lambda$ is injective). It follows that

$$H^{p_0+1}(\mathcal{C}, \mathcal{I}) = H^{p_0+1}\left(\ldots \to H^0(K^{p_0}, \mathcal{I}) \to H^0(K^{p_0+1}, \mathcal{I}) \to H^0(K^{p_0+2}, \mathcal{I}) \to \ldots\right)$$

Again using the induction hypothesis, the complex depicted on the right hand side is the colimit over Λ of the complexes

$$\ldots \to H^0(K^{p_0}, \mathcal{I}_\lambda) \to H^0(K^{p_0+1}, \mathcal{I}_\lambda) \to H^0(K^{p_0+2}, \mathcal{I}_\lambda) \to \ldots$$

These complexes are exact as $\mathcal{I}_\lambda$ is an injective abelian sheaf (follows from the spectral sequence for example). Since filtered colimits are exact in the category of abelian groups we obtain the desired vanishing.

We still have to show that the induction hypothesis applies to the site $\mathcal{C}/K^n$ for all $n \geq 1$. Recall that $Sh(\mathcal{C}/K^n) = Sh(\mathcal{C})/K^n$, see Sites, Lemma 7.30.3. Thus we may work in $Sh(\mathcal{C})/K^n$. Denote $S_n \subset \mathrm{Ob}(Sh(\mathcal{C}/K^n)$ the set of objects of the form $K' \to K^n$. We check each property in turn:

(1) By (3) and induction there exists a surjection $K' \to K^n$ with $K' \in S$. Then $(K' \to K^n) \to (K^n \to K^n)$ is a surjection in $Sh(\mathcal{C})/K^n$ and $K^n \to K^n$ is the final object of $Sh(\mathcal{C})/K^n$. Hence (1) holds for S_n,

(2) Property (2) for S_n is an immediate consequence of (2) for S.

(3) Let $a : K_1 \to K^n$ and $b : K_2 \to K^n$ be in S_n. Then $(K_1 \to K^n) \times (K_2 \to K^n)$ is the object $K_1 \times_{K^n} K_2 \to K^n$ of $Sh(\mathcal{C})/K^n$. The subsheaf $K_1 \times_{K^n} K_2 \subset K_1 \times K_2$ is the equalizer of $a \circ \mathrm{pr}_1$ and $b \circ \mathrm{pr}_2$. Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Pick $K_3 \to K_1 \times K_2$ surjective with $K_3 \in S$; this is possibly by assumption (3) for $\mathcal{C}$. Pick

$$K_4 \longrightarrow \mathrm{Equalizer}(K_3 \to K_1 \times K_2 \xrightarrow{a_1, b_1} K)$$

surjective with $K_4 \in S$. This is possible by assumption (4) for $\mathcal{C}$. Pick

$$K_5 \longrightarrow \mathrm{Equalizer}(K_4 \to K_1 \times K_2 \xrightarrow{a_2, b_2} K)$$

surjective with $K_5 \in S$. Again this is possible. Continue in this fashion until we get

$$K_{3+n} \longrightarrow \mathrm{Equalizer}(K_{2+n} \to K_1 \times K_2 \xrightarrow{a_n, b_n} K)$$

surjective with $K_{3+n} \in S$. By construction $K_{3+n} \to K_1 \times_{K^n} K_2$ is surjective. Hence $(K_{3+n} \to K^n)$ is in S_n and surjects onto the product $(K_1 \to K^n) \times (K_2 \to K^n)$. Thus (3) holds for S_n.
(4) Property (4) for S_n is an immediate consequence of property (4) for S.
(5) Property (5) for S_n is a consequence of property (5) for S. Namely, an object $\mathcal{F} \to K^n$ of $Sh(\mathcal{C})/K^n$ corresponds to a quasi-compact object of $Sh(\mathcal{C}/K^n)$ if and only if $\mathcal{F}$ is a quasi-compact object of $Sh(\mathcal{C})$.

This finishes the proof of the lemma. □

0GN4 **Remark 21.16.3.** Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ be the set of sheaves K which have the form

$$K = \coprod\nolimits_{i=1,\ldots,n} h^\#_{U_i}$$

with $U_1, \ldots, U_n \in \mathcal{B}$. Then we can ask: when does this set satisfy the assumptions of Lemma 21.16.2? One answer is that it suffices if

(1) for some $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$ the map $\coprod_{i=1,\ldots,n} h^\#_{U_i} \to *$ is surjective,
(2) every covering of $U \in \mathcal{B}$ can be refined by a covering of the form $\{U_i \to U\}_{i=1,\ldots,n}$ with $U_i \in \mathcal{B}$,
(3) given $U, U' \in \mathcal{B}$ there exist $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ and $U_i \to U'$ such that $\coprod_{i=1,\ldots,n} h^\#_{U_i} \to h^\#_U \times h^\#_{U'}$ is surjective,
(4) given morphisms $a, b : U \to U'$ in $\mathcal{C}$ with $U, U' \in \mathcal{B}$, there exist $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ equalizing a, b such that $\coprod_{i=1,\ldots,n} h^\#_{U_i} \to \mathrm{Equalizer}(h^\#_a, h^\#_b : h^\#_U \to h^\#_{U'})$ is surjective.

We omit the detailed verification, except to mention that part (2) above insures that every element of $\mathcal{B}$ is quasi-compact and hence every $K \in S$ is quasi-compact as well by Sites, Lemma 7.17.6.

0H7B **Lemma 21.16.4.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of abelian sheaves on $\mathcal{C}$. Set $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$. Let $p \geq 0$ be an integer. Denote $\mathcal{B}$ the set of $V \in \mathrm{Ob}(\mathcal{D})$ such that $H^p(u(V), \mathcal{F}) = \operatorname{colim} H^p(u(V), \mathcal{F}_i)$. If every object of $\mathcal{D}$ has a covering by elements of $\mathcal{B}$, then $R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i$.*

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheafification of the presheaf $\mathcal{G}$ sending V to $H^p(u(V), \mathcal{F})$, see Lemma 21.7.4. Similarly, $R^p f_* \mathcal{F}_i$ is the sheafification of the presheaf $\mathcal{G}_i$ sending V to $H^p(u(V), \mathcal{F}_i)$. Recall that sheafification is the left adjoint to the inclusion from sheaves to presheaves, see Sites, Section 7.10. Hence sheafification commutes with colimits, see Categories, Lemma 4.24.5. Hence it suffices to show that the map of presheaves (with colimit in the category of presheaves)

$$\operatorname{colim} \mathcal{G}_i \longrightarrow \mathcal{G}$$

induces an isomorphism on sheafifications. This follows from Sites, Lemma 7.10.17 and our assumption on $\mathcal{B}$. □

0EXZ **Lemma 21.16.5.** *Let $\mathcal{I}$ be a cofiltered index category and let $(\mathcal{C}_i, f_a)$ be an inverse system of sites over $\mathcal{I}$ as in Sites, Situation 7.18.1. Set $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$ as in Sites, Lemmas 7.18.2 and 7.18.3. Moreover, assume given*

(1) *an abelian sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,*
(2) *for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves on $\mathcal{C}_j$*

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ *whenever* $c = a \circ b$. *Then there exists a map of systems* $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ *such that* $\mathcal{F}_i \to \mathcal{G}_i$ *is injective and* $\mathcal{G}_i$ *is an injective abelian sheaf.*

Proof. For each i we pick an injection $\mathcal{F}_i \to \mathcal{A}_i$ where $\mathcal{A}_i$ is an injective abelian sheaf on $\mathcal{C}_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k\to i} f_{b,*}\mathcal{A}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{A}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1}f_{a\circ b,*}\mathcal{A}_k \to f_{b,*}\mathcal{A}_k$ for $b : k \to j$. Thus we find an injection $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 21.14.2 and Homology, Lemma 12.27.3. This finishes the construction. □

09YP **Lemma 21.16.6.** *In the situation of Lemma 21.16.5 set* $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$. *Let* $i \in \mathrm{Ob}(\mathcal{I})$, $X_i \in Ob(\mathcal{C}_i)$. *Then*

$$\operatorname{colim}_{a:j\to i} H^p(u_a(X_i), \mathcal{F}_j) = H^p(u_i(X_i), \mathcal{F})$$

for all $p \geq 0$.

Proof. The case $p = 0$ is Sites, Lemma 7.18.4.

Choose $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ as in Lemma 21.16.5. Arguing exactly as in the proof of Lemma 21.16.1 we see that it suffices to prove that $H^p(X, \operatorname{colim} f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every object of $\mathcal{C}$ we show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of $\mathcal{C}$ is zero (Lemma 21.10.9). The covering $\mathcal{U}$ comes from a covering $\mathcal{U}_i$ of $\mathcal{C}_i$ for some i. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{a:j\to i} \check{\mathcal{C}}^\bullet(u_a(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 21.10.2. See Algebra, Lemma 10.8.8. □

21.17. Flat resolutions

06YL In this section we redo the arguments of Cohomology, Section 20.26 in the setting of ringed sites and ringed topoi.

06YM **Lemma 21.17.1.** *Let* $(\mathcal{C}, \mathcal{O})$ *be a ringed site. Let* $\mathcal{G}^\bullet$ *be a complex of* $\mathcal{O}$*-modules. The functors*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$$

and

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. □

06YN **Definition 21.17.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is acyclic.

06YP **Lemma 21.17.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 21.17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

0E8K **Lemma 21.17.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules, then $\mathcal{K}^\bullet|_U$ is a K-flat complex of $\mathcal{O}_U$-modules.*

Proof. Let $\mathcal{G}^\bullet$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Modules on Sites, Lemma 18.19.3) and $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules we see that the complex

$$j_{U!}(\mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_U} \mathcal{K}^\bullet|_U)) = \mathrm{Tot}(j_{U!}\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is exact. Here the equality comes from Modules on Sites, Lemma 18.27.9 and the fact that $j_{U!}$ commutes with direct sums (as a left adjoint). We conclude because $j_{U!}$ reflects exactness by Modules on Sites, Lemma 18.19.4. □

07A2 **Lemma 21.17.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$$

and the definition. □

07A3 **Lemma 21.17.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 21.17.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

0G7B **Lemma 21.17.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. By Modules on Sites, Lemma 18.28.9 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_1^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_2^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_3^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. □

06YQ **Lemma 21.17.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.*

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = \operatorname{colim}_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) = \operatorname{colim}_m \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.25.3 has

$${}'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. □

06YR **Lemma 21.17.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\operatorname{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\operatorname{colim}_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \operatorname{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

077J **Lemma 21.17.10.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules*

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 18.28.8 and Derived Categories, Lemma 13.29.1. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

06YS **Lemma 21.17.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.*

Proof. Choose a diagram as in Lemma 21.17.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.7. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.8. Thus $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.9. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 21.17.10 shows that $\operatorname{colim} \mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. □

06YT **Lemma 21.17.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 21.17.11. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \end{array}$$

The result follows as by Lemma 21.17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 21.17.11. By Lemma 21.17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

By Lemma 21.17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 21.17.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

06YU **Definition 21.17.13.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

08FF **Definition 21.17.14.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are defined by the formula

$$\mathrm{Tor}^{\mathcal{O}}_p(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{array}{ccccccc} \mathcal{F}_1 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_2 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_3 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & 0 \\ & \searrow & & & & & \\ \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_1, \mathcal{G}) & \longrightarrow & \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_2, \mathcal{G}) & \longrightarrow & \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_3, \mathcal{G}) & & \end{array}$$

for every $\mathcal{O}$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

08FG **Lemma 21.17.15.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}$-module, and*
(2) *$Tor_1^{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}$ is a quotient of $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. □

0G7C **Lemma 21.17.16.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat, acyclic complex with flat terms. Then $\mathcal{F} = \mathrm{Ker}(\mathcal{K}^n \to \mathcal{K}^{n+1})$ is a flat $\mathcal{O}$-module.*

Proof. Observe that

$$\ldots \to \mathcal{K}^{n-2} \to \mathcal{K}^{n-1} \to \mathcal{F} \to 0$$

is a flat resolution of our module $\mathcal{F}$. Since a bounded above complex of flat modules is K-flat (Lemma 21.17.8) we may use this resolution to compute $\mathrm{Tor}_i(\mathcal{F}, \mathcal{G})$ for any $\mathcal{O}$-module $\mathcal{G}$. On the one hand $\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G}$ is zero in $D(\mathcal{O})$ because $\mathcal{K}^\bullet$ is acyclic and on the other hand it is represented by $\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{G}$. Hence we see that

$$\mathcal{K}^{n-3} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-2} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-1} \otimes_{\mathcal{O}} \mathcal{G}$$

is exact. Thus $\mathrm{Tor}_1(\mathcal{F}, \mathcal{G}) = 0$ and we conclude by Lemma 21.17.15. □

0G7D **Lemma 21.17.17.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that*

(1) *$\mathcal{N}^\bullet$ is K-flat,*
(2) *c is a quasi-isomorphism,*
(3) *a is homotopic to $c \circ b$.*

If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, b, and c such that the same is true for $\mathcal{N}^\bullet$.

Proof. We will use that the homotopy category $K(Mod(\mathcal{O}))$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 21.17.11. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 21.17.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{K}^\bullet & \longrightarrow & \mathcal{N}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{C}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1]
\end{array}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O})$ and use

Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. □

21.18. Derived pullback

06YV Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

0G7E **Lemma 21.18.1.** *Let $f : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules whose terms are flat $\mathcal{O}$-modules. Then $f^*\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules whose terms are flat $\mathcal{O}'$-modules.*

Proof. The terms $f^*\mathcal{K}^n$ are flat $\mathcal{O}'$-modules by Modules on Sites, Lemma 18.39.1. Choose a diagram

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{K}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{K}^\bullet & \longrightarrow & \ldots \end{array}$$

as in Lemma 21.17.10. We will use all of the properties stated in the lemma without further mention. Each $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.7. Consider the short exact sequence of complexes

$$0 \to \mathcal{M}^\bullet \to \operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{K}^\bullet \to 0$$

defining $\mathcal{M}^\bullet$. By Lemmas 21.17.8 and 21.17.9 the complex $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat and by Modules on Sites, Lemma 18.28.5 it has flat terms. By Modules on Sites, Lemma 18.28.10 $\mathcal{M}^\bullet$ has flat terms, by Lemma 21.17.7 $\mathcal{M}^\bullet$ is K-flat, and by the long exact cohomology sequence $\mathcal{M}^\bullet$ is acyclic (because the second arrow is a quasi-isomorphism). The pullback $f^*(\operatorname{colim} \mathcal{K}_n^\bullet) = \operatorname{colim} f^*\mathcal{K}_n^\bullet$ is a colimit of bounded below complexes of flat $\mathcal{O}'$-modules and hence is K-flat (by the same lemmas as above). The pullback of our short exact sequence

$$0 \to f^*\mathcal{M}^\bullet \to f^*(\operatorname{colim} \mathcal{K}_n^\bullet) \to f^*\mathcal{K}^\bullet \to 0$$

is a short exact sequence of complexes by Modules on Sites, Lemma 18.39.4. Hence by Lemma 21.17.7 it suffices to show that $f^*\mathcal{M}^\bullet$ is K-flat. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ is acyclic as well as K-flat and with flat terms. Then Lemma 21.17.16 guarantees that all terms of $\tau_{\leq n}\mathcal{K}^\bullet$ are flat $\mathcal{O}$-modules. We choose a diagram as above and we will use all the properties proven above for this diagram. Denote $\mathcal{M}_n^\bullet$ the kernel of the map of complexes $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet$ so that we have short exact sequences of complexes

$$0 \to \mathcal{M}_n^\bullet \to \mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet \to 0$$

By Modules on Sites, Lemma 18.28.10 we see that the terms of the complex $\mathcal{M}_n^\bullet$ are flat. Hence we see that $\mathcal{M} = \operatorname{colim} \mathcal{M}_n^\bullet$ is a filtered colimit of bounded below complexes of flat modules in this case. Thus $f^*\mathcal{M}^\bullet$ is K-flat (same argument as above) and we win. □

06YY **Lemma 21.18.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any K-flat complex $\mathcal{K}^\bullet$ with flat terms and in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes $\mathcal{K}^\bullet$ with flat terms. Namely, (1) follows from Lemma 21.17.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by Lemma 21.18.1 because we can consider the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$. Hence Lemma 21.17.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. The proof is finished by observing that bounded above complexes of flat modules are in $\mathcal{P}$ by Lemma 21.17.8. □

0D6D **Lemma 21.18.3.** *Consider morphisms of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_\mathcal{E}) \to D(\mathcal{O}_\mathcal{C})$.*

Proof. Let E be an object of $D(\mathcal{O}_\mathcal{E})$. We may represent E by a K-flat complex $\mathcal{K}^\bullet$ with flat terms, see Lemma 21.17.11. By construction Lg^*E is computed by $g^*\mathcal{K}^\bullet$, see Lemma 21.18.2. By Lemma 21.18.1 the complex $g^*\mathcal{K}^\bullet$ is K-flat with flat terms. Hence Lf^*Lg^*E is represented by $f^*g^*\mathcal{K}^\bullet$. Since also $L(g \circ f)^*E$ is represented by $(g \circ f)^*\mathcal{K}^\bullet = f^*g^*\mathcal{K}^\bullet$ we conclude. □

07A4 **Lemma 21.18.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. By our construction of derived pullback in Lemma 21.18.2. and the existence of resolutions in Lemma 21.17.11 we may replace $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ by complexes of $\mathcal{O}'$-modules which are K-flat and have flat terms. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. The complex

$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat with flat terms by Lemma 21.17.5 and Modules on Sites, Lemma 18.28.12. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 18.26.2. □

08I6 **Lemma 21.18.5.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O})$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}')$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module and let $\mathcal{G}$ be an $\mathcal{O}'$-module. Then $\mathcal{F} \otimes_{\mathcal{O}} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

0DEN **Lemma 21.18.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}$-modules.*

(1) *If $\mathcal{K}^\bullet$ is K-flat, then for every point p of the site $\mathcal{C}$ the complex of $\mathcal{O}_p$-modules $\mathcal{K}^\bullet_p$ is K-flat in the sense of More on Algebra, Definition 15.59.1*
(2) *If $\mathcal{C}$ has enough points, then the converse is true.*

Proof. Proof of (2). If $\mathcal{C}$ has enough points and $\mathcal{K}^\bullet_p$ is K-flat for all points p of $\mathcal{C}$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules on Sites, Lemma 18.14.4.

Proof of (1). Assume $\mathcal{K}^\bullet$ is K-flat. Choose a quasi-isomorphism $a : \mathcal{L}^\bullet \to \mathcal{K}^\bullet$ such that $\mathcal{L}^\bullet$ is K-flat with flat terms, see Lemma 21.17.11. Any pullback of $\mathcal{L}^\bullet$ is K-flat, see Lemma 21.18.1. In particular the stalk $\mathcal{L}^\bullet_p$ is a K-flat complex of $\mathcal{O}_p$-modules. Thus the cone $C(a)$ on a is a K-flat (Lemma 21.17.6) acyclic complex of $\mathcal{O}$-modules and it suffuces to show the stalk of $C(a)$ is K-flat (by More on Algebra, Lemma 15.59.5). Thus we may assume that $\mathcal{K}^\bullet$ is K-flat and acyclic.

Assume $\mathcal{K}^\bullet$ is acyclic and K-flat. Before continuing we replace the site $\mathcal{C}$ by another one as in Sites, Lemma 7.29.5 to insure that $\mathcal{C}$ has all finite limits. This implies the category of neighbourhoods of p is filtered (Sites, Lemma 7.33.2) and the colimit defining the stalk of a sheaf is filtered. Let M be a finitely presented $\mathcal{O}_p$-module. It suffices to show that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic, see More on Algebra, Lemma 15.59.9. Since $\mathcal{O}_p$ is the filtered colimit of $\mathcal{O}(U)$ where U runs over the neighbourhoods of p, we can find a neighbourhood (U, x) of p and a finitely presented $\mathcal{O}(U)$-module M' whose base change to $\mathcal{O}_p$ is M, see Algebra, Lemma 10.127.6. By Lemma 21.17.4 we may replace $\mathcal{C}, \mathcal{O}, \mathcal{K}^\bullet$ by $\mathcal{C}/U, \mathcal{O}_U, \mathcal{K}^\bullet|_U$. We conclude that we may assume there exists an $\mathcal{O}$-module $\mathcal{F}$ such that $M \cong \mathcal{F}_p$. Since $\mathcal{K}^\bullet$ is K-flat and acyclic, we see that $\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$ is acyclic (as it computes the derived tensor product by definition). Taking stalks is an exact functor, hence we get that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic as desired. □

0DEP **Lemma 21.18.7.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. If $\mathcal{C}$ has enough points, then the pullback of a K-flat complex of $\mathcal{O}'$-modules is a K-flat complex of $\mathcal{O}$-modules.*

Proof. This follows from Lemma 21.18.6, Modules on Sites, Lemma 18.36.4, and More on Algebra, Lemma 15.59.3. □

0FPH **Lemma 21.18.8.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{D}$-modules. The diagram*

$$\begin{array}{ccc} Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) & \longrightarrow & Lf^* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*\mathcal{M}^\bullet & & f^* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet & \longrightarrow & Tot(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet) \end{array}$$

commutes.

Proof. We will use the existence of K-flat resolutions with flat terms (Lemma 21.17.11), we will use that derived pullback is computed by such complexes (Lemma 21.18.2), and that pullbacks preserve these properties (Lemma 21.18.1). If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$\begin{array}{ccc} Lf^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) & \longrightarrow & Lf^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ f^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) & \longrightarrow & f^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{Q}^\bullet) & \longrightarrow & \mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet) \end{array}$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 21.17.5. □

21.19. Cohomology of unbounded complexes

07A5 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $Mod(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 18.14 and Lemmas 18.28.7 and 18.28.8. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules and moreover this embedding can be chosen functorial in $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 13.31.7 that

(1) any exact functor $F : K(Mod(\mathcal{O})) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}) \to \mathcal{D}$,
(2) for any additive functor $F : Mod(\mathcal{O}) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(Mod(\mathcal{O})) \to K(\mathcal{A})$ induced by F and we obtain a right derived functor $RF : D(\mathcal{O}) \to D(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(\mathcal{C}, -) : \mathit{Mod}(\mathcal{O}) \to \mathrm{Mod}_{\Gamma(\mathcal{C},\mathcal{O})}$ gives rise to

$$R\Gamma(\mathcal{C}, -) : D(\mathcal{O}) \longrightarrow D(\Gamma(\mathcal{C}, \mathcal{O}))$$

We shall use the notation $H^i(\mathcal{C}, K) = H^i(R\Gamma(\mathcal{C}, K))$ for cohomology.

(2) For an object U of $\mathcal{C}$ we consider the functor $\Gamma(U, -) : \mathit{Mod}(\mathcal{O}) \to \mathrm{Mod}_{\Gamma(U,\mathcal{O})}$. This gives rise to

$$R\Gamma(U, -) : D(\mathcal{O}) \to D(\Gamma(U, \mathcal{O}))$$

We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.

(3) For a morphism of ringed topoi $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}')$ we consider the functor $f_* : \mathit{Mod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}')$ which gives rise to the total direct image

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on unbounded derived categories.

07A6 **Lemma 21.19.1.** *Let $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. The functor Rf_* defined above and the functor Lf^* defined in Lemma 21.18.2 are adjoint:*

$$\mathrm{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.30.3. □

0D6E **Lemma 21.19.2.** *Let $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (\mathit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (\mathit{Sh}(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{E})$.*

Proof. By Lemma 21.19.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 21.18.3 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. □

07A7 **Remark 21.19.3.** The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} (\mathit{Sh}(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (\mathit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (\mathit{Sh}(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (\mathit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{array}$$

is a commutative diagram of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Then there exists a canonical base change map

$$Lg^*Rf_*K \longrightarrow R(f')_*L(g')^*K$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*K \to L(g')^*K$. Since $L(f')^* \circ Lg^* = L(g')^* \circ Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*K \to L(g')^*K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*K \to K$.

0E46 **Remark 21.19.4.** Consider a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) & \xrightarrow{k} & (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{l} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{m} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Then the base change maps of Remark 21.19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* &= Lm^* \circ Rg_* \circ Rf_* \\ &\to Rg'_* \circ Ll^* \circ Rf_* \\ &\to Rg'_* \circ Rf'_* \circ Lk^* \\ &= R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0E47 **Remark 21.19.5.** Consider a commutative diagram

$$\begin{array}{ccccc} (Sh(\mathcal{C}''), \mathcal{O}_{\mathcal{C}''}) & \xrightarrow{g'} & (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}''), \mathcal{O}_{\mathcal{D}''}) & \xrightarrow{h'} & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{h} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Then the base change maps of Remark 21.19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} L(h \circ h')^* \circ Rf_* &= L(h')^* \circ Lh^* \circ Rf_* \\ &\to L(h')^* \circ Rf'_* \circ Lg^* \\ &\to Rf''_* \circ L(g')^* \circ Lg^* \\ &= Rf''_* \circ L(g \circ g')^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0FPI **Lemma 21.19.6.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_{\mathcal{C}}$-modules. The diagram*

$$\begin{array}{ccc} Lf^*f_*\mathcal{K}^\bullet & \longrightarrow & f^*f_*\mathcal{K}^\bullet \\ \downarrow & & \downarrow \\ Lf^*Rf_*\mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet \end{array}$$

coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to id$ commutes in $D(\mathcal{O}_{\mathcal{C}})$.

Proof. We will use the existence of K-flat resolutions and K-injective resolutions, see Lemmas 21.17.11, 21.18.2, and 21.18.1 and the discussion above. Choose a

quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_\mathcal{C}$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_*\mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules with flat terms. We can choose a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{P}^\bullet$ with flat terms and a diagram of morphisms of complexes

$$\begin{array}{ccc} \mathcal{P}^\bullet & \longrightarrow & f_*\mathcal{K}^\bullet \\ \downarrow & & \downarrow \\ \mathcal{Q}^\bullet & \longrightarrow & f_*\mathcal{I}^\bullet \end{array}$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can choose a resolution of $\mathcal{P}^\bullet$ by a K-flat complex with flat terms. Taking pullbacks we obtain a diagram of morphisms of complexes

$$\begin{array}{ccccc} f^*\mathcal{P}^\bullet & \longrightarrow & f^*f_*\mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ f^*\mathcal{Q}^\bullet & \longrightarrow & f^*f_*\mathcal{I}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. □

0B6C **Remark 21.19.7.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The adjointness of Lf^* and Rf_* allows us to construct a relative cup product

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L \longrightarrow Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L)$$

in $D(\mathcal{O}_\mathcal{D})$ for all K, L in $D(\mathcal{O}_\mathcal{C})$. Namely, this map is adjoint to a map $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L) = Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*Rf_*L$ (Lemma 21.18.4) with the map $Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*Rf_*L \to K \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L$ coming from the counit $Lf^* \circ Rf_* \to \mathrm{id}$.

0DD7 **Lemma 21.19.8.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \subset Ab(\mathcal{C})$ denote the Serre subcategory consisting of torsion abelian sheaves. Then the functor $D(\mathcal{A}) \to D_\mathcal{A}(\mathcal{C})$ is an equivalence.*

Proof. A key observation is that an injective abelian sheaf $\mathcal{I}$ is divisible. Namely, if $s \in \mathcal{I}(U)$ is a local section, then we interpret s as a map $s : j_{U!}\mathbf{Z} \to \mathcal{I}$ and we apply the defining property of an injective object to the injective map of sheaves $n : j_{U!}\mathbf{Z} \to j_{U!}\mathbf{Z}$ to see that there exists an $s' \in \mathcal{I}(U)$ with $ns' = s$.

For a sheaf $\mathcal{F}$ denote $\mathcal{F}_{tor}$ its torsion subsheaf. We claim that if $\mathcal{I}^\bullet$ is a complex of injective abelian sheaves whose cohomology sheaves are torsion, then

$$\mathcal{I}^\bullet_{tor} \to \mathcal{I}^\bullet$$

is a quasi-isomorphism. Namely, by flatness of $\mathbf{Q}$ over $\mathbf{Z}$ we have

$$H^p(\mathcal{I}^\bullet) \otimes_\mathbf{Z} \mathbf{Q} = H^p(\mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q})$$

which is zero because the cohomology sheaves are torsion. By divisibility (shown above) we see that $\mathcal{I}^\bullet \to \mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q}$ is surjective with kernel $\mathcal{I}^\bullet_{tor}$. The claim follows from the long exact sequence of cohomology sheaves associated to the short exact sequence you get.

To prove the lemma we will construct right adjoint $T : D(\mathcal{C}) \to D(\mathcal{A})$. Namely, given K in $D(\mathcal{C})$ we can represent K by a K-injective complex $\mathcal{I}^\bullet$ whose cohomology sheaves are injective, see Injectives, Theorem 19.12.6. Then we set $T(K) = \mathcal{I}^\bullet_{tor}$, in other words, T is the right derived functor of taking torsion. The functor T is a right adjoint to $i : D(\mathcal{A}) \to D_{\mathcal{A}}(\mathcal{C})$. This readily follows from the observation that if $\mathcal{F}^\bullet$ is a complex of torsion sheaves, then

$$\mathrm{Hom}_{K(\mathcal{A})}(\mathcal{F}^\bullet, I^\bullet_{tor}) = \mathrm{Hom}_{K(Ab(\mathcal{C}))}(\mathcal{F}^\bullet, I^\bullet)$$

in particular $\mathcal{I}^\bullet_{tor}$ is a K-injective complex of $\mathcal{A}$. Some details omitted; in case of doubt, it also follows from the more general Derived Categories, Lemma 13.30.3. Our claim above gives that $L = T(i(L))$ for L in $D(\mathcal{A})$ and $i(T(K)) = K$ if K is in $D_{\mathcal{A}}(\mathcal{C})$. Using Categories, Lemma 4.24.4 the result follows. □

21.20. Some properties of K-injective complexes

08FH Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ (see Modules on Sites, Definition 18.19.1) is an exact functor (Modules on Sites, Lemma 18.19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

08FI **Lemma 21.20.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The restriction of a K-injective complex of $\mathcal{O}$-modules to $\mathcal{C}/U$ is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 13.31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. See discussion above. □

0D6F **Lemma 21.20.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For K in $D(\mathcal{O})$ we have $H^p(U, K) = H^p(\mathcal{C}/U, K|_{\mathcal{C}/U})$.*

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(\mathcal{C}/U, \mathcal{I}^\bullet|_{\mathcal{C}/U}))$$

by construction of cohomology. By Lemma 21.20.1 the complex $\mathcal{I}^\bullet|_{\mathcal{C}/U}$ is a K-injective complex representing $K|_{\mathcal{C}/U}$ and the lemma follows. □

0BKV **Lemma 21.20.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. The sheafification of*

$$U \mapsto H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$$

is the qth cohomology sheaf $H^q(K)$ of K.

Proof. The equality $H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$ holds by Lemma 21.20.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Then

$$H^q(U, K) = \frac{\mathrm{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\mathrm{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \mathrm{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1})/\mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. □

0D6G **Lemma 21.20.4.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Given $V \in \mathcal{D}$, set $U = u(V)$ and denote $g : (\mathcal{C}/U, \mathcal{O}_U) \to (\mathcal{D}/V, \mathcal{O}_V)$ the induced morphism of ringed sites (Modules on Sites, Lemma 18.20.1). Then $(Rf_*E)|_{\mathcal{D}/V} = Rg_*(E|_{\mathcal{C}/U})$ for E in $D(\mathcal{O}_\mathcal{C})$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_{\mathcal{C}/U}) = g_*(\mathcal{I}^\bullet|_{\mathcal{C}/U})$ by Lemma 21.20.1. Since it is clear that $(f_*\mathcal{F})|_{\mathcal{D}/V} = g_*(\mathcal{F}|_{\mathcal{C}/U})$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$ (see Modules on Sites, Lemma 18.20.1 or the more basic Sites, Lemma 7.28.1) the result follows. □

0D6H **Lemma 21.20.5.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Then $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\Gamma(\mathcal{O}_\mathcal{D}))$. More generally, for $V \in \mathcal{D}$ with $U = u(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.*

Proof. Consider the punctual topos pt endowed with $\mathcal{O}_{pt}$ given by the ring $\Gamma(\mathcal{O}_\mathcal{D})$. There is a canonical morphism $(\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$ of ringed topoi inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_{pt}) = D(\Gamma(\mathcal{O}_\mathcal{D}))$. The assertion $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ follows from Lemma 21.19.2 applied to

$$(\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$$

The second (more general) statement follows from the first statement after applying Lemma 21.20.4. □

0D6I **Lemma 21.20.6.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let K be in $D(\mathcal{O}_\mathcal{C})$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(u(V), K) = H^i(V, Rf_*K)$$

Proof. The equality $H^i(u(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 21.20.5. Then the statement on sheafification follows from Lemma 21.20.3. □

0D6J **Lemma 21.20.7.** *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Let K be an object of $D(\mathcal{O}_\mathcal{C})$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_\mathcal{C})$.*

1. *There is a canonical map $R\Gamma(\mathcal{C}, K) \to R\Gamma(\mathcal{C}, K_{ab})$ which is an isomorphism in $D(Ab)$.*
2. *For any $U \in \mathcal{C}$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(Ab)$.*
3. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_\mathcal{D})$.*

Proof. The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(\mathcal{C}, \mathcal{I}^\bullet) \to \Gamma(\mathcal{C}, \mathcal{J}^\bullet)$ (2) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (3) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 21.20.2 and 21.20.6 it suffices to show that the map

$$H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$$

is an isomorphism. Observe that

$$H^0(\mathcal{C}, K) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules representing K. By construction of the derived category as a localization we have

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K) = \mathrm{colim}_{s:\mathcal{F}^\bullet \to \mathcal{O}_\mathcal{C}} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms s of complexes of $\mathcal{O}_\mathcal{C}$-modules. Similarly, we have

$$\mathrm{Hom}_{D(\underline{\mathbf{Z}}_\mathcal{C})}(\underline{\mathbf{Z}}_\mathcal{C}, K) = \mathrm{colim}_{s:\mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}} \mathrm{Hom}_{K(\underline{\mathbf{Z}}_\mathcal{C})}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules is cofinal in the system. (This follows from Modules on Sites, Lemma 18.28.8 and Derived Categories, Lemma 13.15.4; see discussion in Section 21.17.) Hence we can construct an inverse to the map $H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$ by representing an element $\xi \in H^0(\mathcal{C}, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 21.17.12 and 21.17.8. We omit the detailed verification that this construction is indeed an inverse. □

08FJ **Lemma 21.20.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The restriction functor $D(\mathcal{O}) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O})$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 21.20.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Modules on Sites, Lemma 18.19.2) and Derived Categories, Lemma 13.31.2. □

0GL1 **Lemma 21.20.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For L in $D(\mathcal{O}_U)$ and K in $D(\mathcal{O})$ we have $j_!L \otimes^{\mathbf{L}}_{\mathcal{O}} K = j_!(L \otimes^{\mathbf{L}}_{\mathcal{O}_U} K|_U)$.*

Proof. Represent L by a complex of $\mathcal{O}_U$-modules and K by a K-flat complexe of $\mathcal{O}$-modules and apply Modules on Sites, Lemma 18.27.9. Details omitted. □

093Y **Lemma 21.20.10.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a flat morphism of ringed topoi. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_\mathcal{C}$-modules, then $f_*\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_\mathcal{D}$-modules.*

Proof. This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Modules on Sites, Lemma 18.13.2 and the fact that f^* is exact as f is assumed to be flat. □

093Z **Lemma 21.20.11.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules, then $\mathcal{H}om_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}'$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 18.27.8. □

21.21. Localization and cohomology

0EYZ Let $\mathcal{C}$ be a site. Let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then we obtain a morphism of topoi

$$j_{X/Y} : Sh(\mathcal{C}/X) \longrightarrow Sh(\mathcal{C}/Y)$$

See Sites, Sections 7.25 and 7.27. Some questions about cohomology are easier for this type of morphisms of topoi. Here is an example where we get a trivial type of base change theorem.

0EZ0 **Lemma 21.21.1.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j^{-1}_{Y'/Y} \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j^{-1}_{X'/X}$ as functors $D(\mathcal{C}/X) \to D(\mathcal{C}/Y')$.

Proof. Let $E \in D(\mathcal{C}/X)$. Choose a K-injective complex $\mathcal{I}^\bullet$ of abelian sheaves on $\mathcal{C}/X$ representing E. By Lemma 21.20.1 we see that $j^{-1}_{X'/X}\mathcal{I}^\bullet$ is K-injective too. Hence we may compute $Rj_{X'/Y'}(j^{-1}_{X'/X}E)$ by $j_{X'/Y',*}j^{-1}_{X'/X}\mathcal{I}^\bullet$. Thus we see that the equality holds by Sites, Lemma 7.27.5. □

If we have a ringed site $(\mathcal{C}, \mathcal{O})$ and a morphism $f : X \to Y$ of $\mathcal{C}$, then $j_{X/Y}$ becomes a morphism of ringed topoi

$$j_{X/Y} : (Sh(\mathcal{C}/X), \mathcal{O}_X) \longrightarrow (Sh(\mathcal{C}/Y), \mathcal{O}_Y)$$

See Modules on Sites, Lemma 18.19.5.

0FN5 **Lemma 21.21.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j_{Y'/Y}^* \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j_{X'/X}^*$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_{Y'})$.

Proof. Since $j_{Y'/Y}^{-1}\mathcal{O}_Y = \mathcal{O}_{Y'}$ we have $j_{Y'/Y}^* = Lj_{Y'/Y}^* = j_{Y'/Y}^{-1}$. Similarly we have $j_{X'/X}^* = Lj_{X'/X}^* = j_{X'/X}^{-1}$. Thus by Lemma 21.20.7 it suffices to prove the result on derived categories of abelian sheaves which we did in Lemma 21.21.1. □

21.22. Inverse systems and cohomology

0GYP We prove some results on inverse systems of sheaves of modules.

0GYQ **Lemma 21.22.1.** *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[2].

Proof. Set $N_n = H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$. Then $\bigoplus \mathrm{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & I^{n+m}\mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m} & \longrightarrow & 0 \end{array}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \mathrm{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j\delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_j s_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_j s_j$ in M_n. It follows that

$$\mathrm{Im}(M_{n+1} \to M_{n-c}) = \mathrm{Im}(M_n \to M_{n-c})$$

Namely, the elements $f_j s_j$ map to zero in M_{n-c}. This proves the lemma. □

[2]In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

0GYR **Lemma 21.22.2.** *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[3].

Proof. The proof is exactly the same as the proof of Lemma 21.22.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ have image contained in N_n.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I^{n+k}\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{n+k} & \longrightarrow & 0
\end{array}$$

constructed as in the proof of Lemma 21.22.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(\mathcal{C}, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in N_{n+k} as desired.

To see the boundary maps δ_n have image contained in N_n we consider the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0
\end{array}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. □

0GYS **Lemma 21.22.3.** *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \mathrm{Ker}(M \to H^p(\mathcal{C}, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$. Hence the I-adic topology is finer than the limit topology. For the converse, we will show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$[4]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(\mathcal{C}, \mathcal{F}_{n+1})$$

[3] In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

[4] In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^nM$ for all n.

whose image is contained in the image of $H^p(\mathcal{C}, I^n\mathcal{F}_{n+1}) \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

the inverse image of F^n/F^{n+1}. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^nF^c \subset I^nM$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_ia_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j}h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j}a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_ix_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 15.36.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

0GYT **Lemma 21.22.4.** *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. The proof is exactly the same as the proof of Lemma 21.22.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(\mathcal{C}, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. In the proof of Lemma 21.22.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ which maps to the image of t in $H^p(\mathcal{C}, \mathcal{F}_{n+1})$. We have to show that s is in N_n. Now F^n is the kernel of the map from $M \to H^p(\mathcal{C}, \mathcal{F}_n)$ hence for all $m \geq n$ we can map t to an element $t_m \in H^p(\mathcal{C}, \mathcal{F}_{m+1})$ which maps to zero in $H^p(\mathcal{C}, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n\mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(\mathcal{C}, I^n\mathcal{F}_{m+1})$ mapping to t_m. Comparing the sequence above with the one for $m = n$ we see that s_m maps to s up to an element in the image

of $H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$. However, this map factors through the map $H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and we see that s is in the image as desired. $\square$

21.23. Derived and homotopy limits

0940 Let $\mathcal{C}$ be a site. Consider the category $\mathcal{C} \times \mathbf{N}$ with $\mathrm{Mor}((U,n),(V,m)) = \emptyset$ if $n > m$ and $\mathrm{Mor}((U,n),(V,m)) = \mathrm{Mor}(U,V)$ else. We endow this with the structure of a site by letting coverings be families $\{(U_i,n) \to (U,n)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then the reader verifies immediately that sheaves on $\mathcal{C} \times \mathbf{N}$ are the same thing as inverse systems of sheaves on $\mathcal{C}$. In particular $Ab(\mathcal{C} \times \mathbf{N})$ is inverse systems of abelian sheaves on $\mathcal{C}$. Consider now the functor

$$\lim : Ab(\mathcal{C} \times \mathbf{N}) \to Ab(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but g_* where $g : Sh(\mathcal{C} \times \mathbf{N}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to the continuous and cocontinuous functor $\mathcal{C} \times \mathbf{N} \to \mathcal{C}$. (Observe that g^{-1} assigns to a sheaf on $\mathcal{C}$ the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \to D(\mathcal{C}).$$

As indicated this functor is often denoted $R\lim$.

On the other hand, the continuous and cocontinuous functors $\mathcal{C} \to \mathcal{C} \times \mathbf{N}$, $U \mapsto (U,n)$ define morphisms of topoi $i_n : Sh(\mathcal{C}) \to Sh(\mathcal{C} \times \mathbf{N})$. Of course i_n^{-1} is the functor which picks the nth term of the inverse system. Thus there are transformations of functors $i_{n+1}^{-1} \to i_n^{-1}$. Hence given $K \in D(\mathcal{C} \times \mathbf{N})$ we get $K_n = i_n^{-1}K \in D(\mathcal{C})$ and maps $K_{n+1} \to K_n$. In Derived Categories, Definition 13.34.1 we have defined the notion of a homotopy limit

$$R\lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

0941 **Lemma 21.23.1.** *Let $\mathcal{C}$ be a site. Let K be an object of $D(\mathcal{C} \times \mathbf{N})$. Set $K_n = i_n^{-1}K$ as above. Then*

$$R\lim K \cong R\lim K_n$$

in $D(\mathcal{C})$.

Proof. To calculate $R\lim$ on an object K of $D(\mathcal{C} \times \mathbf{N})$ we choose a K-injective representative $\mathcal{I}^\bullet$ whose terms are injective objects of $Ab(\mathcal{C} \times \mathbf{N})$, see Injectives, Theorem 19.12.6. We may and do think of $\mathcal{I}^\bullet$ as an inverse system of complexes $(\mathcal{I}_n^\bullet)$ and then we see that

$$R\lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let $\mathcal{J} = (\mathcal{J}_n)$ be an injective object of $Ab(\mathcal{C} \times \mathbf{N})$. The morphisms $(U,n) \to (U,n+1)$ are monomorphisms of $\mathcal{C} \times \mathbf{N}$, hence $\mathcal{J}(U,n+1) \to \mathcal{J}(U,n)$ is surjective (Lemma 21.12.6). It follows that $\mathcal{J}_{n+1} \to \mathcal{J}_n$ is surjective as a map of presheaves.

Note that the functor i_n^{-1} has an exact left adjoint $i_{n,!}$. Namely, $i_{n,!}\mathcal{F}$ is the inverse system $\ldots 0 \to 0 \to \mathcal{F} \to \ldots \to \mathcal{F}$. Thus the complexes $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ are K-injective by Derived Categories, Lemma 13.31.9.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \to \lim \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in $D(\mathcal{C})$, see Injectives, Lemma 19.13.4 and its proof (this is where we use that $\mathcal{I}_n^\bullet$ is K-injective). Thus $R\lim K$ is a homotopy limit of the inverse system (K_n) by definition of homotopy limits in triangulated categories. □

0D6K **Lemma 21.23.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The functors $R\Gamma(\mathcal{C}, -)$ and $R\Gamma(U, -)$ for $U \in \mathrm{Ob}(\mathcal{C})$ commute with $R\lim$. Moreover, there are short exact sequences*

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

for any inverse system (K_n) in $D(\mathcal{O})$ and $m \in \mathbf{Z}$. Similar for $H^m(\mathcal{C}, R\lim K_n)$.

Proof. The first statement follows from Injectives, Lemma 19.13.6. Then we may apply More on Algebra, Remark 15.87.10 to $R\lim R\Gamma(U, K_n) = R\Gamma(U, R\lim K_n)$ to get the short exact sequences. □

0A07 **Lemma 21.23.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.*

Proof. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. By induction on n we may choose actual complexes $\mathcal{K}_n^\bullet$ of $\mathcal{O}$-modules and maps of complexes $\mathcal{K}_{n+1}^\bullet \to \mathcal{K}_n^\bullet$ representing the maps $K_{n+1} \to K_n$ in $D(\mathcal{O})$. In other words, there exists an object K in $D(\mathcal{C} \times \mathbf{N})$ whose associated inverse system is the given one. Next, consider the commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C} \times \mathbf{N}) & \xrightarrow{g} & Sh(\mathcal{C}) \\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{C}' \times \mathbf{N}) & \xrightarrow{g'} & Sh(\mathcal{C}') \end{array}$$

of morphisms of topoi. It follows that $R\lim R(f \times 1)_* K = Rf_* R\lim K$. Working through the definitions and using Lemma 21.23.1 we obtain that $R\lim(Rf_* K_n) = Rf_*(R\lim K_n)$.

Alternate proof in case $\mathcal{C}$ has enough points. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O})$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}')$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 19.13.4). However, since Rf_* is a right adjoint by Lemma 21.19.1 this follows formally (see Categories, Lemma 4.24.5). Caution: Note that we cannot apply Categories, Lemma 4.24.5 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O})$. □

0BKW **Remark 21.23.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Set $K = R\lim K_n$. For each n and m let $\mathcal{H}^m_n = H^m(K_n)$ be the mth cohomology sheaf of K_n and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}^m_n$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}^m_n(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 21.20.3 we see that $\mathcal{H}^m_n$ is the sheafification of $\underline{\mathcal{H}}^m_n$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$\begin{array}{ccc} K & & \underline{\mathcal{H}}^m \longrightarrow \mathcal{H}^m \\ \| & & \downarrow \quad\quad \downarrow \\ R\lim K_n & & \lim \underline{\mathcal{H}}^m_n \longrightarrow \lim \mathcal{H}^m_n \end{array}$$

In general it may not be the case that $\lim \mathcal{H}^m_n$ is the sheafification of $\lim \underline{\mathcal{H}}^m_n$. If $U \in \mathcal{C}$, then we have short exact sequences

0BKX (21.23.4.1) $$0 \to R^1 \lim \underline{\mathcal{H}}^{m-1}_n(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}^m_n(U) \to 0$$

by Lemma 21.23.2.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on an algebraic space or an algebraic stack.

0BKY **Lemma 21.23.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}$-modules. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *$H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,*
3. *the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1 \lim$ for $U \in \mathcal{B}$.*

Then $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.

Proof. Set $K_n = \mathcal{F}_n$ and $K = R\lim \mathcal{F}_n$. Using the notation of Remark 21.23.4 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}^m_n(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}^0_n(U) = \mathcal{F}_n(U)$. From Equation (21.23.4.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. □

0D6L **Lemma 21.23.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Let $V \in \mathrm{Ob}(\mathcal{C})$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(V)$ and a cofinal system Cov_V of coverings of V such that for $\{V_i \to V\} \in Cov_V$*

1. *$R^1 \lim H^{m-1}(V_i, K_n) = 0$, and*
2. *$H^m(V_i, K_n) \to H^m(V_i, K_{n(V)})$ is injective for $n \geq n(V)$.*

Then the map on sections $H^m(R\lim K_n)(V) \to H^m(K_{n(V)})(V)$ is injective.

Proof. Let $\gamma \in H^m(R\lim K_n)(V)$ map to zero in $H^m(K_{n(V)})(V)$. Since $H^m(R\lim K_n)$ is the sheafification of $U \mapsto H^m(U, R\lim K_n)$ (by Lemma 21.20.3) we can choose $\{V_i \to V\} \in \mathrm{Cov}_V$ and elements $\tilde{\gamma}_i \in H^m(V_i, R\lim K_n)$ mapping to $\gamma|_{V_i}$. Then $\tilde{\gamma}_i$ maps to $\tilde{\gamma}_{i,n(V)} \in H^m(V_i, K_{n(V)})$. Using that $H^m(K_{n(V)})$ is the sheafification of $U \mapsto H^m(U, K_{n(V)})$ (by Lemma 21.20.3 again) we see that after replacing $\{V_i \to V\}$

by a refinement we may assume that $\tilde{\gamma}_{i,n(V)} = 0$ for all i. For this covering we consider the short exact sequences

$$0 \to R^1 \lim H^{m-1}(V_i, K_n) \to H^m(V_i, R\lim K_n) \to \lim H^m(V_i, K_n) \to 0$$

of Lemma 21.23.2. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(V_i, K_{n(V)})$. We conclude $\tilde{\gamma}_i = 0$ and hence $\gamma = 0$ as desired. □

0D6M **Lemma 21.23.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and*
(2) *for every $V \in \mathcal{B}$ there exist a function $p(V, -) : \mathbf{Z} \to \mathbf{Z}$ and a cofinal system Cov_V of coverings of V such that*

$$H^p(V_i, H^{m-p}(E)) = 0$$

for all $\{V_i \to V\} \in Cov_V$ and all integers p, m satisfying $p > p(V, m)$.

Then the map $E \to R\lim \tau_{\geq -n} E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O})$.

Proof. Set $K_n = \tau_{\geq -n} E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n} E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix m. If $n \geq -m$, then the map $E \to \tau_{\geq -n} E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $V \in \mathcal{B}$ there exists an integer $n(V) \geq -m$ such that the map $H^m(K)(V) \to H^m(K_{n(V)})(V)$ is injective. Namely, then the composition

$$H^m(E)(V) \to H^m(K)(V) \to H^m(K_{n(V)})(V)$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. By property (1) this will imply $H^m(E) \to H^m(K)$ is an isomorphism. Set

$$n(V) = 1 + \max\{-m, p(V, m-1) - m, -1 + p(V, m) - m, -2 + p(V, m+1) - m\}.$$

so that in any case $n(V) \geq -m$. Claim: the maps

$$H^{m-1}(V_i, K_{n+1}) \to H^{m-1}(V_i, K_n) \quad \text{and} \quad H^m(V_i, K_{n+1}) \to H^m(V_i, K_n)$$

are isomorphisms for $n \geq n(V)$ and $\{V_i \to V\} \in \mathrm{Cov}_V$. The claim implies conditions (1) and (2) of Lemma 21.23.6 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 13.12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n+2]$$

Looking at the associated long exact cohomology sequence the claim follows if

$$H^{m+n}(V_i, H^{-n-1}(E)), \quad H^{m+n+1}(V_i, H^{-n-1}(E)), \quad H^{m+n+2}(V_i, H^{-n-1}(E))$$

are zero for $n \geq n(V)$ and $\{V_i \to V\} \in \mathrm{Cov}_V$. This follows from our choice of $n(V)$ and the assumption in the lemma. □

0D6N **Lemma 21.23.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and*

(2) *for every $V \in \mathcal{B}$ there exist an integer $d_V \geq 0$ and a cofinal system Cov_V of coverings of V such that*

$$H^p(V_i, H^q(E)) = 0 \text{ for } \{V_i \to V\} \in Cov_V,\ p > d_V,\ \text{and } q < 0$$

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O})$.

Proof. This follows from Lemma 21.23.7 with $p(V, m) = d_V + \max(0, m)$. □

08U3 **Lemma 21.23.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(V, H^{m-p}(E)) = 0$ for $p > p(m)$ and $V \in \mathcal{B}$.*

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 21.23.7 with $p(V, m) = p(m)$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. □

0D6P **Lemma 21.23.10.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists an integer $d \geq 0$ and a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(V, H^q(E)) = 0$ for $p > d$, $q < 0$, and $V \in \mathcal{B}$.*

Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 21.23.8 with $d_V = d$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. □

The lemmas above can be used to compute cohomology in certain situations.

0BKZ **Lemma 21.23.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.*

Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.

Proof. Observe that $K = R\lim \tau_{\geq -n}K$ by Lemma 21.23.10 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (21.23.4.1) we get a short exact sequence

$$0 \to R^1 \lim H^{q-1}(U, \tau_{\geq -n}K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n}K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n}K) = H^0(U, H^q(\tau_{\geq -n}K))$ for all q by using the spectral sequence of Derived Categories, Lemma 13.21.3. The spectral sequence converges because $\tau_{\geq -n}K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n}K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$ and the lemma follows. □

Here is another case where we can describe the derived limit.

0A09 **Lemma 21.23.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*

(2) *for all* $U \in \mathcal{B}$ *and all* $q \in \mathbf{Z}$ *we have*
   (a) $H^p(U, H^q(K_n)) = 0$ *for* $p > 0$,
   (b) *the inverse system* $H^0(U, H^q(K_n))$ *has vanishing* $R^1\lim$.

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ *for* $q \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. We will use notation as in Remark 21.23.4. Let $U \in \mathcal{B}$. By Lemma 21.23.11 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 15.87.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma. □

21.24. Producing K-injective resolutions

070N Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $Mod(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 13.29.3 produce a diagram

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet \end{array}$$

in the category of complexes of $\mathcal{O}$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.31.4 and 13.31.8 this is a K-injective complex. In general the canonical map

070P (21.24.0.1) $$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

070Q **Lemma 21.24.1.** *In the situation described above. Denote* $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ *the* m*th cohomology sheaf. Let* $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ *be a subset. Let* $d \in \mathbf{N}$. *Assume*

(1) *every object of* $\mathcal{C}$ *has a covering whose members are elements of* $\mathcal{B}$,
(2) *for every* $U \in \mathcal{B}$ *we have* $H^p(U, \mathcal{H}^q) = 0$ *for* $p > d$ *and* $q < 0$[5].

Then (21.24.0.1) is a quasi-isomorphism.

Proof. By Derived Categories, Lemma 13.34.5 it suffices to show that the map $\mathcal{F}^\bullet \to R\lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This follows from Lemma 21.23.10. □

[5]It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 21.23.9.

Here is a technical lemma about cohomology sheaves of termwise limits of inverse systems of complexes of modules. We should avoid using this lemma as much as possible and instead use arguments with derived inverse limits.

08CT **Lemma 21.24.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$. Suppose given $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ and an integer n_0 such that*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$*
   (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler property),*
   (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler property), and*
   (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 15.87.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

21.25. Bounded cohomological dimension

0D6Q In this section we ask when a functor Rf_* has bounded cohomological dimension. This is a rather subtle question when we consider unbounded complexes.

0D6R **Situation 21.25.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A} \subset \mathit{Mod}(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that

(1) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and
(2) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \mathrm{Cov}_V,\ p > d_V, \text{ and } \mathcal{F} \in \mathrm{Ob}(\mathcal{A})$$

0D6S **Lemma 21.25.2.** *In Situation 21.25.1 for any $E \in D_{\mathcal{A}}(\mathcal{O})$ the map $E \to R\lim \tau_{\geq -n} E$ of Derived Categories, Remark 13.34.4 is an isomorphism in $D(\mathcal{O})$.*

This is [**LO08a**, Proposition 2.1.4] with slightly changed hypotheses; it is the analogue of [**Spa88**, Proposition 3.13] for sites.

Proof. Follows immediately from Lemma 21.23.8. □

0D6T **Lemma 21.25.3.** *In Situation 21.25.1 let (K_n) be an inverse system in $D^+_{\mathcal{A}}(\mathcal{O})$. Assume that for every j the inverse system $(H^j(K_n))$ in $\mathcal{A}$ is eventually constant with value $\mathcal{H}^j$. Then $H^j(R\lim K_n) = \mathcal{H}^j$ for all j.*

Proof. Let $V \in \mathcal{B}$. Let $\{V_i \to V\}$ be in the set Cov_V of Situation 21.25.1. Because K_n is bounded below there is a spectral sequence

$$E_2^{p,q} = H^p(V_i, H^q(K_n))$$

converging to $H^{p+q}(V_i, K_n)$. See Derived Categories, Lemma 13.21.3. Observe that $E_2^{p,q} = 0$ for $p > d_V$ by assumption. Pick n_0 such that

$$\begin{aligned} \mathcal{H}^{j+1} &= H^{j+1}(K_n), \\ \mathcal{H}^j &= H^j(K_n), \\ &\ldots, \\ \mathcal{H}^{j-d_V-2} &= H^{j-d_V-2}(K_n) \end{aligned}$$

for all $n \geq n_0$. Comparing the spectral sequences above for K_n and K_{n_0}, we see that for $n \geq n_0$ the cohomology groups $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$ are independent of n. It follows that the map on sections $H^j(R\lim K_n)(V) \to H^j(K_n)(V)$ is injective for n large enough (depending on V), see Lemma 21.23.6. Since every object of $\mathcal{C}$ can be covered by elements of $\mathcal{B}$, we conclude that the map $H^j(R\lim K_n) \to \mathcal{H}^j$ is injective.

Surjectivity is shown in a similar manner. Namely, pick $U \in \mathrm{Ob}(\mathcal{C})$ and $\gamma \in \mathcal{H}^j(U)$. We want to lift γ to a section of $H^j(R\lim K_n)$ after replacing U by the members of a covering. Hence we may assume $U = V \in \mathcal{B}$ by property (1) of Situation 21.25.1. Pick n_0 such that

$$\begin{aligned} \mathcal{H}^{j+1} &= H^{j+1}(K_n), \\ \mathcal{H}^j &= H^j(K_n), \\ &\ldots, \\ \mathcal{H}^{j-d_V-2} &= H^{j-d_V-2}(K_n) \end{aligned}$$

for all $n \geq n_0$. Choose an element $\{V_i \to V\}$ of Cov_V such that $\gamma|_{V_i} \in \mathcal{H}^j(V_i) = H^j(K_{n_0})(V_i)$ lifts to an element $\gamma_{n_0,i} \in H^j(V_i, K_{n_0})$. This is possible because $H^j(K_{n_0})$ is the sheafification of $U \mapsto H^j(U, K_{n_0})$ by Lemma 21.20.3. By the discussion in the first paragraph of the proof we have that $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$ are independent of $n \geq n_0$. Hence $\gamma_{n_0,i}$ lifts to an element $\gamma_i \in H^j(V_i, R\lim K_n)$ by Lemma 21.23.2. This finishes the proof. □

0D6U **Lemma 21.25.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume there is an integer N such that*

(1) *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,*
(2) *$\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 21.25.1,*
(3) *$R^p f_* \mathcal{F} \in \mathrm{Ob}(\mathcal{A}')$ for $p \geq 0$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*
(4) *$R^p f_* \mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*

Then for K in $D_{\mathcal{A}}(\mathcal{O})$ we have

(a) *Rf_*K is in $D_{\mathcal{A}'}(\mathcal{O}')$,*
(b) *the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N - n$.*

This is a version of [**LO08a**, Lemma 2.1.10] with slightly changed hypotheses.

Proof. By Lemma 21.25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 21.23.3 we have $Rf_*K = R\lim Rf_*\tau_{\geq -n}K$. The complexes $Rf_*\tau_{\geq -n}K$ are bounded below. The spectral sequence

$$E_2^{p,q} = R^p f_* H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*\tau_{\geq -n}K)$ (Derived Categories, Lemma 13.21.3) and assumption (3) show that $Rf_*\tau_{\geq -n}K$ lies in $D^+_{\mathcal{A}'}(\mathcal{O}')$, see Homology, Lemma 12.24.11. Observe that for $m \geq n$ the map

$$Rf_*(\tau_{\geq -m}K) \longrightarrow Rf_*(\tau_{\geq -n}K)$$

induces an isomorphism on cohomology sheaves in degrees $j \geq -n + N$ by the spectral sequences above. Hence we may apply Lemma 21.25.3 to conclude. □

It turns out that we sometimes need a variant of the lemma above where the assumptions are slightly different.

0D6V **Situation 21.25.5.** Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $u : \mathcal{C}' \to \mathcal{C}$ be the corresponding continuous functor of sites. Let $\mathcal{A} \subset Mod(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B}' \subset \mathrm{Ob}(\mathcal{C}')$ such that

(1) every object of $\mathcal{C}'$ has a covering whose members are in $\mathcal{B}'$, and
(2) for every $V' \in \mathcal{B}'$ there exists an integer $d_{V'}$ and a cofinal system $\mathrm{Cov}_{V'}$ of coverings of V' such that

$$H^p(u(V'_i), \mathcal{F}) = 0 \text{ for } \{V'_i \to V'\} \in \mathrm{Cov}_{V'},\ p > d_{V'}, \text{ and } \mathcal{F} \in \mathrm{Ob}(\mathcal{A})$$

0D6W **Lemma 21.25.6.** *Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. assume moreover there is an integer N such that*

(1) *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,*
(2) *$f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 21.25.5,*
(3) *$R^p f_*\mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*

Then for K in $D_{\mathcal{A}}(\mathcal{O})$ the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N - n$.

This is a version of [**LO08a**, Lemma 2.1.10] with slightly changed hypotheses.

Proof. Let K be in $D_{\mathcal{A}}(\mathcal{O})$. By Lemma 21.25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 21.23.3 we have $Rf_*K = R\lim Rf_*(\tau_{\geq -n}K)$. Let $V' \in \mathcal{B}'$ and let $\{V'_i \to V'\}$ be an element of $\mathrm{Cov}_{V'}$. Then we consider

$$H^j(V'_i, Rf_*K) = H^j(u(V'_i), K) \quad \text{and} \quad H^j(V'_i, Rf_*(\tau_{\geq -n}K)) = H^j(u(V'_i), \tau_{\geq -n}K)$$

The assumption in Situation 21.25.5 implies that the last group is independent of n for n large enough depending on j and $d_{V'}$. Some details omitted. We apply this for j and $j - 1$ and via Lemma 21.23.2 this gives that

$$H^j(V'_i, Rf_*K) = \lim H^j(V'_i, Rf_*(\tau_{\geq -n}K))$$

and the system on the right is constant for n larger than a constant depending only on $d_{V'}$ and j. Thus Lemma 21.23.6 implies that

$$H^j(Rf_*K)(V') \longrightarrow \left(\lim H^j(Rf_*(\tau_{\geq -n}K))\right)(V')$$

is injective. Since the elements $V' \in \mathcal{B}'$ cover every object of $\mathcal{C}'$ we conclude that the map $H^j(Rf_*K) \to \lim H^j(Rf_*(\tau_{\geq -n}K))$ is injective. The spectral sequence

$$E_2^{p,q} = R^p f_* H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*(\tau_{\geq -n}K))$ (Derived Categories, Lemma 13.21.3) and assumption (3) show that $H^j(Rf_*(\tau_{\geq -n}K))$ is constant for $n \geq N - j$. Hence $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is injective for $j \geq N - n$.

Thus we proved the lemma with "isomorphism" in the last line of the lemma replaced by "injective". However, now choose j and n with $j \geq N - n$. Then consider the distinguished triangle

$$\tau_{\leq -n-1}K \to K \to \tau_{\geq -n}K \to (\tau_{\leq -n-1}K)[1]$$

See Derived Categories, Remark 13.12.4. Since $\tau_{\geq -n}\tau_{\leq -n-1}K = 0$, the injectivity already proven for $\tau_{-n-1}K$ implies

$$0 = H^j(Rf_*(\tau_{\leq -n-1}K)) = H^{j+1}(Rf_*(\tau_{\leq -n-1}K)) = H^{j+2}(Rf_*(\tau_{\leq -n-1}K)) = \ldots$$

By the long exact cohomology sequence associated to the distinguished triangle

$$Rf_*(\tau_{\leq -n-1}K) \to Rf_*K \to Rf_*(\tau_{\geq -n}K) \to Rf_*(\tau_{\leq -n-1}K)[1]$$

this implies that $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism. □

21.26. Mayer-Vietoris

0EVX For the usual statement and proof of Mayer-Vietoris, please see Cohomology, Section 20.8.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

in the category $\mathcal{C}$. In this situation, given an object K of $D(\mathcal{O})$ we get what looks like the beginning of a distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K)$$

In the following lemma we make this more precise.

0F16 **Lemma 21.26.1.** *In the situation above, choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing K. Using -1 times the canonical map for one of the four arrows we get maps of complexes*

$$\mathcal{I}^\bullet(X) \xrightarrow{\alpha} \mathcal{I}^\bullet(Z) \oplus \mathcal{I}^\bullet(Y) \xrightarrow{\beta} \mathcal{I}^\bullet(E)$$

with $\beta \circ \alpha = 0$. Thus a canonical map

$$c^K_{X,Z,Y,E} : \mathcal{I}^\bullet(X) \longrightarrow C(\beta)^\bullet[-1]$$

This map is canonical in the sense that a different choice of K-injective complex representing K determines an isomorphic arrow in the derived category of abelian groups. If $c^K_{X,Z,Y,E}$ is an isomorphism, then using its inverse we obtain a canonical distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K) \to R\Gamma(X, K)[1]$$

All of these constructions are functorial in K.

Proof. This lemma proves itself. For example, if $\mathcal{J}^\bullet$ is a second K-injective complex representing K, then we can choose a quasi-isomorphism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ which determines quasi-isomorphisms between all the complexes in sight. Details omitted. For the construction of cones and the relationship with distinguished triangles see Derived Categories, Sections 13.9 and 13.10. □

0EWP **Lemma 21.26.2.** *In the situation above, let $K_1 \to K_2 \to K_3 \to K_1[1]$ be a distinguished triangle in $D(\mathcal{O})$. If $c^{K_i}_{X,Z,Y,E}$ is a quasi-isomorphism for two i out of $\{1,2,3\}$, then it is a quasi-isomorphism for the third i.*

Proof. By rotating the triangle we may assume $c^{K_1}_{X,Z,Y,E}$ and $c^{K_2}_{X,Z,Y,E}$ are quasi-isomorphisms. Choose a map $f : \mathcal{I}_1^\bullet \to \mathcal{I}_2^\bullet$ of K-injective complexes of $\mathcal{O}$-modules representing $K_1 \to K_2$. Then K_3 is represented by the K-injective complex $C(f)^\bullet$, see Derived Categories, Lemma 13.31.3. Then the morphism $c^{K_3}_{X,Z,Y,E}$ is an isomorphism as it is the third leg in a map of distinguished triangles in $K(Ab)$ whose other two legs are quasi-isomorphisms. Some details omitted; use Derived Categories, Lemma 13.4.3. □

Let us give a criterion for when this does produce a distinguished triangle.

0EVY **Lemma 21.26.3.** *In the situation above assume*

(1) $h^\#_X = h^\#_Y \amalg_{h^\#_E} h^\#_Z$, *and*
(2) $h^\#_E \to h^\#_Y$ *is injective.*

Then the construction of Lemma 21.26.1 produces a distinguished triangle

$$R\Gamma(X,K) \to R\Gamma(Z,K) \oplus R\Gamma(Y,K) \to R\Gamma(E,K) \to R\Gamma(X,K)[1]$$

functorial for K in $D(\mathcal{C})$.

Proof. We can represent K by a K-injective complex whose terms are injective abelian sheaves, see Section 21.19. Thus it suffices to show: if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(Z) \oplus \mathcal{I}(Y) \to \mathcal{I}(E) \to 0$$

is a short exact sequence. The first arrow is injective because by condition (1) the map $h_Y \amalg h_Z \to h_X$ becomes surjective after sheafification, which means that $\{Y \to X, Z \to X\}$ can be refined by a covering of X. The last arrow is surjective because $\mathcal{I}(Y) \to \mathcal{I}(E)$ is surjective. Namely, we have $\mathcal{I}(E) = \mathrm{Hom}(\mathbf{Z}^\#_E, \mathcal{I})$, $\mathcal{I}(Y) = \mathrm{Hom}(\mathbf{Z}^\#_Y, \mathcal{I})$, the map $\mathbf{Z}^\#_E \to \mathbf{Z}^\#_Y$ is injective by (2), and $\mathcal{I}$ is an injective abelian sheaf. Please compare with Modules on Sites, Section 18.5. Finally, suppose we have $s \in \mathcal{I}(Y)$ and $t \in \mathcal{F}(Z)$ mapping to the same element of $\mathcal{I}(E)$. Then s and t define a map

$$s \amalg t : h^\#_Y \amalg h^\#_Z \longrightarrow \mathcal{I}$$

which by assumption factors through $h^\#_Y \amalg_{h^\#_E} h^\#_Z$. Thus by assumption (1) we obtain a unique map $h^\#_X \to \mathcal{I}$ which corresponds to an element of $\mathcal{I}(X)$ restricting to s on Y and t on Z. □

0EVZ **Lemma 21.26.4.** *Let $\mathcal{C}$ be a site. Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{G} \end{array}$$

of presheaves of sets on $\mathcal{C}$ and assume that

(1) $\mathcal{G}^\# = \mathcal{E}^\# \amalg_{\mathcal{D}^\#} \mathcal{F}^\#$, *and*
(2) $\mathcal{D}^\# \to \mathcal{F}^\#$ *is injective.*

Then there is a canonical distinguished triangle

$$R\Gamma(\mathcal{G}, K) \to R\Gamma(\mathcal{E}, K) \oplus R\Gamma(\mathcal{F}, K) \to R\Gamma(\mathcal{D}, K) \to R\Gamma(\mathcal{G}, K)[1]$$

functorial in $K \in D(\mathcal{C})$ where $R\Gamma(\mathcal{G}, -)$ is the cohomology discussed in Section 21.13.

Proof. Since sheafification is exact and since $R\Gamma(\mathcal{G}, -) = R\Gamma(\mathcal{G}^\#, -)$ we may assume $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ are sheaves of sets. Moreover, the cohomology $R\Gamma(\mathcal{G}, -)$ only depends on the topos, not on the underlying site. Hence by Sites, Lemma 7.29.5 we may replace $\mathcal{C}$ by a "larger" site with a subcanonical topology such that $\mathcal{G} = h_X$, $\mathcal{F} = h_Y$, $\mathcal{E} = h_Z$, and $\mathcal{D} = h_E$ for some objects X, Y, Z, E of $\mathcal{C}$. In this case the result follows from Lemma 21.26.3. □

21.27. Comparing two topologies

0EWK Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\mathrm{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\mathrm{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf. See Sites, Examples 7.14.3 and 7.22.3. In the situation above we have the following

(1) $\epsilon_* : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \mathrm{id}$,
(2) $\epsilon_* : Ab(\mathcal{C}_\tau) \to Ab(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \mathrm{id}$,
(3) $R\epsilon_* : D(\mathcal{C}_\tau) \to D(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ R\epsilon_* = \mathrm{id}$,
(4) if $\mathcal{O}$ is a sheaf of rings for the τ-topology, then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a flat morphism of ringed sites
$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$
(5) $\epsilon_* : Mod(\mathcal{O}_\tau) \to Mod(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ \epsilon_* = \mathrm{id}$
(6) $R\epsilon_* : D(\mathcal{O}_\tau) \to D(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \mathrm{id}$.

Here are some explanations.

Ad (1). Let $\mathcal{F}$ be a sheaf of sets in the τ-topology. Then $\epsilon_*\mathcal{F}$ is just $\mathcal{F}$ viewed as a sheaf in the τ'-topology. Applying ϵ^{-1} means taking the τ-sheafification of $\mathcal{F}$, which doesn't do anything as $\mathcal{F}$ is already a τ-sheaf. Thus $\epsilon^{-1}(\epsilon_*\mathcal{F})) = \mathcal{F}$. The fully faithfulness follows by Categories, Lemma 4.24.4.

Ad (2). This is a consequence of (1) since pullback and pushforward of abelian sheaves is the same as doing those operations on the underlying sheaves of sets.

Ad (3). Let K be an object of $D(\mathcal{C}_\tau)$. To compute $R\epsilon_*K$ we choose a K-injective complex $\mathcal{I}^\bullet$ representing K and we set $R\epsilon_*K = \epsilon_*\mathcal{I}^\bullet$. Since $\epsilon^{-1} : D(\mathcal{C}_{\tau'}) \to D(\mathcal{C}_\tau)$ is computed on an object L by applying the exact functor ϵ^{-1} to any complex of abelian sheaves representing L, we find that $\epsilon^{-1}R\epsilon_*K$ is represented by $\epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. By Part (1) we have $\mathcal{I}^\bullet = \epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. In other words, we have $\epsilon^{-1} \circ R\epsilon_* = \mathrm{id}$ and we conclude as before.

Ad (4). Observe that $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$, see discussion in part (1). Hence ϵ is a flat morphism of ringed sites, see Modules on Sites, Definition 18.31.1. Not only that,

it is moreover clear that $\epsilon^* = \epsilon^{-1}$ on $\mathcal{O}_{\tau'}$-modules (the pullback as a module has the same underlying abelian sheaf as the pullback of the underlying abelian sheaf).

Ad (5). This is clear from (2) and what we said in (4).

Ad (6). This is analogous to (3). We omit the details.

21.28. Formalities on cohomological descent

0D7N In this section we discuss only to what extent a morphism of ringed topoi determines an embedding from the derived category downstairs to the derived category upstairs. Here is a typical result.

0D7Q **Lemma 21.28.1.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ consisting of objects K such that*

$$K \longrightarrow Rf_*Lf^*K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{D})$ and the functor $Lf^* : D' \to D(\mathcal{O}_\mathcal{C})$ is fully faithful.

Proof. See Derived Categories, Definition 13.6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 13.4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the unit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 21.19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 4.24.4. □

0D7R **Lemma 21.28.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ consisting of objects K such that*

$$Lf^*Rf_*K \longrightarrow K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{C})$ and the functor $Rf_* : D' \to D(\mathcal{O}_\mathcal{D})$ is fully faithful.

Proof. See Derived Categories, Definition 13.6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 13.4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the counit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 21.19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 4.24.4. □

0D7S **Lemma 21.28.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Assume*

(1) *f is flat,*
(2) *K is bounded below,*
(3) *$f^*Rf_*H^q(K) \to H^q(K)$ is an isomorphism.*

Then $f^*Rf_*K \to K$ is an isomorphism.

Proof. Observe that $f^*Rf_*K \to K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(f^*Rf_*K) = f^*H^j(Rf_*K) = f^*H^j(Rf_*\tau_{\leq j}K) = H^j(f^*Rf_*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ of Lemma 21.28.2. Hence K is in it too. □

0D7T **Lemma 21.28.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{D})$. Assume*

(1) *f is flat,*
(2) *K is bounded below,*
(3) *$H^q(K) \to Rf_*f^*H^q(K)$ is an isomorphism.*

Then $K \to Rf_*f^*K$ is an isomorphism.

Proof. Observe that $K \to Rf_*f^*K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(Rf_*f^*K) = H^j(Rf_*\tau_{\leq j}f^*K) = H^j(Rf_*f^*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ of Lemma 21.28.1. Hence K is in it too. □

0D7U **Lemma 21.28.5.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

(1) *f is flat,*
(2) *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
(3) *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$.*

Then $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$.

Proof. By assumptions (2) and (3) and Lemmas 21.28.4 and 21.28.1 we see that $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is fully faithful. Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$. Then we can write $\mathcal{F} = f^*\mathcal{F}'$. Then $Rf_*\mathcal{F} = Rf_*f^*\mathcal{F}' = \mathcal{F}'$. In particular, we have $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F} \in \mathrm{Ob}(\mathcal{A}')$. Thus for any $K \in D^+_\mathcal{A}(\mathcal{O})$ we see, using the spectral sequence $E_2^{p,q} = R^pf_*H^q(K)$ converging to $R^{p+q}f_*K$, that Rf_*K is in $D^+_{\mathcal{A}'}(\mathcal{O}')$. Of course, it also follows from Lemmas 21.28.3 and 21.28.2 that $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$ is fully faithful. Since f^* and Rf_* are adjoint we then get the result of the lemma, for example by Categories, Lemma 4.24.4. □

0D7V **Lemma 21.28.6.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

(1) *f is flat,*
(2) *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
(3) *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$,*
(4) *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,*
(5) *$\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 21.25.1.*

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_\mathcal{A}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_\mathcal{A}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

This is analogous to [**LO08a**, Theorem 2.2.3].

Proof. Since f^* is exact, it is clear that f^* defines a functor $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_\mathcal{A}(\mathcal{O})$ as in the statement of the lemma and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. We already know that f^* and Rf_* are quasi-inverse equivalence on the corresponding bounded below categories, see Lemma

21.28.5. By Lemma 21.25.4 with $N = 0$ we see that Rf_* indeed defines a functor $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$ and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. Thus for K in $D_{\mathcal{A}}(\mathcal{O})$ the map $f^*Rf_*K \to K$ is an isomorphism as this is true on trunctions. Similarly, for K' in $D_{\mathcal{A}'}(\mathcal{O}')$ the map $K' \to Rf_*f^*K'$ is an isomorphism as this is true on trunctions. This finishes the proof. □

0D7W **Lemma 21.28.7.** *Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

(1) *f is flat,*
(2) *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
(3) *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$,*
(4) *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,*
(5) *$f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 21.25.5.*

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

This is analogous to [**LO08a**, Theorem 2.2.3].

Proof. The proof of this lemma is exactly the same as the proof of Lemma 21.28.6 except the reference to Lemma 21.25.4 is replaced by a reference to Lemma 21.25.6. □

21.29. Comparing two topologies, II

0F17 Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\mathrm{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\mathrm{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

For more discussion, see Section 21.27.

07A8 **Lemma 21.29.1.** *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset PMod(\mathcal{O})$ be a full subcategory. Assume*

(1) *every object of $\mathcal{A}$ is a sheaf for the τ-topology,*
(2) *$\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$,*
(3) *every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and*
(4) *for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.*

Then $\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\tau'})$ and there is an equivalence of triangulated categories $D_{\mathcal{A}}(\mathcal{O}_\tau) = D_{\mathcal{A}}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Since $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$ we see that ϵ is a flat morphism of ringed sites and that in fact $\epsilon^{-1} = \epsilon^*$ on sheaves of modules. By property (1) we can think of every object of $\mathcal{A}$ as a sheaf of $\mathcal{O}_\tau$-modules and as a sheaf of $\mathcal{O}_{\tau'}$-modules. In other words, we have fully faithful inclusion functors

$$\mathcal{A} \to Mod(\mathcal{O}_\tau) \to Mod(\mathcal{O}_{\tau'})$$

To avoid confusion we will denote $\mathcal{A}' \subset Mod(\mathcal{O}_{\tau'})$ the image of $\mathcal{A}$. Then it is clear that $\epsilon_* : \mathcal{A} \to \mathcal{A}'$ and $\epsilon^* : \mathcal{A}' \to \mathcal{A}$ are quasi-inverse equivalences (see discussion preceding the lemma and use that objects of $\mathcal{A}'$ are sheaves in the τ topology).

Conditions (3) and (4) imply that $R^p\epsilon_*\mathcal{F} = 0$ for $p > 0$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$. This is true because $R^p\epsilon_*$ is the sheaf associated to the presheave $U \mapsto H^p_\tau(U, \mathcal{F})$, see Lemma 21.7.4. Thus any exact complex in $\mathcal{A}$ (which is the same thing as an exact complex in $Mod(\mathcal{O}_\tau)$ whose terms are in $\mathcal{A}$, see Homology, Lemma 12.10.3) remains exact upon applying the functor ϵ_*.

Consider an exact sequence

$$\mathcal{F}'_0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to \mathcal{F}'_4$$

in $Mod(\mathcal{O}_{\tau'})$ with $\mathcal{F}'_0, \mathcal{F}'_1, \mathcal{F}'_3, \mathcal{F}'_4$ in $\mathcal{A}'$. Apply the exact functor ϵ^* to get an exact sequence

$$\epsilon^*\mathcal{F}'_0 \to \epsilon^*\mathcal{F}'_1 \to \epsilon^*\mathcal{F}'_2 \to \epsilon^*\mathcal{F}'_3 \to \epsilon^*\mathcal{F}'_4$$

in $Mod(\mathcal{O}_\tau)$. Since $\mathcal{A}$ is a weak Serre subcategory and since $\epsilon^*\mathcal{F}'_0, \epsilon^*\mathcal{F}'_1, \epsilon^*\mathcal{F}'_3, \epsilon^*\mathcal{F}'_4$ are in $\mathcal{A}$, we conclude that $\epsilon^*\mathcal{F}_2$ is in $\mathcal{A}$ by Homology, Definition 12.10.1. Consider the map of sequences

$$\begin{array}{ccccccccc}
\mathcal{F}'_0 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F}'_3 & \longrightarrow & \mathcal{F}'_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\epsilon_*\epsilon^*\mathcal{F}'_0 & \longrightarrow & \epsilon_*\epsilon^*\mathcal{F}'_1 & \longrightarrow & \epsilon_*\epsilon^*\mathcal{F}'_2 & \longrightarrow & \epsilon_*\epsilon^*\mathcal{F}'_3 & \longrightarrow & \epsilon_*\epsilon^*\mathcal{F}'_4
\end{array}$$

The lower row is exact by the discussion in the preceding paragraph. The vertical arrows with index 0, 1, 3, 4 are isomorphisms by the discussion in the first paragraph. By the 5 lemma (Homology, Lemma 12.5.20) we find that $\mathcal{F}'_2 \cong \epsilon_*\epsilon^*\mathcal{F}'_2$ and hence $\mathcal{F}'_2$ is in $\mathcal{A}'$. In this way we see that $\mathcal{A}'$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\tau'})$, see Homology, Definition 12.10.1.

At this point it makes sense to talk about the derived categories $D_{\mathcal{A}}(\mathcal{O}_\tau)$ and $D_{\mathcal{A}'}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 13.17. To finish the proof we show that conditions (1) – (5) of Lemma 21.28.7 apply. We have already seen (1), (2), (3) above. Note that since every object has a τ'-covering by objects of $\mathcal{B}$, a fortiori every object has a τ-covering by objects of $\mathcal{B}$. Hence condition (4) of Lemma 21.28.7 is satisfied. Similarly, condition (5) is satisfied as well. □

0F18 **Lemma 21.29.2.** *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let A be a set and for $\alpha \in A$ let*

$$\begin{array}{ccc}
E_\alpha & \longrightarrow & Y_\alpha \\
\downarrow & & \downarrow \\
Z_\alpha & \longrightarrow & X_\alpha
\end{array}$$

be a commutative diagram in the category $\mathcal{C}$. Assume that

1. *a τ'-sheaf $\mathcal{F}'$ is a τ-sheaf if $\mathcal{F}'(X_\alpha) = \mathcal{F}'(Z_\alpha) \times_{\mathcal{F}'(E_\alpha)} \mathcal{F}'(Y_\alpha)$ for all α,*
2. *for K' in $D(\mathcal{O}_{\tau'})$ in the essential image of $R\epsilon_*$ the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ of Lemma 21.26.1 are isomorphisms for all α.*

Then $K' \in D^+(\mathcal{O}_{\tau'})$ is in the essential image of $R\epsilon_*$ if and only if the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ are isomorphisms for all α.

Proof. The "only if" direction is implied by assumption (2). On the other hand, if K' has a unique nonzero cohomology sheaf, then the "if" direction follows from assumption (1). In general we will use an induction argument to prove the "if" direction. Let us say an object K' of $D^+(\mathcal{O}_{\tau'})$ satisfies (P) if the maps $c^{K'}_{X_\alpha,Z_\alpha,Y_\alpha,E_\alpha}$ are isomorphisms for all $\alpha \in A$.

Namely, let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P). Choose a distinguished triangle

$$K' \to R\epsilon_*\epsilon^{-1}K' \to M' \to K'[1]$$

in $D^+(\mathcal{O}_{\tau'})$ where the first arrow is the adjuntion map. By (2) and Lemma 21.26.2 we see that M' has (P). On the other hand, applying ϵ^{-1} and using that $\epsilon^{-1}R\epsilon_* = \mathrm{id}$ by Section 21.27 we find that $\epsilon^{-1}M' = 0$. In the next paragraph we will show $M' = 0$ which finishes the proof.

Let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P) with $\epsilon^{-1}K' = 0$. We will show $K' = 0$. Namely, given $n \in \mathbf{Z}$ such that $H^i(K') = 0$ for $i < n$ we will show that $H^n(K') = 0$. For $\alpha \in A$ we have a distinguished triangle

$$R\Gamma_{\tau'}(X_\alpha, K') \to R\Gamma_{\tau'}(Z_\alpha, K') \oplus R\Gamma_{\tau'}(Y_\alpha, K') \to R\Gamma_{\tau'}(E_\alpha, K') \to R\Gamma_{\tau'}(X_\alpha, K')[1]$$

by Lemma 21.26.1. Taking cohomology in degree n and using the assumed vanishing of cohomology sheaves of K' we obtain an exact sequence

$$0 \to H^n_{\tau'}(X_\alpha, K') \to H^n_{\tau'}(Z_\alpha, K') \oplus H^n_{\tau'}(Y_\alpha, K') \to H^n_{\tau'}(E_\alpha, K')$$

which is the same as the exact sequence

$$0 \to \Gamma(X_\alpha, H^n(K')) \to \Gamma(Z_\alpha, H^n(K')) \oplus \Gamma(Y_\alpha, H^n(K')) \to \Gamma(E_\alpha, H^n(K'))$$

We conclude that $H^n(K')$ is a a τ-sheaf by assumption (1). However, since the τ-sheafification $\epsilon^{-1}H^n(K') = H^n(\epsilon^{-1}K')$ is 0 as $\epsilon^{-1}K' = 0$ we conclude that $H^n(K') = 0$ as desired. □

0F19 **Lemma 21.29.3.** *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let*

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a commutative diagram in the category $\mathcal{C}$ such that

(1) $h^\#_X = h^\#_Y \amalg_{h^\#_E} h^\#_Z$, *and*

(2) $h^\#_E \to h^\#_Y$ *is injective*

where $^\#$ denotes τ-sheafification. Then for $K' \in D(\mathcal{O}_{\tau'})$ in the essential image of $R\epsilon_*$ the map $c^{K'}_{X,Z,Y,E}$ of Lemma 21.26.1 (using the τ'-topology) is an isomorphism.

Proof. This helper lemma is an almost immediate consequence of Lemma 21.26.3 and we strongly urge the reader skip the proof. Say $K' = R\epsilon_*K$. Choose a K-injective complex of $\mathcal{O}_\tau$-modules $\mathcal{J}^\bullet$ representing K. Then $\epsilon_*\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}_{\tau'}$-modules representing K', see Lemma 21.20.10. Next,

$$0 \to \mathcal{J}^\bullet(X) \xrightarrow{\alpha} \mathcal{J}^\bullet(Z) \oplus \mathcal{J}^\bullet(Y) \xrightarrow{\beta} \mathcal{J}^\bullet(E) \to 0$$

is a short exact sequence of complexes of abelian groups, see Lemma 21.26.3 and its proof. Since this is the same as the sequence of complexes of abelian groups which is used to define $c^{K'}_{X,Z,Y,E}$, we conclude. □

21.30. Comparing cohomology

0EZ1 We develop some general theory which will help us compare cohomology in different topologies. Given $\mathcal{C}$, τ, and τ' as in Section 21.27 and a morphism $f : X \to Y$ in $\mathcal{C}$ we obtain a commutative diagram of morphisms of topoi

0EZ2
$$\begin{array}{ccc} Sh(\mathcal{C}_\tau/X) & \xrightarrow{f_\tau} & Sh(\mathcal{C}_\tau/Y) \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh(\mathcal{C}_{\tau'}/X) & \xrightarrow{f_{\tau'}} & Sh(\mathcal{C}_{\tau'}/X) \end{array} \tag{21.30.0.1}$$

Here the morphism ϵ_X, resp. ϵ_Y is the comparison morphism of Section 21.27 for the category $\mathcal{C}/X$ endowed with the two topologies τ and τ'. The morphisms f_τ and $f_{\tau'}$ are "relocalization" morphisms (Sites, Lemma 7.25.8). The commutativity of the diagram is a special case of Sites, Lemma 7.28.1 (applied with $\mathcal{C} = \mathcal{C}_\tau/Y$, $\mathcal{D} = \mathcal{C}_{\tau'}/Y$, $u = \mathrm{id}$, $U = X$, and $V = X$). We also get $\epsilon_{X,*} \circ f_\tau^{-1} = f_{\tau'}^{-1} \circ \epsilon_{Y,*}$ either from the lemma or because it is obvious.

0EZ3 **Situation 21.30.1.** With $\mathcal{C}$, τ, and τ' as in Section 21.27. Assume we are given a subset $\mathcal{P} \subset \mathrm{Arrows}(\mathcal{C})$ and for every object X of $\mathcal{C}$ we are given a weak Serre subcategory $\mathcal{A}'_X \subset Ab(\mathcal{C}_{\tau'}/X)$. We make the following assumption:

0EZ4 (1) given $f : X \to Y$ in $\mathcal{P}$ and $Y' \to Y$ general, then $X \times_Y Y'$ exists and $X \times_Y Y' \to Y'$ is in $\mathcal{P}$,

0EZ5 (2) $f_{\tau'}^{-1}$ sends $\mathcal{A}'_Y$ into $\mathcal{A}'_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$,

0EZ6 (3) given X in $\mathcal{C}$ and $\mathcal{F}'$ in $\mathcal{A}'_X$, then $\mathcal{F}'$ satisfies the sheaf condition for τ-coverings, i.e., $\mathcal{F}' = \epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$,

0EZ7 (4) if $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}'_X)$, then $R^i f_{\tau',*}\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}'_Y)$ for $i \geq 0$.

0EZ8 (5) if $\{U_i \to U\}_{i\in I}$ is a τ-covering, then there exist
   (a) a τ'-covering $\{V_j \to U\}_{j\in J}$,
   (b) a τ-covering $\{f_j : W_j \to V_j\}$ consisting of a single $f_j \in \mathcal{P}$, and
   (c) a τ'-covering $\{W_{jk} \to W_j\}_{k\in K_j}$
   such that $\{W_{jk} \to U\}_{j\in J, k\in K_j}$ is a refinement of $\{U_i \to U\}_{i\in I}$.

0EZ9 **Lemma 21.30.2.** *In Situation 21.30.1 for X in $\mathcal{C}$ denote $\mathcal{A}_X$ the objects of $Ab(\mathcal{C}_\tau/X)$ of the form $\epsilon_X^{-1}\mathcal{F}'$ with $\mathcal{F}'$ in $\mathcal{A}'_X$. Then*

(1) *for $\mathcal{F}$ in $Ab(\mathcal{C}_\tau/X)$ we have $\mathcal{F} \in \mathcal{A}_X \Leftrightarrow \epsilon_{X,*}\mathcal{F} \in \mathcal{A}'_X$, and*
(2) *f_τ^{-1} sends $\mathcal{A}_Y$ into $\mathcal{A}_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$.*

Proof. Part (1) follows from (3) and part (2) follows from (2) and the commutativity of (21.30.0.1) which gives $\epsilon_X^{-1} \circ f_{\tau'}^{-1} = f_\tau^{-1} \circ \epsilon_Y^{-1}$. □

Our next goal is to prove Lemmas 21.30.10 and 21.30.9. We will do this by an induction argument using the following induction hypothesis.

(V_n) For X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^i\epsilon_{X,*}\mathcal{F} = 0$ for $1 \leq i \leq n$.

0EZA **Lemma 21.30.3.** *In Situation 21.30.1 assume (V_n) holds. For $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^i f_{\tau,*}\mathcal{F}$ for $i \leq n$.*

Proof. We will use the commutative diagram (21.30.0.1) without further mention. In particular have

$$Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$$

Assumption (V_n) tells us that $\epsilon_{X,*}\mathcal{F} \to R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. Hence $Rf_{\tau',*}\epsilon_{X,*}\mathcal{F} \to Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. We conclude that

$$R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to H^i(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F})$$

is an isomorphism for $i \leq n$. We will prove the lemma by looking at the second page of the spectral sequence of Lemma 21.14.7 for $R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$. Here is a picture:

$$\begin{array}{cccc} \ldots & \ldots & \ldots & \ldots \\ \epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & \ldots \\ \epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & \ldots \\ \epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & \ldots \end{array}$$

Let (C_m) be the hypothesis: $R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^i f_{\tau,*}\mathcal{F}$ for $i \leq m$. Observe that (C_0) holds. We will show that $(C_{m-1}) \Rightarrow (C_m)$ for $m < n$. Namely, if (C_{m-1}) holds, then for $n \geq p > 0$ and $q \leq m-1$ we have

$$\begin{aligned} R^p\epsilon_{Y,*}R^q f_{\tau,*}\mathcal{F} &= R^p\epsilon_{Y,*}\epsilon_Y^{-1}\epsilon_{Y,*}R^q f_{\tau,*}\mathcal{F} \\ &= R^p\epsilon_{Y,*}\epsilon_Y^{-1}R^q f_{\tau',*}\epsilon_{X,*}\mathcal{F} = 0 \end{aligned}$$

First equality as $\epsilon_Y^{-1}\epsilon_{Y,*} = \mathrm{id}$, the second by (C_{m-1}), and the final by by (V_n) because $\epsilon_Y^{-1}R^q f_{\tau',*}\epsilon_{X,*}\mathcal{F}$ is in $\mathcal{A}_Y$ by (4). Looking at the spectral sequence we see that $E_2^{0,m} = \epsilon_{Y,*}R^m f_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p+q = m$. Recall that $\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Hence there are no nonzero differentials $\mathrm{d}_r^{p,q}$, $r \geq 2$ either emanating or entering this spot. We conclude that $H^m(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}) = \epsilon_{Y,*}R^m f_{\tau,*}\mathcal{F}$ which implies (C_m) by the discussion above.

Finally, assume (C_{n-1}). The same analysis shows that $E_2^{0,n} = \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p+q = n$. We do still have no nonzero differentials entering this spot, but there can be a nonzero differential emanating it. Namely, the map $d_{n+1}^{0,n} : \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$. We conclude that there is an exact sequence

$$0 \to R^n f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$$

By (4) and (3) the sheaf $R^n f_{\tau',*}\epsilon_{X,*}\mathcal{F}$ satisfies the sheaf property for τ-coverings as does $\epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F}$ (use the description of ϵ_* in Section 21.27). However, the τ-sheafification of the τ'-sheaf $R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$ is zero (by locality of cohomology; use Lemmas 21.7.3 and 21.7.4). Thus $R^n f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F}$ has to be an isomorphism and the proof is complete. □

If E', resp. E is an object of $D(\mathcal{C}_{\tau'}/X)$, resp. $D(\mathcal{C}_\tau/X)$ then we will write $H^n_{\tau'}(U, E')$, resp. $H^n_\tau(U, E)$ for the cohomology of E', resp. E over an object U of $\mathcal{C}/X$.

0EZB **Lemma 21.30.4.** *In Situation 21.30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $L \in D(\mathcal{C}_{\tau'}/X)$ with $H^i(L) = 0$ for $i < 0$ and $H^i(L)$ in $\mathcal{A}'_X$ for $0 \leq i \leq n$ we have $H^n_{\tau'}(X, L) = H^n_\tau(X, \epsilon_X^{-1}L)$.*

Proof. By Lemma 21.20.5 we have $H^n_\tau(X, \epsilon_X^{-1}L) = H^n_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$. There is a spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}\epsilon_X^{-1}H^q(L)$$

converging to $H^{p+q}(R\epsilon_{X,*}\epsilon_X^{-1}L)$. By (V_n) we have the vanishing of $E_2^{p,q}$ for $0 < p \leq n$ and $0 \leq q \leq n$. Thus $E_2^{0,q} = \epsilon_{X,*}\epsilon_X^{-1}H^q(L) = H^q(L)$ are the only nonzero

terms $E_2^{p,q}$ with $p+q \leq n$. It follows that the map

$$L \longrightarrow R\epsilon_{X,*}\epsilon_X^{-1}L$$

is an isomorphism in degrees $\leq n$ (small detail omitted). Hence we find that $H^i_{\tau'}(X,L) = H^i_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$ for $i \leq n$. Thus the lemma is proved. □

0EZC **Lemma 21.30.5.** *In Situation 21.30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ the map $H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X,\mathcal{F})$ is injective with image those classes which become trivial on a τ'-covering of X.*

Proof. Recall that $\epsilon_X^{-1}\epsilon_{X,*}\mathcal{F} = \mathcal{F}$ hence the map is given by pulling back cohomology classes by ϵ_X. The Leray spectral sequence (Lemma 21.14.5)

$$E_2^{p,q} = H^p_{\tau'}(X, R^q\epsilon_{X,*}\mathcal{F}) \Rightarrow H^{p+q}_\tau(X,\mathcal{F})$$

combined with the assumed vanishing gives an exact sequence

$$0 \to H^{n+1}_{\tau'}(X,\epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X,\mathcal{F}) \to H^0_{\tau'}(X, R^{n+1}\epsilon_{X,*}\mathcal{F})$$

This is a restatement of the lemma. □

0EZD **Lemma 21.30.6.** *In Situation 21.30.1 let $f : X \to Y$ be in $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering. Let $\mathcal{F}'$ be in $\mathcal{A}'_Y$. If $n \geq 0$ and*

$$\theta \in \textit{Equalizer}\Big(H^{n+1}_{\tau'}(X,\mathcal{F}') \rightrightarrows H^{n+1}_{\tau'}(X \times_Y X, \mathcal{F}') \Big)$$

then there exists a τ'-covering $\{Y_i \to Y\}$ such that θ restricts to zero in $H^{n+1}_{\tau'}(Y_i \times_Y X, \mathcal{F}')$.

Proof. Observe that $X \times_Y X$ exists by (1). For Z in $\mathcal{C}/Y$ denote $\mathcal{F}'|_Z$ the restriction of $\mathcal{F}'$ to $\mathcal{C}_{\tau'}/Z$. Recall that $H^{n+1}_{\tau'}(X,\mathcal{F}') = H^{n+1}(\mathcal{C}_{\tau'}/X, \mathcal{F}'|_X)$, see Lemma 21.7.1. The lemma asserts that the image $\overline{\theta} \in H^0(Y, R^{n+1}f_{\tau',*}\mathcal{F}'|_X)$ of θ is zero. Consider the cartesian diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

By trivial base change (Lemma 21.21.1) we have

$$f^{-1}_{\tau'}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X) = R^{n+1}\mathrm{pr}_{1,\tau',*}(\mathcal{F}'|_{X\times_Y X})$$

If $\mathrm{pr}_1^{-1}\theta = \mathrm{pr}_2^{-1}\theta$, then the section $f^{-1}_{\tau'}\overline{\theta}$ of $f^{-1}_{\tau'}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is zero, because it is clear that $\mathrm{pr}_1^{-1}\theta$ maps to the zero element in $H^0(X, R^{n+1}\mathrm{pr}_{1,\tau',*}(\mathcal{F}'|_{X\times_Y X}))$. By (2) we have $\mathcal{F}'|_X$ in $\mathcal{A}'_X$. Thus $\mathcal{G}' = R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is an object of $\mathcal{A}'_Y$ by (4). Thus $\mathcal{G}'$ satisfies the sheaf property for τ-coverings by (3). Since $\{X \to Y\}$ is a τ-covering we conclude that restriction $\mathcal{G}'(Y) \to \mathcal{G}'(X)$ is injective. It follows that $\overline{\theta}$ is zero. □

0EZE **Lemma 21.30.7.** *In Situation 21.30.1 we have $(V_n) \Rightarrow (V_{n+1})$.*

Proof. Let X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$. Let $\xi \in H^{n+1}_\tau(U,\mathcal{F})$ for some U/X. We have to show that ξ restricts to zero on the members of a τ'-covering of U. See Lemma 21.7.4. It follows from this that we may replace U by the members of a τ'-covering of U.

By locality of cohomology (Lemma 21.7.3) we can choose a τ-covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. Choose $\{V_j \to V\}$, $\{f_j : W_j \to V_j\}$, and $\{W_{jk} \to W_j\}$ as in (5). After replacing both U by V_j and $\mathcal{F}$ by its restriction to $\mathcal{C}_\tau/V_j$, which is allowed by (1), we reduce to the case discussed in the next paragraph.

Here $f : X \to Y$ is an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ is such that there exists a τ'-covering $\{X_i \to X\}_{i \in I}$ such that ξ restricts to zero on X_i for all $i \in I$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

By Lemma 21.30.5 there exists a unique τ'-cohomology class $\theta \in H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F})$ whose image is $\xi|_X$. Since $\xi|_X$ pulls back to the same class on $X \times_Y X$ via the two projections, we find that the same is true for θ (by uniqueness). By Lemma 21.30.6 we see that after replacing Y by the members of a τ'-covering, we may assume that $\theta = 0$. Consequently, we may assume that $\xi|_X$ is zero.

Let $f : X \to Y$ be an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ maps to zero in $H^{n+1}_\tau(X, \mathcal{F})$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

The assumptions tell us ξ maps to zero under the map

$$\mathcal{F} \longrightarrow Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Use Lemma 21.20.5. A simple argument using the distinguished triangle of truncations (Derived Categories, Remark 13.12.4) shows that ξ maps to zero under the map

$$\mathcal{F} \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

We will compare this with the map $\epsilon_{Y,*}\mathcal{F} \to K$ where

$$K = \tau_{\leq n}Rf_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F} = \tau_{\leq n}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F}$$

The equality $\epsilon_{X,*}f_\tau^{-1} = f_{\tau'}^{-1}\epsilon_{Y,*}$ is a property of (21.30.0.1). Consider the map

$$Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \longrightarrow Rf_{\tau',*}R\epsilon_{X,*}f_\tau^{-1}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

used in the proof of Lemma 21.30.3 which induces by adjunction a map

$$\epsilon_Y^{-1}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \to Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Taking trunctions we find a map

$$\epsilon_Y^{-1}K \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

which is an isomorphism by Lemma 21.30.3; the lemma applies because $f_\tau^{-1}\mathcal{F}$ is in $\mathcal{A}_X$ by Lemma 21.30.2. Choose a distinguished triangle

$$\epsilon_{Y,*}\mathcal{F} \to K \to L \to \epsilon_{Y,*}\mathcal{F}[1]$$

The map $\mathcal{F} \to f_{\tau,*}f_\tau^{-1}\mathcal{F}$ is injective as $\{X \to Y\}$ is a τ-covering. Thus $\epsilon_{Y,*}\mathcal{F} \to \epsilon_{Y,*}f_{\tau,*}f_\tau^{-1}\mathcal{F} = f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is injective too. Hence L only has nonzero cohomology sheaves in degrees $0, \ldots, n$. As $f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4) we conclude that

$$H^0(L) = \mathrm{Coker}(\epsilon_{Y,*}\mathcal{F} \to f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F})$$

is in the weak Serre subcategory $\mathcal{A}'_Y$. For $1 \leq i \leq n$ we see that $H^i(L) = R^i f_{\tau',*} f_{\tau'}^{-1} \epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4). Pulling back the distinguished triangle above by ϵ_Y we get the distinguished triangle

$$\mathcal{F} \to \tau_{\leq n} Rf_{\tau,*} f_\tau^{-1}\mathcal{F} \to \epsilon_Y^{-1} L \to \mathcal{F}[1]$$

Since ξ maps to zero in the middle term we find that ξ is the image of an element $\xi' \in H^n_\tau(Y, \epsilon_Y^{-1}L)$. By Lemma 21.30.4 we have

$$H^n_{\tau'}(Y, L) = H^n_\tau(Y, \epsilon_Y^{-1}L),$$

Thus we may lift ξ' to an element of $H^n_{\tau'}(Y, L)$ and take the boundary into $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$ to see that ξ is in the image of the canonical map $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F}) \to H^{n+1}_\tau(Y, \mathcal{F})$. By locality of cohomology for $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$, see Lemma 21.7.3, we conclude. □

0EZF **Lemma 21.30.8.** *In Situation 21.30.1 we have that (V_n) is true for all n. Moreover:*

(1) *For X in $\mathcal{C}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ the map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.*
(2) *For $f : X \to Y$ in $\mathcal{P}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/Y)$ and $\epsilon_Y^{-1}(Rf_{\tau',*}K') = Rf_{\tau,*}(\epsilon_X^{-1}K')$.*

Proof. Observe that (V_0) holds as it is the empty condition. Then we get (V_n) for all n by Lemma 21.30.7.

Proof of (1). The object $K = \epsilon_X^{-1}K'$ has cohomology sheaves $H^i(K) = \epsilon_X^{-1}H^i(K')$ in $\mathcal{A}_X$. Hence the spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}H^q(K) \Rightarrow H^{p+q}(R\epsilon_{X,*}K)$$

degenerates by (V_n) for all n and we find

$$H^n(R\epsilon_{X,*}K) = \epsilon_{X,*}H^n(K) = \epsilon_{X,*}\epsilon_X^{-1}H^i(K') = H^i(K').$$

again because $H^i(K')$ is in $\mathcal{A}'_X$. Thus the canonical map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.

Proof of (2). Using the spectral sequence

$$E_2^{p,q} = R^p f_{\tau',*}H^q(K') \Rightarrow R^{p+q}f_{\tau',*}K'$$

the fact that $R^p f_{\tau',*}H^q(K')$ is in $\mathcal{A}'_Y$ by (4), the fact that $\mathcal{A}'_Y$ is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/Y)$, and Homology, Lemma 12.24.11 we conclude that $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$. To finish the proof we have to show the base change map

$$\epsilon_Y^{-1}(Rf_{\tau',*}K') \longrightarrow Rf_{\tau,*}(\epsilon_X^{-1}K')$$

is an isomorphism. Comparing the spectral sequence above to the spectral sequence

$$E_2^{p,q} = R^p f_{\tau,*}H^q(\epsilon_X^{-1}K') \Rightarrow R^{p+q}f_{\tau,*}\epsilon_X^{-1}K'$$

we reduce this to the case where K' has a single nonzero cohomology sheaf $\mathcal{F}'$ in $\mathcal{A}'_X$; details omitted. Then Lemma 21.30.3 gives $\epsilon_Y^{-1}R^i f_{\tau',*}\mathcal{F}' = R^i f_{\tau,*}\epsilon_X^{-1}\mathcal{F}'$ for all i and the proof is complete. □

0EZG **Lemma 21.30.9.** *In Situation 21.30.1. For any X in $\mathcal{C}$ the category $\mathcal{A}_X \subset Ab(\mathcal{C}_\tau/X)$ is a weak Serre subcategory and the functor*

$$R\epsilon_{X,*} : D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X) \longrightarrow D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$$

is an equivalence with quasi-inverse given by ϵ_X^{-1}.

Proof. We need to check the conditions listed in Homology, Lemma 12.10.3 for $\mathcal{A}_X$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map in $\mathcal{A}_X$, then $\epsilon_{X,*}\varphi : \epsilon_{X,*}\mathcal{F} \to \epsilon_{X,*}\mathcal{G}$ is a map in $\mathcal{A}'_X$. Hence $\mathrm{Ker}(\epsilon_{X,*}\varphi)$ and $\mathrm{Coker}(\epsilon_{X,*}\varphi)$ are objects of $\mathcal{A}'_X$ as this is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/X)$. Applying ϵ_X^{-1} we obtain an exact sequence

$$0 \to \epsilon_X^{-1}\mathrm{Ker}(\epsilon_{X,*}\varphi) \to \mathcal{F} \to \mathcal{G} \to \epsilon_X^{-1}\mathrm{Coker}(\epsilon_{X,*}\varphi) \to 0$$

and we see that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are in $\mathcal{A}_X$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence in $Ab(\mathcal{C}_\tau/X)$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ in $\mathcal{A}_X$. Then applying $\epsilon_{X,*}$ we obtain an exact sequence

$$0 \to \epsilon_{X,*}\mathcal{F}_1 \to \epsilon_{X,*}\mathcal{F}_2 \to \epsilon_{X,*}\mathcal{F}_3 \to R^1\epsilon_{X,*}\mathcal{F}_1 = 0$$

Vanishing by Lemma 21.30.8. Hence $\epsilon_{X,*}\mathcal{F}_2$ is in $\mathcal{A}'_X$ as this is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/X)$. Pulling back by ϵ_X we conclude that $\mathcal{F}_2$ is in $\mathcal{A}_X$.

Thus $\mathcal{A}_X$ is a weak Serre subcategory of $Ab(\mathcal{C}_\tau/X)$ and it makes sense to consider the category $D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X)$. Observe that $\epsilon_X^{-1} : \mathcal{A}'_X \to \mathcal{A}_X$ is an equivalence and that $\mathcal{F}' \to R\epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$ is an isomorphism for $\mathcal{F}'$ in $\mathcal{A}'_X$ since we have (V_n) for all n by Lemma 21.30.8. Thus we conclude by Lemma 21.28.5. □

0EZH **Lemma 21.30.10.** *In Situation 21.30.1. Let X be in $\mathcal{C}$.*

(1) *for $\mathcal{F}'$ in $\mathcal{A}'_X$ we have $H^n_{\tau'}(X, \mathcal{F}') = H^n_\tau(X, \epsilon_X^{-1}\mathcal{F}')$,*
(2) *for $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $H^n_{\tau'}(X, K') = H^n_\tau(X, \epsilon_X^{-1}K')$.*

Proof. This follows from Lemma 21.30.8 by Remark 21.14.4. □

21.31. Cohomology on Hausdorff and locally quasi-compact spaces

09WY We continue our convention to say "Hausdorff and locally quasi-compact" instead of saying "locally compact" as is often done in the literature. Let LC denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

09WZ **Lemma 21.31.1.** *The category LC has fibre products and a final object and hence has arbitrary finite limits. Given morphisms $X \to Z$ and $Y \to Z$ in LC with X and Y quasi-compact, then $X \times_Z Y$ is quasi-compact.*

Proof. The final object is the singleton space. Given morphisms $X \to Z$ and $Y \to Z$ of LC the fibre product $X \times_Z Y$ is a subspace of $X \times Y$. Hence $X \times_Z Y$ is Hausdorff as $X \times Y$ is Hausdorff by Topology, Section 5.3.

If X and Y are quasi-compact, then $X \times Y$ is quasi-compact by Topology, Theorem 5.14.4. Since $X \times_Z Y$ is a closed subset of $X \times Y$ (Topology, Lemma 5.3.4) we find that $X \times_Z Y$ is quasi-compact by Topology, Lemma 5.12.3.

Finally, returning to the general case, if $x \in X$ and $y \in Y$ we can pick quasi-compact neighbourhoods $x \in E \subset X$ and $y \in F \subset Y$ and we find that $E \times_Z F$ is

a quasi-compact neighbourhood of (x, y) by the result above. Thus $X \times_Z Y$ is an object of LC by Topology, Lemma 5.13.2. □

We can endow LC with a stronger topology than the usual one.

09X0 **Definition 21.31.2.** Let $\{f_i : X_i \to X\}$ be a family of morphisms with fixed target in the category LC. We say this family is a *qc covering*[6] if for every $x \in X$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x.

Observe that an open covering $X = \bigcup U_i$ of an object of LC gives a qc covering $\{U_i \to X\}$ because X is locally quasi-compact. We start with the obligatory lemma.

09X1 **Lemma 21.31.3.** *Let X be a Hausdorff and locally quasi-compact space, in other words, an object of LC.*

(1) *If $X' \to X$ is an isomorphism in LC then $\{X' \to X\}$ is a qc covering.*
(2) *If $\{f_i : X_i \to X\}_{i \in I}$ is a qc covering and for each i we have a qc covering $\{g_{ij} : X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a qc covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a qc covering and $X' \to X$ is a morphism of LC then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a qc covering.*

Proof. Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let $x \in X$. Choose $i_1, \ldots, i_n \in I$ and $E_a \subset X_{i_a}$ quasi-compact such that $\bigcup f_{i_a}(E_a)$ is a neighbourhood of x. For every $e \in E_a$ we can find a finite subset $J_e \subset J_{i_a}$ and quasi-compact $F_{e,j} \subset X_{ij}$, $j \in J_e$ such that $\bigcup g_{ij}(F_{e,j})$ is a neighbourhood of e. Since E_a is quasi-compact we find a finite collection $e_1, \ldots, e_{m_a}$ such that

$$E_a \subset \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} g_{ij}(F_{e_k,j})$$

Then we find that

$$\bigcup\nolimits_{a=1,\ldots,n} \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k,j}))$$

is a neighbourhood of x.

Proof of (3). Let $x' \in X'$ be a point. Let $x \in X$ be its image. Choose $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. Choose a quasi-compact neighbourhood $F \subset X'$ of x' which maps into the quasi-compact neighbourhood $\bigcup f_{i_j}(E_j)$ of x. Then $F \times_X E_j \subset X' \times_X X_{i_j}$ is a quasi-compact subset and F is the image of the map $\coprod F \times_X E_j \to F$. Hence the base change is a qc covering and the proof is finished. □

Since all objects of LC are Hausdorff any morphism $f : X \to Y$ of LC is a separated continuous map of topological spaces. Hence f is a proper map of topological spaces if and only if f is universally closed. See discussion in Topology, Section 5.17.

09X5 **Lemma 21.31.4.** *Let $f : X \to Y$ be a morphism of LC. If f is proper and surjective, then $\{f : X \to Y\}$ is a qc covering.*

Proof. Let $y \in Y$ be a point. For each $x \in X_y$ choose a quasi-compact neighbourhood $E_x \subset X$. Choose $x \in U_x \subset E_x$ open. Since f is proper the fibre X_y is quasi-compact and we find $x_1, \ldots, x_n \in X_y$ such that $X_y \subset U_{x_1} \cup \ldots \cup U_{x_n}$. We

[6] This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

claim that $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ is a neighbourhood of y. Namely, as f is closed (Topology, Theorem 5.17.5) we see that $Z = f(X \setminus U_{x_1} \cup \ldots \cup U_{x_n})$ is a closed subset of Y not containing y. As f is surjective we see that $Y \setminus Z$ is contained in $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ as desired. □

Besides some set theoretic issues Lemma 21.31.3 shows that LC with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues) LC_{qc}.

09X2 **Remark 21.31.5** (Set theoretic issues)**.** The category LC is a "big" category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the *size* of a topological space X to be the cardinality of the set of points of X. Choose a function $Bound$ on cardinals, for example as in Sets, Equation (3.9.1.1). Finally, let S_0 be an initial set of objects of LC, for example $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$. Exactly as in Sets, Lemma 3.9.2 we can choose a limit ordinal α such that $LC_\alpha = LC \cap V_\alpha$ contains S_0 and is preserved under all countable limits and colimits which exist in LC. Moreover, if $X \in LC_\alpha$ and if $Y \in LC$ and $\text{size}(Y) \leq Bound(\text{size}(X))$, then Y is isomorphic to an object of LC_α. Next, we apply Sets, Lemma 3.11.1 to choose set Cov of qc covering on LC_α such that every qc covering in LC_α is combinatorially equivalent to a covering this set. In this way we obtain a site (LC_α, Cov) which we will denote LC_{qc}.

There is a second topology on the site LC_{qc} of Remark 21.31.5. Namely, given an object X we can consider all coverings $\{X_i \to X\}$ of LC_{qc} such that $X_i \to X$ is an open immersion. We denote this site LC_{Zar}. The identity functor $LC_{Zar} \to LC_{qc}$ is continuous and defines a morphism of sites

$$\epsilon : LC_{qc} \longrightarrow LC_{Zar}$$

See Section 21.27. For a Hausdorff and locally quasi-compact topological space X, more precisely for $X \in \text{Ob}(LC_{qc})$, we denote the induced morphism

$$\epsilon_X : LC_{qc}/X \longrightarrow LC_{Zar}/X$$

(see Sites, Lemma 7.28.1). Let X_{Zar} be the site whose objects are opens of X, see Sites, Example 7.6.4. There is a morphism of sites

$$\pi_X : LC_{Zar}/X \longrightarrow X_{Zar}$$

given by the continuous functor $X_{Zar} \to LC_{Zar}/X$, $U \mapsto U$. Namely, X_{Zar} has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies. We often think of π as a morphism of topoi

$$\pi_X : Sh(LC_{Zar}/X) \longrightarrow Sh(X)$$

using the equality $Sh(X_{Zar}) = Sh(X)$.

09X3 **Lemma 21.31.6.** *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be a sheaf on X. The rule*

$$LC_{qc}/X \longrightarrow Sets, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^{-1}\mathcal{F})$$

is a sheaf and a fortiori also a sheaf on LC_{Zar}/X. This sheaf is equal to $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}/X and $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}/X.

Proof. Denote $\mathcal{G}$ the presheaf given by the formula in the lemma. Of course the pullback f^{-1} in the formula denotes usual pullback of sheaves on topological spaces. It is immediate from the definitions that $\mathcal{G}$ is a sheaf for the Zar topology.

Let $Y \to X$ be a morphism in LC_{qc}. Let $\mathcal{V} = \{g_i : Y_i \to Y\}_{i \in I}$ be a qc covering. To prove $\mathcal{G}$ is a sheaf for the qc topology it suffices to show that $\mathcal{G}(Y) \to H^0(\mathcal{V}, \mathcal{G})$ is an isomorphism, see Sites, Section 7.10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use Sheaves, Lemmas 6.11.1 and 6.21.4). Thus $\mathcal{G}$ is a separated presheaf on LC_{qc} hence it suffices to show that any element $(s_i) \in H^0(\mathcal{V}, \mathcal{G})$ maps to an element in the image of $\mathcal{G}(Y)$ after replacing $\mathcal{V}$ by a refinement (Sites, Theorem 7.10.10).

Identifying sheaves on $Y_{i,Zar}$ and sheaves on Y_i we find that $\mathcal{G}|_{Y_{i,Zar}}$ is the pullback of $f^{-1}\mathcal{F}$ under the continuous map $g_i : Y_i \to Y$. Thus we can choose an open covering $Y_i = \bigcup V_{ij}$ such that for each j there is an open $W_{ij} \subset Y$ and a section $t_{ij} \in \mathcal{G}(W_{ij})$ such that V_{ij} maps into W_{ij} and such that $s|_{V_{ij}}$ is the pullback of t_{ij}. In other words, after refining the covering $\{Y_i \to Y\}$ we may assume there are opens $W_i \subset Y$ such that $Y_i \to Y$ factors through W_i and sections t_i of $\mathcal{G}$ over W_i which restrict to the given sections s_i. Moreover, if $y \in Y$ is in the image of both $Y_i \to Y$ and $Y_j \to Y$, then the images $t_{i,y}$ and $t_{j,y}$ in the stalk $f^{-1}\mathcal{F}_y$ agree (because s_i and s_j agree over $Y_i \times_Y Y_j$). Thus for $y \in Y$ there is a well defined element t_y of $f^{-1}\mathcal{F}_y$ agreeing with $t_{i,y}$ whenever y is in the image of $Y_i \to Y$. We will show that the element (t_y) comes from a global section of $f^{-1}\mathcal{F}$ over Y which will finish the proof of the lemma.

It suffices to show that this is true locally on Y, see Sheaves, Section 6.17. Let $y_0 \in Y$. Pick $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset Y_{i_j}$ such that $\bigcup g_{i_j}(E_j)$ is a neighbourhood of y_0. Let $V \subset Y$ be an open neighbourhood of y_0 contained in $\bigcup g_{i_j}(E_j)$ and contained in $W_{i_1} \cap \ldots \cap W_{i_n}$. Since $t_{i_1,y_0} = \ldots = t_{i_n,y_0}$, after shrinking V we may assume the sections $t_{i_j}|_V$, $j = 1, \ldots, n$ of $f^{-1}\mathcal{F}$ agree. As $V \subset \bigcup g_{i_j}(E_j)$ we see that $(t_y)_{y \in V}$ comes from this section.

We still have to show that $\mathcal{G}$ is equal to $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}, resp. $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}. In both cases the pullback is defined by taking the presheaf

$$(f : Y \to X) \longmapsto \operatorname{colim}_{f(Y) \subset U \subset X} \mathcal{F}(U)$$

and then sheafifying. Sheafifying in the Zar topology exactly produces our sheaf $\mathcal{G}$ and the fact that $\mathcal{G}$ is a qc sheaf, shows that it works as well in the qc topology. □

Let $X \in \mathrm{Ob}(LC_{Zar})$ and let $\mathcal{H}$ be an abelian sheaf on LC_{Zar}/X. Then we will write $H^n_{Zar}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of LC_{Zar}/X.

0DCU **Lemma 21.31.7.** *Let X be an object of LC_{Zar}. Then*

(1) *for $\mathcal{F} \in Ab(X)$ we have $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$,*
(2) *$\pi_{X,*} : Ab(LC_{Zar}/X) \to Ab(X)$ is exact,*
(3) *the unit $id \to \pi_{X,*} \circ \pi_X^{-1}$ of the adjunction is an isomorphism, and*
(4) *for $K \in D(X)$ the canonical map $K \to R\pi_{X,*}\pi_X^{-1}K$ is an isomorphism.*

Let $f : X \to Y$ be a morphism of LC_{Zar}. Then

(5) *there is a commutative diagram*

$$\begin{array}{ccc} Sh(LC_{Zar}/X) & \xrightarrow{f_{Zar}} & Sh(LC_{Zar}/Y) \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ Sh(X_{Zar}) & \xrightarrow{f} & Sh(Y_{Zar}) \end{array}$$

of topoi,

(6) *for $L \in D^+(Y)$ we have $H^n_{Zar}(X, \pi_Y^{-1}L) = H^n(X, f^{-1}L)$,*

(7) *if f is proper, then we have*

(a) *$\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$ as functors $Sh(X) \to Sh(LC_{Zar}/Y)$,*

(b) *$\pi_Y^{-1} \circ Rf_* = Rf_{Zar,*} \circ \pi_X^{-1}$ as functors $D^+(X) \to D^+(LC_{Zar}/Y)$.*

Proof. Proof of (1). The equality $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$ is a general fact coming from the trivial observation that coverings of X in LC_{Zar} are the same thing as open coverings of X. The reader who wishes to see a detailed proof should apply Lemma 21.7.2 to the functor $X_{Zar} \to LC_{Zar}$.

Proof of (2). This is true because $\pi_{X,*} = \tau_X^{-1}$ for some morphism of topoi $\tau_X : Sh(X_{Zar}) \to Sh(LC_{Zar})$ as follows from Sites, Lemma 7.21.8 applied to the functor $X_{Zar} \to LC_{Zar}/X$ used to define π_X.

Proof of (3). This is true because $\tau_X^{-1} \circ \pi_X^{-1}$ is the identity functor by Sites, Lemma 7.21.8. Or you can deduce it from the explicit description of π_X^{-1} in Lemma 21.31.6.

Proof of (4). Apply (3) to an complex of abelian sheaves representing K.

Proof of (5). The morphism of topoi f_{Zar} comes from an application of Sites, Lemma 7.25.8 and in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$ by Sites, Lemma 7.27.3. The diagram commutes simply because the corresponding continuous functors compose correctly (see Sites, Lemma 7.14.4).

Proof of (6). We have $H^n_{Zar}(X, \pi_Y^{-1}\mathcal{G}) = H^n_{Zar}(X, f_{Zar}^{-1}\pi_Y^{-1}\mathcal{G})$ for $\mathcal{G}$ in $Ab(Y)$, see Lemma 21.7.1. This is equal to $H^n_{Zar}(X, \pi_X^{-1}f^{-1}\mathcal{G})$ by the commutativity of the diagram in (5). Hence we conclude by (1) in the case L consists of a single sheaf in degree 0. The general case follows by representing L by a bounded below complex of abelian sheaves.

Proof of (7a). Let $\mathcal{F}$ be a sheaf on X. Let $g : Z \to Y$ be an object of LC_{Zar}/Y. Consider the fibre product

$$\begin{array}{ccc} Z' & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then we have

$$(f_{Zar,*}\pi_X^{-1}\mathcal{F})(Z/Y) = (\pi_X^{-1}\mathcal{F})(Z'/X) = \Gamma(Z', (g')^{-1}\mathcal{F}) = \Gamma(Z, f'_*(g')^{-1}\mathcal{F})$$

the second equality by Lemma 21.31.6. On the other hand

$$(\pi_Y^{-1}f_*\mathcal{F})(Z/Y) = \Gamma(Z, g^{-1}f_*\mathcal{F})$$

again by Lemma 21.31.6. Hence by proper base change for sheaves of sets (Cohomology, Lemma 20.18.3) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_Y^{-1}f_*\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$.

Proof of (7b). Let $K \in D^+(X)$. By Lemma 21.20.6 the nth cohomology sheaf of $Rf_{Zar,*}\pi_X^{-1}K$ is the sheaf associated to the presheaf

$$(g : Z \to Y) \longmapsto H^n_{Zar}(Z', \pi_X^{-1}K)$$

with notation as above. Observe that

$$\begin{aligned} H^n_{Zar}(Z', \pi_X^{-1}K) &= H^n(Z', (g')^{-1}K) \\ &= H^n(Z, Rf'_*(g')^{-1}K) \\ &= H^n(Z, g^{-1}Rf_*K) \\ &= H^n_{Zar}(Z, \pi_Y^{-1}Rf_*K) \end{aligned}$$

The first equality is (6) applied to K and $g' : Z' \to X$. The second equality is Leray for $f' : Z' \to Z$ (Cohomology, Lemma 20.13.1). The third equality is the proper base change theorem (Cohomology, Theorem 20.18.2). The fourth equality is (6) applied to $g : Z \to Y$ and Rf_*K. Thus $Rf_{Zar,*}\pi_X^{-1}K$ and $\pi_Y^{-1}Rf_*K$ have the same cohomology sheaves. We omit the verification that the canonical base change map $\pi_Y^{-1}Rf_*K \to Rf_{Zar,*}\pi_X^{-1}K$ induces this isomorphism. □

In the situation of Lemma 21.31.6 the composition of ϵ and π and the equality $Sh(X) = Sh(X_{Zar})$ determine a morphism of topoi

$$a_X : Sh(LC_{qc}/X) \longrightarrow Sh(X)$$

0D92 **Lemma 21.31.8.** *Let $f : X \to Y$ be a morphism of LC_{qc}. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh(LC_{qc}/X) & \xrightarrow{f_{qc}} & Sh(LC_{qc}/Y) \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh(LC_{Zar}/X) & \xrightarrow{f_{Zar}} & Sh(LC_{Zar}/Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} Sh(LC_{qc}/X) & \xrightarrow{f_{qc}} & Sh(LC_{qc}/Y) \\ {\scriptstyle a_X}\downarrow & & \downarrow{\scriptstyle a_Y} \\ Sh(X) & \xrightarrow{f} & Sh(Y) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$, $a_Y = \pi_X \circ \epsilon_X$. If f is proper, then $a_Y^{-1} \circ f_* = f_{qc,*} \circ a_X^{-1}$.

Proof. The morphism of topoi f_{qc} is the one from Sites, Lemma 7.25.8 which in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$, see Sites, Lemma 7.27.3. The diagram on the left commutes because the corresponding continuous functors compose correctly (see Sites, Lemma 7.14.4). The diagram on the right commutes because the one on the left does and because of part (5) of Lemma 21.31.7.

Proof of the final assertion. The reader may repeat the proof of part (7a) of Lemma 21.31.7; we will instead deduce this from it. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 21.31.6 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1}f_*\mathcal{F} \to f_{qc,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1}f_*\mathcal{F} = \epsilon_{Y,*}a_Y^{-1}f_*\mathcal{F} \to \epsilon_{Y,*}f_{qc,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$$

is an isomorphism. This is part (7a) of Lemma 21.31.7. □

0EZI **Lemma 21.31.9.** *Consider the comparison morphism $\epsilon : LC_{qc} \to LC_{Zar}$. Let $\mathcal{P}$ denote the class of proper maps of topological spaces. For X in LC_{Zar} denote $\mathcal{A}'_X \subset Ab(LC_{Zar}/X)$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $Ab(X)$. Then (1), (2), (3), (4), and (5) of Situation 21.30.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab(LC_{Zar}/X)$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful by Lemma 21.31.7 part

(3). If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab(LC_{Zar}/X)$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 21.31.7 part (2). Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Property (1) holds by Lemma 21.31.1 and the fact that the base change of a proper map is a proper map (see Topology, Theorem 5.17.5 and Lemma 5.4.4).

Property (2) follows from the commutative diagram (5) in Lemma 21.31.7.

Property (3) is Lemma 21.31.6.

Property (4) is Lemma 21.31.7 part (7)(b).

Proof of (5). Suppose given a qc covering $\{U_i \to U\}$. For $u \in U$ pick $i_1, \ldots, i_m \in I$ and quasi-compact subsets $E_j \subset U_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of u. Observe that $Y = \coprod_{j=1,\ldots,m} E_j \to U$ is proper as a continuous map between Hausdorff quasi-compact spaces (Topology, Lemma 5.17.7). Choose an open neighbourhood $u \in V$ contained in $\bigcup f_{i_j}(E_j)$. Then $Y \times_U V \to V$ is a surjective proper morphism and hence a *qc* covering by Lemma 21.31.4. Since we can do this for every $u \in U$ we see that (5) holds. □

0DCY **Lemma 21.31.10.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}(LC_{qc})$ and an abelian sheaf $\mathcal{F}$ on X we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a proper morphism $f : X \to Y$ in LC_{qc} and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_*\mathcal{F}) = R^i f_{qc,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For $X \in \mathrm{Ob}(LC_{qc})$ and K in $D^+(X)$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a proper morphism $f : X \to Y$ in LC_{qc} and K in $D^+(X)$ we have $a_Y^{-1}(Rf_*K) = Rf_{qc,*}(a_X^{-1}K)$.*

Proof. By Lemma 21.31.9 the lemmas in Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Lemma 21.30.2 is the essential image of $a_X^{-1} : Ab(X) \to Ab(LC_{qc}/X)$.

Part (1) is equivalent to (V_n) for all n which holds by Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Lemma 21.30.3.

Part (3) follows from Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}(LC_{Zar}/X)$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Lemma 21.30.8 part (2) for the same reason. □

0D91 **Lemma 21.31.11.** *Let X be an object of LC_{qc}. For $K \in D^+(X)$ the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh(LC_{qc}/X) \to Sh(X)$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 21.31.6 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence Lemma 21.14.7. By Lemma 21.31.10 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 21.31.7 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

09X4 **Lemma 21.31.12.** *With $X \in \mathrm{Ob}(LC_{qc})$ and $a_X : Sh(LC_{qc}/X) \to Sh(X)$ as above:*

(1) *for an abelian sheaf $\mathcal{F}$ on X we have $H^n(X, \mathcal{F}) = H^n_{qc}(X, a_X^{-1}\mathcal{F})$,*
(2) *for $K \in D^+(X)$ we have $H^n(X, K) = H^n_{qc}(X, a_X^{-1}K)$.*

For example, if A is an abelian group, then we have $H^n(X, \underline{A}) = H^n_{qc}(X, \underline{A})$.

Proof. This follows from Lemma 21.31.11 by Remark 21.14.4. □

21.32. Spectral sequences for Ext

07A9 In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathcal{G}^\bullet, \mathcal{F}^\bullet$ of complexes of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 13.27.

07AA **Example 21.32.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}}(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{O}}(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$, see Homology, Section 12.25.

21.33. Cup product

0FPJ Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, M be objects of $D(\mathcal{O})$. Set $A = \Gamma(\mathcal{C}, \mathcal{O})$. The (global) cup product in this setting is a map

$$R\Gamma(\mathcal{C}, K) \otimes^{\mathbf{L}}_A R\Gamma(\mathcal{C}, M) \longrightarrow R\Gamma(\mathcal{C}, K \otimes^{\mathbf{L}}_{\mathcal{O}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed topoi $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(pt), A)$ as in Remark 21.19.7.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{C}$-modules. There is a naive cup product

$$\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \longrightarrow f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

0FPK **Lemma 21.33.1.** *In the situation above the following diagram commutes*

$$\begin{array}{ccc}
f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet & \longrightarrow & Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*\mathcal{M}^\bullet \\
\downarrow & & \downarrow{\scriptstyle \text{Remark 21.19.7}} \\
Tot(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) & & Rf_*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
\downarrow{\scriptstyle \text{naive cup product}} & & \downarrow \\
f_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) & \longrightarrow & Rf_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{array}$$

Proof. By the construction in Remark 21.19.7 we see that going around the diagram clockwise the map

$$f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet \longrightarrow Rf_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

is adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to Lf^*Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*Rf_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 21.19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f^*f_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Going around anti-clockwise we obtain the map adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to Lf^*Rf_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 21.19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to f^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Now the proof is finished by a contemplation of the diagram

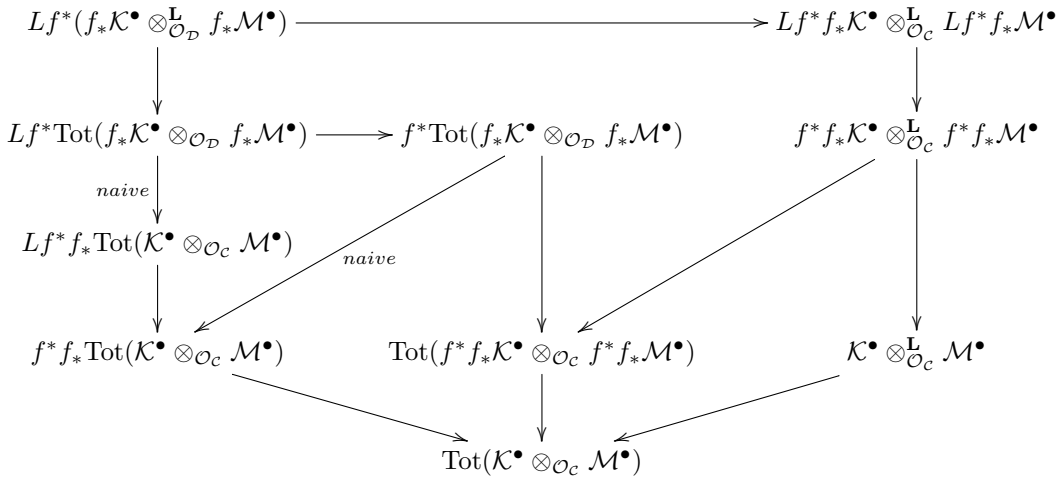

All of the polygons in this diagram commute. The top one commutes by Lemma 21.18.8. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \mathrm{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definition of the naive cup product (by the adjointness of f^* and f_*). The proof is finished because going around the diagram on the outside are the two maps given above. □

0FPL **Lemma 21.33.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 21.19.7 is associative in the sense that the diagram*

$$\begin{array}{ccc} Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M \\ \downarrow & & \downarrow \\ Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} M) & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M) \end{array}$$

is commutative in $D(\mathcal{O}')$ for all K, L, M in $D(\mathcal{O})$.

Proof. Going around either side we obtain the map adjoint to the obvious map

$$\begin{aligned} Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M) &= Lf^*(Rf_*K) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*L) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*M) \\ &\to K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M \end{aligned}$$

in $D(\mathcal{O})$. □

0FPM **Lemma 21.33.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 21.19.7 is commutative in the sense that the diagram*

$$\begin{array}{ccc} Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle Rf_*\psi} \\ Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*K & \longrightarrow & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} K) \end{array}$$

is commutative in $D(\mathcal{O}')$ for all K, L in $D(\mathcal{O})$. Here ψ is the commutativity constraint on the derived category (Lemma 21.48.5).

Proof. Omitted. □

0FPN **Lemma 21.33.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $f' : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. The relative cup product of Remark 21.19.7 is compatible with compositions in the sense that the diagram*

$$\begin{array}{ccccc} R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L & = & & = & Rf'_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} Rf'_*Rf_*L \\ \downarrow & & & & \downarrow \\ R(f' \circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) & = & Rf'_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) & \leftarrow & Rf'_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L) \end{array}$$

is commutative in $D(\mathcal{O}'')$ for all K, L in $D(\mathcal{O})$.

Proof. This is true because going around the diagram either way we obtain the map adjoint to the map

$$\begin{aligned} & L(f' \circ f)^* \left(R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L \right) \\ & = L(f' \circ f)^* R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}} L(f' \circ f)^* R(f' \circ f)_*L) \\ & \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L \end{aligned}$$

in $D(\mathcal{O})$. To see this one uses that the composition of the counits like so

$$L(f' \circ f)^* R(f' \circ f)_* = Lf^* L(f')^* Rf'_* Rf_* \to Lf^* Rf_* \to \mathrm{id}$$

is the counit for $L(f' \circ f)^*$ and $R(f' \circ f)_*$. See Categories, Lemma 4.24.9. □

0H9A **Lemma 21.33.5.** *Consider a commutative square*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Let K, L in $D(\mathcal{O}_{\mathcal{C}})$. The relative cup product is compatible with the square in the sense that the diagram

$$\begin{array}{ccc} Lg^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}}} Rf_*L) & \longrightarrow & Lg^*(Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}}} L)) \\ \| & & \downarrow \\ Lg^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*Rf_*L & & R(f')_*L(g')^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}}} L) \\ \downarrow & & \| \\ R(f')_*(L(g')^*K \otimes_{\mathcal{O}_{\mathcal{D}'}} R(f')_*(L(g')^*L & \longrightarrow & R(f')_*(L(g')^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}'}} L(g')^*L) \end{array}$$

is commutative in $D(\mathcal{O}_{\mathcal{D}'})$. The horizontal arrows are given by the relative cup product (Remark 21.19.7) and the vertical arrows are given by the base change map (Remark 21.19.3) and Lemma 21.18.4.

Proof. Omitted. □

21.34. Hom complexes

0A8X Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}$-modules. We construct a complex of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}$-modules of all $\mathcal{O}$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}$-modules) which are homogeneous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A8Y (21.34.0.1) $$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{M}^\bullet[n]|_U)$$

for all $n \in \mathbf{Z}$ and every $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, we have

0A8Z (21.34.0.2) $$H^n(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$$

for the complex of global sections.

0A90 **Lemma 21.34.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.1. □

0A91 **Lemma 21.34.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.3. □

0BYT **Lemma 21.34.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.4. □

0A93 **Lemma 21.34.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.5. □

0A92 **Lemma 21.34.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.72.6. □

0A94 **Lemma 21.34.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L and M be objects of $D(\mathcal{O})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing M. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}$-modules representing L. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, $H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (21.34.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 21.20.1. The proof of the last equation is similar except that it uses (21.34.0.2). □

0A95 **Lemma 21.34.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O})$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O})$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 21.34.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

0A96 **Lemma 21.34.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}$-modules.*

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}$-modules, then

$$\begin{aligned} \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\ &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\ &= \mathrm{Hom}_{K(\mathcal{O})}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\ &= 0 \end{aligned}$$

The first equality by (21.34.0.2). The second equality by Lemma 21.34.1. The third equality by (21.34.0.2). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 21.17.2) and because $\mathcal{I}^\bullet$ is K-injective. □

21.35. Internal hom in the derived category

08J7 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$ such that for every third object K of $D(\mathcal{O})$ there exists a canonical bijection

08J8 (21.35.0.1) $$\mathrm{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex of $\mathcal{O}$-modules $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Then we set Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}$-modules constructed in Section 21.34. This is well defined by Lemma 21.34.7. We get a functor

$$D(\mathcal{O})^{opp} \times D(\mathcal{O}) \longrightarrow D(\mathcal{O}), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (21.35.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

08JA **Lemma 21.35.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and we have $H^0(\mathcal{C}, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 21.34.8. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 21.34.6. □

08J9 **Lemma 21.35.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$$

in $D(\mathcal{O})$ functorial in K, L, M which recovers (21.35.0.1) on taking $H^0(\mathcal{C}, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. For any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 21.34.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 21.34.8) and that the right hand side represents $R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 21.35.1 we obtain (21.35.0.1). □

08JB **Lemma 21.35.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions, i.e., for every object U of $\mathcal{C}$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. This is clear from the construction and Lemma 21.20.1. □

08JC **Lemma 21.35.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The bifunctor $R\mathcal{H}om(-,-)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. □

0A97 **Lemma 21.35.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 21.34.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O})$. □

0A98 **Lemma 21.35.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O})$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ representing K. By Lemma 21.34.2 there is a map of complexes

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}^\bullet\right) \longrightarrow \mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right)$$

whose source represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. □

0BYU **Lemma 21.35.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}} L)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K, and a K-injective complex $\mathcal{I}^\bullet$ representing L, and choose any complex of $\mathcal{O}$-modules $\mathcal{M}^\bullet$ representing M. Choose a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\right) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 21.34.3. □

0A99 **Lemma 21.35.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L in $D(\mathcal{O})$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes^{\mathbf{L}}_{\mathcal{O}} L)$$

in $D(\mathcal{O})$ functorial in both K and L.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 21.34.4. □

08JD **Lemma 21.35.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L be an object of $D(\mathcal{O})$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O})$. For M in $D(\mathcal{O})$ there is a canonical map*

08JE (21.35.9.1) $$M \otimes^{\mathbf{L}}_{\mathcal{O}} L^\vee \longrightarrow R\mathcal{H}om(L, M)$$

which induces a canonical map

$$H^0(\mathcal{C}, M \otimes^{\mathbf{L}}_{\mathcal{O}} L^\vee) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(L, M)$$

functorial in M in $D(\mathcal{O})$.

Proof. The map (21.35.9.1) is a special case of Lemma 21.35.6 using the identification $M = R\mathcal{H}om(\mathcal{O}, M)$. □

0B6D **Remark 21.35.10.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there is a canonical map

$$Rf_* R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_* L, Rf_* K)$$

Namely, by (21.35.0.1) this is the same thing as a map $Rf_* R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_* L \to Rf_* K$. For this we can use the composition

$$Rf_* R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_* L \to Rf_*(R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L) \to Rf_* K$$

where the first arrow is the relative cup product (Remark 21.19.7) and the second arrow is Rf_* applied to the canonical map $R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L \to K$ coming from Lemma 21.35.6 (with $\mathcal{O}_\mathcal{C}$ in one of the spots).

08JF **Remark 21.35.11.** Let $h : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}')$. We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^* K, Lh^* L)$$

in $D(\mathcal{O})$. Namely, by (21.35.0.1) proved in Lemma 21.35.2 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^* K \longrightarrow Lh^* L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 21.18.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (21.35.0.1).

08JG **Remark 21.35.12.** Suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[h]{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let K, L be objects of $D(\mathcal{O}_{\mathcal{C}})$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K, L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K, L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K, L) \\ &\to Lh^*R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 21.35.11.

21.36. Global derived hom

0B6E Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $K, L \in D(\mathcal{O})$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\mathrm{Hom}_{\mathcal{O}}(K, L) = R\Gamma(\mathcal{C}, R\mathcal{H}om(K, L))$$

in $D(\Gamma(\mathcal{C}, \mathcal{O}))$. By Lemma 21.35.1 we have

$$H^0(R\mathrm{Hom}_{\mathcal{O}}(K, L)) = \mathrm{Hom}_{D(\mathcal{O})}(K, L)$$

and

$$H^p(R\mathrm{Hom}_{\mathcal{O}}(K, L)) = \mathrm{Ext}^p_{D(\mathcal{O})}(K, L)$$

If $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ is a morphism of ringed topoi, then there is a canonical map

$$R\mathrm{Hom}_{\mathcal{O}}(K, L) \longrightarrow R\mathrm{Hom}_{\mathcal{O}'}(Lf^*K, Lf^*L)$$

in $D(\Gamma(\mathcal{O}))$ by taking global sections of the map defined in Remark 21.35.11.

21.37. Derived lower shriek

07AB In this section we study morphisms g of ringed topoi where besides Lg^* and Rg_* there also exists a derived functor $Lg_!$.

0D6X **Lemma 21.37.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. Then $H^p(U, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. The vanishing of the lemma follows from Lemma 21.10.9 if we can prove vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ for any covering $\mathcal{U} = \{U_i \to U\}$ of $\mathcal{C}$. Since u is continuous, $u(\mathcal{U}) = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and $u(U_{i_0} \times_U \ldots \times_U U_{i_n}) = u(U_{i_0}) \times_{u(U)} \ldots \times_{u(U)} u(U_{i_n})$. Thus we have

$$\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$$

because $g^{-1} = u^p$ by Sites, Lemma 7.21.5. Since $\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module these Čech cohomology groups vanish, see Lemma 21.12.3. □

07AC **Lemma 21.37.2.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 18.41.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*. Moreover, for $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where $j_{U!}$ and $j_{u(U)!}$ are extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$ and $j_{u(U)} : \mathcal{D}/u(U) \to \mathcal{D}$.

Proof. We are going to use Derived Categories, Proposition 13.29.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form $j_{U!}\mathcal{O}_U$. Note that $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$, see proof of Modules on Sites, Lemma 18.41.1. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 18.28.8. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})) \end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Here the first equality follows from $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors and the vanishing of $H^p(U, g^{-1}\mathcal{I})$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$ follows from Lemma 21.37.1. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 21.32.1)

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-p}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{p,0} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-p}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. It is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., we have

07AD (21.37.2.1) $$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(K, g^*L) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(Lg_!K, L)$$

by Derived Categories, Lemma 13.30.3. This finishes the proof. □

07AE **Remark 21.37.3.** Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 21.37.2. In general it is **not** the case that the diagram

$$\begin{array}{ccc} D(\mathcal{O}_\mathcal{C}) & \xrightarrow[Lg_!]{} & D(\mathcal{O}_\mathcal{D}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ D(\mathcal{C}) & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{D}) \end{array}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 21.37.2 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 18.41.2), but this phenomenon **can occur even if** $g_! = g_!^{Ab}$! Namely, the construction of $Lg_!$ in the proof of Lemma 21.37.2 shows that $Lg_!$ agrees with $Lg_!^{Ab}$ if and only if the canonical maps

$$Lg_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

are isomorphisms in $D(\mathcal{D})$ for all objects U in $\mathcal{C}$. In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

0D6Y **Lemma 21.37.4.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. If $g_!^{Sh} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ commutes with fibre products[7], then $g^{-1}\mathcal{I}$ is totally acyclic.*

[7]Holds if $\mathcal{C}$ has finite connected limits and u commutes with them, see Sites, Lemma 7.21.6.

Proof. We will use the criterion of Lemma 21.13.5. Condition (1) holds by Lemma 21.37.1. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Since $g_!^{Sh}$ is a left adjoint, we see that $g_!^{Sh}K' \to g_!^{Sh}K$ is surjective. Observe that

$$\begin{aligned} H^0(K' \times_K \ldots \times_K K', g^{-1}\mathcal{I}) &= H^0(g_!^{Sh}(K' \times_K \ldots \times_K K'), \mathcal{I}) \\ &= H^0(g_!^{Sh}K' \times_{g_!^{Sh}K} \ldots \times_{g_!^{Sh}K} g_!^{Sh}K', \mathcal{I}) \end{aligned}$$

by our assumption on $g_!^{Sh}$. Since $\mathcal{I}$ is an injective module it is totally acyclic by Lemma 21.14.1 (applied to the identity). Hence we can use the converse of Lemma 21.13.5 to see that the complex

$$0 \to H^0(K, g^{-1}\mathcal{I}) \to H^0(K', g^{-1}\mathcal{I}) \to H^0(K' \times_K K', g^{-1}\mathcal{I}) \to \ldots$$

is exact as desired. □

0DD8 **Lemma 21.37.5.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

1. *For M in $D(\mathcal{D})$ we have $R\Gamma(U, g^{-1}M) = R\Gamma(u(U), M)$.*
2. *If $\mathcal{O}_\mathcal{D}$ is a sheaf of rings and $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$, then for M in $D(\mathcal{O}_\mathcal{D})$ we have $R\Gamma(U, g^*M) = R\Gamma(u(U), M)$.*

Proof. In the bounded below case (1) and (2) can be seen by representing K by a bounded below complex of injectives and using Lemma 21.37.1 as well as Leray's acyclicity lemma. In the unbounded case, first note that (1) is a special case of (2). For (2) we can use

$$R\Gamma(U, g^*M) = R\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^*M) = R\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, M) = R\Gamma(u(U), M)$$

where the middle equality is a consequence of Lemma 21.37.2. □

0FN6 **Lemma 21.37.6.** *Assume given a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{(g', (g')^\sharp)} & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ \downarrow{\scriptstyle (f', (f')^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{(g, g^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{array}$$

of ringed topoi. Assume

1. *f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 7.21.1,*
2. *$v \circ u' = u \circ v'$,*
3. *v and v' are continuous as well as cocontinuous,*
4. *for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^{u}_{v(V')}\mathcal{I}$ given by v is cofinal,*
5. *$g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$, and*
6. *$g'_! : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ is exact*[8].

Then we have $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 21.20.7 it suffices to prove the result on the derived category $D(\mathcal{C})$ of abelian sheaves. Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 13.31.9 and

[8] Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 18.16.3.

assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 18.41.3 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. □

0FN7 **Lemma 21.37.7.** *Consider a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[(g',(g')^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ \downarrow{\scriptstyle (f',(f')^\sharp)} & & \downarrow{\scriptstyle (f,f^\sharp)} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{(g,g^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi and suppose we have functors

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow[v']{} & \mathcal{C} \\ \uparrow{\scriptstyle u'} & & \uparrow{\scriptstyle u} \\ \mathcal{D}' & \xrightarrow{v} & \mathcal{D} \end{array}$$

such that (with notation as in Sites, Sections 7.14 and 7.21) we have

(1) *u and u' are continuous and give rise to the morphisms f and f',*
(2) *v and v' are cocontinuous giving rise to the morphisms g and g',*
(3) *$u \circ v = v' \circ u'$,*
(4) *v and v' are continuous as well as cocontinuous, and*
(5) *$g^{-1}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$.*

Then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D^+(\mathcal{O}_{\mathcal{C}}) \to D^+(\mathcal{O}_{\mathcal{D}'})$. If in addition

(6) *$g'_! : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ is exact*[9],

then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_{\mathcal{C}}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 21.20.7 it suffices to prove the result on the derived category $D^+(\mathcal{C})$ or $D(\mathcal{C})$ of abelian sheaves.

Choose an object $K \in D^+(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective abelian sheaves on $\mathcal{C}$ representing K. By Lemma 21.37.1 we see that $H^p(U', (g')^{-1}\mathcal{I}^q) = 0$ for all $p > 0$ and any q and any $U' \in \mathrm{Ob}(\mathcal{C}')$. Recall that $R^p f'_*(g')^{-1}\mathcal{I}^q$ is the sheaf associated to the presheaf $V' \mapsto H^p(u'(V'), (g')^{-1}\mathcal{I}^q)$, see Lemma 21.7.4. Thus we see that $(g')^{-1}\mathcal{I}^q$ is right acyclic for the functor f'_*. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we find that $f'_*(g')^*\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$. By Modules on Sites, Lemma 18.41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $g^{-1}f_*\mathcal{I}^\bullet$ represents $g^{-1}Rf_*K$ we conclude.

Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 13.31.9 and assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 18.41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. □

[9]Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 18.16.3.

21.38. Derived lower shriek for fibred categories

08RV In this section we work out some special cases of the situation discussed in Section 21.37. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

08P8 **Situation 21.38.1.** Here $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site and $p : \mathcal{C} \to \mathcal{D}$ is a fibred category. We endow $\mathcal{C}$ with the topology inherited from $\mathcal{D}$ (Stacks, Section 8.10). We denote $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the morphism of topoi associated to p (Stacks, Lemma 8.10.3). We set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

08P9 **Lemma 21.38.2.** *Assumptions and notation as in Situation 21.38.1. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the induced morphism of topoi*

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

Then there exists a morphism of topoi

$$\sigma : Sh(\mathcal{D}/p(U)) \to Sh(\mathcal{C}/U)$$

such that $\pi_U \circ \sigma = id$ and $\sigma^{-1} = \pi_{U,*}$.

Proof. Observe that π_U is the restriction of π to the localizations, see Sites, Lemma 7.28.4. For an object $V \to p(U)$ of $\mathcal{D}/p(U)$ denote $V \times_{p(U)} U \to U$ the strongly cartesian morphism of $\mathcal{C}$ over $\mathcal{D}$ which exists as p is a fibred category. The functor

$$v : \mathcal{D}/p(U) \to \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on $\mathcal{C}$. Moreover, it is a right adjoint to p by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 7.22 and we see that the sheaf $\pi_{U,*}\mathcal{F}$ is equal to $V \mapsto \mathcal{F}(V \times_{p(U)} U)$ (see especially Sites, Lemma 7.22.2).

But here we have more. Namely, the functor v is also cocontinuous (as all morphisms in coverings of $\mathcal{C}$ are strongly cartesian). Hence v defines a morphism σ as indicated in the lemma. The equality $\sigma^{-1} = \pi_{U,*}$ is immediate from the definition. Since $\pi_U^{-1}\mathcal{G}$ is given by the rule $U'/U \mapsto \mathcal{G}(p(U')/p(U))$ it follows that $\sigma^{-1} \circ \pi_U^{-1} = \mathrm{id}$ which proves the equality $\pi_U \circ \sigma = \mathrm{id}$. □

08PA **Situation 21.38.3.** Let $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site. Let $u : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibred categories over $\mathcal{D}$ (Categories, Definition 4.33.9). Endow $\mathcal{C}$ and $\mathcal{C}'$ with their inherited topologies (Stacks, Definition 8.10.2) and let $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, $\pi' : Sh(\mathcal{C}') \to Sh(\mathcal{D})$, and $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be the corresponding morphisms of topoi (Stacks, Lemma 8.10.3). Set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ and $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_\mathcal{D}$. Observe that $g^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$ so that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{\quad g \quad} & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) & \end{array}$$

is a commutative diagram of morphisms of ringed topoi.

08PB **Lemma 21.38.4.** *Assumptions and notation as in Situation 21.38.3. For $U' \in \mathrm{Ob}(\mathcal{C}')$ set $U = u(U')$ and $V = p'(U')$ and consider the induced morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_U) \\ & {\scriptstyle \pi'_{U'}} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & (Sh(\mathcal{D}/V), \mathcal{O}_V) & \end{array}$$

Then there exists a morphism of topoi

$$\sigma' : Sh(\mathcal{D}/V) \to Sh(\mathcal{C}'/U'),$$

such that setting $\sigma = g' \circ \sigma'$ we have $\pi'_{U'} \circ \sigma' = id$, $\pi_U \circ \sigma = id$, $(\sigma')^{-1} = \pi'_{U',*}$, and $\sigma^{-1} = \pi_{U,*}$.

Proof. Let $v' : \mathcal{D}/V \to \mathcal{C}'/U'$ be the functor constructed in the proof of Lemma 21.38.2 starting with $p' : \mathcal{C}' \to \mathcal{D}'$ and the object U'. Since u is a 1-morphism of fibred categories over $\mathcal{D}$ it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor $v = u \circ v'$ is the functor of the proof of Lemma 21.38.2 relative to $p : \mathcal{C} \to \mathcal{D}$ and U. Thus our lemma follows from that lemma. □

08PC **Lemma 21.38.5.** *Assumption and notation as in Situation 21.38.3.*

(1) *There are left adjoints $g_! : Mod(\mathcal{O}_{\mathcal{C}'}) \to Mod(\mathcal{O}_{\mathcal{C}})$ and $g_!^{Ab} : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ to $g^* = g^{-1}$ on modules and on abelian sheaves.*

(2) *The diagram*

$$\begin{array}{ccc} Mod(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g_!} & Mod(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ Ab(\mathcal{C}') & \xrightarrow{g_!^{Ab}} & Ab(\mathcal{C}) \end{array}$$

commutes.

(3) *There are left adjoints $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \to D(\mathcal{O}_{\mathcal{C}})$ and $Lg_!^{Ab} : D(\mathcal{C}') \to D(\mathcal{C})$ to $g^* = g^{-1}$ on derived categories of modules and abelian sheaves.*

(4) *The diagram*

$$\begin{array}{ccc} D(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{Lg_!} & D(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ D(\mathcal{C}') & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{C}) \end{array}$$

commutes.

Proof. The functor u is continuous and cocontinuous Stacks, Lemma 8.10.3. Hence the existence of the functors $g_!$, $g_!^{Ab}$, $Lg_!$, and $Lg_!^{Ab}$ can be found in Modules on Sites, Sections 18.16 and 18.41 and Section 21.37.

To prove (2) it suffices to show that the canonical map

$$g_!^{Ab} j_{U'!} \mathcal{O}_{U'} \to j_{u(U')!} \mathcal{O}_{u(U')}$$

is an isomorphism for all objects U' of $\mathcal{C}'$, see Modules on Sites, Remark 18.41.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{Ab} j_{U'!} \mathcal{O}_{U'} \to j_{u(U')!} \mathcal{O}_{u(U')}$$

is an isomorphism in $D(\mathcal{C})$ for all objects U' of $\mathcal{C}'$, see Remark 21.37.3. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism j the functors $j_!$ and $j_!^{Ab}$ agree (see Modules on Sites, Remark 18.19.6) and that $j_!$ is exact (Modules on Sites, Lemma 18.19.3). Let us adopt the notation of Lemma 21.38.4. Since $Lg_!^{Ab} \circ j_{U'!} = j_{U!} \circ L(g')_!^{Ab}$ (by commutativity of Sites, Lemma 7.28.4 and uniqueness of adjoint functors) it suffices to prove that $L(g')_!^{Ab}\mathcal{O}_{U'} = \mathcal{O}_U$. Using the results of Lemma 21.38.4 we have for any object E of $D(\mathcal{C}/u(U'))$ the following sequence of equalities

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{Ab}\mathcal{O}_{U'}, E) &= \mathrm{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1}\mathcal{O}_V, (g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*}(g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1}(g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*}E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1}\mathcal{O}_V, E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E)
\end{aligned}$$

By Yoneda's lemma we conclude. □

09CY **Remark 21.38.6.** Assumptions and notation as in Situation 21.38.1. Note that setting $\mathcal{C}' = \mathcal{D}$ and u equal to the structure functor of $\mathcal{C}$ gives a situation as in Situation 21.38.3. Hence Lemma 21.38.5 tells us we have functors $\pi_!$, $\pi_!^{Ab}$, $L\pi_!$, and $L\pi_!^{Ab}$ such that $forget \circ \pi_! = \pi_!^{Ab} \circ forget$ and $forget \circ L\pi_! = L\pi_!^{Ab} \circ forget$.

08PD **Remark 21.38.7.** Assumptions and notation as in Situation 21.38.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$, let $\mathcal{F}'$ be an abelian sheaf on $\mathcal{C}'$, and let $t : \mathcal{F}' \to g^{-1}\mathcal{F}$ be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint $g_!\mathcal{F}' \to \mathcal{F}$ of t, the map $Lg_!(\mathcal{F}') \to g_!\mathcal{F}'$, and the equality $L\pi'_! = L\pi_! \circ Lg_!$.

08PE **Lemma 21.38.8.** *Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ the sheaf $\pi_!\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto \mathrm{colim}_{\mathcal{C}_V^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

Proof. Denote $\mathcal{H}$ be the rule of the lemma. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.33.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map $\mathcal{H}(V) \to \mathcal{H}(V')$ between colimits, see Categories, Lemma 4.14.8.

To prove the lemma we show that

$$\mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf $\mathcal{G}$ on $\mathcal{C}$. An element of the left hand side is a compatible system of maps $\mathcal{F}(U) \to \mathcal{G}(p(U))$ for all U in $\mathcal{C}$. Since $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$ by our choice of topology on $\mathcal{C}$ we see the same thing is true for the right hand side and we win. $\square$

21.39. Homology on a category

08RW In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

08PF **Example 21.39.1** (Category over point)**.** Let $\mathcal{C}$ be a category. Endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6). Thus presheaves and sheaves agree on $\mathcal{C}$. The functor $p : \mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism is cocontinuous and continuous. Let $\pi : Sh(\mathcal{C}) \to Sh(*)$ be the corresponding morphism of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}$ with $\mathcal{O}_\mathcal{C} = \pi^{-1}B$ which we will denote $\underline{B}$. In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \to (Sh(*), B)$$

is an example of Situation 21.38.1. By Remark 21.38.6 we do not need to distinguish between $\pi_!$ on modules or abelian sheaves. By Lemma 21.38.8 we see that $\pi_!\mathcal{F} = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}$. Thus $L_n\pi_!$ is the nth left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the *nth homology group of $\mathcal{F}$* on $\mathcal{C}$.

08PG **Example 21.39.2** (Computing homology)**.** In Example 21.39.1 we can compute the functors $H_n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the chain complex

$$K_\bullet(\mathcal{F}) : \ \ldots \to \bigoplus\nolimits_{U_2 \to U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \to U_1 \to U_0, s) \longmapsto (U_1 \to U_0, s) - (U_2 \to U_0, s) + (U_2 \to U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.12. The construction of $K_\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H_n(K_\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.12.1 and Lemma 12.13.12. For $\mathcal{F} = j_{U!}\mathbf{Z}_U$ the complex $K_\bullet(\mathcal{F})$ is the complex associated to the free $\mathbf{Z}$-module on the simplicial set $X_\bullet$ with terms

$$X_n = \coprod\nolimits_{U_n \to \ldots \to U_1 \to U_0} \mathrm{Mor}_\mathcal{C}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_n \to \ldots \to U_0, f) \longmapsto (U_n \to \ldots \to U_i \to U \to \ldots \to U, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. This implies that $K_\bullet(j_{U!}\mathbf{Z}_U)$ has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 14.26.4 and the result of Simplicial, Lemma 14.27.1). Thus $K_\bullet(\mathcal{F})$ computes the left derived functors $H_n(\mathcal{C}, -)$ of $H_0(\mathcal{C}, -)$ for example by (the duals of) Homology, Lemma 12.12.4 and Derived Categories, Lemma 13.16.6.

08PH **Example 21.39.3.** Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor. Endow $\mathcal{C}'$ and $\mathcal{C}$ with the chaotic topology as in Example 21.39.1. The functors u, $\mathcal{C}' \to *$, and $\mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism are cocontinuous and continuous. Let $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$, $\pi' : Sh(\mathcal{C}') \to Sh(*)$, and $\pi : Sh(\mathcal{C}) \to Sh(*)$, be the corresponding morphisms of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}'$, $\mathcal{C}$ with the constant sheaf $\underline{B}$. In this way

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \underline{B}) & \xrightarrow{\quad g \quad} & (Sh(\mathcal{C}), \underline{B}) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (Sh(*), B) & \end{array}$$

is an example of Situation 21.38.3. Thus Lemma 21.38.5 applies to g so we do not need to distinguish between $g_!$ on modules or abelian sheaves. In particular Remark 21.38.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have $\mathcal{F}$ in $Ab(\mathcal{C})$, $\mathcal{F}'$ in $Ab(\mathcal{C}')$, and a map $t : \mathcal{F}' \to g^{-1}\mathcal{F}$. In terms of the computation of homology given in Example 21.39.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \to \ldots \to U'_0, s') \longmapsto (u(U'_n) \to \ldots \to u(U'_0), t(s'))$$

with obvious notation.

08Q6 **Remark 21.39.4.** Notation and assumptions as in Example 21.39.1. Let $\mathcal{F}^\bullet$ be a bounded complex of abelian sheaves on $\mathcal{C}$. For any object U of $\mathcal{C}$ there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in $D(Ab)$. If $\mathcal{F}^\bullet$ is a complex of $\underline{B}$-modules then this map is in $D(B)$. To prove this, note that we compute $L\pi_!(\mathcal{F}^\bullet)$ by taking a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{P}^\bullet$ is a complex of projectives. However, since the topology is chaotic this means that $\mathcal{P}^\bullet(U) \to \mathcal{F}^\bullet(U)$ is a quasi-isomorphism hence can be inverted in $D(Ab)$, resp. $D(B)$. Composing with the canonical map $\mathcal{P}^\bullet(U) \to \pi_!(\mathcal{P}^\bullet)$ coming from the computation of $\pi_!$ as a colimit we obtain the desired arrow.

08Q7 **Lemma 21.39.5.** *Notation and assumptions as in Example 21.39.1. If $\mathcal{C}$ has either an initial or a final object, then $L\pi_! \circ \pi^{-1} = id$ on $D(Ab)$, resp. $D(B)$.*

Proof. If $\mathcal{C}$ has an initial object, then $\pi_!$ is computed by evaluating on this object and the statement is clear. If $\mathcal{C}$ has a final object, then $R\pi_*$ is computed by evaluating on this object, hence $R\pi_* \circ \pi^{-1} \cong \mathrm{id}$ on $D(Ab)$, resp. $D(B)$. This implies that $\pi^{-1} : D(Ab) \to D(\mathcal{C})$, resp. $\pi^{-1} : D(B) \to D(\underline{B})$ is fully faithful, see

Categories, Lemma 4.24.4. Then the same lemma implies that $L\pi_! \circ \pi^{-1} = \mathrm{id}$ as desired. $\square$

08Q8 **Lemma 21.39.6.** *Notation and assumptions as in Example 21.39.1. Let $B \to B'$ be a ring map. Consider the commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}), \underline{B}) & \xleftarrow{h} & (Sh(\mathcal{C}), \underline{B'}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ (*, B) & \xleftarrow{f} & (*, B') \end{array}$$

Then $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$.

Proof. Both functors are right adjoint to the obvious functor $D(B') \to D(\underline{B})$. $\square$

08Q9 **Lemma 21.39.7.** *Notation and assumptions as in Example 21.39.1. Let $U_\bullet$ be a cosimplicial object in $\mathcal{C}$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ the simplicial set $\mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$ is homotopy equivalent to the constant simplicial set on a singleton. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. As $L\pi_!$ agrees for modules and abelian sheaves by Lemma 21.38.5 it suffices to prove this when $\mathcal{F}$ is an abelian sheaf. For $U \in \mathrm{Ob}(\mathcal{C})$ the abelian sheaf $j_{U!}\mathbf{Z}_U$ is a projective object of $Ab(\mathcal{C})$ since $\mathrm{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$ and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of $j_{U!}\mathbf{Z}_U$ by Modules on Sites, Lemma 18.28.8. Thus we can compute $L\pi_!(\mathcal{F})$ by choosing a resolution

$$\ldots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{F} \to 0$$

whose terms are direct sums of sheaves of the form above and taking $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$. Consider the double complex $A^{\bullet,\bullet} = \mathcal{G}^\bullet(U_\bullet)$. The map $\mathcal{G}^0 \to \mathcal{F}$ gives a map of complexes $A^{0,\bullet} \to \mathcal{F}(U_\bullet)$. Since $\pi_!$ is computed by taking the colimit over $\mathcal{C}^{opp}$ (Lemma 21.38.8) we see that the two compositions $\mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m$ are equal. Thus we obtain a canonical map of complexes

$$\mathrm{Tot}(A^{\bullet,\bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\ldots \to \mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m \to 0$$

are exact, see Homology, Lemma 12.25.4. Since the sheaves $\mathcal{G}^m$ are direct sums of the sheaves $j_{U!}\mathbf{Z}_U$ we reduce to $\mathcal{G} = j_{U!}\mathbf{Z}_U$. The complex $j_{U!}\mathbf{Z}_U(U_\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$-module on the simplicial set $\mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$ which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \to \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 14.26.4 and Lemma 14.27.1). This finishes the proof since $\pi_! j_{U!}\mathbf{Z}_U = \mathbf{Z}$ as was shown in the proof of Lemma 21.38.5. $\square$

08QA **Lemma 21.39.8.** *Notation and assumptions as in Example 21.39.3. If there exists a cosimplicial object $U'_\bullet$ of $\mathcal{C}'$ such that Lemma 21.39.7 applies to both $U'_\bullet$ in $\mathcal{C}'$ and $u(U'_\bullet)$ in $\mathcal{C}$, then we have $L\pi'_! \circ g^{-1} = L\pi_!$ as functors $D(\mathcal{C}) \to D(Ab)$, resp. $D(\mathcal{C}, \underline{B}) \to D(B)$.*

Proof. Follows immediately from Lemma 21.39.7 and the fact that g^{-1} is given by precomposing with u. □

08QB **Lemma 21.39.9.** *Let $\mathcal{C}_i$, $i = 1, 2$ be categories. Let $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ be the projection functors. Let B be a ring. Let $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \to (Sh(\mathcal{C}_i), \underline{B})$ be the corresponding morphisms of ringed topoi, see Example 21.39.3. For $K_i \in D(\mathcal{C}_i, B)$ we have*

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes^{\mathbf{L}}_{\underline{B}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes^{\mathbf{L}}_B L\pi_{2,!}(K_2)$$

in $D(B)$ with obvious notation.

Proof. As both sides commute with colimits, it suffices to prove this for $K_1 = j_{U!}\underline{B}_U$ and $K_2 = j_{V!}\underline{B}_V$ for $U \in \mathrm{Ob}(\mathcal{C}_1)$ and $V \in \mathrm{Ob}(\mathcal{C}_2)$. See construction of $L\pi_!$ in Lemma 21.37.2. In this case

$$g_1^{-1}K_1 \otimes^{\mathbf{L}}_{\underline{B}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to B, see construction of $L\pi_!$ in Lemma 21.37.2. □

08QC **Lemma 21.39.10.** *Notation and assumptions as in Example 21.39.1. If there exists a cosimplicial object $U_\bullet$ of $\mathcal{C}$ such that Lemma 21.39.7 applies, then*

$$L\pi_!(K_1 \otimes^{\mathbf{L}}_{\underline{B}} K_2) = L\pi_!(K_1) \otimes^{\mathbf{L}}_B L\pi_!(K_2)$$

for all $K_i \in D(\underline{B})$.

Proof. Consider the diagram of categories and functors

$$\begin{array}{ccccc} & & & & \mathcal{C} \\ & & & \overset{u_1}{\nearrow} & \\ \mathcal{C} & \xrightarrow{u} & \mathcal{C} \times \mathcal{C} & & \\ & & & \underset{u_2}{\searrow} & \\ & & & & \mathcal{C} \end{array}$$

where u is the diagonal functor and u_i are the projection functors. This gives morphisms of ringed topoi g, g_1, g_2. For any object (U_1, U_2) of $\mathcal{C}$ we have

$$\mathrm{Mor}_{\mathcal{C}\times\mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_1) \times \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 14.26.10. Thus Lemma 21.39.8 gives $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$ for any K in $D(\mathcal{C} \times \mathcal{C}, B)$. Take $K = g_1^{-1}K_1 \otimes^{\mathbf{L}}_B g_2^{-1}K_2$. Then $g^{-1}K = K_1 \otimes^{\mathbf{L}}_{\underline{B}} K_2$ because $g^{-1} = g^* = Lg^*$ commutes with derived tensor product (Lemma 21.18.4). To finish we apply Lemma 21.39.9. □

08QD **Remark 21.39.11** (Simplicial modules)**.** Let $\mathcal{C} = \Delta$ and let B be any ring. This is a special case of Example 21.39.1 where the assumptions of Lemma 21.39.7 hold. Namely, let $U_\bullet$ be the cosimplicial object of Δ given by the identity functor. To verify the condition we have to show that for $[m] \in \mathrm{Ob}(\Delta)$ the simplicial set $\Delta[m] : n \mapsto \mathrm{Mor}_\Delta([n], [m])$ is homotopy equivalent to a point. This is explained in Simplicial, Example 14.26.7.

In this situation the category $Mod(\underline{B})$ is just the category of simplicial B-modules and the functor $L\pi_!$ sends a simplicial B-module $M_\bullet$ to its associated complex $s(M_\bullet)$ of B-modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 21.39.10 is: if $M_\bullet$, $M'_\bullet$ are flat simplicial B-modules, then the complex $s(M_\bullet \otimes_B M'_\bullet)$ is quasi-isomorphic to the total complex associated to the double complex $s(M_\bullet) \otimes_B s(M'_\bullet)$. (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [**EZ53**].

08RX **Lemma 21.39.12.** *Let $\mathcal{C}$ be a category (endowed with chaotic topology). Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathcal{C}$. Assume*

(1) *there exists a cosimplicial object $U_\bullet$ in $\mathcal{C}$ as in Lemma 21.39.7, and*
(2) *$L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism.*

For K in $D(\mathcal{O})$ we have

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}')$$

in $D(Ab)$.

Proof. Note: in this proof $L\pi_!$ denotes the left derived functor of $\pi_!$ on abelian sheaves. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of $j_{U!}\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$, see Modules on Sites, Lemma 18.28.8. Thus it suffices to prove the lemma for $j_{U!}\mathcal{O}_U$. By assumption

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$$

is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set $P_n = \mathcal{O}(U_n)$ and $P'_n = \mathcal{O}'(U_n)$. Observe that the complex associated to the simplicial abelian group

$$X_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P_n$$

computes $L\pi_!(j_{U!}\mathcal{O}_U)$ by Lemma 21.39.7. Since $j_{U!}\mathcal{O}_U$ is a flat $\mathcal{O}$-module we have $j_{U!}\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}' = j_{U!}\mathcal{O}'_U$ and $L\pi_!$ of this is computed by the complex associated to the simplicial abelian group

$$X'_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P'_n$$

As the rule which to a simplicial set $T_\bullet$ associated the simplicial abelian group with terms $\bigoplus_{t \in T_n} P_n$ is a functor, we see that $X_\bullet \to P_\bullet$ is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set $T_\bullet$ associates the simplicial abelian group with terms $\bigoplus_{t \in T_n} P'_n$ is a functor. Hence $X'_\bullet \to P'_\bullet$ is a homotopy equivalence of simplicial abelian groups. By assumption $P_\bullet \to P'_\bullet$ is a quasi-isomorphism (since $P_\bullet$, resp. $P'_\bullet$ computes $L\pi_!\mathcal{O}$, resp. $L\pi_!\mathcal{O}'$ by Lemma 21.39.7). We conclude that $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic as desired. $\square$

09CZ **Remark 21.39.13.** Let $\mathcal{C}$ and B be as in Example 21.39.1. Assume there exists a cosimplicial object as in Lemma 21.39.7. Let $\mathcal{O} \to \underline{B}$ be a map sheaf of rings on $\mathcal{C}$ which induces an isomorphism $L\pi_!\mathcal{O} \to L\pi_!\underline{B}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object K of $D(\mathcal{O})$ we have $L\pi_!^{Ab}(K) = L\pi_!^{Ab}(K \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$ by Lemma 21.39.12. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}$ with the functor $L\pi_! : D(\underline{B}) \to D(B)$. In other words, we obtain a B-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$ of the lemma.

21.40. Calculating derived lower shriek

08P7 In this section we apply the results from Section 21.39 to compute $L\pi_!$ in Situation 21.38.1 and $Lg_!$ in Situation 21.38.3.

08PI **Lemma 21.40.1.** *Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $PAb(\mathcal{C})$ and $n \geq 0$ consider the abelian sheaf $L_n(\mathcal{F})$ on $\mathcal{D}$ which is the sheaf associated to the presheaf*

$$V \longmapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$.

Proof. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.33.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 21.39.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \longrightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote $L_{n,p}(\mathcal{F})$ this presheaf, so that $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^\#$. The canonical map $\gamma : \mathcal{F} \to \mathcal{F}^+$ (Sites, Theorem 7.10.10) defines a canonical map $L_{n,p}(\mathcal{F}) \to L_{n,p}(\mathcal{F}^+)$. We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 21.39.2. Denote $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$ the complex associated to the restriction of $\mathcal{F}$ to the fibre category $\mathcal{C}_V$. By the remarks above we obtain a presheaf $K_\bullet(\mathcal{F})$ of complexes

$$V \longmapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves $L_{n,p}(\mathcal{F})$. Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \longrightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

Injectivity. Let V be an object of $\mathcal{D}$ and let $\xi \in K_n(\mathcal{F})(V)$ be an element which maps to zero in $K_n(\mathcal{F}^+)(V)$. We have to show there exists a covering $\{V_j \to V\}$ such that $\xi|_{V_j}$ is zero in $K_n(\mathcal{F})(V_j)$. We write

$$\xi = \sum (U_{i,n+1} \to \ldots \to U_{i,0}, \sigma_i)$$

with $\sigma_i \in \mathcal{F}(U_{i,0})$. We arrange it so that each sequence of morphisms $U_n \to \ldots \to U_0$ of $\mathcal{C}_V$ occurs are most once. Since the sums in the definition of the complex $K_\bullet$ are direct sums, the only way this can map to zero in $K_\bullet(\mathcal{F}^+)(V)$ is if all σ_i map to zero in $\mathcal{F}^+(U_{i,0})$. By construction of $\mathcal{F}^+$ there exist coverings $\{U_{i,0,j} \to U_{i,0}\}$ such that $\sigma_i|_{U_{i,0,j}}$ is zero. By our construction of the topology on $\mathcal{C}$ we can write $U_{i,0,j} \to U_{i,0}$ as the pullback (Categories, Definition 4.33.6) of some morphisms $V_{i,j} \to V$ and moreover each $\{V_{i,j} \to V\}$ is a covering. Choose a covering $\{V_j \to V\}$ dominating each of the coverings $\{V_{i,j} \to V\}$. Then it is clear that $\xi|_{V_j} = 0$.

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity. □

08PJ **Lemma 21.40.2.** *Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ and $n \geq 0$ the sheaf $L_n\pi_!(\mathcal{F})$ is equal to the sheaf $L_n(\mathcal{F})$ constructed in Lemma 21.40.1.*

Proof. Consider the sequence of functors $\mathcal{F} \mapsto L_n(\mathcal{F})$ from $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. Since for each $V \in \mathrm{Ob}(\mathcal{D})$ the sequence of functors $H_n(\mathcal{C}_V, -)$ forms a δ-functor so do the functors $\mathcal{F} \mapsto L_n(\mathcal{F})$. Our goal is to show these form a universal δ-functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For $U' \in \mathrm{Ob}(\mathcal{C})$ consider the abelian presheaf $\mathcal{F}_{U'} = j_{U'!}^{PAb}\mathbf{Z}_{U'}$ (Modules on Sites, Remark 18.19.7). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus\nolimits_{\mathrm{Mor}_\mathcal{C}(U,U')} \mathbf{Z}$$

If U lies over $V = p(U)$ in $\mathcal{D}$) and U' lies over $V' = p(U')$ then any morphism $a : U \to U'$ factors uniquely as $U \to h^*U' \to U'$ where $h = p(a) : V \to V'$ (see Categories, Definition 4.33.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus\nolimits_{h \in \mathrm{Mor}_\mathcal{D}(V,V')} j_{h^*U'!}\mathbf{Z}_{h^*U'}$$

where $j_{h^*U'} : Sh(\mathcal{C}_V/h^*U') \to Sh(\mathcal{C}_V)$ is the localization morphism. The sheaves $j_{h^*U'!}\mathbf{Z}_{h^*U'}$ have vanishing higher homology groups (see Example 21.39.2). We conclude that $L_n(\mathcal{F}_{U'}) = 0$ for all $n > 0$ and all U'. It follows that any abelian presheaf $\mathcal{F}$ is a quotient of an abelian presheaf $\mathcal{G}$ with $L_n(\mathcal{G}) = 0$ for all $n > 0$ (Modules on Sites, Lemma 18.28.8). Since $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ we see that the same thing is true for abelian sheaves. Thus the sequence of functors $L_n(-)$ is a universal delta functor on $Ab(\mathcal{C})$ (Homology, Lemma 12.12.4). Since we have agreement with $H^{-n}(L\pi_!(-))$ for $n = 0$ by Lemma 21.38.8 we conclude by uniqueness of universal δ-functors (Homology, Lemma 12.12.5) and Derived Categories, Lemma 13.16.6. □

08PK **Lemma 21.40.3.** *Assumptions and notation as in Situation 21.38.3. For an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}'$ the sheaf $L_ng_!(\mathcal{F}')$ is the sheaf associated to the presheaf*

$$U \longmapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

For notation and restriction maps see proof.

Proof. Say $p(U) = V$. The category $\mathcal{I}_U$ is the category of pairs (U', φ) where $\varphi : U \to u(U')$ is a morphism of $\mathcal{C}$ with $p(\varphi) = \mathrm{id}_V$, i.e., φ is a morphism of the fibre category $\mathcal{C}_V$. Morphisms $(U'_1, \varphi_1) \to (U'_2, \varphi_2)$ are given by morphisms $a : U'_1 \to U'_2$ of the fibre category $\mathcal{C}'_V$ such that $\varphi_2 = u(a) \circ \varphi_1$. The presheaf $\mathcal{F}'_U$ sends (U', φ) to $\mathcal{F}'(U')$. We will construct the restriction mappings below.

Choose a factorization

$$\mathcal{C}' \xrightarrow[w]{u'} \mathcal{C}'' \xrightarrow{u''} \mathcal{C}$$

of u as in Categories, Lemma 4.33.14. Then $g_! = g''_! \circ g'_!$ and similarly for derived functors. On the other hand, the functor $g'_!$ is exact, see Modules on Sites, Lemma 18.16.6. Thus we get $Lg_!(\mathcal{F}') = Lg''_!(\mathcal{F}'')$ where $\mathcal{F}'' = g'_!\mathcal{F}'$. Note that $\mathcal{F}'' = h^{-1}\mathcal{F}'$ where $h : Sh(\mathcal{C}'') \to Sh(\mathcal{C}')$ is the morphism of topoi associated to w, see Sites, Lemma 7.23.1. The functor u'' turns $\mathcal{C}''$ into a fibred category over $\mathcal{C}$, hence Lemma 21.40.2 applies to the computation of $L_n g''_!$. The result follows as the construction of $\mathcal{C}''$ in the proof of Categories, Lemma 4.33.14 shows that the fibre category $\mathcal{C}''_U$ is equal to $\mathcal{I}_U$. Moreover, $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$ is given by the rule described above (as w is continuous and cocontinuous by Stacks, Lemma 8.10.3 so we may apply Sites, Lemma 7.21.5). □

21.41. Simplicial modules

09D0 Let $A_\bullet$ be a simplicial ring. Recall that we may think of $A_\bullet$ as a sheaf on Δ (endowed with the chaotic topology), see Simplicial, Section 14.4. Then a simplicial module $M_\bullet$ over $A_\bullet$ is just a sheaf of $A_\bullet$-modules on Δ. In other words, for every $n \geq 0$ we have an A_n-module M_n and for every map $\varphi : [n] \to [m]$ we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is $A_\bullet(\varphi)$-linear such that these maps compose in the usual manner.

Let $\mathcal{C}$ be a site. A *simplicial sheaf of rings* $\mathcal{A}_\bullet$ on $\mathcal{C}$ is a simplicial object in the category of sheaves of rings on $\mathcal{C}$. In this case the assignment $U \mapsto \mathcal{A}_\bullet(U)$ is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form $([n], U) \to ([n], V)$. We endow the category $\Delta \times \mathcal{C}$ with the topology inherited from $\mathcal{C}$ (see Stacks, Section 8.10). The simple description of the coverings in $\Delta \times \mathcal{C}$ (Stacks, Lemma 8.10.1) immediately implies that a simplicial sheaf of rings on $\mathcal{C}$ is the same thing as a sheaf of rings on $\Delta \times \mathcal{C}$.

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

09D1 **Definition 21.41.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet$ be a simplicial sheaf of rings on $\mathcal{C}$. A *simplicial $\mathcal{A}_\bullet$-module* $\mathcal{F}_\bullet$ (sometimes called a *simplicial sheaf of $\mathcal{A}_\bullet$-modules*) is a sheaf of modules over the sheaf of rings on $\Delta \times \mathcal{C}$ associated to $\mathcal{A}_\bullet$.

We obtain a category $Mod(\mathcal{A}_\bullet)$ of simplicial modules and a corresponding derived category $D(\mathcal{A}_\bullet)$. Given a map $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial sheaves of rings we obtain a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module $\mathcal{F}_\bullet$ the object $L\pi_!(\mathcal{F}_\bullet)$ is represented by the associated chain complex $s(\mathcal{F}_\bullet)$ (Simplicial, Section 14.23). This follows from Lemmas 21.40.2 and 21.39.7.

09D2 **Lemma 21.41.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a homomorphism of simplicial sheaves of rings on $\mathcal{C}$. If $L\pi_!\mathcal{A}_\bullet \to L\pi_!\mathcal{B}_\bullet$ is an isomorphism in $D(\mathcal{C})$, then we have*

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet)$$

for all K in $D(\mathcal{A}_\bullet)$.

Proof. Let $([n], U)$ be an object of $\Delta \times \mathcal{C}$. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 18.28.8. Thus it suffices to prove the lemma for a flat $\mathcal{A}_\bullet$-module $\mathcal{F}$. In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 21.40.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 21.40.1. Thus it suffices to prove the result for the restriction of $\mathcal{F}$ to the fibre categories (i.e., to $\Delta \times U$). In this case the result follows from Lemma 21.39.12. □

09D3 **Remark 21.41.3.** Let $\mathcal{C}$ be a site. Let $\epsilon : \mathcal{A}_\bullet \to \mathcal{O}$ be an augmentation (Simplicial, Definition 14.20.1) in the category of sheaves of rings. Assume ϵ induces a quasi-isomorphism $s(\mathcal{A}_\bullet) \to \mathcal{O}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object K of $D(\mathcal{A}_\bullet)$ we have $L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ by Lemma 21.41.2. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O}$ with the functor $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \to D(\mathcal{O})$ of Remark 21.38.6. In other words, we obtain a $\mathcal{O}$-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ of the lemma.

21.42. Cohomology on a category

08RY In the situation of Example 21.39.1 in addition to the derived functor $L\pi_!$, we also have the functor $R\pi_*$. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$ and $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$.

08RZ **Example 21.42.1** (Computing cohomology)**.** In Example 21.39.1 we can compute the functors $H^n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod\nolimits_{U_0} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1 \to U_2} \mathcal{F}(U_0) \to \ldots$$

where the transition maps are given by

$$(s_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto s_{U_0 \to U_1} - s_{U_0 \to U_2} + s_{U_1 \to U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.11. The construction of $K^\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.12.1 and Lemma 12.13.12. For an object U of $\mathcal{C}$ denote $p_U : Sh(*) \to Sh(\mathcal{C})$ the corresponding point with p_U^{-1} equal to evaluation at U, see Sites, Example 7.33.8. Let A be an abelian group and set $\mathcal{F} = p_{U,*}A$. In this case the complex $K^\bullet(\mathcal{F})$ is the complex with terms $\mathrm{Map}(X_n, A)$ where

$$X_n = \coprod_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_0 \to \ldots \to U_n, f) \longmapsto (U \to \ldots \to U \to U_i \to \ldots \to U_n, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. Since $\mathrm{Map}(-, A)$ is a contravariant functor, implies that $K^\bullet(p_{U,*}A)$ has trivial cohomology in positive degrees (by the functoriality of Simplicial, Remark 14.26.4 and the result of Simplicial, Lemma 14.28.6). This implies that $K^\bullet(\mathcal{F})$ is acyclic in positive degrees also if $\mathcal{F}$ is a product of sheaves of the form $p_{U,*}A$. As every abelian sheaf on $\mathcal{C}$ embeds into such a product we conclude that $K^\bullet(\mathcal{F})$ computes the left derived functors $H^n(\mathcal{C}, -)$ of $H^0(\mathcal{C}, -)$ for example by Homology, Lemma 12.12.4 and Derived Categories, Lemma 13.16.6.

08S0 **Example 21.42.2** (Computing Exts)**.** In Example 21.39.1 assume we are moreover given a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. Consider the complex $K^\bullet(\mathcal{G}, \mathcal{F})$ with degree n term

$$\prod_{U_0 \to U_1 \to \ldots \to U_n} \mathrm{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto \varphi_{U_0 \to U_1} \circ \rho_{U_1}^{U_2} - \varphi_{U_0 \to U_2} + \rho_{U_0}^{U_1} \circ \varphi_{U_1 \to U_2})$$

and similarly in other degrees. Here the ρ's indicate restriction maps. By construction

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$. The assignment $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$ is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for $i \geq 0$ provided either

(1) $\mathcal{G}(U)$ is a projective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$, or
(2) $\mathcal{F}(U)$ is an injective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$.

Namely, case (1) the functor $K^\bullet(\mathcal{G}, -)$ is an exact functor from the category of $\mathcal{O}$-modules to the category of cochain complexes of abelian groups. Thus, arguing as

in Example 21.42.1, it suffices to show that $K^\bullet(\mathcal{G}, \mathcal{F})$ is acyclic in positive degrees when $\mathcal{F}$ is $p_{U,*}A$ for an $\mathcal{O}(U)$-module A. Choose a short exact sequence

08S1 (21.42.2.1) $$0 \to \mathcal{G}' \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{G} \to 0$$

see Modules on Sites, Lemma 18.28.8. Since (1) holds for the middle and right sheaves, it also holds for $\mathcal{G}'$ and evaluating (21.42.2.1) on an object of $\mathcal{C}$ gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \to K^\bullet(\mathcal{G}, \mathcal{F}) \to \prod K^\bullet(j_{U_i!}\mathcal{O}_{U_i}, \mathcal{F}) \to K^\bullet(\mathcal{G}', \mathcal{F}) \to 0$$

for any $\mathcal{F}$, in particular $\mathcal{F} = p_{U,*}A$. On H^0 we obtain

$$0 \to \mathrm{Hom}(\mathcal{G}, p_{U,*}A) \to \mathrm{Hom}(\prod j_{U_i!}\mathcal{O}_{U_i}, p_{U,*}A) \to \mathrm{Hom}(\mathcal{G}', p_{U,*}A) \to 0$$

which is exact as $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$ and the sequence of sections of (21.42.2.1) over U is split exact. Thus we can use dimension shifting to see that it suffices to prove $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is acyclic in positive degrees for all $U, U' \in \mathrm{Ob}(\mathcal{C})$. In this case $K^n(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is equal to

$$\prod\nolimits_{U \to U_0 \to U_1 \to \ldots \to U_n \to U'} A$$

In other words, $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is the complex with terms $\mathrm{Map}(X_\bullet, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0) \times \mathrm{Mor}_{\mathcal{C}}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$ as can be proved in exactly the same way as the corresponding statement in Example 21.42.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

21.43. Modules on a category

0GYU The material in this section will be used to define a variant of the derived category of quasi-coherent modules on a stack in groupoids over the category of schemes. See Sheaves on Stacks, Section 96.26.

Let $\mathcal{C}$ be a category. We think of $\mathcal{C}$ as a site with the chaotic topology. As in Example 21.42.2 we let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. In other words, $\mathcal{O}$ is a presheaf of rings on the category $\mathcal{C}$, see Categories, Definition 4.3.3.

0GYV **Definition 21.43.1.** In the situation above, we denote $QC(\mathcal{C}, \mathcal{O})$ or simply $QC(\mathcal{O})$ the full subcategory of $D(\mathcal{O}) = D(\mathcal{C}, \mathcal{O})$ consisting of objects K such that for all $U \to V$ in $\mathcal{C}$ the canonical map

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow R\Gamma(U, K)$$

is an isomorphism in $D(\mathcal{O}(U))$.

0GYW **Lemma 21.43.2.** *In the situation above, the subcategory $QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums.*

Proof. Let U be an object of $\mathcal{C}$. Since the topology on $\mathcal{C}$ is chaotic, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact and commutes with direct sums. Hence the exact functor $K \mapsto R\Gamma(U, K)$ is computed by representing K by any complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and taking $\mathcal{F}^\bullet(U)$. Thus $R\Gamma(U, -)$ commutes with direct sums, see Injectives, Lemma 19.13.4. Similarly, given a morphism $U \to V$ of $\mathcal{C}$ the derived tensor

product functor $- \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) : D(\mathcal{O}(V)) \to D(\mathcal{O}(U))$ is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted. □

0GZQ **Lemma 21.43.3.** *In the situation above, suppose that M is an object of $QC(\mathcal{O})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{C}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1.*

Proof. By Modules on Sites, Lemma 18.24.3 it suffices to show that for every morphism $U \to V$ of $\mathcal{C}$ the map

$$H^p(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to H^b(M)(U)$$

is an isomorphism. We are given that the map

$$R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, M)$$

is an isomorphism. Thus the result by the Tor spectral sequence for example. Details omitted. □

0H0R **Lemma 21.43.4.** *In the situation above, suppose that $\mathcal{C}$ has a final object X. Set $R = \mathcal{O}(X)$ and denote $f : (\mathcal{C}, \mathcal{O}) \to (pt, R)$ the obvious morphism of sites. Then $QC(\mathcal{O}) = D(R)$ given by Lf^* and Rf_*.*

Proof. Omitted. □

0H0S **Lemma 21.43.5.** *In the situation above, suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We may replace $\mathcal{C}$ by $\mathcal{C}/U$. Thus we may assume $U = X$ is a final object of $\mathcal{C}$. By Lemma 21.43.4 we see that $K = Lf^*P$ where $P = R\Gamma(U, K) = R\Gamma(X, K) = Rf_*K$. Thus the result because Lf^* is the left adjoint to $Rf_*(-) = R\Gamma(U, -)$. □

Let $(\mathcal{C}, \mathcal{O})$ be as above. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules we define the *size* $|\mathcal{F}^\bullet|$ of $\mathcal{F}^\bullet$ as

$$|\mathcal{F}^\bullet| = \left| \coprod\nolimits_{i \in \mathbf{Z},\ U \in \mathrm{Ob}(\mathcal{C})} \mathcal{F}^i(U) \right|$$

For an object K of $D(\mathcal{O})$ we define the *size* $|K|$ of K to be the cardinal

$$|K| = \min\{|\mathcal{F}^\bullet| \text{ where } \mathcal{F}^\bullet \text{ represents } K\}$$

By properties of cardinals the minimum exists.

0GYX **Lemma 21.43.6.** *In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and subsets $\Omega^i_U \subset \mathcal{F}^i(U)$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ with $\Omega^i_U \subset \mathcal{H}^i(U)$ and $|\mathcal{H}^\bullet| \leq \max(\kappa, |\bigcup \Omega^i_U|)$.*

Proof. Define $\mathcal{H}^i(U)$ to be the $\mathcal{O}(U)$-submodule of $\mathcal{F}^i(U)$ generated by the images of Ω^i_V and $\mathrm{d}(\Omega^{i-1}_U)$ by restriction along any morphism $f : U \to V$. The cardinality of $\mathcal{H}^i(U)$ is bounded by the maximum of $\aleph_0$, the cardinality of the $\mathcal{O}(U)$, the cardinality of $\mathrm{Arrows}(\mathcal{C})$, and $|\bigcup \Omega^i_U|$. Details omitted. □

0GYY **Lemma 21.43.7.** *In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules representing an object K of $D(\mathcal{O})$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ such that $\mathcal{H}^\bullet$ represents K and such that $|\mathcal{H}^\bullet| \leq \max(\kappa, |K|)$.*

Proof. First, for every i and U we choose a subset $\Omega^i_U \subset \mathrm{Ker}(\mathrm{d} : \mathcal{F}^i(U) \to \mathcal{F}^{i+1}(U))$ mapping bijectively onto $H^i(K)(U) = H^i(\mathcal{F}^\bullet(U))$. Hence $|\Omega^i_U| \leq |K|$ as we may represent K by a complex whose size is $|K|$. Applying Lemma 21.43.6 we find a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ of size at most $\max(\kappa, |K|)$ containing Ω^i_U and hence such that $H^i(\mathcal{S}^\bullet) \to H^i(\mathcal{F}^\bullet)$ is a surjection of sheaves.

We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}^\bullet_0 \subset \mathcal{S}^\bullet_1 \subset \mathcal{S}^\bullet_2 \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \max(\kappa, |K|)$ such that the kernel of $H^i(\mathcal{S}^\bullet_n) \to H^i(\mathcal{F}^\bullet)$ is the same as the kernel of $H^i(\mathcal{S}^\bullet_n) \to H^i(\mathcal{S}^\bullet_{n+1})$. Once this is done we can take $\mathcal{H}^\bullet = \bigcup \mathcal{S}^\bullet_n$ as our solution.

Construction of $\mathcal{S}^\bullet_{n+1}$ given $\mathcal{S}^\bullet_n$. For ever U and i let $\Omega^{i-1}_U \subset \mathcal{F}^{i-1}(U)$ be a subset such that $\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U)$ maps Ω^{i-1}_U bijectively onto

$$\mathcal{S}^i_n(U) \cap \mathrm{Im}(\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U))$$

Observe that $|\Omega^i_U| \leq |K|$ because $\mathcal{S}^i_n(U)$ is so bounded. Then we get $\mathcal{S}^\bullet_{n+1}$ by an application of Lemma 21.43.6 to the subsets

$$\mathcal{S}^i(U) \cup \Omega^i_U \subset \mathcal{F}^i(U)$$

and everything is clear. □

0GYZ **Lemma 21.43.8.** *In the situation above, there exists a cardinal κ with the following properties:*

1. *for every nonzero object K of $QC(\mathcal{O})$ there exists a nonzero morphism $E \to K$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$,*
2. *for every morphism $\alpha : E \to \bigoplus_n K_n$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$, there exist morphisms $E_n \to K_n$ in $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ such that α factors through $\bigoplus E_n \to \bigoplus K_n$.*

Proof. Let κ be an upper bound for the following set of cardinals:

1. $|\coprod_V j_{U!}\mathcal{O}_U(V)|$ for all $U \in \mathrm{Ob}(\mathcal{C})$,
2. the cardinals $\kappa(\mathcal{O}(V) \to \mathcal{O}(U))$ found in More on Algebra, Lemma 15.103.5 for all morphisms $U \to V$ in $\mathcal{C}$,
3. the cardinal found in Lemma 21.43.7.

We claim that for any complex $\mathcal{F}^\bullet$ representing an object of $QC(\mathcal{O})$ and any subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$ there exists a subcomplex $\mathcal{H}^\bullet$ of $\mathcal{F}^\bullet$ containing $\mathcal{S}^\bullet$ such that $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ and such that $|\mathcal{H}^\bullet| \leq \kappa$. In the next two paragraphs we show that the claim implies the lemma.

As in (1) let K be a nonzero object of $QC(\mathcal{O})$. Say K is represented by the complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$. Then $H^i(\mathcal{F}^\bullet)$ is nonzero for some i. Hence there exists an object U of $\mathcal{C}$ and a section $s \in \mathcal{F}^i(U)$ with $d(s) = 0$ which determines a nonzero section of $H^i(\mathcal{F}^\bullet)$ over U. Then the image of $s : j_{U!}\mathcal{O}_U[-i] \to \mathcal{F}^\bullet$ is a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$. Applying the claim we get $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ in $QC(\mathcal{O})$ nonzero with $|\mathcal{H}^\bullet| \leq \kappa$. Thus (1) holds.

Let $\alpha : E \to \bigoplus K_n$ be as in (2). Choose any complexes $\mathcal{K}^\bullet_n$ representing K_n. Then $\bigoplus \mathcal{K}^\bullet_n$ represents $\bigoplus K_n$. By the construction of the derived category we can represent E by a complex $\mathcal{E}^\bullet$ such that α is represented by a morphism $a : \mathcal{E}^\bullet \to \bigoplus \mathcal{K}^\bullet_n$ of complexes. By Lemma 21.43.7 and our choice of κ above we may assume

$|\mathcal{E}^\bullet| \leq \kappa$. By the claim we get subcomplexes $\mathcal{E}^\bullet_n \subset \mathcal{K}^\bullet_n$ representing objects E_n of $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ containing the image of $a_n : \mathcal{E}^\bullet \to \mathcal{K}^\bullet_n$ as desired.

Proof of the claim. Let $\mathcal{F}^\bullet$ be a complex representing an object of $QC(\mathcal{O})$ and let $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ be a subcomplex of size $\leq \kappa$. We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}^\bullet_0 \subset \mathcal{S}^\bullet_1 \subset \mathcal{S}^\bullet_2 \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \kappa$ such that for every morphism $f : U \to V$ of $\mathcal{C}$ and every $i \in \mathbf{Z}$

(1) the kernel of the arrow $H^i(\mathcal{S}^\bullet_n(V) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U)) \to H^i(\mathcal{S}^\bullet_n(U))$ maps to zero in $H^i(\mathcal{S}^\bullet_{n+1}(V) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U))$,
(2) the image of the arrow $H^i(\mathcal{S}^\bullet_n(U)) \to H^i(\mathcal{S}^\bullet_{n+1}(U))$ is contained in the image of $H^i(\mathcal{S}^\bullet_{n+1}(V) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U)) \to H^i(\mathcal{S}^\bullet_{n+1}(U))$,

Once this is done we can set $\mathcal{H}^\bullet = \bigcup \mathcal{S}^\bullet_n$. Namely, since derived tensor product and taking cohomology of complexes of modules over rings commute with filtered colimits, the conditions (1) and (2) together will guarantee that

$$\mathcal{H}^\bullet(V) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{H}^\bullet(U)$$

is an isomorphism on cohomology in all degrees and hence an isomorphism in $D(\mathcal{O}(U))$ for all $f : U \to V$ in $\mathcal{C}$. Hence $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ as desired.

Construction of $\mathcal{S}_{n+1}$ given $\mathcal{S}_n$. For every morphism $f : U \to V$ of $\mathcal{C}$ we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{S}^\bullet_n(V) & \longrightarrow & \mathcal{S}^\bullet_n(U) \\ \downarrow & & \downarrow \\ \mathcal{F}^\bullet(V) & \longrightarrow & \mathcal{F}^\bullet(U) \end{array}$$

This is a diagram as in More on Algebra, Lemma 15.103.5 for the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$, i.e., the bottom row induces an isomorphism

$$\mathcal{F}^\bullet(V) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{F}^\bullet(U)$$

in $D(\mathcal{O}(U))$. Thus we may choose subcomplexes

$$\mathcal{S}^\bullet_n(V) \subset M^\bullet_f \subset \mathcal{F}^\bullet(V) \quad \text{and} \quad \mathcal{S}^\bullet_n(U) \subset N^\bullet_f \subset \mathcal{F}^\bullet(U)$$

as in More on Algebra, Lemma 15.103.5 and in particular we see that $|N^i_f|, |M^i_f| \leq \kappa$. Next, we apply Lemma 21.43.6 using the subsets

$$\mathcal{S}^i_n(U) \amalg \coprod\nolimits_{f:U\to V} N^i_f \amalg \coprod\nolimits_{g:W\to U} M^i_g \subset \mathcal{F}^i(U)$$

to find a subcomplex

$$\mathcal{S}^\bullet_n \subset \mathcal{S}^\bullet_{n+1} \subset \mathcal{F}^\bullet$$

with containing those subsets and such that $|\mathcal{S}^\bullet_{n+1}| \leq \kappa$. Conditions (1) and (2) hold because the corresponding statements hold for $\mathcal{S}^\bullet_n(V) \subset M^\bullet_f$ and $\mathcal{S}^\bullet_n(U) \subset N^\bullet_f$ by the construction in More on Algebra, Lemma 15.103.5. Thus the proof is complete. □

0GZ0 **Proposition 21.43.9.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. With $QC(\mathcal{O})$ as in Definition 21.43.1 we have*

(1) *$QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums,*
(2) *any contravariant cohomological functor $H : QC(\mathcal{O}) \to Ab$ which transforms direct sums into products is representable,*
(3) *any exact functor $F : QC(\mathcal{O}) \to \mathcal{D}$ of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and*
(4) *the inclusion functor $QC(\mathcal{O}) \to D(\mathcal{O})$ has an exact right adjoint.*

Proof. Part (1) is Lemma 21.43.2. Part (2) follows from Lemma 21.43.8 and Derived Categories, Lemma 13.39.1. Part (3) follows from Lemma 21.43.8 and Derived Categories, Proposition 13.39.2. Part (4) is a special case of (3). □

Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor between categories. If we view $\mathcal{C}$ and $\mathcal{C}'$ as sites with the chaotic topology, then u is a continuous and cocontinuous functor. Hence we obtain a morphism $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ of topoi, see Sites, Lemma 7.21.1. Additionally, suppose given sheaves of rings $\mathcal{O}$ on $\mathcal{C}$ and $\mathcal{O}'$ on $\mathcal{C}'$ and a map $g^\sharp : g^{-1}\mathcal{O} \to \mathcal{O}'$. We denote the corresponding morphism of ringed topoi simply $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$, see Modules on Sites, Section 18.7.

0GZ1 **Lemma 21.43.10.** *Let $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be as above. Then the functor $Lg^* : D(\mathcal{O}) \to D(\mathcal{O}')$ maps $QC(\mathcal{O})$ into $QC(\mathcal{O}')$.*

Proof. Let $U' \in \mathrm{Ob}(\mathcal{C}')$ with image $U = u(U')$ in $\mathcal{C}$. Let pt denote the category with a single object and a single morphism. Denote $(Sh(pt), \mathcal{O}'(U'))$ and $(Sh(pt), \mathcal{O}(U))$ the ringed topoi as indicated. Of course we identify the derived category of modules on these ringed topoi with $D(\mathcal{O}'(U'))$ and $D(\mathcal{O}(U))$. Then we have a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(pt), \mathcal{O}'(U')) & \xrightarrow[U']{} & (Sh(\mathcal{C}'), \mathcal{O}') \\ \downarrow & & \downarrow g \\ (Sh(pt), \mathcal{O}(U)) & \xrightarrow{U} & (Sh(\mathcal{C}), \mathcal{O}) \end{array}$$

Pullback along the lower horizontal morphism sends K in $D(\mathcal{O})$ to $R\Gamma(U, K)$. Pullback by the left vertical arrow sends M to $M \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$. Going around the diagram either direction produces the same result (Lemma 21.18.3) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$$

Finally, let $f' : U' \to V'$ be a morphism in $\mathcal{C}'$ and denote $f = u(f') : U = u(U') \to V = u(V')$ the image in $\mathcal{C}$. If K is in $QC(\mathcal{O})$ then we have

$$\begin{aligned} R\Gamma(V', Lg^*K) \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(V') \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\ &= R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\ &= R\Gamma(U', Lg^*K) \end{aligned}$$

as desired. Here we have used the observation above both for U' and V'. □

0GZR **Lemma 21.43.11.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction*

map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then $QC(\mathcal{O})$ agrees with the subcategory $D_{QCoh}(\mathcal{O}) \subset D(\mathcal{O})$ of complexes whose cohomology sheaves are quasi-coherent.

Proof. Recall that $QCoh(\mathcal{O}) \subset Mod(\mathcal{O})$ is a weak Serre subcategory under our assumptions, see Modules on Sites, Lemma 18.24.4. Thus taking the full subcategory

$$D_{QCoh}(\mathcal{O}) = D_{QCoh(\mathcal{O})}(Mod(\mathcal{O}))$$

of $D(\mathcal{O})$ makes sense, see Derived Categories, Section 13.17. (Strictly speaking we don't need this in the proof of the lemma.)

Let M be an object of $QC(\mathcal{O})$. Since for every morphism $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat, we see that

$$\begin{aligned} H^i(M)(U) &= H^i(R\Gamma(U, M)) \\ &= H^i(R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U)) \\ &= H^i(R\Gamma(V, M)) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \\ &= H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \end{aligned}$$

and hence $H^i(M)$ is quasi-coherent by Modules on Sites, Lemma 18.24.3. The first and last equality above follow from the fact that taking sections over an object of $\mathcal{C}$ is an exact functor due to the fact that the topology on $\mathcal{C}$ is chaotic.

Conversely, if M is an object of $D_{QCoh}(\mathcal{O})$, then due to Modules on Sites, Lemma 18.24.3 we see that the map $R\Gamma(V, M) \to R\Gamma(U, M)$ induces isomorphisms $H^i(M)(U) \to H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. Whence $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism in $D(\mathcal{O}(U))$ by the flatness of $\mathcal{O}(V) \to \mathcal{O}(U)$ and we conclude that M is in $QC(\mathcal{O})$. □

0GZS **Lemma 21.43.12.** *Let $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ be as in Section 21.27. Assume*

1. *τ' is the chaotic topology on the category $\mathcal{C}$,*
2. *for all $U \in \mathrm{Ob}(\mathcal{C})$ and all K-flat complexes of $\mathcal{O}(U)$-modules $M^\bullet$ the map*

$$M^\bullet \longrightarrow R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

is a quasi-isomorphism (see proof for an explanation).

Then ϵ^* and $R\epsilon_*$ define mutually quasi-inverse equivalences between $QC(\mathcal{O})$ and the full subcategory of $D(\mathcal{C}_\tau, \mathcal{O}_\tau)$ consisting of objects K such that $R\epsilon_* K$ is in $QC(\mathcal{O})$[10].

Proof. We will use the observations made in Section 21.27 without further mention. Since $R\epsilon_*$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \mathrm{id}$, to prove the lemma it suffices to show that for M in $QC(\mathcal{O})$ we have $R\epsilon_*(\epsilon^* M) = M$. Condition (2) is exactly the condition needed to see this. Namely, we choose a K-flat complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules with flat terms representing M. Then we see that $\epsilon^* M$ is represented by the τ-sheafification $(\mathcal{M}^\bullet)^\#$ of $\mathcal{M}^\bullet$. Let $U \in \mathrm{Ob}(\mathcal{C})$. By Leray we get

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet)^\#|_{\mathcal{C}/U}) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet|_{\mathcal{C}/U})^\#)$$

The last equality since sheafification commutes with restriction to $\mathcal{C}/U$. As usual, denote $\mathcal{O}_U$ the restriction of $\mathcal{O}$ to $\mathcal{C}/U$. Consider the map

$$\mathcal{M}^\bullet(U) \otimes_{\mathcal{O}(U)} \mathcal{O}_U \longrightarrow \mathcal{M}^\bullet|_{\mathcal{C}/U}$$

[10] This means that $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism for all $U \to V$ in $\mathcal{C}$.

of complexes of $\mathcal{O}_U$-modules (in τ'-topology). By our choice of $\mathcal{M}^\bullet$ the complex $\mathcal{M}^\bullet(U)$ is a K-flat complex of $\mathcal{O}(U)$-modules; see Lemma 21.18.1 and use that the inclusion of U into $\mathcal{C}$ defines a morphism of ringed topoi $(Sh(pt), \mathcal{O}(U)) \to (Sh(\mathcal{C}_{\tau'}), \mathcal{O})$. Since M is in $QC(\mathcal{O})$ we conclude that the displayed arrow is a quasi-isomorphism. Since sheafification is exact, we see that the same remains true after sheafification. Hence

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

and assumption (2) tells us this is equal to $R\Gamma(U, M) = \mathcal{M}^\bullet(U)$ as desired. □

0H0T **Lemma 21.43.13.** *Notation and assumptions as in Lemma 21.43.12. Suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O}_\tau)$. For every object U of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We have

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = \mathrm{Hom}_{D((\mathcal{O}_U)_{\tau'})}(K|_U, R\epsilon_* M|_U)$$

by adjunction. Hence the result by Lemma 21.43.5 and the fact that

$$R\Gamma(U, M) = R\Gamma(U, R\epsilon_* M)$$

by Leray. □

21.44. Strictly perfect complexes

08FK This section is the analogue of Cohomology, Section 20.46.

08FL **Definition 21.44.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module for all i.

Let U be an object of $\mathcal{C}$. We will often say "Let $\mathcal{E}^\bullet$ be a strictly perfect complex of $\mathcal{O}_U$-modules" to mean $\mathcal{E}^\bullet$ is a strictly perfect complex of modules on the ringed site $(\mathcal{C}/U, \mathcal{O}_U)$, see Modules on Sites, Definition 18.19.1.

08FM **Lemma 21.44.2.** *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

09J8 **Lemma 21.44.3.** *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

08H3 **Lemma 21.44.4.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{C}$-modules.*

Proof. We have seen in Modules on Sites, Lemma 18.17.2 that the pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

08FN **Lemma 21.44.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of $\mathcal{O}_U$-modules*

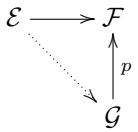

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_U$-module and p surjective, then there exists a covering $\{U_i \to U\}$ such that a dotted arrow making the diagram commute exists over each U_i.

Proof. We may assume $\mathcal{E} = \mathcal{O}_U^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(U, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 7.11). □

08FP **Lemma 21.44.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

(1) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*
(2) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}$-module and integer $n \geq a$, then the result follows from Lemma 21.44.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_U)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is homotopic to zero on the members of a covering of U by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of U by induction hypothesis. □

08FQ **Lemma 21.44.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of complexes of $\mathcal{O}_U$-modules*

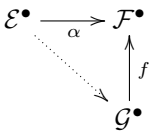

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then there exists a covering $\{U_i \to U\}$ and for each i a dotted arrow over U_i making the diagram commute up to homotopy.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 21.44.6 guarantees there is a covering $\{U_i \to U\}$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

08FR **Lemma 21.44.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

1. *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
2. *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 21.44.7. We omit the proof of (2). □

08JH **Lemma 21.44.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 21.35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 21.35. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object U of $\mathcal{C}$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{I}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{I}^\bullet|_U)$$

By Lemma 21.44.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

08JI **Lemma 21.44.10.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 21.35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.4). Hence the construction in Section 21.35 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.25.3)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

21.45. Pseudo-coherent modules

08FS In this section we discuss pseudo-coherent complexes.

08FT **Definition 21.45.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i^\bullet$ is a strictly perfect complex of $\mathcal{O}_{U_i}$-modules and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O})$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}$-modules.

If $\mathcal{C}$ has a final object X which is quasi-compact (for example if every covering of X can be refined by a finite covering), then an m-pseudo-coherent object of $D(\mathcal{O})$ is in $D^-(\mathcal{O})$. But this need not be the case in general.

08FU **Lemma 21.45.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

(1) *If $\mathcal{C}$ has a final object X and if there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
(2) *If E is m-pseudo-coherent, then any complex of $\mathcal{O}$-modules representing E is m-pseudo-coherent.*

(3) *If for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is m-pseudo-coherent, then E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let X, $\{U_i \to X\}$, and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) in case $\mathcal{C}$ has a final object. By Lemma 21.44.8 we can after refining the covering $\{U_i \to X\}$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (2). By the above we see that $\mathcal{F}^\bullet|_U$ is m-pseudo-coherent as a complex of $\mathcal{O}_U$-modules for all objects U of $\mathcal{C}$. It is a formal consequence of the definitions that $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (3). Follows from the definitions and Sites, Definition 7.6.2 part (2). □

08H4 **Lemma 21.45.3.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{C})$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Say f is given by the functor $u : \mathcal{D} \to \mathcal{C}$. Let U be an object of $\mathcal{C}$. By Sites, Lemma 7.14.10 we can find a covering $\{U_i \to U\}$ and for each i a morphism $U_i \to u(V_i)$ for some object V_i of $\mathcal{D}$. By Lemma 21.45.2 it suffices to show that $Lf^*E|_{U_i}$ is m-pseudo-coherent. To do this it is enough to show that $Lf^*E|_{u(V_i)}$ is m-pseudo-coherent, since $Lf^*E|_{U_i}$ is the restriction of $Lf^*E|_{u(V_i)}$ to $\mathcal{C}/U_i$ (via Modules on Sites, Lemma 18.19.5). By the commutative diagram of Modules on Sites, Lemma 18.20.1 it suffices to prove the lemma for the morphism of ringed sites $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Thus we may assume $\mathcal{D}$ has a final object Y such that $X = u(Y)$ is a final object of $\mathcal{C}$.

Let $\{V_i \to Y\}$ be a covering such that for each i there exists a strictly perfect complex $\mathcal{F}_i^\bullet$ of $\mathcal{O}_{V_i}$-modules and a morphism $\alpha_i : \mathcal{F}_i^\bullet \to E|_{V_i}$ of $D(\mathcal{O}_{V_i})$ such that $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective. Arguing as above it suffices to prove the result for $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Hence we may assume that there exists a strictly perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_\mathcal{D}$-modules and a morphism $\alpha : \mathcal{F}^\bullet \to E$ of $D(\mathcal{O}_\mathcal{D})$ such that $H^j(\alpha)$ is an isomorphism for $j > m$ and $H^m(\alpha)$ is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \to E \to C \to \mathcal{F}^\bullet[1]$$

The assumption on α means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Applying Lf^* we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \to Lf^*E \to Lf^*C \to Lf^*\mathcal{F}^\bullet[1]$$

By the construction of Lf^* as a left derived functor we see that $H^j(Lf^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf^*\alpha)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha)$ is surjective. On the other hand, since $\mathcal{F}^\bullet$ is a bounded above complex of flat $\mathcal{O}_\mathcal{D}$-modules we see that $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$. Applying Lemma 21.44.4 we conclude. □

08FV **Lemma 21.45.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$.*

(1) *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*

(2) *If K and M are m-pseudo-coherent, then L is m-pseudo-coherent.*
(3) *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. Choose a covering $\{U_i \to U\}$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 21.44.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{array}{ccc} K|_{U_i} & \longrightarrow & L|_{U_i} \\ \uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \beta_i} \\ \mathcal{K}_i^\bullet & \xrightarrow{\gamma_i} & \mathcal{L}_i^\bullet \end{array}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 21.44.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 21.45.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

09J9 **Lemma 21.45.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$.*

(1) *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
(2) *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ is pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. By replacing U by the members of a covering and replacing $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let U be an object of $\mathcal{C}$. We may first replace U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

08FW **Lemma 21.45.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O})$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume $K \oplus L \in D^-(\mathcal{O}_U)$, hence $L \in D^-(\mathcal{O}_U)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 21.45.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

08FX **Lemma 21.45.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $m \in \mathbf{Z}$.*

(1) *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}$-module.*
(2) *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}$-module.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. We have to show that $H^m(K)$ is can be generated by finitely many sections over the members of a covering of U (see Modules on Sites, Definition 18.23.1). Thus during the proof we may (finitely often) choose a covering $\{U_i \to U\}$ and replace $\mathcal{C}$ by $\mathcal{C}/U_i$ and U by U_i. In particular, by our definitions we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 21.44.5 we can (after replacing U by the members of a covering) find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Pick an object U of $\mathcal{C}$. As in the proof of (1) we may work locally on U. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

21.46. Tor dimension

08FY In this section we take a closer look at resolutions by flat modules.

08FZ **Definition 21.46.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.

(3) We say E *locally has finite tor dimension* if for any object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ has finite tor dimension for all i.

An $\mathcal{O}$-module $\mathcal{F}$ has *tor dimension* $\leq d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O})$ has tor-amplitude in $[-d, 0]$.

Note that if E as in the definition has finite tor dimension, then E is an object of $D^b(\mathcal{O})$ as can be seen by taking $\mathcal{F} = \mathcal{O}$ in the definition above.

08G0 **Lemma 21.46.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$. Hence for every $\mathcal{O}$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}^{\mathcal{O}}_1(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 21.17.15. □

08G1 **Lemma 21.46.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}$-modules $\mathcal{K}^\bullet$, see Section 21.17. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules on Sites, Lemma 18.28.10). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 21.46.2 and the definitions. □

0F1M **Lemma 21.46.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \in \mathbf{Z}$. The following are equivalent*

(1) *E has tor-amplitude in $[a, \infty]$.*
(2) *E can be represented by a K-flat complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, \infty]$.*

Moreover, we can choose $\mathcal{E}^\bullet$ such that any pullback by a morphism of ringed sites is a K-flat complex with flat terms.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $\mathcal{K}^\bullet$ with flat terms representing E, see Lemma 21.17.11. For any $\mathcal{O}$-module $\mathcal{M}$ the cohomology of

$$\mathcal{K}^{n-1} \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{K}^n \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{K}^{n+1} \otimes_{\mathcal{O}} \mathcal{M}$$

computes $H^n(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{M})$. This is always zero for $n < a$. Hence if we apply Lemma 21.46.2 to the complex $\ldots \to \mathcal{K}^{a-1} \to \mathcal{K}^a \to \mathcal{K}^{a+1}$ we conclude that $\mathcal{N} = \mathrm{Coker}(\mathcal{K}^{a-1} \to \mathcal{K}^a)$ is a flat $\mathcal{O}$-module. We set

$$\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet = (\ldots \to 0 \to \mathcal{N} \to \mathcal{K}^{a+1} \to \ldots)$$

The kernel $\mathcal{L}^\bullet$ of $\mathcal{K}^\bullet \to \mathcal{E}^\bullet$ is the complex

$$\mathcal{L}^\bullet = (\ldots \to \mathcal{K}^{a-1} \to \mathcal{I} \to 0 \to \ldots)$$

where $\mathcal{I} \subset \mathcal{K}^a$ is the image of $\mathcal{K}^{a-1} \to \mathcal{K}^a$. Since we have the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ we see that $\mathcal{I}$ is a flat $\mathcal{O}$-module. Thus $\mathcal{L}^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 21.17.8. It follows that $\mathcal{E}^\bullet$ is K-flat by Lemma 21.17.7.

Proof of the final assertion. Let $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ be a morphism of ringed sites. By Lemma 21.18.1 the complex $f^*\mathcal{K}^\bullet$ is K-flat with flat terms. The complex $f^*\mathcal{L}^\bullet$ is K-flat as it is a bounded above complex of flat $\mathcal{O}'$-modules. We have a short exact sequence of complexes of $\mathcal{O}'$-modules

$$0 \to f^*\mathcal{L}^\bullet \to f^*\mathcal{K}^\bullet \to f^*\mathcal{E}^\bullet \to 0$$

because the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ of flat modules pulls back to a short exact sequence. By Lemma 21.17.7. the complex $f^*\mathcal{E}^\bullet$ is K-flat and the proof is complete. □

08H5 **Lemma 21.46.5.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 21.46.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules on Sites, Lemma 18.39.1 the module $f^*\mathcal{E}^i$ are flat. Thus by Lemma 21.46.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

08G2 **Lemma 21.46.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

(1) *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
(2) *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
(3) *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

09JA **Lemma 21.46.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

08G3 **Lemma 21.46.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O})$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

0942 **Lemma 21.46.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let K be an object of $D(\mathcal{O})$.*

(1) *If $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ is bounded above, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ is uniformly bounded above for all n.*
(2) *If $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ as an object of $D(\mathcal{O}/\mathcal{I}^n)$ has tor amplitude in $[a, b]$ for all n.*

Proof. Proof of (1). Assume that $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ is bounded above, say $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}) = 0$ for $i > b$. Note that we have distinguished triangles

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1} \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^{n+1} \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1} = \left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n) = 0$ for $i > b$ for all n.

Proof of (2). Assume $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}^n$-modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of $\mathcal{O}/\mathcal{I}$-modules. Thus to prove that $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ has tor amplitude in $[a, b]$ it suffices to show $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}^n} \mathcal{G})$ is zero for $i \notin [a, b]$ for all $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$. Since

$$\left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}^n} \mathcal{G} = \left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}} \mathcal{G}$$

for every sheaf of $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$ the result follows. □

0DJJ **Lemma 21.46.10.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

(1) *If E has tor amplitude in $[a, b]$, then for every point p of the site $\mathcal{C}$ the object E_p of $D(\mathcal{O}_p)$ has tor amplitude in $[a, b]$.*
(2) *If $\mathcal{C}$ has enough points, then the converse is true.*

Proof. Proof of (1). This follows because taking stalks at p is the same as pulling back by the morphism of ringed sites $(p, \mathcal{O}_p) \to (\mathcal{C}, \mathcal{O})$ and hence we can apply Lemma 21.46.5.

Proof of (2). If $\mathcal{C}$ has enough points, then we can check vanishing of $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F})$ at stalks, see Modules on Sites, Lemma 18.14.4. Since $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F})_p = H^i(E_p \otimes^{\mathbf{L}}_{\mathcal{O}_p} \mathcal{F}_p)$ we conclude. □

21.47. Perfect complexes

08G4 In this section we discuss properties of perfect complexes on ringed sites.

08G5 **Definition 21.47.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of $D(\mathcal{O})$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}$-modules.

If $Sh(\mathcal{C})$ is quasi-compact (Sites, Section 7.17), then a perfect object of $D(\mathcal{O})$ is in $D^b(\mathcal{O})$. But this need not be the case otherwise.

08G6 **Lemma 21.47.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

(1) *If $\mathcal{C}$ has a final object X and there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
(2) *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 21.45.2. □

08G7 **Lemma 21.47.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[2-a]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Applying $- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 21.17.15). Thus there exists a covering $\{U_i \to U\}$ such that $\mathcal{E}'|_{U_i}$ is a direct summand of a finite free module by Modules on Sites, Lemma 18.29.3. Thus the complex

$$\mathcal{E}'|_{U_i} \to \mathcal{E}^{a+1}|_{U_i} \to \ldots \to \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to $E|_{U_i}$ and E is perfect. □

08G8 **Lemma 21.47.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. The following are equivalent*

(1) *E is perfect, and*
(2) *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). Let U be an object of $\mathcal{C}$. By definition there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 21.45.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 21.46.3. Thus (2) holds.

Assume (2). Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume there exist integers $a \leq b$ such that $E|_U$ has tor amplitude in $[a, b]$. Since $E|_U$ is m-pseudo-coherent for all m we conclude using Lemma 21.47.3. □

08H6 **Lemma 21.47.5.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E is perfect in $D(\mathcal{O}_\mathcal{D})$, then Lf^*E is perfect in $D(\mathcal{O}_\mathcal{C})$.*

Proof. This follows from Lemma 21.47.4, 21.46.5, and 21.45.3. □

08G9 **Lemma 21.47.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 21.47.4, 21.45.4, and 21.46.6. Second proof (sketch): Say K and L are perfect. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume that $K|_U$ and $L|_U$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing U by the members of a covering we may assume the map $K|_U \to L|_U$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 21.44.8. Then $M|_U$ is isomorphic to the cone of α which is strictly perfect by Lemma 21.44.2. □

09JB **Lemma 21.47.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If K, L are perfect objects of $D(\mathcal{O})$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$.*

Proof. Follows from Lemmas 21.47.4, 21.45.5, and 21.46.7. □

08GA **Lemma 21.47.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $K \oplus L$ is a perfect object of $D(\mathcal{O})$, then so are K and L.*

Proof. Follows from Lemmas 21.47.4, 21.45.6, and 21.46.8. □

21.48. Duals

0FPP In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O})$ has a dual if and only if it is perfect (this follows from Example 21.48.6 and Lemma 21.48.7).

0FPQ **Lemma 21.48.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category of complexes of $\mathcal{O}$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$ is a symmetric monoidal category.*

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}$ in degree 0 and zero in other degrees with obvious isomorphisms $\mathrm{Tot}(\mathbf{1} \otimes_{\mathcal{O}} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. □

0FPR **Example 21.48.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules such that for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O})$$

as in Section 21.34. Let

$$\eta : \mathcal{O} \to \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \mathcal{O}$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_{\mathcal{O}} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}} \mathcal{F}^n \to \mathcal{O}$ are as in Modules on Sites, Example 18.29.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 15.73.2.

0FPS **Lemma 21.48.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}$-modules (Categories, Definition 4.43.5) then for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect and the left dual is as constructed in Example 21.48.2.*

Proof. By uniqueness of left duals (Categories, Remark 4.43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_\mathcal{O} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_\mathcal{O} \mathcal{F}^n \to \mathcal{O}$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{\mathcal{G}^{-n}}$. In other words, we see that $\mathcal{G}^{-n}$ is a left dual of $\mathcal{F}^n$ and we see that Modules on Sites, Lemma 18.29.2 applies to each $\mathcal{F}^n$. Let U be an object of $\mathcal{C}$. There exists a covering $\{U_i \to U\}$ such that for every i only a finite number of $\eta_n|_{U_i}$ are nonzero. Thus after replacing U by U_i we may assume only a finite number of $\eta_n|_U$ are nonzero and by the lemma cited this implies only a finite number of $\mathcal{F}^n|_U$ are nonzero. Using the lemma again we can then find a covering $\{U_i \to U\}$ such that each $\mathcal{F}^n|_{U_i}$ is a direct summand of a finite free $\mathcal{O}$-module and the proof is complete. □

08JJ **Lemma 21.48.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms*

$$M \otimes^{\mathbf{L}}_\mathcal{O} K^\vee = R\mathcal{H}om_\mathcal{O}(K, M)$$

and

$$H^0(\mathcal{C}, M \otimes^{\mathbf{L}}_\mathcal{O} K^\vee) = \mathrm{Hom}_{D(\mathcal{O})}(K, M)$$

for M in $D(\mathcal{O})$.

Proof. We will us without further mention that formation of internal hom commutes with restriction (Lemma 21.35.3). Let U be an arbitrary object of $\mathcal{C}$. To check that $K^\vee$ is perfect, it suffices to show that there exists a covering $\{U_i \to U\}$ such that $K^\vee|_{U_i}$ is perfect for all i. There is a canonical map

$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$

see Lemma 21.35.5. It suffices to prove there is a covering $\{U_i \to U\}$ such that the restriction of this map to $\mathcal{C}/U_i$ is an isomorphism for all i. By Lemma 21.35.9 to see the final statement it suffices to check that the map (21.35.9.1)

$$M \otimes^{\mathbf{L}}_\mathcal{O} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is a local question as well (in the sense above). Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 21.44.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^n)^\vee = \mathcal{H}om_\mathcal{O}(\mathcal{E}^n, \mathcal{O})$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}$-module, so is $(\mathcal{E}^n)^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. The map $K \to (K^\vee)^\vee$ is an isomorphism as it is given up to sign by the evaluation maps $\mathcal{E}^n \to ((\mathcal{E}^n)^\vee)^\vee$ which are isomorphisms. To see that (21.35.9.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 21.44.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $M \otimes^{\mathbf{L}}_\mathcal{O} K^\vee$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_\mathcal{O} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (21.35.9.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module and $\mathcal{F}$ is any $\mathcal{O}$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

0FPT **Lemma 21.48.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The derived category $D(\mathcal{O})$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 15.73).*

Proof. Omitted. Compare with Lemma 21.48.1. □

0FPU **Example 21.48.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ as in Lemma 21.48.4. Then the map

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O} \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee$$

the map sending 1 to the section corresponding to id_K under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}} K \longrightarrow \mathcal{O}$$

the evaluation map (to construct it you can use Lemma 21.35.6 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for K as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{K^\vee}$.

0FPV **Lemma 21.48.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. If M has a left dual in the monoidal category $D(\mathcal{O})$ (Categories, Definition 4.43.5) then M is perfect and the left dual is as constructed in Example 21.48.6.*

Proof. Let N, η, ϵ be a left dual. Observe that for any object U of $\mathcal{C}$ the restriction $N|_U, \eta|_U, \epsilon|_U$ is a left dual for $M|_U$.

Let U be an object of $\mathcal{C}$. It suffices to find a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that $M|_{U_i}$ is a perfect object of $D(\mathcal{O}_{U_i})$. Hence we may replace $\mathcal{C}, \mathcal{O}, M, N, \eta, \epsilon$ by $\mathcal{C}/U, \mathcal{O}_U, M|_U, N|_U, \eta|_U, \epsilon|_U$ and assume $\mathcal{C}$ has a final object X. Moreover, during the proof we can (finitely often) replace X by the members of a covering $\{U_i \to X\}$ of X.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules representing M. Choose a K-flat complex $\mathcal{N}^\bullet$ representing N whose terms are flat $\mathcal{O}$-modules, see Lemma 21.17.11. Consider the map

$$\eta : \mathcal{O} \to \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)$$

After replacing X by the members of a covering, we can find an integer N and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections f_i and g_i of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum\nolimits_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}$-modules containing the sections f_i for $i = 1, \ldots, N$. Since $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet) \subset \mathrm{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that η factors through

$$\tilde{\eta} : \mathcal{O} \to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet)$$

Denoting K the object of $D(\mathcal{O})$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\eta\otimes 1} & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1\otimes\epsilon} & M \\ & \searrow^{\tilde{\eta}\otimes 1} & \uparrow & & \uparrow \\ & & K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1\otimes\epsilon} & K \end{array}$$

Since the composition of the upper row is the identity on M we conclude that M is a direct summand of K in $D(\mathcal{O})$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a}\tau_{\leq b}\mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus M is a direct summand in $D(\mathcal{O})$ of a bounded complex and we conclude we may assume M is in $D^b(\mathcal{O})$. (Recall that the process above involves replacing X by the members of a covering.)

Since M is in $D^b(\mathcal{O})$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.17.7 and Derived Categories, Lemma 13.15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a}\mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume M is a direct summand in $D(\mathcal{O})$ of a bounded complex of flat modules. In particular, we find M has finite tor amplitude.

Say M has tor amplitude in $[a, b]$. Assuming M is m-pseudo-coherent we are going to show that (after replacing X by the members of a covering) we may assume M is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 21.47.3 and the fact that M is $(b+1)$-pseudo-coherent in any case. After replacing X by the members of a covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O})$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent L by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 13.27.3. Then M is represented by the complex

$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections f_i of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}$-submodule generated by the sections f_i for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathcal{E}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & \longrightarrow & \sigma_{\leq m-1}\mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \end{array}$$

where all the morphisms are as indicated above. Denote K the object of $D(\mathcal{O})$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O})$ such that the composition $M \to K \to M$ is the identity on M. We don't know that the diagram

$$\begin{array}{ccccc} \mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & = & K \\ {\scriptstyle \mathrm{id}}\uparrow & & & & \uparrow{\scriptstyle s} \\ \mathcal{E}^\bullet & \xrightarrow{i} & \mathcal{M}^\bullet & = & M \end{array}$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 21.44.8 after replacing X by the members of a covering, we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of σ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that σ itself is homotopic to zero! This means that we have a commutative solid diagram

$$\begin{array}{ccccccc} \mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{E}^\bullet & \longrightarrow & K & \longrightarrow & \sigma_{\leq m-1}\mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\ \uparrow & & \uparrow{\scriptstyle s} & & \uparrow & & \uparrow \\ \mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \end{array}$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m-1$ we see that we obtain

$$\begin{array}{ccccc} H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ H^{m-1}(K) & \longrightarrow & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \end{array}$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m-1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m-1$ and the proof is complete. $\square$

0A0A **Lemma 21.48.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O})$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(\mathcal{O})$ we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 21.48.4 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (21.35.0.1) and the fact that derived tensor product commutes with direct sums. □

21.49. Invertible objects in the derived category

0FPW We characterize invertible objects in the derived category of a ringed site (both in the case of a locally ringed topos and in the general case).

0FPX **Lemma 21.49.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. The category of $\mathcal{O}$-modules which are summands of finite free $\mathcal{O}$-modules is equivalent to the category of finite projective R-modules.*

Proof. Observe that a finite projective R-module is the same thing as a summand of a finite free R-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(\mathcal{C}, \mathcal{E})$. The inverse functor is given by the following construction. Consider the morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathrm{pt})$ with f_* given by taking global sections and f^{-1} by sending a set S, i.e., an object of $Sh(\mathrm{pt})$, to the constant sheaf with value S. We obtain a morphism $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathrm{pt}), R)$ of ringed topoi by using the identity map $R \to f_*\mathcal{O}$. Then the inverse functor is given by f^*. □

0FPY **Lemma 21.49.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. The following are equivalent*

1. *M is invertible in $D(\mathcal{O})$, see Categories, Definition 4.43.4, and*
2. *there is a locally finite[11] direct product decomposition*

$$\mathcal{O} = \prod\nolimits_{n\in\mathbf{Z}} \mathcal{O}_n$$

and for each n there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules on Sites, Definition 18.32.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O})$.

If (1) and (2) hold, then M is a perfect object of $D(\mathcal{O})$. If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site these condition are also equivalent to

3. *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i an integer n_i such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree n_i.*

[11] This means that for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}$ such that for every i the sheaf $\mathcal{O}_n|_{U_i}$ is nonzero for only a finite number of n.

Proof. Assume (2). Consider the object $R\mathcal{H}om(M, \mathcal{O})$ and the composition map

$$R\mathcal{H}om(M, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} M \to \mathcal{O}$$

To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O} = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that

$$R\mathcal{H}om(\mathcal{H}^m, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{H}^n$$

is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}$-modules (Modules on Sites, Lemma 18.32.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O})$ is symmetric monoidal, we conclude that M is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O} = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}$ coming from the product decomposition. This shows that in order to prove (2) we may work locally in the following sense. Let U be an object of $\mathcal{C}$. We have to show there exists a covering $\{U_i \to U\}$ of U such that with $\mathcal{O}_n$ as above we have the statements above and those of (2) after restriction to $\mathcal{C}/U_i$. Thus we may assume $\mathcal{C}$ has a final object X and during the proof of (2) we may finitely many times replace X by the members of a covering of X.

Choose an object N of $D(\mathcal{O})$ and an isomorphism $M \otimes^{\mathbf{L}}_{\mathcal{O}} N \cong \mathcal{O}$. Then N is a left dual for M in the monoidal category $D(\mathcal{O})$ and we conclude that M is perfect by Lemma 21.48.7. By symmetry we see that N is perfect. After replacing X by the members of a covering, we may assume M and N are represented by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes^{\mathbf{L}}_{\mathcal{O}} N$ is represented by $\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$. After replacing X by the members of a covering of X we may assume the mutually inverse isomorphisms $\mathcal{O} \to M \otimes^{\mathbf{L}}_{\mathcal{O}} N$ and $M \otimes^{\mathbf{L}}_{\mathcal{O}} N \to \mathcal{O}$ are given by maps of complexes

$$\alpha : \mathcal{O} \to \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \mathcal{O}$$

See Lemma 21.44.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O})$. After replacing X by the members of a covering of X we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy θ with components

$$\theta^n : \mathrm{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \mathrm{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. By Lemma 21.49.1 we find that we obtain

(1) $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective R-modules,
(2) $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective R-modules,
(3) α and β correspond to maps of complexes $a : R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \mathrm{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
(4) θ^n corresponds to a map $h^n : \mathrm{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \mathrm{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
(5) $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 15.127.4 we find a product decomposition $R = \prod_{a \leq n \leq b} R_n$ and invertible R_n-modules H^n such that $M^\bullet \cong \bigoplus_{a \leq n \leq b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each H^n is projective as an R-module. Correspondingly, using Lemma 21.49.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O})$. Here $M \otimes_R \mathcal{O}$ denotes the functor from finite projective R-modules to $\mathcal{O}$-modules constructed in the proof of Lemma 21.49.1. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}$ we conclude (2) is true.

If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then given an object U and a finite product decomposition $\mathcal{O}|_U = \prod_{a \leq n \leq b} \mathcal{O}_n|_U$ we can find a covering $\{U_i \to U\}$ such that for every i there is at most one n with $\mathcal{O}_n|_{U_i}$ nonzero. This follows readily from part (2) of Modules on Sites, Lemma 18.40.1 and the definition of locally ringed sites as given in Modules on Sites, Definition 18.40.4. From this the implication (2) $\Rightarrow$ (3) is easily seen. The implication (3) $\Rightarrow$ (2) holds without any assumptions on the ringed site. We omit the details. □

21.50. Projection formula

0943 Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. Without any further assumptions there is a map

0B56 (21.50.0.1) $$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$ and Lemmas 21.18.4 and 21.19.1. A reasonably general version of the projection formula is the following.

0944 **Lemma 21.50.1.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K)$$

in $D(\mathcal{O}_\mathcal{D})$.

Proof. To check (21.50.0.1) is an isomorphism we may work locally on $\mathcal{D}$, i.e., for any object V of $\mathcal{D}$ we have to find a covering $\{V_j \to V\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_\mathcal{D}$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_\mathcal{D}$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_\mathcal{D}[n]$ in which case it is trivial. □

0E48 **Remark 21.50.2.** The map (21.50.0.1) is compatible with the base change map of Remark 21.19.3 in the following sense. Namely, suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[g']{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let $E \in D(\mathcal{O}_{\mathcal{C}})$ and $K \in D(\mathcal{O}_{\mathcal{D}})$. Then the diagram

$$\begin{array}{ccc} Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}}} K) & \xrightarrow[p]{} & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}}} Lf^*K) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle b} \\ Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K & & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}}} Lf^*K) \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle t} \\ Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K & & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(g')^*Lf^*K) \\ & \searrow^{p} & \downarrow{\scriptstyle c} \\ & & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(f')^*Lg^*K) \end{array}$$

is commutative. Here arrows labeled t are gotten by an application of Lemma 21.18.4, arrows labeled b by an application of Remark 21.19.3, arrows labeled p by an application of (21.50.0.1), and c comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

21.51. Weakly contractible objects

0945 An object U of a site is *weakly contractible* if every surjection $\mathcal{F} \to \mathcal{G}$ of sheaves of sets gives rise to a surjection $\mathcal{F}(U) \to \mathcal{G}(U)$, see Sites, Definition 7.40.2.

0946 **Lemma 21.51.1.** *Let $\mathcal{C}$ be a site. Let U be a weakly contractible object of $\mathcal{C}$. Then*

(1) *the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor $Ab(\mathcal{C}) \to Ab$,*
(2) *$H^p(U, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ and all $p \geq 1$, and*
(3) *for any sheaf of groups $\mathcal{G}$ any $\mathcal{G}$-torsor has a section over U.*

Proof. The first statement follows immediately from the definition (see also Homology, Section 12.7). The higher derived functors vanish by Derived Categories, Lemma 13.16.9. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F} \to *$ is a surjective map of sheaves. Hence (3) follows from the definition as well. □

It is convenient to list some consequences of having enough weakly contractible objects here.

0947 **Proposition 21.51.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Then*

(1) *A complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules is exact, if and only if $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$.*

(2) *Every object K of $D(\mathcal{O})$ is a derived limit of its canonical truncations: $K = R\lim \tau_{\geq -n} K$.*
(3) *Given an inverse system $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ with surjective transition maps, the projection $\lim \mathcal{F}_n \to \mathcal{F}_1$ is surjective.*
(4) *Products are exact on $Mod(\mathcal{O})$.*
(5) *Products on $D(\mathcal{O})$ can be computed by taking products of any representative complexes.*
(6) *If $(\mathcal{F}_n)$ is an inverse system of $\mathcal{O}$-modules, then $R^p \lim \mathcal{F}_n = 0$ for all $p > 1$ and*
$$R^1 \lim \mathcal{F}_n = \mathrm{Coker}(\prod \mathcal{F}_n \to \prod \mathcal{F}_n)$$
where the map is $(x_n) \mapsto (x_n - f(x_{n+1}))$.
(7) *If (K_n) is an inverse system of objects of $D(\mathcal{O})$, then there are short exact sequences*
$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R \lim K_n) \to \lim H^p(K_n) \to 0$$

Proof. Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of $\mathcal{C}$ gives an exact sequence by Lemma 21.51.1. Conversely, assume that $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$. Let V be an object of $\mathcal{C}$ and let $s \in \mathcal{F}_2(V)$ be an element of the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By assumption there exists a covering $\{U_i \to V\}$ with $U_i \in \mathcal{B}$. Then $s|_{U_i}$ lifts to a section $s_i \in \mathcal{F}_1(U_i)$. Thus s is a section of the image sheaf $\mathrm{Im}(\mathcal{F}_1 \to \mathcal{F}_2)$. In other words, the sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact.

Proof of (2). This holds by Lemma 21.23.10 with $d = 0$.

Proof of (3). Let $(\mathcal{F}_n)$ be a system as in (2) and set $\mathcal{F} = \lim \mathcal{F}_n$. If $U \in \mathcal{B}$, then $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$ surjects onto $\mathcal{F}_1(U)$ as all the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective. Thus $\mathcal{F} \to \mathcal{F}_1$ is surjective by Sites, Definition 7.11.1 and the assumption that every object has a covering by elements of $\mathcal{B}$.

Proof of (4). Let $\mathcal{F}_{i,1} \to \mathcal{F}_{i,2} \to \mathcal{F}_{i,3}$ be a family of exact sequences of $\mathcal{O}$-modules. We want to show that $\prod \mathcal{F}_{i,1} \to \prod \mathcal{F}_{i,2} \to \prod \mathcal{F}_{i,3}$ is exact. We use the criterion of (1). Let $U \in \mathcal{B}$. Then
$$(\prod \mathcal{F}_{i,1})(U) \to (\prod \mathcal{F}_{i,2})(U) \to (\prod \mathcal{F}_{i,3})(U)$$
is the same as
$$\prod \mathcal{F}_{i,1}(U) \to \prod \mathcal{F}_{i,2}(U) \to \prod \mathcal{F}_{i,3}(U)$$
Each of the sequences $\mathcal{F}_{i,1}(U) \to \mathcal{F}_{i,2}(U) \to \mathcal{F}_{i,3}(U)$ are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 12.32.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 13.34.2.

Proof of (6) and (7). We refer to Section 21.23 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 13.34.1 we have a distinguished triangle
$$R \lim K_n \to \prod K_n \to \prod K_n \to R \lim K_n[1]$$

Taking the long exact sequence of cohomology sheaves we obtain

$$H^{p-1}(\prod K_n) \to H^{p-1}(\prod K_n) \to H^p(R\lim K_n) \to H^p(\prod K_n) \to H^p(\prod K_n)$$

Since products are exact by (4) this becomes

$$\prod H^{p-1}(K_n) \to \prod H^{p-1}(K_n) \to H^p(R\lim K_n) \to \prod H^p(K_n) \to \prod H^p(K_n)$$

Now we first apply this to the case $K_n = \mathcal{F}_n[0]$ where $(\mathcal{F}_n)$ is as in (6). We conclude that (6) holds. Next we apply it to (K_n) as in (7) and we conclude (7) holds. □

21.52. Compact objects

0948 In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 13.37.1.

094B **Lemma 21.52.1.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $S \subset \mathrm{Ob}(\mathcal{A})$ be a set of objects such that*

(1) *any object of $\mathcal{A}$ is a quotient of a direct sum of elements of S, and*
(2) *for any $E \in S$ the functor $\mathrm{Hom}_{\mathcal{A}}(E, -)$ commutes with direct sums.*

Then every compact object of $D(\mathcal{A})$ is a direct summand in $D(\mathcal{A})$ of a finite complex of finite direct sums of elements of S.

Proof. Assume $K \in D(\mathcal{A})$ is a compact object. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

We may represent K by a bounded above complex $K^\bullet$ each of whose terms is a direct sum of objects from S, see Derived Categories, Lemma 13.15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.15. Hence by Derived Categories, Lemmas 13.33.7 and 13.33.9 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(\mathcal{A})$ for some complex $L^\bullet$ which is bounded and whose terms are direct sums of elements of S. Say L^i is zero for $i \notin [a, b]$. Let c be the largest integer $\leq b + 1$ such that L^i a finite direct sum of elements of S for $i < c$. Claim: if $c < b + 1$, then we can modify $L^\bullet$ to increase c. By induction this claim will show we have a factorization of 1_K as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(\mathcal{A})$ where L can be represented by a finite complex of finite direct sums of elements of S. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(\mathcal{A})}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.14 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$ and we conclude.

Proof of the claim. Write $L^c = \bigoplus_{\lambda\in\Lambda} E_\lambda$. Since L^{c-1} is a finite direct sum of elements of S we can by assumption (2) find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda\in\Lambda'} E_\lambda \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda\in\Lambda\setminus\Lambda'} E_\lambda)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(\mathcal{A})$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(\mathcal{A})$ (see Derived Categories, Lemma 13.12.1). Since $\mathrm{Hom}_{D(\mathcal{A})}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(\mathcal{A})$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda\in\Lambda'} E_\lambda$ the claim is proved. □

094C **Lemma 21.52.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume every object of $\mathcal{C}$ has a covering by quasi-compact objects. Then every compact object of $D(\mathcal{O})$ is a direct summand in $D(\mathcal{O})$ of a finite complex whose terms are finite direct sums of $\mathcal{O}$-modules of the form $j_!\mathcal{O}_U$ where U is a quasi-compact object of $\mathcal{C}$.*

Proof. Apply Lemma 21.52.1 where $S \subset \mathrm{Ob}(\mathit{Mod}(\mathcal{O}))$ is the set of modules of the form $j_!\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 18.28.8 and the assumption that every U can be covered by quasi-compact objects. Assumption (2) follows as

$$\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 7.17.7. □

In the situation of the lemma above it is not always true that the modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O})$ (even if U is a quasi-compact object of $\mathcal{C}$). Here are two lemmas addressing this issue.

0G21 **Lemma 21.52.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Assume the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums. Then $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i\in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\mathrm{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i\in I} \mathrm{Hom}(j_{U!}\mathcal{O}_U, M_i)$.*

Proof. Since $\mathrm{Hom}(j_{U!}\mathcal{O}_U, -)$ is the same as the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ by Modules on Sites, Equation (18.19.2.1) it suffices to prove that $H^p(U, M) = \bigoplus H^p(U, M_i)$. Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}$-modules. By assumption we have

$$H^p(U, \bigoplus\nolimits_{i\in I} \mathcal{I}_i) = \bigoplus\nolimits_{i\in I} H^p(U, \mathcal{I}_i) = 0$$

for all p. Since $M = \bigoplus M_i$ is bounded below, we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}$-modues $\mathcal{I}_i^\bullet$ representing M_i with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 13.18.3. By Injectives, Lemma 19.13.4 we see that the direct sum complex $\bigoplus \mathcal{I}_i^\bullet$ represents M. By Leray acyclicity (Derived Categories, Lemma 13.16.7) we see that

$$R\Gamma(U, M) = \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(U, M_i)$$

as desired. □

0G22 **Lemma 21.52.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with set of coverings $Cov_{\mathcal{C}}$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Assume that*

(1) *For every $\mathcal{U} \in Cov$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then for $U \in \mathcal{B}$ the object $j_{U!}\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\mathrm{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i \in I} \mathrm{Hom}(j_{U!}\mathcal{O}_U, M_i)$.

Proof. This follows from Lemma 21.52.3 and Lemma 21.16.1. □

094D **Lemma 21.52.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$ if there exists an integer d such that*

(1) *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
(2) *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

Proof. Assume (1) and (2). Recall that $\mathrm{Hom}(j_!\mathcal{O}_U, K) = R\Gamma(U, K)$ for K in $D(\mathcal{O})$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second assumption implies any direct sum of injective modules is acyclic for F. Let K_i be a family of objects of $D(\mathcal{O})$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing K_i, see Injectives, Theorem 19.12.6. Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 13.32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 19.13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. □

094E **Lemma 21.52.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$ which is quasi-compact and weakly contractible. Then $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$.*

Proof. Combine Lemmas 21.52.5 and 21.51.1 with Modules on Sites, Lemma 18.30.3. □

09JC **Lemma 21.52.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume $\mathcal{C}$ has the following properties*

(1) *$\mathcal{C}$ has a quasi-compact final object X,*
(2) *every quasi-compact object of $\mathcal{C}$ has a cofinal system of coverings which are finite and consist of quasi-compact objects,*
(3) *for a finite covering $\{U_i \to U\}_{i \in I}$ with U, U_i quasi-compact the fibre products $U_i \times_U U_j$ are quasi-compact.*

Let K be a perfect object of $D(\mathcal{O})$. Then

(a) *K is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\mathrm{Hom}(K, M) = \bigoplus_{i \in I} \mathrm{Hom}(K, M_i)$.*
(b) *If $(\mathcal{C}, \mathcal{O})$ has finite cohomological dimension, i.e., if there exists a d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$ for any $\mathcal{O}$-module $\mathcal{F}$, then K is a compact object of $D(\mathcal{O})$.*

Proof. Let $K^\vee$ be the dual of K, see Lemma 21.48.4. Then we have

$$\mathrm{Hom}_{D(\mathcal{O})}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}} M)$$

functorially in M in $D(\mathcal{O})$. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (a). After reformulation as above this is a special case of Lemma 21.52.4 with $U = X$.

Proof of (b). Since $R\Gamma(X, K) = R\operatorname{Hom}(\mathcal{O}, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 21.16.1 this is a special case of Lemma 21.52.5. □

21.53. Complexes with locally constant cohomology sheaves

094F Locally constant sheaves are introduced in Modules on Sites, Section 18.43. Let $\mathcal{C}$ be a site. Let Λ be a ring. We denote $D(\mathcal{C}, \Lambda)$ the derived category of the abelian category of $\underline{\Lambda}$-modules on $\mathcal{C}$.

094G **Lemma 21.53.1.** *Let $\mathcal{C}$ be a site with final object X. Let Λ be a Noetherian ring. Let $K \in D^b(\mathcal{C}, \Lambda)$ with $H^i(K)$ locally constant sheaves of Λ-modules of finite type. Then there exists a covering $\{U_i \to X\}$ such that each $K|_{U_i}$ is represented by a complex of locally constant sheaves of Λ-modules of finite type.*

Proof. Let $a \leq b$ be such that $H^i(K) = 0$ for $i \notin [a, b]$. By induction on $b - a$ we will prove there exists a covering $\{U_i \to X\}$ such that $K|_{U_i}$ can be represented by a complex $\underline{M^\bullet}_{U_i}$ with M^p a finite type Λ-module and $M^p = 0$ for $p \notin [a, b]$. If $b = a$, then this is clear. In general, we may replace X by the members of a covering and assume that $H^b(K)$ is constant, say $H^b(K) = \underline{M}$. By Modules on Sites, Lemma 18.42.5 the module M is a finite Λ-module. Choose a surjection $\Lambda^{\oplus r} \to M$ given by generators $x_1, \ldots, x_r$ of M.

By a slight generalization of Lemma 21.7.3 (details omitted) there exists a covering $\{U_i \to X\}$ such that $x_i \in H^0(X, H^b(K))$ lifts to an element of $H^b(U_i, K)$. Thus, after replacing X by the U_i we reach the situation where there is a map $\underline{\Lambda^{\oplus r}}[-b] \to K$ inducing a surjection on cohomology sheaves in degree b. Choose a distinguished triangle

$$\underline{\Lambda^{\oplus r}}[-b] \to K \to L \to \underline{\Lambda^{\oplus r}}[-b+1]$$

Now the cohomology sheaves of L are nonzero only in the interval $[a, b-1]$, agree with the cohomology sheaves of K in the interval $[a, b-2]$ and there is a short exact sequence

$$0 \to H^{b-1}(K) \to H^{b-1}(L) \to \underline{\operatorname{Ker}(\Lambda^{\oplus r} \to M)} \to 0$$

in degree $b-1$. By Modules on Sites, Lemma 18.43.5 we see that $H^{b-1}(L)$ is locally constant of finite type. By induction hypothesis we obtain an isomorphism $\underline{M^\bullet} \to L$ in $D(\mathcal{C}, \underline{\Lambda})$ with M^p a finite Λ-module and $M^p = 0$ for $p \notin [a, b-1]$. The map $L \to \Lambda^{\oplus r}[-b+1]$ gives a map $\underline{M^{b-1}} \to \underline{\Lambda^{\oplus r}}$ which locally is constant (Modules on Sites, Lemma 18.43.3). Thus we may assume it is given by a map $M^{b-1} \to \Lambda^{\oplus r}$. The distinguished triangle shows that the composition $M^{b-2} \to M^{b-1} \to \Lambda^{\oplus r}$ is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M^a \to \ldots \to M^{b-1} \to \Lambda^{\oplus r}} \longrightarrow K$$

in $D(\mathcal{C}, \Lambda)$. □

Let $\mathcal{C}$ be a site. Let Λ be a ring. Using the morphism $Sh(\mathcal{C}) \to Sh(pt)$ we see that there is a functor $D(\Lambda) \to D(\mathcal{C}, \Lambda)$, $K \mapsto \underline{K}$.

09BD **Lemma 21.53.2.** *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let*

(1) K *a perfect object of* $D(\Lambda)$,
(2) *a finite complex* $K^\bullet$ *of finite projective* Λ*-modules representing* K,
(3) $\mathcal{L}^\bullet$ *a complex of sheaves of* Λ*-modules, and*
(4) $\varphi : \underline{K} \to \mathcal{L}^\bullet$ *a map in* $D(\mathcal{C}, \Lambda)$.

Then there exists a covering $\{U_i \to X\}$ *and maps of complexes* $\alpha_i : \underline{K^\bullet}|_{U_i} \to \mathcal{L}^\bullet|_{U_i}$ *representing* $\varphi|_{U_i}$.

Proof. Follows immediately from Lemma 21.44.8. □

09BE **Lemma 21.53.3.** *Let* $\mathcal{C}$ *be a site with final object* X. *Let* Λ *be a ring. Let* K, L *be objects of* $D(\Lambda)$ *with* K *perfect. Let* $\varphi : \underline{K} \to \underline{L}$ *be map in* $D(\mathcal{C}, \Lambda)$. *There exists a covering* $\{U_i \to X\}$ *such that* $\varphi|_{U_i}$ *is equal to* $\underline{\alpha_i}$ *for some map* $\alpha_i : K \to L$ *in* $D(\Lambda)$.

Proof. Follows from Lemma 21.53.2 and Modules on Sites, Lemma 18.43.3. □

094H **Lemma 21.53.4.** *Let* $\mathcal{C}$ *be a site. Let* Λ *be a Noetherian ring. Let* $K, L \in D^-(\mathcal{C}, \Lambda)$. *If the cohomology sheaves of* K *and* L *are locally constant sheaves of* Λ*-modules of finite type, then the cohomology sheaves of* $K \otimes^{\mathbf{L}}_\Lambda L$ *are locally constant sheaves of* Λ*-modules of finite type.*

Proof. We'll prove this as an application of Lemma 21.53.1. Note that $H^i(K \otimes^{\mathbf{L}}_\Lambda L)$ is the same as $H^i(\tau_{\geq i-1}K \otimes^{\mathbf{L}}_\Lambda \tau_{\geq i-1}L)$. Thus we may assume K and L are bounded. By Lemma 21.53.1 we may assume that K and L are represented by complexes of locally constant sheaves of Λ-modules of finite type. Then we can replace these complexes by bounded above complexes of finite free Λ-modules. In this case the result is clear. □

094I **Lemma 21.53.5.** *Let* $\mathcal{C}$ *be a site. Let* Λ *be a Noetherian ring. Let* $I \subset \Lambda$ *be an ideal. Let* $K \in D^-(\mathcal{C}, \Lambda)$. *If the cohomology sheaves of* $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I}$ *are locally constant sheaves of* Λ/I*-modules of finite type, then the cohomology sheaves of* $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ *are locally constant sheaves of* Λ/I^n*-modules of finite type for all* $n \geq 1$.

Proof. Recall that the locally constant sheaves of Λ-modules of finite type form a weak Serre subcategory of all $\underline{\Lambda}$-modules, see Modules on Sites, Lemma 18.43.5. Thus the subcategory of $D(\mathcal{C}, \Lambda)$ consisting of complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type forms a strictly full, saturated triangulated subcategory of $D(\mathcal{C}, \Lambda)$, see Derived Categories, Lemma 13.17.1. Next, consider the distinguished triangles

$$K \otimes^{\mathbf{L}}_\Lambda \underline{I^n/I^{n+1}} \to K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^{n+1}} \to K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n} \to K \otimes^{\mathbf{L}}_\Lambda \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes^{\mathbf{L}}_\Lambda \underline{I^n/I^{n+1}} = \left(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I}\right) \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

Combined with Lemma 21.53.4 we obtain the result. □

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CHAPTER 22

Differential Graded Algebra

09JD

22.1. Introduction

09JE In this chapter we talk about differential graded algebras, modules, categories, etc. A basic reference is [**Kel94**]. A survey paper is [**Kel06**].

Since we do not worry about length of exposition in the Stacks project we first develop the material in the setting of categories of differential graded modules. After that we redo the constructions in the setting of differential graded modules over differential graded categories.

22.2. Conventions

09JF In this chapter we hold on to the convention that *ring* means commutative ring with 1. If R is a ring, then an *R-algebra* A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has a unit. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

Sign rules. In this chapter we will work with graded algebras and graded modules often equipped with differentials. The sign rules on underlying complexes will always be (compatible with) those introduced in More on Algebra, Section 15.73. This will occasionally cause the multiplicative structure to be twisted in unexpected ways especially when considering left modules or the relationship between left and right modules.

22.3. Differential graded algebras

061U Just the definitions.

061V **Definition 22.3.1.** Let R be a commutative ring. A *differential graded algebra over R* is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}_{n+m}(ab) = \mathrm{d}_n(a)b + (-1)^n a\mathrm{d}_m(b)$$

and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$

and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "Let (A, d) be a differential graded algebra".

The Leibniz rule relating differentials and multiplication on a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes

$$\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) \to A^\bullet$$

Here $A^\bullet$ denote the underlying cochain complex of A.

061X **Definition 22.3.2.** A *homomorphism of differential graded algebras* $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ is an algebra map $f : A \to B$ compatible with the gradings and d.

061W **Definition 22.3.3.** A differential graded algebra (A, d) is *commutative* if $ab = (-1)^{nm}ba$ for a in degree n and b in degree m. We say A is *strictly commutative* if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps "correct" only when tensoring commutative differential graded algebras.

065W **Definition 22.3.4.** Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product differential graded algebra* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

endowed with differential d defined by the rule $\mathrm{d}(a \otimes b) = \mathrm{d}(a) \otimes b + (-1)^m a \otimes \mathrm{d}(b)$ where $m = \deg(a)$.

065X **Lemma 22.3.5.** *Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have*

$$(A \otimes_R B)^\bullet = \mathit{Tot}(A^\bullet \otimes_R B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $\mathrm{d}_1^{p,q} + (-1)^p \mathrm{d}_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 22.3.4. □

22.4. Differential graded modules

09JH Our default in this chapter is right modules; we discuss left modules in Section 22.11.

09JI **Definition 22.4.1.** Let R be a ring. Let (A, d) be a differential graded algebra over R. A (right) *differential graded module* M over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(ma) = \mathrm{d}(m)a + (-1)^n m\mathrm{d}(a)$$

for $a \in A$ and $m \in M^n$. A *homomorphism of differential graded modules* $f : M \to N$ is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Note that we can think of M as a cochain complex $M^\bullet$ of (right) R-modules. Namely, for $r \in R$ we have $\mathrm{d}(r) = 0$ and r maps to a degree 0 element of A, hence $\mathrm{d}(mr) = \mathrm{d}(m)r$.

The Leibniz rule relating differentials and multiplication on a differential graded R-module M over a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes

$$\mathrm{Tot}(M^\bullet \otimes_R A^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the underlying cochain complexes of A and M.

09JJ **Lemma 22.4.2.** *Let (A, d) be a differential graded algebra. The category $Mod_{(A,d)}$ is abelian and has arbitrary limits and colimits.*

Proof. Kernels and cokernels commute with taking underlying A-modules. Similarly for direct sums and colimits. In other words, these operations in $\mathrm{Mod}_{(A,\mathrm{d})}$ commute with the forgetful functor to the category of A-modules. This is not the case for products and limits. Namely, if N_i, $i \in I$ is a family of differential graded A-modules, then the product $\prod N_i$ in $\mathrm{Mod}_{(A,\mathrm{d})}$ is given by setting $(\prod N_i)^n = \prod N_i^n$ and $\prod N_i = \bigoplus_n (\prod N_i)^n$. Thus we see that the product does commute with the forgetful functor to the category of graded A-modules. A category with products and equalizers has limits, see Categories, Lemma 4.14.11. □

Thus, if (A, d) is a differential graded algebra over R, then there is an exact functor

$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Comp}(R)$$

of abelian categories. For a differential graded module M the cohomology groups $H^n(M)$ are defined as the cohomology of the corresponding complex of R-modules. Therefore, a short exact sequence $0 \to K \to L \to M \to 0$ of differential graded modules gives rise to a long exact sequence

09JK (22.4.2.1) $$H^n(K) \to H^n(L) \to H^n(M) \to H^{n+1}(K)$$

of cohomology modules, see Homology, Lemma 12.13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded A-module M is *acyclic* if $H^k(M) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $M \to N$ of differential graded A-modules is a *quasi-isomorphism* if it induces isomorphisms $H^k(M) \to H^k(N)$ for all $k \in \mathbf{Z}$. And so on and so forth.

09JL **Definition 22.4.3.** Let (A, d) be a differential graded algebra. Let M be a differential graded module whose underlying complex of R-modules is $M^\bullet$. For any $k \in \mathbf{Z}$ we define the *k-shifted module* $M[k]$ as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$, i.e., we have $M[k]^n = M^{n+k}$ and $\mathrm{d}_{M[k]} = (-1)^k \mathrm{d}_M$ and
(2) as A-module the multiplication

$$(M[k])^n \times A^m \longrightarrow (M[k])^{n+m}$$

is equal to the given multiplication $M^{n+k} \times A^m \to M^{n+k+m}$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \mathrm{Mod}_{(A,\mathrm{d})} \to \mathrm{Mod}_{(A,\mathrm{d})}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $x \in M[k]^n = M^{n+k}$ and $a \in A^m$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$\begin{aligned} \mathrm{d}_{M[k]}(x \cdot_{M[k]} a) &= (-1)^k \mathrm{d}_M(xa) \\ &= (-1)^k \mathrm{d}_M(x)a + (-1)^{k+n+k} x\mathrm{d}(a) \\ &= \mathrm{d}_{M[k]}(x)a + (-1)^n x\mathrm{d}(a) \\ &= \mathrm{d}_{M[k]}(x) \cdot_{M[k]} a + (-1)^n x \cdot_{M[k]} \mathrm{d}(a) \end{aligned}$$

This is what we want as x has degree n as a homogeneous element of $M[k]$. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$\mathrm{Tot}(M^\bullet[k] \otimes_R A^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R A^\bullet)[k] \to M^\bullet[k]$$

where the first arrow is More on Algebra, Section 15.73 (7) which in this case does not involve a sign. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

The remarks in Homology, Section 12.14 apply. In particular, we will identify the cohomology groups of all shifts $M[k]$ without the intervention of signs.

At this point we have enough structure to talk about *triangles*, see Derived Categories, Definition 13.3.1. In fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category of differential graded modules is a triangulated category. First we define the homotopy category.

22.5. The homotopy category

09JM Our homotopies take into account the A-module structure and the grading, but not the differential (of course).

09JN **Definition 22.5.1.** Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of differential graded A-modules. A *homotopy between f and g* is an A-module map $h : M \to N$ such that

(1) $h(M^n) \subset N^{n-1}$ for all n, and
(2) $f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$ for all $x \in M$.

If a homotopy exists, then we say f and g are *homotopic*.

Thus h is compatible with the A-module structure and the grading but not with the differential. If $f = g$ and h is a homotopy as in the definition, then h defines a morphism $h : M \to N[-1]$ in $\mathrm{Mod}_{(A,\mathrm{d})}$.

09JP **Lemma 22.5.2.** *Let (A, d) be a differential graded algebra. Let $f, g : L \to M$ be homomorphisms of differential graded A-modules. Suppose given further homomorphisms $a : K \to L$, and $c : M \to N$. If $h : L \to M$ is an A-module map which defines a homotopy between f and g, then $c \circ h \circ a$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Immediate from Homology, Lemma 12.13.7. □

This lemma allows us to define the homotopy category as follows.

09JQ **Definition 22.5.3.** Let (A, d) be a differential graded algebra. The *homotopy category*, denoted $K(\mathrm{Mod}_{(A,\mathrm{d})})$, is the category whose objects are the objects of $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose morphisms are homotopy classes of homomorphisms of differential graded A-modules.

The notation $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

09JR **Lemma 22.5.4.** *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ has direct sums and products.*

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 22.4.2. This works because we saw that these functors commute with the forgetful functor to the category of graded A-modules and because $\prod$ is an exact functor on the category of families of abelian groups. □

22.6. Cones

09K9 We introduce cones for the category of differential graded modules.

09KA **Definition 22.6.1.** Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. The *cone* of f is the differential graded A-module $C(f)$ given by $C(f) = L \oplus K$ with grading $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_L & f \\ 0 & -\mathrm{d}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L \to C(f)$ and $p : C(f) \to K[1]$ induced by the obvious maps $L \to C(f)$ and $C(f) \to K$.

The formation of the cone triangle is functorial in the following sense.

09KD **Lemma 22.6.2.** *Let (A, d) be a differential graded algebra. Suppose that*

$$\begin{array}{ccc} K_1 & \xrightarrow{f_1} & L_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K_2 & \xrightarrow{f_2} & L_2 \end{array}$$

is a diagram of homomorphisms of differential graded A-modules which is commutative up to homotopy. Then there exists a morphism $c : C(f_1) \to C(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \to (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

in $K(Mod_{(A,d)})$.

Proof. Let $h : K_1 \to L_2$ be a homotopy between $f_2 \circ a$ and $b \circ f_1$. Define c by the matrix

$$c = \begin{pmatrix} b & h \\ 0 & a \end{pmatrix} : L_1 \oplus K_1 \to L_2 \oplus K_2$$

A matrix computation show that c is a morphism of differential graded modules. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

22.7. Admissible short exact sequences

09JS An admissible short exact sequence is the analogue of termwise split exact sequences in the setting of differential graded modules.

09JT **Definition 22.7.1.** Let (A, d) be a differential graded algebra.

(1) A homomorphism $K \to L$ of differential graded A-modules is an *admissible monomorphism* if there exists a graded A-module map $L \to K$ which is left inverse to $K \to L$.
(2) A homomorphism $L \to M$ of differential graded A-modules is an *admissible epimorphism* if there exists a graded A-module map $M \to L$ which is right inverse to $L \to M$.
(3) A short exact sequence $0 \to K \to L \to M \to 0$ of differential graded A-modules is an *admissible short exact sequence* if it is split as a sequence of graded A-modules.

Thus the splittings are compatible with all the data except for the differentials. Given an admissible short exact sequence we obtain a triangle; this is the reason that we require our splittings to be compatible with the A-module structure.

09JU **Lemma 22.7.2.** *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ and $\pi : L \to K$ be splittings such that $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$. Then we obtain a morphism*

$$\delta = \pi \circ d_L \circ s : M \to K[1]$$

of $Mod_{(A,d)}$ which induces the boundary maps in the long exact sequence of cohomology (22.4.2.1).

Proof. The map $\pi \circ \mathrm{d}_L \circ s$ is compatible with the A-module structure and the gradings by construction. It is compatible with differentials by Homology, Lemmas 12.14.10. Let R be the ring that A is a differential graded algebra over. The equality of maps is a statement about R-modules. Hence this follows from Homology, Lemmas 12.14.10 and 12.14.11. □

09JV **Lemma 22.7.3.** *Let (A, d) be a differential graded algebra. Let*

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ M & \xrightarrow{g} & N \end{array}$$

be a diagram of homomorphisms of differential graded A-modules commuting up to homotopy.

(1) *If f is an admissible monomorphism, then b is homotopic to a homomorphism which makes the diagram commute.*
(2) *If g is an admissible epimorphism, then a is homotopic to a morphism which makes the diagram commute.*

Proof. Let $h : K \to N$ be a homotopy between bf and ga, i.e., $bf - ga = \mathrm{d}h + h\mathrm{d}$. Suppose that $\pi : L \to K$ is a graded A-module map left inverse to f. Take $b' = b - \mathrm{d}h\pi - h\pi\mathrm{d}$. Suppose $s : N \to M$ is a graded A-module map right inverse to g. Take $a' = a + \mathrm{d}sh + sh\mathrm{d}$. Computations omitted. □

09JW **Lemma 22.7.4.** *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ be a homomorphism of differential graded A-modules. There exists a factorization*

$$K \xrightarrow{\tilde{\alpha}} \tilde{L} \xrightarrow{\pi} L, \quad K \xrightarrow{\alpha} L$$

in $Mod_{(A,d)}$ such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism (see Definition 22.7.1),*
(2) *there is a morphism $s : L \to \tilde{L}$ such that $\pi \circ s = id_L$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}}$.*

Proof. The proof is identical to the proof of Derived Categories, Lemma 13.9.6. Namely, we set $\tilde{L} = L \oplus C(1_K)$ and we use elementary properties of the cone construction. □

09JX **Lemma 22.7.5.** *Let (A, d) be a differential graded algebra. Let $L_1 \to L_2 \to \ldots \to L_n$ be a sequence of composable homomorphisms of differential graded A-modules. There exists a commutative diagram*

$$\begin{array}{ccccccc} L_1 & \longrightarrow & L_2 & \longrightarrow & \ldots & \longrightarrow & L_n \\ \uparrow & & \uparrow & & & & \uparrow \\ M_1 & \longrightarrow & M_2 & \longrightarrow & \ldots & \longrightarrow & M_n \end{array}$$

in $Mod_{(A,d)}$ such that each $M_i \to M_{i+1}$ is an admissible monomorphism and each $M_i \to L_i$ is a homotopy equivalence.

Proof. The case $n = 1$ is without content. Lemma 22.7.4 is the case $n = 2$. Suppose we have constructed the diagram except for M_n. Apply Lemma 22.7.4 to the composition $M_{n-1} \to L_{n-1} \to L_n$. The result is a factorization $M_{n-1} \to M_n \to L_n$ as desired. □

09JY **Lemma 22.7.6.** *Let (A, d) be a differential graded algebra. Let $0 \to K_i \to L_i \to M_i \to 0$, $i = 1, 2, 3$ be admissible short exact sequence of differential graded A-modules. Let $b : L_1 \to L_2$ and $b' : L_2 \to L_3$ be homomorphisms of differential graded modules such that*

$$\begin{array}{ccccc} K_1 & \longrightarrow & L_1 & \longrightarrow & M_1 \\ {\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\ K_2 & \longrightarrow & L_2 & \longrightarrow & M_2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} K_2 & \longrightarrow & L_2 & \longrightarrow & M_2 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\ K_3 & \longrightarrow & L_3 & \longrightarrow & M_3 \end{array}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.7.3 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b) \subset \mathrm{Im}(K_2 \to L_2)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(K_2 \to L_2)$. Then $b \circ b' = 0$ as a map of modules. □

22.8. Distinguished triangles

09K5 The following lemma produces our distinguished triangles.

09K6 **Lemma 22.8.1.** *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. The triangle*

09K7 (22.8.1.1) $$K \to L \to M \xrightarrow{\delta} K[1]$$

with δ as in Lemma 22.7.2 is, up to canonical isomorphism in $K(Mod_{(A,d)})$, independent of the choices made in Lemma 22.7.2.

Proof. Namely, let (s', π') be a second choice of splittings as in Lemma 22.7.2. Then we claim that δ and δ' are homotopic. Namely, write $s' = s + \alpha \circ h$ and $\pi' = \pi + g \circ \beta$ for some unique homomorphisms of A-modules $h : M \to K$ and $g : M \to K$ of degree -1. Then $g = -h$ and g is a homotopy between δ and δ'. The computations are done in the proof of Homology, Lemma 12.14.12. □

09K8 **Definition 22.8.2.** Let (A, d) be a differential graded algebra.

(1) If $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules, then the *triangle associated to* $0 \to K \to L \to M \to 0$ is the triangle (22.8.1.1) of $K(\mathrm{Mod}_{(A,\mathrm{d})})$.
(2) A triangle of $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is called a *distinguished triangle* if it is isomorphic to a triangle associated to an admissible short exact sequence of differential graded A-modules.

22.9. Cones and distinguished triangles

09P1 Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. Then $(K, L, C(f), f, i, p)$ forms a triangle:

$$K \to L \to C(f) \to K[1]$$

in $\mathrm{Mod}_{(A,\mathrm{d})}$ and hence in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Cones are **not** distinguished triangles in general, but the difference is a sign or a rotation (your choice). Here are two precise statements.

09KB **Lemma 22.9.1.** *Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded modules. The triangle $(L, C(f), K[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$0 \to L \to C(f) \to K[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

09KC **Lemma 22.9.2.** *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ and $\beta : L \to M$ define an admissible short exact sequence*

$$0 \to K \to L \to M \to 0$$

of differential graded A-modules. Let $(K, L, M, \alpha, \beta, \delta)$ be the associated triangle. Then the triangles

$$(M[-1], K, L, \delta[-1], \alpha, \beta) \quad \text{and} \quad (M[-1], K, C(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. Using a choice of splittings we write $L = K \oplus M$ and we identify α and β with the natural inclusion and projection maps. By construction of δ we have

$$d_B = \begin{pmatrix} d_K & \delta \\ 0 & d_M \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : M[-1] \to K$ is given as $C(\delta[-1]) = K \oplus M$ with differential identical with the matrix above! Whence the lemma. □

09KE **Lemma 22.9.3.** *Let (A, d) be a differential graded algebra. Let $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ be homomorphisms of differential graded A-modules. Let*

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \longrightarrow (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

be any morphism of triangles of $K(Mod_{(A,d)})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2 \to K_1$ be a homomorphism of differential graded A-modules which is inverse to a in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Let $b^{-1} : L_2 \to L_1$ be a homomorphism of differential graded A-modules which is inverse to b in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Let $c' : C(f_2) \to C(f_1)$ be the morphism from Lemma 22.6.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K, L, C(f), f, i, p)$ in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ the morphism c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By assumption the two squares in the diagram

$$\begin{array}{ccccc} L & \longrightarrow & C(f) & \longrightarrow & K[1] \\ {\scriptstyle 1}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle 1}\downarrow \\ L & \longrightarrow & C(f) & \longrightarrow & K[1] \end{array}$$

commute up to homotopy. By construction of $C(f)$ the rows form admissible short exact sequences. Thus we see that $(c-1)^2 = 0$ in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ by Lemma 22.7.6. Hence c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ with inverse $2 - c$. □

The following lemma shows that the collection of triangles of the homotopy category given by cones and the distinguished triangles are the same up to isomorphisms, at least up to sign!

09KF **Lemma 22.9.4.** *Let (A, d) be a differential graded algebra.*

(1) *Given an admissible short exact sequence $0 \to K \xrightarrow{\alpha} L \to M \to 0$ of differential graded A-modules there exists a homotopy equivalence $C(\alpha) \to M$ such that the diagram*

$$\begin{array}{ccccccc} K & \longrightarrow & L & \longrightarrow & C(\alpha) & \xrightarrow[-p]{} & K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \xrightarrow{\delta} & K[1] \end{array}$$

defines an isomorphism of triangles in $K(Mod_{(A,d)})$.

(2) *Given a morphism of complexes $f : K \to L$ there exists an isomorphism of triangles*

$$\begin{array}{ccccccc} K & \longrightarrow & \tilde{L} & \longrightarrow & M & \xrightarrow[\delta]{} & K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & L & \longrightarrow & C(f) & \xrightarrow{-p} & K[1] \end{array}$$

where the upper triangle is the triangle associated to a admissible short exact sequence $K \to \tilde{L} \to M$.

Proof. Proof of (1). We have $C(\alpha) = L \oplus K$ and we simply define $C(\alpha) \to M$ via the projection onto L followed by β. This defines a morphism of differential graded modules because the compositions $K^{n+1} \to L^{n+1} \to M^{n+1}$ are zero. Choose splittings $s : M \to L$ and $\pi : L \to K$ with $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$ and set $\delta = \pi \circ \mathrm{d}_L \circ s$ as usual. To get a homotopy inverse we take $M \to C(\alpha)$ given by $(s, -\delta)$. This is compatible with differentials because δ^n can be characterized as the unique map $M^n \to K^{n+1}$ such that $\mathrm{d} \circ s^n - s^{n+1} \circ \mathrm{d} = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $M \to C(f) \to M$ is the identity. The composition $C(f) \to M \to C(f)$ is equal to the morphism

$$\begin{pmatrix} s \circ \beta & 0 \\ -\delta \circ \beta & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h : C(\alpha) \to C(\alpha)$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} : C(\alpha) = L \oplus K \to L \oplus K = C(\alpha)$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s \\ -\delta \end{pmatrix} \begin{pmatrix} \beta & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha) \to K[1]$ (see Definition 22.6.1) and $C(\alpha) \to M \to K[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : M \to C(\alpha)$ and check instead that the two maps $M \to K[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K \to \tilde{L}$, $s : L \to \tilde{L}$ and $\pi : L \to L$ be as in Lemma 22.7.4. By Lemmas 22.6.2 and 22.9.3 the triangles $(K, L, C(f), i, p)$ and $(K, \tilde{L}, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace L by $\tilde{L}$ and f by $\tilde{f}$. In other words we may assume that f is an admissible monomorphism. In this case the result follows from part (1). □

22.10. The homotopy category is triangulated

09KG We first prove that it is pre-triangulated.

09KH **Lemma 22.10.1.** *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(K, K, 0, 1, 0, 0)$ is distinguished since $0 \to K \to K \to 0 \to 0$ is an admissible short exact sequence. Finally, given any homomorphism $f : K \to L$ of differential graded A-modules the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 22.9.4.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to K \to L \to M \to 0$ such that the associated triangle $(K, L, M, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, L, -\delta[-1], \alpha, \beta)$. It follows from Lemma 22.9.2 that the triangle $(M[-1], K, L, \delta[-1], \alpha, \beta)$ is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, -p)$. Hence it is distinguished by Lemma 22.9.4. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 22.9.4 this means that it is isomorphic to a triangle of the form $(K, L, C(f), f, i, -p)$ for some morphism f of $\mathrm{Mod}_{(A,\mathrm{d})}$. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L, C(f), K[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L, C(f), K[1], i, p, f[1])$. By Lemma 22.9.1 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 22.9.4 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 22.6.2 to the commutative diagram given by f, f', a, b. □

Before we prove TR4 in general we prove it in a special case.

09KI **Lemma 22.10.2.** *Let (A, d) be a differential graded algebra. Suppose that $\alpha : K \to L$ and $\beta : L \to M$ are admissible monomorphisms of differential graded A-modules. Then there exist distinguished triangles $(K, L, Q_1, \alpha, p_1, d_1)$, $(K, M, Q_2, \beta \circ \alpha, p_2, d_2)$ and $(L, M, Q_3, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1 : L \to K$ and $\pi_3 : M \to L$ are homomorphisms of graded A-modules which are left inverse to α and β. Then also $K \to M$ is an admissible monomorphism with left inverse $\pi_2 = \pi_1 \circ \pi_3$. Let us write Q_1, Q_2 and Q_3 for the cokernels of $K \to L$, $K \to M$, and $L \to M$. Then we obtain identifications (as graded A-modules) $Q_1 = \mathrm{Ker}(\pi_1)$, $Q_3 = \mathrm{Ker}(\pi_3)$ and $Q_2 = \mathrm{Ker}(\pi_2)$. Then $L = K \oplus Q_1$ and $M = L \oplus Q_3$ as graded A-modules. This implies $M = K \oplus Q_1 \oplus Q_3$. Note that $\pi_2 = \pi_1 \circ \pi_3$ is zero on both Q_1 and Q_3. Hence $Q_2 = Q_1 \oplus Q_3$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & K & \to & L & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M & \to & Q_3 & \to & 0
\end{array}$$

The rows of this diagram are admissible short exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of

triangles

$$(K \to L \to Q_1 \to K[1]) \longrightarrow (K \to M \to Q_2 \to K[1])$$

and

$$(K \to M \to Q_2 \to K[1]) \longrightarrow (L \to M \to Q_3 \to L[1]).$$

Note that the splittings $Q_3 \to M$ of the bottom sequence in the diagram provides a splitting for the split sequence $0 \to Q_1 \to Q_2 \to Q_3 \to 0$ upon composing with $M \to Q_2$. It follows easily from this that the morphism $\delta : Q_3 \to Q_1[1]$ in the corresponding distinguished triangle

$$(Q_1 \to Q_2 \to Q_3 \to Q_1[1])$$

is equal to the composition $Q_3 \to L[1] \to Q_1[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

Here is the final result.

09KJ **Proposition 22.10.3.** *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ of differential graded A-modules with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. We know that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a pre-triangulated category. Hence it suffices to prove TR4 and to prove it we can use Derived Categories, Lemma 13.4.15. Let $K \to L$ and $L \to M$ be composable morphisms of $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By Lemma 22.7.5 we may assume that $K \to L$ and $L \to M$ are admissible monomorphisms. In this case the result follows from Lemma 22.10.2. □

22.11. Left modules

0FPZ Everything we have said sofar has an analogue in the setting of left differential graded modules, except that one has to take care with some sign rules.

Let (A, d) be a differential graded R-algebra. Exactly analogous to right modules, we define a *left differential graded A-module* M as a left A-module M which has a grading $M = \bigoplus M^n$ and a differential d, such that $A^n M^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(am) = \mathrm{d}(a)m + (-1)^{\deg(a)} a\mathrm{d}(m)$$

for homogeneous elements $a \in A$ and $m \in M$. As before this Leibniz rule exactly signifies that the multiplication defines a map of complexes

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the complexes of R-modules underlying A and M.

09JG **Definition 22.11.1.** Let R be a ring. Let (A, d) be a differential graded algebra over R. The *opposite differential graded algebra* is the differential graded algebra (A^{opp}, d) over R where $A^{opp} = A$ as a graded R-module, $\mathrm{d} = \mathrm{d}$, and multiplication is given by

$$a \cdot_{opp} b = (-1)^{\deg(a)\deg(b)} ba$$

for homogeneous elements $a, b \in A$.

This makes sense because

$$\begin{aligned}
\mathrm{d}(a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)}\mathrm{d}(ba) \\
&= (-1)^{\deg(a)\deg(b)}\mathrm{d}(b)a + (-1)^{\deg(a)\deg(b)+\deg(b)}b\mathrm{d}(a) \\
&= (-1)^{\deg(a)}a \cdot_{opp} \mathrm{d}(b) + \mathrm{d}(a) \cdot_{opp} b
\end{aligned}$$

as desired. In terms of underlying complexes of R-modules this means that the diagram

$$\begin{array}{ccc}
\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication of } A^{opp}} & A^\bullet \\
{\scriptstyle \text{commutativity constraint}}\downarrow & & \downarrow{\scriptstyle \text{id}} \\
\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication of } A} & A^\bullet
\end{array}$$

commutes. Here the commutativity constraint on the symmetric monoidal category of complexes of R-modules is given in More on Algebra, Section 15.73.

Let (A, d) be a differential graded algebra over R. Let M be a left differential graded A-module. We will denote M^{opp} the module M viewed as a right A^{opp}-module with multiplication $\cdot_{opp}$ defined by the rule

$$m \cdot_{opp} a = (-1)^{\deg(a)\deg(m)}am$$

for a and m homogeneous. This is compatible with differentials because we could have used the diagram

$$\begin{array}{ccc}
\mathrm{Tot}(M^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication on } M^{opp}} & M^\bullet \\
{\scriptstyle \text{commutativity constraint}}\downarrow & & \downarrow{\scriptstyle \text{id}} \\
\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) & \xrightarrow{\text{multiplication on } M} & M^\bullet
\end{array}$$

to define the multiplication $\cdot_{opp}$ on M^{opp}. To see that it is an associative multiplication we compute for homogeneous elements $a, b \in A$ and $m \in M$ that

$$\begin{aligned}
m \cdot_{opp} (a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} m \cdot_{opp} (ba) \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)} bam \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)} (am) \cdot_{opp} b \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)+\deg(a)\deg(m)} (m \cdot_{opp} a) \cdot_{opp} b \\
&= (m \cdot_{opp} a) \cdot_{opp} b
\end{aligned}$$

Of course, we could have been shown this using the compatibility between the associativity and commutativity constraint on the symmetric monoidal category of complexes of R-modules as well.

0FQ0 **Lemma 22.11.2.** *Let (A, d) be a differential graded R-algebra. The functor $M \mapsto M^{opp}$ from the category of left differential graded A-modules to the category of right differential graded A^{opp}-modules is an equivalence.*

Proof. Omitted. □

Mext, we come to shifts. Let (A, d) be a differential graded algebra. Let M be a left differential graded A-module whose underlying complex of R-modules is denoted $M^\bullet$. For any $k \in \mathbf{Z}$ we define the *k-shifted module* $M[k]$ as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$

(2) as A-module the multiplication

$$A^n \times (M[k])^m \longrightarrow (M[k])^{n+m}$$

is equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $a \in A^n$ and $x \in M[k]^m = M^{m+k}$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$\begin{aligned} \mathrm{d}_{M[k]}(a \cdot_{M[k]} x) &= (-1)^{k+nk}\mathrm{d}_M(ax) \\ &= (-1)^{k+nk}\mathrm{d}(a)x + (-1)^{k+nk+n}a\mathrm{d}_M(x) \\ &= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^{nk+n}a\mathrm{d}_{M[k]}(x) \\ &= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^n a \cdot_{M[k]} \mathrm{d}_{M[k]}(x) \end{aligned}$$

This is what we want as a has degree n as a homogeneous element of A. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet[k]) \to \mathrm{Tot}(A^\bullet \otimes_R M^\bullet)[k] \to M^\bullet[k]$$

where the first arrow is More on Algebra, Section 15.73 (7) which in this case involves exactly the sign we chose above. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

With the rule above we have canonical identifications

$$(M[k])^{opp} = M^{opp}[k]$$

of right differential graded A^{opp}-modules defined without the intervention of signs, in other words, the equivalence of Lemma 22.11.2 is compatible with shift functors.

Our choice above necessitates the following definition.

0FQ1 **Definition 22.11.3.** Let R be a ring. Let A be a $\mathbf{Z}$-graded R-algebra.

(1) Given a right graded A-module M we define the *kth shifted A-module* $M[k]$ as the same as a right A-module but with grading $(M[k])^n = M^{n+k}$.
(2) Given a left graded A-module M we define the *kth shifted A-module* $M[k]$ as the module with grading $(M[k])^n = M^{n+k}$ and multiplication $A^n \times (M[k])^m \to (M[k])^{n+m}$ equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of left differential graded A-modules. A *homotopy between f and g* is a graded A-module map $h : M \to N[-1]$ (observe the shift!) such that

$$f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$$

for all $x \in M$. If a homotopy exists, then we say f and g are *homotopic*. Thus h is compatible with the A-module structure (with the shifted one on N) and the grading (with shifted grading on N) but not with the differential. If $f = g$ and h is a homotopy, then h defines a morphism $h : M \to N[-1]$ of left differential graded A-modules.

With the rule above we find that $f, g : M \to N$ are homotopic if and only if the induced morphisms $f^{opp}, g^{opp} : M^{opp} \to N^{opp}$ are homotopic as right differential graded A^{opp}-module homomorphisms (with the same homotopy).

The homotopy category, cones, admissible short exact sequences, distinguished triangles are all defined in exactly the same manner as for right differential graded modules (and everything agrees on underlying complexes of R-modules with the constructions for complexes of R-modules). In this manner we obtain the analogue of Proposition 22.10.3 for left modules as well, or we can deduce it by working with right modules over the opposite algebra.

22.12. Tensor product

09LL Let R be a ring. Let A be an R-algebra (see Section 22.2). Given a right A-module M and a left A-module N there is a *tensor product*

$$M \otimes_A N$$

This tensor product is a module over R. As an R-module $M \otimes_A N$ is generated by symbols $x \otimes y$ with $x \in M$ and $y \in N$ subject to the relations

$$\begin{gathered}(x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y, \\ x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2, \\ xa \otimes y - x \otimes ay\end{gathered}$$

for $a \in A$, $x, x_1, x_2 \in M$ and $y, y_1, y_2 \in N$. We list some properties of the tensor product

In each variable the tensor product is right exact, in fact commutes with direct sums and arbitrary colimits.

The tensor product $M \otimes_A N$ is the receptacle of the universal A-bilinear map $M \times N \to M \otimes_A N$, $(x, y) \mapsto x \otimes y$. In a formula

$$\text{Bilinear}_A(M \times N, Q) = \text{Hom}_R(M \otimes_A N, Q)$$

for any R-module Q.

If A is a $\mathbf{Z}$-graded algebra and M, N are graded A-modules then $M \otimes_A N$ is a graded R-module. Then nth graded piece $(M \otimes_A N)^n$ of $M \otimes_A N$ is equal to

$$\text{Coker}\left(\bigoplus\nolimits_{r+t+s=n} M^r \otimes_R A^t \otimes_R N^s \to \bigoplus\nolimits_{p+q=n} M^p \otimes_R N^q\right)$$

where the map sends $x \otimes a \otimes y$ to $x \otimes ay - xa \otimes y$ for $x \in M^r$, $y \in N^s$, and $a \in A^t$ with $r + s + t = n$. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear and compatible with gradings and universal in the sense that

$$\text{GradedBilinear}_A(M \times N, Q) = \text{Hom}_{\text{graded } R\text{-modules}}(M \otimes_A N, Q)$$

for any graded R-module Q with an obvious notion of graded bilinar map.

If (A, d) is a differential graded algebra and M and N are left and right differential graded A-modules, then $M \otimes_A N$ is a differential graded R-module with differential

$$\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{\deg(x)} x \otimes \mathrm{d}(y)$$

for $x \in M$ and $y \in N$ homogeneous. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear, compatible with gradings, and compatible with differentials and universal in the sense that

$$\text{DifferentialGradedBilinear}_A(M \times N, Q) = \text{Hom}_{\text{Comp}(R)}(M \otimes_A N, Q)$$

for any differential graded R-module Q with an obvious notion of differential graded bilinar map.

22.13. Hom complexes and differential graded modules

0FQ2 We urge the reader to skip this section.

Let R be a ring and let $M^\bullet$ be a complex of R-modules. Consider the complex of R-modules

$$E^\bullet = \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

introduced in More on Algebra, Section 15.72. By More on Algebra, Lemma 15.72.3 there is a canonical composition law

$$\mathrm{Tot}(E^\bullet \otimes_R E^\bullet) \to E^\bullet$$

which is a map of complexes. Thus we see that $E^\bullet$ with this multiplication is a differential graded R-algebra which we will denote (E, d). Moreover, viewing $M^\bullet$ as $\mathrm{Hom}^\bullet(R, M^\bullet)$ we see that composition defines a multiplication

$$\mathrm{Tot}(E^\bullet \otimes_R M^\bullet) \to M^\bullet$$

which turns $M^\bullet$ into a **left** differential graded E-module which we will denote M.

0FQ3 **Lemma 22.13.1.** *In the situation above, let A be a differential graded R-algebra. To give a left A-module structure on M is the same thing as giving a homomorphism $A \to E$ of differential graded R-algebras.*

Proof. Proof omitted. Observe that no signs intervene in this correspondence. □

We continue with the discussion above and we assume given another complex $N^\bullet$ of R-modules. Consider the complex of R-modules $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ introduced in More on Algebra, Section 15.72. As above we see that composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R E^\bullet) \to \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$$

defines a multiplication which turns $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ into a **right** differential graded E-module. Using Lemma 22.13.1 we conclude that given a left differential graded A-module M and a complex of R-modules $N^\bullet$ there is a canonical right differential graded A-module whose underlying complex of R-modules is $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ and where multiplication

$$\mathrm{Hom}^n(M^\bullet, N^\bullet) \times A^m \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)$$

sends $f = (f_{p,q})_{p+q=n}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ and $a \in A^m$ to the element $f \cdot a = (f_{p,q} \circ a)$ where $f_{p,q} \circ a$ is the map

$$M^{-q-m} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(ax)$$

without the intervention of signs. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this right differential graded A-module.

0FQ4 **Lemma 22.13.2.** *Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M' be a right differential graded A-module and let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{Comp(R)}(M' \otimes_A M, N^\bullet)$$

where $M \otimes_A M$ is viewed as a complex of R-modules as in Section 22.12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$M' \times M \longrightarrow N^\bullet$$

compatible with differentials. We have seen this is true for the right hand side in Section 22.12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 15.72.1 determines a map of complexes of R-modules $g' : \mathrm{Tot}(M' \otimes_R M) \to N^\bullet$. In other words, we obtain a graded R-bilinear map $g'' : M' \times M \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. □

Let R, $M^\bullet$, $E^\bullet$, E, and M be as above. However, now suppose given a differential graded R-algebra A and a **right** differential graded A-module structure on M. Then we can consider the map

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \xrightarrow{\psi} \mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

where the first arrow is the commutativity constraint on the differential graded category of complexes of R-modules. This corresponds to a map

$$\tau : A^\bullet \longrightarrow E^\bullet$$

of complexes of R-modules. Recall that $E^n = \prod_{p+q=n} \mathrm{Hom}_R(M^{-q}, M^p)$ and write $\tau(a) = (\tau_{p,q}(a))_{p+q=n}$ for $a \in A^n$. Then we see

$$\tau_{p,q}(a) : M^{-q} \longrightarrow M^p, \quad x \longmapsto (-1)^{\deg(a)\deg(x)} xa = (-1)^{-nq} xa$$

This is not compatible with the product on A as the reader should expect from the discussion in Section 22.11. Namely, we have

$$\tau(aa') = (-1)^{\deg(a)\deg(a')} \tau(a')\tau(a)$$

We conclude the following lemma is true

0FQ5 **Lemma 22.13.3.** *In the situation above, let A be a differential graded R-algebra. To give a right A-module structure on M is the same thing as giving a homomorphism $\tau : A \to E^{opp}$ of differential graded R-algebras.*

Proof. See discussion above and note that the construction of τ from the multiplication map $M^n \times A^m \to M^{n+m}$ uses signs. □

Let R, $M^\bullet$, $E^\bullet$, E, A and M be as above and let a right differential graded A-module structure on M be given as in the lemma. In this case there is a canonical left differential graded A-module whose underlying complex of R-modules is $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$. Namely, for multiplication we can use

$$\begin{aligned} \mathrm{Tot}(A^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)) &\xrightarrow{\psi} \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R A^\bullet) \\ &\xrightarrow{\tau} \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)) \\ &\to \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \end{aligned}$$

The first arrow uses the commutativity constraint on the category of complexes of R-modules, the second arrow is described above, and the third arrow is the composition law for the Hom complex. Each map is a map of complexes, hence the result is a map of complexes. In fact, this construction turns $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ into a left differential graded A-module (associativity of the multiplication can be

shown using the symmetric monoidal structure or by a direct calculation using the formulae below). Let us explicate the multiplication

$$A^n \times \mathrm{Hom}^m(M^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)$$

It sends $a \in A^n$ and $f = (f_{p,q})_{p+q=m}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ to the element $a \cdot f$ with constituents

$$(-1)^{nm} f_{p,q} \circ \tau_{-q,q+n}(a) = (-1)^{nm-n(q+n)} f_{p,q} \circ a = (-1)^{np+n} f_{p,q} \circ a$$

in $\mathrm{Hom}_R(M^{-q-n}, N^p)$ where $f_{p,q} \circ a$ is the map

$$M^{-q-n} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(xa)$$

Here a sign of $(-1)^{np+n}$ does intervene. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this left differential graded A-module.

0FQ6 **Lemma 22.13.4.** *Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M be a right differential graded A-module and let M' be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have*

$$\mathrm{Hom}_{\textit{left diff graded } A\textit{-modules}}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{Comp(R)}(M \otimes_A M', N^\bullet)$$

where $M \otimes_A M'$ is viewed as a complex of R-modules as in Section 22.12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$M \times M' \longrightarrow N^\bullet$$

compatible with differentials. We have seen this is true for the right hand side in Section 22.12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 15.72.1 determines a map of complexes $g' : \mathrm{Tot}(M' \otimes_R M) \to N^\bullet$. We precompose with the commutativity constraint to get

$$\mathrm{Tot}(M \otimes_R M') \xrightarrow{\psi} \mathrm{Tot}(M' \otimes_R M) \xrightarrow{g'} N^\bullet$$

which corresponds to a graded R-bilinear map $g'' : M \times M' \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. Namely, say $x \in M^e$ and $x' \in (M')^{e'}$ and $a \in A^n$. Then on the one hand we have

$$\begin{aligned} g''(x, ax') &= (-1)^{e(n+e')} g'(ax' \otimes x) \\ &= (-1)^{e(n+e')} g(ax')(x) \\ &= (-1)^{e(n+e')} (a \cdot g(x'))(x) \\ &= (-1)^{e(n+e')+n(n+e+e')+n} g(x')(xa) \end{aligned}$$

and on the other hand we have

$$g''(xa, x') = (-1)^{(e+n)e'} g'(x' \otimes xa) = (-1)^{(e+n)e'} g(x')(xa)$$

which is the same thing by a trivial mod 2 calculation of the exponents. □

0FQ7 **Remark 22.13.5.** Let R be a ring. Let A be a differential graded R-algebra. Let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a right differential graded A-module $\mathrm{Hom}(M, N^\bullet)$ and then a leftt differential graded A-module $\mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$ev : M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$$

in the category of left differential graded A-modules. To define it, by Lemma 22.13.2 it suffices to construct an A-bilinear pairing

$$\mathrm{Hom}(M, N^\bullet) \times M \longrightarrow N^\bullet$$

compatible with grading and differentials. For this we take

$$(f, x) \longmapsto f(x)$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

0FQ8 **Remark 22.13.6.** Let R be a ring. Let A be a differential graded R-algebra. Let M be a right differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a left differential graded A-module $\mathrm{Hom}(M, N^\bullet)$ and then a right differential graded A-module $\mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$ev : M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$$

in the category of right differential graded A-modules. To define it, by Lemma 22.13.2 it suffices to construct an A-bilinear pairing

$$M \times \mathrm{Hom}(M, N^\bullet) \longrightarrow N^\bullet$$

compatible with grading and differentials. For this we take

$$(x, f) \longmapsto (-1)^{\deg(x)\deg(f)} f(x)$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

0FQ9 **Remark 22.13.7.** Let R be a ring. Let A be a differential graded R-algebra. Let $M^\bullet$ and $N^\bullet$ be complexes of R-modules. Let $k \in \mathbf{Z}$ and consider the isomorphism

$$\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)[-k] \longrightarrow \mathrm{Hom}^\bullet(M^\bullet[k], N^\bullet)$$

of complexes of R-modules defined in More on Algebra, Item (18). If $M^\bullet$ has the structure of a left, resp. right differential graded A-module, then this is a map of right, resp. left differential graded A-modules (with the module structures as defined in this section). We omit the verification; we warn the reader that the A-module structure on the shift of a left graded A-module is defined using a sign, see Definition 22.11.3.

22.14. Projective modules over algebras

09JZ In this section we discuss projective modules over algebras analogous to Algebra, Section 10.77. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. It is clear that A is a projective right A-module since $\mathrm{Hom}_A(A, M) = M$ for any right A-module M (and thus $\mathrm{Hom}_A(A, -)$ is exact). Conversely, let P be a projective right A-module. Then we can choose a surjection $\bigoplus_{i \in I} A \to P$ by choosing a set $\{p_i\}_{i \in I}$ of generators of P over A. Since P is projective there is a left inverse to the surjection, and we find that P is isomorphic to a direct summand of a free module, exactly as in the commutative case (Algebra, Lemma 10.77.2).

We conclude

(1) the category of A-modules has enough projectives,
(2) A is a projective A-module,
(3) every A-module is a quotient of a direct sum of copies of A,
(4) every projective A-module is a direct summand of a direct sum of copies of A.

22.15. Projective modules over graded algebras

0FQA In this section we discuss projective graded modules over graded algebras analogous to Algebra, Section 10.77.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 22.2 for our conventions. Let Mod_A denote the category of graded right A-modules. For an integer k let $A[k]$ denote the shift of A. For a graded right A-module we have

$$\text{Hom}_{\text{Mod}_A}(A[k], M) = M^{-k}$$

As the functor $M \mapsto M^{-k}$ is exact on Mod_A we conclude that $A[k]$ is a projective object of Mod_A. Conversely, suppose that P is a projective object of Mod_A. By choosing a set of homogeneous generators of P as an A-module, we can find a surjection

$$\bigoplus\nolimits_{i \in I} A[k_i] \longrightarrow P$$

Thus we conclude that a projective object of Mod_A is a direct summand of a direct sum of the shifts $A[k]$.

We conclude

(1) the category of graded A-modules has enough projectives,
(2) $A[k]$ is a projective A-module for every $k \in \mathbf{Z}$,
(3) every graded A-module is a quotient of a direct sum of copies of the modules $A[k]$ for varying k,
(4) every projective A-module is a direct summand of a direct sum of copies of the modules $A[k]$ for varying k.

22.16. Projective modules and differential graded algebras

0FQB If (A, d) is a differential graded algebra and P is an object of $\text{Mod}_{(A,\mathrm{d})}$ then we say *P is projective as a graded A-module* or sometimes *P is graded projective* to mean that P is a projective object of the abelian category Mod_A of graded A-modules as in Section 22.15.

09K0 **Lemma 22.16.1.** *Let (A, d) be a differential graded algebra. Let $M \to P$ be a surjective homomorphism of differential graded A-modules. If P is projective as a graded A-module, then $M \to P$ is an admissible epimorphism.*

Proof. This is immediate from the definitions. □

09K1 **Lemma 22.16.2.** *Let (A, d) be a differential graded algebra. Then we have*

$$\text{Hom}_{Mod_{(A,d)}}(A[k], M) = \text{Ker}(d : M^{-k} \to M^{-k+1})$$

and

$$\text{Hom}_{K(Mod_{(A,d)})}(A[k], M) = H^{-k}(M)$$

for any differential graded A-module M.

Proof. Immediate from the definitions. □

22.17. Injective modules over algebras

04JD In this section we discuss injective modules over algebras analogous to More on Algebra, Section 15.55. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. For a right A-module M we set

$$M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$$

which we think of as a left A-module by the multiplication $(af)(x) = f(xa)$. Namely, $((ab)f)(x) = f(xab) = (bf)(xa) = (a(bf))(x)$. Conversely, if M is a left A-module, then $M^\vee$ is a right A-module. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group (More on Algebra, Lemma 15.54.1), the functor $M \mapsto M^\vee$ is exact (More on Algebra, Lemma 15.55.6). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective for all modules M (More on Algebra, Lemma 15.55.7).

We claim that $A^\vee$ is an injective right A-module. Namely, given a right A-module N we have

$$\mathrm{Hom}_A(N, A^\vee) = \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and we conclude because the functor $N \mapsto N^\vee$ is exact. The second equality holds because

$$\mathrm{Hom}_{\mathbf{Z}}(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_{\mathbf{Z}} A, \mathbf{Q}/\mathbf{Z})$$

by Algebra, Lemma 10.12.8. Inside this module A-linearity exactly picks out the bilinear maps $\varphi : N \times A \to \mathbf{Q}/\mathbf{Z}$ which have the same value on $x \otimes a$ and $xa \otimes 1$, i.e., come from elements of $N^\vee$.

Finally, for every right A-module M we can choose a surjection $\bigoplus_{i \in I} A \to M^\vee$ to get an injection $M \to (M^\vee)^\vee \to \prod_{i \in I} A^\vee$.

We conclude

(1) the category of A-modules has enough injectives,
(2) $A^\vee$ is an injective A-module, and
(3) every A-module injects into a product of copies of $A^\vee$.

22.18. Injective modules over graded algebras

0FQC In this section we discuss injective graded modules over graded algebras analogous to More on Algebra, Section 15.55.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 22.2 for our conventions. If M is a graded R-module we set

$$M^\vee = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z}) = \bigoplus\nolimits_{n \in \mathbf{Z}} (M^{-n})^\vee$$

as a graded R-module (no signs in the actions of R on the homogeneous parts). If M has the structure of a left graded A-module, then we define a right graded A-module structure on $M^\vee$ by letting $a \in A^m$ act by

$$(M^{-n})^\vee \to (M^{-n-m})^\vee, \quad f \mapsto f \circ a$$

as in Section 22.13. If M has the structure of a right graded A-module, then we define a left graded A-module structure on $M^\vee$ by letting $a \in A^n$ act by

$$(M^{-m})^\vee \to (M^{-m-n})^\vee, \quad f \mapsto (-1)^{nm} f \circ a$$

as in Section 22.13 (the sign is forced on us because we want to use the same formula for the case when working with differential graded modules — if you only care about graded modules, then you can omit the sign here). On the category of (left or right) graded A-modules the functor $M \mapsto M^\vee$ is exact (check on graded pieces). Moreover, there is an injective evaluation map

$$ev : M \longrightarrow (M^\vee)^\vee, \quad ev^n = (-1)^n \text{ the evaluation map } M^n \to ((M^n)^\vee)^\vee$$

of graded R-modules, see More on Algebra, Item (17). This evaluation map is a left, resp. right A-module homomorphism if M is a left, resp. right A-module, see Remarks 22.13.5 and 22.13.6. Finally, given $k \in \mathbf{Z}$ there is a canonical isomorphism

$$M^\vee[-k] \longrightarrow (M[k])^\vee$$

of graded R-modules which uses a sign and which, if M is a left, resp. right A-module, is an isomorphism of right, resp. left A-modules. See Remark 22.13.7.

We claim that $A^\vee$ is an injective object of the category Mod_A of graded right A-modules. Namely, given a graded right A-module N we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee) = \mathrm{Hom}_{\mathrm{Comp}(\mathbf{Z})}(N \otimes_A A, \mathbf{Q}/\mathbf{Z})) = (N^0)^\vee$$

by Lemma 22.13.2 (applied to the case where all the differentials are zero). We conclude because the functor $N \mapsto (N^0)^\vee = (N^\vee)^0$ is exact.

Finally, for every graded right A-module M we can choose a surjection of graded left A-modules

$$\bigoplus\nolimits_{i \in I} A[k_i] \to M^\vee$$

where $A[k_i]$ denotes the shift of A by $k_i \in \mathbf{Z}$. We do this by choosing homogeneous generators for $M^\vee$. In this way we get an injection

$$M \to (M^\vee)^\vee \to \prod A[k_i]^\vee = \prod A^\vee[-k_i]$$

Observe that the products in the formula above are products in the category of graded modules (in other words, take products in each degree and then take the direct sum of the pieces).

We conclude that

(1) the category of graded A-modules has enough injectives,
(2) for every $k \in \mathbf{Z}$ the module $A^\vee[k]$ is injective, and
(3) every A-module injects into a product in the category of graded modules of copies of shifts $A^\vee[k]$.

22.19. Injective modules and differential graded algebras

0FQD If (A, d) is a differential graded algebra and I is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say *I is injective as a graded A-module* or sometimes *I is graded injective* to mean that I is an injective object of the abelian category Mod_A of graded A-modules.

09K2 **Lemma 22.19.1.** *Let (A, d) be a differential graded algebra. Let $I \to M$ be an injective homomorphism of differential graded A-modules. If I is graded injective, then $I \to M$ is an admissible monomorphism.*

Proof. This is immediate from the definitions. □

Let (A, d) be a differential graded algebra. If M is a left, resp. right differential graded A-module, then

$$M^\vee = \mathrm{Hom}^\bullet(M^\bullet, \mathbf{Q}/\mathbf{Z})$$

with A-module structure constructed in Section 22.18 is a right, resp. left differential graded A-module by the discussion in Section 22.13. By Remarks 22.13.5 and 22.13.6 there evaluation map of Section 22.18

$$M \longrightarrow (M^\vee)^\vee$$

is a homomorphism of left, resp. right differential graded A-modules

09K3 **Lemma 22.19.2.** *Let (A, d) be a differential graded algebra. If M is a left differential graded A-module and N is a right differential graded A-module, then*

$$\begin{aligned}\mathrm{Hom}_{Mod_{(A,d)}}(N, M^\vee) &= \mathrm{Hom}_{Comp(\mathbf{Z})}(N \otimes_A M, \mathbf{Q}/\mathbf{Z}) \\ &= \mathit{DifferentialGradedBilinear}_A(N \times M, \mathbf{Q}/\mathbf{Z})\end{aligned}$$

Proof. The first equality is Lemma 22.13.2 and the second equality was shown in Section 22.12. □

09K4 **Lemma 22.19.3.** *Let (A, d) be a differential graded algebra. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(M, A^\vee[k]) = \mathrm{Ker}(d : (M^\vee)^k \to (M^\vee)^{k+1})$$

and

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(M, A^\vee[k]) = H^k(M^\vee)$$

as functors in the differential graded A-module M.

Proof. This is clear from the discussion above. □

22.20. P-resolutions

09KK This section is the analogue of Derived Categories, Section 13.29.

Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. We say P *has property (P)* if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a direct sum of $A[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.16.1. Moreover, the reader can verify that as a graded A-module P will be isomorphic to a direct sum of shifts of A.

09KL **Lemma 22.20.1.** *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. If $F_\bullet$ is a filtration as in property (P), then we obtain an admissible short exact sequence*

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded A-modules.

Proof. The second map is the direct sum of the inclusion maps. The first map on the summand F_iP of the source is the sum of the identity $F_iP \to F_iP$ and the negative of the inclusion map $F_iP \to F_{i+1}P$. Choose homomorphisms $s_i : F_{i+1}P \to F_iP$ of graded A-modules which are left inverse to the inclusion maps. Composing gives maps $s_{j,i} : F_jP \to F_iP$ for all $j > i$. Then a left inverse of the first arrow maps $x \in F_jP$ to $(s_{j,0}(x), s_{j,1}(x), \ldots, s_{j,j-1}(x), 0, \ldots)$ in $\bigoplus F_iP$. □

The following lemma shows that differential graded modules with property (P) are the dual notion to K-injective modules (i.e., they are K-projective in some sense). See Derived Categories, Definition 13.31.1.

09KM **Lemma 22.20.2.** *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module with property (P). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(P, N) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on P as in property (P). The short exact sequence of Lemma 22.20.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(F_iP, N) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules F_iP has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of shifts $A[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(A[k], N) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.16.2. □

09KN **Lemma 22.20.3.** *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules with the following properties*

1. *$P \to M$ is surjective,*
2. *$\mathrm{Ker}(d_P) \to \mathrm{Ker}(d_M)$ is surjective, and*
3. *P sits in an admissible short exact sequence $0 \to P' \to P \to P'' \to 0$ where P', P'' are direct sums of shifts of A.*

Proof. Let P_k be the free A-module with generators x, y in degrees k and $k+1$. Define the structure of a differential graded A-module on P_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. For every element $m \in M^k$ there is a homomorphism $P_k \to M$ sending x to m and y to $\mathrm{d}(m)$. Thus we see that there is a surjection from a direct sum of copies of P_k to M. This clearly produces $P \to M$ having properties (1) and (3). To obtain property (2) note that if $m \in \mathrm{Ker}(\mathrm{d}_M)$ has degree k, then there is a map $A[k] \to M$ mapping 1 to m. Hence we can achieve (2) by adding a direct sum of copies of shifts of A. □

09KP **Lemma 22.20.4.** *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules such that*

1. *$P \to M$ is a quasi-isomorphism, and*

(2) *P has property (P).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

where the maps $P_i \to M_i$ are chosen as in Lemma 22.20.3. This gives a "resolution"

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

Then we set

$$P = \bigoplus\nolimits_{i \geq 0} P_i$$

as an A-module with grading given by $P^n = \bigoplus_{a+b=n} P^b_{-a}$ and differential (as in the construction of the total complex associated to a double complex) by

$$\mathrm{d}_P(x) = f_{-a}(x) + (-1)^a \mathrm{d}_{P_{-a}}(x)$$

for $x \in P^b_{-a}$. With these conventions P is indeed a differential graded A-module. Recalling that each P_i has a two step filtration $0 \to P_i' \to P_i \to P_i'' \to 0$ we set

$$F_{2i}P = \bigoplus\nolimits_{i \geq j \geq 0} P_j \subset \bigoplus\nolimits_{i \geq 0} P_i = P$$

and we add P'_{i+1} to $F_{2i}P$ to get F_{2i+1}. These are differential graded submodules and the successive quotients are direct sums of shifts of A. By Lemma 22.16.1 we see that the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms. Finally, we have to show that the map $P \to M$ (given by the augmentation $P_0 \to M$) is a quasi-isomorphism. This follows from Homology, Lemma 12.26.2. □

22.21. I-resolutions

09KQ This section is the dual of the section on P-resolutions.

Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. We say *I has property (I)* if it there exists a filtration

$$I = F_0I \supset F_1I \supset F_2I \supset \ldots \supset 0$$

by differential graded submodules such that

(1) $I = \lim I/F_pI$,
(2) the maps $I/F_{i+1}I \to I/F_iI$ are admissible epimorphisms,
(3) the quotients $F_iI/F_{i+1}I$ are isomorphic as differential graded A-modules to products of the modules $A^\vee[k]$ constructed in Section 22.19.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.19.1. The reader can verify that as a graded module I will be isomorphic to a product of $A^\vee[k]$.

09KR **Lemma 22.21.1.** *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. If $F_\bullet$ is a filtration as in property (I), then we obtain an admissible short exact sequence*

$$0 \to I \to \prod I/F_iI \to \prod I/F_iI \to 0$$

of differential graded A-modules.

Proof. Omitted. Hint: This is dual to Lemma 22.20.1. □

The following lemma shows that differential graded modules with property (I) are the analogue of K-injective modules. See Derived Categories, Definition 13.31.1.

09KS **Lemma 22.21.2.** *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module with property (I). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(N, I) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on I as in property (I). The short exact sequence of Lemma 22.21.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, I/F_iI) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules I/F_iI has a finite filtration by admissible monomorphisms, whose graded pieces are products of $A^\vee[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, A^\vee[k]) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.19.3 and the fact that $(-)^\vee$ is an exact functor. □

09KT **Lemma 22.21.3.** *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules with the following properties*

1. *$M \to I$ is injective,*
2. *$\mathrm{Coker}(d_M) \to \mathrm{Coker}(d_I)$ is injective, and*
3. *I sits in an admissible short exact sequence $0 \to I' \to I \to I'' \to 0$ where I', I'' are products of shifts of $A^\vee$.*

Proof. We will use the functors $N \mapsto N^\vee$ (from left to right differential graded modules and from right to left differential graded modules) constructed in Section 22.19 and all of their properties. For every $k \in \mathbf{Z}$ let Q_k be the free left A-module with generators x, y in degrees k and $k+1$. Define the structure of a left differential graded A-module on Q_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. Arguing exactly as in the proof of Lemma 22.20.3 we find a surjection

$$\bigoplus\nolimits_{i \in I} Q_{k_i} \longrightarrow M^\vee$$

of left differential graded A-modules. Then we can consider the injection

$$M \to (M^\vee)^\vee \to (\bigoplus\nolimits_{i \in I} Q_{k_i})^\vee = \prod\nolimits_{i \in I} I_{k_i}$$

where $I_k = Q_{-k}^\vee$ is the "dual" right differential graded A-module. Further, the short exact sequence $0 \to A[-k-1] \to Q_k \to A[-k] \to 0$ produces a short exact sequence $0 \to A^\vee[k] \to I_k \to A^\vee[k+1] \to 0$.

The result of the previous paragraph produces $M \to I$ having properties (1) and (3). To obtain property (2), suppose $\overline{m} \in \mathrm{Coker}(\mathrm{d}_M)$ is a nonzero element of degree k. Pick a map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ which vanishes on $\mathrm{Im}(M^{k-1} \to M^k)$ but not on m. By Lemma 22.19.3 this corresponds to a homomorphism $M \to A^\vee[k]$ of differential graded A-modules which does not vanish on m. Hence we can achieve (2) by adding a product of copies of shifts of $A^\vee$. □

09KU **Lemma 22.21.4.** *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules such that*

(1) *$M \to I$ is a quasi-isomorphism, and*
(2) *I has property (I).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_i \to I_i \to M_{i+1} \to 0$$

where the maps $M_i \to I_i$ are chosen as in Lemma 22.21.3. This gives a "resolution"

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_1 \to \ldots$$

Denote I the differential graded A-module with graded parts

$$I^n = \prod\nolimits_{i \geq 0} I_i^{n-i}$$

and differential defined by

$$\mathrm{d}_I(x) = f_i(x) + (-1)^i \mathrm{d}_{I_i}(x)$$

for $x \in I_i^{n-i}$. With these conventions I is indeed a differential graded A-module. Recalling that each I_i has a two step filtration $0 \to I_i' \to I_i \to I_i'' \to 0$ we set

$$F_{2i} I^n = \prod\nolimits_{j \geq i} I_j^{n-j} \subset \prod\nolimits_{i \geq 0} I_i^{n-i} = I^n$$

and we add a factor I_{i+1}' to $F_{2i}I$ to get $F_{2i+1}I$. These are differential graded submodules and the successive quotients are products of shifts of $A^\vee$. By Lemma 22.19.1 we see that the inclusions $F_{i+1}I \to F_iI$ are admissible monomorphisms. Finally, we have to show that the map $M \to I$ (given by the augmentation $M \to I_0$) is a quasi-isomorphism. This follows from Homology, Lemma 12.26.3. □

22.22. The derived category

09KV Recall that the notions of acyclic differential graded modules and quasi-isomorphism of differential graded modules make sense (see Section 22.4).

09KW **Lemma 22.22.1.** *Let (A, d) be a differential graded algebra. The full subcategory Ac of $K(Mod_{(A,d)})$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(Mod_{(A,d)})$. The corresponding saturated multiplicative system (see Derived Categories, Lemma 13.6.10) of $K(Mod_{(A,d)})$ is the class Qis of quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Mod_{(A,d)}) \to Qis^{-1} K(Mod_{(A,d)})$$

is Ac. Moreover, the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by the long exact sequence of homology (22.4.2.1). The kernel of H^0 is the subcategory of acyclic objects and the arrows with induce isomorphisms on all H^i are the quasi-isomorphisms. Thus this lemma is a special case of Derived Categories, Lemma 13.6.11.

Set theoretical remark. The construction of the localization in Derived Categories, Proposition 13.5.6 assumes the given triangulated category is "small", i.e., that the underlying collection of objects forms a set. Let V_α be a partial universe (as in Sets, Section 3.5) containing (A, d) and where the cofinality of α is bigger than $\aleph_0$ (see Sets, Proposition 3.7.2). Then we can consider the category $\mathrm{Mod}_{(A,\mathrm{d}),\alpha}$

of differential graded A-modules contained in V_α. A straightforward check shows that all the constructions used in the proof of Proposition 22.10.3 work inside of $\mathrm{Mod}_{(A,\mathrm{d}),\alpha}$ (because at worst we take finite direct sums of differential graded modules). Thus we obtain a triangulated category $\mathrm{Qis}_\alpha^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\alpha})$. We will see below that if $\beta > \alpha$, then the transition functors

$$\mathrm{Qis}_\alpha^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\alpha}) \longrightarrow \mathrm{Qis}_\beta^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\beta})$$

are fully faithful as the morphism sets in the quotient categories are computed by maps in the homotopy categories from P-resolutions (the construction of a P-resolution in the proof of Lemma 22.20.4 takes countable direct sums as well as direct sums indexed over subsets of the given module). The reader should therefore think of the category of the lemma as the union of these subcategories. □

Taking into account the set theoretical remark at the end of the proof of the preceding lemma we define the derived category as follows.

09KX **Definition 22.22.2.** Let (A, d) be a differential graded algebra. Let Ac and Qis be as in Lemma 22.22.1. The *derived category of* (A, d) is the triangulated category

$$D(A, \mathrm{d}) = K(\mathrm{Mod}_{(A,\mathrm{d})})/\mathrm{Ac} = \mathrm{Qis}^{-1}K(\mathrm{Mod}_{(A,\mathrm{d})}).$$

We denote $H^0 : D(A, \mathrm{d}) \to \mathrm{Mod}_R$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above.

Here is the promised lemma computing morphism sets in the derived category.

09KY **Lemma 22.22.3.** *Let (A, d) be a differential graded algebra. Let M and N be differential graded A-modules.*

(1) *Let $P \to M$ be a P-resolution as in Lemma 22.20.4. Then*

$$\mathrm{Hom}_{D(A,d)}(M, N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(P, N)$$

(2) *Let $N \to I$ be an I-resolution as in Lemma 22.21.4. Then*

$$\mathrm{Hom}_{D(A,d)}(M, N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(M, I)$$

Proof. Let $P \to M$ be as in (1). Since $P \to M$ is a quasi-isomorphism we see that

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, N) = \mathrm{Hom}_{D(A,\mathrm{d})}(M, N)$$

by definition of the derived category. A morphism $f : P \to N$ in $D(A, \mathrm{d})$ is equal to $s^{-1}f'$ where $f' : P \to N'$ is a morphism and $s : N \to N'$ is a quasi-isomorphism. Choose a distinguished triangle

$$N \to N' \to Q \to N[1]$$

As s is a quasi-isomorphism, we see that Q is acyclic. Thus $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P, Q[k]) = 0$ for all k by Lemma 22.20.2. Since $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P, -)$ is cohomological, we conclude that we can lift $f' : P \to N'$ uniquely to a morphism $f : P \to N$. This finishes the proof.

The proof of (2) is dual to that of (1) using Lemma 22.21.2 in stead of Lemma 22.20.2. □

09QI **Lemma 22.22.4.** *Let (A, d) be a differential graded algebra. Then*

(1) *$D(A, d)$ has both direct sums and products,*

(2) *direct sums are obtained by taking direct sums of differential graded modules,*
(3) *products are obtained by taking products of differential graded modules.*

Proof. We will use that $\mathrm{Mod}_{(A,\mathrm{d})}$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. See Lemmas 22.4.2 and 22.5.4.

Let M_j be a family of differential graded A-modules. Consider the graded direct sum $M = \bigoplus M_j$ which is a differential graded A-module with the obvious. For a differential graded A-module N choose a quasi-isomorphism $N \to I$ where I is a differential graded A-module with property (I). See Lemma 22.21.4. Using Lemma 22.22.3 we have

$$\begin{aligned}\mathrm{Hom}_{D(A,\mathrm{d})}(M, N) &= \mathrm{Hom}_{K(A,\mathrm{d})}(M, I) \\ &= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(M_j, I) \\ &= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(M_j, N)\end{aligned}$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (2) of the lemma.

Let M_j be a family of differential graded A-modules. Consider the product $M = \prod M_j$ of differential graded A-modules. For a differential graded A-module N choose a quasi-isomorphism $P \to N$ where P is a differential graded A-module with property (P). See Lemma 22.20.4. Using Lemma 22.22.3 we have

$$\begin{aligned}\mathrm{Hom}_{D(A,\mathrm{d})}(N, M) &= \mathrm{Hom}_{K(A,\mathrm{d})}(P, M) \\ &= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(P, M_j) \\ &= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(N, M_j)\end{aligned}$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (3) of the lemma. □

0FQE **Remark 22.22.5.** Let R be a ring. Let (A, d) be a differential graded R-algebra. Using P-resolutions we can sometimes reduce statements about general objects of $D(A, \mathrm{d})$ to statements about $A[k]$. Namely, let T be a property of objects of $D(A, \mathrm{d})$ and assume that

(1) if K_i, $i \in I$ is a family of objects of $D(A, \mathrm{d})$ and $T(K_i)$ holds for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle of $D(A, \mathrm{d})$ and T holds for two, then T holds for the third object, and
(3) $T(A[k])$ holds for all $k \in \mathbf{Z}$.

Then T holds for all objects of $D(A, \mathrm{d})$. This is clear from Lemmas 22.20.1 and 22.20.4.

22.23. The canonical delta-functor

09KZ Let (A, d) be a differential graded algebra. Consider the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to K(\mathrm{Mod}_{(A,\mathrm{d})})$. This functor is **not** a δ-functor in general. However, it turns out that the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to D(A, \mathrm{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to K \xrightarrow{a} L \xrightarrow{b} M \to 0$$

in the abelian category $\mathrm{Mod}_{(A,\mathrm{d})}$. Consider the cone $C(a)$ of the morphism a. We have $C(a) = L \oplus K$ and we define $q : C(a) \to M$ via the projection to L followed by b. Hence a homomorphism of differential graded A-modules

$$q : C(a) \longrightarrow M.$$

It is clear that $q \circ i = b$ where i is as in Definition 22.6.1. Note that, as a is injective, the kernel of q is identified with the cone of id_K which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 22.9.4 the triangle

$$(K, L, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As the localization functor $K(\mathrm{Mod}_{(A,\mathrm{d})}) \to D(A, \mathrm{d})$ is exact we see that $(K, L, C(a), a, i, -p)$ is a distinguished triangle in $D(A, \mathrm{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(A, \mathrm{d})$. Hence we deduce that

$$(K, L, M, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(A, \mathrm{d})$. This suggests the following lemma.

09L0 **Lemma 22.23.1.** *Let (A, d) be a differential graded algebra. The functor $Mod_{(A,d)} \to D(A, d)$ defined has the natural structure of a δ-functor, with*

$$\delta_{K \to L \to M} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 13.3.6. This follows from Lemma 22.6.2 with a bit of work. Compare with Derived Categories, Lemma 13.12.1. □

0CRL **Lemma 22.23.2.** *Let (A, d) be a differential graded algebra. Let M_n be a system of differential graded modules. Then the derived colimit $hocolim M_n$ in $D(A, d)$ is represented by the differential graded module* $\operatorname{colim} M_n$.

Proof. Set $M = \operatorname{colim} M_n$. We have an exact sequence of differential graded modules

$$0 \to \bigoplus M_n \to \bigoplus M_n \to M \to 0$$

by Derived Categories, Lemma 13.33.6 (applied the underlying complexes of abelian groups). The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 22.22.4. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 22.23.1). □

22.24. Linear categories

09MI Just the definitions.

09MJ **Definition 22.24.1.** Let R be a ring. An *R-linear category* $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition law

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \times \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

is R-bilinear.

Thus composition determines an R-linear map

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x,z)$$

of R-modules. Note that we do not assume R-linear categories to be additive.

09MK **Definition 22.24.2.** Let R be a ring. A *functor of R-linear categories*, or an *R-linear functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{B}}(F(x), F(y))$ is a homomorphism of R-modules.

22.25. Graded categories

09L1 Just some definitions.

09L2 **Definition 22.25.1.** Let R be a ring. A *graded category $\mathcal{A}$ over R* is a category where every morphism set is given the structure of a graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x,z)$$

of graded R-modules (i.e., preserving degrees).

In this situation we denote $\mathrm{Hom}^i_{\mathcal{A}}(x,y)$ the degree i part of the graded object $\mathrm{Hom}_{\mathcal{A}}(x,y)$, so that

$$\mathrm{Hom}_{\mathcal{A}}(x,y) = \bigoplus\nolimits_{i \in \mathbf{Z}} \mathrm{Hom}^i_{\mathcal{A}}(x,y)$$

is the direct sum decomposition into graded parts.

09L3 **Definition 22.25.2.** Let R be a ring. A *functor of graded categories over R*, or a *graded functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of graded R-modules.

Given a graded category we are often interested in the corresponding "usual" category of maps of degree 0. Here is a formal definition.

09ML **Definition 22.25.3.** Let R be a ring. Let $\mathcal{A}$ be a graded category over R. We let $\mathcal{A}^0$ be the category with the same objects as $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathcal{A}^0}(x,y) = \mathrm{Hom}^0_{\mathcal{A}}(x,y)$$

the degree 0 graded piece of the graded module of morphisms of $\mathcal{A}$.

09P2 **Definition 22.25.4.** Let R be a ring. Let $\mathcal{A}$ be a graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a *graded direct sum* if i, j, p, q are homogeneous of degree 0.

09MM **Example 22.25.5** (Graded category of graded objects)**.** Let $\mathcal{B}$ be an additive category. Recall that we have defined the category $\mathrm{Gr}(\mathcal{B})$ of graded objects of $\mathcal{B}$ in Homology, Definition 12.16.1. In this example, we will construct a graded category $\mathrm{Gr}^{gr}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category $\mathrm{Gr}^{gr}(\mathcal{B})^0$ recovers $\mathrm{Gr}(\mathcal{B})$. As objects of $\mathrm{Gr}^{gr}(\mathcal{B})$ we take graded objects of $\mathcal{B}$. Then, given graded objects $A = (A^i)$ and $B = (B^i)$ of $\mathcal{B}$ we set

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A,B) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(A,B)$$

where the graded piece of degree n is the abelian group of homogeneous maps of degree n from A to B. Explicitly we have

$$\mathrm{Hom}^n(A,B) = \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe that we have a product here and not a direct sum). In other words, a degree n morphism f from A to B can be seen as a system $f = (f_{p,q})$ where $p, q \in \mathbf{Z}$, $p + q = n$ with $f_{p,q} : A^{-q} \to B^p$ a morphism of $\mathcal{B}$. Given graded objects A, B, C of $\mathcal{B}$ composition of morphisms in $\mathrm{Gr}^{gr}(\mathcal{B})$ is defined via the maps

$$\mathrm{Hom}^m(B, C) \times \mathrm{Hom}^n(A, B) \longrightarrow \mathrm{Hom}^{n+m}(A, C)$$

by simple composition $(g, f) \mapsto g \circ f$ of homogeneous maps of graded objects. In terms of components we have

$$(g \circ f)_{p,r} = g_{p,q} \circ f_{-q,r}$$

where q is such that $p + q = m$ and $-q + r = n$.

09MN **Example 22.25.6** (Graded category of graded modules)**.** Let A be a $\mathbf{Z}$-graded algebra over a ring R. We will construct a graded category Mod_A^{gr} over R whose associated category $(\mathrm{Mod}_A^{gr})^0$ is the category of graded A-modules. As objects of Mod_A^{gr} we take right graded A-modules (see Section 22.14). Given graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}_A^{gr}}(L, M) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(L, M)$$

where $\mathrm{Hom}^n(L, M)$ is the set of right A-module maps $L \to M$ which are homogeneous of degree n, i.e., $f(L^i) \subset M^{i+n}$ for all $i \in \mathbf{Z}$. In terms of components, we have that

$$\mathrm{Hom}^n(L, M) \subset \prod\nolimits_{p+q=n} \mathrm{Hom}_R(L^{-q}, M^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for $a \in A^i$ and $m \in L^{-q-i}$. For graded A-modules K, L, M we define composition in Mod_A^{gr} via the maps

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by simple composition of right A-module maps: $(g, f) \mapsto g \circ f$.

09P3 **Remark 22.25.7.** Let R be a ring. Let $\mathcal{D}$ be an R-linear category endowed with a collection of R-linear functors $[n] : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$ indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}}$ (equality as functors). This allows us to construct a graded category $\mathcal{D}^{gr}$ over R with the same objects of $\mathcal{D}$ setting

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(x, y[n])$$

for x, y in $\mathcal{D}$. Observe that $(\mathcal{D}^{gr})^0 = \mathcal{D}$ (see Definition 22.25.3). Moreover, the graded category $\mathcal{D}^{gr}$ inherits R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}^{gr}}$ with the property that

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y[n]) = \mathrm{Hom}_{\mathcal{D}^{gr}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms.

Conversely, suppose given a graded category $\mathcal{A}$ over R endowed with a collection of R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{A}}$ which are moreover equipped with isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms. Then the reader easily shows that $\mathcal{A} = (\mathcal{A}^0)^{gr}$.

Here are two examples of the relationship $\mathcal{D} \leftrightarrow \mathcal{A}$ we established above:

(1) Let $\mathcal{B}$ be an additive category. If $\mathcal{D} = \mathrm{Gr}(\mathcal{B})$, then $\mathcal{A} = \mathrm{Gr}^{gr}(\mathcal{B})$ as in Example 22.25.5.
(2) If A is a graded ring and $\mathcal{D} = \mathrm{Mod}_A$ is the category of graded right A-modules, then $\mathcal{A} = \mathrm{Mod}_A^{gr}$, see Example 22.25.6.

22.26. Differential graded categories

09L4 Note that if R is a ring, then R is a differential graded algebra over itself (with $R = R^0$ of course). In this case a differential graded R-module is the same thing as a complex of R-modules. In particular, given two differential graded R-modules M and N we denote $M \otimes_R N$ the differential graded R-module corresponding to the total complex associated to the double complex obtained by the tensor product of the complexes of R-modules associated to M and N.

09L5 **Definition 22.26.1.** Let R be a ring. A *differential graded category $\mathcal{A}$ over R* is a category where every morphism set is given the structure of a differential graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of differential graded R-modules.

The final condition of the definition signifies the following: if $f \in \mathrm{Hom}^n_{\mathcal{A}}(x, y)$ and $g \in \mathrm{Hom}^m_{\mathcal{A}}(y, z)$ are homogeneous of degrees n and m, then

$$\mathrm{d}(g \circ f) = \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)$$

in $\mathrm{Hom}^{n+m+1}_{\mathcal{A}}(x, z)$. This follows from the sign rule for the differential on the total complex of a double complex, see Homology, Definition 12.18.3.

09L6 **Definition 22.26.2.** Let R be a ring. A *functor of differential graded categories over R* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of differential graded R-modules.

Given a differential graded category we are often interested in the corresponding categories of complexes and homotopy category. Here is a formal definition.

09L7 **Definition 22.26.3.** Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Then we let

(1) the *category of complexes of* $\mathcal{A}$[1] be the category $\mathrm{Comp}(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with
$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = \mathrm{Ker}(d : \mathrm{Hom}^0_{\mathcal{A}}(x, y) \to \mathrm{Hom}^1_{\mathcal{A}}(x, y))$$
(2) the *homotopy category of* $\mathcal{A}$ be the category $K(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with
$$\mathrm{Hom}_{K(\mathcal{A})}(x, y) = H^0(\mathrm{Hom}_{\mathcal{A}}(x, y))$$

[1] This may be nonstandard terminology.

Our use of the symbol $K(\mathcal{A})$ is nonstandard, but at least is compatible with the use of $K(-)$ in other chapters of the Stacks project.

09P4 **Definition 22.26.4.** Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a *differential graded direct sum* if i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc.

09L8 **Lemma 22.26.5.** *Let R be a ring. A functor $F : \mathcal{A} \to \mathcal{B}$ of differential graded categories over R induces functors $Comp(\mathcal{A}) \to Comp(\mathcal{B})$ and $K(\mathcal{A}) \to K(\mathcal{B})$.*

Proof. Omitted. □

09L9 **Example 22.26.6** (Differential graded category of complexes)**.** Let $\mathcal{B}$ be an additive category. We will construct a differential graded category $\mathrm{Comp}^{dg}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category of complexes is $\mathrm{Comp}(\mathcal{B})$ and whose associated homotopy category is $K(\mathcal{B})$. As objects of $\mathrm{Comp}^{dg}(\mathcal{B})$ we take complexes of $\mathcal{B}$. Given complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{B}$, we sometimes also denote $A^\bullet$ and $B^\bullet$ the corresponding graded objects of $\mathcal{B}$ (i.e., forget about the differential). Using this abuse of notation, we set

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A^\bullet, B^\bullet) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(A, B)$$

as a graded $\mathbf{Z}$-module with notation and definitions as in Example 22.25.5. In other words, the nth graded piece is the abelian group of homogeneous morphism of degree n of graded objects

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

Observe reversal of indices and observe we have a direct product and not a direct sum. For an element $f \in \mathrm{Hom}^n(A^\bullet, B^\bullet)$ of degree n we set

$$\mathrm{d}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A$$

The sign is exactly as in More on Algebra, Section 15.73. To make sense of this we think of d_B and d_A as maps of graded objects of $\mathcal{B}$ homogeneous of degree 1 and we use composition in the category $\mathrm{Gr}^{gr}(\mathcal{B})$ on the right hand side. In terms of components, if $f = (f_{p,q})$ with $f_{p,q} : A^{-q} \to B^p$ we have

09LA (22.26.6.1) $$\mathrm{d}(f_{p,q}) = \mathrm{d}_B \circ f_{p,q} - (-1)^{p+q} f_{p,q} \circ \mathrm{d}_A$$

Note that the first term of this expression is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^{p+1})$ and the second term is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q-1}, B^p)$. The reader checks that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(A^\bullet, B^\bullet)$ has $\mathrm{d}(f) = 0$ if and only if the morphism $f : A^\bullet \to B^\bullet[n]$ of graded objects of $\mathcal{B}$ is actually a map of complexes,
(3) in particular, the category of complexes of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $\mathrm{Comp}(\mathcal{B})$,
(4) the morphism of complexes defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(A^\bullet, B^\bullet)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet))$$

and the homotopy category of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $K(\mathcal{B})$.

Given complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ we define composition

$$\mathrm{Hom}^m(B^\bullet, C^\bullet) \times \mathrm{Hom}^n(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(A^\bullet, C^\bullet)$$

by composition $(g, f) \mapsto g \circ f$ in the graded category $\mathrm{Gr}^{gr}(\mathcal{B})$, see Example 22.25.5. This defines a map of differential graded modules

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(B^\bullet, C^\bullet) \otimes_R \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, C^\bullet)$$

as required in Definition 22.26.1 because

$$\begin{aligned} \mathrm{d}(g \circ f) &= \mathrm{d}_C \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_A \\ &= (\mathrm{d}_C \circ g - (-1)^m g \circ \mathrm{d}_B) \circ f + (-1)^m g \circ (\mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A) \\ &= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f) \end{aligned}$$

as desired.

09LB **Lemma 22.26.7.** *Let $F : \mathcal{B} \to \mathcal{B}'$ be an additive functor between additive categories. Then F induces a functor of differential graded categories*

$$F : \mathit{Comp}^{dg}(\mathcal{B}) \to \mathit{Comp}^{dg}(\mathcal{B}')$$

of Example 22.26.6 inducing the usual functors on the category of complexes and the homotopy categories.

Proof. Omitted. □

Example 22.26.8 (Differential graded category of differential graded modules)**.**
09LC Let (A, d) be a differential graded algebra over a ring R. We will construct a differential graded category $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ over R whose category of complexes is $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose homotopy category is $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As objects of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ we take the differential graded A-modules. Given differential graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) = \mathrm{Hom}_{\mathrm{Mod}^{gr}_A}(L, M) = \bigoplus \mathrm{Hom}^n(L, M)$$

as a graded R-module where the right hand side is defined as in Example 22.25.6. In other words, the nth graded piece $\mathrm{Hom}^n(L, M)$ is the R-module of right A-module maps homogeneous of degree n. For an element $f \in \mathrm{Hom}^n(L, M)$ we set

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

To make sense of this we think of d_M and d_L as graded R-module maps and we use composition of graded R-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n + 1$ as a graded R-module map, and it is A-linear because

$$\begin{aligned} \mathrm{d}(f)(xa) &= \mathrm{d}_M(f(x)a) - (-1)^n f(\mathrm{d}_L(xa)) \\ &= \mathrm{d}_M(f(x))a + (-1)^{\deg(x)+n} f(x)\mathrm{d}(a) - (-1)^n f(\mathrm{d}_L(x))a - (-1)^{n+\deg(x)} f(x)\mathrm{d}(a) \\ &= \mathrm{d}(f)(x)a \end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom). Similar formulae to those of Example 22.26.6 hold for the differential of f in terms of components. The reader checks (in the same way as in Example 22.26.6) that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(L, M)$ has $\mathrm{d}(f) = 0$ if and only if $f : L \to M[n]$ is a homomorphism of differential graded A-modules,

(3) in particular, the category of complexes of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $\mathrm{Mod}_{(A,\mathrm{d})}$,
(4) the homomorphism defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(L, M)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(L, M) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M))$$

and the homotopy category of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Given differential graded A-modules K, L, M we define composition

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by composition of homogeneous right A-module maps $(g, f) \mapsto g \circ f$. This defines a map of differential graded modules

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) \otimes_R \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K, L) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K, M)$$

as required in Definition 22.26.1 because

$$\begin{aligned} \mathrm{d}(g \circ f) &= \mathrm{d}_M \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_K \\ &= (\mathrm{d}_M \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_L \circ f - (-1)^n f \circ \mathrm{d}_K) \\ &= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f) \end{aligned}$$

as desired.

09LD **Lemma 22.26.9.** *Let $\varphi : (A, d) \to (E, d)$ be a homomorphism of differential graded algebras. Then φ induces a functor of differential graded categories*

$$F : Mod^{dg}_{(E,d)} \longrightarrow Mod^{dg}_{(A,d)}$$

of Example 22.26.8 inducing obvious restriction functors on the categories of differential graded modules and homotopy categories.

Proof. Omitted. □

09LE **Lemma 22.26.10.** *Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Let x be an object of $\mathcal{A}$. Let*

$$(E, d) = \mathrm{Hom}_{\mathcal{A}}(x, x)$$

be the differential graded R-algebra of endomorphisms of x. We obtain a functor

$$\mathcal{A} \longrightarrow Mod^{dg}_{(E,d)}, \quad y \longmapsto \mathrm{Hom}_{\mathcal{A}}(x, y)$$

of differential graded categories by letting E act on $\mathrm{Hom}_{\mathcal{A}}(x, y)$ via composition in $\mathcal{A}$. This functor induces functors

$$Comp(\mathcal{A}) \to Mod_{(A,d)} \quad \text{and} \quad K(\mathcal{A}) \to K(Mod_{(A,d)})$$

by an application of Lemma 22.26.5.

Proof. This lemma proves itself. □

22.27. Obtaining triangulated categories

09P5 In this section we discuss the most general setup to which the arguments proving Derived Categories, Proposition 13.10.3 and Proposition 22.10.3 apply.

Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. To make our argument work, we impose some axioms on $\mathcal{A}$:

(A) $\mathcal{A}$ has a zero object and differential graded direct sums of two objects (as in Definition 22.26.4).

(B) there are functors $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ of differential graded categories such that $[0] = \mathrm{id}_{\mathcal{A}}$ and $[n + m] = [n] \circ [m]$ and given isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

of differential graded R-modules compatible with composition.

Given our differential graded category $\mathcal{A}$ we say

(1) a sequence $x \to y \to z$ of morphisms of $\mathrm{Comp}(\mathcal{A})$ is an *admissible short exact sequence* if there exists an isomorphism $y \cong x \oplus z$ in the underlying graded category such that $x \to z$ and $y \to z$ are (co)projections.

(2) a morphism $x \to y$ of $\mathrm{Comp}(\mathcal{A})$ is an *admissible monomorphism* if it extends to an admissible short exact sequence $x \to y \to z$.

(3) a morphism $y \to z$ of $\mathrm{Comp}(\mathcal{A})$ is an *admissible epimorphism* if it extends to an admissible short exact sequence $x \to y \to z$.

The next lemma tells us an admissible short exact sequence gives a triangle, provided we have axioms (A) and (B).

09P6 **Lemma 22.27.1.** *Let $\mathcal{A}$ be a differential graded category satisfying axioms (A) and (B). Given an admissible short exact sequence $x \to y \to z$ we obtain (see proof) a triangle*

$$x \to y \to z \to x[1]$$

in $\mathrm{Comp}(\mathcal{A})$ with the property that any two compositions in $z[-1] \to x \to y \to z \to x[1]$ are zero in $K(\mathcal{A})$.

Proof. Choose a diagram

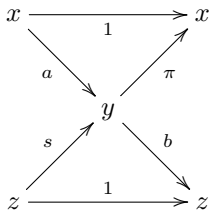

giving the isomorphism of graded objects $y \cong x \oplus z$ as in the definition of an admissible short exact sequence. Here are some equations that hold in this situation

(1) $1 = \pi a$ and hence $\mathrm{d}(\pi)a = 0$,

(2) $1 = bs$ and hence $b\mathrm{d}(s) = 0$,

(3) $1 = a\pi + sb$ and hence $a\mathrm{d}(\pi) + \mathrm{d}(s)b = 0$,

(4) $\pi s = 0$ and hence $\mathrm{d}(\pi)s + \pi\mathrm{d}(s) = 0$,

(5) $\mathrm{d}(s) = a\pi\mathrm{d}(s)$ because $\mathrm{d}(s) = (a\pi + sb)\mathrm{d}(s)$ and $b\mathrm{d}(s) = 0$,

(6) $\mathrm{d}(\pi) = \mathrm{d}(\pi)sb$ because $\mathrm{d}(\pi) = \mathrm{d}(\pi)(a\pi + sb)$ and $\mathrm{d}(\pi)a = 0$,

(7) $\mathrm{d}(\pi\mathrm{d}(s)) = 0$ because if we postcompose it with the monomorphism a we get $\mathrm{d}(a\pi\mathrm{d}(s)) = \mathrm{d}(\mathrm{d}(s)) = 0$, and

(8) $\mathrm{d}(\mathrm{d}(\pi)s) = 0$ as by (4) it is the negative of $\mathrm{d}(\pi\mathrm{d}(s))$ which is 0 by (7).

We've used repeatedly that $\mathrm{d}(a) = 0$, $\mathrm{d}(b) = 0$, and that $\mathrm{d}(1) = 0$. By (7) we see that

$$\delta = \pi\mathrm{d}(s) = -\mathrm{d}(\pi)s : z \to x[1]$$

is a morphism in $\mathrm{Comp}(\mathcal{A})$. By (5) we see that the composition $a\delta = a\pi\mathrm{d}(s) = \mathrm{d}(s)$ is homotopic to zero. By (6) we see that the composition $\delta b = -\mathrm{d}(\pi)sb = \mathrm{d}(-\pi)$ is homotopic to zero. □

Besides axioms (A) and (B) we need an axiom concerning the existence of cones. We formalize everything as follows.

09QJ **Situation 22.27.2.** Here R is a ring and $\mathcal{A}$ is a differential graded category over R having axioms (A), (B), and

(C) given an arrow $f : x \to y$ of degree 0 with $\mathrm{d}(f) = 0$ there exists an admissible short exact sequence $y \to c(f) \to x[1]$ in $\mathrm{Comp}(\mathcal{A})$ such that the map $x[1] \to y[1]$ of Lemma 22.27.1 is equal to $f[1]$.

We will call $c(f)$ a *cone* of the morphism f. If (A), (B), and (C) hold, then cones are functorial in a weak sense.

09P7 **Lemma 22.27.3.** *In Situation 22.27.2 suppose that*

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ x_2 & \xrightarrow{f_2} & y_2 \end{array}$$

is a diagram of $\mathrm{Comp}(\mathcal{A})$ commutative up to homotopy. Then there exists a morphism $c : c(f_1) \to c(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (x_1, y_1, c(f_1)) \to (x_1, y_1, c(f_1))$$

in $K(\mathcal{A})$.

Proof. The assumption means there exists a morphism $h : x_1 \to y_2$ of degree -1 such that $\mathrm{d}(h) = bf_1 - f_2a$. Choose isomorphisms $c(f_i) = y_i \oplus x_i[1]$ of graded objects compatible with the morphisms $y_i \to c(f_i) \to x_i[1]$. Let's denote $a_i : y_i \to c(f_i)$, $b_i : c(f_i) \to x_i[1]$, $s_i : x_i[1] \to c(f_i)$, and $\pi_i : c(f_i) \to y_i$ the given morphisms. Recall that $x_i[1] \to y_i[1]$ is given by $\pi_i\mathrm{d}(s_i)$. By axiom (C) this means that

$$f_i = \pi_i\mathrm{d}(s_i) = -\mathrm{d}(\pi_i)s_i$$

(we identify $\mathrm{Hom}(x_i, y_i)$ with $\mathrm{Hom}(x_i[1], y_i[1])$ using the shift functor [1]). Set $c = a_2b\pi_1 + s_2ab_1 + a_2hb$. Then, using the equalities found in the proof of Lemma 22.27.1 we obtain

$$\begin{aligned} \mathrm{d}(c) &= a_2b\mathrm{d}(\pi_1) + \mathrm{d}(s_2)ab_1 + a_2\mathrm{d}(h)b_1 \\ &= -a_2bf_1b_1 + a_2f_2ab_1 + a_2(bf_1 - f_2a)b_1 \\ &= 0 \end{aligned}$$

(where we have used in particular that $\mathrm{d}(\pi_1) = \mathrm{d}(\pi_1)s_1b_1 = f_1b_1$ and $\mathrm{d}(s_2) = a_2\pi_2\mathrm{d}(s_2) = a_2f_2$). Thus c is a degree 0 morphism $c : c(f_1) \to c(f_2)$ of $\mathcal{A}$ compatible with the given morphisms $y_i \to c(f_i) \to x_i[1]$. □

In Situation 22.27.2 we say that a triangle (x, y, z, f, g, h) in $K(\mathcal{A})$ is a *distinguished triangle* if there exists an admissible short exact sequence $x' \to y' \to z'$ such that (x, y, z, f, g, h) is isomorphic as a triangle in $K(\mathcal{A})$ to the triangle $(x', y', z', x' \to y', y' \to z', \delta)$ constructed in Lemma 22.27.1. We will show below that

$$\boxed{K(\mathcal{A}) \text{ is a triangulated category}}$$

This result, although not as general as one might think, applies to a number of natural generalizations of the cases covered so far in the Stacks project. Here are some examples:

(1) Let $(X, \mathcal{O}_X)$ be a ringed space. Let (A, d) be a sheaf of differential graded $\mathcal{O}_X$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.
(2) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (A, d) be a sheaf of differential graded $\mathcal{O}$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category. See Differential Graded Sheaves, Proposition 24.22.4.
(3) Two examples with a different flavor may be found in Examples, Section 110.70.

The following simple lemma is a key to the construction.

09QK **Lemma 22.27.4.** *In Situation 22.27.2 given any object x of $\mathcal{A}$, and the cone $C(1_x)$ of the identity morphism $1_x : x \to x$, the identity morphism on $C(1_x)$ is homotopic to zero.*

Proof. Consider the admissible short exact sequence given by axiom (C).

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} C(1_x) \underset{s}{\overset{b}{\rightleftarrows}} x[1]$$

Then by Lemma 22.27.1, identifying hom-sets under shifting, we have $1_x = \pi d(s) = -d(\pi)s$ where s is regarded as a morphism in $\mathrm{Hom}^{-1}_{\mathcal{A}}(x, C(1_x))$. Therefore $a = a\pi d(s) = d(s)$ using formula (5) of Lemma 22.27.1, and $b = -d(\pi)sb = -d(\pi)$ by formula (6) of Lemma 22.27.1. Hence

$$1_{C(1_x)} = a\pi + sb = d(s)\pi - sd(\pi) = d(s\pi)$$

since s is of degree -1. □

A more general version of the above lemma will appear in Lemma 22.27.13. The following lemma is the analogue of Lemma 22.7.3.

09QL **Lemma 22.27.5.** *In Situation 22.27.2 given a diagram*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ z & \xrightarrow{g} & w \end{array}$$

in Comp($\mathcal{A}$) commuting up to homotopy. Then

(1) *If f is an admissible monomorphism, then b is homotopic to a morphism b' which makes the diagram commute.*
(2) *If g is an admissible epimorphism, then a is homotopic to a morphism a' which makes the diagram commute.*

Proof. To prove (1), observe that the hypothesis implies that there is some $h \in \mathrm{Hom}_{\mathcal{A}}(x, w)$ of degree -1 such that $bf - ga = d(h)$. Since f is an admissible monomorphism, there is a morphism $\pi : y \to x$ in the category $\mathcal{A}$ of degree 0. Let $b' = b - d(h\pi)$. Then

$$\begin{aligned} b'f = bf - d(h\pi)f &= bf - d(h\pi f) \quad (\text{since } d(f) = 0) \\ &= bf - d(h) \\ &= ga \end{aligned}$$

as desired. The proof for (2) is omitted. □

The following lemma is the analogue of Lemma 22.7.4.

09QM **Lemma 22.27.6.** *In Situation 22.27.2 let $\alpha : x \to y$ be a morphism in Comp($\mathcal{A}$). Then there exists a factorization in Comp($\mathcal{A}$):*

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism, and $\pi\tilde{\alpha} = \alpha$.*
(2) *There exists a morphism $s : y \to \tilde{y}$ in Comp($\mathcal{A}$) such that $\pi s = 1_y$ and $s\pi$ is homotopic to $1_{\tilde{y}}$.*

Proof. By axiom (A), we may let $\tilde{y}$ be the differential graded direct sum of y and $C(1_x)$, i.e., there exists a diagram

$$y \underset{\pi}{\overset{s}{\rightleftarrows}} y \oplus C(1_x) \underset{t}{\overset{p}{\rightleftarrows}} C(1_x)$$

where all morphisms are of degree zero, and in $\mathrm{Comp}(\mathcal{A})$. Let $\tilde{y} = y \oplus C(1_x)$. Then $1_{\tilde{y}} = s\pi + tp$. Consider now the diagram

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

where $\tilde{\alpha}$ is induced by the morphism $x \xrightarrow{\alpha} y$ and the natural morphism $x \to C(1_x)$ fitting in the admissible short exact sequence

$$x \rightleftarrows C(1_x) \rightleftarrows x[1]$$

So the morphism $C(1_x) \to x$ of degree 0 in this diagram, together with the zero morphism $y \to x$, induces a degree-0 morphism $\beta : \tilde{y} \to x$. Then $\tilde{\alpha}$ is an admissible monomorphism since it fits into the admissible short exact sequence

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow x[1]$$

Furthermore, $\pi\tilde{\alpha} = \alpha$ by the construction of $\tilde{\alpha}$, and $\pi s = 1_y$ by the first diagram. It remains to show that $s\pi$ is homotopic to $1_{\tilde{y}}$. Write 1_x as $d(h)$ for some degree -1 map. Then, our last statement follows from

$$\begin{aligned} 1_{\tilde{y}} - s\pi &= tp \\ &= t(dh)p \quad (\text{by Lemma 22.27.4}) \\ &= d(thp) \end{aligned}$$

since $dt = dp = 0$, and t is of degree zero. □

The following lemma is the analogue of Lemma 22.7.5.

09QN **Lemma 22.27.7.** *In Situation 22.27.2 let $x_1 \to x_2 \to \ldots \to x_n$ be a sequence of composable morphisms in Comp($\mathcal{A}$). Then there exists a commutative diagram in Comp($\mathcal{A}$):*

$$\begin{array}{ccccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & \ldots & \longrightarrow & x_n \\ \uparrow & & \uparrow & & & & \uparrow \\ y_1 & \longrightarrow & y_2 & \longrightarrow & \ldots & \longrightarrow & y_n \end{array}$$

such that each $y_i \to y_{i+1}$ is an admissible monomorphism and each $y_i \to x_i$ is a homotopy equivalence.

Proof. The case for $n = 1$ is trivial: one simply takes $y_1 = x_1$ and the identity morphism on x_1 is in particular a homotopy equivalence. The case $n = 2$ is given by Lemma 22.27.6. Suppose we have constructed the diagram up to x_{n-1}. We apply Lemma 22.27.6 to the composition $y_{n-1} \to x_{n-1} \to x_n$ to obtain y_n. Then $y_{n-1} \to y_n$ will be an admissible monomorphism, and $y_n \to x_n$ a homotopy equivalence. □

The following lemma is the analogue of Lemma 22.7.6.

09QP **Lemma 22.27.8.** *In Situation 22.27.2 let $x_i \to y_i \to z_i$ be morphisms in $\mathcal{A}$ ($i = 1, 2, 3$) such that $x_2 \to y_2 \to z_2$ is an admissible short exact sequence. Let $b : y_1 \to y_2$ and $b' : y_2 \to y_3$ be morphisms in Comp($\mathcal{A}$) such that*

$$\begin{array}{ccccc} x_1 & \longrightarrow & y_1 & \longrightarrow & z_1 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 0} \\ x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\ x_3 & \longrightarrow & y_3 & \longrightarrow & z_3 \end{array}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.27.5, we can replace b and b' by homotopic maps $\tilde{b}$ and $\tilde{b}'$, such that the right square of the left diagram commutes and the left square of the right diagram commutes. Say $b = \tilde{b} + d(h)$ and $b' = \tilde{b}' + d(h')$ for degree -1 morphisms h and h' in $\mathcal{A}$. Hence

$$b'b = \tilde{b}'\tilde{b} + d(\tilde{b}'h + h'\tilde{b} + h'd(h))$$

since $d(\tilde{b}) = d(\tilde{b}') = 0$, i.e. $b'b$ is homotopic to $\tilde{b}'\tilde{b}$. We now want to show that $\tilde{b}'\tilde{b} = 0$. Because $x_2 \xrightarrow{f} y_2 \xrightarrow{g} z_2$ is an admissible short exact sequence, there exist degree 0 morphisms $\pi : y_2 \to x_2$ and $s : z_2 \to y_2$ such that $\mathrm{id}_{y_2} = f\pi + sg$. Therefore

$$\tilde{b}'\tilde{b} = \tilde{b}'(f\pi + sg)\tilde{b} = 0$$

since $g\tilde{b} = 0$ and $\tilde{b}'f = 0$ as consequences of the two commuting squares. □

The following lemma is the analogue of Lemma 22.8.1.

09QQ **Lemma 22.27.9.** *In Situation 22.27.2 let $0 \to x \to y \to z \to 0$ be an admissible short exact sequence in Comp($\mathcal{A}$). The triangle*

$$x \longrightarrow y \longrightarrow z \xrightarrow{\delta} x[1]$$

with $\delta : z \to x[1]$ as defined in Lemma 22.27.1 is up to canonical isomorphism in $K(\mathcal{A})$, independent of the choices made in Lemma 22.27.1.

Proof. Suppose δ is defined by the splitting

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} y \underset{s}{\overset{b}{\rightleftarrows}} z$$

and δ' is defined by the splitting with π', s' in place of π, s. Then

$$s' - s = (a\pi + sb)(s' - s) = a\pi s'$$

since $bs' = bs = 1_z$ and $\pi s = 0$. Similarly,

$$\pi' - \pi = (\pi' - \pi)(a\pi + sb) = \pi' sb$$

Since $\delta = \pi d(s)$ and $\delta' = \pi' d(s')$ as constructed in Lemma 22.27.1, we may compute

$$\delta' = \pi' d(s') = (\pi + \pi' sb) d(s + a\pi s') = \delta + d(\pi s')$$

using $\pi a = 1_x$, $ba = 0$, and $\pi' sbd(s') = \pi' sba\pi d(s') = 0$ by formula (5) in Lemma 22.27.1. □

The following lemma is the analogue of Lemma 22.9.1.

09QR **Lemma 22.27.10.** *In Situation 22.27.2 let $f : x \to y$ be a morphism in Comp($\mathcal{A}$). The triangle $(y, c(f), x[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$y \longrightarrow c(f) \longrightarrow x[1]$$

where the cone $c(f)$ is defined as in Lemma 22.27.1.

Proof. This follows from axiom (C). □

The following lemma is the analogue of Lemma 22.9.2.

09QS **Lemma 22.27.11.** *In Situation 22.27.2 let $\alpha : x \to y$ and $\beta : y \to z$ define an admissible short exact sequence*

$$x \longrightarrow y \longrightarrow z$$

in Comp($\mathcal{A}$). Let $(x, y, z, \alpha, \beta, \delta)$ be the associated triangle in $K(\mathcal{A})$. Then, the triangles

$$(z[-1], x, y, \delta[-1], \alpha, \beta) \quad \text{and} \quad (z[-1], x, c(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. We have a diagram of the form

$$\begin{array}{ccccccc}
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{\alpha}}{\overset{\alpha}{\rightleftarrows}} & y & \underset{\tilde{\beta}}{\overset{\beta}{\rightleftarrows}} & z \\
\downarrow 1 & & \downarrow 1 & & \vdots & & \downarrow 1 \\
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{i}}{\overset{i}{\rightleftarrows}} & c(\delta[-1]) & \underset{\tilde{p}}{\overset{p}{\rightleftarrows}} & z
\end{array}$$

with splittings to α, β, i, and p given by $\tilde{\alpha}, \tilde{\beta}, \tilde{i}$, and $\tilde{p}$ respectively. Define a morphism $y \to c(\delta[-1])$ by $i\tilde{\alpha} + \tilde{p}\beta$ and a morphism $c(\delta[-1]) \to y$ by $\alpha\tilde{i} + \tilde{\beta}p$. Let us first check that these define morphisms in Comp($\mathcal{A}$). We remark that by identities from Lemma 22.27.1, we have the relation $\delta[-1] = \tilde{\alpha} d(\tilde{\beta}) = -d(\tilde{\alpha})\tilde{\beta}$ and the relation $\delta[-1] = \tilde{i} d(\tilde{p})$. Then

$$\begin{aligned} d(\tilde{\alpha}) &= d(\tilde{\alpha})\tilde{\beta}\beta \\ &= -\delta[-1]\beta \end{aligned}$$

where we have used equation (6) of Lemma 22.27.1 for the first equality and the preceding remark for the second. Similarly, we obtain $d(\tilde{p}) = i\delta[-1]$. Hence

$$\begin{aligned} d(i\tilde{\alpha} + \tilde{p}\beta) &= d(i)\tilde{\alpha} + id(\tilde{\alpha}) + d(\tilde{p})\beta + \tilde{p}d(\beta) \\ &= id(\tilde{\alpha}) + d(\tilde{p})\beta \\ &= -i\delta[-1]\beta + i\delta[-1]\beta \\ &= 0 \end{aligned}$$

so $i\tilde{\alpha} + \tilde{p}\beta$ is indeed a morphism of $\mathrm{Comp}(\mathcal{A})$. By a similar calculation, $\alpha\tilde{i} + \tilde{\beta}p$ is also a morphism of $\mathrm{Comp}(\mathcal{A})$. It is immediate that these morphisms fit in the commutative diagram. We compute:

$$\begin{aligned} (i\tilde{\alpha} + \tilde{p}\beta)(\alpha\tilde{i} + \tilde{\beta}p) &= i\tilde{\alpha}\alpha\tilde{i} + i\tilde{\alpha}\tilde{\beta}p + \tilde{p}\beta\alpha\tilde{i} + \tilde{p}\beta\tilde{\beta}p \\ &= i\tilde{i} + \tilde{p}p \\ &= 1_{c(\delta[-1])} \end{aligned}$$

where we have freely used the identities of Lemma 22.27.1. Similarly, we compute $(\alpha\tilde{i} + \tilde{\beta}p)(i\tilde{\alpha} + \tilde{p}\beta) = 1_y$, so we conclude $y \cong c(\delta[-1])$. Hence, the two triangles in question are isomorphic. □

The following lemma is the analogue of Lemma 22.9.3.

09QT **Lemma 22.27.12.** *In Situation 22.27.2 let $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ be morphisms in Comp($\mathcal{A}$). Let*

$$(a, b, c) : (x_1, y_1, c(f_1), f_1, i_1, p_1) \to (x_2, y_2, c(f_2), f_2, i_1, p_1)$$

be any morphism of triangles in $K(\mathcal{A})$. If a and b are homotopy equivalences, then so is c.

Proof. Since a and b are homotopy equivalences, they are invertible in $K(\mathcal{A})$ so let a^{-1} and b^{-1} denote their inverses in $K(\mathcal{A})$, giving us a commutative diagram

$$\begin{array}{ccccc} x_2 & \xrightarrow{f_2} & y_2 & \xrightarrow{i_2} & c(f_2) \\ \downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle b^{-1}} & & \downarrow{\scriptstyle c'} \\ x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{i_1} & c(f_1) \end{array}$$

where the map c' is defined via Lemma 22.27.3 applied to the left commutative box of the above diagram. Since the diagram commutes in $K(\mathcal{A})$, it suffices by Lemma 22.27.8 to prove the following: given a morphism of triangle $(1, 1, c) : (x, y, c(f), f, i, p) \to (x, y, c(f), f, i, p)$ in $K(\mathcal{A})$, the map c is an isomorphism in $K(\mathcal{A})$. We have the commutative diagrams in $K(\mathcal{A})$:

$$\begin{array}{ccccc} y & \longrightarrow & c(f) & \longrightarrow & x[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\ y & \longrightarrow & c(f) & \longrightarrow & x[1] \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} y & \longrightarrow & c(f) & \longrightarrow & x[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c-1} & & \downarrow{\scriptstyle 0} \\ y & \longrightarrow & c(f) & \longrightarrow & x[1] \end{array}$$

Since the rows are admissible short exact sequences, we obtain the identity $(c-1)^2 = 0$ by Lemma 22.27.8, from which we conclude that $2 - c$ is inverse to c in $K(\mathcal{A})$ so that c is an isomorphism. □

The following lemma is the analogue of Lemma 22.9.4.

09QU **Lemma 22.27.13.** *In Situation 22.27.2.*

(1) *Given an admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then there exists a homotopy equivalence $e : C(\alpha) \to z$ such that the diagram*

09QV (22.27.13.1)
$$\begin{array}{ccccccc} x & \xrightarrow{\alpha} & y & \xrightarrow{b} & C(\alpha) & \xrightarrow{-c} & x[1] \\ \downarrow & & \downarrow & & \downarrow e & & \downarrow \\ x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\delta} & x[1] \end{array}$$

defines an isomorphism of triangles in $K(\mathcal{A})$. Here $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is the admissible short exact sequence given as in axiom (C).

(2) *Given a morphism $\alpha : x \to y$ in $\mathit{Comp}(\mathcal{A})$, let $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ be the factorization given as in Lemma 22.27.6, where the admissible monomorphism $x \xrightarrow{\tilde{\alpha}} y$ extends to the admissible short exact sequence*

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow z$$

Then there exists an isomorphism of triangles

$$\begin{array}{ccccccc} x & \xrightarrow{\tilde{\alpha}} & \tilde{y} & \longrightarrow & z & \xrightarrow{\delta} & x[1] \\ \downarrow & & \downarrow & & \downarrow e & & \downarrow \\ x & \xrightarrow{\alpha} & y & \longrightarrow & C(\alpha) & \xrightarrow{-c} & x[1] \end{array}$$

where the upper triangle is the triangle associated to the sequence $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to z$.

Proof. For (1), we consider the more complete diagram, *without* the sign change on c:

$$\begin{array}{ccccccccc} x & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y & \underset{p}{\overset{b}{\rightleftarrows}} & C(\alpha) & \underset{\sigma}{\overset{c}{\rightleftarrows}} & x[1] & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y[1] \\ \downarrow & & \downarrow & & \uparrow f \ \downarrow e & & \downarrow & & \\ x & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y & \underset{s}{\overset{\beta}{\rightleftarrows}} & z & \xrightarrow{\delta} & x[1] & & \end{array}$$

where the admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ is given the splitting π, s, and the admissible short exact sequence $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is given the splitting p, σ. Note that (identifying hom-sets under shifting)

$$\alpha = pd(\sigma) = -d(p)\sigma, \quad \delta = \pi d(s) = -d(\pi)s$$

by the construction in Lemma 22.27.1.

We define $e = \beta p$ and $f = bs - \sigma\delta$. We first check that they are morphisms in $\mathrm{Comp}(\mathcal{A})$. To show that $d(e) = \beta d(p)$ vanishes, it suffices to show that $\beta d(p)b$ and $\beta d(p)\sigma$ both vanish, whereas

$$\beta d(p)b = \beta d(pb) = \beta d(1_y) = 0, \quad \beta d(p)\sigma = -\beta\alpha = 0$$

Similarly, to check that $d(f) = bd(s) - d(\sigma)\delta$ vanishes, it suffices to check the post-compositions by p and c both vanish, whereas

$$\begin{aligned} pbd(s) - pd(\sigma)\delta =& d(s) - \alpha\delta = d(s) - \alpha\pi d(s) = 0 \\ cbd(s) - cd(\sigma)\delta =& - cd(\sigma)\delta = -d(c\sigma)\delta = 0 \end{aligned}$$

The commutativity of left two squares of the diagram 22.27.13.1 follows directly from definition. Before we prove the commutativity of the right square (up to homotopy), we first check that e is a homotopy equivalence. Clearly,

$$ef = \beta p(bs - \sigma\delta) = \beta s = 1_z$$

To check that fe is homotopic to $1_{C(\alpha)}$, we first observe

$$b\alpha = bpd(\alpha) = d(\sigma), \quad \alpha c = -d(p)\sigma c = -d(p), \quad d(\pi)p = d(\pi)s\beta p = -\delta\beta p$$

Using these identities, we compute

$$\begin{aligned} 1_{C(\alpha)} =& bp + \sigma c \quad (\text{from } y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]) \\ =& b(\alpha\pi + s\beta)p + \sigma(\pi\alpha)c \quad (\text{from } x \xrightarrow{\alpha} y \xrightarrow{\beta} z) \\ =& d(\sigma)\pi p + bs\beta p - \sigma\pi d(p) \quad (\text{by the first two identities above}) \\ =& d(\sigma)\pi p + bs\beta p - \sigma\delta\beta p + \sigma\delta\beta p - \sigma\pi d(p) \\ =& (bs - \sigma\delta)\beta p + d(\sigma)\pi p - \sigma d(\pi)p - \sigma\pi d(p) \quad (\text{by the third identity above}) \\ =& fe + d(\sigma\pi p) \end{aligned}$$

since $\sigma \in \mathrm{Hom}^{-1}(x, C(\alpha))$ (cf. proof of Lemma 22.27.4). Hence e and f are homotopy inverses. Finally, to check that the right square of diagram 22.27.13.1 commutes up to homotopy, it suffices to check that $-cf = \delta$. This follows from

$$-cf = -c(bs - \sigma\delta) = c\sigma\delta = \delta$$

since $cb = 0$.

For (2), consider the factorization $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ given as in Lemma 22.27.6, so the second morphism is a homotopy equivalence. By Lemmas 22.27.3 and 22.27.12, there exists an isomorphism of triangles between

$$x \xrightarrow{\alpha} y \to C(\alpha) \to x[1] \quad \text{and} \quad x \xrightarrow{\tilde{\alpha}} \tilde{y} \to C(\tilde{\alpha}) \to x[1]$$

Since we can compose isomorphisms of triangles, by replacing α by $\tilde{\alpha}$, y by $\tilde{y}$, and $C(\alpha)$ by $C(\tilde{\alpha})$, we may assume α is an admissible monomorphism. In this case, the result follows from (1). □

The following lemma is the analogue of Lemma 22.10.1.

09QW **Lemma 22.27.14.** *In Situation 22.27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. We will verify each of TR1, TR2, and TR3.

Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Since

$$x \xrightarrow{1_x} x \longrightarrow 0$$

is an admissible short exact sequence, $(x, x, 0, 1_x, 0, 0)$ is a distinguished triangle. Moreover, given a morphism $\alpha : x \to y$ in $\mathrm{Comp}(\mathcal{A})$, the triangle given by $(x, y, c(\alpha), \alpha, i, -p)$ is distinguished by Lemma 22.27.13.

Proof of TR2. Let $(x, y, z, \alpha, \beta, \gamma)$ be a triangle and suppose $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to x' \to y' \to z' \to 0$ such that the associated triangle $(x', y', z', \alpha', \beta', \gamma')$ is isomorphic to $(y, z, x[1], \beta, \gamma, -\alpha[1])$. After rotating, we conclude that $(x, y, z, \alpha, \beta, \gamma)$ is isomorphic to $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$. By Lemma 22.27.11, we deduce that $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$ is isomorphic to $(z'[-1], x', c(\gamma'[-1]), \gamma'[-1], i, p)$. Composing the two isomorphisms with sign changes as indicated in the following diagram:

$$\begin{array}{ccccccc}
x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\gamma} & x[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z'[-1] & \xrightarrow{-\gamma'[-1]} & x & \xrightarrow{\alpha'} & y' & \xrightarrow{\beta'} & z' \\
\downarrow{\scriptstyle -1_{z'[-1]}} & & \| & & \downarrow & & \downarrow{\scriptstyle -1_{z'}} \\
z'[-1] & \xrightarrow{\gamma'[-1]} & x & \xrightarrow{\alpha'} & c(\gamma'[-1]) & \xrightarrow{-p} & z'
\end{array}$$

We conclude that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished by Lemma 22.27.13 (2). Conversely, suppose that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished, so that by Lemma 22.27.13 (1), it is isomorphic to a triangle of the form $(x', y', c(\alpha'), \alpha', i, -p)$ for some morphism $\alpha' : x' \to y'$ in $\mathrm{Comp}(\mathcal{A})$. The rotated triangle $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is isomorphic to the triangle $(y', c(\alpha'), x'[1], i, -p, -\alpha[1])$ which is isomorphic to $(y', c(\alpha'), x'[1], i, p, \alpha[1])$. By Lemma 22.27.10, this triangle is distinguished, from which it follows that $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished.

Proof of TR3: Suppose $(x, y, z, \alpha, \beta, \gamma)$ and $(x', y', z', \alpha', \beta', \gamma')$ are distinguished triangles of $\mathrm{Comp}(\mathcal{A})$ and let $f : x \to x'$ and $g : y \to y'$ be morphisms such that $\alpha' \circ f = g \circ \alpha$. By Lemma 22.27.13, we may assume that $(x, y, z, \alpha, \beta, \gamma) = (x, y, c(\alpha), \alpha, i, -p)$ and $(x', y', z', \alpha', \beta', \gamma') = (x', y', c(\alpha'), \alpha', i', -p')$. Now apply Lemma 22.27.3 and we are done. □

The following lemma is the analogue of Lemma 22.10.2.

09QX **Lemma 22.27.15.** *In Situation 22.27.2 given admissible monomorphisms $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ in $\mathcal{A}$, there exist distinguished triangles $(x, y, q_1, \alpha, p_1, \delta_1)$, $(x, z, q_2, \beta\alpha, p_2, \delta_2)$ and $(y, z, q_3, \beta, p_3, \delta_3)$ for which TR4 holds.*

Proof. Given admissible monomorphisms $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$, we can find distinguished triangles, via their extensions to admissible short exact sequences,

$$x \underset{\pi_1}{\overset{\alpha}{\rightleftarrows}} y \underset{s_1}{\overset{p_1}{\rightleftarrows}} q_1 \xrightarrow{\delta_1} x[1]$$

$$x \underset{\pi_1\pi_3}{\overset{\beta\alpha}{\rightleftarrows}} z \underset{s_2}{\overset{p_2}{\rightleftarrows}} q_2 \xrightarrow{\delta_2} x[1]$$

$$y \underset{\pi_3}{\overset{\beta}{\rightleftarrows}} z \underset{s_3}{\overset{p_3}{\rightleftarrows}} q_3 \xrightarrow{\delta_3} x[1]$$

In these diagrams, the maps δ_i are defined as $\delta_i = \pi_i d(s_i)$ analogous to the maps defined in Lemma 22.27.1. They fit in the following solid commutative diagram

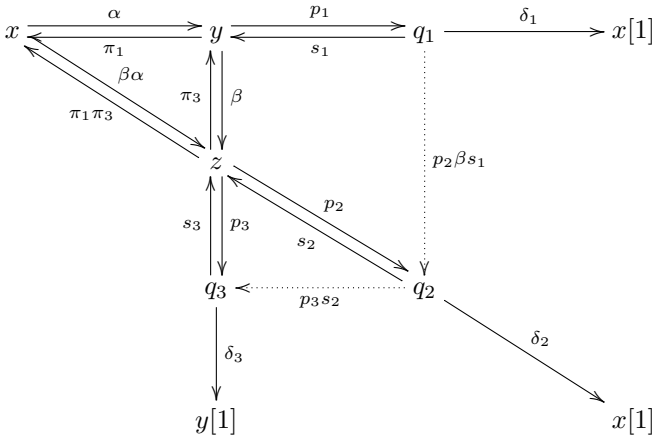

where we have defined the dashed arrows as indicated. Clearly, their composition $p_3 s_2 p_2 \beta s_1 = 0$ since $s_2 p_2 = 0$. We claim that they both are morphisms of $\mathrm{Comp}(\mathcal{A})$. We can check this using equations in Lemma 22.27.1:

$$d(p_2 \beta s_1) = p_2 \beta d(s_1) = p_2 \beta \alpha \pi_1 d(s_1) = 0$$

since $p_2 \beta \alpha = 0$, and

$$d(p_3 s_2) = p_3 d(s_2) = p_3 \beta \alpha \pi_1 \pi_3 d(s_2) = 0$$

since $p_3 \beta = 0$. To check that $q_1 \to q_2 \to q_3$ is an admissible short exact sequence, it remains to show that in the underlying graded category, $q_2 = q_1 \oplus q_3$ with the above two morphisms as coprojection and projection. To do this, observe that in the underlying graded category $\mathcal{C}$, there hold

$$y = x \oplus q_1, \quad z = y \oplus q_3 = x \oplus q_1 \oplus q_3$$

where $\pi_1 \pi_3$ gives the projection morphism onto the first factor: $x \oplus q_1 \oplus q_3 \to z$. By axiom (A) on $\mathcal{A}$, $\mathcal{C}$ is an additive category, hence we may apply Homology, Lemma 12.3.10 and conclude that

$$\mathrm{Ker}(\pi_1 \pi_3) = q_1 \oplus q_3$$

in $\mathcal{C}$. Another application of Homology, Lemma 12.3.10 to $z = x \oplus q_2$ gives $\mathrm{Ker}(\pi_1 \pi_3) = q_2$. Hence $q_2 \cong q_1 \oplus q_3$ in $\mathcal{C}$. It is clear that the dashed morphisms defined above give coprojection and projection.

Finally, we have to check that the morphism $\delta : q_3 \to q_1[1]$ induced by the admissible short exact sequence $q_1 \to q_2 \to q_3$ agrees with $p_1 \delta_3$. By the construction in Lemma 22.27.1, the morphism δ is given by

$$\begin{aligned} p_1 \pi_3 s_2 d(p_2 s_3) &= p_1 \pi_3 s_2 p_2 d(s_3) \\ &= p_1 \pi_3 (1 - \beta \alpha \pi_1 \pi_3) d(s_3) \\ &= p_1 \pi_3 d(s_3) \quad (\text{since } \pi_3 \beta = 0) \\ &= p_1 \delta_3 \end{aligned}$$

as desired. The proof is complete. □

Putting everything together we finally obtain the analogue of Proposition 22.10.3.

09QY **Proposition 22.27.16.** *In Situation 22.27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. By Lemma 22.27.14 we know that $K(\mathcal{A})$ is pre-triangulated. Combining Lemmas 22.27.7 and 22.27.15 with Derived Categories, Lemma 13.4.15, we conclude that $K(\mathcal{A})$ is a triangulated category. □

0FQF **Lemma 22.27.17.** *Let R be a ring. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between differential graded categories over R satisfying axioms (A), (B), and (C) such that $F(x[1]) = F(x)[1]$. Then F induces an exact functor $K(\mathcal{A}) \to K(\mathcal{B})$ of triangulated categories.*

Proof. Namely, if $x \to y \to z$ is an admissible short exact sequence in $\text{Comp}(\mathcal{A})$, then $F(x) \to F(y) \to F(z)$ is an admissible short exact sequence in $\text{Comp}(\mathcal{B})$. Moreover, the "boundary" morphism $\delta = \pi \text{d}(s) : z \to x[1]$ constructed in Lemma 22.27.1 produces the morphism $F(\delta) : F(z) \to F(x[1]) = F(x)[1]$ which is equal to the boundary map $F(\pi)\text{d}(F(s))$ for the admissible short exact sequence $F(x) \to F(y) \to F(z)$. □

22.28. Bimodules

0FQG We continue the discussion started in Section 22.12.

0FQH **Definition 22.28.1.** Bimodules. Let R be a ring.

(1) Let A and B be R-algebras. An (A, B)-*bimodule* is an R-module M equippend with R-bilinear maps
$$A \times M \to M, (a, x) \mapsto ax \quad \text{and} \quad M \times B \to M, (x, b) \mapsto xb$$
such that the following hold
   (a) $a'(ax) = (a'a)x$ and $(xb)b' = x(bb')$,
   (b) $a(xb) = (ax)b$, and
   (c) $1x = x = x1$.

(2) Let A and B be $\mathbf{Z}$-graded R-algebras. A *graded* (A, B)-*bimodule* is an (A, B)-bimodule M which has a grading $M = \bigoplus M^n$ such that $A^n M^m \subset M^{n+m}$ and $M^n B^m \subset M^{n+m}$.

(3) Let A and B be differential graded R-algebras. A *differential graded* (A, B)-*bimodule* is a graded (A, B)-bimodule which comes equipped with a differential $\text{d} : M \to M$ homogeneous of degree 1 such that $\text{d}(ax) = \text{d}(a)x + (-1)^{\deg(a)} a\text{d}(x)$ and $\text{d}(xb) = \text{d}(x)b + (-1)^{\deg(x)} x\text{d}(b)$ for homogeneous elements $a \in A$, $x \in M$, $b \in B$.

Observe that a differential graded (A, B)-bimodule M is the same thing as a right differential graded B-module which is also a left differential graded A-module such that the grading and differentials agree and such that the A-module structure commutes with the B-module structure. Here is a precise statement.

0FQI **Lemma 22.28.2.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let M be a right differential graded B-module. There is a 1-to-1 correspondence between (A, B)-bimodule structures on M compatible with the given differential graded B-module structure and homomorphisms*
$$A \longrightarrow \text{Hom}_{Mod^{dg}_{(B,d)}}(M, M)$$
of differential graded R-algebras.

Proof. Let $\mu : A \times M \to M$ define a left differential graded A-module structure on the underlying complex of R-modules $M^\bullet$ of M. By Lemma 22.13.1 the structure μ corresponds to a map $\gamma : A \to \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$ of differential graded R-algebras. The assertion of the lemma is simply that μ commutes with the B-action, if and only if γ ends up inside

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, M) \subset \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

We omit the detailed calculation. □

Let M be a differential graded (A, B)-bimodule. Recall from Section 22.11 that the left differential graded A-module structure corresponds to a right differential graded A^{opp}-module structure. Since the A and B module structures commute this gives M the structure of a differential graded $A^{opp} \otimes_R B$-module:

$$x \cdot (a \otimes b) = (-1)^{\deg(a)\deg(x)} axb$$

Conversely, if we have a differential graded $A^{opp} \otimes_R B$-module M, then we can use the formula above to get a differential graded (A, B)-bimodule.

0FQJ **Lemma 22.28.3.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. The construction above defines an equivalence of categories*

$$\begin{array}{c} \textit{differential graded} \\ (A,B)\textit{-bimodules} \end{array} \longleftrightarrow \begin{array}{c} \textit{right differential graded} \\ A^{opp} \otimes_R B\textit{-modules} \end{array}$$

Proof. Immediate from discussion the above. □

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule. We say P *has property (P)* if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded (A, B)-bimodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are split as graded (A, B)-bimodule maps,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded (A, B)-bimodules to a direct sum of $(A \otimes_R B)[k]$.

0FQK **Lemma 22.28.4.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let M be a differential graded (A, B)-bimodule. There exists a homomorphism $P \to M$ of differential graded (A, B)-bimodules which is a quasi-isomorphism such that P has property (P) as defined above.*

Proof. Immediate from Lemmas 22.28.3 and 22.20.4. □

0FQL **Lemma 22.28.5.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule having property (P) with corresponding filtration $F_\bullet$, then we obtain a short exact sequence*

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules.

Proof. Immediate from Lemmas 22.28.3 and 22.20.1. □

22.29. Bimodules and tensor product

0FQM Let R be a ring. Let A and B be R-algebras. Let M be a right A-module. Let N be a (A, B)-bimodule. Then $M \otimes_A N$ is a right B-module.

If in the situation of the previous paragraph A and B are $\mathbf{Z}$-graded algebras, M is a graded A-module, and N is a graded (A, B)-bimodule, then $M \otimes_A N$ is a right graded B-module. The construction is functorial in M and defines a functor

$$- \otimes_A N : \mathrm{Mod}_A^{gr} \longrightarrow \mathrm{Mod}_B^{gr}$$

of graded categories as in Example 22.25.6. Namely, if M and M' are graded A-modules and $f : M \to M'$ is an A-module homomorphism homogeneous of degree n, then $f \otimes \mathrm{id}_N : M \otimes_A N \to M' \otimes_A N$ is a B-module homomorphism homogeneous of degree n.

If in the situation of the previous paragraph (A, d) and (B, d) are differential graded algebras, M is a differential graded A-module, and N is a differential graded (A, B)-bimodule, then $M \otimes_A N$ is a right differential graded B-module.

09LM **Lemma 22.29.1.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Then $M \mapsto M \otimes_A N$ defines a functor*

$$- \otimes_A N : Mod_{(A,d)}^{dg} \longrightarrow Mod_{(B,d)}^{dg}$$

of differential graded categories. This functor induces functors

$$Mod_{(A,d)} \to Mod_{(B,d)} \quad \text{and} \quad K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$$

by an application of Lemma 22.26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let M and M' be differential graded A-modules and let $f : M \to M'$ be an A-module homomorphism which is homogeneous of degree n. Then we have

$$\mathrm{d}(f) = \mathrm{d}_{M'} \circ f - (-1)^n f \circ \mathrm{d}_M$$

On the other hand, we have

$$\mathrm{d}(f \otimes \mathrm{id}_N) = \mathrm{d}_{M' \otimes_A N} \circ (f \otimes \mathrm{id}_N) - (-1)^n (f \otimes \mathrm{id}_N) \circ \mathrm{d}_{M \otimes_A N}$$

Applying this to an element $x \otimes y$ with $x \in M$ and $y \in N$ homogeneous we get

$$\begin{aligned} \mathrm{d}(f \otimes \mathrm{id}_N)(x \otimes y) =& \mathrm{d}_{M'}(f(x)) \otimes y + (-1)^{n+\deg(x)} f(x) \otimes \mathrm{d}_N(y) \\ & - (-1)^n f(\mathrm{d}_M(x)) \otimes y - (-1)^{n+\deg(x)} f(x) \otimes \mathrm{d}_N(y) \\ =& \mathrm{d}(f)(x \otimes y) \end{aligned}$$

Thus we see that $\mathrm{d}(f) \otimes \mathrm{id}_N = \mathrm{d}(f \otimes \mathrm{id}_N)$ and the proof is complete. □

0FQN **Remark 22.29.2.** Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let M be a right differential graded A-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$(M \otimes_A N)[k] \longrightarrow M[k] \otimes_A N$$

of right differential graded B-modules defined without the intervention of signs, see More on Algebra, Section 15.73.

If we have a ring R and R-algebras A, B, and C, a right A-module M, an (A, B)-bimodule N, and a (B, C)-bimodule N', then $N \otimes_B N'$ is a (A, C)-bimodule and we have

$$(M \otimes_A N) \otimes_B N' = M \otimes_A (N \otimes_B N')$$

This equality continuous to hold in the graded and in the differential graded case. See More on Algebra, Section 15.73 for sign rules.

22.30. Bimodules and internal hom

0FQP Let R be a ring. If A is an R-algebra (see our conventions in Section 22.2) and M, M' are right A-modules, then we define

$$\mathrm{Hom}_A(M, M') = \{f : M \to M' \mid f \text{ is } A\text{-linear}\}$$

as usual.

Let R-be a ring. Let A and B be R-algebras. Let N be an (A, B)-bimodule. Let N' be a right B-module. In this situation we will think of

$$\mathrm{Hom}_B(N, N')$$

as a right A-module using precomposition.

Let R-be a ring. Let A and B be $\mathbf{Z}$-graded R-algebras. Let N be a graded (A, B)-bimodule. Let N' be a right graded B-module. In this situation we will think of the graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N')$$

defined in Example 22.25.6 as a right graded A-module using precomposition. The construction is functorial in N' and defines a functor

$$\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, -) : \mathrm{Mod}_B^{gr} \longrightarrow \mathrm{Mod}_A^{gr}$$

of graded categories as in Example 22.25.6. Namely, if N_1 and N_2 are graded B-modules and $f : N_1 \to N_2$ is a B-module homomorphism homogeneous of degree n, then the induced map $\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N_1) \to \mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N_2)$ is an A-module homomorphism homogeneous of degree n.

Let R be a ring. Let A and B be differential $\mathbf{Z}$-graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a right differential graded B-module. In this situation we will think of the differential graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N')$$

defined in Example 22.26.8 as a right differential graded A-module using precomposition as in the graded case. This is compatible with differentials because multiplication is the composition

$$\mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N') \otimes_R A \to \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N') \otimes_R \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N) \to \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N')$$

The first arrow uses the map of Lemma 22.28.2 and the second arrow is the composition in the differential graded category $\mathrm{Mod}^{dg}_{(B,\mathrm{d})}$.

0FQQ **Lemma 22.30.1.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. The construction above defines a functor*

$$\mathrm{Hom}_{Mod^{dg}_{(B,d)}}(N, -) : Mod^{dg}_{(B,d)} \longrightarrow Mod^{dg}_{(A,d)}$$

of differential graded categories. This functor induces functors

$$Mod_{(B,d)} \to Mod_{(A,d)} \quad and \quad K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$$

by an application of Lemma 22.26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let N_1 and N_2 be differential graded B-modules. Write

$$H_{12} = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N_1, N_2), \quad H_1 = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N_1), \quad H_2 = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N_2)$$

Consider the composition

$$c : H_{12} \otimes_R H_1 \longrightarrow H_2$$

in the differential graded category $\mathrm{Mod}^{dg}_{(B,\mathrm{d})}$. Let $f : N_1 \to N_2$ be a B-module homomorphism which is homogeneous of degree n, in other words, $f \in H^n_{12}$. The functor in the lemma sends f to $c_f : H_1 \to H_2$, $g \mapsto c(f, g)$. Similarly for $\mathrm{d}(f)$. On the other hand, the differential on

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(H_1, H_2)$$

sends c_f to $\mathrm{d}_{H_2} \circ c_f - (-1)^n c_f \circ \mathrm{d}_{H_1}$. As c is a morphism of complexes of R-modules we have $\mathrm{d}c(f, g) = c(\mathrm{d}f, g) + (-1)^n c(f, \mathrm{d}g)$. Hence we see that

$$\begin{aligned}(\mathrm{d}c_f)(g) &= \mathrm{d}c(f, g) - (-1)^n c(f, \mathrm{d}g) \\ &= c(\mathrm{d}f, g) + (-1)^n c(f, \mathrm{d}g) - (-1)^n c(f, \mathrm{d}g) \\ &= c(\mathrm{d}f, g) = c_{\mathrm{d}f}(g)\end{aligned}$$

and the proof is complete. □

0FQR **Remark 22.30.2.** Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let N' be a right differential graded B-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, N')[k] \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, N'[k])$$

of right differential graded A-modules defined without the intervention of signs, see More on Algebra, Section 15.73.

09LN **Lemma 22.30.3.** *Let R be a ring. Let A and B be R-algebras. Let M be a right A-module, N an (A, B)-bimodule, and N' a right B-module. Then we have a canonical isomorphism*

$$\mathrm{Hom}_B(M \otimes_A N, N') = \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, N'))$$

of R-modules. If A, B, M, N, N' are compatibly graded, then we have a canonical isomorphism

$$\mathrm{Hom}_{Mod^{gr}_B}(M \otimes_A N, N') = \mathrm{Hom}_{Mod^{gr}_A}(M, \mathrm{Hom}_{Mod^{gr}_B}(N, N'))$$

of graded R-modules If A, B, M, N, N' are compatibly differential graded, then we have a canonical isomorphism

$$\mathrm{Hom}_{Mod^{dg}_{(B,d)}}(M \otimes_A N, N') = \mathrm{Hom}_{Mod^{dg}_{(A,d)}}(M, \mathrm{Hom}_{Mod^{dg}_{(B,d)}}(N, N'))$$

of complexes of R-modules.

Proof. Omitted. Hint: in the ungraded case interpret both sides as A-bilinear maps $\psi : M \times N \to N'$ which are B-linear on the right. In the (differential) graded case, use the isomorphism of More on Algebra, Lemma 15.72.1 and check it is compatible with the module structures. Alternatively, use the isomorphism of Lemma 22.13.2 and show that it is compatible with the B-module structures. □

22.31. Derived Hom

09LF This section is analogous to More on Algebra, Section 15.74.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

09LG (22.31.0.1) $$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, -) : \mathrm{Mod}_{(B,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(A,\mathrm{d})}$$

of Section 22.30.

09LH **Lemma 22.31.1.** *The functor (22.31.0.1) defines an exact functor $K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$ of triangulated categories.*

Proof. Via Lemma 22.30.1 and Remark 22.30.2 this follows from the general principle of Lemma 22.27.17. □

Recall that we have an exact functor of triangulated categories

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, -) : K(\mathrm{Mod}_{(B,\mathrm{d})}) \to K(\mathrm{Mod}_{(A,\mathrm{d})})$$

see Lemma 22.31.1. Consider the diagram

$$\begin{array}{ccc} K(\mathrm{Mod}_{(B,\mathrm{d})}) & \xrightarrow[\text{see above}]{} & K(\mathrm{Mod}_{(A,\mathrm{d})}) \\ \downarrow & \searrow^{F} & \downarrow \\ D(B, \mathrm{d}) & \dashrightarrow & D(A, \mathrm{d}) \end{array}$$

We would like to construct a dotted arrow as the *right derived functor* of the composition F. (*Warning*: in most interesting cases the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the right derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

09LI **Lemma 22.31.2.** *In the situation above, the right derived functor of F exists. We denote it $R\mathrm{Hom}(N, -) : D(B, d) \to D(A, d)$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{I}$ of objects we will use the objects with property (I). Property (1) was shown in Lemma 22.21.4. Property (2) holds because if $s : I \to I'$ is a quasi-isomorphism of modules with property (I), then s is a homotopy equivalence by Lemma 22.22.3. □

0BYV **Lemma 22.31.3.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors*

$$- \circ f : R\mathrm{Hom}(N', -) \longrightarrow R\mathrm{Hom}(N, -)$$

If f is a quasi-isomorphism, then $f \circ -$ is an isomorphism of functors.

Proof. Write $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$ the differential graded category of differential graded B-modules, see Example 22.26.8. Let I be a differential graded B-module with property (I). Then $f \circ - : \mathrm{Hom}_{\mathcal{B}}(N', I) \to \mathrm{Hom}_{\mathcal{B}}(N, I)$ is a map of differential graded A-modules. Moreover, this is functorial with respect to I. Since the functors $R\mathrm{Hom}(N', -)$ and $R\mathrm{Hom}(N, -)$ are computed by applying $\mathrm{Hom}_{\mathcal{B}}$ into objects with property (I) (Lemma 22.31.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on I. Since $I = \lim I/F_pI$ we see that $\mathrm{Hom}_{\mathcal{B}}(N', I) = \lim \mathrm{Hom}_{\mathcal{B}}(N', I/F_pI)$ and $\mathrm{Hom}_{\mathcal{B}}(N, I) = \lim \mathrm{Hom}_{\mathcal{B}}(N, I/F_pI)$. Since the transition maps in the system I/F_pI are split as graded modules, we see that the transition maps in the systems $\mathrm{Hom}_{\mathcal{B}}(N', I/F_pI)$ and $\mathrm{Hom}_{\mathcal{B}}(N, I/F_pI)$ are surjective. Hence $\mathrm{Hom}_{\mathcal{B}}(N', I)$, resp. $\mathrm{Hom}_{\mathcal{B}}(N, I)$ viewed as a complex of abelian groups computes $R\lim$ of the system of complexes $\mathrm{Hom}_{\mathcal{B}}(N', I/F_pI)$, resp. $\mathrm{Hom}_{\mathcal{B}}(N, I/F_pI)$. See More on Algebra, Lemma 15.87.1. Thus it suffices to prove each

$$\mathrm{Hom}_{\mathcal{B}}(N', I/F_pI) \to \mathrm{Hom}_{\mathcal{B}}(N, I/F_pI)$$

is a quasi-isomorphism. Since the surjections $I/F_{p+1}I \to I/F_pI$ are split as maps of graded B-modules we see that

$$0 \to \mathrm{Hom}_{\mathcal{B}}(N', F_pI/F_{p+1}I) \to \mathrm{Hom}_{\mathcal{B}}(N', I/F_{p+1}I) \to \mathrm{Hom}_{\mathcal{B}}(N', I/F_pI) \to 0$$

is a short exact sequence of differential graded A-modules. There is a similar sequence for N and f induces a map of short exact sequences. Hence by induction on p (starting with $p = 0$ when $I/F_0I = 0$) we conclude that it suffices to show that the map $\mathrm{Hom}_{\mathcal{B}}(N', F_pI/F_{p+1}I) \to \mathrm{Hom}_{\mathcal{B}}(N, F_pI/F_{p+1}I)$ is a quasi-isomorphism. Since $F_pI/F_{p+1}I$ is a product of shifts of $A^\vee$ it suffice to prove $\mathrm{Hom}_{\mathcal{B}}(N', B^\vee[k]) \to \mathrm{Hom}_{\mathcal{B}}(N, B^\vee[k])$ is a quasi-isomorphism. By Lemma 22.19.3 it suffices to show $(N')^\vee \to N^\vee$ is a quasi-isomorphism. This is true because f is a quasi-isomorphism and $(\ )^\vee$ is an exact functor. □

0CS5 **Lemma 22.31.4.** *Let (A, d) and (B, d) be differential graded algebras over a ring R. Let N be a differential graded (A, B)-bimodule. Then for every $n \in \mathbf{Z}$ there are isomorphisms*

$$H^n(R\mathrm{Hom}(N, M)) = \mathrm{Ext}^n_{D(B,d)}(N, M)$$

of R-modules functorial in M. It is also functorial in N with respect to the operation described in Lemma 22.31.3.

Proof. In the proof of Lemma 22.31.2 we have seen

$$R\mathrm{Hom}(N, M) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, I)$$

as a differential graded A-module where $M \to I$ is a quasi-isomorphism of M into a differential graded B-module with property (I). Hence this complex has the correct cohomology modules by Lemma 22.22.3. We omit a discussion of the functorial nature of these identifications. □

0BYW **Lemma 22.31.5.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. If $\mathrm{Hom}_{D(B,d)}(N, N') = \mathrm{Hom}_{K(Mod_{(B,d)})}(N, N')$ for all $N' \in K(B, d)$, for example if N has property (P) as a differential graded B-module, then*

$$R\mathrm{Hom}(N, M) = \mathrm{Hom}_{Mod^{dg}_{(B,d)}}(N, M)$$

functorially in M in $D(B, d)$.

Proof. By construction (Lemma 22.31.2) to find $R\mathrm{Hom}(N, M)$ we choose a quasi-isomorphism $M \to I$ where I is a differential graded B-module with property (I) and we set $R\mathrm{Hom}(N, M) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, I)$. By assumption the map

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, M) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, I)$$

induced by $M \to I$ is a quasi-isomorphism, see discussion in Example 22.26.8. This proves the lemma. If N has property (P) as a B-module, then we see that the assumption is satisfied by Lemma 22.22.3. □

22.32. Variant of derived Hom

09LJ Let $\mathcal{A}$ be an abelian category. Consider the differential graded category $\mathrm{Comp}^{dg}(\mathcal{A})$ of complexes of $\mathcal{A}$, see Example 22.26.6. Let $K^\bullet$ be a complex of $\mathcal{A}$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, K^\bullet)$$

and consider the functor of differential graded categories

$$\mathrm{Comp}^{dg}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{dg}_{(E,\mathrm{d})}, \quad X^\bullet \longmapsto \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, X^\bullet)$$

of Lemma 22.26.10.

09LK **Lemma 22.32.1.** *In the situation above. If the right derived functor $R\mathrm{Hom}(K^\bullet, -)$ of $\mathrm{Hom}(K^\bullet, -) : K(\mathcal{A}) \to D(Ab)$ is everywhere defined on $D(\mathcal{A})$, then we obtain a canonical exact functor*

$$R\mathrm{Hom}(K^\bullet, -) : D(\mathcal{A}) \longrightarrow D(E, d)$$

of triangulated categories which reduces to the usual one on taking associated complexes of abelian groups.

Proof. Note that we have an associated functor $K(\mathcal{A}) \to K(\mathrm{Mod}_{(E,\mathrm{d})})$ by Lemma 22.26.10. We claim this functor is an exact functor of triangulated categories. Namely, let $f : A^\bullet \to B^\bullet$ be a map of complexes of $\mathcal{A}$. Then a computation shows that

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, C(f)^\bullet) = C\left(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, B^\bullet)\right)$$

where the right hand side is the cone in $\mathrm{Mod}_{(E,\mathrm{d})}$ defined earlier in this chapter. This shows that our functor is compatible with cones, hence with distinguished triangles. Let $X^\bullet$ be an object of $K(\mathcal{A})$. Consider the category of quasi-isomorphisms $s : X^\bullet \to Y^\bullet$. We are given that the functor $(s : X^\bullet \to Y^\bullet) \mapsto \mathrm{Hom}_{\mathcal{A}}(K^\bullet, Y^\bullet)$ is essentially constant when viewed in $D(Ab)$. But since the forgetful functor $D(E, \mathrm{d}) \to D(Ab)$ is compatible with taking cohomology, the same thing is true in $D(E, \mathrm{d})$. This proves the lemma. □

Warning: Although the lemma holds as stated and may be useful as stated, the differential algebra E isn't the "correct" one unless $H^n(E) = \mathrm{Ext}^n_{D(\mathcal{A})}(K^\bullet, K^\bullet)$ for all $n \in \mathbf{Z}$.

22.33. Derived tensor product

09LP This section is analogous to More on Algebra, Section 15.60.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

09LQ (22.33.0.1) $$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(B,\mathrm{d})}, \quad M \longmapsto M \otimes_A N$$

defined in Section 22.29.

09LR **Lemma 22.33.1.** *The functor (22.33.0.1) defines an exact functor of triangulated categories $K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$.*

Proof. Via Lemma 22.29.1 and Remark 22.29.2 this follows from the general principle of Lemma 22.27.17. □

At this point we can consider the diagram

$$\begin{array}{ccc} K(\mathrm{Mod}_{(A,\mathrm{d})}) & \xrightarrow[-\otimes_A N]{} & K(\mathrm{Mod}_{(B,\mathrm{d})}) \\ \downarrow & \searrow^{F} & \downarrow \\ D(A, \mathrm{d}) & \dashrightarrow & D(B, \mathrm{d}) \end{array}$$

The dotted arrow that we will construct below will be the *left derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the left derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

09LS **Lemma 22.33.2.** *In the situation above, the left derived functor of F exists. We denote it $- \otimes^{\mathbf{L}}_A N : D(A, d) \to D(B, d)$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.22.3. □

09S3 **Lemma 22.33.3.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors*

$$1 \otimes f : - \otimes^{\mathbf{L}}_A N \longrightarrow - \otimes^{\mathbf{L}}_A N'$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Let M be a differential graded A-module with property (P). Then $1 \otimes f : M \otimes_A N \to M \otimes_A N'$ is a map of differential graded B-modules. Moreover, this is functorial with respect to M. Since the functors $- \otimes^{\mathbf{L}}_A N$ and $- \otimes^{\mathbf{L}}_A N'$ are computed by tensoring on objects with property (P) (Lemma 22.33.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on M. Observe that $M \otimes_A N = \operatorname{colim} F_i(M) \otimes_A N$ and $M \otimes_A N' = \operatorname{colim} F_i(M) \otimes_A N'$. Hence it suffices to show that $F_n(M) \otimes_A N \to F_n(M) \otimes_A N'$ is a quasi-isomorphism (filtered

colimits are exact, see Algebra, Lemma 10.8.8). Since the inclusions $F_n(M) \to F_{n+1}(M)$ are split as maps of graded A-modules we see that

$$0 \to F_n(M) \otimes_A N \to F_{n+1}(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N \to 0$$

is a short exact sequence of differential graded B-modules. There is a similar sequence for N' and f induces a map of short exact sequences. Hence by induction on n (starting with $n = -1$ when $F_{-1}(M) = 0$) we conclude that it suffices to show that the map $F_{n+1}(M)/F_n(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N'$ is a quasi-isomorphism. This is true because $F_{n+1}(M)/F_n(M)$ is a direct sum of shifts of A and the result is true for $A[k]$ as $f : N \to N'$ is a quasi-isomorphism. □

0GZ2 **Lemma 22.33.4.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule which has property (P) as a left differential graded A-module. Then $M \otimes^{\mathbf{L}}_A N$ is computed by $M \otimes_A N$ for all differential graded A-modules M.*

Proof. Let $f : M \to M'$ be a homomorphism of differential graded A-modules which is a quasi-isomorphism. We claim that $f \otimes \mathrm{id} : M \otimes_A N \to M' \otimes_A N$ is a quasi-isomorphism. If this is true, then by the construction of the derived tensor product in the proof of Lemma 22.33.2 we obtain the desired result. The construction of the map $f \otimes \mathrm{id}$ only depends on the left differential graded A-module structure on N. Moreover, we have $M \otimes_A N = N \otimes_{A^{opp}} M = N \otimes^{\mathbf{L}}_{A^{opp}} M$ because N has property (P) as a differential graded A^{opp}-module. Hence the claim follows from Lemma 22.33.3. □

09LT **Lemma 22.33.5.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Then the functor*

$$- \otimes^{\mathbf{L}}_A N : D(A, d) \longrightarrow D(B, d)$$

of Lemma 22.33.2 is a left adjoint to the functor

$$R\mathrm{Hom}(N, -) : D(B, d) \longrightarrow D(A, d)$$

of Lemma 22.31.2.

Proof. This follows from Derived Categories, Lemma 13.30.1 and the fact that $- \otimes_A N$ and $\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, -)$ are adjoint by Lemma 22.30.3. □

0BYX **Example 22.33.6.** Let R be a ring. Let $(A, \mathrm{d}) \to (B, \mathrm{d})$ be a homomorphism of differential graded R-algebras. Then we can view B as a differential graded (A, B)-bimodule and we get a functor

$$- \otimes_A B : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

By Lemma 22.33.5 the left adjoint of this is the functor $R\mathrm{Hom}(B, -)$. For a differential graded B-module let us denote N_A the differential graded A-module obtained from N by restriction via $A \to B$. Then we clearly have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(B, N) \longrightarrow N_A, \quad f \longmapsto f(1)$$

functorial in the B-module N. Thus we see that $R\mathrm{Hom}(B, -)$ is the restriction functor and we obtain

$$\mathrm{Hom}_{D(A,\mathrm{d})}(M, N_A) = \mathrm{Hom}_{D(B,\mathrm{d})}(M \otimes^{\mathbf{L}}_A B, N)$$

bifunctorially in M and N exactly as in the case of commutative rings. Finally, observe that restriction is a tensor functor as well, since $N_A = N \otimes_B {}_BB_A = N \otimes^{\mathbf{L}}_B {}_BB_A$ where ${}_BB_A$ is B viewed as a differential graded (B, A)-bimodule.

09R9 **Lemma 22.33.7.** *With notation and assumptions as in Lemma 22.33.5. Assume*

(1) *N defines a compact object of $D(B, d)$, and*
(2) *the map $H^k(A) \to \mathrm{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then the functor $- \otimes^{\mathbf{L}}_A N$ is fully faithful.

Proof. Our functor has a left adjoint given by $R\,\mathrm{Hom}(N, -)$ by Lemma 22.33.5. By Categories, Lemma 4.24.4 it suffices to show that for a differential graded A-module M the map

$$M \longrightarrow R\,\mathrm{Hom}(N, M \otimes^{\mathbf{L}}_A N)$$

is an isomorphism in $D(A, \mathrm{d})$. For this it suffices to show that

$$H^n(M) \longrightarrow \mathrm{Ext}^n_{D(B,\mathrm{d})}(N, M \otimes^{\mathbf{L}}_A N)$$

is an isomorphism, see Lemma 22.31.4. Since N is a compact object the right hand side commutes with direct sums. Thus by Remark 22.22.5 it suffices to prove this map is an isomorphism for $M = A[k]$. Since $(A[k] \otimes^{\mathbf{L}}_A N) = N[k]$ by Remark 22.29.2, assumption (2) on N is that the result holds for these. □

0BYZ **Lemma 22.33.8.** *Let $R \to R'$ be a ring map. Let (A, d) be a differential graded R-algebra. Let (A', d) be the base change, i.e., $A' = A \otimes_R R'$. If A is K-flat as a complex of R-modules, then*

(1) *$- \otimes^{\mathbf{L}}_A A' : D(A, d) \to D(A', d)$ is equal to the right derived functor of*

$$K(A, d) \longrightarrow K(A', d), \quad M \longmapsto M \otimes_R R'$$

(2) *the diagram*

$$\begin{array}{ccc} D(A, d) & \xrightarrow[-\otimes^{\mathbf{L}}_A A']{} & D(A', d) \\ {\scriptstyle restriction}\downarrow & & \downarrow{\scriptstyle restriction} \\ D(R) & \xrightarrow{-\otimes^{\mathbf{L}}_R R'} & D(R') \end{array}$$

commutes, and

(3) *if M is K-flat as a complex of R-modules, then the differential graded A'-module $M \otimes_R R'$ represents $M \otimes^{\mathbf{L}}_A A'$.*

Proof. For any differential graded A-module M there is a canonical map

$$c_M : M \otimes_R R' \longrightarrow M \otimes_A A'$$

Let P be a differential graded A-module with property (P). We claim that c_P is an isomorphism and that P is K-flat as a complex of R-modules. This will prove all the results stated in the lemma by formal arguments using the definition of derived tensor product in Lemma 22.33.2 and More on Algebra, Section 15.59.

Let $F_\bullet$ be the filtration on P showing that P has property (P). Note that c_A is an isomorphism and A is K-flat as a complex of R-modules by assumption. Hence the same is true for direct sums of shifts of A (you can use More on Algebra, Lemma 15.59.8 to deal with direct sums if you like). Hence this holds for the complexes $F_{p+1}P/F_pP$. Since the short exact sequences

$$0 \to F_pP \to F_{p+1}P \to F_{p+1}P/F_pP \to 0$$

are split exact as sequences of graded modules, we can argue by induction that c_{F_pP} is an isomorphism for all p and that F_pP is K-flat as a complex of R-modules (use More on Algebra, Lemma 15.59.5). Finally, using that $P = \operatorname{colim} F_pP$ we conclude that c_P is an isomorphism and that P is K-flat as a complex of R-modules (use More on Algebra, Lemma 15.59.8). □

0BZ0 **Lemma 22.33.9.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let T be a differential graded (A, B)-bimodule. Assume*

(1) *T defines a compact object of $D(B, d)$, and*
(2) *$S = \mathrm{Hom}_{Mod^{dg}_{(B,d)}}(T, B)$ represents $R\operatorname{Hom}(T, B)$ in $D(A, d)$.*

Then S has a structure of a differential graded (B, A)-bimodule and there is an isomorphism

$$N \otimes^{\mathbf{L}}_B S \longrightarrow R\operatorname{Hom}(T, N)$$

functorial in N in $D(B, d)$.

Proof. Write $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$. The right A-module structure on S comes from the map $A \to \mathrm{Hom}_{\mathcal{B}}(T, T)$ and the composition $\mathrm{Hom}_{\mathcal{B}}(T, B) \otimes \mathrm{Hom}_{\mathcal{B}}(T, T) \to \mathrm{Hom}_{\mathcal{B}}(T, B)$ defined in Example 22.26.8. Using this multiplication a second time there is a map

$$c_N : N \otimes_B S = \mathrm{Hom}_{\mathcal{B}}(B, N) \otimes_B \mathrm{Hom}_{\mathcal{B}}(T, B) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(T, N)$$

functorial in N. Given N we can choose quasi-isomorphisms $P \to N \to I$ where P, resp. I is a differential graded B-module with property (P), resp. (I). Then using c_N we obtain a map $P \otimes_B S \to \mathrm{Hom}_{\mathcal{B}}(T, I)$ between the objects representing $S \otimes^{\mathbf{L}}_B N$ and $R\operatorname{Hom}(T, N)$. Clearly this defines a transformation of functors c as in the lemma.

To prove that c is an isomorphism of functors, we may assume N is a differential graded B-module which has property (P). Since T defines a compact object in $D(B, \mathrm{d})$ and since both sides of the arrow define exact functors of triangulated categories, we reduce using Lemma 22.20.1 to the case where N has a finite filtration whose graded pieces are direct sums of $B[k]$. Using again that both sides of the arrow are exact functors of triangulated categories and compactness of T we reduce to the case $N = B[k]$. Assumption (2) is exactly the assumption that c is an isomorphism in this case. □

22.34. Composition of derived tensor products

0BZ1 We encourage the reader to skip this section.

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. We denote N_B the bimodule N viewed as a differential graded B-module (forgetting about the A-structure). There is a canonical map

0BZ2 (22.34.0.1) $$N_B \otimes^{\mathbf{L}}_B N' \longrightarrow (N \otimes_B N')_C$$

in $D(C, \mathrm{d})$. Here $(N \otimes_B N')_C$ denotes the (A, C)-bimodule $N \otimes_B N'$ viewed as a differential graded C-module. Namely, this map comes from the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

0BZ3 **Lemma 22.34.1.** *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. Assume (22.34.0.1) is an isomorphism. Then the composition*

$$D(A, d) \xrightarrow{-\otimes^{\mathbf{L}}_A N} D(B, d) \xrightarrow{-\otimes^{\mathbf{L}}_B N'} D(C, d)$$

is isomorphic to $- \otimes^{\mathbf{L}}_A N''$ with $N'' = N \otimes_B N'$ viewed as (A, C)-bimodule.

Proof. Let us define a transformation of functors

$$(- \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B N' \longrightarrow - \otimes^{\mathbf{L}}_A N''$$

To do this, let M be a differential graded A-module with property (P). According to the construction of the functor $- \otimes^{\mathbf{L}}_A N''$ of the proof of Lemma 22.33.2 the plain tensor product $M \otimes_A N''$ represents $M \otimes^{\mathbf{L}}_A N''$ in $D(C, \mathrm{d})$. Then we write

$$M \otimes_A N'' = M \otimes_A (N \otimes_B N') = (M \otimes_A N) \otimes_B N'$$

The module $M \otimes_A N$ represents $M \otimes^{\mathbf{L}}_A N$ in $D(B, \mathrm{d})$. Choose a quasi-isomorphism $Q \to M \otimes_A N$ where Q is a differential graded B-module with property (P). Then $Q \otimes_B N'$ represents $(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B N'$ in $D(C, \mathrm{d})$. Thus we can define our map via

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B N' = Q \otimes_B N' \to M \otimes_A N \otimes_B N' = M \otimes^{\mathbf{L}}_A N''$$

The construction of this map is functorial in M and compatible with distinguished triangles and direct sums; we omit the details. Consider the property T of objects M of $D(A, \mathrm{d})$ expressing that this map is an isomorphism. Then

(1) if T holds for M_i then T holds for $\bigoplus M_i$,
(2) if T holds for 2-out-of-3 in a distinguished triangle, then it holds for the third, and
(3) T holds for $A[k]$ because here we obtain a shift of the map (22.34.0.1) which we have assumed is an isomorphism.

Thus by Remark 22.22.5 property T always holds and the proof is complete. □

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. We temporarily denote $(A \otimes_R B)_B$ the differential graded algebra $A \otimes_R B$ viewed as a (right) differential graded B-module, and ${}_B(B \otimes_R C)_C$ the differential graded algebra $B \otimes_R C$ viewed as a differential graded (B, C)-bimodule. Then there is a canonical map

0BZ4 (22.34.1.1) $$(A \otimes_R B)_B \otimes^{\mathbf{L}}_B {}_B(B \otimes_R C)_C \longrightarrow (A \otimes_R B \otimes_R C)_C$$

in $D(C, \mathrm{d})$ where $(A \otimes_R B \otimes_R C)_C$ denotes the differential graded R-algebra $A \otimes_R B \otimes_R C$ viewed as a (right) differential graded C-module. Namely, this map comes from the identification

$$(A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C = (A \otimes_R B \otimes_R C)_C$$

and the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

0BZ5 **Lemma 22.34.2.** *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume that (22.34.1.1) is an isomorphism. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition*

$$D(A, d) \xrightarrow{-\otimes^{\mathbf{L}}_A N} D(B, d) \xrightarrow{-\otimes^{\mathbf{L}}_B N'} D(C, d)$$

is isomorphic to $- \otimes^{\mathbf{L}}_A N''$ for a differential graded (A, C)-bimodule N'' described in the proof.

Proof. By Lemma 22.33.3 we may replace N and N' by quasi-isomorphic bimodules. Thus we may assume N, resp. N' has property (P) as differential graded (A, B)-bimodule, resp. (B, C)-bimodule, see Lemma 22.28.4. We claim the lemma holds with the (A, C)-bimodule $N'' = N \otimes_B N'$. To prove this, it suffices to show that

$$N_B \otimes^{\mathbf{L}}_B N' \longrightarrow (N \otimes_B N')_C$$

is an isomorphism in $D(C, \mathrm{d})$, see Lemma 22.34.1.

Let $F_\bullet$ be the filtration on N as in property (P) for bimodules. By Lemma 22.28.5 there is a short exact sequence

$$0 \to \bigoplus F_i N \to \bigoplus F_i N \to N \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules. A fortiori this is an admissible short exact sequence of differential graded B-modules and this produces a distinguished triangle

$$\bigoplus F_i N_B \to \bigoplus F_i N_B \to N_B \to \bigoplus F_i N_B[1]$$

in $D(B, \mathrm{d})$. Using that $- \otimes^{\mathbf{L}}_B N'$ is an exact functor of triangulated categories and commutes with direct sums and using that $- \otimes_B N'$ transforms admissible exact sequences into admissible exact sequences and commutes with direct sums we reduce to proving that

$$(F_p N)_B \otimes^{\mathbf{L}}_B N' \longrightarrow (F_p N)_B \otimes_B N'$$

is a quasi-isomorphism for all p. Repeating the argument with the short exact sequences of (A, B)-bimodules

$$0 \to F_p N \to F_{p+1} N \to F_{p+1} N / F_p N \to 0$$

which are split as graded (A, B)-bimodules we reduce to showing the same statement for $F_{p+1} N / F_p N$. Since these modules are direct sums of shifts of $(A \otimes_R B)_B$ we reduce to showing that

$$(A \otimes_R B)_B \otimes^{\mathbf{L}}_B N' \longrightarrow (A \otimes_R B)_B \otimes_B N'$$

is a quasi-isomorphism.

Choose a filtration $F_\bullet$ on N' as in property (P) for bimodules. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules where P has property (P). We have to show that $P \otimes_B N' \to (A \otimes_R B)_B \otimes_B N'$ is a quasi-isomorphism because $P \otimes_B N'$ represents $(A \otimes_R B)_B \otimes^{\mathbf{L}}_B N'$ in $D(C, \mathrm{d})$ by the construction in Lemma 22.33.2. As $N' = \operatorname{colim} F_p N'$ we find that it suffices to show that $P \otimes_B F_p N' \to (A \otimes_R B)_B \otimes_B F_p N'$ is a quasi-isomorphism. Using the short exact sequences $0 \to F_p N' \to F_{p+1} N' \to F_{p+1} N' / F_p N' \to 0$ which are

split as graded (B, C)-bimodules we reduce to showing $P \otimes_B F_{p+1}N'/F_pN' \to (A \otimes_R B)_B \otimes_B F_{p+1}N'/F_pN'$ is a quasi-isomorphism for all p. Then finally using that $F_{p+1}N'/F_pN'$ is a direct sum of shifts of ${}_B(B \otimes_R C)_C$ we conclude that it suffices to show that

$$P \otimes_B {}_B(B \otimes_R C)_C \to (A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C$$

is a quasi-isomorphism. Since $P \to (A \otimes_R B)_B$ is a resolution by a module satisfying property (P) this map of differential graded C-modules represents the morphism (22.34.1.1) in $D(C, \mathrm{d})$ and the proof is complete. □

09S4 **Lemma 22.34.3.** *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. If C is K-flat as a complex of R-modules, then (22.34.1.1) is an isomorphism and the conclusion of Lemma 22.34.2 is valid.*

Proof. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules, where P has property (P). Then we have to show that

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

is a quasi-isomorphism. Equivalently we are looking at

$$P \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

This is a quasi-isomorphism if C is K-flat as a complex of R-modules by More on Algebra, Lemma 15.59.2. □

22.35. Variant of derived tensor product

09LU Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then we have the functors

$$\mathrm{Comp}(\mathcal{O}) \to K(\mathcal{O}) \to D(\mathcal{O})$$

and as we've seen above we have differential graded enhancement $\mathrm{Comp}^{dg}(\mathcal{O})$. Namely, this is the differential graded category of Example 22.26.6 associated to the abelian category $Mod(\mathcal{O})$. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules in other words, an object of $\mathrm{Comp}^{dg}(\mathcal{O})$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$$

This is a differential graded $\mathbf{Z}$-algebra. We claim there is an analogue of the derived base change in this situation.

09LV **Lemma 22.35.1.** *In the situation above there is a functor*

$$- \otimes_E K^\bullet : Mod^{dg}_{(E,d)} \longrightarrow Comp^{dg}(\mathcal{O})$$

of differential graded categories. This functor sends E to $K^\bullet$ and commutes with direct sums.

Proof. Let M be a differential graded E-module. For every object U of $\mathcal{C}$ the complex $K^\bullet(U)$ is a left differential graded E-module as well as a right $\mathcal{O}(U)$-module. The actions commute, so we have a bimodule. Thus, by the constructions in Sections 22.12 and 22.28 we can form the tensor product

$$M \otimes_E K^\bullet(U)$$

which is a differential graded $\mathcal{O}(U)$-module, i.e., a complex of $\mathcal{O}(U)$-modules. This construction is functorial with respect to U, hence we can sheafify to get a complex of $\mathcal{O}$-modules which we denote

$$M \otimes_E K^\bullet$$

Moreover, for each U the construction determines a functor $\mathrm{Mod}^{dg}_{(E,\mathrm{d})} \to \mathrm{Comp}^{dg}(\mathcal{O}(U))$ of differential graded categories by Lemma 22.29.1. It is therefore clear that we obtain a functor as stated in the lemma. □

09LW **Lemma 22.35.2.** *The functor of Lemma 22.35.1 defines an exact functor of triangulated categories $K(Mod_{(E,d)}) \to K(\mathcal{O})$.*

Proof. The functor induces a functor between homotopy categories by Lemma 22.26.5. We have to show that $- \otimes_E K^\bullet$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded E-modules. Let $s : M \to L$ be a graded E-module homomorphism which is left inverse to $L \to M$. Then s defines a map $M \otimes_E K^\bullet \to L \otimes_E K^\bullet$ of graded $\mathcal{O}$-modules (i.e., respecting $\mathcal{O}$-module structure and grading, but not differentials) which is left inverse to $L \otimes_E K^\bullet \to M \otimes_E K^\bullet$. Thus we see that

$$0 \to K \otimes_E K^\bullet \to L \otimes_E K^\bullet \to M \otimes_E K^\bullet \to 0$$

is a termwise split short exact sequences of complexes, i.e., a defines a distinguished triangle in $K(\mathcal{O})$. □

09LX **Lemma 22.35.3.** *The functor $K(Mod_{(E,d)}) \to K(\mathcal{O})$ of Lemma 22.35.2 has a left derived version defined on all of $D(E, d)$. We denote it $-\otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \to D(\mathcal{O})$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.22.3. □

0CS6 **Lemma 22.35.4.** *Let R be a ring. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of commutative R-algebras. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. The functor of Lemma 22.35.3 has the following property: For every M, N in $D(E, d)$ there is a canonical map*

$$R\mathrm{Hom}(M, N) \longrightarrow R\mathrm{Hom}_{\mathcal{O}}(M \otimes^{\mathbf{L}}_E K^\bullet, N \otimes^{\mathbf{L}}_E K^\bullet)$$

in $D(R)$ which on cohomology modules gives the maps

$$\mathrm{Ext}^n_{D(E,d)}(M, N) \to \mathrm{Ext}^n_{D(\mathcal{O})}(M \otimes^{\mathbf{L}}_E K^\bullet, N \otimes^{\mathbf{L}}_E K^\bullet)$$

induced by the functor $- \otimes^{\mathbf{L}}_E K^\bullet$.

Proof. The right hand side of the arrow is the global derived hom introduced in Cohomology on Sites, Section 21.36 which has the correct cohomology modules. For the left hand side we think of M as a (R, A)-bimodule and we have the derived Hom introduced in Section 22.31 which also has the correct cohomology modules. To prove the lemma we may assume M and N are differential graded E-modules with property (P); this does not change the left hand side of the arrow by Lemma 22.31.3. By Lemma 22.31.5 this means that the left hand side of the arrow becomes

$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, N)$. In Lemmas 22.35.1, 22.35.2, and 22.35.3 we have constructed a functor

$$- \otimes_E K^\bullet : \mathrm{Mod}^{dg}_{(E,\mathrm{d})} \longrightarrow \mathrm{Comp}^{dg}(\mathcal{O})$$

of differential graded categories and we have shown that $- \otimes^{\mathbf{L}}_E K^\bullet$ is computed by evaluating this functor on differential graded E-modules with property (P). Hence we obtain a map of complexes of R-modules

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(M \otimes_E K^\bullet, N \otimes_E K^\bullet)$$

For any complexes of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ there is a canonical map

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \longrightarrow R\,\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Combining these maps we obtain the desired map of the lemma. □

09LY **Lemma 22.35.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 22.35.3 is a left adjoint of the functor

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}) \longrightarrow D(E, d)$$

of Lemma 22.32.1.

Proof. The statement means that we have

$$\mathrm{Hom}_{D(E,\mathrm{d})}(M, R\,\mathrm{Hom}(K^\bullet, L^\bullet)) = \mathrm{Hom}_{D(\mathcal{O})}(M \otimes^{\mathbf{L}}_E K^\bullet, L^\bullet)$$

bifunctorially in M and $L^\bullet$. To see this we may replace M by a differential graded E-module P with property (P). We also may replace $L^\bullet$ by a K-injective complex of $\mathcal{O}$-modules $I^\bullet$. The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 22.22.3 translates the above into

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(E,\mathrm{d})})}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

where $\mathcal{B} = \mathrm{Comp}^{dg}(\mathcal{O})$. There is an evaluation map from right to left functorial in P and $I^\bullet$ (details omitted). Choose a filtration $F_\bullet$ on P as in the definition of property (P). By Lemma 22.20.1 and the fact that both sides of the equation are homological functors in P on $K(\mathrm{Mod}_{(E,\mathrm{d})})$ we reduce to the case where P is replaced by the differential graded E-module $\bigoplus F_i P$. Since both sides turn direct sums in the variable P into direct products we reduce to the case where P is one of the differential graded E-modules $F_i P$. Since each $F_i P$ has a finite filtration (given by admissible monomorphisms) whose graded pieces are graded projective E-modules we reduce to the case where P is a graded projective E-module. In this case we clearly have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(E,\mathrm{d})}}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

as graded $\mathbf{Z}$-modules (because this statement reduces to the case $P = E[k]$ where it is obvious). As the isomorphism is compatible with differentials we conclude. □

09LZ **Lemma 22.35.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Assume*

(1) *$K^\bullet$ represents a compact object of $D(\mathcal{O})$, and*
(2) *$E = \mathrm{Hom}_{Comp^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$ computes the ext groups of $K^\bullet$ in $D(\mathcal{O})$.*

Then the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 22.35.3 is fully faithful.

Proof. Because our functor has a left adjoint given by $R\mathrm{Hom}(K^\bullet, -)$ by Lemma 22.35.5 it suffices to show for a differential graded E-module M that the map

$$H^0(M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K^\bullet, M \otimes^{\mathbf{L}}_E K^\bullet)$$

is an isomorphism. We may assume that $M = P$ is a differential graded E-module which has property (P). Since $K^\bullet$ defines a compact object, we reduce using Lemma 22.20.1 to the case where P has a finite filtration whose graded pieces are direct sums of $E[k]$. Again using compactness we reduce to the case $P = E[k]$. The assumption on $K^\bullet$ is that the result holds for these. □

22.36. Characterizing compact objects

09QZ Compact objects of additive categories are defined in Derived Categories, Definition 13.37.1. In this section we characterize compact objects of the derived category of a differential graded algebra.

09R0 **Remark 22.36.1.** Let (A, d) be a differential graded algebra. Is there a characterization of those differential graded A-modules P for which we have

$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{D(A,\mathrm{d})}(P, M)$$

for all differential graded A-modules M? Let $\mathcal{D} \subset K(A, \mathrm{d})$ be the full subcategory whose objects are the objects P satisfying the above. Then $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $K(A, \mathrm{d})$. If P is projective as a graded A-module, then to see where P is an object of $\mathcal{D}$ it is enough to check that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = 0$ whenever M is acyclic. However, in general it is not enough to assume that P is projective as a graded A-module. Example: take $A = R = k[\epsilon]$ where k is a field and $k[\epsilon] = k[x]/(x^2)$ is the ring of dual numbers. Let P be the object with $P^n = R$ for all $n \in \mathbf{Z}$ and differential given by multiplication by ϵ. Then $\mathrm{id}_P \in \mathrm{Hom}_{K(A,\mathrm{d})}(P, P)$ is a nonzero element but P is acyclic.

09R1 **Remark 22.36.2.** Let (A, d) be a differential graded algebra. Let us say a differential graded A-module M is *finite* if M is generated, as a right A-module, by finitely many elements. If P is a differential graded A-module which is finite graded projective, then we can ask: Does P give a compact object of $D(A, \mathrm{d})$? Presumably, this is not true in general, but we do not know a counter example. However, if P is also an object of the category $\mathcal{D}$ of Remark 22.36.1, then this is the case (this follows from the fact that direct sums in $D(A, \mathrm{d})$ are given by direct sums of modules; details omitted).

09R2 **Lemma 22.36.3.** *Let (A, d) be a differential graded algebra. Let E be a compact object of $D(A, d)$. Let P be a differential graded A-module which has a finite filtration*

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset F_nP = P$$

by differential graded submodules such that

$$F_{i+1}P/F_iP \cong \bigoplus\nolimits_{j \in J_i} A[k_{i,j}]$$

as differential graded A-modules for some sets J_i and integers $k_{i,j}$. Let $E \to P$ be a morphism of $D(A, d)$. Then there exists a differential graded submodule $P' \subset P$

such that $F_{i+1}P \cap P'/(F_iP \cap P')$ *is equal to* $\bigoplus_{j\in J_i'} A[k_{i,j}]$ *for some finite subsets* $J_i' \subset J_i$ *and such that* $E \to P$ *factors through* P'.

Proof. We will prove by induction on $-1 \leq m \leq n$ that there exists a differential graded submodule $P' \subset P$ such that

(1) $F_mP \subset P'$,
(2) for $i \geq m$ the quotient $F_{i+1}P \cap P'/(F_iP \cap P')$ is isomorphic to $\bigoplus_{j\in J_i'} A[k_{i,j}]$ for some finite subsets $J_i' \subset J_i$, and
(3) $E \to P$ factors through P'.

The base case is $m = n$ where we can take $P' = P$.

Induction step. Assume P' works for m. For $i \geq m$ and $j \in J_i'$ let $x_{i,j} \in F_{i+1}P \cap P'$ be a homogeneous element of degree $k_{i,j}$ whose image in $F_{i+1}P \cap P'/(F_iP \cap P')$ is the generator in the summand corresponding to $j \in J_i$. The $x_{i,j}$ generate P'/F_mP as an A-module. Write

$$\mathrm{d}(x_{i,j}) = \sum x_{i',j'} a_{i,j}^{i',j'} + y_{i,j}$$

with $y_{i,j} \in F_mP$ and $a_{i,j}^{i',j'} \in A$. There exists a finite subset $J_{m-1}' \subset J_{m-1}$ such that each $y_{i,j}$ maps to an element of the submodule $\bigoplus_{j\in J_{m-1}'} A[k_{m-1,j}]$ of $F_mP/F_{m-1}P$. Let $P'' \subset F_mP$ be the inverse image of $\bigoplus_{j\in J_{m-1}'} A[k_{m-1,j}]$ under the map $F_mP \to F_mP/F_{m-1}P$. Then we see that the A-submodule

$$P'' + \sum x_{i,j}A$$

is a differential graded submodule of the type we are looking for. Moreover

$$P'/(P'' + \sum x_{i,j}A) = \bigoplus\nolimits_{j\in J_{m-1}\setminus J_{m-1}'} A[k_{m-1,j}]$$

Since E is compact, the composition of the given map $E \to P'$ with the quotient map, factors through a finite direct subsum of the module displayed above. Hence after enlarging J_{m-1}' we may assume $E \to P'$ factors through $P'' + \sum x_{i,j}A$ as desired. □

It is not true that every compact object of $D(A, \mathrm{d})$ comes from a finite graded projective differential graded A-module, see Examples, Section 110.69.

09R3 **Proposition 22.36.4.** *Let* (A, d) *be a differential graded algebra. Let* E *be an object of* $D(A, d)$. *Then the following are equivalent*

(1) *E is a compact object,*
(2) *E is a direct summand of an object of $D(A, d)$ which is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A.*

Proof. Assume E is compact. By Lemma 22.20.4 we may assume that E is represented by a differential graded A-module P with property (P). Consider the distinguished triangle

$$\bigoplus F_iP \to \bigoplus F_iP \to P \xrightarrow{\delta} \bigoplus F_iP[1]$$

coming from the admissible short exact sequence of Lemma 22.20.1. Since E is compact we have $\delta = \sum_{i=1,\ldots,n} \delta_i$ for some $\delta_i : P \to F_iP[1]$. Since the composition

of δ with the map $\bigoplus F_iP[1] \to \bigoplus F_iP[1]$ is zero (Derived Categories, Lemma 13.4.1) it follows that $\delta = 0$ (follows as $\bigoplus F_iP \to \bigoplus F_iP$ maps the summand F_iP via the difference of id and the inclusion map into $F_{i-1}P$). Thus we see that the identity on E factors through $\bigoplus F_iP$ in $D(A, \mathrm{d})$ (by Derived Categories, Lemma 13.4.11). Next, we use that P is compact again to see that the map $E \to \bigoplus F_iP$ factors through $\bigoplus_{i=1,\ldots,n} F_iP$ for some n. In other words, the identity on E factors through $\bigoplus_{i=1,\ldots,n} F_iP$. By Lemma 22.36.3 we see that the identity of E factors as $E \to P \to E$ where P is as in part (2) of the statement of the lemma. In other words, we have proven that (1) implies (2).

Assume (2). By Derived Categories, Lemma 13.37.2 it suffices to show that P gives a compact object. Observe that P has property (P), hence we have

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$$

for any differential graded module M by Lemma 22.22.3. As direct sums in $D(A, \mathrm{d})$ are given by direct sums of graded modules (Lemma 22.22.4) we reduce to showing that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$ commutes with direct sums. Using that $K(A, \mathrm{d})$ is a triangulated category, that Hom is a cohomological functor in the first variable, and the filtration on P, we reduce to the case that P is a finite direct sum of shifts of A. Thus we reduce to the case $P = A[k]$ which is clear. □

09RA **Lemma 22.36.5.** *Let (A, d) be a differential graded algebra. For every compact object E of $D(A, d)$ there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(A,d)}(E, M) = 0$ if $H^i(M) = 0$ for $i \in [a, b]$.*

Proof. Observe that the collection of objects of $D(A, \mathrm{d})$ for which such a pair of integers exists is a saturated, strictly full triangulated subcategory of $D(A, \mathrm{d})$. Thus by Proposition 22.36.4 it suffices to prove this when E is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. Using the compatibility with triangles, we see that it suffices to prove it for $P = A$. In this case $\mathrm{Hom}_{D(A,\mathrm{d})}(A, M) = H^0(M)$ and the result holds with $a = b = 0$. □

If (A, d) is just an algebra placed in degree 0 with zero differential or more generally lives in only a finite number of degrees, then we do obtain the more precise description of compact objects.

09RB **Lemma 22.36.6.** *Let (A, d) be a differential graded algebra. Assume that $A^n = 0$ for $|n| \gg 0$. Let E be an object of $D(A, d)$. The following are equivalent*

(1) *E is a compact object, and*
(2) *E can be represented by a differential graded A-module P which is finite projective as a graded A-module and satisfies $\mathrm{Hom}_{K(A,d)}(P, M) = \mathrm{Hom}_{D(A,d)}(P, M)$ for every differential graded A-module M.*

Proof. Let $\mathcal{D} \subset K(A, \mathrm{d})$ be the triangulated subcategory discussed in Remark 22.36.1. Let P be an object of $\mathcal{D}$ which is finite projective as a graded A-module. Then P represents a compact object of $D(A, \mathrm{d})$ by Remark 22.36.2.

To prove the converse, let E be a compact object of $D(A, \mathrm{d})$. Fix $a \leq b$ as in Lemma 22.36.5. After decreasing a and increasing b if necessary, we may also assume that $H^i(E) = 0$ for $i \notin [a, b]$ (this follows from Proposition 22.36.4 and our assumption on A). Moreover, fix an integer $c > 0$ such that $A^n = 0$ if $|n| \geq c$.

By Proposition 22.36.4 we see that E is a direct summand, in $D(A, \mathrm{d})$, of a differential graded A-module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. In particular, P has property (P) and we have $\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$ for any differential graded module M by Lemma 22.22.3. In other words, P is an object of the triangulated subcategory $\mathcal{D} \subset K(A, \mathrm{d})$ discussed in Remark 22.36.1. Note that P is finite free as a graded A-module.

Choose $n > 0$ such that $b + 4c - n < a$. Represent the projector onto E by an endomorphism $\varphi : P \to P$ of differential graded A-modules. Consider the distinguished triangle

$$P \xrightarrow{1-\varphi} P \to C \to P[1]$$

in $K(A, \mathrm{d})$ where C is the cone of the first arrow. Then C is an object of $\mathcal{D}$, we have $C \cong E \oplus E[1]$ in $D(A, \mathrm{d})$, and C is a finite graded free A-module. Next, consider a distinguished triangle

$$C[1] \to C \to C' \to C[2]$$

in $K(A, \mathrm{d})$ where C' is the cone on a morphism $C[1] \to C$ representing the composition

$$C[1] \cong E[1] \oplus E[2] \to E[1] \to E \oplus E[1] \cong C$$

in $D(A, \mathrm{d})$. Then we see that C' represents $E \oplus E[2]$. Continuing in this manner we see that we can find a differential graded A-module P which is an object of $\mathcal{D}$, is a finite free as a graded A-module, and represents $E \oplus E[n]$.

Choose a basis x_i, $i \in I$ of homogeneous elements for P as an A-module. Let $d_i = \deg(x_i)$. Let P_1 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \leq a - c - 1$. Let P_2 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \geq b - n + c$. We observe

(1) P_1 and P_2 are differential graded submodules of P,
(2) $P_1^t = 0$ for $t \geq a$,
(3) $P_1^t = P^t$ for $t \leq a - 2c$,
(4) $P_2^t = 0$ for $t \leq b - n$,
(5) $P_2^t = P^t$ for $t \geq b - n + 2c$.

As $b - n + 2c \geq a - 2c$ by our choice of n we obtain a short exact sequence of differential graded A-modules

$$0 \to P_1 \cap P_2 \to P_1 \oplus P_2 \xrightarrow{\pi} P \to 0$$

Since P is projective as a graded A-module this is an admissible short exact sequence (Lemma 22.16.1). Hence we obtain a boundary map $\delta : P \to (P_1 \cap P_2)[1]$ in $K(A, \mathrm{d})$, see Lemma 22.7.2. Since $P = E \oplus E[n]$ and since $P_1 \cap P_2$ lives in degrees $(b-n, a)$ we find that $\mathrm{Hom}_{D(A,\mathrm{d})}(E \oplus E[n], (P_1 \cap P_2)[1])$ is zero. Therefore $\delta = 0$ as a morphism in $K(A, \mathrm{d})$ as P is an object of $\mathcal{D}$. By Derived Categories, Lemma 13.4.11 we can find a map $s : P \to P_1 \oplus P_2$ such that $\pi \circ s = \mathrm{id}_P + \mathrm{d}h + h\mathrm{d}$ for some $h : P \to P$ of degree -1. Since $P_1 \oplus P_2 \to P$ is surjective and since P is projective as a graded A-module we can choose a homogeneous lift $\tilde{h} : P \to P_1 \oplus P_2$ of h. Then we change s into $s + \mathrm{d}\tilde{h} + \tilde{h}\mathrm{d}$ to get $\pi \circ s = \mathrm{id}_P$. This means we obtain a direct sum decomposition $P = s^{-1}(P_1) \oplus s^{-1}(P_2)$. Since $s^{-1}(P_2)$ is equal to P in degrees $\geq b - n + 2c$ we see that $s^{-1}(P_2) \to P \to E$ is a quasi-isomorphism, i.e., an isomorphism in $D(A, \mathrm{d})$. This finishes the proof. $\square$

22.37. Equivalences of derived categories

09S5 Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. A natural question that arises in nature is what it means that $D(A, \mathrm{d})$ is equivalent to $D(B, \mathrm{d})$ as an R-linear triangulated category. This is a rather subtle question and it will turn out it isn't always the correct question to ask. Nonetheless, in this section we collection some conditions that guarantee this is the case.

We strongly urge the reader to take a look at the groundbreaking paper [**Ric89b**] on this topic.

09S6 **Lemma 22.37.1.** *Let R be a ring. Let $(A, d) \to (B, d)$ be a homomorphism of differential graded algebras over R, which induces an isomorphism on cohomology algebras. Then*

$$- \otimes^{\mathbf{L}}_A B : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories with quasi-inverse the restriction functor $N \mapsto N_A$.

Proof. By Lemma 22.33.7 the functor $M \longmapsto M \otimes^{\mathbf{L}}_A B$ is fully faithful. By Lemma 22.33.5 the functor $N \longmapsto R\mathrm{Hom}(B, N) = N_A$ is a right adjoint, see Example 22.33.6. It is clear that the kernel of $R\mathrm{Hom}(B, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.7.2. □

When we analyze the proof above we see that we obtain the following generalization for free.

09S7 **Lemma 22.37.2.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Assume that*

1. *N defines a compact object of $D(B, d)$,*
2. *if $N' \in D(B, d)$ and $\mathrm{Hom}_{D(B,d)}(N, N'[n]) = 0$ for $n \in \mathbf{Z}$, then $N' = 0$, and*
3. *the map $H^k(A) \to \mathrm{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then

$$- \otimes^{\mathbf{L}}_A N : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories.

Proof. By Lemma 22.33.7 the functor $M \longmapsto M \otimes^{\mathbf{L}}_A N$ is fully faithful. By Lemma 22.33.5 the functor $N' \longmapsto R\mathrm{Hom}(N, N')$ is a right adjoint. By assumption (3) the kernel of $R\mathrm{Hom}(N, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.7.2. □

09SS **Remark 22.37.3.** In Lemma 22.37.2 we can replace condition (2) by the condition that N is a classical generator for $D_{compact}(B, d)$, see Derived Categories, Proposition 13.37.6. Moreover, if we knew that $R\mathrm{Hom}(N, B)$ is a compact object of $D(A, \mathrm{d})$, then it suffices to check that N is a weak generator for $D_{compact}(B, \mathrm{d})$. We omit the proof; we will add it here if we ever need it in the Stacks project.

Sometimes the B-module P in the lemma below is called an "(A, B)-tilting complex".

09S8 **Lemma 22.37.4.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Assume that $A = H^0(A)$. The following are equivalent*

(1) *$D(A, d)$ and $D(B, d)$ are equivalent as R-linear triangulated categories, and*
(2) *there exists an object P of $D(B, d)$ such that*
   (a) *P is a compact object of $D(B, d)$,*
   (b) *if $N \in D(B, d)$ with $\mathrm{Hom}_{D(B,d)}(P, N[i]) = 0$ for $i \in \mathbf{Z}$, then $N = 0$,*
   (c) *$\mathrm{Hom}_{D(B,d)}(P, P[i]) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

The equivalence $D(A, d) \to D(B, d)$ constructed in (2) sends A to P.

Proof. Let $F : D(A, \mathrm{d}) \to D(B, \mathrm{d})$ be an equivalence. Then F maps compact objects to compact objects. Hence $P = F(A)$ is compact, i.e., (2)(a) holds. Conditions (2)(b) and (2)(c) are immediate from the fact that F is an equivalence.

Let P be an object as in (2). Represent P by a differential graded module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(P, P)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.22.3 and assumption (2)(c). Viewing P as a (E, B)-bimodule and using Lemma 22.37.2 and assumption (2)(b) we obtain an equivalence

$$D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

sending E to P. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$. Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 22.37.1. □

09S9 **Remark 22.37.5.** Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Suppose given an R-linear equivalence

$$F : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

of triangulated categories. Set $N = F(A)$. Then N is a differential graded B-module. Since F is an equivalence and A is a compact object of $D(A, \mathrm{d})$, we conclude that N is a compact object of $D(B, \mathrm{d})$. Since A generates $D(A, \mathrm{d})$ and F is an equivalence, we see that N generates $D(B, \mathrm{d})$. Finally, $H^k(A) = \mathrm{Hom}_{D(A,\mathrm{d})}(A, A[k])$ and as F an equivalence we see that F induces an isomorphism $H^k(A) = \mathrm{Hom}_{D(B,\mathrm{d})}(N, N[k])$ for all k. In order to conclude that there is an equivalence $D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$ which arises from the construction in Lemma 22.37.2 all we need is a left A-module structure on N compatible with derivation and commuting with the given right B-module structure. In fact, it suffices to do this after replacing N by a quasi-isomorphic differential graded B-module. The module structure can be constructed in certain cases. For example, if we assume that F can be lifted to a differential graded functor

$$F^{dg} : \mathrm{Mod}^{dg}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$$

(for notation see Example 22.26.8) between the associated differential graded categories, then this holds. Another case is discussed in the proposition below.

09SA **Proposition 22.37.6.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $F : D(A, d) \to D(B, d)$ be an R-linear equivalence of triangulated categories. Assume that*

(1) *$A = H^0(A)$, and*
(2) *B is K-flat as a complex of R-modules.*

Then there exists an (A, B)-bimodule N as in Lemma 22.37.2.

Proof. As in Remark 22.37.5 above, we set $N = F(A)$ in $D(B, \mathrm{d})$. We may assume that N is a differential graded B-module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.22.3. Moreover, by the discussion in Remark 22.37.5 and by Lemma 22.37.2 we see that N as a (E, B)-bimodule induces an equivalence $- \otimes^{\mathbf{L}}_E N : D(E, \mathrm{d}) \to D(B, \mathrm{d})$. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$. Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 22.37.1. Note that the quasi-inverse $D(A, \mathrm{d}) \to D(E', \mathrm{d})$ of the left vertical arrow is given by $M \mapsto M \otimes^{\mathbf{L}}_A A$ where A is viewed as a (A, E')-bimodule, see Example 22.33.6. On the other hand the functor $D(E', \mathrm{d}) \to D(B, \mathrm{d})$ is given by $M \mapsto M \otimes^{\mathbf{L}}_{E'} N$ where N is as above. We conclude by Lemma 22.34.3. □

09SB **Remark 22.37.7.** Let A, B, F, N be as in Proposition 22.37.6. It is not clear that F and the functor $G(-) = - \otimes^{\mathbf{L}}_A N$ are isomorphic. By construction there is an isomorphism $N = G(A) \to F(A)$ in $D(B, \mathrm{d})$. It is straightforward to extend this to a functorial isomorphism $G(M) \to F(M)$ for M is a differential graded A-module which is graded projective (e.g., a sum of shifts of A). Then one can conclude that $G(M) \cong F(M)$ when M is a cone of a map between such modules. We don't know whether more is true in general.

09SC **Lemma 22.37.8.** *Let R be a ring. Let A and B be R-algebras. The following are equivalent*

(1) *there is an R-linear equivalence $D(A) \to D(B)$ of triangulated categories,*
(2) *there exists an object P of $D(B)$ such that*
   (a) *P can be represented by a finite complex of finite projective B-modules,*
   (b) *if $K \in D(B)$ with $\mathrm{Ext}^i_B(P, K) = 0$ for $i \in \mathbf{Z}$, then $K = 0$, and*
   (c) *$\mathrm{Ext}^i_B(P, P) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Moreover, if B is flat as an R-module, then this is also equivalent to

(3) *there exists an (A, B)-bimodule N such that $- \otimes^{\mathbf{L}}_A N : D(A) \to D(B)$ is an equivalence.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 22.37.4 combined with the result of Lemma 22.36.6 characterizing compact objects of $D(B)$ (small

detail omitted). The equivalence with (3) if B is R-flat follows from Proposition 22.37.6. □

09SD **Remark 22.37.9.** Let R be a ring. Let A and B be R-algebras. If $D(A)$ and $D(B)$ are equivalent as R-linear triangulated categories, then the centers of A and B are isomorphic as R-algebras. In particular, if A and B are commutative, then $A \cong B$. The rather tricky proof can be found in [**Ric89b**, Proposition 9.2] or [**KZ98**, Proposition 6.3.2]. Another approach might be to use Hochschild cohomology (see remark below).

09ST **Remark 22.37.10.** Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras which are derived equivalent, i.e., such that there exists an R-linear equivalence $D(A, \mathrm{d}) \to D(B, \mathrm{d})$ of triangulated categories. We would like to show that certain invariants of (A, d) and (B, d) coincide. In many situations one has more control of the situation. For example, it may happen that there is an equivalence of the form

$$- \otimes_A \Omega : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

for some differential graded (A, B)-bimodule Ω (this happens in the situation of Proposition 22.37.6 and is often true if the equivalence comes from a geometric construction). If also the quasi-inverse of our functor is given as

$$- \otimes^{\mathbf{L}}_B \Omega' : D(B, \mathrm{d}) \longrightarrow D(A, \mathrm{d})$$

for a differential graded (B, A)-bimodule Ω' (and as before such a module Ω' often exists in practice). In this case we can consider the functor

$$D(A^{opp} \otimes_R A, \mathrm{d}) \longrightarrow D(B^{opp} \otimes_R B, \mathrm{d}), \quad M \longmapsto \Omega' \otimes^{\mathbf{L}}_A M \otimes^{\mathbf{L}}_A \Omega$$

on derived categories of bimodules (use Lemma 22.28.3 to turn bimodules into right modules). Observe that this functor sends the (A, A)-bimodule A to the (B, B)-bimodule B. Under suitable conditions (e.g., flatness of A, B, Ω over R, etc) this functor will be an equivalence as well. If this is the case, then it follows that we have isomorphisms of Hochschild cohomology groups

$$HH^i(A, \mathrm{d}) = \mathrm{Hom}_{D(A^{opp} \otimes_R A, \mathrm{d})}(A, A[i]) \longrightarrow \mathrm{Hom}_{D(B^{opp} \otimes_R B, \mathrm{d})}(B, B[i]) = HH^i(B, \mathrm{d}).$$

For example, if $A = H^0(A)$, then $HH^0(A, \mathrm{d})$ is equal to the center of A, and this gives a conceptual proof of the result mentioned in Remark 22.37.9. If we ever need this remark we will provide a precise statement with a detailed proof here.

22.38. Resolutions of differential graded algebras

0BZ6 Let R be a ring. Under our assumptions the free R-algebra $R\langle S \rangle$ on a set S is the algebra with R-basis the expressions

$$s_1 s_2 \ldots s_n$$

where $n \geq 0$ and $s_1, \ldots, s_n \in S$ is a sequence of elements of S. Multiplication is given by concatenation

$$(s_1 s_2 \ldots s_n) \cdot (s'_1 s'_2 \ldots s'_m) = s_1 \ldots s_n s'_1 \ldots s'_m$$

This algebra is characterized by the property that the map

$$\mathrm{Mor}_{R\text{-alg}}(R\langle S \rangle, A) \to \mathrm{Map}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every R-algebra A.

In the category of graded R-algebras our set S should come with a grading, which we think of as a map $\deg : S \to \mathbf{Z}$. Then $R\langle S\rangle$ has a grading such that the monomials have degree

$$\deg(s_1 s_2 \ldots s_n) = \deg(s_1) + \ldots + \deg(s_n)$$

In this setting we have

$$\mathrm{Mor}_{\text{graded } R\text{-alg}}(R\langle S\rangle, A) \to \mathrm{Map}_{\text{graded sets}}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every graded R-algebra A.

If A is a graded R-algebra and S is a graded set, then we can similarly form $A\langle S\rangle$. Elements of $A\langle S\rangle$ are sums of elements of the form

$$a_0 s_1 a_1 s_2 \ldots a_{n-1} s_n a_n$$

with $a_i \in A$ modulo the relations that these expressions are R-multilinear in $(a_0, \ldots, a_n)$. Thus for every sequence $s_1, \ldots, s_n$ of elements of S there is an inclusion

$$A \otimes_R \ldots \otimes_R A \subset A\langle S\rangle$$

and the algebra is the direct sum of these. With this definition the reader shows that the map

$$\mathrm{Mor}_{\text{graded } R\text{-alg}}(A\langle S\rangle, B) \to \mathrm{Mor}_{\text{graded } R\text{-alg}}(A, B) \times \mathrm{Map}_{\text{graded sets}}(S, B),$$

sending φ to $(\varphi|_A, (s \mapsto \varphi(s)))$ is a bijection for every graded R-algebra A. We observe that if A was a free graded R-algebra, then so is $A\langle S\rangle$.

Suppose that A is a differential graded R-algebra and that S is a graded set. Suppose moreover for every $s \in S$ we are given a homogeneous element $f_s \in A$ with $\deg(f_s) = \deg(s) + 1$ and $\mathrm{d}f_s = 0$. Then there exists a unique structure of differential graded algebra on $A\langle S\rangle$ with $\mathrm{d}(s) = f_s$. For example, given $a, b, c \in A$ and $s, t \in S$ we would define

$$\begin{aligned}\mathrm{d}(asbtc) &= \mathrm{d}(a)sbtc + (-1)^{\deg(a)} a f_s btc + (-1)^{\deg(a)+\deg(s)} as\mathrm{d}(b)tc \\ &\quad + (-1)^{\deg(a)+\deg(s)+\deg(b)} asbf_t c + (-1)^{\deg(a)+\deg(s)+\deg(b)+\deg(t)} asbt\mathrm{d}(c)\end{aligned}$$

We omit the details.

0BZ7 **Lemma 22.38.1.** *Let R be a ring. Let (B, d) be a differential graded R-algebra. There exists a quasi-isomorphism $(A, d) \to (B, d)$ of differential graded R-algebras with the following properties*

(1) *A is K-flat as a complex of R-modules,*
(2) *A is a free graded R-algebra.*

Proof. First we claim we can find $(A_0, \mathrm{d}) \to (B, \mathrm{d})$ having (1) and (2) inducing a surjection on cohomology. Namely, take a graded set S and for each $s \in S$ a homogeneous element $b_s \in \mathrm{Ker}(d : B \to B)$ of degree $\deg(s)$ such that the classes $\overline{b}_s$ in $H^*(B)$ generate $H^*(B)$ as an R-module. Then we can set $A_0 = R\langle S\rangle$ with zero differential and $A_0 \to B$ given by mapping s to b_s.

Given $A_0 \to B$ inducing a surjection on cohomology we construct a sequence

$$A_0 \to A_1 \to A_2 \to \ldots B$$

by induction. Given $A_n \to B$ we set S_n be a graded set and for each $s \in S_n$ we let $a_s \in \mathrm{Ker}(\mathrm{d} : A_n \to A_n)$ be a homogeneous element of degree $\deg(s) + 1$ mapping

to a class $\overline{a}_s$ in $H^*(A_n)$ which maps to zero in $H^*(B)$. We choose S_n large enough so that the elements $\overline{a}_s$ generate $\mathrm{Ker}(H^*(A_n) \to H^*(B))$ as an R-module. Then we set

$$A_{n+1} = A_n\langle S_n \rangle$$

with differential given by $\mathrm{d}(s) = a_s$ see discussion above. Then each (A_n, d) satisfies (1) and (2), we omit the details. The map $H^*(A_n) \to H^*(B)$ is surjective as this was true for $n = 0$.

It is clear that $A = \operatorname{colim} A_n$ is a free graded R-algebra. It is K-flat by More on Algebra, Lemma 15.59.8. The map $H^*(A) \to H^*(B)$ is an isomorphism as it is surjective and injective: every element of $H^*(A)$ comes from an element of $H^*(A_n)$ for some n and if it dies in $H^*(B)$, then it dies in $H^*(A_{n+1})$ hence in $H^*(A)$. □

As an application we prove the "correct" version of Lemma 22.34.2.

0BZ8 **Lemma 22.38.2.** *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume $A \otimes_R C$ represents $A \otimes^{\mathbf{L}}_R C$ in $D(R)$. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition*

$$D(A, d) \xrightarrow{-\otimes^{\mathbf{L}}_A N} D(B, d) \xrightarrow{-\otimes^{\mathbf{L}}_B N'} D(C, d)$$

is isomorphic to $- \otimes^{\mathbf{L}}_A N''$ for some differential graded (A, C)-bimodule N''.

Proof. Using Lemma 22.38.1 we choose a quasi-isomorphism $(B', \mathrm{d}) \to (B, \mathrm{d})$ with B' K-flat as a complex of R-modules. By Lemma 22.37.1 the functor $- \otimes^{\mathbf{L}}_{B'} B : D(B', \mathrm{d}) \to D(B, \mathrm{d})$ is an equivalence with quasi-inverse given by restriction. Note that restriction is canonically isomorphic to the functor $- \otimes^{\mathbf{L}}_B B : D(B, \mathrm{d}) \to D(B', \mathrm{d})$ where B is viewed as a (B, B')-bimodule. Thus it suffices to prove the lemma for the compositions

$$D(A) \to D(B) \to D(B'), \quad D(B') \to D(B) \to D(C), \quad D(A) \to D(B') \to D(C).$$

The first one is Lemma 22.34.3 because B' is K-flat as a complex of R-modules. The second one is true because $B \otimes^{\mathbf{L}}_B N' = N' = B \otimes_B N'$ and hence Lemma 22.34.1 applies. Thus we reduce to the case where B is K-flat as a complex of R-modules.

Assume B is K-flat as a complex of R-modules. It suffices to show that (22.34.1.1) is an isomorphism, see Lemma 22.34.2. Choose a quasi-isomorphism $L \to A$ where L is a differential graded R-module which has property (P). Then it is clear that $P = L \otimes_R B$ has property (P) as a differential graded B-module. Hence we have to show that $P \to A \otimes_R B$ induces a quasi-isomorphism

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

We can rewrite this as

$$P \otimes_R B \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

Since B is K-flat as a complex of R-modules, it follows from More on Algebra, Lemma 15.59.2 that it is enough to show that

$$P \otimes_R C \to A \otimes_R C$$

is a quasi-isomorphism, which is exactly our assumption. □

The following lemma does not really belong in this section, but there does not seem to be a good natural spot for it.

0CRM **Lemma 22.38.3.** *Let (A, d) be a differential graded algebra with $H^i(A)$ countable for each i. Let M be an object of $D(A, d)$. Then the following are equivalent*

(1) *$M = hocolim E_n$ with E_n compact in $D(A, d)$, and*
(2) *$H^i(M)$ is countable for each i.*

Proof. Assume (1) holds. Then we have $H^i(M) = \operatorname{colim} H^i(E_n)$ by Derived Categories, Lemma 13.33.8. Thus it suffices to prove that $H^i(E_n)$ is countable for each n. By Proposition 22.36.4 we see that E_n is isomorphic in $D(A, \mathrm{d})$ to a direct summand of a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_jP/F_{j-1}P$ are finite direct sums of shifts of A. By assumption the groups $H^i(F_jP/F_{j-1}P)$ are countable. Arguing by induction on the length of the filtration and using the long exact cohomology sequence we conclude that (2) is true. The interesting implication is the other one.

We claim there is a countable differential graded subalgebra $A' \subset A$ such that the inclusion map $A' \to A$ defines an isomorphism on cohomology. To construct A' we choose countable differential graded subalgebras

$$A_1 \subset A_2 \subset A_3 \subset \ldots$$

such that (a) $H^i(A_1) \to H^i(A)$ is surjective, and (b) for $n > 1$ the kernel of the map $H^i(A_{n-1}) \to H^i(A_n)$ is the same as the kernel of the map $H^i(A_{n-1}) \to H^i(A)$. To construct A_1 take any countable collection of cochains $S \subset A$ generating the cohomology of A (as a ring or as a graded abelian group) and let A_1 be the differential graded subalgebra of A generated by S. To construct A_n given A_{n-1} for each cochain $a \in A^i_{n-1}$ which maps to zero in $H^i(A)$ choose $s_a \in A^{i-1}$ with $\mathrm{d}(s_a) = a$ and let A_n be the differential graded subalgebra of A generated by A_{n-1} and the elements s_a. Finally, take $A' = \bigcup A_n$.

By Lemma 22.37.1 the restriction map $D(A, \mathrm{d}) \to D(A', \mathrm{d})$, $M \mapsto M_{A'}$ is an equivalence. Since the cohomology groups of M and $M_{A'}$ are the same, we see that it suffices to prove the implication (2) $\Rightarrow$ (1) for (A', d).

Assume A is countable. By the exact same type of argument as given above we see that for M in $D(A, \mathrm{d})$ the following are equivalent: $H^i(M)$ is countable for each i and M can be represented by a countable differential graded module. Hence in order to prove the implication (2) $\Rightarrow$ (1) we reduce to the situation described in the next paragraph.

Assume A is countable and that M is a countable differential graded module over A. We claim there exists a homomorphism $P \to M$ of differential graded A-modules such that

(1) $P \to M$ is a quasi-isomorphism,
(2) P has property (P), and
(3) P is countable.

Looking at the proof of the construction of P-resolutions in Lemma 22.20.4 we see that it suffices to show that we can prove Lemma 22.20.3 in the setting of countable differential graded modules. This is immediate from the proof.

Assume that A is countable and that M is a countable differential graded module with property (P). Choose a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that we have

(1) $P = \bigcup F_pP$,
(2) $F_iP \to F_{i+1}P$ is an admissible monomorphism,
(3) isomorphisms of differential graded modules $F_iP/F_{i-1}P \to \bigoplus_{j\in J_i} A[k_j]$ for some sets J_i and integers k_j.

Of course J_i is countable for each i. For each i and $j \in J_i$ choose $x_{i,j} \in F_iP$ of degree k_j whose image in $F_iP/F_{i-1}P$ generates the summand corresponding to j.

Claim: Given n and finite subsets $S_i \subset J_i$, $i = 1, \ldots, n$ there exist finite subsets $S_i \subset T_i \subset J_i$, $i = 1, \ldots, n$ such that $P' = \bigoplus_{i\leq n} \bigoplus_{j\in T_i} Ax_{i,j}$ is a differential graded submodule of P. This was shown in the proof of Lemma 22.36.3 but it is also easily shown directly: the elements $x_{i,j}$ freely generate P as a right A-module. The structure of P shows that

$$\mathrm{d}(x_{i,j}) = \sum\nolimits_{i'<i} x_{i',j'}a_{i',j'}$$

where of course the sum is finite. Thus given $S_0, \ldots, S_n$ we can first choose $S_0 \subset S_0', \ldots, S_{n-1} \subset S_{n-1}'$ with $\mathrm{d}(x_{n,j}) \in \bigoplus_{i'<n, j'\in S_{i'}'} x_{i',j'}A$ for all $j \in S_n$. Then by induction on n we can choose $S_0' \subset T_0, \ldots, S_{n-1}' \subset T_{n-1}$ to make sure that $\bigoplus_{i'<n, j'\in T_{i'}} x_{i',j'}A$ is a differential graded A-submodule. Setting $T_n = S_n$ we find that $P' = \bigoplus_{i\leq n, j\in T_i} x_{i,j}A$ is as desired.

From the claim it is clear that $P = \bigcup P_n'$ is a countable rising union of P_n' as above. By construction each P_n' is a differential graded module with property (P) such that the filtration is finite and the successive quotients are finite direct sums of shifts of A. Hence P_n' defines a compact object of $D(A, \mathrm{d})$, see for example Proposition 22.36.4. Since $P = \mathrm{hocolim} P_n'$ in $D(A, \mathrm{d})$ by Lemma 22.23.2 the proof of the implication (2) $\Rightarrow$ (1) is complete. □

22.39. Other chapters

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(7) Sites and Sheaves
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Schemes

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CHAPTER 23

Divided Power Algebra

09PD

23.1. Introduction

09PE In this chapter we talk about divided power algebras and what you can do with them. A reference is the book [**Ber74**].

23.2. Divided powers

07GK In this section we collect some results on divided power rings. We will use the convention $0! = 1$ (as empty products should give 1).

07GL **Definition 23.2.1.** Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a *divided power structure* on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

(1) $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
(2) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(3) $\gamma_n(ax) = a^n\gamma_n(x)$,
(4) $\gamma_n(x+y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Note that the rational numbers $\frac{(n+m)!}{n!m!}$ and $\frac{(nm)!}{n!(m!)^n}$ occurring in the definition are in fact integers; the first is the number of ways to choose n out of $n+m$ and the second counts the number of ways to divide a group of nm objects into n groups of m. We make some remarks about the definition which show that $\gamma_n(x)$ is a replacement for $x^n/n!$ in I.

07GM **Lemma 23.2.2.** *Let A be a ring. Let I be an ideal of A.*

(1) *If γ is a divided power structure[1] on I, then $n!\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.*

Assume A is torsion free as a $\mathbf{Z}$-module.

(2) *A divided power structure on I, if it exists, is unique.*
(3) *If $\gamma_n : I \to I$ are maps then*

$$\gamma \text{ is a divided power structure} \Leftrightarrow n!\gamma_n(x) = x^n \ \forall x \in I, n \geq 1.$$

(4) *The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.*

Proof. Proof of (1). If γ is a divided power structure, then condition (2) (applied to 1 and $n-1$ instead of n and m) implies that $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$. Hence by induction and condition (1) we get $n!\gamma_n(x) = x^n$.

[1] Here and in the following, γ stands short for a sequence of maps $\gamma_1, \gamma_2, \gamma_3, \ldots$ from I to I.

Assume A is torsion free as a $\mathbf{Z}$-module. Proof of (2). This is clear from (1).

Proof of (3). Assume that $n!\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$. Since $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ it suffices to prove the axioms (1) – (5) of Definition 23.2.1 in case A is a $\mathbf{Q}$-algebra. In this case $\gamma_n(x) = x^n/n!$ and it is straightforward to verify (1) – (5); for example, (4) corresponds to the binomial formula

$$(x+y)^n = \sum_{i=0,\ldots,n} \frac{n!}{i!(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators x_i of I as an ideal such that $x_i^n \in (n!)I$ for all $n \geq 1$. We claim that for all $x \in I$ we have $x^n \in (n!)I$. If the claim holds then we can set $\gamma_n(x) = x^n/n!$ which is a divided power structure by (3). To prove the claim we note that it holds for $x = ax_i$. Hence we see that the claim holds for a set of generators of I as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for $x + y$ if it holds for x and y. This follows immediately from the binomial theorem. □

07GN **Example 23.2.3.** Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in I$ we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that $p^n/n! \in p\mathbf{Z}_{(p)}$ for $n \geq 1$ (for instance, this can be derived from the fact that the exponent of p in the prime factorization of $n!$ is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$), so that the definition makes sense and gives us a sequence of maps $\gamma_n : I \to I$. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 23.2.1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Lemma 23.4.2.

We notice that $\gamma_n(0) = 0$ for any ideal I of A and any divided power structure γ on I. (This follows from axiom (3) in Definition 23.2.1, applied to $a = 0$.)

07GP **Lemma 23.2.4.** *Let A be a ring. Let I be an ideal of A. Let $\gamma_n : I \to I$, $n \geq 1$ be a sequence of maps. Assume*

(a) *(1), (3), and (4) of Definition 23.2.1 hold for all $x, y \in I$, and*
(b) *properties (2) and (5) hold for x in some set of generators of I as an ideal.*

Then γ is a divided power structure on I.

Proof. The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 23.2.1. Applying (3) we see that if (2) and (5) hold for x then (2) and (5) hold for ax for all $a \in A$. Hence we see (b) implies (2) and (5) hold for a set of generators of I as an abelian group. Hence, by induction of the length of an expression in terms of these it suffices to prove that, given $x, y \in I$ such that (2) and (5) hold for x and y, then (2) and (5) hold for $x + y$.

Proof of (2) for $x + y$. By (4) we have

$$\gamma_n(x+y)\gamma_m(x+y) = \sum\nolimits_{i+j=n,\ k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for x and y this equals

$$\sum \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!}\gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x+y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given $a+b = n+m$ we have

$$\sum\nolimits_{i+k=a,\ j+l=b,\ i+j=n,\ k+l=m} \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!} = \frac{(n+m)!}{n!m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal $I = (x, y)$ in the polynomial ring $\mathbf{Q}[x, y]$ because $\gamma_n(f) = f^n/n!$, $f \in I$ defines a divided power structure on I. Hence the equality of rational numbers above is true.

Proof of (5) for $x + y$ given that (1) – (4) hold and that (5) holds for x and y. We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write $\gamma_n(\gamma_m(x+y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$. Using (4) we can write $\gamma_n(\gamma_m(x+y))$ as a sum of terms which are products of factors of the form $\gamma_k(\gamma_i(x)\gamma_j(y))$. If $i > 0$ then

$$\begin{aligned}\gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k\gamma_k(\gamma_i(x)) \\ &= \frac{(ki)!}{k!(i!)^k}\gamma_j(y)^k\gamma_{ki}(x) \\ &= \frac{(ki)!}{k!(i!)^k}\frac{(kj)!}{(j!)^k}\gamma_{ki}(x)\gamma_{kj}(y)\end{aligned}$$

using (3) in the first equality, (5) for x in the second, and (2) exactly k times in the third. Using (5) for y we see the same equality holds when $i = 0$. Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x+y)) = \sum\nolimits_{i+j=nm} c_{ij}\gamma_i(x)\gamma_j(y)$$

for certain universal constants $c_{ij} \in \mathbf{Z}$. Again the fact that the equality is valid in the polynomial ring $\mathbf{Q}[x, y]$ implies that the coefficients c_{ij} are all equal to $(nm)!/n!(m!)^n$ as desired. □

07GQ **Lemma 23.2.5.** *Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then*

1. *γ and δ agree on IJ,*
2. *if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I + J$.*

Proof. Let $x \in I$ and $y \in J$. Then

$$\gamma_n(xy) = y^n\gamma_n(x) = n!\delta_n(y)\gamma_n(x) = \delta_n(y)x^n = \delta_n(xy).$$

Hence γ and δ agree on a set of (additive) generators of IJ. By property (4) of Definition 23.2.1 it follows that they agree on all of IJ.

Assume γ and δ agree on $I \cap J$. Let $z \in I + J$. Write $z = x + y$ with $x \in I$ and $y \in J$. Then we set

$$\epsilon_n(z) = \sum \gamma_i(x)\delta_{n-i}(y)$$

for all $n \geq 1$. To see that this is well defined, suppose that $z = x' + y'$ is another representation with $x' \in I$ and $y' \in J$. Then $w = x - x' = y' - y \in I \cap J$. Hence

$$\begin{aligned}
\sum\nolimits_{i+j=n} \gamma_i(x)\delta_j(y) &= \sum\nolimits_{i+j=n} \gamma_i(x' + w)\delta_j(y) \\
&= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\gamma_l(w)\delta_j(y) \\
&= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\delta_l(w)\delta_j(y) \\
&= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y + w) \\
&= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y')
\end{aligned}$$

as desired. Hence, we have defined maps $\epsilon_n : I + J \to I + J$ for all $n \geq 1$; it is easy to see that $\epsilon_n \mid_I = \gamma_n$ and $\epsilon_n \mid_J = \delta_n$. Next, we prove conditions (1) – (5) of Definition 23.2.1 for the collection of maps ϵ_n. Properties (1) and (3) are clear. To see (4), suppose that $z = x + y$ and $z' = x' + y'$ with $x, x' \in I$ and $y, y' \in J$ and compute

$$\begin{aligned}
\epsilon_n(z + z') &= \sum\nolimits_{a+b=n} \gamma_a(x + x')\delta_b(y + y') \\
&= \sum\nolimits_{i+i'+j+j'=n} \gamma_i(x)\gamma_{i'}(x')\delta_j(y)\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \sum\nolimits_{i+j=k} \gamma_i(x)\delta_j(y) \sum\nolimits_{i'+j'=n-k} \gamma_{i'}(x')\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \epsilon_k(z)\epsilon_{n-k}(z')
\end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of I or J, see Lemma 23.2.4. This is clear because γ and δ are divided power structures.

The existence of a divided power structure ϵ on $I + J$ whose restrictions to I and J are γ and δ is thus proven; its uniqueness is rather clear. □

07GR **Lemma 23.2.6.** *Let p be a prime number. Let A be a ring, let $I \subset A$ be an ideal, and let γ be a divided power structure on I. Assume p is nilpotent in A/I. Then I is locally nilpotent if and only if p is nilpotent in A.*

Proof. If $p^N = 0$ in A, then for $x \in I$ we have $x^{pN} = (pN)!\gamma_{pN}(x) = 0$ because $(pN)!$ is divisible by p^N. Conversely, assume I is locally nilpotent. We've also assumed that p is nilpotent in A/I, hence $p^r \in I$ for some r, hence p^r nilpotent, hence p nilpotent. □

23.3. Divided power rings

07GT There is a category of divided power rings. Here is the definition.

07GU **Definition 23.3.1.** A *divided power ring* is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n \geq 1}$ is a divided power structure on I. A *homomorphism of divided power rings* $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$ and $n \geq 1$.

We sometimes say "let (B, J, δ) be a divided power algebra over (A, I, γ)" to indicate that (B, J, δ) is a divided power ring which comes equipped with a homomorphism of divided power rings $(A, I, \gamma) \to (B, J, \delta)$.

07GV **Lemma 23.3.2.** *The category of divided power rings has all limits and they agree with limits in the category of rings.*

Proof. The empty limit is the zero ring (that's weird but we need it). The product of a collection of divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \gamma)$ where $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (A, I, \gamma) \to (B, J, \delta)$ is just $C = \{a \in A \mid \alpha(a) = \beta(a)\}$ with ideal $C \cap I$ and induced divided powers. It follows that all limits exist, see Categories, Lemma 4.14.11. □

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

07GW **Lemma 23.3.3.** *Let $\mathcal{C}$ be the category of divided power rings. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *there exists a cardinal κ such that for every $f \in F(A, I, \gamma)$ there exists a morphism $(A', I', \gamma') \to (A, I, \gamma)$ of $\mathcal{C}$ such that f is the image of $f' \in F(A', I', \gamma')$ and $|A'| \leq \kappa$, and*
2. *F commutes with limits.*

Then F is representable, i.e., there exists an object (B, J, δ) of $\mathcal{C}$ such that

$$F(A, I, \gamma) = \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

functorially in (A, I, γ).

Proof. This is a special case of Categories, Lemma 4.25.1. □

07GX **Lemma 23.3.4.** *The category of divided power rings has all colimits.*

Proof. The empty colimit is $\mathbf{Z}$ with divided power ideal (0). Let's discuss general colimits. Let $\mathcal{C}$ be a category and let $c \mapsto (A_c, I_c, \gamma_c)$ be a diagram. Consider the functor

$$F(B, J, \delta) = \lim_{c \in \mathcal{C}} Hom((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any $f = (f_c)_{c \in C} \in F(B, J, \delta)$ has the property that all the images $f_c(A_c)$ generate a subring B' of B of bounded cardinality κ and that all the images $f_c(I_c)$ generate a divided power sub ideal J' of B'. And we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Also, F commutes with limits. Hence we may apply Lemma 23.3.3 to see that F is representable and we win. □

0H86 **Remark 23.3.5.** Let

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

be a pushout in the category of divided power rings (pushouts exist by Lemma 23.3.4). Then

1. $B''/J'' = B/J \otimes_{A/I} B'/J'$,
2. the map $\varphi : B \otimes_A B' \to B''$ is surjective, and
3. the map $J \otimes_A B' + B \otimes_A J' \to J''$ induced by φ is surjective.

To see (1) consider maps $(B'', J'', \delta'') \to (C, (0), \emptyset)$ and apply the definition of a pushout. To see (2) consider the image $\tilde{B} = \varphi(B \otimes_A B')$ and the ideal $\tilde{J} = \varphi(J \otimes_A B' + B \otimes_A J')$. Then it is clear that $\delta''_n(\tilde{J}) \subset \tilde{J}$ for all $n \geq 1$ because of the compatibility of δ'' with δ and δ'. Hence we must have $\tilde{B} = B''$ and $\tilde{J} = J''$ by the universality of the pushout.

07GY **Remark 23.3.6.** The forgetful functor $(A, I, \gamma) \mapsto A$ does not commute with colimits. For example, let

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

be a pushout in the category of divided power rings as discussed in Remark 23.3.5. Then in general the map $B \otimes_A B' \to B''$ isn't an isomorphism. An explicit example is given by $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$, $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$, and $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$ where $\delta_2(2) = 2$ and $\delta'_2(2) = 0$. More precisely, using Lemma 23.5.3 we let δ, resp. δ' be the unique divided power structure on J, resp. J' such that $\delta_2 : J \to J$, resp. $\delta'_2 : J' \to J'$ is the map $0 \mapsto 0, 2 \mapsto 2$, resp. $0 \mapsto 0, 2 \mapsto 0$. Then $(B'', J'', \delta'') = (\mathbf{F}_2, (0), \emptyset)$ which doesn't agree with the tensor product.

23.4. Extending divided powers

07GZ Here is the definition.

07H0 **Definition 23.4.1.** Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ *extends* to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings.

07H1 **Lemma 23.4.2.** *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds*

(1) *$IB = 0$,*
(2) *I is principal, or*
(3) *$A \to B$ is flat.*

Then γ extends to B.

Proof. Any element of IB can be written as a finite sum $\sum_{i=1}^t b_i x_i$ with $b_i \in B$ and $x_i \in I$. If γ extends to $\bar{\gamma}$ on IB then $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$. Thus, conditions (3) and (4) in Definition 23.2.1 imply that

$$\bar{\gamma}_n(\sum\nolimits_{i=1}^t b_i x_i) = \sum\nolimits_{n_1+\ldots+n_t=n} \prod\nolimits_{i=1}^t b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that $\bar{\gamma}$ is unique if it exists.

If $IB = 0$ then setting $\bar{\gamma}_n(0) = 0$ works. If $I = (x)$ then we define $\bar{\gamma}_n(bx) = b^n \gamma_n(x)$. This is well defined: if $b'x = bx$, i.e., $(b - b')x = 0$ then

$$\begin{aligned} b^n\gamma_n(x) - (b')^n\gamma_n(x) &= (b^n - (b')^n)\gamma_n(x) \\ &= (b^{n-1} + \ldots + (b')^{n-1})(b - b')\gamma_n(x) = 0 \end{aligned}$$

because $\gamma_n(x)$ is divisible by x (since $\gamma_n(I) \subset I$) and hence annihilated by $b - b'$. Next, we prove conditions (1) – (5) of Definition 23.2.1. Parts (1), (2), (3), (5) are

obvious from the construction. For (4) suppose that $y, z \in IB$, say $y = bx$ and $z = cx$. Then $y + z = (b + c)x$ hence

$$\begin{aligned}\bar{\gamma}_n(y+z) &= (b+c)^n\gamma_n(x) \\ &= \sum \frac{n!}{i!(n-i)!} b^i c^{n-i}\gamma_n(x) \\ &= \sum b^i c^{n-i}\gamma_i(x)\gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y)\bar{\gamma}_{n-i}(z)\end{aligned}$$

as desired.

Assume $A \to B$ is flat. Suppose that $b_1, \ldots, b_r \in B$ and $x_1, \ldots, x_r \in I$. Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$$

where the sum is over $e_1 + \ldots + e_r = n$ if $\bar{\gamma}_n$ exists. Next suppose that we have $c_1, \ldots, c_s \in B$ and $a_{ij} \in A$ such that $b_i = \sum a_{ij} c_j$. Setting $y_j = \sum a_{ij} x_i$ we claim that

$$\sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r) = \sum c_1^{d_1} \ldots c_s^{d_s} \gamma_{d_1}(y_1) \ldots \gamma_{d_s}(y_s)$$

in B where on the right hand side we are summing over $d_1 + \ldots + d_s = n$. Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in $\mathbf{Z}[a_{ij}]$ of terms of the form $c_1^{d_1} \ldots c_s^{d_s} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$. To see that the coefficients agree we note that the result is true in $\mathbf{Q}[x_1, \ldots, x_r, c_1, \ldots, c_s, a_{ij}]$ with γ the unique divided power structure on $(x_1, \ldots, x_r)$. By Lazard's theorem (Algebra, Theorem 10.81.4) we can write B as a directed colimit of finite free A-modules. In particular, if $z \in IB$ is written as $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then we can find $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ such that $b_i = \sum a_{ij} c_j$ and $b'_{i'} = \sum a'_{i'j} c_j$ such that $y_j = \sum x_i a_{ij} = \sum x'_{i'} a'_{i'j}$ holds[2]. Hence the procedure above gives a well defined map $\bar{\gamma}_n$ on IB. By construction $\bar{\gamma}$ satisfies conditions (1), (3), and (4). Moreover, for $x \in I$ we have $\bar{\gamma}_n(x) = \gamma_n(x)$. Hence it follows from Lemma 23.2.4 that $\bar{\gamma}$ is a divided power structure on IB. □

07H2 **Lemma 23.4.3.** *Let (A, I, γ) be a divided power ring.*

(1) *If $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then $\mathrm{Ker}(\varphi) \cap I$ is preserved by γ_n for all $n \geq 1$.*
(2) *Let $\mathfrak{a} \subset A$ be an ideal and set $I' = I \cap \mathfrak{a}$. The following are equivalent*
   (a) *I' is preserved by γ_n for all $n > 0$,*
   (b) *γ extends to $A/\mathfrak{a}$, and*
   (c) *there exist a set of generators x_i of I' as an ideal such that $\gamma_n(x_i) \in I'$ for all $n > 0$.*

Proof. Proof of (1). This is clear. Assume (2)(a). Define $\bar{\gamma}_n(x \bmod I') = \gamma_n(x) \bmod I'$ for $x \in I$. This is well defined since $\gamma_n(x + y) = \gamma_n(x) \bmod I'$ for $y \in I'$ by Definition 23.2.1 (4) and the fact that $\gamma_j(y) \in I'$ by assumption. It is clear that $\bar{\gamma}$ is a divided power structure as γ is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that

[2]This can also be proven without recourse to Algebra, Theorem 10.81.4. Indeed, if $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then $\sum x_i b_i - \sum x'_{i'} b'_{i'} = 0$ is a relation in the A-module B. Thus, Algebra, Lemma 10.39.11 (applied to the x_i and $x'_{i'}$ taking the place of the f_i, and the b_i and $b'_{i'}$ taking the role of the x_i) yields the existence of the $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ as required.

$\gamma_n(x) = a^n\gamma_n(x_i) \in I'$ for $x = ax_i$. Hence we see that $\gamma_n(x) \in I'$ for a set of generators of I' as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove $\forall n : \gamma_n(x+y) \in I'$ if $\forall n : \gamma_n(x), \gamma_n(y) \in I'$. This follows immediately from the fourth axiom of a divided power structure. □

07H3 **Lemma 23.4.4.** *Let (A, I, γ) be a divided power ring. Let $E \subset I$ be a subset. Then the smallest ideal $J \subset I$ preserved by γ and containing all $f \in E$ is the ideal J generated by $\gamma_n(f)$, $n \geq 1$, $f \in E$.*

Proof. Follows immediately from Lemma 23.4.3. □

07KD **Lemma 23.4.5.** *Let (A, I, γ) be a divided power ring. Let p be a prime. If p is nilpotent in A/I, then*

(1) *the p-adic completion $A^\wedge = \lim_e A/p^eA$ surjects onto A/I,*
(2) *the kernel of this map is the p-adic completion $I^\wedge$ of I, and*
(3) *each γ_n is continuous for the p-adic topology and extends to $\gamma_n^\wedge : I^\wedge \to I^\wedge$ defining a divided power structure on $I^\wedge$.*

If moreover A is a $\mathbf{Z}_{(p)}$-algebra, then

(4) *for e large enough the ideal $p^eA \subset I$ is preserved by the divided power structure γ and*
$$(A^\wedge, I^\wedge, \gamma^\wedge) = \lim_e (A/p^eA, I/p^eA, \bar{\gamma})$$
in the category of divided power rings.

Proof. Let $t \geq 1$ be an integer such that $p^tA/I = 0$, i.e., $p^tA \subset I$. The map $A^\wedge \to A/I$ is the composition $A^\wedge \to A/p^tA \to A/I$ which is surjective (for example by Algebra, Lemma 10.96.1). As $p^eI \subset p^eA \cap I \subset p^{e-t}I$ for $e \geq t$ we see that the kernel of the composition $A^\wedge \to A/I$ is the p-adic completion of I. The map γ_n is continuous because
$$\gamma_n(x + p^ey) = \sum\nolimits_{i+j=n} p^{je}\gamma_i(x)\gamma_j(y) = \gamma_n(x) \bmod p^eI$$
by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps $\gamma_n^\wedge$ from the maps γ_n. Finally, to see the last statement say $e > t$. Then $p^eA \subset I$ and $\gamma_1(p^eA) \subset p^eA$ and for $n > 1$ we have
$$\gamma_n(p^ea) = p^n\gamma_n(p^{e-1}a) = \frac{p^n}{n!}p^{n(e-1)}a^n \in p^eA$$
as $p^n/n! \in \mathbf{Z}_{(p)}$ and as $n \geq 2$ and $e \geq 2$ so $n(e-1) \geq e$. This proves that γ extends to A/p^eA, see Lemma 23.4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 23.3.2. □

23.5. Divided power polynomial algebras

07H4 A very useful example is the *divided power polynomial algebra*. Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t\rangle$ the following A-algebra: As an A-module we set
$$A\langle x_1, \ldots, x_t\rangle = \bigoplus\nolimits_{n_1,\ldots,n_t \geq 0} Ax_1^{[n_1]} \ldots x_t^{[n_t]}$$
with multiplication given by
$$x_i^{[n]}x_i^{[m]} = \frac{(n+m)!}{n!m!}x_i^{[n+m]}.$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \ldots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \ldots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \ldots, x_t\rangle_+$.

07H5 **Lemma 23.5.1.** *Let (A, I, γ) be a divided power ring. There exists a unique divided power structure δ on*

$$J = IA\langle x_1, \ldots, x_t\rangle + A\langle x_1, \ldots, x_t\rangle_+$$

such that

(1) $\delta_n(x_i) = x_i^{[n]}$, *and*
(2) $(A, I, \gamma) \to (A\langle x_1, \ldots, x_t\rangle, J, \delta)$ *is a homomorphism of divided power rings.*

Moreover, $(A\langle x_1, \ldots, x_t\rangle, J, \delta)$ has the following universal property: A homomorphism of divided power rings $\varphi : (A\langle x_1, \ldots, x_t\rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \ldots, k_t \in K$.

Proof. We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that $A\langle x_1, \ldots, x_t\rangle$ is isomorphic to the ring obtained by adjoining the divided power variables $x_1, \ldots, x_t$ one by one.

Let $A\langle x\rangle_+$ be the ideal generated by $x, x^{[2]}, x^{[3]}, \ldots$. Note that $J = IA\langle x\rangle + A\langle x\rangle_+$ and that

$$IA\langle x\rangle \cap A\langle x\rangle_+ = IA\langle x\rangle \cdot A\langle x\rangle_+$$

Hence by Lemma 23.2.5 it suffices to show that there exist divided power structures on the ideals $IA\langle x\rangle$ and $A\langle x\rangle_+$. The existence of the first follows from Lemma 23.4.2 as $A \to A\langle x\rangle$ is flat. For the second, note that if A is torsion free, then we can apply Lemma 23.2.2 (4) to see that δ exists. Namely, choosing as generators the elements $x^{[m]}$ we see that $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$ and $n!$ divides the integer $\frac{(nm)!}{(m!)^n}$. In general write $A = R/\mathfrak{a}$ for some torsion free ring R (e.g., a polynomial ring over $\mathbf{Z}$). The kernel of $R\langle x\rangle \to A\langle x\rangle$ is $\bigoplus \mathfrak{a} x^{[m]}$. Applying criterion (2)(c) of Lemma 23.4.3 we see that the divided power structure on $R\langle x\rangle_+$ extends to $A\langle x\rangle$ as desired.

Proof of the universal property. Given a homomorphism $\varphi : A \to C$ of divided power rings and $k_1, \ldots, k_t \in K$ we consider

$$A\langle x_1, \ldots, x_t\rangle \to C, \quad x_1^{[n_1]} \ldots x_t^{[n_t]} \longmapsto \epsilon_{n_1}(k_1) \ldots \epsilon_{n_t}(k_t)$$

using φ on coefficients. The only thing to check is that this is an A-algebra homomorphism (details omitted). The inverse construction is clear. □

07H6 **Remark 23.5.2.** Let (A, I, γ) be a divided power ring. There is a variant of Lemma 23.5.1 for infinitely many variables. First note that if $s < t$ then there is a canonical map

$$A\langle x_1, \ldots, x_s\rangle \to A\langle x_1, \ldots, x_t\rangle$$

Hence if W is any set, then we set

$$A\langle x_w : w \in W\rangle = \operatorname{colim}_{E \subset W} A\langle x_e : e \in E\rangle$$

(colimit over E finite subset of W) with transition maps as above. By the definition of a colimit we see that the universal mapping property of $A\langle x_w : w \in W\rangle$ is completely analogous to the mapping property stated in Lemma 23.5.1.

The following lemma can be found in [**BO83**].

07GS **Lemma 23.5.3.** *Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Let $I \subset A$ be an ideal. Two divided power structures γ, γ' on I are equal if and only if $\gamma_p = \gamma'_p$. Moreover, given a map $\delta : I \to I$ such that*

(1) $p!\delta(x) = x^p$ *for all* $x \in I$[3],
(2) $\delta(ax) = a^p\delta(x)$ *for all* $a \in A$, $x \in I$, *and*
(3) $\delta(x+y) = \delta(x) + \sum_{i+j=p, i,j\geq 1} \frac{1}{i!j!}x^i y^j + \delta(y)$ *for all* $x, y \in I$,

then there exists a unique divided power structure γ on I such that $\gamma_p = \delta$.

Proof. If n is not divisible by p, then $\gamma_n(x) = cx\gamma_{n-1}(x)$ where c is a unit in $\mathbf{Z}_{(p)}$. Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where c is a unit in $\mathbf{Z}_{(p)}$. Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all γ_n. More precisely, if $n = a_0 + a_1 p + \ldots + a_e p^e$ with $a_i \in \{0, \ldots, p-1\}$ then we set

$$\gamma_n(x) = c_n x^{a_0} \delta(x)^{a_1} \ldots \delta^e(x)^{a_e}$$

for $c_n \in \mathbf{Z}_{(p)}$ defined by

$$c_n = (p!)^{a_1 + a_2(1+p) + \ldots + a_e(1+\ldots+p^{e-1})}/n!.$$

Now we have to show the axioms (1) – (5) of a divided power structure, see Definition 23.2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let $x, y \in I$. We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle u, v\rangle \longrightarrow A$$

which maps $u^{[n]}$ to $\gamma_n(x)$ and $v^{[n]}$ to $\gamma_n(y)$. By construction of $\mathbf{Z}_{(p)}\langle u, v\rangle$ this means we have to check that

$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$$

in A and similarly for y. This is true because (2) holds for γ. Let ϵ denote the divided power structure on the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ of $\mathbf{Z}_{(p)}\langle u, v\rangle$. Next, we claim that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ for $f \in \mathbf{Z}_{(p)}\langle u, v\rangle_+$ and all n. This is clear for $n = 0, 1, \ldots, p-1$. For $n = p$ it suffices to prove it for a set of generators of the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ because both ϵ_p and $\gamma_p = \delta$ satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p}\gamma_{pn}(x)$ and similarly for y, which follows as (5) holds for γ. Now, if $n = a_0 + a_1 p + \ldots + a_e p^e$ is an arbitrary integer written in p-adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0} \gamma_p(f)^{a_1} \ldots \gamma_p^e(f)^{a_e}$$

because ϵ is a divided power structure. Hence we see that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ holds for all n. Applying this for $f = u + v$ we see that axiom (4) for γ follows from the fact that ϵ is a divided power structure. □

[3] This condition follows from the other two, see Remark 23.5.4.

0H87 **Remark 23.5.4.** With $A \supset I$ as in Lemma 23.5.3, let $\delta : I \to I$ be a map satisfying (2) and (3) of the lemma. Then (1) holds too. First, we prove $\delta(nx) = n\delta(x) + \frac{n^p-n}{p!}x^p$ for $n \geq 1$ by induction on n. The case $n = 1$ is OK. Assume that it is OK for some n, then $\delta((n+1)x) = \delta(nx) + \delta(x) + (\sum_{i=1}^{p-1} \frac{n^i}{i!(p-i)!})x^p = (n+1)\delta(x) + \left(\frac{n^p-n}{p!} + \frac{(n+1)^p}{p!} - \frac{n^p+1}{p!}\right)x^p = (n+1)\delta(x) + \frac{(n+1)^p-(n+1)}{p!}x^p$, so it is also OK for $n+1$. On the other hand, we have $\delta(nx) = n^p\delta(x)$, hence $(n^p - n)\delta(x) = \frac{n^p-n}{p!}x^p$. But for any p there exists n such that $p^2 \not| n^p - n$. Namely, we have $(\mathbb{Z}/p^2\mathbb{Z})^\times \cong \mathbb{Z}/p(p-1)\mathbb{Z}$. So there exists n such that $n^{p-1} \not\equiv 1 \mod p^2$. Hence we get $p!\delta(x) = x^p$

23.6. Tate resolutions

09PF In this section we briefly discuss the resolutions constructed in [**Tat57**] and [**AH86**] which combine divided power structures with differential graded algebras. In this section we will use *homological notation* for differential graded algebras. Our differential graded algebras will sit in nonnegative homological degrees. Thus our differential graded algebras (A, d) will be given as chain complexes

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots$$

endowed with a multiplication.

Let R be a ring (commutative, as usual). In this section we will often consider graded R-algebras $A = \bigoplus_{d\geq 0} A_d$ whose components are zero in negative degrees. We will set $A_+ = \bigoplus_{d>0} A_d$. We will write $A_{even} = \bigoplus_{d\geq 0} A_{2d}$ and $A_{odd} = \bigoplus_{d\geq 0} A_{2d+1}$. Recall that A is graded commutative if $xy = (-1)^{\deg(x)\deg(y)}yx$ for homogeneous elements x, y. Recall that A is strictly graded commutative if in addition $x^2 = 0$ for homogeneous elements x of odd degree. Finally, to understand the following definition, keep in mind that $\gamma_n(x) = x^n/n!$ if A is a $\mathbf{Q}$-algebra.

09PG **Definition 23.6.1.** Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a graded R-algebra which is strictly graded commutative. A collection of maps $\gamma_n : A_{even,+} \to A_{even,+}$ defined for all $n > 0$ is called a *divided power structure* on A if we have

(1) $\gamma_n(x) \in A_{2nd}$ if $x \in A_{2d}$,
(2) $\gamma_1(x) = x$ for any x, we also set $\gamma_0(x) = 1$,
(3) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(4) $\gamma_n(xy) = x^n\gamma_n(y)$ for all $x \in A_{even}$ and $y \in A_{even,+}$,
(5) $\gamma_n(xy) = 0$ if $x, y \in A_{odd}$ homogeneous and $n > 1$
(6) if $x, y \in A_{even,+}$ then $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(7) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$ for $x \in A_{even,+}$.

Observe that conditions (2), (3), (4), (6), and (7) imply that γ is a "usual" divided power structure on the ideal $A_{even,+}$ of the (commutative) ring A_{even}, see Sections 23.2, 23.3, 23.4, and 23.5. In particular, we have $n!\gamma_n(x) = x^n$ for all $x \in A_{even,+}$. Condition (1) states that γ is compatible with grading and condition (5) tells us γ_n for $n > 1$ vanishes on products of homogeneous elements of odd degree. But note that it may happen that

$$\gamma_2(z_1z_2 + z_3z_4) = z_1z_2z_3z_4$$

is nonzero if z_1, z_2, z_3, z_4 are homogeneous elements of odd degree.

09PH **Example 23.6.2** (Adjoining odd variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an odd integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT$$

with grading given by $A\langle T\rangle_m = A_m \oplus A_{m-d}T$. We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A. Namely, we set

$$\gamma_n(x + yT) = \gamma_n(x) + \gamma_{n-1}(x)yT$$

for $x \in A_{even,+}$ and $y \in A_{odd}$.

09PI **Example 23.6.3** (Adjoining even variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an even integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT \oplus AT^{(2)} \oplus AT^{(3)} \oplus \ldots$$

with multiplication given by

$$T^{(n)}T^{(m)} = \frac{(n+m)!}{n!m!}T^{(n+m)}$$

and with grading given by

$$A\langle T\rangle_m = A_m \oplus A_{m-d}T \oplus A_{m-2d}T^{(2)} \oplus \ldots$$

We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A such that $\gamma_n(T^{(i)}) = T^{(ni)}$. To define the divided power structure we first set

$$\gamma_n\left(\sum\nolimits_{i>0} x_i T^{(i)}\right) = \sum \prod\nolimits_{n=\sum e_i} x_i^{e_i} T^{(ie_i)}$$

if x_i is in A_{even}. If $x_0 \in A_{even,+}$ then we take

$$\gamma_n\left(\sum\nolimits_{i\geq 0} x_i T^{(i)}\right) = \sum\nolimits_{a+b=n} \gamma_a(x_0)\gamma_b\left(\sum\nolimits_{i>0} x_i T^{(i)}\right)$$

where γ_b is as defined above.

0F4I **Remark 23.6.4.** We can also adjoin a set (possibly infinite) of exterior or divided power generators in a given degree $d > 0$, rather than just one as in Examples 23.6.2 and 23.6.3. Namely, following Remark 23.5.2: for (A, γ) as above and a set J, let $A\langle T_j : j \in J\rangle$ be the directed colimit of the algebras $A\langle T_j : j \in S\rangle$ over all finite subsets S of J. It is immediate that this algebra has a unique divided power structure, compatible with the given structure on A and on each generator T_j.

At this point we tie in the definition of divided power structures with differentials. To understand the definition note that $\mathrm{d}(x^n/n!) = \mathrm{d}(x)x^{n-1}/(n-1)!$ if A is a $\mathbf{Q}$-algebra and $x \in A_{even,+}$.

09PJ **Definition 23.6.5.** Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a differential graded R-algebra which is strictly graded commutative. A divided power structure γ on A is *compatible with the differential graded structure* if $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for all $x \in A_{even,+}$.

Warning: Let (A, d, γ) be as in Definition 23.6.5. It may not be true that $\gamma_n(x)$ is a boundary, if x is a boundary. Thus γ in general does not induce a divided power structure on the homology algebra $H(A)$. In some papers the authors put an additional compatibility condition in order to ensure that this is the case, but we elect not to do so.

09PK **Lemma 23.6.6.** *Let (A, d, γ) and (B, d, γ) be as in Definition 23.6.5. Let $f : A \to B$ be a map of differential graded algebras compatible with divided power structures. Assume*

(1) *$H_k(A) = 0$ for $k > 0$, and*
(2) *f is surjective.*

Then γ induces a divided power structure on the graded R-algebra $H(B)$.

Proof. Suppose that x and x' are homogeneous of the same degree $2d$ and define the same cohomology class in $H(B)$. Say $x' - x = \mathrm{d}(w)$. Choose a lift $y \in A_{2d}$ of x and a lift $z \in A_{2d+1}$ of w. Then $y' = y + \mathrm{d}(z)$ is a lift of x'. Hence

$$\gamma_n(y') = \sum \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z)) = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z))$$

Since A is acyclic in positive degrees and since $\mathrm{d}(\gamma_j(\mathrm{d}(z))) = 0$ for all j we can write this as

$$\gamma_n(y') = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\mathrm{d}(z_i)$$

for some z_i in A. Moreover, for $0 < i < n$ we have

$$\mathrm{d}(\gamma_i(y)z_i) = \mathrm{d}(\gamma_i(y))z_i + \gamma_i(y)\mathrm{d}(z_i) = \mathrm{d}(y)\gamma_{i-1}(y)z_i + \gamma_i(y)\mathrm{d}(z_i)$$

and the first term maps to zero in B as $\mathrm{d}(y)$ maps to zero in B. Hence $\gamma_n(x')$ and $\gamma_n(x)$ map to the same element of $H(B)$. Thus we obtain a well defined map $\gamma_n : H_{2d}(B) \to H_{2nd}(B)$ for all $d > 0$ and $n > 0$. We omit the verification that this defines a divided power structure on $H(B)$. □

09PL **Lemma 23.6.7.** *Let (A, d, γ) be as in Definition 23.6.5. Let $R \to R'$ be a ring map. Then d and γ induce similar structures on $A' = A \otimes_R R'$ such that (A', d, γ) is as in Definition 23.6.5.*

Proof. Observe that $A'_{even} = A_{even} \otimes_R R'$ and $A'_{even,+} = A_{even,+} \otimes_R R'$. Hence we are trying to show that the divided powers γ extend to A'_{even} (terminology as in Definition 23.4.1). Once we have shown γ extends it follows easily that this extension has all the desired properties.

Choose a polynomial R-algebra P (on any set of generators) and a surjection of R-algebras $P \to R'$. The ring map $A_{even} \to A_{even} \otimes_R P$ is flat, hence the divided powers γ extend to $A_{even} \otimes_R P$ uniquely by Lemma 23.4.2. Let $J = \mathrm{Ker}(P \to R')$. To show that γ extends to $A \otimes_R R'$ it suffices to show that $I' = \mathrm{Ker}(A_{even,+} \otimes_R P \to A_{even,+} \otimes_R R')$ is generated by elements z such that $\gamma_n(z) \in I'$ for all $n > 0$. This is clear as I' is generated by elements of the form $x \otimes f$ with $x \in A_{even,+}$ and $f \in \mathrm{Ker}(P \to R')$. □

09PM **Lemma 23.6.8.** *Let (A, d, γ) be as in Definition 23.6.5. Let $d \geq 1$ be an integer. Let $A\langle T\rangle$ be the graded divided power polynomial algebra on T with $\deg(T) = d$ constructed in Example 23.6.2 or 23.6.3. Let $f \in A_{d-1}$ be an element with $d(f) = 0$. There exists a unique differential d on $A\langle T\rangle$ such that $d(T) = f$ and such that d is compatible with the divided power structure on $A\langle T\rangle$.*

Proof. This is proved by a direct computation which is omitted. □

In Lemma 23.12.3 we will compute the cohomology of $A\langle T\rangle$ in some special cases. Here is Tate's construction, as extended by Avramov and Halperin.

09PN **Lemma 23.6.9.** *Let $R \to S$ be a homomorphism of commutative rings. There exists a factorization*

$$R \to A \to S$$

with the following properties:

(1) *(A, d, γ) is as in Definition 23.6.5,*
(2) *$A \to S$ is a quasi-isomorphism (if we endow S with the zero differential),*
(3) *$A_0 = R[x_j : j \in J] \to S$ is any surjection of a polynomial ring onto S, and*
(4) *A is a graded divided power polynomial algebra over R.*

The last condition means that A is constructed out of A_0 by successively adjoining a set of variables T in each degree > 0 as in Example 23.6.2 or 23.6.3. Moreover, if R is Noetherian and $R \to S$ is of finite type, then A can be taken to have only finitely many generators in each degree.

Proof. We write out the construction for the case that R is Noetherian and $R \to S$ is of finite type. Without those assumptions, the proof is the same, except that we have to use some set (possibly infinite) of generators in each degree.

Start of the construction: Let $A(0) = R[x_1, \ldots, x_n]$ be a (usual) polynomial ring and let $A(0) \to S$ be a surjection. As grading we take $A(0)_0 = A(0)$ and $A(0)_d = 0$ for $d \neq 0$. Thus $\mathrm{d} = 0$ and γ_n, $n > 0$, is zero as well.

Choose generators $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ for the kernel of the given map $A(0) = R[x_1, \ldots, x_n] \to S$. We apply Example 23.6.2 m times to get

$$A(1) = A(0)\langle T_1, \ldots, T_m\rangle$$

with $\deg(T_i) = 1$ as a graded divided power polynomial algebra. We set $\mathrm{d}(T_i) = f_i$. Since $A(1)$ is a divided power polynomial algebra over $A(0)$ and since $\mathrm{d}(f_i) = 0$ this extends uniquely to a differential on $A(1)$ by Lemma 23.6.8.

Induction hypothesis: Assume we are given factorizations

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to S$$

where $A(0)$ and $A(1)$ are as above and each $R \to A(m') \to S$ for $2 \leq m' \leq m$ satisfies properties (1) and (4) of the statement of the lemma and (2) replaced by the condition that $H_i(A(m')) \to H_i(S)$ is an isomorphism for $m' > i \geq 0$. The base case is $m = 1$.

Induction step: Assume we have $R \to A(m) \to S$ as in the induction hypothesis. Consider the group $H_m(A(m))$. This is a module over $H_0(A(m)) = S$. In fact, it is a subquotient of $A(m)_m$ which is a finite type module over $A(m)_0 = R[x_1, \ldots, x_n]$. Thus we can pick finitely many elements

$$e_1, \ldots, e_t \in \mathrm{Ker}(\mathrm{d} : A(m)_m \to A(m)_{m-1})$$

which map to generators of this module. Applying Example 23.6.2 or 23.6.3 t times we get

$$A(m+1) = A(m)\langle T_1, \ldots, T_t\rangle$$

with $\deg(T_i) = m+1$ as a graded divided power algebra. We set $\mathrm{d}(T_i) = e_i$. Since $A(m+1)$ is a divided power polynomial algebra over $A(m)$ and since $\mathrm{d}(e_i) = 0$ this extends uniquely to a differential on $A(m+1)$ compatible with the divided power structure. Since we've added only material in degree $m+1$ and higher we see that $H_i(A(m+1)) = H_i(A(m))$ for $i < m$. Moreover, it is clear that $H_m(A(m+1)) = 0$ by construction.

To finish the proof we observe that we have shown there exists a sequence of maps

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to A(m+1) \to \ldots \to S$$

and to finish the proof we set $A = \operatorname{colim} A(m)$. □

0BZ9 **Lemma 23.6.10.** *Let $R \to S$ be a pseudo-coherent ring map (More on Algebra, Definition 15.83.1). Then Lemma 23.6.9 holds, with the resolution A of S having finitely many generators in each degree.*

Proof. This is proved in exactly the same way as Lemma 23.6.9. The only additional twist is that, given $A(m) \to S$ we have to show that $H_m = H_m(A(m))$ is a finite $R[x_1, \ldots, x_m]$-module (so that in the next step we need only add finitely many variables). Consider the complex

$$\ldots \to A(m)_{m-1} \to A(m)_m \to A(m)_{m-1} \to \ldots \to A(m)_0 \to S \to 0$$

Since S is a pseudo-coherent $R[x_1, \ldots, x_n]$-module and since $A(m)_i$ is a finite free $R[x_1, \ldots, x_n]$-module we conclude that this is a pseudo-coherent complex, see More on Algebra, Lemma 15.65.9. Since the complex is exact in (homological) degrees $> m$ we conclude that H_m is a finite R-module by More on Algebra, Lemma 15.65.3. □

09PP **Lemma 23.6.11.** *Let R be a commutative ring. Suppose that (A, d, γ) and (B, d, γ) are as in Definition 23.6.5. Let $\overline{\varphi} : H_0(A) \to H_0(B)$ be an R-algebra map. Assume*

1. *A is a graded divided power polynomial algebra over R.*
2. *$H_k(B) = 0$ for $k > 0$.*

Then there exists a map $\varphi : A \to B$ of differential graded R-algebras compatible with divided powers that lifts $\overline{\varphi}$.

Proof. The assumption means that A is obtained from R by successively adjoining some set of polynomial generators in degree zero, exterior generators in positive odd degrees, and divided power generators in positive even degrees. So we have a filtration $R \subset A(0) \subset A(1) \subset \ldots$ of A such that $A(m+1)$ is obtained from $A(m)$ by adjoining generators of the appropriate type (which we simply call "divided power generators") in degree $m+1$. In particular, $A(0) \to H_0(A)$ is a surjection from a (usual) polynomial algebra over R onto $H_0(A)$. Thus we can lift $\overline{\varphi}$ to an R-algebra map $\varphi(0) : A(0) \to B_0$.

Write $A(1) = A(0)\langle T_j : j \in J\rangle$ for some set J of divided power variables T_j of degree 1. Let $f_j \in B_0$ be $f_j = \varphi(0)(\mathrm{d}(T_j))$. Observe that f_j maps to zero in $H_0(B)$ as $\mathrm{d}T_j$ maps to zero in $H_0(A)$. Thus we can find $b_j \in B_1$ with $\mathrm{d}(b_j) = f_j$. By the universal property of divided power polynomial algebras from Lemma 23.5.1, we find a lift $\varphi(1) : A(1) \to B$ of $\varphi(0)$ mapping T_j to f_j.

Having constructed $\varphi(m)$ for some $m \geq 1$ we can construct $\varphi(m+1) : A(m+1) \to B$ in exactly the same manner. We omit the details. □

09PQ **Lemma 23.6.12.** *Let R be a commutative ring. Let S and T be commutative R-algebras. Then there is a canonical structure of a strictly graded commutative R-algebra with divided powers on*

$$\mathrm{Tor}_*^R(S, T).$$

Proof. Choose a factorization $R \to A \to S$ as above. Since $A \to S$ is a quasi-isomorphism and since A_d is a free R-module, we see that the differential graded algebra $B = A \otimes_R T$ computes the Tor groups displayed in the lemma. Choose a surjection $R[y_j : j \in J] \to T$. Then we see that B is a quotient of the differential graded algebra $A[y_j : j \in J]$ whose homology sits in degree 0 (it is equal to $S[y_j : j \in J]$). By Lemma 23.6.7 the differential graded algebras B and $A[y_j : j \in J]$ have divided power structures compatible with the differentials. Hence we obtain our divided power structure on $H(B)$ by Lemma 23.6.6.

The divided power algebra structure constructed in this way is independent of the choice of A. Namely, if A' is a second choice, then Lemma 23.6.11 implies there is a map $A \to A'$ preserving all structure and the augmentations towards S. Then the induced map $B = A \otimes_R T \to A' \otimes_R T' = B'$ also preserves all structure and is a quasi-isomorphism. The induced isomorphism of Tor algebras is therefore compatible with products and divided powers. □

23.7. Application to complete intersections

09PR Let R be a ring. Let (A, d, γ) be as in Definition 23.6.5. A *derivation* of degree 2 is an R-linear map $\theta : A \to A$ with the following properties

(1) $\theta(A_d) \subset A_{d-2}$,
(2) $\theta(xy) = \theta(x)y + x\theta(y)$,
(3) θ commutes with d,
(4) $\theta(\gamma_n(x)) = \theta(x)\gamma_{n-1}(x)$ for all $x \in A_{2d}$ all d.

In the following lemma we construct a derivation.

09PS **Lemma 23.7.1.** *Let R be a ring. Let (A, d, γ) be as in Definition 23.6.5. Let $R' \to R$ be a surjection of rings whose kernel has square zero and is generated by one element f. If A is a graded divided power polynomial algebra over R with finitely many variables in each degree, then we obtain a derivation $\theta : A/IA \to A/IA$ where I is the annihilator of f in R.*

Proof. Since A is a divided power polynomial algebra, we can find a divided power polynomial algebra A' over R' such that $A = A' \otimes_{R'} R$. Moreover, we can lift d to an R-linear operator d on A' such that

(1) $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{\deg(x)}x\mathrm{d}(y)$ for $x, y \in A'$ homogeneous, and
(2) $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for $x \in A'_{even,+}$.

We omit the details (hint: proceed one variable at the time). However, it may not be the case that d^2 is zero on A'. It is clear that d^2 maps A' into $fA' \cong A/IA$. Hence d^2 annihilates fA' and factors as a map $A \to A/IA$. Since d^2 is R-linear we obtain our map $\theta : A/IA \to A/IA$. The verification of the properties of a derivation is immediate. □

09PT **Lemma 23.7.2.** *Assumption and notation as in Lemma 23.7.1. Suppose $S = H_0(A)$ is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in R[x_1, \ldots, x_n]$.*

Moreover, suppose given a relation

$$\sum r_j f_j = 0$$

with $r_j \in R[x_1, \ldots, x_n]$. *Choose* $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ *lifting* r_j, f_j. *Write* $\sum r'_j f'_j = gf$ *for some* $g \in R/I[x_1, \ldots, x_n]$. *If* $H_1(A) = 0$ *and all the coefficients of each* r_j *are in* I, *then there exists an element* $\xi \in H_2(A/IA)$ *such that* $\theta(\xi) = g$ *in* S/IS.

Proof. Let $A(0) \subset A(1) \subset A(2) \subset \ldots$ be the filtration of A such that $A(m)$ is gotten from $A(m-1)$ by adjoining divided power variables of degree m. Then $A(0)$ is a polynomial algebra over R equipped with an R-algebra surjection $A(0) \to S$. Thus we can choose a map

$$\varphi : R[x_1, \ldots, x_n] \to A(0)$$

lifting the augmentations to S. Next, $A(1) = A(0)\langle T_1, \ldots, T_t\rangle$ for some divided power variables T_i of degree 1. Since $H_0(A) = S$ we can pick $\xi_j \in \sum A(0)T_i$ with $\mathrm{d}(\xi_j) = \varphi(f_j)$. Then

$$\mathrm{d}\left(\sum \varphi(r_j)\xi_j\right) = \sum \varphi(r_j)\varphi(f_j) = \sum \varphi(r_j f_j) = 0$$

Since $H_1(A) = 0$ we can pick $\xi \in A_2$ with $\mathrm{d}(\xi) = \sum \varphi(r_j)\xi_j$. If the coefficients of r_j are in I, then the same is true for $\varphi(r_j)$. In this case $\mathrm{d}(\xi)$ dies in A_1/IA_1 and hence ξ defines a class in $H_2(A/IA)$.

The construction of θ in the proof of Lemma 23.7.1 proceeds by successively lifting $A(i)$ to $A'(i)$ and lifting the differential d. We lift φ to $\varphi' : R'[x_1, \ldots, x_n] \to A'(0)$. Next, we have $A'(1) = A'(0)\langle T_1, \ldots, T_t\rangle$. Moreover, we can lift ξ_j to $\xi'_j \in \sum A'(0)T_i$. Then $\mathrm{d}(\xi'_j) = \varphi'(f'_j) + fa_j$ for some $a_j \in A'(0)$. Consider a lift $\xi' \in A'_2$ of ξ. Then we know that

$$\mathrm{d}(\xi') = \sum \varphi'(r'_j)\xi'_j + \sum f b_i T_i$$

for some $b_i \in A(0)$. Applying d again we find

$$\theta(\xi) = \sum \varphi'(r'_j)\varphi'(f'_j) + \sum f\varphi'(r'_j)a_j + \sum f b_i \mathrm{d}(T_i)$$

The first term gives us what we want. The second term is zero because the coefficients of r_j are in I and hence are annihilated by f. The third term maps to zero in H_0 because $\mathrm{d}(T_i)$ maps to zero. □

The method of proof of the following lemma is apparently due to Gulliksen.

09PU **Lemma 23.7.3.** *Let* $R' \to R$ *be a surjection of Noetherian rings whose kernel has square zero and is generated by one element* f. *Let* $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. *Let* $\sum r_j f_j = 0$ *be a relation in* $R[x_1, \ldots, x_n]$. *Assume that*

(1) *each* r_j *has coefficients in the annihilator* I *of* f *in* R,
(2) *for some lifts* $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ *we have* $\sum r'_j f'_j = gf$ *where* g *is not nilpotent in* S/IS.

Then S *does not have finite tor dimension over* R *(i.e.,* S *is not a perfect* R*-algebra).*

Proof. Choose a Tate resolution $R \to A \to S$ as in Lemma 23.6.9. Let $\xi \in H_2(A/IA)$ and $\theta : A/IA \to A/IA$ be the element and derivation found in Lemmas 23.7.1 and 23.7.2. Observe that

$$\theta^n(\gamma_n(\xi)) = g^n$$

in $H_0(A/IA) = S/IS$. Hence if g is not nilpotent in S/IS, then ξ^n is nonzero in $H_{2n}(A/IA)$ for all $n > 0$. Since $H_{2n}(A/IA) = \mathrm{Tor}_{2n}^R(S, R/I)$ we conclude. □

The following result can be found in [**Rod88**].

09PV **Lemma 23.7.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. If A/J has finite tor dimension over A/I, then $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.*

Proof. Let $f \in I$ be an element mapping to a nonzero element of $I/\mathfrak{m}I$ which is mapped to zero in $J/\mathfrak{m}J$. We can choose an ideal I' with $\mathfrak{m}I \subset I' \subset I$ such that I/I' is generated by the image of f. Set $R = A/I$ and $R' = A/I'$. Let $J = (a_1, \ldots, a_m)$ for some $a_j \in A$. Then $f = \sum b_j a_j$ for some $b_j \in \mathfrak{m}$. Let $r_j, f_j \in R$ resp. $r'_j, f'_j \in R'$ be the image of b_j, a_j. Then we see we are in the situation of Lemma 23.7.3 (with the ideal I of that lemma equal to $\mathfrak{m}_R$) and the lemma is proved. □

09PW **Lemma 23.7.5.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. Assume*

(1) *A/J has finite tor dimension over A/I, and*
(2) *J is generated by a regular sequence.*

Then I is generated by a regular sequence and J/I is generated by a regular sequence.

Proof. By Lemma 23.7.4 we see that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective. Thus we can find $s \leq r$ and a minimal system of generators $f_1, \ldots, f_r$ of J such that $f_1, \ldots, f_s$ are in I and form a minimal system of generators of I. The lemma follows as any minimal system of generators of J is a regular sequence by More on Algebra, Lemmas 15.30.15 and 15.30.7. □

09PX **Lemma 23.7.6.** *Let $R \to S$ be a local ring map of Noetherian local rings. Let $I \subset R$ and $J \subset S$ be ideals with $IS \subset J$. If $R \to S$ is flat and $S/\mathfrak{m}_R S$ is regular, then the following are equivalent*

(1) *J is generated by a regular sequence and S/J has finite tor dimension as a module over R/I,*
(2) *J is generated by a regular sequence and $Tor_p^{R/I}(S/J, R/\mathfrak{m}_R)$ is nonzero for only finitely many p,*
(3) *I is generated by a regular sequence and J/IS is generated by a regular sequence in S/IS.*

Proof. If (3) holds, then J is generated by a regular sequence, see for example More on Algebra, Lemmas 15.30.13 and 15.30.7. Moreover, if (3) holds, then $S/J = (S/I)/(J/I)$ has finite projective dimension over S/IS because the Koszul complex will be a finite free resolution of S/J over S/IS. Since $R/I \to S/IS$ is flat, it then follows that S/J has finite tor dimension over R/I by More on Algebra, Lemma 15.67.11. Thus (3) implies (1).

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). By More on Algebra, Lemma 15.78.6 we find that S/J has finite tor dimension over S/IS. Thus we can apply Lemma 23.7.5 to conclude that IS and J/IS are generated by regular sequences. Let $f_1, \ldots, f_r \in I$ be a minimal system of generators of I. Since $R \to S$ is flat, we see that $f_1, \ldots, f_r$ form a minimal system of generators for IS in S. Thus $f_1, \ldots, f_r \in R$ is a sequence of elements whose images in S form a regular sequence by More on Algebra, Lemmas 15.30.15 and 15.30.7. Thus $f_1, \ldots, f_r$ is a regular sequence in R by Algebra, Lemma 10.68.5. □

23.8. Local complete intersection rings

09PY Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. By the Cohen structure theorem (see Algebra, Theorem 10.160.8) we can write A as the quotient of a regular Noetherian complete local ring R. Let us say that A is a *complete intersection* if there exists some surjection $R \to A$ with R a regular local ring such that the kernel is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the surjection.

09PZ **Lemma 23.8.1.** *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. The following are equivalent*

(1) *for every surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence, and*
(2) *for some surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence.*

Proof. Let k be the residue field of A. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.160.5) with residue field k (Algebra, Lemma 10.160.6). If the characteristic of k is 0 we set $\Lambda = k$. Recall that $\Lambda[[x_1, \ldots, x_n]]$ for any n is formally smooth over $\mathbf{Z}$, resp. $\mathbf{Q}$ in the $\mathfrak{m}$-adic topology, see More on Algebra, Lemma 15.39.1. Fix a surjection $\Lambda[[x_1, \ldots, x_n]] \to A$ as in the Cohen structure theorem (Algebra, Theorem 10.160.8).

Let $R \to A$ be a surjection from a regular local ring R. Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\mathrm{Ker}(R \to A)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in R if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $R^\wedge$ by Algebra, Lemmas 10.68.5 and 10.97.2. Moreover, we have

$$R^\wedge/(f_1, \ldots, f_r)R^\wedge = (R/(f_1, \ldots, f_n))^\wedge = A^\wedge = A$$

because A is $\mathfrak{m}_A$-adically complete (first equality by Algebra, Lemma 10.97.1). Finally, the ring $R^\wedge$ is regular since R is regular (More on Algebra, Lemma 15.43.4). Hence we may assume R is complete.

If R is complete we can choose a map $\Lambda[[x_1, \ldots, x_n]] \to R$ lifting the given map $\Lambda[[x_1, \ldots, x_n]] \to A$, see More on Algebra, Lemma 15.37.5. By adding some more variables $y_1, \ldots, y_m$ mapping to generators of the kernel of $R \to A$ we may assume that $\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \to R$ is surjective (some details omitted). Then we can consider the commutative diagram

$$\begin{array}{ccc} \Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Lambda[[x_1, \ldots, x_n]] & \longrightarrow & A \end{array}$$

By Algebra, Lemma 10.135.6 we see that the condition for $R \to A$ is equivalent to the condition for the fixed chosen map $\Lambda[[x_1, \ldots, x_n]] \to A$. This finishes the proof of the lemma. □

The following two lemmas are sanity checks on the definition given above.

09Q0 **Lemma 23.8.2.** *Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $f_1, \ldots, f_r \in \mathfrak{p}$ be a regular sequence. Then the completion of*

$$A = (R/(f_1, \ldots, f_r))_\mathfrak{p} = R_\mathfrak{p}/(f_1, \ldots, f_r)R_\mathfrak{p}$$

is a complete intersection in the sense defined above.

Proof. The completion of A is equal to $A^\wedge = R_\mathfrak{p}^\wedge/(f_1, \ldots, f_r)R_\mathfrak{p}^\wedge$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.97.1). The image of the sequence $f_1, \ldots, f_r$ in $R_\mathfrak{p}$ is a regular sequence by Algebra, Lemmas 10.97.2 and 10.68.5. Moreover, $R_\mathfrak{p}^\wedge$ is a regular local ring by More on Algebra, Lemma 15.43.4. Hence the result holds by our definition of complete intersection for complete local rings. □

The following lemma is the analogue of Algebra, Lemma 10.135.4.

09Q1 **Lemma 23.8.3.** *Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $I \subset \mathfrak{p}$ be an ideal. Set $A = (R/I)_\mathfrak{p} = R_\mathfrak{p}/I_\mathfrak{p}$. The following are equivalent*

(1) *the completion of A is a complete intersection in the sense above,*
(2) *$I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence,*
(3) *the module $(I/I^2)_\mathfrak{p}$ can be generated by $\dim(R_\mathfrak{p}) - \dim(A)$ elements,*
(4) *add more here.*

Proof. We may and do replace R by its localization at $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal of R and $A = R/I$. Let $f_1, \ldots, f_r \in I$ be a minimal sequence of generators. The completion of A is equal to $A^\wedge = R^\wedge/(f_1, \ldots, f_r)R^\wedge$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.97.1).

If (1) holds, then the image of the sequence $f_1, \ldots, f_r$ in $R^\wedge$ is a regular sequence by assumption. Hence it is a regular sequence in R by Algebra, Lemmas 10.97.2 and 10.68.5. Thus (1) implies (2).

Assume (3) holds. Set $c = \dim(R) - \dim(A)$ and let $f_1, \ldots, f_c \in I$ map to generators of I/I^2. by Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $I = (f_1, \ldots, f_c)$. Since R is regular and hence Cohen-Macaulay (Algebra, Proposition 10.103.4) we see that $f_1, \ldots, f_c$ is a regular sequence by Algebra, Proposition 10.103.4. Thus (3) implies (2). Finally, (2) implies (1) by Lemma 23.8.2. □

The following result is due to Avramov, see [**Avr75**].

09Q2 **Proposition 23.8.4.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

(1) *$B^\wedge$ is a complete intersection,*
(2) *$A^\wedge$ and $(B/\mathfrak{m}_A B)^\wedge$ are complete intersections.*

Proof. Consider the diagram

$$\begin{array}{ccc} B & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & A^\wedge \end{array}$$

Since the horizontal maps are faithfully flat (Algebra, Lemma 10.97.3) we conclude that the right vertical arrow is flat (for example by Algebra, Lemma 10.99.15).

Moreover, we have $(B/\mathfrak{m}_A B)^\wedge = B^\wedge/\mathfrak{m}_{A^\wedge} B^\wedge$ by Algebra, Lemma 10.97.1. Thus we may assume A and B are complete local Noetherian rings.

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 15.39.3. Let $I = \mathrm{Ker}(R \to A)$ and $J = \mathrm{Ker}(S \to B)$. Note that since $R/I = A \to B = S/J$ is flat the map $J/IS \otimes_R R/\mathfrak{m}_R \to J/J \cap \mathfrak{m}_R S$ is an isomorphism. Hence a minimal system of generators of J/IS maps to a minimal system of generators of $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$. Finally, $S/\mathfrak{m}_R S$ is a regular local ring.

Assume (1) holds, i.e., J is generated by a regular sequence. Since $A = R/I \to B = S/J$ is flat we see Lemma 23.7.6 applies and we deduce that I and J/IS are generated by regular sequences. We have $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ and $\dim(S/IS) = \dim(A) + \dim(S/\mathfrak{m}_R S)$ (Algebra, Lemma 10.112.7). Thus J/IS is generated by

$$\dim(S/IS) - \dim(S/J) = \dim(S/\mathfrak{m}_R S) - \dim(B/\mathfrak{m}_A B)$$

elements (Algebra, Lemma 10.60.13). It follows that $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by the same number of elements (see above). Hence $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by a regular sequence, see for example Lemma 23.8.3. In this way we see that (2) holds.

If (2) holds, then I and $J/J \cap \mathfrak{m}_R S$ are generated by regular sequences. Lifting these generators (see above), using flatness of $R/I \to S/IS$, and using Grothendieck's lemma (Algebra, Lemma 10.99.3) we find that J/IS is generated by a regular sequence in S/IS. Thus Lemma 23.7.6 tells us that J is generated by a regular sequence, whence (1) holds. □

09Q3 **Definition 23.8.5.** Let A be a Noetherian ring.

(1) If A is local, then we say A is a *complete intersection* if its completion is a complete intersection in the sense above.
(2) In general we say A is a *local complete intersection* if all of its local rings are complete intersections.

We will check below that this does not conflict with the terminology introduced in Algebra, Definitions 10.135.1 and 10.135.5. But first, we show this "makes sense" by showing that if A is a Noetherian local complete intersection, then A is a local complete intersection, i.e., all of its local rings are complete intersections.

09Q4 **Lemma 23.8.6.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{p} \subset A$ be a prime ideal. If A is a complete intersection, then $A_\mathfrak{p}$ is a complete intersection too.*

Proof. Choose a prime $\mathfrak{q}$ of $A^\wedge$ lying over $\mathfrak{p}$ (this is possible as $A \to A^\wedge$ is faithfully flat by Algebra, Lemma 10.97.3). Then $A_\mathfrak{p} \to (A^\wedge)_\mathfrak{q}$ is a flat local ring homomorphism. Thus by Proposition 23.8.4 we see that $A_\mathfrak{p}$ is a complete intersection if and only if $(A^\wedge)_\mathfrak{q}$ is a complete intersection. Thus it suffices to prove the lemma in case A is complete (this is the key step of the proof).

Assume A is complete. By definition we may write $A = R/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r$ in a regular local ring R. Let $\mathfrak{q} \subset R$ be the prime corresponding to $\mathfrak{p}$. Observe that $f_1, \ldots, f_r \in \mathfrak{q}$ and that $A_\mathfrak{p} = R_\mathfrak{q}/(f_1, \ldots, f_r)R_\mathfrak{q}$. Hence $A_\mathfrak{p}$ is a complete intersection by Lemma 23.8.2. □

09Q5 **Lemma 23.8.7.** *Let A be a Noetherian ring. Then A is a local complete intersection if and only if $A_\mathfrak{m}$ is a complete intersection for every maximal ideal $\mathfrak{m}$ of A.*

Proof. This follows immediately from Lemma 23.8.6 and the definitions. □

09Q6 **Lemma 23.8.8.** *Let S be a finite type algebra over a field k.*

(1) *for a prime $\mathfrak{q} \subset S$ the local ring $S_\mathfrak{q}$ is a complete intersection in the sense of Algebra, Definition 10.135.5 if and only if $S_\mathfrak{q}$ is a complete intersection in the sense of Definition 23.8.5, and*
(2) *S is a local complete intersection in the sense of Algebra, Definition 10.135.1 if and only if S is a local complete intersection in the sense of Definition 23.8.5.*

Proof. Proof of (1). Let $k[x_1, \ldots, x_n] \to S$ be a surjection. Let $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of our surjection. Note that $k[x_1, \ldots, x_n]_\mathfrak{p} \to S_\mathfrak{q}$ is surjective with kernel $I_\mathfrak{p}$. Observe that $k[x_1, \ldots, x_n]$ is a regular ring by Algebra, Proposition 10.114.2. Hence the equivalence of the two notions in (1) follows by combining Lemma 23.8.3 with Algebra, Lemma 10.135.7.

Having proved (1) the equivalence in (2) follows from the definition and Algebra, Lemma 10.135.9. □

09Q7 **Lemma 23.8.9.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

(1) *B is a complete intersection,*
(2) *A and $B/\mathfrak{m}_A B$ are complete intersections.*

Proof. Now that the definition makes sense this is a trivial reformulation of the (nontrivial) Proposition 23.8.4. □

23.9. Local complete intersection maps

09Q9 Let $A \to B$ be a local homomorphism of Noetherian complete local rings. A consequence of the Cohen structure theorem is that we can find a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \nwarrow & \uparrow \\ & & A \end{array}$$

of Noetherian complete local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. This follows from More on Algebra, Lemma 15.39.3. Let us (temporarily) say $A \to S \to B$ is a *good factorization* of $A \to B$ if S is a Noetherian local ring, $A \to S \to B$ are local ring maps, $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ regular. Let us say that $A \to B$ is a *complete intersection homomorphism* if there exists some good factorization $A \to S \to B$ such that the kernel of $S \to B$

is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the diagram.

09QA **Lemma 23.9.1.** *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent*

(1) *for some good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence, and*
(2) *for every good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence.*

Proof. Let $A \to S \to B$ be a good factorization. As B is complete we obtain a factorization $A \to S^\wedge \to B$ where $S^\wedge$ is the completion of S. Note that this is also a good factorization: The ring map $S \to S^\wedge$ is flat (Algebra, Lemma 10.97.2), hence $A \to S^\wedge$ is flat. The ring $S^\wedge/\mathfrak{m}_A S^\wedge = (S/\mathfrak{m}_A S)^\wedge$ is regular since $S/\mathfrak{m}_A S$ is regular (More on Algebra, Lemma 15.43.4). Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\mathrm{Ker}(S \to B)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in S if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $S^\wedge$ by Algebra, Lemma 10.68.5. Moreover, we have

$$S^\wedge/(f_1, \ldots, f_r)R^\wedge = (S/(f_1, \ldots, f_n))^\wedge = B^\wedge = B$$

because B is $\mathfrak{m}_B$-adically complete (first equality by Algebra, Lemma 10.97.1). Thus the kernel of $S \to B$ is generated by a regular sequence if and only if the kernel of $S^\wedge \to B$ is generated by a regular sequence. Hence it suffices to consider good factorizations where S is complete.

Assume we have two factorizations $A \to S \to B$ and $A \to S' \to B$ with S and S' complete. By More on Algebra, Lemma 15.39.4 the ring $S \times_B S'$ is a Noetherian complete local ring. Hence, using More on Algebra, Lemma 15.39.3 we can choose a good factorization $A \to S'' \to S \times_B S'$ with S'' complete. Thus it suffices to show: If $A \to S' \to S \to B$ are comparable good factorizations, then $\mathrm{Ker}(S \to B)$ is generated by a regular sequence if and only if $\mathrm{Ker}(S' \to B)$ is generated by a regular sequence.

Let $A \to S' \to S \to B$ be comparable good factorizations. First, since $S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S$ is a surjection of regular local rings, the kernel is generated by a regular sequence $\overline{x}_1, \ldots, \overline{x}_c \in \mathfrak{m}_{S'}/\mathfrak{m}_R S'$ which can be extended to a regular system of parameters for the regular local ring $S'/\mathfrak{m}_R S'$, see (Algebra, Lemma 10.106.4). Set $I = \mathrm{Ker}(S' \to S)$. By flatness of S over R we have

$$I/\mathfrak{m}_R I = \mathrm{Ker}(S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S) = (\overline{x}_1, \ldots, \overline{x}_c).$$

Choose lifts $x_1, \ldots, x_c \in I$. These lifts form a regular sequence generating I as S' is flat over R, see Algebra, Lemma 10.99.3.

We conclude that if also $\mathrm{Ker}(S \to B)$ is generated by a regular sequence, then so is $\mathrm{Ker}(S' \to B)$, see More on Algebra, Lemmas 15.30.13 and 15.30.7.

Conversely, assume that $J = \mathrm{Ker}(S' \to B)$ is generated by a regular sequence. Because the generators $x_1, \ldots, x_c$ of I map to linearly independent elements of $\mathfrak{m}_{S'}/\mathfrak{m}_{S'}^2$ we see that $I/\mathfrak{m}_{S'} I \to J/\mathfrak{m}_{S'} J$ is injective. Hence there exists a minimal system of generators $x_1, \ldots, x_c, y_1, \ldots, y_d$ for J. Then $x_1, \ldots, x_c, y_1, \ldots, y_d$ is a

regular sequence and it follows that the images of $y_1, \ldots, y_d$ in S form a regular sequence generating $\mathrm{Ker}(S \to B)$. This finishes the proof of the lemma. □

In the following proposition observe that the condition on vanishing of Tor's applies in particular if B has finite tor dimension over A and thus in particular if B is flat over A.

09QB **Proposition 23.9.2.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the following are equivalent*

(1) *B is a complete intersection and $Tor_p^A(B, A/\mathfrak{m}_A)$ is nonzero for only finitely many p,*
(2) *A is a complete intersection and $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Let $F_\bullet \to A/\mathfrak{m}_A$ be a resolution by finite free A-modules. Observe that $\mathrm{Tor}_p^A(B, A/\mathfrak{m}_A)$ is the pth homology of the complex $F_\bullet \otimes_A B$. Let $F_\bullet^\wedge = F_\bullet \otimes_A A^\wedge$ be the completion. Then $F_\bullet^\wedge$ is a resolution of $A^\wedge/\mathfrak{m}_{A^\wedge}$ by finite free $A^\wedge$-modules (as $A \to A^\wedge$ is flat and completion on finite modules is exact, see Algebra, Lemmas 10.97.1 and 10.97.2). It follows that

$$F_\bullet^\wedge \otimes_{A^\wedge} B^\wedge = F_\bullet \otimes_A B \otimes_B B^\wedge$$

By flatness of $B \to B^\wedge$ we conclude that

$$\mathrm{Tor}_p^{A^\wedge}(B^\wedge, A^\wedge/\mathfrak{m}_{A^\wedge}) = \mathrm{Tor}_p^A(B, A/\mathfrak{m}_A) \otimes_B B^\wedge$$

In this way we see that the condition in (1) on the local ring map $A \to B$ is equivalent to the same condition for the local ring map $A^\wedge \to B^\wedge$. Thus we may assume A and B are complete local Noetherian rings (since the other conditions are formulated in terms of the completions in any case).

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 15.39.3. Let $I = \mathrm{Ker}(R \to A)$ and $J = \mathrm{Ker}(S \to B)$. The proposition now follows from Lemma 23.7.6. □

09QC **Remark 23.9.3.** It appears difficult to define an good notion of "local complete intersection homomorphisms" for maps between general Noetherian rings. The reason is that, for a local Noetherian ring A, the fibres of $A \to A^\wedge$ are not local complete intersection rings. Thus, if $A \to B$ is a local homomorphism of local Noetherian rings, and the map of completions $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above, then $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is in general **not** a complete intersection homomorphism in the sense defined above. A solution can be had by working exclusively with excellent Noetherian rings. More generally, one could work with those Noetherian rings whose formal fibres are complete intersections, see [**Rod87**]. We will develop this theory in Dualizing Complexes, Section 47.23.

To finish of this section we compare the notion defined above with the notion introduced in More on Algebra, Section 23.8.

09QD **Lemma 23.9.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \nwarrow & \uparrow \\ & & A \end{array}$$

of Noetherian local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. The following are equivalent

(1) $\mathrm{Ker}(S \to B)$ *is generated by a regular sequence, and*
(2) $A^\wedge \to B^\wedge$ *is a complete intersection homomorphism as defined above.*

Proof. Omitted. Hint: the proof is identical to the argument given in the first paragraph of the proof of Lemma 23.9.1. □

09QE **Lemma 23.9.5.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. The following are equivalent*

(1) *$A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 15.33.2,*
(2) *for every prime $\mathfrak{q} \subset B$ and with $\mathfrak{p} = A \cap \mathfrak{q}$ the ring map $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Choose a surjection $R = A[x_1, \ldots, x_n] \to B$. Observe that $A \to R$ is flat with regular fibres. Let I be the kernel of $R \to B$. Assume (2). Then we see that I is locally generated by a regular sequence by Lemma 23.9.4 and Algebra, Lemma 10.68.6. In other words, (1) holds. Conversely, assume (1). Then after localizing on R and B we can assume that I is generated by a Koszul regular sequence. By More on Algebra, Lemma 15.30.7 we find that I is locally generated by a regular sequence. Hence (2) hold by Lemma 23.9.4. Some details omitted. □

09QF **Lemma 23.9.6.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map such that the image of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *contains all closed points of* $\mathrm{Spec}(A)$*. Then the following are equivalent*

(1) *B is a complete intersection and $A \to B$ has finite tor dimension,*
(2) *A is a complete intersection and $A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 15.33.2.*

Proof. This is a reformulation of Proposition 23.9.2 via Lemma 23.9.5. We omit the details. □

23.10. Smooth ring maps and diagonals

0FCV In this section we use the material above to characterize smooth ring maps as those whose diagonal is perfect.

0FCW **Lemma 23.10.1.** *Let $A \to B$ be a local ring homomorphism of Noetherian local rings such that B is flat and essentially of finite type over A. If*

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is the localization of a smooth A-algebra.

Proof. As B is essentially of finite type over A, so is $B \otimes_A B$ and in particular $B \otimes_A B$ is Noetherian. Hence the quotient B of $B \otimes_A B$ is pseudo-coherent over $B \otimes_A B$ (More on Algebra, Lemma 15.65.17) which explains why perfectness of the ring map (More on Algebra, Definition 15.83.1) agrees with the condition of finite tor dimension.

We may write $B = R/K$ where R is the localization of $A[x_1, \ldots, x_n]$ at a prime ideal and $K \subset R$ is an ideal. Denote $\mathfrak{m} \subset R \otimes_A R$ the maximal ideal which is the inverse image of the maximal ideal of B via the surjection $R \otimes_A R \to B \otimes_A B \to B$. Then we have surjections

$$(R \otimes_A R)_{\mathfrak{m}} \to (B \otimes_A B)_{\mathfrak{m}} \to B$$

and hence ideals $I \subset J \subset (R \otimes_A R)_{\mathfrak{m}}$ as in Lemma 23.7.4. We conclude that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.

Let $K = (f_1, \ldots, f_r)$ with r minimal. We may and do assume that $f_i \in R$ is the image of an element of $A[x_1, \ldots, x_n]$ which we also denote f_i. Observe that I is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and $1 \otimes f_1, \ldots, 1 \otimes f_r$. We claim that this is a minimal set of generators of I. Namely, if κ is the common residue field of R, B, $(R \otimes_A R)_{\mathfrak{m}}$, and $(B \otimes_A B)_{\mathfrak{m}}$ then we have a map $R \otimes_A R \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ which factors through $(R \otimes_A R)_{\mathfrak{m}}$. Since B is flat over A and since we have the short exact sequence $0 \to K \to R \to B \to 0$ we see that $K \otimes_A \kappa \subset R \otimes_A \kappa$, see Algebra, Lemma 10.39.12. Thus restricting the map $(R \otimes_A R)_{\mathfrak{m}} \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ to I we obtain a map

$$I \to K \otimes_A \kappa \oplus \kappa \otimes_A K \to K \otimes_B \kappa \oplus \kappa \otimes_B K.$$

The elements $f_1 \otimes 1, \ldots, f_r \otimes 1, 1 \otimes f_1, \ldots, 1 \otimes f_r$ map to a basis of the target of this map, since by Nakayama's lemma (Algebra, Lemma 10.20.1) $f_1, \ldots, f_r$ map to a basis of $K \otimes_B \kappa$. This proves our claim.

The ideal J is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and the elements $x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n$ (for the proof it suffices to see that these elements are contained in the ideal J). Now we can write

$$f_i \otimes 1 - 1 \otimes f_i = \sum g_{ij}(x_j \otimes 1 - 1 \otimes x_j)$$

for some g_{ij} in $(R \otimes_A R)_{\mathfrak{m}}$. This is a general fact about elements of $A[x_1, \ldots, x_n]$ whose proof we omit. Denote $a_{ij} \in \kappa$ the image of g_{ij}. Another computation shows that a_{ij} is the image of $\partial f_i/\partial x_j$ in κ. The injectivity of $I/\mathfrak{m}I \to J/\mathfrak{m}J$ and the remarks made above forces the matrix (a_{ij}) to have maximal rank r. Set

$$C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

and consider the naive cotangent complex $NL_{C/A} \cong (C^{\oplus r} \to C^{\oplus n})$ where the map is given by the matrix of partial derivatives. Thus $NL_{C/A} \otimes_A B$ is isomorphic to a free B-module of rank $n - r$ placed in degree 0. Hence C_g is smooth over A for some $g \in C$ mapping to a unit in B, see Algebra, Lemma 10.137.11. This finishes the proof. □

0FCX **Lemma 23.10.2.** *Let $A \to B$ be a flat finite type ring map of Noetherian rings. If*

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is a smooth A-algebra.

Proof. This follows from Lemma 23.10.1 and general facts about smooth ring maps, see Algebra, Lemmas 10.137.11 and 10.137.12. Alternatively, the reader can slightly modify the proof of Lemma 23.10.1 to prove this lemma. □

23.11. Freeness of the conormal module

0FJP Tate resolutions and derivations on them can be used to prove (stronger) versions of the results in this section, see [**Iye01**]. Two more elementary references are [**Vas67**] and [**Fer67b**].

0FJQ **Lemma 23.11.1.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand isomorphic to R/I, then there exists a nonzerodivisor $x \in I$ such that the image of x in I/I^2 generates F.* [**Vas67**]

Proof. By assumption we may choose a finite free resolution

$$0 \to R^{\oplus n_e} \to R^{\oplus n_{e-1}} \to \ldots \to R^{\oplus n_1} \to R \to R/I \to 0$$

Then $\varphi_1 : R^{\oplus n_1} \to R$ has rank 1 and we see that I contains a nonzerodivisor y by Algebra, Proposition 10.102.9. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of R, see Algebra, Lemma 10.63.5. Let $I^2 \subset J \subset I$ be an ideal such that $J/I^2 = F$. Then $J \not\subset \mathfrak{p}_i$ for all i as $y^2 \in J$ and $y^2 \notin \mathfrak{p}_i$, see Algebra, Lemma 10.63.9. By Nakayama's lemma (Algebra, Lemma 10.20.1) we have $J \not\subset \mathfrak{m}J + I^2$. By Algebra, Lemma 10.15.2 we can pick $x \in J$, $x \notin \mathfrak{m}J + I^2$ and $x \notin \mathfrak{p}_i$ for $i = 1, \ldots, n$. Then x is a nonzerodivisor and the image of x in I/I^2 generates (by Nakayama's lemma) the summand $J/I^2 \cong R/I$. □

0FJR **Lemma 23.11.2.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand free of rank r, then there exists a regular sequence $x_1, \ldots, x_r \in I$ such that $x_1 \bmod I^2, \ldots, x_r \bmod I^2$ generate F.* Local version of [**Vas67**, Theorem 1.1]

Proof. If $r = 0$ there is nothing to prove. Assume $r > 0$. We may pick $x \in I$ such that x is a nonzerodivisor and $x \bmod I^2$ generates a summand of F isomorphic to R/I, see Lemma 23.11.1. Consider the ring $R' = R/(x)$ and the ideal $I' = I/(x)$. Of course $R'/I' = R/I$. The short exact sequence

$$0 \to R/I \xrightarrow{x} I/xI \to I' \to 0$$

splits because the map $I/xI \to I/I^2$ sends xR/xI to a direct summand. Now $I/xI = I \otimes^{\mathbf{L}}_R R'$ has finite projective dimension over R', see More on Algebra, Lemmas 15.75.3 and 15.75.9. Hence the summand I' has finite projective dimension over R'. On the other hand, we have the short exact sequence $0 \to xR/xI \to I/I^2 \to I'/(I')^2 \to 0$ and we conclude $I'/(I')^2$ has the free direct summand $F' = F/(R/I \cdot x)$ of rank $r - 1$. By induction on r we may we pick a regular sequence $x'_2, \ldots, x'_r \in I'$ such that there congruence classes freely generate F'. If $x_1 = x$ and $x_2, \ldots, x_r$ are any elements lifting $x'_1, \ldots, x'_r$ in R, then we see that the lemma holds. □

0FJS **Proposition 23.11.3.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal which has finite projective dimension and such that I/I^2 is finite locally free over R/I. Then I is a regular ideal (More on Algebra, Definition 15.32.1).*

Variant of [**Vas67**, Corollary 1]. See also [**Iye01**] and [**Fer67b**].

Proof. By Algebra, Lemma 10.68.6 it suffices to show that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence for every $\mathfrak{p} \supset I$. Thus we may assume R is local. If I/I^2 has rank r, then by Lemma 23.11.2 we find a regular sequence $x_1, \ldots, x_r \in I$ generating I/I^2. By Nakayama (Algebra, Lemma 10.20.1) we conclude that I is generated by $x_1, \ldots, x_r$. □

For any local complete intersection homomorphism $A \to B$ of rings, the naive cotangent complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$, see More on Algebra, Lemma 15.86.4. Using the above, we can show that this sometimes characterizes local complete intersection homomorphisms.

0FJT **Lemma 23.11.4.** *Let $A \to B$ be a perfect (More on Algebra, Definition 15.83.1) ring homomorphism of Noetherian rings. Then the following are equivalent*

1. *$NL_{B/A}$ has tor-amplitude in $[-1, 0]$,*
2. *$NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$, and*
3. *$A \to B$ is a local complete intersection (More on Algebra, Definition 15.33.2).*

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B\mathrm{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see More on Algebra, Lemma 15.67.2. Since I/I^2 is a finite B-module and B is Noetherian, this is true if and only if I/I^2 is a finite locally free B-module (Algebra, Lemma 10.78.2). Thus the equivalence of (1) and (2) is clear. Moreover, the equivalence of (1) and (3) also follows if we apply Proposition 23.11.3 (and the observation that a regular ideal is a Koszul regular ideal as well as a quasi-regular ideal, see More on Algebra, Section 15.32). □

0FJV **Lemma 23.11.5.** *Let $A \to B$ be a flat ring map of finite presentation. Then the following are equivalent*

1. *$NL_{B/A}$ has tor-amplitude in $[-1, 0]$,*
2. *$NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$,*
3. *$A \to B$ is syntomic (Algebra, Definition 10.136.1), and*
4. *$A \to B$ is a local complete intersection (More on Algebra, Definition 15.33.2).*

Proof. The equivalence of (3) and (4) is More on Algebra, Lemma 15.33.5.

If $A \to B$ is syntomic, then we can find a cocartesian diagram

$$\begin{array}{ccc} B_0 & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_0 & \longrightarrow & A \end{array}$$

such that $A_0 \to B_0$ is syntomic and A_0 is Noetherian, see Algebra, Lemmas 10.127.18 and 10.168.9. By Lemma 23.11.4 we see that NL_{B_0/A_0} is perfect of tor-amplitude in $[-1, 0]$. By More on Algebra, Lemma 15.86.3 we conclude the same thing is true for $NL_{B/A} = NL_{B_0/A_0} \otimes^{\mathbf{L}}_{B_0} B$ (see also More on Algebra, Lemmas 15.67.13 and 15.75.9). This proves that (3) implies (2).

Assume (1). By More on Algebra, Lemma 15.86.3 for every ring map $A \to k$ where k is a field, we see that $NL_{B \otimes_A k/k}$ has tor-amplitude in $[-1, 0]$ (see More on Algebra, Lemma 15.67.13). Hence by Lemma 23.11.4 we see that $k \to B \otimes_A k$ is a local complete intersection homomorphism. Thus $A \to B$ is syntomic by definition. This proves (1) implies (3) and finishes the proof. □

23.12. Koszul complexes and Tate resolutions

0GZ3 In this section we "lift" the result of More on Algebra, Lemma 15.95.1 to the category of differential graded algebras endowed with divided powers compatible with the differential graded structure (beware that in this section we represent Koszul complexes as chain complexes whereas in locus citatus we use cochain complexes).

Let R be a ring. Let $I \subset R$ be an ideal generated by $f_1, \ldots, f_r \in R$. For $n \geq 1$ we denote

$$K_n = K_{n,\bullet} = R\langle \xi_1, \ldots, \xi_r \rangle$$

the differential graded Koszul algebra with ξ_i in degree 1 and $\mathrm{d}(\xi_i) = f_i^n$. There exists a unique divided power structure on this (as in Definition 23.6.5), see Example 23.6.2. For $m > n$ the transition map $K_m \to K_n$ is the differential graded algebra map compatible with divided powers given by sending ξ_i to $f_i^{m-n}\xi_i$.

0GZ4 **Lemma 23.12.1.** *In the situation above, if R is Noetherian, then for every n there exists an $N \geq n$ and maps*

$$K_N \to A \to R/(f_1^N, \ldots, f_r^N) \quad \text{and} \quad A \to K_n$$

with the following properties

(1) *(A, d, γ) is as in Definition 23.6.5,*
(2) *$A \to R/(f_1^N, \ldots, f_r^N)$ is a quasi-isomorphism,*
(3) *the composition $K_N \to A \to R/(f_1^N, \ldots, f_r^N)$ is the canonical map,*
(4) *the composition $K_N \to A \to K_n$ is the transition map,*
(5) *$A_0 = R \to R/(f_1^N, \ldots, f_r^N)$ is the canonical surjection,*
(6) *A is a graded divided power polynomial algebra over R with finitely many generators in each degree, and*
(7) *$A \to K_n$ is a homomorphism of differential graded R-algebras compatible with divided powers which induces the canonical map $R/(f_1^N, \ldots, f_r^N) \to R/(f_1^n, \ldots, f_r^n)$ on homology in degree 0.*

Condition (4) means that A is constructed out of A_0 by successively adjoining a finite set of variables T in each degree > 0 as in Example 23.6.2 or 23.6.3.

Proof. Fix n. If $r = 0$, then we can just pick $A = R$. Assume $r > 0$. By More on Algebra, Lemma 15.95.1 (translated into the language of chain complexes) we can choose

$$n_r > n_{r-1} > \ldots > n_1 > n_0 = n$$

such that the transition maps $K_{n_{i+1}} \to K_{n_i}$ on Koszul algebras (see above) induce the zero map on homology in degrees > 0. We will prove the lemma with $N = n_r$.

We will construct A exactly as in the statement and proof of Lemma 23.6.9. Thus we will have

$$A = \operatorname{colim} A(m), \quad \text{and} \quad A(0) \to A(1) \to A(2) \to \ldots \to R/(f_1^N, \ldots, f_r^N)$$

This will immediately give us properties (1), (2), (5), and (6). To finish the proof we will construct the R-algebra maps $K_N \to A \to K_n$. To do this we will construct

(1) an isomorphism $A(1) \to K_N = K_{n_r}$,
(2) a map $A(2) \to K_{n_{r-1}}$,
(3) ...
(4) a map $A(r) \to K_{n_1}$,
(5) a map $A(r+1) \to K_{n_0} = K_n$, and
(6) a map $A \to K_n$.

In each of these steps the map constructed will be between differential graded algebras compatibly endowed with divided powers and each of the maps will be compatible with the previous one via the transition maps between the Koszul algebras and each of the maps will induce the obvious canonical map on homology in degree 0.

Recall that $A(0) = R$. For $m = 1$, the proof of Lemma 23.6.9 chooses $A(1) = R\langle T_1, \ldots, T_r\rangle$ with T_i of degree 1 and with $\mathrm{d}(T_i) = f_i^N$. Namely, the f_i^N are generators of the kernel of $A(0) \to R/(f_1^N, \ldots, f_r^N)$. Thus for $A(1) \to K_N = K_{n_r}$ we use the map

$$\varphi_1 : A(1) \longrightarrow K_{n_r}, \quad T_i \longmapsto \xi_i$$

which is an isomorphism.

For $m = 2$, the construction in the proof of Lemma 23.6.9 chooses generators $e_1, \ldots, e_t \in \operatorname{Ker}(\mathrm{d} : A(1)_1 \to A(1)_0)$. The construction proceeds by taking $A(2) = A(1)\langle T_1, \ldots, T_t\rangle$ as a divided power polynomial algebra with T_i of degree 2 and with $\mathrm{d}(T_i) = e_i$. Since $\varphi_1(e_i)$ is a cocycle in K_{n_r} we see that its image in $K_{n_{r-1}}$ is a coboundary by our choice of n_r and n_{r-1} above. Hence we can construct the following commutative diagram

$$\begin{array}{ccc} A(1) & \xrightarrow{\varphi_1} & K_{n_r} \\ \downarrow & & \downarrow \\ A(2) & \xrightarrow{\varphi_2} & K_{n_{r-1}} \end{array}$$

by sending T_i to an element in degree 2 whose boundary is the image of $\varphi_1(e_i)$. The map φ_2 exists and is compatible with the differential and the divided powers by the universal of the divided power polynomial algebra.

The algebra $A(m)$ and the map $\varphi_m : A(m) \to K_{n_{r+1-m}}$ are constructed in exactly the same manner for $m = 2, \ldots, r$.

Given the map $A(r) \to K_{n_1}$ we see that the composition $H_r(A(r)) \to H_r(K_{n_1}) \to H_r(K_{n_0}) \subset (K_{n_0})_r$ is zero, hence we can extend this to $A(r+1) \to K_{n_0} = K_n$ by sending the new polynomial generators of $A(r+1)$ to zero.

Having constructed $A(r+1) \to K_{n_0} = K_n$ we can simply extend to $A(r+2), A(r+3), \ldots$ in the only possible way by sending the new polynomial generators to zero. This finishes the proof. $\square$

0GZ5 **Remark 23.12.2.** In the situation above, if R is Noetherian, we can inductively choose a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ such that for $i = 1, 2, 3, \ldots$ we have maps $K_{n_i} \to A_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ and $A_i \to K_{n_{i-1}}$ as in Lemma 23.12.1. Denote $A_{i+1} \to A_i$ the composition $A_{i+1} \to K_{n_i} \to A_i$. Then the diagram

$$\begin{array}{ccccccc} K_{n_1} & \longleftarrow & K_{n_2} & \longleftarrow & K_{n_3} & \longleftarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ A_1 & \longleftarrow & A_2 & \longleftarrow & A_3 & \longleftarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_1 & \longleftarrow & K_{n_1} & \longleftarrow & K_{n_2} & \longleftarrow & \ldots \end{array}$$

commutes. In this way we see that the inverse systems (K_n) and (A_n) are pro-isomorphic in the category of differential graded R-algebras with compatible divided powers.

0GZ6 **Lemma 23.12.3.** *Let (A, d, γ), $d \geq 1$, $f \in A_{d-1}$, and $A\langle T\rangle$ be as in Lemma 23.6.8.*

(1) *If $d = 1$, then there is a long exact sequences*
$$\ldots \to H_0(A) \xrightarrow{f} H_0(A) \to H_0(A\langle T\rangle) \to 0$$
(2) *For $d = 2$ there is a bounded spectral sequence $(E_1)_{i,j} = H_{j-i}(A) \cdot T^{[i]}$ converging to $H_{i+j}(A\langle T\rangle)$. The differential $(d_1)_{i,j} : H_{j-i}(A) \cdot T^{[i]} \to H_{j-i+1}(A) \cdot T^{[i-1]}$ sends $\xi \cdot T^{[i]}$ to the class of $f\xi \cdot T^{[i-1]}$.*
(3) *Add more here for other degrees as needed.*

Proof. For $d = 1$, we have a short exact sequence of complexes
$$0 \to A \to A\langle T\rangle \to A \cdot T \to 0$$
and the result (1) follows easily from this. For $d = 2$ we view $A\langle T\rangle$ as a filtered chain complex with subcomplexes
$$F^p A\langle T\rangle = \bigoplus\nolimits_{i \leq p} A \cdot T^{[i]}$$
Applying the spectral sequence of Homology, Section 12.24 (translated into chain complexes) we obtain (2). □

The following lemma will be needed later.

0GZ7 **Lemma 23.12.4.** *In the situation above, for all $n \geq t \geq 1$ there exists an $N > n$ and a map*
$$K_t \longrightarrow K_n \otimes_R K_t$$
in the derived category of left differential graded K_N-modules whose composition with the multiplication map is the transition map (in either direction).

Proof. We first prove this for $r = 1$. Set $f = f_1$. Write $K_t = R\langle x\rangle$, $K_n = R\langle y\rangle$, and $K_N = R\langle z\rangle$ with x, y, z of degree 1 and $\mathrm{d}(x) = f^t$, $\mathrm{d}(y) = f^n$, and $\mathrm{d}(z) = f^N$. For all $N > t$ we claim there is a quasi-isomorphism
$$B_{N,t} = R\langle x, z, u\rangle \longrightarrow K_t, \quad x \mapsto x, \quad z \mapsto f^{N-t}x, \quad u \mapsto 0$$
Here the left hand side denotes the divided power polynomial algebra in variables x and z of degree 1 and u of degree 2 with $\mathrm{d}(x) = f^t$, $\mathrm{d}(z) = f^N$, and $\mathrm{d}(u) =$

$z - f^{N-t}x$. To prove the claim, we observe that the following three submodules of $H_*(R\langle x, z\rangle)$ are the same

(1) the kernel of $H_*(R\langle x, z\rangle) \to H_*(K_t)$,
(2) the image of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$, and
(3) the kernel of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$.

This observation is proved by a direct computation[4] which we omit. Then we can apply Lemma 23.12.3 part (2) to see that the claim is true.

Via the homomorphism $K_N \to B_{N,t}$ of differential graded R-algebras sending z to z, we may view $B_{N,t} \to K_t$ as a quasi-isomorphism of left differential graded K_N-modules. To define the arrow in the statement of the lemma we use the homomorphism

$$B_{N,t} = R\langle x, z, u\rangle \to K_n \otimes_R K_t, \quad x \mapsto 1 \otimes x, \quad z \mapsto f^{N-n} y \otimes 1, \quad u \mapsto -f^{N-n-t} y \otimes x$$

This makes sense as long as we assume $N \geq n + t$. It is a pleasant computation to show that the (pre or post) composition with the multiplication map is the transition map.

For $r > 1$ we proceed by writing each of the Koszul algebras as a tensor product of Koszul algebras in 1 variable and we apply the previous construction. In other words, we write

$$K_t = R\langle x_1, \ldots, x_r\rangle = R\langle x_1\rangle \otimes_R \ldots \otimes_R R\langle x_r\rangle$$

where x_i is in degree 1 and $\mathrm{d}(x_i) = f_i^t$. In the case $r > 1$ we then use

$$B_{N,t} = R\langle x_1, z_1, u_1\rangle \otimes_R \ldots \otimes_R R\langle x_r, z_r, u_r\rangle$$

where x_i, z_i have degree 1 and u_i has degree 2 and we have $\mathrm{d}(x_i) = f_i^t$, $\mathrm{d}(z_i) = f_i^N$, and $\mathrm{d}(u_i) = z_i - f_i^{N-t} x_i$. The tensor product map $B_{N,t} \to K_t$ will be a quasi-isomorphism as it is a tensor product of quasi-isomorphisms between bounded above complexes of free R-modules. Finally, we define the map

$$B_{N,t} \to K_n \otimes_R K_t = R\langle y_1, \ldots, y_r\rangle \otimes_R R\langle x_1, \ldots, x_r\rangle$$

as the tensor product of the maps constructed in the case of $r = 1$ or simply by the rules $x_i \mapsto 1 \otimes x_i$, $z_i \mapsto f_i^{N-n} y_i \otimes 1$, and $u_i \mapsto -f_i^{N-n-t} y_i \otimes x_i$ which makes sense as long as $N \geq n + t$. We omit the details. □

23.13. Other chapters

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[4] Hint: setting $z' = z - f^{N-t}x$ we see that $R\langle x, z\rangle = R\langle x, z'\rangle$ with $\mathrm{d}(z') = 0$ and moreover the map $R\langle x, z'\rangle \to K_t$ is the map killing z'.

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CHAPTER 24

Differential Graded Sheaves

0FQS

24.1. Introduction

0FQT This chapter is a continuation of the discussion started in Differential Graded Algebra, Section 22.1. A survey paper is [**Kel06**].

24.2. Conventions

0FQU In this chapter we hold on to the convention that *ring* means commutative ring with 1. If R is a ring, then an *R-algebra* A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has an identity. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

24.3. Sheaves of graded algebras

0FQV Please skip this section.

0FQW **Definition 24.3.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A *sheaf of graded $\mathcal{O}$-algebras* or a *sheaf of graded algebras* on $(\mathcal{C}, \mathcal{O})$ is given by a family $\mathcal{A}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \to \mathcal{A}^{n+m}, \quad (a, b) \longmapsto ab$$

called the multiplication maps with the following properties

(1) multiplication is associative, and
(2) there is a global section 1 of $\mathcal{A}^0$ which is a two-sided identity for multiplication.

We often denote such a structure $\mathcal{A}$. A *homomorphism of graded $\mathcal{O}$-algebras* $f : \mathcal{A} \to \mathcal{B}$ is a family of maps $f^n : \mathcal{A}^n \to \mathcal{B}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a graded $\mathcal{O}$-algebra $\mathcal{A}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a graded $\mathcal{O}(U)$-algebra.

0FQX **Remark 24.3.2.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. We have

(1) Let $\mathcal{A}$ be a graded $\mathcal{O}_\mathcal{C}$-algebra. The multiplication maps of $\mathcal{A}$ induce multiplication maps $f_*\mathcal{A}^n \times f_*\mathcal{A}^m \to f_*\mathcal{A}^{n+m}$ and via $f^\sharp$ we may view these as $\mathcal{O}_\mathcal{D}$-bilinear maps. We will denote $f_*\mathcal{A}$ the graded $\mathcal{O}_\mathcal{D}$-algebra we so obtain.

(2) Let $\mathcal{B}$ be a graded $\mathcal{O}_\mathcal{D}$-algebra. The multiplication maps of $\mathcal{B}$ induce multiplication maps $f^*\mathcal{B}^n \times f^*\mathcal{B}^m \to f^*\mathcal{B}^{n+m}$ and using $f^\sharp$ we may view these as $\mathcal{O}_\mathcal{C}$-bilinear maps. We will denote $f^*\mathcal{B}$ the graded $\mathcal{O}_\mathcal{C}$-algebra we so obtain.
(3) The set of homomorphisms $f^*\mathcal{B} \to \mathcal{A}$ of graded $\mathcal{O}_\mathcal{C}$-algebras is in 1-to-1 correspondence with the set of homomorphisms $\mathcal{B} \to f_*\mathcal{A}$ of graded $\mathcal{O}_\mathcal{C}$-algebras.

Part (3) follows immediately from the usual adjunction between f^* and f_* on sheaves of modules.

24.4. Sheaves of graded modules

0FQY Please skip this section.

0FQZ **Definition 24.4.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. A (right) *graded $\mathcal{A}$-module* or (right) *graded module* over $\mathcal{A}$ is given by a family $\mathcal{M}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}, \quad (x, a) \longmapsto xa$$

called the multiplication maps with the following properties

(1) multiplication satisfies $(xa)a' = x(aa')$,
(2) the identity section 1 of $\mathcal{A}^0$ acts as the identity on $\mathcal{M}^n$ for all n.

We often say "let $\mathcal{M}$ be a graded $\mathcal{A}$-module" to indicate this situation. A *homomorphism of graded $\mathcal{A}$-modules* $f : \mathcal{M} \to \mathcal{N}$ is a family of maps $f^n : \mathcal{M}^n \to \mathcal{N}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps. The category of (right) graded $\mathcal{A}$-modules is denoted $Mod(\mathcal{A})$.

We can define *left graded modules* in exactly the same manner but our default in the chapter will be right modules.

Given a graded $\mathcal{A}$-module $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) graded $\mathcal{A}(U)$-module.

0FR0 **Lemma 24.4.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. The category $Mod(\mathcal{A})$ is an abelian category with the following properties*

(1) *$Mod(\mathcal{A})$ has arbitrary direct sums,*
(2) *$Mod(\mathcal{A})$ has arbitrary colimits,*
(3) *filtered colimit in $Mod(\mathcal{A})$ are exact,*
(4) *$Mod(\mathcal{A})$ has arbitrary products,*
(5) *$Mod(\mathcal{A})$ has arbitrary limits.*

The functor

$$Mod(\mathcal{A}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{M} \longmapsto \mathcal{M}^n$$

sending a graded $\mathcal{A}$-module to its nth term commutes with all limits and colimits.

The lemma says that we may take limits and colimits termwise. It also says (or implies if you like) that the forgetful functor

$$Mod(\mathcal{A}) \longrightarrow \text{graded } \mathcal{O}\text{-modules}$$

commutes with all limits and colimits.

Proof. Let us denote $\mathrm{gr}^n : \mathit{Mod}(\mathcal{A}) \to \mathit{Mod}(\mathcal{O})$ the functor in the statement of the lemma. Consider a homomorphism $f : \mathcal{M} \to \mathcal{N}$ of graded $\mathcal{A}$-modules. The kernel and cokernel of f as maps of graded $\mathcal{O}$-modules are additionally endowed with multiplication maps as in Definition 24.4.1. Hence these are also the kernel and cokernel in $\mathit{Mod}(\mathcal{A})$. Thus $\mathit{Mod}(\mathcal{A})$ is an abelian category and taking kernels and cokernels commutes with gr^n.

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 4.14.11 and 4.14.12. The same lemmas show that proving the commutation of limits and colimits with gr^n follows if gr^n commutes with direct sums and products.

Let $\mathcal{M}_t$, $t \in T$ be a set of graded $\mathcal{A}$-modules. Then we can consider the graded $\mathcal{A}$-module whose degree n term is $\bigoplus_{t\in T} \mathcal{M}_t^n$ (with obvious multiplication maps). The reader easily verifies that this is a direct sum in $\mathit{Mod}(\mathcal{A})$. Similarly for products.

Observe that gr^n is an exact functor for all n and that a complex $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ of $\mathit{Mod}(\mathcal{A})$ is exact if and only if $\mathrm{gr}^n\mathcal{M}_1 \to \mathrm{gr}^n\mathcal{M}_2 \to \mathrm{gr}^n\mathcal{M}_3$ is exact in $\mathit{Mod}(\mathcal{O})$ for all n. Hence we conclude that (3) holds as filtered colimits are exact in $\mathit{Mod}(\mathcal{O})$; it is a Grothendieck abelian category, see Cohomology on Sites, Section 21.19. □

24.5. The graded category of sheaves of graded modules

0FR1 Please skip this section. This section is the analogue of Differential Graded Algebra, Example 22.25.6. For our conventions on graded categories, please see Differential Graded Algebra, Section 22.25.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct a graded category $\mathit{Mod}^{gr}(\mathcal{A})$ over $R = \Gamma(\mathcal{C}, \mathcal{O})$ whose associated category $(\mathit{Mod}^{gr}(\mathcal{A}))^0$ is the category of graded $\mathcal{A}$-modules. As objects of $\mathit{Mod}^{gr}(\mathcal{A})$ we take right graded $\mathcal{A}$-modules (see Section 24.4). Given graded $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$ we set

$$\mathrm{Hom}_{\mathit{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus\nolimits_{n\in\mathbf{Z}} \mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$$

where $\mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$ is the set of right $\mathcal{A}$-module maps $f : \mathcal{L} \to \mathcal{M}$ which are homogeneous of degree n. More precisely, f is given by a family of maps $f : \mathcal{L}^i \to \mathcal{M}^{i+n}$ for $i \in \mathbf{Z}$ compatible with the multiplication maps. In terms of components, we have that

$$\mathrm{Hom}^n(\mathcal{L}, \mathcal{M}) \subset \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for local sections a of $\mathcal{A}^i$ and m of $\mathcal{L}^{-q-i}$. For graded $\mathcal{A}$-modules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ we define composition in $\mathit{Mod}^{gr}(\mathcal{A})$ via the maps

$$\mathrm{Hom}^m(\mathcal{L}, \mathcal{M}) \times \mathrm{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathrm{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

by simple composition of right $\mathcal{A}$-module maps: $(g, f) \mapsto g \circ f$.

24.6. Tensor product for sheaves of graded modules

0FR2 Please skip this section. This section is the analogue of part of Differential Graded Algebra, Section 22.12.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module and let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then we define the *tensor product* $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to be the graded $\mathcal{O}$-module whose degree n term is

$$(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^n = \mathrm{Coker}\left(\bigoplus\nolimits_{r+s+t=n} \mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t \longrightarrow \bigoplus\nolimits_{p+q=n} \mathcal{M}^p \otimes_{\mathcal{O}} \mathcal{N}^q\right)$$

where the map sends the local section $x \otimes a \otimes y$ of $\mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t$ to $xa \otimes y - x \otimes ay$. With this definition we have that $(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^n$ is the sheafification of the presheaf $U \mapsto (\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))^n$ where the tensor product of graded modules is as defined in Differential Graded Algebra, Section 22.12.

If we fix the left graded $\mathcal{A}$-module $\mathcal{N}$ we obtain a functor

$$- \otimes_{\mathcal{A}} \mathcal{N} : \mathit{Mod}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathit{Mod}(\mathcal{O})) = \text{graded } \mathcal{O}\text{-modules}$$

For the notation $\mathrm{Gr}(-)$ please see Homology, Definition 12.16.1. The graded category of graded $\mathcal{O}$-modules is denoted $\mathrm{Gr}^{gr}(\mathit{Mod}(\mathcal{O}))$, see Differential Graded Algebra, Example 22.25.5. The functor above can be upgraded to a functor of graded categories

$$- \otimes_{\mathcal{A}} \mathcal{N} : \mathit{Mod}^{gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}^{gr}(\mathit{Mod}(\mathcal{O}))$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$.

24.7. Internal hom for sheaves of graded modules

0FR3 We urge the reader to skip this section.

We are going to need the sheafified version of the construction in Section 24.5. Let $(\mathcal{C}, \mathcal{O})$, $\mathcal{A}$, $\mathcal{M}$, $\mathcal{L}$ be as in Section 24.5. Then we define

$$\mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

as the graded $\mathcal{O}$-module whose degree n term

$$\mathcal{H}om^n_{\mathcal{A}}(\mathcal{M}, \mathcal{L}) \subset \prod\nolimits_{p+q=n} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

is the subsheaf consisting of those local sections $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for local sections a of $\mathcal{A}^i$ and m of $\mathcal{L}^{-q-i}$. As in Section 24.5 there is a composition map

$$\mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of graded $\mathcal{O}$-modules defined in Section 24.6. This map is given by the composition map

$$\mathcal{H}om^m_{\mathcal{A}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^n_{\mathcal{A}}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{n+m}_{\mathcal{A}}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally).

With these definitions we have

$$\mathrm{Hom}_{\mathit{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}))$$

as graded R-modules compatible with composition.

24.8. Sheaves of graded bimodules and tensor-hom adjunction

0FR4 Please skip this section.

0FR5 **Definition 24.8.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. A *graded* $(\mathcal{A}, \mathcal{B})$*-bimodule* is given by a family $\mathcal{M}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \to \mathcal{M}^{n+m}, \quad (x, b) \longmapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \to \mathcal{M}^{n+m}, \quad (a, x) \longmapsto ax$$

called the multiplication maps with the following properties

(1) multiplication satisfies $a(a'x) = (aa')x$ and $(xb)b' = x(bb')$,
(2) $(ax)b = a(xb)$,
(3) the identity section 1 of $\mathcal{A}^0$ acts as the identity by multiplication, and
(4) the identity section 1 of $\mathcal{B}^0$ acts as the identity by multiplication.

We often denote such a structure $\mathcal{M}$. A *homomorphism of graded* $(\mathcal{A}, \mathcal{B})$*-bimodules* $f : \mathcal{M} \to \mathcal{N}$ is a family of maps $f^n : \mathcal{M}^n \to \mathcal{N}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a graded $(\mathcal{A}(U), \mathcal{B}(U))$-bimodule.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module and let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. In this case the graded tensor product defined in Section 24.6

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

is a right graded $\mathcal{B}$-module with obvious multiplication maps. This construction defines a functor and a functor of graded categories

$$\otimes_{\mathcal{A}} \mathcal{N} : \mathit{Mod}(\mathcal{A}) \longrightarrow \mathit{Mod}(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A}} \mathcal{N} : \mathit{Mod}^{gr}(\mathcal{A}) \longrightarrow \mathit{Mod}^{gr}(\mathcal{B})$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right graded $\mathcal{B}$-module. In this case the graded internal hom defined in Section 24.7

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

is a right graded $\mathcal{A}$-module with multiplication maps[1]

$$\mathcal{H}om^n_{\mathcal{B}}(\mathcal{N}, \mathcal{L}) \times \mathcal{A}^m \longrightarrow \mathcal{H}om^{n+m}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

sending a section $f = (f_{p,q})$ of $\mathcal{H}om^n_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ over U and a section a of $\mathcal{A}^m$ over U to the section fa if $\mathcal{H}om^{n+m}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ over U defined as the family of maps

$$\mathcal{N}^{-q-m}|_U \xrightarrow{a \cdot -} \mathcal{N}^{-q}|_U \xrightarrow{f_{p,q}} \mathcal{M}^p|_U$$

[1] Our conventions are here that this does not involve any signs.

We omit the verification that this is well defined. This construction defines a functor and a functor of graded categories

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, -) : Mod(\mathcal{B}) \longrightarrow Mod(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, -) : Mod^{gr}(\mathcal{B}) \longrightarrow Mod^{gr}(\mathcal{A})$$

by sending homomorphisms of degree n from $\mathcal{L} \to \mathcal{L}'$ to the induced map of degree n from $\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ to $\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}')$.

0FR6 **Lemma 24.8.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module. Let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right graded $\mathcal{B}$-module. With conventions as above we have*

$$\mathrm{Hom}_{Mod^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{M}, \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

functorially in $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$.

Proof. Omitted. Hint: This follows by interpreting both sides as $\mathcal{A}$-bilinear graded maps $\psi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ which are $\mathcal{B}$-linear on the right. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. As a special case of the above, suppose we are given a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of graded $\mathcal{O}$-algebras. Then we obtain a functor and a functor of graded categories

$$\otimes_{\mathcal{A},\varphi}\mathcal{B} : Mod(\mathcal{A}) \longrightarrow Mod(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A},\varphi} \mathcal{B} : Mod^{gr}(\mathcal{A}) \longrightarrow Mod^{gr}(\mathcal{B})$$

On the other hand, we have the restriction functors

$$res_\varphi : Mod(\mathcal{B}) \longrightarrow Mod(\mathcal{A}) \quad \text{and} \quad res_\varphi : Mod^{gr}(\mathcal{B}) \longrightarrow Mod^{gr}(\mathcal{A})$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ for $\mathcal{B}$ viewed as a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then for any right graded $\mathcal{B}$-module $\mathcal{L}$ we have

$$\mathcal{H}om^{gr}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = res_\varphi(\mathcal{L})$$

as right graded $\mathcal{A}$-modules. Thus Lemma 24.8.2 tells us that we have a functorial isomorphism

$$\mathrm{Hom}_{Mod^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A},\varphi} \mathcal{B}, \mathcal{L}) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{M}, res_\varphi(\mathcal{L}))$$

We usually drop the dependence on φ in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

of graded $\mathcal{O}$-modules.

24.9. Pull and push for sheaves of graded modules

0FR7 We advise the reader to skip this section.

Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.3.2.

Let us define a functor

$$f_* : Mod(\mathcal{A}) \longrightarrow Mod(\mathcal{B})$$

Given a graded $\mathcal{A}$-module $\mathcal{M}$ we define $f_*\mathcal{M}$ to be the graded $\mathcal{B}$-module whose degree n term is $f_*\mathcal{M}^n$. As multiplication we use

$$f_*\mathcal{M}^n \times \mathcal{B}^m \xrightarrow{(\mathrm{id},\varphi^m)} f_*\mathcal{M}^n \times f_*\mathcal{A}^m \xrightarrow{f_*\mu_{n,m}} f_*\mathcal{M}^{n+m}$$

where $\mu_{n,m} : \mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}$ is the multiplication map for $\mathcal{M}$ over $\mathcal{A}$. This uses that f_* commutes with products. The construction is clearly functorial in $\mathcal{M}$ and we obtain our functor.

Let us define a functor

$$f^* : Mod(\mathcal{B}) \longrightarrow Mod(\mathcal{A})$$

We will define this functor as a composite of functors

$$Mod(\mathcal{B}) \xrightarrow{f^{-1}} Mod(f^{-1}\mathcal{B}) \xrightarrow{-\otimes_{f^{-1}\mathcal{B}}\mathcal{A}} Mod(\mathcal{A})$$

First, given a graded $\mathcal{B}$-module $\mathcal{N}$ we define $f^{-1}\mathcal{N}$ to be the graded $f^{-1}\mathcal{B}$-module whose degree n term is $f^{-1}\mathcal{N}^n$. As multiplication we use

$$f^{-1}\nu_{n,m} : f^{-1}\mathcal{N}^n \times f^{-1}\mathcal{B}^m \longrightarrow f^{-1}\mathcal{N}^{n+m}$$

where $\nu_{n,m} : \mathcal{N}^n \times \mathcal{B}^m \to \mathcal{N}^{n+m}$ is the multiplication map for $\mathcal{N}$ over $\mathcal{B}$. This uses that f^{-1} commutes with products. The construction is clearly functorial in $\mathcal{N}$ and we obtain our functor f^{-1}. Having said this, we can use the tensor product discussion in Section 24.8 to define the functor

$$- \otimes_{f^{-1}\mathcal{B}} \mathcal{A} : Mod(f^{-1}\mathcal{B}) \longrightarrow Mod(\mathcal{A})$$

Finally, we set

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B},\varphi} \mathcal{A}$$

as already foretold above.

The functors f_* and f^* are readily enhanced to give functors of graded categories

$$f_* : Mod^{gr}(\mathcal{A}) \longrightarrow Mod^{gr}(\mathcal{B}) \quad \text{and} \quad f^* : Mod^{gr}(\mathcal{B}) \longrightarrow Mod^{gr}(\mathcal{A})$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogeneous morphisms of degree n.

0FR8 **Lemma 24.9.1.** *In the situation above we have*

$$\mathrm{Hom}_{Mod^{gr}(\mathcal{B})}(\mathcal{N}, f_*\mathcal{M}) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(f^*\mathcal{N}, \mathcal{M})$$

Proof. Omitted. Hints: First prove that f^{-1} and f_* are adjoint as functors between $Mod(\mathcal{B})$ and $Mod(f^{-1}\mathcal{B})$ using the adjunction between f^{-1} and f_* on sheaves of abelian groups. Next, use the adjunction between base change and restriction given in Section 24.8. □

24.10. Localization and sheaves of graded modules

0FR9 We advise the reader to skip this section.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$ and denote

$$j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 18.19). Below we will use the following fact: for $\mathcal{O}_U$-modules $\mathcal{M}_i$, $i = 1, 2$ and a $\mathcal{O}$-module $\mathcal{A}$ there is a canonical map

$$j_! : \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$ by Modules on Sites, Lemma 18.27.9.

Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. We will denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to $\mathcal{C}/U$, in other words, we have $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$. In Section 24.9 we have constructed adjoint functors

$$j_* : Mod^{gr}(\mathcal{A}_U) \longrightarrow Mod^{gr}(\mathcal{A}) \quad \text{and} \quad j^* : Mod^{gr}(\mathcal{A}) \longrightarrow Mod^{gr}(\mathcal{A}_U)$$

with j^* left adjoint to j_*. We claim there is in addition an exact functor

$$j_! : Mod^{gr}(\mathcal{A}_U) \longrightarrow Mod^{gr}(\mathcal{A})$$

left adjoint to j^*. Namely, given a graded $\mathcal{A}_U$-module $\mathcal{M}$ we define $j_!\mathcal{M}$ to be the graded $\mathcal{A}$-module whose degree n term is $j_!\mathcal{M}^n$. As multiplication map we use

$$j_!\mu_{n,m} : j_!\mathcal{M}^n \times \mathcal{A}^m \to j_!\mathcal{M}^{n+m}$$

where $\mu_{m,n} : \mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}$ is the given multiplication map. Given a homogeneous map $f : \mathcal{M} \to \mathcal{M}'$ of degree n of graded $\mathcal{A}_U$-modules, we obtain a homogeneous map $j_!f : j_!\mathcal{M} \to j_!\mathcal{M}'$ of degree n. Thus we obtain our functor.

0FRA **Lemma 24.10.1.** *In the situation above we have*

$$\mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A}_U)}(\mathcal{M}, j^*\mathcal{N})$$

Proof. By the discussion in Modules on Sites, Section 18.19 the functors $j_!$ and j^* on $\mathcal{O}$-modules are adjoint. Thus if we only look at the $\mathcal{O}$-module structures we know that

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(Mod(\mathcal{O}))}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(Mod(\mathcal{O}_U))}(\mathcal{M}, j^*\mathcal{N})$$

(Recall that $\mathrm{Gr}^{gr}(Mod(\mathcal{O}))$ denotes the graded category of graded $\mathcal{O}$-modules.) Then one has to check that these identifications map the $\mathcal{A}$-module maps on the left hand side to the $\mathcal{A}_U$-module maps on the right hand side. To check this, given $\mathcal{O}_U$-linear maps $f^n : \mathcal{M}^n \to j^*\mathcal{N}^{n+d}$ corresponding to $\mathcal{O}$-linear maps $g^n : j_!\mathcal{M}^n \to \mathcal{N}^{n+d}$ it suffices to show that

$$\begin{array}{ccc} \mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m & \xrightarrow[f^n \otimes 1]{} & j^*\mathcal{N}^{n+d} \otimes_{\mathcal{O}_U} \mathcal{A}_U^m \\ \downarrow & & \downarrow \\ \mathcal{M}^{n+m} & \xrightarrow{f^{n+m}} & j^*\mathcal{N}^{n+m+d} \end{array}$$

commutes if and only if

$$\begin{array}{ccc} j_!\mathcal{M}^n \otimes_{\mathcal{O}} \mathcal{A}^m & \xrightarrow[g^n \otimes 1]{} & \mathcal{N}^{n+d} \otimes_{\mathcal{O}} \mathcal{A}_U^m \\ \downarrow & & \downarrow \\ j_!\mathcal{M}^{n+m} & \xrightarrow{g^{n+m}} & \mathcal{N}^{n+m+d} \end{array}$$

commutes. However, we know that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m, j^*\mathcal{N}^{n+d+m}) &= \mathrm{Hom}_{\mathcal{O}}(j_!(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m), \mathcal{N}^{n+d+m}) \\ &= \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}^n \otimes_{\mathcal{O}} \mathcal{A}^m, \mathcal{N}^{n+d+m}) \end{aligned}$$

by the already used Modules on Sites, Lemma 18.27.9. We omit the verification that shows that the obstruction to the commutativity of the first diagram in the first group maps to the obstruction to the commutativity of the second diagram in the last group. □

0FRB **Lemma 24.10.2.** *In the situation above, let $\mathcal{M}$ be a right graded $\mathcal{A}_U$-module and let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then*

$$j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

as graded $\mathcal{O}$-modules functorially in $\mathcal{M}$ and $\mathcal{N}$.

Proof. Recall that the degree n component of $j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is the cokernel of the canonical map

$$\bigoplus\nolimits_{r+s+t=n} j_!\mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t \longrightarrow \bigoplus\nolimits_{p+q=n} j_!\mathcal{M}^p \otimes_{\mathcal{O}} \mathcal{N}^q$$

See Section 24.6. By Modules on Sites, Lemma 18.27.9 this is the same thing as the cokernel of

$$\bigoplus\nolimits_{r+s+t=n} j_!(\mathcal{M}^r \otimes_{\mathcal{O}_U} \mathcal{A}^s|_U \otimes_{\mathcal{O}_U} \mathcal{N}^t|_U) \longrightarrow \bigoplus\nolimits_{p+q=n} j_!(\mathcal{M}^p \otimes_{\mathcal{O}_U} \mathcal{N}^q|_U)$$

and we win. An alternative proof would be to redo the Yoneda argument given in the proof of the lemma cited above. □

24.11. Shift functors on sheaves of graded modules

0FRC We urge the reader to skip this section. It turns out that sheaves of graded modules over a graded algebra are an example of the phenomenon discussed in Differential Graded Algebra, Remark 22.25.7.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a graded $\mathcal{A}$-module. Let $k \in \mathbf{Z}$. We define the *kth shift of $\mathcal{M}$*, denoted $\mathcal{M}[k]$, to be the graded $\mathcal{A}$-module whose nth part is given by

$$(\mathcal{M}[k])^n = \mathcal{M}^{n+k}$$

is the $(n+k)$th part of $\mathcal{M}$. As multiplication maps

$$(\mathcal{M}[k])^n \times \mathcal{A}^m \longrightarrow (\mathcal{M}[k])^{n+m}$$

we simply use the multiplication maps

$$\mathcal{M}^{n+k} \times \mathcal{A}^m \longrightarrow \mathcal{M}^{n+m+k}$$

of $\mathcal{M}$. It is clear that we have defined a functor $[k]$, that we have $[k+l] = [k] \circ [l]$, and that we have

$$\mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M})[k]$$

(without the intervention of signs) functorially in $\mathcal{M}$ and $\mathcal{L}$. Thus we see indeed that the graded category of graded $\mathcal{A}$-modules can be recovered from the ordinary category of graded $\mathcal{A}$-modules and the shift functors as discussed in Differential Graded Algebra, Remark 22.25.7.

0FRD **Lemma 24.11.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. The category $Mod(\mathcal{A})$ is a Grothendieck abelian category.*

Proof. By Lemma 24.4.2 and the definition of a Grothendieck abelian category (Injectives, Definition 19.10.1) it suffices to show that $Mod(\mathcal{A})$ has a generator. We claim that

$$\mathcal{G} = \bigoplus\nolimits_{k,U} j_{U!}\mathcal{A}_U[k]$$

is a generator where the sum is over all objects U of $\mathcal{C}$ and $k \in \mathbf{Z}$. Indeed, given a graded $\mathcal{A}$-module $\mathcal{M}$ if there are no nonzero maps from $\mathcal{G}$ to $\mathcal{M}$, then we see that for all k and U we have

$$\mathrm{Hom}_{Mod(\mathcal{A})}(j_{U!}\mathcal{A}_U[k], \mathcal{M}) = \mathrm{Hom}_{Mod(\mathcal{A}_U)}(\mathcal{A}_U[k], \mathcal{M}|_U) = \Gamma(U, \mathcal{M}^{-k})$$

is equal to zero. Hence $\mathcal{M}$ is zero. □

24.12. Sheaves of differential graded algebras

0FRE This section is the analogue of Differential Graded Algebra, Section 22.3.

0FRF **Definition 24.12.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A *sheaf of differential graded $\mathcal{O}$-algebras* or a *sheaf of differential graded algebras* on $(\mathcal{C}, \mathcal{O})$ is a cochain complex $\mathcal{A}^\bullet$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \to \mathcal{A}^{n+m}, \quad (a, b) \longmapsto ab$$

called the multiplication maps with the following properties

(1) multiplication is associative,
(2) there is a global section 1 of $\mathcal{A}^0$ which is a two-sided identity for multiplication,
(3) for $U \in \mathrm{Ob}(\mathcal{C})$, $a \in \mathcal{A}^n(U)$, and $b \in \mathcal{A}^m(U)$ we have
$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$

We often denote such a structure $(\mathcal{A}, \mathrm{d})$. A *homomorphism of differential graded $\mathcal{O}$-algebras* from $(\mathcal{A}, \mathrm{d})$ to $(\mathcal{B}, \mathrm{d})$ is a map $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ of complexes of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a differential graded $\mathcal{O}$-algebra $(\mathcal{A}, \mathrm{d})$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a differential graded $\mathcal{O}(U)$-algebra.

As much as possible, we will think of a differential graded $\mathcal{O}$-algebra $(\mathcal{A}, \mathrm{d})$ as a graded $\mathcal{O}$-algebra $\mathcal{A}$ endowed with the operator $\mathrm{d} : \mathcal{A} \to \mathcal{A}$ of degree 1 (where $\mathcal{A}$ is viewed as a graded $\mathcal{O}$-module) satisfying the Leibniz rule given in the definition.

0FRG **Remark 24.12.2.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.

(1) Let $(\mathcal{A}, \mathrm{d})$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. The pushforward will be the differential graded $\mathcal{O}_\mathcal{D}$-algebra $(f_*\mathcal{A}, \mathrm{d})$ where $f_*\mathcal{A}$ is as in Remark 24.3.2 and $\mathrm{d} = f_*\mathrm{d}$ as maps $f_*\mathcal{A}^n \to f_*\mathcal{A}^{n+1}$. We omit the verification that the Leibniz rule is satisfied.
(2) Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. The pullback will be the differential graded $\mathcal{O}_\mathcal{C}$-algebra $(f^*\mathcal{B}, \mathrm{d})$ where $f^*\mathcal{B}$ is as in Remark 24.3.2 and $\mathrm{d} = f^*\mathrm{d}$ as maps $f^*\mathcal{B}^n \to f^*\mathcal{B}^{n+1}$. We omit the verification that the Leibniz rule is satisfied.
(3) The set of homomorphisms $f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras is in 1-to-1 correspondence with the set of homomorphisms $\mathcal{B} \to f_*\mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{D}$-algebras.

Part (3) follows immediately from the usual adjunction between f^* and f_* on sheaves of modules.

24.13. Sheaves of differential graded modules

0FRH This section is the analogue of Differential Graded Algebra, Section 22.4.

0FRI **Definition 24.13.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A (right) *differential graded $\mathcal{A}$-module* or (right) *differential graded module* over $\mathcal{A}$ is a cochain complex $\mathcal{M}^\bullet$ endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}, \quad (x, a) \longmapsto xa$$

called the multiplication maps with the following properties

(1) multiplication satisfies $(xa)a' = x(aa')$,
(2) the identity section 1 of $\mathcal{A}^0$ acts as the identity on $\mathcal{M}^n$ for all n,
(3) for $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathcal{M}^n(U)$, and $a \in \mathcal{A}^m(U)$ we have

$$\mathrm{d}^{n+m}(xa) = \mathrm{d}^n(x)a + (-1)^n x\mathrm{d}^m(a)$$

We often say "let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module" to indicate this situation. A *homomorphism of differential graded $\mathcal{A}$-modules* from $\mathcal{M}$ to $\mathcal{N}$ is a map $f : \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ of complexes of $\mathcal{O}$-modules compatible with the multiplication maps. The category of (right) differential graded $\mathcal{A}$-modules is denoted $Mod(\mathcal{A}, \mathrm{d})$.

We can define *left differential graded modules* in exactly the same manner but our default in the chapter will be right modules.

Given a differential graded $\mathcal{A}$-module $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) differential graded $\mathcal{A}(U)$-module.

0FRJ **Lemma 24.13.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a differential graded $\mathcal{O}$-algebra. The category $Mod(\mathcal{A}, d)$ is an abelian category with the following properties*

(1) *$Mod(\mathcal{A}, d)$ has arbitrary direct sums,*
(2) *$Mod(\mathcal{A}, d)$ has arbitrary colimits,*
(3) *filtered colimit in $Mod(\mathcal{A}, d)$ are exact,*
(4) *$Mod(\mathcal{A}, d)$ has arbitrary products,*

(5) *Mod*$(\mathcal{A}, d)$ *has arbitrary limits.*

The forgetful functor

$$Mod(\mathcal{A}, d) \longrightarrow Mod(\mathcal{A})$$

sending a differential graded $\mathcal{A}$-module to its underlying graded module commutes with all limits and colimits.

Proof. Let us denote $F : Mod(\mathcal{A}, \mathrm{d}) \to Mod(\mathcal{A})$ the functor in the statement of the lemma. Observe that the category *Mod*$(\mathcal{A})$ has properties (1) – (5), see Lemma 24.4.2.

Consider a homomorphism $f : \mathcal{M} \to \mathcal{N}$ of graded $\mathcal{A}$-modules. The kernel and cokernel of f as maps of graded $\mathcal{A}$-modules are additionally endowed with differentials as in Definition 24.13.1. Hence these are also the kernel and cokernel in *Mod*$(\mathcal{A}, \mathrm{d})$. Thus *Mod*$(\mathcal{A}, \mathrm{d})$ is an abelian category and taking kernels and cokernels commutes with F.

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 4.14.11 and 4.14.12. The same lemmas show that proving the commutation of limits and colimits with F follows if F commutes with direct sums and products.

Let $\mathcal{M}_t$, $t \in T$ be a set of differential graded $\mathcal{A}$-modules. Then we can consider the direct sum $\bigoplus \mathcal{M}_t$ as a graded $\mathcal{A}$-module. Since the direct sum of graded modules is done termwise, it is clear that $\bigoplus \mathcal{M}_t$ comes endowed with a differential. The reader easily verifies that this is a direct sum in *Mod*$(\mathcal{A}, \mathrm{d})$. Similarly for products.

Observe that F is an exact functor and that a complex $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ of *Mod*$(\mathcal{A}, \mathrm{d})$ is exact if and only if $F(\mathcal{M}_1) \to F(\mathcal{M}_2) \to F(\mathcal{M}_3)$ is exact in *Mod*$(\mathcal{A})$. Hence we conclude that (3) holds as filtered colimits are exact in *Mod*$(\mathcal{A})$. □

Combining Lemmas 24.13.2 and 24.4.2 we find that there is an exact and faithful functor

$$Mod(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Comp}(\mathcal{O})$$

of abelian categories. For a differential graded $\mathcal{A}$-module $\mathcal{M}$ the cohomology $\mathcal{O}$-modules, denoted $H^i(\mathcal{M})$, are defined as the cohomology of the complex of $\mathcal{O}$-modules corresponding to $\mathcal{M}$. Therefore, a short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{M} \to 0$ of differential graded $\mathcal{A}$-modules gives rise to a long exact sequence

0FRK (24.13.2.1) $$H^n(\mathcal{K}) \to H^n(\mathcal{L}) \to H^n(\mathcal{M}) \to H^{n+1}(\mathcal{K})$$

of cohomology modules, see Homology, Lemma 12.13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded $\mathcal{A}$-module $\mathcal{M}$ is *acyclic* if $H^k(\mathcal{M}) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $\mathcal{M} \to \mathcal{N}$ of differential graded $\mathcal{A}$-modules is a *quasi-isomorphism* if it induces isomorphisms $H^k(\mathcal{M}) \to H^k(\mathcal{N})$ for all $k \in \mathbf{Z}$. And so on and so forth.

24.14. The differential graded category of modules

0FRL This section is the analogue of Differential Graded Algebra, Example 22.26.8. For our conventions on differential graded categories, please see Differential Graded Algebra, Section 22.26.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct a differential graded category

$$Mod^{dg}(\mathcal{A}, \mathrm{d})$$

over $R = \Gamma(\mathcal{C}, \mathcal{O})$ whose associated category of complexes is the category of differential graded $\mathcal{A}$-modules:

$$Mod(\mathcal{A}, \mathrm{d}) = \mathrm{Comp}(Mod^{dg}(\mathcal{A}, \mathrm{d}))$$

As objects of $Mod^{dg}(\mathcal{A}, \mathrm{d})$ we take right differential graded $\mathcal{A}$-modules, see Section 24.13. Given differential graded $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$ we set

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}) = \mathrm{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$$

as a graded R-module, see Section 24.5. In other words, the nth graded piece $\mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$ is the R-module of right $\mathcal{A}$-module maps homogeneous of degree n. For an element $f \in \mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$ we set

$$\mathrm{d}(f) = \mathrm{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}$$

To make sense of this we think of $\mathrm{d}_{\mathcal{M}}$ and $\mathrm{d}_{\mathcal{L}}$ as graded $\mathcal{O}$-module maps and we use composition of graded $\mathcal{O}$-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n+1$ as a graded $\mathcal{O}$-module map, and it is $\mathcal{A}$-linear because for homogeneous local sections x and a of $\mathcal{M}$ and $\mathcal{A}$ we have

$$\begin{aligned}
\mathrm{d}(f)(xa) &= \mathrm{d}_{\mathcal{M}}(f(x)a) - (-1)^n f(\mathrm{d}_{\mathcal{L}}(xa)) \\
&= \mathrm{d}_{\mathcal{M}}(f(x))a + (-1)^{\deg(x)+n} f(x)\mathrm{d}(a) - (-1)^n f(\mathrm{d}_{\mathcal{L}}(x))a - (-1)^{n+\deg(x)} f(x)\mathrm{d}(a) \\
&= \mathrm{d}(f)(x)a
\end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom).

For differential graded $\mathcal{A}$-modules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ we have already defined the composition

$$\mathrm{Hom}^m(\mathcal{L}, \mathcal{M}) \times \mathrm{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathrm{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

in Section 24.5 by the usual composition of maps of sheaves. This defines a map of differential graded modules

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}) \otimes_R \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}, \mathcal{M})$$

as required in Differential Graded Algebra, Definition 22.26.1 because

$$\begin{aligned}
\mathrm{d}(g \circ f) &= \mathrm{d}_{\mathcal{M}} \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_{\mathcal{K}} \\
&= (\mathrm{d}_{\mathcal{M}} \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_{\mathcal{L}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{K}}) \\
&= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)
\end{aligned}$$

if f has degree n and g has degree m as desired.

24.15. Tensor product for sheaves of differential graded modules

0FRM This section is the analogue of part of Differential Graded Algebra, Section 22.12.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module and let $\mathcal{N}$ be a left differential graded $\mathcal{A}$-module. In this situation we define the *tensor product* $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$

as follows. As a graded $\mathcal{O}$-module it is given by the construction in Section 24.6. It comes endowed with a differential

$$\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}} : (\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N})^n \longrightarrow (\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N})^{n+1}$$

defined by the rule that

$$\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}(x\otimes y) = \mathrm{d}_{\mathcal{M}}(x)\otimes y + (-1)^{\deg(x)} x\otimes \mathrm{d}_{\mathcal{N}}(y)$$

for homogeneous local sections x and y of $\mathcal{M}$ and $\mathcal{N}$. To see that this is well defined we have to show that $\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}$ annihilates elements of the form $xa\otimes y - x\otimes ay$ for homogeneous local sections x, a, y of $\mathcal{M}$, $\mathcal{A}$, $\mathcal{N}$. We compute

$$\begin{aligned}
&\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}(xa\otimes y - x\otimes ay)\\
&= \mathrm{d}_{\mathcal{M}}(xa)\otimes y + (-1)^{\deg(x)+\deg(a)} xa\otimes \mathrm{d}_{\mathcal{N}}(y) - \mathrm{d}_{\mathcal{M}}(x)\otimes ay - (-1)^{\deg(x)} x\otimes \mathrm{d}_{\mathcal{N}}(ay)\\
&= \mathrm{d}_{\mathcal{M}}(x)a\otimes y + (-1)^{\deg(x)} x\mathrm{d}(a)\otimes y + (-1)^{\deg(x)+\deg(a)} xa\otimes \mathrm{d}_{\mathcal{N}}(y)\\
&- \mathrm{d}_{\mathcal{M}}(x)\otimes ay - (-1)^{\deg(x)} x\otimes \mathrm{d}(a)y - (-1)^{\deg(x)+\deg(a)} x\otimes a\mathrm{d}_{\mathcal{N}}(y)
\end{aligned}$$

then we observe that the elements

$$\mathrm{d}_{\mathcal{M}}(x)a\otimes y - \mathrm{d}_{\mathcal{M}}(x)\otimes ay, \quad x\mathrm{d}(a)\otimes y - x\otimes \mathrm{d}(a)y, \quad \text{and} \quad xa\otimes \mathrm{d}_{\mathcal{N}}(y) - x\otimes a\mathrm{d}_{\mathcal{N}}(y)$$

map to zero in $\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}$ and we conclude. We omit the verification that $\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}\circ \mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}} = 0$.

If we fix the left differential graded $\mathcal{A}$-module $\mathcal{N}$ we obtain a functor

$$-\otimes_{\mathcal{A}}\mathcal{N} : \mathit{Mod}(\mathcal{A},\mathrm{d}) \longrightarrow \mathrm{Comp}(\mathcal{O})$$

where on the right hand side we have the category of complexes of $\mathcal{O}$-modules. This can be upgraded to a functor of differential graded categories

$$-\otimes_{\mathcal{A}}\mathcal{N} : \mathit{Mod}^{dg}(\mathcal{A},\mathrm{d}) \longrightarrow \mathrm{Comp}^{dg}(\mathcal{O})$$

On underlying graded objects, we send a homomorphism $f : \mathcal{M}\to\mathcal{M}'$ of degree n to the degree n map $f\otimes \mathrm{id}_{\mathcal{N}} : \mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}\to\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{N}$, because this is what we did in Section 24.6. To show that this works, we have to verify that the map

$$\mathrm{Hom}_{\mathit{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M},\mathcal{M}') \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N},\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{N})$$

is compatible with differentials. To see this for f as above we have to show that

$$(\mathrm{d}_{\mathcal{M}'}\circ f - (-1)^n f\circ \mathrm{d}_{\mathcal{M}})\otimes \mathrm{id}_{\mathcal{N}}$$

is equal to

$$\mathrm{d}_{\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{N}}\circ (f\otimes \mathrm{id}_{\mathcal{N}}) - (-1)^n (f\otimes \mathrm{id}_{\mathcal{N}})\circ \mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}$$

Let us compute the effect of these operators on a local section of the form $x\otimes y$ with x and y homogeneous local sections of $\mathcal{M}$ and $\mathcal{N}$. For the first we obtain

$$(\mathrm{d}_{\mathcal{M}'}(f(x)) - (-1)^n f(\mathrm{d}_{\mathcal{M}}(x)))\otimes y$$

and for the second we obtain

$$\begin{aligned}
&\mathrm{d}_{\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{N}}(f(x)\otimes y) - (-1)^n (f\otimes \mathrm{id}_{\mathcal{N}})(\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}(x\otimes y)\\
&= \mathrm{d}_{\mathcal{M}'}(f(x))\otimes y + (-1)^{\deg(x)+n} f(x)\otimes \mathrm{d}_{\mathcal{N}}(y)\\
&- (-1)^n f(\mathrm{d}_{\mathcal{M}}(x))\otimes y - (-1)^n (-1)^{\deg(x)} f(x)\otimes \mathrm{d}_{\mathcal{N}}(y)
\end{aligned}$$

which is indeeed the same local section.

24.16. Internal hom for sheaves of differential graded modules

0FRN We are going to need the sheafified version of the construction in Section 24.14. Let $(\mathcal{C}, \mathcal{O})$, $\mathcal{A}$, $\mathcal{M}$, $\mathcal{L}$ be as in Section 24.14. Then we define

$$\mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{L}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{H}om^n_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

as a graded $\mathcal{O}$-module, see Section 24.7. In other words, a section f of the nth graded piece $\mathcal{H}om^n_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ over U is a map of right $\mathcal{A}_U$-module map $\mathcal{L}|_U \to \mathcal{M}|_U$ homogeneous of degree n. For such f we set

$$\mathrm{d}(f) = \mathrm{d}_{\mathcal{M}}|_U \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}|_U$$

To make sense of this we think of $\mathrm{d}_{\mathcal{M}}|_U$ and $\mathrm{d}_{\mathcal{L}}|_U$ as graded $\mathcal{O}_U$-module maps and we use composition of graded $\mathcal{O}_U$-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n + 1$ as a graded $\mathcal{O}_U$-module map. Using the exact same computation as in Section 24.14 we see that $\mathrm{d}(f)$ is $\mathcal{A}_U$-linear.

As in Section 24.14 there is a composition map

$$\mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of differential graded $\mathcal{O}$-modules defined in Section 24.15. This map is given by the composition map

$$\mathcal{H}om^m(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{n+m}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally). Using the exact same computation as in Section 24.14 on local sections we see that the composition map is a morphism of differential graded $\mathcal{O}$-modules.

With these definitions we have

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}))$$

as graded R-modules compatible with composition.

24.17. Sheaves of differential graded bimodules and tensor-hom adjunction

0FRP This section is the analogue of part of Differential Graded Algebra, Section 22.12.

0FRQ **Definition 24.17.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A *differential graded* $(\mathcal{A}, \mathcal{B})$*-bimodule* is given by a complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \to \mathcal{M}^{n+m}, \quad (x, b) \longmapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \to \mathcal{M}^{n+m}, \quad (a, x) \longmapsto ax$$

called the multiplication maps with the following properties

(1) multiplication satisfies $a(a'x) = (aa')x$ and $(xb)b' = x(bb')$,
(2) $(ax)b = a(xb)$,
(3) $\mathrm{d}(ax) = \mathrm{d}(a)x + (-1)^{\deg(a)}a\mathrm{d}(x)$ and $\mathrm{d}(xb) = \mathrm{d}(x)b + (-1)^{\deg(x)}x\mathrm{d}(b)$,
(4) the identity section 1 of $\mathcal{A}^0$ acts as the identity by multiplication, and
(5) the identity section 1 of $\mathcal{B}^0$ acts as the identity by multiplication.

We often denote such a structure $\mathcal{M}$ and sometimes we write ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$. A *homomorphism of differential graded* $(\mathcal{A}, \mathcal{B})$*-bimodules* $f : \mathcal{M} \to \mathcal{N}$ is a map of complexes $f : \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a differential graded $(\mathcal{A}(U), \mathcal{B}(U))$-bimodule.

Observe that a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is the same thing as a right differential graded $\mathcal{B}$-module which is also a left differential graded $\mathcal{A}$-module such that the grading and differentials agree and such that the $\mathcal{A}$-module structure commutes with the $\mathcal{B}$-module structure. Here is a precise statement.

0FRR **Lemma 24.17.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a right differential graded $\mathcal{B}$-module. There is a* 1*-to-*1 *correspondence between $(\mathcal{A}, \mathcal{B})$-bimodule structures on $\mathcal{N}$ compatible with the given differential graded $\mathcal{B}$-module structure and homomorphisms*

$$\mathcal{A} \longrightarrow \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{N})$$

of differential graded $\mathcal{O}$-algebras.

Proof. Omitted. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. In this case the differential graded tensor product defined in Section 24.15

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

is a right differential graded $\mathcal{B}$-module with multiplication maps as in Section 24.8. This construction defines a functor and a functor of graded categories

$$\otimes_{\mathcal{A}}\mathcal{N} : \mathit{Mod}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathit{Mod}(\mathcal{B}, \mathrm{d}) \quad \text{and} \quad \otimes_{\mathcal{A}} \mathcal{N} : \mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathit{Mod}^{dg}(\mathcal{B}, \mathrm{d})$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right differential graded $\mathcal{B}$-module. In this case the differential graded internal hom defined in Section 24.16

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

is a right differential graded $\mathcal{A}$-module where the right graded $\mathcal{A}$-module structure is the one defined in Section 24.8. Another way to define the multiplication is the use the composition

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}) \otimes_{\mathcal{O}} \mathcal{A} \to \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}) \otimes_{\mathcal{O}} \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{N}) \to \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

where the first arrow comes from Lemma 24.17.2 and the second arrow is the composition of Section 24.16. Since these arrows are both compatible with differentials, we conclude that we indeed obtain a differential graded $\mathcal{A}$-module. This construction defines a functor and a functor of differential graded categories

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -) : \mathit{Mod}(\mathcal{B}, \mathrm{d}) \longrightarrow \mathit{Mod}(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -) : \mathit{Mod}^{dg}(\mathcal{B}, \mathrm{d}) \longrightarrow \mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d})$$

by sending homomorphisms of degree n from $\mathcal{L} \to \mathcal{L}'$ to the induced map of degree n from $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ to $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}')$.

0FRS **Lemma 24.17.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right differential graded $\mathcal{B}$-module. With conventions as above we have*

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{B},d)}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathrm{Hom}_{Mod^{dg}(\mathcal{A},d)}(\mathcal{M}, \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

functorially in $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$.

Proof. Omitted. Hint: On the graded level we have seen this is true in Lemma 24.8.2. Thus it suffices to check the isomorphisms are compatible with differentials which can be done by a computation on the level of local sections. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. As a special case of the above, suppose we are given a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of differential graded $\mathcal{O}$-algebras. Then we obtain a functor and a functor of differential graded categories

$$\otimes_{\mathcal{A},\varphi}\mathcal{B} : Mod(\mathcal{A}, \mathrm{d}) \longrightarrow Mod(\mathcal{B}, \mathrm{d}) \quad \text{and} \quad \otimes_{\mathcal{A},\varphi} \mathcal{B} : Mod^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{B}, \mathrm{d})$$

On the other hand, we have the restriction functors

$$res_\varphi : Mod(\mathcal{B}, \mathrm{d}) \longrightarrow Mod(\mathcal{A}, \mathrm{d}) \quad \text{and} \quad res_\varphi : Mod^{dg}(\mathcal{B}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}, \mathrm{d})$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ for $\mathcal{B}$ viewed as a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then for any right differential graded $\mathcal{B}$-module $\mathcal{L}$ we have

$$\mathcal{H}om^{dg}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = res_\varphi(\mathcal{L})$$

as right differential graded $\mathcal{A}$-modules. Thus Lemma 24.8.2 tells us that we have a functorial isomorphism

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{B},\mathrm{d})}(\mathcal{M} \otimes_{\mathcal{A},\varphi} \mathcal{B}, \mathcal{L}) = \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M}, res_\varphi(\mathcal{L}))$$

We usually drop the dependence on φ in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

of graded $\mathcal{O}$-modules.

24.18. Pull and push for sheaves of differential graded modules

0FRT Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

Let us define a functor

$$f_* : Mod(\mathcal{A}, \mathrm{d}) \longrightarrow Mod(\mathcal{B}, \mathrm{d})$$

Given a differential graded $\mathcal{A}$-module $\mathcal{M}$ we define $f_*\mathcal{M}$ to be the graded $\mathcal{B}$-module constructed in Section 24.9 with differential given by the maps $f_*d : f_*\mathcal{M}^n \to f_*\mathcal{M}^{n+1}$. The construction is clearly functorial in $\mathcal{M}$ and we obtain our functor.

Let us define a functor

$$f^* : Mod(\mathcal{B}, \mathrm{d}) \longrightarrow Mod(\mathcal{A}, \mathrm{d})$$

Given a differential graded $\mathcal{B}$-module $\mathcal{N}$ we define $f^*\mathcal{N}$ to be the graded $\mathcal{A}$-module constructed in Section 24.9. Recall that

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B}} \mathcal{A}$$

Since $f^{-1}\mathcal{N}$ comes with the differentials $f^{-1}\mathrm{d} : f^{-1}\mathcal{N}^n \to f^{-1}\mathcal{N}^{n+1}$ we can view this tensor product as an example of the tensor product discussed in Section 24.17 which provides us with a differential. The construction is clearly functorial in $\mathcal{N}$ and we obtain our functor f^*.

The functors f_* and f^* are readily enhanced to give functors of differential graded categories

$$f_* : Mod^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{B}, \mathrm{d}) \quad \text{and} \quad f^* : Mod^{dg}(\mathcal{B}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}, \mathrm{d})$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogeneous morphisms of degree n.

0FRU **Lemma 24.18.1.** *In the situation above we have*

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{B},d)}(\mathcal{N}, f_*\mathcal{M}) = \mathrm{Hom}_{Mod^{dg}(\mathcal{A},d)}(f^*\mathcal{N}, \mathcal{M})$$

Proof. Omitted. Hints: This is true for the underlying graded categories by Lemma 24.9.1. A calculation shows that these isomorphisms are compatible with differentials. □

24.19. Localization and sheaves of differential graded modules

0FRV Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$ and denote

$$j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 18.19). Below we will use the following fact: for $\mathcal{O}_U$-modules $\mathcal{M}_i$, $i = 1, 2$ and a $\mathcal{O}$-module $\mathcal{A}$ there is a canonical map

$$j_! : \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$ by Modules on Sites, Lemma 18.27.9.

Let $\mathcal{A}$ be a differential graded $\mathcal{O}$-algebra. We will denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to $\mathcal{C}/U$, in other words, we have $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$. In Section 24.18 we have constructed adjoint functors

$$j_* : Mod^{dg}(\mathcal{A}_U, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}, \mathrm{d}) \quad \text{and} \quad j^* : Mod^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}_U, \mathrm{d})$$

with j^* left adjoint to j_*. We claim there is in addition an exact functor

$$j_! : Mod^{dg}(\mathcal{A}_U, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}, \mathrm{d})$$

right adjoint to j_*. Namely, given a differential graded $\mathcal{A}_U$-module $\mathcal{M}$ we define $j_!\mathcal{M}$ to be the graded $\mathcal{A}$-module constructed in Section 24.10 with differentials $j_!\mathrm{d} : j_!\mathcal{M}^n \to j_!\mathcal{M}^{n+1}$. Given a homogeneous map $f : \mathcal{M} \to \mathcal{M}'$ of degree n of differential graded $\mathcal{A}_U$-modules, we obtain a homogeneous map $j_!f : j_!\mathcal{M} \to j_!\mathcal{M}'$ of degree n of differential graded $\mathcal{A}$-modules. We omit the straightforward verification that this construction is compatible with differentials. Thus we obtain our functor.

0FRW **Lemma 24.19.1.** *In the situation above we have*

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},d)}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{Mod^{dg}(\mathcal{A}_U,d)}(\mathcal{M}, j^*\mathcal{N})$$

Proof. Omitted. Hint: We have seen in Lemma 24.10.1 that the lemma is true on graded level. Thus all that needs to be checked is that the resulting isomorphism is compatible with differentials. □

0FRX **Lemma 24.19.2.** *In the situation above, let $\mathcal{M}$ be a right differential graded $\mathcal{A}_U$-module and let $\mathcal{N}$ be a left differential graded $\mathcal{A}$-module. Then*

$$j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

as complexes of $\mathcal{O}$-modules functorially in $\mathcal{M}$ and $\mathcal{N}$.

Proof. As graded modules, this follows from Lemma 24.10.2. We omit the verification that this isomorphism is compatible with differentials. □

24.20. Shift functors on sheaves of differential graded modules

0FRY Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. Let $k \in \mathbf{Z}$. We define the *kth shift of $\mathcal{M}$*, denoted $\mathcal{M}[k]$, as follows

(1) as a graded $\mathcal{A}$-module we let $\mathcal{M}[k]$ be as defined in Section 24.11,
(2) the differential $d_{\mathcal{M}[k]} : (\mathcal{M}[k])^n \to (\mathcal{M}[k])^{n+1}$ is defined to be $(-1)^k\mathrm{d}_{\mathcal{M}} : \mathcal{M}^{n+k} \to \mathcal{M}^{n+k+1}$.

For a homomorphism $f : \mathcal{L} \to \mathcal{M}$ of $\mathcal{A}$-modules homogeneous of degree n, we let $f[k] : \mathcal{L}[k] \to \mathcal{M}[k]$ be given by the same component maps as f. Then $f[k]$ is a homogeneous $\mathcal{A}$-module map of degree n. This gives a map

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}) \longrightarrow \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}[k], \mathcal{M}[k])$$

compatible with differentials (it follows from the fact that the signs of the differentials of $\mathcal{L}$ and $\mathcal{M}$ are changed by the same amount). These choices are compatible with the choice in Differential Graded Algebra, Definition 22.4.3. It is clear that we have defined a functor

$$[k] : Mod^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}, \mathrm{d})$$

of differential graded categories and that we have $[k + l] = [k] \circ [l]$.

We claim that the isomorphism

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M})[k]$$

defined in Section 24.11 on underlying graded modules is compatible with the differentials. To see this, suppose we have a right $\mathcal{A}$-module map $f : \mathcal{L} \to \mathcal{M}[k]$ homogeneous of degree n; this is an element of degree n of the LHS. Denote $f' : \mathcal{L} \to \mathcal{M}$

the homogeneous $\mathcal{A}$-module map of degree $n+k$ with the **same** component maps as f. By our conventions, this is the corresponding element of degree n of the RHS. By definition of the differential of LHS we obtain

$$\mathrm{d}_{LHS}(f) = \mathrm{d}_{\mathcal{M}[k]} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}} = (-1)^k \mathrm{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}$$

and for the differential on the RHS we obtain

$$\mathrm{d}_{RHS}(f') = (-1)^k \left(\mathrm{d}_{\mathcal{M}} \circ f' - (-1)^{n+k} f' \circ \mathrm{d}_{\mathcal{L}} \right) = (-1)^k \mathrm{d}_{\mathcal{M}} \circ f' - (-1)^n f' \circ \mathrm{d}_{\mathcal{L}}$$

These maps have the same component maps and the proof is complete.

24.21. The homotopy category

0FRZ This section is the analogue of Differential Graded Algebra, Section 22.5.

0FS0 **Definition 24.21.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $f, g : \mathcal{M} \to \mathcal{N}$ be homomorphisms of differential graded $\mathcal{A}$-modules. A *homotopy between f and g* is a graded $\mathcal{A}$-module map $h : \mathcal{M} \to \mathcal{N}$ homogeneous of degree -1 such that

$$f - g = \mathrm{d}_{\mathcal{N}} \circ h + h \circ \mathrm{d}_{\mathcal{M}}$$

If a homotopy exists, then we say f and g are *homotopic*.

In the situation of the definition, if we have maps $a : \mathcal{K} \to \mathcal{M}$ and $c : \mathcal{N} \to \mathcal{L}$ then we see that

$$\begin{array}{c} h \text{ is a homotopy} \\ \text{between } f \text{ and } g \end{array} \quad \Rightarrow \quad \begin{array}{c} c \circ h \circ a \text{ is a homotopy} \\ \text{between } c \circ f \circ a \text{ and } c \circ g \circ a \end{array}$$

Thus we can define composition of homotopy classes of morphisms in $\mathit{Mod}(\mathcal{A}, \mathrm{d})$.

0FS1 **Definition 24.21.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The *homotopy category*, denoted $K(\mathit{Mod}(\mathcal{A}, \mathrm{d}))$, is the category whose objects are the objects of $\mathit{Mod}(\mathcal{A}, \mathrm{d})$ and whose morphisms are homotopy classes of homomorphisms of differential graded $\mathcal{A}$-modules.

The notation $K(\mathit{Mod}(\mathcal{A}, \mathrm{d}))$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

In Differential Graded Algebra, Definition 22.26.3 we have defined what we mean by the category of complexes $\mathrm{Comp}(\mathcal{S})$ and the homotopy category $K(\mathcal{S})$ of a differential graded category $\mathcal{S}$. Applying this to the differential graded category $\mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d})$ we obtain

$$\mathit{Mod}(\mathcal{A}, \mathrm{d}) = \mathrm{Comp}(\mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d}))$$

(see discussion in Section 24.14) and we obtain

$$K(\mathit{Mod}(\mathcal{A}, \mathrm{d})) = K(\mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d}))$$

To see that this last equality is true, note that we have the equality

$$\mathrm{d}_{\mathrm{Hom}_{\mathit{Mod}^{dg}(\mathcal{A}, \mathrm{d})}(\mathcal{M}, \mathcal{N})}(h) = \mathrm{d}_{\mathcal{N}} \circ h + h \circ \mathrm{d}_{\mathcal{M}}$$

when h is as in Definition 24.21.1. We omit the details.

0FS2 **Lemma 24.21.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The homotopy category $K(\mathit{Mod}(\mathcal{A}, d))$ has direct sums and products.*

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 24.13.2. This works because we saw that these functors commute with the forgetful functor to the category of graded $\mathcal{A}$-modules and because $\prod$ and $\bigoplus$ are exact functors on the category of families of abelian groups. □

24.22. Cones and triangles

0FS3 In this section we use the material from Differential Graded Algebra, Section 22.27 to conclude that the homotopy category of the category of differential graded $\mathcal{A}$-modules is a triangulated category.

0FS4 **Lemma 24.22.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The differential graded category $Mod^{dg}(\mathcal{A}, d)$ satisfies axioms (A) and (B) of Differential Graded Algebra, Section 22.27.*

Proof. Suppose given differential graded $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$. Consider the differential graded $\mathcal{A}$-module $\mathcal{M} \oplus \mathcal{N}$ defined in the obvious manner. Then the coprojections $i : \mathcal{M} \to \mathcal{M} \oplus \mathcal{N}$ and $j : \mathcal{N} \to \mathcal{M} \oplus \mathcal{N}$ and the projections $p : \mathcal{M} \oplus \mathcal{N} \to \mathcal{N}$ and $q : \mathcal{M} \oplus \mathcal{N} \to \mathcal{M}$ are morphisms of differential graded $\mathcal{A}$-modules. Hence i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc. Thus this direct sum is a differential graded sum in the sense of Differential Graded Algebra, Definition 22.26.4. This proves axiom (A).

Axiom (B) was shown in Section 24.20. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Recall that a sequence

$$0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{N} \to 0$$

in $Mod(\mathcal{A}, \mathrm{d})$ is called an *admissible short exact sequence* (in Differential Graded Algebra, Section 22.27) if it is split in $Mod(\mathcal{A})$. In other words, if it is split as a sequence of graded $\mathcal{A}$-modules. Denote $s : \mathcal{N} \to \mathcal{L}$ and $\pi : \mathcal{L} \to \mathcal{K}$ graded $\mathcal{A}$-module splittings. Combining Lemma 24.22.1 and Differential Graded Algebra, Lemma 22.27.1 we obtain a triangle

$$\mathcal{K} \to \mathcal{L} \to \mathcal{N} \to \mathcal{K}[1]$$

where the arrow $\mathcal{N} \to \mathcal{K}[1]$ in the proof of Differential Graded Algebra, Lemma 22.27.1 is constructed as

$$\delta = \pi \circ \mathrm{d}_{\mathrm{Hom}_{Mod^{dg}(\mathcal{A}, \mathrm{d})}(\mathcal{L}, \mathcal{M})}(s) = \pi \circ \mathrm{d}_{\mathcal{L}} \circ s - \pi \circ s \circ \mathrm{d}_{\mathcal{N}} = \pi \circ \mathrm{d}_{\mathcal{L}} \circ s$$

with apologies for the horrendous notation. In any case, we see that in our setting the boundary map δ as constructed in Differential Graded Algebra, Lemma 22.27.1 agrees on underlying complexes of $\mathcal{O}$-modules with the usual boundary map used throughout the Stacks project for termwise split short exact sequences of complexes, see Derived Categories, Definition 13.9.9.

0FS5 **Definition 24.22.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of differential graded $\mathcal{A}$-modules. The *cone* of f is the differential graded $\mathcal{A}$-mod $C(f)$ defined as follows:

(1) the underlying complex of $\mathcal{O}$-modules is the cone of the corresponding map $f : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ of complexes of $\mathcal{A}$-modules, i.e., we have $C(f)^n = \mathcal{L}^n \oplus \mathcal{K}^{n+1}$ and differential
$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_\mathcal{L} & f \\ 0 & -\mathrm{d}_\mathcal{K} \end{pmatrix}$$
(2) the multiplication map
$$C(f)^n \times \mathcal{A}^m \to C(f)^{n+m}$$
is the direct sum of the multiplication map $\mathcal{L}^n \times \mathcal{A}^m \to \mathcal{L}^{n+m}$ and the multiplication map $\mathcal{K}^{n+1} \times \mathcal{A}^m \to \mathcal{K}^{n+1+m}$.

It comes equipped with canonical hommorphisms of differential graded $\mathcal{A}$-modules $i : \mathcal{L} \to C(f)$ and $p : C(f) \to \mathcal{K}[1]$ induced by the obvious maps.

Observe that in the situation of the definition the sequence
$$0 \to \mathcal{L} \to C(f) \to \mathcal{K}[1] \to 0$$
is an addmissible short exact sequence.

0FS6 **Lemma 24.22.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The differential graded category $Mod^{dg}(\mathcal{A}, d)$ satisfies axiom (C) formulated in Differential Graded Algebra, Situation 22.27.2.*

Proof. Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of differential graded $\mathcal{A}$-modules. By the above we have an admissible short exact sequence
$$0 \to \mathcal{L} \to C(f) \to \mathcal{K}[1] \to 0$$
To finish the proof we have to show that the boundary map
$$\delta : \mathcal{K}[1] \to \mathcal{L}[1]$$
associated to this (see discussion above) is equal to $f[1]$. For the section $s : \mathcal{K}[1] \to C(f)$ we use in degree n the embedding $\mathcal{K}^{n+1} \to C(f)^n$. Then in degree n the map π is given by the projections $C(f)^n \to \mathcal{L}^n$. Then finally we have to compute
$$\delta = \pi \circ \mathrm{d}_{C(f)} \circ s$$
(see discussion above). In matrix notation this is equal to
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}_\mathcal{L} & f \\ 0 & -\mathrm{d}_\mathcal{K} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f$$
as desired. □

At this point we know that all lemmas proved in Differential Graded Algebra, Section 22.27 are valid for the differential graded category $Mod^{dg}(\mathcal{A}, \mathrm{d})$. In particular, we have the following.

0FS7 **Proposition 24.22.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The homotopy category $K(Mod(\mathcal{A}, d))$ is a triangulated category where*

(1) *the shift functors are those constructed in Section 24.20,*

(2) *the distinguished triangles are those triangles in $K(Mod(\mathcal{A}, d))$ which are isomorphic as a triangle to a triangle*

$$\mathcal{K} \to \mathcal{L} \to \mathcal{N} \xrightarrow{\delta} \mathcal{K}[1], \qquad \delta = \pi \circ d_{\mathcal{L}} \circ s$$

constructed from an admissible short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{N} \to 0$ in $Mod(\mathcal{A}, d)$ above.

Proof. Recall that $K(Mod(\mathcal{A}, \mathrm{d})) = K(Mod^{dg}(\mathcal{A}, \mathrm{d}))$, see Section 24.21. Having said this, the proposition follows from Lemmas 24.22.1 and 24.22.3 and Differential Graded Algebra, Proposition 22.27.16. □

0FS8 **Remark 24.22.5.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $C = C(\mathrm{id}_{\mathcal{A}})$ be the cone on the identity map $\mathcal{A} \to \mathcal{A}$ viewed as a map of differential graded $\mathcal{A}$-modules. Then

$$\mathrm{Hom}_{Mod(\mathcal{A},\mathrm{d})}(C, \mathcal{M}) = \{(x, y) \in \Gamma(\mathcal{C}, \mathcal{M}^0) \times \Gamma(\mathcal{C}, \mathcal{M}^{-1}) \mid x = \mathrm{d}(y)\}$$

where the map from left to right sends f to the pair (x, y) where x is the image of the global section $(0, 1)$ of $C^{-1} = \mathcal{A}^{-1} \oplus \mathcal{A}^0$ and where y is the image of the global section $(1, 0)$ of $C^0 = \mathcal{A}^0 \oplus \mathcal{A}^1$.

0FS9 **Lemma 24.22.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a differential graded $\mathcal{O}$-algebra. The category $Mod(\mathcal{A}, d)$ is a Grothendieck abelian category.*

Proof. By Lemma 24.13.2 and the definition of a Grothendieck abelian category (Injectives, Definition 19.10.1) it suffices to show that $Mod(\mathcal{A}, \mathrm{d})$ has a generator. For every object U of $\mathcal{C}$ we denote C_U the cone on the identity map $\mathcal{A}_U \to \mathcal{A}_U$ as in Remark 24.22.5. We claim that

$$\mathcal{G} = \bigoplus\nolimits_{k,U} j_{U!} C_U[k]$$

is a generator where the sum is over all objects U of $\mathcal{C}$ and $k \in \mathbf{Z}$. Indeed, given a differential graded $\mathcal{A}$-module $\mathcal{M}$ if there are no nonzero maps from $\mathcal{G}$ to $\mathcal{M}$, then we see that for all k and U we have

$$\begin{aligned} \mathrm{Hom}_{Mod(\mathcal{A})}(j_{U!} C_U[k], \mathcal{M}) & \\ &= \mathrm{Hom}_{Mod(\mathcal{A}_U)}(C_U[k], \mathcal{M}|_U) \\ &= \{(x, y) \in \mathcal{M}^{-k}(U) \times \mathcal{M}^{-k-1}(U) \mid x = \mathrm{d}(y)\} \end{aligned}$$

is equal to zero. Hence $\mathcal{M}$ is zero. □

24.23. Flat resolutions

0FSA This section is the analogue of Differential Graded Algebra, Section 22.20.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let us call a right differential graded $\mathcal{A}$-module $\mathcal{P}$ *good* if

(1) the functor $\mathcal{N} \mapsto \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ is exact on the category of graded left $\mathcal{A}$-modules,
(2) if $\mathcal{N}$ is an acyclic differential graded left $\mathcal{A}$-module, then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic,
(3) for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi and any differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$ and any map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}_{\mathcal{D}}$-algebras we have properties (1) and (2) for the pullback $f^*\mathcal{P}$ (Section 24.18) viewed as a differential graded $\mathcal{A}'$-module.

The first condition means that $\mathcal{P}$ is flat as a right graded $\mathcal{A}$-module, the second condition means that $\mathcal{P}$ is K-flat in the sense of Spaltenstein (see Cohomology on Sites, Section 21.17), and the third condition is that this holds after arbitrary base change.

Perhaps surprisingly, there are many good modules.

0FSB **Lemma 24.23.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then $j_!\mathcal{A}_U$ is a good differential graded $\mathcal{A}$-module.*

Proof. Let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. By Lemma 24.10.2 we have

$$j_!\mathcal{A}_U \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{A}_U \otimes_{\mathcal{A}_U} \mathcal{N}|_U) = j_!(\mathcal{N}_U)$$

as graded modules. Since both restriction to U and $j_!$ are exact this proves condition (1). The same argument works for (2) using Lemma 24.19.2.

Consider a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras. We have to show that

$$f^* j_!\mathcal{A}_U = f^{-1} j_!\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$$

satisfies (1) and (2) for the ringed topos $(Sh(\mathcal{C}'), \mathcal{O}')$ endowed with the sheaf of differential graded $\mathcal{O}'$-algebras $\mathcal{A}'$. To prove this we may replace $(Sh(\mathcal{C}), \mathcal{O})$ and $(Sh(\mathcal{C}'), \mathcal{O}')$ by equivalent ringed topoi. Thus by Modules on Sites, Lemma 18.7.2 we may assume that f comes from a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ given by the continuous functor $u : \mathcal{C} \to \mathcal{C}'$. In this case, set $U' = u(U)$ and denote $j' : Sh(\mathcal{C}'/U') \to Sh(\mathcal{C}')$ the corresponding localization morphism. We obtain a commutative square of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}'/U'), \mathcal{O}'_{U'}) & \xrightarrow{(j', (j')^\sharp)} & (Sh(\mathcal{C}'), \mathcal{O}') \\ \downarrow{\scriptstyle (f', (f')^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow{(j, j^\sharp)} & (Sh(\mathcal{C}), \mathcal{O}). \end{array}$$

and we have $f'_*(j')^{-1} = j^{-1}f_*$. See Modules on Sites, Lemma 18.20.1. By uniqueness of adjoints we obtain $f^{-1}j_! = j'_!(f')^{-1}$. Thus we obtain

$$\begin{aligned} f^* j_!\mathcal{A}_U &= f^{-1} j_!\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!(f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!\left((f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}|_{U'}} \mathcal{A}'|_{U'}\right) \\ &= j'_!\mathcal{A}'_{U'} \end{aligned}$$

The first equation is the definition of the pullback of $j_!\mathcal{A}_U$ to a differential graded module over $\mathcal{A}'$. The second equation because $f^{-1}j_! = j'_!(f')^{-1}$. The third equation by Lemma 24.19.2 applied to the ringed site $(\mathcal{C}', f^{-1}\mathcal{O})$ with sheaf of differential graded algebras $f^{-1}\mathcal{A}$ and with differential graded modules $(f')^{-1}\mathcal{A}_U$ on $\mathcal{C}'/U'$ and $\mathcal{A}'$ on $\mathcal{C}'$. The fourth equation holds because of course we have $(f')^{-1}\mathcal{A}_U = f^{-1}\mathcal{A}|_{U'}$. Hence we see that the pullback is another module of the same kind and we've proven conditions (1) and (2) for it above. $\square$

0FSC **Lemma 24.23.2.** *et $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $0 \to \mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to 0$ be an admissible short exact sequence of differential graded $\mathcal{A}$-modules. If two-out-of-three of these modules are good, so is the third.*

Proof. For condition (1) this is immediate as the sequence is a direct sum at the graded level. For condition (2) note that for any left differential graded $\mathcal{A}$-module, the sequence

$$0 \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N} \to 0$$

is an admissible short exact sequence of differential graded $\mathcal{O}$-modules (since forgetting the differential the tensor product is just taken in the category of graded modules). Hence if two out of three are exact as complexes of $\mathcal{O}$-modules, so is the third. Finally, the same argument shows that given a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras we have that

$$0 \to f^*\mathcal{P} \to f^*\mathcal{P}' \to f^*\mathcal{P}'' \to 0$$

is an admissible short exact sequence of differential graded $\mathcal{A}'$-modules and the same argument as above applies here. □

0FSD **Lemma 24.23.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. An arbitrary direct sum of good differential graded $\mathcal{A}$-modules is good. A filtered colimit of good differential graded $\mathcal{A}$-modules is good.*

Proof. Omitted. Hint: direct sums and filtered colimits commute with tensor products and with pullbacks. □

0FSE **Lemma 24.23.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. There exists a homomorphism $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules with the following properties*

1. *$\mathcal{P} \to \mathcal{M}$ is surjective,*
2. *$\mathrm{Ker}(d_{\mathcal{P}}) \to \mathrm{Ker}(d_{\mathcal{M}})$ is surjective, and*
3. *$\mathcal{P}$ is good.*

Proof. Consider triples (U, k, x) where U is an object of $\mathcal{C}$, $k \in \mathbf{Z}$, and x is a section of $\mathcal{M}^k$ over U with $\mathrm{d}_{\mathcal{M}}(x) = 0$. Then we obtain a unique morphism of differential graded $\mathcal{A}_U$-modules $\varphi_x : \mathcal{A}_U[-k] \to \mathcal{M}|_U$ mapping 1 to x. This is adjoint to a morphism $\psi_x : j_{U!}\mathcal{A}_U[-k] \to \mathcal{M}$. Observe that $1 \in \mathcal{A}_U(U)$ corresponds to a section $1 \in j_{U!}\mathcal{A}_U[-k](U)$ of degree k whose differential is zero and which is mapped to x by ψ_x. Thus if we consider the map

$$\bigoplus\nolimits_{(U,k,x)} j_{U!}\mathcal{A}_U[-k] \longrightarrow \mathcal{M}$$

then we will have conditions (2) and (3). Namely, the objects $j_{U!}\mathcal{A}_U[-k]$ are good (Lemma 24.23.1) and any direct sum of good objects is good (Lemma 24.23.3).

Next, consider triples (U, k, x) where U is an object of $\mathcal{C}$, $k \in \mathbf{Z}$, and x is a section of $\mathcal{M}^k$ (not necessarily annihilated by the differential). Then we can consider the cone C_U on the identity map $\mathcal{A}_U \to \mathcal{A}_U$ as in Remark 24.22.5. The element x will

determine a map $\varphi_x : C_U[-k-1] \to \mathcal{A}_U$, see Remark 24.22.5. Now, since we have an admissible short exact sequence

$$0 \to \mathcal{A}_U \to C_U \to \mathcal{A}_U[1] \to 0$$

we conclude that $j_{U!}C_U$ is a good module by Lemma 24.23.2 and the already used Lemma 24.23.1. As above we conclude that the direct sum of the maps ψ_x : $j_{U!}C_U \to \mathcal{M}$ adjoint to the φ_x

$$\bigoplus\nolimits_{(U,k,x)} j_{U!}C_U \longrightarrow \mathcal{M}$$

is surjective. Taking the direct sum with the map produced in the first paragraph we conclude. □

0FSF **Remark 24.23.5.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A *sheaf of graded sets* on $\mathcal{C}$ is a sheaf of sets $\mathcal{S}$ endowed with a map $\deg : \mathcal{S} \to \underline{\mathbf{Z}}$ of sheaves of sets. Let us denote $\mathcal{O}[\mathcal{S}]$ the graded $\mathcal{O}$-module which is the free $\mathcal{O}$-module on the graded sheaf of sets $\mathcal{S}$. More precisely, the nth graded part of $\mathcal{O}[\mathcal{S}]$ is the sheafification of the rule

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U),\ \deg(s)=n} s \cdot \mathcal{O}(U)$$

With zero differential we also may consider this as a differential graded $\mathcal{O}$-module. Let $\mathcal{A}$ be a sheaf of graded $\mathcal{O}$-algebras Then we similarly define $\mathcal{A}[\mathcal{S}]$ to be the graded $\mathcal{A}$-module whose nth graded part is the sheafification of the rule

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U)} s \cdot \mathcal{A}^{n-\deg(s)}(U)$$

If $\mathcal{A}$ is a differential graded $\mathcal{O}$-algebra, the we turn this into a differential graded $\mathcal{O}$-module by setting $\mathrm{d}(s) = 0$ for all $s \in \mathcal{S}(U)$ and sheafifying.

0FSG **Lemma 24.23.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a differential graded $\mathcal{A}$-algebra. Let $\mathcal{S}$ be a sheaf of graded sets on $\mathcal{C}$. Then the free graded module $\mathcal{A}[\mathcal{S}]$ on $\mathcal{S}$ endowed with differential as in Remark 24.23.5 is a good differential graded $\mathcal{A}$-module.*

Proof. Let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then we have

$$\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N} = \mathcal{O}[\mathcal{S}] \otimes_{\mathcal{O}} \mathcal{N} = \mathcal{N}[\mathcal{S}]$$

where $\mathcal{N}[\mathcal{S}$ is the graded $\mathcal{O}$-module whose degree n part is the sheaf associated to the presheaf

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U)} s \cdot \mathcal{N}^{n-\deg(s)}(U)$$

It is clear that $\mathcal{N} \to \mathcal{N}[\mathcal{S}]$ is an exact functor, hence $\mathcal{A}[\mathcal{S}$ is flat as a graded $\mathcal{A}$-module. Next, suppose that $\mathcal{N}$ is a differential graded left $\mathcal{A}$-module. Then we have

$$H^*(\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N}) = H^*(\mathcal{O}[\mathcal{S}] \otimes_{\mathcal{O}} \mathcal{N})$$

as graded sheaves of $\mathcal{O}$-modules, which by the flatness (over $\mathcal{O}$) is equal to

$$H^*(\mathcal{N})[\mathcal{S}]$$

as a graded $\mathcal{O}$-module. Hence if $\mathcal{N}$ is acyclic, then $\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic.

Finally, consider a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras. Then it is straightforward to see that

$$f^*\mathcal{A}[\mathcal{S}] = \mathcal{A}'[f^{-1}\mathcal{S}]$$

which finishes the proof that our module is good. □

0FSH **Lemma 24.23.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. There exists a homomorphism $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules with the following properties*

(1) *$\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism, and*
(2) *$\mathcal{P}$ is good.*

First proof. Let $\mathcal{S}_0$ be the sheaf of graded sets (Remark 24.23.5) whose degree n part is $\mathrm{Ker}(\mathrm{d}^n_{\mathcal{M}})$. Consider the homomorphism of differential graded modules

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{S}_0] \longrightarrow \mathcal{M}$$

where the left hand side is as in Remark 24.23.5 and the map sends a local section s of $\mathcal{S}_0$ to the corresponding local section of $\mathcal{M}^{\deg(s)}$ (which is in the kernel of the differential, so our map is a map of differential graded modules indeed). By construction the induced maps on cohomology sheaves $H^n(\mathcal{P}_0) \to H^n(\mathcal{M})$ are surjective. We are going to inductively construct maps

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \ldots \to \mathcal{M}$$

Observe that of course $H^*(\mathcal{P}_i) \to H^*(\mathcal{M})$ will be surjective for all i. Given $\mathcal{P}_i \to \mathcal{M}$ denote $\mathcal{S}_{i+1}$ the sheaf of graded sets whose degree n part is

$$\mathrm{Ker}(\mathrm{d}^{n+1}_{\mathcal{P}_i}) \times_{\mathcal{M}^{n+1},\mathrm{d}} \mathcal{M}^n$$

Then we set

$$\mathcal{P}_{i+1} = \mathcal{P}_i \oplus \mathcal{A}[\mathcal{S}_{i+1}]$$

as graded $\mathcal{A}$-module with differential and map to $\mathcal{M}$ defined as follows

(1) for local sections of $\mathcal{P}_i$ use the differential on $\mathcal{P}_i$ and the given map to $\mathcal{M}$,
(2) for a local section $s = (p, m)$ of $\mathcal{S}_{i+1}$ we set $\mathrm{d}(s)$ equal to p viewed as a section of $\mathcal{P}_i$ of degree $\deg(s) + 1$ and we map s to m in $\mathcal{M}$, and
(3) extend the differential uniquely so that the Leibniz rule holds.

This makes sense because $\mathrm{d}(m)$ is the image of p and $\mathrm{d}(p) = 0$. Finally, we set $\mathcal{P} = \mathrm{colim}\, \mathcal{P}_i$ with the induced map to $\mathcal{M}$.

The map $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism: we have $H^n(\mathcal{P}) = \mathrm{colim}\, H^n(\mathcal{P}_i)$ and for each i the map $H^n(\mathcal{P}_i) \to H^n(\mathcal{M})$ is surjective with kernel annihilated by the map $H^n(\mathcal{P}_i) \to H^n(\mathcal{P}_{i+1})$ by construction. Each $\mathcal{P}_i$ is good because $\mathcal{P}_0$ is good by Lemma 24.23.6 and each $\mathcal{P}_{i+1}$ is in the middle of the admissible short exact sequence $0 \to \mathcal{P}_i \to \mathcal{P}_{i+1} \to \mathcal{A}[\mathcal{S}_{i+1}] \to 0$ whose outer terms are good by induction. Hence $\mathcal{P}_{i+1}$ is good by Lemma 24.23.2. Finally, we conclude that $\mathcal{P}$ is good by Lemma 24.23.3. □

Second proof. We urge the reader to read the proof of Differential Graded Algebra, Lemma 22.20.4 before reading this proof. Set $\mathcal{M} = \mathcal{M}_0$. We inductively choose short exact sequences

$$0 \to \mathcal{M}_{i+1} \to \mathcal{P}_i \to \mathcal{M}_i \to 0$$

where the maps $\mathcal{P}_i \to \mathcal{M}_i$ are chosen as in Lemma 24.23.4. This gives a "resolution"

$$\ldots \to \mathcal{P}_2 \xrightarrow{f_2} \mathcal{P}_1 \xrightarrow{f_1} \mathcal{P}_0 \to \mathcal{M} \to 0$$

Then we let $\mathcal{P}$ be the differential graded $\mathcal{A}$-module defined as follows

(1) as a graded $\mathcal{A}$-module we set $\mathcal{P} = \bigoplus_{a \leq 0} \mathcal{P}_{-a}[-a]$, i.e., the degree n part is given by $\mathcal{P}^n = \bigoplus_{a+b=n} \mathcal{P}^b_{-a}$,
(2) the differential on $\mathcal{P}$ is as in the construction of the total complex associated to a double complex given by

$$\mathrm{d}_{\mathcal{P}}(x) = f_{-a}(x) + (-1)^a \mathrm{d}_{\mathcal{P}_{-a}}(x)$$

for x a local section of $\mathcal{P}^b_{-a}$.

With these conventions $\mathcal{P}$ is indeed a differential graded $\mathcal{A}$-module; we omit the details. There is a map $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules which is zero on the summands $\mathcal{P}_{-a}[-a]$ for $a < 0$ and the given map $\mathcal{P}_0 \to \mathcal{M}$ for $a = 0$. Observe that we have

$$\mathcal{P} = \mathrm{colim}_i F_i\mathcal{P}$$

where $F_i\mathcal{P} \subset \mathcal{P}$ is the differential graded $\mathcal{A}$-submodule whose underlying graded $\mathcal{A}$-module is

$$F_i\mathcal{P} = \bigoplus\nolimits_{i \geq -a \geq 0} \mathcal{P}_{-a}[-a]$$

It is immediate that the maps

$$0 \to F_1\mathcal{P} \to F_2\mathcal{P} \to F_3\mathcal{P} \to \ldots \to \mathcal{P}$$

are all admissible monomorphisms and we have admissible short exact sequences

$$0 \to F_i\mathcal{P} \to F_{i+1}\mathcal{P} \to \mathcal{P}_{i+1}[i+1] \to 0$$

By induction and Lemma 24.23.2 we find that $F_i\mathcal{P}$ is a good differential graded $\mathcal{A}$-module. Since $\mathcal{P} = \mathrm{colim}\, F_i\mathcal{P}$ we find that $\mathcal{P}$ is good by Lemma 24.23.3.

Finally, we have to show that $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism. If $\mathcal{C}$ has enough points, then this follows from the elementary Homology, Lemma 12.26.2 by checking on stalks. In general, we can argue as follows (this proof is far too long — there is an alternative argument by working with local sections as in the elementary proof but it is also rather long). Since filtered colimits are exact on the category of abelian sheaves, we have

$$H^d(\mathcal{P}) = \mathrm{colim}\, H^d(F_i\mathcal{P})$$

We claim that for each $i \geq 0$ and $d \in \mathbf{Z}$ we have (a) a short exact sequence

$$0 \to H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(\mathcal{M}) \to 0$$

where the second arrow comes from $F_i\mathcal{P} \to \mathcal{P} \to \mathcal{M}$ and (b) the composition

$$H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(F_{i+1}\mathcal{P})$$

is zero. It is clear that the claim suffices to finish the proof.

Proof of the claim. For any $i \geq 0$ there is a map $\mathcal{M}_{i+1}[i] \to F_i\mathcal{P}$ coming from the inclusion of $\mathcal{M}_{i+1}$ into $\mathcal{P}_i$ as the kernel of f_i. Consider the short exact sequence

$$0 \to \mathcal{M}_{i+1}[i] \to F_i\mathcal{P} \to C_i \to 0$$

of complexes of $\mathcal{O}$-modules defining C_i. Observe that $C_0 = \mathcal{M}_0 = \mathcal{M}$. Also, observe that C_i is the total complex associated to the double complex $C_i^{\bullet,\bullet}$ with columns

$$\mathcal{M}_i = \mathcal{P}_i/\mathcal{M}_{i+1}, \mathcal{P}_{i-1}, \ldots, \mathcal{P}_0$$

in degree $-i, -i+1, \ldots, 0$. There is a map of double complexes $C_i^{\bullet,\bullet} \to C_{i-1}^{\bullet,\bullet}$ which is 0 on the column in degree $-i$, is the surjection $\mathcal{P}_{i-1} \to \mathcal{M}_{i-1}$ in degree $-i+1$, and is the identity on the other columns. Hence there are maps of complexes

$$C_i \longrightarrow C_{i-1}$$

These maps are surjective quasi-isomorphisms because the kernel is the total complex on the double complex with columns $\mathcal{M}_i, \mathcal{M}_i$ in degrees $-i, -i+1$ and the identity map between these two columns. Using the resulting identifications $H^d(C_i) = H^d(C_{i-1} = \ldots = H^d(\mathcal{M})$ this already shows we get a long exact sequence

$$H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(\mathcal{M}) \to H^{d+1}(\mathcal{M}_{i+1}[i])$$

from the short exact sequence of complexes above. However, we also have the commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}_{i+2}[i+1] & \xrightarrow{a} & T_{i+1} & \longrightarrow & F_{i+1}\mathcal{P} & \longrightarrow & C_{i+1} \\ & & \uparrow{\scriptstyle b} & & \uparrow & & \downarrow \\ & & \mathcal{M}_{i+1}[i] & \longrightarrow & F_i\mathcal{P} & \longrightarrow & C_i \end{array}$$

where T_{i+1} is the total complex on the double complex with columns $\mathcal{P}_{i+1}, \mathcal{M}_{i+1}$ placed in degrees $-i-1$ and $-i$. In other words, T_{i+1} is a shift of the cone on the map $\mathcal{P}_{i+1} \to \mathcal{M}_{i+1}$ and we find that a is a quasi-isomorphism and the map $a^{-1} \circ b$ is a shift of the third map of the distinguished triangle in $D(\mathcal{O})$ associated to the short exact sequence

$$0 \to \mathcal{M}_{i+2} \to \mathcal{P}_{i+1} \to \mathcal{M}_{i+1} \to 0$$

The map $H^d(\mathcal{P}_{i+1}) \to H^d(\mathcal{M}_{i+1})$ is surjective because we chose our maps such that $\mathrm{Ker}(\mathrm{d}_{\mathcal{P}_{i+1}}) \to \mathrm{Ker}(\mathrm{d}_{\mathcal{M}_{i+1}})$ is surjective. Thus we see that $a^{-1} \circ b$ is zero on cohomology sheaves. This proves part (b) of the claim. Since T_{i+1} is the kernel of the surjective map of complexes $F_{i+1}\mathcal{P} \to C_i$ we find a map of long exact cohomology sequences

$$\begin{array}{ccccccc} H^d(T_{i+1}) & \longrightarrow & H^d(F_{i+1}\mathcal{P}) & \longrightarrow & H^d(\mathcal{M}) & \longrightarrow & H^{d+1}(T_{i+1}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^d(\mathcal{M}_{i+1}[i]) & \longrightarrow & H^d(F_i\mathcal{P}) & \longrightarrow & H^d(\mathcal{M}) & \longrightarrow & H^{d+1}(\mathcal{M}_{i+1}[i]) \end{array}$$

Here we know, by the discussion above, that the vertical maps on the outside are zero. Hence the maps $H^d(F_{i+1}\mathcal{P}) \to H^d(\mathcal{M})$ are surjective and part (a) of the claim follows. More precisely, the claim follows for $i > 0$ and we leave the claim for $i = 0$ to the reader (actually it suffices to prove the claim for all $i \gg 0$ in order to get the lemma). □

0FSI **Lemma 24.23.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{P}$ be a good acyclic right differential graded $\mathcal{A}$-module.*

1. *for any differential graded left $\mathcal{A}$-module $\mathcal{N}$ the tensor product $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic,*
2. *for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi and any differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$ and any map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras the pullback $f^*\mathcal{P}$ is acyclic and good.*

Proof. Proof of (1). By Lemma 24.23.7 we can choose a good left differential graded $\mathcal{Q}$ and a quasi-isomorphism $\mathcal{Q} \to \mathcal{N}$. Then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q}$ is acyclic because $\mathcal{Q}$ is good. Let $\mathcal{N}'$ be the cone on the map $\mathcal{Q} \to \mathcal{N}$. Then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$ is acyclic because $\mathcal{P}$ is good and because $\mathcal{N}'$ is acyclic (as the cone on a quasi-isomorphism). We have a distinguished triangle

$$\mathcal{Q} \to \mathcal{N} \to \mathcal{N}' \to \mathcal{Q}[1]$$

in $K(\mathit{Mod}(\mathcal{A}, \mathrm{d}))$ by our construction of the triangulated structure. Since $\mathcal{P} \otimes_{\mathcal{A}} -$ sends distinguished triangles to distinguished triangles, we obtain a distinguished triangle

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}' \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q}[1]$$

in $K(\mathit{Mod}(\mathcal{O}))$. Thus we conclude.

Proof of (2). Observe that $f^*\mathcal{P}$ is good by our definition of good modules. Recall that $f^*\mathcal{P} = f^{-1}\mathcal{P} \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$. Then $f^{-1}\mathcal{P}$ is a good acyclic (because f^{-1} is exact) differential graded $f^{-1}\mathcal{A}$-module. Hence we see that $f^*\mathcal{P}$ is acyclic by part (1). □

24.24. The differential graded hull of a graded module

0FSJ The differential graded hull of a graded module $\mathcal{N}$ is the result of applying the functor G in the following lemma.

0FSK **Lemma 24.24.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The forgetful functor $F : \mathit{Mod}(\mathcal{A}, d) \to \mathit{Mod}(\mathcal{A})$ has a left adjoint $G : \mathit{Mod}(\mathcal{A}) \to \mathit{Mod}(\mathcal{A}, d)$.*

Proof. To prove the existence of G we can use the adjoint functor theorem, see Categories, Theorem 4.25.3 (observe that we have switched the roles of F and G). The exactness conditions on F are satisfied by Lemma 24.13.2. The set theoretic condition can be seen as follows: suppose given a graded $\mathcal{A}$-module $\mathcal{N}$. Then for any map

$$\varphi : \mathcal{N} \longrightarrow F(\mathcal{M})$$

we can consider the smallest differential graded $\mathcal{A}$-submodule $\mathcal{M}' \subset \mathcal{M}$ with $\mathrm{Im}(\varphi) \subset F(\mathcal{M}')$. It is clear that $\mathcal{M}'$ is the image of the map of graded $\mathcal{A}$-modules

$$\mathcal{N} \oplus \mathcal{N}[-1] \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{M}$$

defined by

$$(n, \sum n_i \otimes a_i) \longmapsto \varphi(n) + \sum \mathrm{d}(\varphi(n_i))a_i$$

because the image of this map is easily seen to be a differential graded submodule of $\mathcal{M}$. Thus the number of possible isomorphism classes of these $\mathcal{M}'$ is bounded and we conclude. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module and suppose we have a short exact sequence

$$0 \to \mathcal{N} \to F(\mathcal{M}) \to \mathcal{N}' \to 0$$

in $\mathit{Mod}(\mathcal{A})$. Then we obtain a canonical graded $\mathcal{A}$-module homomorphism

$$\overline{\mathrm{d}} : \mathcal{N} \to \mathcal{N}'[1]$$

as follows: given a local section x of $\mathcal{N}$ denote $\overline{\mathrm{d}}(x)$ the image in $\mathcal{N}'$ of $\mathrm{d}_{\mathcal{M}}(x)$ when x is viewed as a local section of $\mathcal{M}$.

0FSL **Lemma 24.24.2.** *The functors F, G of Lemma 24.24.1 have the following properties. Given a graded $\mathcal{A}$-module $\mathcal{N}$ we have*

(1) *the counit $\mathcal{N} \to F(G(\mathcal{N}))$ is injective,*
(2) *the map $\overline{d} : \mathcal{N} \to \mathrm{Coker}(\mathcal{N} \to F(G(\mathcal{N})))[1]$ is an isomorphism, and*
(3) *$G(\mathcal{N})$ is an acyclic differential graded $\mathcal{A}$-module.*

Proof. We observe that property (3) is a consequence of properties (1) and (2). Namely, if s is a nonzero local section of $F(G(\mathcal{N}))$ with $\mathrm{d}(s) = 0$, then s cannot be in the image of $\mathcal{N} \to F(G(\mathcal{N}))$. Hence we can write the image $\overline{s}$ of s in the cokernel as $\overline{\mathrm{d}}(s')$ for some local section s' of $\mathcal{N}$. Then we see that $s = \mathrm{d}(s')$ because the difference $s - \mathrm{d}(s')$ is still in the kernel of d and is contained in the image of the counit.

Let us write temporarily $\mathcal{A}_{gr}$, respectively $\mathcal{A}_{dg}$ the sheaf $\mathcal{A}$ viewed as a (right) graded module over itself, respectively as a (right) differential graded module over itself. The most important case of the lemma is to understand what is $G(\mathcal{A}_{gr})$. Of course $G(\mathcal{A}_{gr})$ is the object of $Mod(\mathcal{A}, \mathrm{d})$ representing the functor

$$\mathcal{M} \longmapsto \mathrm{Hom}_{Mod(\mathcal{A})}(\mathcal{A}_{gr}, F(\mathcal{M})) = \Gamma(\mathcal{C}, \mathcal{M})$$

By Remark 24.22.5 we see that this functor represented by $C[-1]$ where C is the cone on the identity of $\mathcal{A}_{dg}$. We have a short exact sequence

$$0 \to \mathcal{A}_{dg}[-1] \to C[-1] \to \mathcal{A}_{dg} \to 0$$

in $Mod(\mathcal{A}, \mathrm{d})$ which is split by the counit $\mathcal{A}_{gr} \to F(C[-1])$ in $Mod(\mathcal{A})$. Thus $G(\mathcal{A}_{gr})$ satisfies properties (1) and (2).

Let U be an object of $\mathcal{C}$. Denote $j_U : \mathcal{C}/U \to \mathcal{C}$ the localization morphism. Denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to U. We will use the notation $\mathcal{A}_{U,gr}$ to denote $\mathcal{A}_U$ viewed as a graded $\mathcal{A}_U$-module. Denote $F_U : Mod(\mathcal{A}_U, \mathrm{d}) \to Mod(\mathcal{A}_U)$ the forgetful functor and denote G_U its adjoint. Then we have the commutative diagrams

$$\begin{array}{ccc} Mod(\mathcal{A}, \mathrm{d}) & \xrightarrow{F} & Mod(\mathcal{A}) \\ {\scriptstyle j_U^*}\downarrow & & \downarrow{\scriptstyle j_U^*} \\ Mod(\mathcal{A}_U, \mathrm{d}) & \xrightarrow{F_U} & Mod(\mathcal{A}_U) \end{array} \quad \text{and} \quad \begin{array}{ccc} Mod(\mathcal{A}_U, \mathrm{d}) & \xrightarrow{F_U} & Mod(\mathcal{A}_U) \\ {\scriptstyle j_{U!}}\downarrow & & \downarrow{\scriptstyle j_{U!}} \\ Mod(\mathcal{A}, \mathrm{d}) & \xrightarrow{F} & Mod(\mathcal{A}) \end{array}$$

by the construction of j_U^* and $j_{U!}$ in Sections 24.9, 24.18, 24.10, and 24.19. By uniqueness of adjoints we obtain $j_{U!} \circ G_U = G \circ j_{U!}$. Since $j_{U!}$ is an exact functor, we see that the properties (1) and (2) for the counit $\mathcal{A}_{U,gr} \to F_U(G_U(\mathcal{A}_{U,gr}))$ which we've seen in the previous part of the proof imply properties (1) and (2) for the counit $j_{U!}\mathcal{A}_{U,gr} \to F(G(j_{U!}\mathcal{A}_{U,gr})) = j_{U!}F_U(G_U(\mathcal{A}_{U,gr}))$.

In the proof of Lemma 24.11.1 we have seen that any object of $Mod(\mathcal{A})$ is a quotient of a direct sum of copies of $j_{U!}\mathcal{A}_{U,gr}$. Since G is a left adjoint, we see that G commutes with direct sums. Thus properties (1) and (2) hold for direct sums of objects for which they hold. Thus we see that every object $\mathcal{N}$ of $Mod(\mathcal{A})$ fits into an exact sequence

$$\mathcal{N}_1 \to \mathcal{N}_0 \to \mathcal{N} \to 0$$

such that (1) and (2) hold for $\mathcal{N}_1$ and $\mathcal{N}_0$. We leave it to the reader to deduce (1) and (2) for $\mathcal{N}$ using that G is right exact. □

24.25. K-injective differential graded modules

0FSM This section is the analogue of Injectives, Section 19.12 in the setting of sheaves of differential graded modules over a sheaf of differential graded algebras.

0FSN **Lemma 24.25.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set T and for each $t \in T$ an injective map $\mathcal{N}_t \to \mathcal{N}'_t$ of graded $\mathcal{A}$-modules such that an object $\mathcal{I}$ of $Mod(\mathcal{A})$ is injective if and only if for every solid diagram*

$$\begin{array}{ccc} \mathcal{N}_t & \longrightarrow & \mathcal{I} \\ \downarrow & \nearrow & \\ \mathcal{N}'_t & & \end{array}$$

a dotted arrow exists in $Mod(\mathcal{A})$ making the diagram commute.

Proof. This is true in any Grothendieck abelian category, see Injectives, Lemma 19.11.6. By Lemma 24.11.1 the category $Mod(\mathcal{A})$ is a Grothendieck abelian category. □

0FSP **Definition 24.25.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A diffential graded $\mathcal{A}$-module $\mathcal{I}$ is said to be *graded injective*[2] if $\mathcal{M}$ viewed as a graded $\mathcal{A}$-module is an injective object of the category $Mod(\mathcal{A})$ of graded $\mathcal{A}$-modules.

0FSQ **Remark 24.25.3.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a graded injective diffential graded $\mathcal{A}$-module. Let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence of differential graded $\mathcal{A}$-modules. Since $\mathcal{I}$ is graded injective we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M}_3, \mathcal{I}) \to \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M}_2, \mathcal{I}) \to \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M}_1, \mathcal{I}) \to 0$$

of $\Gamma(\mathcal{C}, \mathcal{O})$-modules. Taking cohomology we obtain a long exact sequence

$$\begin{array}{ccc} \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_3, \mathcal{I}) & & \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_3, \mathcal{I})[1] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_2, \mathcal{I}) & & \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_2, \mathcal{I})[1] \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_1, \mathcal{I}) & & \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}_1, \mathcal{I})[1] \end{array}$$

of groups of homomorphisms in the homotopy category. The point is that we get this even though we didn't assume that our short exact sequence is admissible (so the short exact sequence in general does not define a distinguished triangle in the homotopy category).

[2] This may be nonstandard terminology.

0FSR **Lemma 24.25.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let T be a set and for each $t \in T$ let $\mathcal{I}_t$ be a graded injective diffential graded $\mathcal{A}$-module. Then $\prod \mathcal{I}_t$ is a graded injective differential graded $\mathcal{A}$-module.*

Proof. This is true because products of injectives are injectives, see Homology, Lemma 12.27.3, and because products in $Mod(\mathcal{A}, \mathrm{d})$ are compatible with products in $Mod(\mathcal{A})$ via the forgetful functor. □

0FSS **Lemma 24.25.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set T and for each $t \in T$ an injective map $\mathcal{M}_t \to \mathcal{M}'_t$ of acyclic differential graded $\mathcal{A}$-modules such that for an object $\mathcal{I}$ of $Mod(\mathcal{A}, d)$ the following are equivalent*

(1) *$\mathcal{I}$ is graded injective, and*
(2) *for every solid diagram*

$$\begin{array}{ccc} \mathcal{M}_t & \longrightarrow & \mathcal{I} \\ \downarrow & \nearrow & \\ \mathcal{M}'_t & & \end{array}$$

a dotted arrow exists in $Mod(\mathcal{A}, d)$ making the diagram commute.

Proof. Let T and $\mathcal{N}_t \to \mathcal{N}'_t$ be as in Lemma 24.25.1. Denote $F : Mod(\mathcal{A}, \mathrm{d}) \to Mod(\mathcal{A})$ the forgetful functor. Let G be the left adjoint functor to F as in Lemma 24.24.1. Set

$$\mathcal{M}_t = G(\mathcal{N}_t) \to G(\mathcal{N}'_t) = \mathcal{M}'_t$$

This is an injective map of acyclic differential graded $\mathcal{A}$-modules by Lemma 24.24.2. Since G is the left adjoint to F we see that there exists a dotted arrow in the diagram

$$\begin{array}{ccc} \mathcal{M}_t & \longrightarrow & \mathcal{I} \\ \downarrow & \nearrow & \\ \mathcal{M}'_t & & \end{array}$$

if and only if there exists a dotted arrow in the diagram

$$\begin{array}{ccc} \mathcal{N}_t & \longrightarrow & F(\mathcal{I}) \\ \downarrow & \nearrow & \\ \mathcal{N}'_t & & \end{array}$$

Hence the result follows from the choice of our collection of arrows $\mathcal{N}_t \to \mathcal{N}'_t$. □

0FST **Lemma 24.25.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set S and for each s an acyclic differential graded $\mathcal{A}$-module $\mathcal{M}_s$ such that for every nonzero acyclic differential graded $\mathcal{A}$-module $\mathcal{M}$ there is an $s \in S$ and an injective map $\mathcal{M}_s \to \mathcal{M}$ in $Mod(\mathcal{A}, d)$.*

Proof. Before we start recall that our conventions guarantee the site $\mathcal{C}$ has a set of objects and morphisms and a set $\mathrm{Cov}(\mathcal{C})$ of coverings. If $\mathcal{F}$ is a differential graded

$\mathcal{A}$-module, let us define $|\mathcal{F}|$ to be the sum of the cardinality of

$$\coprod_{(U,n)} \mathcal{F}^n(U)$$

as U ranges over the objects of $\mathcal{C}$ and $n \in \mathbf{Z}$. Choose an infinite cardinal κ bigger than the cardinals $|\operatorname{Ob}(\mathcal{C})|$, $|\operatorname{Arrows}(\mathcal{C})|$, $|\operatorname{Cov}(\mathcal{C})|$, $\sup|I|$ for $\{U_i \to U\}_{i\in I} \in \operatorname{Cov}(\mathcal{C})$, and $|\mathcal{A}|$.

Let $\mathcal{F} \subset \mathcal{M}$ be an inclusion of differential graded $\mathcal{A}$-modules. Suppose given a set K and for each $k \in K$ a triple (U_k, n_k, x_k) consisting of an object U_k of $\mathcal{C}$, integer n_k, and a section $x_k \in \mathcal{M}^{n_k}(U_k)$. Then we can consider the smallest differential graded $\mathcal{A}$-submodule $\mathcal{F}' \subset \mathcal{M}$ containing $\mathcal{F}$ and the sections x_k for $k \in K$. We can describe

$$(\mathcal{F}')^n(U) \subset \mathcal{M}^n(U)$$

as the set of elements $x \in \mathcal{M}^n(U)$ such that there exists $\{f_i : U_i \to U\}_{i\in I} \in \operatorname{Cov}(\mathcal{C})$ such that for each $i \in I$ there is a finite set T_i and morphisms $g_t : U_i \to U_{k_t}$

$$f_i^* x = y_i + \sum\nolimits_{t\in T_i} a_{it} g_t^* x_{k_t} + b_{it} g_t^* \mathrm{d}(x_{k_t})$$

for some section $y_i \in \mathcal{F}^n(U)$ and sections $a_{it} \in \mathcal{A}^{n-n_{k_t}}(U_i)$ and $b_{it} \in \mathcal{A}^{n-n_{k_t}-1}(U_i)$. (Details omitted; hints: these sections are certainly in $\mathcal{F}'$ and you show conversely that this rule defines a differential graded $\mathcal{A}$-submodule.) It follows from this description that $|\mathcal{F}'| \leq \max(|\mathcal{F}|, |K|, \kappa)$.

Let $\mathcal{M}$ be a nonzero acyclic differential graded $\mathcal{A}$-module. Then we can find an integer n and a nonzero section x of $\mathcal{M}^n$ over some object U of $\mathcal{C}$. Let

$$\mathcal{F}_0 \subset \mathcal{M}$$

be the smallest differential graded $\mathcal{A}$-submodule containing x. By the previous paragraph we have $|\mathcal{F}_0| \leq \kappa$. By induction, given $\mathcal{F}_0, \ldots, \mathcal{F}_n$ define $\mathcal{F}_{n+1}$ as follows. Consider the set

$$L = \{(U, n, x)\}\{U_i \to U\}_{i\in I}, (x_i)_{i\in I})\}$$

of triples where U is an object of $\mathcal{C}$, $n \in \mathbf{Z}$, and $x \in \mathcal{F}_n(U)$ with $\mathrm{d}(x) = 0$. Since $\mathcal{M}$ is acyclic for each triple $l = (U_l, n_l, x_l) \in L$ we can choose $\{(U_{l,i} \to U_l\}_{i\in I_l} \in \operatorname{Cov}(\mathcal{C})$ and $x_{l,i} \in \mathcal{M}^{n_l-1}(U_{l,i})$ such that $\mathrm{d}(x_{l,i}) = x|_{U_{l,i}}$. Then we set

$$K = \{(U_{l,i}, n_l - 1, x_{l,i}) \mid l \in L, i \in I_l\}$$

and we let $\mathcal{F}_{n+1}$ be the smallest differential graded $\mathcal{A}$-submodule of $\mathcal{M}$ containing $\mathcal{F}_n$ and the sections $x_{l,i}$. Since $|K| \leq \max(\kappa, |\mathcal{F}_n|)$ we conclude that $|\mathcal{F}_{n+1}| \leq \kappa$ by induction.

By construction the inclusion $\mathcal{F}_n \to \mathcal{F}_{n+1}$ induces the zero map on cohomology sheaves. Hence we see that $\mathcal{F} = \bigcup \mathcal{F}_n$ is a nonzero acyclic submodule with $|\mathcal{F}| \leq \kappa$. Since there is only a set of isomorphism classes of differential graded $\mathcal{A}$-modules $\mathcal{F}$ with $|\mathcal{F}|$ bounded, we conclude. □

0FSU **Definition 24.25.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A diffential graded $\mathcal{A}$-module $\mathcal{I}$ is *K-injective* if for every acyclic differential graded $\mathcal{M}$ we have

$$\operatorname{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}, \mathcal{I}) = 0$$

Please note the similarity with Derived Categories, Definition 13.31.1.

0FSV **Lemma 24.25.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let T be a set and for each $t \in T$ let $\mathcal{I}_t$ be a K-injective diffential graded $\mathcal{A}$-module. Then $\prod \mathcal{I}_t$ is a K-injective differential graded $\mathcal{A}$-module.*

Proof. Let $\mathcal{K}$ be an acyclic differential graded $\mathcal{A}$-module. Then we have

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}, \prod\nolimits_{t\in T} \mathcal{I}_t) = \prod\nolimits_{t\in T} \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}, \mathcal{I}_t)$$

because taking products in $Mod(\mathcal{A}, \mathrm{d})$ commutes with the forgetful functor to graded $\mathcal{A}$-modules. Since taking products is an exact functor on the category of abelian groups we conclude. □

0FSW **Lemma 24.25.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a K-injective and graded injective object of $Mod(\mathcal{A}, d)$. For every solid diagram in $Mod(\mathcal{A}, d)$*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a} & \mathcal{I} \\ {\scriptstyle b}\downarrow & \nearrow & \\ \mathcal{M}' & & \end{array}$$

where b is injective and $\mathcal{M}$ is acyclic a dotted arrow exists making the diagram commute.

Proof. Since $\mathcal{M}$ is acyclic and $\mathcal{I}$ is K-injective, there exists a graded $\mathcal{A}$-module map $h : \mathcal{M} \to \mathcal{I}$ of degree -1 such that $a = \mathrm{d}(h)$. Since $\mathcal{I}$ is graded injective and b is injective, there exists a graded $\mathcal{A}$-module map $h' : \mathcal{M}' \to \mathcal{I}$ of degree -1 such that $h = h' \circ b$. Then we can take $a' = \mathrm{d}(h')$ as the dotted arrow. □

0FSX **Lemma 24.25.10.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a K-injective and graded injective object of $Mod(\mathcal{A}, d)$. For every solid diagram in $Mod(\mathcal{A}, d)$*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a} & \mathcal{I} \\ {\scriptstyle b}\downarrow & \nearrow & \\ \mathcal{M}' & & \end{array}$$

where b is a quasi-isomorphism a dotted arrow exists making the diagram commute up to homotopy.

Proof. After replacing $\mathcal{M}'$ by the direct sum of $\mathcal{M}'$ and the cone on the identity on $\mathcal{M}$ (which is acyclic) we may assume b is also injective. Then the cokernel $\mathcal{Q}$ of b is acyclic. Thus we see that

$$\mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{Q}, \mathcal{I}) = \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{Q}, \mathcal{I})[1] = 0$$

as $\mathcal{I}$ is K-injective. As $\mathcal{I}$ is graded injective by Remark 24.25.3 we see that

$$\mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}', \mathcal{I}) \longrightarrow \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}, \mathcal{I})$$

is bijective and the proof is complete. □

0FSY **Lemma 24.25.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set R and for each $r \in R$ an injective map $\mathcal{M}_r \to \mathcal{M}'_r$ of acyclic differential graded $\mathcal{A}$-modules such that for an object $\mathcal{I}$ of $Mod(\mathcal{A}, d)$ the following are equivalent*

(1) *$\mathcal{I}$ is K-injective and graded injective, and*
(2) *for every solid diagram*

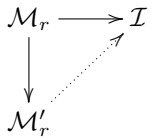

a dotted arrow exists in $Mod(\mathcal{A}, d)$ making the diagram commute.

Proof. Let T and $\mathcal{M}_t \to \mathcal{M}'_t$ be as in Lemma 24.25.5. Let S and $\mathcal{M}_s$ be as in Lemma 24.25.6. Choose an injective map $\mathcal{M}_s \to \mathcal{M}'_s$ of acyclic differential graded $\mathcal{A}$-modules which is homotopic to zero. This is possible because we may take $\mathcal{M}'_s$ to be the cone on the identity; in that case it is even true that the identity on $\mathcal{M}'_s$ is homotopic to zero, see Differential Graded Algebra, Lemma 22.27.4 which applies by the discussion in Section 24.22. We claim that $R = T \coprod S$ with the given maps works.

The implication (1) $\Rightarrow$ (2) holds by Lemma 24.25.9.

Assume (2). First, by Lemma 24.25.5 we see that $\mathcal{I}$ is graded injective. Next, let $\mathcal{M}$ be an acyclic differential graded $\mathcal{A}$-module. We have to show that

$$\mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}, \mathcal{I}) = 0$$

The proof will be exactly the same as the proof of Injectives, Lemma 19.12.3.

We are going to construct by induction on the ordinal α an acyclic differential graded submodule $\mathcal{K}_\alpha \subset \mathcal{M}$ as follows. For $\alpha = 0$ we set $\mathcal{K}_0 = 0$. For $\alpha > 0$ we proceed as follows:

(1) If $\alpha = \beta + 1$ and $\mathcal{K}_\beta = \mathcal{M}$ then we choose $\mathcal{K}_\alpha = \mathcal{K}_\beta$.
(2) If $\alpha = \beta + 1$ and $\mathcal{K}_\beta \neq \mathcal{M}$ then $\mathcal{M}/\mathcal{K}_\beta$ is a nonzero acyclic differential graded $\mathcal{A}$-module. We choose a differential graded $\mathcal{A}$ submodule $\mathcal{N}_\alpha \subset \mathcal{M}/\mathcal{K}_\beta$ isomorphic to $\mathcal{M}_s$ for some $s \in S$, see Lemma 24.25.6. Finally, we let $\mathcal{K}_\alpha \subset \mathcal{M}$ be the inverse image of $\mathcal{N}_\alpha$.
(3) If α is a limit ordinal we set $\mathcal{K}_\beta = \mathrm{colim}\, \mathcal{K}_\alpha$.

It is clear that $\mathcal{M} = \mathcal{K}_\alpha$ for a suitably large ordinal α. We will prove that

$$\mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{K}_\alpha, \mathcal{I})$$

is zero by transfinite induction on α. It holds for $\alpha = 0$ since $\mathcal{K}_0$ is zero. Suppose it holds for β and $\alpha = \beta + 1$. In case (1) of the list above the result is clear. In case (2) there is a short exact sequence

$$0 \to \mathcal{K}_\beta \to \mathcal{K}_\alpha \to \mathcal{N}_\alpha \to 0$$

By Remark 24.25.3 and since we've seen that $\mathcal{I}$ is graded injective, we obtain an exact sequence

$$\mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{K}_\beta, \mathcal{I}) \to \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{K}_\alpha, \mathcal{I}) \to \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{N}_\alpha, \mathcal{I})$$

By induction the term on the left is zero. By assumption (2) the term on the right is zero: any map $\mathcal{M}_s \to \mathcal{I}$ factors through $\mathcal{M}'_s$ and hence is homotopic to zero. Thus the middle group is zero too. Finally, suppose that α is a limit ordinal. Because we also have $\mathcal{K}_\alpha = \mathrm{colim}\, \mathcal{K}_\alpha$ as graded $\mathcal{A}$-modules we see that

$$\mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}_\alpha, \mathcal{I}) = \lim\nolimits_{\beta<\alpha} \mathrm{Hom}_{Mod^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{K}_\beta, \mathcal{I})$$

as complexes of abelian groups. The cohomology groups of these complexes compute morphisms in $K(Mod(\mathcal{A}, \mathrm{d}))$ between shifts. The transition maps in the system of complexes are surjective by Remark 24.25.3 because $\mathcal{I}$ is graded injective. Moreover, for a limit ordinal $\beta \leq \alpha$ we have equality of limit and value. Thus we may apply Homology, Lemma 12.31.8 to conclude. □

0FSZ **Lemma 24.25.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let R be a set and for each $r \in R$ let an injective map $\mathcal{M}_r \to \mathcal{M}'_r$ of acyclic differential graded $\mathcal{A}$-modules be given. There exists a functor $M : Mod(\mathcal{A}, d) \to Mod(\mathcal{A}, d)$ and a natural transformation $j : id \to M$ such that*

(1) *$j_{\mathcal{M}} : \mathcal{M} \to M(\mathcal{M})$ is injective and a quasi-isomorphism,*
(2) *for every solid diagram*

$$\begin{array}{ccc} \mathcal{M}_r & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle j_{\mathcal{M}}} \\ \mathcal{M}'_r & \dashrightarrow & M(\mathcal{M}) \end{array}$$

a dotted arrow exists in $Mod(\mathcal{A}, d)$ making the diagram commute.

Proof. We define $M(\mathcal{M})$ as the pushout in the following diagram

$$\begin{array}{ccc} \bigoplus_{(r,\varphi)} \mathcal{M}_r & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \bigoplus_{(r,\varphi)} \mathcal{M}'_r & \longrightarrow & M(\mathcal{M}) \end{array}$$

where the direct sum is over all pairs (r, φ) with $r \in R$ and $\varphi \in \mathrm{Hom}_{Mod(\mathcal{A},\mathrm{d})}(\mathcal{M}_r, \mathcal{M})$. Since the pushout of an injective map is injective, we see that $\mathcal{M} \to M(\mathcal{M})$ is injective. Since the cokernel of the left vertical arrow is acyclic, we see that the (isomorphic) cokernel of $\mathcal{M} \to M(\mathcal{M})$ is acyclic, hence $\mathcal{M} \to M(\mathcal{M})$ is a quasi-isomorphism. Property (2) holds by construction. We omit the verification that this procedure can be turned into a functor. □

0FT0 **Theorem 24.25.13.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. For every differential graded $\mathcal{A}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{M} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module. Moreover, the construction is functorial in $\mathcal{M}$.*

Proof. Let R and $\mathcal{M}_r \to \mathcal{M}'_r$ be a set of morphisms of $Mod(\mathcal{A}, \mathrm{d})$ found in Lemma 24.25.11. Let M with transformation $\mathrm{id} \to M$ be as constructed in Lemma 24.25.12 using R and $\mathcal{M}_r \to \mathcal{M}'_r$. Using transfinite recursion we define a sequence of functors M_α and natural transformations $M_\beta \to M_\alpha$ for $\alpha < \beta$ by setting

(1) $M_0 = \mathrm{id}$,
(2) $M_{\alpha+1} = M \circ M_\alpha$ with natural transformation $M_\beta \to M_{\alpha+1}$ for $\beta < \alpha + 1$ coming from the already constructed $M_\beta \to M_\alpha$ and the maps $M_\alpha \to M \circ M_\alpha$ coming from $\mathrm{id} \to M$, and
(3) $M_\alpha = \mathrm{colim}_{\beta<\alpha} M_\beta$ if α is a limit ordinal with the coprojections as transformations $M_\beta \to M_\alpha$ for $\alpha < \beta$.

Observe that for every differential graded $\mathcal{A}$-module the maps $\mathcal{M} \to M_\beta(\mathcal{M}) \to M_\alpha(\mathcal{M})$ are injective quasi-isomorphisms (as filtered colimits are exact).

Recall that $Mod(\mathcal{A}, \mathrm{d})$ is a Grothendieck abelian category. Thus by Injectives, Proposition 19.11.5 (applied to the direct sum of $\mathcal{M}_r$ for all $r \in R$) there is a limit ordinal α such that $\mathcal{M}_r$ is α-small with respect to injections for every $r \in R$. We claim that $\mathcal{M} \to M_\alpha(\mathcal{M})$ is the desired functorial embedding of $\mathcal{M}$ into a graded injective K-injective module.

Namely, any map $\mathcal{M}_r \to M_\alpha(\mathcal{M})$ factors through $M_\beta(\mathcal{M})$ for some $\beta < \alpha$. However, by the construction of M we see that this means that $\mathcal{M}_r \to M_{\beta+1}(\mathcal{M}) = M(M_\beta(\mathcal{M}))$ factors through $\mathcal{M}'_r$. Since $M_\beta(\mathcal{M}) \subset M_{\beta+1}(\mathcal{M}) \subset M_\alpha(\mathcal{M})$ we get the desired factorizaton into $M_\alpha(\mathcal{M})$. We conclude by our choice of R and $\mathcal{M}_r \to \mathcal{M}'_r$ in Lemma 24.25.11. □

24.26. The derived category

0FT1 This section is the analogue of Differential Graded Algebra, Section 22.22.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct the derived category $D(\mathcal{A}, \mathrm{d})$ by inverting the quasi-isomorphisms in $K(Mod(\mathcal{A}, \mathrm{d}))$.

0FT2 **Lemma 24.26.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The functor $H^0 : Mod(\mathcal{A}, d) \to Mod(\mathcal{O})$ of Section 24.13 factors through a functor*

$$H^0 : K(Mod(\mathcal{A}, d)) \to Mod(\mathcal{O})$$

which is homological in the sense of Derived Categories, Definition 13.3.5.

Proof. It follows immediately from the definitions that there is a commutative diagram

$$\begin{array}{ccc} Mod(\mathcal{A}, \mathrm{d}) & \longrightarrow & K(Mod(\mathcal{A}, \mathrm{d})) \\ \downarrow & & \downarrow \\ \mathrm{Comp}(\mathcal{O}) & \longrightarrow & K(Mod(\mathcal{O})) \end{array}$$

Since $H^0(\mathcal{M})$ is defined as the zeroth cohomology sheaf of the underlying complex of $\mathcal{O}$-modules of $\mathcal{M}$ the lemma follows from the case of complexes of $\mathcal{O}$-modules which is a special case of Derived Categories, Lemma 13.11.1. □

0FT3 **Lemma 24.26.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The full subcategory Ac of the homotopy category $K(Mod(\mathcal{A}, d))$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(Mod(\mathcal{A}, d))$.*

Proof. Of course an object $\mathcal{M}$ of $K(Mod(\mathcal{A}, \mathrm{d}))$ is in Ac if and only if $H^i(\mathcal{M}) = H^0(\mathcal{M}[i])$ is zero for all i. The lemma follows from this, Lemma 24.26.1, and Derived Categories, Lemma 13.6.3. See also Derived Categories, Definitions 13.6.1 and 13.3.4 and Lemma 13.4.16. □

0FT4 **Lemma 24.26.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Consider the subclass $Qis \subset Arrows(K(Mod(\mathcal{A}, d)))$ consisting of quasi-isomorphisms. This is a saturated multiplicative system compatible with the triangulated structure on $K(Mod(\mathcal{A}, d))$.*

Proof. Observe that if $f, g : \mathcal{M} \to \mathcal{N}$ are morphisms of $Mod(\mathcal{A}, \mathrm{d})$ which are homotopic, then f is a quasi-isomorphism if and only if g is a quasi-isomorphism. Namely, the maps $H^i(f) = H^0(f[i])$ and $H^i(g) = H^0(g[i])$ are the same by Lemma 24.26.1. Thus it is unambiguous to say that a morphism of the homotopy category $K(Mod(\mathcal{A}, \mathrm{d}))$ is a quasi-isomorphism. For definitions of "multiplicative system", "saturated", and "compatible with the triangulated structure" see Derived Categories, Definition 13.5.1 and Categories, Definitions 4.27.1 and 4.27.20.

To actually prove the lemma consider the composition of exact functors of triangulated categories

$$K(Mod(\mathcal{A}, \mathrm{d})) \longrightarrow K(Mod(\mathcal{O})) \longrightarrow D(\mathcal{O})$$

and observe that a morphism $f : \mathcal{M} \to \mathcal{N}$ of $K(Mod(\mathcal{A}, \mathrm{d}))$ is in Qis if and only if it maps to an isomorphism in $D(\mathcal{O})$. Thus the lemma follows from Derived Categories, Lemma 13.5.4. □

In the situation of Lemma 24.26.3 we can apply Derived Categories, Proposition 13.5.6 to obtain an exact functor of triangulated categories

$$Q : K(Mod(\mathcal{A}, \mathrm{d})) \longrightarrow \mathrm{Qis}^{-1} K(Mod(\mathcal{A}, \mathrm{d}))$$

However, as $Mod(\mathcal{A}, \mathrm{d})$ is a "big" category, i.e., its objects form a proper class, it isn't immediately clear that given $\mathcal{M}$ and $\mathcal{N}$ the construction of $\mathrm{Qis}^{-1} K(Mod(\mathcal{A}, \mathrm{d}))$ produces a *set*

$$\mathrm{Mor}_{\mathrm{Qis}^{-1} K(Mod(\mathcal{A}, \mathrm{d}))}(\mathcal{M}, \mathcal{N})$$

of morphisms. However, this is true thanks to our construction of K-injective complexes. Namely, by Theorem 24.25.13 we can choose a quasi-isomorphism $s_0 : \mathcal{N} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module. Next, recall that elements of the displayed set are equivalence classes of pairs $(f : \mathcal{M} \to \mathcal{N}', s : \mathcal{N} \to \mathcal{N}')$ where f is an arbitrary morphism of $K(Mod(\mathcal{A}, \mathrm{d}))$ and s is a quasi-isomorphsm, see the description of the left calculus of fractions in Categories, Section 4.27. By Lemma 24.25.10 we can choose the dotted arrow

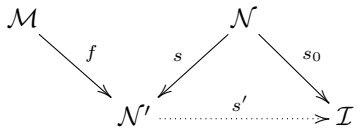

making the diagram commute (in the homotopy category). Thus the pair (f, s) is equivalent to the pair $(s' \circ f, s_0)$ and we find that the collection of equivalence classes forms a set.

0FT5 **Definition 24.26.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let Qis be as in Lemma 24.26.3. The *derived category of* $(\mathcal{A}, d)$ is the triangulated category

$$D(\mathcal{A}, \mathrm{d}) = \mathrm{Qis}^{-1} K(Mod(\mathcal{A}, \mathrm{d}))$$

discussed in more detail above.

We prove some facts about this construction.

0FT6 **Lemma 24.26.5.** *In Definition 24.26.4 the kernel of the localization functor $Q : K(Mod(\mathcal{A}, d)) \to D(\mathcal{A}, d)$ is the category Ac of Lemma 24.26.2.*

Proof. This is immediate from Derived Categories, Lemma 13.5.9 and the fact that $0 \to \mathcal{M}$ is a quasi-isomorphism if and only if $\mathcal{M}$ is acyclic. □

0FT7 **Lemma 24.26.6.** *In Definition 24.26.4 the functor $H^0 : K(Mod(\mathcal{A}, d)) \to Mod(\mathcal{O})$ factors through a homological functor $H^0 : D(\mathcal{A}, d) \to Mod(\mathcal{O})$.*

Proof. Follows immediately from Derived Categories, Lemma 13.5.7. □

Here is the promised lemma computing morphism sets in the derived category.

0FT8 **Lemma 24.26.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ and $\mathcal{N}$ be differential graded $\mathcal{A}$-modules. Let $\mathcal{N} \to \mathcal{I}$ be a quasi-isomorphism with $\mathcal{I}$ a graded injective and K-injective differential graded $\mathcal{A}$-module. Then*

$$\mathrm{Hom}_{D(\mathcal{A},d)}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{K(Mod(\mathcal{A},d))}(\mathcal{M}, \mathcal{I})$$

Proof. Since $\mathcal{N} \to \mathcal{I}$ is a quasi-isomorphism we see that

$$\mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{M}, \mathcal{I})$$

In the discussion preceding Definition 24.26.4 we found, using Lemma 24.25.10, that any morphism $\mathcal{M} \to \mathcal{I}$ in $D(\mathcal{A}, \mathrm{d})$ can be represented by a morphism $f : \mathcal{M} \to \mathcal{I}$ in $K(Mod(\mathcal{A}, \mathrm{d}))$. Now, if $f, f' : \mathcal{M} \to \mathcal{I}$ are two morphism in $K(Mod(\mathcal{A}, \mathrm{d}))$, then they define the same morphism in $D(\mathcal{A}, \mathrm{d})$ if and only if there exists a quasi-isomorphism $g : \mathcal{I} \to \mathcal{K}$ in $K(Mod(\mathcal{A}, \mathrm{d}))$ such that $g \circ f = g \circ f'$, see Categories, Lemma 4.27.6. However, by Lemma 24.25.10 there exists a map $h : \mathcal{K} \to \mathcal{I}$ such that $h \circ g = \mathrm{id}_{\mathcal{I}}$ in in $K(Mod(\mathcal{A}, \mathrm{d}))$. Thus $g \circ f = g \circ f'$ implies $f = f'$ and the proof is complete. □

0FT9 **Lemma 24.26.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Then*

1. *$D(\mathcal{A}, d)$ has both direct sums and products,*
2. *direct sums are obtained by taking direct sums of differential graded $\mathcal{A}$-modules,*
3. *products are obtained by taking products of K-injective differential graded modules.*

Proof. We will use that $Mod(\mathcal{A}, \mathrm{d})$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(Mod(\mathcal{A}, \mathrm{d}))$. See Lemmas 24.13.2 and 24.21.3.

Let $\mathcal{M}_j$ be a family of differential graded $\mathcal{A}$-modules. Consider the direct sum $\mathcal{M} = \bigoplus \mathcal{M}_j$ as a differential graded $\mathcal{A}$-module. For a differential graded $\mathcal{A}$-module $\mathcal{N}$ choose a quasi-isomorphism $\mathcal{N} \to \mathcal{I}$ where $\mathcal{I}$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. See Theorem 24.25.13. Using Lemma 24.26.7 we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{M}, \mathcal{N}) &= \mathrm{Hom}_{K(\mathcal{A},\mathrm{d})}(\mathcal{M}, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{K(\mathcal{A},\mathrm{d})}(\mathcal{M}_j, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{M}_j, \mathcal{I}) \end{aligned}$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (2) of the lemma.

Let $\mathcal{M}_j$ be a family of differential graded $\mathcal{A}$-modules. For each j choose a quasi-isomorphism $\mathcal{M} \to \mathcal{I}_j$ where $\mathcal{I}_j$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. Consider the product $\mathcal{I} = \prod \mathcal{I}_j$ of differential graded $\mathcal{A}$-modules. By Lemmas 24.25.8 and 24.25.4 we see that $\mathcal{I}$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. For a differential graded $\mathcal{A}$-module $\mathcal{N}$ using Lemma 24.26.7 we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{N}, \mathcal{I}) &= \mathrm{Hom}_{K(\mathcal{A},\mathrm{d})}(\mathcal{N}, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{K(\mathcal{A},\mathrm{d})}(\mathcal{N}, \mathcal{I}_j) \\ &= \prod \mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{N}, \mathcal{M}_j) \end{aligned}$$

whence the existence of products in $D(\mathcal{A}, \mathrm{d})$ as given in part (3) of the lemma. □

24.27. The canonical delta-functor

0FTA Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Consider the functor $Mod(\mathcal{A}, \mathrm{d}) \to K(Mod(\mathcal{A}, \mathrm{d}))$. This functor is **not** a δ-functor in general. However, it turns out that the functor $Mod(\mathcal{A}, \mathrm{d}) \to D(A, \mathrm{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to \mathcal{K} \xrightarrow{a} \mathcal{L} \xrightarrow{b} \mathcal{M} \to 0$$

in the abelian category $Mod(\mathcal{A}, \mathrm{d})$. Consider the cone $C(a)$ of the morphism a together with its canonical morphisms $i : \mathcal{L} \to C(a)$ and $p : C(a) \to \mathcal{K}[1]$, see Definition 24.22.2. There is a homomorphism of differential graded $\mathcal{A}$-modules

$$q : C(a) \longrightarrow \mathcal{M}$$

by Differential Graded Algebra, Lemma 22.27.3 (which we may use by the discussion in Section 24.22) applied to the diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow[a]{} & \mathcal{L} \\ \downarrow & & \downarrow{\scriptstyle b} \\ 0 & \longrightarrow & \mathcal{M} \end{array}$$

The map q is a quasi-isomorphism for example because this is true in the category of morphisms of complexes of $\mathcal{O}$-modules, see discussion in Derived Categories, Section 13.12. According to Differential Graded Algebra, Lemma 22.27.13 (which we may use by the discussion in Section 24.22) the triangle

$$(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$$

is a distinguished triangle in $K(Mod(\mathcal{A}, \mathrm{d}))$. As the localization functor $K(Mod(\mathcal{A}, \mathrm{d})) \to D(\mathcal{A}, \mathrm{d})$ is exact we see that $(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A}, \mathrm{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A}, \mathrm{d})$. Hence we deduce that

$$(\mathcal{K}, \mathcal{L}, \mathcal{M}, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A}, \mathrm{d})$. This suggests the following lemma.

0FTB **Lemma 24.27.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The localization functor $Mod(\mathcal{A}, d) \to D(\mathcal{A}, d)$ has the natural structure of a δ-functor, with*

$$\delta_{\mathcal{K} \to \mathcal{L} \to \mathcal{M}} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 13.3.6. This follows from Differential Graded Algebra, Lemma 22.27.3 (which we may use by the discussion in Section 24.22) with a bit of work. Compare with Derived Categories, Lemma 13.12.1. □

0FTC **Lemma 24.27.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}_n$ be a system of differential graded $\mathcal{A}$-modules. Then the derived colimit hocolim$\mathcal{M}_n$ in $D(\mathcal{A}, d)$ is represented by the differential graded module* $\operatorname{colim} \mathcal{M}_n$.

Proof. Set $\mathcal{M} = \operatorname{colim} \mathcal{M}_n$. We have an exact sequence of differential graded $\mathcal{A}$-modules

$$0 \to \bigoplus \mathcal{M}_n \to \bigoplus \mathcal{M}_n \to \mathcal{M} \to 0$$

by Derived Categories, Lemma 13.33.6 (applied the underlying complexes of $\mathcal{O}$-modules). The direct sums are direct sums in $D(\mathcal{A}, \mathrm{d})$ by Lemma 24.26.8. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 24.27.1). □

24.28. Derived pullback

0FTD Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

In addition to the above, let $\mathcal{A}'$ be a second differential graded $\mathcal{O}_\mathcal{C}$-algebra and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule. In this setting we can consider the functor

$$Mod(\mathcal{B}, \mathrm{d}) \longrightarrow Mod(\mathcal{A}', \mathrm{d}), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

Observe that this extends to a functor

$$Mod^{dg}(\mathcal{B}, \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{A}', \mathrm{d}), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

of differential graded categories by the discussion in Sections 24.18 and 24.17. It follows formally that we also obtain an exact functor

0FTE (24.28.0.1) $$K(Mod(\mathcal{B}, \mathrm{d})) \longrightarrow K(Mod(\mathcal{A}', \mathrm{d})), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

of triangulated categories.

0FTF **Lemma 24.28.1.** *In the situation above, the functor (24.28.0.1) composed with the localization functor $K(Mod(\mathcal{A}', d)) \to D(\mathcal{A}', d)$ has a left derived extension $D(\mathcal{B}, d) \to D(\mathcal{A}', d)$ whose value on a good right differential graded $\mathcal{B}$-module $\mathcal{P}$ is $f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$.*

Proof. Recall that for any (right) differential graded $\mathcal{B}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{P} \to \mathcal{M}$ with $\mathcal{P}$ a good differential graded $\mathcal{B}$-module. See Lemma 24.23.7. Hence by Derived Categories, Lemma 13.14.15 it suffices to show that given a quasi-isomorphism $\mathcal{P} \to \mathcal{P}'$ of good differential graded $\mathcal{B}$-modules the induced map

$$f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \longrightarrow f^*\mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N}$$

is a quasi-isomorphism. The cone $\mathcal{P}''$ on $\mathcal{P} \to \mathcal{P}'$ is a good differential graded $\mathcal{A}$-module by Lemma 24.23.2. Since we have a distinguished triangle

$$\mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to \mathcal{P}[1]$$

in $K(Mod(\mathcal{B}, \mathrm{d}))$ we obtain a distinguished triangle

$$f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to f^*\mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N} \to f^*\mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N} \to f^*\mathcal{P}[1] \otimes_{\mathcal{A}} \mathcal{N}$$

in $K(Mod(\mathcal{A}', \mathrm{d}))$. By Lemma 24.23.8 the differential graded module $f^*\mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic and the proof is complete. □

0FTG **Definition 24.28.2.** Derived tensor product and derived pullback.

(1) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. The functor $D(\mathcal{A}, \mathrm{d}) \to D(\mathcal{B}, \mathrm{d})$ constructed in Lemma 24.28.1 is called the *derived tensor product* and denoted $- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}$.

(2) Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_{\mathcal{C}}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_{\mathcal{D}}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_{\mathcal{D}}$-algebras. The functor $D(\mathcal{B}, \mathrm{d}) \to D(\mathcal{A}, \mathrm{d})$ constructed in Lemma 24.28.1 is called *derived pullback* and denote Lf^*.

With this language in place we can express some obvious compatibilities.

0FTH **Lemma 24.28.3.** *In Lemma 24.28.1 the functor $D(\mathcal{B}, d) \to D(\mathcal{A}', d)$ is equal to $\mathcal{M} \mapsto Lf^*\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}$.*

Proof. Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because $f^*\mathcal{P}$ is a good differential graded $\mathcal{A}$-module if $\mathcal{P}$ is a good differential graded $\mathcal{B}$-module. □

0FTI **Lemma 24.28.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $(g, g^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be a differential graded $\mathcal{O}$-algebra, $\mathcal{O}'$-algebra, and $\mathcal{O}''$-algebra. Let $\varphi : \mathcal{A}' \to f_*\mathcal{A}$ and $\varphi' : \mathcal{A}'' \to g_*\mathcal{A}'$ be a homomorphism of differential graded $\mathcal{O}'$-algebras and $\mathcal{O}''$-algebras. Then we have $L(g \circ f)^* = Lf^* \circ Lg^* : D(\mathcal{A}'', d) \to D(\mathcal{A}, d)$.*

Proof. Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because $f^*\mathcal{P}$ is a good differential graded $\mathcal{A}'$-module of $\mathcal{P}$ is a good differential graded $\mathcal{A}$-module. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N} \to \mathcal{N}'$ be a homomorphism of differential graded $(\mathcal{A}, \mathcal{B})$-bimodules. Then we obtain canonical maps

$$t : \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N} \longrightarrow \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}'$$

functorial in $\mathcal{M}$ in $D(\mathcal{A}, \mathrm{d})$ which define a natural transformation between exact functors $D(\mathcal{A}, \mathrm{d}) \to D(\mathcal{B}, \mathrm{d})$ of triangulated categories. The value of t on a good differential graded $\mathcal{A}$-module $\mathcal{P}$ is the obvious map

$$\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N} = \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}' = \mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}'$$

0FTJ **Lemma 24.28.5.** *In the situation above, if $\mathcal{N} \to \mathcal{N}'$ is an isomorphism on cohomology sheaves, then t is an isomorphism of functors $(- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}) \to (- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}')$.*

Proof. It is enough to show that $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$ is an isomorphism on cohomology sheaves for any good differential graded $\mathcal{A}$-module $\mathcal{P}$. To do this, let $\mathcal{N}''$ be the cone on the map $\mathcal{N} \to \mathcal{N}'$ as a left differential graded $\mathcal{A}$-module, see Definition 24.22.2. (To be sure, $\mathcal{N}''$ is a bimodule too but we don't need this.) By functoriality of the tensor construction (it is a functor of differential graded categories) we see that $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}''$ is the cone (as a complex of $\mathcal{O}$-modules) on the map $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$. Hence it suffices to show that $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}''$ is acyclic. This follows from the fact that $\mathcal{P}$ is good and the fact that $\mathcal{N}''$ is acyclic as a cone on a quasi-isomorphism. □

0FTK **Lemma 24.28.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. If $\mathcal{N}$ is good as a left differential graded $\mathcal{A}$-module, then we have $\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N} = \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ for all differential graded $\mathcal{A}$-modules $\mathcal{M}$.*

Proof. Let $\mathcal{P} \to \mathcal{M}$ be a quasi-isomorphism where $\mathcal{P}$ is a good (right) differential graded $\mathcal{A}$-module. To prove the lemma we have to show that $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is a quasi-isomorphism. The cone C on the map $\mathcal{P} \to \mathcal{M}$ is an acyclic right differential graded $\mathcal{A}$-module. Hence $C \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic as $\mathcal{N}$ is assumed good as a left differential graded $\mathcal{A}$-module. Since $C \otimes_{\mathcal{A}} \mathcal{N}$ is the cone on the maps $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ as a complex of $\mathcal{O}$-modules we conclude. □

0FTL **Lemma 24.28.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ and $\mathcal{N}'$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule and $(\mathcal{A}', \mathcal{A}'')$-bimodule. Assume that the canonical map*

$$\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

in $D(\mathcal{A}'', d)$ is a quasi-isomorphism. Then we have

$$(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' = \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$$

as functors $D(\mathcal{A}, d) \to D(\mathcal{A}'', d)$.

Proof. Choose a good differential graded $\mathcal{A}$-module $\mathcal{P}$ and a quasi-isomorphism $\mathcal{P} \to \mathcal{M}$, see Lemma 24.23.7. Then

$$\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}') = \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

and we have

$$(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' = (\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}'$$

Thus we have to show the canonical map

$$(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

is a quasi-isomorphism. Choose a quasi-isomorphism $\mathcal{Q} \to \mathcal{N}'$ where $\mathcal{Q}$ is a good left differential graded $\mathcal{A}'$-module (Lemma 24.23.7). By Lemma 24.28.6 the map above as a map in the derived category of $\mathcal{O}$-modules is the map

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

Since $\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \to \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$ is a quasi-isomorphism by assumption and $\mathcal{P}$ is a good differential graded $\mathcal{A}$-module this map is an quasi-isomorphism by Lemma 24.28.5 (the left and right hand side compute $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q})$ and $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$ or you can just repeat the argument in the proof of the lemma). □

24.29. Derived pushforward

0FTM The existence of enough K-injective guarantees that we can take the right derived functor of any exact functor on the homotopy category.

0FTN **Lemma 24.29.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, d)$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Then any exact functor*

$$T : K(\mathit{Mod}(\mathcal{A}, d)) \longrightarrow \mathcal{D}$$

of triangulated categories has a right derived extension $RT : D(\mathcal{A}, d) \to \mathcal{D}$ whose value on a graded injective and K-injective differential graded $\mathcal{A}$-module $\mathcal{I}$ is $T(\mathcal{I})$.

Proof. By Theorem 24.25.13 for any (right) differential graded $\mathcal{A}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{M} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module. Hence by Derived Categories, Lemma 13.14.15 it suffices to show that given a quasi-isomorphism $\mathcal{I} \to \mathcal{I}'$ of differential graded $\mathcal{A}$-modules which are both graded injective and K-injective then $T(\mathcal{I}) \to T(\mathcal{I}')$ is an isomorphism. This is true because the map $\mathcal{I} \to \mathcal{I}'$ is an isomorphism in $K(\mathit{Mod}(\mathcal{A}, \mathrm{d}))$ as follows for example from Lemma 24.26.7 (or one can deduce it from Lemma 24.25.10). □

There are a number of functors we have already seen to which this applies. Here are two examples.

0FTP **Definition 24.29.2.** Derived internal hom and derived pushforward.

(1) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. The right derived extension

$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, \mathrm{d}) \longrightarrow D(\mathcal{A}, \mathrm{d})$$

of the internal hom functor $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -)$ is called *derived internal hom*.

(2) Let $(f, f^\sharp) : (\mathcal{S}h(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (\mathcal{S}h(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_{\mathcal{C}}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_{\mathcal{D}}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_{\mathcal{D}}$-algebras. The right derived extension

$$Rf_* : D(\mathcal{A}, \mathrm{d}) \longrightarrow D(\mathcal{B}, \mathrm{d})$$

of the pushforward f_* is called *derived pushforward*.

It turns out that $Rf_* : D(\mathcal{A}, \mathrm{d}) \to D(\mathcal{B}, \mathrm{d})$ agrees with derived pusforward on underlying complexes of $\mathcal{O}$-modules, see Lemma 24.29.8.

These functors are the adjoints of derived pullback and derived tensor product.

0FTQ **Lemma 24.29.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then*

$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, d) \longrightarrow D(\mathcal{A}, d)$$

is right adjoint to

$$- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N} : D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d)$$

Proof. This follows from Derived Categories, Lemma 13.30.1 and Lemma 24.17.3. □

0FTR **Lemma 24.29.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. Then*

$$Rf_* : D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d)$$

is right adjoint to

$$Lf^* : D(\mathcal{B}, d) \longrightarrow D(\mathcal{A}, d)$$

Proof. This follows from Derived Categories, Lemma 13.30.1 and Lemma 24.18.1. □

Next, we discuss what happens in the situation considered in Section 24.28.

Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

In addition to the above, let $\mathcal{A}'$ be a second differential graded $\mathcal{O}_\mathcal{C}$-algebra and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule. In this setting we can consider the functor

$$Mod(\mathcal{A}', \mathrm{d}) \longrightarrow Mod(\mathcal{B}, \mathrm{d}), \quad \mathcal{M} \longmapsto f_*\mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

Observe that this extends to a functor

$$Mod^{dg}(\mathcal{A}', \mathrm{d}) \longrightarrow Mod^{dg}(\mathcal{B}, \mathrm{d}), \quad \mathcal{M} \longmapsto f_*\mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

of differential graded categories by the discussion in Sections 24.18 and 24.17. It follows formally that we also obtain an exact functor

0FTS (24.29.4.1) $$K(Mod(\mathcal{A}', \mathrm{d})) \longrightarrow K(Mod(\mathcal{B}, \mathrm{d})), \quad \mathcal{M} \longmapsto f_*\mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

of triangulated categories.

0FTT **Lemma 24.29.5.** *In the situation above, denote $RT : D(\mathcal{A}', d) \to D(\mathcal{B}, d)$ the right derived extension of (24.29.4.1). Then we have*

$$RT(\mathcal{M}) = Rf_* R\mathcal{H}om(\mathcal{N}, \mathcal{M})$$

functorially in $\mathcal{M}$.

Proof. By Lemmas 24.17.3 and 24.18.1 the functor (24.29.4.1) is right adjoint to the functor (24.28.0.1). By Derived Categories, Lemma 13.30.1 the functor RT is right adjoint to the functor of Lemma 24.28.1 which is equal to $Lf^*(-) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}$ by Lemma 24.28.3. By Lemmas 24.29.3 and 24.29.4 the functor $Lf^*(-) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}$ is left adjoint to $Rf_* R\mathcal{H}om(\mathcal{N}, -)$ Thus we conclude by uniqueness of adjoints. □

0FTU **Lemma 24.29.6.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $(g, g^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be a differential graded $\mathcal{O}$-algebra, $\mathcal{O}'$-algebra, and $\mathcal{O}''$-algebra. Let $\varphi : \mathcal{A}' \to f_*\mathcal{A}$ and $\varphi' : \mathcal{A}'' \to g_*\mathcal{A}'$ be a homomorphism of differential graded $\mathcal{O}'$-algebras and $\mathcal{O}''$-algebras. Then we have $R(g \circ f)_* = Rg_* \circ Rf_* : D(\mathcal{A}, d) \to D(\mathcal{A}'', d)$.*

Proof. Follows from Lemmas 24.28.4 and 24.29.4 and uniqueness of adjoints. □

0FTV **Lemma 24.29.7.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ and $\mathcal{N}'$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule and $(\mathcal{A}', \mathcal{A}'')$-bimodule. Assume that the canonical map*

$$\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

in $D(\mathcal{A}'', d)$ is a quasi-isomorphism. Then we have

$$R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}', -) = R\mathcal{H}om_{\mathcal{A}'}(\mathcal{N}, R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N}', -))$$

as functors $D(\mathcal{A}'', d) \to D(\mathcal{A}, d)$.

Proof. Follows from Lemmas 24.28.7 and 24.29.3 and uniqueness of adjoints. □

0FTW **Lemma 24.29.8.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. The diagram*

$$\begin{array}{ccc} D(\mathcal{A}, d) & \xrightarrow{forget} & D(\mathcal{O}_\mathcal{C}) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(\mathcal{B}, d) & \xrightarrow{forget} & D(\mathcal{O}_\mathcal{D}) \end{array}$$

commutes.

Proof. Besides identifying some categories, this lemma follows immediately from Lemma 24.29.6.

We may view $(\mathcal{O}_\mathcal{C}, 0)$ as a differential graded $\mathcal{O}_\mathcal{C}$-algebra by placing $\mathcal{O}_\mathcal{C}$ in degree 0 and endowing it with the zero differential. It is clear that we have

$$Mod(\mathcal{O}_\mathcal{C}, 0) = \mathrm{Comp}(\mathcal{O}_\mathcal{C}) \quad \text{and} \quad D(\mathcal{O}_\mathcal{C}, 0) = D(\mathcal{O}_\mathcal{C})$$

Via this identification the forgetful functor $Mod(\mathcal{A}, \mathrm{d}) \to \mathrm{Comp}(\mathcal{O}_\mathcal{C})$ is the "push-forward" $\mathrm{id}_{\mathcal{C},*}$ defined in Section 24.18 corresponding to the identity morphism

$\mathrm{id}_{\mathcal{C}} : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ of ringed topoi and the map $(\mathcal{O}_{\mathcal{C}}, 0) \to (\mathcal{A}, \mathrm{d})$ of differential graded $\mathcal{O}_{\mathcal{C}}$-algebras. Since $\mathrm{id}_{\mathcal{C},*}$ is exact, we immediately see that

$$R\mathrm{id}_{\mathcal{C},*} = forget : D(\mathcal{A}, \mathrm{d}) \longrightarrow D(\mathcal{O}_{\mathcal{C}}, 0) = D(\mathcal{O}_{\mathcal{C}})$$

The exact same reasoning shows that

$$R\mathrm{id}_{\mathcal{D},*} = forget : D(\mathcal{B}, \mathrm{d}) \longrightarrow D(\mathcal{O}_{\mathcal{D}}, 0) = D(\mathcal{O}_{\mathcal{D}})$$

Moreover, the construction of $Rf_* : D(\mathcal{O}_{\mathcal{C}}) \to D(\mathcal{O}_{\mathcal{D}})$ of Cohomology on Sites, Section 21.19 agrees with the construction of $Rf_* : D(\mathcal{O}_{\mathcal{C}}, 0) \to D(\mathcal{O}_{\mathcal{D}}, 0)$ in Definition 24.29.2 as both functors are defined as the right derived extension of pushforward on underlying complexes of modules. By Lemma 24.29.6 we see that both $Rf_* \circ R\mathrm{id}_{\mathcal{C},*}$ and $R\mathrm{id}_{\mathcal{D},*} \circ Rf_*$ are the derived functors of $f_* \circ forget = forget \circ f_*$ and hence equal by uniqueness of adjoints. □

0FTX **Lemma 24.29.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a differential graded $\mathcal{O}$-algebra. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. Let $n \in \mathbf{Z}$. We have*

$$H^n(\mathcal{C}, \mathcal{M}) = \mathrm{Hom}_{D(\mathcal{A},d)}(\mathcal{A}, \mathcal{M}[n])$$

where on the left hand side we have the cohomology of $\mathcal{M}$ viewed as a complex of $\mathcal{O}$-modules.

Proof. To prove the formula, observe that

$$R\Gamma(\mathcal{C}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{I})$$

where $\mathcal{M} \to \mathcal{I}$ is a quasi-isomorphism to a graded injective and K-injective differential graded $\mathcal{A}$-module $\mathcal{I}$ (combine Lemmas 24.29.1 and 24.29.8). By Lemma 24.26.7 we have

$$\mathrm{Hom}_{D(\mathcal{A},\mathrm{d})}(\mathcal{A}, \mathcal{M}[n]) = \mathrm{Hom}_{K(Mod(\mathcal{A},\mathrm{d}))}(\mathcal{M}, \mathcal{I}[n]) = H^0(\Gamma(\mathcal{C}, \mathcal{I}[n])) = H^n(\Gamma(\mathcal{C}, \mathcal{I}))$$

Combining these two results we obtain our equality. □

24.30. Equivalences of derived categories

0FTY This section is the analogue of Differential Graded Algebra, Section 22.37.

0FTZ **Lemma 24.30.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of differential graded $\mathcal{O}$-algebras which induces an isomorphism on cohomology sheaves, then*

$$D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d), \quad \mathcal{M} \longmapsto \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B}$$

is an equivalence of categories.

Proof. Recall that the restriction functor

$$Mod^{dg}(\mathcal{B}, \mathrm{d}) \to Mod^{dg}(\mathcal{A}, \mathrm{d}), \quad \mathcal{N} \mapsto res_{\varphi}\mathcal{N}$$

is a right adjoint to

$$Mod^{dg}(\mathcal{A}, \mathrm{d}) \to Mod^{dg}(\mathcal{B}, \mathrm{d}), \quad \mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$$

See Section 24.17. Since restriction sends quasi-isomorphisms to quasi-isomorphisms, we see that it trivially has a left derived extension (given by restriction). This functor will be right adjoint to $- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B}$ by Derived Categories, Lemma 13.30.1. The adjunction map

$$\mathcal{M} \to res_{\varphi}(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B})$$

is an isomorphism in $D(\mathcal{A}, \mathrm{d})$ by our assumption that $\mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of (left) differential graded $\mathcal{A}$-modules. In particular, the functor of the lemma is fully faithful, see Categories, Lemma 4.24.4. It is clear that the kernel of the restriction functor $D(\mathcal{B}, \mathrm{d}) \to D(\mathcal{A}, \mathrm{d})$ is zero. Thus we conclude by Derived Categories, Lemma 13.7.2. □

24.31. Resolutions of differential graded algebras

0FU0 This section is the analogue of Differential Graded Algebra, Section 22.38.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. As in Remark 24.23.5 consider a sheaf of graded sets $\mathcal{S}$ on $\mathcal{C}$. Let us think of the r-fold self product $\mathcal{S} \times \ldots \times \mathcal{S}$ as a sheaf of graded sets with the rule $\deg(s_1 \cdot \ldots \cdot s_r) = \sum \deg(s_i)$. Here given local sections $s_i \in \mathcal{S}(U)$, $i = 1, \ldots, r$ we use $s_1 \cdot \ldots \cdot s_r$ to denote the corresponding section of $\mathcal{S} \times \ldots \times \mathcal{S}$ over U. Let us denote $\mathcal{O}\langle\mathcal{S}\rangle$ the free graded $\mathcal{O}$-algebra on $\mathcal{S}$. More precisely, we set

$$\mathcal{O}\langle\mathcal{S}\rangle = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \mathcal{O}[\mathcal{S} \times \ldots \times \mathcal{S}]$$

with notation as in Remark 24.23.5. This becomes a sheaf of graded $\mathcal{O}$-algebras by concatenation

$$(s_1 \cdot \ldots \cdot s_r)(s'_1 \cdot \ldots \cdot s'_{r'}) = s_1 \cdot \ldots s_r \cdot s'_1 \cdot \ldots \cdot s'_{r'}$$

We may endow $\mathcal{O}\langle\mathcal{S}\rangle$ with a differential by setting $\mathrm{d}(s) = 0$ for all local sections s of $\mathcal{S}$ and extending uniquely using the Leibniz rule although it is important to also consider other differentials.

Indeed, suppose that we are given a system of the following kind

(1) for $i = 0, 1, 2, \ldots$ sheaves of graded sets $\mathcal{S}_i$,
(2) for $i = 0, 1, 2, \ldots$ maps
$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i = \mathcal{O}\langle\mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_i\rangle$$
of sheaves of graded sets of degree 1 whose image is contained in the kernel of the inductively defined differential on the target.

More precisely, we first set $\mathcal{A}_0 = \mathcal{O}\langle\mathcal{S}_0\rangle$ and we endow it with the unique differential satisfying the Leibniz rule where $\mathrm{d}(s) = 0$ for any local section s of $\mathcal{S}$. By induction, assume given a differential d on $\mathcal{A}_i$. Then we extend it to the unique differential on $\mathcal{A}_{i+1}$ satisfying the Leibniz rule and with

$$\mathrm{d}(s) = \delta(s)$$

where $\delta(s) = \delta_j(s)$ if s is in the summand $\mathcal{S}_j$ of $\mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_{i+1}$. This makes sense exactly because $\delta(s)$ is in the kernel of the inductively defined differential.

0FU1 **Lemma 24.31.1.** *In the situation above the differential graded $\mathcal{O}$-algebra*

$$\mathcal{A} = \operatorname{colim} \mathcal{A}_i$$

has the following property: for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, the pullback $f^*\mathcal{A}$ is flat as a graded $\mathcal{O}'$-module and is K-flat as a complex of $\mathcal{O}'$-modules.

Proof. Observe that $f^*\mathcal{A} = \operatorname{colim} f^*\mathcal{A}_i$ and that

$$f^*\mathcal{A}_i = \mathcal{O}'\langle f^{-1}\mathcal{S}_0 \amalg \ldots \amalg f^{-1}\mathcal{S}_i\rangle$$

with differential given by the inductive procedure above using $f^{-1}\delta_{i+1}$. Thus it suffices to prove that $\mathcal{A}$ is flat as a graded $\mathcal{O}$-module and is K-flat as a complex of

$\mathcal{O}$-modules. For this it suffices to prove that each $\mathcal{A}_i$ is flat as a graded $\mathcal{O}$-module and is K-flat as a complex of $\mathcal{O}$-modules, compare with Lemma 24.23.3.

For $i \geq 1$ write $\mathcal{S} = \mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_i$ so that we have $\mathcal{A}_i = \mathcal{O}\langle\mathcal{S}\rangle$ as a graded $\mathcal{O}$-algebra. We are going to construct a filtration of this algebra by differential graded $\mathcal{O}$-submodules.

Set $W = \mathbf{Z}_{\geq 0}^{i+1}$ considered with lexicographical ordering. Namely, given $w = (w_0, \ldots w_i)$ and $w' = (w'_0, \ldots, w'_i)$ in W we say

$$w > w' \Leftrightarrow \exists j,\ 0 \leq j \leq i : w_i = w'_i,\ w_{i-1} = w'_{i-1},\ \ldots,\ w_{j+1} = w'_{j+1},\ w_j > w'_j$$

and so on. Suppose given a section $s = s_1 \cdot \ldots \cdot s_r$ of $\mathcal{S} \times \ldots \times \mathcal{S}$ over U. We say that the *weight of s is defined* if we have $s_a \in \mathcal{S}_{j_a}(U)$ for a unique $0 \leq j_a \leq i$. In this case we define the weight

$$w(s) = (w_0(s), \ldots, w_i(s)) \in W, \quad w_j(s) = |\{a \mid j_a = j\}|$$

The weight of any section of $\mathcal{S} \times \ldots \times \mathcal{S}$ is defined locally. The reader checks easily that we obtain a disjoint union decomposition

$$\mathcal{S} \times \ldots \times \mathcal{S} = \coprod\nolimits_{w \in W} (\mathcal{S} \times \ldots \times \mathcal{S})_w$$

into the subsheaves of sections of a given weight. Of course only $w \in W$ with $\sum_{0 \leq j \leq i} w_j = r$ show up for a given r. We correspondingly obtain a decomposition

$$\mathcal{A}_i = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \bigoplus\nolimits_{w \in W} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_w]$$

The rest of the proof relies on the following trivial observation: given r, w and local section $s = s_1 \cdot \ldots \cdot s_r$ of $(\mathcal{S} \times \ldots \times \mathcal{S})_w$ we have

$$\mathrm{d}(s) \text{ is a local section of } \mathcal{O} \oplus \bigoplus\nolimits_{r' \geq 1} \bigoplus\nolimits_{w' \in W,\ w' < w} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_{w'}]$$

The reason is that in each of the expressions

$$(-1)^{\deg(s_1) + \ldots + \deg(s_{a-1})} s_1 \cdot \ldots s_{a-1} \cdot \delta(s_a) \cdot s_{a+1} \cdot \ldots \cdot s_r$$

whose sum give the element $\mathrm{d}(s)$ the element $\delta(s_a)$ is locally a $\mathcal{O}$-linear combination of elements $s'_1 \cdot \ldots \cdot s'_{r'}$ with $s'_{a'}$ in $\mathcal{S}_{j'_a}$ for some $0 \leq j'_{a'} < j_a$ where j_a is such that s_a is section of $\mathcal{S}_{j_a}$.

What this means is the following. Suppose for $w \in W$ we set

$$F_w\mathcal{A}_i = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \bigoplus\nolimits_{w' \in W,\ w' \leq w} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_{w'}]$$

By the observation above this is a differential graded $\mathcal{O}$-submodule. We get admissible short exact sequences

$$0 \to \mathrm{colim}_{w' < w} F_{w'}\mathcal{A}_i \to F_w\mathcal{A}_i \to \bigoplus\nolimits_{r \geq 1} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_w] \to 0$$

of differential graded $\mathcal{A}$-modules where the differential on the right hand side is zero.

Now we finish the proof by transfinite induction over the ordered set W. The differential graded complex $F_0\mathcal{A}_0$ is the summand $\mathcal{O}$ and this is K-flat and graded flat. For $w \in W$ if the result is true for $F_{w'}\mathcal{A}_i$ for $w' < w$, then by Lemmas 24.23.3, 24.23.2, and 24.23.6 we obtain the result for w. Finally, we have $\mathcal{A}_i = \mathrm{colim}_{w \in W} F_w\mathcal{A}_i$ and we conclude. $\square$

0FU2 **Lemma 24.31.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{B}, d)$ be a differential graded $\mathcal{O}$-algebra. There exists a quasi-isomorphism of differential graded $\mathcal{O}$-algebras $(\mathcal{A}, d) \to (\mathcal{B}, d)$ such that $\mathcal{A}$ is graded flat and K-flat as a complex of $\mathcal{O}$-modules and such that the same is true after pullback by any morphism of ringed topoi.*

Proof. The proof is exactly the same as the first proof of Lemma 24.23.7 but now working with free graded algebras instead of free graded modules.

We will construct $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as in Lemma 24.31.1 by constructing

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{B}$$

Let $\mathcal{S}_0$ be the sheaf of graded sets (Remark 24.23.5) whose degree n part is $\mathrm{Ker}(\mathrm{d}^n_{\mathcal{B}})$. Consider the homomorphism of differential graded modules

$$\mathcal{A}_0 = \mathcal{O}\langle \mathcal{S}_0 \rangle \longrightarrow \mathcal{B}$$

where map sends a local section s of $\mathcal{S}_0$ to the corresponding local section of $\mathcal{A}^{\deg(s)}$ (which is in the kernel of the differential, so our map is a map of differential graded algebras indeed). By construction the induced maps on cohomology sheaves $H^n(\mathcal{A}_0) \to H^n(\mathcal{B})$ are surjective and hence the same will remain true for all i.

Induction step of the construction. Given $\mathcal{A}_i \to \mathcal{B}$ denote $\mathcal{S}_{i+1}$ the sheaf of graded sets whose degree n part is

$$\mathrm{Ker}(\mathrm{d}^{n+1}_{\mathcal{A}_i}) \times_{\mathcal{B}^{n+1}, \mathrm{d}} \mathcal{B}^n$$

This comes equipped with a canonical map

$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i$$

whose image is contained in the kernel of $\mathrm{d}_{\mathcal{A}_i}$ by construction. Hence $\mathcal{A}_{i+1} = \mathcal{O}\langle \mathcal{S}_0 \amalg \ldots \mathcal{S}_{i+1} \rangle$ has a differential exteding the differential on $\mathcal{A}_i$, see discussion at the start of this section. The map from $\mathcal{A}_{i+1}$ to $\mathcal{B}$ is the unique map of graded algebras which restricts to the given map on $\mathcal{A}_i$ and sends a local section $s = (a, b)$ of $\mathcal{S}_{i+1}$ to b in $\mathcal{B}$. This is compatible with differentials exactly because $\mathrm{d}(b)$ is the image of a in $\mathcal{B}$.

The map $\mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism: we have $H^n(\mathcal{A}) = \operatorname{colim} H^n(\mathcal{A}_i)$ and for each i the map $H^n(\mathcal{A}_i) \to H^n(\mathcal{B})$ is surjective with kernel annihilated by the map $H^n(\mathcal{A}_i) \to H^n(\mathcal{A}_{i+1})$ by construction. Finally, the flatness condition for $\mathcal{A}$ where shown in Lemma 24.31.1. □

24.32. Miscellany

0FU3 Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a sheaf of differential graded $\mathcal{O}$-algebras. Using the composition[3]

$$\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{A}$$

[3]It would be more precise to write $F(\mathcal{A}) \otimes^{\mathbf{L}}_{\mathcal{O}} F(\mathcal{A}) \to F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \to F(\mathcal{A})$ were F denotes the forgetful functor to complexes of $\mathcal{O}$-modules. Also, note that $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}$ indicates the tensor product of Section 24.15 so that $F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) = \mathrm{Tot}(F(\mathcal{A}) \otimes_{\mathcal{O}} F(\mathcal{A}))$. The first arrow of the sequence is the canonical map from the derived tensor product of two complexes of $\mathcal{O}$-modules to the usual tensor product of complexes of $\mathcal{O}$-modules.

and the relative cup product (see Cohomology on Sites, Remark 21.19.7 and Section 21.33) we obtain a multiplication[4]

$$\mu : Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{A}$$

in $D(\mathcal{O}')$. This multiplication is associative in the sense that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow[\mu\otimes 1]{} & Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \\ \downarrow{\scriptstyle 1\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu} & Rf_*\mathcal{A} \end{array}$$

commutes in $D(\mathcal{O}')$; this follows from Cohomology on Sites, Lemma 21.33.2. In exactly the same way, given a right differential graded $\mathcal{A}$-module $\mathcal{M}$ we obtain a multiplication

$$\mu_{\mathcal{M}} : Rf_*\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{M}$$

in $D(\mathcal{O}')$. This multiplication is compatible with μ above in the sense that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow[\mu_{\mathcal{M}}\otimes 1]{} & Rf_*\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \\ \downarrow{\scriptstyle 1\otimes\mu} & & \downarrow{\scriptstyle \mu_{\mathcal{M}}} \\ Rf_*\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}}} & Rf_*\mathcal{M} \end{array}$$

commutes in $D(\mathcal{O}')$; again this follows from Cohomology on Sites, Lemma 21.33.2.

A particular example of the above is when one takes f to be the morphism to the punctual topos $Sh(pt)$. In that case μ is just the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{A}) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C},\mathcal{O})} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{A}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

and similarly $\mu_{\mathcal{M}}$ is the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{M}) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C},\mathcal{O})} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{M}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

In general, via the identifications

$$R\Gamma(\mathcal{C}, \mathcal{A}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{A}) \quad \text{and} \quad R\Gamma(\mathcal{C}, \mathcal{M}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{M})$$

of Cohomology on Sites, Remark 21.14.4 the map $\mu_{\mathcal{M}}$ induces the cup product on cohomology. To see this use Cohomology on Sites, Lemma 21.33.4 where the second morphism of topoi is the morphism from $Sh(\mathcal{C}')$ to the punctual topos as above.

If $\mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism of right differential graded $\mathcal{A}$-modules, then the diagram

$$\begin{array}{ccc} Rf_*\mathcal{M}_1 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow[\mu_{\mathcal{M}_1}]{} & Rf_*\mathcal{M}_1 \\ \downarrow & & \downarrow \\ Rf_*\mathcal{M}_2 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_2}} & Rf_*\mathcal{M}_2 \end{array}$$

commutes in $D(\mathcal{O}')$; this follows from the fact that the relative cup product is functorial. Suppose we have a short exact sequence

$$0 \to \mathcal{M}_1 \xrightarrow{a} \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

[4]Here and below $Rf_* : D(\mathcal{O}) \to D(\mathcal{O}')$ is the derived functor studied in Cohomology on Sites, Section 21.19 ff.

of right differential graded $\mathcal{A}$-modules. Then we claim that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{M}_3 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow[\mu_{\mathcal{M}_3}]{} & Rf_*\mathcal{M}_3 \\ \downarrow{\scriptstyle Rf_*\delta\otimes\mathrm{id}} & & \downarrow{\scriptstyle Rf_*\delta} \\ Rf_*\mathcal{M}_1[1] \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_1[1]}} & Rf_*\mathcal{M}_1[1] \end{array}$$

commutes in $D(\mathcal{O}')$ where $\delta : \mathcal{M}_3 \to \mathcal{M}_1[1]$ is the morphism of $D(\mathcal{O})$ coming from the given short exact sequence (see Derived Categories, Section 13.12). This is clear if our sequence is split as a sequence of graded right $\mathcal{A}$-modules, because in this case δ can be represented by a map of right $\mathcal{A}$-modules and the discussion above applies. In general we argue using the cone on a and the diagram

$$\begin{array}{ccccccc} \mathcal{M}_1 & \xrightarrow{a} & \mathcal{M}_2 & \xrightarrow{i} & C(a) & \xrightarrow{-p} & \mathcal{M}_1[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_3 & \xrightarrow{\delta} & \mathcal{M}_1[1] \end{array}$$

where the right square is commutative in $D(\mathcal{O})$ by the definition of δ in Derived Categories, Lemma 13.12.1. Now the cone $C(a)$ has the structure of a right differential graded $\mathcal{A}$-module such that i, p, q are homomorphisms of right differential graded $\mathcal{A}$-modules, see Definition 24.22.2. Hence by the above we know that the corresponding diagrams commute for the morphisms q and $-p$. Since q is an isomorphism in $D(\mathcal{O})$ we conclude the same is true for δ as desired.

In the situation above given a right differential graded $\mathcal{A}$-module $\mathcal{M}$ let

$$\xi \in H^n(\mathcal{C}, \mathcal{M})$$

In other words, ξ is a degree n cohomology class in the cohomology of $\mathcal{M}$ viewed as a complex of $\mathcal{O}$-modules. By Lemma 24.29.9 we can construct maps

$$x : \mathcal{A} \to \mathcal{M}'[n] \quad \text{and} \quad s : \mathcal{M} \to \mathcal{M}'$$

of right differential graded $\mathcal{A}$-modules where s is a quasi-isomorphism and such that ξ is the image of $1 \in H^0(\mathcal{C}, \mathcal{A})$ via the morphism $s[n]^{-1} \circ x$ in the derived category $D(\mathcal{A}, \mathrm{d})$ and a fortiori in the derived category $D(\mathcal{O})$. It follows that the corresponding map

$$\xi' = (s[n])^{-1} \circ x : \mathcal{A} \longrightarrow \mathcal{M}[n]$$

in $D(\mathcal{O})$ is uniquely characterized by the following two properties

(1) ξ' can be lifted to a morphism in $D(\mathcal{A}, \mathrm{d})$, and
(2) $\xi = \xi'(1)$ in $H^0(\mathcal{C}, \mathcal{M}[n]) = H^n(\mathcal{C}, \mathcal{M})$.

Using the compatibilities of x and s with the relative cup product discussed above it follows that for every[5] morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ the derived pushforward

$$Rf_*\xi' : Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{M}[n]$$

[5]For example the identity morphism.

of ξ' is compatible with the maps μ and $\mu_{\mathcal{M}[n]}$ constructed above in the sense that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu} & Rf_*\mathcal{A} \\ \downarrow{\scriptstyle Rf_*\xi'\otimes \mathrm{id}} & & \downarrow{\scriptstyle Rf_*\xi'} \\ Rf_*\mathcal{M}[n] \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}[n]}} & Rf_*\mathcal{M}[n] \end{array}$$

commutes in $D(\mathcal{O}')$. Using this compatibility for the map to the punctual topos, we see in particular that

$$\begin{array}{ccc} R\Gamma(\mathcal{C},\mathcal{A}) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C},\mathcal{O})} R\Gamma(\mathcal{C},\mathcal{A}) & \longrightarrow & R\Gamma(\mathcal{C},\mathcal{A}) \\ \downarrow{\scriptstyle \xi'\otimes \mathrm{id}} & & \downarrow{\scriptstyle \xi'} \\ R\Gamma(\mathcal{C},\mathcal{M}[n]) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C},\mathcal{O})} R\Gamma(\mathcal{C},\mathcal{A}) & \longrightarrow & R\Gamma(\mathcal{C},\mathcal{M}[n]) \end{array}$$

commutes. Combined with $\xi'(1) = \xi$ this implies that the induced map on cohomology

$$\xi' : R\Gamma(\mathcal{C},\mathcal{A}) \to R\Gamma(\mathcal{C},\mathcal{M}[n]), \quad \eta \mapsto \xi \cup \eta$$

is given by left cup product by ξ as indicated.

24.33. Differential graded modules on a category

0GZ8 This section is the continuation of Cohomology on Sites, Section 21.43.

Let $\mathcal{C}$ be a category. We think of $\mathcal{C}$ as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded $\mathcal{O}$-algebras. In other words, $\mathcal{O}$ is a presheaf of rings on the category $\mathcal{C}$ and $(\mathcal{A}, \mathrm{d})$ is a presheaf of differential graded $\mathcal{O}$-algebras on $\mathcal{C}$, see Categories, Definition 4.3.3.

0GZ9 **Definition 24.33.1.** In the situation above, we denote $QC(\mathcal{A}, d)$ the full subcategory of $D(\mathcal{A}, \mathrm{d})$ consisting of objects M such that for all $U \to V$ in $\mathcal{C}$ the canonical map

$$R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \longrightarrow R\Gamma(U, M)$$

is an isomorphism in $D(\mathcal{A}(U), \mathrm{d})$.

0GZA **Lemma 24.33.2.** *In the situation above, the subcategory $QC(\mathcal{A}, d)$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{A}, d)$ preserved by arbitrary direct sums.*

Proof. Let U be an object of $\mathcal{C}$. Since the topology on $\mathcal{C}$ is chaotic, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact and commutes with direct sums. Hence the exact functor $M \mapsto R\Gamma(U, M)$ is computed by representing K by any differential graded $\mathcal{A}$-module $\mathcal{M}$ and taking $\mathcal{M}(U)$. Thus $R\Gamma(U, -)$ commutes with direct sums, see Lemma 24.26.8. Similarly, given a morphism $U \to V$ of $\mathcal{C}$ the derived tensor product functor $- \otimes^{\mathbf{L}}_{\mathcal{O}(A)} \mathcal{A}(U) : D(\mathcal{A}(V)) \to D(\mathcal{A}(U))$ is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted. □

0GZB **Remark 24.33.3.** As above, let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$, and let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded $\mathcal{O}$-algebras. Then the analogue of Cohomology on Sites, Proposition 21.43.9 holds for $QC(\mathcal{A}, \mathrm{d})$ with almost exactly the same proof:

(1) any contravariant cohomological functor $H : QC(\mathcal{A}, \mathrm{d}) \to Ab$ which transforms direct sums into products is representable,
(2) any exact functor $F : QC(\mathcal{A}, \mathrm{d}) \to \mathcal{D}$ of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and
(3) the inclusion functor $QC(\mathcal{A}, \mathrm{d}) \to D(\mathcal{A}, \mathrm{d})$ has an exact right adjoint.

If we ever need this we will precisely formulate and prove this here.

Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor between categories. If we view $\mathcal{C}$ and $\mathcal{C}'$ as sites with the chaotic topology, then u is a continuous and cocontinuous functor. Hence we obtain a morphism $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ of topoi, see Sites, Lemma 7.21.1. Additionally, suppose given sheaves of rings $\mathcal{O}$ on $\mathcal{C}$ and $\mathcal{O}'$ on $\mathcal{C}'$ and a map $g^\sharp : g^{-1}\mathcal{O} \to \mathcal{O}'$. We denote the corresponding morphism of ringed topoi simply $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$, see Modules on Sites, Section 18.7. Finally, suppose that $(\mathcal{A}, \mathrm{d})$ is a sheaf of differential graded $\mathcal{O}$-algebras and that $(\mathcal{A}', \mathrm{d})$ is a sheaf of differential graded $\mathcal{O}'$-algebras and moreover that we are given a map $\varphi : g^*\mathcal{A} \to \mathcal{A}'$ of differential graded $\mathcal{O}'$-algebras (see Section 24.18).

0GZC **Lemma 24.33.4.** *Let $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ and $\varphi : g^*\mathcal{A} \to \mathcal{A}'$ be as above. Then the functor $Lg^* : D(\mathcal{A}, d) \to D(\mathcal{A}', d)$ maps $QC(\mathcal{A}, d)$ into $QC(\mathcal{A}', d)$.*

Proof. Let $U' \in \mathrm{Ob}(\mathcal{C}')$ with image $U = u(U')$ in $\mathcal{C}$. Let pt denote the category with a single object and a single morphism. Denote $(Sh(pt), \mathcal{O}'(U'))$ and $(Sh(pt), \mathcal{O}(U))$ the ringed topoi as indicated endowed with the differential graded algebras $\mathcal{A}'(U)$ and $\mathcal{A}(U)$. Of course we identify the derived category of differential graded modules on these with $D(\mathcal{A}'(U'), \mathrm{d})$ and $D(\mathcal{A}(U), \mathrm{d})$. Then we have a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(pt), \mathcal{O}'(U')) & \xrightarrow[U']{} & (Sh(\mathcal{C}'), \mathcal{O}') \\ \downarrow & & \downarrow{\scriptstyle g} \\ (Sh(pt), \mathcal{O}(U)) & \xrightarrow{U} & (Sh(\mathcal{C}), \mathcal{O}) \end{array}$$

each endowed with corresponding differential graded algebras. Pullback along the lower horizontal morphism sends M in $D(\mathcal{A}, \mathrm{d})$ to $R\Gamma(U, K)$ viewed as an object in $D(\mathcal{A}(U), \mathrm{d})$. Pullback by the left vertical arrow sends M to $M \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U')$. Going around the diagram either direction produces the same result (Lemma 24.28.4) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U')$$

Finally, let $f' : U' \to V'$ be a morphism in $\mathcal{C}'$ and denote $f = u(f') : U = u(U') \to V = u(V')$ the image in $\mathcal{C}$. If K is in $QC(\mathcal{A}, \mathrm{d})$ then we have

$$\begin{aligned} R\Gamma(V', Lg^*K) \otimes^{\mathbf{L}}_{\mathcal{A}'(V')} \mathcal{A}'(U') &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}'(V') \otimes^{\mathbf{L}}_{\mathcal{A}'(V')} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U') \\ &= R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U') \\ &= R\Gamma(U', Lg^*K) \end{aligned}$$

as desired. Here we have used the observation above both for U' and V'. □

24.34. Differential graded modules on a category, bis

0GZD We develop a few more results on the notion of quasi-coherent modules introduced in Section 24.33.

0GZE **Lemma 24.34.1.** *Let $\mathcal{C}, \mathcal{O}, \mathcal{A}$ be as in Section 24.33. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory with the following property: for every $U \in \mathrm{Ob}(\mathcal{C})$ the category $U/\mathcal{C}'$ of arrows $U \to U'$ is cofiltered. Denote $\mathcal{O}', \mathcal{A}'$ the restrictions of $\mathcal{O}, \mathcal{A}$ to $\mathcal{C}'$. Then restrictions induces an equivalence $QC(\mathcal{A}, d) \to QC(\mathcal{A}', d)$.*

Proof. We will construct a quasi-inverse of the functor. Namely, let M' be an object of $QC(\mathcal{A}', \mathrm{d})$. We may represent M' by a good differential graded module $\mathcal{M}'$, see Lemma 24.23.7. Then for every $U' \in \mathrm{Ob}(\mathcal{C}')$ the differential graded $\mathcal{A}'(U')$-module $\mathcal{M}'(U)$ is K-flat and graded flat and for every morphism $U'_1 \to U'_2$ of $\mathcal{C}'$ the map

$$\mathcal{M}'(U'_2) \otimes_{\mathcal{A}'(U'_2)} \mathcal{A}'(U'_1) \longrightarrow \mathcal{M}'(U'_1)$$

is a quasi-isomorphism (as the source represents the derived tensor product). Consider the differential graded $\mathcal{A}$-module $\mathcal{M}$ defined by the rule

$$\mathcal{M}(U) = \mathrm{colim}_{U \to U' \in U/\mathcal{C}'} \mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

This is a filtered colimit of complexes by our assumption in the lemma. Since M' is in $QC(\mathcal{A}', \mathrm{d})$ all the transition maps in the system are quasi-isomorphisms. Since filtered colimits are exact, we see that $\mathcal{M}(U)$ in $D(\mathcal{A}(U), \mathrm{d})$ is isomorphic to $\mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$ for any morphism $U \to U'$ with $U' \in \mathrm{Ob}(\mathcal{C}')$.

We claim that $\mathcal{M}$ is in $QC(\mathcal{A}, \mathrm{d})$: namely, given $U \to V$ in $\mathcal{C}$ we choose a map $V \to V'$ with $V' \in \mathrm{Ob}(\mathcal{C}')$. By the above we see that the map $\mathcal{M}(V) \to \mathcal{M}(U)$ is identified with the map

$$\mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(V) \longrightarrow \mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(U)$$

Since $\mathcal{M}'(V')$ is K-flat as differential gradede $\mathcal{A}'(V')$-module, we conclude the claim is true.

The natural map $\mathcal{M}|_{\mathcal{C}'} \to \mathcal{M}'$ is an isomorphism in $D(\mathcal{A}', d)$ as follows immediately from the above.

Conversely, if we have an object E of $QC(\mathcal{A}, \mathrm{d})$, then we represent it by a good differential graded module $\mathcal{E}$. Setting $\mathcal{M}' = \mathcal{E}|_{\mathcal{C}'}$ (this is another good differential graded module) we see that there is a map

$$\mathcal{E} \to \mathcal{M}$$

which over U in $\mathcal{C}$ is given by the map

$$\mathcal{E}(U) \longrightarrow \mathrm{colim}_{U \to U' \in U/\mathcal{C}'} \mathcal{E}(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

which is a quasi-isomorphism by the same reason. Thus restriction and the construction above are quasi-inverse functors as desired. □

0GZF **Lemma 24.34.2.** *Let $\mathcal{C}, \mathcal{O}$ be as in Section 24.33. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded $\mathcal{O}$-algebras which induces an isomorphism on cohomology sheaves, then the equivalence $D(\mathcal{A}, d) \to D(\mathcal{B}, d)$ of Lemma 24.30.1 induces an equivalence $QC(\mathcal{A}, d) \to QC(\mathcal{B}, d)$.*

Proof. It suffices to show the following: given a morphism $U \to V$ of $\mathcal{C}$ and M in $D(\mathcal{A}, \mathrm{d})$ the following are equivalent

(1) $R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \to \Gamma(U, M)$ is an isomorphism in $D(\mathcal{A}(U), \mathrm{d})$, and
(2) $R\Gamma(V, M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B}) \otimes^{\mathbf{L}}_{\mathcal{B}(V)} \mathcal{B}(U) \to \Gamma(U, M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B})$ is an isomorphism in $D(\mathcal{B}(U), \mathrm{d})$.

Since the topology on $\mathcal{C}$ is chaotic, this simply boils down to fact that $\mathcal{A}(U) \to \mathcal{B}(U)$ and $\mathcal{A}(V) \to \mathcal{B}(V)$ are quasi-isomorphisms. Details omitted. □

24.35. Inverse systems of differential graded algebras

0GZG In this section we consider the following special case of the situation discussed in Section 24.33:

(1) $\mathcal{C}$ is the category $\mathbf{N}$ with a unique morphism $i \to j$ if and only if $i \leq j$,
(2) $\mathcal{O}$ is the constant (pre)sheaf of rings with value a given ring R.

In this setting a sheaf $\mathcal{A}$ of differential graded $\mathcal{O}$-algebras is the same thing as an inverse system (A_n) of differential graded R-algebras. A sheaf $\mathcal{M}$ of differential graded $\mathcal{A}$-modules is the same thing as an inverse system (M_n) where M_n is a differential graded A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps.

Suppose that $\mathcal{B} = (B_n)$ is a second inverse system of differential graded R-algebras. Given a morphism $\varphi : (A_n) \to (B_n)$ of pro-objects we will construct an exact functor from $QC(\mathcal{A}, \mathrm{d})$ to $QC(\mathcal{B}, \mathrm{d})$. Namely, according to Categories, Example 4.22.6 the morphism φ is given by a sequence $\ldots \geq m(3) \geq m(2) \geq m(1)$ of integers and a commutative diagram

$$\begin{array}{ccccccc} \ldots & \longrightarrow & A_{m(3)} & \longrightarrow & A_{m(2)} & \longrightarrow & A_{m(1)} \\ & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} \\ \ldots & \longrightarrow & B_3 & \longrightarrow & B_2 & \longrightarrow & B_1 \end{array}$$

of differential graded R-algebras. Then given a good sheaf of differential graded $\mathcal{A}$-modules $\mathcal{M} = (M_n)$ representing an object of $QC(\mathcal{A}, \mathrm{d})$ we can set

$$N_n = M_{m(n)} \otimes_{A_{m(n)}} B_n$$

This inverse system determines an object of $QC(\mathcal{B}, \mathrm{d})$ because the $A_{m(n)}$-modules $M_{m(n)}$ are K-flat; details omitted. We also leave it to the reader to show that the resulting functor is independent of the choices made in its construction.

0GZH **Lemma 24.35.1.** *In the situation above, suppose that $\mathcal{A} = (A_n)$ and $\mathcal{B} = (B_n)$ are inverse systems of differential graded R-algebras. If $\varphi : (A_n) \to (B_n)$ is an isomorphism of pro-objects, then the functor $QC(\mathcal{A}, d) \to QC(\mathcal{B}, d)$ constructed above is an equivalence.*

Proof. Let $\psi : (B_n) \to (A_n)$ be a morphism of pro-objects which is inverse to φ. According to the discussion in Categories, Example 4.22.6 we may assume that φ is given by a system of maps as above and ψ is given $n(1) < n(2) < \ldots$ and a commutative diagram

$$\begin{array}{ccccccc} \ldots & \longrightarrow & B_{n(3)} & \longrightarrow & B_{n(2)} & \longrightarrow & B_{n(1)} \\ & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_1} \\ \ldots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 \end{array}$$

of differential graded R-algebras. Since $\varphi \circ \psi = \mathrm{id}$ we may, after possibly increasing the values of the functions $n(\cdot)$ and $m(\cdot)$ assume that $B_{n(m(i))} \to A_{m(i)} \to B_i$ is the identity. It follows that the composition of the functors

$$QC(\mathcal{B}, \mathrm{d}) \to QC(\mathcal{A}, \mathrm{d}) \to QC(\mathcal{B}, \mathrm{d})$$

sends a good sheaf of differential graded $\mathcal{B}$-modules $\mathcal{N} = (N_n)$ to the inverse system $\mathcal{N}' = (N'_i)$ with values

$$N'_i = N_{n(m(i))} \otimes_{B_{n(m(i))}} B_i$$

which is canonically quasi-isomorphic to $\mathcal{N}$ exactly because $\mathcal{N}$ is an object of $QC(\mathcal{B}, \mathrm{d})$ and because N_j is a K-flat differential graded module for all j. Since the same is true for the composition the other way around we conclude. □

Let $\mathcal{C} = \mathbf{N}$ and $\mathcal{O}$ the constant sheaf with value a ring R and let $\mathcal{A}$ be given by an inverse system (A_n) of differential graded R-algebras. Suppose given two left differential graded $\mathcal{A}$-modules $\mathcal{N}$ and $\mathcal{N}'$ given by inverse systems (N_n) and (N'_n). Thus each N_n and N'_n is a left differential graded A_n-module. Let us temporarily say that (N_n) and (N'_n) are *pro-isomorphic in the derived category* if there exist a sequence of integers

$$1 = n_0 < n_1 < n_2 < n_3 < \ldots$$

and maps

$$N_{n_{2i}} \to N'_{n_{2i-1}} \quad \text{in} \quad D(A^{opp}_{n_{2i}}, \mathrm{d})$$

and

$$N'_{n_{2i+1}} \to N'_{n_{2i}} \quad \text{in} \quad D(A^{opp}_{n_{2i+1}}, \mathrm{d})$$

such that the compositions $N_{n_{2i}} \to N_{n_{2i-2}}$ and $N'_{n_{2i+1}} \to N'_{2i-1}$ are given by the transition maps of the respective systems.

0GZI **Lemma 24.35.2.** *If (N_n) and (N'_n) are pro-isomorphic in the derived category as defined above, then for every object (M_n) of $D(\mathbf{N}, \mathcal{A})$ we have*

$$R\lim(M_n \otimes^{\mathbf{L}}_{A_n} N_n) = R\lim(M_n \otimes^{\mathbf{L}}_{A_n} N'_n)$$

in $D(R)$.

Proof. The assumption implies that the inverse system $(M_n \otimes^{\mathbf{L}}_{A_n} N_n)$ of $D(R)$ is pro-isomorphic (in the usual sense) to the inverse system $(M_n \otimes^{\mathbf{L}}_{A_n} N'_n)$ of $D(R)$. Hence the result follows from the fact that taking $R\lim$ is well defined for inverse systems in the derived category, see discussion in More on Algebra, Section 15.88. □

0GZJ **Lemma 24.35.3.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let K_n be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a differential graded R-algebra. Let (M_n) be an object of $D(\mathbf{N}, (K_n))$. Then for any $t \geq 1$ we have*

This is a variant of [**BS13**, Lemma 3.5.4]

$$R\lim(M_n \otimes^{\mathbf{L}}_R K_t) = R\lim(M_n \otimes^{\mathbf{L}}_{K_n} K_t)$$

in $D(R)$.

Proof. We fix $t \geq 1$. For $n \geq t$ let us denote ${}_nK_t$ the differential graded R-algebra K_t viewed as a left differential graded K_n-module. Observe that

$$M_n \otimes^{\mathbf{L}}_R K_t = M_n \otimes^{\mathbf{L}}_{K_n} (K_n \otimes^{\mathbf{L}}_R K_t) = M_n \otimes^{\mathbf{L}}_{K_n} (K_n \otimes_R K_t)$$

Hence by Lemma 24.35.2 it suffices to show that $({}_nK_t)$ and $(K_n \otimes_R K_t)$ are pro-isomorphic in the derived category. The multiplication maps

$$K_n \otimes_R K_t \longrightarrow {}_nK_t$$

are maps of left differential graded K_n-modules. Thus to finish the proof it suffices to show that for all $n \geq 1$ there exists an $N > n$ and a map

$${}_NK_t \longrightarrow {}_NK_n \otimes_R K_t$$

in $D(K_N^{opp}, \mathrm{d})$ whose composition with the multiplication map is the transition map (in either direction). This is done in Divided Power Algebra, Lemma 23.12.4 by an explicit construction. □

0GZK **Proposition 24.35.4.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. The following three categories are canonically equivalent:*

1. *Let $\mathcal{A}$ be the sheaf of R-algebras on $\mathbf{N}$ corresponding to the inverse system of R-algebras $A_n = R/I^n$. The category $QC(\mathcal{A})$.*
2. *Choose generators $f_1, \ldots, f_r$ of I. Let $\mathcal{B}$ be the sheaf of differential graded R-algebras on $\mathbf{N}$ corresponding to the inverse system of Koszul algebras on $f_1^n, \ldots, f_r^n$. The category $QC(\mathcal{B})$.*
3. *The full subcategory $D_{comp}(R, I) \subset D(R)$ of derived complete objects, see More on Algebra, Definition 15.92.4 and text following.*

Proof. Consider the obvious morphism $f : (Sh(\mathbf{N}), \mathcal{A}) \to (Sh(pt), R)$ of ringed topoi and let us consider the adjoint functors Lf^* and Rf_*. The first restricts to a functor

$$F : D_{comp}(R, I) \longrightarrow QC(\mathcal{A})$$

which sends an object K of $D_{comp}(R, I)$ represented by a K-flat complex $K^\bullet$ to the object $(K^\bullet \otimes_R R/I^n)$ of $QC(\mathcal{A})$. The second restricts to a functor

$$G : QC(\mathcal{A}) \longrightarrow D_{comp}(R, I)$$

which sends an object $(M_n^\bullet)$ of $QC(\mathcal{A})$ to $R\lim M_n^\bullet$. The output is derived complete for example by More on Algebra, Lemma 15.92.14. Also, it follows from More on Algebra, Proposition 15.95.2 that $G \circ F = \mathrm{id}$. Thus to see that F and G are quasi-inverse equivalences it suffices to see that the kernel of G is zero (see Derived Categories, Lemma 13.7.2). However, it does not appear easy to show this directly!

In this paragraph we will show that $QC(\mathcal{A})$ and $QC(\mathcal{B})$ are equivalent. Write $\mathcal{B} = (B_n)$ where B_n is the Koszul complex viewed as a cochain complex in degrees $-r, -r+1, \ldots, 0$. By Divided Power Algebra, Remark 23.12.2 (but with chain complexes turned into cochain complexes) we can find $1 < n_1 < n_2 < \ldots$ and maps of differential graded R-algebras $B_{n_i} \to E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ and $E_i \to B_{n_{i-1}}$ such that

$$\begin{array}{ccccccc}
B_{n_1} & \longleftarrow & B_{n_2} & \longleftarrow & B_{n_3} & \longleftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
E_1 & \longleftarrow & E_2 & \longleftarrow & E_3 & \longleftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longleftarrow & B_{n_1} & \longleftarrow & B_{n_2} & \longleftarrow & \ldots
\end{array}$$

is a commutative diagram of differential graded R-algebras and such that $E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ is a quasi-isomorphism. We conclude

(1) there is an equivalence between $QC(\mathcal{B})$ and $QC((E_i))$,
(2) there is an equivalence between $QC((E_i))$ and $QC((R/(f_1^{n_i}, \ldots, f_r^{n_i})))$,
(3) there is an equivalence between $QC((R/(f_1^{n_i}, \ldots, f_r^{n_i})))$ and $QC(\mathcal{A})$.

Namely, for (1) we can apply Lemma 24.35.1 to the diagram above which shows that (E_i) and (B_n) are pro-isomorphic. For (2) we can apply Lemma 24.34.2 to the inverse system of quasi-isomorphisms $E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$. For (3) we can apply Lemma 24.35.1 and the elementary fact that the inverse systems (R/I^n) and $(R/(f_1^{n_i}, \ldots, f_r^{n_i})$ are pro-isomorphic.

Exactly as in the first paragraph of the proof we can define adjoint functors[6]

$$F' : D_{comp}(R, I) \longrightarrow QC(\mathcal{B}) \quad \text{and} \quad G' : QC(\mathcal{B}) \longrightarrow D_{comp}(R, I).$$

The first sends an object K of $D_{comp}(R, I)$ represented by a K-flat complex $K^\bullet$ to the object $(K^\bullet \otimes_R B_n)$ of $QC(\mathcal{B})$. The second sends an object (M_n) of $QC(\mathcal{B})$ to $R\lim M_n$. Arguing as above it suffices to show that the kernel of G' is zero. So let $\mathcal{M} = (M_n)$ be a good sheaf of differential graded modules over $\mathcal{B}$ which represents an object of $QC(\mathcal{B})$ in the kernel of G'. Then

$$0 = R\lim M_n \Rightarrow 0 = (R\lim M_n) \otimes^{\mathbf{L}}_R B_t = R\lim(M_n \otimes^{\mathbf{L}}_R B_t)$$

By Lemma 24.35.3 we have $R\lim(M_n \otimes^{\mathbf{L}}_R B_t) = R\lim(M_n \otimes^{\mathbf{L}}_{B_n} B_t)$. Since (M_n) is an object of $QC(\mathcal{B})$ we see that the inverse system $M_n \otimes^{\mathbf{L}}_{B_n} B_t$ is eventually constant with value M_t. Hence $M_t = 0$ as desired. □

0H1E **Remark 24.35.5.** Let R be a ring and let $f_1, \ldots, f_r \in R$ be a sequence of elements generating an ideal I. Let K_n be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a differential graded R-algebra. We say $f_1, \ldots, f_r$ is a *weakly proregular sequence* if for all n there is an $m > n$ such that $K_m \to K_n$ induces the zero map on cohomology except in degree 0. If so, then the arguments in the proof of Proposition 24.35.4 continue to work even when R is not Noetherian. In particular we see that $QC(\{R/I^n\})$ is equivalent as an R-linear triangulated category to the category $D_{comp}(R, I)$ of derived complete objects, provided I can be generated by a weakly proregular sequence. If the need arises, we will precisely state and prove this here.

24.36. Other chapters

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(16) Smoothing Ring Maps
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(20) Cohomology of Sheaves
(21) Cohomology on Sites

[6] It can be shown that these functors are, via the equivalences above, compatible with F and G defined before.

(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
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CHAPTER 25

Hypercoverings

01FX

25.1. Introduction

01FY Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let X be an object of $\mathcal{C}$. Given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we would like to compute its cohomology groups

$$H^i(X, \mathcal{F}).$$

According to our general definitions (Cohomology on Sites, Section 21.2) this cohomology group is computed by choosing an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$ and setting

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \ldots)$$

The goal of this chapter is to show that we may also compute these cohomology groups without choosing an injective resolution (in the case that $\mathcal{C}$ has fibre products). To do this we will use hypercoverings.

A hypercovering in a site is a generalization of a covering, see [**AGV71**, Exposé V, Sec. 7]. Given a hypercovering K of an object X, there is a Čech to cohomology spectral sequence expressing the cohomology of an abelian sheaf $\mathcal{F}$ over X in terms of the cohomology of the sheaf over the components K_n of K. It turns out that there are always enough hypercoverings, so that taking the colimit over all hypercoverings, the spectral sequence degenerates and the cohomology of $\mathcal{F}$ over X is computed by the colimit of the Čech cohomology groups.

A more general gadget one can consider is a simplicial augmentation where one has cohomological descent, see [**AGV71**, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see `https://math.stanford.edu/~conrad/papers/hypercover.pdf`. We will come back to these issue in the chapter on simplicial spaces where we will show, for example, that proper hypercoverings of "locally compact" topological spaces are of cohomological descent (Simplicial Spaces, Section 85.25). Our method of attack will be to reduce this statement to the Čech to cohomology spectral sequence constructed in this chapter.

25.2. Semi-representable objects

0DBB In order to start we make the following definition. The letters "SR" stand for Semi-Representable.

01G0 **Definition 25.2.1.** Let $\mathcal{C}$ be a category. We denote $\mathrm{SR}(\mathcal{C})$ the category of *semi-representable objects* defined as follows

(1) objects are families of objects $\{U_i\}_{i \in I}$, and
(2) morphisms $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The category of *semi-representable objects over* X is the category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$.

This definition is essentially equivalent to [**AGV71**, Exposé V, Subsection 7.3.0]. Note that this is a "big" category. We will later "bound" the size of the index sets I that we need for hypercoverings of X. We can then redefine $\mathrm{SR}(\mathcal{C}, X)$ to become a category. Let's spell out the objects and morphisms $\mathrm{SR}(\mathcal{C}, X)$:

(1) objects are families of morphisms $\{U_i \to X\}_{i \in I}$, and
(2) morphisms $\{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

There is a forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

01G1 **Definition 25.2.2.** Let $\mathcal{C}$ be a category. We denote F the functor *which associates a presheaf to a semi-representable object*. In a formula

$$\begin{aligned} F : \mathrm{SR}(\mathcal{C}) &\longrightarrow PSh(\mathcal{C}) \\ \{U_i\}_{i \in I} &\longmapsto \amalg_{i \in I} h_{U_i} \end{aligned}$$

where h_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus we often think of F on $\mathrm{SR}(\mathcal{C}, X)$ as a functor into the category of presheaves of sets over h_X, namely $PSh(\mathcal{C})/h_X$. Here is a picture:

$$\begin{array}{ccc} \mathrm{SR}(\mathcal{C}, X) & \xrightarrow[F]{} & PSh(\mathcal{C})/h_X \\ \downarrow & & \downarrow \\ \mathrm{SR}(\mathcal{C}) & \xrightarrow{F} & PSh(\mathcal{C}) \end{array}$$

Next we discuss the existence of limits in the category of semi-representable objects.

01G2 **Lemma 25.2.3.** *Let $\mathcal{C}$ be a category.*

(1) *the category $SR(\mathcal{C})$ has coproducts and F commutes with them,*
(2) *the functor $F : SR(\mathcal{C}) \to PSh(\mathcal{C})$ commutes with limits,*
(3) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C})$ has fibre products,*
(4) *if $\mathcal{C}$ has products of pairs, then $SR(\mathcal{C})$ has products of pairs,*
(5) *if $\mathcal{C}$ has equalizers, so does $SR(\mathcal{C})$, and*
(6) *if $\mathcal{C}$ has a final object, so does $SR(\mathcal{C})$.*

Let $X \in \mathrm{Ob}(\mathcal{C})$.

(1) *the category $SR(\mathcal{C}, X)$ has coproducts and F commutes with them,*
(2) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C}, X)$ has finite limits and $F : SR(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$ commutes with them.*

Proof. Proof of the results on $\mathrm{SR}(\mathcal{C})$. Proof of (1). The coproduct of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i\}_{i \in I} \amalg \{V_j\}_{j \in J}$, in other words, the family of objects whose index set is $I \amalg J$ and for an element $k \in I \amalg J$ gives U_i if $k = i \in I$ and gives V_j if $k = j \in J$. Similarly for coproducts of families of objects. It is clear that F commutes with these.

Proof of (2). For U in $\mathrm{Ob}(\mathcal{C})$ consider the object $\{U\}$ of $\mathrm{SR}(\mathcal{C})$. It is clear that $\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\}, K)) = F(K)(U)$ for $K \in \mathrm{Ob}(\mathrm{SR}(\mathcal{C}))$. Since limits of presheaves are computed at the level of sections (Sites, Section 7.4) we conclude that F commutes with limits.

Proof of (3). Suppose given a morphism $(\alpha, f_i) : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ and a morphism $(\beta, g_k) : \{W_k\}_{k\in K} \to \{V_j\}_{j\in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j,k)\in I\times J\times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist if $\mathcal{C}$ has fibre products.

Proof of (4). The product of $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ is $\{U_i \times V_j\}_{i\in I, j\in J}$. The products exist if $\mathcal{C}$ has products.

Proof of (5). The equalizer of two maps $(\alpha, f_i), (\alpha', f_i') : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ is

$$\{\mathrm{Eq}(f_i, f_i' : U_i \to V_{\alpha(i)})\}_{i\in I,\ \alpha(i)=\alpha'(i)}$$

The equalizers exist if $\mathcal{C}$ has equalizers.

Proof of (6). If X is a final object of $\mathcal{C}$, then $\{X\}$ is a final object of $\mathrm{SR}(\mathcal{C})$.

Proof of the statements about $\mathrm{SR}(\mathcal{C}, X)$. These follow from the results above applied to the category $\mathcal{C}/X$ using that $\mathrm{SR}(\mathcal{C}/X) = \mathrm{SR}(\mathcal{C}, X)$ and that $PSh(\mathcal{C}/X) = PSh(\mathcal{C})/h_X$ (Sites, Lemma 7.25.4 applied to $\mathcal{C}$ endowed with the chaotic topology). However we also argue directly as follows. It is clear that the coproduct of $\{U_i \to X\}_{i\in I}$ and $\{V_j \to X\}_{j\in J}$ is $\{U_i \to X\}_{i\in I} \amalg \{V_j \to X\}_{j\in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X \to X\}$ is a final object of $\mathrm{SR}(\mathcal{C}, X)$. Suppose given a morphism $(\alpha, f_i) : \{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$ and a morphism $(\beta, g_k) : \{W_k \to X\}_{k\in K} \to \{V_j \to X\}_{j\in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j,k)\in I\times J\times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $\mathrm{SR}(\mathcal{C}, X)$ has finite limits, see Categories, Lemma 4.18.4. We omit verifying the statements on the functor F in this case. □

25.3. Hypercoverings

01FZ If we assume our category is a site, then we can make the following definition.

01G3 **Definition 25.3.1.** Let $\mathcal{C}$ be a site. Let $f = (\alpha, f_i) : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C})$. We say that f is a *covering* if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i\in I, \alpha(i)=j}$ is a covering for the site $\mathcal{C}$. Let X be an object of $\mathcal{C}$. A morphism $K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$ is a *covering* if its image in $\mathrm{SR}(\mathcal{C})$ is a covering.

01G4 **Lemma 25.3.2.** *Let $\mathcal{C}$ be a site.*

(1) *A composition of coverings in $SR(\mathcal{C})$ is a covering.*
(2) *If $K \to L$ is a covering in $SR(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times_L K$ exists and $L' \times_L K \to L'$ is a covering.*
(3) *If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $SR(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.*

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $SR(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Proof. Part (1) is immediate from the axioms of a site. Part (2) follows by the construction of fibre products in $\mathrm{SR}(\mathcal{C})$ in the proof of Lemma 25.2.3 and the requirement that the morphisms in a covering of $\mathcal{C}$ are representable. Part (3)

follows by thinking of $A \times K \to B \times L$ as the composition $A \times K \to B \times K \to B \times L$ and hence a composition of basechanges of coverings. The final statement follows because $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. □

By Lemma 25.2.3 and Simplicial, Lemma 14.19.2 the coskeleton of a truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$ exists if $\mathcal{C}$ has fibre products. Hence the following definition makes sense.

01G5 **Definition 25.3.3.** Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. A *hypercovering of* X is a simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ such that

(1) The object K_0 is a covering of X for the site $\mathcal{C}$.
(2) For every $n \geq 0$ the canonical morphism

$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$$

is a covering in the sense defined above.

Condition (1) makes sense since each object of $\mathrm{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X. It could also be formulated as saying that the morphism of K_0 to the final object of $\mathrm{SR}(\mathcal{C}, X)$ is a covering.

01G6 **Example 25.3.4** (Čech hypercoverings)**.** Let $\mathcal{C}$ be a site with fibre products. Let $\{U_i \to X\}_{i \in I}$ be a covering of $\mathcal{C}$. Set $K_0 = \{U_i \to X\}_{i \in I}$. Then K_0 is a 0-truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. Hence we may form

$$K = \mathrm{cosk}_0 K_0.$$

Clearly K passes condition (1) of Definition 25.3.3. Since all the morphisms $K_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ are isomorphisms by Simplicial, Lemma 14.19.10 it also passes condition (2). Note that the terms K_n are the usual

$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0, i_1, \ldots, i_n) \in I^{n+1}}$$

A hypercovering of X of this form is called a *Čech hypercovering* of X.

0GM9 **Example 25.3.5** (Hypercovering by a simplicial object of the site)**.** Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$. Finally, assume given an augmentation

$$a : U \to X$$

In this situation we can consider the simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ with terms $K_n = \{U_n \to X\}$. Then K is a hypercovering of X in the sense of Definition 25.3.3 if and only if the following three conditions[1] hold:

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

We omit the straightforward verification.

[1] As $\mathcal{C}$ has fibre products, the category $\mathcal{C}/X$ has all finite limits. Hence the required coskeleta exist by Simplicial, Lemma 14.19.2.

0GMA **Example 25.3.6** (Čech hypercovering associated to a cover)**.** Let $\mathcal{C}$ be a site with fibre products. Let $U \to X$ be a morphism of $\mathcal{C}$ such that $\{U \to X\}$ is a covering of $\mathcal{C}$[2]. Consider the simplical object K of $\mathrm{SR}(\mathcal{C}, X)$ with terms

$$K_n = \{U \times_X U \times_X \ldots \times_X U \to X\} \quad (n+1 \text{ factors})$$

Then K is a hypercovering of X. This example is a special case of both Example 25.3.4 and of Example 25.3.5.

01G7 **Lemma 25.3.7.** *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The collection of all hypercoverings of X forms a set.*

Proof. Since $\mathcal{C}$ is a site, the set of all coverings of X forms a set. Thus we see that the collection of possible K_0 forms a set. Suppose we have shown that the collection of all possible $K_0, \ldots, K_n$ form a set. Then it is enough to show that given $K_0, \ldots, K_n$ the collection of all possible K_{n+1} forms a set. And this is clearly true since we have to choose K_{n+1} among all possible coverings of $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$. □

01G8 **Remark 25.3.8.** The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on $\mathcal{C}$ has finite (co)limits. Hence the functors cosk_n exists for presheaves of sets.

01G9 **Lemma 25.3.9.** *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. Let K be a hypercovering of X. Consider the simplicial object $F(K)$ of $PSh(\mathcal{C})$, endowed with its augmentation to the constant simplicial presheaf h_X.*

(1) *The morphism of presheaves $F(K)_0 \to h_X$ becomes a surjection after sheafification.*
(2) *The morphism*
$$(d_0^1, d_1^1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$
becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*
$$F(K)_{n+1} \longrightarrow (cosk_n sk_n F(K))_{n+1}$$
turns into a surjection after sheafification.

Proof. We will use the fact that if $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism

$$\amalg_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification, see Sites, Lemma 7.12.4. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 14.19.1 the simplicial object $\mathrm{cosk}_0 \mathrm{sk}_0 K$ has terms $K_0 \times \ldots \times K_0$. Thus according to the definition of a hypercovering we see that $(d_0^1, d_1^1) : K_1 \to K_0 \times K_0$ is a covering. Hence (2) follows from the claim above and the fact that F transforms products into fibred products over h_X.

For the third, we claim that $\mathrm{cosk}_n \mathrm{sk}_n F(K) = F(\mathrm{cosk}_n \mathrm{sk}_n K)$ for $n \geq 1$. To prove this, denote temporarily F' the functor $\mathrm{SR}(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$. By Lemma 25.2.3

[2] A morphism of $\mathcal{C}$ with this property is sometimes called a "cover".

the functor F' commutes with finite limits. By our description of the cosk_n functor in Simplicial, Section 14.12 we see that $\mathrm{cosk}_n\mathrm{sk}_n F'(K) = F'(\mathrm{cosk}_n\mathrm{sk}_n K)$. Recall that the category used in the description of $(\mathrm{cosk}_n U)_m$ in Simplicial, Lemma 14.19.2 is the category $(\Delta/[m])^{opp}_{\leq n}$. It is an amusing exercise to show that $(\Delta/[m])_{\leq n}$ is a connected category (see Categories, Definition 4.16.1) as soon as $n \geq 1$. Hence, Categories, Lemma 4.16.2 shows that $\mathrm{cosk}_n\mathrm{sk}_n F'(K) = \mathrm{cosk}_n\mathrm{sk}_n F(K)$. Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor F to a covering as in Definition 25.3.1, and the result follows from the first fact mentioned in this proof. □

25.4. Acyclicity

01GA Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_\mathcal{F}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_\mathcal{F}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the *free abelian presheaf on* $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_\mathcal{F}$ is a functor and it is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Of course the sheafification $\mathbf{Z}^\#_\mathcal{F}$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}^\#_\mathcal{F}$ is a left adjoint as well. We sometimes call $\mathbf{Z}^\#_\mathcal{F}$ the *free abelian sheaf on* $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on h_X, and we denote $\mathbf{Z}^\#_X$ its sheafification.

01GB **Definition 25.4.1.** Let $\mathcal{C}$ be a site. Let K be a simplicial object of $PSh(\mathcal{C})$. By the above we get a simplicial object $\mathbf{Z}^\#_K$ of $Ab(\mathcal{C})$. We can take its associated complex of abelian presheaves $s(\mathbf{Z}^\#_K)$, see Simplicial, Section 14.23. The *homology of* K is the homology of the complex of abelian sheaves $s(\mathbf{Z}^\#_K)$.

In other words, the *ith homology* $H_i(K)$ *of* K is the sheaf of abelian groups $H_i(K) = H_i(s(\mathbf{Z}^\#_K))$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

01GC **Lemma 25.4.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of sets. Denote K the simplicial object of $PSh(\mathcal{C})$ whose nth term is the $(n+1)$st fibre product of $\mathcal{F}$ over $\mathcal{G}$, see Simplicial, Example 14.3.5. Then, if $\mathcal{F} \to \mathcal{G}$ is surjective after sheafification, we have*

$$H_i(K) = \begin{cases} 0 & \text{if} \quad i > 0 \\ \mathbf{Z}^\#_\mathcal{G} & \text{if} \quad i = 0 \end{cases}$$

The isomorphism in degree 0 is given by the morphism $H_0(K) \to \mathbf{Z}^\#_\mathcal{G}$ coming from the map $(\mathbf{Z}^\#_K)_0 = \mathbf{Z}^\#_\mathcal{F} \to \mathbf{Z}^\#_\mathcal{G}$.

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be the image of the morphism $\mathcal{F} \to \mathcal{G}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Set $A = \mathcal{F}(U)$ and $B = \mathcal{G}'(U)$. Then the simplicial set $K(U)$ is equal to the simplicial set with n-simplices given by

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \text{ factors}).$$

By Simplicial, Lemma 14.32.3 the morphism $K(U) \to B$ is a trivial Kan fibration. Thus it is a homotopy equivalence (Simplicial, Lemma 14.30.8). Hence applying

the functor "free abelian group on" to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that $s(\mathbf{Z}_B)$ is the complex

$$\ldots \to \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{1} \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B} \mathbf{Z} \to 0$$

see Simplicial, Lemma 14.23.3. Thus we see that $H_i(s(\mathbf{Z}_K(U))) = 0$ for $i > 0$, and $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b\in B} \mathbf{Z} = \bigoplus_{s\in\mathcal{G}'(U)} \mathbf{Z}$. These identifications are compatible with restriction maps.

We conclude that $H_i(s(\mathbf{Z}_K)) = 0$ for $i > 0$ and $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$, where here we compute homology groups in $PAb(\mathcal{C})$. Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that $H_0(s(\mathbf{Z}_K))^\# = H_0(s(\mathbf{Z}_K^\#))$, and similarly for other indices. □

01GD **Lemma 25.4.3.** *Let $\mathcal{C}$ be a site. Let $f : L \to K$ be a morphism of simplicial objects of $PSh(\mathcal{C})$. Let $n \geq 0$ be an integer. Assume that*

(1) *For $i < n$ the morphism $L_i \to K_i$ is an isomorphism.*
(2) *The morphism $L_n \to K_n$ is surjective after sheafification.*
(3) *The canonical map $L \to cosk_n sk_n L$ is an isomorphism.*
(4) *The canonical map $K \to cosk_n sk_n K$ is an isomorphism.*

Then $H_i(f) : H_i(L) \to H_i(K)$ is an isomorphism.

Proof. This proof is exactly the same as the proof of Lemma 25.4.2 above. Namely, we first let $K'_n \subset K_n$ be the sub presheaf which is the image of the map $L_n \to K_n$. Assumption (2) means that the sheafification of K'_n is equal to the sheafification of K_n. Moreover, since $L_i = K_i$ for all $i < n$ we see that get an n-truncated simplicial presheaf U by taking $U_0 = L_0 = K_0, \ldots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K'_n$. Denote $K' = \text{cosk}_n U$, a simplicial presheaf. Because we can construct K'_m as a finite limit, and since sheafification is exact, we see that $(K'_m)^\# = K_m$. In other words, $(K')^\# = K^\#$. We conclude, by exactness of sheafification once more, that $H_i(K) = H_i(K')$. Thus it suffices to prove the lemma for the morphism $L \to K'$, in other words, we may assume that $L_n \to K_n$ is a surjective morphism of *presheaves*!

In this case, for any object U of $\mathcal{C}$ we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 14.32.1. Hence it is a trivial Kan fibration. In particular it is a homotopy equivalence (Simplicial, Lemma 14.30.8). Thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all U. The result follows. □

01GE **Lemma 25.4.4.** *Let $\mathcal{C}$ be a site. Let K be a simplicial presheaf. Let $\mathcal{G}$ be a presheaf. Let $K \to \mathcal{G}$ be an augmentation of K towards $\mathcal{G}$. Assume that*

(1) *The morphism of presheaves $K_0 \to \mathcal{G}$ becomes a surjection after sheafification.*
(2) *The morphism*

$$(d_0^1, d_1^1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$

becomes a surjection after sheafification.

(3) *For every $n \geq 1$ the morphism*

$$K_{n+1} \longrightarrow (cosk_n sk_n K)_{n+1}$$

turns into a surjection after sheafification.

Then $H_i(K) = 0$ for $i > 0$ and $H_0(K) = \mathbf{Z}^\#_{\mathcal{G}}$.

Proof. Denote $K^n = \mathrm{cosk}_n \mathrm{sk}_n K$ for $n \geq 1$. Define K^0 as the simplicial object with terms $(K^0)_n$ equal to the $(n+1)$-fold fibred product $K_0 \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} K_0$, see Simplicial, Example 14.3.5. We have morphisms

$$K \longrightarrow \ldots \to K^n \to K^{n-1} \to \ldots \to K^1 \to K^0.$$

The morphisms $K \to K^i$, $K^j \to K^i$ for $j \geq i \geq 1$ come from the universal properties of the cosk_n functors. The morphism $K^1 \to K^0$ is the canonical morphism from Simplicial, Remark 14.20.4. We also recall that $K^0 \to \mathrm{cosk}_1 \mathrm{sk}_1 K^0$ is an isomorphism, see Simplicial, Lemma 14.20.3.

By Lemma 25.4.2 we see that $H_i(K^0) = 0$ for $i > 0$ and $H_0(K^0) = \mathbf{Z}^\#_{\mathcal{G}}$.

Pick $n \geq 1$. Consider the morphism $K^n \to K^{n-1}$. It is an isomorphism on terms of degree $< n$. Note that $K^n \to \mathrm{cosk}_n \mathrm{sk}_n K^n$ and $K^{n-1} \to \mathrm{cosk}_n \mathrm{sk}_n K^{n-1}$ are isomorphisms. Note that $(K^n)_n = K_n$ and that $(K^{n-1})_n = (\mathrm{cosk}_{n-1} \mathrm{sk}_{n-1} K)_n$. Hence by assumption, we have that $(K^n)_n \to (K^{n-1})_n$ is a morphism of presheaves which becomes surjective after sheafification. By Lemma 25.4.3 we conclude that $H_i(K^n) = H_i(K^{n-1})$. Combined with the above this proves the lemma. □

01GF **Lemma 25.4.5.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees > 0 and equal to $\mathbf{Z}^\#_X$ in degree 0.*

Proof. Combine Lemmas 25.4.4 and 25.3.9. □

25.5. Čech cohomology and hypercoverings

01GU Let $\mathcal{C}$ be a site. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor

$$\begin{aligned} \mathcal{F} : \mathrm{SR}(\mathcal{C})^{opp} &\longrightarrow Ab \\ \{U_i\}_{i \in I} &\longmapsto \prod\nolimits_{i \in I} \mathcal{F}(U_i) \end{aligned}$$

Thus a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. The cochain complex $s(\mathcal{F}(K))$ associated to $\mathcal{F}(K)$ (Simplicial, Section 14.25) is called the Čech complex of $\mathcal{F}$ with respect to the simplicial object K. We set

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$

and we call it the ith Čech cohomology group of $\mathcal{F}$ with respect to K. In this section we prove analogues of some of the results for Čech cohomology of open coverings proved in Cohomology, Sections 20.9, 20.10 and 20.11.

01GV **Lemma 25.5.1.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. We have

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$

Write $K_0 = \{U_i \to X\}$. It is a covering in the site $\mathcal{C}$. As well, we have that $K_1 \to K_0 \times K_0$ is a covering in $\mathrm{SR}(\mathcal{C}, X)$. Hence we may write $K_1 = \amalg_{i_0, i_1 \in I}\{V_{i_0 i_1 j} \to X\}$ so that the morphism $K_1 \to K_0 \times K_0$ is given by coverings $\{V_{i_0 i_1 j} \to U_{i_0} \times_X U_{i_1}\}$ of the site $\mathcal{C}$. Thus we can further identify

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\prod\nolimits_i \mathcal{F}(U_i) \longrightarrow \prod\nolimits_{i_0 i_1 j} \mathcal{F}(V_{i_0 i_1 j}))$$

with obvious map. The sheaf property of $\mathcal{F}$ implies that $\check{H}^0(K, \mathcal{F}) = H^0(X, \mathcal{F})$. □

In fact this property characterizes the abelian sheaves among all abelian presheaves on $\mathcal{C}$ of course. The analogue of Cohomology, Lemma 25.5.2 in this case is the following.

01GW **Lemma 25.5.2.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\mathrm{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\ &= \prod \mathrm{Mor}_{PSh(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\ &= \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\ &= \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^{\#}_{F(Z)}, \mathcal{F}) \end{aligned}$$

Thus we see, for any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ that we have

01GX (25.5.2.1) $$s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}^{\#}_{F(K)}), \mathcal{F})$$

see Definition 25.4.1 for notation. The complex of sheaves $s(\mathbf{Z}^{\#}_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^{\#}_X$ if K is a hypercovering, see Lemma 25.4.5. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, and K a hypercovering, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for $i > 0$. Combined with Lemma 25.5.1 the lemma is proved. □

Next we come to the analogue of Cohomology on Sites, Lemma 21.10.6. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Recall that $\underline{H}^i(\mathcal{F})$ indicates the presheaf of abelian groups on $\mathcal{C}$ which is defined by the rule $\underline{H}^i(\mathcal{F}) : U \longmapsto H^i(U, \mathcal{F})$. We extend this to $\mathrm{SR}(\mathcal{C})$ as in the introduction to this section.

01GY **Lemma 25.5.3.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces natural transformations

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

as functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential on the complex $s(\mathcal{I}^q(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\mathrm{Tot}(A^{\bullet,\bullet})$ the total complex associated to the double complex $A^{\bullet,\bullet}$, see Homology, Section 12.18. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

By Lemma 25.5.2 the complexes $s(\mathcal{I}^q(K))$ are acyclic in positive degrees and have H^0 equal to $\mathcal{I}^q(X)$. Hence by Homology, Lemma 12.25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\mathrm{Tot}(A^{\bullet,\bullet})) = H^n(X, \mathcal{F})$.

The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$ of the lemma is the composition of the map $s(\mathcal{F}(K)) \to \mathrm{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $\mathcal{I}^\bullet(X)$ is a representative of $R\Gamma(X, \mathcal{F})$.

Consider the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$. By Homology, Lemma 12.25.1 we see that

$${}'E_2^{p,q} = H_I^p(H_{II}^q(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_2 which gives the groups ${}'E_1^{p,q} = \underline{H}^q(\mathcal{F})(K_p)$. Hence it is indeed the case (by the description of the differential ${}'d_1$) that ${}'E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. By the above and Homology, Lemma 12.25.3 we see that this converges to $H^n(X, \mathcal{F})$ as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf $\mathcal{F}$ and the hypercovering K. □

25.6. Hypercoverings a la Verdier

09VT The astute reader will have noticed that all we need in order to get the Čech to cohomology spectral sequence for a hypercovering of an object X, is the conclusion of Lemma 25.3.9. Therefore the following definition makes sense.

09VU **Definition 25.6.1.** Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets. A *hypercovering of* $\mathcal{G}$ is a simplicial object K of $\mathrm{SR}(\mathcal{C})$ endowed with an augmentation $F(K) \to \mathcal{G}$ such that

(1) $F(K_0) \to \mathcal{G}$ becomes surjective after sheafification,
(2) $F(K_1) \to F(K_0) \times_{\mathcal{G}} F(K_0)$ becomes surjective after sheafification, and
(3) $F(K_{n+1}) \longrightarrow F((\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1})$ for $n \geq 1$ becomes surjective after sheafification.

We say that a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is a *hypercovering* if K is a hypercovering of the final object $*$ of $PSh(\mathcal{C})$.

The assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\mathrm{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 25.2.3) which suffices to define the coskeleton functors used (see Simplicial, Remark 14.19.11 and Categories, Lemma 4.18.2). If $\mathcal{C}$ is general, we can replace the condition (3) by the condition that $F(K_{n+1}) \longrightarrow ((\mathrm{cosk}_n \mathrm{sk}_n F(K))_{n+1})$ for $n \geq 1$ becomes surjective after sheafification and the results of this section remain valid.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. In the previous section, we defined the Čech complex of $\mathcal{F}$ with respect to a simplicial object K of $\mathrm{SR}(\mathcal{C})$. Next, given a presheaf $\mathcal{G}$ we set

$$H^0(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, \mathcal{F}) = H^0(\mathcal{G}^\#, \mathcal{F})$$

with notation as in Cohomology on Sites, Section 21.13. This is a left exact functor and its higher derived functors (briefly studied in Cohomology on Sites, Section 21.13) are denoted $H^i(\mathcal{G}, \mathcal{F})$. We will show that given a hypercovering K of $\mathcal{G}$, there is a Čech to cohomology spectral sequence converging to the cohomology $H^i(\mathcal{G}, \mathcal{F})$. Note that if $\mathcal{G} = *$, then $H^i(*, \mathcal{F}) = H^i(\mathcal{C}, \mathcal{F})$ recovers the cohomology of $\mathcal{F}$ on the site $\mathcal{C}$.

09VV **Lemma 25.6.2.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = H^0(\mathcal{G}, \mathcal{F})$.*

Proof. This follows from the definition of $H^0(\mathcal{G}, \mathcal{F})$ and the fact that

$$F(K_1) \rightrightarrows F(K_0) \longrightarrow \mathcal{G}$$

becomes an coequalizer diagram after sheafification. □

09VW **Lemma 25.6.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} H^0(\mathcal{G}, \mathcal{I}) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By (25.5.2.1) we have

$$s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}^\#_{F(K)}), \mathcal{F})$$

The complex $s(\mathbf{Z}^\#_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^\#_{\mathcal{G}}$, see Lemma 25.4.4. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have $\check{H}^i(K, \mathcal{I}) = 0$ for $i > 0$. Combined with Lemma 25.6.2 the lemma is proved. □

09VX **Lemma 25.6.4.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces a natural transformation

$$\check{H}^i(K, -) \longrightarrow H^i(\mathcal{G}, -)$$

of functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(\mathcal{G}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

By Lemma 25.6.3 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $H^0(\mathcal{G}, \mathcal{I}^p)$. Hence by Homology, Lemma 12.25.4 and its proof the spectral sequence $({}'E_r, {}'d_r)$ degenerates, and the natural map

$$H^0(\mathcal{G}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to \mathrm{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $H^0(\mathcal{G}, \mathcal{I}^\bullet)$ is a representative of $R\Gamma(\mathcal{G}, \mathcal{F})$.

Consider the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$. By Homology, Lemma 12.25.1 we see that

$${}''E_2^{p,q} = H_{II}^p(H_I^q(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential ${}''d_1$) that ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. Since this spectral sequence converges to the cohomology of $\mathrm{Tot}(A^{\bullet,\bullet})$ the proof is finished. □

09VY **Lemma 25.6.5.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{F}$ be an abelian sheaf. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to the global cohomology groups $H^{p+q}(\mathcal{F})$.

Proof. This is a special case of Lemma 25.6.4. □

25.7. Covering hypercoverings

01GG Here are some ways to construct hypercoverings. We note that since the category $\mathrm{SR}(\mathcal{C}, X)$ has fibre products the category of simplicial objects of $\mathrm{SR}(\mathcal{C}, X)$ has fibre products as well, see Simplicial, Lemma 14.7.2.

01GH **Lemma 25.7.1.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L, M be simplicial objects of $SR(\mathcal{C}, X)$. Let $a : K \to L$, $b : M \to L$ be morphisms. Assume*

(1) *K is a hypercovering of X,*
(2) *the morphism $M_0 \to L_0$ is a covering, and*
(3) *for all $n \geq 0$ in the diagram*

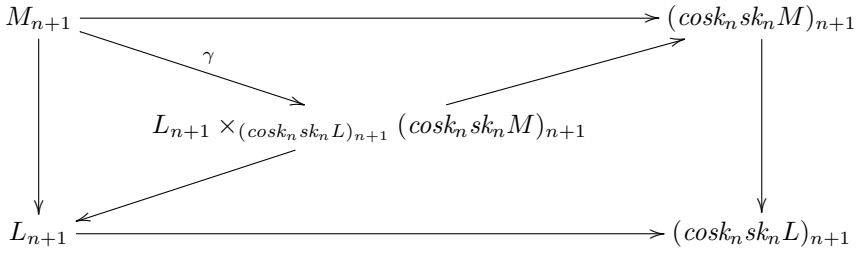

the arrow γ is a covering.

Then the fibre product $K \times_L M$ is a hypercovering of X.

Proof. The morphism $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \to K_0$ is a base change of a covering by (2), hence a covering, see Lemma 25.3.2. And $K_0 \to \{X \to X\}$ is a covering by (1). Thus $(K \times_L M)_0 \to \{X \to X\}$ is a covering by Lemma 25.3.2. Hence $K \times_L M$ satisfies the first condition of Definition 25.3.3.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n (K \times_L M))_{n+1}$$

is a covering for all $n \geq 0$. We abbreviate as follows: $A = (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$, $B = (\mathrm{cosk}_n \mathrm{sk}_n L)_{n+1}$, and $C = (\mathrm{cosk}_n \mathrm{sk}_n M)_{n+1}$. The functor $\mathrm{cosk}_n \mathrm{sk}_n$ commutes with fibre products, see Simplicial, Lemma 14.19.13. Thus the right hand side above is equal to $A \times_B C$. Consider the following commutative diagram

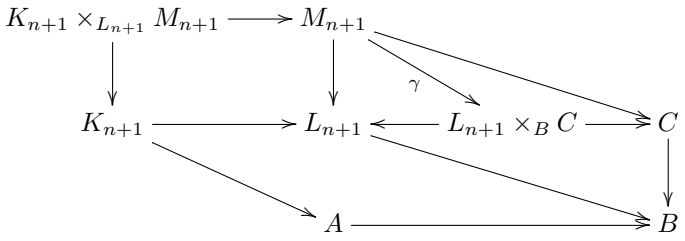

This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now, $K_{n+1} \times_B C \to A \times_B C$ is a base change of the covering $K_{n+1} \to A$ via the morphism $A \times_B C \to A$, hence is a covering. By assumption (3) the morphism γ is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering. □

01GI **Lemma 25.7.2.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.*

Proof. You can either verify this directly, or use Lemma 25.7.1 above and check that $L \to \{X \to X\}$ has property (3). □

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Since the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and finite limits, it is permissible to speak about the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U (for example those with finitely many nondegenerate simplices) and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$. See Simplicial, Sections 14.13 and 14.17.

01GJ **Lemma 25.7.3.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in $SR(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.*

Proof. Denote $Y = K_k$. Let $C[k]$ be the cosimplicial set defined in Simplicial, Example 14.5.6. We will use the description of $\mathrm{Hom}(C[k], Y)$ and $\mathrm{Hom}(C[k], Z)$ given in Simplicial, Lemma 14.15.2. There is a canonical morphism $K \to \mathrm{Hom}(C[k], Y)$ corresponding to $\mathrm{id} : K_k = Y \to Y$. Consider the morphism $\mathrm{Hom}(C[k], Z) \to \mathrm{Hom}(C[k], Y)$ which on degree n terms is the morphism

$$\prod\nolimits_{\alpha:[k]\to[n]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[n]} Y$$

using the given morphism $Z \to Y$ on each factor. Set

$$L = K \times_{\mathrm{Hom}(C[k],Y)} \mathrm{Hom}(C[k], Z).$$

The morphism $L_k \to K_k$ sits in to a commutative diagram

$$\begin{array}{ccccc} L_k & \longrightarrow & \prod_{\alpha:[k]\to[k]} Z & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Z \\ \downarrow & & \downarrow & & \downarrow \\ K_k & \longrightarrow & \prod_{\alpha:[k]\to[k]} Y & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Y \end{array}$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that L is a hypercovering of X. To see this we will use Lemma 25.7.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$\mathrm{Hom}(C[k], Z)_0 \to \mathrm{Hom}(C[k], Y)_0$$

is a covering because it is isomorphic to $Z \to Y$ as there is only one morphism $[k] \to [0]$.

Let us consider condition (3) for $n = 0$. Then, since $(\mathrm{cosk}_0 T)_1 = T \times T$ (Simplicial, Example 14.19.1) and since $\mathrm{Hom}(C[k], Z)_1 = \prod_{\alpha:[k]\to[1]} Z$ we obtain the diagram

$$\begin{array}{ccc} \prod_{\alpha:[k]\to[1]} Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ \prod_{\alpha:[k]\to[1]} Y & \longrightarrow & Y \times Y \end{array}$$

with horizontal arrows corresponding to the projection onto the factors corresponding to the two nonsurjective α. Thus the arrow γ is the morphism

$$\prod\nolimits_{\alpha:[k]\to[1]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[1] \text{ not onto}} Z \times \prod\nolimits_{\alpha:[k]\to[1] \text{ onto}} Y$$

which is a product of coverings and hence a covering by Lemma 25.3.2.

Let us consider condition (3) for $n > 0$. We claim there is an injective map $\tau : S' \to S$ of finite sets, such that for any object T of $\mathrm{SR}(\mathcal{C}, X)$ the morphism

0B16 (25.7.3.1) $$\mathrm{Hom}(C[k], T)_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(C[k], T))_{n+1}$$

is isomorphic to the projection $\prod_{s\in S} T \to \prod_{s'\in S'} T$ functorially in T. If this is true, then we see, arguing as in the previous paragraph, that the arrow γ is the morphism

$$\prod\nolimits_{s\in S} Z \longrightarrow \prod\nolimits_{s\in S'} Z \times \prod\nolimits_{s\notin\tau(S')} Y$$

which is a product of coverings and hence a covering by Lemma 25.3.2. By construction, we have $\mathrm{Hom}(C[k], T)_{n+1} = \prod_{\alpha:[k]\to[n+1]} T$ (see Simplicial, Lemma 14.15.2). Correspondingly we take $S = \mathrm{Map}([k], [n+1])$. On the other hand, Simplicial, Lemma 14.19.5, provides a description of points of $(\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(C[k], T))_{n+1}$ as sequences $(f_0, \ldots, f_{n+1})$ of points of $\mathrm{Hom}(C[k], T)_n$ satisfying $d^n_{j-1} f_i = d^n_i f_j$ for $0 \leq i < j \leq n+1$. We can write $f_i = (f_{i,\alpha})$ with $f_{i,\alpha}$ a point of T and $\alpha \in \mathrm{Map}([k], [n])$. The conditions translate into

$$f_{i,\delta^n_{j-1}\circ\beta} = f_{j,\delta^n_i\circ\beta}$$

for any $0 \leq i < j \leq n+1$ and $\beta : [k] \to [n-1]$. Thus we see that

$$S' = \{0, \ldots, n+1\} \times \mathrm{Map}([k], [n]) / \sim$$

where the equivalence relation is generated by the equivalences

$$(i, \delta^n_{j-1} \circ \beta) \sim (j, \delta^n_i \circ \beta)$$

for $0 \leq i < j \leq n+1$ and $\beta : [k] \to [n-1]$. A computation (omitted) shows that the morphism (25.7.3.1) corresponds to the map $S' \to S$ which sends (i, α) to $\delta^{n+1}_i \circ \alpha \in S$. (It may be a comfort to the reader to see that this map is well defined by part (1) of Simplicial, Lemma 14.2.3.) To finish the proof it suffices to show that if $\alpha, \alpha' : [k] \to [n]$ and $0 \leq i < j \leq n+1$ are such that

$$\delta^{n+1}_i \circ \alpha = \delta^{n+1}_j \circ \alpha'$$

then we have $\alpha = \delta^n_{j-1} \circ \beta$ and $\alpha' = \delta^n_i \circ \beta$ for some $\beta : [k] \to [n-1]$. This is easy to see and omitted. □

01GK **Lemma 25.7.4.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $n \geq 0$ be an integer. Let $u : \mathcal{F} \to F(K_n)$ be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings $f : L \to K$ such that $F(f_n) : F(L_n) \to F(K_n)$ factors through u.*

Proof. Write $K_n = \{U_i \to X\}_{i \in I}$. Thus the map u is a morphism of presheaves of sets $u : \mathcal{F} \to \amalg h_{u_i}$. The assumption on u means that for every $i \in I$ there exists a covering $\{U_{ij} \to U_i\}_{j \in I_i}$ of the site $\mathcal{C}$ and a morphism of presheaves $t_{ij} : h_{U_{ij}} \to \mathcal{F}$ such that $u \circ t_{ij}$ is the map $h_{U_{ij}} \to h_{U_i}$ coming from the morphism $U_{ij} \to U_i$. Set $J = \amalg_{i \in I} I_i$, and let $\alpha : J \to I$ be the obvious map. For $j \in J$ denote $V_j = U_{\alpha(j)j}$. Set $Z = \{V_j \to X\}_{j \in J}$. Finally, consider the morphism $u' : Z \to K_n$ given by $\alpha : J \to I$ and the morphisms $V_j = U_{\alpha(j)j} \to U_{\alpha(j)}$ above. Clearly, this is a covering in the category $\mathrm{SR}(\mathcal{C}, X)$, and by construction $F(u') : F(Z) \to F(K_n)$ factors through u. Thus the result follows from Lemma 25.7.3 above. □

25.8. Adding simplices

01GL In this section we prove some technical lemmas which we will need later. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. As we pointed out in Section 25.7 above, the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ are defined. See Simplicial, Sections 14.13 and 14.17.

01GM **Lemma 25.8.1.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U has finitely many nondegenerate simplices. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) *$V_i = U_i$ for $i < n$,*
(2) *$V_n = U_n \cup \{x\}$,*
(3) *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Then the morphism

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. If $n = 0$, then it follows easily that $V = U \amalg \Delta[0]$ (see below). In this case $\mathrm{Hom}(V, K)_0 = \mathrm{Hom}(U, K)_0 \times K_0$. The result, in this case, then follows from Lemma 25.3.2.

Let $a : \Delta[n] \to V$ be the morphism associated to x as in Simplicial, Lemma 14.11.3. Let us write $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for the $(n-1)$-skeleton of $\Delta[n]$. Let $b : \partial\Delta[n] \to U$ be the restriction of a to the $(n-1)$ skeleton of $\Delta[n]$. By Simplicial, Lemma 14.21.7 we have $V = U \amalg_{\partial\Delta[n]} \Delta[n]$. By Simplicial, Lemma 14.17.5 we get that

$$\begin{array}{ccc} \mathrm{Hom}(V, K)_0 & \longrightarrow & \mathrm{Hom}(U, K)_0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Delta[n], K)_0 & \longrightarrow & \mathrm{Hom}(\partial\Delta[n], K)_0 \end{array}$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By Simplicial, Lemma 14.21.11 this arrow is identified with

$$K_n \to (\mathrm{cosk}_{n-1}\mathrm{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering. □

01GN **Lemma 25.8.2.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then the morphism*

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. By Lemma 25.8.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets $U \subset V$ as in the lemma. And this is exactly the result of Simplicial, Lemma 14.21.8. □

0DEQ **Lemma 25.8.3.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Then*

(1) *K_n is a covering of X for each $n \geq 0$,*
(2) *$d_i^n : K_n \to K_{n-1}$ is a covering for all $n \geq 1$ and $0 \leq i \leq n$.*

Proof. Recall that K_0 is a covering of X by Definition 25.3.3 and that this is equivalent to saying that $K_0 \to \{X \to X\}$ is a covering in the sense of Definition 25.3.1. Hence (1) follows from (2) because it will prove that the composition $K_n \to K_{n-1} \to \ldots \to K_0 \to \{X \to X\}$ is a covering by Lemma 25.3.2.

Proof of (2). Observe that $\mathrm{Mor}(\Delta[n], K)_0 = K_n$ by Simplicial, Lemma 14.17.4. Therefore (2) follows from Lemma 25.8.2 applied to the $n+1$ different inclusions $\Delta[n-1] \to \Delta[n]$. □

0DER **Remark 25.8.4.** A useful special case of Lemmas 25.8.2 and 25.8.3 is the following. Suppose we have a category $\mathcal{C}$ having fibre products. Let $P \subset \mathrm{Arrows}(\mathcal{C})$ be a subset stable under base change, stable under composition, and containing all isomorphisms. Then one says a *P-hypercovering* is an augmentation $a : U \to X$ from a simplicial object of $\mathcal{C}$ such that

(1) $U_0 \to X$ is in P,
(2) $U_1 \to U_0 \times_X U_0$ is in P,
(3) $U_{n+1} \to (\mathrm{cosk}_n\mathrm{sk}_n U)_{n+1}$ is in P for $n \geq 1$.

The category $\mathcal{C}/X$ has all finite limits, hence the coskeleta used in the formulation above exist (see Categories, Lemma 4.18.4). Then we claim that the morphisms $U_n \to X$ and $d_i^n : U_n \to U_{n-1}$ are in P. This follows from the aforementioned lemmas by turning $\mathcal{C}$ into a site whose coverings are $\{f : V \to U\}$ with $f \in P$ and taking K given by $K_n = \{U_n \to X\}$.

25.9. Homotopies

01GO Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. According to Simplicial, Lemma 14.17.4 there exists an object $\mathrm{Hom}(\Delta[1], L)$ in the category $\mathrm{Simp}(\mathrm{SR}(\mathcal{C}, X))$ which represents the functor

$$T \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathrm{SR}(\mathcal{C},X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$\mathrm{Hom}(\Delta[1], L) \to L \times L$$

coming from $e_i : \Delta[0] \to \Delta[1]$ and the identification $\mathrm{Hom}(\Delta[0], L) = L$.

01GP **Lemma 25.9.1.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $SR(\mathcal{C}, X)$. Let $n \geq 0$. Consider the commutative diagram*

01GQ (25.9.1.1)
$$\begin{array}{ccc} \mathrm{Hom}(\Delta[1], L)_{n+1} & \longrightarrow & (cosk_n sk_n \mathrm{Hom}(\Delta[1], L))_{n+1} \\ \downarrow & & \downarrow \\ (L \times L)_{n+1} & \longrightarrow & (cosk_n sk_n (L \times L))_{n+1} \end{array}$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where $\partial\Delta[n+1] = i_{n!} sk_n \Delta[n+1]$ is the n-skeleton of the $(n+1)$-simplex:

$$\begin{aligned} \mathrm{Hom}(\Delta[1], L)_{n+1} &= \mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \\ (cosk_n sk_n \mathrm{Hom}(\Delta[1], L))_{n+1} &= \mathrm{Hom}(\Delta[1] \times \partial\Delta[n+1], L)_0 \\ (L \times L)_{n+1} &= \mathrm{Hom}((\Delta[n+1] \amalg \Delta[n+1], L)_0 \\ (cosk_n sk_n (L \times L))_{n+1} &= \mathrm{Hom}(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0 \end{aligned}$$

and the morphism between these objects of $SR(\mathcal{C}, X)$ come from the commutative diagram of simplicial sets

01GR (25.9.1.2)
$$\begin{array}{ccc} \Delta[1] \times \Delta[n+1] & \longleftarrow & \Delta[1] \times \partial\Delta[n+1] \\ \uparrow & & \uparrow \\ \Delta[n+1] \amalg \Delta[n+1] & \longleftarrow & \partial\Delta[n+1] \amalg \partial\Delta[n+1] \end{array}$$

Moreover the fibre product of the bottom arrow and the right arrow in (25.9.1.1) is equal to

$$\mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. The first and third equalities are Simplicial, Lemma 14.17.4. The second and fourth follow from the cited lemma combined with Simplicial, Lemma 14.21.11. The last assertion follows from the fact that U is the push-out of the bottom and right arrow of the diagram (25.9.1.2), via Simplicial, Lemma 14.17.5. To see that U is equal to this push-out it suffices to see that the intersection of $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ in $\Delta[1] \times \Delta[n+1]$ is equal to $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$. This we leave to the reader. □

01GS **Lemma 25.9.2.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} K' & \overset{def}{=\!=} & K \times_{(L\times L)} \mathrm{Hom}(\Delta[1], L) & \longrightarrow & \mathrm{Hom}(\Delta[1], L) \\ & \searrow^{c} & \downarrow & & \downarrow \\ & & K & \xrightarrow{(a,b)} & L \times L \end{array}$$

By the functorial property of $\mathrm{Hom}(\Delta[1], L)$ the composition of the horizontal morphisms corresponds to a morphism $K' \times \Delta[1] \to L$ which defines a homotopy between $c \circ a$ and $c \circ b$. Thus if we can show that K' is a hypercovering of X, then we obtain the lemma. To see this we will apply Lemma 25.7.1 to the pair of morphisms $K \to L \times L$ and $\mathrm{Hom}(\Delta[1], L) \to L \times L$. Condition (1) of Lemma 25.7.1 is satisfied. Condition (2) of Lemma 25.7.1 is true because $\mathrm{Hom}(\Delta[1], L)_0 = L_1$, and the morphism $(d_0^1, d_1^1) : L_1 \to L_0 \times L_0$ is a covering of $\mathrm{SR}(\mathcal{C}, X)$ by our assumption that L is a hypercovering. To prove condition (3) of Lemma 25.7.1 we use Lemma 25.9.1 above. According to this lemma the morphism γ of condition (3) of Lemma 25.7.1 is the morphism

$$\mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow \mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$. According to Lemma 25.8.2 this is a covering and hence the claim has been proven. □

01GT **Remark 25.9.3.** Note that the crux of the proof is to use Lemma 25.8.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism $a : K \times \partial\Delta[k] \to L$, with K and L hypercoverings, there exists a morphism of hypercoverings $c : K' \to K$ and a morphism $g : K' \times \Delta[k] \to L$ such that $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \mathrm{id}_{\partial\Delta[k]})$. In other words, the category of hypercoverings is in a suitable sense contractible.

25.10. Cohomology and hypercoverings

01GZ Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps, see Simplicial, Lemma 14.28.4. Hence have the same effect on cohomology groups of the associated cochain complexes, see Simplicial, Lemma 14.28.6. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote $\mathrm{HC}(\mathcal{C}, X)$ the category whose objects are hypercoverings of X and whose morphisms are maps between hypercoverings of X up to homotopy. We have seen that this is a category and not a "big" category, see Lemma 25.3.7. The opposite to $\mathrm{HC}(\mathcal{C}, X)$ will be the index category for our diagram, see Categories, Section 4.14 for terminology. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \mathrm{HC}(\mathcal{C}, X)^{opp} \longrightarrow Ab.$$

By Lemmas 25.7.2 and 25.9.2 and the remark on homotopies above, this diagram is directed, see Categories, Definition 4.19.1. Thus the colimit

$$\check{H}^i_{\mathrm{HC}}(X, \mathcal{F}) = \mathrm{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple description (see location cited).

01H0 **Theorem 25.10.1.** *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $i \geq 0$. The functors*

$$\begin{aligned} Ab(\mathcal{C}) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \\ \mathcal{F} &\longmapsto \check{H}^i_{HC}(X, \mathcal{F}) \end{aligned}$$

are canonically isomorphic.

Proof using spectral sequences. Suppose that $\xi \in H^p(X, \mathcal{F})$ for some $p \geq 0$. Let us show that ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 25.5.3 for some hypercovering K of X.

This is true if $p = 0$ by Lemma 25.5.1. If $p = 1$, choose a Čech hypercovering K of X as in Example 25.3.4 starting with a covering $K_0 = \{U_i \to X\}$ in the site $\mathcal{C}$ such that $\xi|_{U_i} = 0$, see Cohomology on Sites, Lemma 21.7.3. It follows immediately from the spectral sequence in Lemma 25.5.3 that ξ comes from an element of $\check{H}^1(K, \mathcal{F})$ in this case. In general, choose any hypercovering K of X such that ξ maps to zero in $\underline{H}^p(\mathcal{F})(K_0)$ (using Example 25.3.4 and Cohomology on Sites, Lemma 21.7.3 again). By the spectral sequence of Lemma 25.5.3 the obstruction for ξ to come from an element of $\check{H}^p(K, \mathcal{F})$ is a sequence of elements $\xi_1, \ldots, \xi_{p-1}$ with $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of the ξ_q in certain subquotients of these groups).

We can inductively replace the hypercovering K by refinements such that the obstructions $\xi_1, \ldots, \xi_{p-1}$ restrict to zero (and not just the images in the subquotients – so no subtlety here). Indeed, suppose we have already managed to reach the situation where $\xi_{q+1}, \ldots, \xi_{p-1}$ are zero. Note that $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathcal{F})(K_{p-q}) = \prod H^q(U_i, \mathcal{F})$$

if $K_{p-q} = \{U_i \to X\}_{i \in I}$. Let $\xi_{q,i}$ be the component of $\tilde{\xi}_q$ in $H^q(U_i, \mathcal{F})$. As $q \geq 1$ we can use Cohomology on Sites, Lemma 21.7.3 yet again to choose coverings $\{U_{i,j} \to U_i\}$ of the site such that each restriction $\xi_{q,i}|_{U_{i,j}} = 0$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_{p-q}$. It is clear that u is a covering, see Definition 25.3.1. By Lemma 25.7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_{p-q} \to K_{p-q}$ factors through u. Then clearly the image of ξ_q in $\underline{H}^q(\mathcal{F})(L_{p-q})$. is zero. Since the spectral sequence of Lemma 25.5.3 is functorial this means that after replacing K by L we reach the situation where $\xi_q, \ldots, \xi_{p-1}$ are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Suppose that K is a hypercovering of X, that $\xi \in \check{H}^p(K, \mathcal{F})$ and that the image of ξ under the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 25.5.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings $L \to K$ such that ξ restricts to zero in $\check{H}^p(L, \mathcal{F})$. By the spectral sequence of Lemma 25.5.3 the vanishing of the image of ξ in $H^p(X, \mathcal{F})$ means that there exist elements $\xi_1, \ldots, \xi_{p-2}$ with $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of these in certain subquotients) such that the images $d_{q+1}^{p-1-q,q}\xi_q$ (in the spectral sequence) add up to ξ. Hence by exactly the same mechanism as above we can find

a morphism of hypercoverings $L \to K$ such that the restrictions of the elements ξ_q, $q = 1, \ldots, p-2$ in $\check{H}^{p-1-q}(L, \underline{H}^q(\mathcal{F}))$ are zero. Then it follows that ξ is zero since the morphism $L \to K$ induces a morphism of spectral sequences according to Lemma 25.5.3. □

Proof without using spectral sequences. We have seen the result for $i = 0$, see Lemma 25.5.1. We know that the functors $H^i(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 13.20.4. In order to prove the theorem it suffices to show that the sequence of functors $\check{H}^i_{HC}(X, -)$ forms a δ-functor. Namely we know that Čech cohomology is zero on injective sheaves (Lemma 25.5.2) and then we can apply Homology, Lemma 12.12.4.

Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to s(\mathcal{F}(K)) \to s(\mathcal{G}(K)) \to s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering, see Definition 25.3.1. By Lemma 25.7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify: (a) $\delta(\xi)$ does not depend on the choice of τ, (b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and (c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ. We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 25.7.2 and 25.9.2. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Čech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(\upsilon)$ for some $\upsilon \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + \upsilon$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. □

Next, we deduce Verdier's case of Theorem 25.10.1 by a sleight of hand.

09VZ **Proposition 25.10.2.** *Let $\mathcal{C}$ be a site with fibre products and products of pairs. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $i \geq 0$. Then*

1. *for every $\xi \in H^i(\mathcal{F})$ there exists a hypercovering K such that ξ is in the image of the canonical map $\check{H}^i(K, \mathcal{F}) \to H^i(\mathcal{F})$, and*
2. *if K, L are hypercoverings and $\xi_K \in \check{H}^i(K, \mathcal{F})$, $\xi_L \in \check{H}^i(L, \mathcal{F})$ are elements mapping to the same element of $H^i(\mathcal{F})$, then there exists a hypercovering M and morphisms $M \to K$ and $M \to L$ such that ξ_K and ξ_L map to the same element of $\check{H}^i(M, \mathcal{F})$.*

In other words, modulo set theoretical issues, the cohomology groups of $\mathcal{F}$ on $\mathcal{C}$ are the colimit of the Čech cohomology groups of $\mathcal{F}$ over all hypercoverings.

Proof. This result is a trivial consequence of Theorem 25.10.1. Namely, we can artificially replace $\mathcal{C}$ with a slightly bigger site $\mathcal{C}'$ such that (I) $\mathcal{C}'$ has a final object X and (II) hypercoverings in $\mathcal{C}$ are more or less the same thing as hypercoverings of X in $\mathcal{C}'$. But due to the nature of things, there is quite a bit of bookkeeping to do.

Let us call a family of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ with fixed target a *weak covering* if the sheafification of the map $\coprod_{i \in I} h_{U_i} \to h_U$ becomes surjective. We construct a new site $\mathcal{C}'$ as follows

1. as a category set $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C}) \amalg \{X\}$ and add a unique morphism to X from every object of $\mathcal{C}'$,
2. $\mathcal{C}'$ has fibre products as fibre products and products of pairs exist in $\mathcal{C}$,
3. coverings of $\mathcal{C}'$ are weak coverings of $\mathcal{C}$ together with those $\{U_i \to X\}_{i \in I}$ such that either $U_i = X$ for some i, or $U_i \neq X$ for all i and the map $\coprod h_{U_i} \to *$ of presheaves on $\mathcal{C}$ becomes surjective after sheafification on $\mathcal{C}$,
4. we apply Sets, Lemma 3.11.1 to restrict the coverings to obtain our site $\mathcal{C}'$.

Then $Sh(\mathcal{C}') = Sh(\mathcal{C})$ because the inclusion functor $\mathcal{C} \to \mathcal{C}'$ is a special cocontinuous functor (see Sites, Definition 7.29.2). We omit the straightforward verifications.

Choose a covering $\{U_i \to X\}$ of $\mathcal{C}'$ such that U_i is an object of $\mathcal{C}$ for all i (possible because $\mathcal{C} \to \mathcal{C}'$ is special cocontinuous). Then $K_0 = \{U_i \to X\}$ is a covering in the site $\mathcal{C}'$ constructed above. We view K_0 as an object of $\mathrm{SR}(\mathcal{C}', X)$ and we set $K_{init} = \mathrm{cosk}_0(K_0)$. Then K_{init} is a hypercovering of X, see Example 25.3.4. Note that every $K_{init,n}$ has the shape $\{W_j \to X\}$ with $W_j \in \mathrm{Ob}(\mathcal{C})$.

Proof of (1). Choose $\xi \in H^i(\mathcal{F}) = H^i(X, \mathcal{F}')$ where $\mathcal{F}'$ is the abelian sheaf on $\mathcal{C}'$ corresponding to $\mathcal{F}$ on $\mathcal{C}$. By Theorem 25.10.1 there exists a morphism of hypercoverings $K' \to K_{init}$ of X in $\mathcal{C}'$ such that ξ comes from an element of $\check{H}^i(K', \mathcal{F})$. Write $K'_n = \{U_{n,j} \to X\}$. Now since K'_n maps to $K_{init,n}$ we see that $U_{n,j}$ is an object of $\mathcal{C}$. Hence we can define a simplicial object K of $\mathrm{SR}(\mathcal{C})$ by setting $K_n = \{U_{n,j}\}$. Since coverings in $\mathcal{C}'$ consisting of families of morphisms of $\mathcal{C}$ are weak coverings, we see that K is a hypercovering in the sense of Definition 25.6.1. Finally, since $\mathcal{F}'$ is the unique sheaf on $\mathcal{C}'$ whose restriction to $\mathcal{C}$ is equal to $\mathcal{F}$ we

see that the Čech complexes $s(\mathcal{F}(K))$ and $s(\mathcal{F}'(K'))$ are identical and (1) follows. (Compatibility with map into cohomology groups omitted.)

Proof of (2). Let K and L be hypercoverings in $\mathcal{C}$. Let K' and L' be the simplicial objects of $\mathrm{SR}(\mathcal{C}', X)$ gotten from K and L by the functor $\mathrm{SR}(\mathcal{C}) \to \mathrm{SR}(\mathcal{C}', X)$, $\{U_i\} \mapsto \{U_i \to X\}$. As before we have equality of Čech complexes and hence we obtain $\xi_{K'}$ and $\xi_{L'}$ mapping to the same cohomology class of $\mathcal{F}'$ over $\mathcal{C}'$. After possibly enlarging our choice of coverings in $\mathcal{C}'$ (due to a set theoretical issue) we may assume that K' and L' are hypercoverings of X in $\mathcal{C}'$; this is true by our definition of hypercoverings in Definition 25.6.1 and the fact that weak coverings in $\mathcal{C}$ give coverings in $\mathcal{C}'$. By Theorem 25.10.1 there exists a hypercovering M' of X in $\mathcal{C}'$ and morphisms $M' \to K'$, $M' \to L'$, and $M' \to K_{init}$ such that $\xi_{K'}$ and $\xi_{L'}$ restrict to the same element of $\check{H}^i(M', \mathcal{F})$. Unwinding this statement as above we find that (2) is true. □

25.11. Hypercoverings of spaces

01H1 The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what a hypercovering is and see what the result actually says.

Let X be a topological space. Consider the site X_{Zar} of Sites, Example 7.6.4. Recall that an object of X_{Zar} is simply an open of X and that morphisms of X_{Zar} correspond simply to inclusions. So what is a hypercovering of X for the site X_{Zar}?

Let us first unwind Definition 25.2.1. An object of $\mathrm{SR}(X_{Zar}, X)$ is simply given by a set I and for each $i \in I$ an open $U_i \subset X$. Let us denote this by $\{U_i\}_{i\in I}$ since there can be no confusion about the morphism $U_i \to X$. A morphism $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ between two such objects is given by a map of sets $\alpha : I \to J$ such that $U_i \subset V_{\alpha(i)}$ for all $i \in I$. When is such a morphism a covering? This is the case if and only if for every $j \in J$ we have $V_j = \bigcup_{i\in I,\ \alpha(i)=j} U_i$ (and is a covering in the site X_{Zar}).

Using the above we get the following description of a hypercovering in the site X_{Zar}. A hypercovering of X in X_{Zar} is given by the following data

(1) a simplicial set I (see Simplicial, Section 14.11), and
(2) for each $n \geq 0$ and every $i \in I_n$ an open set $U_i \subset X$.

We will denote such a collection of data by the notation $(I, \{U_i\})$. In order for this to be a hypercovering of X we require the following properties

- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i \subset U_{d^n_a(i)}$,
- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i = U_{s^n_a(i)}$,
- we have

01H2 (25.11.0.1) $$X = \bigcup\nolimits_{i\in I_0} U_i,$$

- for every $i_0, i_1 \in I_0$, we have

01H3 (25.11.0.2) $$U_{i_0} \cap U_{i_1} = \bigcup\nolimits_{i\in I_1,\ d^1_0(i)=i_0,\ d^1_1(i)=i_1} U_i,$$

- for every $n \geq 1$ and every $(i_0, \ldots, i_{n+1}) \in (I_n)^{n+2}$ such that $d^n_{b-1}(i_a) = d^n_a(i_b)$ for all $0 \leq a < b \leq n+1$ we have

01H4 (25.11.0.3) $$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i\in I_{n+1},\ d^{n+1}_a(i)=i_a,\ a=0,\ldots,n+1} U_i,$$

- each of the open coverings (25.11.0.1), (25.11.0.2), and (25.11.0.3) is an element of $\mathrm{Cov}(X_{Zar})$ (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Conditions (25.11.0.1) and (25.11.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (25.11.0.3) is the natural generalization.

01H5 **Remark 25.11.1.** One feature of this description is that if one of the multiple intersections $U_{i_0} \cap \ldots \cap U_{i_{n+1}}$ is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps $I_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n I)_{n+1}$ are surjective. This means that the geometric realization of I may be an interesting (non-contractible) space.

In fact, let $I'_n \subset I_n$ be the subset consisting of those simplices $i \in I_n$ such that $U_i \neq \emptyset$. It is easy to see that $I' \subset I$ is a subsimplicial set, and that $(I', \{U_i\})$ is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens U_i is empty.

02N9 **Remark 25.11.2.** Let us repackage this information in yet another way. Namely, suppose that $(I, \{U_i\})$ is a hypercovering of the topological space X. Given this data we can construct a simplicial topological space $U_\bullet$ by setting

$$U_n = \coprod\nolimits_{i \in I_n} U_i,$$

and where for given $\varphi : [n] \to [m]$ we let morphisms $U(\varphi) : U_n \to U_m$ be the morphism coming from the inclusions $U_i \subset U_{\varphi(i)}$ for $i \in I_n$. This simplicial topological space comes with an augmentation $\epsilon : U_\bullet \to X$. With this morphism the simplicial space $U_\bullet$ becomes a hypercovering of X along which one has cohomological descent in the sense of [**AGV71**, Exposé Vbis]. In other words, $H^n(U_\bullet, \epsilon^*\mathcal{F}) = H^n(X, \mathcal{F})$. (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that $\mathcal{F}$ is an abelian sheaf on X. In this case the spectral sequence of Lemma 25.5.3 becomes the spectral sequence with E_1-term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^*\mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^*\mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of $\epsilon^*\mathcal{F}$ to the cohomology groups of $\mathcal{F}$ over the pieces of $U_\bullet$. (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of X which have the property that all the U_i come from a given basis for the topology of X and that all the coverings (25.11.0.2) and (25.11.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

01H6 **Lemma 25.11.3.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. There exists a hypercovering $(I, \{U_i\})$ of X such that each U_i is an element of $\mathcal{B}$.*

Proof. Let $n \geq 0$. Let us say that an *n-truncated hypercovering* of X is given by an n-truncated simplicial set I and for each $i \in I_a$, $0 \leq a \leq n$ an open U_i of X such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are a-simplices with $a \leq n$. The lemma follows if we can prove that given a n-truncated hypercovering $(I, \{U_i\})$ with all $U_i \in \mathcal{B}$ we can extend it to an $(n+1)$-truncated hypercovering without adding any a-simplices

for $a \leq n$. This we do as follows. First we consider the $(n+1)$-truncated simplicial set I' defined by $I' = \mathrm{sk}_{n+1}(\mathrm{cosk}_n I)$. Recall that

$$I'_{n+1} = \left\{ \begin{array}{c} (i_0, \ldots, i_{n+1}) \in (I_n)^{n+2} \text{ such that} \\ d^n_{b-1}(i_a) = d^n_a(i_b) \text{ for all } 0 \leq a < b \leq n+1 \end{array} \right\}$$

If $i' \in I'_{n+1}$ is degenerate, say $i' = s^n_a(i)$ then we set $U_{i'} = U_i$ (this is forced on us anyway by the second condition). We also set $J_{i'} = \{i'\}$ in this case. If $i' \in I'_{n+1}$ is nondegenerate, say $i' = (i_0, \ldots, i_{n+1})$, then we choose a set $J_{i'}$ and an open covering

071K (25.11.3.1) $$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in J_{i'}} U_i,$$

with $U_i \in \mathcal{B}$ for $i \in J_{i'}$. Set

$$I_{n+1} = \coprod\nolimits_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map $\pi : I_{n+1} \to I'_{n+1}$ which is a bijection over the set of degenerate simplices in I'_{n+1} by construction. For $i \in I_{n+1}$ we define $d^{n+1}_a(i) = d^{n+1}_a(\pi(i))$. For $i \in I_n$ we define $s^n_a(i) \in I_{n+1}$ as the unique simplex lying over the degenerate simplex $s^n_a(i) \in I'_{n+1}$. We omit the verification that this defines an $(n+1)$-truncated hypercovering of X. □

01H7 **Lemma 25.11.4.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. Assume that*

(1) *X is quasi-compact,*
(2) *each $U \in \mathcal{B}$ is quasi-compact open, and*
(3) *the intersection of any two quasi-compact opens in X is quasi-compact.*

Then there exists a hypercovering $(I, \{U_i\})$ of X with the following properties

(1) *each U_i is an element of the basis $\mathcal{B}$,*
(2) *each of the I_n is a finite set, and in particular*
(3) *each of the coverings (25.11.0.1), (25.11.0.2), and (25.11.0.3) is finite.*

Proof. This follows directly from the construction in the proof of Lemma 25.11.3 if we choose finite coverings by elements of $\mathcal{B}$ in (25.11.3.1). Details omitted. □

25.12. Constructing hypercoverings

094J Let $\mathcal{C}$ be a site. In this section we will think of a simplicial object of $\mathrm{SR}(\mathcal{C})$ as follows. As usual, we set $K_n = K([n])$ and we denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. We may write $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. The fact that K is a simplicial object of $\mathrm{SR}(\mathcal{C})$ implies that $(I_n, \alpha(\varphi))$ is a simplicial set and that $f_{\psi,\alpha(\varphi)(i)} \circ f_{\varphi,i} = f_{\varphi\circ\psi,i}$ when $\psi : [l] \to [m]$.

0DAU **Lemma 25.12.1.** *Let $\mathcal{C}$ be a site. Let K be an r-truncated simplicial object of $SR(\mathcal{C})$. The following are equivalent*

(1) *K is split (Simplicial, Definition 14.18.1),*
(2) *$f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ is an isomorphism for $r \geq n \geq 0$, $\varphi : [m] \to [n]$ surjective, $i \in I_n$, and*
(3) *$f_{\sigma^n_j,i} : U_{n,i} \to U_{n+1,\alpha(\sigma^n_j)(i)}$ is an isomorphism for $0 \leq j \leq n < r$, $i \in I_n$.*

The same holds for simplicial objects if in (2) and (3) we set $r = \infty$.

Proof. The splitting of a simplicial set is unique and is given by the nondegenerate indices $N(I_n)$ in each degree n, see Simplicial, Lemma 14.18.2. The coproduct of two objects $\{U_i\}_{i\in I}$ and $\{U_j\}_{j\in J}$ of $\mathrm{SR}(\mathcal{C})$ is given by $\{U_l\}_{l\in I \amalg J}$ with obvious notation. Hence a splitting of K must be given by $N(K_n) = \{U_i\}_{i\in N(I_n)}$. The equivalence of (1) and (2) now follows by unwinding the definitions. The equivalence of (2) and (3) follows from the fact that any surjection $\varphi : [m] \to [n]$ is a composition of morphisms σ_j^k with $k = n, n+1, \ldots, m-1$. □

0DAV **Lemma 25.12.2.** *Let $\mathcal{C}$ be a site with fibre products. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *any object U of $\mathcal{C}$ has a covering $\{U_j \to U\}_{j\in J}$ with $U_j \in \mathcal{B}$, and*
(2) *if $\{U_j \to U\}_{j\in J}$ is a covering with $U_j \in \mathcal{B}$ and $\{U' \to U\}$ is a morphism with $U' \in \mathcal{B}$, then $\{U_j \to U\}_{j\in J} \amalg \{U' \to U\}$ is a covering.*

Then for any X in $\mathcal{C}$ there is a hypercovering K of X such that $K_n = \{U_{n,i}\}_{i\in I_n}$ with $U_{n,i} \in \mathcal{B}$ for all $i \in I_n$.

Proof. A warmup for this proof is the proof of Lemma 25.11.3 and we encourage the reader to read that proof first.

First we replace $\mathcal{C}$ by the site $\mathcal{C}/X$. After doing so we may assume that X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits (Categories, Lemma 4.18.4).

Let $n \geq 0$. Let us say that an *n-truncated $\mathcal{B}$-hypercovering of X* is given by an n-truncated simplicial object K of $\mathrm{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \leq a \leq n$ we have $U_{a,i} \in \mathcal{B}$ and such that K_0 is a covering of X and $K_{a+1} \to (\mathrm{cosk}_a \mathrm{sk}_a K)_{a+1}$ for $a = 0, \ldots, n-1$ is a covering as in Definition 25.3.1.

Since X has a covering $\{U_{0,i} \to X\}_{i\in I_0}$ with $U_i \in \mathcal{B}$ by assumption, we get a 0-truncated $\mathcal{B}$-hypercovering of X. Observe that any 0-truncated $\mathcal{B}$-hypercovering of X is split, see Lemma 25.12.1.

The lemma follows if we can prove for $n \geq 0$ that given a split n-truncated $\mathcal{B}$-hypercovering K of X we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X.

Construction of the extension. Consider the $(n+1)$-truncated simplicial object $K' = \mathrm{sk}_{n+1}(\mathrm{cosk}_n K)$ of $\mathrm{SR}(\mathcal{C})$. Write

$$K'_{n+1} = \{U'_{n+1,i}\}_{i\in I'_{n+1}}$$

Since $K = \mathrm{sk}_n K'$ we have $K_a = K'_a$ for $0 \leq a \leq n$. For every $i' \in I'_{n+1}$ we choose a covering

0DAW (25.12.2.1) $$\{g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}\}_{j\in J_{i'}}$$

with $U_{n+1,j} \in \mathcal{B}$ for $j \in J_{i'}$. This is possible by our assumption on $\mathcal{B}$ in the lemma. For $0 \leq m \leq n$ denote $N_m \subset I_m$ the subset of nondegenerate indices. We set

$$I_{n+1} = \coprod\nolimits_{\varphi:[n+1]\to[m] \text{ surjective, } 0\leq m\leq n} N_m \amalg \coprod\nolimits_{i'\in I'_{n+1}} J_{i'}$$

For $j \in I_{n+1}$ we set

$$U_{n+1,j} = \begin{cases} U_{m,i} & \text{if } j = (\varphi, i) \text{ where } \varphi : [n+1] \to [m], i \in N_m \\ U_{n+1,j} & \text{if } j \in J_{i'} \text{ where } i' \in I'_{n+1} \end{cases}$$

with obvious notation. We set $K_{n+1} = \{U_{n+1,j}\}_{j \in I_{n+1}}$. By construction $U_{n+1,j}$ is an element of $\mathcal{B}$ for all $j \in I_{n+1}$. Let us define compatible maps

$$I_{n+1} \to I'_{n+1} \quad \text{and} \quad K_{n+1} \to K'_{n+1}$$

Namely, the first map is given by $(\varphi, i) \mapsto \alpha'(\varphi)(i)$ and $(j \in J_{i'}) \mapsto i'$. For the second map we use the morphisms

$$f'_{\varphi,i} : U_{m,i} \to U'_{n+1,\alpha'(\varphi)(i)} \quad \text{and} \quad g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}$$

We claim the morphism

$$K_{n+1} \to K'_{n+1} = (\mathrm{cosk}_n \mathrm{sk}_n K')_{n+1} = (\mathrm{cosk}_n K)_{n+1}$$

is a covering as in Definition 25.3.1. Namely, if $i' \in I'_{n+1}$, then either i' is nondegenerate and the inverse image of i' in I_{n+1} is equal to $J_{i'}$ and we get a covering of $U'_{n+1,i'}$ by our choice (25.12.2.1), or i' is degenerate and the inverse image of i' in I_{n+1} is $J_{i'} \amalg \{(\varphi, i)\}$ for a unique pair (φ, i) and we get a covering by our choice (25.12.2.1) and assumption (2) of the lemma.

To finish the proof we have to define the morphisms $K(\varphi) : K_{n+1} \to K_m$ corresponding to morphisms $\varphi : [m] \to [n+1]$, $0 \leq m \leq n$ and the morphisms $K(\varphi) : K_m \to K_{n+1}$ corresponding to morphisms $\varphi : [n+1] \to [m]$, $0 \leq m \leq n$ satisfying suitable composition relations. For the first kind we use the composition

$$K_{n+1} \to K'_{n+1} \xrightarrow{K'(\varphi)} K'_m = K_m$$

to define $K(\varphi) : K_{n+1} \to K_m$. For the second kind, suppose given $\varphi : [n+1] \to [m]$, $0 \leq m \leq n$. We define the corresponding morphism $K(\varphi) : K_m \to K_{n+1}$ as follows:

(1) for $i \in I_m$ there is a unique surjective map $\psi : [m] \to [m_0]$ and a unique $i_0 \in I_{m_0}$ nondegenerate such that $\alpha(\psi)(i_0) = i$[3],
(2) we set $\varphi_0 = \psi_0 \circ \varphi : [n+1] \to [m_0]$ and we map $i \in I_m$ to $(\varphi_0, i_0) \in I_{n+1}$, in other words, $\alpha(\varphi)(i) = (\varphi_0, i_0)$, and
(3) the morphism $f_{\varphi,i} : U_{m,i} \to U_{n+1,\alpha(\varphi)(i)} = U_{m_0,i_0}$ is the inverse of the isomorphism $f_{\psi,i_0} : U_{m_0,i_0} \to U_{m,i}$ (see Lemma 25.12.1).

We omit the straightforward but cumbersome verification that this defines a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X extending the given n-truncated one. In fact, everything is clear from the above, except for the verification that the morphisms $K(\varphi)$ compose correctly for all $\varphi : [a] \to [b]$ with $0 \leq a, b \leq n+1$. □

0DAX **Lemma 25.12.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume that any object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$. Then there is a hypercovering K such that $K_n = \{U_i\}_{i \in I_n}$ with $U_i \in \mathcal{B}$ for all $i \in I_n$.*

Proof. This proof is almost the same as the proof of Lemma 25.12.2. We will only explain the differences.

Let $n \geq 1$. Let us say that an *n-truncated $\mathcal{B}$-hypercovering* is given by an n-truncated simplicial object K of $\mathrm{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \leq a \leq n$ we have $U_{a,i} \in \mathcal{B}$ and such that

(1) $F(K_0)^\# \to *$ is surjective,
(2) $F(K_1)^\# \to F(K_0)^\# \times F(K_0)^\#$ is surjective,

[3]For example, if i is nondegenerate, then $m = m_0$ and $\psi = \mathrm{id}_{[m]}$.

(3) $F(K_{a+1})^\# \to F((\mathrm{cosk}_a \mathrm{sk}_a K)_{a+1})^\#$ for $a = 1, \ldots, n-1$ is surjective.

We first explicitly construct a split 1-truncated $\mathcal{B}$-hypercovering.

Take $I_0 = \mathcal{B}$ and $K_0 = \{U\}_{U \in \mathcal{B}}$. Then (1) holds by our assumption on $\mathcal{B}$. Set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

Then we set $I_1 = I_0 \amalg \Omega$. For $i \in I_1$ we set $U_{1,i} = U_{0,i}$ if $i \in I_0$ and $U_{1,i} = U$ if $i = (U, V, W, a, b) \in \Omega$. The map $K(\sigma_0^0) : K_0 \to K_1$ corresponds to the inclusion $\alpha(\sigma_0^0) : I_0 \to I_1$ and the identity $f_{\sigma_0^0, i} : U_{0,i} \to U_{1,i}$ on objects. The maps $K(\delta_0^1), K(\delta_1^1) : K_1 \to K_0$ correspond to the two maps $I_1 \to I_0$ which are the identity on $I_0 \subset I_1$ and map $(U, V, W, a, b) \in \Omega \subset I_1$ to V, resp. W. The corresponding morphisms $f_{\delta_0^1, i}, f_{\delta_1^1, i} : U_{1,i} \to U_{0,i}$ are the identity if $i \in I_0$ and a, b in case $i = (U, V, W, a, b) \in \Omega$. The reason that (2) holds is that any section of $F(K_0)^\# \times F(K_0)^\#$ over an object U of $\mathcal{C}$ comes, after replacing U by the members of a covering, from a map $U \to F(K_0) \times F(K_0)$. This in turn means we have $V, W \in \mathcal{B}$ and two morphisms $U \to V$ and $U \to W$. Further replacing U by the members of a covering we may assume $U \in \mathcal{B}$ as desired.

The lemma follows if we can prove that given a split n-truncated $\mathcal{B}$-hypercovering K for $n \geq 1$ we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering. Here the argument proceeds exactly as in the proof of Lemma 25.12.2. We omit the precise details, except for the following comments. First, we do not need assumption (2) in the proof of the current lemma as we do not need the morphism $K_{n+1} \to (\mathrm{cosk}_n K)_{n+1}$ to be covering; we only need it to induce a surjection on associated sheaves of sets which follows from Sites, Lemma 7.12.4. Second, the assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\mathrm{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 25.2.3). This suffices assure us the coskeleton functors used exist (see Simplicial, Remark 14.19.11 and Categories, Lemma 4.18.2). □

0DAY **Lemma 25.12.4.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the functor $u : \mathcal{D} \to \mathcal{C}$. Assume $\mathcal{D}$ and $\mathcal{C}$ have equalizers and fibre products and u commutes with them. If a simplicial object K of $SR(\mathcal{D})$ is a hypercovering, then $u(K)$ is a hypercovering.*

Proof. If we write $K_n = \{U_{n,i}\}_{i \in I_n}$ as in the introduction to this section, then $u(K)$ is the object of $\mathrm{SR}(\mathcal{C})$ given by $u(K_n) = \{u(U_i)\}_{i \in I_n}$. By Sites, Lemma 7.13.5 we have $f^{-1} h_U^\# = h_{u(U)}^\#$ for $U \in \mathrm{Ob}(\mathcal{D})$. This means that $f^{-1} F(K_n)^\# = F(u(K_n))^\#$ for all n. Let us check the conditions (1), (2), (3) for $u(K)$ to be a hypercovering from Definition 25.6.1. Since f^{-1} is an exact functor, we find that

$$F(u(K_0))^\# = f^{-1} F(K_0)^\# \to f^{-1} * = *$$

is surjective as a pullback of a surjective map and we get (1). Similarly,

$$F(u(K_1))^\# = f^{-1} F(K_1)^\# \to f^{-1}(F(K_0) \times F(K_0))^\# = F(u(K_0))^\# \times F(u(K_0))^\#$$

is surjective as a pullback and we get (2). For condition (3), in order to conclude by the same method it suffices if

$$F((\mathrm{cosk}_n \mathrm{sk}_n u(K))_{n+1})^\# = f^{-1} F((\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1})^\#$$

The above shows that $f^{-1} F(-) = F(u(-))$. Thus it suffices to show that u commutes with the limits used in defining $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ for $n \geq 1$. By Simplicial,

Remark 14.19.11 these limits are finite connected limits and u commutes with these by assumption. □

0DAZ **Lemma 25.12.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor. Assume $\mathcal{D}$ and $\mathcal{C}$ have fibre products and u commutes with them. Let $Y \in \mathcal{D}$ and $K \in SR(\mathcal{D}, Y)$ a hypercovering of Y. Then $u(K)$ is a hypercovering of $u(Y)$.*

Proof. This is easier than the proof of Lemma 25.12.4 because the notion of being a hypercovering of an object is stronger, see Definitions 25.3.3 and 25.3.1. Namely, u sends coverings to coverings by the definition of a morphism of sites. It suffices to check u commutes with the limits used in defining $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ for $n \geq 1$. This is clear because the induced functor $\mathcal{D}/Y \to \mathcal{C}/X$ commutes with all finite limits (and source and target have all finite limits by Categories, Lemma 4.18.4). □

094K **Lemma 25.12.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *$\mathcal{C}$ has fibre products,*
(2) *for all $X \in \mathrm{Ob}(\mathcal{C})$ there exists a finite covering $\{U_i \to X\}_{i \in I}$ with $U_i \in \mathcal{B}$,*
(3) *if $\{U_i \to X\}_{i \in I}$ is a finite covering with $U_i \in \mathcal{B}$ and $U \to X$ is a morphism with $U \in \mathcal{B}$, then $\{U_i \to X\}_{i \in I} \amalg \{U \to X\}$ is a covering.*

Then for every X there exists a hypercovering K of X such that each $K_n = \{U_{n,i} \to X\}_{i \in I_n}$ with I_n finite and $U_{n,i} \in \mathcal{B}$.

Proof. This lemma is the analogue of Lemma 25.11.4 for sites. To prove the lemma we follow exactly the proof of Lemma 25.12.2 paying attention to the following two points

(a) We choose our initial covering $\{U_{0,i} \to X\}_{i \in I_0}$ with $U_{0,i} \in \mathcal{B}$ such that the index set I_0 is finite, and
(b) in choosing the coverings (25.12.2.1) we choose $J_{i'}$ finite.

The reader sees easily that with these modifications we end up with finite index sets I_n for all n. □

0DB0 **Remark 25.12.7.** Let $\mathcal{C}$ be a site. Let K and L be objects of $\mathrm{SR}(\mathcal{C})$. Write $K = \{U_i\}_{i \in I}$ and $L = \{V_j\}_{j \in J}$. Assume $U = \coprod_{i \in I} U_i$ and $V = \coprod_{j \in J} V_j$ exist. Then we get

$$\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(K, L) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(U, V)$$

as follows. Given $f : K \to L$ given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ we obtain a transformation of functors

$$\mathrm{Mor}_{\mathcal{C}}(V, -) = \prod\nolimits_{j \in J} \mathrm{Mor}_{\mathcal{C}}(V_j, -) \to \prod\nolimits_{i \in I} \mathrm{Mor}_{\mathcal{C}}(U_i, -) = \mathrm{Mor}_{\mathcal{C}}(U, -)$$

sending $(g_j)_{j \in J}$ to $(g_{\alpha(i)} \circ f_i)_{i \in I}$. Hence the Yoneda lemma produces the corresponding map $U \to V$. Of course, $U \to V$ maps the summand U_i into the summand $V_{\alpha(i)}$ via the morphism f_i.

0DB1 **Remark 25.12.8.** Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products and equalizers and let K be a hypercovering. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists, and
(b) $\coprod_{i \in I_n} h_{U_{n,i}} \to h_{U_n}$ induces an isomorphism on sheafifications.

Then we get another simplicial object L of $\mathrm{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 25.12.7. Now we claim that L is a hypercovering. To see this we check conditions

(1), (2), (3) of Definition 25.6.1. Condition (1) follows from (b) and (1) for K. Condition (2) follows in exactly the same way. Condition (3) follows because

$$\begin{aligned} F((\text{cosk}_n\text{sk}_n L)_{n+1})^\# &= ((\text{cosk}_n\text{sk}_n F(L)^\#)_{n+1}) \\ &= ((\text{cosk}_n\text{sk}_n F(K)^\#)_{n+1}) \\ &= F((\text{cosk}_n\text{sk}_n K)_{n+1})^\# \end{aligned}$$

for $n \geq 1$ and hence the condition for K implies the condition for L exactly as in (1) and (2). Note that F commutes with connected limits and sheafification is exact proving the first and last equality; the middle equality follows as $F(K)^\# = F(L)^\#$ by (b).

0DB2 **Remark 25.12.9.** Let $\mathcal{C}$ be a site. Let $X \in \text{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has fibre products and let K be a hypercovering of X. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists,
(b) given morphisms $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ and $(\beta, g_k) : \{W_k\}_{k \in K} \to \{V_j\}_{j \in J}$ in $\text{SR}(\mathcal{C})$ such that $U = \coprod U_i$, $V = \coprod V_j$, and $W = \coprod W_j$ exist, then $U \times_V W = \coprod_{(i,j,k), \alpha(i) = j = \beta(k)} U_i \times_{V_j} W_k$,
(c) if $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ is a covering in the sense of Definition 25.3.1 and $U = \coprod U_i$ and $V = \coprod V_j$ exist, then the corresponding morphism $U \to V$ of Remark 25.12.7 is a covering of $\mathcal{C}$.

Then we get another simplicial object L of $\text{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 25.12.7. Now we claim that L is a hypercovering of X. To see this we check conditions (1), (2) of Definition 25.3.3. Condition (1) follows from (c) and (1) for K because (1) for K says $K_0 = \{U_{0,i}\}_{i \in I_0}$ is a covering of $\{X\}$ in the sense of Definition 25.3.1. Condition (2) follows because $\mathcal{C}/X$ has all finite limits hence $\text{SR}(\mathcal{C}/X)$ has all finite limits, and condition (b) says the construction of "taking disjoint unions" commutes with these fimite limits. Thus the morphism

$$L_{n+1} \longrightarrow (\text{cosk}_n\text{sk}_n L)_{n+1}$$

is a covering as it is the consequence of applying our "taking disjoint unions" functor to the morphism

$$K_{n+1} \longrightarrow (\text{cosk}_n\text{sk}_n K)_{n+1}$$

which is assumed to be a covering in the sense of Definition 25.3.1 by (2) for K. This makes sense because property (b) in particular assures us that if we start with a finite diagram of semi-representable objects over X for which we can take disjoint unions, then the limit of the diagram in $\text{SR}(\mathcal{C}/X)$ still is a semi-representable object over X for which we can take disjoint unions.

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Part 2

Schemes

CHAPTER 26

Schemes

01H8

26.1. Introduction

01H9 In this document we define schemes. A basic reference is [**DG67**].

26.2. Locally ringed spaces

01HA Recall that we defined ringed spaces in Sheaves, Section 6.25. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, see Sheaves, Definition 6.21.7 and Lemma 6.21.8.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example 6.25.2. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example 6.11.4. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section 10.17) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $\mathrm{Spec}(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

01HB **Definition 26.2.1.** Locally ringed spaces.

(1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.

(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to "the locally ringed space X". We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the *structure sheaf of X*. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say "let $f : X \to Y$ be a morphism of locally ringed spaces" thereby suppressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition 6.25.3. Let $x \in X$. By Sheaves, Lemma 6.21.10 the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^\sharp} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\textit{Locally ringed spaces} \longrightarrow \textit{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

01HC **Lemma 26.2.2.** *Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.*

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring isomorphism. □

26.3. Open immersions of locally ringed spaces

01HD
01HE **Definition 26.3.1.** Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition 6.31.2 (3).

01HF **Example 26.3.2.** Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

01HG **Definition 26.3.3.** Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 26.3.2 above is the *open subspace of X associated to U*.

01HH **Lemma 26.3.4.** *Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.*

Proof. Let f' be the homeomorphism between X and V induced by f. Then $f = j \circ f'$ as maps of topological spaces. Since there is an isomorphism of sheaves $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, there is an isomorphism of rings $f^\sharp : \Gamma(U, f^{-1}(\mathcal{O}_Y)) \to \Gamma(U, \mathcal{O}_X)$ for each open subset $U \subset X$. Since $\mathcal{O}_V = j^{-1}\mathcal{O}_Y$ and $f^{-1} = f'^{-1}j^{-1}$ (Sheaves, Lemma 6.21.6) we see that $f^{-1}\mathcal{O}_Y = f'^{-1}\mathcal{O}_V$, hence $\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, f^{-1}(\mathcal{O}_Y))$ is an isomorphism for every $U \subset X$ open. By composing these we get an isomorphism of rings

$$\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, \mathcal{O}_X)$$

for each open subset $U \subset X$, and therefore an isomorphism of sheaves $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$. In other words, we have an isomorphism $f'^\sharp : f'^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$ and therefore an isomorphism of locally ringed spaces $(f', f'^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ (use Lemma 26.2.2). Note that $f = j \circ f'$ as morphisms of locally ringed spaces by construction.

Suppose we have another morphism $f'' : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ such that $f = j \circ f''$. At any point $x \in X$, we have $j(f'(x)) = j(f''(x))$ from which it follows that $f'(x) = f''(x)$ since j is the inclusion map; therefore f' and f'' are the same as morphisms of topological spaces. On structure sheaves, for each open subset $U \subset X$ we have a commutative diagram

$$\begin{array}{ccc} \Gamma(U, f^{-1}(\mathcal{O}_Y)) & \xrightarrow{\cong} & \Gamma(U, \mathcal{O}_X) \\ \downarrow \cong & \nearrow_{f'^\sharp} \quad \nearrow_{f''^\sharp} & \\ \Gamma(U, f'^{-1}(\mathcal{O}_V)) & & \end{array}$$

from which we see that $f'^\sharp$ and $f''^\sharp$ define the same morphism of sheaves. □

From now on we do not distinguish between open subsets and their associated subspaces.

01HI **Lemma 26.3.5.** *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following*

diagram is a commutative square of locally ringed spaces

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle f|_U}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

Proof. Omitted. □

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

26.4. Closed immersions of locally ringed spaces

01HJ We follow our conventions introduced in Modules, Definition 17.13.1.

01HK **Definition 26.4.1.** Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is a *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

01HL **Lemma 26.4.2.** *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices that there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. □

01HM **Example 26.4.3.** Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma 17.5.3. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma 17.6.1 there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the stalk $\mathcal{O}_{Z,z}$ is equal to a quotient $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ of a local ring and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

01HN **Definition 26.4.4.** Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 26.4.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$*.

01HO **Lemma 26.4.5.** *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. □

01HP **Lemma 26.4.6.** *Let X, Y be locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f_y^\sharp(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma 6.32.4 we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. □

01HQ **Lemma 26.4.7.** *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by sections. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma 17.8.2. The rest of the lemma follows from the characterization, in Lemma 26.4.6 above, of what it means for a morphism to factor through a closed subspace. □

26.5. Affine schemes

01HR Let R be a ring. Consider the topological space $\mathrm{Spec}(R)$ associated to R, see Algebra, Section 10.17. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ and the resulting pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ will be an affine scheme.

Recall that $\mathrm{Spec}(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition 10.17.3. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

01HS **Lemma 26.5.1.** *Let R be a ring. Let $f \in R$.*

(1) *If $g \in R$ and $D(g) \subset D(f)$, then*
   (a) *f is invertible in R_g,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in R$,*
   (c) *there is a canonical ring map $R_f \to R_g$, and*

(d) *there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.*
(2) *Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^n D(g_i)$.*
(3) *If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Recall that $D(g) = \mathrm{Spec}(R_g)$ (see Algebra, Lemma 10.17.6). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.17.2. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma 10.29.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.17.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

01HT **Definition 26.5.2.** Let R be a ring.

(1) A *standard open covering* of $\mathrm{Spec}(R)$ is a covering $\mathrm{Spec}(R) = \bigcup_{i=1}^n D(f_i)$, where $f_1, \ldots, f_n \in R$.
(2) Suppose that $D(f) \subset \mathrm{Spec}(R)$ is a standard open. A *standard open covering* of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^n D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Spec}(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 26.5.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 26.5.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \mathrm{Spec}(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = \mathrm{colim}_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma 10.9.9 we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{fg_i} \to \bigoplus M_{fg_i g_j}.$$

In addition, by Lemma 26.5.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma 10.24.1 to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

01HU **Definition 26.5.3.** Let R be a ring.

(1) The *structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *of the spectrum of* R is the unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
(2) The locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called the *spectrum* of R and denoted $\mathrm{Spec}(R)$.
(3) The sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Spec}(R)$ is called the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

01HV **Lemma 26.5.4.** *Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M.*

(1) *We have $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R$.*
(2) *We have $\Gamma(\mathrm{Spec}(R), \widetilde{M}) = M$ as an R-module.*
(3) *For every $f \in R$ we have $\Gamma(D(f), \mathcal{O}_{\mathrm{Spec}(R)}) = R_f$.*
(4) *For every $f \in R$ we have $\Gamma(D(f), \widetilde{M}) = M_f$ as an R_f-module.*
(5) *Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 26.5.1.*
(6) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\mathcal{O}_{\mathrm{Spec}(R),x} = R_{\mathfrak{p}}$.*
(7) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\widetilde{M}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak{p}}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. □

01HW **Definition 26.5.5.** An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $\mathrm{Spec}(R)$ for some ring R. A *morphism of affine schemes* is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

26.6. The category of affine schemes

01HX Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

01HY **Lemma 26.6.1.** *Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in \mathrm{Mor}(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps*

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak{p} \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y,f(x)} \end{array}$$

(see the discussion of f-maps below Sheaves, Definition 6.21.7). Since the right vertical arrow is local we see that $\mathfrak{m}_{f(x)}$ is the inverse image of $\mathfrak{m}_x$. The result follows. □

01HZ **Lemma 26.6.2.** *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set*

$$D(f) = \{x \in X \mid \text{image } f \notin \mathfrak{m}_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma 17.25.10, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U \cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \notin \mathfrak{m}_x$ implies that $f \in \mathcal{O}_{X,x}$ has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. □

01I0 **Lemma 26.6.3.** *In Lemma 26.6.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition 10.17.1).*

Proof. Omitted. □

01I1 **Lemma 26.6.4.** *Let X be a locally ringed space. Let Y be an affine scheme. The map*

$$\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

A reference for this fact is [**DG67**, II, Err 1, Prop. 1.8.1] where it is attributed to J. Tate.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 26.5.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 26.6.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the topology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 26.6.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma 6.30.14 it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 26.5.4 and 26.5.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y,\Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 26.6.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 26.6.1 and given the uniqueness

of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. □

01I2 **Lemma 26.6.5.** *The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.*

Proof. This is now clear from Definition 26.5.5 and Lemma 26.6.4. □

01I3 **Lemma 26.6.6.** *Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.*

Proof. We may assume that $Y = \mathrm{Spec}(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = \mathrm{Spec}(R_f) \to \mathrm{Spec}(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma 10.17.6 we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 26.3.4. □

01I4 **Lemma 26.6.7.** *The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)*

$$\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B).$$

Proof. This is just an application of Lemma 26.6.4. First of all, by that lemma, the affine scheme $\mathrm{Spec}(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$\begin{aligned} \mathrm{Mor}(X, \mathrm{Spec}(A \otimes_R B)) &= \mathrm{Hom}(A \otimes_R B, \mathcal{O}_X(X)) \\ &= \mathrm{Hom}(A, \mathcal{O}_X(X)) \times_{\mathrm{Hom}(R, \mathcal{O}_X(X))} \mathrm{Hom}(B, \mathcal{O}_X(X)) \\ &= \mathrm{Mor}(X, \mathrm{Spec}(A)) \times_{\mathrm{Mor}(X, \mathrm{Spec}(R))} \mathrm{Mor}(X, \mathrm{Spec}(B)) \end{aligned}$$

which proves the formula. See Categories, Section 4.6 for the relevant definitions. □

01I5 **Lemma 26.6.8.** *Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 26.6.4 there is a canonical morphism of locally ringed spaces $X \to \mathrm{Spec}(R)$. By Algebra, Lemma 10.21.2 we see that as a topological space $\mathrm{Spec}(\mathcal{O}_X(U)) \amalg \mathrm{Spec}(\mathcal{O}_X(V)) = \mathrm{Spec}(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $\mathrm{Spec}(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to \mathrm{Spec}(R)$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to \mathrm{Spec}(R)$ is an isomorphism. □

26.7. Quasi-coherent sheaves on affines

01I6 Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition 17.10.1. In this section we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$ corresponds to the sheaf $\widetilde{M}$ associated to an R-module M.

01I7 **Lemma 26.7.1.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 26.5.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition 17.10.6). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property*

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma 17.10.5 we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. □

01I8 **Lemma 26.7.2.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. There are canonical isomorphisms*

1. *$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, see Modules, Section 17.16.*
2. *$\widetilde{T^n(M)} \cong T^n(\widetilde{M})$, $\widetilde{Sym^n(M)} \cong Sym^n(\widetilde{M})$, and $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$, see Modules, Section 17.21.*
3. *if M is a finitely presented R-module, then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$, see Modules, Section 17.22.*

First proof. Using Lemma 26.7.1 and Modules, Lemma 17.10.5 we see that the functor $M \mapsto \widetilde{M}$ can be viewed as π^* for a morphism π of ringed spaces. And pulling back modules commutes with tensor constructions by Modules, Lemmas 17.16.4 and 17.21.3. The morphism $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ is flat for example because the stalks of $\mathcal{O}_X$ are localizations of R (Lemma 26.5.4) and hence flat over R. Thus pullback by π commutes with internal hom if the first module is finitely presented by Modules, Lemma 17.22.5. □

Second proof. Proof of (1). By Lemma 26.7.1 to give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (either in Lemma 26.5.4 or in Modules, Lemma 17.10.5), the fact that tensor product commutes with localization (Algebra, Lemma 10.12.16) and Modules, Lemma 17.16.1.

Proof of (2). This is similar to the proof of (1), using Algebra, Lemma 10.13.6 and Modules, Lemma 17.21.2.

Proof of (3). Since the construction $M \mapsto \widetilde{M}$ is functorial there is an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. The target of this map is the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. Hence by Lemma 26.7.1 we obtain a map of $\mathcal{O}_X$-modules $\widetilde{\mathrm{Hom}_R(M, N)} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. We check that this is an isomorphism by comparing stalks. If M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence we conclude using Algebra, Lemma 10.10.2 and Modules, Lemma 17.22.4. □

Third proof of part (1). For any $\mathcal{O}_X$-module $\mathcal{F}$ we have the following isomorphisms functorial in M, N, and $\mathcal{F}$

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, \Gamma(X, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M \otimes_R N, \Gamma(X, \mathcal{F})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M \otimes_R N}, \mathcal{F})
\end{aligned}$$

The first equality is Modules, Lemma 17.22.1. The second equality is the universal property of $\widetilde{M}$, see Lemma 26.7.1. The third equality holds by definition of $\mathcal{H}om$. The fourth equality is the universal property of $\widetilde{N}$. Then fifth equality is Algebra, Lemma 10.12.8. The final equality is the universal property of $\widetilde{M \otimes_R N}$. By the Yoneda lemma (Categories, Lemma 4.3.5) we obtain (1). □

01I9 **Lemma 26.7.3.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$, $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be affine schemes. Let $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of affine schemes, corresponding to the ring map $\psi^\sharp : R \to S$ (see Lemma 26.6.5).*

(1) *We have $\psi^*\widetilde{M} = \widetilde{S \otimes_R M}$ functorially in the R-module M.*
(2) *We have $\psi_*\widetilde{N} = \widetilde{N_R}$ functorially in the S-module N.*

Proof. The first assertion follows from the identification in Lemma 26.7.1 and the result of Modules, Lemma 17.10.7. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$\psi_*\widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))$$

as desired. □

Lemma 26.7.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

01IA **Lemma 26.7.4.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is isomorphic to the sheaf associated to the R-module $\Gamma(X, \mathcal{F})$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition 5.13.1. Hence by Modules, Lemma 17.10.8 for every prime $\mathfrak{p} \subset R$ corresponding to

$x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 26.5.1 for example we can refine this covering to a standard open covering. Thus we get a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_if_j)} \xrightarrow{\varphi_i^{-1}|_{D(f_if_j)}} \mathcal{F}|_{D(f_if_j)} \xrightarrow{\varphi_j|_{D(f_if_j)}} \mathcal{F}_{M_j}|_{D(f_if_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_if_jf_k)} \circ \psi_{ij}|_{D(f_if_jf_k)} = \psi_{ik}|_{D(f_if_jf_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_if_j)$, $D(f_if_jf_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 26.7.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_if_j)) = (M_i)_{f_j}$, etc. Also by Lemma 26.7.1 above we see that ψ_{ij} corresponds to a unique $R_{f_if_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_if_j)$. Moreover these then satisfy the cocycle condition that

$$\begin{array}{ccc} (M_i)_{f_jf_k} & \xrightarrow{\psi_{ik}} & (M_k)_{f_if_j} \\ & {\scriptstyle \psi_{ij}} \searrow \quad \nearrow {\scriptstyle \psi_{jk}} & \\ & (M_j)_{f_if_k} & \end{array}$$

commutes (for any triple i, j, k).

Now Algebra, Lemma 10.24.5 shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section 6.33 and in particular Lemma 6.33.4. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 26.7.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism. $\square$

01IB **Lemma 26.7.5.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories*

$$QCoh(\mathcal{O}_X) \rightleftarrows Mod_R$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 26.7.1 and 26.7.4 above. □

From now on we will not distinguish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

01IC **Lemma 26.7.6.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{\ }$ since by Lemma 26.7.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.) □

01ID **Lemma 26.7.7.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.*

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 26.7.5 above we can write $\mathcal{F}_i = \widetilde{M}_i$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \left(\bigoplus M_i\right)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits. □

01IE **Lemma 26.7.8.** *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 26.5.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M}_1$ and $\mathcal{F}_3 = \widetilde{M}_3$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 26.7.1) we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M}_1 & \longrightarrow & \widetilde{M}_2 & \longrightarrow & \widetilde{M}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

and we win by the snake lemma.

The "correct" argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{f \in R \mid \text{the element } fm \text{ comes from } M_2\}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \notin \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \notin \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \notin \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f) \cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f) \cup D(g_j g_k)} - \sigma_k|_{D(f) \cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements $f^N \sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup (D(f) \cup D(g_j)) = X$ as desired. □

26.8. Closed subspaces of affine schemes

01IF

01IG **Example 26.8.1.** Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = \operatorname{Spec}(R/I) \to \operatorname{Spec}(R) = X$. By Algebra, Lemma 10.17.7 this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 26.4.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\widetilde{I}$, as in Example 26.4.3.

01IH **Lemma 26.8.2.** *Let $(X, \mathcal{O}_X) = (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists a unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ constructed in Example 26.8.1 above.*

Proof. This is kind of silly! Namely, by Lemma 26.4.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 26.4.4 and Example 26.4.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j\in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 26.7 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 26.7.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \widetilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. □

26.9. Schemes

01II

01IJ **Definition 26.9.1.** A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an *affine open*, or an *open affine* if the open subspace U is an affine scheme. We will often write $U = \mathrm{Spec}(R)$ to indicate that U is isomorphic to $\mathrm{Spec}(R)$ and moreover that we will identify (temporarily) U and $\mathrm{Spec}(R)$.

01IK **Lemma 26.9.2.** *Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.*

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $u \in D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 26.6.6. This proves that every point of U has an open neighbourhood which is affine. □

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section 5.5) for the topology consisting of affine opens.

01IL **Example 26.9.3.** Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = \mathrm{Spec}(k[x,y]) \setminus \{(x,y)\}$ of the affine scheme $X = \mathrm{Spec}(k[x,y])$. It is covered by two affines, namely $D(x) = \mathrm{Spec}(k[x,y,1/x])$ and $D(y) = \mathrm{Spec}(k[x,y,1/y])$ whose intersection is $D(xy) = \mathrm{Spec}(k[x,y,1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x,y,1/x] \times k[x,y,1/y] \to k[x,y,1/xy]$$

We conclude that the map $k[x,y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 26.6.5 we would have $U \cong X$.

26.10. Immersions of schemes

01IM In Lemma 26.9.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 26.7 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

01IN **Lemma 26.10.1.** *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

(1) *The locally ringed space Z is a scheme,*
(2) *the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,*
(3) *for any affine open $U = \mathrm{Spec}(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal $I \subset R$, and*
(4) *we have $\mathcal{I}|_U = \widetilde{I}$.*

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = \mathrm{Spec}(R)$. By Lemma 26.8.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\widetilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. □

01IO **Definition 26.10.2.** Let X be a scheme.

(1) A morphism of schemes is called an *open immersion* if it is an open immersion of locally ringed spaces (see Definition 26.3.1).
(2) An *open subscheme* of X is an open subspace of X in the sense of Definition 26.3.3; an open subscheme of X is a scheme by Lemma 26.9.2.
(3) A morphism of schemes is called a *closed immersion* if it is a closed immersion of locally ringed spaces (see Definition 26.4.1).
(4) A *closed subscheme* of X is a closed subspace of X in the sense of Definition 26.4.4; a closed subscheme is a scheme by Lemma 26.10.1.
(5) A morphism of schemes $f : X \to Y$ is called an *immersion*, or a *locally closed immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 26.3 and 26.4 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target.

Our definition of a closed immersion is halfway between Hartshorne and EGA. Hartshorne defines a closed immersion as a morphism $f : X \to Y$ of schemes which induces a homeomorphism of X onto a closed subset of Y such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see [**Har77**, Page 85]. We will show this is equivalent to our notion in Lemma 26.24.2. In [**DG67**], Grothendieck and Dieudonné first define closed subschemes via the construction of Example 26.4.3 using quasi-coherent sheaves of ideals and then define a closed immersion as a morphism $f : X \to Y$ which induces an isomorphism with a closed subscheme. It follows from Lemma 26.10.1 that this agrees with our notion.

Pedagogically speaking the definition above is a disaster/nightmare. In teaching this material to students, we have found it often convenient to define a closed immersion as an affine morphism $f : X \to Y$ of schemes such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. Namely, it turns out that the notion of an affine morphism (Morphisms, Section 29.11) is quite natural and easy to understand.

For more information on closed immersions we suggest the reader visit Morphisms, Sections 29.2 and 29.4.

We will discuss locally closed subschemes and immersions at the end of this section.

01IP **Remark 26.10.3.** If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example 29.3.4.

01IQ **Lemma 26.10.4.** *Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.*

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immersion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section 6.33) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction $\mathcal{O}_X/\mathcal{J}$ is supported on U and, restricted there, equal to $\mathcal{O}_U/\mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 26.4.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 26.4.5, we conclude that $Y \to U \to X$ is a closed immersion. □

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. We define a *locally closed subscheme of* X as a pair (Z, U) consisting of a closed subscheme Z of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say "let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z *is contained in* Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

For more information on immersions, we refer the reader to Morphisms, Section 29.3.

26.11. Zariski topology of schemes

01IR See Topology, Section 5.1 for some basic material in topology adapted to the Zariski topology of schemes.

01IS **Lemma 26.11.1.** *Let X be a scheme. Any irreducible closed subset of X has a unique generic point. In other words, X is a sober topological space, see Topology, Definition 5.8.6.*

Proof. Let $Z \subset X$ be an irreducible closed subset. For every affine open $U \subset X$, $U = \mathrm{Spec}(R)$ we know that $Z \cap U = V(I)$ for a unique radical ideal $I \subset R$. Note that $Z \cap U$ is either empty or irreducible. In the second case (which occurs for at least one U) we see that $I = \mathfrak{p}$ is a prime ideal, which is a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\{\xi\}}$, in other words ξ is a generic point of Z. If ξ' was a second generic point, then $\xi' \in Z \cap U$ and it follows immediately that $\xi' = \xi$. □

01IT **Lemma 26.11.2.** *Let X be a scheme. The collection of affine opens of X forms a basis for the topology on X.*

Proof. This follows from the discussion on open subschemes in Section 26.9. □

01IU **Remark 26.11.3.** In general the intersection of two affine opens in X is not affine open. See Example 26.14.3.

01IV **Lemma 26.11.4.** *The underlying topological space of any scheme is locally quasi-compact, see Topology, Definition 5.13.1.*

Proof. This follows from Lemma 26.11.2 above and the fact that the spectrum of a ring is quasi-compact, see Algebra, Lemma 10.17.8. □

01IW **Lemma 26.11.5.** *Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. There exists an affine open neighbourhood W of x such that W is a standard open of both U and V.*

Proof. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Say x corresponds to the prime $\mathfrak{p} \subset A$ and the prime $\mathfrak{q} \subset B$. We may choose an $f \in A$, $f \notin \mathfrak{p}$ such that $D(f) \subset U \cap V$. Note that any standard open of $D(f)$ is a standard open of $\mathrm{Spec}(A) = U$. Hence we may assume that $U \subset V$. In other words, now we may think of U as an affine open of V. Next we choose a $g \in B$, $g \notin \mathfrak{q}$ such that $D(g) \subset U$. In this case we see that $D(g) = D(g_A)$ where $g_A \in A$ denotes the image of g by the map $B \to A$. Thus the lemma is proved. □

01IX **Lemma 26.11.6.** *Let X be a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. Let $V \subset X$ be an affine open. There exists a standard open covering $V = \bigcup_{j=1,\ldots,m} V_j$ (see Definition 26.5.2) such that each V_j is a standard open in one of the U_i.*

Proof. Pick $v \in V$. Then $v \in U_i$ for some i. By Lemma 26.11.5 above there exists an open $v \in W_v \subset V \cap U_i$ such that W_v is a standard open in both V and U_i. Since V is quasi-compact the lemma follows. □

0F1A **Lemma 26.11.7.** *Let X be a scheme. Let $\mathcal{B}$ be the set of affine opens of X. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$, see Sheaves, Definition 6.30.1. The following are equivalent*

(1) *$\mathcal{F}$ is the restriction of a sheaf on X to $\mathcal{B}$,*

(2) $\mathcal{F}$ *is a sheaf on* $\mathcal{B}$, *and*
(3) $\mathcal{F}(\emptyset)$ *is a singleton and whenever* $U = V \cup W$ *with* $U, V, W \in \mathcal{B}$ *and* $V, W \subset U$ *standard open (Algebra, Definition 10.17.3) the map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \times \mathcal{F}(W)$$

is injective with image the set of pairs (s, t) *such that* $s|_{V \cap W} = t|_{V \cap W}$.

Proof. The equivalence of (1) and (2) is Sheaves, Lemma 6.30.7. It is clear that (2) implies (3). Hence it suffices to prove that (3) implies (2). By Sheaves, Lemma 6.30.4 and Lemma 26.5.1 it suffices to prove the sheaf condition holds for standard open coverings (Definition 26.5.2) of elements of $\mathcal{B}$. Let $U = U_1 \cup \ldots \cup U_n$ be a standard open covering with $U \subset X$ affine open. We will prove the sheaf condition for this covering by induction on n. If $n = 0$, then U is empty and we get the sheaf condition by assumption. If $n = 1$, then there is nothing to prove. If $n = 2$, then this is assumption (3). If $n > 2$, then we write $U_i = D(f_i)$ for $f_i \in A = \mathcal{O}_X(U)$. Suppose that $s_i \in \mathcal{F}(U_i)$ are sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $1 \leq i < j \leq n$. Since $U = U_1 \cup \ldots \cup U_n$ we have $1 = \sum_{i=1,\ldots,n} a_i f_i$ in A for some $a_i \in A$, see Algebra, Lemma 10.17.2. Set $g = \sum_{i=1,\ldots,n-1} a_i f_i$. Then $U = D(g) \cup D(f_n)$. Observe that $D(g) = D(gf_1) \cup \ldots \cup D(gf_{n-1})$ is a standard open covering. By induction there is a unique section $s' \in \mathcal{F}(D(g))$ which agrees with $s_i|_{D(gfi)}$ for $i = 1, \ldots, n-1$. We claim that s' and s_n have the same restriction to $D(gf_n)$. This is true by induction and the covering $D(gf_n) = D(gf_nf_1) \cup \ldots \cup D(gf_nf_{n-1})$. Thus there is a unique section $s \in \mathcal{F}(U)$ whose restriction to $D(g)$ is s' and whose restriction to $D(f_n)$ is s_n. We omit the verification that s restricts to s_i on $D(f_i)$ for $i = 1, \ldots, n-1$ and we omit the verification that s is unique. □

02O0 **Lemma 26.11.8.** *Let* X *be a scheme whose underlying topological space is a finite discrete set. Then* X *is affine.*

Proof. Say $X = \{x_1, \ldots, x_n\}$. Then $U_i = \{x_i\}$ is an open neighbourhood of x_i. By Lemma 26.11.2 it is affine. Hence X is a finite disjoint union of affine schemes, and hence is affine by Lemma 26.6.8. □

01IY **Example 26.11.9.** There exists a scheme without closed points. Namely, let R be a local domain whose spectrum looks like $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{m}$. Then the open subscheme $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ does not have a closed point. To see that such a ring R exists, we use that given any totally ordered group $(\Gamma, \geq)$ there exists a valuation ring A with valuation group $(\Gamma, \geq)$, see [**Kru32**]. See Algebra, Section 10.50 for notation. We take $\Gamma = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3 \oplus \ldots$ and we define $\sum_i a_i x_i \geq 0$ if and only if the first nonzero a_i is > 0, or all $a_i = 0$. So $x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0$. The subsets $x_i + \Gamma_{\geq 0}$ are prime ideals of $(\Gamma, \geq)$, see Algebra, notation above Lemma 10.50.17. These together with $\emptyset$ and $\Gamma_{\geq 0}$ are the only prime ideals. Hence A is an example of a ring with the given structure of its spectrum, by Algebra, Lemma 10.50.17.

26.12. Reduced schemes

01IZ

01J0 **Definition 26.12.1.** Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

01J1 **Lemma 26.12.2.** *A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma 6.11.1. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. □

01J2 **Lemma 26.12.3.** *An affine scheme $\mathrm{Spec}(R)$ is reduced if and only if R is reduced.*

Proof. The direct implication follows immediately from Lemma 26.12.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. □

01J3 **Lemma 26.12.4.** *Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = \mathrm{Spec}(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma 10.17.2. Moreover, for any standard open $D(g) \subset \mathrm{Spec}(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\widetilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z \cap U = \mathrm{Spec}(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 26.12.3 above). By construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z' \cap U = \mathrm{Spec}(R/I')$ for some ideal $I' \subset R$. By Lemma 26.12.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma 10.17.2. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. □

01J4 **Definition 26.12.5.** Let X be a scheme. Let $Z \subset X$ be a closed subset. A *scheme structure on Z* is given by a closed subscheme Z' of X whose underlying set is equal to Z. We often say "let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The *reduced induced scheme structure* on Z is the one constructed in Lemma 26.12.4. The *reduction X_{red} of X* is the reduced induced scheme structure on X itself.

Often when we say "let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

0F2L **Remark 26.12.6.** Let X be a scheme. Let $T \subset X$ be a locally closed subset. In this situation we sometimes also use the phrase "reduced induced scheme structure on T". It refers to the reduced induced scheme structure from Definition 26.12.5 when we view T as a closed subset of the open subscheme $X \setminus \partial T$ of X. Here $\partial T = \overline{T} \setminus T$ is the "boundary" of T in the topological space of X.

0356 **Lemma 26.12.7.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.*

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $\mathrm{Spec}(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $b \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 26.12.2, and Algebra, Lemma 10.17.2). Hence f factors through Z by Lemma 26.4.6. □

26.13. Points of schemes

01J5 Given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

See Categories, Example 4.3.4. This is called the *functor of points of X*. A fun part of scheme theory is to find descriptions of the internal geometry of X in terms of this functor h_X. In this section we find a simple way to describe points of X.

Let X be a scheme. Let R be a local ring with maximal ideal $\mathfrak{m} \subset R$. Suppose that $f : \mathrm{Spec}(R) \to X$ is a morphism of schemes. Let $x \in X$ be the image of the closed point $\mathfrak{m} \in \mathrm{Spec}(R)$. Then we obtain a local homomorphism of local rings

$$f^\sharp : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{m}} = R.$$

01J6 **Lemma 26.13.1.** *Let X be a scheme. Let R be a local ring. The construction above gives a bijective correspondence between morphisms $\mathrm{Spec}(R) \to X$ and pairs (x, φ) consisting of a point $x \in X$ and a local homomorphism of local rings $\varphi : \mathcal{O}_{X,x} \to R$.*

Proof. Let A be a ring. For any ring homomorphism $\psi : A \to R$ there exists a unique prime ideal $\mathfrak{p} \subset A$ and a factorization $A \to A_\mathfrak{p} \to R$ where the last map is a local homomorphism of local rings. Namely, $\mathfrak{p} = \psi^{-1}(\mathfrak{m})$. Via Lemma 26.6.4 this proves that the lemma holds if X is an affine scheme.

Let X be a general scheme. Any $x \in X$ is contained in an open affine $U \subset X$. By the affine case we conclude that every pair (x, φ) occurs as the end product of the construction above the lemma.

To finish the proof it suffices to show that any morphism $f : \mathrm{Spec}(R) \to X$ has image contained in any affine open containing the image x of the closed point of $\mathrm{Spec}(R)$. In fact, let $x \in V \subset X$ be any open neighbourhood containing x. Then $f^{-1}(V) \subset \mathrm{Spec}(R)$ is an open containing the unique closed point and hence equal to $\mathrm{Spec}(R)$. □

As a special case of the lemma above we obtain for every point x of a scheme X a canonical morphism

02NA (26.13.1.1) $$\operatorname{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$$

corresponding to the identity map on the local ring of X at x. We may reformulate the lemma above as saying that for any morphism $f : \operatorname{Spec}(R) \to X$ there exists a unique point $x \in X$ such that f factors as $\operatorname{Spec}(R) \to \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ where the first map comes from a local homomorphism $\mathcal{O}_{X,x} \to R$.

In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

where the left vertical map corresponds to the local ring map $f_x^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$.

01J7 **Lemma 26.13.2.** *Let X be a scheme. Let $x, x' \in X$ be points of X. Then $x' \in X$ is a generalization of x if and only if x' is in the image of the canonical morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X$.*

Proof. A continuous map preserves the relation of specialization/generalization. Since every point of $\operatorname{Spec}(\mathcal{O}_{X,x})$ is a generalization of the closed point we see every point in the image of $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is a generalization of x. Conversely, suppose that x' is a generalization of x. Choose an affine open neighbourhood $U = \operatorname{Spec}(R)$ of x. Then $x' \in U$. Say $\mathfrak{p} \subset R$ and $\mathfrak{p}' \subset R$ are the primes corresponding to x and x'. Since x' is a generalization of x we see that $\mathfrak{p}' \subset \mathfrak{p}$. This means that $\mathfrak{p}'$ is in the image of the morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) = \operatorname{Spec}(R_\mathfrak{p}) \to \operatorname{Spec}(R) = U \subset X$ as desired. □

Now, let us discuss morphisms from spectra of fields. Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let K be a field. A local homomorphism $R \to K$ by definition factors as $R \to \kappa \to K$, i.e., is the same thing as a morphism $\kappa \to K$. Thus we see that morphisms

$$\operatorname{Spec}(K) \longrightarrow X$$

correspond to pairs $(x, \kappa(x) \to K)$. We may define a preorder on morphisms of spectra of fields to X by saying that $\operatorname{Spec}(K) \to X$ dominates $\operatorname{Spec}(L) \to X$ if $\operatorname{Spec}(K) \to X$ factors through $\operatorname{Spec}(L) \to X$. This suggests the following notion: Let us temporarily say that two morphisms $p : \operatorname{Spec}(K) \to X$ and $q : \operatorname{Spec}(L) \to X$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\Omega) & \longrightarrow & \operatorname{Spec}(L) \\ \downarrow & & \downarrow q \\ \operatorname{Spec}(K) & \xrightarrow{p} & X \end{array}$$

Of course this immediately implies that the unique points of all three of the schemes $\operatorname{Spec}(K)$, $\operatorname{Spec}(L)$, and $\operatorname{Spec}(\Omega)$ map to the same $x \in X$. Thus a diagram (by the

remarks above) corresponds to a point $x \in X$ and a commutative diagram

$$\begin{array}{ccc} \Omega & \longleftarrow & L \\ \uparrow & & \uparrow \\ K & \longleftarrow & \kappa(x) \end{array}$$

of fields. This defines an equivalence relation, because given any set of field extensions K_i/κ there exists some field extension Ω/κ such that all the field extensions K_i are contained in the extension Ω.

01J9 **Lemma 26.13.3.** *Let X be a scheme. Points of X correspond bijectively to equivalence classes of morphisms from spectra of fields into X. Moreover, each equivalence class contains a (unique up to unique isomorphism) smallest element $\mathrm{Spec}(\kappa(x)) \to X$.*

Proof. Follows from the discussion above. □

Of course the morphisms $\mathrm{Spec}(\kappa(x)) \to X$ factor through the canonical morphisms $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$. And the content of Lemma 26.13.2 is in this setting that the morphism $\mathrm{Spec}(\kappa(x')) \to X$ factors as $\mathrm{Spec}(\kappa(x')) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ whenever x' is a generalization of x. In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S. \end{array}$$

26.14. Glueing schemes

01JA Let I be a set. For each $i \in I$ let $(X_i, \mathcal{O}_i)$ be a locally ringed space. (Actually the construction that follows works equally well for ringed spaces.) For each pair $i, j \in I$ let $U_{ij} \subset X_i$ be an open subspace. For each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

be an isomorphism of locally ringed spaces. For convenience we assume that $U_{ii} = X_i$. For each triple $i, j, k \in I$ assume that

(1) we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and
(2) the diagram

$$\begin{array}{ccc} U_{ij} \cap U_{ik} & \xrightarrow{\varphi_{ik}} & U_{ki} \cap U_{kj} \\ & {\scriptstyle \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \varphi_{jk}} & \\ & U_{ji} \cap U_{jk} & \end{array}$$

is commutative.

(In particular, it follows that $\varphi_{ii} = \mathrm{id}_{X_i}$.) Let us call a collection $(I, (X_i)_{i \in I}, (U_{ij})_{i,j \in I}, (\varphi_{ij})_{i,j \in I})$ satisfying the conditions above a glueing data.

01JB **Lemma 26.14.1.** *Given any glueing data of locally ringed spaces there exists a locally ringed space X and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that*

(1) $X = \bigcup_{i \in I} U_i$,
(2) $\varphi_i(U_{ij}) = U_i \cap U_j$, *and*
(3) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

The locally ringed space X is characterized by the following mapping properties: Given a locally ringed space Y we have

$$\begin{aligned}
\mathrm{Mor}(X, Y) &= \{(f_i)_{i \in I} \mid f_i : X_i \to Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\} \\
f &\mapsto (f|_{U_i} \circ \varphi_i)_{i \in I} \\
\mathrm{Mor}(Y, X) &= \left\{ \begin{array}{c} \text{open covering } Y = \bigcup_{i \in I} V_i \text{ and } (g_i : V_i \to X_i)_{i \in I} \text{ such that} \\ g_i^{-1}(U_{ij}) = V_i \cap V_j \text{ and } g_j|_{V_i \cap V_j} = \varphi_{ij} \circ g_i|_{V_i \cap V_j} \end{array} \right\} \\
g &\mapsto V_i = g^{-1}(U_i),\ g_i = \varphi_i^{-1} \circ g|_{V_i}
\end{aligned}$$

Proof. We construct X in stages. As a set we take

$$X = (\coprod X_i)/\sim .$$

Here given $x \in X_i$ and $x' \in X_j$ we say $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$ and $\varphi_{ij}(x) = x'$. This is an equivalence relation since if $x \in X_i$, $x' \in X_j$, $x'' \in X_k$, and $x \sim x'$ and $x' \sim x''$, then $x' \in U_{ji} \cap U_{jk}$, hence by condition (1) of a glueing data also $x \in U_{ij} \cap U_{ik}$ and $x'' \in U_{ki} \cap U_{kj}$ and by condition (2) we see that $\varphi_{ik}(x) = x''$. (Reflexivity and symmetry follows from our assumptions that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$.) Denote $\varphi_i : X_i \to X$ the natural maps. Denote $U_i = \varphi_i(X_i) \subset X$. Note that $\varphi_i : X_i \to U_i$ is a bijection.

The topology on X is defined by the rule that $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all i. We leave it to the reader to verify that this does indeed define a topology. Note that in particular U_i is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in X_j for all j. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$. Therefore $\varphi_i : X_i \to U_i$ is a homeomorphism.

To obtain a locally ringed space we have to construct the sheaf of rings $\mathcal{O}_X$. We do this by glueing the sheaves of rings $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_i$. Namely, in the commutative diagram

$$\begin{array}{ccc}
U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} \\
& {\scriptstyle \varphi_i|_{U_{ij}}} \searrow \quad \swarrow {\scriptstyle \varphi_j|_{U_{ji}}} & \\
& U_i \cap U_j &
\end{array}$$

the arrow on top is an isomorphism of ringed spaces, and hence we get unique isomorphisms of sheaves of rings

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

These satisfy a cocycle condition as in Sheaves, Section 6.33. By the results of that section we obtain a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ compatibly with the glueing maps displayed above. In particular $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_i$ at corresponding points.

The proof of the mapping properties is omitted. □

01JC **Lemma 26.14.2.** *In Lemma 26.14.1 above, assume that all X_i are schemes. Then the resulting locally ringed space X is a scheme.*

Proof. This is clear since each of the U_i is a scheme and hence every $x \in X$ has an affine neighbourhood. □

It is customary to think of X_i as an open subspace of X via the isomorphisms φ_i. We will do this in the next two examples.

01JD **Example 26.14.3** (Affine space with zero doubled)**.** Let k be a field. Let $n \geq 1$. Let $X_1 = \mathrm{Spec}(k[x_1, \ldots, x_n])$, let $X_2 = \mathrm{Spec}(k[y_1, \ldots, y_n])$. Let $0_1 \in X_1$ be the point corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. Let $0_2 \in X_2$ be the point corresponding to the maximal ideal $(y_1, \ldots, y_n) \subset k[y_1, \ldots, y_n]$. Let $U_{12} = X_1 \setminus \{0_1\}$ and let $U_{21} = X_2 \setminus \{0_2\}$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ mapping y_i to x_i (which induces $X_1 \cong X_2$ mapping 0_1 to 0_2). Let X be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_1, X_2 \subset X$ as open subschemes. There is a morphism $f : X \to \mathrm{Spec}(k[t_1, \ldots, t_n])$ which on X_1 (resp. X_2) corresponds to k algebra map $k[t_1, \ldots, t_n] \to k[x_1, \ldots, x_n]$ (resp. $k[t_1, \ldots, t_n] \to k[y_1, \ldots, y_n]$) mapping t_i to x_i (resp. t_i to y_i). It is easy to see that this morphism identifies $k[t_1, \ldots, t_n]$ with $\Gamma(X, \mathcal{O}_X)$. Since $f(0_1) = f(0_2)$ we see that X is not affine.

Note that X_1 and X_2 are affine opens of X. But, if $n = 2$, then $X_1 \cap X_2$ is the scheme described in Example 26.9.3 and hence not affine. Thus in general the intersection of affine opens of a scheme is not affine. (This fact holds more generally for any $n > 1$.)

Another curious feature of this example is the following. If $n > 1$ there are many irreducible closed subsets $T \subset X$ (take the closure of any non closed point in X_1 for example). But unless $T = \{0_1\}$, or $T = \{0_2\}$ we have $0_1 \in T \Leftrightarrow 0_2 \in T$. Proof omitted.

01JE **Example 26.14.4** (Projective line)**.** Let k be a field. Let $X_1 = \mathrm{Spec}(k[x])$, let $X_2 = \mathrm{Spec}(k[y])$. Let $0 \in X_1$ be the point corresponding to the maximal ideal $(x) \subset k[x]$. Let $\infty \in X_2$ be the point corresponding to the maximal ideal $(y) \subset k[y]$. Let $U_{12} = X_1 \setminus \{0\} = D(x) = \mathrm{Spec}(k[x, 1/x])$ and let $U_{21} = X_2 \setminus \{\infty\} = D(y) = \mathrm{Spec}(k[y, 1/y])$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y, 1/y] \to k[x, 1/x]$ mapping y to $1/x$. Let $\mathbf{P}^1_k$ be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset \mathbf{P}^1_k$ as open subschemes. In this case we see that $\Gamma(\mathbf{P}^1_k, \mathcal{O}) = k$ because the only polynomials $g(x)$ in x such that $g(1/y)$ is also a polynomial in y are constant polynomials. Since $\mathbf{P}^1_k$ is infinite we see that $\mathbf{P}^1_k$ is not affine.

We claim that there exists an affine open $U \subset \mathbf{P}^1_k$ which contains both 0 and ∞. Namely, let $U = \mathbf{P}^1_k \setminus \{1\}$, where 1 is the point of X_1 corresponding to the maximal ideal $(x - 1)$ and also the point of X_2 corresponding to the maximal ideal $(y - 1)$. Then it is easy to see that $s = 1/(x - 1) = y/(1 - y) \in \Gamma(U, \mathcal{O}_U)$. In fact you can show that $\Gamma(U, \mathcal{O}_U)$ is equal to the polynomial ring $k[s]$ and that the corresponding morphism $U \to \mathrm{Spec}(k[s])$ is an isomorphism of schemes. Details omitted.

26.15. A representability criterion

01JF

In this section we reformulate the glueing lemma of Section 26.14 in terms of functors. We recall some of the material from Categories, Section 4.3. Recall that given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

This is called the *functor of points of* X.

Let F be a contravariant functor from the category of schemes to the category of sets. In a formula

$$F : Sch^{opp} \longrightarrow Sets.$$

We will use the same terminology as in Sites, Section 7.2. Namely, given a scheme T, an element $\xi \in F(T)$, and a morphism $f : T' \to T$ we will denote $f^*\xi$ the element $F(f)(\xi)$, and sometimes we will even use the notation $\xi|_{T'}$

01JG **Definition 26.15.1.** (See Categories, Definition 4.3.6.) Let F be a contravariant functor from the category of schemes to the category of sets (as above). We say that F is *representable by a scheme* or *representable* if there exists a scheme X such that $h_X \cong F$.

Suppose that F is representable by the scheme X and that $s : h_X \to F$ is an isomorphism. By Categories, Yoneda Lemma 4.3.5 the pair $(X, s : h_X \to F)$ is unique up to unique isomorphism if it exists. Moreover, the Yoneda lemma says that given any contravariant functor F as above and any scheme Y, we have a bijection

$$\mathrm{Mor}_{\mathrm{Fun}(Sch^{opp}, Sets)}(h_Y, F) \longrightarrow F(Y), \quad s \longmapsto s(\mathrm{id}_Y).$$

Here is the reverse construction. Given any $\xi \in F(Y)$ the transformation of functors $s_\xi : h_Y \to F$ associates to any morphism $f : T \to Y$ the element $f^*\xi \in F(T)$.

In particular, in the case that F is representable, there exists a scheme X and an element $\xi \in F(X)$ such that the corresponding morphism $h_X \to F$ is an isomorphism. In this case we also say *the pair* (X, ξ) *represents* F. The element $\xi \in F(X)$ is often called the *"universal family"* for reasons that will become more clear when we talk about algebraic stacks (insert future reference here). For the moment we simply observe that the fact that if the pair (X, ξ) represents F, then every element $\xi' \in F(T)$ for any T is of the form $\xi' = f^*\xi$ for a unique morphism $f : T \to X$.

01JH **Example 26.15.2.** Consider the rule which associates to every scheme T the set $F(T) = \Gamma(T, \mathcal{O}_T)$. We can turn this into a contravariant functor by using for a morphism $f : T' \to T$ the pullback map $f^\sharp : \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$. Given a ring R and an element $t \in R$ there exists a unique ring homomorphism $\mathbf{Z}[x] \to R$ which maps x to t. Thus, using Lemma 26.6.4, we see that

$$\mathrm{Mor}(T, \mathrm{Spec}(\mathbf{Z}[x])) = \mathrm{Hom}(\mathbf{Z}[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

This does indeed give an isomorphism $h_{\mathrm{Spec}(\mathbf{Z}[x])} \to F$. What is the "universal family" ξ? To get it we have to apply the identifications above to $\mathrm{id}_{\mathrm{Spec}(\mathbf{Z}[x])}$. Clearly under the identifications above this gives that $\xi = x \in \Gamma(\mathrm{Spec}(\mathbf{Z}[x]), \mathcal{O}_{\mathrm{Spec}(\mathbf{Z}[x])}) = \mathbf{Z}[x]$ as expected.

01JI **Definition 26.15.3.** Let F be a contravariant functor on the category of schemes with values in sets.

(1) We say that F *satisfies the sheaf property for the Zariski topology* if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) A *subfunctor* $H \subset F$ is a rule that associates to every scheme T a subset $H(T) \subset F(T)$ such that the maps $F(f) : F(T) \to F(T')$ maps $H(T)$ into $H(T')$ for all morphisms of schemes $f : T' \to T$.
(3) Let $H \subset F$ be a subfunctor. We say that $H \subset F$ is *representable by open immersions* if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$ there exists an open subscheme $U_\xi \subset T$ with the following property:
   (*) A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.
(4) Let I be a set. For each $i \in I$ let $H_i \subset F$ be a subfunctor. We say that the collection $(H_i)_{i \in I}$ *covers* F if and only if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in H_i(U_i)$.

In condition (4), if $H_i \subset F$ is representable by open immersions for all i, then to check $(H_i)_{i \in I}$ covers F, it suffices to check $F(T) = \bigcup H_i(T)$ whenever T is the spectrum of a field.

01JJ **Lemma 26.15.4.** *Let F be a contravariant functor on the category of schemes with values in the category of sets. Suppose that*

(1) *F satisfies the sheaf property for the Zariski topology,*
(2) *there exists a set I and a collection of subfunctors $F_i \subset F$ such that*
   (a) *each F_i is representable,*
   (b) *each $F_i \subset F$ is representable by open immersions, and*
   (c) *the collection $(F_i)_{i \in I}$ covers F.*

Then F is representable.

Proof. Let X_i be a scheme representing F_i and let $\xi_i \in F_i(X_i) \subset F(X_i)$ be the "universal family". Because $F_j \subset F$ is representable by open immersions, there exists an open $U_{ij} \subset X_i$ such that $T \to X_i$ factors through U_{ij} if and only if $\xi_i|_T \in F_j(T)$. In particular $\xi_i|_{U_{ij}} \in F_j(U_{ij})$ and therefore we obtain a canonical morphism $\varphi_{ij} : U_{ij} \to X_j$ such that $\varphi_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji} this implies that φ_{ij} factors through U_{ji}. Since $(\varphi_{ij} \circ \varphi_{ji})^*\xi_j = \varphi_{ji}^*(\varphi_{ij}^*\xi_j) = \varphi_{ji}^*\xi_i = \xi_j$ we conclude that $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{U_{ji}}$ because the pair (X_j, ξ_j) represents F_j. In particular the maps $\varphi_{ij} : U_{ij} \to U_{ji}$ are isomorphisms of schemes. Next we have to show that $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$. This is true because (a) $U_{ji} \cap U_{jk}$ is the largest open of U_{ji} such that ξ_j restricts to an element of F_k, (b) $U_{ij} \cap U_{ik}$ is the largest open of U_{ij} such that ξ_i restricts to an element of F_k, and (c) $\varphi_{ij}^*\xi_j = \xi_i$. Moreover, the cocycle condition in Section 26.14 follows because both $\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}}$ and $\varphi_{ik}|_{U_{ij} \cap U_{ik}}$ pullback ξ_k to the element ξ_i. Thus we may apply Lemma 26.14.2 to obtain a scheme X with an open covering $X = \bigcup U_i$ and isomorphisms $\varphi_i : X_i \to U_i$ with properties as in Lemma 26.14.1. Let $\xi_i' = (\varphi_i^{-1})^*\xi_i$. The conditions of Lemma 26.14.1 imply that $\xi_i'|_{U_i \cap U_j} = \xi_j'|_{U_i \cap U_j}$. Therefore, by the condition that F satisfies the sheaf condition in the Zariski topology we see that there exists an element $\xi' \in F(X)$ such that $\xi_i = \varphi_i^*\xi'|_{U_i}$ for all i. Since φ_i is an isomorphism we also get that $(U_i, \xi'|_{U_i})$ represents the functor F_i.

We claim that the pair (X, ξ') represents the functor F. To show this, let T be a scheme and let $\xi \in F(T)$. We will construct a unique morphism $g : T \to X$ such that $g^*\xi' = \xi$. Namely, by the condition that the subfunctors F_i cover F there exists an open covering $T = \bigcup V_i$ such that for each i the restriction $\xi|_{V_i} \in F_i(V_i)$. Moreover, since each of the inclusions $F_i \subset F$ are representable by open immersions we may assume that each $V_i \subset T$ is maximal open with this property. Because, $(U_i, \xi'|_{U_i})$ represents the functor F_i we get a unique morphism $g_i : V_i \to U_i$ such that $g_i^*\xi'|_{U_i} = \xi|_{V_i}$. On the overlaps $V_i \cap V_j$ the morphisms g_i and g_j agree, for example because they both pull back $\xi'|_{U_i \cap U_j} \in F_i(U_i \cap U_j)$ to the same element. Thus the morphisms g_i glue to a unique morphism from $T \to X$ as desired. □

01JK **Remark 26.15.5.** Suppose the functor F is defined on all locally ringed spaces, and if conditions of Lemma 26.15.4 are replaced by the following:

(1) F satisfies the sheaf property on the category of locally ringed spaces,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
   (a) each F_i is representable by a scheme,
   (b) each $F_i \subset F$ is representable by open immersions on the category of locally ringed spaces, and
   (c) the collection $(F_i)_{i \in I}$ covers F as a functor on the category of locally ringed spaces.

We leave it to the reader to spell this out further. Then the end result is that the functor F is representable in the category of locally ringed spaces and that the representing object is a scheme.

26.16. Existence of fibre products of schemes

01JL A very basic question is whether or not products and fibre products exist on the category of schemes. We first prove abstractly that products and fibre products exist, and in the next section we show how we may think in a reasonable way about fibre products of schemes.

01JM **Lemma 26.16.1.** *The category of schemes has a final object, products and fibre products. In other words, the category of schemes has finite limits, see Categories, Lemma 4.18.4.*

Proof. Please skip this proof. It is more important to learn how to work with the fibre product which is explained in the next section.

By Lemma 26.6.4 the scheme $\mathrm{Spec}(\mathbf{Z})$ is a final object in the category of locally ringed spaces. Thus it suffices to prove that fibred products exist.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We have to show that the functor

$$\begin{aligned} F : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto \mathrm{Mor}(T, X) \times_{\mathrm{Mor}(T,S)} \mathrm{Mor}(T, Y) \end{aligned}$$

is representable. We claim that Lemma 26.15.4 applies to the functor F. If we prove this then the lemma is proved.

First we show that F satisfies the sheaf property in the Zariski topology. Namely, suppose that T is a scheme, $T = \bigcup_{i \in I} U_i$ is an open covering, and $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all pairs i, j. By definition ξ_i corresponds to a

pair (a_i, b_i) where $a_i : U_i \to X$ and $b_i : U_i \to Y$ are morphisms of schemes such that $f \circ a_i = g \circ b_i$. The glueing condition says that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ and $b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$. Thus by glueing the morphisms a_i we obtain a morphism of locally ringed spaces (i.e., a morphism of schemes) $a : T \to X$ and similarly $b : T \to Y$ (see for example the mapping property of Lemma 26.14.1). Moreover, on the members of an open covering the compositions $f \circ a$ and $g \circ b$ agree. Therefore $f \circ a = g \circ b$ and the pair (a, b) defines an element of $F(T)$ which restricts to the pairs (a_i, b_i) on each U_i. The sheaf condition is verified.

Next, we construct the family of subfunctors. Choose an open covering by open affines $S = \bigcup_{i \in I} U_i$. For every $i \in I$ choose open coverings by open affines $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ and $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$. Note that $X = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is an open covering and similarly for Y. For any $i \in I$ and each pair $(j, k) \in J_i \times K_i$ we have a commutative diagram

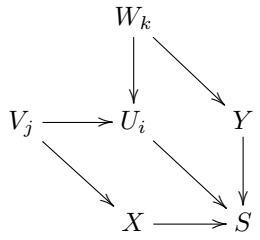

where all the skew arrows are open immersions. For such a triple we get a functor

$$\begin{aligned} F_{i,j,k} : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto \mathrm{Mor}(T, V_j) \times_{\mathrm{Mor}(T, U_i)} \mathrm{Mor}(T, W_k). \end{aligned}$$

There is an obvious transformation of functors $F_{i,j,k} \to F$ (coming from the huge commutative diagram above) which is injective, so we may think of $F_{i,j,k}$ as a subfunctor of F.

We check condition (2)(a) of Lemma 26.15.4. This follows directly from Lemma 26.6.7. (Note that we use here that the fibre products in the category of affine schemes are also fibre products in the whole category of locally ringed spaces.)

We check condition (2)(b) of Lemma 26.15.4. Let T be a scheme and let $\xi \in F(T)$. In other words, $\xi = (a, b)$ where $a : T \to X$ and $b : T \to Y$ are morphisms of schemes such that $f \circ a = g \circ b$. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$. For any further morphism $h : T' \to T$ we have $h^*\xi = (a \circ h, b \circ h)$. Hence we see that $h^*\xi \in F_{i,j,k}(T')$ if and only if $a(h(T')) \subset V_j$ and $b(h(T')) \subset W_k$. In other words, if and only if $h(T') \subset V_{i,j,k}$. This proves condition (2)(b).

We check condition (2)(c) of Lemma 26.15.4. Let T be a scheme and let $\xi = (a, b) \in F(T)$ as above. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$ as above. Condition (2)(c) just means that $T = \bigcup V_{i,j,k}$ which is evident. Thus the lemma is proved and fibre products exist. □

01JN **Remark 26.16.2.** Using Remark 26.15.5 you can show that the fibre product of morphisms of schemes exists in the category of locally ringed spaces and is a scheme.

26.17. Fibre products of schemes

01JO Here is a review of the general definition, even though we have already shown that fibre products of schemes exist.

01JP **Definition 26.17.1.** Given morphisms of schemes $f : X \to S$ and $g : Y \to S$ the *fibre product* is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ sitting into the following commutative diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

which is universal among all diagrams of this sort, see Categories, Definition 4.6.1.

In other words, given any solid commutative diagram of morphisms of schemes

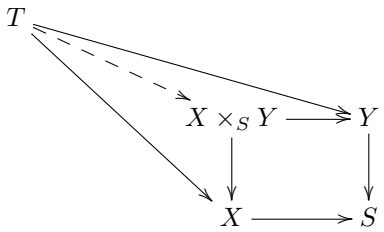

there exists a unique dotted arrow making the diagram commute. We will prove some lemmas which will tell us how to think about fibre products.

01JQ **Lemma 26.17.2.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. If X, Y, S are all affine then $X \times_S Y$ is affine.*

Proof. Suppose that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. By Lemma 26.6.7 the affine scheme $\mathrm{Spec}(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces. Hence it is a fortiori the fibre product in the category of schemes. □

01JR **Lemma 26.17.3.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $X \times_S Y$, p, q be the fibre product. Suppose that $U \subset S$, $V \subset X$, $W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. Let T be a scheme. Suppose $a : T \to V$ and $b : T \to W$ are morphisms such that $f \circ a = g \circ b$ as morphisms into U. Then they agree as morphisms into S. By the universal property of the fibre product we get a unique morphism $T \to X \times_S Y$. Of course this morphism has image contained in the open $p^{-1}(V) \cap q^{-1}(W)$. Thus $p^{-1}(V) \cap q^{-1}(W)$ is a fibre product of V and W over U. The result follows from the uniqueness of fibre products, see Categories, Section 4.6. □

In particular this shows that $V \times_U W = V \times_S W$ in the situation of the lemma. Moreover, if U, V, W are all affine, then we know that $V \times_U W$ is affine. And of course we may cover $X \times_S Y$ by such affine opens $V \times_U W$. We formulate this as a lemma.

01JS **Lemma 26.17.4.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $g^{-1}(U_i)$. Then*

$$X \times_S Y = \bigcup\nolimits_{i \in I} \bigcup\nolimits_{j \in J_i,\ k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

Proof. See discussion above the lemma. □

In other words, we might have used the previous lemma to construct the fibre product directly by glueing the affine schemes. (Which is of course exactly what we did in the proof of Lemma 26.16.1 anyway.) Here is a way to describe the set of points of a fibre product of schemes.

01JT **Lemma 26.17.5.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples*

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

Proof. Let z be a point of $X \times_S Y$ and let us construct a quadruple as above. Recall that we may think of z as a morphism $\mathrm{Spec}(\kappa(z)) \to X \times_S Y$, see Lemma 26.13.3. This morphism corresponds to morphisms $a : \mathrm{Spec}(\kappa(z)) \to X$ and $b : \mathrm{Spec}(\kappa(z)) \to Y$ such that $f \circ a = g \circ b$. By the same lemma again we get points $x \in X$, $y \in Y$ lying over the same point $s \in S$ as well as field maps $\kappa(x) \to \kappa(z)$, $\kappa(y) \to \kappa(z)$ such that the compositions $\kappa(s) \to \kappa(x) \to \kappa(z)$ and $\kappa(s) \to \kappa(y) \to \kappa(z)$ are the same. In other words we get a ring map $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$. We let $\mathfrak{p}$ be the kernel of this map.

Conversely, given a quadruple $(x, y, s, \mathfrak{p})$ we get a commutative solid diagram

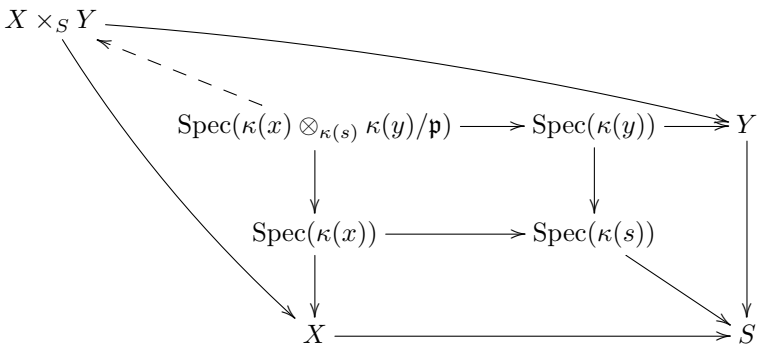

see the discussion in Section 26.13. Thus we get the dotted arrow. The corresponding point z of $X \times_S Y$ is the image of the generic point of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p})$. We omit the verification that the two constructions are inverse to each other. □

01JU **Lemma 26.17.6.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target.*

(1) *If $f : X \to S$ is a closed immersion, then $X \times_S Y \to Y$ is a closed immersion. Moreover, if $X \to S$ corresponds to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$, then $X \times_S Y \to Y$ corresponds to the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$.*
(2) *If $f : X \to S$ is an open immersion, then $X \times_S Y \to Y$ is an open immersion.*
(3) *If $f : X \to S$ is an immersion, then $X \times_S Y \to Y$ is an immersion.*

Proof. Assume that $X \to S$ is a closed immersion corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$. By Lemma 26.4.7 the closed subspace $Z \subset Y$ defined by the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$ is the fibre product in the category of locally ringed spaces. By Lemma 26.10.1 Z is a scheme. Hence $Z = X \times_S Y$ and the first statement follows. The second follows from Lemma 26.17.3 for example. The third is a combination of the first two. □

01JV **Definition 26.17.7.** Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be a closed subscheme of Y. The *inverse image $f^{-1}(Z)$ of the closed subscheme Z* is the closed subscheme $Z \times_Y X$ of X. See Lemma 26.17.6 above.

We may occasionally also use this terminology with locally closed and open subschemes.

26.18. Base change in algebraic geometry

01JW One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (Varieties, Definition 33.3.1) with $X \to \mathrm{Spec}(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given *base scheme*, often denoted S. We use the language: "let X be a scheme over S" to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the *structure morphism* $X \to S$ as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, and so on.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 4.2.13.

01JX **Definition 26.18.1.** Let S be a scheme.

(1) We say X is a *scheme over S* to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the *structure morphism*.
(2) If R is a ring we say X is a *scheme over R* instead of X is a scheme over $\mathrm{Spec}(R)$.
(3) A *morphism $f : X \to Y$ of schemes over S* is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.

[1] Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

(4) We denote $\mathrm{Mor}_S(X, Y)$ the set of all morphisms from X to Y over S.
(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The *base change* of X is the scheme $X_{S'} = S' \times_S X$ over S'.
(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The *base change* of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphism $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$).
(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The *base change* $X_{R'}$ is the scheme $\mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} X$ over R'.

Here is a typical result.

01JY **Lemma 26.18.2.** *Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).*

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:

$$\begin{array}{ccc} X_{S'} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_{S'} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma 26.17.6. □

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

01JZ **Definition 26.18.3.** Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that $\mathcal{P}$ is *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that "being a closed immersion" is preserved under arbitrary base change.

01K0 **Definition 26.18.4.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s*, is the scheme fitting in the following fibre product diagram

$$\begin{array}{ccc} X_s = \mathrm{Spec}(\kappa(s)) \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

01K1 **Lemma 26.18.5.** *Let $f : X \to S$ be a morphism of schemes. Consider the diagrams*

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\kappa(s)) & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{S,s}) \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma 26.13.1 for example. Thus we may assume that S is affine. If X is also affine, then the result follows from Algebra, Remark 10.18.5. In the general case the result follows by covering X by open affines. □

26.19. Quasi-compact morphisms

01K2 A scheme is *quasi-compact* if its underlying topological space is quasi-compact. There is a relative notion which is defined as follows.

01K3 **Definition 26.19.1.** A morphism of schemes is called *quasi-compact* if the underlying map of topological spaces is quasi-compact, see Topology, Definition 5.12.1.

01K4 **Lemma 26.19.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *$f : X \to S$ is quasi-compact,*
(2) *the inverse image of every affine open is quasi-compact, and*
(3) *there exists some affine open covering $S = \bigcup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.*

Proof. Suppose we are given a covering $S = \bigcup_{i \in I} U_i$ as in (3). First, let $U \subset S$ be any affine open. For any $u \in U$ we can find an index $i(u) \in I$ such that $u \in U_{i(u)}$. As standard opens form a basis for the topology on $U_{i(u)}$ we can find $W_u \subset U \cap U_{i(u)}$ which is standard open in $U_{i(u)}$. By compactness we can find finitely many points $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{j=1}^n W_{u_j}$. For each j write $f^{-1}U_{i(u_j)} = \bigcup_{k \in K_j} V_{jk}$ as a finite union of affine opens. Since $W_{u_j} \subset U_{i(u_j)}$ is a standard open we see that $f^{-1}(W_{u_j}) \cap V_{jk}$ is a standard open of V_{jk}, see Algebra, Lemma 10.17.4. Hence $f^{-1}(W_{u_j}) \cap V_{jk}$ is affine, and so $f^{-1}(W_{u_j})$ is a finite union of affines. This proves that the inverse image of any affine open is a finite union of affine opens.

Next, assume that the inverse image of every affine open is a finite union of affine opens. Let $K \subset S$ be any quasi-compact open. Since S has a basis of the topology consisting of affine opens we see that K is a finite union of affine opens. Hence the inverse image of K is a finite union of affine opens. Hence f is quasi-compact.

Finally, assume that f is quasi-compact. In this case the argument of the previous paragraph shows that the inverse image of any affine is a finite union of affine opens. □

01K5 **Lemma 26.19.3.** *Being quasi-compact is a property of morphisms of schemes over a base which is preserved under arbitrary base change.*

Proof. Omitted. □

01K6 **Lemma 26.19.4.** *The composition of quasi-compact morphisms is quasi-compact.*

Proof. This follows from the definitions and Topology, Lemma 5.12.2. □

01K7 **Lemma 26.19.5.** *A closed immersion is quasi-compact.*

Proof. Follows from the definitions and Topology, Lemma 5.12.3. □

01K8 **Example 26.19.6.** An open immersion is in general not quasi-compact. The standard example of this is the open subspace $U \subset X$, where $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$, where U is $X \setminus \{0\}$, and where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, x_3, \ldots)$.

05JL **Lemma 26.19.7.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *$f(X) \subset S$ is closed, and*
(2) *$f(X) \subset S$ is stable under specialization.*

Proof. We have (1) $\Rightarrow$ (2) by Topology, Lemma 5.19.2. Assume (2). Let $U \subset S$ be an affine open. It suffices to prove that $f(X) \cap U$ is closed. Since $U \cap f(X)$ is stable under specializations in U, we have reduced to the case where S is affine. Because f is quasi-compact we deduce that $X = f^{-1}(S)$ is quasi-compact as S is affine. Thus we may write $X = \bigcup_{i=1}^n U_i$ with $U_i \subset X$ open affine. Say $S = \mathrm{Spec}(R)$ and $U_i = \mathrm{Spec}(A_i)$ for some R-algebra A_i. Then $f(X) = \mathrm{Im}(\mathrm{Spec}(A_1 \times \ldots \times A_n) \to \mathrm{Spec}(R))$. Thus the lemma follows from Algebra, Lemma 10.41.5. □

01K9 **Lemma 26.19.8.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if specializations lift along f, see Topology, Definition 5.19.4.*

Proof. According to Topology, Lemma 5.19.7 if f is closed then specializations lift along f. Conversely, suppose that specializations lift along f. Let $Z \subset X$ be a closed subset. We may think of Z as a scheme with the reduced induced scheme structure, see Definition 26.12.5. Since $Z \subset X$ is closed the restriction of f to Z is still quasi-compact. Moreover specializations lift along $Z \to S$ as well, see Topology, Lemma 5.19.5. Hence it suffices to prove $f(X)$ is closed if specializations lift along f. In particular $f(X)$ is stable under specializations, see Topology, Lemma 5.19.6. Thus $f(X)$ is closed by Lemma 26.19.7. □

26.20. Valuative criterion for universal closedness

01KA In Topology, Section 5.17 there is a discussion of proper maps as closed maps of topological spaces all of whose fibres are quasi-compact, or as maps such that all base changes are closed maps. Here is the corresponding notion in algebraic geometry.

01KB **Definition 26.20.1.** A morphism of schemes $f : X \to S$ is said to be *universally closed* if every base change $f' : X_{S'} \to S'$ is closed.

In fact the adjective "universally" is often used in this way. In other words, given a property $\mathcal{P}$ of morphisms then we say that "$X \to S$ is universally $\mathcal{P}$" if and only if every base change $X_{S'} \to S'$ has $\mathcal{P}$.

Please take a look at Morphisms, Section 29.41 for a more detailed discussion of the properties of universally closed morphisms. In this section we restrict the discussion to the relationship between universal closed morphisms and morphisms satisfying the existence part of the valuative criterion.

01KC **Lemma 26.20.2.** *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is universally closed then specializations lift along any base change of f, see Topology, Definition 5.19.4.*
(2) *If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.*

Proof. Part (1) is a direct consequence of Topology, Lemma 5.19.7. Part (2) follows from Lemmas 26.19.8 and 26.19.3. □

01KD **Definition 26.20.3.** Let $f : X \to S$ be a morphism of schemes. We say *f satisfies the existence part of the valuative criterion* if given any commutative solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a valuation ring with field of fractions K, the dotted arrow exists. We say *f satisfies the uniqueness part of the valuative criterion* if there is at most one dotted arrow given any diagram as above (without requiring existence of course).

A *valuation ring* is a local domain maximal among the relation of domination in its fraction field, see Algebra, Definition 10.50.1. Hence the spectrum of a valuation ring has a unique generic point η and a unique closed point 0, and of course we have the specialization $\eta \rightsquigarrow 0$. The significance of valuation rings is that any specialization of points in any scheme is the image of $\eta \rightsquigarrow 0$ under some morphism from the spectrum of some valuation ring. Here is the precise result.

01J8 **Lemma 26.20.4.** *Let S be a scheme. Let $s' \rightsquigarrow s$ be a specialization of points of S. Then*

(1) *there exists a valuation ring A and a morphism $f : \mathrm{Spec}(A) \to S$ such that the generic point η of $\mathrm{Spec}(A)$ maps to s' and the special point maps to s, and*
(2) *given a field extension $K/\kappa(s')$ we may arrange it so that the extension $\kappa(\eta)/\kappa(s')$ induced by f is isomorphic to the given extension.*

Proof. Let $s' \rightsquigarrow s$ be a specialization in S, and let $K/\kappa(s')$ be an extension of fields. By Lemma 26.13.2 and the discussion following Lemma 26.13.3 this leads to ring maps $\mathcal{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{S,s} \to K$, see Algebra, Lemma 10.50.2. The ring map $\mathcal{O}_{S,s} \to A$ induces the morphism $f : \mathrm{Spec}(A) \to S$, see Lemma 26.13.1. This morphism has all the desired properties by construction. □

01KE **Lemma 26.20.5.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *Specializations lift along any base change of f*
(2) *The morphism f satisfies the existence part of the valuative criterion.*

Proof. Assume (1) holds. Let a solid diagram as in Definition 26.20.3 be given. In order to find the dotted arrow we may replace $X \to S$ by $X_{\mathrm{Spec}(A)} \to \mathrm{Spec}(A)$ since after all the assumption is stable under base change. Thus we may assume $S = \mathrm{Spec}(A)$. Let $x' \in X$ be the image of $\mathrm{Spec}(K) \to X$, so that we have

$\kappa(x') \subset K$, see Lemma 26.13.3. By assumption there exists a specialization $x' \rightsquigarrow x$ in X such that x maps to the closed point of $S = \mathrm{Spec}(A)$. We get a local ring map $A \to \mathcal{O}_{X,x}$ and a ring map $\mathcal{O}_{X,x} \to \kappa(x')$, see Lemma 26.13.2 and the discussion following Lemma 26.13.3. The composition $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ is the given injection $A \to K$. Since $A \to \mathcal{O}_{X,x}$ is local, the image of $\mathcal{O}_{X,x} \to K$ dominates A and hence is equal to A, by Algebra, Definition 10.50.1. Thus we obtain a ring map $\mathcal{O}_{X,x} \to A$ and hence a morphism $\mathrm{Spec}(A) \to X$ (see Lemma 26.13.1 and discussion following it). This proves (2).

Conversely, assume (2) holds. It is immediate that the existence part of the valuative criterion holds for any base change $X_{S'} \to S'$ of f by considering the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & X_{S'} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Namely, the more horizontal dotted arrow will lead to the other one by definition of the fibre product. OK, so it clearly suffices to show that specializations lift along f. Let $s' \rightsquigarrow s$ be a specialization in S, and let $x' \in X$ be a point lying over s'. Apply Lemma 26.20.4 to $s' \rightsquigarrow s$ and the extension of fields $K = \kappa(x')/\kappa(s')$. We get a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & \nearrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

and by condition (2) we get the dotted arrow. The image x of the closed point of $\mathrm{Spec}(A)$ in X will be a solution to our problem, i.e., x is a specialization of x' and maps to s. □

01KF **Proposition 26.20.6** (Valuative criterion of universal closedness)**.** *Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion.*

Proof. This is a formal consequence of Lemmas 26.20.2 and 26.20.5 above. □

01KG **Example 26.20.7.** Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 26.14.4. Let us use the valuative criterion above to prove that p is universally closed. By construction $\mathbf{P}^1_k$ is covered by two affine opens and hence p is quasi-compact. Let a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow[\xi]{} & \mathbf{P}^1_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{\varphi} & \mathrm{Spec}(k) \end{array}$$

be given, where A is a valuation ring and K is its field of fractions. Recall that $\mathbf{P}^1_k$ is gotten by glueing $\mathrm{Spec}(k[x])$ to $\mathrm{Spec}(k[y])$ by glueing $D(x)$ to $D(y)$ via $x = y^{-1}$ (or more symmetrically $xy = 1$). To show there is a morphism $\mathrm{Spec}(A) \to \mathbf{P}^1_k$ fitting diagonally into the diagram above we may assume that ξ maps into the

open $\mathrm{Spec}(k[x])$ (by symmetry). This gives the following commutative diagram of rings

$$\begin{array}{ccc} K & \xleftarrow{\xi^\sharp} & k[x] \\ \uparrow & & \uparrow \\ A & \xleftarrow{\varphi^\sharp} & k \end{array}$$

By Algebra, Lemma 10.50.4 we see that either $\xi^\sharp(x) \in A$ or $\xi^\sharp(x)^{-1} \in A$. In the first case we get a ring map

$$k[x] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ x \mapsto \xi^\sharp(x)$$

fitting into the diagram of rings above, and we win. In the second case we see that we get a ring map

$$k[y] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ y \mapsto \xi^\sharp(x)^{-1}.$$

This gives a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k[y]) \to \mathbf{P}^1_k$ which fits diagonally into the initial commutative diagram of this example (check omitted).

26.21. Separation axioms

01KH A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

01KI **Lemma 26.21.1.** *The diagonal morphism of a morphism between affines is closed.*

Proof. The diagonal morphism associated to the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the morphism on spectra corresponding to the ring map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. This map is clearly surjective, so $S \cong S \otimes_R S/J$ for some ideal $J \subset S \otimes_R S$. Hence Δ is a closed immersion according to Example 26.8.1. □

01KJ **Lemma 26.21.2.** *Let X be a scheme over S. The diagonal morphism $\Delta_{X/S}$ is an immersion.*

Proof. Recall that if $V \subset X$ is affine open and maps into $U \subset S$ affine open, then $V \times_U V$ is affine open in $X \times_S X$, see Lemmas 26.17.2 and 26.17.3. Consider the open subscheme W of $X \times_S X$ which is the union of these affine opens $V \times_U V$. By Lemma 26.4.2 it is enough to show that each morphism $\Delta_{X/S}^{-1}(V \times_U V) \to V \times_U V$ is a closed immersion. Since $V = \Delta_{X/S}^{-1}(V \times_U V)$ we are just checking that $\Delta_{V/U}$ is a closed immersion, which is Lemma 26.21.1. □

01KK **Definition 26.21.3.** Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.
(2) We say f is *quasi-separated* if the diagonal morphism $\Delta_{X/S}$ is a quasi-compact morphism.
(3) We say a scheme Y is *separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is separated.
(4) We say a scheme Y is *quasi-separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated.

By Lemmas 26.21.2 and 26.10.4 we see that $\Delta_{X/S}$ is a closed immersion if an only if $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. Moreover, by Lemma 26.19.5 we see that a separated morphism is quasi-separated. The reason for introducing quasi-separated morphisms is that nonseparated morphisms come up naturally in studying algebraic varieties (especially when doing moduli, algebraic stacks, etc). But most often they are still quasi-separated.

01KL **Example 26.21.4.** Here is an example of a non-quasi-separated morphism. Suppose $X = X_1 \cup X_2 \to S = \mathrm{Spec}(k)$ with $X_1 = X_2 = \mathrm{Spec}(k[t_1, t_2, t_3, \ldots])$ glued along the complement of $\{0\} = \{(t_1, t_2, t_3, \ldots)\}$ (glued as in Example 26.14.3). In this case the inverse image of the affine scheme $X_1 \times_S X_2$ under $\Delta_{X/S}$ is the scheme $\mathrm{Spec}(k[t_1, t_2, t_3, \ldots]) \setminus \{0\}$ which is not quasi-compact.

01KM **Lemma 26.21.5.** *Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.*

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{array}{ccc} Z = Y \times_{(Y \times_S Y)} X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (a,b)} \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

Thus the lemma follows from Lemmas 26.18.2, 26.21.2 and Definition 26.21.3. □

01KO **Lemma 26.21.6.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is quasi-separated.*
2. *For every pair of affine opens $U, V \subset X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X.*
3. *There exists an affine open covering $S = \bigcup_{i \in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \bigcup_{j \in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X.*

Proof. Let us prove that (3) implies (1). By Lemma 26.17.4 the covering $X \times_S X = \bigcup_i \bigcup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\Delta_{X/S}^{-1}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from Lemma 26.19.2.

The implication (1) $\Rightarrow$ (2) follows from the fact that under the hypotheses of (2) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) $\Rightarrow$ (3) is trivial. □

01KP **Lemma 26.21.7.** *Let $f : X \to S$ be a morphism of schemes.*

1. *If f is separated then for every pair of affine opens (U, V) of X which map into a common affine open of S we have*
   (a) *the intersection $U \cap V$ is affine.*
   (b) *the ring map $\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*
2. *If any pair of points $x_1, x_2 \in X$ lying over a common point $s \in S$ are contained in affine opens $x_1 \in U$, $x_2 \in V$ which map into a common affine open of S such that (a), (b) hold, then f is separated.*

Proof. Assume f separated. Suppose (U, V) is a pair as in (1). Let $W = \mathrm{Spec}(R)$ be an affine open of S containing both $f(U)$ and $f(V)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ for R-algebras A and B. By Lemma 26.17.3 we see that $U \times_S V = U \times_W V = \mathrm{Spec}(A \otimes_R B)$ is an affine open of $X \times_S X$. Hence, by Lemma 26.10.1 we see that $\Delta^{-1}(U \times_S V) \to U \times_S V$ can be identified with $\mathrm{Spec}((A \otimes_R B)/J) \to \mathrm{Spec}(A \otimes_R B)$ for some ideal $J \subset A \otimes_R B$. Thus $U \cap V = \Delta^{-1}(U \times_S V)$ is affine. Assertion (1)(b) holds because $A \otimes_{\mathbf{Z}} B \to (A \otimes_R B)/J$ is surjective.

Assume the hypothesis formulated in (2) holds. Clearly the collection of affine opens $U \times_S V$ for pairs (U, V) as in (2) form an affine open covering of $X \times_S X$ (see e.g. Lemma 26.17.4). Hence it suffices to show that each morphism $U \cap V = \Delta_{X/S}^{-1}(U \times_S V) \to U \times_S V$ is a closed immersion, see Lemma 26.4.2. By assumption (a) we have $U \cap V = \mathrm{Spec}(C)$ for some ring C. After choosing an affine open $W = \mathrm{Spec}(R)$ of S into which both U and V map and writing $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ we see that the assumption (b) means that the composition

$$A \otimes_{\mathbf{Z}} B \to A \otimes_R B \to C$$

is surjective. Hence $A \otimes_R B \to C$ is surjective and we conclude that $\mathrm{Spec}(C) \to \mathrm{Spec}(A \otimes_R B)$ is a closed immersion. □

01KQ **Example 26.21.8.** Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 26.14.4. Let us use the lemma above to prove that p is separated. By construction $\mathbf{P}^1_k$ is covered by two affine opens $U = \mathrm{Spec}(k[x])$ and $V = \mathrm{Spec}(k[y])$ with intersection $U \cap V = \mathrm{Spec}(k[x, y]/(xy-1))$ (using obvious notation). Thus it suffices to check that conditions (2)(a) and (2)(b) of Lemma 26.21.7 hold for the pairs of affine opens (U, U), (U, V), (V, U) and (V, V). For the pairs (U, U) and (V, V) this is trivial. For the pair (U, V) this amounts to proving that $U \cap V$ is affine, which is true, and that the ring map

$$k[x] \otimes_{\mathbf{Z}} k[y] \longrightarrow k[x, y]/(xy - 1)$$

is surjective. This is clear because any element in the right hand side can be written as a sum of a polynomial in x and a polynomial in y.

01KR **Lemma 26.21.9.** *Let $f : X \to T$ and $g : Y \to T$ be morphisms of schemes with the same target. Let $h : T \to S$ be a morphism of schemes. Then the induced morphism $i : X \times_T Y \to X \times_S Y$ is an immersion. If $T \to S$ is separated, then i is a closed immersion. If $T \to S$ is quasi-separated, then i is a quasi-compact morphism.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_T Y & \longrightarrow & X \times_S Y \\ \downarrow & & \downarrow \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T \end{array}$$

is a fibre product diagram. The lemma follows from Lemmas 26.21.2, 26.17.6 and 26.19.3. □

01KS **Lemma 26.21.10.** *Let $g : X \to Y$ be a morphism of schemes over S. The morphism $i : X \to X \times_S Y$ is an immersion. If Y is separated over S it is a closed immersion. If Y is quasi-separated over S it is quasi-compact.*

Proof. This is a special case of Lemma 26.21.9 applied to the morphism $X = X \times_Y Y \to X \times_S Y$. □

01KT **Lemma 26.21.11.** *Let $f : X \to S$ be a morphism of schemes. Let $s : S \to X$ be a section of f (in a formula $f \circ s = id_S$). Then s is an immersion. If f is separated then s is a closed immersion. If f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 26.21.10 applied to $g = s$ so the morphism $i = s : S \to S \times_S X$. □

01KU **Lemma 26.21.12.** *Permanence properties.*

(1) *A composition of separated morphisms is separated.*
(2) *A composition of quasi-separated morphisms is quasi-separated.*
(3) *The base change of a separated morphism is separated.*
(4) *The base change of a quasi-separated morphism is quasi-separated.*
(5) *A (fibre) product of separated morphisms is separated.*
(6) *A (fibre) product of quasi-separated morphisms is quasi-separated.*

Proof. Let $X \to Y \to Z$ be morphisms. Assume that $X \to Y$ and $Y \to Z$ are separated. The composition

$$X \to X \times_Y X \to X \times_Z X$$

is closed because the first one is by assumption and the second one by Lemma 26.21.9. The same argument works for "quasi-separated" (with the same references).

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is a morphism

$$\Delta_{f'} : X_{S'} = S' \times_S X \longrightarrow X_{S'} \times_{Y_{S'}} X_{S'} = S' \times_S (X \times_Y X)$$

which is easily seen to be the base change of Δ_f. Thus (3) and (4) follow from the fact that closed immersions and quasi-compact morphisms are preserved under arbitrary base change (Lemmas 26.17.6 and 26.19.3).

If $f : X \to Y$ and $g : U \to V$ are morphisms of schemes over a base S, then $f \times g$ is the composition of $X \times_S U \to X \times_S V$ (a base change of g) and $X \times_S V \to Y \times_S V$ (a base change of f). Hence (5) and (6) follow from (1) – (4). □

01KV **Lemma 26.21.13.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f. If $g \circ f$ is quasi-separated then so is f.*

Proof. Assume that $g \circ f$ is separated. Consider the factorization $X \to X \times_Y X \to X \times_Z X$ of the diagonal morphism of $g \circ f$. By Lemma 26.21.9 the last morphism is an immersion. By assumption the image of X in $X \times_Z X$ is closed. Hence it is also closed in $X \times_Y X$. Thus we see that $X \to X \times_Y X$ is a closed immersion by Lemma 26.10.4.

Assume that $g \circ f$ is quasi-separated. Let $V \subset Y$ be an affine open which maps into an affine open of Z. Let $U_1, U_2 \subset X$ be affine opens which map into V. Then $U_1 \cap U_2$ is a finite union of affine opens because U_1, U_2 map into a common affine open of Z. Since we may cover Y by affine opens like V we deduce the lemma from Lemma 26.21.6. □

03GI **Lemma 26.21.14.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 26.21.11 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 26.21.12). The second map is quasi-compact as it is the base change of $g \circ f$, see Lemma 26.19.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 26.19.4. □

01KN **Lemma 26.21.15.** *An affine scheme is separated. A morphism from an affine scheme to another scheme is separated.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme. Then $U \to \mathrm{Spec}(\mathbf{Z})$ has closed diagonal by Lemma 26.21.1. Thus U is separated by Definition 26.21.3. If $U \to X$ is a morphism of schemes, then we can apply Lemma 26.21.13 to the morphisms $U \to X \to \mathrm{Spec}(\mathbf{Z})$ to conclude that $U \to X$ is separated. □

You may have been wondering whether the condition of only considering pairs of affine opens whose image is contained in an affine open is really necessary to be able to conclude that their intersection is affine. Often it isn't!

01KW **Lemma 26.21.16.** *Let $f : X \to S$ be a morphism. Assume f is separated and S is a separated scheme. Suppose $U \subset X$ and $V \subset X$ are affine. Then $U \cap V$ is affine (and a closed subscheme of $U \times V$).*

Proof. In this case X is separated by Lemma 26.21.12. Hence $U \cap V$ is affine by applying Lemma 26.21.7 to the morphism $X \to \mathrm{Spec}(\mathbf{Z})$. □

On the other hand, the following example shows that we cannot expect the image of an affine to be contained in an affine.

01KX **Example 26.21.17.** Consider the nonaffine scheme $U = \mathrm{Spec}(k[x, y]) \setminus \{(x, y)\}$ of Example 26.9.3. On the other hand, consider the scheme

$$\mathbf{GL}_{2,k} = \mathrm{Spec}(k[a, b, c, d, 1/ad - bc]).$$

There is a morphism $\mathbf{GL}_{2,k} \to U$ corresponding to the ring map $x \mapsto a$, $y \mapsto b$. It is easy to see that this is a surjective morphism, and hence the image is not contained in any affine open of U. In fact, the affine scheme $\mathbf{GL}_{2,k}$ also surjects onto $\mathbf{P}^1_k$, and $\mathbf{P}^1_k$ does not even have an immersion into *any* affine scheme.

0816 **Remark 26.21.18.** The category of quasi-compact and quasi-separated schemes $\mathcal{C}$ has the following properties. If $X, Y \in \mathrm{Ob}(\mathcal{C})$, then any morphism of schemes $f : X \to Y$ is quasi-compact and quasi-separated by Lemmas 26.21.14 and 26.21.13 with $Z = \mathrm{Spec}(\mathbf{Z})$. Moreover, if $X \to Y$ and $Z \to Y$ are morphisms in $\mathcal{C}$, then $X \times_Y Z$ is an object of $\mathcal{C}$ too. Namely, the projection $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of the morphism $Z \to Y$, see Lemmas 26.21.12 and 26.19.3. Hence the composition $X \times_Y Z \to Z \to \mathrm{Spec}(\mathbf{Z})$ is quasi-compact and quasi-separated, see Lemmas 26.21.12 and 26.19.4.

26.22. Valuative criterion of separatedness

01KY

01KZ **Lemma 26.22.1.** *Let $f : X \to S$ be a morphism of schemes. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 26.20.3 be given. Suppose there are two morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. By Lemma 26.21.5 this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. □

01L0 **Lemma 26.22.2** (Valuative criterion separatedness)**.** *Let $f : X \to S$ be a morphism. Assume* [**DG67**, II Proposition 7.2.3]

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. By assumption (1), Proposition 26.20.6, and Lemmas 26.21.2 and 26.10.4 we see that it suffices to prove the morphism $\Delta_{X/S} : X \to X \times_S X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \times_S X \end{array}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over S. By (2) we see that $a = b$. Hence using a as the dotted arrow works. □

26.23. Monomorphisms

01L1

01L2 **Definition 26.23.1.** A morphism of schemes is called a *monomorphism* if it is a monomorphism in the category of schemes, see Categories, Definition 4.13.1.

01L3 **Lemma 26.23.2.** *Let $j : X \to Y$ be a morphism of schemes. Then j is a monomorphism if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. This is true in any category with fibre products. □

01L4 **Lemma 26.23.3.** *A monomorphism of schemes is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 26.23.2 above. □

01L5 **Lemma 26.23.4.** *A composition of monomorphisms is a monomorphism.*

Proof. True in any category. □

02YC **Lemma 26.23.5.** *The base change of a monomorphism is a monomorphism.*

Proof. True in any category with fibre products. □

0DVA **Lemma 26.23.6.** *Let $j : X \to Y$ be a morphism of schemes. If j is injective on points, then j is separated.*

Proof. Let z be a point of $X \times_Y X$. Then $x = \mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are the same because j maps these points to the same point y of Y. Then we can choose an affine open neighbourhood $V \subset Y$ of y and an affine open neighbourhood $U \subset X$ of x with $j(U) \subset V$. Then $z \in U \times_V U \subset X \times_Y X$. Hence $X \times_Y X$ is the union of the affine opens $U \times_V U$. Since $\Delta_{X/Y}^{-1}(U \times_V U) = U$ and since $U \to U \times_V U$ is a closed immersion, we conclude that $\Delta_{X/Y}$ is a closed immersion (see argument in the proof of Lemma 26.21.2). □

01L6 **Lemma 26.23.7.** *Let $j : X \to Y$ be a morphism of schemes. If*

(1) *j is injective on points, and*
(2) *for any $x \in X$ the ring map $j_x^\sharp : \mathcal{O}_{Y,j(x)} \to \mathcal{O}_{X,x}$ is surjective,*

then j is a monomorphism.

Proof. Let $a, b : Z \to X$ be two morphisms of schemes such that $j \circ a = j \circ b$. Then (1) implies $a = b$ as underlying maps of topological spaces. For any $z \in Z$ we have $a_z^\sharp \circ j_{a(z)}^\sharp = b_z^\sharp \circ j_{b(z)}^\sharp$ as maps $\mathcal{O}_{Y,j(a(z))} \to \mathcal{O}_{Z,z}$. The surjectivity of the maps $j_x^\sharp$ forces $a_z^\sharp = b_z^\sharp$, $\forall z \in Z$. This implies that $a^\sharp = b^\sharp$. Hence we conclude $a = b$ as morphisms of schemes as desired. □

01L7 **Lemma 26.23.8.** *An immersion of schemes is a monomorphism. In particular, any immersion is separated.*

Proof. We can see this by checking that the criterion of Lemma 26.23.7 applies. More elegantly perhaps, we can use that Lemmas 26.3.5 and 26.4.6 imply that open and closed immersions are monomorphisms and hence any immersion (which is a composition of such) is a monomorphism. □

01L8 **Lemma 26.23.9.** *Let $f : X \to S$ be a separated morphism. Any locally closed subscheme $Z \subset X$ is separated over S.*

Proof. Follows from Lemma 26.23.8 and the fact that a composition of separated morphisms is separated (Lemma 26.21.12). □

01L9 **Example 26.23.10.** The morphism $\mathrm{Spec}(\mathbf{Q}) \to \mathrm{Spec}(\mathbf{Z})$ is a monomorphism. This is true because $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. More generally, for any scheme S and any point $s \in S$ the canonical morphism

$$\mathrm{Spec}(\mathcal{O}_{S,s}) \longrightarrow S$$

is a monomorphism.

03DP **Lemma 26.23.11.** *Let $k_1, \ldots, k_n$ be fields. For any monomorphism of schemes $X \to \mathrm{Spec}(k_1 \times \ldots \times k_n)$ there exists a subset $I \subset \{1, \ldots, n\}$ such that $X \cong \mathrm{Spec}(\prod_{i \in I} k_i)$ as schemes over $\mathrm{Spec}(k_1 \times \ldots \times k_n)$. More generally, if $X = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields and $Y \to X$ is a monomorphism, then there exists a subset $J \subset I$ such that $Y = \coprod_{i \in J} \mathrm{Spec}(k_i)$.*

Proof. First reduce to the case $n = 1$ (or $\#I = 1$) by taking the inverse images of the open and closed subschemes $\mathrm{Spec}(k_i)$. In this case X has only one point hence is affine. The corresponding algebra problem is this: If $k \to R$ is an algebra map with $R \otimes_k R \cong R$, then $R \cong k$ or $R = 0$. This holds for dimension reasons. See also Algebra, Lemma 10.107.8 □

26.24. Functoriality for quasi-coherent modules

01LA Let X be a scheme. We denote $QCoh(\mathcal{O}_X)$ the category of quasi-coherent $\mathcal{O}_X$-modules as defined in Modules, Definition 17.10.1. We have seen in Section 26.7 that the category $QCoh(\mathcal{O}_X)$ has a lot of good properties when X is affine. Since the property of being quasi-coherent is local on X, these properties are inherited by the category of quasi-coherent sheaves on any scheme X. We enumerate them here.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each affine open $U = \mathrm{Spec}(R)$ is of the form $\widetilde{M}$ for some R-module M.
(2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each of the members of an affine open covering is quasi-coherent.
(3) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(4) Any colimit of quasi-coherent sheaves is quasi-coherent.

01LB (5) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.

(6) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(7) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Modules, Lemma 17.10.4.
(8) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see Modules, Lemma 17.16.6.
(9) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Modules, Lemma 17.21.6.
(10) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Modules, Lemma 17.22.6 and (5) above.

On the other hand, it is in general not the case that the pushforward of a quasi-coherent module is quasi-coherent. Here is a case where this does hold.

01LC **Lemma 26.24.1.** *Let $f : X \to S$ be a morphism of schemes. If f is quasi-compact and quasi-separated then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_S$-modules.*

Proof. The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^{n} U_i$ with each U_i open affine. Because f is quasi-separated we may write $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ for some affine open U_{ijk}, see Lemma 26.21.6. Denote $f_i : U_i \to S$ and $f_{ijk} : U_{ijk} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus\nolimits_{i,j,k} \mathcal{F}(f^{-1}V \cap U_{ijk})\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})(V)\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \end{aligned}$$

In other words there is an exact sequence of sheaves

$$0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ijk,*}\mathcal{F}_{ijk}$$

where $\mathcal{F}_i, \mathcal{F}_{ijk}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{U_i}$-module and $\mathcal{F}_{ijk}$ is a quasi-coherent $\mathcal{O}_{U_{ijk}}$-module. Hence by Lemma 26.7.3 we see that the second and third term of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. □

Using this we can characterize (closed) immersions of schemes as follows.

01LD **Lemma 26.24.2.** *Let $f : X \to Y$ be a morphism of schemes. Suppose that*

(1) *f induces a homeomorphism of X with a closed subset of Y, and*
(2) *$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*

Then f is a closed immersion of schemes.

Proof. Assume (1) and (2). By (1) the morphism f is quasi-compact (see Topology, Lemma 5.12.3). Conditions (1) and (2) imply conditions (1) and (2) of Lemma 26.23.7. Hence $f : X \to Y$ is a monomorphism. In particular, f is separated, see Lemma 26.23.3. Hence Lemma 26.24.1 above applies and we conclude that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Therefore the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent by Lemma 26.7.8. Since a quasi-coherent sheaf is locally generated by sections (see Modules, Definition 17.10.1) this implies that f is a closed immersion, see Definition 26.4.1. □

We can use this lemma to prove the following lemma.

02V0 **Lemma 26.24.3.** *A composition of immersions of schemes is an immersion, a composition of closed immersions of schemes is a closed immersion, and a composition of open immersions of schemes is an open immersion.*

Proof. This is clear for the case of open immersions since an open subspace of an open subspace is also an open subspace.

Suppose $a : Z \to Y$ and $b : Y \to X$ are closed immersions of schemes. We will verify that $c = b \circ a$ is also a closed immersion. The assumption implies that a and b are homeomorphisms onto closed subsets, and hence also $c = b \circ a$ is a homeomorphism onto a closed subset. Moreover, the map $\mathcal{O}_X \to c_*\mathcal{O}_Z$ is surjective since it factors as the composition of the surjective maps $\mathcal{O}_X \to b_*\mathcal{O}_Y$ and $b_*\mathcal{O}_Y \to b_*a_*\mathcal{O}_Z$ (surjective as b_* is exact, see Modules, Lemma 17.6.1). Hence by Lemma 26.24.2 above c is a closed immersion.

Finally, we come to the case of immersions. Suppose $a : Z \to Y$ and $b : Y \to X$ are immersions of schemes. This means there exist open subschemes $V \subset Y$ and $U \subset X$ such that $a(Z) \subset V$, $b(Y) \subset U$ and $a : Z \to V$ and $b : Y \to U$ are closed immersions. Since the topology on Y is induced from the topology on U we can find an open $U' \subset U$ such that $V = b^{-1}(U')$. Then we see that $Z \to V = b^{-1}(U') \to U'$ is a composition of closed immersions and hence a closed immersion. This proves that $Z \to X$ is an immersion and we win. □

26.25. Other chapters

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(7) Sites and Sheaves
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Algebraic Spaces

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CHAPTER 27

Constructions of Schemes

01LE

27.1. Introduction

01LF In this chapter we introduce ways of constructing schemes out of others. A basic reference is [**DG67**].

27.2. Relative glueing

01LG The following lemma is relevant in case we are trying to construct a scheme X over S, and we already know how to construct the restriction of X to the affine opens of S. The actual result is completely general and works in the setting of (locally) ringed spaces, although our proof is written in the language of schemes.

01LH **Lemma 27.2.1.** *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

(1) *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
(2) *For $U, V \in \mathcal{B}$ with $V \subset U$ a morphism $\rho_V^U : X_V \to X_U$ over U.*

Assume that

(a) *each ρ_V^U induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho_W^U = \rho_V^U \circ \rho_W^V$.*

Then there exists a morphism $f : X \to S$ of schemes and isomorphisms $i_U : f^{-1}(U) \to X_U$ over $U \in \mathcal{B}$ such that for $V, U \in \mathcal{B}$ with $V \subset U$ the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ_V^U. Moreover X is unique up to unique isomorphism over S.

Proof. To prove this we will use Schemes, Lemma 26.15.4. First we define a contravariant functor F from the category of schemes to the category of sets. Namely, for a scheme T we set

$$F(T) = \left\{ \begin{array}{c} (g, \{h_U\}_{U \in \mathcal{B}}),\ g : T \to S,\ h_U : g^{-1}(U) \to X_U, \\ f_U \circ h_U = g|_{g^{-1}(U)},\ h_U|_{g^{-1}(V)} = \rho_V^U \circ h_V\ \forall\ V, U \in \mathcal{B}, V \subset U \end{array} \right\}.$$

The restriction mapping $F(T) \to F(T')$ given a morphism $T' \to T$ is just gotten by composition. For any $W \in \mathcal{B}$ we consider the subfunctor $F_W \subset F$ consisting of those systems $(g, \{h_U\})$ such that $g(T) \subset W$.

First we show F satisfies the sheaf property for the Zariski topology. Suppose that T is a scheme, $T = \bigcup V_i$ is an open covering, and $\xi_i \in F(V_i)$ is an element such that $\xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j}$. Say $\xi_i = (g_i, \{h_{i,U}\})$. Then we immediately see that the morphisms g_i glue to a unique global morphism $g : T \to S$. Moreover, it is clear that $g^{-1}(U) = \bigcup g_i^{-1}(U)$. Hence the morphisms $h_{i,U} : g_i^{-1}(U) \to X_U$ glue to a

unique morphism $h_U : g^{-1}(U) \to X_U$. It is easy to verify that the system $(g, \{h_U\})$ is an element of $F(T)$. Hence F satisfies the sheaf property for the Zariski topology.

Next we verify that each F_W, $W \in \mathcal{B}$ is representable. Namely, we claim that the transformation of functors

$$F_W \longrightarrow \mathrm{Mor}(-, X_W),\ (g, \{h_U\}) \longmapsto h_W$$

is an isomorphism. To see this suppose that T is a scheme and $\alpha : T \to X_W$ is a morphism. Set $g = f_W \circ \alpha$. For any $U \in \mathcal{B}$ such that $U \subset W$ we can define $h_U : g^{-1}(U) \to X_U$ be the composition $(\rho_U^W)^{-1} \circ \alpha|_{g^{-1}(U)}$. This works because the image $\alpha(g^{-1}(U))$ is contained in $f_W^{-1}(U)$ and condition (a) of the lemma. It is clear that $f_U \circ h_U = g|_{g^{-1}(U)}$ for such a U. Moreover, if also $V \in \mathcal{B}$ and $V \subset U \subset W$, then $\rho_V^U \circ h_V = h_U|_{g^{-1}(V)}$ by property (b) of the lemma. We still have to define h_U for an arbitrary element $U \in \mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on S we can find an open covering $U \cap W = \bigcup U_i$ with $U_i \in \mathcal{B}$. Since g maps into W we have $g^{-1}(U) = g^{-1}(U \cap W) = \bigcup g^{-1}(U_i)$. Consider the morphisms $h_i = \rho_{U_i}^U \circ h_{U_i} : g^{-1}(U_i) \to X_U$. It is a simple matter to use condition (b) of the lemma to prove that $h_i|_{g^{-1}(U_i) \cap g^{-1}(U_j)} = h_j|_{g^{-1}(U_i) \cap g^{-1}(U_j)}$. Hence these morphisms glue to give the desired morphism $h_U : g^{-1}(U) \to X_U$. We omit the (easy) verification that the system $(g, \{h_U\})$ is an element of $F_W(T)$ which maps to α under the displayed arrow above.

Next, we verify each $F_W \subset F$ is representable by open immersions. This is clear from the definitions.

Finally we have to verify the collection $(F_W)_{W \in \mathcal{B}}$ covers F. This is clear by construction and the fact that $\mathcal{B}$ is a basis for the topology of S.

Let X be a scheme representing the functor F. Let $(f, \{i_U\}) \in F(X)$ be a "universal family". Since each F_W is representable by X_W (via the morphism of functors displayed above) we see that $i_W : f^{-1}(W) \to X_W$ is an isomorphism as desired. The lemma is proved. □

01LI **Lemma 27.2.2.** *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

(1) *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
(2) *For every $U \in \mathcal{B}$ a quasi-coherent sheaf $\mathcal{F}_U$ over X_U.*
(3) *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho_V^U : X_V \to X_U$.*
(4) *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\theta_V^U : (\rho_V^U)^* \mathcal{F}_U \to \mathcal{F}_V$.*

Assume that

(a) *each ρ_V^U induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *each θ_V^U is an isomorphism,*
(c) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho_W^U = \rho_V^U \circ \rho_W^V$,*
(d) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\theta_W^U = \theta_W^V \circ (\rho_W^V)^* \theta_V^U$.*

Then there exists a morphism of schemes $f : X \to S$ together with a quasi-coherent sheaf $\mathcal{F}$ on X and isomorphisms $i_U : f^{-1}(U) \to X_U$ and $\theta_U : i_U^* \mathcal{F}_U \to \mathcal{F}|_{f^{-1}(U)}$ over $U \in \mathcal{B}$ such that for $V, U \in \mathcal{B}$ with $V \subset U$ the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{\text{inclusion}} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ_V^U, and the composition

01LJ (27.2.2.1) $$(\rho_V^U)^*\mathcal{F}_U = (i_V^{-1})^*((i_U^*\mathcal{F}_U)|_{f^{-1}(V)}) \xrightarrow{\theta_U|_{f^{-1}(V)}} (i_V^{-1})^*(\mathcal{F}|_{f^{-1}(V)}) \xrightarrow{\theta_V^{-1}} \mathcal{F}_V$$

is equal to θ_V^U. Moreover $(X, \mathcal{F})$ is unique up to unique isomorphism over S.

Proof. By Lemma 27.2.1 we get the scheme X over S and the isomorphisms i_U. Set $\mathcal{F}'_U = i_U^*\mathcal{F}_U$ for $U \in \mathcal{B}$. This is a quasi-coherent $\mathcal{O}_{f^{-1}(U)}$-module. The maps

$$\mathcal{F}'_U|_{f^{-1}(V)} = i_U^*\mathcal{F}_U|_{f^{-1}(V)} = i_V^*(\rho_V^U)^*\mathcal{F}_U \xrightarrow{i_V^*\theta_V^U} i_V^*\mathcal{F}_V = \mathcal{F}'_V$$

define isomorphisms $(\theta')_V^U : \mathcal{F}'_U|_{f^{-1}(V)} \to \mathcal{F}'_V$ whenever $V \subset U$ are elements of $\mathcal{B}$. Condition (d) says exactly that this is compatible in case we have a triple of elements $W \subset V \subset U$ of $\mathcal{B}$. This allows us to get well defined isomorphisms

$$\varphi_{12} : \mathcal{F}'_{U_1}|_{f^{-1}(U_1 \cap U_2)} \longrightarrow \mathcal{F}'_{U_2}|_{f^{-1}(U_1 \cap U_2)}$$

whenever $U_1, U_2 \in \mathcal{B}$ by covering the intersection $U_1 \cap U_2 = \bigcup V_j$ by elements V_j of $\mathcal{B}$ and taking

$$\varphi_{12}|_{V_j} = \left((\theta')_{V_j}^{U_2}\right)^{-1} \circ (\theta')_{V_j}^{U_1}.$$

We omit the verification that these maps do indeed glue to a φ_{12} and we omit the verification of the cocycle condition of a glueing datum for sheaves (as in Sheaves, Section 6.33). By Sheaves, Lemma 6.33.2 we get our $\mathcal{F}$ on X. We omit the verification of (27.2.2.1). □

01LK **Remark 27.2.3.** There is a functoriality property for the constructions explained in Lemmas 27.2.1 and 27.2.2. Namely, suppose given two collections of data $(f_U : X_U \to U, \rho_V^U)$ and $(g_U : Y_U \to U, \sigma_V^U)$ as in Lemma 27.2.1. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U. Functoriality means that this gives rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, where $f : X \to S$ is obtained from the datum $(f_U : X_U \to U, \rho_V^U)$ and $g : Y \to S$ is obtained from the datum $(g_U : Y_U \to U, \sigma_V^U)$.

Similarly, suppose given two collections of data $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$ as in Lemma 27.2.2. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U, and a morphism $\tau_U : h_U^*\mathcal{G}_U \to \mathcal{F}_U$ compatible with the maps θ_V^U and η_V^U. Functoriality means that these give rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, and a morphism $h^*\mathcal{G} \to \mathcal{F}$ restricting back to the maps h_U where $(f : X \to S, \mathcal{F})$ is obtained from the datum $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and where $(g : Y \to S, \mathcal{G})$ is obtained from the datum $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$.

We omit the verifications and we omit a suitable formulation of "equivalence of categories" between relative glueing data and relative objects.

27.3. Relative spectrum via glueing

01LL

01LM **Situation 27.3.1.** Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra. This means that $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the spectra $\mathrm{Spec}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 27.2.1.

01LN **Lemma 27.3.2.** *In Situation 27.3.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of rings $A' \to A$ induces a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and the diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

is cartesian.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 26.7.3 for example). The result follows from the description of the fibre product of affine schemes in Schemes, Lemma 26.6.7. □

In particular the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ of the lemma is an open immersion.

01LO **Lemma 27.3.3.** *In Situation 27.3.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and $\mathrm{Spec}(A') \to \mathrm{Spec}(A'')$ of Lemma 27.3.2 gives the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A'')$ of Lemma 27.3.2.*

Proof. This follows as the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$ (because $\mathcal{A}$ is a sheaf). □

01LP **Lemma 27.3.4.** *In Situation 27.3.1. There exists a morphism of schemes*

$$\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \mathrm{Spec}(\mathcal{A}(U))$ over U, and*
2. *for $U \subset U' \subset S$ affine open the composition*

$$\mathrm{Spec}(\mathcal{A}(U)) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} \mathrm{Spec}(\mathcal{A}(U'))$$

is the open immersion of Lemma 27.3.2 above.

Moreover, $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is unique up to unique isomorphism over S.

Proof. Follows immediately from Lemmas 27.2.1, 27.3.2, and 27.3.3. Uniqueness is stated in the last sentence of Lemma 27.2.1. □

27.4. Relative spectrum as a functor

01LQ We place ourselves in Situation 27.3.1, i.e., S is a scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras.

For any $f : T \to S$ the pullback $f^*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. We are going to consider pairs $(f : T \to S, \varphi)$ where f is a morphism of schemes and $\varphi : f^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of $\mathcal{O}_T$-algebras. Note that this is the same as giving a $f^{-1}\mathcal{O}_S$-algebra homomorphism $\varphi : f^{-1}\mathcal{A} \to \mathcal{O}_T$, see Sheaves, Lemma 6.20.2. This is also the same as giving an $\mathcal{O}_S$-algebra map $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, see Sheaves, Lemma 6.24.7. We will use all three ways of thinking about φ, without further mention.

Given such a pair $(f : T \to S, \varphi)$ and a morphism $a : T' \to T$ we get a second pair $(f' = f \circ a, \varphi' = a^*\varphi)$ which we call the pullback of (f, φ). One way to describe $\varphi' = a^*\varphi$ is as the composition $\mathcal{A} \to f_*\mathcal{O}_T \to f'_*\mathcal{O}_{T'}$ where the second map is $f_*a^\sharp$ with $a^\sharp : \mathcal{O}_T \to a_*\mathcal{O}_{T'}$. In this way we have defined a functor

01LR (27.4.0.1)
$$\begin{aligned} F : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto F(T) = \{\text{pairs } (f, \varphi) \text{ as above}\} \end{aligned}$$

01LS **Lemma 27.4.1.** *In Situation 27.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F' \cong h_{S'} \times_{h_S} F$$

of functors.

Proof. A pair $(f' : T \to S', \varphi' : (f')^*\mathcal{A}' \to \mathcal{O}_T)$ is the same as a pair $(f, \varphi : f^*\mathcal{A} \to \mathcal{O}_T)$ together with a factorization of f as $f = g \circ f'$. Namely with this notation we have $(f')^*\mathcal{A}' = (f')^*g^*\mathcal{A} = f^*\mathcal{A}$. Hence the lemma. □

01LT **Lemma 27.4.2.** *In Situation 27.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme* $\mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is an R-algebra and $\mathcal{A} = \widetilde{A}$. The ring map $R \to A$ gives rise to a canonical map

$$f_{univ} : \mathrm{Spec}(A) \longrightarrow S = \mathrm{Spec}(R).$$

We have $f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A}$ by Schemes, Lemma 26.7.3. Hence there is a canonical map

$$\varphi_{univ} : f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A} \longrightarrow \widetilde{A} = \mathcal{O}_{\mathrm{Spec}(A)}$$

coming from the A-module map $A \otimes_R A \to A$, $a \otimes a' \mapsto aa'$. We claim that the pair $(f_{univ}, \varphi_{univ})$ represents F in this case. In other words we claim that for any scheme T the map

$$\mathrm{Mor}(T, \mathrm{Spec}(A)) \longrightarrow \{\text{pairs } (f, \varphi)\}, \quad a \longmapsto (f_{univ} \circ a, a^*\varphi_{univ})$$

is bijective.

Let us construct the inverse map. For any pair $(f : T \to S, \varphi)$ we get the induced ring map

$$A = \Gamma(S, \mathcal{A}) \xrightarrow{f^*} \Gamma(T, f^*\mathcal{A}) \xrightarrow{\varphi} \Gamma(T, \mathcal{O}_T)$$

This induces a morphism of schemes $T \to \mathrm{Spec}(A)$ by Schemes, Lemma 26.6.4.

The verification that this map is inverse to the map displayed above is omitted. □

01LU **Lemma 27.4.3.** *In Situation 27.3.1. The functor F is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 26.15.4.

First we check that F satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i\in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i\cap U_j} = (f_j, \varphi_j)|_{U_i\cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{U_i} = f_i$, see Schemes, Section 26.14. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$ and by assumption the morphisms φ_i agree on $U_i \cap U_j$. Hence by Sheaves, Section 6.33 these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i\in I} U_i$ be an affine open covering. Let $F_i \subset F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each F_i is representable. This is the case because F_i is identified with the functor associated to U_i equipped with the quasi-coherent $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$, by Lemma 27.4.1. Thus the result follows from Lemma 27.4.2.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_i(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_i)_{i\in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i\in I} U_i$ is an open covering of S we see that $T = \bigcup_{i\in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_i(V_i)$. This finishes the proof of the lemma. □

01LV **Lemma 27.4.4.** *In Situation 27.3.1. The scheme $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ constructed in Lemma 27.3.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let $X \to S$ be the scheme representing the functor F. Consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R} = \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$. By construction of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ we have isomorphisms $\mathcal{A}(U) \to \mathcal{R}(U)$ for every affine open $U \subset S$; this follows from Lemma 27.3.4 part (1). For $U \subset U' \subset S$ open these isomorphisms are compatible with the restriction mappings; this follows from Lemma 27.3.4 part (2). Hence by Sheaves, Lemma 6.30.13 these isomorphisms result from an isomorphism of $\mathcal{O}_S$-algebras $\varphi : \mathcal{A} \to \mathcal{R}$. Hence this gives an element $(\pi, \varphi) \in F(\underline{\mathrm{Spec}}_S(\mathcal{A}))$. Since X represents the functor F we get a corresponding morphism of schemes $can : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to X$ over S.

Let $U \subset S$ be any affine open. Let $F_U \subset F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subset U$. Clearly the base change X_U represents F_U. Moreover, F_U is represented by $\mathrm{Spec}(\mathcal{A}(U)) = \pi^{-1}(U)$ according to Lemma 27.4.2. In other words $X_U \cong \pi^{-1}(U)$. We omit the verification that this identification is brought about by the base change of the morphism can to U. □

01LW **Definition 27.4.5.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The *relative spectrum of $\mathcal{A}$ over S*, or simply the *spectrum of $\mathcal{A}$ over S* is the scheme constructed in Lemma 27.3.4 which represents the functor F (27.4.0.1),

see Lemma 27.4.4. We denote it $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$. The "universal family" is a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

The following lemma says among other things that forming the relative spectrum commutes with base change.

01LX **Lemma 27.4.6.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ be the relative spectrum of $\mathcal{A}$ over S.*

(1) *For every affine open $U \subset S$ the inverse image $\pi^{-1}(U)$ is affine.*
(2) *For every morphism $g : S' \to S$ we have $S' \times_S \underline{\mathrm{Spec}}_S(\mathcal{A}) = \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$.*
(3) *The universal map*

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. Part (1) comes from the description of the relative spectrum by glueing, see Lemma 27.3.4. Part (2) follows immediately from Lemma 27.4.1. Part (3) follows because it is local on S and it is clear in case S is affine by Lemma 27.4.2 for example. □

01LY **Lemma 27.4.7.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. By Schemes, Lemma 26.24.1 the sheaf $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There is a canonical morphism*

$$can : X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$$

of schemes over S. For any affine open $U \subset S$ the restriction $can|_{f^{-1}(U)}$ is identified with the canonical morphism

$$f^{-1}(U) \longrightarrow \mathrm{Spec}(\Gamma(f^{-1}(U), \mathcal{O}_X))$$

coming from Schemes, Lemma 26.6.4.

Proof. The morphism comes, via the definition of $\underline{\mathrm{Spec}}$ as the scheme representing the functor F, from the canonical map $\varphi : f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ (which by adjointness of push and pull corresponds to id : $f_*\mathcal{O}_X \to f_*\mathcal{O}_X$). The statement on the restriction to $f^{-1}(U)$ follows from the description of the relative spectrum over affines, see Lemma 27.4.2. □

27.5. Affine n-space

01LZ As an application of the relative spectrum we define affine n-space over a base scheme S as follows. For any integer $n \geq 0$ we can consider the quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n]$. It is quasi-coherent because as a sheaf of $\mathcal{O}_S$-modules it is just the direct sum of copies of $\mathcal{O}_S$ indexed by multi-indices.

01M0 **Definition 27.5.1.** Let S be a scheme and $n \geq 0$. The scheme

$$\mathbf{A}^n_S = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[T_1, \ldots, T_n])$$

over S is called *affine n-space over S*. If $S = \mathrm{Spec}(R)$ is affine then we also call this *affine n-space over R* and we denote it $\mathbf{A}^n_R$.

Note that $\mathbf{A}^n_R = \mathrm{Spec}(R[T_1, \ldots, T_n])$. For any morphism $g : S' \to S$ of schemes we have $g^*\mathcal{O}_S[T_1, \ldots, T_n] = \mathcal{O}_{S'}[T_1, \ldots, T_n]$ and hence $\mathbf{A}^n_{S'} = S' \times_S \mathbf{A}^n_S$ is the base change. Therefore an alternative definition of affine n-space is the formula

$$\mathbf{A}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}.$$

Also, a morphism from an S-scheme $f : X \to S$ to $\mathbf{A}^n_S$ is given by a homomorphism of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n] \to f_*\mathcal{O}_X$. This is clearly the same thing as giving the images of the T_i. In other words, a morphism from X to $\mathbf{A}^n_S$ over S is the same as giving n elements $h_1, \ldots, h_n \in \Gamma(X, \mathcal{O}_X)$.

27.6. Vector bundles

01M1 Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.21.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Hence it makes sense to apply the relative spectrum.

01M2 **Definition 27.6.1.** Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[1]. The *vector bundle associated to* $\mathcal{E}$ is

$$\mathbf{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E})).$$

The vector bundle associated to $\mathcal{E}$ comes with a bit of extra structure. Namely, we have a grading

$$\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})} = \bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}).$$

which turns $\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})}$ into a graded $\mathcal{O}_S$-algebra. Conversely, we can recover $\mathcal{E}$ from the degree 1 part of this. Thus we define an abstract vector bundle as follows.

062M **Definition 27.6.2.** Let S be a scheme. A *vector bundle* $\pi : V \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_V = \bigoplus_{n \geq 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_S$ and such that the maps

$$\mathrm{Sym}^n(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

are isomorphisms for all $n \geq 0$. A *morphism of vector bundles over* S is a morphism $f : V \to V'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{V'} \longrightarrow \pi_*\mathcal{O}_V$$

is compatible with the given gradings.

An example of a vector bundle over S is affine n-space $\mathbf{A}^n_S$ over S, see Definition 27.5.1. This is true because $\mathcal{O}_S[T_1, \ldots, T_n] = \mathrm{Sym}(\mathcal{O}_S^{\oplus n})$.

062N **Lemma 27.6.3.** *The category of vector bundles over a scheme S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-modules.*

Proof. Omitted. Hint: In one direction one uses the functor $\underline{\mathrm{Spec}}_S(\mathrm{Sym}^*_{\mathcal{O}_S}(-))$ and in the other the functor $(\pi : V \to S) \rightsquigarrow (\pi_*\mathcal{O}_V)_1$ where the subscript indicates we take the degree 1 part. □

[1] The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 1.7.8].

27.7. Cones

062P In algebraic geometry cones correspond to graded algebras. By our conventions a graded ring or algebra A comes with a grading $A = \bigoplus_{d \geq 0} A_d$ by the nonnegative integers, see Algebra, Section 10.56.

062Q **Definition 27.7.1.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Assume that $\mathcal{O}_S \to \mathcal{A}_0$ is an isomorphism[2]. The *cone associated to* $\mathcal{A}$ or the *affine cone associated to* $\mathcal{A}$ is

$$C(\mathcal{A}) = \underline{\mathrm{Spec}}_S(\mathcal{A}).$$

The cone associated to a graded sheaf of $\mathcal{O}_S$-algebras comes with a bit of extra structure. Namely, we obtain a grading

$$\pi_*\mathcal{O}_{C(\mathcal{A})} = \bigoplus\nolimits_{n \geq 0} \mathcal{A}_n$$

Thus we can define an abstract cone as follows.

062R **Definition 27.7.2.** Let S be a scheme. A *cone* $\pi : C \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_C$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_C = \bigoplus_{n \geq 0} \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{O}_S$. A *morphism of cones* from $\pi : C \to S$ to $\pi' : C' \to S$ is a morphism $f : C \to C'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C$$

is compatible with the given gradings.

Any vector bundle is an example of a cone. In fact the category of vector bundles over S is a full subcategory of the category of cones over S.

27.8. Proj of a graded ring

01M3 In this section we construct Proj of a graded ring following [**DG67**, II, Section 2].

Let S be a graded ring. Consider the topological space $\mathrm{Proj}(S)$ associated to S, see Algebra, Section 10.57. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Proj}(S)}$ such that the resulting pair $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ will be a scheme.

Recall that $\mathrm{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call *standard opens*, see Algebra, Section 10.57. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition, the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

01M4 **Lemma 27.8.1.** *Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.*

(1) *If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then*
   (a) *f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,*
   (c) *there is a canonical S-algebra map $S_f \to S_g$,*
   (d) *there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,*
   (e) *the map $S_{(f)} \to S_{(g)}$ induces an isomorphism*

$$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$

[2]Often one imposes the assumption that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{O}_S$. We do not assume this in order to be consistent with [**DG67**, II, (8.3.1)].

(f) *these maps induce a commutative diagram of topological spaces*

$$\begin{array}{ccccc} D_+(g) & \longleftarrow & \{\mathbf{Z}\text{-graded primes of } S_g\} & \longrightarrow & \mathrm{Spec}(S_{(g)}) \\ \downarrow & & \downarrow & & \downarrow \\ D_+(f) & \longleftarrow & \{\mathbf{Z}\text{-graded primes of } S_f\} & \longrightarrow & \mathrm{Spec}(S_{(f)}) \end{array}$$

where the horizontal maps are homeomorphisms and the vertical maps are open immersions,

(g) *there are compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and*

(h) *the map $M_{(f)} \to M_{(g)}$ induces an isomorphism*

$$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$

(2) *Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.*

(3) *Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.*

Proof. Recall that $D_+(g) = \mathrm{Spec}(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma 10.57.3. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.17.2. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma 10.57.3. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $\mathrm{Spec}(S_{(f)})$, see Algebra, Lemma 10.17.8. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.17.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

01M5 **Definition 27.8.2.** Let S be a graded ring. Suppose that $D_+(f) \subset \text{Proj}(S)$ is a standard open. A *standard open covering* of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \text{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 27.8.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define

$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 27.8.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = \text{colim}_{f \in S_d, d > 0, f \notin \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \notin \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section 10.57 we defined $M_{(\mathfrak{p})}$ as the module whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \notin \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \notin \mathfrak{p}$. Hence $x/f = f_0 x / f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(f g_i)} \to \bigoplus M_{(f g_i g_j)}.$$

By Lemma 27.8.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(f g_i)}$, $M_{(f g_i g_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma 10.24.1 to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note

that by our computation of stalks above and Algebra, Lemma 10.57.5 the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

01M6 **Definition 27.8.3.** Let S be a graded ring.

(1) The *structure sheaf* $\mathcal{O}_{Proj(S)}$ *of the homogeneous spectrum of* S is the unique sheaf of rings $\mathcal{O}_{\mathrm{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is called the *homogeneous spectrum* of S and denoted $\mathrm{Proj}(S)$.
(3) The sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

01M7 **Lemma 27.8.4.** *Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Proj(S)}$-modules associated to M.*

(1) *For every $f \in S$ homogeneous of positive degree we have*
$$\Gamma(D_+(f), \mathcal{O}_{Proj(S)}) = S_{(f)}.$$
(2) *For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.*
(3) *Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{Proj(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 27.8.1.*
(4) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{O}_{Proj(S),x} = S_{(\mathfrak{p})}$.*
(5) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $(\widetilde{M})_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.*

01M8 (6) *There is a canonical ring map $S_0 \longrightarrow \Gamma(Proj(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(Proj(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens and stalks above.*

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{Proj(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak{p})}$ is exact (see Algebra, Lemma 10.57.5) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. □

01M9 **Remark 27.8.5.** The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

01MA **Lemma 27.8.6.** *Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset \mathrm{Spec}(S_{(f)})$ is a standard open. Then there exists an $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma 10.57.3. In fact we can take h such that $g = h/f^n$ for some n.*

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n+1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset \mathrm{Spec}(S_{(f)})$. □

01MB **Lemma 27.8.7.** *Let S be a graded ring. The locally ringed space $Proj(S)$ is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Proj(S)}$-modules.*

Proof. Consider a standard open $D_+(f) \subset \mathrm{Proj}(S)$. By Lemmas 27.8.1 and 27.8.4 we have $\Gamma(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to \mathrm{Spec}(S_{(f)})$. For any standard open $D(g) \subset \mathrm{Spec}(S_{(f)})$ we may pick an $h \in S_+$ as in Lemma 27.8.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 27.8.4 and 27.8.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\mathrm{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{\mathrm{Spec}(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\mathrm{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{\mathrm{Spec}(S_{(f)})}$ as defined in Schemes, Section 26.5. We conclude that $D_+(f)$ is an affine scheme isomorphic to $\mathrm{Spec}(S_{(f)})$ via φ and hence that $\mathrm{Proj}(S)$ is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^* \left(\widetilde{M_{(f)}} \right)$$

which shows that $\widetilde{M}$ is quasi-coherent. □

01MC **Lemma 27.8.8.** *Let S be a graded ring. The scheme $Proj(S)$ is separated.*

Proof. We have to show that the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma 26.21.7. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n \deg(f) + m \deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n' \deg(g)$, and $m = m' \deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m' \deg(g)}/g^{m' \deg(f)}$. So s is indeed in the image of the displayed arrow. □

01MD **Lemma 27.8.9.** *Let S be a graded ring. The scheme $Proj(S)$ is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$. In this case $Proj(S) = D_+(f_1) \cup \ldots \cup D_+(f_n)$.*

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\mathrm{Proj}(S)$ by Algebra, Lemma 10.57.3. Conversely, if $\mathrm{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. □

01ME **Lemma 27.8.10.** *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ has a canonical morphism towards the affine scheme $\mathrm{Spec}(S_0)$, agreeing with the map on topological spaces coming from Algebra, Definition 10.57.1.*

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma 26.6.4. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 27.8.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. □

01MF **Lemma 27.8.11.** *Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition 26.20.3.*

Proof. The uniqueness part follows from the fact that $\mathrm{Proj}(S)$ is separated (Lemma 27.8.8 and Schemes, Lemma 26.22.1). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \mathrm{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Proj}(S) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(S_0) \end{array}$$

as in Schemes, Definition 26.20.3. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition 10.50.13. We may choose an $f \in S_+$ homogeneous such that $\mathrm{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{array}{ccc} K & \xleftarrow{\varphi} & S_{(f)} \\ \uparrow & & \uparrow \\ A & \longleftarrow & S_0 \end{array}$$

After renumbering we may assume that $\varphi(x_i^{\deg(f)}/f^{d_i})$ is nonzero for $i = 1, \ldots, r$ and zero for $i = r+1, \ldots, n$. Since the open sets $D_+(x_i)$ cover $\mathrm{Proj}(S)$ we see that $r \geq 1$. Let $i_0 \in \{1, \ldots, r\}$ be an index minimizing $\gamma_i = (d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ in Γ. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. The ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$ which gives a ring map $S_{(x_0)} \to S_{(fx_0)} \to K$. The algebra $S_{(x_0)}$ is generated over S_0 by the elements $x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}$, where $\sum e_i d_i = e_0 d_0$. If $e_i > 0$ for some $i > r$, then $\varphi'(x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}) = 0$. If $e_i = 0$ for $i > r$, then we

have

$$\begin{aligned} d\deg(f)v(\varphi'(x_1^{e_1}\ldots x_r^{e_r}/x_0^{e_0})) &= dv(\varphi'(x_1^{e_1\deg(f)}\ldots x_r^{e_r\deg(f)}/x_0^{e_0\deg(f)})) \\ &= d\sum e_i v(\varphi'(x_i^{\deg(f)}/f^{d_i})) - e_0 v(\varphi'(x_0^{\deg(f)}/f^{d_0})) \\ &= \sum e_i d_i\gamma_i - e_0 d_0\gamma_0 \\ &\geq \sum e_i d_i\gamma_0 - e_0 d_0\gamma_0 = 0 \end{aligned}$$

because γ_0 is minimal among the γ_i. This implies that $S_{(x_0)}$ maps into A via φ'. The corresponding morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(S_{(x_0)}) = D_+(x_0) \subset \mathrm{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. □

We saw in the proof of Lemma 27.8.11 that, under the hypotheses of that lemma, the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition 26.20.6) we see that $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is universally closed in the situation of the lemma. We give several examples showing these results do not hold without some assumption on the graded ring S.

01MG **Example 27.8.12.** Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 1. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. Let K be the fraction field of A. The above gives a morphism $\mathrm{Spec}(K) \to \mathrm{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $\mathrm{Spec}(A)$. The reason is that the image of $\mathrm{Spec}(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

01MH **Example 27.8.13.** Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \mathrm{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\mathrm{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is $D(t)$. Hence the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition 26.20.6).

01MI **Example 27.8.14.** Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ (see Lemma 27.8.10) is an isomorphism.

0G5W **Example 27.8.15.** Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let $U \subset X$ be an open subscheme. Grade $A[T]$ by setting $\deg T = 1$. Define S to be the subring of $A[T]$ generated by A and all fT^i, where $i \geq 0$ and where $f \in A$ is such that $D(f) \subset U$. We claim that S is a graded ring with $S_0 = A$ such that $\mathrm{Proj}(S) \cong U$, and this isomorphism identifies the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ of Lemma 27.8.10 with the inclusion $U \subset X$.

Suppose $\mathfrak{p} \in \mathrm{Proj}(S)$ is such that every $fT \in S_1$ is in $\mathfrak{p}$. Then every generator fT^i with $i \geq 1$ is in $\mathfrak{p}$ because $(fT^i)^2 = (fT)(fT^{2i-1}) \in \mathfrak{p}$ and $\mathfrak{p}$ is radical. But then $\mathfrak{p} \supset S_+$, which is impossible. Consequently $\mathrm{Proj}(S)$ is covered by the standard open affine subsets $\{D_+(fT)\}_{fT \in S_1}$.

Observe that, if $fT \in S_1$, then the inclusion $S \subset A[T]$ induces a graded isomorphism of $S[(fT)^{-1}]$ with $A[T, T^{-1}, f^{-1}]$. Hence the standard open subset $D_+(fT) \cong \mathrm{Spec}(S_{(fT)})$ is isomorphic to $\mathrm{Spec}(A[T, T^{-1}, f^{-1}]_0) = \mathrm{Spec}(A[f^{-1}])$. It is clear that this isomorphism is a restriction of the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$. If in addition $gT \in S_1$, then $S[(fT)^{-1}, (gT)^{-1}] \cong A[T, T^{-1}, f^{-1}, g^{-1}]$ as graded rings, so $D_+(fT) \cap D_+(gT) \cong \mathrm{Spec}(A[f^{-1}, g^{-1}])$. Therefore $\mathrm{Proj}(S)$ is the union of open subschemes $D_+(fT)$ which are isomorphic to the open subschemes $D(f) \subset X$ under the canonical morphism, and these open subschemes intersect in $\mathrm{Proj}(S)$ in the same way they do in X. We conclude that the canonical morphism is an isomorphism of $\mathrm{Proj}(S)$ with the union of all $D(f) \subset U$, which is U.

27.9. Quasi-coherent sheaves on Proj

01MJ Let S be a graded ring. Let M be a graded S-module. We saw in Lemma 27.8.4 how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\mathrm{Proj}(S)$ and a map

0AG1 (27.9.0.1) $$M_0 \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{M})$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section 10.56) is equal to M_n. Hence we can get maps

0AG2 (27.9.0.2) $$M_n \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{M(n)}).$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{Proj}(S)$. We will do this by tensoring with the nth twist of the structure sheaf, see Definition 27.10.1. In order to relate the two notions we will use the following lemma.

01MK **Lemma 27.9.1.** *Let S be a graded ring. Let $(X, \mathcal{O}_X) = (Proj(S), \mathcal{O}_{Proj(S)})$ be the scheme of Lemma 27.8.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{Proj(S)}$-modules*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{array}{ccc} M_0 \otimes_{S_0} N_0 & \longrightarrow & (M \otimes_S N)_0 \\ \downarrow & & \downarrow \\ \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) & \longrightarrow & \Gamma(X, \widetilde{M \otimes_S N}) \end{array}$$

is commutative where the vertical maps are given by (27.9.0.1).

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

see Modules, Section 17.16. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. □

01ML **Remark 27.9.2.** In general the map constructed in Lemma 27.9.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section 10.56 for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{aligned} S_z &= k[x, y, z, 1/z] \\ S_{(z)} &= k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\ M_{(z)} &= S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\ N_{(z)} &= S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\ S(3)_{(z)} &= S_{(z)} \cdot z \subset S_z \end{aligned}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

27.10. Invertible sheaves on Proj

01MM Recall from Algebra, Section 10.56 the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

01MN **Definition 27.10.1.** Let S be a graded ring. Let $X = \operatorname{Proj}(S)$.

(1) We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth *twist of the structure sheaf of Proj*(S).
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 27.9.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n + m)$ we see that there are canonical maps

01MO (27.10.1.1) $$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n + m).$$

These maps are not isomorphisms in general, see the example in Remark 27.9.2. The same example shows that $\mathcal{O}_X(n)$ is *not* an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

03GJ (27.10.1.2) $$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n + m).$$

The maps (27.9.0.2) on global sections give a map of graded rings

01MP (27.10.1.3) $$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)).$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (27.10.1.2) give a graded module structure

03GK (27.10.1.4) $$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m))$$

and via (27.10.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

01MQ (27.10.1.5) $$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}.$$

On global sections (27.9.0.2) defines a map of graded S-modules

01MR (27.10.1.6) $$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}).$$

Here is an important fact which follows basically immediately from the definitions.

01MS **Lemma 27.10.2.** *Let S be a graded ring. Set $X = Proj(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition 17.25.1). The maps (27.10.1.1) restricted to $D_+(f)$*

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)},$$

the maps (27.10.1.2) restricted to $D_+(f)$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{F}(m)|_{D_+(f)} \longrightarrow \mathcal{F}(nd+m)|_{D_+(f)},$$

and the maps (27.10.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(nd)$, and $M \to M(nd)$, given by $x \mapsto f^n x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(nd)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(nd)|_{D_+(f)}$. The second shows that the map $S(nd)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(nd)_{(f)}$ is an isomorphism. The case of the map (27.10.1.2) is a consequence of the case of the map (27.10.1.1). □

01MT **Lemma 27.10.3.** *Let S be a graded ring. Let M be a graded S-module. Set $X = Proj(S)$. Assume X is covered by the standard opens $D_+(f)$ with $f \in S_1$, e.g., if S is generated by S_1 over S_0. Then the sheaves $\mathcal{O}_X(n)$ are invertible and the maps (27.10.1.1), (27.10.1.2), and (27.10.1.5) are isomorphisms. In particular, these maps induce isomorphisms*

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n) \quad \text{and} \quad \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M}(n) \cong \widetilde{M(n)}$$

Thus (27.9.0.2) becomes a map

0AG3 (27.10.3.1) $$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

and (27.10.1.6) becomes a map

0AG4 (27.10.3.2) $$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n)).$$

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 27.10.2 above. □

01MU **Lemma 27.10.4.** *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. The following open subsets of X are equal:*

(1) *The largest open subset $W = W_d \subset X$ such that each $\mathcal{O}_X(dn)|_W$ is invertible and all the multiplication maps $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd+md)|_W$ (see 27.10.1.1) are isomorphisms.*

(2) *The union of the open subsets $D_+(fg)$ with $f, g \in S$ homogeneous and $\deg(f) = \deg(g) + d$.*

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 27.10.1.5) are isomorphisms.

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g) + d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 27.10.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd+md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $x \in D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$ so $s = f/g$. Note that $x \in D_+(g)$ by construction. Note that $g^d \in (S_h)_{d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 27.10.1.5) are isomorphisms. Compare with the proof of Lemma 27.10.2. □

Recall from Modules, Lemma 17.25.10 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open.

01MV **Lemma 27.10.5.** *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 27.10.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (27.10.1.3) restricted to W. Then*

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n (a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 27.10.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. □

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

01MW **Lemma 27.10.6.** *Let S be a graded ring. Let $X = Proj(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that*

(1) *the subscheme Y is contained in the open W_d defined in Lemma 27.10.4,*
(2) *the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,*
(3) *all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd+m)$ of Equation (27.10.1.1) are isomorphisms,*

(4) *all the maps* $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(nd)}|_Y$ *(see 27.10.1.5) are isomorphisms,*
(5) *given* $f \in S_{nd}$ *denote* $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ *the image of* f *via (27.10.1.3) restricted to* Y*, then* $D_+(f) \cap Y = Y_s$*,*
(6) *a basis for the topology on* Y *is given by the collection of opens* Y_s*, where* $s \in \Gamma(Y, \mathcal{O}_Y(nd))$*,* $n \geq 1$*, and*
(7) *a basis for the topology of* Y *is given by those opens* $Y_s \subset Y$*, for* $s \in \Gamma(Y, \mathcal{O}_Y(nd))$*,* $n \geq 1$ *which are affine.*

Proof. Since Y is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of Y. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 27.10.4 or by (1) using Lemma 27.10.2. Note that (1) implies (2), (3) and (4) by Lemma 27.10.4. (Note that (3) is a special case of (4).) Assertion (5) follows from Lemma 27.10.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. □

0B5I **Lemma 27.10.7.** *Let* S *be a graded ring. Set* $X = Proj(S)$*. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Set* $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ *as a graded* S*-module, using (27.10.1.4) and (27.10.1.3). Then there is a canonical* $\mathcal{O}_X$*-module map*

$$\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ *such that the induced map* $M_0 \to \Gamma(X, \mathcal{F})$ *is the identity.*

Proof. Let $f \in S$ be homogeneous of degree $d > 0$. Recall that $\widetilde{M}|_{D_+(f)}$ corresponds to the $S_{(f)}$-module $M_{(f)}$ by Lemma 27.8.4. Thus we can define a canonical map

$$M_{(f)} \longrightarrow \Gamma(D_+(f), \mathcal{F}), \quad m/f^n \longmapsto m|_{D_+(f)} \otimes f|_{D_+(f)}^{-n}$$

which makes sense because $f|_{D_+(f)}$ is a trivializing section of the invertible sheaf $\mathcal{O}_X(d)|_{D_+(f)}$, see Lemma 27.10.2 and its proof. Since $\widetilde{M}$ is quasi-coherent, this leads to a canonical map

$$\widetilde{M}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}$$

via Schemes, Lemma 26.7.1. We obtain a global map if we prove that the displayed maps glue on overlaps. Proof of this is omitted. We also omit the proof of the final statement. □

27.11. Functoriality of Proj

01MX A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

01MY **Lemma 27.11.1.** *Let* A*,* B *be two graded rings. Set* $X = Proj(A)$ *and* $Y = Proj(B)$*. Let* $\psi : A \to B$ *be a graded ring map. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$*-graded* $\mathcal{O}_{U(\psi)}$*-algebras*

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ *is characterized by the following properties:*

(1) *For every* $d \geq 0$ *the diagram*

$$\begin{array}{ccccc} A_d & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & B_d \\ \downarrow & & & & \downarrow \\ \Gamma(X, \mathcal{O}_X(d)) & \xrightarrow{\theta} & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \longleftarrow & \Gamma(Y, \mathcal{O}_Y(d)) \end{array}$$

is commutative.

(2) *For any* $f \in A_+$ *homogeneous we have* $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ *and the restriction of* r_ψ *to* $D_+(\psi(f))$ *corresponds to the ring map* $A_{(f)} \to B_{(\psi(f))}$ *induced by* ψ*.*

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. □

01MZ **Lemma 27.11.2.** *Let* A*,* B*, and* C *be graded rings. Set* $X = Proj(A)$*,* $Y = Proj(B)$ *and* $Z = Proj(C)$*. Let* $\varphi : A \to B$*,* $\psi : B \to C$ *be graded ring maps. Then we have*

$$U(\psi \circ \varphi) = r_\psi^{-1}(U(\varphi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

01N0 **Lemma 27.11.3.** *With hypotheses and notation as in Lemma 27.11.1 above. Assume* $A_d \to B_d$ *is surjective for all* $d \gg 0$*. Then*

(1) $U(\psi) = Y$*,*
(2) $r_\psi : Y \to X$ *is a closed immersion, and*
(3) *the maps* $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are surjective but not isomorphisms in general (even if* $A \to B$ *is surjective).*

Proof. Part (1) follows from the definition of $U(\psi)$ and the fact that $D_+(f) = D_+(f^n)$ for any $n > 0$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(\psi(f))}$ is surjective because any element of $B_{(\psi(f))}$ can be represented by a fraction $b/\psi(f)^n$ with n arbitrarily large (which forces the degree of $b \in B$ to be large). This proves (2). The same argument shows the map

$$A_f \to B_{\psi(f)}$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$ in this case. But it is easy to see that $r_\psi^*\mathcal{O}_X(1)$ is not zero. (There are less silly examples.) □

07ZE **Lemma 27.11.4.** *With hypotheses and notation as in Lemma 27.11.1 above. Assume $A_d \to B_d$ is an isomorphism for all $d \gg 0$. Then*

(1) *$U(\psi) = Y$,*
(2) *$r_\psi : Y \to X$ is an isomorphism, and*
(3) *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. We have (1) by Lemma 27.11.3. Let $f \in A_+$ be homogeneous. The assumption on ψ implies that $A_f \to B_f$ is an isomorphism (details omitted). Thus it is clear that r_ψ and θ restrict to isomorphisms over $D_+(f)$. The lemma follows. □

01N1 **Lemma 27.11.5.** *With hypotheses and notation as in Lemma 27.11.1 above. Assume $A_d \to B_d$ is surjective for $d \gg 0$ and that A is generated by A_1 over A_0. Then*

(1) *$U(\psi) = Y$,*
(2) *$r_\psi : Y \to X$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. By Lemmas 27.11.4 and 27.11.2 we may replace B by the image of $A \to B$ without changing X or the sheaves $\mathcal{O}_X(n)$. Thus we may assume that $A \to B$ is surjective. By Lemma 27.11.3 we get (1) and (2) and surjectivity in (3). By Lemma 27.10.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. □

01N2 **Lemma 27.11.6.** *With hypotheses and notation as in Lemma 27.11.1 above. Assume there exists a ring map $R \to A_0$ and a ring map $R \to R'$ such that $B = R' \otimes_R A$. Then*

(1) *$U(\psi) = Y$,*
(2) *the diagram*

$$\begin{array}{ccc} Y = Proj(B) & \xrightarrow[r_\psi]{} & Proj(A) = X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

is a fibre product square, and
(3) *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. □

01N3 **Lemma 27.11.7.** *With hypotheses and notation as in Lemma 27.11.1 above. Assume there exists a $g \in A_0$ such that ψ induces an isomorphism $A_g \to B$. Then $U(\psi) = Y$, $r_\psi : Y \to X$ is an open immersion which induces an isomorphism of Y with the inverse image of $D(g) \subset \mathrm{Spec}(A_0)$. Moreover the map θ is an isomorphism.*

Proof. This is a special case of Lemma 27.11.6 above. □

0B5J **Lemma 27.11.8.** *Let S be a graded ring. Let $d \geq 1$. Set $S' = S^{(d)}$ with notation as in Algebra, Section 10.56. Set $X = Proj(S)$ and $X' = Proj(S')$. There is a canonical isomorphism $i : X \to X'$ of schemes such that*

(1) *for any graded S-module M setting $M' = M^{(d)}$, we have a canonical isomorphism $\widetilde{M} \to i^*\widetilde{M'}$,*
(2) *we have canonical isomorphisms $\mathcal{O}_X(nd) \to i^*\mathcal{O}_{X'}(n)$*

and these isomorphisms are compatible with the multiplication maps of Lemma 27.9.1 and hence with the maps (27.10.1.1), (27.10.1.2), (27.10.1.3), (27.10.1.4), (27.10.1.5), and (27.10.1.6) (see proof for precise statements.

Proof. The injective ring map $S' \to S$ (which is not a homomorphism of graded rings due to our conventions), induces a map $j : \mathrm{Spec}(S) \to \mathrm{Spec}(S')$. Given a graded prime ideal $\mathfrak{p} \subset S$ we see that $\mathfrak{p}' = j(\mathfrak{p}) = S' \cap \mathfrak{p}$ is a graded prime ideal of S'. Moreover, if $f \in S_+$ is homogeneous and $f \notin \mathfrak{p}$, then $f^d \in S'_+$ and $f^d \notin \mathfrak{p}'$. Conversely, if $\mathfrak{p}' \subset S'$ is a graded prime ideal not containing some homogeneous element $f \in S'_+$, then $\mathfrak{p} = \{g \in S \mid g^d \in \mathfrak{p}'\}$ is a graded prime ideal of S not containing f whose image under j is $\mathfrak{p}'$. To see that $\mathfrak{p}$ is an ideal, note that if $g, h \in \mathfrak{p}$, then $(g+h)^{2d} \in \mathfrak{p}'$ by the binomial formula and hence $g + h \in \mathfrak{p}'$ as $\mathfrak{p}'$ is a prime. In this way we see that j induces a homeomorphism $i : X \to X'$. Moreover, given $f \in S_+$ homogeneous, then we have $S_{(f)} \cong S'_{(f^d)}$. Since these isomorphisms are compatible with the restrictions mappings of Lemma 27.8.1, we see that there exists an isomorphism $i^\sharp : i^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$ of structure sheaves on X and X', hence i is an isomorphism of schemes.

Let M be a graded S-module. Given $f \in S_+$ homogeneous, we have $M_{(f)} \cong M'_{(f^d)}$, hence in exactly the same manner as above we obtain the isomorphism in (1). The isomorphisms in (2) are a special case of (1) for $M = S(nd)$ which gives $M' = S'(n)$. Let M and N be graded S-modules. Then we have

$$M' \otimes_{S'} N' = (M \otimes_S N)^{(d)} = (M \otimes_S N)'$$

as can be verified directly from the definitions. Having said this the compatibility with the multiplication maps of Lemma 27.9.1 is the commutativity of the diagram

$$\begin{array}{ccc} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} & \longrightarrow & \widetilde{M \otimes_S N} \\ \downarrow{\scriptstyle (1)\otimes(1)} & & \downarrow{\scriptstyle (1)} \\ i^*(\widetilde{M'} \otimes_{\mathcal{O}_{X'}} \widetilde{N'}) & \longrightarrow & i^*\widetilde{(M' \otimes_{S'} N')} \end{array}$$

This can be seen by looking at the construction of the maps over the open $D_+(f) = D_+(f^d)$ where the top horizontal arrow is given by the map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$ and the lower horizontal arrow by the map $M'_{(f^d)} \times N'_{(f^d)} \to (M' \otimes_{S'} N')_{(f^d)}$. Since these maps agree via the identifications $M_{(f)} = M'_{(f^d)}$, etc, we get the desired compatibility. We omit the proof of the other compatibilities. □

27.12. Morphisms into Proj

01N4 Let S be a graded ring. Let $X = \text{Proj}(S)$ be the homogeneous spectrum of S. Let $d \geq 1$ be an integer. Consider the open subscheme

01N5 (27.12.0.1) $$U_d = \bigcup\nolimits_{f \in S_d} D_+(f) \quad \subset \quad X = \text{Proj}(S)$$

Note that $d|d' \Rightarrow U_d \subset U_{d'}$ and $X = \bigcup_d U_d$. Neither X nor U_d need be quasi-compact, see Algebra, Lemma 10.57.3. Let us write $\mathcal{O}_{U_d}(n) = \mathcal{O}_X(n)|_{U_d}$. By Lemma 27.10.2 we know that $\mathcal{O}_{U_d}(nd)$, $n \in \mathbf{Z}$ is an invertible $\mathcal{O}_{U_d}$-module and that all the multiplication maps $\mathcal{O}_{U_d}(nd) \otimes_{\mathcal{O}_{U_d}} \mathcal{O}_{U_d}(m) \to \mathcal{O}_{U_d}(nd+m)$ of (27.10.1.1) are isomorphisms. In particular we have $\mathcal{O}_{U_d}(nd) \cong \mathcal{O}_{U_d}(d)^{\otimes n}$. The graded ring map (27.10.1.3) on global sections combined with restriction to U_d give a homomorphism of graded rings

01N6 (27.12.0.2) $$\psi^d : S^{(d)} \longrightarrow \Gamma_*(U_d, \mathcal{O}_{U_d}(d)).$$

For the notation $S^{(d)}$, see Algebra, Section 10.56. For the notation Γ_* see Modules, Definition 17.25.7. Moreover, since U_d is covered by the opens $D_+(f)$, $f \in S_d$ we see that $\mathcal{O}_{U_d}(d)$ is globally generated by the sections in the image of $\psi_1^d : S_1^{(d)} = S_d \to \Gamma(U_d, \mathcal{O}_{U_d}(d))$, see Modules, Definition 17.4.1.

Let Y be a scheme, and let $\varphi : Y \to X$ be a morphism of schemes. Assume the image $\varphi(Y)$ is contained in the open subscheme U_d of X. By the discussion following Modules, Definition 17.25.7 we obtain a homomorphism of graded rings

$$\Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)).$$

The composition of this and ψ^d gives a graded ring homomorphism

01N7 (27.12.0.3) $$\psi_\varphi^d : S^{(d)} \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d))$$

which has the property that the invertible sheaf $\varphi^*\mathcal{O}_X(d)$ is globally generated by the sections in the image of $(S^{(d)})_1 = S_d \to \Gamma(Y, \varphi^*\mathcal{O}_X(d))$.

01N8 **Lemma 27.12.1.** *Let S be a graded ring, and $X = Proj(S)$. Let $d \geq 1$ and $U_d \subset X$ as above. Let Y be a scheme. Let $\mathcal{L}$ be an invertible sheaf on Y. Let $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ be a graded ring homomorphism such that $\mathcal{L}$ is generated by the sections in the image of $\psi|_{S_d} : S_d \to \Gamma(Y, \mathcal{L})$. Then there exist a morphism $\varphi : Y \to X$ such that $\varphi(Y) \subset U_d$ and an isomorphism $\alpha : \varphi^*\mathcal{O}_{U_d}(d) \to \mathcal{L}$ such that ψ_φ^d agrees with ψ via α:*

$$\begin{array}{ccccc} \Gamma_*(Y, \mathcal{L}) & \xleftarrow{\alpha} & \Gamma_*(Y, \varphi^*\mathcal{O}_{U_d}(d)) & \xleftarrow{\varphi^*} & \Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \\ \uparrow{\scriptstyle \psi} & & & \nwarrow{\scriptstyle \psi_\varphi^d} & \uparrow{\scriptstyle \psi^d} \\ S^{(d)} & & \xleftarrow{id} & & S^{(d)} \end{array}$$

commutes. Moreover, the pair (φ, α) is unique.

Proof. Pick $f \in S_d$. Denote $s = \psi(f) \in \Gamma(Y, \mathcal{L})$. On the open set Y_s where s does not vanish multiplication by s induces an isomorphism $\mathcal{O}_{Y_s} \to \mathcal{L}|_{Y_s}$, see Modules, Lemma 17.25.10. We will denote the inverse of this map $x \mapsto x/s$, and similarly for powers of $\mathcal{L}$. Using this we define a ring map $\psi_{(f)} : S_{(f)} \to \Gamma(Y_s, \mathcal{O})$ by mapping the fraction a/f^n to $\psi(a)/s^n$. By Schemes, Lemma 26.6.4 this corresponds to a morphism $\varphi_f : Y_s \to \text{Spec}(S_{(f)}) = D_+(f)$. We also introduce the isomorphism $\alpha_f : \varphi_f^*\mathcal{O}_{D_+(f)}(d) \to \mathcal{L}|_{Y_s}$ which maps the pullback of the trivializing section f over

$D_+(f)$ to the trivializing section s over Y_s. With this choice the commutativity of the diagram in the lemma holds with Y replaced by Y_s, φ replaced by φ_f, and α replaced by α_f; verification omitted.

Suppose that $f' \in S_d$ is a second element, and denote $s' = \psi(f') \in \Gamma(Y, \mathcal{L})$. Then $Y_s \cap Y_{s'} = Y_{ss'}$ and similarly $D_+(f) \cap D_+(f') = D_+(ff')$. In Lemma 27.10.6 we saw that $D_+(f') \cap D_+(f)$ is the same as the set of points of $D_+(f)$ where the section of $\mathcal{O}_X(d)$ defined by f' does not vanish. Hence $\varphi_f^{-1}(D_+(f') \cap D_+(f)) = Y_s \cap Y_{s'} = \varphi_{f'}^{-1}(D_+(f') \cap D_+(f))$. On $D_+(f) \cap D_+(f')$ the fraction f/f' is an invertible section of the structure sheaf with inverse f'/f. Note that $\psi_{(f')}(f/f') = \psi(f)/s' = s/s'$ and $\psi_{(f)}(f'/f) = \psi(f')/s = s'/s$. We claim there is a unique ring map $S_{(ff')} \to \Gamma(Y_{ss'}, \mathcal{O})$ making the following diagram commute

$$\begin{array}{ccccc} \Gamma(Y_s, \mathcal{O}) & \longrightarrow & \Gamma(Y_{ss'}, \mathcal{O}) & \longleftarrow & \Gamma(Y_{s,'}\mathcal{O}) \\ {\scriptstyle \psi_{(f)}}\uparrow & & \uparrow & & \uparrow{\scriptstyle \psi_{(f')}} \\ S_{(f)} & \longrightarrow & S_{(ff')} & \longleftarrow & S_{(f')} \end{array}$$

It exists because we may use the rule $x/(ff')^n \mapsto \psi(x)/(ss')^n$, which "works" by the formulas above. Uniqueness follows as $\mathrm{Proj}(S)$ is separated, see Lemma 27.8.8 and its proof. This shows that the morphisms φ_f and $\varphi_{f'}$ agree over $Y_s \cap Y_{s'}$. The restrictions of α_f and $\alpha_{f'}$ agree over $Y_s \cap Y_{s'}$ because the regular functions s/s' and $\psi_{(f')}(f)$ agree. This proves that the morphisms ψ_f glue to a global morphism from Y into $U_d \subset X$, and that the maps α_f glue to an isomorphism satisfying the conditions of the lemma.

We still have to show the pair (φ, α) is unique. Suppose (φ', α') is a second such pair. Let $f \in S_d$. By the commutativity of the diagrams in the lemma we have that the inverse images of $D_+(f)$ under both φ and φ' are equal to $Y_{\psi(f)}$. Since the opens $D_+(f)$ are a basis for the topology on X, and since X is a sober topological space (see Schemes, Lemma 26.11.1) this means the maps φ and φ' are the same on underlying topological spaces. Let us use $s = \psi(f)$ to trivialize the invertible sheaf $\mathcal{L}$ over $Y_{\psi(f)}$. By the commutativity of the diagrams we have that $\alpha^{\otimes n}(\psi_\varphi^d(x)) = \psi(x) = (\alpha')^{\otimes n}(\psi_{\varphi'}^d(x))$ for all $x \in S_{nd}$. By construction of ψ_φ^d and $\psi_{\varphi'}^d$ we have $\psi_\varphi^d(x) = \varphi^\sharp(x/f^n)\psi_\varphi^d(f^n)$ over $Y_{\psi(f)}$, and similarly for $\psi_{\varphi'}^d$. By the commutativity of the diagrams of the lemma we deduce that $\varphi^\sharp(x/f^n) = (\varphi')^\sharp(x/f^n)$. This proves that φ and φ' induce the same morphism from $Y_{\psi(f)}$ into the affine scheme $D_+(f) = \mathrm{Spec}(S_{(f)})$. Hence φ and φ' are the same as morphisms. Finally, it remains to show that the commutativity of the diagram of the lemma singles out, given φ, a unique α. We omit the verification. □

We continue the discussion from above the lemma. Let S be a graded ring. Let Y be a scheme. We will consider *triples* $(d, \mathcal{L}, \psi)$ where

(1) $d \geq 1$ is an integer,
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(3) $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_d$.

Given a morphism $h : Y' \to Y$ and a triple $(d, \mathcal{L}, \psi)$ over Y we can pull it back to the triple $(d, h^*\mathcal{L}, h^* \circ \psi)$. Given two triples $(d, \mathcal{L}, \psi)$ and $(d, \mathcal{L}', \psi')$ with the

same integer d we say they are *strictly equivalent* if there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded ring maps $S^{(d)} \to \Gamma_*(Y, \mathcal{L}')$.

For each integer $d \geq 1$ we define

$$\begin{aligned} F_d : Sch^{opp} &\longrightarrow Sets, \\ Y &\longmapsto \{\text{strict equivalence classes of triples } (d, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks as defined above.

01N9 **Lemma 27.12.2.** *Let S be a graded ring. Let $X = Proj(S)$. The open subscheme $U_d \subset X$ (27.12.0.1) represents the functor F_d and the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$ defined above is the universal family (see Schemes, Section 26.15).*

Proof. This is a reformulation of Lemma 27.12.1 □

01NA **Lemma 27.12.3.** *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. In this case the scheme $X = Proj(S)$ represents the functor which associates to a scheme Y the set of pairs $(\mathcal{L}, \psi)$, where*

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
2. *$\psi : S \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_1$*

up to strict equivalence as above.

Proof. Under the assumptions of the lemma we have $X = U_1$ and the lemma is a reformulation of Lemma 27.12.2 above. □

We end this section with a discussion of a functor corresponding to $\mathrm{Proj}(S)$ for a general graded ring S. We advise the reader to skip the rest of this section.

Fix an arbitrary graded ring S. Let T be a scheme. We will say two triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ over T with possibly different integers d, d' are *equivalent* if there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ of invertible sheaves over T such that $\beta \circ \psi|_{S^{(dd')}}$ and $\psi'|_{S^{(dd')}}$ agree as graded ring maps $S^{(dd')} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes dd'})$.

01NB **Lemma 27.12.4.** *Let S be a graded ring. Set $X = Proj(S)$. Let T be a scheme. Let $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ be two triples over T. The following are equivalent:*

1. *Let $n = lcm(d, d')$. Write $n = ad = a'd'$. There exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
2. *The triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equivalent.*
3. *For some positive integer $n = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
4. *The morphisms $\varphi : T \to X$ and $\varphi' : T \to X$ associated to $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equal.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Also (3) implies (4) by the uniqueness statement in Lemma 27.12.1. Thus we have to prove that (4) implies (1). Assume (4), in other words $\varphi = \varphi'$. Note that this implies that we may write $\mathcal{L} = \varphi^*\mathcal{O}_X(d)$

and $\mathcal{L}' = \varphi^*\mathcal{O}_X(d')$. Moreover, via these identifications we have that the graded ring maps ψ and ψ' correspond to the restriction of the canonical graded ring map

$$S \longrightarrow \bigoplus\nolimits_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$$

to $S^{(d)}$ and $S^{(d')}$ composed with pullback by φ (by Lemma 27.12.1 again). Hence taking β to be the isomorphism

$$(\varphi^*\mathcal{O}_X(d))^{\otimes a} = \varphi^*\mathcal{O}_X(n) = (\varphi^*\mathcal{O}_X(d'))^{\otimes a'}$$

works. □

Let S be a graded ring. Let $X = \text{Proj}(S)$. Over the open subscheme scheme $U_d \subset X = \text{Proj}(S)$ (27.12.0.1) we have the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$. Clearly, if $d|d'$ the triples $(d, \mathcal{O}_{U_d}(d), \psi^d)$ and $(d', \mathcal{O}_{U_{d'}}(d'), \psi^{d'})$ are equivalent when restricted to the open U_d (which is a subset of $U_{d'}$). This, combined with Lemma 27.12.1 shows that morphisms $Y \to X$ correspond roughly to equivalence classes of triples over Y. This is not quite true since if Y is not quasi-compact, then there may not be a single triple which works. Thus we have to be slightly careful in defining the corresponding functor.

Here is one possible way to do this. Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the triple $(d, \mathcal{L}, \psi)$ over T the triple $(d', \mathcal{L}^{\otimes a}, \psi|_{S^{(d')}})$. One of the implications of Lemma 27.12.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d\in\mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V\subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \quad \longrightarrow \quad Sets$$

01NC **Lemma 27.12.5.** *Let S be a graded ring. Let $X = Proj(S)$. The functor F defined above is representable by the scheme X.*

Proof. We have seen above that the functor F_d corresponds to the open subscheme $U_d \subset X$. Moreover the transformation of functors $F_d \to F_{d'}$ (if $d|d'$) defined above corresponds to the inclusion morphism $U_d \to U_{d'}$ (see discussion above). Hence to show that F is represented by X it suffices to show that $T \to X$ for a quasi-compact scheme T ends up in some U_d, and that for a general scheme T we have

$$\text{Mor}(T, X) = \lim\nolimits_{V\subset T \text{ quasi-compact open}} \text{Mor}(V, X).$$

These verifications are omitted. □

27.13. Projective space

01ND Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

01NE **Lemma 27.13.1.** *Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme*

$$\mathbf{P}^n_{\mathbf{Z}} = Proj(S)$$

represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
(2) *$s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$*

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma 27.12.3 above. Namely, for any graded ring A we have

$$\begin{aligned} \mathrm{Mor}_{gradedrings}(\mathbf{Z}[T_0, \ldots, T_n], A) &= A_1 \times \ldots \times A_1 \\ \psi &\mapsto (\psi(T_0), \ldots, \psi(T_n)) \end{aligned}$$

and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. □

01NF **Definition 27.13.2.** The scheme $\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over $\mathbf{Z}$*. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 27.13.1 the induced morphism to $\mathbf{P}^n_S$ is denoted

$$\varphi_{(\mathcal{L}, (s_0, \ldots, s_n))} : Y \longrightarrow \mathbf{P}^n_S$$

This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} \mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} T_i$$

where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (27.10.1.3).

01NG **Lemma 27.13.3.** *Projective n-space over $\mathbf{Z}$ is covered by $n + 1$ standard opens*

$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since

$$\mathrm{Spec}\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$

in an obvious way. □

01NH **Lemma 27.13.4.** *Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is*

(1) *separated,*
(2) *quasi-compact,*

(3) *satisfies the existence and uniqueness parts of the valuative criterion, and*
(4) *universally closed.*

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas 26.21.12 and 26.19.3). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$. Separatedness is Lemma 27.8.8. Quasi-compactness follows from Lemma 27.13.3. Existence and uniqueness of the valuative criterion follow from Lemma 27.8.11. Universally closed follows from the above and Schemes, Proposition 26.20.6. □

01NI **Remark 27.13.5.** What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is "smoothness". And I'm sure there are many more you can think of.)

01WD **Lemma 27.13.6** (Segre embedding)**.** *Let S be a scheme. There exists a closed immersion*

$$\mathbf{P}^n_S \times_S \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$$

called the Segre embedding.

Proof. It suffices to prove this when $S = \mathrm{Spec}(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, $\mathbf{P}^m = \mathrm{Proj}(\mathbf{Z}[Y_0, \ldots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \mathrm{Proj}(\mathbf{Z}[Z_0, \ldots, Z_{nm+n+m}])$. In order to map into $\mathbf{P}^{nm+n+m}$ we have to write down an invertible sheaf $\mathcal{L}$ on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Lemma 27.13.1. The invertible sheaf we take is

$$\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0Y_0,\ s_1 = X_1Y_0, \ldots,\ s_n = X_nY_0,\ s_{n+1} = X_0Y_1, \ldots,\ s_{nm+n+m} = X_nY_m.$$

These generate $\mathcal{L}$ since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i, j) = (0, 0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \ldots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \ldots, X_n/X_0, Y_1/Y_0, \ldots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0. Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion (see Schemes, Lemma 26.4.2 and Example 26.8.1.) □

The following two lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

03GL **Lemma 27.13.7.** *Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let*

$$I_d = \mathrm{Ker}\left(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\right)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = Proj(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \text{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 27.11.5 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$ is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i}F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)}F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. □

The following lemma is a special case of the more general Properties, Lemmas 28.28.3 or 28.28.5.

03GM **Lemma 27.13.8.** *Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set*

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e + d)$$

see Equation (27.10.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$.

By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean "principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s_a' glue to a global section $m \in \Gamma(\mathbf{P}_R^n, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. □

0B3B **Lemma 27.13.9.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf and let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}$ which generate it. Let $\mathcal{F}$ be the kernel of the induced map $\mathcal{O}_X^{\oplus n+1} \to \mathcal{L}$. Then $\mathcal{F} \otimes \mathcal{L}$ is globally generated.*

Proof. In fact the result is true if X is any locally ringed space. The sheaf $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank n. The elements

$$s_{ij} = (0, \ldots, 0, s_j, 0, \ldots, 0, -s_i, 0, \ldots, 0) \in \Gamma(X, \mathcal{L}^{\oplus n+1})$$

with s_j in the ith spot and $-s_i$ in the jth spot map to zero in $\mathcal{L}^{\otimes 2}$. Hence $s_{ij} \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L})$. A local computation shows that these sections generate $\mathcal{F} \otimes \mathcal{L}$.

Alternative proof. Consider the morphism $\varphi : X \to \mathbf{P}_{\mathbf{Z}}^n$ associated to the pair $(\mathcal{L}, (s_0, \ldots, s_n))$. Since the pullback of $\mathcal{O}(1)$ is $\mathcal{L}$ and since the pullback of T_i is s_i, it suffices to prove the lemma in the case of $\mathbf{P}_{\mathbf{Z}}^n$. In this case the sheaf $\mathcal{F}$ corresponds to the graded $S = \mathbf{Z}[T_0, \ldots, T_n]$ module M which fits into the short exact sequence

$$0 \to M \to S^{\oplus n+1} \to S(1) \to 0$$

where the second map is given by $T_0, \ldots, T_n$. In this case the statement above translates into the statement that the elements

$$T_{ij} = (0, \ldots, 0, T_j, 0, \ldots, 0, -T_i, 0, \ldots, 0) \in M(1)_0$$

generate the graded module $M(1)$ over S. We omit the details. □

27.14. Invertible sheaves and morphisms into Proj

01NJ Let T be a scheme and let $\mathcal{L}$ be an invertible sheaf on T. For a section $s \in \Gamma(T, \mathcal{L})$ we denote T_s the open subset of points where s does not vanish. See Modules, Lemma 17.25.10. We can view the following lemma as a slight generalization of Lemma 27.12.3. It also is a generalization of Lemma 27.11.1.

01NK **Lemma 27.14.1.** *Let A be a graded ring. Set $X = Proj(A)$. Let T be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_T$-module. Let $\psi : A \to \Gamma_*(T, \mathcal{L})$ be a homomorphism of graded rings. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} T_{\psi(f)}$$

The morphism ψ induces a canonical morphism of schemes

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow X$$

together with a map of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r_{\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{L}^{\otimes d}|_{U(\psi)}.$$

The triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ is characterized by the following properties:

(1) *For $f \in A_+$ homogeneous we have $r_{\mathcal{L},\psi}^{-1}(D_+(f)) = T_{\psi(f)}$.*
(2) *For every $d \geq 0$ the diagram*

$$\begin{array}{ccc} A_d & \xrightarrow{\psi} & \Gamma(T, \mathcal{L}^{\otimes d}) \\ {\scriptstyle (27.10.1.3)}\downarrow & & \downarrow{\scriptstyle restrict} \\ \Gamma(X, \mathcal{O}_X(d)) & \xrightarrow{\theta} & \Gamma(U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

Moreover, for any $d \geq 1$ and any open subscheme $V \subset T$ such that the sections in $\psi(A_d)$ generate $\mathcal{L}^{\otimes d}|_V$ the morphism $r_{\mathcal{L},\psi}|_V$ agrees with the morphism $\varphi : V \to Proj(A)$ and the map $\theta|_V$ agrees with the map $\alpha : \varphi^*\mathcal{O}_X(d) \to \mathcal{L}^{\otimes d}|_V$ where (φ, α) is the pair of Lemma 27.12.1 associated to $\psi|_{A^{(d)}} : A^{(d)} \to \Gamma_*(V, \mathcal{L}^{\otimes d})$.

Proof. Suppose that we have two triples $(U, r : U \to X, \theta)$ and $(U', r' : U' \to X, \theta')$ satisfying (1) and (2). Property (1) implies that $U = U' = U(\psi)$ and that $r = r'$ as maps of underlying topological spaces, since the opens $D_+(f)$ form a basis for the topology on X, and since X is a sober topological space (see Algebra, Section 10.57 and Schemes, Lemma 26.11.1). Let $f \in A_+$ be homogeneous. Note that $\Gamma(D_+(f), \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)) = A_f$ as a $\mathbf{Z}$-graded algebra. Consider the two $\mathbf{Z}$-graded ring maps

$$\theta, \theta' : A_f \longrightarrow \Gamma(T_{\psi(f)}, \bigoplus \mathcal{L}^{\otimes n}).$$

We know that multiplication by f (resp. $\psi(f)$) is an isomorphism on the left (resp. right) hand side. We also know that $\theta(x/1) = \theta'(x/1) = \psi(x)|_{T_{\psi(f)}}$ by (2) for all $x \in A$. Hence we deduce easily that $\theta = \theta'$ as desired. Considering the degree 0 parts we deduce that $r^\sharp = (r')^\sharp$, i.e., that $r = r'$ as morphisms of schemes. This proves the uniqueness.

Now we come to existence. By the uniqueness just proved, it is enough to construct the pair (r, θ) locally on T. Hence we may assume that $T = \mathrm{Spec}(R)$ is affine, that $\mathcal{L} = \mathcal{O}_T$ and that for some $f \in A_+$ homogeneous we have $\psi(f)$ generates

$\mathcal{O}_T = \mathcal{O}_T^{\otimes \deg(f)}$. In other words, $\psi(f) = u \in R^*$ is a unit. In this case the map ψ is a graded ring map

$$A \longrightarrow R[x] = \Gamma_*(T, \mathcal{O}_T)$$

which maps f to $ux^{\deg(f)}$. Clearly this extends (uniquely) to a $\mathbf{Z}$-graded ring map $\theta : A_f \to R[x, x^{-1}]$ by mapping $1/f$ to $u^{-1}x^{-\deg(f)}$. This map in degree zero gives the ring map $A_{(f)} \to R$ which gives the morphism $r : T = \operatorname{Spec}(R) \to \operatorname{Spec}(A_{(f)}) = D_+(f) \subset X$. Hence we have constructed (r, θ) in this special case.

Let us show the last statement of the lemma. According to Lemma 27.12.1 the morphism constructed there is the unique one such that the displayed diagram in its statement commutes. The commutativity of the diagram in the lemma implies the commutativity when restricted to V and $A^{(d)}$. Whence the result. □

01NL **Remark 27.14.2.** Assumptions as in Lemma 27.14.1 above. The image of the morphism $r_{\mathcal{L},\psi}$ need not be contained in the locus where the sheaf $\mathcal{O}_X(1)$ is invertible. Here is an example. Let k be a field. Let $S = k[A, B, C]$ graded by $\deg(A) = 1$, $\deg(B) = 2$, $\deg(C) = 3$. Set $X = \operatorname{Proj}(S)$. Let $T = \mathbf{P}^2_k = \operatorname{Proj}(k[X_0, X_1, X_2])$. Recall that $\mathcal{L} = \mathcal{O}_T(1)$ is invertible and that $\mathcal{O}_T(n) = \mathcal{L}^{\otimes n}$. Consider the composition ψ of the maps

$$S \to k[X_0, X_1, X_2] \to \Gamma_*(T, \mathcal{L}).$$

Here the first map is $A \mapsto X_0$, $B \mapsto X_1^2$, $C \mapsto X_2^3$ and the second map is (27.10.1.3). By the lemma this corresponds to a morphism $r_{\mathcal{L},\psi} : T \to X = \operatorname{Proj}(S)$ which is easily seen to be surjective. On the other hand, in Remark 27.9.2 we showed that the sheaf $\mathcal{O}_X(1)$ is not invertible at all points of X.

27.15. Relative Proj via glueing

01NM

01NN **Situation 27.15.1.** Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\operatorname{Proj}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the homogeneous spectra $\operatorname{Proj}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the homogeneous spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 27.2.1.

01NO **Lemma 27.15.2.** *In Situation 27.15.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of graded rings $A' \to A$ induces a morphism $r : Proj(A) \to Proj(A')$, and the diagram*

$$\begin{array}{ccc} Proj(A) & \longrightarrow & Proj(A') \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

is cartesian. Moreover there are canonical isomorphisms $\theta : r^*\mathcal{O}_{Proj(A')}(n) \to \mathcal{O}_{Proj(A)}(n)$ compatible with multiplication maps.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 26.7.3 for example). Hence the lemma follows from Lemma 27.11.6. □

In particular the morphism $\mathrm{Proj}(A) \to \mathrm{Proj}(A')$ of the lemma is an open immersion.

01NP **Lemma 27.15.3.** *In Situation 27.15.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $r : Proj(A) \to Proj(A')$, and $r' : Proj(A') \to Proj(A'')$ of Lemma 27.15.2 gives the morphism $r'' : Proj(A) \to Proj(A'')$ of Lemma 27.15.2. A similar statement holds for the isomorphisms θ.*

Proof. This follows from Lemma 27.11.2 since the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$. □

01NQ **Lemma 27.15.4.** *In Situation 27.15.1. There exists a morphism of schemes*

$$\pi : \underline{Proj}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to Proj(A)$ with $A = \mathcal{A}(U)$, and*
2. *for $U \subset U' \subset S$ affine open the composition*

$$Proj(A) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} Proj(A')$$

with $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ is the open immersion of Lemma 27.15.2 above.

Proof. Follows immediately from Lemmas 27.2.1, 27.15.2, and 27.15.3. □

01NR **Lemma 27.15.5.** *In Situation 27.15.1. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ of Lemma 27.15.4 comes with the following additional structure. There exists a quasi-coherent $\mathbf{Z}$-graded sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)$, and a morphism of graded $\mathcal{O}_S$-algebras*

$$\psi : \mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \pi_* \left(\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)$$

uniquely determined by the following property: For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ there is an isomorphism

$$\theta_U : i_U^* \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{Proj(A)}(n)\right) \longrightarrow \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)|_{\pi^{-1}(U)}$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\pi^{-1}(U)}$-algebras such that

$$\begin{array}{ccc} A_n & \xrightarrow{\quad\psi\quad} & \Gamma(\pi^{-1}(U), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)) \\ & {\scriptstyle (27.10.1.3)}\searrow \quad \nearrow{\scriptstyle \theta_U} & \\ & \Gamma(Proj(A), \mathcal{O}_{Proj(A)}(n)) & \end{array}$$

is commutative.

Proof. We are going to use Lemma 27.2.2 to glue the sheaves of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\mathrm{Proj}(A)}(n)$ for $A = \mathcal{A}(U)$, $U \subset S$ affine open over the scheme $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We have constructed the data necessary for this in Lemma 27.15.2 and we have checked condition (d) of Lemma 27.2.2 in Lemma 27.15.3. Hence we get the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ together with the isomorphisms θ_U for all $U \subset S$ affine open and all $n \in \mathbf{Z}$. For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ we have a map $A \to \Gamma(\mathrm{Proj}(A), \bigoplus_{n\geq 0} \mathcal{O}_{\mathrm{Proj}(A)}(n))$. Hence the map

ψ exists by functoriality of relative glueing, see Remark 27.2.3. The diagram of the lemma commutes by construction. This characterizes the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ because the proof of Lemma 27.11.1 shows that having these diagrams commute uniquely determines the maps θ_U. Some details omitted. □

27.16. Relative Proj as a functor

01NS We place ourselves in Situation 27.15.1. So S is a scheme and $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. In this section we relativize the construction of Proj by constructing a functor which the relative homogeneous spectrum will represent. As a result we will construct a morphism of schemes

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow S$$

which above affine opens of S will look like the homogeneous spectrum of a graded ring. The discussion will be modeled after our discussion of the relative spectrum in Section 27.4. The easier method using glueing schemes of the form $\mathrm{Proj}(A)$, $A = \Gamma(U, \mathcal{A})$, $U \subset S$ affine open, is explained in Section 27.15, and the result in this section will be shown to be isomorphic to that one.

Fix for the moment an integer $d \geq 1$. We denote $\mathcal{A}^{(d)} = \bigoplus_{n\geq 0} \mathcal{A}_{nd}$ similarly to the notation in Algebra, Section 10.56. Let T be a scheme. Let us consider *quadruples* $(d, f : T \to S, \mathcal{L}, \psi)$ *over* T where

(1) d is the integer we fixed above,
(2) $f : T \to S$ is a morphism of schemes,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

Given a morphism $h : T' \to T$ and a quadruple $(d, f, \mathcal{L}, \psi)$ over T we can pull it back to the quadruple $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$ over T'. Given two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d, f', \mathcal{L}', \psi')$ over T with the same integer d we say they are *strictly equivalent* if $f = f'$ and there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded $\mathcal{O}_T$-algebra maps $f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} (\mathcal{L}')^{\otimes n}$.

For each integer $d \geq 1$ we define

$$\begin{aligned} F_d : Sch^{opp} &\longrightarrow Sets, \\ T &\longmapsto \{\text{strict equivalence classes of } (d, f : T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks as defined above.

01NT **Lemma 27.16.1.** *In Situation 27.15.1. Let $d \geq 1$. Let F_d be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F'_d be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F'_d \cong h_{S'} \times_{h_S} F_d$$

of functors.

Proof. A quadruple $(d, f' : T \to S', \mathcal{L}', \psi' : (f')^*(\mathcal{A}')^{(d)} \to \bigoplus_{n\geq 0} (\mathcal{L}')^{\otimes n})$ is the same as a quadruple $(d, f, \mathcal{L}, \psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n})$ together with a factorization of f as $f = g \circ f'$. Namely, the correspondence is $f = g \circ f'$, $\mathcal{L} = \mathcal{L}'$ and $\psi = \psi'$ via the identifications $(f')^*(\mathcal{A}')^{(d)} = (f')^*g^*(\mathcal{A}^{(d)}) = f^*\mathcal{A}^{(d)}$. Hence the lemma. □

01NU **Lemma 27.16.2.** *In Situation 27.15.1. Let F_d be the functor associated to $(d, S, \mathcal{A})$ above. If S is affine, then F_d is representable by the open subscheme U_d (27.12.0.1) of the scheme Proj$(\Gamma(S, \mathcal{A}))$.*

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is a graded R-algebra and $\mathcal{A} = \widetilde{A}$. To prove the lemma we have to identify the functor F_d with the functor $F_d^{triples}$ of triples defined in Section 27.12.

Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple. We may think of ψ as a $\mathcal{O}_S$-module map $\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n}$. Since $\mathcal{A}^{(d)}$ is quasi-coherent this is the same thing as an R-linear homomorphism of graded rings $A^{(d)} \to \Gamma(S, \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n})$. Clearly, $\Gamma(S, \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n}) = \Gamma_*(T, \mathcal{L})$. Thus we may associate to the quadruple the triple $(d, \mathcal{L}, \psi)$.

Conversely, let $(d, \mathcal{L}, \psi)$ be a triple. The composition $R \to A_0 \to \Gamma(T, \mathcal{O}_T)$ determines a morphism $f : T \to S = \mathrm{Spec}(R)$, see Schemes, Lemma 26.6.4. With this choice of f the map $A^{(d)} \to \Gamma(S, \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n})$ is R-linear, and hence corresponds to a ψ which we can use for a quadruple $(d, f : T \to S, \mathcal{L}, \psi)$. We omit the verification that this establishes an isomorphism of functors $F_d = F_d^{triples}$. □

01NV **Lemma 27.16.3.** *In Situation 27.15.1. The functor F_d is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 26.15.4.

First we check that F_d satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i\in I} U_i$ is an open covering, and that $(d, f_i, \mathcal{L}_i, \psi_i) \in F_d(U_i)$ such that $(d, f_i, \mathcal{L}_i, \psi_i)|_{U_i\cap U_j}$ and $(d, f_j, \mathcal{L}_j, \psi_j)|_{U_i\cap U_j}$ are strictly equivalent. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 26.14. Thus $f_i^*\mathcal{A}^{(d)} = f^*\mathcal{A}^{(d)}|_{U_i}$. It also implies there exist isomorphisms $\beta_{ij} : \mathcal{L}_i|_{U_i\cap U_j} \to \mathcal{L}_j|_{U_i\cap U_j}$ such that $\beta_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$. Note that the isomorphisms β_{ij} are uniquely determined by this requirement because the maps $f_i^*\mathcal{A}_d \to \mathcal{L}_i$ are surjective. In particular we see that $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Hence by Sheaves, Section 6.33 the invertible sheaves $\mathcal{L}_i$ glue to an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and the morphisms ψ_i glue to morphism of $\mathcal{O}_T$-algebras $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. This proves that F_d satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i\in I} U_i$ be an affine open covering. Let $F_{d,i} \subset F_d$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each $F_{d,i}$ is representable. This is the case because $F_{d,i}$ is identified with the functor associated to U_i equipped with the quasi-coherent graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$ by Lemma 27.16.1. Thus the result follows from Lemma 27.16.2.

Next we show that $F_{d,i} \subset F_d$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of $F_{d,i}$ that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_{d,i}(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_{d,i})_{i\in I}$ covers F_d. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i\in I} U_i$ is an open covering of S we see that $T = \bigcup_{i\in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_{d,i}(V_i)$. This finishes the proof of the lemma. □

At this point we can redo the material at the end of Section 27.12 in the current relative setting and define a functor which is representable by $\underline{\mathrm{Proj}}_S(\mathcal{A})$. To do this we introduce the notion of equivalence between two quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ and $(d', f' : T \to S, \mathcal{L}', \psi')$ with possibly different values of the integers d, d'. Namely, we say these are *equivalent* if $f = f'$, and there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ such that $\beta \circ \psi|_{f^*\mathcal{A}^{(dd')}} = \psi'|_{f^*\mathcal{A}^{(dd')}}$. The following lemma implies that this defines an equivalence relation. (This is not a complete triviality.)

01NW **Lemma 27.16.4.** *In Situation 27.15.1. Let T be a scheme. Let $(d, f, \mathcal{L}, \psi)$, $(d', f', \mathcal{L}', \psi')$ be two quadruples over T. The following are equivalent:*

1. *Let $m = lcm(d, d')$. Write $m = ad = a'd'$. We have $f = f'$ and there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$.*
2. *The quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent.*
3. *We have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Assume (3) for some integer $m = ad = a'd'$. Let $m_0 = \mathrm{lcm}(d, d')$ and write it as $m_0 = a_0 d = a'_0 d'$. We are given an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property described in (3). We want to find an isomorphism $\beta_0 : \mathcal{L}^{\otimes a_0} \to (\mathcal{L}')^{\otimes a'_0}$ having that property as well. Since by assumption the maps $\psi : f^*\mathcal{A}_d \to \mathcal{L}$ and $\psi' : (f')^*\mathcal{A}_{d'} \to \mathcal{L}'$ are surjective the same is true for the maps $\psi : f^*\mathcal{A}_{m_0} \to \mathcal{L}^{\otimes a_0}$ and $\psi' : (f')^*\mathcal{A}_{m_0} \to (\mathcal{L}')^{\otimes a_0}$. Hence if β_0 exists it is uniquely determined by the condition that $\beta_0 \circ \psi = \psi'$. This means that we may work locally on T. Hence we may assume that $f = f' : T \to S$ maps into an affine open, in other words we may assume that S is affine. In this case the result follows from the corresponding result for triples (see Lemma 27.12.4) and the fact that triples and quadruples correspond in the affine base case (see proof of Lemma 27.16.2). □

Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the quadruple $(d, f, \mathcal{L}, \psi)$ over T the quadruple $(d', f, \mathcal{L}^{\otimes a}, \psi|_{f^*\mathcal{A}^{(d')}})$. One of the implications of Lemma 27.16.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d\in\mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V\subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

01NX (27.16.4.1) $$F : Sch^{opp} \longrightarrow Sets$$

01NY **Lemma 27.16.5.** *In Situation 27.15.1. The functor F above is representable by a scheme.*

Proof. Let $U_d \to S$ be the scheme representing the functor F_d defined above. Let $\mathcal{L}_d$, $\psi^d : \pi_d^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}_d^{\otimes n}$ be the universal object. If $d|d'$, then we may consider the quadruple $(d', \pi_d, \mathcal{L}_d^{\otimes d'/d}, \psi^d|_{\mathcal{A}^{(d')}})$ which determines a canonical morphism $U_d \to U_{d'}$ over S. By construction this morphism corresponds to the transformation of functors $F_d \to F_{d'}$ defined above.

For every affine open $\mathrm{Spec}(R) = V \subset S$ setting $A = \Gamma(V, \mathcal{A})$ we have a canonical identification of the base change $U_{d,V}$ with the corresponding open subscheme of $\mathrm{Proj}(A)$, see Lemma 27.16.2. Moreover, the morphisms $U_{d,V} \to U_{d',V}$ constructed above correspond to the inclusions of opens in $\mathrm{Proj}(A)$. Thus we conclude that $U_d \to U_{d'}$ is an open immersion.

This allows us to construct X by glueing the schemes U_d along the open immersions $U_d \to U_{d'}$. Technically, it is convenient to choose a sequence $d_1|d_2|d_3|\ldots$ such that every positive integer divides one of the d_i and to simply take $X = \bigcup U_{d_i}$ using the open immersions above. It is then a simple matter to prove that X represents the functor F. □

01NZ **Lemma 27.16.6.** *In Situation 27.15.1. The scheme $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ constructed in Lemma 27.15.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let X be the scheme representing the functor F. Note that X is a scheme over S since the functor F comes equipped with a natural transformation $F \to h_S$. Write $Y = \underline{\mathrm{Proj}}_S(\mathcal{A})$. We have to show that $X \cong Y$ as S-schemes. We give two arguments.

The first argument uses the construction of X as the union of the schemes U_d representing F_d in the proof of Lemma 27.16.5. Over each affine open of S we can identify X with the homogeneous spectrum of the sections of $\mathcal{A}$ over that open, since this was true for the opens U_d. Moreover, these identifications are compatible with further restrictions to smaller affine opens. On the other hand, Y was constructed by glueing these homogeneous spectra. Hence we can glue these isomorphisms to an isomorphism between X and $\underline{\mathrm{Proj}}_S(\mathcal{A})$ as desired. Details omitted.

Here is the second argument. Lemma 27.15.5 shows that there exists a morphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_Y(n)$$

over Y which on sections over affine opens of S agrees with (27.10.1.3). Hence for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y and an integer $d \geq 1$ such that for $d|n$ the sheaf $\mathcal{O}_Y(n)|_V$ is invertible and the multiplication maps $\mathcal{O}_Y(n)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y(m)|_V \to \mathcal{O}_Y(n+m)|_V$ are isomorphisms. Thus ψ restricted to the sheaf $\pi^*\mathcal{A}^{(d)}|_V$ gives an element of $F_d(V)$. Since the opens V cover Y we see "ψ" gives rise to an element of $F(Y)$. Hence a canonical morphism $Y \to X$ over S. Because this construction is completely canonical to see that it is an isomorphism we may work locally on S. Hence we reduce to the case S affine where the result is clear. □

01O0 **Definition 27.16.7.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over S*, or the *homogeneous spectrum of $\mathcal{A}$ over S*, or the *relative Proj of $\mathcal{A}$ over S* is the scheme constructed in Lemma 27.15.4 which represents the functor F (27.16.4.1), see Lemma 27.16.6. We denote it $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$.

The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ (the twists of the structure sheaf) and a "universal" homomorphism of graded algebras

$$\psi_{univ} : \mathcal{A} \longrightarrow \pi_* \left(\bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n) \right)$$

see Lemma 27.15.5. We may also think of this as a homomorphism

$$\psi_{univ} : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$$

if we like. The following lemma is a formulation of the universality of this object.

01O1 **Lemma 27.16.8.** *In Situation 27.15.1. Let $(f : T \to S, d, \mathcal{L}, \psi)$ be a quadruple. Let $r_{d,\mathcal{L},\psi} : T \to \underline{Proj}_S(\mathcal{A})$ be the associated S-morphism. There exists an isomorphism of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras*

$$\theta : r^*_{d,\mathcal{L},\psi} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd) \right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$$

such that the following diagram commutes

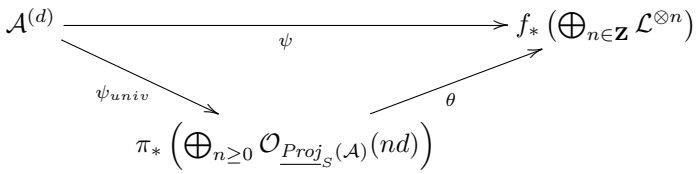

The commutativity of this diagram uniquely determines θ.

Proof. Note that the quadruple $(f : T \to S, d, \mathcal{L}, \psi)$ defines an element of $F_d(T)$. Let $U_d \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ be the locus where the sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(d)$ is invertible and generated by the image of $\psi_{univ} : \pi^*\mathcal{A}_d \to \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(d)$. Recall that U_d represents the functor F_d, see the proof of Lemma 27.16.5. Hence the result will follow if we can show the quadruple $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal family, i.e., the representing object in $F_d(U_d)$. We may do this after restricting to an affine open of S because (a) the formation of the functors F_d commutes with base change (see Lemma 27.16.1), and (b) the pair $(\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n), \psi_{univ})$ is constructed by glueing over affine opens in S (see Lemma 27.15.5). Hence we may assume that S is affine. In this case the functor of quadruples F_d and the functor of triples F_d agree (see proof of Lemma 27.16.2) and moreover Lemma 27.12.2 shows that $(d, \mathcal{O}_{U_d}(d), \psi^d)$ is the universal triple over U_d. Going backwards through the identifications in the proof of Lemma 27.16.2 shows that $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal quadruple as desired. □

01O2 **Lemma 27.16.9.** *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ is separated.*

Proof. To prove a morphism is separated we may work locally on the base, see Schemes, Section 26.21. By construction $\underline{\mathrm{Proj}}_S(\mathcal{A})$ is over any affine $U \subset S$ isomorphic to $\mathrm{Proj}(A)$ with $A = \mathcal{A}(U)$. By Lemma 27.8.8 we see that $\mathrm{Proj}(A)$ is separated. Hence $\mathrm{Proj}(A) \to U$ is separated (see Schemes, Lemma 26.21.13) as desired. □

01O3 **Lemma 27.16.10.** *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $g : S' \to S$ be any morphism of schemes. Then there is a canonical isomorphism*

$$r : \underline{Proj}_{S'}(g^*\mathcal{A}) \longrightarrow S' \times_S \underline{Proj}_S(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^* pr_2^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}$-algebras.

Proof. This follows from Lemma 27.16.1 and the construction of $\underline{\mathrm{Proj}}_S(\mathcal{A})$ in Lemma 27.16.5 as the union of the schemes U_d representing the functors F_d. In terms of the construction of relative Proj via glueing this isomorphism is given by the isomorphisms constructed in Lemma 27.11.6 which provides us with the isomorphism θ. Some details omitted. □

01O4 **Lemma 27.16.11.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X = \underline{Proj}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where*

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and*
(2) *$\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective*

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{Proj}(\mathcal{A})}$-modules and the multiplication maps induce isomorphisms $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{Proj}(\mathcal{A})}} \mathcal{O}_{\underline{Proj}(\mathcal{A})}(m) = \mathcal{O}_{\underline{Proj}(\mathcal{A})}(n+m)$.

Proof. Under the assumptions of the lemma the sheaves $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n)$ are invertible and the multiplication maps isomorphisms by Lemma 27.16.5 and Lemma 27.12.3 over affine opens of S. Thus X actually represents the functor F_1, see proof of Lemma 27.16.5. □

27.17. Quasi-coherent sheaves on relative Proj

01O5 We briefly discuss how to deal with graded modules in the relative setting.

We place ourselves in Situation 27.15.1. So S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{M} = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}_n$ be a graded $\mathcal{A}$-module, quasi-coherent as an $\mathcal{O}_S$-module. We are going to describe the associated quasi-coherent sheaf of modules on $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We first describe the value of this sheaf on schemes T mapping into the relative Proj.

Let T be a scheme. Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple over T, as in Section 27.16. We define a quasi-coherent sheaf $\widetilde{\mathcal{M}}_T$ of $\mathcal{O}_T$-modules as follows

01O6 (27.17.0.1)
$$\widetilde{\mathcal{M}}_T = \left(f^*\mathcal{M}^{(d)} \otimes_{f^*\mathcal{A}^{(d)}} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) \right)_0$$

So $\widetilde{\mathcal{M}}_T$ is the degree 0 part of the tensor product of the graded $f^*\mathcal{A}^{(d)}$-modules $\mathcal{M}^{(d)}$ and $\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that the sheaf $\widetilde{\mathcal{M}}_T$ depends on the quadruple even though we suppressed this in the notation. This construction has the pleasing property that given any morphism $g : T' \to T$ we have $\widetilde{\mathcal{M}}_{T'} = g^*\widetilde{\mathcal{M}}_T$ where $\widetilde{\mathcal{M}}_{T'}$ denotes the quasi-coherent sheaf associated to the pullback quadruple $(d, f \circ g, g^*\mathcal{L}, g^*\psi)$.

Since all sheaves in (27.17.0.1) are quasi-coherent we can spell out the construction over an affine open $\mathrm{Spec}(C) = V \subset T$ which maps into an affine open $\mathrm{Spec}(R) = U \subset S$. Namely, suppose that $\mathcal{A}|_U$ corresponds to the graded R-algebra A, that $\mathcal{M}|_U$ corresponds to the graded A-module M, and that $\mathcal{L}|_V$ corresponds to the invertible C-module L. The map ψ gives rise to a graded R-algebra map $\gamma : A^{(d)} \to \bigoplus_{n\geq 0} L^{\otimes n}$. (Tensor powers of L over C.) Then $(\widetilde{\mathcal{M}}_T)|_V$ is the quasi-coherent sheaf associated to the C-module

$$N_{R,C,A,M,\gamma} = \left(M^{(d)} \otimes_{A^{(d)},\gamma} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} L^{\otimes n}\right)\right)_0$$

By assumption we may even cover T by affine opens V such that there exists some $a \in A_d$ such that $\gamma(a) \in L$ is a C-basis for the module L. In that case any element of $N_{R,C,A,M,\gamma}$ is a sum of pure tensors $\sum m_i \otimes \gamma(a)^{-n_i}$ with $m_i \in M_{n_i d}$. In fact we may multiply each m_i with a suitable positive power of a and collect terms to see that each element of $N_{R,C,A,M,\gamma}$ can be written as $m \otimes \gamma(a)^{-n}$ with $m \in M_{nd}$ and $n \gg 0$. In other words we see that in this case

$$N_{R,C,A,M,\gamma} = M_{(a)} \otimes_{A_{(a)}} C$$

where the map $A_{(a)} \to C$ is the map $x/a^n \mapsto \gamma(x)/\gamma(a)^n$. In other words, this is the value of $\widetilde{M}$ on $D_+(a) \subset \mathrm{Proj}(A)$ pulled back to $\mathrm{Spec}(C)$ via the morphism $\mathrm{Spec}(C) \to D_+(a)$ coming from γ.

01O7 **Lemma 27.17.1.** *In Situation 27.15.1. For any quasi-coherent sheaf of graded $\mathcal{A}$-modules $\mathcal{M}$ on S, there exists a canonical associated sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-modules $\widetilde{\mathcal{M}}$ with the following properties:*

(1) *Given a scheme T and a quadruple $(T \to S, d, \mathcal{L}, \psi)$ over T corresponding to a morphism $h : T \to \underline{Proj}_S(\mathcal{A})$ there is a canonical isomorphism $\widetilde{\mathcal{M}}_T = h^*\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}_T$ is defined by (27.17.0.1).*
(2) *The isomorphisms of (1) are compatible with pullbacks.*
(3) *There is a canonical map*
$$\pi^*\mathcal{M}_0 \longrightarrow \widetilde{\mathcal{M}}.$$
(4) *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is functorial in $\mathcal{M}$.*
(5) *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is exact.*
(6) *There are canonical maps*
$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}} \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$
as in Lemma 27.9.1.
(7) *There exist canonical maps*
$$\pi^*\mathcal{M} \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \widetilde{\mathcal{M}(n)}$$
generalizing (27.10.1.6).
(8) *The formation of $\widetilde{\mathcal{M}}$ commutes with base change.*

Proof. Omitted. We should split this lemma into parts and prove the parts separately. □

27.18. Functoriality of relative Proj

07ZF This section is the analogue of Section 27.11 for the relative Proj. Let S be a scheme. A graded $\mathcal{O}_S$-algebra map $\psi : \mathcal{A} \to \mathcal{B}$ does not always give rise to a morphism of associated relative Proj. The correct result is stated as follows.

07ZG **Lemma 27.18.1.** *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$ be two graded quasi-coherent $\mathcal{O}_S$-algebras. Set $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ and $q : Y = \underline{Proj}_S(\mathcal{B}) \to S$. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of graded $\mathcal{O}_S$-algebras. There is a canonical open $U(\psi) \subset Y$ and a canonical morphism of schemes*

$$r_\psi : U(\psi) \longrightarrow X$$

over S and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any affine open $W \subset S$ the triple

$$(U(\psi) \cap p^{-1}W, \quad r_\psi|_{U(\psi)\cap p^{-1}W} : U(\psi) \cap p^{-1}W \to q^{-1}W, \quad \theta|_{U(\psi)\cap p^{-1}W})$$

is equal to the triple associated to $\psi : \mathcal{A}(W) \to \mathcal{B}(W)$ in Lemma 27.11.1 via the identifications $p^{-1}W = Proj(\mathcal{A}(W))$ and $q^{-1}W = Proj(\mathcal{B}(W))$ of Section 27.15.

Proof. This lemma proves itself by glueing the local triples. □

07ZH **Lemma 27.18.2.** *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_S$-algebras. Set $X = \underline{Proj}_S(\mathcal{A})$, $Y = \underline{Proj}_S(\mathcal{B})$ and $Z = \underline{Proj}_S(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_S$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

07ZI **Lemma 27.18.3.** *With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).*

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.3. □

07ZJ **Lemma 27.18.4.** *With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.4. □

07ZK **Lemma 27.18.5.** *With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

(1) *$U(\psi) = Y$,*
(2) *$r_\psi : Y \to X$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.5. □

27.19. Invertible sheaves and morphisms into relative Proj

01O8 It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra.
(3) Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the relative homogeneous spectrum over S.
(4) Let $f : X \to S$ be a morphism of schemes.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data set

$$U(\psi) = \bigcup\nolimits_{(U,V,a)} U_{\psi(a)}$$

where (U, V, a) satisfies:

(1) $V \subset S$ affine open,
(2) $U = f^{-1}(V)$, and
(3) $a \in \mathcal{A}(V)_+$ is homogeneous.

Namely, then $\psi(a) \in \Gamma(U, \mathcal{L}^{\otimes \deg(a)})$ and $U_{\psi(a)}$ is the corresponding open (see Modules, Lemma 17.25.10).

01O9 **Lemma 27.19.1.** *With assumptions and notation as above. The morphism ψ induces a canonical morphism of schemes over S*

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{Proj}_S(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r_{\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d) \right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) *For every open $V \subset S$ and every $d \geq 0$ the diagram*

$$\begin{array}{ccc} \mathcal{A}_d(V) & \xrightarrow{\psi} & \Gamma(f^{-1}(V), \mathcal{L}^{\otimes d}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle restrict} \\ \Gamma(\pi^{-1}(V), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)) & \xrightarrow{\theta} & \Gamma(f^{-1}(V) \cap U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

(2) *For any $d \geq 1$ and any open subscheme $W \subset X$ such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective the restriction of the morphism $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{Proj}_S(\mathcal{A})$ which exists by the construction of the relative homogeneous spectrum, see Definition 27.16.7.*

(3) *For any affine open $V \subset S$, the restriction*

$$(U(\psi) \cap f^{-1}(V), r_{\mathcal{L},\psi}|_{U(\psi)\cap f^{-1}(V)}, \theta|_{U(\psi)\cap f^{-1}(V)})$$

agrees via i_V (see Lemma 27.15.4) with the triple $(U(\psi'), r_{\mathcal{L},\psi'}, \theta')$ of Lemma 27.14.1 associated to the map $\psi' : A = \mathcal{A}(V) \to \Gamma_*(f^{-1}(V), \mathcal{L}|_{f^{-1}(V)})$ induced by ψ.

Proof. Use characterization (3) to construct the morphism $r_{\mathcal{L},\psi}$ and θ locally over S. Use the uniqueness of Lemma 27.14.1 to show that the construction glues. Details omitted. □

27.20. Twisting by invertible sheaves and relative Proj

02NB Let S be a scheme. Let $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S. In this situation we obtain another quasi-coherent graded $\mathcal{O}_S$-algebra, namely

$$\mathcal{B} = \bigoplus\nolimits_{d\geq 0} \mathcal{A}_d \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes d}$$

It turns out that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic relative homogeneous spectra.

02NC **Lemma 27.20.1.** *With notation S, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{B}$ as above. There is a canonical isomorphism*

$$\begin{array}{ccc} P = \underline{Proj}_S(\mathcal{A}) & \xrightarrow{g} & \underline{Proj}_S(\mathcal{B}) = P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & S & \end{array}$$

with the following properties

(1) *There are isomorphisms $\theta_n : g^*\mathcal{O}_{P'}(n) \to \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$ which fit together to give an isomorphism of $\mathbf{Z}$-graded algebras*

$$\theta : g^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{P'}(n)\right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$$

(2) *For every open $V \subset S$ the diagrams*

$$\begin{array}{ccc} \mathcal{A}_n(V) \otimes \mathcal{L}^{\otimes n}(V) & \xrightarrow[multiply]{} & \mathcal{B}_n(V) \\ \downarrow{\scriptstyle\psi\otimes\pi^*} & & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n)) \otimes \Gamma(\pi^{-1}V, \pi^*\mathcal{L}^{\otimes n}) & & \downarrow{\scriptstyle\psi} \\ \downarrow{\scriptstyle multiply} & & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}) & \xleftarrow{\theta_n} & \Gamma(\pi'^{-1}V, \mathcal{O}_{P'}(n)) \end{array}$$

are commutative.

(3) *Add more here as necessary.*

Proof. This is the identity map when $\mathcal{L} \cong \mathcal{O}_S$. In general choose an open covering of S such that $\mathcal{L}$ is trivialized over the pieces and glue the corresponding maps. Details omitted. □

27.21. Projective bundles

01OA Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.21.6 the symmetric algebra $\text{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Note that it is generated in degree 1 over $\mathcal{O}_S$. Hence it makes sense to apply the construction of the previous section to it, specifically Lemmas 27.16.5 and 27.16.11.

01OB **Definition 27.21.1.** Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[3]. We denote

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and we call it the *projective bundle associated to* $\mathcal{E}$. The symbol $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$ indicates the invertible $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-module of Lemma 27.16.11 and is called the nth *twist of the structure sheaf*.

According to Lemma 27.15.5 there are canonical $\mathcal{O}_S$-module homomorphisms

$$\text{Sym}^n(\mathcal{E}) \longrightarrow \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \quad \text{equivalently} \quad \pi^*\text{Sym}^n(\mathcal{E}) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$$

for all $n \geq 0$. In particular, for $n = 1$ we have

$$\mathcal{E} \longrightarrow \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \quad \text{equivalently} \quad \pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and the map $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is a surjection by Lemma 27.16.11. This is a good way to remember how we have normalized our construction of $\mathbf{P}(\mathcal{E})$.

Warning: In some references the scheme $\mathbf{P}(\mathcal{E})$ is only defined for $\mathcal{E}$ finite locally free on S. Moreover sometimes $\mathbf{P}(\mathcal{E})$ is actually defined as our $\mathbf{P}(\mathcal{E}^\vee)$ where $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$ (and this is done only when $\mathcal{E}$ is finite locally free).

Let S, $\mathcal{E}$, $\mathbf{P}(\mathcal{E}) \to S$ be as in Definition 27.21.1. Let $f : T \to S$ be a scheme over S. Let $\psi : f^*\mathcal{E} \to \mathcal{L}$ be a surjection where $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module. The induced graded $\mathcal{O}_T$-algebra map

$$f^*\text{Sym}(\mathcal{E}) = \text{Sym}(f^*\mathcal{E}) \to \text{Sym}(\mathcal{L}) = \bigoplus\nolimits_{n\geq 0} \mathcal{L}^{\otimes n}$$

corresponds to a morphism

$$\varphi_{\mathcal{L},\psi} : T \longrightarrow \mathbf{P}(\mathcal{E})$$

over S by our construction of the relative Proj as the scheme representing the functor F in Section 27.16. On the other hand, given a morphism $\varphi : T \to \mathbf{P}(\mathcal{E})$ over S we can set $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\psi : f^*\mathcal{E} \to \mathcal{L}$ equal to the pullback by φ of the canonical surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By Lemma 27.16.11 these constructions are inverse bijections between the set of isomorphism classes of pairs $(\mathcal{L}, \psi)$ and the set of morphisms $\varphi : T \to \mathbf{P}(\mathcal{E})$ over S. Thus we see that $\mathbf{P}(\mathcal{E})$ represents the functor which associates to $f : T \to S$ the set of $\mathcal{O}_T$-module quotients of $f^*\mathcal{E}$ which are locally free of rank 1.

0FCY **Example 27.21.2** (Projective space of a vector space)**.** Let k be a field. Let V be a k-vector space. The corresponding *projective space* is the k-scheme

$$\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$$

where $\text{Sym}(V)$ is the symmetric algebra on V over k. Of course we have $\mathbf{P}(V) \cong \mathbf{P}^n_k$ if $\dim(V) = n + 1$ because then the symmetric algebra on V is isomorphic to a

[3]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 4.1.1].

polynomial ring in $n+1$ variables. If we think of V as a quasi-coherent module on $\mathrm{Spec}(k)$, then $\mathbf{P}(V)$ is the corresponding projective space bundle over $\mathrm{Spec}(k)$. By the discussion above a k-valued point p of $\mathbf{P}(V)$ corresponds to a surjection of k-vector spaces $V \to L_p$ with $\dim(L_p) = 1$. More generally, let X be a scheme over k, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let $\psi : V \to \Gamma(X, \mathcal{L})$ be a k-linear map such that $\mathcal{L}$ is generated as an $\mathcal{O}_X$-module by the sections in the image of ψ. Then the discussion above gives a canonical morphism

$$\varphi_{\mathcal{L},\psi} : X \longrightarrow \mathbf{P}(V)$$

of schemes over k such that there is an isomorphism $\theta : \varphi_{\mathcal{L},\psi}^*\mathcal{O}_{\mathbf{P}(V)}(1) \to \mathcal{L}$ and such that ψ agrees with the composition

$$V \to \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \to \Gamma(X, \varphi_{\mathcal{L},\psi}^*\mathcal{O}_{\mathbf{P}(V)}(1)) \to \Gamma(X, \mathcal{L})$$

See Lemma 27.14.1. If $V \subset \Gamma(X, \mathcal{L})$ is a subspace, then we will denote the morphism constructed above simply as $\varphi_{\mathcal{L},V}$. If $\dim(V) = n+1$ and we choose a basis $v_0, \ldots, v_n$ of V then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{\mathcal{L},\psi}} & \mathbf{P}(V) \\ \| & & \downarrow \cong \\ X & \xrightarrow{\varphi_{(\mathcal{L},(s_0,\ldots,s_n))}} & \mathbf{P}^n_k \end{array}$$

is commutative, where $s_i = \psi(v_i) \in \Gamma(X, \mathcal{L})$, where $\varphi_{(\mathcal{L},(s_0,\ldots,s_n))}$ is as in Section 27.13, and where the right vertical arrow corresponds to the isomorphism $k[T_0, \ldots, T_n] \to \mathrm{Sym}(V)$ sending T_i to v_i.

01OC **Example 27.21.3.** The map $\mathrm{Sym}^n(\mathcal{E}) \to \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$ is an isomorphism if $\mathcal{E}$ is locally free, but in general need not be an isomorphism. In fact we will give an example where this map is not injective for $n = 1$. Set $S = \mathrm{Spec}(A)$ with

$$A = k[u, v, s_1, s_2, t_1, t_2]/I$$

where k is a field and

$$I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1).$$

Denote $\overline{u}$ the class of u in A and similarly for the other variables. Let $M = (Ax \oplus Ay)/A(\overline{u}x + \overline{v}y)$ so that

$$\mathrm{Sym}(M) = A[x, y]/(\overline{u}x + \overline{v}y) = k[x, y, u, v, s_1, s_2, t_1, t_2]/J$$

where

$$J = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1, ux + vy).$$

In this case the projective bundle associated to the quasi-coherent sheaf $\mathcal{E} = \widetilde{M}$ on $S = \mathrm{Spec}(A)$ is the scheme

$$P = \mathrm{Proj}(\mathrm{Sym}(M)).$$

Note that this scheme as an affine open covering $P = D_+(x) \cup D_+(y)$. Consider the element $m \in M$ which is the image of the element $us_1x + vt_2y$. Note that

$$x(us_1x + vt_2y) = (s_1x + s_2y)(ux + vy) \bmod I$$

and

$$y(us_1x + vt_2y) = (t_1x + t_2y)(ux + vy) \bmod I.$$

The first equation implies that m maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(x)$ and the second that it maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(y)$. This shows that m maps to zero in $\Gamma(P, \mathcal{O}_P(1))$. On the other hand we claim that $m \neq 0$, so that m gives an example of a nonzero global section of $\mathcal{E}$ mapping to zero in $\Gamma(P, \mathcal{O}_P(1))$. Assume $m = 0$ to get a contradiction. In this case there exists an element $f \in k[u, v, s_1, s_2, t_1, t_2]$ such that

$$us_1x + vt_2y = f(ux + vy) \bmod I$$

Since I is generated by homogeneous polynomials of degree 2 we may decompose f into its homogeneous components and take the degree 1 component. In other words we may assume that

$$f = au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2$$

for some $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. The resulting conditions are that

$$\begin{aligned} us_1 - u(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \\ vt_2 - v(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \end{aligned}$$

There are no terms u^2, uv, v^2 in the generators of I and hence we see $a = b = 0$. Thus we get the relations

$$\begin{aligned} us_1 - u(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \\ vt_2 - v(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \end{aligned}$$

We may use the first generator of I to replace any occurrence of us_1 by $vt_1 + ut_2$, the second generator of I to replace any occurrence of vs_1 by $-us_2 + vt_2$, the third generator to remove occurrences of vs_2 and the third to remove occurrences of ut_1. Then we get the relations

$$\begin{aligned} (1 - \alpha_1)vt_1 + (1 - \alpha_1)ut_2 - \alpha_2 us_2 - \beta_2 ut_2 = 0 \\ (1 - \alpha_1)vt_2 + \alpha_1 us_2 - \beta_1 vt_1 - \beta_2 vt_2 = 0 \end{aligned}$$

This implies that α_1 should be both 0 and 1 which is a contradiction as desired.

01OD **Lemma 27.21.4.** *Let S be a scheme. The structure morphism $\mathbf{P}(\mathcal{E}) \to S$ of a projective bundle over S is separated.*

Proof. Immediate from Lemma 27.16.9. □

01OE **Lemma 27.21.5.** *Let S be a scheme. Let $n \geq 0$. Then $\mathbf{P}^n_S$ is a projective bundle over S.*

Proof. Note that

$$\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n]) = \underline{\mathrm{Proj}}_{\mathrm{Spec}(\mathbf{Z})} \left(\widetilde{\mathbf{Z}[T_0, \ldots, T_n]} \right)$$

where the grading on the ring $\mathbf{Z}[T_0, \ldots, T_n]$ is given by $\deg(T_i) = 1$ and the elements of $\mathbf{Z}$ are in degree 0. Recall that $\mathbf{P}^n_S$ is defined as $\mathbf{P}^n_{\mathbf{Z}} \times_{\mathrm{Spec}(\mathbf{Z})} S$. Moreover, forming the relative homogeneous spectrum commutes with base change, see Lemma 27.16.10. For any scheme $g : S \to \mathrm{Spec}(\mathbf{Z})$ we have $g^*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})}[T_0, \ldots, T_n] = \mathcal{O}_S[T_0, \ldots, T_n]$. Combining the above we see that

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

Finally, note that $\mathcal{O}_S[T_0, \ldots, T_n] = \mathrm{Sym}(\mathcal{O}_S^{\oplus n+1})$. Hence we see that $\mathbf{P}^n_S$ is a projective bundle over S. □

27.22. Grassmannians

089R In this section we introduce the standard Grassmannian functors and we show that they are represented by schemes. Pick integers k, n with $0 < k < n$. We will construct a functor

089S (27.22.0.1) $$G(k,n) : Sch \longrightarrow Sets$$

which will loosely speaking parametrize k-dimensional subspaces of n-space. However, for technical reasons it is more convenient to parametrize $(n-k)$-dimensional quotients and this is what we will do.

More precisely, $G(k,n)$ associates to a scheme S the set $G(k,n)(S)$ of isomorphism classes of surjections

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$$

where $\mathcal{Q}$ is a finite locally free $\mathcal{O}_S$-module of rank $n-k$. Note that this is indeed a set, for example by Modules, Lemma 17.9.8 or by the observation that the isomorphism class of the surjection q is determined by the kernel of q (and given a sheaf there is a set of subsheaves). Given a morphism of schemes $f : T \to S$ we let $G(k,n)(f) : G(k,n)(S) \to G(k,n)(T)$ which sends the isomorphism class of $q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$ to the isomorphism class of $f^*q : \mathcal{O}_T^{\oplus n} \longrightarrow f^*\mathcal{Q}$. This makes sense since (1) $f^*\mathcal{O}_S = \mathcal{O}_T$, (2) f^* is additive, (3) f^* preserves locally free modules (Modules, Lemma 17.14.3), and (4) f^* is right exact (Modules, Lemma 17.3.3).

089T **Lemma 27.22.1.** *Let $0 < k < n$. The functor $G(k,n)$ of (27.22.0.1) is representable by a scheme.*

Proof. Set $F = G(k,n)$. To prove the lemma we will use the criterion of Schemes, Lemma 26.15.4. The reason F satisfies the sheaf property for the Zariski topology is that we can glue sheaves, see Sheaves, Section 6.33 (some details omitted).

The family of subfunctors F_i. Let I be the set of subsets of $\{1,\ldots,n\}$ of cardinality $n-k$. Given a scheme S and $j \in \{1,\ldots,n\}$ we denote e_j the global section

$$e_j = (0,\ldots,0,1,0,\ldots,0) \quad (1 \text{ in } j\text{th spot})$$

of $\mathcal{O}_S^{\oplus n}$. Of course these sections freely generate $\mathcal{O}_S^{\oplus n}$. Similarly, for $j \in \{1,\ldots,n-k\}$ we denote f_j the global section of $\mathcal{O}_S^{\oplus n-k}$ which is zero in all summands except the jth where we put a 1. For $i \in I$ we let

$$s_i : \mathcal{O}_S^{\oplus n-k} \longrightarrow \mathcal{O}_S^{\oplus n}$$

which is the direct sum of the coprojections $\mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ corresponding to elements of I. More precisely, if $i = \{i_1,\ldots,i_{n-k}\}$ with $i_1 < i_2 < \ldots < i_{n-k}$ then s_i maps f_j to e_{i_j} for $j \in \{1,\ldots,n-k\}$. With this notation we can set

$$F_i(S) = \{q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q} \in F(S) \mid q \circ s_i \text{ is surjective}\} \subset F(S)$$

Given a morphism $f : T \to S$ of schemes the pullback f^*s_i is the corresponding map over T. Since f^* is right exact (Modules, Lemma 17.3.3) we conclude that F_i is a subfunctor of F.

Representability of F_i. To prove this we may assume (after renumbering) that $i = \{1,\ldots,n-k\}$. This means s_i is the inclusion of the first $n-k$ summands. Observe that if $q \circ s_i$ is surjective, then $q \circ s_i$ is an isomorphism as a surjective map between finite locally free modules of the same rank (Modules, Lemma 17.14.5).

Thus if $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ is an element of $F_i(S)$, then we can use $q \circ s_i$ to identify $\mathcal{Q}$ with $\mathcal{O}_S^{\oplus n-k}$. After doing so we obtain

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus n-k}$$

mapping e_j to f_j (notation as above) for $j = 1, \ldots, n-k$. To determine q completely we have to fix the images $q(e_{n-k+1}), \ldots, q(e_n)$ in $\Gamma(S, \mathcal{O}_S^{\oplus n-k})$. It follows that F_i is isomorphic to the functor

$$S \longmapsto \prod\nolimits_{j=n-k+1,\ldots,n} \Gamma(S, \mathcal{O}_S^{\oplus n-k})$$

This functor is isomorphic to the $k(n-k)$-fold self product of the functor $S \mapsto \Gamma(S, \mathcal{O}_S)$. By Schemes, Example 26.15.2 the latter is representable by $\mathbf{A}^1_{\mathbf{Z}}$. It follows F_i is representable by $\mathbf{A}^{k(n-k)}_{\mathbf{Z}}$ since fibred product over $\mathrm{Spec}(\mathbf{Z})$ is the product in the category of schemes.

The inclusion $F_i \subset F$ is representable by open immersions. Let S be a scheme and let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. By Modules, Lemma 17.9.4. the set $U_i = \{s \in S \mid (q \circ s_i)_s \text{ surjective}\}$ is open in S. Since $\mathcal{O}_{S,s}$ is a local ring and $\mathcal{Q}_s$ a finite $\mathcal{O}_{S,s}$-module by Nakayama's lemma (Algebra, Lemma 10.20.1) we have

$$s \in U_i \Leftrightarrow \big(\text{the map } \kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s \text{ induced by } (q \circ s_i)_s \text{ is surjective}\big)$$

Let $f : T \to S$ be a morphism of schemes and let $t \in T$ be a point mapping to $s \in S$. We have $(f^*\mathcal{Q})_t = \mathcal{Q}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}$ (Sheaves, Lemma 6.26.4) and so on. Thus the map

$$\kappa(t)^{\oplus n-k} \to (f^*\mathcal{Q})_t/\mathfrak{m}_t(f^*\mathcal{Q})_t$$

induced by $(f^*q \circ f^*s_i)_t$ is the base change of the map $\kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ above by the field extension $\kappa(t)/\kappa(s)$. It follows that $s \in U_i$ if and only if t is in the corresponding open for f^*q. In particular $T \to S$ factors through U_i if and only if $f^*q \in F_i(T)$ as desired.

The collection F_i, $i \in I$ covers F. Let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. We have to show that for every point s of S there exists an $i \in I$ such that s_i is surjective in a neighbourhood of s. Thus we have to show that one of the compositions

$$\kappa(s)^{\oplus n-k} \xrightarrow{s_i} \kappa(s)^{\oplus n} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$$

is surjective (see previous paragraph). As $\mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ is a vector space of dimension $n-k$ this follows from the theory of vector spaces. □

089U **Definition 27.22.2.** Let $0 < k < n$. The scheme $\mathbf{G}(k, n)$ representing the functor $G(k, n)$ is called *Grassmannian over* $\mathbf{Z}$. Its base change $\mathbf{G}(k, n)_S$ to a scheme S is called *Grassmannian over* S. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{G}(k, n)_R$ and called *Grassmannian over* R.

The definition makes sense as we've shown in Lemma 27.22.1 that these functors are indeed representable.

089V **Lemma 27.22.3.** *Let $n \geq 1$. There is a canonical isomorphism $\mathbf{G}(n, n+1) = \mathbf{P}^n_{\mathbf{Z}}$.*

Proof. According to Lemma 27.13.1 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ represents the functor which assigns to a scheme S the set of isomorphisms classes of pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ consisting of an invertible module $\mathcal{L}$ and an $(n+1)$-tuple of global sections generating $\mathcal{L}$. Given such a pair we obtain a quotient

$$\mathcal{O}_S^{\oplus n+1} \longrightarrow \mathcal{L}, \quad (h_0, \ldots, h_n) \longmapsto \sum h_i s_i.$$

Conversely, given an element $q : \mathcal{O}_S^{\oplus n+1} \to \mathcal{Q}$ of $G(n, n+1)(S)$ we obtain such a pair, namely $(\mathcal{Q}, (q(e_1), \ldots, q(e_{n+1})))$. Here e_i, $i = 1, \ldots, n+1$ are the standard generating sections of the free module $\mathcal{O}_S^{\oplus n+1}$. We omit the verification that these constructions define mutually inverse transformations of functors. □

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CHAPTER 28

Properties of Schemes

01OH

28.1. Introduction

01OI In this chapter we introduce some absolute properties of schemes. A foundational reference is [**DG67**].

28.2. Constructible sets

054B Constructible and locally constructible sets are introduced in Topology, Section 5.15. We may characterize locally constructible subsets of schemes as follows.

054C **Lemma 28.2.1.** *Let X be a scheme. A subset E of X is locally constructible in X if and only if $E \cap U$ is constructible in U for every affine open U of X.*

Proof. Assume E is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in U_i for each i. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each j we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 5.15.4 we see that each $E \cap V_j$ is constructible in V_j. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 26.19.2) we conclude that $E \cap V$ is constructible in V by Topology, Lemma 5.15.6. The converse implication is immediate. □

0AAW **Lemma 28.2.2.** *Let X be a scheme and let $E \subset X$ be a locally constructible subset. Let $\xi \in X$ be a generic point of an irreducible component of X.*

(1) *If $\xi \in E$, then an open neighbourhood of ξ is contained in E.*
(2) *If $\xi \notin E$, then an open neighbourhood of ξ is disjoint from E.*

Proof. As the complement of a locally constructible subset is locally constructible it suffices to show (2). We may assume X is affine and hence E constructible (Lemma 28.2.1). In this case X is a spectral space (Algebra, Lemma 10.26.2). Then $\xi \notin E$ implies $\xi \notin \overline{E}$ by Topology, Lemma 5.23.6 and the fact that there are no points of X different from ξ which specialize to ξ. □

054D **Lemma 28.2.3.** *Let X be a quasi-separated scheme. The intersection of any two quasi-compact opens of X is a quasi-compact open of X. Every quasi-compact open of X is retrocompact in X.*

Proof. If U and V are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As X is quasi-separated we see that Δ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 26.17.4 to see $U \times V$ is a finite union of affines). The other assertions follow from the first and Topology, Lemma 5.27.1. □

094L **Lemma 28.2.4.** *Let X be a quasi-compact and quasi-separated scheme. Then the underlying topological space of X is a spectral space.*

Proof. By Topology, Definition 5.23.1 we have to check that X is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. This follows from Schemes, Lemma 26.11.1 and 26.11.2 and Lemma 28.2.3 above. □

054E **Lemma 28.2.5.** *Let X be a quasi-compact and quasi-separated scheme. Any locally constructible subset of X is constructible.*

Proof. As X is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As X is quasi-separated each V_i is retrocompact in X by Lemma 28.2.3. Hence by Topology, Lemma 5.15.6 we see that $E \subset X$ is constructible in X if and only if $E \cap V_j$ is constructible in V_j. Thus we win by Lemma 28.2.1. □

07ZL **Lemma 28.2.6.** *Let X be a scheme. A subset E of X is retrocompact in X if and only if $E \cap U$ is quasi-compact for every affine open U of X.*

Proof. Immediate from the fact that every quasi-compact open of X is a finite union of affine opens. □

0F2M **Lemma 28.2.7.** *A partition $X = \coprod_{i \in I} X_i$ of a scheme X with retrocompact parts is locally finite if and only if the parts are locally constructible.*

Proof. See Topology, Definitions 5.12.1, 5.28.1, and 5.28.4 for the definitions of retrocompact, partition, and locally finite.

If the partition is locally finite and $U \subset X$ is an affine open, then we see that $U = \coprod_{i \in I} U \cap X_i$ is a finite partition (more precisely, all but a finite number of its parts are empty). Hence $U \cap X_i$ is quasi-compact and its complement is retrocompact in U as a finite union of retrocompact parts. Thus $U \cap X_i$ is constructible by Topology, Lemma 5.15.13. It follows that X_i is locally constructible by Lemma 28.2.1.

Assume the parts are locally constructible. Then for any affine open $U \subset X$ we obtain a covering $U = \coprod X_i \cap U$ by constructible subsets. Since the constructible topology is quasi-compact, see Topology, Lemma 5.23.2, this covering has a finite refinement, i.e., the partition is locally finite. □

28.3. Integral, irreducible, and reduced schemes

01OJ

01OK **Definition 28.3.1.** Let X be a scheme. We say X is *integral* if it is nonempty and for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is an integral domain.

01OL **Lemma 28.3.2.** *Let X be a scheme. The following are equivalent.*

(1) *The scheme X is reduced, see Schemes, Definition 26.12.1.*
(2) *There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
(3) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*
(4) *For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*

Proof. See Schemes, Lemmas 26.12.2 and 26.12.3. □

01OM **Lemma 28.3.3.** *Let X be a scheme. The following are equivalent.*

(1) *The scheme X is irreducible.*

(2) *There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.*
(3) *The scheme X is nonempty and every nonempty affine open $U \subset X$ is irreducible.*

Proof. Assume (1). By Schemes, Lemma 26.11.1 we see that X has a unique generic point η. Then $X = \overline{\{\eta\}}$. Hence η is an element of every nonempty affine open $U \subset X$. This implies that $\eta \in U$ is dense hence U is irreducible. It also implies any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every i we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering Z_1, Z_2). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each U_i is nonempty. Since X is nonempty we see that I is not empty. By assumption each U_i is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \amalg U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 26.6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. □

01ON **Lemma 28.3.4.** *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. If X is irreducible, then every affine open $\mathrm{Spec}(R) = U \subset X$ is irreducible. If X is reduced, then R is reduced, by Lemma 28.3.2 above. Hence R is reduced and (0) is a prime ideal, i.e., R is an integral domain.

If X is integral, then for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is reduced and hence X is reduced by Lemma 28.3.2. Moreover, every nonempty affine open is irreducible. Hence X is irreducible, see Lemma 28.3.3. □

In Examples, Section 110.6 we construct a connected affine scheme all of whose local rings are domains, but which is not integral.

28.4. Types of schemes defined by properties of rings

01OO In this section we study what properties of rings allow one to define local properties of schemes.

01OP **Definition 28.4.1.** Let P be a property of rings. We say that P is *local* if the following hold:

(1) For any ring R, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring R, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

01OQ **Definition 28.4.2.** Let P be a property of rings. Let X be a scheme. We say X is *locally P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has property P.

This is only a good notion if the property is local. Even if P is a local property we will not automatically use this definition to say that a scheme is "locally P" unless we also explicitly state the definition elsewhere.

01OR **Lemma 28.4.3.** *Let X be a scheme. Let P be a local property of rings. The following are equivalent:*

(1) *The scheme X is locally P.*
(2) *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.*

Moreover, if X is locally P then every open subscheme is locally P.

Proof. Of course (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). If (3) $\Rightarrow$ (2), then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show (3) $\Rightarrow$ (2).

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = \mathrm{Spec}(R_i)$. Assume $P(R_i)$. Let $\mathrm{Spec}(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 26.11.6 there exists a standard covering of $U = \mathrm{Spec}(R)$ by standard opens $D(f_j)$ such that each ring R_{f_j} is a principal localization of one of the rings R_i. By Definition 28.4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 28.4.1 (2). □

Here is a sample application.

01OS **Lemma 28.4.4.** *Let X be a scheme. Then X is reduced if and only if X is "locally reduced" in the sense of Definition 28.4.2.*

Proof. This is clear from Lemma 28.3.2. □

01OT **Lemma 28.4.5.** *The following properties of a ring R are local.*

(1) *(Cohen-Macaulay.) The ring R is Noetherian and CM, see Algebra, Definition 10.104.6.*
(2) *(Regular.) The ring R is Noetherian and regular, see Algebra, Definition 10.110.7.*
(3) *(Absolutely Noetherian.) The ring R is of finite type over Z.*
(4) *Add more here as needed.*[1]

Proof. Omitted. □

28.5. Noetherian schemes

01OU Recall that a ring R is *Noetherian* if it satisfies the ascending chain condition of ideals. Equivalently every ideal of R is finitely generated.

01OV **Definition 28.5.1.** Let X be a scheme.

(1) We say X is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is Noetherian.
(2) We say X is *Noetherian* if X is locally Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

01OW **Lemma 28.5.2.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is locally Noetherian.*

[1] But we only list those properties here which we have not already dealt with separately somewhere else.

(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally Noetherian.*

Moreover, if X is locally Noetherian then every open subscheme is locally Noetherian.

Proof. To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 28.4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 10.31.1. By Algebra, Lemma 10.23.2 we see the second property to Definition 28.4.1. □

01OX **Lemma 28.5.3.** *Any immersion $Z \to X$ with X locally Noetherian is quasi-compact.*

Proof. A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 5.12.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 26.19.2 we reduce to the case where X is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 10.31.5 and Topology, Lemmas 5.9.2 and 5.12.13). □

01OY **Lemma 28.5.4.** *A locally Noetherian scheme is quasi-separated.*

Proof. By Schemes, Lemma 26.21.6 we have to show that the intersection $U \cap V$ of two affine opens of X is quasi-compact. This follows from Lemma 28.5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) □

01OZ **Lemma 28.5.5.** *A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 5.9.1.*

Proof. This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 10.31.5 and Topology, Lemma 5.9.4. □

02IK **Lemma 28.5.6.** *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

Proof. Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). □

0BA8 **Lemma 28.5.7.** *A Noetherian scheme has a finite number of irreducible components.*

Proof. The underlying topological space of a Noetherian scheme is Noetherian (Lemma 28.5.5) and we conclude because a Noetherian topological space has only finitely many irreducible components (Topology, Lemma 5.9.2). □

01P0 **Lemma 28.5.8.** *Any morphism of schemes $f : X \to Y$ with X Noetherian is quasi-compact.*

Proof. Use Lemma 28.5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas 5.9.2 and 5.12.13). □

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

02IL **Lemma 28.5.9.** *Any nonempty locally Noetherian scheme has a closed point. Any nonempty closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.*

Proof. The second assertion follows from the first (using Schemes, Lemma 26.12.4 and Lemma 28.5.6). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If x is a closed point of X then we are done. If not, let $X_0 \subset X$ be the reduced induced closed subscheme structure on $\overline{\{x\}}$. Then $U_0 = U \cap X_0$ is an affine open of X_0 by Schemes, Lemma 26.10.1 and $U_0 = \{x\}$. Let $y \in X_0$, $y \neq x$ be a specialization of x. Consider the local ring $R = \mathcal{O}_{X_0,y}$. This is a Noetherian local ring as X_0 is Noetherian by Lemma 28.5.6. Denote $V \subset \mathrm{Spec}(R)$ the inverse image of U_0 in $\mathrm{Spec}(R)$ by the canonical morphism $\mathrm{Spec}(R) \to X_0$ (see Schemes, Section 26.13.) By construction V is a singleton with unique point corresponding to x (use Schemes, Lemma 26.13.2). By Algebra, Lemma 10.61.1 we see that $\dim(R) = 1$. In other words, we see that y is an immediate specialization of x (see Topology, Definition 5.20.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of x. Clearly each of these points is a closed point as desired. □

054F **Lemma 28.5.10.** *Let X be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of X. Then*

(1) *there exists a discrete valuation ring R and a morphism $f : \mathrm{Spec}(R) \to X$ such that the generic point η of $\mathrm{Spec}(R)$ maps to x' and the special point maps to x, and*
(2) *provided $x \neq x'$ given a finitely generated field extension $K/\kappa(x')$, we may arrange it so that the extension $\kappa(\eta)/\kappa(x')$ induced by f is isomorphic to the given one.*

Proof. First assume $x' \rightsquigarrow x$ is a specialization in X with $x \neq x'$ and let $K/\kappa(x')$ be a finitely generated extension of fields. By Schemes, Lemma 26.13.2 and the discussion following Schemes, Lemma 26.13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Since $x \neq x'$ the image of $\mathcal{O}_{X,x}$ in K is not a field (details omitted). Let $R \subset K$ be any discrete valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 10.119.13. The ring map $\mathcal{O}_{X,x} \to R$ induces the morphism $f : \mathrm{Spec}(R) \to X$, see Schemes, Lemma 26.13.1. This morphism has all the desired properties by construction. If $x = x'$ then we can set $R = \kappa(x)[t]_{(t)}$. □

0CXG **Lemma 28.5.11.** *Let S be a Noetherian scheme. Let $T \subset S$ be an infinite subset. Then there exists an infinite subset $T' \subset T$ such that there are no nontrivial specializations among the points T'.*

Proof. Let $T_0 \subset T$ be the set of $t \in T$ which do not specialize to another point of T. If T_0 is infinite, then $T' = T_0$ works. Hence we may and do assume T_0 is finite. Inductively, for $i > 0$, consider the set $T_i \subset T$ of $t \in T$ such that

(1) $t \notin T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$,
(2) there exist a nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T_{i-1}$, and
(3) for any nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T$ we have $t' \in T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$.

Again, if T_i is infinite, then $T' = T_i$ works. Let d be the maximum of the dimensions of the local rings $\mathcal{O}_{S,t}$ for $t \in T_0$; then d is an integer because T_0 is finite and the dimensions of the local rings are finite by Algebra, Proposition 10.60.9. Then $T_i = \emptyset$ for $i > d$. Namely, if $t \in T_i$ then we can find a sequence of nontrivial specializations $t = t_i \rightsquigarrow t_{i-1} \rightsquigarrow \ldots \rightsquigarrow t_0$ with $t_0 \in T_0$. As the points $t = t_i, t_{i-1}, \ldots, t_0$ are in $\mathrm{Spec}(\mathcal{O}_{S,t_0})$ (Schemes, Lemma 26.13.2), we see that $i \leq d$. Thus $\bigcup T_i = T_d \cup \ldots \cup T_0$ is a finite subset of T.

Suppose $t \in T$ is not in $\bigcup T_i$. Then there must be a specialization $t \rightsquigarrow t'$ with $t' \in T$ and $t' \notin \bigcup T_i$. (Namely, if every specialization of t is in the finite set $T_d \cup \ldots \cup T_0$, then there is a maximum i such that there is some specialization $t \rightsquigarrow t'$ with $t' \in T_i$ and then $t \in T_{i+1}$ by construction.) Hence we get an infinite sequence

$$t \rightsquigarrow t' \rightsquigarrow t'' \rightsquigarrow \ldots$$

of nontrivial specializations between points of $T \setminus \bigcup T_i$. This is impossible because the underlying topological space of S is Noetherian by Lemma 28.5.4. □

0G2R **Lemma 28.5.12.** *Let S be a Noetherian scheme. Let $T \subset S$ be a subset. Let $T_0 \subset T$ be the set of $t \in T$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$. Then (a) there are no specializations among the points of T_0, (b) every point of T is a specialization of a point of T_0, and (c) the closures of T and T_0 are the same.*

Proof. Recall that $\dim(\mathcal{O}_{S,s}) < \infty$ for any $s \in S$, see Algebra, Proposition 10.60.9. Let $t \in T$. If $t' \rightsquigarrow t$, then by dimension theory $\dim(\mathcal{O}_{S,t'}) \leq \dim(\mathcal{O}_{S,t})$ with equality if and only if $t' = t$. Thus if we pick $t' \rightsquigarrow t$ with $\dim(\mathcal{O}_{T,t'})$ minimal, then $t' \in T_0$. In other words, every $t \in T$ is the specialization of an element of T_0. □

0G2F **Lemma 28.5.13.** *Let S be a Noetherian scheme. Let $T \subset S$ be an infinite dense subset. Then there exist a countable subset $E \subset T$ which is dense in S.*

Proof. Let T' be the set of points $s \in S$ such that $\overline{\{s\}} \cap T$ contains a countable subset whose closure is $\overline{\{s\}}$. Since a finite set is countable we have $T \subset T'$. For $s \in T'$ choose such a countable subset $E_s \subset \overline{\{s\}} \cap T$. Let $E' = \{s_1, s_2, s_3, \ldots\} \subset T'$ be a countable subset. Then the closure of E' in S is the closure of the countable subset $\bigcup_n E_{s_n}$ of T. It follows that if Z is an irreducible component of the closure of E', then the generic point of Z is in T'.

Denote $T'_0 \subset T'$ the subset of $t \in T'$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$ as in Lemma 28.5.12 whose results we will use without further mention. If T'_0 is infinite, then we choose a countable subset $E' \subset T'_0$. By the argument in the first paragraph, the generic points of the irreducible components of the closure of E' are in T'. However, since one of these points specializes to infinitely many distinct elements of $E' \subset T'_0$ this is a contradiction. Thus T'_0 is finite, say $T'_0 = \{s_1, \ldots, s_m\}$. Then it follows that S, which is the closure of T, is contained in the closure of $\{s_1, \ldots, s_m\}$, which in turn is contained in the closure of the countable subset $E_{s_1} \cup \ldots \cup E_{s_m} \subset T$ as desired. □

28.6. Jacobson schemes

01P1 Recall that a space is said to be *Jacobson* if the closed points are dense in every closed subset, see Topology, Section 5.18.

01P2 **Definition 28.6.1.** A scheme S is said to be *Jacobson* if its underlying topological space is Jacobson.

Recall that a ring R is Jacobson if every radical ideal of R is the intersection of maximal ideals, see Algebra, Definition 10.35.1.

01P3 **Lemma 28.6.2.** *An affine scheme* $\mathrm{Spec}(R)$ *is Jacobson if and only if the ring* R *is Jacobson.*

Proof. This is Algebra, Lemma 10.35.4. □

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson $\Leftrightarrow$ locally Jacobson.

01P4 **Lemma 28.6.3.** *Let* X *be a scheme. The following are equivalent:*

(1) *The scheme* X *is Jacobson.*
(2) *The scheme* X *is "locally Jacobson" in the sense of Definition 28.4.2.*
(3) *For every affine open* $U \subset X$ *the ring* $\mathcal{O}_X(U)$ *is Jacobson.*
(4) *There exists an affine open covering* $X = \bigcup U_i$ *such that each* $\mathcal{O}_X(U_i)$ *is Jacobson.*
(5) *There exists an open covering* $X = \bigcup X_j$ *such that each open subscheme* X_j *is Jacobson.*

Moreover, if X *is Jacobson then every open subscheme is Jacobson.*

Proof. The final assertion of the lemma holds by Topology, Lemma 5.18.5. The equivalence of (5) and (1) is Topology, Lemma 5.18.4. Hence, using Lemma 28.6.2, we see that (1) $\Leftrightarrow$ (2). To finish proving the lemma it suffices to show that "Jacobson" is a local property of rings, see Lemma 28.4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 10.35.14. Suppose R is a ring, $f_1, \ldots, f_n \in R$ generate the unit ideal and each R_{f_i} is Jacobson. Then we see that $\mathrm{Spec}(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $\mathrm{Spec}(R)$ is Jacobson by Topology, Lemma 5.18.4 again. This proves the second property of Definition 28.4.1. □

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 29.16.10. We mention here the following interesting case.

02IM **Lemma 28.6.4.** *Examples of Noetherian Jacobson schemes.*

(1) *If* $(R, \mathfrak{m})$ *is a Noetherian local ring, then the punctured spectrum* $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ *is a Jacobson scheme.*
(2) *If* R *is a Noetherian ring with Jacobson radical* $\mathrm{rad}(R)$ *then* $\mathrm{Spec}(R) \setminus V(\mathrm{rad}(R))$ *is a Jacobson scheme.*
(3) *If* (R, I) *is a Zariski pair (More on Algebra, Definition 15.10.1) with* R *Noetherian, then* $\mathrm{Spec}(R) \setminus V(I)$ *is a Jacobson scheme.*

Proof. Proof of (3). Observe that $\mathrm{Spec}(R) - V(I)$ has a covering by the affine opens $\mathrm{Spec}(R_f)$ for $f \in I$. The rings R_f are Jacobson by More on Algebra, Lemma 15.10.5. Hence $\mathrm{Spec}(R) \setminus V(I)$ is Jacobson by Lemma 28.6.3. Parts (1) and (2) are special cases of (3).

Direct proof of case (1). Since $\mathrm{Spec}(R)$ is a Noetherian scheme, S is a Noetherian scheme (Lemma 28.5.6). Hence S is a sober, Noetherian topological space (use Schemes, Lemma 26.11.1). Assume S is not Jacobson to get a contradiction. By

Topology, Lemma 5.18.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $\mathrm{Spec}(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 10.61.1. □

28.7. Normal schemes

033H Recall that a ring R is said to be normal if all its local rings are normal domains, see Algebra, Definition 10.37.11. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 10.37.1. Thus it makes sense to define a normal scheme as follows.

033I **Definition 28.7.1.** A scheme X is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [**DG67**, 0, 4.1.4]. Suppose $X = \mathrm{Spec}(A)$, and A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is Noetherian then this is the case (see Algebra, Lemma 10.37.16).

033J **Lemma 28.7.2.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is normal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is normal.*

Moreover, if X is normal then every open subscheme is normal.

Proof. This is clear from the definitions. □

033K **Lemma 28.7.3.** *A normal scheme is reduced.*

Proof. Immediate from the definitions. □

033L **Lemma 28.7.4.** *Let X be an integral scheme. Then X is normal if and only if for every nonempty affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.*

Proof. This follows from Algebra, Lemma 10.37.10. □

0357 **Lemma 28.7.5.** *Let X be a scheme such that any quasi-compact open has a finite number of irreducible components. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a disjoint union of normal integral schemes.*

Proof. It is immediate from the definitions that (2) implies (1). Let X be a normal scheme such that every quasi-compact open has a finite number of irreducible components. If X is affine then X satisfies (2) by Algebra, Lemma 10.37.16. For a general X, let $X = \bigcup X_i$ be an affine open covering. Note that also each X_i has but a finite number of irreducible components, and the lemma holds for each X_i. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$ is open in X_i and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that X is the disjoint union of its irreducible components, which are integral normal schemes. □

033M **Lemma 28.7.6.** *Let X be a Noetherian scheme. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a finite disjoint union of normal integral schemes.*

Proof. This is a special case of Lemma 28.7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 28.5.5 and Topology, Lemma 5.9.2). □

033N **Lemma 28.7.7.** *Let X be a locally Noetherian scheme. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a disjoint union of integral normal schemes.*

Proof. Omitted. Hint: This is purely topological from Lemma 28.7.6. □

033O **Remark 28.7.8.** Let X be a normal scheme. If X is locally Noetherian then we see that X is integral if and only if X is connected, see Lemma 28.7.7. But there exists a connected affine scheme X such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but X is not irreducible, see Examples, Section 110.6. This example is even a normal scheme (proof omitted), so beware!

0358 **Lemma 28.7.9.** *Let X be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that R is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over R. Say we have $f^d + \sum_{i=0,\ldots,d-1} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be a nonempty affine open. Since $b \in R$ is not zero and since X is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence a/b is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$ (because we can use the same polynomial $f^d + \sum_{i=0,\ldots,d-1} a_i|_U f^i = 0$ on U). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 28.7.4), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is clear that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are nonempty affine open. Hence the local sections f_U glue to an element $g \in R = \Gamma(X, \mathcal{O}_X)$. Then bg and a restrict to the same element of $\mathcal{O}_X(U)$ for all U as above, hence $bg = a$, in other words, g maps to f in the fraction field of R. □

28.8. Cohen-Macaulay schemes

02IN Recall, see Algebra, Definition 10.104.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be Cohen-Macaulay if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Recall that a Noetherian ring R is said to be Cohen-Macaulay if every local ring $R_{\mathfrak{p}}$ of R is Cohen-Macaulay, see Algebra, Definition 10.104.6.

02IO **Definition 28.8.1.** Let X be a scheme. We say X is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

02IP **Lemma 28.8.2.** *Let X be a scheme. The following are equivalent:*

(1) *X is Cohen-Macaulay,*
(2) *X is locally Noetherian and all of its local rings are Cohen-Macaulay, and*
(3) *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

Proof. Algebra, Lemma 10.104.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 28.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 28.5.9), and the definitions. □

02IQ **Lemma 28.8.3.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Cohen-Macaulay.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Cohen-Macaulay.*

Moreover, if X is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.

Proof. Combine Lemmas 28.5.2 and 28.8.2. □

More information on Cohen-Macaulay schemes and depth can be found in Cohomology of Schemes, Section 30.11.

28.9. Regular schemes

02IR Recall, see Algebra, Definition 10.60.10, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring R is said to be *regular* if every local ring $R_\mathfrak{p}$ of R is regular, see Algebra, Definition 10.110.7.

02IS **Definition 28.9.1.** Let X be a scheme. We say X is *regular*, or *nonsingular* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

02IT **Lemma 28.9.2.** *Let X be a scheme. The following are equivalent:*

(1) *X is regular,*
(2) *X is locally Noetherian and all of its local rings are regular, and*
(3) *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof. By the discussion in Algebra preceding Algebra, Definition 10.110.7 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 28.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 28.5.9), and the definitions. □

02IU **Lemma 28.9.3.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is regular.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is regular.*

Moreover, if X is regular then every open subscheme is regular.

Proof. Combine Lemmas 28.5.2 and 28.9.2. □

0569 **Lemma 28.9.4.** *A regular scheme is normal.*

Proof. See Algebra, Lemma 10.157.5. □

28.10. Dimension

04MS The dimension of a scheme is just the dimension of its underlying topological space.

04MT **Definition 28.10.1.** Let X be a scheme.

(1) The *dimension* of X is just the dimension of X as a topological spaces, see Topology, Definition 5.10.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of X at x as in Topology, Definition 5.10.1. We say $\dim_x(X)$ is the *dimension of X at x*.

As a scheme has a sober underlying topological space (Schemes, Lemma 26.11.1) we may compute the dimension of X as the supremum of the lengths n of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of X, or as the supremum of the lengths n of chains of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X.

04MU **Lemma 28.10.2.** *Let X be a scheme. The following are equal*

(1) *The dimension of X.*
(2) *The supremum of the dimensions of the local rings of X.*
(3) *The supremum of $\dim_x(X)$ for $x \in X$.*

Proof. Note that given a chain of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X all of the points ξ_i correspond to prime ideals of the local ring of X at ξ_0 by Schemes, Lemma 26.13.2. Hence we see that the dimension of X is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of x. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x \in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of X. □

02IZ **Lemma 28.10.3.** *Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then*

$$\mathit{codim}(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

where the codimension is as defined in Topology, Definition 5.11.1.

Proof. By Topology, Lemma 5.11.2 we may replace X by an affine open neighbourhood of ξ. In this case the result follows easily from Algebra, Lemma 10.26.3. □

0BA9 **Lemma 28.10.4.** *Let X be a scheme. Let $x \in X$. Then x is a generic point of an irreducible component of X if and only if $\dim(\mathcal{O}_{X,x}) = 0$.*

Proof. This follows from Lemma 28.10.3 for example. □

0AAX **Lemma 28.10.5.** *A locally Noetherian scheme of dimension 0 is a disjoint union of spectra of Artinian local rings.*

Proof. A Noetherian ring of dimension 0 is a finite product of Artinian local rings, see Algebra, Proposition 10.60.7. Hence an affine open of a locally Noetherian scheme X of dimension 0 has discrete underlying topological space. This implies that the topology on X is discrete. The lemma follows easily from these remarks. □

0CKV **Lemma 28.10.6.** *Let X be a scheme of dimension zero. The following are equivalent* Email from Ofer Gabber dated June 4, 2016

(1) *X is quasi-separated,*
(2) *X is separated,*
(3) *X is Hausdorff,*
(4) *every affine open is closed.*

In this case the connected components of X are points and every quasi-compact open of X is affine. In particular, if X is quasi-compact, then X is affine.

Proof. As the dimension of X is zero, we see that for any affine open $U \subset X$ the space U is profinite and satisfies a bunch of other properties which we will use freely below, see Algebra, Lemma 10.26.5. We choose an affine open covering $X = \bigcup U_i$.

If (4) holds, then $U_i \cap U_j$ is a closed subset of U_i, hence quasi-compact, hence X is quasi-separated, by Schemes, Lemma 26.21.6, hence (1) holds.

If (1) holds, then $U_i \cap U_j$ is a quasi-compact open of U_i hence closed in U_i. Then $U_i \cap U_j \to U_i$ is an open immersion whose image is closed, hence it is a closed immersion. In particular $U_i \cap U_j$ is affine and $\mathcal{O}(U_i) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective. Thus X is separated by Schemes, Lemma 26.21.7, hence (2) holds.

Assume (2) and let $x, y \in X$. Say $x \in U_i$. If $y \in U_i$ too, then we can find disjoint open neighbourhoods of x and y because U_i is Hausdorff. Say $y \notin U_i$ and $y \in U_j$. Then $y \notin U_i \cap U_j$ which is an affine open of U_j and hence closed in U_j. Thus we can find an open neighbourhood of y not meeting U_i and we conclude that X is Hausdorff, hence (3) holds.

Assume (3). Let $U \subset X$ be affine open. Then U is closed in X by Topology, Lemma 5.12.4. This proves (4) holds.

Assume X satisfies the equivalent conditions (1) – (4). We prove the final statements of the lemma. Say $x, y \in X$ with $x \neq y$. Since y does not specialize to x we can choose $U \subset X$ affine open with $x \in U$ and $y \notin U$. Then we see that $X = U \amalg (X \setminus U)$ is a decomposistion into open and closed subsets which shows that x and y do not belong to the same connected component of X. Next, assume $U \subset X$ is a quasi-compact open. Write $U = U_1 \cup \ldots \cup U_n$ as a union of affine opens. We will prove by induction on n that U is affine. This immediately reduces us to the case $n = 2$. In this case we have $U = (U_1 \setminus U_2) \amalg (U_1 \cap U_2) \amalg (U_2 \setminus U_1)$ and the arguments above show that each of the pieces is affine. □

0H7C **Lemma 28.10.7.** *Let x be a point of a locally Noetherian scheme X. Then $\dim_x(X) = 0$ if and only if x is an isolated point of X.*

Proof. If x is an isolated point, then $\{x\}$ is an open subset of X (Topology, Definition 5.27.2) and hence $\dim_x(X) = \dim(\{x\}) = 0$ by definition. Conversely, if $\dim_x(X) = 0$, then there exists an open neighbourhood $U \subset X$ of x such that $\dim(U) = 0$ (Topology, Definition 5.10.1). By Lemma 28.10.5 we see that the topology on U is discrete and hence x is an isolated point. □

28.11. Catenary schemes

02IV Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.11.4.

02IW **Definition 28.11.1.** Let S be a scheme. We say S is *catenary* if the underlying topological space of S is catenary.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.105.1.

02IX **Lemma 28.11.2.** *Let S be a scheme. The following are equivalent*

(1) *S is catenary,*
(2) *there exists an open covering of S all of whose members are catenary schemes,*
(3) *for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a catenary ring.*

Moreover, in this case any locally closed subscheme of S is catenary as well.

Proof. Combine Topology, Lemma 5.11.5, and Algebra, Lemma 10.105.2. □

02IY **Lemma 28.11.3.** *Let S be a locally Noetherian scheme. The following are equivalent:*

(1) *S is catenary, and*
(2) *locally in the Zariski topology there exists a dimension function on S (see Topology, Definition 5.20.1).*

Proof. This follows from Topology, Lemmas 5.11.5, 5.20.2, and 5.20.4, Schemes, Lemma 26.11.1 and finally Lemma 28.5.5. □

It turns out that a scheme is catenary if and only if its local rings are catenary.

02J0 **Lemma 28.11.4.** *Let X be a scheme. The following are equivalent*

(1) *X is catenary, and*
(2) *for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.*

Proof. Assume X is catenary. Let $x \in X$. By Lemma 28.11.2 we may replace X by an affine open neighbourhood of x, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 10.105.4 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of X are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of X. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of Y', see Schemes, Lemma 26.13.2. By that same lemma the irreducible closed subsets of X in between Y and Y' correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. □

28.12. Serre's conditions

033P Here are two technical notions that are often useful. See also Cohomology of Schemes, Section 30.11.

033Q **Definition 28.12.1.** Let X be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say X is *regular in codimension k*, or we say X has property (R_k) if for every $x \in X$ we have

$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$

(2) We say X has property (S_k) if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase "regular in codimension k" makes sense since we have seen in Section 28.11 that if $Y \subset X$ is irreducible closed with generic point x, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition (R_0) means that for every generic point $\eta \in X$ of an irreducible component of X the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of X is badly behaved, so care has to be taken.

0B3C **Lemma 28.12.2.** *Let X be a locally Noetherian scheme. Then X is regular if and only if X has (R_k) for all $k \geq 0$.*

Proof. Follows from Lemma 28.9.2 and the definitions. □

0342 **Lemma 28.12.3.** *Let X be a locally Noetherian scheme. Then X is Cohen-Macaulay if and only if X has (S_k) for all $k \geq 0$.*

Proof. By Lemma 28.8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. □

0344 **Lemma 28.12.4.** *Let X be a locally Noetherian scheme. Then X is reduced if and only if X has properties (S_1) and (R_0).*

Proof. This is Algebra, Lemma 10.157.3. □

0345 **Lemma 28.12.5.** *Let X be a locally Noetherian scheme. Then X is normal if and only if X has properties (S_2) and (R_1).*

Proof. This is Algebra, Lemma 10.157.4. □

0BX2 **Lemma 28.12.6.** *Let X be a locally Noetherian scheme which is normal and has dimension ≤ 1. Then X is regular.*

Proof. This follows from Lemma 28.12.5 and the definitions. □

0B3D **Lemma 28.12.7.** *Let X be a locally Noetherian scheme which is normal and has dimension ≤ 2. Then X is Cohen-Macaulay.*

Proof. This follows from Lemma 28.12.5 and the definitions. □

28.13. Japanese and Nagata schemes

033R The notions considered in this section are not prominently defined in EGA. A "universally Japanese scheme" is mentioned and defined in [**DG67**, IV Corollary 5.11.4]. A "Japanese scheme" is mentioned in [**DG67**, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [**Liu02**, Definition 8.2.30] and [**Gre76**, Page 142]).

We briefly recall that a domain R is called *Japanese* if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called *universally Japanese* if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R.

033S **Definition 28.13.1.** Let X be a scheme.

(1) Assume X integral. We say X is *Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 10.161.1).
(2) We say X is *universally Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 10.162.1).
(3) We say X is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 10.162.1).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 28.13.8.

033T **Remark 28.13.2.** In [**Hoo72**] a (locally Noetherian) scheme X is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since there exists a one dimensional Noetherian domain with excellent (in particular Japanese) local rings whose normalization is not finite. See [**Hoc73**, Example 1] or [**HL07**] or [**ILO14**, Exposé XIX]. On the other hand, we could circumvent this problem by calling a scheme X Japanese if for every affine open $\mathrm{Spec}(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of A.

033U **Lemma 28.13.3.** *A Nagata scheme is locally Noetherian.*

Proof. This is true because a Nagata ring is Noetherian by definition. □

033V **Lemma 28.13.4.** *Let X be an integral scheme. The following are equivalent:*

(1) *The scheme X is Japanese.*
(2) *For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Japanese.*

Moreover, if X is Japanese then every open subscheme is Japanese.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.161.3 and 10.161.4. □

033W **Lemma 28.13.5.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is universally Japanese.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is universally Japanese.*

Moreover, if X is universally Japanese then every open subscheme is universally Japanese.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.162.4 and 10.162.7. □

033X **Lemma 28.13.6.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Nagata.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Nagata.*

Moreover, if X is Nagata then every open subscheme is Nagata.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.162.6 and 10.162.7. □

033Y **Lemma 28.13.7.** *Let X be a locally Noetherian scheme. Then X is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.*

Proof. Assume X is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open containing z such that A is Nagata. Then $Z \cap U \cong \mathrm{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 26.10.1 (and Definition 28.3.1). By Algebra, Definition 10.162.1 we see that $A/\mathfrak{p}$ is Japanese. Hence Z is Japanese by definition.

Assume every integral closed subscheme of X is Japanese. Let $\mathrm{Spec}(A) = U \subset X$ be any affine open. As X is locally Noetherian we see that A is Noetherian (Lemma 28.5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset \mathrm{Spec}(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 26.12.4. In other words, $\mathrm{Spec}(A/\mathfrak{p})$ is an affine open of an integral closed subscheme of X. This subscheme is Japanese by assumption and by Lemma 28.13.4 we see that $A/\mathfrak{p}$ is Japanese. □

033Z **Lemma 28.13.8.** *Let X be a scheme. The following are equivalent:*

(1) *X is Nagata, and*
(2) *X is locally Noetherian and universally Japanese.*

Proof. This is Algebra, Proposition 10.162.15. □

This discussion will be continued in Morphisms, Section 29.18.

28.14. The singular locus

07R0 Here is the definition.

07R1 **Definition 28.14.1.** Let X be a locally Noetherian scheme. The *regular locus* $\mathrm{Reg}(X)$ of X is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The *singular locus* $\mathrm{Sing}(X)$ is the complement $X \setminus \mathrm{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 10.110.7. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 15.47 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 29.19.

28.15. Local irreducibility

0BQ1 Recall that in More on Algebra, Section 15.107 we introduced the notion of a (geometrically) unibranch local ring.

0BQ2 **Definition 28.15.1.** Let X be a scheme. Let $x \in X$. We say X is *unibranch at x* if the local ring $\mathcal{O}_{X,x}$ is unibranch. We say X is *geometrically unibranch at x* if the local ring $\mathcal{O}_{X,x}$ is geometrically unibranch. We say X is *unibranch* if X is unibranch at all of its points. We say X is *geometrically unibranch* if X is geometrically unibranch at all of its points. [**GD67**, Chapter IV (6.15.1)]

To be sure, it can happen that a local ring A is geometrically unibranch (in the sense of More on Algebra, Definition 15.107.1) but the scheme $\mathrm{Spec}(A)$ is not geometrically unibranch in the sense of Definition 28.15.1. For example this happens if A is the local ring at the vertex of the cone over an irreducible plane curve which has ordinary double point singularity (a node).

0BQ3 **Lemma 28.15.2.** *A normal scheme is geometrically unibranch.*

Proof. This follows from the definitions. Namely, a scheme is normal if the local rings are normal domains. It is immediate from the More on Algebra, Definition 15.107.1 that a local normal domain is geometrically unibranch. □

0BQ4 **Lemma 28.15.3.** *Let X be a Noetherian scheme. The following are equivalent*

(1) *X is geometrically unibranch (Definition 28.15.1),*
(2) *for every point $x \in X$ which is not the generic point of an irreducible component of X, the punctured spectrum of the strict henselization $\mathcal{O}^{sh}_{X,x}$ is connected.*

Compare with [**Art66**, Proposition 2.3]

Proof. More on Algebra, Lemma 15.107.5 shows that (1) implies that the punctured spectra in (2) are irreducible and in particular connected.

Assume (2). Let $x \in X$. We have to show that $\mathcal{O}_{X,x}$ is geometrically unibranch. By induction on $\dim(\mathcal{O}_{X,x})$ we may assume that the result holds for every nontrivial generalization of x. We may replace X by $\mathrm{Spec}(\mathcal{O}_{X,x})$. In other words, we may assume that $X = \mathrm{Spec}(A)$ with A local and that $A_\mathfrak{p}$ is geometrically unibranch for each nonmaximal prime $\mathfrak{p} \subset A$.

Let A^{sh} be the strict henselization of A. If $\mathfrak{q} \subset A^{sh}$ is a prime lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p} \to A^{sh}_\mathfrak{q}$ is a filtered colimit of étale algebras. Hence the strict henselizations of $A_\mathfrak{p}$

and $A^{sh}_{\mathfrak{q}}$ are isomorphic. Thus by More on Algebra, Lemma 15.107.5 we conclude that $A^{sh}_{\mathfrak{q}}$ has a unique minimal prime ideal for every nonmaximal prime $\mathfrak{q}$ of A^{sh}.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of A^{sh}. We have to show that $r = 1$. By the above we see that $V(\mathfrak{q}_1) \cap V(\mathfrak{q}_j) = \{\mathfrak{m}^{sh}\}$ for $j = 2, \ldots, r$. Hence $V(\mathfrak{q}_1) \setminus \{\mathfrak{m}^{sh}\}$ is an open and closed subset of the punctured spectrum of A^{sh} which is a contradiction with the assumption that this punctured spectrum is connected unless $r = 1$. □

0C38 **Definition 28.15.4.** Let X be a scheme. Let $x \in X$. The *number of branches of X at x* is the number of branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 15.107.6. The *number of geometric branches of X at x* is the number of geometric branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 15.107.6.

Often we want to compare this with the branches of the complete local ring, but the comparison is not straightforward in general; some information on this topic can be found in More on Algebra, Section 15.109.

0E20 **Lemma 28.15.5.** *Let X be a scheme and $x \in X$. Let X_i, $i \in I$ be the irreducible components of X passing through x. Then the number of (geometric) branches of X at x is the sum over $i \in I$ of the number of (geometric) branches of X_i at x.*

Proof. We view the X_i as integral closed subschemes of X, see Schemes, Definition 26.12.5 and Lemma 28.3.4. Observe that the number of (geometric) branches of X_i at x is at least 1 for all i (essentially by definition). Recall that the X_i correspond 1-to-1 with the minimal prime ideals $\mathfrak{p}_i \subset \mathcal{O}_{X,x}$, see Algebra, Lemma 10.26.3. Thus, if I is infinite, then $\mathcal{O}_{X,x}$ has infinitely many minimal primes, whence both $\mathcal{O}^h_{X,x}$ and $\mathcal{O}^{sh}_{X,x}$ have infinitely many minimal primes (combine Algebra, Lemmas 10.30.5 and 10.30.7 and the injectivity of the maps $\mathcal{O}_{X,x} \to \mathcal{O}^h_{X,x} \to \mathcal{O}^{sh}_{X,x}$). In this case the number of (geometric) branches of X at x is defined to be ∞ which is also true for the sum. Thus we may assume I is finite. Let A' be the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions Q of $(\mathcal{O}_{X,x})_{red}$. Let A'_i be the integral closure of $\mathcal{O}_{X,x}/\mathfrak{p}_i$ in the total ring of fractions Q_i of $\mathcal{O}_{X,x}/\mathfrak{p}_i$. By Algebra, Lemma 10.25.4 we have $Q = \prod_{i \in I} Q_i$. Thus $A' = \prod A'_i$. Then the equality of the lemma follows from More on Algebra, Lemma 15.107.7 which expresses the number of (geometric) branches in terms of the maximal ideals of A'. □

0C39 **Lemma 28.15.6.** *Let X be a scheme. Let $x \in X$.*

(1) *The number of branches of X at x is 1 if and only if X is unibranch at x.*
(2) *The number of geometric branches of X at x is 1 if and only if X is geometrically unibranch at x.*

Proof. This lemma follows immediately from the definitions and the corresponding result for rings, see More on Algebra, Lemma 15.107.7. □

28.16. Characterizing modules of finite type and finite presentation

01PA Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 17.9.1) if and only if $\mathcal{F}$ is on each open affine $\mathrm{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finite A-module M. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 17.11.1) if and

only if $\mathcal{F}$ is on each open affine $\mathrm{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented A-module M.

01PB **Lemma 28.16.1.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if M is a finite R-module.*

Proof. Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of X such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus M_{f_i} is finitely generated for each i. Hence we conclude by Algebra, Lemma 10.23.2. □

01PC **Lemma 28.16.2.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is an R-module of finite presentation.*

Proof. Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 28.16.1 we see that M is a finite R-module. Choose a surjection $R^n \to M$ with kernel K. By Schemes, Lemma 26.5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 17.11.4 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 28.16.1 again we see that K is a finite R-module. Hence M is an R-module of finite presentation. □

28.17. Sections over principal opens

0B5K Here is a typical result of this kind. We will use a more naive but more direct method of proof in later lemmas.

01P7 **Lemma 28.17.1.** *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote $X_f \subset X$ the open where f is invertible, see Schemes, Lemma 26.6.2. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

is an isomorphism.

Proof. Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow \mathrm{Spec}(R)$$

of schemes, see Schemes, Lemma 26.6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is X_f on the left hand side. Moreover, since X is assumed quasi-compact and quasi-separated the morphism φ is quasi-compact and quasi-separated, see Schemes, Lemma 26.19.2 and 26.21.13. Hence by Schemes, Lemma 26.24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an R-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The first displayed isomorphism of the lemma follows by taking $\mathcal{F} = \mathcal{O}_X$. □

Recall that given a scheme X, an invertible sheaf $\mathcal{L}$ on X, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and a graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ see Modules, Definition 17.25.7. If we have moreover a section $s \in \Gamma(X, \mathcal{L})$, then we obtain a map

0B5L (28.17.1.1) $$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F}|_{X_s})$$

which sends t/s^n where $t \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ to $t|_{X_s} \otimes s|_{X_s}^{-n}$. This makes sense because $X_s \subset X$ is by definition the open over which s has an inverse, see Modules, Lemma 17.25.10.

01PW **Lemma 28.17.2.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If X is quasi-compact, then (28.17.1.1) is injective, and*
(2) *if X is quasi-compact and quasi-separated, then (28.17.1.1) is an isomorphism.*

In particular, the canonical map

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O}_X), \quad a/s^n \longmapsto a \otimes s^{-n}$$

is an isomorphism if X is quasi-compact and quasi-separated.

Proof. Assume X is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with U_j affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$.

Proof of (1). Let t/s^n be an element in the kernel of (28.17.1.1). Then $t|_{X_s} = 0$. Hence $(t|_{U_j})|_{D(f_j)} = 0$. By Lemma 28.17.1 we conclude that $f_j^{e_j} t|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $t \otimes s^e$ restricts to zero on U_j for all j, hence is zero. Since t/s^n is equal to $t \otimes s^e / s^{n+e}$ in $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)}$ we conclude that $t/s^n = 0$ as desired.

Proof of (2). Assume X is quasi-compact and quasi-separated. Then $U_j \cap U_{j'}$ is quasi-compact for all pairs j, j', see Schemes, Lemma 26.21.6. By part (1) we know (28.17.1.1) is injective. Let $t' \in \Gamma(X_s, \mathcal{F}|_{X_s})$. For every j, there exist an integer $e_j \geq 0$ and $t'_j \in \Gamma(U_j, \mathcal{F}|_{U_j})$ such that $t'|_{D(f_j)}$ corresponds to $t'_j / f_j^{e_j}$ via the isomorphism of Lemma 28.17.1. Set $e = \max(e_j)$ and

$$t_j = f_j^{e - e_j} t'_j \otimes q_j^e \in \Gamma(U_j, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})|_{U_j})$$

where $q_j \in \Gamma(U_j, \mathcal{L}|_{U_j})$ is the trivializing section coming from the isomorphism $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. In particular we have $s|_{U_j} = f_j q_j$. Using this a calculation shows that $t_j|_{U_j \cap U_{j'}}$ and $t_{j'}|_{U_j \cap U_{j'}}$ map to the same section of $\mathcal{F}$ over $U_j \cap U_{j'} \cap X_s$. By quasi-compactness of $U_j \cap U_{j'}$ and part (1) there exists an integer $e' \geq 0$ such that

$$t_j|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}} = t_{j'}|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}}$$

as sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes e + e'}$ over $U_j \cap U_{j'}$. We may choose the same e' to work for all pairs j, j'. Then the sheaf conditions implies there is a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes e + e'})$ whose restriction to U_j is $t_j \otimes s^{e'}|_{U_j}$. A simple computation shows that $t/s^{e+e'}$ maps to t' as desired. □

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Consider the graded $\Gamma_*(X, \mathcal{L})$-module

$$M = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

Next, let $s \in \Gamma(X, \mathcal{L})$ be a section. Then there is a canonical map

0B5M (28.17.2.1) $$M_{(s)} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(\mathcal{F}|_{X_s}, \mathcal{G}|_{X_s})$$

which sends α/s^n to the map $\alpha|_{X_s} \otimes s|_{X_s}^{-n}$. The following lemma, combined with Lemma 28.22.4, says roughly that, if X is quasi-compact and quasi-separated, the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules with the multiplicative system of maps $s^n : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ inverted.

01XQ **Lemma 28.17.3.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules.*

(1) *If X is quasi-compact and $\mathcal{F}$ is of finite type, then (28.17.2.1) is injective, and*
(2) *if X is quasi-compact and quasi-separated and $\mathcal{F}$ is of finite presentation, then (28.17.2.1) is bijective.*

Proof. We first prove the lemma in case $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. In this case s corresponds to an element $f \in A$. Say $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ for some A-modules M and N. Then the lemma translates (via Lemmas 28.16.1 and 28.16.2) into the following algebra statements

(1) If M is a finite A-module and $\varphi : M \to N$ is an A-module map such that the induced map $M_f \to N_f$ is zero, then $f^n\varphi = 0$ for some n.
(2) If M is a finitely presented A-module, then $\mathrm{Hom}_A(M, N)_f = \mathrm{Hom}_{A_f}(M_f, N_f)$.

The second statement is Algebra, Lemma 10.10.2 and we omit the proof of the first statement.

Next, we prove (1) for general X. Assume X is quasi-compact and hoose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with U_j affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$. Let α/s^n be an element in the kernel of (28.17.2.1). Then $\alpha|_{X_s} = 0$. Hence $(\alpha|_{U_j})|_{D(f_j)} = 0$. By the affine case treated above we conclude that $f_j^{e_j}\alpha|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $\alpha \otimes s^e$ restricts to zero on U_j for all j, hence is zero. Since α/s^n is equal to $\alpha \otimes s^e/s^{n+e}$ in $M_{(s)}$ we conclude that $\alpha/s^n = 0$ as desired.

Proof of (2). Since $\mathcal{F}$ is of finite presentation, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Schemes, Section 26.24. Moreover, it is clear that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

for all n. Hence in this case the statement follows from Lemma 28.17.2 applied to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. □

28.18. Quasi-affine schemes

01P5

01P6 **Definition 28.18.1.** A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

0EHM **Lemma 28.18.2.** *Let A be a ring and let $U \subset \mathrm{Spec}(A)$ be a quasi-compact open subscheme. For $\mathcal{F}$ quasi-coherent on U the canonical map*

$$\widetilde{H^0(U, \mathcal{F})}|_U \to \mathcal{F}$$

is an isomorphism.

Proof. Denote $j : U \to \mathrm{Spec}(A)$ the inclusion morphism. Then $H^0(U, \mathcal{F}) = H^0(\mathrm{Spec}(A), j_*\mathcal{F})$ and $j_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 26.24.1. Hence $j_*\mathcal{F} = \widetilde{H^0(U, \mathcal{F})}$ by Schemes, Lemma 26.7.5. Restricting back to U we get the lemma. □

01P8 **Lemma 28.18.3.** *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume X is quasi-compact and quasi-separated and assume that X_f is affine. Then the canonical morphism*

$$j : X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 26.6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$.

Proof. This is clear as j induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 28.17.1 above. □

01P9 **Lemma 28.18.4.** *Let X be a scheme. Then X is quasi-affine if and only if the canonical morphism*

$$X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 26.6.4 is a quasi-compact open immersion.

Proof. If the displayed morphism is a quasi-compact open immersion then X is isomorphic to a quasi-compact open subscheme of $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and clearly X is quasi-affine.

Assume X is quasi-affine, say $X \subset \mathrm{Spec}(R)$ is quasi-compact open. This in particular implies that X is separated, see Schemes, Lemma 26.23.9. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$. By Schemes, Lemma 26.6.4 we obtain a factorization:

$$X \longrightarrow \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of r in A. The open X_f of Lemma 28.17.1 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to \mathrm{Spec}(A)$ is an isomorphism onto the standard affine open $D(f)$ of $\mathrm{Spec}(A)$. Since X can be covered by such affine opens $D(f)$ we win. □

0ARY **Lemma 28.18.5.** *Let $U \to V$ be an open immersion of quasi-affine schemes. Then*

$$\begin{array}{ccccc} U & & \xrightarrow{j} & & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \\ \downarrow & & & & \downarrow \\ U & \longrightarrow & V & \xrightarrow{j'} & \mathrm{Spec}(\Gamma(V, \mathcal{O}_V)) \end{array}$$

is cartesian.

Proof. The diagram is commutative by Schemes, Lemma 26.6.4. Write $A = \Gamma(U, \mathcal{O}_U)$ and $B = \Gamma(V, \mathcal{O}_V)$. Let $g \in B$ be such that V_g is affine and contained in U. This means that if f is the image of g in A, then $U_f = V_g$. By Lemma 28.18.3 we see that j' induces an isomorphism of V_g with the standard open $D(g)$ of $\mathrm{Spec}(B)$. Thus $V_g \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is an isomorphism onto $D(f) \subset \mathrm{Spec}(A)$. By Lemma 28.18.3 again j maps U_f isomorphically to $D(f)$. Thus we see that $U_f = U_f \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. Since by Lemma 28.18.4 we can cover U by $V_g = U_f$ as above, we see that $U \to U \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$ is an isomorphism. □

0F82 **Lemma 28.18.6.** *Let X be a quasi-affine scheme. There exists an integer $n \geq 0$, an affine scheme T, and a morphism $T \to X$ such that for every morphism $X' \to X$ with X' affine the fibre product $X' \times_X T$ is isomorphic to $\mathbf{A}^n_{X'}$ over X'.*

Proof. By definition, there exists a ring A such that X is isomorphic to a quasi-compact open subscheme $U \subset \mathrm{Spec}(A)$. Recall that the standard opens $D(f) \subset \mathrm{Spec}(A)$ form a basis for the topology, see Algebra, Section 10.17. Since U is quasi-compact we can choose $f_1, \ldots, f_n \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_n)$. Thus we may assume $X = \mathrm{Spec}(A) \setminus V(I)$ where $I = (f_1, \ldots, f_n)$. We set

$$T = \mathrm{Spec}(A[t, x_1, \ldots, x_n]/(f_1 x_1 + \ldots + f_n x_n - 1))$$

The structure morphism $T \to \mathrm{Spec}(A)$ factors through the open X to give the morphism $T \to X$. If $X' = \mathrm{Spec}(A')$ and the morphism $X' \to X$ corresponds to the ring map $A \to A'$, then the images $f'_1, \ldots, f'_n \in A'$ of $f_1, \ldots, f_n$ generate the unit ideal in A'. Say $1 = f'_1 a'_1 + \ldots + f'_n a'_n$. The base change $X' \times_X T$ is the spectrum of $A'[t, x_1, \ldots, x_n]/(f'_1 x_1 + \ldots + f'_n x_n - 1)$. We claim the A'-algebra homomorphism

$$\varphi : A'[y_1, \ldots, y_n] \longrightarrow A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1 x_1 + \ldots + f'_n x_n - 1)$$

sending y_i to $a'_i t + x_i$ is an isomorphism. The claim finishes the proof of the lemma. The inverse of φ is given by the A'-algebra homomorphism

$$\psi : A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1 x_1 + \ldots + f'_n x_n - 1) \longrightarrow A'[y_1, \ldots, y_n]$$

sending t to $-1 + f'_1 y_1 + \ldots + f'_n y_n$ and x_i to $y_i + a'_i - a'_i(f'_1 y_1 + \ldots + f'_n y_n)$ for $i = 1, \ldots, n$. This makes sense because $\sum f'_i x_i$ is mapped to

$$\sum f'_i (y_i + a'_i - a'_i(\sum f'_j y_j)) = (\sum f'_i y_i) + 1 - (\sum f'_j y_j) = 1$$

To see the maps are mutually inverse one computes as follows:

$$\begin{gathered} \varphi(\psi(t) = \varphi(-1 + \sum f'_i y_i) = -1 + \sum f'_i (a'_i t + x_i) = t \\ \varphi(\psi(x_i)) = \varphi(y_i + a'_i - a'_i(\sum f'_j y_j)) = a'_i t + x_i + a'_i - a'_i(\sum f'_j a'_j t + f'_j x_j) = x_i \\ \psi(\varphi(y_i)) = \psi(a'_i t + x_i) = a'_i(-1 + \sum f'_j y_j) + y_i + a'_i - a'_i(\sum f'_j y_j) = y_i \end{gathered}$$

This finishes the proof. □

28.19. Flat modules

05NZ On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 17.17.1 (Definition 17.17.3). For quasi-coherent sheaves on an affine scheme this matches the notion defined in the algebra chapter.

05P0 **Lemma 28.19.1.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if M is a flat R-module.*

Proof. Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 17.17.2. The same is true in the case of modules over a ring, see Algebra, Lemma 10.39.18. And since $\mathcal{F}_x = M_{\mathfrak{p}}$ if x corresponds to $\mathfrak{p}$ the lemma is true. □

28.20. Locally free modules

05P1 On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

05JM **Lemma 28.20.1.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if M is a (finite) locally free R-module.*

Proof. Follows from the definitions, see Modules, Definition 17.14.1 and Algebra, Definition 10.78.1. □

We can characterize finite locally free modules in many different ways.

05P2 **Lemma 28.20.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

1. *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,*
2. *$\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,*
3. *$\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,*
4. *$\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and*
5. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function*

$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 10.78.2. The translation uses Lemmas 28.16.1, 28.16.2, 28.19.1, and 28.20.1. □

0FWH **Lemma 28.20.3.** *Let X be a reduced scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the equivalent conditions of Lemma 28.20.2 are also equivalent to*

6. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type and the function*

$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 10.78.3. □

28.21. Locally projective modules

05JN A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

05JP **Definition 28.21.1.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

05JQ **Lemma 28.21.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is locally projective, and*
(2) *there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every i.*

In particular, if $X = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if M is a projective A-module.

Proof. First, note that if M is a projective A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective B-module, see Algebra, Lemma 10.94.1. Hence if U is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each j the open $D(f_j)$ is a standard open of some U_i, see Schemes, Lemma 26.11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if M is an A-module such that $\mathcal{F}|_U$ corresponds to M, then we see that M_{f_j} is a projective A_{f_j}-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \otimes_A B$ is a projective B-module. Hence M is projective by Algebra, Theorem 10.95.6. □

060M **Lemma 28.21.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. See Algebra, Lemma 10.94.1. □

28.22. Extending quasi-coherent sheaves

01PD It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

01PE **Lemma 28.22.1.** *Let $j : U \to X$ be a quasi-compact open immersion of schemes.*

(1) *Any quasi-coherent sheaf on U extends to a quasi-coherent sheaf on X.*
(2) *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.*
(3) *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on U. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.*

Proof. An immersion is separated (see Schemes, Lemma 26.23.8) and j is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on U the sheaf $j_*\mathcal{G}$ is an extension to X. See Schemes, Lemma 26.24.1 and Sheaves, Section 6.31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on X (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 26.24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 6.31 again).

Part (3) is proved in the same manner as (2). Just take $\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over U. □

01PF **Lemma 28.22.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U, V are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U\cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \mathrm{Spec}(R)$. Write $\mathcal{F} = \widetilde{M}$ for some R-module M. By Lemma 28.22.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over U. Write $\mathcal{H} = \widetilde{N}$ for some R-module N. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that N_f is finitely generated, see Lemma 28.16.1. Since U is quasi-compact we can cover it by finitely many $D(f_i)$ such that N_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G}' = \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. □

01PG **Lemma 28.22.3.** *Let X be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.*

Proof. The colimit is directed because if $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then the image of $\mathcal{G}_1 \oplus \mathcal{G}_2 \to \mathcal{F}$ is a quasi-coherent submodule of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by s. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 28.22.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since X has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. □

01PI **Lemma 28.22.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which is of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism*

of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.

Proof. The beginning of the proof is a repeat of the beginning of the proof of Lemma 28.22.2. We write it out carefully anyway.

Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U\cap V}, \varphi|_{U\cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over V. Then we can glue $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$, and similarly we can glue the map φ' to the map φ along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \mathrm{Spec}(R)$. By Lemma 28.22.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over X which restricts to $\mathcal{G}$ and φ over U. By Lemma 28.22.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and ψ by the restriction of ψ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 28.16.2 we conclude that $\mathcal{H} = \widetilde{N}$ with N a finitely generated R-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 17.11.4 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 28.22.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on U with φ' equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. □

0G41 **Lemma 28.22.5.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module.*

(1) *If $\mathcal{G}$ is quasi-coherent and of finite type, then there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*
(2) *If $\mathcal{G}$ is of finite presentation, then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Part (2) is the special case of Lemma 28.22.4 where $\mathcal{F} = 0$. For part (1) we first write $\mathcal{G} = \mathcal{F}|_U$ for some quasi-coherent $\mathcal{O}_X$-module by Lemma 28.22.1 and then we apply Lemma 28.22.2 with $\mathcal{G} = \mathcal{F}|_U$. □

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over directed sets in the next lemma.

01PJ **Lemma 28.22.6.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

(1) *a filtered index category $\mathcal{I}$ (see Categories, Definition 4.19.1),*
(2) *a diagram $\mathcal{I} \to Mod(\mathcal{O}_X)$ (see Categories, Section 4.14), $i \mapsto \mathcal{F}_i$,*
(3) *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. Choose a set I and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 17.9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $\mathrm{Ob}(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$\mathrm{Mor}_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ is a filtered category and that $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of φ_i and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{array}{ccc} \mathcal{F}_i & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \operatorname{Coker}(\mathcal{F}_i \xrightarrow{\alpha - \beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some i''. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes α and β. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : \operatorname{colim}_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that φ_x is an isomorphism. Recall that

$$(\operatorname{colim}_i \mathcal{F}_i)_x = \operatorname{colim}_i \mathcal{F}_{i,x},$$

see Sheaves, Section 6.29. First we show that the map φ_x is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that s maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open U such that s comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 28.22.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \operatorname{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by s: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map φ_i factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section s maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(\operatorname{colim}_i \mathcal{F}_i)_x = \operatorname{colim}_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map φ_x is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of x such that s corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by s). By Lemma 28.22.4 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. □

01PK **Lemma 28.22.7.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

(1) *a directed set I (see Categories, Definition 4.21.1),*
(2) *a system $(\mathcal{F}_i, \varphi_{ii'})$ over I in Mod$(\mathcal{O}_X)$ (see Categories, Definition 4.21.2)*
(3) *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. This is a direct consequence of Lemma 28.22.6 and Categories, Lemma 4.21.5 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 6.29). □

086M **Lemma 28.22.8.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 28.22.7). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$. Choose sections $s_{jl} \in \mathcal{F}(U_j)$ generating $\mathcal{F}|_{U_j}$, see Lemma 28.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \operatorname{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 28.22.3). Then $\mathcal{F} = \operatorname{colim} \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

080V **Lemma 28.22.9.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open such that $\mathcal{F}|_U$ is of finite presentation. Then there exists a map of $\mathcal{O}_X$-modules $\varphi : \mathcal{G} \to \mathcal{F}$ with (a) $\mathcal{G}$ of finite presentation, (b) φ is surjective, and (c) $\varphi|_U$ is an isomorphism.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a directed colimit with each $\mathcal{F}_i$ of finite presentation, see Lemma 28.22.7. Choose a finite affine open covering $X = \bigcup V_j$ and choose finitely many sections $s_{jl} \in \mathcal{F}(V_j)$ generating $\mathcal{F}|_{V_j}$, see Lemma 28.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{F}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{F}_i$ be the kernel of the map $\mathcal{F}_i \to \mathcal{F}$. Since $\mathcal{F}_U$ is of finite presentation, we see that $\mathcal{K}|_U$ is of finite type, see Modules, Lemma 17.11.4. Hence we can find a finite type quasi-coherent submodule $\mathcal{K}' \subset \mathcal{K}$ with $\mathcal{K}'|_U = \mathcal{K}|_U$, see Lemma 28.22.2. Then $\mathcal{G} = \mathcal{F}_i/\mathcal{K}'$ with the given map $\mathcal{G} \to \mathcal{F}$ is a solution. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

05JS **Lemma 28.22.10.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist*

(1) *a directed set I (see Categories, Definition 4.21.1),*
(2) *a system $(\mathcal{A}_i, \varphi_{ii'})$ over I in the category of $\mathcal{O}_X$-algebras,*
(3) *morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$*

such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{A}_i = \mathcal{A}.$$

Proof. First we write $\mathcal{A} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 28.22.7. For each i let $\mathcal{B}_i = \operatorname{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \operatorname{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \operatorname{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 28.22.3. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \operatorname{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

05JT **Lemma 28.22.11.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. If $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are quasi-coherent $\mathcal{O}_X$-subalgebras of finite type, then the image of $\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2 \to \mathcal{A}$ is also a quasi-coherent $\mathcal{O}_X$-subalgebra of finite type (some details omitted) which contains both $\mathcal{A}_1$ and $\mathcal{A}_2$. In this way we see that the

system is directed. To show that $\mathcal{A}$ is the colimit of this system, write $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ as a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras as in Lemma 28.22.10. Then the images $\mathcal{A}'_i = \operatorname{Im}(\mathcal{A}_i \to \mathcal{A})$ are quasi-coherent subalgebras of $\mathcal{A}$ of finite type. Since $\mathcal{A}$ is the colimit of these the result follows. □

Let X be a scheme. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

086N **Lemma 28.22.12.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras such that all transition maps $\mathcal{A}_{i'} \to \mathcal{A}_i$ are surjective.*

Proof. By Lemma 28.22.8 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \operatorname{colim} \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 28.22.3). Set

$$\mathcal{A}_i = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjections, and $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an affine open covering $X = U_1 \cup \ldots \cup U_m$. Take generators $f_{j,1}, \ldots, f_{j,N_j} \in \Gamma(U_i, \mathcal{F})$. As $\mathcal{A}(U_j)$ is a finite $\mathcal{O}_X(U_j)$-algebra we see that for each k there exists a monic polynomial $P_{j,k} \in \mathcal{O}(U_j)[T]$ such that $P_{j,k}(f_{j,k})$ is zero in $\mathcal{A}(U_j)$. Since $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ by construction, we have $P_{j,k}(f_{j,k}) = 0$ in $\mathcal{A}_i(U_j)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

0817 **Lemma 28.22.13.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

(1) *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
(2) *$\mathcal{A}$ is a direct colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

Proof. By Lemma 28.22.11 we have $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (apply Algebra, Lemma 10.36.5 to $\mathcal{A}_i(U) \subset \mathcal{A}(U)$ for affine opens U in X). This proves (1).

To prove (2), write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 28.22.7. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above).

Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Modules, Lemma 17.22.8. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 28.22.12). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 28.22.12 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

28.23. Gabber's result

077K In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent κ-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

077L **Definition 28.23.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *κ-generated* if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most κ.

Note that a direct sum of at most κ κ-generated modules is again κ-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. In particular this holds for the direct sum of two κ-generated modules. Moreover, a quotient of a κ-generated sheaf is κ-generated. (But the same needn't be true for submodules.)

077M **Lemma 28.23.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of κ-generated $\mathcal{O}_X$-modules such that every κ-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.*

Proof. There is a set of coverings of X (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\oplus_{k \in \kappa} \mathcal{O}_X \to \mathcal{F}$ for any ringed space X. This is clear. □

Here is the result the title of this section refers to.

077N **Lemma 28.23.3.** *Let X be a scheme. There exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent κ-generated submodules.*

[2]Nicely explained in a blog post by Akhil Mathew.

Proof. Choose an affine open covering $X = \bigcup_{i \in I} U_i$. For each pair i, j choose an affine open covering $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

see Schemes, Lemma 26.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S_i')$ where

$$S_j' = \bigcup\nolimits_{(i,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S_i'$. Note that S_i' has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of M_{ijk}. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is κ-generated.

Let $\{\mathcal{G}_t\}_{t \in T}$ be the set of κ-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\mathrm{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

077P **Proposition 28.23.4.** *Let X be a scheme.*

(1) *The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.*
(2) *The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*[3]

$$Q : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

[3]This functor is sometimes called the *coherator*.

Proof. Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Schemes, Section 26.24 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules, Lemma 17.3.2 filtered colimits are exact. Hence (a) and (b) hold. To construct a generator U, pick a cardinal κ as in Lemma 28.23.3. Pick a collection $(\mathcal{F}_t)_{t\in T}$ of κ-generated quasi-coherent sheaves as in Lemma 28.23.2. Set $U = \bigoplus_{t\in T} \mathcal{F}_t$. Since every object of $QCoh(\mathcal{O}_X)$ is a filtered colimit of κ-generated quasi-coherent modules, i.e., of objects isomorphic to $\mathcal{F}_t$, it is clear that U is a generator. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful. □

28.24. Sections with support in a closed subset

07ZM Given any topological space X, a closed subset $Z \subset X$, and an abelian sheaf $\mathcal{F}$ you can take the subsheaf of sections whose support is contained in Z. If X is a scheme, Z a closed subscheme, and $\mathcal{F}$ a quasi-coherent module there is a variant where you take sections which are scheme theoretically supported on Z. However, in the scheme setting you have to be careful because the resulting $\mathcal{O}_X$-module may not be quasi-coherent.

01PH **Lemma 28.24.1.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be an open subscheme. The following are equivalent:*

(1) *U is retrocompact in X,*
(2) *U is quasi-compact,*
(3) *U is a finite union of affine opens, and*
(4) *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $X \setminus U = V(\mathcal{I})$ (set theoretically).*

Proof. The equivalence of (1), (2), and (3) follows from Lemma 28.2.3. Assume (1), (2), (3). Let $T = X \setminus U$. By Schemes, Lemma 26.12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on T. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 28.22.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $X \setminus U = V(\mathcal{I})$ and we obtain (4). Conversely, if $\mathcal{I}$ is as in (4) and $W = \mathrm{Spec}(R) \subset X$ is an affine open, then $\mathcal{I}|_W = \widetilde{I}$ for some finitely generated ideal $I \subset R$, see Lemma 28.16.1. It follows that $U \cap W = \mathrm{Spec}(R) \setminus V(I)$ is quasi-compact, see Algebra, Lemma 10.29.1. Hence $U \subset X$ is retrocompact by Lemma 28.2.6. □

01PO **Lemma 28.24.2.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

(1) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
(2) *on any affine open $U \subset X$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
(3) *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on X to verify the other statements. In other words we may assume that $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\widetilde{I}$ is generated by its global sections I which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 28.16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every i there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one m which works for all i (and this is where we use that I is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. □

01PP **Definition 28.24.3.** Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 28.24.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

07ZN **Lemma 28.24.4.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. (Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent so that Lemma 28.24.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$.) □

For an abelian sheaf on a topological space we have discussed the subsheaf of sections with support in a closed subset in Modules, Remark 17.6.2. For quasi-coherent modules this submodule isn't always a quasi-coherent module, but if the closed subset has a retrocompact complement, then it is.

07ZP **Lemma 28.24.5.** *Let X be a scheme. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \text{the support of } s \text{ is contained in } Z \cap U\}$$

If $X \setminus Z$ is a retrocompact open of X, then

(1) *for an affine open $U \subset X$ there exist a finitely generated ideal $I \subset \mathcal{O}_X(U)$ such that $Z \cap U = V(I)$,*
(2) *for U and I as in (1) we have $\mathcal{F}'(U) = \{x \in \mathcal{F}(U) \mid I^n x = 0 \text{ for some } n\}$,*
(3) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. Part (1) is Algebra, Lemma 10.29.1. Let $U = \mathrm{Spec}(A)$ and I be as in (1). Then $\mathcal{F}|_U$ is the quasi-coherent sheaf associated to some A-module M. We have

$$\mathcal{F}'(U) = \{x \in M \mid x = 0 \text{ in } M_\mathfrak{p} \text{ for all } \mathfrak{p} \notin Z\}.$$

by Modules, Definition 17.5.1. Thus $x \in \mathcal{F}'(U)$ if and only if $V(\mathrm{Ann}(x)) \subset V(I)$, see Algebra, Lemma 10.40.7. Since I is finitely generated this is equivalent to $I^n x = 0$ for some n. This proves (2).

Proof of (3). Observe that given $U \subset X$ open there is an exact sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$$

If we denote $j : X \setminus Z \to X$ the inclusion morphism, then we observe that $\mathcal{F}(U \setminus Z)$ is the sections of the module $j_*(\mathcal{F}|_{X\setminus Z})$ over U. Thus we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to j_*(\mathcal{F}|_{X\setminus Z})$$

The restriction $\mathcal{F}|_{X\setminus Z}$ is quasi-coherent. Hence $j_*(\mathcal{F}|_{X\setminus Z})$ is quasi-coherent by Schemes, Lemma 26.24.1 and our assumption that j is quasi-compact (any open immersion is separated). Hence $\mathcal{F}'$ is quasi-coherent as a kernel of a map of quasi-coherent modules, see Schemes, Section 26.24. □

084L **Definition 28.24.6.** Let X be a scheme. Let $T \subset X$ be a closed subset whose complement is retrocompact in X. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 28.24.5 is called the *subsheaf of sections supported on* T.

07ZQ **Lemma 28.24.7.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $Z \subset Y$ be a closed subset such that $Y \setminus Z$ is retrocompact in Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported in $f^{-1}Z$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported in Z.*

Proof. Omitted. (Hint: First show that $X \setminus f^{-1}Z$ is retrocompact in X as $Y \setminus Z$ is retrocompact in Y. Hence Lemma 28.24.5 applies to $f^{-1}Z$ and $\mathcal{F}$. As f is quasi-compact and quasi-separated we see that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 28.24.5 applies to Z and $f_*\mathcal{F}$. Finally, match the sheaves directly.) □

28.25. Sections of quasi-coherent sheaves

01PL Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

01PM **Lemma 28.25.1.** *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Then there is a canonical map*

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

Proof. Since $I^{n+1} \subset I^n$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of A-modules

$$\mathrm{Hom}_A(I^n, M) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M)$$

and

$$\mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an A-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \widetilde{I^n} \to \widetilde{M}$ which we can restrict to the open $\mathrm{Spec}(A) \setminus V(I)$. Since $\widetilde{I^n}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $\mathrm{Hom}_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $\mathrm{Spec}(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $\mathrm{Spec}(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an A-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if φ maps to zero, then for each i the element $\varphi(f_i^n)/f_i^n$ is zero in M_{f_i}. In other words we see that for each i we have $f_i^m \varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single m which works for all i. Then we see that $\varphi(f_i^{n+m}) = 0$ for all i. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that φ maps to zero in the $t(n+m-1)+1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector x_i/f_i^n with $x_i \in M$ of the first direct sum of the exact sequence above. Hence for each i, j there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in M. We may choose a single m which works for all pairs i, j. After replacing x_i by $f_i^m x_i$ and n by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in M for all i, j. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an A-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1)+1$ to the class modulo K_n of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where i is chosen such that $e_i \geq n$ (note that there is at least one such i). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients a_E in A. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)} - n} \ldots f_t^{e_t} x_{i(E)}$$

where for each multiindex E we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by f_j^n for any j, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j+n} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_{i(E)} \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in M/K_n. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map φ in particular gives rise to an A-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. □

01PN **Example 28.25.2.** We will give two examples showing that the first displayed map of Lemma 28.25.1 is not an isomorphism.

Let k be a field. Consider the ring

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n).$$

Set $I = (x)$ and let $M = A$. Then the element y/x defines a section of the structure sheaf of $\mathrm{Spec}(A)$ over $D(x) = \mathrm{Spec}(A) \setminus V(I)$. We claim that y/x is not in the image of the canonical map $\mathrm{colim} \mathrm{Hom}_A(I^n, A) \to A_x = \mathcal{O}(D(x))$. Namely, if so it would come from a homomorphism $\varphi : I^n \to A$ for some n. Set $a = \varphi(x^n)$. Then we would have $x^m(xa - x^n y) = 0$ for some $m > 0$. This would mean that $x^{m+1}a = x^{m+n}y$. This would mean that $\varphi(x^{n+m+1}) = x^{m+n}y$. This leads to a contradiction because it would imply that

$$0 = \varphi(0) = \varphi(z_{n+m+1}x^{n+m+1}) = x^{m+n}yz_{n+m+1}$$

which is not true in the ring A.

Let k be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n\geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n\geq 1}).$$

Set $I = (f, g)$ and let $M = A$. Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of A_{fg}. Hence these define a section s of the structure sheaf of $\mathrm{Spec}(A)$ over $D(f) \cup D(g) = \mathrm{Spec}(A) \setminus V(I)$. However, there is no $n \geq 0$ such that s comes from an A-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 28.25.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1}x$ and $g^m y_n = g^{n+m-1}y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m}f^{n+m} + b_{n+m}g^{n+m}) = a_{n+m}f^{n+m-1}x + b_{n+m}g^{n+m-1}y$ which is not the case in the ring A.

We will improve on the following lemma in the Noetherian case, see Cohomology of Schemes, Lemma 30.10.5.

01PQ **Lemma 28.25.3.** *Let X be a quasi-compact scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The canonical map*

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

is injective. Assume further that X is quasi-separated. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is an isomorphism.

Proof. Let $\operatorname{Spec}(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some A-module M and $\mathcal{I}|_W = \widetilde{I}$ for some finite type ideal $I \subset A$. Restricting the first displayed map of the lemma to W we obtain the first displayed map of Lemma 28.25.1. Since we can cover X by a finite number of affine opens this proves the first displayed map of the lemma is injective.

We have $\mathcal{F}_n|_W = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 28.25.1 (details omitted). The lemma guarantees that we have a bijection

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open W.

To see the second displayed arrow of the lemma is bijective, we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity follows immediately from the above and the finiteness of the covering. If X is quasi-separated, then for each pair j, j' we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each j there exists an n_j and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{U \cap W_j}$. By the same token for each triple (j, j', k) there exists an integer $n_{jj'k}$ such that the restriction of φ_j and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$ agree over $W_{jj'k}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the φ_j glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over X. This proves surjectivity of the map. □

28.26. Ample invertible sheaves

01PR Recall from Modules, Lemma 17.25.10 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

01PS **Definition 28.26.1.** Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if [**DG67**, II Definition 4.5.3]

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

01PT **Lemma 28.26.2.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.* [**DG67**, II Proposition 4.5.6(i)]

Proof. This follows from the fact that $X_{s^n} = X_s$. □

01PU **Lemma 28.26.3.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to Z is ample.*

Proof. This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 26.8.2). □

01PV **Lemma 28.26.4.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.*

Proof. This translates into the following algebra problem. Let R be a ring. Let N be an invertible R-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $V = \{\mathfrak{p} \mid s \notin \mathfrak{p}N\}$ is an affine open subset of $\mathrm{Spec}(R)$.

Let $A = \bigoplus_{n \geq 0} A_n$ be the symmetric algebra of N (which is commutative) and view s as an element of A_1. Set $B = A/(s-1)A$. This is an R-algebra whose construction commutes with any base change $R \to R'$. Thus $B' = B \otimes_R R'$ is the zero ring if s maps to zero in $N' = N \otimes_R R'$. It follows that if $x \in \mathrm{Spec}(R) \setminus V$, then $B \otimes_R \kappa(x) = 0$. We conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ factors through V as the fibres over $x \notin V$ are empty. On the other hand, if $\mathrm{Spec}(R') \subset V$ is an affine open, then s maps to a basis element of N' and we see that $B' = R'[s]/(s-1) \cong R'$. It follows that $\mathrm{Spec}(B) \to V$ is an isomorphism and V is indeed affine. □

0890 **Lemma 28.26.5.** *Let X be a scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules. If* [**DG67**, II Proposition 4.5.6(ii)]

(1) *$\mathcal{L}$ is ample, and*
(2) *the open sets X_t where $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ for $m > 0$ cover X,*

then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Proof. We check the conditions of Definition 28.26.1. As $\mathcal{L}$ is ample we see that X is quasi-compact. Let $x \in X$. Choose $n \geq 1$, $m \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$, and $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ such that $x \in X_s$, $x \in X_t$ and X_s is affine. Then $s^m t^n \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^{\otimes nm})$, $x \in X_{s^m t^n}$, and $X_{s^m t^n}$ is affine by Lemma 28.26.4. □

01PX **Lemma 28.26.6.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on X. Then among those opens, the open sets X_s which are affine form a basis for the topology on X.*

Proof. Let $x \in X$. Choose an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ of x. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 28.26.4 above the intersection $X_s = U \cap X_s$ is affine. Since U can be chosen arbitrarily small we win. □

01PY **Lemma 28.26.7.** *Let X be a scheme and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point x of X there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine. Then X is separated.*

Proof. We show first that X is quasi-separated. By assumption we can find a covering of X by affine opens of the form X_s. By Lemma 28.26.4, the intersection of any two such sets is affine, so Schemes, Lemma 26.21.6 implies that X is quasi-separated.

To show that X is separated, we can use the valuative criterion, Schemes, Lemma 26.22.2. Thus, let A be a valuation ring with fraction field K and consider two morphisms $f, g : \mathrm{Spec}(A) \to X$ such that the two compositions $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \to X$ agree. As A is local, there exists $p, q \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes p})$, and $t \in \Gamma(X, \mathcal{L}^{\otimes q})$ such that X_s and X_t are affine, $f(\mathrm{Spec}\, A) \subseteq X_s$, and $g(\mathrm{Spec}\, A) \subseteq X_t$. We now replace s by s^q, t by t^p, and $\mathcal{L}$ by $\mathcal{L}^{\otimes pq}$. This is harmless as $X_s = X_{s^q}$ and $X_t = X_{t^p}$, and now s and t are both sections of the same sheaf $\mathcal{L}$.

The quasi-coherent module $f^*\mathcal{L}$ corresponds to an A-module M and $g^*\mathcal{L}$ corresponds to an A-module N by our classification of quasi-coherent modules over affine schemes (Schemes, Lemma 26.7.4). The A-modules M and N are locally free of rank 1 (Lemma 28.20.1) and as A is local they are free (Algebra, Lemma 10.55.8). Therefore we may identify M and N with A-submodules of $M \otimes_A K$ and $N \otimes_A K$. The equality $f|_{\mathrm{Spec}(K)} = g|_{\mathrm{Spec}(K)}$ determines an isomorphism $\phi\colon M \otimes_A K \to N \otimes_A K$.

Let $x \in M$ and $y \in N$ be the elements corresponding to the pullback of s along f and g, respectively. These satisfy $\phi(x \otimes 1) = y \otimes 1$. The image of f is contained in X_s, so $x \notin \mathfrak{m}_A M$, that is, x generates M. Hence ϕ determines an isomorphism of M with the submodule of N generated by y. Arguing symmetrically using t, ϕ^{-1} determines an isomorphism of N with a submodule of M. Consequently ϕ restricts to an isomorphism of M and N. Since x generates M, its image y generates N, implying $y \notin \mathfrak{m}_A N$. Therefore $g(\mathrm{Spec}(A)) \subseteq X_s$. Because X_s is affine, it is separated by Schemes, Lemma 26.21.15, and we conclude $f = g$. □

09MP **Lemma 28.26.8.** *Let X be a scheme. If there exists an ample invertible sheaf on X then X is separated.*

Proof. Follows immediately from Lemma 28.26.7 and Definition 28.26.1. □

01PZ **Lemma 28.26.9.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = \mathit{Proj}(S),$$

to the homogeneous spectrum of S (see Constructions, Section 27.8). This morphism has the following properties

1. *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
2. *there are $\mathcal{O}_X$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (27.10.1.1),*
3. *the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and*
4. *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 27.14.1. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 27.14.1 since the map α mentioned there is an isomorphism. □

01Q0 **Lemma 28.26.10.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow \mathit{Proj}(S)$ of Lemma 28.26.9 above is quasi-compact with dense image.*

Proof. To prove f is quasi-compact it suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 28.26.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. Assume that the image of f is not dense to get a contradiction. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\text{Proj}(S)$, we can find such an s with $D_+(s) \neq \emptyset$ and $f(X) \cap D_+(s) = \emptyset$. By Lemma 28.26.9 this means $X_s = \emptyset$. By Lemma 28.17.2 this means that a power s^n is the zero section of $\mathcal{L}^{\otimes n \deg(s)}$. This in turn means that $D_+(s) = \emptyset$ which is the desired contradiction. □

01Q1 **Lemma 28.26.11.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 28.26.9 is an open immersion with dense image.*

Proof. By Lemma 28.26.7 we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 28.26.9. By Lemma 28.17.2 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\text{Proj}(S)$. The image is dense by Lemma 28.26.10. □

01Q2 **Lemma 28.26.12.** *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma 27.10.6. □

01Q3 **Proposition 28.26.13.** *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

01Q4 (1) *$\mathcal{L}$ is ample,*

01Q5 (2) *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*

01Q6 (3) *the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,*

01Q7 (4) *the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,*

01Q8 (5) *for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps*

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

with $n \geq 1$ equals $\mathcal{F}$,

01Q9 (6) *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*

01QA (7) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*

01QB (8) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and*

01QC (9) *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.*

Proof. Lemma 28.26.11 is (1) $\Rightarrow$ (2). Lemmas 28.26.2 and 28.26.12 provide the implication (1) $\Leftarrow$ (2). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear from Constructions, Section 27.8. Lemma 28.26.6 is (3) $\Rightarrow$ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\mathrm{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma 28.17.2 we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section 26.12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma 28.26.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 17.4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 17.9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally

generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 17.9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 28.22.3 (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

0B3E **Lemma 28.26.14.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $i : X' \to X$ be a morphism of schemes. Assume at least one of the following conditions holds*

(1) *i is a quasi-compact immersion,*
(2) *X' is quasi-compact and i is an immersion,*
(3) *i is quasi-compact and induces a homeomorphism between X' and $i(X')$,*
(4) *X' is quasi-compact and i induces a homeomorphism between X' and $i(X')$.*

Then $i^*\mathcal{L}$ is ample on X'.

Proof. Observe that in cases (1) and (3) the scheme X' is quasi-compact as X is quasi-compact by Definition 28.26.1. Thus it suffices to prove (2) and (4). Since (2) is a special case of (4) it suffices to prove (4).

Assume condition (4) holds. For $s \in \Gamma(X, \mathcal{L}^{\otimes d})$ denote $s' = i^*s$ the pullback of s to X'. Note that s' is a section of $(i^*\mathcal{L})^{\otimes d}$. By Proposition 28.26.13 the opens X_s, for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, form a basis for the topology on X. Since $X'_{s'} = i^{-1}(X_s)$ by Modules, Remark 17.25.11 and since $X' \to i(X')$ is a homeomorphism, we conclude the opens $X'_{s'}$ form a basis for the topology of X'. Hence $i^*\mathcal{L}$ is ample by Proposition 28.26.13. □

0DNK **Lemma 28.26.15.** *Let S be a quasi-separated scheme. Let X, Y be schemes over S. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $\mathcal{N}$ be an ample invertible $\mathcal{O}_Y$-module. Then $\mathcal{M} = pr_1^*\mathcal{L} \otimes_{\mathcal{O}_{X \times_S Y}} pr_2^*\mathcal{N}$ is an ample invertible sheaf on $X \times_S Y$.*

Proof. The morphism $i : X \times_S Y \to X \times Y$ is a quasi-compact immersion, see Schemes, Lemma 26.21.9. On the other hand, $\mathcal{M}$ is the pullback by i of the corresponding invertible module on $X \times Y$. By Lemma 28.26.14 it suffices to prove the lemma for $X \times Y$. We check (1) and (2) of Definition 28.26.1 for $\mathcal{M}$ on $X \times Y$.

Since X and Y are quasi-compact, so is $X \times Y$. Let $z \in X \times Y$ be a point. Let $x \in X$ and $y \in Y$ be the projections. Choose $n > 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x. Choose $m > 0$ and $t \in \Gamma(Y, \mathcal{N}^{\otimes m})$ such that Y_t is an affine open neighbourhood of y. Then $r = \mathrm{pr}_1^* s \otimes \mathrm{pr}_2^* t$ is a section of $\mathcal{M}$ with $(X \times Y)_r = X_s \times Y_t$. This is an affine open neighbourhood of z and the proof is complete. □

28.27. Affine and quasi-affine schemes

01QD

01QE **Lemma 28.27.1.** *Let X be a scheme. Then X is quasi-affine if and only if $\mathcal{O}_X$ is ample.*

Proof. Suppose that X is quasi-affine. Set $A = \Gamma(X, \mathcal{O}_X)$. Consider the open immersion

$$j : X \longrightarrow \mathrm{Spec}(A)$$

from Lemma 28.18.4. Note that $\mathrm{Spec}(A) = \mathrm{Proj}(A[T])$, see Constructions, Example 27.8.14. Hence we can apply Lemma 28.26.12 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong A[T]$ as graded rings. Hence the result follows from Lemmas 28.26.11 and 28.18.4 taking into account that $\mathrm{Spec}(A) = \mathrm{Proj}(A[T])$ for any ring A as seen above. □

0BCK **Lemma 28.27.2.** *Let X be a quasi-affine scheme. For any quasi-compact immersion $i : X' \to X$ the scheme X' is quasi-affine.*

Proof. This can be proved directly without making use of the material on ample invertible sheaves; we urge the reader to do this on a napkin. Since X is quasi-affine, we have that $\mathcal{O}_X$ is ample by Lemma 28.27.1. Then $\mathcal{O}_{X'}$ is ample by Lemma 28.26.14. Then X' is quasi-affine by Lemma 28.27.1. □

01QF **Lemma 28.27.3.** *Let X be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

(1) *each X_{f_i} is an affine open of X, and*
(2) *the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.*

Then X is affine.

Proof. Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The f_i's cannot all vanish at a point.) Since each X_{f_i} is quasi-compact (being affine) it follows that X is quasi-compact. Hence we see that X is quasi-affine by Lemma 28.27.1 above. Consider the open immersion

$$j : X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 28.18.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to X_{f_i} on the left hand side and the morphism j induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 28.18.3. Since the f_i generate the unit ideal we see that $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus j is an isomorphism. □

28.28. Quasi-coherent sheaves and ample invertible sheaves

01QG Theme of this section: in the presence of an ample invertible sheaf every quasi-coherent sheaf comes from a graded module.

01QH **Situation 28.28.1.** Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \text{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 28.26.9. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

01QI **Lemma 28.28.2.** *In Situation 28.28.1. The canonical morphism $f : X \to Y$ maps X into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms (see Constructions, Lemma 27.10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.*

Proof. By Proposition 28.26.13 there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that a and b do not vanish at x. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 27.10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 27.14.1 which was used in the construction of the map f the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of x. By compatibility with the algebra structure and the fact that f maps into W we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of x. Hence we win. □

Recall from Modules, Definition 17.25.7 that given a locally ringed space X, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 28.28.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 27.13.8 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

01QJ **Lemma 28.28.3.** *In Situation 28.28.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded S-module. There are isomorphisms*

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(\text{Proj}(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.

Proof. Let $s \in S_+$ be homogeneous such that X_s is affine open in X. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 27.8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As X carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 28.26. By Lemma 28.17.2 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on X we know that X is covered by the affine opens of the form X_s. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. □

01QK **Remark 28.28.4.** With assumptions and notation of Lemma 28.28.3. Denote the displayed map of the lemma by $\theta_{\mathcal{F}}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 28.28.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (27.10.1.5). Pull this back to X and consider

$$\begin{array}{ccc} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) & \longrightarrow & f^*\widetilde{M(n)} \\ \downarrow{\scriptstyle \theta_{\mathcal{F}} \otimes \theta_{\mathcal{L}^{\otimes n}}} & & \downarrow{\scriptstyle \theta_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}} \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} & \xrightarrow{\text{id}} & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{array}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

It should be possible to deduce the following lemma from Lemma 28.28.3 (or conversely) but it seems simpler to just repeat the proof.

0AG5 **Lemma 28.28.5.** *Let S be a graded ring such that $X = Proj(S)$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ as a graded S-module, see Constructions, Section 27.10. The map*

$$\widetilde{M} \longrightarrow \mathcal{F}$$

of Constructions, Lemma 27.10.7 is an isomorphism. If X is covered by standard opens $D_+(f)$ where f has degree 1, then the induced maps $M_n \to \Gamma(X, \mathcal{F}(n))$ are the identity maps.

Proof. Since X is quasi-compact we can find homogeneous elements $f_1, \ldots, f_n \in S$ of positive degrees such that $X = D_+(f_1) \cup \ldots \cup D_+(f_n)$. Let d be the least common multiple of the degrees of $f_1, \ldots, f_n$. After replacing f_i by a power we may assume that each f_i has degree d. Then we see that $\mathcal{L} = \mathcal{O}_X(d)$ is invertible, the multiplication maps $\mathcal{O}_X(ad) \otimes \mathcal{O}_X(bd) \to \mathcal{O}_X((a+b)d)$ are isomorphisms, and each f_i determines a global section s_i of $\mathcal{L}$ such that $X_{s_i} = D_+(f_i)$, see Constructions, Lemmas 27.10.4 and 27.10.5. Thus $\Gamma(X, \mathcal{F}(ad)) = \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$. Recall that $\widetilde{M}|_{D_+(f_i)}$ corresponds to the $S_{(f_i)}$-module $M_{(f_i)}$, see Constructions, Lemma 27.8.4. Since the degree of f_i is d, the isomorphism class of $M_{(f_i)}$ depends only on the homogeneous summands of M of degree divisible by d. More precisely, the isomorphism class of $M_{(f_i)}$ depends only on the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F})$ and the image s_i of f_i in $\Gamma_*(X, \mathcal{L})$. The scheme X is quasi-compact by assumption and separated by Constructions, Lemma 27.8.8. By Lemma 28.17.2 there is a canonical isomorphism

$$M_{(f_i)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s_i)} \to \Gamma(X_{s_i}, \mathcal{F}).$$

The construction of the map in Constructions, Lemma 27.10.7 then shows that it is an isomorphism over $D_+(f_i)$ hence an isomorphism as X is covered by these opens. We omit the proof of the final statement. □

28.29. Finding suitable affine opens

01ZU In this section we collect some results on the existence of affine opens in more and less general situations.

01ZV **Lemma 28.29.1.** *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X, see Topology, Section 5.8. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 26.11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.*

Proof. Let V_i be any affine open containing η_i and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since X is quasi-separated for each i the union $W_i = \bigcup_{j,j\neq i} V_i \cap V_j$ is a quasi-compact open of V_i not containing η_i. We can find open neighbourhoods $U_i \subset V_i$ containing η_i and disjoint from W_i by Algebra, Lemma 10.26.4. Finally, U is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 26.6.8. □

01ZW **Remark 28.29.2.** Lemma 28.29.1 above is false if X is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x-i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of R. Glue two copies of $\mathrm{Spec}(R)$ along the (not quasi-compact) open $\mathrm{Spec}(R) \setminus V(\mathfrak{p})$ to get a scheme X (glueing as in Schemes, Example 26.14.3). Then the two maximal points of X corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$ contains infinitely many of the "lines" $x = i$, $y_j = 0$, $j \neq i$ with parameter y_i. Details omitted.

Notwithstanding the example above, for "most" finite sets of irreducible closed subsets one can apply Lemma 28.29.1 above, at least if X is quasi-compact. This is true because X contains a dense open which is separated.

03J1 **Lemma 28.29.3.** *Let X be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.*

Proof. Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of n affine open subschemes. We will prove the lemma by induction on n. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \amalg (U_n \setminus \overline{V'}).$$

It is clear that V is separated and a dense open subscheme of X. □

It turns out that, even if X is quasi-separated as well as quasi-compact, there does not exist a separated, quasi-compact dense open, see Examples, Lemma 110.27.2. Here is a slight refinement of Lemma 28.29.1 above.

01ZX **Lemma 28.29.4.** *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing x and all the η_i.*

Proof. Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \notin Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 28.29.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since X is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain η_i for

$i = r+1, \ldots, n$. Hence by Algebra, Lemma 10.26.4 we may shrink U_i such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 26.6.8) a suitable affine open. □

01ZY **Lemma 28.29.5.** *Let X be a scheme. Assume either*

(1) *The scheme X is quasi-affine.*
(2) *The scheme X is isomorphic to a locally closed subscheme of an affine scheme.*
(3) *There exists an ample invertible sheaf on X.*
(4) *The scheme X is isomorphic to a locally closed subscheme of $Proj(S)$ for some graded ring S.*

Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.

Proof. By Properties, Definition 28.18.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $\mathrm{Spec}(R)$ is isomorphic to $\mathrm{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Proposition 28.26.13 if X has an ample invertible sheaf then X is isomorphic to an open subscheme of $\mathrm{Proj}(S)$ for some graded ring S. Hence, it suffices to prove the lemma in case (4). (We urge the reader to prove case (2) directly for themselves.)

Thus assume $X \subset \mathrm{Proj}(S)$ is a locally closed subscheme where S is some graded ring. Let $T = \overline{X} \setminus X$. Recall that the standard opens $D_+(f)$ form a basis of the topology on $\mathrm{Proj}(S)$. Since E is finite we may choose finitely many homogeneous elements $f_i \in S_+$ such that

$$E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset \mathrm{Proj}(S) \setminus T$$

Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\mathrm{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 10.57.6 that there exists a homogeneous element $f \in I$, $f \notin \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$. Since $D_+(f)$ does not meet T we see that $X \cap D_+(f)$ is a closed subscheme of the affine scheme $D_+(f)$, hence is an affine open of X as desired. □

09NV **Lemma 28.29.6.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let*

$$E \subset W \subset X$$

with E finite and W open in X. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $E \subset X_s \subset W$.

Proof. The reader can modify the proof of Lemma 28.29.5 to prove this lemma; we will instead deduce the lemma from it. By Lemma 28.29.5 we can choose an affine open $U \subset W$ such that $E \subset U$. Consider the graded ring $S = \Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. For each $x \in E$ let $\mathfrak{p}_x \subset S$ be the graded ideal of sections vanishing at x. It is clear that $\mathfrak{p}_x$ is a prime ideal and since some power of $\mathcal{L}$ is globally generated, it is clear that $S_+ \not\subset \mathfrak{p}_x$. Let $I \subset S$ be the graded ideal of sections vanishing on all points of $X \setminus U$. Since the sets X_s form a basis for the topology we see that $I \not\subset \mathfrak{p}_x$ for all $x \in E$. By (graded) prime avoidance (Algebra, Lemma 10.57.6) we can find $s \in I$ homogeneous with $s \notin \mathfrak{p}_x$ for all $x \in E$. Then $E \subset X_s \subset U$ and X_s is affine by Lemma 28.26.4. □

0F20 **Lemma 28.29.7.** *Let X be a quasi-affine scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $E \subset W \subset X$ with E finite and W open. Then there exists an $s \in \Gamma(X, \mathcal{L})$ such that X_s is affine and $E \subset X_s \subset W$.*

Proof. The proof of this lemma has a lot in common with the proof of Algebra, Lemma 10.15.2. Say $E = \{x_1, \ldots, x_n\}$. If $E = W = \emptyset$, then $s = 0$ works. If $W \neq \emptyset$, then we may assume $E \neq \emptyset$ by adding a point if necessary. Thus we may assume $n \geq 1$. We will prove the lemma by induction on n.

Base case: $n = 1$. After replacing W by an affine open neighbourhood of x_1 in W, we may assume W is affine. Combining Lemmas 28.27.1 and Proposition 28.26.13 we see that every quasi-coherent $\mathcal{O}_X$-module is globally generated. Hence there exists a global section s of $\mathcal{L}$ which does not vanish at x_1. On the other hand, let $Z \subset X$ be the reduced induced closed subscheme on $X \setminus W$. Applying global generation to the quasi-coherent ideal sheaf $\mathcal{I}$ of Z we find a global section f of $\mathcal{I}$ which does not vanish at x_1. Then $s' = fs$ is a global section of $\mathcal{L}$ which does not vanish at x_1 such that $X_{s'} \subset W$. Then $X_{s'}$ is affine by Lemma 28.26.4.

Induction step for $n > 1$. If there is a specialization $x_i \rightsquigarrow x_j$ for $i \neq j$, then it suffices to prove the lemma for $\{x_1, \ldots, x_n\} \setminus \{x_i\}$ and we are done by induction. Thus we may assume there are no specializations among the x_i. By either Lemma 28.29.5 or Lemma 28.29.6 we may assume W is affine. By induction we can find a global section s of $\mathcal{L}$ such that $X_s \subset W$ is affine and contains $x_1, \ldots, x_{n-1}$. If $x_n \in X_s$ then we are done. Assume s is zero at x_n. By the case $n = 1$ we can find a global section s' of $\mathcal{L}$ with $\{x_n\} \subset X_{s'} \subset W \setminus \overline{\{x_1, \ldots, x_{n-1}\}}$. Here we use that x_n is not a specialization of $x_1, \ldots, x_{n-1}$. Then $s + s'$ is a global section of $\mathcal{L}$ which is nonvanishing at $x_1, \ldots, x_n$ with $X_{s+s'} \subset W$ and we conclude as before. □

0BX3 **Lemma 28.29.8.** *Let X be a scheme and $x \in X$ a point. There exists an affine open neighbourhood $U \subset X$ of x such that the canonical map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective in each of the following cases:*

1. *X is integral,*
2. *X is locally Noetherian,*
3. *X is reduced and has a finite number of irreducible components.*

Proof. After translation into algebra, this follows from Algebra, Lemma 10.31.9. □

0H9B **Lemma 28.29.9.** *Let U, V be affine schemes and let $W \to U$ and $W \to V$ be open immersions. For any $w \in W$ there exists an affine open neighbourhood $W' \subset W$ of w such that W' maps to a standard open of both U and V.*

Proof. Let X be the glueing of U and V along W and apply Schemes, Lemma 26.11.5. □

28.30. Other chapters

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CHAPTER 29

Morphisms of Schemes

01QL

29.1. Introduction

01QM In this chapter we introduce some types of morphisms of schemes. A basic reference is [**DG67**].

29.2. Closed immersions

01QN In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \to X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X, (b) $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\sharp$ is locally generated by sections, see Schemes, Definitions 26.10.2 and 26.4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

01QO **Lemma 29.2.1.** *Let $i : Z \to X$ be a morphism of schemes. The following are equivalent:*

1. *The morphism i is a closed immersion.*
2. *For every affine open $\mathrm{Spec}(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \mathrm{Spec}(R/I)$ as schemes over $U = \mathrm{Spec}(R)$.*
3. *There exists an affine open covering $X = \bigcup_{j \in J} U_j$, $U_j = \mathrm{Spec}(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = \mathrm{Spec}(R_j/I_j)$ as schemes over $U_j = \mathrm{Spec}(R_j)$.*
4. *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
5. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*
6. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 26.4.1 and 26.10.2. So (6) $\Leftrightarrow$ (1). We have (1) $\Rightarrow$ (2) by Schemes, Lemma 26.10.1. Trivially (2) $\Rightarrow$ (3).

Assume (3). Each of the morphisms $\mathrm{Spec}(R_j/I_j) \to \mathrm{Spec}(R_j)$ is a closed immersion, see Schemes, Example 26.8.1. Hence $i^{-1}(U_j) \to U_j$ is a homeomorphism onto its image and $i^\sharp|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\sharp$ is surjective since this may be checked locally. We conclude that (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is Schemes, Lemma 26.24.2. The implication (5) $\Rightarrow$ (6) is trivial. And the implication (6) $\Rightarrow$ (5) follows from Schemes, Lemma 26.10.1. $\square$

01QP **Lemma 29.2.2.** *Let X be a scheme. Let $i : Z \to X$ and $i' : Z' \to X$ be closed immersions and consider the ideal sheaves $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ and $\mathcal{I}' = \mathrm{Ker}((i')^\sharp)$ of $\mathcal{O}_X$.*

(1) *The morphism $i : Z \to X$ factors as $Z \to Z' \to X$ for some $a : Z \to Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.*
(2) *We have $Z \cong Z'$ over X if and only if $\mathcal{I} = \mathcal{I}'$.*

Proof. This follows from our discussion of closed subspaces in Schemes, Section 26.4 especially Schemes, Lemmas 26.4.5 and 26.4.6. It also follows in a straightforward way from characterization (3) in Lemma 29.2.1 above. $\square$

01QQ **Lemma 29.2.3.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:*

(1) *$\mathcal{I}$ is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules,*
(2) *$\mathcal{I}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules, and*
(3) *there exists a closed immersion $i : Z \to X$ of schemes whose corresponding sheaf of ideals $\mathrm{Ker}(i^\sharp)$ is equal to $\mathcal{I}$.*

Proof. The equivalence of (1) and (2) is immediate from Schemes, Lemma 26.10.1. If (1) holds, then there is a closed subspace $i : Z \to X$ with $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ by Schemes, Definition 26.4.4 and Example 26.4.3. By Schemes, Lemma 26.10.1 this is a closed immersion of schemes and (3) holds. Conversely, if (3) holds, then (2) holds by Schemes, Lemma 26.10.1 (which applies because a closed immersion of schemes is a fortiori a closed immersion of locally ringed spaces). $\square$

01QR **Lemma 29.2.4.** *The base change of a closed immersion is a closed immersion.*

Proof. See Schemes, Lemma 26.18.2. $\square$

01QS **Lemma 29.2.5.** *A composition of closed immersions is a closed immersion.*

Proof. We have seen this in Schemes, Lemma 26.24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 29.2.1. Since if $I \subset R$ is an ideal, and $\overline{J} \subset R/I$ is an ideal, then $\overline{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\overline{J} = R/J$. $\square$

01QT **Lemma 29.2.6.** *A closed immersion is quasi-compact.*

Proof. This lemma is a duplicate of Schemes, Lemma 26.19.5. $\square$

01QU **Lemma 29.2.7.** *A closed immersion is separated.*

Proof. This lemma is a special case of Schemes, Lemma 26.23.8. $\square$

29.3. Immersions

07RJ In this section we collect some facts on immersions.

07RK **Lemma 29.3.1.** *Let $Z \to Y \to X$ be morphisms of schemes.*

(1) *If $Z \to X$ is an immersion, then $Z \to Y$ is an immersion.*
(2) *If $Z \to X$ is a quasi-compact immersion and $Y \to X$ is quasi-separated, then $Z \to Y$ is a quasi-compact immersion.*

(3) *If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Y \times_X Z & \longrightarrow & Z \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & X \end{array}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence an immersion by Schemes, Lemma 26.21.11. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence an immersion (Schemes, Lemma 26.18.2). Finally, a composition of immersions is an immersion (Schemes, Lemma 26.24.3). This proves (1). The other two results are proved in exactly the same manner. □

01QV **Lemma 29.3.2.** *Let $h : Z \to X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 26.23.8). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules by Schemes, Lemma 26.24.1. This implies that $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 26.24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Lemma 29.2.3. By Schemes, Lemma 26.4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \to X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U. Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \to U$. Hence we see that $Z = \overline{Z} \cap U$. □

03DQ **Lemma 29.3.3.** *Let $h : Z \to X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 26.12.5. By Schemes, Lemma 26.12.7 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \to X$ the inclusion morphism and $j : Z \to \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U. Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 26.12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. □

01QW **Example 29.3.4.** Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if

$n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

0FCZ **Lemma 29.3.5.** *Let $f : Y \to X$ be a morphism of schemes. If for all $y \in Y$ there is an open subscheme $f(y) \in U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an immersion, then f is an immersion.*

Proof. This statement follows readily from the discussion of closed subschemes at the end of Schemes, Section 26.10 but we will also give a detailed proof. Let $Z \subset X$ be the closure of $f(Y)$. Since taking closures commutes with restricting to opens, we see from the assumption that $f(Y) \subset Z$ is open. Hence $Z' = Z \setminus f(Y)$ is closed. Hence $X' = X \setminus Z'$ is an open subscheme of X and f factors as $f : Y \to X'$ followed by the inclusion. If $y \in Y$ and $U \subset X$ is as in the statement of the lemma, then $U' = X' \cap U$ is an open neighbourhood of $f'(y)$ such that $(f')^{-1}(U') \to U'$ is an immersion (Lemma 29.3.1) with closed image. Hence it is a closed immersion, see Schemes, Lemma 26.10.4. Since being a closed immersion is local on the target (for example by Lemma 29.2.1) we conclude that f' is a closed immersion as desired. □

29.4. Closed immersions and quasi-coherent sheaves

01QX The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 17.6.1 does for abelian sheaves. See also the discussion in Modules, Section 17.13.

01QY **Lemma 29.4.1.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 29.2.6 and 29.2.7. Hence Schemes, Lemma 26.24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X.

By Modules, Lemma 17.13.4 the functor i_* is fully faithful.

Now we turn to the description of the essential image of the functor i_*. We have $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent $\mathcal{O}_Z$-module, for example by Modules, Lemma 17.13.4. Next, suppose that $\mathcal{G}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism[1]. In the case of schemes and quasi-coherent modules, working affine locally on X and using Lemma 29.2.1 and Schemes, Lemma 26.7.3 it suffices

[1]This was proved in a more general situation in the proof of Modules, Lemma 17.13.4.

to prove the following algebraic statement: Given a ring R, an ideal I and an R-module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R-modules. Proof of this easy algebra fact is omitted. □

Let $i : Z \to X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

01QZ **Lemma 29.4.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 26.24. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

01R0 **Lemma 29.4.3.** *Let $i : Z \to X$ be a closed immersion of schemes. There is a functor[2] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 29.4.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subschemes (see Lemma 29.2.3) we can define scheme theoretic intersections and unions of closed subschemes.

0C4H **Definition 29.4.4.** Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The *scheme theoretic intersection* of Z and Y is the closed subscheme of X cut out by $\mathcal{I} + \mathcal{J}$. The *scheme theoretic union* of Z and Y is the closed subscheme of X cut out by $\mathcal{I} \cap \mathcal{J}$.

[2]This is likely nonstandard notation.

0C4I **Lemma 29.4.5.** *Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and*

$$\begin{array}{ccc} Z \cap Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is a cartesian diagram of schemes, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 29.2.2. Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $Z \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $Z \cap Y \cap U$ corresponds to $I + J \subset A$. Since $A/I \otimes_A A/J = A/(I + J)$ we see that the diagram is cartesian by our description of fibre products of schemes in Schemes, Section 26.17. □

0C4J **Lemma 29.4.6.** *Let S be a scheme. Let $X, Y \subset S$ be closed subschemes. Let $X \cup Y$ be the scheme theoretic union of X and Y. Let $X \cap Y$ be the scheme theoretic intersection of X and Y. Then $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions, there is a short exact sequence*

$$0 \to \mathcal{O}_{X \cup Y} \to \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_{X \cap Y} \to 0$$

of $\mathcal{O}_S$-modules, and the diagram

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup Y \end{array}$$

is cocartesian in the category of schemes, i.e., $X \cup Y = X \amalg_{X \cap Y} Y$.

Proof. The morphisms $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions by Lemma 29.2.2. In the short exact sequence we use the equivalence of Lemma 29.4.1 to think of quasi-coherent modules on closed subschemes of S as quasi-coherent modules on S. For the first map in the sequence we use the canonical maps $\mathcal{O}_{X \cup Y} \to \mathcal{O}_X$ and $\mathcal{O}_{X \cup Y} \to \mathcal{O}_Y$ and for the second map we use the canonical map $\mathcal{O}_X \to \mathcal{O}_{X \cap Y}$ and the negative of the canonical map $\mathcal{O}_Y \to \mathcal{O}_{X \cap Y}$. Then to check exactness we may work affine locally. Let $U = \mathrm{Spec}(A)$ be an affine open of S and let $X \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $(X \cup Y) \cap U$ corresponds to $I \cap J \subset A$ and $X \cap Y \cap U$ corresponds to $I + J \subset A$. Thus exactness follows from the exactness of

$$0 \to A/I \cap J \to A/I \times A/J \to A/(I + J) \to 0$$

To show the diagram is cocartesian, suppose we are given a scheme T and morphisms of schemes $f : X \to T$, $g : Y \to T$ agreeing as morphisms $X \cap Y \to T$. Goal: Show there exists a unique morphism $h : X \cup Y \to T$ agreeing with f and g. To construct h we may work affine locally on $X \cup Y$, see Schemes, Section 26.14. If $s \in X$, $s \notin Y$, then $X \to X \cup Y$ is an isomorphism in a neighbourhood of s and it is clear how to construct h. Similarly for $s \in Y$, $s \notin X$. For $s \in X \cap Y$ we can pick an affine open $V = \mathrm{Spec}(B) \subset T$ containing $f(s) = g(s)$. Then we can choose

an affine open $U = \mathrm{Spec}(A) \subset S$ containing s such that $f(X \cap U)$ and $g(Y \cap U)$ are contained in V. The morphisms $f|_{X\cap U}$ and $g|_{Y\cap V}$ into V correspond to ring maps

$$B \to A/I \quad \text{and} \quad B \to A/J$$

which agree as maps into $A/(I + J)$. By the short exact sequence displayed above there is a unique lift of these ring homomorphism to a ring map $B \to A/I \cap J$ as desired. □

29.5. Supports of modules

056H In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 17.5. On the other hand, the support of a module was defined in Algebra, Section 10.62. These match.

056I **Lemma 29.5.1.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

(1) *$\mathfrak{p}$ is in the support of M, and*
(2) *x is in the support of $\mathcal{F}$.*

Proof. This follows from the equality $\mathcal{F}_x = M_{\mathfrak{p}}$, see Schemes, Lemma 26.5.4 and the definitions. □

05AC **Lemma 29.5.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The support of $\mathcal{F}$ is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module $\mathcal{F}_x$. Hence the complement of $\mathrm{Supp}(\mathcal{F})$ is closed under generalization. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

056J **Lemma 29.5.3.** *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. Then*

(1) *The support of $\mathcal{F}$ is closed.*
(2) *For $x \in X$ we have*
$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$
(3) *For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$.*

Proof. Part (1) is a reformulation of Modules, Lemma 17.9.6. You can also combine Lemma 29.5.1, Properties, Lemma 28.16.1, and Algebra, Lemma 10.40.5 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.20.1. Let $f : Y \to X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 17.9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$

see Sheaves, Lemma 6.26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \mathrm{Supp}(\mathcal{F})$ if and only if $y \in \mathrm{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

05JU **Lemma 29.5.4.** *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. There exists a smallest closed subscheme $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

(1) *If $\mathrm{Spec}(A) \subset X$ is any affine open, and $\mathcal{F}|_{\mathrm{Spec}(A)} = \widetilde{M}$ then $Z \cap \mathrm{Spec}(A) = \mathrm{Spec}(A/I)$ where $I = Ann_A(M)$.*
(2) *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
(3) *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
(4) *The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.*

Proof. Suppose that $i' : Z' \to X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Lemma 29.4.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf $\mathcal{G}'$ on Z'. Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 28.16.1 and Algebra, Lemma 10.5.5 it follows that $\mathcal{G}'$ is of finite type. We have $\mathrm{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 10.40.5. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Lemma 29.2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A-module, then $\mathrm{Ann}_A(M)_f = \mathrm{Ann}_{A_f}(M_f)$. We omit the proof. □

05JV **Definition 29.5.5.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The *scheme theoretic support of $\mathcal{F}$* is the closed subscheme $Z \subset X$ constructed in Lemma 29.5.4.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 29.4.1).

29.6. Scheme theoretic image

01R5 Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

01R6 **Lemma 29.6.1.** *Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 29.2.3. This works by Lemma 29.2.2. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 29.4.2. □

01R7 **Definition 29.6.2.** Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 29.6.1 above.

For a morphism $f : X \to Y$ of schemes with scheme theoretic image Z we often denote $f : X \to Z$ the factorization of f through its scheme theoretic image. If the morphism f is not quasi-compact, then (in general)

(1) the set theoretic inclusion $\overline{f(X)} \subset Z$ is not an equality, i.e., $f(X) \subset Z$ is not a dense subset, and

(2) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y.

In Examples, Section 110.24 the reader finds an example for both phenomena. These phenomena can arise even for immersions, see Examples, Section 110.26. However, the next lemma shows that both disasters are avoided when the morphism is quasi-compact.

01R8 **Lemma 29.6.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

(1) *the sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
(2) *the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,*
(3) *for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and*
(4) *the image $f(X) \subset Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant (see Definition 29.8.1).*

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with restriction to open subschemes of Y. And if (1) holds then in the proof of Lemma 29.6.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f'_*\mathcal{O}_{X'})$. By Schemes, Lemma 26.24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. □

056A **Example 29.6.4.** If $A \to B$ is a ring map with kernel I, then the scheme theoretic image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the closed subscheme $\mathrm{Spec}(A/I)$ of $\mathrm{Spec}(A)$. This follows from Lemma 29.6.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

02JQ **Lemma 29.6.5.** *Let $f : X \to Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f. Let $z \in Z$[3]. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z. In particular any point of Z is the specialization of a point of $f(X)$.

Proof. Let $z \in \mathrm{Spec}(R) = V \subset Y$ be an affine open neighbourhood of z. By Lemma 29.6.3 the intersection $Z \cap V$ is the scheme theoretic image of $f^{-1}(V) \to V$. Hence we may replace Y by V and assume $Y = \mathrm{Spec}(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \ldots \cup U_n$ is a finite affine

[3]By Lemma 29.6.3 set-theoretically Z agrees with the closure of $f(X)$ in Y.

open covering. Write $U_i = \mathrm{Spec}(A_i)$. Let $I = \mathrm{Ker}(R \to A_1 \times \ldots \times A_n)$. By Lemma 29.6.3 again we see that Z corresponds to the closed subscheme $\mathrm{Spec}(R/I)$ of Y. If $\mathfrak{p} \subset R$ is the prime corresponding to z, then we see that $I_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 10.9.12) we see that $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} \times \ldots \times (A_n)_{\mathfrak{p}}$ is not zero. Hence one of the rings $(A_i)_{\mathfrak{p}}$ is not zero. Hence there exists an i and a prime $\mathfrak{q}_i \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. By Algebra, Lemma 10.50.2 we can choose a valuation ring $A \subset K = \kappa(\mathfrak{q}_i)$ dominating the local ring $R_{\mathfrak{p}}/\mathfrak{p}_i R_{\mathfrak{p}} \subset \kappa(\mathfrak{q}_i)$. This gives the desired diagram. Some details omitted. □

01R9 **Lemma 29.6.6.** *Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a commutative diagram of schemes. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z_2 & \longrightarrow & Y_2 \end{array}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subscheme of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. □

056B **Lemma 29.6.7.** *Let $f : X \to Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced induced scheme structure on $\overline{f(X)}$.*

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 26.12.7. □

0CNG **Lemma 29.6.8.** *Let $f : X \to Y$ be a separated morphism of schemes. Let $V \subset Y$ be a retrocompact open. Let $s : V \to X$ be a morphism such that $f \circ s = id_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.*

Proof. The assumption that V is retrocompact in Y (Topology, Definition 5.12.1) means that $V \to Y$ is a quasi-compact morphism. By Schemes, Lemma 26.21.14 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 29.6.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Schemes, Lemma 26.21.11), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. □

29.7. Scheme theoretic closure and density

01RA We take the following definition from [**DG67**, IV, Definition 11.10.2].

01RB **Definition 29.7.1.** Let X be a scheme. Let $U \subset X$ be an open subscheme.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.

(2) We say U is *scheme theoretically dense in* X if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X, see Example 29.7.2. This is somewhat inelegant; but see Lemmas 29.7.3 and 29.7.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is injective, see Lemma 29.7.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

01RC **Example 29.7.2.** Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \ldots]/(x^n z_n)$ Set $I = (z_1, z_2, \ldots) \subset A$. Consider the affine scheme $X = \mathrm{Spec}(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to \mathrm{Spec}(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X.

01RD **Lemma 29.7.3.** *Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 29.6.3 part (3). □

056C **Example 29.7.4.** Let A be a ring and $X = \mathrm{Spec}(A)$. Let $f_1, \ldots, f_n \in A$ and let $U = D(f_1) \cup \ldots \cup D(f_n)$. Let $I = \mathrm{Ker}(A \to \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $\mathrm{Spec}(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence by Lemma 29.7.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

01RE **Lemma 29.7.5.** *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X. Hence the scheme theoretic closure of $U \cap V$ in V is equal to V, see proof of Lemma 29.6.1. Conversely, suppose that the scheme theoretic closure of $U \cap V$ is equal to V for all opens V. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $\mathrm{Spec}(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

01RF **Lemma 29.7.6.** *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is $U \cap V$.*

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. By Lemma 29.7.5 both maps are injective. Hence the composite is injective. Hence by Lemma 29.7.5 $U \cap V$ is scheme theoretically dense in X. □

01RG **Lemma 29.7.7.** *Let $h : Z \to X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. By Lemma 29.3.2 or Lemma 29.3.3 we can factor $Z \to X$ as $Z \to \overline{Z}_1 \to X$ with $Z \to \overline{Z}_1$ open and $\overline{Z}_1 \to X$ closed. On the other hand, let $Z \to \overline{Z} \subset X$ be the scheme theoretic image of $Z \to X$. We conclude that $\overline{Z} \subset \overline{Z}_1$. Since Z is an open subscheme of $\overline{Z}_1$ it follows that Z is an open subscheme of $\overline{Z}$ as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of $\overline{Z}_1$ in the proof of Lemma 29.3.3. Hence $\overline{Z}_1$ and $\overline{Z}$ have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \to X$ is quasi-compact we argue as follows: The assertion that Z is scheme theoretically dense in $\overline{Z}$ follows from Lemma 29.6.3 part (3). The last assertion follows from Lemma 29.6.3 part (4). □

056D **Lemma 29.7.8.** *Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent*

(1) *U is topologically dense in X,*
(2) *the scheme theoretic closure of U in X is X, and*
(3) *U is scheme theoretically dense in X.*

Proof. This follows from Lemma 29.7.7 and the fact that a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. □

056E **Lemma 29.7.9.** *Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent*

(1) *the scheme theoretic closure of U in X is X, and*
(2) *U is scheme theoretically dense in X.*

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 29.7.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 29.6.7. □

01RH **Lemma 29.7.10.** *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$.*

Proof. Follows from the definitions and Schemes, Lemma 26.21.5. □

29.8. Dominant morphisms

01RI The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

01RJ **Definition 29.8.1.** A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \ldots$ is a countable collection of distinct elements of k, then the morphism

$$\coprod\nolimits_{i=1,2,\ldots} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k[x])$$

with ith factor mapping to the point $x = \lambda_i$ is dominant.

01RK **Lemma 29.8.2.** *Let $f : X \to S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f, then f is dominant.*

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 26.11.1, and that every point of S is contained in an irreducible component of S, see Topology, Lemma 5.8.3. □

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

01RL **Lemma 29.8.3.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is dominant if and only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f.*

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \ldots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod\nolimits_{i=1,\ldots,n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 26.6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V. Also, f is dominant if and only if f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 10.30.6. □

0H3F **Lemma 29.8.4.** *Let $f : X \to S$ be a quasi-compact dominant morphism of schemes. Let $g : S' \to S$ be a morphism of schemes, and let $f' : X' \to S'$ be the base change of f by g. If generalizations lift along g, then f' is dominant.*

We will see that generalizations lift along g if g is open (Lemma 29.23.5) or flat (Lemma 29.25.9).

Proof. Observe that f' is quasi-compact by Schemes, Lemma 26.19.3. Let $\eta' \in S'$ be the generic point of an irreducible component of S'. If generalizations lift along g, then $\eta = g(\eta')$ is the generic point of an irreducible component of S. By Lemma 29.8.3 we see that η is in the image of f. Hence η' is in the image of f' by Schemes, Lemma 26.17.5. It follows that f' is dominant by Lemma 29.8.3. □

Part of this holds for arbitrary (not necessarily quasi-compact) dominant morphisms.

0H8F **Lemma 29.8.5.** *Let $f : X \to S$ be a dominant morphism of schemes. Let $g : S' \to S$ be an open morphism of schemes, and let $f' : X' \to S'$ be the base change of f by g. Then f' is dominant.*

Proof. If f' is not dominant, then $f'(X')$ is contained in some closed subset B not equal to S'. Then $S' - B$ is open in S' and nonempty. Since g is open, $g(S' - B)$ is nonempty and open in S. Since $f(X)$ is dense in S, there is a point $x \in X$ with $f(x) \in g(S' - B)$. Let $W = \mathrm{Spec}(\kappa(f(x)))$, and write X_W, S_W, S'_W for the pullbacks of these schemes from S to W. Then X_W and $(S' - B)_W$ are nonempty, and so $X_W \times_W (S' - B)_W$ is nonempty, as W is the spectrum of a field. This contradicts that $f'(X \times_S S')$ is contained in $B \subset S'$. So in fact f' is dominant. □

02NE **Lemma 29.8.6.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.*

Proof. Let $Z \subset S$ be the scheme theoretic image of f. We have to show that $\eta \notin Z$. This follows from Lemma 29.6.5 but can also be seen as follows. By Lemma 29.6.3 the morphism $X \to Z$ is dominant, which by Lemma 29.8.3 means all the generic points of all irreducible components of Z are in the image of $X \to Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z. □

There is another case where dominant is the same as having all generic points of irreducible components in the image.

01RM **Lemma 29.8.7.** *Let $f : X \to S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f. If so, then S has finitely many irreducible components as well.*

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S. Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f. The lemma follows. □

0CC1 **Lemma 29.8.8.** *Let $f : X \to Y$ be a morphism of integral schemes. The following are equivalent*

(1) *f is dominant,*
(2) *f maps the generic point of X to the generic point of Y,*
(3) *for some nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,*
(4) *for all nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,*
(5) *for some $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective, and*
(6) *for all $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective.*

Proof. The equivalence of (1) and (2) follows from Lemma 29.8.7. Let $U \subset X$ and $V \subset Y$ be nonempty affine opens with $f(U) \subset V$. Recall that the rings $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(V)$ are integral domains. The morphism $f|_U : U \to V$ corresponds to a ring map $\varphi : B \to A$. The generic points of X and Y correspond to the prime ideals $(0) \subset A$ and $(0) \subset B$. Thus (2) is equivalent to the condition $(0) = \varphi^{-1}((0))$, i.e., to the condition that φ is injective. In this way we see that (2), (3), and (4) are equivalent. Similarly, given x and y as in (5) the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are domains and the prime ideals $(0) \subset \mathcal{O}_{X,x}$ and $(0) \subset \mathcal{O}_{Y,y}$ correspond to the generic points of X and Y (via the identification of the spectrum of the local ring at x with the set of points specializing to x, see Schemes, Lemma 26.13.2). Thus we can argue in the exact same manner as above to see that (2), (5), and (6) are equivalent. □

29.9. Surjective morphisms

01RY

01RZ **Definition 29.9.1.** A morphism of schemes is said to be *surjective* if it is surjective on underlying topological spaces.

01S0 **Lemma 29.9.2.** *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

0495 **Lemma 29.9.3.** *Let X and Y be schemes over a base scheme S. Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S.*

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 26.17.5. □

01S1 **Lemma 29.9.4.** *The base change of a surjective morphism is surjective.*

Proof. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. If $S' \to S$ is a morphism of schemes, let $p : X_{S'} \to X$ and $q : Y_{S'} \to Y$ be the canonical projections. The commutative square

$$\begin{array}{ccc} X_{S'} & \xrightarrow{p} & X \\ {\scriptstyle f_{S'}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{S'} & \xrightarrow{q} & Y. \end{array}$$

identifies $X_{S'}$ as a fibre product of $X \to Y$ and $Y_{S'} \to Y$. Let Z be a subset of the underlying topological space of X. Then $q^{-1}(f(Z)) = f_{S'}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 29.9.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \to Y_{S'}$. □

0496 **Example 29.9.5.** Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$. The base change $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$ (the decomposition comes from the idempotent $\frac{1 \otimes 1 + i \otimes i}{2}$) and hence $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ has two points.

04ZD **Lemma 29.9.6.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 5.12.7 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. □

29.10. Radicial and universally injective morphisms

01S2 In this section we define what it means for a morphism of schemes to be *radicial* and what it means for a morphism of schemes to be *universally injective*. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

01S3 **Definition 29.10.1.** Let $f : X \to S$ be a morphism.

(1) We say that f is *universally injective* if and only if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is injective (on underlying topological spaces).
(2) We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is purely inseparable.

01S4 **Lemma 29.10.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *For every field K the induced map* $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ *is injective.*
(2) *The morphism f is universally injective.*
(3) *The morphism f is radicial.*
(4) *The diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$ is surjective.*

Proof. Let K be a field, and let $s : \mathrm{Spec}(K) \to S$ be a morphism. Giving a morphism $x : \mathrm{Spec}(K) \to X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = \mathrm{Spec}(K) \times_S X \to \mathrm{Spec}(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K. Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{array}{ccc} K & \longleftarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \kappa(x') & \longleftarrow & \kappa(s') \end{array}$$

of fields. By Schemes, Lemma 26.13.3 we get two morphisms $a, a' : \mathrm{Spec}(K) \to X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \to X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(x)/\kappa(s)$ is not purely inseparable, then we may find a field extension $K/\kappa(s)$ such that $\kappa(x)$ has two $\kappa(s)$-homomorphisms into K. By Schemes, Lemma 26.13.3 this implies that the map $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radicial, i.e., they imply (3).

Assume (3). By Schemes, Lemma 26.13.3 a morphism $\mathrm{Spec}(K) \to X$ is given by a pair $(x, \kappa(x) \to K)$. Property (3) says exactly that associating to the pair

$(x, \kappa(x) \to K)$ the pair $(s, \kappa(s) \to \kappa(x) \to K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 26.17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \longrightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one such prime ideal $\mathfrak{p}$, which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radicial. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \to S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. □

05VE **Lemma 29.10.3.** *A universally injective morphism is separated.*

Proof. Combine Lemma 29.10.2 with the remark that $X \to S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 26.21.3 and the discussion following it. □

0472 **Lemma 29.10.4.** *A base change of a universally injective morphism is universally injective.*

Proof. This is formal. □

02V1 **Lemma 29.10.5.** *A composition of radicial morphisms is radicial, and so the same holds for the equivalent condition of being universally injective.*

Proof. Omitted. □

29.11. Affine morphisms

01S5

01S6 **Definition 29.11.1.** A morphism of schemes $f : X \to S$ is called *affine* if the inverse image of every affine open of S is an affine open of X.

01S7 **Lemma 29.11.2.** *An affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 26.19.2. We will show f is separated using Schemes, Lemma 26.21.7. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. □

01S8 **Lemma 29.11.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent* [**DG67**, II, Corollary 1.3.2]

(1) *The morphism f is affine.*
(2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.*
(3) *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and an isomorphism $X \cong \underline{\mathrm{Spec}}_S(\mathcal{A})$ of schemes over S. See Constructions, Section 27.4 for notation.*

Moreover, in this case $X = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 26.19.2 we see that f is quasi-compact. By Schemes, Lemma 26.21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 26.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 27.4.7. By assumption and the lemma just cited the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 27.4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. □

01S9 **Remark 29.11.4.** We can also argue directly that (2) implies (1) in Lemma 29.11.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $\mathrm{Spec}(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 26.6.4 there is a canonical morphism $\psi : f^{-1}(V) \to \mathrm{Spec}(A)$ over $\mathrm{Spec}(R) = V$. By Schemes, Lemma 26.11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1,\ldots,n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \ldots, n\} \to J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 10.17.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because $\mathcal{A}$ is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i}. It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $\mathrm{Spec}(A_{h_i})$ of $\mathrm{Spec}(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $\mathrm{Spec}(A) = \bigcup \mathrm{Spec}(A_{h_i})$ we win.

01SA **Lemma 29.11.5.** *Let S be a scheme. There is an anti-equivalence of categories*

$$\begin{array}{c}\text{Schemes affine}\\ \text{over } S\end{array} \longleftrightarrow \begin{array}{c}\text{quasi-coherent sheaves}\\ \text{of } \mathcal{O}_S\text{-algebras}\end{array}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. The functor from right to left is given by $\underline{\mathrm{Spec}}_S$. The two functors are mutually inverse by Lemma 29.11.3 and Constructions, Lemma 27.4.6 part (3). The final statement is Constructions, Lemma 27.4.6 part (2). □

0H88 **Lemma 29.11.6.** *Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. An $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.*

Proof. Let $\mathcal{F}$ be an $\mathcal{A}$-module. If $\mathcal{F}$ is quasi-coherent as an $\mathcal{A}$-module, then for every $s \in S$ there exists an open neighbourhood U of s and an exact sequence

$$\bigoplus\nolimits_J \mathcal{A}|_U \to \bigoplus\nolimits_I \mathcal{A}|_U \to \mathcal{F}|_U$$

of $\mathcal{A}|_U$-modules. Then this is also an exact sequence of $\mathcal{O}_U$-modules. Hence $\mathcal{F}|_U$ is quasi-coherent as the cokernel of a morphism of quasi-coherent $\mathcal{O}_U$-modules on a scheme. It follows that $\mathcal{F}$ is quasi-coherent as an $\mathcal{O}_X$-module.

Conversely, assume $\mathcal{F}$ is quasi-coherent as an $\mathcal{O}_X$-module. Pick an open affine $\mathrm{Spec}(R) = U \subset S$. We have isomorphisms of $\mathcal{O}_U$-modules $\mathcal{A}|_U \cong \widetilde{A}$ and $\mathcal{F}|_U \cong \widetilde{M}$, for some R-algebra A and some R-module M. The $\mathcal{A}$-module structure on $\mathcal{F}$ translates into an A-module structure on M compatible with the given R-module structure (details omitted). Choose an exact sequence

$$\bigoplus\nolimits_J A \to \bigoplus\nolimits_I A \to M \to 0$$

of A-modules. Since the functor $\widetilde{\ }$ is exact, this produces an exact sequence

$$\bigoplus\nolimits_J \widetilde{A} \to \bigoplus\nolimits_I \widetilde{A} \to \widetilde{M} \to 0$$

of $\widetilde{A}$-modules. This means that $\mathcal{F}$ is quasi-coherent as an $\mathcal{A}$-module. □

01SB **Lemma 29.11.7.** *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{array}{c}\text{category of quasi-coherent}\\ \mathcal{O}_X\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of quasi-coherent}\\ \mathcal{A}\text{-modules}\end{array}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. The final statement is Lemma 29.11.6. By Lemma 29.11.2 and Schemes, Lemma 26.24.1 the pushforward $f_*\mathcal{F}$ of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. Hence a functor as in the statement of the lemma.

We will construct an quasi-inverse h to this functor. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{A}$-module. Then we set

$$h(\mathcal{G}) = f^*\mathcal{G} \otimes_{f^*\mathcal{A}} \mathcal{O}_X = f^*\mathcal{G} \otimes_{f^*f_*\mathcal{O}_X} \mathcal{O}_X$$

Elucidation: the pullback $f^*\mathcal{A} = f^*f_*\mathcal{O}_X$ is an $\mathcal{O}_X$-algebra, the adjunction map $f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ is an algebra homomorphism, and the pullback $f^*\mathcal{G}$ is an $f^*\mathcal{A}$-module. Observe that $h(\mathcal{G})$ is quasi-coherent as quasi-coherence is preserved by pullbacks and change of rings. Observe that there is a functorial map

$$h(f_*\mathcal{F}) = f^*f_*\mathcal{F} \otimes_{f^*f_*\mathcal{O}_X} \mathcal{O}_X \to \mathcal{F}$$

coming from the adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$ and a functorial map

$$\mathcal{G} \to f_*h(\mathcal{G}) \quad \text{adjoint to the map} \quad f^*\mathcal{G} \to f^*\mathcal{G} \otimes_{f^*f_*\mathcal{O}_X} \mathcal{O}_X$$

which sends a local section s of $f^*\mathcal{F}$ to $s \otimes 1$. To finish the proof it suffices to show that these maps are isomorphisms for $\mathcal{F}$ and $\mathcal{G}$ as above. This may be checked on the members of an affine covering, i.e., when X and S are affine.

The key algebra observation which makes this work is the following: Let $R \to A$ be a ring map. Let N be an A-module. Then

$$(N \otimes_R A) \otimes_{(A \otimes_R A)} A = N$$

Namely, the left hand side of this equality is the effect of applying h to the quasi-coherent $\widetilde{A}$-module $\widetilde{N}$ on $\mathrm{Spec}(R)$. We omit the details. □

01SC **Lemma 29.11.8.** *The composition of affine morphisms is affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g. Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f. Hence $(g \circ f)^{-1}(U)$ is affine. □

01SD **Lemma 29.11.9.** *The base change of an affine morphism is affine.*

Proof. Let $f : X \to S$ be an affine morphism. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 26.17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 29.11.3 above. □

01SE **Lemma 29.11.10.** *A closed immersion is affine.*

Proof. The first indication of this is Schemes, Lemma 26.8.2. See Schemes, Lemma 26.10.1 for a complete statement. □

01SF **Lemma 29.11.11.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \to X$ is affine.*

Proof. This follows from Properties, Lemma 28.26.4 and the definition. □

01SG **Lemma 29.11.12.** *Suppose $g : X \to Y$ is a morphism of schemes over S.*

(1) *If X is affine over S and $\Delta : Y \to Y \times_S Y$ is affine, then g is affine.*
(2) *If X is affine over S and Y is separated over S, then g is affine.*
(3) *A morphism from an affine scheme to a scheme with affine diagonal is affine.*
(4) *A morphism from an affine scheme to a separated scheme is affine.*

Proof. Proof of (1). The base change $X \times_S Y \to Y$ is affine by Lemma 29.11.9. The morphism $(1, g) : X \to X \times_S Y$ is the base change of $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$. Hence it is affine by Lemma 29.11.9. The composition of affine morphisms is affine (see Lemma 29.11.8) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 29.11.10) and Y/S separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

01SH **Lemma 29.11.13.** *A morphism between affine schemes is affine.*

Proof. Immediate from Lemma 29.11.12 with $S = \mathrm{Spec}(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 29.11.3. □

01SI **Lemma 29.11.14.** *Let S be a scheme. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to S$ is affine.*

Proof. Omitted. □

0C3A **Lemma 29.11.15.** *Let $j : Y \to X$ be an immersion of schemes. Assume there exists an open $U \subset X$ with complement $Z = X \setminus U$ such that*

(1) *$U \to X$ is affine,*
(2) *$j^{-1}(U) \to U$ is affine, and*
(3) *$j(Y) \cap Z$ is closed.*

Then j is affine. In particular, if X is affine, so is Y.

Proof. By Schemes, Definition 26.10.2 there exists an open subscheme $W \subset X$ such that j factors as a closed immersion $i : Y \to W$ followed by the inclusion morphism $W \to X$. Since a closed immersion is affine (Lemma 29.11.10), we see that for every $x \in W$ there is an affine open neighbourhood of x in X whose inverse image under j is affine. If $x \in U$, then the same thing is true by assumption (2). Finally, assume $x \in Z$ and $x \notin W$. Then $x \notin j(Y) \cap Z$. By assumption (3) we can find an affine open neighbourhood $V \subset X$ of x which does not meet $j(Y) \cap Z$. Then $j^{-1}(V) = j^{-1}(V \cap U)$ which is affine by assumptions (1) and (2). It follows that j is affine by Lemma 29.11.3. □

29.12. Families of ample invertible modules

0FXQ A short section on the notion of a family of ample invertible modules.

0FXR **Definition 29.12.1.** Let X be a scheme. Let $\{\mathcal{L}_i\}_{i\in I}$ be a family of invertible $\mathcal{O}_X$-modules. We say $\{\mathcal{L}_i\}_{i\in I}$ is an *ample family of invertible modules on X* if [**BGI71**, II Definition 2.2.4]

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $i \in I$, an $n \geq 1$, and $s \in \Gamma(X, \mathcal{L}_i^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

If $\{\mathcal{L}_i\}_{i\in I}$ is an ample family of invertible modules on a scheme X, then there exists a finite subset $I' \subset I$ such that $\{\mathcal{L}_i\}_{i\in I'}$ is an ample family of invertible modules on X (follows immediately from quasi-compactness). A scheme having an ample family of invertible modules has an affine diagonal by the next lemma and hence is a fortiori quasi-separated.

0FXS **Lemma 29.12.2.** *Let X be a scheme such that for every point $x \in X$ there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a global section $s \in \Gamma(X, \mathcal{L})$ such that $x \in X_s$ and X_s is affine. Then the diagonal of X is an affine morphism.*

Proof. Given invertible $\mathcal{O}_X$-modules $\mathcal{L}$, $\mathcal{M}$ and global sections $s \in \Gamma(X, \mathcal{L})$, $t \in \Gamma(X, \mathcal{M})$ such that X_s and X_t are affine we have to prove $X_s \cap X_t$ is affine. Namely, then Lemma 29.11.3 applied to $\Delta : X \to X \times X$ and the fact that $\Delta^{-1}(X_s \times X_t) = X_s \cap X_t$ shows that Δ is affine. The fact that $X_s \cap X_t$ is affine follows from Properties, Lemma 28.26.4. □

0FXT **Remark 29.12.3.** In Properties, Lemma 28.26.7 we see that a scheme which has an ample invertible module is separated. This is wrong for schemes having an ample family of invertible modules. Namely, let X be as in Schemes, Example 26.14.3 with $n = 1$, i.e., the affine line with zero doubled. We use the notation of that example except that we write x for x_1 and y for y_1. There is, for every integer n, an invertible sheaf $\mathcal{L}_n$ on X which is trivial on X_1 and X_2 and whose transition function $U_{12} \to U_{21}$ is $f(x) \mapsto y^n f(y)$. The global sections of $\mathcal{L}_n$ are pairs $(f(x), g(y)) \in k[x] \oplus k[y]$ such that $y^n f(y) = g(y)$. The sections $s = (1, y)$ of $\mathcal{L}_1$ and $t = (x, 1)$ of $\mathcal{L}_{-1}$ determine an open affine cover because $X_s = X_1$ and $X_t = X_2$. Therefore X has an ample family of invertible modules but it is not separated.

29.13. Quasi-affine morphisms

01SJ Recall that a scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 28.18.1.

01SK **Definition 29.13.1.** A morphism of schemes $f : X \to S$ is called *quasi-affine* if the inverse image of every affine open of S is a quasi-affine scheme.

01SL **Lemma 29.13.2.** *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 26.19.2. Let $U \subset S$ be an affine open. If we can show that $f^{-1}(U)$ is a separated scheme, then f is separated (Schemes, Lemma 26.21.7 shows that being separated is local on the base). By assumption $f^{-1}(U)$ is isomorphic to an open subscheme of an affine scheme. An affine scheme is separated and hence every open subscheme of an affine scheme is separated as desired. □

01SM **Lemma 29.13.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is quasi-affine.*
(2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.*
(3) *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion*

$$\begin{array}{ccc} X & \longrightarrow & \underline{\mathrm{Spec}}_S(\mathcal{A}) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

over S.
(4) *Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 27.4.7.*

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 26.19.2 we see that f is quasi-compact. By Schemes, Lemma 26.21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 26.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 27.4.7. By assumption, the lemma just cited, and Properties, Lemma 28.18.4 the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 27.4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). □

01SN **Lemma 29.13.4.** *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion into an affine scheme V. By Lemma 29.13.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative

spectrum $\underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$-algebras $\mathcal{A}$. By Schemes, Lemma 26.24.1 the sheaf $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_V$-algebras with the property that $j^*\mathcal{A}' = \mathcal{A}$. Hence we get a commutative diagram

$$\begin{array}{ccccc} f^{-1}(g^{-1}(U)) & \longrightarrow & \underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A}) & \longrightarrow & \underline{\mathrm{Spec}}_V(\mathcal{A}') \\ & & \downarrow & & \downarrow \\ & & g^{-1}(U) & \xrightarrow{j} & V \end{array}$$

with the square being a fibre square, see Constructions, Lemma 27.4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. □

01SO **Lemma 29.13.5.** *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \to S$ be a quasi-affine morphism. By Lemma 29.13.3 above we can find a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion $X \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ over S. Let $g : S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \to \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 26.19.3 and 26.18.2 and Constructions, Lemma 27.4.6). By Lemma 29.13.3 again we conclude that $X_{S'} \to S'$ is quasi-affine. □

02JR **Lemma 29.13.6.** *A quasi-compact immersion is quasi-affine.*

Proof. Let $X \to S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 29.7.7 the morphism $X \to S$ factors as $X \to Z \to S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 29.2.1 implies Z is affine. Hence we win. □

01SP **Lemma 29.13.7.** *Let S be a scheme. Let X be an affine scheme. A morphism $f : X \to S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X, hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \to S$ where X is affine and S quasi-separated follows from Schemes, Lemma 26.21.14 applied to $X \to S \to \mathrm{Spec}(\mathbf{Z})$. □

054G **Lemma 29.13.8.** *Suppose $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \to Y$ is quasi-affine by Lemma 29.13.5. The morphism $X \to X \times_S Y$ is a quasi-compact immersion as $Y \to S$ is quasi-separated, see Schemes, Lemma 26.21.11. A quasi-compact immersion is quasi-affine by Lemma 29.13.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 29.13.4). Thus we win. □

29.14. Types of morphisms defined by properties of ring maps

01SQ In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

01SR **Definition 29.14.1.** Let P be a property of ring maps.

(1) We say that P is *local* if the following hold:
   (a) For any ring map $R \to A$, and any $f \in R$ we have $P(R \to A) \Rightarrow P(R_f \to A_f)$.
   (b) For any rings R, A, any $f \in R$, $a \in A$, and any ring map $R_f \to A$ we have $P(R_f \to A) \Rightarrow P(R \to A_a)$.
   (c) For any ring map $R \to A$, and $a_i \in A$ such that $(a_1, \ldots, a_n) = A$ then $\forall i, P(R \to A_{a_i}) \Rightarrow P(R \to A)$.

(2) We say that P is *stable under base change* if for any ring maps $R \to A$, $R \to R'$ we have $P(R \to A) \Rightarrow P(R' \to R' \otimes_R A)$.

(3) We say that P is *stable under composition* if for any ring maps $A \to B$, $B \to C$ we have $P(A \to B) \wedge P(B \to C) \Rightarrow P(A \to C)$.

01SS **Definition 29.14.2.** Let P be a property of ring maps. Let $f : X \to S$ be a morphism of schemes. We say f is *locally of type* P if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property P.

This is not a "good" definition unless the property P is a local property. Even if P is a local property we will not automatically use this definition to say that a morphism is "locally of type P" unless we also explicitly state the definition elsewhere.

01ST **Lemma 29.14.3.** *Let $f : X \to S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X, and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds.*

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u, see Schemes, Lemma 26.11.5. By Definition 29.14.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u, see Schemes, Lemma 26.11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \to \mathcal{O}_X(U'_u))$ by Definition 29.14.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 29.14.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \ldots, u_n \in U$ such that

$$U = U_{u_1} \cup \ldots \cup U_{u_n}.$$

By Definition 29.14.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds. □

01SU **Lemma 29.14.4.** *Let P be a local property of ring maps. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of type P.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$.*

(3) *There exists an open covering $S = \bigcup_{j\in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of type P.*
(4) *There exists an affine open covering $S = \bigcup_{j\in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that $P(\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i))$ holds, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of type P then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of type P.

Proof. This follows from Lemma 29.14.3 above. □

01SV **Lemma 29.14.5.** *Let P be a property of ring maps. Assume P is local and stable under composition. The composition of morphisms locally of type P is locally of type P.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of type P. Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood $U \subset f^{-1}(V)$ of x. By Lemma 29.14.4 the ring maps $\mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ satisfy P. Hence $\mathcal{O}_Z(W) \to \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. □

01SW **Lemma 29.14.6.** *Let P be a property of ring maps. Assume P is local and stable under base change. The base change of a morphism locally of type P is locally of type P.*

Proof. Let $f : X \to S$ be a morphism locally of type P. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 29.14.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \to \mathcal{O}_X(U_i)$ all satisfy P. By the material in Schemes, Section 26.17 we see that $f^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \to \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. □

01SX **Lemma 29.14.7.** *The following properties of a ring map $R \to A$ are local.*

(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
(4) *(Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.*
(5) *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
(6) *Add more here as needed*[4].

Proof. Omitted. □

[4]But only those properties that are not already dealt with separately elsewhere.

01SY **Lemma 29.14.8.** *The following properties of ring maps are stable under base change.*

(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *Add more here as needed*[5].

Proof. Omitted. □

01SZ **Lemma 29.14.9.** *The following properties of ring maps are stable under composition.*

(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
(4) *Add more here as needed*[6].

Proof. Omitted. □

29.15. Morphisms of finite type

01T0 Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 10.6.1.

01T1 **Definition 29.15.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is *locally of finite type* if it is of finite type at every point of X.
(3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

01T2 **Lemma 29.15.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite type.*
(2) *For all affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type.*
(3) *There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite type.*
(4) *There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.*

[5] But only those properties that are not already dealt with separately elsewhere.
[6] But only those properties that are not already dealt with separately elsewhere.

Moreover, if f is locally of finite type then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite type.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is of finite type" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.14.2 being of finite type is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.6.2 being of finite type is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite type. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.23.3. □

01T3 **Lemma 29.15.3.** *The composition of two morphisms which are locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 29.15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 26.19.4 we see that the composition of morphisms of finite type is of finite type. □

01T4 **Lemma 29.15.4.** *The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 29.15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.6 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 10.14.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3 we see that the base change of a morphism of finite type is a morphism of finite type. □

01T5 **Lemma 29.15.5.** *A closed immersion is of finite type. An immersion is locally of finite type.*

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. □

01T6 **Lemma 29.15.6.** *Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 10.31.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) □

01T7 **Lemma 29.15.7.** *Let $f : X \to S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.*

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 28.5.4 and Lemma 29.15.6 above. Then apply Schemes, Lemma 26.21.13 to conclude that f is quasi-separated. □

01T8 **Lemma 29.15.8.** *Let $X \to Y$ be a morphism of schemes over a base scheme S. If X is locally of finite type over S, then $X \to Y$ is locally of finite type.*

Proof. Via Lemma 29.15.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 10.6.2). □

29.16. Points of finite type and Jacobson schemes

01T9 Let S be a scheme. A finite type point s of S is a point such that the morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

01TA **Lemma 29.16.1.** *Let S be a scheme. Let k be a field. Let $f : \mathrm{Spec}(k) \to S$ be a morphism. The following are equivalent:*

1. *The morphism f is of finite type.*
2. *The morphism f is locally of finite type.*
3. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that f corresponds to a finite ring map $R \to k$.*
4. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that the image of f consists of a closed point u in U and the field extension $k/\kappa(u)$ is finite.*

Proof. The equivalence of (1) and (2) is obvious as $\mathrm{Spec}(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $\mathrm{Spec}(R) = U \subset S$ such that the image of f is contained in U, and the ring map $R \to k$ is of finite type. Let $\mathfrak{p} \subset R$ be the kernel. Then $R/\mathfrak{p} \subset k$ is of finite type. By Algebra, Lemma 10.34.2 there exist a $\overline{f} \in R/\mathfrak{p}$ such that $(R/\mathfrak{p})_{\overline{f}}$ is a field and $(R/\mathfrak{p})_{\overline{f}} \to k$ is a finite field extension. If $f \in R$ is a lift of $\overline{f}$, then we see that k is a finite R_f-module. Thus (2) $\Rightarrow$ (3).

Suppose that $\mathrm{Spec}(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \to k$. Then f is locally of finite type by Lemma 29.15.2. Thus (3) $\Rightarrow$ (2).

Suppose $R \to k$ is finite. The image of $R \to k$ is a field over which k is finite by Algebra, Lemma 10.36.18. Hence the kernel of $R \to k$ is a maximal ideal. Thus (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is immediate. □

02HV **Lemma 29.16.2.** *Let S be a scheme. Let A be an Artinian local ring with residue field κ. Let $f : \mathrm{Spec}(A) \to S$ be a morphism of schemes. Then f is of finite type if and only if the composition $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A) \to S$ is of finite type.*

Proof. Since the morphism $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A)$ is of finite type it is clear that if f is of finite type so is the composition $\mathrm{Spec}(\kappa) \to S$ (see Lemma 29.15.3). For the converse, note that $\mathrm{Spec}(A) \to S$ maps into some affine open $U = \mathrm{Spec}(B)$ of S as $\mathrm{Spec}(A)$ has only one point. To finish apply Algebra, Lemma 10.54.4 to $B \to A$. □

Recall that given a point s of a scheme S there is a canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$, see Schemes, Section 26.13.

02J1 **Definition 29.16.3.** Let S be a scheme. Let us say that a point s of S is a *finite type point* if the canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. We denote $S_{\text{ft-pts}}$ the set of finite type points of S.

We can describe the set of finite type points as follows.

02J2 **Lemma 29.16.4.** *Let S be a scheme. We have*

$$S_{ft\text{-}pts} = \bigcup\nolimits_{U \subset S \text{ open}} U_0$$

where U_0 is the set of closed points of U. Here we may let U range over all opens or over all affine opens of S.

Proof. Immediate from Lemma 29.16.1. □

02J3 **Lemma 29.16.5.** *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{ft\text{-}pts}) \subset S_{ft\text{-}pts}$.*

Proof. If T is the spectrum of a field this is Lemma 29.16.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). □

06EB **Lemma 29.16.6.** *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{ft\text{-}pts}) = S_{ft\text{-}pts}$.*

Proof. We have $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$ by Lemma 29.16.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = \mathrm{Spec}(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 29.16.4. Now $T_s \to T$ is a morphism of finite type as a base change of $s \to S$ (Lemma 29.15.4). Hence the image of t in T is a finite type point by Lemma 29.16.5 which maps to s by construction. □

02J4 **Lemma 29.16.7.** *Let S be a scheme. For any locally closed subset $T \subset S$ we have*

$$T \neq \emptyset \Rightarrow T \cap S_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{ft\text{-}pts}$ is dense in T.

Proof. Note that T carries a scheme structure (see Schemes, Lemma 26.12.4) such that $T \to S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 29.15.5. Hence by Lemma 29.16.5 we see $T_{\text{ft-pts}} \subset S_{\text{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 29.16.4. □

It follows that most of the material from Topology, Section 5.18 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

01TB **Lemma 29.16.8.** *Let S be a scheme. The following are equivalent:*

1. *the scheme S is Jacobson,*
2. *$S_{ft\text{-}pts}$ is the set of closed points of S,*
3. *for all $T \to S$ locally of finite type closed points map to closed points, and*
4. *for all $T \to S$ locally of finite type closed points $t \in T$ map to closed points $s \in S$ with $\kappa(s) \subset \kappa(t)$ finite.*

Proof. We have trivially (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Lemma 29.16.7 shows that (2) implies (1). Hence it suffices to show that (1) implies (4). Suppose that $T \to S$ is locally of finite type. Choose $t \in T$ closed and let $s \in S$ be the image. Choose affine open neighbourhoods $\mathrm{Spec}(R) = U \subset S$ of s and $\mathrm{Spec}(A) = V \subset T$ of t with V mapping into U. The induced ring map $R \to A$ is of finite type (see Lemma 29.15.2) and R is Jacobson by Properties, Lemma 28.6.3. Thus the result follows from Algebra, Proposition 10.35.19. □

02J5 **Lemma 29.16.9.** *Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.*

Proof. This is clear from Algebra, Proposition 10.35.19 (and Properties, Lemma 28.6.3 and Lemma 29.15.2). □

02J6 **Lemma 29.16.10.** *The following types of schemes are Jacobson.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over $\mathbf{Z}$.*
(3) *Any scheme locally of finite type over a 1-dimensional Noetherian domain with infinitely many primes.*
(4) *A scheme of the form $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ where $(R, \mathfrak{m})$ is a Noetherian local ring. Also any scheme locally of finite type over it.*

Proof. We will use Lemma 29.16.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of $\mathbf{Z}$ is Jacobson, see Algebra, Lemma 10.35.6. For (3) see Algebra, Lemma 10.61.4. For (4) see Properties, Lemma 28.6.4. □

29.17. Universally catenary schemes

02J7 Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.11.4. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 28.11.1.

02J8 **Definition 29.17.1.** Let S be a scheme. Assume S is locally Noetherian. We say S is *universally catenary* if for every morphism $X \to S$ locally of finite type the scheme X is catenary.

This is a "better" notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 110.19. Many schemes are universally catenary, see Lemma 29.17.5 below.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.105.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section 28.11. Recall that a ring A is called *universally catenary* if A is Noetherian and for every finite type ring map $A \to B$ the ring B is catenary. See Algebra, Definition 10.105.3. Many interesting rings which come up in algebraic geometry satisfy this property.

02J9 **Lemma 29.17.2.** *Let S be a locally Noetherian scheme. The following are equivalent*

(1) *S is universally catenary,*
(2) *there exists an open covering of S all of whose members are universally catenary schemes,*
(3) *for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is universally catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.*

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 29.15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $\mathrm{Spec}(R)$ is a universally catenary scheme, then every scheme $\mathrm{Spec}(A)$ with A a finite type R-algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 10.105.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \to S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \to S$ maps into one of the U_i. By Lemma 29.15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 28.11.2. □

02JA **Lemma 29.17.3.** *Let S be a locally Noetherian scheme. The following are equivalent:*

(1) *S is universally catenary, and*
(2) *all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.*

Proof. Assume that all local rings of S are universally catenary. Let $f : X \to S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversely, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 29.17.2. Say $S = \mathrm{Spec}(R)$ and s corresponds to the prime ideal $\mathfrak{p}$. Any finite type $R_\mathfrak{p}$-algebra A' is of the form $A_\mathfrak{p}$ for some finite type R-algebra A. By assumption (and Lemma 29.17.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_\mathfrak{p}$ is universally catenary. □

0G42 **Lemma 29.17.4.** *Let S be a locally Noetherian scheme. Then S is universally catenary if and only if the irreducible components of S are universally catenary.*

Proof. Omitted. For the affine case, please see Algebra, Lemma 10.105.8. □

02JB **Lemma 29.17.5.** *The following types of schemes are universally catenary.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Cohen-Macaulay scheme.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$.
(4) *Any scheme locally of finite type over a* 1*-dimensional Noetherian domain.*
(5) *And so on.*

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 10.105.9. Also, use the last assertion of Lemma 29.17.2. Some details omitted. □

29.18. Nagata schemes, reprise

0359 See Properties, Section 28.13 for the definitions and basic properties of Nagata and universally Japanese schemes.

035A **Lemma 29.18.1.** *Let* $f : X \to S$ *be a morphism. If* S *is Nagata and* f *locally of finite type then* X *is Nagata. If* S *is universally Japanese and* f *locally of finite type then* X *is universally Japanese.*

Proof. For "universally Japanese" this follows from Algebra, Lemma 10.162.4. For "Nagata" this follows from Algebra, Proposition 10.162.15. □

035B **Lemma 29.18.2.** *The following types of schemes are Nagata.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$.
(4) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(5) *And so on.*

Proof. By Lemma 29.18.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 10.162.16. □

29.19. The singular locus, reprise

07R2 We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

07R3 **Definition 29.19.1.** Let X be a locally Noetherian scheme. We say X is *J-2* if for every morphism $Y \to X$ which is locally of finite type the regular locus $\mathrm{Reg}(Y)$ is open in Y.

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 15.47.1.

07R4 **Lemma 29.19.2.** *Let* X *be a locally Noetherian scheme. The following are equivalent*

(1) X *is J-2,*
(2) *there exists an open covering of* X *all of whose members are J-2 schemes,*
(3) *for every affine open* $\mathrm{Spec}(R) = U \subset X$ *the ring* R *is J-2, and*
(4) *there exists an affine open covering* $S = \bigcup U_i$ *such that each* $\mathcal{O}(U_i)$ *is J-2 for all* i.

Moreover, in this case any scheme locally of finite type over X *is J-2 as well.*

Proof. By Lemma 29.15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). Thus it is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $\mathrm{Spec}(R)$ is J-2, then for every finite type R-algebra A the regular locus of the scheme $\mathrm{Spec}(A)$ is open. Hence R is J-2, by definition (see More on Algebra, Definition 15.47.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \to X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \to X$ maps into one of the U_i. By Lemma 29.15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = \mathrm{Spec}(\mathcal{O}(V_j))$ is open. Since $\mathrm{Reg}(Y) \cap V_j = \mathrm{Reg}(V_j)$ we conclude that $\mathrm{Reg}(Y)$ is open as desired. □

07R5 **Lemma 29.19.3.** *The following types of schemes are J-2.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$.
(4) *Any scheme locally of finite type over a Noetherian local ring of dimension* 1.
(5) *Any scheme locally of finite type over a Nagata ring of dimension* 1.
(6) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(7) *And so on.*

Proof. By Lemma 29.19.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 15.48.7. □

29.20. Quasi-finite morphisms

01TC A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 10.122.

Recall that a finite type ring map $R \to A$ is quasi-finite at a prime $\mathfrak{q}$ if $\mathfrak{q}$ defines an isolated point of its fibre, see Algebra, Definition 10.122.3.

01TD **Definition 29.20.1.** Let $f : X \to S$ be a morphism of schemes. [**DG67**, II Definition 6.2.3]

(1) We say that f is *quasi-finite at a point* $x \in X$ if there exist an affine neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \to A$ is of finite type, and $R \to A$ is quasi-finite at the prime of A corresponding to x (see above).
(2) We say f is *locally quasi-finite* if f is quasi-finite at every point x of X.
(3) We say that f is *quasi-finite* if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is

isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 29.56 on Zariski's Main Theorem.

01TE **Lemma 29.20.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(x)/\kappa(s)$ is an algebraic field extension, then*

(1) *x is a closed point of its fibre, and*
(2) *if in addition s is a closed point of S, then x is a closed point of X.*

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 26.18.5 to prove the first statement we may replace X by X_s and S by $\mathrm{Spec}(\kappa(s))$. Thus we may assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field. In this case, let $\mathrm{Spec}(A) = U \subset X$ be any affine open containing x. The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $\kappa(\mathfrak{q})/k$ is an algebraic field extension. By Algebra, Lemma 10.35.9 we see that $\mathfrak{q}$ is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. □

The following lemma is a version of the Hilbert Nullstellensatz.

01TF **Lemma 29.20.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(x)/\kappa(s)$ is a finite field extension.*

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 29.20.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 29.15.2 the ring map $R \to A$ is of finite type. Let $\mathfrak{q} \subset A$, resp. $\mathfrak{p} \subset R$ be the prime ideal corresponding to x, resp. s. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. The assumption that x is a closed point of its fibre implies that $\overline{\mathfrak{q}}$ is a maximal ideal of $\overline{A}$. Since $\overline{A}$ is an algebra of finite type over the field $\kappa(\mathfrak{p})$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1, that $\kappa(\overline{\mathfrak{q}})$ is a finite extension of $\kappa(\mathfrak{p})$. Since $\kappa(s) = \kappa(\mathfrak{p})$ and $\kappa(x) = \kappa(\mathfrak{q}) = \kappa(\overline{\mathfrak{q}})$ we win. □

053M **Lemma 29.20.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $g : S' \to S$ be any morphism. Denote $f' : X' \to S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(x)}$ then x' is closed in $X'_{f'(x')}$.*

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 26.17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 29.20.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. □

01TG **Lemma 29.20.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x, then the residue field extension $\kappa(x)/\kappa(s)$ is finite.*

Proof. This is clear from Algebra, Definition 10.122.3. □

01TH **Lemma 29.20.6.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s. Assume f is locally of finite type. The following are equivalent:*

(1) *The morphism f is quasi-finite at x.*
(2) *The point x is isolated in X_s.*
(3) *The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x.*
(4) *For any pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak{q} \subset A$ the ring map $R \to A$ is quasi-finite at $\mathfrak{q}$.*

Proof. Assume f is quasi-finite at x. By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s. Hence $\{x\} \subset U_s$ is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s.

Note that X_s is a Jacobson scheme by Lemma 29.16.10 (and Lemma 29.15.4). If x is isolated in X_s, i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s. It is clear that there is no point $x' \in X_s$, distinct from x, specializing to x.

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x, specializing to x. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s. By Lemma 29.15.2 the ring map $R \to A$ is of finite type. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. Since $\mathrm{Spec}(\overline{A})$ is an open subscheme of the fibre X_s we see that $\overline{q}$ is a maximal ideal of $\overline{A}$ and that there is no point of $\mathrm{Spec}(\overline{A})$ specializing to $\overline{\mathfrak{q}}$. This implies that $\dim(\overline{A}_{\overline{q}}) = 0$. Hence by Algebra, Definition 10.122.3 we see that $R \to A$ is quasi-finite at $\mathfrak{q}$, i.e., $X \to S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) – (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. □

02NG **Lemma 29.20.7.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *f is locally of finite type, and*
(2) *$f^{-1}(\{s\})$ is a finite set.*

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s.

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k, and (b) has finitely many points. Then Lemma 29.16.10 shows T is a Jacobson scheme. A finite Jacobson space is discrete, see Topology, Lemma 5.18.6. Apply this remark to the fibre X_s which is locally of finite type over $\mathrm{Spec}(\kappa(s))$ to see the first statement. Finally, apply Lemma 29.20.6 to see the second. □

06RT **Lemma 29.20.8.** *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every $s \in S$ the fibre X_s is a discrete topological space, and*
(3) *for every morphism $\mathrm{Spec}(k) \to S$ where k is a field the base change X_k has an underlying discrete topological space.*

Proof. It is immediate that (3) implies (2). Lemma 29.20.6 shows that (2) is equivalent to (1). Assume (2) and let $\mathrm{Spec}(k) \to S$ be as in (3). Denote $s \in S$ the image of $\mathrm{Spec}(k) \to S$. Then X_k is the base change of X_s via $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(s))$. Hence every point of X_k is closed by Lemma 29.20.4. As $X_k \to \mathrm{Spec}(k)$ is locally of finite type (by Lemma 29.15.4), we may apply Lemma 29.20.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. □

01TJ **Lemma 29.20.9.** *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.*

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 29.15.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 29.20.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 29.20.6 we see that x is an isolated point of its fibre. The lemma is proved. □

02NH **Lemma 29.20.10.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is quasi-finite, and*
(2) *f is locally of finite type, quasi-compact, and has finite fibres.*

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 29.20.7. Hence it is quasi-finite by Lemma 29.20.9. □

Recall that a ring map $R \to A$ is quasi-finite if it is of finite type and quasi-finite at *all* primes of A, see Algebra, Definition 10.122.3.

01TK **Lemma 29.20.11.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally quasi-finite.*
(2) *For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is quasi-finite.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally quasi-finite.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.*

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally quasi-finite.

Proof. For a ring map $R \to A$ let us define $P(R \to A)$ to mean "$R \to A$ is quasi-finite" (see remark above lemma). We claim that P is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 29.14.1. In the proof of Lemma

29.15.2 we have seen that (a), (b) and (c) hold for the property of being "of finite type". Note that, for a finite type ring map $R \to A$, the property $R \to A$ is quasi-finite at $\mathfrak{q}$ depends only on the local ring $A_\mathfrak{q}$ as an algebra over $R_\mathfrak{p}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 29.14.1 follow immediately. For example, suppose $R \to A$ is a ring map such that all of the ring maps $R \to A_{a_i}$ are quasi-finite for $a_1, \ldots, a_n \in A$ generating the unit ideal. We conclude that $R \to A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_\mathfrak{q}$ is isomorphic as an R-algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \to A$ is quasi-finite at $\mathfrak{q}$.

We conclude that Lemma 29.14.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P. Since $P(R \to A)$ is "$R \to A$ is quasi-finite" which means $R \to A$ is quasi-finite at every prime of A, this follows from Lemma 29.20.6. □

01TL **Lemma 29.20.12.** *The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.*

Proof. In the proof of Lemma 29.20.11 we saw that $P =$"quasi-finite" is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 29.14.2. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 10.122.8. By the above, Lemma 29.20.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 26.19.4 we see that the composition of quasi-finite morphisms is quasi-finite. □

We will see later (Lemma 29.56.2) that the set U of the following lemma is open.

01TM **Lemma 29.20.13.** *Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Denote $f' : X' \to S'$ the base change of f by g and denote $g' : X' \to X$ the projection. Assume X is locally of finite type over S.*

(1) *Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U \times_S S' = (g')^{-1}(U)$.*
(2) *The base change of a locally quasi-finite morphism is locally quasi-finite.*
(3) *The base change of a quasi-finite morphism is quasi-finite.*

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 10.122.9 (combined with the fact that f' is also locally of finite type by Lemma 29.15.4). By the above, Lemma 29.20.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3 we see that the base change of a quasi-finite morphism is quasi-finite. □

0AAY **Lemma 29.20.14.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $s \in S$. There are at most finitely many points of X lying over s at which f is quasi-finite.*

Proof. The fibre X_s is a scheme of finite type over a field, hence Noetherian (Lemma 29.15.6). Hence the topology on X_s is Noetherian (Properties, Lemma 28.5.5) and can have at most a finite number of isolated points (by elementary topology). Thus our lemma follows from Lemma 29.20.6. □

0CT8 **Lemma 29.20.15.** *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.*

Proof. A monomorphism is separated by Schemes, Lemma 26.23.3. A monomorphism is injective, hence we get f is quasi-finite at every $x \in X$ for example by Lemma 29.20.6. □

01TN **Lemma 29.20.16.** *Any immersion is locally quasi-finite.*

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. □

03WR **Lemma 29.20.17.** *Let $X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$. If $X \to S$ is quasi-finite at x, then $X \to Y$ is quasi-finite at x. If X is locally quasi-finite over S, then $X \to Y$ is locally quasi-finite.*

Proof. Via Lemma 29.20.11 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is quasi-finite, then $B \to C$ is quasi-finite. This follows from Algebra, Lemma 10.122.7 with $R = A$, $S = S' = C$ and $R' = B$. □

0GWS **Lemma 29.20.18.** *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If f is surjective, $g \circ f$ locally quasi-finite, and g locally of finite type, then $g : Y \to S$ is locally quasi-finite.*

Proof. Let $x \in X$ with images $y \in Y$ and $s \in S$. Since $g \circ f$ is locally quasi-finite by Lemma 29.20.5 the extension $\kappa(x)/\kappa(s)$ is finite. Hence $\kappa(y)/\kappa(s)$ is finite. Hence y is a closed point of Y_s by Lemma 29.20.2. Since f is surjective, we see that every point of Y is closed in its fibre over S. Thus by Lemma 29.20.6 we conclude that g is quasi-finite at every point. □

29.21. Morphisms of finite presentation

01TO Recall that a ring map $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j, see Algebra, Definition 10.6.1.

01TP **Definition 29.21.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *of finite presentation at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
(2) We say that f is *locally of finite presentation* if it is of finite presentation at every point of X.
(3) We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$\mathrm{colim}\, \mathrm{Mor}_S(T_i, X) = \mathrm{Mor}_S(\lim T_i, X)$$

for any directed system of affine schemes T_i over S. See Limits, Proposition 32.6.1. In Limits, Section 32.10 we show that, if $S = \lim_i S_i$ is a limit of affine schemes,

any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i.

01TQ **Lemma 29.21.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite presentation.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation.*
(3) *There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite presentation.*
(4) *There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite presentation.

Proof. This follows from Lemma 29.14.4 if we show that the property "$R \to A$ is of finite presentation" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.14.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.6.2 being of finite presentation is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.23.3. □

01TR **Lemma 29.21.3.** *The composition of two morphisms which are locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 29.21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 26.19.4 and 26.21.12 we see that the composition of morphisms of finite presentation is of finite presentation. □

01TS **Lemma 29.21.4.** *The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 29.21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 10.14.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 26.19.3 and 26.21.12 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. □

01TT **Lemma 29.21.5.** *Any open immersion is locally of finite presentation.*

Proof. This is true because an open immersion is a local isomorphism. □

01TU **Lemma 29.21.6.** *Any open immersion is of finite presentation if and only if it is quasi-compact.*

Proof. We have seen (Lemma 29.21.5) that an open immersion is locally of finite presentation. We have seen (Schemes, Lemma 26.23.8) that an immersion is separated and hence quasi-separated. From this and Definition 29.21.1 the lemma follows. □

01TV **Lemma 29.21.7.** *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. On any affine open $\mathrm{Spec}(R) \subset X$ we have $i^{-1}(\mathrm{Spec}(R)) = \mathrm{Spec}(R/I)$ and $\mathcal{I} = \widetilde{I}$. Moreover, $\mathcal{I}$ is of finite type if and only if I is a finite R-module for every such affine open (see Properties, Lemma 28.16.1). And R/I is of finite presentation over R if and only if I is a finite R-module. Hence we win. □

01TW **Lemma 29.21.8.** *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Omitted. □

01TX **Lemma 29.21.9.** *Let $f : X \to S$ be a morphism.*

(1) *If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If S is locally Noetherian and f of finite type then f is of finite presentation.*

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 10.31.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 29.15.7 (and Lemma 29.21.8). □

01TY **Lemma 29.21.10.** *Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

02FV **Lemma 29.21.11.** *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is locally of finite presentation over S and Y is locally of finite type over S, then f is locally of finite presentation.*
(2) *If X is of finite presentation over S and Y is quasi-separated and locally of finite type over S, then f is of finite presentation.*

Proof. Proof of (1). Via Lemma 29.21.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite presentation and $A \to B$ is of finite type, then $B \to C$ is of finite presentation. See Algebra, Lemma 10.6.2.

Part (2) follows from (1) and Schemes, Lemmas 26.21.13 and 26.21.14. □

0818 **Lemma 29.21.12.** *Let $f : X \to Y$ be a morphism of schemes with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism of schemes over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is locally of finite type over X (by Lemma 29.15.4). Thus the first statement holds by Lemma 29.21.11. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is a monomorphism, hence separated (Schemes, Lemma 26.23.3). □

29.22. Constructible sets

054H Constructible and locally constructible sets of schemes have been discussed in Properties, Section 28.2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

054I **Lemma 29.22.1.** *Let $f : X \to Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) constructible in Y, then $f^{-1}(E)$ is (locally) constructible in X.*

Proof. To show that the inverse image of every constructible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X, see Topology, Lemma 5.15.3. The significance of V being retrocompact in Y is just that the open immersion $V \to Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \to X$ is quasi-compact too, see Schemes, Lemma 26.19.3. Hence we see $f^{-1}(V)$ is retrocompact in X. Suppose E is locally constructible in Y. Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \to V$ as a morphism into V. By Properties, Lemma 28.2.1 we see that $E \cap V$ is constructible in V. By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U. Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. □

054J **Lemma 29.22.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *f is quasi-compact and locally of finite presentation, and*
(2) *Y is quasi-compact and quasi-separated.*

Then the image of every constructible subset of X is constructible in Y.

Proof. By Properties, Lemma 28.2.5 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 5.15.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.29.10. □

054K **Theorem 29.22.3** (Chevalley's Theorem)**.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible.* [**DG67**, IV, Theorem 1.8.4]

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 28.2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.29.10. □

05LW **Lemma 29.22.4.** *Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \rightsquigarrow x\} \subset E$, then E contains an open neighbourhood of x.*

Proof. Assume $\{x' \mid x' \rightsquigarrow x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 28.2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 10.29.4 we can find a morphism of affine schemes $f : Y \to X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f. By Lemma 29.6.5 and the assumption $\{x' \mid x' \rightsquigarrow x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E. □

29.23. Open morphisms

01TZ

01U0 **Definition 29.23.1.** Let $f : X \to S$ be a morphism.

(1) We say f is *open* if the map on underlying topological spaces is open.
(2) We say f is *universally open* if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is open.

According to Topology, Lemma 5.19.7 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 29.23.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 29.25.9). This sometimes in turn implies that the morphism is open.

01U1 **Lemma 29.23.2.** *Let $f : X \to S$ be a morphism.*

(1) *If f is locally of finite presentation and generalizations lift along f, then f is open.*
(2) *If f is locally of finite presentation and generalizations lift along every base change of f, then f is universally open.*

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 10.41.3 and Proposition 10.41.8. □

See also Lemma 29.25.10 for the case of a morphism flat of finite presentation.

02V2 **Lemma 29.23.3.** *A composition of (universally) open morphisms is (universally) open.*

Proof. Omitted. □

0383 **Lemma 29.23.4.** *Let k be a field. Let X be a scheme over k. The structure morphism $X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Let $S \to \mathrm{Spec}(k)$ be a morphism. We have to show that the base change $X_S \to S$ is open. The question is local on S and X, hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 10.41.10. □

040F **Lemma 29.23.5.** *Let $\varphi : X \to Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.*

Follows from the implication (a) ⇒ (b) in [**DG67**, IV, Corollary 1.10.4]

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y. Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 10.41.2 (combined with Algebra, Lemma 10.41.3). □

04ZE **Lemma 29.23.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be open and surjective such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 29.9.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 5.12.7. Thus f is quasi-compact. □

29.24. Submersive morphisms

040G

040H **Definition 29.24.1.** Let $f : X \to Y$ be a morphism of schemes.

(1) We say f is *submersive*[7] if the continuous map of underlying topological spaces is submersive, see Topology, Definition 5.6.3.
(2) We say f is *universally submersive* if for every morphism of schemes $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CES **Lemma 29.24.2.** *The base change of a universally submersive morphism of schemes by any morphism of schemes is universally submersive.*

Proof. This is immediate from the definition. □

0CET **Lemma 29.24.3.** *The composition of a pair of (universally) submersive morphisms of schemes is (universally) submersive.*

Proof. Omitted. □

29.25. Flat morphisms

01U2 Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 29.25.10. A very important class of results, namely criteria for flatness, are discussed in Algebra, Sections 10.99, 10.101, 10.128, and More on Morphisms, Section 37.16. There is a chapter dedicated

[7] This is very different from the notion of a submersion of differential manifolds.

to advanced material on flat morphisms of schemes, namely More on Flatness, Section 38.1.

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module. See Algebra, Definition 10.39.1.

01U3 **Definition 29.25.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(1) We say f is *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
(2) We say that $\mathcal{F}$ is *flat over* S *at a point* $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
(3) We say f is *flat* if f is flat at every point of X.
(4) We say that $\mathcal{F}$ is *flat over* S if $\mathcal{F}$ is flat over S at every point x of X.

Thus we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

01U4 **Lemma 29.25.2.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

(1) *The sheaf $\mathcal{F}$ is flat over S.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_S(V)$-module $\mathcal{F}(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the modules $\mathcal{F}|_{U_i}$ is flat over V_j, for all $j \in J, i \in I_j$.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}_S(V_j)$-module, for all $j \in J, i \in I_j$.*

Moreover, if $\mathcal{F}$ is flat over S then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is flat over V.

Proof. Let $R \to A$ be a ring map. Let M be an A-module. If M is R-flat, then for all primes $\mathfrak{q}$ the module $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$. Conversely, if $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ for all primes $\mathfrak{q}$ of A, then M is flat over R. See Algebra, Lemma 10.39.18. This equivalence easily implies the statements of the lemma. □

01U5 **Lemma 29.25.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is flat.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is flat.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is flat, for all $j \in J, i \in I_j$.*

Moreover, if f is flat then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is flat.

Proof. This is a special case of Lemma 29.25.2 above. □

0FLM **Lemma 29.25.4.** *Let $f : X \to Y$ be an affine morphism of schemes over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat over S if and only if $f_*\mathcal{F}$ is flat over S.*

Proof. By Lemma 29.25.2 and the fact that f is an affine morphism, this reduces us to the affine case. Say $X \to Y \to S$ corresponds to the ring maps $C \leftarrow B \leftarrow A$. Let N be the C-module corresponding to $\mathcal{F}$. Recall that $f_*\mathcal{F}$ corresponds to N viewed as a B-module, see Schemes, Lemma 26.7.3. Thus the result is clear. □

01U6 **Lemma 29.25.5.** *Let $X \to Y \to Z$ be morphisms of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image y in Y. If $\mathcal{F}$ is flat over Y at x, and Y is flat over Z at y, then $\mathcal{F}$ is flat over Z at x.*

Proof. See Algebra, Lemma 10.39.4. □

01U7 **Lemma 29.25.6.** *The composition of flat morphisms is flat.*

Proof. This is a special case of Lemma 29.25.5. □

01U8 **Lemma 29.25.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let $g : S' \to S$ be a morphism of schemes. Denote $g' : X' = X_{S'} \to X$ the projection. Let $x' \in X'$ be a point with image $x = g'(x') \in X$. If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'. In particular, if $\mathcal{F}$ is flat over S, then $(g')^*\mathcal{F}$ is flat over S'.*

Proof. See Algebra, Lemma 10.39.7. □

01U9 **Lemma 29.25.8.** *The base change of a flat morphism is flat.*

Proof. This is a special case of Lemma 29.25.7. □

03HV **Lemma 29.25.9.** *Let $f : X \to S$ be a flat morphism of schemes. Then generalizations lift along f, see Topology, Definition 5.19.4.*

Proof. See Algebra, Section 10.41. □

01UA **Lemma 29.25.10.** *A flat morphism locally of finite presentation is universally open.*

Proof. This follows from Lemmas 29.25.9 and Lemma 29.23.2 above. We can also argue directly as follows.

Let $f : X \to S$ be flat and locally of finite presentation. By Lemmas 29.25.8 and 29.21.4 any base change of f is flat and locally of finite presentation. Hence it suffices to show f is open. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \to S$ is open. We may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_i)$ is flat and of finite presentation (Lemmas 29.25.3 and 29.21.2). Then $U_i \to V_i$ is open by Algebra, Proposition 10.41.8 and the proof is complete. □

0CVT **Lemma 29.25.11.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = Supp(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.*

Proof. By Lemmas 29.25.7, 29.21.4, and 29.5.3 the assumptions are preserved under base change. By Lemma 29.23.2 it suffices to show that generalizations lift along f. This follows from Algebra, Lemma 10.41.12. □

02JY **Lemma 29.25.12.** *Let $f : X \to Y$ be a quasi-compact, surjective, flat morphism. A subset $T \subset Y$ is open (resp. closed) if and only $f^{-1}(T)$ is open (resp. closed). In other words, f is a submersive morphism.* [**Gro71**, Expose VIII, Corollaire 4.3] and [**DG67**, IV, Corollaire 2.3.12]

Proof. The question is local on Y, hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \amalg \ldots \amalg X_n \to Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \amalg \ldots \amalg f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \to B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) = V(J)$ for $J \subset B$ an ideal. Then $T = f(f^{-1}(T)) = f(V(J))$ is the image of $\mathrm{Spec}(B/J) \to \mathrm{Spec}(A)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 29.25.9). Hence by Topology, Lemma 5.19.6 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 5.19.2). Thus T is closed by Algebra, Lemma 10.41.5. □

02JZ **Lemma 29.25.13.** *Let $h : X \to Y$ be a morphism of schemes over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x, then*

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then $\mathcal{G}$ is flat over S, if and only if $h^*\mathcal{G}$ is flat over S. If h is surjective and flat, and X is flat over S, then Y is flat over S.

Proof. You can prove this by applying Algebra, Lemma 10.39.9. Here is a direct proof. Let $s \in S$ be the image of y. Consider the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 10.39.17. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 6.26.4. Let $M' \to M$ be an injection of $\mathcal{O}_{S,s}$-modules. By the faithful flatness mentioned above we have

$$\begin{aligned} &\mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \to M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ &= \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 10.39.5. □

07T9 **Lemma 29.25.14.** *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of scheme theoretic support on affines as given in Lemma 29.5.4 we reduce to Algebra, Lemma 10.40.4. □

081H **Lemma 29.25.15.** *Let $f : X \to Y$ be a flat morphism of schemes. Let $V \subset Y$ be a retrocompact open which is scheme theoretically dense. Then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Lemma 29.7.5. We have to show that for any open $U \subset X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap f^{-1}V)$ is injective. It suffices to prove this when U is an affine open which maps into an affine open $W \subset Y$. Say $W = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$. Then $V \cap W = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.29.1. Thus we have to show that $B \to B_{f_1} \times \ldots \times B_{f_n}$ is injective. We are given that $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective and that $A \to B$ is flat. Since $B_{f_i} = A_{f_i} \otimes_A B$ we win. □

081I **Lemma 29.25.16.** *Let $f : X \to Y$ be a flat morphism of schemes. Let $g : V \to Y$ be a quasi-compact morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Recall that Z is cut out by $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to g_*\mathcal{O}_V)$ and Z' is cut out by $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_X \to (V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X})$, see Lemma 29.6.3. Hence the question is local on X and Y and we may assume X and Y affine. Note that we may replace V by $\coprod V_i$ where $V = V_1 \cup \ldots \cup V_n$ is a finite affine open covering. Hence we may assume g is affine. In this case $(V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X}$ is the pullback of $g_*\mathcal{O}_V$ by f. Since f is flat we conclude that $f^*\mathcal{I} = \mathcal{I}'$ and the lemma holds. □

29.26. Flat closed immersions

04PV Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

04PW **Lemma 29.26.1.** *Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection*

$$\left\{\begin{array}{c}\text{closed subschemes } Z \subset X \\ \text{such that } Z \to X \text{ is flat}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{closed subsets } Z \subset X \\ \text{closed under generalizations}\end{array}\right\}$$

If $Z \subset X$ is such a closed subscheme, every morphism of schemes $g : Y \to X$ with $g(Y) \subset Z$ set theoretically factors (scheme theoretically) through Z.

Proof. The affine case of the bijection is Algebra, Lemma 10.108.4. For general schemes X the bijection follows by covering X by affines and glueing. Details omitted. For the final assertion, observe that the projection $Z \times_{X,g} Y \to Y$ is a flat (Lemma 29.25.8) closed immersion which is bijective on underlying topological spaces and hence must be an isomorphism by the bijection established in the first part of the proof. □

0819 **Lemma 29.26.2.** *A flat closed immersion of finite presentation is the open immersion of an open and closed subscheme.*

Proof. The affine case is Algebra, Lemma 10.108.5. In general the lemma follows by covering X by affines. Details omitted. □

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

04PX **Definition 29.26.3.** Let X be a scheme. Let $T \subset X$ be a connected component. The *canonical scheme structure on T* is the unique scheme structure on T such that the closed immersion $T \to X$ is flat, see Lemma 29.26.1.

It turns out that we can determine when every finite flat $\mathcal{O}_X$-module is finite locally free using the previous lemma.

053N **Lemma 29.26.4.** *Let X be a scheme. The following are equivalent*

(1) *every finite flat quasi-coherent $\mathcal{O}_X$-module is finite locally free, and*
(2) *every closed subset $Z \subset X$ which is closed under generalizations is open.*

Proof. In the affine case this is Algebra, Lemma 10.108.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \to X$ such that i is flat. Then $i_*\mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \text{Supp}(i_*\mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) $\Rightarrow$ (2) follows from the characterization of flat closed immersions in Lemma 29.26.1.

For the converse assume that X satisfies (2). Let $\mathcal{F}$ be a finite flat quasi-coherent $\mathcal{O}_X$-module. The support $Z = \text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is closed, see Modules, Lemma 17.9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 10.78.5 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of $\mathcal{F}$ is open and closed. The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \ldots$ are finite flat quasi-coherent $\mathcal{O}_X$-modules also, see Modules, Section 17.21. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \amalg U_1 \amalg U_2 \amalg \ldots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \ldots$ for all i. Let x be a point of X, and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r. By Nakayama's lemma, see Algebra, Lemma 10.20.1 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \ldots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent $\mathcal{O}_U$-module whose support is all of U. By the above it is generated by a single element, namely $s_1 \wedge \ldots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$ such that $\mathcal{O}_U/\mathcal{I}$ is flat over $\mathcal{O}_U$ and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 29.26.1. Thus $s_1 \wedge \ldots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that $\mathcal{F}$ is finite locally free as desired. □

29.27. Generic flatness

0529 A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 10.118 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 29.27.2 which shows the result holds if we only assume the base is reduced.

052A **Proposition 29.27.1** (Generic flatness)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *S is integral,*

(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. As S is integral it is irreducible (see Properties, Lemma 28.3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = \mathrm{Spec}(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \mathrm{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 29.15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 28.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \mathrm{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.118.3 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f, i.e., perform the base change by $U = \mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 29.21.2 and Properties, Lemma 28.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 26.21.6. This proves the proposition. □

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

052B **Proposition 29.27.2** (Generic flatness, reduced case)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *S is reduced,*
(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 29.27.1 using Algebra, Lemma 10.118.7 instead of Algebra, Lemma 10.118.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 29.25.2 and 29.21.2, we may work affine locally on S. Thus we may assume that

$S = \mathrm{Spec}(A)$, where A is a reduced ring (see Properties, Lemma 28.3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \mathrm{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 29.15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 28.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \mathrm{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.118.7 to the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain dense opens $U(A \to B_i, M_{ij}) \subset S$ and dense opens $U(A \to B_i, M_i) \subset S$ with notation as in Algebra, Equation (10.118.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap\nolimits_{i,j} U(A \to B_i, M_{ij}) \quad \cap \quad \bigcap\nolimits_i U(A \to B_i, M_i)$$

is open and dense in S. We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \to B_i, M_{ij})$ and $U(A \to B, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_{f_i} and M_{i,f_i} is of finite presentation over B_{i,f_i} and flat over A_{f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \to S$ is flat and of finite presentation over $D(f)$ and that $\mathcal{F}$ restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f, i.e., perform the base change by $\mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 29.21.2 and Properties, Lemma 28.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 26.21.6. This proves the proposition. □

052C **Remark 29.27.3.** The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [**GR71**] by Raynaud and Gruson contains many wonderful results in this direction.

29.28. Morphisms and dimensions of fibres

02FW Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X. See Topology, Definition 5.10.1.

02FX **Lemma 29.28.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then*

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + trdeg_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 10.116.3. □

02JS **Lemma 29.28.2.** *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then*

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s,x}$ is flat over $\mathcal{O}_{Y_s,y}$, which holds for example if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$.

Proof. Note that $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \mathrm{trdeg}_{\kappa(y)}(\kappa(x)) + \mathrm{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 29.28.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s,x}) \leq \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y_s,y}).$$

For this see Algebra, Lemma 10.112.6. For the flat case see Algebra, Lemma 10.112.7. □

02FY **Lemma 29.28.3.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then

(1) $\dim_x(X_s) = \dim_{x'}(X'_{s'})$,
(2) *if F is the fibre of the morphism $X'_{s'} \to X_s$ over x, then*

$$\dim(\mathcal{O}_{F,x'}) = \dim(\mathcal{O}_{X'_{s'},x'}) - \dim(\mathcal{O}_{X_s,x}) = \mathit{trdeg}_{\kappa(s)}(\kappa(x)) - \mathit{trdeg}_{\kappa(s')}(\kappa(x'))$$

In particular $\dim(\mathcal{O}_{X'_{s'},x'}) \geq \dim(\mathcal{O}_{X_s,x})$ and $\mathit{trdeg}_{\kappa(s)}(\kappa(x)) \geq \mathit{trdeg}_{\kappa(s')}(\kappa(x'))$.

(3) *given s', s, x there exists a choice of x' such that $\dim(\mathcal{O}_{X'_{s'},x'}) = \dim(\mathcal{O}_{X_s,x})$ and $\mathit{trdeg}_{\kappa(s)}(\kappa(x)) = \mathit{trdeg}_{\kappa(s')}(\kappa(x'))$.*

Proof. Part (1) follows immediately from Algebra, Lemma 10.116.6. Parts (2) and (3) from Algebra, Lemma 10.116.7. □

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

02FZ **Lemma 29.28.4.** *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set* [**DG67**, IV Theorem 13.1.3]

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X.

Proof. This is immediate from Algebra, Lemma 10.125.6 □

0A3V **Lemma 29.28.5.** *Let $f : X \to Y$ be a morphism of finite type with Y quasi-compact. Then the dimension of the fibres of f is bounded.*

Proof. By Lemma 29.28.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since f is of finite type, every point is contained in some U_n (because the dimension of a finite type algebra over a field is finite). Since Y is quasi-compact and f is of finite type, we see that X is quasi-compact. Hence $X = U_n$ for some n. □

02G0 **Lemma 29.28.6.** *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 29.28.4 is retrocompact in X. (See Topology, Definition 5.12.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the induced morphism $f|_U : U \to S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U. Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 10.125.8 (and Lemma 29.21.2). □

06RU **Lemma 29.28.7.** *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then*

(1) $\dim_x(X_s) \leq \dim_{x'}(X_s)$,
(2) $\dim(\mathcal{O}_{X_s,x}) < \dim(\mathcal{O}_{X_s,x'})$, *and*
(3) $trdeg_{\kappa(s)}(\kappa(x)) > trdeg_{\kappa(s)}(\kappa(x'))$.

Proof. Part (1) follows from the fact that any open of X_s containing x' also contains x. Part (2) follows since $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{X_s,x'}$ at a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s,x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s,x'}$. The last inequality follows from Algebra, Lemma 10.116.2. □

29.29. Morphisms of given relative dimension

02NI In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

02NJ **Definition 29.29.1.** Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type.

(1) We say f is of *relative dimension $\leq d$ at x* if $\dim_x(X_{f(x)}) \leq d$.
(2) We say f is of *relative dimension $\leq d$* if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
(3) We say f is of *relative dimension d* if all nonempty fibres X_s are equidimensional of dimension d.

This is not a particularly well behaved notion, but it works well in a number of situations.

02NK **Lemma 29.29.2.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d, then so does any base change of f. Same for relative dimension $\leq d$.*

Proof. This is immediate from Lemma 29.28.3. □

02NL **Lemma 29.29.3.** *Let $f : X \to Y$, $g : Y \to Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If*

(1) *f has relative dimension d,*
(2) *g has relative dimension e, and*
(3) *f is flat,*

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 29.28.2. □

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

02NM **Lemma 29.29.4.** *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *f is flat,*
(2) *f is locally of finite presentation, and*
(3) *for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 28.8.1)*

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to S$ has relative dimension d.

Proof. This is immediate from Algebra, Lemma 10.130.8. □

0397 **Lemma 29.29.5.** *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.*

Proof. First proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 29.20.5) and x is isolated in X_s (by Lemma 29.20.6), hence $\dim_x(X_s) = 0$ by Lemma 29.28.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 29.28.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x. Hence x is closed in its fibre by Lemma 29.20.2 and by Lemma 29.20.6 (3) we conclude that f is quasi-finite at x.

Second proof. The fibre X_s is a scheme locally of finite type over a field, hence locally Noetherian (Lemma 29.15.6). The result now follows from Lemma 29.20.6 and Properties, Lemma 28.10.7. □

0AFE **Lemma 29.29.6.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is flat, locally of finite type and of relative dimension d. For every point x in X with image y in Y we have $\dim_x(X) = \dim_y(Y) + d$.*

Proof. After shrinking X and Y to open neighborhoods of x and y, we can assume that $\dim(X) = \dim_x(X)$ and $\dim(Y) = \dim_y(Y)$, by definition of the dimension of a scheme at a point (Properties, Definition 28.10.1). The morphism f is open by Lemmas 29.21.9 and 29.25.10. Hence we can shrink Y to arrange that f is surjective. It remains to show that $\dim(X) = \dim(Y) + d$.

Let a be a point in X with image b in Y. By Algebra, Lemma 10.112.7,

$$\dim(\mathcal{O}_{X,a}) = \dim(\mathcal{O}_{Y,b}) + \dim(\mathcal{O}_{X_b,a}).$$

Taking the supremum over all points a in X, it follows that $\dim(X) = \dim(Y) + d$, as we want, see Properties, Lemma 28.10.2. □

29.30. Syntomic morphisms

01UB An algebra A over a field k is called a *global complete intersection over k* if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $\mathrm{Spec}(A)$ can be covered by standard opens each of which are global complete intersections over k. See Algebra, Section 10.135. Recall

that a ring map $R \to A$ is *syntomic* if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 10.136.1.

01UC **Definition 29.30.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *syntomic at* $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is syntomic.
(2) We say that f is *syntomic* if it is syntomic at every point of X.
(3) If $S = \mathrm{Spec}(k)$ and f is syntomic, then we say that X is a *local complete intersection over* k.
(4) A morphism of affine schemes $f : X \to S$ is called *standard syntomic* if there exists a global relative complete intersection $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.136.5) such that $X \to S$ is isomorphic to
$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 37.62 we will consider morphisms $X \to S$ which locally are of the form
$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$
for some Koszul-regular sequence $f_1, \ldots, f_r$ in $R[x_1, \ldots, x_n]$. Such a morphism will be called a *local complete intersection morphism*. Once we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

01UD **Lemma 29.30.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is syntomic.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is syntomic.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is syntomic.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is syntomic, for all $j \in J, i \in I_j$.*

Moreover, if f is syntomic then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is syntomic.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is syntomic" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.136.3 being syntomic is stable under base change and hence

we conclude (a) holds. By Algebra, Lemma 10.136.17 being syntomic is stable under composition and trivially for any ring R the ring map $R \to R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.136.4. □

01UH **Lemma 29.30.3.** *The composition of two morphisms which are syntomic is syntomic.*

Proof. In the proof of Lemma 29.30.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 10.136.17. □

01UI **Lemma 29.30.4.** *The base change of a morphism which is syntomic is syntomic.*

Proof. In the proof of Lemma 29.30.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 10.136.3. □

01UJ **Lemma 29.30.5.** *Any open immersion is syntomic.*

Proof. This is true because an open immersion is a local isomorphism. □

01UK **Lemma 29.30.6.** *A syntomic morphism is locally of finite presentation.*

Proof. True because a syntomic ring map is of finite presentation by definition. □

01UL **Lemma 29.30.7.** *A syntomic morphism is flat.*

Proof. True because a syntomic ring map is flat by definition. □

056F **Lemma 29.30.8.** *A syntomic morphism is universally open.*

Proof. Combine Lemmas 29.30.6, 29.30.7, and 29.25.10. □

Let k be a field. Let A be a local k-algebra essentially of finite type over k. Recall that A is called a *complete intersection over k* if we can write $A \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring essentially of finite type over k, and $f_1, \ldots, f_c$ is a regular sequence in R, see Algebra, Definition 10.135.5.

01UG **Lemma 29.30.9.** *Let k be a field. Let X be a scheme locally of finite type over k. The following are equivalent:*

(1) *X is a local complete intersection over k,*
(2) *for every $x \in X$ there exists an affine open $U = \mathrm{Spec}(R) \subset X$ neighbourhood of x such that $R \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k, and*
(3) *for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a complete intersection over k.*

Proof. The corresponding algebra results can be found in Algebra, Lemmas 10.135.8 and 10.135.9. □

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a *local model* for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

01UE **Lemma 29.30.10.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$. Let $V \subset S$ be an affine open neighbourhood of s. The following are equivalent*

(1) *The morphism f is syntomic at x.*
(2) *There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that $f|_U : U \to V$ is standard syntomic.*
(3) *The morphism f is of finite presentation at x, the local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a complete intersection over $\kappa(s)$ (see Algebra, Definition 10.135.5).*

Proof. Follows from the definitions and Algebra, Lemma 10.136.15. □

01UF **Lemma 29.30.11.** *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are local complete intersections, then f is syntomic.*

Proof. Clear from Lemmas 29.30.9 and 29.30.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

02V3 **Lemma 29.30.12.** *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is syntomic.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 10.135.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 29.30.10. □

02K0 **Lemma 29.30.13.** *Let R be a ring. Let $R \to A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Set $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Consider the morphism $f : X \to S$ associated to the ring map $R \to A$. The function $x \mapsto \dim_x(X_{f(x)})$ is constant with value $n - c$.*

Proof. By Algebra, Definition 10.136.5 $R \to A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime $\mathfrak{p}$ of R the fibre ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 10.135.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. □

02K1 **Lemma 29.30.14.** *Let $f : X \to S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X.*

Proof. By Lemma 29.30.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 29.30.13. □

Lemma 29.30.14 says that the following definition makes sense.

02K2 **Definition 29.30.15.** Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *syntomic of relative dimension d* if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d.

02K3 **Lemma 29.30.16.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective and syntomic,*
(2) *p is syntomic, and*
(3) *q is locally of finite presentation*[8].

Then q is syntomic.

Proof. By Lemma 29.25.13 we see that q is flat. Hence it suffices to show that the fibres of $Y \to S$ are local complete intersections, see Lemma 29.30.11. Let $s \in S$. Consider the morphism $X_s \to Y_s$. This is a base change of the morphism $X \to Y$ and hence surjective, and syntomic (Lemma 29.30.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 29.15.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = \mathrm{Spec}(k)$. Let $y \in Y$. Choose an affine open $\mathrm{Spec}(A) \subset Y$ neighbourhood of y. Let $\mathrm{Spec}(B) \subset X$ be an affine open such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \ldots, x_n] \to A$ with kernel I. Choose a surjection $A[y_1, \ldots, y_m] \to B$, which gives rise in turn to a surjection $k[x_i, y_j] \to B$ with kernel J. Let $\mathfrak{q} \subset k[x_i, y_j]$ be the prime corresponding to $y \in \mathrm{Spec}(B)$ and let $\mathfrak{p} \subset k[x_i]$ the prime corresponding to $x \in \mathrm{Spec}(A)$. Since x maps to y we have $\mathfrak{p} = \mathfrak{q} \cap k[x_i]$. Consider the following commutative diagram of local rings:

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & = & B_{\mathfrak{q}} & \longleftarrow & k[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\mathfrak{q}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & = & A_{\mathfrak{p}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{p}} \end{array}$$

We claim that the hypotheses of Algebra, Lemma 10.135.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \to Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 29.30.10. The output of Algebra, Lemma 10.135.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 29.30.10 again we see that Y is syntomic over k at y as desired. □

[8] In fact, if f is surjective, flat, and locally of finite presentation and p is syntomic, then both q and f are syntomic, see Descent, Lemma 35.14.7.

29.31. Conormal sheaf of an immersion

01R1 Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Lemma 29.4.1. This quasi-coherent $\mathcal{O}_Z$-module is called the *conormal sheaf of Z in X* and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 29.4.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

01R2 **Definition 29.31.1.** Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

01R3 **Lemma 29.31.2.** *Let $i : Z \to X$ be an immersion. The conormal sheaf of i has the following properties:*

(1) *Let $U \subset X$ be any open subscheme such that i factors as $Z \xrightarrow{i'} U \to X$ where i' is a closed immersion. Let $\mathcal{I} = \mathrm{Ker}((i')^\sharp) \subset \mathcal{O}_U$. Then*

$$\mathcal{C}_{Z/X} = (i')^*\mathcal{I} \quad \text{and} \quad i'_*\mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$$

(2) *For any affine open $\mathrm{Spec}(R) = U \subset X$ such that $Z \cap U = \mathrm{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. □

01R4 **Lemma 29.31.3.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ *with* $g(U) \subset U'$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *and* $Z' \cap U' = \mathrm{Spec}(R'/I')$ *the induced map*

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $g^\sharp : R' \to R$ *which has the property* $g^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \left(g^{-1}(\partial Z') \cup \partial Z\right)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 26.4.6 and 26.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \to i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \to I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

0473 **Lemma 29.31.4.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram in the category of schemes with i, i' *immersions. Then the canonical map* $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ *of Lemma 29.31.3 is surjective. If* g *is flat, then it is an isomorphism.*

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \to I/I^2$ is surjective. If $R' \to R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. □

062S **Lemma 29.31.5.** *Let* $Z \to Y \to X$ *be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 29.31.3 and $i : Z \to Y$ *is the first morphism.*

Proof. Via Lemma 29.31.3 this translates into the following algebra fact. Suppose that $C \to B \to A$ are surjective ring maps. Let $I = \mathrm{Ker}(B \to A)$, $J = \mathrm{Ker}(C \to A)$ and $K = \mathrm{Ker}(C \to B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0.$$

This follows immediately from the observation that $I = J/K$. □

29.32. Sheaf of differentials of a morphism

01UM We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131) and the corresponding section in the chapter on sheaves of modules (Modules, Section 17.28).

01UQ **Definition 29.32.1.** Let $f : X \to S$ be a morphism of schemes. The *sheaf of differentials* $\Omega_{X/S}$ *of* X *over* S is the sheaf of differentials of f viewed as a morphism of ringed spaces (Modules, Definition 17.28.10) equipped with its *universal* S*-derivation*

$$\mathrm{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

It turns out that $\Omega_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module for example as it is isomorphic to the conormal sheaf of the diagonal morphism $\Delta : X \to X \times_S X$ (Lemma 29.32.7). We have defined the module of differentials of X over S using a universal property, namely as the receptacle of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above. For convenience we restate the universal property here.

01UR **Lemma 29.32.2.** *Let $f : X \to S$ be a morphism of schemes. The map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow Der_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ d_{X/S}$$

is an isomorphism of functors $Mod(\mathcal{O}_X) \to Sets$.

Proof. This is just a restatement of the definition. □

01US **Lemma 29.32.3.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$, $V \subset S$ be open subschemes such that $f(U) \subset V$. Then there is a unique isomorphism $\Omega_{X/S}|_U = \Omega_{U/V}$ of $\mathcal{O}_U$-modules such that $d_{X/S}|_U = d_{U/V}$.*

Proof. This is a special case of Modules, Lemma 17.28.5 if we use the canonical identification $f^{-1}\mathcal{O}_S|_U = (f|_U)^{-1}\mathcal{O}_V$. □

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U.

01UO **Lemma 29.32.4.** *Let $R \to A$ be a ring map. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(A)$. Set $S = \mathrm{Spec}(R)$. The rule which associates to an S-derivation on $\mathcal{F}$ its action on global sections defines a bijection between the set of S-derivations of $\mathcal{F}$ and the set of R-derivations on $M = \Gamma(X, \mathcal{F})$.*

Proof. Let $D : A \to M$ be an R-derivation. We have to show there exists a unique S-derivation on $\mathcal{F}$ which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n}D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. □

01UT **Lemma 29.32.5.** *Let $f : X \to S$ be a morphism of schemes. For any pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ there is a unique isomorphism*

$$\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}.$$

compatible with $d_{X/S}$ and $d : A \to \Omega_{A/R}$.

Proof. By Lemma 29.32.3 we may replace X and S by U and V. Thus we may assume $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ and we have to show the lemma with $U = X$ and $V = S$. Consider the A-module $M = \Gamma(X, \Omega_{X/S})$ together with the R-derivation $\mathrm{d}_{X/S} : A \to M$. Let N be another A-module and denote $\widetilde{N}$ the quasi-coherent $\mathcal{O}_X$-module associated to N, see Schemes, Section 26.7. Precomposing by $\mathrm{d}_{X/S} : A \to M$ we get an arrow

$$\alpha : \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Der}_R(A, N)$$

Using Lemmas 29.32.2 and 29.32.4 we get identifications

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) = \mathrm{Der}_S(\mathcal{O}_X, \widetilde{N}) = \mathrm{Der}_R(A, N)$$

Taking global sections determines an arrow $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) \to \mathrm{Hom}_R(M, N)$. Combining this arrow and the identifications above we get an arrow

$$\beta : \mathrm{Der}_R(A, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

Checking what happens on global sections, we find that α and β are each others inverse. Hence we see that $\mathrm{d}_{X/S} : A \to M$ satisfies the same universal property as $\mathrm{d} : A \to \Omega_{A/R}$, see Algebra, Lemma 10.131.3. Thus the Yoneda lemma (Categories, Lemma 4.3.5) implies there is a unique isomorphism of A-modules $M \cong \Omega_{A/R}$ compatible with derivations. □

01UU **Remark 29.32.6.** The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \to S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S. These form a basis for the topology on X. Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U. We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 29.32.5 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

The following lemma gives yet another way to define the sheaf of differentials and it in particular shows that $\Omega_{X/S}$ is quasi-coherent if X and S are schemes.

08S2 **Lemma 29.32.7.** *Let $f : X \to S$ be a morphism of schemes. There is a canonical isomorphism between $\Omega_{X/S}$ and the conormal sheaf of the diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$.*

Proof. We first establish the existence of a couple of "global" sheaves and global maps of sheaves, and further down we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \to X \times_S X$ is an immersion, see Schemes, Lemma 26.21.2. Let $\mathcal{J}$ be the ideal sheaf of the immersion which lives over some open subscheme W of $X \times_S X$ such that $\Delta(X) \subset W$ is closed. Let us take the one that was found in the proof of Schemes, Lemma 26.21.2. Note that the sheaf of

rings $\mathcal{O}_W/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_W/\mathcal{J}^2 \to \Delta_*\mathcal{O}_X \to 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras with kernel the conormal sheaf of Δ (see Definition 29.31.1 and Lemma 29.31.2).

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X \to 0$$

This places us in the situation of Modules, Lemma 17.28.11. The projection morphisms $p_i : X \times_S X \to X$, $i = 1, 2$ induce maps of sheaves of rings $(p_i)^\sharp : (p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$. We may restrict to W and quotient by $\mathcal{J}^2$ to get $(p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_W/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X$, as in Modules, Lemma 17.28.11. Thus we get an S-derivation $\mathrm{d} = s_2 - s_1 : \mathcal{O}_X \to \mathcal{C}_{X/X\times_S X}$. By the universal property of the module of differentials we find a unique $\mathcal{O}_X$-linear map

$$\Omega_{X/S} \longrightarrow \mathcal{C}_{X/X\times_S X}, \quad f\mathrm{d}g \longmapsto fs_2(g) - fs_1(g)$$

To see the map is an isomorphism, let us work this out over suitable affine opens. We can cover X by affine opens $\mathrm{Spec}(A) = U \subset X$ whose image is contained in an affine open $\mathrm{Spec}(R) = V \subset S$. According to the proof of Schemes, Lemma 26.21.2 $U \times_V U \subset X \times_S X$ is an affine open contained in the open W mentioned above. Also $U \times_V U = \mathrm{Spec}(A \otimes_R A)$. The sheaf $\mathcal{J}$ corresponds to the ideal $J = \mathrm{Ker}(A \otimes_R A \to A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$-modules

$$0 \to J/J^2 \to (A \otimes_R A)/J^2 \to A \to 0$$

The sections s_i correspond to the ring maps

$$A \longrightarrow (A \otimes_R A)/J^2, \quad s_1 : a \mapsto a \otimes 1, \quad s_2 : a \mapsto 1 \otimes a.$$

By Lemma 29.31.2 we have $\Gamma(U, \mathcal{C}_{X/X\times_S X}) = J/J^2$ and by Lemma 29.32.5 we have $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$. The map above is the map $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ which is shown to be an isomorphism in Algebra, Lemma 10.131.13. □

01UV **Lemma 29.32.8.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*d_{X/S}(h)$ maps to $d_{X'/S'}(f^*h)$ for any local section h of $\mathcal{O}_X$.

Proof. This is a special case of Modules, Lemma 17.28.12. In the case of schemes we can also use the functoriality of the conormal sheaves (see Lemma 29.31.3) and Lemma 29.32.7 to define c_f. Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (10.131.4.1)). □

01UX **Lemma 29.32.9.** *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 29.32.8.

Proof. This is the sheafified version of Algebra, Lemma 10.131.7. Alternatively, there is a general version for morphisms of ringed spaces, see Modules, Lemma 17.28.14. □

01V0 **Lemma 29.32.10.** *Let $X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Let $X' = X_{S'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/S} \to \Omega_{X'/S'}$$

of Lemma 29.32.8 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 10.131.12. □

01V1 **Lemma 29.32.11.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms. The maps from Lemma 29.32.8*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 29.32.10 the composition $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y}$ is an isomorphism, and similarly for q. Moreover, the cokernel of $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S}$ is $\Omega_{X\times_S Y/X}$ by Lemma 29.32.9. The result follows. □

01V2 **Lemma 29.32.12.** *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Immediate from Algebra, Lemma 10.131.16, Lemma 29.32.5, Lemma 29.15.2, and Properties, Lemma 28.16.1. □

01V3 **Lemma 29.32.13.** *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite presentation, then $\Omega_{X/S}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Immediate from Algebra, Lemma 10.131.15, Lemma 29.32.5, Lemma 29.21.2, and Properties, Lemma 28.16.2. □

01UY **Lemma 29.32.14.** *If $X \to S$ is an immersion, or more generally a monomorphism, then $\Omega_{X/S}$ is zero.*

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. Hence $\Omega_{X/S} = 0$ by Lemma 29.32.7. The algebraic version is Algebra, Lemma 10.131.4. □

01UZ **Lemma 29.32.15.** *Let $i : Z \to X$ be an immersion of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

where the first arrow is induced by $d_{X/S}$ and the second arrow comes from Lemma 29.32.8.

Proof. This is the sheafified version of Algebra, Lemma 10.131.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/S}$" here. Namely, we may assume that i is a closed immersion by shrinking X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Lemma 29.4.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

0474 **Lemma 29.32.16.** *Let $i : Z \to X$ be an immersion of schemes over S, and assume i (locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 29.32.15 is (locally) split exact. In particular, if $s : S \to X$ is a section of the structure morphism $X \to S$ then the map $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$ induced by $d_{X/S}$ is an isomorphism.

Proof. Follows from Algebra, Lemma 10.131.10. Clarification: if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \to \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \to X$ over S (see Schemes, Lemma 26.21.11) and in that case $\Omega_{Z/S} = 0$. □

060N **Remark 29.32.17.** Let $X \to S$ be a morphism of schemes. According to Lemma 29.32.11 we have

$$\Omega_{X \times_S X/S} = \mathrm{pr}_1^*\Omega_{X/S} \oplus \mathrm{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 29.32.16 we obtain a canonical short exact sequence

$$0 \to \mathcal{C}_{X/X \times_S X} \to \Omega_{X/S} \oplus \Omega_{X/S} \to \Omega_{X/S} \to 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by Lemma 29.32.7 we have an identification $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $\mathrm{d}_{X/S}(f) = s_2(f) - s_1(f)$ in this identification it turns out that the left arrow is the map $(-1, 1)$[9].

067L **Lemma 29.32.18.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

[9]Namely, the local section $\mathrm{d}_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $\mathrm{d}_{X \times_S X/X}$ to the local section $1 \otimes 1 \otimes 1 \otimes f - 1 \otimes f \otimes 1 \otimes 1 - 1 \otimes 1 \otimes f \otimes 1 + f \otimes 1 \otimes 1 \otimes 1 = \mathrm{pr}_2^*\mathrm{d}_{X/S}(f) - \mathrm{pr}_1^*\mathrm{d}_{X/S}(f)$.

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 29.31.3 and the second from Lemma 29.32.15.

Proof. The algebraic version of this is Algebra, Lemma 10.134.7. □

29.33. Finite order differential operators

0G43 We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133) and the corresponding section in the chapter on sheaves of modules (Modules, Section 17.29).

0G44 **Lemma 29.33.1.** *Let $R \to A$ be a ring map. Denote $f : X \to S$ the corresponding morphism of affine schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is quasi-coherent then the map*

$$Diff^k_{X/S}(\mathcal{F}, \mathcal{G}) \to Diff^k_{A/R}(\Gamma(X, \mathcal{F}), \Gamma(X, \mathcal{G}))$$

sending a differential operator to its action on global sections is bijective.

Proof. Write $\mathcal{F} = \widetilde{M}$ for some A-module M. Set $N = \Gamma(X, \mathcal{G})$. Let $D : M \to N$ be a differential operator of order k. We have to show there exists a unique differential operator $\mathcal{F} \to \mathcal{G}$ of order k which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Then $\mathcal{F}(U) = M_f$ is the localization. By Algebra, Lemma 10.133.12 the differential operator D extends to a unique differential operator

$$D_f : \mathcal{F}(U) = \widetilde{M}(U) = M_f \to N_f = \widetilde{N}(U)$$

The uniqueness shows that these maps D_f glue to give a map of sheaves $\widetilde{M} \to \widetilde{N}$ on the basis of all standard opens of X. Hence we get a unique map of sheaves $\widetilde{D} : \widetilde{M} \to \widetilde{N}$ agreeing with these maps by the material in Sheaves, Section 6.30. Since $\widetilde{D}$ is given by differential operators of order k on the standard opens, we find that $\widetilde{D}$ is a differential operator of order k (small detail omitted). Finally, we can post-compose with the canonical $\mathcal{O}_X$-module map $c : \widetilde{N} \to \mathcal{G}$ (Schemes, Lemma 26.7.1) to get $c \circ \widetilde{D} : \mathcal{F} \to \mathcal{G}$ which is a differential operator of order k by Modules, Lemma 17.29.2. This proves existence. We omit the proof of uniqueness. □

0G45 **Lemma 29.33.2.** *Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$ and $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $D : \mathcal{F} \to \mathcal{F}'$ be a differential operator of order k on X/S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Then there is a unique differential operator*

$$D' : pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\mathcal{G} \longrightarrow pr_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\mathcal{G}$$

of order k on $X \times_S Y/Y$ such that $D'(s \otimes t) = D(s) \otimes t$ for local sections s of $\mathcal{F}$ and t of $\mathcal{G}$.

Proof. In case X, Y, and S are affine, this follows, via Lemma 29.33.1, from the corresponding algebra result, see Algebra, Lemma 10.133.13. In general, one uses coverings by affines (for example as in Schemes, Lemma 26.17.4) to construct D' globally. Details omitted. □

0G46 **Remark 29.33.3.** Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Denote $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ the projections. In this remark, given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\mathcal{G}$$

Denote $\mathcal{A}_{X/S}$ the additive category whose objects are quasi-coherent $\mathcal{O}_X$-modules and whose morphisms are differential operators of finite order on X/S. Similarly for $\mathcal{A}_{Y/S}$ and $\mathcal{A}_{X\times_S Y/S}$. The construction of Lemma 29.33.2 determines a functor

$$\boxtimes : \mathcal{A}_{X/S} \times \mathcal{A}_{Y/S} \longrightarrow \mathcal{A}_{X\times_S Y/S}, \quad (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$$

which is bilinear on morphisms. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$, then via the identification of quasi-coherent sheaves with modules this functor is given by $(M, N) \mapsto M \otimes_R N$ on objects and sends the morphism $(D, D') : (M, N) \to (M', N')$ to $D \otimes D' : M \otimes_R N \to M' \otimes_R N'$.

29.34. Smooth morphisms

01V4 Let $f : X \to Y$ be a continuous map of topological spaces. Consider the following condition: For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) \subset V$ and such that we obtain a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[\pi]{} & V \times \mathbf{R}^d \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

where π is a homeomorphism onto an open subset. Smooth morphisms of schemes are the analogue of these maps in the category of schemes. See Lemma 29.34.11 and Lemma 29.36.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \to A$ is a *smooth ring map* if A is of finite presentation over R and if the naive cotangent complex of A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 10.137.1.

01V5 **Definition 29.34.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *smooth at* $x \in X$ if there exist an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth.
(2) We say that f is *smooth* if it is smooth at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard smooth* if there exists a standard smooth ring map $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.137.5) such that $X \to S$ is isomorphic to

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

01V6 **Lemma 29.34.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is smooth.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is smooth.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.*

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is smooth.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is smooth" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.137.3 being smooth is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.137.13 being smooth is stable under composition and for any ring R the ring map $R \to R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.137.12. □

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 33.25.

01V8 **Lemma 29.34.3.** *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.*

Proof. Follows from Algebra, Lemma 10.137.16. □

01VA **Lemma 29.34.4.** *The composition of two morphisms which are smooth is smooth.*

Proof. In the proof of Lemma 29.34.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 10.137.13. □

01VB **Lemma 29.34.5.** *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 29.34.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 10.137.3. □

01VC **Lemma 29.34.6.** *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. □

01VD **Lemma 29.34.7.** *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 10.137.9. □

01VE **Lemma 29.34.8.** *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. □

01VF **Lemma 29.34.9.** *A smooth morphism is flat.*

Proof. Combine Lemmas 29.30.7 and 29.34.7. □

056G **Lemma 29.34.10.** *A smooth morphism is universally open.*

Proof. Combine Lemmas 29.34.9, 29.34.8, and 29.25.10. Or alternatively, combine Lemmas 29.34.7, 29.30.8. □

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a *local model* for a smooth morphism.

01V7 **Lemma 29.34.11.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent*

(1) *The morphism f is smooth at x.*
(2) *There exists an affine open $U \subset X$, with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 10.137.6 and 10.137.9. □

02G1 **Lemma 29.34.12.** *Let $f : X \to S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$rank_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S. By Lemma 29.34.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 10.137.6 (and the definition of a relative global complete intersection, see Algebra, Definition 10.136.5). □

Lemma 29.34.12 says that the following definition makes sense.

02G2 **Definition 29.34.13.** Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. By Lemma 29.34.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d, and (d) $\Omega_{X/S}$ finite locally free of rank d. It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d. A counter example is given by $\mathrm{Spec}(\mathbf{F}_p[t]) \to \mathrm{Spec}(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

01V9 **Lemma 29.34.14.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is smooth at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is smooth at x.*

(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.*
(4) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
can be generated by at most $\dim_x(X_{f(x)})$ elements.
(5) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*
(6) *There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.

Proof. Note that if f is smooth at x, then we see from Lemma 29.34.11 that (5) holds, and (6) is a slightly weakened version of (5). Moreover, f smooth implies that the ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat (see Lemma 29.34.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 29.34.12). Thus (1) implies (3) and (4). By Lemma 29.34.5 we also see that (1) implies (2).

By Lemma 29.32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. Hence the displayed equality in part (4) of the lemma. By Lemma 29.32.12 these modules are of finite type. Hence the minimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$-vector space by Nakayama's Lemma (Algebra, Lemma 10.20.1). This in particular shows that (3) and (4) are equivalent.

Algebra, Lemma 10.137.16 shows that (2) implies (1). Algebra, Lemma 10.140.3 shows that (3) and (4) imply (2). Finally, (6) implies (5) see for example Algebra, Example 10.137.7 and (5) implies (1) by Algebra, Lemma 10.137.6. □

02V4 **Lemma 29.34.15.** *Let*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$
be a cartesian diagram of schemes. Let $W \subset X$, resp. $W' \subset X'$ be the open subscheme of points where f, resp. f' is smooth. Then $W' = (g')^{-1}(W)$ if

(1) *f is flat and locally of finite presentation, or*
(2) *f is locally of finite presentation and g is flat.*

Proof. Assume first that f locally of finite type. Consider the set
$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$

and the corresponding set $T' \subset X'$ for f'. Then we claim $T' = (g')^{-1}(T)$. Namely, let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the claim is equivalent to Algebra, Lemma 10.137.18.

Thus case (1) follows because in case (1) T is the (open) set of points where f is smooth by Lemma 29.34.14.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 29.25.7) and $g \circ f$ is flat at x' (Lemma 29.25.5). It follows that f is flat at $x = g'(x')$ by Lemma 29.25.13. On the other hand, since $x' \in T'$ (Lemma 29.34.5) we see that $x \in T$. Hence f is smooth at x by Lemma 29.34.14. □

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

02K4 **Lemma 29.34.16.** *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Lemma 29.32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.131.7 is exact. This is Algebra, Lemma 10.139.1. □

06AA **Lemma 29.34.17.** *Let $i : Z \to X$ be an immersion of schemes over S. Assume that Z is smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 29.32.15 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ smooth and $B \to C$ surjective with kernel J, then the sequence

$$0 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

of Algebra, Lemma 10.131.9 is exact. This is Algebra, Lemma 10.139.2. □

06AB **Lemma 29.34.18.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are immersions and $X \to Y$ is smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 29.32.18 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ smooth, then the sequence

$$0 \to I/I^2 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to 0$$

of Algebra, Lemma 10.134.7 is exact. This is Algebra, Lemma 10.139.3. □

02K5 **Lemma 29.34.19.** *Let*

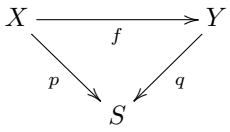

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and smooth,*
(2) *p is smooth, and*
(3) *q is locally of finite presentation*[10].

Then q is smooth.

Proof. By Lemma 29.25.13 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y. Suppose f has relative dimension a at x and p has relative dimension b at x. By Lemma 29.34.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. By the short exact sequence of Lemma 29.34.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 29.28.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 29.34.14 part (2). □

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 31.22.8.

05D9 **Lemma 29.34.20.** *Let $f : X \to S$ be a morphism of schemes. Let $\sigma : S \to X$ be a section of f. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $\mathrm{Spec}(A) = U \subset S$ of s and $\mathrm{Spec}(B) = V \subset X$ of x such that*

(1) *$f(V) \subset U$ and $\sigma(U) \subset V$,*
(2) *with $I = \mathrm{Ker}(\sigma^\# : B \to A)$ the module I/I^2 is a free A-module, and*
(3) *$B^\wedge \cong A[[x_1, \ldots, x_d]]$ as A-algebras where $B^\wedge$ denotes the completion of B with respect to I.*

Proof. Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset \sigma^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = \mathrm{Spec}(A')$ and $V' = \mathrm{Spec}(B')$. By Algebra, Lemma 10.139.4 the module $I'/(I')^2$ is finite locally free as a A'-module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 10.139.4. □

[10] In fact this is implied by (1) and (2), see Descent, Lemma 35.14.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 35.14.5.

The dimension of a scheme X at a point x (Properties, Definition 28.10.1) is just the dimension of X at x as a topological space, see Topology, Definition 5.10.1. This is not the dimension of the local ring $\mathcal{O}_{X,x}$, in general.

0AFF **Lemma 29.34.21.** *Let $f : X \to Y$ be a smooth morphism of locally Noetherian schemes. For every point x in X with image y in Y,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Lemma 29.34.12. Then f is flat (Lemma 29.34.9), locally of finite type (Lemma 29.34.8), and of relative dimension d. Hence the result follows from Lemma 29.29.6. □

29.35. Unramified morphisms

02G3 We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \to A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [**Ray70**]). A ring map $R \to A$ is called *G-unramified* if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [**DG67**]). See Algebra, Definition 10.151.1.

02G4 **Definition 29.35.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *unramified at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified.
(2) We say that f is *G-unramified at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
(3) We say that f is *unramified* if it is unramified at every point of X.
(4) We say that f is *G-unramified* if it is G-unramified at every point of X.

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 29.35.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

02G5 **Lemma 29.35.2.** *Let $f : X \to S$ be a morphism of schemes. Then*

(1) *f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and*
(2) *f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.*

Proof. By definition a ring map $R \to A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 29.32.5. □

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

02G6 **Lemma 29.35.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is unramified (resp. G-unramified).*
(2) *For every affine open $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is unramified (resp. G-unramified).*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is unramified (resp. G-unramified).*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is unramified (resp. G-unramified), for all $j \in J, i \in I_j$.*

Moreover, if f is unramified (resp. G-unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is unramified (resp. G-unramified).

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is unramified" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. These properties are proved in Algebra, Lemma 10.151.3. □

02G9 **Lemma 29.35.4.** *The composition of two morphisms which are unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 29.35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 10.151.3. □

02GA **Lemma 29.35.5.** *The base change of a morphism which is unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 29.35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the lemma follows from Lemma 29.14.6 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 10.151.3. □

04EV **Lemma 29.35.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G-unramified.*

Proof. Follows from the definitions and Lemma 29.21.9. □

02GB **Lemma 29.35.7.** *Any open immersion is G-unramified.*

Proof. This is true because an open immersion is a local isomorphism. □

02GC **Lemma 29.35.8.** *A closed immersion $i : Z \to X$ is unramified. It is G-unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Follows from Lemma 29.21.7 and Algebra, Lemma 10.151.3. □

02GD **Lemma 29.35.9.** *An unramified morphism is locally of finite type. A G-unramified morphism is locally of finite presentation.*

Proof. An unramified ring map is of finite type by definition. A G-unramified ring map is of finite presentation by definition. □

02V5 **Lemma 29.35.10.** *Let $f : X \to S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. See Algebra, Lemma 10.151.6. □

02G7 **Lemma 29.35.11.** *Fibres of unramified morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. Part (2) follows from part (1) and Lemma 29.35.5. Let us prove part (1). We first use Algebra, Lemma 10.151.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is unramified. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is unramified. By Algebra, Lemma 10.151.5 for every $x \in X$ the residue field extension $\kappa(x)/k$ is finite separable. Since $X \to \mathrm{Spec}(k)$ is locally quasi-finite (Lemma 29.35.10) we see that all points of X are isolated closed points, see Lemma 29.20.6. Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 10.151.5 again these local rings are fields, and we win. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

02G8 **Lemma 29.35.12.** *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is unramified then for any $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is finite separable.*
(2) *If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.*
(3) *If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G-unramified.*

Proof. Follows from Algebra, Lemmas 10.151.5 and 10.151.7. □

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

02GE **Lemma 29.35.13.** *Let $f : X \to S$ be a morphism.*

(1) *If f is unramified, then the diagonal morphism $\Delta : X \to X \times_S X$ is an open immersion.*
(2) *If f is locally of finite type and Δ is an open immersion, then f is unramified.*
(3) *If f is locally of finite presentation and Δ is an open immersion, then f is G-unramified.*

Proof. The first statement follows from Algebra, Lemma 10.151.4. The second statement from the fact that $\Omega_{X/S}$ is the conormal sheaf of the diagonal morphism (Lemma 29.32.7) and hence clearly zero if Δ is an open immersion. □

02GF **Lemma 29.35.14.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified) at x.*
(2) *The fibre X_s is unramified over $\kappa(s)$ at x.*
(3) *The $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(4) *The $\mathcal{O}_{X_s,x}$-module $\Omega_{X_s/s,x}$ is zero.*
(5) *The $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) *We have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*

Proof. Note that if f is unramified at x, then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 29.32. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 29.32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 29.32.12 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$-vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 10.20.1). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X, see Modules, Lemma 17.9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x, which implies (1). The equivalence of (1) and (3) applied to $X_s \to s$ implies the equivalence of (2) and (4). At this point we have seen that (1) – (5) are equivalent.

Alternatively you can use Algebra, Lemma 10.151.3 to see the equivalence of (1) – (5) more directly.

The equivalence of (1) and (6) follows from Lemma 29.35.12. It also follows more directly from Algebra, Lemmas 10.151.5 and 10.151.7. □

0475 **Lemma 29.35.15.** *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set*
$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$
commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G-unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'},x'}} \kappa(x')$$

see Lemma 29.32.10. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 29.35.14. The second part follows from the first because in that case T is the (open) set of points where f is G-unramified according to Lemma 29.35.14. □

02GG **Lemma 29.35.16.** *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is unramified over S, then f is unramified.*
(2) *If X is G-unramified over S and Y is locally of finite type over S, then f is G-unramified.*

Proof. Assume that X is unramified over S. By Lemma 29.15.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 29.32.9. Thus f is unramified. If X is G-unramified over S and Y is locally of finite type over S, then by Lemma 29.21.11 we see that f is locally of finite presentation and we conclude that f is G-unramified. □

04HB **Lemma 29.35.17.** *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms over S. Let $x \in X$. Assume that*

(1) *the structure morphism $Y \to S$ is unramified,*
(2) *$f(x) = g(x)$ in Y, say $y = f(x) = g(x)$, and*
(3) *the induced maps $f^\sharp, g^\sharp : \kappa(y) \to \kappa(x)$ are equal.*

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \to Y \times_S Y$. By assumption (1) and Lemma 29.35.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X. And assumptions (2) and (3) imply that x is in this open subset. □

29.36. Étale morphisms

02GH The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over $\mathbf{C}$. It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over $\mathbf{C}$ this topology gives rise to the "correct" Betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \to Y$ is an étale morphism of varieties over $\mathbf{C}$, and if x is a closed point of X, then f induces an isomorphism $\mathcal{O}^\wedge_{Y,f(x)} \to \mathcal{O}^\wedge_{X,x}$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \to A$ is said to be *étale* if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 10.143.1.

02GI **Definition 29.36.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *étale at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.

(2) We say that f is *étale* if it is étale at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard étale* if $X \to S$ is isomorphic to

$$\mathrm{Spec}(R[x]_h/(g)) \to \mathrm{Spec}(R)$$

where $R \to R[x]_h/(g)$ is a standard étale ring map, see Algebra, Definition 10.144.1, i.e., g is monic and g' invertible in $R[x]_h/(g)$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 29.34.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

02GJ **Lemma 29.36.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is étale.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.*

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is étale.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is étale" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. These all follow from Algebra, Lemma 10.143.3. □

02GN **Lemma 29.36.3.** *The composition of two morphisms which are étale is étale.*

Proof. In the proof of Lemma 29.36.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 10.143.3. □

02GO **Lemma 29.36.4.** *The base change of a morphism which is étale is étale.*

Proof. In the proof of Lemma 29.36.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 10.143.3. □

02GK **Lemma 29.36.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x.*

Proof. This follows immediately from the definitions. □

03WS **Lemma 29.36.6.** *An étale morphism is locally quasi-finite.*

Proof. By Lemma 29.36.5 an étale morphism is unramified. By Lemma 29.35.10 an unramified morphism is locally quasi-finite. □

02GL **Lemma 29.36.7.** *Fibres of étale morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. You can deduce this from Lemma 29.35.11 via Lemma 29.36.5 above. Here is a direct proof.

We will use Algebra, Lemma 10.143.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is étale. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k. Hence all points of X are closed points (see Lemma 29.20.2 for example). Thus X is a discrete space and we win. □

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with "étale fibres".

02GM **Lemma 29.36.8.** *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.*

Proof. You can deduce this from Algebra, Lemma 10.143.7. Here is another proof.

By Lemma 29.36.7 a fibre X_s is étale and hence smooth over s. By Lemma 29.34.3 we see that $X \to S$ is smooth. By Lemma 29.35.12 we see that f is unramified. We conclude by Lemma 29.36.5. □

02GP **Lemma 29.36.9.** *Any open immersion is étale.*

Proof. This is true because an open immersion is a local isomorphism. □

02GQ **Lemma 29.36.10.** *An étale morphism is syntomic.*

Proof. See Algebra, Lemma 10.137.9 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. □

02GR **Lemma 29.36.11.** *An étale morphism is locally of finite presentation.*

Proof. True because an étale ring map is of finite presentation by definition. □

02GS **Lemma 29.36.12.** *An étale morphism is flat.*

Proof. Combine Lemmas 29.30.7 and 29.36.10. □

03WT **Lemma 29.36.13.** *An étale morphism is open.*

Proof. Combine Lemmas 29.36.12, 29.36.11, and 29.25.10. □

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a *local model* for a general étale morphism.

02GT **Lemma 29.36.14.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent*

(1) *The morphism f is étale at x.*
(2) *There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard étale (see Definition 29.36.1).*

Proof. Follows from the definitions and Algebra, Proposition 10.144.4. □

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

02GU **Lemma 29.36.15.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is étale at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \operatorname{Spec}(\kappa(s))$ is étale at x.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \operatorname{Spec}(\kappa(s))$ is unramified at x.*
(4) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(5) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*
(7) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.*
(8) *There exist affine opens $\operatorname{Spec}(A) = U \subset X$ and $\operatorname{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.
(9) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.*
(10) *There exist affine opens $\operatorname{Spec}(A) = U \subset X$ and $\operatorname{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$

with $P, Q \in R[x]$, P monic and $P' = dP/dx$ mapping to an element of A not in $\mathfrak{q}$.

Proof. Use Lemma 29.36.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) – (10) combine Lemmas 29.34.14 and 29.35.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 29.36.5. Some details omitted. □

02GV **Lemma 29.36.16.** *A morphism is étale at a point if and only if it is flat and G-unramified at that point. A morphism is étale if and only if it is flat and G-unramified.*

Proof. This is clear from Lemmas 29.36.15 and 29.35.14. □

0476 **Lemma 29.36.17.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $W \subset X$, resp. $W' \subset X'$ be the open subscheme of points where f, resp. f' is étale. Then $W' = (g')^{-1}(W)$ if

(1) *f is flat and locally of finite presentation, or*
(2) *f is locally of finite presentation and g is flat.*

Proof. Assume first that f locally of finite type. Consider the set

$$T = \{x \in X \mid f \text{ is unramified at } x\}$$

and the corresponding set $T' \subset X'$ for f'. Then $T' = (g')^{-1}(T)$ by Lemma 29.35.15.

Thus case (1) follows because in case (1) T is the (open) set of points where f is étale by Lemma 29.36.16.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 29.25.7) and $g \circ f'$ is flat at x' (Lemma 29.25.5). It follows that f is flat at $x = g'(x')$ by Lemma 29.25.13. On the other hand, since $x' \in T'$ (Lemma 29.34.5) we see that $x \in T$. Hence f is étale at x by Lemma 29.36.15. □

02GW **Lemma 29.36.18.** *Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.*

Proof. As a first proof one may reduce to the affine case and then use Algebra, Lemma 10.143.8. This proof is somewhat complicated as it uses the "Critère de platitude par fibres" to see that a morphism $X \to Y$ over S between schemes étale over S is automatically flat.

We give a second proof using the graph argument. Namely, consider the factorization $X \to X \times_S Y \to Y$, where the first arrow is given by id_X and f and the second arrow is the projection. We claim both arrows are étale and hence f is étale by Lemma 29.36.3. Namely, the projection is étale as it is the base change of $X \to S$,

see Lemma 29.36.4. The first arrow is the base change of the diagonal morphism $Y \to Y \times_S Y$ because the square

$$\begin{array}{ccc} X & \longrightarrow & X \times_S Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_S Y \end{array}$$

is cartesian. The diagonal $Y \to Y \times_S Y$ is an open immersion because $Y \to S$ is étale and hence unramified (Lemma 29.36.5) and we may use Lemma 29.35.13. The base change of an open immersion is an open immersion (Schemes, Lemma 26.18.2) and an open immersion is étale (Lemma 29.36.9). This finishes the second proof. □

02K6 **Lemma 29.36.19.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and étale,*
(2) *p is étale, and*
(3) *q is locally of finite presentation*[11].

Then q is étale.

Proof. By Lemma 29.34.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 29.28.2 and the fact that f and p have relative dimension 0. □

A final characterization of smooth morphisms is that a smooth morphism $f : X \to S$ is locally the composition of an étale morphism by a projection $\mathbf{A}^d_S \to S$.

054L **Lemma 29.36.20.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is smooth at x, then there exists an integer $d \geq 0$ and an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\pi} & \mathbf{A}^d_V \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

where π is étale.

Proof. By Lemma 29.34.11 we can find an affine open U as in the lemma such that $\varphi|_U : U \to V$ is standard smooth. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ so that we can write

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

[11] In fact this is implied by (1) and (2), see Descent, Lemma 35.14.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 35.14.5.

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A. Then it is clear that $R[x_{c+1}, \ldots, x_n] \to A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \ldots, x_n] \to A$ is étale. As $\mathbf{A}^{n-c}_V = \mathrm{Spec}(R[x_{c+1}, \ldots, x_n])$ the lemma with $d = n - c$. □

29.37. Relatively ample sheaves

01VG Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 28.26.1. We turn this into a relative notion as follows.

01VH **Definition 29.37.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/S*, or *f-ample* if $f : X \to S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample. [**DG67**, II Definition 4.6.1]

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \to S$ to be of finite type.

02NN **Lemma 29.37.2.** *Let $X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is f-ample if and only if $\mathcal{L}^{\otimes n}$ is f-ample.*

Proof. This follows from Properties, Lemma 28.26.2. □

01VI **Lemma 29.37.3.** *Let $f : X \to S$ be a morphism of schemes. If there exists an f-ample invertible sheaf, then f is separated.*

Proof. Being separated is local on the base (see Schemes, Lemma 26.21.7 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 28.26.8. □

There are many ways to characterize relatively ample invertible sheaves, analogous to the equivalent conditions in Properties, Proposition 28.26.13. We will add these here as needed.

01VJ **Lemma 29.37.4.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The following are equivalent:* [**DG67**, II, Proposition 4.6.3]

1. *The invertible sheaf $\mathcal{L}$ is f-ample.*
2. *There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \to V_i$.*
3. *There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.*
4. *There exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and*

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{Proj}_S(\mathcal{A})$$

is an open immersion (see Constructions, Lemma 27.19.1 for notation).

(5) *The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d\geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.*
(6) *Same as (4) but just requiring $r_{\mathcal{L},\psi}$ to be an immersion.*

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (Properties, Lemma 28.26.8). Hence f is separated. By Schemes, Lemma 26.24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d\geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Denote $\psi : f^*\mathcal{A} \to \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 27.19 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 27.19.1 part (3) shows that the restriction of $r_{\mathcal{L},\psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 28.26.9 which is an open immersion according to Properties, Lemma 28.26.11. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 27.16.7 we see that $\pi^{-1}(V) \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ is equal to $\mathrm{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L},\psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\mathrm{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 27.19.1 and the final statement of Constructions, Lemma 27.14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 28.26.12 and 28.26.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S. Hence we may assume that S is affine, and we have to show that $\mathcal{L}$ is ample on X. In this case the morphism $r_{\mathcal{L},\psi}$ is identified with the morphism, also denoted $r_{\mathcal{L},\psi} : X \to \mathrm{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \to \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \mathrm{Proj}(A)$. By Constructions, Lemma 27.10.6 (see also Properties, Lemma 28.26.12) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 28.26.3 we conclude that the pullback of $\mathcal{O}_Y(d')$ is ample. Combining these results with Properties, Lemma 28.26.2 we conclude that $\mathcal{L}$ is ample as desired. □

01VK **Lemma 29.37.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine. Then $\mathcal{L}$ is f-relatively ample if and only if $\mathcal{L}$ is ample on X.* [**DG67**, II Corollary 4.6.6]

Proof. Immediate from Lemma 29.37.4 and the definitions. □

0891 **Lemma 29.37.6.** *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample.* [**DG67**, II Proposition 5.1.6]

Proof. Follows from Properties, Lemma 28.27.1 and the definitions. □

0892 **Lemma 29.37.7.** *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{M}$ an invertible $\mathcal{O}_Y$-module, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(1) *If $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample for $a \gg 0$.*
(2) *If $\mathcal{M}$ is ample and f quasi-affine, then $f^*\mathcal{M}$ is ample.*

Proof. Assume $\mathcal{L}$ is f-ample and $\mathcal{M}$ ample. By assumption Y and f are quasi-compact (see Definition 29.37.1 and Properties, Definition 28.26.1). Hence X is quasi-compact. By Properties, Lemma 28.26.8 the scheme Y is separated and by Lemma 29.37.3 the morphism f is separated. Hence X is separated by Schemes, Lemma 26.21.12. Pick $x \in X$. We can choose $m \geq 1$ and $t \in \Gamma(Y, \mathcal{M}^{\otimes m})$ such that Y_t is affine and $f(x) \in Y_t$. Since $\mathcal{L}$ restricts to an ample invertible sheaf on $f^{-1}(Y_t) = X_{f^*t}$ we can choose $n \geq 1$ and $s \in \Gamma(X_{f^*t}, \mathcal{L}^{\otimes n})$ with $x \in (X_{f^*t})_s$ with $(X_{f^*t})_s$ affine. By Properties, Lemma 28.17.2 part (2) whose assumptions are satisfied by the above, there exists an integer $e \geq 1$ and a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes em})$ which restricts to $s(f^*t)^e$ on X_{f^*t}. For any $b > 0$ consider the section $s'' = s'(f^*t)^b$ of $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes(e+b)m}$. Then $X_{s''} = (X_{f^*t})_s$ is an affine open of X containing x. Picking b such that n divides $e + b$ we see $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes(e+b)m}$ is the nth power of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ for some a and we can get any a divisible by m and big enough. Since X is quasi-compact a finite number of these affine opens cover X. We conclude that for some a sufficiently divisible and large enough the invertible sheaf $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on X. On the other hand, we know that $\mathcal{M}^{\otimes c}$ (and hence its pullback to X) is globally generated for all $c \gg 0$ by Properties, Proposition 28.26.13. Thus $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a+c}$ is ample (Properties, Lemma 28.26.5) for $c \gg 0$ and (1) is proved.

Part (2) follows from Lemma 29.37.6, Properties, Lemma 28.26.2, and part (1). □

0C4K **Lemma 29.37.8.** *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{M}$ be an invertible $\mathcal{O}_Y$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If S is quasi-compact, $\mathcal{M}$ is g-ample, and $\mathcal{L}$ is f-ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is $g \circ f$-ample for $a \gg 0$.*

Proof. Let $S = \bigcup_{i=1,\ldots,n} V_i$ be a finite affine open covering. By Lemma 29.37.4 it suffices to prove that $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on $(g \circ f)^{-1}(V_i)$ for $i = 1, \ldots, n$. Thus the lemma follows from Lemma 29.37.7. □

0893 **Lemma 29.37.9.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.*

Proof. By Lemma 29.37.4 it suffices to find an affine open covering $S' = \bigcup U'_i$ such that $\mathcal{L}'$ restricts to an ample invertible sheaf on $(f')^{-1}(U'_i)$ for all i. We may choose U'_i mapping into an affine open $U_i \subset S$. In this case the morphism $(f')^{-1}(U'_i) \to f^{-1}(U_i)$ is affine as a base change of the affine morphism $U'_i \to U_i$ (Lemma 29.11.9). Thus $\mathcal{L}'|_{(f')^{-1}(U'_i)}$ is ample by Lemma 29.37.7. □

0C4L **Lemma 29.37.10.** *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is $g \circ f$-ample and f is quasi-compact[12] then $\mathcal{L}$ is f-ample.*

[12]This follows if g is quasi-separated by Schemes, Lemma 26.21.14.

Proof. Assume f is quasi-compact and $\mathcal{L}$ is $g \circ f$-ample. Let $U \subset S$ be an affine open and let $V \subset Y$ be an affine open with $g(V) \subset U$. Then $\mathcal{L}|_{(g\circ f)^{-1}(U)}$ is ample on $(g\circ f)^{-1}(U)$ by assumption. Since $f^{-1}(V) \subset (g\circ f)^{-1}(U)$ we see that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$ by Properties, Lemma 28.26.14. Namely, $f^{-1}(V) \to (g \circ f)^{-1}(U)$ is a quasi-compact open immersion by Schemes, Lemma 26.21.14 as $(g \circ f)^{-1}(U)$ is separated (Properties, Lemma 28.26.8) and $f^{-1}(V)$ is quasi-compact (as f is quasi-compact). Thus we conclude that $\mathcal{L}$ is f-ample by Lemma 29.37.4. □

29.38. Very ample sheaves

01VL Recall that given a quasi-coherent sheaf $\mathcal{E}$ on a scheme S the *projective bundle* associated to $\mathcal{E}$ is the morphism $\mathbf{P}(\mathcal{E}) \to S$, where $\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E}))$, see Constructions, Definition 27.21.1.

01VM **Definition 29.38.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively very ample* or more precisely *f-relatively very ample*, or *very ample on X/S*, or *f-very ample* if there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

01VN **Lemma 29.38.2.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If f is quasi-compact and $\mathcal{L}$ is a relatively very ample invertible sheaf, then $\mathcal{L}$ is a relatively ample invertible sheaf.* [**DG67**, II, Proposition 4.6.2]

Proof. By definition there exists quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \mathrm{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathcal{A})$ by definition. The graded $\mathcal{O}_S$-algebra $\mathcal{A}$ comes equipped with a map

$$\psi : \mathcal{A} \to \bigoplus\nolimits_{n\geq 0} \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \to \bigoplus\nolimits_{n\geq 0} f_*\mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L},\psi}$ associated to this map is exactly the immersion i. Hence the result follows from part (6) of Lemma 29.37.4. □

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf $\mathcal{L}$ there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

(1) Even if S is affine, it can happen that no finite integer n works because $X \to S$ is not of finite type, see Example 29.38.4.
(2) The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the dth power. See Example 29.38.5. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $\mathrm{Spec}(k)$ gives an example.

To see our version of the converse take a look at Lemma 29.39.5 below. We will do some preliminary work before proving it.

07ZR **Example 29.38.3.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Set $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$. In this case $\mathcal{O}_X(1)$ is invertible and very ample on X/S. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 27.18.5.

01VO **Example 29.38.4.** Let k be a field. Consider the graded k-algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \ldots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \ldots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \mathrm{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f-ample for the morphism $f : X \to \mathrm{Spec}(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f-relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A. Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k. On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \ldots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \ldots)$ which is not of finite type over k.

01VP **Example 29.38.5.** Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be pairwise distinct elements of k^*. (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k-algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod\nolimits_{i=1}^{2d}(U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \mathrm{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d, and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

01VQ **Lemma 29.38.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is relatively very ample on X/S then f is separated.*

Proof. Being separated is local on the base (see Schemes, Section 26.21). An immersion is separated (see Schemes, Lemma 26.23.8). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 27.8.8. □

01VR **Lemma 29.38.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is quasi-compact. The following are equivalent*

1. *$\mathcal{L}$ is relatively very ample on X/S,*
2. *there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j,*
3. *there exists a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ generated in degree 1 over $\mathcal{O}_S$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L},\psi} : X \to \underline{Proj}_S(\mathcal{A})$ is an immersion, and*

(4) *f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ is an immersion.*

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-separation in (4) is used to guarantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 26.24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \to V_j$ the restriction of f. We see that f is separated by Lemma 29.38.6 (as being separated is local on the base). By assumption there exists a quasi-coherent $\mathcal{O}_{V_j}$-module $\mathcal{E}_j$ and an immersion $i_j : X_j \to \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. The morphism i_j corresponds to a surjection $f_j^*\mathcal{E}_j \to \mathcal{L}|_{X_j}$, see Constructions, Section 27.21. This map is adjoint to a map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ such that the composition

$$f_j^*\mathcal{E}_j \to (f^*f_*\mathcal{L})|_{X_j} \to \mathcal{L}|_{X_j}$$

is surjective. We conclude that $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. Let $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. We still have to show that $r_{\mathcal{L},\psi}$ is an immersion; we urge the reader to prove this for themselves. The $\mathcal{O}_{V_j}$-module map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ determines a homomorphism on symmetric algebras, which in turn defines a morphism

$$\mathbf{P}(f_*\mathcal{L}|_{V_j}) \supset U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

where U_j is the open subscheme of Constructions, Lemma 27.18.1. The compatibility of ψ with $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ shows that $r_{\mathcal{L},\psi}(X_j) \subset U_j$ and that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L},\psi}} U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

We omit the verification. This shows that $r_{\mathcal{L},\psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras generated in degree 1 over $\mathcal{O}_S$. Consider the map of graded $\mathcal{O}_S$-algebras $\mathrm{Sym}(\mathcal{A}_1) \to \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see Constructions, Lemma 27.18.5. Hence it is clear that (3) implies (1). □

0B3F **Lemma 29.38.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-very ample, then $\mathcal{L}'$ is f'-very ample.*

Proof. By Definition 29.38.1 there exists there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. The base change of $\mathbf{P}(\mathcal{E})$ to S' is the projective bundle associated to the pullback $\mathcal{E}'$ of $\mathcal{E}$ and the pullback of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)$, see Constructions, Lemma 27.16.10. Finally, the base change of an immersion is an immersion (Schemes, Lemma 26.18.2). □

29.39. Ample and very ample sheaves relative to finite type morphisms

02NO In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

02NP **Lemma 29.39.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *the invertible sheaf $\mathcal{L}$ is very ample on X/S,*
(2) *the morphism $X \to S$ is of finite type, and*
(3) *S is affine.*

Then there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Assume (1), (2) and (3). Condition (3) means $S = \mathrm{Spec}(R)$ for some ring R. Condition (1) means by definition there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $\alpha : X \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R-module M. Thus we have

$$\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\mathrm{Proj}(\mathrm{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \to D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \mathrm{Sym}^{d_j}_R(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = \mathrm{Spec}(A_j)$. Condition (2) also implies that A_j is of finite type over R, see Lemma 29.15.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R-algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \mathrm{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \ldots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \mathrm{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R. Consider the graded ring map

$$\psi : R[Y_0, \ldots, Y_n] \longrightarrow \mathrm{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote F_j, $F_{j,k}$ the elements of $R[Y_0, \ldots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 27.11.1 we obtain an open subscheme

$$U(\psi) \subset \mathrm{Proj}(\mathrm{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \to \mathbf{P}^n_R$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}^n_R$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^*\mathcal{O}_{\mathbf{P}^n_R}(1) \to \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see

lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)$ we win. □

04II **Lemma 29.39.2.** *Let $\pi : X \to S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \to \mathbf{A}^n_S$ over S.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $\mathrm{Spec}(A)$, see Properties, Lemma 28.18.4. Moreover, the set of opens X_f, for those $f \in A$ such that X_f is affine, forms a basis for the topology of X, see the proof of Properties, Lemma 28.18.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \ldots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 29.15.2 the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \ldots, a_N \in A$ such that the elements $a_1, \ldots, a_N, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j. Take $n = m + N$ and let

$$i : X \longrightarrow \mathbf{A}^n_S$$

be the morphism given by the global sections $f_1, \ldots, f_m, a_1, \ldots, a_N$ of the structure sheaf of X. Let $D(x_j) \subset \mathbf{A}^n_S$ be the open subscheme where the jth coordinate function is nonzero. Then for $1 \leq j \leq m$ we have $i^{-1}(D(x_j)) = X_{f_j}$ and the induced morphism $X_{f_j} \to D(x_j)$ factors through the affine open $\mathrm{Spec}(\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j] \to A_{f_j}$ is surjective by construction we conclude that $i^{-1}(D(x_j)) \to D(x_j)$ is an immersion as desired. □

01VS **Lemma 29.39.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

1. *the invertible sheaf $\mathcal{L}$ is ample on X, and*
2. *the morphism $X \to S$ is locally of finite type.*

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 28.26.13 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X. Hence we can find finitely many such elements $a_0, \ldots, a_n \in A_+$ such that

1. we have $X = \bigcup_{i=0,\ldots,n} X_{a_i}$,
2. each X_{a_i} is affine, and
3. each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 29.15.2 we see that the ring maps $\mathcal{O}_S(V_i) \to \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \ldots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$-algebra. By Properties, Lemma 28.17.2 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer which is a common multiple of all the degrees of the elements a_i, a_{ij}. Consider the elements

$$a_i^{N/\deg(a_i)},\ a_{ij}a_i^{(N/\deg(a_i))-e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X. Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}^m_S \quad \text{with } m = n + \sum n_i$$

over S, see Constructions, Lemma 27.13.1 and Definition 27.13.2. Moreover, $j^*\mathcal{O}_{\mathbf{P}^N_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates $T_0, \ldots, T_n, T_{ij}$ instead of $T_0, \ldots, T_m$. For $i = 0, \ldots, n$ we have $j^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via $j^\sharp$ we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}^m_{V_i}$ of $\mathbf{P}^N_S$. Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \to \mathbf{P}^m_S$ given by global sections $s'_0, \ldots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 28.26.13 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0. Namely, given an integer $d \geq d_0$ we may choose $s''_1, \ldots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d - d_2})$ which generate $\mathcal{L}^{\otimes d - d_2}$ over X. Set $k = (m+1)t$ and denote $s_0, \ldots, s_k$ the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \ldots, m$, $\beta = 1, \ldots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}^{k-1}_S$$

such that $i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong \mathcal{L}^{\otimes d}$. To see that i is an immersion, observe that i is the composition

$$X \longrightarrow \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \longrightarrow \mathbf{P}^{k-1}_S$$

where the first morphism is (j, j') with j' given by $s''_1, \ldots, s''_t$ and the second morphism is the Segre embedding (Constructions, Lemma 27.13.6). Since j is an immersion, so is (j, j') (apply Lemma 29.3.1 to $X \to \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \to \mathbf{P}^m_S$). Thus i is a composition of immersions and hence an immersion (Schemes, Lemma 26.24.3). □

01VT **Lemma 29.39.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine and f of finite type. The following are equivalent*

1. *$\mathcal{L}$ is ample on X,*
2. *$\mathcal{L}$ is f-ample,*
3. *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
4. *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$,*
5. *for some $d \geq 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$, and*
6. *for all $d \gg 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.*

Proof. The equivalence of (1) and (2) is Lemma 29.37.5. The implication (2) $\Rightarrow$ (6) is Lemma 29.39.3. Trivially (6) implies (5). As $\mathbf{P}^n_S$ is a projective bundle over S (see Constructions, Lemma 27.21.5) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 29.38.2 and Lemma 29.37.2. □

01VU **Lemma 29.39.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S quasi-compact and f of finite type. The following are equivalent*

1. *$\mathcal{L}$ is f-ample,*
2. *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
3. *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.*

Proof. Trivially (3) implies (2). Lemma 29.38.2 guarantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that $\mathcal{L}$ is f-ample. Choose a finite affine open covering $S = V_1 \cup \ldots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 29.39.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 29.38.7. □

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

02NQ **Lemma 29.39.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively very ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integer n_j, and immersions*
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 29.39.1 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 29.38.7. □

02NR **Lemma 29.39.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integers $d_j \geq 1$, $n_j \geq 0$, and immersions*
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 29.39.4 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 29.37.4. □

0FVC **Lemma 29.39.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{N}$, $\mathcal{L}$ be invertible $\mathcal{O}_X$-modules. Assume S is quasi-compact, f is of finite type, and $\mathcal{L}$ is f-ample. Then $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.*

Proof. By Lemma 29.39.6 we reduce to the case S is affine. Combining Lemma 29.39.4 and Properties, Proposition 28.26.13 we can find an integer d_0 such that $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ is globally generated. Choose global sections $s_0, \ldots, s_n$ of $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ which generate it. This determines a morphism $j : X \to \mathbf{P}^n_S$ over S. By Lemma 29.39.4 we can also pick an integer d_1 such that for all $d \geq d_1$ there exist sections $t_{d,0}, \ldots, t_{d,n(d)}$ of $\mathcal{L}^{\otimes d}$ which generate it and define an immersion
$$j_d = \varphi_{\mathcal{L}^{\otimes d}, t_{d,0}, \ldots, t_{d,n(d)}} : X \longrightarrow \mathbf{P}^{n(d)}_S$$
over S. Then for $d \geq d_0 + d_1$ we can consider the morphism
$$\varphi_{\mathcal{N} \otimes \mathcal{L}^{\otimes d}, s_j \otimes t_{d-d_0,i}} : X \longrightarrow \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

This morphism is an immersion as it is the composition

$$X \to \mathbf{P}^n_S \times_S \mathbf{P}^{n(d-d_0)}_S \to \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

where the first morphism is (j, j_{d-d_0}) and the second is the Segre embedding (Constructions, Lemma 27.13.6). Since j is an immersion, so is (j, j_{d-d_0}) (apply Lemma 29.3.1). We have a composition of immersions and hence an immersion (Schemes, Lemma 26.24.3). □

29.40. Quasi-projective morphisms

01VV The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [**DG67**]. The version with the letter "H" is the definition in [**Har77**].

01VW **Definition 29.40.1.** Let $f : X \to S$ be a morphism of schemes. [**DG67**, II, Definition 5.3.1] and [**Har77**, page 103]

(1) We say f is *quasi-projective* if f is of finite type and there exists an f-relatively ample invertible $\mathcal{O}_X$-module.
(2) We say f is *H-quasi-projective* if there exists a quasi-compact immersion $X \to \mathbf{P}^n_S$ over S for some n.[13]
(3) We say f is *locally quasi-projective* if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S. At a later stage we will be able to say more about the category of quasi-projective schemes, see More on Morphisms, Section 37.49.

0B3G **Lemma 29.40.2.** *A base change of a quasi-projective morphism is quasi-projective.*

Proof. This follows from Lemmas 29.15.4 and 29.37.9. □

0C4M **Lemma 29.40.3.** *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and f and g are quasi-projective, then $g \circ f$ is quasi-projective.*

Proof. This follows from Lemmas 29.15.3 and 29.37.8. □

01VX **Lemma 29.40.4.** *Let $f : X \to S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.*

Proof. Omitted. □

01VY **Lemma 29.40.5.** *A H-quasi-projective morphism is quasi-projective.*

Proof. Omitted. □

01VZ **Lemma 29.40.6.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally quasi-projective.*
(2) *There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is H-quasi-projective.*

[13] This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 29.43.1), which does not imply f is quasi-compact. See Lemma 29.43.11 for the implication in the other direction.

Proof. By Lemma 29.40.5 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and $\mathcal{L}$ is a relatively ample invertible sheaf on X/S. By Lemma 29.39.4 we may assume $\mathcal{L}$ is ample on X. By Lemma 29.39.3 we see that there exists an immersion of X into a projective space over S, i.e., X is H-quasi-projective over S as desired. □

0B3H **Lemma 29.40.7.** *A quasi-affine morphism of finite type is quasi-projective.* [**DG67**, II, Proposition 5.3.4 (i)]

Proof. This follows from Lemma 29.37.6. □

0C4N **Lemma 29.40.8.** *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is quasi-projective and f is quasi-compact*[14]*, then f is quasi-projective.*

Proof. Observe that f is of finite type by Lemma 29.15.8. Thus the lemma follows from Lemma 29.37.10 and the definitions. □

29.41. Proper morphisms

01W0 The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}^n_S \to S$ of projective n-space, and this is in fact the motivating example leading to the definition.

01W1 **Definition 29.41.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *proper* if f is separated, finite type, and universally closed.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

02K7 **Lemma 29.41.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is universally closed.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is universally closed for all indices j.*

Proof. This is clear from the definition. □

01W2 **Lemma 29.41.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is proper for all indices j.*

Proof. Omitted. □

01W3 **Lemma 29.41.4.** *The composition of proper morphisms is proper. The same is true for universally closed morphisms.*

[14] This follows if g is quasi-separated by Schemes, Lemma 26.21.14.

Proof. A composition of closed morphisms is closed. If $X \to Y \to Z$ are universally closed morphisms and $Z' \to Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \to Z' \times_Z Y$ is closed and $Z' \times_Z Y \to Z'$ is closed. Hence the result for universally closed morphisms. We have seen that "separated" and "finite type" are preserved under compositions (Schemes, Lemma 26.21.12 and Lemma 29.15.3). Hence the result for proper morphisms. □

01W4 **Lemma 29.41.5.** *The base change of a proper morphism is proper. The same is true for universally closed morphisms.*

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 26.21.12). It is true for morphisms of finite type (Lemma 29.15.4). Hence it is true for proper morphisms. □

01W5 **Lemma 29.41.6.** *A closed immersion is proper, hence a fortiori universally closed.*

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 26.18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 26.23.8). A closed immersion is of finite type (Lemma 29.15.5). Hence a closed immersion is proper. □

01W6 **Lemma 29.41.7.** *Suppose given a commutative diagram of schemes*

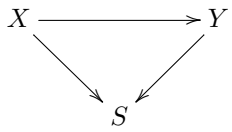

with Y separated over S.

(1) *If $X \to S$ is universally closed, then the morphism $X \to Y$ is universally closed.*
(2) *If X is proper over S, then the morphism $X \to Y$ is proper.*

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \to S$ is universally closed (resp. proper). We factor the morphism as $X \to X \times_S Y \to Y$. The first morphism is a closed immersion, see Schemes, Lemma 26.21.10. Hence the first morphism is proper (Lemma 29.41.6). The projection $X \times_S Y \to Y$ is the base change of a universally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 29.41.5. Thus $X \to Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 29.41.4). □

The proof of the following lemma is due to Bjorn Poonen, see this location.

04XU **Lemma 29.41.8.** *A universally closed morphism of schemes is quasi-compact.* Due to Bjorn Poonen.

Proof. Let $f : X \to S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 26.19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \to V$ is not universally closed, hence we may assume that $S = \mathrm{Spec}(A)$ for some ring A.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X. Let $T = \mathrm{Spec}(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) -$

$\bigcup_{i\in I}(X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \to T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 26.19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i. Then $t \in T_i$ for all i, and the fiber of $Z_s \to T_s$ above t is isomorphic to $(X - \bigcup_{i\in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T. Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible on some neighborhood U of s. Let J be the finite set of $j \in I$ such that y_j appears in g. Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j\in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U, we can find a point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = \mathrm{Spec}(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $\mathrm{Spec}(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 26.17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(Z)$ closed" is wrong and we conclude indeed that f_T is not closed, as desired. □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

03GN **Lemma 29.41.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X is universally closed over S and f is surjective then Y is universally closed over S. In particular, if also Y is separated and locally of finite type over S, then Y is proper over S.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to S$, $q : Y \to S$ the structure morphisms. Let $S' \to S$ be a morphism of schemes. The base change $f' : X_{S'} \to Y_{S'}$ is surjective (Lemma 29.9.4), and the base change $p' : X_{S'} \to S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. This proves the first statement. Thus $Y \to S$ is quasi-compact by Lemma 29.41.8 and hence $Y \to S$ is proper by definition if in addition $Y \to S$ is locally of finite type and separated. □

0AH6 **Lemma 29.41.10.** *Suppose given a commutative diagram of schemes*

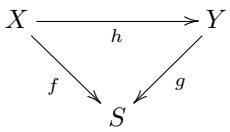

Assume

(1) *$X \to S$ is a universally closed (for example proper) morphism, and*
(2) *$Y \to S$ is separated and locally of finite type.*

Then the scheme theoretic image $Z \subset Y$ of h is proper over S and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 29.6. Since f is quasi-compact (Lemma 29.41.8) we find that h is quasi-compact (Schemes, Lemma 26.21.14). Hence $h(X) \subset Z$ is dense (Lemma 29.6.3). On the other hand $h(X)$ is closed in Y (Lemma 29.41.7) hence $X \to Z$ is surjective. Thus $Z \to S$ is a proper (Lemma 29.41.9). □

The target of a separated scheme under a surjective universally closed morphism is separated.

09MQ **Lemma 29.41.11.** *Let S be a scheme. Let $f : X \to Y$ be a surjective universally closed morphism of schemes over S.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over S, then Y is quasi-separated over S.*
(4) *If X is separated over S, then Y is separated over S.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = \mathrm{Spec}(\mathbf{Z})$ (see Schemes, Definition 26.21.3). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/S}} & X \times_S X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_S X \to X \times_S Y \to Y \times_S Y$. Hence it is also quasi-compact, see Lemma 29.41.8.

Assume X is quasi-separated over S, i.e., $\Delta_{X/S}$ is quasi-compact. If $V \subset Y \times_S Y$ is a quasi-compact open, then $V \times_{Y \times_S Y} X \to \Delta_{Y/S}^{-1}(V)$ is surjective and $V \times_{Y \times_S Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/S}$ is quasi-compact, i.e., Y is quasi-separated over S.

Assume X is separated over S, i.e., $\Delta_{X/S}$ is a closed immersion. Then $X \to Y \times_S Y$ is closed as a composition of closed morphisms. Since $X \to Y$ is surjective, it follows that $\Delta_{Y/S}(Y)$ is closed in $Y \times_S Y$. Hence Y is separated over S by the discussion following Schemes, Definition 26.21.3. □

29.42. Valuative criteria

0BX4 We have already discussed the valuative criterion for universal closedness and for separatedness in Schemes, Sections 26.20 and 26.22. In this section we will discuss some consequences and variants. In Limits, Section 32.15 we will show that it suffices to consider discrete valuation rings when working with locally Noetherian schemes and morphisms of finite type.

0BX5 **Lemma 29.42.1** (Valuative criterion for properness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume f is of finite type and quasi-separated. Then the following are equivalent* [**DG67**, II Theorem 7.3.8]

(1) *f is proper,*

(2) *f satisfies the valuative criterion (Schemes, Definition 26.20.3),*
(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute.

Proof. Part (3) is a reformulation of (2). Thus the lemma is a formal consequence of Schemes, Proposition 26.20.6 and Lemma 26.22.2 and the definitions. □

One usually does not have to consider all possible diagrams when testing the valuative criterion. We will call a valuative criterion as in the next lemma a "refined valuative criterion".

0894 **Lemma 29.42.2.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that f and h are quasi-compact and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & & \longrightarrow & & S \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is universally closed. If moreover f is quasi-separated, then f is separated.

Proof. To prove f is universally closed we will verify the existence part of the valuative criterion for f which suffices by Schemes, Proposition 26.20.6. To do this, consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Schemes, Lemma 26.12.7. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 29.6.7. We may also replace S by an affine open through which the morphism $\mathrm{Spec}(A) \to S$ factors. Thus we may assume that $S = \mathrm{Spec}(R)$.

Let $\mathrm{Spec}(B) \subset X$ be an affine open through which the morphism $\mathrm{Spec}(K) \to X$ factors. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 29.25.16. By Lemma 29.6.5 we

can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.50.10. As C is an R-algebra with fraction field K', we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(C) & & \longrightarrow & & S \end{array}$$

as in the statement of the lemma. Thus a dotted arrow fitting into the diagram as indicated. By the uniqueness assumption of the lemma the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(C) \to X$ agrees with the given morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(P) \to \mathrm{Spec}(B) \subset X$. Hence the restriction of the morphism to the spectrum of $C/\mathfrak{m}_{A'} = A''$ induces the given morphism $\mathrm{Spec}(K'') = \mathrm{Spec}(A'/\mathfrak{m}_{A'}) \to \mathrm{Spec}(K) \to X$. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(A'') \to X$. The image of the induced ring map $\mathcal{O}_{X,x} \to A''$ is a local subring which is contained in $K \subset K''$. Since A is maximal for the relation of domination in K and since $A \subset A''$, we have $A = K \cap A''$. We conclude that $\mathcal{O}_{X,x} \to A''$ factors through $A \subset A''$. In this way we obtain our desired arrow $\mathrm{Spec}(A) \to X$.

Finally, assume f is quasi-separated. Then $\Delta : X \to X \times_S X$ is quasi-compact. Given a solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow \Delta \\ \mathrm{Spec}(A) & & \longrightarrow & & X \times_S X \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute. Namely, the lower horizontal arrow is the same thing as a pair of morphisms $\mathrm{Spec}(A) \to X$ which can serve as the dotted arrow in the diagram of the lemma. Thus the required uniqueness shows that the lower horizontal arrow factors through Δ. Hence we can apply the result we just proved to $\Delta : X \to X \times_S X$ and $h : U \to X$ and conclude that Δ is universally closed. Clearly this means that f is separated. □

0895 **Remark 29.42.3.** The assumption on uniqueness of the dotted arrows in Lemma 29.42.2 is necessary (details omitted). Of course, uniqueness is guaranteed if f is separated (Schemes, Lemma 26.22.1).

0BX6 **Lemma 29.42.4.** *Let S be a scheme. Let X, Y be schemes over S. Let $s \in S$ and $x \in X$, $y \in Y$ points over s.*

(1) *Let $f, g : X \to Y$ be morphisms over S such that $f(x) = g(x) = y$ and $f_x^\sharp = g_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Then there is an open neighbourhood $U \subset X$ with $f|_U = g|_U$ in the following cases*
   (a) *Y is locally of finite type over S,*
   (b) *X is integral,*
   (c) *X is locally Noetherian, or*
   (d) *X is reduced with finitely many irreducible components.*

(2) *Let $\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a local $\mathcal{O}_{S,s}$-algebra map. Then there exists an open neighbourhood $U \subset X$ of x and a morphism $f : U \to Y$ mapping x to y with $f_x^\sharp = \varphi$ in the following cases*
   (a) *Y is locally of finite presentation over S,*
   (b) *Y is locally of finite type and X is integral,*
   (c) *Y is locally of finite type and X is locally Noetherian, or*
   (d) *Y is locally of finite type and X is reduced with finitely many irreducible components.*

Proof. Proof of (1). We may replace X, Y, S by suitable affine open neighbourhoods of x, y, s and reduce to the following algebra problem: given a ring R, two R-algebra maps $\varphi, \psi : B \to A$ such that

(1) $R \to B$ is of finite type, or A is a domain, or A is Noetherian, or A is reduced and has finitely many minimal primes,
(2) the two maps $B \to A_\mathfrak{p}$ are the same for some prime $\mathfrak{p} \subset A$,

show that φ, ψ define the same map $B \to A_g$ for a suitable $g \in A$, $g \notin \mathfrak{p}$. If $R \to B$ is of finite type, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. For each j we can find $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j)$ and $\psi(t_j)$ have the same image in A_{g_j}. Then we set $g = \prod g_j$. In the other cases (if A is a domain, Noetherian, or reduced with finitely many minimal primes), we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \subset A_\mathfrak{p}$. See Algebra, Lemma 10.31.9. Thus the maps $B \to A_g$ are equal as desired.

Proof of (2). To do this we may replace X, Y, and S by suitable affine opens. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. Let $\mathfrak{q} \subset B$ be the prime corresponding to y. Then φ is a local R-algebra map $\varphi : B_\mathfrak{q} \to A_\mathfrak{p}$. If $R \to B$ is a ring map of finite presentation, then there exists a $g \in A \setminus \mathfrak{p}$ and an R-algebra map $B \to A_g$ such that

$$\begin{array}{ccc} B_\mathfrak{q} & \xrightarrow{\varphi} & A_\mathfrak{p} \\ \uparrow & & \uparrow \\ B & \longrightarrow & A_g \end{array}$$

commutes, see Algebra, Lemmas 10.127.3 and 10.9.9. The induced morphism $\mathrm{Spec}(A_g) \to \mathrm{Spec}(B)$ works. If B is of finite type over R, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. Then we can choose $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j) \in \mathrm{Im}(A_{g_j} \to A_\mathfrak{p})$. Thus after replacing A by $A[1/\prod g_j]$ we may assume that B maps into the image of $A \to A_\mathfrak{p}$. If we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \to A_\mathfrak{p}$ is injective, then we'll get the desired R-algebra map $B \to A_g$. Thus the proof is finished by another application of See Algebra, Lemma 10.31.9. $\square$

0BX7 **Lemma 29.42.5.** *Let S be a scheme. Let X, Y be schemes over S. Let $x \in X$. Let $U \subset X$ be an open and let $f : U \to Y$ be a morphism over S. Assume*

(1) *x is in the closure of U,*
(2) *X is reduced with finitely many irreducible components or X is Noetherian,*
(3) *$\mathcal{O}_{X,x}$ is a valuation ring,*
(4) *$Y \to S$ is proper*

Then there exists an open $U \subset U' \subset X$ containing x and an S-morphism $f' : U' \to Y$ extending f.

Proof. It is harmless to replace X by an open neighbourhood of x in X (small detail omitted). By Properties, Lemma 28.29.8 we may assume X is affine with $\Gamma(X, \mathcal{O}_X) \subset \mathcal{O}_{X,x}$. In particular X is integral with a unique generic point ξ whose residue field is the fraction field K of the valuation ring $\mathcal{O}_{X,x}$. Since x is in the closure of U we see that U is not empty, hence U contains ξ. Thus by the valuative criterion of properness (Lemma 29.42.1) there is a morphism $t : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,x}) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle t} \\ U & \xrightarrow{\quad f \quad} & Y \end{array}$$

of morphisms of schemes over S. Applying Lemma 29.42.4 with $y = t(x)$ and $\varphi = t_x^\sharp$ we obtain an open neighbourhood $V \subset X$ of x and a morphism $g : V \to Y$ over S which sends x to y and such that $g_x^\sharp = t_x^\sharp$. As $Y \to S$ is separated, the equalizer E of $f|_{U \cap V}$ and $g|_{U \cap V}$ is a closed subscheme of $U \cap V$, see Schemes, Lemma 26.21.5. Since f and g determine the same morphism $\mathrm{Spec}(K) \to Y$ by construction we see that E contains the generic point of the integral scheme $U \cap V$. Hence $E = U \cap V$ and we conclude that f and g glue to a morphism $U' = U \cup V \to Y$ as desired. □

29.43. Projective morphisms

01W7 We will use the definition of a projective morphism from [**DG67**]. The version of the definition with the "H" is the one from [**Har77**]. The resulting definitions are different. Both are useful.

01W8 **Definition 29.43.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.
(2) We say f is *H-projective* if there exists an integer n and a closed immersion $X \to \mathbf{P}^n_S$ over S.
(3) We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is projective.

As expected, a projective morphism is quasi-projective, see Lemma 29.43.10. Conversely, quasi-projective morphisms are often compositions of open immersions and projective morphisms, see Lemma 29.43.12. For an overview of properties of projective morphisms over a quasi-projective base, see More on Morphisms, Section 37.50.

07ZS **Example 29.43.2.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Assume furthermore that $\mathcal{A}_1$ is of finite type over $\mathcal{O}_S$. Set $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$. In this case $X \to S$ is projective. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion, see Constructions, Lemma 27.18.5.

01W9 **Lemma 29.43.3.** *An H-projective morphism is H-quasi-projective. An H-projective morphism is projective.*

Proof. The first statement is immediate from the definitions. The second holds as $\mathbf{P}^n_S$ is a projective bundle over S, see Constructions, Lemma 27.21.5. □

01WB **Lemma 29.43.4.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally projective.*
(2) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is H-projective.*

Proof. By Lemma 29.43.3 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $\mathrm{Spec}(R) = U \subset S$ an affine open neighbourhood of s such that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{E}$. Write $\mathcal{E} = \widetilde{M}$ for some finite type R-module M (see Properties, Lemma 28.16.1). Choose generators $x_0, \ldots, x_n \in M$ of M as an R-module. Consider the surjective graded R-algebra map

$$R[X_0, \ldots, X_n] \longrightarrow \mathrm{Sym}_R(M).$$

According to Constructions, Lemma 27.11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n_R$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of $\mathbf{P}^n_U$ (as a scheme over U). In other words: (2) holds. □

01WC **Lemma 29.43.5.** *A locally projective morphism is proper.*

Proof. Let $f : X \to S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 29.41.3. Hence, by Lemma 29.43.4 above we may assume there exists a closed immersion $X \to \mathbf{P}^n_S$. By Lemmas 29.41.4 and 29.41.6 it suffices to prove that $\mathbf{P}^n_S \to S$ is proper. Since $\mathbf{P}^n_S \to S$ is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is proper, see Lemma 29.41.5. By Constructions, Lemma 27.8.8 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is separated. By Constructions, Lemma 27.8.9 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is quasi-compact. It is clear that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite type since $\mathbf{P}^n_{\mathbf{Z}}$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type $\mathbf{Z}$-algebra

$$\mathbf{Z}[X_0/X_i, \ldots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 27.8.11 and the valuative criterion (see Schemes, Proposition 26.20.6). □

0B5N **Lemma 29.43.6.** *Let $f : X \to S$ be a proper morphism of schemes. If there exists an f-ample invertible sheaf on X, then f is locally projective.*

Proof. If there exists an f-ample invertible sheaf, then we can locally on S find an immersion $i : X \to \mathbf{P}^n_S$, see Lemma 29.39.4. Since $X \to S$ is proper the morphism i is a closed immersion, see Lemma 29.41.7. □

01WE **Lemma 29.43.7.** *A composition of H-projective morphisms is H-projective.*

Proof. Suppose $X \to Y$ and $Y \to Z$ are H-projective. Then there exist closed immersions $X \to \mathbf{P}^n_Y$ over Y, and $Y \to \mathbf{P}^m_Z$ over Z. Consider the following diagram

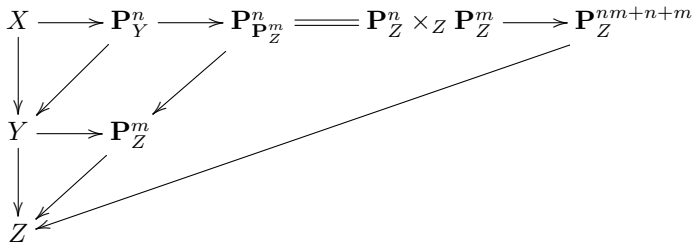

Here the rightmost top horizontal arrow is the Segre embedding, see Constructions, Lemma 27.13.6. The diagram identifies X as a closed subscheme of $\mathbf{P}^{nm+n+m}_Z$ as desired. □

01WF **Lemma 29.43.8.** *A base change of a H-projective morphism is H-projective.*

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 26.18.2. □

02V6 **Lemma 29.43.9.** *A base change of a (locally) projective morphism is (locally) projective.*

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type $\mathcal{O}$-module is of finite type (Modules, Lemma 17.9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 26.18.2. Some details omitted. □

07RL **Lemma 29.43.10.** *A projective morphism is quasi-projective.*

Proof. Let $f : X \to S$ be a projective morphism. Choose a closed immersion $i : X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is f-very ample. Since f is proper (Lemma 29.43.5) it is quasi-compact. Hence Lemma 29.38.2 implies that $\mathcal{L}$ is f-ample. Since f is proper it is of finite type. Thus we've checked all the defining properties of quasi-projective holds and we win. □

01WA **Lemma 29.43.11.** *Let $f : X \to S$ be a H-quasi-projective morphism. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is H-projective.*

Proof. By definition we can factor f as a quasi-compact immersion $i : X \to \mathbf{P}^n_S$ followed by the projection $\mathbf{P}^n_S \to S$. By Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_S$ such that i factors through an open immersion $X \to X'$. The lemma follows. □

07RM **Lemma 29.43.12.** *Let $f : X \to S$ be a quasi-projective morphism with S quasi-compact and quasi-separated. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is projective.*

Proof. Let $\mathcal{L}$ be f-ample. Since f is of finite type and S is quasi-compact $\mathcal{L}^{\otimes n}$ is f-very ample for some $n > 0$, see Lemma 29.39.5. Replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$. Write $\mathcal{F} = f_*\mathcal{L}$. This is a quasi-coherent $\mathcal{O}_S$-module by Schemes, Lemma 26.24.1 (quasi-projective morphisms are quasi-compact and separated, see Lemma 29.40.4). By Properties, Lemma 28.22.7 we can find a directed set I and a system of finite type quasi-coherent $\mathcal{O}_S$-modules $\mathcal{E}_i$ over I such that $\mathcal{F} = \operatorname{colim} \mathcal{E}_i$. Consider the compositions $\psi_i : f^*\mathcal{E}_i \to f^*\mathcal{F} \to \mathcal{L}$. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,m} V_j$. For each j we can choose sections

$$s_{j,0}, \ldots, s_{j,n_j} \in \Gamma(f^{-1}(V_j), \mathcal{L}) = f_*\mathcal{L}(V_j) = \mathcal{F}(V_j)$$

which generate $\mathcal{L}$ over $f^{-1}V_j$ and define an immersion

$$f^{-1}V_j \longrightarrow \mathbf{P}^{n_j}_{V_j},$$

see Lemma 29.39.1. Choose i such that there exist sections $e_{j,t} \in \mathcal{E}_i(V_j)$ mapping to $s_{j,t}$ in $\mathcal{F}$ for all $j = 1, \ldots, m$ and $t = 1, \ldots, n_j$. Then the map ψ_i is surjective as the sections $f^*e_{j,t}$ have the same image as the sections $s_{j,t}$ which generate $\mathcal{L}|_{f^{-1}V_j}$. Whence we obtain a morphism

$$r_{\mathcal{L},\psi_i} : X \longrightarrow \mathbf{P}(\mathcal{E}_i)$$

over S such that over V_j we have a factorization

$$f^{-1}V_j \to \mathbf{P}(\mathcal{E}_i)|_{V_j} \to \mathbf{P}^{n_j}_{V_j}$$

of the immersion given above. It follows that $r_{\mathcal{L},\psi_i}|_{V_j}$ is an immersion, see Lemma 29.3.1. Since $S = \bigcup V_j$ we conclude that $r_{\mathcal{L},\psi_i}$ is an immersion. Note that $r_{\mathcal{L},\psi_i}$ is quasi-compact as $X \to S$ is quasi-compact and $\mathbf{P}(\mathcal{E}_i) \to S$ is separated (see Schemes, Lemma 26.21.14). By Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}(\mathcal{E}_i)$ such that i factors through an open immersion $X \to X'$. Then $X' \to S$ is projective by definition and we win. □

0BCL **Lemma 29.43.13.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Then*

(1) *f is projective if and only if f is quasi-projective and proper, and*
(2) *f is H-projective if and only if f is H-quasi-projective and proper.*

Proof. If f is projective, then f is quasi-projective by Lemma 29.43.10 and proper by Lemma 29.43.5. Conversely, if $X \to S$ is quasi-projective and proper, then we can choose an open immersion $X \to X'$ with $X' \to S$ projective by Lemma 29.43.12. Since $X \to S$ is proper, we see that X is closed in X' (Lemma 29.41.7), i.e., $X \to X'$ is a (open and) closed immersion. Since X' is isomorphic to a closed subscheme of a projective bundle over S (Definition 29.43.1) we see that the same thing is true for X, i.e., $X \to S$ is a projective morphism. This proves (1). The proof of (2) is the same, except it uses Lemmas 29.43.3 and 29.43.11. □

0C4P **Lemma 29.43.14.** *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and quasi-separated and f and g are projective, then $g \circ f$ is projective.*

Proof. By Lemmas 29.43.10 and 29.43.5 we see that f and g are quasi-projective and proper. By Lemmas 29.41.4 and 29.40.3 we see that $g \circ f$ is proper and quasi-projective. Thus $g \circ f$ is projective by Lemma 29.43.13. □

0C4Q **Lemma 29.43.15.** *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is projective and g is separated, then f is projective.*

Proof. Choose a closed immersion $X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then we get a morphism $X \to \mathbf{P}(\mathcal{E}) \times_S Y$. This morphism is a closed immersion because it is the composition

$$X \to X \times_S Y \to \mathbf{P}(\mathcal{E}) \times_S Y$$

where the first morphism is a closed immersion by Schemes, Lemma 26.21.10 (and the fact that g is separated) and the second as the base change of a closed immersion. Finally, the fibre product $\mathbf{P}(\mathcal{E}) \times_S Y$ is isomorphic to $\mathbf{P}(g^*\mathcal{E})$ and pullback preserves quasi-coherent, finite type modules. □

087S **Lemma 29.43.16.** *Let S be a scheme which admits an ample invertible sheaf. Then*

(1) *any projective morphism $X \to S$ is H-projective, and*
(2) *any quasi-projective morphism $X \to S$ is H-quasi-projective.*

Proof. The assumptions on S imply that S is quasi-compact and separated, see Properties, Definition 28.26.1 and Lemma 28.26.11 and Constructions, Lemma 27.8.8. Hence Lemma 29.43.12 applies and we see that (1) implies (2). Let $\mathcal{E}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. By our definition of projective morphisms it suffices to show that $\mathbf{P}(\mathcal{E}) \to S$ is H-projective. If $\mathcal{E}$ is generated by finitely many global sections, then the corresponding surjection $\mathcal{O}_S^{\oplus n} \to \mathcal{E}$ induces a closed immersion

$$\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}(\mathcal{O}_S^{\oplus n}) = \mathbf{P}_S^{n-1}$$

as desired. In general, let $\mathcal{L}$ be an ample invertible sheaf on S. By Properties, Proposition 28.26.13 there exists an integer n such that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n}$ is globally generated by finitely many sections. Since $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n})$ by Constructions, Lemma 27.20.1 this finishes the proof. □

0C6J **Lemma 29.43.17.** *Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Then the canonical morphism*

$$r : X \longrightarrow \underline{Proj}_S \left(\bigoplus\nolimits_{d \geq 0} f_* \mathcal{L}^{\otimes d} \right)$$

of Lemma 29.37.4 is an isomorphism.

Proof. Observe that f is quasi-compact because the existence of an f-ample invertible module forces f to be quasi-compact. By the lemma cited the morphism r is an open immersion. On the other hand, the image of r is closed by Lemma 29.41.7 (the target of r is separated over S by Constructions, Lemma 27.16.9). Finally, the image of r is dense by Properties, Lemma 28.26.11 (here we also use that it was shown in the proof of Lemma 29.37.4 that the morphism r over affine opens of S is given by the canonical morphism of Properties, Lemma 28.26.9). Thus we conclude that r is a surjective open immersion, i.e., an isomorphism. □

0EKE **Lemma 29.43.18.** *Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then $X_s \to S$ is an affine morphism.*

Proof. The question is local on S (Lemma 29.11.3) hence we may assume S is affine. By Lemma 29.43.17 we can write $X = \text{Proj}(A)$ where A is a graded ring and s corresponds to $f \in A_1$ and $X_s = D_+(f)$ (Properties, Lemma 28.26.9) which proves the lemma by construction of $\text{Proj}(A)$, see Constructions, Section 27.8. □

29.44. Integral and finite morphisms

01WG Recall that a ring map $R \to A$ is said to be *integral* if every element of A satisfies a monic equation with coefficients in R. Recall that a ring map $R \to A$ is said to be *finite* if A is finite as an R-module. See Algebra, Definition 10.36.1.

01WH **Definition 29.44.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *integral* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is integral.
(2) We say that f is *finite* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 29.44.7 at the end of the section.

02K8 **Lemma 29.44.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is integral.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is integral.*
(3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is integral.*

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is integral.

Proof. See Algebra, Lemma 10.36.14. Some details omitted. □

01WI **Lemma 29.44.3.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is finite.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is finite.*
(3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is finite.*

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is finite.

Proof. See Algebra, Lemma 10.36.14. Some details omitted. □

01WJ **Lemma 29.44.4.** *A finite morphism is integral. An integral morphism which is locally of finite type is finite.*

Proof. See Algebra, Lemma 10.36.3 and Lemma 10.36.5. □

01WK **Lemma 29.44.5.** *A composition of finite morphisms is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemmas 10.7.3 and 10.36.6. □

01WL **Lemma 29.44.6.** *A base change of a finite morphism is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemma 10.36.13. □

01WM **Lemma 29.44.7.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 10.36.22 and 10.41.6.

Assume (2). We may assume f is the morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ coming from a ring map $R \to A$. Let a be an element of A. We have to show that a is integral over R, i.e. that in the kernel I of the map $R[x] \to A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax-1)$ and let J be the kernel of the composition $R[x] \to A[x] \to B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax-1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum\nolimits_{i\geq 0} f_i x^{n-i} = \sum\nolimits_{i\geq 0} (aq_{i-1} - q_i)x^{n-i} = (a-x)\sum\nolimits_{i\geq 0} q_i x^{n-i-1}$$

is clearly in I; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $\mathrm{Spec}(R[x]/(J + (x)))$ is empty.

Since f is universally closed the base change $\mathrm{Spec}(A[x]) \to \mathrm{Spec}(R[x])$ is closed. Hence the image of the closed subset $\mathrm{Spec}(B) \subset \mathrm{Spec}(A[x])$ is the closed subset $\mathrm{Spec}(R[x]/J) \subset \mathrm{Spec}(R[x])$, see Example 29.6.4 and Lemma 29.6.3. In particular $\mathrm{Spec}(B) \to \mathrm{Spec}(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{array}{ccccc}
\mathrm{Spec}(B) & \xrightarrow{g} & \mathrm{Spec}(R[x]/J) & \longrightarrow & \mathrm{Spec}(R[x]) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle 0} \\
\emptyset & \longrightarrow & \mathrm{Spec}(R[x]/(J + (x))) & \longrightarrow & \mathrm{Spec}(R)
\end{array}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \to B$ sends x to an invertible element and $R[x] \to R$ sends x to 0. Since g is surjective this implies $\mathrm{Spec}(R[x]/(J + (x)))$ is empty, as we wanted to show. □

02NT **Lemma 29.44.8.** *Let $f : X \to S$ be an integral morphism. Then every point of X is closed in its fibre.*

Proof. See Algebra, Lemma 10.36.20. □

0ECG **Lemma 29.44.9.** *Let $f : X \to Y$ be an integral morphism. Then* $\dim(X) \leq \dim(Y)$. *If f is surjective then* $\dim(X) = \dim(Y)$.

Proof. Since the dimension of X and Y is the supremum of the dimensions of the members of an affine open covering, we may assume Y and X are affine. The inequality follows from Algebra, Lemma 10.112.3. The equality then follows from Algebra, Lemmas 10.112.1 and 10.36.22. □

02NU **Lemma 29.44.10.** *A finite morphism is quasi-finite.*

Proof. This is implied by Algebra, Lemma 10.122.5 and Lemma 29.20.9. Alternatively, all points in fibres are closed points by Lemma 29.44.8 (and the fact that a finite morphism is integral) and use Lemma 29.20.6 (3) to see that f is quasi-finite at x for all $x \in X$. □

01WN **Lemma 29.44.11.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is finite, and*
(2) *f is affine and proper.*

Proof. This follows formally from Lemma 29.44.7, the fact that a finite morphism is integral and separated, the fact that a proper morphism is the same thing as a finite type, separated, universally closed morphism, and the fact that an integral morphism of finite type is finite (Lemma 29.44.4). □

035C **Lemma 29.44.12.** *A closed immersion is finite (and a fortiori integral).*

Proof. True because a closed immersion is affine (Lemma 29.11.10) and a surjective ring map is finite and integral. □

0CYI **Lemma 29.44.13.** *Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of schemes. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.*

Proof. Follows from the algebra fact that if $R \to A_i$, $i = 1, \ldots, n$ are finite ring maps, then $R \to A_1 \times \ldots \times A_n$ is finite too. □

035D **Lemma 29.44.14.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 29.44.6. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Schemes, Lemma 26.21.11. A closed immersion is finite (resp. integral), see Lemma 29.44.12. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 29.44.5. Thus we win. □

03BB **Lemma 29.44.15.** *Let $f : X \to Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.*

Proof. This reduces to Algebra, Lemma 10.107.6. □

0B3I **Lemma 29.44.16.** *A finite morphism is projective.*

Proof. Let $f : X \to S$ be a finite morphism. Then $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-module (Lemma 29.11.5) of finite type (by our definition of finite morphisms and Properties, Lemma 28.16.1). We claim there is a closed immersion

$$\sigma : X \longrightarrow \mathbf{P}(f_*\mathcal{O}_X) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}^*_{\mathcal{O}_S}(f_*\mathcal{O}_X))$$

over S, which finishes the proof. Namely, we let σ be the morphism which corresponds (via Constructions, Lemma 27.16.11) to the surjection

$$f^*f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$$

coming from the adjunction map $f^*f_* \to \mathrm{id}$. Then σ is a closed immersion. Namely, affine locally on S we can write $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. Since X is finite over S we may choose $a_1, \ldots, a_n \in A$ generating A as an R-module. Then the R-algebra map $R[T_0, T_1, \ldots, T_n] \to \mathrm{Sym}^*_R(A)$ sending T_0 to 1 and T_i to a_i for $1 \leq i \leq n$ is surjective. Whence $\mathbf{P}(f_*\mathcal{O}_X)$ is a closed subscheme of $\mathbf{P}^n_S$ by Constructions, Lemma 27.11.3. Thus it suffices to prove that the induced morphism $X \to \mathbf{P}^n_S$ is a closed immersion (see for example, Lemma 29.2.2). The reader checks that $X \to \mathbf{P}^n_S$ has image contained in the open $D_+(T_0) \cong \mathbf{A}^n_S$ and that $X \to \mathbf{A}^n_S$ corresponds to the surjective R-algebra map $R[x_1, \ldots, x_n] \to \mathrm{Sym}^*_R(A)$ sending x_i to a_i. Whence $X \to \mathbf{P}^n_S$ is an immersion (as a composition of a closed immersion and an open immersion). Since X is finite over S the image of this immersion is closed (this uses Lemma 29.44.11, Lemma 29.41.7, and Constructions, Lemma 27.13.4). We conclude by Schemes, Lemma 26.10.4. □

29.45. Universal homeomorphisms

04DC The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 29.45.5.

04DD **Definition 29.45.1.** A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \to Y'$ is a homeomorphism for every morphism $Y' \to Y$.

First we state the obligatory lemmas.

0CEU **Lemma 29.45.2.** *The base change of a universal homeomorphism of schemes by any morphism of schemes is a universal homeomorphism.*

Proof. This is immediate from the definition. □

0CEV **Lemma 29.45.3.** *The composition of a pair of universal homeomorphisms of schemes is a universal homeomorphism.*

Proof. Omitted. □

The following simple lemma is the key to characterizing universal homeomorphisms.

04DE **Lemma 29.45.4.** *Let $f : X \to Y$ be a morphism of schemes. If f is a homeomorphism onto a closed subset of Y then f is affine.*

Proof. Let $y \in Y$ be a point. If $y \notin f(X)$, then there exists an affine neighbourhood of y which is disjoint from $f(X)$. If $y \in f(X)$, let $x \in X$ be the unique point of X mapping to y. Let $y \in V$ be an affine open neighbourhood. Let $U \subset X$ be an affine open neighbourhood of x which maps into V. Since $f(U) \subset V \cap f(X)$ is open in the induced topology by our assumption on f we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap f(X) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U. Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 29.11.3. □

04DF **Lemma 29.45.5.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is a universal homeomorphism, and*
(2) *f is integral, universally injective and surjective.*

Proof. Assume f is a universal homeomorphism. By Lemma 29.45.4 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 29.44.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 29.44.7 f is universally closed. Since it is also universally bijective (see Lemma 29.9.4) we see that it is a universal homeomorphism. □

054M **Lemma 29.45.6.** *Let X be a scheme. The canonical closed immersion $X_{red} \to X$ (see Schemes, Definition 26.12.5) is a universal homeomorphism.*

Proof. Omitted. □

0896 **Lemma 29.45.7.** *Let $f : X \to S$ and $S' \to S$ be morphisms of schemes. Assume*

(1) *$S' \to S$ is a closed immersion,*
(2) *$S' \to S$ is bijective on points,*
(3) *$X \times_S S' \to S'$ is a closed immersion, and*
(4) *$X \to S$ is of finite type or $S' \to S$ is of finite presentation.*

Then $f : X \to S$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $S' \to S$ is a universal homeomorphism (for example because $S_{red} = S'_{red}$ and using Lemma 29.45.6). Hence (3) implies that $X \to S$ is homeomorphism onto a closed subset of S. Then $X \to S$ is affine by Lemma 29.45.4. Let $U \subset S$ be an affine open, say $U = \mathrm{Spec}(A)$. Then $S' = \mathrm{Spec}(A/I)$ by (1) for a locally nilpotent ideal I by (2). As f is affine we see that $f^{-1}(U) = \mathrm{Spec}(B)$. Assumption (4) tells us B is a finite type A-algebra (Lemma 29.15.2) or that I is finitely generated (Lemma 29.21.7). Assumption (3) is that $A/I \to B/IB$ is surjective. From Algebra, Lemma 10.126.9 if $A \to B$ is of finite type or Algebra, Lemma 10.20.1 if I is finitely generated and hence nilpotent we deduce that $A \to B$ is surjective. This means that f is a closed immersion, see Lemma 29.2.1. □

0H2M **Lemma 29.45.8.** *Let $f : X \to Z$ be the composition of two morphisms $g : X \to Y$ and $h : Y \to Z$. If two of the morphisms $\{f, g, h\}$ are universal homeomorphisms, so is the third morphism.*

Proof. If both of g and h are universal homeomorphisms, so is f by Lemma 29.45.3.

Suppose both of f and g are universal homeomorphisms. We want to show that h is also. Now base change the diagram along an arbitrary morphism $\alpha : Z' \to Z$ of schemes, we get the following diagram with all squares Cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & Y' & \xrightarrow{h'} & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Z. \end{array}$$

Our assumption implies that the composition $f' = h' \circ g' : X' \to Z'$ and $g' : X' \to Y'$ are homeomorphisms, therefore so is h'. This finishes the proof of h being a universal homeomorphism.

Finally, assume f and h are universal homeomorphisms. We want to show that g is a universal homeomorphism. Let $\beta : Y' \to Y$ be an arbitrary morphism of schemes. We get the following diagram with all squares Cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & Y' & & \\ \downarrow & & \downarrow{\scriptstyle \gamma} & & \\ X'' & \xrightarrow{g''} & Y'' & \xrightarrow{h''} & Y' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle h\circ\beta} \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Z. \end{array}$$

Here the morphism $\gamma : Y' \to Y''$ is defined by the universal property of fiber products and the two morphisms $id_{Y'} : Y' \to Y'$ and $\beta : Y' \to Y$. We shall prove that g' is a homeomorphism. Since the property of being a homeomorphism has 2-out-of-3 property, we see that g'' is a homeomorphism. Staring at the top square, it suffices to prove that γ is a universal homeomorphism. Since h'' is a homeomorphism, we see that it is an affine morphism by Lemma 29.45.4 and a fortiori separated (Lemma 29.11.2). Since $h'' \circ \gamma$ is the identity, we see that γ is a closed immersion by Schemes, Lemma 26.21.11. Since h'' is bijective, it follows that γ is a bijective closed immersion and hence a universal homeomorphism (for example by the characterization in Lemma 29.45.5) as desired. □

29.46. Universal homeomorphisms of affine schemes

0CN6 In this section we characterize universal homeomorphisms of affine schemes.

0CN7 **Lemma 29.46.1.** *Let $A \to B$ be a ring map such that the induced morphism of schemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective. Then for any A-subalgebra $B' \subset B$ the same thing is true for $f' : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$.*

Proof. If f is universally closed, then B is integral over A by Lemma 29.44.7. Hence B' is integral over A and f' is universally closed (by the same lemma). This proves the case where f is universally closed.

Continuing, we see that B is integral over B' (Algebra, Lemma 10.36.15) which implies $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is surjective (Algebra, Lemma 10.36.17). Thus if

$A \to B$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B'$. This proves the case where f is universally closed and universally injective, see Lemma 29.10.2.

The case where f is a universal homeomorphism follows from the remarks above, Lemma 29.45.5, and the obvious observation that if f is surjective, then so is f'.

If $A \to B$ induces isomorphisms on residue fields, then so does $A \to B'$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. □

0CN8 **Lemma 29.46.2.** *Let A be a ring. Let $B = \operatorname{colim} B_\lambda$ be a filtered colimit of A-algebras. If each $f_\lambda : \operatorname{Spec}(B_\lambda) \to \operatorname{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective, then the same thing is true for $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$.*

Proof. If f_λ is universally closed, then B_λ is integral over A by Lemma 29.44.7. Hence B is integral over A and f is universally closed (by the same lemma). This proves the case where each f_λ is universally closed.

For a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ denote $\mathfrak{q}_\lambda \subset B_\lambda$ the inverse image. Then $\kappa(\mathfrak{q}) = \operatorname{colim} \kappa(\mathfrak{q}_\lambda)$. Thus if $A \to B_\lambda$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B$. This proves the case where f_λ is universally closed and universally injective, see Lemma 29.10.2.

The case where f is a universal homeomorphism follows from the remarks above and Lemma 29.45.5 combined with the fact that prime ideals in B are the same thing as compatible sequences of prime ideals in all of the B_λ.

If $A \to B_\lambda$ induces isomorphisms on residue fields, then so does $A \to B$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. □

0CN9 **Lemma 29.46.3.** *Let $A \subset B$ be a ring extension. Let $S \subset A$ be a multiplicative subset. Let $n \geq 1$ and $b_i \in B$ for $1 \leq i \leq n$. Any $x \in S^{-1}B$ such that*

$$x \notin S^{-1}A \text{ and } b_i x^i \in S^{-1}A \text{ for } i = 1, \ldots, n$$

is equal to $s^{-1}y$ with $s \in S$ and $y \in B$ such that

$$y \notin A \text{ and } b_i y^i \in A \text{ for } i = 1, \ldots, n$$

Proof. Omitted. Hint: clear denominators. □

0CNA **Lemma 29.46.4.** *Let $A \subset B$ be a ring extension. If there exists $b \in B$, $b \notin A$ and an integer $n \geq 2$ with $b^n \in A$ and $b^{n+1} \in A$, then there exists a $b' \in B$, $b' \notin A$ with $(b')^2 \in A$ and $(b')^3 \in A$.*

Proof. Let b and n be as in the lemma. Then all sufficiently large powers of b are in A. Namely, $(b^n)^k(b^{n+1})^i = b^{(k+i)n+i}$ which implies any power b^m with $m \geq n^2$ is in A. Hence if $i \geq 1$ is the largest integer such that $b^i \notin A$, then $(b^i)^2 \in A$ and $(b^i)^3 \in A$. □

0CNB **Lemma 29.46.5.** *Let $A \subset B$ be a ring extension such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. If $A \neq B$, then there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$.*

Proof. Recall that $A \subset B$ is integral (Lemma 29.44.7). By Lemma 29.46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 10.36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 10.40.5 and 10.40.2). Let $\mathfrak{p} \subset A$ be a minimal prime of the support. After replacing $A \subset B$ by $A_\mathfrak{p} \subset B_\mathfrak{p}$ (permissible by Lemma 29.46.3) we may assume that $(A, \mathfrak{m})$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak{m}\}$ as an A-module. Since B/A is a finite module, we see that $I = \mathrm{Ann}_A(B/A)$ satisfies $\mathfrak{m} = \sqrt{I}$ (Algebra, Lemma 10.40.5). Let $\mathfrak{m}' \subset B$ be the unique prime ideal lying over $\mathfrak{m}$. Because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism, we find that $\mathfrak{m}' = \sqrt{IB}$. For $f \in \mathfrak{m}'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 29.46.4 we conclude our lemma is true if $\mathfrak{m}' \not\subset A$. However, if $\mathfrak{m}' \subset A$, then $\mathfrak{m}' = \mathfrak{m}$ and we conclude that $A = B$ as the residue fields are isomorphic as well by assumption. This contradiction finishes the proof. □

0CNC **Lemma 29.46.6.** *Let $A \subset B$ be a ring extension such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism. If $A \neq B$, then either there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$ or there exists a prime number p and a $b \in B$, $b \notin A$ with $pb \in A$ and $b^p \in A$.*

Proof. The argument is almost exactly the same as in the proof of Lemma 29.46.5 but we write everything out to make sure it works.

Recall that $A \subset B$ is integral (Lemma 29.44.7). By Lemma 29.46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 10.36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 10.40.5 and 10.40.2). Let $\mathfrak{p} \subset A$ be a minimal prime of the support. After replacing $A \subset B$ by $A_\mathfrak{p} \subset B_\mathfrak{p}$ (permissible by Lemma 29.46.3) we may assume that $(A, \mathfrak{m})$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak{m}\}$ as an A-module. Since B/A is a finite module, we see that $I = \mathrm{Ann}_A(B/A)$ satisfies $\mathfrak{m} = \sqrt{I}$ (Algebra, Lemma 10.40.5). Let $\mathfrak{m}' \subset B$ be the unique prime ideal lying over $\mathfrak{m}$. Because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism, we find that $\mathfrak{m}' = \sqrt{IB}$. For $f \in \mathfrak{m}'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 29.46.4 we conclude our lemma is true if $\mathfrak{m}' \not\subset A$. If $\mathfrak{m}' \subset A$, then $\mathfrak{m}' = \mathfrak{m}$. Since $A \neq B$ we conclude the map $\kappa = A/\mathfrak{m} \to B/\mathfrak{m}' = \kappa'$ of residue fields cannot be an isomorphism. By Lemma 29.10.2 we conclude that the characteristic of κ is a prime number p and that the extension κ'/κ is purely inseparable. Pick $b \in B$ whose image in κ' is an element not contained in κ but whose pth power is in κ. Then $b \notin A$, $b^p \in A$, and $pb \in A$ (because $pb \in \mathfrak{m}' = \mathfrak{m} \subset A$) as desired. □

0CND **Proposition 29.46.7.** *Let $A \subset B$ be a ring extension. The following are equivalent*

1. *$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields, and*
2. *every finite subset $E \subset B$ is contained in an extension*

$$A[b_1, \ldots, b_n] \subset B$$

such that $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$ for $i = 1, \ldots, n$.

Proof. Assume (1). Using transfinite recursion we construct for each ordinal α an A-subalgebra $B_\alpha \subset B$ as follows. Set $B_0 = A$. If α is a limit ordinal, then we set $B_\alpha = \operatorname{colim}_{\beta<\alpha} B_\beta$. If $\alpha = \beta+1$, then either $B_\beta = B$ in which case we set $B_\alpha = B_\beta$ or $B_\beta \neq B$, in which case we apply Lemma 29.46.5 to choose a $b_\alpha \in B$, $b_\alpha \notin B_\beta$ with $b_\alpha^2, b_\alpha^3 \in B_\beta$ and we set $B_\alpha = B_\beta[b_\alpha] \subset B$. Clearly, $B = \operatorname{colim} B_\alpha$ (in fact $B = B_\alpha$ for some ordinal α as one sees by looking at cardinalities). We will prove, by transfinite induction, that (2) holds for $A \to B_\alpha$ for every ordinal α. It is clear for $\alpha = 0$. Assume the statement holds for every $\beta < \alpha$ and let $E \subset B_\alpha$ be a finite subset. If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and we see that $E \subset B_\beta$ for some $\beta < \alpha$ which proves the result in this case. If $\alpha = \beta + 1$, then $B_\alpha = B_\beta[b_\alpha]$. Thus any $e \in E$ can be written as a polynomial $e = \sum d_{e,i} b_\alpha^i$ with $d_{e,i} \in B_\beta$. Let $D \subset B_\beta$ be the set $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$. By induction assumption there exists an A-subalgebra $A[b_1, \ldots, b_n] \subset B_\beta$ as in the statement of the lemma containing D. Then $A[b_1, \ldots, b_n, b_\alpha] \subset B_\alpha$ is an A-subalgebra of B_α as in the statement of the lemma containing E.

Assume (2). Write $B = \operatorname{colim} B_\lambda$ as the colimit of its finite A-subalgebras. By Lemma 29.46.2 it suffices to show that $\operatorname{Spec}(B_\lambda) \to \operatorname{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. Compositions of universally closed morphisms are universally closed and the same thing for morphisms which induce isomorphisms on residue fields. Thus it suffices to show that if $A \subset B$ and B is generated by a single element b with $b^2, b^3 \in A$, then (1) holds. Such an extension is integral and hence $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is universally closed and surjective (Lemma 29.44.7 and Algebra, Lemma 10.36.17). Note that $(b^2)^3 = (b^3)^2$ in A. For any ring map $\varphi : A \to K$ to a field K we see that there exists a $\lambda \in K$ with $\varphi(b^2) = \lambda^2$ and $\varphi(b^3) = \lambda^3$. Namely, $\lambda = 0$ if $\varphi(b^2) = 0$ and $\lambda = \varphi(b^3)/\varphi(b^2)$ if not. Thus $B \otimes_A K$ is a quotient of $K[x]/(x^2 - \lambda^2, x^3 - \lambda^3)$. This ring has exactly one prime with residue field K. This implies that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is bijective and induces isomorphisms on residue fields. Combined with universal closedness this shows (1) is true, see Lemmas 29.45.5 and 29.10.2. □

0CNE **Proposition 29.46.8.** *Let $A \subset B$ be a ring extension. The following are equivalent*

(1) $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is a universal homeomorphism, and*
(2) *every finite subset $E \subset B$ is contained in an extension*

$$A[b_1, \ldots, b_n] \subset B$$

such that for $i = 1, \ldots, n$ we have
   (a) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, *or*
   (b) *there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.*

Proof. The proof is exactly the same as the proof of Proposition 29.46.7 except for the following changes:

(1) Use Lemma 29.46.6 instead of Lemma 29.46.5 which means that for each successor ordinal $\alpha = \beta + 1$ we either have $b_\alpha^2, b_\alpha^3 \in B_\beta$ or we have a prime p and $pb_\alpha, b_\alpha^p \in B_\beta$.
(2) If α is a successor ordinal, then take $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$ or take $D = \{d_{e,i}\} \cup \{pb_\alpha, b_\alpha^p\}$ depending on which case α falls into.
(3) In the proof of (2) $\Rightarrow$ (1) we also need to consider the case where B is generated over A by a single element b with $pb, b^p \in B$ for some prime

number p. Here $A \subset B$ induces a universal homeomorphism on spectra for example by Algebra, Lemma 10.46.7.

This finishes the proof. □

0CNF **Lemma 29.46.9.** *Let p be a prime number. Let $A \to B$ be a ring map which induces an isomorphism $A[1/p] \to B[1/p]$ (for example if p is nilpotent in A). The following are equivalent*

1. *$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism, and*
2. *the kernel of $A \to B$ is a locally nilpotent ideal and for every $b \in B$ there exists a p-power q with qb and b^q in the image of $A \to B$.*

Proof. If (2) holds, then (1) holds by Algebra, Lemma 10.46.7. Assume (1). Then the kernel of $A \to B$ consists of nilpotent elements by Algebra, Lemma 10.30.6. Thus we may replace A by the image of $A \to B$ and assume that $A \subset B$. By Algebra, Lemma 10.46.5 the set

$$B' = \{b \in B \mid p^n b, b^{p^n} \in A \text{ for some } n \geq 0\}$$

is an A-subalgebra of B (being closed under products is trivial). We have to show $B' = B$. If not, then according to Lemma 29.46.6 there exists a $b \in B$, $b \notin B'$ with either $b^2, b^3 \in B'$ or there exists a prime number ℓ with $\ell b, b^\ell \in B'$. We will show both cases lead to a contradiction, thereby proving the lemma.

Since $A[1/p] = B[1/p]$ we can choose a p-power q such that $qb \in A$.

If $b^2, b^3 \in B'$ then also $b^q \in B'$. By definition of B' we find that $(b^q)^{q'} \in A$ for some p-power q'. Then $qq'b, b^{qq'} \in A$ whence $b \in B'$ which is a contradiction.

Assume now there exists a prime number ℓ with $\ell b, b^\ell \in B'$. If $\ell \neq p$ then $\ell b \in B'$ and $qb \in A \subset B'$ imply $b \in B'$ a contradiction. Thus $\ell = p$ and $b^p \in B'$ and we get a contradiction exactly as before. □

0EUI **Lemma 29.46.10.** *Let A be a ring. Let $x, y \in A$.*

1. *If $x^3 = y^2$ in A, then $A \to B = A[t]/(t^2 - x, t^3 - y)$ induces bijections on residue fields and a universal homeomorphism on spectra.*
2. *If there is a prime number p such that $p^p x = y^p$ in A, then $A \to B = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra.*

Proof. We will use the criterion of Lemma 29.45.5 to check this. In both cases the ring map is integral. Thus it suffices to show that given a field k and a ring map $\varphi : A \to k$ the k-algebra $B \otimes_A k$ has a unique prime ideal whose residue field is equal to k in case (1) and purely inseparable over k in case (2). See Lemma 29.10.2.

In case (1) set $\lambda = 0$ if $\varphi(x) = 0$ and set $\lambda = \varphi(y)/\varphi(x)$ if not. Then $B = k[t]/(t^2 - \lambda^2, t^3 - \lambda^2)$. Thus the result is clear.

In case (2) if the characteristic of k is p, then we obtain $\varphi(y) = 0$ and $B = k[t]/(t^p - \varphi(x))$ which is a local Artinian k-algebra whose residue field is either k or a degree p purely inseparable extension of k. If the characteristic of k is not p, then setting $\lambda = \varphi(y)/p$ we see $B = k[t]/(t - \lambda) = k$ and we conclude as well. □

0EUJ **Lemma 29.46.11.** *Let $A \to B$ be a ring map.*

1. *If $A \to B$ induces a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces a universal homeomorphism on spectra.*

(2) *If $A \to B$ induces isomorphisms on residue fields and a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces isomorphisms on residue fields and a universal homeomorphism on spectra.*

Proof. Proof of (1). We will use the criterion of Algebra, Lemma 10.127.4. Let $A \to C$ be of finite presentation and let $\varphi : C \to B$ be an A-algebra map. Let $B' = \varphi(C) \subset B$ be the image. Then $A \to B'$ induces a universal homeomorphism on spectra by Lemma 29.46.1. By Algebra, Lemma 10.127.2 we can write $B' = \operatorname{colim}_{i \in I} B_i$ with $A \to B_i$ of finite presentation and surjective transition maps. By Algebra, Lemma 10.127.3 we can choose an index $0 \in I$ and a factorization $C \to B_0 \to B'$ of the map $C \to B'$. We claim that $\operatorname{Spec}(B_i) \to \operatorname{Spec}(A)$ is a universal homeomorphism for i sufficiently large. The claim finishes the proof of (1).

Proof of the claim. By Lemma 29.45.6 the ring map $A_{red} \to B'_{red}$ induces a universal homeomorphism on spectra. Thus $A_{red} \subset B'_{red}$ by Algebra, Lemma 10.30.6. Setting $A' = \operatorname{Im}(A \to B')$ we have surjections $A \to A' \to A_{red}$ inducing bijections $\operatorname{Spec}(A_{red}) = \operatorname{Spec}(A') = \operatorname{Spec}(A)$. Thus $A' \subset B'$ induces a universal homeomorphism on spectra. By Proposition 29.46.8 and the fact that B' is finite type over A' we can find n and $b'_1, \ldots, b'_n \in B'$ such that $B' = A'[b'_1, \ldots, b'_n]$ and such that for $j = 1, \ldots, n$ we have

(1) $(b'_j)^2, (b'_j)^3 \in A'[b'_1, \ldots, b'_{j-1}]$, or
(2) there exists a prime number p with $pb'_j, (b'_j)^p \in A'[b'_1, \ldots, b'_{j-1}]$.

Choose $b_1, \ldots, b_n \in B_0$ lifting $b'_1, \ldots, b'_n$. For $i \geq 0$ denote $b_{j,i}$ the image of b_j in B_i. For large enough i we will have for $j = 1, \ldots, n$

(1) $b_{j,i}^2, b_{j,i}^3 \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$, or
(2) there exists a prime number p with $pb_{j,i}, b_{j,i}^p \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$.

Here $A_i \subset B_i$ is the image of $A \to B_i$. Observe that $A \to A_i$ is a surjective ring map whose kernel is a locally nilpotent ideal. After increasing i more if necessary, we may assume B_i is generated by $b_1, \ldots, b_n$ over A_i, in other words $B_i = A_i[b_1, \ldots, b_n]$. By Algebra, Lemmas 10.46.7 and 10.46.4 we conclude that $A \to A_i \to A_i[b_1] \to \ldots \to A_i[b_1, \ldots, b_n] = B_i$ induce universal homeomorphisms on spectra. This finishes the proof of the claim.

The proof of (2) is exactly the same. □

29.47. Absolute weak normalization and seminormalization

0EUK Motivated by the results proved in the previous section we give the following definition.

0EUL **Definition 29.47.1.** Let A be a ring.

(1) We say A is *seminormal* if for all $x, y \in A$ with $x^3 = y^2$ there is a unique $a \in A$ with $x = a^2$ and $y = a^3$.
(2) We say A is *absolutely weakly normal* if (a) A is seminormal and (b) for any prime number p and $x, y \in A$ with $p^p x = y^p$ there is a unique $a \in A$ with $x = a^p$ and $y = pa$.

An amusing observation, see [**Cos82**], is that in the definition of seminormal rings it suffices[15] to assume the existence of a. Absolutely weakly normal schemes were defined in [**Ryd07b**, Appendix B].

0EUM **Lemma 29.47.2.** *Being seminormal or being absolutely weakly normal is a local property of rings, see Properties, Definition 28.4.1.*

Proof. Suppose that A is seminormal and $f \in A$. Let $x', y' \in A_f$ with $(x')^3 = (y')^2$. Write $x' = x/f^{2n}$ and $y' = y/f^{3n}$ for some $n \geq 0$ and $x, y \in A$. After replacing x, y by $f^{2m}x, f^{3m}y$ and n by $n + m$, we see that $x^3 = y^2$ in A. Then we find a unique $a \in A$ with $x = a^2$ and $y = a^3$. Setting $a' = a/f^n$ we get $x' = (a')^2$ and $y' = (a')^3$ as desired. Uniqueness of a' follows from uniqueness of a. In exactly the same manner the reader shows that if A is absolutely weakly normal, then A_f is absolutely weakly normal.

Assume A is a ring and $f_1, \ldots, f_n \in A$ generate the unit ideal. Assume A_{f_i} is seminormal for each i. Let $x, y \in A$ with $x^3 = y^2$. For each i we find a unique $a_i \in A_{f_i}$ with $x = a_i^2$ and $y = a_i^3$ in A_{f_i}. By the uniqueness and the result of the first paragraph (which tells us that $A_{f_if_j}$ is seminormal) we see that a_i and a_j map to the same element of $A_{f_if_j}$. By Algebra, Lemma 10.24.2 we find a unique $a \in A$ mapping to a_i in A_{f_i} for all i. Then $x = a^2$ and $y = a^3$ by the same token. Clearly this a is unique. Thus A is seminormal. If we assume A_{f_i} is absolutely weakly normal, then the exact same argument shows that A is absolutely weakly normal. □

Next we define seminormal schemes and absolutely weakly normal schemes.

0EUN **Definition 29.47.3.** Let X be a scheme.

(1) We say X is *seminormal* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is seminormal.
(2) We say X is *absolutely weakly normal* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is absolutely weakly normal.

Here is the obligatory lemma.

0EUP **Lemma 29.47.4.** *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is seminormal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is seminormal.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is seminormal.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is seminormal.*

Moreover, if X is seminormal then every open subscheme is seminormal. The same statements are true with "seminormal" replaced by "absolutely weakly normal".

Proof. Combine Properties, Lemma 28.4.3 and Lemma 29.47.2. □

[15] Let A be a ring such that for all $x, y \in A$ with $x^3 = y^2$ there is an $a \in A$ with $x = a^2$ and $y = a^3$. Then A is reduced: if $x^2 = 0$, then $x^2 = x^3$ and hence there exists an a such that $x = a^3$ and $x = a^2$. Then $x = a^3 = ax = a^4 = x^2 = 0$. Finally, if $a_1^2 = a_2^2$ and $a_1^3 = a_2^3$ for a_1, a_2 in a reduced ring, then $(a_1 - a_2)^3 = a_1^3 - 3a_1^2a_2 + 3a_1a_2^2 - a_2^3 = (1 - 3 + 3 - 1)a_1^3 = 0$ and hence $a_1 = a_2$.

0EUQ **Lemma 29.47.5.** *A seminormal scheme or ring is reduced. A fortiori the same is true for absolutely weakly normal schemes or rings.*

Proof. Let A be a ring. If $a \in A$ is nonzero but $a^2 = 0$, then $a^2 = 0^2$ and $a^3 = 0^3$ and hence A is not seminormal. □

0EUR **Lemma 29.47.6.** *Let A be a ring.*

(1) *The category of ring maps $A \to B$ inducing a universal homeomorphism on spectra has a final object $A \to A^{awn}$.*
(2) *Given $A \to B$ in the category of (1) the resulting map $B \to A^{awn}$ is an isomorphism if and only if B is absolutely weakly normal.*
(3) *The category of ring maps $A \to B$ inducing isomorphisms on residue fields and a universal homeomorphism on spectra has a final object $A \to A^{sn}$.*
(4) *Given $A \to B$ in the category of (3) the resulting map $B \to A^{sn}$ is an isomorphism if and only if B is seminormal.*

For any ring map $\varphi : A \to A'$ there are unique maps $\varphi^{awn} : A^{awn} \to (A')^{awn}$ and $\varphi^{sn} : A^{sn} \to (A')^{sn}$ compatible with φ.

Proof. We prove (1) and (2) and we omit the proof of (3) and (4) and the final statement. Consider the category of A-algebras of the form

$$B = A[x_1, \ldots, x_n]/J$$

where J is a finitely generated ideal such that $A \to B$ defines a universal homeomorphism on spectra. We claim this category is directed (Categories, Definition 4.19.1). Namely, given

$$B = A[x_1, \ldots, x_n]/J \quad \text{and} \quad B' = A[x_1, \ldots, x_{n'}]/J'$$

then we can consider

$$B'' = A[x_1, \ldots, x_{n+n'}]/J''$$

where J'' is generated by the elements of J and the elements $f(x_{n+1}, \ldots, x_{n+n'})$ where $f \in J'$. Then we have A-algebra homomorphisms $B \to B''$ and $B' \to B''$ which induce an isomorphism $B \otimes_A B' \to B''$. It follows from Lemmas 29.45.2 and 29.45.3 that $\mathrm{Spec}(B'') \to \mathrm{Spec}(A)$ is a universal homeomorphism and hence $A \to B''$ is in our category. Finally, given $\varphi, \varphi' : B \to B'$ in our category with B as displayed above, then we consider the quotient B'' of B' by the ideal generated by $\varphi(x_i) - \varphi'(x_i)$, $i = 1, \ldots, n$. Since $\mathrm{Spec}(B') = \mathrm{Spec}(B)$ we see that $\mathrm{Spec}(B'') \to \mathrm{Spec}(B')$ is a bijective closed immersion hence a universal homeomorphism. Thus B'' is in our category and φ, φ' are equalized by $B' \to B''$. This completes the proof of our claim. We set

$$A^{awn} = \mathrm{colim}\, B$$

where the colimit is over the category just described. Observe that $A \to A^{awn}$ induces a universal homeomorphism on spectra by Lemma 29.46.2 (this is where we use the category is directed).

Given a ring map $A \to B$ of finite presentation inducing a universal homeomorphism on spectra, we get a canonical map $B \to A^{awn}$ by the very construction of A^{awn}. Since every $A \to B$ as in (1) is a filtered colimit of $A \to B$ as in (1) of finite presentation (Lemma 29.46.11), we see that $A \to A^{awn}$ is final in the category (1).

Let $x, y \in A^{awn}$ be elements such that $x^3 = y^2$. Then $A^{awn} \to A^{awn}[t]/(t^2 - x, t^3 - y)$ induces a universal homeomorphism on spectra by Lemma 29.46.10. Thus

$A \to A^{awn}[t]/(t^2 - x, t^3 - y)$ is in the category (1) and we obtain a unique A-algebra map $A^{awn}[t]/(t^2 - x, t^3 - y) \to A^{awn}$. The image $a \in A^{awn}$ of t is therefore the unique element such that $a^2 = x$ and $a^3 = y$ in A^{awn}. In exactly the same manner, given a prime p and $x, y \in A^{awn}$ with $p^p x = y^p$ we find a unique $a \in A^{awn}$ with $a^p = x$ and $pq = y$. Thus A^{awn} is absolutely weakly normal by definition.

Finally, let $A \to B$ be in the category (1) with B absolutely weakly normal. Since $A^{awn} \to B^{awn}$ induces a universal homeomorphism on spectra and since A^{awn} is reduced (Lemma 29.47.5) we find $A^{awn} \subset B^{awn}$ (see Algebra, Lemma 10.30.6). If this inclusion is not an equality, then Lemma 29.46.6 implies there is an element $b \in B^{awn}$, $b \notin A^{awn}$ such that either $b^2, b^3 \in A^{awn}$ or $pb, b^p \in A^{awn}$ for some prime number p. However, by the existence and uniqueness in Definition 29.47.1 this forces $b \in A^{awn}$ and hence we obtain the contradiction that finishes the proof. □

0EUS **Lemma 29.47.7.** *Let X be a scheme.*

(1) *The category of universal homeomorphisms $Y \to X$ has an initial object $X^{awn} \to X$.*
(2) *Given $Y \to X$ in the category of (1) the resulting morphism $X^{awn} \to Y$ is an isomorphism if and only if Y is absolutely weakly normal.*
(3) *The category of universal homeomorphisms $Y \to X$ which induce ismomorphisms on residue fields has an initial object $X^{sn} \to X$.*
(4) *Given $Y \to X$ in the category of (3) the resulting morphism $X^{sn} \to Y$ is an isomorphism if and only if Y is seminormal.*

For any morphism $h : X' \to X$ of schemes there are unique morphisms $h^{awn} : (X')^{awn} \to X^{awn}$ and $h^{sn} : (X')^{sn} \to X^{sn}$ compatible with h.

Proof. We will prove (1) and (2) and omit the proof of (3) and (4). Let $h : X' \to X$ be a morphism of schemes. If (1) holds for X and X', then $X' \times_X X^{awn} \to X'$ is a universal homeomorphism and hence we get a unique morphism $(X')^{awn} \to X' \times_X X^{awn}$ over X' by the universal property of $(X')^{awn} \to X'$. Composed with the projection $X' \times_X X^{awn} \to X^{awn}$ we obtain h^{awn}. If in addition (2) holds for X and X' and h is an open immersion, then $X' \times_X X^{awn}$ is absolutely weakly normal (Lemma 29.47.4) and we deduce that $(X')^{awn} \to X' \times_X X^{awn}$ is an isomorphism.

Recall that any universal homeomorphism is affine, see Lemma 29.45.4. Thus if X is affine then (1) and (2) follow immediately from Lemma 29.47.6. Let X be a scheme and let $\mathcal{B}$ be the set of affine opens of X. For each $U \in \mathcal{B}$ we obtain $U^{awn} \to U$ and for $V \subset U$, $V, U \in \mathcal{B}$ we obtain a canonical isomorphism $\rho_{V,U} : V^{awn} \to V \times_U U^{awn}$ by the discussion in the previous paragraph. Thus by relative glueing (Constructions, Lemma 27.2.1) we obtain a morphism $X^{awn} \to X$ which restricts to U^{awn} over U compatibly with the $\rho_{V,U}$. Next, let $Y \to X$ be a universal homeomorphism. Then $U \times_X Y \to U$ is a universal homeomorphism for $U \in \mathcal{B}$ and we obtain a unique morphism $g_U : U^{awn} \to U \times_X Y$ over U. These g_U are compatible with the morphisms $\rho_{V,U}$; details omitted. Hence there is a unique morphism $g : X^{awn} \to Y$ over X agreeing with g_U over U, see Constructions, Remark 27.2.3. This proves (1) for X. Part (2) follows because it holds affine locally. □

0EUT **Definition 29.47.8.** Let X be a scheme.

(1) The morphism $X^{sn} \to X$ constructed in Lemma 29.47.7 is the *seminormalization* of X.

(2) The morphism $X^{awn} \to X$ constructed in Lemma 29.47.7 is the *absolute weak normalization* of X.

To be sure, the seminormalization X^{sn} of X is a seminormal scheme and the absolute weak normalization X^{awn} is an absolutely weakly normal scheme. Moreover, for any morphism $h : Y \to X$ of schemes we obtain a canonical commutative diagram

$$\begin{array}{ccccc} Y^{awn} & \longrightarrow & Y^{sn} & \longrightarrow & Y \\ \downarrow{\scriptstyle h^{awn}} & & \downarrow{\scriptstyle h^{sn}} & & \downarrow{\scriptstyle h} \\ X^{awn} & \longrightarrow & X^{sn} & \longrightarrow & X \end{array}$$

of schemes; the arrows h^{sn} and h^{awn} are the unique ones compatible with h.

0H3G **Lemma 29.47.9.** *Let X be a scheme. The following are equivalent*

(1) *X is seminormal,*
(2) *X is equal to its own seminormalization, i.e., the morphism $X^{sn} \to X$ is an isomorphism,*
(3) *if $\pi : Y \to X$ is a universal homeomorphism inducing isomorphisms on residue fields with Y reduced, then π is an isomorphism.*

Proof. The equivalence of (1) and (2) is clear from Lemma 29.47.7. If (3) holds, then $X^{sn} \to X$ is an isomorphism and we see that (2) holds.

Assume (2) holds and let $\pi : Y \to X$ be a universal homeomorphism inducing isomorphisms on residue fields with Y reduced. Then there exists a factorization $X \to Y \to X$ of id_X by Lemma 29.47.7. Then $X \to Y$ is a closed immersion (by Schemes, Lemma 26.21.11 and the fact that π is separated for example by Lemma 29.10.3). Since $X \to Y$ is also a bijection on points, the reducedness of Y shows that it has to be an isomorphism. This finishes the proof. □

0H3H **Lemma 29.47.10.** *Let X be a scheme. The following are equivalent*

(1) *X is absolutely weakly normal,*
(2) *X is equal to its own absolute weak normalization, i.e., the morphism $X^{awn} \to X$ is an isomorphism,*
(3) *if $\pi : Y \to X$ is a universal homeomorphism with Y reduced, then π is an isomorphism.*

Proof. This is proved in exactly the same manner as Lemma 29.47.9. □

29.48. Finite locally free morphisms

02K9 In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 111.5.3.

02KA **Definition 29.48.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d.

Note that if $f : X \to S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \to S_d$ is finite locally free of degree d.

02KB **Lemma 29.48.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If S is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. Let $V \subset S$ be affine open. In all three cases the morphism is affine hence $f^{-1}(V)$ is affine. Thus we may write $V = \mathrm{Spec}(R)$ and $f^{-1}(V) = \mathrm{Spec}(A)$ for some R-algebra A. Assume (1). This means we can cover S by affine opens $V = \mathrm{Spec}(R)$ such that A is finite free as an R-module. Then $R \to A$ is of finite presentation by Algebra, Lemma 10.7.4. Thus (2) holds. Conversely, assume (2). For every affine open $V = \mathrm{Spec}(R)$ of S the ring map $R \to A$ is finite and of finite presentation and A is flat as an R-module. By Algebra, Lemma 10.36.23 we see that A is finitely presented as an R-module. Thus Algebra, Lemma 10.78.2 implies A is finite locally free. Thus (1) holds. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 10.31.4. □

02KC **Lemma 29.48.3.** *A composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

02KD **Lemma 29.48.4.** *A base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

04MH **Lemma 29.48.5.** *Let $f : X \to S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \to S_d$ of degree d.*

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omitted. □

03HW **Lemma 29.48.6.** *Let $f : Y \to X$ be a finite morphism with X affine. There exists a diagram*

$$\begin{array}{ccccc} Z' & \xleftarrow[i]{} & Y' & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow \\ & & X' & \longrightarrow & X \end{array}$$

where

(1) *$Y' \to Y$ and $X' \to X$ are surjective finite locally free,*
(2) *$Y' = X' \times_X Y$,*
(3) *$i : Y' \to Z'$ is a closed immersion,*
(4) *$Z' \to X'$ is finite locally free, and*
(5) *$Z' = \bigcup_{j=1,\ldots,m} Z'_j$ is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to X'.*

Proof. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. See also More on Algebra, Section 15.21. Let $x_1, \ldots, x_n \in B$ be generators of B over A. For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B. By Algebra, Lemma

10.136.14 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{k=1,\ldots,d_i} (T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \to B'$, $T_i \mapsto 1 \otimes x_i$ is an A'-algebra surjection. Setting $X' = \mathrm{Spec}(A')$, $Y' = \mathrm{Spec}(B')$ and $Z' = \mathrm{Spec}(C)$ we see that (1) – (4) hold. Part (5) holds because set theoretically $\mathrm{Spec}(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \ldots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. □

The following lemma is stated in the correct generality in Lemma 29.56.4 below.

03HX **Lemma 29.48.7.** *Let $f : Y \to X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a closed nowhere dense subset of X.*

Proof. By Lemma 29.44.11 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y, we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 5.21.4.

Assume X is affine. Choose a diagram

$$\begin{array}{ccccc} Z' & \xleftarrow{i} & Y' & \xrightarrow{a} & Y \\ & \searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & X' & \xrightarrow{b} & X \end{array}$$

as in Lemma 29.48.6. The morphisms a, b are open since they are finite locally free (Lemmas 29.48.2 and 29.25.10). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 5.21.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 5.21.6. As i is a closed immersion, by Topology, Lemma 5.21.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X. Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i, then each $f(T_i)$ is nowhere dense in X as $Y_i \to X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 5.21.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V. In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2. In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2. Repeat sequentially with $i = 3, \ldots, n$. The result is a disjoint union decomposition

$$\{1, \ldots, n\} = I_1 \amalg I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \to X$ is an isomorphism. Then

$$f^{-1}(U) = V \amalg \bigcup\nolimits_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y, as desired. □

29.49. Rational maps

01RR Let X be a scheme. Note that if U, V are dense open in X, then so is $U \cap V$.

01RS **Definition 29.49.1.** Let X, Y be schemes.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of schemes defined on dense open subsets U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an *S-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is an S-morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In some cases rational maps are determined by maps on local rings at generic points.

0BX8 **Lemma 29.49.2.** *Let S be a scheme. Let X and Y be schemes over S. Assume X has finitely many irreducible components with generic points $x_1, \ldots, x_n$. Let $s_i \in S$ be the image of x_i. Consider the map*

$$\left\{\begin{array}{c} S\text{-rational maps} \\ \text{from } X \text{ to } Y \end{array}\right\} \longrightarrow \left\{\begin{array}{l} (y_1, \varphi_1, \ldots, y_n, \varphi_n) \text{ where } y_i \in Y \text{ lies over } s_i \text{ and} \\ \varphi_i : \mathcal{O}_{Y,y_i} \to \mathcal{O}_{X,x_i} \text{ is a local } \mathcal{O}_{S,s_i}\text{-algebra map} \end{array}\right\}$$

which sends $f : U \to Y$ to the $2n$-tuple with $y_i = f(x_i)$ and $\varphi_i = f^\sharp_{x_i}$. Then

(1) *If $Y \to S$ is locally of finite type, then the map is injective.*
(2) *If $Y \to S$ is locally of finite presentation, then the map is bijective.*
(3) *If $Y \to S$ is locally of finite type and X reduced, then the map is bijective.*

Proof. Observe that any dense open of X contains the points x_i so the construction makes sense. To prove (1) or (2) we may replace X by any dense open. Thus if $Z_1, \ldots, Z_n$ are the irreducible components of X, then we may replace X by $X \setminus \bigcup_{i \neq j} Z_i \cap Z_j$. After doing this X is the disjoint union of its irreducible components (viewed as open and closed subschemes). Then both the right hand side and the left hand side of the arrow are products over the irreducible components and we reduce to the case where X is irreducible.

Assume X is irreducible with generic point x lying over $s \in S$. Part (1) follows from part (1) of Lemma 29.42.4. Parts (2) and (3) follow from part (2) of the same lemma. □

01RT **Definition 29.49.3.** Let X be a scheme. A *rational function on X* is a rational map from X to $\mathbf{A}^1_{\mathbf{Z}}$.

See Constructions, Definition 27.5.1 for the definition of the affine line $\mathbf{A}^1$. Let X be a scheme over S. For any open $U \subset X$ a morphism $U \to \mathbf{A}^1_{\mathbf{Z}}$ is the same as a morphism $U \to \mathbf{A}^1_S$ over S. Hence a rational function is also the same as a S-rational map from X into $\mathbf{A}^1_S$.

Recall that we have the canonical identification $\mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$ for any scheme T, see Schemes, Example 26.15.2. Hence $\mathbf{A}^1_{\mathbf{Z}}$ is a ring-object in the category of schemes. More precisely, the morphisms

$$\begin{aligned} + : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} &\longrightarrow \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) &\longmapsto f + g \\ * : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} &\longrightarrow \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) &\longmapsto fg \end{aligned}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_{\mathbf{Z}}$ has a natural ring structure.

01RU **Definition 29.49.4.** Let X be a scheme. The *ring of rational functions on* X is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

For schemes with finitely many irreducible components we can compute this.

01RV **Lemma 29.49.5.** *Let X be a scheme with finitely many irreducible components $X_1, \ldots, X_n$. If $\eta_i \in X_i$ is the generic point, then*

$$R(X) = \mathcal{O}_{X,\eta_1} \times \ldots \times \mathcal{O}_{X,\eta_n}$$

If X is reduced this is equal to $\prod \kappa(\eta_i)$. If X is integral then $R(X) = \mathcal{O}_{X,\eta} = \kappa(\eta)$ is a field.

Proof. Let $U \subset X$ be an open dense subset. Then $U_i = (U \cap X_i) \setminus (\bigcup_{j \neq i} X_j)$ is nonempty open as it contained η_i, contained in X_i, and $\bigcup U_i \subset U \subset X$ is dense. Thus the identification in the lemma comes from the string of equalities

$$\begin{aligned} R(X) &= \mathrm{colim}_{U \subset X \text{ open dense}} \mathrm{Mor}(U, \mathbf{A}^1_{\mathbf{Z}}) \\ &= \mathrm{colim}_{U \subset X \text{ open dense}} \mathcal{O}_X(U) \\ &= \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \prod \mathcal{O}_X(U_i) \\ &= \prod \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \mathcal{O}_X(U_i) \\ &= \prod \mathcal{O}_{X,\eta_i} \end{aligned}$$

where the second equality is Schemes, Example 26.15.2. The final statement follows from Algebra, Lemma 10.25.1. □

01RW **Definition 29.49.6.** Let X be an integral scheme. The *function field*, or the *field of rational functions* of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$. We can use the notion of the function field to elucidate the separation condition on an integral scheme. Note that by Lemma 29.49.5 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

02NF **Lemma 29.49.7.** *Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X. Let η_i be the generic point of Z_i. If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x, there exists a function regular in a neighborhood of x which is not in $\mathcal{O}_{X,\eta_2}$.*

Proof. First observe that under the assumption of X being separated, there is a unique map of schemes $\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$ over X such that the composition

$$\mathrm{Spec}(R(X)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $\mathrm{Spec}(R(X)) \to X$. Namely, there is the canonical map $can : \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$, see Schemes, Equation (26.13.1.1). Given a second morphism a to X, we have that a agrees with can on the generic point of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ by assumption. Now X being separated guarantees that the subset in $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 26.21.5. Hence $a = can$ on all of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to can. This implies that η_2 specializes to η_1 (see Schemes, Lemma 26.13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

0A1X **Definition 29.49.8.** Let φ be a rational map between two schemes X and Y. We say φ is *defined in a point* $x \in X$ if there exists a representative (U, f) of φ with $x \in U$. The *domain of definition* of φ is the set of all points where φ is defined.

With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

0A1Y **Lemma 29.49.9.** *Let X and Y be schemes. Assume X reduced and Y separated. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ representing φ. If X and Y are schemes over a separated scheme S and if φ is an S-rational map, then f is a morphism over S.*

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subscheme $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V\cap V'}$ and $g'|_{V\cap V'}$ is a closed subscheme of $V \cap V'$ (Schemes, Lemma 26.21.5). Now $W \subset E$ implies that $E = V \cap V'$ set theoretically. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V\cap V'} = g'|_{V\cap V'}$. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$, see Schemes, Lemma 26.14.1. We omit the proof of the final statement. □

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our schemes are irreducible and we look at dominant rational maps, then we can compose rational maps.

0A1Z **Definition 29.49.10.** Let X and Y be irreducible schemes. A rational map from X to Y is called *dominant* if any representative $f : U \to Y$ is a dominant morphism of schemes.

By Lemma 29.8.7 it is equivalent to require that the generic point $\eta \in X$ maps to the generic point ξ of Y, i.e., $f(\eta) = \xi$ for any representative $f : U \to Y$. We can compose a dominant rational map φ between irreducible schemes X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $U \subset X$ open dense and $g : V \to Z$ with $V \subset Y$ open dense. Then $W = f^{-1}(V) \subset X$ is open nonempty (because it contains the generic point of X) and we let $\psi \circ \varphi$ be the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible schemes and whose morphisms are dominant rational maps. Given a base scheme S we can similarly define a category whose objects are irreducible schemes over S and whose morphisms are dominant S-rational maps.

0A20 **Definition 29.49.11.** Let X and Y be irreducible schemes.

(1) We say X and Y are *birational* if X and Y are isomorphic in the category of irreducible schemes and dominant rational maps.
(2) Assume X and Y are schemes over a base scheme S. We say X and Y are *S-birational* if X and Y are isomorphic in the category of irreducible schemes over S and dominant S-rational maps.

If X and Y are birational irreducible schemes, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all schemes Z (functorially in Z). For "general" irreducible schemes this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions. For varieties these conditions are equivalent, see Lemma 29.50.6.

0BAA **Lemma 29.49.12.** *Let X and Y be irreducible schemes.*

(1) *The schemes X and Y are birational if and only if they have isomorphic nonempty opens.*
(2) *Assume X and Y are schemes over a base scheme S. Then X and Y are S-birational if and only if there are nonempty opens $U \subset X$ and $V \subset Y$ which are S-isomorphic.*

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. We may assume V affine. We may replace U by an affine open of $f^{-1}(V)$. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller affine open we may assume that $U \to V \to X$ is the inclusion of U into X. It follows that $U \to V$ is an immersion (apply Schemes, Lemma 26.21.11 to $U \to g^{-1}(U) \to U$). However, switching the roles of U and V and redoing the argument above, we see that there exists a nonempty affine open $V' \subset V$ such that the inclusion factors as $V' \to U \to V$. Then $V' \to U$ is necessarily an open immersion. Namely, $V' \to f^{-1}(V') \to V'$ are monomorphisms (Schemes, Lemma 26.23.8) composing to the identity, hence isomorphisms. Thus V' is isomorphic to an open of both X and Y. In the S-rational maps case, the exact same argument works. □

0BX9 **Remark 29.49.13.** Here is a generalization of the category of irreducible schemes and dominant rational maps. For a scheme X denote X^0 the set of points $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 0$, in other words, X^0 is the set of generic points of irreducible components of X. Then we can consider the category with

(1) objects are schemes X such that every quasi-compact open has finitely many irreducible components, and
(2) morphisms from X to Y are rational maps $f : U \to Y$ from X to Y such that $f(U^0) = Y^0$.

If $U \subset X$ is a dense open of a scheme, then $U^0 \subset X^0$ need not be an equality, but if X is an object of our category, then this is the case. Thus given two morphisms in our category, the composition is well defined and a morphism in our category.

01RX **Remark 29.49.14.** There is a variant of Definition 29.49.1 where we consider only those morphism $U \to Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 29.7.6 to see that we obtain an equivalence relation. An equivalence class of these is called a *pseudo-morphism from X to Y*. If X is reduced the two notions coincide.

29.50. Birational morphisms

01RN You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 29.50.4. Here is the formal definition.

01RO **Definition 29.50.1.** Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if [**GD60**, (2.2.9)]

(1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
(2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

We will see below that the fibres of a birational morphism over generic points are singletons. Moreover, we will see that in most cases one encounters in practice the existence of a birational morphism between irreducible schemes X and Y implies X and Y are birational schemes.

01RP **Lemma 29.50.2.** *Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows from Lemma 29.8.2 and the definition. □

0BAB **Lemma 29.50.3.** *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.*

Proof. We may assume $Y = \mathrm{Spec}(B)$ is affine and irreducible. Then X is irreducible too. If we prove the result for any nonempty affine open $U \subset X$, then the result holds for X (small argument omitted). Hence we may assume X is affine too, say $X = \mathrm{Spec}(A)$. Let $y \in Y$ correspond to the minimal prime $\mathfrak{q} \subset B$. By assumption A has a unique minimal prime $\mathfrak{p}$ lying over $\mathfrak{q}$ and $B_\mathfrak{q} \to A_\mathfrak{p}$ is an isomorphism. It follows that $A_\mathfrak{q} \to \kappa(\mathfrak{p})$ is surjective, hence $\mathfrak{p}A_\mathfrak{q}$ is a maximal ideal. On the other hand $\mathfrak{p}A_\mathfrak{q}$ is the unique minimal prime of $A_\mathfrak{q}$. We conclude that $\mathfrak{p}A_\mathfrak{q}$ is the unique prime of $A_\mathfrak{q}$ and that $A_\mathfrak{q} = A_\mathfrak{p}$. Since $A_\mathfrak{q} = A \otimes_B B_\mathfrak{q}$ the lemma follows. □

01RQ **Example 29.50.4.** Here are two examples of birational morphisms which are not isomorphisms over any open of the target.

First example. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x - \alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are birational but not an isomorphism over any open.

Second example. Let A be a domain. Let $S \subset A$ be a multiplicative subset not containing 0. With $B = S^{-1}A$ the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is birational. If there exists an open U of $\mathrm{Spec}(A)$ such that $f^{-1}(U) \to U$ is an isomorphism, then there exists an $a \in A$ such that each every element of S becomes invertible in the principal localization A_a. Taking $A = \mathbf{Z}$ and S the set of odd integers give a counter example.

0BAC **Lemma 29.50.5.** *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components over a base scheme S. Assume one of the following conditions is satisfied*

(1) *f is locally of finite type and Y reduced,*
(2) *f is locally of finite presentation.*

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism. In particular if X and Y are irreducible, then X and Y are S-birational.

Proof. There is an immediate reduction to the case where X and Y are irreducible which we omit. Moreover, after shrinking further and we may assume X and Y are affine, say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. By assumption A, resp. B has a unique minimal prime $\mathfrak{p}$, resp. $\mathfrak{q}$, the prime $\mathfrak{p}$ lies over $\mathfrak{q}$, and $B_\mathfrak{q} = A_\mathfrak{p}$. By Lemma 29.50.3 we have $B_\mathfrak{q} = A_\mathfrak{q} = A_\mathfrak{p}$.

Suppose $B \to A$ is of finite type, say $A = B[x_1, \ldots, x_n]$. There exist a $b_i \in B$ and $g_i \in B \setminus \mathfrak{q}$ such that b_i/g_i maps to the image of x_i in $A_\mathfrak{q}$. Hence $b_i - g_i x_i$ maps to zero in $A_{g_i'}$ for some $g_i' \in B \setminus \mathfrak{q}$. Setting $g = \prod g_i g_i'$ we see that $B_g \to A_g$ is surjective. If moreover Y is reduced, then the map $B_g \to B_\mathfrak{q}$ is injective and hence $B_g \to A_g$ is injective as well. This proves case (1).

Proof of (2). By the argument given in the previous paragraph we may assume that $B \to A$ is surjective. As f is locally of finite presentation the kernel $J \subset B$ is a finitely generated ideal. Say $J = (b_1, \ldots, b_r)$. Since $B_\mathfrak{q} = A_\mathfrak{q}$ there exist $g_i \in B \setminus \mathfrak{q}$ such that $g_i b_i = 0$. Setting $g = \prod g_i$ we see that $B_g \to A_g$ is an isomorphism. □

0BAD **Lemma 29.50.6.** *Let S be a scheme. Let X and Y be irreducible schemes locally of finite presentation over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

(1) *X and Y are S-birational,*
(2) *there exist nonempty opens of X and Y which are S-isomorphic, and*
(3) *x and y map to the same point s of S and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as $\mathcal{O}_{S,s}$-algebras.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 29.49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). By Lemma 29.49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and

hence induces an isomorphism on nonempty opens by Lemma 29.50.5. This finishes the proof. □

0552 **Lemma 29.50.7.** *Let S be a scheme. Let X and Y be integral schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

(1) *X and Y are S-birational,*
(2) *there exist nonempty opens of X and Y which are S-isomorphic, and*
(3) *x and y map to the same point $s \in S$ and $\kappa(x) \cong \kappa(y)$ as $\kappa(s)$-extensions.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 29.49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). Observe that $\mathcal{O}_{X,x} = \kappa(x)$ and $\mathcal{O}_{Y,y} = \kappa(y)$ by Algebra, Lemma 10.25.1. By Lemma 29.49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma 29.50.5. This finishes the proof. □

29.51. Generically finite morphisms

02NV In this section we characterize maps between schemes which are locally of finite type and which are "generically finite" in some sense.

02NW **Lemma 29.51.1.** *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y. The following are equivalent:*

(1) *the set $f^{-1}(\{\eta\})$ is finite,*
(2) *there exist affine opens $U_i \subset X$, $i = 1, \ldots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \to V$ is finite.*

If f is quasi-separated, then these are also equivalent to

(3) *there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \to V$ is finite.*

If f is quasi-compact and quasi-separated, then these are also equivalent to

(4) *there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \to V$ is finite.*

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η, and we may and do assume throughout the proof below that Y is affine, say $Y = \mathrm{Spec}(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 10.122.11 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \to Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume f is quasi-separated and (1). Write $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$. There are no specializations among the ξ_i by Lemma 29.20.7. Since each ξ_i maps to the generic point η of an irreducible component of Y, there cannot be a nontrivial specialization $\xi \rightsquigarrow \xi_i$ in X (since ξ would map to η as well). We conclude each ξ_i is a generic point of an irreducible component of X. Since Y is affine and f quasi-separated we see X is quasi-separated. By Properties, Lemma 28.29.1 we can find an affine open $U \subset X$ containing each ξ_i.

By Algebra, Lemma 10.122.11 we see that after replacing Y by a standard open in Y the morphisms $U \to Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) – (3) with no further assumptions on f. Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) – (3) imply (4). Let U, V be as in (3). Replace Y by V. Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \to Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 29.8.6 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \to V'$ is finite. □

03HY **Example 29.51.2.** Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field $\mathbf{F}_2$. Thus the topology on $X = \mathrm{Spec}(A)$ is totally disconnected and quasi-compact. The projection maps $A \to \mathbf{F}_2$ define open points of $\mathrm{Spec}(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 26.14.3. The morphism $f : Y \to X$ is quasi-compact, finite type and has finite fibres but is not quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 29.51.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separate the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

03HZ **Remark 29.51.3.** An alternative to Lemma 29.51.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Lemma 37.45.2.

0BAH **Lemma 29.51.4.** *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Let $\eta \in Y^0$. The following are equivalent*

(1) *$f^{-1}(\{\eta\}) \subset X^0$,*
(2) *f is quasi-finite at all points lying over η,*
(3) *f is quasi-finite at all $\xi \in X^0$ lying over η.*

Proof. Condition (1) implies there are no specializations among the points of the fibre X_η. Hence (2) holds by Lemma 29.20.6. The implication (2) $\Rightarrow$ (3) is immediate. Since η is a generic point of Y, the generic points of X_η are generic points of X. Hence (3) and Lemma 29.20.6 imply the generic points of X_η are also closed. Thus all points of X_η are generic and we see that (1) holds. □

0BAI **Lemma 29.51.5.** *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Assume*

(1) *X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,*
(2) *either f is quasi-compact or f is separated.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible affine with generic point η. Then the fibre over η is finite as X^0 is finite.

Assume f is separated and Y irreducible affine. Choose $V \subset Y$ and $U \subset X$ as in Lemma 29.51.1 part (3). Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Lemmas 29.41.7 and 29.44.11). Thus $f^{-1}(V) = U \amalg W$ for some open subscheme W of X. However, since U contains all the generic points of X we conclude that $W = \emptyset$ as desired.

Assume f is quasi-compact and Y irreducible affine. Then X is quasi-compact, hence there exists a dense open subscheme $U \subset X$ which is separated (Properties, Lemma 28.29.3). Since the set of generic points X^0 is finite, we see that $X^0 \subset U$. Thus $\eta \notin f(X \setminus U)$. Since $X \setminus U \to Y$ is quasi-compact, we conclude that there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \subset U$, see Lemma 29.8.3. After replacing X by $f^{-1}(V)$ and Y by V we reduce to the separated case which we dealt with in the preceding paragraph. □

0BAJ **Lemma 29.51.6.** *Let X, Y be schemes. Let $f : X \to Y$ be a birational morphism between schemes which have finitely many irreducible components. Assume*

(1) *either f is quasi-compact or f is separated, and*
(2) *either f is locally of finite type and Y is reduced or f is locally of finite presentation.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Proof. By Lemma 29.51.5 we may assume that f is finite. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible. By Lemma 29.50.5 we find a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is an open immersion. After removing the closed (as f finite) subset $f(X \setminus U)$ from Y we see that f is an isomorphism. □

02NX **Lemma 29.51.7.** *Let X, Y be integral schemes. Let $f : X \to Y$ be locally of finite type. Assume f is dominant. The following are equivalent:*

(1) *the extension $R(Y) \subset R(X)$ has transcendence degree 0,*
(2) *the extension $R(Y) \subset R(X)$ is finite,*
(3) *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite, and*
(4) *the generic point of X is the only point of X mapping to the generic point of Y.*

If f is separated or if f is quasi-compact, then these are also equivalent to

(5) *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. Choose any affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \to A$ is of finite type (Lemma 29.15.2). By Lemma 29.8.7 the generic point of X maps to the generic point of Y hence $R \to A$ is injective. Let $K = R(Y)$ be the fraction field of R and $L = R(X)$ the fraction field of A. Then L/K is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \ldots, x_n \in A$ be generators of A over R. By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i. Then we conclude that $R_{f_1 \ldots f_n} \to A_{f_1 \ldots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X, and let $\eta' \in Y$ be the generic point of Y. Assume (4). Then $\dim_\eta(X_{\eta'}) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 29.28.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η'. As x is a specialization of η, this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 10.36.19. Hence $x = \eta$. Thus we see that (1) – (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f. What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) – (4) imply (5). This follows from Lemma 29.51.5. □

02NY **Definition 29.51.8.** Let X and Y be integral schemes. Let $f : X \to Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) – (4) of Lemma 29.51.7. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the *degree of X over Y*.

It is possible to extend this notion to a morphism $f : X \to Y$ if (a) Y is integral with generic point η, (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. In this case we can define

$$\deg(X/Y) = \sum\nolimits_{\xi \in X,\ f(\xi) = \eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 29.49.5) the dimensions above are finite by Lemma 29.51.1 above. However, for most applications the definition given above is the right one.

02NZ **Lemma 29.51.9.** *Let X, Y, Z be integral schemes. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then*

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 9.7.7. □

073A **Remark 29.51.10.** Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define *generically finite* morphisms. These correspond to whether you want the property to be local on the source or local on the target:

(1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 29.51.1.
(2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \to Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y, but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

0AAZ **Definition 29.51.11.** Let X be an integral scheme. A *modification of* X is a birational proper morphism $f : X' \to X$ with X' integral.

Let $f : X' \to X$ be a modification as in the definition. By Lemma 29.51.7 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Proposition 29.27.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Lemma 29.48.2). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. Thus we can *redefine* a modification to be a proper morphism $f : X' \to X$ of integral schemes such that $f^{-1}(U) \to U$ is an isomorphism for some nonempty open $U \subset X$.

0AB0 **Definition 29.51.12.** Let X be an integral scheme. An *alteration of* X is a proper dominant morphism $f : Y \to X$ with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$. [**dJ96**, Definition 2.20]

This is the definition as given in [**dJ96**], except that here we do not require X and Y to be Noetherian. Arguing as above we see that an alteration is a proper dominant morphism $f : Y \to X$ of integral schemes which induces a finite extension of function fields, i.e., such that the equivalent conditions of Lemma 29.51.7 hold.

29.52. The dimension formula

02JT For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X.

02JU **Lemma 29.52.1.** *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume*

(1) *S is locally Noetherian,*
(2) *f is locally of finite type,*
(3) *X and S integral, and*
(4) *f dominant.*

We have

02JV (29.52.1.1) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + trdeg_{R(S)}R(X) - trdeg_{\kappa(s)}\kappa(x).$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 10.113.1. □

0BAE **Lemma 29.52.2.** *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume S is locally Noetherian and f is locally of finite type, We have*

0BAF (29.52.2.1) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + E - trdeg_{\kappa(s)}\kappa(x).$$

where E is the maximum of $trdeg_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs over the generic points of irreducible components of X containing x.

Proof. Let $X_1, \ldots, X_n$ be the irreducible components of X containing x endowed with their reduced induced scheme structure. These correspond to the minimal primes $\mathfrak{q}_i$ of $\mathcal{O}_{X,x}$ and hence there are finitely many of them (Schemes, Lemma 26.13.2 and Algebra, Lemma 10.31.6). Then $\dim(\mathcal{O}_{X,x}) = \max \dim(\mathcal{O}_{X,x}/\mathfrak{q}_i) = \max \dim(\mathcal{O}_{X_i,x})$. The ξ's occurring in the definition of E are exactly the generic points $\xi_i \in X_i$. Let $Z_i = \overline{\{f(\xi_i)\}} \subset S$ endowed with the reduced induced scheme structure. The composition $X_i \to X \to S$ factors through Z_i (Schemes, Lemma 26.12.7). Thus we may apply the dimension formula (Lemma 29.52.1) to see that $\dim(\mathcal{O}_{X_i,x}) \leq \dim(\mathcal{O}_{Z_i,x}) + \mathrm{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) - \mathrm{trdeg}_{\kappa(s)}\kappa(x)$. Putting everything together we obtain the lemma. □

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 5.20.1.

02JW **Lemma 29.52.3.** *Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Then the map*

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + trdeg_{\kappa(f(x))}\kappa(x) \end{aligned}$$

is a dimension function on X.

Proof. Let $f : X \to S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X. We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x.

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y. We may clearly replace X by U and S by V. Thus we may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ and that f is given by a ring map $R \to A$. The ring R is universally catenary (Lemma 29.17.2) and the map $R \to A$ is of finite type (Lemma 29.15.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = \mathrm{Spec}(R/\mathfrak{p}) \subset S$ is a dimension function. The ring $R/\mathfrak{p}$ is universally catenary. The restriction of $\delta_{X/S}$ to $X' = \mathrm{Spec}(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ'. Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes, and that x is the generic point of X and $f(x)$ is the generic point of S.

Note that $\mathcal{O}_{X,x} = R(X)$ and that since $x \rightsquigarrow y$, $x \neq y$, the spectrum of $\mathcal{O}_{X,y}$ has at least two points (Schemes, Lemma 26.13.2) hence $\dim(\mathcal{O}_{X,y}) > 0$. If y is an immediate specialization of x, then $\mathrm{Spec}(\mathcal{O}_{X,y}) = \{x, y\}$ and $\dim(\mathcal{O}_{X,y}) = 1$.

Write $s = f(x)$ and $t = f(y)$. We compute

$$\begin{aligned} \delta_{X/S}(x) - \delta_{X/S}(y) &= \delta(s) + \mathrm{trdeg}_{\kappa(s)}\kappa(x) - \delta(t) - \mathrm{trdeg}_{\kappa(t)}\kappa(y) \\ &= \delta(s) - \delta(t) + \mathrm{trdeg}_{R(S)}R(X) - \mathrm{trdeg}_{\kappa(t)}\kappa(y) \\ &= \delta(s) - \delta(t) + \dim(\mathcal{O}_{X,y}) - \dim(\mathcal{O}_{S,t}) \end{aligned}$$

where we use equality in (29.52.1.1) in the last step. Since δ is a dimension function on the scheme S and $s \in S$ is the generic point, the difference $\delta(s) - \delta(t)$ is equal to

$\text{codim}(\overline{\{t\}}, S)$ by Topology, Lemma 5.20.2. This is equal to $\dim(\mathcal{O}_{S,t})$ by Properties, Lemma 28.10.3. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = \dim(\mathcal{O}_{X,y})$$

and the lemma follows from what we said above about $\dim(\mathcal{O}_{X,y})$. □

Another application of the dimension formula is that the dimension does not change under "alterations" (to be defined later).

02JX **Lemma 29.52.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *Y is locally Noetherian,*
(2) *X and Y are integral schemes,*
(3) *f is dominant, and*
(4) *f is locally of finite type.*

Then we have

$$\dim(X) \leq \dim(Y) + trdeg_{R(Y)}R(X).$$

If f is closed[16] *then equality holds.*

Proof. Let $f : X \to Y$ be as in the lemma. Let $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_e$ be a sequence of specializations in X. Set $x = \xi_e$ and $y = f(x)$. Observe that $e \leq \dim(\mathcal{O}_{X,x})$ as the given specializations occur in the spectrum of $\mathcal{O}_{X,x}$, see Schemes, Lemma 26.13.2. By the dimension formula, Lemma 29.52.1, we see that

$$\begin{aligned} e &\leq \dim(\mathcal{O}_{X,x}) \\ &\leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)}R(X) - \text{trdeg}_{\kappa(y)}\kappa(x) \\ &\leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)}R(X) \end{aligned}$$

Hence we conclude that $e \leq \dim(Y) + \text{trdeg}_{R(Y)}R(X)$ as desired.

Next, assume f is also closed. Say $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$ is a sequence of specializations in Y. We want to show that $\dim(X) \geq d + r$. We may assume that $\overline{\xi}_0 = \eta$ is the generic point of Y. The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is nonempty as f is dominant. Hence by Lemma 29.16.10 it is a Jacobson scheme. Thus by Lemma 29.16.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y. Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 29.52.1 applied to the scheme X_η over the universally catenary scheme $\text{Spec}(\kappa(\eta))$ (see Lemma 29.17.5) and the point ξ_0. This means that we can find $\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0$ in X. On the other hand, as f is closed specializations lift along f, see Topology, Lemma 5.19.7. Thus, as ξ_0 lies over $\eta = \overline{\xi}_0$ we can find specializations $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$ lying over $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$. In other words we have

$$\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$$

which means that $\dim(X) \geq d + r$ as desired. □

0BAG **Lemma 29.52.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is locally Noetherian and f is locally of finite type. Then*

$$\dim(X) \leq \dim(Y) + E$$

[16] For example if f is proper, see Definition 29.41.1.

where E is the supremum of $trdeg_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs through the generic points of the irreducible components of X.

Proof. Immediate consequence of Lemma 29.52.2 and Properties, Lemma 28.10.2. □

29.53. Relative normalization

0BAK In this section we construct the *normalization of one scheme in another*.

035F **Lemma 29.53.1.** *Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule*

$$U \longmapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \text{ integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\}$$

is a quasi-coherent $\mathcal{O}_X$-algebra, the stalk $\mathcal{A}'_x$ is the integral closure of $\mathcal{O}_{X,x}$ in $\mathcal{A}_x$, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. This is a subsheaf by the local nature of the conditions. It is an $\mathcal{O}_X$-algebra by Algebra, Lemma 10.36.7. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(R)$ and say $\mathcal{A}$ is the quasi-coherent sheaf associated to the R-algebra A. Then according to Algebra, Lemma 10.36.12 the value of $\mathcal{A}'$ over U is given by the integral closure A' of R in A. This proves the last assertion of the lemma. To prove that $\mathcal{A}'$ is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 10.36.11. The same fact shows that the stalks are as advertised. □

035G **Definition 29.53.2.** Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 29.53.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{array}{ccc} Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) & \longrightarrow & X' = \underline{\mathrm{Spec}}_X(\mathcal{A}') \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

see Lemma 29.11.5. The scheme $X' \to X$ will be the normalization of X in the scheme Y. Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \to X$ of schemes. In this case the sheaf of $\mathcal{O}_X$-algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 26.24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of $\mathcal{O}_X$-algebras whose relative spectrum is the normalization of X in Y. Here is the formal definition.

035H **Definition 29.53.3.** Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[17]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

[17]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 27.4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

035I **Lemma 29.53.4.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*
(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f'}\downarrow & \nearrow_{h} & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\nu} & X \end{array}$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 29.53.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 29.44.1 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : Y \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 29.44.1) we see from Lemma 29.53.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 29.11.5 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category.

From the universal property in (2) we see that f' does not factor through a proper closed subscheme of X'. Hence the scheme theoretic image of f' is X'. Since f' is quasi-compact (by Schemes, Lemma 26.21.14 and the fact that ν is separated as an affine morphism) we see that $f'(Y)$ is dense in X'. Hence f' is dominant.

Observe that g is quasi-compact and quasi-separated by Schemes, Lemmas 26.21.13 and 26.21.14. Thus the last statement of the lemma makes sense. The morphism h in (2) is integral by Lemma 29.44.14. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the final assertion of the lemma. □

035J **Lemma 29.53.5.** *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \longrightarrow & X_1 \end{array}$$

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 quasi-compact and quasi-separated. Let $f_i = \nu_i \circ f_i'$, $i = 1, 2$ be the canonical factorizations, where $\nu_i : X_i' \to X_i$ is the normalization of X_i in Y_i. Then there exists a unique arrow $X_2' \to X_1'$ fitting into a commutative diagram

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2'}\downarrow & & \downarrow{\scriptstyle f_1'} \\ X_2' & \longrightarrow & X_1' \\ {\scriptstyle \nu_2}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X_2 & \longrightarrow & X_1 \end{array}$$

Proof. By Lemmas 29.53.4 (1) and 29.44.6 the base change $X_2 \times_{X_1} X_1' \to X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a unique morphism $X_2' \to X_2 \times_{X_1} X_1'$ from Lemma 29.53.4 (2). This gives the arrow $X_2' \to X_1'$ fitting into the commutative diagram and uniqueness follows as well. □

035K **Lemma 29.53.6.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $U \subset X$ be an open subscheme and set $V = f^{-1}(U)$. Then the normalization of U in V is the inverse image of U in the normalization of X in Y.*

Proof. Clear from the construction. □

0BXA **Lemma 29.53.7.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Then the normalization of X' in Y is X'.*

Proof. If $Y \to X'' \to X'$ is the normalization of X' in Y, then we can apply Lemma 29.53.4 to the composition $X'' \to X$ to get a canonical morphism $h : X' \to X''$ over X. We omit the verification that the morphisms h and $X'' \to X'$ are mutually inverse (using uniqueness of the factorization in the lemma). □

0AXN **Lemma 29.53.8.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.*

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. □

0AXP **Lemma 29.53.9.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. Every generic point of an irreducible component of X' is the image of a generic point of an irreducible component of Y.*

Proof. By Lemma 29.53.6 we may assume $X = \mathrm{Spec}(A)$ is affine. Choose a finite affine open covering $Y = \bigcup \mathrm{Spec}(B_i)$. Then $X' = \mathrm{Spec}(A')$ and the morphisms $\mathrm{Spec}(B_i) \to Y \to X'$ jointly define an injective A-algebra map $A' \to \prod B_i$. Thus the lemma follows from Algebra, Lemma 10.30.5. □

03GO **Lemma 29.53.10.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two schemes. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.*

Proof. In terms of integral closures this corresponds to the following fact: Let $A \to B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A_i' be the integral closure of A in B_i. Then $A_1' \times A_2'$ is the integral closure of A in B. The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A. Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A_i' as described above (some details omitted). □

03GQ **Lemma 29.53.11.** *Let $f : X \to S$ be a quasi-compact, quasi-separated and universally closed morphisms of schemes. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_S$. In other words, the normalization of S in X is equal to the factorization*

$$X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 27.4.7.

Proof. The question is local on S, hence we may assume $S = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

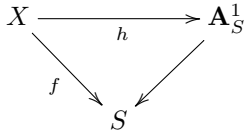

Let $Z \subset \mathbf{A}^1_S$ be the scheme theoretic image of h, see Definition 29.6.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_S \to S$ is separated, see Schemes, Lemma 26.21.14. By Lemma 29.6.3 the morphism $X \to Z$ is dominant. By Lemma 29.41.7 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 29.41.9 to conclude that $Z \to S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_S$, we see that $Z \to S$ is affine and proper, hence integral by Lemma 29.44.7. Writing $\mathbf{A}^1_S = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

03GP **Lemma 29.53.12.** *Let $f : Y \to X$ be an integral morphism. Then the normalization of X in Y is equal to Y.*

Proof. By Lemma 29.44.7 this is a special case of Lemma 29.53.11. □

035L **Lemma 29.53.13.** *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Assume*

(1) *Y is a normal scheme,*
(2) *quasi-compact opens of Y have finitely many irreducible components.*

Then X' is a disjoint union of integral normal schemes. Moreover, the morphism $Y \to X'$ is dominant and induces a bijection of irreducible components.

Proof. Let $U \subset X$ be an affine open. Consider the inverse image U' of U in X'. Set $V = f^{-1}(U)$. By Lemma 29.53.6 we $V \to U' \to U$ is the normalization of U in V. Say $U = \mathrm{Spec}(A)$. Then V is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $V = \coprod_{i=1,\ldots n} V_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 28.7.5. By Lemma 29.53.10 we see that $U' = \coprod_{i=1,\ldots,n} U'_i$, where U'_i is the normalization of U in V_i. By Properties, Lemma 28.7.9 we see that $B_i = \Gamma(V_i, \mathcal{O}_{V_i})$ is a normal domain. Note that $U'_i = \mathrm{Spec}(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i, see Lemma 29.53.1. By Algebra, Lemma 10.37.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $U' = \coprod U'_i$ is a finite union of normal integral schemes and hence is normal.

As X' has an open covering by the schemes U' we conclude from Properties, Lemma 28.7.2 that X' is normal. On the other hand, each U' is a finite disjoint union of irreducible schemes, hence every quasi-compact open of X' has finitely many irreducible components (by a topological argument which we omit). Thus X' is a disjoint union of normal integral schemes by Properties, Lemma 28.7.5. It is clear from the description of X' above that $Y \to X'$ is dominant and induces a bijection on irreducible components $V \to U'$ for every affine open $U \subset X$. The bijection of irreducible components for the morphism $Y \to X'$ follows from this by a topological argument (omitted). □

0AVK **Lemma 29.53.14.** *Let $f : X \to S$ be a morphism. Assume that*

(1) *S is a Nagata scheme,*
(2) *f is quasi-compact and quasi-separated,*
(3) *quasi-compact opens of X have finitely many irreducible components,*
(4) *if $x \in X$ is a generic point of an irreducible component, then the field extension $\kappa(x)/\kappa(f(x))$ is finitely generated, and*
(5) *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = \mathrm{Spec}(R)$ where R is a Nagata ring by assumption (1). We have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact by assumption (2) we can write $X = \bigcup_{i=1,\ldots,n} U_i$ with each U_i affine. Say $U_i = \mathrm{Spec}(B_i)$. Each B_i is reduced by assumption (5) and has finitely many minimal primes $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{im_i}$ by assumption (3) and Algebra, Lemma 10.26.1. We have

$$\Gamma(X, \mathcal{O}_X) \subset B_1 \times \ldots \times B_n \subset \prod\nolimits_{i=1,\ldots,n} \prod\nolimits_{j=1,\ldots,m_i} (B_i)_{\mathfrak{q}_{ij}}$$

the second inclusion by Algebra, Lemma 10.25.2. We have $\kappa(\mathfrak{q}_{ij}) = (B_i)_{\mathfrak{q}_{ij}}$ by Algebra, Lemma 10.25.1. Hence the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is contained in the product of the integral closures A_{ij} of R in $\kappa(\mathfrak{q}_{ij})$. Since R is Noetherian it suffices to show that A_{ij} is a finite R-module for each i, j. Let $\mathfrak{p}_{ij} \subset R$ be the image of $\mathfrak{q}_{ij}$. As $\kappa(\mathfrak{q}_{ij})/\kappa(\mathfrak{p}_{ij})$ is a finitely generated field extension by assumption (4), we see that $R \to \kappa(\mathfrak{q}_{ij})$ is essentially of finite type. Thus $R \to A_{ij}$ is finite by Algebra, Lemma 10.162.2. □

03GR **Lemma 29.53.15.** *Let $f : X \to S$ be a morphism. Assume that*

(1) *S is a Nagata scheme,*
(2) *f is of finite type,*
(3) *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. This is a special case of Lemma 29.53.14. Namely, (2) holds as the finite type morphism f is quasi-compact by definition and quasi-separated by Lemma 29.15.7. Condition (3) holds because X is locally Noetherian by Lemma 29.15.6. Finally, condition (4) holds because a finite type morphism induces finitely generated residue field extensions. □

0BXB **Lemma 29.53.16.** *Let $f : Y \to X$ be a finite type morphism of schemes with Y reduced and X Nagata. Let X' be the normalization of X in Y. Let $x' \in X'$ be a point such that*

(1) *$\dim(\mathcal{O}_{X',x'}) = 1$, and*
(2) *the fibre of $Y \to X'$ over x' is empty.*

Then $\mathcal{O}_{X',x'}$ is a discrete valuation ring.

Proof. We can replace X by an affine neighbourhood of the image of x'. Hence we may assume $X = \mathrm{Spec}(A)$ with A Nagata. By Lemma 29.53.15 the morphism $X' \to X$ is finite. Hence we can write $X' = \mathrm{Spec}(A')$ for a finite A-algebra A'. By Lemma 29.53.7 after replacing X by X' we reduce to the case described in the next paragraph.

The case $X = X' = \mathrm{Spec}(A)$ with A Noetherian. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to our point x'. Choose $g \in \mathfrak{p}$ not contained in any minimal prime of A (use prime avoidance and the fact that A has finitely many minimal primes, see Algebra, Lemmas 10.15.2 and 10.31.6). Set $Z = f^{-1}V(g) \subset Y$; it is a closed subscheme of Y. Then $f(Z)$ does not contain any generic point by choice of g and does not contain x' because x' is not in the image of f. The closure of $f(Z)$ is the set of specializations of points of $f(Z)$ by Lemma 29.6.5. Thus the closure of $f(Z)$ does not contain x' because the condition $\dim(\mathcal{O}_{X',x'}) = 1$ implies only the generic points of $X = X'$ specialize to x'. In other words, after replacing X by an affine open neighbourhood of x' we may assume that $f^{-1}V(g) = \emptyset$. Thus g maps to an invertible global function on Y and we obtain a factorization

$$A \to A_g \to \Gamma(Y, \mathcal{O}_Y)$$

Since $X = X'$ this implies that A is equal to the integral closure of A in A_g. By Algebra, Lemma 10.36.11 we conclude that $A_\mathfrak{p}$ is the integral closure of $A_\mathfrak{p}$ in $A_\mathfrak{p}[1/g]$. By our choice of g, since $\dim(A_\mathfrak{p}) = 1$ and since A is reduced we see that $A_\mathfrak{p}[1/g]$ is a finite product of fields (the product of the residue fields of the minimal primes contained in $\mathfrak{p}$). Hence $A_\mathfrak{p}$ is normal (Algebra, Lemma 10.37.16) and the proof is complete. Some details omitted. □

29.54. Normalization

035E Next, we come to the normalization of a scheme X. We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

035M (29.54.0.1) $$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 26.13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 26.21).

035N **Definition 29.54.1.** Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *normalization* of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

which is the normalization of X in the morphism $f : Y \to X$ (29.54.0.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

035O **Lemma 29.54.2.** *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. Let $f : Y \to X$ be the morphism (29.54.0.1). We get a factorization $Y \to X_{red} \to X$ of f from Schemes, Lemma 26.12.7. By Lemma 29.53.4 we obtain a canonical morphism $X^\nu \to X_{red}$ and that X^ν is the normalization of X_{red} in Y. The lemma follows as $Y \to X_{red}$ is identical to the morphism (29.54.0.1) constructed for X_{red}. □

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions $\mathcal{K}_X$ (see Divisors, Section 31.23). Namely, $\mathcal{K}_X = f_*\mathcal{O}_Y$ in this case, see Divisors, Lemma 31.25.1 and its proof. We describe this here explicitly.

035P **Lemma 29.54.3.** *Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then*

1. *A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,*
2. *the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,*
3. *the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and*
4. *$\nu^{-1}(U)$ is identified with the spectrum of A' where $\nu : X^\nu \to X$ is the normalization morphism.*

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 10.26.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 10.17.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 10.25.4 and 10.25.1. This proves (2). Part (3) follows from Algebra, Lemma 10.37.16, or Lemma 29.53.10. Part (4) holds because it is clear that $f^{-1}(U) \to U$ is the morphism

$$\mathrm{Spec}\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow \mathrm{Spec}(A)$$

where $f : Y \to X$ is the morphism (29.54.0.1). □

0C3B **Lemma 29.54.4.** *Let X be a scheme such that every quasi-compact open has a finite number of irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$. Then the following are canonically isomorphic as $\mathcal{O}_{X,x}$-algebras*

(1) *the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$,*
(2) *the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red}$,*
(3) *the integral closure of $\mathcal{O}_{X,x}$ in the product of the residue fields of the minimal primes of $\mathcal{O}_{X,x}$ (and there are finitely many of these).*

Proof. After replacing X by an affine open neighbourhood of x we may assume that X has finitely many irreducible components and that x is contained in each of them. Then the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the product L of the residue fields of the minimal primes of A. This follows from the construction of the normalization and Lemma 29.53.1. Alternatively, you can use Lemma 29.54.3 and the fact that normalization commutes with localization (Algebra, Lemma 10.36.11). Since A_{red} has finitely many minimal primes (because these correspond exactly to the generic points of the irreducible components of X passing through x) we see that L is the total ring of fractions of A_{red} (Algebra, Lemma 10.25.4). Thus our ring is also the integral closure of A in the total ring of fractions of A_{red}. □

035Q **Lemma 29.54.5.** *Let X be a scheme such that every quasi-compact open has finitely many irreducible components.*

(1) *The normalization X^ν is a disjoint union of integral normal schemes.*
(2) *The morphism $\nu : X^\nu \to X$ is integral, surjective, and induces a bijection on irreducible components.*
(3) *For any integral morphism $\alpha : X' \to X$ such that for $U \subset X$ quasi-compact open the inverse image $\alpha^{-1}(U)$ has finitely many irreducible components and $\alpha|_{\alpha^{-1}(U)} : \alpha^{-1}(U) \to U$ is birational[18] there exists a factorization $X^\nu \to X' \to X$ and $X^\nu \to X'$ is the normalization of X'.*
(4) *For any morphism $Z \to X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Let $f : Y \to X$ be as in (29.54.0.1). The scheme X^ν is a disjoint union of normal integral schemes because Y is normal and every affine open of Y has finitely many irreducible components, see Lemma 29.53.13. This proves (1). Alternatively one can deduce (1) from Lemmas 29.54.2 and 29.54.3.

The morphism ν is integral by Lemma 29.53.4. By Lemma 29.53.13 the morphism $Y \to X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \to X$ induces a bijection on irreducible components. By construction $f : Y \to X$ is dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 29.44.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \to X$ is as in (3). It is clear that X' satisfies the assumptions under which the normalization is defined. Let $f' : Y' \to X'$ be the morphism (29.54.0.1) constructed starting with X'. As α is locally birational it is clear that

[18] This awkward formulation is necessary as we've only defined what it means for a morphism to be birational if the source and target have finitely many irreducible components. It suffices if $X'_{red} \to X_{red}$ satisfies the condition.

$Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \to X' \to X$ exists and $X^\nu \to X'$ is the normalization of X' by Lemma 29.53.4. This proves (3).

Let $g : Z \to X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X. By Lemma 29.54.2 we have $X^\nu = X^\nu_{red}$ and by Schemes, Lemma 26.12.7 $Z \to X$ factors through X_{red}. Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z. Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = \mathrm{Spec}(A)$ and $W = \mathrm{Spec}(B)$, with $g|_W$ given by $\varphi : A \to B$. We will use the results of Lemma 29.54.3 freely. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of A. As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak{q} \subset B$ we have that $\varphi^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence if $x \in A$ is a nonzerodivisor, i.e., $x \notin \bigcup \mathfrak{p}_i$, then $\varphi(x)$ is a nonzerodivisor in B. Thus we obtain a canonical ring map $Q(A) \to Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$ (see Algebra, Lemma 10.37.12). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \to Q(B)$. Since $\nu^{-1}(U) = \mathrm{Spec}(A')$ this gives the canonical factorization $W \to \nu^{-1}(U) \to U$ of $\nu|_W$. We omit the verification that it is unique. □

0CDV **Lemma 29.54.6.** *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $Z_i \subset X$, $i \in I$ be the irreducible components of X endowed with the reduced induced structure. Let $Z_i^\nu \to Z_i$ be the normalization. Then $\coprod_{i \in I} Z_i^\nu \to X$ is the normalization of X.*

Proof. We may assume X is reduced, see Lemma 29.54.2. Then the lemma follows either from the local description in Lemma 29.54.3 or from Lemma 29.54.5 part (3) because $\coprod Z_i \to X$ is integral and locally birational (as X is reduced and has locally finitely many irreducible components). □

0BXC **Lemma 29.54.7.** *Let X be a reduced scheme with finitely many irreducible components. Then the normalization morphism $X^\nu \to X$ is birational.*

Proof. The normalization induces a bijection of irreducible components by Lemma 29.54.5. Let $\eta \in X$ be a generic point of an irreducible component of X and let $\eta^\nu \in X^\nu$ be the generic point of the corresponding irreducible component of X^ν. Then $\eta^\nu \mapsto \eta$ and to finish the proof we have to show that $\mathcal{O}_{X,\eta} \to \mathcal{O}_{X^\nu,\eta^\nu}$ is an isomorphism, see Definition 29.50.1. Because X and X^ν are reduced, we see that both local rings are equal to their residue fields (Algebra, Lemma 10.25.1). On the other hand, by the construction of the normalization as the normalization of X in $Y = \coprod \mathrm{Spec}(\kappa(\eta))$ we see that we have $\kappa(\eta) \subset \kappa(\eta^\nu) \subset \kappa(\eta)$ and the proof is complete. □

0AB1 **Lemma 29.54.8.** *A finite (or even integral) birational morphism $f : X \to Y$ of integral schemes with Y normal is an isomorphism.*

Proof. Let $V \subset Y$ be an affine open with inverse image $U \subset X$ which is an affine open too. Since f is a birational morphism of integral schemes, the homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is an injective map of domains which induces an isomorphism of fraction fields. As Y is normal, the ring $\mathcal{O}_Y(V)$ is integrally closed in the fraction field. Since f is finite (or integral) every element of $\mathcal{O}_X(U)$ is integral over $\mathcal{O}_Y(V)$. We conclude that $\mathcal{O}_Y(V) = \mathcal{O}_X(U)$. This proves that f is an isomorphism as desired. □

0H7D **Lemma 29.54.9.** *Let X be a scheme with locally finitely many irreducible components. The normalization morphism $\nu : X^\nu \to X$ is an isomorphism if and only if X is normal.*

Proof. If ν is an isomorphism, then X is normal by Lemma 29.54.5. Conversely, suppose X is normal. By Lemma 29.53.6 and Properties, Lemma 28.7.5 we may assume X is integral. By Lemma 29.54.5 the morphism ν is integral and X^ν has a unique irreducible component (so it is an integral scheme). We conclude from Lemmas 29.54.7 and 29.54.8. □

035R **Lemma 29.54.10.** *Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Follows from the definition (Properties, Definition 28.13.1) and Lemma 29.54.3. Namely, in this case the lemma says that $\nu^{-1}(\mathrm{Spec}(A))$ is the spectrum of the integral closure of A in its field of fractions. □

035S **Lemma 29.54.11.** *Let X be a Nagata scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 29.54.1 does apply. The lemma is now a special case of Lemma 29.53.14 but we can also prove it directly as follows. Write $X^\nu \to X$ as the composition $X^\nu \to X_{red} \to X$. As $X_{red} \to X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 29.44.5). Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. By Lemma 29.54.3 we have $\nu^{-1}(U) = \mathrm{Spec}(\prod A'_i)$ where A'_i is the integral closure of $A/\mathfrak{q}_i$ in its fraction field. As A is a Nagata ring (see Properties, Lemma 28.13.6) each of the ring extensions $A/\mathfrak{q}_i \subset A'_i$ are finite. Hence $A \to \prod A'_i$ is a finite ring map and we win. □

0GIQ **Lemma 29.54.12.** *Let X be an irreducible, geometrically unibranch scheme. The normalization morphism $\nu : X^\nu \to X$ is a universal homeomorphism.*

Proof. We have to show that ν is integral, universally injective, and surjective, see Lemma 29.45.5. By Lemma 29.54.5 the morphism ν is integral. Let $x \in X$ and set $A = \mathcal{O}_{X,x}$. Since X is irreducible we see that A has a single minimal prime $\mathfrak{p}$ and $A_{red} = A/\mathfrak{p}$. By Lemma 29.54.4 the stalk $A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of A in the fraction field of A_{red}. By More on Algebra, Definition 15.107.1 we see that A' has a single prime $\mathfrak{m}'$ lying over $\mathfrak{m}_x \subset A$ and $\kappa(\mathfrak{m}')/\kappa(x)$ is purely inseparable. Hence ν is bijective (hence surjective) and universally injective by Lemma 29.10.2. □

29.55. Weak normalization

0H3I We will only define the weak normalization of a scheme when it locally has finitely many irreducible components; similar to the case of normalization.

0H3J **Lemma 29.55.1.** *Let $A \to B$ be a ring map inducing a dominant morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of spectra. There exists an A-subalgebra $B' \subset B$ such that*

(1) *$\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is a universal homeomorphism,*
(2) *given a factorization $A \to C \to B$ such that $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is a universal homeomorphism, the image of $C \to B$ is contained in B'.*

Proof. We will use Lemma 29.45.6 without further mention. Consider the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B_{red} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_{red} \end{array}$$

For any factorization $A \to C \to B$ of $A \to B$ as in (2), we see that $A_{red} \to C_{red} \to B_{red}$ is a factorization of $A_{red} \to B_{red}$ as in (2). It follows that if the lemma holds for $A_{red} \to B_{red}$ and produces the A_{red}-subalgebra $B'_{red} \subset B_{red}$, then setting $B' \subset B$ equal to the inverse image of B'_{red} solves the lemma for $A \to B$. This reduces us to the case discussed in the next paragraph.

Assume A and B are reduced. In this case $A \subset B$ by Algebra, Lemma 10.30.6. Let $A \to C \to B$ be a factorization as in (2). Then we may apply Proposition 29.46.8 to $A \subset C$ to see that every element of C is contained in an extension $A[c_1, \ldots, c_n] \subset C$ such that for $i = 1, \ldots, n$ we have

(1) $c_i^2, c_i^3 \in A[c_1, \ldots, c_{i-1}]$, or
(2) there exists a prime number p with $pc_i, c_i^p \in A[c_1, \ldots, c_{i-1}]$.

Thus property (2) holds if we define $B' \subset B$ to be the subset of elements $b \in B$ which are contained in an extension $A[b_1, \ldots, b_n] \subset B$ such that (*) holds: for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

There are only two things to check: (a) B' is an A-subalgebra, and (b) $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is a universal homeomorphism. Part (a) follows because given $n \geq 0$ and $b_1, \ldots, b_n \in B$ satisfying (*) and $m \geq 0$ and $b'_1, \ldots, b'_m \in B$ satisfying (*), the integer $n + m$ and $b_1, \ldots, b_n, b'_1, \ldots, b'_m \in B$ also satisfies (*). Finally, part (b) holds by Proposition 29.46.8 and our construction of B'. □

0H3K **Lemma 29.55.2.** *Let $A \to B$ be a ring map inducing a dominant morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 29.55.1 commutes with localization (see proof for explanation).*

Proof. Let $S \subset A$ be a multiplicative subset. Then $S^{-1}A \to S^{-1}B$ is a ring map which induces a dominant morphism $\mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ as well (see Lemmas 29.8.4 and 29.25.9). Hence Lemma 29.55.1 produces an $S^{-1}A$-subalgebra $(S^{-1}B)' \subset S^{-1}B$. The statement means that $S^{-1}B' = (S^{-1}B)'$ as $S^{-1}A$-subalgebras of $S^{-1}B$.

To see this is true, we will use the construction of B' and $(S^{-1}B)'$ in the proof of Lemma 29.55.1. In the first step, we see that B' is the inverse image of the A_{red}-subalgebra $B'_{red} \subset B_{red}$ constructed for the ring map $A_{red} \to B_{red}$ and similarly for $(S^{-1}B)'$. Noting that $S^{-1}B_{red} = (S^{-1}B)_{red}$ this reduces us to the case discussed in the next paragraph.

If A and B are reduced, we have constructed B' as the union of the subalgebras $A[b_1, \ldots, b_n]$ such that for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

Similarly for $(S^{-1}B)' \subset S^{-1}B$. Thus it is clear that the image of $B' \to B \to S^{-1}B$ is contained in $(S^{-1}B)'$. To show that the corresponding map $S^{-1}B' \to (S^{-1}B)'$ is surjective, one uses Lemma 29.46.3 to clear denominators successively; we omit the details. □

0H3L **Lemma 29.55.3.** *Let $A \to B$ be a ring map inducing a dominant morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of spectra. There exists an A-subalgebra $B' \subset B$ such that*

(1) *$\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields,*
(2) *given a factorization $A \to C \to B$ such that $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields, the image of $C \to B$ is contained in B'.*

Proof. This proof is exactly the same as the proof of Lemma 29.55.1 except we use Proposition 29.46.7 in stead of Proposition 29.46.8 □

0H3M **Lemma 29.55.4.** *Let $A \to B$ be a ring map inducing a dominant morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 29.55.3 commutes with localization (see proof for explanation).*

Proof. The proof is the same as the proof of Lemma 29.55.2. □

0H3N **Lemma 29.55.5.** *Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.*

(1) *The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism has an initial object $Y \to X^{Y/wn} \to X$.*
(2) *The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism inducing isomorphisms on residue fields has an initial object $Y \to X^{Y/sn} \to X$.*

Moreover, formation of the factorization $Y \to X^{Y/wn} \to X$ and $Y \to X^{Y/sn} \to X$ commutes with base change to open subschemes of X.

Proof. We will prove (1) and omit the proof of (2); also the final assertion will follow from the construction of the factorization. We will use Lemma 29.45.5 without further mention. First, let $Y \to X^{Y/n} \to X$ be the normalization of X in Y, see Definition 29.53.3. For $Y \to X' \to X$ as in (1), we obtain a unique morphism $X^{Y/n} \to X'$ compatible with the given morphisms, see Lemma 29.53.4. Thus it suffices to prove the lemma with f replaced by $X^{Y/n} \to X$. This reduces us to the case studied in the next paragraph.

Assume f is integral (the rest of the proof works more generally if f is affine). Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $V = f^{-1}(U) = \mathrm{Spec}(B)$ be the inverse image in Y. Then $A \to B$ is a ring map which induces a dominant morphism on spectra. By Lemma 29.55.1 we obtain an A-subalgebra $B' \subset B$ such that setting $U^{V/wn} = \mathrm{Spec}(B')$ the factorization $V \to U^{V/wn} \to U$ is initial in the category of factorizations $V \to U' \to U$ where $U' \to U$ is a universal homeomorphism.

If $U_1 \subset U_2 \subset X$ are affine opens, then setting $V_i = f^{-1}(U_i)$ we obtain a canonical morphism

$$\rho^{U_2}_{U_1} : U_1^{V_1/wn} \to U_1 \times_{U_2} U_2^{V_2/wn}$$

over U_1 by the universal property of $U_1^{V_1/wn}$. These morphisms satisfy a natural functoriality which we leave to the reader to formulate and prove. Furthermore,

the morphism $\rho_{U_1}^{U_2}$ is an isomorphism; this follows from Lemma 29.55.2 provided that $U_1 \subset U_2$ is a standard open and in the general case can be reduced to this case by the functorial nature of these maps and Schemes, Lemma 26.11.5 (details omitted). Thus by relative glueing (Constructions, Lemma 27.2.1) we obtain a morphism $X^{Y/wn} \to X$ which restricts to $U^{V/wn} \to U$ over U compatibly with the $\rho_{U_1}^{U_2}$. Of course, the morphisms $V \to U^{V/wn}$ glue to a morphism $Y \to X^{Y/wn}$ (see Constructions, Remark 27.2.3) and we get our factorization $Y \to X^{Y/wn} \to X$ where the second morphism is a universal homeomorphism.

Finally, let $Y \to X' \to X$ be a factorization as in (1). With $V \to U^{V/wn} \to U \subset X$ as above, we obtain a factorization $V \to U \times_X X' \to U$ where the second arrow is a universal homeomorphism and we obtain a unique morphism $g_U : U^{V/wn} \to U \times_X X'$ over U by the universal property of $U^{V/wn}$. These g_U are compatible with the morphisms $\rho_{U_1}^{U_2}$; details omitted. Hence there is a unique morphism $g : X^{Y/wn} \to X'$ over X agreeing with g_U over U, see Constructions, Remark 27.2.3. This proves that $Y \to X^{Y/wn} \to X$ is initial in our category and the proof is complete. □

0H3P **Definition 29.55.6.** Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.

(1) The factorization $Y \to X^{Y/sn} \to X$ constructed in Lemma 29.55.5 part (2) is the *seminormalization of X in Y*.
(2) The factorization $Y \to X^{Y/wn} \to X$ constructed in Lemma 29.55.5 part (1) is the *weak normalization of X in Y*.

Here is a way to reinterpret the seminormalization of a scheme which locally has finitely many irreducible components.

0H3Q **Lemma 29.55.7.** *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Then the seminormalization of X in X^ν is is the seminormalization of X. In a formula: $X^{sn} = X^{X^\nu/sn}$.*

Proof. Let $f : Y \to X$ be as in (29.54.0.1) so that X^ν is the normalization of X in Y. The seminormalization $X^{sn} \to X$ of X is the initial object in the category of universal homeomorphisms $X' \to X$ inducing isomorphisms on residue fields. Since Y is the disjoint union of the spectra of the residue fields at the generic points of irreducible components of X, we see that for any $X' \to X$ in this category we obtain a canonical lift $f' : Y \to X'$ of f. Then by Lemma 29.53.4 we obtain a canonical morphism $X^\nu \to X'$. Whence in turn a canonical morphism $X^{X^\nu/sn} \to X'$ by the universal property of $X^{X^\nu/sn}$. In this way we see that $X^{X^\nu/sn}$ satisfies the same universal property that X^{sn} has and we conclude. □

Lemma 29.55.7 motivates the following definition. Since we have only constructed the normalization in case X locally has finitely many irreducible components, we will also restrict ourselves to that case for the weak normalization.

0H3R **Definition 29.55.8.** Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *weak normalization* of X as the weak normalization

$$X^\nu \longrightarrow X^{wn} \longrightarrow X$$

of X in the normalization X^ν of X (Definition 29.54.1). In a formula: $X^{wn} = X^{X^\nu/wn}$.

Combined with Lemma 29.55.7 we see that for a scheme X which locally has finitely many irreducible components there are canonical morphisms

$$X^\nu \to X^{wn} \to X^{sn} \to X$$

Having made this definition, we can say what it means for a scheme to be weakly normal (provided it has locally finitely many irreducible components).

0H3S **Definition 29.55.9.** Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We say X is *weakly normal* if the weak normalization $X^{wn} \to X$ is an isomorphism (Definition 29.55.8).

It follows immediately from the definitions that for a scheme X such that every quasi-compact open has finitely many irreducible components we have

$$X \text{ normal} \Rightarrow X \text{ weakly normal} \Rightarrow X \text{ seminormal}$$

We can work out the meaning of weak normality in the affine case as follows.

0H3T **Lemma 29.55.10.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme which has finitely many irreducible components. Then X is weakly normal if and only if*

(1) *A is seminormal (Definition 29.47.1),*
(2) *for a prime number p and $z, w \in A$ such that (a) z is a nonzerodivisor, (b) w^p is divisible by z^p, and (c) pw is divisible by z, then w is divisible by z.*

Proof. Assume X is weakly normal. Since a weakly normal scheme is seminormal, we see that (1) holds (by our definition of weakly normal schemes). In particular A is reduced. Let p, z, w be as in (2). Choose $x, y \in A$ such that $z^p x = w^p$ and $zy = pw$. Then $p^p x = y^p$. The ring map $A \to C = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra. The normalization X^ν of X is the spectrum of the integral closure A' of A in the total ring of fractions of A, see Lemma 29.54.3. Note that $a = w/z \in A'$ because $a^p = x$. Hence we have an A-algebra homomorphism $A \to C \to A'$ sending t to a. At this point the defining property $X = X^{wn} = X^{X^\nu/wn}$ of being weakly normal tells us that $C \to A'$ maps into A. Thus we find $a \in A$ as desired.

Conversely, assume (1) and (2). Let A' be as in the previous paragraph. We have to show that $X^{X^\nu/wn} = X$. By construction in the proof of Lemma 29.55.1, the scheme $X^{X^\nu/wn}$ is the spectrum of the subring of A' which is the union of the subrings $A[a_1, \ldots, a_n] \subset A'$ such that for $i = 1, \ldots, n$ we have

(a) $a_i^2, a_i^3 \in A[a_1, \ldots, a_{i-1}]$, or
(b) there exists a prime number p with $pa_i, a_i^p \in A[a_1, \ldots, a_{i-1}]$.

Then we can use (1) and (2) to inductively see that $a_1, \ldots, a_n \in A$; we omit the details. Consequently, we have $X = X^{X^\nu/wn}$ and hence X is weakly normal. □

Here is the obligatory lemma.

0H3U **Lemma 29.55.11.** *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The following are equivalent:*

(1) *The scheme X is weakly normal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ satisfies conditions (1) and (2) of Lemma 29.55.10.*

(3) *There exists an affine open covering $X = \bigcup U_i$ such that each ring $\mathcal{O}_X(U_i)$ satisfies conditions (1) and (2) of Lemma 29.55.10.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is weakly normal.*

Moreover, if X is weakly normal then every open subscheme is weakly normal.

Proof. The condition to X be weakly normal is that the morphism $X^{wn} = X^{X^\nu/wn} \to X$ is an isomorphism. Since the construction of $X^\nu \to X$ commutes with base change to open subschemes and since the construction of $X^{X^\nu/wn}$ commutes with base change to open subschemes of X (Lemma 29.55.5) the lemma is clear. □

29.56. Zariski's Main Theorem (algebraic version)

03GS This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Cohomology of Schemes, Section 30.21 and More on Morphisms, Section 37.43.

03GT **Theorem 29.56.1** (Algebraic version of Zariski's Main Theorem)**.** *Let $f : Y \to X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X in Y. Picture:*

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & X & \end{array}$$

Then there exists an open subscheme $U' \subset X'$ such that
(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset Y$ is the set of points at which f is quasi-finite.*

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$. Then $X' = \mathrm{Spec}(A')$, where A' is the integral closure of R in A, see Definitions 29.53.2 and 29.53.3. By Algebra, Theorem 10.123.12 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \to U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \to X'_x$ is an isomorphism in a neighbourhood of y. Hence there is no point of Y_x which specializes to y, since this is true for $f'(y)$ in X'_x, see Lemma 29.44.8. By Lemma 29.20.6 part (3) this implies f is quasi-finite at y. □

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

01TI **Lemma 29.56.2.** *Let $f : X \to S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \to S$ is locally quasi-finite.*

Proof. Suppose f is quasi-finite at x. Let $x \in U = \mathrm{Spec}(A) \subset X$, $V = \mathrm{Spec}(R) \subset S$ be affine opens as in Definition 29.20.1. By either Theorem 29.56.1 above or Algebra, Lemma 10.123.13, the set of primes $\mathfrak{q}$ at which $R \to A$ is quasi-finite is open in $\mathrm{Spec}(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. □

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 37.43.3.

03GU **Lemma 29.56.3.** *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *X and Y are affine, and*
(2) *f is quasi-finite.*

Then there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 10.123.14 reformulated in the language of schemes. □

03J2 **Lemma 29.56.4.** *Let $f : Y \to X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a nowhere dense subset of X.*

Proof. As in the proof of Lemma 29.48.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 5.21.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X. This reduces us to the case where both X and Y are affine. At this point we apply Lemma 29.56.3 above to get a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion. Set $\overline{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 5.21.3 we see $\overline{T}$ is nowhere dense in Z. Since $f(T) \subset \pi(\overline{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 29.48.7. □

29.57. Universally bounded fibres

03J3 Let X be a scheme over a field k. If X is finite over k, then $X = \mathrm{Spec}(A)$ where A is a finite k-algebra. Another way to say this is that X is finite locally free over $\mathrm{Spec}(k)$, see Definition 29.48.1. Hence $X \to \mathrm{Spec}(k)$ has a *degree* which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the *degree* of the (finite) scheme X over k.

03J4 **Definition 29.57.1.** Let $f : X \to Y$ be a morphism of schemes.

(1) We say the integer n *bounds the degrees of the fibres of f* if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
(2) We say the *fibres of f are universally bounded*[19] if there exists an integer n which bounds the degrees of the fibres of f.

[19] This is probably nonstandard notation.

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

03J5 **Lemma 29.57.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:*

(1) *the integer n bounds the degrees of the fibres of f, and*
(2) *for every morphism $\mathrm{Spec}(k) \to Y$, where k is a field, the fibre product $X_k = \mathrm{Spec}(k) \times_Y X$ is finite over k of degree $\leq n$.*

In this case the fibres of f are universally bounded and the schemes X_k have at most n points. More precisely, if $X_k = \{x_1, \ldots, x_t\}$, then we have

$$n \geq \sum\nolimits_{i=1,\ldots,t} [\kappa(x_i) : k]$$

Proof. The implication (2) $\Rightarrow$ (1) is trivial. The other implication holds because if the image of $\mathrm{Spec}(k) \to Y$ is y, then $X_k = \mathrm{Spec}(k) \times_{\mathrm{Spec}(\kappa(y))} X_y$. By definition the fibres of f being universally bounded means that some n exists. Finally, suppose that $X_k = \mathrm{Spec}(A)$. Then $\dim_k A = n$. Hence A is Artinian, all prime ideals are maximal ideals $\mathfrak{m}_i$, and A is the product of the localizations at these maximal ideals. See Algebra, Lemmas 10.53.2 and 10.53.6. Then $\mathfrak{m}_i$ corresponds to x_i, we have $A_{\mathfrak{m}_i} = \mathcal{O}_{X_k, x_i}$ and hence there is a surjection $A \to \bigoplus \kappa(\mathfrak{m}_i) = \bigoplus \kappa(x_i)$ which implies the inequality in the statement of the lemma by linear algebra. □

0CC2 **Lemma 29.57.3.** *If f is a finite locally free morphism of degree d, then d bounds the degree of the fibres of f.*

Proof. This is true because any base change of f is finite locally free of degree d (Lemma 29.48.4) and hence the fibres of f all have degree d. □

03J6 **Lemma 29.57.4.** *A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \to Y$ and m bounds the degrees of $g : Y \to Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \to Z$.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $\mathrm{Spec}(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \to Y_z$ are the fibres of f at the corresponding points of Y, which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 28.29.1 applied to the set of all points of X_z). Write $X_z = \mathrm{Spec}(A)$, $Y_z = \mathrm{Spec}(B)$, and $k = \kappa(z)$. Then $k \to B \to A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak{m} \subset B$ we have $\dim_{\kappa(\mathfrak{m})}(A/\mathfrak{m}A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_{\mathfrak{m}}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 10.53.2 and 10.53.6. So we see that $\dim_k(B) = \sum_{\mathfrak{m}} \dim_k(B_{\mathfrak{m}})$ and $\dim_k(A) = \sum_{\mathfrak{m}} \dim_k(A_{\mathfrak{m}})$ where in both cases $\mathfrak{m}$ runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 10.20.1) we see that each $A_{\mathfrak{m}}$ is a quotient of $B_{\mathfrak{m}}^{\oplus n}$ as a $B_{\mathfrak{m}}$-module. Hence $\dim_k(A_{\mathfrak{m}}) \leq n \dim_k(B_{\mathfrak{m}})$.

Putting everything together we see that

$$\dim_k(A) = \sum\nolimits_{\mathfrak{m}} \dim_t a(A_{\mathfrak{m}}) \leq \sum\nolimits_{\mathfrak{m}} n \dim_k(B_{\mathfrak{m}}) = n \dim_k(B) \leq nm$$

as desired. □

03J7 **Lemma 29.57.5.** *A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \to Y$ and $Y' \to Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \to Y'$ is also bounded by n.*

Proof. This is clear from the result of Lemma 29.57.2. □

03J8 **Lemma 29.57.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $Y' \to Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \to Y'$ be the base change of f. If $Y' \to Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f', then also n bounds the degrees of the fibres of f.*

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f'. We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = \mathrm{Spec}(\kappa(y')) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) □

03J9 **Lemma 29.57.7.** *An immersion has universally bounded fibres.*

Proof. The integer $n = 1$ works in the definition. □

03WU **Lemma 29.57.8.** *Let $f : X \to Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent*

1. *the integer n bounds the degrees of the fibres,*
2. *for every field k and morphism $\mathrm{Spec}(k) \to Y$ the base change $X_k = \mathrm{Spec}(k) \times_Y X$ has at most n points, and*
3. *for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{sep}$ the scheme $X_{\kappa(y)^{sep}}$ has at most n points.*

Proof. This follows from Lemma 29.57.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 29.36.7. □

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

03JA **Lemma 29.57.9.** *Let $f : X \to Y$ be a morphism of schemes. Assume that*

1. *f is locally quasi-finite, and*
2. *X is quasi-compact.*

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \ldots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of "local quasi-finiteness" (see Lemma 29.20.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \to V_i$ are locally quasi-finite morphisms of affines, hence

quasi-finite, see Lemma 29.20.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \to V_i$, then $\sum n_i$ works for f).

Assume $f : X \to Y$ is a quasi-finite morphism of affines. By Lemma 29.56.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & Y & \end{array}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 29.57.7) this reduces us to showing that π has universally bounded fibres (Lemma 29.57.4).

This reduces us to a morphism of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $A \to B$ is finite. Say B is generated by $x_1, \ldots, x_n$ over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 10.36.3). With these notations it is clear that

$$\bigoplus\nolimits_{0 \leq e_i < d_i, i=1,\ldots n} A \longrightarrow B, \quad (a_{(e_1,\ldots,e_n)}) \longmapsto \sum a_{(e_1,\ldots,e_n)} x_1^{e_1} \ldots x_n^{e_n}$$

is a surjective A-module map. Thus for any prime $\mathfrak{p} \subset A$ this induces a surjective map $\kappa(\mathfrak{p})$-vector spaces

$$\kappa(\mathfrak{p})^{\oplus d_1 \ldots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak{p})$$

In other words, the integer $d_1 \ldots d_n$ works in the definition of a morphism with universally bounded fibres. □

03JB **Lemma 29.57.10.** *Consider a commutative diagram of morphisms of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle h} & \\ & Z & \end{array}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g, then also n bounds the degree of the fibres of h.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h. Let $z \in Z$. Consider the morphism of schemes $X_z \to Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an Artinian ring (by Algebra, Lemma 10.53.2). By Lemma 29.11.14 the morphism $X_z \to Y_z$ is affine in particular quasi-compact. It follows from Lemma 29.25.12 that Y_z is a finite discrete as this holds for X_z. Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 28.29.1 applied to the set of all points of Y_z). Write $Y_z = \mathrm{Spec}(B)$ and $X_z = \mathrm{Spec}(A)$. Then A is faithfully flat over B, so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. □

29.58. Miscellany

0H1L Results which do not fit elsewhere.

0H1M **Lemma 29.58.1.** *Let $f : Y \to X$ be a morphism of schemes. Let $x \in X$ be a point. Assume that Y is reduced and $f(Y)$ is set-theoretically contained in $\{x\}$. Then f factors through the canonical morphism $x = \mathrm{Spec}(\kappa(x)) \to X$.*

Proof. Omitted. Hints: working affine locally one reduces to a commutative algebra lemma. Given a ring map $A \to B$ with B reduced such that there exists a unique prime ideal $\mathfrak{p} \subset A$ in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then $A \to B$ factors through $\kappa(\mathfrak{p})$. This is a nice exercise. □

0H1N **Lemma 29.58.2.** *Let $f : Y \to X$ be a morphism of schemes. Let $E \subset X$. Assume X is locally Noetherian, there are no nontrivial specializations among the elements of E, Y is reduced, and $f(Y) \subset E$. Then f factors through $\coprod_{x \in E} x \to X$.*

Proof. When E is a singleton this follows from Lemma 29.58.1. If E is finite, then E (with the induced topology of X) is a finite discrete space by our assumption on specializations. Hence this case reduces to the singleton case. In general, there is a reduction to the case where X and Y are affine schemes. Say $f : Y \to X$ corresponds to the ring map $\varphi : A \to B$. Denote $A' \subset B$ the image of φ. Let $E' \subset \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$ be the set of minimal primes of A'. By Algebra, Lemma 10.30.5 the set E' is contained in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$. We conclude that $E' \subset E$. Since A' is Noetherian we have E' is finite by Algebra, Lemma 10.31.6. Since any other point in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a specialization of an element of E' and in E, we conclude that the image is contained in E' (by our assumption on specializations between points of E). Thus we reduce to the case where E is finite which we dealt with above. □

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CHAPTER 30

Cohomology of Schemes

01X6 **30.1. Introduction**

01X7 In this chapter we first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [**DG67**]. Having done this we will elaborate on cohomology of coherent sheaves in the Noetherian setting. See [**Ser55b**].

30.2. Čech cohomology of quasi-coherent sheaves

01X8 Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a *standard open covering* of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 26.5.2.

01X9 **Lemma 30.2.1.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Write $U = \mathrm{Spec}(A)$ for some ring A. In other words, $f_1, \ldots, f_n$ are elements of A which generate the unit ideal of A. Write $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. Clearly the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

We are asked to show that the extended complex

01XA (30.2.1.1) $$0 \to M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

(whose truncation we have studied in Algebra, Lemma 10.24.1) is exact. It suffices to show that (30.2.1.1) is exact after localizing at a prime $\mathfrak{p}$, see Algebra, Lemma 10.23.1. In fact we will show that the extended complex localized at $\mathfrak{p}$ is homotopic to zero.

There exists an index i such that $f_i \notin \mathfrak{p}$. Choose and fix such an element i_{fix}. Note that $M_{f_{i_{\mathrm{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$. Similarly for a localization at a product $f_{i_0} \ldots f_{i_p}$ and $\mathfrak{p}$ we can drop any f_{i_j} for which $i_j = i_{\mathrm{fix}}$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} M_{f_{i_0} \ldots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod\nolimits_{i_0 \ldots i_p} M_{f_{i_0} \ldots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\mathrm{fix}} i_0 \ldots i_p}$$

(This is "dual" to the homotopy in the proof of Cohomology, Lemma 20.10.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \to M_{\mathfrak{p}}$ is projection onto the factor $M_{f_{i_{\mathrm{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$ and in

general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned}(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p}\end{aligned}$$

This proves the identity map is homotopic to zero as desired. □

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

01XB **Lemma 30.2.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma 20.11.9. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 20.11.9 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma 26.5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 30.2.1. □

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

01XC **Lemma 30.2.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

Proof. According to Cohomology, Lemma 20.7.3 the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 29.11.1. By Lemma 30.2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. □

089W **Lemma 30.2.4.** *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(S, f_* \mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 30.2.3 and the Leray spectral sequence. See Cohomology, Lemma 20.13.6. □

The following two lemmas explain when Čech cohomology can be used to compute cohomology of quasi-coherent modules.

0BDX **Lemma 30.2.5.** *Let X be a scheme. The following are equivalent*

1. *X has affine diagonal $\Delta : X \to X \times X$,*
2. *for $U, V \subset X$ affine open, the intersection $U \cap V$ is affine, and*
3. *there exists an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$.*

In particular this holds if X is separated.

Proof. Assume X has affine diagonal. Let $U, V \subset X$ be affine opens. Then $U \cap V = \Delta^{-1}(U \times V)$ is affine. Thus (2) holds. It is immediate that (2) implies (3). Conversely, if there is a covering of X as in (3), then $X \times X = \bigcup U_i \times U_{i'}$ is an affine open covering, and we see that $\Delta^{-1}(U_i \times U_{i'}) = U_i \cap U_{i'}$ is affine. Then Δ is an affine morphism by Morphisms, Lemma 29.11.3. The final assertion follows from Schemes, Lemma 26.21.7. □

01XD **Lemma 30.2.6.** *Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. In this case for any quasi-coherent sheaf $\mathcal{F}$ we have*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$-modules for all p.

Proof. In view of Lemma 30.2.2 this is a special case of Cohomology, Lemma 20.11.6. □

30.3. Vanishing of cohomology

01XE We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 30.2.2). It turns out that this also characterizes affine schemes. We give two versions.

01XF **Lemma 30.3.1.** *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.*

Then X is affine.

[**Ser57**], [**DG67**, II, Theorem 5.2.1 (d') and IV (1.7.17)]

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = \mathrm{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 26.12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$. Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.12.8). This would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \ldots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties,

Lemma 28.27.3 to finish the proof it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1,\ldots,f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. We let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take $\mathcal{F}_i$ to be the intersection of $\mathcal{F}$ with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\textstyle\bigoplus_{i=1,\ldots,n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1,\ldots,f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. □

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

01XG **Lemma 30.3.2.** *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *X is quasi-separated, and*
(3) *$H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type.*

Then X is affine.

[**Ser57**], [**DG67**, II, Theorem 5.2.1]

Proof. By Properties, Lemma 28.22.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 20.19.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X. In other words we see that Lemma 30.3.1 applies. □

We can use the arguments given above to find a sufficient condition to see when an invertible sheaf is ample. However, we warn the reader that this condition is not necessary.

0B5P **Lemma 30.3.3.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that*

(1) *X is quasi-compact,*
(2) *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$.*

Then $\mathcal{L}$ is ample.

Proof. This is proved in exactly the same way as Lemma 30.3.1. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \mathrm{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 26.12.4 there are quasi-coherent

sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

For every $n \geq 1$ we obtain a short exact sequence

$$0 \to \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to 0.$$

By our assumption we may pick n such that $H^1(X, \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Since $\mathcal{L}$ is trivial on U we see that the restriction of $\mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ to U also corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = A/\mathfrak{m}$. By our choice of n we see there exists a global section $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ which maps to the element $1 \in A/\mathfrak{m}$. Clearly we have $x \in X_s \subset U$ because s vanishes at points of Z. This implies that $X_s = D(f)$ where $f \in A$ is the image of s in $A \cong \Gamma(U, \mathcal{L}^{\otimes n})$. In particular X_s is affine.

Consider the union $W = \bigcup X_s$ over all $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for $n \geq 1$ such that X_s is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.12.8). This would contradict the fact that all closed points are in W. Hence we conclude $X = W$. This means that $\mathcal{L}$ is ample by Properties, Definition 28.26.1. □

There is a variant of Lemma 30.3.3 with finite type ideal sheaves which we will formulate and prove here if we ever need it.

0F83 **Lemma 30.3.4.** *Let $f : X \to Y$ be a quasi-compact morphism with X and Y quasi-separated. If $R^1 f_* \mathcal{I} = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ on X, then f is affine.*

Proof. Let $V \subset Y$ be an affine open subscheme. We have to show that $U = f^{-1}(V)$ is affine. The inclusion morphism $V \to Y$ is quasi-compact by Schemes, Lemma 26.21.14. Hence the base change $U \to X$ is quasi-compact, see Schemes, Lemma 26.19.3. Thus any quasi-coherent sheaf of ideals $\mathcal{I}$ on U extends to a quasi-coherent sheaf of ideals on X, see Properties, Lemma 28.22.1. Since the formation of $R^1 f_*$ is local on Y (Cohomology, Section 20.7) we conclude that $R^1(U \to V)_* \mathcal{I} = 0$ by the assumption in the lemma. Hence by the Leray Spectral sequence (Cohomology, Lemma 20.13.4) we conclude that $H^1(U, \mathcal{I}) = H^1(V, (U \to V)_* \mathcal{I})$. Since $(U \to V)_* \mathcal{I}$ is quasi-coherent by Schemes, Lemma 26.24.1, we have $H^1(V, (U \to V)_* \mathcal{I}) = 0$ by Lemma 30.2.2. Thus we find that U is affine by Lemma 30.3.1. □

30.4. Quasi-coherence of higher direct images

01XH We have seen that the higher cohomology groups of a quasi-coherent module on an affine are zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of *all* $\mathcal{O}_X$-modules.

08DR **Lemma 30.4.1** (Induction Principle)**.** *Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X. Assume that* [**BV03**, Proposition 3.3.1]

(1) *P holds for every affine open of X,*
(2) *if U is quasi-compact open, V affine open, P holds for U, V, and $U \cap V$, then P holds for $U \cup V$.*

Then P holds for every quasi-compact open of X and in particular for X.

Proof. First we argue by induction that P holds for *separated* quasi-compact opens $W \subset X$. Namely, such an open can be written as $W = U_1 \cup \ldots \cup U_n$ and we can do induction on n using property (2) with $U = U_1 \cup \ldots \cup U_{n-1}$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \ldots \cup (U_{n-1} \cap U_n)$ is also a union of $n-1$ affine open subschemes by Schemes, Lemma 26.21.7 applied to the affine opens U_i and U_n of W. Having said this, for any quasi-compact open $W \subset X$ we can do induction on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n. □

01XI **Lemma 30.4.2.** *Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Let $t = t(X)$ be the minimal number of affine opens needed to cover X. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves $\mathcal{F}$.*

Proof. First proof. By induction on t. If $t = 1$ the result follows from Lemma 30.2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes. Namely, since the diagonal is affine, the intersection of two affine opens is affine, see Lemma 30.2.5. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 20.8.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(V, \mathcal{F})$, $H^i(U \cap V, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X is has affine diagonal the multiple intersections $U_{i_0 \ldots i_p}$ are all affine, see Lemma 30.2.5. By Lemma 30.2.6 the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 20.23.6 the Čech cohomology groups may be computed using the alternating Čech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. □

0BDY **Lemma 30.4.3.** *Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Then*

(1) *given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an embedding $\mathcal{F} \to \mathcal{F}'$ of quasi-coherent $\mathcal{O}_X$-modules such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$, and*
(2) *$\{H^n(X, -)\}_{n \geq 0}$ is a universal δ-functor from $QCoh(\mathcal{O}_X)$ to Ab.*

Proof. Let $X = \bigcup U_i$ be a finite affine open covering. Set $U = \coprod U_i$ and denote $j : U \to X$ the morphism inducing the given open immersions $U_i \to X$. Since U is an affine scheme and X has affine diagonal, the morphism j is affine, see Morphisms,

Lemma 29.11.12. For every $\mathcal{O}_X$-module $\mathcal{F}$ there is a canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$. This map is injective as can be seen by checking on stalks: if $x \in U_i$, then we have a factorization

$$\mathcal{F}_x \to (j_*j^*\mathcal{F})_x \to (j^*\mathcal{F})_{x'} = \mathcal{F}_x$$

where $x' \in U$ is the point x viewed as a point of $U_i \subset U$. Now if $\mathcal{F}$ is quasi-coherent, then $j^*\mathcal{F}$ is quasi-coherent on the affine scheme U hence has vanishing higher cohomology by Lemma 30.2.2. Then $H^p(X, j_*j^*\mathcal{F}) = 0$ for $p > 0$ by Lemma 30.2.4 as j is affine. This proves (1). Finally, we see that the map $H^p(X, \mathcal{F}) \to H^p(X, j_*j^*\mathcal{F})$ is zero and part (2) follows from Homology, Lemma 12.12.4. □

071L **Lemma 30.4.4.** *Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \ldots \cup U_n$ be an open covering with each U_i quasi-compact and separated (for example affine). Set*

$$d = \max_{I \subset \{1,\ldots,n\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_i) - 1 \right)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U. Then $H^p(X, \mathcal{F}) = 0$ for all $p \geq d$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. Note that since X is quasi-separated and U_i quasi-compact the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Proof using induction on n. If $n = 1$ then the result follows from Lemma 30.4.2. If $n > 1$, write $X = U \cup V$ with $U = U_1 \cup \ldots \cup U_{n-1}$ and $V = U_n$. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 20.8.2. To finish the proof for $q \geq d$ we will show that $H^q(V, \mathcal{F})$, $H^q(U, \mathcal{F})$, and $H^{q-1}(U \cap V, \mathcal{F})$ vanish. By the case $n = 1$ we have $H^q(V, \mathcal{F}) = 0$ for $q \geq t(V) = t(U_n)$. Since $t(V) \leq d$ this proves what we want. By induction hypothesis we have $H^q(U, \mathcal{F}) = 0$ for

$$q \geq \max_{I \subset \{1,\ldots,n-1\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_i) - 1 \right)$$

Since the integer on the right is less than or equal to d, this proves what we want. Finally we may use our induction hypothesis for the open $U \cap V = (U_1 \cap U_n) \cup \ldots \cup (U_{n-1} \cap U_n)$ to get the vanishing of $H^q(U \cap V, \mathcal{F}) = 0$ for

$$q \geq \max_{I \subset \{1,\ldots,n-1\}} \left(|I| + t(U_n \cap \bigcap\nolimits_{i \in I} U_i) - 1 \right)$$

Since the integer on the right is strictly less than d the lemma follows. □

01XJ **Lemma 30.4.5.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

(1) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the higher direct images $R^pf_*\mathcal{F}$ are quasi-coherent on S.*
(2) *If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^pf_*\mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.*
(3) *In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \to S$ we have $R^p(f')_*\mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}'$ on X'. Here $f' : X' = S' \times_S X \to S'$ is the base change of f.*

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0 f_* \mathcal{F} = f_* \mathcal{F}$ is quasi-coherent by Schemes, Lemma 26.24.1. Using Cohomology, Lemma 20.7.4 we see that forming higher direct images commutes with restriction to open subschemes. Since being quasi-coherent is local on S we reduce to the case discussed in the next paragraph.

Proof of (1) in case S is affine. We will use the induction principle. Since f quasi-compact and quasi-separated we see that X is quasi-compact and quasi-separated. For $U \subset X$ quasi-compact open and $a = f|_U$ we let $P(U)$ be the property that $R^p a_* \mathcal{F}$ is quasi-coherent on S for all quasi-coherent modules $\mathcal{F}$ on U and all $p \geq 0$. Since $P(X)$ is (1), it suffices the prove conditions (1) and (2) of Lemma 30.4.1 hold. If U is affine, then $P(U)$ holds because $R^p a_* \mathcal{F} = 0$ for $p \geq 1$ (by Lemma 30.2.3 and Morphisms, Lemma 29.11.13) and we've already observed the result holds for $p = 0$ in the first paragraph. Next, let $U \subset X$ be a quasi-compact open, $V \subset X$ an affine open, and assume $P(U)$, $P(V)$, $P(U \cap V)$ hold. Let $a = f|_U$, $b = f|_V$, $c = f|_{U \cap V}$, and $g = f|_{U \cup V}$. Then for any quasi-coherent $\mathcal{O}_{U \cup V}$-module $\mathcal{F}$ we have the relative Mayer-Vietoris sequence

$$0 \to g_* \mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 g_* \mathcal{F} \to \ldots$$

see Cohomology, Lemma 20.8.3. By $P(U)$, $P(V)$, $P(U \cap V)$ we see that $R^p a_*(\mathcal{F}|_U)$, $R^p b_*(\mathcal{F}|_V)$ and $R^p c_*(\mathcal{F}|_{U \cap V})$ are all quasi-coherent. Using the results on quasi-coherent sheaves in Schemes, Section 26.24 this implies that each of the sheaves $R^p g_* \mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Whence $P(U \cup V)$ and the proof of (1) is complete.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1,\ldots m} S_j$. For each j choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1,\ldots t_j} U_{ji}$. Let

$$d_j = \max\nolimits_{I \subset \{1,\ldots,t_j\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_{ji}) \right)$$

be the integer found in Lemma 30.4.4. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \to S$ be a morphism of schemes and let $\mathcal{F}'$ be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^p f'_* \mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j. Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \ldots t_j$ and the intersections

$$\bigcap\nolimits_{i \in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap\nolimits_{i \in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 30.4.4 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^q f'_* \mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S', thus we see that $H^0(S', R^p f'_* \mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 20.13.6. Since $R^p f'_* \mathcal{F}'$ is quasi-coherent we conclude that $R^p f'_* \mathcal{F}' = 0$. □

01XK **Lemma 30.4.6.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_* \mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q f_* \mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 20.13.4. By Lemma 30.4.5 we see that the sheaves $R^q f_* \mathcal{F}$ are quasi-coherent. By Lemma 30.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 20.13.6 (2) for the general principle. □

30.5. Cohomology and base change, I

02KE Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^* \mathcal{F}$. Here is a diagram representing the situation:

02KF (30.5.0.1)
$$\begin{array}{ccccc} \mathcal{F}' = (g')^*\mathcal{F} & & X' & \xrightarrow[g']{} & X & & \mathcal{F} \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ R f'_* \mathcal{F}' & & S' & \xrightarrow{g} & S & & R f_* \mathcal{F} \end{array}$$

Here is the simplest case of the base change property we have in mind.

02KG **Lemma 30.5.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is affine. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (30.5.0.1) we have*

$$g^* f_* \mathcal{F} = f'_* (g')^* \mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 30.2.3. The statement is local on S and S'. Hence we may assume $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$ and $\mathcal{F} = \widetilde{M}$ for some A-module M. We use Schemes, Lemma 26.7.3 to describe pullbacks and pushforwards of $\mathcal{F}$. Namely, $X' = \mathrm{Spec}(R' \otimes_R A)$ and $\mathcal{F}'$ is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R'-modules. □

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in Section 30.7, but since it is so important it deserves its own proof.

02KH **Lemma 30.5.2** (Flat base change)**.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow[g']{} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

(1) *the base change map of Cohomology, Lemma 20.17.1 is an isomorphism*

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* \mathcal{F}',$$

(2) *if* $S = \mathrm{Spec}(A)$ *and* $S' = \mathrm{Spec}(B)$*, then* $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.

Proof. Using Cohomology, Lemma 20.17.1 in (1) is allowed since g' is flat by Morphisms, Lemma 29.25.8. Having said this, part (1) follows from part (2). Namely, part (1) is local on S' and hence we may assume S and S' are affine. In other words, we have $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ as in (2). Then since $R^i f_* \mathcal{F}$ is quasi-coherent (Lemma 30.4.5), it is the quasi-coherent $\mathcal{O}_S$-module associated to the A-module $H^0(S, R^i f_* \mathcal{F}) = H^i(X, \mathcal{F})$ (equality by Lemma 30.4.6). Similarly, $R^i f'_* \mathcal{F}'$ is the quasi-coherent $\mathcal{O}_{S'}$-module associated to the B-module $H^i(X', \mathcal{F}')$. Since pullback by g corresponds to $- \otimes_A B$ on modules (Schemes, Lemma 26.7.3) we see that it suffices to prove (2).

Let $A \to B$ be a flat ring homomorphism. Let X be a quasi-compact and quasi-separated scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B$ the pullback of $\mathcal{F}$. We are trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 30.2.6. If $\mathcal{U}_B : X_B = (U_1)_B \cup \ldots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 30.5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 20.11.5. The reader who wishes to avoid this spectral sequence can use Mayer-Vietoris and induction on t as in the proof of Lemma 30.4.5. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ which converges to $H^{p+q}(X_B, \mathcal{F}_B)$. Since the intersections $U_{i_0 \ldots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. □

0CKW **Lemma 30.5.3** (Finite locally free base change)**.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{G} = h^*\mathcal{F}$. If B is a finite locally free A-module, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(Y, \mathcal{G})$.

Warning: Do not use this lemma unless you understand the difference between this and Lemma 30.5.2.

Proof. In case X is separated, choose an affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 30.2.6. Let $\mathcal{V} : Y = \bigcup_{i \in I} g^{-1}(U_i)$ be the corresponding affine open covering of Y. The opens $V_i = g^{-1}(U_i) = U_i \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ are affine and Y is separated. Thus we obtain

$$\check{H}^p(\mathcal{V}, \mathcal{G}) = H^p(Y, \mathcal{G}).$$

We claim the map of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$$

is an isomorphism. Namely, as B is finitely presented as an A-module we see that tensoring with B over A commutes with products, see Algebra, Proposition 10.89.3. Thus it suffices to show that the maps $\Gamma(U_{i_0 \ldots i_p}, \mathcal{F}) \otimes_A B \to \Gamma(V_{i_0 \ldots i_p}, \mathcal{G})$ are isomorphisms which follows from Lemma 30.5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In the general case we argue in exactly the same way using affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and the corresponding covering $\mathcal{V} : Y = \bigcup_{i \in I} V_i$ with $V_i = g^{-1}(U_i)$ as above. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 20.11.5. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ which converges to $H^{p+q}(Y, \mathcal{G})$. Since the intersections $U_{i_0 \ldots i_p}$ are separated, the result of the previous paragraph gives isomorphisms $\Gamma(U_{i_0 \ldots i_p}, \underline{H}^q(\mathcal{F})) \otimes_A B \to \Gamma(V_{i_0 \ldots i_p}, \underline{H}^q(\mathcal{G}))$. Using that $- \otimes_A B$ commutes with products and is exact, we conclude that $\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B \to \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ is an isomorphism. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(Y, \mathcal{G})$ is an isomorphism for all i and we win. □

30.6. Colimits and higher direct images

07TA General results of this nature can be found in Cohomology, Section 20.19, Sheaves, Lemma 6.29.1, and Modules, Lemma 17.22.8.

07TB **Lemma 30.6.1.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be a filtered colimit of abelian sheaves on X. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i.$$

Proof. We will apply Cohomology, Lemma 20.19.2. Since the affine opens form a basis for the topology of S, it suffices to show that for $U \subset S$ affine open we have $H^p(f^{-1}U, \mathcal{F}) = \operatorname{colim} H^p(f^{-1}U, \mathcal{F}_i)$. Since $f^{-1}U$ is quasi-compact and quasi-separated we conclude using Cohomology, Lemma 20.19.1. (Because the basis of affine opens in $f^{-1}U$ satisfies the assumptions of that lemma.) □

30.7. Cohomology and base change, II

071M Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is quasi-compact and quasi-separated we would like to represent $Rf_*\mathcal{F}$ by a complex of quasi-coherent sheaves on S. This follows from the fact that the sheaves $R^if_*\mathcal{F}$ are quasi-coherent if S is quasi-compact and has affine diagonal, using that $D_{QCoh}(S)$ is equivalent to $D(QCoh(\mathcal{O}_S))$, see Derived Categories of Schemes, Proposition 36.7.5.

In this section we will use a different approach which produces an explicit complex having a good base change property. The construction is particularly easy if f and S are separated, or more generally have affine diagonal. Since this is the case which by far the most often used we treat it separately.

01XL **Lemma 30.7.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume X is quasi-compact and X and S have affine diagonal (e.g., if X and S are separated). In this case we can compute $Rf_*\mathcal{F}$ as follows:*

1. *Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$.*
2. *For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ denote $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ the restriction of f to the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$.*
3. *Set $\mathcal{F}_{i_0\ldots i_p}$ equal to the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$.*
4. *Set*
$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$$
and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (20.9.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$
in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf $\mathcal{F}$.

Proof. Consider the resolution $\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 20.24.1. We have an equality of complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of quasi-coherent $\mathcal{O}_S$-modules. The morphisms $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ and the morphisms $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ are affine by Morphisms, Lemma 29.11.12 and Lemma 30.2.5. Hence $R^qj_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ as well as $R^qf_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ are zero for $q > 0$ (Lemma 30.2.3). Using $f \circ j_{i_0\ldots i_p} = f_{i_0\ldots i_p}$ and the spectral sequence of Cohomology, Lemma 20.13.8 we conclude that $R^qf_*(j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}) = 0$ for $q > 0$. Since the terms of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ are finite direct sums of the sheaves $j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ we conclude using Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) that
$$Rf_*\mathcal{F} = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$
as desired. □

Next, we are going to consider what happens if we do a base change.

01XM **Lemma 30.7.2.** *With notation as in diagram (30.5.0.1). Assume $f : X \to S$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 30.7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X. There is a canonical isomorphism*
$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$. Moreover, if $S' \to S$ is affine, then in fact

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X'. In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 30.7.1. This is clear from the case of affine morphisms (Lemma 30.5.1) by working locally on S'. Moreover, exactly as in the proof of Lemma 30.7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \to X'$ and $U'_i \to S'$ are still affine (being base changes of affine morphisms). Details omitted. □

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which *universally* computes the higher direct images of $f : X \to S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is *not* equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

01XN **Lemma 30.7.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules with the following property: For every morphism $g : S' \to S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (30.5.0.1).*

Proof. (If f is separated as well, please see Lemma 30.7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 25.11.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 25.5.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the pth cohomology group of the complex

$$\prod\nolimits_{i \in I_0} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_1} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_2} H^q(U_i, \mathcal{F}) \to \ldots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod_{i\in I_0} \mathcal{F}(U_i) \to \prod_{i\in I_1} \mathcal{F}(U_i) \to \prod_{i\in I_2} \mathcal{F}(U_i) \to \ldots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \to S$ the restriction of f to U_i. As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod_{i\in I_0} f_{i,*}\mathcal{F}|_{U_i} \to \prod_{i\in I_1} f_{i,*}\mathcal{F}|_{U_i} \to \prod_{i\in I_2} f_{i,*}\mathcal{F}|_{U_i} \to \ldots)$$

on S. As in Hypercoverings, Lemma 25.5.3 we obtain a map $\mathcal{K}^\bullet \to Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of $\mathcal{F}$ (details omitted). Consider any affine scheme V and a morphism $g : V \to S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^* f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S'. □

The following lemma is a variant to flat base change.

0GN5 **Lemma 30.7.4.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_{S'}$-module flat over S. Assume f is quasi-compact and quasi-separated. For any $i \geq 0$ there is an identification

$$\mathcal{G} \otimes_{\mathcal{O}_{S'}} g^* R^i f_*\mathcal{F} = R^i f'_* \left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^*\mathcal{F}\right)$$

Proof. Let us construct a map from left to right. First, we have the base change map $Lg^*Rf_*\mathcal{F} \to Rf'_*L(g')^*\mathcal{F}$. There is also the adjunction map $\mathcal{G} \to Rf'_*L(f')^*\mathcal{G}$. Using the relative cup product We obtain

$$\begin{aligned} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Lg^*Rf_*\mathcal{F} &\to Rf'_*L(f')^*\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Rf'_*L(g')^*\mathcal{F} \\ &\to Rf'_*\left(L(f')^*\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*\mathcal{F}\right) \\ &\to Rf'_*\left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^*\mathcal{F}\right) \end{aligned}$$

where for the middle arrow we used the relative cup product, see Cohomology, Remark 20.28.7. The source of the composition is

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Lg^*Rf_*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_S} g^{-1}Rf_*\mathcal{F}$$

by Cohomology, Lemma 20.27.4. Since $\mathcal{G}$ is flat as a sheaf of $g^{-1}\mathcal{O}_S$-modules and since g^{-1} is an exact functor, this is a complex whose ith cohomology sheaf is $\mathcal{G} \otimes_{g^{-1}\mathcal{O}_S} g^{-1}R^i f_*\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_{S'}} g^* R^i f_*\mathcal{F}$. In this way we obtain global maps from left to right in the equality of the lemma. To show this map is an isomorphism we may work locally on S'. Thus we may and do assume that S and S' are affine schemes.

Proof in case S and S' are affine. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ and say $\mathcal{G}$ corresponds to the B-module N which is assumed to be A-flat. Since S is affine,

X is quasi-compact and quasi-separated. We will use a hypercovering argument to finish the proof; if X is separated or has affine diagonal, then you can use a Čech covering. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 25.11.4 we can find a hypercovering $K = (I, \{U_i\})$ of X such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 25.5.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Since each U_i is affine and $\mathcal{F}$ is quasi-coherent the value of $\underline{H}^q(\mathcal{F})$ is zero on U_i for $q > 0$. Thus the spectral sequence degenerates and we conclude that the cohomology modules $H^q(X, \mathcal{F})$ are computed by

$$\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots$$

Next, note that the base change of our hypercovering to S' is a hypercovering of $X' = S' \times_S X$. The schemes $S' \times_S U_i$ are affine too and we have

$$\left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F}\right)(S' \times_S U_i) = N \otimes_A \mathcal{F}(U_i)$$

In this way we conclude that the cohomology modules $H^q(X', (f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F})$ are computed by

$$N \otimes_A \left(\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots\right)$$

Since N is flat over A, we conclude that

$$H^q(X', (f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F}) = N \otimes_A H^q(X, \mathcal{F})$$

Since this is the translation into algebra of the statement we had to show the proof is complete. □

30.8. Cohomology of projective space

01XS In this section we compute the cohomology of the twists of the structure sheaf on $\mathbf{P}^n_S$ over a scheme S. Recall that $\mathbf{P}^n_S$ was defined as the fibre product $\mathbf{P}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ in Constructions, Definition 27.13.2. It was shown to be equal to

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n])$$

in Constructions, Lemma 27.21.5. In particular, projective space is a particular case of a projective bundle. If $S = \mathrm{Spec}(R)$ is affine then we have

$$\mathbf{P}^n_S = \mathbf{P}^n_R = \mathrm{Proj}(R[T_0, \ldots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n_S}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \ldots, T_n]$ is a graded R-algebra where each T_i is homogeneous of degree 1. Denote $(R[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite free R-module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ denotes the $\mathbf{Z}$-graded ring with $\frac{1}{T_i}$ in degree -1. In particular the $\mathbf{Z}$-graded $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \ldots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n \binom{n+d}{d}$ for $d \leq -n-1$.

01XT **Lemma 30.8.1.** *Let R be a ring. Let $n \geq 0$ be an integer. We have* [**DG67**, III Proposition 2.1.12]

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0,\ d \geq 0 \\ \left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} \quad q = n,\ d < 0 \\ 0 & \text{if} \quad q, n, d \text{ not as above} \end{cases}$$

as R-modules.

Proof. We will use the standard affine open covering

$$\mathcal{U} : \mathbf{P}^n_R = \bigcup\nolimits_{i=0}^{n} D_+(T_i)$$

to compute the cohomology using the Čech complex. This is permissible by Lemma 30.2.6 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 27.8). In fact, we can use the alternating or ordered Čech complex according to Cohomology, Lemmas 20.23.3 and 20.23.6.

The ordering we will use on $\{0, \ldots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{i_0 < \ldots < i_p} (R[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

see Cohomology, Section 20.23. Note that each term of this complex has a natural $\mathbf{Z}^{n+1}$-grading. Namely, we get this by declaring a monomial $T_0^{e_0} \ldots T_n^{e_n}$ to be homogeneous with weight $(e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summands on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \ldots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this, set

$$NEG(\vec{e}) = \{i \in \{0, \ldots, n\} \mid e_i < 0\}.$$

With this notation the weight $\vec{e}$ summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Čech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus\nolimits_{i_0 < \ldots < i_p,\ NEG(\vec{e}) \subset \{i_0, \ldots, i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \ldots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \ldots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \ldots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T_0^{-e_0} \ldots T_n^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T_0^{-e_0} \ldots T_n^{-e_n}} & \text{if} \quad q = n \\ 0 & \text{if} \quad \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \ldots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open covering

$$\mathcal{V} : \mathrm{Spec}(R) = \bigcup\nolimits_{i \in \{0,\ldots,n\}} V_i$$

where $V_i = \mathrm{Spec}(R)$ for all i. Consider the alternating Čech complex

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $\mathrm{Spec}(R)$. Hence we see that $H^p(\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 30.2.1 and its proof. Note that also $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})$ has a summand R for every $i_0 < \ldots < i_p$ and has exactly the same differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T_0^{e_0} \ldots T_n^{e_n} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \ldots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \ldots, n\}$. Pick an index $i_{\mathrm{fix}} \notin NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \to \check{\mathcal{C}}^p(\vec{e})$$

given by the rule that for $i_0 < \ldots < i_p$ we have

$$h(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad p \notin \{0, \ldots, n-1\} \\ 0 & \text{if} \quad i_{\mathrm{fix}} \in \{i_0, \ldots, i_p\} \\ s_{i_{\mathrm{fix}} i_0 \ldots i_p} & \text{if} \quad i_{\mathrm{fix}} < i_0 \\ (-1)^a s_{i_0 \ldots i_{a-1} i_{\mathrm{fix}} i_a \ldots i_p} & \text{if} \quad i_{a-1} < i_{\mathrm{fix}} < i_a \\ (-1)^{p+1} s_{i_0 \ldots i_p i_{\mathrm{fix}}} & \text{if} \quad i_p < i_{\mathrm{fix}} \end{cases}$$

Please compare with the proof of Lemma 30.2.1. This makes sense because we have

$$NEG(\vec{e}) \subset \{i_0, \ldots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\mathrm{fix}}, i_0, \ldots, i_p\}$$

The exact same (combinatorial) computation[1] as in the proof of Lemma 30.2.1 shows that

$$(hd + dh)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. □

In the following lemma we are going to use the pairing of free R-modules

$$R[T_0, \ldots, T_n] \times \frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \ldots T_n} \text{ in } fg.$$

In other words, the basis element $T_0^{e_0} \ldots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \ldots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i, and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \right)_d = \text{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 30.8.1 as saying that

01XU (30.8.1.1) $$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n \\ \text{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R) & \text{if } q = n \end{cases}$$

01XV **Lemma 30.8.2.** *The identifications of Equation (30.8.1.1) are compatible with base change w.r.t. ring maps $R \to R'$. Moreover, for any $f \in R[T_0, \ldots, T_n]$ homogeneous of degree m the map multiplication by f*

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + m)$$

induces the map on the cohomology group via the identifications of Equation (30.8.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \ldots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

on H^n.

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[1] For example, suppose that $i_0 < \ldots < i_p$ is such that $i_{\text{fix}} \notin \{i_0, \ldots, i_p\}$ and that $i_{a-1} < i_{\text{fix}} < i_a$ for some $1 \leq a \leq p$. Then we have

$$\begin{aligned}
&(dh + hd)(s)_{i_0 \ldots i_p} \\
&= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + (-1)^a d(s)_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} \\
&= \sum\nolimits_{j=0}^{a-1} (-1)^{j+a-1} s_{i_0 \ldots \hat{i}_j \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} + \sum\nolimits_{j=a}^{p} (-1)^{j+a} s_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots \hat{i}_j \ldots i_p} + \\
&\sum\nolimits_{j=0}^{a-1} (-1)^{a+j} s_{i_0 \ldots \hat{i}_j \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} + (-1)^{2a} s_{i_0 \ldots i_p} + \sum\nolimits_{j=a}^{p} (-1)^{a+j+1} s_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}$$

as desired. The other cases are similar.

Proof. Suppose that $R \to R'$ is a ring map. Let $\mathcal{U}$ be the standard affine open covering of $\mathbf{P}^n_R$, and let $\mathcal{U}'$ be the standard affine open covering of $\mathbf{P}^n_{R'}$. Note that $\mathcal{U}'$ is the pullback of the covering $\mathcal{U}$ under the canonical morphism $\mathbf{P}^n_{R'} \to \mathbf{P}^n_R$. Hence there is a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}^n_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}', \mathcal{O}_{\mathbf{P}^n_{R'}}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 20.15.1. It is clear from the computations in the proof of Lemma 30.8.1 that this map of Čech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Čech complex over R simply map to the corresponding basis elements for the Čech complex over R'.) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \ldots, T_n]$ both of the same degree m. Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g. Moreover, since cohomology is a functor, a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology groups or complex in sight has the structure of an R-module or complex of R-modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}^n_R}$-linear map

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d+1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Čech complexes

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}^n_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}^n_R}(d+1))$$

We are going to use the notation introduced in the proof of Lemma 30.8.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}^n_R}(d+1)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e}+\vec{b}_i)$ where $\vec{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0))$ the ith basis vector. In other words, it maps the summand of H^n corresponding to $\vec{e}$ with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$ (which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \ldots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

This proves the lemma. □

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \ldots, T_n]$ is a graded $\mathcal{O}_S$-module where each T_i is homogeneous of degree 1. Denote $(\mathcal{O}_S[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S.

01XW **Lemma 30.8.3.** *Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism*

$$f : \mathbf{P}^n_S \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}^n_S}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \ldots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (30.8.1.1) are compatible with affine base change by Lemma 30.8.2. □

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$, see Modules, Section 17.14. In this case we think of $\mathrm{Sym}(\mathcal{E})$ as a graded $\mathcal{O}_S$-module where $\mathcal{E}$ is the graded part of degree 1. And $\mathrm{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S. Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

01XX **Lemma 30.8.4.** *Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. Consider the structure morphism*

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} Sym^d(\mathcal{E}) & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(Sym^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1} \mathcal{E}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n+1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (More on Algebra, Lemma 15.28.5). In other words there is an exact complex

$$0 \to \pi^*(\wedge^{n+1}\mathcal{E})(-n-1) \to \ldots \to \pi^*(\wedge^i \mathcal{E})(-i) \to \ldots \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to 0$$

We will think of the term $\pi^*(\wedge^i \mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 13.21.3. Namely, we see that there is a spectral sequence with terms

$$E_1^{p,q} = R^q \pi_* \left(\pi^*(\wedge^{-p}\mathcal{E})(p)\right)$$

converging to zero! By the projection formula (Cohomology, Lemma 20.54.2) we have

$$E_1^{p,q} = \wedge^{-p}\mathcal{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(p)\right).$$

Note that locally on S the sheaf $\mathcal{E}$ is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \to S$ can be identified with $\mathbf{P}^n_S \to S$. Hence locally on S we can use the result of Lemmas 30.8.1, 30.8.2, or 30.8.3. It follows that $E_1^{p,q} = 0$ unless (p,q) is $(0,0)$ or $(-n-1,n)$. The nonzero terms are

$$E_1^{0,0} = \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S$$
$$E_1^{-n-1,n} = R^n\pi_*\left(\pi^*(\wedge^{n+1}\mathcal{E})(-n-1)\right) = \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right)$$

Hence there can only be one nonzero differential in the spectral sequence namely the map $d_{n+1}^{-n-1,n} : E_{n+1}^{-n-1,n} \to E_{n+1}^{0,0}$ which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Thus $E_1^{p,q} = E_{n+1}^{p,q}$ and we obtain a canonical isomorphism

$$\wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right) = R^n\pi_*\left(\pi^*(\wedge^{n+1}\mathcal{E})(-n-1)\right) \xrightarrow{d_{n+1}^{-n-1,n}} \mathcal{O}_S$$

Since $\wedge^{n+1}\mathcal{E}$ is an invertible sheaf, this implies that $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1}\mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 30.8.1, 30.8.2, or 30.8.3 the statements on vanishing of the lemma. Moreover the result on $R^0\pi_*$ is clear as well, since there are canonical maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d. It remains to show that the description of $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In order to do this we consider the map

$$\pi^*(\mathrm{Sym}^{-d-n-1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n\pi_*$ and the projection formula (see above) we get a map

$$\mathrm{Sym}^{-d-n-1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1}\mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 30.8.1, 30.8.2, or 30.8.3 it follows that this map induces a perfect pairing between $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\mathrm{Sym}^{-d-n-1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. □

30.9. Coherent sheaves on locally Noetherian schemes

01XY We have defined the notion of a coherent module on any ringed space in Modules, Section 17.12. Although it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

01XZ **Lemma 30.9.1.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is coherent,*
2. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
3. *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
4. *for any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module, and*

(5) *there exists an affine open covering* $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ *such that each* $\mathcal{F}|_{U_i} = \widetilde{M_i}$ *with* M_i *a finite* A_i*-module.*

In particular $\mathcal{O}_X$ *is coherent, any invertible* $\mathcal{O}_X$*-module is coherent, and more generally any finite locally free* $\mathcal{O}_X$*-module is coherent.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) hold in general, see Modules, Lemma 17.12.2. If $\mathcal{F}$ is finitely presented then $\mathcal{F}$ is quasi-coherent, see Modules, Lemma 17.11.2. Hence also (3) $\Rightarrow$ (2).

Assume $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module. By Properties, Lemma 28.16.1 we see that on any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0.$$

Hence $\mathcal{F}$ is of finite presentation by Properties, Lemma 28.16.2. In other words (2) $\Rightarrow$ (3).

By Modules, Lemma 17.12.5 it suffices to show that $\mathcal{O}_X$ is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, n$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = \mathrm{Spec}(A)$ is affine. In this case $f_1, \ldots, f_n \in A$ are elements of A. Since A is Noetherian, see Properties, Lemma 28.5.2 the kernel K of the map $\bigoplus_{i=1,\ldots,n} A \to A$ is a finite A-module. See for example Algebra, Lemma 10.51.1. As the functor $\widetilde{}$ is exact, see Schemes, Lemma 26.5.4 we get an exact sequence

$$\widetilde{K} \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$$

and by Properties, Lemma 28.16.1 again we see that $\widetilde{K}$ is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 28.16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 26.24 show that (5) implies that $\mathcal{F}$ is quasi-coherent and it is clear that (5) implies that $\mathcal{F}$ is of finite type. Hence (5) implies (2) and we win. □

01Y0 **Lemma 30.9.2.** *Let* X *be a locally Noetherian scheme. The category of coherent* $\mathcal{O}_X$*-modules is abelian. More precisely, the kernel and cokernel of a map of coherent* $\mathcal{O}_X$*-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. This is a restatement of Modules, Lemma 17.12.4 in a particular case. □

The following lemma does not always hold for the category of coherent $\mathcal{O}_X$-modules on a general ringed space X.

01Y1 **Lemma 30.9.3.** *Let* X *be a locally Noetherian scheme. Let* $\mathcal{F}$ *be a coherent* $\mathcal{O}_X$*-module. Any quasi-coherent submodule of* $\mathcal{F}$ *is coherent. Any quasi-coherent quotient module of* $\mathcal{F}$ *is coherent.*

Proof. We may assume that X is affine, say $X = \mathrm{Spec}(A)$. Properties, Lemma 28.5.2 implies that A is Noetherian. Lemma 30.9.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A-module is a finite A-module, see for example Algebra, Lemma 10.51.1. □

01Y2 **Lemma 30.9.4.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. It is shown in Modules, Lemma 17.22.6 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 17.16.6 □

01Y3 **Lemma 30.9.5.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$.*

(1) *If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.*
(2) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.*
(3) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.*
(4) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.*

Proof. See Modules, Lemmas 17.9.4, 17.9.5, and 17.12.6. □

01Y4 **Lemma 30.9.6.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \to \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$-modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi_x = \psi$.*

Proof. In view of Lemma 30.9.1 this is a reformulation of Modules, Lemma 17.22.4. □

01Y5 **Lemma 30.9.7.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{Supp}(\mathcal{F})$ is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5.*

Proof. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and let $\mathcal{G}$ be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \mathrm{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 29.15.5 and 29.15.6. Finally, $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module by Lemma 30.9.1 □

087T **Lemma 30.9.8.** *Let $i : Z \to X$ be a closed immersion of locally Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms, Lemma 29.4.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms, Lemma 29.4.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. By Modules, Lemma 17.13.3 we see that $\mathcal{G}$ is of finite type. Hence $\mathcal{G}$ is coherent by Lemma 30.9.1. Thus the functor is also essentially surjective as desired. □

01Y6 **Lemma 30.9.9.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^p f_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.*

Proof. The higher direct images vanish by Lemma 30.2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 29.15.6) and hence the statement makes

sense. Let $\mathrm{Spec}(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 29.44.1 we see that $f^{-1}(V) = \mathrm{Spec}(B)$ with $A \to B$ finite. Lemma 30.9.1 turns the statement of the lemma into the following algebra fact: If M is a finite B-module, then M is also finite viewed as a A-module, see Algebra, Lemma 10.7.2. □

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (Proposition 30.19.1).

0B3J **Lemma 30.9.10.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\mathit{Supp}(\mathcal{F})) \leq 0$. Then $\mathcal{F}$ is generated by global sections and $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. By Lemma 30.9.7 we see that $\mathcal{F} = i_*\mathcal{G}$ where $i : Z \to X$ is the inclusion of the scheme theoretic support of $\mathcal{F}$ and where $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. Since the dimension of Z is 0, we see Z is a disjoint union of affines (Properties, Lemma 28.10.5). Hence $\mathcal{G}$ is globally generated and the higher cohomology groups of $\mathcal{G}$ are zero (Lemma 30.2.2). Hence $\mathcal{F} = i_*\mathcal{G}$ is globally generated. Since the cohomologies of $\mathcal{F}$ and $\mathcal{G}$ agree (Lemma 30.2.4 applies as a closed immersion is affine) we conclude that the higher cohomology groups of $\mathcal{F}$ are zero. □

0CYJ **Lemma 30.9.11.** *Let X be a scheme. Let $j : U \to X$ be the inclusion of an open. Let $T \subset X$ be a closed subset contained in U. If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module with $\mathit{Supp}(\mathcal{F}) \subset T$, then $j_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.*

Proof. Consider the open covering $X = U \cup (X \setminus T)$. Then $j_*\mathcal{F}|_U = \mathcal{F}$ is coherent and $j_*\mathcal{F}|_{X\setminus T} = 0$ is also coherent. Hence $j_*\mathcal{F}$ is coherent. □

30.10. Coherent sheaves on Noetherian schemes

01Y7 In this section we mention some properties of coherent sheaves on Noetherian schemes.

01Y8 **Lemma 30.10.1.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 10.51.1. Hence the lemma follows. □

01Y9 **Lemma 30.10.2.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $\mathit{Supp}(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. This follows immediately from Algebra, Lemma 10.62.4 because X has a finite covering by spectra of Noetherian rings. □

01YA **Lemma 30.10.3** (Artin-Rees)**.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. This follows immediately from Algebra, Lemma 10.51.2 because X has a finite covering by spectra of Noetherian rings. □

0GN6 **Lemma 30.10.4.** *Let X be a Noetherian scheme. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.*

Proof. This is a reformulation of Properties, Lemma 28.22.3 in view of the fact that a finite type quasi-coherent $\mathcal{O}_X$-module is coherent by Lemma 30.9.1. □

01YB **Lemma 30.10.5.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second map is an isomorphism. It is injective by Properties, Lemma 28.25.3. Since $\mathcal{F}$ is the union of its coherent submodules, see Properties, Lemma 28.22.3 (and Lemma 30.9.1) we may and do assume that $\mathcal{F}$ is coherent to prove surjectivity. Let $\mathcal{F}_n$ denote the quasi-coherent subsheaf of $\mathcal{F}$ consisting of sections annihilated by $\mathcal{I}^n$, see Properties, Lemma 28.25.3. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$ by Lemma 30.10.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n. By Artin-Rees (Lemma 30.10.3) there exists an $c \geq 0$ such that $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^{m-c}\mathcal{H}$. Picking $m = n + c$ we get $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^n\mathcal{H} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m\mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by $\mathcal{I}^N$ is zero for all $N \geq 0$. Hence by Properties, Lemma 28.25.3 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{array}{ccc} \operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}') \\ \downarrow & & \downarrow \\ \operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \end{array}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective as desired.

Next, we prove the first arrow of the lemma is a bijection. By Lemma 30.9.1 the sheaf $\mathcal{G}$ is of finite presentation and hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is quasi-coherent, see Schemes, Section 26.24. By definition we have

$$\mathcal{H}(U) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to $\mathcal{H}$ and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \to \mathcal{H}$ which recovers ψ on restriction to U. By Modules, Lemma 17.22.1 φ corresponds to a map

$$\varphi : \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of $\mathcal{I}^n$ and $\mathcal{G}$ and not the product. We will show that, at the cost of increasing n, the difference is irrelevant. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{I}^n\mathcal{G} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Note that $\mathcal{I}^n\mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m}\mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{m+n}\mathcal{G} \to \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have proved surjectivity of the arrow. We omit the proof of injectivity. □

0FD0 **Lemma 30.10.6.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $U \subset X$ be open and let $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ be an $\mathcal{O}_U$-module map. Then there exists a coherent submodule $\mathcal{F}' \subset \mathcal{F}$ agreeing with $\mathcal{F}$ over U such that φ extends to $\varphi' : \mathcal{F}' \to \mathcal{G}$.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the reduced induced scheme structure on $X \setminus U$. If X is Noetherian, then Lemma 30.10.5 tells us that we can take $\mathcal{F}' = \mathcal{I}^n\mathcal{F}$ for some n. The general case will follow from this using Zorn's lemma.

Consider the set of triples $(U', \mathcal{F}', \varphi')$ where $U \subset U' \subset X$ is open, $\mathcal{F}' \subset \mathcal{F}|_{U'}$ is a coherent subsheaf agreeing with $\mathcal{F}$ over U, and $\varphi' : \mathcal{F}' \to \mathcal{G}|_{U'}$ restricts to φ over U. We say $(U'', \mathcal{F}'', \varphi'') \geq (U', \mathcal{F}', \varphi')$ if and only if $U'' \supset U'$, $\mathcal{F}''|_{U'} = \mathcal{F}'$, and $\varphi''|_{U'} = \varphi'$. It is clear that if we have a totally ordered collection of triples $(U_i, \mathcal{F}_i, \varphi_i)$, then we can glue the $\mathcal{F}_i$ to a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ over $U' = \bigcup U_i$ and extend φ to a map $\varphi' : \mathcal{F}' \to \mathcal{G}|_{U'}$. Hence any totally ordered subset of triples has an upper bound. Finally, suppose that $(U', \mathcal{F}', \varphi')$ is any triple but $U' \neq X$. Then we can choose an affine open $W \subset X$ which is not contained in U'. By the result of the first paragraph we can extend the subsheaf $\mathcal{F}'|_{W \cap U'}$ and the restriction $\varphi'|_{W \cap U'}$ to some subsheaf $\mathcal{F}'' \subset \mathcal{F}|_W$ and map $\varphi'' : \mathcal{F}'' \to \mathcal{G}|_W$. Of course the agreement between $(\mathcal{F}', \varphi')$ and $(\mathcal{F}'', \varphi'')$ over $W \cap U'$ exactly means that we can extend this to a triple $(U' \cup W, \mathcal{F}''', \varphi''')$. Hence any maximal triple $(U', \mathcal{F}', \varphi')$ (which exist by Zorn's lemma) must have $U' = X$ and the proof is complete. □

30.11. Depth

0340 In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 28.12.

0341 **Definition 30.11.1.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer.

(1) We say $\mathcal{F}$ has *depth k at a point x* of X if $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
(2) We say X has *depth k at a point x* of X if $\text{depth}(\mathcal{O}_{X,x}) = k$.

(3) We say $\mathcal{F}$ has property (S_k) if
$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\text{Supp}(\mathcal{F}_x)))$$
for all $x \in X$.
(4) We say X has property (S_k) if $\mathcal{O}_X$ has property (S_k).

Any coherent sheaf satisfies condition (S_0). Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 31.4.3.

0EBC **Lemma 30.11.2.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules and $x \in X$.*

(1) *If $\mathcal{G}_x$ has depth ≥ 1, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 1.*
(2) *If $\mathcal{G}_x$ has depth ≥ 2, then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 2.*

Proof. Observe that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_X$-module by Lemma 30.9.4. Coherent modules are of finite presentation (Lemma 30.9.1) hence taking stalks commutes with taking $\mathcal{H}om$ and Hom, see Modules, Lemma 17.22.4. Thus we reduce to the case of finite modules over local rings which is More on Algebra, Lemma 15.23.10. □

0AXQ **Lemma 30.11.3.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{G}$ has property (S_1), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_1).*
(2) *If $\mathcal{G}$ has property (S_2), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_2).*

Proof. Follows immediately from Lemma 30.11.2 and the definitions. □

We have seen in Properties, Lemma 28.12.3 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k. Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

0343 **Definition 30.11.4.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *Cohen-Macaulay* if and only if (S_k) holds for all $k \geq 0$.

0B3K **Lemma 30.11.5.** *Let X be a regular scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is Cohen-Macaulay and Supp$(\mathcal{F}) = X$,*
(2) *$\mathcal{F}$ is finite locally free of rank > 0.*

Proof. Let $x \in X$. If (2) holds, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank > 0. Hence $\text{depth}(\mathcal{F}_x) = \dim(\mathcal{O}_{X,x})$ because a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3). Conversely, if (1) holds, then $\mathcal{F}_x$ is a maximal Cohen-Macaulay module over $\mathcal{O}_{X,x}$ (Algebra, Definition 10.103.8). Hence $\mathcal{F}_x$ is free by Algebra, Lemma 10.106.6. □

30.12. Devissage of coherent sheaves

01YC Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \to X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where $\mathcal{G}$ is a coherent sheaf on Z. In particular we are interested in these sheaves when $\mathcal{G}$ is a torsion free rank 1 sheaf. For example $\mathcal{G}$ could be a nonzero sheaf of ideals on Z, or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 30.9. The support of a coherent sheaf is closed, see Modules, Lemma 17.9.6.

01YD **Lemma 30.12.1.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\text{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\text{Supp}(\mathcal{G}') \subset Z'$ and $\text{Supp}(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z, see Schemes, Lemma 26.12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 30.10.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

01YE **Lemma 30.12.2.** *Let X be a Noetherian scheme. Let $i : Z \to X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exist an integer $r \geq 0$ and a coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ.

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of $\mathcal{F}$ which are annihilated by $\mathcal{J}$. It is a quasi-coherent sheaf by Properties, Lemma 28.24.2. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 28.24.2. By Lemma 30.9.5 we see that $\mathcal{F}' \to \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ. Hence we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\mathcal{F}$ is annihilated by $\mathcal{J}$.

Assume $\mathcal{J}\mathcal{F} = 0$. By Lemma 30.9.8 we can write $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf $\mathcal{G}$ on Z. Suppose we can find a morphism $\mathcal{I}^{\oplus r} \to \mathcal{G}$ which is an isomorphism in a neighbourhood of the generic point ξ of Z. Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ. Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since $\mathcal{F}_\xi$ is a finite $\mathcal{O}_\xi$-module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and $\mathcal{F}$ have isomorphic stalks at ξ. By Lemma 30.9.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ which is an isomorphism at ξ, and hence an isomorphism in a neighbourhood of ξ by Lemma 30.9.5. By Schemes, Lemma 26.12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Y \subset X$ is the complement of U. By Lemma 30.10.5 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our ψ over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a

map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). $\square$

01YF **Lemma 30.12.3.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exist an integral closed subscheme $Z_j \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z. Then we can apply Lemma 30.12.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let $\mathcal{J}$ be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z, see Schemes, Lemma 26.12.4. By Lemma 30.10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 30.12.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined as the quotient. Since $\mathcal{Q}$ is zero in a neighbourhood of ξ by the lemma just cited we see that the support of $\mathcal{Q}$ is strictly smaller than Z. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of Z. But then clearly $\mathcal{F}$ does too, which is our final contradiction. $\square$

01YG **Lemma 30.12.4.** *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) *For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 30.12.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. □

01YH **Lemma 30.12.5.** *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X with support contained in Z_0 such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
(2) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
(3) *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$\mathrm{Supp}(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with support contained in Z_0 with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. Or, if $\mathcal{F}$ has property $\mathcal{P}$ and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$ then so does the last one. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 30.12.3) whose subquotients have property $\mathcal{P}$ according to (2).

Let $\mathcal{G}$ be as in (3). By Lemma 30.12.2 there exist a sheaf of ideals $\mathcal{I}$ on Z_0, an integer $r \geq 1$, and a short exact sequence
$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$
where the support of $\mathcal{Q}$ is strictly contained in Z_0. By (3)(c) we see that $r = 1$. Since $\mathcal{Q}$ has property $\mathcal{P}$ too we conclude that $(Z_0 \to X)_*\mathcal{I}$ has property $\mathcal{P}$.

Next, suppose that $\mathcal{I}' \neq 0$ is another quasi-coherent sheaf of ideals on Z_0. Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences
$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q} \to 0$$

and

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I}' \to \mathcal{Q}' \to 0.$$

Note that the support of the coherent sheaves $\mathcal{Q}$ and $\mathcal{Q}'$ are strictly contained in Z_0. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ have property $\mathcal{P}$ (see above). Hence we conclude using (1) that $(Z_0 \to X)_*\mathcal{I}''$ and $(Z_0 \to X)_*\mathcal{I}'$ both have $\mathcal{P}$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 30.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

01YI **Lemma 30.12.6.** *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
(2) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. According to Lemma 30.12.4 it suffices to show that for all integral closed subschemes $Z \subset X$ and all quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that $\mathcal{P}$ fails for $(Z_0 \to X)_*\mathcal{I}_0$ for some quasi-coherent sheaf of ideals $\mathcal{I}_0 \subset \mathcal{O}_{Z_0}$, but $\mathcal{P}$ does hold for $(Z \to X)_*\mathcal{I}$ for all integral closed subschemes $Z \subset Z_0$, $Z \neq Z_0$ and quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$. Since we have the existence of $\mathcal{G}$ for Z_0 by part (2), according to Lemma 30.12.5 this cannot happen. □

01YL **Lemma 30.12.7.** *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

   if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*
(3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
(4) *There exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. Note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 30.12.3) whose subquotients have property $\mathcal{P}$ according to (3).

Let us denote $i : Z_0 \to X$ the closed immersion. Consider a coherent sheaf $\mathcal{G}$ as in (4). By Lemma 30.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to i_*\mathcal{I}^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z_0. Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z_0 contained in $\mathcal{I}$. Then we also get a short exact sequence

$$0 \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{G} \to \mathcal{Q}' \to 0$$

where $\mathcal{Q}'$ has support properly contained in Z_0. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the support of $\mathcal{Q}'$ (for example the ideal corresponding to the reduced induced closed subscheme structure on the support of $\mathcal{Q}'$). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 30.10.2 we see that $\mathcal{J}^n\mathcal{Q}' = 0$ for some n. Hence $\mathcal{J}^n\mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property $\mathcal{P}$ holds. Hence we get a short exact sequence

$$0 \to \mathcal{G}' \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{Q}'' \to 0$$

where $\mathcal{Q}''$ has support properly contained in Z_0. Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies $\mathcal{P}$. Hence we see that $i_*\mathcal{I}'$ satisfies $\mathcal{P}$ by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I}' = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \to i_*(\mathcal{I}') \to i_*(\mathcal{I}'') \to \mathcal{Q}''' \to 0$$

where $\mathcal{Q}'''$ has support properly contained in Z_0. Hence we conclude that property $\mathcal{P}$ holds for $i_*\mathcal{I}''$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 30.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

01YM **Lemma 30.12.8.** *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*

(3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$\mathrm{Supp}(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*

(c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Follows from Lemma 30.12.7 in exactly the same way that Lemma 30.12.6 follows from Lemma 30.12.5. □

30.13. Finite morphisms and affines

01YN In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 32.11.1 and Proposition 32.11.2).

01YO **Lemma 30.13.1.** *Let $f : Y \to X$ be a morphism of schemes. Assume f is finite, surjective and X locally Noetherian. Let $Z \subset X$ be an integral closed subscheme with generic point ξ. Then there exists a coherent sheaf $\mathcal{F}$ on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$.*

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 29.15.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ. Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 26.12.4. Hence the sheaf $\mathcal{F} = (Z' \to Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 30.9.9). Look at the commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ. Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ, see Algebra, Lemmas 10.36.3 and 10.36.20. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*$(something) implies the lemma. □

01YP **Lemma 30.13.2.** *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If the morphism f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus

we may write $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \widetilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A-module associated to the B-module M. □

01YQ **Lemma 30.13.3.** *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *f finite,*
(2) *f surjective,*
(3) *Y affine, and*
(4) *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then it follows that X is affine from either Lemma 30.3.1 or Lemma 30.3.2.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 30.12.8. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 30.13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 30.12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 30.13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 30.13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 30.2.2. Since f is finite, hence affine, we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$$

by Lemma 30.2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies $\mathcal{P}$. This verifies property (3)(c) of Lemma 30.12.8 as desired. □

30.14. Coherent sheaves on Proj, I

01YR In this section we discuss coherent sheaves on $\mathrm{Proj}(A)$ where A is a Noetherian graded ring generated by A_1 over A_0. In the next section we discuss what happens if A is not generated by degree 1 elements. First, we formulate an all-in-one result for projective space over a Noetherian ring.

01YS **Lemma 30.14.1.** *Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have the following:*

(1) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \longrightarrow \mathcal{F}.$$

(2) *We have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.*

(3) *For any i the cohomology group $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module.*
(4) *If $i > 0$, then $H^i(\mathbf{P}^n_R, \mathcal{F}(d)) = 0$ for all d large enough.*
(5) *For any $k \in \mathbf{Z}$ the graded $R[T_0, \ldots, T_n]$-module*

$$\bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is a finite $R[T_0, \ldots, T_n]$-module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}^n_R}(1)$ is an ample invertible sheaf on the scheme $\mathbf{P}^n_R$. This follows directly from the definition since $\mathbf{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 28.26.13 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}^n_R}(1)$. On the other hand coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 30.9.1. Thus we see (1).

Projective n-space $\mathbf{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \ldots, n$, see Constructions, Lemma 27.13.3. Hence we see that for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$, see Lemma 30.4.2. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on $\mathbf{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$ by (2). Suppose we know the result for $i+1$ and we want to show the result for i. (If $i = 0$, then part (4) is vacuous.) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_R$. Choose a surjection as in (1) and denote $\mathcal{G}$ the kernel so that we have a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \to \mathcal{F} \to 0$$

By Lemma 30.9.2 we see that $\mathcal{G}$ is coherent. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j)) \to H^i(\mathbf{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}).$$

By induction assumption the right R-module is finite and by Lemma 30.8.1 the left R-module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}^n_R}(d)$ is invertible we see that twisting on $\mathbf{P}^n_R$ is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 27.10.1). This means that for all $d \in \mathbf{Z}$ the sequence

$$0 \to \mathcal{G}(d) \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d) \to \mathcal{F}(d) \to 0$$

is short exact. The resulting cohomology sequence is

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^i(\mathbf{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}(d)).$$

By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 30.8.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map

$$H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is surjective by the vanishing of $H^1(\mathbf{P}^n_R, \mathcal{G}(d))$ we just proved. In other words, the module

$$M_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus\nolimits_{j=1,\ldots,r} R[T_0, \ldots, T_n](d_j)$$

by our computations in Section 30.8. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{array}{ccc} N_{k_-} & \longleftarrow & N_{k_+} \\ & & \downarrow \\ M_k & \longleftarrow & M_{k_+} \end{array}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \ldots, T_n]$-module. Hence N_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module because it is a submodule of a finitely generated module and the ring $R[T_0, \ldots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module. The quotient M_k/M_{k_+} is finite as an R-module since it is a finite direct sum of the finite R-modules $H^0(\mathbf{P}^n_R, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \ldots, T_n]$-module. In other words, we have sandwiched M_k between two finite $R[T_0, \ldots, T_n]$-modules and we win. □

0AG6 **Lemma 30.14.2.** *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = Proj(A)$. Then X is a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$

(2) *For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0-module.*
(3) *If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.*
(4) *For any $k \in \mathbf{Z}$ the graded A-module*

$$\bigoplus\nolimits_{d \geq k} H^0(X, \mathcal{F}(d))$$

is a finite A-module.

Proof. By assumption there exists a surjection of graded A_0-algebras

$$A_0[T_0, \ldots, T_n] \longrightarrow A$$

where $\deg(T_j) = 1$ for $j = 0, \ldots, n$. By Constructions, Lemma 27.11.5 this defines a closed immersion $i : X \to \mathbf{P}^n_{A_0}$ such that $i^*\mathcal{O}_{\mathbf{P}^n_{A_0}}(1) = \mathcal{O}_X(1)$. In particular, X is Noetherian as a closed subscheme of the Noetherian scheme $\mathbf{P}^n_{A_0}$. We claim that the results of the lemma for $\mathcal{F}$ follow from the corresponding results of Lemma

30.14.1 for the coherent sheaf $i_*\mathcal{F}$ (Lemma 30.9.8) on $\mathbf{P}^n_{A_0}$. For example, by this lemma there exists a surjection

$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_{A_0}}(d_j) \longrightarrow i_*\mathcal{F}.$$

By adjunction this corresponds to a map $\bigoplus_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}$ which is surjective as well. The statements on cohomology follow from the fact that $H^p(X, \mathcal{F}(d)) = H^p(\mathbf{P}^n_{A_0}, i_*\mathcal{F}(d))$ by Lemma 30.2.4. □

0AG7 **Lemma 30.14.3.** *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Let M be a finite graded A-module. Set $X = Proj(A)$ and let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_X$-module on X associated to M. The maps*

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

from Constructions, Lemma 27.10.3 are isomorphisms for all sufficiently large n.

Proof. Because M is a finite A-module we see that $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module, i.e., a coherent $\mathcal{O}_X$-module. Set $N = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \widetilde{M}(n))$. We have to show that the map $M \to N$ of graded A-modules is an isomorphism in all sufficiently large degrees. By Properties, Lemma 28.28.5 we have a canonical isomorphism $\widetilde{N} \to \widetilde{M}$ such that the induced maps $N_n \to N_n = \Gamma(X, \widetilde{M}(n))$ are the identity maps. Thus we have maps $\widetilde{M} \to \widetilde{N} \to \widetilde{M}$ such that for all n the diagram

$$\begin{array}{ccccc} M_n & \longrightarrow & N_n & & \\ \downarrow & & \downarrow & \searrow^{=} & \\ \Gamma(X, \widetilde{M}(n)) & \longrightarrow & \Gamma(X, \widetilde{N}(n)) & \xrightarrow{\cong} & \Gamma(X, \widetilde{M}(n)) \end{array}$$

is commutative. This means that the composition

$$M_n \to \Gamma(X, \widetilde{M}(n)) \to \Gamma(X, \widetilde{N}(n)) \to \Gamma(X, \widetilde{M}(n))$$

is equal to the canonical map $M_n \to \Gamma(X, \widetilde{M}(n))$. Clearly this implies that the composition $\widetilde{M} \to \widetilde{N} \to \widetilde{M}$ is the identity. Hence $\widetilde{M} \to \widetilde{N}$ is an isomorphism. Let $K = \mathrm{Ker}(M \to N)$ and $Q = \mathrm{Coker}(M \to N)$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 27.8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. Recall that A is a Noetherian ring, M is a finitely generated A-module, and N is a graded A-module such that $N' = \bigoplus_{n\geq 0} N_n$ is finitely generated by the last part of Lemma 30.14.2. Hence $K' = \bigoplus_{n\geq 0} K_n$ and $Q' = \bigoplus_{n\geq 0} Q_n$ are finite A-modules. Observe that $\widetilde{Q} = \widetilde{Q'}$ and $\widetilde{K} = \widetilde{K'}$. Thus to finish the proof it suffices to show that a finite A-module K with $\widetilde{K} = 0$ has only finitely many nonzero homogeneous parts K_d with $d \geq 0$. To do this, let $x_1, \ldots, x_r \in K$ be homogeneous generators say sitting in degrees $d_1, \ldots, d_r$. Let $f_1, \ldots, f_n \in A_1$ be elements generating A over A_0. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j+n_{ij}}$: if not then $x_i/f_i^{d_i} \in K_{(f_i)}$ would not be zero, i.e., $\widetilde{K}$ would not be zero. Then we see that K_d is zero for $d > \max_j(d_j + \sum_i n_{ij})$ as every element of K_d is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree d. □

Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Recall that $A_+ = \bigoplus_{n>0} A_n$ is the irrelevant ideal. Let M be a graded A-module. Recall that M is an A_+-power torsion module if

for all $x \in M$ there is an $n \geq 1$ such that $(A_+)^n x = 0$, see More on Algebra, Definition 15.89.1. If M is finitely generated, then we see that this is equivalent to $M_n = 0$ for $n \gg 0$. Sometimes A_+-power torsion modules are called torsion modules. Sometimes a graded A-module M is called torsion free if $x \in M$ with $(A_+)^n x = 0$, $n > 0$ implies $x = 0$. Denote Mod_A the category of graded A-modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,torsion}^{fg}$ the full subcategory of modules M such that $M_n = 0$ for $n \gg 0$.

0BXD **Proposition 30.14.4.** *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = Proj(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence*

$$Mod_A^{fg}/Mod_{A,torsion}^{fg} \longrightarrow Coh(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. The subcategory $\text{Mod}_{A,torsion}^{fg}$ is a Serre subcategory of Mod_A^{fg}, see Homology, Definition 12.10.1. This is clear from the description of objects given above but it also follows from More on Algebra, Lemma 15.89.5. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 12.10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends torsion modules to 0. This is clear because for any $f \in A_+$ homogeneous the module M_f is zero and hence the value $M_{(f)}$ of $\widetilde{M}$ on $D_+(f)$ is zero too.

By Lemma 30.14.2 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $Coh(\mathcal{O}_X) \to \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \to \text{Mod}_A^{fg}/\text{Mod}_{A,torsion}^{fg}$.

By Lemma 30.14.3 the composite left to right to left is isomorphic to the identity functor.

Finally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A-module, so that our functor sends $\mathcal{F}$ to $M_{\geq 0} = \bigoplus_{n \geq 0} M_n$. By Properties, Lemma 28.28.5 the canonical map $\widetilde{M} \to \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \to M$ defines an isomorphism $\widetilde{M_{\geq 0}} \to \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. □

30.15. Coherent sheaves on Proj, II

0BXE In this section we discuss coherent sheaves on $\text{Proj}(A)$ where A is a Noetherian graded ring. Most of the results will be deduced by sleight of hand from the corresponding result in the previous section where we discussed what happens if A is generated by degree 1 elements.

0B5Q **Lemma 30.15.1.** *Let A be a Noetherian graded ring. Set $X = Proj(A)$. Then X is a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$

(2) *For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0-module.*
(3) *If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.*

(4) *For any $k \in \mathbf{Z}$ the graded A-module*

$$\bigoplus\nolimits_{d \geq k} H^0(X, \mathcal{F}(d))$$

is a finite A-module.

Proof. We will prove this by reducing the statement to Lemma 30.14.2. By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let d be a sufficiently divisible integer. Set $A' = A^{(d)}$ with notation as in Algebra, Section 10.56. Then A' is generated over $A'_0 = A_0$ by elements of degree 1, see Algebra, Lemma 10.56.2. Thus Lemma 30.14.2 applies to $X' = \text{Proj}(A')$.

By Constructions, Lemma 27.11.8 there exist an isomorphism of schemes $i : X \to X'$ and isomorphisms $\mathcal{O}_X(nd) \to i^*\mathcal{O}_{X'}(n)$ compatible with the map $A' \to A$ and the maps $A_n \to H^0(X, \mathcal{O}_X(n))$ and $A'_n \to H^0(X', \mathcal{O}_{X'}(n))$. Thus Lemma 30.14.2 implies X is Noetherian and that (1) and (2) hold. To see (3) and (4) we can use that for any fixed k, p, and q we have

$$\bigoplus\nolimits_{dn+q \geq k} H^p(X, \mathcal{F}(dn+q)) = \bigoplus\nolimits_{dn+q \geq k} H^p(X', (i_*\mathcal{F}(q))(n)$$

by the compatibilities above. If $p > 0$, we have the vanishing of the right hand side for k depending on q large enough by Lemma 30.14.2. Since there are only a finite number of congruence classes of integers modulo d, we see that (3) holds for $\mathcal{F}$ on X. If $p = 0$, then we have that the right hand side is a finite A'-module by Lemma 30.14.2. Using the finiteness of congruence classes once more, we find that $\bigoplus_{n \geq k} H^0(X, \mathcal{F}(n))$ is a finite A'-module too. Since the A'-module structure comes from the A-module structure (by the compatibilities mentioned above), we conclude it is finite as an A-module as well. □

0B5R **Lemma 30.15.2.** *Let A be a Noetherian graded ring and let d be the lcm of generators of A over A_0. Let M be a finite graded A-module. Set $X = Proj(A)$ and let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_X$-module on X associated to M. Let $k \in \mathbf{Z}$.*

(1) *$N' = \bigoplus_{n \geq k} H^0(X, \widetilde{M(n)})$ is a finite A-module,*
(2) *$N = \bigoplus_{n \geq k} H^0(X, \widetilde{M}(n))$ is a finite A-module,*
(3) *there is a canonical map $N \to N'$,*
(4) *if k is small enough there is a canonical map $M \to N'$,*
(5) *the map $M_n \to N'_n$ is an isomorphism for $n \gg 0$,*
(6) *$N_n \to N'_n$ is an isomorphism for $d|n$.*

Proof. The map $N \to N'$ in (3) comes from Constructions, Equation (27.10.1.5) by taking global sections.

By Constructions, Equation (27.10.1.6) there is a map of graded A-modules $M \to \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M(n)})$. If the generators of M sit in degrees $\geq k$, then the image is contained in the submodule $N' \subset \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M(n)})$ and we get the map in (4).

By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let $d = \text{lcm}(\deg(f_i))$. Then we see that (6) holds for example by Constructions, Lemma 27.10.4.

Because M is a finite A-module we see that $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module, i.e., a coherent $\mathcal{O}_X$-module. Thus part (2) follows from Lemma 30.15.1.

We will deduce (1) from (2) using a trick. For $q \in \{0, \ldots, d-1\}$ write

$$^qN = \bigoplus\nolimits_{n+q \geq k} H^0(X, \widetilde{M(q)}(n))$$

By part (2) these are finite A-modules. The Noetherian ring A is finite over $A^{(d)} = \bigoplus_{n \geq 0} A_{dn}$, because it is generated by f_i over $A^{(d)}$ and $f_i^d \in A^{(d)}$. Hence ^qN is a finite $A^{(d)}$-module. Moreover, $A^{(d)}$ is Noetherian (follows from Algebra, Lemma 10.57.9). It follows that the $A^{(d)}$-submodule $^qN^{(d)} = \bigoplus_{n \in \mathbf{Z}} {}^qN_{dn}$ is a finite module over $A^{(d)}$. Using the isomorphisms $\widetilde{M(dn+q)} = \widetilde{M(q)}(dn)$ we can write

$$N' = \bigoplus\nolimits_{q \in \{0,\ldots,d-1\}} \bigoplus\nolimits_{dn+q \geq k} H^0(X, \widetilde{M(q)}(dn)) = \bigoplus\nolimits_{q \in \{0,\ldots,d-1\}} {}^qN^{(d)}$$

Thus N' is finite over $A^{(d)}$ and a fortiori finite over A. Thus (1) is true.

Let K be a finite A-module such that $\widetilde{K} = 0$. We claim that $K_n = 0$ for $d|n$ and $n \gg 0$. Arguing as above we see that $K^{(d)}$ is a finite $A^{(d)}$-module. Let $x_1, \ldots, x_m \in K$ be homogeneous generators of $K^{(d)}$ over $A^{(d)}$, say sitting in degrees $d_1, \ldots, d_m$ with $d|d_j$. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j + n_{ij}}$: if not then $x_j/f_i^{d_i/\deg(f_i)} \in K_{(f_i)}$ would not be zero, i.e., $\widetilde{K}$ would not be zero. Here we use that $\deg(f_i)|d|d_j$ for all i, j. We conclude that K_n is zero for n with $d|n$ and $n > \max_j(d_j + \sum_i n_{ij} \deg(f_i))$ as every element of K_n is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree n.

To finish the proof, we have to show that $M \to N'$ is an isomorphism in all sufficiently large degrees. The map $N \to N'$ induces an isomorphism $\widetilde{N} \to \widetilde{N'}$ because on the affine opens $D_+(f_i) = D_+(f_i^d)$ the corresponding modules are isomorphic: $N_{(f_i)} \cong N_{(f_i^d)} \cong N'_{(f_i^d)} \cong N'_{(f_i)}$ by property (6). By Properties, Lemma 28.28.5 we have a canonical isomorphism $\widetilde{N} \to \widetilde{M}$. The composition $\widetilde{N} \to \widetilde{M} \to \widetilde{N'}$ is the isomorphism above (proof omitted; hint: look on standard affine opens to check this). Thus the map $M \to N'$ induces an isomorphism $\widetilde{M} \to \widetilde{N'}$. Let $K = \mathrm{Ker}(M \to N')$ and $Q = \mathrm{Coker}(M \to N')$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 27.8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. By the result of the previous paragraph we see that $K_n = 0$ and $Q_n = 0$ for $d|n$ and $n \gg 0$. At this point we finally see the advantage of using N' over N: the functor $M \rightsquigarrow N'$ is compatible with shifts (immediate from the construction). Thus, repeating the whole argument with M replaced by $M(q)$ we find that $K_n = 0$ and $Q_n = 0$ for $n \equiv q \bmod d$ and $n \gg 0$. Since there are only finitely many congruence classes modulo n the proof is finished. □

Let A be a Noetherian graded ring. Recall that $A_+ = \bigoplus_{n>0} A_n$ is the irrelevant ideal. By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let $d = \mathrm{lcm}(\deg(f_i))$. Let M be a graded A-module. In this situation we

say a homogeneous element $x \in M$ is *irrelevant*[2] if

$$(A_+x)_{nd} = 0 \text{ for all } n \gg 0$$

If $x \in M$ is homogeneous and irrelevant and $f \in A$ is homogeneous, then fx is irrelevant too. Hence the set of irrelevant elements generate a graded submodule $M_{irrelevant} \subset M$. We will say M is *irrelevant* if every homogeneous element of M is irrelevant, i.e., if $M_{irrelevant} = M$. If M is finitely generated, then we see that this is equivalent to $M_{nd} = 0$ for $n \gg 0$. Denote Mod_A the category of graded A-modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,irrelevant}^{fg}$ the full subcategory of irrelevant modules.

0BXF **Proposition 30.15.3.** *Let A be a Noetherian graded ring. Set $X = Proj(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence*

$$Mod_A^{fg}/Mod_{A,irrelevant}^{fg} \longrightarrow Coh(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \longmapsto \bigoplus_{n\geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. We urge the reader to read the proof in the case where A is generated in degree 1 first, see Proposition 30.14.4. Let $f_1, \ldots, f_r \in A$ be homogeneous elements of positive degree which generate A over A_0. Let d be the lcm of the degrees d_i of f_i. Let M be a finite A-module. Let us show that $\widetilde{M}$ is zero if and only if M is an irrelevant graded A-module (as defined above the statement of the proposition). Namely, let $x \in M$ be a homogeneous element. Choose $k \in \mathbf{Z}$ sufficiently small and let $N \to N'$ and $M \to N'$ be as in Lemma 30.15.2. We may also pick l sufficiently large such that $M_n \to N_n$ is an isomorphism for $n \geq l$. If $\widetilde{M}$ is zero, then $N = 0$. Thus for any $f \in A_+$ homogeneous with $\deg(f) + \deg(x) = nd$ and $nd > l$ we see that fx is zero because $N_{nd} \to N'_{nd}$ and $M_{nd} \to N'_{nd}$ are isomorphisms. Hence x is irrelevant. Conversely, assume M is irrelevant. Then M_{nd} is zero for $n \gg 0$ (see discussion above proposition). Clearly this implies that $M_{(f_i)} = M_{(f_i^{d/\deg(f_i)})} = 0$, whence $\widetilde{M} = 0$ by construction.

It follows that the subcategory $\text{Mod}_{A,irrelevant}^{fg}$ is a Serre subcategory of Mod_A^{fg} as the kernel of the exact functor $M \mapsto \widetilde{M}$, see Homology, Lemma 12.10.4 and Constructions, Lemma 27.8.4. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 12.10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends irrelevant modules to 0 which we have shown above.

By Lemma 30.15.1 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \longmapsto \bigoplus_{n\geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $Coh(\mathcal{O}_X) \to \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \to \text{Mod}_A^{fg}/\text{Mod}_{A,irrelevant}^{fg}$.

By Lemma 30.15.2 the composite left to right to left is isomorphic to the identity functor. Namely, let M be a finite graded A-module and let $k \in \mathbf{Z}$ sufficiently small and let $N \to N'$ and $M \to N'$ be as in Lemma 30.15.2. Then the kernel and cokernel of $M \to N'$ are nonzero in only finitely many degrees, hence are irrelevant. Moreover, the kernel and cokernel of the map $N \to N'$ are zero in all sufficiently large degrees divisible by d, hence these are irrelevant modules too. Thus $M \to N'$ and $N \to N'$ are both isomorphisms in the quotient category, as desired.

[2] This is nonstandard notation.

Finally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A-module, so that our functor sends $\mathcal{F}$ to $M_{\geq 0} = \bigoplus_{n\geq 0} M_n$. By Properties, Lemma 28.28.5 the canonical map $\widetilde{M} \to \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \to M$ defines an isomorphism $\widetilde{M_{\geq 0}} \to \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. □

30.16. Higher direct images along projective morphisms

0B5S We first state and prove a result for when the base is affine and then we deduce some results for projective morphisms.

0B5T **Lemma 30.16.1.** *Let R be a Noetherian ring. Let $X \to \mathrm{Spec}(R)$ be a proper morphism. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *The graded ring $A = \bigoplus_{d\geq 0} H^0(X, \mathcal{L}^{\otimes d})$ is a finitely generated R-algebra.*
(2) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{L}^{\otimes d_j} \longrightarrow \mathcal{F}.$$
(3) *For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite R-module.*
(4) *If $p > 0$, then $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all d large enough.*
(5) *For any $k \in \mathbf{Z}$ the graded A-module*
$$\bigoplus\nolimits_{d\geq k} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$$
is a finite A-module.

Proof. By Morphisms, Lemma 29.39.4 there exists a $d > 0$ and an immersion $i : X \to \mathbf{P}^n_R$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_R}(1)$. Since X is proper over R the morphism i is a closed immersion (Morphisms, Lemma 29.41.7). Thus we have $H^i(X, \mathcal{G}) = H^i(\mathbf{P}^n_R, i_*\mathcal{G})$ for any quasi-coherent sheaf $\mathcal{G}$ on X (by Lemma 30.2.4 and the fact that closed immersions are affine, see Morphisms, Lemma 29.11.10). Moreover, if $\mathcal{G}$ is coherent, then $i_*\mathcal{G}$ is coherent as well (Lemma 30.9.8). We will use these facts without further mention.

Proof of (1). Set $S = R[T_0, \ldots, T_n]$ so that $\mathbf{P}^n_R = \mathrm{Proj}(S)$. Observe that A is an S-algebra (but the ring map $S \to A$ is not a homomorphism of graded rings because S_n maps into A_{dn}). By the projection formula (Cohomology, Lemma 20.54.2) we have
$$i_*(\mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(n)$$
for all $n \in \mathbf{Z}$. We conclude that $\bigoplus_{n\geq 0} A_{nd+q}$ is a finite graded S-module by Lemma 30.14.1. Since $A = \bigoplus_{q\in\{0,\ldots,d-1} \bigoplus_{n\geq 0} A_{nd+q}$ we see that A is finite as an S-algebra, hence (1) is true.

Proof of (2). This follows from Properties, Proposition 28.26.13.

Proof of (3). Apply Lemma 30.14.1 and use $H^p(X, \mathcal{F}) = H^p(\mathbf{P}^n_R, i_*\mathcal{F})$.

Proof of (4). Fix $p > 0$. By the projection formula we have
$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(n)$$

for all $n \in \mathbf{Z}$. By Lemma 30.14.1 we conclude that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{nd+q}) = 0$ for $n \gg 0$. Since there are only finitely many congruence classes of integers modulo d this proves (4).

Proof of (5). Fix an integer k. Set $M = \bigoplus_{n \geq k} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Arguing as above we conclude that $\bigoplus_{nd+q \geq k} M_{nd+q}$ is a finite graded S-module. Since $M = \bigoplus_{q \in \{0,\ldots,d-1\}} \bigoplus_{nd+q \geq k} M_{nd+q}$ we see that M is finite as an S-module. Since the S-module structure factors through the ring map $S \to A$, we conclude that M is finite as an A-module. □

02O1 **Lemma 30.16.2.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *S is Noetherian,*
(2) *f is proper,*
(3) *$\mathcal{F}$ is coherent, and*
(4) *$\mathcal{L}$ is relatively ample on X/S.*

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \right) = 0$$

for all $p > 0$.

Proof. Choose a finite affine open covering $S = \bigcup V_j$ and set $X_j = f^{-1}(V_j)$. Clearly, if we solve the question for each of the finitely many systems $(X_j \to V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Thus we may assume S is affine. In this case the vanishing of $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 30.4.6. Thus the required vanishing follows from Lemma 30.16.1 (which applies because $\mathcal{L}$ is ample on X by Morphisms, Lemma 29.39.4). □

02O4 **Lemma 30.16.3.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally projective morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 29.43.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 29.15.6) and hence the statement makes sense. Moreover, by Lemma 30.4.5 the sheaves $R^p f_* \mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 20.7.4). Hence we may assume $S = \operatorname{Spec}(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of $\mathbf{P}^n_R$ for some n, see Morphisms, Lemma 29.43.4. In this case, the sheaves $R^p f_* \mathcal{F}$ are the quasi-coherent sheaves associated to the R-modules $H^p(X, \mathcal{F})$, see Lemma 30.4.6. Hence it suffices to show that R-modules $H^p(X, \mathcal{F})$ are finite R-modules (Lemma 30.9.1). This follows from Lemma 30.16.1 (because the restriction of $\mathcal{O}_{\mathbf{P}^n_R}(1)$ to X is ample on X). □

30.17. Ample invertible sheaves and cohomology

01XO Here is a criterion for ampleness on proper schemes over affine bases in terms of vanishing of cohomology after twisting.

0B5U **Lemma 30.17.1.** *Let R be a Noetherian ring. Let $f : X \to \operatorname{Spec}(R)$ be a proper morphism. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent* [**DG67**, III Proposition 2.6.1]

(1) $\mathcal{L}$ *is ample on* X *(this is equivalent to many other things, see Properties, Proposition 28.26.13 and Morphisms, Lemma 29.39.4),*
(2) *for every coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ *there exists an* $n_0 \geq 0$ *such that* $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ *for all* $n \geq n_0$ *and* $p > 0$*, and*
(3) *for every quasi-coherent sheaf of ideals* $\mathcal{I} \subset \mathcal{O}_X$*, there exists an* $n \geq 1$ *such that* $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$*.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 30.16.1. The implication (2) $\Rightarrow$ (3) is trivial. The implication (3) $\Rightarrow$ (1) is Lemma 30.3.3. □

0B5V **Lemma 30.17.2.** *Let* R *be a Noetherian ring. Let* $f : Y \to X$ *be a morphism of schemes proper over* R*. Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Assume* f *is finite and surjective. Then* $\mathcal{L}$ *is ample if and only if* $f^*\mathcal{L}$ *is ample.*

Proof. The pullback of an ample invertible sheaf by a quasi-affine morphism is ample, see Morphisms, Lemma 29.37.7. This proves one of the implications as a finite morphism is affine by definition.

Assume that $f^*\mathcal{L}$ is ample. Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which implies $\mathcal{L}$ is ample by Lemma 30.17.1. We are going to apply Lemma 30.12.8. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 30.13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 30.12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 30.13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 30.13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module. As $\mathcal{L}$ is ample we see from Lemma 30.17.1 that there exists an n_0 such that

$$H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0,$$

for $n \geq n_0$ and $p > 0$. Since f is finite, hence affine, we see that

$$\begin{aligned} H^p(X, \mathcal{J}\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \\ &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

Here we have used the projection formula (Cohomology, Lemma 20.54.2) and Lemma 30.2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies P. This verifies property (3)(c) of Lemma 30.12.8 as desired. □

Cohomology is functorial. In particular, given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s. We set $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ as a graded ring, see Modules, Definition 17.25.7. Given a sheaf

of $\mathcal{O}_X$-modules $\mathcal{F}$ and an integer $p \geq 0$ we set

$$H^p_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

This is a graded $\Gamma_*(X, \mathcal{L})$-module by the multiplication defined above. Warning: the notation $H^p_*(X, \mathcal{L}, \mathcal{F})$ is nonstandard.

09MR **Lemma 30.17.3.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is quasi-compact and quasi-separated, the canonical map*

$$H^p_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(X_s, \mathcal{F})$$

which maps ξ/s^n to $s^{-n}\xi$ is an isomorphism.

Proof. Note that for $p = 0$ this is Properties, Lemma 28.17.2. We will prove the statement using the induction principle (Lemma 30.4.1) where for $U \subset X$ quasi-compact open we let $P(U)$ be the property: for all $p \geq 0$ the map

$$H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(U_s, \mathcal{F})$$

is an isomorphism.

If U is affine, then both sides of the arrow displayed above are zero for $p > 0$ by Lemma 30.2.2 and Properties, Lemma 28.26.4 and the statement is true. If P is true for U, V, and $U \cap V$, then we can use the Mayer-Vietoris sequences (Cohomology, Lemma 20.8.2) to obtain a map of long exact sequences

$$\begin{array}{ccccc}
H^{p-1}_*(U \cap V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U \cup V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \oplus H^p_*(V, \mathcal{L}, \mathcal{F})_{(s)} \\
\downarrow & & \downarrow & & \downarrow \\
H^{p-1}(U_s \cap V_s, \mathcal{F}) & \longrightarrow & H^p(U_s \cup V_s, \mathcal{F}) & \longrightarrow & H^p(U_s, \mathcal{F}) \oplus H^p(V_s, \mathcal{F})
\end{array}$$

(only a snippet shown). Observe that $U_s \cap V_s = (U \cap V)_s$ and that $U_s \cup V_s = (U \cup V)_s$. Thus the left and right vertical maps are isomorphisms (as well as one more to the right and one more to the left which are not shown in the diagram). We conclude that $P(U \cup V)$ holds by the 5-lemma (Homology, Lemma 12.5.20). This finishes the proof. □

01XR **Lemma 30.17.4.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *X_s is affine.*

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n\xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. Recall that $H^p(X_s, \mathcal{G})$ is zero for every quasi-coherent module $\mathcal{G}$ by Lemma 30.2.2. Hence the lemma follows from Lemma 30.17.3. □

For a more general version of the following lemma see Limits, Lemma 32.11.4.

09MS **Lemma 30.17.5.** *Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. Then $i^*\mathcal{L}$ is ample on Z, if and only if $\mathcal{L}$ is ample on X.*

Proof. If $\mathcal{L}$ is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 29.37.7. Assume $i^*\mathcal{L}$ is ample. We have to show that $\mathcal{L}$ is ample on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the closed subscheme Z. Since $i(Z) = X$ set theoretically we see that $\mathcal{I}^n = 0$ for some n by Lemma 30.10.2. Consider the sequence

$$X = Z_n \supset Z_{n-1} \supset Z_{n-2} \supset \ldots \supset Z_1 = Z$$

of closed subschemes cut out by $0 = \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I}$. Then each of the closed immersions $Z_i \to Z_{i-1}$ is defined by a coherent sheaf of ideals of square zero. In this way we reduce to the case that $\mathcal{I}^2 = 0$.

Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes n}$ we obtain short exact sequences

0B8T (30.17.5.1)
$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to i_*i^*\mathcal{L}^{\otimes n} \to 0$$

As $\mathcal{I}^2 = 0$, we can use Morphisms, Lemma 29.4.1 to think of $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_Z$-module and then $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}$ with obvious abuse of notation. Moreover, the cohomology of this sheaf over Z is canonically the same as the cohomology of this sheaf over X (as i is a homeomorphism).

Let $x \in X$ be a point and denote $z \in Z$ the corresponding point. Because $i^*\mathcal{L}$ is ample there exists an n and a section $s \in \Gamma(Z, i^*\mathcal{L}^{\otimes n})$ with $z \in Z_s$ and with Z_s affine. The obstruction to lifting s to a section of $\mathcal{L}^{\otimes n}$ over X is the boundary

$$\xi = \partial s \in H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})$$

coming from the short exact sequence of sheaves (30.17.5.1). If we replace s by s^{e+1} then ξ is replaced by $\partial(s^{e+1}) = (e+1)s^e\xi$ in $H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes (e+1)n})$ because the boundary map for

$$0 \to \bigoplus\nolimits_{m \geq 0} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \bigoplus\nolimits_{m \geq 0} \mathcal{L}^{\otimes m} \to \bigoplus\nolimits_{m \geq 0} i_*i^*\mathcal{L}^{\otimes m} \to 0$$

is a derivation by Cohomology, Lemma 20.25.5. By Lemma 30.17.4 we see that $s^e\xi$ is zero for e large enough. Hence, after replacing s by a power, we can assume s is the image of a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n})$. Then $X_{s'}$ is an open subscheme and $Z_s \to X_{s'}$ is a surjective closed immersion of Noetherian schemes with Z_s affine. Hence X_s is affine by Lemma 30.13.3 and we conclude that $\mathcal{L}$ is ample. □

For a more general version of the following lemma see Limits, Lemma 32.11.5.

0B7K **Lemma 30.17.6.** *Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.*

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 28.27.1. Hence if Z is quasi-affine, then $\mathcal{O}_Z$ is ample, hence $\mathcal{O}_X$ is ample by Lemma 30.17.5, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. □

0EBD **Lemma 30.17.7.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let n_0 be an integer. If $H^p(X, \mathcal{L}^{\otimes -n}) = 0$ for $n \geq n_0$ and $p > 0$, then X is affine.*

Proof. We claim $H^p(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module and $p > 0$. Since X is quasi-compact by Properties, Definition 28.26.1 the claim finishes the proof by Lemma 30.3.1. The scheme X is separated by Properties, Lemma 28.26.8. Say X is covered by $e + 1$ affine opens. Then $H^p(X, \mathcal{F}) = 0$ for $p > e$, see Lemma 30.4.2. Thus we may use descending induction on p to prove the claim. Writing $\mathcal{F}$ as a filtered colimit of finite type quasi-coherent modules (Properties, Lemma 28.22.3) and using Cohomology, Lemma 20.19.1 we may assume $\mathcal{F}$ is of finite type. Then we can choose $n > n_0$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 28.26.13. This means there is a short exact sequence

$$0 \to \mathcal{F}' \to \bigoplus\nolimits_{i \in I} \mathcal{L}^{\otimes -n} \to \mathcal{F} \to 0$$

for some set I (in fact we can choose I finite). By induction hypothesis we have $H^{p+1}(X, \mathcal{F}') = 0$ and by assumption (combined with the already used commutation of cohomology with colimits) we have $H^p(X, \bigoplus_{i \in I} \mathcal{L}^{\otimes -n}) = 0$. From the long exact cohomology sequence we conclude that $H^p(X, \mathcal{F}) = 0$ as desired. □

0EBE **Lemma 30.17.8.** *Let X be a quasi-affine scheme. If $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$, then X is affine.*

Proof. Since $\mathcal{O}_X$ is ample by Properties, Lemma 28.27.1 this follows from Lemma 30.17.7. □

30.18. Chow's Lemma

02O2 In this section we prove Chow's lemma in the Noetherian case (Lemma 30.18.1). In Limits, Section 32.12 we prove some variants for the non-Noetherian case.

0200 **Lemma 30.18.1.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exist an $n \geq 0$ and a diagram* [**DG67**, II Theorem 5.6.1(a)]

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 29.15.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 29.15.8 and Properties, Lemmas 28.5.4 and 28.5.8.

The scheme X has only finitely many irreducible components (Properties, Lemma 28.5.7). Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. See Properties, Lemma 28.29.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X are all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$, see Morphisms, Definition 29.6.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X, see Morphisms, Definition 29.7.1.

By Morphisms, Lemma 29.39.3 we can find an immersion $j_i : U_i \to \mathbf{P}_S^{n_i}$ for each i. By Morphisms, Lemma 29.7.7 we can find closed subschemes $Z_i \subset \mathbf{P}_S^{n_i}$ such that $j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper, see Morphisms, Lemma 29.43.5. Consider the morphism

$$j = (j_1|_U, \ldots, j_m|_U) : U \longrightarrow \mathbf{P}_S^{n_1} \times_S \ldots \times_S \mathbf{P}_S^{n_m}.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}_S^{n_1} \times_S \ldots \times_S \mathbf{P}_S^{n_m}$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbf{P}_S^{n_i}.$$

This morphism factors through Z_i (see Morphisms, Lemma 29.6.6). Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 29.41.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the "diagonal" morphism $U \to Z_1 \times_S \ldots \times_S Z_m$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_m$. By construction X' has an immersion into the scheme $\mathbf{P}_S^{n_1} \times_S \ldots \times_S \mathbf{P}_S^{n_m}$. Thus by the Segre embedding (see Constructions, Lemma 27.13.6) we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X, see Morphisms, Lemma 29.7.10. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{array}{ccc} V_i & \longrightarrow & \pi^{-1}(U_i) \\ & \searrow^{p_i|_{V_i}} & \downarrow \\ & & U_i \end{array}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 29.41.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 29.41.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{array}{ccc} U & \longrightarrow & \pi^{-1}(U) \\ & \searrow & \downarrow \\ & & U \end{array}$$

and using that $\mathrm{id}_U : U \to U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. □

0201 **Remark 30.18.2.** In the situation of Chow's Lemma 30.18.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 29.43.3) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 29.41.7).
(2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X'. Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X'. See Morphisms, Section 29.6.
(3) If X is reduced then we may choose X' reduced. This is clear from (2).

30.19. Higher direct images of coherent sheaves

02O3 In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

02O5 **Proposition 30.19.1.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.* [**DG67**, III Theorem 3.2.1]

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^p f_* \mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 30.12.6 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1} f_* \mathcal{F}_3 \to R^p f_* \mathcal{F}_1 \to R^p f_* \mathcal{F}_2 \to R^p f_* \mathcal{F}_3 \to R^{p+1} f_* \mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 30.9.2 and 30.9.3. Hence property $\mathcal{P}$ holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf $\mathcal{F}$ on X whose support is contained in Z, whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that $\mathcal{P}$ holds for $\mathcal{F}$. Denote

$g = f|_Z : Z \to S$ the restriction of f. Suppose we can find a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{G}_\xi$ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^p g_*\mathcal{G} = 0$ for $p > 0$, and (c) $g_*\mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \to X)_*\mathcal{G}$. As $Z \to X$ is a closed immersion we see that $(Z \to X)_*\mathcal{G}$ is coherent on X and $R^p(Z \to X)_*\mathcal{G} = 0$ for $p > 0$ (Lemma 30.9.9). Hence by the relative Leray spectral sequence (Cohomology, Lemma 20.13.8) we will have $R^p f_*\mathcal{F} = R^p g_*\mathcal{G} = 0$ for $p > 0$ and $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \to X)_*\mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ. Hence everything depends on finding a coherent sheaf $\mathcal{G}$ on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 30.18.1 to the morphism $Z \to S$. Thus we get a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\pi} & Z' & \xrightarrow{i} & \mathbf{P}^m_S \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle g'} & \swarrow & \\ & & S & & \end{array}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. By the discussion in Remark 30.18.2 we see that $i' = (i, \pi) : Z' \to \mathbf{P}^m_Z$ is a closed immersion. Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^m_S}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^m_Z}(1)$$

is g'-relatively ample and π-relatively ample (for example by Morphisms, Lemma 29.39.7). Hence by Lemma 30.16.2 there exists an $n \geq 0$ such that both $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_*\mathcal{L}^{\otimes n}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \to U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 20.13.8) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes n} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^p g_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 30.16.3 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$ is coherent as desired. □

02O6 **Lemma 30.19.2.** *Let $S = \mathrm{Spec}(A)$ with A a Noetherian ring. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is a finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Proposition 30.19.1. Namely, by Lemmas 30.4.5 and 30.4.6 we know that $R^i f_*\mathcal{F}$ is the quasi-coherent sheaf associated to the A-module $H^i(X, \mathcal{F})$ and by Lemma 30.9.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. □

0897 **Lemma 30.19.3.** *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type.*

(1) *For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*
(2) *If $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module, then there exists an integer d_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$ and $d \geq d_0$.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms, Lemma 29.41.5. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.31.1). This implies that X' is a Noetherian scheme (Morphisms, Lemma 29.15.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Constructions, Lemma 27.4.6. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms, Lemma 29.11.7 Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 30.9.1). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 30.2.4. Thus (1) follows from Lemma 30.19.2. Given $\mathcal{L}$ as in (2) we set $\mathcal{L}' = \pi^*\mathcal{L}$. Note that $\mathcal{L}'$ is ample on X' by Morphisms, Lemma 29.37.7. By the projection formula (Cohomology, Lemma 20.54.2) we have $\pi_*(\mathcal{F}' \otimes \mathcal{L}') = \mathcal{F} \otimes \mathcal{L}$. Thus part (2) follows by the same reasoning as above from Lemma 30.16.2. □

30.20. The theorem on formal functions

02O7 In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n\mathcal{F}\}_{n\geq 0}$ or of the form $\{\mathcal{F}/I^n\mathcal{F}\}_{n\geq 0}$ as n varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote $\mathcal{I}^n\mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and $\mathcal{F}$. In a formula

$$\mathcal{I}^n\mathcal{F} = \mathrm{Im}\left(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}\right).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{I}^m\mathcal{F} \longrightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m\mathcal{F} \longrightarrow \mathcal{I}^{n+m}\mathcal{F}$$

Hence a section of $\mathcal{I}^n$ will give rise to a map $R^pf_*(\mathcal{I}^m\mathcal{F}) \to R^pf_*(\mathcal{I}^{n+m}\mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that there are $\mathcal{O}_Y$-module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^pf_*(\mathcal{I}^m\mathcal{F}) \longrightarrow R^pf_*(\mathcal{I}^{n+m}\mathcal{F}).$$

In other words we see that $\bigoplus_{n\geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$ is a graded $\bigoplus_{n\geq 0}\mathcal{I}^n$-module.

If $Y = \mathrm{Spec}(A)$ and $\mathcal{I} = \widetilde{I}$ we denote $\mathcal{I}^n\mathcal{F}$ simply $I^n\mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n\mathcal{F})$ the structure of a graded $S = \bigoplus I^n$-module. If f is proper, A is Noetherian and $\mathcal{F}$ is coherent, then this turns out to be a module of finite type.

02O8 **Lemma 30.20.1.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $B = \bigoplus_{n\geq 0} I^n$. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.* [**DG67**, III Cor 3.3.2]

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 30.19.3 part (1). □

02O9 **Lemma 30.20.2.** *Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and $\mathcal{F}$ coherent. Then*

$$\mathcal{M} = \bigoplus\nolimits_{n\geq 0} R^p f_*(\mathcal{I}^n\mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n\geq 0}\mathcal{I}^n$-module which is quasi-coherent and of finite type.

Proof. The statement is local on Y, hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 30.20.1. Details omitted. □

02OA **Lemma 30.20.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that*

1. *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$,*
2. *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-e}H^p(X, I^n\mathcal{F})$ where $e = \max(0, c-n)$ for $n + m \geq c$, $n, m \geq 0$,*
3. *we have*
$$\mathrm{Ker}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F})) = \mathrm{Ker}(H^p(X, I^n\mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F}))$$
for $n \geq c$,
4. *there are maps $I^nH^p(X, \mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F})$ for $n \geq c$ such that the compositions*
$$H^p(X, I^n\mathcal{F}) \to I^{n-c}H^p(X, \mathcal{F}) \to H^p(X, I^{n-2c}\mathcal{F})$$
and
$$I^nH^p(X, \mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F}) \to I^{n-2c}H^p(X, \mathcal{F})$$
for $n \geq 2c$ are the canonical ones, and
5. *the inverse systems $(H^p(X, I^n\mathcal{F}))$ and $(I^nH^p(X, \mathcal{F}))$ are pro-isomorphic.*

Proof. Write $M_n = H^p(X, I^n\mathcal{F})$ for $n \geq 1$ and $M_0 = H^p(X, \mathcal{F})$ so that we have maps $\ldots \to M_3 \to M_2 \to M_1 \to M_0$. Setting $B = \bigoplus_{n\geq 0} I^n$, then $M = \bigoplus_{n\geq 0} M_n$ is a finite graded B-module, see Lemma 30.20.1. Observe that the products $B_n \otimes M_m \to M_{m+n}$, $a \otimes m \mapsto a \cdot m$ are compatible with the maps in our inverse system in the sense that the diagrams

$$\begin{array}{ccc} B_n \otimes_A M_m & \longrightarrow & M_{n+m} \\ \downarrow & & \downarrow \\ B_n \otimes_A M_{m'} & \longrightarrow & M_{n+m'} \end{array}$$

commute for $n, m' \geq 0$ and $m \geq m'$.

Proof of (1). Choose $d_1, \ldots, d_t \geq 0$ and $x_i \in M_{d_i}$ such that M is generated by $x_1, \ldots, x_t$ over B. For any $c \geq \max\{d_i\}$ we conclude that $B_{n-c} \cdot M_c = M_n$ for $n \geq c$ and we conclude (1) is true.

Proof of (2). Let c be as in the proof of (1). Let $n + m \geq c$. We have $M_{n+m} = B_{n+m-c} \cdot M_c$. If $c > n$ then we use $M_c \to M_n$ and the compatibility of products with

transition maps pointed out above to conclude that the image of $M_{n+m} \to M_n$ is contained in $I^{n+m-c}M_n$. If $c \leq n$, then we write $M_{n+m} = B_m \cdot B_{n-c} \cdot M_c = B_m \cdot M_n$ to see that the image is contained in $I^m M_n$. This proves (2).

Let $K_n \subset M_n$ be the kernel of the map $M_n \to M_0$. The compatibility of products with transition maps pointed out above shows that $K = \bigoplus K_n \subset M$ is a graded B-submodule. As B is Noetherian and M is a finitely generated graded B-module, this shows that K is a finitely generated graded B-module. Choose $d'_1, \ldots, d'_{t'} \geq 0$ and $y_i \in K_{d'_i}$ such that K is generated by $y_1, \ldots, y_{t'}$ over B. Set $c = \max(d'_i, d'_j)$. Since $y_i \in \mathrm{Ker}(M_{d'_i} \to M_0)$ we see that $B_n \cdot y_i \subset \mathrm{Ker}(M_{n+d'_i} \to M_n)$. In this way we see that $K_n = \mathrm{Ker}(M_n \to M_{n-c})$ for $n \geq c$. This proves (3).

Consider the following commutative solid diagram

$$\begin{array}{ccccc} I^n \otimes_A M_0 & \longrightarrow & I^n M_0 & \longrightarrow & M_0 \\ \downarrow & & \vdots & & \downarrow \\ M_n & \longrightarrow & M_{n-c} & \longrightarrow & M_0 \end{array}$$

Since the kernel of the surjective arrow $I^n \otimes_A M_0 \to I^n M_0$ maps into K_n by the above we obtain the dotted arrow and the composition $I^n M_0 \to M_{n-c} \to M_0$ is the canonical map. Then clearly the composition $I^n M_0 \to M_{n-c} \to I^{n-2c} M_0$ is the canonical map for $n \geq 2c$. Consider the composition $M_n \to I^{n-c} M_0 \to M_{n-2c}$. The first map sends an element of the form $a \cdot m$ with $a \in I^{n-c}$ and $m \in M_c$ to am' where m' is the image of m in M_0. Then the second map sends this to $a \cdot m'$ in M_{n-2c} and we see (4) is true.

Part (5) is an immediate consequence of (4) and the definition of morphisms of pro-objects. □

In the situation of Lemmas 30.20.1 and 30.20.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \ldots$$

We would like to know what happens to the cohomology groups. Here is a first result.

02OB **Lemma 30.20.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. There exists a $c \geq 0$ such that*

(1) *for all $n \geq c$ we have*

$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c}H^p(X, \mathcal{F}).$$

(2) *the inverse system*

$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n \in \mathbf{N}}$$

satisfies the Mittag-Leffler condition (see Homology, Definition 12.31.2), and

(3) *we have*

$$\mathrm{Im}(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

for all $k \geq n + c$.

Proof. Let $c = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 30.20.3 for H^p and H^{p+1}.

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by Lemma 30.20.3 part (2) we see that this is contained in $I^{n-c}H^p(X, \mathcal{F})$ for $n \geq c$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler systems.

Let us prove part (3). Fix an n. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^n\mathcal{F} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{n+m}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^{n+m}\mathcal{F} & \longrightarrow & 0
\end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccc}
H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^n\mathcal{F}) & \xrightarrow{\delta} & H^{p+1}(X, I^n\mathcal{F}) & \longrightarrow & H^{p+1}(X, \mathcal{F}) \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle 1} \\
H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) & \longrightarrow & H^{p+1}(X, I^{n+m}\mathcal{F}) & \xrightarrow{\beta} & H^{p+1}(X, \mathcal{F})
\end{array}$$

with exact rows. By Lemma 30.20.3 part (4) the kernel of β is equal to the kernel of α for $m \geq c$. By a diagram chase this shows that the image of γ is contained in the kernel of δ which shows that part (3) is true (set $k = n + m$ to get it). □

02OC **Theorem 30.20.5** (Theorem on formal functions)**.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^nH^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. This follows from Lemma 30.20.4 as follows. Set $M = H^p(X, \mathcal{F})$, $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$, and denote $N_n = \mathrm{Im}(M \to M_n)$. By Lemma 30.20.4 parts (2) and (3) we see that (M_n) is a Mittag-Leffler system with $N_n \subset M_n$ equal to the image of M_k for all $k \gg n$. It follows that $\lim M_n = \lim N_n$ as topological modules (with limit topologies). On the other hand, the N_n form an inverse system of quotients of the module M and hence $\lim N_n$ is the completion of M with respect to the topology given by the kernels $K_n = \mathrm{Ker}(M \to N_n)$. By Lemma 30.20.4 part (1) we have $K_n \subset I^{n-c}M$ and since $N_n \subset M_n$ is annihilated by I^n we have $I^nM \subset K_n$. Thus the topology defined using the submodules K_n as a fundamental system of open neighbourhoods of 0 is the same as the I-adic topology and we find

that the induced map $M^\wedge = \lim M/I^nM \to \lim N_n = \lim M_n$ is an isomorphism of topological modules[3]. □

087U **Lemma 30.20.6.** *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 30.20.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 30.19.2) hence I-adically complete (Algebra, Lemma 10.97.1) and we see that completion on the left hand side is not necessary. □

02OD **Lemma 30.20.7.** *Given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ on X. Assume*

(1) *Y locally Noetherian,*
(2) *f proper, and*
(3) *$\mathcal{F}$ coherent.*

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{array}{ccc} X_n = \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X & \xrightarrow[i_n]{} & X \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) & \xrightarrow{c_n} & Y \end{array}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})^\wedge_y \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}^\wedge_{Y,y}$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 30.20.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 30.5.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $y \in Y$ corresponds to the closed point (i.e., to $\mathfrak{m}$). In particular it follows that

$$(R^pf_*\mathcal{F})_y = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F}).$$

[3]To be sure, the limit topology on $M^\wedge$ is the same as its I-adic topology as follows from Algebra, Lemma 10.96.3. See More on Algebra, Section 15.36.

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 30.9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}_n) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

02OE **Lemma 30.20.8.** *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$f^{-1}(\{y\})$ is finite.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > 0$.

Proof. The fibre X_y is finite, and by Morphisms, Lemma 29.20.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 26.11.8. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^pf_*\mathcal{F})^\wedge_y = 0$ by Lemma 30.20.7. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 30.19.1 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Nakayama's lemma (Algebra, Lemma 10.20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^pf_*\mathcal{F})_y = 0$. □

02V7 **Lemma 30.20.9.** *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$\dim(X_y) = d$.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > d$.

Proof. The fibre X_y is of finite type over $\mathrm{Spec}(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 29.15.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 28.5.5. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of X_y. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Proposition 20.20.7. Hence we see that $(R^pf_*\mathcal{F})^\wedge_y = 0$ by Lemma 30.20.7 for $p > d$. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 30.19.1 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Nakayama's lemma (Algebra, Lemma 10.20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^pf_*\mathcal{F})_y = 0$. □

30.21. Applications of the theorem on formal functions

02OF We will add more here as needed. For the moment we need the following characterization of finite morphisms in the Noetherian case.

02OG **Lemma 30.21.1.** *(For a more general version see More on Morphisms, Lemma 37.44.1.) Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent*

(1) *f is finite, and*
(2) *f is proper with finite fibres.*

Proof. A finite morphism is proper according to Morphisms, Lemma 29.44.11. A finite morphism is quasi-finite according to Morphisms, Lemma 29.44.10. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 29.20.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms, Lemma 29.44.7 whereupon it follows from Morphisms, Lemma 29.44.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 30.3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular $\mathcal{I}$ is a coherent sheaf on X. By Lemma 30.20.8 we conclude that $R^1 f_* \mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1 f_* \mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_* \mathcal{I})$. Since S is affine, and $f_* \mathcal{I}$ is quasi-coherent (Schemes, Lemma 26.24.1) we conclude $H^1(S, f_* \mathcal{I}) = 0$ from Lemma 30.2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. □

As a consequence we have the following useful result.

02OH **Lemma 30.21.2.** *(For a more general version see More on Morphisms, Lemma 37.44.2.) Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

(1) *S is locally Noetherian,*
(2) *f is proper, and*
(3) *$f^{-1}(\{s\})$ is a finite set.*

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 29.20.7. By Morphisms, Lemma 29.56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 29.20.9), hence has finite fibres (Morphisms, Lemma 29.20.10), hence is finite by Lemma 30.21.1. □

0D2M **Lemma 30.21.3.** *Let $f : X \to Y$ be a proper morphism of schemes with Y Noetherian. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there exists a d_0 such that for all $d \geq d_0$ we have*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^* R^p f_* \mathcal{F} = R^p f'_* \mathcal{F}'$ by Lemma 30.5.2. Moreover,

the fibres X_y and X'_y are identified. Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to $\mathfrak{m}$. In this case $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \to \mathrm{Spec}(\kappa)$ the projection.

Let $B = \mathrm{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^* \widetilde{B}$ of quasi-coherent graded $\mathcal{O}_{X_y}$-algebras. We will use notation as in Section 30.20 with I replaced by $\mathfrak{m}$. Since X_y is the closed subscheme of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ as a coherent $\mathcal{O}_{X_y}$-module, see Lemma 30.9.8. Then $\bigoplus_{n \geq 0} \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is a quasi-coherent graded $\mathcal{B}$-module of finite type because it is generated in degree zero over $\mathcal{B}$ and because the degree zero part is $\mathcal{F}_y = \mathcal{F}/\mathfrak{m}\mathcal{F}$ which is a coherent $\mathcal{O}_{X_y}$-module. Hence by Lemma 30.19.3 part (2) we see that

$$H^p(X_y, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Lemma 30.2.4 this is the same as the statement that $H^p(X, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

(1) $H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
(2) $H^0(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Theorem 30.20.5) we find that the $\mathfrak{m}$-adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A-module by Lemma 30.19.2 it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Lemma 30.20.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. □

0D2N **Lemma 30.21.4.** *(For a more general version see More on Morphisms, Lemma 37.50.3.) Let $f : X \to Y$ be a proper morphism of schemes with Y Noetherian. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$.*

Proof. Pick d_0 as in Lemma 30.21.3 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \ldots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \ldots, s_{y,r})} : X_y \to \mathbf{P}^r_{\kappa(y)}.$$

This is possible by Morphisms, Lemma 29.39.4 but we also use Morphisms, Lemma 29.41.7 to see that φ_y is a closed immersion and Constructions, Section 27.13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0, after replacing Y by an open neighbourhood of

y, we can choose $s_0, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \ldots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subset where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $X_y \subset U = \bigcup X_{s_i}$. Since $X \to Y$ is closed, we see that there is an open neighbourhood $y \in V \subset Y$ such that $f^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \ldots, s_r)} : X \longrightarrow \mathbf{P}^r_Y$$

with $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}^r_Y}(1)$ whose base change to y gives φ_y.

We will finish the proof by a sleight of hand; the "correct" proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y. Namely, by Lemma 30.21.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}^r_{\kappa(y)}$ of $\mathbf{P}^r_Y \to Y$ above y. Using that $\mathbf{P}^r_Y \to Y$ is closed, after shrinking Y we may assume that φ is finite. Then $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}^r_Y}(1)$ is ample by the very general Morphisms, Lemma 29.37.7. □

30.22. Cohomology and base change, III

07VJ In this section we prove the simplest case of a very general phenomenon that will be discussed in Derived Categories of Schemes, Section 36.22. Please see Remark 30.22.2 for a translation of the following lemma into algebra.

07VK **Lemma 30.22.1.** *Let A be a Noetherian ring and set $S = \operatorname{Spec}(A)$. Let $f : X \to S$ be a proper morphism of schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. Then*

(1) *$R\Gamma(X, \mathcal{F})$ is a perfect object of $D(A)$, and*
(2) *for any ring map $A \to A'$ the base change map*

$$R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

is an isomorphism.

Proof. Choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. By Lemmas 30.7.1 and 30.7.2 the Čech complex $K^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ satisfies

$$K^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

for all ring maps $A \to A'$. Let $K^\bullet_{alt} = \check{C}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ be the alternating Čech complex. By Cohomology, Lemma 20.23.6 there is a homotopy equivalence $K^\bullet_{alt} \to K^\bullet$ of A-modules. In particular, we have

$$K^\bullet_{alt} \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

as well. Since $\mathcal{F}$ is flat over A we see that each K^n_{alt} is flat over A (see Morphisms, Lemma 29.25.2). Since moreover $K^\bullet_{alt}$ is bounded above (this is why we switched to the alternating Čech complex) $K^\bullet_{alt} \otimes_A A' = K^\bullet_{alt} \otimes^{\mathbf{L}}_A A'$ by the definition of derived tensor products (see More on Algebra, Section 15.59). By Lemma 30.19.2 the cohomology groups $H^i(K^\bullet_{alt})$ are finite A-modules. As $K^\bullet_{alt}$ is bounded, we conclude that $K^\bullet_{alt}$ is pseudo-coherent, see More on Algebra, Lemma 15.65.17. Given any A-module M set $A' = A \oplus M$ where M is a square zero ideal, i.e., $(a, m) \cdot (a', m') = (aa', am' + a'm)$. By the above we see that $K^\bullet_{alt} \otimes^{\mathbf{L}}_A A'$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt} \otimes^{\mathbf{L}}_A M$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt}$ has finite Tor dimension, see More on Algebra, Definition 15.67.1. We win by More on Algebra, Lemma 15.75.2. □

07VL **Remark 30.22.2.** A consequence of Lemma 30.22.1 is that there exists a finite complex of finite projective A-modules $M^\bullet$ such that we have

$$H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$$

functorially in A'. The condition that $\mathcal{F}$ is flat over A is essential, see [**Har98**].

30.23. Coherent formal modules

0EHN As we do not yet have the theory of formal schemes to our disposal, we develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal scheme".

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism of such inverse systems is defined as usual. Let us denote the category of these inverse systems with $Coh(X, \mathcal{I})$. We are going to proceed by proving a bunch of lemmas about objects in this category. In fact, most of the lemmas that follow are straightforward consequences of the following description of the category in the affine case.

087W **Lemma 30.23.1.** *If $X = \mathrm{Spec}(A)$ is the spectrum of a Noetherian ring and $\mathcal{I}$ is the quasi-coherent sheaf of ideals associated to the ideal $I \subset A$, then $Coh(X, \mathcal{I})$ is equivalent to the category of finite $A^\wedge$-modules where $A^\wedge$ is the completion of A with respect to I.*

Proof. Let $\mathrm{Mod}^{fg}_{A,I}$ be the category of inverse systems (M_n) of finite A-modules satisfying: (1) M_n is annihilated by I^n and (2) $M_{n+1}/I^nM_{n+1} = M_n$. By the correspondence between coherent sheaves on X and finite A-modules (Lemma 30.9.1) it suffices to show $\mathrm{Mod}^{fg}_{A,I}$ is equivalent to the category of finite $A^\wedge$-modules. To see this it suffices to prove that given an object (M_n) of $\mathrm{Mod}^{fg}_{A,I}$ the module

$$M = \lim M_n$$

is a finite $A^\wedge$-module and that $M/I^nM = M_n$. As the transition maps are surjective, we see that $M \to M_1$ is surjective. Pick $x_1, \ldots, x_t \in M$ which map to generators of M_1. This induces a map of systems $(A/I^n)^{\oplus t} \to M_n$. By Nakayama's lemma (Algebra, Lemma 10.20.1) these maps are surjective. Let $K_n \subset (A/I^n)^{\oplus t}$ be the kernel. Property (2) implies that $K_{n+1} \to K_n$ is surjective, in particular the system (K_n) satisfies the Mittag-Leffler condition. By Homology, Lemma 12.31.3 we obtain an exact sequence $0 \to K \to (A^\wedge)^{\oplus t} \to M \to 0$ with $K = \lim K_n$. Hence M is a finite $A^\wedge$-module. As $K \to K_n$ is surjective it follows that

$$M/I^nM = \mathrm{Coker}(K \to (A/I^n)^{\oplus t}) = (A/I^n)^{\oplus t}/K_n = M_n$$

as desired. □

087X **Lemma 30.23.2.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $Coh(X, \mathcal{I})$ is abelian.*
(2) *For $U \subset X$ open the restriction functor $Coh(X, \mathcal{I}) \to Coh(U, \mathcal{I}|_U)$ is exact.*

(3) *Exactness in $Coh(X, \mathcal{I})$ may be checked by restricting to the members of an open covering of X.*

Proof. Let $\alpha = (\alpha_n) : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. The cokernel of α is the inverse system $(\mathrm{Coker}(\alpha_n))$ (details omitted). To describe the kernel let

$$\mathcal{K}'_{l,m} = \mathrm{Im}(\mathrm{Ker}(\alpha_l) \to \mathcal{F}_m)$$

for $l \geq m$. We claim:

(a) the inverse system $(\mathcal{K}'_{l,m})_{l \geq m}$ is eventually constant, say with value $\mathcal{K}'_m$,
(b) the system $(\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m)_{m \geq n}$ is eventually constant, say with value $\mathcal{K}_n$,
(c) the system $(\mathcal{K}_n)$ forms an object of $Coh(X, \mathcal{I})$, and
(d) this object is the kernel of α.

To see (a), (b), and (c) we may work affine locally, say $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. By Lemma 30.23.1 α corresponds to a map $f : M \to N$ of finite $A^\wedge$-modules. Denote $K = \mathrm{Ker}(f)$. Note that $A^\wedge$ is a Noetherian ring (Algebra, Lemma 10.97.6). Choose an integer $c \geq 0$ such that $K \cap I^n M \subset I^{n-c}K$ for $n \geq c$ (Algebra, Lemma 10.51.2) and which satisfies Algebra, Lemma 10.51.3 for the map f and the ideal $I^\wedge = IA^\wedge$. Then $\mathcal{K}'_{l,m}$ corresponds to the A-module

$$K'_{l,m} = \frac{a^{-1}(I^l N) + I^m M}{I^m M} = \frac{K + I^{l-c} f^{-1}(I^c N) + I^m M}{I^m M} = \frac{K + I^m M}{I^m M}$$

where the last equality holds if $l \geq m + c$. So $\mathcal{K}'_m$ corresponds to the A-module $K/K \cap I^m M$ and $\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ corresponds to

$$\frac{K}{K \cap I^m M + I^n K} = \frac{K}{I^n K}$$

for $m \geq n + c$ by our choice of c above. Hence $\mathcal{K}_n$ corresponds to $K/I^n K$.

We prove (d). It is clear from the description on affines above that the composition $(\mathcal{K}_n) \to (\mathcal{F}_n) \to (\mathcal{G}_n)$ is zero. Let $\beta : (\mathcal{H}_n) \to (\mathcal{F}_n)$ be a morphism such that $\alpha \circ \beta = 0$. Then $\mathcal{H}_l \to \mathcal{F}_l$ maps into $\mathrm{Ker}(\alpha_l)$. Since $\mathcal{H}_m = \mathcal{H}_l/\mathcal{I}^m\mathcal{H}_l$ for $l \geq m$ we obtain a system of maps $\mathcal{H}_m \to \mathcal{K}'_{l,m}$. Thus a map $\mathcal{H}_m \to \mathcal{K}'_m$. Since $\mathcal{H}_n = \mathcal{H}_m/\mathcal{I}^n\mathcal{H}_m$ we obtain a system of maps $\mathcal{H}_n \to \mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ and hence a map $\mathcal{H}_n \to \mathcal{K}_n$ as desired.

To finish the proof of (1) we still have to show that Coim = Im in $Coh(X, \mathcal{I})$. We have seen above that taking kernels and cokernels commutes, over affines, with the description of $Coh(X, \mathcal{I})$ as a category of modules. Since Im = Coim holds in the category of modules this gives Coim = Im in $Coh(X, \mathcal{I})$. Parts (2) and (3) of the lemma are immediate from our construction of kernels and cokernels. □

087Y **Lemma 30.23.3.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. Omitted. Hint: Look on affine opens, use Lemma 30.23.1, and use Algebra, Lemma 10.20.1. □

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

0880 (30.23.3.1) $$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

0881 **Lemma 30.23.4.** *The functor (30.23.3.1) is exact.*

Proof. It suffices to check this locally on X. Hence we may assume X is affine, i.e., we have a situation as in Lemma 30.23.1. The functor is the functor $\mathrm{Mod}_A^{fg} \to \mathrm{Mod}_{A^\wedge}^{fg}$ which associates to a finite A-module M the completion $M^\wedge$. Thus the result follows from Algebra, Lemma 10.97.2. □

0882 **Lemma 30.23.5.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

Proof. To prove this we may work affine locally on X. Hence we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{F}$, $\mathcal{G}$ given by finite A-module M and N. Then $\mathcal{H}$ corresponds to the finite A-module $H = \mathrm{Hom}_A(M, N)$. The statement of the lemma becomes the statement

$$H^\wedge = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

via the equivalence of Lemma 30.23.1. By Algebra, Lemma 10.97.2 (used 3 times) we have

$$H^\wedge = \mathrm{Hom}_A(M, N) \otimes_A A^\wedge = \mathrm{Hom}_{A^\wedge}(M \otimes_A A^\wedge, N \otimes_A A^\wedge) = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

where the second equality uses that $A^\wedge$ is flat over A (see More on Algebra, Lemma 15.66.4). The lemma follows. □

Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We say an object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is *$\mathcal{I}$-power torsion* or is *annihilated by a power of $\mathcal{I}$* if there exists a $c \geq 1$ such that $\mathcal{F}_n = \mathcal{F}_c$ for all $n \geq c$. If this is the case we will say that $(\mathcal{F}_n)$ is *annihilated by $\mathcal{I}^c$*. If $X = \mathrm{Spec}(A)$ is affine, then, via the equivalence of Lemma 30.23.1, these objects corresponds exactly to the finite A-modules annihilated by a power of I or by I^c.

0889 **Lemma 30.23.6.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$.*

1. *If $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ is a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$, then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*
   (a) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
   (b) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
   (c) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
   (d) *$\alpha = a^\wedge \circ \beta$.*
2. *If $\alpha : \mathcal{G}^\wedge \to (\mathcal{F}_n)$ is a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$, then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*
   (a) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
   (b) *$a : \mathcal{G} \to \mathcal{F}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
   (c) *$\beta : \mathcal{F}^\wedge \to (\mathcal{F}_n)$ is an isomorphism, and*
   (d) *$\alpha = \beta \circ a^\wedge$.*

Proof. Proof of (1). The uniqueness implies it suffices to construct $(\mathcal{F}, a, \beta)$ Zariski locally on X. Thus we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. In this situation Lemma 30.23.1 applies. Let M' be the finite $A^\wedge$-module corresponding to $(\mathcal{F}_n)$. Let N be the finite A-module corresponding to $\mathcal{G}$. Then α corresponds to a map

$$\varphi : M' \longrightarrow N^\wedge$$

whose kernel and cokernel are annihilated by I^t for some t. Recall that $N^\wedge = N \otimes_A A^\wedge$ (Algebra, Lemma 10.97.1). By More on Algebra, Lemma 15.90.17 there is an A-module map $\psi : M \to N$ whose kernel and cokernel are I-power torsion and an isomorphism $M \otimes_A A^\wedge = M'$ compatible with φ. As N and M' are finite modules, we conclude that M is a finite A-module, see More on Algebra, Remark 15.90.20. Hence $M \otimes_A A^\wedge = M^\wedge$. We omit the verification that the triple $(M, N \to M, M^\wedge \to M')$ so obtained is unique up to unique isomorphism.

The proof of (2) is exactly the same and we omit it. □

0EHP **Lemma 30.23.7.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Any object of $Coh(X, \mathcal{I})$ which is annihilated by a power of $\mathcal{I}$ is in the essential image of (30.23.3.1). Moreover, if $\mathcal{F}$, $\mathcal{G}$ are in $Coh(\mathcal{O}_X)$ and either $\mathcal{F}$ or $\mathcal{G}$ is annihilated by a power of $\mathcal{I}$, then the maps*

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) & & \mathrm{Ext}_X(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) & & \mathrm{Ext}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) \end{array}$$

are isomorphisms.

Proof. Suppose $(\mathcal{F}_n)$ is an object of $Coh(X, \mathcal{I})$ which is annihilated by $\mathcal{I}^c$ for some $c \geq 1$. Then $\mathcal{F}_n \to \mathcal{F}_c$ is an isomorphism for $n \geq c$. Hence if we set $\mathcal{F} = \mathcal{F}_c$, then we see that $\mathcal{F}^\wedge \cong (\mathcal{F}_n)$. This proves the first assertion.

Let $\mathcal{F}$, $\mathcal{G}$ be objects of $Coh(\mathcal{O}_X)$ such that either $\mathcal{F}$ or $\mathcal{G}$ is annihilated by $\mathcal{I}^c$ for some $c \geq 1$. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}^c$. Hence we see that

$$\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = H^0(X, \mathcal{H}) = \lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

see Lemma 30.23.5. This proves the statement on homomorphisms.

The notation Ext refers to extensions as defined in Homology, Section 12.6. The injectivity of the map on Ext's follows immediately from the bijectivity of the map on Hom's. For surjectivity, assume $\mathcal{F}$ is annihilated by a power of I. Then part (1) of Lemma 30.23.6 shows that given an extension

$$0 \to \mathcal{G}^\wedge \to (\mathcal{E}_n) \to \mathcal{F}^\wedge \to 0$$

in $Coh(U, I\mathcal{O}_U)$ the morphism $\mathcal{G}^\wedge \to (\mathcal{E}_n)$ is isomorphic to $\mathcal{G} \to \mathcal{E}^\wedge$ for some $\mathcal{G} \to \mathcal{E}$ in $Coh(\mathcal{O}_U)$. Similarly in the other case. □

087Z **Lemma 30.23.8.** *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If $(\mathcal{F}_n)$ is an object of $Coh(X, \mathcal{I})$ then $\bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is a finite type, graded, quasi-coherent $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$-module.*

Proof. The question is local on X hence we may assume X is affine, i.e., we have a situation as in Lemma 30.23.1. In this case, if $(\mathcal{F}_n)$ corresponds to the finite $A^\wedge$ module M, then $\bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ corresponds to $\bigoplus I^n M/I^{n+1}M$ which is clearly a finite module over $\bigoplus I^n/I^{n+1}$. □

0887 **Lemma 30.23.9.** *Let $f : X \to Y$ be a morphism of Noetherian schemes. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals and set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Then there is a right exact functor*

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. If f is flat, then f^* is an exact functor.

Proof. Since $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is right exact we have

$$f^*\mathcal{G}_n = f^*(\mathcal{G}_{n+1}/\mathcal{I}^n\mathcal{G}_{n+1}) = f^*\mathcal{G}_{n+1}/f^{-1}\mathcal{I}^n f^*\mathcal{G}_{n+1} = f^*\mathcal{G}_{n+1}/\mathcal{J}^n f^*\mathcal{G}_{n+1}$$

hence the pullback of a system is a system. The construction of cokernels in the proof of Lemma 30.23.2 shows that $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ is always right exact. If f is flat, then $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is an exact functor. It follows from the construction of kernels in the proof of Lemma 30.23.2 that in this case $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ also transforms kernels into kernels. □

0EHQ **Lemma 30.23.10.** *Let $f : X' \to X$ be a morphism of Noetherian schemes. Let $Z \subset X$ be a closed subscheme and denote $Z' = f^{-1}Z$ the scheme theoretic inverse image. Let $\mathcal{I} \subset \mathcal{O}_X$, $\mathcal{I}' \subset \mathcal{O}_{X'}$ be the corresponding quasi-coherent sheaves of ideals. If f is flat and the induced morphism $Z' \to Z$ is an isomorphism, then the pullback functor $f^* : Coh(X, \mathcal{I}) \to Coh(X', \mathcal{I}')$ (Lemma 30.23.9) is an equivalence.*

Proof. If X and X' are affine, then this follows immediately from More on Algebra, Lemma 15.90.3. To prove it in general we let $Z_n \subset X$, $Z'_n \subset X'$ be the nth infinitesimal neighbourhoods of Z, Z'. The induced morphism $Z_n \to Z'_n$ is a homeomorphism on underlying topological spaces. On the other hand, if $z' \in Z'$ maps to $z \in Z$, then the ring map $\mathcal{O}_{X,z} \to \mathcal{O}_{X',z'}$ is flat and induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z \to \mathcal{O}_{X',z'}/\mathcal{I}'_{z'}$. Hence it induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z^n \to \mathcal{O}_{X',z'}/(\mathcal{I}'_{z'})^n$ for all $n \geq 1$ for example by More on Algebra, Lemma 15.90.2. Thus $Z'_n \to Z_n$ is an isomorphism of schemes. Thus f^* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}^n$ and the category of coherent $\mathcal{O}_{X'}$-modules annihilated by $(\mathcal{I}')^n$, see Lemma 30.9.8. This clearly implies the lemma. □

0EHR **Lemma 30.23.11.** *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. If $V(\mathcal{I}) = V(\mathcal{J})$ is the same closed subset of X, then $Coh(X, \mathcal{I})$ and $Coh(X, \mathcal{J})$ are equivalent.*

Proof. First, assume $X = \mathrm{Spec}(A)$ is affine. Let $I, J \subset A$ be the ideals corresponding to $\mathcal{I}, \mathcal{J}$. Then $V(I) = V(J)$ implies we have $I^c \subset J$ and $J^d \subset I$ for some $c, d \geq 1$ by elementary properties of the Zariski topology (see Algebra, Section 10.17 and Lemma 10.32.5). Hence the I-adic and J-adic completions of A agree, see Algebra, Lemma 10.96.9. Thus the equivalence follows from Lemma 30.23.1 in this case.

In general, using what we said above and the fact that X is quasi-compact, to choose $c, d \geq 1$ such that $\mathcal{I}^c \subset \mathcal{J}$ and $\mathcal{J}^d \subset \mathcal{I}$. Then given an object $(\mathcal{F}_n)$ in $Coh(X, \mathcal{I})$ we claim that the inverse system

$$(\mathcal{F}_{cn}/\mathcal{J}^n\mathcal{F}_{cn})$$

is in $Coh(X, \mathcal{J})$. This may be checked on the members of an affine covering; we omit the details. In the same manner we can construct an object of $Coh(X, \mathcal{I})$ starting with an object of $Coh(X, \mathcal{J})$. We omit the verification that these constructions define mutually quasi-inverse functors. □

30.24. Grothendieck's existence theorem, I

087V In this section we discuss Grothendieck's existence theorem for the projective case. We will use the notion of coherent formal modules developed in Section 30.23. The reader who is familiar with formal schemes is encouraged to read the statement and proof of the theorem in [**DG67**].

0883 **Lemma 30.24.1.** *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is fully faithful.*

Proof. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module, see Modules, Lemma 17.22.6. By Lemma 30.23.5 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Hence fully faithfulness of (30.23.3.1) follows from the theorem on formal functions (Lemma 30.20.6) for the coherent sheaf $\mathcal{H}$. □

0884 **Lemma 30.24.2.** *Let A be Noetherian ring and $I \subset A$ an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism and let $\mathcal{L}$ be an f-ample invertible sheaf. Let $\mathcal{I} = I\mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Then there exists an integer d_0 such that*

$$H^1(X, \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n) \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $n \geq 0$ and all $d \geq d_0$.

Proof. Set $B = \bigoplus I^n/I^{n+1}$ and $\mathcal{B} = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} = f^*\widetilde{B}$. By Lemma 30.23.8 the graded quasi-coherent $\mathcal{B}$-module $\mathcal{G} = \bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is of finite type. Hence the lemma follows from Lemma 30.19.3 part (2). □

0885 **Lemma 30.24.3.** *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a projective morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is an equivalence.*

Proof. We have already seen that (30.23.3.1) is fully faithful in Lemma 30.24.1. Thus it suffices to show that the functor is essentially surjective.

We first show that every object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is the quotient of an object in the image of (30.23.3.1). Let $\mathcal{L}$ be an f-ample invertible sheaf on X. Choose d_0 as in Lemma 30.24.2. Choose a $d \geq d_0$ such that $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes d}$ is globally generated by some sections $s_{1,1}, \ldots, s_{t,1}$. Since the transition maps of the system

$$H^0(X, \mathcal{F}_{n+1} \otimes \mathcal{L}^{\otimes d}) \longrightarrow H^0(X, \mathcal{F}_n \otimes \mathcal{L}^{\otimes d})$$

are surjective by the vanishing of H^1 we can lift $s_{1,1}, \ldots, s_{t,1}$ to a compatible system of global sections $s_{1,n}, \ldots, s_{t,n}$ of $\mathcal{F}_n \otimes \mathcal{L}^{\otimes d}$. These determine a compatible system of maps

$$(s_{1,n}, \ldots, s_{t,n}) : (\mathcal{L}^{\otimes -d})^{\oplus t} \longrightarrow \mathcal{F}_n$$

Using Lemma 30.23.3 we deduce that we have a surjective map

$$\left((\mathcal{L}^{\otimes -d})^{\oplus t}\right)^\wedge \longrightarrow (\mathcal{F}_n)$$

as desired.

The result of the previous paragraph and the fact that $Coh(X, \mathcal{I})$ is abelian (Lemma 30.23.2) implies that every object of $Coh(X, \mathcal{I})$ is a cokernel of a map between objects coming from $Coh(\mathcal{O}_X)$. As (30.23.3.1) is fully faithful and exact by Lemmas 30.24.1 and 30.23.4 we conclude. □

30.25. Grothendieck's existence theorem, II

0886 In this section we discuss Grothendieck's existence theorem in the proper case. Before we give the statement and proof, we need to develop a bit more theory regarding the categories $Coh(X, \mathcal{I})$ of coherent formal modules introduced in Section 30.23.

0888 **Remark 30.25.1.** Let X be a Noetherian scheme and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine open $\mathrm{Spec}(A) = U \subset X$ with $\mathcal{I}|_U, \mathcal{K}|_U$ corresponding to ideals $I, K \subset A$ denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Lemma 30.23.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$, and
(3) there exists a finite affine open covering $X = \bigcup U_i$ such that the conclusion of (2) holds for α_{U_i}.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we use the following commutative algebra fact: suppose given an exact sequence

$$0 \to T \to M \to N \to Q \to 0$$

of A-modules with T and Q annihilated by K^t for some ideal $K \subset A$. Then for every $f, g \in K^t$ there exists a canonical map "fg" : $N \to M$ such that $M \to N \to M$ is equal to multiplication by fg. Namely, for $y \in N$ we can pick $x \in M$ mapping to fy in N and then we can set "fg"$(y) = gx$. Thus it is clear that $\mathrm{Ker}(M/JM \to N/JN)$ and $\mathrm{Coker}(M/JM \to N/JN)$ are annihilated by K^{2t} for any ideal $J \subset A$.

Applying the commutative algebra fact to α_{U_i} and $J = I^n$ we see that (3) implies (1). Conversely, suppose (1) holds and $M \to N$ is equal to α_U. Then there is a $t \geq 1$ such that $\mathrm{Ker}(M/I^nM \to N/I^nN)$ and $\mathrm{Coker}(M/I^nM \to N/I^nN)$ are annihilated by K^t for all n. We obtain maps "fg" : $N/I^nN \to M/I^nM$ which in the limit induce a map $N \to M$ as N and M are I-adically complete. Since the composition with $N \to M \to N$ is multiplication by fg we conclude that fg annihilates T and Q. In other words T and Q are annihilated by K^{2t} as desired.

088A **Lemma 30.25.2.** *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $X_e \subset X$ be the closed subscheme cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Assume*

(1) *the functor $Coh(\mathcal{O}_{X_e}) \to Coh(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
(2) *there exists a coherent sheaf $\mathcal{H}$ on X and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (30.23.3.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Lemma 30.9.8. In particular we may identify $Coh(\mathcal{O}_{X_e})$ with the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{K}^e$ and $Coh(X_e, \mathcal{I}_e)$ as the full subcategory of $Coh(X, \mathcal{I})$ of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (30.23.3.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. Hence $(\mathcal{G}_e)$ is an object of $Coh(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$, see Remark 30.25.1. Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

see Remark 30.25.1. We apply Lemma 30.23.6 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 30.23.6 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight. This finishes the proof of the lemma. □

088B **Lemma 30.25.3.** *Let Y be a Noetherian scheme. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Let $f : X \to Y$ be a proper morphism which is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Proposition 30.19.1). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness we may assume that $Y = \mathrm{Spec}(B)$ is affine. Let $J \subset B$ corresponds to the ideal $\mathcal{J}$. Set

$$M_n = H^0(X, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \quad \text{and} \quad M = H^0(X, \mathcal{F})$$

By Lemma 30.20.4 and Theorem 30.20.5 the inverse limit of the modules M_n equals the completion $M^\wedge = \lim M/J^nM$. Set $N_n = H^0(Y, \mathcal{G}_n)$ and $N = \lim N_n$. Via the equivalence of categories of Lemma 30.23.1 the finite $B^\wedge$ modules N and $M^\wedge$ correspond to $(\mathcal{G}_n)$ and $f_*\mathcal{F}^\wedge$. It follows from this that α has to be the morphism of $Coh(Y, \mathcal{J})$ corresponding to the homomorphism

$$\lim \gamma_n : N = \lim_n N_n \longrightarrow \lim M_n = M^\wedge$$

of finite $B^\wedge$-modules.

We still have to show that the kernel and cokernel of α are annihilated by a power of $\mathcal{K}$. Set $Y' = \mathrm{Spec}(B^\wedge)$ and $X' = Y' \times_Y X$. Let $\mathcal{K}'$, $\mathcal{J}'$, $\mathcal{G}'_n$ and $\mathcal{I}'$, $\mathcal{F}'$ be the pullback of $\mathcal{K}$, $\mathcal{J}$, $\mathcal{G}_n$ and $\mathcal{I}$, $\mathcal{F}$, to Y' and X'. The projection morphism $f' : X' \to Y'$ is the base change of f by $Y' \to Y$. Note that $Y' \to Y$ is a flat morphism of schemes as $B \to B^\wedge$ is flat by Algebra, Lemma 10.97.2. Hence $f'_*\mathcal{F}'$, resp. $f'_*(f')^*\mathcal{G}'_n$ is the pullback of $f_*\mathcal{F}$, resp. $f_*f^*\mathcal{G}_n$ to Y' by Lemma 30.5.2. The uniqueness of our construction shows the pullback of α to Y' is the corresponding map α' constructed for the situation on Y'. Moreover, to check that the kernel and cokernel of α are annihilated by $\mathcal{K}^t$ it suffices to check that the kernel and cokernel of α' are annihilated by $(\mathcal{K}')^t$. Namely, to see this we need to check this for kernels and cokernels of the maps α_n and α'_n (see Remark 30.25.1) and the ring map $B \to B^\wedge$ induces an equivalence of categories between modules annihilated by J^n and $(J')^n$, see More on Algebra, Lemma 15.90.3. Thus we may assume B is complete with respect to J.

Assume $Y = \mathrm{Spec}(B)$ is affine, $\mathcal{J}$ corresponds to the ideal $J \subset B$, and B is complete with respect to J. In this case $(\mathcal{G}_n)$ is in the essential image of the functor $Coh(\mathcal{O}_Y) \to Coh(Y, \mathcal{J})$. Say $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module such that $(\mathcal{G}_n) = \mathcal{G}^\wedge$. Note that $f^*(\mathcal{G}^\wedge) = (f^*\mathcal{G})^\wedge$. Hence Lemma 30.24.1 tells us that β comes from an isomorphism $b : f^*\mathcal{G} \to \mathcal{F}$ and α is the completion functor applied to

$$\mathcal{G} \to f_*f^*\mathcal{G} \cong f_*\mathcal{F}$$

Hence we are trying to verify that the kernel and cokernel of the adjunction map $c : \mathcal{G} \to f_*f^*\mathcal{G}$ are annihilated by a power of $\mathcal{K}$. However, since the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism we see that $c|_V$ is an isomorphism. Thus the coherent sheaves $\mathrm{Ker}(c)$ and $\mathrm{Coker}(c)$ are supported on $V(\mathcal{K})$ hence are annihilated by a power of $\mathcal{K}$ (Lemma 30.10.2) as desired. □

The following proposition is the form of Grothendieck's existence theorem which is most often used in practice.

088C **Proposition 30.25.4.** *Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is an equivalence.*

Proof. We have already seen that (30.23.3.1) is fully faithful in Lemma 30.24.1. Thus it suffices to show that the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that every object $(\mathcal{F}_n)$ annihilated by $\mathcal{K}$ is in the essential image. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 30.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ.

(This uses the correspondence by coherent modules annihilated by $\mathcal{K}$ and coherent modules on the closed subscheme corresponding to $\mathcal{K}$, see Lemma 30.9.8.) Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. We will show that this object is in the essential image of the functor (30.23.3.1), thereby completion the proof of the proposition.

Apply Chow's lemma (Lemma 30.18.1) to find a proper surjective morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is projective over A. Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By the projective case of Grothendieck's existence theorem (Lemma 30.24.3) there exists a coherent module $\mathcal{F}'$ on X' such that $(\mathcal{F}')^\wedge \cong (f^*\mathcal{F}_n)$. By Proposition 30.19.1 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 30.25.3 there exists a morphism $(\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. The powers $\mathcal{K}^e$ are all in Ξ so that (30.23.3.1) is an equivalence for the closed subschemes $X_e = V(\mathcal{K}^e)$. We conclude by Lemma 30.25.2. □

30.26. Being proper over a base

0CYK This is just a short section to point out some useful features of closed subsets proper over a base and finite type, quasi-coherent modules with support proper over a base.

0CYL **Lemma 30.26.1.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. The following are equivalent*

(1) *the morphism $Z \to S$ is proper if Z is endowed with the reduced induced closed subscheme structure (Schemes, Definition 26.12.5),*
(2) *for some closed subscheme structure on Z the morphism $Z \to S$ is proper,*
(3) *for any closed subscheme structure on Z the morphism $Z \to S$ is proper.*

Proof. The implications (3) ⇒ (1) and (1) ⇒ (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let Z' and Z'' be closed subscheme structures on Z such that $Z' \to S$ is proper. We have to show that $Z'' \to S$ is proper. Let $Z''' = Z' \cup Z''$ be the scheme theoretic union, see Morphisms, Definition 29.4.4. Then Z''' is another closed subscheme structure on Z. This follows for example from the description of scheme theoretic unions in Morphisms, Lemma 29.4.6. Since $Z'' \to Z'''$ is a closed immersion it suffices to prove that $Z''' \to S$ is proper (see Morphisms, Lemmas 29.41.6 and 29.41.4). The morphism $Z' \to Z'''$ is a bijective closed immersion and in particular surjective and universally closed. Then the fact that $Z' \to S$ is separated implies that $Z''' \to S$ is separated, see Morphisms, Lemma 29.41.11. Moreover $Z''' \to S$ is locally of finite type as $X \to S$ is locally of finite type (Morphisms, Lemmas 29.15.5 and 29.15.3). Since $Z' \to S$ is quasi-compact and $Z' \to Z'''$ is a homeomorphism we see that $Z''' \to S$ is quasi-compact. Finally, since $Z' \to S$ is universally closed, we see that the same thing is true for $Z''' \to S$ by Morphisms, Lemma 29.41.9. This finishes the proof. □

0CYM **Definition 30.26.2.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. We say *Z is proper over S* if the equivalent conditions of Lemma 30.26.1 are satisfied.

The lemma used in the definition above is false if the morphism $f : X \to S$ is not locally of finite type. Therefore we urge the reader not to use this terminology if f is not locally of finite type.

0CYN **Lemma 30.26.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Y \subset Z \subset X$ be closed subsets. If Z is proper over S, then the same is true for Y.*

Proof. Omitted. □

0CYP **Lemma 30.26.4.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

with f locally of finite type. If Z is a closed subset of X proper over S, then $(g')^{-1}(Z)$ is a closed subset of X' proper over S'.

Proof. Observe that the statement makes sense as f' is locally of finite type by Morphisms, Lemma 29.15.4. Endow Z with the reduced induced closed subscheme structure. Denote $Z' = (g')^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z' = X' \times_X Z = (S' \times_S X) \times_X Z = S' \times_S Z$ is proper over S' as a base change of Z over S (Morphisms, Lemma 29.41.5). □

0CYQ **Lemma 30.26.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes which are locally of finite type over S.*

1. *If Y is separated over S and $Z \subset X$ is a closed subset proper over S, then $f(Z)$ is a closed subset of Y proper over S.*
2. *If f is universally closed and $Z \subset X$ is a closed subset proper over S, then $f(Z)$ is a closed subset of Y proper over S.*
3. *If f is proper and $Z \subset Y$ is a closed subset proper over S, then $f^{-1}(Z)$ is a closed subset of X proper over S.*

Proof. Proof of (1). Assume Y is separated over S and $Z \subset X$ is a closed subset proper over S. Endow Z with the reduced induced closed subscheme structure and apply Morphisms, Lemma 29.41.10 to $Z \to Y$ over S to conclude.

Proof of (2). Assume f is universally closed and $Z \subset X$ is a closed subset proper over S. Endow Z and $Z' = f(Z)$ with their reduced induced closed subscheme structures. We obtain an induced morphism $Z \to Z'$. Denote $Z'' = f^{-1}(Z')$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z'' \to Z'$ is universally closed as a base change of f (Morphisms, Lemma 29.41.5). Hence $Z \to Z'$ is universally closed as a composition of the closed immersion $Z \to Z''$ and $Z'' \to Z'$ (Morphisms, Lemmas 29.41.6 and 29.41.4). We conclude that $Z' \to S$ is separated by Morphisms, Lemma 29.41.11. Since $Z \to S$ is quasi-compact and $Z \to Z'$ is surjective we see that $Z' \to S$ is quasi-compact. Since $Z' \to S$ is the composition of $Z' \to Y$ and $Y \to S$ we see that $Z' \to S$ is locally of finite type (Morphisms, Lemmas 29.15.5 and 29.15.3). Finally, since $Z \to S$ is universally closed, we see that the same thing is true for $Z' \to S$ by Morphisms, Lemma 29.41.9. This finishes the proof.

Proof of (3). Assume f is proper and $Z \subset Y$ is a closed subset proper over S. Endow Z with the reduced induced closed subscheme structure. Denote $Z' = f^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z' \to Z$ is proper as a base change of f (Morphisms, Lemma 29.41.5). Whence $Z' \to S$ is

proper as the composition of $Z' \to Z$ and $Z \to S$ (Morphisms, Lemma 29.41.4). This finishes the proof. □

0CYR **Lemma 30.26.6.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z_i \subset X$, $i = 1, \ldots, n$ be closed subsets. If Z_i, $i = 1, \ldots, n$ are proper over S, then the same is true for $Z_1 \cup \ldots \cup Z_n$.*

Proof. Endow Z_i with their reduced induced closed subscheme structures. The morphism

$$Z_1 \amalg \ldots \amalg Z_n \longrightarrow X$$

is finite by Morphisms, Lemmas 29.44.12 and 29.44.13. As finite morphisms are universally closed (Morphisms, Lemma 29.44.11) and since $Z_1 \amalg \ldots \amalg Z_n$ is proper over S we conclude by Lemma 30.26.5 part (2) that the image $Z_1 \cup \ldots \cup Z_n$ is proper over S. □

Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Then the support $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of X, see Morphisms, Lemma 29.5.3. Hence it makes sense to say "the support of $\mathcal{F}$ is proper over S".

0CYS **Lemma 30.26.7.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *the support of $\mathcal{F}$ is proper over S,*
2. *the scheme theoretic support of $\mathcal{F}$ (Morphisms, Definition 29.5.5) is proper over S, and*
3. *there exists a closed subscheme $Z \subset X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $Z \to S$ is proper, and (b) $(Z \to X)_*\mathcal{G} = \mathcal{F}$.*

Proof. The support $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of X, see Morphisms, Lemma 29.5.3. Hence we can apply Definition 30.26.2. Since the scheme theoretic support of $\mathcal{F}$ is a closed subscheme whose underlying closed subset is $\mathrm{Supp}(\mathcal{F})$ we see that (1) and (2) are equivalent by Definition 30.26.2. It is clear that (2) implies (3). Conversely, if (3) is true, then $\mathrm{Supp}(\mathcal{F}) \subset Z$ (an inclusion of closed subsets of X) and hence $\mathrm{Supp}(\mathcal{F})$ is proper over S for example by Lemma 30.26.3. □

0CYT **Lemma 30.26.8.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

with f locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. If the support of $\mathcal{F}$ is proper over S, then the support of $(g')^*\mathcal{F}$ is proper over S'.

Proof. Observe that the statement makes sense because $(g')*\mathcal{F}$ is of finite type by Modules, Lemma 17.9.2. We have $\mathrm{Supp}((g')^*\mathcal{F}) = (g')^{-1}(\mathrm{Supp}(\mathcal{F}))$ by Morphisms, Lemma 29.5.3. Thus the lemma follows from Lemma 30.26.4. □

0CYU **Lemma 30.26.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$, $\mathcal{G}$ be finite type, quasi-coherent $\mathcal{O}_X$-module.*

(1) *If the supports of $\mathcal{F}$, $\mathcal{G}$ are proper over S, then the same is true for $\mathcal{F} \oplus \mathcal{G}$, for any extension of $\mathcal{G}$ by $\mathcal{F}$, for* $\mathrm{Im}(u)$ *and* $\mathrm{Coker}(u)$ *given any $\mathcal{O}_X$-module map $u : \mathcal{F} \to \mathcal{G}$, and for any quasi-coherent quotient of $\mathcal{F}$ or $\mathcal{G}$.*
(2) *If S is locally Noetherian, then the category of coherent $\mathcal{O}_X$-modules with support proper over S is a Serre subcategory (Homology, Definition 12.10.1) of the abelian category of coherent $\mathcal{O}_X$-modules.*

Proof. Proof of (1). Let Z, Z' be the support of $\mathcal{F}$ and $\mathcal{G}$. Then all the sheaves mentioned in (1) have support contained in $Z \cup Z'$. Thus the assertion itself is clear from Lemmas 30.26.3 and 30.26.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Schemes, Section 26.24. For "finite type" we suggest the reader take a look at Modules, Section 17.9.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as X is locally Noetherian by Morphisms, Lemma 29.15.6 and by the description of the category of coherent modules in Section 30.9. □

08DS **Lemma 30.26.10.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with support proper over S. Then $R^p f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $p \geq 0$.*

Proof. By Lemma 30.26.7 there exists a closed immersion $i : Z \to X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $g = f \circ i : Z \to S$ is proper, and (b) $i_* \mathcal{G} = \mathcal{F}$. We see that $R^p g_* \mathcal{G}$ is coherent on S by Proposition 30.19.1. On the other hand, $R^q i_* \mathcal{G} = 0$ for $q > 0$ (Lemma 30.9.9). By Cohomology, Lemma 20.13.8 we get $R^p f_* \mathcal{F} = R^p g_* \mathcal{G}$ which concludes the proof. □

0CYV **Lemma 30.26.11.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a finite type morphism. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The following are Serre subcategories of $Coh(X, \mathcal{I})$*

(1) *the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over S,*
(2) *the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that there exists a closed subscheme $Z \subset X$ proper over S with $\mathcal{I}_Z \mathcal{F}_n = 0$ for all $n \geq 1$.*

Proof. We will use the criterion of Homology, Lemma 12.10.2. Moreover, we will use that if $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ is a short exact sequence of $Coh(X, \mathcal{I})$, then (a) $\mathcal{G}_n \to \mathcal{F}_n \to \mathcal{H}_n \to 0$ is exact for all $n \geq 1$ and (b) $\mathcal{G}_n$ is a quotient of $\mathrm{Ker}(\mathcal{F}_m \to \mathcal{H}_m)$ for some $m \geq n$. See proof of Lemma 30.23.2.

Proof of (1). Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Then $\mathrm{Supp}(\mathcal{F}_n) = \mathrm{Supp}(\mathcal{F}_1)$ for all $n \geq 1$. Hence by remarks (a) and (b) above we see that for any short exact sequence $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ of $Coh(X, \mathcal{I})$ we have $\mathrm{Supp}(\mathcal{G}_1) \cup \mathrm{Supp}(\mathcal{H}_1) = \mathrm{Supp}(\mathcal{F}_1)$. This proves that the category defined in (1) is a Serre subcategory of $Coh(X, \mathcal{I})$.

Proof of (2). Here we argue the same way. Let $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ be a short exact sequence of $Coh(X, \mathcal{I})$. If $Z \subset X$ is a closed subscheme and $\mathcal{I}_Z$ annihilates $\mathcal{F}_n$ for all n, then $\mathcal{I}_Z$ annihilates $\mathcal{G}_n$ and $\mathcal{H}_n$ for all n by (a) and (b) above. Hence if $Z \to S$ is proper, then we conclude that the category defined in (2) is closed under taking sub and quotient objects inside of $Coh(X, \mathcal{I})$. Finally,

suppose that $Z \subset X$ and $Y \subset X$ are closed subschemes proper over S such that $\mathcal{I}_Z\mathcal{G}_n = 0$ and $\mathcal{I}_Y\mathcal{H}_n = 0$ for all $n \geq 1$. Then it follows from (a) above that $\mathcal{I}_{Z\cup Y} = \mathcal{I}_Z \cdot \mathcal{I}_Y$ annihilates $\mathcal{F}_n$ for all n. By Lemma 30.26.6 (and via Definition 30.26.2 which tells us we may choose an arbitrary scheme structure used on the union) we see that $Z \cup Y \to S$ is proper and the proof is complete. □

30.27. Grothendieck's existence theorem, III

0CYW To state the general version of Grothendieck's existence theorem we introduce a bit more notation. Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a separated finite type morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. In this situation we let

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose support is proper over $\mathrm{Spec}(A)$. This is a Serre subcategory of $Coh(\mathcal{O}_X)$, see Lemma 30.26.9. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. This is a Serre subcategory of $Coh(X, \mathcal{I})$ by Lemma 30.26.11 part (1). Since the support of a quotient module is contained in the support of the module, it follows that (30.23.3.1) induces a functor

088D (30.27.0.1) $$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I})$$

We are now ready to state the main theorem of this section.

088E **Theorem 30.27.1** (Grothendieck's existence theorem)**.** *Let A be a Noetherian ring complete with respect to an ideal I. Let X be a separated, finite type scheme over A. Then the functor (30.27.0.1)* [**DG67**, III Theorem 5.1.5]

$$Coh_{\textit{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\textit{support proper over } A}(X, \mathcal{I})$$

is an equivalence.

Proof. We will use the equivalence of categories of Lemma 30.9.8 without further mention. For a closed subscheme $Z \subset X$ proper over A in this proof we will say a coherent module on X is "supported on Z" if it is annihilated by the ideal sheaf of Z or equivalently if it is the pushforward of a coherent module on Z. By Proposition 30.25.4 we know that the result is true for the functor between coherent modules and systems of coherent modules supported on Z. Hence it suffices to show that every object of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$ and every object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ is supported on a closed subscheme $Z \subset X$ proper over A. This holds by definition for objects of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$. We will prove this statement for objects of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ using the method of proof of Proposition 30.25.4. We urge the reader to read that proof first.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 30.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We

will show that this object is supported on a closed subscheme $Z \subset X$ proper over A, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 30.18.1) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 29.43.11. Observe that

$$\mathrm{Supp}(f^*\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_1)$$

The first equality by Morphisms, Lemma 29.5.3. By assumption and Lemma 30.26.5 part (3) we see that $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ is proper over A. Hence the image of $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ under j is closed in Y' by Lemma 30.26.5 part (1). Thus $\mathcal{F}'_n = j_*f^*\mathcal{F}_n$ is coherent on Y' by Lemma 30.9.11. It follows that $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Lemma 30.24.3) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that

$$\mathrm{Supp}(\mathcal{F}') \cap V(I\mathcal{O}_{Y'}) = \mathrm{Supp}(\mathcal{F}'_1) = j(f^{-1}\mathrm{Supp}(\mathcal{F}_1))$$

The structure morphism $p' : Y' \to \mathrm{Spec}(A)$ is proper, hence $p'(\mathrm{Supp}(\mathcal{F}') \setminus j(Y))$ is closed in $\mathrm{Spec}(A)$. A nonempty closed subset of $\mathrm{Spec}(A)$ contains a point of $V(I)$ as I is contained in the Jacobson radical of A by Algebra, Lemma 10.96.6. The displayed equation shows that $\mathrm{Supp}(\mathcal{F}') \cap (p')^{-1}V(I) \subset j(Y)$ hence we conclude that $\mathrm{Supp}(\mathcal{F}') \subset j(Y)$. Thus $\mathcal{F}'|_Y = j^*\mathcal{F}'$ is supported on a closed subscheme Z' of Y proper over A and $(\mathcal{F}'|_Y)^\wedge = (f^*\mathcal{F}_n)$.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Proposition 30.19.1 the $\mathcal{O}_X$-module $\mathcal{H} = f_*(\mathcal{F}'|_Y)$ is coherent and by Lemma 30.25.3 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power $\mathcal{K}^t$ of $\mathcal{K}$. We obtain an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to (\mathcal{F}_n) \to \mathcal{H}^\wedge \to \mathrm{Coker}(\alpha) \to 0$$

in $Coh(X, \mathcal{I})$. If $Z_0 \subset X$ is the scheme theoretic support of $\mathcal{H}$, then it is clear that $Z_0 \subset f(Z')$ set-theoretically. Hence Z_0 is proper over A by Lemma 30.26.3 and Lemma 30.26.5 part (2). Hence $\mathcal{H}^\wedge$ is in the subcategory defined in Lemma 30.26.11 part (2) and a fortiori in $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We conclude that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in $Coh_{\text{support proper over } A}(X, \mathcal{I})$ by Lemma 30.26.11 part (1). By induction hypothesis, more precisely because $\mathcal{K}^t$ is in Ξ, we see that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in the subcategory defined in Lemma 30.26.11 part (2). Since this is a Serre subcategory by the lemma, we conclude that the same is true for $(\mathcal{F}_n)$ which is what we wanted to show. □

088F **Remark 30.27.2** (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{array}{ccccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots & & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & & S \end{array}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\mathrm{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 30.27.1 says that the completion functor

$$\begin{array}{c}\text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A\end{array} \longrightarrow \begin{array}{c}\text{systems } (\mathcal{F}_n) \\ \text{as above}\end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

30.28. Grothendieck's algebraization theorem

0898 Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

0899 **Lemma 30.28.1.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of schemes $Z \to X$ such that $Z_n = Z \times_S S_n$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Morphisms, Lemma 29.45.7 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 30.27.2, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 30.23.4) we see that the cokernel $\mathcal{Q}$ of $\mathcal{O}_X \to \mathcal{F}$ has vanishing completion. Since $\mathcal{F}$ has support proper over S and so does $\mathcal{Q}$ this implies that $\mathcal{Q} = 0$ for example because the functor (30.27.0.1) is an equivalence by Grothendieck's existence theorem. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. □

In the following lemma it is actually enough to assume that $Y_1 \to X_1$ is finite as it will imply that $Y_n \to X_n$ is finite too (see More on Morphisms, Lemma 37.3.3).

09ZT **Lemma 30.28.2.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of*

finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Assume that

(1) *$Y_n \to X_n$ is a finite morphism, and*
(2) *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of schemes $Y \to X$ such that $Y_n = Y \times_S S_n$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. This is a coherent $\mathcal{O}_{X_n}$-module as f_n is finite (Lemma 30.9.9). Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 30.27.2, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fully faithful (Theorem 30.27.1) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. □

0A42 **Lemma 30.28.3.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be schemes over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

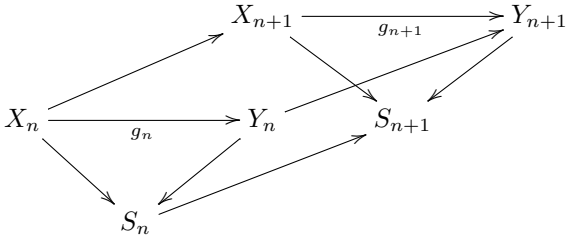

Assume that

(1) *$X \to S$ is proper, and*
(2) *$Y \to S$ is separated of finite type.*

Then there exists a unique morphism of schemes $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Schemes, Lemma 26.21.11). Thus by Lemma 30.28.1 there exists a closed subscheme $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms, Lemma 29.41.7) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is finite over an open neighbourhood of X_0 by Lemma 30.21.2. As X is proper over S this open neighbourhood is all of X and we conclude $p : Z \to X$ is finite. Applying the equivalence of Proposition 30.25.4 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an

isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. □

In order to prove an "abstract" algebraization theorem we need to assume we have an ample invertible sheaf, as the result is false without such an assumption.

089A **Theorem 30.28.4** (Grothendieck's algebraization theorem)**.** *Let A be a Noetherian ring complete with respect to an ideal I. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Consider a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

(1) *$X_1 \to S_1$ is proper, and*
(2) *$\mathcal{L}_1$ is ample on X_1*

then there exists a proper morphism of schemes $X \to S$ and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Since the squares in the diagram are cartesian and since the morphisms $S_n \to S_{n+1}$ are closed immersions, we see that the morphisms i_n are closed immersions too. In particular we may think of X_m as a closed subscheme of X_n for $m < n$. In fact X_m is the closed subscheme cut out by the quasi-coherent sheaf of ideals $I^m\mathcal{O}_{X_n}$. Moreover, the underlying topological spaces of the schemes $X_1, X_2, X_3, \ldots$ are all identified, hence we may (and do) think of sheaves $\mathcal{O}_{X_n}$ as living on the same underlying topological space; similarly for coherent $\mathcal{O}_{X_n}$-modules. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$$

so that we obtain short exact sequences

$$0 \to \mathcal{F}_n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$$

By the above we have $\mathcal{F}_n = I^n\mathcal{O}_{X_{n+1}}$. It follows $\mathcal{F}_n$ is a coherent sheaf on X_{n+1} annihilated by I, hence we may (and do) think of it as a coherent module $\mathcal{O}_{X_1}$-module. Observe that for $m > n$ the sheaf

$$I^n\mathcal{O}_{X_m}/I^{n+1}\mathcal{O}_{X_m}$$

maps isomorphically to $\mathcal{F}_n$ under the map $\mathcal{O}_{X_m} \to \mathcal{O}_{X_{n+1}}$. Hence given $n_1, n_2 \geq 0$ we can pick an $m > n_1 + n_2$ and consider the multiplication map

$$I^{n_1}\mathcal{O}_{X_m} \times I^{n_2}\mathcal{O}_{X_m} \longrightarrow I^{n_1+n_2}\mathcal{O}_{X_m} \to \mathcal{F}_{n_1+n_2}$$

This induces an $\mathcal{O}_{X_1}$-bilinear map

$$\mathcal{F}_{n_1} \times \mathcal{F}_{n_2} \longrightarrow \mathcal{F}_{n_1+n_2}$$

which in turn defines the structure of a graded $\mathcal{O}_{X_1}$-algebra on $\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{F}_n$.

Set $B = \bigoplus I^n/I^{n+1}$; this is a finitely generated graded A/I-algebra. Set $\mathcal{B} = (X_1 \to S_1)^*\widetilde{B}$. The discussion above provides us with a canonical surjection

$$\mathcal{B} \longrightarrow \mathcal{F}$$

of graded $\mathcal{O}_{X_1}$-algebras. In particular we see that $\mathcal{F}$ is a finite type quasi-coherent graded $\mathcal{B}$-module. By Lemma 30.19.3 we can find an integer d_0 such that $H^1(X_1, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $d \geq d_0$. Pick a $d \geq d_0$ such that there exist sections $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ which induce an immersion

$$\psi_1 : X_1 \to \mathbf{P}^N_{S_1}$$

over S_1, see Morphisms, Lemma 29.39.4. As X_1 is proper over S_1 we see that ψ_1 is a closed immersion, see Morphisms, Lemma 29.41.7 and Schemes, Lemma 26.10.4. We are going to "lift" ψ_1 to a compatible system of closed immersions of X_n into $\mathbf{P}^N$.

Upon tensoring the short exact sequences of the first paragraph of the proof by $\mathcal{L}_{n+1}^{\otimes d}$ we obtain short exact sequences

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to 0$$

Using the isomorphisms φ_n we obtain isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{X_l} = \mathcal{L}_l$ for $l \leq n$. Whence the sequence above becomes

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_n^{\otimes d} \to 0$$

The vanishing of $H^1(X, \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d})$ implies we can inductively lift $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ to sections $s_{0,n}, \ldots, s_{N,n} \in \Gamma(X_n, \mathcal{L}_n^{\otimes d})$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\ \mathbf{P}^N_{S_1} & \longrightarrow & \mathbf{P}^N_{S_2} & \longrightarrow & \mathbf{P}^N_{S_3} & \longrightarrow & \ldots \end{array}$$

where $\psi_n = \varphi_{(\mathcal{L}_n, (s_{0,n}, \ldots, s_{N,n}))}$ in the notation of Constructions, Section 27.13. As the squares in the statement of the theorem are cartesian we see that the squares in the above diagram are cartesian. We win by applying Lemma 30.28.1. □

30.29. Other chapters

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CHAPTER 31

Divisors

01WO

31.1. Introduction

01WP In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [**DG67**].

31.2. Associated points

02OI Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *associated* to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 10.63.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

02OJ **Definition 31.2.1.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\mathrm{Ass}(\mathcal{F})$ or $\mathrm{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.
(3) The *associated points of* X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 31.2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 31.5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

02OK **Lemma 31.2.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.*

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.*
(2) *If $\mathfrak{p}$ is finitely generated, then the converse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in Ass(M) \Leftrightarrow x \in Ass(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 10.63.15. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_\mathfrak{p}$ is the annihilator of $m/1 \in M_\mathfrak{p}$. Since $M_\mathfrak{p} = \mathcal{F}_x$ (Schemes, Lemma 26.5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}A_\mathfrak{p}$. Write

$m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_\mathfrak{p} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 28.5.2) and $\mathfrak{p}$ is always finitely generated. □

05AD **Lemma 31.2.3.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \subset Supp(\mathcal{F})$.*

Proof. This is immediate from the definitions. □

05AE **Lemma 31.2.4.** *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $Ass(\mathcal{F}_2) \subset Ass(\mathcal{F}_1) \cup Ass(\mathcal{F}_3)$ and $Ass(\mathcal{F}_1) \subset Ass(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.63.3. □

05AF **Lemma 31.2.5.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.*

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 10.63.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 28.5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Cohomology of Schemes, Lemma 30.9.1), and finally use Lemma 31.2.2. □

05AG **Lemma 31.2.6.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = 0 \Leftrightarrow Ass(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\mathrm{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\mathrm{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 10.63.7. To translate from schemes to algebra, restrict to any affine and use Lemma 31.2.2. □

05AI **Example 31.2.7.** Let k be a field. The ring $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\mathrm{Ass}(R) = \emptyset$, and $X = \mathrm{Spec}(R)$ is an example of a scheme which has no associated points.

0B3L **Lemma 31.2.8.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $Ass(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\mathrm{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\mathrm{Ass}(\mathcal{F}') \subset \mathrm{Ass}(\mathcal{F})$ by Lemma 31.2.4. Since also $\mathrm{Ass}(\mathcal{F}') \subset \mathrm{Supp}(\mathcal{F}')$ (Lemma 31.2.3) we conclude $\mathrm{Ass}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 31.2.6. □

05AH **Lemma 31.2.9.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in Ass(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.*

Proof. Since $x \in \mathrm{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\mathrm{Ass}(\mathcal{F}_x) \subset \mathrm{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.63.7. On the other hand, by assumption $\mathrm{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\mathrm{Ass}(\mathcal{F}_x) \subset \mathrm{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.63.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. □

The following lemma is the analogue of More on Algebra, Lemma 15.23.12.

0AVL **Lemma 31.2.10.** *Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens*

(1) *$\mathcal{F}_x \to \mathcal{G}_x$ is injective, or*
(2) *$x \notin Ass(\mathcal{F})$.*

Then φ is injective.

Proof. The assumptions imply that $\mathrm{Ass}(\mathrm{Ker}(\varphi)) = \emptyset$ and hence $\mathrm{Ker}(\varphi) = 0$ by Lemma 31.2.6. □

0AVM **Lemma 31.2.11.** *Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent and that for every $x \in X$ one of the following happens*

(1) *$\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, or*
(2) *$depth(\mathcal{F}_x) \geq 2$ and $x \notin Ass(\mathcal{G})$.*

Then φ is an isomorphism.

Proof. This is a translation of More on Algebra, Lemma 15.23.13 into the language of schemes. □

31.3. Morphisms and associated points

05DA Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $s \in S$ is a point, then it is often convenient to denote $\mathcal{F}_s$ the $\mathcal{O}_{X_s}$-module one gets by pulling back $\mathcal{F}$ by the morphism $i_s : X_s \to X$. Here X_s is the scheme theoretic fibre of f over s. In a formula

$$\mathcal{F}_s = i_s^*\mathcal{F}$$

Of course, this notation clashes with the already existing notation for the stalk of $\mathcal{F}$ at a point $x \in X$ if $f = \mathrm{id}_X$. However, the notation is often convenient, as in the formulation of the following lemma.

05DB **Lemma 31.3.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in Ass_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian (for the notation $\mathcal{F}_s$ see above).

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \mathrm{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\mathrm{Ass}_B(M \otimes_A N)$. Similarly $s \in \mathrm{Ass}_S(\mathcal{G})$ and $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \mathrm{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \mathrm{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 10.65.5. □

31.4. Embedded points

05AJ Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an *embedded associated prime* of M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 10.67.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

05AK **Definition 31.4.1.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An *embedded associated point* of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.
(2) A point x of X is called an *embedded point* if x is an embedded associated point of $\mathcal{O}_X$.
(3) An *embedded component* of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

05AL **Lemma 31.4.2.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(1) *the generic points of irreducible components of $\mathrm{Supp}(\mathcal{F})$ are associated points of $\mathcal{F}$, and*
(2) *an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F})$.*

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \mathrm{Supp}(\mathcal{F})$ is closed, see Morphisms, Lemma 29.5.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 31.2.9. Finally, we have $\mathrm{Ass}(\mathcal{F}) \subset Z$, by Lemma 31.2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). □

0346 **Lemma 31.4.3.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:*

(1) *$\mathcal{F}$ has no embedded associated points, and*
(2) *$\mathcal{F}$ has property (S_1).*

Proof. This is Algebra, Lemma 10.157.2, combined with Lemma 31.2.2 above. □

0BXG **Lemma 31.4.4.** *Let X be a locally Noetherian scheme of dimension ≤ 1. The following are equivalent*

(1) *X is Cohen-Macaulay, and*
(2) *X has no embedded points.*

Proof. Follows from Lemma 31.4.3 and the definitions. □

083P **Lemma 31.4.5.** *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme. The following are equivalent*

(1) *U is scheme theoretically dense in X (Morphisms, Definition 29.7.1),*
(2) *U is dense in X and U contains all embedded points of X, and*
(3) *U contains all associated points of X.*

Proof. The question is local on X, hence we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring. Then U is quasi-compact (Properties, Lemma 28.5.3) hence $U = D(f_1) \cup \ldots \cup D(f_n)$ (Algebra, Lemma 10.29.1). In this situation U is scheme theoretically dense in X if and only if $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective, see Morphisms, Example 29.7.4. Condition (3) translated into algebra means that for every associated prime $\mathfrak{p}$ of A there exists an i with $f_i \notin \mathfrak{p}$.

Assume (1), i.e., $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Say $a \in A$ has annihilator a prime $\mathfrak{p}$. Since a maps to a nonzero element of A_{f_i} for some i we see that $f_i \notin \mathfrak{p}$. Thus (3) holds.

Note that U is dense in $\mathrm{Spec}(A)$ if and only if it contains all minimal prime ideals. Since the set of associated primes of A is equal to the union of the set of minimal primes and the set of embedded associated primes (see Algebra, Proposition 10.63.6), we see that (2) and (3) are equivalent.

Assume (3). This means that every associated prime $\mathfrak{p}$ of A corresponds to a prime of A_{f_i} for some i. Then $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective because $A \to \prod_{\mathfrak{p} \in \mathrm{Ass}(A)} A_{\mathfrak{p}}$ is injective by Algebra, Lemma 10.63.19. In this way we see that (3) implies (1). □

02OL **Lemma 31.4.6.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves*

$$\{\mathcal{K} \subset \mathcal{F} \mid \mathit{Supp}(\mathcal{K}) \text{ is nowhere dense in } \mathit{Supp}(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

1. *$\mathit{Supp}(\mathcal{F}') = \mathit{Supp}(\mathcal{F})$,*
2. *$\mathcal{F}'$ has no embedded associated points, and*
3. *there exists a dense open $U \subset X$ such that $U \cap \mathit{Supp}(\mathcal{F})$ is dense in $\mathit{Supp}(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.*

Proof. This follows from Algebra, Lemmas 10.67.2 and 10.67.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. □

02OM **Lemma 31.4.7.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set*

$$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $\mathit{Supp}(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Cohomology of Schemes, Lemma 30.9.7. The others follow from Algebra, Lemma 10.67.4. □

31.5. Weakly associated points

056K Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *weakly associated* to M if there exists an element m of M such that $\mathfrak{p}$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 10.66.1. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is weakly associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak{m}$, see Algebra, Lemma 10.66.2.

056L **Definition 31.5.1.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *weakly associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
(3) The *weakly associated points of* X are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

056M **Lemma 31.5.2.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\text{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

(1) *$\mathfrak{p}$ is weakly associated to M, and*
(2) *x is weakly associated to $\mathcal{F}$.*

Proof. This follows from Algebra, Lemma 10.66.2. □

05AM **Lemma 31.5.3.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\textit{Ass}(\mathcal{F}) \subset \textit{WeakAss}(\mathcal{F}) \subset \textit{Supp}(\mathcal{F}).$$

Proof. This is immediate from the definitions. □

05AN **Lemma 31.5.4.** *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\textit{WeakAss}(\mathcal{F}_2) \subset \textit{WeakAss}(\mathcal{F}_1) \cup \textit{WeakAss}(\mathcal{F}_3)$ and $\textit{WeakAss}(\mathcal{F}_1) \subset \textit{WeakAss}(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.66.4. □

05AP **Lemma 31.5.5.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow \textit{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 31.5.2 and Algebra, Lemma 10.66.5 □

0B3M **Lemma 31.5.6.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $\textit{WeakAss}(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\text{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\text{WeakAss}(\mathcal{F}') \subset \text{WeakAss}(\mathcal{F})$ by Lemma 31.5.4. Since also $\text{WeakAss}(\mathcal{F}') \subset \text{Supp}(\mathcal{F}')$ (Lemma 31.5.3) we conclude $\text{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 31.5.5. □

05AQ **Lemma 31.5.7.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in \textit{Supp}(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $\textit{Supp}(\mathcal{F})$. Then $x \in \textit{WeakAss}(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.66.5. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{WeakAss}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.66.6) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. □

05AR **Lemma 31.5.8.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then*

$$x \in Ass(\mathcal{F}) \Leftrightarrow x \in WeakAss(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $Ass(\mathcal{F}) = WeakAss(\mathcal{F})$.

Proof. See Algebra, Lemma 10.66.9. □

0AVN **Lemma 31.5.9.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point which is not in the image of f. Then s is not weakly associated to $f_*\mathcal{F}$.*

Proof. Consider the base change $f' : X' \to \mathrm{Spec}(\mathcal{O}_{S,s})$ of f by the morphism $g : \mathrm{Spec}(\mathcal{O}_{S,s}) \to S$ and denote $g' : X' \to X$ the other projection. Then

$$(f_*\mathcal{F})_s = (g^* f_*\mathcal{F})_s = (f'_*(g')^*\mathcal{F})_s$$

The first equality because g induces an isomorphism on local rings at s and the second by flat base change (Cohomology of Schemes, Lemma 30.5.2). Of course $s \in \mathrm{Spec}(\mathcal{O}_{S,s})$ is not in the image of f'. Thus we may assume S is the spectrum of a local ring $(A, \mathfrak{m})$ and s corresponds to $\mathfrak{m}$. By Schemes, Lemma 26.24.1 the sheaf $f_*\mathcal{F}$ is quasi-coherent, say corresponding to the A-module M. As s is not in the image of f we see that $X = \bigcup_{a \in \mathfrak{m}} f^{-1}D(a)$ is an open covering. Since X is quasi-compact we can find $a_1, \ldots, a_n \in \mathfrak{m}$ such that $X = f^{-1}D(a_1) \cup \ldots \cup f^{-1}D(a_n)$. It follows that

$$M \to M_{a_1} \oplus \ldots \oplus M_{a_r}$$

is injective. Hence for any nonzero element m of the stalk $M_{\mathfrak{p}}$ there exists an i such that $a_i^n m$ is nonzero for all $n \geq 0$. Thus $\mathfrak{m}$ is not weakly associated to M. □

0AVP **Lemma 31.5.10.** *Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens*

(1) *$\mathcal{F}_x \to \mathcal{G}_x$ is injective, or*
(2) *$x \notin WeakAss(\mathcal{F})$.*

Then φ is injective.

Proof. The assumptions imply that $\mathrm{WeakAss}(\mathrm{Ker}(\varphi)) = \emptyset$ and hence $\mathrm{Ker}(\varphi) = 0$ by Lemma 31.5.5. □

0E9I **Lemma 31.5.11.** *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $j : U \to X$ be an open subscheme such that for $x \in X \setminus U$ we have $depth(\mathcal{F}_x) \geq 2$. Then*

$$\mathcal{F} \longrightarrow j_*(\mathcal{F}|_U)$$

is an isomorphism and consequently $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is an isomorphism too.

Proof. We claim Lemma 31.2.11 applies to the map displayed in the lemma. Let $x \in X$. If $x \in U$, then the map is an isomorphism on stalks as $j_*(\mathcal{F}|_U)|_U = \mathcal{F}|_U$. If $x \in X \setminus U$, then $x \notin \mathrm{Ass}(j_*(\mathcal{F}|_U))$ (Lemmas 31.5.9 and 31.5.3). Since we've assumed $\mathrm{depth}(\mathcal{F}_x) \geq 2$ this finishes the proof. □

0EME **Lemma 31.5.12.** *Let X be a reduced scheme. Then the weakly associated points of X are exactly the generic points of the irreducible components of X.*

Proof. Follows from Algebra, Lemma 10.66.3. □

31.6. Morphisms and weakly associated points

05EW

05EX **Lemma 31.6.1.** *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$\mathit{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.66.11. □

05EY **Lemma 31.6.2.** *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$f(\mathit{Ass}_X(\mathcal{F})) = \mathit{Ass}_S(f_*\mathcal{F}) = \mathit{WeakAss}_S(f_*\mathcal{F}) = f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 28.5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\mathrm{Ass}_X(\mathcal{F})) \subset \mathrm{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 10.63.13. We have the inclusion

$$\mathrm{Ass}_S(f_*\mathcal{F}) \subset \mathrm{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 31.5.3. We have the inclusion

$$\mathrm{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathrm{WeakAss}_X(\mathcal{F}))$$

by Lemma 31.6.1. The outer sets are equal by Lemma 31.5.8 hence we have equality everywhere. □

05EZ **Lemma 31.6.3.** *Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathit{WeakAss}(f_*\mathcal{F}) = f(\mathit{WeakAss}(\mathcal{F}))$.*

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.66.13. □

05F0 **Lemma 31.6.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in \mathit{WeakAss}_S(\mathcal{G})$, then $x \in \mathit{WeakAss}(f^*\mathcal{G})$.*

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 10.25.1) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. □

0CUC **Lemma 31.6.5.** *Let K/k be a field extension. Let X be a scheme over k. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback $\mathcal{F}_K$, then x is a weakly associated point of $\mathcal{F}$.*

Proof. This is the translation of Algebra, Lemma 10.66.19 into the language of schemes. □

Here is a simple lemma where we find that pushforwards often have depth at least 2.

0EY0 **Lemma 31.6.6.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$.*

(1) *If $s \notin f(X)$, then s is not weakly associated to $f_*\mathcal{F}$.*
(2) *If $s \notin f(X)$ and $\mathcal{O}_{S,s}$ is Noetherian, then s is not associated to $f_*\mathcal{F}$.*
(3) *If $s \notin f(X)$, $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, and $\mathcal{O}_{S,s}$ is Noetherian, then $\mathit{depth}((f_*\mathcal{F})_s) \geq 2$.*
(4) *If $\mathcal{F}$ is flat over S and $a \in \mathfrak{m}_s$ is a nonzerodivisor, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$.*
(5) *If $\mathcal{F}$ is flat over S and $a, b \in \mathfrak{m}_s$ is a regular sequence, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$ and b is a nonzerodivisor on $(f_*\mathcal{F})_s/a(f_*\mathcal{F})_s$.*
(6) *If $\mathcal{F}$ is flat over S and $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, then $\mathit{depth}((f_*\mathcal{F})_s) \geq \min(2, \mathit{depth}(\mathcal{O}_{S,s}))$.*

Proof. Part (1) is Lemma 31.5.9. Part (2) follows from (1) and Lemma 31.5.8.

Proof of part (3). To show the depth is ≥ 2 it suffices to show that $\mathrm{Hom}_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$ and $\mathrm{Ext}^1_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$, see Algebra, Lemma 10.72.5. Using the exact sequence $0 \to \mathfrak{m}_s \to \mathcal{O}_{S,s} \to \kappa(s) \to 0$ it suffices to prove that the map

$$\mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}, (f_*\mathcal{F})_s) \to \mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s)$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and X by $\mathrm{Spec}(\mathcal{O}_{S,s}) \times_S X$. Denote $\mathfrak{m} \subset \mathcal{O}_S$ the ideal sheaf of s. Then we see that

$$\mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s) = \mathrm{Hom}_{\mathcal{O}_S}(\mathfrak{m}, f_*\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathfrak{m}, \mathcal{F})$$

the first equality because S is local with closed point s and the second equality by adjunction for f^*, f_* on quasi-coherent modules. However, since $s \notin f(X)$ we see that $f^*\mathfrak{m} = \mathcal{O}_X$. Working backwards through the arguments we get the desired equality.

For the proof of (4), (5), and (6) we use flat base change (Cohomology of Schemes, Lemma 30.5.2) to reduce to the case where S is the spectrum of $\mathcal{O}_{S,s}$. Then a nonzerodivisor $a \in \mathcal{O}_{S,s}$ deterimines a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{a} \mathcal{O}_S \to \mathcal{O}_S/a\mathcal{O}_S \to 0$$

Since $\mathcal{F}$ is flat over S, we obtain an exact sequence

$$0 \to \mathcal{F} \xrightarrow{a} \mathcal{F} \to \mathcal{F}/a\mathcal{F} \to 0$$

Pushing forward we obtain an exact sequence

$$0 \to f_*\mathcal{F} \xrightarrow{a} f_*\mathcal{F} \to f_*(\mathcal{F}/a\mathcal{F})$$

This proves (4) and it shows that $f_*\mathcal{F}/af_*\mathcal{F} \subset f_*(\mathcal{F}/a\mathcal{F})$. If b is a nonzerodivisor on $\mathcal{O}_{S,s}/a\mathcal{O}_{S,s}$, then the exact same argument shows $b : \mathcal{F}/a\mathcal{F} \to \mathcal{F}/a\mathcal{F}$ is injective. Pushing forward we conclude

$$b : f_*(\mathcal{F}/a\mathcal{F}) \to f_*(\mathcal{F}/a\mathcal{F})$$

is injective and hence also $b : f_*\mathcal{F}/af_*\mathcal{F} \to f_*\mathcal{F}/af_*\mathcal{F}$ is injective. This proves (5). Part (6) follows from (4) and (5) and the definitions. □

31.7. Relative assassin

05AS Let $A \to B$ be a ring map. Let N be a B-module. Recall that a prime $\mathfrak{q} \subset B$ is said to be in the relative assassin of N over B/A if $\mathfrak{q}$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p})$. Here $\mathfrak{p} = A \cap \mathfrak{q}$. See Algebra, Definition 10.65.2. Here is the definition of the relative assassin for quasi-coherent sheaves over a morphism of schemes.

05AT **Definition 31.7.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section). Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for morphisms of schemes whose fibres are locally Noetherian.

0CU5 **Lemma 31.7.2.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $U \subset X$ and $V \subset S$ be affine opens with $f(U) \subset V$. Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. Then*

$$\mathfrak{p} \in Ass_{A/R}(M) \Rightarrow x \in Ass_{X/S}(\mathcal{F})$$

If all fibres X_s of f are locally Noetherian, then $\mathfrak{p} \in Ass_{A/R}(M) \Leftrightarrow x \in Ass_{X/S}(\mathcal{F})$ for all pairs $(\mathfrak{p}, x)$ as above.

Proof. The set $\mathrm{Ass}_{A/R}(M)$ is defined in Algebra, Definition 10.65.2. Choose a pair $(\mathfrak{p}, x)$. Let $s = f(x)$. Let $\mathfrak{r} \subset R$ be the prime lying under $\mathfrak{p}$, i.e., the prime corresponding to s. Let $\mathfrak{p}' \subset A \otimes_R \kappa(\mathfrak{r})$ be the prime whose inverse image is $\mathfrak{p}$, i.e., the prime corresponding to x viewed as a point of its fibre X_s. Then $\mathfrak{p} \in \mathrm{Ass}_{A/R}(M)$ if and only if $\mathfrak{p}'$ is an associated prime of $M \otimes_R \kappa(\mathfrak{r})$, see Algebra, Lemma 10.65.1. Note that the ring $A \otimes_R \kappa(\mathfrak{r})$ corresponds to U_s and the module $M \otimes_R \kappa(\mathfrak{r})$ corresponds to the quasi-coherent sheaf $\mathcal{F}_s|_{U_s}$. Hence x is an associated point of $\mathcal{F}_s$ by Lemma 31.2.2. The reverse implication holds if $\mathfrak{p}'$ is finitely generated which is how the last sentence is seen to be true. □

05DC **Lemma 31.7.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. *Let* $x' \in X'$ *be a point with images* $x \in X$, $s' \in S'$ *and* $s \in S$. *Assume* f *locally of finite type. Then* $x' \in Ass_{X'/S'}(\mathcal{F}')$ *if and only if* $x \in Ass_{X/S}(\mathcal{F})$ *and* x' *corresponds to a generic point of an irreducible component of* $\mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $\mathrm{Spec}(\kappa(s))$ we have that X_s is locally Noetherian, see Morphisms, Lemma 29.15.6. Thus we may apply Lemma 31.3.1 and we see that

$$\mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \mathrm{Ass}(\mathcal{F}_s)} \mathrm{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = \mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 10.167.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 10.63.6 (minimal primes are associated) and Algebra, Lemma 10.157.2 (no embedded primes). □

05KL **Remark 31.7.4.** With notation and assumptions as in Lemma 31.7.3 we see that it is always the case that $(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \supset \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) = \mathrm{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s')/\kappa(s)$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s')/\kappa(s)$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 10.36.19.

31.8. Relative weak assassin

05AU

05AV **Definition 31.8.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of* $\mathcal{F}$ *in* X *over* S is the set

$$\mathrm{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

05F2 **Lemma 31.8.2.** *Let* $f : X \to S$ *be a morphism of schemes which is locally of finite type. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Then* $WeakAss_{X/S}(\mathcal{F}) = Ass_{X/S}(\mathcal{F})$.

Proof. This is true because the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 31.5.8. □

0CUD **Lemma 31.8.3.** *Let* $f : X \to S$ *be a morphism of schemes. Let* $i : Z \to X$ *be a finite morphism. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_Z$*-module. Then* $WeakAss_{X/S}(i_*\mathcal{F}) = i(WeakAss_{Z/S}(\mathcal{F}))$.

Proof. Let $i_s : Z_s \to X_s$ be the induced morphism between fibres. Then $(i_*\mathcal{F})_s = i_{s,*}(\mathcal{F}_s)$ by Cohomology of Schemes, Lemma 30.5.1 and the fact that i is affine. Hence we may apply Lemma 31.6.3 to conclude. □

31.9. Fitting ideals

0C3C This section is the continuation of the discussion in More on Algebra, Section 15.8. Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. In this situation we can construct the Fitting ideals

$$0 = \mathrm{Fit}_{-1}(\mathcal{F}) \subset \mathrm{Fit}_0(\mathcal{F}) \subset \mathrm{Fit}_1(\mathcal{F}) \subset \ldots \subset \mathcal{O}_S$$

as the sequence of quasi-coherent ideals characterized by the following property: for every affine open $U = \mathrm{Spec}(A)$ of S if $\mathcal{F}|_U$ corresponds to the A-module M, then $\mathrm{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\mathrm{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $f \in A$, then the ith Fitting ideal of M_f over A_f is equal to $\mathrm{Fit}_i(M)A_f$ by More on Algebra, Lemma 15.8.4.

Alternatively, we can construct the Fitting ideals in terms of local presentations of $\mathcal{F}$. Namely, if $U \subset X$ is open, and

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

is a presentation of $\mathcal{F}$ over U, then $\mathrm{Fit}_r(\mathcal{F})|_U$ is generated by the $(n-r) \times (n-r)$-minors of the matrix defining the first arrow of the presentation. This is compatible with the construction above because this is how the Fitting ideal of a module over a ring is actually defined. Some details omitted.

0C3D **Lemma 31.9.1.** *Let $f : T \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Then $f^{-1}Fit_i(\mathcal{F}) \cdot \mathcal{O}_T = Fit_i(f^*\mathcal{F})$.*

Proof. Follows immediately from More on Algebra, Lemma 15.8.4 part (3). □

0C3E **Lemma 31.9.2.** *Let S be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_S$-module. Then $Fit_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.*

Proof. Follows immediately from More on Algebra, Lemma 15.8.4 part (4). □

0CYX **Lemma 31.9.3.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $Z_0 \subset S$ be the closed subscheme cut out by $Fit_0(\mathcal{F})$. Let $Z \subset S$ be the scheme theoretic support of $\mathcal{F}$. Then*

1. *$Z \subset Z_0 \subset S$ as closed subschemes,*
2. *$Z = Z_0 = Supp(\mathcal{F})$ as closed subsets,*
3. *there exists a finite type, quasi-coherent $\mathcal{O}_{Z_0}$-module $\mathcal{G}_0$ with*

$$(Z_0 \to X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that Z is locally cut out by the annihilator of $\mathcal{F}$, see Morphisms, Definition 29.5.5 (which uses Morphisms, Lemma 29.5.4 to define Z). Hence we see that $Z \subset Z_0$ scheme theoretically by More on Algebra, Lemma 15.8.4 part (6). On the other hand we have $Z = \mathrm{Supp}(\mathcal{F})$ set theoretically by Morphisms, Lemma 29.5.4 and we have $Z_0 = Z$ set theoretically by More on Algebra, Lemma 15.8.4 part (7). Finally, to get $\mathcal{G}_0$ as in part (3) we can either use that we have $\mathcal{G}$ on Z as in Morphisms, Lemma 29.5.4 and set $\mathcal{G}_0 = (Z \to Z_0)_*\mathcal{G}$ or we can use Morphisms, Lemma 29.4.1 and the fact that $\mathrm{Fit}_0(\mathcal{F})$ annihilates $\mathcal{F}$ by More on Algebra, Lemma 15.8.4 part (6). □

0C3F **Lemma 31.9.4.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $s \in S$. Then $\mathcal{F}$ can be generated by r elements in a neighbourhood of s if and only if $Fit_r(\mathcal{F})_s = \mathcal{O}_{S,s}$.*

Proof. Follows immediately from More on Algebra, Lemma 15.8.7. □

0C3G **Lemma 31.9.5.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $r \geq 0$. The following are equivalent*

(1) *$\mathcal{F}$ is finite locally free of rank r*
(2) *$Fit_{r-1}(\mathcal{F}) = 0$ and $Fit_r(\mathcal{F}) = \mathcal{O}_S$, and*
(3) *$Fit_k(\mathcal{F}) = 0$ for $k < r$ and $Fit_k(\mathcal{F}) = \mathcal{O}_S$ for $k \geq r$.*

Proof. Follows immediately from More on Algebra, Lemma 15.8.8. □

05P8 **Lemma 31.9.6.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. The closed subschemes*

$$S = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$

defined by the Fitting ideals of $\mathcal{F}$ have the following properties

(1) *The intersection $\bigcap Z_r$ is empty.*
(2) *The functor $(Sch/S)^{opp} \to Sets$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subscheme $S \setminus Z_r$.
(3) *The functor $F_r : (Sch/S)^{opp} \to Sets$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subscheme $Z_{r-1} \setminus Z_r$ of S.

If $\mathcal{F}$ is of finite presentation, then $Z_r \to S$, $S \setminus Z_r \to S$, and $Z_{r-1} \setminus Z_r \to S$ are of finite presentation.

Proof. Part (1) is true because over every affine open U there is an integer n such that $\mathrm{Fit}_n(\mathcal{F})|_U = \mathcal{O}_U$. Namely, we can take n to be the number of generators of $\mathcal{F}$ over U, see More on Algebra, Section 15.8.

For any morphism $g : T \to S$ we see from Lemmas 31.9.1 and 31.9.4 that $\mathcal{F}_T$ is locally generated by $\leq r$ sections if and only if $\mathrm{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$. This proves (2).

For any morphism $g : T \to S$ we see from Lemmas 31.9.1 and 31.9.5 that $\mathcal{F}_T$ is free of rank r if and only if $\mathrm{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\mathrm{Fit}_{r-1}(\mathcal{F}) \cdot \mathcal{O}_T = 0$. This proves (3).

Assume $\mathcal{F}$ is of finite presentation. Then each of the morphisms $Z_r \to S$ is of finite presentation as $\mathrm{Fit}_r(\mathcal{F})$ is of finite type (Lemma 31.9.2 and Morphisms, Lemma 29.21.7). This implies that $Z_{r-1} \setminus Z_r$ is a retrocompact open in Z_r (Properties, Lemma 28.24.1) and hence the morphism $Z_{r-1} \setminus Z_r \to Z_r$ is of finite presentation as well. □

Lemma 31.9.6 notwithstanding the following lemma does not hold if $\mathcal{F}$ is a finite type quasi-coherent module. Namely, the stratification still exists but it isn't true that it represents the functor F_{flat} in general.

05P9 **Lemma 31.9.7.** *Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module of finite presentation. Let $S = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ be as in Lemma 31.9.6. Set $S_r = Z_{r-1} \setminus Z_r$. Then $S' = \coprod_{r \geq 0} S_r$ represents the functor*

$$F_{flat} : Sch/S \longrightarrow Sets, \qquad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{S_r}$ is locally free of rank r and the morphisms $S_r \to S$ and $S' \to S$ are of finite presentation.

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^*\mathcal{F}$ is flat. Then $\mathcal{F}_T$ is a flat $\mathcal{O}_T$-module of finite presentation. Hence $\mathcal{F}_T$ is finite locally free, see Properties, Lemma 28.20.2. Thus $T = \coprod_{r\geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r\geq 0} F_r$$

in the category of Zariski sheaves on Sch/S where F_r is as in Lemma 31.9.6. It follows that F_{flat} is represented by $\coprod_{r\geq 0}(Z_{r-1} \setminus Z_r)$ where Z_r is as in Lemma 31.9.6. The other statements also follow from the lemma. □

0FJ0 **Example 31.9.8.** Let $R = \prod_{n\in\mathbf{N}} \mathbf{F}_2$. Let $I \subset R$ be the ideal of elements $a = (a_n)_{n\in\mathbf{N}}$ almost all of whose components are zero. Let $\mathfrak{m}$ be a maximal ideal containing I. Then $M = R/\mathfrak{m}$ is a finite flat R-module, because R is absolutely flat (More on Algebra, Lemma 15.105.6). Set $S = \mathrm{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$. The closed subschemes of Lemma 31.9.6 are $S = Z_{-1}$, $Z_0 = \mathrm{Spec}(R/\mathfrak{m})$, and $Z_i = \emptyset$ for $i > 0$. But $\mathrm{id} : S \to S$ does not factor through $(S \setminus Z_0) \amalg Z_0$ because $\mathfrak{m}$ is a nonisolated point of S. Thus Lemma 31.9.7 does not hold for finite type modules.

31.10. The singular locus of a morphism

0C3H Let $f : X \to S$ be a finite type morphism of schemes. The set U of points where f is smooth is an open of X (by Morphisms, Definition 29.34.1). In many situations it is useful to have a canonical closed subscheme $\mathrm{Sing}(f) \subset X$ whose complement is U and whose formation commutes with arbitrary change of base.

If f is of finite presentation, then one choice would be to consider the closed subscheme Z cut out by functions which are affine locally "strictly standard" in the sense of Smoothing Ring Maps, Definition 16.2.3. It follows from Smoothing Ring Maps, Lemma 16.2.7 that if $f' : X' \to S'$ is the base change of f by a morphism $S' \to S$, then $Z' \subset S' \times_S Z$ where Z' is the closed subscheme of X' cut out by functions which are affine locally strictly standard. However, equality isn't clear. The notion of a strictly standard element was useful in the chapter on Popescu's theorem. The closed subscheme defined by these elements is (as far as we know) not used in the literature[1].

If f is flat, of finite presentation, and the fibres of f all are equidimensional of dimension d, then the dth fitting ideal of $\Omega_{X/S}$ is used to get a good closed subscheme. For any morphism of finite type the closed subschemes of X defined by the fitting ideals of $\Omega_{X/S}$ define a stratification of X in terms of the rank of $\Omega_{X/S}$ whose formation commutes with base change. This can be helpful; it is related to embedding dimensions of fibres, see Varieties, Section 33.46.

0C3I **Lemma 31.10.1.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $X = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ be the closed subschemes defined by the fitting ideals of $\Omega_{X/S}$. Then the formation of Z_i commutes with arbitrary base change.*

[1] If f is a local complete intersection morphism (More on Morphisms, Definition 37.62.2) then the closed subscheme cut out by the locally strictly standard elements is the correct thing to look at.

Proof. Observe that $\Omega_{X/S}$ is a finite type quasi-coherent $\mathcal{O}_X$-module (Morphisms, Lemma 29.32.12) hence the fitting ideals are defined. If $f' : X' \to S'$ is the base change of f by $g : S' \to S$, then $\Omega_{X'/S'} = (g')^*\Omega_{X/S}$ where $g' : X' \to X$ is the projection (Morphisms, Lemma 29.32.10). Hence $(g')^{-1}\mathrm{Fit}_i(\Omega_{X/S}) \cdot \mathcal{O}_{X'} = \mathrm{Fit}_i(\Omega_{X'/S'})$. This means that

$$Z'_i = (g')^{-1}(Z_i) = Z_i \times_X X'$$

scheme theoretically and this is the meaning of the statement of the lemma. □

The 0th fitting ideal of Ω cuts out the "ramified locus" of the morphism.

0C3J **Lemma 31.10.2.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The closed subscheme $Z \subset X$ cut out by the 0th fitting ideal of $\Omega_{X/S}$ is exactly the set of points where f is not unramified.*

Proof. By Lemma 31.9.3 the complement of Z is exactly the locus where $\Omega_{X/S}$ is zero. This is exactly the set of points where f is unramified by Morphisms, Lemma 29.35.2. □

0C3K **Lemma 31.10.3.** *Let $f : X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. Assume*

(1) *f is flat,*
(2) *f is locally of finite presentation, and*
(3) *every nonempty fibre of f is equidimensional of dimension d.*

Let $Z \subset X$ be the closed subscheme cut out by the dth fitting ideal of $\Omega_{X/S}$. Then Z is exactly the set of points where f is not smooth.

Proof. By Lemma 31.9.6 the complement of Z is exactly the locus where $\Omega_{X/S}$ can be generated by at most d elements. Hence the lemma follows from Morphisms, Lemma 29.34.14. □

31.11. Torsion free modules

0AVQ This section is the analogue of More on Algebra, Section 15.22 for quasi-coherent modules.

0AXR **Lemma 31.11.1.** *Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be nonempty open and $s \in \mathcal{F}(U)$. The following are equivalent*

(1) *for some $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
(2) *for all $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
(3) *the image of s in $\mathcal{F}_\eta$ is zero,*
(4) *the image of s in $j_*\mathcal{F}_\eta$ is zero, where $j : \eta \to X$ is the inclusion morphism.*

Proof. Omitted. □

0AVR **Definition 31.11.2.** Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) We say a local section of $\mathcal{F}$ is *torsion* if it satisfies the equivalent conditions of Lemma 31.11.1.
(2) We say $\mathcal{F}$ is *torsion free* if every torsion section of $\mathcal{F}$ is 0.

Here is the obligatory lemma comparing this to the usual algebraic notion.

0AXS **Lemma 31.11.3.** *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is torsion free,*
(2) *for $U \subset X$ affine open $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module.*

Proof. Omitted. □

0AXT **Lemma 31.11.4.** *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The torsion sections of $\mathcal{F}$ form a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{F}_{tors} \subset \mathcal{F}$. The quotient module $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 15.22.2 for the algebraic analogue. □

0AXU **Lemma 31.11.5.** *Let X be an integral scheme. Any flat quasi-coherent $\mathcal{O}_X$-module is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 15.22.9. □

0AXV **Lemma 31.11.6.** *Let $f : X \to Y$ be a flat morphism of integral schemes. Let $\mathcal{G}$ be a torsion free quasi-coherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a torsion free $\mathcal{O}_X$-module.*

Proof. Omitted. See More on Algebra, Lemma 15.22.4 for the algebraic analogue. □

0BCM **Lemma 31.11.7.** *Let $f : X \to Y$ be a flat morphism of schemes. If Y is integral and the generic fibre of f is integral, then X is integral.*

Proof. The algebraic analogue is this: let A be a domain with fraction field K and let B be a flat A-algebra such that $B \otimes_A K$ is a domain. Then B is a domain. This is true because B is torsion free by More on Algebra, Lemma 15.22.9 and hence $B \subset B \otimes_A K$. □

0AXW **Lemma 31.11.8.** *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is torsion free if and only if $\mathcal{F}_x$ is a torsion free $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Omitted. See More on Algebra, Lemma 15.22.6. □

0AXX **Lemma 31.11.9.** *Let X be an integral scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{F}''$ are torsion free, then $\mathcal{F}'$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 15.22.5 for the algebraic analogue. □

0AXY **Lemma 31.11.10.** *Let X be a locally Noetherian integral scheme with generic point η. Let $\mathcal{F}$ be a nonzero coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is torsion free,*
(2) *η is the only associated prime of $\mathcal{F}$,*
(3) *η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has property (S_1), and*
(4) *η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has no embedded associated prime.*

Proof. This is a translation of More on Algebra, Lemma 15.22.8 into the language of schemes. We omit the translation. □

0CC4 **Lemma 31.11.11.** *Let X be an integral regular scheme of dimension ≤ 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is torsion free,*
(2) *$\mathcal{F}$ is finite locally free.*

Proof. It is clear that a finite locally free module is torsion free. For the converse, we will show that if $\mathcal{F}$ is torsion free, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 10.78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 15.22.11. □

0AXZ **Lemma 31.11.12.** *Let X be an integral scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is torsion free and $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is torsion free.*

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent by Schemes, Section 26.24. To see the statement is true, see More on Algebra, Lemma 15.22.12. Some details omitted. □

0AVS **Lemma 31.11.13.** *Let X be an integral locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent, $\mathcal{G}$ is torsion free, and that for every $x \in X$ one of the following happens*

(1) *$\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, or*
(2) *$\text{depth}(\mathcal{F}_x) \geq 2$.*

Then φ is an isomorphism.

Proof. This is a translation of More on Algebra, Lemma 15.23.14 into the language of schemes. □

31.12. Reflexive modules

0AVT This section is the analogue of More on Algebra, Section 15.23 for coherent modules on locally Noetherian schemes. The reason for working with coherent modules is that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent for every pair of coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, see Modules, Lemma 17.22.6.

0AVU **Definition 31.12.1.** Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

It follows from Lemma 31.12.8 that the reflexive hull is a reflexive $\mathcal{O}_X$-module. You can use the same definition to define reflexive modules in more general situations, but this does not seem to be very useful. Here is the obligatory lemma comparing this to the usual algebraic notion.

0AY0 **Lemma 31.12.2.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is reflexive,*
(2) *for $U \subset X$ affine open $\mathcal{F}(U)$ is a reflexive $\mathcal{O}(U)$-module.*

Proof. Omitted. □

0AY1 **Remark 31.12.3.** If X is a scheme of finite type over a field, then sometimes a different notion of reflexive modules is used (see for example [**HL97**, bottom of page 5 and Definition 1.1.9]). This other notion uses $R\mathcal{H}om$ into a dualizing complex $\omega_X^\bullet$ instead of into $\mathcal{O}_X$ and should probably have a different name because it can be different when X is not Gorenstein. For example, if $X = \mathrm{Spec}(k[t^3, t^4, t^5])$, then a computation shows the dualizing sheaf ω_X is not reflexive in our sense, but it is reflexive in the other sense as $\omega_X \to \mathcal{H}om(\mathcal{H}om(\omega_X, \omega_X), \omega_X)$ is an isomorphism.

0AY2 **Lemma 31.12.4.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.*
(2) *The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 15.23.2. □

0AY3 **Lemma 31.12.5.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is reflexive,*
(2) *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all $x \in X$,*
(3) *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all closed points $x \in X$.*

Proof. By Modules, Lemma 17.22.4 we see that (1) and (2) are equivalent. Since every point of X specializes to a closed point (Properties, Lemma 28.5.9) we see that (2) and (3) are equivalent. □

0EBF **Lemma 31.12.6.** *Let $f : X \to Y$ be a flat morphism of integral locally Noetherian schemes. Let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module.*

Proof. Omitted. See More on Algebra, Lemma 15.22.4 for the algebraic analogue. □

0EBG **Lemma 31.12.7.** *Let X be an integral locally Noetherian scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ be an exact sequence of coherent $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is reflexive and $\mathcal{F}''$ is torsion free, then $\mathcal{F}$ is reflexive.*

Proof. Omitted. See More on Algebra, Lemma 15.23.5. □

0AY4 **Lemma 31.12.8.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is reflexive, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.*

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent by Cohomology of Schemes, Lemma 30.9.4. To see the statement is true, see More on Algebra, Lemma 15.23.8. Some details omitted. □

0EBH **Remark 31.12.9.** Let X be an integral locally Noetherian scheme. Thanks to Lemma 31.12.8 we know that the reflexive hull $\mathcal{F}^{**}$ of a coherent $\mathcal{O}_X$-module is coherent reflexive. Consider the category $\mathcal{C}$ of coherent reflexive $\mathcal{O}_X$-modules. Taking reflexive hulls gives a left adjoint to the inclusion functor $\mathcal{C} \to Coh(\mathcal{O}_X)$. Observe that $\mathcal{C}$ is an additive category with kernels and cokernels. Namely, given $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$, the usual kernel $\mathrm{Ker}(\varphi)$ is reflexive (Lemma 31.12.7) and the reflexive hull $\mathrm{Coker}(\varphi)^{**}$ of the usual cokernel is the cokernel in $\mathcal{C}$. Moreover $\mathcal{C}$ inherits a tensor product

$$\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$$

which is associative and symmetric. There is an internal Hom in the sense that for any three objects $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of $\mathcal{C}$ we have the identity

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

see Modules, Lemma 17.22.1. In $\mathcal{C}$ every object $\mathcal{F}$ has a *dual object* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Without further conditions on X it can happen that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \ncong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{C}} \mathcal{G} \quad \text{and} \quad \mathcal{F} \otimes_{\mathcal{C}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \ncong \mathcal{O}_X$$

for $\mathcal{F}, \mathcal{G}$ of rank 1 in $\mathcal{C}$. To make an example let $X = \mathrm{Spec}(R)$ where R is as in More on Algebra, Example 15.23.17 and let $\mathcal{F}, \mathcal{G}$ be the modules corresponding to M. Computation omitted.

0AY5 **Lemma 31.12.10.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *for each $x \in X$ one of the following happens*
   (a) *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module, or*
   (b) *$\mathrm{depth}(\mathcal{F}_x) \geq 2$.*

Proof. Omitted. See More on Algebra, Lemma 15.23.15. □

0EBI **Lemma 31.12.11.** *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. Let $x \in X$.*

1. *If $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$, then $\mathrm{depth}(\mathcal{F}_x) \geq 2$.*
2. *If X is (S_2), then $\mathcal{F}$ is (S_2).*

Proof. Omitted. See More on Algebra, Lemma 15.23.16. □

0EBJ **Lemma 31.12.12.** *Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be an open subscheme with complement Z. Assume $\mathcal{O}_{X,z}$ has depth ≥ 2 for all $z \in Z$. Then j^* and j_* define an equivalence of categories between the category of coherent reflexive $\mathcal{O}_X$-modules and the category of coherent reflexive $\mathcal{O}_U$-modules.*

Proof. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. For $z \in Z$ the stalk $\mathcal{F}_z$ has depth ≥ 2 by Lemma 31.12.11. Thus $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism by Lemma 31.5.11. Conversely, let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_U$-module. It suffices to show that $j_*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module. To prove this we may assume X is affine. By Properties, Lemma 28.22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathcal{G} = j^*\mathcal{F}$. After replacing $\mathcal{F}$ by its reflexive hull, we may assume $\mathcal{F}$ is reflexive (see discussion above and in particular Lemma 31.12.8). By the above $j_*\mathcal{G} = j_*j^*\mathcal{F} = \mathcal{F}$ as desired. □

If the scheme is normal, then reflexive is the same thing as torsion free and (S_2).

0AY6 **Lemma 31.12.13.** *Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *$\mathcal{F}$ is torsion free and has property (S_2), and*
3. *there exists an open subscheme $j : U \to X$ such that*
   (a) *every irreducible component of $X \setminus U$ has codimension ≥ 2 in X,*
   (b) *$j^*\mathcal{F}$ is finite locally free, and*
   (c) *$\mathcal{F} = j_*j^*\mathcal{F}$.*

Proof. Using Lemma 31.12.2 the equivalence of (1) and (2) follows from More on Algebra, Lemma 15.23.18. Let $U \subset X$ be as in (3). By Properties, Lemma 28.12.5 we see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Since a finite locally free module is reflexive, we conclude (3) implies (1) by Lemma 31.12.12.

Assume (1). Let $U \subset X$ be the maximal open subscheme such that $j^*\mathcal{F} = \mathcal{F}|_U$ is finite locally free. So (3)(b) holds. Let $x \in X$ be a point. If $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, then $x \in U$, see Modules, Lemma 17.11.7. If $\dim(\mathcal{O}_{X,x}) \leq 1$, then $\mathcal{O}_{X,x}$ is either a field or a discrete valuation ring (Properties, Lemma 28.12.5) and hence $\mathcal{F}_x$ is free (More on Algebra, Lemma 15.22.11). Thus $x \notin U \Rightarrow \dim(\mathcal{O}_{X,x}) \geq 2$. Then Properties, Lemma 28.10.3 shows (3)(a) holds. By the already used Properties, Lemma 28.12.5 we also see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$ and hence (3)(c) follows from Lemma 31.12.12. □

0AY7 **Lemma 31.12.14.** *Let X be an integral locally Noetherian normal scheme with generic point η. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $T : \mathcal{G}_\eta \to \mathcal{F}_\eta$ be a linear map. Then T extends to a map $\mathcal{G} \to \mathcal{F}^{**}$ of $\mathcal{O}_X$-modules if and only if*

(*) *for every $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$ we have*

$$T\left(\text{Im}(\mathcal{G}_x \to \mathcal{G}_\eta)\right) \subset \text{Im}(\mathcal{F}_x \to \mathcal{F}_\eta).$$

Proof. Because $\mathcal{F}^{**}$ is torsion free and $\mathcal{F}_\eta = \mathcal{F}_\eta^{**}$ an extension, if it exists, is unique. Thus it suffices to prove the lemma over the members of an open covering of X, i.e., we may assume X is affine. In this case we are asking the following algebra question: Let R be a Noetherian normal domain with fraction field K, let M, N be finite R-modules, let $T : M \otimes_R K \to N \otimes_R K$ be a K-linear map. When does T extend to a map $N \to M^{**}$? By More on Algebra, Lemma 15.23.19 this happens if and only if $N_\mathfrak{p}$ maps into $(M/M_{tors})_\mathfrak{p}$ for every height 1 prime $\mathfrak{p}$ of R. This is exactly condition $(*)$ of the lemma. □

0B3N **Lemma 31.12.15.** *Let X be a regular scheme of dimension ≤ 2. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is reflexive,*
(2) *$\mathcal{F}$ is finite locally free.*

Proof. It is clear that a finite locally free module is reflexive. For the converse, we will show that if $\mathcal{F}$ is reflexive, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 10.78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 15.22.11. If $\dim(\mathcal{O}_{X,x}) = 2$, then $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. By More on Algebra, Lemma 15.23.18 we see that $\mathcal{F}_x$ has depth ≥ 2. Hence $\mathcal{F}_x$ is free by Algebra, Lemma 10.106.6. □

31.13. Effective Cartier divisors

01WQ We define the notion of an effective Cartier divisor before any other type of divisor.

01WR **Definition 31.13.1.** Let S be a scheme.

(1) A *locally principal closed subscheme* of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.

(2) An *effective Cartier divisor* on S is a closed subscheme $D \subset S$ whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_S$ is an invertible $\mathcal{O}_S$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is a nonzerodivisor. In particular they are nowhere dense.

01WS **Lemma 31.13.2.** *Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:*

(1) *The subscheme D is an effective Cartier divisor on S.*
(2) *For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.*

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = \mathrm{Spec}(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{S \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. □

07ZT **Lemma 31.13.3.** *Let S be a scheme. Let $Z \subset S$ be a locally principal closed subscheme. Let $U = S \setminus Z$. Then $U \to S$ is an affine morphism.*

Proof. The question is local on S, see Morphisms, Lemmas 29.11.3. Thus we may assume $S = \mathrm{Spec}(A)$ and $Z = V(f)$ for some $f \in A$. In this case $U = D(f) = \mathrm{Spec}(A_f)$ is affine hence $U \to S$ is affine. □

07ZU **Lemma 31.13.4.** *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $U = S \setminus D$. Then $U \to S$ is an affine morphism and U is scheme theoretically dense in S.*

Proof. Affineness is Lemma 31.13.3. The density question is local on S, see Morphisms, Lemma 29.7.5. Thus we may assume $S = \mathrm{Spec}(A)$ and D corresponding to the nonzerodivisor $f \in A$, see Lemma 31.13.2. Thus $A \subset A_f$ which implies that $U \subset S$ is scheme theoretically dense, see Morphisms, Example 29.7.4. □

056N **Lemma 31.13.5.** *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.*

Proof. Assume $\dim_s(S) < \infty$. Let $U = \mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of s such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzerodivisor $f \in A$ (see Lemma 31.13.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. □

01WT **Definition 31.13.6.** Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

01WU **Lemma 31.13.7.** *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzerodivisors, then also $f_1 f_2 \in A$ is a nonzerodivisor. □

02ON **Lemma 31.13.8.** *Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.*

Proof. Omitted. □

07ZV **Lemma 31.13.9.** *Let X be a scheme. Let Z, Y be two closed subschemes of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. Applying Lemma 31.13.2 we obtain the following algebra situation: A is a ring, $I, J \subset A$ ideals and $f \in A$ a nonzerodivisor such that $IJ = (f)$. Thus the result follows from Algebra, Lemma 10.120.16. □

0C4R **Lemma 31.13.10.** *Let X be a scheme. Let $D, D' \subset X$ be effective Cartier divisors such that the scheme theoretic intersection $D \cap D'$ is an effective Cartier divisor on D'. Then $D + D'$ is the scheme theoretic union of D and D'.*

Proof. See Morphisms, Definition 29.4.4 for the definition of scheme theoretic intersection and union. To prove the lemma working locally (using Lemma 31.13.2) we obtain the following algebra problem: Given a ring A and nonzerodivisors $f_1, f_2 \in A$ such that f_1 maps to a nonzerodivisor in A/f_2A, show that $f_1A \cap f_2A = f_1 f_2 A$. We omit the straightforward argument. □

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 26.17.7.

053P **Lemma 31.13.11.** *Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.*

Proof. Omitted. □

01WV **Definition 31.13.12.** Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

02OO **Lemma 31.13.13.** *Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

(1) *$f(x) \notin D$ for any weakly associated point x of X,*
(2) *X, Y integral and f dominant,*
(3) *X reduced and $f(\xi) \notin D$ for any generic point ξ of any irreducible component of X,*
(4) *X is locally Noetherian and $f(x) \notin D$ for any associated point x of X,*
(5) *X is locally Noetherian, has no embedded points, and $f(\xi) \notin D$ for any generic point ξ of an irreducible component of X,*
(6) *f is flat, and*
(7) *add more here as needed.*

Proof. The question is local on X, and hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$, f is given by $\varphi : R \to A$ and $D = \mathrm{Spec}(R/(t))$ where $t \in R$ is a nonzerodivisor. The goal in each case is to show that $\varphi(t) \in A$ is a nonzerodivisor.

In case (1) this follows from Algebra, Lemma 10.66.7. Case (4) is a special case of (1) by Lemma 31.5.8. Case (5) follows from (4) and the definitions. Case (3) is a special case of (1) by Lemma 31.5.12. Case (2) is a special case of (3). If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (6). □

01WW **Lemma 31.13.14.** *Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. □

31.14. Effective Cartier divisors and invertible sheaves

0C4S Since an effective Cartier divisor has an invertible ideal sheaf (Definition 31.13.1) the following definition makes sense.

01WX **Definition 31.14.1.** Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.

(1) The *invertible sheaf $\mathcal{O}_S(D)$ associated to D* is defined by
$$\mathcal{O}_S(D) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$
(2) The *canonical section*, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.
(3) We write $\mathcal{O}_S(-D) = \mathcal{O}_S(D)^{\otimes -1} = \mathcal{I}_D$.
(4) Given a second effective Cartier divisor $D' \subset S$ we define $\mathcal{O}_S(D - D') = \mathcal{O}_S(D) \otimes_{\mathcal{O}_S} \mathcal{O}_S(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_S(D)$ turns addition of effective Cartier divisors (Definition 31.13.6) into addition in the Picard group of S (Lemma 31.14.4). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on S as a commutative monoid $\mathrm{EffCart}(S)$ whose zero element is

the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_S(D - D')$ defines a group homomorphism

$$\mathrm{EffCart}(S)^{gp} \longrightarrow \mathrm{Pic}(S)$$

where the left hand side is the group completion of $\mathrm{EffCart}(S)$. In other words, when we write $\mathcal{O}_S(D - D')$ we may think of $D - D'$ as an element of $\mathrm{EffCart}(S)^{gp}$.

0B3P **Lemma 31.14.2.** *Let S be a scheme and let $D \subset S$ be an effective Cartier divisor. Then the conormal sheaf is $\mathcal{C}_{D/S} = \mathcal{I}_D|_D = \mathcal{O}_S(-D)|_D$ and the normal sheaf is $\mathcal{N}_{D/S} = \mathcal{O}_S(D)|_D$.*

Proof. This follows from Morphisms, Lemma 29.31.2. □

0C4T **Lemma 31.14.3.** *Let X be a scheme. Let $D, C \subset X$ be effective Cartier divisors with $C \subset D$ and let $D' = D + C$. Then there is a short exact sequence*

$$0 \to \mathcal{O}_X(-D)|_C \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

of $\mathcal{O}_X$-modules.

Proof. In the statement of the lemma and in the proof we use the equivalence of Morphisms, Lemma 29.4.1 to think of quasi-coherent modules on closed subschemes of X as quasi-coherent modules on X. Let $\mathcal{I}$ be the ideal sheaf of D in D'. Then there is a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

because $D \to D'$ is a closed immersion. There is a canonical surjection $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 = \mathcal{C}_{D/D'}$. We have $\mathcal{C}_{D/X} = \mathcal{O}_X(-D)|_D$ by Lemma 31.14.2 and there is a canonical surjective map

$$\mathcal{C}_{D/X} \longrightarrow \mathcal{C}_{D/D'}$$

see Morphisms, Lemmas 29.31.3 and 29.31.4. Thus it suffices to show: (a) $\mathcal{I}^2 = 0$ and (b) $\mathcal{I}$ is an invertible $\mathcal{O}_C$-module. Both (a) and (b) can be checked locally, hence we may assume $X = \mathrm{Spec}(A)$, $D = \mathrm{Spec}(A/fA)$ and $C = \mathrm{Spec}(A/gA)$ where $f, g \in A$ are nonzerodivisors (Lemma 31.13.2). Since $C \subset D$ we see that $f \in gA$. Then $I = fA/fgA$ has square zero and is invertible as an A/gA-module as desired. □

02OP **Lemma 31.14.4.** *Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

0C4U **Lemma 31.14.5.** *Let $f : S' \to S$ be a morphism of schemes. Let D be a effective Cartier divisors on S. If the pullback of D is defined then $f^*\mathcal{O}_S(D) = \mathcal{O}_{S'}(f^*D)$ and the canonical section 1_D pulls back to the canonical section 1_{f^*D}.*

Proof. Omitted. □

01WY **Definition 31.14.6.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

01WZ **Lemma 31.14.7.** *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor.*

If X is a scheme these are also equivalent to

(3) *for any affine open $\mathrm{Spec}(A) = U \subset X$ the image $f \in A$ is a nonzerodivisor,*
(4) *there exists an affine open covering $X = \bigcup \mathrm{Spec}(A_i)$ such that the image of f in A_i is a nonzerodivisor for all i.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 17.25.6 for the definition of the dual invertible sheaf.)

02OQ **Definition 31.14.8.** Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$ be a global section. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

02OR **Lemma 31.14.9.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L})$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
(2) *For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subscheme.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

01X0 **Lemma 31.14.10.** *Let X be a scheme.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\textit{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of pairs } (\mathcal{L}, s)\\ \textit{consisting of an invertible } \mathcal{O}_X\textit{-module}\\ \mathcal{L} \textit{ and a regular global section } s\end{array}\right\}$$

Proof. Omitted. □

0C6K **Remark 31.14.11.** Let X be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and s a regular section of $\mathcal{L}$. Then the zero scheme $D = Z(s)$ is an effective Cartier divisor on X and there are short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0 \quad \text{and} \quad 0 \to \mathcal{L}^{\otimes -1} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

Given an effective Cartier divisor $D \subset X$ using Lemmas 31.14.10 and 31.14.2 we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*(\mathcal{N}_{D/X}) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D) \to 0$$

31.15. Effective Cartier divisors on Noetherian schemes

0B3Q In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

0AYL **Lemma 31.15.1.** *Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then s is a regular section if and only if s does not vanish in the associated points of X.*

Proof. Omitted. Hint: reduce to the affine case and $\mathcal{L}$ trivial and then use Lemma 31.14.7 and Algebra, Lemma 10.63.9. □

0AG8 **Lemma 31.15.2.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme corresponding to the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.*

(1) *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by one element, then D is locally principal.*
(2) *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by a single nonzerodivisor, then D is an effective Cartier divisor.*

Proof. Let $\mathrm{Spec}(A)$ be an affine neighbourhood of a point $x \in D$. Let $\mathfrak{p} \subset A$ be the prime corresponding to x. Let $I \subset A$ be the ideal defining the trace of D on $\mathrm{Spec}(A)$. Since A is Noetherian (as X is locally Noetherian) the ideal I is generated by finitely many elements, say $I = (f_1, \ldots, f_r)$. Under the assumption of (1) we have $I_\mathfrak{p} = (f)$ for some $f \in A_\mathfrak{p}$. Then $f_i = g_i f$ for some $g_i \in A_\mathfrak{p}$. Write $g_i = a_i/h_i$ and $f = f'/h$ for some $a_i, h_i, f', h \in A$, $h_i, h \notin \mathfrak{p}$. Then $I_{h_1 \ldots h_r h} \subset A_{h_1 \ldots h_r h}$ is principal, because it is generated by f'. This proves (1). For (2) we may assume $I = (f)$. The assumption implies that the image of f in $A_\mathfrak{p}$ is a nonzerodivisor. Then f is a nonzerodivisor on a neighbourhood of x by Algebra, Lemma 10.68.6. This proves (2). □

0BCN **Lemma 31.15.3.** *Let X be a locally Noetherian scheme.*

(1) *Let $D \subset X$ be a locally principal closed subscheme. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) \leq 1$.*
(2) *Let $D \subset X$ be an effective Cartier divisor. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) = 1$.*

Proof. Proof of (1). By assumption we may assume $X = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$. Let ξ correspond to the prime ideal $\mathfrak{p} \subset A$. The assumption that ξ is a generic point of an irreducible component of D signifies $\mathfrak{p}$ is minimal over (f). Thus $\dim(A_\mathfrak{p}) \leq 1$ by Algebra, Lemma 10.60.11.

Proof of (2). By part (1) we see that $\dim(\mathcal{O}_{X,\xi}) \leq 1$. On the other hand, the local equation f is a nonzerodivisor in $A_\mathfrak{p}$ by Lemma 31.13.2 which implies the dimension is at least 1 (because there must be a prime in $A_\mathfrak{p}$ not containing f by the elementary Algebra, Lemma 10.17.2). □

0AG9 **Lemma 31.15.4.** *Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme which is also an effective Cartier divisor. Then the local ring of X at the generic point of D is a discrete valuation ring.*

Proof. By Lemma 31.13.2 we may assume $X = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$ is a nonzerodivisor. The assumption that D is integral signifies that (f) is prime. Hence the local ring of X at the generic point is $A_{(f)}$ which is a Noetherian local ring whose maximal ideal is generated by a nonzerodivisor. Thus it is a discrete valuation ring by Algebra, Lemma 10.119.7. □

0B3R **Lemma 31.15.5.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. If X is (S_k), then D is (S_{k-1}).*

Proof. Let $x \in D$. Then $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(f)$ where $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor. By assumption we have $\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min(\dim(\mathcal{O}_{X,x}), k)$. By Algebra, Lemma 10.72.7 we have $\mathrm{depth}(\mathcal{O}_{D,x}) = \mathrm{depth}(\mathcal{O}_{X,x}) - 1$ and by Algebra, Lemma 10.60.13 $\dim(\mathcal{O}_{D,x}) = \dim(\mathcal{O}_{X,x}) - 1$. It follows that $\mathrm{depth}(\mathcal{O}_{D,x}) \geq \min(\dim(\mathcal{O}_{D,x}), k-1)$ as desired. □

0B3S **Lemma 31.15.6.** *Let X be a locally Noetherian normal scheme. Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1).*

Proof. By Properties, Lemma 28.12.5 we see that X is (S_2). Thus we conclude by Lemma 31.15.5. □

0AGA **Lemma 31.15.7.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an integral closed subscheme. Assume that*

(1) *D has codimension 1 in X, and*
(2) *$\mathcal{O}_{X,x}$ is a UFD for all $x \in D$.*

Then D is an effective Cartier divisor.

Proof. Let $x \in D$ and set $A = \mathcal{O}_{X,x}$. Let $\mathfrak{p} \subset A$ correspond to the generic point of D. Then $A_{\mathfrak{p}}$ has dimension 1 by assumption (1). Thus $\mathfrak{p}$ is a prime ideal of height 1. Since A is a UFD this implies that $\mathfrak{p} = (f)$ for some $f \in A$. Of course f is a nonzerodivisor and we conclude by Lemma 31.15.2. □

0AGB **Lemma 31.15.8.** *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Assume there exist*

(1) *a collection of integral effective Cartier divisors $D_i \subset X$, $i \in I$*
(2) *a closed subset $Z' \subset X$ all of whose irreducible components have codimension ≥ 2 in X (Topology, Definition 5.11.1)*

such that $Z \subset Z' \cup \bigcup_{i \in I} D_i$ set-theoretically. Then there exist integers $a_i \geq 0$ with $a_i = 0$ for almost all $i \in I$ such that the effective Cartier divisor

$$D = \sum a_i D_i$$

is contained in Z and such that the inclusion morphism $D \to Z$ is an isomorphism away from codimension 2 in X (in the sense that there exists an open $U \subset Z$ such that $D \cap U \to Z \cap U$ is an isomorphism and such that every irreducible component of $Z \setminus U$ has codimension ≥ 2 in X). When Z is nowhere dense in X existence of the D_i, $i \in I$ and Z' is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in Z$ or if X is regular.

Proof. Let $\xi_i \in D_i$ be the generic point and let $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ be the local ring which is a discrete valuation ring by Lemma 31.15.4. Let $a_i \geq 0$ be the minimal valuation of an element of $\mathcal{I}_{Z,\xi_i} \subset \mathcal{O}_i$. Of course $a_i > 0$ only if D_i is an irreducible component of Z and hence $a_i > 0$ only for a finite number of $i \in I$. We claim that the effective Cartier divisor $D = \sum a_i D_i$ works.

Namely, suppose that $x \in X$. Let $A = \mathcal{O}_{X,x}$. Let $D_1, \ldots, D_n$ be the pairwise distinct divisors D_i such that $x \in D_i$ and $a_i > 0$. For $1 \leq i \leq n$ let $f_i \in A$ be a local equation for D_i. Then f_i is a prime element of A and $\mathcal{O}_i = A_{(f_i)}$. Let $I = \mathcal{I}_{Z,x} \subset A$ be the stalk of the ideal sheaf of Z. By our choice of a_i we have $IA_{(f_i)} = f_i^{a_i} A_{(f_i)}$. We claim that $I \subset (\prod_{i=1,\ldots,n} f_i^{a_i})$.

Proof of the claim. The localization map $\varphi : A/(f_i) \to A_{(f_i)}/f_i A_{(f_i)}$ is injective as the prime ideal (f_i) is the inverse image of the maximal ideal $f_i A_{(f_i)}$. By induction on n we deduce that $\varphi_n : A/(f_i^n) \to A_{(f_i)}/f_i^n A_{(f_i)}$ is also injective. Since $\varphi_{a_i}(I) = 0$, we have $I \subset (f_i^{a_i})$. Thus, for any $x \in I$, we may write $x = f_1^{a_1} x_1$ for some $x_1 \in A$. Since $D_1, \ldots, D_n$ are pairwise distinct, f_i is a unit in $A_{(f_j)}$ for $i \neq j$. Comparing x and x_1 at $A_{(f_i)}$ for $n \geq i > 1$, we still have $x_1 \in (f_i^{a_i})$. Repeating the previous process, we inductively write $x_i = f_{i+1}^{a_{i+1}} x_{i+1}$ for any $n > i \geq 1$. In conclusion, $x \in (\prod_{i=1,\ldots n} f_i^{a_i})$ for any $x \in I$ as desired.

The claim shows that $\mathcal{I}_Z \subset \mathcal{I}_D$, i.e., that $D \subset Z$. Moreover, we also see that D and Z agree at the ξ_i, which proves that $D \to Z$ is an isomorphism away from codimension 2 on X.

To see the final statements we argue as follows. A regular local ring is a UFD (More on Algebra, Lemma 15.122.2) hence it suffices to argue in the UFD case. In that case, let D_i be the irreducible components of Z which have codimension 1 in X. By Lemma 31.15.7 each D_i is an effective Cartier divisor. □

0BXH **Lemma 31.15.9.** *Let $Z \subset X$ be a closed subscheme of a Noetherian scheme. Assume*

(1) *Z has no embedded points,*
(2) *every irreducible component of Z has codimension 1 in X,*
(3) *every local ring $\mathcal{O}_{X,x}$, $x \in Z$ is a UFD or X is regular.*

Then Z is an effective Cartier divisor.

Proof. Let $D = \sum a_i D_i$ be as in Lemma 31.15.8 where $D_i \subset Z$ are the irreducible components of Z. If $D \to Z$ is not an isomorphism, then $\mathcal{O}_Z \to \mathcal{O}_D$ has a nonzero kernel sitting in codimension ≥ 2. This would mean that Z has embedded points, which is forbidden by assumption (1). Hence $D \cong Z$ as desired. □

0BXI **Lemma 31.15.10.** *Let R be a Noetherian UFD. Let $I \subset R$ be an ideal such that R/I has no embedded primes and such that every minimal prime over I has height 1. Then $I = (f)$ for some $f \in R$.*

Proof. By Lemma 31.15.9 the ideal sheaf $\tilde{I}$ is invertible on $\mathrm{Spec}(R)$. By More on Algebra, Lemma 15.118.3 it is generated by a single element. □

0BCP **Lemma 31.15.11.** *Let X be a Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Assume that there exist integral effective Cartier divisors $D_i \subset X$ such that $D \subset \bigcup D_i$ set theoretically. Then $D = \sum a_i D_i$ for some $a_i \geq 0$. The existence of the D_i is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in D$ or if X is regular.*

Proof. Choose a_i as in Lemma 31.15.8 and set $D' = \sum a_i D_i$. Then $D' \to D$ is an inclusion of effective Cartier divisors which is an isomorphism away from codimension 2 on X. Pick $x \in X$. Set $A = \mathcal{O}_{X,x}$ and let $f, f' \in A$ be the nonzerodivisor generating the ideal of D, D' in A. Then $f = gf'$ for some $g \in A$. Moreover, for every prime $\mathfrak{p}$ of height ≤ 1 of A we see that g maps to a unit of $A_\mathfrak{p}$. This implies that g is a unit because the minimal primes over (g) have height 1 (Algebra, Lemma 10.60.11). □

0AYM **Lemma 31.15.12.** *Let X be a Noetherian scheme which has an ample invertible sheaf. Then every invertible $\mathcal{O}_X$-module is isomorphic to*

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X. Moreover, given a finite subset $E \subset X$ we may choose D, D' such that $E \cap D = \emptyset$ and $E \cap D' = \emptyset$. If X is quasi-affine, then we may choose $D' = \emptyset$.

Proof. Let $x_1, \ldots, x_n$ be the associated points of X (Lemma 31.2.5).

If X is quasi-affine and $\mathcal{N}$ is any invertible $\mathcal{O}_X$-module, then we can pick a section t of $\mathcal{N}$ which does not vanish at any of the points of $E \cup \{x_1, \ldots, x_n\}$, see Properties, Lemma 28.29.7. Then t is a regular section of $\mathcal{N}$ by Lemma 31.15.1. Hence $\mathcal{N} \cong \mathcal{O}_X(D)$ where $D = Z(t)$ is the effective Cartier divisor corresponding to t, see Lemma 31.14.10. Since $E \cap D = \emptyset$ by construction we are done in this case.

Returning to the general case, let $\mathcal{L}$ be an ample invertible sheaf on X. There exists an $m > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_s is affine and such that $E \cup \{x_1, \ldots, x_n\} \subset X_s$ (Properties, Lemma 28.29.6).

Let $\mathcal{N}$ be an arbitrary invertible $\mathcal{O}_X$-module. By the quasi-affine case, we can find a section $t \in \mathcal{N}(X_s)$ which does not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. By Properties, Lemma 28.17.2 we see that for some $e \geq 0$ the section $s^e|_{X_s} t$ extends to a global section τ of $\mathcal{L}^{\otimes me} \otimes \mathcal{N}$. Thus both $\mathcal{L}^{\otimes me} \otimes \mathcal{N}$ and $\mathcal{L}^{\otimes me}$ are invertible sheaves which have global sections which do not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. Thus these are regular sections by Lemma 31.15.1. Hence $\mathcal{L}^{\otimes me} \otimes \mathcal{N} \cong \mathcal{O}_X(D)$ and $\mathcal{L}^{\otimes me} \cong \mathcal{O}_X(D')$ for some effective Cartier divisors D and D', see Lemma 31.14.10. By construction $E \cap D = \emptyset$ and $E \cap D' = \emptyset$ and the proof is complete. □

0B3T **Lemma 31.15.13.** *Let X be an integral regular scheme of dimension 2. Let $i : D \to X$ be the immersion of an effective Cartier divisor. Let $\mathcal{F} \to \mathcal{F}' \to i_*\mathcal{G} \to 0$ be an exact sequence of coherent $\mathcal{O}_X$-modules. Assume*

(1) *$\mathcal{F}, \mathcal{F}'$ are locally free of rank r on a nonempty open of X,*
(2) *D is an integral scheme,*
(3) *$\mathcal{G}$ is a finite locally free $\mathcal{O}_D$-module of rank s.*

Then $\mathcal{L} = (\wedge^r \mathcal{F})^{**}$ and $\mathcal{L}' = (\wedge^r \mathcal{F}')^{**}$ are invertible $\mathcal{O}_X$-modules and $\mathcal{L}' \cong \mathcal{L}(kD)$ for some $k \in \{0, \ldots, \min(s, r)\}$.

Proof. The first statement follows from Lemma 31.12.15 as assumption (1) implies that $\mathcal{L}$ and $\mathcal{L}'$ have rank 1. Taking $\wedge^r$ and double duals are functors, hence we obtain a canonical map $\sigma : \mathcal{L} \to \mathcal{L}'$ which is an isomorphism over the nonempty open of (1), hence nonzero. To finish the proof, it suffices to see that σ viewed as a global section of $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ does not vanish at any codimension point of X, except at the generic point of D and there with vanishing order at most $\min(s, r)$.

Translated into algebra, we arrive at the following problem: Let $(A, \mathfrak{m}, \kappa)$ be a discrete valuation ring with fraction field K. Let $M \to M' \to N \to 0$ be an exact sequence of finite A-modules with $\dim_K(M \otimes K) = \dim_K(M' \otimes K) = r$ and with $N \cong \kappa^{\oplus s}$. Show that the induced map $L = \wedge^r(M)^{**} \to L' = \wedge^r(M')^{**}$ vanishes to order at most $\min(s, r)$. We will use the structure theorem for modules over A, see More on Algebra, Lemma 15.125.3 or 15.125.9. Dividing out a finite A-module by a torsion submodule does not change the double dual. Thus we may replace M by M/M_{tors} and M' by $M'/\operatorname{Im}(M_{tors} \to M')$ and assume that M is torsion free. Then $M \to M'$ is injective and $M'_{tors} \to N$ is injective. Hence we may replace M' by M'/M'_{tors} and N by N/M'_{tors}. Thus we reduce to the case where M and M' are free of rank r and $N \cong \kappa^{\oplus s}$. In this case σ is the determinant of $M \to M'$ and vanishes to order s for example by Algebra, Lemma 10.121.7. □

31.16. Complements of affine opens

0BCQ In this section we discuss the result that the complement of an affine open in a variety has pure codimension 1.

0BCR **Lemma 31.16.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The punctured spectrum $U = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ of A is affine if and only if $\dim(A) \leq 1$.*

Proof. If $\dim(A) = 0$, then U is empty hence affine (equal to the spectrum of the 0 ring). If $\dim(A) = 1$, then we can choose an element $f \in \mathfrak{m}$ not contained in any of the finite number of minimal primes of A (Algebra, Lemmas 10.31.6 and 10.15.2). Then $U = \operatorname{Spec}(A_f)$ is affine.

The converse is more interesting. We will give a somewhat nonstandard proof and discuss the standard argument in a remark below. Assume $U = \operatorname{Spec}(B)$ is affine. Since affineness and dimension are not affected by going to the reduction we may replace A by the quotient by its ideal of nilpotent elements and assume A is reduced. Set $Q = B/A$ viewed as an A-module. The support of Q is $\{\mathfrak{m}\}$ as $A_\mathfrak{p} = B_\mathfrak{p}$ for all nonmaximal primes $\mathfrak{p}$ of A. We may assume $\dim(A) \geq 1$, hence as above we can pick $f \in \mathfrak{m}$ not contained in any of the minimal ideals of A. Since A is reduced this implies that f is a nonzerodivisor. In particular $\dim(A/fA) = \dim(A) - 1$, see Algebra, Lemma 10.60.13. Applying the snake lemma to multiplication by f on the short exact sequence $0 \to A \to B \to Q \to 0$ we obtain

$$0 \to Q[f] \to A/fA \to B/fB \to Q/fQ \to 0$$

where $Q[f] = \operatorname{Ker}(f : Q \to Q)$. This implies that $Q[f]$ is a finite A-module. Since the support of $Q[f]$ is $\{\mathfrak{m}\}$ we see $l = \operatorname{length}_A(Q[f]) < \infty$ (Algebra, Lemma 10.62.3). Set $l_n = \operatorname{length}_A(Q[f^n])$. The exact sequence

$$0 \to Q[f^n] \to Q[f^{n+1}] \xrightarrow{f^n} Q[f]$$

shows inductively that $l_n < \infty$ and that $l_n \leq l_{n+1}$. Considering the exact sequence

$$0 \to Q[f] \to Q[f^{n+1}] \xrightarrow{f} Q[f^n] \to Q/fQ$$

and we see that the image of $Q[f^n]$ in Q/fQ has length $l_n - l_{n+1} + l \leq l$. Since $Q = \bigcup Q[f^n]$ we find that the length of Q/fQ is at most l, i.e., bounded. Thus Q/fQ is a finite A-module. Hence $A/fA \to B/fB$ is a finite ring map, in particular induces a closed map on spectra (Algebra, Lemmas 10.36.22 and 10.41.6). On the other hand $\operatorname{Spec}(B/fB)$ is the punctured spectrum of $\operatorname{Spec}(A/fA)$. This is

a contradiction unless $\mathrm{Spec}(B/fB) = \emptyset$ which means that $\dim(A/fA) = 0$ as desired. □

0BCS **Remark 31.16.2.** If $(A, \mathfrak{m})$ is a Noetherian local normal domain of dimension ≥ 2 and U is the punctured spectrum of A, then $\Gamma(U, \mathcal{O}_U) = A$. This algebraic version of Hartogs's theorem follows from the fact that $A = \bigcap_{\mathrm{height}(\mathfrak{p})=1} A_{\mathfrak{p}}$ we've seen in Algebra, Lemma 10.157.6. Thus in this case U cannot be affine (since it would force $\mathfrak{m}$ to be a point of U). This is often used as the starting point of the proof of Lemma 31.16.1. To reduce the case of a general Noetherian local ring to this case, we first complete (to get a Nagata local ring), then replace A by $A/\mathfrak{q}$ for a suitable minimal prime, and then normalize. Each of these steps does not change the dimension and we obtain a contradiction. You can skip the completion step, but then the normalization in general is not a Noetherian domain. However, it is still a Krull domain of the same dimension (this is proved using Krull-Akizuki) and one can apply the same argument.

0BCT **Remark 31.16.3.** It is not clear how to characterize the non-Noetherian local rings $(A, \mathfrak{m})$ whose punctured spectrum is affine. Such a ring has a finitely generated ideal I with $\mathfrak{m} = \sqrt{I}$. Of course if we can take I generated by 1 element, then A has an affine puncture spectrum; this gives lots of non-Noetherian examples. Conversely, it follows from the argument in the proof of Lemma 31.16.1 that such a ring cannot possess a nonzerodivisor $f \in \mathfrak{m}$ with $H^0_I(A/fA) = 0$ (so A cannot have a regular sequence of length 2). Moreover, the same holds for any ring A' which is the target of a local homomorphism of local rings $A \to A'$ such that $\mathfrak{m}_{A'} = \sqrt{\mathfrak{m}A'}$.

0BCU **Lemma 31.16.4.** *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme such that the inclusion morphism $U \to X$ is affine. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then $\dim(\mathcal{O}_{X,\xi}) = 1$.* [**GD67**, EGA IV, Corollaire 21.12.7]

Proof. Since ξ is a generic point of $X \setminus U$, we see that

$$U_\xi = U \times_X \mathrm{Spec}(\mathcal{O}_{X,\xi}) \subset \mathrm{Spec}(\mathcal{O}_{X,\xi})$$

is the punctured spectrum of $\mathcal{O}_{X,\xi}$ (hint: use Schemes, Lemma 26.13.2). As $U \to X$ is affine, we see that $U_\xi \to \mathrm{Spec}(\mathcal{O}_{X,\xi})$ is affine (Morphisms, Lemma 29.11.9) and we conclude that U_ξ is affine. Hence $\dim(\mathcal{O}_{X,\xi}) \leq 1$ by Lemma 31.16.1. If $\xi \in \overline{U}$, then there is a specialization $\eta \to \xi$ where $\eta \in U$ (just take η a generic point of an irreducible component of $\overline{U}$ which contains ξ; since $\overline{U}$ is locally Noetherian, hence locally has finitely many irreducible components, we see that $\eta \in U$). Then $\eta \in \mathrm{Spec}(\mathcal{O}_{X,\xi})$ and we see that the dimension cannot be 0. □

0BCV **Lemma 31.16.5.** *Let X be a separated locally Noetherian scheme. Let $U \subset X$ be an affine open. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then $\dim(\mathcal{O}_{X,\xi}) = 1$.*

Proof. This follows from Lemma 31.16.4 because the morphism $U \to X$ is affine by Morphisms, Lemma 29.11.12. □

The following lemma can sometimes be used to produce effective Cartier divisors.

0BCW **Lemma 31.16.6.** *Let X be a Noetherian separated scheme. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 28.5.7 applied to the reduced induced subscheme structure on $X \setminus U$). Each $D_i \subset X$ has codimension 1 by Lemma 31.16.5 (and Properties, Lemma 28.10.3). Thus D_i is an effective Cartier divisor by Lemma 31.15.7. Hence we can take $D = D_1 + \ldots + D_r$. □

0EGJ **Lemma 31.16.7.** *Let X be a Noetherian scheme with affine diagonal. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 28.5.7 applied to the reduced induced subscheme structure on $X \setminus U$). We view D_i as a reduced closed subscheme of X. Let $X = \bigcup_{j \in J} X_j$ be an affine open covering of X. For all j in J, set $U_j = U \cap X_j$. Since X has affine diagonal, the scheme

$$U_j = X \times_{(X \times X)} (U \times X_j)$$

is affine. Therefore, as X_j is separated, it follows from Lemma 31.16.6 and its proof that for all $j \in J$ and $1 \leq i \leq r$ the intersection $D_i \cap X_j$ is either empty or an effective Cartier divisor in X_j. Thus $D_i \subset X$ is an effective Cartier divisor (as this is a local property). Hence we can take $D = D_1 + \ldots + D_r$. □

0GML **Lemma 31.16.8.** *Let X be a quasi-compact, regular scheme with affine diagonal. Then X has an ample family of invertible modules (Morphisms, Definition 29.12.1).*

Proof. Observe that X is a finite disjoint union of integral schemes (Properties, Lemmas 28.9.4 and 28.7.6). Thus we may assume that X is integral as well as Noetherian, regular, and having affine diagonal. Let $x \in X$. Choose an affine open neighbourhood $U \subset X$ of x. Since X is integral, U is dense in X. By More on Algebra, Lemma 15.122.2 the local rings of X are UFDs. Hence by Lemma 31.16.7 we can find an effective Cartier divisor $D \subset X$ whose complement is U. Then the canonical section $s = 1_D$ of $\mathcal{L} = \mathcal{O}_X(D)$, see Definition 31.14.1, vanishes exactly along D hence $U = X_s$. Thus both conditions in Morphisms, Definition 29.12.1 hold and we are done. □

31.17. Norms

0BCX Let $\pi : X \to Y$ be a finite morphism of schemes and let $d \geq 1$ be an integer. Let us say there exists a *norm of degree d for π*[2] if there exists a multiplicative map

$$\mathrm{Norm}_\pi : \pi_* \mathcal{O}_X \to \mathcal{O}_Y$$

of sheaves such that

(1) the composition $\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_* \mathcal{O}_X \xrightarrow{\mathrm{Norm}_\pi} \mathcal{O}_Y$ equals $g \mapsto g^d$, and
(2) for $V \subset Y$ open if $f \in \mathcal{O}_X(\pi^{-1}V)$ is zero at $x \in \pi^{-1}(V)$, then $\mathrm{Norm}_\pi(f)$ is zero at $\pi(x)$.

[2] This is nonstandard notation.

We observe that condition (1) forces π to be surjective. Since Norm_π is multiplicative it sends units to units hence, given $y \in Y$, if f is a regular function on X defined at but nonvanishing at any $x \in X$ with $\pi(x) = y$, then $\mathrm{Norm}_\pi(f)$ is defined and does not vanish at y. This holds without requiring (2); in fact, the constructions in this section will only require condition (1) and only certain vanishing properties (which are used in particular in the proof of Lemma 31.17.4) will require property (2).

0BUT **Lemma 31.17.1.** *Let $\pi : X \to Y$ be a finite morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$. There exists an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{\pi^{-1}(V)}$ is trivial.*

Proof. Clearly we may assume Y and hence X affine. Since π is finite the fibre $\pi^{-1}(\{y\})$ over y is finite. Since X is affine, we can pick $s \in \Gamma(X, \mathcal{L})$ not vanishing in any point of $\pi^{-1}(\{y\})$. This follows from Properties, Lemma 28.29.7 but we also give a direct argument. Namely, we can pick a finite set $E \subset X$ of closed points such that every $x \in \pi^{-1}(\{y\})$ specializes to some point of E. For $x \in E$ denote $i_x : x \to X$ the closed immersion. Then $\mathcal{L} \to \bigoplus_{x \in E} i_{x,*} i_x^* \mathcal{L}$ is a surjective map of quasi-coherent $\mathcal{O}_X$-modules, and hence the map

$$\Gamma(X, \mathcal{L}) \to \bigoplus\nolimits_{x \in E} \mathcal{L}_x / \mathfrak{m}_x \mathcal{L}_x$$

is surjective (as taking global sections is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules, see Schemes, Lemma 26.7.5). Thus we can find an $s \in \Gamma(X, \mathcal{L})$ not vanishing at any point specializing to a point of E. Then $X_s \subset X$ is an open neighbourhood of $\pi^{-1}(\{y\})$. Since π is finite, hence closed, we conclude that there is an open neighbourhood $V \subset Y$ of y whose inverse image is contained in X_s as desired. □

0BCY **Lemma 31.17.2.** *Let $\pi : X \to Y$ be a finite morphism of schemes. If there exists a norm of degree d for π, then there exists a homomorphism of abelian groups*

$$\mathit{Norm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

such that $\mathit{Norm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$.

Proof. We will use the correspondence between isomorphism classes of invertible $\mathcal{O}_X$-modules and elements of $H^1(X, \mathcal{O}_X^*)$ given in Cohomology, Lemma 20.6.1 without further mention. We explain how to take the norm of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Namely, by Lemma 31.17.1 there exists an open covering $Y = \bigcup V_j$ such that $\mathcal{L}|_{\pi^{-1}V_j}$ is trivial. Choose a generating section $s_j \in \mathcal{L}(\pi^{-1}V_j)$ for each j. On the overlaps $\pi^{-1}V_j \cap \pi^{-1}V_{j'}$ we can write

$$s_j = u_{jj'} s_{j'}$$

for a unique $u_{jj'} \in \mathcal{O}_X^*(\pi^{-1}V_j \cap \pi^{-1}V_{j'})$. Thus we can consider the elements

$$v_{jj'} = \mathrm{Norm}_\pi(u_{jj'}) \in \mathcal{O}_Y^*(V_j \cap V_{j'})$$

These elements satisfy the cocycle condition (because the $u_{jj'}$ do and Norm_π is multiplicative) and therefore define an invertible $\mathcal{O}_Y$-module. We omit the verification that: this is well defined, additive on Picard groups, and satisfies the property $\mathrm{Norm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$. □

0BCZ **Lemma 31.17.3.** *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume there exists a norm of degree d for π. For any $\mathcal{O}_X$-linear map $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible $\mathcal{O}_X$-modules there is an $\mathcal{O}_Y$-linear map*

$$Norm_\pi(\varphi) : Norm_\pi(\mathcal{L}) \longrightarrow Norm_\pi(\mathcal{L}')$$

with $Norm_\pi(\mathcal{L})$, $Norm_\pi(\mathcal{L}')$ as in Lemma 31.17.2. Moreover, for $y \in Y$ the following are equivalent

(1) *φ is zero at a point of $x \in X$ with $\pi(x) = y$, and*
(2) *$Norm_\pi(\varphi)$ is zero at y.*

Proof. We choose an open covering $Y = \bigcup V_j$ such that $\mathcal{L}$ and $\mathcal{L}'$ are trivial over the opens $\pi^{-1}V_j$. This is possible by Lemma 31.17.1. Choose generating sections s_j and s'_j of $\mathcal{L}$ and $\mathcal{L}'$ over the opens $\pi^{-1}V_j$. Then $\varphi(s_j) = f_j s'_j$ for some $f_j \in \mathcal{O}_X(\pi^{-1}V_j)$. Define $\mathrm{Norm}_\pi(\varphi)$ to be multiplication by $\mathrm{Norm}_\pi(f_j)$ on V_j. A simple calculation involving the cocycles used to construct $\mathrm{Norm}_\pi(\mathcal{L})$, $\mathrm{Norm}_\pi(\mathcal{L}')$ in the proof of Lemma 31.17.2 shows that this defines a map as stated in the lemma. The final statement follows from condition (2) in the definition of a norm map of degree d. Some details omitted. □

0BD0 **Lemma 31.17.4.** *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X has an ample invertible sheaf and there exists a norm of degree d for π. Then Y has an ample invertible sheaf.*

Proof. Let $\mathcal{L}$ be the ample invertible sheaf on X given to us by assumption. We will prove that $\mathcal{N} = \mathrm{Norm}_\pi(\mathcal{L})$ is ample on Y.

Since X is quasi-compact (Properties, Definition 28.26.1) and $X \to Y$ surjective (by the existence of Norm_π) we see that Y is quasi-compact. Let $y \in Y$ be a point. To finish the proof we will show that there exists a section t of some positive tensor power of $\mathcal{N}$ which does not vanish at y such that Y_t is affine. To do this, choose an affine open neighbourhood $V \subset Y$ of y. Choose $n \gg 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that

$$\pi^{-1}(\{y\}) \subset X_s \subset \pi^{-1}V$$

by Properties, Lemma 28.29.6. Then $t = \mathrm{Norm}_\pi(s)$ is a section of $\mathcal{N}^{\otimes n}$ which does not vanish at x and with $Y_t \subset V$, see Lemma 31.17.3. Then Y_t is affine by Properties, Lemma 28.26.4. □

0BD1 **Lemma 31.17.5.** *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X is quasi-affine and there exists a norm of degree d for π. Then Y is quasi-affine.*

Proof. By Properties, Lemma 28.27.1 we see that $\mathcal{O}_X$ is an ample invertible sheaf on X. The proof of Lemma 31.17.4 shows that $\mathrm{Norm}_\pi(\mathcal{O}_X) = \mathcal{O}_Y$ is an ample invertible $\mathcal{O}_Y$-module. Hence Properties, Lemma 28.27.1 shows that Y is quasi-affine. □

0BD2 **Lemma 31.17.6.** *Let $\pi : X \to Y$ be a finite locally free morphism of degree $d \geq 1$. Then there exists a canonical norm of degree d whose formation commutes with arbitrary base change.*

Proof. Let $V \subset Y$ be an affine open such that $(\pi_*\mathcal{O}_X)|_V$ is finite free of rank d. Choosing a basis we obtain an isomorphism

$$\mathcal{O}_V^{\oplus d} \cong (\pi_*\mathcal{O}_X)|_V$$

For every $f \in \pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$ multiplication by f defines a $\mathcal{O}_V$-linear endomorphism m_f of the displayed free vector bundle. Thus we get a $d \times d$ matrix $M_f \in \mathrm{Mat}(d \times d, \mathcal{O}_Y(V))$ and we can set

$$\mathrm{Norm}_\pi(f) = \det(M_f)$$

Since the determinant of a matrix is independent of the choice of the basis chosen we see that this is well defined which also means that this construction will glue to a global map as desired. Compatibility with base change is straightforward from the construction.

Property (1) follows from the fact that the determinant of a $d \times d$ diagonal matrix with entries $g, g, \ldots, g$ is g^d. To see property (2) we may base change and assume that Y is the spectrum of a field k. Then $X = \mathrm{Spec}(A)$ with A a k-algebra with $\dim_k(A) = d$. If there exists an $x \in X$ such that $f \in A$ vanishes at x, then there exists a map $A \to \kappa$ into a field such that f maps to zero in κ. Then $f : A \to A$ cannot be surjective, hence $\det(f : A \to A) = 0$ as desired. □

0BD3 **Lemma 31.17.7.** *Let $\pi : X \to Y$ be a finite surjective morphism with X and Y integral and Y normal. Then there exists a norm of degree $[R(X) : R(Y)]$ for π.*

Proof. Let $\mathrm{Spec}(B) \subset Y$ be an affine open subset and let $\mathrm{Spec}(A) \subset X$ be its inverse image. Then A and B are domains. Let K be the fraction field of A and L the fraction field of B. Picture:

$$\begin{array}{ccc} L & \longrightarrow & K \\ \uparrow & & \uparrow \\ B & \longrightarrow & A \end{array}$$

Since K/L is a finite extension, there is a norm map $\mathrm{Norm}_{K/L} : K^* \to L^*$ of degree $d = [K : L]$; this is given by mapping $f \in K$ to $\det_L(f : K \to K)$ as in the proof of Lemma 31.17.6. Observe that the characteristic polynomial of $f : K \to K$ is a power of the minimal polynomial of f over L; in particular $\mathrm{Norm}_{K/L}(f)$ is a power of the constant coefficient of the minimal polynomial of f over L. Hence by Algebra, Lemma 10.38.6 $\mathrm{Norm}_{K/L}$ maps A into B. This determines a compatible system of maps on sections over affines and hence a global norm map Norm_π of degree d.

Property (1) is immediate from the construction. To see property (2) let $f \in A$ be contained in the prime ideal $\mathfrak{p} \subset A$. Let $f^m + b_1 f^{m-1} + \ldots + b_m$ be the minimal polynomial of f over L. By Algebra, Lemma 10.38.6 we have $b_i \in B$. Hence $b_0 \in B \cap \mathfrak{p}$. Since $\mathrm{Norm}_{K/L}(f) = b_0^{d/m}$ (see above) we conclude that the norm vanishes in the image point of $\mathfrak{p}$. □

0BDZ **Lemma 31.17.8.** *Let X be a Noetherian scheme. Let p be a prime number such that $p\mathcal{O}_X = 0$. Then for some $e > 0$ there exists a norm of degree p^e for $X_{red} \to X$ where X_{red} is the reduction of X.*

Proof. Let A be a Noetherian ring with $pA = 0$. Let $I \subset A$ be the ideal of nilpotent elements. Then $I^n = 0$ for some n (Algebra, Lemma 10.32.5). Pick e such that $p^e \geq n$. Then

$$A/I \longrightarrow A, \quad f \bmod I \longmapsto f^{p^e}$$

is well defined. This produces a norm of degree p^e for $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. Now if X is obtained by glueing some affine schemes $\mathrm{Spec}(A_i)$ then for some $e \gg 0$ these maps glue to a norm map for $X_{red} \to X$. Details omitted. □

0BD4 **Proposition 31.17.9.** *Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X has an ample invertible $\mathcal{O}_X$-module. If*

(1) *π is finite locally free, or*
(2) *Y is an integral normal scheme, or*
(3) *Y is Noetherian, $p\mathcal{O}_Y = 0$, and $X = Y_{red}$,*

then Y has an ample invertible $\mathcal{O}_Y$-module.

Proof. Case (1) follows from a combination of Lemmas 31.17.6 and 31.17.4. Case (3) follows from a combination of Lemmas 31.17.8 and 31.17.4. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 31.17.7. □

0BD5 **Lemma 31.17.10.** *Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X is quasi-affine. If either*

(1) *π is finite locally free, or*
(2) *Y is an integral normal scheme*

then Y is quasi-affine.

Proof. Case (1) follows from a combination of Lemmas 31.17.6 and 31.17.5. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 31.17.7. □

31.18. Relative effective Cartier divisors

056P The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

056Q **Lemma 31.18.1.** *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume*

(1) *D is an effective Cartier divisor, and*
(2) *$D \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$ where $g' : X' \to X$ is the projection.

Proof. Using Lemma 31.13.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 10.39.12 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. □

This lemma is the motivation for the following definition.

062T **Definition 31.18.2.** Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [**DG67**, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

0B8U **Lemma 31.18.3.** *Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S then so is $D_1 + D_2$ (Definition 31.13.6).*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors and that B/h_iB is flat over A. Then h_1h_2 is a nonzerodivisor and B/h_1h_2B is flat over A. The reason is that we have a short exact sequence

$$0 \to B/h_1B \to B/h_1h_2B \to B/h_2B \to 0$$

where the first arrow is given by multiplication by h_2. Since the outer two are flat modules over A, so is the middle one, see Algebra, Lemma 10.39.13. □

0B8V **Lemma 31.18.4.** *Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S and $D_1 \subset D_2$ as closed subschemes, then the effective Cartier divisor D such that $D_2 = D_1 + D$ (Lemma 31.13.8) is a relative effective Cartier divisor on X/S.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors, that B/h_iB is flat over A, and that $(h_2) \subset (h_1)$. Then we can write $h_2 = hh_1$ where $h \in B$ is a nonzerodivisor. We get a short exact sequence

$$0 \to B/hB \to B/h_2B \to B/h_1B \to 0$$

where the first arrow is given by multiplication by h_1. Since the right two are flat modules over A, so is the middle one, see Algebra, Lemma 10.39.13. □

062U **Lemma 31.18.5.** *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzerodivisor in B such that B/hB is a flat local A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{h} & B & \longrightarrow & B/hB & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & B \otimes_A I & \xrightarrow{h} & B \otimes_A I & \longrightarrow & B/hB \otimes_A I & \longrightarrow & 0 \end{array}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 10.39.12. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 10.39.5. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 10.20.1 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 10.39.5. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 37.15.

062V **Lemma 31.18.6.** *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.*

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that D is given by a nonzerodivisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$ and $B/hB = B_0/h_0B_0 \otimes_{A_0} A$. By Algebra, Lemma 10.168.1 we may, after enlarging A_0, assume that B_0/h_0B_0 is flat over A_0. Let $K_0 = \mathrm{Ker}(h_0 : B_0 \to B_0)$. As B_0 is of finite type over $\mathbf{Z}$ we see that K_0 is a finitely generated ideal. Let $A_1 \subset A$ be a finite type $\mathbf{Z}$-subalgebra containing A_0 and denote B_1, h_1, K_1 the corresponding objects over A_1. By More on Algebra, Lemma 15.31.3 the map $K_0 \otimes_{A_0} A_1 \to K_1$ is surjective. On the other hand, the kernel of $h : B \to B$ is zero by assumption. Hence every element of K_0 maps to zero in K_1 for sufficiently large subrings $A_1 \subset A$. Since K_0 is finitely generated, we conclude that $K_1 = 0$ for a suitable choice of A_1.

Set $f_1 : X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$. Set $D_1 = \mathrm{Spec}(B_1/h_1B_1)$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, it now suffices to prove the lemma for $X_1 \to S_1$ and the relative effective Cartier divisor D_1, see Morphisms, Lemma 29.25.7. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 31.18.5. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.129.4 we win. □

There is also the following lemma (whose idea is apparently due to Michael Artin, see [**Nob77**]) which needs no finiteness assumptions at all.

062W **Lemma 31.18.7.** *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to \mathrm{colim}_n (1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 10.39.13). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 10.39.3) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. □

062X **Example 31.18.8.** Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzerodivisor and hence defines a relative effective Cartier divisor in $\mathrm{Spec}(B)$ over $\mathrm{Spec}(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 37.23.1 for a slightly different take on this lemma.

062Y **Lemma 31.18.9.** *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

(1) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$ and $Z_s \cap U = D_s$.*
(2) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.*
(3) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.*

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzerodivisor in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset \mathrm{Spec}(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 10.168.1 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = (B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzerodivisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 10.99.2 we see that f_0 is a nonzerodivisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 10.68.6 and Algebra, Theorem 10.129.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzerodivisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset \mathrm{Spec}((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 31.18.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.126.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzerodivisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 29.21.3 and 29.21.11. □

31.19. The normal cone of an immersion

062Z Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 29.4.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 29.4.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

0630 **Definition 31.19.1.** Let $f : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of f* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

0631 (31.19.1.1) $$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

0632 (31.19.1.2) $$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

0633 **Lemma 31.19.2.** *Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*

$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

(2) *For any affine open* $\mathrm{Spec}(R) = U \subset X$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *there is a canonical isomorphism* $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. □

0634 **Lemma 31.19.3.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram in the category of schemes. Assume i, i' *immersions. There is a canonical map of graded* $\mathcal{O}_Z$*-algebras*

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ *with* $g(U) \subset U'$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *and* $Z' \cap U' = \mathrm{Spec}(R'/I')$ *the induced map*

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n \geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ *which has the property* $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \left(g^{-1}(\partial Z') \cup \partial Z\right)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 26.4.6 and 26.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

0635 **Lemma 31.19.4.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram in the category of schemes with i, i' *immersions. Then the canonical map* $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ *of Lemma 31.19.3 is surjective. If* g *is flat, then it is an isomorphism.*

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. □

0636 **Definition 31.19.5.** Let $i : Z \to X$ be an immersion of schemes. The *normal cone* $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 27.7.1 and 27.7.2. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 27.6.1 and 27.6.2.

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (31.19.1.2) of graded algebras defines a canonical closed immersion

0637 (31.19.5.1) $$C_Z X \longrightarrow N_Z X$$

of cones over Z.

31.20. Regular ideal sheaves

067M In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a *regular sequence* if for each $i = 1, \ldots, r$ the element f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 10.68.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a *Koszul-regular sequence*, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 15.30.1. The sequence is called an *H_1-regular sequence* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to f_i mod J^2 is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a *quasi-regular sequence*, see Algebra, Definition 10.69.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a *regular sequence* if for each $i = 1, \ldots, r$ the map

0639 (31.20.0.1) $$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1})$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a *Koszul-regular sequence* if the Koszul complex

063A (31.20.0.2) $$K_\bullet(\mathcal{O}_X, f_\bullet),$$

see Modules, Definition 17.24.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a *H_1-regular sequence* if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a *quasi-regular* sequence if the map

063B (31.20.0.3) $$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

063C **Lemma 31.20.1.** *Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a quasi-regular sequence.*

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 15.30.2, 15.30.3, and 15.30.6. □

063D **Definition 31.20.2.** Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

(1) We say $\mathcal{J}$ is *regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(2) We say $\mathcal{J}$ is *Koszul-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(3) We say $\mathcal{J}$ is *H_1-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(4) We say $\mathcal{J}$ is *quasi-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

The concept of a Koszul-regular ideal sheaf was introduced in [**BGI71**, Expose VII, Definition 1.4] where it was called a regular ideal sheaf.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

063E **Lemma 31.20.3.** *Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow$ $\mathcal{J}$ is Koszul-regular $\Rightarrow$ $\mathcal{J}$ is H_1-regular $\Rightarrow$ $\mathcal{J}$ is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 31.20.1. □

063H **Lemma 31.20.4.** *Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:*

(1) *$\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,*
(2) *$\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and*
(3) *the canonical maps*

$$Sym^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$

are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (31.20.0.3). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma 17.9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. □

067N **Lemma 31.20.5.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.*

(1) *If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.*
(2) *If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.*
(3) *If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.*

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 31.20.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write

$$f_i = \sum a_{ij} g_j$$

for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij}) is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 15.28.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. □

063F **Lemma 31.20.6.** *Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.*

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 26.10.1. □

063G **Lemma 31.20.7.** *Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = \mathrm{Spec}(A)$ such that $\mathcal{J}|_U = \widetilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.*

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = \mathrm{Spec}(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 31.20.6 we see that $\mathcal{J}|_U = \widetilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\widetilde{\ }: \mathrm{Mod}_A \longrightarrow QCoh(\mathcal{O}_U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (31.20.0.1) is injective (resp. (31.20.0.2) is exact, (31.20.0.2) is exact in degree 1, (31.20.0.3) is an isomorphism) implies the corresponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. □

063I **Lemma 31.20.8.** *Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent*

(1) *$\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,*
(2) *$\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,*
(3) *$\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,*
(4) *$\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,*
(5) *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a regular sequence in A, and*
(6) *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a Koszul-regular sequence in A, and*
(7) *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by an H_1-regular sequence in A, and*
(8) *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a quasi-regular sequence in A,*
(9) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is regular, and*
(10) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is Koszul-regular, and*
(11) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is H_1-regular, and*
(12) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is quasi-regular.*

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1-regular, or quasi-regular ideal sheaf all agree.

Proof. It follows from Lemma 31.20.7 that (5) ⇔ (9), (6) ⇔ (10), (7) ⇔ (11), and (8) ⇔ (12). It is clear that (5) ⇒ (1), (6) ⇒ (2), (7) ⇒ (3), and (8) ⇒ (4). We have (1) ⇒ (5) by Algebra, Lemma 10.68.6. We have (9) ⇒ (10) ⇒ (11) ⇒ (12) by Lemma 31.20.3. Finally, (4) ⇒ (1) by Algebra, Lemma 10.69.6. Now all 12 statements are equivalent. □

31.21. Regular immersions

0638 Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U \cap U'} = \mathcal{I}'|_{U \cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$

is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 31.20.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

063J **Definition 31.21.1.** Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

(1) We say i is a *regular immersion* if $\mathcal{I}$ is regular.
(2) We say i is a *Koszul-regular immersion* if $\mathcal{I}$ is Koszul-regular.
(3) We say i is a *H_1-regular immersion* if $\mathcal{I}$ is H_1-regular.
(4) We say i is a *quasi-regular immersion* if $\mathcal{I}$ is quasi-regular.

The concept of a Koszul-regular immersion was introduced in [**BGI71**, Expose VII, Definition 1.4] where it was called a regular immersion.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 31.21.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 31.21.11. In the locally Noetherian case all four notions agree, see Lemma 31.20.8.

063K **Lemma 31.21.2.** *Let $i : Z \to X$ be an immersion of schemes. We have the following implications: i is regular $\Rightarrow$ i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 31.20.3. □

063L **Lemma 31.21.3.** *Let $i : Z \to X$ be an immersion of schemes. Assume X is locally Noetherian. Then i is regular $\Leftrightarrow$ i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Follows immediately from Lemma 31.21.2 and Lemma 31.20.8. □

067P **Lemma 31.21.4.** *Let $i : Z \to X$ be a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion. Let $X' \to X$ be a flat morphism. Then the base change $i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.*

Proof. Via Lemma 31.20.7 this translates into the algebraic statements in Algebra, Lemmas 10.68.5 and 10.69.3 and More on Algebra, Lemma 15.30.5. □

063M **Lemma 31.21.5.** *Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (31.19.1.2) is an isomorphism.*

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 29.21.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 31.20.4 and unwinding the definitions. □

063N **Lemma 31.21.6.** *Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 29.31.5*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 31.21.5 and Lemma 31.20.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 29.31.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 15.30.9. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.69.8. The other types of regular immersions are preserved under composition.

067Q **Lemma 31.21.7.** *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.*

(1) *If i and j are regular immersions, so is $j \circ i$.*
(2) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(3) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(4) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. The algebraic version of (1) is Algebra, Lemma 10.68.7. The algebraic version of (2) is More on Algebra, Lemma 15.30.13. The algebraic version of (3) is More on Algebra, Lemma 15.30.11. The algebraic version of (4) is More on Algebra, Lemma 15.30.10. □

068Z **Lemma 31.21.8.** *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 29.31.5 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 31.20.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 10.69.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 15.30.12.

Proof of (3). By Lemma 31.20.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by

More on Algebra, Lemma 15.30.14 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. □

0690 **Lemma 31.21.9.** *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent*

(1) *i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,*
(2) *i and $j \circ i$ are regular immersions in a neighbourhood of z,*
(3) *$j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence*
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 31.20.8. The implication (1) ⇒ (2) is Lemma 31.21.7. The implication (2) ⇒ (3) is Lemma 31.21.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any minimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 31.20.5. By assumption the conormal sequence
$$0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$$
is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. □

0691 **Remark 31.21.10.** In the situation of Lemma 31.21.9 parts (1), (2), (3) are **not** equivalent to "$j \circ i$ and j are regular immersions at z and y". An example is $X = \mathbf{A}^1_k = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[x]/(x^2))$ and $Z = \mathrm{Spec}(k[x]/(x))$.

0692 **Lemma 31.21.11.** *Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.*

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 31.20.7). The affine case is More on Algebra, Lemma 15.30.17. □

0E9J **Lemma 31.21.12.** *Let $i : Z \to X$ be an immersion. If Z and X are regular schemes, then i is a regular immersion.*

Proof. Let $z \in Z$. By Lemma 31.20.8 it suffices to show that the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is generated by a regular sequence. This follows from Algebra, Lemmas 10.106.4 and 10.106.3. □

31.22. Relative regular immersions

063P In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 110.15.2.

063R **Lemma 31.22.1.** *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume*

(1) *i is an H_1-regular (resp. quasi-regular) immersion, and*
(2) *$Z \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 31.20.7 this translates into More on Algebra, Lemma 15.31.4. □

This lemma is the motivation for the following definition.

063S **Definition 31.22.2.** Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

(1) We say i is a *relative quasi-regular immersion* if $Z \to S$ is flat and i is a quasi-regular immersion.
(2) We say i is a *relative H_1-regular immersion* if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 31.22.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 31.21.2. Please take a look at Lemma 31.22.6 (or Lemma 31.22.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

063T **Lemma 31.22.3.** *Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 10.69.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzerodivisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 31.18.5 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

063U **Lemma 31.22.4.** *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume*

(1) *$X \to S$ is flat and locally of finite presentation,*
(2) *$Z \to X$ is a relative quasi-regular immersion.*

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By More on Algebra, Lemma 15.31.4 the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 10.69.6 and 10.68.6. By Lemma 31.18.9 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 31.18.1, we also see that the final statement of the lemma holds. □

0FUD **Remark 31.22.5.** The codimension of a relative quasi-regular immersion, if it is constant, does not change after a base change. In fact, if we have a ring map $A \to B$ and a quasi-regular sequence $f_1, \ldots, f_r \in B$ such that $B/(f_1, \ldots, f_r)$ is flat over A, then for any ring map $A \to A'$ we have a quasi-regular sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ in $B' = B \otimes_A A'$ by More on Algebra, Lemma 15.31.4 (which was used in the proof of Lemma 31.22.1 above). Now the proof of Lemma 31.22.4 shows that if $A \to B$ is flat and locally of finite presentation, then for every prime ideal $\mathfrak{q}' \subset B'$ the sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ is even a regular sequence in the local ring $B'_{\mathfrak{q}'}$.

063V **Lemma 31.22.6.** *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then*

(1) *there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and*
(2) *$Z \to X$ is a regular immersion and the same remains true after any base change.*

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f'_i \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f'_1, \ldots, f'_r, g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f'_i in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 10.168.1 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 15.31.3 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$ which is acyclic in degree 1. Hence

$$\operatorname{colim}_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 10.8.8. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$ and $Z_1 = \operatorname{Spec}(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 29.25.7. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 31.22.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.129.4 we see that (1) holds. Part (2) then follows from an application of Lemma 31.22.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

063W **Lemma 31.22.7.** *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.*

(1) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.*
(2) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.*
(3) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

In each case $Z \to U$ is a regular immersion by Lemma 31.22.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $\operatorname{Spec}(A)$ of s and $\operatorname{Spec}(B)$ of x such that $\varphi(\operatorname{Spec}(B)) \subset \operatorname{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\operatorname{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \notin \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 10.69.6 and 10.68.6 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 31.18.9 it follows that $Z_1 = V(f_1) \subset \operatorname{Spec}(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to \operatorname{Spec}(B)$

is a regular immersion and Z is flat over S, in particular $Z \to \mathrm{Spec}(B)$ is a relative quasi-regular immersion over $\mathrm{Spec}(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.126.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.116.3 and the second by r times applying Algebra, Lemma 10.60.13. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix (b_{ij}) mod $\mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 29.21.3 and 29.21.11. □

The following lemma is an enhancement of Morphisms, Lemma 29.34.20.

067R **Lemma 31.22.8.** *Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.*

Proof. By Schemes, Lemma 26.21.10 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.34.9 and 29.34.8. Thus, according to Lemma 31.22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 29.34.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak{m}$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$. In Algebra, Lemma 10.140.3 we proved that $A_\mathfrak{m}$ is a regular local ring, hence $\mathfrak{m}A_\mathfrak{m}$ is generated by a regular sequence. This does indeed imply that $\mathfrak{m}$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$), see Algebra, Lemma 10.68.6. □

The following lemma has a kind of converse, see Lemma 31.22.12.

067S **Lemma 31.22.9.** *Let*

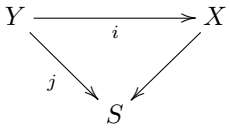

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 31.22.8 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 31.21.4. We conclude by an application of Lemma 31.21.7. □

067T **Lemma 31.22.10.** *Let*

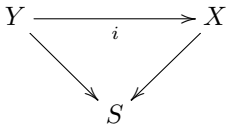

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 29.30.6 and 29.30.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 29.34.9 and 29.34.8). Thus, according to Lemma 31.22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 29.15.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 31.20.8. Take $t \in T_s$. By Morphisms, Lemma 29.30.9 the local ring $\mathcal{O}_{T_s,t}$ is a local complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 10.140.3. By Algebra, Lemma 10.135.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. □

067U **Lemma 31.22.11.** *Let*

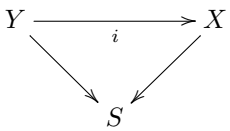

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 31.22.10 because a smooth morphism is syntomic, see Morphisms, Lemma 29.34.7. □

0693 **Lemma 31.22.12.** *Let*

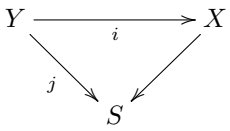

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth and i and j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. We will use Lemma 31.21.2 without further mention. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X and S by open neighbourhoods U and V of x and s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$.

We first prove the result for $X = \mathbf{A}^n_S$. After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that j is a closed immersions and S is affine. Write $S = \mathrm{Spec}(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $\mathrm{Spec}(A/I)$ for some ideal $I \subset A$. The map $i : Y = \mathrm{Spec}(A/I) \to \mathbf{A}^n_S = \mathrm{Spec}(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\mathrm{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\mathrm{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular sequence in $A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 31.21.8.

Next, we prove the result holds if i is H_1-regular or quasi-regular. Namely, shrinking as in the first paragraph of the proof, we may assume that Y, X, and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 31.22.11. Hence $h \circ i$ is a H_1-regular or quasi-regular immersion, see Lemma 31.21.7 (note that this step does not work in the "quasi-regular case"). Thus we reduce to the case $X = \mathbf{A}^n_S$ and S affine we proved above.

Finally, assume i is quasi-regular. After shrinking as in the first paragraph of the proof, we may use Morphisms, Lemma 29.36.20 to factor f as $X \to \mathbf{A}^n_S \to S$ where the first morphism $X \to \mathbf{A}^n_S$ is étale. This reduces the problem to the the two cases

(a) $X = \mathbf{A}^n_S$ and (b) f is étale. Case (a) was handled in the second paragraph of the proof. Case (b) is handled by the next paragraph.

Assume f is étale. After shrinking we may assume X, Y, and S affine i and j closed immersions (small detail omitted). Say $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(B/J) = \mathrm{Spec}(A/I)$. Shrinking further we may assume J is generated by a quasi-regular sequence. The ring map $A \to B$ is étale, hence formally étale (Algebra, Lemma 10.150.3). Thus $\bigoplus I^n/I^{n+1} \cong \bigoplus J^n/J^{n+1}$ by Algebra, Lemma 10.150.6. Since J is generated by a quasi-regular sequence, so is I. This finishes the proof. □

31.23. Meromorphic functions and sections

01X1 This section contains only the general definitions and some elementary results. See [**Kle79**] for some possible pitfalls[3].

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 31.14.6. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 17.27.1.

01X2 **Definition 31.23.1.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *sheaf of meromorphic functions on* X is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

01X3 **Example 31.23.2.** Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x - \alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = \mathrm{Spec}(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzerodivisor is a unit (proof omitted).

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 17.27.2.

01X4 **Definition 31.23.3.** Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
(2) A *meromorphic section of* $\mathcal{F}$ is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$

[3] Danger, Will Robinson!

for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzerodivisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map

$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$

to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzerodivisors in $\mathcal{O}_{X,x}$. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

02OT **Definition 31.23.4.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pullbacks of meromorphic functions are defined for f* if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{array}{ccc} (X, \mathcal{K}_X) & \longrightarrow & (X, \mathcal{O}_X) \\ \downarrow f & & \downarrow f \\ (Y, \mathcal{K}_Y) & \longrightarrow & (Y, \mathcal{O}_Y) \end{array}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

02OU **Lemma 31.23.5.** *Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic functions are defined.*

1. *every weakly associated point of X maps to a generic point of an irreducible component of Y,*
2. *X, Y are integral and f is dominant,*
3. *X is integral and the generic point of X maps to a generic point of an irreducible component of Y,*
4. *X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,*
5. *X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y,*
6. *X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y, and*
7. *f is flat.*

Proof. The question is local on X and Y. Hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and f is given by a ring map $\varphi : R \to A$. By the characterization of regular sections of the structure sheaf in Lemma 31.14.7 we have to show that $R \to A$ maps nonzerodivisors to nonzerodivisors. Let $t \in R$ be a nonzerodivisor.

If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (7).

In the other cases we note that t is not contained in any of the minimal primes of R (because every element of a minimal prime in a ring is a zerodivisor). Hence in case (1) we see that $\varphi(t)$ is not contained in any weakly associated prime of A.

Thus this case follows from Algebra, Lemma 10.66.7. Case (5) is a special case of (1) by Lemma 31.5.8. Case (6) follows from (5) and the definitions. Case (4) is a special case of (1) by Lemma 31.5.12. Cases (2) and (3) are special cases of (4). □

0EMF **Lemma 31.23.6.** *Let X be a scheme such that*

(a) *every weakly associated point of X is a generic point of an irreducible component of X, and*
(b) *any quasi-compact open has a finite number of irreducible components.*

Let X^0 be the set of generic points of irreducible components of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

(1) *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
(2) *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*
$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$
of meromorphic sections of $\mathcal{F}$ is quasi-coherent,
(3) *$\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,*
(4) *$\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,*
(5) *$\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,*
(6) *the ring of rational functions of X (Morphisms, Definition 29.49.3) is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. Observe that a locally finite direct sum of sheaves of modules is equal to the product since you can check this on stalks for example. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that (2) follows from the other statements. Also, observe that part (6) follows from the initial statement of the lemma and Morphisms, Lemma 29.49.5 when X^0 is finite; the general case of (6) follows from this by glueing (argument omitted).

Let $j : Y = \coprod_{\eta \in X^0} \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ be the coproduct of the morphisms j_η. We have to show that $\mathcal{K}_X = j_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 31.23.5. This gives a map

$$\mathcal{K}_X \longrightarrow j_*\mathcal{O}_Y.$$

Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then A is a ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ and every weakly associated prime of A is one of the $\mathfrak{q}_i$. We obtain $Q(A) = \prod A_{\mathfrak{q}_i}$ by Algebra, Lemmas 10.25.4 and 10.66.7. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $j_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism which proves the first displayed equality in the statement of the lemma.

Finally, we prove (1), (3), (4), and (5). Part (5) we saw during the course of the proof that $\mathcal{K}_X = j_*\mathcal{O}_Y$. The morphism j is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence j is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 26.24.1 $j_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. We may choose an affine

open neighbourhood $U = \mathrm{Spec}(A)$ of x all of whose irreducible components pass through x. Then $A \subset A_{\mathfrak{p}}$ because every weakly associated prime of A is contained in $\mathfrak{p}$ hence elements of $A \setminus \mathfrak{p}$ are nonzerodivisors by Algebra, Lemma 10.66.7. It follows easily that any nonzerodivisor of $A_{\mathfrak{p}}$ is the image of a nonzerodivisor on a (possibly smaller) affine open neighbourhood of x. This proves (3). Part (4) follows from part (3) by computing stalks. □

02OX **Definition 31.23.7.** Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. In other words, s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$, see Definition 31.14.6.

Let us spell out when (regular) meromorphic sections can be pulled back.

02OY **Lemma 31.23.8.** *Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 31.23.4).*

(1) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
(2) *Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. □

02P0 **Lemma 31.23.9.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule*

$$\mathcal{I}(V) = \{f \in \mathcal{O}_X(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = \mathrm{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking further we may assume that $s = a/b$ with $a, b \in A$ *both* nonzerodivisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzerodivisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokernel of s since $s(b) = (a) \subset sI \subset A$. □

02P1 **Definition 31.23.10.** Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 31.23.9 is called the *ideal sheaf of denominators of s*.

31.24. Meromorphic functions and sections; Noetherian case

0EMG For locally Noetherian schemes we can prove some results about the sheaf of meromorphic functions. However, there is an example in [**Kle79**] showing that $\mathcal{K}_X$ need not be quasi-coherent for a Noetherian scheme X.

08I7 **Lemma 31.24.1.** *Let X be a quasi-compact scheme. Let $h \in \Gamma(X, \mathcal{O}_X)$ and $f \in \Gamma(X, \mathcal{K}_X)$ such that f restricts to zero on X_h. Then $h^n f = 0$ for some $n \gg 0$.*

Proof. We can find a covering of X by affine opens U such that $f|_U = s^{-1}a$ with $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{S}(U)$. Since X is quasi-compact we can cover it by finitely many affine opens of this form. Thus it suffices to prove the lemma when $X = \mathrm{Spec}(A)$ and $f = s^{-1}a$. Note that $s \in A$ is a nonzerodivisor hence it suffices to prove the result when $f = a$. The condition $f|_{X_h} = 0$ implies that a maps to zero in $A_h = \mathcal{O}_X(X_h)$ as $\mathcal{O}_X \subset \mathcal{K}_X$. Thus $h^n a = 0$ for some $n > 0$ as desired. □

02OV **Lemma 31.24.2.** *Let X be a locally Noetherian scheme.*

(1) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors, and hence $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(2) *For any affine open $U \subset X$ the ring $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$.*

Proof. To prove this lemma we may assume X is the spectrum of a Noetherian ring A. Say $x \in X$ corresponds to $\mathfrak{p} \subset A$.

Proof of (1). It is clear that $\mathcal{S}_x$ is contained in the set of nonzerodivisors of $\mathcal{O}_{X,x} = A_\mathfrak{p}$. For the converse, let $f, g \in A$, $g \notin \mathfrak{p}$ and assume f/g is a nonzerodivisor in $A_\mathfrak{p}$. Let $I = \{a \in A \mid af = 0\}$. Then we see that $I_\mathfrak{p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \notin \mathfrak{p}$. Hence f is a nonzerodivisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak{p}$. Thus f/g is an element of $\mathcal{S}_x$.

Proof of (2). Let $f \in \Gamma(X, \mathcal{K}_X)$ be a meromorphic function. Set $I = \{a \in A \mid af \in A\}$. Fix a prime $\mathfrak{p} \subset A$ corresponding to the point $x \in X$. By (1) we can write the image of f in the stalk at $\mathfrak{p}$ as a/b, $a, b \in A_\mathfrak{p}$ with $b \in A_\mathfrak{p}$ not a zerodivisor. Write $b = c/d$ with $c, d \in A$, $d \notin \mathfrak{p}$. Then $ad - cf$ is a section of $\mathcal{K}_X$ which vanishes in an open neighbourhood of x. Say it vanishes on $D(e)$ with $e \in A$, $e \notin \mathfrak{p}$. Then $e^n(ad - cf) = 0$ for some $n \gg 0$ by Lemma 31.24.1. Thus $e^n c \in I$ and $e^n c$ maps to a nonzerodivisor in $A_\mathfrak{p}$. Let $\mathrm{Ass}(A) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak{q}_i}$ and using Algebra, Lemma 10.63.15 the above says that $I \not\subset \mathfrak{q}_i$ for each i. By Algebra, Lemma 10.15.2 there exists an element $z \in I$, $z \notin \bigcup \mathfrak{q}_i$. By Algebra, Lemma 10.63.9 we see that z is not a zerodivisor on A. Hence $f = (zf)/z$ is an element of the total ring of fractions of A. This proves (2). □

0EMH **Lemma 31.24.3.** *Let X be a locally Noetherian scheme having no embedded points. Let X^0 be the set of generic points of irreducible components of X. Then we have*

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

(1) *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*

(2) *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

of meromorphic sections of $\mathcal{F}$ is quasi-coherent, and
(3) *the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. This lemma is a special case of Lemma 31.23.6 because in the locally Noetherian case weakly associated points are the same thing as associated points by Lemma 31.5.8. □

0EMI **Lemma 31.24.4.** *Let X be a locally Noetherian scheme having no embedded points. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ has a regular meromorphic section.*

Proof. For each generic point η of X pick a generator s_η of the free rank 1 module $\mathcal{L}_\eta$ over the artinian local ring $\mathcal{O}_{X,\eta}$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 31.24.3 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. □

02P2 **Lemma 31.24.5.** *Suppose given*

(1) *X a locally Noetherian scheme,*
(2) *$\mathcal{L}$ an invertible $\mathcal{O}_X$-module,*
(3) *s a regular meromorphic section of $\mathcal{L}$, and*
(4) *$\mathcal{F}$ coherent on X without embedded associated points and $Supp(\mathcal{F}) = X$.*

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 31.23.9. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzerodivisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \widetilde{I}$ (see proof of Lemma 31.23.9). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zerodivisor in A, and since M has support $\mathrm{Spec}(A)$ and no embedded primes we see that b is a nonzerodivisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzerodivisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. □

31.25. Meromorphic functions and sections; reduced case

0EMJ For a scheme which is reduced and which locally has finitely many irreducible components, the sheaf of meromorphic functions is quasi-coherent.

02OW **Lemma 31.25.1.** *Let X be a reduced scheme such that any quasi-compact open has a finite number of irreducible components. Let X^0 be the set of generic points of irreducible components of X. Then we have*

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta) = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta)$$

where $j_\eta : \mathrm{Spec}(\kappa(\eta)) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

(1) *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
(2) *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

of meromorphic sections of $\mathcal{F}$ is quasi-coherent,
(3) *$\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,*
(4) *$\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,*
(5) *$\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,*
(6) *the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. This lemma is a special case of Lemma 31.23.6 because on a reduced scheme the weakly associated points are the generic points by Lemma 31.5.12. □

035T **Lemma 31.25.2.** *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism*

$$\underline{\mathrm{Spec}}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 29.54.1) with the description of $\mathcal{K}_X$ in Lemma 31.25.1 above. □

01X5 **Lemma 31.25.3.** *Let X be an integral scheme with generic point η. We have*

(1) *the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 29.49.6) of X.*
(2) *for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.*

Proof. Omitted. □

In some cases we can show regular meromorphic sections exist.

02OZ **Lemma 31.25.4.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:*

(1) *X is integral,*
(2) *X is reduced and any quasi-compact open has a finite number of irreducible components,*
(3) *X is locally Noetherian and has no embedded points.*

Proof. In case (1) let $\eta \in X$ be the generic point. We have seen in Lemma 31.25.3 that $\mathcal{K}_X$, resp. $\mathcal{K}_X(\mathcal{L})$ is the constant sheaf with value $\kappa(\eta)$, resp. $\mathcal{L}_\eta$. Since $\dim_{\kappa(\eta)} \mathcal{L}_\eta = 1$ we can pick a nonzero element $s \in \mathcal{L}_\eta$. Clearly s is a regular meromorphic section of $\mathcal{L}$. In case (2) pick $s_\eta \in \mathcal{L}_\eta$ nonzero for all generic points η of X; this is possible as $\mathcal{L}_\eta$ is a 1-dimensional vector space over $\kappa(\eta)$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 31.25.1 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. Case (3) is Lemma 31.24.4. □

31.26. Weil divisors

0BE0 We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral schemes. Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. For quasi-compact schemes our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subschemes are locally finite.

0BE1 **Lemma 31.26.1.** *Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. The collection of irreducible components of Z is locally finite in X.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 28.5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 5.9.2). □

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 28.10.3.

0BE2 **Definition 31.26.2.** Let X be a locally Noetherian integral scheme.

(1) A *prime divisor* is an integral closed subscheme $Z \subset X$ of codimension 1.
(2) A *Weil divisor* is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{Z \mid n_Z \neq 0\}$ is locally finite (Topology, Definition 5.28.4).

The group of all Weil divisors on X is denoted $\mathrm{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we use the order of vanishing of a rational function along a prime divisor which is defined as follows.

02RJ **Definition 31.26.3.** Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the *order of vanishing of f along Z* as the integer

$$\mathrm{ord}_Z(f) = \mathrm{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 10.121.2 and ξ is the generic point of Z.

Note that for $f, g \in R(X)^*$ we have

$$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g).$$

Of course it can happen that $\mathrm{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\mathrm{ord}_Z(f) > 0$ is the *order of pole of f along Z*. It is important

to note that the condition $\text{ord}_Z(f) \geq 0$ is **not** equivalent to the condition $f \in \mathcal{O}_{X,\xi}$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

02RL **Lemma 31.26.4.** *Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. Then the collections*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } f \text{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \text{ a prime divisor and } ord_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 31.26.1 gives the desired result. □

This lemma allows us to make the following definition.

0BE3 **Definition 31.26.5.** Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. The *principal Weil divisor associated to f* is the Weil divisor

$$\text{div}(f) = \text{div}_X(f) = \sum \text{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\text{ord}_Z(f)$ is as in Definition 31.26.3. This makes sense by Lemma 31.26.4.

02RP **Lemma 31.26.6.** *Let X be a locally Noetherian integral scheme. Let $f, g \in R(X)^*$. Then*

$$div_X(fg) = div_X(f) + div_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. □

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

0BE4 **Definition 31.26.7.** Let X be a locally Noetherian integral scheme. The *Weil divisor class group* of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\text{Cl}(X)$.

By construction we obtain an exact complex

0BE5 (31.26.7.1) $$R(X)^* \xrightarrow{\text{div}} \text{Div}(X) \to \text{Cl}(X) \to 0$$

which we can think of as a presentation of $\text{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

31.27. The Weil divisor class associated to an invertible module

02SE In this section we go through exactly the same progression as in Section 31.26 to define a canonical map $\text{Pic}(X) \to \text{Cl}(X)$ on a locally Noetherian integral scheme.

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}_{X,\xi}^*$ such that $s_\xi = us'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in

the stalk at x can be written as $s = fs_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a regular meromorphic section, then actually $s/s_\xi \in R(X)^*$. On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section. Finally, we see that s/s_ξ is independent of the choice of s_ξ up to multiplication by a unit of the local ring $\mathcal{O}_{X,x}$. Putting everything together we see the following definition makes sense.

02SF **Definition 31.27.1.** Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ we define the *order of vanishing of s along Z* as the integer

$$\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 10.121.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

As in the case of principal divisors we have the following lemma.

02SG **Lemma 31.27.2.** *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } ord_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 31.26.1. gives the desired result. □

02SH **Lemma 31.27.3.** *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 31.27.2 guarantees that the sums are indeed Weil divisors. □

0BE6 **Definition 31.27.4.** Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\text{div}_\mathcal{L}(s) = \sum \text{ord}_{Z,\mathcal{L}}(s)[Z] \in \text{Div}(X)$$

where the sum is over prime divisors.

(2) We define *Weil divisor class associated to $\mathcal{L}$* as the image of $\text{div}_\mathcal{L}(s)$ in $\text{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 31.27.3.

As expected this construction is additive in the invertible module.

02SL **Lemma 31.27.5.** *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and*

$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$

in $Div(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that

$$\text{ord}_{Z,\mathcal{L}\otimes\mathcal{N}}(st) = \text{ord}_{Z,\mathcal{L}}(s) + \text{ord}_{Z,\mathcal{N}}(t)$$

by the additivity of the ord_Z function. □

In this way we obtain a homomorphism of abelian groups

0BE7 (31.27.5.1) $$\text{Pic}(X) \longrightarrow \text{Cl}(X)$$

which assigns to an invertible module its Weil divisor class.

0BE8 **Lemma 31.27.6.** *Let X be a locally Noetherian integral scheme. If X is normal, then the map (31.27.5.1) $\text{Pic}(X) \to Cl(X)$ is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\text{div}_{\mathcal{L}}(s) = \text{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\text{div}_{\mathcal{L}}(t) = 0$, see Lemma 31.27.3. We will show that t defines a trivialization of $\mathcal{L}$ which finishes the proof of the lemma. In order to prove this we may work locally on X. Hence we may assume that $X = \text{Spec}(A)$ is affine and that $\mathcal{L}$ is trivial. Then A is a Noetherian normal domain and t is an element of its fraction field such that $\text{ord}_{A_\mathfrak{p}}(t) = 0$ for all height 1 primes $\mathfrak{p}$ of A. Our goal is to show that t is a unit of A. Since $A_\mathfrak{p}$ is a discrete valuation ring for height one primes of A (Algebra, Lemma 10.157.4), the condition signifies that $t \in A_\mathfrak{p}^*$ for all primes $\mathfrak{p}$ of height 1. This implies $t \in A$ and $t^{-1} \in A$ by Algebra, Lemma 10.157.6 and the proof is complete. □

0BE9 **Lemma 31.27.7.** *Let X be a locally Noetherian integral scheme. Consider the map (31.27.5.1) $\text{Pic}(X) \to Cl(X)$. The following are equivalent*

(1) *the local rings of X are UFDs, and*
(2) *X is normal and $\text{Pic}(X) \to Cl(X)$ is surjective.*

In this case $\text{Pic}(X) \to Cl(X)$ is an isomorphism.

Proof. If (1) holds, then X is normal by Algebra, Lemma 10.120.11. Hence the map (31.27.5.1) is injective by Lemma 31.27.6. Moreover, every prime divisor $D \subset X$ is an effective Cartier divisor by Lemma 31.15.7. In this case the canonical section 1_D of $\mathcal{O}_X(D)$ (Definition 31.14.1) vanishes exactly along D and we see that the class of D is the image of $\mathcal{O}_X(D)$ under the map (31.27.5.1). Thus the map is surjective as well.

Assume (2) holds. Pick a prime divisor $D \subset X$. Since (31.27.5.1) is surjective there exists an invertible sheaf $\mathcal{L}$, a regular meromorphic section s, and $f \in R(X)^*$ such that $\mathrm{div}_{\mathcal{L}}(s) + \mathrm{div}(f) = [D]$. In other words, $\mathrm{div}_{\mathcal{L}}(fs) = [D]$. Let $x \in X$ and let $A = \mathcal{O}_{X,x}$. Thus A is a Noetherian local normal domain with fraction field $K = R(X)$. Every height 1 prime of A corresponds to a prime divisor on X and every invertible $\mathcal{O}_X$-module restricts to the trivial invertible module on $\mathrm{Spec}(A)$. It follows that for every height 1 prime $\mathfrak{p} \subset A$ there exists an element $f \in K$ such that $\mathrm{ord}_{A_\mathfrak{p}}(f) = 1$ and $\mathrm{ord}_{A_{\mathfrak{p}'}}(f) = 0$ for every other height one prime $\mathfrak{p}'$. Then $f \in A$ by Algebra, Lemma 10.157.6. Arguing in the same fashion we see that every element $g \in \mathfrak{p}$ is of the form $g = af$ for some $a \in A$. Thus we see that every height one prime ideal of A is principal and A is a UFD by Algebra, Lemma 10.120.6. □

31.28. More on invertible modules

0BD6 In this section we discuss some properties of invertible modules.

0BD7 **Lemma 31.28.1.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that*

(1) *X is locally Noetherian,*
(2) *Y is locally Noetherian, integral, and normal,*
(3) *φ is flat with integral (hence nonempty) fibres,*
(4) *φ is either quasi-compact or locally of finite type,*
(5) *$\mathcal{L}$ is trivial when restricted to the generic fibre of φ.*

Then $\mathcal{L} \cong \varphi^*\mathcal{N}$ for some invertible $\mathcal{O}_Y$-module $\mathcal{N}$.

Proof. Let $\xi \in Y$ be the generic point. Let X_ξ be the scheme theoretic fibre of φ over ξ. Denote $\mathcal{L}_\xi$ the pullback of $\mathcal{L}$ to X_ξ. Assumption (5) means that $\mathcal{L}_\xi$ is trivial. Choose a trivializing section $s \in \Gamma(X_\xi, \mathcal{L}_\xi)$. Observe that X is integral by Lemma 31.11.7. Hence we can think of s as a regular meromorphic section of $\mathcal{L}$. Pullbacks of meromorphic functions are defined for φ by Lemma 31.23.5. Let $\mathcal{N} \subset \mathcal{K}_Y$ be the $\mathcal{O}_Y$-module whose sections over an open $V \subset Y$ are those meromorphic functions $g \in \mathcal{K}_Y(V)$ such that $\varphi^*(g)s \in \mathcal{L}(\varphi^{-1}V)$. A priori $\varphi^*(g)s$ is a section of $\mathcal{K}_X(\mathcal{L})$ over $\varphi^{-1}V$. We claim that $\mathcal{N}$ is an invertible $\mathcal{O}_Y$-module and that the map

$$\varphi^*\mathcal{N} \longrightarrow \mathcal{L}, \quad g \longmapsto gs$$

is an isomorphism.

We first prove the claim in the following situation: X and Y are affine and $\mathcal{L}$ trivial. Say $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$ and s given by the element $s \in A \otimes_R K$ where K is the fraction field of R. We can write $s = a/r$ for some nonzero $r \in R$ and $a \in A$. Since s generates $\mathcal{L}$ on the generic fibre we see that there exists an $s' \in A \otimes_R K$ such that $ss' = 1$. Thus we see that $s = r'/a'$ for some nonzero $r' \in R$ and $a' \in A$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset R$ be the minimal primes over rr'. Each $R_{\mathfrak{p}_i}$ is a discrete valuation ring (Algebra, Lemmas 10.60.11 and 10.157.4). By assumption $\mathfrak{q}_i = \mathfrak{p}_i A$ is a prime. Hence $\mathfrak{q}_i A_{\mathfrak{q}_i}$ is generated by a single element and we find that $A_{\mathfrak{q}_i}$ is a discrete valuation ring as well (Algebra, Lemma 10.119.7). Of course $R_{\mathfrak{p}_i} \to A_{\mathfrak{q}_i}$ has ramification index 1. Let $e_i, e_i' \geq 0$ be the valuation of a, a' in $A_{\mathfrak{q}_i}$. Then $e_i + e_i'$ is the valuation of rr' in $R_{\mathfrak{p}_i}$. Note that

$$\mathfrak{p}_1^{(e_1+e_1')} \cap \ldots \cap \mathfrak{p}_i^{(e_n+e_n')} = (rr')$$

in R by Algebra, Lemma 10.157.6. Set

$$I = \mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)} \quad \text{and} \quad I' = \mathfrak{p}_1^{(e_1')} \cap \ldots \cap \mathfrak{p}_i^{(e_n')}$$

so that $II' \subset (rr')$. Observe that

$$IA = (\mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)})A = (\mathfrak{p}_1 A)^{(e_1)} \cap \ldots \cap (\mathfrak{p}_i A)^{(e_n)}$$

by Algebra, Lemmas 10.64.3 and 10.39.2. Similarly for $I'A$. Hence $a \in IA$ and $a' \in I'A$. We conclude that $IA \otimes_A I'A \to rr'A$ is surjective. By faithful flatness of $R \to A$ we find that $I \otimes_R I' \to (rr')$ is surjective as well. It follows that $II' = (rr')$ and I and I' are finite locally free of rank 1, see Algebra, Lemma 10.120.16. Thus Zariski locally on R we can write $I = (g)$ and $I' = (g')$ with $gg' = rr'$. Then $a = ug$ and $a' = u'g'$ for some $u, u' \in A$. We conclude that u, u' are units. Thus Zariski locally on R we have $s = ug/r$ and the claim follows in this case.

Let $y \in Y$ be a point. Pick $x \in X$ mapping to y. We may apply the result of the previous paragraph to $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$. We conclude there exists an element $g \in R(Y)^*$ well defined up to multiplication by an element of $\mathcal{O}_{Y,y}^*$ such that $\varphi^*(g)s$ generates $\mathcal{L}_x$. Hence $\varphi^*(g)s$ generates $\mathcal{L}$ in a neighbourhood U of x. Suppose x' is a second point lying over y and $g' \in R(Y)^*$ is such that $\varphi^*(g')s$ generates $\mathcal{L}$ in an open neighbourhood U' of x'. Then we can choose a point x'' in $U \cap U' \cap \varphi^{-1}(\{y\})$ because the fibre is irreducible. By the uniqueness for the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x''}$ we find that g and g' differ (multiplicatively) by an element in $\mathcal{O}_{Y,y}^*$. Hence we see that $\varphi^*(g)s$ is a generator for $\mathcal{L}$ on an open neighbourhood of $\varphi^{-1}(y)$. Let $Z \subset X$ be the set of points $z \in X$ such that $\varphi^*(g)s$ does not generate $\mathcal{L}_z$. The arguments above show that Z is closed and that $Z = \varphi^{-1}(T)$ for some subset $T \subset Y$ with $y \notin T$. If we can show that T is closed, then g will be a generator for $\mathcal{N}$ as an $\mathcal{O}_Y$-module in the open neighbourhood $Y \setminus T$ of y thereby finishing the proof (some details omitted).

If φ is quasi-compact, then T is closed by Morphisms, Lemma 29.25.12. If φ is locally of finite type, then φ is open by Morphisms, Lemma 29.25.10. Then $Y \setminus T$ is open as the image of the open $X \setminus Z$. □

0BD8 **Lemma 31.28.2.** *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $D \subset U$ be an effective Cartier divisor. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D' \subset X$ with $D = U \cap D'$.*

Proof. Let $D' \subset X$ be the scheme theoretic image of the morphism $D \to X$. Since X is locally Noetherian the morphism $D \to X$ is quasi-compact, see Properties, Lemma 28.5.3. Hence the formation of D' commutes with passing to opens in X by Morphisms, Lemma 29.6.3. Thus we may assume $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal corresponding to D'. Let $\mathfrak{p} \subset A$ be a prime ideal corresponding to a point of $X \setminus U$. To finish the proof it is enough to show that $I_\mathfrak{p}$ is generated by one element, see Lemma 31.15.2. Thus we may replace X by $\mathrm{Spec}(A_\mathfrak{p})$, see Morphisms, Lemma 29.25.16. In other words, we may assume that X is the spectrum of a local UFD A. Then all local rings of A are UFD's. It follows that $D = \sum a_i D_i$ with $D_i \subset U$ an integral effective Cartier divisor, see Lemma 31.15.11. The generic points ξ_i of D_i correspond to prime ideals $\mathfrak{p}_i \subset A$ of height 1, see Lemma 31.15.3. Then $\mathfrak{p}_i = (f_i)$ for some prime element $f_i \in A$ and we conclude that D' is cut out by $\prod f_i^{a_i}$ as desired. □

0BD9 **Lemma 31.28.3.** *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}'$ with $\mathcal{L} \cong \mathcal{L}'|_U$.*

Proof. Choose $x \in X$, $x \notin U$. We will show there exists an affine open neighbourhood $W \subset X$, such that $\mathcal{L}|_{W \cap U}$ extends to an invertible sheaf on W. This implies by glueing of sheaves (Sheaves, Section 6.33) that we can extend $\mathcal{L}$ to the strictly bigger open $U \cup W$. Let $W = \mathrm{Spec}(A)$ be an affine open neighbourhood. Since $U \cap W$ is quasi-affine, we see that we can write $\mathcal{L}|_{W \cap U}$ as $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2)^{\otimes -1}$ for some effective Cartier divisors $D_1, D_2 \subset W \cap U$, see Lemma 31.15.12. Then D_1 and D_2 extend to effective Cartier divisors of W by Lemma 31.28.2 which gives us the extension of the invertible sheaf.

If X is Noetherian (which is the case most used in practice), the above combined with Noetherian induction finishes the proof. In the general case we argue as follows. First, because every local ring of a point outside of U is a domain and X is locally Noetherian, we see that the closure of U in X is open. Thus we may assume that $U \subset X$ is dense and schematically dense. Now we consider the set T of triples $(U', \mathcal{L}', \alpha)$ where $U \subset U' \subset X$ is an open subscheme, $\mathcal{L}'$ is an invertible $\mathcal{O}_{U'}$-module, and $\alpha : \mathcal{L}'|_U \to \mathcal{L}$ is an isomorphism. We endow T with a partial ordering $\leq$ defined by the rule $(U', \mathcal{L}', \alpha) \leq (U'', \mathcal{L}'', \alpha')$ if and only if $U' \subset U''$ and there exists an isomorphism $\beta : \mathcal{L}''|_{U'} \to \mathcal{L}'$ compatible with α and α'. Observe that β is unique (if it exists) because $U \subset X$ is dense. The first part of the proof shows that for any element $t = (U', \mathcal{L}', \alpha)$ of T with $U' \neq X$ there exists a $t' \in T$ with $t' > t$. Hence to finish the proof it suffices to show that Zorn's lemma applies. Thus consider a totally ordered subset $I \subset T$. If $i \in I$ corresponds to the triple $(U_i, \mathcal{L}_i, \alpha_i)$, then we can construct an invertible module $\mathcal{L}'$ on $U' = \bigcup U_i$ as follows. For $W \subset U'$ open and quasi-compact we see that $W \subset U_i$ for some i and we set

$$\mathcal{L}'(W) = \mathcal{L}_i(W)$$

For the transition maps we use the β's (which are unique and hence compose correctly). This defines an invertible $\mathcal{O}$-module $\mathcal{L}'$ on the basis of quasi-compact opens of U' which is sufficient to define an invertible module (Sheaves, Section 6.30). We omit the details. □

0BDA **Lemma 31.28.4.** *Let R be a UFD. The Picard groups of the following are trivial.*

(1) $\mathrm{Spec}(R)$ *and any open subscheme of it.*
(2) $\mathbf{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n])$ *and any open subscheme of it.*

In particular, the Picard group of any open subscheme of affine n-space $\mathbf{A}^n_k$ over a field k is trivial.

Proof. Any polynomial ring over R is a UFD by Algebra, Lemma 10.120.10. Thus the question is to show that $\mathrm{Pic}(U)$ is trivial if $U \subset \mathrm{Spec}(R)$ is a (nonempty) open.

In the Noetherian case, we may use Lemma 31.28.3 to reduce to showing $\mathrm{Pic}(\mathrm{Spec}(R))$ is trivial. This in turn is proved in More on Algebra, Lemma 15.118.3. We urge the reader to skip the rest of the proof.

Let $U \subset \mathrm{Spec}(R)$ be a nonempty open. Choose a nonzero $f \in R$ with $D(f) \subset U$. Since f is a product of prime elements of R we conclude that there are finitely many prime elements $a_1, \ldots, a_n \in R$ such that $(a_i) \notin U$ (or equivalentaly $U \cap V(a_i) = \emptyset$). After replacing R by $R_{a_1 \ldots a_n}$ we may assume that U contains all principal prime

ideals. (Note that a localization of a UFD is a UFD, for example by Algebra, Lemma 10.120.7.)

Assume U contains all principal prime ideals. Write $U = \mathrm{Spec}(R) \setminus V(I)$ for some ideal $I \subset R$. Pick $f \in I$ nonzero and write $f = a_1 \ldots a_n$ as a product of prime elements. Since $(a_i) \notin V(I)$ we may choose $g \in I$ with $g \notin (a_i)$ for all i (prime avoidance). Picture

$$U' = D(f) \cup D(g) \subset U \subset \mathrm{Spec}(R)$$

By More on Algebra, Lemma 15.118.3 the Picard groups of $D(f)$ and $D(g)$ are trivial and by an elementary argument about UFDs the map $R_f^* \times R_g^* \to R_{fg}^*$, $(u, v) \mapsto uv^{-1}$ is surjective. We conclude that $\mathrm{Pic}(U')$ is trivial.

Thus it suffices to show that an invertible module $\mathcal{L}$ on U whose pullback by $j : U' \to U'$ is trivial, is trivial. To do this it suffices to show that the map $\mathcal{L} \to j_* j^* \mathcal{L}$ is an isomorphism. For this in turn, let $u \in U \setminus U'$. Then $f, g \in \mathfrak{m}_u \subset \mathcal{O}_{U,u} = A$ are elements of a local UFD which do not have a prime factor in common. The reader easily shows that this means that

$$0 \to A \to A_f \times A_g \to A_{fg}$$

is exact by looking at prime factors. Since $A = \mathcal{L}_u$ because $\mathcal{L}$ is invertible and since $(j_* j^* \mathcal{L})_u$ is the kernel of $A_f \times A_g \to A_{fg}$ we conclude. □

0BXJ **Lemma 31.28.5.** *Let R be a UFD. The Picard group of $\mathbf{P}^n_R$ is $\mathbf{Z}$. More precisely, there is an isomorphism*

$$\mathbf{Z} \longrightarrow \mathrm{Pic}(\mathbf{P}^n_R), \quad m \longmapsto \mathcal{O}_{\mathbf{P}^n_R}(m)$$

In particular, the Picard group of $\mathbf{P}^n_k$ of projective space over a field k is $\mathbf{Z}$.

Proof. By Lemma 31.28.4 the Picard groups of the opens $D_+(T_i) \cong \mathbf{A}^n_R$ are trivial. Thus if $\mathcal{L}$ is an invertible module on $\mathbf{P}^n_R$, then it is given by a 1-cocycle with values in the sheaf of invertible functors $\mathcal{O}^*$ for the open covering

$$\mathbf{P}^n_R = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

Observe that for $i \neq j$ we have

$$\mathcal{O}^*(D_+(T_i) \cap D_+(T_j)) = \mathcal{O}^*(D_+(T_i T_j)) = \left(R[T_0, \ldots, T_n]_{(T_i T_j)}\right)^* = R^* \times (T_i/T_j)^{\mathbf{Z}}$$

Thus such a cocycle (g_{ij}) is given by units

$$g_{ij} = u_{ij}(T_i/T_j)^{e_{ij}}$$

with $u_{ij} \in R^*$ and $e_{ij} \in \mathbf{Z}$ satisfying the cocycle condition. The cocycle condition over $D_+(T_i T_j T_k)$ for $\#\{i, j, k\} = 3$ tell us that

$$u_{ik} = u_{ij} u_{jk} \quad \text{and} \quad e_{ik} = e_{ij} = e_{jk}$$

Whence $u_{ij} = u_{i1} u_{j1}^{-1}$ is a boundary. Thus all isomorphism classes of invertible modules are given by taking the cocycle with

$$g_{ij} = (T_i/T_j)^e$$

for some $e \in \mathbf{Z}$. Since $\mathcal{O}(n)$ has trivializing section T_i^n over $D_+(T_i)$ we see that the corresponding cocycle of $\mathcal{O}(n)$ is $(T_i/T_j)^n$ and the proof is complete. □

31.29. Weil divisors on normal schemes

0EBK First we discuss properties of reflexive modules.

0EBL **Lemma 31.29.1.** *Let X be an integral locally Noetherian normal scheme. For $\mathcal{F}$ and $\mathcal{G}$ coherent reflexive $\mathcal{O}_X$-modules the map*

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G})^{**} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism. The rule $\mathcal{F}, \mathcal{G} \mapsto (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$ defines an abelian group law on the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules.

Proof. Although not strictly necessary, we recommend reading Remark 31.12.9 before proceeding with the proof. Choose an open subscheme $j : U \to X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $j^*\mathcal{F}$ and $j^*\mathcal{G}$ are finite locally free, see Lemma 31.12.13. The map

$$\mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, \mathcal{O}_U) \otimes_{\mathcal{O}_U} j^*\mathcal{G} \to \mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, j^*\mathcal{G})$$

is an isomorphism, because we may check it locally and it is clear when the modules are finite free. Observe that j^* applied to the displayed arrow of the lemma gives the arrow we've just shown is an isomorphism (small detail omitted). Since j^* defines an equivalence between coherent reflexive modules on U and coherent reflexive modules on X (by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5), we conclude that the arrow of the lemma is an isomorphism too. If $\mathcal{F}$ has rank 1, then $j^*\mathcal{F}$ is an invertible $\mathcal{O}_U$-module and the reflexive module $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ restricts to its inverse. It follows in the same manner as before that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee)^{**} = \mathcal{O}_X$. In this way we see that we have inverses for the group law given in the statement of the lemma. □

0EBM **Lemma 31.29.2.** *Let X be an integral locally Noetherian normal scheme. The group of rank 1 coherent reflexive $\mathcal{O}_X$-modules is isomorphic to the Weil divisor class group $Cl(X)$ of X.*

Proof. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Choose an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $\mathcal{F}|_U$ is invertible, see Lemma 31.12.13. Observe that $\mathrm{Cl}(U) = \mathrm{Cl}(X)$ as the Weil divisor class group of X only depends on its field of rational functions and the points of codimension 1 and their local rings. Thus we can define the Weil divisor class of $\mathcal{F}$ to be the Weil divisor class of $\mathcal{F}|_U$ in $\mathrm{Cl}(U)$. We omit the verification that this is independent of the choice of U.

Denote $\mathrm{Cl}'(X)$ the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules. The construction above gives a group homorphism

$$\mathrm{Cl}'(X) \longrightarrow \mathrm{Cl}(X)$$

because for any pair $\mathcal{F}, \mathcal{G}$ of elements of $\mathrm{Cl}'(X)$ we can choose a U which works for both and the assignment (31.27.5.1) sending an invertible module to its Weil divisor class is a homorphism. If $\mathcal{F}$ is in the kernel of this map, then we find that $\mathcal{F}|_U$ is trivial (Lemma 31.27.6) and hence $\mathcal{F}$ is trivial too by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. To finish the proof it suffices to check the map is surjective.

Let $D = \sum n_Z Z$ be a Weil divisor on X. We claim that there is an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and

such that $Z|_U$ is an effective Cartier divisor for $n_Z \neq 0$. To prove the claim we may assume X is affine. Then we may assume $D = n_1Z_1 + \ldots + n_rZ_r$ is a finite sum with $Z_1, \ldots, Z_r$ pairwise distinct. After throwing out $Z_i \cap Z_j$ for $i \neq j$ we may assume $Z_1, \ldots, Z_r$ are pairwise disjoint. This reduces us to the case of a single prime divisor Z on X. As X is (R_1) by Properties, Lemma 28.12.5 the local ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of Z is a discrete valuation ring. Let $f \in \mathcal{O}_{X,\xi}$ be a uniformizer. Let $V \subset X$ be an open neighbourhood of ξ such that f is the image of an element $f \in \mathcal{O}_X(V)$. After shrinking V we may assume that $Z \cap V = V(f)$ scheme theoretically, since this is true in the local ring at ξ. In this case taking

$$U = X \setminus (Z \setminus V) = (X \setminus Z) \cup V$$

gives the desired open, thereby proving the claim.

In order to show that the divisor class of D is in the image, we may write $D = \sum_{n_Z<0} n_Z Z - \sum_{n_Z>0}(-n_Z)Z$. By additivity of the map constructed above, we may and do assume $n_Z \leq 0$ for all prime divisors Z (this step may be avoided if the reader so desires). Let $U \subset X$ be as in the claim above. If U is quasi-compact, then we write $D|_U = -n_1Z_1 - \ldots - n_rZ_r$ for pairwise distinct prime divisors Z_i and $n_i > 0$ and we consider the invertible $\mathcal{O}_U$-module

$$\mathcal{L} = \mathcal{I}_1^{n_1} \ldots \mathcal{I}_r^{n_r} \subset \mathcal{O}_U$$

where $\mathcal{I}_i$ is the ideal sheaf of Z_i. This is invertible by our choice of U and Lemma 31.13.7. Also $\mathrm{div}_{\mathcal{L}}(1) = D|_U$. Since $\mathcal{L} = \mathcal{F}|_U$ for some rank 1 coherent reflexive $\mathcal{O}_X$-module $\mathcal{F}$ by Lemma 31.12.12 we find that D is in the image of our map.

If U is not quasi-compact, then we define $\mathcal{L} \subset \mathcal{O}_U$ locally by the displayed formula above. The reader shows that the construction glues and finishes the proof exactly as before. Details omitted. □

0EBN **Lemma 31.29.3.** *Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{F})$. Let*

$$U = \{x \in X \mid s : \mathcal{O}_{X,x} \to \mathcal{F}_x \text{ is an isomorphism}\}$$

Then $j : U \to X$ is an open subscheme of X and

$$j_*\mathcal{O}_U = \mathrm{colim}(\mathcal{O}_X \xrightarrow{s} \mathcal{F} \xrightarrow{s} \mathcal{F}^{[2]} \xrightarrow{s} \mathcal{F}^{[3]} \xrightarrow{s} \ldots)$$

where $\mathcal{F}^{[1]} = \mathcal{F}$ and inductively $\mathcal{F}^{[n+1]} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{[n]})^{**}$.

Proof. The set U is open by Modules, Lemmas 17.9.4 and 17.12.6. Observe that j is quasi-compact by Properties, Lemma 28.5.3. To prove the final statement it suffices to show for every quasi-compact open $W \subset X$ there is an isomorphism

$$\mathrm{colim}\, \Gamma(W, \mathcal{F}^{[n]}) \longrightarrow \Gamma(U \cap W, \mathcal{O}_U)$$

of $\mathcal{O}_X(W)$-modules compatible with restriction maps. We will omit the verification of compatibilities. After replacing X by W and rewriting the above in terms of homs, we see that it suffices to construct an isomorphism

$$\mathrm{colim}\, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^{[n]}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U)$$

Choose an open $V \subset X$ such that every irreducible component of $X \setminus V$ has codimension ≥ 2 in X and such that $\mathcal{F}|_V$ is invertible, see Lemma 31.12.13. Then restriction defines an equivalence of categories between rank 1 coherent reflexive modules on X and V and between rank 1 coherent reflexive modules on U and

$V \cap U$. See Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. Thus it suffices to construct an isomorphism

$$\operatorname{colim} \Gamma(V, (\mathcal{F}|_V)^{\otimes n}) \longrightarrow \Gamma(V \cap U, \mathcal{O}_U)$$

Since $\mathcal{F}|_V$ is invertible and since $U \cap V$ is equal to the set of points where $s|_V$ generates this invertible module, this is a special case of Properties, Lemma 28.17.2 (there is an explicit formula for the map as well). □

0EBP **Lemma 31.29.4.** *Assumptions and notation as in Lemma 31.29.3. If s is nonzero, then every irreducible component of $X \setminus U$ has codimension 1 in X.*

Proof. Let $\xi \in X$ be a generic point of an irreducible component Z of $X \setminus U$. After replacing X by an open neighbourhood of ξ we may assume that $Z = X \setminus U$ is irreducible. Since $s : \mathcal{O}_U \to \mathcal{F}|_U$ is an isomorphism, if the codimension of Z in X is ≥ 2, then $s : \mathcal{O}_X \to \mathcal{F}$ is an isomorphism by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. This would mean that $Z = \emptyset$, a contradiction. □

0EBQ **Remark 31.29.5.** Let A be a Noetherian normal domain. Let M be a rank 1 finite reflexive A-module. Let $s \in M$ be nonzero. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height 1 primes of A in the support of M/As. Then the open U of Lemma 31.29.3 is

$$U = \operatorname{Spec}(A) \setminus (V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r))$$

by Lemma 31.29.4. Moreover, if $M^{[n]}$ denotes the reflexive hull of $M \otimes_A \ldots \otimes_A M$ (n-factors), then

$$\Gamma(U, \mathcal{O}_U) = \operatorname{colim} M^{[n]}$$

according to Lemma 31.29.3.

0EBR **Lemma 31.29.6.** *Assumptions and notation as in Lemma 31.29.3. The following are equivalent*

(1) *the inclusion morphism $j : U \to X$ is affine, and*
(2) *for every $x \in X \setminus U$ there is an $n > 0$ such that $s^n \in \mathfrak{m}_x \mathcal{F}_x^{[n]}$.*

Proof. Assume (1). Then for $x \in X \setminus U$ the inverse image U_x of U under the canonical morphism $f_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is affine and does not contain x. Thus $\mathfrak{m}_x \Gamma(U_x, \mathcal{O}_{U_x})$ is the unit ideal. In particular, we see that we can write

$$1 = \sum f_i g_i$$

with $f_i \in \mathfrak{m}_x$ and $g_i \in \Gamma(U_x, \mathcal{O}_{U_x})$. By Lemma 31.29.3 we have $\Gamma(U_x, \mathcal{O}_{U_x}) = \operatorname{colim} \mathcal{F}_x^{[n]}$ with transition maps given by multiplication by s. Hence for some $n > 0$ we have

$$s^n = \sum f_i t_i$$

for some $t_i = s^n g_i \in \mathcal{F}_x^{[n]}$. Thus (2) holds.

Conversely, assume that (2) holds. To prove j is affine is local on X, see Morphisms, Lemma 29.11.3. Thus we may and do assume that X is affine. Our goal is to show that U is affine. By Cohomology of Schemes, Lemma 30.17.8 it suffices to show that $H^p(U, \mathcal{O}_U) = 0$ for $p > 0$. Since $H^p(U, \mathcal{O}_U) = H^0(X, R^p j_* \mathcal{O}_U)$ (Cohomology of Schemes, Lemma 30.4.6) and since $R^p j_* \mathcal{O}_U$ is quasi-coherent (Cohomology of

Schemes, Lemma 30.4.5) it is enough to show the stalk $(R^p j_* \mathcal{O}_U)_x$ at a point $x \in X$ is zero. Consider the base change diagram

$$\begin{array}{ccc} U_x & \longrightarrow & U \\ {\scriptstyle j_x}\downarrow & & \downarrow{\scriptstyle j} \\ \mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \end{array}$$

By Cohomology of Schemes, Lemma 30.5.2 we have $(R^p j_* \mathcal{O}_U)_x = R^p j_{x,*} \mathcal{O}_{U_x}$. Hence we may assume X is local with closed point x and we have to show U is affine (because this is equivalent to the desired vanishing by the reference given above). In particular $d = \dim(X)$ is finite (Algebra, Proposition 10.60.9). If $x \in U$, then $U = X$ and the result is clear. If $d = 0$ and $x \notin U$, then $U = \emptyset$ and the result is clear. Now assume $d > 0$ and $x \notin U$. Since $j_* \mathcal{O}_U = \mathrm{colim}\, \mathcal{F}^{[n]}$ our assumption means that we can write

$$1 = \sum f_i g_i$$

for some $n > 0$, $f_i \in \mathfrak{m}_x$, and $g_i \in \mathcal{O}(U)$. By induction on d we know that $D(f_i) \cap U$ is affine for all i: going through the whole argument just given with X replaced by $D(f_i)$ we end up with Noetherian local rings whose dimension is strictly smaller than d. Hence U is affine by Properties, Lemma 28.27.3 as desired. □

31.30. Relative Proj

07ZW Some results on relative Proj. First some very basic results. Recall that a relative Proj is always separated over the base, see Constructions, Lemma 27.16.9.

07ZX **Lemma 31.30.1.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If one of the following holds*

1. *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
2. *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
3. *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then p is quasi-compact.

Proof. The question is local on the base, see Schemes, Lemma 26.19.2. Thus we may assume S is affine. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. Then $X = \mathrm{Proj}(A)$, see Constructions, Section 27.15. In case (1) we may after possibly localizing more assume that A is generated by homogeneous elements $f_1, \ldots, f_n \in A_+$ over A_0. Then $A_+ = (f_1, \ldots, f_n)$ by Algebra, Lemma 10.58.1. In case (3) we see that $\mathcal{F} = \widetilde{M}$ for some finite type A_0-module $M \subset A_+$. Say $M = \sum A_0 f_i$. Say $f_i = \sum f_{i,j}$ is the decomposition into homogeneous pieces. The condition in (3) signifies that $A_+ \subset \sqrt{(f_{i,j})}$. Thus in both cases we conclude that $\mathrm{Proj}(A)$ is quasi-compact by Constructions, Lemma 27.8.9. Finally, (2) follows from (1). □

07ZY **Lemma 31.30.2.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_S$-algebras, then p is of finite type and $\mathcal{O}_X(d)$ is a finite type $\mathcal{O}_X$-module.*

Proof. The assumption implies that p is quasi-compact, see Lemma 31.30.1. Hence it suffices to show that p is locally of finite type. Thus the question is local on the base and target, see Morphisms, Lemma 29.15.2. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. After further localizing on S we may assume that A is a finite type R-algebra. The scheme X is constructed out of glueing the spectra of the rings $A_{(f)}$ for $f \in A_+$ homogeneous. Each of these is of finite type over R by Algebra, Lemma 10.57.9 part (1). Thus $\mathrm{Proj}(A)$ is of finite type over R. To see the statement on $\mathcal{O}_X(d)$ use part (2) of Algebra, Lemma 10.57.9. □

07ZZ **Lemma 31.30.3.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{O}_S \to \mathcal{A}_0$ is an integral algebra map[4] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then p is universally closed.*

Proof. The question is local on the base. Thus we may assume that $X = \mathrm{Spec}(R)$ is affine. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_X$-algebra associated to the graded R-algebra A. The assumption is that $R \to A_0$ is integral and A is of finite type over A_0. Write $X \to \mathrm{Spec}(R)$ as the composition $X \to \mathrm{Spec}(A_0) \to \mathrm{Spec}(R)$. Since $R \to A_0$ is an integral ring map, we see that $\mathrm{Spec}(A_0) \to \mathrm{Spec}(R)$ is universally closed, see Morphisms, Lemma 29.44.7. The quasi-compact (see Constructions, Lemma 27.8.9) morphism

$$X = \mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$$

satisfies the existence part of the valuative criterion by Constructions, Lemma 27.8.11 and hence it is universally closed by Schemes, Proposition 26.20.6. Thus $X \to \mathrm{Spec}(R)$ is universally closed as a composition of universally closed morphisms. □

0800 **Lemma 31.30.4.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. The following conditions are equivalent*

(1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
(2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_S$-algebra*

If these conditions hold, then p is locally projective and in particular proper.

Proof. Assume that $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module. Choose an affine open $U = \mathrm{Spec}(R) \subset X$ such that $\mathcal{A}$ corresponds to a graded R-algebra A with A_0 a finite R-module. Condition (1) means that (after possibly localizing further on S) that A is a finite type A_0-algebra and condition (2) means that (after possibly localizing further on S) that A is a finite type R-algebra. Thus these conditions imply each other by Algebra, Lemma 10.6.2.

A locally projective morphism is proper, see Morphisms, Lemma 29.43.5. Thus we may now assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Proj}(A)$ and that A_0 is finite over R and A of finite type over R. We will show that $X = \mathrm{Proj}(A) \to \mathrm{Spec}(R)$ is projective. We urge the reader to prove this for themselves, by directly constructing a closed immersion of X into a projective space over R, instead of reading the argument we give below.

[4] In other words, the integral closure of $\mathcal{O}_S$ in $\mathcal{A}_0$, see Morphisms, Definition 29.53.2, equals $\mathcal{A}_0$.

By Lemma 31.30.2 we see that X is of finite type over $\mathrm{Spec}(R)$. Constructions, Lemma 27.10.6 tells us that $\mathcal{O}_X(d)$ is ample on X for some $d \geq 1$ (see Properties, Section 28.26). Hence $X \to \mathrm{Spec}(R)$ is quasi-projective (by Morphisms, Definition 29.40.1). By Morphisms, Lemma 29.43.12 we conclude that X is isomorphic to an open subscheme of a scheme projective over $\mathrm{Spec}(R)$. Therefore, to finish the proof, it suffices to show that $X \to \mathrm{Spec}(R)$ is universally closed (use Morphisms, Lemma 29.41.7). This follows from Lemma 31.30.3. □

0B3U **Lemma 31.30.5.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ and $\mathcal{A}_1$ is a finite type $\mathcal{O}_S$-module, then p is projective.*

Proof. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$, see Constructions, Lemma 27.18.5. □

0D4C **Lemma 31.30.6.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}_d$ is a flat $\mathcal{O}_S$-module for $d \gg 0$, then p is flat and $\mathcal{O}_X(d)$ is flat over S.*

Proof. Affine locally flatness of X over S reduces to the following statement: Let R be a ring, let A be a graded R-algebra with A_d flat over R for $d \gg 0$, let $f \in A_d$ for some $d > 0$, then $A_{(f)}$ is flat over R. Since $A_{(f)} = \mathrm{colim}\, A_{nd}$ where the transition maps are given by multiplication by f, this follows from Algebra, Lemma 10.39.3. Argue similarly to get flatness of $\mathcal{O}_X(d)$ over S. □

0D4D **Lemma 31.30.7.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is a finitely presented $\mathcal{O}_S$-algebra, then p is of finite presentation and $\mathcal{O}_X(d)$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Affine locally this reduces to the following statement: Let R be a ring and let A be a finitely presented graded R-algebra. Then $\mathrm{Proj}(A) \to \mathrm{Spec}(R)$ is of finite presentation and $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ is a $\mathcal{O}_{\mathrm{Proj}(A)}$-module of finite presentation. The finite presentation condition implies we can choose a presentation

$$A = R[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$$

where $R[X_1, \ldots, X_n]$ is a polynomial ring graded by giving weights d_i to X_i and $F_1, \ldots, F_m$ are homogeneous polynomials of degree e_j. Let $R_0 \subset R$ be the subring generated by the coefficients of the polynomials $F_1, \ldots, F_m$. Then we set $A_0 = R_0[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$. By construction $A = A_0 \otimes_{R_0} R$. Thus by Constructions, Lemma 27.11.6 it suffices to prove the result for $X_0 = \mathrm{Proj}(A_0)$ over R_0. By Lemma 31.30.2 we know X_0 is of finite type over R_0 and $\mathcal{O}_{X_0}(d)$ is a quasi-coherent $\mathcal{O}_{X_0}$-module of finite type. Since R_0 is Noetherian (as a finitely generated $\mathbf{Z}$-algebra) we see that X_0 is of finite presentation over R_0 (Morphisms, Lemma 29.21.9) and $\mathcal{O}_{X_0}(d)$ is of finite presentation by Cohomology of Schemes, Lemma 30.9.1. This finishes the proof. □

31.31. Closed subschemes of relative proj

084M Some auxiliary lemmas about closed subschemes of relative proj.

0801 **Lemma 31.31.1.** *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} p_* \left((i_* \mathcal{O}_Z)(d) \right).$$

If p is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{I})$.

Proof. The morphism p is separated by Constructions, Lemma 27.16.9. As p is quasi-compact, p_* transforms quasi-coherent modules into quasi-coherent modules, see Schemes, Lemma 26.24.1. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_S$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. The functoriality morphism $Z' = \underline{\mathrm{Proj}}_S(\mathcal{B}) \to \underline{\mathrm{Proj}}_S(\mathcal{A})$ is everywhere defined and a closed immersion, see Constructions, Lemma 27.18.3. Hence it suffices to prove $Z = Z'$ as closed subschemes of X.

Having said this, the question is local on the base and we may assume that $S = \mathrm{Spec}(R)$ and that $X = \mathrm{Proj}(A)$ for some graded R-algebra A. Assume $\mathcal{I} = \widetilde{I}$ for $I \subset A$ a graded ideal. By Constructions, Lemma 27.8.9 there exist $f_0, \ldots, f_n \in A_+$ such that $A_+ \subset \sqrt{(f_0, \ldots, f_n)}$ in other words $X = \bigcup D_+(f_i)$. Therefore, it suffices to check that $Z \cap D_+(f_i) = Z' \cap D_+(f_i)$ for each i. By renumbering we may assume $i = 0$. Say $Z \cap D_+(f_0)$, resp. $Z' \cap D_+(f_0)$ is cut out by the ideal J, resp. J' of $A_{(f_0)}$.

The inclusion $J' \subset J$. Let d be the least common multiple of $\deg(f_0), \ldots, \deg(f_n)$. Note that each of the twists $\mathcal{O}_X(nd)$ is invertible, trivialized by $f_i^{nd/\deg(f_i)}$ over $D_+(f_i)$, and that for any quasi-coherent module $\mathcal{F}$ on X the multiplication maps $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \to \mathcal{F}(nd + m)$ are isomorphisms, see Constructions, Lemma 27.10.2. Observe that J' is the ideal generated by the elements g/f_0^e where $g \in I$ is homogeneous of degree $e \deg(f_0)$ (see proof of Constructions, Lemma 27.11.3). Of course, by replacing g by $f_0^l g$ for suitable l we may always assume that $d | e$. Then, since g vanishes as a section of $\mathcal{O}_X(e \deg(f_0))$ restricted to Z we see that g/f_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $g/f_0^e \in J$. Again we may assume $d | e$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(f_i)$ is cut out by some ideal $J_i \subset A_{(f_i)}$. Moreover,

$$J \cdot A_{(f_0 f_i)} = J_i \cdot A_{(f_0 f_i)}.$$

The right hand side is the localization of J_i with respect to $f_0^{\deg(f_i)}/f_i^{\deg(f_0)}$. It follows that

$$f_0^{e_i} g / f_i^{(e_i + e)\deg(f_0)/\deg(f_i)} \in J_i$$

for some $e_i \gg 0$ sufficiently divisible. This proves that $f_0^{\max(e_i)} g$ is an element of I, because its restriction to each affine open $D_+(f_i)$ vanishes on the closed subscheme $Z \cap D_+(f_i)$. Hence $g/f_0^e \in J'$ and we conclude $J \subset J'$ as desired. □

0BXK **Example 31.31.2.** Let A be a graded ring. Let $X = \mathrm{Proj}(A)$ and $S = \mathrm{Spec}(A_0)$. Given a graded ideal $I \subset A$ we obtain a closed subscheme $V_+(I) = \mathrm{Proj}(A/I) \to X$ by Constructions, Lemma 27.11.3. Translating the result of Lemma 31.31.1 we see

that if X is quasi-compact, then any closed subscheme Z is of the form $V_+(I(Z))$ where the graded ideal $I(Z) \subset A$ is given by the rule

$$I(Z) = \mathrm{Ker}(A \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(Z, \mathcal{O}_Z(n)))$$

Then we can ask the following two natural questions:

(1) Which ideals I are of the form $I(Z)$?
(2) Can we describe the operation $I \mapsto I(V_+(I))$?

We will answer this when A is Noetherian.

First, assume that A is generated by A_1 over A_0. In this case, for any ideal $I \subset A$ the kernel of the map $A/I \to \bigoplus \Gamma(\mathrm{Proj}(A/I), \mathcal{O})$ is the set of torsion elements of A/I, see Cohomology of Schemes, Proposition 30.14.4. Hence we conclude that

$$I(V_+(I)) = \{x \in A \mid A_n x \subset I \text{ for some } n \geq 0\}$$

The ideal on the right is sometimes called the saturation of I. This answers (2) and the answer to (1) is that an ideal is of the form $I(Z)$ if and only if it is saturated, i.e., equal to its own saturation.

If A is a general Noetherian graded ring, then we use Cohomology of Schemes, Proposition 30.15.3. Thus we see that for d equal to the lcm of the degrees of generators of A over A_0 we get

$$I(V_+(I)) = \{x \in A \mid (Ax)_{nd} \subset I \text{ for all } n \gg 0\}$$

This can be different from the saturation of I if $d \neq 1$. For example, suppose that $A = \mathbf{Q}[x, y]$ with $\deg(x) = 2$ and $\deg(y) = 3$. Then $d = 6$. Let $I = (y^2)$. Then we see $y \in I(V_+(I))$ because for any homogeneous $f \in A$ such that $6 | \deg(fy)$ we have $y | f$, hence $fy \in I$. It follows that $I(V_+(I)) = (y)$ but $x^n y \notin I$ for all n hence $I(V_+(I))$ is not equal to the saturation.

0BXL **Lemma 31.31.3.** *Let R be a Noetherian UFD. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme which has no embedded points such that every irreducible component of Z has codimension 1 in $\mathbf{P}^n_R$. Then the ideal $I(Z) \subset R[T_0, \ldots, T_n]$ corresponding to Z is principal.*

Proof. Observe that the local rings of $X = \mathbf{P}^n_R$ are UFDs because X is covered by affine pieces isomorphic to $\mathbf{A}^n_R$ and $R[x_1, \ldots, x_n]$ is a UFD (Algebra, Lemma 10.120.10). Thus Z is an effective Cartier divisor by Lemma 31.15.9. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to Z. Choose an isomorphism $\mathcal{O}(m) \to \mathcal{I}$ for some $m \in \mathbf{Z}$, see Lemma 31.28.5. Then the composition

$$\mathcal{O}_X(m) \to \mathcal{I} \to \mathcal{O}_X$$

is nonzero. We conclude that $m \leq 0$ and that the corresponding section of $\mathcal{O}_X(m)^{\otimes -1} = \mathcal{O}_X(-m)$ is given by some $F \in R[T_0, \ldots, T_n]$ of degree $-m$, see Cohomology of Schemes, Lemma 30.8.1. Thus on the ith standard open $U_i = D_+(T_i)$ the closed subscheme $Z \cap U_i$ is cut out by the ideal

$$(F(T_0/T_i, \ldots, T_n/T_i)) \subset R[T_0/T_i, \ldots, T_n/T_i]$$

Thus the homogeneous elements of the graded ideal $I(Z) = \mathrm{Ker}(R[T_0, \ldots, T_n] \to \bigoplus \Gamma(\mathcal{O}_Z(m)))$ is the set of homogeneous polynomials G such that

$$G(T_0/T_i, \ldots, T_n/T_i) \in (F(T_0/T_i, \ldots, T_n/T_i))$$

for $i = 0, \ldots, n$. Clearing denominators, we see there exist $e_i \geq 0$ such that

$$T_i^{e_i} G \in (F)$$

for $i = 0, \ldots, n$. As R is a UFD, so is $R[T_0, \ldots, T_n]$. Then $F | T_0^{e_0} G$ and $F | T_1^{e_1} G$ implies $F | G$ as $T_0^{e_0}$ and $T_1^{e_1}$ have no factor in common. Thus $I(Z) = (F)$. □

In case the closed subscheme is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

0802 **Lemma 31.31.4.** *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. If p is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. By Lemma 31.31.1 we know there exists a quasi-coherent graded sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ such that $Z = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{I})$. Since S is quasi-compact we can choose a finite affine open covering $S = U_1 \cup \ldots \cup U_n$. Say $U_i = \mathrm{Spec}(R_i)$. Let $\mathcal{A}|_{U_i}$ correspond to the graded R_i-algebra A_i and $\mathcal{I}|_{U_i}$ to the graded ideal $I_i \subset A_i$. Note that $p^{-1}(U_i) = \mathrm{Proj}(A_i)$ as schemes over R_i. Since p is quasi-compact we can choose finitely many homogeneous elements $f_{i,j} \in A_{i,+}$ such that $p^{-1}(U_i) = \bigcup_j D_+(f_{i,j})$. The condition on $Z \to X$ means that the ideal sheaf of Z in $\mathcal{O}_X$ is of finite type, see Morphisms, Lemma 29.21.7. Hence we can find finitely many homogeneous elements $h_{i,j,k} \in I_i \cap A_{i,+}$ such that the ideal of $Z \cap D_+(f_{i,j})$ is generated by the elements $h_{i,j,k}/f_{i,j}^{e_{i,j,k}}$. Choose $d > 0$ to be a common multiple of all the integers $\deg(f_{i,j})$ and $\deg(h_{i,j,k})$. By Properties, Lemma 28.22.3 there exists a finite type quasi-coherent $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

The following version of Lemma 31.31.4 will be used in the proof of Lemma 31.34.2.

0803 **Lemma 31.31.5.** *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Let $U \subset S$ be an open. Assume that*

(1) *p is quasi-compact,*
(2) *i of finite presentation,*
(3) *$U \cap p(i(Z)) = \emptyset$,*
(4) *U is quasi-compact,*
(5) *$\mathcal{A}_n$ is a finite type $\mathcal{O}_S$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be the sheaf of quasi-coherent graded ideals constructed in Lemma 31.31.1. Let U_i, R_i, A_i, I_i, $f_{i,j}$, $h_{i,j,k}$, and d be as constructed in the proof of Lemma 31.31.4. Since $U \cap p(i(Z)) = \emptyset$ we see that $\mathcal{I}_d|_U = \mathcal{A}_d|_U$ (by our construction of $\mathcal{I}$ as a kernel). Since U is quasi-compact we can choose a finite affine open covering $U = W_1 \cup \ldots \cup W_m$. Since $\mathcal{A}_d$ is of finite type we can find finitely many sections $g_{t,s} \in \mathcal{A}_d(W_t)$ which generate $\mathcal{A}_d|_{W_t} = \mathcal{I}_d|_{W_t}$ as an $\mathcal{O}_{W_t}$-module. To

finish the proof, note that by Properties, Lemma 28.22.3 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d-\deg(h_{i,j,k}))/\deg(f_{i,j})} \quad \text{and} \quad g_{t,s}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

0B3V **Lemma 31.31.6.** *Let X be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There is a bijection*

$$\left\{ \begin{array}{c} \text{sections } \sigma \text{ of the} \\ \text{morphism } \mathbf{P}(\mathcal{E}) \to X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{surjections } \mathcal{E} \to \mathcal{L} \text{ where} \\ \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module} \end{array} \right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\mathrm{Ker}(\mathcal{E} \to \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Recall that $\pi : \mathbf{P}(\mathcal{E}) \to X$ is the relative proj of the symmetric algebra on $\mathcal{E}$, see Constructions, Definition 27.21.1. Hence the descriptions of sections σ follows immediately from the description of the functor of points of $\mathbf{P}(\mathcal{E})$ in Constructions, Lemma 27.16.11. Since π is separated, any section is a closed immersion (Constructions, Lemma 27.16.9 and Schemes, Lemma 26.21.11). Let $U \subset X$ be an affine open and $k \in \mathcal{E}(U)$ and $s \in \mathcal{E}(U)$ be local sections such that k maps to zero in $\mathcal{L}$ and s maps to a generator $\overline{s}$ of $\mathcal{L}$. Then $f = k/s$ is a section of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$ defined in an open neighbourhood $D_+(s)$ of $s(U)$ in $\pi^{-1}(U)$. Moreover, since k maps to zero in $\mathcal{L}$ we see that f is a section of the ideal sheaf of $s(U)$ in $\pi^{-1}(U)$. Thus we can take the image $\overline{f}$ of f in $\mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}(U)$. We claim (1) that the image $\overline{f}$ depends only on the sections k and $\overline{s}$ and not on the choice of s and (2) that we get an isomorphism over U in this manner (see below). However, once (1) and (2) are established, we see that the construction is compatible with base change by $U' \to U$ where U' is affine, which proves that these local maps glue and are compatible with arbitrary base change.

To prove (1) and (2) we make explicit what is going on. Namely, say $U = \mathrm{Spec}(A)$ and say $\mathcal{E} \to \mathcal{L}$ corresponds to the map of A-modules $M \to N$. Then $k \in K = \mathrm{Ker}(M \to N)$ and $s \in M$ maps to a generator $\overline{s}$ of N. Hence $M = K \oplus As$. Thus

$$\mathrm{Sym}(M) = \mathrm{Sym}(K)[s]$$

Consider the identification $\mathrm{Sym}(K) \to \mathrm{Sym}(M)_{(s)}$ via the rule $g \mapsto g/s^n$ for $g \in \mathrm{Sym}^n(K)$. This gives an isomorphism $D_+(s) = \mathrm{Spec}(\mathrm{Sym}(K))$ such that σ corresponds to the ring map $\mathrm{Sym}(K) \to A$ mapping K to zero. Via this isomorphism we see that the quasi-coherent module corresponding to K is identified with $\mathcal{C}_{\sigma(U)/D_+(s)}$ proving (2). Finally, suppose that $s' = k' + s$ for some $k' \in K$. Then

$$k/s' = (k/s)(s/s') = (k/s)(s'/s)^{-1} = (k/s)(1 + k'/s)^{-1}$$

in an open neighbourhood of $\sigma(U)$ in $D_+(s)$. Thus we see that s'/s restricts to 1 on $\sigma(U)$ and we see that k/s' maps to the same element of the conormal sheaf as does k/s thereby proving (1). □

31.32. Blowing up

01OF Blowing up is an important tool in algebraic geometry.

01OG **Definition 31.32.1.** Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 26.10.2. The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\mathrm{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n \right) \longrightarrow X$$

The *exceptional divisor* of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" scheme over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blowup of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Constructions, Lemma 27.16.11. Note that $\mathcal{O}_{X'}(1)$ is b-relatively very ample, even though b need not be of finite type or even quasi-compact, because X' comes equipped with a closed immersion into $\mathbf{P}(\mathcal{I})$, see Morphisms, Example 29.38.3.

0804 **Lemma 31.32.2.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $b : X' \to X$ is the blowup of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$b^{-1}(U) = \mathit{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of $b^{-1}(U)$ with the homogeneous spectrum of the Rees algebra of I in A. Moreover, $b^{-1}(U)$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. The first statement is clear from the construction of the relative Proj via glueing, see Constructions, Section 27.15. For $a \in I$ denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra $\bigoplus_{n \geq 0} I^n$. Since these elements generate the Rees algebra over A we see that $\mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$ is covered by the affine opens $D_+(a^{(1)})$. The affine scheme $D_+(a^{(1)})$ is the spectrum of the affine blowup algebra $A' = A[\frac{I}{a}]$, see Algebra, Definition 10.70.1. This finishes the proof. □

0805 **Lemma 31.32.3.** *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X_i' be the blowup of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_1' & \longrightarrow & X_2' \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of schemes.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (by definition of the inverse image, see Schemes, Definition 26.17.7). By Constructions, Lemma 27.16.10 we see that $X_1 \times_{X_2} X_2'$ is the relative Proj of $\bigoplus_{n\geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X_2' = X_1'$. □

02OS **Lemma 31.32.4.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blowing up $b : X' \to X$ of Z in X has the following properties:*

1. *$b|_{b^{-1}(X\setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
2. *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
3. *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. As blowing up commutes with restrictions to open subschemes (Lemma 31.32.3) the first statement just means that $X' = X$ if $Z = \emptyset$. In this case we are blowing up in the ideal sheaf $\mathcal{I} = \mathcal{O}_X$ and the result follows from Constructions, Example 27.8.14.

The second statement is local on X, hence we may assume X affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. By Lemma 31.32.2 we see that X' is covered by the spectra of the affine blowup algebras $A' = A[\frac{I}{a}]$. Then $IA' = aA'$ and a maps to a nonzerodivisor in A' according to Algebra, Lemma 10.70.2. This proves the lemma as the inverse image of Z in $\mathrm{Spec}(A')$ corresponds to $\mathrm{Spec}(A'/IA') \subset \mathrm{Spec}(A')$.

Consider the canonical map $\psi_{univ,1} : b^*\mathcal{I} \to \mathcal{O}_{X'}(1)$, see discussion following Constructions, Definition 27.16.7. We claim that this factors through an isomorphism $\mathcal{I}_E \to \mathcal{O}_{X'}(1)$ (which proves the final assertion). Namely, on the affine open corresponding to the blowup algebra $A' = A[\frac{I}{a}]$ mentioned above $\psi_{univ,1}$ corresponds to the A'-module map

$$I \otimes_A A' \longrightarrow \Big(\Big(\bigoplus\nolimits_{d\geq 0} I^d\Big)_{a^{(1)}}\Big)_1$$

where $a^{(1)}$ is as in Algebra, Definition 10.70.1. We omit the verification that this is the map $I \otimes_A A' \to IA' = aA'$. □

0806 **Lemma 31.32.5** (Universal property blowing up)**.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{C}$ be the full subcategory of (Sch/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 31.32.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By the material in Constructions, Section 27.16 the triple $(1, f : Y \to X, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 31.13.4. Thus the morphism $Y \to X'$ is unique by Morphisms, Lemma 29.7.10 (also b is separated by Constructions, Lemma 27.16.9). □

0BFL **Lemma 31.32.6.** *Let $b : X' \to X$ be the blowing up of the scheme X along a closed subscheme Z. Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $I \subset A$ be the ideal corresponding to $Z \cap U$. Let $a \in I$ and let $x' \in X'$ be a point mapping to a point of U. Then x' is a point of the affine open $U' = \mathrm{Spec}(A[\frac{I}{a}])$ if and only if the image of a in $\mathcal{O}_{X',x'}$ cuts out the exceptional divisor.*

Proof. Since the exceptional divisor over U' is cut out by the image of a in $A' = A[\frac{I}{a}]$ one direction is clear. Conversely, assume that the image of a in $\mathcal{O}_{X',x'}$ cuts out E. Since every element of I maps to an element of the ideal defining E over $b^{-1}(U)$ we see that elements of I become divisible by a in $\mathcal{O}_{X',x'}$. Thus for $f \in I^n$ we can write $f = \psi(f)a^n$ for some $\psi(f) \in \mathcal{O}_{X',x'}$. Observe that since a maps to a nonzerodivisor of $\mathcal{O}_{X',x'}$ the element $\psi(f)$ is uniquely characterized by this. Then we define

$$A' \longrightarrow \mathcal{O}_{X',x'}, \quad f/a^n \longmapsto \psi(f)$$

Here we use the description of blowup algebras given following Algebra, Definition 31.32.1. The uniqueness mentioned above shows that this is an A-algebra homomorphism. This gives a morphism $\mathrm{Spec}(\mathcal{O}_{X',x''}) \to \mathrm{Spec}(A') = U'$. By the universal property of blowing up (Lemma 31.32.5) this is a morphism over X', which of course implies that $x' \in U'$. □

0807 **Lemma 31.32.7.** *Let X be a scheme. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 31.32.5). □

0808 **Lemma 31.32.8.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in $\mathcal{I}$ is reduced.*

Proof. Combine Lemma 31.32.2 with Algebra, Lemma 10.70.9. □

02ND **Lemma 31.32.9.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a nonzero quasi-coherent sheaf of ideals. If X is integral, then the blowup X' of X in $\mathcal{I}$ is integral.*

Proof. Combine Lemma 31.32.2 with Algebra, Lemma 10.70.10. □

0BFM **Lemma 31.32.10.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $b : X' \to X$ be the blowing up of X along Z. Then b induces an bijective map from the set of generic points of irreducible components of X' to the set of generic points of irreducible components of X which are not in Z.*

Proof. The exceptional divisor $E \subset X'$ is an effective Cartier divisor and $X' \setminus E \to X \setminus Z$ is an isomorphism, see Lemma 31.32.4. Thus it suffices to show the following: given an effective Cartier divisor $D \subset S$ of a scheme S none of the generic points of irreducible components of S are contained in D. To see this, we may replace S by the members of an affine open covering. Hence by Lemma 31.13.2 we may assume $S = \mathrm{Spec}(A)$ and $D = V(f)$ where $f \in A$ is a nonzerodivisor. Then we have to show f is not contained in any minimal prime ideal $\mathfrak{p} \subset A$. If so, then f would map to a nonzerodivisor contained in the maximal ideal of $R_\mathfrak{p}$ which is a contradiction with Algebra, Lemma 10.25.1. □

0809 **Lemma 31.32.11.** *Let X be a scheme. Let $b : X' \to X$ be a blowup of X in a closed subscheme. The pullback $b^{-1}D$ is defined for all effective Cartier divisors $D \subset X$ and pullbacks of meromorphic functions are defined for b (Definitions 31.13.12 and 31.23.4).*

Proof. By Lemmas 31.32.2 and 31.13.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

080A **Lemma 31.32.12.** *Let X be a scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 31.32.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 31.32.11. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 31.32.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 31.13.9. Thus a morphism $c' : Y \to X'$ over X by Lemma 31.32.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

02NS **Lemma 31.32.13.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then*

(1) *$b : X' \to X$ is a projective morphism, and*
(2) *$\mathcal{O}_{X'}(1)$ is a b-relatively ample invertible sheaf.*

Proof. The surjection of graded $\mathcal{O}_X$-algebras

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{I}) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{I}^d$$

defines via Constructions, Lemma 27.18.5 a closed immersion

$$X' = \underline{\mathrm{Proj}}_X(\bigoplus\nolimits_{d \geq 0} \mathcal{I}^d) \longrightarrow \mathbf{P}(\mathcal{I}).$$

Hence b is projective, see Morphisms, Definition 29.43.1. The second statement follows for example from the characterization of relatively ample invertible sheaves in Morphisms, Lemma 29.37.4. Some details omitted. □

080B **Lemma 31.32.14.** *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subscheme of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subscheme $Y \subset X$ of finite presentation, such that*

(1) *$Y = Z \cup b(Z')$ set theoretically, and*
(2) *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms, Lemma 29.21.7. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open of X by Properties, Lemma 28.24.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is

an isomorphism (Lemma 31.32.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open in X. By Lemma 31.31.5 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $X \setminus U$ we see that $V(\mathcal{J})$ is contained in $X \setminus U$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $Z \subset V(\mathcal{J})$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $V(\mathcal{J})$ equals $Z \cup b(Z')$. It follows that also $V(\mathcal{I}\mathcal{J}) = Z \cup b(Z')$ set theoretically. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 31.32.12. Hence it suffices to show that the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 31.32.7 and 31.32.12.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 31.32.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \mathrm{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. □

31.33. Strict transform

080C In this section we briefly discuss strict transform under blowing up. Let S be a scheme and let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of S in Z and denote $E \subset S'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider a scheme X over S and form the cartesian diagram

$$\begin{array}{ccccc} \mathrm{pr}_{S'}^{-1}E & \longrightarrow & X \times_S S' & \xrightarrow[\mathrm{pr}_X]{} & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{S'}} & & \downarrow{\scriptstyle f} \\ E & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Since E is an effective Cartier divisor (Lemma 31.32.4) we see that $\mathrm{pr}_{S'}^{-1}E \subset X \times_S S'$ is locally principal (Lemma 31.13.11). Thus the complement of $\mathrm{pr}_{S'}^{-1}E$ in $X \times_S S'$ is retrocompact (Lemma 31.13.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_S S'}$-module $\mathcal{G}$ the subsheaf of sections supported on $\mathrm{pr}_{S'}^{-1}E$ is a quasi-coherent submodule, see Properties, Lemma 28.24.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_S S'}$, then this subsheaf is an ideal of $\mathcal{G}$.

080D **Definition 31.33.1.** With $Z \subset S$ and $f : X \to S$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of S in Z is the quotient $\mathcal{F}'$ of $\mathrm{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $\mathrm{pr}_{S'}^{-1}E$.
(2) The *strict transform* of X is the closed subscheme $X' \subset X \times_S S'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X\times_S S'}$ supported on $\mathrm{pr}_{S'}^{-1}E$.

Note that taking the strict transform along a blowup depends on the closed subscheme used for the blowup (and not just on the morphism $S' \to S$). This notion is often used for closed subschemes of S. It turns out that the strict transform of X is a blowup of X.

080E **Lemma 31.33.2.** *In the situation of Definition 31.33.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subscheme $f^{-1}Z$ of X.*
(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_S S'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 31.32.5) there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

whence a morphism $X'' \to X \times_S S'$. Thus the first assertion is that this morphism is a closed immersion with image X'. The question is local on X. Thus we may assume X and S are affine. Say that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and Z is cut out by the ideal $I \subset A$. Set $J = IB$. The map $B \otimes_A \bigoplus_{n\geq 0} I^n \to \bigoplus_{n\geq 0} J^n$ defines a closed immersion $X'' \to X \times_S S'$, see Constructions, Lemmas 27.11.6 and 27.11.5. We omit the verification that this morphism is the same as the one constructed above from the universal property. Pick $a \in I$ corresponding to the affine open $\mathrm{Spec}(A[\frac{I}{a}]) \subset S'$, see Lemma 31.32.2. The inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in the strict transform X' of X is the spectrum of

$$B' = (B \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. On the other hand, letting $b \in J$ be the image of a we see that $\mathrm{Spec}(B[\frac{J}{b}])$ is the inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in X''. By Algebra, Lemma 10.70.3 the open $\mathrm{Spec}(B[\frac{J}{b}])$ maps isomorphically to the open subscheme $\mathrm{pr}_{S'}^{-1}(\mathrm{Spec}(A[\frac{I}{a}]))$ of X'. Thus $X'' \to X'$ is an isomorphism.

In the notation above, let $\mathcal{F}$ correspond to the B-module N. The strict transform of $\mathcal{F}$ corresponds to the $B \otimes_A A[\frac{I}{a}]$-module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. The strict transform of $\mathcal{F}$ relative to the blowup of X in $f^{-1}Z$ corresponds to the $B[\frac{J}{b}]$-module $N \otimes_B B[\frac{J}{b}]/b$-power-torsion. In exactly the same way as above one proves that these two modules are isomorphic. Details omitted. □

080F **Lemma 31.33.3.** *In the situation of Definition 31.33.1.*

(1) *If X is flat over S at all points lying over Z, then the strict transform of X is equal to the base change $X \times_S S'$.*
(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over S at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. We will prove part (2) as it implies part (1) by the definition of the strict transform of a scheme over S. The question is local on X. Thus we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and that $\mathcal{F}$ corresponds to the B-module N. Then $\mathcal{F}'$ over the open $\mathrm{Spec}(B \otimes_A A[\frac{I}{a}])$ of $X \times_S S'$ corresponds to the module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. Thus we have to show that the a-power-torsion of $N \otimes_A A[\frac{I}{a}]$ is zero. Let $y \in N \otimes_A A[\frac{I}{a}]$ with $a^n y = 0$. If $\mathfrak{q} \subset B$ is a prime and $a \notin \mathfrak{q}$, then y maps to zero in $(N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$. on the other hand, if $a \in \mathfrak{q}$, then $N_\mathfrak{q}$ is a flat A-module and we see that $N_\mathfrak{q} \otimes_A A[\frac{I}{a}] = (N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$ has no a-power torsion (as $A[\frac{I}{a}]$ doesn't). Hence y maps to zero in this localization as well. We conclude that y is zero by Algebra, Lemma 10.23.1. □

080G **Lemma 31.33.4.** *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of Z in S. Let $g : X \to Y$ be an affine morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_S S' \to Y \times_S S'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Observe that $g'_*\mathrm{pr}_X^*\mathcal{F} = \mathrm{pr}_Y^* g_*\mathcal{F}$ by Cohomology of Schemes, Lemma 30.5.1. Let $\mathcal{K} \subset \mathrm{pr}_X^*\mathcal{F}$ be the subsheaf of sections supported in the inverse image of Z in $X \times_S S'$. By Properties, Lemma 28.24.7 the pushforward $g'_*\mathcal{K}$ is the subsheaf of sections of $\mathrm{pr}_Y^* g_*\mathcal{F}$ supported in the inverse image of Z in $Y \times_S S'$. As g' is affine (Morphisms, Lemma 29.11.9) we see that g'_* is exact, hence we conclude. □

080H **Lemma 31.33.5.** *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $D \subset S$ be an effective Cartier divisor. Let $Z' \subset S$ be the closed subscheme cut out by the product of the ideal sheaves of Z and D. Let $S' \to S$ be the blowup of S in Z.*

(1) *The blowup of S in Z' is isomorphic to $S' \to S$.*
(2) *Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ has no nonzero local sections supported in $f^{-1}D$, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of S in Z'.*

Proof. The first statement follows on combining Lemmas 31.32.12 and 31.32.7. Using Lemma 31.32.2 the second statement translates into the following algebra problem. Let A be a ring, $I \subset A$ an ideal, $x \in A$ a nonzerodivisor, and $a \in I$. Let M be an A-module whose x-torsion is zero. To show: the a-power torsion in $M \otimes_A A[\frac{I}{a}]$ is equal to the xa-power torsion. The reason for this is that the kernel and cokernel of the map $A \to A[\frac{I}{a}]$ is a-power torsion, so this map becomes an isomorphism after inverting a. Hence the kernel and cokernel of $M \to M \otimes_A A[\frac{I}{a}]$ are a-power torsion too. This implies the result. □

080I **Lemma 31.33.6.** *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up with center Z. Let $Z' \subset S'$ be a closed subscheme. Let $S'' \to S'$ be the blowing up with center Z'. Let $Y \subset S$ be a closed subscheme such that $Y = Z \cup b(Z')$ set theoretically and the composition $S'' \to S$ is isomorphic to the blowing up of S in Y. In this situation, given any scheme X over S and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

(1) *the strict transform of $\mathcal{F}$ with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z, and*
(2) *the strict transform of X with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of X with respect to the blowup $S' \to S$ of S in Z.*

Proof. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z. Let $\mathcal{F}''$ be the strict transform of $\mathcal{F}'$ with respect to the blowup $S'' \to S'$ of S' in Z'. Let $\mathcal{G}$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S'' \to S$ of S in Y. We also label the morphisms

$$\begin{array}{ccccc} X \times_S S'' & \xrightarrow{q} & X \times_S S' & \xrightarrow{p} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

By definition there is a surjection $p^*\mathcal{F} \to \mathcal{F}'$ and a surjection $q^*\mathcal{F}' \to \mathcal{F}''$ which combine by right exactness of q^* to a surjection $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$. Also we have the surjection $(p \circ q)^*\mathcal{F} \to \mathcal{G}$. Thus it suffices to prove that these two surjections have the same kernel.

The kernel of the surjection $p^*\mathcal{F} \to \mathcal{F}'$ is supported on $(f \circ p)^{-1}Z$, so this map is an isomorphism at points in the complement. Hence the kernel of $q^*p^*\mathcal{F} \to q^*\mathcal{F}'$ is supported on $(f \circ p \circ q)^{-1}Z$. The kernel of $q^*\mathcal{F}' \to \mathcal{F}''$ is supported on $(f' \circ q)^{-1}Z'$. Combined we see that the kernel of $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$ is supported on $(f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z' = (f \circ p \circ q)^{-1}Y$. By construction of $\mathcal{G}$ we see that we obtain a factorization $(p \circ q)^*\mathcal{F} \to \mathcal{F}'' \to \mathcal{G}$. To finish the proof it suffices to show that $\mathcal{F}''$ has no nonzero (local) sections supported on $(f \circ p \circ q)^{-1}(Y) = (f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z'$. This follows from Lemma 31.33.5 applied to $\mathcal{F}'$ on $X \times_S S'$ over S', the closed subscheme Z' and the effective Cartier divisor $b^{-1}Z$. □

080W **Lemma 31.33.7.** *In the situation of Definition 31.33.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to S$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $S' \to S$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. We may localize on S and X and assume both are affine. Then we may push $\mathcal{F}_i$ to S, see Lemma 31.33.4. We may assume that our blowup is the morphism $1 : S \to S$ associated to an effective Cartier divisor $D \subset S$. Then the translation into algebra is the following: Suppose that A is a ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ is a universally exact sequence of A-modules. Let $a \in A$. Then the sequence

$$0 \to M_1/a\text{-power torsion} \to M_2/a\text{-power torsion} \to M_3/a\text{-power torsion} \to 0$$

is exact too. Namely, surjectivity of the last map and injectivity of the first map are immediate. The problem is exactness in the middle. Suppose that $x \in M_2$ maps to zero in M_3/a-power torsion. Then $y = a^n x \in M_1$ for some n. Then y maps to zero in $M_2/a^n M_2$. Since $M_1 \to M_2$ is universally injective we see that y maps to zero in $M_1/a^n M_1$. Thus $y = a^n z$ for some $z \in M_1$. Thus $a^n(x - y) = 0$. Hence y maps to the class of x in M_2/a-power torsion as desired. □

31.34. Admissible blowups

080J To have a bit more control over our blowups we introduce the following standard terminology.

080K **Definition 31.34.1.** Let X be a scheme. Let $U \subset X$ be an open subscheme. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms, Lemma 29.21.7. In particular, a U-admissible blowup is a projective morphism, see Lemma 31.32.13. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 31.32.4) we will often abuse notation and think of U as an open subscheme of X' as well.

080L **Lemma 31.34.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 31.32.14. □

080M **Lemma 31.34.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $U, V \subset X$ be quasi-compact open subschemes. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$ (see Sheaves, Section 6.33). By Properties, Lemma 28.22.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

080N **Lemma 31.34.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 31.32.12. □

080P **Lemma 31.34.5.** *Let X be a quasi-compact and quasi-separated scheme. Let U, V be quasi-compact disjoint open subschemes of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Properties, Lemma 28.24.1. Then $V(\mathcal{I}\mathcal{J}) = X$ set theoretically, hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $V(\mathcal{I} + \mathcal{J}) = X \setminus U \cup V$. We will show that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ such that $b^{-1}\mathcal{I}|_{X'_2} = 0$ and $b^{-1}\mathcal{J}|_{X'_1} = 0$ which will prove the lemma.

We will use the description of the blowing up in Lemma 31.32.2. Suppose that $U = \mathrm{Spec}(A) \subset X$ is an affine open such that $\mathcal{I}|_U$, resp. $\mathcal{J}|_U$ corresponds to the finitely generated ideal $I \subset A$, resp. $J \subset A$. Then

$$b^{-1}(U) = \mathrm{Proj}(A \oplus (I + J) \oplus (I + J)^2 \oplus \ldots)$$

This is covered by the affine open subsets $A[\frac{I+J}{x}]$ and $A[\frac{I+J}{y}]$ with $x \in I$ and $y \in J$. Since $x \in I$ is a nonzerodivisor in $A[\frac{I+J}{x}]$ and $IJ = 0$ we see that $JA[\frac{I+J}{x}] = 0$. Since $y \in J$ is a nonzerodivisor in $A[\frac{I+J}{y}]$ and $IJ = 0$ we see that $IA[\frac{I+J}{y}] = 0$. Moreover,

$$\mathrm{Spec}(A[\tfrac{I+J}{x}]) \cap \mathrm{Spec}(A[\tfrac{I+J}{y}]) = \mathrm{Spec}(A[\tfrac{I+J}{xy}]) = \emptyset$$

because xy is both a nonzerodivisor and zero. Thus $b^{-1}(U)$ is the disjoint union of the open subscheme U_1 defined as the union of the standard opens $\mathrm{Spec}(A[\frac{I+J}{x}])$ for $x \in I$ and the open subscheme U_2 which is the union of the affine opens $\mathrm{Spec}(A[\frac{I+J}{y}])$ for $y \in J$. We have seen that $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_2 and $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_1. We omit the verification that these open subschemes glue to global open subschemes X'_1 and X'_2. □

0ESL **Lemma 31.34.6.** *Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. The blowup $b : X' \to X$ in the ideal of denominators of s is U-admissible. There exists an effective Cartier divisor $D \subset X'$ and an isomorphism*

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

where $E \subset X'$ is the exceptional divisor such that the meromorphic section b^*s corresponds, via the isomorphism, to the meromorphic section $1_D \otimes (1_E)^{-1}$.

Proof. From the definition of the ideal of denominators in Definition 31.23.10 we immediately see that b is a U-admissible blowup. For the notation 1_D, 1_E, and $\mathcal{O}_{X'}(D - E)$ please see Definition 31.14.1. The pullback b^*s is defined by Lemmas 31.32.11 and 31.23.8. Thus the statement of the lemma makes sense. We can reinterpret the final assertion as saying that b^*s is a global regular section of $b^*\mathcal{L}(E)$ whose zero scheme is D. This uniquely defines D hence to prove the lemma we may work affine locally on X and X'. Assume $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. Then s is a regular meromorphic function and shrinking further we may assume $s = a'/a$ with $a', a \in A$ nonzerodivisors. Then the ideal of denominators of s corresponds to

the ideal $I = \{x \in A \mid xa' \in aA\}$. Recall that X' is covered by spectra of affine blowup algebras $A' = A[\frac{I}{x}]$ with $x \in I$ (Lemma 31.32.2). Fix $x \in I$ and write $xa' = aa''$ for some $a'' \in A$. The divisor $E \subset X'$ is cut out by $x \in A'$ over the spectrum of A' and hence $1/x$ is a generator of $\mathcal{O}_{X'}(E)$ over $\mathrm{Spec}(A')$. Finally, in the total quotient ring of A' we have $a'/a = a''/x$. Hence $b^*s = a'/a$ restricts to a regular section of $\mathcal{O}_{X'}(E)$ which is over $\mathrm{Spec}(A')$ given by a''/x. This finishes the proof. (The divisor $D \cap \mathrm{Spec}(A')$ is cut out by the image of a'' in A'.) □

31.35. Blowing up and flatness

0F84 We continue the discussion started in More on Algebra, Section 15.26. We will prove further results in More on Flatness, Section 38.30.

0CZP **Lemma 31.35.1.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 31.9. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ can locally be generated by $\leq k$ sections.*

Proof. Recall that $\mathcal{F}'$ can locally be generated by $\leq k$ sections if and only if $\mathrm{Fit}_k(\mathcal{F}') = \mathcal{O}_{S'}$, see Lemma 31.9.4. Hence this lemma is a translation of More on Algebra, Lemma 15.26.3. □

0CZQ **Lemma 31.35.2.** *Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 31.9. Assume that $\mathcal{F}$ is locally free of rank k on $S \setminus Z_k$. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ is locally free of rank k.*

Proof. Translation of More on Algebra, Lemma 15.26.4. □

0ESN **Lemma 31.35.3.** *Let X be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then*

1. *the blowup $b : X' \to X$ of X in the rth Fitting ideal of $\mathcal{F}$ is U-admissible,*
2. *the strict transform $\mathcal{F}'$ of $\mathcal{F}$ with respect to b is locally free of rank r,*
3. *the kernel $\mathcal{K}$ of the surjection $b^*\mathcal{F} \to \mathcal{F}'$ is finitely presented and $\mathcal{K}|_U = 0$,*
4. *$b^*\mathcal{F}$ and $\mathcal{K}$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.*

Proof. The ideal $\mathrm{Fit}_r(\mathcal{F})$ is of finite type by Lemma 31.9.2 and its restriction to U is equal to $\mathcal{O}_U$ by Lemma 31.9.5. Hence $b : X' \to X$ is U-admissible, see Definition 31.34.1.

By Lemma 31.9.5 the restriction of $\mathrm{Fit}_{r-1}(\mathcal{F})$ to U is zero, and since U is scheme theoretically dense we conclude that $\mathrm{Fit}_{r-1}(\mathcal{F}) = 0$ on all of X. Thus it follows from Lemma 31.9.5 that $\mathcal{F}$ is locally free of rank r on the complement of subscheme cut out by the rth Fitting ideal of $\mathcal{F}$ (this complement may be bigger than U which is why we had to do this step in the argument). Hence by Lemma 31.35.2 the strict transform

$$b^*\mathcal{F} \longrightarrow \mathcal{F}'$$

is locally free of rank r. The kernel $\mathcal{K}$ of this map is supported on the exceptional divisor of the blowup b and hence $\mathcal{K}|_U = 0$. Finally, since $\mathcal{F}'$ is finite locally free and since the displayed arrow is surjective, we can locally on X' write $b^*\mathcal{F}$ as the direct sum of $\mathcal{K}$ and $\mathcal{F}'$. Since $b^*\mathcal{F}'$ is finitely presented (Modules, Lemma 17.11.5) the same is true for $\mathcal{K}$.

The statement on tor dimension follows from More on Algebra, Lemma 15.8.10. □

31.36. Modifications

0AYN In this section we will collect results of the type: after a modification such and such are true. We will later see that a modification can be dominated by a blowup (More on Flatness, Lemma 38.31.4).

0AYP **Lemma 31.36.1.** *Let X be an integral scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. There exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules.*

Proof. We prove this by induction on the rank r of $\mathcal{E}$. If $r = 1$ or $r = 0$ the lemma is obvious. Assume $r > 1$. Let $P = \mathbf{P}(\mathcal{E})$ with structure morphism $\pi : P \to X$, see Constructions, Section 27.21. Then π is proper (Lemma 31.30.4). There is a canonical surjection

$$\pi^*\mathcal{E} \to \mathcal{O}_P(1)$$

whose kernel is finite locally free of rank $r-1$. Choose a nonempty open subscheme $U \subset X$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$. Then $P_U = \pi^{-1}(U)$ is isomorphic to $\mathbf{P}_U^{r-1}$. In particular, there exists a section $s : U \to P_U$ of π. Let $X' \subset P$ be the scheme theoretic image of the morphism $U \to P_U \to P$. Then X' is integral (Morphisms, Lemma 29.6.7), the morphism $f = \pi|_{X'} : X' \to X$ is proper (Morphisms, Lemmas 29.41.6 and 29.41.4), and $f^{-1}(U) \to U$ is an isomorphism. Hence f is a modification (Morphisms, Definition 29.51.11). By construction the pullback $f^*\mathcal{E}$ has a two step filtration whose quotient is invertible because it is equal to $\mathcal{O}_P(1)|_{X'}$ and whose sub $\mathcal{E}'$ is locally free of rank $r-1$. By induction we can find a modification $g : X'' \to X'$ such that $g^*\mathcal{E}'$ has a filtration as in the statement of the lemma. Thus $f \circ g : X'' \to X$ is the required modification. □

0C4V **Lemma 31.36.2.** *Let S be a scheme. Let X, Y be schemes over S. Assume X is Noetherian and Y is proper over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a morphism $p : X' \to X$ and an S-morphism $f' : X' \to Y$ such that*

(1) *p is proper and $p^{-1}(U) \to U$ is an isomorphism,*
(2) *$f'|_{p^{-1}(U)}$ is equal to $f \circ p|_{p^{-1}(U)}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $X' \subset Y \times_S X$ be the scheme theoretic image of $(f, j) : U \to Y \times_S X$ (Morphisms, Definition 29.6.2). The projection $g : Y \times_S X \to X$ is proper (Morphisms, Lemma 29.41.5). The composition $p : X' \to X$ of $X' \to Y \times_S X$ and g is proper (Morphisms, Lemmas 29.41.6 and 29.41.4). Since g is separated and $U \subset X$ is retrocompact (as X is Noetherian) we conclude that $p^{-1}(U) \to U$ is an isomorphism by Morphisms, Lemma 29.6.8. On the other hand, the composition $f' : X' \to Y$ of $X' \to Y \times_S X$ and the projection $Y \times_S X \to Y$ agrees with f on $p^{-1}(U)$. □

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CHAPTER 32

Limits of Schemes

01YT

32.1. Introduction

01YU In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed sets (Categories, Definition 4.21.1) with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [**DG67**].

32.2. Directed limits of schemes with affine transition maps

01YV In this section we construct the limit.

01YW **Lemma 32.2.1.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \mathrm{Spec}(\mathrm{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.*

Proof. Just define $S = \mathrm{Spec}(\mathrm{colim}_i R_i)$. It follows from Schemes, Lemma 26.6.4 that S is the limit even in the category of locally ringed spaces. □

01YX **Lemma 32.2.2.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*
$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$
in the category of schemes.

Proof. Choose an element $0 \in I$. Note that I is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$, see Morphisms, Lemma 29.11.3. Set $\mathcal{A} = \mathrm{colim}_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 26.24. Set $S = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 29.11.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 29.11.12 for example. By Lemma 32.2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$

with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. □

01YZ **Lemma 32.2.3.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then*

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

Proof. The right hand side is a scheme by Lemma 32.2.2. The equality is formal, see Categories, Lemma 4.14.10. □

32.3. Infinite products

0CNH Infinite products of schemes usually do not exist. For example in Examples, Section 110.56 it is shown that an infinite product of copies of $\mathbf{P}^1$ is not even an algebraic space.

On the other hand, infinite products of affine schemes do exist and are affine. Using Schemes, Lemma 26.6.4 this corresponds to the fact that in the category of rings we have infinite coproducts: if I is a set and R_i is a ring for each i, then we can consider the ring

$$R = \otimes R_i = \operatorname{colim}_{\{i_1,\ldots,i_n\} \subset I} R_{i_1} \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} R_{i_n}$$

Given another ring A a map $R \to A$ is the same thing as a collection of ring maps $R_i \to A$ for all $i \in I$ as follows from the corresponding property of finite tensor products.

0CNI **Lemma 32.3.1.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an affine morphism. Then the product $T = \prod T_i$ exists in the category of schemes over S. In fact, we have*

$$T = \lim_{\{i_1,\ldots,i_n\} \subset I} T_{i_1} \times_S \ldots \times_S T_{i_n}$$

and the projection morphisms $T \to T_{i_1} \times_S \ldots \times_S T_{i_n}$ are affine.

Proof. Omitted. Hint: Argue as in the discussion preceding the lemma and use Lemma 32.2.2 for existence of the limit. □

0CNJ **Lemma 32.3.2.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be a surjective affine morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 32.3.1) maps surjectively to S.*

Proof. Let $s \in S$. Choose $t_i \in T_i$ mapping to s. Choose a huge field extension $K/\kappa(s)$ such that $\kappa(s_i)$ embeds into K for each i. Then we get morphisms $\operatorname{Spec}(K) \to T_i$ with image s_i agreeing as morphisms to S. Whence a morphism $\operatorname{Spec}(K) \to T$ which proves there is a point of T mapping to s. □

0CNK **Lemma 32.3.3.** *Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an integral morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 32.3.1) is integral over S.*

Proof. Omitted. Hint: On affine pieces this reduces to the following algebra fact: if $A \to B_i$ is integral for all i, then $A \to \otimes_A B_i$ is integral. □

32.4. Descending properties

081A First some basic lemmas describing the topology of a limit.

0CUE **Lemma 32.4.1.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Then $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*

Proof. Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Here $f_{i'i} : S_{i'} \to S_i$ is the transition morphism and $f_i : S \to S_i$ is the projection. By Lemma 32.2.2 we have $U = \lim_{i' \geq i} U_i$. Suppose we can show that $U_{set} = \lim_{i' \geq i} U_{i',set}$. Then the lemma follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \operatorname{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$\operatorname{Spec}(A) = \lim_i \operatorname{Spec}(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. □

0CUF **Lemma 32.4.2.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Then $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S.* [**DG67**, IV, Proposition 8.2.9]

Proof. We will use the criterion of Topology, Lemma 5.14.3. We have seen that $S_{set} = \lim_i S_{i,set}$ in Lemma 32.4.1. The maps $f_i : S \to S_i$ are morphisms of schemes hence continuous. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset S_i$. Finally, let $s \in S$ and let $s \in V \subset S$ be an open neighbourhood. Choose $0 \in I$ and choose an affine open neighbourhood $U_0 \subset S_0$ of the image of s. Then $f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$, see Lemma 32.2.2. Then $f_0^{-1}(U_0)$ and $f_{i0}^{-1}(U_0)$ are affine and

$$\mathcal{O}_S(f_0^{-1}(U_0)) = \operatorname{colim}_{i \geq 0} \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$$

either by the proof of Lemma 32.2.2 or by Lemma 32.2.1. Choose $a \in \mathcal{O}_S(f_0^{-1}(U_0))$ such that $s \in D(a) \subset V$. This is possible because the principal opens form a basis for the topology on the affine scheme $f_0^{-1}(U_0)$. Then we can pick an $i \geq 0$ and $a_i \in \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$ mapping to a. It follows that $D(a_i) \subset f_{i0}^{-1}(U_0) \subset S_i$ is an open subset whose inverse image in S is $D(a)$. This finishes the proof. □

01Z2 **Lemma 32.4.3.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). If all the schemes S_i are nonempty and quasi-compact, then the limit $S = \lim_i S_i$ is nonempty.*

Proof. Choose $0 \in I$. Note that I is nonempty as the limit is directed. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. □

0CUG **Lemma 32.4.4.** *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Let $s \in S$ with images $s_i \in S_i$. Then*

(1) *$s = \lim s_i$ as schemes, i.e., $\kappa(s) = \operatorname{colim} \kappa(s_i)$,*
(2) *$\overline{\{s\}} = \lim \overline{\{s_i\}}$ as sets, and*
(3) *$\overline{\{s\}} = \lim \overline{\{s_i\}}$ as schemes where $\overline{\{s\}}$ and $\overline{\{s_i\}}$ are endowed with the reduced induced scheme structure.*

Proof. Choose $0 \in I$ and an affine open covering $S_0 = \bigcup_{j \in J} U_{0,j}$. For $i \geq 0$ let $U_{i,j} = f_{i,0}^{-1}(U_{0,j})$ and set $U_j = f_0^{-1}(U_{0,j})$. Here $f_{i'i} : S_{i'} \to S_i$ is the transition morphism and $f_i : S \to S_i$ is the projection. For $j \in J$ the following are equivalent: (a) $s \in U_j$, (b) $s_0 \in U_{0,j}$, (c) $s_i \in U_{i,j}$ for all $i \geq 0$. Let $J' \subset J$ be the set of indices for which (a), (b), (c) are true. Then $\overline{\{s\}} = \bigcup_{j \in J'} (\overline{\{s\}} \cap U_j)$ and similarly for $\overline{\{s_i\}}$ for $i \geq 0$. Note that $\overline{\{s\}} \cap U_j$ is the closure of the set $\{s\}$ in the topological space U_j. Similarly for $\overline{\{s_i\}} \cap U_{i,j}$ for $i \geq 0$. Hence it suffices to prove the lemma in the case S and S_i affine for all i. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \operatorname{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Let $\mathfrak{p} \subset A$ be a prime and set $\mathfrak{p}_i = \varphi_i^{-1}(\mathfrak{p})$. Then

$$V(\mathfrak{p}) = \lim_i V(\mathfrak{p}_i)$$

This follows from Lemma 32.4.1 because $A/\mathfrak{p} = \operatorname{colim} A_i/\mathfrak{p}_i$. This equality of rings also shows the final statement about reduced induced scheme structures holds true. The equality $\kappa(\mathfrak{p}) = \operatorname{colim} \kappa(\mathfrak{p}_i)$ follows from the statement as well. □

In the rest of this section we work in the following situation.

086P **Situation 32.4.5.** Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$ (Lemma 32.2.2). We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also choose an element $0 \in I$.

In this situation the morphism $S \to S_0$ is affine. It follows that S is quasi-compact and quasi-separated[1]. The type of result we are looking for is the following: If we have an object over S, then for some i there is a similar object over S_i.

01YY **Lemma 32.4.6.** *In Situation 32.4.5.*

(1) *We have $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*
(2) *We have $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S.*

[1] Follows from Morphisms, Lemma 29.11.2, Topology, Definition 5.12.1, and Schemes, Lemma 26.21.12.

(3) *If $s, s' \in S$ and s' is not a specialization of s then for some $i \in I$ the image $s'_i \in S_i$ of s' is not a specialization of the image $s_i \in S_i$ of s.*
(4) *Add more easy facts on topology of S here. (Requirement: whatever is added should be easy in the affine case.)*

Proof. Part (1) is a special case of Lemma 32.4.1.

Part (2) is a special case of Lemma 32.4.2.

Part (3) is a special case of Lemma 32.4.4. □

01Z0 **Lemma 32.4.7.** *In Situation 32.4.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on S_0. Set $\mathcal{F}_i = f_{i0}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(S, \mathcal{F}) = \operatorname{colim}_{i\geq 0} \Gamma(S_i, \mathcal{F}_i)$$

Proof. Write $\mathcal{A}_j = f_{i0,*}\mathcal{O}_{S_i}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras (see Morphisms, Lemma 29.11.5) and S_i is the relative spectrum of $\mathcal{A}_i$ over S_0. In the proof of Lemma 32.2.2 we constructed S as the relative spectrum of $\mathcal{A} = \operatorname{colim}_{i\geq 0} \mathcal{A}_i$ over S_0. Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

Then we have $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$ and $f_{0,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_i$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = \operatorname{colim}_{i\geq 0} \mathcal{M}_i$. Since S_0 is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \operatorname{colim}_{i\geq 0} \mathcal{M}_i) \\ &= \operatorname{colim}_{i\geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \operatorname{colim}_{i\geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 6.29.1 and Topology, Lemma 5.27.1 for the middle equality. □

01Z3 **Lemma 32.4.8.** *In Situation 32.4.5. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{i'i}(Z_{i'}) \subset Z_i$ for all $i' \geq i$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.*

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 26.12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 29.11.10 and 29.11.8), and hence $Z_{i'} \to S_i$ is affine when $i' \geq i$. We conclude that the morphism $f_{i'i} : Z_{i'} \to Z_i$ is affine by Morphisms, Lemma 29.11.12. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 32.4.3 to see that $Z = \lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. □

05F3 **Lemma 32.4.9.** *In Situation 32.4.5. Suppose we are given an i and a morphism $T \to S_i$ such that*

(1) *$T \times_{S_i} S = \emptyset$, and*
(2) *T is quasi-compact.*

Then $T \times_{S_i} S_{i'} = \emptyset$ for all sufficiently large i'.

Proof. By Lemma 32.2.3 we see that $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 32.4.3. □

05F4 **Lemma 32.4.10.** *In Situation 32.4.5. Suppose we are given an i and a locally constructible subset $E \subset S_i$ such that $f_i(S) \subset E$. Then $f_{i'i}(S_{i'}) \subset E$ for all sufficiently large i'.*

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 32.2.2 and Properties, Lemma 28.2.1. In this case the complement $S_i \setminus E$ is constructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 10.29.4. By Lemma 32.4.9 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{i'i}(S_{i'}) \subset E$ for all sufficiently large i'. □

01Z4 **Lemma 32.4.11.** *In Situation 32.4.5 we have the following:*

(1) *Given any quasi-compact open $V \subset S = \lim_i S_i$ there exists an $i \in I$ and a quasi-compact open $V_i \subset S_i$ such that $f_i^{-1}(V_i) = V$.*
(2) *Given $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ there exists an index $i'' \geq i, i'$ such that $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.*
(3) *If $V_{1,i}, \ldots, V_{n,i} \subset S_i$ are quasi-compact opens and $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ then $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ for some $i' \geq i$.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = \lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$ become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = \operatorname{Spec}(R)$ and $S_i = \operatorname{Spec}(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = \operatorname{colim}_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence: If S_0 is affine, then $S_i = \operatorname{Spec}(R_i)$ for all $i \geq 0$ and $S = \operatorname{Spec}(R)$ with $R = \operatorname{colim} R_i$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$. If S_0 is general, then the opens $V \cap U_j$ are quasi-compact because S is quasi-separated. Hence by the affine case we see that for each $j = 1, \ldots, m$ there exists an $i_j \in I$ and a quasi-compact open $V_{i_j} \subset U_{j,i_j}$ whose inverse image in U_j is $V \cap U_j$. Set $i = \max(i_1, \ldots, i_m)$ and let $V_i = \bigcup f_{ii_j}^{-1}(V_{i_j})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. □

01Z5 **Lemma 32.4.12.** *In Situation 32.4.5 if S is quasi-affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are quasi-affine.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 32.4.7 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 28.26.4 for example. Hence we may apply Lemma 32.4.11 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 32.4.11 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 28.26.1) and hence that S_i is quasi-affine, see Properties, Lemma 28.27.1. Hence we win. □

01Z6 **Lemma 32.4.13.** *In Situation 32.4.5 if S is affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.*

Proof. By Lemma 32.4.12 we may assume that S_0 is quasi-affine for some $0 \in I$. Set $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. Then S_0 is a quasi-compact open of $T_0 = \mathrm{Spec}(R_0)$. Denote $j_0 : S_0 \to T_0$ the corresponding quasi-compact open immersion. For $i \geq 0$ set $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Since f_{i0} is affine we see that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$. Set $T_i = \underline{\mathrm{Spec}}_{T_0}(j_{0,*}\mathcal{A}_i)$. Then $T_i \to T_0$ is affine, hence T_i is affine. Thus T_i is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*}\mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write $S = \mathrm{Spec}(R)$. We have $R = \mathrm{colim}_i R_i$ by Lemma 32.4.7. Hence also $S = \lim_i T_i$. As formation of the relative spectrum commutes with base change, the inverse image of the open $S_0 \subset T_0$ in T_i is S_i. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of Z_0. As $S_i = T_i \setminus Z_i$, it suffices to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 32.4.8 there exists a point s of $S = \lim T_i$ which maps to a point of Z_i for every i. But $S = \lim_i S_i$, and hence we arrive at a contradiction by Lemma 32.4.6. □

086Q **Lemma 32.4.14.** *In Situation 32.4.5 if S is separated, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are separated.*

Proof. Choose a finite affine open covering $S_0 = U_{0,1} \cup \ldots \cup U_{0,m}$. Set $U_{i,j} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0,j}$. Note that $U_{i,j}$ and U_j are affine. As S is separated the intersections $U_{j_1} \cap U_{j_2}$ are affine. Since $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$

we see that $U_{i,j_1} \cap U_{i,j_2}$ is affine for large i by Lemma 32.4.13. To show that S_i is separated for large i it now suffices to show that

$$\mathcal{O}_{S_i}(U_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(U_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(U_{i,j_1} \cap U_{i,j_2})$$

is surjective for large i (Schemes, Lemma 26.21.7).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for i large enough and we know that $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \to \mathcal{O}(U_0 \cap V_0)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma 26.21.2) hence locally of finite type (Morphisms, Lemmas 29.15.2 and 29.15.5). Thus we can choose elements $f_{0,1}, \ldots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes

$$\begin{array}{ccc} U_i \cap V_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_0 \cap V_0 & \longrightarrow & U_0 \end{array}$$

is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_0)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \operatorname{colim} \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved. □

09MT **Lemma 32.4.15.** *In Situation 32.4.5 let $\mathcal{L}_0$ be an invertible sheaf of modules on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.*

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that S_{s_j} is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition 28.26.1. By Lemma 32.4.7 we can find an $i \in I$ and sections $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to s_j. By Lemma 32.4.13 we may, after increasing i, assume that $(S_i)_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma 32.4.11 we may, after increasing i a last time, assume that $S_i = \bigcup (S_i)_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition. □

081B **Lemma 32.4.16.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S.*

1. *If $Y \to X$ is a closed immersion, X_i quasi-compact, and Y locally of finite type over S, then $Y \to X_i$ is a closed immersion for i large enough.*
2. *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to S$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*
3. *If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to S$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to S$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \ldots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let

$V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for i sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \operatorname{colim} A_i$ is a directed colimit of R-algebras, $A \to B$ is a surjection of R-algebras, and B is a finitely generated R-algebra, then $A_i \to B$ is surjective for i sufficiently large.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 32.4.11 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Working affine locally on X_0 for some $0 \in I$ as in the proof of (1) we reduce to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$. □

01ZH **Lemma 32.4.17.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ separated.*

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $0 \in I$. Note that I is nonempty as the limit is directed. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine. In this case we have to show that X_i is separated and we know that X is separated. Thus the lemma follows from Lemma 32.4.14. □

09ZM **Lemma 32.4.18.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ affine.*

Then $X_i \to S$ is affine for i large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each j the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma 32.4.13 there exists an $i \in I$ such that each $f_i^{-1}(V_j)$ is affine. In other words, $f_i : X_i \to S$ is affine for i large enough, see Morphisms, Lemma 29.11.3. □

09ZN **Lemma 32.4.19.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are finite,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ integral.*

Then $X_i \to S$ is finite for i large enough.

Proof. By Lemma 32.4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is finite over V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 29.44.3). Namely, for $i' \geq i$ the composition $X_{i'} \to X_i \to S$ will be finite as a composition of finite morphisms (Morphisms, Lemma 29.44.5). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ integral and $A_i \to A_{i'}$ finite for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that A_i is finite over R for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since A is integral over R we can find monic polynomials $P_j \in R[T]$ such that $P_j(x_j) = 0$ in A. Thus there exists an $i' \geq i$ such that $P_j(x_j) = 0$ in $A_{i'}$ for $j = 1, \ldots, m$. Then the image A_i' of A_i in $A_{i'}$ is finite over R by Algebra, Lemma 10.36.5. Since $A_i' \subset A_{i'}$ is finite too we conclude that $A_{i'}$ is finite over R by Algebra, Lemma 10.7.3. □

0A0N **Lemma 32.4.20.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are closed immersions,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ a closed immersion.*

Then $X_i \to S$ is a closed immersion for i large enough.

Proof. By Lemma 32.4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is a closed subscheme of V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 29.2.1). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ surjective and $A_i \to A_{i'}$ surjective for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that $R \to A_i$ is surjective for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since $R \to A$ is surjective we can find $r_j \in R$ such that r_j maps to x_j in A. Thus there exists an $i' \geq i$ such that r_j maps to the image of x_j in $A_{i'}$ for $j = 1, \ldots, m$. Since $A_i \to A_{i'}$ is surjective this implies that $R \to A_{i'}$ is surjective. □

0GIH **Lemma 32.4.21.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*

(3) *the transition morphisms $X_{i'} \to X_i$ are closed immersions,*
(4) *$X_i \to S$ locally of finite type, and*
(5) *$X \to S$ an immersion.*

Then $X_i \to S$ is an immersion for i large enough.

Proof. Choose an open subscheme $U \subset S$ such that $X \to S$ factors as a closed immersion $X \to U$ composed with the inclusion morphism $U \to S$. Since X is quasi-compact, we may shrink U and assume U is quasi-compact. Denote $V_i \subset X_i$ the inverse image of U. Since V_i pulls back to X we see that $V_i = X_i$ for all i large enough by Lemma 32.4.11. Thus we may assume $X = \lim X_i$ in the category of schemes over U. Then we see that $X_i \to U$ is a closed immersion for i large enough by Lemma 32.4.20. This proves the lemma. □

32.5. Absolute Noetherian Approximation

01Z1 A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [**TT90**]. See Categories, Section 4.21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 32.2 without further mention.

01Z7 **Lemma 32.5.1.** *Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to \mathrm{Spec}(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to \mathrm{Spec}(A)$. Moreover, R is the directed colimit of such subalgebras.*

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $\mathrm{Spec}(R)$. In other words, if we write $W_i = \mathrm{Spec}(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. □

01Z9 **Lemma 32.5.2.** *Suppose given a cartesian diagram of rings*

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow{\scriptstyle t} \\ B' & \longrightarrow & R' \end{array}$$

Let $W' \subset \mathrm{Spec}(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $\mathrm{Spec}(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. More on Algebra, Lemma 15.5.3 shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. □

The following lemma is a precise statement of Noetherian approximation.

07RN **Lemma 32.5.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $V \subset S$ be a quasi-compact open. Let I be a directed set and let $(V_i, f_{ii'})$ be an inverse system of schemes over I with affine transition maps, with each V_i of finite type over $\mathbf{Z}$, and with $V = \lim V_i$. Then there exist*

(1) *a directed set J,*

(2) *an inverse system of schemes* $(S_j, g_{jj'})$ *over* J,
(3) *an order preserving map* $\alpha : J \to I$,
(4) *open subschemes* $V'_j \subset S_j$, *and*
(5) *isomorphisms* $V'_j \to V_{\alpha(j)}$

such that

(1) *the transition morphisms* $g_{jj'} : S_j \to S_{j'}$ *are affine,*
(2) *each* S_j *is of finite type over* $\mathbf{Z}$,
(3) $g_{jj'}^{-1}(V'_{j'}) = V'_j$,
(4) $S = \lim S_j$ *and* $V = \lim V'_j$, *and*
(5) *the diagrams*

$$\begin{array}{ccc} V & & \\ \downarrow & \searrow & \\ V'_j & \longrightarrow & V_{\alpha(j)} \end{array} \quad \text{and} \quad \begin{array}{ccc} V'_j & \longrightarrow & V_{\alpha(j)} \\ \downarrow & & \downarrow \\ V'_{j'} & \longrightarrow & V_{\alpha(j')} \end{array}$$

are commutative.

Proof. Set $Z = S \setminus V$. Choose affine opens $U_1, \ldots, U_m \subset S$ such that $Z \subset \bigcup_{l=1,\ldots,m} U_l$. Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup\nolimits_{l=1,\ldots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \ldots \cup U_l \subset V \cup U_1 \cup \ldots \cup U_{l+1}$$

then the lemma will follow for $V \subset S$. In each case we are adding one affine open. Thus we may assume

(1) $S = U \cup V$,
(2) U affine open in S,
(3) V quasi-compact open in S, and
(4) $V = \lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Denote $f_i : V \to V_i$ the projections. Set $W = U \cap V$. As S is quasi-separated, this is a quasi-compact open of V. By Lemma 32.4.11 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ii'}^{-1}(W_{i'}) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 32.4.12.

Write $U = \mathrm{Spec}(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 32.4.7 we have $R = \mathrm{colim}_i R_i$. Now we have the maps of rings

$$\begin{array}{ccc} B & \xrightarrow[s]{} & R \\ & & \uparrow{\scriptstyle t_i} \\ & & R_i \end{array}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(r)\}$ so that we have a cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow{\scriptstyle t_i} \\ B_i & \longrightarrow & R_i \end{array}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = \operatorname{colim}_i B_i$. In the next paragraph we show that for all sufficiently large i the composition $W_i \to \operatorname{Spec}(R_i) \to \operatorname{Spec}(B_i)$ is an open immersion.

As W is a quasi-compact open of $U = \operatorname{Spec}(B)$ we can find a finitely many elements $g_l \in B$, $l = 1, \ldots, m$ such that $D(g_l) \subset W$ and such that $W = \bigcup_{l=1,\ldots,m} D(g_l)$. Note that this implies $D(g_l) = W_{s(g_l)}$ as open subsets of U, where $W_{s(g_l)}$ denotes the largest open subset of W on which $s(g_l)$ is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 28.17.1. Since $W_{s(g_l)}$ is affine this also implies that $D(s(g_l)) = W_{s(g_l)}$ as open subsets of $\operatorname{Spec}(R)$. Since $R = \operatorname{colim}_i R_i$ we can (after shrinking I) assume there exist $g_{l,i} \in R_i$ for all $i \in I$ such that $s(g_l) = t_i(g_{l,i})$. Of course we choose the $g_{l,i}$ such that $g_{l,i}$ maps to $g_{l,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 32.4.11 we can (after shrinking I again) assume the corresponding opens $D(g_{l,i}) \subset \operatorname{Spec}(R_i)$ are contained in W_i for $l = 1, \ldots, m$ and cover W_i. We conclude that the morphism $W_i \to \operatorname{Spec}(R_i) \to \operatorname{Spec}(B_i)$ is an open immersion, see Lemma 32.5.2.

By Lemma 32.5.1 we can write B_i as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $\operatorname{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing V_i and $\operatorname{Spec}(A_{i,p})$ along the open W_i, see Schemes, Section 26.14. Here is the resulting commutative diagram of schemes:

$$\begin{array}{ccccc} & & V & \leftarrow & W \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ V_i & \leftarrow & W_i & & S \leftarrow U \\ \downarrow & & \downarrow & \swarrow & \\ S_{i,p} & \leftarrow & \operatorname{Spec}(A_{i,p}) & & \end{array}$$

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a preorder on $J = \coprod_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\operatorname{Spec}(A_{i',p'}) \to \operatorname{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i, p) \mapsto i$. The

open subscheme V_j' is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. □

01ZA **Proposition 32.5.4.** *Let S be a quasi-compact and quasi-separated scheme. There exist a directed set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each S_i is of finite type over $\mathbf{Z}$, and*
(3) *$S = \lim_i S_i$.*

Proof. This is a special case of Lemma 32.5.3 with $V = \emptyset$. □

32.6. Limits and morphisms of finite presentation

01ZB The following is a generalization of Algebra, Lemma 10.127.3.

01ZC **Proposition 32.6.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:* [**DG67**, IV, Proposition 8.14.2]

(1) *The morphism f is locally of finite presentation.*
(2) *For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$
(3) *For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \mathrm{colim}_i A_i$. According to Algebra, Lemma 10.127.3 we have to show that
$$\mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$
Consider the schemes $T_i = \mathrm{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \mathrm{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes
$$\mathrm{Mor}_V(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_V(T_i, U).$$
We first observe that $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$ and $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$. Hence we have to show that
$$\mathrm{Mor}_S(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, U)$$
and we are given that
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X).$$
Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 32.4.8 there exists a point

t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = \lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g_i' : T_i \to X$ over S such that $g_i \circ f_i = g_i' \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g_i' \circ f_{i'i}$.
(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g_i' : T_i \to X$ be morphisms such that $g_i \circ f_i = g_i' \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g_{i'}' = g_i' \circ f_{i'i}$. We also set $g = g_i \circ f_i = g_i' \circ f_i$. Consider the morphism $(g_i, g_i') : T_i \to X \times_S X$. Set

$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 26.21.2. Note that the composition $(g_i, g_i') \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g_{i'}')^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 32.4.8 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g_i') : T_i \to X \times_S X$ is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g_i')|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g_{i'}'$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 10.127.3 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 29.21.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 32.4.11 and 32.4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = \lim_i W_{j,i}$. We apply Algebra, Lemma 10.127.3 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 29.21.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that $g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. □

05LX **Remark 32.6.2.** Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to Sets$ is *limit preserving* if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \operatorname{colim}_i F(T_i)$. Let X be a scheme over S, and let

$h_X : (Sch/S)^{opp} \to Sets$ be its functor of points, see Schemes, Section 26.15. In this terminology Proposition 32.6.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

0CM0 **Lemma 32.6.3.** *Let $f : X \to S$ be a morphism of schemes. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map*

$$\operatorname{colim} \operatorname{Mor}_S(T_i, X) \longrightarrow \operatorname{Mor}_S(T, X)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 32.6.1 parts (2) and (3) it suffices to check surjectivity of the map.

Proof. The proof is exactly the same as the proof of the implication "(2) implies (1)" in Proposition 32.6.1. Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \operatorname{colim}_i A_i$. According to Algebra, Lemma 10.127.3 it suffices to show that

$$\operatorname{colim}_i \operatorname{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i) \to \operatorname{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A)$$

is surjective. Consider the schemes $T_i = \operatorname{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \operatorname{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes

$$\operatorname{colim}_i \operatorname{Mor}_V(T_i, U) \to \operatorname{Mor}_V(\lim_i T_i, U)$$

We first observe that $\operatorname{Mor}_V(T_i, U) = \operatorname{Mor}_S(T_i, U)$ and $\operatorname{Mor}_V(T, U) = \operatorname{Mor}_S(T, U)$. Hence we have to show that

$$\operatorname{colim}_i \operatorname{Mor}_S(T_i, U) \to \operatorname{Mor}_S(\lim_i T_i, U)$$

is surjective and we are given that

$$\operatorname{colim}_i \operatorname{Mor}_S(T_i, X) \to \operatorname{Mor}_S(\lim_i T_i, X)$$

is surjective. Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 32.4.8 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction. □

The following is an example application of Proposition 32.6.1.

0GWT **Lemma 32.6.4.** *Let S be a scheme. Let X and Y be schemes over S. Assume Y is locally of finite presentation over S. Let $x \in X$ be a closed point such that $U = X \setminus \{x\} \to X$ is quasi-compact. With $V = \operatorname{Spec}(\mathcal{O}_{X,x}) \setminus \{x\}$ there is a bijection*

$$\left\{\text{morphisms } X \to Y \text{ over } S\right\} \longrightarrow \left\{\begin{array}{c}(a, b) \text{ where } a : U \to Y \text{ and } b : \operatorname{Spec}(\mathcal{O}_{X,x}) \to Y \\ \text{are morphisms over } S \text{ which agree over } V\end{array}\right\}$$

Proof. Let $W \subset X$ be an open neighbourhood of x. By glueing of schemes, see Schemes, Section 26.14 the result holds if we consider pairs of morphisms $a : U \to Y$ and $c : W \to Y$ which agree over $U \cap W$. We have $\mathcal{O}_{X,x} = \operatorname{colim} \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of x in X. Hence $\operatorname{Spec}(\mathcal{O}_{X,x}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus by Proposition 32.6.1 any morphism $b : \operatorname{Spec}(\mathcal{O}_{X,x}) \to Y$ over S comes from a morphism $c : W \to Y$ for

some W as above (and c is unique up to further shrinking W). For every affine open $x \in W$ we see that $U \cap W$ is quasi-compact as $U \to X$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{x\}$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus if a and b agree over V, then after shrinking W we see that a and c agree over $U \cap W$ (by the same proposition). The lemma follows. □

32.7. Relative approximation

09MU We discuss variants of Proposition 32.5.4 over a base.

0GS1 **Lemma 32.7.1.** *Let $f : X \to S$ be a morphism of quasi-compact and quasi-separated schemes. Then there exists a direct set I and an inverse system $(f_i : X_i \to S_i)$ of morphisms schemes over I, such that the transition morphisms $X_i \to X_{i'}$ and $S_i \to S_{i'}$ are affine, such that X_i and S_i are of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim(X_i \to S_i)$.*

Proof. Write $X = \lim_{a \in A} X_a$ and $S = \lim_{b \in B} S_b$ as in Proposition 32.5.4, i.e., with X_a and S_b of finite type over $\mathbf{Z}$ and with affine transition morphisms.

Fix $b \in B$. By Proposition 32.6.1 applied to S_b and $X = \lim X_a$ over $\mathbf{Z}$ we find there exists an $a \in A$ and a morphism $f_{a,b} : X_a \to S_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & S_b \end{array}$$

commute. Let I be the set of triples $(a, b, f_{a,b})$ we obtain in this manner.

Let $(a, b, f_{a,b})$ and $(a', b', f_{a',b'})$ be in I. Let $b'' \leq \min(b, b')$. By Proposition 32.6.1 again, there exists an $a'' \geq \max(a, a')$ such that the compositions $X_{a''} \to X_a \to S_b \to S_{b''}$ and $X_{a''} \to X_{a'} \to S_{b'} \to S_{b''}$ are equal. We endow I with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b \geq b',\ \text{and } g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b,b'} : S_b \to S_{b'}$ are the transition morphisms. The remarks above show that I is directed and that the maps $I \to A$, $(a, b, f_{a,b}) \mapsto a$ and $I \to B$, $(a, b, f_{a,b})$ are cofinal. If for $i = (a, b, f_{a,b})$ we set $X_i = X_a$, $S_i = S_b$, and $f_i = f_{a,b}$, then we get an inverse system of morphisms over I and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} S_i = \lim_{b \in B} S_b = S$$

by Categories, Lemma 4.17.4 (recall that limits over I are really limits over the opposite category associated to I and hence cofinal turns into initial). This finishes the proof. □

09MV **Lemma 32.7.2.** *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *S is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of schemes X_i of finite presentation over S with affine transition morphisms over S.

Proof. Since $f(X)$ is quasi-compact we may replace S by a quasi-compact open containing $f(X)$. Hence we may assume S is quasi-compact. By Lemma 32.7.1 we can write $(X \to S) = \lim(X_i \to S_i)$ for some directed inverse system of morphisms of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. Since limits commute with limits (Categories, Lemma 4.14.10) we have $X = \lim X_i \times_{S_i} S$. Let $i \geq i'$ in I. The morphism $X_i \times_{S_i} S \to X_{i'} \times_{S_{i'}} S$ is affine as the composition

$$X_i \times_{S_i} S \to X_i \times_{S_{i'}} S \to X_{i'} \times_{S_{i'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 26.21.9) and the second is a base change of an affine morphism (Morphisms, Lemma 29.11.9) and the composition of affine morphisms is affine (Morphisms, Lemma 29.11.8). The morphisms f_i are of finite presentation (Morphisms, Lemmas 29.21.9 and 29.21.11) and hence the base changes $X_i \times_{f_i, S_i} S \to S$ are of finite presentation (Morphisms, Lemma 29.21.4). □

09YZ **Lemma 32.7.3.** *Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.*

Proof. Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 26.24.1. Combining Properties, Lemma 28.22.13 we can write $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \underline{\operatorname{Spec}}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim_i X_i$ which proves the lemma. □

32.8. Descending properties of morphisms

081C This section is the analogue of Section 32.4 for properties of morphisms over S. We will work in the following situation.

081D **Situation 32.8.1.** Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 32.2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

01ZN **Lemma 32.8.2.** *Notation and assumptions as in Situation 32.8.1. If f is affine, then there exists an index $i \geq 0$ such that f_i is affine.*

Proof. Let $Y_0 = \bigcup_{j=1,\ldots,m} V_{j,0}$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_{j,0})$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in Y_i and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 32.2.2). Since U_j is affine by assumption we see that each $U_{j,i}$ is affine for i large enough, see Lemma 32.4.13. As there are finitely many j we can pick an i which works for all j. Thus f_i is affine for i large enough, see Morphisms, Lemma 29.11.3. □

01ZO **Lemma 32.8.3.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is a finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is finite.

Proof. A finite morphism is affine, see Morphisms, Definition 29.44.1. Hence by Lemma 32.8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement follows from Algebra, Lemma 10.168.3. □

0C4W **Lemma 32.8.4.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is unramified, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is unramified.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being unramified is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite type. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is unramified, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is unramified. This follows from Algebra, Lemma 10.168.5. □

01ZP **Lemma 32.8.5.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is a closed immersion, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is a closed immersion.

Proof. A closed immersion is affine, see Morphisms, Lemma 29.11.10. Hence by Lemma 32.8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement is a consequence of Algebra, Lemma 10.168.4. □

01ZQ **Lemma 32.8.6.** *Notation and assumptions as in Situation 32.8.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 32.8.5 to the diagonal morphism $\Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0$. (This is permissible as diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{S_0} X_0$ is quasi-compact and quasi-separated, see Schemes, Lemma 26.21.2, Morphisms, Lemma 29.15.5, and Schemes, Remark 26.21.18. □

04AI **Lemma 32.8.7.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is flat,*
(2) *f_0 is locally of finite presentation,*

then f_i is flat for some $i \geq 0$.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which

are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 10.168.1 part (3). □

06AC **Lemma 32.8.8.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is finite locally free (of degree d),*
(2) *f_0 is locally of finite presentation,*

then f_i is finite locally free (of degree d) for some $i \geq 0$.

Proof. By Lemmas 32.8.7 and 32.8.3 we find an i such that f_i is flat and finite. On the other hand, f_i is locally of finite presentation. Hence f_i is finite locally free by Morphisms, Lemma 29.48.2. If moreover f is finite locally free of degree d, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which f_i has degree d. By Lemma 32.4.10 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then $f_{i'}$ will be finite locally free of degree d. □

0C0C **Lemma 32.8.9.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is smooth,*
(2) *f_0 is locally of finite presentation,*

then f_i is smooth for some $i \geq 0$.

Proof. Being smooth is local on the source and the target (Morphisms, Lemma 29.34.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 10.168.8. □

07RP **Lemma 32.8.10.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is étale,*
(2) *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 29.36.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 10.168.7. □

081E **Lemma 32.8.11.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is an isomorphism, and*
(2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 32.8.10 and 32.8.5 we can find an i such that f_i is flat and a closed immersion. Then f_i identifies X_i with an open and closed subscheme of Y_i, see Morphisms, Lemma 29.26.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$. Thus by Lemma 32.4.10 we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. □

0EUU **Lemma 32.8.12.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is an open immersion, and*

(2) f_0 *is locally of finite presentation,*

then f_i *is an open immersion for some* $i \geq 0$.

Proof. By Lemma 32.8.10 we can find an i such that f_i is étale. Then $V_i = f_i(X_i)$ is a quasi-compact open subscheme of Y_i (Morphisms, Lemma 29.36.13). let V and $V_{i'}$ for $i' \geq i$ be the inverse image of V_i in Y and $Y_{i'}$. Then $f : X \to V$ is an isomorphism (namely it is a surjective open immersion). Hence by Lemma 32.8.11 we see that $X_{i'} \to V_{i'}$ is an isomorphism for some $i' \geq i$ as desired. □

0GTB **Lemma 32.8.13.** *Notation and assumptions as in Situation 32.8.1. If*

(1) f *is an immersion, and*
(2) f_0 *is locally of finite type,*

then f_i *is an immersion for some* $i \geq 0$.

Proof. There exists an open $V \subset Y$ such that the morphism f factors as $X \to V \to Y$ and such that $X \to V$ is a closed immersion, see discussion in Schemes, Section 26.10. Since X is quasi-compact, we may and do assume V is a quasi-compact open of Y. By Lemma 32.4.11 after increasing 0 we can find a quasi-compact open $V_0 \subset Y_0$ such that V is the inverse image of V_0. Then the inverse image of V_0 in X_0 is a quasi-compact open whose inverse image in X is X. Hence by the same lemma applied to $X = \lim X_i$ we may assume after increasing 0 that we have the factorization $X_0 \to V_0 \to Y_0$. Then for large enough $i \geq 0$ the morphism $X_i \to V_i$ where $V_i = Y_i \times_{Y_0} V_0$ is a closed immersion by Lemma 32.8.5 and the proof is complete. □

07RQ **Lemma 32.8.14.** *Notation and assumptions as in Situation 32.8.1. If*

(1) f *is a monomorphism, and*
(2) f_0 *is locally of finite type,*

then f_i *is a monomorphism for some* $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 26.23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 29.21.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 26.21.18) we conclude from Lemma 32.8.11 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

07RR **Lemma 32.8.15.** *Notation and assumptions as in Situation 32.8.1. If*

(1) f *is surjective, and*
(2) f_0 *is locally of finite presentation,*

then there exists an $i \geq 0$ *such that* f_i *is surjective.*

Proof. The morphism f_0 is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of Y_0, see Morphisms, Lemma 29.22.2. Since f_i is the base change of f_0 by $Y_i \to Y_0$ we see that the image of f_i is the inverse image of E in Y_i. Moreover, we know that $Y \to Y_0$ maps into E. Hence we win by Lemma 32.4.10. □

0C3L **Lemma 32.8.16.** *Notation and assumptions as in Situation 32.8.1. If*

(1) f *is syntomic, and*
(2) f_0 *is locally of finite presentation,*

then there exists an $i \geq 0$ *such that* f_i *is syntomic.*

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being syntomic is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i\in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is syntomic, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is syntomic. This follows from Algebra, Lemma 10.168.9. □

32.9. Finite type closed in finite presentation

01ZD A result of this type is [**Kie72**, Satz 2.10]. Another reference is [**Con07b**].

01ZE **Lemma 32.9.1.** *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 32.5.4 we can write $X = \lim_i X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,S} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & \operatorname{Spec}(\mathbf{Z}) \end{array}$$

Note that X_i is of finite presentation over $\operatorname{Spec}(\mathbf{Z})$, see Morphisms, Lemma 29.21.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 29.21.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 29.15.2. By Lemmas 32.4.11 and 32.4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = \lim_i V_{j,i}$. By Lemma 32.4.7 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. Thus the arrow in

$$V_j \longrightarrow U_j \times_{\operatorname{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since $\bigcup (V_{j,i})_{U_j}$ forms an open of $X_{i,S}$ and since the inverse image of $(V_{j,i})_{U_j}$ in X is V_j it follows that $X \to X_{i,S}$ is an immersion. □

01ZF **Remark 32.9.2.** We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a *closed* immersion $X \to X'$ as in the lemma. The reason is that this would

imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

01ZG **Lemma 32.9.3.** *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 32.9.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 26.21.12. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 29.21.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 29.21.3). □

09ZP **Lemma 32.9.4.** *Let $X \to Y$ be a closed immersion of schemes. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ of schemes over Y where $X_i \to Y$ is a closed immersion of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 28.22.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \operatorname{colim}_{i \in I} \mathcal{I}_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $\mathcal{I}_i$. These form an inverse system of schemes indexed by I. The transition morphisms $X_i \to X_{i'}$ are affine because they are closed immersions. Each X_i is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_i X_i$ as follows directly from the fact that $\mathcal{I} = \operatorname{colim}_{i \in I} \mathcal{I}_a$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma 29.21.7. □

09ZQ **Lemma 32.9.5.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the X_i are quasi-compact and quasi-separated, and the transition morphisms $X_{i'} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all i).

Proof. By Lemma 32.9.3 there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then Y is quasi-separated by Schemes, Lemma 26.21.12. Since X is quasi-compact, we may assume Y is quasi-compact by replacing Y with a quasi-compact open containing X. We see that $X = \lim X_i$ with $X_i \to Y$ a closed

immersion of finite presentation by Lemma 32.9.4. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma 29.21.3. □

01ZJ **Proposition 32.9.6.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is of finite type and separated, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. Apply Lemma 32.9.5 and note that $X_i \to S$ is separated for large i by Lemma 32.4.17 as we have assumed that $X \to S$ is separated. □

01ZK **Lemma 32.9.7.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We may write $X = \lim X_i$ as in Lemma 32.9.5. Applying Lemma 32.4.19 we see that $X_i \to S$ is finite for large enough i. □

09YY **Lemma 32.9.8.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S quasi-compact and quasi-separated.*

Then X is a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over S.

Proof. We may write $X = \lim X_i$ as in Lemma 32.9.5. Applying Lemma 32.4.19 we see that $X_i \to S$ is finite for large enough i. □

32.10. Descending relative objects

01ZL The following lemma is typical of the type of results in this section. We write out the "standard" proof completely. It may be faster to convince yourself that the result is true than to read this proof.

01ZM **Lemma 32.10.1.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.*
(2) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i, and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S, there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.*
(3) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$ whose base changes $\varphi_{i,S} = \psi_{i,S}$ are equal, there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.*

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 10.127.8. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 32.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S_i. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 32.4.11 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 32.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S_i. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of X. Hence by Lemma 32.4.11 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10).

Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 32.4.11 and 32.4.13 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 32.4.11 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1 j_2} = U_{j_2 j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \mathrm{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 26.14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_2,j_1,i} \cap W_{j_2,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 32.4.11 and the fact that $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$. The second from part (3) of the lemma proved above and the fact that the cocycle condition holds for the maps $\mathrm{id} : U_{j_1 j_2} \to U_{j_2 j_1}$. Ok, so now we can use Schemes, Lemma 26.14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1 j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. □

01ZR **Lemma 32.10.2.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ of finite presentation there exists an index $i \in I$ and a sheaf of $\mathcal{O}_{S_i}$-modules of finite presentation $\mathcal{F}_i$ such that $\mathcal{F} \cong f_i^* \mathcal{F}_i$.*
(2) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a morphism $\varphi : f_i^* \mathcal{F}_i \to f_i^* \mathcal{G}_i$ over S. Then there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : f_{i'i}^* \mathcal{F}_i \to f_{i'i}^* \mathcal{G}_i$ whose base change to S is φ.*
(3) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a pair of morphisms $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. Assume that the base changes are equal: $f_i^* \varphi_i = f_i^* \psi_i$. Then there exists an index $i' \geq i$ such that $f_{i'i}^* \varphi_i = f_{i'i}^* \psi_i$.*

In other words, the category of modules of finite presentation over S is the colimit over I of the categories modules of finite presentation over S_i.

Proof. We sketch two proofs, but we omit the details.

First proof. If S and S_i are affine schemes, then this lemma is equivalent to Algebra, Lemma 10.127.6. In the general case, use Zariski glueing to deduce it from the affine case.

Second proof. We use

(1) there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 27.6, and
(2) a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Having said this, we can use Lemma 32.10.1 to show that the category of vector bundles of finite presentation over S is the colimit over I of the categories of vector bundles over S_i. □

0B8W **Lemma 32.10.3.** *Let $S = \lim S_i$ be the limit of a directed system of quasi-compact and quasi-separated schemes S_i with affine transition morphisms. Then*

(1) *any finite locally free $\mathcal{O}_S$-module is the pullback of a finite locally free $\mathcal{O}_{S_i}$-module for some i,*
(2) *any invertible $\mathcal{O}_S$-module is the pullback of an invertible $\mathcal{O}_{S_i}$-module for some i, and*
(3) *any finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ is of the form $\mathcal{I}_i \cdot \mathcal{O}_S$ for some i and some finite type quasi-coherent ideal $\mathcal{I}_i \subset \mathcal{O}_{S_i}$.*

Proof. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module. Since finite locally free modules are of finite presentation we can find an i and an $\mathcal{O}_{S_i}$-module $\mathcal{E}_i$ of finite presentation such that $f_i^*\mathcal{E}_i \cong \mathcal{E}$, see Lemma 32.10.2. After increasing i we may assume $\mathcal{E}_i$ is a flat $\mathcal{O}_{S_i}$-module, see Algebra, Lemma 10.168.1. (Using this lemma is not necessary, but it is convenient.) Then $\mathcal{E}_i$ is finite locally free by Algebra, Lemma 10.78.2.

If $\mathcal{L}$ is an invertible $\mathcal{O}_S$-module, then by the above we can find an i and finite locally free $\mathcal{O}_{S_i}$-modules $\mathcal{L}_i$ and $\mathcal{N}_i$ pulling back to $\mathcal{L}$ and $\mathcal{L}^{\otimes -1}$. After possible increasing i we see that the map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ descends to a map $\mathcal{L}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{N}_i \to \mathcal{O}_{S_i}$. And after increasing i further, we may assume it is an isomorphism. It follows that $\mathcal{L}_i$ is an invertible module (Modules, Lemma 17.25.2) and the proof of (2) is complete.

Given $\mathcal{I}$ as in (3) we see that $\mathcal{O}_S \to \mathcal{O}_S/\mathcal{I}$ is a map of finitely presented $\mathcal{O}_S$-modules. Hence by Lemma 32.10.2 this is the pullback of some map $\mathcal{O}_{S_i} \to \mathcal{F}_i$ of finitely presented $\mathcal{O}_{S_i}$-modules. After increasing i we may assume this map is surjective (details omitted; hint: use Algebra, Lemma 10.127.5 on affine open cover). Then the kernel of $\mathcal{O}_{S_i} \to \mathcal{F}_i$ is a finite type quasi-coherent ideal in $\mathcal{O}_{S_i}$ whose pullback gives $\mathcal{I}$. □

05LY **Lemma 32.10.4.** *With notation and assumptions as in Lemma 32.10.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i and that $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite presentation. If the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S$ is flat over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.*

Proof. (This lemma is the analogue of Lemma 32.8.7 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^*\mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^*\mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and

$V_j = \lim_{i' \geq i} V_j$ (see Lemma 32.2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 10.168.1 part (3). □

0EY1 **Lemma 32.10.5.** *For a scheme T denote $\mathcal{C}_T$ the full subcategory of schemes W over T such that W is quasi-compact and quasi-separated and such that the structure morphism $W \to T$ is locally of finite presentation. Let $S = \lim S_i$ be a directed limit of schemes with affine transition morphisms. Then there is an equivalence of categories*

$$\operatorname{colim} \mathcal{C}_{S_i} \longrightarrow \mathcal{C}_S$$

given by the base change functors.

Warning: do not use this lemma if you do not understand the difference between this lemma and Lemma 32.10.1.

Proof. Fully faithfulness. Suppose we have $i \in I$ and objects X_i, Y_i of $\mathcal{C}_{S_i}$. Denote $X = X_i \times_{S_i} S$ and $Y = Y_i \times_{S_i} S$. Suppose given a morphism $f : X \to Y$ over S. We can choose a finite affine open covering $Y_i = V_{i,1} \cup \ldots \cup V_{i,m}$ such that $V_{i,j} \to Y_i \to S_i$ maps into an affine open $W_{i,j}$ of S_i. Denote $Y = V_1 \cup \ldots \cup V_m$ the induced affine open covering of Y. Since $f : X \to Y$ is quasi-compact (Schemes, Lemma 26.21.14) after increasing i we may assume that there is a finite open covering $X_i = U_{i,1} \cup \ldots \cup U_{i,m}$ by quasi-compact opens such that the inverse image of $U_{i,j}$ in Y is $f^{-1}(V_j)$, see Lemma 32.4.11. By Lemma 32.10.1 applied to $f|_{f^{-1}(V_j)}$ over W_j we may assume, after increasing i, that there is a morphism $f_{i,j} : V_{i,j} \to U_{i,j}$ over S whose base change to S is $f|_{f^{-1}(V_j)}$. Increasing i more we may assume $f_{i,j}$ and $f_{i,j'}$ agree on the quasi-compact open $U_{i,j} \cap U_{i,j'}$. Then we can glue these morphisms to get the desired morphism $f_i : X_i \to Y_i$. This morphism is unique (up to increasing i) because this is true for the morphisms $f_{i,j}$.

To show that the functor is essentially surjective we argue in exactly the same way. Namely, suppose that X is an object of $\mathcal{C}_S$. Pick $i \in I$. We can choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that $U_j \to X \to S \to S_i$ factors through an affine open $W_{i,j} \subset S_i$. Set $W_j = W_{i,j} \times_{S_i} S$. This is an affine open of S. By Lemma 32.10.1, after increasing i, we may assume there exist $U_{i,j} \to W_{i,j}$ of finite presentation whose base change to W_j is U_j. After increasing i we may assume there exist quasi-compact opens $U_{i,j,j'} \subset U_{i,j}$ whose base changes to S are equal to $U_j \cap U_{j'}$. Claim: after increasing i we may assume the image of the morphism $U_{i,j,j'} \to U_{i,j} \to W_{i,j}$ ends up in $W_{i,j} \cap W_{i,j'}$. Namely, because the complement of $W_{i,j} \cap W_{i,j'}$ is closed in the affine scheme $W_{i,j}$ it is affine. Since $U_j \cap U_{j'} = \lim U_{i,j,j'}$ does map into $W_{i,j} \cap W_{i,j'}$ we can apply Lemma 32.4.9 to get the claim. Thus we can view both

$$U_{i,j,j'} \quad \text{and} \quad U_{i,j',j}$$

as schemes over $W_{i,j'}$ whose base changes to $W_{j'}$ recover $U_j \cap U_{j'}$. Hence after increasing i, using Lemma 32.10.1, we may assume there are isomorphisms $U_{i,j,j'} \to$

$U_{i,j',j}$ over $W_{i,j'}$ and hence over S_i. Increasing i further (details omitted) we may assume these isomorphisms satisfy the cocycle condition mentioned in Schemes, Section 26.14. Applying Schemes, Lemma 26.14.1 we obtain an object X_i of $\mathcal{C}_{S_i}$ whose base change to S is isomorphic to X; we omit some of the verifications. □

32.11. Characterizing affine schemes

01ZS If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [**Con07b**, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 30.13.3. See also [**DG67**, II 6.7.1].

01ZT **Lemma 32.11.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 29.44.7), we see that S is separated (Morphisms, Lemma 29.41.11).

By Lemma 32.9.8 we can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. By Lemma 32.4.13 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 32.5.4 we may write $S = \lim_{i \in I} S_i$ as a directed limits of schemes of finite type over $\mathbf{Z}$. By Lemma 32.10.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. By Lemma 32.8.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 32.4.13 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 30.13.3. □

05YU **Proposition 32.11.2.** *Let $f : X \to S$ be a morphism of schemes. Assume X is affine and that f is surjective and universally closed[2]. Then S is affine.*

Proof. By Morphisms, Lemma 29.41.11 the scheme S is separated. Then by Morphisms, Lemma 29.11.12 we find that f is affine. Whereupon by Morphisms, Lemma 29.44.7 we see that f is integral.

By the preceding paragraph, we may assume $f : X \to S$ is surjective and integral, X is affine, and S is separated. Since f is surjective and X is quasi-compact we also deduce that S is quasi-compact.

By Lemma 32.7.3 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 32.4.13 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 32.11.1. □

09NL **Lemma 32.11.3.** *Let X be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then X is affine.*

Proof. Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod Z_i \to X$ is surjective and integral with affine source. Hence X is affine by Proposition 32.11.2. □

[2] An integral morphism is universally closed, see Morphisms, Lemma 29.44.7.

09MW **Lemma 32.11.4.** *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. Then $i^*\mathcal{L}$ is ample on Z, if and only if $\mathcal{L}$ is ample on X.*

Proof. If $\mathcal{L}$ is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 29.37.7. Assume $i^*\mathcal{L}$ is ample. Then Z is quasi-compact (Properties, Definition 28.26.1) and separated (Properties, Lemma 28.26.8). Since i is surjective, we see that X is quasi-compact. Since i is universally closed and surjective, we see that X is separated (Morphisms, Lemma 29.41.11).

By Proposition 32.5.4 we can write $X = \lim X_i$ as a directed limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. We can find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X is isomorphic to $\mathcal{L}$, see Lemma 32.10.2.

For each i let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X_i$. If $\mathrm{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image $\mathrm{Spec}(A)$ in X and if $Z \cap \mathrm{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \mathrm{Spec}(A_i)$ is defined by the ideal $I_i \subset A_i$ which is the inverse image of I in A_i under the ring map $A_i \to A$, see Morphisms, Example 29.6.4. Since $\mathrm{colim}\, A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 32.4.15 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some i. Since Z and hence X maps into Z_i set theoretically, we see that $X_{i'} \to X_i$ maps into Z_i set theoretically for some $i' \geq i$, see Lemma 32.4.10. (Observe that since X_i is Noetherian, every closed subset of X_i is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of Z_i in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 29.37.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 30.17.5. Pulling back to X (using the same lemma as above) we find that $\mathcal{L}$ is ample. □

0B7L **Lemma 32.11.5.** *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.*

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 28.27.1. Hence if Z is quasi-affine, then $\mathcal{O}_Z$ is ample, hence $\mathcal{O}_X$ is ample by Lemma 32.11.4, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. □

The following lemma does not really belong in this section.

0E21 **Lemma 32.11.6.** *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Assume we have morphisms of schemes*

$$\mathrm{Spec}(k) \leftarrow \mathrm{Spec}(A) \to W \subset X$$

where k is a field, A is an integral k-algebra, W is open in X. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine, $X_s \subset W$, and $\mathrm{Spec}(A) \to W$ factors through X_s

Proof. Since $\mathrm{Spec}(A)$ is quasi-compact, we may replace W by a quasi-compact open still containing the image of $\mathrm{Spec}(A) \to X$. Recall that X is quasi-separated and quasi-compact by dint of having an ample invertible sheaf, see Properties, Definition 28.26.1 and Lemma 28.26.7. By Proposition 32.5.4 we can write $X = \lim X_i$ as a limit of a directed system of schemes of finite type over $\mathbf{Z}$ with affine

transition morphisms. For some i the ample invertible sheaf $\mathcal{L}$ on X descends to an ample invertible sheaf $\mathcal{L}_i$ on X_i and the open W is the inverse image of a quasi-compact open $W_i \subset X_i$, see Lemmas 32.4.15, 32.10.3, and 32.4.11. We may replace $X, W, \mathcal{L}$ by $X_i, W_i, \mathcal{L}_i$ and assume X is of finite presentation over $\mathbf{Z}$. Write $A = \operatorname{colim} A_j$ as the colimit of its finite k-subalgebras. Then for some j the morphism $\operatorname{Spec}(A) \to X$ factors through a morphism $\operatorname{Spec}(A_j) \to X$, see Proposition 32.6.1. Since $\operatorname{Spec}(A_j)$ is finite this reduces the lemma to Properties, Lemma 28.29.6. □

32.12. Variants of Chow's Lemma

01ZZ In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 30.18.

0202 **Lemma 32.12.1.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 32.9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 32.5.4. By Lemma 32.10.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 32.8.6 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 30.18.1. □

0GII **Remark 32.12.2.** In the situation of Chow's Lemma 32.12.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 29.43.3) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 29.41.7).
(2) We may assume that X' is reduced as we can replace X' by its reduction without changing the other assertions of the lemma.
(3) We may assume that $X' \to X$ is of finite presentation without changing the other assertions of the lemma. This can be deduced from the proof of Lemma 32.12.1 but we can also prove this directly as follows. By (1) we have a closed immersion $X' \to \mathbf{P}^n_X$. By Lemma 32.9.4 we can write $X' = \lim X'_i$ where $X'_i \to \mathbf{P}^n_X$ is a closed immersion of finite presentation. In particular $X'_i \to X$ is of finite presentation, proper, and surjective. For large enough i the morphism $X'_i \to \mathbf{P}^n_S$ is an immersion by Lemma 32.4.16. Replacing X' by X'_i we get what we want.

Of course in general we can't simultaneously achieve both (2) and (3).

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

0203 **Lemma 32.12.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 30.18.1 using the general Properties, Lemma 28.29.4 to find suitable affine opens in X. (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 32.12.1 above. This is what we will do here.

By Proposition 32.9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 32.5.4. By Lemma 32.10.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 32.8.6 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$\begin{array}{ccccccc} \eta_j \in Z_j & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Y_i \\ & & & \searrow & \downarrow & & \downarrow \\ & & & & S & \longrightarrow & S_i \end{array}$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, see Lemma 32.2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 32.4.6 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 29.6.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 29.6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 29.8.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Cohomology of Schemes, Lemma 30.18.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$\begin{array}{ccccc} Y' & \xleftarrow{\pi} & Y^* & \longrightarrow & \mathbf{P}^n_{S_i} \\ & \searrow & \downarrow & \swarrow & \\ & & S_i & & \end{array}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

$$\begin{array}{ccccccc} X' & = & X \times_{Y'} Y^* & \longrightarrow & Y^* & \longrightarrow & \mathbf{P}^n_{S_i} \\ & & \downarrow & & \downarrow & \nearrow & \\ & & X & \longrightarrow & Y' & & \\ & & \downarrow & & \downarrow & & \\ & & S & \longrightarrow & S_i & & \end{array}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. □

32.13. Applications of Chow's lemma

0204 Here is a first application of Chow's lemma.

081F **Lemma 32.13.1.** *Assumptions and notation as in Situation 32.8.1. If*

(1) *f is proper, and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. By Lemma 32.8.6 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact, see Schemes, Lemma 26.21.14. By Lemma 32.12.1 we can choose a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{\pi} & X'_0 & \longrightarrow & \mathbf{P}^n_{Y_0} \\ & \searrow & \downarrow & \swarrow & \\ & & Y_0 & & \end{array}$$

where $X'_0 \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X'_0 \to X_0$ is proper and surjective. Introduce $X' = X'_0 \times_{Y_0} Y$ and $X'_i = X'_0 \times_{Y_0} Y_i$. By Morphisms, Lemmas 29.41.4 and 29.41.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 29.41.7). By Morphisms, Lemma 29.41.9 it suffices to prove that $X'_i \to Y_i$ is proper for some i. By Lemma 32.8.5 we find that $X'_i \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X'_i \to Y_i$ is proper and we win. □

09ZR **Lemma 32.13.2.** *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of schemes X_i proper and of finite presentation over S such that all transition morphisms and the morphisms $X \to X_i$ are closed immersions.*

Proof. By Proposition 32.9.6 we can find a closed immersion $X \to Y$ with Y separated and of finite presentation over S. By Lemma 32.12.1 we can find a diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & Y' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 32.9.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X'_i \subset Y'$, resp. $X' \subset Y'$ the scheme theoretic inverse image of $X_i \subset Y$, resp. $X \subset Y$. Then $\lim X'_i = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 29.41.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 29.41.7). Hence for i large enough we find that $X'_i \to \mathbf{P}^n_S$ is a closed immersion by Lemma 32.4.20. Thus X'_i is proper over S. For such i the morphism $X_i \to S$ is proper by Morphisms, Lemma 29.41.9. □

0A0P **Lemma 32.13.3.** *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to S_i)$ of morphisms of schemes over I, such that the transition morphisms $X_i \to X_{i'}$ and $S_i \to S_{i'}$ are affine, such that f_i is proper, such that S_i is of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim(X_i \to S_i)$.*

Proof. By Lemma 32.13.2 we can write $X = \lim_{k \in K} X_k$ with $X_k \to S$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 32.5.4) we can write $S = \lim_{j \in J} S_j$ with S_j of finite type over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to S_j$ of finite presentation with $X_k \cong S \times_{S_j} X_{k,j}$ as schemes over S, see Lemma 32.10.1. After increasing j we may assume $X_{k,j} \to S_j$ is proper, see Lemma 32.13.1. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to S_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $S_{j'} \to S_j$ whose base change to S gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 32.10.1). These morphisms form the transition morphisms of the system. Some details omitted. □

0EX1 **Lemma 32.13.4.** *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S. If $Y \to X$ is proper, X_i quasi-compact and quasi-separated, and Y locally of finite type over S, then $Y \to X_i$ is proper for i large enough.*

Proof. Choose a closed immersion $Y \to Y'$ with Y' proper and of finite presentation over X, see Lemma 32.13.2. Then choose an i and a proper morphism $Y'_i \to X_i$ such that $Y' = X \times_{X_i} Y'_i$. This is possible by Lemmas 32.10.1 and 32.13.1. Then after replacing i by a larger index we have that $Y \to Y'_i$ is a closed immersion, see Lemma 32.4.16. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 29.5.5.

081G **Lemma 32.13.5.** *Assumptions and notation as in Situation 32.8.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

(1) *f_0 is locally of finite type,*

(2) $\mathcal{F}_0$ *is of finite type,*
(3) *the scheme theoretic support of* $\mathcal{F}$ *is proper over* Y.

Then the scheme theoretic support of $\mathcal{F}_i$ *is proper over* Y_i *for some* i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms, Lemma 29.5.3 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 29.41.9. By Lemma 32.13.1 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms, Lemmas 29.41.6 and 29.41.4. □

32.14. Universally closed morphisms

05JW In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

05BD **Lemma 32.14.1.** *Let* $f : X \to S$ *be a quasi-compact morphism of schemes. Let* $g : T \to S$ *be a morphism of schemes. Let* $t \in T$ *be a point and* $Z \subset X_T$ *be a closed subscheme such that* $Z \cap X_t = \emptyset$. *Then there exists an open neighbourhood* $V \subset T$ *of* t, *a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ *such that*

(1) *the morphism* $b : T' \to S$ *is locally of finite presentation,*
(2) *with* $t' = a(t)$ *we have* $Z' \cap X_{t'} = \emptyset$, *and*
(3) $Z \cap X_V$ *maps into* Z' *via the morphism* $X_V \to X_{T'}$.

Moreover, we may assume V *and* T' *are affine.*

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = \mathrm{Spec}(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots n} \mathrm{Spec}(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} (C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \notin \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \notin \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = \operatorname{Spec}(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic image of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\operatorname{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 29.6.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. □

05JX **Lemma 32.14.2.** *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and*
(3) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 26.19.8 this means that there exists some specialization $t_1 \rightsquigarrow t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 32.14.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \rightsquigarrow a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed

implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion $T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 29.21.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{array}{ccc} X_T & \longrightarrow & \mathbf{A}^n \times X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f_n} \\ T & \longrightarrow & \mathbf{A}^n \times S \end{array}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. □

0205 **Lemma 32.14.3.** *Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.*
(3) *For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

First proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 32.14.2. □

Second proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 29.41.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 32.12.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 29.41.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma

29.41.7) and hence a closed immersion (Schemes, Lemma 26.10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

05LZ (32.14.3.1)
$$\begin{array}{ccc} X' & \longrightarrow & \mathbf{P}^n_S \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 29.41.9. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine n-space, see Constructions, Lemma 27.13.3. By Morphisms, Lemma 29.41.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 26.21.11). Composing these morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we find that the immersion $X' \to \mathbf{P}^n_S$ is closed, and hence a closed immersion. □

32.15. Noetherian valuative criterion

0CM1 If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CM2 **Lemma 32.15.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume f finite type and Y locally Noetherian. Let $y \in Y$ be a point in the closure of the image of f. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. Moreover, we can assume that the image point of $\mathrm{Spec}(K) \to X$ is a generic point η of an irreducible component of X and that $K = \kappa(\eta)$.

Proof. By the non-Noetherian version of this lemma (Morphisms, Lemma 29.6.5) there exists a point $x \in X$ such that $f(x)$ specializes to y. We may replace x by any point specializing to x, hence we may assume that x is a generic point of an irreducible component of X. This produces a ring map $\mathcal{O}_{Y,y} \to \kappa(x)$ (see Schemes,

Section 26.13). Let $R \subset \kappa(x)$ be the image. Then R is Noetherian as a quotient of the Noetherian local ring $\mathcal{O}_{Y,y}$. On the other hand, the extension $\kappa(x)$ is a finitely generated extension of the fraction field of R as f is of finite type. Thus there exists a discrete valuation ring $A \subset \kappa(x)$ with fraction field $\kappa(x)$ dominating R by Algebra, Lemma 10.119.13. Then

$$\begin{array}{ccccccc} \operatorname{Spec}(\kappa(x)) & & & & & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(R) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{Y,y}) & \longrightarrow & Y \end{array}$$

gives the desired diagram. □

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

0206 (32.15.1.1)
$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & S \end{array}$$

with A a valuation ring and K its field of fractions.

0207 **Lemma 32.15.2.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:*

1. *The morphism f is separated.*
2. *For any diagram (32.15.1.1) there is at most one dotted arrow.*
3. *For all diagrams (32.15.1.1) with A a discrete valuation ring there is at most one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (32.15.1.1) such that the morphism $\operatorname{Spec}(K) \to X$ is the canonical one (see Schemes, Section 26.13) there is at most one dotted arrow.*

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 26.21.7). Hence, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

05M0 (32.15.2.1)
$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X_\alpha \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & S_\alpha \end{array}$$

would then give two distinct arrows in diagram (32.15.1.1) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine.

We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 29.15.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 28.5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 26.22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (32.15.1.1), the image of the closed points of $\operatorname{Spec} A$ will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of $\operatorname{Spec}(A)$ under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 29.15.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0, y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 10.119.13), there exists a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}_0, y}$ with fraction field K. This allows to construct a diagram:

05M1 (32.15.2.2)
$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X_0 \\ \downarrow & \nearrow & \downarrow{\scriptstyle \Delta} \\ \operatorname{Spec}(A) & \longrightarrow & X_0 \times_S X_0 \end{array}$$

which sends $\operatorname{Spec} K$ to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 26.13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

05M2 (32.15.2.3)
$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X_0 \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & S \end{array}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). □

0208 **Lemma 32.15.3.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *For any diagram (32.15.1.1) there exists exactly one dotted arrow.*
(3) *For all diagrams (32.15.1.1) with A a discrete valuation ring there exists exactly one dotted arrow.*
(4) *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (32.15.1.1) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 26.13) there exists exactly one dotted arrow.*

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 32.15.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 32.14.3, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 10.119.13 or Properties, Lemma 28.5.10 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0, y}$ and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

05M3 (32.15.3.1)

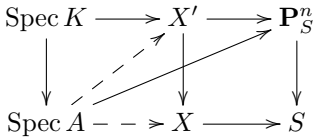

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $\mathrm{Spec}(A) \to X$ exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\mathrm{Spec}(A) \to X'$ (use Schemes, Proposition 26.20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\mathrm{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\mathrm{Spec}(A) \to \mathbf{P}^n_S$ to X'. □

05JY **Lemma 32.15.4.** *Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent*

(1) *f is universally closed,*
(2) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,*
(3) *for any diagram (32.15.1.1) there exists some dotted arrow,*
(4) *for all diagrams (32.15.1.1) with A a discrete valuation ring there exists some dotted arrow.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 32.14.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 26.20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 26.19.8. Pick n and a nontrivial specialization $z \rightsquigarrow z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 28.5.10 or Algebra, Lemma 10.119.13 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(y)) & \longrightarrow & \mathbf{A}^n \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathbf{A}^n \times S & \longrightarrow & S \end{array}$$

Now property (4) implies that there exists a morphism $\mathrm{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\mathrm{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\mathrm{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. □

32.16. Refined Noetherian valuative criteria

0H1P One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria. An example is given by Morphisms, Lemma 29.42.2. In the Noetherian setting, we have also seen this in Lemmas 32.15.2 and 32.15.3. Here is another variant.

0CM3 **Lemma 32.16.1.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, that f is separated, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \longrightarrow & & S \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. There is an immediate reduction to the case where S is affine. Then U is quasi-compact. Let $U = U_1 \cup \ldots \cup U_n$ be an affine open covering. We may replace

U by $U_1 \amalg \ldots \amalg U_n$ without changing the assumptions, hence we may assume U is affine. Thus we can find an open immersion $U \to Y$ over X with Y proper over X. (First put U inside $\mathbf{A}^n_X$ using Morphisms, Lemma 29.39.2 and then take the closure inside $\mathbf{P}^n_X$, or you can directly use Morphisms, Lemma 29.43.12.) We can assume U is dense in Y (replace Y by the scheme theoretic closure of U if necessary, see Morphisms, Section 29.7). Note that $g : Y \to X$ is surjective as the image is closed and contains the dense subset $h(U)$. We will show that $Y \to S$ is proper. This will imply that $X \to S$ is proper by Morphisms, Lemma 29.41.9 thereby finishing the proof. To show that $Y \to S$ is proper we will use part (4) of Lemma 32.15.3. To do this consider a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{y} & Y \\ \downarrow & \nearrow & \downarrow{\scriptstyle f\circ g} \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a discrete valuation ring with fraction field K and where $y : \mathrm{Spec}(K) \to Y$ is the inclusion of a generic point. We have to show there exists a unique dotted arrow. Uniqueness holds by the converse to the valuative criterion for separatedness (Schemes, Lemma 26.22.1) since $Y \to S$ is separated as the composition of the separated morphisms $Y \to X$ and $X \to S$ (Schemes, Lemma 26.21.12). Existence can be seen as follows. As y is a generic point of Y, it is contained in U. By assumption of the lemma there exists a morphism $a : \mathrm{Spec}(A) \to X$ such that

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{y} & U & \longrightarrow & X \\ \downarrow & & & \nearrow_{a} & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \longrightarrow & & S \end{array}$$

is commutative. Then since $Y \to X$ is proper, we can apply the valuative criterion for properness (Morphisms, Lemma 29.42.1) to find a morphism $b : \mathrm{Spec}(A) \to Y$ such that

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{y} & Y \\ \downarrow & \nearrow_{b} & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(A) & \xrightarrow{a} & X \end{array}$$

is commutative. This finishes the proof since b can serve as the dotted arrow above. □

0CM4 **Lemma 32.16.2.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f is locally of finite type, that h is of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \longrightarrow & & S \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 32.16.1 to the morphisms $U \to X$ and $\Delta : X \to X \times_S X$. We check the conditions. Observe that Δ is quasi-compact by Properties, Lemma 28.5.4 (and Schemes, Lemma 26.21.13). Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow \Delta \\ \mathrm{Spec}(A) & & \xrightarrow{(a,b)} & & X \times_S X \end{array}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. □

0CM5 **Lemma 32.16.3.** *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & & & & S \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 32.16.2 and 32.16.1. □

32.17. Valuative criteria over a Nagata base

0GWU When working with schemes locally of finite type over a Nagata base we can reduce to discrete valuation rings which are essentially of finite type over the base. The following are just some example results one can get.

0GWV **Lemma 32.17.1.** *Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a quasi-compact morphism of schemes locally of finite type over S. The following are equivalent*

(1) *f is universally closed,*
(2) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ is closed,*
(3) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over S such that
(a) *C is a normal integral scheme of finite type over S,*
(b) *$U = C \setminus \{c\}$ for some closed point $c \in C$,*
(c) *$A = \mathcal{O}_{C,c}$ has dimension 1*[3]

[3]It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $K = Frac(A)$ some dotted arrow exists[4] *making the diagram commute.*

Proof. We have seen the equivalence of (1) and (2) and the fact that these imply (3) in Lemma 32.15.4. Thus it suffices to prove that (3) implies (2). Observe that if condition (3) holds for $f : X \to Y$, then condition (3) holds for $1 \times f : \mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ (see argument in the proof of Lemma 32.15.4). Hence it suffices to show that (3) implies that f is closed.

Reduction to the case where Y and S are affine; we suggest skipping this paragraph. Let $S' \subset S$ be an affine open and let $Y' \subset Y$ be an affine open mapping into S'. Set $X' = f^{-1}(Y')$. Then we claim that the restriction $f' : X' \to Y'$ of f viewed as a morphism of schemes over S' has property (3) also. We omit the details. Now if we can prove that f' is closed for all choices of S' and Y', then it follows that f is closed. This reduces us to the case discussed in the next paragraph.

Assume S and Y affine. Let $Z \subset X$ be a closed subset. We may and do view Z as a reduced closed subscheme of X. We have to show that $E = f(Z)$ is closed. Pick $y \in Y$ a closed point contained in the closure of $f(Z)$. It suffices to show $y \in E$. We assume $y \notin E$ to get a contradiction. The image $s \in S$ of y is a finite type point of S, see Morphisms, Lemma 29.16.5. Recall that E is constructible (Morphisms, Lemma 29.22.2). Consider the intersection $\mathrm{Spec}(\mathcal{O}_{Y,y}) \cap E$. This is a constructible subset of the spectrum (Morphisms, Lemma 29.22.1) which doesn't contain the closed point. Since the punctured spectrum $\mathrm{Spec}(\mathcal{O}_{Y,y})\backslash\{y\}$ is Jacobson (Morphisms, Lemma 29.16.10), we find a closed point $t \in \mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{y\}$ with $t \in E$ (see Topology, Lemma 5.18.5). In other words, $t \in E$ is a point of Y which has an immediate specialization $t \rightsquigarrow y$. As $t \in E$ the scheme theoretic fibre Z_t is nonempty. Choose a closed point $x \in Z_t$. In particular we have $[\kappa(x) : \kappa(t)] < \infty$ by the Hilbert Nullstellensatz (Morphisms, Lemma 29.20.3).

Denote $T = \overline{\{t\}} \subset Y$ the integral closed subscheme whose underlying topological space is as indicated (Schemes, Definition 26.12.5). Then $t \in T$ is the generic point. Denote $C \to T$ the normalization of T in $\kappa(x)$, see Morphisms, Section 29.53 (more precisely, $C \to T$ is the normalization of T in x where we view $x = \mathrm{Spec}(\kappa(x)) \to T$ as a scheme over T). Since S is a Nagata scheme, so is T (Morphisms, Lemma 29.18.1). Hence we see that $C \to T$ is finite (Morphisms, Lemma 29.53.14). As t is in the image we see that $C \to T$ is surjective (because the image is closed and T is the closure of t in Y). Choose a point $c \in C$ mapping to $y \in T$. Since y is a closed point of T we see that c is a closed point of C. Since $\dim(\mathcal{O}_{T,y}) = 1$ we see that $\dim(\mathcal{O}_{C,c}) = 1$ (the dimension is at least 1 as c is not the generic point of C and at most 1 as $C \to T$ is finite). As the function field of C is $\kappa(x)$ and as x is a point of X, we have a Y-rational map from C to X (see for example Morphisms, Lemma

[4]By Lemma 32.6.4 this is equivalent to asking for the existence of dotted arrow making the first commutative diagram commute.

29.49.2). Let $C \supset U \to X$ be a representative (in particular U is nonempty). We may assume $c \notin U$ (replace U by $U \setminus \{c\}$). Since c is a closed point of codimension 1 in the integral scheme C we have $C = U \amalg \{c\} \amalg \Sigma$ for some proper closed subset $\Sigma \subset C$. After replacing C by $C \setminus \Sigma$ we have constructed a commutative diagram as in part (3). By the 2nd footnote in the statement of the lemma, the existence of the dotted arrow produces an extension of the rational map to all of C and we get the contradiction because the image of c will be a point of Z mapping to y. □

0GWW **Lemma 32.17.2.** *Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. The following are equivalent*

(1) *f separated,*
(2) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over S such that
(a) *C is a normal integral scheme of finite type over S,*
(b) *$U = C \setminus \{c\}$ for some closed point $c \in C$,*
(c) *$A = \mathcal{O}_{C,c}$ has dimension 1*[5]
then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $K = Frac(A)$ there exists at most one dotted arrow[6] *making the diagram commute.*

Proof. By Lemma 32.15.2 we see that (1) implies (2). Assume (2). In order to show that f is separated, we have to show that $\Delta : X \to X \times_Y X$ is closed. By Morphisms, Lemma 29.15.7 the morphism Δ is quasi-compact. By Lemma 32.17.1 it suffices to show: for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \Delta \\ C & \xrightarrow{(a_1,a_2)} & X \times_Y X \end{array}$$

of schemes over S such that

(1) C is a normal integral scheme of finite type over S,
(2) $U = C \setminus \{c\}$ for some closed point $c \in C$,
(3) $A = \mathcal{O}_{C,c}$ has dimension 1.

[5]It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

[6]By Lemma 32.6.4 this is equivalent to asking there to be at most one dotted arrow making the first commutative diagram commute.

then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \Delta \\ \mathrm{Spec}(A) & \longrightarrow & X \times_Y X \end{array}$$

where $K = \mathrm{Frac}(A)$ there exists some dotted arrow making the diagram commute. By Lemma 32.6.4 the existence of the dotted arrow in the second diagram is equivalent to the existence of the dotted arrow in the first diagram. Moreover, the existence there is the same as asking $a_1 = a_2$. However $a_1|_U = a_2|_U$, so by the uniqueness assumption (2) we see that this is true and the proof is complete. □

0GWX **Lemma 32.17.3.** *Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a quasi-compact morphism of schemes locally of finite type over S. The following are equivalent*

(1) *f proper,*
(2) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over S such that
(a) *C is a normal integral scheme of finite type over S,*
(b) *$U = C \setminus \{c\}$ for some closed point $c \in C$,*
(c) *$A = \mathcal{O}_{C,c}$ has dimension 1*[7]
then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $K = Frac(A)$ there exists exactly one dotted arrow[8] *making the diagram commute.*

Proof. This is formal from Lemmas 32.17.1 and 32.17.2 and the definition of proper morphisms as being finite type, separated, and universally closed. □

32.18. Limits and dimensions of fibres

05M4 The following lemma is most often used in the situation of Lemma 32.10.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

05M5 **Lemma 32.18.1.** *Let I be a directed set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume*

[7] It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

[8] By Lemma 32.6.4 this is equivalent to asking for the existence and uniqueness of the dotted arrow making the first commutative diagram commute.

(1) *all the morphisms $S_{i'} \to S_i$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated,*
(3) *the morphisms f_i are of finite type, and*
(4) *the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.*

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 29.28.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = \lim Z_i \neq \emptyset$, see Lemma 32.4.3. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. □

094M **Lemma 32.18.2.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f is a quasi-finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is quasi-finite.

Proof. Follows immediately from Lemma 32.18.1. □

0H3V **Lemma 32.18.3.** *Assumptions and notation as in Situation 32.8.1. Let $d \geq 0$. If*

(1) *f has relative dimension $\leq d$ (Morphisms, Definition 29.29.1), and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Follows immediately from Lemma 32.18.1. □

0EY2 **Lemma 32.18.4.** *Notation and assumptions as in Situation 32.8.1. If*

(1) *f has relative dimension d, and*
(2) *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i has relative dimension d.

Proof. By Lemma 32.18.1 we may assume all fibres of f_0 have dimension $\leq d$. By Morphisms, Lemma 29.28.6 the set $U_0 \subset X_0$ of points $x \in X_0$ such that the dimension of the fibre of $X_0 \to Y_0$ at x is $\leq d - 1$ is open and retrocompact in X_0. Hence the complement $E = X_0 \setminus U_0$ is constructible. Moreover the image of $X \to X_0$ is contained in E by Morphisms, Lemma 29.28.3. Thus for $i \gg 0$ we have that the image of $X_i \to X_0$ is contained in E (Lemma 32.4.10). Then all fibres of $X_i \to Y_i$ have dimension d by the aforementioned Morphisms, Lemma 29.28.3. □

05M6 **Lemma 32.18.5.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that*

(1) *$Z \subset Z'$,*
(2) *$Z' \to X$ is of finite presentation, and*
(3) *$\dim(Z'_s) \leq d$ for all $s \in S$.*

Proof. By Proposition 32.5.4 we can write $S = \lim S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 32.10.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 32.18.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. □

32.19. Base change in top degree

0EX2 For a proper morphism and a finite type quasi-coherent module the base change map is an isomorphism in top degree.

0EX3 **Lemma 32.19.1.** *Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Assume*

(1) *X and Y are quasi-compact and quasi-separated, and*
(2) *$R^i f_* \mathcal{F} = 0$ for $i > d$ and every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Then we have

(a) *for any base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

we have $R^i f'_* \mathcal{F}' = 0$ for $i > d$ and any quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$,
(b) *$R^d f'_*(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \mathcal{G}') = R^d f'_* \mathcal{F}' \otimes_{\mathcal{O}_{Y'}} \mathcal{G}'$ for any quasi-coherent $\mathcal{O}_{Y'}$-module $\mathcal{G}'$,*
(c) *formation of $R^d f'_* \mathcal{F}'$ commutes with arbitrary further base change (see proof for explanation).*

Proof. Before giving the proofs, we explain the meaning of (c). Suppose we have an additional cartesian square

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

tacked onto our given diagram. If (a) holds, then there is a canonical map $\gamma : h^*R^df'_*\mathcal{F}' \to R^df''_*(h')^*\mathcal{F}'$. Namely, γ is the map on degree d cohomology sheaves induced by the composition

$$Lh^*Rf'_*\mathcal{F}' \longrightarrow Rf''_*L(h')^*\mathcal{F}' \longrightarrow Rf''_*(h')^*\mathcal{F}'$$

Here the first arrow is the base change map (Cohomology, Remark 20.28.3) and the second arrow complex from the canonical map $L(g')^*\mathcal{F} \to (g')^*\mathcal{F}$. Similarly, since $Rf'_*\mathcal{F}$ has no nonzero cohomology sheaves in degrees $> d$ by (a) we have $H^d(Lh^*Rf_*\mathcal{F}') = h^*R^df_*\mathcal{F}$. The content of (c) is that γ is an isomorphism.

Having said this, we can check (a), (b), and (c) locally on Y' and Y''. Suppose that $V \subset Y$ is a quasi-compact open subscheme. Then we claim (1) and (2) hold for $f|_{f^{-1}(V)} : f^{-1}(V) \to V$. Namely, (1) is immediate and (2) follows because any quasi-coherent module on $f^{-1}(V)$ is the restriction of a quasi-coherent module on X (Properties, Lemma 28.22.1) and formation of higher direct images commutes with restriction to opens. Thus we may also work locally on Y. In other words, we may assume Y'', Y', and Y are affine schemes.

Proof of (a) when Y' and Y are affine. In this case the morphisms g and g' are affine. Thus $g_* = Rg_*$ and $g'_* = Rg'_*$ (Cohomology of Schemes, Lemma 30.2.3) and g_* is identified with the restriction functor on modules (Schemes, Lemma 26.7.3). Then

$$g_*(R^if'_*\mathcal{F}') = H^i(Rg_*Rf'_*\mathcal{F}') = H^i(Rf_*Rg'_*\mathcal{F}') = H^i(Rf_*g'_*\mathcal{F}') = Rf^i_*g'_*\mathcal{F}'$$

which is zero by assumption (2). Hence (a) by our description of g_*.

Proof of (b) when Y' is affine, say $Y' = \mathrm{Spec}(R')$. By part (a) we have $H^{d+1}(X', \mathcal{F}') = 0$ for any quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$, see Cohomology of Schemes, Lemma 30.4.6. Consider the functor F on R'-modules defined by the rule

$$F(M) = H^d(X', \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M})$$

By Cohomology, Lemma 20.19.1 this functor commutes with direct sums (this is where we use that X and hence X' is quasi-compact and quasi-separated). On the other hand, if $M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then

$$\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_1} \to \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_2} \to \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_3} \to 0$$

is an exact sequence of quasi-coherent modules on X' and by the vanishing of higher cohomology given above we get an exact sequence

$$F(M_1) \to F(M_2) \to F(M_3) \to 0$$

In other words, F is right exact. Any right exact R'-linear functor $F : \mathrm{Mod}_{R'} \to \mathrm{Mod}_{R'}$ which commutes with direct sums is given by tensoring with an R'-module (omitted; left as exercise for the reader). Thus we obtain $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$. Since $R^d(f')_*\mathcal{F}'$ and $R^d(f')_*(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M})$ are quasi-coherent (Cohomology of Schemes, Lemma 30.4.5), the fact that $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$ translates into the statement given in (b).

Proof of (c) when $Y'' \to Y' \to Y$ are morphisms of affine schemes. Say $Y'' = \mathrm{Spec}(R'')$ and $Y' = \mathrm{Spec}(R')$. Then we see that $R^df''_*(h')^*\mathcal{F}'$ is the quasi-coherent module on Y' associated to the R''-module $H^d(X'', (h')^*\mathcal{F}')$. Now $h' : X'' \to X'$ is

affine hence $H^d(X'', (h')^*\mathcal{F}') = H^d(X, h'_*(h')^*\mathcal{F}')$ by the already used Cohomology of Schemes, Lemma 30.2.4. We have

$$h'_*(h')^*\mathcal{F}' = \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{R''}$$

as the reader sees by checking on an affine open covering. Thus $H^d(X'', (h')^*\mathcal{F}') = H^d(X', \mathcal{F}') \otimes_{R'} R''$ by part (b) applied to f' and the proof is complete. □

0E7D **Lemma 32.19.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume f is proper and $\dim(X_y) = d$. Then*

(1) *for $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have $(R^i f_*\mathcal{F})_y = 0$ for all $i > d$,*
(2) *there is an affine open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ and d satisfy the assumptions and conclusions of Lemma 32.19.1.*

Proof. By Morphisms, Lemma 29.28.4 and the fact that f is closed, we can find an affine open neighbourhood V of y such that the fibres over points of V all have dimension $\leq d$. Thus we may assume $X \to Y$ is a proper morphism all of whose fibres have dimension $\leq d$ with Y affine. We will show that (2) holds, which will immediately imply (1) for all $y \in Y$.

By Lemma 32.13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some i we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 32.18.1. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $R^p f_*\mathcal{F} = R^p f_{i,*}(X \to X_i)_*\mathcal{F}$ by Cohomology of Schemes, Lemma 30.2.3 and Leray (Cohomology, Lemma 20.13.8). Thus we may replace X by X_i and reduce to the case discussed in the next paragraph.

Assume Y is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. It suffices to show that $H^p(X, \mathcal{F}) = 0$ for $p > d$. Namely, by Cohomology of Schemes, Lemma 30.4.6 we have $H^p(X, \mathcal{F}) = H^0(Y, R^p f_*\mathcal{F})$. On the other hand, $R^p f_*\mathcal{F}$ is quasi-coherent on Y by Cohomology of Schemes, Lemma 30.4.5, hence vanishing of global sections implies vanishing. Write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of affine schemes with Y_i the spectrum of a Noetherian ring (for example a finite type $\mathbf{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 32.10.1. After increasing 0 we may assume $X_0 \to Y_0$ is proper (Lemma 32.13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 32.18.1). Since $X \to X_0$ is affine, we find that $H^p(X, \mathcal{F}) = H^p(X_0, (X \to X_0)_*\mathcal{F})$ by Cohomology of Schemes, Lemma 30.2.4. This reduces us to the case discussed in the next paragraph.

Assume Y is affine Noetherian and $f : X \to Y$ is proper and all fibres have dimension $\leq d$. In this case we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Properties, Lemma 28.22.7. Then $H^p(X, \mathcal{F}) = \operatorname{colim} H^p(X, \mathcal{F}_i)$ by Cohomology, Lemma 20.19.1. Thus we may assume $\mathcal{F}$ is coherent. In this case we see that $(R^p f_*\mathcal{F})_y = 0$ for all $y \in Y$ by Cohomology of Schemes, Lemma 30.20.9. Thus $R^p f_*\mathcal{F} = 0$ and therefore $H^p(X, \mathcal{F}) = 0$ (see above) and we win. □

0EX4 **Lemma 32.19.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume*

(1) *f is a proper morphism all of whose fibres have dimension $\leq d$,*
(2) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.*

Then $R^d f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.

Proof. The module $R^d f_* \mathcal{F}$ is quasi-coherent by Cohomology of Schemes, Lemma 30.4.5. The question is local on Y hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. Then it suffices to prove that $H^d(X, \mathcal{F})$ is a finite R-module.

By Lemma 32.13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some i we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 32.18.1. We have $R^p f_* \mathcal{F} = R^p f_{i,*}(X \to X_i)_* \mathcal{F}$ by Cohomology of Schemes, Lemma 30.2.3 and Leray (Cohomology, Lemma 20.13.8). Thus we may replace X by X_i and reduce to the case discussed in the next paragraph.

Assume Y is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. We can write $\mathcal{F}$ as a quotient of a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$, see Properties, Lemma 28.22.8. The map $H^d(X, \mathcal{F}') \to H^d(X, \mathcal{F})$ is surjective, as we have $H^{d+1}(X, \mathrm{Ker}(\mathcal{F}' \to \mathcal{F})) = 0$ by the vanishing of higher cohomology seen in Lemma 32.19.2 (or its proof). Thus we reduce to the case discussed in the next paragraph.

Assume $Y = \mathrm{Spec}(R)$ is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$ and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation. Write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of affine schemes with $Y_i = \mathrm{Spec}(R_i)$ the spectrum of a Noetherian ring (for example a finite type $\mathbf{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 32.10.1. After increasing 0 we may assume $X_0 \to Y_0$ is proper (Lemma 32.13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 32.18.1). After increasing 0 we can assume there is a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$, see Lemma 32.10.2. By Lemma 32.19.1 we have

$$H^d(X, \mathcal{F}) = H^d(X_0, \mathcal{F}_0) \otimes_{R_0} R$$

This finishes the proof because the cohomology module $H^d(X_0, \mathcal{F}_0)$ is finite by Cohomology of Schemes, Lemma 30.19.2. □

0EX5 **Lemma 32.19.4.** *Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume*

(1) *f is a proper morphism of finite presentation all of whose fibres have dimension $\leq d$,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Then $R^d f_* \mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. The proof is exactly the same as the proof of Lemma 32.19.3 except that the third paragraph can be skipped. We omit the details. □

32.20. Glueing in closed fibres

0E8P Applying our theory above to the spectrum of a local ring we obtain the following pleasing glueing result for relative schemes.

0BPA **Lemma 32.20.1.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories*

$$\{X \to S \text{ of finite presentation}\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) \end{array} \right\}$$

where on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 27.2, the functor

$$\{X \to S \text{ of finite presentation}\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & W \setminus \{s\} & \longrightarrow & W \end{array} \right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 32.10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0F21 **Lemma 32.20.2.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories*

$$\{\mathcal{O}_S\text{-modules } \mathcal{F} \text{ of finite presentation}\} \longrightarrow \{(\mathcal{G}, \mathcal{H}, \alpha)\}$$

where on the right hand side we consider triples consisting of a $\mathcal{O}_U$-module $\mathcal{G}$ of finite presentation, a $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{S,s})}$-module $\mathcal{H}$ of finite presentation, and an isomorphism $\alpha : \mathcal{G}|_V \to \mathcal{H}|_V$ of $\mathcal{O}_V$-modules.

Proof. You can either prove this by redoing the proof of Lemma 32.20.1 using Lemma 32.10.2 or you can deduce it from Lemma 32.20.1 using the equivalence between quasi-coherent modules and "vector bundles" from Constructions, Section 27.6. We omit the details. □

0BQ5 **Lemma 32.20.3.** *Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an*

equivalence of categories

$$\mathrm{colim}_{s\in U'\supset U \text{ open}} \left\{ \begin{array}{c} X \\ \downarrow \\ U' \end{array} \right\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) \end{array} \right\}$$

where on the left hand side the vertical arrow is of finite presentation and on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 27.2, the functor

$$\big\{X \to U' = U \cup W \text{ of finite presentation}\big\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & W \cap U & \longrightarrow & W \end{array} \right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 32.10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0EY3 **Lemma 32.20.4.** *Notation and assumptions as in Lemma 32.20.3. Let $U \subset U' \subset X$ be an open containing s.*

(1) *Let $f' : X \to U'$ correspond to $f : X' \to U$ and $g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s})$ via the equivalence. If f and g are separated, proper, finite, étale, then after possibly shrinking U' the morphism f' has the same property.*

(2) *Let $a : X_1 \to X_2$ be a morphism of schemes of finite presentation over U' with base change $a' : X_1' \to X_2'$ over U and $b : Y_1 \to Y_2$ over $\mathrm{Spec}(\mathcal{O}_{S,s})$. If a' and b are separated, proper, finite, étale, then after possibly shrinking U' the morphism a has the same property.*

Proof. Proof of (1). Recall that $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the affine open neighbourhoods of s in S. Since g has the property in question, then the restriction of f' to one of these affine open neighbourhoods does too, see Lemmas 32.8.6, 32.13.1, 32.8.3, and 32.8.10. Since f' has the given property over U as f does, we conclude as one can check the property locally on the base.

Proof of (2). If we write $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s in S, then we have $Y_i = \lim W \times_S X_i$. Thus we can use exactly the same arguments as in the proof of (1). □

0E8Q **Lemma 32.20.5.** *Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where FP_T is the category of schemes of finite presentation over the scheme T.

Proof. For $n = 1$ this is Lemma 32.20.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of FP_{S_i} and $f : X \to U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 32.20.1 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i, which is isomorphic to X over U, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_n} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n-1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. □

32.21. Application to modifications

0B3W Using the results from Section 32.20 we can describe the category of modifications of a scheme over a closed point in terms of the local ring.

0B3X **Lemma 32.21.1.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ the base change functor*

$$\left\{\begin{array}{c} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array}\right\} \longrightarrow \left\{\begin{array}{c} g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{array}\right\}$$

is an equivalence of categories.

Proof. This is a special case of Lemma 32.20.1. □

0BFN **Lemma 32.21.2.** *Notation and assumptions as in Lemma 32.21.1. Let $f : X \to S$ correspond to $g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s})$ via the equivalence. Then f is separated, proper, finite, étale and add more here if and only if g is so.*

Proof. The property of being separated, proper, integral, finite, etc is stable under base change. See Schemes, Lemma 26.21.12 and Morphisms, Lemmas 29.41.5 and 29.44.6. Hence if f has the property, then so does g. The converse follows from Lemma 32.20.4 but we also give a direct proof here. Namely, if g has to property, then f does in a neighbourhood of s by Lemmas 32.8.6, 32.13.1, 32.8.3, and 32.8.10. Since f clearly has the given property over $S \setminus \{s\}$ we conclude as one can check the property locally on the base. □

0B3Y **Remark 32.21.3.** The lemma above can be generalized as follows. Let S be a scheme and let $T \subset S$ be a closed subset. Assume there exists a cofinal system of open neighbourhoods $T \subset W_i$ such that (1) $W_i \setminus T$ is quasi-compact and (2) $W_i \subset W_j$ is an affine morphism. Then $W = \lim W_i$ is a scheme which contains T as a closed subscheme. Set $U = X \setminus T$ and $V = W \setminus T$. Then the base change functor

$$\left\{\begin{array}{c} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array}\right\} \longrightarrow \left\{\begin{array}{c} g : Y \to W \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{array}\right\}$$

is an equivalence of categories. If we ever need this we will change this remark into a lemma and provide a detailed proof.

32.22. Descending finite type schemes

0CNL This section continues the theme of Section 32.9 in the spirit of the results discussed in Section 32.10.

0CNM **Situation 32.22.1.** Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of Noetherian schemes with affine transition morphisms $S_{i'} \to S_i$ for $i' \geq i$.

0CNN **Lemma 32.22.2.** *In Situation 32.22.1. Let $X \to S$ be quasi-separated and of finite type. Then there exists an $i \in I$ and a diagram*

0CNP (32.22.2.1)
$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_i \end{array}$$

such that $W \to S_i$ is of finite type and such that the induced morphism $X \to S \times_{S_i} W$ is a closed immersion.

Proof. By Lemma 32.9.3 we can find a closed immersion $X \to X'$ over S where X' is a scheme of finite presentation over S. By Lemma 32.10.1 we can find an i and a morphism of finite presentation $X'_i \to S_i$ whose pull back is X'. Set $W = X'_i$. □

0CNQ **Lemma 32.22.3.** *In Situation 32.22.1. Let $X \to S$ be quasi-separated and of finite type. Given $i \in I$ and a diagram*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_i \end{array}$$

as in (32.22.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$ commutes with restriction to open subschemes (Morphisms, Lemma 29.6.3). Hence we may and do assume W is affine and maps into an affine open U_i of S_i. Let $U \subset S$, $U_{i'} \subset S_{i'}$ be the inverse image of U_i. Then U, $U_{i'}$, $S_{i'} \times_{S_i} W = U_{i'} \times_{U_i} W$, and $S \times_{S_i} W = U \times_{U_i} W$ are all affine. This implies X is affine because $X \to S \times_{S_i} W$ is a closed immersion. This also shows the ring map

$$\mathcal{O}(U) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W) \to \mathcal{O}(X)$$

is surjective. Let I be the kernel. Then we see that $X_{i'}$ is the spectrum of the ring

$$\mathcal{O}(X_{i'}) = \mathcal{O}(U_{i'}) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W)/I_{i'}$$

where $I_{i'}$ is the inverse image of the ideal I (see Morphisms, Example 29.6.4). Since $\mathcal{O}(U) = \operatorname{colim} \mathcal{O}(U_{i'})$ we see that $I = \operatorname{colim} I_{i'}$ and we conclude that $\operatorname{colim} \mathcal{O}(X_{i'}) = \mathcal{O}(X)$. □

0CNR **Lemma 32.22.4.** *In Situation 32.22.1. Let $f : X \to Y$ be a morphism of schemes quasi-separated and of finite type over S. Let*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_2} \end{array}$$

be diagrams as in (32.22.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 32.22.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to S_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 32.6.1 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over S_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_i & \longrightarrow & X_{i_0} \\ \downarrow & \swarrow & \vdots & & \downarrow h \\ Y & \longrightarrow & Y_i & \longrightarrow & V \\ \downarrow & \searrow & \downarrow & & \downarrow \\ S & \longrightarrow & S_i & \longrightarrow & S_{i_0} \end{array}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to S_i \times_{S_{i_2}} V$ we find that the morphism $X_i \to S_i \times_{S_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms, Lemma 29.6.6). This proves existence.

Uniqueness. Let $E_i \subset X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. By Schemes, Lemma 26.21.5 E_i is a locally closed subscheme of X_i. Since X_i is a closed subscheme of $S_i \times_{S_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(S_i \times_{S_{i_0}} X_{i_0})} (S_i \times_{S_{i_0}} E_{i_0})$$

Thus to finish the proof it suffices to show that $X_i \to X_{i_0}$ factors through E_{i_0} for some $i \geq i_0$. To do this we will use that $X \to X_{i_0}$ factors through E_{i_0} as both f_{i_0} and g_{i_0} are compatible with f. Since X_i is Noetherian, we see that the underlying topological space $|E_{i_0}|$ is a constructible subset of $|X_{i_0}|$ (Topology, Lemma 5.16.1). Hence $X_i \to X_{i_0}$ factors through E_{i_0} set theoretically for large enough i by Lemma 32.4.10. For such an i the scheme theoretic inverse image $(X_i \to X_{i_0})^{-1}(E_{i_0})$ is a closed subscheme of X_i through which X factors and hence equal to X_i since $X \to X_i$ has scheme theoretically dense image by construction. This concludes the proof. □

0CNS **Remark 32.22.5.** In Situation 32.22.1 Lemmas 32.22.2, 32.22.3, and 32.22.4 tell us that the category of schemes quasi-separated and of finite type over S is

equivalent to certain types of inverse systems of schemes over $(S_i)_{i\in I}$, namely the ones produced by applying Lemma 32.22.3 to a diagram of the form (32.22.2.1). For example, given $X \to S$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to S_{i_1}$ and $X \to V_2 \to S_{i_2}$ as in (32.22.2.1), then applying Lemma 32.22.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CNT **Lemma 32.22.6.** *Notation and assumptions as in Lemma 32.22.4. If f is flat and of finite presentation, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.*

Proof. By Lemma 32.10.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 32.8.7. As discussed in Remark 32.22.5 we may and do replace the initial diagram used to define the system $(X_i)_{i\geq i_1}$ by the system corresponding to $X \to U \to S_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms, Lemma 29.25.16). Observe that $Y_{i'} \to S_{i'} \times_{S_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$ is equal to the closed subscheme $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. □

0CNU **Lemma 32.22.7.** *Notation and assumptions as in Lemma 32.22.4. If f is smooth, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is smooth.*

Proof. Combine Lemmas 32.22.6 and 32.8.9. □

0CNV **Lemma 32.22.8.** *Notation and assumptions as in Lemma 32.22.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.*

Proof. By the discussion in Remark 32.22.5 the choice of i_1 and W fitting into a diagram as in (32.22.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to S$ proper and of finite presentation, see Lemma 32.13.2. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i\geq i_2} Y_i$ and Lemmas 32.10.1 and 32.13.1. With this choice of W it is immediate from the construction that for $i \geq i_3$ the scheme X_i is a closed subscheme of $Y_i \times_{Y_{i_3}} W \subset S_i \times_{S_{i_3}} W$ and hence proper over Y_i. □

0CNW **Lemma 32.22.9.** *In Situation 32.22.1 suppose that we have a cartesian diagram*

$$\begin{array}{ccc} X^1 & \xrightarrow{p} & X^3 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle a} \\ X^2 & \xrightarrow{b} & X^4 \end{array}$$

of schemes quasi-separated and of finite type over S. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{array}{ccc} X^j & \longrightarrow & W^j \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_j} \end{array}$$

as in (32.22.2.1). Let $X^j = \lim_{i \geq i_j} X^j_i$ be the corresponding limit descriptions as in Lemma 32.22.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding morphisms of systems constructed in Lemma 32.22.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X^1_i \longrightarrow X^2_i \times_{b_i, X^4_i, a_i} X^3_i$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 32.22.5 the choice of W^1 fitting into a diagram as in (32.22.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X^1_i that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X^1_i \longrightarrow X^2_i \times_{b_i, X^4_i, a_i} X^3_i$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 32.22.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X^1_i \to X^2_i \times_{X^4_i} X^3_i$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X^4_i \to X^4_{i_9}$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 32.8.11. □

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CHAPTER 33

Varieties

0209 **33.1. Introduction**

020A In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [**DG67**].

33.2. Notation

020B Throughout this chapter we use the letter k to denote the ground field.

33.3. Varieties

020C In the Stacks project we will use the following as our definition of a variety.

020D **Definition 33.3.1.** Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that k'/k is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

020G **Example 33.3.2.** Let $k = \mathbf{Q}$. Let $X = \mathrm{Spec}(\mathbf{Q}(i))$ and $Y = \mathrm{Spec}(\mathbf{Q}(i))$. Then the product $X \times_{\mathrm{Spec}(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

05P3 **Lemma 33.3.3.** *Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{\mathrm{Spec}(k)} Y$ is a variety over k.*

Proof. The morphism $X \times_{\mathrm{Spec}(k)} Y \to \mathrm{Spec}(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{\mathrm{Spec}(k)} Y \to Y \to \mathrm{Spec}(k)$ which are separated and of finite type, see Morphisms, Lemmas 29.15.4 and 29.15.3 and Schemes, Lemma 26.21.12. To finish the proof it suffices to show that $X \times_{\mathrm{Spec}(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{\mathrm{Spec}(k)} V_j$ is integral, then we are done by Properties, Lemmas 28.3.2, 28.3.3, and 28.3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1c_1) \subset \mathrm{Spec}(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) we can find a maximal ideal $\mathfrak{m} \subset A$ contained in $D(a_1c_1)$ and $A/\mathfrak{m} = k$ as k is algebraically closed. Denote $\overline{a}_i, \overline{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak{m} = k$. The equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \overline{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \overline{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak{m} \in D(a_1c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. □

33.4. Varieties and rational maps

0BXM Let k be a field. Let X and Y be varieties over k. We will use the phrase *rational map of varieties from X to Y* to mean a $\mathrm{Spec}(k)$-rational map from the scheme X to the scheme Y as defined in Morphisms, Definition 29.49.1. As is customary, the phrase "rational map of varieties" does not refer to the (common) base field of the varieties, even though for general schemes we make the distinction between rational maps and rational maps over a given base.

The title of this section refers to the following fundamental theorem.

0BXN **Theorem 33.4.1.** *Let k be a field. The category of varieties and dominant rational maps is equivalent to the category of finitely generated field extensions K/k.*

Proof. Let X and Y be varieties with generic points $x \in X$ and $y \in Y$. Recall that dominant rational maps from X to Y are exactly those rational maps which map x to y (Morphisms, Definition 29.49.10 and discussion following). Thus given a dominant rational map $X \supset U \to Y$ we obtain a map of function fields

$$k(Y) = \kappa(y) = \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x} = \kappa(x) = k(X)$$

Conversely, such a k-algebra map (which is automatically local as the source and target are fields) determines (uniquely) a dominant rational map by Morphisms, Lemma 29.49.2. In this way we obtain a fully faithful functor. To finish the proof it suffices to show that every finitely generated field extension K/k is in the essential image. Since K/k is finitely generated, there exists a finite type k-algebra $A \subset K$ such that K is the fraction field of A. Then $X = \mathrm{Spec}(A)$ is a variety whose function field is K. □

Let k be a field. Let X and Y be varieties over k. We will use the phrase X *and* Y *are birational varieties* to mean X and Y are $\mathrm{Spec}(k)$-birational as defined in Morphisms, Definition 29.50.1. As is customary, the phrase "birational varieties" does not refer to the (common) base field of the varieties, even though for general irreducible schemes we make the distinction between being birational and being birational over a given base.

0BXP **Lemma 33.4.2.** *Let X and Y be varieties over a field k. The following are equivalent*

(1) *X and Y are birational varieties,*
(2) *the function fields $k(X)$ and $k(Y)$ are isomorphic,*
(3) *there exist nonempty opens of X and Y which are isomorphic as varieties,*
(4) *there exists an open $U \subset X$ and a birational morphism $U \to Y$ of varieties.*

Proof. This is a special case of Morphisms, Lemma 29.50.6. □

33.5. Change of fields and local rings

0C4X Some preliminary results on what happens to local rings under an extension of ground fields.

0C4Y **Lemma 33.5.1.** *Let K/k be an extension of fields. Let X be scheme over k and set $Y = X_K$. If $y \in Y$ with image $x \in X$, then*

(1) *$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a faithfully flat local ring homomorphism,*
(2) *with $\mathfrak{p}_0 = \mathrm{Ker}(\kappa(x) \otimes_k K \to \kappa(y))$ we have $\kappa(y) = \kappa(\mathfrak{p}_0)$,*
(3) *$\mathcal{O}_{Y,y} = (\mathcal{O}_{X,x} \otimes_k K)_{\mathfrak{p}}$ where $\mathfrak{p} \subset \mathcal{O}_{X,x} \otimes_k K$ is the inverse image of $\mathfrak{p}_0$.*
(4) *we have $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} = (\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine. Then $Y = \mathrm{Spec}(A \otimes_k K)$. Since K is flat over k, we see that $A \to A \otimes_k K$ is flat. Hence $Y \to X$ is flat and we get the first statement if we also use Algebra, Lemma 10.39.17. The second statement follows from Schemes, Lemma 26.17.5. Now y corresponds to a prime ideal $\mathfrak{q} \subset A \otimes_k K$ and x to $\mathfrak{r} = A \cap \mathfrak{q}$. Then $\mathfrak{p}_0$ is the kernel of the induced map $\kappa(\mathfrak{r}) \otimes_k K \to \kappa(\mathfrak{q})$. The map on local rings is

$$A_{\mathfrak{r}} \longrightarrow (A \otimes_k K)_{\mathfrak{q}}$$

We can factor this map through $A_{\mathfrak{r}} \otimes_k K = (A \otimes_k K)_{\mathfrak{r}}$ to get

$$A_{\mathfrak{r}} \longrightarrow A_{\mathfrak{r}} \otimes_k K \longrightarrow (A \otimes_k K)_{\mathfrak{q}}$$

and then the second arrow is a localization at some prime. This prime ideal is the inverse image of $\mathfrak{p}_0$ (details omitted) and this proves (3). To see (4) use (3) and that localization and $- \otimes_k K$ are exact functors. □

0C4Z **Lemma 33.5.2.** *Notation as in Lemma 33.5.1. Assume X is locally of finite type over k. Then*

$$\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}) = \mathit{trdeg}_k(\kappa(x)) - \mathit{trdeg}_K(\kappa(y)) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x})$$

Proof. This is a restatement of Algebra, Lemma 10.116.7. □

0C50 **Lemma 33.5.3.** *Notation as in Lemma 33.5.1. Assume X is locally of finite type over k, that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x) \otimes_k K$ is reduced (for example if $\kappa(x)/k$ is separable or K/k is separable). Then $\mathfrak{m}_x\mathcal{O}_{Y,y} = \mathfrak{m}_y$.*

Proof. (The parenthetical statement follows from Algebra, Lemma 10.43.6.) Combining Lemmas 33.5.1 and 33.5.2 we see that $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ has dimension 0 and is reduced. Hence it is a field. □

33.6. Geometrically reduced schemes

035U If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

035V **Definition 33.6.1.** Let k be a field. Let X be a scheme over k.

(1) Let $x \in X$ be a point. We say X is *geometrically reduced at* x if for any field extension k'/k and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
(2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection. See also Lemma 33.25.7, which implies that "geometrically reduced" is equivalent to "generically smooth" for varieties over a field.

035W **Lemma 33.6.2.** *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically reduced at x, and*
(2) *the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 10.43.1).*

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.3). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 10.44.4.

Assume (2). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. □

The notion isn't interesting in characteristic zero.

020I **Lemma 33.6.3.** *Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Lemma 33.6.2 and Algebra, Lemma 10.43.6 and the definition of a perfect field (Algebra, Definition 10.45.1). The second statement follows from the first. □

035X **Lemma 33.6.4.** *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically reduced,*
(2) *$X_{k'}$ is reduced for every field extension k'/k,*
(3) *$X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,*
(4) *$X_{k^{1/p}}$ is reduced,*
(5) *$X_{k^{perf}}$ is reduced,*

(6) $X_{\bar{k}}$ *is reduced,*
(7) *for every affine open* $U \subset X$ *the ring* $\mathcal{O}_X(U)$ *is geometrically reduced (see Algebra, Definition 10.43.1).*

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension k'/k the scheme $X_{k'}$ is reduced, hence $U_{k'} = \mathrm{Spec}(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 28.3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 26.17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 10.44.4 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. □

035Y **Lemma 33.6.5.** *Let* k *be a field of characteristic* $p > 0$*. Let* X *be a scheme over* k*. Let* $x \in X$*. The following are equivalent*

(1) X *is geometrically reduced at* x*,*
(2) $\mathcal{O}_{X_{k'},x'}$ *is reduced for every finite purely inseparable field extension* k' *of* k *and* $x' \in X_{k'}$ *the unique point lying over* x*,*
(3) $\mathcal{O}_{X_{k^{1/p}},x'}$ *is reduced for* $x' \in X_{k^{1/p}}$ *the unique point lying over* x*, and*
(4) $\mathcal{O}_{X_{k^{perf}},x'}$ *is reduced for* $x' \in X_{k^{perf}}$ *the unique point lying over* x*.*

Proof. Note that if k'/k is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 10.46.7. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'},x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 33.6.2 above and Algebra, Lemma 10.44.4. □

0384 **Lemma 33.6.6.** *Let* k *be a field. Let* X *be a scheme over* k*. Let* k'/k *be a field extension. Let* $x \in X$ *be a point, and let* $x' \in X_{k'}$ *be a point lying over* x*. The following are equivalent*

(1) X *is geometrically reduced at* x*,*
(2) $X_{k'}$ *is geometrically reduced at* x'*.*

In particular, X *is geometrically reduced over* k *if and only if* $X_{k'}$ *is geometrically reduced over* k'*.*

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). Choose a common field extension k'''/k of k''/k and k'/k, see Fields, Lemma 9.6.9. Choose a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 10.164.2 we conclude that $\mathcal{O}_{X_{k''},x''}$ is reduced. Thus by Lemma 33.6.5 we conclude that X is geometrically reduced at x. □

035Z **Lemma 33.6.7.** *Let k be a field. Let X, Y be schemes over k.*

(1) *If X is geometrically reduced at x, and Y reduced, then $X \times_k Y$ is reduced at every point lying over x.*
(2) *If X geometrically reduced over k and Y reduced, then $X \times_k Y$ is reduced.*
(3) *If X and Y are geometrically reduced over k, then $X \times_k Y$ is geometrically reduced.*
(4) *If k is perfect and X and Y are reduced, then $X \times_k Y$ is reduced.*
(5) *Add more here.*

Proof. To prove (1) combine Lemma 33.6.2 with Algebra, Lemma 10.43.5. To prove (2) combine Lemma 33.6.4 with Algebra, Lemma 10.43.5. To prove (3) note that $(X \times_k Y)_{\overline{k}} = X_{\overline{k}} \times_{\overline{k}} Y_{\overline{k}}$ and use (2) as well as Lemma 33.6.4. To prove (4) use (3) combined with Lemma 33.6.3. □

04KS **Lemma 33.6.8.** *Let k be a field. Let X be a scheme over k.*

(1) *If $x' \rightsquigarrow x$ is a specialization and X is geometrically reduced at x, then X is geometrically reduced at x'.*
(2) *If $x \in X$ such that (a) $\mathcal{O}_{X,x}$ is reduced, and (b) for each specialization $x' \rightsquigarrow x$ where x' is a generic point of an irreducible component of X the scheme X is geometrically reduced at x', then X is geometrically reduced at x.*
(3) *If X is reduced and geometrically reduced at all generic points of irreducible components of X, then X is geometrically reduced.*
(4) *If X is a variety over k, then X is geometrically reduced if and only if its function field is a separable extension of k.*

Proof. Part (1) follows from Lemma 33.6.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 10.43.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_{\mathfrak{q}}$ is geometrically reduced over k for every minimal prime $\mathfrak{q}$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 10.43.7. Thus X is geometrically reduced at x, see Lemma 33.6.2.

Part (3) follows trivially from part (2). Finally, (4) follows from (3) by Algebra, Lemma 10.44.2. □

0360 **Lemma 33.6.9.** *Let k be a field. Let X be a scheme over k. Let $x \in X$. Assume X locally Noetherian and geometrically reduced at x. Then there exists an open neighbourhood $U \subset X$ of x which is geometrically reduced over k.*

Proof. Assume X locally Noetherian and geometrically reduced at x. By Properties, Lemma 28.29.8 we can find an affine open neighbourhood $U \subset X$ of x such that $R = \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective. By Lemma 33.6.2 the assumption means that $\mathcal{O}_{X,x}$ is geometrically reduced over k. By Algebra, Lemma 10.43.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. □

020F **Example 33.6.10.** Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = \mathrm{Spec}(k[x,y]/(1 + sx^p + ty^p))$. Let k'/k be any extension such that both s and t have a pth root in k'. Then the base change

$X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1+sx^p+ty^p)$ contains the element $1+s^{1/p}x+t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1+sx^p+ty^p)$ certainly does not contain any nonzero element of degree $< p$).

04KT **Lemma 33.6.11.** *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'.*

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension k'/k the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 10.46.7. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 10.25.1. As X is locally of finite type over k the field extensions K_i/k are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{(X_{k'})_{red},x'_i}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 10.45.3 we can find a finite purely inseparable extension k'/k such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 10.44.2. At this point Lemma 33.6.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. □

33.7. Geometrically connected schemes

0361 If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

0362 **Definition 33.7.1.** Let X be a scheme over the field k. We say X is *geometrically connected* over k if the scheme $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected schemes are nonempty. Here is an example of a variety which is not geometrically connected.

020E **Example 33.7.2.** Let $k = \mathbf{Q}$. The scheme $X = \mathrm{Spec}(\mathbf{Q}(i))$ is a variety over $\mathrm{Spec}(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $\mathrm{Spec}(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

054N **Lemma 33.7.3.** *Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, for any field extension k''/k there exists a common field extension k'''/k' and k'''/k'', see Fields, Lemma 9.6.9. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. □

0385 **Lemma 33.7.4.** *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 26.18.5. By Morphisms, Lemma 29.23.4 the map p is open. Thus we may apply Topology, Lemma 5.7.6 to conclude. □

0386 **Lemma 33.7.5.** *Let k be a field. Let A be a k-algebra. Then $X = \mathrm{Spec}(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 10.48.3).*

Proof. Immediate from the definitions. □

0363 **Lemma 33.7.6.** *Let k'/k be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 10.48.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Lemma 33.7.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 33.7.4. □

0387 **Lemma 33.7.7.** *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 33.7.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. □

0388 **Lemma 33.7.8.** *Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.*

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 32.4.11. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = \mathrm{Spec}(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. □

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

038A (33.7.8.1) $$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

04KU **Lemma 33.7.9.** *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

1. *there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
2. *there exists an open subgroup $H \subset Gal(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. By Lemma 33.7.8 there exists a finite subextension $k'/k \subset \overline{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\mathrm{Gal}(\overline{k}/k')$ is open in $\mathrm{Gal}(\overline{k}/k)$ part (2) is clear as well. □

038B **Lemma 33.7.10.** *Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\overline{k}}$ have the following properties*

1. *$\overline{T}$ is a closed subset of $X_{\overline{k}}$,*
2. *for every $\sigma \in Gal(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{\overline{k}}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \overline{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension k'/k contained in $\overline{k}$ such that $x \in A \otimes_k k'$. Set $G = \mathrm{Gal}(k'/k)$. Set

$$P(T) = \prod\nolimits_{\sigma \in G} (T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ the we see that $a_i \in I := A \cap \overline{I}$. Combining $P(x) = 0$ and $a_i \in I$ we find $x^d = -a_1 x^{d-1} - \ldots - a_d \in I(A \otimes_k \overline{k})$. Thus x is contained in the radical of $I(A \otimes_k \overline{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \overline{k})$ and setting $T = V(I)$ is a solution. □

0389 **Lemma 33.7.11.** *Let k be a field. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically connected,*
2. *for every finite separable field extension k'/k the scheme $X_{k'}$ is connected.*

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 33.7.7 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed in $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 33.7.9 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\overline{k}/k'/k$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$. Let $H = \mathrm{Gal}(\overline{k}/k') \subset \mathrm{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 33.7.10 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. □

038C **Lemma 33.7.12.** *Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\overline{k}}$ connected and $X_{\overline{k}}$ disconnected. Then X is disconnected.*

Proof. Write $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\overline{k}} \to X_{\overline{k}}$ the base change of f. Since $T_{\overline{k}}$ is connected we see that $T_{\overline{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\overline{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $\overline{k}/k'/k$ such that $\overline{U} \cap W_{\overline{k}}$ and $\overline{V} \cap W_{\overline{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \amalg V'$. Consider

$$U'' = \bigcap\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \amalg V''$. By Lemma 33.7.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \amalg V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. □

056R **Lemma 33.7.13.** *Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is geometrically connected and X connected. Then X is geometrically connected.* [**DG67**, IV Corollary 4.5.13.1(i)]

Proof. This is a reformulation of Lemma 33.7.12. □

04KV **Lemma 33.7.14.** *Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.*

Proof. Set $T = \mathrm{Spec}(\kappa(x))$. Let $\overline{k}$ be a separable algebraic closure of k. The assumption on $\kappa(x)/k$ implies that $T_{\overline{k}}$ is irreducible, see Algebra, Lemma 10.47.8. Hence by Lemma 33.7.13 we see that $X_{\overline{k}}$ is connected. By Lemma 33.7.7 we conclude that X is geometrically connected. □

04PY **Lemma 33.7.15.** *Let K/k be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.*

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. □

The following lemma will be superseded by the stronger Lemma 33.7.17 below.

07VM **Lemma 33.7.16.** *Let K/k be a finite extension of fields and let X be a scheme over k. Denote by $p : X_K \to X$ the projection morphism. For every connected component T of X_K the image $p(T)$ is a connected component of X.*

Proof. The image $p(T)$ is contained in some connected component X' of X. Consider X' as a closed subscheme of X in any way. Then T is also a connected component of $X'_K = p^{-1}(X')$ and we may therefore assume that X is connected. The morphism p is open (Morphisms, Lemma 29.23.4), closed (Morphisms, Lemma 29.44.7) and the fibers of p are finite sets (Morphisms, Lemma 29.44.10). Thus we may apply Topology, Lemma 5.7.7 to conclude. □

04PZ **Lemma 33.7.17** (Gabber)**.** *Let K/k be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Then $p(\overline{T})$ is a connected component of X.* Email from Ofer Gabber dated June 4, 2016

Proof. When K/k is finite this is Lemma 33.7.16. In general the proof is more difficult.

Let $T \subset X$ be the connected component of X containing the image of $\overline{T}$. We may replace X by T (with the induced reduced subscheme structure). Thus we may assume X is connected. Let $A = H^0(X, \mathcal{O}_X)$. Let $L \subset A$ be the maximal weakly étale k-subalgebra, see More on Algebra, Lemma 15.106.2. Since A does not have any nontrivial idempotents we see that L is a field and a separable algebraic extension of k by More on Algebra, Lemma 15.106.1. Observe that L is also the maximal weakly étale L-subalgebra of A (because any weakly étale L-algebra is weakly étale over k by More on Algebra, Lemma 15.105.9). By Schemes, Lemma 26.6.4 we obtain a factorization $X \to \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ of the structure morphism.

Let L'/L be a finite separable extension. By Cohomology of Schemes, Lemma 30.5.3 we have

$$A \otimes_L L' = H^0(X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L'), \mathcal{O}_{X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L')})$$

The maximal weakly étale L'-subalgebra of $A \otimes_L L'$ is $L \otimes_L L' = L'$ by More on Algebra, Lemma 15.106.4. In particular $A \otimes_L L'$ does not have nontrivial idempotents (such an idempotent would generate a weakly étale subalgebra) and we conclude that $X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L')$ is connected. By Lemma 33.7.11 we conclude that X is geometrically connected over L.

Let's give $\overline{T}$ the reduced induced scheme structure and consider the composition

$$\overline{T} \xrightarrow{i} X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{\pi} \mathrm{Spec}(L \otimes_k K)$$

The image is contained in a connected component of $\mathrm{Spec}(L \otimes_k K)$. Since $K \to L \otimes_k K$ is integral we see that the connected components of $\mathrm{Spec}(L \otimes_k K)$ are points and all points are closed, see Algebra, Lemma 10.36.19. Thus we get a quotient field $L \otimes_k K \to E$ such that $\overline{T}$ maps into $\mathrm{Spec}(E) \subset \mathrm{Spec}(L \otimes_k K)$. Hence $i(\overline{T}) \subset \pi^{-1}(\mathrm{Spec}(E))$. But

$$\pi^{-1}(\mathrm{Spec}(E)) = (X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)) \times_{\mathrm{Spec}(L \otimes_k K)} \mathrm{Spec}(E) = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$$

which is connected because X is geometrically connected over L. Then we get the equality $\overline{T} = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$ (set theoretically) and we conclude that $\overline{T} \to X$ is surjective as desired. □

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

038D **Lemma 33.7.18.** *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

1. *An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.*
2. *For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) & \longrightarrow & \pi_0(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times \pi_0(X_{\overline{k}}) & \longrightarrow & \pi_0(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 33.7.6).

Proof. The action (33.7.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 5.7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 33.7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 33.7.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{k'}$ strictly contains $\overline{T}$

and hence is disconnected. By Lemma 33.7.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). □

038E **Lemma 33.7.19.** *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume*

(1) *X is quasi-compact, and*
(2) *the connected components of $X_{\overline{k}}$ are open.*

Then

(a) *$\pi_0(X_{\overline{k}})$ is finite, and*
(b) *the action of $Gal(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.*

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 5.7.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 33.7.8 these connected components are each defined over a finite subextension of $\overline{k}/k$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 29.15.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 29.15.6). Thus $X_{\overline{k}}$ has finitely many irreducible components (Properties, Lemma 28.5.7) and a fortiori finitely many connected components (which are therefore open). □

33.8. Geometrically irreducible schemes

0364 If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

0365 **Definition 33.8.1.** Let X be a scheme over the field k. We say X is *geometrically irreducible* over k if the scheme $X_{k'}$ is irreducible[1] for any field extension k' of k.

054P **Lemma 33.8.2.** *Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.*

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, for any field extension k''/k there exists a common field extension k'''/k' and k'''/k'', see Fields, Lemma 9.6.9. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. □

020J **Lemma 33.8.3.** *Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension K/k. I.e., X is geometrically irreducible over k.*

Proof. Use Properties, Lemma 28.3.3 and Algebra, Lemma 10.47.2. □

[1] An irreducible space is nonempty.

038F **Lemma 33.8.4.** *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 26.18.5. By Morphisms, Lemma 29.23.4 the map p is open. Thus we may apply Topology, Lemma 5.8.15 to conclude. □

038G **Lemma 33.8.5.** *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically irreducible over k,*
(2) *for every nonempty affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 10.47.4),*
(3) *X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and*
(4) *there exists an open covering $X = \bigcup_{i \in I} X_i$ with $I \neq \emptyset$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.*

Moreover, if X is geometrically irreducible so is every nonempty open subscheme of X.

Proof. An affine scheme $\mathrm{Spec}(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 28.3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). □

054Q **Lemma 33.8.6.** *Let X be an irreducible scheme over the field k. Let $\xi \in X$ be its generic point. The following are equivalent:*

(1) *X is geometrically irreducible over k;*
(2) *$\kappa(\xi)$ is geometrically irreducible over k;*
(3) *the separable algebraic closure of k in $\kappa(\xi)$ is equal to k.*

Proof. By Algebra, Lemma 10.47.12, (2) and (3) are equivalent.

Assume (1). Recall that $\mathcal{O}_{X,\xi}$ is the filtered colimit of $\mathcal{O}_X(U)$ where U runs over the nonempty open affine subschemes of X. Combining Lemma 33.8.5 and Algebra, Lemma 10.47.6 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 10.25.1) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 10.46.7.

Assume (2). We may assume that X is reduced. Let $U \subset X$ be a nonempty affine open. Then $U = \mathrm{Spec}(A)$ where A is a domain with fraction field $\kappa(\xi)$. Thus A is a k-subalgebra of a geometrically irreducible k-algebra. Hence by Algebra, Lemma

10.47.6 we see that A is geometrically irreducible over k. By Lemma 33.8.5 we conclude that X is geometrically irreducible over k. □

0H8G **Lemma 33.8.7.** *A scheme X over a field k is geometrically irreducible over k if and only if $X \times_k X$ is irreducible.*

Proof. If X is geometrically irreducible over k, then $X \times_k X$ is irreducible by Lemma 33.8.4.

Conversely, suppose that $X \times_k X$ is irreducible. Replacing X by the underlying reduced scheme, we can assume that X is reduced without changing the problem. Since the first projection $X \times_k X \to X$ is surjective, X is also irreducible, hence integral. Suppose that there is an element α in the function field $k(X)$ that is separably algebraic over k but not in k. Let $E = k(\alpha)$.

Then there is a nonempty affine open subscheme U of X such that $k \subset E \subset O(U)$, and so $E \otimes_k E \subset O(U) \otimes_k O(U) = O(U \times_k U)$. Thus we have a dominant morphism $U \times_k U \to \mathrm{Spec}(E \otimes_k E)$. Since $X \times_k X$ is irreducible, so is $U \times_k U$, and hence $\mathrm{Spec}(E \otimes_k E)$ is connected. If E is strictly bigger than k, then $E \otimes_k E$ is a product of fields including E as one factor. It has dimension $(\dim_k(E))^2 > \dim_k(E)$ as a k-vector space, and so $E \otimes_k E$ is a product of at least two fields, contradicting that $\mathrm{Spec}(E \otimes_k E)$ is connected. So in fact E is equal to k. By Lemma 33.8.6, X is geometrically irreducible over k. □

038H **Lemma 33.8.8.** *Let k'/k be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.*

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 10.47.5. Hence $Z = \mathrm{Spec}(k')$ is geometrically irreducible over k. by Lemma 33.8.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 33.8.4. □

038I **Lemma 33.8.9.** *Let k be a field. Let X be a scheme over k. The following are equivalent:*

1. *X is geometrically irreducible over k,*
2. *for every finite separable field extension k'/k the scheme $X_{k'}$ is irreducible, and*
3. *$X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.*

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 33.8.8 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $\overline{k}/k'/k$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 28.3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 10.47.3 this means

that $U_{i,k'}$ is reducible for some finite separable field extension k'/k. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. □

04KW **Lemma 33.8.10.** *Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.*

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. □

For a scheme X we denote $\mathrm{IrredComp}(X)$ the set of irreducible components of X.

04KX **Lemma 33.8.11.** *Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map*

$$\mathit{IrredComp}(X_K) \longrightarrow \mathit{IrredComp}(X)$$

which is surjective.

Proof. Consider the diagram

$$\begin{array}{ccc} X_K & \longleftarrow & X_{\overline{K}} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X_{\overline{k}} \end{array}$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 33.8.8 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 29.25.9. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of X. Integrality implies that p is universally closed, see Morphisms, Lemma 29.44.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 33.8.10. Each of these necessarily maps onto T by the first part. Hence the map is surjective. □

0G69 **Lemma 33.8.12.** *Let k be a field. Let X be a scheme over k. If X is irreducible and has a dense set of k-rational points, then X is geometrically irreducible.*

Proof. Let k'/k be a finite extension of fields and let $Z, Z' \subset X_{k'}$ be irreducible components. It suffices to show $Z = Z'$, see Lemma 33.8.9. By Lemma 33.8.11 we have $p(Z) = p(Z') = X$ where $p : X_{k'} \to X$ is the projection. If $Z \neq Z'$ then $Z \cap Z'$ is nowhere dense in $X_{k'}$ and hence $p(Z \cap Z')$ is not dense by Morphisms, Lemma 29.48.7; here we also use that p is a finite morphism as the base change of the finite morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, see Morphisms, Lemma 29.44.6. Thus we can pick a k-rational point $x \in X$ with $x \notin p(Z \cap Z')$. Since the residue field of x is k we see that $p^{-1}(\{x\}) = \{x'\}$ where $x' \in X_{k'}$ is a point whose residue field is k'. Since $x \in p(Z) = p(Z')$ we conclude that $x' \in Z \cap Z'$ which is the contradiction we were looking for. □

038J **Lemma 33.8.13.** *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in IrredComp(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times IrredComp(X_{\overline{k}'}) & \longrightarrow & IrredComp(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times IrredComp(X_{\overline{k}}) & \longrightarrow & IrredComp(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 33.8.8).

Proof. The action (33.7.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 5.7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 33.7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma 33.8.9. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to $\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). □

04KY **Lemma 33.8.14.** *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map*

$$IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X)$$

of Lemma 33.8.11 are exactly the orbits of $Gal(\overline{k}/k)$ under the action of Lemma 33.8.13.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 33.8.10 and 33.8.11 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 10.47.14 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. □

04KZ **Lemma 33.8.15.** *Let k be a field. Assume $X \to \mathrm{Spec}(k)$ locally of finite type. In this case*

(1) *the action*
$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$
is continuous if we give $IrredComp(X_{\overline{k}})$ the discrete topology,
(2) *every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and*
(3) *given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $Gal(\overline{k}/k)$-orbit.*

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = \mathrm{Spec}(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 29.15.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to $\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $\overline{k}/k'/k$ such that each $f_i \in A_{k'}$. It is clear that $\mathrm{Gal}(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 33.8.13 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 33.8.11. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $\mathrm{Gal}(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 33.7.10. Clearly $W = T$ and we win. □

054R **Lemma 33.8.16.** *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension k'/k such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.*

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 33.8.15 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 33.8.15 (2) there exists a finite extension $\overline{k}/k'/k$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. □

054S **Lemma 33.8.17.** *Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.*

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. □

33.9. Geometrically integral schemes

0366 If X is an integral scheme over a field, then it can happen that X becomes either nonreduced or reducible after extending the ground field. This does not happen for geometrically integral schemes.

020H **Definition 33.9.1.** Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically pointwise integral at* x if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is integral.
(2) We say X is *geometrically pointwise integral* if X is geometrically pointwise integral at every point.
(3) We say X is *geometrically integral* over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = \mathrm{Spec}(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

038K **Lemma 33.9.2.** *Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. See Properties, Lemma 28.3.4. □

0BUG **Lemma 33.9.3.** *Let k be a field. Let X be a proper scheme over k.*

(1) *$A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,*
(2) *$A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of X,*
(3) *if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,*
(4) *if X is geometrically reduced, then k_i is finite separable over k,*
(5) *if X is geometrically connected, then A is geometrically irreducible over k,*
(6) *if X is geometrically irreducible, then A is geometrically irreducible over k,*
(7) *if X is geometrically reduced and geometrically connected, then $A = k$, and*
(8) *if X is geometrically integral, then $A = k$.*

Proof. By Cohomology of Schemes, Lemma 30.19.2 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local Artinian k-algebras by Algebra, Lemma 10.53.2 and Proposition 10.60.7. If $X = Y \amalg Z$ with Y and Z open in X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of X. Finally, as X has a Noetherian underlying topological space its connected components are open. Hence the connected components of X correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced. Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 10.25.1). This proves (3).

If X is geometrically reduced, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (equality by Cohomology of Schemes, Lemma 30.5.2) is reduced. This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 10.44.2 and 10.44.4. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and geometrically connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). □

Here is a baby version of Stein factorization; actual Stein factorization will be discussed in More on Morphisms, Section 37.53.

0FD1 **Lemma 33.9.4.** *Let X be a proper scheme over a field k. Set $A = H^0(X, \mathcal{O}_X)$. The fibres of the canonical morphism $X \to \mathrm{Spec}(A)$ are geometrically connected.*

Proof. Set $S = \mathrm{Spec}(A)$. The canonical morphism $X \to S$ is the morphism corresponding to $\Gamma(S, \mathcal{O}_S) = A = \Gamma(X, \mathcal{O}_X)$ via Schemes, Lemma 26.6.4. The k-algebra A is a finite product $A = \prod A_i$ of local Artinian k-algebras finite over k, see Lemma 33.9.3. Denote $s_i \in S$ the point corresponding to the maximal ideal of A_i. Choose an algebraic closure $\overline{k}$ of k and set $\overline{A} = A \otimes_k \overline{k}$. Choose an embedding $\kappa(s_i) \to \overline{k}$ over k; this determines a $\overline{k}$-algebra map

$$\sigma_i : \overline{A} = A \otimes_k \overline{k} \to \kappa(s_i) \otimes_k \overline{k} \to \overline{k}$$

Consider the base change

$$\begin{array}{ccc} \overline{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \overline{S} & \longrightarrow & S \end{array}$$

of X to $\overline{S} = \mathrm{Spec}(\overline{A})$. By Cohomology of Schemes, Lemma 30.5.2 we have $\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) = \overline{A}$. If $\overline{s}_i \in \mathrm{Spec}(\overline{A})$ denotes the $\overline{k}$-rational point corresponding to σ_i, then we see that $\overline{s}_i$ maps to $s_i \in S$ and $\overline{X}_{\overline{s}_i}$ is the base change of X_{s_i} by $\mathrm{Spec}(\sigma_i)$. Thus we see that it suffices to prove the lemma in case k is algebraically closed.

Assume k is algebraically closed. In this case $\kappa(s_i)$ is algebraically closed and we have to show that X_{s_i} is connected. The product decomposition $A = \prod A_i$ corresponds to a disjoint union decomposition $\mathrm{Spec}(A) = \coprod \mathrm{Spec}(A_i)$, see Algebra, Lemma 10.21.2. Denote X_i the inverse image of $\mathrm{Spec}(A_i)$. It follows from Lemma 33.9.3 part (2) that $A_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that $X_{s_i} \to X_i$ is a closed immersion inducing an isomorphism on underlying topological spaces (because $\mathrm{Spec}(A_i)$ is a singleton). Hence if X_{s_i} isn't connected, then neither is X_i. So either X_i is empty and $A_i = 0$ or X_i can be written as $U \amalg V$ with U and V open and nonempty which would imply that A_i has a nontrivial idempotent. Since A_i is local this is a contradiction and the proof is complete. □

0FD2 **Lemma 33.9.5.** *Let k be a field. Let X be a proper geometrically reduced scheme over k. The following are equivalent*

(1) $H^0(X, \mathcal{O}_X) = k$, *and*
(2) X *is geometrically connected.*

Proof. By Lemma 33.9.4 we have (1) $\Rightarrow$ (2). By Lemma 33.9.3 we have (2) $\Rightarrow$ (1). □

33.10. Geometrically normal schemes

038L In Properties, Definition 28.7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of "geometrically regular" schemes we consider all field extensions of the ground field.

038M **Definition 33.10.1.** Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically normal at* x if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal.
(2) We say X is *geometrically normal* over k if X is geometrically normal at every $x \in X$.

038N **Lemma 33.10.2.** *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 10.165.2).*

Proof. It is clear that (1) implies (2). Assume (2). Let k'/k be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.3). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 10.165.1.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. □

038O **Lemma 33.10.3.** *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically normal,*
(2) *$X_{k'}$ is a normal scheme for every field extension k'/k,*
(3) *$X_{k'}$ is a normal scheme for every finitely generated field extension k'/k,*
(4) *$X_{k'}$ is a normal scheme for every finite purely inseparable field extension k'/k,*
(5) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 10.165.2), and*
(6) *$X_{k^{perf}}$ is a normal scheme.*

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition. Hence (4) implies (5).

Assume (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 26.17). Hence $X_{k'}$ is normal. So (1) holds.

The equivalence of (5) and (6) follows from the definition of geometrically normal algebras and the equivalence (just proved) of (3) and (4). □

038P **Lemma 33.10.4.** *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *$X_{k'}$ is geometrically normal at x'.*

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). Choose a common field extension k'''/k of k'/k and k''/k, see Fields, Lemma 9.6.9. Choose a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 10.164.3 we conclude that $\mathcal{O}_{X_{k''},x''}$ is normal. By Lemma 33.10.2 we see that X is geometrically normal at x. □

06DG **Lemma 33.10.5.** *Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.*

Proof. This reduces to Algebra, Lemma 10.165.5 by Lemma 33.10.3. □

0C3M **Lemma 33.10.6.** *Let k be a field. Let X be a normal scheme over k. Let K/k be a separable field extension. Then X_K is a normal scheme.*

Proof. Follows from Lemma 33.10.5 and Algebra, Lemma 10.165.4. □

0FD3 **Lemma 33.10.7.** *Let k be a field. Let X be a proper geometrically normal scheme over k. The following are equivalent*

(1) *$H^0(X, \mathcal{O}_X) = k$,*
(2) *X is geometrically connected,*
(3) *X is geometrically irreducible, and*
(4) *X is geometrically integral.*

Proof. By Lemma 33.9.5 we have the equivalence of (1) and (2). A locally Noetherian normal scheme (such as $X_{\overline{k}}$) is a disjoint union of its irreducible components (Properties, Lemma 28.7.6). Thus we see that (2) and (3) are equivalent. Since $X_{\overline{k}}$ is assumed reduced, we see that (3) and (4) are equivalent too. □

33.11. Change of fields and locally Noetherian schemes

038Q Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = \mathrm{Spec}(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this property is preserved under base change.)

038R **Lemma 33.11.1.** *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.*

Proof. Using Properties, Lemma 28.5.2 we reduce to the case where X is affine, say $X = \mathrm{Spec}(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 10.164.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 10.31.8. □

33.12. Geometrically regular schemes

038S A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 33.12.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 33.11.1 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

038T **Definition 33.12.1.** Let k be a field. Let X be a locally Noetherian scheme over k.

(1) Let $x \in X$. We say X is *geometrically regular at x* over k if for every finitely generated field extension k'/k and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
(2) We say X is *geometrically regular over k* if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

038U **Lemma 33.12.2.** *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically regular at x,*

(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 10.166.2).*

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.3). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. □

038V **Lemma 33.12.3.** *Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent*

(1) *X is geometrically regular,*
(2) *$X_{k'}$ is a regular scheme for every finitely generated field extension k'/k,*
(3) *$X_{k'}$ is a regular scheme for every finite purely inseparable field extension k'/k,*
(4) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 10.166.2), and*
(5) *there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.*

Proof. Assume (1). Then for every finitely generated field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 28.9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 26.17). Hence $X_{k'}$ is regular. So (1) holds. □

038W **Lemma 33.12.4.** *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *$X_{k'}$ is geometrically regular at x'.*

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension k'''/k of k'/k and k''/k, see Fields, Lemma 9.6.9 and its proof. and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 10.110.9 we conclude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 33.12.2 we see that X is geometrically regular at x. □

The following lemma is a geometric variant of Algebra, Lemma 10.166.3.

05AW **Lemma 33.12.5.** *Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.*

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see Lemma 33.12.3. By Morphisms, Lemma 29.25.7 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homomorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 10.110.9. □

038X **Lemma 33.12.6.** *Let k be a field. Let X be a scheme locally of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to \mathrm{Spec}(k)$ is smooth at x (Morphisms, Definition 29.34.1).*

Proof. The question is local around x, hence we may assume that $X = \mathrm{Spec}(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 10.140.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak{p})$ of x. It is a finitely generated extension of k. By Algebra, Lemma 10.45.3 there exists a finite purely inseparable extension k'/k such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak{p}' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak{p}$. It is the unique prime lying over $\mathfrak{p}$, see Algebra, Lemma 10.46.7. Hence the residue field $K' := \kappa(\mathfrak{p}')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak{p}'}$ is regular. Hence by Algebra, Lemma 10.140.5 we see that $k' \to A'$ is smooth at $\mathfrak{p}'$. This in turn implies that $k \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemma 10.137.18. The lemma is proved. □

038Y **Example 33.12.7.** Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $\mathrm{Spec}(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = \mathrm{Spec}(k[x, y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x, y]$ is irreducible. The morphism $V \to \mathrm{Spec}(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \big(k[x, y]/(y^2 - x^p + t)\big)_{(y,x^p-t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely y. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v'_0 \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in (\text{fraction field of } \mathcal{O}_{V',v'_0})$$

is integral over $\mathcal{O}_{V',v'_0}$ (just compute its square) and not contained in it, so V' is not normal at v'_0. This concludes the example.

33.13. Change of fields and the Cohen-Macaulay property

045O The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

045P **Lemma 33.13.1.** *Let X be a locally Noetherian scheme over the field k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension k'/k.

Proof. The first case of the lemma follows from Algebra, Lemma 10.167.2. The second case of the lemma is equivalent to Algebra, Lemma 10.130.6. □

33.14. Change of fields and the Jacobson property

0477 A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

0478 **Lemma 33.14.1.** *Let X be a scheme which is locally of finite type over k. Then*

(1) *for any closed point $x \in X$ the extension $\kappa(x)/k$ is algebraic, and*
(2) *X is a Jacobson scheme (Properties, Definition 28.6.1).*

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 28.6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1. It also follows from a combination of Morphisms, Lemmas 29.16.8, 29.16.9, and 29.16.10. □

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

0479 **Lemma 33.14.2.** *Let X be a scheme over a field k. For any field extension K/k whose cardinality is large enough we have*

(1) *for any closed point $x \in X_K$ the extension $\kappa(x)/K$ is algebraic, and*
(2) *X_K is a Jacobson scheme (Properties, Definition 28.6.1).*

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 10.35.12 and Properties, Lemma 28.6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 28.6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. □

33.15. Change of fields and ample invertible sheaves

0BDB The following result is typical for the results in this section.

0BDC **Lemma 33.15.1.** *Let k be a field. Let X be a scheme over k. If there exists an ample invertible sheaf on X_K for some field extension K/k, then X has an ample invertible sheaf.*

Proof. Let K/k be a field extension such that X_K has an ample invertible sheaf $\mathcal{L}$. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.26.1). Since X_K is quasi-separated (by Properties, Lemma 28.26.7) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X_K is $\mathcal{L}$ (Limits, Lemma 32.10.3; here and below we use that X is quasi-compact and quasi-separated as just shown). By Limits, Lemma 32.4.15 we may assume $\mathcal{L}_i$ is ample after possibly increasing i. Fix such an i and let $\mathfrak{m} \subset A_i$ be a maximal ideal. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence has an ample invertible sheaf (Properties, Lemma 28.26.3). Since $X_{k'} \to X$ is finite locally free we conclude that X has an ample invertible sheaf by Divisors, Proposition 31.17.9. □

0BDD **Lemma 33.15.2.** *Let k be a field. Let X be a scheme over k. If X_K is quasi-affine for some field extension K/k, then X is quasi-affine.*

Proof. Let K/k be a field extension such that X_K is quasi-affine. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.18.1). Since X_K is quasi-separated (as an open subscheme of an affine scheme) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i such that

X_i is quasi-affine (Limits, Lemma 32.4.12; here we use that X is quasi-compact and quasi-separated as just shown). By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence is quasi-affine (Properties, Lemma 28.27.2). Since $X_{k'} \to X$ is finite locally free we conclude by Divisors, Lemma 31.17.10. □

0BDE **Lemma 33.15.3.** *Let k be a field. Let X be a scheme over k. If X_K is quasi-projective over K for some field extension K/k, then X is quasi-projective over k.*

Proof. By definition a morphism of schemes $g : Y \to T$ is quasi-projective if it is locally of finite type, quasi-compact, and there exists a g-ample invertible sheaf on Y. Let K/k be a field extension such that X_K is quasi-projective over K. Let $\mathrm{Spec}(A) \subset X$ be an affine open. Then U_K is an affine open subscheme of X_K, hence A_K is a K-algebra of finite type. Then A is a k-algebra of finite type by Algebra, Lemma 10.126.1. Hence $X \to \mathrm{Spec}(k)$ is locally of finite type. Since $X_K \to \mathrm{Spec}(K)$ is quasi-compact, we see that X_K is quasi-compact, hence X is quasi-compact, hence $X \to \mathrm{Spec}(k)$ is of finite type. By Morphisms, Lemma 29.39.4 we see that X_K has an ample invertible sheaf. Then X has an ample invertible sheaf by Lemma 33.15.1. Hence $X \to \mathrm{Spec}(k)$ is quasi-projective by Morphisms, Lemma 29.39.4. □

The following lemma is a special case of Descent, Lemma 35.23.16.

0BDF **Lemma 33.15.4.** *Let k be a field. Let X be a scheme over k. If X_K is proper over K for some field extension K/k, then X is proper over k.*

Proof. Let K/k be a field extension such that X_K is proper over K. Recall that this implies X_K is separated and quasi-compact (Morphisms, Definition 29.41.1). The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.26.1). Since X_K is separated we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \mathrm{colim}\, A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A_i)$. By Limits, Lemma 32.13.1 there exists an i such that $X_i \to \mathrm{Spec}(A_i)$ is proper. Here we use that X is quasi-compact and quasi-separated as just shown. Choose a maximal ideal $\mathfrak{m} \subset A_i$. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. The base change $X_{k'} \to \mathrm{Spec}(k')$ is proper (Morphisms, Lemma 29.41.5). Since k'/k is finite both $X_{k'} \to X$ and the composition $X_{k'} \to \mathrm{Spec}(k)$ are proper as well (Morphisms, Lemmas 29.44.11, 29.41.5, and 29.41.4). The first implies that X is separated over k as $X_{k'}$ is separated (Morphisms, Lemma 29.41.11). The second implies that $X \to \mathrm{Spec}(k)$ is proper by Morphisms, Lemma 29.41.9. □

0BDG **Lemma 33.15.5.** *Let k be a field. Let X be a scheme over k. If X_K is projective over K for some field extension K/k, then X is projective over k.*

Proof. A scheme over k is projective over k if and only if it is quasi-projective and proper over k. See Morphisms, Lemma 29.43.13. Thus the lemma follows from Lemmas 33.15.3 and 33.15.4. □

33.16. Tangent spaces

0B28 In this section we define the tangent space of a morphism of schemes at a point of the source using points with values in dual numbers.

0B29 **Definition 33.16.1.** For any ring R the *dual numbers* over R is the R-algebra denoted $R[\epsilon]$. As an R-module it is free with basis 1, ϵ and the R-algebra structure comes from setting $\epsilon^2 = 0$.

Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$ in S. Consider the solid commutative diagram

0B2A (33.16.1.1)
$$\begin{array}{ccccc} \operatorname{Spec}(\kappa(x)) & \longrightarrow & \operatorname{Spec}(\kappa(x)[\epsilon]) & \dashrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \operatorname{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

with the curved arrow being the canonical morphism of $\operatorname{Spec}(\kappa(x))$ into X.

0B2B **Lemma 33.16.2.** *The set of dotted arrows making (33.16.1.1) commute has a canonical $\kappa(x)$-vector space structure.*

Proof. Set $\kappa = \kappa(x)$. Observe that we have a pushout in the category of schemes

$$\operatorname{Spec}(\kappa[\epsilon]) \amalg_{\operatorname{Spec}(\kappa)} \operatorname{Spec}(\kappa[\epsilon]) = \operatorname{Spec}(\kappa[\epsilon_1, \epsilon_2])$$

where $\kappa[\epsilon_1, \epsilon_2]$ is the κ-algebra with basis $1, \epsilon_1, \epsilon_2$ and $\epsilon_1^2 = \epsilon_1\epsilon_2 = \epsilon_2^2 = 0$. This follows immediately from the corresponding result for rings and the description of morphisms from spectra of local rings to schemes in Schemes, Lemma 26.13.1. Given two arrows $\theta_1, \theta_2 : \operatorname{Spec}(\kappa[\epsilon]) \to X$ we can consider the morphism

$$\theta_1 + \theta_2 : \operatorname{Spec}(\kappa[\epsilon]) \to \operatorname{Spec}(\kappa[\epsilon_1, \epsilon_2]) \xrightarrow{\theta_1, \theta_2} X$$

where the first arrow is given by $\epsilon_i \mapsto \epsilon$. On the other hand, given $\lambda \in \kappa$ there is a self map of $\operatorname{Spec}(\kappa[\epsilon])$ corresponding to the κ-algebra endomorphism of $\kappa[\epsilon]$ which sends ϵ to $\lambda\epsilon$. Precomposing $\theta : \operatorname{Spec}(\kappa[\epsilon]) \to X$ by this selfmap gives $\lambda\theta$. The reader can verify the axioms of a vector space by verifying the existence of suitable commutative diagrams of schemes. We omit the details. (An alternative proof would be to express everything in terms of local rings and then verify the vector space axioms on the level of ring maps.) □

0B2C **Definition 33.16.3.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. The set of dotted arrows making (33.16.1.1) commute with its canonical $\kappa(x)$-vector space structure is called the *tangent space of X over S at x* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

Since tangent vectors at $x \in X$ live in the scheme theoretic fibre X_s of $f : X \to S$ over $s = f(x)$, we get a canonical identification

0BEA (33.16.3.1)
$$T_{X/S,x} = T_{X_s/s,x}$$

This pleasing definition involving the functor of points has the following algebraic description, which suggests defining the *cotangent space of X over S at x* as the $\kappa(x)$-vector space

$$T^*_{X/S,x} = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

simply because it is canonically $\kappa(x)$-dual to the tangent space of X over S at x.

0B2D **Lemma 33.16.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. There is a canonical isomorphism*

$$T_{X/S,x} = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X/S,x}, \kappa(x))$$

of vector spaces over $\kappa(x)$.

Proof. Set $\kappa = \kappa(x)$. Given $\theta \in T_{X/S,x}$ we obtain a map

$$\theta^*\Omega_{X/S} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa(s))} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa)}$$

Taking sections we obtain an $\mathcal{O}_{X,x}$-linear map $\xi_\theta : \Omega_{X/S,x} \to \kappa \mathrm{d}\epsilon$, i.e., an element of the right hand side of the formula of the lemma. To show that $\theta \mapsto \xi_\theta$ is an isomorphism we can replace S by s and X by the scheme theoretic fibre X_s. Indeed, both sides of the formula only depend on the scheme theoretic fibre; this is clear for $T_{X/S,x}$ and for the RHS see Morphisms, Lemma 29.32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$ as this does not change $T_{X/S,x}$ (Schemes, Lemma 26.13.1) nor $\Omega_{X/S,x}$ (Modules, Lemma 17.28.7).

Let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k. To finish the proof we have to show that any A-linear map $\xi : \Omega_{A/k} \to \kappa$ comes from a unique k-algebra map $\varphi : A \to \kappa[\epsilon]$ agreeing with the canonical map $c : A \to \kappa$ modulo ϵ. Write $\varphi(a) = c(a) + D(a)\epsilon$ the reader sees that $a \mapsto D(a)$ is a k-derivation. Using the universal property of $\Omega_{A/k}$ we see that each D corresponds to a unique ξ and vice versa. This finishes the proof. □

0B2E **Lemma 33.16.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point and let $s = f(x) \in S$. Assume that $\kappa(x) = \kappa(s)$. Then there are canonical isomorphisms*

$$\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

and

$$T_{X/S,x} = \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}), \kappa(x))$$

This works more generally if $\kappa(x)/\kappa(s)$ is a separable algebraic extension.

Proof. The second isomorphism follows from the first by Lemma 33.16.4. For the first, we can replace S by s and X by X_s, see Morphisms, Lemma 29.32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$, see Modules, Lemma 17.28.7. Thus we have to show the following algebra fact: let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k such that κ/k is separable algebraic. Then the canonical map

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{A/k} \otimes \kappa$$

is an isomorphism. Observe that $\mathfrak{m}/\mathfrak{m}^2 = H_1(NL_{\kappa/A})$. By Algebra, Lemma 10.134.4 it suffices to show that $\Omega_{\kappa/k} = 0$ and $H_1(NL_{\kappa/k}) = 0$. Since κ is the union of its finite separable extensions in k it suffices to prove this when κ is a finite separable extension of k (Algebra, Lemma 10.134.9). In this case the ring map $k \to \kappa$ is étale and hence $NL_{\kappa/k} = 0$ (more or less by definition, see Algebra, Section 10.143). □

0B2F **Lemma 33.16.6.** *Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$. If $\kappa(y) = \kappa(x)$, then f induces a natural linear map*

$$df : T_{X/S,x} \longrightarrow T_{Y/S,y}$$

which is dual to the linear map $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x}$ via the identifications of Lemma 33.16.4.

Proof. Omitted. □

0BEB **Lemma 33.16.7.** *Let X, Y be schemes over a base S. Let $x \in X$ and $y \in Y$ with the same image point $s \in S$ such that $\kappa(s) = \kappa(x)$ and $\kappa(s) = \kappa(y)$. There is a canonical isomorphism*

$$T_{X \times_S Y/S,(x,y)} = T_{X/S,x} \oplus T_{Y/S,y}$$

The map from left to right is induced by the maps on tangent spaces coming from the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$. The map from right to left is induced by the maps $1 \times y : X_s \to X_s \times_s Y_s$ and $x \times 1 : Y_s \to X_s \times_s Y_s$ via the identification (33.16.3.1) of tangent spaces with tangent spaces of fibres.

Proof. The direct sum decomposition follows from Morphisms, Lemma 29.32.11 via Lemma 33.16.5. Compatibility with the maps comes from Lemma 33.16.6. □

0B2G **Lemma 33.16.8.** *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$ and assume that $\kappa(y) = \kappa(x)$. Then the following are equivalent*

(1) *$df : T_{X/S,x} \longrightarrow T_{Y/S,y}$ is injective, and*
(2) *f is unramified at x.*

Proof. The morphism f is locally of finite type by Morphisms, Lemma 29.15.8. The map $\mathrm{d}f$ is injective, if and only if $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x} \otimes \kappa(x)$ is surjective (Lemma 33.16.6). The exact sequence $f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ (Morphisms, Lemma 29.32.9) then shows that this happens if and only if $\Omega_{X/Y,x} \otimes \kappa(x) = 0$. Hence the result follows from Morphisms, Lemma 29.35.14. □

33.17. Generically finite morphisms

0AB5 In this section we revisit the notion of a generically finite morphism of schemes as studied in Morphisms, Section 29.51.

0AB6 **Lemma 33.17.1.** *Let $f : X \to Y$ be locally of finite type. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

(1) *for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(2) *for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(3) *f is quasi-finite at every generic point of an irreducible component of X,*
(4) *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
(5) *add more here.*

Then f is quasi-finite at every point of X lying over y.

Proof. Condition (4) implies X is locally Noetherian (Morphisms, Lemma 29.15.6). The set of points at which morphism is quasi-finite is open (Morphisms, Lemma 29.56.2). A dense open of a locally Noetherian scheme contains all generic point of irreducible components, hence (4) implies (3). Condition (3) implies condition (1) by Morphisms, Lemma 29.20.5. Condition (1) implies condition (2). Thus it suffices to prove the lemma in case (2) holds.

Assume (2) holds. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 26.13.2. Combined with Morphisms, Lemma 29.20.13 this

shows we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$. Thus we may assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ is a discrete space, then $X_y = \bigcup X_{i,y}$ is discrete as well and we conclude that $X \to Y$ is quasi-finite at all points of X_y by Morphisms, Lemma 29.20.6. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is the spectrum of a finite extension of $\kappa(y)$ and the result is true. Assume that X maps η to a point corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. Let $x \in X$ be a point lying over y. By the dimension formula (Morphisms, Lemma 29.52.1) we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

We know that $\dim(B/\mathfrak{q}) = 1$, that the generic point of X is not equal to x and specializes to x and that $R(X)$ is algebraic over $\kappa(\mathfrak{q})$. Thus we get

$$1 \leq 1 - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

Hence every point x of X_y is closed in X_y by Morphisms, Lemma 29.20.2 and hence $X \to Y$ is quasi-finite at every point x of X_y by Morphisms, Lemma 29.20.6 (which also implies that X_y is a discrete topological space). □

0AB7 **Lemma 33.17.2.** *Let $f : X \to Y$ be a proper morphism. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

1. *for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
2. *for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
3. *f is quasi-finite at every generic point of X,*
4. *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
5. *add more here.*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemma 33.17.1 the morphism f is quasi-finite at every point of the fibre X_y. Hence X_y is a discrete topological space (Morphisms, Lemma 29.20.6). As f is proper the fibre X_y is quasi-compact, i.e., finite. Thus we can apply Cohomology of Schemes, Lemma 30.21.2 to conclude. □

0BFP **Lemma 33.17.3.** *Let X be a Noetherian scheme. Let $f : Y \to X$ be a birational proper morphism of schemes with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains*

1. *every point of codimension 0 in X,*
2. *every $x \in X$ of codimension 1 on X such that $\mathcal{O}_{X,x}$ is a discrete valuation ring,*
3. *every $x \in X$ such that the fibre of $Y \to X$ over x is finite and such that $\mathcal{O}_{X,x}$ is normal, and*
4. *every $x \in X$ such that f is quasi-finite at some $y \in Y$ lying over x and $\mathcal{O}_{X,x}$ is normal.*

Proof. Part (1) follows from Morphisms, Lemma 29.51.6. Part (2) follows from part (3) and Lemma 33.17.2 (and the fact that finite morphisms have finite fibres).

Part (3) follows from part (4) and Morphisms, Lemma 29.20.7 but we will also give a direct proof. Let $x \in X$ be as in (3). By Cohomology of Schemes, Lemma 30.21.2 we may assume f is finite. We may assume X affine. This reduces us to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. Since $\mathcal{O}_{X,x}$ is a domain and X is Noetherian, we may replace X by an affine open of x which is integral. Then, since $Y \to X$ is birational and Y is reduced we see that Y is integral. Writing $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ we see that $A \subset B$ is a finite inclusion of domains having the same field of fractions. If $\mathfrak{p} \subset A$ is the prime corresponding to x, then $A_\mathfrak{p}$ being normal implies that $A_\mathfrak{p} \subset B_\mathfrak{p}$ is an equality. Since B is a finite A-module, we see there exists an $a \in A$, $a \notin \mathfrak{p}$ such that $A_a \to B_a$ is an isomorphism.

Let $x \in X$ and $y \in Y$ be as in (4). After replacing X by an affine open neighbourhood we may assume $X = \mathrm{Spec}(A)$ and $A \subset \mathcal{O}_{X,x}$, see Properties, Lemma 28.29.8. Then A is a domain and hence X is integral. Since f is birational and Y is reduced it follows that Y is integral too. Consider the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. This is a ring map which is essentially of finite type, the residue field extension is finite, and $\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}) = 0$ (to see this trace through the definitions of quasi-finite maps in Morphisms, Definition 29.20.1 and Algebra, Definition 10.122.3). By Algebra, Lemma 10.124.2 $\mathcal{O}_{Y,y}$ is the localization of a finite $\mathcal{O}_{X,x}$-algebra B. Of course we may replace B by the image of B in $\mathcal{O}_{Y,y}$ and assume that B is a domain with the same fraction field as $\mathcal{O}_{Y,y}$. Then $\mathcal{O}_{X,x} \subset B$ have the same fraction field as f is birational. Since $\mathcal{O}_{X,x}$ is normal, we conclude that $\mathcal{O}_{X,x} = B$ (because finite implies integral), in particular, we see that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$. By Morphisms, Lemma 29.42.4 after shrinking X we may assume there is a section $X \to Y$ of f mapping x to y and inducing the given isomorphism on local rings. Since $X \to Y$ is closed (by Schemes, Lemma 26.21.11) necessarily maps the generic point of X to the generic point of Y it follows that the image of $X \to Y$ is Y. Then $Y = X$ and we've proved what we wanted to show. □

33.18. Variants of Noether normalization

0CBG Noether normalization is the statement that if k is a field and A is a finite type k algebra of dimension d, then there exists a finite injective k-algebra homomorphism $k[x_1, \ldots, x_d] \to A$. See Algebra, Lemma 10.115.4. Geometrically this means there is a finite surjective morphism $\mathrm{Spec}(A) \to \mathbf{A}^d_k$ over $\mathrm{Spec}(k)$.

0CBH **Lemma 33.18.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and $\dim_x(X_s) = d$, then there exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{\pi} \mathbf{A}^d_V \to V$$

of $f|_U : U \to V$ such that π is quasi-finite.

Proof. This follows from Algebra, Lemma 10.125.2. □

0CBI **Lemma 33.18.2.** *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $s \in S$. If $\dim(X_s) = d$, then there exists a factorization*

$$X \xrightarrow{\pi} \mathbf{A}^d_S \to S$$

of f such that the morphism $\pi_s : X_s \to \mathbf{A}^d_{\kappa(s)}$ of fibres over s is finite.

Proof. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. We can choose a surjection $A[x_1, \ldots, x_r] \to B$. By Algebra, Lemma 10.115.4 there exist elements $y_1, \ldots, y_d \in A$ in the $\mathbf{Z}$-subalgebra of A generated by $x_1, \ldots, x_r$ such that the A-algebra homomorphism $A[t_1, \ldots, t_d] \to B$ sending t_i to y_i induces a finite $\kappa(\mathfrak{p})$-algebra homomorphism $\kappa(\mathfrak{p})[t_1, \ldots, t_d] \to B \otimes_A \kappa(\mathfrak{p})$. This proves the lemma. □

0CBJ **Lemma 33.18.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Let $V = \mathrm{Spec}(A)$ be an affine open neighbourhood of $f(x)$ in S. If f is unramified at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that we have a commutative diagram*

$$\begin{array}{ccccccc} X & \longleftarrow & U & \xrightarrow{j} & \mathrm{Spec}(A[t]_{g'}/(g)) & \longrightarrow & \mathrm{Spec}(A[t]) = \mathbf{A}^1_V \\ \downarrow & & & \searrow & \downarrow & \swarrow & \\ Y & \longleftarrow & & & V & & \end{array}$$

where j is an immersion, $g \in A[t]$ is a monic polynomial, and g' is the derivative of g with respect to t. If f is étale at x, then we may choose the diagram such that j is an open immersion.

Proof. The unramified case is a translation of Algebra, Proposition 10.152.1. In the étale case this is a translation of Algebra, Proposition 10.144.4 or equivalently it follows from Morphisms, Lemma 29.36.14 although the statements differ slightly. □

0CBK **Lemma 33.18.4.** *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $x \in X$ with image $s \in S$. Let*

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

Then there exists a factorization

$$X \xrightarrow{\pi} \mathbf{A}^r_S \to S$$

of f such that π is unramified at x.

Proof. By Morphisms, Lemma 29.32.12 the first dimension is finite. The first equality follows as the restriction of $\Omega_{X/S}$ to the fibre is the module of differentials from Morphisms, Lemma 29.32.10. The last equality follows from Lemma 33.16.4. Thus we see that the statement makes sense.

To prove the lemma write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Choose a surjection of A-algebras $A[x_1, \ldots, x_t] \to B$. Since $\Omega_{B/A}$ is generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_t$ we see that their images in $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ generate this as a $\kappa(x)$-vector space. After renumbering we may assume that $\mathrm{d}x_1, \ldots, \mathrm{d}x_r$ map to a basis of $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. We claim that $P = A[x_1, \ldots, x_r] \to B$ is unramified at

$\mathfrak{q}$. To see this it suffices to show that $\Omega_{B/P,\mathfrak{q}} = 0$ (Algebra, Lemma 10.151.3). Note that $\Omega_{B/P}$ is the quotient of $\Omega_{B/A}$ by the submodule generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_r$. Hence $\Omega_{B/P,\mathfrak{q}} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = 0$ by our choice of $x_1, \ldots, x_r$. By Nakayama's lemma, more precisely Algebra, Lemma 10.20.1 part (2) which applies as $\Omega_{B/P}$ is finite (see reference above), we conclude that $\Omega_{B/P,\mathfrak{q}} = 0$. □

0CBL **Lemma 33.18.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and*

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

then there exist

(1) *an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{j} \mathbf{A}^{r+1}_V \to V$$

of $f|_U$ such that j is an immersion, or

(2) *an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{j} D \to V$$

of $f|_U$ such that j is a closed immersion and $D \to V$ is smooth of relative dimension r.

Proof. Pick any affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$. Apply Lemma 33.18.4 to $U \to V$ to get $U \to \mathbf{A}^r_V \to V$ as in the statement of that lemma. By Lemma 33.18.3 we get a factorization

$$U \xrightarrow{j} D \xrightarrow{j'} \mathbf{A}^{r+1}_V \xrightarrow{p} \mathbf{A}^r_V \to V$$

where j and j' are immersions, p is the projection, and $p \circ j'$ is standard étale. Thus we see in particular that (1) and (2) hold. □

33.19. Dimension of fibres

0B2H We have already seen that dimension of fibres of finite type morphisms typically jump up. In this section we discuss the phenomenon that in codimension 1 this does not happen. More generally, we discuss how much the dimension of a fibre can jump. Here is a list of related results:

(1) For a finite type morphism $X \to S$ the set of $x \in X$ with $\dim_x(X_{f(x)}) \leq d$ is open, see Algebra, Lemma 10.125.6 and Morphisms, Lemma 29.28.4.
(2) We have the dimension formula, see Algebra, Lemma 10.113.1 and Morphisms, Lemma 29.52.1.
(3) Constant fibre dimension for an integral finite type scheme dominating a valuation ring, see Algebra, Lemma 10.125.9.
(4) If $X \to S$ is of finite type and is quasi-finite at every generic point of X, then $X \to S$ is quasi-finite in codimension 1, see Algebra, Lemma 10.113.2 and Lemma 33.17.1.

The last result mentioned above generalizes as follows.

0B2I **Lemma 33.19.1.** *Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) = d$.*

Proof. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 26.13.2. Thus we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point. We may also replace X by an affine neighbourhood of x.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ has dimension d, then $X_y = \bigcup X_{i,y}$ has dimension d as well. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is a scheme over $\kappa(y)$ and the result is true by assumption. Assume that X maps η to a point $\xi \in Y$ corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. By the dimension formula (Morphisms, Lemma 29.52.1) we have

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X))$$

We have $\dim(B/\mathfrak{q}) = 1$. We have $\mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 29.28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_s,x}$ has a kernel (as $\eta \mapsto \xi \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq d$$

(Morphisms, Lemma 29.28.1). On the other hand, we have $\dim_x(X_s) \geq \dim_\eta(X_{f(\eta)}) = d$ by Morphisms, Lemma 29.28.4. □

0B2J **Lemma 33.19.2.** *Let $f : X \to \mathrm{Spec}(R)$ be a morphism from an irreducible scheme to the spectrum of a valuation ring. If f is locally of finite type and surjective, then the special fibre is equidimensional of dimension equal to the dimension of the generic fibre.*

Proof. We may replace X by its reduction because this does not change the dimension of X or of the special fibre. Then X is integral and the lemma follows from Algebra, Lemma 10.125.9. □

The following lemma generalizes Lemma 33.19.1.

0B2K **Lemma 33.19.3.** *Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $f(\eta) \neq y$ and $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) \leq d + \dim(\mathcal{O}_{Y,y}) - 1$.*

Proof. Exactly as in the proof of Lemma 33.19.1 we reduce to the case $X = \mathrm{Spec}(A)$ with A a domain and $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring whose maximal ideal corresponds to y. After replacing B by $B/\operatorname{Ker}(B \to A)$ we may assume that B is a domain and that $B \subset A$. Then we use the dimension formula (Morphisms, Lemma 29.52.1) to get

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B) + \mathrm{trdeg}_B(A)$$

We have $\mathrm{trdeg}_B(A) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 29.28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_y,x}$ has a kernel (as $f(\eta) \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_y) = \dim(\mathcal{O}_{X_y,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) < \dim(B) + d$$

(equality by Morphisms, Lemma 29.28.1) which proves what we want. □

33.20. Algebraic schemes

06LF The following definition is taken from [**DG67**, I Definition 6.4.1].

06LG **Definition 33.20.1.** Let k be a field. An *algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is of finite type. A *locally algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 29.15.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

06LH **Lemma 33.20.2.** *Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to \mathrm{Spec}(k)$ is finite.*

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = \mathrm{Spec}(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 10.115.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 10.53.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 10.53.6. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = \mathrm{Spec}(A_1) \amalg \ldots \amalg \mathrm{Spec}(A_r)$ hence affine (see Schemes, Lemma 26.6.8) and we have seen the finiteness of $X \to \mathrm{Spec}(k)$ in the first paragraph of the proof. □

The following lemma collects some statements on dimension theory for locally algebraic schemes.

0A21 **Lemma 33.20.3.** *Let k be a field. Let X be a locally algebraic k-scheme.*

0B17 (1) *The topological space of X is catenary (Topology, Definition 5.11.4).*

0B18 (2) *For $x \in X$ closed, we have $\dim_x(X) = \dim(\mathcal{O}_{X,x})$.*

0B19 (3) *For X irreducible we have $\dim(X) = \dim(U)$ for any nonempty open $U \subset X$ and $\dim(X) = \dim_x(X)$ for any $x \in X$.*

0B1A (4) *For X irreducible any chain of irreducible closed subsets can be extended to a maximal chain and all maximal chains of irreducible closed subsets have length equal to $\dim(X)$.*

0B1B (5) *For $x \in X$ we have $\dim_x(X) = \max \dim(Z) = \min \dim(\mathcal{O}_{X,x'})$ where the maximum is over irreducible components $Z \subset X$ containing x and the minimum is over specializations $x \rightsquigarrow x'$ with x' closed in X.*

0B1C (6) *If X is irreducible with generic point x, then* $\dim(X) = trdeg_k(\kappa(x))$.

0B1D (7) *If $x \rightsquigarrow x'$ is an immediate specialization of points of X, then we have* $trdeg_k(\kappa(x)) = trdeg_k(\kappa(x')) + 1$.

0B1E (8) *The dimension of X is the supremum of the numbers $trdeg_k(\kappa(x))$ where x runs over the generic points of the irreducible components of X.*

0B1F (9) *If $x \rightsquigarrow x'$ is a nontrivial specialization of points of X, then*
   (a) $\dim_x(X) \leq \dim_{x'}(X)$,
   (b) $\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,x'})$,
   (c) $trdeg_k(\kappa(x)) > trdeg_k(\kappa(x'))$, *and*
   (d) *any maximal chain of nontrivial specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n = x$ has length $n = trdeg_k(\kappa(x)) - trdeg_k(\kappa(x'))$.*

0B1G (10) *For $x \in X$ we have* $\dim_x(X) = trdeg_k(\kappa(x)) + \dim(\mathcal{O}_{X,x})$.

0B1H (11) *If $x \rightsquigarrow x'$ is an immediate specialization of points of X and X is irreducible or equidimensional, then* $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{X,x}) + 1$.

Proof. Instead on relying on the more general results proved earlier we will reduce the statements to the corresponding statements for finite type k-algebras and cite results from the chapter on commutative algebra.

Proof of (1). This is local on X by Topology, Lemma 5.11.5. Thus we may assume $X = \mathrm{Spec}(A)$ where A is a finite type k-algebra. We have to show that A is catenary (Algebra, Lemma 10.105.2). We can reduce to $k[x_1, \ldots, x_n]$ using Algebra, Lemma 10.105.7 and then apply Algebra, Lemma 10.114.3. Alternatively, this holds because k is Cohen-Macaulay (trivially) and Cohen-Macaulay rings are universally catenary (Algebra, Lemma 10.105.9).

Proof of (2). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 10.114.6.

Proof of (3). It suffices to show that any two nonempty affine opens $U, U' \subset X$ have the same dimension (any finite chain of irreducible subsets meets an affine open). Pick a closed point x of X with $x \in U \cap U'$. This is possible because X is irreducible, hence $U \cap U'$ is nonempty, hence there is such a closed point because X is Jacobson by Lemma 33.14.1. Then $\dim(U) = \dim(\mathcal{O}_{X,x}) = \dim(U')$ by Algebra, Lemma 10.114.4 (strictly speaking you have to replace X by its reduction before applying the lemma).

Proof of (4). Given a chain of irreducible closed subsets we can find an affine open $U \subset X$ which meets the smallest one. Thus the statement follows from Algebra, Lemma 10.114.4 and $\dim(U) = \dim(X)$ which we have seen in (3).

Proof of (5). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. The rule $Z \mapsto Z \cap U$ is a bijection between irreducible components of X passing through x and irreducible components of U passing through x. Also, $\dim(Z \cap U) = \dim(Z)$ for such Z by (3). Hence the statement follows from Algebra, Lemma 10.114.5.

Proof of (6). By (3) this reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case it follows from Algebra, Lemma 10.116.1 applied to A_{red}.

Proof of (7). Let $Z = \overline{\{x\}} \supset Z' = \overline{\{x'\}}$. Then it follows from (4) that $Z \supset Z'$ is the start of a maximal chain of irreducible closed subschemes in Z and consequently $\dim(Z) = \dim(Z') + 1$. We conclude by (6).

Proof of (8). A simple topological argument shows that $\dim(X) = \sup \dim(Z)$ where the supremum is over the irreducible components of X (hint: use Topology, Lemma 5.8.3). Thus this follows from (6).

Proof of (9). Part (a) follows from the fact that any open $U \subset X$ containing x' also contains x. Part (b) follows because $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,x'}$ hence any chain of primes in $\mathcal{O}_{X,x}$ corresponds to a chain of primes in $\mathcal{O}_{X,x'}$ which can be extended by adding $\mathfrak{m}_{x'}$ at the end. Both (c) and (d) follow formally from (7).

Proof of (10). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 10.116.3.

Proof of (11). If X is equidimensional (Topology, Definition 5.10.5) then $\dim(X)$ is equal to the dimension of every irreducible component of X, whence $\dim_x(X) = \dim(X) = \dim_{x'}(X)$ by (5). Thus this follows from (7). □

0B2L **Lemma 33.20.4.** *Let k be a field. Let $f : X \to Y$ be a morphism of locally algebraic k-schemes.*

(1) *For $y \in Y$, the fibre X_y is a locally algebraic scheme over $\kappa(y)$ hence all the results of Lemma 33.20.3 apply.*
(2) *Assume X is irreducible. Set $Z = \overline{f(X)}$ and $d = \dim(X) - \dim(Z)$. Then*
   (a) *$\dim_x(X_{f(x)}) \geq d$ for all $x \in X$,*
   (b) *the set of $x \in X$ with $\dim_x(X_{f(x)}) = d$ is dense open,*
   (c) *if $\dim(\mathcal{O}_{Z,f(x)}) \geq 1$, then $\dim_x(X_{f(x)}) \leq d + \dim(\mathcal{O}_{Z,f(x)}) - 1$,*
   (d) *if $\dim(\mathcal{O}_{Z,f(x)}) = 1$, then $\dim_x(X_{f(x)}) = d$,*
(3) *For $x \in X$ with $y = f(x)$ we have $\dim_x(X_y) \geq \dim_x(X) - \dim_y(Y)$.*

Proof. The morphism f is locally of finite type by Morphisms, Lemma 29.15.8. Hence the base change $X_y \to \mathrm{Spec}(\kappa(y))$ is locally of finite type. This proves (1). In the rest of the proof we will freely use the results of Lemma 33.20.3 for X, Y, and the fibres of f.

Proof of (2). Let $\eta \in X$ be the generic point and set $\xi = f(\eta)$. Then $Z = \overline{\{\xi\}}$. Hence

$$d = \dim(X) - \dim(Z) = \mathrm{trdeg}_k\kappa(\eta) - \mathrm{trdeg}_k\kappa(\xi) = \mathrm{trdeg}_{\kappa(\xi)}\kappa(\eta) = \dim_\eta(X_\xi)$$

Thus parts (2)(a) and (2)(b) follow from Morphisms, Lemma 29.28.4. Parts (2)(c) and (2)(d) follow from Lemmas 33.19.3 and 33.19.1.

Proof of (3). Let $x \in X$. Let $X' \subset X$ be an irreducible component of X passing through x of dimension $\dim_x(X)$. Then (2) implies that $\dim_x(X_y) \geq \dim(X') - \dim(Z')$ where $Z' \subset Y$ is the closure of the image of X'. This proves (3). □

0B2M **Lemma 33.20.5.** *Let k be a field. Let X, Y be locally algebraic k-schemes.*

(1) *For $z \in X \times Y$ lying over (x, y) we have $\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$.*
(2) *We have $\dim(X \times Y) = \dim(X) + \dim(Y)$.*

Proof. Proof of (1). Consider the factorization

$$X \times Y \longrightarrow Y \longrightarrow \mathrm{Spec}(k)$$

of the structure morphism. The first morphism $p : X \times Y \to Y$ is flat as a base change of the flat morphism $X \to \mathrm{Spec}(k)$ by Morphisms, Lemma 29.25.8. Moreover, we have $\dim_z(p^{-1}(y)) = \dim_x(X)$ by Morphisms, Lemma 29.28.3. Hence

$\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$ by Morphisms, Lemma 29.28.2. Part (2) is a direct consequence of (1). □

33.21. Complete local rings

0C51 Some results on complete local rings of schemes over fields.

0C52 **Lemma 33.21.1.** *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$ be a point with residue field κ. There is an isomorphism*

0C53 (33.21.1.1) $$\kappa[[x_1, \ldots, x_n]]/I \longrightarrow \mathcal{O}^\wedge_{X,x}$$

inducing the identity on residue fields. In general we cannot choose (33.21.1.1) to be a k-algebra isomorphism. However, if the extension κ/k is separable, then we can choose (33.21.1.1) to be an isomorphism of k-algebras.

Proof. The existence of the isomorphism is an immediate consequence of the Cohen structure theorem[2] (Algebra, Theorem 10.160.8).

Let p be an odd prime number, let $k = \mathbf{F}_p(t)$, and $A = k[x, y]/(y^2 + x^p - t)$. Then the completion $A^\wedge$ of A in the maximal ideal $\mathfrak{m} = (y)$ is isomorphic to $k(t^{1/p})[[z]]$ as a ring but not as a k-algebra. The reason is that $A^\wedge$ does not contain an element whose pth power is t (as the reader can see by computing modulo y^2). This also shows that any isomorphism (33.21.1.1) cannot be a k-algebra isomorphism.

If κ/k is separable, then there is a k-algebra homomorphism $\kappa \to \mathcal{O}^\wedge_{X,x}$ inducing the identity on residue fields by More on Algebra, Lemma 15.38.3. Let $f_1, \ldots, f_n \in \mathfrak{m}_x$ be generators. Consider the map

$$\kappa[[x_1, \ldots, x_n]] \longrightarrow \mathcal{O}^\wedge_{X,x}, \quad x_i \longmapsto f_i$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete (the right hand side by Algebra, Lemmas 10.96.3) this map is surjective by Algebra, Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. □

0C54 **Lemma 33.21.2.** *Let K/k be an extension of fields. Let X be a locally algebraic k-scheme. Set $Y = X_K$. Let $y \in Y$ be a point with image $x \in X$. Assume that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x)/k$ is separable. Choose an isomorphism*

$$\kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}^\wedge_{X,x}$$

of k-algebras as in (33.21.1.1). Then we have an isomorphism

$$\kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}^\wedge_{Y,y}$$

of K-algebras as in (33.21.1.1). Here we use $\kappa(x) \to \kappa(y)$ to view g_j as a power series over $\kappa(y)$.

Proof. The local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces a local ring map $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$. The induced map

$$\kappa(x) \to \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$$

composed with the projection to $\kappa(y)$ is the canonical homomorphism $\kappa(x) \to \kappa(y)$. By Lemma 33.5.1 the residue field $\kappa(y)$ is a localization of $\kappa(x) \otimes_k K$ at the kernel

[2] Note that if κ has characteristic p, then the theorem just says we get a surjection $\Lambda[[x_1, \ldots, x_n]] \to \mathcal{O}^\wedge_{X,x}$ where Λ is a Cohen ring for κ. But of course in this case the map factors through $\Lambda/p\Lambda[[x_1, \ldots, x_n]]$ and $\Lambda/p\Lambda = \kappa$.

$\mathfrak{p}_0$ of $\kappa(x) \otimes_k K \to \kappa(y)$. On the other hand, by Lemma 33.5.3 the local ring $(\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$ is equal to $\kappa(y)$. Hence the map

$$\kappa(x) \otimes_k K \to \mathcal{O}^\wedge_{Y,y}$$

factors canonically through $\kappa(y)$. We obtain a commutative diagram

$$\begin{array}{ccccc} \kappa(y) & & \longrightarrow & & \mathcal{O}^\wedge_{Y,y} \\ \uparrow & & & & \uparrow \\ \kappa(x) & \longrightarrow & \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) & \longrightarrow & \mathcal{O}^\wedge_{X,x} \end{array}$$

Let $f_i \in \mathfrak{m}^\wedge_x \subset \mathcal{O}^\wedge_{X,x}$ be the image of x_i. Observe that $\mathfrak{m}^\wedge_x = (f_1, \ldots, f_n)$ as the map is surjective. Consider the map

$$\kappa(y)[[x_1, \ldots, x_n]] \longrightarrow \mathcal{O}^\wedge_{Y,y}, \quad x_i \longmapsto f_i$$

where here f_i really means the image of f_i in $\mathfrak{m}^\wedge_y$. Since $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y$ by Lemma 33.5.3 we see that the right hand side is complete with respect to $(x_1, \ldots, x_n)$ (use Algebra, Lemma 10.96.3 to see that it is a complete local ring). Since both sides are $(x_1, \ldots, x_n)$-adically complete our map is surjective by Algebra, Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$. Of course the power series $g_1, \ldots, g_m$ are mapped to zero under this map, as they already map to zero in $\mathcal{O}^\wedge_{X,x}$. Thus we have the commutative diagram

$$\begin{array}{ccc} \kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) & \longrightarrow & \mathcal{O}^\wedge_{Y,y} \\ \uparrow & & \uparrow \\ \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) & \longrightarrow & \mathcal{O}^\wedge_{X,x} \end{array}$$

We still need to show that the top horizontal arrow is an isomorphism. We already know that it is surjective. We know that $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat (Lemma 33.5.1), which implies that $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$ is flat (More on Algebra, Lemma 15.43.8). Thus we may apply Algebra, Lemma 10.99.1 with $R = \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m)$, with $S = \kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m)$, with $M = \mathcal{O}^\wedge_{Y,y}$, and with $N = S$ to conclude that the map is injective. □

33.22. Global generation

0B5W Some lemmas related to global generation of quasi-coherent modules.

0B57 **Lemma 33.22.1.** *Let $X \to \mathrm{Spec}(A)$ be a morphism of schemes. Let $A \subset A'$ be a faithfully flat ring map. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is globally generated if and only if the base change $\mathcal{F}_{A'}$ is globally generated.*

Proof. More precisely, set $X_{A'} = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$. Let $\mathcal{F}_{A'} = p^*\mathcal{F}$ where $p : X_{A'} \to X$ is the projection. By Cohomology of Schemes, Lemma 30.5.2 we have $H^0(X_{k'}, \mathcal{F}_{A'}) = H^0(X, \mathcal{F}) \otimes_A A'$. Thus if s_i, $i \in I$ are generators for $H^0(X, \mathcal{F})$ as an A-module, then their images in $H^0(X_{A'}, \mathcal{F}_{A'})$ are generators for $H^0(X_{A'}, \mathcal{F}_{A'})$ as an A'-module. Thus we have to show that the map $\alpha : \bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$, $(f_i) \mapsto \sum f_i s_i$ is surjective if and only if $p^*\alpha$ is surjective. This we may check over an affine open $U = \mathrm{Spec}(B)$ of X. Then $\mathcal{F}|_U$ corresponds to a B-module M and $s_i|_U$ to elements $x_i \in M$. Thus we have to show that $\bigoplus_{i \in I} B \to M$ is surjective if and only if the

base change $\bigoplus_{i\in I} B \otimes_A A' \to M \otimes_A A'$ is surjective. This is true because $A \to A'$ is faithfully flat. □

0B58 **Lemma 33.22.2.** *Let k be an infinite field. Let X be a scheme of finite type over k. Let $\mathcal{L}$ be a very ample invertible sheaf on X. Let $n \geq 0$ and $x, x_1, \ldots, x_n \in X$ be points with x a k-rational point, i.e., $\kappa(x) = k$, and $x \neq x_i$ for $i = 1, \ldots, n$. Then there exists an $s \in H^0(X, \mathcal{L})$ which vanishes at x but not at x_i.*

Proof. If $n = 0$ the result is trivial, hence we assume $n > 0$. By definition of a very ample invertible sheaf, the lemma immediately reduces to the case where $X = \mathbf{P}^r_k$ for some $r > 0$ and $\mathcal{L} = \mathcal{O}_X(1)$. Write $\mathbf{P}^r_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$. Set $V = H^0(X, \mathcal{L}) = kT_0 \oplus \ldots \oplus kT_r$. Since x is a k-rational point, we see that the set $s \in V$ which vanish at x is a codimension 1 subspace $W \subset V$ and that W generates the homogeneous prime ideal corresponding to x. Since $x_i \neq x$ the corresponding homogeneous prime $\mathfrak{p}_i \subset k[T_0, \ldots, T_r]$ does not contain W. Since k is infinite, we then see that $W \neq \bigcup W \cap \mathfrak{q}_i$ and the proof is complete. □

0B3Z **Lemma 33.22.3.** *Let k be an infinite field. Let X be an algebraic k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \to \Gamma(X, \mathcal{L})$ be a linear map of k-vector spaces whose image generates $\mathcal{L}$. Then there exists a subspace $W \subset V$ with $\dim_k(W) \leq \dim(X) + 1$ which generates $\mathcal{L}$.*

Proof. Throughout the proof we will use that for every $x \in X$ the linear map

$$\psi_x : V \to \Gamma(X, \mathcal{L}) \to \mathcal{L}_x \to \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero. The proof is by induction on $\dim(X)$.

The base case is $\dim(X) = 0$. In this case X has finitely many points $X = \{x_1, \ldots, x_n\}$ (see for example Lemma 33.20.2). Since k is infinite there exists a vector $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Then $W = k \cdot v$ does the job.

Assume $\dim(X) > 0$. Let $X_i \subset X$ be the irreducible components of dimension equal to $\dim(X)$. Since X is Noetherian there are only finitely many of these. For each i pick a point $x_i \in X_i$. As above choose $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Let $Z \subset X$ be the zero scheme of the image of v in $\Gamma(X, \mathcal{L})$, see Divisors, Definition 31.14.8. By construction $\dim(Z) < \dim(X)$. By induction we can find $W \subset V$ with $\dim(W) \leq \dim(X)$ such that W generates $\mathcal{L}|_Z$. Then $W + k \cdot v$ generates $\mathcal{L}$. □

33.23. Separating points and tangent vectors

0E8R This is just the following result.

0E8S **Lemma 33.23.1.** *Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \subset H^0(X, \mathcal{L})$ be a k-subvector space. If*

1. *for every pair of distinct closed points $x, y \in X$ there is a section $s \in V$ which vanishes at x but not at y, and*
2. *for every closed point $x \in X$ and nonzero tangent vector $\theta \in T_{X/k,x}$ there exists a section $s \in V$ which vanishes at x but whose pullback by θ is nonzero,*

then $\mathcal{L}$ is very ample and the canonical morphism $\varphi_{\mathcal{L},V} : X \to \mathbf{P}(V)$ is a closed immersion.

Proof. Condition (1) implies in particular that the elements of V generate $\mathcal{L}$ over X. Hence we get a canonical morphism

$$\varphi = \varphi_{\mathcal{L},V} : X \longrightarrow \mathbf{P}(V)$$

by Constructions, Example 27.21.2. The morphism φ is proper by Morphisms, Lemma 29.41.7. By (1) the map φ is injective on closed points (computation omitted). In particular, the fibre over any closed point of $\mathbf{P}(V)$ is a singleton (small detail omitted). Thus we see that φ is finite, for example use Cohomology of Schemes, Lemma 30.21.2. To finish the proof it suffices to show that the map

$$\varphi^\sharp : \mathcal{O}_{\mathbf{P}(V)} \longrightarrow \varphi_*\mathcal{O}_X$$

is surjective. This we may check on stalks at closed points. Let $x \in X$ be a closed point with image the closed point $p = \varphi(x) \in \mathbf{P}(V)$. Since $\varphi^{-1}(\{p\}) = \{x\}$ by (1) and since φ is proper (hence closed), we see that $\varphi^{-1}(U)$ runs through a fundamental system of open neighbourhoods of x as U runs through a fundamental system of open neighbourhoods of p. We conclude that on stalks at p we obtain the map

$$\varphi_x^\sharp : \mathcal{O}_{\mathbf{P}(V),p} \longrightarrow \mathcal{O}_{X,x}$$

In particular, $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{\mathbf{P}(V),p}$-module. Moreover, the residue fields of x and p are equal to k (as k is algebraically closed – use the Hilbert Nullstellensatz). Finally, condition (2) implies that the map

$$T_{X/k,x} \longrightarrow T_{\mathbf{P}(V)/k,p}$$

is injective since any nonzero θ in the kernel of this map couldn't possibly satisfy the conclusion of (2). In terms of the map of local rings above this means that

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$$

is surjective, see Lemma 33.16.5. Now the proof is finished by applying Algebra, Lemma 10.20.3. □

0E8T **Lemma 33.23.2.** *Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Suppose that for every closed subscheme $Z \subset X$ of dimension 0 and degree 2 over k the map*

$$H^0(X, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}|_Z)$$

is surjective. Then $\mathcal{L}$ is very ample on X over k.

Proof. This is a reformulation of Lemma 33.23.1. Namely, given distinct closed points $x, y \in X$ taking $Z = x \cup y$ (viewed as closed subscheme) we get condition (1) of the lemma. And given a nonzero tangent vector $\theta \in T_{X/k,x}$ the morphism $\theta : \mathrm{Spec}(k[\epsilon]) \to X$ is a closed immersion. Setting $Z = \mathrm{Im}(\theta)$ we obtain condition (2) of the lemma. □

33.24. Closures of products

047A Some results on the relation between closure and products.

047B **Lemma 33.24.1.** *Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set*

$$AB = \{z \in X \times_k Y \mid pr_X(z) \in A,\ pr_Y(z) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 29.23.4 the subset $U = \mathrm{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\mathrm{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 29.23.4 again the morphism $Y_a \to Y$ is open as $\mathrm{Spec}(\kappa(a)) \to \mathrm{Spec}(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. □

04Q0 **Lemma 33.24.2.** *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have*

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 33.24.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. □

04Q1 **Lemma 33.24.3.** *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 29.6.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.*

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 29.6.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z is cut out by $\mathrm{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. □

33.25. Schemes smooth over fields

04QM Here are two lemmas characterizing smooth schemes over fields.

04QN **Lemma 33.25.1.** *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free, and*
3. *k has characteristic zero.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. This follows from Algebra, Lemma 10.140.7. □

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $\mathrm{Spec}(\mathbf{F}_p[t]/(t^p))$ over $\mathrm{Spec}(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $\mathrm{Spec}(k[t]/(t^p - a)$ where $a \in k$ is not a pth power.

04QP **Lemma 33.25.2.** *Let k be a field. Let X be a scheme over k. Assume*

(1) *X is locally of finite type over k,*
(2) *$\Omega_{X/k}$ is locally free,*
(3) *X is reduced, and*
(4) *k is perfect.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 29.15.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 28.5.5 and Topology, Lemma 5.9.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma 10.25.1. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 10.42). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 29.28.1 we conclude that $\dim_{\eta_i}(X_i) = \mathrm{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\mathrm{rank}_x(\Omega_{X/k}) = \mathrm{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \mathrm{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 10.114.5. It follows that $X \to \mathrm{Spec}(k)$ is smooth at x for example by Algebra, Lemma 10.140.3. □

056S **Lemma 33.25.3.** *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.*

Proof. (See also Lemma 33.12.6.) By Algebra, Lemma 10.140.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 28.9.2. □

056T **Lemma 33.25.4.** *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.*

Proof. (See also Lemma 33.12.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 33.12.3, 33.10.3, and 33.6.5. By Morphisms, Lemma 29.34.5 the morphism $X_{k'} \to \mathrm{Spec}(k')$ is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 33.25.3. Hence Properties, Lemma 28.9.4 guarantees that X is normal. □

055T **Lemma 33.25.5.** *Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}^d_k$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.*

Proof. After possible shrinking W we may assume that $W = \mathbf{A}^d_k \setminus V(f)$ for some $f \in k[x_1, \ldots, x_d]$. If the lemma is wrong then $f(a_1, \ldots, a_d) = 0$ for all $(a_1, \ldots, a_d) \in (k^{sep})^d$. This is absurd as k^{sep} is an infinite field. □

056U **Lemma 33.25.6.** *Let k be a field. If X is smooth over $\mathrm{Spec}(k)$ then the set*

$$\{x \in X \text{ closed such that } k \subset \kappa(x) \text{ is finite separable}\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \longleftarrow U \xrightarrow{\pi} \mathbf{A}^d_k$$

with π étale, see Morphisms, Lemma 29.36.20. The morphism $\pi : U \to \mathbf{A}^d_k$ is open, see Morphisms, Lemma 29.36.13. By Lemma 33.25.5 we may choose a closed point $w \in \pi(U)$ whose residue field is finite separable over k. Pick any $x \in U$ with $\pi(x) = w$. By Morphisms, Lemma 29.36.7 the field extension $\kappa(x)/\kappa(w)$ is finite separable. Hence $\kappa(x)/k$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 29.20.2. □

056V **Lemma 33.25.7.** *Let X be a reduced scheme that is locally of finite type over a field k. Then X is geometrically reduced over k if and only if X contains a dense open which is smooth over k.*

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. First suppose that X contains a dense open which is smooth over k. Then X is geometrically reduced at the generic point of each X_i (for example by Lemma 33.25.4). By Lemma 33.6.8, it follows that X is geometrically reduced.

Conversely, suppose that X is geometrically reduced. Since $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X, we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 28.3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta) = \mathcal{O}_{X,\eta}$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 10.44.2. Let $U = \mathrm{Spec}(A) \subset X$ be any nonempty affine open so that $K = A_{(0)}$ is the fraction field of A. Apply Algebra, Lemma 10.140.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. □

0B8X **Lemma 33.25.8.** *Let k be a perfect field. Let X be a locally algebraic reduced k-scheme, for example a variety over k. Then we have*

$$\{x \in X \mid X \to \mathrm{Spec}(k) \text{ is smooth at } x\} = \{x \in X \mid \mathcal{O}_{X,x} \text{ is regular}\}$$

and this is a dense open subscheme of X.

Proof. The equality of the two sets follows immediately from Algebra, Lemma 10.140.5 and the definitions (see Algebra, Definition 10.45.1 for the definition of a perfect field). The set is open because the set of points where a morphism of schemes is smooth is open, see Morphisms, Definition 29.34.1. Finally, we give two arguments to see that it is dense: (1) The generic points of X are in the set as the local rings at generic points are fields (Algebra, Lemma 10.25.1) hence regular. (2) We use that X is geometrically reduced by Lemma 33.6.3 and hence Lemma 33.25.7 applies. □

05AX **Lemma 33.25.9.** *Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to \mathrm{Spec}(k)$ is smooth at x and f is flat at x then $Y \to \mathrm{Spec}(k)$ is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.*

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 33.12.6. This follows from Lemma 33.12.5 (and Lemma 33.12.6 applied to (X, x)). □

0CDW **Lemma 33.25.10.** *Let k be a field. Let X be a variety over k which has a k-rational point x such that X is smooth at x. Then X is geometrically integral over k.*

Proof. Let $U \subset X$ be the smooth locus of X. By assumption U is nonempty and hence dense and scheme theoretically dense. Then $U_{\overline{k}} \subset X_{\overline{k}}$ is dense and scheme theoretically dense as well (some details omitted). Thus it suffices to show that U is geometrically integral. Because U has a k-rational point it is geometrically connected by Lemma 33.7.14. On the other hand, $U_{\overline{k}}$ is reduced and normal (Lemma 33.25.4. Since a connected normal Noetherian scheme is integral (Properties, Lemma 28.7.6) the proof is complete. □

0H3W **Lemma 33.25.11.** *Let X be a scheme of finite type over a field k. There exists a finite purely inseparable extension k'/k, an integer $t \geq 0$, and closed subschemes*

$$X_{k'} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_0 = (X_{k'})_{red}$ and $Z_i \setminus Z_{i+1}$ is smooth over k' for all i.

Proof. We may use induction on $\dim(X)$. By Lemma 33.6.11 we can find a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'. By Lemma 33.25.7 there is a nowhere dense closed subscheme $X' \subset (X_{k'})_{red}$ such that $(X_{k'})_{red} \setminus X'$ is smooth over k'. Then $\dim(X') < \dim(X)$. By induction hypothesis there exists a finite purely inseparable extension k''/k', an integer $t' \geq 0$, and closed subschemes

$$X'_{k''} \supset Y_0 \supset Y_1 \supset \ldots \supset Y_{t'} = \emptyset$$

such that $Y_0 = (X'_{k''})_{red}$ and $Y_i \setminus Y_{i+1}$ is smooth over k'' for all i. Then we let $t = t' + 1$ and we consider

$$X_{k''} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

given by $Z_0 = (X_{k''})_{red}$ and $Z_i = Y_{i-1}$ for $i > 0$; this makes sense as $X'_{k''}$ is a closed subscheme of $X_{k''}$. We omit the verification that all the stated properties hold. □

33.26. Types of varieties

04L0 Short section discussion some elementary global properties of varieties.

04L1 **Definition 33.26.1.** Let k be a field. Let X be a variety over k.

(1) We say X is an *affine variety* if X is an affine scheme. This is equivalent to requiring X to be isomorphic to a closed subscheme of $\mathbf{A}^n_k$ for some n.
(2) We say X is a *projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is projective. By Morphisms, Lemma 29.43.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}^n_k$ for some n.

(3) We say X is a *quasi-projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is quasi-projective. By Morphisms, Lemma 29.40.6 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ for some n.
(4) A *proper variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is proper.
(5) A *smooth variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is smooth.

Note that a projective variety is a proper variety, see Morphisms, Lemma 29.43.5. Also, an affine variety is quasi-projective as $\mathbf{A}^n_k$ is isomorphic to an open subscheme of $\mathbf{P}^n_k$, see Constructions, Lemma 27.13.3.

04L2 **Lemma 33.26.2.** *Let X be a proper variety over k. Then*

(1) *$K = H^0(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k,*
(2) *if X is geometrically reduced, then K/k is separable,*
(3) *if X is geometrically irreducible, then K/k is purely inseparable,*
(4) *if X is geometrically integral, then $K = k$.*

Proof. This is a special case of Lemma 33.9.3. □

33.27. Normalization

0BXQ Some issues associated to normalization.

0BXR **Lemma 33.27.1.** *Let k be a field. Let X be a locally algebraic scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 29.54.1. Then*

(1) *ν is finite, dominant, and X^ν is a disjoint union of normal irreducible locally algebraic schemes over k,*
(2) *ν factors as $X^\nu \to X_{red} \to X$ and the first morphism is the normalization morphism of X_{red},*
(3) *if X is a reduced algebraic scheme, then ν is birational,*
(4) *if X is a variety, then X^ν is a variety and ν is a finite birational morphism of varieties.*

Proof. Since X is locally of finite type over a field, we see that X is locally Noetherian (Morphisms, Lemma 29.15.6) hence every quasi-compact open has finitely many irreducible components (Properties, Lemma 28.5.7). Thus Morphisms, Definition 29.54.1 applies. The normalization X^ν is always a disjoint union of normal integral schemes and the normalization morphism ν is always dominant, see Morphisms, Lemma 29.54.5. Since X is universally Nagata (Morphisms, Lemma 29.18.2) we see that ν is finite (Morphisms, Lemma 29.54.11). Hence X^ν is locally algebraic too. At this point we have proved (1).

Part (2) is Morphisms, Lemma 29.54.2.

Part (3) is Morphisms, Lemma 29.54.7.

Part (4) follows from (1), (2), (3), and the fact that X^ν is separated as a scheme finite over a separated scheme. □

0GK4 **Lemma 33.27.2.** *Let k be a field. Let X be a proper scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 29.54.1. Then X^ν is proper over k. If X is projective over k, then X^ν is projective over k.*

Proof. By Lemma 33.27.1 the morphism ν is finite. Hence X^ν is proper over k by Morphisms, Lemmas 29.44.11 and 29.41.4. The morphism ν is projective by Morphisms, Lemma 29.44.16 and hence if X is projective over k, then X^ν is projective over k by Morphisms, Lemma 29.43.14. □

0BXS **Lemma 33.27.3.** *Let k be a field. Let $f : Y \to X$ be a quasi-compact morphism of locally algebraic schemes over k. Let X' be the normalization of X in Y. If Y is reduced, then $X' \to X$ is finite.*

Proof. Since Y is quasi-separated (by Properties, Lemma 28.5.4 and Morphisms, Lemma 29.15.6) the morphism f is quasi-separated (Schemes, Lemma 26.21.13). Hence Morphisms, Definition 29.53.3 applies. The result follows from Morphisms, Lemma 29.53.14. This uses that locally algebraic schemes are locally Noetherian (hence have locally finitely many irreducible components) and that locally algebraic schemes are Nagata (Morphisms, Lemma 29.18.2). Some small details omitted. □

0BXT **Lemma 33.27.4.** *Let k be a field. Let X be an algebraic k-scheme. Then there exists a finite purely inseparable extension k'/k such that the normalization Y of $X_{k'}$ is geometrically normal over k'.*

Proof. Let $K = k^{perf}$ be the perfect closure. Let Y_K be the normalization of X_K, see Lemma 33.27.1. By Limits, Lemma 32.10.1 there exists a finite sub extension $K/k'/k$ and a morphism $\nu : Y \to X_{k'}$ of finite presentation whose base change to K is the normalization morphism $\nu_K : Y_K \to X_K$. Observe that Y is geometrically normal over k' (Lemma 33.10.3). After increasing k' we may assume $Y \to X_{k'}$ is finite (Limits, Lemma 32.8.3). Since $\nu_K : Y_K \to X_K$ is the normalization morphism, it induces a birational morphism $Y_K \to (X_K)_{red}$. Hence there is a dense open $V_K \subset X_K$ such that $\nu_K^{-1}(V_K) \to V_K$ is a closed immersion (inducing an isomorphism of $\nu_K^{-1}(V_K)$ with $V_{K,red}$, see for example Morphisms, Lemma 29.51.6). After increasing k' we find V_K is the base change of a dense open $V \subset Y$ and the morphism $\nu^{-1}(V) \to V$ is a closed immersion (Limits, Lemmas 32.4.11 and 32.8.5). It follows readily from this that ν is the normalization morphism and the proof is complete. □

0C3N **Lemma 33.27.5.** *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $\nu : X^\nu \to X$ be the normalization of X and let $Y^\nu \to X_K$ be the normalization of the base change. Then the canonical morphism*

$$Y^\nu \longrightarrow X^\nu \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$$

is an isomorphism if K/k is separable and a universal homeomorphism in general.

Proof. Set $Y = X_K$. Let $X^{(0)}$, resp. $Y^{(0)}$ be the set of generic points of irreducible components of X, resp. Y. Then the projection morphism $\pi : Y \to X$ satisfies $\pi(Y^{(0)}) = X^{(0)}$. This is true because π is surjective, open, and generizing, see Morphisms, Lemmas 29.23.4 and 29.23.5. If we view $X^{(0)}$, resp. $Y^{(0)}$ as (reduced) schemes, then X^ν, resp. Y^ν is the normalization of X, resp. Y in $X^{(0)}$, resp. $Y^{(0)}$. Thus Morphisms, Lemma 29.53.5 gives a canonical morphism $Y^\nu \to X^\nu$

over $Y \to X$ which in turn gives the canonical morphism of the lemma by the universal property of the fibre product.

To prove this morphism has the properties stated in the lemma we may assume $X = \mathrm{Spec}(A)$ is affine. Let $Q(A_{red})$ be the total ring of fractions of A_{red}. Then X^ν is the spectrum of the integral closure A' of A in $Q(A_{red})$, see Morphisms, Lemmas 29.54.2 and 29.54.3. Similarly, Y^ν is the spectrum of the integral closure B' of $A \otimes_k K$ in $Q((A \otimes_k K)_{red})$. There is a canonical map $Q(A_{red}) \to Q((A \otimes_k K)_{red})$, a canonical map $A' \to B'$, and the morphism of the lemma corresponds to the induced map

$$A' \otimes_k K \longrightarrow B'$$

of K-algebras. The kernel consists of nilpotent elements as the kernel of $Q(A_{red}) \otimes_k K \to Q((A \otimes_k K)_{red})$ is the set of nilpotent elements.

If K/k is separable, then $A' \otimes_k K$ is normal by Lemma 33.10.6. In particular it is reduced, whence $Q((A \otimes_k K)_{red}) = Q(A' \otimes_k K)$ and $B' = A' \otimes_k K$ by Algebra, Lemma 10.37.16.

Assume K/k is not separable. Then the characteristic of k is $p > 0$. We will show that for every $b \in B'$ there is a power q of p such that b^q is in the image of $A' \otimes_k K$. This will prove that the displayed map is a universal homeomorphism by Algebra, Lemma 10.46.7. For a given b there is a subfield $F \subset K$ with F/k finitely generated such that b is contained in $Q((A \otimes_k F)_{red})$ and is integral over $A \otimes_k F$. Choose a monic polynomial $P = T^d + \alpha_1 T^{d-1} + \ldots + \alpha_d$ with $P(b) = 0$ and $\alpha_i \in A \otimes_k F$. Choose a transcendence basis $t_1, \ldots, t_r$ for F over k. Let $F/F'/k(t_1, \ldots, t_r)$ be the maximal separable subextension (Fields, Lemma 9.14.6). Since F/F' is finite purely inseparable, there is a q such that $\lambda^q \in F'$ for all $\lambda \in F$. Then b^q is in $Q((A \otimes_k F')_{red})$ and satisfies the polynomial $T^d + \alpha_1^q T^{d-1} + \ldots + \alpha_d^q$ with $\alpha_i^q \in A \otimes_k F'$. By the separable case we see that $b^q \in A' \otimes_k F'$ and the proof is complete. □

0C3P **Lemma 33.27.6.** *Let k be a field. Let X be a locally algebraic k-scheme. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$ be a point such that (a) $\mathcal{O}_{X,x}$ is reduced, (b) $\dim(\mathcal{O}_{X,x}) = 1$, and (c) for every $x' \in X^\nu$ with $\nu(x') = x$ the extension $\kappa(x')/k$ is separable. Then X is geometrically reduced at x and X^ν is geometrically regular at x' with $\nu(x') = x$.*

Proof. We will use the results of Lemma 33.27.1 without further mention. Let $x' \in X^\nu$ be a point over x. By dimension theory (Section 33.20) we have $\dim(\mathcal{O}_{X^\nu,x'}) = 1$. Since X^ν is normal, we see that $\mathcal{O}_{X^\nu,x'}$ is a discrete valuation ring (Properties, Lemma 28.12.5). Thus $\mathcal{O}_{X^\nu,x'}$ is a regular local k-algebra whose residue field is separable over k. Hence $k \to \mathcal{O}_{X^\nu,x'}$ is formally smooth in the $\mathfrak{m}_{x'}$-adic topology, see More on Algebra, Lemma 15.38.5. Then $\mathcal{O}_{X^\nu,x'}$ is geometrically regular over k by More on Algebra, Theorem 15.40.1. Thus X^ν is geometrically regular at x' by Lemma 33.12.2.

Since $\mathcal{O}_{X,x}$ is reduced, the family of maps $\mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu,x'}$ is injective. Since $\mathcal{O}_{X^\nu,x'}$ is a geometrically reduced k-algebra, it follows immediately that $\mathcal{O}_{X,x}$ is a geometrically reduced k-algebra. Hence X is geometrically reduced at x by Lemma 33.6.2. □

33.28. Groups of invertible functions

04L3 It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

04L4 **Lemma 33.28.1.** *Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Since the statement only concerns X, we may replace $\overline{X}$ by a different proper variety over k. Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism. By Lemma 33.27.1 we have that ν is finite and $\overline{X}^\nu$ is a variety. Since X is normal, we see that $\nu^{-1}(X) \to X$ is an isomorphism (tiny detail omitted). Finally, we see that $\overline{X}^\nu$ is proper over k as a finite morphism is proper (Morphisms, Lemma 29.44.11) and compositions of proper morphisms are proper (Morphisms, Lemma 29.41.4). Thus we may and do assume $\overline{X}$ is normal.

We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 10.31.1, Properties, Definition 28.3.1, Properties, Lemmas 28.5.2 and 28.7.2, Morphisms, Lemma 29.15.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 29.15.6, Properties, Lemma 28.5.5, and Topology, Lemma 5.9.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 10.119.7. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod\nolimits_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = \operatorname{Spec}(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\operatorname{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 28.10.3 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_\mathfrak{q}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\operatorname{ht}(\mathfrak{q})=1} B_\mathfrak{q} = B$, see Algebra, Lemma 10.157.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 33.26.2. □

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

04L5 **Lemma 33.28.2.** *Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}^n_k$ and denote $\overline{X}$ the closure of X in $\mathbf{P}^n_k$ via the usual immersion $\mathbf{A}^n_k \to \mathbf{P}^n_k$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 29.54.1. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 29.54.5, 29.54.10, and 29.18.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to \mathrm{Spec}(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 29.41 and 29.44). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 33.28.1. □

The preceding lemma implies the following slight generalization.

04L6 **Lemma 33.28.3.** *Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 33.28.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. □

04MI **Lemma 33.28.4.** *Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.*

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k')$, see Schemes, Lemma 26.6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $\mathrm{Spec}(k')$ is totally disconnected, see Algebra, Lemmas 10.36.20 and 10.26.5. As X is connected the morphism $X \to \mathrm{Spec}(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 10.18.3. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 10.25.1. □

04L7 **Proposition 33.28.5.** *Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:*

(1) *the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,*
(2) *X has a k-rational point, or*
(3) *X is geometrically integral.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 29.23.4) and closed (by Morphisms, Lemma 29.44.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 33.8.15 we see that Y has finitely many irreducible components. Thus Lemma 33.28.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \notin k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 33.28.4 we see that k' is a field. If $e : \mathrm{Spec}(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if k'/k is not purely inseparable) or that X is not geometrically reduced (if k'/k is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). □

04L8 **Lemma 33.28.6.** *Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:*

(1) *k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,*
(2) *k is algebraically closed in $k(X)$,*
(3) *X is geometrically integral over k, or*
(4) *k is the "intersection" of the field extensions $\kappa(x)/k$ where x runs over the closed points of x.*

Proof. We see that (1) is enough by Proposition 33.28.5. We omit the verification that each of (2), (3), (4) implies (1). □

33.29. Künneth formula, I

0BEC In this section we prove the Künneth formula when the base is a field and we are considering cohomology of quasi-coherent modules. For a more general version, please see Derived Categories of Schemes, Section 36.23.

0BED **Lemma 33.29.1.** *Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism*

$$H^n(X \times_{\mathrm{Spec}(k)} Y, pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\mathrm{Spec}(k)} Y}} pr_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and have affine diagonal[3] *(for example if X and Y are separated).*

Proof. In this proof unadorned products and tensor products are over k. As maps

$$H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G}) \longrightarrow H^n(X \times Y, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G})$$

we use functoriality of cohomology to get maps $H^p(X, \mathcal{F}) \to H^p(X \times Y, \mathrm{pr}_1^*\mathcal{F})$ and $H^p(Y, \mathcal{G}) \to H^p(X \times Y, \mathrm{pr}_2^*\mathcal{G})$ and then we use the cup product

$$\cup : H^p(X \times Y, \mathrm{pr}_1^*\mathcal{F}) \otimes H^q(X \times Y, \mathrm{pr}_2^*\mathcal{G}) \longrightarrow H^n(X \times Y, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G})$$

The result is true when X and Y are affine by the vanishing of higher cohomology groups on affines (Cohomology of Schemes, Lemma 30.2.2) and the definitions (of pullbacks of quasi-coherent modules and tensor products of quasi-coherent modules).

Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i\in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j\in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times Y = \bigcup_{(i,j)\in I\times J} U_i \times V_j$. Note that $\mathcal{W}$ is a refinement of $\mathrm{pr}_1^{-1}\mathcal{U}$ and of $\mathrm{pr}_2^{-1}\mathcal{V}$. Thus by Cohomology, Lemma 20.15.1 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_2^*\mathcal{G})$$

compatible with pullback maps on cohomology. In Cohomology, Equation (20.25.3.2) we have constructed a map of complexes

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_2^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G})$$

defining the cup product on cohomology. Combining the above we obtain a map of complexes

0BEE (33.29.1.1)
$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G})$$

We warn the reader that this map is not an isomorphism of complexes. Recall that we may compute the cohomologies of our quasi-coherent sheaves using our coverings (Cohomology of Schemes, Lemmas 30.2.5 and 30.2.6). Thus on cohomology (33.29.1.1) reproduces the map of the lemma.

Consider a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of quasi-coherent modules. Since the construction of (33.29.1.1) is functorial in $\mathcal{F}$ and since the formation of the relevant Čech complexes is exact in the variable $\mathcal{F}$ (because we are taking sections over affine opens) we find a map between short exact sequence of complexes

$$\begin{array}{ccccc}
\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \\
\downarrow & & \downarrow & & \downarrow \\
\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F}'' \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^*\mathcal{G})
\end{array}$$

(we have dropped the outer zeros). Looking at long exact cohomology sequences we find that if the result of the lemma holds for 2-out-of-3 of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$, then it holds for the third.

Observe that X has finite cohomological dimension for quasi-coherent modules, see Cohomology of Schemes, Lemma 30.4.2. Using induction on $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$ we will reduce to the case $d(\mathcal{F}) = 0$. Assume $d(\mathcal{F}) > 0$. By

[3] The case where X and Y are quasi-separated will be discussed in Lemma 33.29.2 below.

Cohomology of Schemes, Lemma 30.4.3 we have seen that there exists an embedding $\mathcal{F} \to \mathcal{F}'$ such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$. Setting $\mathcal{F}'' = \text{Coker}(\mathcal{F} \to \mathcal{F}')$ we see that $d(\mathcal{F}'') < d(\mathcal{F})$. Then we can apply the result from the previous paragraph to see that it suffices to prove the lemma for $\mathcal{F}'$ and $\mathcal{F}''$ thereby proving the induction step.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times V = \bigcup_{i \in I} U_i \times V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

(equality of complexes). We conclude that

$$R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_k \mathcal{G} \cong \bigoplus\nolimits_{\alpha \in A} \mathcal{G}$$

on Y. Here A is a basis for the k-vector space $\Gamma(X, \mathcal{F})$. Cohomology on Y commutes with direct sums (Cohomology, Lemma 20.19.1). Using the Leray spectral sequence for pr_2 (via Cohomology, Lemma 20.13.6) we conclude that $H^n(X \times Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$ for $n = 0$. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). □

0BEF **Lemma 33.29.2.** *Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism*

$$H^n(X \times_{\text{Spec}(k)} Y, pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\text{Spec}(k)} Y}} pr_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and quasi-separated.

Proof. If X and Y are separated or more generally have affine diagonal, then please see Lemma 33.29.1 for "better" proof (the feature it has over this proof is that it identifies the maps as pullbacks followed by cup products). Let X', resp. Y' be the infinitesimal thickening of X, resp. Y whose structure sheaf is $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$, resp. $\mathcal{O}_{Y'} = \mathcal{O}_Y \oplus \mathcal{G}$ where $\mathcal{F}$, resp. $\mathcal{G}$ is an ideal of square zero. Then

$$\mathcal{O}_{X' \times Y'} = \mathcal{O}_{X \times Y} \oplus \text{pr}_1^*\mathcal{F} \oplus \text{pr}_2^*\mathcal{G} \oplus \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}$$

as sheaves on $X \times Y$. In this way we see that it suffices to prove that

$$H^n(X \times Y, \mathcal{O}_{X \times Y}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)$$

for any pair of quasi-compact and quasi-separated schemes over k. Some details omitted.

To prove this statement we use cohomology and base change in the form of Cohomology of Schemes, Lemma 30.7.3. This lemma tells us there exists a bounded below complex of k-vector spaces, i.e., a complex $\mathcal{K}^\bullet$ of quasi-coherent modules on $\text{Spec}(k)$, which universally computes the cohomology of Y over $\text{Spec}(k)$. In particular, we see that

$$R\text{pr}_{1,*}(\mathcal{O}_{X \times Y}) \cong (X \to \text{Spec}(k))^*\mathcal{K}^\bullet$$

in $D(\mathcal{O}_X)$. Up to homotopy the complex $\mathcal{K}^\bullet$ is isomorphic to $\bigoplus_{q\geq 0} H^q(Y, \mathcal{O}_Y)[-q]$ because this is true for every complex of vector spaces over a field. We conclude that

$$R\mathrm{pr}_{1,*}(\mathcal{O}_{X\times Y}) \cong \bigoplus\nolimits_{q\geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X$$

in $D(\mathcal{O}_X)$. Then we have

$$\begin{aligned} R\Gamma(X \times Y, \mathcal{O}_{X\times Y}) &= R\Gamma(X, R\mathrm{pr}_{1,*}(\mathcal{O}_{X\times Y})) \\ &= R\Gamma(X, \bigoplus\nolimits_{q\geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X) \\ &= \bigoplus\nolimits_{q\geq 0} R\Gamma(X, H^q(Y, \mathcal{O}_Y) \otimes \mathcal{O}_X)[-q] \\ &= \bigoplus\nolimits_{q\geq 0} R\Gamma(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)[-q] \\ &= \bigoplus\nolimits_{p,q\geq 0} H^p(X, \mathcal{O}_X)[-p] \otimes_k H^q(Y, \mathcal{O}_Y)[-q] \end{aligned}$$

as desired. The first equality by Leray for pr_1 (Cohomology, Lemma 20.13.1). The second by our decomposition of the total direct image given above. The third because cohomology always commutes with finite direct sums (and cohomology of Y vanishes in sufficiently large degree by Cohomology of Schemes, Lemma 30.4.4). The fourth because cohomology on X commutes with infinite direct sums by Cohomology, Lemma 20.19.1. The final equality by our remark on the derived category of a field above. □

33.30. Picard groups of varieties

0BEG In this section we collect some elementary results on Picard groups of algebraic varieties.

0CDX **Lemma 33.30.1.** *Let $A \to B$ be a faithfully flat ring map. Let X be a quasi-compact and quasi-separated scheme over A. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback to X_B is trivial. Then $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are invertible $H^0(X, \mathcal{O}_X)$-modules and the multiplication map induces an isomorphism*

$$H^0(X, \mathcal{L}) \otimes_{H^0(X,\mathcal{O}_X)} H^0(X, \mathcal{L}^{\otimes -1}) \longrightarrow H^0(X, \mathcal{O}_X)$$

Proof. Denote $\mathcal{L}_B$ the pullback of $\mathcal{L}$ to X_B. Choose an isomorphism $\mathcal{L}_B \to \mathcal{O}_{X_B}$. Set $R = H^0(X, \mathcal{O}_X)$, $M = H^0(X, \mathcal{L})$ and think of M as an R-module. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ with pullback $\mathcal{F}_B$ on X_B there is a canonical isomorphism $H^0(X_B, \mathcal{F}_B) = H^0(X, \mathcal{F}) \otimes_A B$, see Cohomology of Schemes, Lemma 30.5.2. Thus we have

$$M \otimes_R (R \otimes_A B) = M \otimes_A B = H^0(X_B, \mathcal{L}_B) \cong H^0(X_B, \mathcal{O}_{X_B}) = R \otimes_A B$$

Since $R \to R \otimes_A B$ is faithfully flat (as the base change of the faithfully flat map $A \to B$), we conclude that M is an invertible R-module by Algebra, Proposition 10.83.3. Similarly $N = H^0(X, \mathcal{L}^{\otimes -1})$ is an invertible R-module. To see that the statement on tensor products is true, use that it is true after pulling back to X_B and faithful flatness of $R \to R \otimes_A B$. Some details omitted. □

0CDY **Lemma 33.30.2.** *Let $A \to B$ be a faithfully flat ring map. Let X be a scheme over A such that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *$R = H^0(X, \mathcal{O}_X)$ is a semi-local ring.*

Then the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_B)$ *is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback $\mathcal{L}'$ to X_B is trivial. Set $M = H^0(X, \mathcal{L})$ and $N = H^0(X, \mathcal{L}^{\otimes -1})$. By Lemma 33.30.1 the R-modules M and N are invertible. Since R is semi-local $M \cong R$ and $N \cong R$, see Algebra, Lemma 10.78.7. Choose generators $s \in M$ and $t \in N$. Then $st \in R = H^0(X, \mathcal{O}_X)$ is a unit by the last part of Lemma 33.30.1. We conclude that s and t define trivializations of $\mathcal{L}$ and $\mathcal{L}^{\otimes -1}$ over X. □

0CC5 **Lemma 33.30.3.** *Let k'/k be a field extension. Let X be a scheme over k such that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *$R = H^0(X, \mathcal{O}_X)$ is semi-local, e.g., if $\dim_k R < \infty$.*

Then the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k'})$ *is injective.*

Proof. Special case of Lemma 33.30.2. If $\dim_k R < \infty$, then R is Artinian and hence semi-local (Algebra, Lemmas 10.53.2 and 10.53.3). □

0CDP **Example 33.30.4.** Lemma 33.30.3 is not true without some condition on the scheme X over the field k. Here is an example. Let k be a field. Let $t \in \mathbf{P}^1_k$ be a closed point. Set $X = \mathbf{P}^1 \setminus \{t\}$. Then we have a surjection

$$\mathbf{Z} = \mathrm{Pic}(\mathbf{P}^1_k) \longrightarrow \mathrm{Pic}(X)$$

The first equality by Divisors, Lemma 31.28.5 and surjective by Divisors, Lemma 31.28.3 (as $\mathbf{P}^1_k$ is smooth of dimension 1 over k and hence all its local rings are discrete valuation rings). If $\mathcal{L}$ is in the kernel of the displayed map, then $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^1_k}(nt)$ for some $n \in \mathbf{Z}$. We leave it to the reader to show that $\mathcal{O}_{\mathbf{P}^1_k}(t) \cong \mathcal{O}_{\mathbf{P}^1_k}(d)$ where $d = [\kappa(t) : k]$. Hence

$$\mathrm{Pic}(X) = \mathbf{Z}/d\mathbf{Z}$$

Thus if t is not a k-rational point, then $d > 1$ and this Picard group is nonzero. On the other hand, if we extend the ground field k to any field extension k' such that there exists a k-embedding $\kappa(t) \to k'$, then $\mathbf{P}^1_{k'} \setminus X_{k'}$ has a k'-rational point t'. Hence $\mathcal{O}_{\mathbf{P}^1_{k'}}(1) = \mathcal{O}_{\mathbf{P}^1_{k'}}(t')$ will be in the kernel of the map $\mathbf{Z} \to \mathrm{Pic}(X_{k'})$ and it will follow in the same manner as above that $\mathrm{Pic}(X_{k'}) = 0$.

The following lemma tells us that "rationally equivalence invertible modules" are isomorphic on normal varieties.

0BEH **Lemma 33.30.5.** *Let k be a field. Let X be a normal variety over k. Let $U \subset \mathbf{A}^n_k$ be an open subscheme with k-rational points $p, q \in U(k)$. For every invertible module $\mathcal{L}$ on $X \times_{\mathrm{Spec}(k)} U$ the restrictions $\mathcal{L}|_{X \times p}$ and $\mathcal{L}|_{X \times q}$ are isomorphic.*

Proof. The fibres of $X \times_{\mathrm{Spec}(k)} U \to X$ are open subschemes of affine n-space over fields. Hence these fibres have trivial Picard groups by Divisors, Lemma 31.28.4. Applying Divisors, Lemma 31.28.1 we see that $\mathcal{L}$ is the pullback of an invertible module $\mathcal{N}$ on X. □

33.31. Uniqueness of base field

04MJ The phrase "let X be a scheme over k" means that X is a scheme which comes equipped with a morphism $X \to \mathrm{Spec}(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for

example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to \mathrm{Spec}(k)$ does not factor as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for any nontrivial field extension k'/k? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not guarantee uniqueness of k is the scheme

$$X = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted). In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to \mathrm{Spec}(\mathbf{Q}(x))$ and $X \to \mathrm{Spec}(\mathbf{Q}(y))$ are "essentially of finite type" (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field. As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is *not* the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

04MK **Proposition 33.31.1.** *Let X be a scheme. Let $a : X \to \mathrm{Spec}(k_1)$ and $b : X \to \mathrm{Spec}(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.*

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 29.15.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 28.5.5, Topology, Lemma 5.7.11, Topology, Section 5.9, and Topology, Lemma 5.16.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 33.28.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k_1' and k_2' both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k_1' is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 33.28.4 after replacing k_i by k_i' we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 33.28.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 10.34.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k_{12}' of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k_{12}' is a field by Lemma 33.28.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. □

33.32. Automorphisms

0GWY A section on automorphisms of schemes over fields. For some information on (infinitesimal) automorphisms of curves, see Algebraic Curves, Section 53.26 and Moduli of Curves, Section 109.7.

0G05 **Lemma 33.32.1.** *Let X be a reduced scheme of finite type over a field k. Let $f : X \to X$ be an automorphism over k which induces the identity map on the underlying topological space of X. Then*

(1) *$f^*\mathcal{F} \cong \mathcal{F}$ for every coherent $\mathcal{O}_X$-module, and*
(2) *if $\dim(Z) > 0$ for every irreducible component $Z \subset X$, then f is the identity.*

Proof. Part (1) follows from part (2) and the fact that the connected components of X of dimension 0 are spectra of fields.

Let $Z \subset X$ be an irreducible component viewed as an integral closed subscheme. Clearly $f(Z) \subset Z$ and $f|_Z : Z \to Z$ is an automorphism over k which induces the identity map on the underlying topological space of Z. Since X is reduced, it suffices to show that the arrows $f|_Z : Z \to Z$ are the identity. This reduces us to the case discussed in the next paragraph.

Assume X is irreducible of dimension > 0. Choose a nonempty affine open $U \subset X$. Since $f(U) \subset U$ and since $U \subset X$ is scheme theoretically dense it suffices to prove that $f|_U : U \to U$ is the identity.

Assume $X = \mathrm{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is an infinite field. Let $g \in A$ be nonconstant. The set

$$S = \bigcup\nolimits_{\lambda \in k} V(g - \lambda)$$

is dense in X because it is the inverse image of the dense subset $\mathbf{A}^1_k(k)$ by the nonconstant morphism $g : X \to \mathbf{A}^1_k$. If $x \in S$, then the image $g(x)$ of g in $\kappa(x)$ is in the image of $k \to \kappa(x)$. Hence $f^\sharp : \kappa(x) \to \kappa(x)$ fixes $g(x)$. Thus the image of $f^\sharp(g)$ in $\kappa(x)$ is equal to $g(x)$. We conclude that

$$S \subset V(g - f^\sharp(g))$$

and since X is reduced and S is dense we conclude $g = f^\sharp(g)$. This proves $f^\sharp = \mathrm{id}_A$ as A is generated as a k-algebra by elements g as above (details omitted; hint: the set of constant functions is a finite dimensional k-subvector space of A). We conclude that $f = \mathrm{id}_X$.

Assume $X = \mathrm{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is a finite field. If for every 1-dimensional integral closed subscheme $C \subset X$ the restriction $f|_C : C \to C$ is the identity, then f is the identity. This reduces us to the case where X is a curve. A curve over a finite field has a finite automorphism group (details omitted). Hence f has finite order, say n. Then we pick $g : X \to \mathbf{A}^1_k$ nonconstant as above and we consider

$$S = \{x \in X \text{ closed such that } [\kappa(g(x)) : k] \text{ is prime to } n\}$$

Arguing as before we find that S is dense in X. Since for $x \in X$ closed the map $f^\sharp : \kappa(x) \to \kappa(x)$ is an automorphism of order dividing n we see that for $x \in S$ this automorphism acts trivially on the subfield generated by the image of g in $\kappa(x)$. Thus we conclude that $S \subset V(g - f^\sharp(g))$ and we win as before. □

33.33. Euler characteristics

0BEI In this section we prove some elementary properties of Euler characteristics of coherent sheaves on schemes proper over fields.

0BEJ **Definition 33.33.1.** Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of* $\mathcal{F}$ is the integer

$$\chi(X, \mathcal{F}) = \sum\nolimits_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Schemes, Lemma 30.4.5) and each of these spaces is finite dimensional (Cohomology of Schemes, Lemma 30.19.2). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

08AA **Lemma 33.33.2.** *Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. □

0AYT **Lemma 33.33.3.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\mathit{Supp}(\mathcal{F})) \leq 0$. Then*

(1) *$\mathcal{F}$ is generated by global sections,*
(2) *$H^0(X, \mathcal{F}) = \bigoplus_{x \in \mathit{Supp}(\mathcal{F})} \mathcal{F}_x$,*
(3) *$H^i(X, \mathcal{F}) = 0$ for $i > 0$,*
(4) *$\chi(X, \mathcal{F}) = \dim_k H^0(X, \mathcal{F})$, and*
(5) *$\chi(X, \mathcal{F} \otimes \mathcal{E}) = n\chi(X, \mathcal{F})$ for every locally free module $\mathcal{E}$ of rank n.*

Proof. By Cohomology of Schemes, Lemma 30.9.7 we see that $\mathcal{F} = i_*\mathcal{G}$ where $i : Z \to X$ is the inclusion of the scheme theoretic support of $\mathcal{F}$ and where $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. By definition of the scheme theoretic support the underlying topological space of Z is $\mathrm{Supp}(\mathcal{F})$. Since the dimension of Z is 0, we see Z is affine (Properties, Lemma 28.10.5). Hence $\mathcal{G}$ is globally generated and the higher cohomology groups of $\mathcal{G}$ are zero (Cohomology of Schemes, Lemma 30.2.2). In fact, by Lemma 33.20.2 the scheme Z is a finite disjoint union of spectra of local Artinian rings. Thus correspondingly $H^0(Z, \mathcal{G}) = \bigoplus_{z \in Z} \mathcal{G}_z$. The cohomologies of $\mathcal{F}$ and $\mathcal{G}$ agree by Cohomology of Schemes, Lemma 30.2.4. Thus $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $H^0(X, \mathcal{F}) = H^0(Z, \mathcal{G})$. In particular we have (3) is true. For $z \in Z$ corresponding to $x \in \mathrm{Supp}(\mathcal{F})$ we have $\mathcal{G}_z = (i_*\mathcal{G})_x = \mathcal{F}_x$. We conclude that (2) holds. Of course (2) implies (1). We have (4) by definition of the Euler characteristic $\chi(X, \mathcal{F})$ and (3). By the projection formula (Cohomology, Lemma 20.54.2) we have

$$i_*(\mathcal{G} \otimes i^*\mathcal{E}) = \mathcal{F} \otimes \mathcal{E}.$$

Since Z has dimension 0 the locally free sheaf $i^*\mathcal{E}$ is isomorphic to $\mathcal{O}_Z^{\oplus n}$ and arguing as above we see that (5) holds. □

08AB **Lemma 33.33.4.** *Let k'/k be an extension of fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X_{k'}$. Then $\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}')$.*

Proof. This is true because

$$H^i(X_{k'}, \mathcal{F}') = H^i(X, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

0BEK **Lemma 33.33.5.** *Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Schemes, Proposition 30.19.1 and Lemma 30.4.5. By Cohomology, Lemma 20.13.4 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + q = n$ (this is the meaning of convergence of the spectral sequence), we conclude. Compare also with the more general Homology, Lemma 12.24.12. □

33.34. Projective space

0B2N Some results on projective space over a field.

0B2P **Lemma 33.34.1.** *Let k be a field and $n \geq 0$. Then $\mathbf{P}^n_k$ is a smooth projective variety of dimension n over k.*

Proof. Omitted. □

0B2Q **Lemma 33.34.2.** *Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{A}^n_k$ be closed subsets. Assume that X and Y are equidimensional, $\dim(X) = r$ and $\dim(Y) = s$. Then every irreducible component of $X \cap Y$ has dimension $\geq r + s - n$.*

Proof. Consider the closed subscheme $X \times Y \subset \mathbf{A}^{2n}_k$ where we use coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then $X \cap Y = X \times Y \cap V(x_1 - y_1, \ldots, x_n - y_n)$. Let $t \in X \cap Y \subset X \times Y$ be a closed point. By Lemma 33.20.5 we have $\dim_t(X \times Y) = \dim(X) + \dim(Y)$. Thus $\dim(\mathcal{O}_{X \times Y, t}) = r + s$ by Lemma 33.20.3. By Algebra, Lemma 10.60.13 we conclude that

$$\dim(\mathcal{O}_{X \cap Y, t}) = \dim(\mathcal{O}_{X \times Y, t}/(x_1 - y_1, \ldots, x_n - y_n)) \geq r + s - n$$

This implies the result by Lemma 33.20.3. □

0B2R **Lemma 33.34.3.** *Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{P}^n_k$ be nonempty closed subsets. If $\dim(X) = r$ and $\dim(Y) = s$ and $r + s \geq n$, then $X \cap Y$ is nonempty and $\dim(X \cap Y) \geq r + s - n$.*

Proof. Write $\mathbf{A}^n = \mathrm{Spec}(k[x_0, \ldots, x_n])$ and $\mathbf{P}^n = \mathrm{Proj}(k[T_0, \ldots, T_n])$. Consider the morphism $\pi : \mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ which sends $(x_0, \ldots, x_n)$ to the point $[x_0 : \ldots : x_n]$. More precisely, it is the morphism associated to the pair $(\mathcal{O}_{\mathbf{A}^{n+1} \setminus \{0\}}, (x_0, \ldots, x_n))$,

see Constructions, Lemma 27.13.1. Over the standard affine open $D_+(T_i)$ we get the morphism associated to the ring map

$$k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] \longrightarrow k\left[T_0, \ldots, T_n, \frac{1}{T_i}\right] \cong k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\left[T_i, \frac{1}{T_i}\right]$$

which is surjective and smooth of relative dimension 1 with irreducible fibres (details omitted). Hence $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ are nonempty closed subsets of dimension $r+1$ and $s+1$. Choose an irreducible component $V \subset \pi^{-1}(X)$ of dimension $r+1$ and an irreducible component $W \subset \pi^{-1}(Y)$ of dimension $s+1$. Observe that this implies V and W contain every fibre of π they meet (since π has irreducible fibres of dimension 1 and since Lemma 33.20.4 says the fibres of $V \to \pi(V)$ and $W \to \pi(W)$ have dimension ≥ 1). Let $\overline{V}$ and $\overline{W}$ be the closure of V and W in $\mathbf{A}^{n+1}$. Since $0 \in \mathbf{A}^{n+1}$ is in the closure of every fibre of π we see that $0 \in \overline{V} \cap \overline{W}$. By Lemma 33.34.2 we have $\dim(\overline{V} \cap \overline{W}) \geq r+s-n+1$. Arguing as above using Lemma 33.20.4 again, we conclude that $\pi(V \cap W) \subset X \cap Y$ has dimension at least $r+s-n$ as desired. □

0BXU **Lemma 33.34.4.** *Let k be a field. Let $Z \subset \mathbf{P}^n_k$ be a closed subscheme which has no embedded points such that every irreducible component of Z has dimension $n-1$. Then the ideal $I(Z) \subset k[T_0, \ldots, T_n]$ corresponding to Z is principal.*

Proof. This is a special case of Divisors, Lemma 31.31.3. □

33.35. Coherent sheaves on projective space

089X In this section we prove some results on the cohomology of coherent sheaves on $\mathbf{P}^n$ over a field which can be found in [**Mum66**]. These will be useful later when discussing Quot and Hilbert schemes.

089Y **33.35.1. Preliminaries.** Let k be a field, $n \geq 1$, $d \geq 1$, and let $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(d))$ be a nonzero section. In this section we will write $\mathcal{O}(d)$ for the dth twist of the structure sheaf on projective space (Constructions, Definitions 27.10.1 and 27.13.2). Since $\mathbf{P}^n_k$ is a variety this section is regular, hence s is a regular section of $\mathcal{O}(d)$ and defines an effective Cartier divisor $H = Z(s) \subset \mathbf{P}^n_k$, see Divisors, Section 31.13. Such a divisor H is called a *hypersurface* and if $d = 1$ it is called a *hyperplane*.

089Z **Lemma 33.35.2.** *Let k be a field. Let $n \geq 1$. Let $i : H \to \mathbf{P}^n_k$ be a hyperplane. Then there exists an isomorphism*

$$\varphi : \mathbf{P}^{n-1}_k \longrightarrow H$$

such that $i^*\mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$.

Proof. We have $\mathbf{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$. The section s corresponds to a homogeneous form in $T_0, \ldots, T_n$ of degree 1, see Cohomology of Schemes, Section 30.8. Say $s = \sum a_i T_i$. Constructions, Lemma 27.13.7 gives that $H = \mathrm{Proj}(k[T_0, \ldots, T_n]/I)$ for the graded ideal I defined by setting I_d equal to the kernel of the map $\Gamma(\mathbf{P}^n_k, \mathcal{O}(d)) \to \Gamma(H, i^*\mathcal{O}(d))$. By our construction of $Z(s)$ in Divisors, Definition 31.14.8 we see that on $D_+(T_j)$ the ideal of H is generated by $\sum a_i T_i/T_j$ in the polynomial ring $k[T_0/T_j, \ldots, T_n/T_j]$. Thus it is clear that I is the ideal generated by $\sum a_i T_i$. Note that

$$k[T_0, \ldots, T_n]/I = k[T_0, \ldots, T_n]/(\sum a_i T_i) \cong k[S_0, \ldots, S_{n-1}]$$

as graded rings. For example, if $a_n \neq 0$, then mapping S_i equal to the class of T_i works. We obtain the desired isomorphism by functoriality of Proj. Equality of twists of structure sheaves follows for example from Constructions, Lemma 27.11.5. □

08A0 **Lemma 33.35.3.** *Let k be an infinite field. Let $n \geq 1$. Let $\mathcal{F}$ be a coherent module on $\mathbf{P}^n_k$. Then there exist a nonzero section $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ and a short exact sequence*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{G} \to 0$$

where $i : H \to \mathbf{P}^n_k$ is the hyperplane H associated to s and $\mathcal{G} = i^*\mathcal{F}$.

Proof. The map $\mathcal{F}(-1) \to \mathcal{F}$ comes from Constructions, Equation (27.10.1.2) with $n = 1$, $m = -1$ and the section s of $\mathcal{O}(1)$. Let's work out what this map looks like if we restrict it to $D_+(T_0)$. Write $D_+(T_0) = \mathrm{Spec}(k[x_1, \ldots, x_n])$ with $x_i = T_i/T_0$. Identify $\mathcal{O}(1)|_{D_+(T_0)}$ with $\mathcal{O}$ using the section T_0. Hence if $s = \sum a_i T_i$ then $s|_{D_+(T_0)} = a_0 + \sum a_i x_i$ with the identification chosen above. Furthermore, suppose $\mathcal{F}|_{D_+(T_0)}$ corresponds to the finite $k[x_1, \ldots, x_n]$-module M. Via the identification $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}(-1)$ and our chosen trivialization of $\mathcal{O}(1)$ we see that $\mathcal{F}(-1)$ corresponds to M as well. Thus restricting $\mathcal{F}(-1) \to \mathcal{F}$ to $D_+(T_0)$ gives the map

$$M \xrightarrow{a_0 + \sum a_i x_i} M$$

To see that the arrow is injective, it suffices to pick $a_0 + \sum a_i x_i$ outside any of the associated primes of M, see Algebra, Lemma 10.63.9. By Algebra, Lemma 10.63.5 the set $\mathrm{Ass}(M)$ of associated primes of M is finite. Note that for $\mathfrak{p} \in \mathrm{Ass}(M)$ the intersection $\mathfrak{p} \cap \{a_0 + \sum a_i x_i\}$ is a proper k-subvector space. We conclude that there is a finite family of proper sub vector spaces $V_1, \ldots, V_m \subset \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that if we take s outside of $\bigcup V_i$, then multiplication by s is injective over $D_+(T_0)$. Similarly for the restriction to $D_+(T_j)$ for $j = 1, \ldots, n$. Since k is infinite, a finite union of proper sub vector spaces is never equal to the whole space, hence we may choose s such that the map is injective. The cokernel of $\mathcal{F}(-1) \to \mathcal{F}$ is annihilated by $\mathrm{Im}(s : \mathcal{O}(-1) \to \mathcal{O})$ which is the ideal sheaf of H by Divisors, Definition 31.14.8. Hence we obtain $\mathcal{G}$ on H using Cohomology of Schemes, Lemma 30.9.8. □

08A1 **Remark 33.35.4.** Let k be an infinite field. Let $n \geq 1$. Given a finite number of coherent modules $\mathcal{F}_i$ on $\mathbf{P}^n_k$ we can choose a single $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that the statement of Lemma 33.35.3 works for each of them. To prove this, just apply the lemma to $\bigoplus \mathcal{F}_i$.

0EGK **Remark 33.35.5.** In the situation of Lemmas 33.35.2 and 33.35.3 we have $H \cong \mathbf{P}^{n-1}_k$ with Serre twists $\mathcal{O}_H(d) = i^*\mathcal{O}_{\mathbf{P}^n_k}(d)$. For every $d \in \mathbf{Z}$ we have a short exact sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*(\mathcal{G}(d)) \to 0$$

Namely, tensoring by $\mathcal{O}_{\mathbf{P}^n_k}(d)$ is an exact functor and by the projection formula (Cohomology, Lemma 20.54.2) we have $i_*(\mathcal{G}(d)) = i_*\mathcal{G} \otimes \mathcal{O}_{\mathbf{P}^n_k}(d)$. We obtain corresponding long exact sequences

$$H^i(\mathbf{P}^n_k, \mathcal{F}(d-1)) \to H^i(\mathbf{P}^n_k, \mathcal{F}(d)) \to H^i(H, \mathcal{G}(d)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(d-1))$$

This follows from the above and the fact that we have $H^i(\mathbf{P}^n_k, i_*\mathcal{G}(d)) = H^i(H, \mathcal{G}(d))$ by Cohomology of Schemes, Lemma 30.2.4 (closed immersions are affine).

08A2 **33.35.6. Regularity.** Here is the definition.

08A3 **Definition 33.35.7.** Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. We say $\mathcal{F}$ is *m-regular* if

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0$$

for $i = 1, \ldots, n$.

Note that $\mathcal{F} = \mathcal{O}(d)$ is m-regular if and only if $d \geq -m$. This follows from the computation of cohomology groups in Cohomology of Schemes, Equation (30.8.1.1). Namely, we see that $H^n(\mathbf{P}^n_k, \mathcal{O}(d)) = 0$ if and only if $d \geq -n$.

08A4 **Lemma 33.35.8.** *Let k'/k be an extension of fields. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $\mathbf{P}^n_{k'}$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}'$ is m-regular.*

Proof. This is true because

$$H^i(\mathbf{P}^n_{k'}, \mathcal{F}') = H^i(\mathbf{P}^n_k, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

08A5 **Lemma 33.35.9.** *In the situation of Lemma 33.35.3, if $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular on $H \cong \mathbf{P}^{n-1}_k$.*

Proof. Recall that $H^i(\mathbf{P}^n_k, i_*\mathcal{G}) = H^i(H, \mathcal{G})$ by Cohomology of Schemes, Lemma 30.2.4. Hence we see that for $i \geq 1$ we get

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) \to H^i(H, \mathcal{G}(m-i)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(m-1-i))$$

by Remark 33.35.5. The lemma follows. □

08A6 **Lemma 33.35.10.** *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular.*

Proof. We prove this by induction on n. If $n = 0$ every sheaf is m-regular for all m and there is nothing to prove. By Lemma 33.35.8 we may replace k by an infinite overfield and assume k is infinite. Thus we may apply Lemma 33.35.3. By Lemma 33.35.9 we know that $\mathcal{G}$ is m-regular. By induction on n we see that $\mathcal{G}$ is $(m+1)$-regular. Considering the long exact cohomology sequence associated to the sequence

$$0 \to \mathcal{F}(m-i) \to \mathcal{F}(m+1-i) \to i_*\mathcal{G}(m+1-i) \to 0$$

as in Remark 33.35.5 the reader easily deduces for $i \geq 1$ the vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m+1-i))$ from the (known) vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i))$ and $H^i(\mathbf{P}^n_k, \mathcal{G}(m+1-i))$. □

08A7 **Lemma 33.35.11.** *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then the multiplication map*

$$H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Proof. Let k'/k be an extension of fields. Let $\mathcal{F}'$ be as in Lemma 33.35.8. By Cohomology of Schemes, Lemma 30.5.2 the base change of the linear map of the lemma to k' is the same linear map for the sheaf $\mathcal{F}'$. Since $k \to k'$ is faithfully flat it suffices to prove the lemma over k', i.e., we may assume k is infinite.

Assume k is infinite. We prove the lemma by induction on n. The case $n = 0$ is trivial as $\mathcal{O}(1) \cong \mathcal{O}$ is generated by T_0. For $n > 0$ apply Lemma 33.35.3 and tensor the sequence by $\mathcal{O}(m+1)$ to get

$$0 \to \mathcal{F}(m) \xrightarrow{s} \mathcal{F}(m+1) \to i_*\mathcal{G}(m+1) \to 0$$

see Remark 33.35.5. Let $t \in H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$. By induction the image $\bar{t} \in H^0(H, \mathcal{G}(m+1))$ is the image of $\sum g_i \otimes \overline{s}_i$ with $\overline{s}_i \in \Gamma(H, \mathcal{O}(1))$ and $g_i \in H^0(H, \mathcal{G}(m))$. Since $\mathcal{F}$ is m-regular we have $H^1(\mathbf{P}^n_k, \mathcal{F}(m-1)) = 0$, hence long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{F}(m-1) \xrightarrow{s} \mathcal{F}(m) \to i_*\mathcal{G}(m) \to 0$$

shows we can lift g_i to $f_i \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$. We can also lift $\overline{s}_i$ to $s_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$ (see proof of Lemma 33.35.2 for example). After subtracting the image of $\sum f_i \otimes s_i$ from t we see that we may assume $\bar{t} = 0$. But this exactly means that t is the image of $f \otimes s$ for some $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$ as desired. □

08A8 **Lemma 33.35.12.** *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated.*

Proof. For all $d \gg 0$ the sheaf $\mathcal{F}(d)$ is globally generated. This follows for example from the first part of Cohomology of Schemes, Lemma 30.14.1. Pick $d \geq m$ such that $\mathcal{F}(d)$ is globally generated. Choose a basis $f_1, \ldots, f_r \in H^0(\mathbf{P}^n_k, \mathcal{F})$. By Lemma 33.35.11 every element $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(d))$ can be written as $f = \sum P_i f_i$ for some $P_i \in k[T_0, \ldots, T_n]$ homogeneous of degree $d-m$. Since the sections f generate $\mathcal{F}(d)$ it follows that the sections f_i generate $\mathcal{F}(m)$. □

08A9 **33.35.13. Hilbert polynomials.** The following lemma will be made obsolete by the more general Lemma 33.45.1.

08AC **Lemma 33.35.14.** *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma 33.33.4 we may assume k is infinite. Apply Lemma 33.35.3. Applying Lemma 33.33.2 to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

See Remark 33.35.5. Since $H \cong \mathbf{P}^{n-1}_k$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma 10.58.5). □

08AD **Definition 33.35.15.** Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

08AE **Lemma 33.35.16.** *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma 30.14.1. □

08AF **33.35.17. Boundedness of quotients.** In this subsection we bound the regularity of quotients of a given coherent sheaf on $\mathbf{P}^n$ in terms of the Hilbert polynomial.

08AG **Lemma 33.35.18.** *Let k be a field. Let $n \geq 0$. Let $r \geq 1$. Let $P \in \mathbf{Q}[t]$. There exists an integer m depending on n, r, and P with the following property: if*

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$$

is a short exact sequence of coherent sheaves on $\mathbf{P}^n_k$ and $\mathcal{F}$ has Hilbert polynomial P, then $\mathcal{K}$ is m-regular.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and any coherent module is 0-regular and any surjective map is surjective on global sections. Assume $n > 0$. Consider an exact sequence as in the lemma. Let $P' \in \mathbf{Q}[t]$ be the polynomial $P'(t) = P(t) - P(t-1)$. Let m' be the integer which works for $n-1$, r, and P'. By Lemmas 33.35.8 and 33.33.4 we may replace k by a field extension, hence we may assume k is infinite. Apply Lemma 33.35.3 to the coherent sheaf $\mathcal{F}$. The Hilbert polynomial of $\mathcal{F}' = i^*\mathcal{F}$ is P' (see proof of Lemma 33.35.14). Since i^* is right exact we see that $\mathcal{F}'$ is a quotient of $\mathcal{O}_H^{\oplus r} = i^*\mathcal{O}^{\oplus r}$. Thus the induction hypothesis applies to $\mathcal{F}'$ on $H \cong \mathbf{P}^{n-1}_k$ (Lemma 33.35.2). Note that the map $\mathcal{K}(-1) \to \mathcal{K}$ is injective as $\mathcal{K} \subset \mathcal{O}^{\oplus r}$ and has cokernel $i_*\mathcal{H}$ where $\mathcal{H} = i^*\mathcal{K}$. By the snake lemma (Homology, Lemma 12.5.17) we obtain a commutative diagram

with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(-1) & \to & \mathcal{O}^{\oplus r}(-1) & \to & \mathcal{F}(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K} & \to & \mathcal{O}^{\oplus r} & \to & \mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & i_*\mathcal{H} & \to & i_*\mathcal{O}_H^{\oplus r} & \to & i_*\mathcal{F}' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus the induction hypothesis applies to the exact sequence $0 \to \mathcal{H} \to \mathcal{O}_H^{\oplus r} \to \mathcal{F}' \to 0$ on $H \cong \mathbf{P}_k^{n-1}$ (Lemma 33.35.2) and $\mathcal{H}$ is m'-regular. Recall that this implies that $\mathcal{H}$ is d-regular for all $d \geq m'$ (Lemma 33.35.10).

Let $i \geq 2$ and $d \geq m'$. It follows from the long exact cohomology sequence associated to the left column of the diagram above and the vanishing of $H^{i-1}(H, \mathcal{H}(d))$ that the map

$$H^i(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^i(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. As these groups are zero for $d \gg 0$ (Cohomology of Schemes, Lemma 30.14.1) we conclude $H^i(\mathbf{P}_k^n, \mathcal{K}(d))$ are zero for all $d \geq m'$ and $i \geq 2$.

We still have to control H^1. First we observe that all the maps

$$H^1(\mathbf{P}_k^n, \mathcal{K}(m'-1)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m')) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m'+1)) \to \ldots$$

are surjective by the vanishing of $H^1(H, \mathcal{H}(d))$ for $d \geq m'$. Suppose $d > m'$ is such that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. Then $H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \to H^0(H, \mathcal{H}(d))$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc}
H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \otimes_k H^0(\mathbf{P}_k^n, \mathcal{O}(1)) & \longrightarrow & H^0(\mathbf{P}_k^n, \mathcal{K}(d+1)) \\
\downarrow & & \downarrow \\
H^0(H, \mathcal{H}(d)) \otimes_k H^0(H, \mathcal{O}_H(1)) & \longrightarrow & H^0(H, \mathcal{H}(d+1))
\end{array}$$

By Lemma 33.35.11 we see that the bottom horizontal arrow is surjective. Hence the right vertical arrow is surjective. We conclude that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d+1))$$

is injective. By induction we see that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(d)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(d+1)) \to \ldots$$

are all injective and we conclude that $H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) = 0$ because of the eventual vanishing of these groups. Thus the dimensions of the groups $H^1(\mathbf{P}_k^n, \mathcal{K}(d))$ for $d \geq m'$ are strictly decreasing until they become zero. It follows that the regularity

of $\mathcal{K}$ is bounded by $m' + \dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m'))$. On the other hand, by the vanishing of the higher cohomology groups we have

$$\dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m')) = -\chi(\mathbf{P}^n_k, \mathcal{K}(m')) + \dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m'))$$

Note that the H^0 has dimension bounded by the dimension of $H^0(\mathbf{P}^n_k, \mathcal{O}^{\oplus r}(m'))$ which is at most $r\binom{n+m'}{n}$ if $m' > 0$ and zero if not. Finally, the term $\chi(\mathbf{P}^n_k, \mathcal{K}(m'))$ is equal to $r\binom{n+m'}{n} - P(m')$. This gives a bound of the desired type finishing the proof of the lemma. □

33.36. Frobenii

0CC6 Let p be a prime number. If X is a scheme, then we say "X has characteristic p", or "X is of characteristic p", or "X is in characteristic p" if p is zero in $\mathcal{O}_X$.

03SM **Definition 33.36.1.** Let p be a prime number. Let X be a scheme in characteristic p. The *absolute frobenius of* X is the morphism $F_X : X \to X$ given by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$.

This makes sense because for any ring A of characteristic p the map $F_A : A \to A$, $a \mapsto a^p$ is a ring endomorphism which induces the identity on $\mathrm{Spec}(A)$. Moreover, if A is local, then F_A is a local homomorphism. In this way we see that the absolute frobenius of X is an endomorphism of X in the category of schemes. It turns out that the absolute frobenius defines a self map of the identity functor on the category of schemes in characteristic p.

0CC7 **Lemma 33.36.2.** *Let $p > 0$ be a prime number. Let $f : X \to Y$ be a morphism of schemes in characteristic p. Then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{F_Y} & Y \end{array}$$

commutes.

Proof. This follows from the following trivial algebraic fact: if $\varphi : A \to B$ is a homomorphism of rings of characteristic p, then $\varphi(a^p) = \varphi(a)^p$. □

0CC8 **Lemma 33.36.3.** *Let $p > 0$ be a prime number. Let X be a scheme in characteristic p. Then the absolute frobenius $F_X : X \to X$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.*

Proof. This follows from the corresponding results for the frobenius endomorphism $F_A : A \to A$ of a ring A of characteristic $p > 0$. See the discussion in Algebra, Section 10.46, for example Lemma 10.46.7. □

If we are working with schemes over a fixed base, then there is a relative version of the frobenius morphism.

0CC9 **Definition 33.36.4.** Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. We define

$$X^{(p)} = X^{(p/S)} = X \times_{S, F_S} S$$

viewed as a scheme over S. Applying Lemma 33.36.2 we see there is a unique morphism $F_{X/S} : X \longrightarrow X^{(p)}$ over S fitting into the commutative diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\quad F_X \quad} & & X \\ & \searrow^{F_{X/S}} & X^{(p)} & \nearrow & \\ & & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_S} & S \end{array}$$

where the right square is cartesian. The morphism $F_{X/S}$ is called the *relative Frobenius morphism of X/S*.

Observe that $X \mapsto X^{(p)}$ is a functor; it is the base change functor for the absolute frobenius morphism $F_S : S \to S$. We have the same lemmas as before regarding the relative Frobenius morphism.

0CCA **Lemma 33.36.5.** *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let $f : X \to Y$ be a morphism of schemes over S . Then the diagram*

$$\begin{array}{ccc} X & \xrightarrow[F_{X/S}]{} & X^{(p)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(p)}} \\ Y & \xrightarrow{F_{Y/S}} & Y^{(p)} \end{array}$$

commutes.

Proof. This follows from Lemma 33.36.2 and the definitions. □

0CCB **Lemma 33.36.6.** *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then the relative frobenius $F_{X/S} : X \to X^{(p)}$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.*

Proof. By Lemma 33.36.3 the morphisms $F_X : X \to X$ and the base change $h : X^{(p)} \to X$ of F_S are universal homeomorphisms. Since $h \circ F_{X/S} = F_X$ we conclude that $F_{X/S}$ is a universal homeomorphism (Morphisms, Lemma 29.45.8). By Morphisms, Lemmas 29.45.5 and 29.10.2 we conclude that $F_{X/S}$ has the other properties as well. □

0CCC **Lemma 33.36.7.** *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then $\Omega_{X/S} = \Omega_{X/X^{(p)}}$.*

Proof. This translates into the following algebra fact. Let $A \to B$ be a homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $\Omega_{B/A} = \Omega_{B/B'}$. This is true because $\mathrm{d}(b^p a) = 0$ if $\mathrm{d} : B \to \Omega_{B/A}$ is the universal derivation and hence d is a B'-derivation. □

0CCD **Lemma 33.36.8.** *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. If $X \to S$ is locally of finite type, then $F_{X/S}$ is finite.*

Proof. This translates into the following algebra fact. Let $A \to B$ be a finite type homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $F_{B/A}$ is finite.

Namely, if $x_1, \ldots, x_n \in B$ are generators over A, then x_i is integral over B' because $x_i^p = F_{B/A}(x_i \otimes 1)$. Hence $F_{B/A} : B' \to B$ is finite by Algebra, Lemma 10.36.5. □

0CCE **Lemma 33.36.9.** *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Then X is geometrically reduced if and only if $X^{(p)}$ is reduced.*

Proof. Consider the absolute frobenius $F_k : k \to k$. Then $F_k(k) = k^p$ in other words, $F_k : k \to k$ is isomorphic to the embedding of k into $k^{1/p}$. Thus the lemma follows from Lemma 33.6.4. □

0CCF **Lemma 33.36.10.** *Let k be a field of characteristic $p > 0$. Let X be a variety over k. The following are equivalent*

(1) *$X^{(p)}$ is reduced,*
(2) *X is geometrically reduced,*
(3) *there is a nonempty open $U \subset X$ smooth over k.*

In this case $X^{(p)}$ is a variety over k and $F_{X/k} : X \to X^{(p)}$ is a finite dominant morphism of degree $p^{\dim(X)}$.

Proof. We have seen the equivalence of (1) and (2) in Lemma 33.36.9. We have seen that (2) implies (3) in Lemma 33.25.7. If (3) holds, then U is geometrically reduced (see for example Lemma 33.12.6) and hence X is geometrically reduced by Lemma 33.6.8. In this way we see that (1), (2), and (3) are equivalent.

Assume (1), (2), and (3) hold. Since $F_{X/k}$ is a homeomorphism (Lemma 33.36.6) we see that $X^{(p)}$ is a variety. Then $F_{X/k}$ is finite by Lemma 33.36.8. It is dominant as it is surjective. To compute the degree (Morphisms, Definition 29.51.8) it suffices to compute the degree of $F_{U/k} : U \to U^{(p)}$ (as $F_{U/k} = F_{X/k}|_U$ by Lemma 33.36.5). After shrinking U a bit we may assume there exists an étale morphism $h : U \to \mathbf{A}^n_k$, see Morphisms, Lemma 29.36.20. Of course $n = \dim(U)$ because $\mathbf{A}^n_k \to \mathrm{Spec}(k)$ is smooth of relative dimension n, the étale morphism h is smooth of relative dimension 0, and $U \to \mathrm{Spec}(k)$ is smooth of relative dimension $\dim(U)$ and relative dimensions add up correctly (Morphisms, Lemma 29.29.3). Observe that h is a generically finite dominant morphism of varieties, and hence $\deg(h)$ is defined. By Lemma 33.36.5 we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_{X/k}} & X^{(p)} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^{(p)}} \\ \mathbf{A}^n_k & \xrightarrow{F_{\mathbf{A}^n_k/k}} & (\mathbf{A}^n_k)^{(p)} \end{array}$$

Since $h^{(p)}$ is a base change of h it is étale as well and it follows that $h^{(p)}$ is a generically finite dominant morphism of varieties as well. The degree of $h^{(p)}$ is the degree of the extension $k(X^{(p)})/k((\mathbf{A}^n_k)^{(p)})$ which is the same as the degree of the extension $k(X)/k(\mathbf{A}^n_k)$ because $h^{(p)}$ is the base change of h (small detail omitted). By multiplicativity of degrees (Morphisms, Lemma 29.51.9) it suffices to show that the degree of $F_{\mathbf{A}^n_k/k}$ is p^n. To see this observe that $(\mathbf{A}^n_k)^{(p)} = \mathbf{A}^n_k$ and that $F_{\mathbf{A}^n_k/k}$ is given by the map sending the coordinates to their pth powers. □

0CCG **Remark 33.36.11.** Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. For $n \geq 1$

$$X^{(p^n)} = X^{(p^n/S)} = X \times_{S, F_S^n} S$$

viewed as a scheme over S. Observe that $X \mapsto X^{(p^n)}$ is a functor. Applying Lemma 33.36.2 we see $F_{X/S,n} = (F_X^n, \mathrm{id}_S) : X \longrightarrow X^{(p^n)}$ is a morphism over S fitting into the commutative diagram

$$\begin{array}{ccccc} & & \overset{F_X^n}{\frown} & & \\ X & \xrightarrow{F_{X/S,n}} & X^{(p^n)} & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_S^n} & S \end{array}$$

where the right square is cartesian. The morphism $F_{X/S,n}$ is sometimes called the *n-fold relative Frobenius morphism of X/S*. This makes sense because we have the formula

$$F_{X/S,n} = F_{X^{(p^{n-1})}/S} \circ \ldots \circ F_{X^{(p)}/S} \circ F_{X/S}$$

which shows that $F_{X/S,n}$ is the composition of n relative Frobenii. Since we have

$$F_{X^{(p^m)}/S} = F^{(p)}_{X^{(p^{m-1})}/S} = \ldots = F^{(p^m)}_{X/S}$$

(details omitted) we get also that

$$F_{X/S,n} = F^{(p^{n-1})}_{X/S} \circ \ldots \circ F^{(p)}_{X/S} \circ F_{X/S}$$

33.37. Glueing dimension one rings

09MX This section contains some algebraic preliminaries to proving that a finite set of codimension 1 points of a separated scheme is contained in an affine open.

09MY **Situation 33.37.1.** Here we are given a commutative diagram of rings

$$\begin{array}{ccc} A & \longrightarrow & K \\ \uparrow & & \uparrow \\ R & \longrightarrow & B \end{array}$$

where K is a field and A, B are subrings of K with fraction field K. Finally, $R = A \times_K B = A \cap B$.

09MZ **Lemma 33.37.2.** *In Situation 33.37.1 assume that B is a valuation ring. Then for every unit u of A either $u \in R$ or $u^{-1} \in R$.*

Proof. Namely, if the image c of u in K is in B, then $u \in R$. Otherwise, $c^{-1} \in B$ (Algebra, Lemma 10.50.4) and $u^{-1} \in R$. □

The following lemma explains the meaning of the condition "$A \otimes B \to K$ is surjective" which comes up quite a bit in the following.

09N0 **Lemma 33.37.3.** *In Situation 33.37.1 assume A is a Noetherian ring of dimension 1. The following are equivalent*

(1) *$A \otimes B \to K$ is not surjective,*
(2) *there exists a discrete valuation ring $\mathcal{O} \subset K$ containing both A and B.*

Proof. It is clear that (2) implies (1). On the other hand, if $A \otimes B \to K$ is not surjective, then the image $C \subset K$ is not a field hence C has a nonzero maximal ideal $\mathfrak{m}$. Choose a valuation ring $\mathcal{O} \subset K$ dominating $C_{\mathfrak{m}}$. By Algebra, Lemma 10.119.12

applied to $A \subset \mathcal{O}$ the ring $\mathcal{O}$ is Noetherian. Hence $\mathcal{O}$ is a discrete valuation ring by Algebra, Lemma 10.50.18. □

09N1 **Lemma 33.37.4.** *In Situation 33.37.1 assume*

(1) *A is a Noetherian semi-local domain of dimension 1,*
(2) *B is a discrete valuation ring,*

Then we have the following two possibilities

(a) *If A^* is not contained in R, then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ are open immersions covering $\mathrm{Spec}(R)$ and $K = A \otimes_R B$.*
(b) *If A^* is contained in R, then B dominates one of the local rings of A at a maximal ideal and $A \otimes B \to K$ is not surjective.*

Proof. Assumption (a) implies there is a unit u of A whose image in K lies in the maximal ideal of B. Then u is a nonzerodivisor of R and for every $a \in A$ there exists an n such that $u^n a \in R$. It follows that $A = R_u$.

Let $\mathfrak{m}_A$ be the Jacobson radical of A. Let $x \in \mathfrak{m}_A$ be a nonzero element. Since $\dim(A) = 1$ we see that $K = A_x$. After replacing x by $x^n u^m$ for some $n \geq 1$ and $m \in \mathbf{Z}$ we may assume x maps to a unit of B. We see that for every $b \in B$ we have that $x^n b$ in the image of R for some n. Thus $B = R_x$.

Let $z \in R$. If $z \notin \mathfrak{m}_A$ and z does not map to an element of $\mathfrak{m}_B$, then z is invertible. Thus $x + u$ is invertible in R. Hence $\mathrm{Spec}(R) = D(x) \cup D(u)$. We have seen above that $D(u) = \mathrm{Spec}(A)$ and $D(x) = \mathrm{Spec}(B)$.

Case (b). If $x \in \mathfrak{m}_A$, then $1 + x$ is a unit and hence $1 + x \in R$, i.e, $x \in R$. Thus we see that $\mathfrak{m}_A \subset R \subset A$. In fact, in this case A is integral over R. Namely, write $A/\mathfrak{m}_A = \kappa_1 \times \ldots \times \kappa_n$ as a product of fields. Say $x = (c_1, \ldots, c_r, 0, \ldots, 0)$ is an element with $c_i \neq 0$. Then

$$x^2 - x(c_1, \ldots, c_r, 1, \ldots, 1) = 0$$

Since R contains all units we see that $A/\mathfrak{m}_A$ is integral over the image of R in it, and hence A is integral over R. It follows that $R \subset A \subset B$ as B is integrally closed. Moreover, if $x \in \mathfrak{m}_A$ is nonzero, then $K = A_x = \bigcup x^{-n} A = \bigcup x^{-n} R$. Hence $x^{-1} \notin B$, i.e., $x \in \mathfrak{m}_B$. We conclude $\mathfrak{m}_A \subset \mathfrak{m}_B$. Thus $A \cap \mathfrak{m}_B$ is a maximal ideal of A thereby finishing the proof. □

09N2 **Lemma 33.37.5.** *Let B be a semi-local Noetherian domain of dimension 1. Let B' be the integral closure of B in its fraction field. Then B' is a semi-local Dedekind domain. Let x be a nonzero element of the Jacobson radical of B'. Then for every $y \in B'$ there exists an n such that $x^n y \in B$.*

Proof. Let $\mathfrak{m}_B$ be the Jacobson radical of B. The structure of B' results from Algebra, Lemma 10.120.18. Given $x, y \in B'$ as in the statement of the lemma consider the subring $B \subset A \subset B'$ generated by x and y. Then A is finite over B (Algebra, Lemma 10.36.5). Since the fraction fields of B and A are the same we see that the finite module A/B is supported on the set of closed points of B. Thus $\mathfrak{m}_B^n A \subset B$ for a suitable n. Moreover, $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is surjective (Algebra, Lemma 10.36.17), hence A is semi-local as well. It also follows that x is in the Jacobson radical $\mathfrak{m}_A$ of A. Note that $\mathfrak{m}_A = \sqrt{\mathfrak{m}_B A}$. Thus $x^m y \in \mathfrak{m}_B A$ for some m. Then $x^{nm} y \in B$. □

09N3 **Lemma 33.37.6.** *In Situation 33.37.1 assume*

(1) *A is a Noetherian semi-local domain of dimension* 1,
(2) *B is a Noetherian semi-local domain of dimension* 1,
(3) *$A \otimes B \to K$ is surjective.*

Then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *and* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *are open immersions covering* $\mathrm{Spec}(R)$ *and* $K = A \otimes_R B$.

Proof. Special case: B is integrally closed in K. This means that B is a Dedekind domain (Algebra, Lemma 10.120.17) whence all of its localizations at maximal ideals are discrete valuation rings. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of B. We set

$$R_1 = A \times_K B_{\mathfrak{m}_1}$$

Observing that $A \otimes_{R_1} B_{\mathfrak{m}_1} \to K$ is surjective we conclude from Lemma 33.37.4 that A and $B_{\mathfrak{m}_1}$ define open subschemes covering $\mathrm{Spec}(R_1)$ and that $K = A \otimes_{R_1} B_{\mathfrak{m}_1}$. In particular R_1 is a semi-local Noetherian ring of dimension 1. By induction we define

$$R_{i+1} = R_i \times_K B_{\mathfrak{m}_{i+1}}$$

for $i = 1, \ldots, r-1$. Observe that $R = R_r$ because $B = B_{\mathfrak{m}_1} \cap \ldots \cap B_{\mathfrak{m}_r}$ (see Algebra, Lemma 10.157.6). It follows from the inductive procedure that $R \to A$ defines an open immersion $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. On the other hand, the maximal ideals $\mathfrak{n}_i$ of R not in this open correspond to the maximal ideals $\mathfrak{m}_i$ of B and in fact the ring map $R \to B$ defines an isomorphisms $R_{\mathfrak{n}_i} \to B_{\mathfrak{m}_i}$ (details omitted; hint: in each step we added exactly one maximal ideal to $\mathrm{Spec}(R_i)$). It follows that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is an open immersion as desired.

General case. Let $B' \subset K$ be the integral closure of B. See Lemma 33.37.5. Then the special case applies to $R' = A \times_K B'$. Pick $x \in R'$ which is not contained in the maximal ideals of A and is contained in the maximal ideals of B' (see Algebra, Lemma 10.15.4). By Lemma 33.37.5 there exists an integer n such that $x^n \in R = A \times_K B$. Replace x by x^n so $x \in R$. For every $y \in R'$ there exists an integer n such that $x^n y \in R$. On the other hand, it is clear that $R'_x = A$. Thus $R_x = A$. Exchanging the roles of A and B we also find an $y \in R$ such that $B = R_y$. Note that inverting both x and y leaves no primes except (0). Thus $K = R_{xy} = R_x \otimes_R R_y$. This finishes the proof. □

09N4 **Lemma 33.37.7.** *Let K be a field. Let $A_1, \ldots, A_r \subset K$ be Noetherian semi-local rings of dimension* 1 *with fraction field K. If $A_i \otimes A_j \to K$ is surjective for all $i \neq j$, then there exists a Noetherian semi-local domain $A \subset K$ of dimension* 1 *contained in $A_1, \ldots, A_r$ such that*

(1) *$A \to A_i$ induces an open immersion $j_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$,*
(2) *$\mathrm{Spec}(A)$ is the union of the opens $j_i(\mathrm{Spec}(A_i))$,*
(3) *each closed point of $\mathrm{Spec}(A)$ lies in exactly one of these opens.*

Proof. Namely, we can take $A = A_1 \cap \ldots \cap A_r$. First we note that (3), once (1) and (2) have been proven, follows from the assumption that $A_i \otimes A_j \to K$ is surjective since if $\mathfrak{m} \in j_i(\mathrm{Spec}(A_i)) \cap j_j(\mathrm{Spec}(A_j))$, then $A_i \otimes A_j \to K$ ends up in $A_{\mathfrak{m}}$. To prove (1) and (2) we argue by induction on r. If $r > 1$ by induction we have the results (1) and (2) for $B = A_2 \cap \ldots \cap A_r$. Then we apply Lemma 33.37.6 to see they hold for $A = A_1 \cap B$. □

09N5 **Lemma 33.37.8.** *Let A be a domain with fraction field K. Let $B_1, \ldots, B_r \subset K$ be Noetherian 1-dimensional semi-local domains whose fraction fields are K. If $A \otimes B_i \to K$ are surjective for $i = 1, \ldots, r$, then there exists an $x \in A$ such that x^{-1} is in the Jacobson radical of B_i for $i = 1, \ldots, r$.*

Proof. Let B_i' be the integral closure of B_i in K. Suppose we find a nonzero $x \in A$ such that x^{-1} is in the Jacobson radical of B_i' for $i = 1, \ldots, r$. Then by Lemma 33.37.5, after replacing x by a power we get $x^{-1} \in B_i$. Since $\mathrm{Spec}(B_i') \to \mathrm{Spec}(B_i)$ is surjective we see that x^{-1} is then also in the Jacobson radical of B_i. Thus we may assume that each B_i is a semi-local Dedekind domain.

If B_i is not local, then remove B_i from the list and add back the finite collection of local rings $(B_i)_{\mathfrak{m}}$. Thus we may assume that B_i is a discrete valuation ring for $i = 1, \ldots, r$.

Let $v_i : K \to \mathbf{Z}$, $i = 1, \ldots, r$ be the corresponding discrete valuations (see Algebra, Lemma 10.120.17). We are looking for a nonzero $x \in A$ with $v_i(x) < 0$ for $i = 1, \ldots, r$. We will prove this by induction on r.

If $r = 1$ and the result is wrong, then $A \subset B$ and the map $A \otimes B \to K$ is not surjective, contradiction.

If $r > 1$, then by induction we can find a nonzero $x \in A$ such that $v_i(x) < 0$ for $i = 1, \ldots, r-1$. If $v_r(x) < 0$ then we are done, so we may assume $v_r(x) \geq 0$. By the base case we can find $y \in A$ nonzero such that $v_r(y) < 0$. After replacing x by a power we may assume that $v_i(x) < v_i(y)$ for $i = 1, \ldots, r-1$. Then $x + y$ is the element we are looking for. □

0AB2 **Lemma 33.37.9.** *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak{p}}$ where the product is over the minimal primes of A. Let $a_1, a_2 \in \mathfrak{m}_A$ map to the same element of L. Then $a_1^n = a_2^n$ for some $n > 0$.*

Proof. Write $a_1 = a_2 + x$. Then x maps to zero in L. Hence x is a nilpotent element of A because $\bigcap \mathfrak{p}$ is the radical of (0) and the annihilator I of x contains a power of the maximal ideal because $\mathfrak{p} \notin V(I)$ for all minimal primes. Say $x^k = 0$ and $\mathfrak{m}^n \subset I$. Then

$$a_1^{k+n} = a_2^{k+n} + \binom{n+k}{1} a_2^{n+k-1}x + \binom{n+k}{2} a_2^{n+k-2}x^2 + \ldots + \binom{n+k}{k-1} a_2^{n+1}x^{k-1} = a_2^{n+k}$$

because $a_2 \in \mathfrak{m}_A$. □

0AB3 **Lemma 33.37.10.** *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak{p}}$ and $I = \bigcap \mathfrak{p}$ where the product and intersection are over the minimal primes of A. Let $f \in L$ be an element of the form $f = i + a$ where $a \in \mathfrak{m}_A$ and $i \in IL$. Then some power of f is in the image of $A \to L$.*

Proof. Since A is Noetherian we have $I^t = 0$ for some $t > 0$. Suppose that we know that $f = a + i$ with $i \in I^k L$. Then $f^n = a^n + na^{n-1}i \bmod I^{k+1}L$. Hence it suffices to show that $na^{n-1}i$ is in the image of $I^k \to I^k L$ for some $n \gg 0$. To see this, pick a $g \in A$ such that $\mathfrak{m}_A = \sqrt{(g)}$ (Algebra, Lemma 10.60.8). Then $L = A_g$ for example by Algebra, Proposition 10.60.7. On the other hand, there is an n such that $a^n \in (g)$. Hence we can clear denominators for elements of L by multiplying by a high power of a. □

0AB4 **Lemma 33.37.11.** *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak{p}}$ where the product is over the minimal primes of A. Let $K \to L$ be an integral ring map. Then there exist $a \in \mathfrak{m}_A$ and $x \in K$ which map to the same element of L such that $\mathfrak{m}_A = \sqrt{(a)}$.*

Proof. By Lemma 33.37.10 we may replace A by $A/(\bigcap \mathfrak{p})$ and assume that A is a reduced ring (some details omitted). We may also replace K by the image of $K \to L$. Then K is a reduced ring. The map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is surjective and closed (details omitted). Hence $\mathrm{Spec}(K)$ is a finite discrete space. It follows that K is a finite product of fields.

Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the minimal primes of A. Set L_j be the fraction field of A_j so that $L = \prod_{j=1,\ldots,m} L_j$. Let A_j be the normalization of $A/\mathfrak{p}_j$. Then A_j is a semi-local Dedekind domain with at least one maximal ideal, see Algebra, Lemma 10.120.18. Let n be the sum of the numbers of maximal ideals in $A_1, \ldots, A_m$. For such a maximal ideal $\mathfrak{m} \subset A_j$ we consider the function

$$v_{\mathfrak{m}} : L \to L_j \to \mathbf{Z} \cup \{\infty\}$$

where the second arrow is the discrete valuation corresponding to the discrete valuation ring $(A_j)_{\mathfrak{m}}$ extended by mapping 0 to ∞. In this way we obtain n functions $v_1, \ldots, v_n : L \to \mathbf{Z} \cup \{\infty\}$. We will find an element $x \in K$ such that $v_i(x) < 0$ for all $i = 1, \ldots, n$.

First we claim that for each i there exists an element $x \in K$ with $v_i(x) < 0$. Namely, suppose that v_i corresponds to $\mathfrak{m} \subset A_j$. If $v_i(x) \geq 0$ for all $x \in K$, then K maps into $(A_j)_{\mathfrak{m}}$ inside the fraction field L_j of A_j. The image of K in L_j is a field over L_j is algebraic by Algebra, Lemma 10.36.18. Combined we get a contradiction with Algebra, Lemma 10.50.8.

Suppose we have found an element $x \in K$ such that $v_1(x) < 0, \ldots, v_r(x) < 0$ for some $r < n$. If $v_{r+1}(x) < 0$, then x works for $r+1$. If not, then choose some $y \in K$ with $v_{r+1}(y) < 0$ as is possible by the result of the previous paragraph. After replacing x by x^n for some $n > 0$, we may assume $v_i(x) < v_i(y)$ for $i = 1, \ldots, r$. Then $v_j(x+y) = v_j(x) < 0$ for $j = 1, \ldots, r$ by properties of valuations and similarly $v_{r+1}(x + y) = v_{r+1}(y) < 0$. Arguing by induction, we find $x \in K$ with $v_i(x) < 0$ for $i = 1, \ldots, n$.

In particular, the element $x \in K$ has nonzero projection in each factor of K (recall that K is a finite product of fields and if some component of x was zero, then one of the values $v_i(x)$ would be ∞). Hence x is invertible and $x^{-1} \in K$ is an element with $\infty > v_i(x^{-1}) > 0$ for all i. It follows from Lemma 33.37.5 that for some $e < 0$ the element $x^e \in K$ maps to an element of $\mathfrak{m}_A/\mathfrak{p}_j \subset A/\mathfrak{p}_j$ for all $j = 1, \ldots, m$. Observe that the cokernel of the map $\mathfrak{m}_A \to \prod \mathfrak{m}_A/\mathfrak{p}_j$ is annihilated by a power of $\mathfrak{m}_A$. Hence after replacing e by a more negative e, we find an element $a \in \mathfrak{m}_A$ whose image in $\mathfrak{m}_A/\mathfrak{p}_j$ is equal to the image of x^e. The pair (a, x^e) satisfies the conclusions of the lemma. □

09N6 **Lemma 33.37.12.** *Let A be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be a finite set of a primes of A. Let $S = A \setminus \bigcup \mathfrak{p}_i$. Then S is a multiplicative system and $S^{-1}A$ is a semi-local ring whose maximal ideals correspond to the maximal elements of the set $\{\mathfrak{p}_i\}$.*

Proof. If $a, b \in A$ and $a, b \in S$, then $a, b \notin \mathfrak{p}_i$ hence $ab \notin \mathfrak{p}_i$, hence $ab \in S$. Also $1 \in S$. Thus S is a multiplicative subset of A. By the description of $\mathrm{Spec}(S^{-1}A)$ in

Algebra, Lemma 10.17.5 and by Algebra, Lemma 10.15.2 we see that the primes of $S^{-1}A$ correspond to the primes of A contained in one of the $\mathfrak{p}_i$. Hence the maximal ideals of $S^{-1}A$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. □

33.38. One dimensional Noetherian schemes

09N7 The main result of this section is that a Noetherian separated scheme of dimension 1 has an ample invertible sheaf. See Proposition 33.38.12.

09N8 **Lemma 33.38.1.** *Let X be a scheme all of whose local rings are Noetherian of dimension ≤ 1. Let $U \subset X$ be a retrocompact open. Denote $j : U \to X$ the inclusion morphism. Then $R^p j_* \mathcal{F} = 0$, $p > 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. We may check the vanishing of $R^p j_* \mathcal{F}$ at stalks. Formation of $R^q j_*$ commutes with flat base change, see Cohomology of Schemes, Lemma 30.5.2. Thus we may assume that X is the spectrum of a Noetherian local ring of dimension ≤ 1. In this case X has a closed point x and finitely many other points $x_1, \ldots, x_n$ which specialize to x but not each other (see Algebra, Lemma 10.31.6). If $x \in U$, then $U = X$ and the result is clear. If not, then $U = \{x_1, \ldots, x_r\}$ for some r after possibly renumbering the points. Then U is affine (Schemes, Lemma 26.11.8). Thus the result follows from Cohomology of Schemes, Lemma 30.2.3. □

09N9 **Lemma 33.38.2.** *Let X be an affine scheme all of whose local rings are Noetherian of dimension ≤ 1. Then any quasi-compact open $U \subset X$ is affine.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. By Lemma 33.38.1 the higher direct images $R^p j_* \mathcal{F}$ are zero. The $\mathcal{O}_X$-module $j_* \mathcal{F}$ is quasi-coherent (Schemes, Lemma 26.24.1). Hence it has vanishing higher cohomology groups by Cohomology of Schemes, Lemma 30.2.2. By the Leray spectral sequence Cohomology, Lemma 20.13.6 we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$. Thus U is affine, for example by Cohomology of Schemes, Lemma 30.3.1. □

09NA **Lemma 33.38.3.** *Let X be a scheme. Let $U \subset X$ be an open. Assume*

(1) *U is a retrocompact open of X,*
(2) *$X \setminus U$ is discrete, and*
(3) *for $x \in X \setminus U$ the local ring $\mathcal{O}_{X,x}$ is Noetherian of dimension ≤ 1.*

Then (1) there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$ and (2) the map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective.

Proof. Let $X \setminus U = \{x_i; i \in I\}$. Choose affine opens $U_i \subset X$ with $x_i \in U_i$ and $x_j \notin U_i$ for $j \neq i$. This is possible by condition (2). Say $U_i = \mathrm{Spec}(A_i)$. Let $\mathfrak{m}_i \subset A_i$ be the maximal ideal corresponding to x_i. By our assumption on the local rings there are only a finite number of prime ideals $\mathfrak{q} \subset \mathfrak{m}_i$, $\mathfrak{q} \neq \mathfrak{m}_i$ (see Algebra, Lemma 10.31.6). Thus by prime avoidance (Algebra, Lemma 10.15.2) we can find $f_i \in \mathfrak{m}_i$ not contained in any of those primes. Then $V(f_i) = \{\mathfrak{m}_i\} \amalg Z_i$ for some closed subset $Z_i \subset U_i$ because Z_i is a retrocompact open subset of $V(f_i)$ closed under specialization, see Algebra, Lemma 10.41.7. After shrinking U_i we may assume $V(f_i) = \{x_i\}$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f_i on $U \cap U_i$ and by f_i/f_j on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ such that the section s defined by 1 on U and f_i on U_i is as in the statement of the lemma.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_U$-module. Let N_i be the invertible $(A_i)_{f_i}$ module such that $\mathcal{N}|_{U \cap U_i}$ is equal to $\tilde{N}_i$. Observe that $(A_{\mathfrak{m}_i})_{f_i}$ is an Artinian ring (as a dimension zero Noetherian ring, see Algebra, Lemma 10.60.5). Thus it is a product of local rings (Algebra, Lemma 10.53.6) and hence has trivial Picard group. Thus, after shrinking U_i (i.e., after replacing A_i by $(A_i)_g$ for some $g \in A_i$, $g \notin \mathfrak{m}_i$) we can assume that $N_i = (A_i)_{f_i}$, i.e., that $\mathcal{N}|_{U \cap U_i}$ is trivial. In this case it is clear how to extend $\mathcal{N}$ to an invertible sheaf over X (by extending it by a trivial invertible module over each U_i). □

09NB **Lemma 33.38.4.** *Let X be an integral separated scheme. Let $U \subset X$ be a nonempty affine open such that $X \setminus U$ is a finite set of points $x_1, \ldots, x_r$ with $\mathcal{O}_{X,x_i}$ Noetherian of dimension 1. Then there exists a globally generated invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$.*

Proof. Say $U = \mathrm{Spec}(A)$ and let K be the function field of X. Write $B_i = \mathcal{O}_{X,x_i}$ and $\mathfrak{m}_i = \mathfrak{m}_{x_i}$. Since $x_i \notin U$ we see that the open $U \times_X \mathrm{Spec}(B_i)$ of $\mathrm{Spec}(B_i)$ has only one point, i.e., $U \times_X \mathrm{Spec}(B_i) = \mathrm{Spec}(K)$. Since X is separated, we find that $\mathrm{Spec}(K)$ is a closed subscheme of $U \times \mathrm{Spec}(B_i)$, i.e., the map $A \otimes B_i \to K$ is a surjection. By Lemma 33.37.8 we can find a nonzero $f \in A$ such that $f^{-1} \in \mathfrak{m}_i$ for $i = 1, \ldots, r$. Pick opens $x_i \in U_i \subset X$ such that $f^{-1} \in \mathcal{O}(U_i)$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f on $U \cap U_i$ and by 1 on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ with two sections:

(1) a section s defined by 1 on U and f^{-1} on U_i is as in the statement of the lemma, and
(2) a section t defined by f on U and 1 on U_i.

Note that $X_t \supset U_1 \cup \ldots \cup U_r$. Hence s, t generate $\mathcal{L}$ and the lemma is proved. □

09NC **Lemma 33.38.5.** *Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.*

Proof. Since X is quasi-compact we can find a finite collection $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ of pairs such that $\mathcal{L}_i$ is globally generated, X_{s_i} is affine and $X = \bigcup X_{s_i}$. Again because X is quasi-compact we can find, for each i, a finite collection of sections $t_{i,j}$ of $\mathcal{L}_i$, $j = 1, \ldots, m_i$ such that $X = \bigcup X_{t_{i,j}}$. Set $t_{i,0} = s_i$. Consider the invertible sheaf

$$\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}_n$$

and the global sections

$$\tau_J = t_{1,j_1} \otimes \ldots \otimes t_{n,j_n}$$

By Properties, Lemma 28.26.4 the open X_{τ_J} is affine as soon as $j_i = 0$ for some i. It is a simple matter to see that these opens cover X. Hence $\mathcal{L}$ is ample by definition. □

09ND **Lemma 33.38.6.** *Let X be a Noetherian integral separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Since X is Noetherian, each of the sets $X \setminus U_i$ is finite. Thus by Lemma 33.38.4 we can find a pair $(\mathcal{L}_i, s_i)$ consisting of a globally generated invertible sheaf $\mathcal{L}_i$ and a global section s_i such that $U_i = X_{s_i}$. We conclude that X has an ample invertible sheaf by Lemma 33.38.5. □

0C0T **Lemma 33.38.7.** *Let $f : X \to Y$ be a finite morphism of schemes. Assume there exists an open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism and $Y \setminus V$ is a discrete space. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.*

Proof. We will use that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, see Cohomology, Lemma 20.6.1. Consider the Leray spectral sequence for the abelian sheaf $\mathcal{O}_X^*$ and f, see Cohomology, Lemma 20.13.4. Consider the induced map

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^0(Y, R^1 f_* \mathcal{O}_X^*)$$

Divisors, Lemma 31.17.1 says exactly that this map is zero. Hence Leray gives $H^1(X, \mathcal{O}_X^*) = H^1(Y, f_* \mathcal{O}_X^*)$. Next we consider the map

$$f^\sharp : \mathcal{O}_Y^* \longrightarrow f_* \mathcal{O}_X^*$$

By assumption the kernel and cokernel of this map are supported on the closed subset $T = Y \setminus V$ of Y. Since T is a discrete topological space by assumption the higher cohomology groups of any abelian sheaf on Y supported on T is zero (follows from Cohomology, Lemma 20.20.1, Modules, Lemma 17.6.1, and the fact that $H^i(T, \mathcal{F}) = 0$ for any $i > 0$ and any abelian sheaf $\mathcal{F}$ on T). Breaking the displayed map into short exact sequences

$$0 \to \mathrm{Ker}(f^\sharp) \to \mathcal{O}_Y^* \to \mathrm{Im}(f^\sharp) \to 0, \quad 0 \to \mathrm{Im}(f^\sharp) \to f_* \mathcal{O}_X^* \to \mathrm{Coker}(f^\sharp) \to 0$$

we first conclude that $H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \mathrm{Im}(f^\sharp))$ is surjective and then that $H^1(Y, \mathrm{Im}(f^\sharp)) \to H^1(Y, f_* \mathcal{O}_X^*)$ is surjective. Combining all the above we find that $H^1(Y, \mathcal{O}_Y^*) \to H^1(X, \mathcal{O}_X^*)$ is surjective as desired. □

09NE **Lemma 33.38.8.** *Let X be a scheme. Let $Z_1, \ldots, Z_n \subset X$ be closed subschemes. Let $\mathcal{L}_i$ be an invertible sheaf on Z_i. Assume that*

1. *X is reduced,*
2. *$X = \bigcup Z_i$ set theoretically, and*
3. *$Z_i \cap Z_j$ is a discrete topological space for $i \neq j$.*

Then there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to Z_i is $\mathcal{L}_i$. Moreover, if we are given sections $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ which are nonvanishing at the points of $Z_i \cap Z_j$, then we can choose $\mathcal{L}$ such that there exists a $s \in \Gamma(X, \mathcal{L})$ with $s|_{Z_i} = s_i$ for all i.

Proof. The existence of $\mathcal{L}$ can be deduced from Lemma 33.38.7 but we will also give a direct proof and we will use the direct proof to see the statement about sections is true. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is reduced we have

$$X \setminus T = \bigcup (Z_i \setminus T)$$

as schemes. Assumption (3) implies T is a discrete subset of X. Thus for each $t \in T$ we can find an open $U_t \subset X$ with $t \in U_t$ but $t' \notin U_t$ for $t' \in T$, $t' \neq t$.

By shrinking U_t if necessary, we may assume that there exist isomorphisms $\varphi_{t,i} : \mathcal{L}_i|_{U_t \cap Z_i} \to \mathcal{O}_{U_t \cap Z_i}$. Furthermore, for each i choose an open covering

$$Z_i \setminus T = \bigcup\nolimits_j U_{ij}$$

such that there exist isomorphisms $\varphi_{i,j} : \mathcal{L}_i|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Observe that

$$\mathcal{U} : X = \bigcup U_t \cup \bigcup U_{ij}$$

is an open covering of X. We claim that we can use the isomorphisms $\varphi_{t,i}$ and $\varphi_{i,j}$ to define a 2-cocycle with values in $\mathcal{O}_X^*$ for this covering that defines $\mathcal{L}$ as in the statement of the lemma.

Namely, if $i \neq i'$, then $U_{i,j} \cap U_{i',j'} = \emptyset$ and there is nothing to do. For $U_{i,j} \cap U_{i,j'}$ we have $\mathcal{O}_X(U_{i,j} \cap U_{i,j'}) = \mathcal{O}_{Z_i}(U_{i,j} \cap U_{i,j'})$ by the first remark of the proof. Thus the transition function for $\mathcal{L}_i$ (more precisely $\varphi_{i,j} \circ \varphi_{i,j'}^{-1}$) defines the value of our cocycle on this intersection. For $U_t \cap U_{i,j}$ we can do the same thing. Finally, for $t \neq t'$ we have

$$U_t \cap U_{t'} = \coprod (U_t \cap U_{t'}) \cap Z_i$$

and moreover the intersection $U_t \cap U_{t'} \cap Z_i$ is contained in $Z_i \setminus T$. Hence by the same reasoning as before we see that

$$\mathcal{O}_X(U_t \cap U_{t'}) = \prod \mathcal{O}_{Z_i}(U_t \cap U_{t'} \cap Z_i)$$

and we can use the transition functions for $\mathcal{L}_i$ (more precisely $\varphi_{t,i} \circ \varphi_{t',i}^{-1}$) to define the value of our cocycle on $U_t \cap U_{t'}$. This finishes the proof of existence of $\mathcal{L}$.

Given sections s_i as in the last assertion of the lemma, in the argument above, we choose U_t such that $s_i|_{U_t \cap Z_i}$ is nonvanishing and we choose $\varphi_{t,i}$ such that $\varphi_{t,i}(s_i|_{U_t \cap Z_i}) = 1$. Then using 1 over U_t and $\varphi_{i,j}(s_i|_{U_{i,j}})$ over $U_{i,j}$ will define a section of $\mathcal{L}$ which restricts to s_i over Z_i. □

09NW **Remark 33.38.9.** Let A be a reduced ring. Let I, J be ideals of A such that $V(I) \cup V(J) = \mathrm{Spec}(A)$. Set $B = A/J$. Then $I \to IB$ is an isomorphism of A-modules. Namely, we have $IB = I + J/J = I/(I \cap J)$ and $I \cap J$ is zero because A is reduced and $\mathrm{Spec}(A) = V(I) \cup V(J) = V(I \cap J)$. Thus for any projective A-module P we also have $IP = I(P/JP)$.

09NX **Lemma 33.38.10.** *Let X be a Noetherian reduced separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let Z_i, $i = 1, \ldots, n$ be the irreducible components of X. We view these as reduced closed subschemes of X. By Lemma 33.38.6 there exist ample invertible sheaves $\mathcal{L}_i$ on Z_i. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is Noetherian of dimension 1, the set T is finite and consists of closed points of X. For each i we may, possibly after replacing $\mathcal{L}_i$ by a power, choose $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ such that $(Z_i)_{s_i}$ is affine and contains $T \cap Z_i$, see Properties, Lemma 28.29.6.

By Lemma 33.38.8 we can find an invertible sheaf $\mathcal{L}$ on X and $s \in \Gamma(X, \mathcal{L})$ such that $(\mathcal{L}, s)|_{Z_i} = (\mathcal{L}_i, s_i)$. Observe that X_s contains T and is set theoretically equal to the affine closed subschemes $(Z_i)_{s_i}$. Thus it is affine by Limits, Lemma 32.11.3. To finish the proof, it suffices to find for every $x \in X$, $x \notin T$ an integer $m > 0$ and a section $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_t is affine and $x \in X_t$. Since $x \notin T$ we see that $x \in Z_i$ for some unique i, say $i = 1$. Let $Z \subset X$ be the reduced closed

subscheme whose underlying topological space is $Z_2 \cup \ldots \cup Z_n$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Denote that $\mathcal{I}_1 \subset \mathcal{O}_{Z_1}$ the inverse image of this ideal sheaf under the inclusion morphism $Z_1 \to X$. Observe that

$$\Gamma(X, \mathcal{I}\mathcal{L}^{\otimes m}) = \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$$

see Remark 33.38.9. Thus it suffices to find $m > 0$ and $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ affine. Since $\mathcal{L}_1$ is ample and since x is not in $Z_1 \cap T = V(\mathcal{I}_1)$ we can find a section $t_1 \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m_1})$ with $x \in (Z_1)_{t_1}$, see Properties, Proposition 28.26.13. Since $\mathcal{L}_1$ is ample we can find a section $t_2 \in \Gamma(Z_1, \mathcal{L}_1^{\otimes m_2})$ with $x \in (Z_1)_{t_2}$ and $(Z_1)_{t_2}$ affine, see Properties, Definition 28.26.1. Set $m = m_1 + m_2$ and $t = t_1t_2$. Then $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ by construction and $(Z_1)_t$ is affine by Properties, Lemma 28.26.4. □

09NY **Lemma 33.38.11.** *Let $i : Z \to X$ be a closed immersion of schemes. If the underlying topological space of X is Noetherian and $\dim(X) \leq 1$, then $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is surjective.*

Proof. Consider the short exact sequence

$$0 \to (1 + \mathcal{I}) \cap \mathcal{O}_X^* \to \mathcal{O}_X^* \to i_*\mathcal{O}_Z^* \to 0$$

of sheaves of abelian groups on X where $\mathcal{I}$ is the quasi-coherent sheaf of ideals corresponding to Z. Since $\dim(X) \leq 1$ we see that $H^2(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$, see Cohomology, Proposition 20.20.7. Hence the map $H^1(X, \mathcal{O}_X^*) \to H^1(X, i_*\mathcal{O}_Z^*)$ is surjective. By Cohomology, Lemma 20.20.1 we have $H^1(X, i_*\mathcal{O}_Z^*) = H^1(Z, \mathcal{O}_Z^*)$. This proves the lemma by Cohomology, Lemma 20.6.1. □

09NZ **Proposition 33.38.12.** *Let X be a Noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let $Z \subset X$ be the reduction of X. By Lemma 33.38.10 the scheme Z has an ample invertible sheaf. Thus by Lemma 33.38.11 there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X whose restriction to Z is ample. Then $\mathcal{L}$ is ample by an application of Cohomology of Schemes, Lemma 30.17.5. □

09P0 **Remark 33.38.13.** In fact, if X is a scheme whose reduction is a Noetherian separated scheme of dimension 1, then X has an ample invertible sheaf. The argument to prove this is the same as the proof of Proposition 33.38.12 except one uses Limits, Lemma 32.11.4 instead of Cohomology of Schemes, Lemma 30.17.5.

The following lemma actually holds for quasi-finite separated morphisms as the reader can see by using Zariski's main theorem (More on Morphisms, Lemma 37.43.3) and Lemma 33.38.3.

0C0U **Lemma 33.38.14.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y is Noetherian of dimension ≤ 1, f is finite, and there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is a closed immersion. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.*

Proof. We factor f as $X \to Z \to Y$ where Z is the scheme theoretic image of f. Then $X \to Z$ is an isomorphism over $V \cap Z$ and Lemma 33.38.7 applies. On the other hand, Lemma 33.38.11 applies to $Z \to Y$. Some details omitted. □

33.39. The delta invariant

0C3Q In this section we define the δ-invariant of a singular point on a reduced 1-dimensional Nagata scheme.

0C3R **Lemma 33.39.1.** *Let $(A, \mathfrak{m})$ be a Noetherian 1-dimensional local ring. Let $f \in \mathfrak{m}$. The following are equivalent*

(1) $\mathfrak{m} = \sqrt{(f)}$,
(2) *f is not contained in any minimal prime of A, and*
(3) *$A_f = \prod_{\mathfrak{p}\ minimal} A_\mathfrak{p}$ as A-algebras.*

Such an $f \in \mathfrak{m}$ exists. If $\text{depth}(A) = 1$ (for example A is reduced), then (1) – (3) are also equivalent to

(4) *f is a nonzerodivisor,*
(5) *A_f is the total ring of fractions of A.*

If A is reduced, then (1) – (5) are also equivalent to

(6) *A_f is the product of the residue fields at the minimal primes of A.*

Proof. The spectrum of A has finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ besides $\mathfrak{m}$ and these are all minimal, see Algebra, Lemma 10.31.6. Then the equivalence of (1) and (2) follows from Algebra, Lemma 10.17.2. Clearly, (3) implies (2). Conversely, if (2) is true, then the spectrum of A_f is the subset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of $\text{Spec}(A)$ with induced topology, see Algebra, Lemma 10.17.5. This is a finite discrete topological space. Hence $A_f = \prod_{\mathfrak{p}\ \text{minimal}} A_\mathfrak{p}$ by Algebra, Proposition 10.60.7. The existence of an f is asserted in Algebra, Lemma 10.60.8.

Assume A has depth 1. (This is the maximum by Algebra, Lemma 10.72.3 and holds if A is reduced by Algebra, Lemma 10.157.3.) Then $\mathfrak{m}$ is not an associated prime of A. Every minimal prime of A is an associated prime (Algebra, Proposition 10.63.6). Hence the set of nonzerodivisors of A is exactly the set of elements not contained in any of the minimal primes by Algebra, Lemma 10.63.9. Thus (4) is equivalent to (2). Part (5) is equivalent to (3) by Algebra, Lemma 10.25.4.

Then $A_\mathfrak{p}$ is a field for $\mathfrak{p} \subset A$ minimal, see Algebra, Lemma 10.25.1. Hence (3) is equivalent to (6). □

0C3S **Lemma 33.39.2.** *Let $(A, \mathfrak{m})$ be a reduced Nagata 1-dimensional local ring. Let A' be the integral closure of A in the total ring of fractions of A. Then A' is a normal Nagata ring, $A \to A'$ is finite, and A'/A has finite length as an A-module.*

Proof. The total ring of fractions is essentially of finite type over A hence $A \to A'$ is finite because A is Nagata, see Algebra, Lemma 10.162.2. The ring A' is normal for example by Algebra, Lemma 10.37.16 and 10.31.6. The ring A' is Nagata for example by Algebra, Lemma 10.162.5. Choose $f \in \mathfrak{m}$ as in Lemma 33.39.1. As $A' \subset A_f$ it is clear that $A_f = A'_f$. Hence the support of the finite A-module A'/A is contained in $\{\mathfrak{m}\}$. It follows that it has finite length by Algebra, Lemma 10.62.3. □

0C3T **Definition 33.39.3.** Let A be a reduced Nagata local ring of dimension 1. The *δ-invariant of A* is $\text{length}_A(A'/A)$ where A' is as in Lemma 33.39.2.

We prove some lemmas about the behaviour of this invariant.

0C3U **Lemma 33.39.4.** *Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is 0 if and only if A is a discrete valuation ring.*

Proof. If A is a discrete valuation ring, then A is normal and the ring A' is equal to A. Conversely, if the δ-invariant of A is 0, then A is integrally closed in its total ring of fractions which implies that A is normal (Algebra, Lemma 10.37.16) and this forces A to be a discrete valuation ring by Algebra, Lemma 10.119.7. □

0C3V **Lemma 33.39.5.** *Let A be a reduced Nagata local ring of dimension 1. Let $A \to A'$ be as in Lemma 33.39.2. Let A^h, A^{sh}, resp. $A^\wedge$ be the henselization, strict henselization, resp. completion of A. Then A^h, A^{sh}, resp. $A^\wedge$ is a reduced Nagata local ring of dimension 1 and $A' \otimes_A A^h$, $A' \otimes_A A^{sh}$, resp. $A' \otimes_A A^\wedge$ is the integral closure of A^h, A^{sh}, resp. $A^\wedge$ in its total ring of fractions.*

Proof. Observe that $A^\wedge$ is reduced, see More on Algebra, Lemma 15.43.6. The rings A^h and A^{sh} are reduced by More on Algebra, Lemma 15.45.4. The dimensions of A, A^h, A^{sh}, and $A^\wedge$ are the same by More on Algebra, Lemmas 15.43.1 and 15.45.7.

Recall that a Noetherian local ring is Nagata if and only if the formal fibres of A are geometrically reduced, see More on Algebra, Lemma 15.52.4. This property is inherited by A^h and A^{sh}, see the material in More on Algebra, Section 15.51 and especially Lemma 15.51.8. The completion is Nagata by Algebra, Lemma 10.162.8.

Now we come to the statement on integral closures. Before continuing let us pick $f \in \mathfrak{m}$ as in Lemma 33.39.1. Then the image of f in A^h, A^{sh}, and $A^\wedge$ clearly is an element satisfying properties (1) – (6) in that ring.

Since $A \to A'$ is finite we see that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ is the product of henselian local rings finite over A^h and A^{sh}, see Algebra, Lemma 10.153.4. Each of these local rings is the henselization of A' at a maximal ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$, see Algebra, Lemma 10.156.1 or 10.156.3. Hence these local rings are normal domains by More on Algebra, Lemma 15.45.6. It follows that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are normal rings. Since $A^h \to A' \otimes_A A^h$ and $A^{sh} \to A' \otimes_A A^{sh}$ are finite (hence integral) and since $A' \otimes_A A^h \subset (A^h)_f = Q(A^h)$ and $A' \otimes_A A^{sh} \subset (A^{sh})_f = Q(A^{sh})$ we conclude that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are the desired integral closures.

For the completion we argue in entirely the same manner. First, by Algebra, Lemma 10.97.8 we have

$$A' \otimes_A A^\wedge = (A')^\wedge = \prod (A'_{\mathfrak{m}'})^\wedge$$

The local rings $A'_{\mathfrak{m}'}$ are normal and have dimension 1 (by Algebra, Lemma 10.113.2 for example or the discussion in Algebra, Section 10.112). Thus $A'_{\mathfrak{m}'}$ is a discrete valuation ring, see Algebra, Lemma 10.119.7. Hence $(A'_{\mathfrak{m}'})^\wedge$ is a discrete valuation ring by More on Algebra, Lemma 15.43.5. It follows that $A' \otimes_A A^\wedge$ is a normal ring and we can conclude in exactly the same manner as before. □

0C3W **Lemma 33.39.6.** *Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is the same as the δ-invariant of the henselization, strict henselization, or the completion of A.*

Proof. Let us do this in case of the completion $B = A^\wedge$; the other cases are proved in exactly the same manner. Let A', resp. B' be the integral closure of A, resp. B in its total ring of fractions. Then $B' = A' \otimes_A B$ by Lemma 33.39.5. Hence

$B'/B = (A'/A) \otimes_A B$. The equality now follows from Algebra, Lemma 10.52.13 and the fact that $B \otimes_A \kappa_A = \kappa_B$. □

0C1T **Definition 33.39.7.** Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point such that $\mathcal{O}_{X,x}$ is reduced and $\dim(\mathcal{O}_{X,x}) = 1$. The *$\delta$-invariant of X at x* is the δ-invariant of $\mathcal{O}_{X,x}$ as defined in Definition 33.39.3.

This makes sense because the local ring of a locally algebraic scheme is Nagata by Algebra, Proposition 10.162.16. Of course, more generally we can make this definition whenever $x \in X$ is a point of a scheme such that the local ring $\mathcal{O}_{X,x}$ is reduced, Nagata of dimension 1. It follows from Lemma 33.39.6 that the δ-invariant of X at x is

$$\delta\text{-invariant of } X \text{ at } x = \delta\text{-invariant of } \mathcal{O}_{X,x}^h = \delta\text{-invariant of } \mathcal{O}_{X,x}^\wedge$$

We conclude that the δ-invariant is an invariant of the complete local ring of the point.

0C3X **Lemma 33.39.8.** *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume X is geometrically reduced at x and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) = 1$. Then*

$$\text{δ-invariant of } X \text{ at } x \leq \text{δ-invariant of } Y \text{ at } y$$

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 33.6.2 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 33.39.2. Then

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is finite over B and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying (1) – (6) of Lemma 33.39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ playes the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. Let B'' be the integral closure of B in its total ring of fractions as in Lemma 33.39.2. Then $B' \subset B''$. Thus the δ-invariant of Y at y is $\text{length}_B(B''/B)$ and

$$\begin{aligned} \text{length}_B(B''/B) &\geq \text{length}_B(B'/B) \\ &= \text{length}_B((A'/A) \otimes_A B) \\ &= \text{length}_B(B/\mathfrak{m}_A B)\text{length}_A(A'/A) \end{aligned}$$

by Algebra, Lemma 10.52.13 since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$). Since $\text{length}_A(A'/A)$ is the δ-invariant of X at x and since $\text{length}_B(B/\mathfrak{m}_A B) \geq 1$ the lemma is proved. □

0C3Y **Lemma 33.39.9.** *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume assumptions (a), (b), (c) of Lemma 33.27.6 hold for $x \in X$ and that $\dim(\mathcal{O}_{Y,y}) = 1$. Then the δ-invariant of X at x is δ-invariant of Y at y.*

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 33.27.6 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 33.39.2. By Lemma 33.27.6 we see that $A' \otimes_k K$ is normal. Hence

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is normal, finite over B, and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying (1) – (6) of Lemma 33.39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ playes the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. It follows that $B \to B'$ is as in Lemma 33.39.2 for B. Thus we have to show that $\text{length}_A(A'/A) = \text{length}_B(B'/B) = \text{length}_B((A'/A) \otimes_A B)$. Since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$) and since $\mathfrak{m}_B = \mathfrak{m}_A B$ (because $B/\mathfrak{m}_A B$ is zero dimensional by the remarks above and a localization of $K \otimes_k \kappa(x)$ which is reduced as $\kappa(x)$ is separable over k) we conclude by Algebra, Lemma 10.52.13. □

33.40. The number of branches

0C3Z We have defined the number of branches of a scheme at a point in Properties, Section 28.15.

0C1S **Lemma 33.40.1.** *Let X be a scheme. Assume every quasi-compact open of X has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$.*

(1) *The number of branches of X at x is the number of inverse images of x in X^ν.*
(2) *The number of geometric branches of X at x is $\sum_{\nu(x^\nu)=x} [\kappa(x^\nu) : \kappa(x)]_s$.*

Proof. First note that the assumption on X exactly means that the normalization is defined, see Morphisms, Definition 29.54.1. Then the stalk $A' = (\nu_* \mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the total ring of fractions of A_{red}, see Morphisms, Lemma 29.54.4. Since ν is an integral morphism, we see that the points of X^ν lying over x correspond to the primes of A' lying over the maximal ideal $\mathfrak{m}$ of A. As $A \to A'$ is integral, this is the same thing as the maximal ideals of A' (Algebra, Lemmas 10.36.20 and 10.36.22). Thus the lemma now follows from its algebraic counterpart: More on Algebra, Lemma 15.107.7. □

0C40 **Lemma 33.40.2.** *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then the number of geometric branches of X at x is the number of geometric branches of X_K at y.*

Proof. Write $Y = X_K$ and let X^ν, resp. Y^ν be the normalization of X, resp. Y. Consider the commutative diagram

$$\begin{array}{ccccc} Y^\nu & \longrightarrow & X^\nu_K & \longrightarrow & X^\nu \\ \downarrow & & \downarrow{\scriptstyle \nu_K} & & \downarrow{\scriptstyle \nu} \\ Y & = & Y & \longrightarrow & X \end{array}$$

By Lemma 33.27.5 we see that the left top horizontal arrow is a universal homeomorphism. Hence it induces purely inseparable residue field extensions, see Morphisms, Lemmas 29.45.5 and 29.10.2. Thus the number of geometric branches of Y at y is $\sum_{\nu_K(y')=y} [\kappa(y') : \kappa(y)]_s$ by Lemma 33.40.1. Similarly $\sum_{\nu(x')=x} [\kappa(x') : \kappa(x)]_s$ is the number of geometric branches of X at x. Using Schemes, Lemma 26.17.5 our statement follows from the following algebra fact: given a field extension l/κ and an algebraic field extension m/κ, then

$$\sum\nolimits_{m \otimes_\kappa l \to m'} [m' : l']_s = [m : \kappa]_s$$

where the sum is over the quotient fields of $m \otimes_\kappa l$. One can prove this in an elementary way, or one can use Lemma 33.7.6 applied to

$$\mathrm{Spec}(m \otimes_\kappa l) \times_{\mathrm{Spec}(l)} \mathrm{Spec}(\bar{l}) = \mathrm{Spec}(m) \otimes_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\bar{l}) \longrightarrow \mathrm{Spec}(m) \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\overline{\kappa})$$

because one can interpret $[m : \kappa]_s$ as the number of connected components of the right hand side and the sum $\sum_{m \otimes_\kappa l \to m'} [m' : l']_s$ as the number of connected components of the left hand side. □

0C55 **Lemma 33.40.3.** *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then X is geometrically unibranch at x if and only if X_K is geometrically unibranch at y.*

Proof. Immediate from Lemma 33.40.2 and More on Algebra, Lemma 15.107.7. □

0C41 **Definition 33.40.4.** Let A and A_i, $1 \leq i \leq n$ be local rings. We say A *is a wedge of* $A_1, \ldots, A_n$ if there exist isomorphisms

$$\kappa_{A_1} \to \kappa_{A_2} \to \ldots \to \kappa_{A_n}$$

and A is isomorphic to the ring consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}.

If we are given a base ring Λ and A and A_i are Λ-algebras, then we require $\kappa_{A_i} \to \kappa_{A_{i+1}}$ to be a Λ-algebra isomorphisms and A to be isomorphic as a Λ-algebra to the Λ-algebra consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}. In particular, if $\Lambda = k$ is a field and the maps $k \to \kappa_{A_i}$ are isomorphisms, then there is a unique choice for the isomorphisms $\kappa_{A_i} \to \kappa_{A_{i+1}}$ and we often speak of *the wedge of* $A_1, \ldots, A_n$.

0C42 **Lemma 33.40.5.** *Let $(A, \mathfrak{m})$ be a strictly henselian 1-dimensional reduced Nagata local ring. Then*

$$\delta\text{-invariant of } A \geq \text{number of geometric branches of } A - 1$$

If equality holds, then A is a wedge of $n \geq 1$ strictly henselian discrete valuation rings.

Proof. The number of geometric branches is equal to the number of branches of A (immediate from More on Algebra, Definition 15.107.6). Let $A \to A'$ be as in Lemma 33.39.2. Observe that the number of branches of A is the number of maximal ideals of A', see More on Algebra, Lemma 15.107.7. There is a surjection

$$A'/A \longrightarrow \left(\prod\nolimits_{\mathfrak{m}'} \kappa(\mathfrak{m}') \right) / \kappa(\mathfrak{m})$$

Since $\dim_{\kappa(\mathfrak{m})} \prod \kappa(\mathfrak{m}')$ is $\geq$ the number of branches, the inequality is obvious.

If equality holds, then $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m})$ for all $\mathfrak{m}' \subset A'$ and the displayed arrow above is an isomorphism. Since A is henselian and $A \to A'$ is finite, we see that A' is a product of local henselian rings, see Algebra, Lemma 10.153.4. The factors are the local rings $A'_{\mathfrak{m}'}$ and as A' is normal, these factors are discrete valuation rings (Algebra, Lemma 10.119.7). Since the displayed arrow is an isomorphism we see that A is indeed the wedge of these local rings. □

0C43 **Lemma 33.40.6.** *Let $(A, \mathfrak{m})$ be a 1-dimensional reduced Nagata local ring. Then*

$$\delta\text{-invariant of } A \geq \text{number of geometric branches of } A - 1$$

Proof. We may replace A by the strict henselization of A without changing the δ-invariant (Lemma 33.39.6) and without changing the number of geometric branches of A (this is immediate from the definition, see More on Algebra, Definition 15.107.6). Thus we may assume A is strictly henselian and we may apply Lemma 33.40.5. □

33.41. Normalization of one dimensional schemes

0C44 The normalization morphism of a Noetherian scheme of dimension 1 has unexpectedly good properties by the Krull-Akizuki result.

0C45 **Lemma 33.41.1.** *Let X be a locally Noetherian scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Then*

(1) *ν is integral, surjective, and induces a bijection on irreducible components,*
(2) *there is a factorization $X^\nu \to X_{red} \to X$ and the morphism $X^\nu \to X_{red}$ is the normalization of X_{red},*
(3) *$X^\nu \to X_{red}$ is birational,*
(4) *for every closed point $x \in X$ the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red} = \mathcal{O}_{X_{red},x}$,*
(5) *the fibres of ν are finite and the residue field extensions are finite,*
(6) *X^ν is a disjoint union of integral normal Noetherian schemes and each affine open is the spectrum of a finite product of Dedekind domains.*

Proof. Many of the results are in fact general properties of the normalization morphism, see Morphisms, Lemmas 29.54.2, 29.54.4, 29.54.5, and 29.54.7. What is not clear is that the fibres are finite, that the induced residue field extensions are finite, and that X^ν locally looks like the spectrum of a Dedekind domain (and hence is Noetherian). To see this we may assume that $X = \mathrm{Spec}(A)$ is affine, Noetherian, dimension 1, and that A is reduced. Then we may use the description in Morphisms, Lemma 29.54.3 to reduce to the case where A is a Noetherian domain of dimension 1. In this case the desired properties follow from Krull-Akizuki in the form stated in Algebra, Lemma 10.120.18. □

Of course there is a variant of the following lemma in case X is not reduced.

0C1R **Lemma 33.41.2.** *Let X be a reduced Nagata scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Let $x \in X$ denote a closed point. Then*

(1) *$\nu : X^\nu \to X$ is finite, surjective, and birational,*
(2) *$\mathcal{O}_X \subset \nu_*\mathcal{O}_{X^\nu}$ and $\nu_*\mathcal{O}_{X^\nu}/\mathcal{O}_X$ is a direct sum of skyscraper sheaves $\mathcal{Q}_x$ in the singular points x of X,*
(3) *$A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in its total ring of fractions,*
(4) *$\mathcal{Q}_x = A'/A$ has finite length equal to the δ-invariant of X at x,*
(5) *A' is a semi-local ring which is a finite product of Dedekind domains,*
(6) *$A^\wedge$ is a reduced Noetherian complete local ring of dimension 1,*
(7) *$(A')^\wedge$ is the integral closure of $A^\wedge$ in its total ring of fractions,*
(8) *$(A')^\wedge$ is a finite product of complete discrete valuation rings, and*
(9) *$A'/A \cong (A')^\wedge/A^\wedge$.*

Proof. We may and will use all the results of Lemma 33.41.1. Finiteness of ν follows from Morphisms, Lemma 29.54.11. Since X is reduced, Nagata, of dimension 1, we see that the regular locus is a dense open $U \subset X$ by More on Algebra, Proposition 15.48.7. Since a regular scheme is normal, this shows that ν is an

isomorphism over U. Since $\dim(X) \leq 1$ this implies that ν is not an isomorphism over a discrete set of closed points $x \in X$. In particular we see that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X^\nu} \to \bigoplus\nolimits_{x \in X \setminus U} \mathcal{Q}_x \to 0$$

As we have the description of the stalks of $\nu_*\mathcal{O}_{X^\nu}$ by Lemma 33.41.1, we conclude that $Q_x = A'/A$ indeed has length equal to the δ-invariant of X at x. Note that $Q_x \neq 0$ exactly when x is a singular point for example by Lemma 33.39.4. The description of A' as a product of semi-local Dedekind domains follows from Lemma 33.41.1 as well. The relationship between A, A', and $(A')^\wedge$ we have see in Lemma 33.39.5 (and its proof). □

33.42. Finding affine opens

09NF We continue the discussion started in Properties, Section 28.29. It turns out that we can find affines containing a finite given set of codimension 1 points on a separated scheme. See Proposition 33.42.7.

We will improve on the following lemma in Descent, Lemma 35.25.4.

09NG **Lemma 33.42.1.** *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is separated,*
(2) *there is an open covering $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to Y$ is an open immersion, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Suppose that $y = f(x) = f(x')$. Pick a specialization $y_0 \rightsquigarrow y$ where y_0 is a generic point of an irreducible component of Y. Since f is locally on the source an isomorphism we can pick specializations $x_0 \rightsquigarrow x$ and $x_0' \rightsquigarrow x'$ mapping to $y_0 \rightsquigarrow y$. Note that $x_0, x_0' \in X^0$. Hence $x_0 = x_0'$ by assumption (3). As f is separated we conclude that $x = x'$. Thus f is an open immersion. □

09NH **Lemma 33.42.2.** *Let $X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. If*

(1) *$\mathcal{O}_{X,x} = \mathcal{O}_{S,s}$,*
(2) *S is reduced,*
(3) *$X \to S$ is of finite type, and*
(4) *S has finitely many irreducible components,*

then there exists an open neighbourhood U of x such that $f|_U$ is an open immersion.

Proof. We may remove the (finitely many) irreducible components of S which do not contain s. We may replace S by an affine open neighbourhood of s. We may replace X by an affine open neighbourhood of x. Say $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, resp. s. As A is a reduced and all of the minimal primes of A are contained in $\mathfrak{p}$ we see that $A \subset A_\mathfrak{p}$. As $X \to S$ is of finite type, B is of finite type over A. Let $b_1, \ldots, b_n \in B$ be elements which generate B over A Since $A_\mathfrak{p} = B_\mathfrak{q}$ we can find $f \in A$, $f \notin \mathfrak{p}$ and $a_i \in A$ such that b_i and a_i/f have the same image in $B_\mathfrak{q}$. Thus we can find $g \in B$, $g \notin \mathfrak{q}$ such that $g(fb_i - a_i) = 0$ in B. It follows that the image of $A_f \to B_{fg}$ contains the images of $b_1, \ldots, b_n$, in particular also the image

of g. Choose $n \geq 0$ and $f' \in A$ such that f'/f^n maps to the image of g in B_{fg}. Since $A_{\mathfrak{p}} = B_{\mathfrak{q}}$ we see that $f' \notin \mathfrak{p}$. We conclude that $A_{ff'} \to B_{fg}$ is surjective. Finally, as $A_{ff'} \subset A_{\mathfrak{p}} = B_{\mathfrak{q}}$ (see above) the map $A_{ff'} \to B_{fg}$ is injective, hence an isomorphism. □

09NI **Lemma 33.42.3.** *Let $f : T \to X$ be a morphism of schemes. Let X^0, resp. T^0 denote the sets of generic points of irreducible components. Let $t_1, \ldots, t_m \in T$ be a finite set of points with images $x_j = f(t_j)$. If*

(1) *T is affine,*
(2) *X is quasi-separated,*
(3) *X^0 is finite*
(4) *$f(T^0) \subset X^0$ and $f : T^0 \to X^0$ is injective, and*
(5) *$\mathcal{O}_{X,x_j} = \mathcal{O}_{T,t_j}$,*

then there exists an affine open of X containing $x_1, \ldots, x_r$.

Proof. Using Limits, Proposition 32.11.2 there is an immediate reduction to the case where X and T are reduced. Details omitted.

Assume X and T are reduced. We may write $T = \lim_{i \in I} T_i$ as a directed limit of schemes of finite presentation over X with affine transition morphisms, see Limits, Lemma 32.7.2. Pick $i \in I$ such that T_i is affine, see Limits, Lemma 32.4.13. Say $T_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(R)$. Let $R' \subset R$ be the image of $R_i \to R$. Then $T' = \mathrm{Spec}(R')$ is affine, reduced, of finite type over X, and $T \to T'$ dominant. For $j = 1, \ldots, r$ let $t'_j \in T'$ be the image of t_j. Consider the local ring maps

$$\mathcal{O}_{X,x_j} \to \mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$$

Denote $(T')^0$ the set of generic points of irreducible components of T'. Let $\xi \rightsquigarrow t'_j$ be a specialization with $\xi \in (T')^0$. As $T \to T'$ is dominant we can choose $\eta \in T^0$ mapping to ξ (warning: a priori we do not know that η specializes to t_j). Assumption (3) applied to η tells us that the image θ of ξ in X corresponds to a minimal prime of $\mathcal{O}_{X,x_j}$. Lifting ξ via the isomorphism of (5) we obtain a specialization $\eta' \rightsquigarrow t_j$ with $\eta' \in T^0$ mapping to $\theta \rightsquigarrow x_j$. The injectivity of (4) shows that $\eta = \eta'$. Thus every minimal prime of $\mathcal{O}_{T',t'_j}$ lies below a minimal prime of $\mathcal{O}_{T,t_j}$. We conclude that $\mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$ is injective, hence both maps above are isomorphisms.

By Lemma 33.42.2 there exists an open $U \subset T'$ containing all the points t'_j such that $U \to X$ is a local isomorphism as in Lemma 33.42.1. By that lemma we see that $U \to X$ is an open immersion. Finally, by Properties, Lemma 28.29.5 we can find an open $W \subset U \subset T'$ containing all the t'_j. The image of W in X is the desired affine open. □

09NJ **Lemma 33.42.4.** *Let X be an integral separated scheme. Let $x_1, \ldots, x_r \in X$ be a finite set of points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. Let K be the field of rational functions of X. Set $A_i = \mathcal{O}_{X,x_i}$. Then $A_i \subset K$ and K is the fraction field of A_i. Since X is separated, and $x_i \neq x_j$ there cannot be a valuation ring $\mathcal{O} \subset K$ dominating both A_i and A_j. Namely, considering

the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}) & \longrightarrow & \mathrm{Spec}(A_1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_2) & \longrightarrow & X \end{array}$$

and applying the valuative criterion of separatedness (Schemes, Lemma 26.22.1) we would get $x_i = x_j$. Thus we see by Lemma 33.37.3 that $A_i \otimes A_j \to K$ is surjective for all $i \neq j$. By Lemma 33.37.7 we see that $A = A_1 \cap \ldots \cap A_r$ is a Noetherian semi-local ring with exactly r maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ such that $A_i = A_{\mathfrak{m}_i}$. Moreover,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \ldots \cup \mathrm{Spec}(A_r)$$

is an open covering and the intersection of any two pieces of this covering is $\mathrm{Spec}(K)$. Thus the given morphisms $\mathrm{Spec}(A_i) \to X$ glue to a morphism of schemes

$$\mathrm{Spec}(A) \longrightarrow X$$

mapping $\mathfrak{m}_i$ to x_i and inducing isomorphisms of local rings. Thus the result follows from Lemma 33.42.3. □

09NK **Lemma 33.42.5.** *Let A be a ring, $I \subset A$ an ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ primes of A, and $\overline{f} \in A/I$ an element. If $I \not\subset \mathfrak{p}_i$ for all i, then there exists an $f \in A$, $f \notin \mathfrak{p}_i$ which maps to $\overline{f}$ in A/I.*

Proof. We may assume there are no inclusion relations among the $\mathfrak{p}_i$ (by removing the smaller primes). First pick any $f \in A$ lifting $\overline{f}$. Let S be the set $s \in \{1, \ldots, r\}$ such that $f \in \mathfrak{p}_s$. If S is empty we are done. If not, consider the ideal $J = I \prod_{i \notin S} \mathfrak{p}_i$. Note that J is not contained in $\mathfrak{p}_s$ for $s \in S$ because there are no inclusions among the $\mathfrak{p}_i$ and because I is not contained in any $\mathfrak{p}_i$. Hence we can choose $g \in J$, $g \notin \mathfrak{p}_s$ for $s \in S$ by Algebra, Lemma 10.15.2. Then $f + g$ is a solution to the problem posed by the lemma. □

09NM **Lemma 33.42.6.** *Let X be a scheme. Let $T \subset X$ be finite set of points. Assume*

1. *X has finitely many irreducible components $Z_1, \ldots, Z_t$, and*
2. *$Z_i \cap T$ is contained in an affine open of the reduced induced subscheme corresponding to Z_i.*

Then there exists an affine open subscheme of X containing T.

Proof. Using Limits, Proposition 32.11.2 there is an immediate reduction to the case where X is reduced. Details omitted. In the rest of the proof we endow every closed subset of X with the induced reduced closed subscheme structure.

We argue by induction that we can find an affine open $U \subset Z_1 \cup \ldots \cup Z_r$ containing $T \cap (Z_1 \cup \ldots \cup Z_r)$. For $r = 1$ this holds by assumption. Say $r > 1$ and let $U \subset Z_1 \cup \ldots \cup Z_{r-1}$ be an affine open containing $T \cap (Z_1 \cup \ldots \cup Z_{r-1})$. Let $V \subset X_r$ be an affine open containing $T \cap Z_r$ (exists by assumption). Then $U \cap V$ contains $T \cap (Z_1 \cup \ldots \cup Z_{r-1}) \cap Z_r$. Hence

$$\Delta = (U \cap Z_r) \setminus (U \cap V)$$

does not contain any element of T. Note that Δ is a closed subset of U. By prime avoidance (Algebra, Lemma 10.15.2), we can find a standard open U' of U containing $T \cap U$ and avoiding Δ, i.e., $U' \cap Z_r \subset U \cap V$. After replacing U by U' we may assume that $U \cap V$ is closed in U.

Using that by the same arguments as above also the set $\Delta' = (U \cap (Z_1 \cup \ldots \cup Z_{r-1})) \setminus (U \cap V)$ does not contain any element of T we find a $h \in \mathcal{O}(V)$ such that $D(h) \subset V$ contains $T \cap V$ and such that $U \cap D(h) \subset U \cap V$. Using that $U \cap V$ is closed in U we can use Lemma 33.42.5 to find an element $g \in \mathcal{O}(U)$ whose restriction to $U \cap V$ equals the restriction of h to $U \cap V$ and such that $T \cap U \subset D(g)$. Then we can replace U by $D(g)$ and V by $D(h)$ to reach the situation where $U \cap V$ is closed in both U and V. In this case the scheme $U \cup V$ is affine by Limits, Lemma 32.11.3. This proves the induction step and thereby the lemma. □

Here is a conclusion we can draw from the material above.

09NN **Proposition 33.42.7.** *Let X be a separated scheme such that every quasi-compact open has a finite number of irreducible components. Let $x_1, \ldots, x_r \in X$ be points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. We can replace X by a quasi-compact open containing $x_1, \ldots, x_r$ hence we may assume that X has finitely many irreducible components. By Lemma 33.42.6 we reduce to the case where X is integral. This case is Lemma 33.42.4. □

33.43. Curves

0A22 In the Stacks project we will use the following as our definition of a curve.

0A23 **Definition 33.43.1.** Let k be a field. A *curve* is a variety of dimension 1 over k.

Two standard examples of curves over k are the affine line $\mathbf{A}^1_k$ and the projective line $\mathbf{P}^1_k$. The scheme $X = \mathrm{Spec}(k[x, y]/(f))$ is a curve if and only if $f \in k[x, y]$ is irreducible.

Our definition of a curve has the same problems as our definition of a variety, see the discussion following Definition 33.3.1. Moreover, it means that every curve comes with a specified field of definition. For example $X = \mathrm{Spec}(\mathbf{C}[x])$ is a curve over $\mathbf{C}$ but we can also view it as a curve over $\mathbf{R}$. The scheme $\mathrm{Spec}(\mathbf{Z})$ isn't a curve, even though the schemes $\mathrm{Spec}(\mathbf{Z})$ and $\mathbf{A}^1_{\mathbf{F}_p}$ behave similarly in many respects.

0A24 **Lemma 33.43.2.** *Let X be a separated, irreducible scheme of dimension > 0 over a field k. Let $x \in X$ be a closed point. The open subscheme $X \setminus \{x\}$ is not proper over k.*

Proof. Since X is irreducible, $U = X \setminus \{x\}$ is not closed in X. In particular, the immersion $U \to X$ is not proper. By Morphisms, Lemma 29.41.7 (here we use X is separated), $U \to \mathrm{Spec}(k)$ is not proper either. □

0A25 **Lemma 33.43.3.** *Let X be a separated finite type scheme over a field k. If $\dim(X) \leq 1$ then X is H-quasi-projective over k.*

Proof. By Proposition 33.38.12 the scheme X has an ample invertible sheaf $\mathcal{L}$. By Morphisms, Lemma 29.39.3 we see that X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ over $\mathrm{Spec}(k)$. This is the definition of being H-quasi-projective over k, see Morphisms, Definition 29.40.1. □

0A26 **Lemma 33.43.4.** *Let X be a proper scheme over a field k. If $\dim(X) \leq 1$ then X is H-projective over k.*

Proof. By Lemma 33.43.3 we see that X is a locally closed subscheme of $\mathbf{P}^n_k$ for some field k. Since X is proper over k it follows that X is a closed subscheme of $\mathbf{P}^n_k$ (Morphisms, Lemma 29.41.7). □

0BXV **Lemma 33.43.5.** *Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ with the following properties*

(1) *$\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}^d_k$ for some d,*
(2) *$j(X) \subset \overline{X}$ is dense and scheme theoretically dense,*
(3) *$\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$.*

Proof. By Lemma 33.43.3 we may assume X is a locally closed subscheme of $\mathbf{P}^d_k$ for some d. Let $\overline{X} \subset \mathbf{P}^d_k$ be the scheme theoretic image of $X \to \mathbf{P}^d_k$, see Morphisms, Definition 29.6.2. The description in Morphisms, Lemma 29.7.7 gives properties (1) and (2). Then $\dim(X) = 1 \Rightarrow \dim(\overline{X}) = 1$ for example by looking at generic points, see Lemma 33.20.3. As $\overline{X}$ is Noetherian, it then follows that $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ is a finite set of closed points. □

0BXW **Lemma 33.43.6.** *Let X be a separated scheme of finite type over k. If X is reduced and $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ such that*

(1) *$\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}^d_k$ for some d,*
(2) *$j(X) \subset \overline{X}$ is dense and scheme theoretically dense,*
(3) *$\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$,*
(4) *the local rings $\mathcal{O}_{\overline{X}, x_i}$ are discrete valuation rings for $i = 1, \ldots, n$.*

Proof. Let $j : X \to \overline{X}$ be as in Lemma 33.43.5. Consider the normalization X' of $\overline{X}$ in X. By Lemma 33.27.3 the morphism $X' \to \overline{X}$ is finite. By Morphisms, Lemma 29.44.16 $X' \to \overline{X}$ is projective. By Morphisms, Lemma 29.43.16 we see that $X' \to \overline{X}$ is H-projective. By Morphisms, Lemma 29.43.7 we see that $X' \to \mathrm{Spec}(k)$ is H-projective. Let $\{x'_1, \ldots, x'_m\} \subset X'$ be the inverse image of $\{x_1, \ldots, x_n\} = \overline{X} \setminus X$. Then $\dim(\mathcal{O}_{X', x'_i}) = 1$ for all $1 \leq i \leq m$. Hence the local rings $\mathcal{O}_{X', x'}$ are discrete valuation rings by Morphisms, Lemma 29.53.16. Then $X \to X'$ and $\{x'_1, \ldots, x'_m\}$ is as desired. □

0GK5 **Lemma 33.43.7.** *Let X be a separated scheme of finite type over k with $\dim(X) \leq 1$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n & \xleftarrow{j} & Y_1 \amalg \ldots \amalg Y_n & \xrightarrow{\nu} & X_{k'} & \longrightarrow & X \\
 & \searrow & \downarrow & & \downarrow & & \downarrow f \\
 & & \mathrm{Spec}(k'_1) \amalg \ldots \amalg \mathrm{Spec}(k'_n) & \longrightarrow & \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k)
\end{array}$$

of schemes with the following properties:

(1) *k'/k is a finite purely inseparable extension of fields,*
(2) *ν is the normalization of $X_{k'}$,*
(3) *j is an open immersion with dense image,*
(4) *k'_i/k' is a finite separable extension for $i = 1, \ldots, n$,*
(5) *$\overline{Y}_i$ is smooth, projective, geometrically irreducible dimension ≤ 1 over k'_i.*

Proof. As we may replace X by its reduction, we may and do assume X is reduced. Choose $X \to \overline{X}$ as in Lemma 33.43.6. If we can show the lemma for $\overline{X}$, then the lemma follows for X (details omitted). Thus we may and do assume X is projective.

Choose k'/k finite purely inseparable such that the normalization of $X_{k'}$ is geometrically normal over k', see Lemma 33.27.4. Denote $Y = (X_{k'})^\nu$ the normalization; for properties of the normalization, see Section 33.27. Then Y is geometrically regular as normal and regular are the same in dimension ≤ 1, see Properties, Lemma 28.12.6. Hence Y is smooth over k' by Lemma 33.12.6. Let $Y = Y_1 \amalg \ldots \amalg Y_n$ be the decomposition of Y into irreducible components. Set $k'_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$. These are finite separable extensions of k' by Lemma 33.9.3. The proof is finished by Lemma 33.9.4. □

0B8Y **Lemma 33.43.8.** *Let k be a field. Let X be a curve over k. Let $x \in X$ be a closed point. We think of x as a (reduced) closed subscheme of X with sheaf of ideals $\mathcal{I}$. The following are equivalent*

(1) *$\mathcal{O}_{X,x}$ is regular,*
(2) *$\mathcal{O}_{X,x}$ is normal,*
(3) *$\mathcal{O}_{X,x}$ is a discrete valuation ring,*
(4) *$\mathcal{I}$ is an invertible $\mathcal{O}_X$-module,*
(5) *x is an effective Cartier divisor on X.*

If k is perfect or if $\kappa(x)$ is separable over k, these are also equivalent to

(6) *$X \to \mathrm{Spec}(k)$ is smooth at x.*

Proof. Since X is a curve, the local ring $\mathcal{O}_{X,x}$ is a Noetherian local domain of dimension 1 (Lemma 33.20.3). Parts (4) and (5) are equivalent by definition and are equivalent to $\mathcal{I}_x = \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ having one generator (Divisors, Lemma 31.15.2). The equivalence of (1), (2), (3), (4), and (5) therefore follows from Algebra, Lemma 10.119.7. The final statement follows from Lemma 33.25.8 in case k is perfect. If $\kappa(x)/k$ is separable, then the equivalence follows from Algebra, Lemma 10.140.5. □

0H1F **Remark 33.43.9.** Let k be a field. Let X be a regular curve over k. By Lemmas 33.43.8 and 33.43.6 there exists a nonsingular projective curve $\overline{X}$ which is a compactification of X, i.e., there exists an open immersion $j : X \to \overline{X}$ such that the complement consists of a finite number of closed points. If k is perfect, then X and $\overline{X}$ are smooth over k and $\overline{X}$ is a smooth projective compactification of X.

Observe that if an affine scheme X over k is proper over k then X is finite over k (Morphisms, Lemma 29.44.11) and hence has dimension 0 (Algebra, Lemma 10.53.2 and Proposition 10.60.7). Hence a scheme of dimension > 0 over k cannot be both affine and proper over k. Thus the possibilities in the following lemma are mutually exclusive.

0A27 **Lemma 33.43.10.** *Let X be a curve over k. Then either X is an affine scheme or X is H-projective over k.*

Proof. Choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_r\}$ as in Lemma 33.43.6. Then $\overline{X}$ is a curve as well. If $r = 0$, then $X = \overline{X}$ is H-projective over k. Thus we may assume $r \geq 1$ and our goal is to show that X is affine. By Lemma 33.38.2 it suffices to show that $\overline{X} \setminus \{x_1\}$ is affine. This reduces us to the claim stated in the next paragraph.

Let X be an H-projective curve over k. Let $x \in X$ be a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring. Claim: $U = X \setminus \{x\}$ is affine. By Lemma 33.43.8 the point x defines an effective Cartier divisor of X. For $n \geq 1$ denote $nx = x + \ldots + x$

the n-fold sum, see Divisors, Definition 31.13.6. Denote $\mathcal{O}_{nx}$ the structure sheaf of nx viewed as a coherent module on X. Since every invertible module on the local scheme nx is trivial the first short exact sequence of Divisors, Remark 31.14.11 reads

$$0 \to \mathcal{O}_X \xrightarrow{1} \mathcal{O}_X(nx) \to \mathcal{O}_{nx} \to 0$$

in our case. Note that $\dim_k H^0(X, \mathcal{O}_{nx}) \geq n$. Namely, by Lemma 33.33.3 we have $H^0(X, \mathcal{O}_{nx}) = \mathcal{O}_{X,x}/(\pi^n)$ where π in $\mathcal{O}_{X,x}$ is a uniformizer and the powers π^i map to k-linearly independent elements in $\mathcal{O}_{X,x}/(\pi^n)$ for $i = 0, 1, \ldots, n-1$. We have $\dim_k H^1(X, \mathcal{O}_X) < \infty$ by Cohomology of Schemes, Lemma 30.19.2. If $n > \dim_k H^1(X, \mathcal{O}_X)$ we conclude from the long exact cohomology sequence that there exists an $s \in \Gamma(X, \mathcal{O}_X(nx))$ which is not a section of $\mathcal{O}_X$. If we take n minimal with this property, then s will map to a generator of the stalk $(\mathcal{O}_X(nx))_x$ since otherwise it would define a section of $\mathcal{O}_X((n-1)x) \subset \mathcal{O}_X(nx)$. For this n we conclude that $s_0 = 1$ and $s_1 = s$ generate the invertible module $\mathcal{L} = \mathcal{O}_X(nx)$.

Consider the corresponding morphism $f = \varphi_{\mathcal{L},(s_0,s_1)} : X \to \mathbf{P}^1_k$ of Constructions, Section 27.13. Observe that the inverse image of $D_+(T_0)$ is $U = X \setminus \{x\}$ as the section s_0 of $\mathcal{L}$ only vanishes at x. In particular, f is non-constant, i.e., $\mathrm{Im}(f)$ has more than one point. Hence f must map the generic point η of X to the generic point of $\mathbf{P}^1_k$. Hence if $y \in \mathbf{P}^1_k$ is a closed point, then $f^{-1}(\{y\})$ is a closed set of X not containing η, hence finite. Finally, f is proper[4]. By Cohomology of Schemes, Lemma 30.21.2[5] we conclude that f is finite. Hence $U = f^{-1}(D_+(T_0))$ is affine. □

The following lemma combined with Lemma 33.43.2 tells us that given a separated scheme X of dimension 1 and of finite type over k, then $X \setminus Z$ is affine, whenever the closed subset Z meets every irreducible component of X.

0A28 **Lemma 33.43.11.** *Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$ and no irreducible component of X is proper of dimension 1, then X is affine.*

Proof. Let $X = \bigcup X_i$ be the decomposition of X into irreducible components. We think of X_i as an integral scheme (using the reduced induced scheme structure, see Schemes, Definition 26.12.5). In particular X_i is a singleton (hence affine) or a curve hence affine by Lemma 33.43.10. Then $\coprod X_i \to X$ is finite surjective and $\coprod X_i$ is affine. Thus we see that X is affine by Cohomology of Schemes, Lemma 30.13.3. □

33.44. Degrees on curves

0AYQ We start defining the degree of an invertible sheaf and more generally a locally free sheaf on a proper scheme of dimension 1 over a field. In Section 33.33 we defined

[4]Namely, a H-projective variety is a proper variety by Morphisms, Lemma 29.43.13. A morphism of varieties whose source is a proper variety is a proper morphism by Morphisms, Lemma 29.41.7.

[5]One can avoid using this lemma which relies on the theorem of formal functions. Namely, X is projective hence it suffices to show a proper morphism $f : X \to Y$ with finite fibres between quasi-projective schemes over k is finite. To do this, one chooses an affine open of X containing the fibre of f over a point y using that any finite set of points of a quasi-projective scheme over k is contained in an affine. Shrinking Y to a small affine neighbourhood of y one reduces to the case of a proper morphism between affines. Such a morphism is finite by Morphisms, Lemma 29.44.7.

the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a proper scheme X over a field k by the formula

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

0AYR **Definition 33.44.1.** Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The *degree* of $\mathcal{L}$ is defined by

$$\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$$

More generally, if $\mathcal{E}$ is a locally free sheaf of rank n we define the *degree* of $\mathcal{E}$ by

$$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - n\chi(X, \mathcal{O}_X)$$

Observe that this depends on the triple $\mathcal{E}/X/k$. If X is disconnected and $\mathcal{E}$ is finite locally free (but not of constant rank), then one can modify the definition by summing the degrees of the restriction of $\mathcal{E}$ to the connected components of X. If $\mathcal{E}$ is just a coherent sheaf, there are several different ways of extending the definition[6]. In a series of lemmas we show that this definition has all the properties one expects of the degree.

0B59 **Lemma 33.44.2.** *Let k'/k be an extension of fields. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank n. Then the degree of $\mathcal{E}/X/k$ is equal to the degree of $\mathcal{E}_{k'}/X_{k'}/k'$.*

Proof. More precisely, set $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Let $\mathcal{E}_{k'} = p^*\mathcal{E}$ where $p : X_{k'} \to X$ is the projection. By Cohomology of Schemes, Lemma 30.5.2 we have $H^i(X_{k'}, \mathcal{E}_{k'}) = H^i(X, \mathcal{E}) \otimes_k k'$ and $H^i(X_{k'}, \mathcal{O}_{X_{k'}}) = H^i(X, \mathcal{O}_X) \otimes_k k'$. Hence we see that the Euler characteristics are unchanged, hence the degree is unchanged. □

0AYS **Lemma 33.44.3.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of locally free $\mathcal{O}_X$-modules each of finite constant rank. Then*

$$\deg(\mathcal{E}_2) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_3)$$

Proof. Follows immediately from additivity of Euler characteristics (Lemma 33.33.2) and additivity of ranks. □

0AYU **Lemma 33.44.4.** *Let k be a field. Let $f : X' \to X$ be a birational morphism of proper schemes of dimension ≤ 1 over k. Then*

$$\deg(f^*\mathcal{E}) = \deg(\mathcal{E})$$

for every finite locally free sheaf of constant rank. More generally it suffices if f induces a bijection between irreducible components of dimension 1 and isomorphisms of local rings at the corresponding generic points.

Proof. The morphism f is proper (Morphisms, Lemma 29.41.7) and has fibres of dimension ≤ 0. Hence f is finite (Cohomology of Schemes, Lemma 30.21.2). Thus

$$Rf_*f^*\mathcal{E} = f_*f^*\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_{X'}$$

Since f induces an isomorphism on local rings at generic points of all irreducible components of dimension 1 we see that the kernel and cokernel

$$0 \to \mathcal{K} \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

[6] If X is a proper curve and $\mathcal{F}$ is a coherent sheaf on X, then one often defines the degree as $\chi(X, \mathcal{F}) - r\chi(X, \mathcal{O}_X)$ where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$ at the generic point ξ of X.

have supports of dimension ≤ 0. Note that tensoring this with $\mathcal{E}$ is still an exact sequence as $\mathcal{E}$ is locally free. We obtain

$$\begin{aligned}\chi(X, \mathcal{E}) - \chi(X', f^*\mathcal{E}) &= \chi(X, \mathcal{E}) - \chi(X, f_*f^*\mathcal{E}) \\ &= \chi(X, \mathcal{E}) - \chi(X, \mathcal{E} \otimes f_*\mathcal{O}_{X'}) \\ &= \chi(X, \mathcal{K} \otimes \mathcal{E}) - \chi(X, \mathcal{Q} \otimes \mathcal{E}) \\ &= n\chi(X, \mathcal{K}) - n\chi(X, \mathcal{Q}) \\ &= n\chi(X, \mathcal{O}_X) - n\chi(X, f_*\mathcal{O}_{X'}) \\ &= n\chi(X, \mathcal{O}_X) - n\chi(X', \mathcal{O}_{X'})\end{aligned}$$

which proves what we want. The first equality as f is finite, see Cohomology of Schemes, Lemma 30.2.4. The second equality by projection formula, see Cohomology, Lemma 20.54.2. The third by additivity of Euler characteristics, see Lemma 33.33.2. The fourth by Lemma 33.33.3. □

0AYV **Lemma 33.44.5.** *Let k be a field. Let X be a proper curve over k with generic point ξ. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

$$\chi(X, \mathcal{E} \otimes \mathcal{F}) = r \deg(\mathcal{E}) + n\chi(X, \mathcal{F})$$

where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$.

Proof. Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X expressing that the formula of the lemma holds. We claim that the assumptions (1) and (2) of Cohomology of Schemes, Lemma 30.12.6 hold for $\mathcal{P}$. Namely, (1) holds because the Euler characteristic and the rank r are additive in short exact sequences of coherent sheaves. And (2) holds too: If $Z = X$ then we may take $\mathcal{G} = \mathcal{O}_X$ and $\mathcal{P}(\mathcal{O}_X)$ is true by the definition of degree. If $i : Z \to X$ is the inclusion of a closed point we may take $\mathcal{G} = i_*\mathcal{O}_Z$ and $\mathcal{P}$ holds by Lemma 33.33.3 and the fact that $r = 0$ in this case. □

Let k be a field. Let X be a finite type scheme over k of dimension ≤ 1. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. We view C_i as a scheme by using the induced reduced scheme structure. Let $\xi_i \in C_i$ be the generic point. The *multiplicity of C_i in X* is defined as the length

$$m_i = \text{length}_{\mathcal{O}_{X,\xi_i}} \mathcal{O}_{X,\xi_i}$$

This makes sense because $\mathcal{O}_{X,\xi_i}$ is a zero dimensional Noetherian local ring and hence has finite length over itself (Algebra, Proposition 10.60.7). See Chow Homology, Section 42.9 for additional information. It turns out the degree of a locally free sheaf only depends on the restriction of the irreducible components.

0AYW **Lemma 33.44.6.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then*

$$\deg(\mathcal{E}) = \sum m_i \deg(\mathcal{E}|_{C_i})$$

where $C_i \subset X$, $i = 1, \ldots, t$ are the irreducible components of dimension 1 with reduced induced scheme structure and m_i is the multiplicity of C_i in X.

Proof. Observe that the statement makes sense because $C_i \to \text{Spec}(k)$ is proper of dimension 1 (Morphisms, Lemmas 29.41.6 and 29.41.4). Consider the open subscheme $U_i = X \setminus (\bigcup_{j \neq i} C_j)$ and let $X_i \subset X$ be the scheme theoretic closure

of U_i. Note that $X_i \cap U_i = U_i$ (scheme theoretically) and that $X_i \cap U_j = \emptyset$ (set theoretically) for $i \neq j$; this follows from the description of scheme theoretic closure in Morphisms, Lemma 29.7.7. Thus we may apply Lemma 33.44.4 to the morphism $X' = \bigcup X_i \to X$. Since it is clear that $C_i \subset X_i$ (scheme theoretically) and that the multiplicity of C_i in X_i is equal to the multiplicity of C_i in X, we see that we reduce to the case discussed in the following paragraph.

Assume X is irreducible with generic point ξ. Let $C = X_{red}$ have multiplicity m. We have to show that $\deg(\mathcal{E}) = m \deg(\mathcal{E}|_C)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal defining the closed subscheme C. Let $e \geq 0$ be minimal such that $\mathcal{I}^{e+1} = 0$ (Cohomology of Schemes, Lemma 30.10.2). We argue by induction on e. If $e = 0$, then $X = C$ and the result is immediate. Otherwise we set $\mathcal{F} = \mathcal{I}^e$ viewed as a coherent $\mathcal{O}_C$-module (Cohomology of Schemes, Lemma 30.9.8). Let $X' \subset X$ be the closed subscheme cut out by the coherent ideal $\mathcal{I}^e$ and let m' be the multiplicity of C in X'. Taking stalks at ξ of the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

we find (use Algebra, Lemmas 10.52.3, 10.52.6, and 10.52.5) that

$$m = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{X,\xi} = \dim_{\kappa(\xi)} \mathcal{F}_\xi + \text{length}_{\mathcal{O}_{X',\xi}} \mathcal{O}_{X',\xi} = r + m'$$

where r is the rank of $\mathcal{F}$ as a coherent sheaf on C. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E}|_C \otimes \mathcal{F} \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_{X'} \to 0$$

By induction we have $\deg(\mathcal{E}|_{X'}) = m' \deg(\mathcal{E}|_C)$. By Lemma 33.44.5 we have $\chi(\mathcal{E}|_C \otimes \mathcal{F}) = r \deg(\mathcal{E}|_C) + n\chi(\mathcal{F})$. Putting everything together we obtain the result. □

0AYX **Lemma 33.44.7.** *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$, $\mathcal{V}$ be locally free $\mathcal{O}_X$-modules of constant finite rank. Then*

$$\deg(\mathcal{E} \otimes \mathcal{V}) = \mathit{rank}(\mathcal{E}) \deg(\mathcal{V}) + \mathit{rank}(\mathcal{V}) \deg(\mathcal{E})$$

Proof. By Lemma 33.44.6 and elementary arithmetic, we reduce to the case of a proper curve. This case follows from Lemma 33.44.5. □

0DJ5 **Lemma 33.44.8.** *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then*

$$\deg(\mathcal{E}) = \deg(\wedge^n(\mathcal{E})) = \deg(\det(\mathcal{E}))$$

Proof. By Lemma 33.44.6 and elementary arithmetic, we reduce to the case of a proper curve. Then there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible modules, see Divisors, Lemma 31.36.1. By Lemma 33.44.4 we may work on X'. Thus we may assume we have a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

by locally free $\mathcal{O}_X$-modules with $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is invertible. By Modules, Lemma 17.26.1 and induction we find $\det(\mathcal{E}) = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n$. Thus the equality follows from Lemma 33.44.7 and additivity (Lemma 33.44.3). □

0AYY **Lemma 33.44.9.** *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k. Let D be an effective Cartier divisor on X. Then D is finite over $\mathrm{Spec}(k)$ of degree $\deg(D) = \dim_k \Gamma(D, \mathcal{O}_D)$. For a locally free sheaf $\mathcal{E}$ of rank n we have*

$$\deg(\mathcal{E}(D)) = n \deg(D) + \deg(\mathcal{E})$$

where $\mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

Proof. Since D is nowhere dense in X (Divisors, Lemma 31.13.4) we see that $\dim(D) \leq 0$. Hence D is finite over k by Lemma 33.20.2. Since k is a field, the morphism $D \to \mathrm{Spec}(k)$ is finite locally free and hence has a degree (Morphisms, Definition 29.48.1), which is clearly equal to $\dim_k \Gamma(D, \mathcal{O}_D)$ as stated in the lemma. By Divisors, Definition 31.14.1 there is a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_* i^* \mathcal{O}_X(D) \to 0$$

where $i : D \to X$ is the closed immersion. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(D) \to i_* i^* \mathcal{E}(D) \to 0$$

The equation of the lemma follows from additivity of the Euler characteristic (Lemma 33.33.2) and Lemma 33.33.3. □

0C6P **Lemma 33.44.10.** *Let k be a field. Let X be a proper scheme over k which is reduced and connected. Let $\kappa = H^0(X, \mathcal{O}_X)$. Then κ/k is a finite extension of fields and $w = [\kappa : k]$ divides*

(1) $\deg(\mathcal{E})$ *for all locally free $\mathcal{O}_X$-modules $\mathcal{E}$,*
(2) $[\kappa(x) : k]$ *for all closed points $x \in X$, and*
(3) $\deg(D)$ *for all closed subschemes $D \subset X$ of dimension zero.*

Proof. See Lemma 33.9.3 for the assertions about κ. For every quasi-coherent $\mathcal{O}_X$-module, the k-vector spaces $H^i(X, \mathcal{F})$ are κ-vector spaces. The divisibilities easily follow from this statement and the definitions. □

0AYZ **Lemma 33.44.11.** *Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of proper curves over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. Then*

$$\deg(f^*\mathcal{E}) = \deg(X/Y) \deg(\mathcal{E})$$

Proof. The degree of X over Y is defined in Morphisms, Definition 29.51.8. Thus $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module of rank $\deg(X/Y)$, i.e., $\deg(X/Y) = \dim_{\kappa(\xi)}(f_*\mathcal{O}_X)_\xi$ where ξ is the generic point of Y. Thus we obtain

$$\begin{aligned} \chi(X, f^*\mathcal{E}) &= \chi(Y, f_* f^* \mathcal{E}) \\ &= \chi(Y, \mathcal{E} \otimes f_*\mathcal{O}_X) \\ &= \deg(X/Y) \deg(\mathcal{E}) + n\chi(Y, f_*\mathcal{O}_X) \\ &= \deg(X/Y) \deg(\mathcal{E}) + n\chi(X, \mathcal{O}_X) \end{aligned}$$

as desired. The first equality as f is finite, see Cohomology of Schemes, Lemma 30.2.4. The second equality by projection formula, see Cohomology, Lemma 20.54.2. The third equality by Lemma 33.44.5. □

The following is a trivial but important consequence of the results on degrees above.

0B40 **Lemma 33.44.12.** *Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

(1) *If $\mathcal{L}$ has a nonzero section, then* $\deg(\mathcal{L}) \geq 0$.
(2) *If $\mathcal{L}$ has a nonzero section s which vanishes at a point, then* $\deg(\mathcal{L}) > 0$.
(3) *If $\mathcal{L}$ and $\mathcal{L}^{-1}$ have nonzero sections, then $\mathcal{L} \cong \mathcal{O}_X$.*
(4) *If $\deg(\mathcal{L}) \leq 0$ and $\mathcal{L}$ has a nonzero section, then $\mathcal{L} \cong \mathcal{O}_X$.*
(5) *If $\mathcal{N} \to \mathcal{L}$ is a nonzero map of invertible $\mathcal{O}_X$-modules, then* $\deg(\mathcal{L}) \geq \deg(\mathcal{N})$ *and if equality holds then it is an isomorphism.*

Proof. Let s be a nonzero section of $\mathcal{L}$. Since X is a curve, we see that s is a regular section. Hence there is an effective Cartier divisor $D \subset X$ and an isomorphism $\mathcal{L} \to \mathcal{O}_X(D)$ mapping s the canonical section 1 of $\mathcal{O}_X(D)$, see Divisors, Lemma 31.14.10. Then $\deg(\mathcal{L}) = \deg(D)$ by Lemma 33.44.9. As $\deg(D) \geq 0$ and $= 0$ if and only if $D = \emptyset$, this proves (1) and (2). In case (3) we see that $\deg(\mathcal{L}) = 0$ and $D = \emptyset$. Similarly for (4). To see (5) apply (1) and (4) to the invertible sheaf

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L})$$

which has degree $\deg(\mathcal{L}) - \deg(\mathcal{N})$ by Lemma 33.44.7. □

0E22 **Lemma 33.44.13.** *Let k be a field. Let X be a proper scheme over k which is reduced, connected, and equidimensional of dimension* 1. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}|_C) \leq 0$ for all irreducible components C of X, then either $H^0(X, \mathcal{L}) = 0$ or $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Let $s \in H^0(X, \mathcal{L})$ be nonzero. Since X is reduced there exists an irreducible component C of X with $s|_C \neq 0$. But if $s|_C$ is nonzero, then s is nonwhere vanishing on C by Lemma 33.44.12. This in turn implies s is nowhere vanishing on every irreducible component of X meeting C. Since X is connected, we conclude that s vanishes nowhere and the lemma follows. □

0B5X **Lemma 33.44.14.** *Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample if and only if* $\deg(\mathcal{L}) > 0$.

Proof. If $\mathcal{L}$ is ample, then there exists an $n > 0$ and a section $s \in H^0(X, \mathcal{L}^{\otimes n})$ with X_s affine. Since X isn't affine (otherwise by Morphisms, Lemma 29.44.11 X would be finite), we see that s vanishes at some point. Hence $\deg(\mathcal{L}^{\otimes n}) > 0$ by Lemma 33.44.12. By Lemma 33.44.7 we conclude that $\deg(\mathcal{L}) = 1/n \deg(\mathcal{L}^{\otimes n}) > 0$.

Assume $\deg(\mathcal{L}) > 0$. Then

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \geq \chi(X, \mathcal{L}^n) = n \deg(\mathcal{L}) + \chi(X, \mathcal{O}_X)$$

grows linearly with n. Hence for any finite collection of closed points $x_1, \ldots, x_t$ of X, we can find an n such that $\dim_k H^0(X, \mathcal{L}^{\otimes n}) > \sum \dim_k \kappa(x_i)$. (Recall that by Hilbert Nullstellensatz, the extension fields $\kappa(x_i)/k$ are finite, see for example Morphisms, Lemma 29.20.3). Hence we can find a nonzero $s \in H^0(X, \mathcal{L}^{\otimes n})$ vanishing in $x_1, \ldots, x_t$. In particular, if we choose $x_1, \ldots, x_t$ such that $X \setminus \{x_1, \ldots, x_t\}$ is affine, then X_s is affine too (for example by Properties, Lemma 28.26.4 although if we choose our finite set such that $\mathcal{L}|_{X \setminus \{x_1, \ldots, x_t\}}$ is trivial, then it is immediate). The conclusion is that we can find an $n > 0$ and a nonzero section $s \in H^0(X, \mathcal{L}^{\otimes n})$ such that X_s is affine.

We will show that for every quasi-coherent sheaf of ideals $\mathcal{I}$ there exists an $m > 0$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes m})$ is zero. This will finish the proof by Cohomology of Schemes, Lemma 30.17.1. To see this we consider the maps

$$\mathcal{I} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes 2n} \xrightarrow{s} \ldots$$

Since $\mathcal{I}$ is torsion free, these maps are injective and isomorphisms over X_s, hence the cokernels have vanishing H^1 (by Cohomology of Schemes, Lemma 30.9.10 for example). We conclude that the maps of vector spaces

$$H^1(X, \mathcal{I}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes 2n}) \to \ldots$$

are surjective. On the other hand, the dimension of $H^1(X, \mathcal{I})$ is finite, and every element maps to zero eventually by Cohomology of Schemes, Lemma 30.17.4. Thus for some $e > 0$ we see that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes en})$ is zero. This finishes the proof. □

0B5Y **Lemma 33.44.15.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. The following are equivalent:*

(1) *$\mathcal{L}$ is ample, and*
(2) *$\deg(\mathcal{L}|_{C_i}) > 0$ for $i = 1, \ldots, t$.*

Proof. Let $x_1, \ldots, x_r \in X$ be the isolated closed points. Think of $x_i = \mathrm{Spec}(\kappa(x_i))$ as a scheme. Consider the morphism of schemes

$$f : C_1 \amalg \ldots \amalg C_t \amalg x_1 \amalg \ldots \amalg x_r \longrightarrow X$$

This is a finite surjective morphism of schemes proper over k (details omitted). Thus $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample (Cohomology of Schemes, Lemma 30.17.2). Thus we conclude by Lemma 33.44.14. □

0B8Z **Lemma 33.44.16.** *Let k be an algebraically closed field. Let X be a proper curve over k. Then there exist*

(1) *an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\dim_k H^0(X, \mathcal{L}) = 1$ and $H^1(X, \mathcal{L}) = 0$, and*
(2) *an invertible $\mathcal{O}_X$-module $\mathcal{N}$ with $\dim_k H^0(X, \mathcal{N}) = 0$ and $H^1(X, \mathcal{N}) = 0$.*

Proof. Choose a closed immersion $i : X \to \mathbf{P}^n_k$ (Lemma 33.43.4). Setting $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n}(d)$ for $d \gg 0$ we see that there exists an invertible sheaf $\mathcal{L}$ with $H^0(X, \mathcal{L}) \neq 0$ and $H^1(X, \mathcal{L}) = 0$ (see Cohomology of Schemes, Lemma 30.17.1 for vanishing and the references therein for nonvanishing). We will finish the proof of (1) by descending induction on $t = \dim_k H^0(X, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Let $U \subset X$ be the nonempty open subset of nonsingular points studied in Lemma 33.25.8. Let $s \in H^0(X, \mathcal{L})$ be nonzero. There exists a closed point $x \in U$ such that s does not vanish in x. Let $\mathcal{I}$ be the ideal sheaf of $i : x \to X$ as in Lemma 33.43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{L} \to i_*i^*\mathcal{L} \to 0$$

Observe that $H^0(X, i_*i^*\mathcal{L}) = H^0(x, i^*\mathcal{L})$ has dimension 1 as x is a k-rational point (k is algebraically closed). Since s does not vanish at x we conclude that

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, i_*i^*\mathcal{L})$$

is surjective. Hence $\dim_k H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$ thereby finishing the proof of the induction step.

To get an invertible sheaf as in (2) take an invertible sheaf $\mathcal{L}$ as in (1) and do the argument in the previous paragraph one more time. □

0B90 **Lemma 33.44.17.** *Let k be an algebraically closed field. Let X be a proper curve over k. Set $g = \dim_k H^1(X, \mathcal{O}_X)$. For every invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}) \geq 2g - 1$ we have $H^1(X, \mathcal{L}) = 0$.*

Proof. Let $\mathcal{N}$ be the invertible module we found in Lemma 33.44.16 part (2). The degree of $\mathcal{N}$ is $\chi(X, \mathcal{N}) - \chi(X, \mathcal{O}_X) = 0 - (1 - g) = g - 1$. Hence the degree of $\mathcal{L} \otimes \mathcal{N}^{\otimes -1}$ is $\deg(\mathcal{L}) - (g - 1) \geq g$. Hence $\chi(X, \mathcal{L} \otimes \mathcal{N}^{\otimes -1}) \geq g + 1 - g = 1$. Thus there is a nonzero global section s whose zero scheme is an effective Cartier divisor D of degree $\deg(\mathcal{L}) - (g - 1)$. This gives a short exact sequence

$$0 \to \mathcal{N} \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0$$

where $i : D \to X$ is the inclusion morphism. We conclude that $H^0(X, \mathcal{L})$ maps isomorphically to $H^0(D, \mathcal{L}|_D)$ which has dimension $\deg(\mathcal{L}) - (g - 1)$. The result follows from the definition of degree. □

33.45. Numerical intersections

0BEL In this section we play around with the Euler characteristic of coherent sheaves on proper schemes to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

0BEM **Lemma 33.45.1.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map*

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the support of $\mathcal{F}$.

Proof. We prove this by induction on $\dim(\mathrm{Supp}(\mathcal{F}))$. If this number is zero, then the function is constant with value $\dim_k \Gamma(X, \mathcal{F})$ by Lemma 33.33.3. Assume $\dim(\mathrm{Supp}(\mathcal{F})) > 0$.

If $\mathcal{F}$ has embedded associated points, then we can consider the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$ constructed in Divisors, Lemma 31.4.6. Since the dimension of the support of $\mathcal{K}$ is strictly less, the result holds for $\mathcal{K}$ by induction hypothesis and with strictly smaller total degree. By additivity of the Euler characteristic (Lemma 33.33.2) it suffices to prove the result for $\mathcal{F}'$. Thus we may assume $\mathcal{F}$ does not have embedded associated points.

If $i : Z \to X$ is a closed immersion and $\mathcal{F} = i_*\mathcal{G}$, then we see that the result for X, $\mathcal{F}$, $\mathcal{L}_1, \ldots, \mathcal{L}_r$ is equivalent to the result for Z, $\mathcal{G}$, $i^*\mathcal{L}_1, \ldots, i^*\mathcal{L}_r$ (since the cohomologies agree, see Cohomology of Schemes, Lemma 30.2.4). Applying Divisors, Lemma 31.4.7 we may assume that X has no embedded components and $X = \mathrm{Supp}(\mathcal{F})$.

Pick a regular meromorphic section s of $\mathcal{L}_1$, see Divisors, Lemma 31.25.4. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s and consider the maps

$$\mathcal{I}\mathcal{F} \to \mathcal{F}, \quad \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes \mathcal{L}_1$$

of Divisors, Lemma 31.24.5. These are injective and have cokernels $\mathcal{Q}$, $\mathcal{Q}'$ supported on nowhere dense closed subschemes of $X = \mathrm{Supp}(\mathcal{F})$. Tensoring with the invertible

module $\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$ is exact, hence using additivity again we see that

$$\begin{aligned}&\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1+1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) \\ &= \chi(\mathcal{Q} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(\mathcal{Q}' \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})\end{aligned}$$

Thus we see that the function $P(n_1, \ldots, n_r)$ of the lemma has the property that

$$P(n_1 + 1, n_2, \ldots, n_r) - P(n_1, \ldots, n_r)$$

is a numerical polynomial of total degree $<$ the dimension of the support of $\mathcal{F}$. Of course by symmetry the same thing is true for

$$P(n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_r) - P(n_1, \ldots, n_r)$$

for any $i \in \{1, \ldots, r\}$. A simple arithmetic argument shows that P is a numerical polynomial of total degree at most $\dim(\text{Supp}(\mathcal{F}))$. □

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

0BEN **Lemma 33.45.2.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(\text{Supp}(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $\text{Supp}(\mathcal{F})$ of dimension d. Let $\xi_i \in Z_i$ be the generic point and set $m_i = \text{length}_{\mathcal{O}_{X,\xi_i}}(\mathcal{F}_{\xi_i})$. Then*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. Consider pairs (ξ, Z) where $Z \subset X$ is an integral closed subscheme of dimension d and ξ is its generic point. Then the finite $\mathcal{O}_{X,\xi}$-module $\mathcal{F}_\xi$ has support contained in $\{\xi\}$ hence the length $m_Z = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi)$ is finite (Algebra, Lemma 10.62.3) and zero unless $Z = Z_i$ for some i. Thus the expression of the lemma can be written as

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum m_Z \ \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_Z)$$

where the sum is over integral closed subschemes $Z \subset X$ of dimension d. The assignment $\mathcal{F} \mapsto E(\mathcal{F})$ is additive in short exact sequences $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_X$-modules whose support has dimension $\leq d$. This follows from additivity of Euler characteristics (Lemma 33.33.2) and additivity of lengths (Algebra, Lemma 10.52.3). Let us apply Cohomology of Schemes, Lemma 30.12.3 to find a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $V_j \subset X$ and a nonzero sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{V_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (V_j \to X)_*\mathcal{I}_j$$

It follows that $V_j \subset \text{Supp}(\mathcal{F})$ and hence $\dim(V_j) \leq d$. By the additivity we remarked upon above it suffices to prove the result for each of the subquotients $\mathcal{F}_j/\mathcal{F}_{j-1}$. Thus it suffices to prove the result when $\mathcal{F} = (V \to X)_*\mathcal{I}$ where $V \subset X$ is an integral closed subscheme of dimension $\leq d$ and $\mathcal{I} \subset \mathcal{O}_V$ is a nonzero coherent

sheaf of ideals. If $\dim(V) < d$ and more generally for $\mathcal{F}$ whose support has dimension $< d$, then the first term in $E(\mathcal{F})$ has total degree $< d$ by Lemma 33.45.1 and the second term is zero. If $\dim(V) = d$, then we can use the short exact sequence

$$0 \to (V \to X)_*\mathcal{I} \to (V \to X)_*\mathcal{O}_V \to (V \to X)_*(\mathcal{O}_V/\mathcal{I}) \to 0$$

The result holds for the middle sheaf because the only Z occurring in the sum is $Z = V$ with $m_Z = 1$ and because

$$H^i(X, ((V \to X)_*\mathcal{O}_V) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = H^i(V, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_V)$$

by the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4; so in this case we actually have $E(\mathcal{F}) = 0$. The result holds for the sheaf on the right because its support has dimension $< d$. Thus the result holds for the sheaf on the left and the lemma is proved. □

0BEP **Definition 33.45.3.** Let k be a field. Let X be a proper scheme over k. Let $i : Z \to X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4. We prove a few lemmas for these intersection numbers.

0BEQ **Lemma 33.45.4.** *In the situation of Definition 33.45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.*

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a $\mathbf{Z}$-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2}\ldots\binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. □

0BER **Lemma 33.45.5.** *In the situation of Definition 33.45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have*

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 33.45.1 the function

$$(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}_i')^{\otimes n_i'} \otimes (\mathcal{L}_i'')^{\otimes n_i''} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d + 1$ variables. □

0BES **Lemma 33.45.6.** *In the situation of Definition 33.45.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = length_{\mathcal{O}_{X,\xi_i}}(\mathcal{O}_{Z,\xi_i})$ where $\xi_i \in Z_i$ is the generic point. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 33.45.2 and the definitions. □

0BET **Lemma 33.45.7.** *Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $Z \subset Y$ be an integral closed subscheme of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ *is as in Morphisms, Definition 29.51.8 or* 0 *if* $\dim(f(Z)) < d$.

Proof. The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 33.33.5 and the projection formula (Cohomology, Lemma 20.54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 33.45.1 and the result is true. Assume $\dim(f(Z)) = d$. Let $\xi \in f(Z)$ be the generic point. By dimension theory (see Lemmas 33.20.3 and 33.20.4) the generic point of Z is the unique point of Z mapping to ξ. Then $f : Z \to f(Z)$ is finite over a nonempty open of $f(Z)$, see Morphisms, Lemma 29.51.1. Thus $\deg(f : Z \to f(Z))$ is defined and in fact it is equal to the length of the stalk of $f_*\mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$. Moreover, the stalk of $R^i f_* \mathcal{O}_X$ at ξ is zero for $i > 0$ because we just saw that $f|_Z$ is finite in a neighbourhood of ξ (so that Cohomology of Schemes, Lemma 30.9.9 gives the vanishing). Thus the terms $\chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z)) \chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 33.45.2. The desired result follows. □

0BEU **Lemma 33.45.8.** *Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. □

0BEV **Lemma 33.45.9.** *Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. If $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are ample, then $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is positive.*

Proof. We will prove this by induction on d. The case $d = 0$ follows from Lemma 33.33.3. Assume $d > 0$. By Lemma 33.45.6 we may assume that Z is an integral closed subscheme. In fact, we may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$ to reduce to the case $Z = X$ is a proper variety of dimension d. By Lemma 33.45.5 we may replace $\mathcal{L}_1$ by a positive tensor power. Thus we may assume there exists a nonzero section $s \in \Gamma(X, \mathcal{L}_1)$ such that X_s is affine (here we use the definition of ample invertible sheaf, see Properties, Definition 28.26.1). Observe that X is not affine because proper and affine implies finite (Morphisms, Lemma 29.44.11) which contradicts $d > 0$. It follows that s has a nonempty vanishing scheme $Z(s) \subset X$. Since X is a variety, s is a regular section of $\mathcal{L}_1$, so $Z(s)$ is an effective Cartier divisor, thus

$Z(s)$ has codimension 1 in X, and hence $Z(s)$ has dimension $d-1$ (here we use material from Divisors, Sections 31.13, 31.14, and 31.15 and from dimension theory as in Lemma 33.20.3). By Lemma 33.45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot Z(s))$$

By induction the right hand side is positive and the proof is complete. □

0BEW **Definition 33.45.10.** Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme the *degree of Z with respect to $\mathcal{L}$*, denoted $\deg_{\mathcal{L}}(Z)$, is the intersection number $(\mathcal{L}^d \cdot Z)$ where $d = \dim(Z)$.

By Lemma 33.45.9 the degree of a subscheme is always a positive integer. We note that $\deg_{\mathcal{L}}(Z) = d$ if and only if

$$\chi(Z, \mathcal{L}^{\otimes n}|_Z) = \frac{d}{\dim(Z)!} n^{\dim(Z)} + l.o.t$$

as can be seen using that

$$(n_1 + \ldots + n_{\dim(Z)})^{\dim(Z)} = \dim(Z)!\ n_1 \ldots n_{\dim(Z)} + \text{other terms}$$

0BEX **Lemma 33.45.11.** *Let k be a field. Let $f : Y \to X$ be a finite dominant morphism of proper varieties over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then*

$$\deg_{f^*\mathcal{L}}(Y) = \deg(f) \deg_{\mathcal{L}}(X)$$

where $\deg(f)$ *is as in Morphisms, Definition 29.51.8.*

Proof. The statement makes sense because $f^*\mathcal{L}$ is ample by Morphisms, Lemma 29.37.7. Having said this the result is a special case of Lemma 33.45.7. □

Finally we relate the intersection number with a curve to the notion of degrees of invertible modules on curves introduced in Section 33.44.

0BEY **Lemma 33.45.12.** *Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension ≤ 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$(\mathcal{L} \cdot Z) = \deg(\mathcal{L}|_Z)$$

where $\deg(\mathcal{L}|_Z)$ *is as in Definition 33.44.1. If $\mathcal{L}$ is ample, then* $\deg_{\mathcal{L}}(Z) = \deg(\mathcal{L}|_Z)$.

Proof. This follows from the fact that the function $n \mapsto \chi(Z, \mathcal{L}|_Z^{\otimes n})$ has degree 1 and hence the leading coefficient is the difference of consecutive values. □

0BJ8 **Proposition 33.45.13** (Asymptotic Riemann-Roch)**.** *Let k be a field. Let X be a proper scheme over k of dimension d. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then*

$$\dim_k \Gamma(X, \mathcal{L}^{\otimes n}) \sim cn^d + l.o.t.$$

where $c = \deg_{\mathcal{L}}(X)/d!$ is a positive constant.

Proof. This follows from the definitions, Lemma 33.45.9, and the vanishing of higher cohomology in Cohomology of Schemes, Lemma 30.17.1. □

33.46. Embedding dimension

0C2G There are several ways to define the embedding dimension, but for closed points on algebraic schemes over algebraically closed fields all definitions are equivalent to the following.

0C1Q **Definition 33.46.1.** Let k be an algebraically closed field. Let X be a locally algebraic k-scheme and let $x \in X$ be a closed point. The *embedding dimension of X at x* is $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$.

Facts about embedding dimension. Let k, X, x be as in Definition 33.46.1.

(1) The embedding dimension of X at x is the dimension of the tangent space $T_{X/k,x}$ (Definition 33.16.3) as a k-vector space.
(2) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a surjection
$$k[[x_1, \ldots, x_d]] \longrightarrow \mathcal{O}^\wedge_{X,x}$$
of k-algebras.
(3) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.
(4) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and an unramified morphism $U \to \mathbf{A}^d_k$.
(5) If we are given a closed embedding $X \to Y$ with Y smooth over k, then the embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a closed subscheme $Z \subset Y$ with $X \subset Z$, with $Z \to \mathrm{Spec}(k)$ smooth at x, and with $\dim_x(Z) = d$.

If we ever need these, we will formulate a precise result and provide a proof.

Non-algebraically closed ground fields or non-closed points. Let k be a field and let X be a locally algebraic k-scheme. If $x \in X$ is a point, then we have several options for the embedding dimension of X at x. Namely, we could use

(1) $\dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$,
(2) $\dim_{\kappa(x)}(T_{X/k,x}) = \dim_{\kappa(x)}(\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$ (Lemma 33.16.4),
(3) the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.

In characteristic zero (1) = (2) if x is a closed point; more generally this holds if $\kappa(x)$ is separable algebraic over k, see Lemma 33.16.5. It seems that the geometric definition (3) corresponds most closely to the geometric intuition the phrase "embedding dimension" invokes. Since one can show that (3) and (2) define the same number (this follows from Lemma 33.18.5) this is what we will use. In our terminology we will make clear that we are taking the embedding dimension relative to the ground field.

0C2H **Definition 33.46.2.** Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point. The *embedding dimension of X/k at x* is $\dim_{\kappa(x)}(T_{X/k,x})$.

If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring the *embedding dimension of A* is sometimes defined as the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over κ. Above we have seen that if A is given as

an algebra over a field k, it may be preferable to use $\dim_\kappa(\Omega_{A/k} \otimes_A \kappa)$. Let us call this quantity the *embedding dimension of* A/k. With this terminology in place we have

$$\text{embed dim of } X/k \text{ at } x = \text{embed dim of } \mathcal{O}_{X,x}/k = \text{embed dim of } \mathcal{O}^\wedge_{X,x}/k$$

if k, X, x are as in Definition 33.46.2.

33.47. Bertini theorems

0FD4 In this section we prove results of the form: given a smooth projective variety X over a field k there exists an ample divisor $H \subset X$ which is smooth.

0FD5 **Lemma 33.47.1.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Then there exists an integer n_0 such that for all $n \geq n_0$ the kernel V_n of $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(Z, \mathcal{L}^{\otimes n}|_Z)$ generates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ and the canonical morphism*

$$X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

is an immersion of schemes over k.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Observe that via the inclusion $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \subset \mathcal{L}^{\otimes n}$ we have $V_n = \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$. Choose n_1 such that for $n \geq n_1$ the sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 28.26.13. It follows that V_n generates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ for $n \geq n_1$.

For $n \geq n_1$ denote $\psi_n : V_n \to \Gamma(X \setminus Z, \mathcal{L}^{\otimes n}|_{X \setminus Z})$ the restriction map. We get a canonical morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes n}|_{X \setminus Z}, \psi_n} : X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

by Constructions, Example 27.21.2. Choose n_2 such that for all $n \geq n_2$ the invertible sheaf $\mathcal{L}^{\otimes n}$ is very ample on X. We claim that $n_0 = n_1 + n_2$ works.

Proof of the claim. Say $n \geq n_0$ and write $n = n_1 + n'$. For $x \in X \setminus Z$ we can choose $s_1 \in V_1$ not vanishing at x. Set $V' = \Gamma(X, \mathcal{L}^{\otimes n'})$. By our choice of n and n' we see that the corresponding morphism $\varphi' : X \to \mathbf{P}(V')$ is a closed immersion. Thus if we choose $s' \in \Gamma(X, \mathcal{L}^{\otimes n'})$ not vanishing at x, then $X_{s'} = (\varphi')^{-1}(D_+(s'))$ (see Constructions, Lemma 27.14.1) is affine and $X_{s'} \to D_+(s')$ is a closed immersion. Then $s = s_1 \otimes s' \in V_n$ does not vanish at x. If $D_+(s) \subset \mathbf{P}(V_n)$ denotes the corresponding open affine space of our projective space, then $\varphi^{-1}(D_+(s)) = X_s \subset X \setminus Z$ (see reference above). The open $X_s = X_{s'} \cap X_{s_1}$ is affine, see Properties, Lemma 28.26.4. Consider the ring map

$$\mathrm{Sym}(V)_{(s)} \longrightarrow \mathcal{O}_X(X_s)$$

defining the morphism $X_s \to D_+(s)$. Because $X_{s'} \to D_+(s')$ is a closed immersion, the images of the elements

$$\frac{s_1 \otimes t'}{s_1 \otimes s'}$$

where $t' \in V'$ generate the image of $\mathcal{O}_X(X_{s'}) \to \mathcal{O}_X(X_s)$. Since $X_s \to X_{s'}$ is an open immersion, this implies that $X_s \to D_+(s)$ is an immersion of affine schemes (see below). Thus φ_n is an immersion by Morphisms, Lemma 29.3.5.

Let $a : A' \to A$ and $c : B \to A$ be ring maps such that $\mathrm{Spec}(a)$ is an immersion and $\mathrm{Im}(a) \subset \mathrm{Im}(c)$. Set $B' = A' \times_A B$ with projections $b : B' \to B$ and $c' : B' \to A'$.

By assumption c' is surjective and hence $\mathrm{Spec}(c')$ is a closed immersion. Whence $\mathrm{Spec}(c') \circ \mathrm{Spec}(a)$ is an immersion (Schemes, Lemma 26.24.3). Then $\mathrm{Spec}(c)$ has to be an immersion because it factors the immersion $\mathrm{Spec}(c') \circ \mathrm{Spec}(a) = \mathrm{Spec}(b) \circ \mathrm{Spec}(c)$, see Morphisms, Lemma 29.3.1. □

0G47 **Situation 33.47.2.** Let k be a field, let X be a scheme over k, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, let V be a finite dimensional k-vector space, and let $\psi : V \to \Gamma(X, \mathcal{L})$ be a k-linear map. Say $\dim(V) = r$ and we have a basis $v_1, \ldots, v_r$ of V. Then we obtain a "universal divisor"

$$H_{univ} = Z(s_{univ}) \subset \mathbf{A}^r \times_k X$$

as the zero scheme (Divisors, Definition 31.14.8) of the section

$$s_{univ} = \sum\nolimits_{i=1,\ldots,r} x_i \psi(v_i) \in \Gamma(\mathbf{A}^r \times_k X, \mathrm{pr}_2^* \mathcal{L})$$

For a field extension k'/k the k'-points $v \in \mathbf{A}^r_k(k')$ correspond to vectors $(a_1, \ldots, a_r)$ of elements of k'. Thus we may on the one hand think of v as the element $v = \sum_{i=1,\ldots,r} a_i v_i \in V \otimes_k k'$ and on the other hand we may assign to v the section

$$\psi(v) = \sum\nolimits_{i=1,\ldots,r} a_i \psi(v_i) \in \Gamma(X_{k'}, \mathcal{L}|_{X_{k'}})$$

With this notation it is clear that the fibre of H_{univ} over $v \in V \otimes k'$ is the zero scheme of $\psi(v)$. In a formula:

$$H_v = H_{univ,v} = Z(\psi(v))$$

We will denote this common value by H_v as indicated. Finally, in this situation let P be a property of vectors $v \in V \otimes_k k'$ for k'/k an arbitrary field extension[7]. We say *P holds for general $v \in V \otimes_k k'$* if there exists a nonempty Zariski open $U \subset \mathbf{A}^r_k$ such that if v corresponds to a k'-point of U for any k'/k then $P(v)$ holds.

0FD6 **Lemma 33.47.3.** *In Situation 33.47.2 assume*

(1) *X is smooth over k,*
(2) *the image of $\psi : V \to \Gamma(X, \mathcal{L})$ generates $\mathcal{L}$,*
(3) *the corresponding morphism $\varphi_{\mathcal{L},\psi} : X \to \mathbf{P}(V)$ is an immersion.*

Then for general $v \in V \otimes_k k'$ the scheme H_v is smooth over k'.

Proof. (We observe that X is separated and finite type as a locally closed subscheme of a projective space.) Let us use the notation introduced above the statement of the lemma. We consider the projections

$$\begin{array}{ccccc} \mathbf{A}^r_k \times_k X & \longleftarrow & H_{univ} & \longrightarrow & \mathbf{A}^r_k \times_k X \\ \downarrow & \swarrow^{p} & & \searrow^{q} & \downarrow \\ X & & & & \mathbf{A}^r_k \end{array}$$

Let $\Sigma \subset H_{univ}$ be the singular locus of the morphsm $q : H_{univ} \to \mathbf{A}^r_k$, i.e., the set of points where q is not smooth. Then Σ is closed because the smooth locus of a morphism is open by definition. Since the fibre of a smooth morphism is smooth, it suffices to prove $q(\Sigma)$ is contained in a proper closed subset of $\mathbf{A}^r_k$. Since Σ (with reduced induced scheme structure) is a finite type scheme over k it suffices to prove

[7] For example we could consider the condition that H_v is smooth over k', or geometrically irreducible over k'.

$\dim(\Sigma) < r$ This follows from Lemma 33.20.4. Since dimensions aren't changed by replacing k by a bigger field (Morphisms, Lemma 29.28.3), we may and do assume k is algebraically closed. By dimension theory (Lemma 33.20.4), it suffices to prove that for $x \in X \setminus Z$ closed we have $p^{-1}(\{x\}) \cap \Sigma$ has dimension $< r - \dim(X')$ where X' is the unique irreducible component of X containing x. As X is smooth over k and x is a closed point we have $\dim(X') = \dim \mathfrak{m}_x/\mathfrak{m}_x^2$ (Morphisms, Lemma 29.34.12 and Algebra, Lemma 10.140.1). Thus we win if

$$\dim p^{-1}(x) \cap \Sigma < r - \dim \mathfrak{m}_x/\mathfrak{m}_x^2$$

for all $x \in X$ closed.

Since V globally generated $\mathcal{L}$, for every irreducible component X' of X there is a nonempty Zariski open of $\mathbf{A}^r$ such that the fibres of q over this open do not contain X'. (For example, if $x' \in X'$ is a closed point, then we can take the open corresponding to those vectors $v \in V$ such that $\psi(v)$ does not vanish at x'. This open will be the complement of a hyperplane in $\mathbf{A}^r_k$.) Let $U \subset \mathbf{A}^r$ be the (nonempty) intersection of these opens. Then the fibres of $q^{-1}(U) \to U$ are effective Cartier divisors on the fibres of $U \times_k X \to U$ (because a nonvanishing section of an invertible module on an integral scheme is a regular section). Hence the morphism $q^{-1}(U) \to U$ is flat by Divisors, Lemma 31.18.9. Thus for $x \in X$ closed and $v \in V = \mathbf{A}^r_k(k)$, if $(x, v) \in H_{univ}$, i.e., if $x \in H_v$ then q is smooth at (x, v) if and only if the fibre H_v is smooth at x, see Morphisms, Lemma 29.34.14.

Consider the image $\psi(v)_x$ in the stalk $\mathcal{L}_x$ of the section corresponding to $v \in V$. We have

$$x \in H_v \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x\mathcal{L}_x$$

If this is true, then we have

$$H_v \text{ singular at } x \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x^2\mathcal{L}_x$$

Namely, $\psi(v)_x$ is not contained in $\mathfrak{m}_x^2\mathcal{L}_x \Leftrightarrow$ the local equation for $H_v \subset X$ at x is not contained in $\mathfrak{m}_x^2 \Leftrightarrow \mathcal{O}_{H_v,x}$ is regular (Algebra, Lemma 10.106.3) $\Leftrightarrow H_v$ is smooth at x over k (Algebra, Lemma 10.140.5). We conclude that the closed points of $p^{-1}(x) \cap \Sigma$ correspond to those $v \in V$ such that $\psi(v)_x \in \mathfrak{m}_x^2\mathcal{L}_x$. However, as $\varphi_{\mathcal{L},\psi}$ is an immersion the map

$$V \longrightarrow \mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$$

is surjective (small detail omitted). By the above, the closed points of the locus $p^{-1}(x) \cap \Sigma$ viewed as a subspace of V is the kernel of this map and hence has dimension $r - \dim \mathfrak{m}_x/\mathfrak{m}_x^2 - 1$ as desired. □

33.48. Enriques-Severi-Zariski

0FVD In this section we prove some results of the form: twisting by a "very negative" invertible module kills low degree cohomology. We also deduce the connectedness of a hypersurface section of a normal proper scheme of dimension ≥ 2.

0FD7 **Lemma 33.48.1.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. If $Ass(\mathcal{F})$ does not contain any closed points, then $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for $n \ll 0$.*

Proof. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ let $\mathcal{P}(\mathcal{F})$ be the property: there exists an $n_0 \in \mathbf{Z}$ such that for $n \leq n_0$ every section s of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ has support consisting only of closed points. Since $\mathrm{Ass}(\mathcal{F}) = \mathrm{Ass}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ we see that it suffices to prove $\mathcal{P}$ holds for all coherent modules on X. To do this we will prove that conditions (1), (2), and (3) of Cohomology of Schemes, Lemma 30.12.8 are satisfied.

To see condition (1) suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

is a short exact sequence of coherent $\mathcal{O}_X$-modules such that we have $\mathcal{P}$ for $\mathcal{F}_i$, $i = 1, 2$. Let n_1, n_2 be the cutoffs we find. Let $\mathcal{F}_2' \subset \mathcal{F}_2$ be the maximal coherent submodule whose support is a finite set of closed points. Let $\mathcal{I} \subset \mathcal{O}_X$ be the annihilator of $\mathcal{F}_2'$. Since $\mathcal{L}$ is ample, we can find an $e > 0$ such that $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e}$ is globally generated. Set $n_0 = \min(n_2, n_1 - e)$. Let $n \leq n_0$ and let t be a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$. The image of t in $\mathcal{F}_2 \otimes \mathcal{L}^{\otimes n}$ falls into $\mathcal{F}_2' \otimes \mathcal{L}^{\otimes n}$ because $n \leq n_2$. Hence for any $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})$ the product $t \otimes s$ lies in $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes n+e}$. Thus $t \otimes s$ has support contained in the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_1)$ because $n + e \leq n_1$. Since by our choice of e we may choose s invertible in any point not in the support of $\mathcal{F}_2'$ we conclude that the support of t is contained in the union of the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_1)$ and the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_2)$. This finishes the proof of condition (1).

Condition (2) is immediate.

For condition (3) we choose $\mathcal{G} = \mathcal{O}_Z$. In this case, if Z is a closed point of X, then there is nothing the show. If $\dim(Z) > 0$, then we will show that $\Gamma(Z, \mathcal{L}^{\otimes n}|_Z) = 0$ for $n < 0$. Namely, let s be a nonzero section of a negative power of $\mathcal{L}|_Z$. Choose a nonzero section t of a positive power of $\mathcal{L}|_Z$ (this is possible as $\mathcal{L}$ is ample, see Properties, Proposition 28.26.13). Then $s^{\deg(t)} \otimes t^{\deg(s)}$ is a nonzero global section of $\mathcal{O}_Z$ (because Z is integral) and hence a unit (Lemma 33.9.3). This implies that t is a trivializing section of a positive power of $\mathcal{L}$. Thus the function $n \mapsto \dim_k \Gamma(X, \mathcal{L}^{\otimes n})$ is bounded on an infinite set of positive integers which contradicts asymptotic Riemann-Roch (Proposition 33.45.13) since $\dim(Z) > 0$. □

0FD8 **Lemma 33.48.2** (Enriques-Severi-Zariski)**.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for $x \in X$ closed we have depth$(\mathcal{F}_x) \geq 2$. Then $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) = 0$ for $m \ll 0$.*

Proof. Choose a closed immersion $i : X \to \mathbf{P}^n_k$ such that $i^*\mathcal{O}(1) \cong \mathcal{L}^{\otimes e}$ for some $e > 0$ (see Morphisms, Lemma 29.39.4). Then it suffices to prove the lemma for

$$\mathcal{G} = i_*(\mathcal{F} \oplus \mathcal{F} \otimes \mathcal{L} \oplus \ldots \oplus \mathcal{F} \otimes \mathcal{L}^{\otimes e-1}) \quad \text{and} \quad \mathcal{O}(1)$$

on $\mathbf{P}^n_k$. Namely, we have

$$H^1(\mathbf{P}^n_k, \mathcal{G}(m)) = \bigoplus\nolimits_{j=0,\ldots,e-1} H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes j+me})$$

by Cohomology of Schemes, Lemma 30.2.4. Also, if $y \in \mathbf{P}^n_k$ is a closed point then $\mathrm{depth}(\mathcal{G}_y) = \infty$ if $y \notin i(X)$ and $\mathrm{depth}(\mathcal{G}_y) = \mathrm{depth}(\mathcal{F}_x)$ if $y = i(x)$ because in this case $\mathcal{G}_y \cong \mathcal{F}_x^{\oplus e}$ as a module over $\mathcal{O}_{\mathbf{P}^n_k, x}$ and we can use for example Algebra, Lemma 10.72.11 to get the equality.

Assume $X = \mathbf{P}^n_k$ and $\mathcal{L} = \mathcal{O}(1)$ and k is infinite. Choose $s \in H^0(\mathbf{P}^1_k, \mathcal{O}(1))$ which determines an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{s} \mathcal{F} \to \mathcal{G} \to 0$$

as in Lemma 33.35.3. Since the map $\mathcal{F}(-1) \to \mathcal{F}$ is affine locally given by multiplying by a nonzerodivisor on $\mathcal{F}$ we see that for $x \in \mathbf{P}^n_k$ closed we have $\text{depth}(\mathcal{G}_x) \geq 1$, see Algebra, Lemma 10.72.7. Hence by Lemma 33.48.1 we have $H^0(\mathcal{G}(m)) = 0$ for $m \ll 0$. Looking at the long exact sequence of cohomology after twisting (see Remark 33.35.5) we find that the sequence of numbers

$$\dim H^1(\mathbf{P}^n_k, \mathcal{F}(m))$$

stabilizes for $m \leq m_0$ for some integer m_0. Let N be the common dimension of these spaces for $m \leq m_0$. We have to show $N = 0$.

For $d > 0$ and $m \leq m_0$ consider the bilinear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{F}(m))$$

By linear algebra, there is a codimension $\leq N^2$ subspace $V_m \subset H^0(\mathbf{P}^n_k, \mathcal{O}(d))$ such that multiplication by $s' \in V_m$ annihilates $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$. Observe that for $m' < m \leq m_0$ the diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m'-d)) & \longrightarrow & H^1(\mathbf{P}^n_k, \mathcal{F}(m')) \\ \downarrow{\scriptstyle 1 \times s^{m'-m}} & & \downarrow{\scriptstyle s^{m'-m}} \\ H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) & \longrightarrow & H^1(\mathbf{P}^n_k, \mathcal{F}(m)) \end{array}$$

commutes with isomorphisms going vertically. Thus $V_m = V$ is independent of $m \leq m_0$. For $x \in \text{Ass}(\mathcal{F})$ set $Z = \overline{\{x\}}$. For d large enough the linear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \to H^0(Z, \mathcal{O}(d)|_Z)$$

has rank $> N^2$ because $\dim(Z) \geq 1$ (for example this follows from asymptotic Riemann-Roch and ampleness $\mathcal{O}(1)$; details omitted). Hence we can find $s' \in V$ such that s' does not vanish in any associated point of $\mathcal{F}$ (use that the set of associated points is finite). Then we obtain

$$0 \to \mathcal{F}(-d) \xrightarrow{s'} \mathcal{F} \to \mathcal{G}' \to 0$$

and as before we conclude as before that multiplication by s' on $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$ is injective for $m \ll 0$. This contradicts the choice of s' unless $N = 0$ as desired.

We still have to treat the case where k is finite. In this case let K/k be any infinite algebraic field extension. Denote $\mathcal{F}_K$ and $\mathcal{L}_K$ the pullbacks of $\mathcal{F}$ and $\mathcal{L}$ to $X_K = \text{Spec}(K) \times_{\text{Spec}(k)} X$. We have

$$H^1(X_K, \mathcal{F}_K \otimes \mathcal{L}_K^{\otimes m}) = H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes_k K$$

by Cohomology of Schemes, Lemma 30.5.2. On the other hand, a closed point x_K of X_K maps to a closed point x of X because K/k is an algebraic extension. The ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_K,x_K}$ is flat (Lemma 33.5.1). Hence we have

$$\text{depth}(\mathcal{F}_{x_K}) = \text{depth}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_K,x_K}) \geq \text{depth}(\mathcal{F}_x)$$

by Algebra, Lemma 10.163.1 (in fact equality holds here but we don't need it). Therefore the result over k follows from the result over the infinite field K and the proof is complete. $\square$

0FD9 **Lemma 33.48.3.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Assume*

(1) *s is a regular section (Divisors, Definition 31.14.6),*
(2) *for every closed point $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq 2$, and*
(3) *X is connected.*

Then the zero scheme $Z(s)$ of s is connected.

Proof. Since s is a regular section, so is $s^n \in \Gamma(X, \mathcal{L}^{\otimes n})$ for all $n > 1$. Moreover, the inclusion morphism $Z(s) \to Z(s^n)$ is a bijection on underlying topological spaces. Hence if $Z(s)$ is disconnected, so is $Z(s^n)$. Now consider the canonical short exact sequence

$$0 \to \mathcal{L}^{\otimes -n} \xrightarrow{s^n} \mathcal{O}_X \to \mathcal{O}_{Z(s^n)} \to 0$$

Consider the k-algebra $R_n = \Gamma(X, \mathcal{O}_{Z(s^n)})$. If $Z(s)$ is disconnected, i.e., $Z(s^n)$ is disconnected, then either R_n is zero in case $Z(s^n) = \emptyset$ or R_n contains a nontrivial idempotent in case $Z(s^n) = U \amalg V$ with $U, V \subset Z(s^n)$ open and nonempty (the reader may wish to consult Lemma 33.9.3). Thus the map $\Gamma(X, \mathcal{O}_X) \to R_n$ cannot be an isomorphism. It follows that either $H^0(X, \mathcal{L}^{\otimes -n})$ or $H^1(X, \mathcal{L}^{\otimes -n})$ is nonzero for infinitely many positive n. This contradicts Lemma 33.48.1 or 33.48.2 and the proof is complete. □

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CHAPTER 34

Topologies on Schemes

34.1. Introduction

020K

020L In this document we explain what the different topologies on the category of schemes are. Some references are [**Gro71**] and [**BLR90**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

34.2. The general procedure

020M In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 7.6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say "consider all schemes with the Zariski topology" since that would give a "big" category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 7.6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an "absolute" big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.
(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ-covering of affines[2] whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

[1] The words big and small here do not relate to bigness/smallness of the corresponding categories.

[2] In the case of the ph topology we deviate very slightly from this approach, see Definition 34.8.11 and the surrounding discussion.

34.3. The Zariski topology

020N

020O **Definition 34.3.1.** Let T be a scheme. A *Zariski covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

020P **Lemma 34.3.2.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a Zariski covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

020Q **Lemma 34.3.3.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is a standard open of T, see Schemes, Definition 26.5.2. Moreover, we may choose each U_j to be an open of one of the T_i.*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 26.5.1. □

Thus we define the corresponding standard coverings of affines as follows.

020R **Definition 34.3.4.** Compare Schemes, Definition 26.5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

020S **Definition 34.3.5.** A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
(2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 7.8.8 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 3.9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 34.12.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen

universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

03WV **Lemma 34.3.6.** *Let Sch_{Zar} be a big Zariski site as in Definition 34.3.5. Let $T \in \mathrm{Ob}(Sch_{Zar})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 7.8.2) to $\{T_i \to T\}_{i\in I}$.*

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 3.9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i\in I}$ is tautologically equivalent to $\{T_i \to T\}_{i\in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.11.1. □

020T **Definition 34.3.7.** Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.

(1) The *big Zariski site of S*, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 7.25.
(2) The *small Zariski site of S*, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$.
(3) The *big affine Zariski site of S*, denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ consisting of objects U/S such that U is an affine scheme. A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}((Aff/S)_{Zar})$ which is a standard Zariski covering.
(4) The *small affine Zariski site of S*, denoted $S_{affine,Zar}$, is the full subcategory of S_{Zar} whose objects are those U/S such that U is an affine scheme. A covering of $S_{affine,Zar}$ is any covering $\{U_i \to U\}$ of S_{Zar} with $U \in \mathrm{Ob}(S_{affine,Zar})$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site, the big affine Zariski site, and the small affine Zariski site are sites. We check this now.

020U **Lemma 34.3.8.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The structures S_{Zar}, $(Aff/S)_{Zar}$, and $S_{affine,Zar}$ defined above are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$ we also have $U_i \in \mathrm{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if

necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \mathrm{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \mathrm{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to \mathrm{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3).

We omit the proof that $S_{affine,Zar}$ is a site. □

020V **Lemma 34.3.9.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, $(Aff/S)_{Zar}$, and $S_{affine,Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.*

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$ and $S_{affine,Zar}$. □

Next, we check that the big, resp. small affine site defines the same topos as the big, resp. small site.

020W **Lemma 34.3.10.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(Aff/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 34.3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

0F1B **Lemma 34.3.11.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $S_{affine,Zar} \to S_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(S_{affine,Zar})$ to $Sh(S_{Zar})$.*

Proof. Omitted. Hint: compare with the proof of Lemma 34.3.10. □

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

020X **Lemma 34.3.12.** *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\mathrm{Im}(U \to S))$. For the converse,

we choose for every open subscheme $U \subset S$ an object $U'/S \in \mathrm{Ob}(S_{Zar})$ with $\mathrm{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 3.9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}'(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_j \to U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 3.11.1, and we use Sites, Lemma 7.8.4. Details omitted. □

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

020Y **Lemma 34.3.13.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 34.3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

020Z **Lemma 34.3.14.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 34.3.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 34.3.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows. □

04BS **Definition 34.3.15.** In the situation of Lemma 34.3.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) = \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G})$$
$$\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) = \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

0210 **Lemma 34.3.16.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor*

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 34.3.13). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0211 **Lemma 34.3.17.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 34.3.13 and i_T as in Lemma 34.3.14.*
(2) *The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of sites*

$$f_{small} : T_{Zar} \longrightarrow S_{Zar}$$

The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* if we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 34.3.12.
(3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} T_{Zar} & \xleftarrow{\pi_T} & (Sch/T)_{Zar} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{Zar} & \xleftarrow{\pi_S} & (Sch/S)_{Zar} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 7.14.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 34.3.15 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

0212 **Lemma 34.3.18.** *Given schemes X, Y, Z in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.3.16. For the functors on the small sites this is Sheaves, Lemma 6.21.2 via the identification of Lemma 34.3.12. □

0DD9 **Lemma 34.3.19.** *Let Sch_{Zar} be a big Zariski site. Consider a cartesian diagram*

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in Sch_{Zar}. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{Zar}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on S'_{Zar} (use Lemmas 34.3.13 and 34.3.17). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

We can think about a sheaf on the big Zariski site of S as a collection of "usual" sheaves on all schemes over S.

0213 **Lemma 34.3.20.** *Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(a) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $c_g \circ g^{-1}c_f$ is equal to $c_{f \circ g}$, and*
(b) *if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 7.26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 34.3.12 and 34.3.13. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map

c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 6.21.7 and the discussion surrounding it. It is clear that $c_{f\circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

34.4. The étale topology

0214 Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

0215 **Definition 34.4.1.** Let T be a scheme. An *étale covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

0216 **Lemma 34.4.2.** *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 29.36.9. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0217 **Lemma 34.4.3.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an étale covering.*

Proof. Omitted. □

0218 **Lemma 34.4.4.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted. □

Thus we define the corresponding standard coverings of affines as follows.

0219 **Definition 34.4.5.** Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff/S, for example because of Algebra, Lemma 10.144.2 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 10.143.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

021A **Definition 34.4.6.** A *big étale site* is any site $Sch_{étale}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $Sch_{étale}$ is in some sense induced from the étale topology on the category of all schemes.

03WW **Lemma 34.4.7.** *Let $Sch_{étale}$ be a big étale site as in Definition 34.4.6. Let $T \in \mathrm{Ob}(Sch_{étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary étale covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{étale}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.4.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of $Sch_{étale}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{étale}$ by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{étale}$ by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{étale}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

021B **Definition 34.4.8.** Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.

(1) The *big étale site of S*, denoted $(Sch/S)_{étale}$, is the site $Sch_{étale}/S$ introduced in Sites, Section 7.25.
(2) The *small étale site of S*, which we denote $S_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$.
(3) The *big affine étale site of S*, denoted $(Aff/S)_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that U is an affine scheme. A covering of $(Aff/S)_{étale}$ is any covering $\{U_i \to U\}$ of

$(Sch/S)_{étale}$ with $U \in \mathrm{Ob}((Aff/S)_{étale})$ which is a standard étale covering.

(4) The *small affine étale site of* S, denoted $S_{affine,étale}$, is the full subcategory of $S_{étale}$ whose objects are those U/S such that U is an affine scheme. A covering of $S_{affine,étale}$ is any covering $\{U_i \to U\}$ of $S_{étale}$ with $U \in \mathrm{Ob}(S_{affine,étale})$ which is a standard étale covering.

It is not completely clear that the big affine étale site, the small étale site, and the small affine étale site are sites. We check this now.

021C **Lemma 34.4.9.** *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The structures $S_{étale}$, $(Aff/S)_{étale}$, and $S_{affine,étale}$ are sites.*

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$ we also have $U_i \in \mathrm{Ob}(S_{étale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(Aff/S)_{étale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard étale covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard étale covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine.

We omit the proof that $S_{affine,étale}$ is a site. □

021D **Lemma 34.4.10.** *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, $(Aff/S)_{étale}$, and $S_{affine,étale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.*

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{étale})$. The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{étale}$ and $S_{affine,étale}$. □

Next, we check that the big, resp. small affine site defines the same topos as the big, resp. small site.

021E **Lemma 34.4.11.** *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(Aff/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{étale})$ to $Sh((Sch/S)_{étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous

just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 34.4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

04HR **Lemma 34.4.12.** *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $S_{affine,étale} \to S_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh(S_{affine,étale})$ to $Sh(S_{étale})$.*

Proof. Omitted. Hint: compare with the proof of Lemma 34.4.11. □

Next, we establish some relationships between the topoi associated to these sites.

021F **Lemma 34.4.13.** *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 34.4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

021G **Lemma 34.4.14.** *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 34.4.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{étale} \to (Sch/S)_{étale}$, in addition to the properties seen in the proof of Lemma 34.4.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

04BT **Definition 34.4.15.** In the situation of Lemma 34.4.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\begin{aligned}\mathrm{Mor}_{Sh(S_{étale})}(\mathcal{F}|_{S_{étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{étale})}(\mathcal{G}, \mathcal{F}|_{S_{étale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{étale}} = \mathcal{G}$.

021H **Lemma 34.4.16.** *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor*

$$u : (Sch/T)_{étale} \longrightarrow (Sch/S)_{étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{étale} \longrightarrow (Sch/T)_{étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 34.4.13). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

021I **Lemma 34.4.17.** *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 34.4.13 and i_T as in Lemma 34.4.14.*
(2) *The functor $S_{étale} \to T_{étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of sites*

$$f_{small} : T_{étale} \longrightarrow S_{étale}$$

We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} T_{étale} & \xleftarrow{\pi_T} & (Sch/T)_{étale} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{étale} & \xleftarrow{\pi_S} & (Sch/S)_{étale} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{étale} \to T_{étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 34.4.3 (3) and 34.4.10. Moreover, both $S_{étale}, T_{étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 7.14.7 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 34.4.15 we have: for $\mathcal{F}$ a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

021J **Lemma 34.4.18.** *Given schemes X, Y, Y in $Sch_{étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.4.16. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 34.4.17. □

0DDA **Lemma 34.4.19.** *Let $Sch_{étale}$ be a big étale site. Consider a cartesian diagram*

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in $Sch_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{étale}$ (use Lemmas 34.4.13 and 34.4.16). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

We can think about a sheaf on the big étale site of S as a collection of "usual" sheaves on all schemes over S.

021K **Lemma 34.4.20.** *Let S be a scheme contained in a big étale site $Sch_{étale}$. A sheaf $\mathcal{F}$ on the big étale site $(Sch/S)_{étale}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{étale})$ a sheaf $\mathcal{F}_T$ on $T_{étale}$,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(a) *given any* $f : T' \to T$ *and* $g : T'' \to T'$ *in* $(Sch/S)_{étale}$ *the composition* $c_g \circ g_{small}^{-1} c_f$ *is equal to* $c_{f \circ g}$*, and*
(b) *if* $f : T' \to T$ *in* $(Sch/S)_{étale}$ *is étale then* c_f *is an isomorphism.*

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 7.26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{étale}$ see Lemma 34.4.13. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{étale} \to T'_{étale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 7.13) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f \circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f \circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

34.5. The smooth topology

021Y In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 37.38) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

021Z **Definition 34.5.1.** Let T be a scheme. A *smooth covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

0220 **Lemma 34.5.2.** *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 29.36.1 and Lemma 34.4.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0221 **Lemma 34.5.3.** *Let* T *be a scheme.*

(1) *If* $T' \to T$ *is an isomorphism then* $\{T' \to T\}$ *is a smooth covering of* T*.*
(2) *If* $\{T_i \to T\}_{i \in I}$ *is a smooth covering and for each* i *we have a smooth covering* $\{T_{ij} \to T_i\}_{j \in J_i}$*, then* $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ *is a smooth covering.*
(3) *If* $\{T_i \to T\}_{i \in I}$ *is a smooth covering and* $T' \to T$ *is a morphism of schemes then* $\{T' \times_T T_i \to T'\}_{i \in I}$ *is a smooth covering.*

Proof. Omitted. □

0222 **Lemma 34.5.4.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a smooth covering of T. Then there exists a smooth covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard smooth, see Morphisms, Definition 29.34.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 10.137.9. □

Thus we define the corresponding standard coverings of affines as follows.

0223 **Definition 34.5.5.** Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

03WY **Definition 34.5.6.** A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

03WZ **Lemma 34.5.7.** *Let Sch_{smooth} be a big smooth site as in Definition 34.5.6. Let $T \in \mathrm{Ob}(Sch_{smooth})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.5.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{smooth}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{smooth} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

03X0 **Definition 34.5.8.** Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The *big smooth site of* S, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 7.25.
(2) The *big affine smooth site of* S, denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

06VC **Lemma 34.5.9.** *Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S. The functor $(Aff/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 34.5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

04HC **Lemma 34.5.10.** *Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

34.6. The syntomic topology

0224 In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

0225 **Definition 34.6.1.** Let T be a scheme. An *syntomic covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

0226 **Lemma 34.6.2.** *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 29.34.7 and Lemma 34.5.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0227 **Lemma 34.6.3.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a syntomic covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

0228 **Lemma 34.6.4.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 29.30.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 10.136.15. □

Thus we define the corresponding standard coverings of affines as follows.

0229 **Definition 34.6.5.** Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

03X1 **Definition 34.6.6.** A *big syntomic site* is any site $Sch_{syntomic}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

03X2 **Lemma 34.6.7.** *Let $Sch_{syntomic}$ be a big syntomic site as in Definition 34.6.6. Let $T \in \mathrm{Ob}(Sch_{syntomic})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary syntomic covering of T.*

(1) *There exists a covering* $\{U_j \to T\}_{j\in J}$ *of* T *in the site* $Sch_{syntomic}$ *which refines* $\{T_i \to T\}_{i\in I}$.
(2) *If* $\{T_i \to T\}_{i\in I}$ *is a standard syntomic covering, then it is tautologically equivalent to a covering in* $Sch_{syntomic}$.
(3) *If* $\{T_i \to T\}_{i\in I}$ *is a Zariski covering, then it is tautologically equivalent to a covering in* $Sch_{syntomic}$.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.6.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of $Sch_{syntomic}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{syntomic}$ by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{syntomic}$ by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{syntomic}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

03X3 **Definition 34.6.8.** Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The *big syntomic site of* S, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 7.25.
(2) The *big affine syntomic site of* S, denoted $(Aff/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(Aff/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

06VD **Lemma 34.6.9.** *Let* S *be a scheme. Let* $Sch_{syntomic}$ *be a big syntomic site containing* S. *The functor* $(Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$ *is special cocontinuous and induces an equivalence of topoi from* $Sh((Aff/S)_{syntomic})$ *to* $Sh((Sch/S)_{syntomic})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 34.6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

04HD **Lemma 34.6.10.** *Let* $Sch_{syntomic}$ *be a big syntomic site. Let* $f : T \to S$ *be a morphism in* $Sch_{syntomic}$. *The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *We have* $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. *Also,* f_{big}^{-1} *has a left adjoint* $f_{big!}$ *which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

34.7. The fppf topology

021L Let S be a scheme. We would like to define the fppf-topology[3] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

021M **Definition 34.7.1.** Let T be a scheme. An *fppf covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

021N **Lemma 34.7.2.** *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.30.6 and 29.30.7, and Lemma 34.6.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

021O **Lemma 34.7.3.** *Let* T *be a scheme.*

(1) *If* $T' \to T$ *is an isomorphism then* $\{T' \to T\}$ *is an fppf covering of* T.
(2) *If* $\{T_i \to T\}_{i \in I}$ *is an fppf covering and for each* i *we have an fppf covering* $\{T_{ij} \to T_i\}_{j \in J_i}$, *then* $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ *is an fppf covering.*
(3) *If* $\{T_i \to T\}_{i \in I}$ *is an fppf covering and* $T' \to T$ *is a morphism of schemes then* $\{T' \times_T T_i \to T'\}_{i \in I}$ *is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 29.25.6) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 29.21.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 29.25.8) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 29.21.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). □

[3]The letters fppf stand for "fidèlement plat de présentation finie".

021P **Lemma 34.7.4.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 29.25.10. □

Thus we define the corresponding standard coverings of affines as follows.

021Q **Definition 34.7.5.** Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

021R **Definition 34.7.6.** A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

03WX **Lemma 34.7.7.** *Let Sch_{fppf} be a big fppf site as in Definition 34.7.6. Let $T \in \mathrm{Ob}(Sch_{fppf})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary fppf covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.7.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{fppf}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{fppf} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

021S **Definition 34.7.8.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 7.25.
(2) The *big affine fppf site of S*, denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

021T **Lemma 34.7.9.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(Aff/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 34.4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard fppf covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard fppf covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. □

021U **Lemma 34.7.10.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S.*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{fppf}$. □

Next, we check that the big affine site defines the same topos as the big site.

021V **Lemma 34.7.11.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(Aff/S)_{fppf} \to (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{fppf} \to (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 34.7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we establish some relationships between the topoi associated to these sites.

021W **Lemma 34.7.12.** *Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor*

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *We have* $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. *Also,* f_{big}^{-1} *has a left adjoint* $f_{big!}$ *which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

021X **Lemma 34.7.13.** *Given schemes* X, Y, Z *in* $(Sch/S)_{fppf}$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *we have* $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.7.12. □

34.8. The ph topology

0DBC In this section we define the ph topology. This is the topology generated by Zariski coverings and proper surjective morphisms, see Lemma 34.8.15.

We borrow our notation/terminology from the paper [**GL01**] by Goodwillie and Lichtenbaum. These authors show that if we restrict to the subcategory of Noetherian schemes, then the ph topology is the same as the "h topology" as originally defined by Voevodsky: this is the topology generated by Zariski open coverings and finite type morphisms which are universally submersive. They also show that the two topologies do not agree on non-Noetherian schemes, see [**GL01**, Example 4.5]. We return to (our version of) the h topology in More on Flatness, Section 38.34.

Before we can define the coverings in our topology we need to do a bit of work.

0DBD **Definition 34.8.1.** Let T be an affine scheme. A *standard ph covering* is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ constructed from a proper surjective morphism $f : U \to T$ and an affine open covering $U = \bigcup_{j=1,\ldots,m} U_j$ by setting $f_j = f|_{U_j}$.

It follows immediately from Chow's lemma that we can refine a standard ph covering by a standard ph covering corresponding to a surjective projective morphism.

0DBE **Lemma 34.8.2.** *Let* $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ *be a standard ph covering. Let* $T' \to T$ *be a morphism of affine schemes. Then* $\{U_j \times_T T' \to T'\}_{j=1,\ldots,m}$ *is a standard ph covering.*

Proof. Let $f : U \to T$ be proper surjective and let an affine open covering $U = \bigcup_{j=1,\ldots,m} U_j$ be given as in Definition 34.8.1. Then $U \times_T T' \to T'$ is proper surjective (Morphisms, Lemmas 29.9.4 and 29.41.5). Also, $U \times_T T' = \bigcup_{j=1,\ldots,m} U_j \times_T T'$ is an affine open covering. This concludes the proof. □

0DBF **Lemma 34.8.3.** *Let* T *be an affine scheme. Each of the following types of families of maps with target* T *has a refinement by a standard ph covering:*

(1) *any Zariski open covering of T,*
(2) $\{W_{ji} \to T\}_{j=1,\ldots,m, i=1,\ldots n_j}$ *where* $\{W_{ji} \to U_j\}_{i=1,\ldots,n_j}$ *and* $\{U_j \to T\}_{j=1,\ldots,m}$ *are standard ph coverings.*

Proof. Part (1) follows from the fact that any Zariski open covering of T can be refined by a finite affine open covering.

Proof of (3). Choose $U \to T$ proper surjective and $U = \bigcup_{j=1,\ldots,m} U_j$ as in Definition 34.8.1. Choose $W_j \to U_j$ proper surjective and $W_j = \bigcup W_{ji}$ as in Definition 34.8.1. By Chow's lemma (Limits, Lemma 32.12.1) we can find $W'_j \to W_j$ proper surjective and closed immersions $W'_j \to \mathbf{P}^{e_j}_{U_j}$. Thus, after replacing W_j by W'_j and $W_j = \bigcup W_{ji}$ by a suitable affine open covering of W'_j, we may assume there is a closed immersion $W_j \subset \mathbf{P}^{e_j}_{U_j}$ for all $j = 1, \ldots, m$.

Let $\overline{W}_j \subset \mathbf{P}^{e_j}_U$ be the scheme theoretic closure of W_j. Then $W_j \subset \overline{W}_j$ is an open subscheme; in fact W_j is the inverse image of $U_j \subset U$ under the morphism $\overline{W}_j \to U$. (To see this use that $W_j \to \mathbf{P}^{e_j}_U$ is quasi-compact and hence formation of the scheme theoretic image commutes with restriction to opens, see Morphisms, Section 29.6.) Let $Z_j = U \setminus U_j$ with reduced induced closed subscheme structure. Then

$$V_j = \overline{W}_j \amalg Z_j \to U$$

is proper surjective and the open subscheme $W_j \subset V_j$ is the inverse image of U_j. Hence for $v \in V_j$, $v \notin W_j$ we can pick an affine open neighbourhood $v \in V_{j,v} \subset V_j$ which maps into $U_{j'}$ for some $1 \leq j' \leq m$.

To finish the proof we consider the proper surjective morphism

$$V = V_1 \times_U V_2 \times_U \ldots \times_U V_m \longrightarrow U \longrightarrow T$$

and the covering of V by the affine opens

$$V_{1,v_1} \times_U \ldots \times_U V_{j-1,v_{j-1}} \times_U W_{ji} \times_U V_{j+1,v_{j+1}} \times_U \ldots \times_U V_{m,v_m}$$

These do indeed form a covering, because each point of U is in some U_j and the inverse image of U_j in V is equal to $V_1 \times \ldots \times V_{j-1} \times W_j \times V_{j+1} \times \ldots \times V_m$. Observe that the morphism from the affine open displayed above to T factors through W_{ji} thus we obtain a refinement. Finally, we only need a finite number of these affine opens as V is quasi-compact (as a scheme proper over the affine scheme T). □

0DBG **Definition 34.8.4.** Let T be a scheme. A *ph covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that f_i is locally of finite type and such that for every affine open $U \subset T$ there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{T_i \times_T U \to U\}_{i \in I}$.

A standard ph covering is a ph covering by Lemma 34.8.2.

0DBH **Lemma 34.8.5.** *A Zariski covering is a ph covering*[4].

Proof. This is true because a Zariski covering of an affine scheme can be refined by a standard ph covering by Lemma 34.8.3. □

0DES **Lemma 34.8.6.** *Let $f : Y \to X$ be a surjective proper morphism of schemes. Then $\{Y \to X\}$ is a ph covering.*

[4]We will see in More on Morphisms, Lemma 37.48.7 that fppf coverings (and hence syntomic, smooth, or étale coverings) are ph coverings as well.

Proof. Omitted. □

0ET9 **Lemma 34.8.7.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms such that f_i is locally of finite type for all i. The following are equivalent*

(1) *$\{T_i \to T\}_{i\in I}$ is a ph covering,*
(2) *there is a ph covering which refines $\{T_i \to T\}_{i\in I}$, and*
(3) *$\{\coprod_{i\in I} T_i \to T\}$ is a ph covering.*

Proof. The equivalence of (1) and (2) follows immediately from Definition 34.8.4 and the fact that a refinement of a refinement is a refinement. Because of the equivalence of (1) and (2) and since $\{T_i \to T\}_{i\in I}$ refines $\{\coprod_{i\in I} T_i \to T\}$ we see that (1) implies (3). Finally, assume (3) holds. Let $U \subset T$ be an affine open and let $\{U_j \to U\}_{j=1,\ldots,m}$ be a standard ph covering which refines $\{U \times_T \coprod_{i\in I} T_i \to U\}$. This means that for each j we have a morphism

$$h_j : U_j \longrightarrow U \times_T \coprod\nolimits_{i\in I} T_i = \coprod\nolimits_{i\in I} U \times_T T_i$$

over U. Since U_j is quasi-compact, we get disjoint union decompositions $U_j = \coprod_{i\in I} U_{j,i}$ by open and closed subschemes almost all of which are empty such that $h_j|_{U_{j,i}}$ maps $U_{j,i}$ into $U \times_T T_i$. It follows that

$$\{U_{j,i} \to U\}_{j=1,\ldots,m,\ i\in I,\ U_{j,i}\neq\emptyset}$$

is a standard ph covering (small detail omitted) refining $\{U \times_T T_i \to U\}_{i\in I}$. Thus (1) holds. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0DBI **Lemma 34.8.8.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a ph covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a ph covering and for each i we have a ph covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a ph covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a ph covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a ph covering.*

Proof. Assertion (1) is clear.

Proof of (3). The base change $T_i \times_T T' \to T'$ is locally of finite type by Morphisms, Lemma 29.15.4. hence we only need to check the condition on affine opens. Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard ph covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 34.8.2 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard ph covering. Note that $\{U'_j \to U'\}$ is a standard ph covering as well. By Lemma 34.8.3 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ can be refined by a standard ph covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Composition preserves being locally of finite type, see Morphisms, Lemma 29.15.3. Hence we only need to check the condition on affine opens. Let $U \subset T$ be affine open. First we pick a standard ph covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j\in J_{i_k}}$$

is a ph covering by part (3). As U_k is affine, we can find a standard ph covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 34.8.3 to see that $\{U_{ka} \to U\}$ can be refined by a standard ph covering. Since $\{U_{ka} \to U\}$ refines $\{T_{ij} \times_T U \to U\}$ this finishes the proof. □

0DBJ **Definition 34.8.9.** A *big ph site* is any site Sch_{ph} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of ph coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of ph coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big ph site of a scheme S, let us point out that the topology on a big ph site Sch_{ph} is in some sense induced from the ph topology on the category of all schemes.

0DBK **Lemma 34.8.10.** *Let Sch_{ph} be a big ph site as in Definition 34.8.9. Let $T \in \mathrm{Ob}(Sch_{ph})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary ph covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{ph} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard ph covering, then it is tautologically equivalent to a covering of Sch_{ph}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{ph}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemmas 34.8.5 and 34.8.8 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a ph covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{ph}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{ph} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{ph} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{ph} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

0DBL **Definition 34.8.11.** Let S be a scheme. Let Sch_{ph} be a big ph site containing S.

(1) The *big ph site of S*, denoted $(Sch/S)_{ph}$, is the site Sch_{ph}/S introduced in Sites, Section 7.25.
(2) The *big affine ph site of S*, denoted $(Aff/S)_{ph}$, is the full subcategory of $(Sch/S)_{ph}$ whose objects are affine U/S. A covering of $(Aff/S)_{ph}$ is any finite covering $\{U_i \to U\}$ of $(Sch/S)_{ph}$ with U_i and U affine.

Observe that the coverings in $(Aff/S)_{ph}$ are *not* given by standard ph coverings. The reason is simply that this would fail the second axiom of Sites, Definition 7.6.2. Rather, the coverings in $(Aff/S)_{ph}$ are those finite families $\{U_i \to U\}$ of finite type morphisms between affine objects of $(Sch/S)_{ph}$ which can be refined by a standard ph covering. We explicitly state and prove that the big affine ph site is a site.

0DBM **Lemma 34.8.12.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. Then $(Aff/S)_{ph}$ is a site.*

Proof. Reasoning as in the proof of Lemma 34.4.9 it suffices to show that the collection of finite ph coverings $\{U_i \to U\}$ with U, U_i affine satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a finite ph covering $\{T_i \to T\}_{i\in I}$ with T_i, T affine, and for each i a finite ph covering $\{T_{ij} \to T_i\}_{j\in J_i}$ with T_{ij} affine , then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a ph covering (Lemma 34.8.8), $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. □

0DBN **Lemma 34.8.13.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The underlying categories of the sites Sch_{ph}, $(Sch/S)_{ph}$, and $(Aff/S)_{ph}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{ph}$ has a final object, namely S/S.*

Proof. For Sch_{ph} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{ph})$. The fibre product $V \times_U W$ in Sch_{ph} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{ph}$. This proves the result for $(Sch/S)_{ph}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{ph}$. □

Next, we check that the big affine site defines the same topos as the big site.

0DBP **Lemma 34.8.14.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The functor $(Aff/S)_{ph} \to (Sch/S)_{ph}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{ph})$ to $Sh((Sch/S)_{ph})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{ph} \to (Sch/S)_{ph}$. Being cocontinuous follows because any ph covering of T/S, T affine, can be refined by a standard ph covering of T by definition. Hence (1) holds. We see u is continuous simply because a finite ph covering of an affine by affines is a ph covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a ph covering). □

0DBQ **Lemma 34.8.15.** *Let $\mathcal{F}$ be a presheaf on $(Sch/S)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and*
(2) *if $f : V \to U$ is proper surjective, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ as in (2) with U affine.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(Sch/S)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_{ph}$ which is proper and surjective. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. □

Next, we establish some relationships between the topoi associated to these sites.

0DBR **Lemma 34.8.16.** *Let Sch_{ph} be a big ph site. Let $f : T \to S$ be a morphism in Sch_{ph}. The functor*

$$u : (Sch/T)_{ph} \longrightarrow (Sch/S)_{ph}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{ph} \longrightarrow (Sch/T)_{ph}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{ph}) \longrightarrow Sh((Sch/S)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DBS **Lemma 34.8.17.** *Given schemes X, Y, Y in $(Sch/S)_{ph}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.8.16. □

34.9. The fpqc topology

022A

022B **Definition 34.9.1.** Let T be a scheme. An *fpqc covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

03L7 **Lemma 34.9.2.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) *$\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering,*
(2) *each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
(3) *each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha\in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

(4) *each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U. By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T. Since T is quasi-separated we see that $U_j = U'_j \cap f_{i_j}^{-1}(U) = (f_{i_j}|_{U'_j})^{-1}(U)$ is quasi-compact by Schemes, Lemma 26.21.14. Thus (2) holds. Since it is clear that (2) implies (4) the proof is finished. □

0H7E **Example 34.9.3.** Let $X = X_1 \cup X_2$ where $X_1 = X_2 = \mathrm{Spec}(k[t_1, t_2, \ldots])$ be the non-quasi-separated scheme from Schemes, Example 26.21.4, and $Y = X_1 \amalg \mathrm{Spec}(\mathcal{O}_{X_2,0})$. Then the obvious morphism $f : Y \to X$ is flat and surjective, and (since Y is quasi-compact) trivially satisfies (4) of Lemma 34.9.2. But $\{f : Y \to X\}$ isn't an fpqc covering. Namely, we claim there is no quasi-compact open W of Y whose image by f is X_2. Namely, $f^{-1}(X_2) = X_1 \setminus \{0\} \amalg \mathrm{Spec}(\mathcal{O}_{X_2,0})$. Hence if W exists, then $W = U \amalg \mathrm{Spec}(\mathcal{O}_{X_2,0})$ for some quasi-compact open U of $X_1 \setminus \{0\} = X_2 \setminus \{0\}$. But then we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in k[t_1, t_2, \ldots]$ vanishing at 0. Suppose that $f_1, \ldots, f_n$ use only the variables $x_1, \ldots, x_m$. Then $p = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the $(m+1)$st spot is a closed point of X_2 not in the image of W.

040I **Lemma 34.9.4.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) $\{f_i : T_i \to T\}_{i\in I}$ *is an fpqc covering, and*
(2) *setting* $T' = \coprod_{i\in I} T_i$, *and* $f = \coprod_{i\in I} f_i$ *the family* $\{f : T' \to T\}$ *is an fpqc covering.*

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$. Then $U_1 \amalg \ldots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 34.9.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j. By Lemma 34.9.2 we see that (1) holds. □

03L8 **Lemma 34.9.5.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *the family $\{f_i : T_i \to T\}_{i\in I}$ can be refined by an fpqc covering of T.*

Then $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering of T.

Proof. Let $\{g_j : X_j \to T\}_{j\in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \to T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f_{i_{j_k}}^{-1}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering by Lemma 34.9.2. □

03L9 **Lemma 34.9.6.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *there exists an fpqc covering $\{g_j : S_j \to T\}_{j\in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i\in I}$ is an fpqc covering.*

Then $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering of T.

Proof. We will use Lemma 34.9.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \mathrm{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 34.9.5. □

022C **Lemma 34.9.7.** *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 34.7.2. Let $\{f_i : U_i \to U\}_{i\in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 34.9.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j\in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 29.25.10), we see that $V = \bigcup_{i\in I} \bigcup_{j\in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. □

The fpqc[5] topology cannot be treated in the same way as the fppf topology[6]. Namely, suppose that R is a nonzero ring. We will see in Lemma 34.9.15 that there does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A. If $R = k$ is a field, then the reason for this unboundedness is that there does not exist a field extension of k such that every field extension of k is contained in it.

If you ignore set theoretic difficulties, then you run into presheaves which do not have a sheafification, see [**Wat75**, Theorem 5.5]. A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

022D **Lemma 34.9.8.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.*

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 29.25.8 and 29.25.6). Thus we can apply Lemma 34.9.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $U_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \mathrm{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering by Lemma 34.9.2. □

022E **Lemma 34.9.9.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement*

[5] The letters fpqc stand for "fidèlement plat quasi-compacte".

[6] A more precise statement would be that the analogue of Lemma 34.7.7 for the fpqc topology does not hold.

of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. This follows directly from the definition. □

022F **Definition 34.9.10.** Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

03LA **Lemma 34.9.11.** *Let T be an affine scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard fpqc covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a standard fpqc covering.*

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 29.25.8 and 29.25.6) and that fibre products of affine schemes are affine (Schemes, Lemma 26.17.2). □

03LB **Lemma 34.9.12.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i\in I}$.*

Then $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering of T.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 34.9.5. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 34.9.6. □

022G **Definition 34.9.13.** Let F be a contravariant functor on the category of schemes with values in sets.

(1) Let $\{U_i \to T\}_{i\in I}$ be a family of morphisms of schemes with fixed target. We say that F *satisfies the sheaf property for the given family* if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i\times_T U_j} = \xi_j|_{U_i\times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) We say that F *satisfies the sheaf property for the fpqc topology* if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of fpqc sheaves as we explained above.

022H **Lemma 34.9.14.** *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies*

(1) *the sheaf property for every Zariski covering, and*
(2) *the sheaf property for any standard fpqc covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

By Lemma 34.9.9 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. □

The following lemma is here just to point out set theoretical difficulties do indeed arise and should be ignored by most readers.

0BBK **Lemma 34.9.15.** *Let R be a nonzero ring. There does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A.*

Proof. Let us first explain this when $R = k$ is a field. For any set I consider the purely transcendental field extension $k_I = k(\{t_i\}_{i \in I})/k$. Since $k \to k_I$ is faithfully flat we see that $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(k)\}_{j \in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ then the morphisms $S_{\alpha,j} \to \mathrm{Spec}(k)$ factor through $\mathrm{Spec}(k_I)$. Since $\mathcal{U}_\alpha$ is a covering, at least some $S_{\alpha,j}$ is nonempty. Pick a point $s \in S_{\alpha,j}$. Since we have the factorization $S_{\alpha,j} \to \mathrm{Spec}(k_I) \to \mathrm{Spec}(k)$ we obtain a homomorphism of fields $k_I \to \kappa(s)$. In particular, we see that the cardinality of $\kappa(s)$ is at least the cardinality of I. Thus if we take I to be a set of cardinality bigger than the cardinalities of the residue fields of all the schemes $S_{\alpha,j}$, then such a factorization does not exist and the lemma holds for $R = k$.

General case. Since R is nonzero it has a maximal prime ideal $\mathfrak{m}$ with residue field κ. Let I be a set and consider $R_I = S_I^{-1} R[\{t_i\}_{i \in I}]$ where $S_I \subset R[\{t_i\}_{i \in I}]$ is the multiplicative subset of $f \in R[\{t_i\}_{i \in I}]$ such that f maps to a nonzero element of $R/\mathfrak{p}[\{t_i\}_{i \in I}]$ for all primes $\mathfrak{p}$ of R. Then R_I is a faithfully flat R-algebra and

$\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$ is an fpqc covering. We leave it as an exercise to the reader to show that $R_I \otimes_R \kappa \cong \kappa(\{t_i\}_{i\in I}) = \kappa_I$ with notation as above (hint: use that $R \to \kappa$ is surjective and that any $f \in R[\{t_i\}_{i\in I}]$ one of whose monomials occurs with coefficient 1 is an element of S_I). Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(R)\}_{j\in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$, then by base change we conclude that $\{S_{\alpha,j} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa)\}$ refines $\{\mathrm{Spec}(\kappa_I) \to \mathrm{Spec}(\kappa)\}$. Hence by the result of the previous paragraph, there exists an I such that this is not the case and the lemma is proved. □

34.10. The V topology

0ETA The V topology is stronger than all other topologies in this chapter. Roughly speaking it is generated by Zariski coverings and by quasi-compact morphisms satisfying a lifting property for specializations (Lemma 34.10.13). However, the procedure we will use to define V coverings is a bit different. We will first define standard V coverings of affines and then use these to define V coverings in general. Typographical point: in the literature sometimes "v-covering" is used instead of "V covering".

0ETB **Definition 34.10.1.** Let T be an affine scheme. A *standard V covering* is a finite family $\{T_j \to T\}_{j=1,\ldots,m}$ with T_j affine such that for every morphism $g : \mathrm{Spec}(V) \to T$ where V is a valuation ring, there is an extension $V \subset W$ of valuation rings (More on Algebra, Definition 15.124.1), an index $1 \leq j \leq m$, and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & T_j \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \xrightarrow{g} & T \end{array}$$

We first prove a few basic lemmas about this notion.

0ETC **Lemma 34.10.2.** *A standard fpqc covering is a standard V covering.*

Proof. Let $\{X_i \to X\}_{i=1,\ldots,n}$ be a standard fpqc covering (Definition 34.9.10). Let $g : \mathrm{Spec}(V) \to X$ be a morphism where V is a valuation ring. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(V)$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\mathrm{Spec}(V) \times_X X_i$ has a point x_i' mapping to the closed point of $\mathrm{Spec}(V)$. Since $\mathrm{Spec}(V) \times_X X_i \to \mathrm{Spec}(V)$ is flat we can find a specialization $x_i'' \rightsquigarrow x_i'$ of points of $\mathrm{Spec}(V) \times_X X_i$ with x_i'' mapping to the generic point of $\mathrm{Spec}(V)$, see Morphisms, Lemma 29.25.9. By Schemes, Lemma 26.20.4 we can choose a valuation ring W and a morphism $h : \mathrm{Spec}(W) \to \mathrm{Spec}(V) \times_X X_i$ such that h maps the generic point of $\mathrm{Spec}(W)$ to x_i'' and the closed point of $\mathrm{Spec}(W)$ to x_i'. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & X \end{array}$$

where $V \to W$ is an extension of valuation rings. This proves the lemma. □

0ETD **Lemma 34.10.3.** *A standard ph covering is a standard V covering.*

Proof. Let T be an affine scheme. Let $f : U \to T$ be a proper surjective morphism. Let $U = \bigcup_{j=1,\ldots,m} U_j$ be a finite affine open covering. We have to show that $\{U_j \to T\}$ is a standard V covering, see Definition 34.8.1. Let $g : \mathrm{Spec}(V) \to T$ be a morphism where V is a valuation ring with fraction field K. Since $U \to T$ is surjective, we may choose a field extension L/K and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(L) & & \longrightarrow & & U \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(V) & \xrightarrow{g} & T \end{array}$$

By Algebra, Lemma 10.50.2 we can choose a valuation ring $W \subset L$ dominating V. By the valuative criterion of properness (Morphisms, Lemma 29.42.1) we can then find the morphism h in the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(W) & \xrightarrow[h]{} & U \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(V) & \xrightarrow{g} & X \end{array}$$

Since $\mathrm{Spec}(W)$ has a unique closed point, we see that $\mathrm{Im}(h)$ is contained in U_j for some j. Thus $h : \mathrm{Spec}(W) \to U_j$ is the desired lift and we conclude $\{U_j \to T\}$ is a standard V covering. □

0ETE **Lemma 34.10.4.** *Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a standard V covering. Let $T' \to T$ be a morphism of affine schemes. Then $\{T_j \times_T T' \to T'\}_{j=1,\ldots,m}$ is a standard V covering.*

Proof. Let $\mathrm{Spec}(V) \to T'$ be a morphism where V is a valuation ring. By assumption we can find an extension of valuation rings $V \subset W$, an i, and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & T_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & T \end{array}$$

By the universal property of fibre products we obtain a morphism $\mathrm{Spec}(W) \to T' \times_T T_i$ as desired. □

0ETF **Lemma 34.10.5.** *Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a standard V covering. Let $\{T_{ji} \to T_j\}_{i=1,\ldots n_j}$ be a standard V covering. Then $\{T_{ji} \to T\}_{i,j}$ is a standard V covering.*

Proof. This follows formally from the observation that if $V \subset W$ and $W \subset \Omega$ are extensions of valuation rings, then $V \subset \Omega$ is an extension of valuation rings. □

0ETG **Lemma 34.10.6.** *Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a family of morphisms with T_j affine for all j. The following are equivalent*

1. *$\{T_j \to T\}_{j=1,\ldots,m}$ is a standard V covering,*
2. *there is a standard V covering which refines $\{T_j \to T\}_{j=1,\ldots,m}$, and*
3. *$\{\coprod_{j=1,\ldots,m} T_j \to T\}$ is a standard V covering.*

Proof. Omitted. Hints: This follows almost immediately from the definition. The only slightly interesting point is that a morphism from the spectrum of a local ring into $\coprod_{j=1,\ldots,m} T_j$ must factor through some T_j. □

0ETH **Definition 34.10.7.** Let T be a scheme. A *V covering of T* is a family of morphisms $\{T_i \to T\}_{i\in I}$ of schemes such that for every affine open $U \subset T$ there exists a standard V covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{T_i \times_T U \to U\}_{i\in I}$.

The V topology has the same set theoretical problems as the fpqc topology. Thus we refrain from defining V sites and we will not consider cohomology with respect to the V topology. On the other hand, given a $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the V topology, see below. Moreover, we can wonder about descent of object in the V topology, etc.

0ETI **Lemma 34.10.8.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms. The following are equivalent*

(1) *$\{T_i \to T\}_{i\in I}$ is a V covering,*
(2) *there is a V covering which refines $\{T_i \to T\}_{i\in I}$, and*
(3) *$\{\coprod_{i\in I} T_i \to T\}$ is a V covering.*

Proof. Omitted. Hint: compare with the proof of Lemma 34.8.7. □

0ETJ **Lemma 34.10.9.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a V covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a V covering and for each i we have a V covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a V covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a V covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a V covering.*

Proof. Assertion (1) is clear.

Proof of (3). Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard V covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 34.10.4 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard V covering. Note that $\{U'_j \to U'\}$ is a standard V covering (for example by Lemma 34.10.2). By Lemma 34.10.5 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ is a standard V covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Let $U \subset T$ be affine open. First we pick a standard V covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j\in J_{i_k}}$$

is a V covering by part (3). As U_k is affine, we can find a standard V covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 34.10.5 to see that $\{U_{ka} \to U\}$ is a standard V covering which refines $\{T_{ij} \times_T U \to U\}$. This finishes the proof. □

0ETK **Lemma 34.10.10.** *Any fpqc covering is a V covering. A fortiori, any fppf, syntomic, smooth, étale or Zariski covering is a V covering. Also, a ph covering is a V covering.*

Proof. An fpqc covering can affine locally be refined by a standard fpqc covering, see Lemmas 34.9.9. A standard fpqc covering is a standard V covering, see Lemma 34.10.2. Hence the first statement follows from our definition of V covers in terms of standard V coverings. The conclusion for fppf, syntomic, smooth, étale or Zariski coverings follows as these are fpqc coverings, see Lemma 34.9.7.

The statement on ph coverings follows from Lemma 34.10.3 in the same manner. □

0ETL **Definition 34.10.11.** Let F be a contravariant functor on the category of schemes with values in sets. We say that F *satisfies the sheaf property for the V topology* if it satisfies the sheaf property for any V covering (see Definition 34.9.13).

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of V sheaves as we explained above.

0ETM **Lemma 34.10.12.** *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the V topology if and only if it satisfies*

(1) *the sheaf property for every Zariski covering, and*
(2) *the sheaf property for any standard V covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for a standard V covering of the form $\{V \to U\}$, i.e., consisting of a single arrow.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be a V covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

We can find a standard V covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard V covering (Lemma 34.10.4) and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}_{i=1,\ldots,n}$ is a standard V covering, then $\coprod_{i=1,\ldots,n} T_i \to T$ is a morphism of affine schemes which is clearly also a standard V covering. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. □

The following lemma shows that being a V covering is related to the possibility of lifting specializations.

0ETN **Lemma 34.10.13.** *Let $X \to Y$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *$\{X \to Y\}$ is a V covering,*
(2) *for any valuation ring V and morphism $g : \mathrm{Spec}(V) \to Y$ there exists an extension of valuation rings $V \subset W$ and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

(3) *for any morphism $Z \to Y$ and specialization $z' \rightsquigarrow z$ of points in Z, there is a specialization $w' \rightsquigarrow w$ of points in $Z \times_Y X$ mapping to $z' \rightsquigarrow z$.*

Proof. Assume (1) and let $g : \mathrm{Spec}(V) \to Y$ be as in (2). Since V is a local ring there is an affine open $U \subset Y$ such that g factors through U. By Definition 34.10.7 we can find a standard V covering $\{U_j \to U\}$ refining $\{X \times_Y U \to U\}$. By Definition 34.10.1 we can find a j, an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(W) & \longrightarrow & U_j & \dashrightarrow & X \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(V) & \longrightarrow & Y & & \end{array}$$

We have the dotted arrow making the diagram commute by the refinement property of the covering and we see that (2) holds.

Assume (2) and let $Z \to Y$ and $z' \rightsquigarrow z$ be as in (3). By Schemes, Lemma 26.20.4 we can find a valuation ring V and a morphism $\mathrm{Spec}(V) \to Z$ such that the closed point of $\mathrm{Spec}(V)$ maps to z and the generic point of $\mathrm{Spec}(V)$ maps to z'. By (2) we can find an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(W) & & \longrightarrow & & X \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

The generic and closed points of $\mathrm{Spec}(W)$ map to points $w' \rightsquigarrow w$ in $Z \times_Y X$ via the induced morphism $\mathrm{Spec}(W) \to Z \times_Y X$. This shows that (3) holds.

Assume (3) holds and let $U \subset Y$ be an affine open. Choose a finite affine open covering $U \times_Y X = \bigcup_{j=1,\ldots,m} U_j$. This is possible as $X \to Y$ is quasi-compact. We claim that $\{U_j \to U\}$ is a standard V covering. The claim implies (1) is true and finishes the proof of the lemma. In order to prove the claim, let V be a valuation ring and let $g : \mathrm{Spec}(V) \to U$ be a morphism. By (3) we find a specialization $w' \rightsquigarrow w$ of points of

$$T = \mathrm{Spec}(V) \times_X Y = \mathrm{Spec}(V) \times_U (U \times_X Y)$$

such that w' maps to the generic point of $\mathrm{Spec}(V)$ and w maps to the closed point of $\mathrm{Spec}(V)$. By Schemes, Lemma 26.20.4 we can find a valuation ring W and a morphism $\mathrm{Spec}(W) \to T$ such that the generic point of $\mathrm{Spec}(W)$ maps to w' and the closed point of $\mathrm{Spec}(W)$ maps to w. The composition $\mathrm{Spec}(W) \to T \to \mathrm{Spec}(V)$

corresponds to an inclusion $V \subset W$ which presents W as an extension of the valuation ring V. Since $T = \bigcup \mathrm{Spec}(V) \times_U U_j$ is an open covering, we see that $\mathrm{Spec}(W) \to T$ factors through $\mathrm{Spec}(V) \times_U U_j$ for some j. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & U \end{array}$$

and the proof of the claim is complete. □

A V covering gives a universally submersive family of maps. The converse of this lemma is false, see Examples, Section 110.79.

0ETP **Lemma 34.10.14.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a V covering. Then*

$$\coprod_{i \in I} f_i : \coprod_{i \in I} X_i \longrightarrow X$$

is a universally submersive morphism of schemes (Morphisms, Definition 29.24.1).

Proof. We will use without further mention that the base change of a V covering is a V covering (Lemma 34.10.9). In particular it suffices to show that the morphism is submersive. Being submersive is clearly Zariski local on the base. Thus we may assume X is affine. Then $\{X_i \to X\}$ can be refined by a standard V covering $\{Y_j \to X\}$. If we can show that $\coprod Y_j \to X$ is submersive, then since there is a factorization $\coprod Y_j \to \coprod X_i \to X$ we conclude that $\coprod X_i \to X$ is submersive. Set $Y = \coprod Y_j$ and consider the morphism of affines $f : Y \to X$. By Lemma 34.10.13 we know that we can lift any specialization $x' \rightsquigarrow x$ in X to some specialization $y' \rightsquigarrow y$ in Y. Thus if $T \subset X$ is a subset such that $f^{-1}(T)$ is closed in Y, then $T \subset X$ is closed under specialization. Since $f^{-1}(T) \subset Y$ with the reduced induced closed subscheme structure is an affine scheme, we conclude that $T \subset X$ is closed by Algebra, Lemma 10.41.5. Hence f is submersive. □

34.11. Change of topologies

03FE Let $f : X \to Y$ be a morphism of schemes over a base scheme S. In this case we have the following morphisms of sites[7] (with suitable choices of sites as in Remark 34.11.1 below):

(1) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
(2) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
(3) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
(4) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
(5) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
(6) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
(7) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
(8) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
(9) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
(10) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
(11) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,

[7] We have not included the comparison between the ph topology and the others; for this see More on Morphisms, Remark 37.48.8.

(12) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
(13) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
(14) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
(15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,
(16) $(Sch/X)_{fppf} \longrightarrow Y_{étale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{étale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{étale}$,
(19) $(Sch/X)_{étale} \longrightarrow Y_{étale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,
(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{étale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar}$,
(25) $X_{étale} \longrightarrow Y_{étale}$,
(26) $X_{étale} \longrightarrow Y_{Zar}$,
(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{étale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

03FF **Remark 34.11.1.** Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, étale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\mathrm{Cov}_\tau \subset \mathrm{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

34.12. Change of big sites

022I In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, étale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that Sch_τ *is contained in* $Sch'_{\tau'}$ if $\mathrm{Ob}(Sch_\tau) \subset \mathrm{Ob}(Sch'_{\tau'})$ and $\mathrm{Cov}(Sch_\tau) \subset \mathrm{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

022J **Lemma 34.12.1.** *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 3.9.2 and 3.11.1 allow one to start with any initially given set of schemes and coverings. □

022K **Lemma 34.12.2.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8. There are morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 7.21.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 34.3.6, 34.4.7, 34.5.7, 34.6.7, or 34.7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. □

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto \operatorname{colim}_{V \to U} \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 7.21.5 and 7.21.8.

34.13. Extending functors

0EUV Let us start with a simple example which explains what we are doing. Let R be a ring. Suppose F is a functor defined on the category $\mathcal{C}$ of R-algebras of the form

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

for $n, m \geq 0$ integers and $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ elements. Then for any R-algebra B we can define

$$F'(B) = \operatorname{colim}_{A \to B,\ A \in \mathcal{C}} F(A)$$

It turns out F' is the unique functor on the category of all R-algebras which extends F and commutes with filtered colimits. The same procedure works in the category of schemes if we impose that our functor is a Zariski sheaf.

0EUW **Lemma 34.13.1.** *Let S be a scheme. Let $\mathcal{C}$ be a full subcategory of the category Sch/S of all schemes over S. Assume*

(1) *if $X \to S$ is an object of $\mathcal{C}$ and $U \subset X$ is an affine open, then $U \to S$ is isomorphic to an object of $\mathcal{C}$,*
(2) *if V is an affine scheme lying over an affine open $U \subset S$ such that $V \to U$ is of finite presentation, then $V \to S$ is isomorphic to an object of $\mathcal{C}$.*

Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume

(a) *for any Zariski covering $\{f_i : X_i \to X\}_{i \in I}$ with X, X_i objects of $\mathcal{C}$ we have the sheaf condition for F and this family*[8]*,*
(b) *if $X = \lim X_i$ is a directed limit of affine schemes over S with X, X_i objects of $\mathcal{C}$, then $F(X) = \operatorname{colim} F(X_i)$.*

Then there is a unique way to extend F to a functor $F' : (Sch/S)^{opp} \to Sets$ satisfying the analogues of (a) and (b), i.e., F' satisfies the sheaf condition for any Zariski covering and $F'(X) = \operatorname{colim} F'(X_i)$ whenever $X = \lim X_i$ is a directed limit of affine schemes over S.

Proof. The idea will be to first extend F to a sufficiently large collection of affine schemes over S and then use the Zariski sheaf property to extend to all schemes.

Suppose that V is an affine scheme over S whose structure morphism $V \to S$ factors through some affine open $U \subset S$. In this case we can write

$$V = \lim V_i$$

as a cofiltered limit with $V_i \to U$ of finite presentation and V_i affine. See Algebra, Lemma 10.127.2. By conditions (1) and (2) we may replace our V_i by objects of $\mathcal{C}$. Observe that $V_i \to S$ is locally of finite presentation (if S is quasi-separated, then these morphisms are actually of finite presentation). Then we set

$$F'(V) = \operatorname{colim} F(V_i)$$

Actually, we can give a more canonical expression, namely

$$F'(V) = \operatorname{colim}_{V \to V'} F(V')$$

where the colimit is over the category of morphisms $V \to V'$ over S where V' is an object of $\mathcal{C}$ whose structure morphism $V' \to S$ is locally of finite presentation. The reason this is the same as the first formula is that by Limits, Proposition 32.6.1 our inverse system V_i is cofinal in this category! Finally, note that if V were an object of $\mathcal{C}$, then $F'(V) = F(V)$ by assumption (b).

The second formula turns F' into a contravariant functor on the category formed by affine schemes V over S whose structure morphism factors through an affine open of S. Let V be such an affine scheme over S and suppose that $V = \bigcup_{k=1,\ldots,n} V_k$ is a finite open covering by affines. Then it makes sense to ask if the sheaf condition holds for F' and this open covering. This is true and easy to show: write $V = \lim V_i$ as in the previous paragraph. By Limits, Lemma 32.4.11 for all sufficiently large i we can find affine opens $V_{i,k} \subset V_i$ compatible with transition maps pulling back to V_k in V. Thus

$$F'(V_k) = \operatorname{colim} F(V_{i,k}) \quad \text{and} \quad F'(V_k \cap V_l) = \operatorname{colim} F(V_{i,k} \cap V_{i,l})$$

Strictly speaking in these formulas we need to replace $V_{i,k}$ and $V_{i,k} \cap V_{i,l}$ by isomorphic affine objects of $\mathcal{C}$ before applying the functor F. Since I is directed the

[8] As we do not know that $X_i \times_X X_j$ is in $\mathcal{C}$ this has to be interpreted as follows: by property (1) there exist Zariski coverings $\{U_{ijk} \to X_i \times_X X_j\}_{k \in K_{ij}}$ with U_{ijk} an object of $\mathcal{C}$. Then the sheaf condition says that $F(X)$ is the equalizer of the two maps from $\prod F(X_i)$ to $\prod F(U_{ijk})$.

colimits pass through equalizers. Hence the sheaf condition (b) for F and the Zariski coverings $\{V_{i,k} \to V_i\}$ implies the sheaf condition for F' and this covering.

Let X be a general scheme over S. Let $\mathcal{B}_X$ denote the collection of affine opens of X whose structure morphism to S maps into an affine open of S. It is clear that $\mathcal{B}_X$ is a basis for the topology of X. By the result of the previous paragraph and Sheaves, Lemma 6.30.4 we see that F' is a sheaf on $\mathcal{B}_X$. Hence F' restricted to $\mathcal{B}_X$ extends uniquely to a sheaf F'_X on X, see Sheaves, Lemma 6.30.6. If X is an object of $\mathcal{C}$ then we have a canonical identification $F'_X(X) = F(X)$ by the agreement of F' and F on the objects for which they are both defined and the fact that F satisfies the sheaf condition for Zariski coverings.

Let $f : X \to Y$ be a morphism of schemes over S. We get a unique f-map from F'_Y to F'_X compatible with the maps $F'(V) \to F'(U)$ for all $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $f(U) \subset V$, see Sheaves, Lemma 6.30.16. We omit the verification that these maps compose correctly given morphisms $X \to Y \to Z$ of schemes over S. We also omit the verification that if f is a morphism of $\mathcal{C}$, then the induced map $F'_Y(Y) \to F'_X(X)$ is the same as the map $F(Y) \to F(X)$ via the identifications $F'_X(X) = F(X)$ and $F'_Y(Y) = F(Y)$ above. In this way we see that the desired extension of F is the functor which sends X/S to $F'_X(X)$.

Property (a) for the functor $X \mapsto F'_X(X)$ is almost immediate from the construction; we omit the details. Suppose that $X = \lim_{i \in I} X_i$ is a directed limit of affine schemes over S. We have to show that

$$F'_X(X) = \operatorname{colim}_{i \in I} F'_{X_i}(X_i)$$

First assume that there is some $i \in I$ such that $X_i \to S$ factors through an affine open $U \subset S$. Then F' is defined on X and on $X_{i'}$ for $i' \geq i$ and we see that $F'_{X_{i'}}(X_{i'}) = F'(X_{i'})$ for $i' \geq i$ and $F'_X(X) = F'(X)$. In this case every arrow $X \to V$ with V locally of finite presentation over S factors as $X \to X_{i'} \to V$ for some $i' \geq i$, see Limits, Proposition 32.6.1. Thus we have

$$\begin{aligned} F'_X(X) &= F'(X) \\ &= \operatorname{colim}_{X \to V} F(V) \\ &= \operatorname{colim}_{i' \geq i} \operatorname{colim}_{X_{i'} \to V} F(V) \\ &= \operatorname{colim}_{i' \geq i} F'(X_{i'}) \\ &= \operatorname{colim}_{i' \geq i} F'_{X_{i'}}(X_{i'}) \\ &= \operatorname{colim}_{i' \in I} F'_{X_{i'}}(X_{i'}) \end{aligned}$$

as desired. Finally, in general we pick any $i \in I$ and we choose a finite affine open covering $V_i = V_{i,1} \cup \ldots \cup V_{i,n}$ such that $V_{i,k} \to S$ factors through an affine open of S. Let $V_k \subset V$ and $V_{i',k}$ for $i' \geq i$ be the inverse images of $V_{i,k}$. By the previous case we see that

$$F'_{V_k}(V_k) = \operatorname{colim}_{i' \geq i} F'_{V_{i',k}}(V_{i',k})$$

and

$$F'_{V_k \cap V_l}(V_k \cap V_l) = \operatorname{colim}_{i' \geq i} F'_{V_{i',k} \cap V_{i',l}}(V_{i',k} \cap V_{i',l})$$

By the sheaf property and exactness of filtered colimits we find that $F'_X(X) = \operatorname{colim}_{i \in I} F'_{X_i}(X_i)$ also in this case. This finishes the proof of property (b) and hence finishes the proof of the lemma. $\square$

049N **Lemma 34.13.2.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.*

(1) *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard τ-covering of T, see Definitions 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5. Then there exists an index i and a standard τ-covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*
(2) *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard τ-coverings of T_i. If $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*
(3) *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard τ-coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard τ-coverings of T is the colimit over I of the categories of standard τ-coverings of T_i.

Proof. Let us prove this for $\tau = fppf$. By Limits, Lemma 32.10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 32.8.2 and 32.8.7 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 29.25.10. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 32.4.11 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win.

For $\tau \in \{Zariski, étale, smooth, syntomic\}$ a standard τ-covering is a standard fppf covering. Hence the fully faithfulness of the functor holds. The only issue is to show that given a standard fppf covering $\mathcal{V}_i$ for some i such that $\mathcal{V}_i \times_{T_i} T$ is a standard τ-covering, then $\mathcal{V}_i \times_{T_i} T_{i'}$ is a standard τ-covering for all $i' \gg i$. This follows immediately from Limits, Lemmas 32.8.12, 32.8.10, 32.8.9, and 32.8.16. □

0GDW **Lemma 34.13.3.** *Let S, $\mathcal{C}$, F satisfy conditions (1), (2), (a), and (b) of Lemma 34.13.1 and denote $F' : (Sch/S)^{opp} \to Sets$ the unique extension constructed in the lemma. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Assume*

(c) *for any standard τ-covering $\{V_i \to V\}_{i=1,\ldots,n}$ of affines in Sch/S such that $V \to S$ factors through an affine open $U \subset S$ and $V \to U$ is of finite presentation, the sheaf condition hold for F and $\{V_i \to V\}_{i=1,\ldots,n}$*[9].

Then F' satisfies the sheaf condition for all τ-coverings.

Proof. Let X be a scheme over S and let $\{X_i \to X\}_{i \in I}$ be a τ-covering. Let $s_i \in F'(X_i)$ be elements such that s_i and s_j map to the same element of $F'(X_i \times_X X_j)$ for all $i, j \in I$. We have to show that there is a unique element $s \in F'(X)$ restricting to $s_i \in F'(X_i)$ for all $i \in I$.

[9]This makes sense as V, V_i, and $V_i \times_V V_j$ are isomorphic to objects of $\mathcal{C}$ by (2).

Special case: X is an affine such that the structure morphism maps into an affine open U of S and the covering $\{X_i \to X\}_{i\in I}$ is a standard τ-covering. In this case we can write

$$X = \lim V_k$$

as a cofiltered limit with $V_k \to U$ of finite presentation and V_k affine. See Algebra, Lemma 10.127.2. By Lemma 34.13.2 there exists a k and a standard τ-covering $\{V_{k,i} \to V_k\}_{i\in I}$ whose base change to X is the given covering. For $k' \geq k$ denote $\{V_{k',i} \to V_{k'}\}_{i\in I}$ the base change to $V_{k'}$ of our covering. Then we see that

$$\begin{aligned}
F'(X) &= \operatorname{colim}_{k'\geq k} F(V_k) \\
&= \operatorname{colim}_{k'\geq k} \text{Equalizer}\Big(\prod F(V_{k',i}) \rightrightarrows \prod F(V_{k',i} \times_{V_{k'}} V_{k',j})\Big) \\
&= \text{Equalizer}\Big(\operatorname{colim}_{k'\geq k} \prod F(V_{k',i}) \rightrightarrows \operatorname{colim}_{k'\geq k} \prod F(V_{k',i} \times_{V_{k'}} V_{k',j})\Big) \\
&= \text{Equalizer}\Big(\prod F'(X_i) \rightrightarrows \prod F'(X_i \times_X X_j)\Big)
\end{aligned}$$

The first equality holds by construction of F'. The second holds by assumption (c). The third holds because filtered colimits are exact. The fourth again holds by construction of F'. In this way we find that the sheaf property holds for F' with respect to $\{X_i \to X\}_{i\in I}$.

General case. Choose an affine open covering $X = \bigcup U_k$ such that each U_k maps into an affine open of S. For every k we may choose a standard τ-covering $\{V_{k,j} \to U_k\}_{j=1,\ldots,m_k}$ which refines $\{X_i \times_X U_k \to U_k\}_{i\in I}$. For each $j \in \{1,\ldots,m_k\}$ choose an index $i_{k,j} \in I$ and a morphism $g_{k,j} : V_{k,j} \to X_{i_{k,j}}$ over X. Let $s_{k,j}$ be the element of $F'(V_{k,j})$ we get by restricting $s_{i_{k,j}}$ via $g_{k,j}$. Observe that $s_{k,j}$ and $s_{k',j'}$ restrict to the same element of $F'(V_{k,j} \times_X V_{k',j'})$ for all k and k' and all $j \in \{1,\ldots,m_k\}$ and $j' \in \{1,\ldots,m_{k'}\}$; verification omitted. In particular, by the result of the previous paragraph there is a unique element $s_k \in F'(U_k)$ restricting to $s_{k,j}$ for all j. With this notation we are ready to finish the proof.

Proof of uniqueness of s: this is true because F' satisfies the sheaf property for Zariski coverings and $s|_{U_k}$ must be equal to s_k because both restrict to $s_{k,j}$ for all j. This uniqueness then shows that s_k and $s_{k'}$ must restrict to the same section of F' over (the non-affine scheme) $U_k \cap U_{k'}$ because these sections restrict to the same section over the τ-covering $\{V_{k,j} \times_X V_{k',j'} \to U_k \cap U_{k'}\}$. Thus by the sheaf property for Zariski coverings, there is a unique section s of F' over X whose restriction to U_k is s_k. We omit the verification (similar to the above) that s restricts to s_i over X_i. □

0EUX **Lemma 34.13.4.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let S be a scheme contained in a big site Sch_τ. Let $F : (Sch/S)^{opp}_\tau \to Sets$ be a τ-sheaf satisfying property (b) of Lemma 34.13.1 with $\mathcal{C} = (Sch/S)_\tau$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all τ-coverings.*

Proof. This follows from Lemma 34.13.3 applied with $\mathcal{C} = (Sch/S)_\tau$. Conditions (1), (2), (a), and (b) of Lemma 34.13.1 hold; we omit the details. Thus we get our unique extension F' to the category of all schemes over S. Finally, observe that any standard τ-covering is tautologically equivalent to a covering in $(Sch/S)_\tau$, see Sets, Lemma 3.9.9 as well as Lemmas 34.3.6, 34.4.7, 34.5.7, 34.6.7, and 34.7.7. By Sites, Lemma 7.8.4 the sheaf property passes through tautological equivalence of

coverings. Hence the fact that F is a τ-sheaf implies that property (c) of Lemma 34.13.3 holds and we conclude. □

34.14. Other chapters

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CHAPTER 35

Descent

0238 ## 35.1. Introduction

0239 In the chapter on topologies on schemes (see Topologies, Section 34.1) we introduced Zariski, étale, fppf, smooth, syntomic and fpqc coverings of schemes. In this chapter we discuss what kind of structures over schemes can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This is also meant to introduce the notions of descent, descent data, effective descent data, in the less formal setting of descent questions for quasi-coherent sheaves, schemes, etc. The formal notion, that of a stack over a site, is discussed in the chapter on stacks (see Stacks, Section 8.1).

35.2. Descent data for quasi-coherent sheaves

023A In this chapter we will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

023B **Definition 35.2.1.** Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times_S S_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow[\mathrm{pr}_{02}^*\varphi_{ik}]{} & \mathrm{pr}_2^*\mathcal{F}_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & \mathrm{pr}_1^*\mathcal{F}_j & \end{array}$$

of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^*\mathcal{F}_j \\ {\scriptstyle \mathrm{pr}_0^*\psi_i}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\psi_j} \\ \mathrm{pr}_0^*\mathcal{F}'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^*\mathcal{F}'_j \end{array}$$

commute.

A good example to keep in mind is the following. Suppose that $S = \bigcup S_i$ is an open covering. In that case we have seen descent data for sheaves of sets in Sheaves,

Section 6.33 where we called them "glueing data for sheaves of sets with respect to the given covering". Moreover, we proved that the category of glueing data is equivalent to the category of sheaves on S. We will show the analogue in the setting above when $\{S_i \to S\}_{i\in I}$ is an fpqc covering.

In the extreme case where the covering $\{S \to S\}$ is given by id_S a descent datum is necessarily of the form $(\mathcal{F}, \mathrm{id}_{\mathcal{F}})$. The cocycle condition guarantees that the identity on $\mathcal{F}$ is the only permitted map in this case. The following lemma shows in particular that to every quasi-coherent sheaf of $\mathcal{O}_S$-modules there is associated a unique descent datum with respect to any given family.

023C **Lemma 35.2.2.** *Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of schemes with fixed target. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.
(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
(3) *Given a second morphism $(g', \alpha' : I \to J, (g_i'))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g_i')^*\mathcal{F}_{\alpha'(i)}, (g_i' \times g_{i'}')^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g_i')^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g_i') : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Any family $\mathcal{U} = \{S_i \to S\}_{i\in I}$ is a refinement of the trivial covering $\{S \to S\}$ in a unique way. For a quasi-coherent sheaf $\mathcal{F}$ on S we denote simply $(\mathcal{F}|_{S_i}, can)$ the descent datum with respect to $\mathcal{U}$ obtained by the procedure above.

023D **Definition 35.2.3.** Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be a family of morphisms with target S.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{S \to S\}$ the *trivial descent datum*.
(2) The pullback of the trivial descent datum to $\{S_i \to S\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{S_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{S_i}, can)$.

023E **Lemma 35.2.4.** *Let S be a scheme. Let $S = \bigcup U_i$ be an open covering. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This follows immediately from Sheaves, Section 6.33 and the fact that being quasi-coherent is a local property, see Modules, Definition 17.10.1. □

To prove more we first need to study the case of modules over rings.

35.3. Descent for modules

023F Let $R \to A$ be a ring map. By Simplicial, Example 14.5.5 this gives rise to a cosimplicial R-algebra

$$A \rightleftarrows A \otimes_R A \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; A \otimes_R A \otimes_R A$$

Let us denote this $(A/R)_\bullet$ so that $(A/R)_n$ is the $(n+1)$-fold tensor product of A over R. Given a map $\varphi : [n] \to [m]$ the R-algebra map $(A/R)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod\nolimits_{\varphi(i)=0} a_i \otimes \prod\nolimits_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod\nolimits_{\varphi(i)=m} a_i$$

where we use the convention that the empty product is 1. Thus the first few maps, notation as in Simplicial, Section 14.5, are

$$\begin{array}{lcccc}
\delta^1_0 & : & a_0 & \mapsto & 1 \otimes a_0 \\
\delta^1_1 & : & a_0 & \mapsto & a_0 \otimes 1 \\
\sigma^0_0 & : & a_0 \otimes a_1 & \mapsto & a_0 a_1 \\
\delta^2_0 & : & a_0 \otimes a_1 & \mapsto & 1 \otimes a_0 \otimes a_1 \\
\delta^2_1 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes 1 \otimes a_1 \\
\delta^2_2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes a_1 \otimes 1 \\
\sigma^1_0 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 a_1 \otimes a_2 \\
\sigma^1_1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 \otimes a_1 a_2
\end{array}$$

and so on.

An R-module M gives rise to a cosimplicial $(A/R)_\bullet$-module $(A/R)_\bullet \otimes_R M$. In other words $M_n = (A/R)_n \otimes_R M$ and using the R-algebra maps $(A/R)_n \to (A/R)_m$ to define the corresponding maps on $M \otimes_R (A/R)_\bullet$.

The analogue to a descent datum for quasi-coherent sheaves in the setting of modules is the following.

023G **Definition 35.3.1.** Let $R \to A$ be a ring map.

(1) A *descent datum* (N, φ) *for modules with respect to* $R \to A$ is given by an A-module N and an isomorphism of $A \otimes_R A$-modules

$$\varphi : N \otimes_R A \to A \otimes_R N$$

such that the *cocycle condition* holds: the diagram of $A \otimes_R A \otimes_R A$-module maps

$$\begin{array}{ccc}
N \otimes_R A \otimes_R A & \xrightarrow{\varphi_{02}} & A \otimes_R A \otimes_R N \\
& {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\
& A \otimes_R N \otimes_R A &
\end{array}$$

commutes (see below for notation).

(2) A *morphism* $(N, \varphi) \to (N', \varphi')$ *of descent data* is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$\begin{array}{ccc}
N \otimes_R A & \xrightarrow{\varphi} & A \otimes_R N \\
{\scriptstyle \psi \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes \psi} \\
N' \otimes_R A & \xrightarrow{\varphi'} & A \otimes_R N'
\end{array}$$

is commutative.

In the definition we use the notation that $\varphi_{01} = \varphi \otimes \mathrm{id}_A$, $\varphi_{12} = \mathrm{id}_A \otimes \varphi$, and $\varphi_{02}(n \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes n_i$ if $\varphi(n \otimes 1) = \sum a_i \otimes n_i$. All three are $A \otimes_R A \otimes_R A$-module homomorphisms. Equivalently we have

$$\varphi_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^2_{ij})} (A/R)_2$$

where $\tau^2_{ij} : [1] \to [2]$ is the map $0 \mapsto i$, $1 \mapsto j$. Namely, $(A/R)_\bullet(\tau^2_{02})(a_0 \otimes a_1) = a_0 \otimes 1 \otimes a_1$, and similarly for the others[1].

We need some more notation to be able to state the next lemma. Let (N, φ) be a descent datum with respect to a ring map $R \to A$. For $n \geq 0$ and $i \in [n]$ we set

$$N_{n,i} = A \otimes_R \ldots \otimes_R A \otimes_R N \otimes_R A \otimes_R \ldots \otimes_R A$$

with the factor N in the ith spot. It is an $(A/R)_n$-module. If we introduce the maps $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ then we see that

$$N_{n,i} = N \otimes_{(A/R)_0,\ (A/R)_\bullet(\tau^n_i)} (A/R)_n$$

For $0 \leq i \leq j \leq n$ we let $\tau^n_{ij} : [1] \to [n]$ be the map such that 0 maps to i and 1 to j. Similarly to the above the homomorphism φ induces isomorphisms

$$\varphi^n_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^n_{ij})} (A/R)_n : N_{n,i} \longrightarrow N_{n,j}$$

of $(A/R)_n$-modules when $i < j$. If $i = j$ we set $\varphi^n_{ij} = \mathrm{id}$. Since these are all isomorphisms they allow us to move the factor N to any spot we like. And the cocycle condition exactly means that it does not matter how we do this (e.g., as a composition of two of these or at once). Finally, for any $\beta : [n] \to [m]$ we define the morphism

$$N_{\beta,i} : N_{n,i} \to N_{m,\beta(i)}$$

as the unique $(A/R)_\bullet(\beta)$-semi linear map such that

$$N_{\beta,i}(1 \otimes \ldots \otimes n \otimes \ldots \otimes 1) = 1 \otimes \ldots \otimes n \otimes \ldots \otimes 1$$

for all $n \in N$. This hints at the following lemma.

023H **Lemma 35.3.2.** *Let $R \to A$ be a ring map. Given a descent datum (N, φ) we can associate to it a cosimplicial $(A/R)_\bullet$-module $N_\bullet$[2] by the rules $N_n = N_{n,n}$ and given $\beta : [n] \to [m]$ setting we define*

$$N_\bullet(\beta) = (\varphi^m_{\beta(n)m}) \circ N_{\beta,n} : N_{n,n} \longrightarrow N_{m,m}.$$

This procedure is functorial in the descent datum.

Proof. Here are the first few maps where $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$

$$\begin{array}{lcccccc}
\delta^1_0 & : & N & \to & A \otimes N & n & \mapsto & 1 \otimes n \\
\delta^1_1 & : & N & \to & A \otimes N & n & \mapsto & \sum \alpha_i \otimes x_i \\
\sigma^0_0 & : & A \otimes N & \to & N & a_0 \otimes n & \mapsto & a_0 n \\
\delta^2_0 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & 1 \otimes a_0 \otimes n \\
\delta^2_1 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & a_0 \otimes 1 \otimes n \\
\delta^2_2 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & \sum a_0 \otimes \alpha_i \otimes x_i \\
\sigma^1_0 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 a_1 \otimes n \\
\sigma^1_1 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 \otimes a_1 n
\end{array}$$

[1] Note that $\tau^2_{ij} = \delta^2_k$, if $\{i, j, k\} = [2] = \{0, 1, 2\}$, see Simplicial, Definition 14.2.1.

[2] We should really write $(N, \varphi)_\bullet$.

with notation as in Simplicial, Section 14.5. We first verify the two properties $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$ and $\sigma_0^0 \circ \delta_1^1 = \mathrm{id}$. The first one, $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$, is clear from the explicit description of the morphisms above. To prove the second relation we have to use the cocycle condition (because it does not hold for an arbitrary isomorphism $\varphi : N \otimes_R A \to A \otimes_R N$). Write $p = \sigma_0^0 \circ \delta_1^1 : N \to N$. By the description of the maps above we deduce that p is also equal to

$$p = \varphi \otimes \mathrm{id} : N = (N \otimes_R A) \otimes_{(A \otimes_R A)} A \longrightarrow (A \otimes_R N) \otimes_{(A \otimes_R A)} A = N$$

Since φ is an isomorphism we see that p is an isomorphism. Write $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$ for certain $\alpha_i \in A$ and $x_i \in N$. Then $p(n) = \sum \alpha_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum \alpha_{ij} \otimes y_j$ for certain $\alpha_{ij} \in A$ and $y_j \in N$. Then the cocycle condition says that

$$\sum \alpha_i \otimes \alpha_{ij} \otimes y_j = \sum \alpha_i \otimes 1 \otimes x_i.$$

This means that $p(n) = \sum \alpha_i x_i = \sum \alpha_i \alpha_{ij} y_j = \sum \alpha_i p(x_i) = p(p(n))$. Thus p is a projector, and since it is an isomorphism it is the identity.

To prove fully that $N_\bullet$ is a cosimplicial module we have to check all 5 types of relations of Simplicial, Remark 14.5.3. The relations on composing σ's are obvious. The relations on composing δ's come down to the cocycle condition for φ. In exactly the same way as above one checks the relations $\sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1} = \mathrm{id}$. Finally, the other relations on compositions of δ's and σ's hold for any φ whatsoever. □

Note that to an R-module M we can associate a canonical descent datum, namely $(M \otimes_R A, can)$ where $can : (M \otimes_R A) \otimes_R A \to A \otimes_R (M \otimes_R A)$ is the obvious map: $(m \otimes a) \otimes a' \mapsto a \otimes (m \otimes a')$.

023I **Lemma 35.3.3.** *Let $R \to A$ be a ring map. Let M be an R-module. The cosimplicial $(A/R)_\bullet$-module associated to the canonical descent datum is isomorphic to the cosimplicial module $(A/R)_\bullet \otimes_R M$.*

Proof. Omitted. □

023J **Definition 35.3.4.** Let $R \to A$ be a ring map. We say a descent datum (N, φ) is *effective* if there exists an R-module M and an isomorphism of descent data from $(M \otimes_R A, can)$ to (N, φ).

Let $R \to A$ be a ring map. Let (N, φ) be a descent datum. We may take the cochain complex $s(N_\bullet)$ associated with $N_\bullet$ (see Simplicial, Section 14.25). It has the following shape:

$$N \to A \otimes_R N \to A \otimes_R A \otimes_R N \to \ldots$$

We can describe the maps. The first map is the map

$$n \longmapsto 1 \otimes n - \varphi(n \otimes 1).$$

The second map on pure tensors has the values

$$a \otimes n \longmapsto 1 \otimes a \otimes n - a \otimes 1 \otimes n + a \otimes \varphi(n \otimes 1).$$

It is clear how the pattern continues.

In the special case where $N = A \otimes_R M$ we see that for any $m \in M$ the element $1 \otimes m$ is in the kernel of the first map of the cochain complex associated to the cosimplicial module $(A/R)_\bullet \otimes_R M$. Hence we get an extended cochain complex

023K (35.3.4.1) $$0 \to M \to A \otimes_R M \to A \otimes_R A \otimes_R M \to \ldots$$

Here we think of the 0 as being in degree -2, the module M in degree -1, the module $A \otimes_R M$ in degree 0, etc. Note that this complex has the shape

$$0 \to R \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A \to \ldots$$

when $M = R$.

023L **Lemma 35.3.5.** *Suppose that $R \to A$ has a section. Then for any R-module M the extended cochain complex (35.3.4.1) is exact.*

Proof. By Simplicial, Lemma 14.28.5 the map $R \to (A/R)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras (here R denotes the constant cosimplicial R-algebra). Hence $M \to (A/R)_\bullet \otimes_R M$ is a homotopy equivalence in the category of cosimplicial R-modules, because $\otimes_R M$ is a functor from the category of R-algebras to the category of R-modules, see Simplicial, Lemma 14.28.4. This implies that the induced map of associated complexes is a homotopy equivalence, see Simplicial, Lemma 14.28.6. Since the complex associated to the constant cosimplicial R-module M is the complex

$$M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \ldots$$

we win (since the extended version simply puts an extra M at the beginning). □

023M **Lemma 35.3.6.** *Suppose that $R \to A$ is faithfully flat, see Algebra, Definition 10.39.1. Then for any R-module M the extended cochain complex (35.3.4.1) is exact.*

Proof. Suppose we can show there exists a faithfully flat ring map $R \to R'$ such that the result holds for the ring map $R' \to A' = R' \otimes_R A$. Then the result follows for $R \to A$. Namely, for any R-module M the cosimplicial module $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ is just the cosimplicial module $R' \otimes_R (M \otimes_R (A/R)_\bullet)$. Hence the vanishing of cohomology of the complex associated to $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ implies the vanishing of the cohomology of the complex associated to $M \otimes_R (A/R)_\bullet$ by faithful flatness of $R \to R'$. Similarly for the vanishing of cohomology groups in degrees -1 and 0 of the extended complex (proof omitted).

But we have such a faithful flat extension. Namely $R' = A$ works because the ring map $R' = A \to A' = A \otimes_R A$ has a section $a \otimes a' \mapsto aa'$ and Lemma 35.3.5 applies. □

Here is how the complex relates to the question of effectivity.

039W **Lemma 35.3.7.** *Let $R \to A$ be a faithfully flat ring map. Let (N, φ) be a descent datum. Then (N, φ) is effective if and only if the canonical map*

$$A \otimes_R H^0(s(N_\bullet)) \longrightarrow N$$

is an isomorphism.

Proof. If (N, φ) is effective, then we may write $N = A \otimes_R M$ with $\varphi = can$. It follows that $H^0(s(N_\bullet)) = M$ by Lemmas 35.3.3 and 35.3.6. Conversely, suppose the map of the lemma is an isomorphism. In this case set $M = H^0(s(N_\bullet))$. This is an R-submodule of N, namely $M = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\}$. The only thing to check is that via the isomorphism $A \otimes_R M \to N$ the canonical descent data agrees with φ. We omit the verification. □

039X **Lemma 35.3.8.** *Let $R \to A$ be a faithfully flat ring map, and let $R \to R'$ be faithfully flat. Set $A' = R' \otimes_R A$. If all descent data for $R' \to A'$ are effective, then so are all descent data for $R \to A$.*

Proof. Let (N, φ) be a descent datum for $R \to A$. Set $N' = R' \otimes_R N = A' \otimes_A N$, and denote $\varphi' = \mathrm{id}_{R'} \otimes \varphi$ the base change of the descent datum φ. Then (N', φ') is a descent datum for $R' \to A'$ and $H^0(s(N'_\bullet)) = R' \otimes_R H^0(s(N_\bullet))$. Moreover, the map $A' \otimes_{R'} H^0(s(N'_\bullet)) \to N'$ is identified with the base change of the A-module map $A \otimes_R H^0(s(N)) \to N$ via the faithfully flat map $A \to A'$. Hence we conclude by Lemma 35.3.7. □

Here is the main result of this section. Its proof may seem a little clumsy; for a more highbrow approach see Remark 35.3.11 below.

023N **Proposition 35.3.9.** *Let $R \to A$ be a faithfully flat ring map. Then*

1. *any descent datum on modules with respect to $R \to A$ is effective,*
2. *the functor $M \mapsto (A \otimes_R M, can)$ from R-modules to the category of descent data is an equivalence, and*
3. *the inverse functor is given by $(N, \varphi) \mapsto H^0(s(N_\bullet))$.*

Proof. We only prove (1) and omit the proofs of (2) and (3). As $R \to A$ is faithfully flat, there exists a faithfully flat base change $R \to R'$ such that $R' \to A' = R' \otimes_R A$ has a section (namely take $R' = A$ as in the proof of Lemma 35.3.6). Hence, using Lemma 35.3.8 we may assume that $R \to A$ has a section, say $\sigma : A \to R$. Let (N, φ) be a descent datum relative to $R \to A$. Set

$$M = H^0(s(N_\bullet)) = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\} \subset N$$

By Lemma 35.3.7 it suffices to show that $A \otimes_R M \to N$ is an isomorphism.

Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in N$. By Lemma 35.3.2 we have $n = \sum a_i x_i$ in N (because $\sigma_0^0 \circ \delta_1^1 = \mathrm{id}$ in any cosimplicial object). Next, write $\varphi(x_i \otimes 1) = \sum a_{ij} \otimes y_j$ for certain $a_{ij} \in A$ and $y_j \in N$. The cocycle condition means that

$$\sum a_i \otimes a_{ij} \otimes y_j = \sum a_i \otimes 1 \otimes x_i$$

in $A \otimes_R A \otimes_R N$. We conclude two things from this:

1. applying σ to the first A we get $\sum \sigma(a_i)\varphi(x_i \otimes 1) = \sum \sigma(a_i) \otimes x_i$,
2. applying σ to the middle A we get $\sum_i a_i \otimes \sum_j \sigma(a_{ij}) y_j = \sum a_i \otimes x_i$.

Part (1) shows that $\sum \sigma(a_i) x_i \in M$. Applying this to x_i we see that $\sum \sigma(a_{ij}) y_i \in M$ for all i. Multiplying out the equation in (2) we conclude that $\sum_i a_i (\sum_j \sigma(a_{ij}) y_j) = \sum a_i x_i = n$. Hence $A \otimes_R M \to N$ is surjective. Finally, suppose that $m_i \in M$ and $\sum a_i m_i = 0$. Then we see by applying φ to $\sum a_i m_i \otimes 1$ that $\sum a_i \otimes m_i = 0$. In other words $A \otimes_R M \to N$ is injective and we win. □

023O **Remark 35.3.10.** Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. The ring $A = \prod_i R_{f_i}$ is a faithfully flat R-algebra. We remark that the cosimplicial ring $(A/R)_\bullet$ has the following ring in degree n:

$$\prod_{i_0, \ldots, i_n} R_{f_{i_0} \ldots f_{i_n}}$$

Hence the results above recover Algebra, Lemmas 10.24.2, 10.24.1 and 10.24.5. But the results above actually say more because of exactness in higher degrees. Namely,

it implies that Čech cohomology of quasi-coherent sheaves on affines is trivial. Thus we get a second proof of Cohomology of Schemes, Lemma 30.2.1.

039Y **Remark 35.3.11.** Let R be a ring. Let $A_\bullet$ be a cosimplicial R-algebra. In this setting a descent datum corresponds to an cosimplicial $A_\bullet$-module $M_\bullet$ with the property that for every $n, m \geq 0$ and every $\varphi : [n] \to [m]$ the map $M(\varphi) : M_n \to M_m$ induces an isomorphism

$$M_n \otimes_{A_n, A(\varphi)} A_m \longrightarrow M_m.$$

Let us call such a cosimplicial module a *cartesian module*. In this setting, the proof of Proposition 35.3.9 can be split in the following steps

(1) If $R \to R'$ and $R \to A$ are faithfully flat, then descent data for A/R are effective if descent data for $(R' \otimes_R A)/R'$ are effective.
(2) Let A be an R-algebra. Descent data for A/R correspond to cartesian $(A/R)_\bullet$-modules.
(3) If $R \to A$ has a section then $(A/R)_\bullet$ is homotopy equivalent to R, the constant cosimplicial R-algebra with value R.
(4) If $A_\bullet \to B_\bullet$ is a homotopy equivalence of cosimplicial R-algebras then the functor $M_\bullet \mapsto M_\bullet \otimes_{A_\bullet} B_\bullet$ induces an equivalence of categories between cartesian $A_\bullet$-modules and cartesian $B_\bullet$-modules.

For (1) see Lemma 35.3.8. Part (2) uses Lemma 35.3.2. Part (3) we have seen in the proof of Lemma 35.3.5 (it relies on Simplicial, Lemma 14.28.5). Moreover, part (4) is a triviality if you think about it right!

35.4. Descent for universally injective morphisms

08WE Numerous constructions in algebraic geometry are made using techniques of *descent*, such as constructing objects over a given space by first working over a somewhat larger space which projects down to the given space, or verifying a property of a space or a morphism by pulling back along a covering map. The utility of such techniques is of course dependent on identification of a wide class of *effective descent morphisms*. Early in the Grothendieckian development of modern algebraic geometry, the class of morphisms which are *quasi-compact* and *faithfully flat* was shown to be effective for descending objects, morphisms, and many properties thereof.

As usual, this statement comes down to a property of rings and modules. For a homomorphism $f : R \to S$ to be an effective descent morphism for modules, Grothendieck showed that it is sufficient for f to be faithfully flat. However, this excludes many natural examples: for instance, any split ring homomorphism is an effective descent morphism. One natural example of this even arises in the proof of faithfully flat descent: for $f : R \to S$ any ring homomorphism, $1_S \otimes f : S \to S \otimes_R S$ is split by the multiplication map whether or not it is flat.

One may then ask whether there is a natural ring-theoretic condition implying effective descent for modules which includes both the case of a faithfully flat morphism and that of a split ring homomorphism. It may surprise the reader (at least it surprised this author) to learn that a complete answer to this question has been known since around 1970! Namely, it is not hard to check that a necessary condition for $f : R \to S$ to be an effective descent morphism for modules is that f must be *universally injective* in the category of R-modules, that is, for any R-module M, the map $1_M \otimes f : M \to M \otimes_R S$ must be injective. This then turns out to be a sufficient

condition as well. For example, if f is split in the category of R-modules (but not necessarily in the category of rings), then f is an effective descent morphism for modules.

The history of this result is a bit involved: it was originally asserted by Olivier [**Oli70**], who called universally injective morphisms *pure*, but without a clear indication of proof. One can extract the result from the work of Joyal and Tierney [**JT84**], but to the best of our knowledge, the first free-standing proof to appear in the literature is that of Mesablishvili [**Mes00**]. The first purpose of this section is to expose Mesablishvili's proof; this requires little modification of his original presentation aside from correcting typos, with the one exception that we make explicit the relationship between the customary definition of a descent datum in algebraic geometry and the one used in [**Mes00**]. The proof turns out to be entirely category-theoretic, and consequently can be put in the language of monads (and thus applied in other contexts); see [**JT04**].

The second purpose of this section is to collect some information about which properties of modules, algebras, and morphisms can be descended along universally injective ring homomorphisms. The cases of finite modules and flat modules were treated by Mesablishvili [**Mes02**].

08WF **35.4.1. Category-theoretic preliminaries.** We start by recalling a few basic notions from category theory which will simplify the exposition. In this subsection, fix an ambient category.

For two morphisms $g_1, g_2 : B \to C$, recall that an *equalizer* of g_1 and g_2 is a morphism $f : A \to B$ which satisfies $g_1 \circ f = g_2 \circ f$ and is universal for this property. This second statement means that any commutative diagram

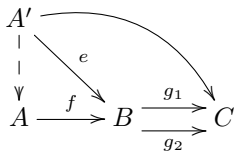

without the dashed arrow can be uniquely completed. We also say in this situation that the diagram

08WG (35.4.1.1) $$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C$$

is an equalizer. Reversing arrows gives the definition of a *coequalizer*. See Categories, Sections 4.10 and 4.11.

Since it involves a universal property, the property of being an equalizer is typically not stable under applying a covariant functor. Just as for monomorphisms and epimorphisms, one can get around this in some cases by exhibiting splittings.

08WH **Definition 35.4.2.** A *split equalizer* is a diagram (35.4.1.1) with $g_1 \circ f = g_2 \circ f$ for which there exist auxiliary morphisms $h : B \to A$ and $i : C \to B$ such that

08WI (35.4.2.1) $$h \circ f = 1_A, \quad f \circ h = i \circ g_1, \quad i \circ g_2 = 1_B.$$

The point is that the equalities among arrows force (35.4.1.1) to be an equalizer: the map e factors uniquely through f by writing $e = f \circ (h \circ e)$. Consequently,

applying a covariant functor to a split equalizer gives a split equalizer; applying a contravariant functor gives a *split coequalizer*, whose definition is apparent.

08WJ **35.4.3. Universally injective morphisms.** Recall that *Rings* denotes the category of commutative rings with 1. For an object R of *Rings* we denote Mod_R the category of R-modules.

08WK **Remark 35.4.4.** Any functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories which is exact and takes nonzero objects to nonzero objects reflects injections and surjections. Namely, exactness implies that F preserves kernels and cokernels (compare with Homology, Section 12.7). For example, if $f : R \to S$ is a faithfully flat ring homomorphism, then $\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ has these properties.

Let R be a ring. Recall that a morphism $f : M \to N$ in Mod_R is *universally injective* if for all $P \in \mathrm{Mod}_R$, the morphism $f \otimes 1_P : M \otimes_R P \to N \otimes_R P$ is injective. See Algebra, Definition 10.82.1.

08WL **Definition 35.4.5.** A ring map $f : R \to S$ is *universally injective* if it is universally injective as a morphism in Mod_R.

08WM **Example 35.4.6.** Any split injection in Mod_R is universally injective. In particular, any split injection in *Rings* is universally injective.

08WN **Example 35.4.7.** For a ring R and $f_1, \ldots, f_n \in R$ generating the unit ideal, the morphism $R \to R_{f_1} \oplus \ldots \oplus R_{f_n}$ is universally injective. Although this is immediate from Lemma 35.4.8, it is instructive to check it directly: we immediately reduce to the case where R is local, in which case some f_i must be a unit and so the map $R \to R_{f_i}$ is an isomorphism.

08WP **Lemma 35.4.8.** *Any faithfully flat ring map is universally injective.*

Proof. This is a reformulation of Algebra, Lemma 10.82.11. □

The key observation from [**Mes00**] is that universal injectivity can be usefully reformulated in terms of a splitting, using the usual construction of an injective cogenerator in Mod_R.

08WQ **Definition 35.4.9.** Let R be a ring. Define the contravariant functor $C : \mathrm{Mod}_R \to \mathrm{Mod}_R$ by setting

$$C(M) = \mathrm{Hom}_{Ab}(M, \mathbf{Q}/\mathbf{Z}),$$

with the R-action on $C(M)$ given by $rf(s) = f(rs)$.

This functor was denoted $M \mapsto M^\vee$ in More on Algebra, Section 15.55.

08WR **Lemma 35.4.10.** *For a ring R, the functor $C : Mod_R \to Mod_R$ is exact and reflects injections and surjections.*

Proof. Exactness is More on Algebra, Lemma 15.55.6 and the other properties follow from this, see Remark 35.4.4. □

08WS **Remark 35.4.11.** We will use frequently the standard adjunction between Hom and tensor product, in the form of the natural isomorphism of contravariant functors

08WT (35.4.11.1) $$C(\bullet_1 \otimes_R \bullet_2) \cong \mathrm{Hom}_R(\bullet_1, C(\bullet_2)) : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

taking $f : M_1 \otimes_R M_2 \to \mathbf{Q}/\mathbf{Z}$ to the map $m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2))$. See Algebra, Lemma 10.14.5. A corollary of this observation is that if

$$C(M) \rightrightarrows C(N) \longrightarrow C(P)$$

is a split coequalizer diagram in Mod_R, then so is

$$C(M \otimes_R Q) \rightrightarrows C(N \otimes_R Q) \longrightarrow C(P \otimes_R Q)$$

for any $Q \in \mathrm{Mod}_R$.

08WU **Lemma 35.4.12.** *Let R be a ring. A morphism $f : M \to N$ in Mod_R is universally injective if and only if $C(f) : C(N) \to C(M)$ is a split surjection.*

Proof. By (35.4.11.1), for any $P \in \mathrm{Mod}_R$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P, C(N)) & \xrightarrow[\mathrm{Hom}_R(P,C(f))]{} & \mathrm{Hom}_R(P, C(M)) \\ \downarrow \cong & & \downarrow \cong \\ C(P \otimes_R N) & \xrightarrow{C(1_P \otimes f)} & C(P \otimes_R M). \end{array}$$

If f is universally injective, then $1_{C(M)} \otimes f : C(M) \otimes_R M \to C(M) \otimes_R N$ is injective, so both rows in the above diagram are surjective for $P = C(M)$. We may thus lift $1_{C(M)} \in \mathrm{Hom}_R(C(M), C(M))$ to some $g \in \mathrm{Hom}_R(C(N), C(M))$ splitting $C(f)$. Conversely, if $C(f)$ is a split surjection, then both rows in the above diagram are surjective, so by Lemma 35.4.10, $1_P \otimes f$ is injective. □

08WV **Remark 35.4.13.** Let $f : M \to N$ be a universally injective morphism in Mod_R. By choosing a splitting g of $C(f)$, we may construct a functorial splitting of $C(1_P \otimes f)$ for each $P \in \mathrm{Mod}_R$. Namely, by (35.4.11.1) this amounts to splitting $\mathrm{Hom}_R(P, C(f))$ functorially in P, and this is achieved by the map $g \circ \bullet$.

08WW **35.4.14. Descent for modules and their morphisms.** Throughout this subsection, fix a ring map $f : R \to S$. As seen in Section 35.3 we can use the language of cosimplicial algebras to talk about descent data for modules, but in this subsection we prefer a more down to earth terminology.

For $i = 1, 2, 3$, let S_i be the i-fold tensor product of S over R. Define the ring homomorphisms $\delta^1_0, \delta^1_1 : S_1 \to S_2$, $\delta^1_{01}, \delta^1_{02}, \delta^1_{12} : S_1 \to S_3$, and $\delta^2_0, \delta^2_1, \delta^2_2 : S_2 \to S_3$ by the formulas

$$\begin{aligned} \delta^1_0(a_0) &= 1 \otimes a_0 \\ \delta^1_1(a_0) &= a_0 \otimes 1 \\ \delta^2_0(a_0 \otimes a_1) &= 1 \otimes a_0 \otimes a_1 \\ \delta^2_1(a_0 \otimes a_1) &= a_0 \otimes 1 \otimes a_1 \\ \delta^2_2(a_0 \otimes a_1) &= a_0 \otimes a_1 \otimes 1 \\ \delta^1_{01}(a_0) &= 1 \otimes 1 \otimes a_0 \\ \delta^1_{02}(a_0) &= 1 \otimes a_0 \otimes 1 \\ \delta^1_{12}(a_0) &= a_0 \otimes 1 \otimes 1. \end{aligned}$$

In other words, the upper index indicates the source ring, while the lower index indicates where to insert factors of 1. (This notation is compatible with the notation introduced in Section 35.3.)

Recall[3] from Definition 35.3.1 that for $M \in \mathrm{Mod}_S$, a *descent datum* on M relative to f is an isomorphism

$$\theta : M \otimes_{S,\delta_0^1} S_2 \longrightarrow M \otimes_{S,\delta_1^1} S_2$$

of S_2-modules satisfying the *cocycle condition*

08WX (35.4.14.1) $$(\theta \otimes \delta_2^2) \circ (\theta \otimes \delta_2^0) = (\theta \otimes \delta_2^1) : M \otimes_{S,\delta_{01}^1} S_3 \to M \otimes_{S,\delta_{12}^1} S_3.$$

Let $DD_{S/R}$ be the category of S-modules equipped with descent data relative to f.

For example, for $M_0 \in \mathrm{Mod}_R$ and a choice of isomorphism $M \cong M_0 \otimes_R S$ gives rise to a descent datum by identifying $M \otimes_{S,\delta_0^1} S_2$ and $M \otimes_{S,\delta_1^1} S_2$ naturally with $M_0 \otimes_R S_2$. This construction in particular defines a functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$.

08WY **Definition 35.4.15.** The functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$ is called *base extension along* f. We say that f is a *descent morphism for modules* if f^* is fully faithful. We say that f is an *effective descent morphism for modules* if f^* is an equivalence of categories.

Our goal is to show that for f universally injective, we can use θ to locate M_0 within M. This process makes crucial use of some equalizer diagrams.

08WZ **Lemma 35.4.16.** *For $(M, \theta) \in DD_{S/R}$, the diagram*
(35.4.16.1)

08X0 $$M \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S,\delta_1^1} S_2 \underset{1_{M \otimes S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3$$

is a split equalizer.

Proof. Define the ring homomorphisms $\sigma_0^0 : S_2 \to S_1$ and $\sigma_0^1, \sigma_1^1 : S_3 \to S_2$ by the formulas

$$\sigma_0^0(a_0 \otimes a_1) = a_0 a_1$$
$$\sigma_0^1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2$$
$$\sigma_1^1(a_0 \otimes a_1 \otimes a_2) = a_0 \otimes a_1 a_2.$$

We then take the auxiliary morphisms to be $1_M \otimes \sigma_0^0 : M \otimes_{S,\delta_1^1} S_2 \to M$ and $1_M \otimes \sigma_0^1 : M \otimes_{S,\delta_{12}^1} S_3 \to M \otimes_{S,\delta_1^1} S_2$. Of the compatibilities required in (35.4.2.1), the first follows from tensoring the cocycle condition (35.4.14.1) with σ_1^1 and the others are immediate. □

08X1 **Lemma 35.4.17.** *For $(M, \theta) \in DD_{S/R}$, the diagram*
(35.4.17.1)

08X2 $$C(M \otimes_{S,\delta_{12}^1} S_3) \underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} C(M).$$

obtained by applying C to (35.4.16.1) is a split coequalizer.

Proof. Omitted. □

[3]To be precise, our θ here is the inverse of φ from Definition 35.3.1.

08X3 **Lemma 35.4.18.** *The diagram*

08X4 (35.4.18.1)
$$S_1 \xrightarrow{\delta_1^1} S_2 \underset{\delta_1^2}{\overset{\delta_2^2}{\rightrightarrows}} S_3$$

is a split equalizer.

Proof. In Lemma 35.4.16, take $(M, \theta) = f^*(S)$. □

This suggests a definition of a potential quasi-inverse functor for f^*.

08X5 **Definition 35.4.19.** Define the functor $f_* : DD_{S/R} \to \mathrm{Mod}_R$ by taking $f_*(M, \theta)$ to be the R-submodule of M for which the diagram
(35.4.19.1)

08X6
$$f_*(M,\theta) \longrightarrow M \underset{1_M \otimes \delta_1^1}{\overset{\theta \circ (1_M \otimes \delta_0^1)}{\rightrightarrows}} M \otimes_{S,\delta_1^1} S_2$$

is an equalizer.

Using Lemma 35.4.16 and the fact that the restriction functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is right adjoint to the base extension functor $\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$, we deduce that f_* is right adjoint to f^*.

We are ready for the key lemma. In the faithfully flat case this is a triviality (see Remark 35.4.21), but in the general case some argument is needed.

08X7 **Lemma 35.4.20.** *If f is universally injective, then the diagram*
(35.4.20.1)

08X8
$$f_*(M,\theta) \otimes_R S \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S,\delta_1^1} S_2 \underset{1_{M \otimes S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3$$

obtained by tensoring (35.4.19.1) over R with S is an equalizer.

Proof. By Lemma 35.4.12 and Remark 35.4.13, the map $C(1_N \otimes f) : C(N \otimes_R S) \to C(N)$ can be split functorially in N. This gives the upper vertical arrows in the commutative diagram

$$\begin{array}{ccccc}
C(M \otimes_{S,\delta_1^1} S_2) & \underset{C(1_M \otimes \delta_1^1)}{\overset{C(\theta \circ (1_M \otimes \delta_0^1))}{\rightrightarrows}} & C(M) & \longrightarrow & C(f_*(M,\theta)) \\
\downarrow & & \downarrow & & \vdots \\
C(M \otimes_{S,\delta_{12}^1} S_3) & \underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\rightrightarrows}} & C(M \otimes_{S,\delta_1^1} S_2) & \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} & C(M) \\
\downarrow & & \downarrow C(1_M \otimes \delta_1^1) & & \downarrow \\
C(M \otimes_{S,\delta_1^1} S_2) & \underset{C(1_M \otimes \delta_1^1)}{\overset{C(\theta \circ (1_M \otimes \delta_0^1))}{\rightrightarrows}} & C(M) & \longrightarrow & C(f_*(M,\theta))
\end{array}$$

in which the compositions along the columns are identity morphisms. The second row is the coequalizer diagram (35.4.17.1); this produces the dashed arrow. From the top right square, we obtain auxiliary morphisms $C(f_*(M,\theta)) \to C(M)$ and $C(M) \to C(M \otimes_{S,\delta_1^1} S_2)$ which imply that the first row is a split coequalizer diagram.

By Remark 35.4.11, we may tensor with S inside C to obtain the split coequalizer diagram

$$C(M \otimes_{S,\delta_2^2\circ\delta_1^1} S_3) \underset{C(1_{M\otimes S_2}\otimes\delta_1^2)}{\overset{C((\theta\otimes\delta_2^2)\circ(1_M\otimes\delta_0^2))}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta\circ(1_M\otimes\delta_0^1))} C(f_*(M,\theta)\otimes_R S).$$

By Lemma 35.4.10, we conclude (35.4.20.1) must also be an equalizer. □

08X9 **Remark 35.4.21.** If f is a split injection in Mod_R, one can simplify the argument by splitting f directly, without using C. Things are even simpler if f is faithfully flat; in this case, the conclusion of Lemma 35.4.20 is immediate because tensoring over R with S preserves all equalizers.

08XA **Theorem 35.4.22.** *The following conditions are equivalent.*

(a) *The morphism f is a descent morphism for modules.*
(b) *The morphism f is an effective descent morphism for modules.*
(c) *The morphism f is universally injective.*

Proof. It is clear that (b) implies (a). We now check that (a) implies (c). If f is not universally injective, we can find $M \in \mathrm{Mod}_R$ such that the map $1_M \otimes f : M \to M \otimes_R S$ has nontrivial kernel N. The natural projection $M \to M/N$ is not an isomorphism, but its image in $DD_{S/R}$ is an isomorphism. Hence f^* is not fully faithful.

We finally check that (c) implies (b). By Lemmas 35.4.16 and 35.4.20, for $(M,\theta) \in DD_{S/R}$, the natural map $f^*f_*(M,\theta) \to M$ is an isomorphism of S-modules. On the other hand, for $M_0 \in \mathrm{Mod}_R$, we may tensor (35.4.18.1) with M_0 over R to obtain an equalizer sequence, so $M_0 \to f_*f^*M_0$ is an isomorphism. Consequently, f_* and f^* are quasi-inverse functors, proving the claim. □

08XB **35.4.23. Descent for properties of modules.** Throughout this subsection, fix a universally injective ring map $f : R \to S$, an object $M \in \mathrm{Mod}_R$, and a ring map $R \to A$. We now investigate the question of which properties of M or A can be checked after base extension along f. We start with some results from [**Mes02**].

08XC **Lemma 35.4.24.** *If $M \in Mod_R$ is flat, then $C(M)$ is an injective R-module.*

Proof. Let $0 \to N \to P \to Q \to 0$ be an exact sequence in Mod_R. Since M is flat,

$$0 \to N \otimes_R M \to P \otimes_R M \to Q \otimes_R M \to 0$$

is exact. By Lemma 35.4.10,

$$0 \to C(Q \otimes_R M) \to C(P \otimes_R M) \to C(N \otimes_R M) \to 0$$

is exact. By (35.4.11.1), this last sequence can be rewritten as

$$0 \to \mathrm{Hom}_R(Q, C(M)) \to \mathrm{Hom}_R(P, C(M)) \to \mathrm{Hom}_R(N, C(M)) \to 0.$$

Hence $C(M)$ is an injective object of Mod_R. □

08XD **Theorem 35.4.25.** *If $M \otimes_R S$ has one of the following properties as an S-module*

(a) *finitely generated;*
(b) *finitely presented;*
(c) *flat;*
(d) *faithfully flat;*

(e) *finite projective;*

then so does M as an R-module (and conversely).

Proof. To prove (a), choose a finite set $\{n_i\}$ of generators of $M \otimes_R S$ in Mod_S. Write each n_i as $\sum_j m_{ij} \otimes s_{ij}$ with $m_{ij} \in M$ and $s_{ij} \in S$. Let F be the finite free R-module with basis e_{ij} and let $F \to M$ be the R-module map sending e_{ij} to m_{ij}. Then $F \otimes_R S \to M \otimes_R S$ is surjective, so $\text{Coker}(F \to M) \otimes_R S$ is zero and hence $\text{Coker}(F \to M)$ is zero. This proves (a).

To see (b) assume $M \otimes_R S$ is finitely presented. Then M is finitely generated by (a). Choose a surjection $R^{\oplus n} \to M$ with kernel K. Then $K \otimes_R S \to S^{\oplus r} \to M \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.5.3 the kernel of $S^{\oplus r} \to M \otimes_R S$ is a finite S-module. Thus we can find finitely many elements $k_1, \ldots, k_t \in K$ such that the images of $k_i \otimes 1$ in $S^{\oplus r}$ generate the kernel of $S^{\oplus r} \to M \otimes_R S$. Let $K' \subset K$ be the submodule generated by $k_1, \ldots, k_t$. Then $M' = R^{\oplus r}/K'$ is a finitely presented R-module with a morphism $M' \to M$ such that $M' \otimes_R S \to M \otimes_R S$ is an isomorphism. Thus $M' \cong M$ as desired.

To prove (c), let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence in Mod_R. Since $\bullet \otimes_R S$ is a right exact functor, $M'' \otimes_R S \to M \otimes_R S$ is surjective. So by Lemma 35.4.10 the map $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is injective. If $M \otimes_R S$ is flat, then Lemma 35.4.24 shows $C(M \otimes_R S)$ is an injective object of Mod_S, so the injection $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is split in Mod_S and hence also in Mod_R. Since $C(M \otimes_R S) \to C(M)$ is a split surjection by Lemma 35.4.12, it follows that $C(M) \to C(M'')$ is a split injection in Mod_R. That is, the sequence

$$0 \to C(M) \to C(M'') \to C(M') \to 0$$

is split exact. For $N \in \text{Mod}_R$, by (35.4.11.1) we see that

$$0 \to C(M \otimes_R N) \to C(M'' \otimes_R N) \to C(M' \otimes_R N) \to 0$$

is split exact. By Lemma 35.4.10,

$$0 \to M' \otimes_R N \to M'' \otimes_R N \to M \otimes_R N \to 0$$

is exact. This implies M is flat over R. Namely, taking M' a free module surjecting onto M we conclude that $\text{Tor}_1^R(M, N) = 0$ for all modules N and we can use Algebra, Lemma 10.75.8. This proves (c).

To deduce (d) from (c), note that if $N \in \text{Mod}_R$ and $M \otimes_R N$ is zero, then $M \otimes_R S \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S$ is zero, so $N \otimes_R S$ is zero and hence N is zero.

To deduce (e) at this point, it suffices to recall that M is finitely generated and projective if and only if it is finitely presented and flat. See Algebra, Lemma 10.78.2. □

There is a variant for R-algebras.

08XE **Theorem 35.4.26.** *If $A \otimes_R S$ has one of the following properties as an S-algebra*

(a) *of finite type;*
(b) *of finite presentation;*
(c) *formally unramified;*
(d) *unramified;*
(e) *étale;*

then so does A as an R-algebra (and of course conversely).

Proof. To prove (a), choose a finite set $\{x_i\}$ of generators of $A \otimes_R S$ over S. Write each x_i as $\sum_j y_{ij} \otimes s_{ij}$ with $y_{ij} \in A$ and $s_{ij} \in S$. Let F be the polynomial R-algebra on variables e_{ij} and let $F \to M$ be the R-algebra map sending e_{ij} to y_{ij}. Then $F \otimes_R S \to A \otimes_R S$ is surjective, so $\mathrm{Coker}(F \to A) \otimes_R S$ is zero and hence $\mathrm{Coker}(F \to A)$ is zero. This proves (a).

To see (b) assume $A \otimes_R S$ is a finitely presented S-algebra. Then A is finite type over R by (a). Choose a surjection $R[x_1, \ldots, x_n] \to A$ with kernel I. Then $I \otimes_R S \to S[x_1, \ldots, x_n] \to A \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.6.3 the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$ is a finitely generated ideal. Thus we can find finitely many elements $y_1, \ldots, y_t \in I$ such that the images of $y_i \otimes 1$ in $S[x_1, \ldots, x_n]$ generate the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$. Let $I' \subset I$ be the ideal generated by $y_1, \ldots, y_t$. Then $A' = R[x_1, \ldots, x_n]/I'$ is a finitely presented R-algebra with a morphism $A' \to A$ such that $A' \otimes_R S \to A \otimes_R S$ is an isomorphism. Thus $A' \cong A$ as desired.

To prove (c), recall that A is formally unramified over R if and only if the module of relative differentials $\Omega_{A/R}$ vanishes, see Algebra, Lemma 10.148.3 or [**GD67**, Proposition 17.2.1]. Since $\Omega_{(A \otimes_R S)/S} = \Omega_{A/R} \otimes_R S$, the vanishing descends by Theorem 35.4.22.

To deduce (d) from the previous cases, recall that A is unramified over R if and only if A is formally unramified and of finite type over R, see Algebra, Lemma 10.151.2.

To prove (e), recall that by Algebra, Lemma 10.151.8 or [**GD67**, Théorème 17.6.1] the algebra A is étale over R if and only if A is flat, unramified, and of finite presentation over R. □

08XF **Remark 35.4.27.** It would make things easier to have a faithfully flat ring homomorphism $g : R \to T$ for which $T \to S \otimes_R T$ has some extra structure. For instance, if one could ensure that $T \to S \otimes_R T$ is split in *Rings*, then it would follow that every property of a module or algebra which is stable under base extension and which descends along faithfully flat morphisms also descends along universally injective morphisms. An obvious guess would be to find g for which T is not only faithfully flat but also injective in Mod_R, but even for $R = \mathbf{Z}$ no such homomorphism can exist.

35.5. Fpqc descent of quasi-coherent sheaves

023R The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

023S **Lemma 35.5.1.** *Let S be an affine scheme. Let $\mathcal{U} = \{f_i : U_i \to S\}_{i=1,\ldots,n}$ be a standard fpqc covering of S, see Topologies, Definition 34.9.10. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This is a restatement of Proposition 35.3.9 in terms of schemes. First, note that a descent datum ξ for quasi-coherent sheaves with respect to $\mathcal{U}$ is exactly the same as a descent datum ξ' for quasi-coherent sheaves with respect to the covering $\mathcal{U}' = \{\coprod_{i=1,\ldots,n} U_i \to S\}$. Moreover, effectivity for ξ is the same as effectivity for

ξ'. Hence we may assume $n = 1$, i.e., $\mathcal{U} = \{U \to S\}$ where U and S are affine. In this case descent data correspond to descent data on modules with respect to the ring map

$$\Gamma(S, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O}).$$

Since $U \to S$ is surjective and flat, we see that this ring map is faithfully flat. In other words, Proposition 35.3.9 applies and we win. □

023T **Proposition 35.5.2.** *Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}$ be an fpqc covering, see Topologies, Definition 34.9.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. Let $S = \bigcup_{j \in J} V_j$ be an affine open covering. For $j, j' \in J$ we denote $V_{jj'} = V_j \cap V_{j'}$ the intersection (which need not be affine). For $V \subset S$ open we denote $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ which is a fpqc-covering (Topologies, Lemma 34.9.8). By definition of an fpqc covering, we can find for each $j \in J$ a finite set K_j, a map $\underline{i} : K_j \to I$, affine opens $U_{\underline{i}(k),k} \subset U_{\underline{i}(k)}$, $k \in K_j$ such that $\mathcal{V}_j = \{U_{\underline{i}(k),k} \to V_j\}_{k \in K_j}$ is a standard fpqc covering of V_j. And of course, $\mathcal{V}_j$ is a refinement of $\mathcal{U}_{V_j}$. Picture

$$\begin{array}{ccccc} \mathcal{V}_j & \longrightarrow & \mathcal{U}_{V_j} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow \\ V_j & = & V_j & \longrightarrow & S \end{array}$$

where the top horizontal arrows are morphisms of families of morphisms with fixed target (see Sites, Definition 7.8.1).

To prove the proposition you show successively the faithfulness, fullness, and essential surjectivity of the functor from quasi-coherent sheaves to descent data.

Faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. Suppose $\varphi_i^*(a) = \varphi_i^*(b)$ for all i. Pick $s \in S$. Then $s = \varphi_i(u)$ for some $i \in I$ and $u \in U_i$. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{U_i,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.39.17) we see that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a_i : \varphi_i^*\mathcal{F} \to \varphi_i^*\mathcal{G}$ be homomorphisms of $\mathcal{O}_{U_i}$-modules such that $\mathrm{pr}_0^* a_i = \mathrm{pr}_1^* a_j$ on $U_i \times_U U_j$. We can pull back these morphisms to get morphisms

$$a_k : \varphi_{i(k)}^*\mathcal{F}|_{U_{\underline{i}(k),k}} \longrightarrow \varphi_{i(k)}^*\mathcal{G}|_{U_{\underline{i}(k),k}}$$

$k \in K_j$ with notation as above. Moreover, Lemma 35.2.2 assures us that these define a morphism between (canonical) descent data on $\mathcal{V}_j$. Hence, by Lemma 35.5.1, we get correspondingly unique morphisms $a_j : \mathcal{F}|_{V_j} \to \mathcal{G}|_{V_j}$. To see that $a_j|_{V_{jj'}} = a_{j'}|_{V_{jj'}}$ we use that both a_j and $a_{j'}$ agree with the pullback of the morphism $(a_i)_{i \in I}$ of (canonical) descent data to any covering refining both $\mathcal{V}_{j,V_{jj'}}$ and $\mathcal{V}_{j',V_{jj'}}$, and using the faithfulness already shown. For example the covering $\mathcal{V}_{jj'} = \{V_k \times_S V_{k'} \to V_{jj'}\}_{k \in K_j, k' \in K_{j'}}$ will do.

Essential surjectivity. Let $\xi = (\mathcal{F}_i, \varphi_{ii'})$ be a descent datum for quasi-coherent sheaves relative to the covering $\mathcal{U}$. Pull back this descent datum to get descent

data ξ_j for quasi-coherent sheaves relative to the coverings $\mathcal{V}_j$ of V_j. By Lemma 35.5.1 once again there exist quasi-coherent sheaves $\mathcal{F}_j$ on V_j whose associated canonical descent datum is isomorphic to ξ_j. By fully faithfulness (proved above) we see there are isomorphisms

$$\phi_{jj'} : \mathcal{F}_j|_{V_{jj'}} \longrightarrow \mathcal{F}_{j'}|_{V_{jj'}}$$

corresponding to the isomorphism of descent data between the pullback of ξ_j and $\xi_{j'}$ to $\mathcal{V}_{jj'}$. To see that these maps $\phi_{jj'}$ satisfy the cocycle condition we use faithfulness (proved above) over the triple intersections $V_{jj'j''}$. Hence, by Lemma 35.2.4 we see that the sheaves $\mathcal{F}_j$ glue to a quasi-coherent sheaf $\mathcal{F}$ as desired. We still have to verify that the canonical descent datum relative to $\mathcal{U}$ associated to $\mathcal{F}$ is isomorphic to the descent datum we started out with. This verification is omitted. □

35.6. Galois descent for quasi-coherent sheaves

0CDQ Galois descent for quasi-coherent sheaves is just a special case of fpqc descent for quasi-coherent sheaves. In this section we will explain how to translate from a Galois descent to an fpqc descent and then apply earlier results to conclude.

Let k'/k be a field extension. Then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let X be a scheme over k. For a k-algebra A we set $X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$. By Topologies, Lemma 34.9.8 we see that $\{X_{k'} \to X\}$ is an fpqc covering. Observe that

$$X_{k'} \times_X X_{k'} = X_{k' \otimes_k k'} \quad \text{and} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

Thus a descent datum for quasi-coherent sheaves with respect to $\{X_{k'} \to X\}$ is given by a quasi-coherent sheaf $\mathcal{F}$ on $X_{k'}$, an isomorphism $\varphi : \mathrm{pr}_0^*\mathcal{F} \to \mathrm{pr}_1^*\mathcal{F}$ on $X_{k' \otimes_k k'}$ which satisfies an obvious cocycle condition on $X_{k' \otimes_k k' \otimes_k k'}$. We will work out what this means in the case of a Galois extension below.

Let k'/k be a finite Galois extension with Galois group $G = \mathrm{Gal}(k'/k)$. Then there are k-algebra isomorphisms

$$k' \otimes_k k' \longrightarrow \prod\nolimits_{\sigma \in G} k', \quad a \otimes b \longrightarrow \prod a\sigma(b)$$

and

$$k' \otimes_k k' \otimes_k k' \longrightarrow \prod\nolimits_{(\sigma,\tau) \in G \times G} k', \quad a \otimes b \otimes c \longrightarrow \prod a\sigma(b)\sigma(\tau(c))$$

The reason for choosing here $a\sigma(b)\sigma(\tau(c))$ and not $a\sigma(b)\tau(c)$ is that the formulas below simplify but it isn't strictly necessary. Given $\sigma \in G$ we denote

$$f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \longrightarrow X_{k'}$$

Please keep in mind that because $\mathrm{Spec}(-)$ is a contravariant functor we have $f_{\sigma\tau} = f_\tau \circ f_\sigma$ and not the other way around. Using the first isomorphism above we obtain an identification

$$X_{k' \otimes_k k'} = \coprod\nolimits_{\sigma \in G} X_{k'}$$

such that pr_0 corresponds to the map

$$\coprod\nolimits_{\sigma \in G} X_{k'} \xrightarrow{\coprod \mathrm{id}} X_{k'}$$

and such that pr_1 corresponds to the map

$$\coprod_{\sigma\in G} X_{k'} \xrightarrow{\coprod f_\sigma} X_{k'}$$

Thus we see that a descent datum φ on $\mathcal{F}$ over $X_{k'}$ corresponds to a family of isomorphisms $\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$. To work out the cocycle condition we use the identification

$$X_{k'\otimes_k k'\otimes_k k'} = \coprod_{(\sigma,\tau)\in G\times G} X_{k'}.$$

we get from our isomorphism of algebras above. Via this identification the map pr_{01} corresponds to the map

$$\coprod_{(\sigma,\tau)\in G\times G} X_{k'} \longrightarrow \coprod_{\sigma\in G} X_{k'}$$

which maps the summand with index (σ,τ) to the summand with index σ via the identity morphism. The map pr_{12} corresponds to the map

$$\coprod_{(\sigma,\tau)\in G\times G} X_{k'} \longrightarrow \coprod_{\sigma\in G} X_{k'}$$

which maps the summand with index (σ,τ) to the summand with index τ via the morphism f_σ. Finally, the map pr_{02} corresponds to the map

$$\coprod_{(\sigma,\tau)\in G\times G} X_{k'} \longrightarrow \coprod_{\sigma\in G} X_{k'}$$

which maps the summand with index (σ,τ) to the summand with index $\sigma\tau$ via the identity morphism. Thus the cocycle condition

$$\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$$

translates into one condition for each pair (σ,τ), namely

$$\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$$

as maps $\mathcal{F} \to f_{\sigma\tau}^*\mathcal{F}$. (Everything works out beautifully; for example the target of φ_σ is $f_\sigma^*\mathcal{F}$ and the source of $f_\sigma^*\varphi_\tau$ is $f_\sigma^*\mathcal{F}$ as well.)

0CDR **Lemma 35.6.1.** *Let k'/k be a (finite) Galois extension with Galois group G. Let X be a scheme over k. The category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma\in G})$ where*

(1) *$\mathcal{F}$ is a quasi-coherent module on $X_{k'}$,*
(2) *$\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$ is an isomorphism of modules,*
(3) *$\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ for all $\sigma,\tau \in G$.*

Here $f_\sigma = id_X \times \mathrm{Spec}(\sigma) : X_{k'} \to X_{k'}$.

Proof. As seen above a datum $(\mathcal{F}, (\varphi_\sigma)_{\sigma\in G})$ as in the lemma is the same thing as a descent datum for the fpqc covering $\{X_{k'} \to X\}$. Thus the lemma follows from Proposition 35.5.2. □

A slightly more general case of the above is the following. Suppose we have a surjective finite étale morphism $X \to Y$ and a finite group G together with a group homomorphism $G^{opp} \to \mathrm{Aut}_Y(X), \sigma \mapsto f_\sigma$ such that the map

$$G \times X \longrightarrow X \times_Y X, \quad (\sigma, x) \longmapsto (x, f_\sigma(x))$$

is an isomorphism. Then the same result as above holds.

0D1V **Lemma 35.6.2.** *Let $X \to Y$, G, and $f_\sigma : X \to X$ be as above. The category of quasi-coherent $\mathcal{O}_Y$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ where*

(1) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,*
(2) *$\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$ is an isomorphism of modules,*
(3) *$\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ for all $\sigma, \tau \in G$.*

Proof. Since $X \to Y$ is surjective finite étale $\{X \to Y\}$ is an fpqc covering. Since $G \times X \to X \times_Y X$, $(\sigma, x) \mapsto (x, f_\sigma(x))$ is an isomorphism, we see that $G \times G \times X \to X \times_Y X \times_Y X$, $(\sigma, \tau, x) \mapsto (x, f_\sigma(x), f_{\sigma\tau}(x))$ is an isomorphism too. Using these identifications, the category of data as in the lemma is the same as the category of descent data for quasi-coherent sheaves for the covering $\{x \to Y\}$. Thus the lemma follows from Proposition 35.5.2. □

35.7. Descent of finiteness properties of modules

05AY In this section we prove that one can check quasi-coherent module has a certain finiteness conditions by checking on the members of a covering.

05AZ **Lemma 35.7.1.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. □

09UB **Lemma 35.7.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. If*

(1) *f is open as a map of topological spaces,*
(2) *f is surjective and flat, and*
(3) *$f^*\mathcal{F}$ is of finite type,*

then $\mathcal{F}$ is of finite type.

Proof. Let $y \in Y$ be a point. Choose a point $x \in X$ mapping to y. Choose an open $x \in U \subset X$ and elements $s_1, \ldots, s_n$ of $f^*\mathcal{F}(U)$ which generate $f^*\mathcal{F}$ over U. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ we can after shrinking U assume $s_i = \sum t_{ij} \otimes a_{ij}$ with $t_{ij} \in f^{-1}\mathcal{F}(U)$ and $a_{ij} \in \mathcal{O}_X(U)$. After shrinking U further we may assume that t_{ij} comes from a section $s_{ij} \in \mathcal{F}(V)$ for some $V \subset Y$ open with $f(U) \subset V$. Let N be the number of sections s_{ij} and consider the map

$$\sigma = (s_{ij}) : \mathcal{O}_V^{\oplus N} \to \mathcal{F}|_V$$

By our choice of the sections we see that $f^*\sigma|_U$ is surjective. Hence for every $u \in U$ the map

$$\sigma_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u} : \mathcal{O}_{X,u}^{\oplus N} \longrightarrow \mathcal{F}_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u}$$

is surjective. As f is flat, the local ring map $\mathcal{O}_{Y,f(u)} \to \mathcal{O}_{X,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.39.17). Hence $\sigma_{f(u)}$ is surjective. Since f is open, $f(U)$ is an open neighbourhood of y and the proof is done. □

05B0 **Lemma 35.7.3.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. □

082U **Lemma 35.7.4.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_X$-module.*

Proof. By Lemma 35.7.1 we see that $\mathcal{F}$ is of finite type. Hence Nakayama's lemma (Algebra, Lemma 10.20.1) implies that $\mathcal{F}$ is generated by r sections in the neighbourhood of a point $x \in X$ if and only if $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) \leq r$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) = \dim_{\kappa(x_i)} (f_i^*\mathcal{F})_{x_i} \otimes \kappa(x_i)$ which is $\leq r$ as $f_i^*\mathcal{F}$ is locally generated by r sections. □

05B1 **Lemma 35.7.5.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. □

05B2 **Lemma 35.7.6.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the fact that a quasi-coherent sheaf is finite locally free if and only if it is of finite presentation and flat, see Algebra, Lemma 10.78.2. Namely, if each $f_i^*\mathcal{F}$ is flat and of finite presentation, then so is $\mathcal{F}$ by Lemmas 35.7.5 and 35.7.3. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties, Section 28.21.

05JZ **Lemma 35.7.7.** *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. Omitted. For Zariski coverings this is Properties, Lemma 28.21.2. For the affine case this is Algebra, Theorem 10.95.6. □

05VF **Remark 35.7.8.** Being locally free is a property of quasi-coherent modules which does not descend in the fpqc topology. Namely, suppose that R is a ring and that M is a projective R-module which is a countable direct sum $M = \bigoplus L_n$ of rank 1 locally free modules, but not locally free, see Examples, Lemma 110.34.5. Then M becomes free on making the faithfully flat base change

$$R \longrightarrow \bigoplus\nolimits_{m\geq 1} \bigoplus\nolimits_{(i_1,\ldots,i_m)\in \mathbf{Z}^{\oplus m}} L_1^{\otimes i_1} \otimes_R \ldots \otimes_R L_m^{\otimes i_m}$$

But we don't know what happens for fppf coverings. In other words, we don't know the answer to the following question: Suppose $A \to B$ is a faithfully flat ring map of finite presentation. Let M be an A-module such that $M \otimes_A B$ is free. Is M a locally free A-module? It turns out that if A is Noetherian, then the answer is yes. This follows from the results of [**Bas63**]. But in general we don't know the answer. If you know the answer, or have a reference, please email stacks.project@gmail.com.

We also add here two results which are related to the results above, but are of a slightly different nature.

05B3 **Lemma 35.7.9.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finite as a B-module if and only if M is finite as an A-module, see Algebra, Lemma 10.7.2. Combined with Properties, Lemma 28.16.1 this proves the lemma. □

05B4 **Lemma 35.7.10.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite and finitely presented ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finitely presented as a B-module if and only if M is finitely presented as an A-module, see Algebra, Lemma 10.36.23. Combined with Properties, Lemma 28.16.2 this proves the lemma. □

35.8. Quasi-coherent sheaves and topologies, I

03DR The results in this section say there is a natural equivalence between the category quasi-coherent modules on a scheme S and the category of quasi-coherent modules on many of the sites associated to S in the chapter on topologies.

Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor

03DS (35.8.0.1) $$(Sch/S)^{opp} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F}).$$

03DT **Lemma 35.8.1.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf, fpqc\}$. The functor defined in (35.8.0.1) satisfies the sheaf condition with respect to any τ-covering $\{T_i \to T\}_{i\in I}$ of any scheme T over S.*

Proof. For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ a τ-covering is also a fpqc-covering, see the results in Topologies, Lemmas 34.4.2, 34.5.2, 34.6.2, 34.7.2, and 34.9.7. Hence it suffices to prove the theorem for a fpqc covering. Assume that $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering where $f : T \to S$ is given. Suppose that we have a family of sections $s_i \in \Gamma(T_i, f_i^*f^*\mathcal{F})$ such that $s_i|_{T_i\times_T T_j} = s_j|_{T_i\times_T T_j}$. We have to find the correspond section $s \in \Gamma(T, f^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_T = \mathcal{O}_{T_i} \to f_i^*f^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_T$ and $f^*\mathcal{F}$ on T. Hence by Proposition 35.5.2 we see that we may (uniquely) descend these to a map $\mathcal{O}_T \to f^*\mathcal{F}$ which gives us our section s. □

We may in particular make the following definition.

03DU **Definition 35.8.2.** Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let S be a scheme. Let Sch_τ be a big site containing S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module.

(1) The *structure sheaf of the big site* $(Sch/S)_\tau$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.

(2) If $\tau = Zariski$ or $\tau = étale$ the *structure sheaf of the small site* S_{Zar} or $S_{étale}$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
(3) The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the big site $(Sch/S)_\tau$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).
(4) If $\tau = Zariski$ or $\tau = étale$ the *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the small site S_{Zar} or $S_{étale}$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

Note how we use the same notation $\mathcal{F}^a$ in each case. No confusion can really arise from this as by definition the rule that defines the sheaf $\mathcal{F}^a$ is independent of the site we choose to look at.

03FG **Remark 35.8.3.** In Topologies, Lemma 34.3.12 we have seen that the small Zariski site of a scheme S is equivalent to S as a topological space in the sense that the categories of sheaves are naturally equivalent. Now that S_{Zar} is also endowed with a structure sheaf $\mathcal{O}$ we see that sheaves of modules on the ringed site $(S_{Zar}, \mathcal{O})$ agree with sheaves of modules on the ringed space $(S, \mathcal{O}_S)$.

070R **Remark 35.8.4.** Let $f : T \to S$ be a morphism of schemes. Each of the morphisms of sites f_{sites} listed in Topologies, Section 34.11 becomes a morphism of ringed sites. Namely, each of these morphisms of sites $f_{sites} : (Sch/T)_\tau \to (Sch/S)_{\tau'}$, or $f_{sites} : (Sch/S)_\tau \to S_{\tau'}$ is given by the continuous functor $S'/S \mapsto T \times_S S'/S$. Hence, given S'/S we let

$$f_{sites}^\sharp : \mathcal{O}(S'/S) \longrightarrow f_{sites,*}\mathcal{O}(S'/S) = \mathcal{O}(T \times_S S'/T)$$

be the usual map $\mathrm{pr}_{S'}^\sharp : \mathcal{O}(S') \to \mathcal{O}(T \times_S S')$. Similarly, the morphism $i_f : Sh(T_\tau) \to Sh((Sch/S)_\tau)$ for $\tau \in \{Zar, étale\}$, see Topologies, Lemmas 34.3.13 and 34.4.13, becomes a morphism of ringed topoi because $i_f^{-1}\mathcal{O} = \mathcal{O}$. Here are some special cases:

(1) The morphism of big sites $f_{big} : (Sch/X)_{fppf} \to (Sch/Y)_{fppf}$, becomes a morphism of ringed sites
$$(f_{big}, f_{big}^\sharp) : ((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow ((Sch/Y)_{fppf}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 18.6.1. Similarly for the big syntomic, smooth, étale and Zariski sites.
(2) The morphism of small sites $f_{small} : X_{étale} \to Y_{étale}$ becomes a morphism of ringed sites
$$(f_{small}, f_{small}^\sharp) : (X_{étale}, \mathcal{O}_X) \longrightarrow (Y_{étale}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 18.6.1. Similarly for the small Zariski site.

Let S be a scheme. It is clear that given an $\mathcal{O}$-module on (say) $(Sch/S)_{Zar}$ the pullback to (say) $(Sch/S)_{fppf}$ is just the fppf-sheafification. To see what happens when comparing big and small sites we have the following.

070S **Lemma 35.8.5.** *Let S be a scheme. Denote*

$$\begin{aligned} id_{\tau,Zar} &: (Sch/S)_\tau \to S_{Zar}, \quad \tau \in \{Zar, étale, smooth, syntomic, fppf\} \\ id_{\tau,étale} &: (Sch/S)_\tau \to S_{étale}, \quad \tau \in \{étale, smooth, syntomic, fppf\} \\ id_{small,étale,Zar} &: S_{étale} \to S_{Zar}, \end{aligned}$$

the morphisms of ringed sites of Remark 35.8.4. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then

(1) *$(id_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $(Sch/S)_\tau$, and

(2) *$(id_{small,étale,Zar})^*\mathcal{F}$ is the étale sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $S_{étale}$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $S_{étale}$. Then

(3) *$(id_{\tau,étale})^*\mathcal{G}$ is the τ-sheafification of the étale sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{G})$$
where $f_{small} : T_{étale} \to S_{étale}$ is the morphism of ringed small étale sites of Remark 35.8.4.

Proof. Proof of (1). We first note that the result is true when $\tau = Zar$ because in that case we have the morphism of topoi $i_f : Sh(T_{Zar}) \to Sh((Sch/S)_{Zar})$ such that $\mathrm{id}_{\tau,Zar} \circ i_f = f_{small}$ as morphisms $T_{Zar} \to S_{Zar}$, see Topologies, Lemmas 34.3.13 and 34.3.17. Since pullback is transitive (see Modules on Sites, Lemma 18.13.3) we see that $i_f^*(\mathrm{id}_{\tau,Zar})^*\mathcal{F} = f^*_{small}\mathcal{F}$ as desired. Hence, by the remark preceding this lemma we see that $(\mathrm{id}_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the presheaf $T \mapsto \Gamma(T, f^*\mathcal{F})$.

The proof of (3) is exactly the same as the proof of (1), except that it uses Topologies, Lemmas 34.4.13 and 34.4.17. We omit the proof of (2). □

03FH **Remark 35.8.6.** Remark 35.8.4 and Lemma 35.8.5 have the following applications:

(1) Let S be a scheme. The construction $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback under the morphism of ringed sites $\mathrm{id}_{\tau,Zar} : ((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$ or the morphism $\mathrm{id}_{small,étale,Zar} : (S_{étale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$.

(2) Let $f : X \to Y$ be a morphism of schemes. For any of the morphisms f_{sites} of ringed sites of Remark 35.8.4 we have
$$(f^*\mathcal{F})^a = f^*_{sites}\mathcal{F}^a.$$
This follows from (1) and the fact that pullbacks are compatible with compositions of morphisms of ringed sites, see Modules on Sites, Lemma 18.13.3.

03DV **Lemma 35.8.7.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.*

(1) *The sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$, as defined in Modules on Sites, Definition 18.23.1.*

(2) *If $\tau = Zariski$ or $\tau = étale$, then the sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on S_{Zar} or $S_{étale}$ as defined in Modules on Sites, Definition 18.23.1.*

Proof. Let $\{S_i \to S\}$ be a Zariski covering such that we have exact sequences

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}_{S_i} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}_{S_i} \longrightarrow \mathcal{F}|_{S_i} \longrightarrow 0$$

for some index sets K_i and J_i. This is possible by the definition of a quasi-coherent sheaf on a ringed space (See Modules, Definition 17.10.1).

Proof of (1). Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. It is clear that $\mathcal{F}^a|_{(Sch/S_i)_\tau}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3.

Proof of (2). Let $\tau = étale$. It is clear that $\mathcal{F}^a|_{(S_i)_{étale}}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(S_i)_{étale}} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(S_i)_{étale}} \longrightarrow \mathcal{F}^a|_{(S_i)_{étale}} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3. The case $\tau = Zariski$ is similar (actually, it is really tautological since the corresponding ringed topoi agree). □

0GN7 **Lemma 35.8.8.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Each of the functors $\mathcal{F} \mapsto \mathcal{F}^a$ of Definition 35.8.2*

$$QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O}) \quad or \quad QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$$

is fully faithful.

Proof. (By Lemma 35.8.7 we do indeed get functors as indicated.) We may and do identify $\mathcal{O}_S$-modules on S with modules on $(S_{Zar}, \mathcal{O}_S)$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ on quasi-coherent modules $\mathcal{F}$ is given by pullback by a morphism f of ringed sites, see Remark 35.8.6. In each case the functor f_* is given by restriction along the inclusion functor $S_{Zar} \to S_\tau$ or $S_{Zar} \to (Sch/S)_\tau$ (see discussion of how these morphisms of sites are defined in Topologies, Section 34.11). Combining this with the description of $f^*\mathcal{F} = \mathcal{F}^a$ we see that $f_*f^*\mathcal{F} = \mathcal{F}$ provided that $\mathcal{F}$ is quasi-coherent. Then we see that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a) = \mathrm{Hom}_{\mathcal{O}}(f^*\mathcal{F}, f^*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, f_*f^*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

as desired. □

03DX **Proposition 35.8.9.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.*

(1) *The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the big τ site of S.

(2) *Let $\tau = Zariski$ or $\tau = étale$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the small τ site of S.

Proof. We have seen in Lemma 35.8.7 that the functor is well defined. By Lemma 35.8.8 the functor is fully faithful. To finish the proof we will show that a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ gives rise to a descent datum for quasi-coherent sheaves relative to a τ-covering of S. Having produced this descent datum we will appeal to Proposition 35.5.2 to get the corresponding quasi-coherent sheaf on S.

Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}$-modules on the big τ site of S. By Modules on Sites, Definition 18.23.1 there exists a τ-covering $\{S_i \to S\}_{i\in I}$ of S such that each of the restrictions $\mathcal{G}|_{(Sch/S_i)_\tau}$ has a global presentation

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau} \longrightarrow 0$$

for some index sets J_i and K_i. We claim that this implies that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is $\mathcal{F}_i^a$ for some quasi-coherent sheaf $\mathcal{F}_i$ on S_i. Namely, this is clear for the direct sums $\bigoplus_{k\in K_i} \mathcal{O}|_{(Sch/S_i)_\tau}$ and $\bigoplus_{j\in J_i} \mathcal{O}|_{(Sch/S_i)_\tau}$. Hence we see that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is a cokernel of a map $\varphi : \mathcal{K}_i^a \to \mathcal{L}_i^a$ for some quasi-coherent sheaves $\mathcal{K}_i$, $\mathcal{L}_i$ on S_i. By the fully faithfulness of $(\ )^a$ we see that $\varphi = \phi^a$ for some map of quasi-coherent sheaves $\phi : \mathcal{K}_i \to \mathcal{L}_i$ on S_i. Then it is clear that $\mathcal{G}|_{(Sch/S_i)_\tau} \cong \mathrm{Coker}(\phi)^a$ as claimed.

Since $\mathcal{G}$ lives on all of the category $(Sch/S)_\tau$ we see that

$$(\mathrm{pr}_0^*\mathcal{F}_i)^a \cong \mathcal{G}|_{(Sch/(S_i\times_S S_j))_\tau} \cong (\mathrm{pr}_1^*\mathcal{F})^a$$

as $\mathcal{O}$-modules on $(Sch/(S_i \times_S S_j))_\tau$. Hence, using fully faithfulness again we get canonical isomorphisms

$$\phi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \longrightarrow \mathrm{pr}_1^*\mathcal{F}_j$$

of quasi-coherent modules over $S_i \times_S S_j$. We omit the verification that these satisfy the cocycle condition. Since they do we see by effectivity of descent for quasi-coherent sheaves and the covering $\{S_i \to S\}$ (Proposition 35.5.2) that there exists a quasi-coherent sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$ compatible with the given descent data. In other words we are given $\mathcal{O}$-module isomorphisms

$$\phi_i : \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau}$$

which agree over $S_i \times_S S_j$. Hence, since $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G})$ is a sheaf (Modules on Sites, Lemma 18.27.1), we conclude that there is a morphism of $\mathcal{O}$-modules $\mathcal{F}^a \to \mathcal{G}$ recovering the isomorphisms ϕ_i above. Hence this is an isomorphism and we win.

The case of the sites $S_{étale}$ and S_{Zar} is proved in the exact same manner. □

05VG **Lemma 35.8.10.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{P}$ be one of the properties of modules[4] defined in Modules on Sites, Definitions 18.17.1, 18.23.1, and 18.28.1. The equivalences of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O}) \quad \text{and} \quad QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 35.8.9 have the property

$$\mathcal{F} \text{ has } \mathcal{P} \Leftrightarrow \mathcal{F}^a \text{ has } \mathcal{P} \text{ as an } \mathcal{O}\text{-module}$$

[4]The list is: free, finite free, generated by global sections, generated by r global sections, generated by finitely many global sections, having a global presentation, having a global finite presentation, locally free, finite locally free, locally generated by sections, locally generated by r sections, finite type, of finite presentation, coherent, or flat.

except (possibly) when $\mathcal{P}$ is "locally free" or "coherent". If $\mathcal{P}$ ="coherent" the equivalence holds for $QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$ when S is locally Noetherian and τ is Zariski or étale.

Proof. This is immediate for the global properties, i.e., those defined in Modules on Sites, Definition 18.17.1. For the local properties we can use Modules on Sites, Lemma 18.23.3 to translate "$\mathcal{F}^a$ has $\mathcal{P}$" into a property on the members of a covering of X. Hence the result follows from Lemmas 35.7.1, 35.7.3, 35.7.4, 35.7.5, and 35.7.6. Being coherent for a quasi-coherent module is the same as being of finite type over a locally Noetherian scheme (see Cohomology of Schemes, Lemma 30.9.1) hence this reduces to the case of finite type modules (details omitted). □

35.9. Cohomology of quasi-coherent modules and topologies

0GN8 In this section we prove that cohomology of quasi-coherent modules is independent of the choice of topology.

03FI **Lemma 35.9.1.** *Let S be a scheme. Let*

(a) *$\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$ and $\mathcal{C} = (Sch/S)_\tau$, or*
(b) *let $\tau = étale$ and $\mathcal{C} = S_{étale}$, or*
(c) *let $\tau = Zariski$ and $\mathcal{C} = S_{Zar}$.*

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ be affine. Let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a standard affine τ-covering in $\mathcal{C}$. Then

(1) *$\mathcal{V} = \{\coprod_{i=1,\ldots,n} U_i \to U\}$ is a τ-covering of U,*
(2) *$\mathcal{U}$ is a refinement of $\mathcal{V}$, and*
(3) *the induced map on Čech complexes (Cohomology on Sites, Equation (21.8.2.1))*

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Proof. This follows because

$$(\coprod\nolimits_{i_0=1,\ldots,n} U_{i_0}) \times_U \ldots \times_U (\coprod\nolimits_{i_p=1,\ldots,n} U_{i_p}) = \coprod\nolimits_{i_0,\ldots,i_p \in \{1,\ldots,n\}} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

and the fact that $\mathcal{F}(\coprod_a V_a) = \prod_a \mathcal{F}(V_a)$ since disjoint unions are τ-coverings. □

03FJ **Lemma 35.9.2.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let τ, $\mathcal{C}$, U, $\mathcal{U}$ be as in Lemma 35.9.1. Then there is an isomorphism of complexes*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a) \cong s((A/R)_\bullet \otimes_R M)$$

(see Section 35.3) where $R = \Gamma(U, \mathcal{O}_U)$, $M = \Gamma(U, \mathcal{F}^a)$ and $R \to A$ is a faithfully flat ring map. In particular

$$\check{H}^p(\mathcal{U}, \mathcal{F}^a) = 0$$

for all $p \geq 1$.

Proof. By Lemma 35.9.1 we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a)$ is isomorphic to $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^a)$ where $\mathcal{V} = \{V \to U\}$ with $V = \coprod_{i=1,\ldots n} U_i$ affine also. Set $A = \Gamma(V, \mathcal{O}_V)$. Since $\{V \to U\}$ is a τ-covering we see that $R \to A$ is faithfully flat. On the other hand, by definition of $\mathcal{F}^a$ we have that the degree p term $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{F}^a)$ is

$$\Gamma(V \times_U \ldots \times_U V, \mathcal{F}^a) = \Gamma(\mathrm{Spec}(A \otimes_R \ldots \otimes_R A), \mathcal{F}^a) = A \otimes_R \ldots \otimes_R A \otimes_R M$$

We omit the verification that the maps of the Čech complex agree with the maps in the complex $s((A/R)_\bullet \otimes_R M)$. The vanishing of cohomology is Lemma 35.3.6. □

03DW **Proposition 35.9.3.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.*

(1) *There is a canonical isomorphism*
$$H^q(S, \mathcal{F}) = H^q((Sch/S)_\tau, \mathcal{F}^a).$$
(2) *There are canonical isomorphisms*
$$H^q(S, \mathcal{F}) = H^q(S_{Zar}, \mathcal{F}^a) = H^q(S_{étale}, \mathcal{F}^a).$$

Proof. The result for $q = 0$ is clear from the definition of $\mathcal{F}^a$. Let $\mathcal{C} = (Sch/S)_\tau$, or $\mathcal{C} = S_{étale}$, or $\mathcal{C} = S_{Zar}$.

We are going to apply Cohomology on Sites, Lemma 21.10.9 with $\mathcal{F} = \mathcal{F}^a$, $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ the set of affine schemes in $\mathcal{C}$, and $\mathrm{Cov} \subset \mathrm{Cov}_\mathcal{C}$ the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Lemma 35.9.2. Hence we conclude that $H^p(U, \mathcal{F}^a) = 0$ for every affine object U of $\mathcal{C}$.

Next, let $U \in \mathrm{Ob}(\mathcal{C})$ be any separated object. Denote $f : U \to S$ the structure morphism. Let $U = \bigcup U_i$ be an affine open covering. We may also think of this as a τ-covering $\mathcal{U} = \{U_i \to U\}$ of U in $\mathcal{C}$. Note that $U_{i_0} \times_U \ldots \times_U U_{i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine as we assumed U separated. By Cohomology on Sites, Lemma 21.10.7 and the result above we see that
$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = H^p(U, f^*\mathcal{F})$$
the last equality by Cohomology of Schemes, Lemma 30.2.6. In particular, if S is separated we can take $U = S$ and $f = \mathrm{id}_S$ and the proposition is proved. We suggest the reader skip the rest of the proof (or rewrite it to give a clearer exposition).

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on S. Choose an injective resolution $\mathcal{F}^a \to \mathcal{J}^\bullet$ on $\mathcal{C}$. Denote $\mathcal{J}^n|_S$ the restriction of $\mathcal{J}^n$ to opens of S; this is a sheaf on the topological space S as open coverings are τ-coverings. We get a complex
$$0 \to \mathcal{F} \to \mathcal{J}^0|_S \to \mathcal{J}^1|_S \to \ldots$$
which is exact since its sections over any affine open $U \subset S$ is exact (by the vanishing of $H^p(U, \mathcal{F}^a)$, $p > 0$ seen above). Hence by Derived Categories, Lemma 13.18.6 there exists map of complexes $\mathcal{J}^\bullet|_S \to \mathcal{I}^\bullet$ which in particular induces a map
$$R\Gamma(\mathcal{C}, \mathcal{F}^a) = \Gamma(S, \mathcal{J}^\bullet) \longrightarrow \Gamma(S, \mathcal{I}^\bullet) = R\Gamma(S, \mathcal{F}).$$
Taking cohomology gives the map $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ which we have to prove is an isomorphism. Let $\mathcal{U} : S = \bigcup U_i$ be an affine open covering which we may think of as a τ-covering also. By the above we get a map of double complexes
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}|_S) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$
This map induces a map of spectral sequences
$${}^\tau E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$
The first spectral sequence converges to $H^{p+q}(\mathcal{C}, \mathcal{F})$ and the second to $H^{p+q}(S, \mathcal{F})$. On the other hand, we have seen that the induced maps ${}^\tau E_2^{p,q} \to E_2^{p,q}$ are bijections (as all the intersections are separated being opens in affines). Whence also the maps $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ are isomorphisms, and we win. □

03LC **Proposition 35.9.4.** *Let $f : T \to S$ be a morphism of schemes.*

(1) *The equivalences of categories of Proposition 35.8.9 are compatible with pullback. More precisely, we have* $f^*(\mathcal{G}^a) = (f^*\mathcal{G})^a$ *for any quasi-coherent sheaf* $\mathcal{G}$ *on* S.
(2) *The equivalences of categories of Proposition 35.8.9 part (1) are* **not** *compatible with pushforward in general.*
(3) *If* f *is quasi-compact and quasi-separated, and* $\tau \in \{Zariski, étale\}$ *then* f_* *and* $f_{small,*}$ *preserve quasi-coherent sheaves and the diagram*

$$\begin{array}{ccc} QCoh(\mathcal{O}_T) & \xrightarrow{f_*} & QCoh(\mathcal{O}_S) \\ {\scriptstyle \mathcal{F} \mapsto \mathcal{F}^a}\downarrow & & \downarrow{\scriptstyle \mathcal{G} \mapsto \mathcal{G}^a} \\ QCoh(T_\tau, \mathcal{O}) & \xrightarrow{f_{small,*}} & QCoh(S_\tau, \mathcal{O}) \end{array}$$

is commutative, i.e., $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$.

Proof. Part (1) follows from the discussion in Remark 35.8.6. Part (2) is just a warning, and can be explained in the following way: First the statement cannot be made precise since f_* does not transform quasi-coherent sheaves into quasi-coherent sheaves in general. Even if this is the case for f (and any base change of f), then the compatibility over the big sites would mean that formation of $f_*\mathcal{F}$ commutes with any base change, which does not hold in general. An explicit example is the quasi-compact open immersion $j : X = \mathbf{A}^2_k \setminus \{0\} \to \mathbf{A}^2_k = Y$ where k is a field. We have $j_*\mathcal{O}_X = \mathcal{O}_Y$ but after base change to $\mathrm{Spec}(k)$ by the 0 map we see that the pushforward is zero.

Let us prove (3) in case $\tau = étale$. Note that f, and any base change of f, transforms quasi-coherent sheaves into quasi-coherent sheaves, see Schemes, Lemma 26.24.1. The equality $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$ means that for any étale morphism $g : U \to S$ we have $\Gamma(U, g^*f_*\mathcal{F}) = \Gamma(U \times_S T, (g')^*\mathcal{F})$ where $g' : U \times_S T \to T$ is the projection. This is true by Cohomology of Schemes, Lemma 30.5.2. □

071N **Lemma 35.9.5.** *Let* $f : T \to S$ *be a quasi-compact and quasi-separated morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* T. *For either the étale or Zariski topology, there are canonical isomorphisms* $R^i f_{small,*}(\mathcal{F}^a) = (R^i f_*\mathcal{F})^a$.

Proof. We prove this for the étale topology; we omit the proof in the case of the Zariski topology. By Cohomology of Schemes, Lemma 30.4.5 the sheaves $R^i f_*\mathcal{F}$ are quasi-coherent so that the assertion makes sense. The sheaf $R^i f_{small,*}\mathcal{F}^a$ is the sheaf associated to the presheaf

$$U \longmapsto H^i(U \times_S T, \mathcal{F}^a)$$

where $g : U \to S$ is an object of $S_{étale}$, see Cohomology on Sites, Lemma 21.7.4. By our conventions the right hand side is the étale cohomology of the restriction of $\mathcal{F}^a$ to the localization $T_{étale}/U \times_S T$ which equals $(U \times_S T)_{étale}$. By Proposition 35.9.3 this is presheaf the same as the presheaf

$$U \longmapsto H^i(U \times_S T, (g')^*\mathcal{F}),$$

where $g' : U \times_S T \to T$ is the projection. If U is affine then this is the same as $H^0(U, R^i f'_*(g')^*\mathcal{F})$, see Cohomology of Schemes, Lemma 30.4.6. By Cohomology of Schemes, Lemma 30.5.2 this is equal to $H^0(U, g^*R^i f_*\mathcal{F})$ which is the value of $(R^i f_*\mathcal{F})^a$ on U. Thus the values of the sheaves of modules $R^i f_{small,*}(\mathcal{F}^a)$ and

$(R^i f_* \mathcal{F})^a$ on every affine object of $S_{étale}$ are canonically isomorphic which implies they are canonically isomorphic. □

35.10. Quasi-coherent sheaves and topologies, II

0GN9 We continue the discussion comparing quasi-coherent modules on a scheme S with quasi-coherent modules on any of the sites associated to S in the chapter on topologies.

0GNA **Lemma 35.10.1.** *In Lemma 35.8.5 the morphism of ringed sites $id_{small,étale,Zar} : S_{étale} \to S_{Zar}$ is flat.*

Proof. Let us denote $\epsilon = \mathrm{id}_{small,étale,Zar}$ and $\mathcal{O}_{étale}$ and $\mathcal{O}_{Zar}$ the structure sheaves on $S_{étale}$ and S_{Zar}. We have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_{Zar}$-module. Recall that étale morphisms are open, see Morphisms, Lemma 29.36.13. It follows (from the construction of pullback on sheaves) that $\epsilon^{-1}\mathcal{O}_{Zar}$ is the sheafification of the presheaf $\mathcal{O}'$ on $S_{étale}$ which sends an étale morphism $f : V \to S$ to $\mathcal{O}_S(f(V))$. If both V and $U = f(V) \subset S$ are affine, then $V \to U$ is an étale morphism of affines, hence corresponds to an étale ring map. Since étale ring maps are flat, we see that $\mathcal{O}_S(U) = \mathcal{O}'(V) \to \mathcal{O}_{étale}(V) = \mathcal{O}_V(V)$ is flat. Finally, for every étale morphism $f : V \to S$, i.e., object of $S_{étale}$, there is an affine open covering $V = \bigcup V_i$ such that $f(V_i)$ is an affine open in S for all i[5]. Thus the result by Modules on Sites, Lemma 18.28.4. □

06VE **Lemma 35.10.2.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The functors*

$$QCoh(\mathcal{O}_S) \longrightarrow Mod((Sch/S)_\tau, \mathcal{O}) \quad and \quad QCoh(\mathcal{O}_S) \longrightarrow Mod(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 35.8.9 are

(1) *fully faithful,*
(2) *commmute with direct sums,*
(3) *commmute with colimits,*
(4) *right exact,*
(5) *exact as a functor $QCoh(\mathcal{O}_S) \to Mod(S_{étale}, \mathcal{O})$,*
(6) **not** *exact as a functor $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$ in general,*
(7) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ we have $(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})^a = \mathcal{F}^a \otimes_{\mathcal{O}} \mathcal{G}^a$,*
(8) *if $\tau = étale$ or $\tau = Zariski$, given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation we have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $Mod(S_\tau, \mathcal{O})$,*
(9) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ we do* **not** *have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $Mod((Sch/S)_\tau, \mathcal{O})$ in general even if $\mathcal{F}$ is of finite presentation, and*
(10) *given a short exact sequence $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ of $\mathcal{O}$-modules then $\mathcal{E}$ is quasi-coherent[6], i.e., $\mathcal{E}$ is in the essential image of the functor.*

[5] Namely, for $y \in V$, we pick an affine open $y \in V' \subset V$ with $f(V')$ contained in an affine open $U \subset S$. Then we pick an affine open $f(y) \in U' \subset f(V')$. Then $V'' = f^{-1}(U') \subset V'$ is affine as it is equal to $U' \times_U V'$ and $f(V'') = U'$ is affine too.

[6] Warning: This is misleading. See part (6).

Proof. Part (1) we saw in Proposition 35.8.9.

We have seen in Schemes, Section 26.24 that a colimit of quasi-coherent sheaves on a scheme is a quasi-coherent sheaf. Moreover, in Remark 35.8.6 we saw that $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites, hence commutes with all colimits, see Modules on Sites, Lemma 18.14.3. Thus (3) and its special case (2) hold.

This also shows that the functor is right exact (i.e., commutes with finite colimits), hence (4).

The functor $QCoh(\mathcal{O}_S) \to Mod(S_{étale}, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is left exact because an étale morphism is flat, see Morphisms, Lemma 29.36.12. This proves (5).

To see (6), suppose that $S = \mathrm{Spec}(\mathbf{Z})$. Then $2 : \mathcal{O}_S \to \mathcal{O}_S$ is injective but the associated map of $\mathcal{O}$-modules on $(Sch/S)_\tau$ isn't injective because $2 : \mathbf{F}_2 \to \mathbf{F}_2$ isn't injective and $\mathrm{Spec}(\mathbf{F}_2)$ is an object of $(Sch/S)_\tau$.

Part (7) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites and such commute with tensor products by Modules on Sites, Lemma 18.26.2.

Part (8) is obvious if $\tau = Zariski$ because the category of $\mathcal{O}$-modules on S_{Zar} is the same as the category of $\mathcal{O}_S$-modules on the topological space S. If $\tau = étale$ then (8) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for the flat morphism of ringed sites $(S_{étale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Lemma 35.10.1. Pullback by flat morphisms of ringed sites commutes with taking internal hom out of a finitely presented module by Modules on Sites, Lemma 18.31.4.

To see (9), suppose that $S = \mathrm{Spec}(\mathbf{Z})$. Let $\mathcal{F} = \mathrm{Coker}(2 : \mathcal{O}_S \to \mathcal{O}_S)$ and $\mathcal{G} = \mathcal{O}_S$. Then $\mathcal{F}^a = \mathrm{Coker}(2 : \mathcal{O} \to \mathcal{O})$ and $\mathcal{G}^a = \mathcal{O}$. Hence $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not zero, see proof of (6). On the other hand, the module $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is zero.

Proof of (10). Let $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_2$ quasi-coherent on S. Consider the restriction

$$0 \to \mathcal{F}_1 \to \mathcal{E}|_{S_{Zar}} \to \mathcal{F}_2$$

to S_{Zar}. By Proposition 35.9.3 we see that on any affine $U \subset S$ we have $H^1(U, \mathcal{F}_1^a) = H^1(U, \mathcal{F}_1) = 0$. Hence the sequence above is also exact on the right. By Schemes, Section 26.24 we conclude that $\mathcal{F} = \mathcal{E}|_{S_{Zar}}$ is quasi-coherent. Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{F}_1^a & \longrightarrow & \mathcal{F}^a & \longrightarrow & \mathcal{F}_2^a & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1^a & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_2^a & \longrightarrow & 0 \end{array}$$

To finish the proof it suffices to show that the top row is also right exact. To do this, denote once more $U = \mathrm{Spec}(A) \subset S$ an affine open of S. We have seen above that $0 \to \mathcal{F}_1(U) \to \mathcal{E}(U) \to \mathcal{F}_2(U) \to 0$ is exact. For any affine scheme V/U, $V = \mathrm{Spec}(B)$ the map $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ is injective. We have $\mathcal{F}_1^a(V) = \mathcal{F}_1(U) \otimes_A B$ by definition. The injection $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ factors as

$$\mathcal{F}_1(U) \otimes_A B \to \mathcal{E}(U) \otimes_A B \to \mathcal{E}(V)$$

Considering A-algebras B of the form $B = A \oplus M$ we see that $\mathcal{F}_1(U) \to \mathcal{E}(U)$ is universally injective (see Algebra, Definition 10.82.1). Since $\mathcal{E}(U) = \mathcal{F}(U)$ we conclude that $\mathcal{F}_1 \to \mathcal{F}$ remains injective after any base change, or equivalently that $\mathcal{F}_1^a \to \mathcal{F}^a$ is injective. □

0GNB **Lemma 35.10.3.** *Let S be a scheme. The category $QCoh(S_{étale}, \mathcal{O})$ of quasi-coherent modules on $S_{étale}$ has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.*
(5) *Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation. then the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 26.24. The proof will be to use Lemma 35.10.2 to transfer these truths to $S_{étale}$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh(S_{étale}, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh(S_{étale}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh(S_{étale}, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\operatorname{Ker}(a) = \operatorname{Ker}(b)^a$ and $\operatorname{Coker}(a) = \operatorname{Coker}(b)^a$ and we conclude.

Proof of (4). This follows from (3) except in the case when we know $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent. In this case write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 35.10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh(S_{étale}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(S_{étale}, \mathcal{O})$ with $\mathcal{F}$ of finite presentation. Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. By Lemma 35.8.10 we see that $\mathcal{H}$ is of finite presentation. By Lemma 35.10.2 part (8) we have $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = (\mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}, \mathcal{I}))^a$ and we conclude. □

0GNC **Lemma 35.10.4.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The category $QCoh((Sch/S)_\tau, \mathcal{O})$ of quasi-coherent modules on $(Sch/S)_\tau$ has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*

(4) *Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent so is $\mathcal{F}_2$.*
(5) *Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is finite locally free, the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 26.24. The proof will be to use Lemma 35.10.2 to transfer these truths to $(Sch/S)_\tau$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh((Sch/S)_\tau, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\operatorname{Coker}(a) = \operatorname{Coker}(b)^a$ (because a cokernel is a colimit) and we conclude.

Proof of (4). Write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 35.10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Write $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent $\mathcal{O}_S$-module. By Lemma 35.8.10 we see that $\mathcal{H}$ is finite locally free. The problem is Zariski local on S (details omitted) hence we may assume $\mathcal{H} = \mathcal{O}_S^{\oplus n}$ is finite free. Then $\mathcal{F} = \mathcal{O}^{\oplus n}$ and $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{G}^{\oplus n}$ is quasi-coherent. □

0GND **Example 35.10.5.** Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent modules on $(Sch/S)_\tau$ for one of the topologies τ considered in Lemma 35.10.4. In general it is not the case that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent even if $\mathcal{F}$ is of finite presentation. Namely, say $S = \operatorname{Spec}(\mathbf{Z})$, $\mathcal{F} = \operatorname{Coker}(2 : \mathcal{O} \to \mathcal{O})$, and $\mathcal{G} = \mathcal{O}$. Then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not quasi-coherent.

0GNE **Lemma 35.10.6.** *Let S be a scheme.*
(1) *The category $QCoh((Sch/S)_{fppf}, \mathcal{O})$ has colimits and they agree with colimits in the categories $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, and $Mod((Sch/S)_{fppf}, \mathcal{O})$.*
(2) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ the tensor products $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ computed in $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, or $Mod((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.*
(3) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ computed in $Mod((Sch/S)_{Zar}, \mathcal{O})$, $Mod((Sch/S)_{étale}, \mathcal{O})$, or $Mod((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.*

Proof. This lemma collects the results shown above in a slightly different manner. First of all, by Lemma 35.10.4 we already know the output of the construction in (1),

(2), or (3) ends up in $QCoh((Sch/S)_\tau, \mathcal{O})$. It remains to show in each case that the result is independent of the topology used. The key to this is that the equivalence $QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ of Proposition 35.8.9 is given by the same formula independent of the choice of the topology $\tau \in \{Zariski, étale, fppf\}$.

Proof of (1). Let $\mathcal{I} \to QCoh((Sch/S)_{fppf}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ in $Mod((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, étale, fppf\}$ by Lemma 35.10.2. This proves (1).

Proof of (2). Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. Then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{H} \otimes_{\mathcal{O}} \mathcal{I})^a$ in $Mod((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, étale, fppf\}$ by Lemma 35.10.2. This proves (2).

Proof of (3). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh((Sch/S)_{fppf}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{H}^a$ with $\mathcal{H}$ quasi-coherent on S. By Lemma 35.8.10 we have

$$\begin{aligned} \mathcal{F} \text{ finite locally free in fppf topology} &\Leftrightarrow \mathcal{H} \text{ finite locally free on } S \\ &\Leftrightarrow \mathcal{F} \text{ finite locally free in étale topology} \\ &\Leftrightarrow \mathcal{H} \text{ finite locally free on } S \\ &\Leftrightarrow \mathcal{F} \text{ finite locally free in Zariski topology} \end{aligned}$$

This explains the parenthetical statement of part (3). Now, if these equivalent conditions hold, then $\mathcal{H}$ is finite locally free. The construction of $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 18.27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules (only whether the output $\mathcal{H}om$ is a sheaf depends on whether $\mathcal{F}$ and $\mathcal{G}$ are sheaves). □

35.11. Quasi-coherent modules and affines

0GZT Let S be a scheme[7]. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Recall that $(Aff/S)_\tau$ is the full subcategory of $(Sch/S)_\tau$ whose objects are affine turned into a site by declaring the coverings to be the standard τ-coverings. By Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11 we have an equivalence of topoi $g : Sh((Aff/S)_\tau) \to Sh((Sch/S)_\tau)$ whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on $(Sch/S)_\tau$, let us temporarily and pedantically denote $\mathcal{O}_{Aff}$ the restriction of $\mathcal{O}$ to $(Aff/S)_\tau$. Then we obtain an equivalence

0GZU (35.11.0.1) $$(Sh((Aff/S)_\tau), \mathcal{O}_{Aff}) \longrightarrow (Sh((Sch/S)_\tau), \mathcal{O})$$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZV **Lemma 35.11.1.** *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{Aff}$-modules on $(Aff/S)_{fppf}$. The following are equivalent*

1. *for every morphism $U \to U'$ of $(Aff/S)_{fppf}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,*
2. *$\mathcal{F}$ is a sheaf on $(Aff/S)_{Zar}$ and a quasi-coherent module on the ringed site $((Aff/S)_{Zar}, \mathcal{O}_{Aff})$ in the sense of Modules on Sites, Definition 18.23.1,*
3. *same as in (2) for the étale topology,*

[7] In this section, as in Topologies, Section 34.11, we choose our sites $(Sch/S)_\tau$ to have the same underlying category for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then also the sites $(Aff/S)_\tau$ have the same underlying category.

(4) *same as in (2) for the smooth topology,*
(5) *same as in (2) for the syntomic topology,*
(6) *same as in (2) for the fppf topology,*
(7) *$\mathcal{F}$ corresponds to a quasi-coherent module on $(Sch/S)_{Zar}$, $(Sch/S)_{étale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, or $(Sch/S)_{fppf}$ via the equivalence (35.11.0.1),*
(8) *$\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 35.8.*

Proof. Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (35.11.0.1) induces an equivalence between categories of quasi-coherent modules. Proposition 35.8.9 says the topology we use to study quasi-coherent modules on Sch/S does not matter and it also tells us that (8) is the same as (7). Hence we see that (2) – (8) are all equivalent.

Assume the equivalent conditions (2) – (8) hold and let $\mathcal{G}$ be as in (8). Let $h : U \to U' \to S$ be a morphism of Aff/S. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3.

Assume (1) holds. To finish the proof it suffices to prove (2). Let U be an object of $(Aff/S)_{Zar}$. Say $U = \mathrm{Spec}(R)$. A standard open covering $U = U_1 \cup \ldots \cup U_n$ is given by $U_i = D(f_i)$ for some elements $f_1, \ldots, f_n \in R$ generating the unit ideal of R. By property (1) we see that

$$\mathcal{F}(U_i) = \mathcal{F}(U) \otimes_R R_{f_i} = \mathcal{F}(U)_{f_i}$$

and

$$\mathcal{F}(U_i \cap U_j) = \mathcal{F}(U) \otimes_R R_{f_i f_j} = \mathcal{F}(U)_{f_i f_j}$$

Thus we conclude from Algebra, Lemma 10.24.1 that $\mathcal{F}$ is a sheaf on $(Aff/S)_{Zar}$. Choose a presentation

$$\bigoplus\nolimits_{k \in K} R \longrightarrow \bigoplus\nolimits_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $(Aff/S)_{Zar}$ we obtain a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}(U') \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $(Aff/S)_{Zar}/U$ has a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}|_{(Aff/S)_{Zar}/U} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}|_{(Aff/S)_{Zar}/U} \longrightarrow \mathcal{F}|_{(Aff/S)_{Zar}/U} \longrightarrow 0$$

With apologies for the horrible notation, this finishes the proof. □

We continue the discussion started in the introduction to this section. Let $\tau \in \{Zariski, étale\}$. Recall that $S_{affine,\tau}$ is the full subcategory of S_τ whose objects are affine turned into a site by declaring the coverings to be the standard τ coverings. See Topologies, Definitions 34.3.7 and 34.4.8. By Topologies, Lemmas 34.3.11, resp. 34.4.12 we have an equivalence of topoi $g : Sh(S_{affine,\tau}) \to Sh(S_\tau)$, whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on S_τ let us temporarily and pedantically denote $\mathcal{O}_{affine}$ the restriction of $\mathcal{O}$ to $S_{affine,\tau}$. Then we obtain an equivalence

0GZW (35.11.1.1) $$(Sh(S_{affine,\tau}), \mathcal{O}_{affine}) \longrightarrow (Sh(S_\tau), \mathcal{O})$$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZX **Lemma 35.11.2.** *Let S be a scheme. Let $\tau \in \{Zariski, étale\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{affine}$-modules on $S_{affine,\tau}$. The following are equivalent*

1. *for every morphism $U \to U'$ of $S_{affine,\tau}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,*
2. *$\mathcal{F}$ is a sheaf on $S_{affine,\tau}$ and a quasi-coherent module on the ringed site $(S_{affine,\tau}, \mathcal{O}_{affine})$ in the sense of Modules on Sites, Definition 18.23.1,*
3. *$\mathcal{F}$ corresponds to a quasi-coherent module on S_τ via the equivalence (35.11.1.1),*
4. *$\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 35.8.*

Proof. Let us prove this in the case of the étale topology.

Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $S_{affine,étale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$. reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}(U) \to \prod \mathcal{O}(U_i)$ is faithfully flat (by Lemma 35.9.1 and the fact that coverings in $S_{affine,étale}$ are standard étale coverings) and we may apply Lemma 35.3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $S_{affine,étale}$. Namely, for U in $S_{affine,étale}$, set $R = \mathcal{O}(U)$ and choose a presentation

$$\bigoplus\nolimits_{k\in K} R \longrightarrow \bigoplus\nolimits_{l\in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $S_{affine,étale}$ we obtain a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}(U') \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $S_{affine,etale}/U$ has a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_{affine}|_{S_{affine,étale}/U} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_{affine}|_{S_{affine,étale}/U} \longrightarrow \mathcal{F}|_{S_{affine,étale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (35.11.1.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

The equivalence of (3) and (4) follows from Proposition 35.8.9.

Let us assume (4) and prove (1). Namely, let $\mathcal{G}$ be as in (4). Let $h : U \to U' \to S$ be a morphism of $S_{affine,étale}$. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3.

We omit the proof in the case of the Zariski topology. □

35.12. Parasitic modules

07AF Parasitic modules are those which are zero when restricted to schemes flat over the base scheme. Here is the formal definition.

06ZL **Definition 35.12.1.** Let S be a scheme. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.

(1) $\mathcal{F}$ is called *parasitic*[8] if for every flat morphism $U \to S$ we have $\mathcal{F}(U) = 0$.
(2) $\mathcal{F}$ is called *parasitic for the τ-topology* if for every τ-covering $\{U_i \to S\}_{i \in I}$ we have $\mathcal{F}(U_i) = 0$ for all i.

If $\tau = fppf$ this means that $\mathcal{F}|_{U_{Zar}} = 0$ whenever $U \to S$ is flat and locally of finite presentation; similar for the other cases.

0755 **Lemma 35.12.2.** *Let S be a scheme. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

(1) *If $\mathcal{G}$ is parasitic for the τ-topology, then $H^p_\tau(U, \mathcal{G}) = 0$ for every U open in S, resp. étale over S, resp. smooth over S, resp. syntomic over S, resp. flat and locally of finite presentation over S.*
(2) *If $\mathcal{G}$ is parasitic then $H^p_\tau(U, \mathcal{G}) = 0$ for every U flat over S.*

Proof. Proof in case $\tau = fppf$; the other cases are proved in the exact same way. The assumption means that $\mathcal{G}(U) = 0$ for any $U \to S$ flat and locally of finite presentation. Apply Cohomology on Sites, Lemma 21.10.9 to the subset $\mathcal{B} \subset \mathrm{Ob}((Sch/S)_{fppf})$ consisting of $U \to S$ flat and locally of finite presentation and the collection Cov of all fppf coverings of elements of $\mathcal{B}$. □

07AG **Lemma 35.12.3.** *Let $f : T \to S$ be a morphism of schemes. For any parasitic $\mathcal{O}$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

Proof. Recall that $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i((Sch/U \times_S T)_\tau, \mathcal{F})$$

see Cohomology on Sites, Lemma 21.7.4. If $U \to S$ is flat, then $U \times_S T \to T$ is flat as a base change. Hence the displayed group is zero by Lemma 35.12.2. If $\{U_i \to U\}$ is a τ-covering then $U_i \times_S T \to T$ is also flat. Hence it is clear that the sheafification of the displayed presheaf is zero on schemes U flat over S. □

0756 **Lemma 35.12.4.** *Let S be a scheme. Let $\tau \in \{Zar, étale\}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$ such that*

(1) *$\mathcal{G}|_{S_\tau}$ is quasi-coherent, and*
(2) *for every flat, locally finitely presented morphism $g : U \to S$ the canonical map $g^*_{\tau,small}(\mathcal{G}|_{S_\tau}) \to \mathcal{G}|_{U_\tau}$ is an isomorphism.*

Then $H^p(U, \mathcal{G}) = H^p(U, \mathcal{G}|_{U_\tau})$ for every U flat and locally of finite presentation over S.

Proof. Let $\mathcal{F}$ be the pullback of $\mathcal{G}|_{S_\tau}$ to the big fppf site $(Sch/S)_{fppf}$. Note that $\mathcal{F}$ is quasi-coherent. There is a canonical comparison map $\varphi : \mathcal{F} \to \mathcal{G}$ which by

[8] This may be nonstandard notation.

assumptions (1) and (2) induces an isomorphism $\mathcal{F}|_{U_\tau} \to \mathcal{G}|_{U_\tau}$ for all $g : U \to S$ flat and locally of finite presentation. Hence in the short exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{Im}(\varphi) \to 0$$

and

$$0 \to \mathrm{Im}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

the sheaves $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are parasitic for the fppf topology. By Lemma 35.12.2 we conclude that $H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$ is an isomorphism for $g : U \to S$ flat and locally of finite presentation. Since the result holds for $\mathcal{F}$ by Proposition 35.9.3 we win. □

35.13. Fpqc coverings are universal effective epimorphisms

023P We apply the material above to prove an interesting result, namely Lemma 35.13.7. By Sites, Section 7.12 this lemma implies that the representable presheaves on any of the sites $(Sch/S)_\tau$ are sheaves for $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. First we prove a helper lemma.

02KI **Lemma 35.13.1.** *For a scheme X denote $|X|$ the underlying set. Let $f : X \to S$ be a morphism of schemes. Then*

$$|X \times_S X| \to |X| \times_{|S|} |X|$$

is surjective.

Proof. Follows immediately from the description of points on the fibre product in Schemes, Lemma 26.17.5. □

0EUA **Lemma 35.13.2.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. The following are equivalent*

(1) *for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$\Gamma(X, \mathcal{F}) = \mathit{Equalizer}\Big(\prod_{i \in I} \Gamma(X_i, f_i^*\mathcal{F}) \rightrightarrows \prod_{i,j \in I} \Gamma(X_i \times_X X_j, (f_i \times f_j)^*\mathcal{F}) \Big)$$

(2) *$\{f_i : X_i \to X\}_{i \in I}$ is a universal effective epimorphism (Sites, Definition 7.12.1) in the category of affine schemes.*

Proof. Assume (2) holds and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the scheme (Constructions, Section 27.4)

$$X' = \underline{\mathrm{Spec}}_X(\mathcal{O}_X \oplus \mathcal{F})$$

where $\mathcal{O}_X \oplus \mathcal{F}$ is an $\mathcal{O}_X$-algebra with multiplication $(f, s)(f', s') = (ff', fs' + f's)$. If $s_i \in \Gamma(X_i, f_i^*\mathcal{F})$ is a section, then s_i determines a unique element of

$$\Gamma(X' \times_X X_i, \mathcal{O}_{X' \times_X X_i}) = \Gamma(X_i, \mathcal{O}_{X_i}) \oplus \Gamma(X_i, f_i^*\mathcal{F})$$

Proof of equality omitted. If $(s_i)_{i \in I}$ is in the equalizer of (1), then, using the equality

$$\mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$$

which holds for any scheme T, we see that these sections define a family of morphisms $h_i : X' \times_X X_i \to \mathbf{A}^1_{\mathbf{Z}}$ with $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ as morphisms $(X' \times_X X_i) \times_{X'} (X' \times_X X_j) \to \mathbf{A}^1_{\mathbf{Z}}$. Since we've assume (2) we obtain a morphism $h : X' \to \mathbf{A}^1_{\mathbf{Z}}$ compatible with the morphisms h_i which in turn determines an element $s \in \Gamma(X, \mathcal{F})$. We omit the verification that s maps to s_i in $\Gamma(X_i, f_i^*\mathcal{F})$.

Assume (1). Let T be an affine scheme and let $h_i : X_i \to T$ be a family of morphisms such that $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $X_i \times_X X_j$ for all $i, j \in I$. Then

$$\prod h_i^\sharp : \Gamma(T, \mathcal{O}_T) \to \prod \Gamma(X_i, \mathcal{O}_{X_i})$$

maps into the equalizer and we find that we get a ring map $\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ by the assumption of the lemma for $\mathcal{F} = \mathcal{O}_X$. This ring map corresponds to a morphism $h : X \to T$ such that $h_i = h \circ f_i$. Hence our family is an effective epimorphism.

Let $p : Y \to X$ be a morphism of affines. We will show the base changes $g_i : Y_i \to Y$ of f_i form an effective epimorphism by applying the result of the previous paragraph. Namely, if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then

$$\Gamma(Y, \mathcal{G}) = \Gamma(X, p_*\mathcal{G}), \quad \Gamma(Y_i, g_i^*\mathcal{G}) = \Gamma(X, f_i^* p_* \mathcal{G}),$$

and

$$\Gamma(Y_i \times_Y Y_j, (g_i \times g_j)^*\mathcal{G}) = \Gamma(X, (f_i \times f_j)^* p_* \mathcal{G})$$

by the trivial base change formula (Cohomology of Schemes, Lemma 30.5.1). Thus we see property (1) lemma holds for the family g_i. □

0EUB **Lemma 35.13.3.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes.*

(1) *If the family is universal effective epimorphism in the category of schemes, then $\coprod f_i$ is surjective.*
(2) *If X and X_i are affine and the family is a universal effective epimorphism in the category of affine schemes, then $\coprod f_i$ is surjective.*

Proof. Omitted. Hint: perform base change by $\mathrm{Spec}(\kappa(x)) \to X$ to see that any $x \in X$ has to be in the image. □

0EUC **Lemma 35.13.4.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes. If for every morphism $Y \to X$ with Y affine the family of base changes $g_i : Y_i \to Y$ forms an effective epimorphism, then the family of f_i forms a universal effective epimorphism in the category of schemes.*

Proof. Let $Y \to X$ be a morphism of schemes. We have to show that the base changes $g_i : Y_i \to Y$ form an effective epimorphism. To do this, assume given a scheme T and morphisms $h_i : Y_i \to T$ with $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Choose an affine open covering $Y = \bigcup V_\alpha$. Set $V_{\alpha,i}$ equal to the inverse image of V_α in Y_i. Then we see that $V_{\alpha,i} \to V_\alpha$ is the base change of f_i by $V_\alpha \to X$. Thus by assumption the family of restrictions $h_i|_{V_{\alpha,i}}$ come from a morphism of schemes $h_\alpha : V_\alpha \to T$. We leave it to the reader to show that these agree on overlaps and define the desired morphism $Y \to T$. See discussion in Schemes, Section 26.14. □

0EUD **Lemma 35.13.5.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. Assume the equivalent assumption of Lemma 35.13.2 hold and that moreover for any morphism of affines $Y \to X$ the map*

$$\coprod X_i \times_X Y \longrightarrow Y$$

is a submersive map of topological spaces (Topology, Definition 5.6.3). Then our family of morphisms is a universal effective epimorphism in the category of schemes.

Proof. By Lemma 35.13.4 it suffices to base change our family of morphisms by $Y \to X$ with Y affine. Set $Y_i = X_i \times_X Y$. Let T be a scheme and let $h_i : Y_i \to T$ be a family of morphisms such that $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Note that Y as a set is the coequalizer of the two maps from $\coprod Y_i \times_Y Y_j$ to $\coprod Y_i$. Namely, surjectivity by the affine case of Lemma 35.13.3 and injectivity by Lemma 35.13.1. Hence there is a set map of underlying sets $h : Y \to T$ compatible with the maps h_i. By the second condition of the lemma we see that h is continuous! Thus if $y \in Y$ and $U \subset T$ is an affine open neighbourhood of $h(y)$, then we can find an affine open $V \subset Y$ such that $h(V) \subset U$. Setting $V_i = Y_i \times_Y V = X_i \times_X V$ we can use the result proved in Lemma 35.13.2 to see that $h|_V : V \to U \subset T$ comes from a unique morphism of affine schemes $h_V : V \to U$ agreeing with $h_i|_{V_i}$ as morphisms of schemes for all i. Glueing these h_V (see Schemes, Section 26.14) gives a morphism $Y \to T$ as desired. □

03N0 **Lemma 35.13.6.** *Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering. Suppose that for each i we have an open subset $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subsets of $T_i \times_T T_j$. Then there exists a unique open subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. Apply Lemma 35.13.1 to the map $\coprod_{i \in I} T_i \to T$. It implies there exists a subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i, namely $W = \bigcup f_i(W_i)$. To see that W is open we may work Zariski locally on T. Hence we may assume that T is affine. Using Topologies, Definition 34.9.1 we may choose a standard fpqc covering $\{g_j : V_j \to T\}_{j \in J}$ which refines $\{T_i \to T\}_{i \in I}$. Let $\alpha : J \to I$ and $h_j : V_j \to T_{\alpha(j)}$ be as in Sites, Definition 7.8.1. Then $g_j^{-1}(W) = h_j^{-1}(W_{\alpha(j)})$. Thus we may assume $\{f_i : T_i \to T\}$ is a standard fpqc covering. In this case we may apply Morphisms, Lemma 29.25.12 to the morphism $\coprod T_i \to T$ to conclude that W is open. □

023Q **Lemma 35.13.7.** *Let $\{T_i \to T\}$ be an fpqc covering, see Topologies, Definition 34.9.1. Then $\{T_i \to T\}$ is a universal effective epimorphism in the category of schemes, see Sites, Definition 7.12.1. In other words, every representable functor on the category of schemes satisfies the sheaf condition for the fpqc topology, see Topologies, Definition 34.9.13.*

Proof. Let S be a scheme. We have to show the following: Given morphisms $\varphi_i : T_i \to S$ such that $\varphi_i|_{T_i \times_T T_j} = \varphi_j|_{T_i \times_T T_j}$ there exists a unique morphism $T \to S$ which restricts to φ_i on each T_i. In other words, we have to show that the functor $h_S = \mathrm{Mor}_{Sch}(-, S)$ satisfies the sheaf property for the fpqc topology.

If $\{T_i \to T\}$ is a Zariski covering, then this follows from Schemes, Lemma 26.14.1. Thus Topologies, Lemma 34.9.14 reduces us to the case of a covering $\{X \to Y\}$ given by a single surjective flat morphism of affines.

First proof. By Lemma 35.8.1 we have the sheaf condition for quasi-coherent modules for $\{X \to Y\}$. By Lemma 35.13.6 the morphism $X \to Y$ is universally submersive. Hence we may apply Lemma 35.13.5 to see that $\{X \to Y\}$ is a universal effective epimorphism.

Second proof. Let $R \to A$ be the faithfully flat ring map corresponding to our surjective flat morphism $\pi : X \to Y$. Let $f : X \to S$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as morphisms $X \times_Y X = \mathrm{Spec}(A \otimes_R A) \to S$. By Lemma 35.13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some

(set theoretic) map $g : \mathrm{Spec}(R) \to S$. By Morphisms, Lemma 29.25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in Y = \mathrm{Spec}(R)$. Choose $U \subset S$ affine open containing $g(y)$. Say $U = \mathrm{Spec}(B)$. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(U)$. The restriction of f to $\pi^{-1}(D(r))$ into U corresponds to a ring map $B \to A_r$. The two induced ring maps $B \to A_r \otimes_{R_r} A_r = (A \otimes_R A)_r$ are equal by assumption on f. Note that $R_r \to A_r$ is faithfully flat. By Lemma 35.3.6 the equalizer of the two arrows $A_r \to A_r \otimes_{R_r} A_r$ is R_r. We conclude that $B \to A_r$ factors uniquely through a map $B \to R_r$. This map in turn gives a morphism of schemes $D(r) \to U \to S$, see Schemes, Lemma 26.6.4.

What have we proved so far? We have shown that for any prime $\mathfrak{p} \subset R$, there exists a standard affine open $D(r) \subset \mathrm{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to S$ factors uniquely through some morphism of schemes $D(r) \to S$. We omit the verification that these morphisms glue to the desired morphism $\mathrm{Spec}(R) \to S$. □

0BMN **Lemma 35.13.8.** *Consider schemes X, Y, Z and morphisms $a, b : X \to Y$ and a morphism $c : Y \to Z$ with $c \circ a = c \circ b$. Set $d = c \circ a = c \circ b$. If there exists an fpqc covering $\{Z_i \to Z\}$ such that*

(1) *for all i the morphism $Y \times_{c,Z} Z_i \to Z_i$ is the coequalizer of $(a, 1) : X \times_{d,Z} Z_i \to Y \times_{c,Z} Z_i$ and $(b, 1) : X \times_{d,Z} Z_i \to Y \times_{c,Z} Z_i$, and*
(2) *for all i and i' the morphism $Y \times_{c,Z} (Z_i \times_Z Z_{i'}) \to (Z_i \times_Z Z_{i'})$ is the coequalizer of $(a, 1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$ and $(b, 1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$*

then c is the coequalizer of a and b.

Proof. Namely, for a scheme T a morphism $Z \to T$ is the same thing as a collection of morphism $Z_i \to T$ which agree on overlaps by Lemma 35.13.7. □

35.14. Descent of finiteness and smoothness properties of morphisms

02KJ In this section we show that several properties of morphisms (being smooth, locally of finite presentation, and so on) descend under faithfully flat morphisms. We start with an algebraic version. (The "Noetherian" reader should consult Lemma 35.14.2 instead of the next lemma.)

02KK **Lemma 35.14.1.** *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite presentation and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite presentation.*

Proof. Consider the algebra $C = B \otimes_A B$ together with the pair of maps $p, q : B \to C$ given by $p(b) = b \otimes 1$ and $q(b) = 1 \otimes b$. Of course the two compositions $A \to B \to C$ are the same. Note that as $p : B \to C$ is flat and of finite presentation (base change of $A \to B$), the ring map $R \to C$ is of finite presentation (as the composite of $R \to B \to C$).

We are going to use the criterion Algebra, Lemma 10.127.3 to show that $R \to A$ is of finite presentation. Let S be any R-algebra, and suppose that $S = \mathrm{colim}_{\lambda \in \Lambda} S_\lambda$ is written as a directed colimit of R-algebras. Let $A \to S$ be an R-algebra homomorphism. We have to show that $A \to S$ factors through one of the S_λ. Consider the rings $B' = S \otimes_A B$ and $C' = S \otimes_A C = B' \otimes_S B'$. As B is faithfully flat of

finite presentation over A, also B' is faithfully flat of finite presentation over S. By Algebra, Lemma 10.168.1 part (2) applied to the pair $(S \to B', B')$ and the system (S_λ) there exists a $\lambda_0 \in \Lambda$ and a flat, finitely presented S_{λ_0}-algebra B_{λ_0} such that $B' = S \otimes_{S_{\lambda_0}} B_{\lambda_0}$. For $\lambda \geq \lambda_0$ set $B_\lambda = S_\lambda \otimes_{S_{\lambda_0}} B_{\lambda_0}$ and $C_\lambda = B_\lambda \otimes_{S_\lambda} B_\lambda$.

We interrupt the flow of the argument to show that $S_\lambda \to B_\lambda$ is faithfully flat for λ large enough. (This should really be a separate lemma somewhere else, maybe in the chapter on limits.) Since $\mathrm{Spec}(B_{\lambda_0}) \to \mathrm{Spec}(S_{\lambda_0})$ is flat and of finite presentation it is open (see Morphisms, Lemma 29.25.10). Let $I \subset S_{\lambda_0}$ be an ideal such that $V(I) \subset \mathrm{Spec}(S_{\lambda_0})$ is the complement of the image. Note that formation of the image commutes with base change. Hence, since $\mathrm{Spec}(B') \to \mathrm{Spec}(S)$ is surjective, and $B' = B_{\lambda_0} \otimes_{S_{\lambda_0}} S$ we see that $IS = S$. Thus for some $\lambda \geq \lambda_0$ we have $IS_\lambda = S_\lambda$. For this and all greater λ the morphism $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(S_\lambda)$ is surjective.

By analogy with the notation in the first paragraph of the proof denote $p_\lambda, q_\lambda : B_\lambda \to C_\lambda$ the two canonical maps. Then $B' = \mathrm{colim}_{\lambda \geq \lambda_0} B_\lambda$ and $C' = \mathrm{colim}_{\lambda \geq \lambda_0} C_\lambda$. Since B and C are finitely presented over R there exist (by Algebra, Lemma 10.127.3 applied several times) a $\lambda \geq \lambda_0$ and an R-algebra maps $B \to B_\lambda$, $C \to C_\lambda$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & C_\lambda \\ {\scriptstyle p}\uparrow\uparrow{\scriptstyle q} & & {\scriptstyle p_\lambda}\uparrow\uparrow{\scriptstyle q_\lambda} \\ B & \longrightarrow & B_\lambda \end{array}$$

is commutative. OK, and this means that $A \to B \to B_\lambda$ maps into the equalizer of p_λ and q_λ. By Lemma 35.3.6 we see that S_λ is the equalizer of p_λ and q_λ. Thus we get the desired ring map $A \to S_\lambda$ and we win. □

Here is an easier version of this dealing with the property of being of finite type.

0367 **Lemma 35.14.2.** *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite type and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite type.*

Proof. By Algebra, Lemma 10.168.2 there exists a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & A_0 & \longrightarrow & B_0 \\ \| & & \downarrow & & \downarrow \\ R & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$. Since $R \to B$ is of finite type by assumption, we may add some elements to A_0 and assume that the map $B_0 \to B$ is surjective! In this case, since $A_0 \to B_0$ is faithfully flat, we see that as

$$(A_0 \to A) \otimes_{A_0} B_0 \cong (B_0 \to B)$$

is surjective, also $A_0 \to A$ is surjective. Hence we win. □

02KL **Lemma 35.14.3.** *Let* [**DG67**, IV, 17.7.5 (i) and (ii)].

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that f is surjective, flat and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. The problem is local on S and Y. Hence we may assume that S and Y are affine. Since f is flat and locally of finite presentation, we see that f is open (Morphisms, Lemma 29.25.10). Hence, since Y is quasi-compact, there exist finitely many affine opens $X_i \subset X$ such that $Y = \bigcup f(X_i)$. Clearly we may replace X by $\coprod X_i$, and hence we may assume X is affine as well. In this case the lemma is equivalent to Lemma 35.14.1 (resp. Lemma 35.14.2) above. □

We use this to improve some of the results on morphisms obtained earlier.

02KM **Lemma 35.14.4.** *Let*

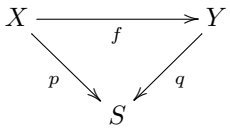

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and syntomic (resp. smooth, resp. étale),*
(2) *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. Combine Morphisms, Lemmas 29.30.16, 29.34.19, and 29.36.19 with Lemma 35.14.3 above. □

Actually we can strengthen this result as follows.

05B5 **Lemma 35.14.5.** *Let*

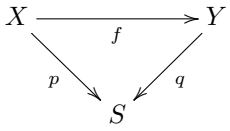

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. Assume (1) and that p is smooth. By Lemma 35.14.3 we see that q is locally of finite presentation. By Morphisms, Lemma 29.25.13 we see that q is flat. Hence now it suffices to show that the fibres of q are smooth, see Morphisms, Lemma 29.34.3. Apply Varieties, Lemma 33.25.9 to the flat surjective morphisms $X_s \to Y_s$ for $s \in S$ to conclude. We omit the proof of the étale case. □

05B6 **Remark 35.14.6.** With the assumptions (1) and p smooth in Lemma 35.14.5 it is not automatically the case that $X \to Y$ is smooth. A counter example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k[s])$, $Y = \mathrm{Spec}(k[t])$ and f given by $t \mapsto s^2$. But see also Lemma 35.14.7 for some information on the structure of f.

05B7 **Lemma 35.14.7.** *Let*

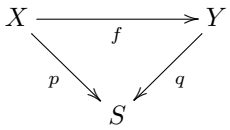

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is syntomic.*

Then both q and f are syntomic.

Proof. By Lemma 35.14.3 we see that q is of finite presentation. By Morphisms, Lemma 29.25.13 we see that q is flat. By Morphisms, Lemma 29.30.10 it now suffices to show that the local rings of the fibres of $Y \to S$ and the fibres of $X \to Y$ are local complete intersection rings. To do this we may take the fibre of $X \to Y \to S$ at a point $s \in S$, i.e., we may assume S is the spectrum of a field. Pick a point $x \in X$ with image $y \in Y$ and consider the ring map

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

This is a flat local homomorphism of local Noetherian rings. The local ring $\mathcal{O}_{X,x}$ is a complete intersection. Thus may use Avramov's result, see Divided Power Algebra, Lemma 23.8.9, to conclude that both $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings. □

The following type of lemma is occasionally useful.

06NB **Lemma 35.14.8.** *Let $X \to Y \to Z$ be morphism of schemes. Let P be one of the following properties of morphisms of schemes: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $\{X \to Y\}$ can be refined by an fppf covering of Y. Then $Y \to Z$ is P.*

Proof. Let $\mathrm{Spec}(C) \subset Z$ be an affine open and let $\mathrm{Spec}(B) \subset Y$ be an affine open which maps into $\mathrm{Spec}(C)$. The assumption on $X \to Y$ implies we can find a standard affine fppf covering $\{\mathrm{Spec}(B_j) \to \mathrm{Spec}(B)\}$ and lifts $x_j : \mathrm{Spec}(B_j) \to X$. Since $\mathrm{Spec}(B_j)$ is quasi-compact we can find finitely many affine opens $\mathrm{Spec}(A_i) \subset X$ lying over $\mathrm{Spec}(B)$ such that the image of each x_j is contained in the union $\bigcup \mathrm{Spec}(A_i)$. Hence after replacing each $\mathrm{Spec}(B_j)$ by a standard affine Zariski coverings of itself we may assume we have a standard affine fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(B)\}$ such that each $\mathrm{Spec}(B_i) \to Y$ factors through an affine open $\mathrm{Spec}(A_i) \subset X$ lying over $\mathrm{Spec}(B)$. In other words, we have ring maps $C \to B \to A_i \to B_i$ for each i. Note that we can also consider

$$C \to B \to A = \prod A_i \to B' = \prod B_i$$

and that the ring map $B \to \prod B_i$ is faithfully flat and of finite presentation.

The case $P = flat$. In this case we know that $C \to A$ is flat and we have to prove that $C \to B$ is flat. Suppose that $N \to N' \to N''$ is an exact sequence of C-modules. We want to show that $N \otimes_C B \to N' \otimes_C B \to N'' \otimes_C B$ is exact. Let H be its cohomology and let H' be the cohomology of $N \otimes_C B' \to N' \otimes_C B' \to N'' \otimes_C B'$. As $B \to B'$ is flat we know that $H' = H \otimes_B B'$. On the other hand $N \otimes_C A \to N' \otimes_C A \to N'' \otimes_C A$ is exact hence has zero cohomology. Hence the

map $H \to H'$ is zero (as it factors through the zero module). Thus $H' = 0$. As $B \to B'$ is faithfully flat we conclude that $H = 0$ as desired.

The case $P = \textit{locally finite type}$. In this case we know that $C \to A$ is of finite type and we have to prove that $C \to B$ is of finite type. Because $B \to B'$ is of finite presentation (hence of finite type) we see that $A \to B'$ is of finite type, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite type and we conclude by Lemma 35.14.2.

The case $P = \textit{locally finite presentation}$. In this case we know that $C \to A$ is of finite presentation and we have to prove that $C \to B$ is of finite presentation. Because $B \to B'$ is of finite presentation and $B \to A$ of finite type we see that $A \to B'$ is of finite presentation, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite presentation and we conclude by Lemma 35.14.1. □

35.15. Local properties of schemes

0347 It often happens one can prove the members of a covering of a scheme have a certain property. In many cases this implies the scheme has the property too. For example, if S is a scheme, and $f : S' \to S$ is a surjective flat morphism such that S' is a reduced scheme, then S is reduced. You can prove this by looking at local rings and using Algebra, Lemma 10.164.2. We say that the property of being reduced *descends through flat surjective morphisms*. Some results of this type are collected in Algebra, Section 10.164 and for schemes in Section 35.19. Some analogous results on descending properties of morphisms are in Section 35.14.

On the other hand, there are examples of surjective flat morphisms $f : S' \to S$ with S reduced and S' not, for example the morphism $\mathrm{Spec}(k[x]/(x^2)) \to \mathrm{Spec}(k)$. Hence the property of being reduced does not *ascend along flat morphisms*. Having infinite residue fields is a property which does ascend along flat morphisms (but does not descend along surjective flat morphisms of course). Some results of this type are collected in Algebra, Section 10.163.

Finally, we say that a property is *local for the flat topology* if it ascends along flat morphisms and descends along flat surjective morphisms. A somewhat silly example is the property of having residue fields of a given characteristic. To be more precise, and to tie this in with the various topologies on schemes, we make the following formal definition.

0348 **Definition 35.15.1.** Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *local in the τ-topology* if for any τ-covering $\{S_i \to S\}_{i \in I}$ (see Topologies, Section 34.2) we have

$$S \text{ has } \mathcal{P} \Leftrightarrow \text{each } S_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for S if and only if it holds for any scheme S' isomorphic to S. In fact, if $\tau = fpqc, fppf, syntomic, smooth, étale$, or $Zariski$, then if S has $\mathcal{P}$ and $S' \to S$ is flat, flat and locally of finite presentation, syntomic, smooth, étale, or an open immersion, then S' has $\mathcal{P}$. This is true because we can always extend $\{S' \to S\}$ to a τ-covering.

We have the following implications: $\mathcal{P}$ is local in the fpqc topology $\Rightarrow$ $\mathcal{P}$ is local in the fppf topology $\Rightarrow$ $\mathcal{P}$ is local in the syntomic topology $\Rightarrow$ $\mathcal{P}$ is local in the smooth

topology $\Rightarrow$ $\mathcal{P}$ is local in the étale topology $\Rightarrow$ $\mathcal{P}$ is local in the Zariski topology. This follows from Topologies, Lemmas 34.4.2, 34.5.2, 34.6.2, 34.7.2, and 34.9.7.

0349 **Lemma 35.15.2.** *Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is local in the Zariski topology,*
(2) *for any morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S' if property $\mathcal{P}$ holds for S, and*
(3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S if property $\mathcal{P}$ holds for S'.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.9, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. □

034A **Remark 35.15.3.** In Lemma 35.15.2 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

35.16. Properties of schemes local in the fppf topology

034B In this section we find some properties of schemes which are local on the base in the fppf topology.

034C **Lemma 35.16.1.** *The property $\mathcal{P}(S) =$"S is locally Noetherian" is local in the fppf topology.*

Proof. We will use Lemma 35.15.2. First we note that "being locally Noetherian" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.5.1. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is locally Noetherian, then S' is locally Noetherian. This is Morphisms, Lemma 29.15.6. Finally, we have to show that if $S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is locally Noetherian, then S is locally Noetherian. This follows from Algebra, Lemma 10.164.1. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

0368 **Lemma 35.16.2.** *The property $\mathcal{P}(S) =$"S is Jacobson" is local in the fppf topology.*

Proof. We will use Lemma 35.15.2. First we note that "being Jacobson" is local in the Zariski topology. This is Properties, Lemma 28.6.3. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is Jacobson, then S' is Jacobson. This is Morphisms, Lemma 29.16.9. Finally, we have to show that if $f : S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is Jacobson, then S is Jacobson. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ and $S' \to S$ given by $A \to B$. Then $A \to B$ is finitely presented and faithfully flat. Moreover, the ring B is Jacobson, see Properties, Lemma 28.6.3.

By Algebra, Lemma 10.168.10 there exists a diagram

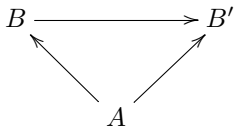

with $A \to B'$ finitely presented, faithfully flat and quasi-finite. In particular, $B \to B'$ is finite type, and we see from Algebra, Proposition 10.35.19 that B' is Jacobson. Hence we may assume that $A \to B$ is quasi-finite as well as faithfully flat and of finite presentation.

Assume A is not Jacobson to get a contradiction. According to Algebra, Lemma 10.35.5 there exists a nonmaximal prime $\mathfrak{p} \subset A$ and an element $f \in A$, $f \notin \mathfrak{p}$ such that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$.

This leads to a contradiction as follows. First let $\mathfrak{p} \subset \mathfrak{m}$ be a maximal ideal of A. Pick a prime $\mathfrak{m}' \subset B$ lying over $\mathfrak{m}$ (exists because $A \to B$ is faithfully flat, see Algebra, Lemma 10.39.16). As $A \to B$ is flat, by going down see Algebra, Lemma 10.39.19, we can find a prime $\mathfrak{q} \subset \mathfrak{m}'$ lying over $\mathfrak{p}$. In particular we see that $\mathfrak{q}$ is not maximal. Hence according to Algebra, Lemma 10.35.5 again the set $V(\mathfrak{q}) \cap D(f)$ is infinite (here we finally use that B is Jacobson). All points of $V(\mathfrak{q}) \cap D(f)$ map to $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$. Hence the fibre over $\mathfrak{p}$ is infinite. This contradicts the fact that $A \to B$ is quasi-finite (see Algebra, Lemma 10.122.5 or more explicitly Morphisms, Lemma 29.20.10). Thus the lemma is proved. □

0BAL **Lemma 35.16.3.** *The property $\mathcal{P}(S) =$"every quasi-compact open of S has a finite number of irreducible components" is local in the fppf topology.*

Proof. We will use Lemma 35.15.2. First we note that $\mathcal{P}$ is local in the Zariski topology. Next, we show that if $T \to S$ is a flat, finitely presented morphism of affines and S has a finite number of irreducible components, then so does T. Namely, since $T \to S$ is flat, the generic points of T map to the generic points of S, see Morphisms, Lemma 29.25.9. Hence it suffices to show that for $s \in S$ the fibre T_s has a finite number of generic points. Note that T_s is an affine scheme of finite type over $\kappa(s)$, see Morphisms, Lemma 29.15.4. Hence T_s is Noetherian and has a finite number of irreducible components (Morphisms, Lemma 29.15.6 and Properties, Lemma 28.5.7). Finally, we have to show that if $T \to S$ is a surjective flat, finitely presented morphism of affines and T has a finite number of irreducible components, then so does S. This follows from Topology, Lemma 5.8.5. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

35.17. Properties of schemes local in the syntomic topology

0369 In this section we find some properties of schemes which are local on the base in the syntomic topology.

036A **Lemma 35.17.1.** *The property $\mathcal{P}(S) =$"S is locally Noetherian and (S_k)" is local in the syntomic topology.*

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. As a syntomic morphism is flat of finite presentation (Morphisms, Lemmas 29.30.7 and 29.30.6) we have already checked this for "being locally Noetherian" in the proof of Lemma 35.16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in

the Zariski topology. This is clear from the definition, see Cohomology of Schemes, Definition 30.11.1. Next, we show that if $S' \to S$ is a syntomic morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemma 10.163.4 (use Morphisms, Lemma 29.30.2 and Algebra, Definition 10.136.1 and Lemma 10.135.3). Finally, we show that if $S' \to S$ is a surjective syntomic morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.164.5. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

036B **Lemma 35.17.2.** *The property $\mathcal{P}(S) =$"S is Cohen-Macaulay" is local in the syntomic topology.*

Proof. This is clear from Lemma 35.17.1 above since a scheme is Cohen-Macaulay if and only if it is locally Noetherian and (S_k) for all $k \geq 0$, see Properties, Lemma 28.12.3. □

35.18. Properties of schemes local in the smooth topology

034D In this section we find some properties of schemes which are local on the base in the smooth topology.

034E **Lemma 35.18.1.** *The property $\mathcal{P}(S) =$"S is reduced" is local in the smooth topology.*

Proof. We will use Lemma 35.15.2. First we note that "being reduced" is local in the Zariski topology. This is clear from the definition, see Schemes, Definition 26.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is reduced, then S' is reduced. This is Algebra, Lemma 10.163.7. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is reduced, then S is reduced. This is Algebra, Lemma 10.164.2. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

034F **Lemma 35.18.2.** *The property $\mathcal{P}(S) =$"S is normal" is local in the smooth topology.*

Proof. We will use Lemma 35.15.2. First we show "being normal" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.7.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is normal, then S' is normal. This is Algebra, Lemma 10.163.9. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is normal, then S is normal. This is Algebra, Lemma 10.164.3. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

036C **Lemma 35.18.3.** *The property $\mathcal{P}(S) =$"S is locally Noetherian and (R_k)" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. As a smooth morphism is flat of finite presentation (Morphisms, Lemmas 29.34.9 and 29.34.8) we have already checked this for "being locally Noetherian" in the proof of Lemma 35.16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemmas 10.163.5 (use Morphisms, Lemma 29.34.2, Algebra, Lemmas 10.137.3 and 10.140.3). Finally, we show that if $S' \to S$

is a surjective smooth morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.164.6. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

036D **Lemma 35.18.4.** *The property $\mathcal{P}(S) =$"S is regular" is local in the smooth topology.*

Proof. This is clear from Lemma 35.18.3 above since a locally Noetherian scheme is regular if and only if it is locally Noetherian and (R_k) for all $k \geq 0$. □

036E **Lemma 35.18.5.** *The property $\mathcal{P}(S) =$"S is Nagata" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. First we note that being Nagata is local in the Zariski topology. This is Properties, Lemma 28.13.6. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is Nagata, then S' is Nagata. This is Morphisms, Lemma 29.18.1. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is Nagata, then S is Nagata. This is Algebra, Lemma 10.164.7. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

35.19. Variants on descending properties

06QL Sometimes one can descend properties, which are not local. We put results of this kind in this section. See also Section 35.14 on descending properties of morphisms, such as smoothness.

06QM **Lemma 35.19.1.** *If $f : X \to Y$ is a flat and surjective morphism of schemes and X is reduced, then Y is reduced.*

Proof. The result follows by looking at local rings (Schemes, Definition 26.12.1) and Algebra, Lemma 10.164.2. □

06QN **Lemma 35.19.2.** *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. This lemma reduces to the following algebra statement: If $A \to B$ is a faithfully flat, finitely presented ring homomorphism with B Noetherian and regular, then A is Noetherian and regular. We see that A is Noetherian by Algebra, Lemma 10.164.1 and regular by Algebra, Lemma 10.110.9. □

35.20. Germs of schemes

04QQ

04QR **Definition 35.20.1.** Germs of schemes.

(1) A pair (X, x) consisting of a scheme X and a point $x \in X$ is called the *germ of X at x*.
(2) A *morphism of germs* $f : (X, x) \to (S, s)$ is an equivalence class of morphisms of schemes $f : U \to S$ with $f(x) = s$ where $U \subset X$ is an open neighbourhood of x. Two such f, f' are said to be equivalent if and only if f and f' agree in some open neighbourhood of x.
(3) We define the *composition of morphisms of germs* by composing representatives (this is well defined).

Before we continue we need one more definition.

04QS **Definition 35.20.2.** Let $f : (X, x) \to (S, s)$ be a morphism of germs. We say f is *étale* (resp. *smooth*) if there exists a representative $f : U \to S$ of f which is an étale morphism (resp. a smooth morphism) of schemes.

35.21. Local properties of germs

04QT

04N1 **Definition 35.21.1.** Let $\mathcal{P}$ be a property of germs of schemes. We say that $\mathcal{P}$ is *étale local* (resp. *smooth local*) if for any étale (resp. smooth) morphism of germs $(U', u') \to (U, u)$ we have $\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u')$.

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$. The following lemma tells us that the assertion $\dim_x(X) = d$ is an étale local property of germs.

04N4 **Lemma 35.21.2.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim_u(U) = \dim_v(V)$.*

Proof. In the statement $\dim_u(U)$ is the dimension of U at u as defined in Topology, Definition 5.10.1 as the minimum of the Krull dimensions of open neighbourhoods of u in U. Similarly for $\dim_v(V)$.

Let us show that $\dim_v(V) \geq \dim_u(U)$. Let V' be an open neighbourhood of v in V. Then there exists an open neighbourhood U' of u in U contained in $f^{-1}(V')$ such that $\dim_u(U) = \dim(U')$. Suppose that $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ is a chain of irreducible closed subschemes of U'. If $\xi_i \in Z_i$ is the generic point then we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. This gives specializations $f(\xi_n) \rightsquigarrow f(\xi_{n-1}) \rightsquigarrow \ldots \rightsquigarrow f(\xi_0)$ in V'. Note that $f(\xi_j) \neq f(\xi_i)$ if $i \neq j$ as the fibres of f are discrete (see Morphisms, Lemma 29.36.7). Hence we see that $\dim(V') \geq n$. The inequality $\dim_v(V) \geq \dim_u(U)$ follows formally.

Let us show that $\dim_u(U) \geq \dim_v(V)$. Let U' be an open neighbourhood of u in U. Note that $V' = f(U')$ is an open neighbourhood of v by Morphisms, Lemma 29.25.10. Hence $\dim(V') \geq \dim_v(V)$. Pick a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subschemes of V'. Let $\xi_i \in Z_i$ be the generic point, so we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. Since $\xi_0 \in f(U')$ we can find a point $\eta_0 \in U'$ with $f(\eta_0) = \xi_0$. Consider the map of local rings

$$\mathcal{O}_{V',\xi_0} \longrightarrow \mathcal{O}_{U',\eta_0}$$

which is a flat local ring map by Morphisms, Lemma 29.36.12. Note that the points ξ_i correspond to primes of the ring on the left by Schemes, Lemma 26.13.2. Hence by going down (see Algebra, Section 10.41) for the displayed ring map we can find a sequence of specializations $\eta_n \rightsquigarrow \eta_{n-1} \rightsquigarrow \ldots \rightsquigarrow \eta_0$ in U' mapping to the sequence $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$ under f. This implies that $\dim_u(U) \geq \dim_v(V)$. □

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property $\dim(\mathcal{O}_{X,x}) = d$ is an étale local property of germs.

04N8 **Lemma 35.21.3.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{V,v})$.*

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$. This is More on Algebra, Lemma 15.44.2. □

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property "$\mathcal{O}_{X,x}$ is regular" is an étale local property of germs.

0AH7 **Lemma 35.21.4.** *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\mathcal{O}_{U,u}$ is a regular local ring if and only if $\mathcal{O}_{V,v}$ is a regular local ring.*

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p}$ is regular if and only if $B_\mathfrak{q}$ is regular. This is More on Algebra, Lemma 15.44.3. □

35.22. Properties of morphisms local on the target

02KN Suppose that $f : X \to Y$ is a morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Let $f' : X' \to Y'$ be the base change of f by g:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{P}$ be a property of morphisms of schemes. Then we can wonder if (a) $\mathcal{P}(f) \Rightarrow \mathcal{P}(f')$, and also whether the converse (b) $\mathcal{P}(f') \Rightarrow \mathcal{P}(f)$ is true. If (a) holds whenever g is flat, then we say $\mathcal{P}$ is preserved under flat base change. If (b) holds whenever g is surjective and flat, then we say $\mathcal{P}$ descends through flat surjective base changes. If $\mathcal{P}$ is preserved under flat base changes and descends through flat surjective base changes, then we say $\mathcal{P}$ is flat local on the target. Compare with the discussion in Section 35.15. This turns out to be a very important notion which we formalize in the following definition.

02KO **Definition 35.22.1.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ (see Topologies, Section 34.2) and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

04QU **Lemma 35.22.2.** *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subscheme $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

06QP **Lemma 35.22.3.** *Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

(1) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then $g(Y') \subset W(f)$,*
(2) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
(3) *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union W of the images $g(Y') \subset Y$ of morphisms $g : Y' \to Y$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 29.25.10) we see that $W \subset Y$ is an open subset of Y. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 35.22.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. □

02KP **Lemma 35.22.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is preserved under flat, flat and locally of finite presentation, étale, smooth, or syntomic base change depending on whether τ is fpqc, fppf, étale, smooth, or syntomic (compare with Schemes, Definition 26.18.3),*
(2) *the property is Zariski local on the base.*
(3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, and any morphism of schemes $f : X \to S$ property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the base change $f' : X' = S' \times_S X \to S'$.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.9, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. □

034G **Remark 35.22.5.** (This is a repeat of Remark 35.15.3 above.) In Lemma 35.22.4 above if $\tau = \textit{smooth}$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = \textit{syntomic}$ or $\tau = \textit{étale}$.

35.23. Properties of morphisms local in the fpqc topology on the target

02YJ In this section we find a large number of properties of morphisms of schemes which are local on the base in the fpqc topology. By contrast, in Examples, Section 110.65 we will show that the properties "projective" and "quasi-projective" are not local on the base even in the Zariski topology.

02KQ **Lemma 35.23.1.** *The property $\mathcal{P}(f) =$"f is quasi-compact" is fpqc local on the base.*

Proof. A base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3. Being quasi-compact is Zariski local on the base, see Schemes, Lemma 26.19.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-compact. Then X' is quasi-compact, and $X' \to X$ is surjective. Hence X is quasi-compact. This implies that f is quasi-compact. Therefore Lemma 35.22.4 applies and we win. □

02KR **Lemma 35.23.2.** *The property $\mathcal{P}(f) =$"f is quasi-separated" is fpqc local on the base.*

Proof. Any base change of a quasi-separated morphism is quasi-separated, see Schemes, Lemma 26.21.12. Being quasi-separated is Zariski local on the base (from the definition or by Schemes, Lemma 26.21.6). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is quasi-compact. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 35.23.1 this implies Δ is quasi-compact, and hence f is quasi-separated. Therefore Lemma 35.22.4 applies and we win. □

02KS **Lemma 35.23.3.** *The property $\mathcal{P}(f) =$"f is universally closed" is fpqc local on the base.*

Proof. A base change of a universally closed morphism is universally closed by definition. Being universally closed is Zariski local on the base (from the definition or by Morphisms, Lemma 29.41.2). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally closed. Let $T \to S$ be any morphism. Consider the diagram

$$\begin{array}{ccccc} X' & \longleftarrow & S' \times_S T \times_S X & \longrightarrow & T \times_S X \\ \downarrow & & \downarrow & & \downarrow \\ S' & \longleftarrow & S' \times_S T & \longrightarrow & T \end{array}$$

in which both squares are cartesian. Thus the assumption implies that the middle vertical arrow is closed. The right horizontal arrows are flat, quasi-compact and surjective (as base changes of $S' \to S$). Hence a subset of T is closed if and only if its inverse image in $S' \times_S T$ is closed, see Morphisms, Lemma 29.25.12. An easy diagram chase shows that the right vertical arrow is closed too, and we conclude $X \to S$ is universally closed. Therefore Lemma 35.22.4 applies and we win. □

02KT **Lemma 35.23.4.** *The property $\mathcal{P}(f) =$"f is universally open" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 35.23.3. □

0CEW **Lemma 35.23.5.** *The property $\mathcal{P}(f) =$"f is universally submersive" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 35.23.3 using that a quasi-compact flat surjective morphism is universally submersive by Morphisms, Lemma 29.25.12. □

02KU **Lemma 35.23.6.** *The property $\mathcal{P}(f) =$"f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Schemes, Lemma 26.21.12. Being separated is Zariski local on the base (from the definition or by Schemes, Lemma 26.21.7). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, hence universally closed. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 35.23.3 this implies Δ is universally closed. Since it is an immersion (Schemes, Lemma 26.21.2) we conclude Δ is a closed immersion. Hence f is separated. Therefore Lemma 35.22.4 applies and we win. □

02KV **Lemma 35.23.7.** *The property $\mathcal{P}(f) =$"f is surjective" is fpqc local on the base.*

Proof. This is clear. □

0H8H **Lemma 35.23.8.** *The property $\mathcal{P}(f) =$"f is quasi-compact and dominant" is fpqc local on the base.*

Proof. By Morphisms, Lemma 29.8.4, quasi-compact dominant morphisms are preserved by flat pullback. The other direction is easier. Indeed, quasi-compactness is fpqc local on the base by Lemma 35.23.1. Being dominant is clearly Zariski local on the base. Finally, let $S' \to S$ be a surjective morphism of schemes (not necessarily flat), and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is dominant. Then $X' \to S' \to S$ is a composite of two dominant morphisms, hence dominant. Since this is also the composite $X' \to X \to S$, it follows that $X \to S$ is dominant. □

Dominant morphisms are not preserved by flat pullback in full generality, but the following special case is worth mentioning. See also Lemma 35.24.2.

0H8I **Lemma 35.23.9.** *Let E/k be a field extension. Then a morphism $X \to Y$ over k is dominant if and only if the pullback $X_E \to Y_E$ is dominant.*

Proof. By Morphisms, Lemma 29.23.4, the morphism $\mathrm{Spec}(E) \to \mathrm{Spec}(k)$ is universally open. So $Y_E \to Y$ is open. Therefore, if $X \to Y$ is dominant, Morphisms, Lemma 29.8.5 gives that $X_E \to Y_E$ is dominant. Conversely, suppose that $X_E \to Y_E$ is dominant. The morphism $Y_E \to Y$ is surjective by Morphisms, Lemma 29.9.4, and so the composition $X_E \to Y_E \to Y$ is dominant. This is also the composition $X_E \to X \to Y$, and so $X \to Y$ is dominant. □

02KW **Lemma 35.23.10.** *The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.*

Proof. A base change of a universally injective morphism is universally injective (this is formal). Being universally injective is Zariski local on the base; this is clear from the definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $S' \to S$ is surjective and by the discussion in Schemes, Section 26.13 there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to S'$ such that the following solid diagram commutes

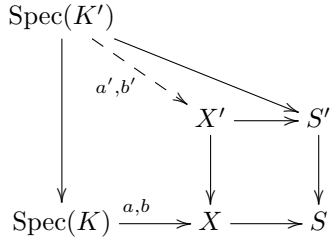

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to S'$ is universally injective we get $a' = b'$, by Morphisms, Lemma 29.10.2. Clearly this forces $a = b$ (by the discussion in Schemes, Section 26.13). Therefore Lemma 35.22.4 applies and we win.

An alternative proof would be to use the characterization of a universally injective morphism as one whose diagonal is surjective, see Morphisms, Lemma 29.10.2. The lemma then follows from the fact that the property of being surjective is fpqc local on the base, see Lemma 35.23.7. (Hint: use that the base change of the diagonal is the diagonal of the base change.) □

0CEX **Lemma 35.23.11.** *The property $\mathcal{P}(f) =$"f is a universal homeomorphism" is fpqc local on the base.*

Proof. This can be proved in exactly the same manner as Lemma 35.23.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. Thus the lemma follows from Lemmas 35.23.7, 35.23.10, and 35.23.4. □

02KX **Lemma 35.23.12.** *The property $\mathcal{P}(f) =$"f is locally of finite type" is fpqc local on the base.*

Proof. Being locally of finite type is preserved under base change, see Morphisms, Lemma 29.15.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 29.15.2. Finally, let $S' \to S$ be a flat surjective morphism of

affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite type. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite type. We have to show that $R \to A$ is of finite type. This is the result of Algebra, Lemma 10.126.1. It follows that f is locally of finite type. Therefore Lemma 35.22.4 applies and we win. □

02KY **Lemma 35.23.13.** *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fpqc local on the base.*

Proof. Being locally of finite presentation is preserved under base change, see Morphisms, Lemma 29.21.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 29.21.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite presentation. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite presentation. We have to show that $R \to A$ is of finite presentation. This is the result of Algebra, Lemma 10.126.2. It follows that f is locally of finite presentation. Therefore Lemma 35.22.4 applies and we win. □

02KZ **Lemma 35.23.14.** *The property $\mathcal{P}(f) =$"f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 35.23.1 and 35.23.12. □

02L0 **Lemma 35.23.15.** *The property $\mathcal{P}(f) =$"f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 35.23.1, 35.23.2 and 35.23.13. □

02L1 **Lemma 35.23.16.** *The property $\mathcal{P}(f) =$"f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 35.23.3, 35.23.6 and 35.23.14. □

02L2 **Lemma 35.23.17.** *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the base.*

Proof. Being flat is preserved under arbitrary base change, see Morphisms, Lemma 29.25.8. Being flat is Zariski local on the base by definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is flat. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is flat. Goal: Show that $R \to A$ is flat. This follows immediately from Algebra, Lemma 10.39.8. Hence f is flat. Therefore Lemma 35.22.4 applies and we win. □

02L3 **Lemma 35.23.18.** *The property $\mathcal{P}(f) =$"f is an open immersion" is fpqc local on the base.*

Proof. The property of being an open immersion is stable under base change, see Schemes, Lemma 26.18.2. The property of being an open immersion is Zariski local on the base (this is obvious).

Let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an open immersion. We claim that f is an open immersion. Then f' is universally open, and universally injective. Hence we conclude that f is universally open by Lemma 35.23.4, and universally injective by Lemma 35.23.10. In particular $f(X) \subset S$ is open. If for every affine open $U \subset f(X)$ we can prove that $f^{-1}(U) \to U$ is an isomorphism, then f is an open immersion and we're done. If $U' \subset S'$ denotes the inverse image of U, then $U' \to U$ is a faithfully flat morphism of affines and $(f')^{-1}(U') \to U'$ is an isomorphism (as $f'(X')$ contains U' by our choice of U). Thus we reduce to the case discussed in the next paragraph.

Let $S' \to S$ be a flat surjective morphism of affine schemes, let $f : X \to S$ be a morphism, and assume that the base change $f' : X' \to S'$ is an isomorphism. We have to show that f is an isomorphism also. It is clear that f is surjective, universally injective, and universally open (see arguments above for the last two). Hence f is bijective, i.e., f is a homeomorphism. Thus f is affine by Morphisms, Lemma 29.45.4. Since

$$\mathcal{O}(S') \to \mathcal{O}(X') = \mathcal{O}(S') \otimes_{\mathcal{O}(S)} \mathcal{O}(X)$$

is an isomorphism and since $\mathcal{O}(S) \to \mathcal{O}(S')$ is faithfully flat this implies that $\mathcal{O}(S) \to \mathcal{O}(X)$ is an isomorphism. Thus f is an isomorphism. This finishes the proof of the claim above. Therefore Lemma 35.22.4 applies and we win. □

02L4 **Lemma 35.23.19.** *The property $\mathcal{P}(f) =$"f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 35.23.7 and 35.23.18. □

02L5 **Lemma 35.23.20.** *The property $\mathcal{P}(f) =$"f is affine" is fpqc local on the base.*

Proof. A base change of an affine morphism is affine, see Morphisms, Lemma 29.11.9. Being affine is Zariski local on the base, see Morphisms, Lemma 29.11.3. Finally, let $g : S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is affine. In other words, X' is affine, say $X' = \mathrm{Spec}(A')$. Also write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. We have to show that X is affine.

By Lemmas 35.23.1 and 35.23.6 we see that $X \to S$ is separated and quasi-compact. Thus $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 26.24.1. Hence $f_*\mathcal{O}_X = \widetilde{A}$ for some R-algebra A. In fact $A = \Gamma(X, \mathcal{O}_X)$ of course. Also, by flat base change (see for example Cohomology of Schemes, Lemma 30.5.2) we have $g^*f_*\mathcal{O}_X = f'_*\mathcal{O}_{X'}$. In other words, we have $A' = R' \otimes_R A$. Consider the canonical morphism

$$X \longrightarrow \mathrm{Spec}(A)$$

over S from Schemes, Lemma 26.6.4. By the above the base change of this morphism to S' is an isomorphism. Hence it is an isomorphism by Lemma 35.23.19. Therefore Lemma 35.22.4 applies and we win. □

02L6 **Lemma 35.23.21.** *The property $\mathcal{P}(f) =$"f is a closed immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is a closed immersion. This implies that each f_i is affine, see Morphisms, Lemma 29.11.10. By Lemma 35.23.20 we conclude that f is affine. It remains to show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. For every $y \in Y$ there exists an i and a point $y_i \in Y_i$ mapping to y. By Cohomology of Schemes, Lemma 30.5.2 the sheaf $f_{i,*}(\mathcal{O}_{Y_i \times_Y X})$ is the pullback of $f_*\mathcal{O}_X$. By assumption it is a quotient of $\mathcal{O}_{Y_i}$. Hence we see that

$$\Big(\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y\Big) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_i,y_i}$$

is surjective. Since $\mathcal{O}_{Y_i,y_i}$ is faithfully flat over $\mathcal{O}_{Y,y}$ this implies the surjectivity of $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ as desired. □

02L7 **Lemma 35.23.22.** *The property $\mathcal{P}(f) =$"f is quasi-affine" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is quasi-affine. This implies that each f_i is quasi-compact and separated. By Lemmas 35.23.1 and 35.23.6 this implies that f is quasi-compact and separated. Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = f_*\mathcal{O}_X$. By Schemes, Lemma 26.24.1 it is a quasi-coherent $\mathcal{O}_Y$-algebra. Consider the canonical morphism

$$j : X \longrightarrow \underline{\mathrm{Spec}}_Y(\mathcal{A})$$

see Constructions, Lemma 27.4.7. By flat base change (see for example Cohomology of Schemes, Lemma 30.5.2) we have $g_i^* f_*\mathcal{O}_X = f_{i,*}\mathcal{O}_{X'}$ where $g_i : Y_i \to Y$ are the given flat maps. Hence the base change j_i of j by g_i is the canonical morphism of Constructions, Lemma 27.4.7 for the morphism f_i. By assumption and Morphisms, Lemma 29.13.3 all of these morphisms j_i are quasi-compact open immersions. Hence, by Lemmas 35.23.1 and 35.23.18 we see that j is a quasi-compact open immersion. Hence by Morphisms, Lemma 29.13.3 again we conclude that f is quasi-affine. □

02L8 **Lemma 35.23.23.** *The property $\mathcal{P}(f) =$"f is a quasi-compact immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Write $X_i = Y_i \times_Y X$ and $f_i : X_i \to Y_i$ the base change of f. Also denote $q_i : Y_i \to Y$ the given flat morphisms. Assume each f_i is a quasi-compact immersion. By Schemes, Lemma 26.23.8 each f_i is separated. By Lemmas 35.23.1 and 35.23.6 this implies that f is quasi-compact and separated. Let $X \to Z \to Y$ be the factorization of f through its scheme theoretic image. By Morphisms, Lemma 29.6.3 the closed subscheme $Z \subset Y$ is cut out by the quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ as f is quasi-compact. By flat base change (see for example Cohomology of Schemes, Lemma 30.5.2; here we use f is separated) we see $f_{i,*}\mathcal{O}_{X_i}$ is the pullback $q_i^* f_*\mathcal{O}_X$. Hence $Y_i \times_Y Z$ is cut out by the quasi-coherent sheaf of ideals $q_i^*\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Y_i} \to f_{i,*}\mathcal{O}_{X_i})$. By Morphisms, Lemma 29.7.7 the morphisms $X_i \to Y_i \times_Y Z$ are open immersions. Hence by Lemma 35.23.18 we see that $X \to Z$ is an open immersion and hence f is an immersion as desired (we already saw it was quasi-compact). □

02L9 **Lemma 35.23.24.** *The property $\mathcal{P}(f) =$"f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms, Lemma 29.44.7. Hence the lemma follows on combining Lemmas 35.23.3 and 35.23.20. □

02LA **Lemma 35.23.25.** *The property $\mathcal{P}(f) =$"f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral morphism which is locally of finite type. See Morphisms, Lemma 29.44.4. Hence the lemma follows on combining Lemmas 35.23.12 and 35.23.24. □

02VI **Lemma 35.23.26.** *The properties $\mathcal{P}(f) =$"f is locally quasi-finite" and $\mathcal{P}(f) =$"f is quasi-finite" are fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes, and let $\{S_i \to S\}$ be an fpqc covering such that each base change $f_i : X_i \to S_i$ is locally quasi-finite. We have already seen (Lemma 35.23.12) that "locally of finite type" is fpqc local on the base, and hence we see that f is locally of finite type. Then it follows from Morphisms, Lemma 29.20.13 that f is locally quasi-finite. The quasi-finite case follows as we have already seen that "quasi-compact" is fpqc local on the base (Lemma 35.23.1). □

02VJ **Lemma 35.23.27.** *The property $\mathcal{P}(f) =$"f is locally of finite type of relative dimension d" is fpqc local on the base.*

Proof. This follows immediately from the fact that being locally of finite type is fpqc local on the base and Morphisms, Lemma 29.28.3. □

02VK **Lemma 35.23.28.** *The property $\mathcal{P}(f) =$"f is syntomic" is fpqc local on the base.*

Proof. A morphism is syntomic if and only if it is locally of finite presentation, flat, and has locally complete intersections as fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 35.23.17, and 35.23.13). Hence the result follows for syntomic from Morphisms, Lemma 29.30.12. □

02VL **Lemma 35.23.29.** *The property $\mathcal{P}(f) =$"f is smooth" is fpqc local on the base.*

Proof. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 35.23.17, and 35.23.13). Hence the result follows for smooth from Morphisms, Lemma 29.34.15. □

02VM **Lemma 35.23.30.** *The property $\mathcal{P}(f) =$"f is unramified" is fpqc local on the base. The property $\mathcal{P}(f) =$"f is G-unramified" is fpqc local on the base.*

Proof. A morphism is unramified (resp. G-unramified) if and only if it is locally of finite type (resp. finite presentation) and its diagonal morphism is an open immersion (see Morphisms, Lemma 29.35.13). We have seen already that being locally of finite type (resp. locally of finite presentation) and an open immersion is fpqc local on the base (Lemmas 35.23.13, 35.23.12, and 35.23.18). Hence the result follows formally. □

02VN **Lemma 35.23.31.** *The property $\mathcal{P}(f) =$"f is étale" is fpqc local on the base.*

Proof. A morphism is étale if and only if it flat and G-unramified. See Morphisms, Lemma 29.36.16. We have seen already that being flat and G-unramified are fpqc local on the base (Lemmas 35.23.17, and 35.23.30). Hence the result follows. □

02VO **Lemma 35.23.32.** *The property $\mathcal{P}(f) =$"f is finite locally free" is fpqc local on the base. Let $d \geq 0$. The property $\mathcal{P}(f) =$"f is finite locally free of degree d" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms, Lemma 29.48.2). Hence this follows from Lemmas 35.23.25, 35.23.17, and 35.23.13. If $f : Z \to U$ is finite locally free, and $\{U_i \to U\}$ is a surjective family of morphisms such that each pullback $Z \times_U U_i \to U_i$ has degree d, then $Z \to U$ has degree d, for example because we can read off the degree in a point $u \in U$ from the fibre $(f_*\mathcal{O}_Z)_u \otimes_{\mathcal{O}_{U,u}} \kappa(u)$. □

02YK **Lemma 35.23.33.** *The property $\mathcal{P}(f) =$"f is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to S_i$ of f is a monomorphism. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : S_i \times_S T \to X_i$ are the base changes. In particular the compositions $S_i \times_S T \to T \to X$ are equal. Since $\coprod S_i \times_S T \to T$ is an epimorphism (see e.g. Lemma 35.13.7) we conclude $a = b$. □

0694 **Lemma 35.23.34.** *The properties*

$$\mathcal{P}(f) =\text{"}f \text{ is a Koszul-regular immersion"},$$
$$\mathcal{P}(f) =\text{"}f \text{ is an } H_1\text{-regular immersion", and}$$
$$\mathcal{P}(f) =\text{"}f \text{ is a quasi-regular immersion"}$$

are fpqc local on the base.

Proof. We will use the criterion of Lemma 35.22.4 to prove this. By Divisors, Definition 31.21.1 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is Zariski local on the base. By Divisors, Lemma 31.21.4 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is preserved under flat base change. The final hypothesis (3) of Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is locally on $\mathrm{Spec}(B)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in B, then $I \subset A$ is locally on $\mathrm{Spec}(A)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in A. This is More on Algebra, Lemma 15.32.4. □

35.24. Properties of morphisms local in the fppf topology on the target

02YL In this section we find some properties of morphisms of schemes for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

02YM **Lemma 35.24.1.** *The property $\mathcal{P}(f) =$"f is an immersion" is fppf local on the base.*

Proof. The property of being an immersion is stable under base change, see Schemes, Lemma 26.18.2. The property of being an immersion is Zariski local on the base. Finally, let $\pi : S' \to S$ be a surjective morphism of affine schemes, which is flat and locally of finite presentation. Note that $\pi : S' \to S$ is open by Morphisms, Lemma 29.25.10. Let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an immersion. In particular we see that $f'(X') = \pi^{-1}(f(X))$ is locally closed. Hence by Topology, Lemma 5.6.4 we see that $f(X) \subset S$ is locally closed. Let $Z \subset S$ be the closed subset $Z = \overline{f(X)} \setminus f(X)$. By Topology, Lemma 5.6.4 again we see that $f'(X')$ is closed in $S' \setminus Z'$. Hence we may apply Lemma 35.23.21 to the fpqc covering $\{S' \setminus Z' \to S \setminus Z\}$ and conclude that $f : X \to S \setminus Z$ is a closed immersion. In other words, f is an immersion. Therefore Lemma 35.22.4 applies and we win. □

0H8J **Lemma 35.24.2.** *The property $\mathcal{P}(f) =$"f is dominant" is fppf local on the base.*

Proof. By Morphisms, Lemma 29.8.5, dominant morphisms are preserved under pullback by open morphisms, hence by flat morphisms locally of finite presentation (Morphisms, Lemma 29.25.10). The other direction is easier. Indeed, being dominant is clearly Zariski local on the base. Next, let $S' \to S$ be a surjective morphism of schemes (not necessarily flat or locally of finite presentation), and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is dominant. Then $X' \to S' \to S$ is a composition of two dominant morphisms, hence dominant. Since this is also the composition $X' \to X \to S$, it follows that $X \to S$ is dominant. □

35.25. Application of fpqc descent of properties of morphisms

02LB The following lemma may seem a bit frivolous but turns out is a useful tool in studying étale and unramified morphisms.

06NC **Lemma 35.25.1.** *Let $f : X \to Y$ be a flat, quasi-compact, surjective monomorphism. Then f is an isomorphism.*

Proof. As f is a flat, quasi-compact, surjective morphism we see $\{X \to Y\}$ is an fpqc covering of Y. The diagonal $\Delta : X \to X \times_Y X$ is an isomorphism (Schemes, Lemma 26.23.2). This implies that the base change of f by f is an isomorphism. Hence we see f is an isomorphism by Lemma 35.23.19. □

We can use this lemma to show the following important result; we also give a proof avoiding fpqc descent. We will discuss this and related results in more detail in Étale Morphisms, Section 41.14.

02LC **Lemma 35.25.2.** *A universally injective étale morphism is an open immersion.*

First proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 29.36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Then f is bijective and open hence a homeomorphism. Hence f is quasi-compact. Thus by Lemma 35.25.1 it suffices to show that f is a monomorphism. As $X \to Y$ is étale the morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion by Morphisms, Lemma 29.35.13 (and Morphisms, Lemma 29.36.16). As f is universally injective $\Delta_{X/Y}$ is also surjective, see Morphisms, Lemma 29.10.2. Hence $\Delta_{X/Y}$ is an isomorphism, i.e., $X \to Y$ is a monomorphism. □

Second proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 29.36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Since the hypotheses remain satisfied after any base change, we conclude that f is a universal homeomorphism. Therefore f is integral, see Morphisms, Lemma 29.45.5. It follows that f is finite by Morphisms, Lemma 29.44.4. It follows that f is finite locally free by Morphisms, Lemma 29.48.2. To finish the proof, it suffices that f is finite locally free of degree 1 (a finite locally free morphism of degree 1 is an isomorphism). There is decomposition of Y into open and closed subschemes V_d such that $f^{-1}(V_d) \to V_d$ is finite locally free of degree d, see Morphisms, Lemma 29.48.5. If V_d is not empty, we can pick a morphism $\mathrm{Spec}(k) \to V_d \subset Y$ where k is an algebraically closed field (just take the algebraic closure of the residue field of some point of V_d). Then $\mathrm{Spec}(k) \times_Y X \to \mathrm{Spec}(k)$ is a disjoint union of copies of $\mathrm{Spec}(k)$, by Morphisms, Lemma 29.36.7 and the fact that k is algebraically closed. However, since f is universally injective, there can only be one copy and hence $d = 1$ as desired. □

We can reformulate the hypotheses in the lemma above a bit by using the following characterization of flat universally injective morphisms.

09NP **Lemma 35.25.3.** *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is flat and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is universally injective.

Proof. We have to show that $\Delta : X \to X \times_Y X$ is surjective, see Morphisms, Lemma 29.10.2. As $X \to Y$ is separated, the image of Δ is closed. Thus if Δ is not surjective, we can find a generic point $\eta \in X \times_S X$ of an irreducible component of $X \times_S X$ which is not in the image of Δ. The projection $\mathrm{pr}_1 : X \times_Y X \to X$ is flat as a base change of the flat morphism $X \to Y$, see Morphisms, Lemma 29.25.8. Hence generalizations lift along pr_1, see Morphisms, Lemma 29.25.9. We conclude that $\xi = \mathrm{pr}_1(\eta) \in X^0$. However, assumptions (2) and (3) guarantee that the scheme $(X \times_Y X)_{f(\xi)}$ has at most one point for every $\xi \in X^0$. In other words, we have $\Delta(\xi) = \eta$ a contradiction. □

Thus we can reformulate Lemma 35.25.2 as follows.

09NQ **Lemma 35.25.4.** *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is étale and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Immediate from Lemmas 35.25.3 and 35.25.2. □

0F4J **Lemma 35.25.5.** *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let Z be a closed subset of X. If there exists an fpqc covering $\{Y_i \to Y\}$ such that the inverse image $Z_i \subset Y_i \times_Y X$ is proper over Y_i (Cohomology of Schemes, Definition 30.26.2) then Z is proper over Y.*

Proof. Endow Z with the reduced induced closed subscheme structure, see Schemes, Definition 26.12.5. For every i the base change $Y_i \times_Y Z$ is a closed subscheme of $Y_i \times_Y X$ whose underlying closed subset is Z_i. By definition (via Cohomology of Schemes, Lemma 30.26.1) we conclude that the projections $Y_i \times_Y Z \to Y_i$ are proper morphisms. Hence $Z \to Y$ is a proper morphism by Lemma 35.23.16. Thus Z is proper over Y by definition. □

0D2P **Lemma 35.25.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : S_i \to S\}_{i\in I}$ be an fpqc covering. Let $f_i : X_i \to S_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/S, and*
(2) *$\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Morphisms, Lemma 29.37.9. Assume $\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$. By Morphisms, Definition 29.37.1 this implies that $X_i \to S_i$ is quasi-compact and by Morphisms, Lemma 29.37.3 this implies $X_i \to S$ is separated. Hence f is quasi-compact and separated by Lemmas 35.23.1 and 35.23.6.

This means that $\mathcal{A} = \bigoplus_{d\geq 0} f_*\mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra (Schemes, Lemma 26.24.1). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Schemes, Lemma 30.5.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d\geq 0} f_{i,*}\mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^*\mathcal{A}$. It follows that the natural maps $\psi_d : f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give the natural maps $\psi_{i,d} : f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/S_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 29.37.4. If $x \in X$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is flat hence faithfully flat, we conclude that for every $x \in X$ there exists a $d \geq 1$ such that $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Constructions, Lemma 27.19.1 corresponding to the map $\psi : f^*\mathcal{A} \to \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\mathrm{Proj}}_S(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to S_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 29.37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 35.23.18 and we conclude $\mathcal{L}$ is ample on X/S by Morphisms, Lemma 29.37.4. □

35.26. Properties of morphisms local on the source

036F It often happens one can prove a morphism has a certain property after precomposing with some other morphism. In many cases this implies the morphism has the property too. We formalize this in the following definition.

036G **Definition 35.26.1.** Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism of schemes $f : X \to Y$ over S, and any τ-covering $\{X_i \to X\}_{i\in I}$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

04QV **Lemma 35.26.2.** *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

0CEY **Lemma 35.26.3.** *Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset X$ such that the restriction $f|_{W(f)} : W(f) \to Y$ has $\mathcal{P}$. Moreover, if $g : X' \to X$ is flat and locally of finite presentation, syntomic, smooth, or étale and $f' = f \circ g : X' \to Y$, then $g^{-1}(W(f)) = W(f')$.*

Proof. Consider the union W of the images $g(X') \subset X$ of morphisms $g : X' \to X$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the composition $X' \to X \to Y$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 29.25.10) we see that $W \subset X$ is an open subset of X. Since $\mathcal{P}$ is local in the τ topology the restriction $f|_W : W \to Y$ has property $\mathcal{P}$ because we are given a τ covering $\{X' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence of $W(f)$. The compatibility stated in the last sentence follows immediately from the construction of $W(f)$. □

036H **Lemma 35.26.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is preserved under precomposing with flat, flat locally of finite presentation, étale, smooth or syntomic morphisms depending on whether τ is fpqc, fppf, étale, smooth, or syntomic,*
(2) *the property is Zariski local on the source,*
(3) *the property is Zariski local on the target,*
(4) *for any morphism of affine schemes $f : X \to Y$, and any surjective morphism of affine schemes $X' \to X$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the composition $f' : X' \to Y$.*

Then $\mathcal{P}$ is τ local on the source.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.9, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. (Hint: Use locality on the source and target to reduce the verification of property $\mathcal{P}$ to the case of a morphism between affines. Then apply (1) and (4).) □

036I **Remark 35.26.5.** (This is a repeat of Remarks 35.15.3 and 35.22.5 above.) In Lemma 35.26.4 above if $\tau = smooth$ then in condition (4) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

35.27. Properties of morphisms local in the fpqc topology on the source

036J Here are some properties of morphisms that are fpqc local on the source.

036K **Lemma 35.27.1.** *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the source.*

Proof. Since flatness is defined in terms of the maps of local rings (Morphisms, Definition 29.25.1) what has to be shown is the following algebraic fact: Suppose $A \to B \to C$ are local homomorphisms of local rings, and assume $B \to C$ is flat. Then $A \to B$ is flat if and only if $A \to C$ is flat. If $A \to B$ is flat, then $A \to C$ is flat by Algebra, Lemma 10.39.4. Conversely, assume $A \to C$ is flat. Note that $B \to C$ is faithfully flat, see Algebra, Lemma 10.39.17. Hence $A \to B$ is flat by Algebra, Lemma 10.39.10. (Also see Morphisms, Lemma 29.25.13 for a direct proof.) □

036L **Lemma 35.27.2.** *Then property $\mathcal{P}(f : X \to Y) =$"for every $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective" is fpqc local on the source.*

Proof. Omitted. This is just a (probably misguided) attempt to be playful. □

35.28. Properties of morphisms local in the fppf topology on the source

036M Here are some properties of morphisms that are fppf local on the source.

036N **Lemma 35.28.1.** *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fppf local on the source.*

Proof. Being locally of finite presentation is Zariski local on the source and the target, see Morphisms, Lemma 29.21.2. It is a property which is preserved under composition, see Morphisms, Lemma 29.21.3. This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) is Lemma 35.14.1. Hence we win. □

036O **Lemma 35.28.2.** *The property $\mathcal{P}(f) =$"f is locally of finite type" is fppf local on the source.*

Proof. Being locally of finite type is Zariski local on the source and the target, see Morphisms, Lemma 29.15.2. It is a property which is preserved under composition, see Morphisms, Lemma 29.15.3, and a flat morphism locally of finite presentation is locally of finite type, see Morphisms, Lemma 29.21.8. This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) is Lemma 35.14.2. Hence we win. □

036P **Lemma 35.28.3.** *The property $\mathcal{P}(f) =$"f is open" is fppf local on the source.*

Proof. Being an open morphism is clearly Zariski local on the source and the target. It is a property which is preserved under composition, see Morphisms, Lemma 29.23.3, and a flat morphism of finite presentation is open, see Morphisms, Lemma 29.25.10 This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) follows from Morphisms, Lemma 29.25.12. Hence we win. □

036Q **Lemma 35.28.4.** *The property $\mathcal{P}(f) =$"f is universally open" is fppf local on the source.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i\in I}$ be an fppf covering. Denote $f_i : X_i \to X$ the compositions. We have to show that f is universally open if and only if each f_i is universally open. If f is universally open, then also each f_i is universally open since the maps $X_i \to X$ are universally open and compositions of universally open morphisms are universally open (Morphisms, Lemmas 29.25.10 and 29.23.3). Conversely, assume each f_i is universally open. Let $Y' \to Y$ be a morphism of schemes. Denote $X' = Y' \times_Y X$ and $X'_i = Y' \times_Y X_i$. Note that $\{X'_i \to X'\}_{i\in I}$ is an fppf covering also. The morphisms $f'_i : X'_i \to Y'$ are open by assumption. Hence by the Lemma 35.28.3 above we conclude that $f' : X' \to Y'$ is open as desired. □

35.29. Properties of morphisms local in the syntomic topology on the source

036R Here are some properties of morphisms that are syntomic local on the source.

036S **Lemma 35.29.1.** *The property $\mathcal{P}(f)$ = "f is syntomic" is syntomic local on the source.*

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.30.2 (local for Zariski on source and target), Morphisms, Lemma 29.30.3 (pre-composing), and Lemma 35.14.4 (part (4)). □

35.30. Properties of morphisms local in the smooth topology on the source

036T Here are some properties of morphisms that are smooth local on the source. Note also the (in some respects stronger) result on descending smoothness via flat morphisms, Lemma 35.14.5.

036U **Lemma 35.30.1.** *The property $\mathcal{P}(f)$ = "f is smooth" is smooth local on the source.*

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.34.2 (local for Zariski on source and target), Morphisms, Lemma 29.34.4 (pre-composing), and Lemma 35.14.4 (part (4)). □

35.31. Properties of morphisms local in the étale topology on the source

036V Here are some properties of morphisms that are étale local on the source.

036W **Lemma 35.31.1.** *The property $\mathcal{P}(f)$ = "f is étale" is étale local on the source.*

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.36.2 (local for Zariski on source and target), Morphisms, Lemma 29.36.3 (pre-composing), and Lemma 35.14.4 (part (4)). □

03X4 **Lemma 35.31.2.** *The property $\mathcal{P}(f)$ = "f is locally quasi-finite" is étale local on the source.*

Proof. We are going to use Lemma 35.26.4. By Morphisms, Lemma 29.20.11 the property of being locally quasi-finite is local for Zariski on source and target. By Morphisms, Lemmas 29.20.12 and 29.36.6 we see the precomposition of a locally quasi-finite morphism by an étale morphism is locally quasi-finite. Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $X' \to X$ is a surjective

étale morphism of affine schemes such that $X' \to Y$ is locally quasi-finite. Then $X' \to Y$ is of finite type, and by Lemma 35.14.2 we see that $X \to Y$ is of finite type also. Moreover, by assumption $X' \to Y$ has finite fibres, and hence $X \to Y$ has finite fibres also. We conclude that $X \to Y$ is quasi-finite by Morphisms, Lemma 29.20.10. This proves the last assumption of Lemma 35.26.4 and finishes the proof. □

03YV **Lemma 35.31.3.** *The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source. The property $\mathcal{P}(f) =$"f is G-unramified" is étale local on the source.*

Proof. We are going to use Lemma 35.26.4. By Morphisms, Lemma 29.35.3 the property of being unramified (resp. G-unramified) is local for Zariski on source and target. By Morphisms, Lemmas 29.35.4 and 29.36.5 we see the precomposition of an unramified (resp. G-unramified) morphism by an étale morphism is unramified (resp. G-unramified). Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $f : X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is unramified (resp. G-unramified). Then $X' \to Y$ is of finite type (resp. finite presentation), and by Lemma 35.14.2 (resp. Lemma 35.14.1) we see that $X \to Y$ is of finite type (resp. finite presentation) also. By Morphisms, Lemma 29.34.16 we have a short exact sequence

$$0 \to f^*\Omega_{X/Y} \to \Omega_{X'/Y} \to \Omega_{X'/X} \to 0.$$

As $X' \to Y$ is unramified we see that the middle term is zero. Hence, as f is faithfully flat we see that $\Omega_{X/Y} = 0$. Hence $X \to Y$ is unramified (resp. G-unramified), see Morphisms, Lemma 29.35.2. This proves the last assumption of Lemma 35.26.4 and finishes the proof. □

35.32. Properties of morphisms étale local on source-and-target

04QW Let $\mathcal{P}$ be a property of morphisms of schemes. There is an intuitive meaning to the phrase "$\mathcal{P}$ is étale local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both étale local on the source and étale local on the target. Before we discuss this further we give two silly examples.

04QX **Example 35.32.1.** Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(X \to Y) =$"Y is locally Noetherian". The reader can verify that this is étale local on the source and étale local on the target (omitted, see Lemma 35.16.1). But it is **not** true that if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$. Namely, f could be the identity on Y and g could be an open immersion of a locally Noetherian scheme Y into a non locally Noetherian scheme Z.

The following example is in some sense worse.

04QY **Example 35.32.2.** Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(f : X \to Y) =$"for every $y \in Y$ which is a specialization of some $f(x)$, $x \in X$ the local ring $\mathcal{O}_{Y,y}$ is Noetherian". Let us verify that this is étale local on the source and étale local on the target. We will freely use Schemes, Lemma 26.13.2.

Local on the target: Let $\{g_i : Y_i \to Y\}$ be an étale covering. Let $f_i : X_i \to Y_i$ be the base change of f, and denote $h_i : X_i \to X$ the projection. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y_i$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow g_i(y_i)$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,g_i(y_i)}$ is Noetherian. Also $\mathcal{O}_{Y,g_i(y_i)} \to \mathcal{O}_{Y_i,y_i}$ is a localization of an étale ring map. Hence $\mathcal{O}_{Y_i,y_i}$ is Noetherian by Algebra, Lemma 10.31.1. Conversely, assume $\mathcal{P}(f_i)$ for all

i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $y_i \in Y_i$ mapping to y. Since x can be viewed as a point of $\mathrm{Spec}(\mathcal{O}_{Y,y}) \times_Y X$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_i,y_i}$ is faithfully flat, there exists a point $x_i \in \mathrm{Spec}(\mathcal{O}_{Y_i,y_i}) \times_Y X$ mapping to x. Then $x_i \in X_i$, and $f_i(x_i)$ specializes to y_i. Thus we see that $\mathcal{O}_{Y_i,y_i}$ is Noetherian by $\mathcal{P}(f_i)$ which implies that $\mathcal{O}_{Y,y}$ is Noetherian by Algebra, Lemma 10.164.1.

Local on the source: Let $\{h_i : X_i \to X\}$ be an étale covering. Let $f_i : X_i \to Y$ be the composition $f \circ h_i$. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow y$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,y}$ is Noetherian. Thus $\mathcal{P}(f_i)$ holds. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $x_i \in X_i$ mapping to x. Then y is a specialization of $f_i(x_i) = f(x)$. Hence $\mathcal{P}(f_i)$ implies $\mathcal{O}_{Y,y}$ is Noetherian as desired.

We claim that there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with surjective étale vertical arrows, such that h has $\mathcal{P}$ and f does not have $\mathcal{P}$. Namely, let

$$Y = \mathrm{Spec}\left(\mathbf{C}[x_n; n \in \mathbf{Z}]/(x_n x_m; n \neq m)\right)$$

and let $X \subset Y$ be the open subscheme which is the complement of the point all of whose coordinates $x_n = 0$. Let $U = X$, let $V = X \amalg Y$, let a, b the obvious map, and let $h : U \to V$ be the inclusion of $U = X$ into the first summand of V. The claim above holds because U is locally Noetherian, but Y is not.

What should be the correct notion of a property which is étale local on the source-and-target? We think that, by analogy with Morphisms, Definition 29.14.1 it should be the following.

04QZ **Definition 35.32.3.** Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is *étale local on source-and-target* if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under étale base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is étale, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,
   (b) for every $x \in X$ there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

   with étale vertical arrows and $u \in U$ with $a(u) = x$ such that h has $\mathcal{P}$.

It turns out this definition excludes the behavior seen in Examples 35.32.1 and 35.32.2. We will compare this to the definition in the paper [**DM69**] by Deligne and Mumford in Remark 35.32.8. Moreover, a property which is étale local on the

source-and-target is étale local on the source and étale local on the target. Finally, the converse is almost true as we will see in Lemma 35.32.6.

04R0 **Lemma 35.32.4.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Then*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) *$\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $x \in X$ we can find an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, and $\mathrm{id}_Y : Y \to Y$ is an étale morphism, and h_i is as in part (3) of Definition 35.32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 35.32.3 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in X$ we pick an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, $Y_i \to Y$ is étale, and h_i is as in part (3) of Definition 35.32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 35.32.3 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. For every $x \in X$ we can think of $(X, x) \to (X, x)$ as an étale morphism of germs, $Y \to Z$ is an étale morphism, and $h = f$ is as in part (3) of Definition 35.32.3. Thus we see that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 35.32.3 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms, Lemma 29.36.18. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 35.32.3 part (1). □

The following lemma is the analogue of Morphisms, Lemma 29.14.4.

04R1 **Lemma 35.32.5.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in X$ there exists an étale morphism of germs $a : (U, u) \to (X, x)$, an étale morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*

(c) *for any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b étale the morphism h has $\mathcal{P}$,
(d) *for some diagram as in (c) with $a : U \to X$ surjective h has $\mathcal{P}$,*
(e) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 35.32.3. The equivalence of (a) and (e) is Lemma 35.32.4 part (2). The equivalence of (a) and (f) is Lemma 35.32.4 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (a). If (a) holds, then for any diagram as in (c) the morphism $f \circ a$ has $\mathcal{P}$ by Definition 35.32.3 part (1), whereupon h has $\mathcal{P}$ by Lemma 35.32.4 part (4). Thus (a) and (c) are equivalent. It is clear that (c) implies (d). To see that (d) implies (a) assume we have a diagram as in (c) with $a : U \to X$ surjective and h having $\mathcal{P}$. Then $b \circ h$ has $\mathcal{P}$ by Lemma 35.32.4 part (3). Since $\{a : U \to X\}$ is an étale covering we conclude that f has $\mathcal{P}$ by Lemma 35.32.4 part (1). □

It seems that the result of the following lemma is not a formality, i.e., it actually uses something about the geometry of étale morphisms.

04R2 **Lemma 35.32.6.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of schemes which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 35.26.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 35.22.2 we see that $\mathcal{P}$ is stable under étale base change. Hence it suffices to prove part (3) of Definition 35.32.3 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of schemes which satisfies Definition 35.32.3 part (3)(b). In other words, for every $x \in X$ there exists an étale morphism $a_x : U_x \to X$, a point $u_x \in U_x$ mapping to x, an étale morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$,

$a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is étale local on the target. Because a is surjective and $\mathcal{P}$ is étale local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This reduces the lemma to proving that $\mathcal{P}$ is stable under postcomposing with an étale morphism.

During the rest of the proof we let $f : X \to Y$ be a morphism with property $\mathcal{P}$ and $g : Y \to Z$ is an étale morphism. Consider the following statements:

(-) With no additional assumptions $g \circ f$ has property $\mathcal{P}$.
(A) Whenever Z is affine $g \circ f$ has property $\mathcal{P}$.
(AA) Whenever X and Z are affine $g \circ f$ has property $\mathcal{P}$.
(AAA) Whenever X, Y, and Z are affine $g \circ f$ has property $\mathcal{P}$.

Once we have proved (-) the proof of the lemma will be complete.

Claim 1: (AAA) $\Rightarrow$ (AA). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with X, Z affine. As X is affine hence quasi-compact we can find finitely many affine open $Y_i \subset Y$, $i = 1, \ldots, n$ such that $X = \bigcup_{i=1,\ldots,n} f^{-1}(Y_i)$. Set $X_i = f^{-1}(Y_i)$. By Lemma 35.22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. Hence $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ has $\mathcal{P}$ as $\mathcal{P}$ is étale local on the target. By (AAA) applied to $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ and the étale morphism $\coprod_{i=1,\ldots,n} Y_i \to Z$ we see that $\coprod_{i=1,\ldots,n} X_i \to Z$ has $\mathcal{P}$. Now $\{\coprod_{i=1,\ldots,n} X_i \to X\}$ is an étale covering, hence as $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$ as desired.

Claim 2: (AAA) $\Rightarrow$ (A). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with Z affine. Choose an affine open covering $X = \bigcup X_i$. As $\mathcal{P}$ is étale local on the source we see that each $f|_{X_i} : X_i \to Y$ has $\mathcal{P}$. By (AA), which follows from (AAA) according to Claim 1, we see that $X_i \to Z$ has $\mathcal{P}$ for each i. Since $\{X_i \to X\}$ is an étale covering and $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$.

Claim 3: (AAA) $\Rightarrow$ (-). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above. Choose an affine open covering $Z = \bigcup Z_i$. Set $Y_i = g^{-1}(Z_i)$ and $X_i = f^{-1}(Y_i)$. By Lemma 35.22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. By (A), which follows from (AAA) according to Claim 2, we see that $X_i \to Z_i$ has $\mathcal{P}$ for each i. Since $\mathcal{P}$ is local on the target and $X_i = (g \circ f)^{-1}(Z_i)$ we conclude that $X \to Z$ has $\mathcal{P}$.

Thus to prove the lemma it suffices to prove (AAA). Let $f : X \to Y$ and $g : Y \to Z$ be as above X, Y, Z affine. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 29.36.6 and Lemma 29.57.9. Hence we can do induction on the integer n bounding the degree of the fibres of $Y \to Z$. See Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $Y \to Z$ is an open immersion, see Lemma 35.25.2, and the result follows from assumption (3) of the lemma. Assume $n > 1$.

Consider the following commutative diagram

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow[f_Y]{} & Y \times_Z Y & \xrightarrow[\text{pr}]{} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

Note that we have a decomposition into open and closed subschemes $Y \times_Z Y = \Delta_{Y/Z}(Y) \amalg Y'$, see Morphisms, Lemma 29.35.13. As a base change the degrees of the fibres of the second projection $\text{pr} : Y \times_Z Y \to Y$ are bounded by n, see Morphisms, Lemma 29.57.5. On the other hand, $\text{pr}|_{\Delta(Y)} : \Delta(Y) \to Y$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 29.57.8 we conclude the degrees of the fibres of the restriction $\text{pr}|_{Y'} : Y' \to Y$ are bounded by $n - 1$. Set $X' = f_Y^{-1}(Y')$. Picture

$$\begin{array}{ccccc} X \amalg X' & \xrightarrow[f \amalg f']{} & \Delta(Y) \amalg Y' & \longrightarrow & Y \\ \| & & \| & & \| \\ X \times_Z Y & \xrightarrow{f_Y} & Y \times_Z Y & \xrightarrow{\text{pr}} & Y \end{array}$$

As $\mathcal{P}$ is étale local on the target and hence stable under étale base change (see Lemma 35.22.2) we see that f_Y has $\mathcal{P}$. Hence, as $\mathcal{P}$ is étale local on the source, $f' = f_Y|_{X'}$ has $\mathcal{P}$. By induction hypothesis we see that $X' \to Y$ has $\mathcal{P}$. As $\mathcal{P}$ is local on the source, and $\{X \to X \times_Z Y, X' \to X \times_Y Z\}$ is an étale covering, we conclude that $\text{pr} \circ f_Y$ has $\mathcal{P}$. Note that $g \circ f$ can be viewed as a morphism $g \circ f : X \to g(Y)$. As $\text{pr} \circ f_Y$ is the pullback of $g \circ f : X \to g(Y)$ via the étale covering $\{Y \to g(Y)\}$, and as $\mathcal{P}$ is étale local on the target, we conclude that $g \circ f : X \to g(Y)$ has property $\mathcal{P}$. Finally, applying assumption (3) of the lemma once more we conclude that $g \circ f : X \to Z$ has property $\mathcal{P}$. □

04R3 **Remark 35.32.7.** Using Lemma 35.32.6 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are étale local on the source-and-target. In each case we list the lemma which implies the property is étale local on the source and the lemma which implies the property is étale local on the target. In each case the third assumption of Lemma 35.32.6 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 35.27.1 and 35.23.17,
(2) locally of finite presentation, see Lemmas 35.28.1 and 35.23.13,
(3) locally finite type, see Lemmas 35.28.2 and 35.23.12,
(4) universally open, see Lemmas 35.28.4 and 35.23.4,
(5) syntomic, see Lemmas 35.29.1 and 35.23.28,
(6) smooth, see Lemmas 35.30.1 and 35.23.29,
(7) étale, see Lemmas 35.31.1 and 35.23.31,
(8) locally quasi-finite, see Lemmas 35.31.2 and 35.23.26,
(9) unramified, see Lemmas 35.31.3 and 35.23.30,
(10) G-unramified, see Lemmas 35.31.3 and 35.23.30, and
(11) add more here as needed.

04R4 **Remark 35.32.8.** At this point we have three possible definitions of what it means for a property $\mathcal{P}$ of morphisms to be "étale local on the source and target":

(ST) $\mathcal{P}$ is étale local on the source and $\mathcal{P}$ is étale local on the target,

(DM) (the definition in the paper [**DM69**, Page 100] by Deligne and Mumford) for every diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with surjective étale vertical arrows we have $\mathcal{P}(h) \Leftrightarrow \mathcal{P}(f)$, and

(SP) $\mathcal{P}$ is étale local on the source-and-target.

In this section we have seen that (SP) $\Rightarrow$ (DM) $\Rightarrow$ (ST). The Examples 35.32.1 and 35.32.2 show that neither implication can be reversed. Finally, Lemma 35.32.6 shows that the difference disappears when looking at properties of morphisms which are stable under postcomposing with open immersions, which in practice will always be the case.

0CEZ **Lemma 35.32.9.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Given a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \quad \text{with points} \quad \begin{array}{ccc} x' & \longrightarrow & y' \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

such that g' is étale at x' and g is étale at y', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 35.26.3.

Proof. Lemma 35.26.3 applies since $\mathcal{P}$ is étale local on the source by Lemma 35.32.4.

Assume $x \in W(f)$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $g'(U') \subset W(f)$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $f \circ g'|_{U'} = g \circ f'|_{U'}$ has $\mathcal{P}$ by property (1) of Definition 35.32.3. Then $f'|_{U'} : U' \to V'$ has property $\mathcal{P}$ by (4) of Lemma 35.32.4. Then by (3) of Lemma 35.32.4 we conclude that $f'_{U'} : U' \to Y'$ has $\mathcal{P}$. Hence $U' \subset W(f')$ by definition. Hence $x' \in W(f')$.

Assume $x' \in W(f')$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $U' \subset W(f')$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $U' \to Y'$ has $\mathcal{P}$ by definition of $W(f')$. Then $U' \to V'$ has $\mathcal{P}$ by (4) of Lemma 35.32.4. Then $U' \to Y$ has $\mathcal{P}$ by (3) of Lemma 35.32.4. Let $U \subset X$ be the image of the étale (hence open) morphism $g'|'_U : U' \to X$. Then $\{U' \to U\}$ is an étale covering and we conclude that $U \to Y$ has $\mathcal{P}$ by (1) of Lemma 35.32.4. Thus $U \subset W(f)$ by definition. Hence $x \in W(f)$. □

0CF0 **Lemma 35.32.10.** *Let k be a field. Let $n \geq 2$. For $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}^n_k$ given in coordinates by*

$$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{i-1}, x_i + x_j^d, x_{i+1}, \ldots, x_n)$$

Let $W \subset \mathbf{A}^n_k$ be a nonempty open subscheme such that $T_{i,j,d}(W) = W$ for all i, j, d as above. Then either $W = \mathbf{A}^n_k$ or the characteristic of k is $p > 0$ and $\mathbf{A}^n_k \setminus W$ is a finite set of closed points whose coordinates are algebraic over $\mathbf{F}_p$.

Proof. We may replace k by any extension field in order to prove this. Let Z be an irreducible component of $\mathbf{A}^n_k \setminus W$. Assume $\dim(Z) \geq 1$, to get a contradiction. Then

there exists an extension field k'/k and a k'-valued point $\xi = (\xi_1, \ldots, \xi_n) \in (k')^n$ of $Z_{k'} \subset \mathbf{A}^n_{k'}$ such that at least one of $x_1, \ldots, x_n$ is transcendental over the prime field. Claim: the orbit of ξ under the group generated by the transformations $T_{i,j,d}$ is Zariski dense in $\mathbf{A}^n_{k'}$. The claim will give the desired contradiction.

If the characteristic of k' is zero, then already the operators $T_{i,j,0}$ will be enough since these transform ξ into the points

$$(\xi_1 + a_1, \ldots, \xi_n + a_n)$$

for arbitrary $(a_1, \ldots, a_n) \in \mathbf{Z}^n_{\geq 0}$. If the characteristic is $p > 0$, we may assume after renumbering that ξ_n is transcendental over $\mathbf{F}_p$. By successively applying the operators $T_{i,n,d}$ for $i < n$ we see the orbit of ξ contains the elements

$$(\xi_1 + P_1(\xi_n), \ldots, \xi_{n-1} + P_{n-1}(\xi_n), \xi_n)$$

for arbitrary $(P_1, \ldots, P_{n-1}) \in \mathbf{F}_p[t]$. Thus the Zariski closure of the orbit contains the coordinate hyperplane $x_n = \xi_n$. Repeating the argument with a different coordinate, we conclude that the Zariski closure contains $x_i = \xi_i + P(\xi_n)$ for any $P \in \mathbf{F}_p[t]$ such that $\xi_i + P(\xi_n)$ is transcendental over $\mathbf{F}_p$. Since there are infinitely many such P the claim follows.

Of course the argument in the preceding paragraph also applies if $Z = \{z\}$ has dimension 0 and the coordinates of z in $\kappa(z)$ are not algebraic over $\mathbf{F}_p$. The lemma follows. □

0CF1 **Lemma 35.32.11.** *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Given a commutative diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \quad \text{with points} \quad \begin{array}{ccc} x' & \longrightarrow & y' \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

such that g is smooth y' and $X' \to X \times_Y Y'$ is étale at x', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 35.26.3.

Proof. Since $\mathcal{P}$ is étale local on the source we see that $x \in W(f)$ if and only if the image of x in $X \times_Y Y'$ is in $W(X \times_Y Y' \to Y')$. Hence we may assume the diagram in the lemma is cartesian.

Assume $x \in W(f)$. Since $\mathcal{P}$ is smooth local on the target we see that $(g')^{-1}W(f) = W(f) \times_Y Y' \to Y'$ has $\mathcal{P}$. Hence $(g')^{-1}W(f) \subset W(f')$. We conclude $x' \in W(f')$.

Assume $x' \in W(f')$. For any open neighbourhood $V' \subset Y'$ of y' we may replace Y' by V' and X' by $U' = (f')^{-1}V'$ because $V' \to Y'$ is smooth and hence the base change $W(f') \cap U' \to V'$ of $W(f') \to Y'$ has property $\mathcal{P}$. Thus we may assume there exists an étale morphism $Y' \to \mathbf{A}^n_Y$ over Y, see Morphisms, Lemma 29.36.20.

Picture

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ \mathbf{A}^n_X & \xrightarrow[f_n]{} & \mathbf{A}^n_Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

By Lemma 35.32.6 (and because étale coverings are smooth coverings) we see that $\mathcal{P}$ is étale local on the source-and-target. By Lemma 35.32.9 we see that $W(f')$ is the inverse image of the open $W(f_n) \subset \mathbf{A}^n_X$. In particular $W(f_n)$ contains a point lying over x. After replacing X by the image of $W(f_n)$ (which is open) we may assume $W(f_n) \to X$ is surjective. Claim: $W(f_n) = \mathbf{A}^n_X$. The claim implies f has $\mathcal{P}$ as $\mathcal{P}$ is local in the smooth topology and $\{\mathbf{A}^n_Y \to Y\}$ is a smooth covering.

Essentially, the claim follows as $W(f_n) \subset \mathbf{A}^n_X$ is a "translation invariant" open which meets every fibre of $\mathbf{A}^n_X \to X$. However, to produce an argument along these lines one has to do étale localization on Y to produce enough translations and it becomes a bit annoying. Instead we use the automorphisms of Lemma 35.32.10 and étale morphisms of affine spaces. We may assume $n \geq 2$. Namely, if $n = 0$, then we are done. If $n = 1$, then we consider the diagram

$$\begin{array}{ccc} \mathbf{A}^2_X & \xrightarrow[f_2]{} & \mathbf{A}^2_Y \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbf{A}^1_X & \xrightarrow{f_1} & \mathbf{A}^1_Y \end{array}$$

We have $p^{-1}(W(f_1)) \subset W(f_2)$ (see first paragraph of the proof). Thus $W(f_2) \to X$ is still surjective and we may work with f_2. Assume $n \geq 2$.

For any $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}^n$ defined in Lemma 35.32.10. Then we get a commutative diagram

$$\begin{array}{ccc} \mathbf{A}^n_X & \xrightarrow[f_n]{} & \mathbf{A}^n_Y \\ {\scriptstyle T_{i,j,d}}\downarrow & & \downarrow{\scriptstyle T_{i,j,d}} \\ \mathbf{A}^n_X & \xrightarrow{f_n} & \mathbf{A}^n_Y \end{array}$$

whose vertical arrows are isomorphisms. We conclude that $T_{i,j,d}(W(f_n)) = W(f_n)$. Applying Lemma 35.32.10 we conclude for any $x \in X$ the fibre $W(f_n)_x \subset \mathbf{A}^n_x$ is either $\mathbf{A}^n_x$ (this is what we want) or $\kappa(x)$ has characteristic $p > 0$ and $W(f_n)_x$ is the complement of a finite set $Z_x \subset \mathbf{A}^n_x$ of closed points. The second possibility cannot occur. Namely, consider the morphism $T_p : \mathbf{A}^n \to \mathbf{A}^n$ given by

$$(x_1, \ldots, x_n) \mapsto (x_1 - x_1^p, \ldots, x_n - x_n^p)$$

As above we get a commutative diagram

$$\begin{array}{ccc} \mathbf{A}^n_X & \xrightarrow[f_n]{} & \mathbf{A}^n_Y \\ {\scriptstyle T_p}\downarrow & & \downarrow{\scriptstyle T_p} \\ \mathbf{A}^n_X & \xrightarrow{f_n} & \mathbf{A}^n_Y \end{array}$$

The morphism $T_p : \mathbf{A}^n_X \to \mathbf{A}^n_X$ is étale at every point lying over x and the morphism $T_p : \mathbf{A}^n_Y \to \mathbf{A}^n_Y$ is étale at every point lying over the image of x in Y. (Details omitted; hint: compute the derivatives.) We conclude that

$$T_p^{-1}(W) \cap \mathbf{A}^n_x = W \cap \mathbf{A}^n_x$$

by Lemma 35.32.9 (we've already seen $\mathcal{P}$ is étale local on the source-and-target). Since $T_p : \mathbf{A}^n_x \to \mathbf{A}^n_x$ is finite étale of degree $p^n > 1$ we see that if Z_x is not empty then it contains $T_p^{-1}(Z_x)$ which is bigger. This contradiction finishes the proof. □

35.33. Properties of morphisms of germs local on source-and-target

04R5 In this section we discuss the analogue of the material in Section 35.32 for morphisms of germs of schemes.

04NB **Definition 35.33.1.** Let $\mathcal{Q}$ be a property of morphisms of germs of schemes. We say $\mathcal{Q}$ is *étale local on the source-and-target* if for any commutative diagram

$$\begin{array}{ccc} (U', u') & \xrightarrow[h']{} & (V', v') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (U, u) & \xrightarrow{h} & (V, v) \end{array}$$

of germs with étale vertical arrows we have $\mathcal{Q}(h) \Leftrightarrow \mathcal{Q}(h')$.

04R6 **Lemma 35.33.2.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs defined by the rule*

$$\mathcal{Q}((X, x) \to (S, s)) \Leftrightarrow \text{there exists a representative } U \to S \text{ which has } \mathcal{P}$$

Then $\mathcal{Q}$ is étale local on the source-and-target as in Definition 35.33.1.

Proof. If a morphism of germs $(X, x) \to (S, s)$ has $\mathcal{Q}$, then there are arbitrarily small neighbourhoods $U \subset X$ of x and $V \subset S$ of s such that a representative $U \to V$ of $(X, x) \to (S, s)$ has $\mathcal{P}$. This follows from Lemma 35.32.4. Let

$$\begin{array}{ccc} (U', u') & \xrightarrow[h']{} & (V', v') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (U, u) & \xrightarrow{h} & (V, v) \end{array}$$

be as in Definition 35.33.1. Choose $U_1 \subset U$ and a representative $h_1 : U_1 \to V$ of h. Choose $V'_1 \subset V'$ and an étale representative $b_1 : V'_1 \to V$ of b (Definition 35.20.2). Choose $U'_1 \subset U'$ and representatives $a_1 : U'_1 \to U_1$ and $h'_1 : U'_1 \to V'_1$ of a and h' with a_1 étale. After shrinking U'_1 we may assume $h_1 \circ a_1 = b_1 \circ h'_1$. By the initial remark of the proof, we are trying to show $u' \in W(h'_1) \Leftrightarrow u \in W(h_1)$ where $W(-)$ is as in Lemma 35.26.3. Thus the lemma follows from Lemma 35.32.9. □

04R7 **Lemma 35.33.3.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let Q be the associated property of morphisms of germs, see Lemma 35.33.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f has property $\mathcal{P}$, and*

(2) *for every $x \in X$ the morphism of germs $(X, x) \to (Y, f(x))$ has property $\mathcal{Q}$.*

Proof. The implication (1) $\Rightarrow$ (2) is direct from the definitions. The implication (2) $\Rightarrow$ (1) also follows from part (3) of Definition 35.32.3. □

A morphism of germs $(X, x) \to (S, s)$ determines a well defined map of local rings. Hence the following lemma makes sense.

04ND **Lemma 35.33.4.** *The property of morphisms of germs*

$$\mathcal{P}((X, x) \to (S, s)) = \mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \text{ is flat}$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{array}{ccc} \mathcal{O}_{U',u'} & \longleftarrow & \mathcal{O}_{V',v'} \\ \uparrow & & \uparrow \\ \mathcal{O}_{U,u} & \longleftarrow & \mathcal{O}_{V,v} \end{array}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are essentially of finite presentation, flat, and unramified (see Algebra, Section 10.143). In particular the vertical maps are faithfully flat, see Algebra, Lemma 10.39.17. Now, if the upper horizontal arrow is flat, then the lower horizontal arrow is flat by an application of Algebra, Lemma 10.39.10 with $R = \mathcal{O}_{V,v}$, $S = \mathcal{O}_{U,u}$ and $M = \mathcal{O}_{U',u'}$. If the lower horizontal arrow is flat, then the ring map

$$\mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \longleftarrow \mathcal{O}_{V',v'}$$

is flat by Algebra, Lemma 10.39.7. And the ring map

$$\mathcal{O}_{U',u'} \longleftarrow \mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u}$$

is a localization of a map between étale ring extensions of $\mathcal{O}_{U,u}$, hence flat by Algebra, Lemma 10.143.8. □

04NI **Lemma 35.33.5.** *Consider a commutative diagram of morphisms of schemes*

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

with étale vertical arrows and a point $v' \in V'$ mapping to $v \in V$. Then the morphism of fibres $U'_{v'} \to U_v$ is étale.

Proof. Note that $U'_v \to U_v$ is étale as a base change of the étale morphism $U' \to U$. The scheme U'_v is a scheme over V'_v. By Morphisms, Lemma 29.36.7 the scheme V'_v is a disjoint union of spectra of finite separable field extensions of $\kappa(v)$. One of these is $v' = \mathrm{Spec}(\kappa(v'))$. Hence $U'_{v'}$ is an open and closed subscheme of U'_v and it follows that $U'_{v'} \to U'_v \to U_v$ is étale (as a composition of an open immersion and an étale morphism, see Morphisms, Section 29.36). □

Given a morphism of germs of schemes $(X, x) \to (S, s)$ we can define the *fibre* as the isomorphism class of germs (U_s, x) where $U \to S$ is any representative. We will often abuse notation and just write (X_s, x).

04NJ **Lemma 35.33.6.** *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) = \textit{the local ring } \mathcal{O}_{X_s, x} \textit{ of the fibre has dimension } d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 35.33.5. Hence the result follows from Lemma 35.21.3. □

04NK **Lemma 35.33.7.** *Let $r \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_r((X, x) \to (S, s)) \Leftrightarrow \textit{trdeg}_{\kappa(s)}\kappa(x) = r$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{array}{ccc} \mathcal{O}_{U',u'} & \longleftarrow & \mathcal{O}_{V',v'} \\ \uparrow & & \uparrow \\ \mathcal{O}_{U,u} & \longleftarrow & \mathcal{O}_{V,v} \end{array}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are unramified (see Algebra, Section 10.143). Hence $\kappa(u')/\kappa(u)$ and $\kappa(v')/\kappa(v)$ are finite separable field extensions. Thus we have $\mathrm{trdeg}_{\kappa(v)}\kappa(u) = \mathrm{trdeg}_{\kappa(v')}\kappa(u)$ which proves the lemma. □

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$.

04NL **Lemma 35.33.8.** *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) \Leftrightarrow \dim_x(X_s) = d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 35.33.5. Hence now the equality $\dim_u(U_v) = \dim_{u'}(U'_{v'})$ follows from Lemma 35.21.2. □

35.34. Descent data for schemes over schemes

023U Most of the arguments in this section are formal relying only on the definition of a descent datum. In Simplicial Spaces, Section 85.27 we will examine the relationship with simplicial schemes which will somewhat clarify the situation.

023V **Definition 35.34.1.** Let $f : X \to S$ be a morphism of schemes.

(1) Let $V \to X$ be a scheme over X. A *descent datum for $V/X/S$* is an isomorphism $\varphi : V \times_S X \to X \times_S V$ of schemes over $X \times_S X$ satisfying the *cocycle condition* that the diagram

$$\begin{array}{ccc} V \times_S X \times_S X & \xrightarrow{\varphi_{02}} & X \times_S X \times_S V \\ & {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\ & X \times_S V \times_S X & \end{array}$$

commutes (with obvious notation).

(2) We also say that the pair $(V/X, \varphi)$ is a *descent datum relative to $X \to S$*.

(3) A *morphism $g : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$* is a morphism $g : V \to V'$ of schemes over X such that the diagram

$$\begin{array}{ccc} V \times_S X & \xrightarrow{\varphi} & X \times_S V \\ {\scriptstyle g \times \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_X \times g} \\ V' \times_S X & \xrightarrow{\varphi'} & X \times_S V' \end{array}$$

commutes.

There are all kinds of "miraculous" identities which arise out of the definition above. For example the pullback of φ via the diagonal morphism $\Delta : X \to X \times_S X$ can be seen as a morphism $\Delta^*\varphi : V \to V$. This because $X \times_{\Delta, X \times_S X} (V \times_S X) = V$ and also $X \times_{\Delta, X \times_S X} (X \times_S V) = V$. In fact, $\Delta^*\varphi$ is equal to the identity. This is a good exercise if you are unfamiliar with this material.

02VP **Remark 35.34.2.** Let $X \to S$ be a morphism of schemes. Let $(V/X, \varphi)$ be a descent datum relative to $X \to S$. We may think of the isomorphism φ as an isomorphism

$$(X \times_S X) \times_{\mathrm{pr}_0, X} V \longrightarrow (X \times_S X) \times_{\mathrm{pr}_1, X} V$$

of schemes over $X \times_S X$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^* V \to \mathrm{pr}_1^* V$[9]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

023W **Definition 35.34.3.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$* is given by a scheme V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_S X_j \to X_i \times_S V_j$ of schemes over $X_i \times_S X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} V_i \times_S X_j \times_S X_k & \xrightarrow{\mathrm{pr}_{02}^*\varphi_{ik}} & X_i \times_S X_j \times_S V_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & X_i \times_S V_j \times_S X_k & \end{array}$$

of schemes over $X_i \times_S X_j \times_S X_k$ commutes (with obvious notation).

[9] Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 35.34.1 and 35.34.3 we should have the opposite direction to what was done in Definition 35.2.1 by the general principle that "functions" and "spaces" are dual.

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of X_i-schemes $\psi_i : V_i \to V'_i$ such that all the diagrams

$$\begin{array}{ccc} V_i \times_S X_j & \xrightarrow{\varphi_{ij}} & X_i \times_S V_j \\ {\scriptstyle \psi_i \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \psi_j} \\ V'_i \times_S X_j & \xrightarrow{\varphi'_{ij}} & X_i \times_S V'_j \end{array}$$

commute.

This is the notion that comes up naturally for example when the question arises whether the fibred category of relative curves is a stack in the fpqc topology (it isn't – at least not if you stick to schemes).

02VQ **Remark 35.34.4.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to S\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_S X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_S X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of schemes over $X_i \times_S X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_S X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi_{ik} = \mathrm{pr}_{12}^* \varphi_{jk} \circ \mathrm{pr}_{01}^* \varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

023X **Lemma 35.34.5.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Set $X = \coprod_{i \in I} X_i$, and consider it as an S-scheme. There is a canonical equivalence of categories*

$$\begin{array}{c} \textit{category of descent data} \\ \textit{relative to the family } \{X_i \to S\}_{i \in I} \end{array} \longrightarrow \begin{array}{c} \textit{category of descent data} \\ \textit{relative to } X/S \end{array}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i \in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $X \times_S X = \coprod_{ij} X_i \times_S X_j$ and similarly for higher fibre products. Giving a morphism $V \to X$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. □

023Y **Lemma 35.34.6.** *Pullback of descent data for schemes over schemes.*

(1) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ S' & \xrightarrow{h} & S \end{array}$$

be a commutative diagram of morphisms of schemes. The construction

$$(V \to X, \varphi) \longmapsto f^*(V \to X, \varphi) = (V' \to X', \varphi')$$

where $V' = X' \times_X V$ and where φ' is defined as the composition

$$\begin{array}{ccccc} V' \times_{S'} X' & = & (X' \times_X V) \times_{S'} X' & = & (X' \times_{S'} X') \times_{X \times_S X} (V \times_S X) \\ & & & & \downarrow{\scriptstyle id \times \varphi} \\ X' \times_{S'} V' & = & X' \times_{S'} (X' \times_X V) & = & (X' \times_{S'} X') \times_{X \times_S X} (X \times_S V) \end{array}$$

defines a functor from the category of descent data relative to $X \to S$ to the category of descent data relative to $X' \to S'$.

(2) *Given two morphisms $f_i : X' \to X$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 35.34.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : X' \to X \times_S X$ with $f_i = \mathrm{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= X' \times_{f_0, X} V \\ &= X' \times_{\mathrm{pr}_0 \circ r, X} V \\ &= X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_0, X} V \\ &\xrightarrow{\varphi} X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_1, X} V \\ &= X' \times_{\mathrm{pr}_1 \circ r, X} V \\ &= X' \times_{f_1, X} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. □

02VR **Definition 35.34.7.** With $S, S', X, X', f, a, a', h$ as in Lemma 35.34.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

02VS **Lemma 35.34.8** (Pullback of descent data for schemes over families)**.** *Let $\mathcal{U} = \{U_i \to S'\}_{i \in I}$ and $\mathcal{V} = \{V_j \to S\}_{j \in J}$ be families of morphisms with fixed target. Let $\alpha : I \to J$, $h : S' \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.*

(1) *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to S'\}$. The system*

$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$

(with notation as in Remark 35.34.4) is a descent datum relative to $\mathcal{V}$.

(2) *This construction defines a functor between descent data relative to $\mathcal{U}$ and descent data relative to $\mathcal{V}$.*

(3) *Given a second $\alpha' : I \to J$, $h' : S' \to S$ and $g_i' : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*

(4) *These functors agree, via Lemma 35.34.5, with the pullback functors constructed in Lemma 35.34.6.*

Proof. This follows from Lemma 35.34.6 via the correspondence of Lemma 35.34.5. □

02VT **Definition 35.34.9.** With $\mathcal{U} = \{U_i \to S'\}_{i\in I}$, $\mathcal{V} = \{V_j \to S\}_{j\in J}$, $\alpha : I \to J$, $h : S' \to S$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 35.34.8 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{V}$ have the same target S, and if $\mathcal{U}$ refines $\mathcal{V}$ (see Sites, Definition 7.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 35.34.8 that the choice of the pair does not matter (up to a canonical isomorphism).

023Z **Definition 35.34.10.** Let S be a scheme. Let $f : X \to S$ be a morphism of schemes.

(1) Given a scheme U over S we have the *trivial descent datum* of U relative to $\mathrm{id} : S \to S$, namely the identity morphism on U.
(2) By Lemma 35.34.6 we get a *canonical descent datum* on $X \times_S U$ relative to $X \to S$ by pulling back the trivial descent datum via f. We often denote $(X \times_S U, can)$ this descent datum.
(3) A descent datum (V, φ) relative to X/S is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(X \times_S U, can)$ for some scheme U over S.

Thus being effective means there exists a scheme U over S and an isomorphism $\psi : V \to X \times_S U$ of X-schemes such that φ is equal to the composition

$$V \times_S X \xrightarrow{\psi \times \mathrm{id}_X} X \times_S U \times_S X = X \times_S X \times_S U \xrightarrow{\mathrm{id}_X \times \psi^{-1}} X \times_S V$$

02VU **Definition 35.34.11.** Let S be a scheme. Let $\{X_i \to S\}$ be a family of morphisms with target S.

(1) Given a scheme U over S we have a *canonical descent datum* on the family of schemes $X_i \times_S U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_S U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists a scheme U over S such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_S U, can)$.

35.35. Fully faithfulness of the pullback functors

02VV It turns out that the pullback functor between descent data for fpqc-coverings is fully faithful. In other words, morphisms of schemes satisfy fpqc descent. The goal of this section is to prove this. The reader is encouraged instead to prove this him/herself. The key is to use Lemma 35.13.7.

02VW **Lemma 35.35.1.** *A surjective and flat morphism is an epimorphism in the category of schemes.*

Proof. Suppose we have $h : X' \to X$ surjective and flat and $a, b : X \to Y$ morphisms such that $a \circ h = b \circ h$. As h is surjective we see that a and b agree on underlying topological spaces. Pick $x' \in X'$ and set $x = h(x')$ and $y = a(x) = b(x)$. Consider the local ring maps

$$a_x^\sharp, b_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

These become equal when composed with the flat local homomorphism $h^\sharp_{x'} : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. Since a flat local homomorphism is faithfully flat (Algebra, Lemma 10.39.17) we conclude that $h^\sharp_{x'}$ is injective. Hence $a^\sharp_x = b^\sharp_x$ which implies $a = b$ as desired. □

02VX **Lemma 35.35.2.** *Let $h : S' \to S$ be a surjective, flat morphism of schemes. The base change functor*

$$Sch/S \longrightarrow Sch/S', \quad X \longmapsto S' \times_S X$$

is faithful.

Proof. Let X_1, X_2 be schemes over S. Let $\alpha, \beta : X_2 \to X_1$ be morphisms over S. If α, β base change to the same morphism then we get a commutative diagram as follows

$$\begin{array}{ccccc} X_2 & \longleftarrow & S' \times_S X_2 & \longrightarrow & X_2 \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \beta} \\ X_1 & \longleftarrow & S' \times_S X_1 & \longrightarrow & X_1 \end{array}$$

Hence it suffices to show that $S' \times_S X_2 \to X_2$ is an epimorphism. As the base change of a surjective and flat morphism it is surjective and flat (see Morphisms, Lemmas 29.9.4 and 29.25.8). Hence the lemma follows from Lemma 35.35.1. □

0240 **Lemma 35.35.3.** *In the situation of Lemma 35.34.6 assume that $f : X' \to X$ is surjective and flat. Then the pullback functor is faithful.*

Proof. Let (V_i, φ_i), $i = 1, 2$ be descent data for $X \to S$. Let $\alpha, \beta : V_1 \to V_2$ be morphisms of descent data. Suppose that $f^*\alpha = f^*\beta$. Our task is to show that $\alpha = \beta$. Note that α, β are morphisms of schemes over X, and that $f^*\alpha$, $f^*\beta$ are simply the base changes of α, β to morphisms over X'. Hence the lemma follows from Lemma 35.35.2. □

Here is the key lemma of this section.

0241 **Lemma 35.35.4.** *In the situation of Lemma 35.34.6 assume*

(1) *$\{f : X' \to X\}$ is an fpqc covering (for example if f is surjective, flat, and quasi-compact), and*
(2) *$S = S'$.*

Then the pullback functor is fully faithful.

Proof. Assumption (1) implies that f is surjective and flat. Hence the pullback functor is faithful by Lemma 35.35.3. Let (V, φ) and (W, ψ) be two descent data relative to $X \to S$. Set $(V', \varphi') = f^*(V, \varphi)$ and $(W', \psi') = f^*(W, \psi)$. Let $\alpha' : V' \to W'$ be a morphism of descent data for X' over S. We have to show there exists a morphism $\alpha : V \to W$ of descent data for X over S whose pullback is α'.

Recall that V' is the base change of V by f and that φ' is the base change of φ by $f \times f$ (see Remark 35.34.2). By assumption the diagram

$$\begin{array}{ccc} V' \times_S X' & \xrightarrow[\varphi']{} & X' \times_S V' \\ {\scriptstyle \alpha' \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \alpha'} \\ W' \times_S X' & \xrightarrow{\psi'} & X' \times_S W' \end{array}$$

commutes. We claim the two compositions

$$V' \times_V V' \xrightarrow{\mathrm{pr}_i} V' \xrightarrow{\alpha'} W' \longrightarrow W \ , \quad i = 0, 1$$

are the same. The reader is advised to prove this themselves rather than read the rest of this paragraph. (Please email if you find a nice clean argument.) Let v_0, v_1 be points of V' which map to the same point $v \in V$. Let $x_i \in X'$ be the image of v_i, and let x be the point of X which is the image of v in X. In other words, $v_i = (x_i, v)$ in $V' = X' \times_X V$. Write $\varphi(v, x) = (x, v')$ for some point v' of V. This is possible because φ is a morphism over $X \times_S X$. Denote $v'_i = (x_i, v')$ which is a point of V'. Then a calculation (using the definition of φ') shows that $\varphi'(v_i, x_j) = (x_i, v'_j)$. Denote $w_i = \alpha'(v_i)$ and $w'_i = \alpha'(v'_i)$. Now we may write $w_i = (x_i, u_i)$ for some point u_i of W, and $w'_i = (x_i, u'_i)$ for some point u'_i of W. The claim is equivalent to the assertion: $u_0 = u_1$. A formal calculation using the definition of ψ' (see Lemma 35.34.6) shows that the commutativity of the diagram displayed above says that

$$((x_i, x_j), \psi(u_i, x)) = ((x_i, x_j), (x, u'_j))$$

as points of $(X' \times_S X') \times_{X \times_S X} (X \times_S W)$ for all $i, j \in \{0, 1\}$. This shows that $\psi(u_0, x) = \psi(u_1, x)$ and hence $u_0 = u_1$ by taking ψ^{-1}. This proves the claim because the argument above was formal and we can take scheme points (in other words, we may take $(v_0, v_1) = \mathrm{id}_{V' \times_V V'}$).

At this point we can use Lemma 35.13.7. Namely, $\{V' \to V\}$ is a fpqc covering as the base change of the morphism $f : X' \to X$. Hence, by Lemma 35.13.7 the morphism $\alpha' : V' \to W' \to W$ factors through a unique morphism $\alpha : V \to W$ whose base change is necessarily α'. Finally, we see the diagram

$$\begin{array}{ccc} V \times_S X & \xrightarrow{\varphi} & X \times_S V \\ {\scriptstyle \alpha \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \alpha} \\ W \times_S X & \xrightarrow{\psi} & X \times_S W \end{array}$$

commutes because its base change to $X' \times_S X'$ commutes and the morphism $X' \times_S X' \to X \times_S X$ is surjective and flat (use Lemma 35.35.2). Hence α is a morphism of descent data $(V, \varphi) \to (W, \psi)$ as desired. □

The following two lemmas have been obsoleted by the improved exposition of the previous material. But they are still true!

0242 **Lemma 35.35.5.** *Let $X \to S$ be a morphism of schemes. Let $f : X \to X$ be a selfmap of X over S. In this case pullback by f is isomorphic to the identity functor on the category of descent data relative to $X \to S$.*

Proof. This is clear from Lemma 35.34.6 since it tells us that $f^* \cong \mathrm{id}^*$. □

0243 **Lemma 35.35.6.** *Let $f : X' \to X$ be a morphism of schemes over a base scheme S. Assume there exists a morphism $g : X \to X'$ over S, for example if f has a section. Then the pullback functor of Lemma 35.34.6 defines an equivalence of categories between the category of descent data relative to X/S and X'/S.*

Proof. Let $g : X \to X'$ be a morphism over S. Lemma 35.35.5 above shows that the functors $f^* \circ g^* = (g \circ f)^*$ and $g^* \circ f^* = (f \circ g)^*$ are isomorphic to the respective identity functors as desired. □

040J **Lemma 35.35.7.** *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $X \to S$ is surjective and flat. Then the pullback functor of Lemma 35.34.6 is a faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.35.6. The second morphism is surjective and flat, hence induces a faithful functor by Lemma 35.35.3. □

040K **Lemma 35.35.8.** *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $\{X \to S\}$ is an fpqc covering (for example if f is surjective, flat and quasi-compact). Then the pullback functor of Lemma 35.34.6 is a fully faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.35.6. The second morphism is an fpqc covering hence induces a fully faithful functor by Lemma 35.35.4. □

02VZ **Lemma 35.35.9.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is an fpqc covering of V_j. Then the pullback functor*

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

of Lemma 35.34.8 is fully faithful.

Proof. Consider the morphism of schemes

$$g : X = \coprod\nolimits_{i \in I} U_i \longrightarrow Y = \coprod\nolimits_{j \in J} V_j$$

over S which on the ith component maps into the $\alpha(i)$th component via the morphism $g_{\alpha(i)}$. We claim that $\{g : X \to Y\}$ is an fpqc covering of schemes. Namely, by Topologies, Lemma 34.9.4 for each j the morphism $\{\coprod_{\alpha(i)=j} U_i \to V_j\}$ is an fpqc covering. Thus for every affine open $V \subset V_j$ (which we may think of as an affine open of Y) we can find finitely many affine opens $W_1, \ldots, W_n \subset \coprod_{\alpha(i)=j} U_i$ (which we may think of as affine opens of X) such that $V = \bigcup_{i=1,\ldots,n} g(W_i)$. This provides enough affine opens of Y which can be covered by finitely many affine opens of X so that Topologies, Lemma 34.9.2 part (3) applies, and the claim follows. Let us write $DD(X/S)$, resp. $DD(\mathcal{U})$ for the category of descent data with respect to X/S, resp. $\mathcal{U}$, and similarly for Y/S and $\mathcal{V}$. Consider the diagram

$$\begin{array}{ccc} DD(Y/S) & \longrightarrow & DD(X/S) \\ \uparrow{\scriptstyle \text{Lemma 35.34.5}} & & \uparrow{\scriptstyle \text{Lemma 35.34.5}} \\ DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U}) \end{array}$$

This diagram is commutative, see the proof of Lemma 35.34.8. The vertical arrows are equivalences. Hence the lemma follows from Lemma 35.35.4 which shows the top horizontal arrow of the diagram is fully faithful. □

The next lemma shows that, in order to check effectiveness, we may always Zariski refine the given family of morphisms with target S.

02VY **Lemma 35.35.10.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i\in I}$, and $\mathcal{V} = \{V_j \to S\}_{j\in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is a Zariski covering (see Topologies, Definition 34.3.1) of V_j. Then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

of Lemma 35.34.8 is an equivalence of categories. In particular, the category of schemes over S is equivalent to the category of descent data relative to any Zariski covering of S.

Proof. The functor is faithful and fully faithful by Lemma 35.35.9. Let us indicate how to prove that it is essentially surjective. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$. Fix $j \in J$ and set $I_j = \{i \in I \mid \alpha(i) = j\}$. For $i, i' \in I_j$ note that there is a canonical morphism

$$c_{ii'} : U_i \times_{g_i, V_j, g_{i'}} U_{i'} \to U_i \times_S U_{i'}.$$

Hence we can pullback $\varphi_{ii'}$ by this morphism and set $\psi_{ii'} = c_{ii'}^* \varphi_{ii'}$ for $i, i' \in I_j$. In this way we obtain a descent datum $(X_i, \psi_{ii'})$ relative to the Zariski covering $\{g_i : U_i \to V_j\}_{i\in I_j}$. Note that $\psi_{ii'}$ is an isomorphism from the open $X_{i,U_i\times_{V_j}U_{i'}}$ of X_i to the corresponding open of $X_{i'}$. It follows from Schemes, Section 26.14 that we may glue $(X_i, \psi_{ii'})$ into a scheme Y_j over V_j. Moreover, the morphisms $\varphi_{ii'}$ for $i \in I_j$ and $i' \in I_{j'}$ glue to a morphism $\varphi_{jj'} : Y_j \times_S V_{j'} \to V_j \times_S Y_{j'}$ satisfying the cocycle condition (details omitted). Hence we obtain the desired descent datum $(Y_j, \varphi_{jj'})$ relative to $\mathcal{V}$. □

02W0 **Lemma 35.35.11.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i\in I}$, and $\mathcal{V} = \{V_j \to S\}_{j\in J}$, be fpqc-coverings of S. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

is fully faithful. In particular, the category of schemes over S is identified with a full subcategory of the category of descent data relative to any fpqc-covering of S.

Proof. Consider the fpqc-covering $\mathcal{W} = \{U_i \times_S V_j \to S\}_{(i,j)\in I\times J}$ of S. It is a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Hence we have a 2-commutative diagram of functors and categories

$$\begin{array}{ccc} DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U}) \\ & \searrow \quad \swarrow & \\ & DD(\mathcal{W}) & \end{array}$$

Notation as in the proof of Lemma 35.35.9 and commutativity by Lemma 35.34.8 part (3). Hence clearly it suffices to prove the functors $DD(\mathcal{V}) \to DD(\mathcal{W})$ and $DD(\mathcal{U}) \to DD(\mathcal{W})$ are fully faithful. This follows from Lemma 35.35.9 as desired. □

040L **Remark 35.35.12.** Lemma 35.35.11 says that morphisms of schemes satisfy fpqc descent. In other words, given a scheme S and schemes X, Y over S the functor

$$(Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_T(X_T, Y_T)$$

satisfies the sheaf condition for the fpqc topology. The simplest case of this is the following. Suppose that $T \to S$ is a surjective flat morphism of affines. Let $\psi_0 : X_T \to Y_T$ be a morphism of schemes over T which is compatible with the canonical descent data. Then there exists a unique morphism $\psi : X \to Y$ whose base change to T is ψ_0. In fact this special case follows in a straightforward manner from Lemma 35.35.4. And, in turn, that lemma is a formal consequence of the following two facts: (a) the base change functor by a faithfully flat morphism is faithful, see Lemma 35.35.2 and (b) a scheme satisfies the sheaf condition for the fpqc topology, see Lemma 35.13.7.

0AP4 **Lemma 35.35.13.** *Let $X \to S$ be a surjective, quasi-compact, flat morphism of schemes. Let (V, φ) be a descent datum relative to X/S. Suppose that for all $v \in V$ there exists an open subscheme $v \in W \subset V$ such that $\varphi(W \times_S X) \subset X \times_S W$ and such that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. Then (V, φ) is effective.*

Proof. Let $V = \bigcup W_i$ be an open covering with $\varphi(W_i \times_S X) \subset X \times_S W_i$ and such that the descent datum $(W_i, \varphi|_{W_i \times_S X})$ is effective. Let $U_i \to S$ be a scheme and let $\alpha_i : (X \times_S U_i, can) \to (W_i, \varphi|_{W_i \times_S X})$ be an isomorphism of descent data. For each pair of indices (i, j) consider the open $\alpha_i^{-1}(W_i \cap W_j) \subset X \times_S U_i$. Because everything is compatible with descent data and since $\{X \to S\}$ is an fpqc covering, we may apply Lemma 35.13.6 to find an open $U_{ij} \subset U_i$ such that $\alpha_i^{-1}(W_i \cap W_j) = X \times_S U_{ij}$. Now the identity morphism on $W_i \cap W_j$ is compatible with descent data, hence comes from a unique morphism $\varphi_{ij} : U_{ij} \to U_{ji}$ over S (see Remark 35.35.12). Then $(U_i, U_{ij}, \varphi_{ij})$ is a glueing data as in Schemes, Section 26.14 (proof omitted). Thus we may assume there is a scheme U over S such that $U_i \subset U$ is open, $U_{ij} = U_i \cap U_j$ and $\varphi_{ij} = \mathrm{id}_{U_i \cap U_j}$, see Schemes, Lemma 26.14.1. Pulling back to X we can use the α_i to get the desired isomorphism $\alpha : X \times_S U \to V$. □

35.36. Descending types of morphisms

02W1 In the following we study the question as to whether descent data for schemes relative to a fpqc-covering are effective. The first remark to make is that this is not always the case. We will see this in Algebraic Spaces, Example 65.14.2. Even projective morphisms do not always satisfy descent for fpqc-coverings, by Examples, Lemma 110.66.1.

On the other hand, if the schemes we are trying to descend are particularly simple, then it is sometime the case that for whole classes of schemes descent data are effective. We will introduce terminology here that describes this phenomenon abstractly, even though it may lead to confusion if not used correctly later on.

02W2 **Definition 35.36.1.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings* if for any τ-covering $\mathcal{U} : \{U_i \to S\}_{i \in I}$ (see Topologies, Section 34.2), any descent datum (X_i, φ_{ij}) relative to $\mathcal{U}$ such that each morphism $X_i \to U_i$ has property $\mathcal{P}$ is effective.

Note that in each of the cases we have already seen that the functor from schemes over S to descent data over $\mathcal{U}$ is fully faithful (Lemma 35.35.11 combined with the results in Topologies that any τ-covering is also a fpqc-covering). We have also seen that descent data are always effective with respect to Zariski coverings (Lemma 35.35.10). It may be prudent to only study the notion just introduced

when $\mathcal{P}$ is either stable under any base change or at least local on the base in the τ-topology (see Definition 35.22.1) in order to avoid erroneous arguments (relying on $\mathcal{P}$ when descending halfway).

Here is the obligatory lemma reducing this question to the case of a covering given by a single morphism of affines.

02W3 **Lemma 35.36.2.** *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Suppose that*

(1) *$\mathcal{P}$ is stable under any base change (see Schemes, Definition 26.18.3),*
(2) *if $Y_j \to V_j$, $j = 1, \ldots, m$ have $\mathcal{P}$, then so does $\coprod Y_j \to \coprod V_j$, and*
(3) *for any surjective morphism of affines $X \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.*

Then morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings.

Proof. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ be a τ-covering of S. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$ and assume that each morphism $X_i \to U_i$ has property $\mathcal{P}$. We have to show there exists a scheme $X \to S$ such that $(X_i, \varphi_{ii'}) \cong (U_i \times_S X, can)$.

Before we start the proof proper we remark that for any family of morphisms $\mathcal{V} : \{V_j \to S\}$ and any morphism of families $\mathcal{V} \to \mathcal{U}$, if we pullback the descent datum $(X_i, \varphi_{ii'})$ to a descent datum $(Y_j, \varphi_{jj'})$ over $\mathcal{V}$, then each of the morphisms $Y_j \to V_j$ has property $\mathcal{P}$ also. This is true because of assumption (1) that $\mathcal{P}$ is stable under any base change and the definition of pullback (see Definition 35.34.9). We will use this without further mention.

First, let us prove the lemma when S is affine. By Topologies, Lemma 34.9.9, 34.7.4, 34.4.4, 34.5.4, or 34.6.4 there exists a standard τ-covering $\mathcal{V} : \{V_j \to S\}_{j=1,\ldots,m}$ which refines $\mathcal{U}$. The pullback functor $DD(\mathcal{U}) \to DD(\mathcal{V})$ between categories of descent data is fully faithful by Lemma 35.35.11. Hence it suffices to prove that the descent datum over the standard τ-covering $\mathcal{V}$ is effective. By assumption (2) we see that $\coprod Y_j \to \coprod V_j$ has property $\mathcal{P}$. By Lemma 35.34.5 this reduces us to the covering $\{\coprod_{j=1,\ldots,m} V_j \to S\}$ for which we have assumed the result in assumption (3) of the lemma. Hence the lemma holds when S is affine.

Assume S is general. Let $V \subset S$ be an affine open. By the properties of site the family $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ is a τ-covering of V. Denote $(X_i, \varphi_{ii'})_V$ the restriction (or pullback) of the given descent datum to $\mathcal{U}_V$. Hence by what we just saw we obtain a scheme X_V over V whose canonical descent datum with respect to $\mathcal{U}_V$ is isomorphic to $(X_i, \varphi_{ii'})_V$. Suppose that $V' \subset V$ is an affine open of V. Then both $X_{V'}$ and $V' \times_V X_V$ have canonical descent data isomorphic to $(X_i, \varphi_{ii'})_{V'}$. Hence, by Lemma 35.35.11 again we obtain a canonical morphism $\rho^V_{V'} : X_{V'} \to X_V$ over S which identifies $X_{V'}$ with the inverse image of V' in X_V. We omit the verification that given affine opens $V'' \subset V' \subset V$ of S we have $\rho^V_{V''} = \rho^V_{V'} \circ \rho^{V'}_{V''}$.

By Constructions, Lemma 27.2.1 the data $(X_V, \rho^V_{V'})$ glue to a scheme $X \to S$. Moreover, we are given isomorphisms $V \times_S X \to X_V$ which recover the maps $\rho^V_{V'}$. Unwinding the construction of the schemes X_V we obtain isomorphisms

$$V \times_S U_i \times_S X \longrightarrow V \times_S X_i$$

compatible with the maps $\varphi_{ii'}$ and compatible with restricting to smaller affine opens in X. This implies that the canonical descent datum on $U_i \times_S X$ is isomorphic to the given descent datum and we win. □

35.37. Descending affine morphisms

0244 In this section we show that "affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0245 **Lemma 35.37.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is affine, then the descent datum is effective.*

Proof. Being affine is a property of morphisms of schemes which is local on the base and preserved under any base change, see Morphisms, Lemmas 29.11.3 and 29.11.9. Hence Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is affine. Then $V \to X$ being affine implies that $V = \mathrm{Spec}(B)$ for some A-algebra B (see Morphisms, Definition 29.11.1). The isomorphism φ corresponds to an isomorphism of rings

$$\varphi^\sharp : B \otimes_R A \longleftarrow A \otimes_R B$$

as $A \otimes_R A$-algebras. The cocycle condition on φ says that

$$\begin{array}{ccc} B \otimes_R A \otimes_R A & \longleftarrow & A \otimes_R A \otimes_R B \\ & \nwarrow \quad \swarrow & \\ & A \otimes_R B \otimes_R A & \end{array}$$

is commutative. Inverting these arrows we see that we have a descent datum for modules with respect to $R \to A$ as in Definition 35.3.1. Hence we may apply Proposition 35.3.9 to obtain an R-module $C = \mathrm{Ker}(B \to A \otimes_R B)$ and an isomorphism $A \otimes_R C \cong B$ respecting descent data. Given any pair $c, c' \in C$ the product cc' in B lies in C since the map φ is an algebra homomorphism. Hence C is an R-algebra whose base change to A is isomorphic to B compatibly with descent data. Applying Spec we obtain a scheme U over S such that $(V, \varphi) \cong (X \times_S U, can)$ as desired. □

03I0 **Lemma 35.37.2.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is a closed immersion, then the descent datum is effective.*

Proof. This is true because a closed immersion is an affine morphism (Morphisms, Lemma 29.11.10), and hence Lemma 35.37.1 applies. □

35.38. Descending quasi-affine morphisms

0246 In this section we show that "quasi-affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0247 **Lemma 35.38.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is quasi-affine, then the descent datum is effective.*

Proof. Being quasi-affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemmas 29.13.3 and 29.13.5. Hence Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is quasi-affine.

According to Morphisms, Lemma 29.13.3 this means that

$$V \longrightarrow \underline{\mathrm{Spec}}_X(\pi_*\mathcal{O}_V) = W$$

is a quasi-compact open immersion of schemes over X. The projections $\mathrm{pr}_i : X \times_S X \to X$ are flat and hence we have

$$\mathrm{pr}_0^*\pi_*\mathcal{O}_V = (\pi \times \mathrm{id}_X)_*\mathcal{O}_{V\times_S X}, \quad \mathrm{pr}_1^*\pi_*\mathcal{O}_V = (\mathrm{id}_X \times \pi)_*\mathcal{O}_{X\times_S V}$$

by flat base change (Cohomology of Schemes, Lemma 30.5.2). Thus the isomorphism $\varphi : V \times_S X \to X \times_S V$ (which is an isomorphism over $X \times_S X$) induces an isomorphism of quasi-coherent sheaves of algebras

$$\varphi^\sharp : \mathrm{pr}_0^*\pi_*\mathcal{O}_V \longrightarrow \mathrm{pr}_1^*\pi_*\mathcal{O}_V$$

on $X \times_S X$. The cocycle condition for φ implies the cocycle condition for $\varphi^\sharp$. Another way to say this is that it produces a descent datum φ' on the affine scheme W relative to X over S, which moreover has the property that the morphism $V \to W$ is a morphism of descent data. Hence by Lemma 35.37.1 (or by effectivity of descent for quasi-coherent algebras) we obtain a scheme $U' \to S$ with an isomorphism $(W, \varphi') \cong (X \times_S U', can)$ of descent data. We note in passing that U' is affine by Lemma 35.23.20.

And now we can think of V as a (quasi-compact) open $V \subset X \times_S U'$ with the property that it is stable under the descent datum

$$can : X \times_S U' \times_S X \to X \times_S X \times_S U', (x_0, u', x_1) \mapsto (x_0, x_1, u').$$

In other words $(x_0, u') \in V \Rightarrow (x_1, u') \in V$ for any x_0, x_1, u' mapping to the same point of S. Because $X \to S$ is surjective we immediately find that V is the inverse image of a subset $U \subset U'$ under the morphism $X \times_S U' \to U'$. Because $X \to S$ is quasi-compact, flat and surjective also $X \times_S U' \to U'$ is quasi-compact flat and surjective. Hence by Morphisms, Lemma 29.25.12 this subset $U \subset U'$ is open and we win. □

35.39. Descent data in terms of sheaves

02W4 Here is another way to think about descent data in case of a covering on a site.

02W5 **Lemma 35.39.1.** *Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$[10]. Let Sch_τ be a big τ-site. Let $S \in \mathrm{Ob}(Sch_\tau)$. Let $\{S_i \to S\}_{i\in I}$ be a covering in the site*

[10]The fact that fpqc is missing is not a typo. See discussion in Topologies, Section 34.9.

$(Sch/S)_\tau$. There is an equivalence of categories

$$\left\{\begin{array}{c}\text{descent data } (X_i, \varphi_{ii'}) \text{ such that} \\ \text{each } X_i \in \mathrm{Ob}((Sch/S)_\tau)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{sheaves } F \text{ on } (Sch/S)_\tau \text{ such that} \\ \text{each } h_{S_i} \times F \text{ is representable}\end{array}\right\}.$$

Moreover,

(1) *the objects representing $h_{S_i} \times F$ on the right hand side correspond to the schemes X_i on the left hand side, and*
(2) *the sheaf F is representable if and only if the corresponding descent datum $(X_i, \varphi_{ii'})$ is effective.*

Proof. We have seen in Section 35.13 that representable presheaves are sheaves on the site $(Sch/S)_\tau$. Moreover, the Yoneda lemma (Categories, Lemma 4.3.5) guarantees that maps between representable sheaves correspond one to one with maps between the representing objects. We will use these remarks without further mention during the proof.

Let us construct the functor from right to left. Let F be a sheaf on $(Sch/S)_\tau$ such that each $h_{S_i} \times F$ is representable. In this case let X_i be a representing object in $(Sch/S)_\tau$. It comes equipped with a morphism $X_i \to S_i$. Then both $X_i \times_S S_{i'}$ and $S_i \times_S X_{i'}$ represent the sheaf $h_{S_i} \times F \times h_{S_{i'}}$ and hence we obtain an isomorphism

$$\varphi_{ii'} : X_i \times_S S_{i'} \to S_i \times_S X_{i'}$$

It is straightforward to see that the maps $\varphi_{ii'}$ are morphisms over $S_i \times_S S_{i'}$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (X_i, \varphi_{ii'})$.

Let us construct a functor from left to right. For each i denote F_i the sheaf h_{X_i}. The isomorphisms $\varphi_{ii'}$ give isomorphisms

$$\varphi_{ii'} : F_i \times h_{S_{i'}} \longrightarrow h_{S_i} \times F_{i'}$$

over $h_{S_i} \times h_{S_{i'}}$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} F_i \times h_{S_{i'}} \xrightarrow[\mathrm{pr}_1 \circ \varphi_{ii'}]{\mathrm{pr}_0} \coprod_i F_i \longrightarrow F$$

The cocycle condition guarantees that $h_{S_i} \times F$ is isomorphic to F_i and hence representable. The functor from left to right is given by this construction $(X_i, \varphi_{ii'}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. □

02W6 **Remark 35.39.2.** In the statement of Lemma 35.39.1 the condition that $h_{S_i} \times F$ is representable is equivalent to the condition that the restriction of F to $(Sch/S_i)_\tau$ is representable.

35.40. Other chapters

Preliminaries

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CHAPTER 36

Derived Categories of Schemes

08CU

36.1. Introduction

08CV In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [**TT90**], [**BN93**], [**BV03**], and [**LN07**]. Of course there are many other references.

36.2. Conventions

08CW If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

36.3. Derived category of quasi-coherent modules

06YZ In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme X. A reference is [**TT90**, Appendix B]. By the discussion in Schemes, Section 26.24 the embedding $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ exhibits $QCoh(\mathcal{O}_X)$ as a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

06VT (36.3.0.1) $$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

08DT **Lemma 36.3.1.** *Let X be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 26.24. □

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh}.

0A0J **Lemma 36.3.2.** *Let X be a scheme. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \to H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then*

(1) *$H^q(K) = \lim H^q(K_n)$,*
(2) *$R\lim H^q(K_n) = \lim H^q(K_n)$, and*
(3) *for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.*

Proof. Let $\mathcal{B}$ be the set of affine opens of X. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by Cohomology of Schemes, Lemma 30.2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by Schemes, Lemma 26.7.5. Part (1) follows from Cohomology, Lemma 20.37.11 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology, Lemma 20.37.4. □

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

08D3 **Lemma 36.3.3.** *Let X be a scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism*[1].

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine open subsets of X. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 30.2.2. Thus the lemma follows from Cohomology, Lemma 20.37.9. □

08D4 **Lemma 36.3.4.** *Let X be a scheme. Let $F : Mod(\mathcal{O}_X) \to Ab$ be an additive functor and $N \geq 0$ an integer. Assume that*

(1) *F commutes with countable direct products,*
(2) *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$

(1) *$H^i(RF(\tau_{\leq a}E)) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,*
(2) *$H^i(RF(E)) \to H^i(RF(\tau_{\geq b-N+1}E))$ is an isomorphism for $i \geq b$,*
(3) *if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + N - 1]$.*

Proof. Statement (1) is Derived Categories, Lemma 13.16.1.

Proof of statement (2). Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 36.3.3. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 19.13.6. Thus for every $i \in \mathbf{Z}$ we have a short exact sequence

$$0 \to R^1 \lim H^{i-1}(RF(E_n)) \to H^i(RF(E)) \to \lim H^i(RF(E_n)) \to 0$$

see More on Algebra, Remark 15.87.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals $H^i(RF(E_{-b+N-1}))$ for any b with $i \geq b$.

For every n we have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

[1] In particular, E has a K-injective representative by Derived Categories, Lemma 13.34.5.

(Derived Categories, Remark 13.12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for $i + n \geq N$ and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

for $i + n + 1 \geq N$. We conclude that

$$H^i(RF(E_n)) \to H^i(RF(E_{n-1}))$$

is an isomorphism for $n \geq N - i$. Thus the systems $H^i(RF(E_n))$ all satisfy the ML condition and the R^1 lim term in our short exact sequence is zero (see discussion in More on Algebra, Section 15.87). Moreover, the system $H^i(RF(E_n))$ is constant starting with $n = N - i - 1$ as desired.

Proof of (3). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from (1). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + N$ from part (2). □

The following lemma is the key ingredient to many of the results in this chapter.

06Z0 **Lemma 36.3.5.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. All the functors in the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \xrightarrow[(36.3.0.1)]{} & D_{QCoh}(\mathcal{O}_X) \\ & {\scriptstyle\sim}\nwarrow \quad \swarrow{\scriptstyle R\Gamma(X,-)} & \\ & D(A) & \end{array}$$

are equivalences of triangulated categories. Moreover, for E in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.

Proof. The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor

06VU (36.3.5.1) $$R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A).$$

We will show this functor is quasi-inverse to (36.3.0.1) via the equivalence between quasi-coherent modules on X and the category of A-modules.

Elucidation. Denote $(Y, \mathcal{O}_Y)$ the one point space with sheaf of rings given by A. Denote $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (36.3.0.1) via the equivalence $Mod(\mathcal{O}_Y) = \mathrm{Mod}_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = {}^\sim$ (see Modules, Lemma 17.10.5 and Schemes, Lemmas 26.7.1 and 26.7.5). Thus the functors

$$D(A) \rightleftarrows D(\mathcal{O}_X)$$

are adjoint (by Cohomology, Lemma 20.28.1). In particular we obtain canonical adjunction mappings

$$a : \widetilde{R\Gamma(X, E)} \longrightarrow E$$

for E in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for $M^\bullet$ a complex of A-modules.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 36.3.4 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 30.2.2. Hence

$$H^0(R\Gamma(X, E)) = H^0(R\Gamma(X, \tau_{\geq 0}E)) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings a and b induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus a and b are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that $\widetilde{\ }$ is an exact functor (Schemes, Lemma 26.5.4). Namely, this implies that

$$H^0\left(\widetilde{R\Gamma(X, E)}\right) = H^0(\widetilde{R\Gamma(X, E)}) = \Gamma(X, \widetilde{H^0(E)})$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. □

08DV **Lemma 36.3.6.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a K-flat complex of A-modules, then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. By More on Algebra, Lemma 15.59.3 we see that $K^\bullet \otimes_A A_\mathfrak{p}$ is a K-flat complex of $A_\mathfrak{p}$-modules for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Hence we conclude from Cohomology, Lemma 20.26.4 (and Schemes, Lemma 26.5.4) that $\widetilde{K^\bullet}$ is K-flat. □

0DJK **Lemma 36.3.7.** *If $f : X \to Y$ is a morphism of affine schemes given by the ring map $A \to B$, then the diagram*

$$\begin{array}{ccc} D(B) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(A) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Follows from Lemma 36.3.5 using that $R\Gamma(Y, Rf_*K) = R\Gamma(X, K)$ by Cohomology, Lemma 20.32.5. □

08DW **Lemma 36.3.8.** *Let $f : Y \to X$ be a morphism of schemes.*

(1) *The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
(2) *If X and Y are affine and f is given by the ring map $A \to B$, then the diagram*

$$\begin{array}{ccc} D(B) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \\ {\scriptstyle -\otimes^{\mathbf{L}}_A B}\uparrow & & \uparrow{\scriptstyle Lf^*} \\ D(A) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \end{array}$$

commutes.

Proof. We first prove the diagram

$$\begin{array}{ccc} D(B) & \longrightarrow & D(\mathcal{O}_Y) \\ {\scriptstyle -\otimes^{\mathbf{L}}_A B}\uparrow & & \uparrow{\scriptstyle Lf^*} \\ D(A) & \longrightarrow & D(\mathcal{O}_X) \end{array}$$

commutes. This is clear from Lemma 36.3.6 and the constructions of the functors in question. To see (1) let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that Lf^*E has quasi-coherent cohomology sheaves we may work locally on X. Note that Lf^* is compatible with restricting to open subschemes. Hence we can assume that f is a morphism of affine schemes as in (2). Then we can apply Lemma 36.3.5 to see that E comes from a complex of A-modules. By the commutativity of the first diagram of the proof the same holds for Lf^*E and we conclude (1) is true. □

08DX **Lemma 36.3.9.** *Let X be a scheme.*

(1) *For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*
(2) *If $X = \mathrm{Spec}(A)$ is affine then*

$$\widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes^{\mathbf{L}}_A K^\bullet}$$

for any pair of complexes of A-modules $K^\bullet$, $M^\bullet$.

Proof. The equality of (2) follows immediately from Lemma 36.3.6 and the construction of the derived tensor product. To see (1) let K, L be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on X, hence we may assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 36.3.5 we may represent K and L by complexes of A-modules. Then part (2) implies the result. □

36.4. Total direct image

08DY The following lemma is the analogue of Cohomology of Schemes, Lemma 30.4.5.

08D5 **Lemma 36.4.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

(1) *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.*
(2) *If S is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.*
(3) *In fact, if S is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. The question is local on S, hence we may assume S is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 30.4.5. Thus $R^p f_* \mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$ and the same remains true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N. In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 13.21.3), the quasi-coherence of $R^pf_*H^q(E)$, and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let $U \subset S$ be affine open. By Cohomology of Schemes, Lemma 30.4.6 and our choice of N we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence we may apply Lemma 36.3.4 to the functor $\Gamma(f^{-1}(U), -)$ to see that

$$R\Gamma(U, Rf_*E) = R\Gamma(f^{-1}(U), E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all $U \subset S$ affine open we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \to E \to \tau_{\geq -n}E \to (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 13.12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that R^qf_*E is equal to $R^qf_*\tau_{\geq -n}E$ for some n and the result above applies. □

0G9N **Lemma 36.4.2.** *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for f_*. Then $f_*\mathcal{F}^\bullet$ represents $Rf_*\mathcal{F}^\bullet$ in $D(\mathcal{O}_S)$.*

Proof. There is always a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. Our task is to show that this is an isomorphism on cohomology sheaves. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \to R^0f_*\mathcal{F}^\bullet$ is an isomorphism. The statement is local on S hence we may assume S affine. By Lemma 36.4.1 we have $R^0f_*\mathcal{F}^\bullet = R^0f_*\tau_{\geq -n}\mathcal{F}^\bullet$ for all sufficiently large n. Thus we may assume $\mathcal{F}^\bullet$ bounded below. As each $\mathcal{F}^n$ is right f_*-acyclic by assumption we see that $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7). □

0G9P **Lemma 36.4.3.** *Let X be a quasi-separated and quasi-compact scheme. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for $\Gamma(X, -)$. Then $\Gamma(X, \mathcal{F}^\bullet)$ represents $R\Gamma(X, \mathcal{F}^\bullet)$ in $D(\Gamma(X, \mathcal{O}_X)$.*

Proof. Apply Lemma 36.4.2 to the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$. Some details omitted. □

0G9Q **Lemma 36.4.4.** *Let X be a quasi-separated and quasi-compact scheme. For any object K of $D_{QCoh}(\mathcal{O}_X)$ the spectral sequence*

$$E_2^{i,j} = H^i(X, H^j(K)) \Rightarrow H^{i+j}(X, K)$$

of Cohomology, Example 20.29.3 is bounded and converges.

Proof. By the construction of the spectral sequence via Cohomology, Lemma 20.29.1 using the filtration given by $\tau_{\leq -p}K$, we see that suffices to show that given $n \in \mathbf{Z}$ we have

$$H^n(X, \tau_{\leq -p}K) = 0 \text{ for } p \gg 0$$

and

$$H^n(X, K) = H^n(X, \tau_{\leq -p}K) \text{ for } p \ll 0$$

The first follows from Lemma 36.3.4 applied with $F = \Gamma(X, -)$ and the bound in Cohomology of Schemes, Lemma 30.4.5. The second holds whenever $-p \leq n$ for any ringed space $(X, \mathcal{O}_X)$ and any $K \in D(\mathcal{O}_X)$. □

08DZ **Lemma 36.4.5.** *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. To prove this we may work locally on S, hence we may and do assume that S is quasi-compact. Choose an integer N as in Lemma 36.4.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited. Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 13.21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 30.6.1. □

36.5. Affine morphisms

0AVV In this section we collect some information about pushforward along an affine morphism of schemes.

0G9R **Lemma 36.5.1.** *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then $f_*\mathcal{F}^\bullet = Rf_*\mathcal{F}^\bullet$.*

Proof. Combine Lemma 36.4.2 with Cohomology of Schemes, Lemma 30.2.3. An alternative proof is to work affine locally on S and use Lemma 36.3.7. □

08I8 **Lemma 36.5.2.** *Let $f : X \to S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 36.3.5. Some details omitted. □

08I9 **Lemma 36.5.3.** *Let $f : X \to S$ be an affine morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_S} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 30.2.3). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor Lf^* given in Lemmas 36.3.5 and 36.3.8. Some details omitted. □

Let Y be a scheme. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_Y$-algebras. We will denote $D_{QCoh}(\mathcal{A})$ the inverse image of $D_{QCoh}(\mathcal{O}_X)$ under the restriction functor $D(\mathcal{A}) \to D(\mathcal{O}_X)$. In other words, $K \in D(\mathcal{A})$ is in $D_{QCoh}(\mathcal{A})$ if and only if its cohomology sheaves are quasi-coherent as $\mathcal{O}_X$-modules. If $\mathcal{A}$ is quasi-coherent itself this is the same as asking the cohomology sheaves to be quasi-coherent as $\mathcal{A}$-modules, see Morphisms, Lemma 29.11.7.

0AVW **Lemma 36.5.4.** *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* induces an equivalence*

$$\Phi : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(f_*\mathcal{O}_X)$$

whose composition with $D_{QCoh}(f_*\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$.

Proof. Recall that Rf_* is computed on an object $K \in D_{QCoh}(\mathcal{O}_X)$ by choosing a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing K and taking $f_*\mathcal{I}^\bullet$. Thus we let $\Phi(K)$ be the complex $f_*\mathcal{I}^\bullet$ viewed as a complex of $f_*\mathcal{O}_X$-modules. Denote $g : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ the obvious morphism of ringed spaces. Then g is a flat morphism of ringed spaces (see below for a description of the stalks) and Φ is the restriction of Rg_* to $D_{QCoh}(\mathcal{O}_X)$. We claim that Lg^* is a quasi-inverse. First, observe that Lg^* sends $D_{QCoh}(f_*\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_X)$ because g^* transforms quasi-coherent modules into quasi-coherent modules (Modules, Lemma 17.10.4). To finish the proof it suffices to show that the adjunction mappings

$$Lg^*\Phi(K) = Lg^*Rg_*K \to K \quad \text{and} \quad M \to Rg_*Lg^*M = \Phi(Lg^*M)$$

are isomorphisms for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(f_*\mathcal{O}_X)$. This is a local question, hence we may assume Y and therefore X are affine.

Assume $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Let $\mathfrak{p} = x \in \mathrm{Spec}(A) = X$ be a point mapping to $\mathfrak{q} = y \in \mathrm{Spec}(B) = Y$. Then $(f_*\mathcal{O}_X)_y = A_\mathfrak{q}$ and $\mathcal{O}_{X,x} = A_\mathfrak{p}$ hence g is flat. Hence g^* is exact and $H^i(Lg^*M) = g^*H^i(M)$ for any M in $D(f_*\mathcal{O}_X)$. For $K \in D_{QCoh}(\mathcal{O}_X)$ we see that

$$H^i(\Phi(K)) = H^i(Rf_*K) = f_*H^i(K)$$

by the vanishing of higher direct images (Cohomology of Schemes, Lemma 30.2.3) and Lemma 36.3.4 (small detail omitted). Thus it suffice to show that

$$g^*g_*\mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{G} \to g_*g^*\mathcal{G}$$

are isomorphisms where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{G}$ is a quasi-coherent $f_*\mathcal{O}_X$-module. This follows from Morphisms, Lemma 29.11.7. □

36.6. Cohomology with support in a closed subset

0G7F We elaborate on the material in Cohomology, Sections 20.21 and 20.34 for schemes and quasi-coherent modules.

08DA **Definition 36.6.1.** Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say E is *supported on* T if the cohomology sheaves $H^i(E)$ are supported on T.

We repeat some of the discussion from Cohomology, Section 20.34 in the situation of the definition. Let X be a scheme. Let $T \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in T is a Serre subcategory of the category

of all $\mathcal{O}_X$-modules, see Homology, Definition 12.10.1 and Modules, Lemma 17.5.2. Thus we may apply the discussion in Derived Categories, Section 13.17 and we obtain a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T; we will denote this category $D_T(\mathcal{O}_X)$ in the following.

In the situation of Definition 36.6.1 denote $i : T \to X$ the inclusion map. Recall from Cohomology, Section 20.34 that in this situation we have a functor $R\mathcal{H}_T : D(\mathcal{O}_X) \to D(i^{-1}\mathcal{O}_X)$ which is right adjoint to $i_* : D(i^{-1}\mathcal{O}_X) \to D(\mathcal{O}_X)$.

0G7G **Lemma 36.6.2.** *Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Let $i : T \to X$ be the inclusion.*

(1) *For E in $D_{QCoh}(\mathcal{O}_X)$ we have $i_*R\mathcal{H}_T(E)$ in $D_{QCoh,T}(\mathcal{O}_X)$.*
(2) *The functor $i_* \circ R\mathcal{H}_T : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh,T}(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_{QCoh,T}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X)$.*

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 20.34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$. By Lemma 36.4.1 the complex $Rj_*(E|_U)$ has quasi-coherent cohomology sheaves (this is where we use that U is retrocompact in X). Thus we see that (1) is true. Part (2) follows from this and the adjointness of functors in Cohomology, Lemma 20.34.2. □

0G7H **Lemma 36.6.3.** *Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Then for a family of objects E_i, $i \in I$ of $D_{QCoh}(\mathcal{O}_X)$ we have $R\mathcal{H}_T(\bigoplus E_i) = \bigoplus R\mathcal{H}_T(E_i)$.*

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 20.34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$ for any E in $D(\mathcal{O}_X)$. The functor $E \mapsto Rj_*(E|_U)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.5. It follows that the same is true for the functor $i_* \circ R\mathcal{H}_T$ (details omitted). Since $i_* : D(i^{-1}\mathcal{O}_X) \to D_T(\mathcal{O}_X)$ is an equivalence (Cohomology, Lemma 20.34.2) we conclude. □

0G7I **Remark 36.6.4.** Let X be a scheme. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Denote $Z \subset X$ the closed subscheme cut out by $f_1, \ldots, f_c$. For $0 \leq p < c$ and $1 \leq i_0 < \ldots < i_p \leq c$ we denote $U_{i_0 \ldots i_p} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible. For any $\mathcal{O}_X$-module $\mathcal{F}$ we set

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_*(\mathcal{F}|_{U_{i_0 \ldots i_p}})$$

In this situation the *extended alternating Čech complex* is the complex of $\mathcal{O}_X$-modules

0G7J (36.6.4.1) $$0 \to \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \ldots \to \bigoplus\nolimits_{i_0 < \ldots < i_p} \mathcal{F}_{i_0 \ldots i_p} \to \ldots \to \mathcal{F}_{1 \ldots c} \to 0$$

where $\mathcal{F}$ is put in degree 0. The maps are constructed as follows. Given $1 \leq i_0 < \ldots < i_{p+1} \leq c$ and $0 \leq j \leq p+1$ we have the canonical map

$$\mathcal{F}_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \to \mathcal{F}_{i_0 \ldots i_p}$$

coming from the inclusion $U_{i_0 \ldots i_p} \subset U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$. The differentials in the extended alternating complex use these canonical maps with sign $(-1)^j$.

0G7K **Lemma 36.6.5.** *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4 the complex (36.6.4.1) restricts to an acyclic complex over $X \setminus Z$.*

We remark that this lemma holds more generally for any extended alternating Čech complex defined as in Remark 36.6.4 starting with a finite open covering $X \setminus Z = U_1 \cup \ldots \cup U_c$.

Proof. Let $W \subset X \setminus Z$ be an open subset. Evaluating the complex of sheaves (36.6.4.1) on W we obtain the complex

$$\mathcal{F}(W) \to \bigoplus\nolimits_{i_0} \mathcal{F}(U_{i_0} \cap W) \to \bigoplus\nolimits_{i_0 < i_1} \mathcal{F}(U_{i_0 i_1} \cap W) \to \ldots$$

In other words, we obtain the extended ordered Čech complex for the covering $W = \bigcup U_i \cap W$ and the standard ordering on $\{1, \ldots, c\}$, see Cohomology, Section 20.23. By Cohomology, Lemma 20.23.7 this complex is homotopic to zero as soon as W is contained in $V(f_i)$ for some $1 \leq i \leq c$. This finishes the proof. □

0G7L **Remark 36.6.6.** Let X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ be as in Remark 36.6.4. Denote $\mathcal{F}^\bullet$ the complex (36.6.4.1). By Lemma 36.6.5 the cohomology sheaves of $\mathcal{F}^\bullet$ are supported on Z hence $\mathcal{F}^\bullet$ is an object of $D_Z(\mathcal{O}_X)$. On the other hand, the equality $\mathcal{F}^0 = \mathcal{F}$ determines a canonical map $\mathcal{F}^\bullet \to \mathcal{F}$ in $D(\mathcal{O}_X)$. As $i_* \circ R\mathcal{H}_Z$ is a right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$, see Cohomology, Lemma 20.34.2, we obtain a canonical commutative diagram

$$\begin{array}{ccc} \mathcal{F}^\bullet & \longrightarrow & \mathcal{F} \\ & \searrow \quad \nearrow & \\ & i_* R\mathcal{H}_Z(\mathcal{F}) & \end{array}$$

in $D(\mathcal{O}_X)$ functorial in the $\mathcal{O}_X$-module $\mathcal{F}$.

0G7M **Lemma 36.6.7.** *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. If $\mathcal{F}$ is quasi-coherent, then the complex (36.6.4.1) represents $i_* R\mathcal{H}_Z(\mathcal{F})$ in $D_Z(\mathcal{O}_X)$.*

Proof. Let us denote $\mathcal{F}^\bullet$ the complex (36.6.4.1). The statement of the lemma means that the map $\mathcal{F}^\bullet \to i_* R\mathcal{H}_Z(\mathcal{F})$ of Remark 36.6.6 is an isomorphism. Since $\mathcal{F}^\bullet$ is in $D_Z(\mathcal{O}_X)$ (see remark cited), we see that $i_* R\mathcal{H}_Z(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ by Cohomology, Lemma 20.34.2. The morphism $U_{i_0 \ldots i_p} \to X$ is affine as it is given over affine opens of X by inverting the function $f_{i_0} \ldots f_{i_p}$. Thus we see that

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}} = R(U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}}$$

by Cohomology of Schemes, Lemma 30.2.3 and the assumption that $\mathcal{F}$ is quasi-coherent. We conclude that $R\mathcal{H}_Z(\mathcal{F}_{i_0 \ldots i_p}) = 0$ by Cohomology, Lemma 20.34.7. Thus $i_* R\mathcal{H}_Z(\mathcal{F}^p) = 0$ for $p > 0$. Putting everything together we obtain

$$\mathcal{F}^\bullet = i_* R\mathcal{H}_Z(\mathcal{F}^\bullet) = i_* R\mathcal{H}_Z(\mathcal{F})$$

as desired. □

0G7N **Lemma 36.6.8.** *Let X be a scheme. Let $T \subset X$ be a closed subset which can locally be cut out by at most c elements of the structure sheaf. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i > c$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. This follows immediately from the local description of $R\mathcal{H}_T(\mathcal{F})$ given in Lemma 36.6.7. □

0G7P **Lemma 36.6.9.** *Let X be a scheme. Let $Z \subset X$ be a closed subset which can locally be cut out by a Koszul regular sequence having c elements. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i \neq c$ for every flat, quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. By the description of $R\mathcal{H}_Z(\mathcal{F})$ given in Lemma 36.6.7 this boils down to the following algebra statement: given a ring R, a Koszul regular sequence $f_1, \ldots, f_c \in R$, and a flat R-module M, the extended alternating Čech complex $M \to \bigoplus_{i_0} M_{f_{i_0}} \to \bigoplus_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1\ldots f_c}$ from More on Algebra, Section 15.29 only has cohomology in degree c. By More on Algebra, Lemma 15.31.1 we obtain the desired vanishing for the extended alternating Čech complex of R. Since the complex for M is obtained by tensoring this with the flat R-module M (More on Algebra, Lemma 15.29.2) we conclude. □

0G7Q **Remark 36.6.10.** With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. There is a canonical $\mathcal{O}_X|_Z$-linear map

$$c_{f_1,\ldots,f_c} : i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in $\mathcal{F}$. Namely, denoting $\mathcal{F}^\bullet$ the extended alternating Čech complex (36.6.4.1) we have the canonical map $\mathcal{F}^\bullet \to i_*R\mathcal{H}_Z(\mathcal{F})$ of Remark 36.6.6. This determines a canonical map

$$\text{Coker}\left(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c}\right) \longrightarrow i_*\mathcal{H}^c_Z(\mathcal{F})$$

on cohomology sheaves in degree c. Given a local section s of $\mathcal{F}$ we can consider the local section

$$\frac{s}{f_1 \ldots f_c}$$

of $\mathcal{F}_{1\ldots c}$. The class of this section in the cokernel displayed above depends only on s modulo the image of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$. Since $i_*i^*\mathcal{F}$ is equal to the cokernel of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$ we see that we get an $\mathcal{O}_X$-module map $i_*i^*\mathcal{F} \to i_*\mathcal{H}^c_Z(\mathcal{F})$. As i_* is fully faithful we get the map $c_{f_1,\ldots,f_c}$.

0G7R **Example 36.6.11.** Let $X = \text{Spec}(A)$ be affine, $f_1, \ldots, f_c \in A$, and let $\mathcal{F} = \widetilde{M}$ for some A-module M. The map $c_{f_1,\ldots,f_c}$ of Remark 36.6.10 can be described as the map

$$M/(f_1, \ldots, f_c)M \longrightarrow \text{Coker}\left(\bigoplus M_{f_1\ldots\hat{f}_i\ldots f_c} \to M_{f_1\ldots f_c}\right)$$

sending the class of $s \in M$ to the class of $s/f_1 \ldots f_c$ in the cokernel.

0G7S **Lemma 36.6.12.** *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. Let $a_{ji} \in \Gamma(X, \mathcal{O}_X)$ for $1 \leq i, j \leq c$ and set $g_j = \sum_{i=1,\ldots,c} a_{ji}f_i$. Assume $g_1, \ldots, g_c$ scheme theoretically cut out Z. If $\mathcal{F}$ is quasi-coherent, then*

$$c_{f_1,\ldots,f_c} = \det(a_{ji})c_{g_1,\ldots,g_c}$$

where $c_{f_1,\ldots,f_c}$ and $c_{g_1,\ldots,g_c}$ are as in Remark 36.6.10.

Proof. We will prove that $c_{f_1,\ldots,f_c}(s) = \det(a_{ij})c_{g_1,\ldots,g_c}(s)$ as global sections of $\mathcal{H}_Z(\mathcal{F})$ for any $s \in \mathcal{F}(X)$. This is sufficient since we then obtain the same result for section over any open subscheme of X. To do this, for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_q \leq c$ we denote $U_{i_0\ldots i_p} \subset X$, $V_{j_0\ldots j_q} \subset X$, and $W_{i_0\ldots i_p, j_0\ldots j_q} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible, $g_{j_0} \ldots g_{j_q}$ is invertible, and where $f_{i_0} \ldots f_{i_p}g_{j_0} \ldots g_{j_q}$ is invertible. We denote $\mathcal{F}_{i_0\ldots i_p}$, resp. $\mathcal{F}'_{j_0\ldots j_q}$ $\mathcal{F}''_{i_0\ldots i_p, j_0\ldots j_q}$ the

pushforward to X of the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$, resp. $V_{j_0\ldots j_q}$, resp. $W_{i_0\ldots i_p,j_0\ldots j_q}$. Then we obtain three extended alternating Čech complexes

$$\mathcal{F}^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \to \ldots$$

and

$$(\mathcal{F}')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

and

$$(\mathcal{F}'')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \oplus \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \oplus \bigoplus\nolimits_{i_0,j_0} \mathcal{F}''_{i_0,j_0} \oplus \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

whose differentials are those used in defining (36.6.4.1). There are maps of complexes

$$(\mathcal{F}'')^\bullet \to \mathcal{F}^\bullet \quad \text{and} \quad (\mathcal{F}'')^\bullet \to (\mathcal{F}')^\bullet$$

given by the projection maps on the terms (and hence inducing the identity map in degree 0). Observe that by Lemma 36.6.7 each of these complexes represents $i_*R\mathcal{H}_Z(\mathcal{F})$ and these maps represent the identity on this object. Thus it suffices to find an element

$$\sigma \in H^c((\mathcal{F}'')^\bullet(X))$$

mapping to $c_{f_1,\ldots,f_c}(s)$ and $\det(a_{ji})c_{g_1,\ldots,g_c}(s)$ by these two maps. It turns out we can explicitly give a cocycle for σ. Namely, we take

$$\sigma_{1\ldots c} = \frac{s}{f_1 \ldots f_c} \in \mathcal{F}_{1\ldots c}(X) \quad \text{and} \quad \sigma'_{1\ldots c} = \frac{\det(a_{ji})s}{g_1 \ldots g_c} \in \mathcal{F}'_{1\ldots c}(X)$$

and we take

$$\sigma_{i_0\ldots i_p,j_0\ldots j_{c-p-2}} = \frac{\lambda(i_0 \ldots i_p, j_0 \ldots j_{c-p-2})s}{f_{i_0} \ldots f_{i_p} g_{j_0} \ldots g_{j_{c-p-2}}} \in \mathcal{F}''_{i_0\ldots i_p,j_0\ldots j_{c-p-2}}(X)$$

where $\lambda(i_0 \ldots i_p, j_0 \ldots j_{c-p-2})$ is the coefficient of $e_1 \wedge \ldots \wedge e_c$ in the formal expression

$$e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_01}e_1 + \ldots + a_{j_0c}e_c) \wedge \ldots \wedge (a_{j_{c-p-2}1}e_1 + \ldots + a_{j_{c-p-2}c}e_c)$$

To verify that σ is a cocycle, we have to show for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_{c-p-1} \leq c$ that we have

$$\begin{aligned} 0 = & \sum\nolimits_{a=0,\ldots,p} (-1)^a f_{i_a} \lambda(i_0 \ldots \hat{i}_a \ldots i_p, j_0 \ldots j_{c-p-1}) \\ & + \sum\nolimits_{b=0,\ldots,c-p-1} (-1)^{p+b+1} g_{j_b} \lambda(i_0 \ldots i_p, j_0 \ldots \hat{j}_b \ldots j_{c-p-1}) \end{aligned}$$

The easiest way to see this is perhaps to argue that the formal expression

$$\xi = e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_01}e_1 + \ldots + a_{j_0c}e_c) \wedge \ldots \wedge (a_{j_{c-p-1}1}e_1 + \ldots + a_{j_{c-p-1}c}e_c)$$

is 0 as it is an element of the $(c+1)$st wedge power of the free module on $e_1, \ldots, e_c$ and that the expression above is the image of ξ under the Koszul differential sending $e_i \to f_i$. Some details omitted. □

0G7T **Lemma 36.6.13.** *Let X be a scheme. Let $Z \to X$ be a closed immersion of finite presentation whose conormal sheaf $\mathcal{C}_{Z/X}$ is locally free of rank c. Then there is a canonical map*

$$c : \wedge^c(\mathcal{C}_{Z/X})^\vee \otimes_{\mathcal{O}_Z} i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in the quasi-coherent module $\mathcal{F}$.

Proof. Follows from the construction in Remark 36.6.10 and the independence of the choice of generators of the ideal sheaf shown in Lemma 36.6.12. Some details omitted. □

0G7U **Remark 36.6.14.** Let $g : X' \to X$ be a morphism of schemes. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Set $f'_i = g^\sharp(f_i) \in \Gamma(X', \mathcal{O}_{X'})$. Denote $Z \subset X$, resp. $Z' \subset X'$ the closed subscheme cut out by $f_1, \ldots, f_c$, resp. $f'_1, \ldots, f'_c$. Then $Z' = Z \times_X X'$. Denote $h : Z' \to Z$ the induced morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Set $\mathcal{F}' = g^*\mathcal{F}$. In this setting, if $\mathcal{F}$ is quasi-coherent, then the diagram

$$\begin{array}{ccc} (i')^{-1}\mathcal{O}_{X'} \otimes_{h^{-1}i^{-1}\mathcal{O}_X} h^{-1}\mathcal{H}^c_Z(\mathcal{F}) & \longrightarrow & \mathcal{H}^c_{Z'}(\mathcal{F}') \\ \uparrow{\scriptstyle c_{f_1,\ldots,f_c}} & & {\scriptstyle c_{f'_1,\ldots,f'_c}}\uparrow \\ h^*i^*\mathcal{F} & \longrightarrow & (i')^*\mathcal{F}' \end{array}$$

is commutative where the top horizontal arrow is the map of Cohomology, Remark 20.34.12 on cohomology sheaves in degree c. Namely, denote $\mathcal{F}^\bullet$, resp. $(\mathcal{F}')^\bullet$ the extended alternating Čech complex constructed in Remark 36.6.4 using $\mathcal{F}, f_1, \ldots, f_c$, resp. $\mathcal{F}', f'_1, \ldots, f'_c$. Note that $(\mathcal{F}')^\bullet = g^*\mathcal{F}^\bullet$. Then, without assuming $\mathcal{F}$ is quasi-coherent, the diagram

$$\begin{array}{ccc} i'_*L(g|_{Z'})^*R\mathcal{H}_Z(\mathcal{F}) & \longrightarrow & i'_*R\mathcal{H}_{Z'}(Lg^*\mathcal{F}) \\ \| & & \downarrow \\ Lg^*i_*R\mathcal{H}_Z(\mathcal{F}) & & i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \\ \uparrow & & \uparrow \\ Lg^*(\mathcal{F}^\bullet) & \longrightarrow & (\mathcal{F}')^\bullet \end{array}$$

is commutative where $g|_{Z'} : (Z', (i')^{-1}\mathcal{O}_{X'}) \to (Z, i^{-1}\mathcal{O}_X)$ is the induced morphism of ringed spaces. Here the top horizontal arrow is given in Cohomology, Remark 20.34.12 as is the explanation for the equal sign. The arrows pointing up are from Remark 36.6.6. The lower horizontal arrow is the map $Lg^*\mathcal{F}^\bullet \to g^*\mathcal{F}^\bullet = (\mathcal{F}')^\bullet$ and the arrow pointing down is induced by $Lg^*\mathcal{F} \to g^*\mathcal{F} = \mathcal{F}'$. The diagram commutes because going around the diagram both ways we obtain two arrows $Lg^*\mathcal{F}^\bullet \to i'_*R\mathcal{H}_{Z'}(\mathcal{F}')$ whose composition with $i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \to \mathcal{F}'$ is the canonical map $Lg^*\mathcal{F}^\bullet \to \mathcal{F}'$. Some details omitted. Now the commutativity of the first diagram follows by looking at this diagram on cohomology sheaves in degree c and using that the construction of the map $i^*\mathcal{F} \to \mathrm{Coker}(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c})$ used in Remark 36.6.10 is compatible with pullbacks.

36.7. The coherator

08D6 Let X be a scheme. The *coherator* is a functor

$$Q_X : \mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$. It exists for any scheme X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties, Proposition 28.23.4.

Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(QCoh(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$ by Derived Categories, Lemma 13.30.3.

In this section we will study the functor RQ_X. In Section 36.21 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ (when it exists).

08D7 **Lemma 36.7.1.** *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 30.2.3. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.9. The diagram commutes by Lemma 36.5.1. □

08D8 **Lemma 36.7.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \to D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Schemes, Lemma 26.24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 28.23.4). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology, Lemma 20.28.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.30.3.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. □

08D9 **Lemma 36.7.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then*

(1) *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,*
(2) *$RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
(3) *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (36.3.0.1).*

Proof. The functor Q_X is the functor

$$\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$$

by Schemes, Lemma 26.7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 36.3.5. □

At this point we are ready to prove a criterion for when the functor $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence.

09T6 **Lemma 36.7.4.** *Let X be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor*

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_U)) & \xrightarrow[i_U]{} & D_{QCoh}(\mathcal{O}_U) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rj_*} \\ D(QCoh(\mathcal{O}_X)) & \xrightarrow{i_X} & D_{QCoh}(\mathcal{O}_X) \end{array}$$

Then the functor (36.3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ and let A be an object of $D(QCoh(\mathcal{O}_X))$. We have to show that the adjunction maps

$$A \to RQ_X(i_X(A)) \quad \text{and} \quad i_X(RQ_X(E)) \to E$$

are isomorphisms. Consider the hypothesis H_n: the adjunction maps above are isomorphisms whenever E and $i_X(A)$ are supported (Definition 36.6.1) on a closed subset of X which is contained in the union of n affine opens of X. We will prove H_n by induction on n.

Base case: $n = 0$. In this case $E = 0$, hence the map $i_X(RQ_X(E)) \to E$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of A are zero, i.e., $A = 0$ and $A \to RQ_X(i_X(A))$ is an isomorphism as well.

Induction step. Suppose that E and $i_X(A)$ are supported on a closed subset T of X contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. Consider the distinguished triangles

$$A \to \Phi(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where Φ is as in the statement of the lemma. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ \Phi = Rj_* \circ i_U$ by assumption and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(\Phi(A|_U))$ is a quasi-isomorphism over U. Hence $i_X(A')$ and E' are supported on the closed subset $T \setminus U$ of X which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for A' and E'. By Derived Categories, Lemma 13.4.3 it suffices to prove the maps

$$\Phi(A|_U) \to RQ_X(i_X(\Phi(A|_U))) \quad \text{and} \quad i_X(RQ_X(Rj_*E|_U)) \to Rj_*(E|_U)$$

are isomorphisms. By assumption and by Lemma 36.7.2 (the inclusion morphism $j : U \to X$ is flat, quasi-compact, and quasi-separated) we have

$$RQ_X(i_X(\Phi(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = \Phi(RQ_U(i_U(A|_U)))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(\Phi(RQ_U(E|_U))) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$A|_U \to RQ_U(i_U(A|_U)) \quad \text{and} \quad i_U(RQ_U(E|_U)) \to E|_U$$

are isomorphisms by Lemma 36.7.3. The result follows. □

08DB **Proposition 36.7.5.** *Let X be a quasi-compact scheme with affine diagonal. Then the functor (36.3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let $U \subset X$ be an affine open. Then the morphism $U \to X$ is affine by Morphisms, Lemma 29.11.12. Thus the assumption of Lemma 36.7.4 holds by Lemma 36.7.1 and we win. □

0CRX **Lemma 36.7.6.** *Let $f : X \to Y$ be a morphism of schemes. Assume X and Y are quasi-compact and have affine diagonal. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 36.7.5. Hence we can identify these categories (and similarly for other quasi-compact schemes with affine diagonal). The statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle

in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $U \subset X$ be an affine open. Since the diagonal of X is affine, the inclusion morphism $j : U \to X$ is affine (Morphisms, Lemma 29.11.12). Similarly, the composition $g = f \circ j : U \to Y$ is affine. Let $\mathcal{I}^\bullet$ be a K-injective complex in $QCoh(\mathcal{O}_U)$. Since $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_X)$, see Derived Categories, Lemma 13.31.9. It follows that

$$\Phi(j_*\mathcal{I}^\bullet) = f_*j_*\mathcal{I}^\bullet = g_*\mathcal{I}^\bullet$$

By Lemma 36.7.1 we see that $j_*\mathcal{I}^\bullet$ represents $Rj_*\mathcal{I}^\bullet$ and $g_*\mathcal{I}^\bullet$ represents $Rg_*\mathcal{I}^\bullet$. On the other hand, we have $Rf_* \circ Rj_* = Rg_*$. Hence $f_*j_*\mathcal{I}^\bullet$ represents $Rf_*(j_*\mathcal{I}^\bullet)$. We conclude that the lemma is true for any complex of the form $j_*\mathcal{G}^\bullet$ with $\mathcal{G}^\bullet$ a complex of quasi-coherent modules on U. (Note that if $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism, then $j_*\mathcal{G}^\bullet \to j_*\mathcal{I}^\bullet$ is a quasi-isomorphism as well since j_* is an exact functor on quasi-coherent modules.)

Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Let $T \subset X$ be a closed subset such that the support of $\mathcal{F}^p$ is contained in T for all p. We will use induction on the minimal number n of affine opens $U_1, \ldots, U_n$ such that $T \subset U_1 \cup \ldots \cup U_n$. The base case $n = 0$ is trivial. If $n \geq 1$, then set $U = U_1$ and denote $j : U \to X$ the open immersion as above. We consider the map of complexes $c : \mathcal{F}^\bullet \to j_*j^*\mathcal{F}^\bullet$. We obtain two short exact sequences of complexes:

$$0 \to \mathrm{Ker}(c) \to \mathcal{F}^\bullet \to \mathrm{Im}(c) \to 0$$

and

$$0 \to \mathrm{Im}(c) \to j_*j^*\mathcal{F}^\bullet \to \mathrm{Coker}(c) \to 0$$

The complexes $\mathrm{Ker}(c)$ and $\mathrm{Coker}(c)$ are supported on $T \setminus U \subset U_2 \cup \ldots \cup U_n$ and the result holds for them by induction. The result holds for $j_*j^*\mathcal{F}^\bullet$ by the discussion in the preceding paragraph. We conclude by looking at the distinguished triangles associated to the short exact sequences and using the initial remark of the proof. □

0CRY **Remark 36.7.7** (Warning)**.** Let X be a quasi-compact scheme with affine diagonal. Even though we know that $D(QCoh(\mathcal{O}_X)) = D_{QCoh}(\mathcal{O}_X)$ by Proposition 36.7.5 strange things can happen and it is easy to make mistakes with this material. One pitfall is to carelessly assume that this equality means derived functors are the same. For example, suppose we have a quasi-compact open $U \subset X$. Then we can consider the higher right derived functors

$$R^i(QCoh)\Gamma(U, -) : QCoh(\mathcal{O}_X) \to Ab$$

of the left exact functor $\Gamma(U, -)$. Since this is a universal δ-functor, and since the functors $H^i(U, -)$ (defined for all abelian sheaves on X) restricted to $QCoh(\mathcal{O}_X)$ form a δ-functor, we obtain canonical transformations

$$t^i : R^i(QCoh)\Gamma(U, -) \to H^i(U, -).$$

These transformations aren't in general isomorphisms even if $X = \mathrm{Spec}(A)$ is affine! Namely, we have $R^1(QCoh)\Gamma(U, \widetilde{I}) = 0$ if I an injective A-module by construction of right derived functors and the equivalence of $QCoh(\mathcal{O}_X)$ and Mod_A. But Examples, Lemma 110.47.2 shows there exists A, I, and U such that $H^1(U, \widetilde{I}) \neq 0$.

36.8. The coherator for Noetherian schemes

09T1 In the case of Noetherian schemes we can use the following lemma.

09T2 **Lemma 36.8.1.** *Let X be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.*

Proof. Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 26.12.5). By Cohomology of Schemes, Lemma 30.10.5 the section s corresponds to a map $\sigma : \mathcal{I}^n \to \mathcal{J}$ for some n. As $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$ we can extend σ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to s. □

09T3 **Lemma 36.8.2.** *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Since X and Y are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 28.5.4 and Schemes, Remark 26.21.18). Thus f_* preserve quasi-coherence, see Schemes, Lemma 26.24.1. Next, let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 28.23.4), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this by Lemma 36.4.2 it suffices to show that $\mathcal{I}^n$ is right f_*-acyclic for all $n \in \mathbf{Z}$. This is true because $\mathcal{I}^n$ is flasque by Lemma 36.8.1 and flasque modules are right f_*-acyclic by Cohomology, Lemma 20.12.5. □

09T4 **Proposition 36.8.3.** *Let X be a Noetherian scheme. Then the functor (36.3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows from Lemma 36.7.4 and Lemma 36.8.2. □

36.9. Koszul complexes

08CX Let A be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of A. We have defined the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ in More on Algebra, Definition 15.28.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \ldots, f_r) \to A \to K^\bullet(f_1, \ldots, f_r) \to I^\bullet(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^{\oplus r} \to A$ is given by multiplication by f_i on the ith factor. Hence $I^\bullet(f_1, \ldots, f_r) \to A$ factors as

$$I^\bullet(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet(f_1, \ldots, f_s)) \to H^0(I^\bullet(f_1, \ldots, f_s)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \ldots, f_r)) = H^{i-1}(K^\bullet(f_1, \ldots, f_r))$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of A there are canonical maps

$$I^\bullet(f_1 g_1, \ldots, f_r g_r) \to I^\bullet(f_1, \ldots, f_r) \quad \text{and} \quad K^\bullet(f_1 g_1, \ldots, f_r g_r) \to K^\bullet(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by g_i on the ith summand of $I^0(f_1 g_1, \ldots, f_r g_r) = A^{\oplus r}$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

08CY (36.9.0.1)
$$I^\bullet(f_1, \ldots, f_r) \leftarrow I^\bullet(f_1^2, \ldots, f_r^2) \leftarrow I^\bullet(f_1^3, \ldots, f_r^3) \leftarrow \ldots$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

08CZ **Situation 36.9.1.** Here A is a ring and $f_1, \ldots, f_r$ is a sequence of elements of A. We set $X = \operatorname{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1,\ldots,r} D(f_i)$ the given open covering of U.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on U.

08D0 **Lemma 36.9.2.** *In Situation 36.9.1. Let M be an A-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes*

$$\Psi : \operatorname{colim}_e \operatorname{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M) \longrightarrow \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

functorial in M where the differentials on the Hom*-complex are the contragredients of the differentials on $I^\bullet(f_1^e, \ldots, f_r^e)$.*

Proof. Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 20.23. As usual we view a p-cochain in $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ as an alternating function s on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0 \ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0} \ldots f_{i_p}} = \mathcal{F}(U_{i_0 \ldots i_p})$. On the other hand,

a p-cochain t in $\mathrm{Hom}^\bullet(I^\bullet(f_1^e, \ldots, f_r^e), M)$ is a map $t : \wedge^{p+1}(A^{\oplus r}) \to M$. Write $[i] \in A^{\oplus r}$ for the ith basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

For t as above we set

$$\Psi(t)_{i_0 \ldots i_p} = (-1)^p \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \ldots f_{i_p}^e}$$

It is clear that $\Psi(t)$ is an alternating cochain. The rule above is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \ldots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \ldots, f_r^{e+1}),$$

of (36.9.0.1) sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p}[i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \ldots f_{i_p}}$ in Algebra, Lemma 10.9.9 that the rule Ψ defines an isomorphism of cochain modules in degree p in the colimit. To finish the proof we have to show that the map is compatible with differentials. To see this, for t as above we compute

$$\begin{aligned} d(\Psi(t))_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j \Psi(t)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \\ &= (-1)^p \sum\nolimits_{j=0}^{p+1} (-1)^j t([i_0 \ldots \hat{i}_j \ldots i_{p+1}])(f_{i_0} \ldots \hat{f}_{i_j} \ldots f_{i_p})^{-e} \end{aligned}$$

Recall that the differentials on $I^\bullet(f_1^e, \ldots, f_r^e)$ are the negative of the differentials on $K^\bullet(f_1, \ldots, f_r)$. Thus

$$\begin{aligned} \Psi(d(t))_{i_0 \ldots i_{p+1}} &= (-1)^{p+1} d(t)([i_0, \ldots, i_{p+1}])(f_{i_0} \ldots f_{i_{p+1}})^{-e} \\ &= (-1)^{p+1} t(d([i_0, \ldots, i_{p+1}]))(f_{i_0} \ldots f_{i_{p+1}})^{-e} \\ &= (-1)^{p+1} t(-\sum\nolimits_{j=0}^{p+1} (-1)^j f_{i_j}^e [i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])(f_{i_0} \ldots f_{i_{p+1}})^{-e} \\ &= -(-1)^{p+1} \sum\nolimits_{j=0}^{p+1} (-1)^j t([i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])(f_{i_0} \ldots \hat{f}_{i_j} \ldots f_{i_p})^{-e} \end{aligned}$$

The two formulas agree concluding the proof. □

Suppose given a finite complex $I^\bullet$ of A-modules and a complex of A-modules $M^\bullet$. Then we have the corresponding Hom complex $\mathrm{Hom}^\bullet(I^\bullet, M^\bullet)$. This is a complex with degree n term given by

$$\bigoplus\nolimits_{p+q=n} \mathrm{Hom}_A(I^{-q}, M^p)$$

and differential as described in More on Algebra, Section 15.72. As our complex $I^\bullet$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 20.25.

08D1 **Lemma 36.9.3.** *In Situation 36.9.1. Let $M^\bullet$ be a complex of A-modules and denote $\mathcal{F}^\bullet$ the associated complex of $\mathcal{O}_X$-modules. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e \mathrm{Hom}^\bullet(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) \longrightarrow Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

functorial in $M^\bullet$.

Proof. Consider the double complex $F^{\bullet,\bullet}$ with terms $F^{p,q} = \mathcal{C}^p_{alt}(\mathcal{U}, \mathcal{F}^q)$ discussed in Cohomology, Section 20.25. Consider the double complex $G^{\bullet,\bullet}$ with terms $G^{p,q} = \operatorname{colim}_e \operatorname{Hom}_A(I^{-p}(f_1^e, \ldots, f_r^e), M^q)$ and differentials given by functoriality (without the intervention of signs). The maps $\psi^{p,q} : G^{p,q} \to F^{p,q}$ constructed in the proof of Lemma 36.9.2 are isomorphisms and compatible with the differentials d_1 (by the lemma) and d_2 (this is clear). However, the differentials d on the complexes on the left and right hand side of the arrow in the lemma have different signs. Namely, for $g \in G^{p,q}$ is given by

$$d(g) = d_2(g) - (-1)^{p+q} d_1(g)$$

(see More on Algebra, Section 15.72) and the differential for $f \in F^{p,q}$ is given by

$$d(f) = d_1(f) + (-1)^p d_2(f)$$

Thus we can fix the signs by multiplying $\psi^{p,q}$ by $(-1)^{pq+p(p-1)/2}$. □

08D2 **Lemma 36.9.4.** *In Situation 36.9.1. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

in $D(A)$ functorial in $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{B}$ be the set of affine opens of U. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 30.2.2) this is a special case of Cohomology, Lemma 20.40.2. □

In Situation 36.9.1 denote I_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $I^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 36.3.5. The maps (36.9.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to U. Thus we see that for every object E of $D(\mathcal{O}_X)$ there is a canonical map

08DC (36.9.4.1) $$\operatorname{colim}_e \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E)$$

constructed by sending a map $I_e \to E$ to its restriction to U and using that $\operatorname{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

08DD **Proposition 36.9.5.** *In Situation 36.9.1. For every object E of $D_{QCoh}(\mathcal{O}_X)$ the map (36.9.4.1) is an isomorphism.*

Proof. By Lemma 36.3.5 we may assume that E is given by a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$. Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of A-modules. By Lemmas 36.9.3 and 36.9.4 we have quasi-isomorphisms

$$\operatorname{colim}_e \operatorname{Hom}^\bullet(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

By More on Algebra, Lemma 15.74.2 and Equation (15.74.0.2) taking H^0 of the complex $\operatorname{Hom}^\bullet(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)$ computes Hom in $D(A)$. Thus taking H^0 on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 36.3.5 the lemma follows. □

In Situation 36.9.1 denote K_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $K^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 36.3.5. Thus we have distinguished triangles

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system (I_e). Moreover, there is a compatible system of maps

$$K_e \to H^0(K_e) = \mathcal{O}_X/(f_1^e, \ldots, f_r^e)$$

08E3 **Lemma 36.9.6.** *In Situation 36.9.1. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \ldots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that s is in the image of $H^0(X, K_e) \to H^0(X, E)$.*

Proof. Since U is covered by r affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 30.4.2). By Lemma 36.3.4 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 13.21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of E. We conclude that $s|_U = 0$. Think of s as a morphism $\mathcal{O}_X \to E$ in $D(\mathcal{O}_X)$. By Proposition 36.9.5 the composition $I_e \to \mathcal{O}_X \to E$ is zero for some e. By the distinguished triangle $I_e \to \mathcal{O}_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that s is the composition $\mathcal{O}_X \to K_e \to E$. □

36.10. Pseudo-coherent and perfect complexes

08E4 In this section we make the connection between the general notions defined in Cohomology, Sections 20.46, 20.47, 20.48, and 20.49 and the corresponding notions for complexes of modules in More on Algebra, Sections 15.65, 15.67, and 15.75.

08E5 **Lemma 36.10.1.** *Let X be a scheme. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$ and $H^m(E)$ is a quotient of a quasi-coherent $\mathcal{O}_X$-module. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally on X there exists a strictly perfect complex $\mathcal{E}^\bullet$ such that $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ for $i > m$ and $H^m(E)$ is a quotient of $H^m(\mathcal{E}^\bullet)$. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

08E7 **Lemma 36.10.2.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is an m-pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of A-modules.*

Proof. It is immediate from the definitions that if $M^\bullet$ is m-pseudo-coherent, so is E. To prove the converse, assume E is m-pseudo-coherent. As $X = \mathrm{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ inducing isomorphisms on H^j for $j > m$ and surjections on H^m. By Cohomology, Lemma 20.46.8 after refining the open

covering we may assume α_i is given by a map of complexes $\mathcal{E}_i^\bullet \to \widetilde{M^\bullet}|_{U_i}$ for each i. By Modules, Lemma 17.14.6 the terms $\mathcal{E}_i^n$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}_i^n$ is a finite free $\mathcal{O}_{U_i}$-module. From the definition it follows that $M_{f_i}^\bullet$ is an m-pseudo-coherent complex of A_{f_i}-modules. We conclude by applying More on Algebra, Lemma 15.65.14.

The case "pseudo-coherent" follows from the fact that E is pseudo-coherent if and only if E is m-pseudo-coherent for all m (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 15.65.5. □

08E8 **Lemma 36.10.3.** *Let X be a Noetherian scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

(1) *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
(2) *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we see that in both (1) and (2) the object E is bounded above. Thus the question is local on X and we may assume X is affine. Say $X = \mathrm{Spec}(A)$ for some Noetherian ring A. In this case E corresponds to a complex of A-modules $M^\bullet$ by Lemma 36.3.5. By Lemma 36.10.2 we see that E is m-pseudo-coherent if and only if $M^\bullet$ is m-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite A-module (Properties, Lemma 28.16.1). Thus the result follows from More on Algebra, Lemma 15.65.17. □

08E9 **Lemma 36.10.4.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

(1) *E has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.*
(2) *E has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.*

Proof. Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 36.3.5 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$. Then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat A-modules with $K^i = 0$ for $i \notin [a, b]$, see More on Algebra, Lemma 15.67.3. Then E is isomorphic to $\widetilde{K^\bullet}$. Since each $\widetilde{K^i}$ is a flat $\mathcal{O}_X$-module, we see that E has tor amplitude in $[a, b]$ by Cohomology, Lemma 20.48.3.

Assume that E has tor amplitude in $[a, b]$. Then E is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of A-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free A-modules. Then for any A-module N we have

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. □

0DHY **Lemma 36.10.5.** *Let $f : X \to S$ be a morphism of affine schemes corresponding to the ring map $R \to A$. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

(1) *E as an object of $D(f^{-1}\mathcal{O}_S)$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$ as an object of $D(R)$.*
(2) *E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ if and only if $M^\bullet$ has finite tor dimension as an object of $D(R)$.*

Proof. Consider a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Let $x \in X$ and $s = f(x) \in S$ be the corresponding points. Then $(f^{-1}\mathcal{O}_S)_x = \mathcal{O}_{S,s} = R_\mathfrak{p}$ and $E_x = M^\bullet_\mathfrak{q}$. Keeping this in mind we can see the equivalence as follows.

If $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules, then the same is true for the localization of $M^\bullet$ at any prime of A. Then we conclude by Cohomology, Lemma 20.48.5 that E has tor amplitude in $[a, b]$ as a complex of sheaves of $f^{-1}\mathcal{O}_S$-modules. Conversely, assume that E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_S)$. We conclude (using the last cited lemma) that $M^\bullet_\mathfrak{q}$ has tor amplitude in $[a, b]$ as a complex of $R_\mathfrak{p}$-modules for every prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. By More on Algebra, Lemma 15.67.15 we find that $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules. This finishes the proof of (1).

Since X is quasi-compact, if E locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_S$-modules, then actually E has tor amplitude in $[a, b]$ for some a, b as a complex of $f^{-1}\mathcal{O}_S$-modules. Thus (2) follows from (1). □

08EA **Lemma 36.10.6.** *Let X be a quasi-separated scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

(1) *E has tor amplitude in $[a, b]$, and*
(2) *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As X is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 26.24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \mathrm{Spec}(A)$ and let $M^\bullet$ be the complex of A-modules corresponding to $E|_U$ by Lemma 36.3.5. We have just shown that $M^\bullet \otimes^{\mathbf{L}}_A N$ has vanishing cohomology groups outside the range $[a, b]$, in other words $M^\bullet$ has tor amplitude in $[a, b]$. By Lemma 36.10.4 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. □

08EB **Lemma 36.10.7.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.*

Proof. This is a logical consequence of Lemmas 36.10.2 and 36.10.4, Cohomology, Lemma 20.49.5, and More on Algebra, Lemma 15.75.2. □

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

0A6H **Lemma 36.10.8.** *Let X be a scheme.*

(1) *If L is in $D^+_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$ and locally bounded below.*
(2) *If L is in $D_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.*
(3) *If $X = \mathrm{Spec}(A)$ is affine and $K, L \in D(A)$ then*

$$R\mathcal{H}om(\widetilde{K}, \widetilde{L}) = \widetilde{R\mathrm{Hom}_A(K, L)}$$

in the following two cases

(a) *K is pseudo-coherent and L is bounded below,*
(b) *K is perfect and L arbitrary.*

(4) *If $X = \mathrm{Spec}(A)$ and K, L are in $D(A)$, then the nth cohomology sheaf of $R\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf*

$$X \supset D(f) \longmapsto \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, L \otimes_A A_f)$$

for $f \in A$.

Proof. The construction of the internal hom in the derived category of $\mathcal{O}_X$ commutes with localization (see Cohomology, Section 20.42). Hence to prove (1) and (2) we may replace X by an affine open. By Lemmas 36.3.5, 36.10.2, and 36.10.7 in order to prove (1) and (2) it suffices to prove (3).

Part (3) follows from the computation of the internal hom of Cohomology, Lemma 20.46.11 by representing K by a bounded above (resp. finite) complex of finite projective A-modules and L by a bounded below (resp. arbitrary) complex of A-modules.

To prove (4) recall that on any ringed space the nth cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{D(U)}(A|_U, B|_U[n]) = \mathrm{Ext}^n_{D(\mathcal{O}_U)}(A|_U, B|_U)$$

See Cohomology, Section 20.42. On the other hand, the restriction of $\widetilde{K}$ to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for L. Hence (4) follows from the equivalence of categories of Lemma 36.3.5. □

0ATN **Lemma 36.10.9.** *Let X be a scheme. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

of Cohomology, Lemma 20.42.6 is an isomorphism in the following cases

(1) *M perfect, or*
(2) *K is perfect, or*
(3) *M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.*

Proof. Lemma 36.10.8 reduces cases (1) and (3) to the affine case which is treated in More on Algebra, Lemma 15.99.3. (You also have to use Lemmas 36.10.2, 36.10.7, and 36.10.4 to do the translation into algebra.) If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because we can work locally and reduce to the case that K is a finite complex of finite free modules in which case it is clear. □

36.11. Derived category of coherent modules

08E0 Let X be a locally Noetherian scheme. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.10 and Cohomology of Schemes, Lemma 30.9.2. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

08E1 (36.11.0.1) $$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

0FDA **Lemma 36.11.1.** *Let X be a Noetherian scheme. Then the functor*

$$D^-(Coh(\mathcal{O}_X)) \longrightarrow D^-_{Coh(\mathcal{O}_X)}(QCoh(\mathcal{O}_X))$$

is an equivalence.

Proof. Observe that $Coh(\mathcal{O}_X) \subset QCoh(\mathcal{O}_X)$ is a Serre subcategory, see Homology, Definition 12.10.1 and Lemma 12.10.2 and Cohomology of Schemes, Lemmas 30.9.2 and 30.9.3. On the other hand, if $\mathcal{G} \to \mathcal{F}$ is a surjection from a quasi-coherent $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X$-module, then there exists a coherent submodule $\mathcal{G}' \subset \mathcal{G}$ which surjects onto $\mathcal{F}$. Namely, we can write $\mathcal{G}$ as the filtered union of its coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job. Thus the lemma follows from Derived Categories, Lemma 13.17.4. □

0FDB **Proposition 36.11.2.** *Let X be a Noetherian scheme. Then the functors*

$$D^-(Coh(\mathcal{O}_X)) \longrightarrow D^-_{Coh}(\mathcal{O}_X) \quad \text{and} \quad D^b(Coh(\mathcal{O}_X)) \longrightarrow D^b_{Coh}(\mathcal{O}_X)$$

are equivalences.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} D^-(Coh(\mathcal{O}_X)) & \longrightarrow & D^-_{Coh}(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ D^-(QCoh(\mathcal{O}_X)) & \longrightarrow & D^-_{QCoh}(\mathcal{O}_X) \end{array}$$

By Lemma 36.11.1 the left vertical arrow is fully faithful. By Proposition 36.8.3 the bottom arrow is an equivalence. By construction the right vertical arrow is fully faithful. We conclude that the top horizontal arrow is fully faithful. If K is an object of $D^-_{Coh}(\mathcal{O}_X)$ then the object K' of $D^-(QCoh(\mathcal{O}_X))$ which corresponds to it by Proposition 36.8.3 will have coherent cohomology sheaves. Hence K' is in the essential image of the left vertical arrow by Lemma 36.11.1 and we find that the top horizontal arrow is essentially surjective. This finishes the proof for the bounded above case. The bounded case follows immediately from the bounded above case. □

08E2 **Lemma 36.11.3.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$.*

Proof. Consider the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

see Derived Categories, Lemma 13.21.3. By assumption and Cohomology of Schemes, Lemma 30.26.10 the sheaves $R^pf_*H^q(E)$ are coherent. Hence $R^{p+q}f_*E$ is coherent, i.e., $Rf_*E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 30.4.5 or from Lemma 36.4.1. □

0D0B **Lemma 36.11.4.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^+_{Coh}(\mathcal{O}_X)$ such*

that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^+_{Coh}(\mathcal{O}_S)$.

Proof. The proof is the same as the proof of Lemma 36.11.3. You can also deduce it from Lemma 36.11.3 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a}E \to E \to \tau_{\geq a+1}E \to \tau_{\leq a}E[1]$. □

0D0C **Lemma 36.11.5.** *Let X be a locally Noetherian scheme. If L is in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D^+_{Coh}(\mathcal{O}_X)$.*

Proof. It suffices to prove this when X is the spectrum of a Noetherian ring A. By Lemma 36.10.3 we see that K is pseudo-coherent. Then we can use Lemma 36.10.8 to translate the problem into the following algebra problem: for $L \in D^+_{Coh}(A)$ and K in $D^-_{Coh}(A)$, then $R\mathrm{Hom}_A(K, L)$ is in $D^+_{Coh}(A)$. Since L is bounded below and K is bounded above there is a convergent spectral sequence

$$\mathrm{Ext}^p_A(K, H^q(L)) \Rightarrow \mathrm{Ext}^{p+q}_A(K, L)$$

and there are convergent spectral sequences

$$\mathrm{Ext}^i_A(H^{-j}(K), H^q(L)) \Rightarrow \mathrm{Ext}^{i+j}_A(K, H^q(L))$$

See Injectives, Remarks 19.13.9 and 19.13.11. This finishes the proof as the modules $\mathrm{Ext}^p_A(M, N)$ are finite for finite A-modules M, N by Algebra, Lemma 10.71.9. □

0FXU **Lemma 36.11.6.** *Let X be a Noetherian scheme. Let E in $D(\mathcal{O}_X)$ be perfect. Then*

1. *E is in $D^b_{Coh}(\mathcal{O}_X)$,*
2. *if L is in $D_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D_{Coh}(\mathcal{O}_X)$,*
3. *if L is in $D^b_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^b_{Coh}(\mathcal{O}_X)$,*
4. *if L is in $D^+_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^+_{Coh}(\mathcal{O}_X)$,*
5. *if L is in $D^-_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^-_{Coh}(\mathcal{O}_X)$.*

Proof. Since X is quasi-compact, each of these statements can be checked over the members of any open covering of X. Thus we may assume E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite free modules, see Cohomology, Lemma 20.49.3. In this case each of the statements is clear as both $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ and $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ can be computed on the level of complexes using $\mathcal{E}^\bullet$, see Cohomology, Lemmas 20.46.9 and 20.26.9. Some details omitted. □

0D0D **Lemma 36.11.7.** *Let A be a Noetherian ring. Let X be a proper scheme over A. For L in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, the A-modules $\mathrm{Ext}^n_{\mathcal{O}_X}(K, L)$ are finite.*

Proof. Recall that

$$\mathrm{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\mathrm{Spec}(A), Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology, Lemma 20.42.1 and Cohomology, Section 20.13. Thus the result follows from Lemmas 36.11.5 and 36.11.4. □

0FDC **Lemma 36.11.8.** *Let X be a locally Noetherian regular scheme. Then every object of $D^b_{Coh}(\mathcal{O}_X)$ is perfect. If X is quasi-compact, i.e., Noetherian regular, then conversely every perfect object of $D(\mathcal{O}_X)$ is in $D^b_{Coh}(\mathcal{O}_X)$.*

Proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_X)$. To check that K is perfect, we may work affine locally on X (see Cohomology, Section 20.49). Then K is perfect by Lemma 36.10.7 and More on Algebra, Lemma 15.75.14. The converse is Lemma 36.11.6. □

36.12. Descent finiteness properties of complexes

09UC This section is the analogue of Descent, Section 35.7 for objects of the derived category of a scheme. The easiest such result is probably the following.

09UD **Lemma 36.12.1.** *Let $f : X \to Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then E has tor-amplitude in $[a, b]$ if and only if Lf^*E has tor-amplitude in $[a, b]$.*

Proof. Pullback always preserves tor-amplitude, see Cohomology, Lemma 20.48.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 20.48.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 10.39.17. Thus the result follows from More on Algebra, Lemma 15.67.17. □

09UE **Lemma 36.12.2.** *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 15.65.15 via Lemmas 36.3.5 and 36.10.2. □

09UF **Lemma 36.12.3.** *Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of finite presentation.

Since f is flat the derived functor Lf^* is just given by f^* and f^* is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since Lf^*E is m-pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since f is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type $\mathcal{O}_Y$-module by Cohomology, Lemma 20.47.9. Then by Descent, Lemma 35.7.2 the $\mathcal{O}_X$-module $H^i(E)$ is of finite type. Thus,

after replacing X by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let C be the cone of α in $D(\mathcal{O}_X)$. Pulling back to Y and using Cohomology, Lemma 20.47.4 we find that Lf^*C is m-pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on i we see that C is m-pseudo-coherent. Using Cohomology, Lemma 20.47.4 again we conclude. □

09UG **Lemma 36.12.4.** *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then E is perfect if and only if each Lf_i^*E is perfect.*

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 20.49.6. Conversely, assume that Lf_i^*E is perfect for all i. Then the cohomology sheaves of each Lf_i^*E are quasi-coherent, see Lemma 36.10.1 and Cohomology, Lemma 20.49.5. Since the morphisms f_i is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of E are quasi-coherent by Descent, Proposition 35.5.2. Having said this the lemma follows formally from Cohomology, Lemma 20.49.5 and Lemmas 36.12.1 and 36.12.2. □

09VA **Lemma 36.12.5.** *Let $i : Z \to X$ be a morphism of ringed spaces such that i is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an $\mathcal{O}_X$-module. Let $E \in D(\mathcal{O}_Z)$. Then E is m-pseudo-coherent if and only if Ri_*E is m-pseudo-coherent.*

Proof. Throughout this proof we will use that i_* is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 17.6.1.

Assume E is m-pseudo-coherent. Let $x \in X$. We will find a neighbourhood of x such that i_*E is m-pseudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of x in Z. After shrinking X we may assume E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an $\mathcal{O}_Z$-module of finite type, see Cohomology, Lemma 20.47.9. Thus we may choose, after shrinking X, a map $\mathcal{O}_Z^{\oplus n}[-p] \to E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that C is m-pseudo-coherent by Cohomology, Lemma 20.47.4. By induction we see that i_*C is m-pseudo-coherent on X. Since $i_*\mathcal{O}_Z$ is m-pseudo-coherent on X as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 20.47.4 that i_*E is m-pseudo-coherent.

Assume that i_*E is m-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of z such that E is m-pseudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of z in Z. After shrinking X we may assume i_*E and hence E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 20.47.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing

E. We may choose, after shrinking X, a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 20.47.4 we conclude that i_*C is m-pseudo-coherent. By induction we conclude that C is m-pseudo-coherent. By Cohomology, Lemma 20.47.4 again we conclude that E is m-pseudo-coherent. □

09VB **Lemma 36.12.6.** *Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module[2]. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent.*

Proof. This is a translation of More on Algebra, Lemma 15.65.11 into the language of schemes. To do the translation, use Lemmas 36.3.5 and 36.10.2. □

36.13. Lifting complexes

08EC Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as Rj_* is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

08ED **Lemma 36.13.1.** *Let X be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors*

$$D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_U)$$

are essentially surjective. If X is quasi-compact, then the functors

$$D^-_{QCoh}(\mathcal{O}_X) \to D^-_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^b_{QCoh}(\mathcal{O}_X) \to D^b_{QCoh}(\mathcal{O}_U)$$

are essentially surjective.

Proof. The argument preceding the lemma applies for the first case because Rj_* maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.1. It is clear that Rj_* maps $D^+_{QCoh}(\mathcal{O}_U)$ into $D^+_{QCoh}(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 36.4.1 guarantees that Rj_* maps $D^-_{QCoh}(\mathcal{O}_U)$ into $D^-_{QCoh}(\mathcal{O}_X)$ if X is quasi-compact. Combining these two we obtain the last statement. □

0G48 **Lemma 36.13.2.** *Let X be a Noetherian scheme and let $j : U \to X$ be an open immersion. The functor $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_U)$ is essentially surjective.*

Proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_U)$. By Proposition 36.11.2 we can represent K by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_U$-modules. Say $\mathcal{F}^i = 0$ for $i \notin [a, b]$ for some $a \leq b$. Since j is quasi-compact and separated, the terms of the bounded complex $j_*\mathcal{F}^\bullet$ are quasi-coherent modules on X, see Schemes, Lemma 26.24.1. We inductively pick a coherent submodule $\mathcal{G}^i \subset j_*\mathcal{F}^i$ as follows. For $i = a$ we pick any coherent submodule $\mathcal{G}^a \subset j_*\mathcal{F}^a$ whose restriction to U is $\mathcal{F}^a$. This is possible

[2]This means that f is pseudo-coherent, see More on Morphisms, Lemma 37.60.8.

by Properties, Lemma 28.22.2. For $i > a$ we first pick any coherent submodule $\mathcal{H}^i \subset j_*\mathcal{F}^i$ whose restriction to U is $\mathcal{F}^i$ and then we set $\mathcal{G}^i = \mathrm{Im}(\mathcal{H}^i \oplus \mathcal{G}^{i-1} \to j_*\mathcal{F}^i)$. It is clear that $\mathcal{G}^\bullet \subset j_*\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules whose restriction to U is $\mathcal{F}^\bullet$ as desired. □

08EE **Lemma 36.13.3.** *Let X be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object E of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free $\mathcal{O}_X$-modules whose restriction to U is isomorphic to E.*

Proof. By Lemma 36.10.1 we see that E is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 36.13.1 we may assume $E = E'|U$ for some object E' of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \mathrm{Spec}(A)$. By Lemma 36.3.5 we can find a complex $M^\bullet$ of A-modules whose associated complex of $\mathcal{O}_X$-modules is a representative of E'.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 36.10.2 the complexes $M^\bullet_{f_j}$ are pseudo-coherent complexes of A_{f_j}-modules. Let n be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each F^i is a finite free R-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{array}{ccccccc} & & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots \end{array}$$

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on n we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all i. The lemma will follow by taking $\widetilde{F^\bullet}$.

The induction step will be to extend the diagram above by adding F^{n-1}. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 15.65.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 15.65.3 we see that $H^{n-1}(C^\bullet_{f_j})$ is a finite A_{f_j}-module. Choose a finite free A-module F^{n-1} and an A-module $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccccc} & & F^{n-1} & \xrightarrow{d^{n-1}} & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc} (F^n \to \ldots) & \longrightarrow & (F^{n-1} \to \ldots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \to \ldots)[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\ (F^n \to \ldots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \to \ldots)[1] \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology localized at f_j in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. □

The following two lemmas should probably go somewhere else.

08EF **Lemma 36.13.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^b_{QCoh}(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0$ for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and every $n \geq n_0$.*

Proof. Recall that $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U \cap V}$ for some bound n_0, then the result holds for E with bound $n_0 + 1$. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 30.4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. Choose a complex of A-modules $M^\bullet$ whose associated complex of quasi-coherent modules represents E, see Lemma 36.3.5. Write $\mathcal{E} = \widetilde{P}$ for some A-module P. Since $\mathcal{E}$ is finite locally free, we see that P is a finite projective A-module. We have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \mathrm{Hom}_{D(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_{K(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_A(P, H^n(M^\bullet)) \end{aligned}$$

The first equality by Lemma 36.3.5, the second equality by Derived Categories, Lemma 13.19.8, and the final equality because $\mathrm{Hom}_A(P, -)$ is an exact functor. As E and hence $M^\bullet$ is bounded we get zero for all sufficiently large n. □

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

09M4 **Lemma 36.13.5.** *Let X be a quasi-compact and quasi-separated scheme. Let K be a perfect object of $D(\mathcal{O}_X)$. Then*

1. *there exist integers $a \leq b$ such that for any $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$ we have $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$, and*
2. *if L is bounded, then $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L)$ is zero for all but finitely many n.*

Proof. Part (2) follows from (1) as $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology, Lemma 20.50.5. Note that $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 36.3.9 and 36.10.1 (to see that a

perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

shows that $H^j(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ is zero if $H^q(L) = 0$ for $q \in [j - b, j - a]$. Let N be the integer d of Cohomology of Schemes, Lemma 30.4.4. Then $H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ H^{-N+1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ \ldots,\ H^0(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

are zero. Namely, by the lemma cited and Lemma 36.3.4, we have

$$H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$ which is zero by Derived Categories, Lemma 13.16.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b - N, -a]$. □

08EG **Lemma 36.13.6.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object E of $D(\mathcal{O}_U)$ there exists an integer r and a finite locally free sheaf $\mathcal{F}$ on U such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Say $X = \mathrm{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 20.49.5. By Lemma 36.13.3 we can find a bounded above complex $\mathcal{F}^\bullet$ of finite free A-modules such that E is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By Cohomology, Lemma 20.49.5 and since U is quasi-compact, we see that E has finite tor dimension, say E has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{K} = \mathrm{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \mathrm{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).$$

Since E has tor amplitude in $[a, b]$ we see that $\mathcal{F} = \mathcal{K}|_U$ is flat (Cohomology, Lemma 20.48.2). Hence $\mathcal{F}$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 28.20.2). It follows that

$$\mathcal{F} \to \mathcal{F}^r|_U \to \mathcal{F}^{r+1}|_U \to \ldots$$

is a strictly perfect complex on U representing E. On the other hand, the complex $P = (\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)$ is a perfect complex on X. Using stupid truncations we obtain a distinguished triangle

$$P|_U \to E \to \mathcal{F}[-r - 1] \to (P|_U)[1]$$

If the map $E \to \mathcal{F}[-r-1]$ is zero in $D(\mathcal{O}_U)$, then $P|_U = \mathcal{F}[-r-2] \oplus E$, see Derived Categories, Lemma 13.4.11. This will be true for $r \ll 0$ for example by Lemma 36.13.5. □

08EH **Lemma 36.13.7.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let E, E' be objects of $D_{QCoh}(\mathcal{O}_X)$ with E perfect. For every map $\alpha : E|_U \to E'|_U$ there exist maps*

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

of complexes on X with E_1 perfect such that $\beta : E_1 \to E$ restricts to an isomorphism on U and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I$ for some perfect complex I on X.

Proof. Write $X = \mathrm{Spec}(A)$. Write $U = D(f_1) \cup \ldots \cup D(f_r)$. Choose finite complex of finite projective A-modules $M^\bullet$ representing E (Lemma 36.10.7). Choose a complex of A-modules $(M')^\bullet$ representing E' (Lemma 36.3.5). In this case the complex $H^\bullet = \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of A-modules whose associated complex of quasi-coherent $\mathcal{O}_X$-modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 20.46.9. Then α determines an element s of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 20.42.1. There exists an e and a map

$$\xi : I^\bullet(f_1^e, \ldots, f_r^e) \to \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to s, see Proposition 36.9.5. Letting E_1 be the object corresponding to complex of quasi-coherent $\mathcal{O}_X$-modules associated to

$$\mathrm{Tot}(I^\bullet(f_1^e, \ldots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \to E$ using the canonical map $I^\bullet(f_1^e, \ldots, f_r^e) \to A$ and $E_1 \to E'$ using ξ and Cohomology, Lemma 20.42.1. □

08EI **Lemma 36.13.8.** *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. By Lemma 36.13.6 we can find a perfect object E of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. By Lemma 36.13.7 we can find a morphism of perfect complexes $\alpha : E_1 \to E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \mathrm{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F$$

Then the cone on α is a solution. □

08EJ **Lemma 36.13.9.** *Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \to E'$ in $D_{QCoh}(\mathcal{O}_X)$ such that*

(1) *E is perfect, and*
(2) *E' is supported on $T = V(f)$*

there exists an $n \geq 0$ such that $f^n\alpha = 0$.

Proof. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U \cap V}$, then the result holds for f. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 30.4.1.

Let $X = \mathrm{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 36.3.5 without further mention. Represent E by a finite complex of finite projective A-modules $P^\bullet$. This is possible by Lemma 36.10.7. Let t be the largest integer such that P^t is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1}P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective A-module P in degree 0. Represent E' by a complex $M^\bullet$ of A-modules. Then α corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2)

we see that every element of $H^0(M^\bullet)$ is annihilated by a power of f. Since P is a finite A-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some n as desired. $\square$

08EK **Lemma 36.13.10.** *Let X be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object E of $D(\mathcal{O}_X)$ supported in T.*

Proof. Say $T = V(g_1, \ldots, g_s)$. After replacing g_j by a power we may assume multiplication by g_j is zero on F, see Lemma 36.13.9. Choose E as in Lemma 36.13.8. Note that $g_j : E \to E$ restricts to zero on U. Choose a distinguished triangle

$$E \xrightarrow{g_1} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 13.4.11 the object C_1 restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on U. Moreover, $g_1 : C_1 \to C_1$ has square zero by Derived Categories, Lemma 13.4.5. Namely, the diagram

$$\begin{array}{ccccc} E & \longrightarrow & C_1 & \longrightarrow & E[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle 0} \\ E & \longrightarrow & C_1 & \longrightarrow & E[1] \end{array}$$

is commutative since the compositions $E \xrightarrow{g_1} E \to C_1$ and $C_1 \to E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting C_{i+1} equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex C_s on X supported on T whose restriction to U gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

Choose morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 36.13.7 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s] \to F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for F where it is zero. By Lemma 36.13.7 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex I on X. Hence the nullity of $g_j^2 \mathrm{id}_{C_s}$ implies the same thing for C'. Thus C' is supported on T as well. Then $\mathrm{Cone}(\gamma)$ is a solution. $\square$

A special case of the following lemma can be found in [**Nee96**].

09IM **Lemma 36.13.11.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retrocompact in X. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where P is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible α and $\beta|_U$.*

Proof. Since X is quasi-compact there exists an integer m such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens V_j of X. Arguing by induction on m we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with V affine. By Lemma 36.13.10 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported

on $T \cap V$ and an isomorphism $Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V}$. By Lemma 36.13.7 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V} \longrightarrow P|_{U\cap V} \longrightarrow E|_{U\cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

09IN **Remark 36.13.12.** The proof of Lemma 36.13.11 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and P above any given degree.

36.14. Approximation by perfect complexes

08EL In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be "approximated" by perfect complexes.

08EM **Definition 36.14.1.** Let X be a scheme. Consider triples (T, E, m) where

(1) $T \subset X$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple (T, E, m), then

(1) E is m-pseudo-coherent, see Cohomology, Definition 20.47.1, and
(2) the cohomology sheaves $H^i(E)$ are supported on T for $i \geq m$.

Moreover, the "support" of a perfect complex is a closed subscheme whose complement is retrocompact in X (details omitted). Hence we cannot expect approximation to hold without this assumption on T. This partly explains the conditions in the following definition.

08EN **Definition 36.14.2.** Let X be a scheme. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in X there exists an integer r such that for every triple (T, E, m) as in Definition 36.14.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to "E is m-pseudo-coherent" by carefully analyzing the arguments below.

08EP **Lemma 36.14.3.** *Let X be a scheme. Let $U \subset X$ be an open subscheme. Let (T, E, m) be a triple as in Definition 36.14.1. If*

(1) *$T \subset U$,*
(2) *approximation holds for $(T, E|_U, m)$, and*
(3) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over U, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of (T, E, m) over X. See Cohomology, Lemmas 20.33.6 and 20.49.10. □

08EQ **Lemma 36.14.4.** *Let X be an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 36.14.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Say $X = \mathrm{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 36.10.2 and the definitions.) Let (T, E, m) be a triple as in the lemma. Let t be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on t. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 20.47.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 28.16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 36.3.5) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that s is in the image of $H^0(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma 36.9.6. Taking a finite direct sum of these maps we obtain a map $P \to E$ where P is a perfect complex supported on T, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then E' is m-pseudo-coherent (Cohomology, Lemma 20.47.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of (T, E', m). Fit the composition $P' \to E' \to P[1]$ into a distinguished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$ and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} P & \longrightarrow & P'' & \longrightarrow & P' & \longrightarrow & P[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & P[1] \end{array}$$

using TR3. Then P'' is a perfect complex (Cohomology, Lemma 20.49.7) supported on T. An easy diagram chase shows that $P'' \to E$ is the desired approximation. □

08ER **Lemma 36.14.5.** *Let X be a scheme. Let $X = U \cup V$ be an open covering with U quasi-compact, V affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation holds on X.*

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let r_U be the integer of Definition 36.14.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on T. Let r' be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m - r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 36.14.4 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 36.14.3 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. By Lemma 36.13.10 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 36.13.7 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m - r)$-pseudo-coherent (Cohomology, Lemma 20.47.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then R'' is a perfect complex (Cohomology, Lemma 20.49.7) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

08ES **Theorem 36.14.6.** *Let X be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 30.4.1 and Lemmas 36.14.5 and 36.14.4. □

36.15. Generating derived categories

09IP In this section we prove that the derived category $D_{QCoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [**BV03**].

09IQ **Lemma 36.15.1.** *Let X be a quasi-compact and quasi-separated scheme. Let U be a quasi-compact open subscheme. Let P be a perfect object of $D(\mathcal{O}_U)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 36.13.11. □

09IR **Lemma 36.15.2.** *In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent* [**BN93**, Proposition 6.1]

(1) $E = Rj_*(E|_U)$, *and*
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ *for all* $n \in \mathbf{Z}$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all n as $K|_U = 0$. Thus it suffices to prove the result for N. In other words, we may assume that E restricts to zero on U. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'+e_i''}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i''}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the K replaced by K_e as in Lemma 36.9.6. Thus our lemma follows immediately from that one and the fact that E is determined by the complex of A-modules $R\Gamma(X, E)$, see Lemma 36.3.5. □

09IS **Theorem 36.15.3.** *Let X be a quasi-compact and quasi-separated scheme. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

(1) $E = 0$, *and*
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ *for all* $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 36.3.5.

Assume that $X = U \cup V$ is an open covering with U quasi-compact such that the theorem holds for U and V is an affine open. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 36.15.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a closed subset of V with $V \setminus Z$ quasi-compact. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on $Z \subset V$ closed, the pushforward $K' = R(V \to X)_* K$

is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 20.49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 20.33.6 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 36.15.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_* E|_{U \cap V}$. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

The following result is an strengthening of Theorem 36.15.3 proved using exactly the same methods. Recall that for a closed subset T of a scheme X we denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition 36.6.1). We similarly denote $D_{QCoh,T}(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of those complexes whose cohomology sheaves are quasi-coherent and are supported on T.

0A9A **Lemma 36.15.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.* [**Rou08**, Theorem 6.8]

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. In this case there exist $f_1, \ldots, f_r \in A$ such that $T = V(f_1, \ldots, f_r)$. Let K be the Koszul complex on $f_1, \ldots, f_r$ as in Lemma 36.15.2. Then K is a perfect object with cohomology supported on T and hence a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$. On the other hand, if $E \in D_{QCoh,T}(\mathcal{O}_X)$ and $\mathrm{Hom}(K, E[n]) = 0$ for all n, then Lemma 36.15.2 tells us that $E = Rj_*(E|_{X \setminus T}) = 0$. Hence K generates $D_{QCoh,T}(\mathcal{O}_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 13.36.3).

Assume that $X = U \cup V$ is an open covering with V affine and U quasi-compact such that the lemma holds for U. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T \cap U}(\mathcal{O}_U)$. Using Lemma 36.13.11 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then Z is a closed subset of V such that $V \setminus Z$ is quasi-compact. As X is quasi-separated, it follows that $Z \cap T$ is a closed subset of V such that $W = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $Z \cap T = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_* K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 20.49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 20.33.6 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 36.15.2 we have $E|_V = Rj_* E|_W$ where $j : W \to V$ is the inclusion. Picture

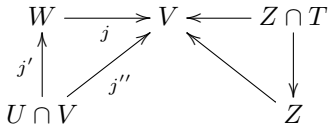

Since E is supported on T we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

where the second equality is part (1) of Cohomology, Lemma 20.33.6. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

36.16. An example generator

0BQQ In this section we prove that the derived category of projective space over a ring is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf.

The following lemma says that $\bigoplus_{n\geq 0} \mathcal{L}^{\otimes -n}$ is a generator if $\mathcal{L}$ is ample.

0BQR **Lemma 36.16.1.** *Let X be a scheme and $\mathcal{L}$ an ample invertible $\mathcal{O}_X$-module. If K is a nonzero object of $D_{QCoh}(\mathcal{O}_X)$, then for some $n \geq 0$ and $p \in \mathbf{Z}$ the cohomology group $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ is nonzero.*

Proof. Recall that as X has an ample invertible sheaf, it is quasi-compact and separated (Properties, Definition 28.26.1 and Lemma 28.26.7). Thus we may apply Proposition 36.7.5 and represent K by a complex $\mathcal{F}^\bullet$ of quasi-coherent modules. Pick any p such that $\mathcal{H}^p = \mathrm{Ker}(\mathcal{F}^p \to \mathcal{F}^{p+1})/\mathrm{Im}(\mathcal{F}^{p-1} \to \mathcal{F}^p)$ is nonzero. Choose a point $x \in X$ such that the stalk $\mathcal{H}^p_x$ is nonzero. Choose an $n \geq 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x. Choose $\tau \in \mathcal{H}^p(X_s)$ which maps to a nonzero element of the stalk $\mathcal{H}^p_x$; this is possible as $\mathcal{H}^p$ is quasi-coherent and X_s is affine. Since taking sections over X_s is an exact functor on quasi-coherent modules, we can find a section $\tau' \in \mathcal{F}^p(X_s)$ mapping to zero in $\mathcal{F}^{p+1}(X_s)$ and mapping to τ in $\mathcal{H}^p(X_s)$. By Properties, Lemma 28.17.2 there exists an m such that $\tau' \otimes s^{\otimes m}$ is the image of a section $\tau'' \in \Gamma(X, \mathcal{F}^p \otimes \mathcal{L}^{\otimes mn})$. Applying the same lemma once more, we find $l \geq 0$ such that $\tau'' \otimes s^{\otimes l}$ maps to zero in $\mathcal{F}^{p+1} \otimes \mathcal{L}^{\otimes (m+l)n}$. Then τ'' gives a nonzero class in $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{(m+l)n})$ as desired. □

0BQS **Lemma 36.16.2.** *Let A be a ring. Let $X = \mathbf{P}^n_A$. For every $a \in \mathbf{Z}$ there exists an exact complex*

$$0 \to \mathcal{O}_X(a) \to \ldots \to \mathcal{O}_X(a+i)^{\oplus \binom{n+1}{i}} \to \ldots \to \mathcal{O}_X(a+n+1) \to 0$$

of vector bundles on X.

Proof. Recall that $\mathbf{P}^n_A$ is $\mathrm{Proj}(A[X_0, \ldots, X_n])$, see Constructions, Definition 27.13.2. Consider the Koszul complex

$$K_\bullet = K_\bullet(A[X_0, \ldots, X_n], X_0, \ldots, X_n)$$

over $S = A[X_0, \ldots, X_n]$ on $X_0, \ldots, X_n$. Since $X_0, \ldots, X_n$ is clearly a regular sequence in the polynomial ring S, we see that (More on Algebra, Lemma 15.30.2) that the Koszul complex $K_\bullet$ is exact, except in degree 0 where the cohomology is $S/(X_0, \ldots, X_n)$. Note that $K_\bullet$ becomes a complex of graded modules if we put the generators of K_i in degree $+i$. In other words an exact complex

$$0 \to S(-n-1) \to \ldots \to S(-n-1+i)^{\oplus \binom{n}{i}} \to \ldots \to S \to S/(X_0, \ldots, X_n) \to 0$$

Applying the exact functor $\tilde{}$ functor of Constructions, Lemma 27.8.4 and using that the last term is in the kernel of this functor, we obtain the exact complex

$$0 \to \mathcal{O}_X(-n-1) \to \ldots \to \mathcal{O}_X(-n-1+i)^{\oplus \binom{n+1}{i}} \to \ldots \to \mathcal{O}_X \to 0$$

Twisting by the invertible sheaves $\mathcal{O}_X(n+a+1)$ we get the exact complexes of the lemma. □

0A9V **Lemma 36.16.3.** *Let A be a ring. Let $X = \mathbf{P}^n_A$. Then*

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$$

is a generator (Derived Categories, Definition 13.36.3) of $D_{QCoh}(X)$.

Proof. Let $K \in D_{QCoh}(\mathcal{O}_X)$. Assume $\mathrm{Hom}(E, K[p]) = 0$ for all $p \in \mathbf{Z}$. We have to show that $K = 0$. By Derived Categories, Lemma 13.36.4 we see that $\mathrm{Hom}(E', K[p])$ is zero for all $E' \in \langle E \rangle$ and $p \in \mathbf{Z}$. By Lemma 36.16.2 applied with $a = -n-1$ we see that $\mathcal{O}_X(-n-1) \in \langle E \rangle$ because it is quasi-isomorphic to a finite complex whose terms are finite direct sums of summands of E. Repeating the argument with $a = -n-2$ we see that $\mathcal{O}_X(-n-2) \in \langle E \rangle$. Arguing by induction we find that $\mathcal{O}_X(-m) \in \langle E \rangle$ for all $m \geq 0$. Since

$$\mathrm{Hom}(\mathcal{O}_X(-m), K[p]) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(m)) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m})$$

we conclude that $K = 0$ by Lemma 36.16.1. (This also uses that $\mathcal{O}_X(1)$ is an ample invertible sheaf on X which follows from Properties, Lemma 28.26.12.) □

0BQT **Remark 36.16.4.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 36.15.3). Then the following are equivalent

(1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
(2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
(3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_*. These conditions hold if f is affine (Lemma 36.5.2) or if f is an open immersion, or if f is a composition of such. We conclude that

(1) if X is a quasi-affine scheme then $\mathcal{O}_X$ is a generator for $D_{QCoh}(\mathcal{O}_X)$,
(2) if $X \subset \mathbf{P}^n_A$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$ is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.16.3.

36.17. Compact and perfect objects

09M0 Let X be a Noetherian scheme of finite dimension. By Cohomology, Proposition 20.20.7 and Cohomology on Sites, Lemma 21.52.5 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect objects of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

09M1 **Proposition 36.17.1.** *Let X be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. If K is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology, Lemma 20.50.5) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums and since $H^0(X, -)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.5 we conclude that K is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 20.49.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 36.4.5. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine.

Assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 36.3.5 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 15.79.3. □

0GEF **Remark 36.17.2.** Let X be a quasi-compact and quasi-separated scheme. Let G be a perfect object of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$. By Theorem 36.15.3 there is at least one of these. Combining Lemma 36.3.1 with Proposition 36.17.1 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf}(\mathcal{O}_X)$.

The following result is a strengthening of Proposition 36.17.1. Let $T \subset X$ be a closed subset of a scheme X. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition 36.6.1). Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0A9B **Lemma 36.17.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 36.17.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 36.15.4.

Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 13.37.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. □

0GEG **Remark 36.17.4.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let G be a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$ which is a generator for $D_{QCoh,T}(\mathcal{O}_X)$. By Lemma 36.15.4 there is at least one of these. Combining the fact that $D_{QCoh,T}(\mathcal{O}_X)$ has direct sums with Lemma 36.17.3 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf,T}(\mathcal{O}_X)$.

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0A9C **Lemma 36.17.5.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

(1) *P is perfect and E supported on T, or*
(2) *P pseudo-coherent, E supported on T, and E bounded below.*

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 36.10.8 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 36.15.4. Write

$$K = \mathrm{hocolim} K_n$$

as in Derived Categories, Lemma 13.37.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \mathrm{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α, see Derived Categories, Lemma 13.33.9. By Derived Categories, Lemma 13.37.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. □

36.18. Derived categories as module categories

09M2 In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

09M3 **Lemma 36.18.1.** *Let X be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property (P) and let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.20. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. □

The following result is taken from [**BV03**].

09M5 **Theorem 36.18.2.** *Let X be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 36.15.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.35. Since $K^\bullet$ is K-injective we have

09M6 (36.18.2.1) $$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 36.13.5. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 36.17.1. Combined with (36.18.2.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.35.6. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 22.35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 36.18.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.7.2. □

0DJL **Remark 36.18.3** (Variant with support)**.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. The analogue of Theorem 36.18.2 holds for $D_{QCoh,T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the theorem, using Lemmas 36.15.4 and 36.17.3 and a variant of Lemma 36.18.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

09SU **Remark 36.18.4** (Uniqueness of dga)**.** Let X be a quasi-compact and quasi-separated scheme over a ring R. By the construction of the proof of Theorem 36.18.2 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \mathrm{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \mathrm{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \mathrm{d})$$

is given by the functor $- \otimes^{\mathbf{L}}_A \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 22.37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

36.19. Characterizing pseudo-coherent complexes, I

0DJM We can use the methods above to characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DJN **Lemma 36.19.1.** *Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(\mathcal{O}_X)$. The following are equivalent*

1. *K is pseudo-coherent, and*
2. *$K = hocolim K_n$ where K_n is perfect and $\tau_{\geq -n}K_n \to \tau_{\geq -n}K$ is an isomorphism for all n.*

Proof. The implication (2) $\Rightarrow$ (1) is true on any ringed space. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology, Lemma 20.49.5. Then it follows from the definitions that $\tau_{\geq -n}K_n$ is $(-n+1)$-pseudo-coherent and hence $\tau_{\geq -n}K$ is $(-n+1)$-pseudo-coherent, hence K is $(-n+1)$-pseudo-coherent. This is true for all n, hence K is pseudo-coherent, see Cohomology, Definition 20.47.1.

Assume (1). We start by choosing an approximation $K_1 \to K$ of $(X, K, -2)$ by a perfect complex K_1, see Definitions 36.14.1 and 36.14.2 and Theorem 36.14.6. Suppose by induction we have

$$K_1 \to K_2 \to \ldots \to K_n \to K$$

with K_i perfect such that such that $\tau_{\geq -i}K_i \to \tau_{\geq -i}K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 36.13.5 for the perfect object K_n. Choose an approximation $K_{n+1} \to K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \to K \to C \to K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \to K \to C$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can factor $K_n \to K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \mathrm{hocolim} K_n$. This follows by an application of Derived Categories, Lemma 13.33.8 to the functors $H^i(-) : D(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ and our choice of K_n. □

0DJP **Lemma 36.19.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T. The following are equivalent*

(1) *K is pseudo-coherent, and*
(2) *$K = \mathit{hocolim} K_n$ where K_n is perfect, supported on T, and $\tau_{\geq -n}K_n \to \tau_{\geq -n}K$ is an isomorphism for all n.*

Proof. The proof of this lemma is exactly the same as the proof of Lemma 36.19.1 except that in the choice of the approximations we use the triples (T, K, m). □

36.20. An example equivalence

0CS7 In Section 36.16 we proved that the derived category of projective space $\mathbf{P}^n_A$ over a ring A is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf. In this section we prove this determines an equivalence of $D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$ with the derived category of an A-algebra.

Before we can state the result we need some notation. Let A be a ring. Let $X = \mathbf{P}^n_A = \mathrm{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. By Lemma 36.16.3 we know that

0CS8 (36.20.0.1) $$P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$$

is a perfect generator of $D_{QCoh}(\mathcal{O}_X)$. Consider the (noncommutative) A-algebra

0CS9 (36.20.0.2) $$R = \mathrm{Hom}_{\mathcal{O}_X}(P, P) = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix}$$

with obvious multiplication and addition. If we view P as a complex of $\mathcal{O}_X$-modules in the usual way (i.e., with P in degree 0 and zero in every other degree), then we have

$$R = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(P, P)$$

where on the right hand side we view R as a differential graded algebra over A with zero differential (i.e., with R in degree 0 and zero in every other degree). According to the discussion in Differential Graded Algebra, Section 22.35 we obtain a derived functor

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D(\mathcal{O}_X),$$

see especially Differential Graded Algebra, Lemma 22.35.3. By Lemma 36.18.1 we see that the essential image of this functor is contained in $D_{QCoh}(\mathcal{O}_X)$.

0BQU **Lemma 36.20.1.** *Let A be a ring. Let $X = \mathbf{P}^n_A = Proj(S)$ where $S = A[X_0, \ldots, X_n]$.* [**Bei78**] *With P as in (36.20.0.1) and R as in (36.20.0.2) the functor*

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an A-linear equivalence of triangulated categories sending R to P.

In words: the derived category of quasi-coherent modules on projective space is equivalent to the derived category of modules over a (noncommutative) algebra. This property of projective space appears to be quite unusual among all projective schemes over A.

Proof. To prove that our functor is fully faithful it suffices to prove that $\mathrm{Ext}^i_X(P, P)$ is zero for $i \neq 0$ and equal to R for $i = 0$, see Differential Graded Algebra, Lemma 22.35.6. As in the proof of Lemma 36.13.5 we see that

$$\mathrm{Ext}^i_X(P, P) = H^i(X, P^\vee \otimes P) = \bigoplus\nolimits_{0 \leq a, b \leq n} H^i(X, \mathcal{O}_X(a - b))$$

By the computation of cohomology of projective space (Cohomology of Schemes, Lemma 30.8.1) we find that these Ext-groups are zero unless $i = 0$. For $i = 0$ we recover R because this is how we defined R in (36.20.0.2). By Differential Graded Algebra, Lemma 22.35.5 our functor has a right adjoint, namely $R\mathrm{Hom}(P, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$. Since P is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.16.3 we see that the kernel of $R\mathrm{Hom}(P, -)$ is zero. Hence our functor is an equivalence of triangulated categories by Derived Categories, Lemma 13.7.2. □

36.21. The coherator revisited

0CQZ In Section 36.7 we constructed and studied the right adjoint RQ_X to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote[3] it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated schemes have such a right adjoint.

0CR0 **Lemma 36.21.1.** *Let X be a quasi-compact and quasi-separated scheme. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint DQ_X.*

First proof. We will use the induction principle as in Cohomology of Schemes, Lemma 30.4.1 to prove this. If $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 36.7 is a right adjoint to the functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine schemes, see Lemma 36.7.3. Thus we see that it suffices to show: if $X = U \cup V$ is a union of two quasi-compact opens and the lemma holds for U, V, and $U \cap V$, then the lemma holds for X.

[3] This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 13.40.7. Consider the distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$ of Cohomology, Lemma 20.33.2. By Derived Categories, Lemma 13.40.5 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U\cap V,*}E|_{U\cap V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an open immersion corresponding with U a quasi-compact open for which the lemma is true. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_*L \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\mathrm{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 13.40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \to Rj_*L \to Rj_*Q \to Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\mathrm{Hom}(M, Rj_*Q) = \mathrm{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 36.3.8. This finishes the proof. □

Second proof. The adjoint exists by Derived Categories, Proposition 13.38.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 36.3.1). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 36.15.3 and because perfect objects are compact by Proposition 36.17.1. □

0CR1 **Lemma 36.21.2.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \to D$ exist for X and Y, then*

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 36.4.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 36.3.8) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. □

0CR2 **Remark 36.21.3.** Let X be a quasi-compact and quasi-separated scheme. Let $X = U \cup V$ with U and V quasi-compact open. By Lemma 36.21.1 the functors DQ_X, DQ_U, DQ_V, $DQ_{U\cap V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \to Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \to Rj_{U\cap V,*}DQ_{U\cap V}(K|_{U\cap V}) \to$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Cohomology, Lemma 20.33.2 and using Lemma 36.21.2 three times.

0CSA **Lemma 36.21.4.** *Let X be a quasi-compact and quasi-separated scheme. The functor DQ_X of Lemma 36.21.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine open in X and $i \notin [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \notin [a, b + N]$.*

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemmas 36.3.5 and 36.7.3.

Assume U, V are quasi-compact open in X and the lemma holds for U, V, and $U\cap V$. Say with integers $N(U)$, $N(V)$, and $N(U \cap V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine open $W \subset X$ and all $i \notin [a, b]$. Then $K|_U$, $K|_V$, $K|_{U\cap V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U\cap V}(K|_{U\cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + \max(N(U), N(V), N(U \cap V))$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U\cap V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 36.4.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U\cap V,*}DQ_{U\cap V}(K|_{U\cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark 36.21.3. □

0CSB **Example 36.21.5.** Let X be a quasi-compact and quasi-separated scheme. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R\lim \mathcal{F}_n$ for this. For any affine open $U \subset X$ we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D^b_{QCoh}(\mathcal{O}_X)$ by Lemma 36.21.4.

36.22. Cohomology and base change, IV

08ET This section continues the discussion of Cohomology of Schemes, Section 30.22. First, we have a very general version of the projection formula for quasi-compact and quasi-separated morphisms of schemes and complexes with quasi-coherent cohomology sheaves.

08EU **Lemma 36.22.1.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ the map*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

defined in Cohomology, Equation (20.54.2.1) is an isomorphism.

Proof. To check the map is an isomorphism we may work locally on Y. Hence we reduce to the case that Y is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 36.4.5. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $^\sim : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 36.3.5). Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.59.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

08IA **Definition 36.22.2.** Let S be a scheme. Let X, Y be schemes over S. We say X and Y are *Tor independent over S* if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 15.61.1).

0FXV **Lemma 36.22.3.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. The following are equivalent*

1. *X and Y are tor independent over S, and*
2. *for every affine opens $U \subset X$, $V \subset Y$, $W \subset S$ with $f(U) \subset W$ and $g(V) \subset W$ the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ are tor independent over $\mathcal{O}_S(W)$.*
3. *there exists an affine open overing $S = \bigcup W_i$ and for each i affine open coverings $f^{-1}(W_i) = \bigcup U_{ij}$ and $g^{-1}(W_i) = \bigcup V_{ik}$ such that the rings $\mathcal{O}_X(U_{ij})$ and $\mathcal{O}_Y(V_{ik})$ are tor independent over $\mathcal{O}_S(W_i)$ for all i, j, k.*

Proof. Omitted. Hint: use More on Algebra, Lemma 15.61.6. □

0FXW **Lemma 36.22.4.** *Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $S' \to S$ be a morphism of schemes and denote $X' = X \times_S S'$ and $Y' = Y \times_S S'$. If X and Y are tor independent over S and $S' \to S$ is flat, then X' and Y' are tor independent over S'.*

Proof. Omitted. Hint: use Lemma 36.22.3 and on affine opens use More on Algebra, Lemma 15.61.4. □

08IB **Lemma 36.22.5.** *Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

If X and S' are Tor independent over S, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ the canonical arrow $Lg^*Rf_*E \to Rf'_*L(g')^*E$ is an isomorphism.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 20.28.3 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*L(g')^*E$. To check this is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 36.5.2 (and use Lemmas 36.3.8, 36.3.9, and 36.4.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that g' is affine as well (Morphisms, Lemma 29.11.9). By Lemma 36.5.3 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_S} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$ (by affine base change, see Cohomology of Schemes, Lemma 30.5.1). Thus by Lemma 36.22.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that X and S' are Tor independent over S. Namely, to check it we may work locally on X, hence we may also assume X is affine. Say $X = \operatorname{Spec}(A)$, $S = \operatorname{Spec}(R)$ and $S' = \operatorname{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (More on Algebra, Lemma 15.61.6), i.e., $\operatorname{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$ (use Lemma 36.3.8). □

The following lemma will be used in the chapter on dualizing complexes.

0AA7 **Lemma 36.22.6.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of quasi-compact and quasi-separated schemes. Assume g and f Tor independent and $S = \operatorname{Spec}(R)$, $S' = \operatorname{Spec}(R')$ affine. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\operatorname{Hom}_X(M, K) \otimes^{\mathbf{L}}_R R' \longrightarrow R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

1. *$M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or*
2. *$M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and R' has finite tor dimension over R.*

Proof. There is a canonical map $R\operatorname{Hom}_X(M, K) \to R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology, Section 20.44. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness

of restriction and $- \otimes^{\mathbf{L}}_R R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\mathrm{Hom}_X(M, R(g')_*L(g')^*K) = R\mathrm{Hom}_X(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

by Lemma 36.5.3. In both cases the complex $R\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.10.8. There is a natural map

$$R\mathcal{H}om(M, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases by Lemma 36.10.9. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_{S'}$ the second equality by Lemma 36.22.5. Using Lemma 36.10.4 and Cohomology, Lemma 20.48.4 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 36.5.3. Hence the result follows from Lemma 36.22.5. □

0BZA **Remark 36.22.7.** With notation as in Lemma 36.22.6. The diagram

$$\begin{array}{ccc} R\mathrm{Hom}_X(M, Rg'_*L) \otimes^{\mathbf{L}}_R R' & \longrightarrow & R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle a} \\ R\mathrm{Hom}_X(M, R(g')_*L) & = & R\mathrm{Hom}_{X'}(L(g')^*M, L) \end{array}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \to L$. The multiplication map is the adjunction map $K' \otimes^{\mathbf{L}}_R R' \to K'$ for any $K' \in D(R')$.

0C0V **Lemma 36.22.8.** *Consider a cartesian square of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Assume g and f Tor independent.

1. *If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_S$-modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_{S'}$-modules.*
2. *If $\mathcal{G}$ is an $\mathcal{O}_X$-module flat over S, then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.*

Proof. We can compute tor dimension at stalks, see Cohomology, Lemma 20.48.5. If $x' \in X'$ with image $x \in X$, then

$$(L(g')^*E)_{x'} = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

Let $s' \in S'$ and $s \in S$ be the image of x' and x. Since X and S' are tor independent over S, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ in $D(\mathcal{O}_{S',s'})$. Thus (1) follows from More on Algebra, Lemma 15.67.13. To see (2) observe that flatness of

$\mathcal{G}$ is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0,0]$. Applying (1) we conclude. □

0E23 **Lemma 36.22.9.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

where i is a closed immersion. If Z and X' are tor independent over X, then $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_{X'})$.

Proof. Note that the lemma is supposed to hold for all $K \in D(\mathcal{O}_Z)$. Observe that i_* and i'_* are exact functors and hence Ri_* and Ri'_* are computed by applying i_* and i'_* to any representatives. Thus the base change map

$$Lf^*(Ri_*(K)) \longrightarrow Ri'_*(Lg^*(K))$$

on stalks at a point $z' \in Z'$ with image $z \in Z$ is given by

$$K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X,z}} \mathcal{O}_{X',z'} \longrightarrow K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{Z,z}} \mathcal{O}_{Z',z'}$$

This map is an isomorphism by More on Algebra, Lemma 15.61.2 and the assumed tor independence. □

36.23. Künneth formula, II

0FLN For the case where the base is a field, please see Varieties, Section 33.29. Consider a cartesian diagram of schemes

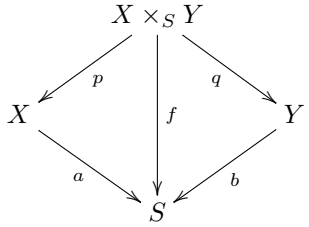

Let $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$. There is a canonical map

0FLP (36.23.0.1) $$Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \longrightarrow Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M)$$

Namely, we can use the maps $Ra_*K \to Ra_*Rp_*Lp^*K = Rf_*Lp^*K$ and $Rb_*M \to Rb_*Rq_*Lq^*M = Rf_*Lq^*M$ and then we can use the relative cup product (Cohomology, Remark 20.28.7).

Set $A = \Gamma(S, \mathcal{O}_S)$. There is a global Künneth map

0G7V (36.23.0.2) $$R\Gamma(X,K) \otimes^{\mathbf{L}}_A R\Gamma(Y,M) \longrightarrow R\Gamma(X \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M)$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X,K) \to R\Gamma(X \times_S Y, Lp^*K)$ and $R\Gamma(Y,M) \to R\Gamma(X \times_S Y, Lq^*M)$ and the cup product constructed in Cohomology, Section 20.31.

0FLQ **Lemma 36.23.1.** *In the situation above, if a and b are quasi-compact and quasi-separated and X and Y are tor-independent over S, then (36.23.0.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(\mathcal{O}_Y)$. If in addition $S = \mathrm{Spec}(A)$ is affine, then the map (36.23.0.2) is an isomorphism.*

First proof. This follows from the following sequence of isomorphisms

$$\begin{aligned} Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M) &= Ra_*Rp_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M) \\ &= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Rp_*Lq^*M) \\ &= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*Rb_*M) \\ &= Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \end{aligned}$$

The first equality holds because $f = a \circ p$. The second equality by Lemma 36.22.1. The third equality by Lemma 36.22.5. The fourth equality by Lemma 36.22.1. We omit the verification that the composition of these isomorphisms is the same as the map (36.23.0.1). If S is affine, then the source and target of the arrow (36.23.0.2) are the result of applying $R\Gamma(S, -)$ to the source and target of (36.23.0.1) and we obtain the final statement; details omitted. □

Second proof. The construction of the arrow (36.23.0.1) is compatible with restricting to open subschemes of S as is immediate from the construction of the relative cup product. Thus it suffices to prove that (36.23.0.1) is an isomorphism when S is affine.

Assume $S = \mathrm{Spec}(A)$ is affine. By Leray we have $R\Gamma(S, Rf_*K) = R\Gamma(X, K)$ and similarly for the other cases. By Cohomology, Lemma 20.31.7 the map (36.23.0.1) induces the map (36.23.0.2) on taking $R\Gamma(S, -)$. On the other hand, by Lemmas 36.4.1 and 36.3.9 the source and target of the map (36.23.0.1) are in $D_{QCoh}(\mathcal{O}_S)$. Thus, by Lemma 36.3.5, it suffices to prove that (36.23.0.2) is an isomorphism.

Assume $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(C)$ are all affine. We will use Lemma 36.3.5 without further mention. In this case we can choose a K-flat complex $K^\bullet$ of B-modules whose terms are flat such that K is represented by $\widetilde{K^\bullet}$. Similarly, we can choose a K-flat complex $M^\bullet$ of C-modules whose terms are flat such that M is represented by $\widetilde{M^\bullet}$. See More on Algebra, Lemma 15.59.10. Then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules and similarly for $\widetilde{M^\bullet}$, see Lemma 36.3.6. Thus La^*K is represented by

$$a^*\widetilde{K^\bullet} = \widetilde{K^\bullet \otimes_A C}$$

and similarly for Lb^*M. This in turn is a K-flat complex of $\mathcal{O}_{X\times_S Y}$-modules by the lemma cited above and More on Algebra, Lemma 15.59.3. Thus we finally see that the complex of $\mathcal{O}_{X\times_S Y}$-modules associated to

$$\mathrm{Tot}((K^\bullet \otimes_A C) \otimes_{B\otimes_A C} (B \otimes_A M^\bullet)) = \mathrm{Tot}(K^\bullet \otimes_A M^\bullet)$$

represents $La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lb^*M$ in the derived category of $X \times_S Y$. Taking global sections we obtain $\mathrm{Tot}(K^\bullet \otimes_A M^\bullet)$ which of course is also the complex representing $R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M)$. The fact that the isomorphism is given by cup product follows from the relationship between the genuine cup product and the naive one in Cohomology, Section 20.31 (and in particular Cohomology, Lemma 20.31.3 and the discussion following it).

Assume $S = \mathrm{Spec}(A)$ and Y are affine. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove the statement. To do this we only have to show: if $X = U \cup V$ is an open covering with U and V quasi-compact and if the map (36.23.0.2)

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(U \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for U and Y over S, the map (36.23.0.2)

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(V \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for V and Y over S, and the map (36.23.0.2)

$$R\Gamma(U \cap V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma((U \cap V) \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for $U \cap V$ and Y over S are isomorphisms, then so is the map (36.23.0.2) for X and Y over S. However, by Cohomology, Lemma 20.33.7 these maps fit into a map of distinguished triangles with (36.23.0.2) the final leg and hence we conclude by Derived Categories, Lemma 13.4.3.

Assume $S = \mathrm{Spec}(A)$ is affine. To finish the proof we can use the induction principle of Cohomology of Schemes, Lemma 30.4.1 on Y. Namely, by the above we already know that our map is an isomorphism when Y is affine. The rest of the argument is exactly the same as in the previous paragraph but with the roles of X and Y switched. □

0FML **Lemma 36.23.2.** *Let $a : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $a^{-1}\mathcal{O}_S$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $a^{-1}\mathcal{O}_S$-module and $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module compatible with the given $a^{-1}\mathcal{O}_S$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the following hold*

(1) *$Ra_*\mathcal{F}^\bullet$ is locally bounded,*
(2) *$Ra_*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_S)$,*
(3) *$Ra_*\mathcal{F}^\bullet$ locally has finite tor dimension,*
(4) *$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ for $\mathcal{G} \in QCoh(\mathcal{O}_S)$, and*
(5) *$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ for $K \in D_{QCoh}(\mathcal{O}_S)$.*

Proof. Parts (1), (2), (3) are local on S hence we may and do assume S is affine. Since a is quasi-compact we conclude that X is quasi-compact. Since $\mathcal{F}^\bullet$ is locally bounded, we conclude that $\mathcal{F}^\bullet$ is bounded.

For (1) and (2) we can use the first spectral sequence $R^p a_*\mathcal{F}^q \Rightarrow R^{p+q}a_*\mathcal{F}^\bullet$ of Derived Categories, Lemma 13.21.3. Combining Cohomology of Schemes, Lemma 30.4.5 and Homology, Lemma 12.24.11 we conclude.

Let us prove (3) by the induction principle of Cohomology of Schemes, Lemma 30.4.1. Namely, for a quasi-compact open of U of X consider the condition that $R(a|_U)_*(\mathcal{F}^\bullet|_U)$ has finite tor dimension. If U, V are quasi-compact open in X, then we have a relative Mayer-Vietoris distinguished triangle

$$R(a|_{U\cup V})_*\mathcal{F}^\bullet|_{U\cup V} \to R(a|_U)_*\mathcal{F}^\bullet|_U \oplus R(a|_V)_*\mathcal{F}^\bullet|_V \to R(a|_{U\cap V})_*\mathcal{F}^\bullet|_{U\cap V} \to$$

by Cohomology, Lemma 20.33.5. By the behaviour of tor amplitude in distinguished triangles (see Cohomology, Lemma 20.48.6) we see that if we know the result for

U, V, $U \cap V$, then the result holds for $U \cup V$. This reduces us to the case where X is affine. In this case we have

$$Ra_*\mathcal{F}^\bullet = a_*\mathcal{F}^\bullet$$

by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and the vanishing of higher direct images of quasi-coherent modules under an affine morphism (Cohomology of Schemes, Lemma 30.2.3). Since $\mathcal{F}^n$ is S-flat by assumption and X affine, the modules $a_*\mathcal{F}^n$ are flat for all n. Hence $a_*\mathcal{F}^\bullet$ is a bounded complex of flat $\mathcal{O}_S$-modules and hence has finite tor dimension.

Proof of part (5). Denote $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ the obvious flat morphism of ringed spaces. Part (5) says that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra'_*\mathcal{F}^\bullet = Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$$

Thus Cohomology, Equation (20.54.2.1) gives a functorial map from the left to the right and we want to show this map is an isomorphism. This question is local on S hence we may and do assume S is affine. The rest of the proof is *exactly* the same as the proof of Lemma 36.22.1 except that we have to show that the functor $K \mapsto Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ commutes with direct sums. This is where we will use $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module. Namely, observe that $K \mapsto L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ commutes with arbitrary direct sums. Next, if $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}^\bullet = \mathcal{F}^n[-n]$ then we have $L(a')^*G \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet = La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^n[-n]$, see Cohomology, Lemma 20.27.4. Hence in this case the commutation with direct sums follows from Lemma 36.4.5. Now, in general, since S is affine (hence X quasi-compact) and $\mathcal{F}^\bullet$ is locally bounded, we see that

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

is bounded. Arguing by induction on $b-a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

the proof of this part is finished. Some details omitted.

Proof of part (4). Let $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ be as above. Since $\mathcal{F}^\bullet$ is a locally bounded complex of flat $a^{-1}\mathcal{O}_S$-modules we see the complex $a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ represents $L(a')^*\mathcal{G} \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ in $D(a^{-1}\mathcal{O}_S)$. Hence (4) follows from (5). □

0FMQ **Lemma 36.23.3.** *Let $f : X \to Y$ be a morphism of schemes with $Y = \mathrm{Spec}(A)$ affine. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite affine open covering such that all the finite intersections $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ are affine. Let $\mathcal{F}^\bullet$ be a bounded complex of $f^{-1}\mathcal{O}_Y$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $f^{-1}\mathcal{O}_Y$-module and $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module compatible with the given $p^{-1}\mathcal{O}_Y$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the complex $Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is K-flat as a complex of A-modules.*

Proof. We may write

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

Arguing by induction on $b - a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

and using More on Algebra, Lemma 15.59.5 we reduce to the case where $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ placed in degree 0. In this case the Čech

complex for $\mathcal{F}$ and $\mathcal{U}$ is homotopy equivalent to the alternating Čech complex, see Cohomology, Lemma 20.23.6. Since $U_{i_0 \ldots i_p}$ is always affine, we see that $\mathcal{F}(U_{i_0 \ldots i_p})$ is A-flat. Hence $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is a bounded complex of flat A-modules and hence K-flat by More on Algebra, Lemma 15.59.7. □

Let X, Y, S, a, b, p, q, f be as in the introduction to this section. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Set $A = \Gamma(S, \mathcal{O}_S)$. Consider the map

0G49 (36.23.3.1) $$R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X, \mathcal{F}) \to R\Gamma(X \times_S Y, p^*\mathcal{F})$ and $R\Gamma(Y, \mathcal{G}) \to R\Gamma(X \times_S Y, q^*\mathcal{G})$, the cup product constructed in Cohomology, Section 20.31, and the canonical map $p^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G} \to p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$.

0FU4 **Lemma 36.23.4.** *In the situation above the map (36.23.3.1) is an isomorphism if S is affine, $\mathcal{F}$ and $\mathcal{G}$ are S-flat and quasi-coherent and X and Y are quasi-compact with affine diagonal.*

Proof. We strongly urge the reader to read the proof of Varieties, Lemma 33.29.1 first. Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times_S Y = \bigcup_{(i,j) \in I \times J} U_i \times_S V_j$. Note that $\mathcal{W}$ is a refinement of $\mathrm{pr}_1^{-1}\mathcal{U}$ and of $\mathrm{pr}_2^{-1}\mathcal{V}$. Thus by the discussion in Cohomology, Section 20.25 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})$$

well defined up to homotopy and compatible with pullback maps on cohomology. In Cohomology, Equation (20.25.3.2) we have constructed a map of complexes

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

which is compatible with the cup product on cohomology by Cohomology, Lemma 20.31.4. Combining the above we obtain a map of complexes

0FLU (36.23.4.1) $$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

We claim this is the map in the statement of the lemma, i.e., the source and target of this arrow are the same as the source and target of (36.23.3.1). Namely, by Cohomology of Schemes, Lemma 30.2.2 and Cohomology, Lemma 20.25.2 the canonical maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to R\Gamma(X, \mathcal{F}), \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to R\Gamma(Y, \mathcal{G})$$

and

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

are isomorphisms. On the other hand, the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is K-flat by Lemma 36.23.3 and we conclude that $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}))$ represents the derived tensor product $R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G})$ as claimed.

We still have to show that (36.23.4.1) is a quasi-isomorphism. We will do this using dimension shifting. Set $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$. Assume $d(\mathcal{F}) > 0$. Set $U = \coprod_{i \in I} U_i$. This is an affine scheme as I is finite. Denote $j : U \to X$ the morphism which is the inclusion $U_i \to X$ on each U_i. Since the diagonal of X is affine, the morphism j is affine, see Morphisms, Lemma 29.11.12. It follows that $\mathcal{F}' = j_*j^*\mathcal{F}$ is S-flat, see Morphisms, Lemma 29.25.4. It also follows that $d(\mathcal{F}') = 0$ by combining Cohomology of Schemes, Lemmas 30.2.4 and 30.2.2. For

all $x \in X$ we have $\mathcal{F}_x \to \mathcal{F}'_x$ is the inclusion of a direct summand: if $x \in U_i$, then $\mathcal{F}' \to (U_i \to X)_*\mathcal{F}|_{U_i}$ gives a splitting. We conclude that $\mathcal{F} \to \mathcal{F}'$ is injective and $\mathcal{F}'' = \mathcal{F}'/\mathcal{F}$ is S-flat as well. The short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of flat quasi-coherent $\mathcal{O}_X$-modules produces a short exact sequence of complexes

$$0 \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to 0$$

and a short exact sequence of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}'' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to 0$$

Moreover, the maps (36.23.4.1) between these are compatible with these short exact sequences. Hence it suffices to prove (36.23.4.1) is an isomorphism for $\mathcal{F}'$ and $\mathcal{F}''$. Finally, we have $d(\mathcal{F}'') < d(\mathcal{F})$. In this way we reduce to the case $d(\mathcal{F}) = 0$.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Observe that this means that $\Gamma(X, \mathcal{F})$ is quasi-isomorphic to the K-flat complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of A-modules sitting in degrees ≥ 0. It follows that $\Gamma(X, \mathcal{F})$ is a flat A-module (because we can compute higher Tor's against this module by tensoring with the Cech complex). Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times_S V = \bigcup_{i \in I} U_i \times_S V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$$

(equality of complexes). By the flatness of $\mathcal{G}(V)$ over A we see that $\Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}(V) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V)$ is a quasi-isomorphism. Since the sheafification of $V \mapsto \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ represents $Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ by Cohomology of Schemes, Lemma 30.7.1 we conclude that

$$Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}$$

on Y where the notation on the right hand side indicates the module

$$b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G}$$

Using the Leray spectral sequence for q we find

$$H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) = H^n(Y, b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G})$$

Using Lemma 36.22.1 for the morphism $b : Y \to S = \mathrm{Spec}(A)$ and using that $\Gamma(X, \mathcal{F})$ is A-flat we conclude that $H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes_A H^0(Y, \mathcal{G})$ for $n = 0$. Of course, here we also use that $\mathcal{G}$ only has cohomology in degree 0. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). □

0G7W **Remark 36.23.5.** Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules and let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\xi \in H^i(X, \mathcal{G})$ with pullback $p^*\xi \in H^i(X \times_S Y, p^*\mathcal{G})$. Then the following diagram is commutative

$$\begin{array}{ccc} R\Gamma(X, \mathcal{F})[-i] \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{E}) & \xrightarrow[\xi \otimes \mathrm{id}]{} & R\Gamma(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{E}) \\ \downarrow & & \downarrow \\ R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes q^*\mathcal{E})[-i] & \xrightarrow{p^*\xi} & R\Gamma(X \times_S Y, p^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes q^*\mathcal{E}) \end{array}$$

where the unadorned tensor products are over $\mathcal{O}_{X\times_S Y}$. The horizontal arrows are from Cohomology, Remark 20.31.2 and the vertical arrows are (36.23.0.2) hence given by pulling back followed by cup product on $X \times_S Y$. The diagram commutes because the global cup product (on $X \times_S Y$ with the sheaves $p^*\mathcal{G}$, $p^*\mathcal{F}$, and $q^*\mathcal{E}$) is associative, see Cohomology, Lemma 20.31.5.

36.24. Künneth formula, III

0G4A Let X, Y, S, a, b, p, q, f be as in the introduction to Section 36.23. In this section, given an $\mathcal{O}_X$-module $\mathcal{F}$ and a $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\mathcal{G}$$

Note that, contrary to what happens in a future section, we take the nonderived tensor product here.

On X let $\mathcal{F}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_X$-modules such that the differentials $d^i_{\mathcal{F}} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are differential operators on X/S of finite order, see Morphisms, Section 29.33. Similarly, on Y let $\mathcal{G}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_Y$-modules such that the differentials $d^j_{\mathcal{G}} : \mathcal{G}^j \to \mathcal{G}^{j+1}$ are differential operators on Y/S of finite order. Applying the construction of Morphisms, Lemma 29.33.2 we obtain a double complex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
\cdots & \uparrow & & \uparrow & & \uparrow & \cdots \\
\cdots & \mathcal{F}^i \boxtimes \mathcal{G}^{j+1} & \xrightarrow{d_1^{i,j+1}} & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^{j+1} & \longrightarrow & \cdots & \\
 & \uparrow{\scriptstyle d_2^{i,j}} & & \uparrow{\scriptstyle d_2^{i+1,j}} & & & \\
\cdots & \mathcal{F}^i \boxtimes \mathcal{G}^{j} & \xrightarrow{d_1^{i,j}} & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^{j} & \longrightarrow & \cdots & \\
\cdots & \uparrow & & \uparrow & & \uparrow & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

of quasi-coherent modules whose maps are differential operators of finite order on $X \times_S Y/S$. Please see the discussion in Morphisms, Remark 29.33.3 and Homology, Example 12.18.2. To be explicit, we set

$$d_1^{i,j} = d^i_{\mathcal{F}} \boxtimes 1 \quad \text{and} \quad d_2^{i,j} = 1 \boxtimes d^j_{\mathcal{G}}$$

In the discussion below the notation

$$\mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

refers to the total complex associated to this double complex. This complex has terms which are quasi-coherent $\mathcal{O}_{X\times_S Y}$-modules and whose differentials are differential operators of finite order on $X \times_S Y/S$.

In the situation above there exists a "relative cup product" map

0G4B (36.24.0.1) $$Ra_*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*(\mathcal{G}^\bullet) \longrightarrow Rf_*\left(\mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)\right)$$

Namely, we can construct this map by combining

(1) $Ra_*(\mathcal{F}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet)$,
(2) $Rb_*(\mathcal{G}^\bullet) \to Rf_*(q^{-1}\mathcal{G}^\bullet)$,
(3) $Rf_*(p^{-1}\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(q^{-1}\mathcal{G}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$,

(4) $p^{-1}\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet \to \mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Maps (1) and (2) are pullback maps, map (3) is the relative cup product, see Cohomology, Remark 20.28.7, map (4) compares the derived and nonderived tensor products, and map (5) is given by the obvious maps $p^{-1}\mathcal{F}^i \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^j \to \mathcal{F}^i \boxtimes \mathcal{G}^j$ on the underlying double complexes.

Set $A = \Gamma(S, \mathcal{O}_S)$. There exists a "global cup product" map

0FLR (36.24.0.2) $$R\Gamma(X, \mathcal{F}^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}^\bullet) \longrightarrow R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet))$$

in $D(A)$. This is constructed similarly to the relative cup product above using

(1) $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet)$
(2) $R\Gamma(Y, \mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet)$,
(3) $R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$,
(4) $p^{-1}\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet \to \mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Here maps (1) and (2) are the pullback maps, map (3) is the cup product constructed in Cohomology, Section 20.31. Maps (4) and (5) are as indicated in the previous paragraph.

0FLT **Lemma 36.24.1.** *In the situation above the cup product (36.24.0.2) is an isomorphism in $D(A)$ if the following assumptions hold*

(1) $S = \mathrm{Spec}(A)$ *is affine,*
(2) X *and* Y *are quasi-compact with affine diagonal,*
(3) $\mathcal{F}^\bullet$ *is bounded,*
(4) $\mathcal{G}^\bullet$ *is bounded below,*
(5) $\mathcal{F}^n$ *is S-flat, and*
(6) $\mathcal{G}^m$ *is S-flat.*

Proof. We will use the notation $\mathcal{A}_{X/S}$ and $\mathcal{A}_{Y/S}$ introduced in Morphisms, Remark 29.33.3. Suppose that we have maps of complexes

$$\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to \mathcal{F}_1^\bullet[1]$$

in the category $\mathcal{A}_{X/S}$. Then by the functoriality of the cup product we obtain a commutative diagram

$$\begin{array}{ccc}
R\Gamma(X, \mathcal{F}_1^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}^\bullet) & \longrightarrow & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet \boxtimes \mathcal{G}^\bullet)) \\
\downarrow & & \downarrow \\
R\Gamma(X, \mathcal{F}_2^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}^\bullet) & \longrightarrow & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_2^\bullet \boxtimes \mathcal{G}^\bullet)) \\
\downarrow & & \downarrow \\
R\Gamma(X, \mathcal{F}_3^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}^\bullet) & \longrightarrow & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_3^\bullet \boxtimes \mathcal{G}^\bullet)) \\
\downarrow & & \downarrow \\
R\Gamma(X, \mathcal{F}_1^\bullet[1]) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}^\bullet) & \longrightarrow & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet[1] \boxtimes \mathcal{G}^\bullet))
\end{array}$$

If the original maps form a distinguished triangle in the homotopy category of $\mathcal{A}_{X/S}$, then the columns of this diagram form distinguished triangles in $D(A)$.

In the situation of the lemma, suppose that $\mathcal{F}^n = 0$ for $n < i$. Then we may consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to 0$$

where the truncation is as in Homology, Section 12.15. This produces the distinguished triangle

$$\sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to (\sigma_{\geq i+1}\mathcal{F}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{X/S}$ where the final arrow is given by the boundary map $\mathcal{F}^i \to \mathcal{F}^{i+1}$. It follows from the discussion above that it suffices to prove the lemma for $\mathcal{F}^i[-i]$ and $\sigma_{\geq i+1}\mathcal{F}^\bullet$. Since $\sigma_{\geq i+1}\mathcal{F}^\bullet$ has fewer nonzero terms, by induction, if we can prove the lemma if $\mathcal{F}^\bullet$ is nonzero only in single degree, then the lemma follows. Thus we may assume $\mathcal{F}^\bullet$ is nonzero only in one degree.

Assume $\mathcal{F}^\bullet$ is the complex which has an S-flat quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ sitting in degree 0 and is zero in other degrees. Observe that $R\Gamma(X, \mathcal{F})$ has finite tor dimension by Lemma 36.23.2 for example. Say it has tor amplitude in $[i, j]$. Pick $N \gg 0$ and consider the distinguished triangle

$$\sigma_{\geq N+1}\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \sigma_{\leq N}\mathcal{G}^\bullet \to (\sigma_{\geq N+1}\mathcal{G}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{Y/S}$. Now observe that both

$$R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \sigma_{\geq N+1}\mathcal{G}^\bullet) \quad \text{and} \quad R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F} \boxtimes \sigma_{\geq N+1}\mathcal{G}^\bullet))$$

have vanishing cohomology in degrees $\leq N + i$. Thus, using the arguments given above, if we want to prove our statement in a given degree, then we may assume $\mathcal{G}^\bullet$ is bounded. Repeating the arguments above one more time we may also assume $\mathcal{G}^\bullet$ is nonzero only in one degree. This case is handled by Lemma 36.23.4. □

36.25. Künneth formula for Ext

0FXX Consider a cartesian diagram of schemes

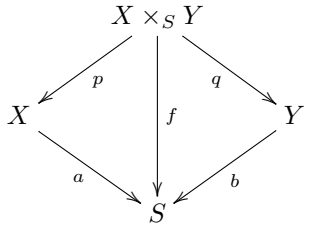

For $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ in this section let us define

$$K \boxtimes M = Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M$$

We claim there is a canonical map

(36.25.0.1)

0FXY $$Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*R\mathcal{H}om(M, M') \longrightarrow Rf_*(R\mathcal{H}om(K \boxtimes M, K' \boxtimes M'))$$

for $K, K' \in D(\mathcal{O}_X)$ and $M, M' \in D(\mathcal{O}_Y)$. Namely, we can take the map adjoint to the map

$$\begin{aligned} Lf^* \left(Ra_* R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_* R\mathcal{H}om(M, M')\right) = \\ Lf^* Ra_* R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lf^* Rb_* R\mathcal{H}om(M, M') = \\ Lp^* La^* Ra_* R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^* Lb^* Rb_* R\mathcal{H}om(M, M') \to \\ Lp^* R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^* R\mathcal{H}om(M, M') \to \\ R\mathcal{H}om(Lp^* K, Lp^* K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} R\mathcal{H}om(Lq^* M, Lq^* M') \to \\ R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') \end{aligned}$$

Here the first equality is compatibility of pullbacks with tensor products, Cohomology, Lemma 20.27.3. The second equality is $f = a \circ p = b \circ q$ and composition of pullbacks, Cohomology, Lemma 20.27.2. The first arrow is given by the adjunction maps $La^* Ra_* \to \mathrm{id}$ and $Lb^* Rb_* \to \mathrm{id}$ because pushforward and pullback are adjoint, Cohomology, Lemma 20.28.1. The second arrow is given by Cohomology, Remark 20.42.13. The third and final arrow is Cohomology, Remark 20.42.10. A simple special case of this is the following result.

0FXZ **Lemma 36.25.1.** *In the situation above, assume a and b are quasi-compact and quasi-separated and X and Y are tor independent over S. If K is perfect, $K' \in D_{QCoh}(\mathcal{O}_X)$, M is perfect, and $M' \in D_{QCoh}(\mathcal{O}_Y)$, then (36.25.0.1) is an isomorphism.*

Proof. In this case we have $R\mathcal{H}om(K, K') = K' \otimes^{\mathbf{L}} K^\vee$, $R\mathcal{H}om(M, M') = M' \otimes^{\mathbf{L}} M^\vee$, and

$$R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') = (K' \otimes^{\mathbf{L}} K^\vee) \boxtimes (M' \otimes^{\mathbf{L}} M^\vee)$$

See Cohomology, Lemma 20.50.5 and we also use that being perfect is preserved by pullback and by tensor products. Hence this case follows from Lemma 36.23.1. (We omit the verification that with these identifications we obtain the same map.) □

36.26. Cohomology and base change, V

0DJ6 In Section 36.22 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 15.61. In this section we analyze when one can get a base change result "one complex at a time".

To make this work, suppose we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes (usually we will assume it is cartesian). Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^* K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. For a point $x' \in X'$ set $x = g'(x') \in X$, $s' = f'(x') \in S'$ and $s = f(x) = g(s')$. Then we can consider the maps

$$K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'} \to K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \to K'_{x'}$$

where the first arrow is More on Algebra, Equation (15.61.0.1) and the second comes from $(L(g')^* K)_{x'} = K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ and the given map $L(g')^* K \to K'$. For

each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$-module structure on $H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})$. Putting everything together we obtain canonical maps

0DJ7 (36.26.0.1) $$H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})} \mathcal{O}_{X',x'} \longrightarrow H^i(K'_{x'})$$

of $\mathcal{O}_{X',x'}$-modules.

0DJ8 **Lemma 36.26.1.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

1. *for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (36.26.0.1) is an isomorphism,*
2. *for $U \subset X$, $V' \subset S'$ affine open both mapping into the affine open $V \subset S$ with $U' = V' \times_V U$ the composition*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(U)} \mathcal{O}_{S'}(V') \to R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') \to R\Gamma(U', K')$$

   is an isomorphism in $D(\mathcal{O}_{S'}(V'))$, and
3. *there is a set I of quadruples U_i, V'_i, V_i, U'_i, $i \in I$ as in (2) with $X' = \bigcup U'_i$.*

Proof. The second arrow in (2) comes from the equality

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') = R\Gamma(U', L(g')^*K)$$

of Lemma 36.3.8 and the given arrow $L(g')^*K \to K'$. The first arrow of (2) is More on Algebra, Equation (15.61.0.1). It is clear that (2) implies (3). Observe that (1) is local on X'. Therefore it suffices to show that if X, S, S', X' are affine, then (1) is equivalent to the condition that

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(S)} \mathcal{O}_{S'}(S') \to R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(X)} \mathcal{O}_{X'}(X') \to R\Gamma(X', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(S'))$. Say $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(R')$, $X' = \mathrm{Spec}(A')$, K corresponds to the complex $M^\bullet$ of A-modules, and K' corresponds to the complex $N^\bullet$ of A'-modules. Note that $A' = A \otimes_R R'$. The condition above is that the composition

$$M^\bullet \otimes^{\mathbf{L}}_R R' \to M^\bullet \otimes^{\mathbf{L}}_A A' \to N^\bullet$$

is an isomorphism in $D(R')$. Equivalently, it is that for all $i \in \mathbf{Z}$ the map

$$H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \to H^i(M^\bullet \otimes^{\mathbf{L}}_A A') \to H^i(N^\bullet)$$

is an isomorphism. Observe that this is a map of $A \otimes_R R'$-modules, i.e., of A'-modules. On the other hand, (1) is the requirement that for compatible primes $\mathfrak{q}' \subset A'$, $\mathfrak{q} \subset A$, $\mathfrak{p}' \subset R'$, $\mathfrak{p} \subset R$ the composition

$$H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) \otimes_{(A_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} \to H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{A_{\mathfrak{q}}} A'_{\mathfrak{q}'}) \to H^i(N^\bullet_{\mathfrak{q}'})$$

is an isomorphism. Since

$$H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) \otimes_{(A_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} = H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \otimes_{A'} A'_{\mathfrak{q}'}$$

is the localization at $\mathfrak{q}'$, we see that these two conditions are equivalent by Algebra, Lemma 10.23.1. □

0DJ9 **Lemma 36.26.2.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

(1) *the equivalent conditions of Lemma 36.26.1 hold, and*
(2) *f is quasi-compact and quasi-separated,*

then the composition $Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K'$ is an isomorphism.

Proof. We could prove this using the same method as in the proof of Lemma 36.22.5 but instead we will prove it using the induction principle and relative Mayer-Vietoris.

To check the map is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular X is a quasi-compact and quasi-separated scheme. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove that for any quasi-compact open $U \subset X$ the similarly constructed map $Lg^*R(U \to S)_*K|_U \to R(U' \to S')_*K'|_{U'}$ is an isomorphism. Here $U' = (g')^{-1}(U)$.

If $U \subset X$ is an affine open, then we find that the result is true by assumption, see Lemma 36.26.1 part (2) and the translation into algebra afforded to us by Lemmas 36.3.5 and 36.3.8.

The induction step. Suppose that $X = U \cup V$ is an open covering with U, V, $U \cap V$ quasi-compact such that the result holds for U, V, and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \cap V}$. Let $a' : U' \to S'$, $b' : V' \to S'$ and $c' : U' \cap V' \to S'$ be the base changes of a, b, and c. Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 20.33.5) we obtain a commutative diagram

$$\begin{array}{ccc} Lg^*Rf_*K & \longrightarrow & Rf'_*K' \\ \downarrow & & \downarrow \\ Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V & \longrightarrow & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \\ \downarrow & & \downarrow \\ Lg^*Rc_*K|_{U\cap V} & \longrightarrow & Rc'_*K'|_{U'\cap V'} \\ \downarrow & & \downarrow \\ Lg^*Rf_*K[1] & \longrightarrow & Rf'_*K'[1] \end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. □

0DJA **Lemma 36.26.3.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 36.26.1 hold, then

(1) *for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \to L(g')^*E \otimes^{\mathbf{L}} K'$,*
(2) *if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$, and*
(3) *if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$.*

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 36.3.8, 36.3.9, and 36.10.8 and Cohomology, Lemmas 20.47.3 and 20.49.6. Having said this, we can check the maps (36.26.0.1) are isomorphisms in case (1) by computing the source and target of (36.26.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 15.59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, L(g')^*K) \to R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology, Remark 20.42.13 and the second arrow comes from the given map $L(g')^*K \to K'$. To prove the maps (36.26.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X,x}$-modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. □

0A1D **Lemma 36.26.4.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a bounded above complex of quasi-coherent $\mathcal{O}_X$-modules flat over S. Then formation of*

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \longrightarrow Rf'_*\left(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} (g')^*\mathcal{G}^\bullet\right)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 36.26.2 and 36.26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 36.26.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}_x^\bullet \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ computes the derived tensor product by our assumptions on the complex $\mathcal{G}^\bullet$. □

08IE **Lemma 36.26.5.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. If*

(1) *E is perfect, $\mathcal{G}^\bullet$ is a bounded above, and $\mathcal{G}^n$ is flat over S, or*
(2) *E is pseudo-coherent, $\mathcal{G}^\bullet$ is bounded, and $\mathcal{G}^n$ is flat over S,*

then formation of

$$Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_*R\mathcal{H}om(L(g')^*E, (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 36.26.2 and 36.26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 36.26.1. This was shown in the proof of Lemma 36.26.4. □

36.27. Producing perfect complexes

0A1E The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 36.30.

08EV **Lemma 36.27.1.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that*

(1) *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
(2) *the support of $H^i(E)$ is proper over S for all i, and*
(3) *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. By Lemma 36.11.3 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$. Hence Rf_*E is pseudo-coherent (Lemma 36.10.3). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology, Lemma 20.49.5. By Lemma 36.10.6 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on S. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over S by assumptions (1) and (2), see Cohomology of Schemes, Lemma

30.26.6. In particular there exists a quasi-compact open $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. Moreover, f is quasi-separated by Morphisms, Lemma 29.15.7. Now

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F} = Rf_* \left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_* \left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}\right)$$

by Lemma 36.22.1 and Cohomology, Lemma 20.27.4. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. □

0DJQ **Lemma 36.27.2.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. The object K is perfect by Lemma 36.27.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus\nolimits_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers $a, b, r_a, \ldots, r_b$. This immediately implies the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are coherent. The hypothesis on the tor dimension also follows as $\mathcal{G}^i$ is flat over $f^{-1}\mathcal{O}_S$. □

0DJR **Lemma 36.27.3.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. Since E is a perfect complex there exists a dual perfect complex $E^\vee$, see Cohomology, Lemma 20.50.5. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 36.27.2. □

We will generalize the following lemma to flat and proper morphisms over general bases in Lemma 36.30.4 and to perfect proper morphisms in More on Morphisms, Lemma 37.61.13.

0B6F **Lemma 36.27.4.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a flat proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.*

Proof. We claim that Lemma 36.27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Since $\mathcal{O}_X$ is flat as a sheaf of $f^{-1}\mathcal{O}_S$-modules we find that condition (3) is satisfied. □

36.28. A projection formula for Ext

08IC Lemma 36.28.3 (or similar results in the literature) is sometimes used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

Suppose that $f : X \to S$ is a proper, flat, finitely presented morphism of schemes and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}^i_X(E, f^*\mathcal{F}) = \mathrm{Ext}^i_S((Rf_*E^\vee)^\vee, \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 36.22.1.

0A1F **Lemma 36.28.1.** *Assumptions and notation as in Lemma 36.27.2. Then there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 20.27.4, the second as $\mathcal{G}^n$ is a flat $f^{-1}\mathcal{O}_S$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \\ &= H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 20.13.1), and the third equality by Lemma 36.22.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of quasi-coherent $\mathcal{O}_S$-modules, the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) & \longrightarrow & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) & \longrightarrow & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes of $\mathcal{O}_X$-modules. This sequence is exact because $\mathcal{G}^n$ is flat over S. We omit the verification of the commutativity of the displayed diagram. □

08ID **Lemma 36.28.2.** *Assumptions and notation as in Lemma 36.27.3. Then there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 36.27.3 let $E^\vee$ be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$. Since we also have

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E^\vee$, the existence of the isomorphisms follows from Lemma 36.28.1 applied to $E^\vee$ and $\mathcal{G}^\bullet$. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}$ is flat over S. We omit the verification of the commutativity of the displayed diagram. □

08IF **Lemma 36.28.3.** *Let $f : X \to S$ be a morphism of schemes, $E \in D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ a complex of $\mathcal{O}_X$-modules. Assume*

1. *S is Noetherian,*
2. *f is locally of finite type,*
3. *$E \in D^-_{Coh}(\mathcal{O}_X)$,*
4. *$\mathcal{G}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S.*

Then the following two statements are true

(A) *for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial maps*

$$\alpha^i_{\mathcal{F}} : \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$

(B) *there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ and functorial isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps.

Proof. Proof of (A). Suppose $\mathcal{G}^i$ is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of $\mathcal{G}^i$. Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m - 1 + a)$ (possible by Theorem 36.14.6). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.27.3. This reduces us to the case that E is a perfect complex which is Lemma 36.28.2. The statement on boundaries is explained in the proof of Lemma 36.28.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 36.10.3. By Lemma 36.19.1 we can write $E = \mathrm{hocolim} E_n$ with E_n perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n^\vee$ be the dual perfect complex (Cohomology, Lemma 20.50.5). We obtain an inverse system $\ldots \to E_3^\vee \to E_2^\vee \to E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system

$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

perfect on S, see Lemma 36.27.2. By Lemma 36.28.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent $\mathcal{F}$ on S we have functorial canonical maps

$$\begin{array}{ccc} & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) & \\ \swarrow & & \searrow \\ H^i(S, K_{n+1} \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) & \longrightarrow & H^i(S, K_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{array}$$

which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(\mathcal{O}_S)$ such that for any quasi-coherent $\mathcal{F}$ on S the maps

$$\mathrm{Ext}^i_{\mathcal{O}_S}(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \mathrm{hocolim} L_n$ is pseudo-coherent, see Lemma 36.19.1. The mapping property of homotopy colimits gives that $\mathrm{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. □

0DJS **Remark 36.28.4.** The pseudo-coherent complex L of part (B) of Lemma 36.28.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes we have canonical functorial isomorphisms

$$\mathrm{Ext}^i_{\mathcal{O}_{S'}}(Lg^*L, \mathcal{F}') \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on S'. Obsere that we do **not** use derived pullback on $\mathcal{G}^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

36.29. Limits and derived categories

09RC In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

09RD **Situation 36.29.1.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

09RE **Lemma 36.29.2.** *In Situation 36.29.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^* E_0$ and $K_i = Lf_{i0}^* K_0$ for $i \geq 0$ and set $E = Lf_0^* E_0$ and $K = Lf_0^* K_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

1. *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or*
2. *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the canonical map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 30.4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 20.33.3. Thus it suffices to prove the lemma when S_0 is affine.

Assume S_0 is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. We will use Lemma 36.3.5 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective A_0-modules, see Lemma 36.10.7. We may represent the object K_0 by a K-flat complex $K_0^\bullet$ of A_0-modules. In this situation we are trying to prove

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules E_0^n and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object E_0 corresponds to a bounded above complex $E_0^\bullet$ of finite free A_0-modules, see Lemma 36.10.2. We may represent K_0 by a finite complex $K_0^\bullet$ of flat A_0-modules, see Lemma 36.10.4 and More on Algebra, Lemma 15.67.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). □

09RF **Lemma 36.29.3.** *In Situation 36.29.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the functor

$$\operatorname{colim}_{i\geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 30.4.1. First, we observe that we already know the functor is fully faithful by Lemma 36.29.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that P holds for U_0, V_0, and $U_0 \cap V_0$. Let E be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i\cap V_i} \to E_{V,i}|_{U_i\cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U\cap V} \to E|_{U\cap V} \to Lf_i^*E_{V,i}|_{U\cap V}.$$

Apply Cohomology, Lemma 20.45.1 to get an object E_i on S_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when S_0 is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. Using Lemmas 36.3.5 and 36.10.7 we see that we have to show that

$$D_{perf}(A) = \operatorname{colim} D_{perf}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. See More on Algebra, Lemma 15.75.18 for details. □

36.30. Cohomology and base change, VI

0A1G A final section on cohomology and base change continuing the discussion of Sections 36.22, 36.26, and 36.27. An easy to grok special case is given in Remark 36.30.2.

0A1H **Lemma 36.30.1.** *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.4. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 36.27.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 32.10.2 we may assume after increasing i,

that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 32.10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 32.13.5 and Cohomology of Schemes, Lemma 30.26.7. Finally, by Lemma 36.29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

0A1I **Remark 36.30.2.** Let R be a ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with support proper over R. By Lemma 36.30.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

0CSC **Lemma 36.30.3.** *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent object. Let $\mathcal{G}^\bullet$ be a bounded above complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 36.30.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is local on S, hence we may assume S is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ has cohomological dimension N as explained in Lemma 36.4.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m-N-1-b)$. This is possible by Theorem 36.14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m-N-1-b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m-N-1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$ are zero in degrees $\geq m-1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \to Rf_*(E \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet \longrightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m-N-1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet) \to Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 36.30.1 the source is a perfect complex. We conclude that K is m-pseudo-coherent as desired. □

0B91 **Lemma 36.30.4.** *Let S be a scheme. Let $f : X \to S$ be a proper morphism of finite presentation.*

(1) *Let $E \in D(\mathcal{O}_X)$ be perfect and f flat. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*
(2) *Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

Proof. Special cases of Lemma 36.30.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. □

0CSD **Lemma 36.30.5.** *Let S be a scheme. Let $f : X \to S$ be a flat proper morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

More generally, if $f : X \to S$ is proper and E on X is pseudo-coherent relative to S (More on Morphisms, Definition 37.59.2), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). See [**Kie72**].

Proof. Special case of Lemma 36.30.3 applied with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0. □

0D2Q **Lemma 36.30.6.** *Let R be a ring. Let X be a scheme and let $f : X \to \mathrm{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R-modules with surjective transition maps. Then the canonical map*

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\mathrm{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \mathrm{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 36.15.3) it suffices to check this for perfect E. By Lemma 36.3.2 we have $\lim \mathcal{O}_X \otimes_R M_n = R\lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$ of Cohomology, Section 20.44 commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R\lim R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology, Lemma 20.50.5. We have

$$R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*M_n) = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n$$

by Lemma 36.22.1. From Lemma 36.30.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 15.103.3 we obtain

$$R\lim R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R \lim M_n$$

as desired. □

0A1J **Lemma 36.30.7.** *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.5. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 36.27.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 32.10.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 32.10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 32.13.5 and Cohomology of Schemes, Lemma 30.26.7. Finally, by Lemma 36.29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.3 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

36.31. Perfect complexes

0BDH We first talk about jumping loci for betti numbers of perfect complexes. Given a complex E on a scheme X and a point x of X we often write $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x)$ instead of the more correct $Li_x^* E$, where $i_x : x \to X$ is the canonical morphism.

0BDI **Lemma 36.31.1.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function*

$$\beta_i : X \longrightarrow \{0, 1, 2, \ldots\}, \quad x \longmapsto \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

Then we have

1. *formation of β_i commutes with arbitrary base change,*
2. *the functions β_i are upper semi-continuous, and*
3. *the level sets of β_i are locally constructible in X.*

Proof. Consider a morphism of schemes $f : Y \to X$ and a point $y \in Y$. Let x be the image of y and consider the commutative diagram

$$\begin{array}{ccc} y & \xrightarrow{j} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i} & X \end{array}$$

Then we see that $Lg^* \circ Li^* = Lj^* \circ Lf^*$. This implies that the function β'_i associated to the pseudo-coherent complex Lf^*E is the pullback of the function β_i, in a formula: $\beta'_i = \beta_i \circ f$. This is the meaning of (1).

Fix i and let $x \in X$. It is enough to prove (2) and (3) holds in an open neighbourhood of x, hence we may assume X affine. Then we can represent E by a bounded above complex $\mathcal{F}^\bullet$ of finite free modules (Lemma 36.13.3). Then $P = \sigma_{\geq i-1}\mathcal{F}^\bullet$ is a perfect object and $P \to E$ induces an isomorphism

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x')) \to H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x'))$$

for all $x' \in X$. Thus we may assume E is perfect. In this case by More on Algebra, Lemma 15.76.7 there exists an affine open neighbourhood U of x and $a \leq b$ such that $E|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus \beta_a(x)} \to \mathcal{O}_U^{\oplus \beta_{a+1}(x)} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 36.3.5 and 36.10.7.) It follows immediately that $\beta_i(x') \leq \beta_i(x)$ for all $x' \in U$. This proves that β_i is upper semi-continuous.

To prove (3) we may assume that X is affine and E is given by a complex of finite free $\mathcal{O}_X$-modules (for example by arguing as in the previous paragraph, or by using Cohomology, Lemma 20.49.3). Thus we have to show that given a complex

$$\mathcal{O}_X^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X^{\oplus c}$$

the function associated to a point $x \in X$ the dimension of the cohomology of $\kappa_x^{\oplus a} \to \kappa_x^{\oplus b} \to \kappa_x^{\oplus c}$ in the middle has constructible level sets. Let $A \in \mathrm{Mat}(a \times b, \Gamma(X, \mathcal{O}_X))$ be the matrix of the first arrow. The rank of the image of A in $\mathrm{Mat}(a \times b, \kappa(x))$ is equal to r if all $(r+1) \times (r+1)$-minors of A vanish at x and there is some $r \times r$-minor of A which does not vanish at x. Thus the set of points where the rank is r is a constructible locally closed set. Arguing similarly for the second arrow and putting everything together we obtain the desired result. □

0BDJ **Lemma 36.31.2.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. The function*

$$\chi_E : X \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

is locally constant on X.

Proof. By Cohomology, Lemma 20.49.3 we see that we can, locally on X, represent E by a finite complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules. On such an open the function χ_E is constant with value $\sum (-1)^i \mathrm{rank}(\mathcal{E}^i)$. □

0BDK **Lemma 36.31.3.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subscheme $U \subset X$ characterized by the following*

1. *$E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank r,*
2. *a morphism $f : Y \to X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i.*

Proof. Let $\beta_j : X \to \{0, 1, 2, \ldots\}$ for $j \in \mathbf{Z}$ be the functions of Lemma 36.31.1. Then the set

$$W = \{x \in X \mid \beta_j(x) \leq 0 \text{ for all } j \neq i\}$$

is open in X and its formation commutes with pullback to any Y over X. This follows from the lemma using that apriori in a neighbourhood of any point only a finite number of the β_j are nonzero. Thus we may replace X by W and assume that $\beta_j(x) = 0$ for all $x \in X$ and all $j \neq i$. In this case $H^i(E)$ is a finite locally free module and $E \cong H^i(E)[-i]$, see for example More on Algebra, Lemma 15.76.7. Thus X is the disjoint union of the open subschemes where the rank of $H^i(E)$ is fixed and we win. □

0BDL **Lemma 36.31.4.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $r \geq 0$. Then there exists a locally closed subscheme $j : Z \to X$ characterized by the following*

(1) $H^a(Lj^*E)$ *is a locally free* $\mathcal{O}_Z$*-module of rank* r*, and*
(2) *a morphism* $f : Y \to X$ *factors through* Z *if and only if for all morphisms* $g : Y' \to Y$ *the* $\mathcal{O}_{Y'}$*-module* $H^a(L(f \circ g)^*E)$ *is locally free of rank* r*.*

Moreover, $j : Z \to X$ *is of finite presentation and we have*

(3) *if* $f : Y \to X$ *factors as* $Y \xrightarrow{g} Z \to X$*, then* $H^a(Lf^*E) = g^*H^a(Lj^*E)$*,*
(4) *if* $\beta_a(x) \leq r$ *for all* $x \in X$*, then* j *is a closed immersion and given* $f : Y \to X$ *the following are equivalent*
   (a) $f : Y \to X$ *factors through* Z*,*
   (b) $H^a(Lf^*E)$ *is a locally free* $\mathcal{O}_Y$*-module of rank* r*,*

   and if $r = 1$ *these are also equivalent to*
   (c) $\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(H^a(Lf^*E), H^a(Lf^*E))$ *is injective.*

Proof. First, let $U \subset X$ be the locally constructible open subscheme where the function β_a of Lemma 36.31.1 has values $\leq r$. Let $f : Y \to X$ be as in (2). Then for any $y \in Y$ we have $\beta_a(Lf^*E) = r$ hence y maps into U by Lemma 36.31.1. Hence f as in (2) factors through U. Thus we may replace X by U and assume that $\beta_a(x) \in \{0, 1, \ldots, r\}$ for all $x \in X$. We will show that in this case there is a closed subscheme $Z \subset X$ cut out by a finite type quasi-coherent ideal characterized by the equivalence of (4) (a), (b) and (4)(c) if $r = 1$ and that (3) holds. This will finish the proof because it will a fortiori show that morphisms as in (2) factor through Z.

If $x \in X$ and $\beta_a(x) < r$, then there is an open neighbourhood of x where $\beta_a < r$ (Lemma 36.31.1). In this way we see that set theoretically at least Z is a closed subset.

To get a scheme theoretic structure, consider a point $x \in X$ with $\beta_a(x) = r$. Set $\beta = \beta_{a+1}(x)$. By More on Algebra, Lemma 15.76.7 there exists an affine open neighbourhood U of x such that $K|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus r} \xrightarrow{(f_{ij})} \mathcal{O}_U^{\oplus \beta} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 36.3.5 and 36.10.7.) Now, if $g : Y \to U$ is any morphism of schemes such that $g^\sharp(f_{ij})$ is nonzero for some pair i, j, then $H^a(Lg^*E)$ is not a locally free $\mathcal{O}_Y$-module of rank r. See More on Algebra, Lemma 15.15.7. Trivially $H^a(Lg^*E)$ is a locally free $\mathcal{O}_Y$-module if $g^\sharp(f_{ij}) = 0$ for all i, j. Thus we see that over U the closed subscheme cut out by all f_{ij} satisfies (3) and we have the equivalence of (4)(a) and (b). The characterization of Z shows that the locally constructed patches glue (details omitted). Finally, if $r = 1$ then (4)(c) is equivalent to (4)(b) because in this case locally $H^a(Lg^*E) \subset \mathcal{O}_Y$ is the annihilator of the ideal generated by the elements $g^\sharp(f_{ij})$. □

36.32. Applications

0BDM Mostly applications of cohomology and base change. In the future we may generalize these results to the situation discussed in Lemma 36.30.1.

0BDN **Lemma 36.32.1.** *Let* $f : X \to S$ *be a proper morphism of finite presentation. Let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module of finite presentation, flat over* S*. For fixed* $i \in \mathbf{Z}$ *consider the function*

$$\beta_i : S \to \{0, 1, 2, \ldots\}, \quad s \longmapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$$

Then we have

(1) *formation of β_i commutes with arbitrary base change,*
(2) *the functions β_i are upper semi-continuous, and*
(3) *the level sets of β_i are locally constructible in S.*

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_s, \mathcal{F}_s) = H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

Thus the lemma follows from Lemma 36.31.1. □

0B9T **Lemma 36.32.2.** *Let $f : X \to S$ be a proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. The function*

$$s \longmapsto \chi(X_s, \mathcal{F}_s)$$

is locally constant on S. Formation of this function commutes with base change.

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus we have to show the map

$$s \longmapsto \sum (-1)^i \dim_{\kappa(s)} H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

is locally constant on S. This is Lemma 36.31.2. □

0B9S **Lemma 36.32.3.** *Let $f : X \to S$ be a proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Fix $i, r \in \mathbf{Z}$. Then there exists an open subscheme $U \subset S$ with the following property: A morphism $T \to S$ factors through U if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i.*

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 36.31.3. □

0D4E **Lemma 36.32.4.** *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S with support proper over S. If $R^i f_*\mathcal{F} = 0$ for $i > 0$, then $f_*\mathcal{F}$ is locally free and its formation commutes with arbitrary base change (see proof for explanation).*

Proof. By Lemma 36.30.1 the object $E = Rf_*\mathcal{F}$ of $D(\mathcal{O}_S)$ is perfect and its formation commutes with arbitrary base change, in the sense that $Rf'_*(g')^*\mathcal{F} = Lg^*E$ for any cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes. Since there is never any cohomology in degrees < 0, we see that E (locally) has tor-amplitude in $[0, b]$ for some b. If $H^i(E) = R^i f_*\mathcal{F} = 0$ for $i > 0$, then E has tor amplitude in $[0, 0]$. Whence $E = H^0(E)[0]$. We conclude $H^0(E) = f_*\mathcal{F}$ is finite locally free by More on Algebra, Lemma 15.75.2 (and the characterization of finite projective modules in Algebra, Lemma 10.78.2). Commutation with base

change means that $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$ for a diagram as above and it follows from the already established commutation of base change for E. □

0E62 **Lemma 36.32.5.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *for all $s \in S$ we have $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$.*

Then we have

(a) *$f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change,*
(b) *locally on S we have*

$$Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$$

in $D(\mathcal{O}_S)$ where P is perfect of tor amplitude in $[1, \infty)$.

Proof. By cohomology and base change (Lemma 36.30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor amplitude in $[0, \infty)$. Second, it implies that for $s \in S$ we have $H^0(E \otimes^{\mathbf{L}} \kappa(s)) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. It follows that the map $\mathcal{O}_S \to Rf_*\mathcal{O}_X = E$ induces an isomorphism $\mathcal{O}_S \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$. Hence $H^0(E) \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$ is surjective and we may apply More on Algebra, Lemma 15.77.2 to see that, after replacing S by an affine open neighbourhood of s, we have a decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ with $\tau_{\geq 1}E$ perfect of tor amplitude in $[1, \infty)$. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. It follows that $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 15.75.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 10.78.2). It follows that the map $\mathcal{O}_S \to H^0(E)$ is an isomorphism as we can check this after tensoring with residue fields (Algebra, Lemma 10.79.4). □

0E0L **Lemma 36.32.6.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the geometric fibres of f are reduced and connected.*

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change.

Proof. By Lemma 36.32.5 it suffices to show that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$. This follows from Varieties, Lemma 33.9.3 and the fact that X_s is geometrically connected and geometrically reduced. □

0G7X **Lemma 36.32.7.** *Let $f : X \to S$ be a proper morphism of schemes. Let $s \in S$ and let $e \in H^0(X_s, \mathcal{O}_{X_s})$ be an idempotent. Then e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$.*

Proof. Let $X_s = T_1 \amalg T_2$ be the disjoint union decomposition with T_1 and T_2 nonempty and open and closed in X_s corresponding to e, i.e., such that e is identitically 1 on T_1 and identically 0 on T_2.

Assume S is Noetherian. We will use the theorem on formal functions in the form of Cohomology of Schemes, Lemma 30.20.7. It tells us that

$$(f_*\mathcal{O}_X)^\wedge_s = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we obtain for all n a disjoint union decomposition of schemes $X_n = T_{1,n} \amalg T_{2,n}$ where the underlying topological space

of $T_{i,n}$ is T_i for $i = 1, 2$. This means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent e_n, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that e_{n+1} restricts to e_n on X_n. Hence $e_\infty = \lim e_n$ is a nontrivial idempotent of the limit. Thus e_∞ is an element of the completion of $(f_*\mathcal{O}_X)_s$ mapping to e in $H^0(X_s, \mathcal{O}_{X_s})$. Since the map $(f_*\mathcal{O}_X)^\wedge_s \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $(f_*\mathcal{O}_X)^\wedge_s/\mathfrak{m}_s(f_*\mathcal{O}_X)^\wedge_s = (f_*\mathcal{O}_X)_s/\mathfrak{m}_s(f_*\mathcal{O}_X)_s$ (Algebra, Lemma 10.96.3) we conclude that e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$ as desired.

General case: we reduce the general case to the Noetherian case by limit arguments. We urge the reader to skip the proof. We may replace S by an affine open neighbourhood of s. Thus we may and do assume that S is affine. By Limits, Lemma 32.13.3 we can write $(f : X \to S) = \lim(f_i : X_i \to S_i)$ with f_i proper and S_i Noetherian. Denote $s_i \in S_i$ the image of s. Then $s = \lim s_i$, see Limits, Lemma 32.4.4. Then $X_s = X \times_S s = \lim X_i \times_{S_i} s_i = \lim X_{i,s_i}$ because limits commute with limits (Categories, Lemma 4.14.10). Hence e is the image of some idempotent $e_i \in H^0(X_{i,s_i}, \mathcal{O}_{X_{i,s_i}})$ by Limits, Lemma 32.4.7. By the Noetherian case there is an element $\tilde{e}_i$ in the stalk $(f_{i,*}\mathcal{O}_{X_i})_{s_i}$ mapping to e_i. Taking the pullback of $\tilde{e}_i$ we get an element $\tilde{e}$ of $(f_*\mathcal{O}_X)_s$ mapping to e and the proof is complete. □

0G7Y **Lemma 36.32.8.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the fibre X_s is geometrically reduced.*

Then, after replacing S by an open neighbourhood of s, there exists a direct sum decomposition $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X \oplus P$ in $D(\mathcal{O}_S)$ where $f_*\mathcal{O}_X$ is a finite étale $\mathcal{O}_S$-algebra and P is a perfect of tor amplitude in $[1, \infty)$.

Proof. The proof of this lemma is similar to the proof of Lemma 36.32.5 which we suggest the reader read first. By cohomology and base change (Lemma 36.30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor amplitude in $[0, \infty)$.

We claim that after replacing S by an open neighbourhood of s we can find a direct sum decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ in $D(\mathcal{O}_S)$ with $\tau_{\geq 1}E$ of tor amplitude in $[1, \infty)$. Assume the claim is true for now and assume we've made the replacement so we have the direct sum decomposition. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. Hence $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 15.75.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 10.78.2). Of course $H^0(E) = f_*\mathcal{O}_X$ is an $\mathcal{O}_S$-algebra. By cohomology and base change we obtain $H^0(E) \otimes \kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$. By Varieties, Lemma 33.9.3 and the assumption that X_s is geometrically reduced, we see that $\kappa(s) \to H^0(E) \otimes \kappa(s)$ is finite étale. By Morphisms, Lemma 29.36.17 applied to the finite locally free morphism $\underline{\mathrm{Spec}}_S(H^0(E)) \to S$, we conclude that after shrinking S the $\mathcal{O}_S$-algebra $H^0(E)$ is finite étale.

It remains to prove the claim. For this it suffices to prove that the map

$$(f_*\mathcal{O}_X)_s \longrightarrow H^0(X_s, \mathcal{O}_{X_s}) = H^0(E \otimes^{\mathbf{L}} \kappa(s))$$

is surjective, see More on Algebra, Lemma 15.77.2. Choose a flat local ring homomorphism $\mathcal{O}_{S,s} \to A$ such that the residue field k of A is algebraically closed, see Algebra, Lemma 10.159.1. By flat base change (Cohomology of Schemes,

Lemma 30.5.2) we get $H^0(X_A, \mathcal{O}_{X_A}) = (f_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} A$ and $H^0(X_k, \mathcal{O}_{X_k}) = H^0(X_s, \mathcal{O}_{X_s}) \otimes_{\kappa(s)} k$. Hence it suffices to prove that $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ is surjective. Since X_k is a reduced proper scheme over k and since k is algebraically closed, we see that $H^0(X_k, \mathcal{O}_{X_k})$ is a finite product of copies of k by the already used Varieties, Lemma 33.9.3. Since by Lemma 36.32.7 the idempotents of this k-algebra are in the image of $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ we conclude. □

36.33. Other applications

0CRN In this section we state and prove some results that can be deduced from the theory worked out above.

0EX6 **Lemma 36.33.1.** *Let R be a coherent ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over R, with support proper over R. Then $H^i(X, \mathcal{G})$ is a coherent R-module.*

Proof. Combine Lemma 36.30.1 with More on Algebra, Lemmas 15.65.18 and 15.75.2. □

0CRP **Lemma 36.33.2.** *Let X be a quasi-compact and quasi-separated scheme. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens. Then for any quasi-compact open $U \subset X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets*

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology, Lemma 20.50.5 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Cohomology of Schemes, Lemma 30.4.1.

First we show that it holds when $U = \mathrm{Spec}(A)$ is affine. Namely, we can represent K by a complex of A-modules $K^\bullet$ and E by a finite complex of finite projective A-modules $P^\bullet$. See Lemmas 36.3.5 and 36.10.7 and our definition of perfect complexes of A-modules (More on Algebra, Definition 15.75.1). Then $(E \otimes^{\mathbf{L}} K)|_U$ is represented by the total complex associated to the double complex $P^\bullet \otimes_A K^\bullet$ (Lemma 36.3.9). Using induction on the length of the complex $P^\bullet$ (or using a suitable spectral sequence) we see that it suffices to show that $H^i(P^a \otimes_A K^\bullet)$ is countable for each a. Since P^a is a direct summand of $A^{\oplus n}$ for some n this follows from the assumption that the cohomology group $H^i(K^\bullet)$ is countable.

To finish the proof it suffices to show: if $U = V \cup W$ and the result holds for V, W, and $V \cap W$, then the result holds for U. This is an immediate consequence of the Mayer-Vietoris sequence, see Cohomology, Lemma 20.33.4. □

0CRQ **Lemma 36.33.3.** *Let X be a quasi-compact and quasi-separated scheme such that the sets of sections of $\mathcal{O}_X$ over affine opens are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$K = hocolim E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and*
2. *the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens.*

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 13.33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on X.

Assume (2). Choose a K-injective complex $\mathcal{K}^\bullet$ representing K. Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex $\mathcal{I}^\bullet$. According to Theorem 36.18.2 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A, \mathrm{d})$ where (A, d) is the differential graded algebra

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \mathrm{Ext}^i(E, E)$ and $H^i(M) = \mathrm{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, \mathrm{d})$. By Differential Graded Algebra, Lemma 22.38.3 it suffices show that $\mathrm{Ext}^i(E, E)$ and $\mathrm{Ext}^i(E, K)$ are countable for each i. This follows from Lemma 36.33.2. □

0CRR **Lemma 36.33.4.** *Let A be a ring. Let X be a scheme of finite presentation over A. Let $f : U \to X$ be a flat morphism of finite presentation. Then*

(1) *there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that*

$$R\Gamma(U, Lf^*K) = \mathit{hocolim}\ R\,\mathrm{Hom}_X(L_n, K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and

(2) *there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that*

$$R\Gamma(U, Lf^*K) = \mathit{hocolim}\ R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 36.22.1 we have

$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$

functorially in K. Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 36.4.5. Similarly, $- \otimes^{\mathbf{L}} K$ commutes with derived colimits because $- \otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove (2) it suffices to write $Rf_*\mathcal{O}_U = \mathrm{hocolim} E_n$ for a system of perfect objects E_n of $D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology, Lemma 20.50.5.

Write $A = \mathrm{colim}\, A_i$ with A_i of finite type over $\mathbf{Z}$. By Limits, Lemma 32.10.1 we can find an i and morphisms $U_i \to X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to $\mathrm{Spec}(A)$ recovers $U \to X \to \mathrm{Spec}(A)$. After increasing i we may assume that $f_i : U_i \to X_i$ is flat, see Limits, Lemma 32.8.7. By Lemma 36.22.5 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \to X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i. Hence we may assume that U and X are of finite type over $\mathbf{Z}$.

Assume $f : U \to X$ is a morphism of schemes of finite type over $\mathbf{Z}$. To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes. To see

this we apply Lemma 36.33.3. Thus it suffices to show that $R^if_*\mathcal{O}_U$ has countable sets of sections over affine opens. This follows from Lemma 36.33.2 applied to the structure sheaf. □

36.34. Characterizing pseudo-coherent complexes, II

0CSE This section is a continuation of Section 36.19. In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. More results of this nature can be found in More on Morphisms, Section 37.69.

0CSF **Lemma 36.34.1.** *Let A be a ring. Let R be a (possibly noncommutative) A-algebra which is finite free as an A-module. Then any object M of $D(R)$ which is pseudo-coherent in $D(A)$ can be represented by a bounded above complex of finite free (right) R-modules.*

Proof. Choose a complex $M^\bullet$ of right R-modules representing M. Since M is pseudo-coherent we have $H^i(M) = 0$ for large enough i. Let m be the smallest index such that $H^m(M)$ is nonzero. Then $H^m(M)$ is a finite A-module by More on Algebra, Lemma 15.65.3. Thus we can choose a finite free R-module F^m and a map $F^m \to M^m$ such that $F^m \to M^m \to M^{m+1}$ is zero and such that $F^m \to H^m(M)$ is surjective. Picture:

$$\begin{array}{ccccccc} & & F^m & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ M^{m-1} & \longrightarrow & M^m & \longrightarrow & M^{m+1} & \longrightarrow & \ldots \end{array}$$

By descending induction on $n \leq m$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > m$. Picture

$$\begin{array}{ccccccccccc} F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots & \longrightarrow & F^i & \longrightarrow & 0 & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots & \longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & \ldots \end{array}$$

The base case is $n = m$ which we've done above. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. Observe that $F^\bullet$ is pseudo-coherent as a complex of A-modules because R is finite free as an A-module. Hence by More on Algebra, Lemma 15.65.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent as a complex of A-modules. By More on Algebra, Lemma 15.65.3 we see that $H^{n-1}(C^\bullet)$ is a finite A-module. Choose a finite free R-module F^{n-1} and a map $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccc} F^{n-1} & \xrightarrow[d^{n-1}]{} & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc} (F^n \to \ldots) & \longrightarrow & (F^{n-1} \to \ldots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \to \ldots)[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\ (F^n \to \ldots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \to \ldots)[1] \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. □

0CSG **Lemma 36.34.2.** *Let A be a ring. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$. The following are equivalent*

1. *K is pseudo-coherent,*
2. *$R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each pseudo-coherent object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
3. *$R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
4. *$R\mathrm{Hom}_{\mathbf{P}^n_A}(E, K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
5. *$R\Gamma(\mathbf{P}^n_A, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(d))$ is pseudo-coherent object of $D(A)$ for $d = 0, 1, \ldots, n$.*

Proof. Recall that

$$R\mathrm{Hom}_{\mathbf{P}^n_A}(E, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K))$$

by definition, see Cohomology, Section 20.44. Thus parts (4) and (3) are equivalent by Cohomology, Lemma 20.50.5.

Since every perfect complex is pseudo-coherent, it is clear that (2) implies (3).

Assume (1) holds. Then $E \otimes^{\mathbf{L}} K$ is pseudo-coherent for every pseudo-coherent E, see Cohomology, Lemma 20.47.5. By Lemma 36.30.5 the direct image of such a pseudo-coherent complex is pseudo-coherent and we see that (2) is true.

Part (3) implies (5) because we can take $E = \mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$.

To finish the proof we have to show that (5) implies (1). Let P be as in (36.20.0.1) and R as in (36.20.0.2). By Lemma 36.20.1 we have an equivalence

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$$

Let $M \in D(R)$ be an object such that $M \otimes^{\mathbf{L}} P = K$. By Differential Graded Algebra, Lemma 22.35.4 there is an isomorphism

$$R\mathrm{Hom}(R, M) = R\mathrm{Hom}_{\mathbf{P}^n_A}(P, K)$$

in $D(A)$. Arguing as above we obtain

$$R\mathrm{Hom}_{\mathbf{P}^n_A}(P, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K)) = R\Gamma(\mathbf{P}^n_A, P^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{P}^n_A}} K).$$

Using that $P^\vee$ is the direct sum of $\mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$ and (5) we conclude $R\mathrm{Hom}(R, M)$ is pseudo-coherent as a complex of A-modules. Of course $M = R\mathrm{Hom}(R, M)$ in $D(A)$. Thus M is pseudo-coherent as a complex of A-modules. By Lemma 36.34.1 we may represent M by a bounded above complex $F^\bullet$ of finite free R-modules. Then $F^\bullet = \bigcup_{p \geq 0} \sigma_{\geq p} F^\bullet$ is a filtration which shows that $F^\bullet$ is a

differential graded R-module with property (P), see Differential Graded Algebra, Section 22.20. Hence $K = M \otimes^{\mathbf{L}}_R P$ is represented by $F^\bullet \otimes_R P$ (follows from the construction of the derived tensor functor, see for example the proof of Differential Graded Algebra, Lemma 22.35.3). Since $F^\bullet \otimes_R P$ is a bounded above complex whose terms are direct sums of copies of P we conclude that the lemma is true. □

0CSH **Lemma 36.34.3.** *Let A be a ring. Let X be a scheme over A which is quasi-compact and quasi-separated. Let $K \in D^-_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.*

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ has cohomological dimension N as explained in Lemma 36.4.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X. It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 36.14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \to R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent as desired. □

36.35. Relatively perfect objects

0DHZ In this section we introduce a notion from [**Lie06a**].

0DI0 **Definition 36.35.1.** Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is *perfect relative to S* or *S-perfect* if E is pseudo-coherent (Cohomology, Definition 20.47.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 20.48.1).

Please see Remark 36.35.14 for a discussion.

0DI1 **Example 36.35.2.** Let k be a field. Let X be a scheme of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{Coh}(X)$ (by Lemma 36.10.3). Thus being relatively perfect does **not** mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 15.84. We can link the notion for schemes with the algebraic notion as follows.

0DI2 **Lemma 36.35.3.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

(1) *E is S-perfect,*

(2) *for any affine open $U \subset X$ mapping into an affine open $V \subset S$ the complex $R\Gamma(U, E)$ is $\mathcal{O}_S(V)$-perfect.*
(3) *there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex $R\Gamma(U_{ij}, E)$ is $\mathcal{O}_S(V_i)$-perfect.*

Proof. Being pseudo-coherent is a local property and "locally having finite tor dimension" is a local property. Hence this lemma immediately reduces to the statement: if X and S are affine, then E is S-perfect if and only if $K = R\Gamma(X, E)$ is $\mathcal{O}_S(S)$-perfect. Say $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and E corresponds to $K \in D(A)$, i.e., $K = R\Gamma(X, E)$, see Lemma 36.3.5.

Observe that K is R-perfect if and only if K is pseudo-coherent and has finite tor dimension as a complex of R-modules (More on Algebra, Definition 15.84.1). By Lemma 36.10.2 we see that E is pseudo-coherent if and only if K is pseudo-coherent. By Lemma 36.10.5 we see that E has finite tor dimension over $f^{-1}\mathcal{O}_S$ if and only if K has finite tor dimension as a complex of R-modules. □

0DI3 **Lemma 36.35.4.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of S-perfect objects is a saturated[4] triangulated subcategory.*

Proof. This follows from Cohomology, Lemmas 20.47.4, 20.47.6, 20.48.6, and 20.48.8. □

0DI4 **Lemma 36.35.5.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is S-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is S-perfect if K is perfect and M is S-perfect.*

Proof. First proof: reduce to the affine case using Lemma 36.35.3 and then apply More on Algebra, Lemma 15.84.3. □

0DI5 **Lemma 36.35.6.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $g : S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is S-perfect, then $L(g')^*K$ is S'-perfect.*

Proof. First proof: reduce to the affine case using Lemma 36.35.3 and then apply More on Algebra, Lemma 15.84.5.

Second proof: $L(g')^*K$ is pseudo-coherent by Cohomology, Lemma 20.47.3 and the bounded tor dimension property follows from Lemma 36.22.8. □

0DI6 **Situation 36.35.7.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : S \to S_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to S_0$. We set $X_i = S_i \times_{S_0} X_0$ and $X = S \times_{S_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

[4] Derived Categories, Definition 13.6.1.

0DI7 **Lemma 36.35.8.** *In Situation 36.35.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map*

$$\operatorname{colim}_{i\geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to S_0)^{-1}\mathcal{O}_{S_0})$

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that

$$\operatorname{colim}_{i\geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. If P holds for U_0, V_0 and $U_0 \cap V_0$, then it holds for $U_0 \cup V_0$ by Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 20.33.3.

Denote $\pi_0 : X_0 \to S_0$ the given morphism. Then we can first consider $U_0 = \pi_0^{-1}(W_0)$ with $W_0 \subset S_0$ quasi-compact open. By the induction principle of Cohomology of Schemes, Lemma 30.4.1 applied to quasi-compact opens of S_0 and the remark above, we find that it is enough to prove P for $U_0 = \pi_0^{-1}(W_0)$ with W_0 affine. In other words, we have reduced to the case where S_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and S_0 are affine, the result follows from More on Algebra, Lemma 15.84.7. Namely, by Lemmas 36.10.1 and 36.3.5 the statement is translated into computations of homs in the derived categories of modules. Then Lemma 36.10.2 shows that the complex of modules corresponding to K_0 is pseudo-coherent. And Lemma 36.10.5 shows that the complex of modules corresponding to L_0 has finite tor dimension over $\mathcal{O}_{S_0}(S_0)$. Thus the assumptions of More on Algebra, Lemma 15.84.7 are satisfied and we win. □

0DI8 **Lemma 36.35.9.** *In Situation 36.35.7 the category of S-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of S_i-perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that the functor

$$\operatorname{colim}_{i\geq 0} D_{S_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{S\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We observe that we already know this functor is fully faithful by Lemma 36.35.8. Thus it suffices to prove essential surjectivity.

Suppose that P holds for quasi-compact opens U_0, V_0 of X_0. We claim that P holds for $U_0 \cup V_0$. We will use the notation $U_i = f_{i0}^{-1}U_0$, $U = f_0^{-1}U_0$, $V_i = f_{i0}^{-1}V_0$, and $V = f_0^{-1}V_0$ and we will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \cap V \to U_i \cap V_i$, and $U \cup V \to U_i \cup V_i$. Suppose E is an S-perfect object of $D(\mathcal{O}_{U\cup V})$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an S_i-perfect object $E_{U,i}$ on U_i, an S_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 20.45.1 to get an object E_i on $U_i \cup V_i$ and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is S_i-perfect (because being relatively perfect is a local property) and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 36.35.8 using the induction principle (Cohomology of Schemes, Lemma 30.4.1) we reduce to the case where both X_0 and S_0 are affine. (First work with opens in S_0 to reduce to S_0 affine, then work with opens in X_0 to reduce to X_0 affine.) In the affine case the result follows from More on Algebra, Lemma 15.84.7. The translation into algebra is done by Lemma 36.35.3. □

0DJT **Lemma 36.35.10.** *Let $f : X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be S-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

Proof. The statement on base change is Lemma 36.22.5. Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where S is Noetherian affine by a limit argument.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemmas 32.13.1 and 32.8.7 we may assume X_i is proper and flat over R_i. By Lemma 36.35.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.1 to $X_i \to \mathrm{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. □

0DJU **Lemma 36.35.11.** *Let $f : X \to S$ be a morphism of schemes. Let $E, K \in D(\mathcal{O}_X)$. Assume*

1. *S is quasi-compact and quasi-separated,*
2. *f is proper, flat, and of finite presentation,*
3. *E is S-perfect,*
4. *K is pseudo-coherent.*

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ such that

$$Rf_*R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_S)$$

and the same is true after arbitrary base change: given

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

cartesian, then we have

$$Rf'_*R\mathcal{H}om(L(g')^*K, L(g')^*E) = R\mathcal{H}om(Lg^*L, \mathcal{O}_{S'})$$

Proof. Since S is quasi-compact and quasi-separated, the same is true for X. By Lemma 36.19.1 we can write $K = \mathrm{hocolim} K_n$ with K_n perfect and $K_n \to K$

inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology, Lemma 20.50.5). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 36.35.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is S-perfect. Thus we may apply Lemma 36.35.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on S with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_* R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^* M_n = Rf'_*((L(g')^* K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^* E) = Rf'_* R\mathcal{H}om(L(g')^* K_n, L(g')^* E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Lemma 36.22.8. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Take S' equal to the spectrum of the residue field at a point $s \in S$ and pull back to see that $C_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 15.76.7 we see that the perfect complex C_n has tor amplitude in $[n + a, m_n]$ for some integer m_n. In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n - a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 36.19.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology, Lemma 20.51.1) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_S) = R\mathcal{H}om(L, \mathcal{O}_S)$$

This proves the formula over S. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. □

0DJV **Remark 36.35.12.** The reader may have noticed the similarity between Lemma 36.35.11 and Lemma 36.28.3. Indeed, the pseudo-coherent complex L of Lemma 36.35.11 may be characterized as the unique pseudo-coherent complex on S such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_S)$. If we ever need this we will formulate a precise result here and give a detailed proof.

0GEH **Lemma 36.35.13.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *E is S-perfect, and*

(2) *E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.*

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 36.35.3. Say $X \to S$ corresponds to $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ and E corresponds to K in $D(A)$. If s corresponds to the prime $\mathfrak{p} \subset R$, then $L(X_s \to X)^*E$ corresponds to $K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ as $R \to A$ is flat, see for example Lemma 36.22.5. Thus we see that our lemma follows from the corresponding algebra result, see More on Algebra, Lemma 15.84.10. □

0DI9 **Remark 36.35.14.** Our Definition 36.35.1 of a relatively perfect complex is equivalent to the one given in [**Lie06a**] whenever our definition applies[5]. Next, suppose that $f : X \to S$ is only assumed to be locally of finite type (not necessarily flat, nor locally of finite presentation). The definition in the paper cited above is that $E \in D(\mathcal{O}_X)$ is relatively perfect if

(A) locally on X the object E should be quasi-isomorphic to a finite complex of S-flat, finitely presented $\mathcal{O}_X$-modules.

On the other hand, the natural generalization of our Definition 36.35.1 is

(B) E is pseudo-coherent relative to S (More on Morphisms, Definition 37.59.2) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 20.48.1).

The advantage of condition (B) is that it clearly defines a triangulated subcategory of $D(\mathcal{O}_X)$, whereas we suspect this is not the case for condition (A). The advantage of condition (A) is that it is easier to work with in particular in regards to limits.

36.36. The resolution property

0F85 This notion is discussed in the paper [**Tot04**]; the discussion is continued in [**Gro10**], [**Gro12**], and [**Gro17**]. It is currently not known if a proper scheme over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com.

We can make the following definition although it scarcely makes sense to consider it for general schemes.

0F86 **Definition 36.36.1.** Let X be a scheme. We say X has the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

If X is a quasi-compact and quasi-separated scheme, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Properties, Lemma 28.22.8. If X is a Noetherian scheme, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Schemes, Lemma 30.9.1.

0F87 **Lemma 36.36.2.** *Let X be a scheme. If X has an ample invertible $\mathcal{O}_X$-module, then X has the resolution property.*

Proof. Immediate consequence of Properties, Proposition 28.26.13. □

0FDD **Lemma 36.36.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

[5]To see this, use Lemma 36.35.3 and More on Algebra, Lemma 15.84.4.

(1) *Y is quasi-compact and quasi-separated and has the resolution property,*
(2) *there exists an f-ample invertible module on X.*

Then X has the resolution property.

Proof. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an f-ample invertible module. Choose an affine open covering $Y = V_1 \cup \ldots \cup V_m$. Set $U_j = f^{-1}(V_j)$. By Properties, Proposition 28.26.13 for each j we know there exists finitely many maps $s_{j,i} : \mathcal{L}^{\otimes n_{j,i}}|_{U_j} \to \mathcal{F}|_{U_j}$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of $s_{j,i}$ as a section over V_j of the sheaf $\mathcal{H}_{-n_{j,i}}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{H}_{-n_{j,i}}$ such that $s_{j,i}$ is in the image. Then the corresponding maps

$$f^*\mathcal{E}_{i,j} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_{i,j}} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Properties, Lemma 28.22.3 for each i, j we can find a finite type quasi-coherent submodule $\mathcal{H}'_{i,j} \subset \mathcal{H}_{-n_{j,i}}$ which contains the section $s_{i,j}$ over V_j. Thus the resolution property of Y produces surjections $\mathcal{E}_{i,j} \to \mathcal{H}'_{j,i}$ and we conclude. □

0F88 **Lemma 36.36.4.** *Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes with Y quasi-compact and quasi-separated. If Y has the resolution property, so does X.*

Proof. By Morphisms, Lemma 29.37.6 this is a special case of Lemma 36.36.3. □

Here is a case where one can prove the resolution property goes down.

0GTC **Lemma 36.36.5.** *Let $f : X \to Y$ be a surjective finite locally free morphism of schemes. If X has the resolution property, so does Y.*

Proof. The condition means that f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of positive rank. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. By assumption there exists a surjection $\mathcal{E} \to f^*\mathcal{G}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Since f_* is exact on quasi-coherent modules (Cohomology of Schemes, Lemma 30.2.3) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since $f_*\mathcal{E}$ is finite locally free[6], we conclude. □

0F89 **Lemma 36.36.6.** *Let X be a scheme. Suppose given*

(1) *a finite affine open covering $X = U_1 \cup \ldots \cup U_m$*
(2) *finite type quasi-coherent ideals $\mathcal{I}_j$ with $V(\mathcal{I}_j) = X \setminus U_j$*

[6]Namely, if $A \to B$ is a finite locally free ring map and N is a finite locally free B-module, then N is a finite locally free A-module. To see this, first note that N finite locally free over B implies N is flat and finitely presented as a B-module, see Algebra, Lemma 10.78.2. Then N is an A-module of finite presentation by Algebra, Lemma 10.36.23 and a flat A-module by Algebra, Lemma 10.39.4. Then conclude by using Algebra, Lemma 10.78.2 over A.

Then X has the resolution property if and only if $\mathcal{I}_j$ is the quotient of a finite locally free $\mathcal{O}_X$-module for $j = 1, \ldots, m$.

Proof. One direction of the lemma is trivial. For the other, say $\mathcal{E}_j \to \mathcal{I}_j$ is a surjection with $\mathcal{E}_j$ finite locally free. In the next paragraph, we reduce to the Noetherian case; we suggest the reader skip it.

The first observation is that $U_j \cap U_{j'}$ is quasi-compact as the complement of the zero scheme of the quasi-coherent finite type ideal $\mathcal{I}_{j'}|U_j$ on the affine scheme U_j, see Properties, Lemma 28.24.1. Hence X is quasi-compact and quasi-separated, see Schemes, Lemma 26.21.6. By Limits, Proposition 32.5.4 we can write $X = \lim X_i$ as the limit of a direct system of Noetherian schemes with affine transition morphisms. For each j we can find an i and a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_{i,j}$ pulling back to $\mathcal{E}_j$, see Limits, Lemma 32.10.3. After increasing i we may assume that the composition $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$ is the pullback of a map $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$, see Limits, Lemma 32.10.2. Denote $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ the image of this map; this is a quasi-coherent ideal sheaf on the Noetherian scheme X_i whose pullback to X is $\mathcal{I}_j$. Denoting $U_{i,j} \subset X_i$ the complementary opens, we may assume these are affine for all i, j, see Limits, Lemma 32.4.13. If we can prove the lemma for the opens $U_{i,j}$ and the ideal sheaves $\mathcal{I}_{i,j}$ on X_i then X, being affine over X_i, will have the resolution property by Lemma 36.36.4. In this way we reduce to the case of a Noetherian scheme.

Assume X is Noetherian. For every coherent module $\mathcal{F}$ we can choose a finite list of sections $s_{jk} \in \mathcal{F}(U_j)$, $k = 1, \ldots, e_j$ which generate the restriction of $\mathcal{F}$ to U_j. By Cohomology of Schemes, Lemma 30.10.5 we can extend s_{jk} to a map $s'_{jk} : \mathcal{I}_i^{n_{jk}} \to \mathcal{F}$ for some $n_{jk} \geq 1$. Then we can consider the compositions

$$\mathcal{E}_j^{\otimes n_{jk}} \to \mathcal{I}_j^{n_{jk}} \to \mathcal{F}$$

to conclude. □

0GMM **Lemma 36.36.7.** *Let X be a scheme. If X has an ample family of invertible modules (Morphisms, Definition 29.12.1), then X has the resolution property.*

Proof. Since X is quasi-compact, there exists n and pairs $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ where $\mathcal{L}_i$ is an invertible $\mathcal{O}_X$-module and $s_i \in \Gamma(X, \mathcal{L}_i)$ is a section such that the set of points $U_i \subset X$ where s_i is nonvanishing is affine and $X = U_1 \cup \ldots \cup U_n$. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the image of $s_i : \mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$. Applying Lemma 36.36.6 we find that X has the resolution property. □

0F8A **Lemma 36.36.8.** *Let X be a quasi-compact, regular scheme with affine diagonal. Then X has the resolution property.*

Proof. Combine Divisors, Lemma 31.16.8 and the above Lemma 36.36.7. □

0F8B **Lemma 36.36.9.** *Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.*

Proof. If X_i has the resolution property, then X does by Lemma 36.36.4. Assume X has the resolution property. Choose $i \in I$. Choose a finite affine open covering $X_i = U_{i,1} \cup \ldots \cup U_{i,m}$. For each j choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ such that $X_i \setminus V(\mathcal{I}_{i,j}) = U_{i,j}$, see Properties, Lemma 28.24.1. Denote $U_j \subset X$ the inverse image of $U_{i,j}$ and denote $\mathcal{I}_j \subset \mathcal{O}_X$ the pullback of $\mathcal{I}_{i,j}$. Since

X has the resolution property, we may choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_j$ and surjections $\mathcal{E}_j \to \mathcal{I}_j$. By Limits, Lemmas 32.10.3 and 32.10.2 after increasing i we can find finite locally free $\mathcal{O}_{X_i}$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ whose base changes to X recover the compositions $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$, $j = 1, \ldots, m$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\mathrm{Coker}(\mathcal{E}_{i,j} \to \mathcal{O}_{X_i})$ and $\mathcal{O}_{X_i}/\mathcal{I}_{i,j}$ to X agree as quotients of $\mathcal{O}_X$. Hence by Limits, Lemma 32.10.2 we may assume that these agree, in other words that the image of $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ is equal to $\mathcal{I}_{i,j}$. Then we conclude that X_i has the resolution property by Lemma 36.36.6. □

0F8C **Lemma 36.36.10.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then X has affine diagonal.*

Special case of [**Tot04**, Proposition 1.3].

Proof. Combining Limits, Proposition 32.5.4 and Lemma 36.36.9 this reduces to the case where X is Noetherian (small detail omitted). Assume X is Noetherian. Recall that $X \times X$ is covered by the affine opens $U \times V$ for affine opens U, V of X, see Schemes, Section 26.17. Hence to show that the diagonal $\Delta : X \to X \times X$ is affine, it suffices to show that $U \cap V = \Delta^{-1}(U \times V)$ is affine for all affine opens U, V of X, see Morphisms, Lemma 29.11.3. In particular, it suffices to show that the inclusion morphism $j : U \to X$ is affine if U is an affine open of X. By Cohomology of Schemes, Lemma 30.3.4 it suffices to show that $R^1 j_* \mathcal{G} = 0$ for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$. By Proposition 36.8.3 (this is where we use that we've reduced to the Noetherian case) we can represent $Rj_*\mathcal{G}$ by a complex $\mathcal{H}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Assume

$$H^1(\mathcal{H}^\bullet) = \mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)/\mathrm{Im}(\mathcal{H}^0 \to \mathcal{H}^1)$$

is nonzero in order to get a contradiction. Then we can find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$$

such that the composition with the projection onto $H^1(\mathcal{H}^\bullet)$ is nonzero. Namely, we can write $\mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$ as the filtered union of its coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F}$ using the resolution property of X. This produces a map in the derived category

$$\mathcal{E}[-1] \longrightarrow Rj_*\mathcal{G}$$

which is nonzero on cohomology sheaves and hence nonzero in $D(\mathcal{O}_X)$. By adjunction, this is the same thing as a map

$$j^*\mathcal{E}[-1] \to \mathcal{G}$$

nonzero in $D(\mathcal{O}_U)$. Since $\mathcal{E}$ is finite locally free this is the same thing as a nonzero element of

$$H^1(U, j^*\mathcal{E}^\vee \otimes_{\mathcal{O}_U} \mathcal{G})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual finite locally free module. However, this group is zero by Cohomology of Schemes, Lemma 30.2.2 which is the desired contradiction. (If in doubt about the step using duals, please see the more general Cohomology, Lemma 20.50.5.) □

36.37. The resolution property and perfect complexes

0F8D In this section we discuss the relationship between perfect complexes and strictly perfect complexes on schemes which have the resolution property.

0F8E **Lemma 36.37.1.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{F}^\bullet$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules representing a perfect object of $D(\mathcal{O}_X)$. Then there exists a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.*

Proof. Let $a, b \in \mathbf{Z}$ be integers such that $\mathcal{F}^\bullet$ has tor amplitude in $[a, b]$ and such that $\mathcal{F}^n = 0$ for $n < a$. The existence of such a pair of integers follows from Cohomology, Lemma 20.49.5 and the fact that X is quasi-compact. If $b < a$, then $\mathcal{F}^\bullet$ is zero in the derived category and the lemma holds. We will prove by induction on $b - a \geq 0$ that there exists a complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^b$ with $\mathcal{E}^i$ finite locally free and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

The base case is the case $b - a = 0$. In this case $H^b(\mathcal{F}^\bullet) = H^a(\mathcal{F}^\bullet) = \mathrm{Ker}(\mathcal{F}^a \to \mathcal{F}^{a+1})$ is finite locally free. Namely, it is a finitely presented $\mathcal{O}_X$-module of tor dimension 0 and hence finite locally free. See Cohomology, Lemmas 20.49.5 and 20.47.9 and Properties, Lemma 28.20.2. Thus we can take $\mathcal{E}^\bullet$ to be $H^b(\mathcal{F}^\bullet)$ sitting in degree b. The rest of the proof is dedicated to the induction step.

Assume $b > a$. Observe that

$$H^b(\mathcal{F}^\bullet) = \mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1}) / \mathrm{Im}(\mathcal{F}^{b-1} \to \mathcal{F}^b)$$

is a finite type quasi-coherent $\mathcal{O}_X$-module, see Cohomology, Lemmas 20.49.5 and 20.47.9. Then we can find a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$$

such that the composition with the projection onto $H^b(\mathcal{F}^\bullet)$ is surjective. Namely, we can write $\mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$ as the filtered union of its finite type quasi-coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}^b$ and a surjection $\mathcal{E}^b \to \mathcal{F}$ using the resolution property of X. Consider the map of complexes

$$\alpha : \mathcal{E}^b[-b] \to \mathcal{F}^\bullet$$

and its cone $C(\alpha)^\bullet$, see Derived Categories, Definition 13.9.1. We observe that $C(\alpha)^\bullet$ is nonzero only in degrees $\geq a$, has tor amplitude in $[a, b]$ by Cohomology, Lemma 20.48.6, and has $H^b(C(\alpha)^\bullet) = 0$ by construction. Thus we actually find that $C(\alpha)^\bullet$ has tor amplitude in $[a, b-1]$. Hence the induction hypothesis applies to $C(\alpha)^\bullet$ and we find a map of complexes

$$(\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-1}) \longrightarrow C(\alpha)^\bullet$$

with properties as stated in the induction hypothesis. Unwinding the definition of the cone this gives a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathcal{E}^{b-2} & \longrightarrow & \mathcal{E}^{b-1} & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \mathcal{F}^{b-2} & \longrightarrow & \mathcal{F}^{b-1} \oplus \mathcal{E}^b & \longrightarrow & \mathcal{F}^b & \longrightarrow & \ldots
\end{array}$$

It is clear that we obtain a map of complexes $(\mathcal{E}^a \to \ldots \to \mathcal{E}^b) \to \mathcal{F}^\bullet$. We omit the verification that this map is a quasi-isomorphism. □

0F8F **Lemma 36.37.2.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then every perfect object of $D(\mathcal{O}_X)$ can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules.*

Proof. Let E be a perfect object of $D(\mathcal{O}_X)$. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 we can represent E by a complex $\mathcal{F}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Observe that E is in $D^b(\mathcal{O}_X)$ because X is quasi-compact. Hence $\tau_{\geq n}\mathcal{F}^\bullet$ is a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules which represents E if $n \ll 0$. Thus we may apply Lemma 36.37.1 to conclude. □

0F8G **Lemma 36.37.3.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Then any $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ can be represented by a diagram*

$$\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules and where $\mathcal{G}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism.

Proof. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 we can represent α by a diagram

$$\mathcal{E}^\bullet \leftarrow \mathcal{H}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{H}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules and where $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism. For $n \ll 0$ the maps $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ and $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ factor through the quasi-isomorphism $\mathcal{H}^\bullet \to \tau_{\geq n}\mathcal{H}^\bullet$ simply because $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes. Thus we may replace $\mathcal{H}^\bullet$ by $\tau_{\geq n}\mathcal{H}^\bullet$ and assume that $\mathcal{H}^\bullet$ is bounded below. Then we may apply Lemma 36.37.1 to conclude. □

0F8H **Lemma 36.37.4.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Let $\alpha^\bullet, \beta^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be two maps of complexes defining the same map in $D(\mathcal{O}_X)$. Then there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes.*

Proof. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 (and the definition of the derived category) there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes. Choose a homotopy $h^i : \mathcal{G}^i \to \mathcal{F}^{i-1}$ witnessing this fact. Choose $n \ll 0$. Then the map $\gamma^\bullet$ factors canonically over the quotient map $\mathcal{G}^\bullet \to \tau_{\geq n}\mathcal{G}^\bullet$ as $\mathcal{E}^\bullet$ is bounded below. For the exact same reason the maps h^i will factor over the surjections $\mathcal{G}^i \to (\tau_{\geq n}\mathcal{G})^i$. Hence we see that we may replace $\mathcal{G}^\bullet$ by $\tau_{\geq n}\mathcal{G}^\bullet$. Then we may apply Lemma 36.37.1 to conclude. □

0F8I **Proposition 36.37.5.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Denote*

(1) *$\mathcal{A}$ the additive category of finite locally free $\mathcal{O}_X$-modules,*

(2) $K^b(\mathcal{A})$ *the homotopy category of bounded complexes in* $\mathcal{A}$*, see Derived Categories, Section 13.8, and*
(3) $D_{perf}(\mathcal{O}_X)$ *the strictly full, saturated, triangulated subcategory of* $D(\mathcal{O}_X)$ *consisting of perfect objects.*

With this notation the obvious functor

$$K^b(\mathcal{A}) \longrightarrow D_{perf}(\mathcal{O}_X)$$

is an exact functor of trianglated categories which factors through an equivalence $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ *of triangulated categories where* S *is the saturated multiplicative system of quasi-isomorphisms in* $K^b(\mathcal{A})$*.*

Proof. If you can parse the statement of the proposition, then please skip this first paragraph. For some of the definitions used, please see Derived Categories, Definition 13.3.4 (triangulated subcategory), Derived Categories, Definition 13.6.1 (saturated triangulated subcategory), Derived Categories, Definition 13.5.1 (multiplicative system compatible with the triangulated structure), and Categories, Definition 4.27.20 (saturated multiplicative system). Observe that $D_{perf}(\mathcal{O}_X)$ is a saturated triangulated subcategory of $D(\mathcal{O}_X)$ by Cohomology, Lemmas 20.49.7 and 20.49.9. Also, note that $K^b(\mathcal{A})$ is a triangulated category, see Derived Categories, Lemma 13.10.5.

It is clear that the functor sends distinguished triangles to distinguished triangles, i.e., is exact. Then S is a saturated multiplicative system compatible with the triangulated structure on $K^b(\mathcal{A})$ by Derived Categories, Lemma 13.5.4. Hence the localization $S^{-1}K^b(\mathcal{A})$ exists and is a triangulated category by Derived Categories, Proposition 13.5.6. We get an exact factorization $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ by Derived Categories, Lemma 13.5.7. By Lemmas 36.37.2, 36.37.3, and 36.37.4 this functor is an equivalence. Then finally the functor $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ is an equivalence of triangulated categories (in the sense that distinguished triangles correspond) by Derived Categories, Lemma 13.4.18. □

36.38. K-groups

0FDE A tiny bit about K_0 of various categories associated to schemes. Previous material can be found in Algebra, Section 10.55, Homology, Section 12.11, Derived Categories, Section 13.28, and More on Algebra, Lemma 15.120.2.

Analogous to Algebra, Section 10.55 we will define two K-groups $K'_0(X)$ and $K_0(X)$ for any Noetherian scheme X. The first will use coherent $\mathcal{O}_X$-modules and the second will use finite locally free $\mathcal{O}_X$-modules.

0FDF **Lemma 36.38.1.** *Let* X *be a Noetherian scheme. Then*

$$K_0(Coh(\mathcal{O}_X)) = K_0(D^b(Coh(\mathcal{O}_X)) = K_0(D^b_{Coh}(\mathcal{O}_X))$$

Proof. The first equality is Derived Categories, Lemma 13.28.2. We have $K_0(Coh(\mathcal{O}_X)) = K_0(D^b_{Coh}(\mathcal{O}_X))$ by Derived Categories, Lemma 13.28.5. This proves the lemma. (We can also use that $D^b(Coh(\mathcal{O}_X)) = D^b_{Coh}(\mathcal{O}_X)$ by Proposition 36.11.2 to see the second equality.) □

Here is the definition.

0FDG **Definition 36.38.2.** Let X be a scheme.

(1) We denote $K_0(X)$ the *Grothendieck group of* X. It is the zeroth K-group of the strictly full, saturated, triangulated subcategory $D_{perf}(\mathcal{O}_X)$ of $D(\mathcal{O}_X)$ consisting of perfect objects. In a formula
$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$$
(2) If X is locally Noetherian, then we denote $K'_0(X)$ the *Grothendieck group of coherent sheaves on* X. It is the is the zeroth K-group of the abelian category of coherent $\mathcal{O}_X$-modules. In a formula
$$K'_0(X) = K_0(Coh(\mathcal{O}_X))$$

We will show that our definition of $K_0(X)$ agrees with the often used definition in terms of finite locally free modules if X has the resolution property (for example if X has an ample invertible module). See Lemma 36.38.5.

0FDH **Lemma 36.38.3.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Then $K_0(X) = K_0(R)$ and if R is Noetherian then $K'_0(X) = K'_0(R)$.*

Proof. Recall that $K'_0(R)$ and $K_0(R)$ have been defined in Algebra, Section 10.55.

By More on Algebra, Lemma 15.120.2 we have $K_0(R) = K_0(D_{perf}(R))$. By Lemmas 36.10.7 and 36.3.5 we have $D_{perf}(R) = D_{perf}(\mathcal{O}_X)$. This proves the equality $K_0(R) = K_0(X)$.

The equality $K'_0(R) = K'_0(X)$ holds because $Coh(\mathcal{O}_X)$ is equivalent to the category of finite R-modules by Cohomology of Schemes, Lemma 30.9.1. Moreover it is clear that $K'_0(R)$ is the zeroth K-group of the category of finite R-modules from the definitions. □

Let X be a Noetherian scheme. Then both $K'_0(X)$ and $K_0(X)$ are defined. In this case there is a canonical map
$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(D^b_{Coh}(\mathcal{O}_X)) = K'_0(X)$$
Namely, perfect complexes are in $D^b_{Coh}(\mathcal{O}_X)$ (by Lemma 36.10.3), the inclusion functor $D_{perf}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$ induces a map on zeroth K-groups (Derived Categories, Lemma 13.28.3), and we have the equality on the right by Lemma 36.38.1.

0FDI **Lemma 36.38.4.** *Let X be a Noetherian regular scheme. Then the map $K_0(X) \to K'_0(X)$ is an isomorphism.*

Proof. Follows immediately from Lemma 36.11.8 and our construction of the map $K_0(X) \to K'_0(X)$ above. □

Let X be a scheme. Let us denote $Vect(X)$ the category of finite locally free $\mathcal{O}_X$-modules. Although $Vect(X)$ isn't an abelian category in general, it is clear what a short exact sequence of $Vect(X)$ is. Denote $K_0(Vect(X))$ the unique abelian group with the following properties[7]:

(1) For every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ there is given an element $[\mathcal{E}]$ in $K_0(Vect(X))$,
(2) for every short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we have the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ in $K_0(Vect(X))$,

[7]The correct generality here would be to define K_0 for any exact category, see Injectives, Remark 19.9.6.

(3) the group $K_0(\mathit{Vect}(X))$ is generated by the elements $[\mathcal{E}]$, and
(4) all relations in $K_0(\mathit{Vect}(X))$ among the generators $[\mathcal{E}]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

We omit the detailed construction of $K_0(\mathit{Vect}(X))$. There is a natural map

$$K_0(\mathit{Vect}(X)) \longrightarrow K_0(X)$$

Namely, given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let us denote $\mathcal{E}[0]$ the perfect complex on X which has $\mathcal{E}$ sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we obtain a distinguished triangle $\mathcal{E}[0] \to \mathcal{E}'[0] \to \mathcal{E}''[0] \to \mathcal{E}[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(\mathit{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ by sending $[\mathcal{E}]$ to $[\mathcal{E}[0]]$ with apologies for the horrendous notation.

0FDJ **Lemma 36.38.5.** *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then the map $K_0(\mathit{Vect}(X)) \to K_0(X)$ is an isomorphism.*

Proof. This lemma will follow in a straightforward manner from Lemmas 36.37.2, 36.37.3, and 36.37.4 whose results we will use without further mention. Let us construct an inverse map

$$c : K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(\mathit{Vect}(X))$$

Namely, any object of $D_{perf}(\mathcal{O}_X)$ can be represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules. Then we set

$$c([\mathcal{E}^\bullet]) = \sum (-1)^i [\mathcal{E}^i]$$

Of course we have to show that this is well defined. For the moment we view c as a map defined on bounded complexes of finite locally free $\mathcal{O}_X$-modules.

Suppose that $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a surjective map of bounded complexes of finite locally free $\mathcal{O}_X$-modules. Let $\mathcal{K}^\bullet$ be the kernel. Then we obtain short exact sequences of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}^n \to \mathcal{E}^n \to \mathcal{F}^n \to 0$$

which are locally split because $\mathcal{F}^n$ is finite locally free. Hence $\mathcal{K}^\bullet$ is also a bounded complex of finite locally free $\mathcal{O}_X$-modules and we have $c(\mathcal{E}^\bullet) = c(\mathcal{K}^\bullet) + c(\mathcal{F}^\bullet)$ in $K_0(\mathit{Vect}(X))$.

Suppose given a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules which is acyclic. Say $\mathcal{E}^n = 0$ for $n \notin [a, b]$. Then we can break $\mathcal{E}^\bullet$ into short exact sequences

$$\begin{gathered} 0 \to \mathcal{E}^a \to \mathcal{E}^{a+1} \to \mathcal{F}^{a+1} \to 0, \\ 0 \to \mathcal{F}^{a+1} \to \mathcal{E}^{a+2} \to \mathcal{F}^{a+3} \to 0, \\ \ldots \\ 0 \to \mathcal{F}^{b-3} \to \mathcal{E}^{b-2} \to \mathcal{F}^{b-2} \to 0, \\ 0 \to \mathcal{F}^{b-2} \to \mathcal{E}^{b-1} \to \mathcal{E}^b \to 0 \end{gathered}$$

Arguing by descending induction we see that $\mathcal{F}^{b-2}, \ldots, \mathcal{F}^{a+1}$ are finite locally free $\mathcal{O}_X$-modules, and

$$c(\mathcal{E}^\bullet) = \sum (-1)[\mathcal{E}^n] = \sum (-1)^n([\mathcal{F}^{n-1}] + [\mathcal{F}^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows from the results of the preceding two paragraphs that c is well defined. Namely, suppose the bounded complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules represent the same object of $D(\mathcal{O}_X)$. Then we can find quasi-isomorphisms $a : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ and $b : \mathcal{G}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{G}^\bullet$ bounded complex of finite locally free $\mathcal{O}_X$-modules. We obtain a short exact sequence of complexes

$$0 \to \mathcal{E}^\bullet \to C(a)^\bullet \to \mathcal{G}^\bullet[1] \to 0$$

see Derived Categories, Definition 13.9.1. Since a is a quasi-isomorphism, the cone $C(a)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 13.12). Hence

$$0 = c(C(f)^\bullet) = c(\mathcal{E}^\bullet) + c(\mathcal{G}^\bullet[1]) = c(\mathcal{E}^\bullet) - c(\mathcal{G}^\bullet)$$

as desired. The same argument using b shows that $0 = c(\mathcal{F}^\bullet) - c(\mathcal{G}^\bullet)$. Hence we find that $c(\mathcal{E}^\bullet) = c(\mathcal{F}^\bullet)$ and c is well defined.

A similar argument using the cone on a map $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ of bounded complexes of finite locally free $\mathcal{O}_X$-modules shows that $c(Y) = c(X) + c(Z)$ if $X \to Y \to Z$ is a distinguished triangle in $D_{perf}(\mathcal{O}_X)$. Details omitted. Thus we get the desired homomorphism of abelian groups $c : K_0(X) \to K_0(\mathit{Vect}(X))$.

It is clear that the composition $K_0(\mathit{Vect}(X)) \to K_0(X) \to K_0(\mathit{Vect}(X))$ is the identity. On the other hand, let $\mathcal{E}^\bullet$ be a bounded complex of finite locally free $\mathcal{O}_X$-modules. Then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \sigma_{\geq n-1}\mathcal{E}^\bullet \to \mathcal{E}^{n-1}[-n+1] \to (\sigma_{\geq n}\mathcal{E}^\bullet)[1]$$

and induction show that

$$[\mathcal{E}^\bullet] = \sum (-1)^i [\mathcal{E}^i[0]]$$

in $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ with apologies for the notation. Hence the map $K_0(\mathit{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ is surjective which finishes the proof. □

0FDK **Remark 36.38.6.** Let X be a scheme. The K-group $K_0(X)$ is canonically a commutative ring. Namely, using the derived tensor product

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X)$$

and Derived Categories, Lemma 13.28.6 we obtain a bilinear multiplication. Since $K \otimes L \cong L \otimes K$ we see that this product is commutative. Since $(K \otimes L) \otimes M = K \otimes (L \otimes M)$ we see that this product is associative. Finally, the unit of $K_0(X)$ is the element $1 = [\mathcal{O}_X]$.

If $\mathit{Vect}(X)$ and $K_0(\mathit{Vect}(X))$ are as above, then it is clearly the case that $K_0(\mathit{Vect}(X))$ also has a ring structure: if $\mathcal{E}$ and $\mathcal{F}$ are finite locally free $\mathcal{O}_X$-modules, then we set

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}]$$

The reader easily verifies that this indeed defines a bilinear commutative, associative product. Details omitted. The map

$$K_0(\mathit{Vect}(X)) \longrightarrow K_0(X)$$

constructed above is a ring map with these definitions.

Now assume X is Noetherian. The derived tensor product also produces a map

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D^b_{Coh}(\mathcal{O}_X) \longrightarrow D^b_{Coh}(\mathcal{O}_X)$$

Again using Derived Categories, Lemma 13.28.6 we obtain a bilinear multiplication $K_0(X) \times K'_0(X) \to K'_0(X)$ since $K'_0(X) = K_0(D^b_{Coh}(\mathcal{O}_X))$ by Lemma 36.38.1. The reader easily shows that this gives $K'_0(X)$ the structure of a module over the ring $K_0(X)$.

0FDL **Remark 36.38.7.** Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. There is a map

$$f_* : K'_0(X) \longrightarrow K'_0(Y)$$

which sends $[\mathcal{F}]$ to

$$[\bigoplus\nolimits_{i\geq 0} R^{2i} f_* \mathcal{F}] - [\bigoplus\nolimits_{i\geq 0} R^{2i+1} f_* \mathcal{F}]$$

This is well defined because the sheaves $R^i f_* \mathcal{F}$ are coherent (Cohomology of Schemes, Proposition 30.19.1), because locally only a finite number are nonzero, and because a short exact sequence of coherent sheaves on X produces a long exact sequence of $R^i f_*$ on Y. If Y is quasi-compact (the only case most often used in practice), then we can rewrite the above as

$$f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}] = [Rf_* \mathcal{F}]$$

where we have used the equality $K'_0(Y) = K_0(D^b_{Coh}(Y))$ from Lemma 36.38.1.

0FDM **Lemma 36.38.8.** *Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Then we have $f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta$ for $\alpha \in K'_0(X)$ and $\beta \in K_0(Y)$.*

Proof. Follows from Lemma 36.22.1, the discussion in Remark 36.38.7, and the definition of the product $K'_0(X) \times K_0(X) \to K'_0(X)$ in Remark 36.38.6. □

0FDN **Remark 36.38.9.** Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-theoretically supported on Z. The zeroth K-group $K_0(D_{Z,perf}(\mathcal{O}_X))$ of this triangulated category is sometimes denoted $K_Z(X)$ or $K_{0,Z}(X)$. Using derived tensor product exactly as in Remark 36.38.6 we see that $K_0(D_{Z,perf}(\mathcal{O}_X))$ has a multiplication which is associative and commutative, but in general $K_0(D_{Z,perf}(\mathcal{O}_X))$ doesn't have a unit.

36.39. Determinants of complexes

0FJW This section is the continuation of More on Algebra, Section 15.123. For any ringed space $(X, \mathcal{O}_X)$ there is a functor

$$\det : \left\{\begin{matrix}\text{category of perfect complexes} \\ \text{morphisms are isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category of invertible modules} \\ \text{morphisms are isomorphisms}\end{matrix}\right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(\mathcal{O}_X)$ whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\mathrm{gr}L) \to \det(L)$. See [**KM76**] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case where X is a scheme and where we consider perfect objects L in $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.

0FJX **Lemma 36.39.1.** *Let X be a scheme. There is a functor*

$$\det : \left\{ \begin{matrix} \text{category of perfect complexes} \\ \text{with tor amplitude in } [-1,0] \\ \text{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix} \right\}$$

In addition, given a rank 0 perfect object L of $D(\mathcal{O}_X)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \Gamma(X, \det(L))$ such that for any isomorphism $a : L \to K$ in $D(\mathcal{O}_X)$ we have $\det(a)(\delta(L)) = \delta(K)$. Moreover, the construction is affine locally given by the construction of More on Algebra, Section 15.123.

Proof. Let L be an object of the left hand side. If $\mathrm{Spec}(A) = U \subset X$ is an affine open, then $L|_U$ corresponds to a perfect complex $L^\bullet$ of A-modules with tor-amplitude in $[-1, 0]$, see Lemmas 36.3.5, 36.10.4, and 36.10.7. Then we can consider the invertible A-module $\det(L^\bullet)$ constructed in More on Algebra, Lemma 15.123.4. If $\mathrm{Spec}(B) = V \subset U$ is another affine open contained in U, then $\det(L^\bullet) \otimes_A B = \det(L^\bullet \otimes_A B)$ and hence this construction is compatible with restriction mappings (see Lemma 36.3.8 and note $A \to B$ is flat). Thus we can glue these invertible modules to obtain an invertible module $\det(L)$ on X. The functoriality and canonical sections are constructed in exactly the same manner. Details omitted. □

0FJY **Remark 36.39.2.** The construction of Lemma 36.39.1 is compatible with pullbacks. More precisely, given a morphism $f : X \to Y$ of schemes and a perfect object K of $D(\mathcal{O}_Y)$ of tor-amplitude in $[-1, 0]$ then Lf^*K is a perfect object K of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$ and we have a canonical identification

$$f^* \det(K) \longrightarrow \det(Lf^*K)$$

Moreover, if K has rank 0, then $\delta(K)$ pulls back to $\delta(Lf^*K)$ via this map. This is clear from the affine local construction of the determinant.

36.40. Detecting Boundedness

0GEI In this section, we show that compact generators of D_{QCoh} of a quasi-compact, quasi-separated scheme, as constructed in Section 36.15, have a special property. We recommend reading that section first as it is very similar to this one.

0GEJ **Lemma 36.40.1.** *In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions*

1. *The canonical map $\tau_{\geq a} E \to \tau_{\geq a} Rj_*(E|_U)$ is an isomorphism.*
2. *We have $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \geq a$.*

Then (2) implies (1) and (1) implies (2) with a replaced by $a + 1$.

Proof. Choose a distinguished triangle $N \to E \to Rj_*(E|_U) \to N[1]$. Then (1) implies $\tau_{\geq a+1} N = 0$ and (1) is implied by $\tau_{\geq a} N = 0$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'+e_i''}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i''}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$ is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \geq a$ and all $e \geq 1$ with K_e as in Lemma 36.9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \geq a$. We conclude that (2) is equivalent to $\tau_{\geq a}N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 36.3.5. Thus we find that our lemma is true. □

0GEK **Lemma 36.40.2.** *In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions*

(1) *The canonical map $\tau_{\leq a}E \to \tau_{\leq a}Rj_*(E|_U)$ is an isomorphism, and*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \leq a$.*

Then (2) implies (1) and (1) implies (2) with a replaced by $a - 1$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Then (1) implies $\tau_{\leq a-1}N = 0$ and (1) is implied by $\tau_{\leq a}N = 0$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'+e_i''}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i''}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$ is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \leq a$ and all $e \geq 1$ with K_e as in Lemma 36.9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \leq a$. We conclude that (2) is equivalent to $\tau_{\leq a}N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 36.3.5. Thus we find that our lemma is true. □

0GEL **Lemma 36.40.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

(1) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \gg 0$, and*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a}E) = 0$ for $i \gg 0$.*

Proof. Using the triangle $\tau_{<a}E \to E \to \tau_{\geq a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a}E)[i]) = 0$$

for $i \gg 0$. As P is perfect this is true by Lemma 36.13.5. □

0GEM **Lemma 36.40.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

(1) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \ll 0$, and*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a}E) = 0$ for $i \ll 0$.*

Proof. Using the triangle $\tau_{\leq a}E \to E \to \tau_{>a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a}E)[i]) = 0$$

for $i \ll 0$. As P is perfect this is true by Lemma 36.13.5. □

0GEN **Proposition 36.40.5.** *Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$E \in D^-_{QCoh}(\mathcal{O}_X)$,*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \gg 0$,*
3. *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \gg 0$,*
4. *$R\mathrm{Hom}_X(G, E)$ is in $D^-(\mathbf{Z})$,*
5. *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
6. *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$,*
7. *for every perfect object P of $D(\mathcal{O}_X)$*
   (a) *the assertions (2), (3), (4) hold with G replaced by P, and*
   (b) *$H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
   (c) *$R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \gg 0$ by Lemma 36.13.5. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 20.44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 20.50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 36.17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\mathrm{Hom}_X(P, E)$ is in $D^-(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. Set $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \gg 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 36.3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) $\Rightarrow$ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 36.15.1 and Theorem 36.15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let

$K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

0GEP (36.40.5.1) $$K' = R(V \to X)_* K = R(V \to X)_! K,$$

see Cohomology, Lemmas 20.33.6 and 20.49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\mathrm{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\mathrm{Hom}_{\mathcal{O}_X}(K', E)$ are bounded above.

By the second description of K' in (36.40.5.1) we have

$$\mathrm{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \gg 0$. Therefore, we may apply Lemma 36.40.1 to $E|_V$ to obtain an integer a such that $\tau_{\geq a}(E|_V) = \tau_{\geq a} R(U \cap V \to V)_*(E|_{U \cap V})$. Then $\tau_{\geq a} E = \tau_{\geq a} R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 36.40.3 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \gg 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^-_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^-_{QCoh} (by Lemma 36.4.1), we get $\tau_{\geq a} E \in D^-_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^-_{QCoh}(\mathcal{O}_X)$. □

0GEQ **Proposition 36.40.6.** *Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$E \in D^+_{QCoh}(\mathcal{O}_X)$,*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \ll 0$,*
3. *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \ll 0$,*
4. *$R\mathrm{Hom}_X(G, E)$ is in $D^+(\mathbf{Z})$,*
5. *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
6. *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$,*
7. *for every perfect object P of $D(\mathcal{O}_X)$*
   (a) *the assertions (2), (3), (4) hold with G replaced by P, and*
   (b) *$H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
   (c) *$R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \ll 0$ by Lemma 36.13.5. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 20.44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 20.50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 36.17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\mathrm{Hom}_X(P, E)$ is in $D^+(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties,

except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. Let $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \ll 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 36.3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) $\Rightarrow$ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 36.15.1 and Theorem 36.15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let $K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

0GER (36.40.6.1) $$K' = R(V \to X)_* K = R(V \to X)_! K,$$

see Cohomology, Lemmas 20.33.6 and 20.49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\mathrm{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\mathrm{Hom}_{\mathcal{O}_X}(K', E)$ are bounded below.

By the second description of K' in (36.40.6.1) we have

$$\mathrm{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \ll 0$. Therefore, we may apply Lemma 36.40.2 to $E|_V$ to obtain an integer a such that $\tau_{\leq a}(E|_V) = \tau_{\leq a} R(U \cap V \to V)_*(E|_{U \cap V})$. Then $\tau_{\leq a} E = \tau_{\leq a} R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 36.40.4 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \ll 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^+_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves bounded below objects (see Cohomology, Section 20.3) we get $\tau_{\leq a} E \in D^+_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^+_{QCoh}(\mathcal{O}_X)$. □

36.41. Quasi-coherent objects in the derived category

0GZY Let X be a scheme. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Topologies, Definition 34.3.7. We remind the reader that the topos of $X_{affine,Zar}$ is the small Zariski topos of X, see Topologies, Lemma 34.3.11. The site $X_{affine,Zar}$ comes with a structure sheaf $\mathcal{O}$ and there is an equivalence of ringed topoi

$$(Sh(X_{affine,Zar}), \mathcal{O}) \longrightarrow (Sh(X_{Zar}), \mathcal{O})$$

See Descent, Equation (35.11.1.1) and the discussion in Descent, Section 35.11 surrounding it where a slightly different notation is used.

In this section we denote X_{affine} the underlying category of $X_{affine,Zar}$ endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf $\mathcal{O}$ becomes a sheaf on X_{affine} as well. We obtain a

morphisms of ringed sites

$$\epsilon : (X_{affine,Zar}, \mathcal{O}) \longrightarrow (X_{affine}, \mathcal{O})$$

as in Cohomology on Sites, Section 21.27. In this section we will identify $D_{QCoh}(\mathcal{O}_X)$ with the category $QC(X_{affine}, \mathcal{O})$ introduced in Cohomology on Sites, Section 21.43.

0GZZ **Lemma 36.41.1.** *In the situation above there are canonical exact equivalences between the following triangulated categories*

(1) $D_{QCoh}(\mathcal{O}_X)$,
(2) $D_{QCoh}(X_{Zar}, \mathcal{O})$,
(3) $D_{QCoh}(X_{affine,Zar}, \mathcal{O})$,
(4) $D_{QCoh}(X_{affine}, \mathcal{O}_X)$, *and*
(5) $QC(X_{affine}, \mathcal{O})$.

Proof. If $U \subset V \subset X$ are affine open, then the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat. Hence the equivalence between (4) and (5) is a special case of Cohomology on Sites, Lemma 21.43.11 (the proof also clarifies the statement).

The ringed site $(X_{Zar}, \mathcal{O})$ and the ringed space $(X, \mathcal{O}_X)$ have the same categories of modules by Descent, Remark 35.8.3. Via this equivalence the quasi-coherent modules correspond by Descent, Proposition 35.8.9. Hence we get a canonical exact equivalence between the triangulated categories in (1) and (2).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi $(Sh(X_{affine,Zar}), \mathcal{O}) \to (Sh(X_{Zar}), \mathcal{O})$ and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 18.23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (2) and (3).

To get an exact equivalence between the triangulated categories in (3) and (4) we will apply Cohomology on Sites, Lemma 21.29.1 to the morphism $\epsilon : (X_{affine,Zar}, \mathcal{O}) \to (X_{affine}, \mathcal{O})$ above. We take $\mathcal{B} = \mathrm{Ob}(X_{affine})$ and we take $\mathcal{A} \subset PMod(X_{affine}, \mathcal{O})$ to be the full subcategory of those presheaves $\mathcal{F}$ such that $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism. Observe that by Descent, Lemma 35.11.2 objects of $\mathcal{A}$ are exactly those sheaves in the Zariski topology which are quasi-coherent modules on $(X_{affine,Zar}, \mathcal{O})$. On the other hand, by Modules on Sites, Lemma 18.24.3, the objects of $\mathcal{A}$ are exactly the quasi-coherent modules on $(X_{affine}, \mathcal{O})$, i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 21.29.1 applies, then we do indeed get the canonical equivalence between the categories of (3) and (4) using ϵ^* and $R\epsilon_*$.

We have to verify 4 conditions:

(1) Every object of $\mathcal{A}$ is a sheaf for the Zariski topology. This we have seen above.
(2) $\mathcal{A}$ is a weak Serre subcategory of $Mod(X_{affine,Zar}, \mathcal{O})$. Above we have seen that $\mathcal{A} = QCoh(X_{affine,Zar}, \mathcal{O})$ and we have seen above that these, via the equivalence $Mod(X_{affine,Zar}, \mathcal{O}) = Mod(X, \mathcal{O}_X)$, correspond to the quasi-coherent modules on X. Thus the result by the discussion in Schemes, Section 26.24.

(3) Every object of X_{affine} has a covering in the chaotic topology whose members are elements of $\mathcal{B}$. This holds because $\mathcal{B}$ contains all objects.
(4) For every object U of X_{affine} and $\mathcal{F}$ in $\mathcal{A}$ we have $H^p_{Zar}(U, \mathcal{F}) = 0$ for $p > 0$. This holds by the vanishing of cohomology of quasi-coherent modules on affines, see Cohomology of Schemes, Lemma 30.2.2.

This finishes the proof. □

0H00 **Remark 36.41.2.** Let S be a scheme. We will later show that also $QC((Aff/S), \mathcal{O})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_S)$. See Sheaves on Stacks, Proposition 96.26.4.

36.42. Other chapters

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CHAPTER 37

More on Morphisms

02GX

37.1. Introduction

02GY In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [**DG67**].

37.2. Thickenings

04EW The following terminology may not be completely standard, but it is convenient.

04EX **Definition 37.2.1.** Thickenings.

(1) We say a scheme X' is a *thickening* of a scheme X if X is a closed subscheme of X' and the underlying topological spaces are equal.
(2) We say a scheme X' is a *first order thickening* of a scheme X if X is a closed subscheme of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) We say a scheme X' is a *finite order thickening* of a scheme X if X is a closed subscheme of X and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X is nilpotent, i.e., there exists an integer $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$.
(4) We say a scheme X' is an *nth order thickening* of a scheme X if X is a closed subscheme of X and $\mathcal{I}^{n+1} = 0$ where $\mathcal{I} \subset \mathcal{O}_{X'}$ is the quasi-coherent sheaf of ideals defining X.
(5) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f'(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(6) Let S be a scheme. We similarly define *thickenings over S*, and *morphisms of thickenings over S*. This means that the schemes X, X', Y, Y' above are schemes over S, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over S.

Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp)$ is locally nilpotent. This gives some control over the structure sheaf of $\mathcal{O}_{X'}$, but technically the class of finite order thickenings $X \subset X'$ is much easier to handle. Namely, if X' is an nth order thinking then we have a filtration

$$0 = \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over S. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickening are described as follows (see Modules, Lemma 17.28.11).

05YV **Lemma 37.2.2.** *Let X be a scheme over a base S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on X where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over S. Moreover, any first order thickening over S is of this form.

Proof. It is clear that X' is a locally ringed space. Let $U = \mathrm{Spec}(B)$ be an affine open of X. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 30.2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring A. By Schemes, Lemma 26.6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \longrightarrow \mathrm{Spec}(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$\begin{array}{ccc} (U, \mathcal{O}_X|_U) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ (U, \mathcal{A}|_U) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\mathfrak{p} & \longrightarrow & A_\mathfrak{p} & \longrightarrow & B_\mathfrak{p} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_u & \longrightarrow & \mathcal{A}_u & \longrightarrow & \mathcal{O}_{X,u} & \longrightarrow & 0. \end{array}$$

The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and X' is a scheme as desired. □

06AD **Lemma 37.2.3.** *Any thickening of an affine scheme is affine.*

Proof. This is a special case of Limits, Proposition 32.11.2. □

The case of a finite order thickening is [**GD60**, Proposition 5.1.9].

Proof for a finite order thickening. Suppose that $X \subset X'$ is a finite order thickening with X affine. Then we may use Serre's criterion to prove X' is affine. More precisely, we will use Cohomology of Schemes, Lemma 30.3.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. It suffices to show that $H^1(X', \mathcal{F}) = 0$. Denote $i : X \to X'$ the given closed immersion and denote $\mathcal{I} = \mathrm{Ker}(i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$. By

our discussion of finite order thickenings (following Definition 37.2.1) there exists an $n \geq 0$ and a filtration

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

by quasi-coherent submodules such that $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$. Namely, we can take $\mathcal{F}_a = \mathcal{I}^a\mathcal{F}$. Then $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}_a$, see Morphisms, Lemma 29.4.1. We obtain

$$H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) = H^1(X', i_*\mathcal{G}_a) = H^1(X, \mathcal{G}_a) = 0$$

The second equality comes from Cohomology of Schemes, Lemma 30.2.4 and the last equality from Cohomology of Schemes, Lemma 30.2.2. Thus $\mathcal{F}$ has a finite filtration whose successive quotients have vanishing first cohomology and it follows by a simple induction argument that $H^1(X', \mathcal{F}) = 0$. □

09ZU **Lemma 37.2.4.** *Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

0BPE **Lemma 37.2.5.** *If $S \subset S'$ and $S' \subset S''$ are thickenings, then so is $S \subset S''$.*

Proof. Omitted. □

0BPF **Lemma 37.2.6.** *The property of being a thickening is fpqc local. Similarly for first order thickenings.*

Proof. The statement means the following: Let $X \to X'$ be a morphism of schemes and let $\{g_i : X'_i \to X'\}$ be an fpqc covering such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent, Lemma 35.23.21 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

37.3. Morphisms of thickenings

0CF2 If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of schemes, then often properties of the morphism f are inherited by f'. There are several variants.

09ZV **Lemma 37.3.1.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*
(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is universally injective if and only if f' is universally injective,*
(8) *f is universally open if and only if f' is universally open,*
(9) *f is quasi-affine if and only if f' is quasi-affine, and*
(10) *add more here.*

Proof. Observe that $S \to S'$ and $X \to X'$ are universal homeomorphisms (see for example Morphisms, Lemma 29.45.6). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 37.2.3 which tells us that there is a 1-to-1 correspondence between affine opens of S and S' and between affine opens of X and X'. Part (9) follows from Limits, Lemma 32.11.5 and the remark just made about affine opens of S and S'. Part (5) follows from (1) and (4) by Morphisms, Lemma 29.44.7. Finally, note that

$$S \times_X S = S \times_{X'} S \to S \times_{X'} S' \to S' \times_{X'} S'$$

is a thickening (the two arrows are thickenings by Lemma 37.2.4). Hence applying (3) and (4) to the morphism $(S \subset S') \to (S \times_X S \to S' \times_{X'} S')$ we obtain (6). □

0D2R **Lemma 37.3.2.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Let $\mathcal{L}'$ be an invertible sheaf on X' and denote $\mathcal{L}$ the restriction to X. Then $\mathcal{L}'$ is f'-ample if and only if $\mathcal{L}$ is f-ample.*

Proof. Recall that being relatively ample is a condition for each affine open in the base, see Morphisms, Definition 29.37.1. By Lemma 37.2.3 there is a 1-to-1 correspondence between affine opens of S and S'. Thus we may assume S and S' are affine and we reduce to proving that $\mathcal{L}'$ is ample if and only if $\mathcal{L}$ is ample. This is Limits, Lemma 32.11.4. □

09ZW **Lemma 37.3.3.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then*

(1) *f is a closed immersion if and only if f' is a closed immersion,*
(2) *f is locally of finite type if and only if f' is locally of finite type,*
(3) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(4) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(5) *$\Omega_{X/S} = 0$ if and only if $\Omega_{X'/S'} = 0$,*
(6) *f is unramified if and only if f' is unramified,*
(7) *f is proper if and only if f' is proper,*
(8) *f is finite if and only if f' is finite,*
(9) *f is a monomorphism if and only if f' is a monomorphism,*
(10) *f is an immersion if and only if f' is an immersion, and*
(11) *add more here.*

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2 and 26.23.5 and Morphisms, Lemmas 29.15.4, 29.20.13, 29.29.2, 29.32.10, 29.35.5, 29.41.5, and 29.44.6.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 37.2.1.

Most of the proofs will use a reduction to the affine case. Let $U' \subset S'$ be an affine open and let $V' \subset X'$ be an affine open lying over U'. Let $U' = \mathrm{Spec}(A')$ and denote $I \subset A'$ be the ideal defining the closed subscheme $U' \cap S$. Say $V' = \mathrm{Spec}(B')$. Then

$V' \cap X = \mathrm{Spec}(B'/IB')$. Setting $A = A'/I$ and $B = B'/IB'$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows and $I^2 = 0$.

The translation of (1) into algebra: If $A \to B$ is surjective, then $A' \to B'$ is surjective. This follows from Nakayama's lemma (Algebra, Lemma 10.20.1).

The translation of (2) into algebra: If $A \to B$ is a finite type ring map, then $A' \to B'$ is a finite type ring map. This follows from Nakayama's lemma (Algebra, Lemma 10.20.1) applied to a map $A'[x_1, \ldots, x_n] \to B'$ such that $A[x_1, \ldots, x_n] \to B$ is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6.

Proof of (4). Follows from (2) and that the additional property of "being of relative dimension d" can be checked on fibres (by definition, see Morphisms, Definition 29.29.1.

The translation of (5) into algebra: If $\Omega_{B/A} = 0$, then $\Omega_{B'/A'} = 0$. By Algebra, Lemma 10.131.12 we have $0 = \Omega_{B/A} = \Omega_{B'/A'}/I\Omega_{B'/A'}$. Hence $\Omega_{B'/A'} = 0$ by Nakayama's lemma (Algebra, Lemma 10.20.1).

The translation of (6) into algebra: If $A \to B$ is unramified map, then $A' \to B'$ is unramified. Since $A \to B$ is of finite type we see that $A' \to B'$ is of finite type by (2) above. Since $A \to B$ is unramified we have $\Omega_{B/A} = 0$. By part (5) we have $\Omega_{B'/A'} = 0$. Thus $A' \to B'$ is unramified.

Proof of (7). Follows by combining (2) with results of Lemma 37.3.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 37.3.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 29.44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_S X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_S X \subset X' \times_{S'} X')$. We conclude $X' \to X' \times_{S'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{S'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{S'}$ cutting out S in S'. Indeed, $X = X \times_S X = (X' \times_{S'} X') \times_{S'} S$ is contained in X'. Hence by Morphisms, Lemma 29.32.7 it suffices to show that $\Omega_{X'/S'} = 0$ which follows from (5) and the corresponding statement for X/S.

Proof of (10). If $f : X \to S$ is an immersion, then it factors as $X \to U \to S$ where $U \to S$ is an open immersion and $X \to U$ is a closed immersion. Let $U' \subset S'$ be the open subscheme whose underlying topological space is the same as U. Then $X' \to S'$ factors through U' and we conclude that $X' \to U'$ is a closed immersion by part (1). This finishes the proof. $\square$

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

0BPG **Lemma 37.3.4.** *Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then*

(1) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(2) *f is finite if and only if f' is finite,*
(3) *f is a closed immersion if and only if f' is a closed immersion,*
(4) *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
(5) *f is unramified if and only if f' is unramified,*
(6) *f is a monomorphism if and only if f' is a monomorphism,*
(7) *f is an immersion if and only if f' is an immersion,*
(8) *f is proper if and only if f' is proper, and*
(9) *add more here.*

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2 and 26.23.5 and Morphisms, Lemmas 29.20.13, 29.29.2, 29.32.10, 29.35.5, 29.41.5, and 29.44.6. Hence in each case we need only to prove that if f has the desired property, so does f'.

A morphism is locally quasi-finite if and only if it is locally of finite type and the scheme theoretic fibres are discrete spaces, see Morphisms, Lemma 29.20.8. Since the underlying topological space is unchanged by passing to a thickening, we see that f' is locally quasi-finite if (and only if) f is. This proves (1).

Case (2) follows from case (5) of Lemma 37.3.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms, Lemma 29.44.4).

Case (3). This follows immediately from Morphisms, Lemma 29.45.7.

Case (4) follows from the following algebra statement: Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Let B be a finite type A-algebra. If $\Omega_{(B/IB)/(A/I)} = 0$, then $\Omega_{B/A} = 0$. Namely, the assumption means that $I\Omega_{B/A} = 0$, see Algebra, Lemma 10.131.12. On the other hand $\Omega_{B/A}$ is a finite B-module, see Algebra, Lemma 10.131.16. Hence the vanishing of $\Omega_{B/A}$ follows from Nakayama's lemma (Algebra, Lemma 10.20.1) and the fact that IB is contained in the Jacobson radical of B.

Case (5) follows immediately from (4) and Morphisms, Lemma 29.35.2.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ is a bijection on underlying sets, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms, Lemma 29.21.12. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero by Morphisms, Lemma 29.32.7. In other words,

$\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 10.21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open immersion and $X \to V$ is a closed immersion. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is the same as V. Then $X' \to V'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 37.3.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. □

37.4. Picard groups of thickenings

0C6Q Some material on Picard groups of thickenings.

0C6R **Lemma 37.4.1.** *Let $X \subset X'$ be a first order thickening with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X', \mathcal{O}_{X'}^*) \longrightarrow H^0(X, \mathcal{O}_X^*) \\ \longrightarrow H^1(X, \mathcal{I}) \longrightarrow \mathrm{Pic}(X') \longrightarrow \mathrm{Pic}(X) \\ \longrightarrow H^2(X, \mathcal{I}) \longrightarrow \ldots \longrightarrow \ldots \end{aligned}$$

of abelian groups.

Proof. This is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed space with the first cohomology group of the sheaf of invertible functions, see Cohomology, Lemma 20.6.1. □

0C6S **Lemma 37.4.2.** *Let $X \subset X'$ be a thickening. Let n be an integer invertible in $\mathcal{O}_X$. Then the map* $\mathrm{Pic}(X')[n] \to \mathrm{Pic}(X)[n]$ *is bijective.*

Proof for a finite order thickening. By the general principle explained following Definition 37.2.1 this reduces to the case of a first order thickening. Then may use Lemma 37.4.1 to see that it suffices to show that $H^1(X, \mathcal{I})[n]$, $H^1(X, \mathcal{I})/n$, and $H^2(X, \mathcal{I})[n]$ are zero. This follows as multiplication by n on $\mathcal{I}$ is an isomorphism as it is an $\mathcal{O}_X$-module. □

Proof in general. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent ideal sheaf cutting out X. Then we have a short exact sequence of abelian groups

$$0 \to (1 + \mathcal{I})^* \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

We obtain a long exact cohomology sequence as in the statement of Lemma 37.4.1 with $H^i(X, \mathcal{I})$ replaced by $H^i(X, (1+\mathcal{I})^*)$. Thus it suffices to show that raising to

the nth power is an isomorphism $(1+\mathcal{I})^* \to (1+\mathcal{I})^*$. Taking sections over affine opens this follows from Algebra, Lemma 10.32.8. □

37.5. Infinitesimal neighbourhoods

05YW A natural construction of finite order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U. For $n \geq 1$ we can consider the closed subscheme $Z_n \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^{n+1}$.

04EY **Definition 37.5.1.** Let $i : Z \to X$ be an immersion of schemes.

(1) The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z_1$ over X described above.
(2) The *nth order infinitesimal neighbourhood* of Z in X is the nth order thickening $Z \subset Z_n$ over X described above.

These thickenings have the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

04EZ **Lemma 37.5.2.** *Let $i : Z \to X$ be an immersion of schemes.*

(1) *The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

(2) *For $n \geq 1$ the nth order infinitesimal neighbourhood Z_n of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is an nth order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z_n)$ of thickenings over X.

Proof. We will only prove (1). Let $U \subset X$ be the open used in the construction of Z', i.e., an open such that Z is identified with a closed subscheme of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of b as a morphism into U. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the ideal cutting out T. Since $b(T) \subset Z$ by the diagram above we see that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Schemes, Lemma 26.4.6 this implies that b factors through Z'. Denote $a' : T' \to Z'$ this factorization and everything is clear. □

04F0 **Lemma 37.5.3.** *Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 29.31.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. □

37.6. Formally unramified morphisms

02H7 Recall that a ring map $R \to A$ is called *formally unramified* (see Algebra, Definition 10.148.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H8 **Definition 37.6.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally unramified* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \swarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

04F1 **Lemma 37.6.2.** *If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \swarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 37.6.1 the condition that T be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. □

02HA **Lemma 37.6.3.** *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

02HB **Lemma 37.6.4.** *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

02HC **Lemma 37.6.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If f is formally unramified, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow{} & T \\ {\scriptstyle f|_U}\downarrow & \overset{a}{\nwarrow} & \downarrow{\scriptstyle i} \\ V & \xleftarrow{} & T' \end{array}$$

as in Definition 37.6.1. If f is formally ramified, then there exists at most one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into U. □

02HD **Lemma 37.6.6.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.*

Proof. This is immediate from the definitions (Definition 37.6.1 and Algebra, Definition 10.148.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. □

Here is a characterization in terms of the sheaf of differentials.

02H9 **Lemma 37.6.7.** *Let $f : X \to S$ be a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/S} = 0$.*

Proof. We recall some of the arguments of the proof of Morphisms, Lemma 29.32.5. Let $W \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into W. Let $\mathcal{J} \subset \mathcal{O}_W$ be the ideal sheaf of this closed immersion. Let $X' \subset W$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that p_1 and p_2 agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by a an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^\sharp(f) - p_2^\sharp(f)$ for f a local section of $\mathcal{O}_X$.

Suppose that $f : X \to S$ is formally unramified. By assumption this means that $p_1 = p_2$ when restricted to any affine open $T' \subset X'$. Hence $p_1 = p_2$. By what was said above we conclude that $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0$.

Conversely, suppose that $\Omega_{X/S} = 0$. Then $X' = X$. Take any pair of morphisms $f_1', f_2' : T' \to X$ fitting as dotted arrows in the diagram of Definition 37.6.1. This gives a morphism $(f_1', f_2') : T' \to X \times_S X$. Since $f_1'|_T = f_2'|_T$ and $|T| = |T'|$ we see that the image of T' under (f_1', f_2') is contained in the open W chosen above. Since $(f_1', f_2')(T) \subset \Delta(X)$ and since T is defined by an ideal of square zero in T' we see that (f_1', f_2') factors through X'. As $X' = X$ we conclude $f_1' = f_2'$ as desired. □

02HE **Lemma 37.6.8.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified), and*
(2) *the morphism f is locally of finite type (resp. locally of finite presentation) and formally unramified.*

Proof. Use Lemma 37.6.7 and Morphisms, Lemma 29.35.2. □

37.7. Universal first order thickenings

04F2 Let $h : Z \to X$ be a morphism of schemes. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

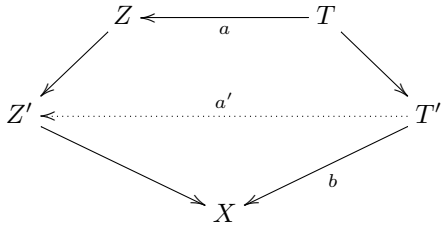

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does.

04F3 **Lemma 37.7.1.** *Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of Z over X if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over X by glueing, starting with the affine case which is Algebra, Lemma 10.149.1. We begin with some general remarks.

If a universal first order thickening of Z over X exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of Z over X. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of V over U. Namely, suppose given any diagram

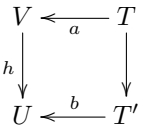

where $T \subset T'$ is a first order thickening over U. By the universal property of Z' we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of (a, a') as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over U. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$

is a universal first order thickening over U, then $Z \subset Z'$ is a universal first order thickening over X.

Before we glue affine pieces let us show that the lemma holds if Z and X are affine. Say $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(S)$. By Algebra, Lemma 10.149.1 there exists a first order thickening $Z \subset Z'$ over X which has the universal property of the lemma for diagrams

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where T, T' are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that a'_i and a'_j agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. Thus the lemma holds if X and Z are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each Z_i maps into an affine open U_i of X. By Lemma 37.6.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over U_i. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of Z_i over X. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$. By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over X. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 26.14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 26.14 to get a first order thickening $Z \subset Z'$ over X which the property that the open subscheme $Z'_i \subset Z'$ with $Z_i = Z'_i \cap Z$ is a universal first order thickening of Z_i over X.

It turns out that this implies formally that Z' is a universal first order thickening of Z over X. Namely, we have the universal property for any diagram

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $a(T)$ is contained in some Z_i. Given a general diagram we can choose an open covering $T' = \bigcup T'_i$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. We see that a'_i and a'_j necessarily agree on $T'_i \cap T'_j$ since both $a'_i|_{T'_i \cap T'_j}$ and $a'_j|_{T'_i \cap T'_j}$ are solutions of the problem of mapping into the universal first order thickening $Z'_i \cap Z'_j$ of $Z_i \cap Z_j$ over X. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. This finishes the proof. □

04F4 **Definition 37.7.2.** Let $h : Z \to X$ be a formally unramified morphism of schemes.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 37.7.1.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on Z. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

04F5 **Lemma 37.7.3.** *Let $i : Z \to X$ be an immersion of schemes. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 37.5.1, and*
(3) *the conormal sheaf of i in the sense of Morphisms, Definition 29.31.1 agrees with the conormal sheaf of i in the sense of Definition 37.7.2.*

Proof. By Morphisms, Lemmas 29.35.7 and 29.35.8 an immersion is unramified, hence formally unramified by Lemma 37.6.8. The other assertions follow by combining Lemmas 37.5.2 and 37.5.3 and the definitions. □

04F6 **Lemma 37.7.4.** *Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening Z' is formally unramified over X.*

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 10.149.5. Then Lemma 37.6.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

$$\begin{array}{ccc} Z' & \xleftarrow{c} & T \\ \downarrow & \nwarrow^{a,b} & \downarrow \\ X & \xleftarrow{} & T' \end{array}$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

04F7 **Lemma 37.7.5.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

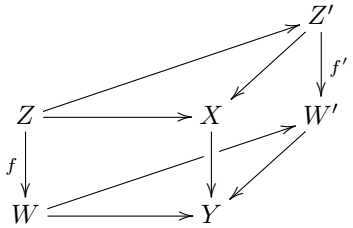

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Morphisms, Lemma 29.31.3 applied to $(Z \subset Z') \to (W \subset W')$. □

04F8 **Lemma 37.7.6.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram in the category of schemes with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 37.7.5 is surjective.

Proof. The morphism h is formally unramified by Lemma 37.6.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 29.31.4 for why this implies that the map of conormal sheaves is surjective. □

04F9 **Lemma 37.7.7.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram in the category of schemes with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms, Lemma 29.25.8. Hence the first statement follows from the description of W' in Lemma 37.7.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 29.31.4 for why this implies that the map of conormal sheaves is an isomorphism. □

04FA **Lemma 37.7.8.** *Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z$, $U \subset X$ be opens such that $h(V) \subset U$. Let Z' be the universal first order thickening of Z over X. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of V over U is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.*

Proof. The first statement is Lemma 37.6.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 37.7.1, or from Algebra, Lemma 10.149.4 or deduced from Lemma 37.7.7. □

04FB **Lemma 37.7.9.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$c_{h'} : (h')^*\Omega_{X/S} \longrightarrow \Omega_{Z'/S}$$

induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Morphisms, Lemma 29.32.8. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 37.7.8 and Morphisms, Lemma 29.32.3. In this case the result is Algebra, Lemma 10.149.5. □

04FC **Lemma 37.7.10.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

The first arrow is induced by $d_{Z'/S}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Morphisms, Lemma 29.32.15. Hence the result follows on applying Lemma 37.7.9. □

067V **Lemma 37.7.11.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 37.7.5 and the second from Lemma 37.7.10.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 37.7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & Z' & \longrightarrow & X \\ & \searrow^{j'} & \downarrow & & \downarrow \\ & & Z'' & \longrightarrow & Y \end{array}$$

Apply Morphisms, Lemma 29.32.18 to the left triangle to get an exact sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

As Z'' is formally unramified over Y (see Lemma 37.7.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 37.6.7 and Morphisms, Lemma 29.32.9). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 37.7.9. □

06AE **Lemma 37.7.12.** *Let $Z \to Y \to X$ be formally unramified morphisms of schemes.*

(1) *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*

(2) *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 37.7.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 37.7.5. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Morphisms, Lemma 29.31.5 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Morphisms, Lemma 29.31.4 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

37.8. Formally étale morphisms

02HF Recall that a ring map $R \to A$ is called *formally étale* (see Algebra, Definition 10.150.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02HG **Definition 37.8.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally étale* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 37.6.7.

04FD **Lemma 37.8.2.** *If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 37.8.1 the condition that T be affine may be dropped.

Proof. Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that a'_i and a'_j agree on any affine open subscheme of $T'_i \cap T'_j$. Hence a'_i and a'_j agree on $T'_i \cap T'_j$. Thus we see that the morphisms a'_i glue to a global morphism $a' : T' \to X$. The uniqueness of a' we have seen in Lemma 37.6.2. □

02HI **Lemma 37.8.3.** *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

02HJ **Lemma 37.8.4.** *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

02HK **Lemma 37.8.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally étale, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow[a]{} & T \\ {\scriptstyle f|_U}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ V & \longleftarrow & T' \end{array}$$

as in Definition 37.8.1. If f is formally ramified, then there exists exactly one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since $|T'| = |T|$ we conclude that $a'(T') \subset U$ which gives our unique morphism from T' into U. □

04FE **Lemma 37.8.6.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is formally étale,*

(2) *f is formally unramified and the universal first order thickening of X over S is equal to X,*
(3) *f is formally unramified and $\mathcal{C}_{X/S} = 0$, and*
(4) *$\Omega_{X/S} = 0$ and $\mathcal{C}_{X/S} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/S} = 0$ implies that $\mathcal{C}_{X/S}$ is defined via Lemma 37.6.7 and Definition 37.7.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0$ since after all by definition $\mathcal{C}_{X/S} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. □

04FF **Lemma 37.8.7.** *An unramified flat morphism is formally étale.*

Proof. Say $X \to S$ is unramified and flat. Then $\Delta : X \to X \times_S X$ is an open immersion, see Morphisms, Lemma 29.35.13. We have to show that $\mathcal{C}_{X/S}$ is zero. Consider the two projections $p, q : X \times_S X \to X$. As f is formally unramified (see Lemma 37.6.8), q is formally unramified (see Lemma 37.6.4). As f is flat, p is flat, see Morphisms, Lemma 29.25.8. Hence $p^*\mathcal{C}_{X/S} = \mathcal{C}_q$ by Lemma 37.7.7 where $\mathcal{C}_q$ denotes the conormal sheaf of the formally unramified morphism $q : X \times_S X \to X$. But $\Delta(X) \subset X \times_S X$ is an open subscheme which maps isomorphically to X via q. Hence by Lemma 37.7.8 we see that $\mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0$. In other words, the pullback of $\mathcal{C}_{X/S}$ to X via the identity morphism is zero, i.e., $\mathcal{C}_{X/S} = 0$. □

02HL **Lemma 37.8.8.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally étale if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally étale ring map.*

Proof. This is immediate from the definitions (Definition 37.8.1 and Algebra, Definition 10.150.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. □

02HM **Lemma 37.8.9.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is étale, and*
(2) *the morphism f is locally of finite presentation and formally étale.*

Proof. Assume f is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 29.36. Hence f is locally of finite presentation and formally étale, see Lemma 37.8.7.

Conversely, suppose that f is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on X and S, see Morphisms, Lemma 29.36.2. By Lemma 37.8.5 we can cover X by affine opens U which map into affine opens V such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 29.21.2). By Lemma 37.8.8 we see that the ring maps $\mathcal{O}(V) \to \mathcal{O}(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 10.150.3. (We will give another proof of this implication when we discuss formally smooth morphisms.) □

37.9. Infinitesimal deformations of maps

04BU In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X.

04FG **Lemma 37.9.1.** *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over S. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over S. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Morphisms, Lemma 29.31.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_S$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}_{X/X'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle (a')^\sharp} \uparrow{\scriptstyle (b')^\sharp} & & \uparrow & & \\ 0 & \longrightarrow & a^{-1}\mathcal{C}_{Y/Y'} & \longrightarrow & a^{-1}\mathcal{O}_{Y'} & \longrightarrow & a^{-1}\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_S$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_S$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

04BV (37.9.1.1) $$D = \mathrm{d}_{Y/S} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/S} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$.

02H5 **Lemma 37.9.2.** *Let S be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over S. Let*

$$\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 37.9.1 hold and the derivation D and θ are related by Equation (37.9.1.1).

Proof. We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (37.9.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 37.9.1 hold. Equation (37.9.1.1) holds by construction. □

0CK1 **Remark 37.9.3.** Assumptions and notation as in Lemma 37.9.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (37.9.1.1).

04FH **Lemma 37.9.4.** *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over S. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that*

(1) *a' is a morphism over S,*
(2) *$a'|_U = a|_U$, and*
(3) *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Here $U = X \cap U'$. Then the rule

04FI (37.9.4.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on X'.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. □

In the following lemma we identify sheaves on X and any thickening of X.

04FJ **Lemma 37.9.5.** *Same notation and assumptions as in Lemma 37.9.4. There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf (37.9.4.1). Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf (37.9.4.1) has a section.

Proof. This is a combination of Lemmas 37.9.1, 37.9.2, and 37.9.4. □

04FK **Remark 37.9.6.** A special case of Lemmas 37.9.1, 37.9.2, 37.9.4, and 37.9.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 37.9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$.

0CK2 **Remark 37.9.7.** Another special case of Lemmas 37.9.1, 37.9.2, 37.9.4, and 37.9.5 is where S itself is a thickening $Z \subset Z' = S$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(a,?)]{} & (Y \subset Y') \\ & {\scriptstyle (g,g')} \searrow \quad \swarrow {\scriptstyle (h,h')} & \\ & (Z \subset Z') & \end{array}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 37.9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

04FL **Lemma 37.9.8.** *Let S be a scheme. Let $X \subset X'$ be a first order thickening over S. Let Y be a scheme over S. Let $a', b' : X' \to Y$ be two morphisms over S with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle (b',a')} \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

Since the horizontal arrows are immersions with conormal sheaves $\mathcal{C}_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 29.31.3, we obtain a map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$. Then this θ and the derivation D of Lemma 37.9.1 are related by Equation (37.9.1.1).

Proof. Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If f is a local section of $\mathcal{O}_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $\mathcal{C}_{Y/(Y \times_S Y)}$ corresponding to $\mathrm{d}_{Y/S}(f)$. It is mapped to the local section $(a')^\sharp(f) - (b')^\sharp(f) = D(f)$ of $\mathcal{C}_{X/X'}$. In other words, $\theta(\mathrm{d}_{Y/S}(f)) = D(f)$. □

For later purposes we need a result that roughly states that the construction of Lemma 37.9.2 is compatible with étale localization.

04BX **Lemma 37.9.9.** *Let*

$$\begin{array}{ccc} X_1 & \xleftarrow[f]{} & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longleftarrow & S_2 \end{array}$$

be a commutative diagram of schemes with $X_2 \to X_1$ and $S_2 \to S_1$ étale. Then the map $c_f : f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2}$ of Morphisms, Lemma 29.32.8 is an isomorphism.

Proof. We recall that an étale morphism $U \to V$ is a smooth morphism with $\Omega_{U/V} = 0$. Using this we see that Morphisms, Lemma 29.32.9 implies $\Omega_{X_2/S_2} = \Omega_{X_2/S_1}$ and Morphisms, Lemma 29.34.16 implies that the map $f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1}$ (for the morphism f seen as a morphism over S_1) is an isomorphism. Hence the lemma follows. □

04BY **Lemma 37.9.10.** *Consider a commutative diagram of first order thickenings*

$$\begin{array}{ccc} (T_2 \subset T_2') & \xrightarrow[(a_2,a_2')]{} & (X_2 \subset X_2') \\ {\scriptstyle (h,h')}\downarrow & & \downarrow{\scriptstyle (f,f')} \\ (T_1 \subset T_1') & \xrightarrow{(a_1,a_1')} & (X_1 \subset X_1') \end{array} \quad \text{and a commutative diagram of schemes} \quad \begin{array}{ccc} X_2' & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ X_1' & \longrightarrow & S_1 \end{array}$$

with $X_2 \to X_1$ and $S_2 \to S_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/S_2} = h^*a_1^*\Omega_{X_1/S_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{array}{ccc} T_2' & \xrightarrow[\theta_2 \cdot a_2']{} & X_2' \\ \downarrow & & \downarrow \\ T_1' & \xrightarrow{\theta_1 \cdot a_1'} & X_1' \end{array}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 37.9.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ of Lemma 37.9.9. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain commutative diagrams

$$\begin{array}{ccc} (B_2', I_2) & \xleftarrow{a_2'} & (A_2', J_2) \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle f'} \\ (B_1', I_1) & \xleftarrow{a_1'} & (A_1', J_1) \end{array} \quad \text{and} \quad \begin{array}{ccc} A_2' & \longleftarrow & R_2 \\ \uparrow & & \uparrow \\ A_1' & \longleftarrow & R_1 \end{array}$$

The notation signifies that I_1, I_2, J_1, J_2 are ideals of square zero and maps of pairs are ring maps sending ideals into ideals. Set $A_1 = A_1'/J_1$, $A_2 = A_2'/J_2$, $B_1 = B_1'/I_1$, and $B_2 = B_2'/I_2$. We are given that

$$A_2 \otimes_{A_1} \Omega_{A_1/R_1} \longrightarrow \Omega_{A_2/R_2}$$

is an isomorphism. Then $\theta_1 : B_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1$ is B_1-linear. This gives an R_1-derivation $D_1 = \theta_1 \circ \mathrm{d}_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2$ gives rise to a R_2-derivation $D_2 = \theta_2 \circ \mathrm{d}_{A_2/R_2} : A_2 \to I_2$. The construction of θ_2 implies the following compatibility between θ_1 and θ_2: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f'(x))$$

as elements of I_2. We may view D_1 as a map $A_1' \to B_1'$ using $A_1' \to A_1 \xrightarrow{D_1} I_1 \to B_1$ similarly we may view D_2 as a map $A_2' \to B_2'$. Then the displayed equality holds for $x \in A_1'$. By the construction of the action in Lemma 37.9.2 and Remark 37.9.3 we know that $\theta_1 \cdot a_1'$ corresponds to the ring map $a_1' + D_1 : A_1' \to B_1'$ and $\theta_2 \cdot a_2'$ corresponds to the ring map $a_2' + D_2 : A_2' \to B_2'$. By the displayed equality we obtain that $h' \circ (a_1' + D_1) = (a_2' + D_2) \circ f'$ as desired. □

04BZ **Remark 37.9.11.** Lemma 37.9.10 can be improved in the following way. Suppose that we have commutative diagrams as in Lemma 37.9.10 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to \mathcal{C}_{T_2/T_2'}$ such that

$$\begin{array}{ccc} f_*\mathcal{O}_{X_2} & \xrightarrow[f_*D_2]{} & f_*a_{2,*}\mathcal{C}_{T_2/T_2'} \\ {\scriptstyle f^\sharp}\uparrow & & \uparrow{\scriptstyle \text{induced by } (h')^\sharp} \\ \mathcal{O}_{X_1} & \xrightarrow{D_1} & a_{1,*}\mathcal{C}_{T_1/T_1'} \end{array}$$

is commutative where D_i corresponds to θ_i as in Equation (37.9.1.1). Then we have the conclusion of Lemma 37.9.10. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a θ_2 from θ_1.

37.10. Infinitesimal deformations of schemes

063X The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

063Y **Lemma 37.10.1.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that f is flat. Then the following are equivalent*

(1) *f' is flat and $X = S \times_{S'} X'$, and*
(2) *the canonical map $f^*\mathcal{C}_{S/S'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. As the problem is local on X' we may assume that X, X', S, S' are affine schemes. Say $S' = \mathrm{Spec}(A')$, $X' = \mathrm{Spec}(B')$, $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. The canonical map of the lemma is the map

$$I \otimes_A B = I \otimes_{A'} B' \longrightarrow J.$$

The assumption that f is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\mathrm{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma 10.75.8). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark 10.75.9). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\mathrm{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence B' is flat over A' by Algebra, Lemma 10.99.8. □

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 37.16.

06AF **Lemma 37.10.2.** *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume

(1) *X' is flat over S',*
(2) *f is flat,*
(3) *$S \subset S'$ is a finite order thickening, and*
(4) *$X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$.*

Then f' is flat and Y' is flat over S' at all points in the image of f'.

Proof. Immediate consequence of Algebra, Lemma 10.101.8. □

Many properties of morphisms of schemes are preserved under flat deformations.

06AG **Lemma 37.10.3.** *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume $S \subset S'$ is a finite order thickening, X' flat over S', $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

06AH (1) *f is flat if and only if f' is flat,*
06AI (2) *f is an isomorphism if and only if f' is an isomorphism,*
06AJ (3) *f is an open immersion if and only if f' is an open immersion,*
06AK (4) *f is quasi-compact if and only if f' is quasi-compact,*
06AL (5) *f is universally closed if and only if f' is universally closed,*
06AM (6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
06AN (7) *f is a monomorphism if and only if f' is a monomorphism,*
06AP (8) *f is surjective if and only if f' is surjective,*
06AQ (9) *f is universally injective if and only if f' is universally injective,*
06AR (10) *f is affine if and only if f' is affine,*
06AS (11) *f is locally of finite type if and only if f' is locally of finite type,*
06AT (12) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
06AU (13) *f is locally of finite presentation if and only if f' is locally of finite presentation,*
06AV (14) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
06AW (15) *f is universally open if and only if f' is universally open,*
06AX (16) *f is syntomic if and only if f' is syntomic,*
06AY (17) *f is smooth if and only if f' is smooth,*
06AZ (18) *f is unramified if and only if f' is unramified,*
06B0 (19) *f is étale if and only if f' is étale,*
06B1 (20) *f is proper if and only if f' is proper,*
06B2 (21) *f is integral if and only if f' is integral,*
06B3 (22) *f is finite if and only if f' is finite,*
06B4 (23) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(24) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (23) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2, 26.19.3, 26.21.12, and 26.23.5 and Morphisms, Lemmas 29.9.4, 29.10.4, 29.11.9, 29.15.4, 29.20.13, 29.21.4, 29.29.2, 29.30.4, 29.34.5, 29.35.5, 29.36.4, 29.41.5, 29.44.6, and 29.48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 37.2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \mathrm{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \mathrm{Spec}(B')$ and $V' = \mathrm{Spec}(A')$. Then we get a commutative

diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. Moreover $IB' \cong I \otimes_R B$, see proof of Lemma 37.10.1. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 37.10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. By (III) we see that $Y' \to S'$ is flat. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I) and moreover $IA \cong I \otimes_R A$ because $R' \to A'$ is flat. Then $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 37.3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 37.3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Hence f' is affine. See also Lemma 37.3.1 for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in B generate B as an A-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma 37.3.3 for a more general statement.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6. See also Lemma 37.3.3 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal K' by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As B' is flat over R' we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since I is nilpotent we see that $f'_1, \ldots, f'_m$ generate K' by Nakayama's lemma, see Algebra, Lemma 10.20.1.

Ad (14). Follows from (11) and general remark (II). See also Lemma 37.3.3 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (16). Assume f is syntomic. By (13) f' is locally of finite presentation, by general remark (III) f' is flat and the fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 29.30.11.

Ad (17). Assume f is smooth. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 29.34.3.

Ad (18). Assume f unramified. By (11) f' is locally of finite type and the fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 29.35.12. See also Lemma 37.3.3 for a more general statement.

Ad (19). Assume f étale. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 29.36.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 37.3.3 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 29.44.7. See also Lemma 37.3.1 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 29.44.4. See also Lemma 37.3.3 for a more general statement.

Ad (23). Assume f finite locally free. By (22) we see that f' is finite, by general remark (III) f' is flat, and by (13) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 29.48.2. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 37.16.

0CF3 **Lemma 37.10.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume

(1) $Y' \to S'$ *is locally of finite type,*
(2) $X' \to S'$ *is flat and locally of finite presentation,*
(3) f *is flat, and*
(4) $X = S \times_{S'} X'$ *and* $Y = S \times_{S'} Y'$.

Then f' is flat and for all $y' \in Y'$ in the image of f' the local ring $\mathcal{O}_{Y',y'}$ is flat and essentially of finite presentation over $\mathcal{O}_{S',s'}$.

Proof. Immediate consequence of Algebra, Lemma 10.128.10. □

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

0CF4 **Lemma 37.10.5.** *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume $Y' \to S'$ locally of finite type, $X' \to S'$ flat and locally of finite presentation, $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

0CF5 (1) f *is flat if and only if f' is flat,*
0CF6 (2) f *is an isomorphism if and only if f' is an isomorphism,*
0CF7 (3) f *is an open immersion if and only if f' is an open immersion,*
0CF8 (4) f *is quasi-compact if and only if f' is quasi-compact,*
0CF9 (5) f *is universally closed if and only if f' is universally closed,*
0CFA (6) f *is (quasi-)separated if and only if f' is (quasi-)separated,*
0CFB (7) f *is a monomorphism if and only if f' is a monomorphism,*
0CFC (8) f *is surjective if and only if f' is surjective,*
0CFD (9) f *is universally injective if and only if f' is universally injective,*
0CFE (10) f *is affine if and only if f' is affine,*
0CFF (11) f *is locally quasi-finite if and only if f' is locally quasi-finite,*
0CFG (12) f *is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
0CFH (13) f *is universally open if and only if f' is universally open,*
0CFI (14) f *is syntomic if and only if f' is syntomic,*
0CFJ (15) f *is smooth if and only if f' is smooth,*
0CFK (16) f *is unramified if and only if f' is unramified,*
0CFL (17) f *is étale if and only if f' is étale,*
0CFM (18) f *is proper if and only if f' is proper,*
0CFN (19) f *is finite if and only if f' is finite,*

0CFP (20) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(21) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (20) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2, 26.19.3, 26.21.12, and 26.23.5 and Morphisms, Lemmas 29.9.4, 29.10.4, 29.11.9, 29.20.13, 29.29.2, 29.30.4, 29.34.5, 29.35.5, 29.36.4, 29.41.5, 29.44.6, and 29.48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \mathrm{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \mathrm{Spec}(B')$ and $V' = \mathrm{Spec}(A')$. Then we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 37.10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I). By Algebra, Lemma 10.126.11 we see that $A' \to B'$ is an isomorphism, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 37.3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. Observe that $X' \times_{Y'} X' \to S'$ is locally of finite type. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 37.3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Hence f' is affine. See also Lemma 37.3.1 for a more general statement.

Ad (11). Follows from the fact that f' is locally of finite type (by Morphisms, Lemma 29.15.8) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6.

Ad (12). Follows from general remark (II) and the fact that f' is locally of finite type (Morphisms, Lemma 29.15.8).

Ad (13). Immediate from general remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (14). Assume f is syntomic. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 29.30.11.

Ad (15). Assume f is smooth. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 29.34.3.

Ad (16). Assume f unramified. By Morphisms, Lemma 29.15.8 f' is locally of finite type. The fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 29.35.12.

Ad (17). Assume f étale. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 29.36.8.

Ad (18). This follows from a combination of (6), the fact that f is locally of finite type (Morphisms, Lemma 29.15.8), (4), and (5).

Ad (19). Combine (5), (10), Morphisms, Lemma 29.44.7, the fact that f is locally of finite type (Morphisms, Lemma 29.15.8), and Morphisms, Lemma 29.44.4.

Ad (20). Assume f finite locally free. By (19) we see that f' is finite. By general remark (III) f' is flat. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 29.48.2. □

0D4F **Lemma 37.10.6** (Deformations of projective schemes)**.** *Let $f : X \to S$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $\mathcal{L}$ be f-ample. Assume S is quasi-compact. There exists a $d_0 \geq 0$ such that for every cartesian diagram*

$$\begin{array}{ccc} X & \xrightarrow{i'} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{i} & S' \end{array} \quad \text{and} \quad \begin{array}{c} \text{invertible } \mathcal{O}_{X'}\text{-module} \\ \mathcal{L}' \text{ with } \mathcal{L} \cong (i')^*\mathcal{L}' \end{array}$$

where $S \subset S'$ is a thickening and f' is proper, flat, of finite presentation we have

(1) $R^p(f')_*(\mathcal{L}')^{\otimes d} = 0$ *for all* $p > 0$ *and* $d \geq d_0$,

(2) $\mathcal{A}'_d = (f')_*(\mathcal{L}')^{\otimes d}$ *is finite locally free for* $d \geq d_0$,
(3) $\mathcal{A}' = \mathcal{O}_{S'} \oplus \bigoplus_{d \geq d_0} \mathcal{A}'_d$ *is a quasi-coherent* $\mathcal{O}_{S'}$*-algebra of finite presentation,*
(4) *there is a canonical isomorphism* $r' : X' \to \underline{Proj}_{S'}(\mathcal{A}')$*, and*
(5) *there is a canonical isomorphism* $\theta' : (r')^* \mathcal{O}_{\underline{Proj}_{S'}(\mathcal{A}')}(1) \to \mathcal{L}'$.

The construction of $\mathcal{A}'$*,* r'*,* θ' *is functorial in the data* $(X', S', i, i', f', \mathcal{L}')$.

Proof. We first describe the maps r' and θ'. Observe that $\mathcal{L}'$ is f'-ample, see Lemma 37.3.2. There is a canonical map of quasi-coherent graded $\mathcal{O}_{S'}$-algebras $\mathcal{A}' \to \bigoplus_{d \geq 0} (f')_*(\mathcal{L}')^{\otimes d}$ which is an isomorphism in degrees $\geq d_0$. Hence this induces an isomorphism on relative Proj compatible with the Serre twists of the structure sheaf, see Constructions, Lemma 27.18.4. Hence we get the morphism r' by Morphisms, Lemma 29.37.4 (which in turn appeals to the construction given in Constructions, Lemma 27.19.1) and it is an isomorphism by Morphisms, Lemma 29.43.17. We get the map θ' from Constructions, Lemma 27.19.1. By Properties, Lemma 28.28.2 we find that θ' is an isomorphism (this also uses that the morphism r' over affine opens of S' is the same as the morphism from Properties, Lemma 28.26.9 as is explained in the proof of Morphisms, Lemma 29.43.17).

Assuming the vanishing and local freeness stated in parts (1) and (2), the functoriality of the construction can be seen as follows. Suppose that $h : T \to S'$ is a morphism of schemes, denote $f_T : X'_T \to T$ the base change of f' and $\mathcal{L}_T$ the pullback of $\mathcal{L}$ to X'_T. By cohomology and base change (as formulated in Derived Categories of Schemes, Lemma 36.22.5 for example) we have the corresponding vanishing over T and moreover $h^*\mathcal{A}'_d = f_{T,*}\mathcal{L}_T^{\otimes d}$ (and thus the local freeness of pushforwards as well as the finite generation of the corresponding graded $\mathcal{O}_T$-algebra $\mathcal{A}_T$). Hence the morphism $r_T : X_T \to \underline{\mathrm{Proj}}_T(\bigoplus f_{T,*}\mathcal{L}_T^{\otimes d})$ is simply the base change of r' to T and the pullback of θ' is the map θ_T.

Having said all of the above, we see that it suffices to prove (1), (2), and (3). Pick d_0 such that $R^p f_* \mathcal{L}^{\otimes d} = 0$ for all $d \geq d_0$ and $p > 0$, see Cohomology of Schemes, Lemma 30.16.1. We claim that d_0 works.

By cohomology and base change (Derived Categories of Schemes, Lemma 36.30.4) we see that $E'_d = Rf'_*(\mathcal{L}')^{\otimes d}$ is a perfect object of $D(\mathcal{O}_{S'})$ and its formation commutes with arbitrary base change. In particular, $E_d = Li^*E'_d = Rf_*\mathcal{L}^{\otimes d}$. By Derived Categories of Schemes, Lemma 36.32.4 we see that for $d \geq d_0$ the complex E_d is isomorphic to the finite locally free $\mathcal{O}_S$-module $f_*\mathcal{L}^{\otimes d}$ placed in cohomological degree 0. Then by Derived Categories of Schemes, Lemma 36.31.3 we conclude that E'_d is isomorphic to a finite locally free module placed in cohomological degree 0. Of course this means that $E'_d = \mathcal{A}'_d[0]$, that $R^p f'_*(\mathcal{L}')^{\otimes d} = 0$ for $p > 0$, and that $\mathcal{A}'_d$ is finite locally free. This proves (1) and (2).

The last thing we have to show is finite presentation of $\mathcal{A}'$ as a sheaf of $\mathcal{O}_{S'}$-algebras (this notion was introduced in Properties, Section 28.22). Let $U' = \mathrm{Spec}(R') \subset S'$ be an affine open. Then $A' = \mathcal{A}'(U')$ is a graded R'-algebra whose graded parts are finite projective R'-modules. We have to show that A' is a finitely presented R'-algebra. We will prove this by reduction to the Noetherian case. Namely, we can find a finite type $\mathbf{Z}$-subalgebra $R'_0 \subset R'$ and a pair[1]

[1]With the same properties as those enjoyed by $X' \to S'$ and $\mathcal{L}'$, i.e., $X'_0 \to \mathrm{Spec}(R'_0)$ is flat and proper and $\mathcal{L}'_0$ is ample.

$(X'_0, \mathcal{L}'_0)$ over R'_0 whose base change is $(X'_{U'}, \mathcal{L}'|_{X'_{U'}})$, see Limits, Lemmas 32.10.2, 32.10.3, 32.13.1, 32.8.7, and 32.4.15. Cohomology of Schemes, Lemma 30.16.1 implies $A'_0 = \bigoplus_{d\geq 0} H^0(X'_0, (\mathcal{L}'_0)^{\otimes d})$ is a finitely generated graded R'_0-algebra and implies there exists a d'_0 such that $H^p(X'_0, (\mathcal{L}'_0)^{\otimes d}) = 0$, $p > 0$ for $d \geq d'_0$. By the arguments given above applied to $X'_0 \to \mathrm{Spec}(R'_0)$ and $\mathcal{L}'_0$ we see that $(A'_0)_d$ is a finite projective R'_0-module and that

$$A'_d = \mathcal{A}'_d(U') = H^0(X'_{U'}, (\mathcal{L}')^{\otimes d}|_{X'_{U'}}) = H^0(X'_0, (\mathcal{L}'_0)^{\otimes d}) \otimes_{R'_0} R' = (A'_0)_d \otimes_{R'_0} R'$$

for $d \geq d'_0$. Now a small twist in the argument is that we don't know that we can choose d'_0 equal to d_0[2]. To get around this we use the following sequence of arguments to finish the proof:

(a) The algebra $B = R'_0 \oplus \bigoplus_{d\geq \max(d_0, d'_0)} (A'_0)_d$ is an R'_0-algebra of finite type: apply the Artin-Tate lemma to $B \subset A'_0$, see Algebra, Lemma 10.51.7.
(b) As R'_0 is Noetherian we see that B is an R'_0-algebra of finite presentation.
(c) By right exactness of tensor product we see that $B \otimes_{R'_0} R'$ is an R'-algebra of finite presentation.
(d) By the displayed equalities this exactly says that $C = R' \oplus \bigoplus_{d\geq \max(d_0, d'_0)} A'_d$ is an R'-algebra of finite presentation.
(e) The quotient A'/C is the direct sum of the finite projective R'-modules A'_d, $d_0 \leq d \leq \max(d_0, d'_0)$, hence finitely presented as R'-module.
(f) The quotient A'/C is finitely presented as a C-module by Algebra, Lemma 10.6.4.
(g) Thus A' is finitely presented as a C-module by Algebra, Lemma 10.5.3.
(h) By Algebra, Lemma 10.7.4 this implies A' is finitely presented as a C-algebra.
(i) Finally, by Algebra, Lemma 10.6.2 applied to $R' \to C \to A'$ this implies A' is finitely presented as an R'-algebra.

This finishes the proof. □

37.11. Formally smooth morphisms

02GZ Michael Artin's position on differential criteria of smoothness (e.g., Morphisms, Lemma 29.34.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism $X \to S$. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 37.11.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called *formally smooth* (see Algebra, Definition 10.138.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

[2]Actually, one can reduce to this case by doing more limit arguments.

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H0 **Definition 37.11.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally smooth* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 37.6.2 and 37.8.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on T'. In fact, analyzing the proof of Lemma 37.11.10 shows that this would be equivalent to the definition as it currently stands. In particular, being formally smooth is Zariski local on the source (and in fact it is smooth local on the source, insert future reference here).

02H1 **Lemma 37.11.2.** *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

02H2 **Lemma 37.11.3.** *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 10.138.2 for the algebraic version. □

02HH **Lemma 37.11.4.** *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

02H3 **Lemma 37.11.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally smooth, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow[a]{} & T \\ {\scriptstyle f|_U}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ V & \longleftarrow & T' \end{array}$$

as in Definition 37.11.1. If f is formally smooth, then there exists an S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since the underlying sets of T and T' are the same we see that a' is a morphism into U (see Schemes, Section 26.3). And it clearly is a V-morphism as well. Hence the dotted arrow above as desired. □

02H4 **Lemma 37.11.6.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally smooth if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally smooth ring map.*

Proof. This is immediate from the definitions (Definition 37.11.1 and Algebra, Definition 10.138.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. □

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 32.6.1) that we can recognize whether a morphism $f : X \to S$ is smooth in terms of "simple" properties of the functor $h_X : Sch/S \to Sets$.

02H6 **Lemma 37.11.7** (Infinitesimal lifting criterion)**.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is smooth, and*
(2) *the morphism f is locally of finite presentation and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 37.11.5 we see that $U \to V$ is formally smooth. By Lemma 37.11.6 we see that $R \to A$ is formally smooth. By Morphisms, Lemma 29.21.2 we see that $R \to A$ is of finite presentation. By Algebra, Proposition 10.138.13 we see that $R \to A$ is smooth. Hence by the definition of a smooth morphism we see that $X \to S$ is smooth.

Conversely, assume that $f : X \to S$ is smooth. Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ S & \xleftarrow{} & T' \end{array}$$

as in Definition 37.11.1. We will show the dotted arrow exists thereby proving that f is formally smooth.

Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 37.9.4 in the special case discussed in Remark 37.9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 37.9.5. Our goal is simply to show that $\mathcal{F}(T) \neq \emptyset$. In other words we are trying to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor on T (see Cohomology, Section 20.4). There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 20.4.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 20.4.3 – we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 20.13.3).

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of t such that $a(U)$ is contained in an affine open $\mathrm{Spec}(A) = W \subset X$ which maps to an affine open $\mathrm{Spec}(R) = V \subset S$. By Morphisms, Lemma 29.34.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 10.138.13 the ring map $R \to A$ is formally smooth. Lemma 37.11.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a V-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 29.32.13 we see that $\Omega_{X/S}$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 29.34.12). Hence

$a^*\Omega_{X/S}$ is of finite presentation (see Modules, Lemma 17.11.5). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$ is quasi-coherent by the discussion in Schemes, Section 26.24. Thus by Cohomology of Schemes, Lemma 30.2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired. □

Locally projective quasi-coherent modules are defined in Properties, Section 28.21.

06B5 **Lemma 37.11.8.** *Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 37.11.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 37.11.6. Hence by Algebra, Lemma 10.138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 28.21. □

0D0E **Lemma 37.11.9.** *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules. Consider $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ is locally projective, then $H^1(T, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on a scheme (see Properties, Definition 28.21.1) we see that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module. Hence we may assume that $\mathcal{F} = \bigoplus_{i\in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i\in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology, Lemma 20.11.12 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Cohomology of Schemes, Lemma 30.2.2. □

0D0F **Lemma 37.11.10.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f is formally smooth,*
2. *for every $x \in X$ there exist opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth,*
3. *for every pair of affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is formally smooth, and*
4. *there exists an affine open covering $Y = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a formally smooth ring map for all j and i.*

Proof. The implications (1) ⇒ (2), (1) ⇒ (3), and (2) ⇒ (4) follow from Lemma 37.11.5. The implication (3) ⇒ (4) is immediate.

Assume (4). The proof that f is formally smooth is the same as the second part of the proof of Lemma 37.11.7. Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ Y & \longleftarrow & T' \end{array}$$

as in Definition 37.11.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 37.9.4 as in the special case discussed in Remark 37.9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on T with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 37.9.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology, Definition 20.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ locally has a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$, see Cohomology, Lemma 20.4.3.

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $W \subset T$ neighbourhood of t such that $a(W)$ is contained in U_{ji} for some i, j. Let $W' \subset T'$ be the corresponding open subscheme. By assumption (4) we can lift $a|_W : W \to U_{ji}$ to a V_j-morphism $a' : W' \to U_{ji}$ showing that $\mathcal{F}(W')$ is nonempty.

Finally we prove (II). By Lemma 37.11.8 we see that Ω_{U_{ji}/V_j} locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Properties, Lemma 28.21.2. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties, Lemma 28.21.3. Hence

$$H^1(T, \mathcal{H}) = H^1(T, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})) = 0$$

by Lemma 37.11.9 as desired. □

06B6 **Lemma 37.11.11.** *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 29.32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.131.7 is exact. This is Algebra, Lemma 10.138.9. □

06B7 **Lemma 37.11.12.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Assume that Z is formally smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 37.7.10 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 37.7.10 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

Since $Z \to S$ is formally smooth we can locally on Z' find a left inverse $Z' \to Z$ over S to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 29.32.16. □

067W **Lemma 37.11.13.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow f \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 37.7.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 37.7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \xrightarrow{a} & X \\ & \searrow^{i'} & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle f} \\ & \searrow_{j'} & Z'' & \xrightarrow{b} & Y \end{array}$$

In the proof of Lemma 37.7.11 we identified the sequence above with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{array}{ccc} U & \longrightarrow & Z' \\ \downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle k} \\ U'' & \longrightarrow & Z'' \end{array}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

37.12. Smoothness over a Noetherian base

02HW It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

02HX **Lemma 37.12.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth at x,*
(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\quad} & \operatorname{Spec}(B) \\ & \overset{\alpha}{\nwarrow} & \\ f\downarrow & & \downarrow{\scriptstyle i} \\ & \overset{\beta}{\swarrow} & \\ S & \xleftarrow{\quad} & \operatorname{Spec}(B') \end{array}$$

where $B' \to B$ is a surjection of local rings with $\operatorname{Ker}(B' \to B)$ of square zero, and α mapping the closed point of $\operatorname{Spec}(B)$ to x there exists a dotted arrow making the diagram commute,
(3) *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 10.141.1), and*
(4) *same as in (2) but with $B' \to B$ ranging over small extensions such that α induces an isomorphism $\kappa(x) \to \kappa(\mathfrak{m})$ where $\mathfrak{m} \subset B$ is the maximal ideal.*

Proof. Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. For any "test" diagram as in (2) the morphism α will map $\mathrm{Spec}(B)$ into U and the morphism β will map $\mathrm{Spec}(B')$ into V (see Schemes, Section 26.13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, V is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 28.5.2 and Morphisms, Lemma 29.15.2.) In this affine case the lemma is identical to Algebra, Lemma 10.141.2. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms, Section 29.16).

02HY **Lemma 37.12.2.** *Let $f : X \to S$ be a morphism of schemes. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*
(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\alpha} & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ S & \xleftarrow{\beta} & \mathrm{Spec}(B') \end{array}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 37.11.7) says f is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where f is not smooth forms a closed subset T of X. By the discussion in Morphisms, Section 29.16, if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$\mathrm{Spec}(\kappa(x)) \to X \to S$$

is of finite type (namely, pick any point x of T which is closed in an affine open of X). By Morphisms, Lemma 29.16.2 given any local Artinian ring B' with residue field $\kappa(x)$ then any morphism $\beta : \mathrm{Spec}(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 37.12.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a x a contradiction. □

Here is a useful application.

0A43 **Lemma 37.12.3.** *Let $f : X \to S$ be a finite type morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$.*

(1) *If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.*
(2) *If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.*

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 37.12.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. □

0D4G **Lemma 37.12.4.** *Let $f : X \to S$ be a morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$. If $X_n \to Z_n$ is flat for all n, then f is flat at every point of $f^{-1}(Z)$.*

Proof. This is a translation of Algebra, Lemma 10.99.11 into the language of schemes. □

37.13. The naive cotangent complex

0D0G This section is the continuation of Modules, Section 17.31 which in turn continues the discussion in Algebra, Section 10.134.

0D0H **Definition 37.13.1.** Let $f : X \to Y$ be a morphism of schemes. The *naive cotangent complex of f* is the complex defined in Modules, Definition 17.31.6. Notation: NL_f or $NL_{X/Y}$.

0D0I **Lemma 37.13.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ be affine opens with $f(U) \subset V$. There is a canonical map*

$$\widetilde{NL_{A/R}} \longrightarrow NL_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$.

Proof. From the construction of $NL_{X/Y}$ in Modules, Section 17.31 we see there is a canonical map of complexes $NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to NL_{X/Y}(U)$ of $A = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $R \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $NL_{A/R} \to NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of A-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 26.7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows from Algebra, Lemma 10.134.11 and 10.134.13 and Modules, Lemma 17.31.4 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(A)$ as $A_\mathfrak{p}$ and $R_\mathfrak{q}$ where $\mathfrak{q} = R \cap \mathfrak{p}$; references used are Schemes, Lemma 26.5.4 and Sheaves, Lemma 6.21.5). □

0D0J **Lemma 37.13.3.** *Let $f : X \to Y$ be a morphism of schemes. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.*

Proof. By construction of the naive cotangent complex in Modules, Section 17.31 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$. The sheaf of differentials is quasi-coherent (by Morphisms, Lemma 29.32.7). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking over affines using Lemma 37.13.2. □

0D0K **Lemma 37.13.4.** *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite presentation, then $NL_{X/Y}$ is locally on X quasi-isomorphic to a complex*

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. By Lemma 37.13.2 it suffices to show that $NL_{A/R}$ has this shape if $R \to A$ is a finitely presented ring map. Write $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. Then I/I^2 is a finite A-module and $NL_{A/R}$ is quasi-isomorphic to

$$\ldots \to 0 \to I/I^2 \to \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \to 0 \to \ldots$$

by Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. □

0D0L **Lemma 37.13.5.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is formally smooth,*
(2) *$H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.*

Proof. This follows from Algebra, Proposition 10.138.8 and Lemma 37.11.10. □

0D0M **Lemma 37.13.6.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is formally étale,*
(2) *$H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. A formally étale morphism is formally smooth and hence we have $H^{-1}(NL_{X/Y}) = 0$ by Lemma 37.13.5. On the other hand, we have $\Omega_{X/Y} = 0$ by Lemma 37.8.6. Conversely, if (2) holds, then f is formally smooth by Lemma 37.13.5 and formally unramified by Lemma 37.6.7 and hence formally étale by Lemmas 37.11.4. □

0D0N **Lemma 37.13.7.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is smooth, and*
(2) *f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.*

Proof. This follows from the definition of a smooth ring homomorphism (Algebra, Definition 10.137.1), Lemma 37.13.2, and the definition of a smooth morphism of schemes (Morphisms, Definition 29.34.1). We also use that finite locally free is the same as finite projective for modules over rings (Algebra, Lemma 10.78.2). □

0G7Z **Lemma 37.13.8.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is étale, and*
(2) *f is locally of finite presentation and $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. This follows from the definition of an étale ring homomorphism (Algebra, Definition 10.143.1), Lemma 37.13.2, and the definition of an étale morphism of schemes (Morphisms, Definition 29.36.1). □

0FV2 **Lemma 37.13.9.** *Let $i : Z \to X$ be an immersion of schemes. Then $NL_{Z/X}$ is isomorphic to $\mathcal{C}_{Z/X}[1]$ in $D(\mathcal{O}_Z)$ where $\mathcal{C}_{Z/X}$ is the conormal sheaf of Z in X.*

Proof. This follows from Algebra, Lemma 10.134.6, Morphisms, Lemma 29.31.2, and Lemma 37.13.2. □

0E44 **Lemma 37.13.10.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a canonical six term exact sequence*

$$H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z}) \to H^{-1}(NL_{X/Y}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of cohomology sheaves.

Proof. Special case of Modules, Lemma 17.31.7. □

0FV3 **Lemma 37.13.11.** *Let $f : X \to Y$ and $Y \to Z$ be morphisms of schemes. Assume $X \to Y$ is a complete intersection morphism. Then there is a canonical distinguished triangle*

$$f^* NL_{Y/Z} \to NL_{X/Z} \to NL_{X/Y} \to f^* NL_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$ which recovers the 6-term exact sequence of Lemma 37.13.10.

Proof. It suffices to show the canonical map

$$f^* NL_{Y/Z} \to \text{Cone}(NL_{X/Z} \to NL_{X/Y})[-1]$$

of Modules, Lemma 17.31.7 is an isomorphism in $D(\mathcal{O}_X)$. In order to show this, it suffices to show that the 6-term sequence has a zero on the left, i.e., that $H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z})$ is injective. Affine locally this follows from the corresponding algebra result in More on Algebra, Lemma 15.33.6. To translate into algebra use Lemma 37.13.2. □

0G80 **Lemma 37.13.12.** *Let $X \to Y \to Z$ be morphisms of schemes. Assume $X \to Z$ smooth and $Y \to Z$ étale. Then $X \to Y$ is smooth.*

Proof. The morphism $X \to Y$ is locally of finite presentation by Morphisms, Lemma 29.21.11. By Lemma 37.13.7 we have $H^{-1}(NL_{X/Z}) = 0$ and the module $\Omega_{X/Z}$ is finite locally free. By Lemma 37.13.8 we have $H^{-1}(NL_{Y/Z}) = H^0(NL_{Y/Z}) = 0$. By Lemma 37.13.10 we get $H^{-1}(NL_{X/Y}) = 0$ and $\Omega_{X/Y} \cong \Omega_{X/Z}$ is finite locally free. By Lemma 37.13.7 the morphism $X \to Y$ is smooth. □

0FV4 **Lemma 37.13.13.** *Let $f : X \to Y$ be a morphism of schemes which factors as $f = g \circ i$ with i an immersion and $g : P \to Y$ formally smooth (for example smooth). Then there is a canonical isomorphism*

$$NL_{X/Y} \cong \left(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\right)$$

in $D(\mathcal{O}_X)$ where the conormal sheaf $\mathcal{C}_{X/P}$ is placed in degree -1.

Proof. (For the parenthetical statement see Lemma 37.11.7.) By Lemmas 37.13.9 and 37.13.5 we have $NL_{X/P} = \mathcal{C}_{X/P}[1]$ and $NL_{P/Y} = \Omega_{P/Y}$ with $\Omega_{P/Y}$ locally projective. This implies that $i^* NL_{P/Y} \to i^*\Omega_{P/Y}$ is a quasi-isomorphism too (small detail omitted; the reason is that $i^* NL_{P/Y}$ is the same thing as $\tau_{\geq -1} L i^* NL_{P/Y}$, see More on Algebra, Lemma 15.86.1). Thus the canonical map

$$i^* NL_{P/Y} \to \text{Cone}(NL_{X/Y} \to NL_{X/P})[-1]$$

of Modules, Lemma 17.31.7 is an isomorphism in $D(\mathcal{O}_X)$ because the cohomology group $H^{-1}(i^* NL_{P/Y})$ is zero by what we said above. In other words, we have a distinguished triangle

$$i^* NL_{P/Y} \to NL_{X/Y} \to NL_{X/P} \to i^* NL_{P/Y}[1]$$

Clearly, this means that $NL_{X/Y}$ is the cone on the map $NL_{X/P}[-1] \to i^* NL_{P/Y}$ which is equivalent to the statement of the lemma by our computation of the cohomology sheaves of these objects in the derived category given above. □

0FV5 **Lemma 37.13.14.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

The canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ induces an isomorphism on H^0 and a surjection on H^{-1}.

Proof. Translated into algebra this is More on Algebra, Lemma 15.86.2. To do the translation use Lemma 37.13.2. □

0FJZ **Lemma 37.13.15.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

If $Y' \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism.

Proof. By Lemma 37.13.2 this follows from Algebra, Lemma 10.134.8. □

0FK0 **Lemma 37.13.16.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

If $X \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism. If in addition $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$ then $L(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism too.

Proof. Translated into algebra this is More on Algebra, Lemma 15.86.3. To do the translation use Lemma 37.13.2 and Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.4. □

37.14. Pushouts in the category of schemes, I

07RS In this section we construct pushouts of $Y \leftarrow X \to X'$ where $X \to Y$ is affine and $X \to X'$ is a thickening. This will actually be an important case for us, hence a detailed discussion is merited. In Section 37.67 we discuss a more interesting and more difficult case. See Categories, Section 4.9 for a general discussion of pushouts in any category.

0ET0 **Lemma 37.14.1.** *Let $A' \to A$ be a surjection of rings and let $B \to A$ be a ring map. Let $B' = B \times_A A'$ be the fibre product of rings. Set $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $T = \mathrm{Spec}(B)$, and $T' = \mathrm{Spec}(B')$. Then*

$$\begin{array}{ccc} S & \xrightarrow[i]{} & S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ T & \xrightarrow[i']{} & T' \end{array} \quad \text{corresponding to} \quad \begin{array}{ccc} A & \longleftarrow & A' \\ \uparrow & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

is a pushout of schemes.

Proof. By More on Algebra, Lemma 15.6.2 we have $T' = T \amalg_S S'$ as topological spaces, i.e., the diagram is a pushout in the category of topological spaces. Next, consider the map

$$((i')^\sharp, (f')^\sharp) : \mathcal{O}_{T'} \longrightarrow i'_*\mathcal{O}_T \times_{g_*\mathcal{O}_S} f'_*\mathcal{O}_{S'}$$

where $g = i' \circ f = f' \circ i$. We claim this map is an isomorphism of sheaves of rings. Namely, we can view both sides as quasi-coherent $\mathcal{O}_{T'}$-modules (use Schemes, Lemmas 26.24.1 for the right hand side) and the map is $\mathcal{O}_{T'}$-linear. Thus it suffices to show the map is an isomorphism on the level of global sections (Schemes, Lemma 26.7.5). On global sections we recover the identification $B' \to B \times_A A'$ from statement of the lemma (this is how we chose B').

Let X be a scheme. Suppose we are given morphisms of schemes $m' : S' \to X$ and $n : T \to X$ such that $m' \circ i = n \circ f$ (call this m). We get a unique map of topological spaces $n' : T' \to X$ compatible with m' and n as $T' = T \amalg_S S'$ (see above). By the description of $\mathcal{O}_{T'}$ in the previous paragraph we obtain a unique homomorphism of sheaves of rings

$$(n')^\sharp : \mathcal{O}_X \longrightarrow (n')_*\mathcal{O}_{T'} = m'_*\mathcal{O}_T \times_{m_*\mathcal{O}_T} n_*\mathcal{O}_S$$

given by $(m')^\sharp$ and $n^\sharp$. Thus $(n', (n')^\sharp)$ is the unique morphism of ringed spaces $T' \to X$ compatible with m' and n. To finish the proof it suffices to show that n' is a morphism of schemes, i.e., a morphism of locally ringed spaces.

Let $t' \in T'$ with image $x \in X$. We have to show that $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ is local. If $t' \notin T$, then t' is the image of a unique point $s' \in S'$ and $\mathcal{O}_{T',t'} = \mathcal{O}_{S',s'}$. Namely, $S' \setminus S \to T' \setminus T$ is an isomorphism of schemes as $B' \to A'$ induces an isomorphism $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$. If t' is the image of $t \in T$, then we know that the composition $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'} \to \mathcal{O}_{T,t}$ is local and we conclude also. □

0BMP **Lemma 37.14.2.** *Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes. Let $(W, X_i \to W)$ be a cocone for the diagram in the category of schemes (Categories, Remark 4.14.5). If there exists a fpqc covering $\{W_a \to W\}_{a \in A}$ of schemes such that*

1. *for all $a \in A$ we have $W_a = \mathrm{colim}\, X_i \times_W W_a$ in the category of schemes, and*
2. *for all $a, b \in A$ we have $W_a \times_W W_b = \mathrm{colim}\, X_i \times_W W_a \times_W W_b$ in the category of schemes,*

then $W = \mathrm{colim}\, X_i$ in the category of schemes.

Proof. Namely, for a scheme T a morphism $W \to T$ is the same thing as collection of morphism $W_a \to T$, $a \in A$ which agree on the overlaps $W_a \times_W W_b$, see Descent, Lemma 35.13.7. □

07RT **Lemma 37.14.3.** *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Then there exists a pushout*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

in the category of schemes. Moreover, $Y \subset Y'$ is a thickening, $X = Y \times_{Y'} X'$, and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $|Y| = |Y'|$.

Proof. We first construct Y' as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals f. As structure sheaf $\mathcal{O}_{Y'}$ we take the right hand side of the equation of the lemma. To see that Y' is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume Y is affine. Then X is affine (as f is affine) and X' is affine as well (see Lemma 37.2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_A A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\mathrm{Spec}(B') = \mathrm{Spec}(B)$ (as topological spaces). We claim that $Y' = \mathrm{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_{g'}$. Namely, by More on Algebra, Lemma 15.5.3 we see that

$$(B')_{g'} = B_g \times_{A_h} A'_{h'}$$

where $h \in A$, $h' \in A'$ are the images of g'. Since B_g, resp. A_h, resp. $A'_{h'}$ is equal to $\mathcal{O}_Y(D(g))$, resp. $f_*\mathcal{O}_X(D(g))$, resp. $f'_*\mathcal{O}_{X'}(D(g))$ the claim follows.

It remains to show that Y' is the pushout. The discussion above shows the scheme Y' has an affine open covering $Y' = \bigcup W'_i$ such that the corresponding opens $U'_i \subset X'$, $W_i \subset Y$, and $U_i \subset X$ are affine open. Moreover, if A'_i, B_i, A_i are the rings corresponding to U'_i, W_i, U_i, then W'_i corresponds to $B_i \times_{A_i} A'_i$. Thus we can apply Lemmas 37.14.1 and 37.14.2 to conclude our construction is a pushout in the category of schemes. □

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.31.3.

07RV **Lemma 37.14.4.** *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 37.14.3). Base change gives a functor*

$$F : (Sch/Y') \longrightarrow (Sch/Y) \times_{(Sch/Y')} (Sch/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which has a left adjoint

$$G : (Sch/Y) \times_{(Sch/Y')} (Sch/X') \longrightarrow (Sch/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$. Finally, $F \circ G$ is isomorphic to the identity functor.

Proof. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 29.11.9. Hence we can apply Lemma 37.14.3 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

Let us prove that G is a left adjoint to F. Let Z be a scheme over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that $F \circ G$ is isomorphic to the identity we have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. To do this we may work affine locally. Say $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, and $Y = \mathrm{Spec}(B)$. Then $A' \to A$ and $B \to A$ are ring maps as in More on Algebra, Lemma 15.6.4 and $Y' = \mathrm{Spec}(B')$ with $B' = B \times_A A'$. Next, suppose that $V = \mathrm{Spec}(D)$, $U' = \mathrm{Spec}(C')$ and φ is given by an A-algebra isomorphism $D \otimes_B A \to C' \otimes_{A'} A = C'/IC'$. Set $D' = D \times_{C'/IC'} C'$. In this case the statement we have to prove is that $D' \otimes_{B'} B \cong D$ and $D' \otimes_{B'} A' \cong C'$. This is a special case of More on Algebra, Lemma 15.6.4. □

08KU **Lemma 37.14.5.** *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 37.14.3). Let $V' \to Y'$ be a morphism of schemes. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between*

(1) *quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and*
(2) *the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where*
   (a) *$\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,*
   (b) *$\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X', and*
   (c) *$\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.*

The equivalence maps $\mathcal{G}'$ to $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

(a) *$\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.*
(b) *if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines we recover the equivalence of More on Algebra, Lemma 15.7.5. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 15.7.4 and 15.7.6. □

07RX **Lemma 37.14.6.** *In the situation of Lemma 37.14.4. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then*

1. *$V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,*
2. *$V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,*
3. *$V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,*
4. *$V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,*
5. *$V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and*
6. *add more here as needed.*

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of schemes flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 15.7.7. □

37.15. Openness of the flat locus

0398 This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

0399 **Theorem 37.15.1.** *Let S be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is locally of finite presentation. Then* [**DG67**, IV Theorem 11.3.1]

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in X.

Proof. We may test for openness locally on X hence we may assume that f is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 10.129.4. □

047C **Lemma 37.15.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = f'(x')$.

1. *If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'.*
2. *If g is flat at s' and $(g')^*\mathcal{F}$ is flat over S' at x', then $\mathcal{F}$ is flat over S at x.*

In particular, if g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 37.15.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{S',s'} & \longleftarrow & \mathcal{O}_{S,s} \end{array}$$

Note that $\mathcal{O}_{X',x'}$ is a localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$, and that $((g')^*\mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$. Hence the lemma follows from Algebra, Lemma 10.100.1. □

37.16. Critère de platitude par fibres

039A Consider a commutative diagram of schemes (left hand diagram)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array} \qquad \begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\kappa(s)) & \end{array}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $\mathcal{F}$ flat over Y at x? If $\mathcal{F}$ is flat over S at x, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$$

is flat over Y_s at x. Below you will find a "Noetherian" version, a "finitely presented" version, and earlier we treated a "nilpotent" version, see Lemma 37.10.2.

039B **Theorem 37.16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Assume S, X, Y locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
(2) *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) ⇒ (2) is Algebra, Lemma 10.99.15. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.39.7. □

Here is the non-Noetherian version.

039C **Theorem 37.16.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over S,*

(2) $\mathcal{F}$ *an* $\mathcal{O}_X$*-module of finite presentation, and*
(3) Y *is locally of finite type over* S.

Let $x \in X$. *Set* $y = f(x)$ *and let* $s \in S$ *be the image of* x *in* S. *If* $\mathcal{F}_x \neq 0$, *then the following are equivalent:*

(1) $\mathcal{F}$ *is flat over* S *at* x, *and* $\mathcal{F}_s$ *is flat over* Y_s *at* x, *and*
(2) Y *is flat over* S *at* y *and* $\mathcal{F}$ *is flat over* Y *at* x.

Moreover, the set of points x *where (1) and (2) hold is open in* $\mathrm{Supp}(\mathcal{F})$.

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 10.128.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.39.7.

By Morphisms, Lemma 29.21.11 the morphism f is locally of finite presentation. Consider the set

05VI (37.16.2.1) $$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S\}.$$

This set is open in X by Theorem 37.15.1. Note that if $x \in U$, then $\mathcal{F}_s$ is flat at x over Y_s as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 29.25.7. Hence at every point of $U \cap \mathrm{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \mathrm{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \mathrm{Supp}(\mathcal{F})$. □

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

039D **Lemma 37.16.3.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of schemes over* S. *Assume*

(1) S, X, Y *are locally Noetherian,*
(2) X *is flat over* S,
(3) *for every* $s \in S$ *the morphism* $f_s : X_s \to Y_s$ *is flat.*

Then f *is flat. If* f *is also surjective, then* Y *is flat over* S.

Proof. This is a special case of Theorem 37.16.1. □

039E **Lemma 37.16.4.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of schemes over* S. *Assume*

(1) X *is locally of finite presentation over* S,
(2) X *is flat over* S,
(3) *for every* $s \in S$ *the morphism* $f_s : X_s \to Y_s$ *is flat, and*
(4) Y *is locally of finite type over* S.

Then f *is flat. If* f *is also surjective, then* Y *is flat over* S.

Proof. This is a special case of Theorem 37.16.2. □

05VJ **Lemma 37.16.5.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of schemes over* S. *Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Assume*

(1) *X is locally of finite presentation over S,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over S, and*
(4) *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and U' is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

Proof. By Morphisms, Lemma 29.21.11 the morphism f is locally of finite presentation. Hence U is open by Theorem 37.15.1. Because we have assumed that $\mathcal{F}$ is flat over S we see that Theorem 37.16.2 implies

$$U = \{x \in X \mid \mathcal{F}_s \text{ flat at } x \text{ over } Y_s\}.$$

where s always denotes the image of x in S. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence U' has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{x' \in X'_{s'} \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_{s'}\}$$

is the inverse image of the corresponding locus in X_s. This is true by Lemma 37.15.2 because in the cartesian diagram

$$\begin{array}{ccc} X'_{s'} & \longrightarrow & X_s \\ \downarrow & & \downarrow \\ Y'_{s'} & \longrightarrow & Y_s \end{array}$$

the horizontal morphisms are flat as they are base changes by the flat morphism $\mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s))$. □

05VK **Lemma 37.16.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *X is locally of finite presentation over S,*
(2) *X is flat over S, and*
(3) *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid X \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change.

Proof. This is a special case of Lemma 37.16.5. □

The following lemma is a variant of Algebra, Lemma 10.99.4. Note that the hypothesis that $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module means that $(\mathcal{F}_s)_x$ is a free $\mathcal{O}_{X_s,x}$-module which is always the case if $x \in X_s$ is a generic point of an irreducible component of X_s and X_s is reduced (namely, in this case $\mathcal{O}_{X_s,x}$ is a field, see Algebra, Lemma 10.25.1).

080Q **Lemma 37.16.7.** *Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ with image $s \in S$. If $\mathcal{F}$ is flat at x over S and $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module, then $\mathcal{F}$ is finite free in a neighbourhood of x.*

Proof. If $\mathcal{F}_x \otimes \kappa(x)$ is zero, then $\mathcal{F}_x = 0$ by Nakayama's lemma (Algebra, Lemma 10.20.1) and hence $\mathcal{F}$ is zero in a neighbourhood of x (Modules, Lemma 17.9.5) and the lemma holds. Thus we may assume $\mathcal{F}_x \otimes \kappa(x)$ is not zero and we see that Theorem 37.16.2 applies with $f = \mathrm{id} : X \to X$. We conclude that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{F}_x$ is free, see Algebra, Lemma 10.78.5 for example. Choose an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ which map to a basis in $\mathcal{F}_x$. The corresponding map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective after shrinking U (Modules, Lemma 17.9.5). Then $\mathrm{Ker}(\psi)$ is of finite type (see Modules, Lemma 17.11.4) and $\mathrm{Ker}(\psi)_x = 0$. Whence after shrinking U once more ψ is an isomorphism. □

0CZR **Lemma 37.16.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over S. Then the set*

$$\{x \in X : \mathcal{F} \text{ free in a neighbourhood of } x\}$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. Openness holds trivially. Let $x \in X$ mapping to $s \in S$. By Lemma 37.16.7 we see that x is in our set if and only if $\mathcal{F}|_{X_s}$ is flat at x over X_s. Clearly this is also equivalent to $\mathcal{F}$ being flat at x over X (because this statement is implied by freeness of $\mathcal{F}_x$ and implies flatness of $\mathcal{F}|_{X_s}$ at x over X_s). Thus the base change statement follows from Lemma 37.16.5 applied to $\mathrm{id} : X \to X$ over S. □

37.17. Closed immersions between smooth schemes

0H1G Some results that do not fit elsewhere very well.

0FUE **Lemma 37.17.1.** *Let S be a scheme. Let $Y \to X$ be a closed immersion of schemes smooth over S. For every $y \in Y$ there exist integers $0 \leq m, n$ and a commutative diagram*

$$\begin{array}{ccccc} Y & \longleftarrow & V & \longrightarrow & \mathbf{A}^m_S \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0\ldots,0)} \\ X & \longleftarrow & U & \xrightarrow{\pi} & \mathbf{A}^{m+n}_S \end{array}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_S)$, and $y \in V$.

Proof. The question is local on X hence we may replace X by an open neighbourhood of y. Since $Y \to X$ is a regular immersion by Divisors, Lemma 31.22.11 we may assume $X = \mathrm{Spec}(A)$ is affine and there exists a regular sequence $f_1, \ldots, f_n \in A$ such that $Y = V(f_1, \ldots, f_n)$. After shrinking X (and hence Y) further we may assume there exists an étale morphism $Y \to \mathbf{A}^m_S$, see Morphisms, Lemma 29.36.20. Let $\overline{g}_1, \ldots, \overline{g}_m$ in $\mathcal{O}_Y(Y)$ be the coordinate functions of this étale morphism. Choose lifts $g_1, \ldots, g_m \in A$ of these functions and consider the morphism

$$(g_1, \ldots, g_m, f_1, \ldots, f_n) : X \longrightarrow \mathbf{A}^{m+n}_S$$

over S. This is a morphism of schemes locally of finite presentation over S and hence is locally of finite presentation (Morphisms, Lemma 29.21.11). The restriction of this morphism to $\mathbf{A}^m_S \subset \mathbf{A}^{m+n}_S$ is étale by construction. Thus, in order to show that $X \to \mathbf{A}^{m+n}_S$ is étale at y it suffices to show that $X \to \mathbf{A}^{m+n}_S$ is flat at y, see Morphisms, Lemma 29.36.15. Let $s \in S$ be the image of y. It suffices to check that $X_s \to \mathbf{A}^{m+n}_s$ is flat at y, see Theorem 37.16.2. Let $z \in \mathbf{A}^{m+n}_s$ be the image of y. The local ring map

$$\mathcal{O}_{\mathbf{A}^{m+n}_s, z} \longrightarrow \mathcal{O}_{X_s, y}$$

is flat by Algebra, Lemma 10.128.1. Namely, schemes smooth over fields are regular and regular rings are Cohen-Macaulay, see Varieties, Lemma 33.25.3 and Algebra, Lemma 10.106.3. Thus both source and target are regular local rings (and hence CM). The source and target have the same dimension: namely, we have $\dim(\mathcal{O}_{Y_s,y}) = \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$ by More on Algebra, Lemma 15.44.2, we have $\dim(\mathcal{O}_{\mathbf{A}^{m+n}_s,z}) = n + \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$, and we have $\dim(\mathcal{O}_{X_s,y}) = n + \dim(\mathcal{O}_{Y_s,y})$ because $\mathcal{O}_{Y_s,y}$ is the quotient of $\mathcal{O}_{X_s,y}$ by the regular sequence $f_1, \ldots, f_n$ of length n (see Divisors, Remark 31.22.5). Finally, the fibre ring of the displayed arrow is finite over $\kappa(z)$ since $Y_s \to \mathbf{A}^m_s$ is étale at y. This finishes the proof. □

0H1H **Remark 37.17.2.** We fix a ring R and we set $S = \mathrm{Spec}(R)$. Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion

$$Z = \mathbf{A}^m_S \longrightarrow \mathbf{A}^{m+n}_S = X, \quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0).$$

We are going to consider the blowing up X' of X along the closed subscheme Z. Write

$$X = \mathrm{Spec}(A) \quad \text{with} \quad A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

Then X' is the Proj of the Rees algebra of A with respect to the ideal $(y_1, \ldots, y_n)$. This Rees algebra is equal to $B = A[T_1, \ldots, T_n]/(y_i T_j - y_j T_i)$; details omitted. Hence $X' = \mathrm{Proj}(B)$ is smooth over S as it is covered by the affine opens

$$\begin{aligned} D_+(T_i) &= \mathrm{Spec}(B_{(T_i)}) \\ &= \mathrm{Spec}(A[t_1, \ldots, \hat{t}_i, \ldots t_n]/(y_j - y_i t_j)) \\ &= \mathrm{Spec}(R[x_1, \ldots, x_m, y_i, t_1, \ldots, \hat{t}_i, \ldots, t_n]) \end{aligned}$$

which are isomorphic to $\mathbf{A}^{n+m}_S$. In this chart the exceptional divisor is cut out by setting $y_i = 0$ hence the exceptional divisor is smooth over S as well.

0FUT **Lemma 37.17.3.** *Let S be a scheme. Let $Z \to X$ be a closed immersion of schemes smooth over S. Let $b : X' \to X$ be the blowing up of Z with exceptional divisor $E \subset X'$. Then X' and E are smooth over S. The morphism $p : E \to Z$ is canonically isomorphic to the projective space bundle*

$$\mathbf{P}(\mathcal{I}/\mathcal{I}^2) \longrightarrow Z$$

where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z. The relative $\mathcal{O}_E(1)$ coming from the projective space bundle structure is isomorphic to the restriction of $\mathcal{O}_{X'}(-E)$ to E.

Proof. By Divisors, Lemma 31.22.11 the immersion $Z \to X$ is a regular immmersion, hence the ideal sheaf $\mathcal{I}$ is of finite type, hence b is a projective morphism with relatively ample invertible sheaf $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-E)$, see Divisors, Lemmas 31.32.4 and 31.32.13. The canonical map $\mathcal{I} \to b_*\mathcal{O}_{X'}(1)$ gives a closed immersion

$$X' \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{I})\right)$$

by the very construction of the blowup. The restriction of this morphism to E gives a canonical map

$$E \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n\geq 0} \mathrm{Sym}^n_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2)\right)$$

over Z. Since $\mathcal{I}/\mathcal{I}^2$ is finite locally free if this canonical map is an isomorphism, then the final part of the lemma holds. Having said all of this, now the question is étale local on X. Namely, blowing up commutes with flat base change by Divisors, Lemma 31.32.3 and we can check smoothness after precomposing with a surjective étale morphism. Thus by the étale local structure of a closed immersion of schemes over S given in Lemma 37.17.1 this reduces us to the case discussed in Remark 37.17.2. □

37.18. Flat modules and relative assassins

0GSF In this section we will prove that the support of a flat module is (in some sense) equidimensional over the base in geometric situations. For the Noetherian case we refer the reader to [**DG67**, IV Proposition 12.1.1.5]. First, we prove two helper lemmas.

0GSG **Lemma 37.18.1.** *Let A be a valuation ring. Let $A \to B$ is a local homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a map of finite B-modules. Assume M is flat over A and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is injective and $M/u(N)$ is flat over A.*

Proof. We will deduce this lemma from Algebra, Lemma 10.128.4 (please note that we exchanged the roles of M and N). To do the reduction we will use More on Algebra, Lemma 15.25.7 to reduce to the finitely presented case.

By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak{q}$. Then we can think of $u : N \to M$ as a map of finite $P_\mathfrak{q}$-modules. Hence we may and do assume that B is essentially of finite presentation over A.

Next, the B-module N is finite but perhaps not of finite presentation. Write $N = \operatorname{colim} N_\lambda$ as a filtered colimit of finitely presented B-modules with surjective transition maps. For example choose a presentation $0 \to K \to B^{\oplus r} \to N \to 0$, write K as the union of its finite submodules K_λ, and set $N_\lambda = \mathrm{Coker}(K_\lambda \to B^{\oplus r})$. The module $N/\mathfrak{m}_A N$ is of finite presentation over the Noetherian ring $B/\mathfrak{m}_A B$. Hence for λ large enough we have $N_\lambda/\mathfrak{m}_A N_\lambda = N/\mathfrak{m}_A N$. Now, if we can show the lemma for the composition $u_\lambda : N_\lambda \to M$, then we conclude that $N_\lambda = N$ and the result holds for u. Hence we may and do assume N is of finite presentation over B.

By More on Algebra, Lemma 15.25.7 the module M is of finite presentation over B. Thus all the assumptions of Algebra, Lemma 10.128.4 hold and we conclude. □

0GSH **Lemma 37.18.2.** *Let $f : X \to S$ be a morphism of schemes. Let $y \in X$ be a point with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ viewed as an integral closed subscheme of X. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. There exists a cartesian diagram*

This can be found in the proof of [**DG67**, IV Proposition 12.1.1.5]

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

with the following properties:

(1) *S' is the spectrum of a valuation ring with generic point t' and closed point s',*
(2) *$g(t') = t$ and $g(s') = s$,*
(3) *there exists a point $y' \in X'_{t'}$ which is a generic point of an irreducible component of $(S' \times_S Y)_{t'} = Y_t \times_t t'$ and satisfies $g'(y') = y$,*
(4) *denoting $Y' \subset X'$ the closure of $\{y'\}$ viewed as an integral closed subscheme of X' there exists a point $x' \in Y'_{s'}$ which is a generic point of an irreducible component of $Y'_{s'}$ with $g'(x') = x$.*

Proof. We choose a valuation ring R, we set $S' = \mathrm{Spec}(R)$ with generic point t' and closed point s', and we choose a morphism $h : S' \to X$ with $h(t') = y$ and $h(s') = x$. See Schemes, Lemma 26.20.4. Set $g = f \circ h$ so that $g(t') = t$ and $g(s') = s$. Consider the base change

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle\sigma}\uparrow\downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

We obtain a section σ of the base change such that $h = g' \circ \sigma$.

Of course σ factors through the base change $S' \times_S Y$ of Y as h factors through Y. Let $y' \in X'_{t'} \subset X'$ be the generic point of an irreducible component of the fibre

$$(S' \times_S Y)_{t'} = Y_t \times_t t'$$

containing the point $\sigma(t')$, i.e., such that $y' \rightsquigarrow \sigma(t')$. Since $g'(y') \in Y_t$ and $g(y') \rightsquigarrow g(\sigma(t')) = y$ we find that $g'(y') = y$ because y is the generic point of the fibre Y_t. Denote $Y' \subset X'$ the closure of $\{y'\}$ in X' viewed as an integral closed subscheme. Then σ factors through Y' as $\sigma(t') \in Y'$. Choose a generic point $x' \in Y'_{s'}$ of an irreducible component of $Y'_{s'}$ which contains $\sigma(s')$, i.e., we get $x' \rightsquigarrow \sigma(s')$ and hence $g'(x') \rightsquigarrow g'(\sigma(s')) = x$. Again as x is a generic point of an irreducible component of Y_s by assumption and as $g'(Y') \subset Y$ we conclude that $g'(x') = x$. □

0GSI **Lemma 37.18.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent finite type $\mathcal{O}_X$-module. Let $y \in Ass_{X/S}(\mathcal{F})$ with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. If $\mathcal{F}$ is flat over S at x, then $x \in Ass_{X/S}(\mathcal{F})$ and $\dim_x(Y_s) = \dim(Y_t)$.*

Proof. Choose a diagram as in Lemma 37.18.2. Set $\mathcal{F}' = (g')^*\mathcal{F}$. Divisors, Lemma 31.7.3 implies that $y' \in \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. By our choice of y' we also see that $\dim(Y'_{t'}) = \dim(Y_t)$, see for example Algebra, Lemma 10.116.7. By Algebra, Lemma 10.125.9 we see that $Y'_{s'}$ is equidimensional of dimension equal to $\dim(Y_t)$. Since $\mathcal{F}$ is flat at x over S we see that $\mathcal{F}'$ is flat at x' over S', see Morphisms, Lemma 29.25.7.

Suppose that we can show $x' \in \mathrm{Ass}_{X'/S}(\mathcal{F}')$. Then Divisors, Lemma 31.7.3 implies that $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and that the irreducible component C' of $Y'_{s'}$ containing x' is an irreducible component of $C \times_s s'$ where $C \subset Y_s$ is the irreducible component

containing x. Whence $\dim(C) = \dim(C') = \dim(Y_t)$ (see above) and the proof is complete. This reduces us to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ where A is a valuation ring and t and s are the generic and closed points of S. We will assume $x \notin \mathrm{Ass}_{X/S}(\mathcal{F})$ in order to get a contradiction. In other words, we assume $x \notin \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ where $\mathcal{F}_s$ is the pullback of $\mathcal{F}$ to X_s. Consider the ring map

$$A \longrightarrow \mathcal{O}_{X,x} = B$$

and the module $N = \mathcal{F}_x$ over $B = \mathcal{O}_{X,x}$. Then $B/\mathfrak{m}_A B = \mathcal{O}_{X_s,x}$ and $N/\mathfrak{m}_A N$ is the stalk of $\mathcal{F}_s$ at the point x. Denote $\mathfrak{q} \subset B$ the prime ideal corresponding to the point y, see Schemes, Lemma 26.13.2. Since x is a generic point of Y_s we see that the radical of $\mathfrak{q} + \mathfrak{m}_A B$ is $\mathfrak{m}_B$. Then $\mathrm{Ass}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N)$ is a finite set of prime ideals (Algebra, Lemma 10.63.5) which doesn't contain the maximal ideal of $B/\mathfrak{m}_A B$ since $x \notin \mathrm{Ass}_{X/S}(\mathcal{F})$. Thus the image of of $\mathfrak{q}$ in $B/\mathfrak{m}_A B$ is not contained in any of those prime ideals. Hence by prime avoidance (Algebra, Lemma 10.15.2) we can find an element $g \in \mathfrak{q}$ whose image in $B/\mathfrak{m}_A B$ is a nonzerodivisor on $N/\mathfrak{m}_A N$ (this uses the description of zerodivisors in Algebra, Lemma 10.63.9). Since $N = \mathcal{F}_x$ is A-flat by Lemma 37.18.1 we see that

$$g : N \longrightarrow N$$

is injective. In particular, if $K = \mathrm{Frac}(A)$ is the fraction field of A, then we see that

$$g : N \otimes_A K \longrightarrow N \otimes_A K$$

is injective. Observe that $\mathfrak{q}$ corresponds to a prime ideal of $B \otimes_A K$. Denote $\mathcal{F}_t$ the restriction of $\mathcal{F}$ to the generic fibre X_t. We have $(B \otimes_A K)_{\mathfrak{q}} = \mathcal{O}_{X_t,y}$ and $(N \otimes_A K)_{\mathfrak{q}}$ is the stalk at y of $\mathcal{F}_t$. Hence we find that $g \in \mathfrak{m}_y \subset \mathcal{O}_{X_t,y}$ is a nonzerodivisor on the stalk $(\mathcal{F}_t)_y$ which contradicts our assumption that $y \in \mathrm{Ass}_{X/S}(\mathcal{F})$. □

0H3X **Lemma 37.18.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module flat over S. Assume S is irreducible with generic point η. If $\dim(\mathit{Supp}(\mathcal{F}_\eta)) \leq r$ then for all $s \in S$ we have $\dim(\mathit{Supp}(\mathcal{F}_s)) \leq r$.*

Proof. Let $x \in \mathrm{Supp}(\mathcal{F}_s)$ be a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_s)$. By Algebra, Lemma 10.41.12 we can find a specialization $y \rightsquigarrow x$ in $\mathrm{Supp}(\mathcal{F})$ with $f(y) = \eta$. Of course we may assume y is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_\eta)$. We conclude from Lemma 37.18.3 that the dimension of $\overline{\{x\}}$ is at most r. □

0GSJ **Lemma 37.18.5.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $y \in \mathit{Ass}_{X/S}(\mathcal{F})$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Denote $T \subset S$ the closure of $\{f(y)\}$ viewed as an integral closed subscheme. We obtain a commutative diagram*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $Y \to T$ is dominant. Assume $\mathcal{F}$ is flat over S at all generic points of irreducible components of fibres of $Y \to T$ (for example if $\mathcal{F}$ is flat over S). Then

(1) *if $s \in S$ and $x \in Y_s$ is the generic point of an irreducible component of Y_s, then $x \in Ass_{X/S}(\mathcal{F})$, and*
(2) *there is an integer $d \geq 0$ such that $Y \to T$ is of relative dimension d, see Morphisms, Definition 29.29.1.*

Proof. This follows immediately from the pointwise version Lemma 37.18.3. Note that to compute the dimension of the locally algebraic schemes Y_s it suffices to look near the generic points, see Varieties, Section 33.20. □

0GSK **Remark 37.18.6.** Here are some cases where the material above, especially Lemma 37.18.5, allows one to conclude that a morphism $f : X \to S$ of schemes has relative dimension d as defined in Morphisms, Definition 29.29.1. For example, this is true if

(1) X is integral with generic point ξ,
(2) the transcendence degree of $\kappa(\xi)$ over $\kappa(f(\xi))$ is d,
(3) f is locally of finite type, and
(4) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Another set of hypotheses that work are the following:

(1) S is irreducible with generic point η,
(2) X_η is dense in X,
(3) every irreducible component of X_η has dimension d,
(4) f is locally of finite type, and
(5) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Of course, we can relax the flatness condition on $\mathcal{F}$ and require only that $\mathcal{F}$ is flat over S in codimension 0, i.e., that $\mathcal{F}$ is flat over S at every generic point of every fibre. If we ever need these results, we will carefully state and prove them here.

37.19. Normalization revisited

081J Normalization commutes with smooth base change.

081K **Lemma 37.19.1.** *Let $f : Y \to X$ be a smooth morphism of schemes. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.*

Proof. This is a translation of Algebra, Lemma 10.147.4 into the language of schemes. Details omitted. □

03GV **Lemma 37.19.2** (Normalization commutes with smooth base change)**.** *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow{\varphi} & X_1 \end{array}$$

be a fibre square in the category of schemes. Assume f_1 is quasi-compact and quasi-separated, and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_2$ and $X_2 \times_{X_1} X_1' \to X_2$ is integral (Morphisms, Lemma 29.44.6). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms, Lemma 29.53.4. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_1}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$, see Constructions, Lemma 27.4.6. By Cohomology of Schemes, Lemma 30.5.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 37.19.1. □

07TD **Lemma 37.19.3** (Normalization and smooth morphisms)**.** *Let $X \to Y$ be a smooth morphism of schemes. Assume every quasi-compact open of Y has finitely many irreducible components. Then the same is true for X and there is a unique isomorphism $X^\nu = X \times_Y Y^\nu$ over X where X^ν, Y^ν are the normalizations of X, Y.*

Proof. By Descent, Lemma 35.16.3 every quasi-compact open of X has finitely many irreducible components. Note that $X_{red} = X \times_Y Y_{red}$ as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 35.18.1. Hence we may assume that X and Y are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that $X' = X \times_Y Y^\nu$ is a normal scheme by Descent, Lemma 35.18.2. The morphism $X' \to Y^\nu$ is smooth (hence flat) thus the generic points of irreducible components of X' lie over generic points of irreducible components of Y^ν. Since $Y^\nu \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y^\nu$ induces an isomorphism on fibres over generic points of Y). We conclude that there exists a factorization $X^\nu \to X' \to X$, see Morphisms, Lemma 29.54.5 which is an isomorphism as X' is normal and integral over X. □

0CBM **Lemma 37.19.4** (Normalization and henselization)**.** *Let X be a locally Noetherian scheme. Let $\nu : X^\nu \to X$ be the normalization morphism. Then for any point $x \in X$ the base change*

$$X^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^h) \to \mathrm{Spec}(\mathcal{O}_{X,x}^h), \quad \textit{resp.} \quad X^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$$

is the normalization of $\mathrm{Spec}(\mathcal{O}_{X,x}^h)$*, resp.* $\mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$*.*

Proof. Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of X passing through x. The base change of the normalization to $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the spectrum of the integral closure of $\mathcal{O}_{X,x}$ in $\prod \kappa(\eta_i)$. This follows from our construction of the normalization of X in Morphisms, Definition 29.54.1 and Morphisms, Lemma 29.53.1; you can also use the description of the normalization in Morphisms, Lemma 29.54.3. Thus we reduce to the following algebra problem. Let A be a Noetherian local ring; recall that this implies the henselization A^h and strict henselization A^{sh} are Noetherian too (More on Algebra, Lemma 15.45.3). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be its minimal primes. Let A' be the integral closure of A in $\prod \kappa(\mathfrak{p}_i)$. Problem: show that $A' \otimes_A A^h$, resp. $A' \otimes_A A^{sh}$ is constructed from the Noetherian local ring A^h, resp. A^{sh} in the same manner.

Since A^h, resp. A^{sh} are colimits of étale A-algebras, we see that the minimal primes of A and A^{sh} are exactly the primes of A^h, resp. A^{sh} lying over the minimal primes of A (by going down, see Algebra, Lemmas 10.39.19 and 10.30.7). Thus More on Algebra, Lemma 15.45.13 tells us that $A^h \otimes_A \prod \kappa(\mathfrak{p}_i)$, resp. $A^{sh} \otimes_A \prod \kappa(\mathfrak{p}_i)$ is the

product of the residue fields at the minimal primes of A^h, resp. A^{sh}. We know that taking the integral closure in an overring commutes with étale base change, see Algebra, Lemma 10.147.2. Writing A^h and A^{sh} as a limit of étale A-algebras we see that the same thing is true for the base change to A^h and A^{sh} (you can also use the more general Algebra, Lemma 10.147.5). □

37.20. Normal morphisms

038Z In the article [**DM69**] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types[3] of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

0390 **Definition 37.20.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *normal at* x if f is flat at x, and the scheme X_y is geometrically normal at x over $\kappa(y)$ (see Varieties, Definition 33.10.1).
(2) We say f is a *normal morphism* if f is normal at every point of X.

So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

0391 **Lemma 37.20.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is normal, and*
(2) *f is flat and its fibres are geometrically normal schemes.*

Proof. This follows directly from the definitions. □

056W **Lemma 37.20.3.** *A smooth morphism is normal.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 29.34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 29.34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 29.34.5) and hence geometrically normal by Varieties, Lemma 33.25.4. Thus f is normal by Lemma 37.20.2. □

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

0392 **Lemma 37.20.4.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian" is local in the fppf topology on the source and the target.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Moreover, as φ_i is of finite presentation the field extension $\kappa(y_i)/\kappa(y)$ is finitely generated. Hence in this situation we have that X_y is locally Noetherian if and

[3] The other types are coprof $\leq k$, Cohen-Macaulay, (S_k), regular, (R_k), and reduced. See [**DG67**, IV Definition 6.8.1.]. Gorenstein morphisms will be defined in Duality for Schemes, Section 48.24.

only if X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. This fact implies locality on the target.

Let $\{X_i \to X\}$ be an fppf covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 35.16.1. □

0393 **Lemma 37.20.5.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is normal" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically normal". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 37.20.4, and Descent, Lemmas 35.23.17 and 35.27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically normal if and only if X_{i,y_i} is geometrically normal, see Varieties, Lemma 33.10.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 35.18.2. Combining the above the lemma follows. □

37.21. Regular morphisms

07R6 Compare with Section 37.20. The algebraic version of this notion is discussed in More on Algebra, Section 15.41.

07R7 **Definition 37.21.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *regular at x* if f is flat at x, and the scheme X_y is geometrically regular at x over $\kappa(y)$ (see Varieties, Definition 33.12.1).
(2) We say f is a *regular morphism* if f is regular at every point of X.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

07R8 **Lemma 37.21.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is regular,*
(2) *f is flat and its fibres are geometrically regular schemes,*
(3) *for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is regular,*
(4) *there exists an open covering $Y = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and*

(5) *there exists an affine open covering $Y = \bigcup_{j\in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that the ring maps $\mathcal{O}_Y(V_j) \to \mathcal{O}_X(U_i)$ are regular.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition f is flat at x if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and X_y is geometrically regular at x over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence Whether or not f is regular at x depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 15.41.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(\mathfrak{p})$ are Noetherian and geometrically regular for all primes $\mathfrak{p} \subset A$. By Varieties, Lemma 33.12.3 this is equivalent to $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ being a geometrically regular scheme over $\kappa(\mathfrak{p})$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that f is flat (see Morphisms, Lemma 29.25.3). Moreover, if $y \in Y$, then $y \in V_j$ for some j and we see that $X_y = \bigcup_{i\in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 33.12.3. Another application of Varieties, Lemma 33.12.3 shows that X_y is geometrically regular. Hence (2) holds and the proof of the lemma is finished. □

07R9 **Lemma 37.21.3.** *A smooth morphism is regular.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 29.34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 29.34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 29.34.5) and hence geometrically regular by Varieties, Lemma 33.25.4. Thus f is regular by Lemma 37.21.2. □

07RA **Lemma 37.21.4.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is regular" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically regular". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 37.20.4, and Descent, Lemmas 35.23.17 and 35.27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i\in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically regular if and only if X_{i,y_i} is geometrically regular, see Varieties, Lemma 33.12.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on

the source follows from Descent, Lemma 35.18.4. Combining the above the lemma follows. □

37.22. Cohen-Macaulay morphisms

045Q Compare with Section 37.20. Note that, as pointed out in Algebra, Section 10.167 and Varieties, Section 33.13 "geometrically Cohen-Macaulay" is the same as plain Cohen-Macaulay.

045R **Definition 37.22.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Cohen-Macaulay at* x if f is flat at x, and the local ring of the scheme X_y at x is Cohen-Macaulay.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

045S **Lemma 37.22.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay, and*
(2) *f is flat and its fibres are Cohen-Macaulay schemes.*

Proof. This follows directly from the definitions. □

0AFG **Lemma 37.22.3.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is locally of finite type and Cohen-Macaulay. For every point x in X with image y in Y,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Morphisms, Lemma 29.29.4. Then f is flat, locally of finite type and of relative dimension d. Hence the result follows from Morphisms, Lemma 29.29.6. □

0C0W **Lemma 37.22.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in X$ with images $y \in Y$ and $z \in Z$.*

(1) *If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.*
(2) *If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.*
(3) *If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.*
(4) *If $g \circ f$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.*

Proof. Consider the map of Noetherian local rings

$$\mathcal{O}_{Y_z,y} \to \mathcal{O}_{X_z,x}$$

and observe that its fibre is

$$\mathcal{O}_{X_z,x}/\mathfrak{m}_{Y_z,y}\mathcal{O}_{X_z,x} = \mathcal{O}_{X_y,x}$$

Thus the lemma this follows from Algebra, Lemma 10.163.3. □

0C0X **Lemma 37.22.5.** *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(x)}$ is Cohen-Macaulay for all $x \in X$.*

Proof. After translating into algebra this follows from Algebra, Lemma 10.163.3. □

045T **Lemma 37.22.6.** *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

1. *If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
2. *If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*
3. *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 33.11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y',y'} & \longleftarrow & \mathcal{O}_{Y,y} \end{array}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Cohen-Macaulay at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f' is Cohen-Macaulay at x'.

Assume f is flat at x and f' is Cohen-Macaulay at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f is Cohen-Macaulay at x.

Assume $Y' \to Y$ is flat at y' and f' is Cohen-Macaulay at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f is Cohen-Macaulay at x. □

045U **Lemma 37.22.7.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let* [**DG67**, IV Corollary 12.1.7(iii)]

$$W = \{x \in X \mid f \text{ is Cohen-Macaulay at } x\}$$

Then

1. *$W = \{x \in X \mid \mathcal{O}_{X_{f(x)},x}$ is Cohen-Macaulay$\}$,*
2. *W is open in X,*
3. *W is dense in every fibre of $X \to S$,*

(4) *the formation of W commutes with arbitrary base change of f: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.*

Proof. As f is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma 10.130.5. Part (4) follows either from Algebra, Lemma 10.130.7 or Varieties, Lemma 33.13.1. □

0BUU **Lemma 37.22.8.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent*

(1) *f is Cohen-Macaulay at x,*
(2) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is flat at x,*
(3) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite flat morphism $U \to \mathbf{A}^d_S$ over S,*
(4) *for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is flat at x.*

Proof. Openness of flatness shows (2) and (3) are equivalent, see Theorem 37.15.1.

Choose affine open $U = \mathrm{Spec}(A) \subset X$ with $x \in U$ and $V = \mathrm{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 10.125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[4] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Claim: Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ we have: $\mathrm{Spec}(\varphi)$ is flat at $\mathfrak{p}$ if and only if $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Cohen-Macaulay at $\mathfrak{p}$. Namely, by Theorem 37.16.2 flatness may be checked on fibres. The same is true for being Cohen-Macaulay (as A is already assumed flat over R). Thus the claim follows from Algebra, Lemma 10.130.1.

The claim shows that (1) is equivalent to (4) and combined with the fact that we have constructed a suitable (U, g) in the second paragraph, the claim also shows that (1) is equivalent to (2). □

054T **Lemma 37.22.9.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties*

(1) *$W = \bigcup_{d \geq 0} U_d$ is dense in every fibre of f, and*
(2) *$U_d \to S$ is of relative dimension d (see Morphisms, Definition 29.29.1).*

Proof. This follows by combining Lemma 37.22.7 with Morphisms, Lemma 29.29.4. □

054U **Lemma 37.22.10.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Suppose $x' \rightsquigarrow x$ is a specialization of points of X with*

[4] If S is quasi-separated, then g will be quasi-finite.

image $s' \rightsquigarrow s$ in S. If x is a generic point of an irreducible component of X_s then $\dim_{x'}(X_{s'}) = \dim_x(X_s)$.

Proof. The point x is contained in U_d for some d, where U_d as in Lemma 37.22.9. □

045V **Lemma 37.22.11.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is Cohen-Macaulay" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$"f is flat", and $\mathcal{P}_2(f) =$"the fibres of f are locally Noetherian and Cohen-Macaulay". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 35.23.17 and 35.27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. And if in addition X_y is Cohen-Macaulay, then X_{i,y_i} is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Descent, Lemma 35.17.2. Combining the above the lemma follows. □

37.23. Slicing Cohen-Macaulay morphisms

056X The results in this section eventually lead to the assertion that the fppf topology is the same as the "finitely presented, flat, quasi-finite" topology. The following lemma is very closely related to Divisors, Lemma 31.18.9.

056Y **Lemma 37.23.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *the image $\overline{h}$ of h in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a nonzerodivisor.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 37.15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that there exists an $h \in \Gamma(X, \mathcal{O}_X)$ which maps to our given h.

We may write $S = \mathrm{Spec}(A)$ and we may write $A = \mathrm{colim}_i A_i$ as a directed colimit of finite type $\mathbf{Z}$ algebras. Then by Algebra, Lemma 10.168.1 or Limits, Lemmas

32.10.1, 32.8.2, and 32.10.1 we can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ S & \longrightarrow & S_0 \end{array}$$

with f_0 flat and of finite presentation, X_0 affine, and S_0 affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of x, resp. s. We may also assume there exists an element $h_0 \in \Gamma(X_0, \mathcal{O}_{X_0})$ which restricts to h on X. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 32.10.1 by thinking of h as a morphism $X \to \mathbf{A}^1_S$.) Note that $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{(X_0)_{s_0},x_0} \otimes_{\kappa(s_0)} \kappa(s)$, so that $\mathcal{O}_{(X_0)_{s_0},x_0} \to \mathcal{O}_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in \mathcal{O}_{(X_0)_{s_0},x_0}$ is contained in $\mathfrak{m}_{(X_0)_{s_0},x_0}$ and is a nonzerodivisor. We claim that after replacing X_0 by a principal open neighbourhood of x_0 the element h_0 is a nonzerodivisor in $B_0 = \Gamma(X_0, \mathcal{O}_{X_0})$ such that B_0/h_0B_0 is flat over $A_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat A_0-modules. Hence this remains exact on tensoring with A (by Algebra, Lemma 10.39.12) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Assume $h \in \mathfrak{q}$ maps to a nonzerodivisor in $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$. Goal: show that after possible replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ the element h becomes a nonzerodivisor and B/hB becomes flat over A. By Algebra, Lemma 10.99.2 we see that h is a nonzerodivisor in $B_\mathfrak{q}$ and that $B_\mathfrak{q}/hB_\mathfrak{q}$ is flat over A. By openness of flatness, see Algebra, Theorem 10.129.4 or Theorem 37.15.1 we see that B/hB is flat over A after replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$. Finally, let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $IB_\mathfrak{q} = 0$ as h is a nonzerodivisor in $B_\mathfrak{q}$. Also I is finitely generated as B is Noetherian. Hence there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = 0$. After replacing B by B_g we see that h is a nonzerodivisor. □

06LI **Lemma 37.23.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume*

1. *f is locally of finite presentation,*
2. *f is flat at x, and*
3. *the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ form a regular sequence.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme S' over S.

Proof. (Our conventions on regular sequences imply that $h_i \in \mathfrak{m}_x$ for each i.) The case $r = 1$ follows from Lemma 37.23.1 combined with Divisors, Lemma 31.18.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case

$r > 1$ follows from a straightforward induction on r (applying the result for $r = 1$ exactly r times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 31.22. Namely, first by openness of flatness (see Theorem 37.15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that X_s is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 33.20) and that the topology on X_s is induced from the topology on X (see Schemes, Lemma 26.18.5). Hence after shrinking X a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the r elements $h_i|_{X_s}$, see Divisors, Lemma 31.20.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(Z_s) &= \dim(\mathcal{O}_{Z_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{Z_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.116.3 and the second by r times applying Algebra, Lemma 10.60.13. Hence Divisors, Lemma 31.22.7 part (3) applies to show that (after Zariski shrinking X) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 31.22.4 applies (which gives the flatness and the statement on base change). □

056Z **Lemma 37.23.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *$\mathcal{O}_{X_s,x}$ has depth ≥ 1.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which maps to a nonzerodivisor in $\mathcal{O}_{X_s,x}$ and apply Lemma 37.23.1. □

0570 **Lemma 37.23.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume* [**DG67**, IV Proposition 17.16.1]

(1) *f is locally of finite presentation,*
(2) *f is Cohen-Macaulay at x, and*
(3) *x is a closed point of X_s.*

Then there exists a regular immersion $Z \to X$ containing x such that

(a) *$Z \to S$ is flat and locally of finite presentation,*
(b) *$Z \to S$ is locally quasi-finite, and*
(c) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We will prove the lemma for affine S by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite,

see Morphisms, Lemma 29.56.2. Thus after replacing X by U we see that the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 26.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 29.20.3). Thus we see

$$\operatorname{depth}(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 29.28.1. Thus we may apply Lemma 37.23.3 to find a diagram

$$\begin{array}{ccccc} D & \longrightarrow & U & \longrightarrow & X \\ & \searrow & & \swarrow & \downarrow \\ & & S & & \end{array}$$

with $x \in D$. Note that $\mathcal{O}_{D_s,x} = \mathcal{O}_{X_s,x}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma 31.18.1. Hence $\mathcal{O}_{D_s,x}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X_s,x}$, see Algebra, Lemma 10.104.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at x with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 31.21.7. This finishes the proof. □

0571 **Lemma 37.23.5.** *Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $s \in S$ be a point in the image of f. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \longrightarrow & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

Proof. The fibre X_s is not empty by assumption. Hence there exists a closed point $x \in X_s$ where f is Cohen-Macaulay, see Lemma 37.22.7. Apply Lemma 37.23.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the "quasi-finite, flat, finite presentation" topology.

0572 **Lemma 37.23.6.** *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be an fppf covering of S, see Topologies, Definition 34.7.1. Then there exists an fppf covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 7.8.1) $\mathcal{U}$ such that each $T_j \to S$ is locally quasi-finite.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 37.23.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of S that refines $\mathcal{U}$. □

37.24. Generic fibres

054V Some results on the relationship between generic fibres and nearby fibres.

054W **Lemma 37.24.1.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.*

Proof. Follows immediately from the more general Morphisms, Lemma 29.8.6. □

05F5 **Lemma 37.24.2.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.*

Proof. This follows, upon taking affine opens, from Algebra, Lemma 10.30.2. (Of course it also follows from generic flatness.) □

054X **Lemma 37.24.3.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $Z \subset X$ is a closed subset with Z_η nowhere dense in X_η, then there exists a nonempty open $V \subset Y$ such that Z_y is nowhere dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 29.45.6 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that Y is integral, see Properties, Lemma 28.3.4. By Morphisms, Proposition 29.27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to V$ are flat and of finite presentation. We claim that V works. Pick $y \in V$. If Z_y has a nonempty interior, then Z_y contains a generic point ξ of an irreducible component of X_y. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 29.25.9. By Lemma 37.22.10 we see that

$$\dim_{\xi'}(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_{\xi'}(X_\eta).$$

Hence some irreducible component of Z_η passing through ξ' has dimension $\dim_{\xi'}(X_\eta)$ which contradicts the assumption that Z_η is nowhere dense in X_η and we win. □

0573 **Lemma 37.24.4.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $U \subset X$ be an open subscheme such that U_η is scheme theoretically dense in X_η. Then there exists a nonempty open $V \subset Y$ such that U_y is scheme theoretically dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace Y by Y' and X by X'. Thus we may assume that Y is integral, see Properties, Lemma 28.3.4. We may also replace Y by a nonempty affine open. In other words we may assume that $Y = \mathrm{Spec}(A)$ where A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i. By Morphisms, Definition 29.7.1 we see that $U_i = X_i \cap U$ is an open subscheme of X_i such that $U_{i,\eta}$ is scheme theoretically dense in $X_{i,\eta}$. Thus it suffices to prove the result for the pairs (X_i, U_i), in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that U_η is scheme theoretically dense in X_η means that $B_K \to \bigoplus_j (B_K)_{g_j}$ is injective, see Morphisms, Example 29.7.4. By Algebra, Lemma 10.24.4 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1,\ldots,m} B_K$, $b \mapsto (g_1 b, \ldots, g_m b)$. Let M be the cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra, and that M is flat over A, see Algebra, Lemma 10.118.3. The flatness of B over A implies that B is torsion free as an A-module, see More on Algebra, Lemma 15.22.9. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to B \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ is scheme theoretically dense in $\mathrm{Spec}(B \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ is scheme theoretically dense in X_κ which is what we wanted to prove. □

Suppose given a morphism of schemes $f : X \to Y$ and a point $y \in Y$. Recall that the fibre X_y is homeomorphic to the subset $f^{-1}(\{y\})$ of X with induced topology, see Schemes, Lemma 26.18.5. Suppose given a closed subset $T(y) \subset X_y$. Let T be the closure of $T(y)$ in X. Endow T with the induced reduced scheme structure. Then T is a closed subscheme of X with the property that $T_y = T(y)$ set-theoretically. In fact T is the smallest closed subscheme of X with this property. Thus it is "harmless" to denote a closed subset of X_y by T_y if we so desire. In the following lemma we apply this to the generic fibre of f.

054Y **Lemma 37.24.5.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$ be a covering of the generic fibre by closed subsets of X_η. Let Z_i be the closure of $Z_{i,\eta}$ in X (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.*

Proof. If Y is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over Y and we can directly apply Lemma 37.24.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace Y by its reduction. Thus Y is integral, see Properties, Lemma 28.3.4. After shrinking Y we may assume that $X \to Y$ is flat, see Morphisms, Proposition 29.27.1. In this case every point x in X_y is a specialization of a point $x' \in X_\eta$ by Morphisms, Lemma 29.25.9. As the Z_i are closed in X and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired. □

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

054Z **Lemma 37.24.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of Y. Then $(X_\eta)_{red} = (X_{red})_\eta$.*

Proof. Choose an affine neighbourhood $\mathrm{Spec}(A) \subset Y$ of η. Choose an affine open $\mathrm{Spec}(B) \subset X$ mapping into $\mathrm{Spec}(A)$ via the morphism f. Let $\mathfrak{p} \subset A$ be the minimal prime corresponding to η. Let B_{red} be the quotient of B by the nilradical $\sqrt{(0)}$. The algebraic content of the lemma is that $C = B_{red} \otimes_A \kappa(\mathfrak{p})$ is reduced. Denote $I \subset A$ the nilradical so that $A_{red} = A/I$. Denote $\mathfrak{p}_{red} = \mathfrak{p}/I$ which is a minimal prime of A_{red} with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}_{red})$. Since $A \to B_{red}$ and $A \to \kappa(\mathfrak{p})$ both factor through $A \to A_{red}$ we have $C = B_{red} \otimes_{A_{red}} \kappa(\mathfrak{p}_{red})$. Now $\kappa(\mathfrak{p}_{red}) = (A_{red})_{\mathfrak{p}_{red}}$ is a localization by Algebra, Lemma 10.25.1. Hence C is a localization of B_{red} (Algebra, Lemma 10.12.15) and hence reduced. □

0550 **Lemma 37.24.7.** *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

where

(1) *V is a nonempty open of Y,*
(2) *$X_V = V \times_Y X$,*
(3) *$g : Y' \to V$ is a finite universal homeomorphism,*
(4) *$X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red}$,*
(5) *g' is a finite universal homeomorphism,*
(6) *Y' is an integral affine scheme,*
(7) *f' is flat and of finite presentation, and*
(8) *the generic fibre of f' is geometrically reduced.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the nilradical $\mathfrak{p} = \sqrt{(0)}$ of A is a prime ideal. Let $K = \kappa(\mathfrak{p})$. Let p be the characteristic of K if positive and 1 if the characteristic is zero. By Varieties, Lemma 33.6.11 there exists a finite purely inseparable field extension K'/K such that $(X_{K'})_{red}$ is geometrically reduced over K'. Choose elements $x_1, \ldots, x_n \in K'$ which generate K' over K and such that some p-power of x_i is in $A/\mathfrak{p}$. Let $A' \subset K'$ be the finite A-subalgebra of K' generated by $x_1, \ldots, x_n$. Note that A' is a domain with fraction field K'. By Algebra, Lemma 10.46.7 we see that $A \to A'$ induces a universal homeomorphism on spectra. Set $Y' = \mathrm{Spec}(A')$. Set $X' = (Y' \times_Y X)_{red}$. The generic fibre of $X' \to Y'$ is $(X_{K'})_{red}$ by Lemma 37.24.6 which is geometrically reduced by construction. Note that $X' \to X_V$ is a finite universal homeomorphism as the composition of the reduction morphism $X' \to Y' \times_Y X$ (see Morphisms, Lemma 29.45.6) and the base change of g. At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking Y' and hence V, see Morphisms, Proposition 29.27.1. □

0551 **Lemma 37.24.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

where

(1) *V is a nonempty open of Y,*
(2) *$X_V = V \times_Y X$,*
(3) *$g : Y' \to V$ is surjective finite étale,*
(4) *$X' = Y' \times_Y X = Y' \times_V X_V$,*
(5) *g' is surjective finite étale,*
(6) *Y' is an irreducible affine scheme, and*
(7) *all irreducible components of the generic fibre of f' are geometrically irreducible.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the nilradical $\mathfrak{p} = \sqrt{(0)}$ of A is a prime ideal. Let $K = \kappa(\mathfrak{p})$. By Varieties, Lemma 33.8.16 there exists a finite separable field extension K'/K such that all irreducible components of $X_{K'}$ are geometrically irreducible over K'. Choose an element $\alpha \in K'$ which generates K' over K, see Fields, Lemma 9.19.1. Let $P(T) \in K[T]$ be the minimal polynomial for α over K. After replacing α by $f\alpha$ for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that there exists a monic polynomial $T^d + a_1 T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $\mathfrak{q}$ lying over $\mathfrak{p}$, such that $K = \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) = K'$ is finite separable, and such that $\mathfrak{p}A'_\mathfrak{q}$ is the maximal ideal of $A'_\mathfrak{q}$. Hence $g : Y' = \mathrm{Spec}(A') \to V = \mathrm{Spec}(A)$ is étale at $\mathfrak{q}$, see Algebra, Lemma 10.143.7. This means that there exists an open $W \subset \mathrm{Spec}(A')$ such that $g|_W : W \to \mathrm{Spec}(A)$ is étale. Since g is finite and since $\mathfrak{q}$ is the only point lying over $\mathfrak{p}$ we see that $Z = g(Y' \setminus W)$ is a closed subset of V not containing $\mathfrak{p}$. Hence after replacing V by a principal affine open of V which does not meet Z we obtain that g is finite étale. □

37.25. Relative assassins

05KM

05F1 **Lemma 37.25.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in Ass_{X/S}(\mathcal{F})$ and set $Z = \overline{\{\xi\}} \subset X$. If f is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of V_s are elements of $Ass_{X/S}(\mathcal{F})$.*

Proof. We may replace S by an affine open neighbourhood of $f(\xi)$ and X by an affine open neighbourhood of ξ. Hence we may assume $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and that f is given by the finite type ring map $A \to B$, see Morphisms, Lemma 29.15.2. Moreover, we may write $\mathcal{F} = \widetilde{M}$ for some finite B-module M, see Properties, Lemma 28.16.1. Let $\mathfrak{q} \subset B$ be the prime corresponding to ξ and let $\mathfrak{p} \subset A$ be the corresponding prime of A. By assumption $\mathfrak{q} \in \mathrm{Ass}_B(M \otimes_A \kappa(\mathfrak{p}))$, see Algebra, Remark 10.65.6 and Divisors, Lemma 31.2.2. With this notation

$Z = V(\mathfrak{q}) \subset \mathrm{Spec}(B)$. In particular $f(Z) \subset V(\mathfrak{p})$. Hence clearly it suffices to prove the lemma after replacing A, B, and M by $A/\mathfrak{p}A$, $B/\mathfrak{p}B$, and $M/\mathfrak{p}M$. In other words we may assume that A is a domain with fraction field K and $\mathfrak{q} \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma 10.118.3 there exists a nonzero $g \in A$ such that M_g is flat as an A_g-module. After replacing A by A_g we may assume that M is flat as an A-module.

In this case, by Algebra, Lemma 10.65.4 we see that $\mathfrak{q}$ is also an associated prime of M. Hence we obtain an injective B-module map $B/\mathfrak{q} \to M$. Let Q be the cokernel so that we obtain a short exact sequence

$$0 \to B/\mathfrak{q} \to M \to Q \to 0$$

of finite B-modules. After applying generic flatness Algebra, Lemma 10.118.3 once more, this time to the B-module Q, we may assume that Q is a flat A-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma 10.39.12. In this situation $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}') \subset M \otimes_A \kappa(\mathfrak{p}')$ for any prime $\mathfrak{p}'$ of A. The lemma follows as a minimal prime $\mathfrak{q}'$ of the support of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ is an associated prime of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ by Divisors, Lemma 31.2.9. □

05KN **Lemma 37.25.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *Y is irreducible with generic point η, and*
(4) *$Ass_{X_\eta}(\mathcal{F}_\eta)$ is not contained in U_η.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $Ass_{X_y}(\mathcal{F}_y)$ is not contained in U_y.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5. Then Z_η is the scheme theoretic support of $\mathcal{F}_\eta$ (Morphisms, Lemma 29.25.14). Hence the generic points of irreducible components of Z_η are contained in $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta)$ by Divisors, Lemma 31.2.9. Hence we see that $Z_\eta \cap U_\eta = \emptyset$. Thus $T = Z \setminus U$ is a closed subset of Z with $T_\eta = \emptyset$. If we endow T with the induced reduced scheme structure then $T \to Y$ is a morphism of finite type. By Lemma 37.24.1 there is a nonempty open $V \subset Y$ with $T_V = \emptyset$. Then V works. □

05KP **Lemma 37.25.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *Y is irreducible with generic point η, and*
(4) *$Ass_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $Ass_{X_y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 37.24.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace Y by its reduction. Hence we may assume that Y is integral, see

Properties, Lemma 28.3.4. We may also assume that $Y = \mathrm{Spec}(A)$ is affine. Then A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of $U_i = X_i \cap U$ contains $\mathrm{Ass}_{X_{i,\eta}}(\mathcal{F}_{i,\eta})$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F} = \widetilde{N}$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. Since $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K \to \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 10.24.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_K$, $n \mapsto (g_1 n, \ldots, g_m n)$. Let I and M be the kernel and cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra and that both M and N are flat over A, see Algebra, Lemma 10.118.3. The flatness of N over A implies that N is torsion free as an A-module, see More on Algebra, Lemma 15.22.9. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\mathrm{Ass}_{B\otimes_A\kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ contains $\mathrm{Ass}_{X\otimes\kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. □

05KQ **Lemma 37.25.4.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of f, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^*\mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case*

$$Ass_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow Ass_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

Proof. This follows immediately from Divisors, Lemma 31.7.3. See also Divisors, Remark 31.7.4. □

05KR **Lemma 37.25.5.** *Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid Ass_{X_y}(\mathcal{F}_y) \subset U_y\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. After possibly increasing i we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to X is isomorphic to $\mathcal{F}$, see Limits, Lemma 32.10.2. After possibly increasing i one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in X is U, see Limits, Lemma 32.4.11. By Lemma 37.25.4 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. This follows from Lemmas 37.25.2 and 37.25.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over Z. □

37.26. Reduced fibres

0574

0575 **Lemma 37.26.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is nonreduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is nonreduced.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 28.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote K the fraction field of A. Pick an affine open $\mathrm{Spec}(B) = U \subset X$ and a section $h_\eta \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) = B_K$ which is nonzero and nilpotent. After shrinking Y we may assume that h comes from $h \in \Gamma(U, \mathcal{O}_U) = B$. After shrinking Y a bit more we may assume that h is nilpotent. Let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $C = B/I$ is a finite type A-algebra whose generic fiber $(B/I)_K$ is nonzero (as $h_\eta \neq 0$). We apply generic flatness to $A \to C$ and $A \to B/hB$, see Algebra, Lemma 10.118.3, and we obtain a $g \in A$, $g \neq 0$ such that C_g is free as an A_g-module and $(B/hB)_g$ is flat as an A_g-module. Replace Y by $D(g) \subset Y$. Now we have the short exact sequence

$$0 \to C \to B \to B/hB \to 0.$$

with B/hB flat over A and with C nonzero free as an A-module. It follows that for any homomorphism $A \to \kappa$ to a field the ring $C \otimes_A \kappa$ is nonzero and the sequence

$$0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0$$

is exact, see Algebra, Lemma 10.39.12. Note that $B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa)$ by right exactness of tensor product. Thus we conclude that multiplication by h is not zero on $B \otimes_A \kappa$. This clearly means that for any point $y \in Y$ the element h restricts to a nonzero element of U_y, whence X_y is nonreduced. □

0576 **Lemma 37.26.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \text{ is geometrically reduced}\} = g^{-1}(\{y \in Y \mid X_y \text{ is geometrically reduced}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically reduced over $\kappa(y')$ if and only if X_y is geometrically reduced over $\kappa(y)$. This follows from Varieties, Lemma 33.6.6. □

0577 **Lemma 37.26.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is not geometrically reduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is not geometrically reduced.*

Proof. Apply Lemma 37.24.7 to get

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

with all the properties mentioned in that lemma. Let η' be the generic point of Y'. Consider the morphism $X' \to X_{Y'}$ (which is the reduction morphism) and the resulting morphism of generic fibres $X'_{\eta'} \to X_{\eta'}$. Since $X'_{\eta'}$ is geometrically reduced, and X_η is not this cannot be an isomorphism, see Varieties, Lemma 33.6.6. Hence $X_{\eta'}$ is nonreduced. Hence by Lemma 37.26.1 the fibres of $X_{Y'} \to Y'$ are nonreduced at all points $y' \in V'$ of a nonempty open $V' \subset Y'$. Since $g : Y' \to V$ is a homeomorphism Lemma 37.26.2 proves that $g(V')$ is the open we are looking for. □

0578 **Lemma 37.26.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically reduced, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically reduced fibres.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 28.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote K the fraction field of A. After shrinking Y a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1.

As X_η is geometrically reduced there exists an open dense subset $V \subset X_\eta$ such that $V \to \mathrm{Spec}(K)$ is smooth, see Varieties, Lemma 33.25.7. Let $U \subset X$ be the set of points where f is smooth. By Morphisms, Lemma 29.34.15 we see that $V \subset U_\eta$. Thus the generic fibre of U is dense in the generic fibre of X. Since X_η is reduced, it follows that U_η is scheme theoretically dense in X_η, see Morphisms, Lemma 29.7.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically

reduced. Thus it suffices to show that, after shrinking Y, for all $y \in Y$ the scheme U_y is scheme theoretically dense in X_y, see Morphisms, Lemma 29.7.9. This follows from Lemma 37.24.4. □

0579 **Lemma 37.26.5.** *Let $f : X \to Y$ be a morphism which is quasi-compact and locally of finite presentation. Then the set*

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Then X is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. Then the fibres of $U_i \to Y$ at y form an affine open covering of the fibre of $X \to Y$ at y. Hence we may assume X is affine as well. Write $Y = \mathrm{Spec}(A)$. Write $A = \operatorname{colim} A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.26.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme with generic point ξ. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. If X_ξ is geometrically reduced, then Lemma 37.26.4 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. If X_ξ is not geometrically reduced, then Lemma 37.26.3 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are not geometrically reduced for a nonempty open $V \subset Z$. Thus we win. □

0C0D **Lemma 37.26.6.** *Let $f : X \to \mathrm{Spec}(R)$ be a proper morphism where R is a discrete valuation ring. Assume every irreducible component of X dominates $\mathrm{Spec}(R)$ (for example if f is flat) and assume the special fibre is reduced. Then both X and the generic fibre X_η are reduced.*

Proof. (The fact that a flat morphism satisfies the assumption on irreducible components follows for example from Divisors, Lemma 31.3.1 or from going down for flat ring maps.) Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X and X_η are reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $\pi \in R$ be a uniformizer. If $a \in A$ is nonzero then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \notin \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 10.51.4). If a is nilpotent the a is contained in all minimal primes of A, and hence so is a', because π is not contained in any of the minimal primes of A by our assumption on the irreducible components of X. Thus a' is nilpotent. But a' maps to a nonzero element of the reduced ring $A/\pi A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show. □

0C0E **Lemma 37.26.7.** *Let $f : X \to Y$ be a flat proper morphism of finite presentation. Then the set $\{y \in Y \mid X_y$ is geometrically reduced$\}$ is open in Y.*

Proof. We may assume Y is affine. Then Y is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits, Lemma 32.10.1, 32.8.7, and 32.13.1. Since the formation of the set of points where the fibres are geometrically reduced commutes with base change (Lemma 37.26.2), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 37.26.5. Hence it suffices to show the set is stable under generalization (Topology, Lemma 5.19.10). By Properties, Lemma 28.5.10 we reduce to the case where $Y = \mathrm{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Varieties, Lemma 33.6.4. There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 10.120.18. After replacing R by R' we reduce to Lemma 37.26.6. □

37.27. Irreducible components of fibres

0553

0554 **Lemma 37.27.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n irreducible components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n irreducible components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 28.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into irreducible components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X_i whose generic fibre $Z_{i,j,\eta}$ is nowhere dense in $X_{i,\eta}$. Hence after shrinking Y we may assume that $Z_{i,j,y}$ is nowhere dense in $X_{i,y}$ for every $y \in Y$, see Lemma 37.24.3. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 37.24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 29.25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are not dense in $X_{i,y}$. Clearly this implies that X_y has at least n irreducible components. □

0555 **Lemma 37.27.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically irreducible}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if X_y is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 33.8.2. □

0556 **Lemma 37.27.3.** *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & K'' & \longleftarrow & K' \\ \uparrow & & & & \uparrow \\ \kappa(y) & & \longrightarrow & & \kappa(y') \end{array}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between irreducible components, see Varieties, Lemma 33.8.8. □

0557 **Lemma 37.27.4.** *Let A be a domain with fraction field K. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of K. Assume P is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.*

Proof. There exists an automorphism Ψ of $A[x_1, \ldots, x_n]$ over A such that $\Psi(P) = ax_n^d +$ lower order terms in x_n with $a \neq 0$, see Algebra, Lemma 10.115.2. We may replace P by $\Psi(P)$ and we may replace A by A_a. Thus we may assume that P is monic in x_n of degree $d > 0$. For $i = 1, \ldots, n-1$ let d_i be the degree of P in x_i. Note that this implies that P^φ is monic of degree d in x_n and has degree $\leq d_i$ in x_i for every homomorphism $\varphi : A \to \kappa$ where κ is a field. Thus if P^φ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with Q_1, Q_2 monic of degree $e_1, e_2 \geq 0$ in x_n with $e_1 + e_2 = d$ and having degree $\leq d_i$ in x_i for $i = 1, \ldots, n-1$. In other words we can write

0558 (37.27.4.1)
$$Q_j = x_n^{e_j} + \sum\nolimits_{0 \leq l < e_j} \left(\sum\nolimits_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l$$

where the sum is over the set $\mathcal{L}$ of multi-indices L of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the A-algebra

$$B_{e_1,e_2} = A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial

$$P - Q_1 Q_2 \in A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$$

with Q_1 and Q_2 defined as in (37.27.4.1). OK, and the assumption that P is irreducible over $\overline{K}$ implies that there does not exist any A-algebra homomorphism $B_{e_1,e_2} \to \overline{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1 this means that $B_{e_1,e_2} \otimes_A K = 0$. As B_{e_1,e_2} is a finitely generated A-algebra this signifies that we can find an $f_{e_1,e_2} \in A$ such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1,e_2}} \to \kappa$ is a homomorphism to a field, then P^φ does not have a factorization $P^\varphi = Q_1 Q_2$ with Q_1 of degree e_1 in x_n and Q_2 of degree e_2 in x_n. Thus taking $f = \prod_{e_1,e_2 \geq 0, e_1+e_2=d} f_{e_1,e_2}$ we win. □

0559 **Lemma 37.27.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically irreducible, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

First proof of Lemma 37.27.5. We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

as in Lemma 37.24.7. Note that the generic fibre of f' is the reduction of the generic fibre of f (see Lemma 37.24.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism f'. Then after shrinking V all the fibres of f' are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 37.27.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \mathrm{Spec}(A)$ with A a domain.

Let $x \in X_\eta$ be the generic point. As X_η is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 33.6.2 and 33.8.6. In particular the field extension L/K is separable, see Algebra, Lemma 10.44.2. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate L over K and such that $x_1, \ldots, x_r$ is a transcendence basis for L over K, see Algebra, Lemma 10.42.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for x_{r+1}. Clearing denominators we may assume that P has coefficients in $A[x_1, \ldots, x_r]$. Note that as L is geometrically reduced and geometrically irreducible over K, the polynomial P is irreducible in $\overline{K}[x_1, \ldots, x_r, T]$ where $\overline{K}$ is the algebraic closure of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$$

and set $X' = \mathrm{Spec}(B')$. By construction the fraction field of B' is isomorphic to $L = \kappa(x)$ as K-extensions. Hence there exists an open $U \subset X$, and open $U' \subset X'$

and a Y-isomorphism $U \to U'$, see Morphisms, Lemma 29.50.7. Here is a diagram:

$$\begin{array}{ccccccc} X & \longleftarrow & U & = & U' & \longrightarrow & X' = \operatorname{Spec}(B') \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & Y & = & Y & & \end{array}$$

Note that $U_\eta \subset X_\eta$ and $U'_\eta \subset X'_\eta$ are dense opens. Thus after shrinking Y by applying Lemma 37.24.3 we obtain that U_y is dense in X_y and U'_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 37.27.4. □

Second proof of Lemma 37.27.5. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to X$ be the reduction of X. Note that $X' \to X \to Y$ factors through Y', see Schemes, Lemma 26.12.7. As $Y' \to Y$ and $X' \to X$ are universal homeomorphisms by Morphisms, Lemma 29.45.6 we see that it suffices to prove the lemma for $X' \to Y'$. Thus we may assume that X and Y are reduced. In particular Y is integral, see Properties, Lemma 28.3.4. Thus by Morphisms, Proposition 29.27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ is flat and of finite presentation. After replacing Y by V we may assume, in addition to (1), (2), (3) that Y is integral affine, X is reduced, and f is flat and of finite presentation. In particular f is universally open, see Morphisms, Lemma 29.25.10.

Pick a nonempty affine open $U \subset X$. Then $U \to Y$ is flat and of finite presentation with geometrically irreducible generic fibre. The complement $X_\eta \setminus U_\eta$ is nowhere dense. Thus after shrinking Y we may assume $U_y \subset X_y$ is open dense for all $y \in Y$, see Lemma 37.24.3. Thus we may replace X by U and we reduce to the case where Y is integral affine and X is reduced affine, flat and of finite presentation over Y with geometrically irreducible generic fibre X_η.

Write $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$. Then A is a domain, B is reduced, $A \to B$ is flat of finite presentation, and B_K is geometrically irreducible over the fraction field K of A. In particular we see that B_K is a domain. Let L be the fraction field of B_K. Note that L is a finitely generated field extension of K as B is an A-algebra of finite presentation. Let K'/K be a finite purely inseparable extension such that $(L \otimes_K K')_{red}$ is a separably generated field extension, see Algebra, Lemma 10.45.3. Choose $x_1, \ldots, x_n \in K'$ which generate the field extension K' over K, and such that $x_i^{q_i} \in A$ for some prime power q_i (proof existence x_i omitted). Let A' be the A-subalgebra of K' generated by $x_1, \ldots, x_n$. Then A' is a finite A-subalgebra $A' \subset K'$ whose fraction field is K'. Note that $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is a universal homeomorphism, see Algebra, Lemma 10.46.7. Hence it suffices to prove the result after base changing to $\operatorname{Spec}(A')$. We are going to replace A by A' and B by $(B \otimes_A A')_{red}$ to arrive at the situation where L is a separably generated field extension of K. Of course it may happen that $(B \otimes_A A')_{red}$ is no longer flat, or of finite presentation over A', but this can be remedied by replacing A' by A'_f for a suitable $f \in A'$, see Algebra, Lemma 10.118.3.

At this point we know that A is a domain, B is reduced, $A \to B$ is flat and of finite presentation, B_K is a domain whose fraction field L is a separably generated field extension of the fraction field K of A. By Algebra, Lemma 10.42.3 we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over K, and x_{r+1} is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume

that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of x_{r+1} over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since L/K is separable and since L is geometrically irreducible over K, the polynomial P is irreducible over the algebraic closure $\overline{K}$ of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).$$

By construction the fraction fields of B and B' are isomorphic as K-extensions. Hence there exists an isomorphism of A-algebras $B_h \cong B'_{h'}$ for suitable $h \in B$ and $h' \in B'$, see Morphisms, Lemma 29.50.7. In other words X and $X' = \mathrm{Spec}(B')$ have a common affine open U. Here is a diagram:

$$\begin{array}{ccccc} X = \mathrm{Spec}(B) & \longleftarrow & U & \longrightarrow & \mathrm{Spec}(B') = X' \\ & \searrow & \downarrow & \swarrow & \\ & & Y = \mathrm{Spec}(A) & & \end{array}$$

After shrinking Y once more (by applying Lemma 37.24.3 to $Z = X \setminus U$ in X and $Z' = X' \setminus U$ in X') we see that U_y is dense in X_y and U_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 37.27.4. □

055A **Lemma 37.27.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 37.27.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 28.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 37.24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 29.36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 37.27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. After shrinking Y some more we see that each fibre of f has at least n irreducible components, see Lemma 37.27.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 37.27.5. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

055B **Lemma 37.27.7.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 37.27.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.27.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.27.6 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

37.28. Connected components of fibres

055C

055D **Lemma 37.28.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n connected components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 28.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking Y we may assume that $Z_{i,j} = \emptyset$, see Lemma 37.24.1. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 37.24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 29.25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that X_y has at least n connected components. □

055E **Lemma 37.28.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \text{ is geometrically connected}\}$$
$$= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if X_y is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 33.7.3. □

055F **Lemma 37.28.3.** *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & K'' & \longleftarrow & K' \\ \uparrow & & & & \uparrow \\ \kappa(y) & & \longrightarrow & & \kappa(y') \end{array}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between connected components, see Varieties, Lemma 33.7.6. □

055G **Lemma 37.28.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically connected, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

Proof. Choose a diagram

$$\begin{array}{ccccc} X' & \xrightarrow[g']{} & X_V & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

as in Lemma 37.24.8. Note that the generic fibre of f' is geometrically connected (for example by Lemma 37.28.3). Suppose that the lemma holds for the morphism f'. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over W is geometrically connected. Then, as g is an open morphism by Morphisms, Lemma 29.36.13 all the fibres of f at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 37.28.3. In this way we see that we may assume that the irreducible components of the generic fibre X_η are geometrically irreducible.

Let Y' be the reduction of Y, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 37.28.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that Y is irreducible and reduced (i.e., integral, see Properties, Lemma 28.3.4) and that the irreducible components of the generic fibre X_η are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let X_i be the closure of

$X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \amalg J$$

where $(i,j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i,j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking Y we may assume that $Z_{i,j} = \emptyset$ for all $(i,j) \in I$, see Lemma 37.24.1. After shrinking Y we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 37.27.5. After shrinking Y some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i,j) \in J$, see Morphisms, Proposition 29.27.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 29.25.10. We claim that

$$V = \bigcap\nolimits_{(i,j) \in J} f(Z_{i,j})$$

works, i.e., that X_y is geometrically connected for each $y \in V$. Namely, the fact that X_η is connected implies that the equivalence relation generated by the pairs in J has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only if $(i,j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. □

055H **Lemma 37.28.5.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 37.28.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 28.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 37.24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 29.36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 37.27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma 33.8.17 this implies that also the connected components of X_η are geometrically connected.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) connected components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. After shrinking Y some more we see that each fibre of f has at least n connected components, see Lemma 37.28.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically connected for each i and all $y \in Y$, see Lemma 37.28.4. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

055I **Lemma 37.28.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometric connected components of fibres*

of f introduced in Lemma 37.28.3. Assume f of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.28.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.28.5 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

055J **Lemma 37.28.7.** *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *S is the spectrum of a discrete valuation ring,*
(2) *f is flat,*
(3) *X is connected,*
(4) *the closed fibre X_s is reduced.*

Then the generic fibre X_η is connected.

Proof. Write $S = \mathrm{Spec}(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that X_η is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = \mathrm{Spec}(A)$ be any affine open in X. Note that π is a nonzerodivisor on A as A is flat over R, see More on Algebra, Lemma 15.22.9 for example. Then $e|_{U_\eta}$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \bmod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of n implies $n = 0$. Thus we conclude that $e \in A$! In other words $e \in \Gamma(X, \mathcal{O}_X)$. As X is connected it follows that e is a trivial idempotent which is a contradiction. □

37.29. Connected components meeting a section

055K The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

055L **Situation 37.29.1.** Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of f. For every $y \in Y$ we denote X_y^0 the connected component of X_y containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X_y^0$.

055M **Lemma 37.29.2.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. If $g : Y' \to Y$ is any morphism, consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \xrightarrow{g} & Y \end{array}$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g')^{-1}(X^0)$.

Proof. Let $y' \in Y'$ with image $y \in Y$. We may think of X^0_y as a closed subscheme of X_y, see for example Morphisms, Definition 29.26.3. As $s(y) \in X^0_y$ we conclude from Varieties, Lemma 33.7.14 that X^0_y is a geometrically connected scheme over $\kappa(y)$. Hence $X^0_y \times_y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$. Thus $X^0_y \times_y y' \subset (X'_{y'})^0$. The other inclusion $X^0_y \times_y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in X_y is a connected subset of X_y which contains $s(y)$. □

055N **Lemma 37.29.3.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that the inverse image of X^0 in the base change X_V is open and closed in X_V.*

Proof. Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of f and s. By Lemma 37.29.2 we have $(X_Z)^0 = (X^0)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of Z. In other words we have reduced the problem to the case where Y is an integral scheme (see Properties, Lemma 28.3.4) with generic point η. Our goal is to show that after shrinking Y the subset X^0 becomes an open and closed subset of X.

Note that the scheme X_η is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X^0_\eta \amalg T_\eta$ for some open and closed subset T_η of X_η. Next, let $T \subset X$ be the closure of T_η and let $X^{00} \subset X$ be the closure of X^0_η. Note that T_η, resp. X^0_η is the generic fibre of T, resp. X^{00}, see discussion preceding Lemma 37.24.5. Moreover, that lemma implies that after shrinking Y we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X^0_\eta = \emptyset$. Hence after shrinking Y we may assume that $T \cap X^{00} = \emptyset$, see Lemma 37.24.1. In particular X^{00} is open in X. Note that X^0_η is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 33.7.14. Thus after shrinking Y we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 37.28.4. At this point it follows that the fibres X^{00}_y are open, closed, and connected subsets of X_y containing $s(y)$. It follows that $X^0 = X^{00}$ and we win. □

055P **Lemma 37.29.4.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. If f is of finite presentation then X^0 is locally constructible in X.*

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood U of x such that $X^0 \cap U$ is constructible in U. This reduces us to the case where Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \mathrm{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. By Lemma 37.29.2 it suffices to

prove the lemma for f_i, s_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is a Noetherian affine scheme. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 29.15.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of Z or is not dense in Z, see Topology, Lemma 5.16.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 37.29.3 there exists a nonempty open subset $V \subset \overline{\{y\}}$ such that $X^0 \cap X_V$ is open and closed in X_V. Since $f(Z) \subset \overline{\{y\}}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of Z. It follows that $X^0 \cap W$ is open and closed in W. Since W is irreducible we see that $X^0 \cap W$ is either empty or equal to W. This proves the lemma. □

055Q **Lemma 37.29.5.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Let $y \in Y$ be a point. Assume*

(1) *f is of finite presentation and flat, and*
(2) *the fibre X_y is geometrically reduced.*

Then X^0 is a neighbourhood of X^0_y in X.

Proof. We may replace Y with an affine open neighbourhood of y. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \mathrm{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. After possibly increasing i we may also assume that f_i is flat, see Limits, Lemma 32.8.7. Let y_i be the image of y in Y_i. Note that $X_y = (X_{i,y_i}) \times_{y_i} y$. Hence X_{i,y_i} is geometrically reduced, see Varieties, Lemma 33.6.6. By Lemma 37.29.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is the spectrum of a Noetherian ring. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 29.15.6. Let $x \in X^0$ be a point lying over y. By Topology, Lemma 5.16.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through x the intersection $X^0 \cap Z$ is dense in Z. In particular it suffices to prove that the generic point $x' \in Z$ is in X^0. By Properties, Lemma 28.5.10 we can find a discrete valuation ring R and a morphism $\mathrm{Spec}(R) \to X$ which maps the special point to x and the generic point to x'. We are going to think of $\mathrm{Spec}(R)$ as a scheme over Y via the composition $\mathrm{Spec}(R) \to X \to Y$. By Lemma 37.29.2 we have that $(X_R)^0$ is the inverse image of X^0. By construction we have a second section $t : \mathrm{Spec}(R) \to X_R$ (besides the base change s_R of s) of the structure morphism $X_R \to \mathrm{Spec}(R)$ such that $t(\eta_R)$ is a point of X_R which maps to x' and $t(0_R)$ is a point of X_R which maps to x. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \rightsquigarrow t(0_R)$. Thus it suffices to prove that this implies that $t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where Y is the spectrum of a discrete valuation ring and y its closed point.

Assume Y is the spectrum of a discrete valuation ring and y is its closed point. Our goal is to prove that X^0 is a neighbourhood of X^0_y. Note that X^0_y is open and closed in X_y as X_y has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in X. Thus $U = X \setminus C$ is an open neighbourhood of X^0_y and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other

words, we may assume that X_y is connected. Suppose that X is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the X_i. Say $s(Y) \subset X_1$. Then $X^0 \subset X_1$. Hence we may replace X by X_1 and assume that X is connected. At this point Lemma 37.28.7 implies that X_η is connected, i.e., $X^0 = X$ and we win. □

055R **Lemma 37.29.6.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Assume*

(1) *f is of finite presentation and flat, and*
(2) *all fibres of f are geometrically reduced.*

Then X^0 is open in X.

Proof. This is an immediate consequence of Lemma 37.29.5. □

37.30. Dimension of fibres

05F6

05F7 **Lemma 37.30.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has dimension n, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has dimension n.*

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 29.28.4 this is a closed subset of X. By assumption $Z_\eta = \emptyset$. Hence by Lemma 37.24.1 we may shrink Y and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n-1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of X. By assumption we have $Z'_\eta \neq \emptyset$. Hence after shrinking Y we may assume that $Z' \to Y$ is surjective, see Lemma 37.24.2. Hence we win. □

05F8 **Lemma 37.30.2.** *Let $f : X \to Y$ be a morphism of finite type. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the dimension of X_y. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. This follows from Morphisms, Lemma 29.28.3. □

05F9 **Lemma 37.30.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.30.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible

closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.30.1 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This implies what we want. □

0D4H **Lemma 37.30.4.** *Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is lower semi-continuous.*

Proof. Let $W \subset X$, $W = \coprod_{d \geq 0} U_d$ be the open constructed in Lemmas 37.22.7 and 37.22.9. Let $y \in Y$ be a point. If $n_{X/Y}(y) = \dim(X_y) = n$, then y is in the image of $U_n \to Y$. By Morphisms, Lemma 29.25.10 we see that $f(U_n)$ is open in Y. Hence there is an open neighbourhood of y where $n_{X/Y}$ is $\geq n$. □

0D4I **Lemma 37.30.5.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is upper semi-continuous.*

Proof. Let $Z_d = \{x \in X \mid \dim_x(X_{f(x)}) > d\}$. Then Z_d is a closed subset of X by Morphisms, Lemma 29.28.4. Since f is proper $f(Z_d)$ is closed. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. □

0D4J **Lemma 37.30.6.** *Let $f : X \to Y$ be a proper, flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is locally constant.*

Proof. Immediate consequence of Lemmas 37.30.4 and 37.30.5. □

37.31. Weak relative Noether normalization

0GTD The goal of this section is to prove Lemma 37.31.3.

0GTE **Lemma 37.31.1.** *Let R be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be prime ideals of R with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ if $i \neq j$. Let $k_i \subset \kappa(\mathfrak{p}_i)$ be subfields such that the extensions $\kappa(\mathfrak{p}_i)/k_i$ are not algebraic. Let $J \subset R$ be an ideal not contained in any of the $\mathfrak{p}_i$. Then there exists an element $x \in J$ such that the image of x in $\kappa(\mathfrak{p}_i)$ is transcendental over k_i for $i = 1, \ldots, r$.*

Proof. The ideal $J_i = J\mathfrak{p}_1 \ldots \hat{\mathfrak{p}}_i \ldots \mathfrak{p}_r$ is not contained in $\mathfrak{p}_i$, see Algebra, Lemma 10.15.1. It follows that every element ξ of $\kappa(\mathfrak{p}_i) = \mathrm{Frac}(B/\mathfrak{p}_i)$ is of the form $\xi = a/b$ with $a, b \in J_i$ and $b \notin \mathfrak{p}_i$. Choosing ξ transcendental over k_i we see that either a or b maps to an element of $\kappa(\mathfrak{p}_i)$ transcendental over k_i. We conclude that for every $i = 1, \ldots, r$ we can find an element $x_i \in J_i = J\mathfrak{p}_1 \ldots \hat{\mathfrak{p}}_i \ldots \mathfrak{p}_r$ which maps to an element of $\kappa(\mathfrak{p}_i)$ transcendental over k_i. Then $x = x_1 + \ldots + x_r$ works. □

0GTF **Lemma 37.31.2.** *Let $R \to S$ be a finite type ring map. Let $d \geq 0$. Let $a, b \in S$. Assume that the fibres of*

$$f_a : \mathrm{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to a have dimension $\leq d$. Then there exists an n_0 such that for $n \geq n_0$ the fibres of

$$f_{a^n+b} : \mathrm{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to $a^n + b$ have dimension $\leq d$.

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[A, B, x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$ where A and B map to a and b in S. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[A, B, x_1, \ldots, x_n]/J_\lambda$ and denote $a_\lambda, b_\lambda \in S_\lambda$ the images of A and B. Then for some λ the fibres of

$$f_{a_\lambda} : \mathrm{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_R$$

have dimension $\leq d$, see Limits, Lemma 32.18.1. Fix such a λ. If we can find n_0 which works for $R \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, the fibres of $f_{a_\lambda^n + b_\lambda} : \mathrm{Spec}(S_\lambda) \to \mathbf{A}^1_R$ contain the fibres of $f_{a^n+b} : \mathrm{Spec}(S) \to \mathbf{A}^1_R$. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 10.127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Choose $\lambda_0 \in \Lambda$ and elements $a_{\lambda_0}, b_{\lambda_0} \in S_{\lambda_0}$ mapping to $a, b \in S$. For $\lambda \geq \lambda_0$ denote $a_\lambda, b_\lambda \in S_\lambda$ the image of $a_{\lambda_0}, b_{\lambda_0}$. Then for $\lambda \geq \lambda_0$ large enough the fibres of

$$f_{a_\lambda} : \mathrm{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_{R_\lambda}$$

have dimension $\leq d$, see Limits, Lemma 32.18.4. Fix such a λ. If we can find n_0 which works for $R_\lambda \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, any fibre of $f_{a^n+b} : \mathrm{Spec}(S) \to \mathbf{A}^1_R$ has the same dimension as a fibre of $f_{a_\lambda^n + b_\lambda} : \mathrm{Spec}(S_\lambda) \to \mathbf{A}^1_{R_\lambda}$ by Morphisms, Lemma 29.28.3. This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. In particular the dimension of R is finite, and we may use induction on $\dim(R)$. Thus we may assume the result holds for all situations with $R' \to S'$, a, b as in the lemma with R' and S' of finite type over $\mathbf{Z}$ but with $\dim(R') < \dim(R)$.

Since the statement is about the topology of the spectrum of S we may assume S is reduced. Let S^ν be the normalization of S. Then $S \subset S^\nu$ is a finite extension as S is excellent, see Algebra, Proposition 10.162.16 and Morphisms, Lemma 29.54.11. Thus $\mathrm{Spec}(S^\nu) \to \mathrm{Spec}(S)$ is surjective and finite (Algebra, Lemma 10.36.17). It follows that if the result holds for $R \to S^\nu$ and the images of a, b in S^ν, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S normal. Then $S = S_1 \times \ldots \times S_r$ for some normal domains S_i. If the result holds for each $R \to S_i$ and the images of a, b in S_i, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S a normal domain. We may replace R by the image of R in S (this does not increase the dimension of R). This reduces us to the case discussed in the next paragraph.

Assume $R \subset S$ are of finite type over $\mathbf{Z}$ and S a normal domain. Consider the morphism

$$f_a : \mathrm{Spec}(S) \to \mathbf{A}^1_R$$

The assumption tells us that f_a has fibres of dimension $\leq d$. Hence the fibres of $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ have dimension $\leq d + 1$ (Morphisms, Lemma 29.28.2). Consider the morphism of integral schemes

$$\phi : \mathrm{Spec}(S) \to \mathbf{A}^2_R = \mathrm{Spec}(R[x, y])$$

corresponding to the R-algebra map $R[x, y] \to S$ sending x to a and y to b. There are two cases to consider

(1) ϕ is dominant, and
(2) ϕ is not dominant.

We claim that in both cases there exists an integer n_0 and a nonempty open $V \subset \mathrm{Spec}(R)$ such that for $n \geq n_0$ the fibres of f_{a^n+b} at points $q \in \mathbf{A}^1_V$ have dimension $\leq d$.

Proof of the claim in case (1). We have $f_{a^n+b} = \pi_n \circ \phi$ where

$$\pi_n : \mathbf{A}^2_R \to \mathbf{A}^1_R$$

is the flat morphism corresponding to the R-algebra map $R[x] \to R[x, y]$ sending x to $x^n + y$. Since ϕ is dominant there is a dense open $U \subset \mathrm{Spec}(S)$ such that $\phi|_U : U \to \mathbf{A}^2_R$ is flat (this follows for example from generic flatness, see Morphisms, Proposition 29.27.1). Then the composition

$$f_{a^n+b}|_U : U \xrightarrow{\phi|_U} \mathbf{A}^2_R \xrightarrow{\pi_n} \mathbf{A}^1_R$$

is flat as well. Hence the fibres of this morphism have at least codimension 1 in the fibres of $f|_U : U \to \mathrm{Spec}(R)$ by Morphisms, Lemma 29.28.2. In other words, the fibres of $f_{a^n+b}|_U$ have dimension $\leq d$. On the other hand, since U is dense in $\mathrm{Spec}(S)$, we can find a nonempty open $V \subset \mathrm{Spec}(R)$ such that $U \cap f^{-1}(p) \subset f^{-1}(p)$ is dense for all $p \in V$ (see for example Lemma 37.24.3). Thus $\dim(f^{-1}(p) \setminus U \cap f^{-1}(p)) \leq d$ and we conclude that our claim is true (as any fibres of $f_{a^n+b} : \mathrm{Spec}(S) \to \mathbf{A}^1_R$ is contained in a fibre of f).

Case (2). In this case we can find a nonzero $g = \sum c_{ij} x^i y^j$ in $R[x, y]$ such that $\mathrm{Im}(\phi) \subset V(g)$. In fact, we may assume g is irreducible over $\mathrm{Frac}(R)$. If $g \in R[x]$, say with leading coefficient c, then over $V = D(c) \subset \mathrm{Spec}(R)$ the fibres of f already have dimension $\leq d$ (because the image of f_a is contained in $V(g) \subset \mathbf{A}^1_R$ which has finite fibres over V). Hence we may assume g is not contained in $R[x]$. Let $s \geq 1$ be the degree of g as a polynomial in y and let t be the degree of $\sum c_{is} x^i$ as a polynomial in x. Then c_{ts} is nonzero and

$$g(x, -x^n) = (-1)^s c_{ts} x^{t+sn} + l.o.t.$$

provided that n is bigger than the degree of g as a polynomial in x (small detail omitted). For such n the polynomial $g(x, -x^n)$ is a nonzero polynomial in x and maps to a nonzero polynomial in $\kappa(\mathfrak{p})[x]$ for all $\mathfrak{p} \subset R$, $c_{st} \notin \mathfrak{p}$. We conclude that our claim is true for V equal to the principal open $D(c_{ts})$ of $\mathrm{Spec}(R)$.

OK, and now we can use induction on $\dim(R)$. Namely, let $I \subset R$ be an ideal such that $V(I) = \mathrm{Spec}(R) \setminus V$. Observe that $\dim(R/I) < \dim(R)$ as R is a domain. Let n_0' be the integer we have by induction on $\dim(R)$ for $R/I \to S/IS$ and the images of a and b in S/IS. Then $\max(n_0, n_0')$ works. $\square$

0GTG **Lemma 37.31.3.** *Let $R \to S$ be a finite type ring map. Let d be the maximum of the dimensions of fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Then there exists a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$.*

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[x_1, \ldots, x_n]/J_\lambda$. Then for some λ the fibres of $\mathrm{Spec}(S_\lambda) \to \mathrm{Spec}(R)$ have dimension $\leq d$, see Limits, Lemma 32.18.1. Fix such a λ. If we can find a quasi-finite $R[t_1, \ldots, t_d] \to S_\lambda$, then of course the composition $R[t_1, \ldots, t_d] \to S$ is quasi-finite. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 10.127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Then for λ large enough the fibres of $\mathrm{Spec}(S_\lambda) \to \mathrm{Spec}(R_\lambda)$ have dimension $\leq d$, see Limits, Lemma 32.18.4. Fix such a λ. If we can find a quasi-finite ring map $R_\lambda[t_1, \ldots, t_d] \to S_\lambda$, then the base change $R[t_1, \ldots, t_d] \to S$ is quasi-finite too (Algebra, Lemma 10.122.9). This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. If $d = 0$, then the ring map is quasi-finite and we are done. Assume $d > 0$. We will find an element $a \in S$ such that the fibres of the R-algebra map $R[x] \to S$, $x \mapsto a$ have dimension $< d$. This will finish the proof by induction.

We will prove the existence of a by induction on $\dim(R)$.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r \subset S$ be those among the minimal primes of S such that $\dim_{\mathfrak{q}_i}(S/R) = d$. For notation, see Algebra, Definition 10.125.1. Say $\mathfrak{q}_i$ lies over the prime $\mathfrak{p}_i \subset R$. We have $\mathrm{trdeg}_{\kappa(\mathfrak{p}_i)}(\kappa(\mathfrak{q}_i)) = d$ as $\mathfrak{q}_i$ is a generic point of its fibre; for example apply Algebra, Lemma 10.116.3 to $S \otimes_R \kappa(\mathfrak{p}_i)$. Hence by Lemma 37.31.1 we can find an element $a \in S$ such that the image of a in $\kappa(\mathfrak{q}_i)$ is transcendental over $\kappa(\mathfrak{p}_i)$ for $i = 1, \ldots, r$. Consider the morphism

$$f_a : \mathrm{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

corresponding the R-algebra homomorphism $R[x] \to S$ to mapping x to a. Let $U \subset \mathrm{Spec}(S)$ be the open subset where the fibres have dimension $\leq d - 1$, see Morphisms, Lemma 29.28.4. By construction U contains all the generic points of $\mathrm{Spec}(S)$. In particular we see that U contains all generic points of all the generic fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ as such points are necessarily generic points of $\mathrm{Spec}(S)$. Set $T = \mathrm{Spec}(S) \setminus U$ viewed as a reduced closed subscheme of $\mathrm{Spec}(S)$. It follows from what we just said and the assumption that $\dim(S/R) \leq d$ that the generic fibres of $T \to \mathrm{Spec}(R)$ have dimension $\leq d - 1$. Hence by Lemma 37.30.1, applied several times to produce open neighbourhoods of the generic points of $\mathrm{Spec}(R)$, we can find a dense open $V \subset \mathrm{Spec}(R)$ such that $T_V \to V$ has fibres of dimension $\leq d - 1$. We conclude that for $q \in \mathbf{A}^1_V$ the fibre of f_a over q has dimension $\leq d - 1$ (as we have bounded the dimension of the fibre of $f_a|_U$ and of the fibre of $f_a|_T$).

By prime avoidance, we may assume that $V = D(f)$ for some $f \in R$. Then we see that the ring map $R_f[x] \to S_f$, $x \mapsto a$ has fibres of dimension $\leq d - 1$. We

may replace a by fa and assume $a \in (f)$. By induction on $\dim(R)$ we can find an element $\overline{b} \in S/fS$ such that the fibres of $\mathrm{Spec}(S/fS) \to \mathrm{Spec}(R/fR[x])$, $x \mapsto \overline{b}$ have dimension $\leq d-1$. Let $b \in S$ be a lift of $\overline{b}$. By Lemma 37.31.2 there exists an $n > 0$ such that $a^n + b$ still works for $R_f \to S_f$. On the other hand, the image of $a^n + b$ in S/fS is $\overline{b}$ and the proof is complete. □

37.32. Bertini theorems

0G4C We continue the discussion started in Varieties, Section 33.47. In this section we prove that general hyperplane sections of geometrically irreducible varieties are geometrically irreducible following the remarkable argument given in [**Jou83**].

0G4D **Lemma 37.32.1.** *Let K/k be a geometrically irreducible and finitely generated field extension. Let $n \geq 1$. Let $g_1, \ldots, g_n \in K$ be elements such that there exist $c_1, \ldots, c_n \in k$ such that the elements*

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in K(x_1, \ldots, x_n)$$

are algebraically independent over k. Then $K(x_1, \ldots, x_n)$ is geometrically irreducible over $k(x_1, \ldots, x_n, \sum g_i x_i)$.

See pages 71 and 72 of [**Jou83**]

Proof. Let $c_1, \ldots, c_n \in k$ be as in the statement of the lemma. Write $\xi = \sum g_i x_i$ and $\delta = \sum c_i g_i$. For $a \in k$ consider the automorphism σ_a of $K(x_1, \ldots, x_n)$ given by the identity on K and the rules

$$\sigma_a(x_i) = x_i + ac_i$$

Observe that $\sigma_a(\xi) = \xi + a\delta$ and $\sigma_a(\delta) = \delta$. Consider the tower of fields

$$K_0 = k(x_1, \ldots, x_n) \subset K_1 = K_0(\xi) \subset K_2 = K_0(\xi, \delta) \subset K(x_1, \ldots, x_n) = \Omega$$

Observe that $\sigma_a(K_0) = K_0$ and $\sigma_a(K_2) = K_2$. Let $\theta \in \Omega$ be separable algebraic over K_1. We have to show $\theta \in K_1$, see Algebra, Lemma 10.47.12.

Denote K_2' the separable algebraic closure of K_2 in Ω. Since K_2'/K_2 is finite (Algebra, Lemma 10.47.13) and separable there are only a finite number of fields in between K_2' and K_2 (Fields, Lemma 9.19.1). If k is infinite[5], then we can find distinct elements a_1, a_2 of k such that

$$K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta))$$

as subfields of Ω. Write $\theta_i = \sigma_{a_i}(\theta)$ and $\xi_i = \sigma_{a_i}(\xi) = \xi + a_i\delta$. Observe that

$$K_2 = K_0(\xi_1, \xi_2)$$

as we have $\xi_i = \xi + a_i\delta$, $\xi = (a_2\xi_1 - a_1\xi_2)/(a_2 - a_1)$, and $\delta = (\xi_1 - \xi_2)/(a_1 - a_2)$. Since K_2/K_0 is purely transcendental of degree 2 we conclude that ξ_1 and ξ_2 are algebraically independent over K_0. Since θ_1 is algebraic over $K_0(\xi_1)$ we conclude that ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$.

By assumption K/k is geometrically irreducible. This implies that $K(x_1, \ldots, x_n)/K_0$ is geometrically irreducible (Algebra, Lemma 10.47.10). This in turn implies that $K_0(\xi_1, \theta_1)/K_0$ is geometrically irreducible as a subextension (Algebra, Lemma 10.47.6). Since ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$ we conclude that $K_0(\xi_1, \xi_2, \theta_1)/K_0(\xi_2)$ is

[5] We will deal with the finite field case in the last paragraph of the proof.

geometrically irreducible (Algebra, Lemma 10.47.11). By our choice of a_1, a_2 above we have

$$K_0(\xi_1, \xi_2, \theta_1) = K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta)) = K_0(\xi_1, \xi_2, \theta_2)$$

Since θ_2 is separably algebraic over $K_0(\xi_2)$ we conclude by Algebra, Lemma 10.47.12 again that $\theta_2 \in K_0(\xi_2)$. Taking $\sigma_{a_2}^{-1}$ of this relation givens $\theta \in K_0(\xi) = K_1$ as desired.

This finishes the proof in case k is infinite. If k is finite, then we can choose a variable t and consider the extension $K(t)/k(t)$ which is geometrically irreducible by Algebra, Lemma 10.47.10. Since it is still be true that $x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i$ in $K(t, x_1, \ldots, x_n)$ are algebraically independent over $k(t)$ we conclude that $K(t, x_1, \ldots, x_n)$ is geometrically irreducible over $k(t, x_1, \ldots, x_n, \sum g_i x_i)$ by the argument already given. Then using Algebra, Lemma 10.47.10 once more finishes the job. □

0G4E **Lemma 37.32.2.** *Let A be a domain of finite type over a field k. Let $n \geq 2$. Let $g_1, \ldots, g_n \in A$ be elements such that $V(g_1, g_2)$ has an irreducible component of dimension $\dim(A) - 2$. Then there exist $c_1, \ldots, c_n \in k$ such that the elements*

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in Frac(A)(x_1, \ldots, x_n)$$

are algebraically independent over k.

Proof. The algebraic independence over k means that the morphism

$$T = \mathrm{Spec}(A[x_1, \ldots, x_n]) \longrightarrow \mathrm{Spec}(k[x_1, \ldots, x_n, y, z]) = S$$

given by $y = \sum g_i x_i$ and $z = \sum c_i g_i$ is dominant. Set $d = \dim(A)$. If $T \to S$ is not dominant, then the image has dimension $< n + 2$ and hence every irreducible component of every fibre has dimension $> d + n - (n + 2) = d - 2$, see Varieties, Lemma 33.20.4. Choose a closed point $u \in V(g_1, g_2)$ contained in an irreducible component of dimension $d-2$ and in no other component of $V(g_1, g_2)$. Consider the closed point $t = (u, 1, 0, \ldots 0)$ of T lying over u. Set $(c_1, \ldots, c_n) = (0, 1, 0, \ldots, 0)$. Then t maps to the point $s = (1, 0, \ldots, 0)$ of S. The fibre of $T \to S$ over s is cut out by

$$x_1 - 1, x_2, \ldots, x_n, \sum x_i g_i, g_2$$

and hence equivalently is cut out by

$$x_1 - 1, x_2, \ldots, x_n, g_1, g_2$$

By our condition on g_1, g_2 this subscheme has an irreducible component of dimension $d - 2$. □

0G4F **Lemma 37.32.3.** *In Varieties, Situation 33.47.2 assume* [**Jou83**, Theorem 6.3 part 4)]

(1) *X is of finite type over k,*
(2) *X is geometrically irreducible over k,*
(3) *there exist $v_1, v_2, v_3 \in V$ and an irreducible component Z of $H_{v_2} \cap H_{v_3}$ such that $Z \not\subset H_{v_1}$ and $codim(Z, X) = 2$, and*
(4) *every irreducible component Y of $\bigcap_{v \in V} H_v$ has $codim(Y, X) \geq 2$.*

Then for general $v \in V \otimes_k k'$ the scheme H_v is geometrically irreducible over k'.

Proof. In order for assumption (3) to hold, the elements v_1, v_2, v_3 must be k-linearly independent in V (small detail omitted). Thus we may choose a basis $v_1, \ldots, v_r$ of V incorporating these elements as the first 3. Recall that $H_{univ} \subset \mathbf{A}^r_k \times_k X$ is the "universal divisor". Consider the projection $q : H_{univ} \to \mathbf{A}^r_k$ whose

scheme theoretic fibres are the divisors H_v. By Lemma 37.27.5 it suffices to show that the generic fibre of q is geometrically irreducible. To prove this we may replace X by its reduction, hence we may assume X is an integral scheme of finite type over k.

Let $U \subset X$ be a nonempty affine open such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \mathrm{Spec}(A)$. Denote $f_i \in A$ the element corresponding to section $\psi(v_i)|_U$ via the isomorphism $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $H_{univ} \cap (\mathbf{A}^r_k \times_k U)$ is given by

$$H_U = \mathrm{Spec}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))$$

By our choice of basis we see that f_1 cannot be zero because this would mean $v_1 = 0$ and hence $H_{v_1} = X$ which contradicts assumption (3). Hence $\sum x_i f_i$ is a nonzerodivisor in $A[x_1, \ldots, x_r]$. It follows that every irreducible component of H_U has dimension $d + r - 1$ where $d = \dim(X) = \dim(A)$. If $U' = U \cap D(f_1)$ then we see that

$$H_{U'} = \mathrm{Spec}(A_{f_1}[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r)) \cong \mathrm{Spec}(A_{f_1}[x_2, \ldots x_r]) = \mathbf{A}^{r-1}_k \times_k U'$$

is irreducible. On the other hand, we have

$$H_U \setminus H_{U'} = \mathrm{Spec}(A/(f_1)[x_1, \ldots, x_r]/(x_2 f_2 + \ldots + x_r f_r))$$

which has dimension at most $d + r - 2$. Namely, for $i \neq 1$ the scheme $(H_U \setminus H_{U'}) \times_U D(f_i)$ is either empty (if $f_i = 0$) or by the same argument as above isomorphic to an $r - 1$ dimensional affine space over an open of $\mathrm{Spec}(A/(f_1))$ and hence has dimension at most $d + r - 2$. On the other hand, $(H_U \setminus H_{U'}) \times_U V(f_2, \ldots, f_r)$ is an r dimensional affine space over $\mathrm{Spec}(A/(f_1, \ldots, f_r))$ and hence assumption (4) tells us this has dimension at most $d + r - 2$. We conclude that H_U is irreducible for every U as above. It follows that H_{univ} is irreducible.

Thus it suffices to show that the generic point of H_{univ} is geometrically irreducible over the generic point of $\mathbf{A}^r_k$, see Varieties, Lemma 33.8.6. Choose a nonempty affine open $U = \mathrm{Spec}(A)$ of X contained in $X \setminus H_{v_1}$ which meets the irreducible component Z of $H_{v_2} \cap H_{v_3}$ whose existence is asserted in assumption (3). With notation as above we have to prove that the field extension

$$\mathrm{Frac}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))/k(x_1, \ldots, x_r)$$

is geometrically irreducible. Observe that f_1 is invertible in A by our choice of U. Set $K = \mathrm{Frac}(A)$ equal to the fraction field of A. Eliminating the variable x_1 as above, we find that we have to show that the field extension

$$K(x_2, \ldots, x_r)/k(x_2, \ldots, x_r, -\sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i)$$

is geometrically irreducible. By Lemma 37.32.1 it suffices to show that for some $c_2, \ldots, c_r \in k$ the elements

$$x_2, \ldots, x_r, \sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i, \sum\nolimits_{i=2,\ldots,r} c_i f_1^{-1} f_i$$

are algebraically independent over k in the fraction field of $A[x_2, \ldots, x_r]$. This follows from Lemma 37.32.2 and the fact that $Z \cap U$ is an irreducible component of $V(f_1^{-1} f_2, f_1^{-1} f_3) \subset U$. □

0G4G **Remark 37.32.4.** Let us sketch a "geometric" proof of a special case of Lemma 37.32.3. Namely, say k is an algebraically closed field and $X \subset \mathbf{P}^n_k$ is smooth and irreducible of dimension ≥ 2. Then we claim there is a hyperplane $H \subset \mathbf{P}^n_k$ such

that $X \cap H$ is smooth and irreducible. Namely, by Varieties, Lemma 33.47.3 for a general $v \in V = kT_0 \oplus \ldots \oplus kT_n$ the corresponding hyperplane section $X \cap H_v$ is smooth. On the other hand, by Enriques-Severi-Zariski the scheme $X \cap H_v$ is connected, see Varieties, Lemma 33.48.3. Hence $X \cap H_v$ is smooth and irreducible.

37.33. Theorem of the cube

0BEZ The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

0BDP **Lemma 37.33.1.** *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : T \to S$ consider the base change diagram*

$$\begin{array}{ccc} X_T & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Assume $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism for all $g : T \to S$. Then there exists an immersion $j : Z \to S$ of finite presentation such that a morphism $g : T \to S$ factors through Z if and only if there exists a finite locally free $\mathcal{O}_T$-module $\mathcal{N}$ with $p^*\mathcal{N} \cong q^*\mathcal{E}$.

Proof. Observe that the fibres X_s of f are connected by our assumption that $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms, Lemma 29.25.10) and closed we conclude that there is a decomposition $S = \coprod S_r$ of S into open and closed subschemes such that $\mathcal{E}$ has constant rank r on the inverse image of S_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Schemes, Lemma 36.30.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of S and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : T \to S$ we have $p_*(q^*\mathcal{E}) = H^0(Lg^*E)$ and $p_*(q^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to S$ and $j' : Z' \to S$ be immersions of finite presentation constructed in Derived Categories of Schemes, Lemma 36.31.4 for E and E' with $a = 0$ and $r = r$; these are roughly speaking characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are finite locally free modules compatible with pullback.

Let $g : T \to S$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology, Lemma 20.54.2, we see that the modules

$$p_*(q^*\mathcal{E}) \cong p_*(p^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_T} p_*\mathcal{O}_{X_T} \cong \mathcal{N} \quad \text{and similarly} \quad p_*(q^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are finite locally free modules of rank r and remain so after any further base change $T' \to T$. Hence in this case $T \to S$ factors through j and through j'. Thus we may replace S by $Z \times_S Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are finite locally free $\mathcal{O}_S$-modules of rank r whose formation commutes with arbitrary change of base (small detail omitted).

In this situation if $g : T \to S$ be a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$p_*(q^*\mathcal{E}) \otimes_{\mathcal{O}_T} p_*(q^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_T$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on T we may choose a basis of sections $\sigma_1, \ldots, \sigma_r$ in $p_*(q^*\mathcal{E})$ and $\tau_1, \ldots, \tau_r$ in $p_*(q^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i\tau_j = \delta_{ij}$. Thinking of σ_i as a section of $q^*\mathcal{E}$ on X_T and τ_j as a section of $q^*\mathcal{E}^\vee$ on X_T, we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X_T}^{\oplus r} \longrightarrow q^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : q^*\mathcal{E} \longrightarrow \mathcal{O}_{X_T}^{\oplus r}$$

In other words, we see that $p^*p_*q^*\mathcal{E} \cong q^*\mathcal{E}$. But the condition that the cup product is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. □

The lemma above in particular tells us, that if a vector bundle is trivial on fibres for a proper flat family of proper spaces, then it is the pull back of a vector bundle. Let's spell this out a bit.

0EX7 **Lemma 37.33.2.** *Let $f : X \to S$ be a flat, proper morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and this remains true after arbitrary base change. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume*

(1) *$\mathcal{E}|_{X_s}$ is isomorphic to $\mathcal{O}_{X_s}^{\oplus r_s}$ for all $s \in S$, and*
(2) *S is reduced.*

Then $\mathcal{E} = f^*\mathcal{N}$ for some finite locally free $\mathcal{O}_S$-module $\mathcal{N}$.

Proof. Namely, in this case the locally closed immersion $j : Z \to S$ of Lemma 37.33.1 is bijective and hence a closed immersion. But since S is reduced, j is an isomorphism. □

0EX8 **Lemma 37.33.3.** *Let $f : X \to S$ be a proper flat morphism of finite presentation. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

(1) *S is the spectrum of a valuation ring,*
(2) *$\mathcal{L}$ is trivial on the generic fibre X_η of f,*
(3) *the closed fibre X_0 of f is integral,*
(4) *$H^0(X_\eta, \mathcal{O}_{X_\eta})$ is equal to the function field of S.*

Then $\mathcal{L}$ is trivial.

Proof. Write $S = \mathrm{Spec}(A)$. We will first prove the lemma when A is a discrete valuation ring (as this is the case most often used in practice). Let $\pi \in A$ be a uniformizer. Take a trivializing section $s \in \Gamma(X_\eta, \mathcal{L}_\eta)$. After replacing s by $\pi^n s$ if necessary we can assume that $s \in \Gamma(X, \mathcal{L})$. If $s|_{X_0} = 0$, then we see that s is divisible by π (because X_0 is the scheme theoretic fibre and X is flat over A). Thus we may assume that $s|_{X_0}$ is nonzero. Then the zero locus $Z(s)$ of s is contained in X_0 but does not contain the generic point of X_0 (because X_0 is integral). This means that the $Z(s)$ has codimension ≥ 2 in X which contradicts Divisors, Lemma 31.15.3 unless $Z(s) = \emptyset$ as desired.

Proof in the general case. Since the valuation ring A is coherent (Algebra, Example 10.90.2) we see that $H^0(X, \mathcal{L})$ is a coherent A-module, see Derived Categories of

Schemes, Lemma 36.33.1. Equivalently, $H^0(X, \mathcal{L})$ is a finitely presented A-module (Algebra, Lemma 10.90.4). Since $H^0(X, \mathcal{L})$ is torsion free (by flatness of X over A), we see from More on Algebra, Lemma 15.125.3 that $H^0(X, \mathcal{L}) = A^{\oplus n}$ for some n. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we have

$$K = H^0(X_\eta, \mathcal{O}_{X_\eta}) \cong H^0(X_\eta, \mathcal{L}_\eta) = H^0(X, \mathcal{L}) \otimes_A K$$

where K is the fraction field of A. Thus $n = 1$. Pick a generator $s \in H^0(X, \mathcal{L})$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Then $\kappa = A/\mathfrak{m} = \operatorname{colim} A/\pi$ where this is a filtered colimit over nonzero $\pi \in \mathfrak{m}$ (here we use that A is a valuation ring). Thus $X_0 = \lim X \times_S \operatorname{Spec}(A/\pi)$. If $s|_{X_0}$ is zero, then for some π we see that s restricts to zero on $X \times_S \operatorname{Spec}(A/\pi)$, see Limits, Lemma 32.4.7. But if this happens, then $\pi^{-1}s$ is a global section of $\mathcal{L}$ which contradicts the fact that s is a generator of $H^0(X, \mathcal{L})$. Thus $s|_{X_0}$ is not zero. Let $Z(s) \subset X$ be the zero scheme of s. Since $s|_{X_0}$ is not zero and since X_0 is integral, we see that $Z(s)_0 \subset X_0$ is an effective Cartier divisor. Since f is proper and S is local, every point of $Z(s)$ specializes to a point of $Z(s)_0$. Thus by Divisors, Lemma 31.18.9 part (3) we see that $Z(s)$ is a relative effective Cartier divisor, in particular $Z(s) \to S$ is flat. Hence if $Z(s)$ were nonemtpy, then $Z(s)_\eta$ would be nonempty which contradicts the fact that $s|_{X_\eta}$ is a trivialization of $\mathcal{L}_\eta$. Thus $Z(s) = \emptyset$ as desired. □

0BF0 **Lemma 37.33.4.** *Let $f : X \to S$ and $\mathcal{E}$ be as in Lemma 37.33.1 and in addition assume $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. If moreover the geometric fibres of f are integral, then Z is closed in S.*

Proof. Since $j : Z \to S$ is of finite presentation, it suffices to show: for any morphism $g : \operatorname{Spec}(A) \to S$ where A is a valuation ring with fraction field K such that $g(\operatorname{Spec}(K)) \in j(Z)$ we have $g(\operatorname{Spec}(A)) \subset j(Z)$. See Morphisms, Lemma 29.6.5. This follows from Lemma 37.33.3 and the characterization of $j : Z \to S$ in Lemma 37.33.1. □

0BF1 **Lemma 37.33.5.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle f} & \\ & S & \end{array}$$

with $f' : X' \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 37.33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}'$ be the pullback to X'. Let $Z \subset S$, resp. $Z' \subset S$ be the locally closed subscheme constructed in Lemma 37.33.1 for $(f, \mathcal{L})$, resp. $(f', \mathcal{L}')$ so that $Z \subset Z'$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow H^1(X'_s, \mathcal{O})$$

is injective, then $Z \cap U = Z' \cap U$ for some open neighbourhood U of s.

Proof. We may replace S by Z'. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}' = \mathcal{O}_{X'}$. Let $E = Rf_*\mathcal{L}$ and $E' = Rf'_*\mathcal{L}' = Rf'_*\mathcal{O}_{X'}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 36.30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_i = \dim_{\kappa(s)} H^i(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 15.76.7 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7). Similarly, we may assume E' is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h'_1} \to \mathcal{O}_S^{\oplus h'_2} \to \ldots$$

where $h'_i = \dim_{\kappa(s)} H^i(X'_s, \mathcal{O}_{X'_s})$. By functoriality of cohomology we have a map

$$E \longrightarrow E'$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{array}{ccccccc}
\mathcal{O}_S & \xrightarrow{d} & \mathcal{O}_S^{\oplus h_1} & \longrightarrow & \mathcal{O}_S^{\oplus h_2} & \longrightarrow & \ldots \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
\mathcal{O}_S & \xrightarrow{d'} & \mathcal{O}_S^{\oplus h'_1} & \longrightarrow & \mathcal{O}_S^{\oplus h'_2} & \longrightarrow & \ldots
\end{array}$$

representing the given map $E \to E'$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}'_s = \mathcal{O}_{X'_s}$. Thus $a \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to H^1(X'_s, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining b which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that b is injective. However, since $\mathcal{L}' = \mathcal{O}_{X'}$ we see that $d' = 0$. It follows that $d = 0$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. □

0BF2 **Lemma 37.33.6.** *Consider n commutative diagrams of schemes*

$$\begin{array}{ccc}
X_i & \longrightarrow & X \\
& {\scriptstyle f_i} \searrow \quad \swarrow {\scriptstyle f} & \\
& S &
\end{array}$$

with $f_i : X_i \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 37.33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}_i$ be the pullback to X_i. Let $Z \subset S$, resp. $Z_i \subset S$ be the locally closed subscheme constructed in Lemma 37.33.1 for $(f, \mathcal{L})$, resp. $(f_i, \mathcal{L}_i)$ so that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$$

is injective, then $Z \cap U = (\bigcap_{i=1,\ldots,n} Z_i) \cap U$ (scheme theoretic intersection) for some open neighbourhood U of s.

Proof. This lemma is a variant of Lemma 37.33.5 and we strongly urge the reader to read that proof first; this proof is basically a copy of that proof with minor modifications. It follows from the description of (scheme valued) points of Z and the

Z_i that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$ where we take the scheme theoretic intersection. Thus we may replace S by the scheme theoretic intersection $\bigcap_{i=1,\ldots,n} Z_i$. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}_i = \mathcal{O}_{X_i}$ for $i = 1, \ldots, n$. Let $E = Rf_*\mathcal{L}$ and $E_i = Rf_{i,*}\mathcal{L}_i = Rf_{i,*}\mathcal{O}_{X_i}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 36.30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_j = \dim_{\kappa(s)} H^j(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 15.76.7 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7). Similarly, we may assume E_i is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_{i,1}} \to \mathcal{O}_S^{\oplus h_{i,2}} \to \ldots$$

where $h_{i,j} = \dim_{\kappa(s)} H^j(X_{i,s}, \mathcal{O}_{X_{i,s}})$. By functoriality of cohomology we have a map

$$E \longrightarrow E_i$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{array}{ccccccc}
\mathcal{O}_S & \xrightarrow{d} & \mathcal{O}_S^{\oplus h_1} & \longrightarrow & \mathcal{O}_S^{\oplus h_2} & \longrightarrow & \ldots \\
\downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle b_i} & & \downarrow{\scriptstyle c_i} & & \\
\mathcal{O}_S & \xrightarrow{d_i} & \mathcal{O}_S^{\oplus h_{i,1}} & \longrightarrow & \mathcal{O}_S^{\oplus h_{i,2}} & \longrightarrow & \ldots
\end{array}$$

representing the given map $E \to E_i$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}_{i,s} = \mathcal{O}_{X_{i,s}}$. Thus $a_i \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a_i is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining $\oplus b_i$ which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that $(b_1, \ldots, b_n) : \mathcal{O}_S^{h_1} \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_S^{h_{i,1}}$ is injective. However, since $\mathcal{L}_i = \mathcal{O}_{X_i}$ we see that $d_i = 0$ for $i = 1, \ldots n$. It follows that $d = 0$ because $(b_1, \ldots, b_n) \circ d = (\oplus d_i) \circ (a_1, \ldots, a_n)$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. □

0BF3 **Lemma 37.33.7.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes satisfying the hypotheses of Lemma 37.33.1. Let $\sigma : S \to X$ and $\tau : S \to Y$ be sections of f and g. Let $s \in S$. Let $\mathcal{L}$ be an invertible sheaf on $X \times_S Y$. If $(1 \times \tau)^*\mathcal{L}$ on X, $(\sigma \times 1)^*\mathcal{L}$ on Y, and $\mathcal{L}|_{(X \times_S Y)_s}$ are trivial, then there is an open neighbourhood U of s such that $\mathcal{L}$ is trivial over $(X \times_S Y)_U$.*

Proof. By Künneth (Varieties, Lemma 33.29.1) the map

$$H^1(X_s \times_{\mathrm{Spec}(\kappa(s))} Y_s, \mathcal{O}) \to H^1(X_s, \mathcal{O}) \oplus H^1(Y_s, \mathcal{O})$$

is injective. Thus we may apply Lemma 37.33.6 to the two morphisms

$$1 \times \tau : X \to X \times_S Y \quad \text{and} \quad \sigma \times 1 : Y \to X \times_S Y$$

to conclude. □

0BF4 **Theorem 37.33.8** (Theorem of the cube)**.** *Let S be a scheme. Let X, Y, and Z be schemes over S. Let $x : S \to X$ and $y : S \to Y$ be sections of the structure morphisms. Let $\mathcal{L}$ be an invertible module on $X \times_S Y \times_S Z$. If*

(1) *$X \to S$ and $Y \to S$ are flat, proper morphisms of finite presentation with geometrically integral fibres,*
(2) *the pullbacks of $\mathcal{L}$ by $x \times id_Y \times id_Z$ and $id_X \times y \times id_Z$ are trivial over $Y \times_S Z$ and $X \times_S Z$,*
(3) *there is a point $z \in Z$ such that $\mathcal{L}$ restricted to $X \times_S Y \times_S z$ is trivial, and*
(4) *Z is connected,*

then $\mathcal{L}$ is trivial.

An often used special case is the following. Let k be a field. Let X, Y, Z be varieties with k-rational points x, y, z. Let $\mathcal{L}$ be an invertible module on $X \times Y \times Z$. If

(1) $\mathcal{L}$ is trivial over $x \times Y \times Z$, $X \times y \times Z$, and $X \times Y \times z$, and
(2) X and Y are geometrically integral and proper over k,

then $\mathcal{L}$ is trivial.

Proof. Observe that the morphism $X \times_S Y \to S$ is a flat, proper morphism of finite presentation whose geometrically integral fibres (see Varieties, Lemmas 33.9.2, 33.8.4, and 33.6.7 for the statement about the fibres). By Derived Categories of Schemes, Lemma 36.32.6 we see that the pushforward of the structure sheaf by $X \to S$, $Y \to S$, or $X \times_S Y \to S$ is the structure sheaf of S and the same remains true after any base change. Thus we may apply Lemma 37.33.1 to the morphism

$$p : X \times_S Y \times_S Z \longrightarrow Z$$

and the invertible module $\mathcal{L}$ to get a "universal" locally closed subscheme $Z' \subset Z$ such that $\mathcal{L}|_{X \times_S Y \times_S Z'}$ is the pullback of an invertible module $\mathcal{N}$ on Z'. The existence of z shows that Z' is nonempty. By Lemma 37.33.4 we see that $Z' \subset Z$ is a closed subscheme. Let $z' \in Z'$ be a point. Observe that we may write p as the product morphism

$$(X \times_S Z) \times_Z (Y \times_S Z) \longrightarrow Z$$

Hence we may apply Lemma 37.33.7 to the morphism p, the point z', and the sections $\sigma : Z \to X \times_S Z$ and $\tau : Z \to Y \times_S Z$ given by x and y. We conclude that Z' is open. Hence $Z' = Z$ and $\mathcal{L} = p^*\mathcal{N}$ for some invertible module $\mathcal{N}$ on Z. Pulling back via $x \times y \times \mathrm{id}_Z : Z \to X \times_S Y \times_S Z$ we obtain on the one hand $\mathcal{N}$ and on the other hand we obtain the trivial invertible module by assumption (2). Thus $\mathcal{N} = \mathcal{O}_Z$ and the proof is complete. □

37.34. Limit arguments

05FA Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

05FB **Lemma 37.34.1.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longleftarrow & S \end{array}$$

such that

(1) *X_0, S_0 are affine schemes,*
(2) *S_0 of finite type over $\mathbf{Z}$,*
(3) *f_0 is of finite type.*

Proof. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. Thus the lemma follows from Algebra, Lemma 10.127.18. □

05FC **Lemma 37.34.2.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 37.34.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 28.16.2. Thus the lemma follows from Algebra, Lemma 10.127.18. □

05FD **Lemma 37.34.3.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over S. Then we may choose a diagram as in Lemma 37.34.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over S_0.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 28.16.2. As $\mathcal{F}$ is flat over S we see that M is flat over A, see Morphisms, Lemma 29.25.2. Thus the lemma follows from Algebra, Lemma 10.168.1. □

05FE **Lemma 37.34.4.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is flat.*

Proof. This is a special case of Lemma 37.34.3. □

05FF **Lemma 37.34.5.** *Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is smooth.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and as f is smooth we see that B is smooth as an A-algebra, see Morphisms, Lemma 29.34.2. Hence the lemma follows from Algebra, Lemma 10.138.14. □

05FG **Lemma 37.34.6.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically reduced fibres.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.26.5 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically reduced is a constructible subset. By Lemma 37.26.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically reduced. □

05FH **Lemma 37.34.7.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically irreducible fibres.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.27.7 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically irreducible is a constructible subset. By Lemma 37.27.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically irreducible. □

05FI **Lemma 37.34.8.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically connected fibres.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.28.6 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically connected is a constructible subset. By Lemma 37.28.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\mathrm{Spec}(A_i) \to S_0$

has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically connected. □

05FJ **Lemma 37.34.9.** *Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension d. Then there exists a diagram as in Lemma 37.34.1 such that in addition all fibres of f_0 have dimension d.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.30.3 the set $E \subset S_0$ of points where the fibre of f_0 has dimension d is a constructible subset. By Lemma 37.30.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 have dimension d. □

05FK **Lemma 37.34.10.** *Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 29.30.1). Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is standard syntomic.*

Proof. This lemma is a copy of Algebra, Lemma 10.136.11. □

05FL **Lemma 37.34.11.** *(Noetherian approximation and combining properties.) Let P, Q be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams*

$$\begin{array}{ccc} X_1 & \longleftarrow & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ S_1 & \longleftarrow & S \end{array} \quad \text{and} \quad \begin{array}{ccc} X_2 & \longleftarrow & X \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f} \\ S_2 & \longleftarrow & S \end{array}$$

of affine schemes, with S_1, S_2 of finite type over $\mathbf{Z}$ and f_1, f_2 of finite type such that f_1 has property P and f_2 has property Q. Then we can find a cartesian diagram

$$\begin{array}{ccc} X_0 & \longleftarrow & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longleftarrow & S \end{array}$$

of affine schemes with S_0 of finite type over $\mathbf{Z}$ and f_0 of finite type such that f_0 has both property P and property Q.

Proof. The given pair of diagrams correspond to cocartesian diagrams of rings

$$\begin{array}{ccc} B_1 & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_1 & \longrightarrow & A \end{array} \quad \text{and} \quad \begin{array}{ccc} B_2 & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_2 & \longrightarrow & A \end{array}$$

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of A containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both A_1 and A_2 are of finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type A_0-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as A-algebras. As A is the directed colimit of its finite type A_0-subalgebras, by Limits, Lemma 32.10.1 we may assume after enlarging A_0 that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as A_0-algebras. Since properties P and Q are assumed stable under base change we conclude that setting $S_0 = \mathrm{Spec}(A_0)$ and

$$X_0 = X_1 \times_{S_1} S_0 = \mathrm{Spec}(B_{0,1}) \cong \mathrm{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0$$

works. □

37.35. Étale neighbourhoods

02LD It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

02LE **Definition 37.35.1.** Let S be a scheme. Let $s \in S$ be a point.

(1) An *étale neighbourhood of* (S, s) is a pair (U, u) together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.
(2) A *morphism of étale neighbourhoods* $f : (V, v) \to (U, u)$ of (S, s) is simply a morphism of S-schemes $f : V \to U$ such that $f(v) = u$.
(3) An *elementary étale neighbourhood* is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

The notion of an elementary étale neighbourhood has many different names in the literature, for example these are sometimes called "étale neighbourhoods" ([**Mil80**, Page 36] or "strongly étale" ([**KPR75**, Page 108]). Here we follow the convention of the paper [**GR71**] by calling them elementary étale neighbourhoods.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then f is automatically étale, see Morphisms, Lemma 29.36.18. Hence it turns (V, v) into an étale neighbourhood of (U, u). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma 29.36.3) we see that conversely any étale neighbourhood (V, v) of (U, u) is an étale neighbourhood of (S, s) as well. We also remark that if $U \subset S$ is an open neighbourhood of s, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 29.36.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(u)/\kappa(s)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma 29.36.15.

02LF **Lemma 37.35.2.** *Let S be a scheme. Let $s \in S$. Let $k/\kappa(s)$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(u)/\kappa(s)$ is isomorphic to $k/\kappa(s)$.*

Proof. We may assume S is affine. In this case the lemma follows from Algebra, Lemma 10.144.3. □

057A **Lemma 37.35.3.** *Let S be a scheme, and let s be a point of S. The category of étale neighborhoods has the following properties:*

(1) *Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of s in S. Then there exists a third étale neighborhood (U, u) and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of s. Assume h_1, h_2 induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood (U'', u'') and a morphism $h : (U'', u'') \to (U, u)$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Morphisms, Lemma 29.36.4. There is a point of U mapping to both u_1 and u_2 for example by the description of points of a fibre product in Schemes, Lemma 26.17.5. For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1,h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_S U'. \end{array}$$

Since h_1 and h_2 induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over u' with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 29.36.4 and 29.36.3). Hence the vertical arrow (h_1, h_2) is étale by Morphisms, Lemma 29.36.18. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus (U'', u'') is a solution to the problem. □

057B **Lemma 37.35.4.** *Let S be a scheme, and let s be a point of S. The category of elementary étale neighborhoods of (S, s) is cofiltered (see Categories, Definition 4.20.1).*

Proof. This is immediate from the definitions and Lemma 37.35.3. □

05KS **Lemma 37.35.5.** *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \operatorname{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods (U, u) of (S, s).

Proof. Let $\operatorname{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods (U, u) such that U is affine and $U \to S$ factors through $\operatorname{Spec}(A)$. In other words, we see that the right hand side is equal to $\operatorname{colim}_{(B,\mathfrak{q})} B$ where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Thus the lemma follows from Algebra, Lemma 10.155.7. □

We can lift étale neighbourhoods of points on fibres to the total space.

0CAS **Lemma 37.35.6.** *Let $X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $(V, v) \to (X_s, x)$ be an étale neighbourhood. Then there exists an étale neighbourhood $(U, u) \to (X, x)$ such that there exists a morphism $(U_s, u) \to (V, v)$ of étale neighbourhoods of (X_s, x) which is an open immersion.*

Proof. We may assume X, V, and S affine. Say the morphism $X \to S$ is given by $A \to B$ the point x by a prime $\mathfrak{q} \subset B$, the point s by $\mathfrak{p} = A \cap \mathfrak{q}$, and the morphism $V \to X_s$ by $B \otimes_A \kappa(\mathfrak{p}) \to C$. Since $\kappa(\mathfrak{p})$ is a localization of $A/\mathfrak{p}$ there exists an $f \in A$, $f \notin \mathfrak{p}$ and an étale ring map $B \otimes_A (A/\mathfrak{p})_f \to D$ such that

$$C = (B \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A (A/\mathfrak{p})_f} D$$

See Algebra, Lemma 10.143.3 part (9). After replacing A by A_f and B by B_f we may assume D is étale over $B \otimes_A A/\mathfrak{p} = B/\mathfrak{p}B$. Then we can apply Algebra, Lemma 10.143.10. This proves the lemma. □

37.36. Étale neighbourhoods and branches

0CB2 The number of (geometric) branches of a scheme at a point was defined in Properties, Section 28.15. In Varieties, Section 33.40 we related this to fibres of the normalization morphism. In this section we discuss a characterization of this number in terms of étale neighbourhoods.

0CB3 **Lemma 37.36.1.** *Let $R = \operatorname{colim} R_i$ be colimit of a directed system of rings whose transition maps are faithfully flat. Then the number of minimal primes of R taken as an element of $\{0, 1, 2, \ldots, \infty\}$ is the supremum of the numbers of minimal primes of the R_i.*

Proof. If $A \to B$ is a flat ring map, then $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ maps minimal primes to minimal primes by going down (Algebra, Lemma 10.39.19). If $A \to B$ is faithfully flat, then every minimal prime is the image of a minimal prime (by Algebra, Lemma 10.39.16 and 10.30.7). Hence the number of minimal primes of R_i is $\geq$ the number of minimal primes of $R_{i'}$ if $i \leq i'$. By Algebra, Lemma 10.39.20 each of the maps $R_i \to R$ is faithfully flat and we also see that the number of minimal primes of R is $\geq$ the number of minimal primes of R_i. Finally, suppose that $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are pairwise distinct minimal primes of R. Then we can find an i such that $R_i \cap \mathfrak{q}_1, \ldots, R_i \cap \mathfrak{q}_n$ are pairwise distinct (as sets and hence as prime ideals). This implies the lemma. □

0CB4 **Lemma 37.36.2.** *Let X be a scheme and $x \in X$ a point. Then*

1. *the number of branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,*
2. *the number of geometric branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,*
3. *X is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u, and*

(4) *X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u.*

Proof. Parts (3) and (4) follow from parts (1) and (2) via Properties, Lemma 28.15.6.

Proof of (1). Let $\mathrm{Spec}(A)$ be an affine open neighbourhood of x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. We may replace X by $\mathrm{Spec}(A)$ and it suffices to consider affine elementary étale neighbourhoods (U, u) in the supremum as they form a cofinal subsystem. Recall that the henselization $A^h_\mathfrak{p}$ is the colimit of the rings $B_\mathfrak{q}$ over the category of pairs $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ with $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$, see Algebra, Lemma 10.155.7. These pairs $(B, \mathfrak{q})$ correspond exactly to the affine elementary étale neighbourhoods (U, u) by the correspondence between rings and affine schemes. Observe that irreducible components of $\mathrm{Spec}(B)$ passing through $\mathfrak{q}$ are exactly the minimal prime ideals of $B_\mathfrak{q}$. The number of minimal primes of $A^h_\mathfrak{p}$ is the number of branches of X at x by Properties, Definition 28.15.4. Observe that the transition maps $B_\mathfrak{q} \to B'_{\mathfrak{q}'}$ in the system are all flat. Since a flat local ring map is faithfully flat (Algebra, Lemma 10.39.17) we see that the lemma follows from Lemma 37.36.1.

Proof of (2). The proof is the same as the proof of (1), except that we use Algebra, Lemma 10.155.11. There is a tiny difference: given a separable algebraic closure κ^{sep} of $\kappa(x)$ for every étale neighbourhood (U, u) we can choose a $\kappa(x)$-embedding $\phi : \kappa(u) \to \kappa^{sep}$ because $\kappa(u)/\kappa(x)$ is finite separable (Morphisms, Lemma 29.36.15). Hence we can look at the supremum over all triples (U, u, ϕ) where $(U, u) \to (X, x)$ is an affine étale neighbourhood and $\phi : \kappa(u) \to \kappa^{sep}$ is a $\kappa(x)$-embedding. These triples correspond exactly to the triples in Algebra, Lemma 10.155.11 and the rest of the proof is exactly the same. □

We will need a relative variant of the lemma above.

0CB5 **Lemma 37.36.3.** *Let $X \to S$ be a morphism of schemes and $x \in X$ a point with image s. Then*

(1) *the number of branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,*
(2) *the number of geometric branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,*
(3) *the fibre X_s is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of the fibre U_s passing through u, and*
(4) *X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U_s passing through u.*

Proof. Combine Lemmas 37.36.2 and 37.35.6. □

0DQ2 **Lemma 37.36.4.** *Let $X \to S$ be a smooth morphism of schemes. Let $x \in X$ with image $s \in S$. Then*

(1) *The number of geometric branches of X at x is equal to the number of geometric branches of S at s.*
(2) *If $\kappa(x)/\kappa(s)$ is a purely inseparable*[6] *extension of fields, then number of branches of X at x is equal to the number of branches of S at s.*

Proof. Follows immediately from More on Algebra, Lemma 15.107.8 and the definitions. □

37.37. Unramified and étale morphisms

0GS7 Sometimes unramified morphisms are automatically étale.

0GS8 **Lemma 37.37.1.** *Let $f : X \to Y$ be a morphism of schemes. Let $x \in X$ with image $y \in Y$. Assume*

(1) *Y is integral and geometrically unibranch at y,*
(2) *f is locally of finite type,*
(3) *there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y,*
(4) *f is unramified at x.*

Then f is étale at x.

Proof. We may replace X and Y by suitable affine open neighbourhoods of x and y. Then Y is the spectrum of a domain A and X is the spectrum of a finite type A-algebra B. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and $\mathfrak{p} \subset A$ the prime ideal corresponding to y. The local ring $A_\mathfrak{p} = \mathcal{O}_{Y,y}$ is geometrically unibranch. The ring map $A \to B$ is unramified at $\mathfrak{q}$. Also, the point x' in (3) corresponds to a prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ such that $A \cap \mathfrak{q}' = (0)$. It follows that $A_\mathfrak{p} \to B_\mathfrak{q}$ is injective. We conclude by More on Algebra, Lemma 15.108.2. □

0GS9 **Lemma 37.37.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume* [**Gro71**, Expose I, Corollary 9.11]

(1) *Y is integral and geometrically unibranch,*
(2) *at least one irreducible component of X dominates Y,*
(3) *f is unramified, and*
(4) *X is connected.*

Then f is étale and X is irreducible.

Proof. Let $X' \subset X$ be the irreducible component which dominates Y. This means that the generic point of X' maps to the generic point of Y (see for example Morphisms, Lemma 29.8.7). By Lemma 37.37.1 we see that f is étale at every point of X'. In particular, the open subscheme $U \subset X$ where f is étale contains X'. Note that every quasi-compact open of U has finitely many irreducible components, see Descent, Lemma 35.16.3. On the other hand since Y is geometrically unibranch and U is étale over Y, the scheme U is geometrically unibranch. In particular, through every point of U there passes at most one irreducible component. A simple topological argument now shows that $X' \subset U$ is both open and closed. Then of course X' is open and closed in X and by connectedness we find $X = U = X'$ as desired. □

0GSA **Lemma 37.37.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Let $x \in X$ with image $y \in Y$. Assume*

[6] In fact, it would suffice if $\kappa(x)$ is geometrically irreducible over $\kappa(s)$. If we ever need this we will add a detailed proof.

(1) *Y is integral and geometrically unibranch at y,*
(2) *$g \circ f$ is étale at x,*
(3) *there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y.*

Then f is étale at x and g is étale at y.

Proof. After replacing X by an open neighbourhood of x we may assume $g \circ f$ is étale. Then we find f is unramified by Morphisms, Lemmas 29.35.16 and 29.36.5. Hence f is étale at x by Lemma 37.37.1. Then by étale descent we see that g is étale at y, see for example Descent, Lemma 35.14.4. □

0GSB **Lemma 37.37.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume*

(1) *Y is integral and geometrically unibranch,*
(2) *$g \circ f$ is étale,*
(3) *every irreducible component of X dominates Y.*

Then f is étale and g is étale at every point in the image of f.

Proof. Immediate from the pointwise version Lemma 37.37.3. □

37.38. Slicing smooth morphisms

055S In this section we explain a result that roughly states that smooth coverings of a scheme S can be refined by étale coverings. The technique to prove this relies on a slicing argument.

057C **Lemma 37.38.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is smooth at x, and*
(2) *the image $\overline{dh}$ of dh in*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ smooth.

Proof. As f is smooth at x we may assume, after replacing X by an open neighbourhood of x that f is smooth. In particular we see that f is flat and locally of finite presentation. By Lemma 37.23.1 we already know there exists an open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 29.32.15 we have a short exact sequence
$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$
where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of D in U. As D is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator h to the section $\mathrm{d}h|_D$ of $i^*\Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes\kappa(x)$ we conclude that
$$\dim_{\kappa(x)} \big(\Omega_{D/S,x} \otimes \kappa(x)\big) = \dim_{\kappa(x)} \big(\Omega_{U/S,x} \otimes \kappa(x)\big) - 1.$$

By Morphisms, Lemma 29.34.14 we see that $\Omega_{U/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{U_s,x}) - 1$ by Algebra, Lemma 10.60.13 (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma 31.18.1) and then using Morphisms, Lemma 29.28.1. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at x by Morphisms, Lemma 29.34.14 again. After shrinking U we get that $D \to S$ is smooth and we win. □

057D **Lemma 37.38.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *the map*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}$$
has a nonzero kernel.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 10.131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By Algebra, Lemma 10.131.9 there is an exact sequence
$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\mathrm{d}} \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0.$$
Hence if (2) holds, there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 37.38.1. □

057E **Remark 37.38.3.** The second condition in Lemma 37.38.2 is necessary even if x is a closed point of a positive dimensional fibre. An example is the following: Let k be a field of characteristic $p > 0$ which is imperfect. Let $a \in k$ be an element which is not a pth power. Let $\mathfrak{m} = (x, y^p - a) \subset k[x, y]$. This corresponds to a closed point w of $X = \mathbf{A}^2_k$. Set $S = \mathbf{A}^1_k$ and let $f : X \to S$ be the morphism corresponding to $k[x] \to k[x, y]$. Then there does not exist any commutative diagram

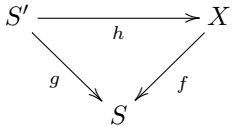

with g étale and w in the image of h. This is clear as the residue field extension $\kappa(w)/\kappa(f(w))$ is purely inseparable, but for any $s' \in S'$ with $g(s') = f(w)$ the extension $\kappa(s')/\kappa(f(w))$ would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark 37.38.3 does not happen. Here is the precise result.

057F **Lemma 37.38.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x,*
(2) *the residue field extension $\kappa(x)/\kappa(s)$ is separable, and*

(3) *x is not a generic point of X_s.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 10.131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 10.140.4 the map

$$\mathrm{d} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective. Assumption (3) implies that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. Thus there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 37.38.1. □

The subscheme Z constructed in the following lemma is really a complete intersection in an affine open neighbourhood of x. If we ever need this we will explicitly formulate a separate lemma stating this fact.

057G **Lemma 37.38.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *x is a closed point of X_s and $\kappa(s) \subset \kappa(x)$ is separable.*

Then there exists an immersion $Z \to X$ containing x such that

(1) *$Z \to S$ is étale, and*
(2) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We may and do replace X by an affine open neighbourhood of x such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 29.56.2. Thus after replacing X by U we see that X is quasi-finite and smooth over S, hence smooth of relative dimension 0 over S, hence étale over S. Moreover, the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 26.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 29.20.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions. Thus we may apply Lemma 37.38.2 to find a diagram

$$\begin{array}{ccccc} D & \longrightarrow & U & \longrightarrow & X \\ & \searrow & & \swarrow & \downarrow \\ & & S & & \end{array}$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 10.60.13 (also $D_s \subset X_s$ is effective Cartier, see

Divisors, Lemma 31.18.1) and then using Morphisms, Lemma 29.28.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof. □

055U **Lemma 37.38.6.** *Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of f. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a S-morphism $S' \to X$.*

First proof of Lemma 37.38.6. By assumption $X_s \neq \emptyset$. By Varieties, Lemma 33.25.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension of $\kappa(s)$. Hence by Lemma 37.38.5 there exists an immersion $Z \to X$ such that $Z \to S$ is étale and such that $x \in Z$. Take $(S', s') = (Z, x)$. □

Second proof of Lemma 37.38.6. Pick a point $x \in X$ with $f(x) = s$. Choose a diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\pi} & \mathbf{A}^d_V \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & V & & \end{array}$$

with π étale, $x \in U$ and $V = \mathrm{Spec}(R)$ affine, see Morphisms, Lemma 29.36.20. In particular $s \in V$. The morphism $\pi : U \to \mathbf{A}^d_V$ is open, see Morphisms, Lemma 29.36.13. Thus $W = \pi(U) \cap \mathbf{A}^d_s$ is a nonempty open subset of $\mathbf{A}^d_s$. Let $w \in W$ be a point with $\kappa(s) \subset \kappa(w)$ finite separable, see Varieties, Lemma 33.25.5. By Algebra, Lemma 10.114.1 there exist d elements $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$ which generate the maximal ideal corresponding to w in $\kappa(s)[x_1, \ldots, x_d]$. After replacing R by a principal localization we may assume there are $f_1, \ldots, f_d \in R[x_1, \ldots, x_d]$ which map to $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$. Consider the R-algebra

$$R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$$

and set $S' = \mathrm{Spec}(R')$. By construction we have a closed immersion $j : S' \to \mathbf{A}^d_V$ over V. By construction the fibre of $S' \to V$ over s is a single point s' whose residue field is finite separable over $\kappa(s)$. Let $\mathfrak{q}' \subset R'$ be the corresponding prime. By Algebra, Lemma 10.136.10 we see that $(R')_g$ is a relative global complete intersection over R for some $g \in R'$, $g \notin \mathfrak{q}$. Thus $S' \to V$ is flat and of finite presentation in a neighbourhood of s', see Algebra, Lemma 10.136.13. By construction the scheme theoretic fibre of $S' \to V$ over s is $\mathrm{Spec}(\kappa(s'))$. Hence it follows from Morphisms, Lemma 29.36.15 that $S' \to S$ is étale at s'. Set

$$S'' = U \times_{\pi, \mathbf{A}^d_V, j} S'.$$

By construction there exists a point $s'' \in S''$ which maps to s' via the projection $p : S'' \to S'$. Note that p is étale as the base change of the étale morphism π, see Morphisms, Lemma 29.36.4. Choose a small affine neighbourhood $S''' \subset S''$ of s'' which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. □

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

055V **Lemma 37.38.7.** *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i\in I}$ be a smooth covering of S, see Topologies, Definition 34.5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j\in J}$ (see Topologies, Definition 34.4.1) which refines (see Sites, Definition 7.8.1) $\mathcal{U}$.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 37.38.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of S that refines $\mathcal{U}$. □

0EY4 **Lemma 37.38.8.** *Let $f : X \to S$ be a smooth morphism of schemes. Then there exists an étale covering $\{U_i \to X\}_{i\in I}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ where $V_i \to S$ is étale and $U_i \to V_i$ is a smooth morphism of affine schemes, which has a section, and has geometrically connected fibres.*

Proof. Let $s \in S$. By Varieties, Lemma 33.25.6 the set of closed points $x \in X_s$ such that $\kappa(x)/\kappa(s)$ is separable is dense in X_s. Thus it suffices to construct an étale morphism $U \to X$ with x in the image such that $U \to S$ factors in the manner described in the lemma. To do this, choose an immersion $Z \to X$ passing through x such that $Z \to S$ is étale (Lemma 37.38.5). After replacing S by Z and X by $Z \times_S X$ we see that we may assume $X \to S$ has a section $\sigma : S \to X$ with $\sigma(s) = x$. Then we can first replace S by an affine open neighbourhood of s and next replace X by an affine open neighbourhood of x. Then finally, we consider the subset $X^0 \subset X$ of Section 37.29. By Lemmas 37.29.6 and 37.29.4 this is a retrocompact open subscheme containing σ such that the fibres $X^0 \to S$ are geometrically connected. If X^0 is not affine, then we choose an affine open $U \subset X^0$ containing x. Since $X^0 \to S$ is smooth, the image of U is open. Choose an affine open neighbourhood $V \subset S$ of s contained in $\sigma^{-1}(U)$ and in the image of $U \to S$. Finally, the reader sees that $U \cap f^{-1}(V) \to V$ has all the desired properties. For example $U \cap f^{-1}(V)$ is equal to $U \times_S V$ is affine as a fibre product of affine schemes. Also, the geometric fibres of $U \cap f^{-1}(V) \to V$ are nonempty open subschemes of the irreducible fibres of $X^0 \to S$ and hence connected. Some details omitted. □

37.39. Étale neighbourhoods and Artin approximation

0CAT In this section we prove results of the form: if two pointed schemes have isomorphic complete local rings, then they have isomorphic étale neighbourhoods. We will rely on Popescu's theorem, see Smoothing Ring Maps, Theorem 16.12.1.

0CAU **Lemma 37.39.1.** *Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map*

$$\varphi : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}^\wedge_{X,x}$$

For every $N \geq 1$ there exists an elementary étale neighbourhood $(U, u) \to (X, x)$ and an S-morphism $f : U \to Y$ mapping u to y such that the diagram

$$\begin{array}{ccc} \mathcal{O}^\wedge_{X,x} & \longrightarrow & \mathcal{O}^\wedge_{U,u} \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & \xrightarrow{f_u^\sharp} & \mathcal{O}_{U,u} \end{array}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. The question is local on X hence we may assume X, Y, S are affine. Say $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. Write $B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. The local ring $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ is a G-ring by More on Algebra, Proposition 15.50.10. The map φ is a map

$$B_{\mathfrak{q}}^\wedge \longrightarrow A_{\mathfrak{p}}^\wedge$$

where $\mathfrak{q} \subset B$ is the prime corresponding to y. Let $a_1, \ldots, a_n \in A_{\mathfrak{p}}^\wedge$ be the images of $x_1, \ldots, x_n$ via $R[x_1, \ldots, x_n] \to B \to B_{\mathfrak{q}}^\wedge \to A_{\mathfrak{p}}^\wedge$. Then we can apply Smoothing Ring Maps, Lemma 16.13.4 to get an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ and $b_1, \ldots, b_n \in A'$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $a_i - b_i \in (\mathfrak{p}')^N (A'_{\mathfrak{p}'})^\wedge$, and $f_j(b_1, \ldots, b_n) = 0$ for $j = 1, \ldots, n$. This determines an R-algebra map $B \to A'$ by sending the class of x_i to $b_i \in A'$. This finishes the proof by taking $U = \mathrm{Spec}(A') \to \mathrm{Spec}(B)$ as the morphism f and $u = \mathfrak{p}'$. □

0CAV **Lemma 37.39.2.** *Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume we have an $\mathcal{O}_{S,s}$-algebra isomorphism*

$$\varphi : \mathcal{O}_{Y,y}^\wedge \longrightarrow \mathcal{O}_{X,x}^\wedge$$

between the complete local rings. Then for every $N \geq 1$ there exists morphisms

$$(X, x) \leftarrow (U, u) \to (Y, y)$$

of pointed schemes over S such that both arrows define elementary étale neighbourhoods and such that the diagram

$$\begin{array}{ccc} & \mathcal{O}_{U,u}^\wedge & \\ \nearrow & & \nwarrow \\ \mathcal{O}_{Y,y}^\wedge & \xrightarrow{\varphi} & \mathcal{O}_{X,x}^\wedge \end{array}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. We may assume $N \geq 2$. Apply Lemma 37.39.1 to get $(U, u) \to (X, x)$ and $f : (U, u) \to (Y, y)$. We claim that f is étale at u which will finish the proof. In fact, we will show that the induced map $\mathcal{O}_{Y,y}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism. Having proved this, Lemma 37.12.1 will show that f is smooth at u and of course f is unramified at u as well, so Morphisms, Lemma 29.36.5 tells us f is étale at u. For a local ring $(R, \mathfrak{m})$ we set $\mathrm{Gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. To prove the claim we look at the induced diagram of graded rings

$$\begin{array}{ccc} & \mathrm{Gr}_{\mathfrak{m}_u}(\mathcal{O}_{U,u}) & \\ \nearrow & & \nwarrow \\ \mathrm{Gr}_{\mathfrak{m}_y}(\mathcal{O}_{Y,y}) & \xrightarrow{\varphi} & \mathrm{Gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \end{array}$$

Since $N \geq 2$ this diagram is actually commutative as the displayed graded algebras are generated in degree 1! By assumption the lower arrow is an isomorphism. By More on Algebra, Lemma 15.43.9 (for example) the map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism and hence the north-west arrow in the diagram is an isomorphism.

We conclude that f induces an isomorphism $\mathrm{Gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \to \mathrm{Gr}_{\mathfrak{m}_y}(\mathcal{O}_{U,u})$. Using induction and the short exact sequences

$$0 \to \mathrm{Gr}^n_{\mathfrak{m}}(R) \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0$$

for both local rings we conclude (from the snake lemma) that f induces isomorphisms $\mathcal{O}_{Y,y}/\mathfrak{m}_y^n \to \mathcal{O}_{U,u}/\mathfrak{m}_u^n$ for all n which is what we wanted to show. □

0GDX **Lemma 37.39.3.** *Let $X \to S$, $Y \to T$, x, s, y, t, σ, y_σ, and φ be given as follows: we have morphisms of schemes*

$$\begin{array}{ccc} X & Y & \\ \downarrow & \downarrow & \text{with points} \\ S & T & \end{array} \qquad \begin{array}{cc} x & y \\ \downarrow & \downarrow \\ s & t \end{array}$$

Here S is locally Noetherian and T is of finite type over $\mathbf{Z}$. The morphisms $X \to S$ and $Y \to T$ are locally of finite type. The local ring $\mathcal{O}_{S,s}$ is a G-ring. The map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}^\wedge_{S,s}$$

is a local homomorphism. Set $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$. Next, y_σ is a point of Y_σ mapping to y and the closed point of $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$. Finally

$$\varphi : \mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma, y_\sigma}$$

is an isomorphism of $\mathcal{O}^\wedge_{S,s}$-algebras. In this situation there exists a commutative diagram

$$\begin{array}{ccccccc} X & \longleftarrow & W & \longrightarrow & Y \times_{T,\tau} V & \longrightarrow & Y \\ \downarrow & & & \swarrow & & & \downarrow \\ S & \longleftarrow & & V & \xrightarrow{\tau} & & T \end{array}$$

of schemes and points $w \in W$, $v \in V$ such that

(1) *$(V, v) \to (S, s)$ is an elementary étale neighbourhood,*
(2) *$(W, w) \to (X, x)$ is an elementary étale neighbourhood, and*
(3) *$\tau(v) = t$.*

Let $y_\tau \in Y \times_T V$ correspond to y_σ via the identification $(Y_\sigma)_s = (Y \times_T V)_v$. Then

(4) *$(W, w) \to (Y \times_{T,\tau} V, y_\tau)$ is an elementary étale neighbourhood.*

Proof. Denote $X_\sigma = X \times_S \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ and $x_\sigma \in X_\sigma$ the unique point lying over x. Observe that $\mathcal{O}^\wedge_{S,s}$ is a G-ring by More on Algebra, Proposition 15.50.6. By Lemma 37.39.2 we can choose

$$(X_\sigma, x_\sigma) \leftarrow (U, u) \to (Y_\sigma, y_\sigma)$$

where both arrows are elementary étale neighbourhoods.

After replacing S by an open neighbourhood of s, we may assume $S = \mathrm{Spec}(R)$ is affine. Since $\mathcal{O}_{S,s}$ is a G-ring by Smoothing Ring Maps, Theorem 16.12.1 the ring $\mathcal{O}^\wedge_{S,s}$ is a filtered colimit of smooth R-algebras. Thus we can write

$$\mathrm{Spec}(\mathcal{O}^\wedge_{S,s}) = \lim S_i$$

as a directed limit of affine schemes S_i smooth over S. Denote $s_i \in S_i$ the image of the closed point of $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$. Observe that $\kappa(s) = \kappa(s_i)$. Set $X_i = X \times_S S_i$ and denote $x_i \in X_i$ the unique point mapping to x. Note that $\kappa(x) = \kappa(x_i)$. Since T is of finite type over $\mathbf{Z}$ by Limits, Proposition 32.6.1 we can choose an i

and a morphism $\sigma_i : (S_i, s_i) \to (T, t)$ of pointed schemes whose composition with $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s}) \to S_i$ is equal to σ. Set $Y_i = Y \times_T S_i$ and denote y_i the image of y_σ. Note that $\kappa(y_i) = \kappa(y_\sigma)$. By Limits, Lemma 32.10.1 we can choose an i and a diagram

$$\begin{array}{ccccc} X_i & \longleftarrow & U_i & \longrightarrow & Y_i \\ & \searrow & \downarrow & \swarrow & \\ & & S_i & & \end{array}$$

whose base change to $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ recovers $X_\sigma \leftarrow U \to Y_\sigma$. By Limits, Lemma 32.8.10 after increasing i we may assume the morphisms $X_i \leftarrow U_i \to Y_i$ are étale. Let $u_i \in U_i$ be the image of u. Then $u_i \mapsto x_i$ hence $\kappa(x) = \kappa(x_\sigma) = \kappa(u) \supset \kappa(u_i) \supset \kappa(x_i) = \kappa(x)$ and we see that $\kappa(u_i) = \kappa(x_i)$. Hence $(X_i, x_i) \leftarrow (U_i, u_i)$ is an elementary étale neighbourhood. Since also $\kappa(y_i) = \kappa(y_\sigma) = \kappa(u)$ we see that also $(U_i, u_i) \to (Y_i, y_i)$ is an elementary étale neighbourhood.

At this point we have constructed a diagram

$$\begin{array}{ccccccccc} X & \longleftarrow & X \times_S S_i & \longleftarrow & U_i & \longrightarrow & Y \times_T S_i & \longrightarrow & Y \\ \downarrow & & & \searrow & \downarrow & \swarrow & & & \downarrow \\ S & \longleftarrow & & & S_i & & \longrightarrow & & T \end{array}$$

as in the statement of the lemma, except that $S_i \to S$ is smooth. By Lemma 37.38.5 and after shrinking S_i we can assume there exists a closed subscheme $V \subset S_i$ passing through s_i such that $V \to S$ is étale. Setting W equal to the scheme theoretic inverse image of V in U_i we conclude. □

We strongly encourage the reader to skip the rest of this section.

0CAW **Lemma 37.39.4.** *Consider a diagram*

$$\begin{array}{ccc} X & & Y \\ \downarrow & & \downarrow \\ S & \longleftarrow & T \end{array} \quad \text{with points} \quad \begin{array}{ccc} x & & y \\ \downarrow & & \downarrow \\ s & \longleftarrow & t \end{array}$$

where S be a locally Noetherian scheme and the morphisms are locally of finite type. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}^\wedge_{S,s}$$

and a local $\mathcal{O}_{S,s}$-algebra map

$$\varphi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma, y_\sigma}$$

where $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ and y_σ is the unique point of Y_σ lying over y. For every $N \geq 1$ there exists a commutative diagram

$$\begin{array}{ccccccccc} X & \longleftarrow & X \times_S V & \xleftarrow[f]{} & W & \longrightarrow & Y \times_{T,\tau} V & \longrightarrow & Y \\ \downarrow & & & \searrow & \downarrow & \swarrow & & & \downarrow \\ S & \longleftarrow & & & V & & \xrightarrow{\tau} & & T \end{array}$$

of schemes over S and points $w \in W$, $v \in V$ such that

(1) *$v \mapsto s$, $\tau(v) = t$, $f(w) = (x, v)$, and $w \mapsto (y, v)$,*

(2) $(V, v) \to (S, s)$ *is an elementary étale neighbourhood,*
(3) *the diagram*

$$\begin{array}{ccc} \mathcal{O}_{S,s}^\wedge & \longrightarrow & \mathcal{O}_{V,v}^\wedge \\ \uparrow{\scriptstyle \sigma} & & \uparrow \\ \mathcal{O}_{T,t} & \xrightarrow{\tau_v^\sharp} & \mathcal{O}_{V,v} \end{array}$$

commutes module $\mathfrak{m}_v^N$,
(4) $(W, w) \to (Y \times_{T,\tau} V, (y, v))$ *is an elementary étale neighbourhood,*
(5) *the diagram*

$$\begin{array}{ccccccc} \mathcal{O}_{X,x} & \xrightarrow[\varphi]{} & \mathcal{O}_{Y_\sigma, y_\sigma}^\wedge & \longrightarrow & \mathcal{O}_{Y_\sigma, y_\sigma}/\mathfrak{m}_{y_\sigma}^N & = & \mathcal{O}_{Y\times_{T,\tau}V,(y,v)}/\mathfrak{m}_{(y,v)}^N \\ \| & & & & & & \cong \downarrow \\ \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X\times_S V,(x,v)} & \xrightarrow{f_w^\sharp} & \mathcal{O}_{W,w} & \longrightarrow & \mathcal{O}_{W,w}/\mathfrak{m}_w^N \end{array}$$

commutes. The equality comes from the fact that Y_σ *and* $Y \times_{T,\tau} V$ *are canonically isomorphic over* $\mathcal{O}_{V,v}/\mathfrak{m}_v^N = \mathcal{O}_{S,s}/\mathfrak{m}_s^N$ *by parts (2) and (3).*

Proof. After replacing X, S, T, Y by affine open subschemes we may assume the diagram in the statement of the lemma comes from applying Spec to a diagram

$$\begin{array}{ccc} A & & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & C \end{array} \quad \text{with primes} \quad \begin{array}{ccc} \mathfrak{p}_A & & \mathfrak{p}_B \\ | & & | \\ \mathfrak{p}_R & \longrightarrow & \mathfrak{p}_C \end{array}$$

of Noetherian rings and finite type ring maps. In this proof every ring E will be a Noetherian R-algebra endowed with a prime ideal $\mathfrak{p}_E$ lying over $\mathfrak{p}_R$ and all ring maps will be R-algebra maps compatible with the given primes. Moreover, if we write $E^\wedge$ we mean the completion of the localization of E at $\mathfrak{p}_E$. We will also use without further mention that an étale ring map $E_1 \to E_2$ such that $\kappa(\mathfrak{p}_{E_1}) = \kappa(\mathfrak{p}_{E_2})$ induces an isomorphism $E_1^\wedge = E_2^\wedge$ by More on Algebra, Lemma 15.43.9.

With this notation σ and φ correspond to ring maps

$$\sigma : C \to R^\wedge \quad \text{and} \quad \varphi : A \longrightarrow (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

Here is a picture

$$\begin{array}{ccccccc} & & & & \overset{\varphi}{\frown} & & \\ A & & B & \longrightarrow & B \otimes_{C,\sigma} R^\wedge & \longrightarrow & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\ \uparrow & & \uparrow & & \uparrow & \nearrow & \\ R & \longrightarrow & C & \xrightarrow{\sigma} & R^\wedge & & \end{array}$$

Observe that $R^\wedge$ is a G-ring by More on Algebra, Proposition 15.50.6. Thus $B \otimes_{C,\sigma} R^\wedge$ is a G-ring by More on Algebra, Proposition 15.50.10. By Lemma 37.39.1 (translated into algebra) there exists an étale ring map $B \otimes_{C,\sigma} R^\wedge \to B'$ inducing an isomorphism $\kappa(\mathfrak{p}_{B\otimes_{C,\sigma}R^\wedge}) \to \kappa(\mathfrak{p}_{B'})$ and an R-algebra map $A \to B'$ such that the composition

$$A \to B' \to (B')^\wedge = (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

is the same as φ modulo $(\mathfrak{p}_{(B\otimes_{C,\sigma}R^\wedge)^\wedge})^N$. Thus we may replace φ by this composition because the only way φ enters the conclusion is via the commutativity requirement in part (5) of the statement of the lemma. Picture:

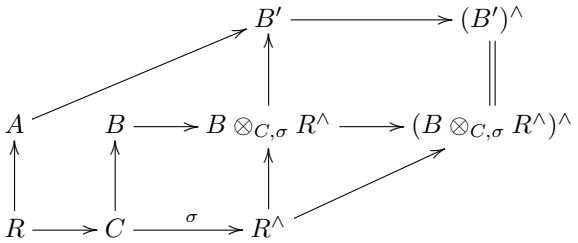

Next, we use that $R^\wedge$ is a filtered colimit of smooth R-algebras (Smoothing Ring Maps, Theorem 16.12.1) because $R_{\mathfrak{p}_R}$ is a G-ring by assumption. Since C is of finite presentation over R we get a factorization

$$C \to R' \to R^\wedge$$

for some $R \to R'$ smooth, see Algebra, Lemma 10.127.3. After increasing R' we may assume there exists an étale $B \otimes_C R'$-algebra B'' whose base change to $B \otimes_{C,\sigma} R^\wedge$ is B', see Algebra, Lemma 10.143.3. Then B' is the filtered colimit of these B'' and we conclude that after increasing R' we may assume there is an R-algebra map $A \to B''$ such that $A \to B'' \to B'$ is the previously constructed map (same reference as above). Picture

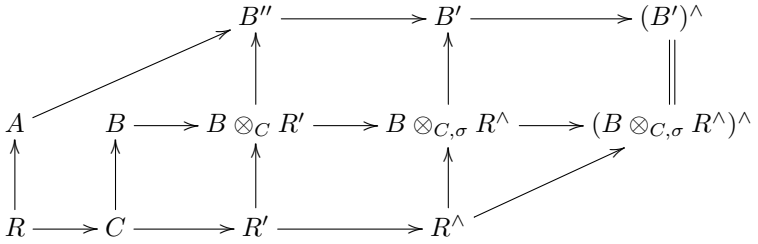

and

$$B' = B'' \otimes_{(B\otimes_C R')} (B \otimes_{C,\sigma} R^\wedge)$$

This means that we may replace C by R', $\sigma : C \to R^\wedge$ by $R' \to R^\wedge$, and B by B'' so that we simplify to the diagram

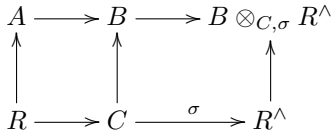

with φ equal to the composition of the horizontal arrows followed by the canonical map from $B \otimes_{C,\sigma} R^\wedge$ to its completion. The final step in the proof is to apply Lemma 37.39.1 (or its proof) one more time to $\mathrm{Spec}(C)$ and $\mathrm{Spec}(R)$ over $\mathrm{Spec}(R)$ and the map $C \to R^\wedge$. The lemma produces a ring map $C \to D$ such that $R \to D$ is étale, such that $\kappa(\mathfrak{p}_R) = \kappa(\mathfrak{p}_D)$, and such that

$$C \to D \to D^\wedge = R^\wedge$$

is equal to $\sigma : C \to R^\wedge$ modulo $(\mathfrak{p}_{R^\wedge})^N$. Then we can take

$$V = \mathrm{Spec}(D) \quad \text{and} \quad W = \mathrm{Spec}(B \otimes_C D)$$

as our solution to the problem posed by the lemma. Namely the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B \otimes_{C,\sigma} R^\wedge & \longrightarrow & B \otimes_{C,\sigma} R^\wedge/(\mathfrak{p}_{R^\wedge})^N & = & B \otimes_C D/(\mathfrak{p}_D)^N \\ \| & & & & & & \| \\ A & \longrightarrow & A \otimes_R D & \longrightarrow & B \otimes_R D & \longrightarrow & B \otimes_C D/(\mathfrak{p}_D)^N \end{array}$$

commutes because $C \to D \to D^\wedge = R^\wedge$ is equal to σ modulo $(\mathfrak{p}_{R^\wedge})^N$. This proves part (5) and the other properties are immediate from the construction. □

0CAX **Lemma 37.39.5.** *Let $T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}^\wedge_{S,s}$ be a local $\mathcal{O}_{S,s}$-algebra map. For every $N \geq 1$ there exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}^\wedge_{S,s}$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the maps σ and σ' agree modulo $\mathfrak{m}_s^N$.*

Proof. We may assume S and T are affine. Say $S = \mathrm{Spec}(R)$ and $T = \mathrm{Spec}(C)$. Let $c_1, \ldots, c_n \in C$ be generators of C as an R-algebra. Let $\mathfrak{p} \subset R$ be the prime ideal corresponding to s. Say $\mathfrak{p} = (f_1, \ldots, f_m)$. After replacing R by a principal localization (to clear denominators in $R_\mathfrak{p}$) we may assume there exist $r_1, \ldots, r_n \in R$ and $a_{i,I} \in \mathcal{O}^\wedge_{S,s}$ where $I = (i_1, \ldots, i_m)$ with $\sum i_j = N$ such that

$$\sigma(c_i) = r_i + \sum\nolimits_I a_{i,I} f_1^{i_1} \ldots f_m^{i_m}$$

in $\mathcal{O}^\wedge_{S,s}$. Then we consider

$$C' = C[t_{i,I}] / \left(c_i - r_i - \sum\nolimits_I t_{i,I} f_1^{i_1} \ldots f_m^{i_m} \right)$$

with $\mathfrak{p}' = \mathfrak{p}C' + (t_{i,I})$ and factorization of $\sigma : C \to \mathcal{O}^\wedge_{S,s}$ through C' given by sending $t_{i,I}$ to $a_{i,I}$. Taking $T' = \mathrm{Spec}(C')$ works because any σ' as in the statement of the lemma will send c_i to r_i modulo the maximal ideal to the power N. □

0CAY **Lemma 37.39.6.** *Let $Y \to T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}^\wedge_{S,s}$ be a local $\mathcal{O}_{S,s}$-algebra map. There exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}^\wedge_{S,s}$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the closed immersions*

$$Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s}) = Y_\sigma \longleftarrow Y_t \longrightarrow Y_{\sigma'} = Y \times_{T,\sigma'} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$$

have isomorphic conormal algebras.

Proof. A useful observation is that $\kappa(s) = \kappa(t)$ by the existence of σ. Observe that the statement makes sense as the fibres of Y_σ and $Y_{\sigma'}$ over $s \in \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ are both canonically isomorphic to Y_t. We will think of the property "σ' factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$" as a constraint on σ'. If we have several such constraints, say given by $(T'_i, t'_i) \to (T, t)$, $i = 1, \ldots, n$ then we can combined them by considering $(T'_1 \times_T \ldots \times_T T'_n, (t'_1, \ldots, t'_n)) \to (T, t)$. We will use this without further mention in the following.

By Lemma 37.39.5 we can assume that any σ' as in the statement of the lemma is the same as σ modulo $\mathfrak{m}_s^2$. Note that the conormal algebra of Y_t in Y_σ is just the

quasi-coherent graded $\mathcal{O}_{Y_t}$-algebra

$$\bigoplus\nolimits_{n\geq 0} \mathfrak{m}_s^n\mathcal{O}_{Y_\sigma}/\mathfrak{m}_s^{n+1}\mathcal{O}_{Y_\sigma}$$

and similarly for $Y_{\sigma'}$. Since σ and σ' agree modulo $\mathfrak{m}_s^2$ we see that these two algebras are the same in degrees 0 and 1. On the other hand, these conormal algebras are generated in degree 1 over degree 0. Hence if there is an isomorphism extending the isomorphism just constructed in degrees 0 and 1, then it is unique.

We may assume S and T are affine. Let $Y = Y_1 \cup \ldots \cup Y_n$ be an affine open covering. If we can construct $(T_i', t_i') \to (T, t)$ as in the lemma such that the desired isomorphism (see previous paragraph) exists for $Y_i \to T \to S$ and σ, then these glue by uniqueness to prove the result for $Y \to T$. Thus we may assume Y is affine.

Write $S = \mathrm{Spec}(R)$, $T = \mathrm{Spec}(C)$, and $Y = \mathrm{Spec}(B)$. Choose a presentation $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Denote $R^\wedge = \mathcal{O}_{S,s}^\wedge$. Let $a_{kj} \in R^\wedge[x_1, \ldots, x_n]$ be polynomials such that

$$\sum\nolimits_{j=1,\ldots,m} a_{kj}\sigma(f_j) = 0, \quad \text{for } k = 1, \ldots, K$$

is a set of generators for the module of relations among the $\sigma(f_j) \in R^\wedge[x_1, \ldots, x_n]$. Thus we have an exact sequence

(37.39.6.1)

0CAZ
$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma} R^\wedge \to 0$$

Let c be an integer which works in the Artin-Rees lemma for both the first and the second map in this sequence and the ideal $\mathfrak{m}_{R^\wedge}R^\wedge[x_1, \ldots, x_n]$ as defined in More on Algebra, Section 15.4. Write

$$a_{kj} = \sum\nolimits_{I\in\Omega} a_{kj,I}x^I \quad \text{and} \quad f_j = \sum\nolimits_{I\in\Omega} f_{j,I}x^I$$

in multiindex notation where $a_{kj,I} \in R^\wedge$, $f_{j,I} \in C$, and Ω a finite set of multiindices. Then we see that

$$\sum\nolimits_{j=1,\ldots,m,\ I,I'\in\Omega,\ I+I'=I''} a_{kj,I}\sigma(f_{j,I'}) = 0, \quad I'' \text{ a multiindex}$$

in $R^\wedge$. Thus we take

$$C' = C[t_{jk,I}]/\left(\sum\nolimits_{j=1,\ldots,m,\ I,I'\in\Omega,\ I+I'=I''} t_{kj,I}f_{j,I'},\ I'' \text{ a multiindex}\right)$$

Then σ factors through a map $\tilde{\sigma} : C' \to R^\wedge$ sending $t_{kj,I}$ to $a_{jk,I}$. Thus $T' = \mathrm{Spec}(C')$ comes with a point $t' \in T'$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$. Let $t_{kj} = \sum t_{kj,I}x^I$ in $C'[x_1, \ldots, x_n]$. Then we see that we have a complex

(37.39.6.2)

0CB0
$$C'[x_1, \ldots, x_n]^{\oplus K} \to C'[x_1, \ldots, x_n]^{\oplus m} \to C'[x_1, \ldots, x_n] \to B \otimes_C C' \to 0$$

which is exact at $C'[x_1, \ldots, x_n]$ and whose base change by $\tilde{\sigma}$ gives (37.39.6.1).

By Lemma 37.39.5 we can find a further morphism $(T'', t'') \to (T', t')$ such that $\tilde{\sigma}$ factors through $\mathcal{O}_{T',t'} \to \mathcal{O}_{T'',t''}$ and such that if $\sigma' : C \to R^\wedge$ factors through $\mathcal{O}_{T'',t''}$, then the induced map $\tilde{\sigma}' : C' \to R^\wedge$ agrees modulo $\mathfrak{m}_s^{c+1}$ with $\tilde{\sigma}$. Thus if σ' is such a map, then we obtain a complex

$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma'} R^\wedge \to 0$$

over $R^\wedge[x_1, \ldots, x_n]$ by applying $\tilde{\sigma}'$ to the polynomials t_{kj} and f_j. In other words, this is the base change of the complex (37.39.6.2) by $\tilde{\sigma}'$. The matrices defining

this complex are congruent modulo $\mathfrak{m}_s^{c+1}$ to the matrices defining the complex (37.39.6.1) because $\tilde{\sigma}$ and $\tilde{\sigma}'$ are congruent modulo $\mathfrak{m}_s^{c+1}$. Since (37.39.6.1) is exact, we can apply More on Algebra, Lemma 15.4.2 to conclude that

$$\mathrm{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma'} R^\wedge) \cong \mathrm{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma} R^\wedge)$$

as desired. □

0CB1 **Lemma 37.39.7.** *With notation an assumptions as in Lemma 37.39.4 assume that φ induces an isomorphism on completions. Then we can choose our diagram such that f is étale.*

Proof. We may assume $N \geq 2$ and we may replace (T, t) with (T', t') as in Lemma 37.39.6. Since $(V, v) \to (S, s)$ is an elementary étale neighbourhood, so is $(X \times_S V, (x, v)) \to (X, x)$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{X \times_S V,(x,v)}$ induces an isomorphism on completions by More on Algebra, Lemma 15.43.9. We claim $\mathcal{O}_{X,x} \to \mathcal{O}_{W,w}$ induces an isomorphism on completions. Having proved this, Lemma 37.12.1 will show that f is smooth at w and of course f is unramified at u as well, so Morphisms, Lemma 29.36.5 tells us f is étale at w.

First we use the commutativity in part (5) of Lemma 37.39.4 to see that for $i \leq N$ there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Gr}^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) & \xrightarrow{\varphi} & \mathrm{Gr}^i_{\mathfrak{m}_{y_\sigma}}(\mathcal{O}^\wedge_{Y_\sigma,y_\sigma}) & = & \mathrm{Gr}^i_{\mathfrak{m}_{(y,v)}}(\mathcal{O}_{Y \times_{T,\tau} V,(y,v)}) \\
\| & & & & \cong \downarrow \\
\mathrm{Gr}^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) & \xrightarrow{\cong} & \mathrm{Gr}^i_{\mathfrak{m}_{(x,v)}}(\mathcal{O}_{X \times_S V,(x,v)}) & \xrightarrow{f_w^\sharp} & \mathrm{Gr}^i_{\mathfrak{m}_w}(\mathcal{O}_{W,w})
\end{array}$$

This implies that $f_w^\sharp$ defines an isomorphism $\kappa(x) \to \kappa(w)$ on residue fields and an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_w/\mathfrak{m}_w^2$ on cotangent spaces. Hence $f_w^\sharp$ defines a surjection $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ on complete local rings.

By Lemma 37.39.6 there is an isomorphism of $\mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}_{(Y \times_{T,\tau} V,(y,v)})$ with $\mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}_{Y_\sigma,y_\sigma})$. This follows by taking stalks of the isomorphism of conormal sheaves at the point y. Since our local rings are Noetherian taking associated graded with respect to $\mathfrak{m}_s$ commutes with completion because completion with respect to an ideal is an exact functor on finite modules over Noetherian rings. This produces the right vertical isomorphism in the diagram of graded rings

$$\begin{array}{ccc}
\mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}^\wedge_{W,w}) & \longleftarrow & \mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}^\wedge_{(Y \times_{T,\tau} V,(y,v)}) \\
\uparrow & & \uparrow \cong \\
\mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}^\wedge_{X,x}) & \xrightarrow{\varphi} & \mathrm{Gr}_{\mathfrak{m}_s}(\mathcal{O}^\wedge_{Y_\sigma,y_\sigma})
\end{array}$$

We do not claim the diagram commutes. By the result of the previous paragraph the left arrow is surjective. The other three arrows are isomorphisms. It follows that the left arrow is a surjective map between isomorphic Noetherian rings. Hence it is an isomorphism by Algebra, Lemma 10.31.10 (you can argue this directly using Hilbert functions as well). In particular $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ must be injective as well as surjective which finishes the proof. □

37.40. Finite free locally dominates étale

04HE In this section we explain a result that roughly states that étale coverings of a scheme S can be refined by Zariski coverings of finite locally free covers of S.

02LG **Lemma 37.40.1.** *Let S be a scheme. Let $s \in S$. Let $f : (U, u) \to (S, s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subset S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subset T$ and a commutative diagram*

$$\begin{array}{ccccc} T & \leftarrow & W_t & \xrightarrow{h_t} & U \\ \downarrow{\scriptstyle \pi} & & & \swarrow & \downarrow \\ V & & \longrightarrow & & S \end{array}$$

with $h_t(t) = u$.

Proof. The problem is local on S hence we may replace S by any open neighbourhood of s. We may also replace U by an open neighbourhood of u. Hence, by Morphisms, Lemma 29.36.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 10.144.5. □

02LH **Lemma 37.40.2.** *Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:*

$$\begin{array}{ccccc} & & \coprod T_i & & \\ & \swarrow & & \searrow & \\ T & & & & U \\ & \searrow^{\pi} & & \swarrow_{f} & \\ & & S & & \end{array}$$

where the south-west arrow is a Zariski-covering.

Proof. This is a restatement of Algebra, Lemma 10.144.6. □

02LI **Remark 37.40.3.** In terms of topologies Lemmas 37.40.1 and 37.40.2 mean the following. Let S be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of S. There exists a Zariski open covering $S = \bigcup V_j$, for each j a finite locally free, surjective morphism $W_j \to V_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with π finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for an affine scheme X, see [**Hoo82**].

37.41. Étale localization of quasi-finite morphisms

04HF Now we come to a series of lemmas around the theme "quasi-finite morphisms become finite after étale localization". The general idea is the following. Suppose

given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of s in S. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

02LJ (37.41.0.1)
$$\begin{array}{ccccc} V & \longrightarrow & X_U & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & U & \xrightarrow{\varphi} & S \end{array}$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens V? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

02LK **Lemma 37.41.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that*

(1) *f is locally of finite type, and*
(2) *$x \in X_s$ is isolated[7].*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *an open subscheme $V \subset X_U$ (see 37.41.0.1)*

such that

(i) *$V \to U$ is a finite morphism,*
(ii) *there is a unique point v of V mapping to u in U, and*
(iii) *the point v maps to x under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.*

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$ the triple (U', u', V') satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that x is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for f. Hence we reduce to the case where f is a morphism of affine schemes. This case is Algebra, Lemma 10.145.2. □

In the preceding and following lemma we do not assume that the morphism f is separated. This means that the opens V, V_i created in them are not necessarily closed in X_U. Moreover, if we choose the neighbourhood U to be affine, then each V_i is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

02LL **Lemma 37.41.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an open subscheme $V_i \subset X_U$,*

[7] In the presence of (1) this means that f is quasi-finite at x, see Morphisms, Lemma 29.20.6.

such that for each i we have

(i) *$V_i \to U$ is a finite morphism,*
(ii) *there is a unique point v_i of V_i mapping to u in U, and*
(iii) *the point v_i maps to x_i in X and $\kappa(x_i) = \kappa(v_i)$.*

Proof. We will use induction on n. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n-1$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also x'_n is isolated in X_u. Hence by Lemma 37.41.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for x'_n and hence for x_n. By the final assertion of Lemma 37.41.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. □

If we allow a nontrivial field extension $\kappa(u)/\kappa(s)$, i.e., general étale neighbourhoods, then we can split the points as follows.

02LM **Lemma 37.41.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an integer m_i and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$*

such that we have

(i) *each $V_{i,j} \to U$ is a finite morphism,*
(ii) *there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to u in U with $\kappa(u) \subset \kappa(v_{i,j})$ finite purely inseparable,*
(iv) *if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and*
(iii) *the points $v_{i,j}$ map to x_i in X and no other points of $(X_U)_u$ map to x_i.*

Proof. This proof is a variant of the proof of Algebra, Lemma 10.145.4 in the language of schemes. By Morphisms, Lemma 29.20.6 the morphism f is quasi-finite at each of the points x_i. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each i (Morphisms, Lemma 29.20.5). For each i, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension $L/\kappa(s)$ such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 37.35.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of X_U lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 26.17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of L (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair i, j. Also, by Morphisms, Lemma 29.20.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all i, j.

Apply Lemma 37.41.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We

claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. □

02LN **Lemma 37.41.4.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V_1 \amalg \ldots \amalg V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over u are singletons $\{v_i\}$, each v_i maps to x_i with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 37.41.2. Since $X_U \to U$ is separated (Schemes, Lemma 26.21.12) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 29.44.11) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 29.41.7. Hence $V_i \cap V_j$ is a closed subset of V_i which does not contain v_i. Hence the image of $V_i \cap V_j$ in U is a closed set (because $V_i \to U$ proper) not containing u. After shrinking U we may therefore assume that $V_i \cap V_j = \emptyset$ for all i, j. This gives the decomposition as in the lemma. □

Here is the variant where we reduce to purely inseparable field extensions.

02LO **Lemma 37.41.5.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over u are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to x_i, $\kappa(u) \subset \kappa(v_{i,j})$ is purely inseparable, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. This is proved in exactly the same way as the proof of Lemma 37.41.4 except that it uses Lemma 37.41.3 instead of Lemma 37.41.2. □

The following version may be a little easier to parse.

02LP **Lemma 37.41.6.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *X_s has at most finitely many isolated points.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre W_u of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 37.41.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of X_s and setting $V = \bigcup V_i$. □

37.42. Étale localization of integral morphisms

0BUH Some variants of the results of Section 37.41 for the case of integral morphisms.

0BSR **Lemma 37.42.1.** *Let $R \to S$ be an integral ring map. Let $\mathfrak{p} \subset R$ be a prime ideal. Assume*

1. *there are finitely many primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{p}$, and*
2. *for each i the maximal separable subextension $\kappa(\mathfrak{q})/\kappa(\mathfrak{q}_i)_{sep}/\kappa(\mathfrak{p})$ (Fields, Lemma 9.14.6) is finite over $\kappa(\mathfrak{p})$.*

Then there exists an étale ring map $R \to R'$ and a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that

$$S \otimes_R R' = A_1 \times \ldots \times A_m$$

with $R' \to A_j$ integral having a unique prime $\mathfrak{r}_j$ over $\mathfrak{p}'$ such that $\kappa(\mathfrak{r}_j)/\kappa(\mathfrak{p}')$ is purely inseparable.

First proof. This proof uses Algebra, Lemma 10.145.4. Namely, choose a generator $\theta_i \in \kappa(\mathfrak{q}_i)_{sep}$ of this field over $\kappa(\mathfrak{p})$ (Fields, Lemma 9.19.1). The spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is finite discrete with points corresponding to $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. By the Chinese remainder theorem (Algebra, Lemma 10.15.4) we see that $S \otimes_R \kappa(\mathfrak{p}) \to \prod \kappa(\mathfrak{q}_i)$ is surjective. Hence after replacing R by R_g for some $g \in R$, $g \notin \mathfrak{p}$ we may assume that $(0, \ldots, 0, \theta_i, 0, \ldots, 0) \in \prod \kappa(\mathfrak{q}_i)$ is the image of some $x_i \in S$. Let $S' \subset S$ be the R-subalgebra generated by our x_i. Since $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is surjective (Algebra, Lemma 10.36.17) we conclude that $\mathfrak{q}_i' = S' \cap \mathfrak{q}_i$ are the primes of S' over $\mathfrak{p}$. By our choice of x_i we conclude these primes are distinct that and $\kappa(\mathfrak{q}_i')_{sep} = \kappa(\mathfrak{q}_i)_{sep}$. In particular the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{q}_i')$ are purely inseparable. Since $R \to S'$ is finite we may apply Algebra, Lemma 10.145.4. and we get $R \to R'$ and $\mathfrak{p}'$ and a decomposition

$$S' \otimes_R R' = A_1' \times \ldots \times A_m' \times B'$$

with $R' \to A_j'$ integral having a unique prime $\mathfrak{r}_j'$ over $\mathfrak{p}'$ such that $\kappa(\mathfrak{r}_j')/\kappa(\mathfrak{p}')$ is purely inseparable and such that B' does not have a prime lying over $\mathfrak{p}'$. Since $R' \to B'$ is finite (as $R \to S'$ is finite) we can after localizing R' at some $g' \in R'$, $g' \notin \mathfrak{p}'$ assume that $B' = 0$. Via the map $S' \otimes_R R' \to S \otimes_R R'$ we get the corresponding decomposition for S. □

Second proof. This proof uses strict henselization. First, assume R is strictly henselization with maximal ideal $\mathfrak{p}$. Then $S/\mathfrak{p}S$ has finitely many primes corresponding to $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, each maximal, each with purely inseparable residue field over $\kappa(\mathfrak{p})$. Hence $S/\mathfrak{p}S$ is equal to $\prod (S/\mathfrak{p}S)_{\mathfrak{p}_i}$. By More on Algebra, Lemma 15.11.6 we can lift this product decomposition to a product composition of S as in the statement.

In the general case, let R^{sh} be the strict henselization of $R_{\mathfrak{p}}$. Then we can apply the result of the first paragraph to $R^{sh} \to S \otimes_R R^{sh}$. Consider the m mutually orthogonal idempotents in $S \otimes_R R^{sh}$ corresponding to the product decomposition. Since R^{sh} is a filtered colimit of étale ring maps $(R, \mathfrak{p}) \to (R', \mathfrak{p}')$ by Algebra, Lemma 10.155.11 we see that these idempotents descend to some R' as desired. □

37.43. Zariski's Main Theorem

02LQ In this section we prove Zariski's main theorem as reformulated by Grothendieck. Often when we say "Zariski's main theorem" in this content we mean either of Lemma 37.43.1, Lemma 37.43.2, or Lemma 37.43.3. In most texts people refer to the last of these as Zariski's main theorem.

We have already proved the algebraic version in Algebra, Theorem 10.123.12 and we have already restated this algebraic version in the language of schemes, see Morphisms, Theorem 29.56.1. The version in this section is more subtle; to get the full result we use the étale localization techniques of Section 37.41 to reduce to the algebraic case.

03GW **Lemma 37.43.1.** *Let $f : X \to S$ be a morphism of schemes. Assume f is of finite type and separated. Let S' be the normalization of S in X, see Morphisms, Definition 29.53.3. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & S' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & S & \end{array}$$

Then there exists an open subscheme $U' \subset S'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms, Lemma 29.56.2 the subset $U \subset X$ of points where f is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = (f')^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over S (as an open subscheme of X) and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset S'$ and S' is integral over S). Hence the fibres of $V \to S$ are discrete (Morphisms, Lemma 29.20.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 29.20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $s = f(x)$. Let $(T, t) \to (S, s)$ be an elementary étale neighbourhood. Denote by a subscript $_T$ the base change to T. Let $y = (x, t) \in X_T$ be the unique point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms, Lemma 29.20.13. Note that

$$X_T \xrightarrow{f'_T} S'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 37.19.2. Suppose that the claim holds for $y \in U_T \subset X_T \to S'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(y) \in V' \subset S'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $S'_T \to S'$ is étale hence the image $V \subset S'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(y) \in V'$. Observe that

$$\begin{array}{ccc} (f'_T)^{-1}V' & \longrightarrow & (f')^{-1}(V) \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is a fibre square (as $S'_T \times_{S'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent, Lemma 35.23.19. In other words, the claim holds for $x \in U \subset X \to S' \to S$.

By the result of the previous paragraph we may replace S by an elementary étale neighbourhood of $s = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subschemes where $V \to S$ is finite and $x \in V$, see Lemma 37.41.4. Since X is a disjoint union of V and W over S and since $V \to S$ is finite we see that the normalization of S in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of S in W, see Morphisms, Lemmas 29.53.10, 29.44.4, and 29.53.12. The claim follows and we win. □

02LR **Lemma 37.43.2.** *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated. Let S' be the normalization of S in X, see Morphisms, Definition 29.53.3. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & S & \end{array}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 37.43.1. Namely, by that lemma there exists an open subscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ and $X \to U'$ is an isomorphism. In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 26.21.14). It follows that f is quasi-affine by Morphisms, Lemma 29.13.3. □

05K0 **Lemma 37.43.3** (Zariski's Main Theorem)**.** *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact* [**DG67**, IV Corollary 18.12.13]

and quasi-separated. Then there exists a factorization

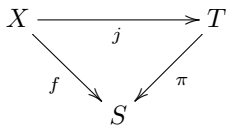

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 37.43.2. By Properties, Lemma 28.22.13 we can write $\nu_*\mathcal{O}_{S'} = \operatorname{colim}_{i\in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{S'}$. Then $\pi_i : T_i = \underline{\operatorname{Spec}}_S(\mathcal{A}_i) \to S$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $S' = \lim T_i$.

By Limits, Lemma 32.4.11 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in S' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i'\geq i} U_{i'}$, see Limits, Lemma 32.2.2. By Limits, Lemma 32.4.16 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms, Lemma 29.3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

0F2N **Lemma 37.43.4.** *With notation and hypotheses as in Lemma 37.43.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over S.*

Proof. By Limits, Lemma 32.9.8 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over S and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits, Lemma 32.4.11 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits, Lemma 32.4.16 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

37.44. Applications of Zariski's Main Theorem, I

0F2P A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

02LS **Lemma 37.44.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite,*
(2) *f is proper with finite fibres,*
(3) *f is proper and locally quasi-finite,*
(4) *f is universally closed, separated, locally of finite type and has finite fibres.*

Proof. We have (1) implies (2) by Morphisms, Lemmas 29.44.11, 29.20.10, and 29.44.10. We have (2) implies (3) by Morphisms, Lemma 29.20.7. We have (3) implies (4) by the definition of proper morphisms and Morphisms, Lemmas 29.20.9 and 29.20.10.

Assume (4). Pick $s \in S$. By Morphisms, Lemma 29.20.7 we see that all the finitely many points of X_s are isolated in X_s. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \amalg W$ as in Lemma 37.41.6. Note that

$W_u = \emptyset$ because all points of X_s are isolated. Since f is universally closed we see that the image of W in U is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover S such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 34.4.1. Hence it is an fpqc covering, see Topologies, Lemma 34.9.7. Hence we conclude f is finite by Descent, Lemma 35.23.25. □

As a consequence we have the following useful results.

02UP **Lemma 37.44.2.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that f is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.*

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 29.20.7. By Morphisms, Lemma 29.56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $f(Z) \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 29.20.9), hence has finite fibres (Morphisms, Lemma 29.20.10), hence is finite by Lemma 37.44.1. □

0AH8 **Lemma 37.44.3.** *Consider a commutative diagram of schemes*

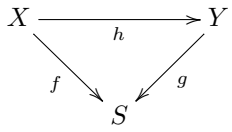

Let $s \in S$. Assume

(1) *$X \to S$ is a proper morphism,*
(2) *$Y \to S$ is separated and locally of finite type, and*
(3) *the image of $X_s \to Y_s$ is finite.*

Then there is an open subspace $U \subset S$ containing s such that $X_U \to Y_U$ factors through a closed subscheme $Z \subset Y_U$ finite over U.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms, Section 29.6. By Morphisms, Lemma 29.41.10 the morphism $X \to Z$ is surjective and $Z \to S$ is proper. Thus $X_s \to Z_s$ is surjective. We see that either (3) implies Z_s is finite. Hence $Z \to S$ is finite in an open neighbourhood of s by Lemma 37.44.2. □

37.45. Applications of Zariski's Main Theorem, II

0F2Q In this section we give a few more consequences of Zariski's main theorem to the structure of quasi-finite morphisms.

07S0 **Lemma 37.45.1.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism with Y affine. Then every finite set of points of X is contained in an open affine of X.*

Proof. Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 37.43.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 28.29.5. □

03I1 **Lemma 37.45.2.** *Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.*

Proof. If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 29.44.3. Thus the problem is local on X and we may assume that X is affine.

Assume X is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine. This is possible since f is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of X. We see from Morphisms, Lemmas 29.20.10 and 29.51.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose U_η such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of X is dense in X. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace X by an affine open subscheme of W and assume that each $V_j \to X$ is finite.

Assume X is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$\Delta_{ij} = \left(\overline{V_i \cap V_j} \setminus V_i \cap V_j\right) \cap V_j.$$

This is a nowhere dense closed subset of V_j because it is the boundary of the open subset $V_i \cap V_j$ in V_j. By Morphisms, Lemma 29.48.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of X. By Topology, Lemma 5.21.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of X. Thus $U = X \setminus T$ is a dense open subset of X. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V'_j = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$f' = f|_{Y'} : Y' \longrightarrow U'$$

of f. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V'_j$ is an affine open covering, each $V'_j \to U'$ is finite, and $V'_i \cap V'_j$ is (open and) closed both in V'_i and V'_j. Hence $V'_i \cap V'_j$ is affine, and the map

$$\mathcal{O}(V'_i) \otimes_{\mathbf{Z}} \mathcal{O}(V'_j) \longrightarrow \mathcal{O}(V'_i \cap V'_j)$$

is surjective. This implies that Y' is separated, see Schemes, Lemma 26.21.7. Finally, consider the commutative diagram

$$\begin{array}{ccc} \coprod_{j=1,\ldots,m} V'_j & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & U' & \end{array}$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 29.41.9 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 37.44.1, and we win. □

07RY **Lemma 37.45.3.** *Let $f : X \to S$ be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod_{r=0,\ldots,n} S_r$ is characterized by the following universal property: Given $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically).

Proof. Let n be an integer bounding the degree of the fibres of $X \to S$. By Morphisms, Lemma 29.57.5 we see that any base change has degrees of fibres bounded by n also. In particular, all the integers r that occur in the statement of the lemma will be $\leq n$. We will prove the lemma by induction on n. The base case is $n = 0$ which is obvious.

We claim the set of points $s \in S$ with $\deg_{\kappa(s)}(X_s) = n$ is an open subset $S_n \subset S$ and that $X \times_S S_n \to S_n$ is finite locally free of degree n. Namely, suppose that $s \in S$ is such a point. Choose an elementary étale morphism $(U, u) \to (S, s)$ and a decomposition $U \times_S X = W \amalg V$ as in Lemma 37.41.6. Since $V \to U$ is finite, flat, and locally of finite presentation, we see that $V \to U$ is finite locally free, see Morphisms, Lemma 29.48.2. After shrinking U to a smaller neighbourhood of u we may assume $V \to U$ is finite locally free of some degree d, see Morphisms, Lemma 29.48.5. As $u \mapsto s$ and $W_u = \emptyset$ we see that $d = n$. Since n is the maximum degree of a fibre we see that $W = \emptyset$! Thus $U \times_S X \to U$ is finite locally free of degree n. By Descent, Lemma 35.23.32 we conclude that $X \to S$ is finite locally free of degree n over $\mathrm{Im}(U \to S)$ which is an open neighbourhood of s (Morphisms, Lemma 29.36.13). This proves the claim.

Let $S' = S \setminus S_n$ endowed with the reduced induced scheme structure and set $X' = X \times_S S'$. Note that the degrees of fibres of $X' \to S'$ are universally bounded by $n - 1$. By induction we find a stratification $S' = S_0 \amalg \ldots \amalg S_{n-1}$ adapted to the morphism $X' \to S'$. We claim that $S = \coprod_{r=0,\ldots,n} S_r$ works for the morphism $X \to S$. Let $g : T \to S$ be a morphism of schemes and assume that $X \times_S T \to T$ is finite locally free of degree r. As remarked above this implies that $r \leq n$. If $r = n$, then it is clear that $T \to S$ factors through S_n. If $r < n$, then $g(T) \subset S' = S \setminus S_d$ (set theoretically) hence $T_{red} \to S$ factors through S', see Schemes, Lemma 26.12.7. Note that $X \times_S T_{red} \to T_{red}$ is also finite locally free of degree r as a base change. By the universal property of the stratification $S' = \coprod_{r=0,\ldots,n-1} S_r$ we see that $g(T) = g(T_{red})$ is contained in S_r. Conversely, suppose that we have $g : T \to S$ such that $g(T) \subset S_r$ (set theoretically). If $r = n$, then g factors through S_n and it is clear that $X \times_S T \to T$ is finite locally free of degree n as a base change. If $r < n$, then $X \times_S T \to T$ is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to T_{red} is finite locally free of degree r. Since $T_{red} \to T$ is a universal homeomorphism, we conclude that $X \times_S T_{red} \to X \times_S T$ is a universal homeomorphism too and hence $X \times_S T \to T$ is universally closed (as this is true for the finite morphism $X \times_S T_{red} \to T_{red}$). It follows that $X \times_S T \to T$ is finite, for example by Lemma 37.44.1. Then we can use Morphisms, Lemma 29.48.2 to see that $X \times_S T \to T$ is finite locally free. Finally, the degree is r as all the fibres have degree r. $\square$

07RZ **Lemma 37.45.4.** *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod S_r$ is characterized by the following universal property: Given a morphism $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically). Moreover, the inclusion maps $S_r \to S$ are quasi-compact.

Proof. The question is local on S, hence we may assume that S is affine. By Morphisms, Lemma 29.57.9 the fibres of f are universally bounded in this case. Hence the existence of the stratification follows from Lemma 37.45.3.

We will show that $U_r = S \setminus Z_r \to S$ is quasi-compact for each $r \geq 0$. This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that $U_r \to U_{r-1}$ is quasi-compact. Choose an affine open $W \subset U_{r-1}$. Write $W = \mathrm{Spec}(A)$. Then $Z_r \cap W = V(I)$ for some ideal $I \subset A$ and $X \times_S \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite locally free of degree r. Note that $A/I = \mathrm{colim}\, A/I_i$ where $I_i \subset I$ runs through the finitely generated ideals. By Limits, Lemma 32.8.8 we see that $X \times_S \mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A/I_i)$ is finite locally free of degree r for some i. (This uses that $X \to S$ is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence $\mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A) = W$ factors (set theoretically) through $Z_r \cap W$. It follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of A. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. □

086R **Lemma 37.45.5.** *Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes*

$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to S$ where k is a field factors through U_d if and only if $X \times_S \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on S and assume S is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 37.45.4. Since $U_d = \bigcup_W U_d(W)$ the result follows. □

082V **Lemma 37.45.6.** *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram*

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S, \end{array}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_* \mathcal{O}_V$ whose image contains $g|_V$.

Proof. We may assume X and S affine. We obtain a filtration $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 37.45.3 and 37.45.4. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\mathrm{Im}(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that T is the support of g as a section of $\mathcal{O}_X$ (Modules, Definition 17.5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let r be the smallest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of T such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace S by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \mathrm{Spec}(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 10.29.1). Since the support of g is contained in $f^{-1}V(I)$ by our choice of r we see that there exists an integer N such that $a_j^N g = 0$ for $j = 1, \ldots, m$. Replacing a_j by a_j^r we may assume that $Ig = 0$. For any A-module M write $M[I]$ for the I-torsion of M, i.e., $M[I] = \{m \in M \mid Im = 0\}$. Write $X = \mathrm{Spec}(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of Z_r, the A/I-module B/IB is finite locally free of rank r. Hence after replacing S by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/IA)^{\oplus r}$ as A/I-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo I to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \longrightarrow B[I]$$

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the A-module $A[I]$ and the map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ the one corresponding to $A[I]^{\oplus r} \subset A^{\oplus r} \to B$. □

09Z0 **Lemma 37.45.7.** *Let $U \to X$ be a surjective étale morphism of schemes. Assume X is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U. In fact, we may assume $Y \to X$ is finite and of finite presentation.*

Proof. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U', we see that we may assume U is affine. In particular $U \to X$ is separated (Schemes, Lemma 26.21.15). Then there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 29.57.9). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 37.43.2 and we obtain a factorization

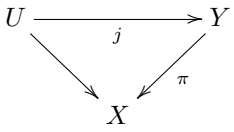

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 26.21.10 and open because it is étale as a morphism between schemes étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of Y for example by Lemma 37.45.3 (the degree of the fibres of a quasi-compact separated étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 37.43.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $\Omega \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through U. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 32.7.3. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Limits, Lemma 32.4.11. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Limits, Proposition 32.6.1. This finishes the proof. □

37.46. Application to morphisms with connected fibres

057H In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

057I **Lemma 37.46.1.** *Consider a diagram of morphisms of schemes*

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & X \\ & \searrow & \downarrow \\ & & Y \end{array}$$

an a point $y \in Y$. Assume

(1) *$X \to Y$ is of finite presentation and flat,*
(2) *$Z \to Y$ is finite locally free,*
(3) *$Z_y \neq \emptyset$,*
(4) *all fibres of $X \to Y$ are geometrically reduced, and*
(5) *X_y is geometrically connected over $\kappa(y)$.*

Then there exists a quasi-compact open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 29.25.8, 29.21.4, 29.48.4, 29.25.6, 29.21.3, 29.48.3, 29.25.10, and 29.44.11.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section s coming from σ. Let X^0_Z denote the subset of X_Z defined in Situation 37.29.1. By Lemma 37.29.6 it is an open subset of X_Z.

The pullback $X_{Z \times_Y Z}$ of X to $Z \times_Y Z$ comes equipped with two sections s_0, s_1, namely the base changes of s by $\mathrm{pr}_0, \mathrm{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 37.29.1 gives two subsets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$. By Lemma 37.29.2 these are the inverse images of X^0_Z under the morphisms $1_X \times \mathrm{pr}_0, 1_X \times \mathrm{pr}_1 : X_{Z \times_Y Z} \to X_Z$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As X_y is geometrically connected, we see that the fibres of $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ over each z_i agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z \times_Y Z})^0_{s_0}.$$

Since the sets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ are open in $X_{Z \times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z \times_Y Z})^0_{s_0}.$$

Then it follows directly from the construction in Situation 37.29.1 that

$$p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_1}$$

where $p : X_{Z \times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an affine open neighbourhood $V \subset Y$ of y such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace Y by V. After we do this we see that $X^0_Z \subset Y_Z$ is an open such that

$$(1_X \times \mathrm{pr}_0)^{-1}(X^0_Z) = (1_X \times \mathrm{pr}_1)^{-1}(X^0_Z).$$

This means that the image $X^0 \subset X$ of X^0_Z is an open such that $(X_Z \to X)^{-1}(X^0) = X^0_Z$, see Descent, Lemma 35.13.6. Finally, X^0 is quasi-compact because X^0_Z is quasi-compact by Lemma 37.29.4 (use that at this point Y is affine, hence X is quasi-compact and quasi-separated, hence locally constructible is the same as constructible and in particular quasi-compact; details omitted). In this way we see that X^0 has all the desired properties. □

055W **Lemma 37.46.2.** *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is flat and of finite presentation,*
(2) *all fibres of h are geometrically reduced, and*
(3) *T is geometrically connected over $\kappa(s)$.*

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) *all fibres of* $V \to S'$ *are geometrically connected,*
(b) $V_{s'} = T \times_s s'$.

Proof. The problem is clearly local on S, hence we may replace S by an affine open neighbourhood of s. The topology on Y_s is induced from the topology on Y, see Schemes, Lemma 26.18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of h to V is quasi-compact (as S affine and V quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing Y by V we may assume, in addition to (1) and (2) that $Y_s = T$ and S affine.

Pick a closed point $y \in Y_s$ such that h is Cohen-Macaulay at y, see Lemma 37.22.7. By Lemma 37.23.4 there exists a diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{y\}$. Apply Lemma 37.41.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over s. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 29.48.2. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of h. Also $Y_{s'} = Y_s$ is geometrically connected. Apply Lemma 37.46.1 to $Z' \to Y_{S'}$ over S' to get $V \subset Y_{S'}$ quasi-compact open satisfying (2) whose fibres over S' are either empty or geometrically connected. As $V \to S'$ is open (Morphisms, Lemma 29.25.10), after replacing S' by an affine open neighbourhood of s' we may assume $V \to S'$ is surjective, whence (1) holds. □

0EY5 **Lemma 37.46.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation and flat with geometrically reduced fibres. Then there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $X_i \to S$ factors as $X_i \to S_i \to S$ where $S_i \to S$ is étale and $X_i \to S_i$ is flat of finite presentation with geometrically connected and geometrically reduced fibres.*

Proof. Pick a point $x \in X$ with image $s \in S$. We will produce a diagram

$$\begin{array}{ccccc} X' & \longrightarrow & S' \times_S X & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S' & \longrightarrow & S \end{array}$$

and points $s' \in S'$, $x' \in X'$, $y \in S' \times_S X$ such that x' maps to x, $(S', s') \to (S, s)$ is an étale neighbourhood, $(X', x') \to (S' \times_S X, y)$ is an étale neighbourhood[8], and $X' \to S'$ has geometrically connected fibres. If we can do this for every $x \in X$, then the lemma follows (with members of the covering given by the collection of étale morphisms $X' \to X$ so produced). The first step is the replace X and S by affine open neighbourhoods of x and s which reduces us to the case that X and S are affine (and hence f of finite presentation).

[8]The proof actually gives an open $X' \subset S' \times_S X$.

Choose a separable algebraic extension $\overline{k}$ of $\kappa(s)$. Denote $X_{\overline{k}}$ the base change of X_s. Choose a point $\overline{x}$ in $X_{\overline{k}}$ mapping to $x \in X_s$. Choose a connected quasi-compact open neighbourhood $\overline{V} \subset X_{\overline{k}}$ of $\overline{x}$. (This is possible because any scheme locally of finite type over a field is locally connected as a locally Noetherian topological space.) By Varieties, Lemma 33.7.9 we can find a finite separable extension $k'/\kappa(s)$ and a quasi-compact open $V' \subset X_{k'}$ whose base change is $\overline{V}$. In particular V' is geometrically connected over k', see Varieties, Lemma 33.7.7. By Lemma 37.35.2 we can find an étale neighbourhood $(S', s') \to (S, s)$ such that $\kappa(s')$ is isomorphic to k' as an extension of $\kappa(s)$. Denote $x' \in (S' \times_S X)_{s'} = X_{k'}$ the image of $\overline{x}$. Thus after replacing (S, s) by (S', s') and (X, x) by $(S' \times_S X, x')$ we reduce to the case handled in the next paragraph.

Assume there is a quasi-compact open $V \subset X_s$ which contains x and is geometrically connected. Then we can apply Lemma 37.46.2 to find an affine étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $X' \subset S' \times_S X$ such that $X' \to S'$ has geometrically connected fibres and such that X' contains a point mapping to x. This finishes the proof. □

057J **Lemma 37.46.4.** *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is of finite presentation,*
(2) *h is normal, and*
(3) *T is geometrically irreducible over $\kappa(s)$.*

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically integral,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. Apply Lemma 37.46.2 to find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that all fibres of $V \to S'$ are geometrically connected and $V_{s'} = T \times_s s'$. As V is an open of the base change of h all fibres of $V \to S'$ are geometrically normal, see Lemma 37.20.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if $t \in S'$ then V_t is of finite type over $\kappa(t)$ and hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is of finite type over $\overline{\kappa(t)}$ hence Noetherian. By choice of $S' \to S$ the scheme $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is connected. Hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is irreducible by Properties, Lemma 28.7.6 and we win. □

37.47. Application to the structure of finite type morphisms

052D The result in this section can be found in [**GR71**]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

052E **Lemma 37.47.1.** *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a*

commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & & \downarrow{\scriptstyle \pi} \\ & & Y \\ & & \downarrow{\scriptstyle h} \\ S & = & S \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ s & = & s \end{array} \qquad \begin{array}{c} x' \\ \downarrow \\ y \\ \downarrow \\ s \end{array}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y = \pi(x')$ we have

(1) *$h : Y \to S$ is smooth of relative dimension n,*
(2) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(3) *π is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and*
(4) *$\kappa(y)$ is a purely transcendental extension of $\kappa(s)$.*

Moreover, if f is locally of finite presentation then π is of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. By Algebra, Lemma 10.125.3 after replacing X by a standard open neighbourhood of x in X we may assume there is a factorization

$$X \xrightarrow{\pi} \mathbf{A}^n_S \longrightarrow S$$

such that π is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma 37.41.1 there exists an elementary étale neighbourhood

$$(Y, y) \to (\mathbf{A}^n_S, \pi(x))$$

and an open $X' \subset X \times_{\mathbf{A}^n_S} Y$ which contains a unique point x' lying over y such that $X' \to Y$ is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma 29.21.11. □

057K **Lemma 37.47.2.** *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & & \downarrow{\scriptstyle \pi} \\ & & Y' \\ & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ s & \longleftarrow & s' \end{array} \qquad \begin{array}{c} x' \\ \downarrow \\ y' \\ \downarrow \\ s' \end{array}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

(1) *$h : Y' \to S'$ is smooth of relative dimension n,*
(2) *all fibres of $Y' \to S'$ are geometrically integral,*
(3) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(4) *π is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,*
(5) *$\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and*
(6) *$e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.*

Moreover, if f is locally of finite presentation, then π is of finite presentation.

Proof. The question is local on S, hence we may replace S by an affine open neighbourhood of s. Next, we apply Lemma 37.47.1 to get a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ & & Y \\ & & \downarrow{\scriptstyle h} \\ S & = & S \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ & & y \\ & & \downarrow \\ s & = & s \end{array}$$

where h is smooth of relative dimension n and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on X also, we may replace Y by an affine neighbourhood of y (and X' by the inverse image of this under π). As S is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let T be the connected component of Y_s containing y. As Y_s is Noetherian we see that T is open. We also see that T is geometrically connected over $\kappa(s)$ by Varieties, Lemma 33.7.14. Since T is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 33.25.4. We conclude that T is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 28.7.6). Finally, note that the smooth morphism h is normal by Lemma 37.20.3. At this point we have verified all assumption of Lemma 37.46.4 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 37.46.4 we obtain $e : S' \to S$, $s' \in S'$, Y' as in the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \longleftarrow & X' \times_Y Y' \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ & & Y & \longleftarrow & Y' \\ & & \downarrow{\scriptstyle h} & & \downarrow \\ S & = & S & \xleftarrow{e} & S' \end{array} \qquad \begin{array}{ccccc} x & \longleftarrow & x' & \longleftarrow & (x', s') \\ \downarrow & & \downarrow & & \downarrow \\ & & y & \longleftarrow & (y, s') \\ & & \downarrow & & \downarrow \\ s & = & s & \longleftarrow & s' \end{array}$$

where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over S' are geometrically irreducible, such that $Y'_{s'} = T$ via the identification $Y_s = Y_{S',s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over y with residue field equal to $\kappa(y)$ which maps to s' in S'. Similarly, let $(x', s') \in X' \times_Y Y' \subset X' \times_S S'$ be the unique point over x' with residue field equal to $\kappa(x')$ lying over s'. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. □

057L **Lemma 37.47.3.** *Assumption and notation as in Lemma 37.47.2. In addition to properties (1) – (6) we may also arrange it so that*

(7) *S', Y', X' are affine.*

Proof. Note that if Y' is affine, then X' is affine as π is finite. Choose an affine open neighbourhood $U' \subset S'$ of s'. Choose an affine open neighbourhood $V' \subset h^{-1}(U')$ of y'. Let $W' = h(V')$. This is an open neighbourhood of s' in S', see Morphisms, Lemma 29.34.10, contained in U'. Choose an affine open neighbourhood $U'' \subset$

W' of s'. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By construction $h^{-1}(U'') \cap V' \to U''$ is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of $Y' \to S'$, and hence geometrically integral. Thus we may replace S' by U'' and Y' by $h^{-1}(U'') \cap V'$. $\square$

The significance of the property $\pi^{-1}(\{y'\}) = \{x'\}$ is partially explained by the following lemma.

05B8 **Lemma 37.47.4.** *Let $\pi : X \to Y$ be a finite morphism. Let $x \in X$ with $y = \pi(x)$ such that $\pi^{-1}(\{y\}) = \{x\}$. Then*

(1) *For every neighbourhood $U \subset X$ of x in X, there exists a neighbourhood $V \subset Y$ of y such that $\pi^{-1}(V) \subset U$.*
(2) *The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.*
(3) *If π is of finite presentation, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is of finite presentation.*
(4) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $\mathcal{F}_x = \pi_*\mathcal{F}_y$ as $\mathcal{O}_{Y,y}$-modules.*

Proof. The first assertion is purely topological; use that π is a continuous and closed map such that $\pi^{-1}(\{y\}) = \{x\}$. To prove the second and third parts we may assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Then $A \to B$ is a finite ring map and y corresponds to a prime $\mathfrak{p}$ of A such that there exists a unique prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Then $B_\mathfrak{q} = B_\mathfrak{p}$, see Algebra, Lemma 10.41.11. In other words, the map $A_\mathfrak{p} \to B_\mathfrak{q}$ is equal to the map $A_\mathfrak{p} \to B_\mathfrak{p}$ you get from localizing $A \to B$ at $\mathfrak{p}$. Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that $\mathcal{F} = \widetilde{M}$ for some B-module M. Then $\mathcal{F} = M_\mathfrak{q}$ and $\pi_*\mathcal{F}_y = M_\mathfrak{p}$. By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that $\mathcal{F}_x = \pi_*\mathcal{F}_y$ directly. $\square$

37.48. Application to the fppf topology

05WM We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology "generated by" Zariski coverings and by coverings of the form $\{f : T \to S\}$ where f is surjective finite locally free.

0DET **Lemma 37.48.1.** *Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist*

(1) *an étale covering $\{S'_a \to S\}$,*
(2) *surjective finite locally free morphisms $V_a \to S'_a$,*

such that the fppf covering $\{V_a \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. We may assume that each $S_i \to S$ is locally quasi-finite, see Lemma 37.23.6.

Fix a point $s \in S$. Pick an $i \in I$ and a point $s_i \in S_i$ mapping to s. Choose an elementary étale neighbourhood $(S', s) \to (S, s)$ such that there exists an open

$$S_i \times_S S' \supset V$$

which contains a unique point $v \in V$ mapping to $s \in S'$ and such that $V \to S'$ is finite, see Lemma 37.41.1. Then $V \to S'$ is finite locally free, because it is finite and because $S_i \times_S S' \to S'$ is flat and locally of finite presentation as a base change of the morphism $S_i \to S$, see Morphisms, Lemmas 29.21.4, 29.25.8, and 29.48.2. Hence $V \to S'$ is open, and after shrinking S' we may assume that $V \to S'$ is surjective finite locally free. Since we can do this for every point of S we conclude

that $\{S_i \to S\}$ can be refined by a covering of the form $\{V_a \to S\}_{a\in A}$ where each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free. □

05WN **Lemma 37.48.2.** *Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be an fppf covering. Then there exist*

(1) *a Zariski open covering $S = \bigcup U_j$,*
(2) *surjective finite locally free morphisms $W_j \to U_j$,*
(3) *Zariski open coverings $W_j = \bigcup_k W_{j,k}$,*
(4) *surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$*

such that the fppf covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. Let $\{V_a \to S\}_{a\in A}$ be the fppf covering found in Lemma 37.48.1. In other words, this covering refines $\{S_i \to S\}$ and each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free.

By Remark 37.40.3 there exists a Zariski open covering $S = \bigcup U_j$, for each j a finite locally free, surjective morphism $W_j \to U_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the étale covering $\{S'_a \to S\}$, i.e., for each pair j, k there exists an $a(j, k)$ and a factorization $W_{j,k} \to S'_a \to S$ of the morphism $W_{j,k} \to S$. Set $T_{j,k} = W_{j,k} \times_{S'_a} V_a$ and everything is clear. □

0CNX **Lemma 37.48.3.** *Let S be a scheme. If $U \subset S$ is open and $V \to U$ is a surjective integral morphism, then there exists a surjective integral morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.*

Proof. Let $V' \to S$ be the normalization of S in U, see Morphisms, Section 29.53. By construction $V' \to S$ is integral. By Morphisms, Lemmas 29.53.6 and 29.53.12 we see that the inverse image of U in V' is V. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. □

0CNY **Lemma 37.48.4.** *Let S be a quasi-compact and quasi-separated scheme. If $U \subset S$ is a quasi-compact open and $V \to U$ is a surjective finite morphism, then there exists a surjective finite morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.*

Proof. By Zariski's Main Theorem (Lemma 37.43.3) we can assume V is a quasi-compact open in a scheme V' finite over S. After replacing V' by the scheme theoretic image of V we may assume that V is dense in V'. It follows that $V' \times_S U = V$ because $V \to V' \times_S U$ is closed as V is finite over U. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. □

0CNZ **Lemma 37.48.5.** *Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be an fppf covering. Then there exists a surjective integral morphism $S' \to S$ and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.*

Proof. Choose $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, and $T_{j,k} \to W_{j,k}$ as in Lemma 37.48.2. By Lemma 37.48.3 we can extend $W_j \to U_j$ to a surjective integral morphism $\overline{W}_j \to S$. After this we can extend $T_{j,k} \to W_{j,k}$ to a surjective integral morphism $\overline{T}_{j,k} \to \overline{W}_j$. We set $\overline{T}_j$ equal to the product of all the schemes $\overline{T}_{j,k}$ over $\overline{W}_j$ (Limits, Lemma 32.3.1). Then we set S' equal to the product of all the schemes $\overline{T}_j$ over S. If $x \in S'$, then there is a j such that the image of x in S lies

in U_j. Hence there is a k such that the image of x under the projection $S' \to \overline{W}_j$ lies in $W_{j,k}$. Hence under the projection $S' \to \overline{T}_j \to \overline{T}_{j,k}$ the point x ends up in $T_{j,k}$. And $T_{j,k} \to S$ factors through S_i for some i. Finally, the morphism $S' \to S$ is integral and surjective by Limits, Lemmas 32.3.3 and 32.3.2. □

0CP0 **Lemma 37.48.6.** *Let S be a quasi-compact and quasi-separated scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exists a surjective finite morphism $S' \to S$ of finite presentation and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.*

Proof. Let $Y \to X$ be the integral surjective morphism found in Lemma 37.48.5. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $S_{i(j)}$. We can write $Y = \lim Y_\lambda$ with $Y_\lambda \to X$ finite and of finite presentation, see Limits, Lemma 32.7.3. For large enough λ we can find affine opens $V_{\lambda,j} \subset Y_\lambda$ whose inverse image in Y recovers V_j, see Limits, Lemma 32.4.11. For even larger λ the morphisms $V_j \to S_{i(j)}$ over X come from morphisms $V_{\lambda,j} \to S_{i(j)}$ over X, see Limits, Proposition 32.6.1. Setting $S' = Y_\lambda$ for this λ finishes the proof. □

0DBT **Lemma 37.48.7.** *An fppf covering of schemes is a ph covering.*

Proof. Let $\{T_i \to T\}$ be an fppf covering of schemes, see Topologies, Definition 34.7.1. Observe that $T_i \to T$ is locally of finite type. Let $U \subset T$ be an affine open. It suffices to show that $\{T_i \times_T U \to U\}$ can be refined by a standard ph covering, see Topologies, Definition 34.8.4. This follows immediately from Lemma 37.48.6 and the fact that a finite morphism is proper (Morphisms, Lemma 29.44.11). □

0DBU **Remark 37.48.8.** As a consequence of Lemma 37.48.7 we obtain a comparison morphism

$$\epsilon : (Sch/S)_{ph} \longrightarrow (Sch/S)_{fppf}$$

This is the morphism of sites given by the identity functor on underlying categories (with suitable choices of sites as in Topologies, Remark 34.11.1). The functor ϵ_* is the identity on underlying presheaves and the functor ϵ^{-1} associated to an fppf sheaf its ph sheafification. By composition we can in addition compare the ph topology with the syntomic, smooth, étale, and Zariski topologies.

37.49. Quasi-projective schemes

0B41 The term "quasi-projective scheme" has not yet been defined. A possible definition could be a scheme which has an ample invertible sheaf. However, if X is a scheme over a base scheme S, then we say that X *is quasi-projective over S* if the morphism $X \to S$ is quasi-projective (Morphisms, Definition 29.40.1). Since the identity morphism of any scheme is quasi-projective, we see that a scheme quasi-projective over S doesn't necessarily have an ample invertible sheaf. For this reason it seems better to leave the term "quasi-projective scheme" undefined.

0B42 **Lemma 37.49.1.** *Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *$X \to S$ is quasi-projective,*
2. *$X \to S$ is H-quasi-projective,*
3. *there exists a quasi-compact open immersion $X \to X'$ of schemes over S with $X' \to S$ projective,*
4. *$X \to S$ is of finite type and X has an ample invertible sheaf, and*

(5) *$X \to S$ is of finite type and there exists an f-very ample invertible sheaf.*

Proof. The implication (2) $\Rightarrow$ (1) is Morphisms, Lemma 29.40.5. The implication (1) $\Rightarrow$ (2) is Morphisms, Lemma 29.43.16. The implication (2) $\Rightarrow$ (3) is Morphisms, Lemma 29.43.11

Assume $X \subset X'$ is as in (3). In particular $X \to S$ is of finite type. By Morphisms, Lemma 29.43.11 the morphism $X \to S$ is H-projective. Thus there exists a quasi-compact immersion $i : X \to \mathbf{P}^n_S$. Hence $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$ is f-very ample. As $X \to S$ is quasi-compact we conclude from Morphisms, Lemma 29.38.2 that $\mathcal{L}$ is f-ample. Thus $X \to S$ is quasi-projective by definition.

The implication (4) $\Rightarrow$ (2) is Morphisms, Lemma 29.39.3.

The implication (5) $\Rightarrow$ (4) follows from Morphisms, Lemma 29.37.7.

The equivalence of (1) and (5) follows from Morphisms, Lemmas 29.38.2 and 29.39.5. □

0B43 **Lemma 37.49.2.** *Let S be a scheme which has an ample invertible sheaf. Let QP_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 37.49.1.*

(1) *if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $QP_S \to QP_{S'}$,*
(2) *if $X \in QP_S$ and $Y \in QP_X$, then $Y \in QP_S$,*
(3) *the category QP_S is closed under fibre products,*
(4) *the category QP_S is closed under finite disjoint unions,*
(5) *if $X \to S$ is projective, then $X \in QP_S$,*
(6) *if $X \to S$ is quasi-affine of finite type, then X is in QP_S,*
(7) *if $X \to S$ is quasi-finite and separated, then $X \in QP_S$,*
(8) *if $X \to S$ is a quasi-compact immersion, then $X \in QP_S$,*
(9) *add more here.*

Proof. Part (1) follows from Morphisms, Lemma 29.40.2.

Part (2) follows from the fourth characterization of Lemma 37.49.1.

If $X \to S$ and $Y \to S$ are quasi-projective, then $X \times_S Y \to Y$ is quasi-projective by Morphisms, Lemma 29.40.2. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fourth characterization of Lemma 37.49.1.

Part (5) follows from Morphisms, Lemma 29.43.10.

Part (6) follows from Morphisms, Lemma 29.40.7.

Part (7) follows from part (6) and Lemma 37.43.2.

Part (8) follows from part (7) and Morphisms, Lemma 29.20.16. □

The following lemma doesn't really belong in this section, but there does not seem to be a good spot for it anywhere else.

0EJY **Lemma 37.49.3.** *Let X be a quasi-affine scheme. Let $f : U \to X$ be an integral morphism. Then U is quasi-affine and the diagram*

$$\begin{array}{ccc} U & \longrightarrow & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \end{array}$$

is cartesian.

Proof. The scheme U is quasi-affine because integral morphisms are affine, affine morphisms are quasi-affine, a scheme is quasi-affine if and only if the structure morphism to $\mathrm{Spec}(\mathbf{Z})$ is quasi-affine, and compositions of quasi-affine morphisms are quasi-affine. The first two statements follow immediately from the definition and the third is Morphisms, Lemma 29.13.4. Set $U' = X \times_{\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))} \mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ and consider the extended diagram

$$\begin{array}{ccccc} U & \xrightarrow[j]{} & U' & \longrightarrow & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \end{array}$$

The morphism j is closed by Morphisms, Lemma 29.41.7 combined with the fact that an integral morphism is universally closed (Morphisms, Lemma 29.44.7) and the fact that the vertical arrows are in the diagram are separated. On the other hand, j is open because the horizontal arrows in the diagram of the lemma are open by Properties, Lemma 28.18.4. Thus j identifies U with an open and closed subscheme of U'. If $U \neq U'$ then U isn't dense in U' and a fortiori not dense in the spectrum of $\Gamma(U, \mathcal{O}_U)$. However, the scheme theoretic image of U in $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ is $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ because any ideal in $\Gamma(U, \mathcal{O}_U)$ cutting out a closed subscheme through which U factors would have to be zero. Hence U is dense in $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ for example by Morphisms, Lemma 29.6.3. Thus $U = U'$ and we win. □

37.50. Projective schemes

0B44 This section is the analogue of Section 37.49 for projective morphisms.

0B45 **Lemma 37.50.1.** *Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *$X \to S$ is projective,*
(2) *$X \to S$ is H-projective,*
(3) *$X \to S$ is quasi-projective and proper,*
(4) *$X \to S$ is H-quasi-projective and proper,*
(5) *$X \to S$ is proper and X has an ample invertible sheaf,*
(6) *$X \to S$ is proper and there exists an f-ample invertible sheaf,*
(7) *$X \to S$ is proper and there exists an f-very ample invertible sheaf,*
(8) *there is a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ with $\mathcal{A}_1$ a finite type $\mathcal{O}_S$-module such that $X = \underline{Proj}_S(\mathcal{A})$.*

Proof. Observe first that in each case the morphism f is proper, see Morphisms, Lemmas 29.43.3 and 29.43.5. Hence it suffices to prove the equivalence of the

notions in case f is a proper morphism. We will use this without further mention in the following.

The equivalences (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are Morphisms, Lemma 29.43.13.

The implication (2) $\Rightarrow$ (1) is Morphisms, Lemma 29.43.3.

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are Morphisms, Lemma 29.43.16.

The implication (1) $\Rightarrow$ (7) is immediate from Morphisms, Definitions 29.43.1 and 29.38.1.

The conditions (3) and (6) are equivalent by Morphisms, Definition 29.40.1.

Thus (1) – (4), (6) are equivalent and imply (7). By Lemma 37.49.1 conditions (3), (5), and (7) are equivalent. Thus we see that (1) – (7) are equivalent.

By Divisors, Lemma 31.30.5 we see that (8) implies (1). Conversely, if (2) holds, then we can choose a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

See Constructions, Lemma 27.21.5 for the equality. By Divisors, Lemma 31.31.1 we see that X is the relative Proj of a quasi-coherent graded quotient algebra $\mathcal{A}$ of $\mathcal{O}_S[T_0, \ldots, T_n]$. Then $\mathcal{A}$ satisfies the conditions of (8). □

0B46 **Lemma 37.50.2.** *Let S be a scheme which has an ample invertible sheaf. Let P_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 37.50.1.*

(1) *if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $P_S \to P_{S'}$,*
(2) *if $X \in P_S$ and $Y \in P_X$, then $Y \in P_S$,*
(3) *the category P_S is closed under fibre products,*
(4) *the category P_S is closed under finite disjoint unions,*
(5) *if $X \to S$ is finite, then X is in P_S,*
(6) *add more here.*

Proof. Part (1) follows from Morphisms, Lemma 29.43.9.

Part (2) follows from the fifth characterization of Lemma 37.50.1 and the fact that compositions of proper morphisms are proper (Morphisms, Lemma 29.41.4).

If $X \to S$ and $Y \to S$ are projective, then $X \times_S Y \to Y$ is projective by Morphisms, Lemma 29.43.9. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fifth characterization of Lemma 37.50.1.

Part (5) follows from Morphisms, Lemma 29.44.16. □

Here is a slightly different type of result.

0D2S **Lemma 37.50.3.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$.* [**DG67**, IV Corollary 9.6.4]

Proof. We may assume Y is affine. Then we find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of schemes with Y_i of finite type over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper, such that $X \to Y = \lim(X_i \to Y_i)$. See Limits, Lemma 32.13.3. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits, Lemma 32.10.3. Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \operatorname{colim} \kappa(y_i)$. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 32.4.15. By Cohomology of Schemes, Lemma 30.21.4 we find an open neighbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 29.37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 29.37.7). □

37.51. Proj and Spec

0EKF In this section we clarify the relationship between the Proj and the spectrum of a graded ring.

Let R be a ring. Let A be a graded R-algebra, see Algebra, Section 10.56. For $m \geq 0$ we denote $A_{\geq m} = \bigoplus_{d \geq m} A_d$. Consider the graded ring

$$B = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

For $d' \geq d$ and $a \in A_{d'}$ let us denote $a^{(d)} \in B$ the element in B_d corresponding to a. Let us denote $\sigma : A \to B$ and $\psi : A \to B$ the two obvious ring maps: if $a \in A_d$, then $\sigma(a) = a^{(0)}$ and $\psi(a) = a^{(d)}$. Then ψ is a graded ring map and σ turns B into a graded algebra over A. There is also a surjective graded ring map $\tau : B \to A$ which for $d' \geq d$ and $a \in A_{d'}$ sends $a^{(d)}$ to 0 if $d' > d$ and to a if $d' = d$.

Affine schemes and spectra. We set $X = \operatorname{Spec}(A)$. The irrelevant ideal A_+ cuts out a closed subscheme $Z = V(A_+) = \operatorname{Spec}(A/A_+) = \operatorname{Spec}(A_0)$. Set $U = X \setminus Z$.

$$U \longrightarrow X \longrightarrow Z$$

Projective schemes and Proj. Set $P = \operatorname{Proj}(A)$. We may and do view P as a scheme over $\operatorname{Spec}(A_0) = Z$. Set $L = \operatorname{Proj}(B)$. We may and do view L as a scheme over $\operatorname{Spec}(B_0) = \operatorname{Spec}(A) = X$; observe that the identification of B_0 with A is given by σ. The surjection τ defines a closed immersion $0 : P \to L$. Since $A \xrightarrow{\sigma} B \to A$ is equal to the map $A \to A_0 \to A$ we conclude that

$$\begin{array}{ccc} P & \xrightarrow[0]{} & L \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is commutative.

We claim that ψ defines a morphism $L \to P$. To see this, by Constructions, Lemma 27.11.1, it suffices to check $\psi(A_+) \not\subset \mathfrak{p}$ for every homogeneous prime ideal $\mathfrak{p} \subset B$ with $B_+ \not\subset \mathfrak{p}$. First, pick $g \in B_+$ homogeneous $g \notin \mathfrak{p}$. Then we can write g as a finite sum $g = \sum a_i^{(d)}$ with $a_i \in A_{d_i}$ for some $d_i \geq d$. We conclude that there exist $d' \geq d$ and $a \in A_{d'}$ such that $a^{(d)} \notin \mathfrak{p}$. Then

$$(a^{(d)})^{d'} = (a^{d'})^{(d'd)} = a^{(d)}(a^{d'-1})^{(d(d'-1))} = \psi(a)(a^{d'-1})^{(d(d'-1))}$$

(the notation leaves something to be desired) is not in $\mathfrak{p}$. Hence $\psi(a) \notin \mathfrak{p}$, proving the claim. Thus we can extend our diagram above to a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow[0]{} & L & \xrightarrow[\pi]{} & P \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & X & \longrightarrow & Z \end{array}$$

where $X \to Z$ is given by $A_0 \to A$. Since $\tau \circ \psi = \mathrm{id}_A$ we see $\pi \circ 0 = \mathrm{id}_P$.

Observe that π is an affine morphism. This is clear from the construction in Constructions, Lemma 27.11.1. In fact, if $f \in A_d$ for some $d > 0$, then setting $g = \psi(f)$ we have $\pi^{-1}(D_+(f)) = D_+(g)$. In this case we have the following equality of homogeneous parts

$$(B[1/g])_{m'} = \bigoplus\nolimits_{m \geq m'} (A[1/f])_m$$

This isomorphism is compatible with further localization. Taking $m' = 0$ we see that $\pi_*\mathcal{O}_L$ is the direct sum of $\mathcal{O}_P(m)$ for $m \geq 0$[9]. We conclude L is identified with the relative spectrum:

$$L = \underline{\mathrm{Spec}}_P \left(\bigoplus\nolimits_{m \geq 0} \mathcal{O}_P(m) \right)$$

In particular $L \to P$ is a cone[10], see Constructions, Section 27.7. Moreover, it is clear that $0 : P \to L$ is the vertex of the cone.

Let $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as in the previous paragraph. Looking at the structure of the ring maps

$$\begin{array}{ccccc} A_0 & \longrightarrow & A & \longrightarrow & A_0 \\ \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow \\ (A[1/f])_0 & \xrightarrow{\psi} & (B[1/g])_0 = \bigoplus_{m \geq 0} (A[1/f])_m & \xrightarrow{\tau} & (A[1/f])_0 \end{array}$$

some compuations[11] in graded rings will show that

(1) $\sigma(A_+)(B[1/g])_0 \subset \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$,
(2) $\sigma(f) \in (B[1/g])_0$ is a nonzerodivisor,
(3) $\sigma(f)(B[1/g])_0 = \sigma(A_d)(B[1/g])_0$ as ideals,
(4) $\sigma(f)(B[1/g])_0$ and $\mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$ have the same radical,
(5) if $d = 1$, then $\sigma(f)(B[1/g])_0 = \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$.

We see in particular that

$$0(D_+(f)) = V(\sigma(f)) \subset D_+(g) = \mathrm{Spec}((B[1/g])_0)$$

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[9] It similarly follows that $\pi_*\mathcal{O}_L(i) = \bigoplus_{m \geq -i} \mathcal{O}_P(m)$.

[10] Often L is a line bundle over P, see below.

[11] Parts (1) and (2) are clear. To see (3), note that if $a \in A_d$, then $\sigma(a) = \sigma(f)\psi(a/f)$. For (4) note that b/g^m is in the kernel of τ if and only if $b \in A_{\geq md}$ maps to zero in A_{md}. Thus it suffices to show if $m' > md$ and $a \in A_{m'}$, then some power of $a^{(md)}/g^m$ is in the ideal generated by $\sigma(f)$. Take e such that $em' - emd \geq d$. Then

$$(a^{(md)}/g^m)^e = (a^e)^{(emd)}/g^{em} = (fa^e)^{(emd+d)}/g^{em+1} = \sigma(f) \cdot (a^e)^{(emd+d)}/g^{em+1}$$

as desired (apologies for the terrible notation). To see (5) argue as before and note that $a^{(md)}/g^m = \sigma(f) \cdot a^{(md+1)}/g^{m+1}$ if $d = 1$.

set theoretically. In other words, the ideal generated by $\sigma(A_d)$ cuts out an effective Cartier divisor on $D_+(g)$ which is set theoretically equal to the image of the closed immersion $0 : P \to L$.

We claim that $L \to X$ is an isomorphism over U. Namely, if $f \in A_d$ for some $d > 0$, then

$$\mathrm{Spec}(A_f) \times_X L = \mathrm{Proj}(A_f \otimes_A B) = \mathrm{Proj}(B_{\sigma(f)})$$

For each e we have $(B_{\sigma(f)})_e = A_f \otimes_B B_e = A_f \otimes_A A_{\geq e} = A_f$, the final equality induced by the injection $A_{\geq e} \subset A$. Hence $B_{\sigma(f)} \cong A_f[T]$ with T in degree 1. This proves the claim as $\mathrm{Proj}(A_f[T]) \to \mathrm{Spec}(A_f)$ is an isomorphism. From now on we identify U with the corresponding open of L.

The identification made in the previous paragraph lets us consider the restriction $\pi|_U : U \to P$. Pick $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as we have done above several times. Then

$$U \cap \pi^{-1}(D_+(f)) = U \cap D_+(g)$$

is the complement of the zero locus of $\sigma(f) \in (B[1/g])_0$ via the identification of $D_+(g)$ with the spectrum of $(B[1/g])_0$. This is assertion (4) above. Therefore $U \cap D_+(g)$ is affine and

$$\mathcal{O}_L(U \cap D_+(g)) = (B[1/g])_0[1/\sigma(f)] = \bigoplus\nolimits_{m \in \mathbf{Z}} (A[1/f])_m$$

where the last equal sign is the natural extension of the identification $(B[1/g])_0 = \bigoplus_{m \geq 0}(A[1/f])_m$ made above. Exactly as we did before with $\pi : L \to P$ we conclude that $\pi|_U : U \to P$ is affine and

$$U = \underline{\mathrm{Spec}}_P \left(\bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m) \right)$$

as schemes over P.

Summarising the above, our constructions produce a commutative diagram

0EKG (37.51.0.1)
$$\begin{array}{ccccc} \underline{\mathrm{Spec}}_P \left(\bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m) \right) & \longrightarrow & L = \underline{\mathrm{Spec}}_P \left(\bigoplus_{m \geq 0} \mathcal{O}_P(m) \right) & \xrightarrow{\pi} & P \\ \| & & \downarrow \sigma & & \downarrow \\ U & \longrightarrow & X & \longrightarrow & Z \end{array}$$

of schemes where π is a cone whose zero section $0 : P \to L$ maps set theoretically onto the inverse image of Z in L.

Let $W \subset P$ be the largest open such that $\mathcal{O}_P(1)|_W$ is invertible and the natural maps induce isomorphisms $\mathcal{O}_P(m)|_W \cong \mathcal{O}_P(1)^{\otimes m}|_W$ for all $m \in \mathbf{Z}$, i.e., the open of Constructions, Lemma 27.10.4 for $d = 1$. Then we see that $L|_W = \pi^{-1}(W) \to W$ is a vector bundle (Constructions, Section 27.6) of rank 1, namely,

$$L|_W = \mathbf{V}(\mathcal{O}_P(1)|_W)$$

in Grothendieckian notation. This is immediate from the above showing that $L|_W$ is equal to the relative spectrum of the symmetric algebra over $\mathcal{O}_W$ on $\mathcal{O}_P(1)|_W$. Then clearly the morphism $0|_W : W \to L|_W$ is the zero section of this vector bundle. In particular $0(W)$ is an effective Cartier divisor on $L|_W$. Moreover, the open $U|_W = (\pi|_U)^{-1}(W)$ is the complement of the zero section.

If A is generated by $f_1, \ldots, f_r \in A_1$ over A_0, then $(f_1, \ldots, f_r)^m = A_{\geq m}$ for all $m \geq 0$ and hence our B above is the Rees algebra for $A_+ = (f_1, \ldots, f_r)$. Thus in this case $L \to X$ is the blowup of Z and $W = P$ where W is as in the preceding paragraph.

If P is quasi-compact, then for d sufficiently divisible, the closed subscheme $D \subset L$ cut out by $\sigma(A_d)\mathcal{O}_L$ is an effective Cartier divisor, $0 : P \to L$ factors through D, and $0(P) = D$ set theoretically. This follows from Constructions, Lemma 27.8.9 and (1), (2), (3), and (4) proved above. (Take any d divisible by the lcm of the degrees of the elements found in the lemma.)

We continue to assume P is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_P$-module. Let us set $\mathcal{F}_U = \pi^*\mathcal{F}|_U$. Then we have

0EKH (37.51.0.2)
$$R\Gamma(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} R\Gamma(P, \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m))$$

Moreover, this direct sum decomposition is functorial in $\mathcal{F}$ and the induced A-module structure on the right is the same as the A-module structure on the left coming from $U \subset X$. To prove the formula, since $\pi|_U$ is affine and $(\pi|_U)_*\mathcal{O}_U = \bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m)$ we get

$$\begin{aligned} R(\pi|_U)_*\mathcal{F}_U &= (\pi|_U)_*\mathcal{F}_U \\ &= (\pi|_U)_*(\pi|_U)^*\mathcal{F} \\ &= \mathcal{F} \otimes_{\mathcal{O}_P} \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m) \\ &= \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) \end{aligned}$$

By Leray we find that $R\Gamma(U, \mathcal{F}_U) = R\Gamma(P, R(\pi|_U)_*\mathcal{F}_U)$, see Cohomology, Lemma 20.13.6. The proof is finished because taking cohomology commutes with direct sums in this case, see Derived Categories of Schemes, Lemma 36.4.5. This is where we use that P is quasi-compact; P is separated by Constructions, Lemma 27.8.8.

0EKI **Lemma 37.51.1.** *Let R be a ring. Let P be a proper scheme over R and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Set $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$. Then $P = Proj(A)$ and diagram (37.51.0.1) becomes the diagram*

$$\begin{array}{ccccc} \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \in \mathbf{Z}} \mathcal{L}^{\otimes m}\right) & \longrightarrow & L = \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}\right) & \xrightarrow{\pi} & P \\ \| & & \downarrow{\scriptstyle \sigma} & & \downarrow \\ U & \longrightarrow & X & \longrightarrow & Z \end{array}$$

having the properties explained above.

Proof. We have $P = \mathrm{Proj}(A)$ by Morphisms, Lemma 29.43.17. Moreover, by Properties, Lemma 28.28.2 via this identification we have $\mathcal{O}_P(m) = \mathcal{L}^{\otimes m}$ for all $m \in \mathbf{Z}$. □

37.52. Closed points in fibres

053Q Some of the material in this section is taken from the preprint [**OP10**].

053R **Lemma 37.52.1.** *Let $f : X \to S$ be a morphism of schemes. Let $Z \subset X$ be a closed subscheme. Let $s \in S$. Assume*

(1) *S is irreducible with generic point η,*

(2) X *is irreducible,*
(3) f *is dominant,*
(4) f *is locally of finite type,*
(5) $\dim(X_s) \leq \dim(X_\eta)$*,*
(6) Z *is locally principal in* X*, and*
(7) $Z_\eta = \emptyset$*.*

Then the fibre Z_s *is (set theoretically) a union of irreducible components of* X_s*.*

Proof. Let X_{red} denote the reduction of X. Then $Z \cap X_{red}$ is a locally principal closed subscheme of X_{red}, see Divisors, Lemma 31.13.11. Hence we may assume that X is reduced. In other words X is integral, see Properties, Lemma 28.3.4. In this case the morphism $X \to S$ factors through S_{red}, see Schemes, Lemma 26.12.7. Thus we may replace S by S_{red} and assume that S is integral too.

The assertion that f is dominant signifies that the generic point of X is mapped to η, see Morphisms, Lemma 29.8.7. Moreover, the scheme X_η is an integral scheme which is locally of finite type over the field $\kappa(\eta)$. Hence $d = \dim(X_\eta) \geq 0$ is equal to $\dim_\xi(X_\eta)$ for every point ξ of X_η, see Algebra, Lemmas 10.114.4 and 10.114.5. In view of Morphisms, Lemma 29.28.4 and condition (5) we conclude that $\dim_x(X_s) = d$ for every $x \in X_s$.

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists $x \in Z_s$ which is a generic point of an irreducible component of Z_s but not a generic point of any irreducible component of X_s. Then we see that $\dim_x(Z_s) \leq d - 1$, because $\dim_x(X_s) = d$ and in a neighbourhood of x in X_s the closed subscheme Z_s does not contain any of the irreducible components of X_s. Hence after replacing X by an open neighbourhood of x we may assume that $\dim_z(Z_{f(z)}) \leq d - 1$ for all $z \in Z$, see Morphisms, Lemma 29.28.4. Let $\xi' \in Z$ be a generic point of an irreducible component of Z and set $s' = f(\xi)$. As $Z \neq X$ is locally principal we see that $\dim(\mathcal{O}_{X,\xi}) = 1$, see Algebra, Lemma 10.60.11 (this is where we use X is Noetherian). Let $\xi \in X$ be the generic point of X and let ξ_1 be a generic point of any irreducible component of $X_{s'}$ which contains ξ'. Then we see that we have the specializations

$$\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.$$

As $\dim(\mathcal{O}_{X,\xi}) = 1$ one of the two specializations has to be an equality. By assumption $s' \neq \eta$, hence the first specialization is not an equality. Hence $\xi' = \xi_1$ is a generic point of an irreducible component of $X_{s'}$. Applying Morphisms, Lemma 29.28.4 one more time this implies $\dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_\eta) = d$ which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point $x \in X$ lying over s such that x is a generic point of an irreducible component of Z_s, but not a generic point of any of the irreducible components of X_s. Let $U \subset S$ be an affine neighbourhood of s and let $V \subset X$ be an affine neighbourhood of x with $f(V) \subset U$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ so that $f|_V$ is given by a ring map $A \to B$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime corresponding to x, resp. s. After possibly shrinking V we may assume $Z \cap V$ is cut out by some element $g \in B$. Denote K the fraction field of A. What we know at this point is the following:

(1) $A \subset B$ is a finitely generated extension of domains,

(2) the element $g \otimes 1$ is invertible in $B \otimes_A K$,
(3) $d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(\mathfrak{p}))$,
(4) $g \otimes 1$ is not a unit of $B \otimes_A \kappa(\mathfrak{p})$, and
(5) $g \otimes 1$ is not in any of the minimal primes of $B \otimes_A \kappa(\mathfrak{p})$.

We are seeking a contradiction.

Pick elements $x_1, \ldots, x_n \in B$ which generate B over A. For a finitely generated $\mathbf{Z}$-algebra $A_0 \subset A$ let $B_0 \subset B$ be the A_0-subalgebra generated by $x_1, \ldots, x_n$, denote K_0 the fraction field of A_0, and set $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$. We claim that when A_0 is large enough then (1) – (5) also hold for the system $(A_0 \subset B_0, g, \mathfrak{p}_0)$.

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write $(g \otimes 1)h = 1$ for some $h \otimes 1/a \in B \otimes_A K$. Write $g = \sum a_I x^I$, $h = \sum a'_I x^I$ (multi-index notation) for some coefficients $a_I, a'_I \in A$. As soon as A_0 contains a and the a_I, a'_I then (2) holds because $B_0 \otimes_{A_0} K_0 \subset B \otimes_A K$ (as localizations of the injective map $B_0 \to B$). To achieve (3) consider the exact sequence

$$0 \to I \to A[X_1, \ldots, X_n] \to B \to 0$$

which defines I where the second map sends X_i to x_i. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(\mathfrak{p})$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(\mathfrak{p})[X_1, \ldots, X_n] \to B \otimes_A \kappa(\mathfrak{p})$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I$, $j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(\mathfrak{p})$. Write $h_j = \sum a_{j,I} X^I$. As soon as A_0 contains all $a_{j,I}$ we get to the situation where

$$B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \otimes_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p}).$$

By either Morphisms, Lemma 29.28.3 or Algebra, Lemma 10.116.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \subset B \otimes_A \kappa(\mathfrak{p})$ each minimal prime of $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$ lies under a minimal prime of $B \otimes_A \kappa(\mathfrak{p})$ by Algebra, Lemma 10.30.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. □

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 37.52.3.

053S **Lemma 37.52.2.** *Let $A \to B$ be a local homomorphism of local rings, and $g \in \mathfrak{m}_B$. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A,*
(3) *g is not contained in any minimal prime over $\mathfrak{m}_A B$, and*
(4) *$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B)$.*

Then $A \subset B/gB$, i.e., the generic point of $\mathrm{Spec}(A)$ *is in the image of the morphism* $\mathrm{Spec}(B/gB) \to \mathrm{Spec}(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma 10.30.6. To start the proof let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_{\mathfrak{q}}$. Of course we may assume that C is a domain and that $g \in C$. After replacing C by a localization we see that $\dim(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \mathrm{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 29.28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Hence assumption (4) means

that the generic and closed fibres of the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_{\mathfrak{q}} \otimes_A K$. As $C_{\mathfrak{q}}$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \notin \mathfrak{q}$. Thus after replacing C by C_h we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of C to make sure that the minimal primes of $C/\mathfrak{m}_A C$ correspond one-to-one with the minimal primes of $B/\mathfrak{m}_A B$. At this point we apply Lemma 37.52.1 to $X = \mathrm{Spec}(C) \to \mathrm{Spec}(A) = S$ and the locally closed subscheme $Z = \mathrm{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(\mathfrak{m}_A)$ has to contain an irreducible component of $X \otimes \kappa(\mathfrak{m}_A) = \mathrm{Spec}(C/\mathfrak{m}_A C)$. But this contradicts the assumption that g is not contained in any prime minimal over $\mathfrak{m}_A B$. The lemma follows. □

053T **Lemma 37.52.3.** *Let $A \to B$ be a local homomorphism of local rings. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A, and*
(3) *B is flat over A.*

Then we have

$$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B).$$

Proof. Let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_{\mathfrak{q}}$. We may assume C is a domain. We have $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \mathrm{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 29.28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$. Choose a valuation ring A' in K dominating A, see Algebra, Lemma 10.50.2. Set $C' = C \otimes_A A'$. Choose a prime $\mathfrak{q}'$ of C' lying over $\mathfrak{q}$; such a prime exists because

$$C'/\mathfrak{m}_{A'} C' = C/\mathfrak{m}_A C \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_{A'})$$

which proves that $C/\mathfrak{m}_A C \to C'/\mathfrak{m}_{A'} C'$ is faithfully flat. This also proves that $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C')$, see Algebra, Lemma 10.116.6. Note that $B' = C'_{\mathfrak{q}'}$ is a localization of $B \otimes_A A'$. Hence B' is flat over A'. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence B' is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where A is a valuation ring.

Let $A \subset B$ be as in the lemma with A a valuation ring. As before write $B = C_{\mathfrak{q}}$ for some domain C of finite type over A. By Algebra, Lemma 10.125.9 we obtain $\dim(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ and we win. □

053U **Lemma 37.52.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a specialization of points in X. Set $s = f(x)$ and $s' = f(x')$. Assume*

(1) *x' is a closed point of $X_{s'}$, and*
(2) *f is locally of finite type.*

Then the set

$$\{x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \rightsquigarrow x_1 \rightsquigarrow x'\}$$

is dense in the closure of x in X_s.

Proof. We apply Schemes, Lemma 26.20.4 to the specialization $x \rightsquigarrow x'$. This produces a morphism $\varphi : \mathrm{Spec}(B) \to X$ where B is a valuation ring such that φ maps the generic point to x and the closed point to x'. We may also assume that $\kappa(x)$ is the fraction field of B. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see Algebra, Lemma 10.50.7) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(B) & \longrightarrow & X_A & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

The generic (resp. closed) point of B maps to a point x_A (resp. x'_A) of X_A lying over the generic (resp. closed) point of $\mathrm{Spec}(A)$. Note that x'_A is a closed point of the special fibre of X_A by Morphisms, Lemma 29.20.4. Note that the generic fibre of $X_A \to \mathrm{Spec}(A)$ is isomorphic to X_s. Thus we have reduced the lemma to the case where S is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of X_η specializing to x and x is closed in its fibre, see Morphisms, Lemma 29.20.6, and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\overline{\{x\}}$ of X, see Schemes, Definition 26.12.5. To prove the lemma we may replace X by X' as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that X is an integral scheme and that x is its generic point. In addition, we may replace X by an affine neighbourhood of x'. Thus we have $X = \mathrm{Spec}(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 10.125.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to "avoid"). As X_s is of finite type over a field we see that W has finitely many irreducible components $W = W_1 \cup \ldots \cup W_n$. Let $\mathfrak{q}_j \subset B$, $j = 1, \ldots, r$ be the corresponding prime ideals. Let $\mathfrak{q} \subset B$ be the maximal ideal corresponding to the point x'. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \subset B$ be the minimal primes lying over $\mathfrak{m}_A B$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension > 0 (see above) hence we see that $\mathfrak{p}_i \neq \mathfrak{q}$ for all i. Now, pick an element $g \in \mathfrak{q}$ such that $g \notin \mathfrak{q}_j$ for all j and $g \notin \mathfrak{p}_i$ for all i, see Algebra, Lemma 10.15.2. Denote $Z \subset X$ the locally principal closed subscheme defined by g. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of Z into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing X by a smaller affine neighbourhood we may assume that $x' \in Z_i$ for each $i = 1, \ldots, n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 37.52.1 we conclude that $Z_\eta \neq \emptyset$! In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension ≥ 2 in X_η whereas $\mathrm{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 10.60.11. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \to S$ and the specialization $x_1 \rightsquigarrow x'$.

This produces a closed point x_2 of $Z_{1,\eta}$ not contained in W which specializes to x'. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point x_2 is closed in X_η, and $x_2 \notin W$ as desired. □

053V **Remark 37.52.5.** The proof of Lemma 37.52.4 actually shows that there exists a sequence of specializations

$$x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_d \rightsquigarrow x'$$

where all x_i are in the fibre X_s, each specialization is immediate, and x_d is a closed point of X_s. The integer $d = \mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \dim(\overline{\{x\}})$ where the closure is taken in X_s. Moreover, the points x_i can be chosen to avoid any closed subset of X_s which does not contain the point x.

Examples, Section 110.39 shows that the following lemma is false if A is not assumed Noetherian.

05GT **Lemma 37.52.6.** *Let $\varphi : A \to B$ be a local ring map of local rings. Let $V \subset \mathrm{Spec}(B)$ be an open subscheme which contains at least one prime not lying over $\mathfrak{m}_A$. Assume A is Noetherian, φ essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite. Then there exists a $\mathfrak{q} \in V$, $\mathfrak{m}_A \neq \mathfrak{q} \cap A$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map.*

Proof. Since A is Noetherian and $A \to B$ is essentially of finite type, we know that B is Noetherian too. By Properties, Lemma 28.6.4 the topological space $\mathrm{Spec}(B)\backslash\{\mathfrak{m}_B\}$ is Jacobson. Hence we can choose a closed point $\mathfrak{q}$ which is contained in the nonempty open

$$V \setminus \{\mathfrak{q} \subset B \mid \mathfrak{m}_A = \mathfrak{q} \cap A\}.$$

(Nonempty by assumption, open because $\{\mathfrak{m}_A\}$ is a closed subset of $\mathrm{Spec}(A)$.) Then $\mathrm{Spec}(B/\mathfrak{q})$ has two points, namely $\mathfrak{m}_B$ and $\mathfrak{q}$ and $\mathfrak{q}$ does not lie over $\mathfrak{m}_A$. Write $B/\mathfrak{q} = C_{\mathfrak{m}}$ for some finite type A-algebra C and prime ideal $\mathfrak{m}$. Then $A \to C$ is quasi-finite at $\mathfrak{m}$ by Algebra, Lemma 10.122.2 (2). Hence by Algebra, Lemma 10.123.13 we see that after replacing C by a principal localization the ring map $A \to C$ is quasi-finite. □

05GU **Lemma 37.52.7.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $U \subset X$ be an open subscheme. Assume f locally of finite type, S locally Noetherian, x a closed point of X_s, and assume there exists a point $x' \in U$ with $x' \rightsquigarrow x$ and $f(x') \neq s$. Then there exists a closed subscheme $Z \subset X$ such that (a) $x \in Z$, (b) $f|_Z : Z \to S$ is quasi-finite at x, and (c) there exists a $z \in Z$, $z \in U$, $z \rightsquigarrow x$ and $f(z) \neq s$.*

Proof. This is a reformulation of Lemma 37.52.6. Namely, set $A = \mathcal{O}_{S,s}$ and $B = \mathcal{O}_{X,x}$. Denote $V \subset \mathrm{Spec}(B)$ the inverse image of U. The ring map $f^\sharp : A \to B$ is essentially of finite type. By assumption there exists at least one point of V which does not map to the closed point of $\mathrm{Spec}(A)$. Hence all the assumptions of Lemma 37.52.6 hold and we obtain a prime $\mathfrak{q} \subset B$ which does not lie over $\mathfrak{m}_A$ and such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. Let $z \in X$ be the image of the point $\mathfrak{q}$ under the canonical morphism $\mathrm{Spec}(B) \to X$. Set $Z = \overline{\{z\}}$ with the induced reduced scheme structure. As $z \rightsquigarrow x$ we see that $x \in Z$ and $\mathcal{O}_{Z,x} = B/\mathfrak{q}$. By construction $Z \to S$ is quasi-finite at x. □

05GV **Remark 37.52.8.** We can use Lemma 37.52.6 or its variant Lemma 37.52.7 to give an alternative proof of Lemma 37.52.4 in case S is locally Noetherian. Here is a rough sketch. Namely, first replace S by the spectrum of the local ring at s'. Then we may use induction on $\dim(S)$. The case $\dim(S) = 0$ is trivial because then $s' = s$. Replace X by the reduced induced scheme structure on $\overline{\{x\}}$. Apply Lemma 37.52.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing x. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at x' and such that $f(z) \neq s'$. Then z is a closed point of $X_{f(z)}$, and $z \rightsquigarrow x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since x is the generic point of X we see $x \rightsquigarrow z$, hence $s = f(x) \rightsquigarrow f(z)$. Apply the induction hypothesis to $s \rightsquigarrow f(z)$ and $z \mapsto f(z)$ to win.

05GW **Lemma 37.52.9.** *Suppose that $f : X \to S$ is locally of finite type, S locally Noetherian, $x \in X$ a closed point of its fibre X_s, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in \overline{U}$, then the conclusions of Lemma 37.52.7 hold.*

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace X and S by affine neighbourhoods of x and s. Then X is Noetherian, in particular U is quasi-compact (see Morphisms, Lemma 29.15.6 and Topology, Lemmas 5.9.2 and 5.12.13). Hence there exists a specialization $x' \rightsquigarrow x$ with $x' \in U$ (see Morphisms, Lemma 29.6.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. □

37.53. Stein factorization

03GX Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

03GY **Lemma 37.53.1.** *Let S be a scheme. Let $f : X \to S$ be a universally closed and quasi-separated morphism. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with the following properties:

1. *the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,*
2. *the morphism $\pi : S' \to S$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X, see Morphisms, Definition 29.53.3.*

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms, Lemma 29.41.8 the morphism f is quasi-compact. Hence the normalization S' of S in X is defined (Morphisms, Definition 29.53.3) and we have the factorization $X \to S' \to S$. By Morphisms, Lemma 29.53.11 we have (2), (4), and (5). The morphism f' is universally closed by Morphisms, Lemma 29.41.7. It is quasi-compact by Schemes, Lemma 26.21.14 and quasi-separated by Schemes, Lemma 26.21.13.

To show the remaining statements we may assume the base scheme S is affine, say $S = \mathrm{Spec}(R)$. Then $S' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 26.24.1, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. As every element of I is nilpotent, we conclude that $V(I) = S'$ as desired.

By Cohomology of Schemes, Lemma 30.5.2 we see that formation of $f_*\mathcal{O}_X$ commutes with flat base change. Formation of the relative spectrum commutes with any base change by Constructions, Lemma 27.4.6. Thus formation of the factorization commutes with flat base change. □

0E0M **Lemma 37.53.2.** *In Lemma 37.53.1 assume in addition that f is locally of finite type. Then for $s \in S$ the fibre $\pi^{-1}(\{s\}) = \{s_1, \ldots, s_n\}$ is finite and the field extensions $\kappa(s_i)/\kappa(s)$ are finite.*

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{s\})$, see Algebra, Lemma 10.36.20. As f' is surjective, we find that $|X_s| \to \pi^{-1}(\{s\})$ is surjective. Observe that X_s is a quasi-separated scheme of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus X_s is Noetherian (Morphisms, Lemma 29.15.6). A topological argument (omitted) now shows that $\pi^{-1}(\{s\})$ is finite. For each i we can pick a finite type point $x_i \in X_s$ mapping to s_i (Morphisms, Lemma 29.16.7). We conclude that $\kappa(s_i)/\kappa(s)$ is finite: x_i can be represented by a morphism $\mathrm{Spec}(k_i) \to X_s$ of finite type (by our definition of finite type points) and hence $\mathrm{Spec}(k_i) \to s = \mathrm{Spec}(\kappa(s))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(s)$ is finite (Morphisms, Lemma 29.16.1). □

03GZ **Lemma 37.53.3.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then X_s is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre X_u.*

Proof. If X_s is geometrically connected, then any base change of it is connected. On the other hand, suppose that X_s is not geometrically connected. Then by Varieties, Lemma 33.7.11 we see that $X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(k)$ is disconnected for some finite separable field extension $k/\kappa(s)$. By Lemma 37.35.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(u)/\kappa(s)$ is identified with $k/\kappa(s)$. In this case X_u is disconnected. □

03H0 **Theorem 37.53.4** (Stein factorization; Noetherian case)**.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

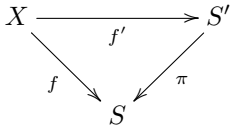

with the following properties:

(1) *the morphism f' is proper with geometrically connected fibres,*
(2) *the morphism $\pi : S' \to S$ is finite,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
(4) *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
(5) *S' is the normalization of S in X, see Morphisms, Definition 29.53.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 37.53.1. Note that besides the conclusions of Lemma 37.53.1 we also have that f' is separated (Schemes, Lemma 26.21.13) and finite type (Morphisms, Lemma 29.15.8). Hence f' is proper. By Cohomology of Schemes, Proposition 30.19.1 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of f' are geometrically connected. It is clear from the discussion above that we may replace S by S', and we may therefore assume that S is Noetherian, affine, $f : X \to S$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of S. We have to show that X_s is geometrically connected. By Lemma 37.53.3 we see that it suffices to show X_u is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume U is affine. Thus U is Noetherian (Morphisms, Lemma 29.15.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 29.41.5), and that also $(f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$ (Cohomology of Schemes, Lemma 30.5.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre X_s is connected when $f_*\mathcal{O}_X = \mathcal{O}_S$. We can deduce this from Derived Categories of Schemes, Lemma 36.32.7 (by looking at idempotents in the structure sheaf of X_s) but we will also give a direct argument below.

Namely, we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 30.20.7. It tells us that

$$\mathcal{O}_{S,s}^\wedge = (f_*\mathcal{O}_X)_s^\wedge = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we see that if $X_s = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}_{S,s}^\wedge$ is a local ring. Thus the assumption was wrong, i.e., X_s is connected, and we win. □

03H2 **Theorem 37.53.5** (Stein factorization; general case)**.** *Let S be a scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

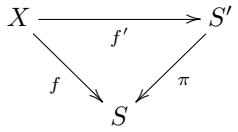

with the following properties:

(1) *the morphism f' is proper with geometrically connected fibres,*
(2) *the morphism $\pi : S' \to S$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
(4) *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*

(5) *S' is the normalization of S in X, see Morphisms, Definition 29.53.3.*

Proof. We may apply Lemma 37.53.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 37.53.1 we also have that f' is separated (Schemes, Lemma 26.21.13) and finite type (Morphisms, Lemma 29.15.8). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has geometrically connected fibres.

We may assume that $S = \mathrm{Spec}(R)$ is affine. Set $R' = \Gamma(X, \mathcal{O}_X)$. Then $S' = \mathrm{Spec}(R')$. Thus we may replace S by S' and assume that $S = \mathrm{Spec}(R)$ is affine $R = \Gamma(X, \mathcal{O}_X)$. Next, let $s \in S$ be a point. Let $U \to S$ be an étale morphism of affine schemes and let $u \in U$ be a point mapping to s. Let $X_U \to U$ be the base change of X. By Lemma 37.53.3 it suffices to show that the fibre of $X_U \to U$ over u is connected. By Cohomology of Schemes, Lemma 30.5.2 we see that $\Gamma(X_U, \mathcal{O}_{X_U}) = \Gamma(U, \mathcal{O}_U)$. Hence we have to show: Given $S = \mathrm{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X, \mathcal{O}_X) = R$ and $s \in S$ is a point, the fibre X_s is connected.

To do this it suffices to show that the only idempotents $e \in H^0(X_s, \mathcal{O}_{X_s})$ are 0 and 1 (we already know that X_s is nonempty by Lemma 37.53.1). By Derived Categories of Schemes, Lemma 36.32.7 after replacing R by a principal localization we may assume e is the image of an element of R. Since $R \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $\kappa(s)$ we conclude. □

Here is an application.

0AY8 **Lemma 37.53.6.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper,*
(2) *S is integral with generic point ξ,*
(3) *S is normal,*
(4) *X is reduced,*
(5) *every generic point of an irreducible component of X maps to ξ,*
(6) *we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.*

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and f has geometrically connected fibres.

Proof. Apply Theorem 37.53.5 to get a factorization $X \to S' \to S$. It is enough to show that $S' = S$. This will follow from Morphisms, Lemma 29.54.8. Namely, S' is reduced because X is reduced (Morphisms, Lemma 29.53.8). The morphism $S' \to S$ is integral by the theorem cited above. Every generic point of S' lies over ξ by Morphisms, Lemma 29.53.9 and assumption (5). On the other hand, since S' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre S'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence S' is an integral scheme with function field equal to the function field of S. This finishes the proof. □

Here is another application.

0BUI **Lemma 37.53.7.** *Let $X \to S$ be a flat proper morphism of finite presentation. Let $n_{X/S}$ be the function on S counting the numbers of geometric connected components of fibres of f introduced in Lemma 37.28.3. Then $n_{X/S}$ is lower semi-continuous.*

Proof. Let $s \in S$. Set $n = n_{X/S}(s)$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at s is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of s

such that $n_{X/S}|_V \geq n$. Let $X \to S' \to S$ be the Stein factorization as in Theorem 37.53.5. By Lemma 37.53.2 there are finitely many points $s'_1, \ldots, s'_m \in S'$ lying over s and the extensions $\kappa(s'_i)/\kappa(s)$ are finite. Then Lemma 37.42.1 tells us that after replacing S by an étale neighbourhood of s we may assume $S' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $s'_i \in V_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. Then the schemes $X_{s'_i}$ are geometrically connected over $\kappa(s)$, hence $m = n$. The schemes $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over S. Hence the image of $X_i \to S$ is open (Morphisms, Lemma 29.25.10). Thus in a neighbourhood of s we see that $n_{X/S}$ is at least n. □

0E0N **Lemma 37.53.8.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the geometric fibres of f are reduced.*

Then the function $n_{X/S} : S \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f is locally constant.

Proof. By Lemma 37.53.7 the function $n_{X/S}$ is lower semincontinuous. For $s \in S$ consider the $\kappa(s)$-algebra

$$A_s = H^0(X_s, \mathcal{O}_{X_s})$$

Define a function $\beta_0 : X \to \mathbf{Z}$ sending s to $\dim_{\kappa(s)} A_s$. By Varieties, Lemma 33.9.3 and the fact that X_s is geometrically reduced A_s is finite product of finite separable extensions of $\kappa(s)$. Hence $A_s \otimes_{\kappa(s)} \kappa(\overline{s})$ is a product of $\beta_0(s)$ copies of $\kappa(\overline{s})$. Thus $X_{\overline{s}}$ has $\beta_0(s)$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on S. Thus $n_{X/S}$ is upper semi-continuous by Derived Categories of Schemes, Lemma 36.32.1. This finishes the proof. □

A final application.

0CT9 **Lemma 37.53.9.** *Let (A, I) be a henselian pair. Let $X \to \mathrm{Spec}(A)$ be separated and of finite type. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subscheme such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then there exists an open and closed subscheme $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.*

A reference for the case of an adic Noetherian base is [**DG67**, III, Proposition 5.5.1]

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By Chow's lemma (in the form of Limits, Lemma 32.12.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. Suppose the lemma holds for (X', Y'). Let $W' \subset X'$ be the open and closed subscheme proper over A such that $Y' = W'_0$. By Morphisms, Lemma 29.41.7 $W = \varphi(W') \subset X$ and $Q = \varphi(X' \setminus W') \subset X$ are closed subsets and by Morphisms, Lemma 29.41.9 W is proper over A. The image of $W \cap Q$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $W \cap Q$ is nonempty, then we find that $W \cap Q$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that W is an open and closed subscheme of X proper over A with $W_0 = Y$. Thus we reduce to the case described in the next paragraph.

Assume there exists an immersion $j : X \to \mathbf{P}^n_A$ over A. Let $\overline{X}$ be the scheme theoretic image of j. Since j is a quasi-compact morphism (Schemes, Lemma 26.21.14) we see that $j : X \to \overline{X}$ is an open immersion (Morphisms, Lemma 29.7.7). Hence the base change $j_0 : X_0 \to \overline{X}_0$ is an open immersion as well. Thus

$j_0(Y) \subset \overline{X}_0$ is open. It is also closed by Morphisms, Lemma 29.41.7. Suppose that the lemma holds for $(\overline{X}, j_0(Y))$. Let $\overline{W} \subset \overline{X}$ be the corresponding open and closed subscheme proper over A such that $j_0(Y) = \overline{W}_0$. Then $T = \overline{W} \setminus j(X)$ is closed in $\overline{W}$, hence has closed image in $\mathrm{Spec}(A)$ by properness of $\overline{W}$ over A. Since (A, I) is henselian, we find that if T is nonempty, then there is a point of T mapping into $\mathrm{Spec}(A/I)$. This is impossible because $j_0(Y) = \overline{W}_0$ is contained in $j(X)$. Hence $\overline{W}$ is contained in $j(X)$ and we can set $W \subset X$ equal to the unique open and closed subscheme mapping isomorphically to $\overline{W}$ via j. Thus we reduce to the case described in the next paragraph.

Assume $X \subset \mathbf{P}^n_A$ is a closed subscheme. Then $X \to \mathrm{Spec}(A)$ is a proper morphism. Let $Z = X_0 \setminus Y$. This is an open and closed subscheme of X_0 and $X_0 = Y \amalg Z$. Let $X \to X' \to \mathrm{Spec}(A)$ be the Stein factorization as in Theorem 37.53.5. Let $Y' \subset X'_0$ and $Z' \subset X'_0$ be the images of Y and Z. Since the fibres of $X \to Z$ are geometrically connected, we see that $Y' \cap Z' = \emptyset$. Hence $X'_0 = Y' \amalg Z'$ as $X \to X'$ is surjective. Since $X' \to \mathrm{Spec}(A)$ is integral, we see that X' is the spectrum of an A-algebra integral over A. Recall that open and closed subsets of spectra correspond 1-to-1 with idempotents in the corresponding ring, see Algebra, Lemma 10.21.3. Hence by More on Algebra, Lemma 15.11.6 we see that we may write $X' = W' \amalg V'$ with W' and V' open and closed and with $Y' = W'_0$ and $Z' = V'_0$. Let W be the inverse image in X to finish the proof. □

37.54. Generic flatness stratification

0H3Y We can use generic flatness to construct a stratification of the base such that a given module becomes flat over the strata.

0ASY **Lemma 37.54.1** (Generic flatness stratification)**.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $\mathcal{F}$ pulled back to $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. We can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_0 \end{array}$$

and a finitely presented $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$ such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 32.5.4 and Lemmas 32.10.1 and 32.10.2. Thus we may assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. By generic flatness as stated in Morphisms, Proposition 29.27.2 there is a dense open $U_0 \subset S_0$ such that $\mathcal{F}$ pulled back to $X \times_S U_0$ is flat over U_0. Let $S_1 \subset S_0$ be the reduced closed subscheme whose underlying closed subset is

$S \setminus U_0$. We continue in this way, provided $S_1 \neq \emptyset$, to find $S_0 \supset S_1 \supset \ldots$. Because S is Noetherian any descending chain of closed subsets stabilizes hence we see that $S_t = \emptyset$ for some $t \geq 0$. □

0H3Z **Lemma 37.54.2.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. Apply Lemma 37.54.1 with $\mathcal{F} = \mathcal{O}_X$. □

0H89 **Lemma 37.54.3** (Longke Tang)**.** *Let $f : X \to S$ be a surjective morphism of finite presentation. Let M be an object of $D_{QCoh}(\mathcal{O}_S)$ such that $H^i(M) = 0$ for $i > 0$. The following are equivalent*

1. *M is isomorphic to a flat $\mathcal{O}_S$-module placed in degree 0,*
2. *Lf^*M is isomorphic to a flat $\mathcal{O}_X$-module placed in degree 0.*

Proof. We omit the proof that (1) implies (2). Assume (2). To prove (1) is a local question, hence we may assume that S is affine. Choose a stratification $S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$ as in Lemma 37.54.2. Set $T_i = S_i \setminus S_{i-1}$ and $Y_i = X \times_S T_i$. Denote $g_i : T_i \to S$ and $h_i : Y_i \to X$ the inclusion morphisms. Since f is surjective, the morphisms $f_i : X_i \to T_i$ are surjective and flat. Since $Lh_i^* \circ Lf^* = Lf_i^* \circ Lg_i^*$ we conclude from (2) that $Lf_i^* Lg_i^* M$ is isomorphic to a flat $\mathcal{O}_{Y_i}$-module placed in degree 0. Since f_i is flat and surjective, we see that $Lg_i^* M$ is isomorphic to a flat $\mathcal{O}_{T_i}$-module placed in degree 0.

We will prove this implies the result by induction on t. We will use the following notation: $S = \mathrm{Spec}(A)$ and M is the object of $D_{QCoh}(\mathcal{O}_S)$ corresponding to N in $D(A)$, see Derived Categories of Schemes, Lemma 36.3.5.

Base case: $t = 1$. Write $S_0 = \mathrm{Spec}(A/I)$ where I is a finite generated ideal. Then I is nilpotent as $S_0 = S$ set theoretically. The assumption is that $N \otimes^{\mathbf{L}}_A A/I$ has tor amplitude in $[0, 0]$. By More on Algebra, Lemma 15.67.20 the same is true for N.

Induction step. Assume $t > 1$. Write $S_{t-1} = \mathrm{Spec}(A/I)$ where $I = (f_1, \ldots, f_r)$ is a finitely generated ideal. We will argue by induction on r. Observe that $N_{f_r} = N \otimes^{\mathbf{L}}_A A_{f_r}$ in $D(A_{f_r})$ has tor amplitude in $[0, 0]$ by induction hypothesis (because the stratification on the principal open $D(f_r)$ has length at most $t - 1$). On the other hand, consider $N' = N \otimes^{\mathbf{L}}_A A/f_r A$ in $D(A/f_r A)$. Setting $S' = \mathrm{Spec}(A/f_r A)$ we obtain a stratification

$$S' \supset S' \cap S_0 \supset S' \cap S_1 \supset \ldots \supset S' \cap S_{t-1} \supset S' \cap S_t = \emptyset$$

where $S' \cap S_{t-1}$ is cut out by $r - 1$ equations in S'. Similarly to before, the derived pullbacks of M to the parts $S' \cap S_i \setminus S' \cap S_{i+1}$ are of tor amplitude in $[0, 0]$. Whence by induction on r we conclude that N' has tor amplitude in $[0, 0]$. Then we finally conclude that N has tor amplitude in $[0, 0]$ by More on Algebra, Lemma 15.103.7. □

0H40 **Lemma 37.54.4.** *Let R be a Noetherian domain. Let $R \to A \to B$ be finite type ring maps. Let M be a finite A-module and let N a finite B-module. Let $M \to N$ be an A-linear map. There exists an nonzero $f \in R$ such that the cokernel of $M_f \to N_f$ is a flat R_f-module.*

Proof. By replacing M by the image of $M \to N$, we may assume $M \subset N$. Choose a filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_t = N$ such that $N_i/N_{i-1} = B/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset B$, see Algebra, Lemma 10.62.1. Set $M_i = M \cap N_i$. Then $Q = N/M$ has a filtration by the submodules $Q_i = N_i/M_i$. It suffices to prove Q_i/Q_{i-1} becomes flat after localizing at a nonzero element of f (since extensions of flat modules are flat by Algebra, Lemma 10.39.13). Since Q_i/Q_{i-1} is isomorphic to the cokernel of the map $M_i/M_{i-1} \to N_i/N_{i-1}$, we reduce to the case discussed in the next paragraph.

Assume B is a domain and $M \subset N = B$. After replacing A by the image of A in B we may assume $A \subset B$. By generic flatness, we may assume A and B are flat over R (Algebra, Lemma 10.118.1). It now suffices to show $M \to B$ becomes R-universally injective after replacing R by a principal localization (Algebra, Lemma 10.82.7). By generic freeness, we can find a nonzero $g \in A$ such that B_g is a free A_g-module (Algebra, Lemma 10.118.1). Thus we may choose a direct summand $M' \subset B_g$ as an A_g-module, which is finite free as an A_g-module, and such that $M \to B \to B_g$ factors through M'. Clearly, it suffices to show that $M \to M'$ becomes R-universally injective after replacing R by a principal localization.

Say $M' = A_g^{\oplus n}$. Since $M \subset M'$ is a finite A-module, we see that M is contained in $(1/g^m)A^{\oplus n}$ for some $m \geq 0$. After changing our basis for M' we may assume $M \subset A^{\oplus n}$. Then it suffices to show that $A^{\oplus n}/M$ and A_g/A become R-flat after replacing R by a principal localization. Namely, then $M \to A^{\oplus n}$ and $A^{\oplus n} \to A_g^{\oplus n}$ are universally injective by Algebra, Lemma 10.39.12 and consequently so is the composition $M \to M' = A_g^{\oplus n}$.

By generic flatness (see reference above), we may assume the module $A^{\oplus n}/M$ is R-flat. For the quotient A_g/A we use the fact that

$$A_g/A = \operatorname{colim}(1/g^m)A/A \cong \operatorname{colim} A/g^m A$$

and the module $A/g^m A$ has a filtration of length m whose successive quotients are isomorphic to A/gA. Again by generic flatness we may assume A/gA is R-flat and hence each $A/g^m A$ is R-flat, and hence so is A_g/A. □

Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $Z \subset Y$ be the scheme theoretic image of f, see Morphisms, Section 29.6. Let $g : S' \to S$ be a morphism of schemes and let $f' : X \times_S S' \to Y \times_S S'$ be the base change of f by g. It is not always true that $Z \times_S S' \subset Y \times_S S'$ is the scheme theoretic image of f'. Let us say that *formation of the scheme theoretic image of f/S commutes with arbitrary base change* if for every g as above the scheme theoretic image of f' is equal to $Z \times_S S'$.

0H41 **Lemma 37.54.5.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with both X and Y of finite presentation over S. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with the following properties:

(1) $S_i \to S$ *is defined by a finite type ideal sheaf,*
(2) $S_0 \subset S$ *is a thickening, and*
(3) *with* $T_i = S_i \setminus S_{i+1}$ *and* f_i *the base change of* f *to* T_i *we have: formation of the scheme theoretic image of* f_i/T_i *commutes with arbitrary base change (see discussion above the lemma).*

Proof. We can find a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & V & \longrightarrow & W \end{array}$$

with cartesian squares such that U, V, W are of finite type over $\mathbf{Z}$. Namely, first write S as a cofiltered limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms using Limits, Proposition 32.5.4 and then descend the morphism $X \to Y$ using Limits, Lemma 32.10.1. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component of S_0. By Noetherian induction on the underlying topological space of S_0, we may assume the result holds for any closed subscheme of S_0 not containing η. Thus it suffices to show that there exists an open neighbourhood $U_0 \subset S_0$ such that the base change f_0 of f to U_0 has property (3).

Let R be a Noetherian domain. Let $f : X \to Y$ be a morphism of finite type schemes over R. By the discussion in the previous paragraph it suffices to show that after replacing R by R_g for some $g \in R$ nonzero and X, Y by their base changes to R_g, formation of the scheme theoretic image of f/R commutes with arbitrary base change.

Let $Y = V_1 \cup \ldots V_n$ be an affine open covering. Let $U_i = f^{-1}(V_i)$. If the statement is true for each of the morphisms $U_i \to V_i$ over R, then it holds for f. Namely, the scheme theoretic image of $U_i \to V_i$ is the intersection of V_i with the scheme theoretic image of $f : X \to Y$ by Morphisms, Lemma 29.6.3. Thus we may assume Y is affine.

Let $X = U_1 \cup \ldots U_n$ be an affine open covering. Then the scheme theoretic image of $X \to Y$ is the same as the scheme theoretic imge of $\coprod U_i \to Y$. Thus we may assume X is affine.

Say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ and f corresponds to the R-algebra map $\varphi : A \to B$. Then the scheme theoretic image of f is $\mathrm{Spec}(A/\operatorname{Ker}(\varphi))$ and similarly after base change (by an affine morphism, but it is enough to check for those). Thus formation of the scheme theoretic image commutes with base change if $\operatorname{Ker}(\varphi \otimes_R R') = \operatorname{Ker}(\varphi) \otimes_R R'$ for all ring maps $R \to R'$.

After replacing R, A, B by R_g, A_g, B_g for a suitable nonzero g in R, we may assume A and B are flat over R. By Lemma 37.54.4 we may also assume B/A is a flat R-module. Then $0 \to \operatorname{Ker}(\varphi) \to A \to B \to B/A \to 0$ is an exact sequence of flat R-modules, which implies the desired base change statement. $\square$

37.55. Stratifying a morphism

0H42 Let $f : X \to S$ be a finitely presented morphism of quasi-compact and quasi-separated schemes. In Section 37.54 we have seen that we can stratify S such that X is flat over the strata. In this section look for stratifications of both S and X such that we obtain smooth strata; this won't quite work and we'll need a base change by finite locally free morphisms as well.

0H43 **Lemma 37.55.1.** *Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume S is reduced. Then there exist*

(1) *an open subscheme $U \subset S$ containing η,*
(2) *a surjective, universally injective, finite locally free morphism $V \to U$,*
(3) *a $t \geq 0$ and closed subschemes*

$$X \times_S V \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_0 \subset X \times_S V$ is a thickening, and such that the morphism $Z_i \setminus Z_{i+1} \to V$ is smooth.

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Thus we may assume $S = \mathrm{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 10.25.1.

Apply Varieties, Lemma 33.25.11 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak{p})$ is zero.)

If the characteristic of $k = \kappa(\mathfrak{p})$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak{p})$ is $p > 0$, then p maps to zero in $A_\mathfrak{p} = \kappa(\mathfrak{p})$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set $A' = A[x]/(x^p - a)$. Then $S' = \mathrm{Spec}(A') \to \mathrm{Spec}(A) = S$ is finite locally free, surjective, and universally injective. Moreover, if $\mathfrak{p}' \subset A'$ denotes the unique prime ideal lying over $\mathfrak{p}$, then $A'_{\mathfrak{p}'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak{p}'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak{p}) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a $t \geq 0$ and closed subschemes

$$X_\eta \supset Z_{\eta,0} \supset Z_{\eta,1} \supset \ldots \supset Z_{\eta,t} = \emptyset$$

such that $Z_{\eta,0} = (X_\eta)_{red}$ and $Z_{\eta,i} \setminus Z_{\eta,i+1}$ is smooth over η for all i. Recall that $\kappa(\eta) = \kappa(\mathfrak{p}) = A_\mathfrak{p}$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak{p}$. See Algebra, Lemma 10.9.9. Thus we can descend the diagram above to a corresponding diagram over $\mathrm{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak{p}$. More precisely, after replacing S by $\mathrm{Spec}(A_a)$ we may assume we have a $t \geq 0$ and closed subschemes

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is a closed immersion of finite presentation, such that $Z_0 \to X$ is a thickening, and such that $Z_i \setminus Z_{i+1}$ is smooth over S. In other words, the lemma holds. More precisely, we first use Limits, Lemma 32.10.1 to obtain morphisms

$$Z_t \to Z_{t-1} \to \ldots \to Z_0 \to X$$

over S, each of finite presentation, and whose base change to η produces the inclusions between the given closed subschemes above. After shrinking S further we may assume each of the morphisms is a closed immersion, see Limits, Lemma 32.8.5. After shrinking S we may assume $Z_0 \to X$ is surjective and hence a thickening, see Limits, Lemma 32.8.15. After shrinking S once more we may assume $Z_i \setminus Z_{i+1} \to S$ is smooth, see Limits, Lemma 32.8.9. This finishes the proof. □

0H44 **Lemma 37.55.2.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that

(1) *$S_i \to S$ is defined by a finite type ideal sheaf,*
(2) *$S_0 \subset S$ is a thickening,*
(3) *for each i there exists a surjective finite locally free morphism $T_i \to S_i \setminus S_{i+1}$,*
(4) *for each i there exists a $t_i \geq 0$ and closed subschemes*

$$X_i = X \times_S T_i \supset Z_{i,0} \supset Z_{i,1} \supset \ldots \supset Z_{i,t_i} = \emptyset$$

such that $Z_{i,j} \to X_i$ is defined by a finite type ideal sheaf, $Z_{i,0} \subset X_i$ is a thickening, and such that the morphism $Z_{i,j} \setminus Z_{i,j+1} \to T_i$ is smooth.

Proof. We can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_0 \end{array}$$

such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 32.5.4 and Lemma 32.10.1. Thus we may assume X and S are of finite type over $\mathbf{Z}$. Namely, a solution of the problem posed by the lemma for $X_0 \to S_0$ will base change to a solution over S; details omitted.

Assume X and S are of finite type over $\mathbf{Z}$. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component. By Lemma 37.55.1 we can find an open subscheme $U \subset S_0$, a surjective, universally injective, finite locally free morphism $V \to U$, a $t_0 \geq 0$ and closed subschemes

$$X \times_S V \supset Z_{0,0} \supset Z_{0,1} \supset \ldots \supset Z_{0,t_0} = \emptyset$$

such that $Z_{0,i} \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_{0,0} \subset X \times_S V$ is a thickening, and such that the morphism $Z_{0,i} \setminus Z_{0,i+1} \to V$ is smooth. Then we let $S_1 \subset S_0$ be the reduced induced subscheme structure on $S_0 \setminus U$. By Noetherian

induction on the underlying topological space of S, we may assume that the lemma holds for $X \times_S S_1 \to S_1$. This produces $t \geq 1$ and

$$S_1 = S_1 \supset S_2 \supset \ldots \supset S_t = \emptyset$$

and t_i and $Z_{i,j}$ as in the statement of the lemma. This proves the lemma. □

37.56. Improving morphisms of relative dimension one

0GK6 We can make any curve be smooth and projective after extending the ground field, compactifying, and normalizing. This also implies results about finite type morphisms whose generic fibres have dimension 1.

0GK7 **Lemma 37.56.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume f is separated, of finite presentation, and $\dim(X_\eta) \leq 1$. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n & \xleftarrow[j]{} & Y_1 \amalg \ldots \amalg Y_n & \xrightarrow[\nu]{} & X_V & \longrightarrow & X_U & \longrightarrow & X \\
 & \searrow & \downarrow & & \downarrow & & \downarrow & & \downarrow f \\
 & & T_1 \amalg \ldots \amalg T_n & \longrightarrow & V & \longrightarrow & U & \longrightarrow & S
\end{array}$$

of schemes with the following properties:

(1) *$U \subset S$ is an open neighbourhood of η,*
(2) *$V \to U$ is a finite, surjective, universally injective morphism,*
(3) *$X_U = U \times_S X$ and $X_V = V \times_S X$ are the base changes,*
(4) *ν is finite, surjective, and there is an open $W \subset X_V$ such that*
   (a) *W is dense in all fibres of $X_V \to V$,*
   (b) *$\nu^{-1}(W) \cap Y_i$ is dense in all fibres of $Y_i \to T_i$, and*
   (c) *$\nu^{-1}(W) \to W$ is a thickening,*
(5) *j is an open immersion,*
(6) *$T_i \to V$ is finite étale,*
(7) *$Y_i \to T_i$ is surjective and smooth,*
(8) *$\overline{Y}_i \to T_i$ is smooth, proper, with geometrically connected fibres of dimension ≤ 1.*

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Moreover, we may replace S by its reduction and X by the base change to this reduction. Thus we may assume $S = \mathrm{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 10.25.1.

Apply Varieties, Lemma 33.43.7 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak{p})$ is zero.)

If the characteristic of $k = \kappa(\mathfrak{p})$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak{p})$ is $p > 0$, then p maps to zero in $A_\mathfrak{p} = \kappa(\mathfrak{p})$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set

$A' = A[x]/(x^p - a)$. Then $S' = \mathrm{Spec}(A') \to \mathrm{Spec}(A) = S$ is finite, surjective, and universally injective. Moreover, if $\mathfrak{p}' \subset A'$ denotes the unique prime ideal lying over $\mathfrak{p}$, then $A'_{\mathfrak{p}'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak{p}'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak{p}) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a diagram

$$\begin{array}{ccccc} \overline{Y}_{1,\eta} \amalg \ldots \amalg \overline{Y}_{n,\eta} & \xleftarrow[j]{} & Y_{1,\eta} \amalg \ldots \amalg Y_{n,\eta} & \xrightarrow[\nu]{} & X_\eta \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(k_1) \amalg \ldots \amalg \mathrm{Spec}(k_n) & \longrightarrow & \eta \end{array}$$

of schemes with the following properties:

(1) ν is the normalization of X_η,
(2) j is an open immersion with dense image,
(3) $k_i/\kappa(\eta)$ is a finite separable extension for $i = 1, \ldots, n$,
(4) $\overline{Y}_{i,\eta}$ is smooth, projective, and geometrically irreducible of dimension ≤ 1 over k_i.

Recall that $\kappa(\eta) = \kappa(\mathfrak{p}) = A_{\mathfrak{p}}$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak{p}$. See Algebra, Lemma 10.9.9. Thus we can descend the diagram above to a corresponding diagram over $\mathrm{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak{p}$. More precisely, after replacing S by $\mathrm{Spec}(A_a)$ we may assume we have a commutative diagram

$$\begin{array}{ccccc} \overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n & \xleftarrow[j]{} & Y_1 \amalg \ldots \amalg Y_n & \xrightarrow[\nu]{} & X \\ & \searrow & \downarrow & & \downarrow \\ & & T_1 \amalg \ldots \amalg T_n & \longrightarrow & S \end{array}$$

of schemes whose base change to η is the diagram above with the following properties

(1) ν is a finite, surjective morphism,
(2) j is an open immersion,
(3) $T_i \to S$ is finite étale for $i = 1, \ldots, n$,
(4) $Y_i \to T_i$ is smooth and surjective,
(5) $\overline{Y}_i \to T_i$ is smooth and proper and has geometrically connected fibres of dimension ≤ 1.

For this we first use Limits, Lemma 32.10.1 to obtain the diagram base changing to the previous diagram. Then we use Limits, Lemmas 32.8.10, 32.8.9, 32.8.3, 32.4.13, 32.8.12, 32.13.1, and 32.8.15 to obtain ν finite, surjective, j open immersion, $T_i \to S$ finite étale, $Y_i \to T$ smooth, $\overline{Y}_i \to T_i$ proper and smooth. Since Y_i cannot be empty, since smooth morphisms are open, and since $T_i \to S$ is finite étale, after shrinking S we may assume $Y_i \to T_i$ is surjective. Finally, the fibre of $\overline{Y}_i \to T_i$ over the unique point $\eta_i = \mathrm{Spec}(k_i)$ of T_i lying over η is geometrically connected. Hence by another shrinking we may assume the same thing is true for all fibres, see Lemma 37.53.8.

It remains to prove the existence of an open $W \subset X$ satisfying (a), (b), and (c). Since $\nu_\eta : \coprod Y_{i,\eta} \to X_\eta$ is the normalization morphism, we know by Varieties,

Lemma 33.27.1 there exists a dense open $W_\eta \subset X_\eta$ such that $\nu^{-1}(W_\eta) \to W_\eta$ is equal to the inclusion of the reduction of W_η into W_η. Let $W \subset X$ be a quasi-compact open whose fibre over η is the open W_η we just found. After replacing $A = \Gamma(S, \mathcal{O}_S)$ by another localization we may assume $\nu^{-1}(W) \to W$ is a closed immersion, see Limits, Lemma 32.8.5. Since ν is also surjective we conclude $\nu^{-1}(W) \to W$ is a thickening. Set $W_i = \nu^{-1}(W) \cap Y_i$. Shrinking S once more we can assume $W_i \to T_i$ is surjective for all i (same argument as above). Then we find that $W_i \subset Y_i$ is dense in all fibres of $Y_i \to T_i$ as $Y_i \to T_i$ has geometrically irreducible fibres. Since ν is finite and surjective, it then follows that $W = \nu(\nu^{-1}(W))$ is dense in all fibres of $X \to S$ too. □

37.57. Descending separated locally quasi-finite morphisms

02W7 In this section we show that "separated locally quasi-finite morphisms satisfy descent for fppf-coverings". See Descent, Definition 35.36.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [**GR71**, Lemma 5.7.1]. It can also be found in [**Mur95**], and [**ABD+66**, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

02W8 **Lemma 37.57.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 34.7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.*

Proof. Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 26.21.12 and Morphisms, Lemma 29.20.13. Hence Descent, Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \mathrm{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

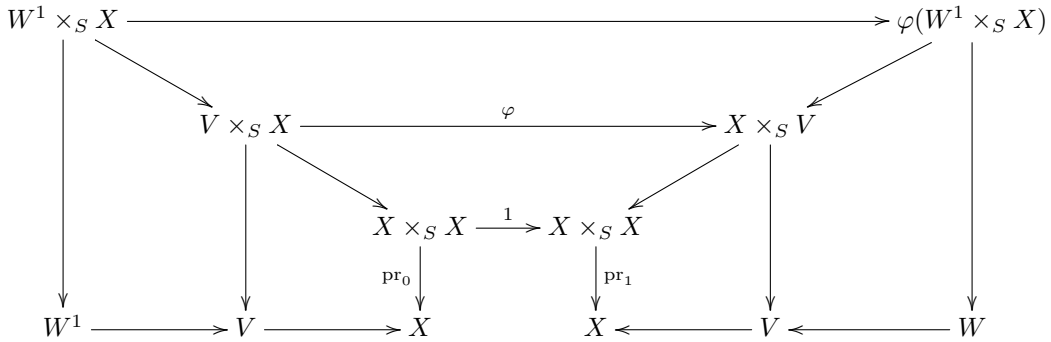

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 29.25.10). Hence we conclude that W is open. Moreover, it is also clearly the case that W is quasi-compact, and $W^1 \subset W$. Moreover, we note

that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for φ. Hence we obtain a new descent datum (W, φ') by restricting φ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 37.43.2. Thus by Descent, Lemma 35.38.1 we see that the descent datum (W, φ') is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens W_i which are stable for the descent morphism φ. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data (W, φ') is effective. This means the original descent datum is effective by glueing the schemes obtained from descending the opens W_i, see Descent, Lemma 35.35.13. □

37.58. Relative finite presentation

05GX Let $R \to A$ be a finite type ring map. Let M be an A-module. In More on Algebra, Section 15.81 we defined what it means for M to be finitely presented relative to R. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

05H1 **Definition 37.58.1.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *finitely presented relative to S* or *of finite presentation relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for every i an affine open covering $f^{-1}(V_i) = \bigcup_j U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to S, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.

09T7 **Lemma 37.58.2.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite presentation relative to S,*
(2) *for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely presented relative to $\mathcal{O}_S(V)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 37.58.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 15.81.8 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 26.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By More on Algebra, Lemma 15.81.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to

$\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V_i)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 15.81.5 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 15.81.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. □

09T8 **Lemma 37.58.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to S if and only if $\mathcal{F}$ is of finite presentation.*
(2) *The morphism f is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.*

Proof. Follows immediately from the definitions, see discussion following More on Algebra, Definition 15.81.2. □

09T9 **Lemma 37.58.4.** *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite presentation relative to S if and only if $\pi_*\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.3 into the language of schemes. □

09TA **Lemma 37.58.5.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes, set $X' = X \times_S S'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. If $\mathcal{F}$ is of finite presentation relative to S, then $\mathcal{F}'$ is of finite presentation relative to S'.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.5 into the language of schemes. □

09TB **Lemma 37.58.6.** *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $f : X \to Y$ is locally of finite presentation and $\mathcal{G}$ of finite presentation relative to S, then $f^*\mathcal{G}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.6 into the language of schemes. □

09TC **Lemma 37.58.7.** *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Y \to S$ is locally of finite presentation and $\mathcal{F}$ is of finite presentation relative to Y, then $\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.7 into the language of schemes. □

09TD **Lemma 37.58.8.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}', \mathcal{F}''$ are finitely presented relative to S, then so is $\mathcal{F}$.*
(2) *If $\mathcal{F}'$ is a finite type $\mathcal{O}_X$-module and $\mathcal{F}$ is finitely presented relative to S, then $\mathcal{F}''$ is finitely presented relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.9 into the language of schemes. □

09TE **Lemma 37.58.9.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F} \oplus \mathcal{F}'$ is finitely presented relative to S, then so are $\mathcal{F}$ and $\mathcal{F}'$.*

Proof. Translation of the result of More on Algebra, Lemma 15.81.10 into the language of schemes. □

37.59. Relative pseudo-coherence

09UH This section is the analogue of More on Algebra, Section 15.82 for schemes. We strongly urge the reader to take a look at that section first. Although we have developed the material in this section and the material on pseudo-coherent complexes in Cohomology, Sections 20.46, 20.47, 20.48, and 20.49 for arbitrary complexes of $\mathcal{O}_X$-modules, if X is a scheme then working exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$ greatly simplifies many of the lemmmas and arguments, often reducing the problem at hand immediately to the algebraic counterpart. Moreover, one of the first thing we do is to show that being relatively pseudo-coherent implies the cohomology sheaves are quasi-coherent, see Lemma 37.59.3. Hence, on a first reading we suggest the reader work exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$.

09VC **Lemma 37.59.1.** *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. The following are equivalent*

1. *for some closed immersion $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent, and*
2. *for all closed immersions $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent.*

Proof. Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Let i correspond to the surjection $\alpha : R[x_1, \ldots, x_n] \to A$ and let $X \to \mathbf{A}^m_S$ correspond to $\beta : R[y_1, \ldots, y_m] \to A$. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

corresponding to the commutative diagram of closed immersions

$$\begin{array}{ccc} \mathbf{A}^{n+m}_S & \longleftarrow & \mathbf{A}^n_S \\ \uparrow & & \uparrow \\ \mathbf{A}^m_S & \longleftarrow & X \end{array}$$

Thus it suffices to show that under a closed immersion

$$f : \mathbf{A}^m_S \to \mathbf{A}^{n+m}_S$$

an object E of $D(\mathcal{O}_{\mathbf{A}^m_S})$ is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 36.12.5 and the fact that $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is a pseudo-coherent $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. The pseudo-coherence

of $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 36.10.2 and More on Algebra, Lemma 15.82.3. $\square$

Recall that if $f : X \to S$ is a morphism of scheme which is locally of finite type, then for every pair of affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type (Morphisms, Lemma 29.15.2). Hence there always exist closed immersions $U \to \mathbf{A}^n_V$ and the following definition makes sense.

09UI **Definition 37.59.2.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is *m-pseudo-coherent relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$.
(2) We say E is *pseudo-coherent relative to S* if E is m-pseudo-coherent relative to S for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is *m-pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S.
(4) We say $\mathcal{F}$ is *pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is pseudo-coherent relative to S.

If X is quasi-compact and E is m-pseudo-coherent relative to S for some m, then E is bounded above. If E is pseudo-coherent relative to S, then E has quasi-coherent cohomology sheaves.

0CSU **Lemma 37.59.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If E in $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent relative to S, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Choose an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$. Since being quasi-coherent is local on X, we may assume that there exists an closed immersion $i : X \to \mathbf{A}^n_S$ such that Ri_*E is m-pseudo-coherent on $\mathbf{A}^n_S$. By Derived Categories of Schemes, Lemma 36.10.1 this means that $H^q(Ri_*E)$ is quasi-coherent for $q > m$. Since i_* is an exact functor, we have $i_*H^q(E) = H^q(Ri_*E)$ is quasi-coherent on $\mathbf{A}^n_S$. By Morphisms, Lemma 29.4.1 this implies that $H^q(E)$ is quasi-coherent as desired (strictly speaking it implies there exists some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $i_*\mathcal{F} = i_*H^q(E)$ and then Modules, Lemma 17.13.4 tells us that $\mathcal{F} \cong H^q(E)$ hence the result). $\square$

Next, we prove the condition of relative pseudo-coherence localizes well.

09VD **Lemma 37.59.4.** *Let S be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let E be an object of $D(\mathcal{O}_X)$. If the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U \to S, E|_U)$.*

Proof. Write $S = \mathrm{Spec}(R)$, $V = D(f)$, $X = \mathrm{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(X \to V, E)$.

Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to \mathbf{A}^n_V \to \mathbf{A}^{n+1}_S$$

and we can apply Derived Categories of Schemes, Lemma 36.12.5 to conclude that the pushforward E' of E to $\mathbf{A}^{n+1}_S$ is m-pseudo-coherent.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $R[x_0, \ldots, x_{n+1}] \to R[x_0, \ldots, x_n, 1/g']$. We obtain

$$\begin{array}{ccccc} X & \longleftarrow & U & & \\ \downarrow & & \downarrow & \searrow & \\ \mathbf{A}^{n+1}_S & \longleftarrow & D(g') & \longrightarrow & \mathbf{A}^{n+2}_S \end{array}$$

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'|_{D(g')}$ to $\mathbf{A}^{n+2}_S$ is m-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbf{A}^{n+2}_S$ we conclude the lemma is true. □

09VE **Lemma 37.59.5.** *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent*

(1) *the equivalent conditions of Lemma 37.59.1 hold for the pairs $(U_i \to S, E|_{U_i})$,*
(2) *the equivalent conditions of Lemma 37.59.1 hold for the pair $(X \to S, E)$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 37.59.4. Assume (1). Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in A. Consider the surjections

$$R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.$$

which sends y_i to f_i and z_i to g_i. Note that $R[x_i, y_i, z_i]/(\sum y_i z_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of E to $T = \mathrm{Spec}(R[x_i, y_i, z_i]/(\sum y_i z_i - 1))$ is m-pseudo-coherent, see Derived Categories of Schemes, Lemma 36.12.5. For each i_0 it suffices to prove the restriction of this pushforward to $W_{i_0} = \mathrm{Spec}(R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1))$ is m-pseudo-coherent. Note that there is a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & U_{i_0} \\ \downarrow & & \downarrow \\ T & \longleftarrow & W_{i_0} \end{array}$$

which implies that the pushforward of E to T restricted to W_{i_0} is the pushforward of $E|_{U_{i_0}}$ to W_{i_0}. Since $R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1)$ is isomorphic to a polynomial ring over R this proves what we want. □

09UJ **Lemma 37.59.6.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

(1) *E is m-pseudo-coherent relative to S,*

(2) *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U \to V, E|_U)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is m-pseudo-coherent relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 37.59.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 37.59.5 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that the equivalent conditions of Lemma 37.59.1 are satisfied for the pairs $(U_k \to V, E|_{U_k})$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 26.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By Lemma 37.59.4 the assumption that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U_{ij} \to V_i, E|_{U_{ij}})$ implies that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. □

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative m-pseudo-coherence to the notion defined in More on Algebra, Definition 15.82.4. We will use the fact that for an affine scheme $U = \mathrm{Spec}(A)$ the functor $R\Gamma(U, -)$ induces an equivalence between $D_{QCoh}(\mathcal{O}_U)$ and $D(A)$, see Derived Categories of Schemes, Lemma 36.3.5. This functor is compatible with pullbacks: if E is an object of $D_{QCoh}(\mathcal{O}_U)$ and $A \to B$ is a ring map corresponding to a morphism of affine schemes $g : V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U$, then $R\Gamma(V, Lg^*E) = R\Gamma(U, E) \otimes^{\mathbf{L}}_A B$. See Derived Categories of Schemes, Lemma 36.3.8.

09VF **Lemma 37.59.7.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

(1) *E is m-pseudo-coherent relative to S,*
(2) *there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex of $\mathcal{O}_X(U_{ij})$-modules $R\Gamma(U_{ij}, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V_i)$, and*
(3) *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the complex of $\mathcal{O}_X(U)$-modules $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$.*

Proof. Let U and V be as in (2) and choose a closed immersion $i : U \to \mathbf{A}^n_V$. A formal argument, using Lemma 37.59.6, shows it suffices to prove that $Ri_*(E|_U)$ is m-pseudo-coherent if and only if $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$. Say $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and $\mathbf{A}^n_V = \mathrm{Spec}(R[x_1, \ldots, x_n]$. By the remarks preceding the lemma, $E|_U$ is quasi-isomorphic to the complex of quasi-coherent sheaves on U associated to the object $R\Gamma(U, E)$ of $D(A)$. Note that $R\Gamma(U, E) = R\Gamma(\mathbf{A}^n_V, Ri_*(E|_U))$ as i is a closed immersion (and hence i_* is exact). Thus Ri_*E is associated to $R\Gamma(U, E)$ viewed as an object of $D(R[x_1, \ldots, x_n])$. We conclude as m-pseudo-coherence of $Ri_*(E|_U)$ is equivalent to m-pseudo-coherence of $R\Gamma(U, E)$

in $D(R[x_1, \ldots, x_n])$ by Derived Categories of Schemes, Lemma 36.10.2 which is equivalent to $R\Gamma(U, E)$ is m-pseudo-coherent relative to $R = \mathcal{O}_S(V)$ by definition. □

09VG **Lemma 37.59.8.** *Let $i : X \to Y$ morphism of schemes locally of finite type over a base scheme S. Assume that i induces a homeomorphism of X with a closed subset of Y. Let E be an object of $D(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if Ri_*E is m-pseudo-coherent relative to S.*

Proof. By Morphisms, Lemma 29.45.4 the morphism i is affine. Thus we may assume S, Y, and X are affine. Say $S = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}(B)$. The condition means that $A/\mathrm{rad}(A) \to B/\mathrm{rad}(B)$ is surjective; here $\mathrm{rad}(A)$ and $\mathrm{rad}(B)$ denote the Jacobson radical of A and B. As B is of finite type over A, we can find $b_1, \ldots, b_m \in \mathrm{rad}(B)$ which generate B as an A-algebra. Say $b_j^N = 0$ for all j. Consider the diagram of rings

$$\begin{array}{ccccc} B & \longleftarrow & R[x_i, y_j]/(y_j^N) & \longleftarrow & R[x_i, y_j] \\ \uparrow & & \uparrow & \nearrow & \\ A & \longleftarrow & R[x_i] & & \end{array}$$

which translates into a diagram

$$\begin{array}{ccccc} X & \longrightarrow & T & \longrightarrow & \mathbf{A}_S^{n+m} \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longrightarrow & \mathbf{A}_S^n & & \end{array}$$

of affine schemes. By Lemma 37.59.6 we see that E is m-pseudo-coherent relative to S if and only if its pushforward to $\mathbf{A}_S^{n+m}$ is m-pseudo-coherent. By Derived Categories of Schemes, Lemma 36.12.5 we see that this is true if and only if its pushforward to T is m-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to $\mathbf{A}_S^n$ is m-pseudo-coherent. Again by Lemma 37.59.6 this holds if and only if Ri_*E is m-pseudo-coherent relative to S. □

09UK **Lemma 37.59.9.** *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if $R\pi_*E$ is m-pseudo-coherent relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.82.5 into the language of schemes. Observe that $R\pi_*$ indeed maps $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Lemma 36.4.1. To do the translation use Lemma 37.59.6. □

09UL **Lemma 37.59.10.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let (E, E', E'') be a distinguished triangle of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

1. *If E is $(m+1)$-pseudo-coherent relative to S and E' is m-pseudo-coherent relative to S then E'' is m-pseudo-coherent relative to S.*
2. *If E, E'' are m-pseudo-coherent relative to S, then E' is m-pseudo-coherent relative to S.*

(3) *If E' is $(m+1)$-pseudo-coherent relative to S and E'' is m-pseudo-coherent relative to S, then E is $(m+1)$-pseudo-coherent relative to S.*

Moreover, if two out of three of E, E', E'' are pseudo-coherent relative to S, the so is the third.

Proof. Immediate from Lemma 37.59.6 and Cohomology, Lemma 20.47.4. □

09UM **Lemma 37.59.11.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then*

(1) *$\mathcal{F}$ is m-pseudo-coherent relative to S for all $m > 0$,*
(2) *$\mathcal{F}$ is 0-pseudo-coherent relative to S if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,*
(3) *$\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $\mathcal{F}$ is quasi-coherent and finitely presented relative to S.*

Proof. Part (1) is immediate from the definition. To see part (3) we may work locally on X (both properties are local). Thus we may assume X and S are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then we see that $\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $i_*\mathcal{F}$ is (-1)-pseudo-coherent, which is true if and only if $i_*\mathcal{F}$ is an $\mathcal{O}_{\mathbf{A}^n_S}$-module of finite presentation, see Cohomology, Lemma 20.47.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 17.11.2. By Morphisms, Lemma 29.4.1 we see that $\mathcal{F}$ is quasi-coherent if and only if $i_*\mathcal{F}$ is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved. □

09UN **Lemma 37.59.12.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E, K be objects of $D(\mathcal{O}_X)$. If $E \oplus K$ is m-pseudo-coherent relative to S so are E and K.*

Proof. Follows from Cohomology, Lemma 20.47.6 and the definitions. □

09UP **Lemma 37.59.13.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent relative to S for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 20.47.7 and the definitions. □

09UQ **Lemma 37.59.14.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to S for all i, then E is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 20.47.8 and the definitions. □

09UR **Lemma 37.59.15.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$ which is m-pseudo-coherent relative to S. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote E' the derived pullback of E to X'. If S' and X are Tor independent over S, then E' is m-pseudo-coherent relative to S'.*

Proof. The problem is local on X and X' hence we may assume X, S, S', and X' are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and denote $i' : X' \to \mathbf{A}^n_{S'}$ the base change to S'. Denote $g : X' \to X$ and $g' : \mathbf{A}^n_{S'} \to \mathbf{A}^n_S$ the projections,

so $E' = Lg^*E$. Since X and S' are tor-independent over S, the base change map (Cohomology, Remark 20.28.3) induces an isomorphism

$$Ri'_*(Lg^*E) = L(g')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at x' is the map

$$E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_{S',x}}} \mathcal{O}_{\mathbf{A}^n_{S',x'}} \longrightarrow E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

coming from the closed immersions i and i'. Note that the source is quasi-isomorphic to a localization of $E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ which is isomorphic to the target as $\mathcal{O}_{X',x'}$ is isomorphic to (the same) localization of $\mathcal{O}_{X,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma 20.47.3. □

09US **Lemma 37.59.16.** *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_Y)$. Assume*

(1) *$\mathcal{O}_X$ is pseudo-coherent relative to Y[12], and*
(2) *E is m-pseudo-coherent relative to S.*

Then Lf^*E is m-pseudo-coherent relative to S.

Proof. The problem is local on X. Thus we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.82.13 we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathbf{A}^d_Y & \xrightarrow{j} & \mathbf{A}^{n+d}_S \\ {\scriptstyle f}\downarrow & \swarrow_{p} & & \swarrow & \\ Y & \longrightarrow & \mathbf{A}^n_S & & \end{array}$$

Observe that

$$Ri_*Lf^*E = Ri_*Li^*Lp^*E = Lp^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_Y}} Ri_*\mathcal{O}_X$$

by Cohomology, Lemma 20.54.4. By assumption and the fact that Y is affine, we can represent $Ri_*\mathcal{O}_X = i_*\mathcal{O}_X$ by a complexes of finite free $\mathcal{O}_{\mathbf{A}^n_Y}$-modules $\mathcal{F}^\bullet$, with $\mathcal{F}^q = 0$ for $q > 0$ (details omitted; use Derived Categories of Schemes, Lemma 36.10.2 and More on Algebra, Lemma 15.82.7). By assumption E is bounded above, say $H^q(E) = 0$ for $q > a$. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules with $\mathcal{E}^q = 0$ for $q > a$. Then the derived tensor product above is represented by $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$.

Since j is a closed immersion, the functor j_* is exact and Rj_* is computed by applying j_* to any representing complex of sheaves. Thus we have to show that $j_*\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ is m-pseudo-coherent as a complex of $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-modules. Note that $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$. Note that for $q \ll 0$ the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent. Hence, applying Lemma 37.59.10 and induction, it suffices to show that $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ is pseudo-coherent relative to S for all q. Note that $\mathcal{F}^q = 0$ for $q > 0$. Since also $\mathcal{F}^q$ is finite free this reduces to proving that $p^*\mathcal{E}^\bullet$ is m-pseudo-coherent relative to S which follows from Lemma 37.59.15 for instance. □

[12]This means f is pseudo-coherent, see Definition 37.60.2.

09UT **Lemma 37.59.17.** *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. Assume $\mathcal{O}_Y$ is pseudo-coherent relative to S[13]. Then the following are equivalent*

(1) *E is m-pseudo-coherent relative to Y, and*
(2) *E is m-pseudo-coherent relative to S.*

Proof. The question is local on X, hence we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.82.13 we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathbf{A}^m_Y & \xrightarrow{j} & \mathbf{A}^{n+m}_S \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & & \swarrow & \\ Y & \longrightarrow & \mathbf{A}^n_S & & \end{array}$$

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to S implies that $\mathcal{O}_{\mathbf{A}^m_Y}$ is pseudo-coherent relative to $\mathbf{A}^m_S$ (by flat base change; this can be seen by using for example Lemma 37.59.15). This in turn implies that $j_*\mathcal{O}_{\mathbf{A}^n_Y}$ is pseudo-coherent as an $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma 36.12.5. □

09UU **Lemma 37.59.18.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of schemes. Assume i is a closed immersion and $P \to S$ flat and locally of finite presentation. Let E be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

(1) *E is m-pseudo-coherent relative to S,*
(2) *Ri_*E is m-pseudo-coherent relative to S, and*
(3) *Ri_*E is m-pseudo-coherent on P.*

Proof. The equivalence of (1) and (2) is Lemma 37.59.9. The equivalence of (2) and (3) follows from Lemma 37.59.17 applied to $\mathrm{id} : P \to P$ provided we can show that $\mathcal{O}_P$ is pseudo-coherent relative to S. This follows from More on Algebra, Lemma 15.83.4 and the definitions. □

37.60. Pseudo-coherent morphisms

067X Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 15.65, 15.82, and 15.83. For now the only thing you need to know is that a ring map $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

067Y **Lemma 37.60.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

[13] This means $Y \to S$ is pseudo-coherent, see Definition 37.60.2.

(1) *there exist an affine open covering* $S = \bigcup V_j$ *and for each* j *an affine open covering* $f^{-1}(V_j) = \bigcup U_{ji}$ *such that* $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ *is a pseudo-coherent ring map,*
(2) *for every pair of affine opens* $U \subset X$, $V \subset S$ *such that* $f(U) \subset V$ *the ring map* $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ *is pseudo-coherent, and*
(3) f *is locally of finite type and* $\mathcal{O}_X$ *is pseudo-coherent relative to* S.

Proof. To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 29.14.1 for the property of being a pseudo-coherent ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 15.82.12, 15.82.11, and 15.82.16.

If (1) holds, then f is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma 37.59.7 and the definitions that $\mathcal{O}_X$ is pseudo-coherent relative to S. Conversely, if (3) holds, then we see that for every U and V as in (2) the ring $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_S(V)$ and $\mathcal{O}_X(U)$ is as a module pseudo-coherent relative to $\mathcal{O}_S(V)$, see Lemmas 37.59.6 and 37.59.7. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold. □

067Z **Definition 37.60.2.** A morphism of schemes $f : X \to S$ is called *pseudo-coherent* if the equivalent conditions of Lemma 37.60.1 are satisfied. In this case we also say that X is pseudo-coherent over S.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

0680 **Lemma 37.60.3.** *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma 15.82.12. □

0681 **Lemma 37.60.4.** *A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma 15.82.13 or More on Algebra, Lemma 15.82.15. □

0682 **Lemma 37.60.5.** *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

0695 **Lemma 37.60.6.** *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 15.83.4. □

0683 **Lemma 37.60.7.** *Let* $f : X \to Y$ *be a morphism of schemes pseudo-coherent over a base scheme* S. *Then* f *is pseudo-coherent.*

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 15.82.15. □

0AVX **Lemma 37.60.8.** *Let $f : X \to S$ be a finite morphism of schemes. Then f is pseudo-coherent if and only if $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_S$-module.*

Proof. Translated into algebra this lemma says the following: If $R \to A$ is a finite ring map, then $R \to A$ is pseudo-coherent as a ring map (which means by definition that A as an A-module is pseudo-coherent relative to R) if and only if A is pseudo-coherent as an R-module. This follows from the more general More on Algebra, Lemma 15.82.5. □

0684 **Lemma 37.60.9.** *Let $f : X \to S$ be a morphism of schemes. If S is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with R Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 15.65.17 that A is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. □

0696 **Lemma 37.60.10.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 35.22.4 to prove this. By Definition 37.60.2 being pseudo-coherent is Zariski local on the base. By Lemma 37.60.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then C is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.65.15. □

0697 **Lemma 37.60.11.** *Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if I is pseudo-coherent as a B-module.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 37.60.6. Note that $A \to B/I$ is pseudo-coherent if and only if B/I is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 15.65.11 we see this is equivalent to the condition that B/I is pseudo-coherent as an B-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that I is pseudo-coherent if and only if B/I is (see More on Algebra, Lemma 15.65.6). □

The following lemma will be obsoleted by the stronger Lemma 37.60.13.

0698 **Lemma 37.60.12.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 35.26.4 to prove this. It follows from Lemmas 37.60.6 and 37.60.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 29.30.7 and 29.30.6). It is clear from Definition 37.60.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 35.26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 35.14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{array}{ccc} X'_i & \longrightarrow & W_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram

$$\begin{array}{ccc} X' & \longrightarrow & W \\ \downarrow & & \downarrow h \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

with $W \to \mathbf{A}^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 29.25.10) and $X' \to X$ is surjective we can find $f \in \Gamma(\mathbf{A}^n_Y, \mathcal{O})$ such that $X \subset D(f) \subset \mathrm{Im}(h)$. Write $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$ and $W = \mathrm{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$\begin{array}{ccc} A' = B/IB & \longleftarrow & B \\ \uparrow & & \uparrow \\ A = R[x_1, \ldots, x_n]/I & \longleftarrow & R[x_1, \ldots, x_n] \end{array}$$

By Lemma 37.60.11 we see that IB is pseudo-coherent as a B-module. The ring map $R[x_1, \ldots, x_n]_f \to B_f$ is faithfully flat by our choice of f above. This implies that $I_f \subset R[x_1, \ldots, x_n]_f$ is pseudo-coherent, see More on Algebra, Lemma 15.65.15. Applying Lemma 37.60.11 one more time we see that $R \to A$ is pseudo-coherent. □

0699 **Lemma 37.60.13.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fppf local on the source.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fppf covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 37.48.2 there exist

(1) a Zariski open covering $X = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,

(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both X_i and $T_{j,k}$ are pseudo-coherent over X by Lemma 37.60.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 37.60.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 37.60.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 35.14.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that X', X and S are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an A-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption that $R \to A'$ is pseudo-coherent means that A' as an A'-module is pseudo-coherent relative to R. By More on Algebra, Lemma 15.82.5 this implies that A' as an A-module is pseudo-coherent relative to R. Since $A' \cong A^{\oplus r}$ as an A-module we see that A as an A-module is pseudo-coherent relative to R, see More on Algebra, Lemma 15.82.8. This by definition means that $R \to A$ is pseudo-coherent and we win. □

37.61. Perfect morphisms

0685 In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and B has finite tor dimension as an A-module.

0686 **Lemma 37.61.1.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.*

Proof. Assume (1) and let U, V be as in (2). It follows from Lemma 37.60.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 29.14.1. These properties follow from More on Algebra, Lemmas 15.67.11, 15.67.14, and 15.67.16. Some details omitted. □

0687 **Definition 37.61.2.** A morphism of schemes $f : X \to S$ is called *perfect* if the equivalent conditions of Lemma 37.61.1 are satisfied. In this case we also say that X is perfect over S.

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

0688 **Lemma 37.61.3.** *A flat base change of a perfect morphism is perfect.*

Proof. This translates into the following algebra result: Let $A \to B$ be a perfect ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 37.60.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 15.67.14. □

0689 **Lemma 37.61.4.** *A composition of perfect morphisms of schemes is perfect.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 37.60.4 and its proof. By assumption there exist integers n, m such that B has tor dimension $\leq n$ over A and C has tor dimension $\leq m$ over B. Then for any A-module M we have

$$M \otimes^{\mathbf{L}}_A C = (M \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_B C$$

and the spectral sequence of More on Algebra, Example 15.62.4 shows that $\mathrm{Tor}^A_p(M, C) = 0$ for $p > n + m$ as desired. □

068A **Lemma 37.61.5.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 15.83.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 37.60.5. □

068B **Lemma 37.61.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S is regular and f is locally of finite type. Then f is perfect.*

Proof. See More on Algebra, Lemma 15.83.5. □

068C **Lemma 37.61.7.** *A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of A. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of A/I by a polynomial algebra over A. Hence it suffices to see that A/I is pseudo-coherent as an A-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex $K(A, f_1, \ldots, f_r)$ is a finite free resolution of A/I. Hence A/I is a perfect complex of A-modules and we win. □

068D **Lemma 37.61.8.** *Let*

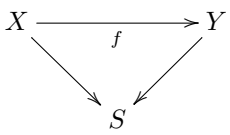

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

Proof. We can factor f as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 29.34.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 37.61.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 29.34.5. Hence i is a section of a smooth morphism, therefore i is a regular immersion, see Divisors, Lemma 31.22.8. This implies that i is perfect, see Lemma 37.61.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 37.61.4. □

069A **Remark 37.61.9.** It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(k[x]/(x^2)$ and $X \to Y$ the unique S-morphism.

069B **Lemma 37.61.10.** *The property $\mathcal{P}(f) =$"f is perfect" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 35.22.4 to prove this. By Definition 37.61.2 being perfect is Zariski local on the base. By Lemma 37.61.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then C is perfect as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.75.13. □

069C **Lemma 37.61.11.** *Let $f : X \to S$ be a pseudo-coherent morphism of schemes. The following are equivalent*

1. *f is perfect,*
2. *$\mathcal{O}_X$ locally has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules, and*
3. *for all $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.*

Proof. The problem is local on X and S. Hence we may assume that $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and f corresponds to a pseudo-coherent ring map $A \to B$.

If (1) holds, then B has finite tor dimension d as A-module. Then $B_\mathfrak{q}$ has tor dimension d as an $A_\mathfrak{p}$-module for all primes $\mathfrak{q} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{q}$, see More on Algebra, Lemma 15.67.15. Then $\mathcal{O}_X$ has tor dimension d as a sheaf of $f^{-1}\mathcal{O}_S$-modules by Cohomology, Lemma 20.48.5. Thus (1) implies (2).

By Cohomology, Lemma 20.48.5 (2) implies (3).

Assume (3). We cannot use More on Algebra, Lemma 15.67.15 to conclude as we are not given that the tor dimension of $B_\mathfrak{q}$ over $A_\mathfrak{p}$ is bounded independent of $\mathfrak{q}$. Choose a presentation $A[x_1, \ldots, x_n] \to B$. Then B is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. Let $\mathfrak{q} \subset A[x_1, \ldots, x_n]$ be a prime ideal lying over $\mathfrak{p} \subset A$. Then either $B_\mathfrak{q}$ is zero or by assumption it has finite tor dimension as an $A_\mathfrak{p}$-module. Since the fibres of $A \to A[x_1, \ldots, x_n]$ have finite global dimension, we can apply More on Algebra, Lemma 15.78.5 to $A_\mathfrak{p} \to A[x_1, \ldots, x_n]_\mathfrak{q}$ to see that $B_\mathfrak{q}$ is a perfect $A[x_1, \ldots, x_n]_\mathfrak{q}$-module. Hence B is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 15.78.3. Thus $A \to B$ is a perfect ring map by definition. □

0G2E **Lemma 37.61.12.** *Let $i : Z \to X$ be a perfect closed immersion of schemes. Then $i_*\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module, i.e., it is a perfect object of $D(\mathcal{O}_X)$.*

Proof. This is more or less immediate from the definition. Namely, let $U = \mathrm{Spec}(A)$ be an affine open of X. Then $i^{-1}(U) = \mathrm{Spec}(A/I)$ for some ideal $I \subset A$ and A/I has a finite resolution by finite projective A-modules by More on Algebra, Lemma 15.83.2. Hence $i_*\mathcal{O}_Z|_U$ can be represented by a finite length complex of finite locally free $\mathcal{O}_U$-modules. This is what we had to show, see Cohomology, Section 20.49. □

0B6G **Lemma 37.61.13.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a perfect proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.*

Proof. We claim that Derived Categories of Schemes, Lemma 36.27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules. This is equivalent to being perfect by Lemma 37.61.11. □

069D **Lemma 37.61.14.** *The property $\mathcal{P}(f) =$"f is perfect" is fppf local on the source.*

Proof. Let $\{g_i : X_i \to X\}_{i\in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 37.60.13 we conclude that f is pseudo-coherent. Hence by Lemma 37.61.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to x. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 15.67.17. □

09RK **Lemma 37.61.15.** *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume*

(1) *X is locally Noetherian,*
(2) *$j \circ i$ is a regular immersion, and*
(3) *i is perfect.*

Then i and j are regular immersions.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 31.20.8. Thus the result follows from Divided Power Algebra, Lemma 23.7.5. □

37.62. Local complete intersection morphisms

068E In Divisors, Section 31.21 we have defined 4 different types of regular immersions: regular, Koszul-regular, H_1-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on X factor as

$$\begin{array}{ccc} X & \xrightarrow[i]{} & \mathbf{A}^n_S \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where i is a $*$-regular immersion for $* \in \{\emptyset, Koszul, H_1, quasi\}$. However, we don't know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require i to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = Koszul$ or $* = \emptyset$. Hence we will define a *local complete intersection morphism* or *Koszul morphism* to be a morphism of schemes $f : X \to S$ that locally on X has a factorization as above with i a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

069E **Lemma 37.62.1.** *Let S be a scheme. Let U, P, P' be schemes over S. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over S. Assume P and P' smooth over S. Then the following are equivalent*

(1) *i is a Koszul-regular immersion in a neighbourhood of u, and*
(2) *i' is a Koszul-regular immersion in a neighbourhood of u.*

Proof. Assume i is a Koszul-regular immersion in a neighbourhood of u. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 31.22.9 that j is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 31.22.12 that i' is a Koszul-regular immersion. □

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbf{A}^n_S$ followed by the projection $\mathbf{A}^n_S \to S$ which is smooth. Picture

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[i]{} & \mathbf{A}^n_S = P \\ & \searrow & \downarrow & \swarrow_{\pi} & \\ & & S & & \end{array}$$

In fact you can do this with any affine open neighbourhood U of x in X, see Morphisms, Lemma 29.39.2.

069F **Definition 37.62.2.** Let $f : X \to S$ be a morphism of schemes.

(1) Let $x \in X$. We say that f is *Koszul at x* if f is of finite type at x and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.
(2) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if f is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion i followed by a smooth morphism π, whether or not i is Koszul regular in a neighbourhood of x is an intrinsic property of f at x. Let us record this here explicitly as a lemma so that we can refer to it

069G **Lemma 37.62.3.** *Let $f : X \to S$ be a local complete intersection morphism. Let P be a scheme smooth over S. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over S. Then i is a Koszul-regular immersion.*

Proof. This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

069H **Lemma 37.62.4.** *Let $f : X \to S$ be a local complete intersection morphism. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that f factors as $\pi \circ i$ where π is smooth and i is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 37.61.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 37.61.5. Finally a composition of perfect morphisms is perfect, see Lemma 37.61.4. □

07DB **Lemma 37.62.5.** *Let $f : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ be a morphism of affine schemes. Then f is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 15.33.2.*

Proof. Follows immediately from the definitions. □

Beware that a base change of a Koszul morphism is not Koszul in general.

069I **Lemma 37.62.6.** *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.3. □

069J **Lemma 37.62.7.** *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Let $g : Y \to S$ and $f : X \to Y$ be local complete intersection morphisms. Let $x \in X$ and set $y = f(x)$. Choose an open neighbourhood $V \subset Y$ of y and a factorization $g|_V = \pi \circ i$ for some Koszul-regular immersion $i : V \to P$ and smooth morphism $\pi : P \to S$. Next choose an open neighbourhood U of $x \in X$ and a factorization $f|_U = \pi' \circ i'$ for some Koszul-regular immersion $i' : U \to P'$ and smooth morphism $\pi' : P' \to Y$. In fact, we may assume that $P' = \mathbf{A}^n_V$, see discussion preceding and following Definition 37.62.2. Picture:

$$\begin{array}{ccccc}
X & \longleftarrow & U & \xrightarrow[i']{} & P' = \mathbf{A}^n_V \\
\downarrow & & & & \downarrow \\
Y & \longleftarrow & & & V \xrightarrow[i]{} P \\
\downarrow & & & & \downarrow \\
S & \longleftarrow & & & S
\end{array}$$

Set $P'' = \mathbf{A}^n_P$. Then $U \to P' \to P''$ is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely i' and the flat base change of i via $P'' \to P$,

see Divisors, Lemma 31.21.3 and Divisors, Lemma 31.21.7. Also $P'' \to P \to S$ is smooth as a composition of smooth morphisms, see Morphisms, Lemma 29.34.4. Hence we conclude that $X \to S$ is Koszul at x as desired. □

069K **Lemma 37.62.8.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Working affine locally this is More on Algebra, Lemma 15.33.5. We also give a more geometric proof.

Assume (2). By Morphisms, Lemma 29.30.10 for every point x of X there exist affine open neighbourhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = \mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \mathrm{Spec}(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. By Algebra, Lemma 10.136.12 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}}$ for every prime $\mathfrak{q} \supset (f_1, \ldots, f_c)$. Consider the Koszul complex $K_\bullet = K_\bullet(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_\bullet)$. By More on Algebra, Lemma 15.30.2 we see that $(H_i)_{\mathfrak{q}} = 0$, $i > 0$ for every $\mathfrak{q}$ as above. On the other hand, by More on Algebra, Lemma 15.28.6 we see that H_i is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0$, $i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then f is a flat and locally of finite presentation (Lemma 37.62.4). Hence, according to Morphisms, Lemma 29.30.10 it suffices to show that the local rings $\mathcal{O}_{X_s,x}$ are local complete intersection rings. Choose, locally on X, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over S, see Divisors, Definition 31.22.2. Hence according to Divisors, Lemma 31.22.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of X are local complete intersections. □

069L **Lemma 37.62.9.** *A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

069M **Lemma 37.62.10.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions. □

0E9K **Lemma 37.62.11.** *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and X and Y are regular, then f is a local complete intersection morphism.*

Proof. We may assume there is a factorization $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is an immersion. As Y is regular also $\mathbf{A}^n_Y$ is regular by Algebra, Lemma 10.163.10. Hence $X \to \mathbf{A}^n_Y$ is a regular immersion by Divisors, Lemma 31.21.12. □

The following lemma is of a different nature.

09RL **Lemma 37.62.12.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume

(1) *S is locally Noetherian,*
(2) *$Y \to S$ is locally of finite type,*
(3) *$f : X \to Y$ is perfect,*
(4) *$X \to S$ is a local complete intersection morphism.*

Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 15.30.7. Moreover, whether an ideal (resp. ideal sheaf) is regular may be checked on local rings (resp. stalks), see Algebra, Lemma 10.68.6 (resp. Divisors, Lemma 31.20.8)

The question is local. Hence we may assume S, X, Y are affine. In this situation we may choose a commutative diagram

$$\begin{array}{ccc} \mathbf{A}^{n+m}_S & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathbf{A}^n_S & \longleftarrow & Y \\ \downarrow & \swarrow & \\ S & & \end{array}$$

whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings

$$\begin{array}{ccccc} J & \longrightarrow & \mathcal{O}_{\mathbf{A}^{n+m}_S, x} & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow & & \uparrow \\ I & \longrightarrow & \mathcal{O}_{\mathbf{A}^n_S, f(x)} & \longrightarrow & \mathcal{O}_{Y,f(x)} \end{array}$$

where J and I are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal J is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X,x}$ has finite tor dimension over $\mathcal{O}_{Y,f(x)}$. Hence we may apply Divided Power Algebra, Lemma 23.7.6 to conclude that I and J/I are generated by regular sequences. By our initial remarks, this finishes the proof. □

0FJ2 **Lemma 37.62.13.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume S is locally Noetherian, $Y \to S$ is locally of finite type, Y is regular, and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. This is a special case of Lemma 37.62.12 in view of Lemma 37.61.6 (and Morphisms, Lemma 29.15.8). □

0FK1 **Lemma 37.62.14.** *Let $i : X \to Y$ be an immersion. If*

(1) *i is perfect,*
(2) *Y is locally Noetherian, and*
(3) *the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free,*

then i is a regular immersion.

Proof. Translated into algebra, this is Divided Power Algebra, Proposition 23.11.3. □

0FV6 **Lemma 37.62.15.** *Let $f : X \to Y$ be a local complete intersection homomorphism. Then the naive cotangent complex $NL_{X/Y}$ is a perfect object of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is More on Algebra, Lemma 15.86.4. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

0FK2 **Lemma 37.62.16.** *Let $f : X \to Y$ be a perfect morphism of locally Noetherian schemes. The following are equivalent*

(1) *f is a local complete intersection morphism,*
(2) *$NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(3) *$NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is Divided Power Algebra, Lemma 23.11.4. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

0FK3 **Lemma 37.62.17.** *Let $f : X \to Y$ be a flat morphism of finite presentation. The following are equivalent*

(1) *f is a local complete intersection morphism,*
(2) *f is syntomic,*
(3) *$NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(4) *$NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is Divided Power Algebra, Lemma 23.11.5. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

The following lemma gives a characterization of smooth morphisms as flat morphisms whose diagonal is perfect.

0FDP **Lemma 37.62.18.** *Let $f : X \to Y$ be a finite type morphism of locally Noetherian schemes. Denote $\Delta : X \to X \times_Y X$ the diagonal morphism. The following are equivalent*

(1) *f is smooth,*
(2) *f is flat and $\Delta : X \to X \times_Y X$ is a regular immersion,*
(3) *f is flat and $\Delta : X \to X \times_Y X$ is a local complete intersection morphism,*
(4) *f is flat and $\Delta : X \to X \times_Y X$ is perfect.*

Proof. Assume (1). Then f is flat by Morphisms, Lemma 29.34.9. The projections $X \times_Y X \to X$ are smooth by Morphisms, Lemma 29.34.5. Hence the diagonal is a section to a smooth morphism and hence a regular immersion, see Divisors, Lemma 31.22.8. Hence (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is Lemma 37.62.9. The implication (3) $\Rightarrow$ (4) is Lemma 37.62.4. The interesting implication (4) $\Rightarrow$ (1) follows immediately from Divided Power Algebra, Lemma 23.10.2. □

069N **Lemma 37.62.19.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of S. Assume that each base change $f_i : X_i \to S_i$ of f is a local complete intersection morphism. Note that this implies in particular that f is locally of finite type, see Lemma 37.62.4 and Descent, Lemma 35.23.12. Let $x \in X$. Choose an open neighbourhood U of x and an immersion $j : U \to \mathbf{A}^n_S$ over S (see discussion preceding Definition 37.62.2). We have to show that j is a Koszul-regular immersion. Since f_i is a local complete intersection morphism, we see that the base change $j_i : U \times_S S_i \to \mathbf{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 37.62.3. Because $\{\mathbf{A}^n_{S_i} \to \mathbf{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 35.23.34 that j is a Koszul-regular immersion as desired. □

069P **Lemma 37.62.20.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 35.26.4 to prove this. It follows from Lemmas 37.62.8 and 37.62.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 37.62.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 35.26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism. Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 35.14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product

diagrams

$$\begin{array}{ccc} X_i' & \longrightarrow & W_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

After replacing X' by $\coprod X_i'$ and setting $W = \coprod W_i$ we obtain a fibre product diagram of affine schemes

$$\begin{array}{ccc} X' & \longrightarrow & W \\ \downarrow & & \downarrow h \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

with $h : W \to \mathbf{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 29.25.10) and $X' \to X$ is surjective we see that X is contained in the image of $W \to \mathbf{A}^n_Y$. Choose a closed immersion $W \to \mathbf{A}^{n+m}_Y$ over $\mathbf{A}^n_Y$. Now the diagram looks like

$$\begin{array}{ccccc} X' & \longrightarrow & W & \longrightarrow & \mathbf{A}^{n+m}_Y \\ \downarrow & & \downarrow h & \swarrow & \\ X & \longrightarrow & \mathbf{A}^n_Y & & \end{array}$$

Because h is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma 31.21.8 that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 35.23.34) we conclude that $X \to \mathbf{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. □

06B8 **Lemma 37.62.21.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume both X and Y are flat and locally of finite presentation over S. Then the set*

$$\{x \in X \mid f \text{ Koszul at } x\}.$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. The set is open by definition (see Definition 37.62.2). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of f. Let $x' \in X'$ be a point such that f' is Koszul at x'. Denote $s' \in S'$, $x \in X$, $y' \in Y'$, $y \in Y$, $s \in S$ the image of x'. Note that f is locally of finite presentation, see Morphisms, Lemma 29.21.11. Hence we may choose an affine neighbourhood $U \subset X$ of x and an immersion $i : U \to \mathbf{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbf{A}^n_{Y'}$ the base change of i. The assumption that f' is Koszul at x' implies that i' is a Koszul-regular immersion in a neighbourhood of x', see Lemma 37.62.3. The scheme X' is flat and locally of finite presentation over S' as a base change of X (see Morphisms, Lemmas 29.25.8 and 29.21.4). Hence i' is a relative H_1-regular immersion over S' in a neighbourhood of x' (see Divisors, Definition 31.22.2). Thus the base change $i'_{s'} : U'_{s'} \to \mathbf{A}^n_{Y'_{s'}}$ is a H_1-regular immersion in an open neighbourhood of x', see Divisors, Lemma 31.22.1 and the discussion following

Divisors, Definition 31.22.2. Since $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma 29.23.4) we conclude that the base change $i_s : U_s \to \mathbf{A}^n_{Y_s}$ is an H_1-regular immersion in a neighbourhood of x, see Descent, Lemma 35.23.34. Finally, note that $\mathbf{A}^n_Y$ is flat and locally of finite presentation over S, hence Divisors, Lemma 31.22.7 implies that i is a (Koszul-)regular immersion in a neighbourhood of x as desired. □

06B9 **Lemma 37.62.22.** *Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. The first assertion follows immediately from Lemma 37.6.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of f we choose, locally on X, a factorization of f as $f = p \circ i$ where $i : X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma 37.11.13 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as i is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 31.21.5. □

06BA **Lemma 37.62.23.** *Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 37.7.12 is short exact.

Proof. The question is local on Z hence we may assume there exists a factorization $Z \to \mathbf{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & \mathbf{A}^n_Y & \longrightarrow & \mathbf{A}^n_X \\ \| & & \downarrow & & \downarrow \\ Z & \xrightarrow{i} & Y & \longrightarrow & X \end{array}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbf{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 31.21.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X}$ by Lemma 37.7.7 and note that the diagram

$$\begin{array}{ccc} (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} & \longrightarrow & \mathcal{C}_{Z/\mathbf{A}^n_X} \\ \cong \uparrow & & \uparrow \\ i^*\mathcal{C}_{Y/X} & \longrightarrow & \mathcal{C}_{Z/X} \end{array}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. □

37.63. Exact sequences of differentials and conormal sheaves

06BB In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of schemes.

(1) There is a canonical exact sequence
$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$
see Morphisms, Lemma 29.32.9. If $g : Z \to Y$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.16 or see Lemma 37.11.11.

(2) If g is an immersion or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Morphisms, Lemma 29.32.15 or see Lemma 37.7.10. If $f \circ g : Z \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.17 or see Lemma 37.11.12.

(3) If g and $f \circ g$ are immersions or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Morphisms, Lemma 29.32.18 or see Lemma 37.7.11. If $f : Y \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.18 or see Lemma 37.11.13.

(4) If g and f are immersions or more generally formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Morphisms, Lemma 29.31.5 or see Lemma 37.7.12. If $g : Z \to Y$ is a regular immersion[14] or more generally a local complete intersection morphism, then this sequence is a short exact sequence, see Divisors, Lemma 31.21.6 or see Lemma 37.62.23.

37.64. Weakly étale morphisms

094N A ring homomorphism $A \to B$ is weakly étale if both $A \to B$ and $B \otimes_A B \to B$ are flat, see More on Algebra, Definition 15.105.1. The analogous notion for morphisms of schemes is the following.

094P **Definition 37.64.1.** A morphism of schemes $X \to Y$ is *weakly étale* or *absolutely flat* if both $X \to Y$ and the diagonal morphism $X \to X \times_Y X$ are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if $X \to Y$ is weakly étale, then $L_{X/Y} = 0$, as follows from Cotangent, Lemma 92.8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 61.9). In this section we stick with the basics.

[14] It suffices for g to be a H_1-regular immersion. Observe that an immersion which is a local complete intersection morphism is Koszul regular.

094Q **Lemma 37.64.2.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *$X \to Y$ is weakly étale, and*
(2) *for every $x \in X$ the ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is weakly étale.*

Proof. Observe that under both assumptions (1) and (2) the morphism f is flat. Thus we may assume f is flat. Let $x \in X$ with image $y = f(x)$ in Y. There are canonical maps of rings

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \longrightarrow \mathcal{O}_{X,x}$$

where the first map is a localization (hence flat) and the second map is a surjection. Condition (1) means that the second arrow is flat for all x. Condition (2) is that the composition is flat for all x. Thus the equivalence by Algebra, Lemma 10.39.18 part (2). □

094R **Lemma 37.64.3.** *Let $X \to Y$ be a morphism of schemes such that $X \to X \times_Y X$ is flat. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over Y, then $\mathcal{F}$ is flat over X.*

Proof. Let $x \in X$ with image $y = f(x)$ in Y. Since $X \to X \times_Y X$ is flat, we see that $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is flat. Hence the result follows from More on Algebra, Lemma 15.105.2 and the definitions. □

094S **Lemma 37.64.4.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is weakly étale.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is weakly étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is weakly étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of weakly étale, for all $j \in J, i \in I_j$.*

Moreover, if f is weakly étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is weakly-étale.

Proof. Suppose given open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$. Then $U \times_V U \subset X \times_Y X$ is open (Schemes, Lemma 26.17.3) and the diagonal $\Delta_{U/V}$ of $f|_U : U \to V$ is the restriction $\Delta_{X/Y}|_U : U \to U \times_V U$. Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 29.25.3) the final statement of the lemma is follows as well as the equivalence of (1) and (3). If X and Y are affine, then $X \to Y$ is weakly étale if and only if $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is weakly étale (use again Morphisms, Lemma 29.25.3). Thus (1) and (3) are also equivalent to (2) and (4). □

094T **Lemma 37.64.5.** *Let $X \to Y \to Z$ be morphisms of schemes.*

(1) *If $X \to X \times_Y X$ and $Y \to Y \times_Z Y$ are flat, then $X \to X \times_Z X$ is flat.*
(2) *If $X \to Y$ and $Y \to Z$ are weakly étale, then $X \to Z$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal of X over Z, the fact that

$$X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y,$$

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 29.25.8 and 29.25.6). Part (2) follows from part (1) and the fact (just used) that the composition of flat morphisms is flat. □

094U **Lemma 37.64.6.** *Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of X.*

(1) *If $X \to X \times_Y X$ is flat, then $X' \to X' \times_{Y'} X'$ is flat.*
(2) *If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.*

Proof. Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_{Y'} X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 29.25.8. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. □

094V **Lemma 37.64.7.** *Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

The top horizontal arrow is flat and the vertical arrows are flat. Hence X is flat over $Y \times_Z Y$. By Morphisms, Lemma 29.25.13 we see that Y is flat over $Y \times_Z Y$. □

094W **Lemma 37.64.8.** *Let $f : X \to Y$ be a weakly étale morphism of schemes. Then f is formally unramified, i.e., $\Omega_{X/Y} = 0$.*

Proof. Recall that f is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 37.6.7. Via Lemma 37.64.4 and Morphisms, Lemma 29.32.5 this follows from the case of rings which is More on Algebra, Lemma 15.105.12. □

094X **Lemma 37.64.9.** *Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases*

(1) *$X \to Y$ is a flat monomorphism,*
(2) *$X \to Y$ is an open immersion,*
(3) *$X \to Y$ is flat and unramified,*
(4) *$X \to Y$ is étale.*

Proof. If (1) holds, then $\Delta_{X/Y}$ is an isomorphism (Schemes, Lemma 26.23.2), hence certainly f is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 29.35.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 29.36.5), hence (4) follows from (3). □

094Y **Lemma 37.64.10.** *Let $f : X \to Y$ be a morphism of schemes. If Y is reduced and f weakly étale, then X is reduced.*

Proof. Via Lemma 37.64.4 this follows from the case of rings which is More on Algebra, Lemma 15.105.8. □

The following lemma uses a nontrivial result about weakly étale ring maps.

094Z **Lemma 37.64.11.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is weakly étale, and*
(2) *for $x \in X$ the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism on strict henselizations.*

Proof. Let $x \in X$ be a point with image $y = f(x)$ in Y. Choose a separable algebraic closure κ^{sep} of $\kappa(x)$. Let $\mathcal{O}^{sh}_{X,x}$ be the strict henselization corresponding to κ^{sep} and $\mathcal{O}^{sh}_{Y,y}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in κ^{sep}. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}^{sh}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & \longrightarrow & \mathcal{O}^{sh}_{Y,y} \end{array}$$

local homomorphisms of local rings, see Algebra, Lemma 10.155.10. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 15.105.14 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 15.45.1.

Assume f weakly étale. By Lemma 37.64.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 15.105.9 and 15.105.11 the right vertical arrow is weakly étale. By More on Algebra, Theorem 15.105.24 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}^{sh}_{X,x}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 15.105.10. Thus (2) implies (1) by Lemma 37.64.2. □

0950 **Lemma 37.64.12.** *Let $f : X \to Y$ be a morphism of schemes. If Y is a normal scheme and f weakly étale, then X is a normal scheme.*

Proof. By More on Algebra, Lemma 15.45.6 a scheme S is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 37.64.11. □

0951 **Lemma 37.64.13.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X, Y are weakly étale over S, then f is weakly étale.*

Proof. We will use Morphisms, Lemmas 29.25.8 and 29.25.6 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an immersion. Thus Lemma 37.64.3 implies, that since X is flat over $X \times_S X$ it follows that X is flat over $X \times_Y X$. □

The following is a scheme theoretic generalization of the observation that a field extension that is simultaneously separable and purely inseparable must be an isomorphism.

0F6V **Lemma 37.64.14.** *Let $f : X \to Y$ be a morphism of schemes. If f is weakly étale and a universal homeomorphism, it is an isomorphism.*

Proof. Since f is a universal homeomorphism, the diagonal $\Delta : X \to X \times_Y X$ is a surjective closed immersion by Morphisms, Lemmas 29.45.4 and 29.10.2. Since Δ is also flat, we see that Δ must be an isomorphism by Morphisms, Lemma 29.26.1. In other words, f is a monomorphism (Schemes, Lemma 26.23.2). Since f is a universal homeomorphism it is certainly quasi-compact. Hence by Descent, Lemma 35.25.1 we find that f is an isomorphism. □

The following is a weakly étale generalization of Étale Morphisms, Lemma 41.14.3.

0F6W **Lemma 37.64.15.** *Let $U \to X$ be a weakly étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.*

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 33.36.6. The morphism $F_{U/X}$ is weakly étale as a morphism between schemes weakly étale over X by Lemma 37.64.13. Hence $F_{U/X}$ is an isomorphism by Lemma 37.64.14. □

37.65. Reduced fibre theorem

09IJ In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in a straightforward manner from Epp's result on eliminating ramification, see More on Algebra, Theorem 15.116.18.

Let A be a Dedekind domain with fraction field K. Let X be a scheme flat and of finite type over A. Let L be a finite extension of K. Let B be the integral closure of A in L. Then B is a Dedekind domain (Algebra, Lemma 10.120.18). Let $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ be the base change. Then $X_B \to \mathrm{Spec}(B)$ is of finite type (Morphisms, Lemma 29.15.4). Hence X_B is Noetherian (Morphisms, Lemma 29.15.6). Thus the normalization $\nu : Y \to X_B$ exists (see Morphisms, Definition 29.54.1 and the discussion following). Picture

09IK (37.65.0.1)
$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & X_B & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

We sometimes call Y the *normalized base change* of X. In general the morphism ν may not be finite. But if A is a Nagata ring (a condition that is virtually always satisfied in practice) then ν is finite and Y is of finite type over B, see Morphisms, Lemmas 29.54.11 and 29.18.1.

Taking the normalized base change commutes with composition. More precisely, if $M/L/K$ are finite extensions of fields with integral closures $A \subset B \subset C$ then the normalized base change Z of $Y \to \mathrm{Spec}(B)$ relative to M/L is equal to the normalized base change of $X \to \mathrm{Spec}(A)$ relative to M/K.

09IL **Theorem 37.65.1.** *Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring. There exists a finite extension L/K such that the normalized base change Y is smooth over* $\mathrm{Spec}(B)$ *at all generic points of all fibres.*

Proof. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 29.34.15.

We first choose a finite extension L/K such that $(X_L)_{red}$ is geometrically reduced over L, see Varieties, Lemma 33.6.11. Since $Y \to (X_B)_{red}$ is birational we see applying Varieties, Lemma 33.6.8 that Y_L is geometrically reduced over L as well. Hence $Y_L \to \mathrm{Spec}(L)$ is smooth on a dense open $V \subset Y_L$ by Varieties, Lemma 33.25.7. Thus the smooth locus $U \subset Y$ of the morphism $Y \to \mathrm{Spec}(B)$ is open (by Morphisms, Definition 29.34.1) and is dense in the generic fibre. Replacing A by B and X by Y we reduce to the case treated in the next paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 37.24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. By More on Algebra, Proposition 15.117.8 there exist finite extensions K_i/K which are solutions for the extension of discrete valuation rings $A_i \to \mathcal{O}_i$. Let L/K be a finite extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 15.117.1.

Consider the diagram (37.65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 10.163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_{\mathfrak{m}} \to \mathcal{O}_{Y,y}$ is formally smooth in the $\mathfrak{m}_y$-adic topology (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable, see More on Algebra, Lemma 15.112.5. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 10.140.5. This finishes the proof. □

0BRQ **Lemma 37.65.2** (Variant over curves)**.** *Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension* 1*. Then there exists a finite extension L/K such that in the diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & X \times_S T & \longrightarrow & X \\ & \searrow^{g} & \downarrow & & \downarrow f \\ & & T & \longrightarrow & S \end{array}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\mathrm{Spec}(L)$ *and $\nu : Y \to X \times_S T$ is the normalization.*

Proof. Choose a finite affine open covering $S = \bigcup \mathrm{Spec}(A_i)$. Then K is equal to the fraction field of A_i for all i. Let $X_i = X \times_S \mathrm{Spec}(A_i)$. Choose L_i/K as in Theorem 37.65.1 for the morphism $X_i \to \mathrm{Spec}(A_i)$. Let $B_i \subset L_i$ be the integral closure of A_i and let Y_i be the normalized base change of X to B_i. Let L/K be a finite extension dominating each L_i. Let $T_i \subset T$ be the inverse image of $\mathrm{Spec}(A_i)$. For each i we get a commutative diagram

$$\begin{array}{ccccc} g^{-1}(T_i) & \longrightarrow & Y_i & \longrightarrow & X \times_S \mathrm{Spec}(A_i) \\ \downarrow & & \downarrow & & \downarrow \\ T_i & \longrightarrow & \mathrm{Spec}(B_i) & \longrightarrow & \mathrm{Spec}(A_i) \end{array}$$

and in fact the left hand square is a normalized base change as discussed at the beginning of the section. In the proof of Theorem 37.65.1 we have seen that the smooth locus of $Y \to T$ contains the inverse image in $g^{-1}(T_i)$ of the set of points where Y_i is smooth over B_i. This proves the lemma. □

0BRR **Lemma 37.65.3** (Variant with separable extension)**.** *Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring and that for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/K$ is separable. Then there exists a finite separable extension L/K such that the normalized base change Y is smooth over $\mathrm{Spec}(B)$ at all generic points of all fibres.*

Proof. This is proved in exactly the same manner as Theorem 37.65.1 with a few minor modifications. The most important change is to use More on Algebra, Lemma 15.117.9 instead of More on Algebra, Proposition 15.117.8. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 29.34.15.

Since X is flat over A every generic point η of X maps to the generic point of $\mathrm{Spec}(A)$. After replacing X by its reduction we may assume X is reduced. In this case X_K is geometrically reduced over K by Varieties, Lemma 33.6.8. Hence $X_K \to \mathrm{Spec}(K)$ is smooth on a dense open by Varieties, Lemma 33.25.7. Thus the smooth locus $U \subset X$ of the morphism $X \to \mathrm{Spec}(A)$ is open (by Morphisms, Definition 29.34.1) and is dense in the generic fibre. This reduces us to the situation of the following paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 37.24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Observe that the fraction field of $\mathcal{O}_i$ is the residue field of a generic point of X. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. We may apply More on Algebra, Lemma 15.117.9 and we find finite separable extensions K_i/K which are solutions for $A_i \to \mathcal{O}_i$. Let L/K be a finite separable extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 15.117.1.

Consider the diagram (37.65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 10.163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_{\mathfrak{m}} \to \mathcal{O}_{Y,y}$ is formally smooth (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 10.140.5. This finishes the proof. □

0BRS **Lemma 37.65.4** (Variant with separable extensions over curves)**.** *Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension 1. Assume the field extensions $\kappa(\eta)/K$ are separable for every generic point η of an irreducible component of X. Then there exists a finite separable extension L/K such that in the diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & X \times_S T & \longrightarrow & X \\ & \searrow^{g} & \downarrow & & \downarrow f \\ & & T & \longrightarrow & S \end{array}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\mathrm{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

Proof. This follows from Lemma 37.65.3 in exactly the same manner that Lemma 37.65.2 follows from Theorem 37.65.1. □

37.66. Ind-quasi-affine morphisms

0AP5 A bit of theory to be used later.

0AP6 **Definition 37.66.1.** A scheme X is *ind-quasi-affine* if every quasi-compact open of X is quasi-affine. Similarly, a morphism of schemes $X \to Y$ is *ind-quasi-affine* if $f^{-1}(V)$ is ind-quasi-affine for each affine open V in Y.

An example of an ind-quasi-affine scheme is an open of an affine scheme. If $X = \bigcup_{i \in I} U_i$ is a union of quasi-affine opens such that any two U_i are contained in a third, then X is ind-quasi-affine. An ind-quasi-affine scheme X is separated because any two affine opens U, V are contained in a separated open subscheme of X, namely $U \cup V$. Similarly an ind-quasi-affine morphism is separated.

0F1U **Lemma 37.66.2.** *For a morphism of schemes $f : X \to Y$, the following are equivalent:*

(1) *f is ind-quasi-affine,*
(2) *for every affine open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.*
(3) *for some cover $\{V_j\}_{j \in J}$ of Y by quasi-compact and quasi-separated open subschemes $V_j \subset Y$, every $j \in J$, and every quasi-compact open subscheme $U \subset f^{-1}(V_j)$, the induced morphism $U \to V_j$ is quasi-affine.*

(4) *for every quasi-compact and quasi-separated open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.*

In particular, the property of being an ind-quasi-affine morphism is Zariski local on the base.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the definitions and Morphisms, Lemma 29.13.3. For (2) $\Rightarrow$ (4), let U and V be as in (4). By Schemes, Lemma 26.21.14, the induced morphism $U \to V$ is quasi-compact. Thus, for every affine open $V' \subset V$, the fiber product $V' \times_V U$ is quasi-compact, so, by (2), the induced map $V' \times_V U \to V'$ is quasi-affine. Thus, $U \to V$ is also quasi-affine by Morphisms, Lemma 29.13.3. This argument also gives (3) $\Rightarrow$ (4): indeed, keeping the same notation, those affine opens $V' \subset V$ that lie in one of the V_j cover V, so one needs to argue that the quasi-compact map $V' \times_V U \to V'$ is quasi-affine. However, by (3), the composition $V' \times_V U \to V' \to V_j$ is quasi-affine and, by Schemes, Lemma 26.21.13, the map $V' \to V_j$ is quasi-separated. Thus, $V' \times_V U \to V'$ is quasi-affine by Morphisms, Lemma 29.13.8. The final implications (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are evident. □

0F1V **Lemma 37.66.3.** *The property of being an ind-quasi-affine morphism is stable under composition.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be ind-quasi-affine morphisms. Let $V \subset Z$ and $U \subset f^{-1}(g^{-1}(V))$ be quasi-compact opens such that V is also quasi-separated. The image $f(U)$ is a quasi-compact subset of $g^{-1}(V)$, so it is contained in some quasi-compact open $W \subset g^{-1}(V)$ (a union of finitely many affines). We obtain a factorization $U \to W \to V$. The map $W \to V$ is quasi-affine by Lemma 37.66.2, so, in particular, W is quasi-separated. Then, by Lemma 37.66.2 again, $U \to W$ is quasi-affine as well. Consequently, by Morphisms, Lemma 29.13.4, the composition $U \to V$ is also quasi-affine, and it remains to apply Lemma 37.66.2 once more. □

0F1W **Lemma 37.66.4.** *Any quasi-affine morphism is ind-quasi-affine. Any immersion is ind-quasi-affine.*

Proof. The first assertion is immediate from the definitions. In particular, affine morphisms, such as closed immersions, are ind-quasi-affine. Thus, by Lemma 37.66.3, it remains to show that an open immersion is ind-quasi-affine. This, however, is immediate from the definitions. □

0F1X **Lemma 37.66.5.** *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes such that $g \circ f$ is ind-quasi-affine, then f is ind-quasi-affine.*

Proof. By Lemma 37.66.2, we may work Zariski locally on Z and then on Y, so we lose no generality by assuming that Z, and then also Y, is affine. Then any quasi-compact open of X is quasi-affine, so Lemma 37.66.2 gives the claim. □

0AP7 **Lemma 37.66.6.** *The property of being ind-quasi-affine is stable under base change.*

Proof. Let $f : X \to Y$ be an ind-quasi-affine morphism. For checking that every base change of f is ind-quasi-affine, by Lemma 37.66.2, we may work Zariski locally on Y, so we assume that Y is affine. Furthermore, we may also assume that in the base change morphism $Z \to Y$ the scheme Z is affine, too. The base change

$X \times_Y Z \to X$ is an affine morphism, so, by Lemmas 37.66.3 and 37.66.4, the map $X \times_Y Z \to Y$ is ind-quasi-affine. Then, by Lemma 37.66.5, the base change $X \times_Y Z \to Z$ is ind-quasi-affine, as desired. □

0AP8 **Lemma 37.66.7.** *The property of being ind-quasi-affine is fpqc local on the base.*

Proof. The stability of ind-quasi-affineness under base change supplied by Lemma 37.66.6 gives one direction. For the other, let $f : X \to Y$ be a morphism of schemes and let $\{g_i : Y_i \to Y\}$ be an fpqc covering such that the base change $f_i : X_i \to Y_i$ is ind-quasi-affine for all i. We need to show f is ind-quasi-affine.

By Lemma 37.66.2, we may work Zariski locally on Y, so we assume that Y is affine. Then we use stability under base change ensured by Lemma 37.66.6 to refine the cover and assume that it is given by a single affine, faithfully flat morphism $g : Y' \to Y$. For any quasi-compact open $U \subset X$, its Y'-base change $U \times_Y Y' \subset X \times_Y Y'$ is also quasi-compact. It remains to observe that, by Descent, Lemma 35.23.22, the map $U \to Y$ is quasi-affine if and only if so is $U \times_Y Y' \to Y'$. □

0AP9 **Lemma 37.66.8.** *A separated locally quasi-finite morphism of schemes is ind-quasi-affine.*

Proof. Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Let $V \subset Y$ be affine and $U \subset f^{-1}(V)$ quasi-compact open. We have to show U is quasi-affine. Since $U \to V$ is a separated quasi-finite morphism of schemes, this follows from Zariski's Main Theorem. See Lemma 37.43.2. □

37.67. Pushouts in the category of schemes, II

0ECH This section is a continuation of Section 37.14. In this section we construct pushouts of $Y \leftarrow Z \to X$ where $Z \to X$ is a closed immersion and $Z \to Y$ is integral and an additional condition is satisfied. Please see the detailed discussion in [**Fer03**].

0ECI **Situation 37.67.1.** Here S is a scheme and $i : Z \to X$ and $j : Z \to Y$ are morphisms of schemes over S. We assume

(1) i is a closed immersion,
(2) j is an integral morphism of schemes,
(3) for $y \in Y$ there exists an affine open $U \subset X$ with $j^{-1}(\{y\}) \subset i^{-1}(U)$.

0ECJ **Lemma 37.67.2.** *In Situation 37.67.1 then for $y \in Y$ there exist affine opens $U \subset X$ and $V \subset Y$ with $i^{-1}(U) = j^{-1}(V)$ and $y \in V$.*

Proof. Let $y \in Y$. Choose an affine open $U \subset X$ such that $j^{-1}(\{y\}) \subset i^{-1}(U)$ (possible by assumption). Choose an affine open $V \subset Y$ neighbourhood of y such that $j^{-1}(V) \subset i^{-1}(U)$. This is possible because $j : Z \to Y$ is a closed morphism (Morphisms, Lemma 29.44.7) and $i^{-1}(U)$ contains the fibre over y. Since j is integral, the scheme theoretic fibre Z_y is the spectrum of an algebra integral over a field. By Limits, Lemma 32.11.6 we can find an $\overline{f} \in \Gamma(i^{-1}(U), \mathcal{O}_{i^{-1}(U)})$ such that $Z_y \subset D(\overline{f}) \subset j^{-1}(V)$. Since $i|_{i^{-1}(U)} : i^{-1}(U) \to U$ is a closed immersion of affines, we can choose an $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is $\overline{f}$. After replacing U by the principal open $D(f) \subset U$ we find affine opens $y \in V \subset Y$ and $U \subset X$ with

$$j^{-1}(\{y\}) \subset i^{-1}(U) \subset j^{-1}(V)$$

Now we (in some sense) repeat the argument. Namely, we choose $g \in \Gamma(V, \mathcal{O}_V)$ such that $y \in D(g)$ and $j^{-1}(D(g)) \subset i^{-1}(U)$ (possible by the same argument as above). Then we can pick $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is the pullback of g by $i^{-1}(U) \to V$ (again possible by the same reason as above). Then we finally have affine opens $y \in V' = D(g) \subset V \subset Y$ and $U' = D(f) \subset U \subset X$ with $j^{-1}(V') = i^{-1}(V')$. □

0E25 **Proposition 37.67.3.** *In Situation 37.67.1 the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture* [**Fer03**, Theorem 7.1 part iii]

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{b} & Y \amalg_Z X \end{array}$$

The diagram is a fibre square, the morphism a is integral, the morphism b is a closed immersion, and

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as sheaves of rings where $c = a \circ i = b \circ j$.

Proof. As a topological space we set $Y \amalg_Z X$ equal to the pushout of the diagram in the category of topological spaces (Topology, Section 5.29). This is just the pushout of the underlying sets (Topology, Lemma 5.29.1) endowed with the quotient topology. On $Y \amalg_Z X$ we have the maps of sheaves of rings

$$b_*\mathcal{O}_Y \longrightarrow c_*\mathcal{O}_Z \longleftarrow a_*\mathcal{O}_X$$

and we can define

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as the fibre product in the category of sheaves of rings. To prove that we obtain a scheme we have to show that every point has an affine open neighbourhood. This is clear for points not in the image of c as the image of c is a closed subset whose complement is isomorphic as a ringed space to $(Y \setminus j(Z)) \amalg (X \setminus i(Z))$.

A point in the image of c corresponds to a unique $y \in Y$ in the image of j. By Lemma 37.67.2 we find affine opens $U \subset X$ and $V \subset Y$ with $y \in V$ and $i^{-1}(U) = j^{-1}(V)$. Since the construction of the first paragraph is clearly compatible with restriction to compatible open subschemes, to prove that it produces a scheme we may assume X, Y, and Z are affine.

If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Z = \mathrm{Spec}(C)$ are affine, then More on Algebra, Lemma 15.6.2 shows that $Y \amalg_Z X = \mathrm{Spec}(B \times_C A)$ as topological spaces. To finish the proof that $Y \times_Z X$ is a scheme, it suffices to show that on $\mathrm{Spec}(B \times_C A)$ the structure sheaf is the fibre product of the pushforwards. This follows by applying More on Algebra, Lemma 15.5.3 to principal affine opens of $\mathrm{Spec}(B \times_C A)$.

The discussion above shows the scheme $Y \amalg_Z X$ has an affine open covering $Y \amalg_Z X = \bigcup W_i$ such that $U_i = a^{-1}(W_i)$, $V_i = b^{-1}(W_i)$, and $\Omega_i = c^{-1}(W_i)$ are affine open in X, Y, and Z. Thus a and b are affine. Moreover, if A_i, B_i, C_i are the rings corresponding to U_i, V_i, Ω_i, then $A_i \to C_i$ is surjective and W_i corresponds to $A_i \times_{C_i} B_i$ which surjects onto B_i. Hence b is a closed immersion. The ring map

$A_i \times_{C_i} B_i \to A_i$ is integral by More on Algebra, Lemma 15.6.3 hence a is integral. The diagram is cartesian because

$$C_i \cong B_i \otimes_{B_i \times_{C_i} A_i} A_i$$

This follows as $B_i \times_{C_i} A_i \to B_i$ and $A_i \to C_i$ are surjective maps whose kernels are the same.

Finally, we can apply Lemmas 37.14.1 and 37.14.2 to conclude our construction is a pushout in the category of schemes. □

0E26 **Lemma 37.67.4.** *In Situation 37.67.1. If X and Y are separated, then the pushout $Y \amalg_Z X$ (Proposition 37.67.3) is separated. Same with "separated over S", "quasi-separated", and "quasi-separated over S".*

Proof. The morphism $Y \amalg X \to Y \amalg_Z X$ is surjective and universally closed. Thus we may apply Morphisms, Lemma 29.41.11. □

0E27 **Lemma 37.67.5.** *In Situation 37.67.1 assume S is a locally Noetherian scheme and X, Y, and Z are locally of finite type over S. Then the pushout $Y \amalg_Z X$ (Proposition 37.67.3) is locally of finite type over S.*

Proof. Looking on affine opens we recover the result of More on Algebra, Lemma 15.5.1. □

0ECK **Lemma 37.67.6.** *In Situation 37.67.1 suppose given a commutative diagram*

$$\begin{array}{ccccc} Y' & \xleftarrow{j'} & Z' & \xrightarrow{i'} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ Y & \longleftarrow & Z & \longrightarrow & X \end{array}$$

with cartesian squares and f, g, h separated and locally quasi-finite. Then

1. *the pushouts $Y \amalg_Z X$ and $Y' \amalg_{Z'} X'$ exist,*
2. *$Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated and locally quasi-finite, and*
3. *the squares*

$$\begin{array}{ccccc} Y' & \longrightarrow & Y' \amalg_{Z'} X' & \longleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y \amalg_Z X & \longleftarrow & X \end{array}$$

are cartesian.

Proof. The pushout $Y \amalg_Z X$ exists by Proposition 37.67.3. To see that the pushout $Y' \amalg_{Z'} X'$ exists, we check condition (3) of Situation 37.67.1 holds for (X', Y', Z', i', j'). Namely, let $y' \in Y'$ and denote $y \in Y$ the image. Choose $U \subset X$ affine open with $i(j^{-1}(y)) \subset U$. Choose a quasi-compact open $U' \subset X'$ contained in $f^{-1}(U)$ containing the quasi-compact subset $i'((j')^{-1}(\{y'\}))$. By Lemma 37.66.8 we see that U' is quasi-affine. Since $Z'_{y'}$ is the spectrum of an algebra integral over a field, we can apply Limits, Lemma 32.11.6 and we find there exists an affine open subscheme of U' containing $i'((j')^{-1}(\{y'\}))$ as desired.

Having verified existence we check the other assertions. Affine locally we are exactly in the situation of More on Algebra, Lemma 15.7.7 with $B \to D$ and $A' \to C'$

locally quasi-finite[15]. In particular, the morphism $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally of finite type. The squares in of the diagram are cartesian by More on Algebra, Lemma 15.6.4. Since being locally quasi-finite can be checked on fibres (Morphisms, Lemma 29.20.6) we conclude that $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally quasi-finite.

We still have to check $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated. Observe that $Y' \amalg X' \to Y' \amalg_{Z'} X'$ is universally closed and surjective by Proposition 37.67.3. Since also the morphism $Y' \amalg X' \to Y \amalg_Z X$ is separated (as it factors as $Y' \amalg X' \to Y \amalg X \to Y \amalg_Z X$) we conclude by Morphisms, Lemma 29.41.11. □

0ECL **Lemma 37.67.7.** *In Situation 37.67.1 the category of schemes flat, separated, and locally quasi-finite over the pushout $Y \amalg_Z X$ is equivalent to the category of $(X', Y', Z', i', j', f, g, h)$ as in Lemma 37.67.6 with f, g, h flat. Similarly with "flat" replaced with "étale".*

Proof. If we start with $(X', Y', Z', i', j', f, g, h)$ as in Lemma 37.67.6 with f, g, h flat or étale, then $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is flat or étale by More on Algebra, Lemma 15.7.7.

For the converse, let $W \to Y \amalg_Z X$ be a separated and locally quasi-finite morphism. Set $X' = W \times_{Y \amalg_Z X} X$, $Y' = W \times_{Y \amalg_Z X} Y$, and $Z' = W \times_{Y \amalg_Z X} Z$ with obvious morphisms i', j', f, g, h. Form the pushout $Y' \amalg_{Z'} X'$. We obtain a morphism

$$Y' \amalg_{Z'} X' \longrightarrow W$$

of schemes over $Y \amalg_X Z$ by the universal property of the pushout. If we do not assume that $W \to Y \amalg_Z X$ is flat, then in general this morphism won't be an isomorphism. (In fact, More on Algebra, Lemma 15.6.5 shows the displayed arrow is a closed immersion but not an isomorphism in general.) However, if $W \to Y \times_Z X$ is flat, then it is an isomorphism by More on Algebra, Lemma 15.7.7. □

Next, we discuss existence in the case where both morphisms are closed immersions.

0B7M **Lemma 37.67.8.** *Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Then the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{b} & Y \amalg_Z X \end{array}$$

The diagram is a fibre square, the morphisms a and b are closed immersions, and there is a short exact sequence

$$0 \to \mathcal{O}_{Y \amalg_Z X} \to a_*\mathcal{O}_X \oplus b_*\mathcal{O}_Y \to c_*\mathcal{O}_Z \to 0$$

where $c = a \circ i = b \circ j$.

Proof. This is a special case of Proposition 37.67.3. Observe that hypothesis (3) in Situation 37.67.1 is immediate because the fibres of j are singletons. Finally, reverse the roles of the arrows to conclude that both a and b are closed immersions. □

[15] To be precise $X, Y, Z, Y \amalg_Z X, X', Y', Z', Y' \amalg_{Z'} X'$ correspond to $A', B, A, B', C', D, C, D'$.

0CYY **Lemma 37.67.9.** *Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes and let $\varphi : X' \times_{X,i} Z \to Y' \times_{Y,j} Z$ be an isomorphism of schemes over Z. Consider the morphism*

$$h : X' \amalg_{X' \times_{X,i} Z, \varphi} Y' \longrightarrow X \amalg_Z Y$$

Then we have

1. *h is locally of finite type if and only if f and g are locally of finite type,*
2. *h is flat if and only if f and g are flat,*
3. *h is flat and locally of finite presentation if and only if f and g are flat and locally of finite presentation,*
4. *h is smooth if and only if f and g are smooth,*
5. *h is étale if and only if f and g are étale, and*
6. *add more here as needed.*

Proof. We know that the pushouts exist by Lemma 37.67.8. In particular we get the morphism h. Hence we may replace all schemes in sight by affine schemes. In this case the assertions of the lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 15.7.7. □

37.68. Relative morphisms

0BL0 In this section we prove a representability result which we will use in Fundamental Groups, Section 58.5 to prove a result on the category of finite étale coverings of a scheme. The material in this section is discussed in the correct generality in Criteria for Representability, Section 97.10.

Let S be a scheme. Let Z and X be schemes over S. Given a scheme T over S we can consider morphisms $b : T \times_S Z \to T \times_S X$ over S. Picture

0BL1 (37.68.0.1)

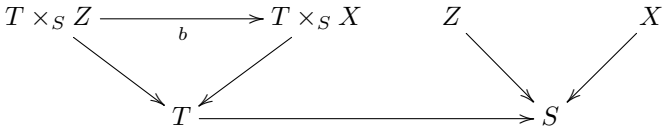

Of course, we can also think of b as a morphism $b : T \times_S Z \to X$ such that

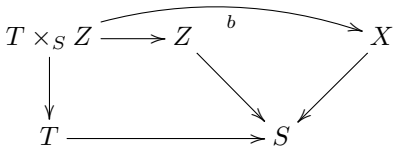

commutes. In this situation we can define a functor

0BL2 (37.68.0.2) $$Mor_S(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{b \text{ as above}\}$$

Here is a basic representability result.

05Y6 **Lemma 37.68.1.** *Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $Mor_S(Z, X)$ is representable by an affine scheme over S.*

Proof. Write $S = \mathrm{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over R. Choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i\in I, 1\leq j\leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i\in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = \mathrm{Spec}(P/J)$ represents the functor $Mor_S(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $W \to S$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_S Z \to W \times_S X$. By the Yoneda lemma b_{univ} determines a transformation of functors $W \to Mor_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_S(Z, X)(T)$ over any affine scheme T.

Suppose $T = \mathrm{Spec}(R')$ is an affine scheme over S and $b \in Mor_S(Z, X)(T)$. The structure morphism $T \to S$ defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that b is the pullback of b_{univ}. Some details omitted. □

0BL3 **Lemma 37.68.2.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $Mor_S(Z, X)$ is representable by a scheme affine over S.*

Proof. Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z\times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset Mor_S(Z, X)$ be the subfunctor which assigns to T/S the empty set if $T \to S$ does not factor through U_i and $Mor_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 26.15.4 which proves the lemma. Condition (2)(a) follows from Lemma 37.68.1 and the other two follow from straightforward arguments. □

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: X is quasi-affine, f is quasi-affine, f is quasi-projective, f is locally projective, there exists an ample invertible sheaf on X, there exists an f-ample invertible sheaf on X, or there exists an f-very ample invertible sheaf on X.

0BL4 **Lemma 37.68.3.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume*

1. *$Z \to S$ is finite locally free, and*
2. *for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.*

Then $Mor_S(Z, X)$ is representable by a scheme.

Proof. Consider the set I of pairs (U, V) where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through V. For $i \in I$ denote (U_i, V_i) the corresponding pair. Set $F_i = Mor_{V_i}(Z_{V_i}, U_i)$. It is immediate that F_i is a subfunctor of $Mor_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 26.15.4 which proves the lemma.

Condition (2)(a) follows from Lemma 37.68.2.

To check condition (2)(b) consider T/S and $b \in Mor_S(Z, X)(T)$. Thinking of b as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i,b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i,b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider T/S and $b \in Mor_S(Z, X)(T)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i,b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some i where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since p is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image s of t in S. Thus our condition on $X \to S$ exactly shows a suitable pair exists. □

0BL5 **Lemma 37.68.4.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume $Z \to S$ is finite locally free and $X \to S$ is separated and locally quasi-finite. Then $Mor_S(Z, X)$ is representable by a scheme.*

Proof. This follows from Lemmas 37.68.3 and 37.45.1. □

37.69. Characterizing pseudo-coherent complexes, III

0CSI In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. This is a continuation of Derived Categories of Schemes, Section 36.34. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 37.69.2.

0CTA **Lemma 37.69.1.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} Z' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X' & \longrightarrow & S' \end{array}$$

Let $S \to S'$ be a morphism. Denote by X and Y the base changes of X' and Y' to S. Assume $Y' \to S'$ and $Z' \to X'$ are flat. Then $X \times_S Y$ and Z' are Tor independent over $X' \times_{S'} Y'$.

Proof. The question is local, hence we may assume all schemes are affine (some details omitted). Observe that

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X' \times_{S'} Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

is cartesian with flat vertical arrows. Write $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $X' \times_{S'} Y' = \mathrm{Spec}(B')$. Then $X \times_S Y = \mathrm{Spec}(A \otimes_{A'} B')$. Write $Z' = \mathrm{Spec}(C')$. We have to show

$$\mathrm{Tor}_p^{B'}(A \otimes_{A'} B', C') = 0, \quad \text{for } p > 0$$

Since $A' \to B'$ is flat we have $A \otimes_{A'} B' = A \otimes^{\mathbf{L}}_{A'} B'$. Hence

$$(A \otimes_{A'} B') \otimes^{\mathbf{L}}_{B'} C' = (A \otimes^{\mathbf{L}}_{A'} B') \otimes^{\mathbf{L}}_{B'} C' = A \otimes^{\mathbf{L}}_{A'} C' = A \otimes_{A'} C'$$

The second equality by More on Algebra, Lemma 15.60.5. The last equality because $A' \to C'$ is flat. This proves the lemma. □

0CSJ **Lemma 37.69.2** (Derived Chow's lemma)**.** *Let A be a ring. Let X be a separated scheme of finite presentation over A. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ of x, an $n \geq 0$, an open $V \subset \mathbf{P}^n_A$, a closed subscheme $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in Z$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that*

(1) *$Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,*
(2) *$b : Z \to X$ is an isomorphism over U and $b(z) = x$,*
(3) *$c : Z \to \mathbf{P}^n_A$ is a closed immersion over V,*
(4) *$b^{-1}(U) = c^{-1}(V)$, in particular $c(z) \in V$,*
(5) *$E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,*
(6) *E is pseudo-coherent and supported on Z.*

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and a scheme X' separated and of finite presentation over A' whose base change to A is X. See Limits, Lemmas 32.10.1 and 32.8.6. Let $x' \in X'$ be the image of x. If we can prove the lemma for $x' \in X'/A'$, then the lemma follows for $x \in X/A$. Namely, if U', n', V', Z', z', E' provide the solution for $x' \in X'/A'$, then we can let $U \subset X$ be the inverse image of U', let $n = n'$, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', let $Z \subset X \times \mathbf{P}^n$ be the scheme theoretic inverse image of Z', let $z \in Z$ be the unique point mapping to x, and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology, Lemma 20.47.3. It only remains to check (5). To see this set $W = b^{-1}(U) = c^{-1}(V)$ and $W' = (b')^{-1}(U) = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{array}{ccc} W & \longrightarrow & W' \\ {\scriptstyle (b,c)}\downarrow & & \downarrow{\scriptstyle (b',c')} \\ X \times_A V & \longrightarrow & X' \times_{A'} V' \end{array}$$

By Lemma 37.69.1 the schemes $X \times_A V$ and W' are Tor independent over $X' \times_{A'} V'$. Hence the derived pullback of $(b', c')_* \mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_* \mathcal{O}_W$ by Derived Categories of Schemes, Lemma 36.22.5. This also uses that $R(b', c')_* \mathcal{O}_{Z'} = (b', c')_* \mathcal{O}_{Z'}$ because (b', c') is a closed immersion and similarly for $(b, c)_* \mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_* \mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_* \mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an affine open neighbourhood $U \subset X$ of x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Since the projection $X \times \mathbf{P}^n \to X$ is separated, we conclude from Morphisms, Lemma 29.6.8 that $b : Z \to X$ is an isomorphism over U. Let $z \in Z$ be the unique point lying over x.

Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by the same lemma we used above. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (3) and (4) hold.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian schemes. Hence we can take $E = (b, c)_* \mathcal{O}_Z$ in this case, see Derived Categories of Schemes, Lemma 36.10.3. This finishes the proof. □

0CSK **Lemma 37.69.3.** *Let A, $x \in X$, and U, n, V, Z, z, E be as in Lemma 37.69.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have*

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(U \to V)_* K|_U$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $U \to V$ is the finitely presented closed immersion $c \circ (b|_U)^{-1}$.

Proof. Since $b^{-1}(U) = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c \circ (b|_U)^{-1}$ is a closed immersion. It is of finite presentation because U and V are of finite presentation over A, see Morphisms, Lemma 29.21.11. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Set $W = b^{-1}(U) = c^{-1}(V)$ and denote $i : W \to X \times_A V$ the closed immersion $i = (b, c)|_W$. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_* \mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_* \mathcal{O}_W = Ri_* \mathcal{O}_W$. Using Derived Categories of Schemes, Lemma 36.22.1 we can rewrite this as

$$Rq'_* Ri_* Li^* Lp^* K|_{X \times_A V} = R(q' \circ i)_* Lb^* K|_W = R(U \to V)_* K|_U$$

which is what we want. □

0CSL **Lemma 37.69.4.** *Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in X$. We will show that K is pseudo-coherent relative to A in a neighbourhood of x and this will prove the lemma.

Choose U, n, V, Z, z, E as in Lemma 37.69.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} & R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^* \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^* \mathcal{O}_{\mathbf{P}^n_A}(i))) \end{aligned}$$

by Derived Categories of Schemes, Lemma 36.22.1. By Derived Categories of Schemes, Lemma 36.30.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 36.34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 37.69.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(U \to V)_* K|_U$$

Since $U \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_U$ is pseudo-coherent relative to A by Lemma 37.59.18. □

0GES **Lemma 37.69.5.** *Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. In view of Lemma 37.69.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 36.40.5 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Schemes, Lemma 36.34.3. □

0GET **Lemma 37.69.6.** *Let A be a ring. Let X be a scheme separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is* $\mathrm{Spec}(A)$*-perfect.*

Proof. By Lemma 37.69.5, K is pseudo-coherent relative to A. By Lemma 37.59.18, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 36.40.6 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Derived Categories of Schemes, Lemma 36.35.13, we will be done if we can show $Li^*(K)$ is bounded below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 36.15.3. We have

$$\begin{aligned} R\operatorname{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) &= R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ &= R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology, Lemma 20.50.5; we omit some details. The second equality follows from Derived Categories of Schemes, Lemma 36.22.5 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Remark 36.16.4. Hence Derived Categories of Schemes, Proposition 36.40.6 now implies that $Li^*(K)$ is bounded below and we win. □

37.70. Descent finiteness properties of complexes

0CSM This section is the continuation of Derived Categories of Schemes, Section 36.12.

0CSN **Lemma 37.70.1.** *Let $X \to S$ be locally of finite type. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lf_i^*E is m-pseudo-coherent relative to S.*

Proof. Assume E is m-pseudo-coherent relative to S. The morphisms f_i are pseudo-coherent by Lemma 37.60.6. Hence Lf_i^*E is m-pseudo-coherent relative to S by Lemma 37.59.16.

Conversely, assume that Lf_i^*E is m-pseudo-coherent relative to S for each i. Pick $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, $T_{j,k} \to W_{j,k}$, and morphisms $\alpha_{j,k} : T_{j,k} \to X_{i(j,k)}$ over S as in Lemma 37.48.2. Since the morphism $T_{j,K} \to S$ is flat and of finite presentation, we see that $\alpha_{j,k}$ is pseudo-coherent by Lemma 37.60.7. Hence

$$L\alpha_{j,k}^* Lf_{i(j,k)}^* E = L(T_{i,k} \to S)^* E$$

is m-pseudo-coherent relative to S by Lemma 37.59.16. Now we want to descend this property through the coverings $\{T_{j,k} \to W_{j,k}\}$, $W_j = \bigcup W_{j,k}$, $\{W_j \to U_j\}$, and $S = \bigcup U_j$. Since for Zariski coverings the result is true (by the definition of m-pseudo-coherence relative to S), this means we may assume we have a single surjective finite locally free morphism $\pi : Y \to X$ such that $L\pi^*E$ is pseudo-coherent relative to S. In this case $R\pi_* L\pi^* E$ is pseudo-coherent relative to S by Lemma 37.59.9 (this is the first time we use that E has quasi-coherent cohomology sheaves). We have $R\pi_* L\pi^* E = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \pi_* \mathcal{O}_Y$ for example by Derived Categories of Schemes, Lemma 36.22.1 and locally on X the map $\mathcal{O}_X \to \pi_* \mathcal{O}_Y$ is the inclusion of a direct summand. Hence we conclude by Lemma 37.59.12. □

0CSP **Lemma 37.70.2.** *Let $X \to T \to S$ be morphisms of schemes. Assume $T \to S$ is flat and locally of finite presentation and $X \to T$ locally of finite type. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if E is m-pseudo-coherent relative to T.*

Proof. Locally on X we can choose a closed immersion $i : X \to \mathbf{A}^n_T$. Then $\mathbf{A}^n_T \to S$ is flat and locally of finite presentation. Thus we may apply Lemma 37.59.17 to see the equivalence holds. □

0CSQ **Lemma 37.70.3.** *Let $f : X \to S$ be locally of finite type. Let $\{S_i \to S\}$ be an fppf covering of schemes. Denote $f_i : X_i \to S_i$ the base change of f and $g_i : X_i \to X$ the projection. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lg_i^*E is m-pseudo-coherent relative to S_i.*

Proof. This follows formally from Lemmas 37.70.1 and 37.70.2. Namely, if E is m-pseudo-coherent relative to S, then Lg_i^*E is m-pseudo-coherent relative to S (by the first lemma), hence Lg_i^*E is m-pseudo-coherent relative to S_i (by the second). Conversely, if Lg_i^*E is m-pseudo-coherent relative to S_i, then Lg_i^*E is m-pseudo-coherent relative to S (by the second lemma), hence E is m-pseudo-coherent relative to S (by the first lemma). □

37.71. Relatively perfect objects

0DJW This section is a continuation of the discussion in Derived Categories of Schemes, Section 36.35.

0DJX **Lemma 37.71.1.** *Let $i : X \to X'$ be a finite order thickening of schemes. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.*

Proof. We first prove K' has quasi-coherent cohomology sheaves. To do this, we may reduce to the case of a first order thickening, see Section 37.2. Let $\mathcal{I} \subset \mathcal{O}_{X'}$

be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology, Lemma 20.54.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Schemes, Lemmas 36.10.1, 36.3.9, and 36.4.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is local on X' hence we may assume X' is affine. Say $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A)$ with $A = A'/I$ and I nilpotent. Then K' comes from an object $M' \in D(A')$, see Derived Categories of Schemes, Lemma 36.3.5. Thus $M = M' \otimes^{\mathbf{L}}_{A'} A$ is a pseudo-coherent object of $D(A)$ by Derived Categories of Schemes, Lemma 36.10.2 and our assumption on K. It follows that M' is pseudo-coherent by More on Algebra, Lemma 15.76.4. This finishes the proof. □

0DJY **Lemma 37.71.2.** *Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y' \end{array}$$

of schemes. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Derived Categories of Schemes, Definition 36.35.1). By Lemma 37.71.1 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Lemma 36.10.1. To prove that E' locally has finite tor dimension we may work locally on X'. Hence we may assume X', S', X, S are affine, say given by rings A', R', A, R. Then we reduce to the commutative algebra version by Derived Categories of Schemes, Lemma 36.35.3. The commutative algebra version in More on Algebra, Lemma 15.84.8. □

0DJZ **Lemma 37.71.3.** *Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$. Let $f : X \to S$ be proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.*

Proof. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. For each i we can choose a closed immersion $U_i \to \mathbf{A}^{d_i}_S$. Set $U_{i,0} = S_0 \times_S U_i$. For each i the complex $E_0|_{U_{i,0}}$ has tor amplitude in $[a_i, b_i]$ for some $a_i, b_i \in \mathbf{Z}$. Let $x \in X$ be a point. We

will show that the tor amplitude of E_x over R is in $[a_i - d_i, b_i]$ for some i. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology, Lemma 20.48.5.

Since f is proper $f(\overline{\{x\}})$ is a closed subset of S. Since I is contained in the Jacobson radical, we see that $f(\overline{\{x\}})$ meeting the closed subset $S_0 \subset S$. Hence there is a specialization $x \rightsquigarrow x_0$ with $x_0 \in X_0$. Pick an i with $x_0 \in U_i$, so $x_0 \in U_{i,0}$. We will fix i for the rest of the proof. Write $U_i = \mathrm{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d_i-variables. The restriction $E|_{U_i}$ corresponds (by Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes^{\mathbf{L}}_A A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $x \rightsquigarrow x_0$. Then $E_x = K_{\mathfrak{q}}$ and $K_{\mathfrak{q}}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a_i - d_i, b_i]$ as a complex of R-modules. Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}_0) &= (K \otimes^{\mathbf{L}}_R R/I) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \\ &= (K \otimes^{\mathbf{L}}_A A/IA) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a_i, b_i this complex has cohomology only in degrees in the interval $[a_i, b_i]$. Thus we may finally apply More on Algebra, Lemma 15.84.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. □

37.72. Contracting rational curves

0E7E In this section we study proper morphisms $f : X \to Y$ whose fibres have dimension ≤ 1 having $R^1 f_* \mathcal{O}_X = 0$. To understand the title of this section, please take a look at Algebraic Curves, Sections 53.23, 53.24, and 53.25.

0E7F **Lemma 37.72.1.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$. If*

(1) *$R^1 f_* \mathcal{O}_X = 0$, or more generally*
(2) *there is a morphism $g : Y' \to Y$ such that y is in the image of g and such that $R' f'_* \mathcal{O}_{X'} = 0$ where $f' : X' \to Y'$ is the base change of f by g.*

Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$.

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 29.28.4. In this case $R^1 f_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 32.19.3 and 32.19.2. The lemma follows immediately. □

0E7G **Lemma 37.72.2.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$. Then there is an open neighbourhood $V \subset Y$ of y such that $R^1 f_* \mathcal{O}_X|_V = 0$ and the same is true after base change by any $Y' \to V$.*

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 29.28.4. In this case $R^1 f_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 32.19.3 and 32.19.2. Say $Y = \mathrm{Spec}(A)$, y corresponds to the prime $\mathfrak{p} \subset A$,

and $R^1 f_* \mathcal{O}_X$ corresponds to the finite A-module M. Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$ means that $\mathfrak{p} M_{\mathfrak{p}} = M_{\mathfrak{p}}$ by the statement on base change. By Nakayama's lemma we conclude $M_{\mathfrak{p}} = 0$. Since M is finite, we find an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f = 0$. Thus taking V the principal open $D(f)$ we obtain the desired result. □

0E7H **Lemma 37.72.3.** *Let $f : X \to Y$ be a proper morphism of schemes such that $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for all $y \in Y$. Let $\mathcal{F}$ be quasi-coherent on X. Then*

(1) *$R^p f_* \mathcal{F} = 0$ for $p > 1$, and*
(2) *$R^1 f_* \mathcal{F} = 0$ if there is a surjection $f^* \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ quasi-coherent on Y.*

If Y is affine, then we also have

(3) *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and*
(4) *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.*

Proof. The vanishing in (1) is Limits, Lemma 32.19.2. To prove (2) we may work locally on Y and assume Y is affine. Then $R^1 f_* \mathcal{F}$ is the quasi-coherent module on Y associated to the module $H^1(X, \mathcal{F})$. Here we use that Y is affine, quasi-coherence of higher direct images (Cohomology of Schemes, Lemma 30.4.5), and Cohomology of Schemes, Lemma 30.4.6. Since Y is affine, the quasi-coherent module $\mathcal{G}$ is globally generated, and hence so is $f^* \mathcal{G}$ and $\mathcal{F}$. In this way we see that (4) implies (2). Part (3) follows from (1) as well as the remarks on quasi-coherence of direct images just made. Thus all that remains is the prove (4). If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ because $R^1 f_* \mathcal{O}_X = 0$ by Lemma 37.72.2, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.19.1) we conclude. □

0E7I **Lemma 37.72.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Assume*

(1) *for all $y \in Y$ we have $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$, and*
(2) *$\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective.*

Then $\mathcal{O}_{Y'} \to f'_* \mathcal{O}_{X'}$ is surjective for any base change $f' : X' \to Y'$ of f.

Proof. We may assume Y and Y' affine. Then we can choose a closed immersion $Y' \to Y''$ with $Y'' \to Y$ a flat morphism of affines. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that the result holds for $X'' \to Y''$. Thus we may assume Y' is a closed subscheme of Y. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal cutting out Y'. Then there is a short exact sequence

$$0 \to \mathcal{I}\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

where we view $\mathcal{O}_{X'}$ as a quasi-coherent module on X. By Lemma 37.72.3 we have $H^1(X, \mathcal{I}\mathcal{O}_X) = 0$. It follows that

$$H^0(Y, \mathcal{O}_Y) \to H^0(Y, f_* \mathcal{O}_X) = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{X'})$$

is surjective as desired. The first arrow is surjective as Y is affine and since we assumed $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective and the second by the long exact sequence of cohomology associated to the short exact sequence above and the vanishing just proved. □

0E7J **Lemma 37.72.5.** *Consider a commutative diagram*

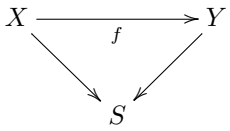

of morphisms of schemes. Let $s \in S$ be a point. Assume

(1) *$X \to S$ is locally of finite presentation and flat at points of X_s,*
(2) *f is proper,*
(3) *the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,*
(4) *$\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is surjective.*

Then there is an open $Y_s \subset V \subset Y$ such that (a) $f^{-1}(V)$ is flat over S, (b) $\dim(X_y) \leq 1$ for $y \in V$, (c) $R^1 f_*\mathcal{O}_X|_V = 0$, (d) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is surjective, and (b), (c), and (d) remain true after base change by any $Y' \to V$.

Proof. Let $y \in Y$ be a point over s. It suffices to find an open neighbourhood of y with the desired properties. As a first step, we replace Y by the open V found in Lemma 37.72.2 so that $R^1 f_*\mathcal{O}_X$ is zero universally (the hypothesis of the lemma holds by Lemma 37.72.1). We also shrink Y so that all fibres of f have dimension ≤ 1 (use Morphisms, Lemma 29.28.4 and properness of f). Thus we may assume we have (b) and (c) with $V = Y$ and after any base change $Y' \to Y$. Thus by Lemma 37.72.4 it now suffices to show (d) over Y. We may still shrink Y further; for example, we may and do assume Y and S are affine.

By Theorem 37.15.1 there is an open subset $U \subset X$ where $X \to S$ is flat which contains X_s by hypothesis. Then $f(X \setminus U)$ is a closed subset not containing y. Thus after shrinking Y we may assume X is flat over S.

Say $S = \mathrm{Spec}(R)$. Choose a closed immersion $Y \to Y'$ where Y' is the spectrum of a polynomial ring $R[x_e; e \in E]$ on a set E. Denote $f' : X \to Y'$ the composition of f with $Y \to Y'$. Then the hypotheses (1) – (4) as well as (b) and (c) hold for f' and s. If we we show $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_X$ is surjective in an open neighbourhood of y, then the same is true for $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Thus we may assume Y is the spectrum of $R[x_e; e \in E]$.

At this point X and Y are flat over S. Then Y_s and X are tor independent over Y. We urge the reader to find their own proof, but it also follows from Lemma 37.69.1 applied to the square with corners X, Y, S, S and its base change by $s \to S$. Hence

$$Rf_{s,*}\mathcal{O}_{X_s} = L(Y_s \to Y)^* Rf_*\mathcal{O}_X$$

by Derived Categories of Schemes, Lemma 36.22.5. Because of the vanishing already established this implies $f_{s,*}\mathcal{O}_{X_s} = (Y_s \to Y)^* f_*\mathcal{O}_X$. We conclude that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of quasi-coherent $\mathcal{O}_Y$-modules whose pullback to Y_s is surjective. We claim $f_*\mathcal{O}_X$ is a finite type $\mathcal{O}_Y$-module. If true, then the cokernel $\mathcal{F}$ of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a finite type quasi-coherent $\mathcal{O}_Y$-module such that $\mathcal{F}_y \otimes \kappa(y) = 0$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we have $\mathcal{F}_y = 0$. Thus $\mathcal{F}$ is zero in an open neighbourhood of y (Modules, Lemma 17.9.5) and the proof is complete.

Proof of the claim. For a finite subset $E' \subset E$ set $Y' = \mathrm{Spec}(R[x_e; e \in E'])$. For large enough E' the morphism $f' : X \to Y \to Y'$ is proper, see Limits, Lemma 32.13.4. We fix E' and Y' in the following. Write $R = \mathrm{colim}\, R_i$ as the colimit of its

finite type $\mathbf{Z}$-subalgebras. Set $S_i = \mathrm{Spec}(R_i)$ and $Y_i' = \mathrm{Spec}(R_i[x_e; e \in E'])$. For i large enough we can find a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f'} & Y' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ X_i & \xrightarrow{f_i'} & Y_i' & \longrightarrow & S_i \end{array}$$

with cartesian squares such that X_i is flat over S_i and $X_i \to Y_i'$ is proper. See Limits, Lemmas 32.10.1, 32.8.7, and 32.13.1. The same argument as above shows Y' and X_i are tor independent over Y_i' and hence

$$R\Gamma(X, \mathcal{O}_X) = R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^{\mathbf{L}}_{R_i[x_e; e \in E']} R[x_e; e \in E']$$

by the same reference as above. By Cohomology of Schemes, Lemma 30.19.2 the complex $R\Gamma(X_i, \mathcal{O}_{X_i})$ is pseudo-coherent in the derived category of the Noetherian ring $R_i[x_e; e \in E']$ (see More on Algebra, Lemma 15.65.17). Hence $R\Gamma(X, \mathcal{O}_X)$ is pseudo-coherent in the derived category of $R[x_e; e \in E']$, see More on Algebra, Lemma 15.65.12. Since the only nonvanishing cohomology module is $H^0(X, \mathcal{O}_X)$ we conclude it is a finite $R[x_e; e \in E']$-module, see More on Algebra, Lemma 15.65.4. This concludes the proof. □

0E7K **Lemma 37.72.6.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of morphisms of schemes. Assume $X \to S$ is flat, f is proper, $\dim(X_y) \leq 1$ for $y \in Y$, and $R^1 f_* \mathcal{O}_X = 0$. Then $f_* \mathcal{O}_X$ is S-flat and formation of $f_* \mathcal{O}_X$ commutes with arbitrary base change $S' \to S$.

Proof. We may assume Y and S are affine, say $S = \mathrm{Spec}(A)$. To show the quasi-coherent $\mathcal{O}_Y$-module $f_* \mathcal{O}_X$ is flat relative to S it suffices to show that $H^0(X, \mathcal{O}_X)$ is flat over A (some details omitted). By Lemma 37.72.3 we have $H^1(X, \mathcal{O}_X \otimes_A M) = 0$ for every A-module M. Since also $\mathcal{O}_X$ is flat over A we deduce the functor $M \mapsto H^0(X, \mathcal{O}_X \otimes_A M)$ is exact. Moreover, this functor commutes with direct sums by Cohomology, Lemma 20.19.1. Then it is an exercise to see that $H^0(X, \mathcal{O}_X \otimes_A M) = M \otimes_A H^0(X, \mathcal{O}_X)$ functorially in M and this gives the desired flatness. Finally, if $S' \to S$ is a morphism of affines given by the ring map $A \to A'$, then in the affine case just discussed we see that

$$H^0(X \times_S S', \mathcal{O}_{X \times_S S'}) = H^0(X, \mathcal{O}_X \otimes_A A') = H^0(X, \mathcal{O}_X) \otimes_A A'$$

This shows that formation of $f_* \mathcal{O}_X$ commutes with any base change $S' \to S$. Some details omitted. □

0E7L **Lemma 37.72.7.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

(1) *$X \to S$ is locally of finite presentation and flat at points of X_s,*
(2) *$Y \to S$ is locally of finite presentation,*
(3) *f is proper,*
(4) *the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,*
(5) *$\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism.*

Then there is an open $Y_s \subset V \subset Y$ such that (a) V is flat over S, (b) $f^{-1}(V)$ is flat over S, (c) $\dim(X_y) \leq 1$ for $y \in V$, (d) $R^1 f_* \mathcal{O}_X|_V = 0$, (e) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is an isomorphism, and (a) – (e) remain true after base change of $f^{-1}(V) \to V$ by any $S' \to S$.

Proof. Let $y \in Y_s$. We may always replace Y by an open neighbourhood of y. Thus we may assume Y and S affine. We may also assume that X is flat over S, $\dim(X_y) \leq 1$ for $y \in Y$, $R^1 f_* \mathcal{O}_X = 0$ universally, and that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see Lemma 37.72.5. (We won't use all of this.)

Assume S and Y affine. Write $S = \lim S_i$ as a cofiltered of affine Noetherian schemes S_i. By Limits, Lemma 32.10.1 there exists an element $0 \in I$ and a diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ & \searrow \quad \swarrow & \\ & S_0 & \end{array}$$

of finite type morphisms of schemes whose base change to S is the diagram of the lemma. After increasing 0 we may assume Y_0 is affine and $X_0 \to S_0$ proper, see Limits, Lemmas 32.13.1 and 32.4.13. Let $s_0 \in S_0$ be the image of s. As Y_s is affine, we see that $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$ is equivalent to $H^1(X_s, \mathcal{O}_{X_s}) = 0$. Since X_s is the base change of X_{0,s_0} by the faithfully flat map $\kappa(s_0) \to \kappa(s)$ we see that $H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0$ and hence $R^1 f_{0,*}\mathcal{O}_{X_{0,s_0}} = 0$. Similarly, as $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism, so is $\mathcal{O}_{Y_{0,s_0}} \to f_{0,*}\mathcal{O}_{X_{0,s_0}}$. Since the dimensions of the fibres of $X_s \to Y_s$ are at most 1, the same is true for the morphism $X_{0,s_0} \to Y_{0,s_0}$. Finally, since $X \to S$ is flat, after increasing 0 we may assume X_0 is flat over S_0, see Limits, Lemma 32.8.7. Thus it suffices to prove the lemma for $X_0 \to Y_0 \to S_0$ and the point s_0.

Combining the reduction arguments above we reduce to the case where S and Y affine, S Noetherian, the fibres of f have dimension ≤ 1, and $R^1 f_* \mathcal{O}_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$$

is an isomorphism. The claim implies the lemma. Namely, since $f_*\mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition 30.19.1) the claim means we can replace Y by an open neighbourhood of y and obtain an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we conclude that Y is flat over S by Lemma 37.72.6. Finally, the isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ remains an isomorphism after any base change $S' \to S$ by the final statement of Lemma 37.72.6.

Proof of the claim. We already know that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is surjective (Lemma 37.72.5) and that $(f_*\mathcal{O}_X)_y$ is $\mathcal{O}_{S,s}$-flat (Lemma 37.72.6) and that the induced map

$$\mathcal{O}_{Y_s,y} = \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y/\mathfrak{m}_s(f_*\mathcal{O}_X)_y \to (f_{s,*}\mathcal{O}_{X_s})_y$$

is injective by the assumption in the lemma. Then it follows from Algebra, Lemma 10.99.1 that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is injective as desired. □

0E24 **Lemma 37.72.8.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, such that the fibres of f have dimension ≤ 1, and such that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Then $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is a bijection onto the subgroup of $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$.*

Proof. By the projection formula (Cohomology, Lemma 20.54.2) we see that $f_*f^*\mathcal{N} \cong \mathcal{N}$ for $\mathcal{N} \in \mathrm{Pic}(Y)$. We claim that for $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$ we have $\mathcal{N} = f_*\mathcal{L}$ is invertible and $\mathcal{L} \cong f^*\mathcal{N}$. This will finish the proof.

The $\mathcal{O}_Y$-module $\mathcal{N} = f_*\mathcal{L}$ is coherent by Cohomology of Schemes, Proposition 30.19.1. Thus to see that it is an invertible $\mathcal{O}_Y$-module, it suffices to check on stalks (Algebra, Lemma 10.78.2). Since the map from a Noetherian local ring to its completion is faithfully flat, it suffices to check the completion $(f_*\mathcal{L})^\wedge_y$ is free (see Algebra, Section 10.97 and Lemma 10.78.6). For this we will use the theorem of formal functions as formulated in Cohomology of Schemes, Lemma 30.20.7. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$ and hence $(f_*\mathcal{O}_X)^\wedge_y \cong \mathcal{O}^\wedge_{Y,y}$, it suffices to show that $\mathcal{L}|_{X_n} \cong \mathcal{O}_{X_n}$ for each n (compatibly for varying n. By Lemma 37.4.1 we have an exact sequence

$$H^1(X_y, \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X) \to \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$$

with notation as in the theorem on formal functions. Observe that we have a surjection

$$\mathcal{O}_{X_y}^{\oplus r_n} \cong \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \otimes_{\kappa(y)} \mathcal{O}_{X_y} \longrightarrow \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X$$

for some integers $r_n \geq 0$. Since $\dim(X_y) \leq 1$ this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees ≥ 2 coming from Cohomology, Proposition 20.20.7). Hence the H^1 in the sequence is zero and the transition maps $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ are injective as desired.

We still have to show that $f^*\mathcal{N} \cong \mathcal{L}$. This is proved by the same method and we omit the details. □

37.73. Affine stratifications

0F2R This material is taken from [**RV04**]. Please read a little bit about stratifications in Topology, Section 5.28 before reading this section.

If X is a scheme, then a stratification of X usually means a stratification of the underlying topological space of X. The strata are locally closed subsets. We will view these strata as reduced locally closed subschemes of X using Schemes, Remark 26.12.6.

0F2S **Definition 37.73.1.** Let X be a scheme. An *affine stratification* is a locally finite stratification $X = \coprod_{i \in I} X_i$ whose strata X_i are affine and such that the inclusion morphisms $X_i \to X$ are affine.

The condition that a stratification $X = \coprod X_i$ is locally finite is, in the presence of the condition that the inclusion morphisms $X_i \to X$ are quasi-compact, equivalent to the condition that the strata are locally constructible subsets of X, see Properties, Lemma 28.2.7.

The condition that $X_i \to X$ is an affine morphism is independent on the scheme structure we put on the locally closed subset X_i, see Lemma 37.3.1. Moreover, if

X is separated (or more generally has affine diagonal) and $X = \coprod X_i$ is a locally finite stratification with affine strata, then the morphisms $X_i \to X$ are affine. See Morphisms, Lemma 29.11.12. This allows us to disregard the condition of affineness of the inclusion morphisms $X_i \to X$ in most cases of interest.

We are often interested in the case where the partially ordered index set I of the stratification is finite. Recall that the *length* of a partially ordered set I is the supremum of the lengths p of chains $i_0 < i_1 < \ldots < i_p$ of elements of I.

0F2T **Lemma 37.73.2.** *Let X be a scheme. Let $X = \coprod_{i \in I} X_i$ be a finite affine stratification. There exists an affine stratification with index set $\{0, \ldots, n\}$ where n is the length of I.*

Proof. Recall that we have a partial ordering on I such that the closure of X_i is contained in $\bigcup_{j \leq i} X_j$ for all $i \in I$. Let $I' \subset I$ be the set of maximal indices of I. If $i \in I'$, then X_i is open in X because the union of the closures of the other strata is the complement of X_i. Let $U = \bigcup_{i \in I'} X_i$ viewed as an open subscheme of X so that $U_{red} = \coprod_{i \in I'} X_i$ as schemes. Then U is an affine scheme by Schemes, Lemma 26.6.8 and Lemma 37.2.3. The morphism $U \to X$ is affine as each $X_i \to X$, $i \in I'$ is affine by the same reasoning using Lemma 37.3.1. The complement $Z = X \setminus U$ endowed with the reduced induced scheme structure has the affine stratification $Z = \bigcup_{i \in I \setminus I'} X_i$. Here we use that a morphism of schemes $T \to Z$ is affine if and only if the composition $T \to X$ is affine; this follows from Morphisms, Lemmas 29.11.10, 29.11.8, and 29.11.12. Observe that the partially ordered set $I \setminus I'$ has length exactly one less than the length of I. Hence by induction we find that Z has an affine stratification $Z = Z_0 \amalg \ldots \amalg Z_{n-1}$ with index set $\{1, \ldots, n\}$. Setting $Z_n = U$ we obtain the desired stratification of X. □

If a scheme X has a finite affine stratification, then of course X is quasi-compact. A bit less obvious is the fact that it forces X to be quasi-separated as well.

0F2U **Lemma 37.73.3.** *Let X be a scheme. The following are equivalent*

(1) *X has a finite affine stratification, and*
(2) *X is quasi-compact and quasi-separated.*

Proof. Let $X = \bigcup X_i$ be a finite affine stratification. Since each X_i is affine hence quasi-compact, we conclude that X is quasi-compact. Let $U, V \subset X$ be affine open. Then $U \cap X_i$ and $V \cap X_i$ are affine open in X_i since $X_i \to X$ is an affine morphism. Hence $U \cap V \cap X_i$ is an affine open of the affine scheme X_i (see Schemes, Lemma 26.21.7 for example). Therefore $U \cap V = \coprod U \cap V \cap X_i$ is quasi-compact as a finite union of affine strata. We conclude that X is quasi-separated by Schemes, Lemma 26.21.6.

Assume X is quasi-compact and quasi-separated. We may use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove the assertion that X has a finite affine stratification. If X is empty, then it has an empty affine stratification. If X is nonempty affine then it has an affine stratification with one stratum. Next, assume $X = U \cup V$ where U is quasi-compact open, V is affine open, and we have a finite affine stratifications $U = \bigcup_{i \in I} U_i$ and $U \cap V = \coprod_{j \in J} W_j$. Denote $Z = X \setminus V$ and $Z' = X \setminus U$. Note that Z is closed in U and Z' is closed in V. Observe that $U_i \cap Z$ and $U_i \cap W_j = U_i \times_U W_j$ are affine schemes affine over U. (Hints: use that

$U_i \times_U W_j \to W_j$ is affine as a base change of $U_i \to U$, hence $U_i \cap W_j$ is affine, hence $U_i \cap W_j \to U_i$ is affine, hence $U_i \cap W_j \to U$ is affine.) It follows that

$$U = \coprod\nolimits_{i \in I} (U_i \cap Z) \amalg \coprod\nolimits_{(i,j) \in I \times J} (U_i \cap W_j)$$

is a finite affine stratification with partial ordering on $I \amalg I \times J$ given by $i' \leq (i,j) \Leftrightarrow i' \leq i$ and $(i',j') \leq (i,j) \Leftrightarrow i' \leq i$ and $j' \leq j$. Observe that $(U_i \cap Z) \times_X V = \emptyset$ and $(U_i \cap W_j) \times_X V = U_i \cap W_j$ are affine. Hence the morphisms $U_i \cap Z \to X$ and $U_i \cap W_j \to X$ are affine because we can check affineness of a morphism locally on the target (Morphisms, Lemma 29.11.3) and we have affineness over both U and V. To finish the proof we take the stratification above and we add one additional stratum, namely Z', whose index we add as a minimal element to the partially ordered set. □

0F2V **Definition 37.73.4.** Let X be a nonempty quasi-compact and quasi-separated scheme. The *affine stratification number* is the smallest integer $n \geq 0$ such that the following equivalent conditions are satisfied

(1) there exists a finite affine stratification $X = \coprod_{i \in I} X_i$ where I has length n,
(2) there exists an affine stratification $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$ with index set $\{0, \ldots, n\}$.

The equivalence of the conditions holds by Lemma 37.73.2. The existence of a finite affine stratification is proven in Lemma 37.73.3.

0F2W **Lemma 37.73.5.** *Let X be a separated scheme which has an open covering by $n+1$ affines. Then the affine stratification number of X is at most n.*

Proof. Say $X = U_0 \cup \ldots \cup U_n$ is an affine open covering. Set

$$X_i = (U_i \cup \ldots \cup U_n) \setminus (U_{i+1} \cup \ldots \cup U_n)$$

Then X_i is affine as a closed subscheme of U_i. The morphism $X_i \to X$ is affine by Morphisms, Lemma 29.11.12. Finally, we have $\overline{X_i} \subset X_i \cup X_{i-1} \cup \ldots X_0$. □

0F2X **Lemma 37.73.6.** *Let X be a Noetherian scheme of dimension $\infty > d \geq 0$. Then the affine stratification number of X is at most d.*

Proof. By induction on d. If $d = 0$, then X is affine, see Properties, Lemma 28.10.5. Assume $d > 0$. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of X (Properties, Lemma 28.5.7). We can cover X by affine opens containing $\eta_1, \ldots, \eta_n$, see Properties, Lemma 28.29.4. Since X is quasi-compact we can find a finite affine open covering $X = \bigcup_{j=1,\ldots,m} U_j$ with $\eta_1, \ldots, \eta_n \in U_j$ for all $j = 1, \ldots, m$. Choose an affine open $U \subset U_1 \cap \ldots \cap U_m$ containing $\eta_1, \ldots, \eta_n$ (possible by the lemma already quoted). Then the morphism $U \to X$ is affine because $U \to U_j$ is affine for all j, see Morphisms, Lemma 29.11.3. Let $Z = X \setminus U$. By construction $\dim(Z) < \dim(X)$. By induction hypothesis we can find an affine stratification $Z = \bigcup_{i \in \{0,\ldots,n\}} Z_i$ of Z with $n \leq \dim(Z)$. Setting $U = X_{n+1}$ and $X_i = Z_i$ for $i \leq n$ we conclude. □

0F2Y **Proposition 37.73.7.** *Let X be a nonempty quasi-compact and quasi-separated scheme with affine stratification number n. Then $H^p(X, \mathcal{F}) = 0$, $p > n$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. We will prove this by induction on the affine stratification number n. If $n = 0$, then X is affine and the result is Cohomology of Schemes, Lemma 30.2.2. Assume $n > 0$. By Definition 37.73.4 there is an affine scheme U and an affine open immersion $j : U \to X$ such that the complement Z has affine stratification number $n - 1$. As U and j are affine we have $H^p(X, j_*(\mathcal{F}|_U)) = 0$ for $p > 0$, see Cohomology of Schemes, Lemmas 30.2.4 and 30.2.3. Denote $\mathcal{K}$ and $\mathcal{Q}$ the kernel and cokernel of the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$. Thus we obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to \mathcal{Q} \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules (see Schemes, Section 26.24). A standard argument, breaking our exact sequence into short exact sequences and using the long exact cohomology sequence, shows it suffices to prove $H^p(X, \mathcal{K}) = 0$ and $H^p(X, \mathcal{Q}) = 0$ for $p \geq n$. Since $\mathcal{F} \to j_*(\mathcal{F}|_U)$ restricts to an isomorphism over U, we see that $\mathcal{K}$ and $\mathcal{Q}$ are supported on Z. By Properties, Lemma 28.22.3 we can write these modules as the filtered colimits of their finite type quasi-coherent submodules. Using the fact that cohomology of sheaves on X commutes with filtered colimits, see Cohomology, Lemma 20.19.1, we conclude it suffices to show that if $\mathcal{G}$ is a finite type quasi-coherent module whose support is contained in Z, then $H^p(X, \mathcal{G}) = 0$ for $p \geq n$. Let $Z' \subset X$ be the scheme theoretic support of $\mathcal{G} \oplus \mathcal{O}_Z$; we may and do think of $\mathcal{G}$ as a quasi-coherent module on Z', see Morphisms, Section 29.5. Then Z' and Z have the same underlying topological space and hence the same affine stratification number, namely $n - 1$. Hence $H^p(X, \mathcal{G}) = H^p(Z', \mathcal{G})$ (equality by Cohomology of Schemes, Lemma 30.2.4) vanishes for $p \geq n$ by induction hypothesis. □

0F2Z **Example 37.73.8.** Let k be a field and let $X = \mathbf{P}^n_k$ be n-dimensional projective space over k. Lemma 37.73.5 applies to this by Constructions, Lemma 27.13.3. Hence the affine stratification number of $\mathbf{P}^n_k$ is at most n. On the other hand, we have nonzero cohomology in degree n for some quasi-coherent modules on $\mathbf{P}^n_k$, see Cohomology of Schemes, Lemma 30.8.1. Using Proposition 37.73.7 we conclude that the affine stratification number of $\mathbf{P}^n_k$ is equal to n.

37.74. Universally open morphisms

0F30 Some material on universally open morphisms.

0F31 **Lemma 37.74.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *f is universally open,*
2. *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is open, and*
3. *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is open.*

Proof. It is clear that (1) implies (2) and (2) implies (3). Let us prove that (3) implies (1). Suppose that the base change $X_T \to T$ is not open for some morphism of schemes $g : T \to S$. Then we can find some affine opens $V \subset S$, $U \subset X$, $W \subset T$ with $f(U) \subset V$ and $g(W) \subset V$ such that $U \times_V W \to W$ is not open. If we can show that this implies $\mathbf{A}^n \times U \to \mathbf{A}^n \times V$ is not open, then $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is not open and the proof is complete. This reduces us to the result proved in the next paragraph.

Let $A \to B$ be a ring map such that $A' \to B' = A' \otimes_A B$ does not induce an open map of spectra for some A-algebra A'. As the principal opens give a basis for the

topology of $\mathrm{Spec}(B')$ we conclude that the image of $D(g)$ in $\mathrm{Spec}(A')$ is not open for some $g \in B'$. Write $g = \sum_{i=1,\ldots,n} a'_i \otimes b_i$ for some n, $a'_i \in A'$, and $b_i \in B$. Consider the element $h = \sum_{i=1,\ldots,n} x_i b_i$ in $B[x_1, \ldots, x_n]$. Assume that $D(h)$ maps to an open subset under the morphism

$$\mathrm{Spec}(B[x_1, \ldots, x_n]) \longrightarrow \mathrm{Spec}(A[x_1, \ldots, x_n])$$

in order to get a contradiction. Then $D(h)$ would map surjectively onto a quasi-compact open $U \subset \mathrm{Spec}(A[x_1, \ldots, x_n])$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to B'$ sending h to g. Since

$$\begin{array}{ccc} \mathrm{Spec}(B[x_1, \ldots, x_n]) & \longleftarrow & \mathrm{Spec}(B') \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[x_1, \ldots, x_n]) & \longleftarrow & \mathrm{Spec}(A') \end{array}$$

is cartesian the image of $D(g)$ in $\mathrm{Spec}(A')$ is equal to the inverse image of U in $\mathrm{Spec}(A')$ and hence open which is the desired contradiction. □

0F32 **Lemma 37.74.2.** *Let $f : X \to Y$ be a morphism of schemes. If*

(1) *f is locally quasi-finite,*
(2) *Y is geometrically unibranch and locally Noetherian, and*
(3) *every irreducible component of X dominates an irreducible component of Y,*

then f is universally open.

Proof. For any n the scheme $\mathbf{A}^n \times Y$ is geometrically unibranch by Lemma 37.36.4 and Properties, Lemma 28.15.6. Hence the hypotheses of the lemma hold for the morphisms $\mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ for all n. By Lemma 37.74.1 it suffices to prove f is open. By Morphisms, Lemma 29.23.2 it suffices to show that generalizations lift along f. Suppose that $y' \rightsquigarrow y$ is a specialization of points in Y and $x \in X$ is a point mapping to y. As in Lemma 37.41.1 choose a diagram

$$\begin{array}{ccc} u & U & \longrightarrow X \\ \downarrow & \downarrow & \downarrow \\ v & V & \longrightarrow Y \end{array}$$

where $(V, v) \to (Y, y)$ is an elementary étale neighbourhood, $U \to V$ is finite, u is the unique point of U mapping to v, $U \subset V \times_Y X$ is open, and $v \mapsto y$ and $u \mapsto x$. Let E be an irreducible component of U passing through u (there is at least one of these). Since $U \to X$ is étale, E maps to an irreducible component of X, which in turn dominates an irreducible component of Y (by assumption). Since $U \to V$ is finite hence closed, we conclude that the image $E' \subset V$ of E is an irreducible closed subset passing through v which dominates an irreducible component of Y. Since $V \to Y$ is étale E' must be an irreducible component of V passing through v. Since Y is geometrically unibranch we see that E' is the unique irreducible component of V passing through v (Lemma 37.36.2). Since V is locally Noetherian we may after shrinking V assume that $E' = V$ (equality of sets).

Since $V \to Y$ is étale we can find a specialization $v' \rightsquigarrow v$ whose image is $y' \rightsquigarrow y$. By the above we can find $u' \in U$ mapping to v'. Then $u' \rightsquigarrow u$ because u is the only

point of U mapping to v and $U \to V$ is closed. Then finally the image $x' \in X$ of u' is a point specializing to x and mapping to y' and the proof is complete. □

0F33 **Lemma 37.74.3.** *Let $A \to B$ be a ring map. Say B is generated as an A-module by $b_1, \ldots, b_d \in B$. Set $h = \sum x_i b_i \in B[x_1, \ldots, x_d]$. Then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally open if and only if the image of $D(h)$ in $\mathrm{Spec}(A[x_1, \ldots, x_d])$ is open.*

Proof. If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally open, then of course the image of $D(h)$ is open. Conversely, assume the image U of $D(h)$ is open. Let $A \to A'$ be a ring map. It suffices to show that the image of any principal open $D(g) \subset \mathrm{Spec}(A' \otimes_A B)$ in $\mathrm{Spec}(A')$ is open. We may write $g = \sum_{i=1,\ldots,d} a'_i \otimes b_i$ for some $a'_i \in A'$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to A' \otimes_A B$ sending h to g. Since

$$\begin{array}{ccc} \mathrm{Spec}(B[x_1, \ldots, x_n]) & \longleftarrow & \mathrm{Spec}(B') \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[x_1, \ldots, x_n]) & \longleftarrow & \mathrm{Spec}(A') \end{array}$$

is cartesian the image of $D(g)$ in $\mathrm{Spec}(A')$ is equal to the inverse image of U in $\mathrm{Spec}(A')$ and hence open. □

0F34 **Lemma 37.74.4.** *Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If*

(1) *f is locally quasi-finite and universally open, and*
(2) *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is locally quasi-finite and universally open.

Proof. By Limits, Lemma 32.18.2 after increasing 0 we may assume f_0 is locally quasi-finite. Let $x \in X$. By étale localization of quasi-finite morphisms we can find a diagram

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ Y & \longleftarrow & V \end{array}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, $U \to V$ is finite, and x is in the image of $U \to X$, see Lemma 37.41.1. After shrinking V we may assume V and U are affine. Since X is quasi-compact, it follows, by taking a finite disjoint union of such V and U, that we can make a diagram as above such that $U \to X$ is surjective. By Limits, Lemmas 32.10.1, 32.4.11, 32.8.15, 32.8.3, 32.8.10, and 32.4.13 after possibly increasing 0 we may assume we have a diagram

$$\begin{array}{ccc} X_0 & \longleftarrow & U_0 \\ \downarrow & & \downarrow \\ Y_0 & \longleftarrow & V_0 \end{array}$$

where V_0 is affine, $V_0 \to Y_0$ is étale, $U_0 \subset (X_0)_{V_0}$ is open, $U_0 \to V_0$ is finite, and $U_0 \to X_0$ is surjective. Since $V_i \to Y_i$ is étale and hence universally open, follows

that it suffices to prove that $U_i \to V_i$ is universally open for large enough i. This reduces us to the case discussed in the next paragraph.

Let $A = \operatorname{colim} A_i$ be a filtered colimit of rings. Let $A_0 \to B_0$ be a ring map. Set $B = A \otimes_{A_0} B_0$ and $B_i = A_i \otimes_{A_0} B_0$. Assume $A_0 \to B_0$ is finite, of finite presentation, and $A \to B$ is universally open. We have to show that $A_i \to B_i$ is universally open for i large enough. Pick $b_{0,1}, \ldots, b_{0,d} \in B_0$ which generate B_0 as an A_0-module. Set $h_0 = \sum_{j=1,\ldots,d} x_j b_{0,j}$ in $B_0[x_1, \ldots, x_d]$. Denote h, resp. h_i the image of h_0 in $B[x_1, \ldots, x_d]$, resp. $B_i[x_1, \ldots, x_d]$. The image U of $D(h)$ in $\operatorname{Spec}(A[x_1, \ldots, x_d])$ is open as $A \to B$ is universally open. Of course U is quasi-compact as the image of an affine scheme. For i large enough there is a quasi-compact open $U_i \subset \operatorname{Spec}(A_i[x_1, \ldots, x_d])$ whose inverse image in $\operatorname{Spec}(A[x_1, \ldots, x_d])$ is U, see Limits, Lemma 32.4.11. After increasing i we may assume that $D(h_i)$ maps into U_i; this follows from the same lemma by considering the pullback of U_i in $D(h_i)$. Finally, for i even larger the morphism of schemes $D(h_i) \to U_i$ will be surjective by an application of the already used Limits, Lemma 32.8.15. We conclude $A_i \to B_i$ is universally open by Lemma 37.74.3. □

0F35 **Lemma 37.74.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Then*

(1) *the functions $n_{X/Y}$ of Lemmas 37.27.3 and 37.28.3 agree,*
(2) *if X is quasi-compact, then $n_{X/Y}$ attains a maximum $d < \infty$.*

Proof. Agreement of the functions is immediate from the fact that the (geometric) fibres of a locally quasi-finite morphism are discrete, see Morphisms, Lemma 29.20.8. Boundedness follows from Morphisms, Lemmas 29.57.2 and 29.57.9. □

0F36 **Lemma 37.74.6.** *Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 37.74.5. If $n_{X/Y}(y) \geq d$ for some $y \in Y$ and $d \geq 0$, then $n_{X/Y} \geq d$ in an open neighbourhood of y.*

Proof. The question is local on Y hence we may assume Y affine. Let K be an algebraic closure of the residue field $\kappa(y)$. Our assumption is that $(X_y)_K$ has $\geq d$ connected components. Then for a suitable quasi-compact open $X' \subset X$ the scheme $(X'_y)_K$ has $\geq d$ connected components; details omitted. After replacing X by X' we may assume X is quasi-compact. Then f is quasi-finite. Let $x_1, \ldots, x_n$ be the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = n_{X/Y}(y) \geq d$. Since the construction of $n_{X/Y}$ is compatible with base change the proof is complete. □

0F37 **Lemma 37.74.7.** *Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 37.74.5. If $n_{X/Y}$ attains a maximum $d < \infty$, then the set*

$$Y_d = \{y \in Y \mid n_{X/Y}(y) = d\}$$

is open in Y and the morphism $f^{-1}(Y_d) \to Y_d$ is finite.

Proof. The openness of Y_d is immediate from Lemma 37.74.6. To prove finiteness over Y_d we redo the argument of the proof of that lemma. Namely, let $y \in Y_d$. Then there are at most d points of X lying over y. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j and we may assume U maps into W. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = d$. Since the construction of $n_{X/Y}$ is compatible with base change we know that $n_{U \times_Y X/U} = d$. This means that W has to be empty and we conclude that $U \times_Y X \to U$ is finite. By Descent, Lemma 35.23.25 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. □

37.75. Weightings

0F38 The material in this section is taken from [**AGV71**, Exposee XVII, 6.2.4].

Let $\pi : U \to V$ be a locally quasi-finite morphism of schemes with finite fibres. Given a function $w : U \to \mathbf{Z}$ we define a function

$$\textstyle\int_\pi w : V \longrightarrow \mathbf{Z}, \quad v \longmapsto \sum_{u \in U,\ \pi(u) = v} w(u)[\kappa(u) : \kappa(v)]_s$$

Note that the field extensions are finite (Morphisms, Lemma 29.20.5), $[\kappa' : \kappa]_s$ is the separable degree (Fields, Definition 9.14.7), and the sum is finite as the fibres of π are assumed finite. Another way to compute the value of $\int_\pi w$ at a point $v \in V$ is as follows. Choose an algebraically closed field k and a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ whose image is v. Then we have

$$(\textstyle\int_\pi w)(v) = \sum_{\overline{u} \in U_{\overline{v}}} w(\overline{u})$$

where of course $w(\overline{u})$ denotes the value of w at the image u of the point $\overline{u}$ under the morphism $U_{\overline{v}} \to U$. Note that we may view $\overline{u} \in U_{\overline{v}}$ as morphisms $\overline{u} : \mathrm{Spec}(k) \to U$ such that $\pi \circ \overline{u} = \overline{v}$. Namely, since $U \to V$ is locally quasi-finite with finite fibres, the scheme $U_{\overline{v}}$ is the spectrum of a finite dimension algebra over k and all of whose prime ideals are maximal ideals with residue field k. To see that the equality holds, note that the number of morphisms $\overline{u}$ lying over a given u is equal to $[\kappa(u) : \kappa(v)]_s$ by Fields, Lemma 9.14.8.

0F39 **Lemma 37.75.1.** *Given a cartesian square*

$$\begin{array}{ccc} U & \xleftarrow{h} & U' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ V & \xleftarrow{g} & V' \end{array}$$

with π locally quasi-finite with finite fibres and a function $w : U \to \mathbf{Z}$ we have $(\int_\pi w) \circ g = \int_{\pi'} (w \circ h)$.

Proof. This follows immediately from the second description of $\int_\pi w$ given above. To prove it from the definition, you use that if E/F is a finite extension of fields and F'/F is another field extension, then writing $(E \otimes_F F')_{red} = \prod E'_i$ as a product of fields finite over F', we have

$$[E : F]_s = \sum [E'_i : F']_s$$

To prove this equality pick an algebraically closed field extension Ω/F' and observe that

$$\begin{aligned} [E : F]_s &= |\operatorname{Mor}_F(E, \Omega)| \\ &= |\operatorname{Mor}_{F'}(E \otimes_F F', \Omega)| \\ &= |\operatorname{Mor}_{F'}((E \otimes_F F')_{red}, \Omega)| \\ &= \sum |\operatorname{Mor}_{F'}(E'_i, \Omega)| \\ &= \sum [E'_i : F']_s \end{aligned}$$

where we have used Fields, Lemma 9.14.8. □

0F3A **Definition 37.75.2.** Let $f : X \to Y$ be a locally quasi-finite morphism. A *weighting* or a *pondération* of f is a map $w : X \to \mathbf{Z}$ such that for any diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xleftarrow{g} & V \end{array}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, and $U \to V$ finite, the function $\int_\pi (w \circ h)$ is locally constant.

Of course taking $w = 0$ we obtain a weighting of any locally quasi-finite morphism f, albeit not a very interesting one. It will turn out that *positive* weightings, i.e., $w : X \to \mathbf{Z}_{>0}$ are the most interesting ones for various purposes.

0F3B **Lemma 37.75.3.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. Let $f' : X' \to Y'$ be the base change of f by a morphism $Y' \to Y$. Then the composition $w' : X' \to \mathbf{Z}$ of w and the projection $X' \to X$ is a weighting of f'.*

Proof. Consider a diagram

$$\begin{array}{ccc} X' & \xleftarrow{h'} & U' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Y' & \xleftarrow{g'} & V' \end{array}$$

as in Definition 37.75.2 for the morphism f'. For any $v' \in V'$ we have to show that $\int_{\pi'}(w' \circ h')$ is constant in an open neighbourhood of v'. By Lemma 37.75.1 (and the fact that étale morphisms are open) we may replace V' by any étale neighbourhood of v'. After replacing V' by an étale neighbourhood of v' we may assume that $U' = U'_1 \amalg \ldots \amalg U'_n$ where each U'_i has a unique point u'_i lying over v' such that $\kappa(u'_i)/\kappa(v')$ is purely inseparable, see Lemma 37.41.5. Clearly, it suffices to prove that $\int_{U'_i \to V'} w'|_{U'_i}$ is constant in a neighbourhood of v'. This reduces us to the case discussed in the next paragraph.

We have $v' \in V'$ and there is a unique point u' of U' lying over v' with $\kappa(u')/\kappa(v')$ purely inseparable. Denote $x \in X$ and $y \in Y$ the image of u' and v'. We can find an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and such that there is a unique point $u \in U$ lying over v which maps to $x \in X$ via the projection $h : U \to X$ such that moreover $\kappa(u)/\kappa(v)$ is purely inseparable. This is possible by the lemma used above. Consider the morphism

$$U'' = U \times_X U' \longrightarrow V \times_Y V' = V''$$

Since u and u' both map to $x \in X$ there is a point $u'' \in U''$ mapping to (u, u'). Denote $v'' \in V''$ the image of u''. After replacing V', v' by V'', v'' we may assume that the composition $V' \to Y' \to Y$ factors through a map of étale neighbourhoods $(V', v') \to (V, v)$ such that the induced morphism $X'_{V'} = X_{V'} \to X_V$ sends u' to u. Inside the base change $X'_{V'} = X_{V'}$ we have two open subschemes, namely U' and the inverse image $U_{V'}$ of $U \subset X_V$. By construction both contain a unique point lying over v', namely u' for both of them. Thus after shrinking V' we may assume these open subsets are the same; namely, $U' \setminus (U' \cap U_{V'})$ and $U_{V'} \setminus (U' \cap U_{V'})$ have a closed image in V' and these images do not contain v'. Thus $U' = U_{V'}$ and we find a cartesian diagram as in Lemma 37.75.1. Since $\int_\pi (w \circ h)$ is locally constant by assumption we conclude. □

0GK8 **Lemma 37.75.4.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. If $X' \subset X$ is open, then $w|_{X'}$ is a weighting of $f|_{X'} : X' \to Y$.*

Proof. Immediate from the definition. □

0GK9 **Lemma 37.75.5.** *Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. Then the function*

$$X \longrightarrow \mathbf{Z}, \quad x \longmapsto w_f(x) w_g(f(x))$$

is a weighting of $g \circ f$.

Proof. Let us set $w_{g\circ f}(x) = w_f(x) w_g(f(x))$ for $x \in X$. Consider a diagram

$$\begin{array}{ccc} X & \longleftarrow & U \\ {\scriptstyle g\circ f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \longleftarrow & W \end{array}$$

where $W \to Z$ is étale, $U \subset X_W$ is open, and $U \to W$ finite. We have to show that $\int_\pi w_{g\circ f}|_U$ is locally constant. Choose a point $w \in W$. By Lemma 37.75.1 (and the fact that étale morphisms are open) it suffices to show that $\int_\pi w_{g\circ f}|_U$ is constant after replacing (W, w) by an étale neighbourhood. After replacing (W, w) by an étale neighbourhood we may assume $U = U_1 \amalg \ldots \amalg U_n$ where each U_i has a unique point u_i lying over w such that $\kappa(u_i)/\kappa(w)$ is purely inseparable, see Lemma 37.41.5. Clearly, it suffices to show that $\int_{U_i \to W} w_{g\circ f}|_{U_i}$ is constant in an étale neighbourhood of w. This reduces us to the case discussed in the next paragraph.

We have $w \in W$ and there is a unique point $u \in U$ lying over w with $\kappa(u)/\kappa(w)$ purely inseparable. Consider the point $v = f(u) \in Y$. After replacing (W, w) by an elementary étale neighbourhood we may assume there is an open neighbourhood

$V \subset Y_W$ of v such that $V \to W$ is finite, see Lemma 37.41.1. Then $f_W^{-1}(V) \cap U$ is an open neighbourhood of u where $f_W : X_W \to Y_W$ is the base change of f to W. Hence after Zariski shrinking W, we may assume $f_W(U) \subset V$. Thus we obtain morphisms

$$U \xrightarrow{a} V \xrightarrow{b} W$$

and $U \to V$ is finite as $V \to W$ is separated (because finite). Since w_f and w_g are weightings of f and g we see that $\int_a w_f|_U$ is locally constant on V and $\int_b w_g|_V$ is locally constant on W. Thus after shrinking W one more time we may assume these functions are constant say with values n and m. It follows immediately that $\int_\pi w_{g\circ f}|_U = \int_{b\circ a} w_{g\circ f}|_U$ is constant with value nm as desired. □

0F3C **Lemma 37.75.6.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. If $w(x) > 0$ for all $x \in X$, then f is universally open.*

Proof. Since the property is preserved by base change, see Lemma 37.75.3, it suffices to prove that f is open. Since we may also replace X by any open of X, it suffices to prove that $f(X)$ is open. Let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. It suffices to prove that $f(X)$ contains an open neighbourhood of y and it suffices to do so after replacing Y by an étale neighbourhood of y. By étale localization of quasi-finite morphisms, see Section 37.41, we may assume there is an open neighbourhood $U \subset X$ of x such that $\pi = f|_U : U \to Y$ is finite. Then $\int_\pi w|_U$ is locally constant and has positive value at y. Hence $\pi(U)$ contains an open neighbourhood of y and the proof is complete. □

0F3D **Lemma 37.75.7.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is locally quasi-finite, locally of finite presentation, and flat. Then there is a positive weighting $w : X \to \mathbf{Z}_{>0}$ of f given by the rule that sends $x \in X$ lying over $y \in Y$ to*

$$w(x) = length_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]_i$$

where $[\kappa' : \kappa]_i$ is the inseparable degree (Fields, Definition 9.14.7).

Proof. Consider a diagram as in Definition 37.75.2. Let $u \in U$ with images x, y, v in X, Y, V. Then we claim that

$$\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

and

$$[\kappa(x) : \kappa(y)]_i = [\kappa(u) : \kappa(v)]_i$$

The first equality follows as $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ is a flat local homomorphism such that $\mathfrak{m}_y\mathcal{O}_{U,u} = \mathfrak{m}_v\mathcal{O}_{U,u}$ and $\mathfrak{m}_x\mathcal{O}_{U,u} = \mathfrak{m}_u$ (because $\mathcal{O}_{Y,y} \to \mathcal{O}_{V,v}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ are unramified) and hence the equality by Algebra, Lemma 10.52.13. The second equality follows because $\kappa(v)/\kappa(y)$ is a finite separable extension and $\kappa(u)$ is a factor of $\kappa(x) \otimes_{\kappa(y)} \kappa(v)$ and hence the inseparable degree is unchanged. Having said this, we see that formation of the function in the lemma commutes with étale base change. This reduces the problem to the discussion of the next paragraph.

Assume that f is a finite, flat morphism of finite presentation. We have to show that $\int_f w$ is locally constant on Y. In fact, f is finite locally free (Morphisms,

Lemma 29.48.2) and we will show that $\int_f w$ is equal to the degree of f (which is a locally constant function on Y). Namely, for $y \in Y$ we see that

$$\begin{aligned}(\textstyle\int_f w)(y) &= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)]_i[\kappa(x):\kappa(y)]_s \\ &= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)] \\ &= \text{length}_{\mathcal{O}_{Y,y}}((f_*\mathcal{O}_X)_y/\mathfrak{m}_y(f_*\mathcal{O}_X)_y)\end{aligned}$$

Last equality by Algebra, Lemma 10.52.12. The final number is the rank of $f_*\mathcal{O}_X$ at y as desired. □

0F3E **Lemma 37.75.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *f is locally quasi-finite, and*
(2) *Y is geometrically unibranch and locally Noetherian.*

Then there is a weighting $w : X \to \mathbf{Z}_{\geq 0}$ given by the rule that sends $x \in X$ lying over $y \in Y$ to the "generic separable degree" of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,y}$.

Proof. It follows from Algebra, Lemma 10.156.3 that $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is finite. Since Y is geometrically unibranch there is a unique minimal prime $\mathfrak{p}$ in $\mathcal{O}^{sh}_{Y,y}$, see More on Algebra, Lemma 15.107.5. Write

$$(\kappa(\mathfrak{p}) \otimes_{\mathcal{O}^{sh}_{Y,y}} \mathcal{O}^{sh}_{X,x})_{red} = \prod K_i$$

as a finite product of fields. We set $w(x) = \sum [K_i : \kappa(\mathfrak{p})]_s$.

Since this definition is clearly insensitive to étale localization, in order to show that w is a weighting we reduce to showing that if f is a finite morphism, then $\int_f w$ is locally constant. Observe that the value of $\int_f w$ in a generic point η of Y is just the number of points of the geometric fibre $X_{\overline{\eta}}$ of $X \to Y$ over η. Moreover, since Y is unibranch a point y of Y is the specialization of a unique generic point η. Hence it suffices to show that $(\int_f w)(y)$ is equal to the number of points of $X_{\overline{\eta}}$. After passing to an affine neighbourhood of y we may assume $X \to Y$ is given by a finite ring map $A \to B$. Suppose $\mathcal{O}^{sh}_{Y,y}$ is constructed using a map $\kappa(y) \to k$ into an algebraically closed field k. Then

$$\mathcal{O}^{sh}_{Y,y} \otimes_A B = \prod\nolimits_{f(x)=y} \prod\nolimits_{\varphi \in \text{Mor}_{\kappa(y)}(\kappa(x),k)} \mathcal{O}^{sh}_{X,x}$$

by Algebra, Lemma 10.153.4 and the lemma used above. Observe that the minimal prime $\mathfrak{p}$ of $\mathcal{O}^{sh}_{Y,y}$ maps to the prime of A corresponding to η. Hence we see that the desired equality holds because the number of points of a geometric fibre is unchanged by a field extension. □

37.76. More on weightings

0F3F We prove a few more basic properties of weightings. Although at first it appears that weightings can be very wild, it actually turns out the condition imposed in Definition 37.75.2 is rather strong.

0F3G **Lemma 37.76.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function w are locally constructible in X.*

Proof. In the proof below we will use Lemmas 37.75.4 and 37.75.3 without further mention. We will also use elementary properties of constructible subsets of schemes and topological spaces, see Topology, Section 5.15 and Properties, Section 28.2. Using this the reader sees question is local on X and Y; details omitted. Hence we may assume X and Y are affine. If we can find a surjective morphism $Y' \to Y$ of finite presentation such that the level sets of w pull back to locally constructible subsets of $X' = Y' \times_Y X$, then we conclude by Morphisms, Theorem 29.22.3.

Assume X and Y affine. We may choose an immersion $X \to T$ where $T \to Y$ is finite, see Lemma 37.43.3. By Morphisms, Lemma 29.48.6 after replacing Y by Y' surjective finite locally free over Y, replacing X by $Y' \times_Y X$ and T by a scheme finite locally free over Y' containing $Y' \times_Y T$ as a closed subscheme, we may assume T is finite locally free over Y, contains closed subschemes T_i mapping isomorphically to Y such that $T = \bigcup_{i=1,\ldots,n} T_i$ (set theoretically). Since $T_i \subset T$ is a constructible closed subset (as the image of a finitely presented morphism $Y \to T$ of schemes), we see that for $I \subset \{1, \ldots, n\}$ the intersection $\bigcap_{i \in I} T_i$ is a constructible closed subset of T and hence maps to a constructible closed subset of Y.

For a disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$ with nonempty parts consider the subset $Y_{I_1,\ldots,I_r} \subset Y$ consisting of points $y \in Y$ such that $T_y = \{x_1, \ldots, x_r\}$ consists of exactly r points with $x_j \in T_i \Leftrightarrow i \in I_j$. By our remarks above this is a constructible partition of Y. There exists an affine scheme Y' of finite presentation over Y such that the image of $Y' \to Y$ is exactly $Y_{I_1,\ldots,I_r}$, see Algebra, Lemma 10.29.4. Hence we may assume that $Y = Y_{I_1,\ldots,I_r}$ for some disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$. In this case $T = T(1) \amalg \ldots \amalg T(r)$ with $T(j) = \bigcap_{i \in I_j} T_i$ is a decomposition of T into disjoint closed (and hence open) subsets. Intersecting with the locally closed subscheme X we obtain an analogous decomposition $X = X(1) \amalg \ldots \amalg X(r)$ into open and closed parts. The morphism $X(j) \to Y$ an immersion. Since w is a weighting, it follows that $w|_{X(j)}$ is locally constant[16] and we conclude. □

0F3H **Lemma 37.76.2.** *Let $f : X \to Y$ be a locally quasi-finite morphism of finite presentation. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function $\int_f w$ are locally constructible in Y.*

Proof. By Lemma 37.75.1 formation of the function $\int_f w$ commutes with arbitrary base change and by Lemma 37.75.3 after base change we still have a weighthing. This means that if we can find $Y' \to Y$ surjective and of finite presentation, then it suffices to prove the result after base change to Y', see Morphisms, Theorem 29.22.3.

The question is local on Y hence we may assume Y is affine. Then X is quasi-compact and quasi-separated (as f is of finite presentation). Suppose that $X = U \cup V$ are quasi-compact open. Then we have

$$\textstyle\int_f w = \int_{f|_U} w|_U + \int_{f|_V} w|_V - \int_{f|_{U \cap V}} w|_{U \cap V}$$

Thus if we know the result for $w|_U$, $w|_V$, $w|_{U \cap V}$ then we know the result for w. By the induction principle (Cohomology of Schemes, Lemma 30.4.1) it suffices to prove the lemma when X is affine.

[16] In fact, if $f : X \to Y$ is an immersion and w is a weighting of f, then f restricts to an open map on the locus where w is nonzero.

Assume X and Y are affine. We may choose an open immersion $X \to T$ where $T \to Y$ is finite, see Lemma 37.43.3. Because we may still base change with a suitable $Y' \to Y$ we can use Morphisms, Lemma 29.48.6 to reduce to the case where all residue field extensions induced by the morphism $T \to Y$ (and a foriori induced by $X \to Y$) are trivial. In this situation $\int_f w$ is just taking the sums of the values of w in fibres. The level sets of w are locally constructible in X (Lemma 37.76.1). The function w only takes a finite number of values by Properties, Lemma 28.2.7. Hence we conclude by Morphisms, Theorem 29.22.3 and some elementary arguments on sums of integers. □

0F3I **Lemma 37.76.3.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then w is upper semi-continuous.*

Proof. Let $x \in X$ with image $y \in Y$. Choose an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and there is a unique point $u \in U$ mapping to v with $\kappa(u)/\kappa(v)$ purely inseparable. See Lemma 37.41.3. Then $(\int_\pi w|_U)(v) = w(u)$. It follows from Definition 37.75.2 that after replacing V by a neighbourhood of v we we have $w|_U(u') \leq w|_U(u) = w(x)$ for all $u' \in U$. Namely, $w|_U(u')$ occurs as a summand in the expression for $(\int_\pi w|_U)(\pi(u'))$. This proves the lemma because the étale morphism $U \to X$ is open. □

0F3J **Lemma 37.76.4.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism with finite fibres. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then $\int_f w$ is lower semi-continuous.*

Proof. Let $y \in Y$. Let $x_1, \ldots, x_r \in X$ be the points lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $(\int_f w)(y) = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j} \to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int_{U \times_Y X \to U} w|_{U \times_Y X} = \int_{W \to U} w|_W + \sum \int_{V_{i,j} \to U} w|_{V_{i,j}} = \int_{W \to U} w|_W + (\int_f w)(y)$$

This is $\geq (\int_f w)(y)$ and we conclude because $U \to Y$ is open and formation of the integral commutes with base change (Lemma 37.75.1). □

0F3K **Lemma 37.76.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism with X quasi-compact. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then $\int_f w$ attains its maximum.*

Proof. It follows from Lemma 37.76.1 and Properties, Lemma 28.2.7 that w only takes a finite number of values on X. It follows from Morphisms, Lemma 29.57.9 that $X \to Y$ has bounded geometric fibres. This shows that $\int_f w$ is bounded. □

0F3L **Lemma 37.76.6.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume $\int_w f$ attains its maximum d and let $Y_d \subset Y$ be the open set of points y with $(\int_f w)(y) = d$. Then the morphism $f^{-1}(Y_d) \to Y_d$ is finite.*

Proof. Observe that Y_d is open by Lemma 37.76.4. Let $y \in Y_d$. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j} \to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int_{U \times_Y X \to U} w|_{U \times_Y X} = \int_{W \to U} w|_W + \sum \int_{V_{i,j} \to U} w|_{V_{i,j}} = \int_{W \to U} w|_W + (\int_f w)(y)$$

This is $\geq (\int_f w)(y) = d$ and we conclude that W must be the emptyset. Thus $U \times_Y X \to U$ is finite. By Descent, Lemma 35.23.25 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. □

0F3M **Lemma 37.76.7.** *Let $A \to B$ be a ring map which is finite and of finite presentation. There exists a finitely presented ring map $A \to A_{univ}$ and an idempotent $e_{univ} \in B \otimes_A A_{univ}$ such that for any ring map $A \to A'$ and idempotent $e \in B \otimes_A A'$ there is a ring map $A_{univ} \to A'$ mapping e_{univ} to e.*

Proof. Choose $b_1, \ldots, b_n \in B$ generating B as an A-module. For each i choose a monic $P_i \in A[x]$ such that $P_i(b_i) = 0$ in B, see Algebra, Lemma 10.36.3. Thus B is a quotient of the finite free A-algebra $B' = A[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$. Let $J \subset B'$ be the kernel of the surjection $B' \to B$. Then $J = (f_1, \ldots, f_m)$ is finitely generated as B is a finitely generated A-algebra, see Algebra, Lemma 10.6.2. Choose an A-basis $b'_1, \ldots, b'_N$ of B'. Consider the algebra

$$A_{univ} = A[z_1, \ldots, z_N, y_1, \ldots, y_m]/I$$

where I is the ideal generated by the coefficients in $A[z_1, \ldots, z_n, y_1, \ldots, y_m]$ of the basis elements $b'_1, \ldots, b'_N$ of the expression

$$(\sum z_j b'_j)^2 - \sum z_j b'_j + \sum y_k f_k$$

in $B'[z_1, \ldots, z_N, y_1, \ldots, y_m]$. By construction the element $\sum z_j b'_j$ maps to an idempotent e_{univ} in the algebra $B \otimes_A A_{univ}$. Moreover, if $e \in B \otimes_A A'$ is an idempotent, then we can lift e to an element of the form $\sum b'_j \otimes a'_j$ in $B' \otimes_A A'$ and we can find $a''_k \in A'$ such that

$$(\sum b'_j \otimes a'_j)^2 - \sum b'_j \otimes a'_j + \sum f_k \otimes a''_k$$

is zero in $B' \otimes_A A'$. Hence we get an A-algebra map $A_{univ} \to A$ sending z_j to a'_j and y_k to a''_k mapping e_{univ} to e. This finishes the proof. □

0F3N **Lemma 37.76.8.** *Let $X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. There exists a morphism $Y_{univ} \to Y$ of finite presentation and an open subscheme $U_{univ} \subset Y_{univ} \times_Y X$ such that $U_{univ} \to Y_{univ}$ is finite with the following property: given any morphism $Y' \to Y$ of affine schemes and an open subscheme $U' \subset Y' \times_Y X$ such that $U' \to Y'$ is finite, there exists a morphism $Y' \to Y_{univ}$ such that the inverse image of U_{univ} is U'.*

Proof. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.29.5. By Lemma 37.34.9 applied with $d = 0$ we reduce to the case where X and Y are Noetherian. We may choose an open immersion $X \to X'$ such that $X' \to Y$ is finite, see Algebra, Lemma 10.123.14. Note that if we have $Y' \to Y$ and U' as in (2), then

$$U' \to Y' \times_Y X \to Y' \times_Y X'$$

is open immersion between schemes finite over Y' and hence is closed as well. We conclude that U' corresponds to an idempotent in

$$\Gamma(Y', \mathcal{O}_{Y'}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

whose corresponding open and closed subset is contained in the open $Y' \times_Y X$. Let $Y'_{univ} \to Y$ and idempotent

$$e'_{univ} \in \Gamma(Y_{univ}, \mathcal{O}_{Y_{univ}}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

be the pair constructed in Lemma 37.76.7 for the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X', \mathcal{O}_{X'})$ (here we use that Y is Noetherian to see that X' is of finite presentation over Y). Let $U'_{univ} \subset Y'_{univ} \times_Y X'$ be the corresponding open and closed subscheme. Then we see that

$$U'_{univ} \setminus Y'_{univ} \times_Y X$$

is a closed subset of U'_{univ} and hence has closed image $T \subset Y'_{univ}$. If we set $Y_{univ} = Y'_{univ} \setminus T$ and U_{univ} the restriction of U'_{univ} to $Y_{univ} \times_Y X$, then we see that the lemma is true. □

0F3P **Lemma 37.76.9.** *Let $Y = \lim Y_i$ be a directed limit of affine schemes. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $f : X \to Y$ and $f_i : X_i \to Y_i$ for $i \geq 0$ be the base changes of f_0. If $w : X \to \mathbf{Z}$ is a weighting of f, then for sufficiently large i there exists a weighting $w_i : X_i \to \mathbf{Z}$ of f_i whose pullback to X is w.*

Proof. By Lemma 37.76.1 the level sets of w are constructible subsets E_k of X. This implies the function w only takes a finite number of values by Properties, Lemma 28.2.7. Thus there exists an i such that E_k descends to a constructible subset $E_{i,k}$ in X_i for all k; moreover, we may assume $X_i = \coprod E_{i,k}$. This follows as the topological space of X is the limit in the category of topological spaces of the spectral spaces X_i along a directed system with spectral transition maps. See Limits, Section 32.4 and Topology, Section 5.24. We define $w_i : X_i \to \mathbf{Z}$ such that its level sets are the constructible sets $E_{i,k}$.

Choose $Y_{i,univ} \to Y_i$ and $U_{i,univ} \subset Y_{i,univ} \times_{Y_i} X_i$ as in Lemma 37.76.8. By the universal property of the construction, in order to show that w_i is a weighting, it would suffice to show that

$$\tau_i = \textstyle\int_{U_{i,univ} \to Y_{i,univ}} w_i|_{U_{i,univ}}$$

is locally constant on $Y_{i,univ}$. By Lemma 37.76.2 this function has constructible level sets but it may not (yet) be locally constant. Set $Y_{univ} = Y_{i,univ} \times_{Y_i} Y$ and let $U_{univ} \subset Y_{univ} \times_Y X$ be the inverse image of $U_{i,univ}$. Then, since the pullback of w to $Y_{univ} \times_Y X$ is a weighting for $Y_{univ} \times_Y X \to Y_{univ}$ (Lemma 37.75.3) we do have that

$$\tau = \textstyle\int_{U_{univ} \to Y_{univ}} w_i|_{U_{univ}}$$

is locally constant on Y_{univ}. Thus the level sets of τ are open and closed. Finally, we have $Y_{univ} = \lim_{i' \geq i} Y_{i',univ}$ and the level sets of τ are the inverse limits of the level sets of $\tau_{i'}$ (similarly defined). Hence the references above imply that for sufficiently large i' the level sets of $\tau_{i'}$ are open as well. For such an index i' we conclude that $w_{i'}$ is a weighting of $f_{i'}$ as desired. □

37.77. Weightings and affine stratification numbers

0F3Q In this section we give a bound for the affine stratification number of a scheme which has a certain kind of cover by an affine scheme.

0F3R **Lemma 37.77.1.** *Let $f : X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Let $d < \infty$ be the maximum value of $\int_f w$. The open*

$$Y_d = \{y \in Y \mid (\textstyle\int_f w)(y) = d\}$$

of Y is affine.

Proof. Observe that $\int_f w$ attains its maximum by Lemma 37.76.5. The set Y_d is open by Lemma 37.76.4. Thus the statement of the lemma makes sense.

Reduction to the Noetherian case; please skip this paragraph. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.29.5. By Lemma 37.34.9 applied with $d = 0$ we can find a quasi-finite morphism $f_0 : X_0 \to Y_0$ of affine Noetherian schemes and a morphism $Y \to Y_0$ such that f is the base change of f_0. Then we can write $Y = \lim Y_i$ as a directed limit of affine schemes of finite type over Y_0, see Algebra, Lemma 10.127.2. By Lemma 37.76.9 we can find an i such that our weighting w descends to a weighting w_i of the base change $f_i : X_i \to Y_i$ of f_0. Now if the lemma holds for f_i, w_i, then it implies the lemma for f as formation of $\int_f w$ commutes with base change, see Lemma 37.75.1.

Assume X and Y Noetherian. Let $X' \to Y'$ be the base change of f by a morphism $g : Y' \to Y$. The formation of $\int_f w$ and hence the open Y_d commute with base change. If g is finite and surjective, then $Y'_d \to Y_d$ is finite and surjective. In this case proving that Y_d is affine is equivalent to showing that Y'_d is affine, see Cohomology of Schemes, Lemma 30.13.3.

We may choose an immersion $X \to T$ with T finite over Y, see Lemma 37.43.3. We are going to apply Morphisms, Lemma 29.48.6 to the finite morphism $T \to Y$. This lemma tells us that there is a finite surjective morphism $Y' \to Y$ such that $Y' \times_Y T$ is a closed subscheme of a scheme T' finite over Y' which has a special form. By the discussion in the first paragraph, we may replace Y by Y', T by T', and X by $Y' \times_Y X$. Thus we may assume there is an immersion $X \to T$ (not necessarily open or closed) and closed subschemes $T_i \subset T$, $i = 1, \ldots, n$ where

(1) $T \to Y$ is finite (and locally free),
(2) $T_i \to Y$ is an isomorphism, and
(3) $T = \bigcup_{i=1,\ldots,n} T_i$ set theoretically.

Let $Y' = \coprod Y_k$ be the disjoint union of the irreducible components of Y (viewed as integral closed subschemes of Y). Then we may base change once more by $Y' \to Y$; here we are using that Y is Noetherian. Thus we may in addition assume Y is integral and Noetherian.

We also may and do assume that $T_i \neq T_j$ if $i \neq j$ by removing repeats. Since Y and hence all T_i are integral, this means that if T_i and T_j intersect, then they intersect in a closed subset which maps to a proper closed subset of Y.

Observe that $V_i = X \cap T_i$ is a locally closed subset which is in addition a closed subscheme of X hence affine. Let $\eta \in Y$ and $\eta_i \in T_i$ be the generic points. If $\eta \notin Y_d$, then $Y_d = \emptyset$ and we're done. Assume $\eta \in Y_d$. Denote $I \in \{1, \ldots, n\}$ the subset of indices i such that $\eta_i \in V_i$. For $i \in I$ the locally closed subset $V_i \subset T_i$ contains the generic point of the irreducible space T_i and hence is open. On the other hand, since f is open (Lemma 37.75.6), for any $x \in X$ we can find an $i \in I$ and a specialization $\eta_i \rightsquigarrow x$. It follows that $x \in T_i$ and hence $x \in V_i$. In other words, we see that $X = \bigcup_{i \in I} V_i$ set theoretically. We claim that $Y_d = \bigcap_{i \in I} \mathrm{Im}(V_i \to Y)$; this will finish the proof as the intersection of affine opens $\mathrm{Im}(V_i \to Y)$ of Y is affine.

For $y \in Y$ let $f^{-1}(\{y\}) = \{x_1, \ldots, x_r\}$ in X. For each $i \in I$ there is at most one $j(i) \in \{1, \ldots, x_r\}$ such that $\eta_i \rightsquigarrow x_{j(i)}$. In fact, $j(i)$ exists and is equal to j if and only if $x_j \in V_i$. If $i \in I$ is such that $j = j(i)$ exists, then $V_i \to Y$ is an isomorphism in a neighbourhood of $x_j \mapsto y$. Hence $\bigcup_{i \in I,\ j(i) = j} V_i \to Y$ is finite after replacing source and target by neighbourhoods of $x_j \mapsto y$. Thus the definition of a weighting tells us that $w(x_j) = \sum_{i \in I,\ j(i) = j} w(\eta_i)$. Thus we see that

$$(\textstyle\int_f w)(\eta) = \sum_{i \in I} w(\eta_i) \geq \sum_{j(i) \text{ exists}} w(\eta_i) = \sum_j w(x_j) = (\int_f w)(y)$$

Thus equality holds if and only if y is contained in $\bigcap_{i \in I} \mathrm{Im}(V_i \to Y)$ which is what we wanted to show. □

0F3S **Proposition 37.77.2.** *Let $f : X \to Y$ be a surjective quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume X affine and Y separated*[17]*. Then the affine stratification number of Y is at most the number of distinct values of $\int_f w$.*

Proof. Note that since Y is separated, the morphism $X \to Y$ is affine (Morphisms, Lemma 29.11.12). The function $\int_f w$ attains its maximum d by Lemma 37.76.5. We will use induction on d. Consider the open subscheme $Y_d = \{y \in Y \mid (\int_f w)(y) = d\}$ of Y and recall that $f^{-1}(Y_d) \to Y_d$ is finite, see Lemma 37.76.6. By Lemma 37.77.1 for every affine open $W \subset Y$ we have that $Y_d \cap W$ is affine (this uses that $W \times_Y X$ is affine, being affine over X). Hence $Y_d \to Y$ is an affine morphism of schemes. We conclude that $f^{-1}(Y_d) = Y_d \times_Y X$ is an affine scheme being affine over X. Then $f^{-1}(Y_d) \to Y_d$ is surjective and hence Y_d is affine by Limits, Lemma 32.11.1. Set $X' = X \setminus f^{-1}(Y_d)$ and $Y' = Y \setminus Y_d$ viewed as closed subschemes of X and Y. Since X' is closed in X it is affine. Since Y' is closed in Y it is separated. The morphism $f' : X' \to Y'$ is surjective and w induces a weighting w' of f', see Lemma 37.75.3. By induction Y' has an affine stratification of length $\leq$ the number of distinct values of $\int_{f'} w'$ and the proof is complete. □

37.78. Completely decomposed morphisms

0GTH Nishnevich studied the notion of a completely decomposed family of étale morphisms, in order to define what is now called the Nishnevich topology, see for example [**Nis89**].

[17] It suffices if the diagonal of Y is affine.

0GTI **Definition 37.78.1.** A morphism $f : X \to Y$ of schemes is said to be *completely decomposed*[18] if for all points $y \in Y$ there is a point $x \in X$ with $f(x) = y$ such that the field extension $\kappa(x)/\kappa(y)$ is trivial. A family of morphisms $\{f_i : X_i \to Y\}_{i \in I}$ of schemes with fixed target is said to be *completely decomposed* if $\coprod f_i : \coprod Y_i \to X$ is completely decomposed.

We start with some basic lemmas.

0GTJ **Lemma 37.78.2.** *The composition of two completely decomposed morphisms of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and for each i we have a family $\{X_{ij} \to X_i\}_{j \in J_i}$ which is completely decomposed, then the family $\{X_{ij} \to Y\}_{i \in I, j \in J_i}$ is completely decomposed.*

Proof. Omitted. □

0GTK **Lemma 37.78.3.** *The base change of a completely decomposed morphism of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and $Y' \to Y$ is a morphism of schemes, then $\{X_i \times_Y Y' \to Y'\}_{i \in I}$ is completely decomposed.*

Proof. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Let $y' \in Y'$ be a point with image $y = g(y')$ in Y. If $x \in X$ is a point such that $f(x) = y$ and $\kappa(x) = \kappa(y)$, then there exists a unique point $x' \in X' = X \times_Y Y'$ which maps to y' in Y' and to x in X and moreover $\kappa(x') = \kappa(y')$, see Schemes, Lemma 26.17.5. From this fact the lemma follows easily; we omit the details. □

0GTL **Lemma 37.78.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is completely decomposed, f is locally of finite presentation, and Y is quasi-compact and quasi-separated. Then there exist $n \geq 0$ and morphisms $Z_i \to Y$, $i = 1, \ldots, n$ with the following properties* [**EHIK21**, Lemma 2.1.2]

(1) *$\coprod Z_i \to Y$ is surjective,*
(2) *$Z_i \to Y$ is an immersion for all i,*
(3) *$Z_i \to Y$ is of finite presentation for all i, and*
(4) *the base change $X \times_Y Z_i \to Z_i$ has a section for all i.*

Proof. Let $y \in Y$. By assumption there is a morphism $\sigma : \mathrm{Spec}(\kappa(y)) \to X$ over Y. We can write $\mathrm{Spec}(\kappa(y))$ as a directed limit of affine schemes Z over Y such that $Z \to Y$ is an immersion of finite presentation. Namely, choose an affine open $y \in \mathrm{Spec}(A) \subset Y$ and say y corresponds to the prime ideal $\mathfrak{p}$ of A. Then $\kappa(\mathfrak{p})$ is the filtered colimit of the rings $(A/I)_f$ where $I \subset \mathfrak{p}$ is a finitely generated ideal and $f \in A$, $f \notin \mathfrak{p}$. The morphisms $Z = \mathrm{Spec}((A/I)_f) \to Y$ are immersions of finite presentation; quasi-compactness of $Z \to Y$ follows as Y is quasi-separated, see Schemes, Lemma 26.21.14. By Limits, Proposition 32.6.1 for some such Z there is a morphism $\sigma' : Z \to X$ over Y agreeing with σ on the spectrum of $\kappa(\mathfrak{p})$. Since σ' is a morphism over Y, we obtain a section of the projection $X \times_Y Z \to Z$

We conclude that Y is the union of the images of immersions $Z \to Y$ of finite presentation such that $X \times_Y Z \to Z$ has a section. Since the image of $Z \to Y$ is constructible (Morphisms, Lemma 29.22.2) and since Y is compact in the constructible topology (Properties, Lemma 28.2.4 and Topology, Lemma 5.23.2), we see that a finite number of these suffice. □

[18] This may be nonstandard terminology.

0GTM **Lemma 37.78.5.** *Let $S = \lim_{\lambda\in\Lambda} S_\lambda$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in \Lambda$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. For $\lambda \geq 0$ let $f_\lambda : X_\lambda \to Y_\lambda$ be the base change of f_0 to S_λ and let $f : X \to Y$ be the base change of f_0 to S. If*

(1) *f is completely decomposed,*
(2) *Y_0 is quasi-compact and quasi-separated, and*
(3) *f_0 is locally of finite presentation,*

then there exists an $\lambda \geq 0$ such that f_λ is completely decomposed.

Proof. Since Y_0 is quasi-compact and quasi-separated, the scheme Y, which is affine over Y_0, is quasi-compact and quasi-separated. Choose $n \geq 0$ and $Z_i \to Y$, $i = 1, \ldots, n$ as in Lemma 37.78.4. Denote $\sigma_i : Z_i \to X$ morphisms over Y which exist by our choice of Z_i. After increasing $0 \in \Lambda$ we may assume there exist morphisms $Z_{i,0} \to Y_0$ of finite presentation whose base changes to S are the morphisms $Z_i \to Y$, see Limits, Lemma 32.10.1. By Limits, Lemma 32.8.13 we may assume, after possibly increasing 0, that $Z_{i,0} \to Y_0$ is an immersion. Since $\coprod Z_i \to Y$ is surjective, we may assume, after possibly increasing 0, that $\coprod Z_{i,0} \to Y_0$ is surjective, see Limits, Lemma 32.8.15. Observe that $Z_i = \lim_{\lambda\geq 0} Z_{i,\lambda}$ where $Z_{i,\lambda} = Y_\lambda \times_{Y_0} Z_{i,0}$. Let us view the compositions

$$Z_i \xrightarrow{\sigma_i} X \to X_0$$

as morphisms over Y_0. Since f_0 is locally of finite presentation by Limits, Proposition 32.6.1 we can find a $\lambda \geq 0$ such that there exist morphisms $\sigma'_{i,\lambda} : Z_{i,\lambda} \to X_0$ over Y_0 whose precomposition with $Z_i \to Z_{i,\lambda}$ are the displayed arrows. Of course, then $\sigma'_{i,\lambda}$ determines a morphism $\sigma_{i,\lambda} : Z_{i,\lambda} \to X_\lambda = X_0 \times_{Y_0} Y_\lambda$ over Y_λ. Since $\coprod Z_{i,\lambda} \to Y_\lambda$ is surjective we conclude that $X_\lambda \to Y_\lambda$ is completely decomposed. □

37.79. Families of ample invertible modules

0GTN We continue the discussion from Morphisms, Section 29.12.

0GTP **Lemma 37.79.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y has an ample family of invertible modules,*
(2) *there exists an f-ample invertible module on X.*

Then X has an ample family of invertible modules.

Proof. Let $\mathcal{L}$ be an f-ample invertible module on X. This in particular implies that f is quasi-compact, see Morphisms, Definition 29.37.1. Since Y is quasi-compact by Morphisms, Definition 29.12.1 we see that X is quasi-compact (and hence X itself satisfies the first condition of Morphisms, Definition 29.12.1). Let $x \in X$ with image $y \in Y$. By assumption (2) we can find an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ and a section $t \in \Gamma(Y, \mathcal{N})$ such that the locus Y_t where t does not vanish is affine. Then $\mathcal{L}$ is ample over $f^{-1}(Y_t) = X_{f^*t}$ and hence we can find a section $s \in \Gamma(X_{f^*t}, \mathcal{L})$ such that $(X_{f^*t})_s$ is affine and contains x. By Properties, Lemma 28.17.2 for some $n \geq 0$ the product $(f^*t)^n s$ extends to a section $s' \in \Gamma(X, f^*\mathcal{N}^{\otimes n} \otimes \mathcal{L})$. Then finally the section $s'' = f^*ts'$ of $f^*\mathcal{N}^{\otimes n+1} \otimes \mathcal{L}$ vanishes at every point of $X \setminus X_{f^*t}$ hence we see that $X_{s''} = (X_{f^*t})_s$ is affine as desired. □

0GTQ **Lemma 37.79.2.** *Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes. If Y has an ample family of invertible modules, so does X.*

Proof. By Morphisms, Lemma 29.37.6 this is a special case of Lemma 37.79.1. □

37.80. Blowing up and ample families of invertible modules

0GTR We prove a result from [**Gro10**].

0GTS **Lemma 37.80.1.** *Let X be a scheme. Suppose given effective Cartier divisors $D_1, \ldots, D_m$ on X and invertible modules $\mathcal{L}_1, \ldots, \mathcal{L}_m$ such that $\bigcap D_i = \emptyset$ and $\mathcal{L}_i|_{X \setminus D_i}$ is ample. Then X has an ample family of invertible modules.*

Proof. Let $x \in X$. Choose an index $i \in \{1, \ldots, m\}$ such that $x \notin D_i$. Set $U_i = X \setminus D_i$. Since $\mathcal{L}_i|_{U_i}$ we can find an $n \geq 1$ and a section $s \in \Gamma(U_i, \mathcal{L}_i^{\otimes n})$ such that the locus $(U_i)_s$ where s doesn't vanish is affine (Properties, Definition 28.26.1). Since U_i is the locus where the canonical section $1 \in \mathcal{O}_X(D_i)$ doesn't vanish, we see from Properties, Lemma 28.17.2 there exists an $N \geq 0$ such that s extends to a section

$$s' \in \Gamma(X, \mathcal{L}_i^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{O}_X(ND_i))$$

After replacing N by $N + 1$ we see that s' vanishes at every point of D_i and hence that $X_{s'} = (U_i)_s$ is affine. This proves that X has an ample family of invertible modules, see Morphisms, Definition 29.12.1. □

0GTT **Lemma 37.80.2.** *Let X be a quasi-compact and quasi-separated scheme with finitely many irreducible components. There exists a quasi-compact dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the scheme X' has an ample family of invertible modules.* [**Gro10**, Proposition 1.3.1]

Proof. Let $\eta_1, \ldots, \eta_n \in X$ be the generic points of the irreducible components of X. By Properties, Lemma 28.29.4 and the fact that X is quasi-compact we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_n$. In particular the quasi-compact open subset $U = U_1 \cap \ldots \cap U_m$ is dense in X. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be a finite type quasi-coherent ideal sheaf such that $U_i = X \setminus Z_i$ where $Z_i = V(\mathcal{I}_i)$, see Properties, Lemma 28.24.1. Let

$$f : X' \longrightarrow X$$

be the blowing up of X in the ideal sheaf $\mathcal{I} = \mathcal{I}_1 \cdots \mathcal{I}_m$. Note that f is a U-admissible blowing up as $V(\mathcal{I})$ is (set theoretically) the union of the Z_i which are disjoint from U. Also, f is a projective morphism and $\mathcal{O}_{X'}(1)$ is f-relatively ample, see Divisors, Lemma 31.32.13. By Divisors, Lemma 31.32.12 for each i the morphism f' factors as $X' \to X'_i \to X$ where $X'_i \to X$ is the blowing up in $\mathcal{I}_i$ and $X' \to X'_i$ is another blowing up (namely in the pullback of the products of the ideals $\mathcal{I}_j$ omitting $\mathcal{I}_i$). It follows from this that $D_i = f^{-1}(Z_i) \subset X'$ is an effective Cartier divisor, see Divisors, Lemmas 31.32.11 and 31.32.4. We have $X' \setminus D_i = f^{-1}(U_i)$. As $\mathcal{O}_{X'}(1)$ is f-ample, the restriction of $\mathcal{O}_{X'}(1)$ to $X' \setminus D_i$ is ample. It follows from Lemma 37.80.1 that X' has an ample family of invertible modules. □

0GTU **Proposition 37.80.3.** *Let X be a quasi-compact and quasi-separated scheme. There exists a morphism $f : Y \to X$ which is of finite presentation, proper, and completely decomposed (Definition 37.78.1) such that the scheme Y has an ample family of invertible modules.*

Proof. By Limits, Proposition 32.5.4 there exists an affine morphism $X \to X_0$ where X_0 is a scheme of finite type over $\mathbf{Z}$. Below we produce a morphism $Y_0 \to X_0$ with all the desired properties. Then setting $Y = X \times_{X_0} Y_0$ and f equal to the

projection $f : Y \to X$ we conclude. To see this observe that f is of finite presentation (Morphisms, Lemma 29.21.4), f is proper (Morphisms, Lemma 29.41.5), f is completely decomposed (Lemma 37.78.3). Finally, since $Y \to Y_0$ is affine (as the base change of $X \to X_0$) we see that Y has an ample family of invertible modules by Lemma 37.79.2. This reduces us to the case discussed in the next paragraph.

Assume X is of finite type over $\mathbf{Z}$. In particular $\dim(X) < \infty$. We will argue by induction on $\dim(X)$. If $\dim(X) = 0$, then X is affine and has the resolution property. In general, there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has an ample family of invertible modules, see Lemma 37.80.2. Since $f : X' \to X$ is an isomorphism over U we see that every point of U lifts to a point of X' with the same residue field. Let $Z = X \setminus U$ with the reduced induced scheme structure. Then $\dim(Z) < \dim(X)$ as U is dense in X (see above). By induction we find a proper, completely decomposed morphism $W \to Z$ such that W has an ample family of invertible modules. Then it follows that $Y = X' \amalg W \to X$ is the desired morphism. □

37.81. The extensive criterion for closed immersions

0H2N In this section, we give a criterion for a morphism of schemes to be a closed immersion.

0H2P **Lemma 37.81.1.** *A morphism $f : X \to Y$ of affine schemes is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square*

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

there exists a lift $\mathrm{Spec}(A) \to X$ making the two triangles commute.

Proof. Let the morphism f be given by the ring map $\phi : R \to S$. Then f is a closed immersion if and only if ϕ is surjective.

First, we assume that ϕ is surjective. Let $\psi : A \to B$ be an injective ring map, and suppose we are given a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \phi} & \nearrow{\scriptstyle \beta} & \downarrow{\scriptstyle \psi} \\ S & \longrightarrow & B \end{array}$$

Then we define a lift $S \to A$ by $s \mapsto \alpha(r)$, where $r \in R$ is such that $\phi(r) = s$. This is well-defined because ψ is injective and the square commutes. Since taking the ring spectrum defines an anti-equivalence between commutative rings and affine schemes, the desired lifting property for f holds.

Next, we assume that ϕ has lifts against all injective ring maps $\psi : A \to B$. Note that $\phi(R)$ is a subring of S, so we obtain a commutative square

$$\begin{array}{ccc} R & \longrightarrow & \phi(R) \\ \downarrow{\scriptstyle \phi} & \nearrow & \downarrow \\ S & = & S \end{array}$$

in which a lift $S \to \phi(R)$ exists. Hence, the inclusion $\phi(R) \to S$ must be an isomorphism, which shows that ϕ is surjective, and we win. □

0H2Q **Lemma 37.81.2.** *Let X be a scheme. If the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ of Schemes, Lemma 26.6.4 has a retraction, then X is an affine scheme.*

Proof. Write $S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and $f : X \to S$ the morphism given in the lemma. Let $s : S \to X$ be a retraction; so $\mathrm{id}_X = sf$. Then $fsf = \mathrm{id}_S f$. Since f induces an isomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ this means that fs and id_S induce the same map on $\Gamma(S, \mathcal{O}_S)$. Whence $fs = \mathrm{id}_S$ as S is affine. Hence f is an isomorphism and X is an affine scheme, as was to be shown. □

0H2R **Lemma 37.81.3.** *Let X be a scheme. Let $f : X \to S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ be the canonical morphism of Schemes, Lemma 26.6.4. The largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp : \mathcal{O}_S \to f_*\mathcal{O}_X$ is zero. If X is quasi-compact, then $f^\sharp$ is injective. In particular, if X is quasi-compact, then f is a dominant morphism.*

Proof. Let $M \subset \Gamma(S, \mathcal{O}_S)$ be the submodule corresponding to the largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp$. Then any element $a \in M$ is mapped to zero by $f^\sharp$. However, $f^\sharp(a)$ is the element of

$$\Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \Gamma(S, \mathcal{O}_S)$$

corresponding to a itself! Thus $a = 0$. Hence $M = 0$ which proves the first assertion. Note that this is equivalent to the morphism $f : X \to S$ being scheme-theoretically surjective.

If X is quasi-compact, then $\mathrm{Ker}(f^\sharp)$ is quasi-coherent by Morphisms, Lemma 29.6.3. Hence $\mathrm{Ker}(f^\sharp) = 0$ and $f^\sharp$ is injective. In this case, f is a dominant morphism by part (4) of Morphisms, Lemma 29.6.3. □

0H2S **Lemma 37.81.4.** *Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then f is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square*

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

there exists a lift $\mathrm{Spec}\, A \to X$ making the diagram commute.

Proof. Assume that f is a closed immersion. Let $A \to B$ be an injective ring map and consider a commutative square

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

Then $\mathrm{Spec}(A) \times_Y X \to \mathrm{Spec}(A)$ is a closed immersion and hence we get an ideal $I \subset A$ and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A/I) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

We obtain a lift by Lemma 37.81.1.

Assume that f has the lifting property stated in the lemma. To prove that f is a closed immersion is local on Y, hence we may and do assume Y is affine. In particular, Y is quasi-compact and therefore X is quasi-compact. Hence there exists a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. The source of the morphism

$$\pi : U = \coprod U_i \longrightarrow X$$

is affine and the induced ring map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$ is injective. By assumption, there exists a lift in the diagram

$$\begin{array}{ccc} U & \xrightarrow{\pi} & X \\ \downarrow & \overset{h}{\nearrow} & \downarrow f \\ \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) & \xrightarrow{f'} & Y \end{array}$$

where f' is the morphism of affine schemes corresponding to the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. It follows from the fact that π is an epimorphism that the morphism h is a retraction of the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$; details omitted. Hence X is affine by Lemma 37.81.2. By Lemma 37.81.1 we conclude that f is a closed immersion. □

37.82. Other chapters

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CHAPTER 38

More on Flatness

057M

38.1. Introduction

057N In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes and applications. Most of the results on flatness can be found in the paper [**GR71**] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 10.39.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 10.75.
(3) Criteria for flatness, see Algebra, Section 10.99 (Noetherian case), Algebra, Section 10.101 (Artinian case), Algebra, Section 10.128 (non-Noetherian case), and finally More on Morphisms, Section 37.16.
(4) Generic flatness, see Algebra, Section 10.118 and Morphisms, Section 29.27.
(5) Openness of the flat locus, see Algebra, Section 10.129 and More on Morphisms, Section 37.15.
(6) Flattening, see More on Algebra, Sections 15.16, 15.17, 15.18, 15.19, and 15.20.
(7) Additional results in More on Algebra, Sections 15.21, 15.22, 15.25, and 15.26.

As applications of the material on flatness we discuss the following topics: a non-Noetherian version of Grothendieck's existence theorem, blowing up and flatness, Nagata's theorem on compactifications, the h topology, blow up squares and descent, weak normalization, descent of vector bundles in positive characteristic, and the local structure of perfect complexes in the h topology.

38.2. Lemmas on étale localization

05FM In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

057R **Lemma 38.2.1.** *Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.*

Proof. See Algebra, Lemma 10.143.10. □

05H2 **Lemma 38.2.2.** *Let*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 29.36.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 29.36.4. □

05B9 **Lemma 38.2.3.** *Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 29.36.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 10.39.4. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via $\mathrm{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 29.25.7. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 29.35.13 and 29.36.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. □

05BA **Lemma 38.2.4.** *Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 26.24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 38.2.3. □

05VL **Lemma 38.2.5.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let M^{sh} be a $\mathcal{O}^{sh}_{S,s}$-module. The following are equivalent*

1. *M^{sh} is flat over $\mathcal{O}_{S,s}$,*
2. *M^{sh} is flat over $\mathcal{O}^h_{S,s}$, and*
3. *M^{sh} is flat over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

4. *M^h is flat over $\mathcal{O}_{S,s}$, and*
5. *M^h is flat over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^h$ this is also equivalent to

(6) *M is flat over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 15.45.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}^h \to \mathcal{O}_{S,s}^{sh}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from Algebra, Lemma 10.39.4. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.39.8. Thus it suffices to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). To prove these we may assume S is an affine scheme.

Assume (1). By Lemma 38.2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T,t) \to (S,s)$. Since $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see Algebra, Lemmas 10.155.7 and 10.155.11) we conclude that M^{sh} is flat over $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ by Algebra, Lemma 10.39.6. Thus (1) implies (2) and (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}_{S,s}^h$. The same argument applies to prove (4) $\Rightarrow$ (5). □

0DK0 **Lemma 38.2.6.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}_{S,s}^h$ (resp. $\mathcal{O}_{S,s}^{sh}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let M^{sh} be an object of $D(\mathcal{O}_{S,s}^{sh})$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$,*
(2) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}^h$, and*
(3) *M^{sh} has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}^{sh}$.*

If $M^{sh} = M^h \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}^h} \mathcal{O}_{S,s}^{sh}$ for $M^h \in D(\mathcal{O}_{S,s}^h)$ this is also equivalent to

(4) *M^h has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$, and*
(5) *M^h has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}^h$.*

If $M^h = M \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^h$ for $M \in D(\mathcal{O}_{S,s})$ this is also equivalent to

(6) *M has tor amplitude in $[a,b]$ over $\mathcal{O}_{S,s}$.*

Proof. By More on Algebra, Lemma 15.45.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}^h \to \mathcal{O}_{S,s}^{sh}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from More on Algebra, Lemma 15.67.11. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from More on Algebra, Lemma 15.67.17. Thus it suffices to show that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5).

Assume (1). In particular M^{sh} has vanishing cohomology in degrees $< a$ and $> b$. Hence we can represent M^{sh} by a complex $P^\bullet$ of free $\mathcal{O}_{X,x}^{sh}$-modules with $P^i = 0$ for $i > b$ (see for example the very general Derived Categories, Lemma 13.15.4). Note that P^n is flat over $\mathcal{O}_{S,s}$ for all n. Consider $\mathrm{Coker}(d_P^{a-1})$. By More on Algebra, Lemma 15.67.2 this is a flat $\mathcal{O}_{S,s}$-module. Hence by Lemma 38.2.5 this is a flat $\mathcal{O}_{S,s}^{sh}$-module. Thus $\tau_{\geq a} P^\bullet$ is a complex of flat $\mathcal{O}_{S,s}^{sh}$-modules representing M^{sh} in $D(\mathcal{O}_{S,s}^{sh}$ and we find that M^{sh} has tor amplitude in $[a,b]$, see More on Algebra, Lemma 15.67.3. Thus (1) implies (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}_{S,s}^h$. The same argument applies to prove (4) $\Rightarrow$ (5). □

05FN **Lemma 38.2.7.** *Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in \mathit{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. The implication "$\Leftarrow$" follows immediately from Divisors, Lemma 31.6.4. Assume $t \in \text{WeakAss}(g^*\mathcal{G})$. Let $\text{Spec}(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. As g is finite flat we have $g^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 10.78.5 we have $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \text{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \text{WeakAss}(g^*\mathcal{G})$ implies that $s \in \text{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 31.6.3 and hence by the above $s \in \text{WeakAss}(\mathcal{G})$. □

05FP **Lemma 38.2.8.** *Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in \textit{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \textit{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 29.36.14. Thus we may assume $S = \text{Spec}(A)$ and $X = \text{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \text{Spec}(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 38.2.7 above. □

0CTU **Lemma 38.2.9.** *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}_{S,s}^h$ (resp. $\mathcal{O}_{S,s}^{sh}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The following are equivalent*

1. *s is a weakly associated point of $\mathcal{F}$,*
2. *$\mathfrak{m}_s$ is a weakly associated prime of $\mathcal{F}_s$,*
3. *$\mathfrak{m}_s^h$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^h$, and*
4. *$\mathfrak{m}_s^{sh}$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^{sh}$.*

Proof. The equivalence of (1) and (2) is the definition, see Divisors, Definition 31.5.1. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from Divisors, Lemma 31.6.4 applied to the flat (More on Algebra, Lemma 15.45.1) morphisms

$$\text{Spec}(\mathcal{O}_{S,s}^{sh}) \to \text{Spec}(\mathcal{O}_{S,s}^h) \to \text{Spec}(\mathcal{O}_{S,s})$$

and the closed points. To prove (4) $\Rightarrow$ (2) we may replace S by an affine neighbourhood. Suppose that $x \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^{sh}$ is an element whose annihilator has radical equal to $\mathfrak{m}_s^{sh}$. (See Algebra, Lemma 10.66.2.) Since $\mathcal{O}_{S,s}^{sh}$ is equal to the limit of $\mathcal{O}_{U,u}$ over étale neighbourhoods $f : (U, u) \to (S, s)$ by Algebra, Lemma 10.155.11 we may assume that x is the image of some $x' \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{U,u}$. The local ring map $\mathcal{O}_{U,u} \to \mathcal{O}_{S,s}^{sh}$ is faithfully flat (as it is the strict henselization), hence universally injective (Algebra, Lemma 10.82.11). It follows that the annihilator of x' is the inverse image of the annihilator of x. Hence the radical of this annihilator is equal to $\mathfrak{m}_u$. Thus u is a weakly associated point of $f^*\mathcal{F}$. By Lemma 38.2.8 we see that s is a weakly associated point of $\mathcal{F}$. □

38.3. The local structure of a finite type module

057P The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 38.3.2.

057Q **Lemma 38.3.1.** *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = Supp(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 29.15.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 28.16.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $\mathrm{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0\ \forall m \in M\}$ of M, see Algebra, Lemma 10.40.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 29.5.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = \mathrm{Spec}(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \mathrm{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. □

057S **Lemma 38.3.2.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(Supp(\mathcal{F}_s))$. Then we can construct*

(1) *elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,*
(2) *a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

(3) *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
(4) *a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,*

such that the following properties hold

(1) *X', Z', Y', S' are affine schemes,*
(2) *i is a closed immersion of finite presentation,*
(3) *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
(4) *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
(5) *the extension $\kappa(y')/\kappa(s')$ is purely transcendental,*

(6) *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \text{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 37.47.2 and 37.47.3. This gives us a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ & & Y' \\ & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' \end{array}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \text{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 38.3.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & Z & \xleftarrow{g} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|_Z} & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & = & S & \xleftarrow{e} & S' \end{array}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 38.2.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 29.36.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 38.2.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & X' & \xleftarrow{i} & Z' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

is a solution of the original problem. □

057T **Lemma 38.3.3.** *Assumptions and notation as in Lemma 38.3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

(1) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.*

Still assuming f locally of finite presentation the following are equivalent to each other

(a) *$\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,*
(b) *$\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.*

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 29.21.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 29.21.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 17.11.5. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 29.36.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 35.7.3. Using Descent, Lemma 35.7.10 and some easy topological arguments (see More on Morphisms, Lemma 37.47.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 10.83.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 10.36.23. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 37.47.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 10.36.23. □

057U **Lemma 38.3.4.** *Assumptions and notation as in Lemma 38.3.2. The following are equivalent*

(1) *$\mathcal{F}$ is flat over S in a neighbourhood of x,*
(2) *$\mathcal{G}$ is flat over S' in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.*

The following are equivalent also

(a) *$\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,*

(b) $\mathcal{G}_{z'}$ *is flat over* $\mathcal{O}_{S',s'}$, *and*
(c) $(\pi_*\mathcal{G})_{y'}$ *is flat over* $\mathcal{O}_{S',s'}$.

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 38.2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 29.36.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 29.25.13. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 37.47.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{array}{ccccccc} \mathcal{O}_{X',x'} & \xrightarrow[\iota]{} & \mathcal{O}_{Z',z'} & \xleftarrow[\alpha]{} & \mathcal{O}_{Y',y'} & \xleftarrow[\beta]{} & \mathcal{O}_{S',s'} \\ \uparrow{\scriptstyle \gamma} & & & & & & \uparrow{\scriptstyle \epsilon} \\ \mathcal{O}_{X,x} & & & \xleftarrow{\quad\varphi\quad} & & & \mathcal{O}_{S,s} \end{array}$$

We will use Lemma 38.2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 10.39.9. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 37.47.4. This finishes the proof. □

38.4. One step dévissage

05H3 In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

05H4 **Definition 38.4.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *one step dévissage of* $\mathcal{F}/X/S$ *over* s is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

(1) X, S, Z and Y are affine,
(2) i is a closed immersion of finite presentation,
(3) $\mathcal{F} \cong i_*\mathcal{G}$,
(4) π is finite, and
(5) the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\mathrm{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [**GR71**] the authors use consistently

the setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it "makes sense" to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it "does not make sense". Please see More on Morphisms, Section 37.58 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$ being "locally of finite presentation relative to S", and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 38.3.3 and the observations in Remark 38.6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

05H5 **Definition 38.4.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

(1) $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}_s))$,
(2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
(3) we have $\pi^{-1}(\{y\}) = \{z\}$,
(4) the extension $\kappa(y)/\kappa(s)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\mathrm{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 38.4.1.

05H6 **Lemma 38.4.3.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ {\scriptstyle f}\downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 38.4.2 and Lemma 38.3.2. □

05H7 **Lemma 38.4.4.** *Let S, X, $\mathcal{F}$, s be as in Definition 38.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. Fibre products of affines are affine, see Schemes, Lemma 26.17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 29.2.4, 29.21.4, 29.44.6, 29.34.5, 29.29.2, and More on Morphisms, Lemma 37.27.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because pushforward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 30.5.1. We have $\dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}'_{s'}))$

by Morphisms, Lemma 29.28.3 because

$$\mathrm{Supp}(\mathcal{F}_s) \times_s s' = \mathrm{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

05H8 **Lemma 38.4.5.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 38.4.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. By Lemma 38.4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) – (4) of Definition 38.4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. □

05H9 **Definition 38.4.6.** Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

05HA **Lemma 38.4.7.** *With assumption and notation as in Definition 38.4.6 we have:*

05HB (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*

05HC (2) *Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.*

05HD (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 38.4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 10.17.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 29.34.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 37.47.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. □

05HE **Lemma 38.4.8.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let*

$$\begin{array}{ccc} (Y, y) & \longleftarrow & (Y', y') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{array}{ccccc} & & (X'', x'') & \longleftarrow & (Z'', z'') \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ (X, x) & \longleftarrow & (Z, z) & (S'', s'') & \longleftarrow & (Y'', y'') \\ \downarrow & & \downarrow & \swarrow & \swarrow \\ (S, s) & \longleftarrow & (Y, y) & & \end{array}$$

of pointed schemes with the following properties:

1. *$(S'', s'') \to (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \to S$ is the composition $S'' \to S' \to S$,*
2. *Y'' is an open subscheme of $Y' \times_{S'} S''$,*
3. *$Z'' = Z \times_Y Y''$,*
4. *$(X'', x'') \to (X, x)$ is an elementary étale neighbourhood, and*
5. *$(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf $\mathcal{F}''$.*

Here $\mathcal{F}''$ (resp. $\mathcal{G}''$) is the pullback of $\mathcal{F}$ (resp. $\mathcal{G}$) via the morphism $X'' \to X$ (resp. $Z'' \to Z$) and $i'' : Z'' \to X''$ and $\pi'' : Z'' \to Y''$ are as in the diagram.

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 38.2.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_*\mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 30.5.1. Set $n = \dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the

point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\mathrm{Supp}(\mathcal{F}''_{s''})) = \dim(\mathrm{Supp}(\mathcal{F}_s)) = n = \dim_x(\mathrm{Supp}(\mathcal{F}_s)) = \dim_{x''}(\mathrm{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 35.21.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 38.4.1 except we have not verified that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 33.7.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 37.46.4 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. □

05HF **Lemma 38.4.9.** *Let S, X, $\mathcal{F}$, s be as in Definition 38.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_*\mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_*\mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(\mathit{Supp}(\mathrm{Coker}(\alpha)_s)) < \dim(\mathit{Supp}(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \mathrm{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\mathrm{Coker}(\alpha)$ the last assertion of the lemma holds. □

38.5. Complete dévissage

05HG In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

05HH **Definition 38.5.1.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *complete dévissage of $\mathcal{F}/X/S$ over s* is given by a diagram

$$\begin{array}{ccccccccc} X & \xleftarrow[i_1]{} & Z_1 & & & & & & \\ & & \downarrow{\scriptstyle \pi_1} & & & & & & \\ & & Y_1 & \xleftarrow[i_2]{} & Z_2 & & & & \\ & & & & \downarrow{\scriptstyle \pi_2} & & & & \\ & & & & Y_2 & \xleftarrow{} & Z_3 & & \\ & & & & & & \downarrow & & \\ & & & & & & \ldots & \xleftarrow{} & \ldots \\ & & & & & & & & \downarrow \\ & & & & & & & & Y_n \end{array}$$

of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,
(2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,
(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,
(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

05HI **Definition 38.5.2.** Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete dévissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,
(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

05HJ **Lemma 38.5.3.** *Let S, X, $\mathcal{F}$, s be as in Definition 38.5.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z_k', Y_k', i_k', \pi_k'$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}_k'$ be*

the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.

Proof. By Lemma 38.4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 38.5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 30.5.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

05HK **Lemma 38.5.4.** *Let S, X, $\mathcal{F}$, x, s be as in Definition 38.5.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 38.5.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. Combine Lemma 38.5.3 and Lemma 38.4.5. □

05HL **Definition 38.5.5.** Let S, X, $\mathcal{F}$, x, s be as in Definition 38.5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

05HM **Lemma 38.5.6.** *With assumption and notation as in Definition 38.5.5 we have:*

05HN (1) *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*

05HP (2) *Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*

05HQ (3) *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 38.5.4 and 38.4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 38.4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\mathrm{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\mathrm{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 38.4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one

step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y_1'', Z_k'', and Y_k'' of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathrm{Coker}(\alpha_1)/Y_1/S$ at x such that $Y_1'' \subset Y_1'$. Using Lemma 38.4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \ldots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. □

05HR **Proposition 38.5.7.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. By Lemma 38.4.3 there exists a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 38.4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\mathrm{Supp}(\mathrm{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 38.4.7 (2) and assume that $\mathrm{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

Assume now that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

05HS (38.5.7.1)
$$\begin{array}{ccc} (Y_1, y_1) & \xleftarrow{h} & (Y_1', y_1') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^* \operatorname{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 38.4.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (38.5.7.1). We obtain a diagram

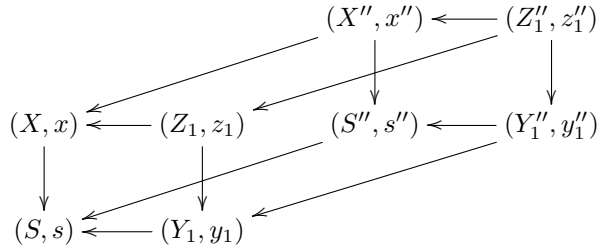

with all the properties as listed in the referenced lemma. In particular $Y_1'' \subset Y_1' \times_{S'} S''$. Set $X_1 = Y_1' \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\operatorname{Coker}(\alpha_1)$. By Lemma 38.5.4 the system

05HT (38.5.7.2) $$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i_k'', \pi_k'', \mathcal{G}_k'', \alpha_k'', z_k'', y_k'')_{k=2,\ldots,n}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \operatorname{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 38.4.7 and 38.5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 38.5.6 (3) to find a standard shrinking $S', X_1', Z_2', Y_2', \ldots, Y_n'$ of the datum (d) so that $X_1' \subset Y_1$. Note that X_1' is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 38.4.7 (1) and then we shrink the result to S'', X'', Z_1'', Y_1'' using Lemma 38.4.7 (2) such that eventually $Y_1'' = X_1' \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z_1'', Y_1'', Z_2' \times_{S'} S'', Y_2' \times_{S'} S'', \ldots, Y_n' \times_{S'} S''$$

gives the complete dévissage we were looking for. □

Some more bookkeeping gives the following consequence.

05HU **Lemma 38.5.8.** *Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_i} & (X'_i, x'_i) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S'_i, s'_i) \end{array} \qquad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \ldots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S'. By Lemma 38.5.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. □

38.6. Translation into algebra

05HV It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

05HW **Definition 38.6.1.** Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. A *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$* is given by a commutative diagram of rings and accompanying primes

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

05HX **Definition 38.6.2.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{r}$ be a prime of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ over $\mathfrak{r}$* is given by R-algebra maps

$$\begin{array}{ccccccccccc} & & A_1 & & A_2 & & \ldots & & A_n & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\ S & & & & B_1 & & \ldots & & \ldots & & B_n \end{array}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,

(5) $N \cong M_1$ as S-modules,
(6) $\mathrm{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
(8) $\mathrm{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak{r}$.

05HY **Remark 38.6.3.** Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 10.6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.36.23. If M_1 is finitely presented, so is $\mathrm{Coker}(\alpha_1)$ (see Algebra, Lemma 10.5.3) and hence M_2, etc.

05HZ **Definition 38.6.4.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{r}$ of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ at $\mathfrak{q}$* is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak{r}$ and prime ideals $\mathfrak{q}_i \subset B_i$ lying over $\mathfrak{r}$ such that

(1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak{q}'_i \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
(3) $\mathfrak{q} = \mathfrak{q}'_1 \cap S$ and $\mathfrak{q}_i = \mathfrak{q}'_{i+1} \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\mathrm{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

05I0 **Remark 38.6.5.** Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak{q}$ be a prime of B lying over the prime $\mathfrak{r}$ of A. Set $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{N}$ on X. Let x be the point corresponding to $\mathfrak{q}$ and let $s \in S$ be the point corresponding to $\mathfrak{p}$. Then

(1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak{p}$, and
(2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak{q}$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of "a complete dévissage of $N/B/A$ over $\mathfrak{p}$" which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak{q}$ does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

05I1 **Lemma 38.6.6.** *Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak{q}$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \to S$ at $\mathfrak{q}$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak{q}'$.*

Proof. This is a reformulation of Proposition 38.5.7 via Remark 38.6.5 $\square$

38.7. Localization and universally injective maps

05DD

05DE **Lemma 38.7.1.** *Let $R \to S$ be a ring map. Let N be a S-module. Assume*

(1) *R is a local ring with maximal ideal $\mathfrak{m}$,*
(2) *$\overline{S} = S/\mathfrak{m}S$ is Noetherian, and*
(3) *$\overline{N} = N/\mathfrak{m}_R N$ is a finite $\overline{S}$-module.*

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained in an element of $Ass_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\mathrm{Ass}_S(\overline{N}) = \mathrm{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 10.63.14. This is a finite set by Algebra, Lemma 10.63.5. Say $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \mathrm{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 10.63.9. By the description of $\mathrm{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 10.17.5 and by Algebra, Lemma 10.15.2 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak{q}_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. This proves the lemma. □

05DF **Lemma 38.7.2.** *Assumption and notation as in Lemma 38.7.1. Assume moreover that*

(1) *S is local and $R \to S$ is a local homomorphism,*
(2) *S is essentially of finite presentation over R,*
(3) *N is finitely presented over S, and*
(4) *N is flat over R.*

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 10.128.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 10.39.12. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

05DG **Lemma 38.7.3.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

(1) *$R \to S$ is a local homomorphism of local rings*
(2) *S is essentially of finite presentation over R,*
(3) *N is of finite presentation over S,*
(4) *N is flat over R,*
(5) *$S \to S'$ is flat, and*
(6) *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & N' \\ \downarrow & & \downarrow \\ \Sigma^{-1}N & \longrightarrow & \Sigma^{-1}N' \end{array}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 10.82.10).

By Lemma 38.7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\mathrm{Spec}(\Sigma^{-1}S') \to \mathrm{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 10.39.16 the ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 10.82.11 and 10.82.8. Finally, we apply Lemma 38.7.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 10.82.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. □

05DH **Lemma 38.7.4.** *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and N is of finite presentation over S,*
(2) *N is flat over R,*
(3) *$S \to S'$ is flat, and*
(4) *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ such that $\mathfrak{q}$ is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. By Algebra, Lemma 10.82.12 it suffices to show that $N_\mathfrak{q} \to (N \otimes_R S')_\mathfrak{q}$ is a $R_\mathfrak{p}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 38.7.3 to the ring maps $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_\mathfrak{q}$ and the module $N_\mathfrak{q}$. □

The reader may want to compare the following lemma to Algebra, Lemmas 10.99.1 and 10.128.4 and the results of Section 38.25. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

05FQ **Lemma 38.7.5.** *Let $(R, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be an R-module map. If M is a projective R-module, N is a flat R-module, and $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is universally injective.*

Proof. By Algebra, Theorem 10.85.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = \mathrm{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 10.81.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. □

05FR **Lemma 38.7.6.** *Assumption and notation as in Lemma 38.7.1. Assume moreover that N is projective as an R-module. Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 38.7.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

38.8. Completion and Mittag-Leffler modules

05DI

05DJ **Lemma 38.8.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. The completion $(\bigoplus_{\alpha\in A} R)^\wedge$ is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 15.27.1 the map $(\bigoplus_{\alpha\in A} R)^\wedge \to \prod_{\alpha\in A} R$ is universally injective. Thus, by Algebra, Lemmas 10.82.7 and 10.89.7 it suffices to show that $\prod_{\alpha\in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 10.90.6 (and Algebra, Lemma 10.90.5) we see that $\prod_{\alpha\in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 10.91.3. □

05DK **Lemma 38.8.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *M is flat over R, and*
(3) *M/IM is a projective R/I-module.*

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 10.97.9 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 38.8.1. □

In Lemmas 38.8.3 and 38.8.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 10.31.1.

05DL **Lemma 38.8.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module, and*
(4) *for any finite R-module Q, any $\mathfrak{q} \in Ass_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 10.82.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \mathrm{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak{q} = 0$ for $\mathfrak{q} \in \mathrm{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q}\in\mathrm{Ass}(N)} N_\mathfrak{q}$, see Algebra, Lemma 10.63.19. Pick $\mathfrak{q} \in \mathrm{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 10.51.4. □

05DM **Lemma 38.8.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module,*
(4) *N is flat over R, and*
(5) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. This follows from Lemma 38.8.3 because Algebra, Lemma 10.65.5 and Remark 10.65.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. □

38.9. Projective modules

05DN The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 38.9.2.

05DP **Lemma 38.9.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *$R \to S$ is of finite type,*
(3) *N is a finite S-module,*
(4) *N is flat over R,*
(5) *N/IN is projective as a R/I-module, and*
(6) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then N is projective as an R-module.

Proof. By Lemma 38.8.4 the map $N \to N^\wedge$ is universally injective. By Lemma 38.8.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 10.89.7 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 10.93.1. □

05FS **Lemma 38.9.2.** *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type and flat, and*
(3) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 10.77.7 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is

a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 10.118.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 10.118.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = \lim R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 10.97.2. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 10.95.6 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 38.9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak{p}^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak{p} \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak{p}^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the Jacobson radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). □

05FT **Lemma 38.9.3.** *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and flat, and*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 37.34.4, 37.34.6, 37.34.7, and 37.34.11. By Lemma 38.9.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 10.94.1. □

05FU **Remark 38.9.4.** Lemma 38.9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [**GR71**] is [**GR71**, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 38.9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [**GR71**].

(1) First reduce to the case where R is Noetherian as above.

(2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 10.95.6 we may work locally in the fppf topology on R, hence we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.
(3) Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 29.34.20. Here we are using that $R \to S$ is smooth.
(4) To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 10.93.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 10.91.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 10.89.7 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.
(5) Apply Lemma 38.7.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\mathrm{Spec}(S^\wedge) \to \mathrm{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 38.9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

38.10. Flat finite type modules, Part I

05I2 In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true "pointwise". We remark that the first proof of Proposition 38.10.3 uses the geometric results of Section 38.3 but not the existence of a complete dévissage.

05I3 **Lemma 38.10.1.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exist $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_\mathfrak{q}$ is R-flat,*
(2) *α is R-universally injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(3) *α is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(4) *$\alpha_\mathfrak{p}$ is an isomorphism and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and*
(5) *$\alpha_\mathfrak{q}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that

$(S/\mathfrak{m}S)^{\oplus r} \to N_\mathfrak{p}/\mathfrak{m}N_\mathfrak{p} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 38.7.5 and 38.9.3. the map $S^{\oplus r} \to N_\mathfrak{p}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 10.82.10. Then also the localization $\alpha_\mathfrak{q}$ is R-universally injective, see Algebra, Lemma 10.82.13. We conclude that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat by Algebra, Lemma 10.82.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_\mathfrak{p}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 10.20.1) $\alpha_\mathfrak{p}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_\mathfrak{p}$ is an isomorphism, so α is injective as $S_\mathfrak{q} \to S_\mathfrak{p}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 38.7.6 and 38.9.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_\mathfrak{q}$ is R-flat as an extension of flat modules, see Algebra, Lemma 10.39.13. Hence (5) $\Rightarrow$ (1) and the proof is finished. □

05I4 **Lemma 38.10.2.** *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_\mathfrak{q}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism*

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 38.6.3. Note that $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1'}$ where $\mathfrak{q}_1'$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}_1'} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 10.41.11, so (c) $(M_1)_{\mathfrak{q}_1'} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 38.10.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\mathrm{Coker}(\alpha_1)_\mathfrak{q}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\mathrm{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 10.5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 38.9.3 hence free as an R-module by Algebra, Theorem 10.85.4. This finishes the proof. □

05I5 **Proposition 38.10.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the proposition by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

(1) elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
(2) a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 38.3.2. We are going to replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{array}{ccccc} X_{\mathcal{O}_{S',s'}} & \xleftarrow{g} & X'_{\mathcal{O}_{S',s'}} & \xleftarrow{i} & Z'_{\mathcal{O}_{S',s'}} \\ & \searrow^{f} & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y'_{\mathcal{O}_{S',s'}} \\ & & & \swarrow_{h} & \\ & & \mathrm{Spec}(\mathcal{O}_{S',s'}) & & \end{array}$$

Here we have base changed the schemes X', Z', Y' over S' via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 38.2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 38.3.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 38.3.3 and 38.3.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 35.25.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 10.78.5 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over $\mathfrak{m}$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Lemma 38.10.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_\mathfrak{q}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is A-flat. Set $Q = \mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain $\mathfrak{p}$. Hence it is certainly the case that $\dim_\mathfrak{q}(\mathrm{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S',s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A, see Algebra, Lemma 10.39.13, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 10.5.3. □

Second proof. We apply Proposition 38.5.7 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 29.25.13 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 38.2.3 we see that it is flat at x' over S'. Via Remark 38.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S',s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 38.10.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$$

is flat over $\mathcal{O}_{S',s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation and flat over $\mathcal{O}_{S',s'}$. Since the morphism $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale (Lemma 38.2.2) its image $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S',s'}$. (Results used: Morphisms, Lemma 29.36.13, Descent, Lemma 35.7.3, and Morphisms, Lemma 29.25.13.) □

05M9 **Lemma 38.10.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 38.10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 29.25.13. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

05KT **Lemma 38.10.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 38.10.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 38.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05KU **Lemma 38.10.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 38.10.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may

view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

05KV **Lemma 38.10.7.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 38.10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 38.10.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

05I6 **Lemma 38.10.8.** *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme*

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that $V \to \operatorname{Spec}(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply

Proposition 38.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05I7 **Lemma 38.10.9.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 38.10.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 10.83.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 10.39.17). □

05I8 **Lemma 38.10.10.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 38.10.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. □

38.11. Extending properties from an open

0B47 In this section we collect a number of results of the form: If $f : X \to S$ is a flat morphism of schemes and f satisfies some property over a dense open of S, then f satisfies the same property over all of S.

081N **Lemma 38.11.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset S$ be open. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type and flat over S,*
3. *$U \subset S$ is retrocompact and scheme theoretically dense,*
4. *$\mathcal{F}|_{f^{-1}U}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. The problem is local on X and S, hence we may assume X and S affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F}$ is the quasi-coherent sheaf associated to N. We have $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.29.1. As U is schematically dense the map $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 38.10.9 the module $N_\mathfrak{q}$ is of finite presentation over $B_\mathfrak{q}$. Choose a surjection $\varphi : B^{\oplus m} \to N$ of B-modules.

Choose $k_1, \ldots, k_t \in \mathrm{Ker}(\varphi)$ and set $N' = B^{\oplus m}/\sum Bk_j$. There is a canonical surjection $N' \to N$ and N is the filtered colimit of the B-modules N' constructed in this manner. Thus we see that we can choose $k_1, \ldots, k_t$ such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \ldots, n$ and (b) $N'_{\mathfrak{q}} \cong N_{\mathfrak{q}}$. This in particular implies that $N'_{\mathfrak{q}}$ is flat over A. By openness of flatness, see Algebra, Theorem 10.129.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak{q}$ such that N'_g is flat over A. Consider the commutative diagram

$$\begin{array}{ccc} N'_g & \longrightarrow & N_g \\ \downarrow & & \downarrow \\ \prod N'_{gf_i} & \longrightarrow & \prod N_{gf_i} \end{array}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $A \to \prod A_{f_i}$ is injective and N'_g is flat over A. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g. We conclude by applying Algebra, Lemma 10.23.2. □

081P **Lemma 38.11.2.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset S$ be open. Assume*

(1) *f is locally of finite type and flat,*
(2) *$U \subset S$ is retrocompact and scheme theoretically dense,*
(3) *$f|_{f^{-1}U} : f^{-1}U \to U$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and apply Lemma 38.11.1 to $i_*\mathcal{O}_X$. Some details omitted. □

081L **Lemma 38.11.3.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite type. Let $U \subset S$ be a dense open such that $X_U \to U$ has relative dimension $\leq e$, see Morphisms, Definition 29.29.1. If also either*

(1) *f is locally of finite presentation, or*
(2) *$U \subset S$ is retrocompact,*

then f has relative dimension $\leq e$.

Proof. Proof in case (1). Let $W \subset X$ be the open subscheme constructed and studied in More on Morphisms, Lemmas 37.22.7 and 37.22.9. Note that every generic point of every fibre is contained in W, hence it suffices to prove the result for W. Since $W = \bigcup_{d \geq 0} U_d$, it suffices to prove that $U_d = \emptyset$ for $d > e$. Since f is flat and locally of finite presentation it is open hence $f(U_d)$ is open (Morphisms, Lemma 29.25.10). Thus if U_d is not empty, then $f(U_d) \cap U \neq \emptyset$ as desired.

Proof in case (2). We may replace S by its reduction. Then U is scheme theoretically dense. Hence f is locally of finite presentation by Lemma 38.11.2. In this way we reduce to case (1). □

0B48 **Lemma 38.11.4.** *Let $f : X \to S$ be a morphism of schemes which is flat and proper. Let $U \subset S$ be a dense open such that $X_U \to U$ is finite. If also either f is locally of finite presentation or $U \subset S$ is retrocompact, then f is finite.*

Proof. By Lemma 38.11.3 the fibres of f have dimension zero. Hence f is quasi-finite (Morphisms, Lemma 29.29.5) whence has finite fibres (Morphisms, Lemma 29.20.10). Hence f is finite by More on Morphisms, Lemma 37.44.1. □

081M **Lemma 38.11.5.** *Let $f : X \to S$ be a morphism of schemes and $U \subset S$ an open. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(U) \to U$ is an isomorphism, and*
(3) *$U \subset S$ is retrocompact and scheme theoretically dense,*

then f is an open immersion.

Proof. By Lemma 38.11.2 the morphism f is locally of finite presentation. The image $f(X) \subset S$ is open (Morphisms, Lemma 29.25.10) hence we may replace S by $f(X)$. Thus we have to prove that f is an isomorphism. We may assume S is affine. We can reduce to the case that X is quasi-compact because it suffices to show that any quasi-compact open $X' \subset X$ whose image is S maps isomorphically to S. Thus we may assume f is quasi-compact. All the fibers of f have dimension 0, see Lemma 38.11.3. Hence f is quasi-finite, see Morphisms, Lemma 29.29.5. Let $s \in S$. Choose an elementary étale neighbourhood $g : (T, t) \to (S, s)$ such that $X \times_S T = V \amalg W$ with $V \to T$ finite and $W_t = \emptyset$, see More on Morphisms, Lemma 37.41.6. Denote $\pi : V \amalg W \to T$ the given morphism. Since π is flat and locally of finite presentation, we see that $\pi(V)$ is open in T (Morphisms, Lemma 29.25.10). After shrinking T we may assume that $T = \pi(V)$. Since f is an isomorphism over U we see that π is an isomorphism over $g^{-1}U$. Since $\pi(V) = T$ this implies that $\pi^{-1}g^{-1}U$ is contained in V. By Morphisms, Lemma 29.25.15 we see that $\pi^{-1}g^{-1}U \subset V \amalg W$ is scheme theoretically dense. Hence we deduce that $W = \emptyset$. Thus $X \times_S T = V$ is finite over T. This implies that f is finite (after replacing S by an open neighbourhood of s), for example by Descent, Lemma 35.23.25. Then f is finite locally free (Morphisms, Lemma 29.48.2) and after shrinking S to a smaller open neighbourhood of s we see that f is finite locally free of some degree d (Morphisms, Lemma 29.48.5). But $d = 1$ as is clear from the fact that the degree is 1 over the dense open U. Hence f is an isomorphism. □

38.12. Flat finitely presented modules

05I9 In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

05IA **Lemma 38.12.1.** *Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_{\mathfrak{q}}$ is R-flat,*
(2) *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathrm{Coker}(\alpha)_g$ is R-flat,*

(3) $\alpha_{\mathfrak{r}}$ *is* $R_{\mathfrak{r}}$*-universally injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is* R*-flat*
(4) $\alpha_{\mathfrak{r}}$ *is injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is* R*-flat,*
(5) $\alpha_{\mathfrak{p}}$ *is an isomorphism and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is* R*-flat, and*
(6) $\alpha_{\mathfrak{q}}$ *is injective and* $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ *is* R*-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 38.10.1 to the map $\alpha_{\mathfrak{r}} : S_{\mathfrak{r}}^{\oplus r} \to N_{\mathfrak{r}}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 10.129.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\mathrm{Spec}(S)$, see Algebra, Lemma 10.79.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, see Algebra, Proposition 10.41.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) $\Rightarrow$ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 10.82.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\mathrm{Coker}(\alpha)_{\mathfrak{q}'}$ is R-flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. □

05IB **Lemma 38.12.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak{r}$. Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 38.6.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 38.12.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\mathrm{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 10.82.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, we may assume that $\mathrm{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 38.9.3. □

05IC **Remark 38.12.3.** There is a variant of Lemma 38.12.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak{q}$ lying over $\mathfrak{r}$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* $\mathfrak{q}$. Compare with Lemma 38.10.2.

The following is the main result of this section.

05ID **Proposition 38.12.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. By openness of flatness, see More on Morphisms, Theorem 37.15.1 we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is flat over S. Next, we apply Proposition 38.5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s'. (In particular S' and X' are affine.) By Morphisms, Lemma 29.25.13 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 38.2.3 we see that it is flat over S'. Via Remark 38.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 38.12.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. □

In the rest of this section we prove a number of variants on this result. The first is a "global" version.

05KW **Lemma 38.12.5.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. For every point $x \in X_s$ we can use Proposition 38.12.4 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow[g_x]{} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S_x, s_x) \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 10.94.1. □

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

05IE **Lemma 38.12.6.** *Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 38.12.4. □

05KX **Lemma 38.12.7.** *Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow[g]{} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 38.12.5. □

The following lemmas explain consequences of Proposition 38.12.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

05KY **Lemma 38.12.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) f *is locally of finite presentation,*
(2) $\mathcal{F}$ *is of finite type, and*
(3) $\mathcal{F}$ *is flat at* x *over* S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ *and a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ *is étale,* $\kappa(x) = \kappa(x')$*, the scheme* X' *is affine of finite presentation over* $\mathcal{O}_{S',s'}$*, the sheaf* $g^*\mathcal{F}$ *is of finite presentation over* $\mathcal{O}_{X'}$*, and such that* $\Gamma(X', g^*\mathcal{F})$ *is a free* $\mathcal{O}_{S',s'}$*-module.*

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Hence by Proposition 38.10.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 38.12.4 because Algebra, Theorem 10.85.4 assures us that projective = free over a local ring. □

05KZ **Lemma 38.12.9.** *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* X*. Let* $x \in X$ *with image* $s \in S$*. Assume that*

(1) f *is locally of finite type,*
(2) $\mathcal{F}$ *is of finite type, and*
(3) $\mathcal{F}$ *is flat at* x *over* S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ *and a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ *is étale,* $\kappa(x) = \kappa(x')$*, the scheme* X' *is affine, and such that* $\Gamma(X', g^*\mathcal{F})$ *is a free* $\mathcal{O}_{S',s'}$*-module.*

Proof. (The only difference with Lemma 38.12.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 38.12.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (\mathbf{A}^n_S, t) & \xleftarrow{h} & (Y, y) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. □

05L0 **Lemma 38.12.10.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 38.12.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x,s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. □

05L1 **Lemma 38.12.11.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 38.12.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 38.12.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. □

38.13. Flat finite type modules, Part II

05IF We will need the following lemma.

0CU6 **Lemma 38.13.1.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Set $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$. Then we can find a $g \in S$ with $g \notin \mathfrak{q}$ such that $\overline{g} \in \mathfrak{r}$ for all $\mathfrak{r} \in Ass_{\overline{S}}(\overline{N})$ such that $\mathfrak{r} \not\subset \overline{\mathfrak{q}}$.*

Proof. Namely, if $\mathrm{Ass}_{\overline{S}}(\overline{N}) = \{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$ (finiteness by Algebra, Lemma 10.63.5), then after renumbering we may assume that

$$\mathfrak{r}_1 \subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_r \subset \overline{\mathfrak{q}}, \quad \mathfrak{r}_{r+1} \not\subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_n \not\subset \overline{\mathfrak{q}}$$

Since $\overline{\mathfrak{q}}$ is a prime ideal we see that the product $\mathfrak{r}_{r+1} \ldots \mathfrak{r}_n$ is not contained in $\overline{\mathfrak{q}}$ and hence we can pick an element a of $\overline{S}$ contained in $\mathfrak{r}_{r+1}, \ldots, \mathfrak{r}_n$ but not in $\overline{\mathfrak{q}}$. If there exists $g \in S$ mapping to a, then g works. In general we can find a nonzero element $\lambda \in \kappa(\mathfrak{p})$ such that λa is the image of a $g \in S$. □

The following lemma has a slightly stronger variant Lemma 38.13.4 below.

05IG **Lemma 38.13.2.** *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module which is flat as an R-module. Let M be an R-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Then*

$$\mathfrak{q} \in \mathit{WeakAss}_S(M \otimes_R N) \Leftrightarrow \Big(\mathfrak{p} \in \mathit{WeakAss}_R(M) \text{ and } \overline{\mathfrak{q}} \in \mathit{Ass}_{\overline{S}}(\overline{N})\Big)$$

Here $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$.

Proof. Pick $g \in S$ as in Lemma 38.13.1. Apply Proposition 38.12.4 to the morphism of schemes $\mathrm{Spec}(S_g) \to \mathrm{Spec}(R)$, the quasi-coherent module associated to N_g, and the points corresponding to the primes $\mathfrak{q}S_g$ and $\mathfrak{p}$. Translating into algebra we obtain a commutative diagram of rings

$$\begin{array}{ccc} S & \longrightarrow & S_g \longrightarrow S' \\ & \nwarrow & \uparrow \quad \uparrow \\ & & R \longrightarrow R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}S_g \longrightarrow \mathfrak{q}' \\ & \searrow & \downarrow \quad \downarrow \\ & & \mathfrak{p} \longrightarrow \mathfrak{p}' \end{array}$$

endowed with primes as shown, the horizontal arrows are étale, and $N \otimes_S S'$ is projective as an R'-module. Set $N' = N \otimes_S S'$, $M' = M \otimes_R R'$, $\overline{S}' = S' \otimes_{R'} \kappa(\mathfrak{q}')$, $\overline{\mathfrak{q}}' = \mathfrak{q}'\overline{S}'$, and

$$\overline{N}' = N' \otimes_{R'} \kappa(\mathfrak{p}') = \overline{N} \otimes_{\overline{S}} \overline{S}'$$

By Lemma 38.2.8 we have

$$\begin{aligned} \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') &= (\mathrm{Spec}(S') \to \mathrm{Spec}(S))^{-1}\mathrm{WeakAss}_S(M \otimes_R N) \\ \mathrm{WeakAss}_{R'}(M') &= (\mathrm{Spec}(R') \to \mathrm{Spec}(R))^{-1}\mathrm{WeakAss}_R(M) \\ \mathrm{Ass}_{\overline{S}'}(\overline{N}') &= (\mathrm{Spec}(\overline{S}') \to \mathrm{Spec}(\overline{S}))^{-1}\mathrm{Ass}_{\overline{S}}(\overline{N}) \end{aligned}$$

Use Algebra, Lemma 10.66.9 for $\overline{N}$ and $\overline{N}'$. In particular we have

$$\begin{aligned} \mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N) &\Leftrightarrow \mathfrak{q}' \in \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N') \\ \mathfrak{p} \in \mathrm{WeakAss}_R(M) &\Leftrightarrow \mathfrak{p}' \in \mathrm{WeakAss}_{R'}(M') \\ \overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N}) &\Leftrightarrow \overline{\mathfrak{q}}' \in \mathrm{WeakAss}_{\overline{S}'}(\overline{N}') \end{aligned}$$

Our careful choice of g and the formula for $\mathrm{Ass}_{\overline{S}'}(\overline{N}')$ above shows that

0CU7 (38.13.2.1) $$\text{if } \mathfrak{r}' \in \mathrm{Ass}_{\overline{S}'}(\overline{N}') \text{ lies over } \mathfrak{r} \subset \overline{S} \text{ then } \mathfrak{r} \subset \overline{\mathfrak{q}}$$

This will be a key observation later in the proof. We will use the characterization of weakly associated primes given in Algebra, Lemma 10.66.2 without further mention.

Suppose that $\overline{\mathfrak{q}} \notin \mathrm{Ass}_{\overline{S}}(\overline{N})$. Then $\overline{\mathfrak{q}}' \notin \mathrm{Ass}_{\overline{S}'}(\overline{N}')$. By Algebra, Lemmas 10.63.9, 10.63.5, and 10.15.2 there exists an element $\overline{a}' \in \overline{\mathfrak{q}}'$ which is not a zerodivisor on $\overline{N}'$. After replacing $\overline{a}'$ by $\lambda\overline{a}'$ for some nonzero $\lambda \in \kappa(\mathfrak{p})$ we can find $a' \in \mathfrak{q}'$ mapping to $\overline{a}'$. By Lemma 38.7.6 the map $a' : N'_{\mathfrak{p}'} \to N'_{\mathfrak{p}'}$ is $R'_{\mathfrak{p}'}$-universally injective. In particular we see that $a' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizing at $\mathfrak{p}'$ and hence after localizing at $\mathfrak{q}'$. Clearly this implies that $\mathfrak{q}' \notin \mathrm{WeakAss}_{S'}(M' \otimes_{R'} N')$. We conclude that $\mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N)$. We want to show $\mathfrak{p} \in \mathrm{WeakAss}_S(M)$. Let $z \in M \otimes_R N$ be an element such that $\mathfrak{q}$ is minimal over $J = \mathrm{Ann}_S(z)$. Let $f_i \in \mathfrak{p}$,

$i \in I$ be a set of generators of the ideal $\mathfrak{p}$. Since $\mathfrak{q}$ lies over $\mathfrak{p}$, for every i we can choose an $n_i \geq 1$ and $g_i \in S$, $g_i \notin \mathfrak{q}$ with $g_i f_i^{n_i} \in J$, i.e., $g_i f_i^{n_i} z = 0$. Let $z' \in (M' \otimes_{R'} N')_{\mathfrak{p}'}$ be the image of z. Observe that z' is nonzero because z has nonzero image in $(M \otimes_R N)_{\mathfrak{q}}$ and because $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ is faithfully flat. We claim that $f_i^{n_i} z' = 0$.

Proof of the claim: Let $g_i' \in S'$ be the image of g_i. By the key observation (38.13.2.1) we find that the image $\overline{g}_i' \in \overline{S}'$ is not contained in $\mathfrak{r}'$ for any $\mathfrak{r}' \in \mathrm{Ass}_{\overline{S}'}(\overline{N})$. Hence by Lemma 38.7.6 we see that $g_i' : N'_{\mathfrak{p}'} \to N'_{\mathfrak{p}'}$ is $R'_{\mathfrak{p}'}$-universally injective. In particular we see that $g_i' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizating at $\mathfrak{p}'$. The claim follows because $g_i f_i^{n_i} z' = 0$.

Our claim shows that the annihilator of z' in $R'_{\mathfrak{p}'}$ contains the elements $f_i^{n_i}$. As $R \to R'$ is étale we have $\mathfrak{p}' R'_{\mathfrak{p}'} = \mathfrak{p} R'_{\mathfrak{p}'}$ by Algebra, Lemma 10.143.5. Hence the annihilator of z' in $R'_{\mathfrak{p}'}$ has radical equal to $\mathfrak{p}' R_{\mathfrak{p}'}$ (here we use z' is not zero). On the other hand

$$z' \in (M' \otimes_{R'} N')_{\mathfrak{p}'} = M'_{\mathfrak{p}'} \otimes_{R'_{\mathfrak{p}'}} N'_{\mathfrak{p}'}$$

The module $N'_{\mathfrak{p}'}$ is projective over the local ring $R'_{\mathfrak{p}'}$ and hence free (Algebra, Theorem 10.85.4). Thus we can find a finite free direct summand $F' \subset N'_{\mathfrak{p}'}$ such that $z' \in M'_{\mathfrak{p}'} \otimes_{R'_{\mathfrak{p}'}} F'$. If F' has rank n, then we deduce that $\mathfrak{p}' R'_{\mathfrak{p}'} \in \mathrm{WeakAss}_{R'_{\mathfrak{p}'}}(M'^{\oplus n}_{\mathfrak{p}'})$. This implies $\mathfrak{p}' R'_{\mathfrak{p}'} \in \mathrm{WeakAss}(M'_{\mathfrak{p}'})$ for example by Algebra, Lemma 10.66.4. Then $\mathfrak{p}' \in \mathrm{WeakAss}_{R'}(M')$ which in turn gives $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$. This finishes the proof of the implication "$\Rightarrow$" of the equivalence of the lemma.

Assume that $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$ and $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$. We want to show that $\mathfrak{q}$ is weakly associated to $M \otimes_R N$. Note that $\overline{\mathfrak{q}}'$ is a maximal element of $\mathrm{Ass}_{\overline{S}'}(\overline{N}')$. This is a consequence of (38.13.2.1) and the fact that there are no inclusions among the primes of $\overline{S}'$ lying over $\overline{\mathfrak{q}}$ (as fibres of étale morphisms are discrete Morphisms, Lemma 29.36.7). Thus, after replacing $R, S, \mathfrak{p}, \mathfrak{q}, M, N$ by $R', S', \mathfrak{p}', \mathfrak{q}', M', N'$ we may assume, in addition to the assumptions of the lemma, that

(1) $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$,
(2) $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$,
(3) N is projective as an R-module, and
(4) $\overline{\mathfrak{q}}$ is maximal in $\mathrm{Ass}_{\overline{S}}(\overline{N})$.

There is one more reduction, namely, we may replace R, S, M, N by their localizations at $\mathfrak{p}$. This leads to one more condition, namely,

(5) R is a local ring with maximal ideal $\mathfrak{p}$.

We will finish by showing that (1) – (5) imply $\mathfrak{q} \in \mathrm{WeakAss}(M \otimes_R N)$.

Since R is local and $\mathfrak{p} \in \mathrm{WeakAss}_R(M)$ we can pick a $y \in M$ whose annihilator I has radical equal to $\mathfrak{p}$. Write $\overline{\mathfrak{q}} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose $g_i \in S$ mapping to $\overline{g}_i$. Then $\mathfrak{q} = \mathfrak{p} S + g_1 S + \ldots + g_n S$. Consider the map

$$\Psi : N/IN \longrightarrow (N/IN)^{\oplus n}, \quad z \longmapsto (g_1 z, \ldots, g_n z).$$

This is a homomorphism of projective R/I-modules. The local ring R/I is autoassociated (More on Algebra, Definition 15.15.1) as $\mathfrak{p}/I$ is locally nilpotent. The map $\Psi \otimes \kappa(\mathfrak{p})$ is not injective, because $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$. Hence More on Algebra, Lemma 15.15.4 implies Ψ is not injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator $J = \mathrm{Ann}_S(z)$ contains IS and g_i by construction. Thus

$\sqrt{J} \subset S$ contains $\mathfrak{q}$. Let $\mathfrak{s} \subset S$ be a prime minimal over J. Then $\mathfrak{q} \subset \mathfrak{s}$, $\mathfrak{s}$ lies over $\mathfrak{p}$, and $\mathfrak{s} \in \text{WeakAss}_S(N/IN)$. The last fact by definition of weakly associated primes. Apply the "$\Rightarrow$" part of the lemma (which we've already proven) to the ring map $R \to S$ and the modules R/I and N to conclude that $\overline{\mathfrak{s}} \in \text{Ass}_{\overline{S}}(\overline{N})$. Since $\overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}$ the maximality of $\overline{\mathfrak{q}}$, see condition (4) above, implies that $\overline{\mathfrak{q}} = \overline{\mathfrak{s}}$. This shows that $\mathfrak{q} = \mathfrak{s}$ and we conlude what we want. □

05IH **Lemma 38.13.3.** *Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. If $\mathcal{F}$ is flat at x over S, then*

$$x \in \textit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in \textit{WeakAss}_S(\mathcal{G}) \textit{ and } x \in \textit{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. In this paragraph we reduce to f being of finite presentation. The question is local on X and S, hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 31.6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$ (see Algebra, Lemma 10.63.14). Hence it suffices to prove the lemma in case $X = \mathbf{A}^n_S$. Thus we may assume that $X \to S$ is of finite presentation.

In this paragraph we reduce to $\mathcal{F}$ being of finite presentation and flat over S. Choose an elementary étale neighbourhood $e : (S', s') \to (S, s)$ and an open $V \subset X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 38.10.3. Let $x' \in X' = X \times_S S'$ be the unique point mapping to x and s'. Then it suffices to prove the statement for $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$, see Lemma 38.2.8. Let $v \in V$ the unique point mapping to x' and let $s' \in \text{Spec}(\mathcal{O}_{S',s'})$ be the closed point. Then $\mathcal{O}_{V,v} = \mathcal{O}_{X',x'}$ and $\mathcal{O}_{\text{Spec}(\mathcal{O}_{S',s'}),s'} = \mathcal{O}_{S',s'}$ and similarly for the stalks of pullbacks of $\mathcal{F}$ and $\mathcal{G}$. Also $V_{s'} \subset X'_{s'}$ is an open subscheme. Since the condition of being a weakly associated point depend only on the stalk of the sheaf, we may replace $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$ by $V \to \text{Spec}(\mathcal{O}_{S',s'})$, v, s', $(V \to X)^*\mathcal{F}$, and $(\text{Spec}(\mathcal{O}_{S',s'}) \to S)^*\mathcal{G}$. Thus we may assume that f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S.

Assume f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S. After shrinking X and S to affine neighbourhoods of x and s, this case is handled by Lemma 38.13.2. □

05II **Lemma 38.13.4.** *Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then*

$$\textit{WeakAss}_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in \textit{WeakAss}_R(M)} \textit{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 38.13.3 into algebra. Details of translation omitted. □

05IJ **Lemma 38.13.5.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$WeakAss_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in WeakAss_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 38.13.3. □

05IK **Theorem 38.13.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *$X \to S$ is locally of finite presentation,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
(3) *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F}$ flat at x over $S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 38.10.9 we conclude from Algebra, Lemma 10.126.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 37.15.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 17.9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 10.20.1). By Lemma 38.13.5 we have

$$\mathrm{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \mathrm{WeakAss}(S)} \mathrm{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\mathrm{WeakAss}(S)$ is finite by assumption and since $\mathrm{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 31.2.5 we conclude that $\mathrm{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 10.15.2 we may, after shrinking X once more, assume that $\mathrm{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \mathrm{Ker}(\varphi)$. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 31.5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 31.5.5. □

05IL **Lemma 38.13.7.** *Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $WeakAss_S(S)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ *and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.*

Proof. Follows immediately from Theorem 38.13.6. □

05IM **Lemma 38.13.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 38.13.7 finishes the proof. □

05IN **Lemma 38.13.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 38.13.8. □

053G **Proposition 38.13.10.** *Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module, then M is finitely presented as an S-module.*

Proof. Part (1) is a special case of Lemma 38.13.9. For Part (2) choose a surjection $R[x_1, \ldots, x_n] \to S$. By Lemma 38.13.7 we find that M is finitely presented as an $R[x_1, \ldots, x_n]$-module. We conclude by Algebra, Lemma 10.6.4. □

05IQ **Lemma 38.13.11** (Finite type version of Theorem 38.13.6)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *$X \to S$ is locally of finite type,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
(3) *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 37.58.1).

Proof. The question is local on X and S. Thus we may assume X and S are affine. Then we may choose a closed immersion $i : X \to \mathbf{A}^n_S$. We apply Theorem 38.13.6 to $X' = \mathbf{A}^n_S \to S$ and the quasi-coherent module $\mathcal{F}' = i_*\mathcal{F}$ of finite type and we find that

$$U' = \{x' \in X' \mid \mathcal{F}' \text{ flat at } x' \text{ over } S\}$$

is open in X' and that $\mathcal{F}'|_{U'}$ is of finite presentation. Since $\mathcal{F}'$ restricts to zero on $X' \setminus i(X)$ and since $\mathcal{F}'_{i(x)} \cong \mathcal{F}_x$ for all $x \in X$ we see that

$$U' = i(U) \amalg (X' \setminus i(X))$$

Hence $U = i^{-1}(U')$ is open. Moreover, it is clear that $\mathcal{F}'|_{U'} = (i|_U)_*(\mathcal{F}|_U)$. Hence we conclude that $\mathcal{F}|_U$ is finitely presented relative to S by More on Morphisms, Lemmas 37.58.3 and 37.58.4. □

05IR **Lemma 38.13.12.** *Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\mathrm{WeakAss}_R(R)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 15.81.2).

Proof. This is Lemma 38.13.11 translated into algebra. □

38.14. Examples of relatively pure modules

05IS In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 38.7.4, 38.8.3, 38.8.4, and 38.9.1. See also Algebra, Lemma 10.65.1 for a discussion.

05FV **Lemma 38.14.1.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume*

(1) *N is projective as an R-module, and*
(2) *$S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.*

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 38.7.1 and 38.7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that $\mathfrak{q}$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where $\mathfrak{q}'$ corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. □

The following lemma gives another (slightly silly) example of this phenomenon.

05IT **Lemma 38.14.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.*

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \mathrm{Ann}_S(z)$. Since $z \neq 0$ we see that $\mathrm{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 10.96.6 we win. □

Note that the following lemma gives an alternative proof of Lemma 38.14.1 as a projective module over a local ring is free, see Algebra, Theorem 10.85.4.

05IU **Lemma 38.14.3.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \mathrm{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the equality

$$f' + \sum x_j g'_j = \mathrm{id}_{M_1} \in \mathrm{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that f' is surjective, hence by Algebra, Lemma 10.16.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. □

05IV **Lemma 38.14.4.** *Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. This lemma reduces to Lemma 38.14.3 by Algebra, Lemma 10.153.13. □

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent module on X. Let $\xi \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{array}{ccc} \xi & & Z \longrightarrow X \\ \downarrow & & \searrow \quad \downarrow f \\ f(\xi) & & S \end{array}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 38.14.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 38.14.2, 38.14.3, and 38.14.4.

38.15. Impurities

05IW We want to formalize the phenomenon of which we gave examples in Section 38.14 in terms of specializations of points of $\mathrm{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

05FW **Situation 38.15.1.** Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

05IX (38.15.1.1)
$$\begin{array}{ccc} \xi & & X_T \longrightarrow X \\ \downarrow & & \downarrow \quad \downarrow \\ t' \rightsquigarrow t \longmapsto s & & T \xrightarrow{g} S \end{array}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

05IY **Definition 38.15.2.** In Situation 38.15.1 we say a diagram (38.15.1.1) defines an *impurity of $\mathcal{F}$ above s* if $\xi \in \mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying "let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s".

05FX **Lemma 38.15.3.** *In Situation 38.15.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 32.14.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$\begin{array}{ccccc} X_T & \longrightarrow & X_{T'} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{a} & T' & \xrightarrow{b} & S \end{array}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 31.7.3 we see that $\xi' \in \mathrm{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 37.24.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 29.16.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. □

05IZ **Lemma 38.15.4.** *In Situation 38.15.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over S. Then for some i the triple $(T_i \to S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.*

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t'_i = p_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 31.7.3 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i.

First proof. Let $Z_i = \overline{\{\xi_i\}} \subset X_{T_i}$ and $Z = \overline{\{\xi\}} \subset X_T$ endowed with the reduced induced scheme structure. Then $Z = \lim Z_i$ by Limits, Lemma 32.4.4. Choose a

field k and a morphism $\mathrm{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \mathrm{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \mathrm{Spec}(k) = \lim Z_i \times_{T_i} \mathrm{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \mathrm{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence $Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \mathrm{Spec}(k)$ is empty for some i by Limits, Lemma 32.4.3. Since the image of the composition $\mathrm{Spec}(k) \to T \to T_i$ is t_i we obtain what we want.

Second proof. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 32.14.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 32.4.11 and 32.4.13 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 32.6.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{V_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t_i' \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. □

05J0 **Lemma 38.15.5.** *In Situation 38.15.1. If there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 37.41.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{array}{ccccc} T & \longleftarrow & T \times_S U & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & U & & \end{array}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \to U$ is finite and such that v is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \rightsquigarrow v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 31.7.3 we see that $\zeta \in \mathrm{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \rightsquigarrow v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \rightsquigarrow u$. By Divisors, Lemma 31.7.3 we see that $\theta \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$. Moreover,

as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \rightsquigarrow u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. □

05J1 **Lemma 38.15.6.** *In Situation 38.15.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.*

Proof. We may replace S by an affine neighbourhood of s. By Lemma 38.15.3 we may assume that we have an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 37.25.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 37.24.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 37.52.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \rightsquigarrow t, \xi')$ is the desired impurity. □

In the following we will use the henselization $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ of S at s, see Étale Cohomology, Definition 59.33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

05J2 **Lemma 38.15.7.** *In Situation 38.15.1. If there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s. Say $S = \mathrm{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}^h_{S,s} = \mathrm{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 37.35.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 38.15.4. □

05J3 **Lemma 38.15.8.** *In Situation 38.15.1 the following are equivalent*

1. *there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s where S^h is the henselization of S at s,*
2. *there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, and*
3. *there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $T \to S$ is quasi-finite at t.*

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 38.15.5. We have seen that (1) implies (2) in Lemma 38.15.7. Finally, if $(T \to S, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \rightsquigarrow s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. □

38.16. Relatively pure modules

05BB The notion of a module pure relative to a base was introduced in [**GR71**].

05J4 **Definition 38.16.1.** Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is *pure along* X_s if there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is *universally pure along* X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is *pure along* X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is *universally S-pure*, or *universally pure relative to* S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is *S-pure*, or *pure relative to* S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is *S-pure* or *pure relative to* S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 38.16.2 for a discussion. In the situation of the definition Lemma 38.15.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is pure along X_s.

05J5 **Remark 38.16.2.** Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 38.15.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [**GR71**]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 38.16.5), or if the sheaf is flat (see Lemmas 38.18.3 and 38.18.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 110.40 for a counter example. First we match our usage of "universally" to the usual notion.

05J6 **Lemma 38.16.3.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent*

(1) *$\mathcal{F}$ is universally pure along X_s, and*
(2) *for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. □

05J7 **Lemma 38.16.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 29.20.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 38.16.1. □

05J8 **Lemma 38.16.5.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.*

Proof. First we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 38.15.6 and characterization (2) of purity given in discussion following Definition 38.16.1 to conclude. □

Purity satisfies flat descent.

05J9 **Lemma 38.16.6.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.*

(1) *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.*
(2) *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.*

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 29.25.8, there exists a specialization $t_1' \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 10.41. Choose a point $\xi_1 \in X_{t_1'}$ which corresponds to a generic point of $\mathrm{Spec}(\kappa(t_1') \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 26.17.5. By Divisors, Lemma 31.7.3 we see that $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence $(T_1 \to S', t_1' \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. □

05K1 **Lemma 38.16.7.** *Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.*

Proof. This follows from Divisors, Lemma 31.8.3. □

38.17. Examples of relatively pure sheaves

05K2 Here are some example cases where it is possible to see what purity means.

05K3 **Lemma 38.17.1.** *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.*

(1) *If the support of $\mathcal{F}$ is proper over S, then $\mathcal{F}$ is universally pure relative to S.*
(2) *If f is proper, then $\mathcal{F}$ is universally pure relative to S.*
(3) *If f is proper, then X is universally pure relative to S.*

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, see Morphisms, Section 29.5. In case (1) $Z \to S$ is proper by assumption. Thus by Lemma 38.16.7 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

05K4 **Lemma 38.17.2.** *Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that $Supp(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

(1) *$\mathcal{F}$ is pure relative to S,*
(2) *the scheme theoretic support of $\mathcal{F}$ is finite over S, and*
(3) *$\mathcal{F}$ is universally pure relative to S.*

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 29.20.10. By Lemma 38.16.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\mathrm{Supp}(\mathcal{F}) = X$.

It follows from Lemma 38.17.1 and Morphisms, Lemma 29.44.11 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 37.43.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 26.19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms,

Lemma 37.41.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 29.25.9 and 29.36.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \mathrm{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. □

05K5 **Lemma 38.17.3.** *Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 29.25.9. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\mathrm{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. □

05K6 **Lemma 38.17.4.** *Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 28.21.1. Then $\mathcal{F}$ is universally pure over S.*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 38.14.1. □

38.18. A criterion for purity

05L2 We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

05L3 **Lemma 38.18.1.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in Ass_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in Ass_{X_s}(\mathcal{F}_s)$.*

Proof. Say $x' \rightsquigarrow t$ with $t \in X_s$. Then we can find specializations $x' \rightsquigarrow x \rightsquigarrow t$ with x corresponding to a generic point of an irreducible component of $\overline{\{x'\}} \cap f^{-1}(\{s\})$. By assumption $\mathcal{F}$ is flat over S at x. By More on Morphisms, Lemma 37.18.3 we see that $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ as desired. □

05L4 **Lemma 38.18.2.** *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of morphisms of schemes. Assume

(1) *$\mathcal{F}$ is flat over S at all points of X_s,*
(2) *$X' \to S'$ is of finite type,*
(3) *$g^*\mathcal{F}$ is pure along $X'_{s'}$,*
(4) *$g : X' \to X$ is étale, and*
(5) *$g(X')$ contains $Ass_{X_s}(\mathcal{F}_s)$.*

In this situation $\mathcal{F}$ is pure along X_s if and only if the image of $X' \to X \times_S S'$ contains the points of $Ass_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 38.16.4. Since purity satisfies flat descent, see Lemma 38.16.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 38.18.1 we see that every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 31.7.3 and Remark 31.7.4 the relative assassin $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\mathrm{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 38.16.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \mathrm{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 38.2.8 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. □

05L5 **Lemma 38.18.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *$\mathcal{F}$ is flat over S at all points of X_s, and*

(4) *$\mathcal{F}$ is pure along X_s.*

Then $\mathcal{F}$ is universally pure along X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 38.16.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 38.16.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 38.12.11. Note that $X' \to \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4.

In this situation we apply Lemma 38.18.2 to deduce that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 31.7.3 and Remark 31.7.4 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 38.18.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. □

05L6 **Lemma 38.18.4.** *Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.*

Proof. Immediate consequence of Lemma 38.18.3 and the definitions. □

05MA **Lemma 38.18.5.** *Let I be a directed set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.*

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 37.15.1 this is an open subscheme of X. By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 32.10.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 32.10.2 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 32.4.11 after possibly increasing i we may assume there exists an open $U_i' \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U_i'$. By Limits, Lemma 32.10.4 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U_i'. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 38.12.5 to choose an elementary étale neighbourhood $(S_i', s_i') \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X_i & \xleftarrow{g_i} & X_i' \\ \downarrow & & \downarrow \\ S_i & \longleftarrow & S_i' \end{array}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X_i')$, the schemes X_i', S_i' are affine, and such that $\Gamma(X_i', g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S_i', \mathcal{O}_{S_i'})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S_i', see Lemma 38.17.4. We may base change the diagram above to a diagram with morphisms $(S_{i'}', s_{i'}') \to (S_{i'}, s_{i'})$ and $g_{i'} : X_{i'}' \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W_i' \subset X_i \times_{S_i} S_i'$ equal to the image of the étale morphism $X_i' \to X_i \times_{S_i} S_i'$. For every $i' \geq i$ we have similarly the image $W_{i'}' \subset X_{i'} \times_{S_{i'}} S_{i'}'$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W_{i'}' = W_i' \times_{S_i'} S_{i'}'$ and $W' = W_i \times_{S_i'} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 38.18.2 implies that

05MB (38.18.5.1) $$f^{-1}(\mathrm{Spec}(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W'$$

By More on Morphisms, Lemma 37.25.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S_{i'}' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W_{i'}'\}$$

are locally constructible subsets of S' and $S_{i'}'$. By More on Morphisms, Lemma 37.25.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S_{i'}' \to S_i'$ and that E is the inverse image of E_i under the morphism $S' \to S_i'$. Thus

Equation (38.18.5.1) is equivalent to the assertion that $\mathrm{Spec}(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = \mathrm{colim}_{i' \geq i} \mathcal{O}_{S'_{i'},s'_{i'}}$ we see that $\mathrm{Spec}(\mathcal{O}_{S'_{i'},s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 32.4.10. Then, applying Lemma 38.18.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. □

05MC **Lemma 38.18.6.** *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set*

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 38.12.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 38.17.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S'. By Lemma 38.18.2 we see that $\mathrm{Spec}(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 38.18.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. □

38.19. How purity is used

05L7 Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

05L8 **Lemma 38.19.1.** *Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent*

1. *the map on stalks φ_x is injective for all $x \in Ass_{X_s}(\mathcal{F}_s)$, and*
2. *φ is injective.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 31.5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 31.5.4. As $\mathcal{F}$ is flat we see from Lemma 38.13.5 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 38.18.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. □

05MD **Proposition 38.19.2.** *Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent*

(1) *$f_*\mathcal{F}$ is locally projective on S, and*
(2) *$\mathcal{F}$ is pure relative to S.*

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $\mathrm{Spec}(B)$ is pure relative to $\mathrm{Spec}(A)$.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 38.17.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 38.18.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 35.7.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S. Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 38.12.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{\quad g \quad} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4. Hence by Lemma 38.18.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 38.7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 10.89.7 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 10.93.1. □

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 38.12.4.

05ME **Lemma 38.19.3.** *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 37.15.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property

is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 38.12.5 to arrive at the situation where there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S, see Lemma 38.18.6, whereupon the projectivity follows from Proposition 38.19.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\mathrm{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 38.18.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \mathrm{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 38.2.8 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. □

The following lemma is an improvement of Lemma 38.12.9.

05MF **Lemma 38.19.4.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S',s'}$-module.*

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 38.10.3. In this case we may appeal to Lemma 38.19.3 and Algebra, Theorem 10.85.4 to conclude. □

05U7 **Lemma 38.19.5.** *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is henselian, then N is a filtered colimit*

$$N = \mathrm{colim}_i F_i$$

of free A-modules F_i such that all transition maps $u_i : F_i \to F_{i'}$ of the system induce injective maps $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$. Also, N is a Mittag-Leffler A-module.

Proof. We can find a morphism of finite type $X \to S = \mathrm{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 38.19.4 we see that there exists a

fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. Since the maps $F_i \to F_{i'}$ are A-universally injective (Lemma 38.7.5) it follows that N is Mittag-Leffler by Algebra, Lemma 10.89.9. □

The following lemma should be skipped if reading through for the first time.

0ASX **Lemma 38.19.6.** *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is a valuation ring, then any element of N has a content ideal $I \subset A$ (More on Algebra, Definition 15.24.1). Also, I is a principal ideal.*

Proof. The final statement follows from the fact that I is a finitely generated ideal by More on Algebra, Lemma 15.24.2 and Algebra, Lemma 10.50.15.

Proof of existence of I. Let $A \subset A^h$ be the henselization. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is flat, hence faithfully flat. Let $N' = N \otimes_B B'$. Let $x \in N$ and let $x' \in N'$ be the image. We claim that for an ideal $I \subset A$ we have $x \in IN \Leftrightarrow x' \in IN'$. Namely, $N/IN \to N'/IN'$ is the tensor product of $B \to B'$ with N/IN and $B \to B'$ is universally injective by Algebra, Lemma 10.82.11. By More on Algebra, Lemma 15.124.6 and Algebra, Lemma 10.50.17 the map $A \to A^h$ defines an inclusion preserving bijection $I \mapsto IA^h$ on sets of ideals. We conclude that x has a content ideal in A if and only if x' has a content ideal in A^h. The assertion for $x' \in N'$ follows from Lemma 38.19.5 and Algebra, Lemma 10.89.6. □

An application is the following.

0H2T **Lemma 38.19.7.** *Let $X \to \mathrm{Spec}(R)$ be a proper flat morphism where R is a valuation ring. If the special fibre is reduced, then X and every fibre of $X \to \mathrm{Spec}(R)$ is reduced.*

Proof. Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X is reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $a \in A$ be nonzero and let $I = (\pi) \subset R$ be its content ideal, see Lemma 38.19.6. Then $a = \pi a'$ and a' maps to a nonzero element of $A/\mathfrak{m}A$ where $\mathfrak{m} \subset R$ is the maximal ideal. If a is nilpotent, so is a', because π is a nonzerodivisor by flatness of A over R. But a' maps to a nonzero element of the reduced ring $A/\mathfrak{m}A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show.

Of course, if X is reduced, so is the generic fibre of X over R. If $\mathfrak{p} \subset R$ is a prime ideal, then $R/\mathfrak{p}$ is a valuation ring by Algebra, Lemma 10.50.9. Hence redoing the argument with the base change of X to $R/\mathfrak{p}$ proves the fibre over $\mathfrak{p}$ is reduced. □

38.20. Flattening functors

05MG Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to Sets$ is called limit preserving if for every directed inverse system $\{T_i\}_{i\in I}$ of affine schemes with limit T we have $F(T) = \operatorname{colim}_i F(T_i)$.

05MH **Situation 38.20.1.** Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

(38.20.1.1)

05MI $$F_{iso} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

05MJ **Lemma 38.20.2.** *In Situation 38.20.1.*

(1) *Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.*
(3) *If f is quasi-compact and $\mathcal{F}$ of finite type, then F_{zero} is limit preserving.*
(4) *If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T, see Topologies, Lemma 34.9.8. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 10.39.17) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = \lim_i U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 28.16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 10.127.5 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module

map. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. If M is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is zero. This is Algebra, Lemma 10.127.5 part (1).

Proof of (4). Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 28.16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 10.127.5 part (3). □

05MK **Situation 38.20.3.** Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in \mathrm{Ob}(\mathcal{C})$, consider the condition

05ML (38.20.3.1) $$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B') \subset \operatorname{Spec}(B') : M'_{\mathfrak{q}} \text{ is flat over } A'.$$

Note the similarity with More on Algebra, Equation (15.19.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (38.20.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 15.19.2. Hence we obtain a functor

05MM (38.20.3.2) $$F_{lf} : \mathcal{C} \longrightarrow \mathit{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (38.20.3.1) holds,} \\ \emptyset & \text{else.} \end{cases}$$

05MN **Lemma 38.20.4.** *In Situation 38.20.3.*

(1) *If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{lf}(A') = F_{lf}(A'')$.*
(2) *If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{lf} is limit preserving: If $\{A_i\}_{i \in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{lf}(\operatorname{colim}_i A_i) = \operatorname{colim}_i F_{lf}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 15.19.3. Part (2) is a special case of More on Algebra, Lemma 15.19.4. □

05P4 **Lemma 38.20.5.** *In Situation 38.20.3. Let $B \to C$ is a local map of local A-algebras and N a C-module. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). If $M \cong N$ as B-modules and $B \to C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 38.20.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C') = (\operatorname{Spec}(C') \to \operatorname{Spec}(B'))^{-1} V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B')$. Then $M'_{\mathfrak{q}}$ is flat over A' if and only if the $C'_{\mathfrak{q}}$-module $N'_{\mathfrak{q}}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_{\mathfrak{q}}$ the module $N'_{\mathfrak{r}}$ is flat over A' (see Algebra, Lemma 10.39.18). As $B'_{\mathfrak{q}} \to C'_{\mathfrak{q}}$ is finite by (b), the maximal ideals of $C'_{\mathfrak{q}}$ correspond exactly

to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 10.36.22) and these primes are all contained in $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. □

05P5 **Lemma 38.20.6.** *In Situation 38.20.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 38.20.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = \mathrm{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_{\mathfrak{q}}$ is flat over A' if and only if $N'_{\mathfrak{r}}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 10.39.9. The lemma follows formally from these remarks. □

05MP **Situation 38.20.7.** Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 29.34.5 and More on Morphisms, Lemma 37.27.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

05MQ (38.20.7.1) $\mathcal{F}_T$ is free of rank p in a neighbourhood of ξ_t

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

(38.20.7.2)

05MR $$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (38.20.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

05MS **Lemma 38.20.8.** *In Situation 38.20.7.*

(1) *The functor H_p satisfies the sheaf property for the fpqc topology.*
(2) *If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (38.20.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 10.78.6. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 17.9.4. After shrinking

[1] It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 38.10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (38.20.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 37.15.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 32.4.11 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 32.10.4. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 32.4.11. Then $\mathcal{F}_{i'}$ satisfies (38.20.7.1) as desired. Some details omitted. □

0CWF **Lemma 38.20.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent*

1. *for $s \in S$ the closed subset $Z \subset X_s$ of points where $\mathcal{F}$ is not flat over S (see Lemma 38.10.4) satisfies $\dim(Z) < n$, and*
2. *for $x \in X$ such that $\mathcal{F}$ is not flat at x over S we have $trdeg_{\kappa(f(x))}(\kappa(x)) < n$.*

If this is true, then it remains true after any base change.

Proof. Let $x \in X$ be a point over $s \in S$. Then the dimension of the closure of $\{x\}$ in X_s is $\mathrm{trdeg}_{\kappa(s)}(\kappa(x))$ by Varieties, Lemma 33.20.3. Conversely, if $Z \subset X_s$ is a closed subset of dimension d, then there exists a point $x \in Z$ with $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = d$ (same reference). Therefore the equivalence of (1) and (2) holds (even fibre by fibre). The statement on base change follows from Morphisms, Lemmas 29.25.7 and 29.28.3. □

0CWG **Definition 38.20.10.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say $\mathcal{F}$ *is flat over S in dimensions $\geq n$* if the equivalent conditions of Lemma 38.20.9 are satisfied.

05MT **Situation 38.20.11.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is

the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms, Lemma 29.15.4 and Modules, Lemma 17.9.2). Let $n \geq 0$. By Definition 38.20.10 and Lemma 38.20.9 we obtain a functor

(38.20.11.1)

05MU $$F_n : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

05MV **Lemma 38.20.12.** *In Situation 38.20.11.*

(1) *The functor F_n satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t. Consider the cartesian diagram

$$\begin{array}{ccc} X_{\mathrm{Spec}(\mathcal{O}_{T,t})} & \longleftarrow & X_{\mathrm{Spec}(\mathcal{O}_{T_i,t_i})} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{T,t}) & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{T_i,t_i}) \end{array}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 37.15.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 29.28.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 37.15.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $< n$. By Limits, Lemma 32.18.5 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 32.10.1 and 32.8.5 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 32.18.1 we may assume all fibres of $Z'_i \to T_i$ have dimension

$< n$. By Limits, Lemma 32.10.4 we may assume that $\mathcal{F}_i|_{X_i \setminus T_i'}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. □

05MW **Situation 38.20.13.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

05MX (38.20.13.1) $$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

05MY **Lemma 38.20.14.** *In Situation 38.20.13.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 37.15.2. Part (2) follows from Limits, Lemma 32.10.4 after reducing to the case where X and S are affine (compare with the proof of Lemma 38.20.12). □

38.21. Flattening stratifications

052F Just the definitions. The reader looking for a "generic flatness stratification", should consult More on Morphisms, Section 37.54.

05P6 **Definition 38.21.1.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 38.20.13 is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a surjective monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

0FJ1 **Example 38.21.2.** Let $X = S = \mathrm{Spec}(k[x, y])$ where k is a field. Let $\mathcal{F} = \widetilde{M}$ where $M = k[x, x^{-1}, y]/(y)$. For a $k[x, y]$-algebra A set $F_{flat}(A) = F_{flat}(\mathrm{Spec}(A))$. Then $F_{flat}(k[x, y]/(x, y)^n) = \{*\}$ for all n, while $F_{flat}(k[[x, y]]) = \emptyset$. This means that F_{flat} isn't representable (even by an algebraic space, see Formal Spaces, Lemma 87.33.3). Thus the universal flattening does not exist in this case.

We define (compare with Topology, Remark 5.28.5) a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

[2]The scheme S' is sometimes called the *universal flatificator*. In [**GR71**] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 15.26.

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

05P7 **Definition 38.21.3.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation 38.20.13 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_X$ has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules. For example if $X = S$ and $\mathcal{F}$ is a finitely presented $\mathcal{O}_S$-module, then the flattening stratification exists and is given by the Fitting ideals of $\mathcal{F}$, see Divisors, Lemma 31.9.7.

38.22. Flattening stratification over an Artinian ring

05PA A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

05PB **Lemma 38.22.1.** *Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 15.17.2. □

38.23. Flattening a map

05PC Theorem 38.23.3 is the key to further flattening statements.

05PD **Lemma 38.23.1.** *Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over S. If $Ass_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_* g_T^* \mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 31.7.3 and Remark 31.7.4. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 31.5.5.

We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 31.5.4. As $\mathcal{F}$ is flat we see from Lemma 38.13.5 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\text{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^*\mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 10.82.11). This implies that $\mathcal{K}_x = 0$ as desired. □

05PE **Lemma 38.23.2.** *Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

(1) *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and*
(2) *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free A-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\text{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\text{Ker}(u \otimes 1)$ is the image of $\text{Ker}(u) \otimes_A B$ in $M \otimes_A B$. □

05PF **Theorem 38.23.3.** *In Situation 38.20.1 assume*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, flat over S, and pure relative to S, and*
(3) *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 38.18.3. By Lemma 38.20.2 and Descent, Lemmas 35.37.2 and 35.39.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 38.12.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

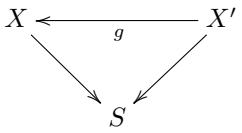

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4. Hence by Lemma 38.18.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $\text{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 38.23.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 38.23.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 38.20.2 part (4) with Limits, Remark 32.6.2. □

07AI **Lemma 38.23.4.** *Let $f : X \to S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X. Let $F = f_*Y$ be the Weil restriction functor of Y along f, defined by*

$$F : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $Y \to S$ is.

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 38.23.3. □

38.24. Flattening in the local case

05MZ In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

05PG **Theorem 38.24.1.** *In Situation 38.20.3 assume A is henselian, B is essentially of finite type over A, and M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

(1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}}$ *is flat over A', and*
(2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say "the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 38.20.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 38.6.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 38.6.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 38.20.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 10.41.11 or More on Morphisms, Lemma 37.47.4). Hence by Lemma 38.20.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \mathrm{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\mathrm{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 38.10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\mathrm{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\mathrm{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 38.10.2 we see that $\mathrm{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence Lemma 38.10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \mathrm{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \mathrm{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\mathrm{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 10.82.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 38.23.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 38.20.4 part (2). Clearly this implies that $I = I_\lambda$. □

05PH **Remark 38.24.2.** Here is a scheme theoretic reformulation of Theorem 38.24.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
(2) $\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

05PI **Lemma 38.24.3.** *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

(1) *$\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and*
(2) *$\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 38.24.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 15.19.4 shows that the functor defined by (1) is "limit preserving". Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 38.24.1. □

052G **Remark 38.24.4.** Tracing the proof of Lemma 38.24.3 to its origins we find a long and winding road. But if we assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $E = X_s$, and
(4) S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 15.20.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

38.25. Variants of a lemma

0ASZ In this section we discuss variants of Algebra, Lemmas 10.128.4 and 10.99.1. The most general version is Proposition 38.25.13; this was stated as [**GR71**, Lemma 4.2.2] but the proof in loc.cit. only gives the weaker result as stated in Lemma 38.25.5. The intricate proof of Proposition 38.25.13 is due to Ofer Gabber. As we currently have no application for the proposition we encourage the reader to skip to the next section after reading the proof of Lemma 38.25.5; this lemma will be used in the next section to prove Theorem 38.26.1.

0AT0 **Situation 38.25.1.** Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective.

In this situation it is our goal to show that u is A-universally injective, N is of finite presentation over B, and N is flat as an A-module. If this is true, we will say *the lemma holds* in the given situation.

0AT1 **Lemma 38.25.2.** *If in Situation 38.25.1 the ring A is Noetherian then the lemma holds.*

Proof. Applying Algebra, Lemma 10.99.1 we see that u is injective and that $N/u(M)$ is flat over A. Then u is A-universally injective (Algebra, Lemma 10.39.12) and N is A-flat (Algebra, Lemma 10.39.13). Since B is Noetherian in this case we see that N is of finite presentation. □

0AT2 **Lemma 38.25.3.** *Let A_0 be a local ring. If the lemma holds for every Situation 38.25.1 with $A = A_0$, with B a localization of a polynomial algebra over A, and N of finite presentation over B, then the lemma holds for every Situation 38.25.1 with $A = A_0$.*

Proof. Let $A \to B$, $u : N \to M$ be as in Situation 38.25.1. Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see Algebra, Lemma 10.6.4). Hence we may assume that B is the localization of a polynomial algebra. Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Now, if the lemma holds for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and $u' = u$. Thus the lemma holds for the original situation. □

0AT3 **Lemma 38.25.4.** *If in Situation 38.25.1 the ring A is henselian then the lemma holds.*

Proof. It suffices to prove this when B is essentially of finite presentation over A and N is of finite presentation over B, see Lemma 38.25.3. Let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = \operatorname{colim}_i F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 38.19.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 38.7.5 wherefore $u = \operatorname{colim} u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 38.24.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for $u \bmod I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_A I/I^2 & \longrightarrow & M/I^2M & \longrightarrow & M/IM \longrightarrow 0 \\ & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} \\ & & N \otimes_A I/I^2 & \longrightarrow & N/I^2N & \longrightarrow & N/IN \longrightarrow 0 \end{array}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\operatorname{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 10.75.9. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 10.99.8 a contradiction unless $I = 0$. □

The following lemma discusses the special case of Situation 38.25.1 where M has a B-module structure and u is B-linear. This is the case most often used in practice and it is significantly easier to prove than the general case.

0AT4 **Lemma 38.25.5.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a B-module map. If N is a finite*

B-module, M is flat over A, and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective, then u is A-universally injective, N is of finite presentation over B, and N is flat over A.

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 10.39.17), we may replace $A \to B$ with $A^h \to B'$, the module M by $M \otimes_B B'$, the module N by $N \otimes_B B'$, and u by $u \otimes \mathrm{id}_{B'}$, see Algebra, Lemmas 10.83.2 and 10.39.9. Thus we may assume that A is a henselian local ring. In this case our lemma follows from the more general Lemma 38.25.4. □

0AT5 **Lemma 38.25.6.** *If in Situation 38.25.1 the ring A is a valuation ring then the lemma holds.*

Proof. Recall that an A-module is flat if and only if it is torsion free, see More on Algebra, Lemma 15.22.10. Let $T \subset N$ be the A-torsion. Then $u(T) = 0$ and N/T is A-flat. Hence N/T is finitely presented over B, see More on Algebra, Lemma 15.25.6. Thus T is a finite B-module, see Algebra, Lemma 10.5.3. Since N/T is A-flat we see that $T/\mathfrak{m}_A T \subset N/\mathfrak{m}_A N$, see Algebra, Lemma 10.39.12. As $\overline{u}$ is injective but $u(T) = 0$, we conclude that $T/\mathfrak{m}_A T = 0$. Hence $T = 0$ by Nakayama's lemma, see Algebra, Lemma 10.20.1. At this point we have proved two out of the three assertions (N is A-flat and of finite presentation over B) and what is left is to show that u is universally injective.

By Algebra, Theorem 10.82.3 it suffices to show that $N \otimes_A Q \to M \otimes_A Q$ is injective for every finitely presented A-module Q. By More on Algebra, Lemma 15.125.3 we may assume $Q = A/(a)$ with $a \in \mathfrak{m}_A$ nonzero. Thus it suffices to show that $N/aN \to M/aM$ is injective. Let $x \in N$ with $u(x) \in aM$. By Lemma 38.19.6 we know that x has a content ideal $I \subset A$. Since I is finitely generated (More on Algebra, Lemma 15.24.2) and A is a valuation ring, we have $I = (b)$ for some b (by Algebra, Lemma 10.50.15). By More on Algebra, Lemma 15.24.3 the element $u(x)$ has content ideal I as well. Since $u(x) \in aM$ we see that $(b) \subset (a)$ by More on Algebra, Definition 15.24.1. Since $x \in bN$ we conclude $x \in aN$ as desired. □

Consider the following situation

0AT6 (38.25.6.1) $$\begin{array}{c} A \to B \text{ of finite presentation, } S \subset B \text{ a multiplicative subset, and} \\ N \text{ a finitely presented } S^{-1}B\text{-module} \end{array}$$

In this situation a *pure spreadout* is an affine open $U \subset \mathrm{Spec}(B)$ with $\mathrm{Spec}(S^{-1}B) \subset U$ and a finitely presented $\mathcal{O}(U)$-module N' extending N such that N' is A-projective and $N' \to N = S^{-1}N'$ is A-universally injective.

In (38.25.6.1) if $A \to A_1$ is a ring map, then we can base change: take $B_1 = B \otimes_A A_1$, let $S_1 \subset B_1$ be the image of S, and let $N_1 = N \otimes_A A_1$. This works because $S_1^{-1}B_1 = S^{-1}B \otimes_A A_1$. We will use this without further mention in the following.

0AT7 **Lemma 38.25.7.** *In (38.25.6.1) if there exists a pure spreadout, then*

(1) *elements of N have content ideals in A, and*
(2) *if $u : N \to M$ is a morphism to a flat A-module M such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective for all maximal ideals $\mathfrak{m}$ of A, then u is A-universally injective.*

Proof. Choose U, N' as in the definition of a pure spreadout. Any element $x' \in N'$ has a content ideal in A because N' is A-projective (this can easily be seen directly,

but it also follows from More on Algebra, Lemma 15.24.4 and Algebra, Example 10.91.1). Since $N' \to N$ is A-universally injective, we see that the image $x \in N$ of any $x' \in N'$ has a content ideal in A (it is the same as the content ideal of x'). For a general $x \in N$ we choose $s \in S$ such that sx is in the image of $N' \to N$ and we use that x and sx have the same content ideal.

Let $u : N \to M$ be as in (2). To show that u is A-universally injective, we may replace A by a localization at a maximal ideal (small detail omitted). Assume A is local with maximal ideal $\mathfrak{m}$. Pick $s \in S$ and consider the composition

$$N' \to N \xrightarrow{1/s} N \xrightarrow{u} M$$

Each of these maps is injective modulo $\mathfrak{m}$, hence the composition is A-universally injective by Lemma 38.7.5. Since $N = \operatorname{colim}_{s \in S}(1/s)N'$ we conclude that u is A-inversally injective as a colimit of universally injective maps. □

0AT8 **Lemma 38.25.8.** *In (38.25.6.1) for every $\mathfrak{p} \in \operatorname{Spec}(A)$ there is a finitely generated ideal $I \subset \mathfrak{p}A_\mathfrak{p}$ such that over $A_\mathfrak{p}/I$ we have a pure spreadout.*

Proof. We may replace A by $A_\mathfrak{p}$. Thus we may assume A is local and $\mathfrak{p}$ is the maximal ideal $\mathfrak{m}$ of A. We may write $N = S^{-1}N'$ for some finitely presented B-module N' by clearing denominators in a presentation of N over $S^{-1}B$. Since $B/\mathfrak{m}B$ is Noetherian, the kernel K of $N'/\mathfrak{m}N' \to N/\mathfrak{m}N$ is finitely generated. Thus we can pick $s \in S$ such that K is annihilated by s. After replacing B by B_s which is allowed as it just means passing to an affine open subscheme of $\operatorname{Spec}(B)$, we find that the elements of S are injective on $N'/\mathfrak{m}N'$. At this point we choose a local subring $A_0 \subset A$ essentially of finite type over $\mathbf{Z}$, a finite type ring map $A_0 \to B_0$ such that $B = A \otimes_{A_0} B_0$, and a finite B_0-module N'_0 such that $N' = B \otimes_{B_0} N'_0 = A \otimes_{A_0} N'_0$. We claim that $I = \mathfrak{m}_{A_0}A$ works. Namely, we have

$$N'/IN' = N'_0/\mathfrak{m}_{A_0}N'_0 \otimes_{\kappa_{A_0}} A/I$$

which is free over A/I. Multiplication by the elements of S is injective after dividing out by the maximal ideal, hence $N'/IN' \to N/IN$ is universally injective for example by Lemma 38.7.6. □

0AT9 **Lemma 38.25.9.** *In (38.25.6.1) assume N is A-flat, M is a flat A-module, and $u : N \to M$ is an A-module map such that $u \otimes id_{\kappa(\mathfrak{p})}$ is injective for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Then u is A-universally injective.*

Proof. By Algebra, Lemma 10.82.14 it suffices to check that $N/IN \to M/IM$ is injective for every ideal $I \subset A$. After replacing A by A/I we see that it suffices to prove that u is injective.

Proof that u is injective. Let $x \in N$ be a nonzero element of the kernel of u. Then there exists a weakly associated prime $\mathfrak{p}$ of the module Ax, see Algebra, Lemma 10.66.5. Replacing A by $A_\mathfrak{p}$ we may assume A is local and we find a nonzero element $y \in Ax$ whose annihilator has radical equal to $\mathfrak{m}_A$, see Algebra, Lemma 10.66.2. Thus $\operatorname{Supp}(y) \subset \operatorname{Spec}(S^{-1}B)$ is nonempty and contained in the closed fibre of $\operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(A)$. Let $I \subset \mathfrak{m}_A$ be a finitely generated ideal so that we have a pure spreadout over A/I, see Lemma 38.25.8. Then $I^n y = 0$ for some n. Now $y \in \operatorname{Ann}_M(I^n) = \operatorname{Ann}_A(I^n) \otimes_R N$ by flatness. Thus, to get the desired contradiction, it suffices to show that

$$\operatorname{Ann}_A(I^n) \otimes_R N \longrightarrow \operatorname{Ann}_A(I^n) \otimes_R M$$

is injective. Since N and M are flat and since $\mathrm{Ann}_A(I^n)$ is annihilated by I^n, it suffices to show that $Q \otimes_A N \to Q \otimes_A M$ is injective for every A-module Q annihilated by I. This holds by our choice of I and Lemma 38.25.7 part (2). □

0ATA **Lemma 38.25.10.** *Let A be a local domain which is not a field. Let S be a set of finitely generated ideals of A. Assume that S is closed under products and such that $\bigcup_{I\in S} V(I)$ is the complement of the generic point of $\mathrm{Spec}(A)$. Then $\bigcap_{I\in S} I = (0)$.*

Proof. Since $\mathfrak{m}_A \subset A$ is not the generic point of $\mathrm{Spec}(A)$ we see that $I \subset \mathfrak{m}_A$ for at least one $I \in S$. Hence $\bigcap_{I\in S} I \subset \mathfrak{m}_A$. Let $f \in \mathfrak{m}_A$ be nonzero. Then $V(f) \subset \bigcup_{I\in S} V(I)$. Since the constructible topology on $V(f)$ is quasi-compact (Topology, Lemma 5.23.2 and Algebra, Lemma 10.26.2) we find that $V(f) \subset V(I_1) \cup \ldots \cup V(I_n)$ for some $I_j \in S$. Because $I_1 \ldots I_n \in S$ we see that $V(f) \subset V(I)$ for some I. As I is finitely generated this implies that $I^m \subset (f)$ for some m and since S is closed under products we see that $I \subset (f^2)$ for some $I \in S$. Then it is not possible to have $f \in I$. □

0ATB **Lemma 38.25.11.** *Let A be a local ring. Let $I, J \subset A$ be ideals. If J is finitely generated and $I \subset J^n$ for all $n \geq 1$, then $V(I)$ contains the closed points of $\mathrm{Spec}(A) \setminus V(J)$.*

Proof. Let $\mathfrak{p} \subset A$ be a closed point of $\mathrm{Spec}(A) \setminus V(J)$. We want to show that $I \subset \mathfrak{p}$. If not, then some $f \in I$ maps to a nonzero element of $A/\mathfrak{p}$. Note that $V(J) \cap \mathrm{Spec}(A/\mathfrak{p})$ is the set of non-generic points. Hence by Lemma 38.25.10 applied to the collection of ideals $J^n A/\mathfrak{p}$ we conclude that the image of f is zero in $A/\mathfrak{p}$. □

0ATC **Lemma 38.25.12.** *Let A be a local ring. Let $I \subset A$ be an ideal. Let $U \subset \mathrm{Spec}(A)$ be quasi-compact open. Let M be an A-module. Assume that*

(1) *M/IM is flat over A/I,*
(2) *M is flat over U,*

Then M/I_2M is flat over A/I_2 where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$.

Proof. It suffices to show that $M \otimes_A I/I_2 \to IM/I_2M$ is injective, see Algebra, Lemma 10.99.9. This is true over U by assumption (2). Thus it suffices to show that $M \otimes_A I/I_2$ injects into its sections over U. We have $M \otimes_A I/I_2 = M/IM \otimes_A I/I_2$ and M/IM is a filtered colimit of finite free A/I-modules (Algebra, Theorem 10.81.4). Hence it suffices to show that I/I_2 injects into its sections over U, which follows from the construction of I_2. □

05U9 **Proposition 38.25.13.** *Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. We may assume that B is the localization of a finitely presented A-algebra B_0 and that N is the localization of a finitely presented B_0-module M_0, see Lemma 38.25.3. By More on Morphisms, Lemma 37.54.1 there exists a "generic flatness stratification" for $\widetilde{M_0}$ on $\mathrm{Spec}(B_0)$ over $\mathrm{Spec}(A)$. Translating back to N we find a sequence of closed subschemes

$$S = \mathrm{Spec}(A) \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with $S_i \subset S$ cut out by a finitely generated ideal of A such that the pullback of $\widetilde{N}$ to $\mathrm{Spec}(B) \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$. We will prove the proposition by induction on t (the base case $t = 1$ will be proved in parallel with the other steps). Let $\mathrm{Spec}(A/J_i)$ be the scheme theoretic closure of $S_i \setminus S_{i+1}$.

Claim 1. N/J_iN is flat over A/J_i. This is immediate for $i = t - 1$ and follows from the induction hypothesis for $i > 0$. Thus we may assume $t > 1$, $S_{t-1} \neq \emptyset$, and $J_0 = 0$ and we have to prove that N is flat. Let $J \subset A$ be the ideal defining S_1. By induction on t again, we also have flatness modulo powers of J. Let A^h be the henselization of A and let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is faithfully flat. Set $N' = N \otimes_B B'$. Note that N' is A^h-flat if and only if N is A-flat. By Theorem 38.24.1 there is a smallest ideal $I \subset A^h$ such that N'/IN' is flat over A^h/I, and I is finitely generated. By the above $I \subset J^n A^h$ for all $n \geq 1$. Let $S_i^h \subset \mathrm{Spec}(A^h)$ be the inverse image of $S_i \subset \mathrm{Spec}(A)$. By Lemma 38.25.11 we see that $V(I)$ contains the closed points of $U = \mathrm{Spec}(A^h) - S_1^h$. By construction N' is A^h-flat over U. By Lemma 38.25.12 we see that N'/I_2N' is flat over A/I_2, where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$. Hence $I = I_2$ by minimality of I. This implies that $I = I^2$ locally on U, i.e., we have $I\mathcal{O}_{U,u} = (0)$ or $I\mathcal{O}_{U,u} = (1)$ for all $u \in U$. Since $V(I)$ contains the closed points of U we see that $I = 0$ on U. Since $U \subset \mathrm{Spec}(A^h)$ is scheme theoretically dense (because replaced A by A/J_0 in the beginning of this paragraph), we see that $I = 0$. Thus N' is A^h-flat and hence Claim 1 holds.

We return to the situation as laid out before Claim 1. With A^h the henselization of A, with B' the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$, and with $N' = N \otimes_B B'$ we now see that the flattening ideal $I \subset A^h$ of Theorem 38.24.1 is nilpotent. If $nil(A^h)$ denotes the ideal of nilpotent elements, then $nil(A^h) = nil(A)A^h$ (More on Algebra, Lemma 15.45.5). Hence there exists a finitely generated nilpotent ideal $I_0 \subset A$ such that N/I_0N is flat over A/I_0.

Claim 2. For every prime ideal $\mathfrak{p} \subset A$ the map $\kappa(\mathfrak{p}) \otimes_A N \to \kappa(\mathfrak{p}) \otimes_A M$ is injective. We say $\mathfrak{p}$ is bad it this is false. Suppose that C is a nonempty chain of bad primes and set $\mathfrak{p}^* = \bigcup_{\mathfrak{p} \in C} \mathfrak{p}$. By Lemma 38.25.8 there is a finitely generated ideal $\mathfrak{a} \subset \mathfrak{p}^* A_{\mathfrak{p}^*}$ such that there is a pure spreadout over $V(\mathfrak{a})$. If $\mathfrak{p}^*$ were good, then it would follow from Lemma 38.25.7 that the points of $V(\mathfrak{a})$ are good. However, since $\mathfrak{a}$ is finitely generated and since $\mathfrak{p}^* A_{\mathfrak{p}^*} = \bigcup_{\mathfrak{p} \in C} A_{\mathfrak{p}^*}$ we see that $V(\mathfrak{a})$ contains a $\mathfrak{p} \in C$, contradiction. Hence $\mathfrak{p}^*$ is bad. By Zorn's lemma, if there exists a bad prime, there exists a maximal one, say $\mathfrak{p}$. In other words, we may assume every $\mathfrak{p}' \supset \mathfrak{p}$, $\mathfrak{p}' \neq \mathfrak{p}$ is good. In this case we see that for every $f \in A$, $f \notin \mathfrak{p}$ the map $u \otimes \mathrm{id}_{A/(\mathfrak{p}+f)}$ is universally injective, see Lemma 38.25.9. Thus it suffices to show that $N/\mathfrak{p}N$ is separated for the topology defined by the submodules $f(N/\mathfrak{p}N)$. Since $B \to B'$ is faithfully flat, it is enough to prove the same for the module $N'/\mathfrak{p}N'$. By Lemma 38.19.5 and More on Algebra, Lemma 15.24.4 elements of $N'/\mathfrak{p}N'$ have content ideals in $A^h/\mathfrak{p}A^h$. Thus it suffices to show that $\bigcap_{f \in A, f \notin \mathfrak{p}} f(A^h/\mathfrak{p}A^h) = 0$. Then it suffices to show the same for $A^h/\mathfrak{q}A^h$ for every prime $\mathfrak{q} \subset A^h$ minimal over $\mathfrak{p}A^h$. Because $A \to A^h$ is the henselization, every $\mathfrak{q}$ contracts to $\mathfrak{p}$ and every $\mathfrak{q}' \supset \mathfrak{q}$, $\mathfrak{q}' \neq \mathfrak{q}$ contracts to a prime $\mathfrak{p}'$ which strictly contains $\mathfrak{p}$. Thus we get the vanishing of the intersections from Lemma 38.25.10.

At this point we can put everything together. Namely, using Claim 1 and Claim 2 we see that $N/I_0N \to M/I_0M$ is A/I_0-universally injective by Lemma 38.25.9.

Then the diagrams

$$\begin{array}{ccc} N \otimes_A (I_0^n/I_0^{n+1}) & \longrightarrow & M \otimes_A (I_0^n/I_0^{n+1}) \\ \downarrow & & \| \\ I_0^n N/I_0^{n+1}N & \longrightarrow & I_0^n M/I_0^{n+1}M \end{array}$$

show that the left vertical arrows are injective. Hence by Algebra, Lemma 10.99.9 we see that N is flat. In a similar way the universal injectivity of u can be reduced (even without proving flatness of N first) to the one modulo I_0. This finishes the proof. □

38.26. Flat finite type modules, Part III

05U8 The following result is one of the main results of this chapter.

05UA **Theorem 38.26.1.** *Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

(1) *$\mathcal{F}$ is flat at x over S, and*
(2) *for every $x' \in Ass_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.*

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_A N$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $\mathrm{Spec}(\Sigma^{-1}N)$ in Lemma 38.7.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 38.25.5 we win. □

Now we apply this directly to obtain the following useful results.

05UB **Lemma 38.26.2.** *Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that*

(1) *every point of $Ass_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s*[3]*,*
(2) *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 38.26.1. □

38.27. Universal flattening

05PS If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

05UC **Lemma 38.27.1.** *In Situation 38.20.7. For each $p \geq 0$ the functor H_p (38.20.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.*

[3] For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 38.20.8. Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemma 35.24.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 10.118.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (38.20.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (38.20.1.1) for the map u (this follows immediately from Lemma 38.19.1 but also from Lemma 38.12.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 38.23.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

Assume $\mathcal{F}$ is of finite presentation. Then $S_p \to S$ is locally of finite presentation by Lemma 38.20.8 part (2) combined with Limits, Remark 32.6.2. Then we redo the induction argument in the paragraph to see that each S_p is quasi-compact when S is affine: first if $p = \max_{s \in S} p(s)$, then $S_p \subset S$ is closed (see above) hence quasi-compact. Then $U = S \setminus S_p$ is quasi-compact open in S because $S_p \to S$ is a closed immersion of finite presentation (see discussion in Morphisms, Section 29.22 for example). Then $S_{p-1} \to U$ is a closed immersion of finite presentation, and so S_{p-1} is quasi-compact and $U' = S \setminus (S_p \cup S_{p-1})$ is quasi-compact. And so on. □

05UD **Lemma 38.27.2.** *In Situation 38.20.11. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (38.20.11.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is

not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 29.25.13. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 35.21.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \mathrm{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \mathrm{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 31.7.3 and Remark 31.7.4. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 29.25.13). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 38.26.1 which is the desired contradiction. □

05UE **Lemma 38.27.3.** *Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (38.20.11.1) and H_p as in (38.20.7.2).*

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t. Namely, then H_r contains the image of U which is an open neighbourhood of t. The existence of U follows from More on Morphisms, Lemma 37.16.7. □

05UF **Lemma 38.27.4.** *In Situation 38.20.11. Let $s \in S$ let $d \geq 0$. Assume*

1. *there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,*
2. *X is of finite presentation over S,*
3. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and*
4. *$\mathcal{F}$ is flat in dimensions $\geq d+1$ over S.*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (38.20.11.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a monomorphism of finite presentation $Z_d \to S$ on the category of affine schemes over S. (Of course we do not require Z_d to be affine.) Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 38.5.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 38.4.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \mathrm{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have

$d_1 = \dim(\mathrm{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\mathrm{Supp}(\mathrm{Coker}(\alpha_{k-1})_s)) = \dim(\mathrm{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for "flat in dimensions $\geq d+1$" and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d+1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 38.12.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \mathrm{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\mathrm{Coker}(\alpha_1)_T$ is flat over S. In this way the question reduces to the sheaf $\mathrm{Coker}(\alpha_1)$ which has a complete dévissage of length $n-1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 38.27.3 and Lemma 38.27.1. □

05UG **Theorem 38.27.5.** *In Situation 38.20.11. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 38.20.12. Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 29.20.15). Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemmas 35.23.33 and 35.23.15 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 38.16.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the

result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 38.5.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 38.27.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after shrinking S we may assume that $E = S$. It follows from Lemma 38.27.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 38.27.4. □

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

05UH **Lemma 38.27.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(2) *If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(3) *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(4) *If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.*

Proof. These statements follow immediately from Theorem 38.27.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 38.17.1. □

38.28. Grothendieck's Existence Theorem, IV

0CTB This section continues the discussion in Cohomology of Schemes, Sections 30.24, 30.25, and 30.27. We will work in the following situation.

0CTC **Situation 38.28.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms, Lemma 29.4.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CTD **Lemma 38.28.2.** *In Situation 38.28.1 consider*

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Schemes, Example 36.21.5. □

0CTE **Lemma 38.28.3.** *In Situation 38.28.1 let K be as in Lemma 38.28.2. For any perfect object E of $D(\mathcal{O}_X)$ we have*

1. *$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$,*
2. *$N = R\mathrm{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\mathrm{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$.*

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\mathrm{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\mathrm{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology, Section 20.44. Hence by Derived Categories of Schemes, Lemma 36.30.7 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.98.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, K) = R\lim R\mathrm{Hom}_X(E, \mathcal{F}_n) = R\lim R\mathrm{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology, Lemma 20.50.5. □

0CTF **Lemma 38.28.4.** *In Situation 38.28.1 let K be as in Lemma 38.28.2. Then K is pseudo-coherent relative to A.*

Proof. Combinging Lemma 38.28.3 and Derived Categories of Schemes, Lemma 36.34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms, Lemma 37.69.4. □

0CTG **Lemma 38.28.5.** *In Situation 38.28.1 let K be as in Lemma 38.28.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 36.33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 38.28.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0CTH **Lemma 38.28.6.** *In Situation 38.28.1 let K be as in Lemma 38.28.2. Denote $X_0 \subset X$ the closed subset consisting of points lying over the closed subset $\text{Spec}(A_1) = \text{Spec}(A_2) = \ldots$ of $\text{Spec}(A)$. There exists an open $W \subset X$ containing X_0 such that*

(1) *$H^i(K)|_W$ is zero unless $i = 0$,*
(2) *$\mathcal{F} = H^0(K)|_W$ is of finite presentation, and*
(3) *$\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.*

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \subset X$ be an affine open neighbourhood of x. Say $U = \text{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 38.28.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \text{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 38.28.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point x is maximal. Hence in an open neighbourhood of x inside $\text{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may

assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. □

0CTI **Lemma 38.28.7.** *In Situation 38.28.1 let K be as in Lemma 38.28.2. Let $W \subset X$ be as in Lemma 38.28.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.*

Proof. Fix $n \geq 1$. Let $I_n = \mathrm{Ker}(A \to A_n)$. By More on Algebra, Lemma 15.11.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the support of $\mathcal{F}$. This is a closed subset as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 38.28.6 we see that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\mathrm{Spec}(A/I)$. By More on Morphisms, Lemma 37.53.9 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, Z_2 does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. □

0CTJ **Lemma 38.28.8.** *Let $A = \lim A_n$ be a limit of a system of rings whose transition maps are surjective and with locally nilpotent kernels. Let $S = \mathrm{Spec}(A)$. Let $T \to S$ be a monomorphism which is locally of finite type. If $\mathrm{Spec}(A_n) \to S$ factors through T for all n, then $T = S$.*

Proof. Set $S_n = \mathrm{Spec}(A_n)$. Let $T_0 \subset T$ be the common image of the factorizations $S_n \to T$. Then T_0 is quasi-compact. Let $T' \subset T$ be a quasi-compact open containing T_0. Then $S_n \to T$ factors through T'. If we can show that $T' = S$, then $T' = T = S$. Hence we may assume T is quasi-compact.

Assume T is quasi-compact. In this case $T \to S$ is separated and quasi-finite (Morphisms, Lemma 29.20.15). Using Zariski's Main Theorem (in the form of More on Morphisms, Lemma 37.43.3) we choose a factorization $T \to W \to S$ with $W \to S$ finite and $T \to W$ an open immersion. Write $W = \mathrm{Spec}(B)$. The (unique) factorizations $S_n \to T$ may be viewed as morphisms into W and we obtain

$$A \longrightarrow B \longrightarrow \lim A_n = A$$

Consider the morphism $h : S = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = W$ coming from the arrow on the right. Then

$$T \times_{W,h} S$$

is an open subscheme of S containing the image of $S_n \to S$ for all n. To finish the proof it suffices to show that any open $U \subset S$ containing the image of $S_n \to S$ for some $n \geq 1$ is equal to S. This is true because $(A, \mathrm{Ker}(A \to A_n))$ is a henselian pair (More on Algebra, Lemma 15.11.3) and hence every closed point of S is contained in the image of $S_n \to S$. □

0CTK **Theorem 38.28.9** (Grothendieck Existence Theorem)**.** *In Situation 38.28.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 38.28.2, 38.28.3, 38.28.4, 38.28.5, 38.28.6, and 38.28.7 to get an open subscheme $W \subset X$ containing all points lying over $\mathrm{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$

for all $n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 38.17.1 we see that $\mathcal{F}$ is universally pure relative to $\mathrm{Spec}(A)$. By Theorem 38.27.5 (for explanation, see Lemma 38.27.6) there exists a universal flattening $S' \to \mathrm{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \mathrm{Spec}(A)$ is a monomorphism of finite presentation. Since the base change of $\mathcal{F}$ to $\mathrm{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)$ factors (uniquely) through S' for each n. By Lemma 38.28.8 we see that $S' = \mathrm{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\mathrm{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. □

38.29. Grothendieck's Existence Theorem, V

0DIA In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 38.28 for quasi-coherent modules. The classical case is discussed in Cohomology of Schemes, Sections 30.24, 30.25, and 30.27. We will work in the following situation.

0DIB **Situation 38.29.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li^*_{n-1}K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIC **Lemma 38.29.2.** *In Situation 38.29.1 consider*

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Schemes, Lemma 36.21.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Schemes, Lemma 36.21.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \subset X$ be an affine open. Then there is a canonical exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology, Lemma 20.37.1. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Schemes, Lemma 36.10.1) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 36.3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology, Lemma 20.47.9). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too. Thus $a_n = a_{n-1}$ for all n and we conclude. □

0DID **Lemma 38.29.3.** *In Situation 38.29.1 let K be as in Lemma 38.29.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology*

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Schemes, Lemma 36.30.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.98.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. □

0DIE **Lemma 38.29.4.** *In Situation 38.29.1 let K be as in Lemma 38.29.2. Then K is pseudo-coherent on X.*

Proof. Combinging Lemma 38.29.3 and Derived Categories of Schemes, Lemma 36.34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms, Lemma 37.69.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms, Lemma 37.59.18. □

0DIF **Lemma 38.29.5.** *In Situation 38.29.1 let K be as in Lemma 38.29.2. For any quasi-compact open $U \subset X$ we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 36.33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \mathrm{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \mathrm{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \mathrm{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 38.29.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0DIG **Theorem 38.29.6** (Derived Grothendieck Existence Theorem)**.** *In Situation 38.29.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 38.29.2, 38.29.3, 38.29.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine opens in Lemma 38.29.5 it follows immediately that K restricts to K_n over X_n. □

0DIH **Remark 38.29.7.** The result in this section can be generalized. It is probably correct if we only assume $X \to \mathrm{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported on a closed subset proper over A. If we ever need this, we will formulate a precise statement and prove it here.

38.30. Blowing up and flatness

080X In this section we continue our discussion of results of the form: "After a blowup the strict transform becomes flat", see More on Algebra, Section 15.26 and Divisors, Section 31.35. We will use the following (more or less standard) notation in this section. If $X \to S$ is a morphism of schemes, $\mathcal{F}$ is a quasi-coherent module on X, and $T \to S$ is a morphism of schemes, then we denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to the base change $X_T = X \times_S T$.

080Y **Remark 38.30.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open subscheme. Given a U-admissible blowup $S' \to S$ we denote X' the strict transform of X and $\mathcal{F}'$ the strict transform of $\mathcal{F}$ which we think of as a quasi-coherent module on X' (via Divisors, Lemma 31.33.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U-admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P.

The general strategy will be to use that a composition of U-admissible blowups is a U-admissible blowup, see Divisors, Lemma 31.34.2. In fact, we will make use of the more precise Divisors, Lemma 31.32.14 and combine it with Divisors, Lemma 31.33.6. The result is that it suffices to find a sequence of U-admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \ldots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting $\mathcal{F}_i/X_i$ equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P.

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 28.24.1. Let $S' \to S$ be the blowup in $\mathcal{I}$ and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 31.32.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 31.13.4).

Suppose that P is local on S: If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 31.34.3 and 31.34.4.

0810 **Lemma 38.30.2.** *Let R be a ring and let $f \in R$. Let $r \geq 0$ be an integer. Let $R \to S$ be a ring map and let M be an S-module. Assume*

(1) *$R \to S$ is of finite presentation and flat,*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R,*
(3) *M is a finite S-module,*
(4) *M_f is a finitely presented S_f-module,*
(5) *for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_\mathfrak{q}$ is free of rank r over $S_\mathfrak{q}$.*

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g-module of rank r.

Proof. This lemma is a generalization of More on Algebra, Lemma 15.26.5; we urge the reader to read that proof first. Choose a surjection $S^{\oplus n} \to M$, which is possible by (1). Choose a finite submodule $K \subset \mathrm{Ker}(S^{\oplus n} \to M)$ such that $S^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible by (4). Set $M_1 = S^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 10.70.7. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S-module.

Assume M is a finitely presented S-module satisfying (3). Then $J = \mathrm{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 38.9.3 we can write S as a direct summand of a free R-module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_\alpha$ in the decomposition above, we say that the a_α are the coefficients of h. Let $I' \subset R$ be the ideal of coefficients of elements of J. Multiplication by an element of S defines an R-linear map $S \to S$, hence I' is generated by the coefficients of the generators of J, i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (5) and More on Algebra, Lemma 15.8.7. Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in $\mathfrak{p}$, so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fa'$ for some $a' \in I'$. By Algebra, Lemmas 10.70.2 and 10.70.8 we see that $I'R' = a'R'$ and a' is a nonzerodivisor in R'. Set $S' = S \otimes_S R'$. Every element g of $JS' = \mathrm{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_\alpha c_\alpha$ for some $c_\alpha \in I'R'$. Since $I'R' = a'R'$ we can write $c_\alpha = a'c'_\alpha$ for

some $c'_\alpha \in R'$ and $g = (\sum c'_\alpha)a' = g'a'$ in S'. Moreover, there is an $g_0 \in J$ such that $a' = c_\alpha$ for some α. For this element we have $g_0 = g'_0 a'$ in S' where g'_0 is a unit in S'. Let $\mathfrak{p}' \subset R'$ be a prime ideal and $\mathfrak{q}' = \mathfrak{p}'S'$. By the above we see that $JS'_{\mathfrak{q}'}$ is the principal ideal generated by the nonzerodivisor a'. It follows from More on Algebra, Lemma 15.8.9 that $M'_{\mathfrak{q}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in S'$, $g \notin \mathfrak{q}'$ such that the corresponding map $(S')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \mathrm{Fit}_{r-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 15.8.8. Since $R'_a = R_f$ (Algebra, Lemma 10.70.7) and $M'_a = M_f$ we see from (5) that J'_a maps to zero in $S_{\mathfrak{q}''}$ for any prime $\mathfrak{q}'' \subset S'$ of the form $\mathfrak{q}'' = \mathfrak{p}''S'$ where $\mathfrak{p}'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak{q}'' \text{ as above}} S'_{\mathfrak{q}''}$ (as $(S'_a)_{\mathfrak{p}''} \subset S'_{\mathfrak{q}''}$ by Lemma 38.7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. □

0811 **Lemma 38.30.3.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is affine, of finite presentation, flat, geometrically integral fibres,*
(2) *$\mathcal{F}$ is a module of finite type,*
(3) *$\mathcal{F}_U$ is of finite presentation,*
(4) *$\mathcal{F}$ is flat over S at all generic points of fibres lying over points of U.*

Then there exists a U-admissible blowup $S' \to S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform $\mathcal{F}'$ of $\mathcal{F}$ restricts to a finitely locally free $\mathcal{O}_V$-module and (b) $V \to S'$ is surjective.

Proof. Given $\mathcal{F}/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property "$\mathcal{F}$ is flat over S at all generic points of fibres". It is clear that P is preserved under strict transform, see Divisors, Lemma 31.33.3 and Morphisms, Lemma 29.25.7. It is also clear that P is local on S. Hence any and all observations of Remark 38.30.1 apply to the problem posed by the lemma.

Consider the function $r : U \to \mathbf{Z}_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal{F}_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u. By More on Morphisms, Lemma 37.16.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \ldots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i. By Divisors, Lemma 31.34.5 we can find a U-admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \ldots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \ldots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r.

By Remark 38.30.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 38.30.1 combined with Divisors, Lemma 31.13.2 tells us we may assume that $S = \mathrm{Spec}(R)$

and $D = \mathrm{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 38.30.2. $\square$

0812 **Lemma 38.30.4.** *Let $A \to C$ be a finite locally free ring map of rank d. Let $h \in C$ be an element such that C_h is étale over A. Let $J \subset C$ be an ideal. Set $I = Fit_0(C/J)$ where we think of C/J as a finite A-module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J'.*

Proof. We will use basic properties of Fitting ideals, see More on Algebra, Lemma 15.8.4. Then IC is the Fitting ideal of $C/J \otimes_A C$. Note that $C \to C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \to C \otimes_A C$ is étale at every prime in the image of $\mathrm{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \to C_h$ is étale in particular flat, see Algebra, Lemma 10.143.8. Hence there exists a C_h-algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h-algebras, see Algebra, Lemma 10.143.9. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h-modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \mathrm{Fit}_0(C'/I')$ where we view C'/J' as a C_h-module. $\square$

0813 **Lemma 38.30.5.** *Let $A \to B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak{q} \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.*

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak{q}$. Thus we may assume that B is standard étale, see Algebra, Proposition 10.144.4. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d. Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \ldots, h_n \in C$ which generate J over B. The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m. Set $h_{n+1} = a^m$. As in Lemma 38.30.4 we take $I = \mathrm{Fit}_0(C/(h_1, \ldots, h_{r+1}))$. Since the module $C/(h_1, \ldots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. $\square$

0814 **Lemma 38.30.6.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams*

$$\begin{array}{ccccc} & & X_i & \xrightarrow[j_i]{} & X \\ & & \downarrow & & \downarrow \\ S_i^* & \longrightarrow & S_i & \xrightarrow{e_i} & S \end{array}$$

where

1. *$e_i : S_i \to S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,*
2. *$j_i : X_i \to X$ are étale morphisms and $X = \bigcup j_i(X_i)$,*
3. *$S_i^* \to S_i$ is an $e_i^{-1}(U)$-admissible blowup such that the strict transform $\mathcal{F}_i^*$ of $j_i^*\mathcal{F}$ is flat over S_i^*.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of $\mathcal{F}$ is flat over S'.

Proof. We claim that the hypotheses of the lemma are preserved under U-admissible blowups. Namely, suppose $b : S' \to S$ is a U-admissible blowup in the quasi-coherent sheaf of ideals $\mathcal{I}$. Moreover, let $S_i^* \to S_i$ be the blowup in the quasi-coherent sheaf of ideals $\mathcal{J}_i$. Then the collection of morphisms $e_i' : S_i' = S_i \times_S S' \to S'$ and $j_i' : X_i' = X_i \times_S S' \to X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ relative to the blowup $S' \to S$. First, observe that S_i' is the blowup of S_i in the pullback of $\mathcal{I}$, see Divisors, Lemma 31.32.3. Second, consider the blowup $S_i'^* \to S_i'$ of S_i' in the pullback of the ideal $\mathcal{J}_i$. By Divisors, Lemma 31.32.12 we get a commutative diagram

$$\begin{array}{ccc} S_i'^* & \longrightarrow & S_i' \\ \downarrow & \searrow & \downarrow \\ S_i^* & \longrightarrow & S_i \end{array}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 31.33.3 and 31.33.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^*\mathcal{F}' \text{ under } S_i'^* \to S_i' \\ = \ & \text{the strict transform of } j_i^*\mathcal{F} \text{ under } S_i'^* \to S_i \\ = \ & \text{the strict transform of } \mathcal{F}_i' \text{ under } S_i'^* \to S_i' \\ = \ & \text{the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \to X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 29.25.7). Having said this, we see that all observations of Remark 38.30.1 apply to the problem of finding a U-admissible blowup such that the strict transform of $\mathcal{F}$ becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark 38.30.1 combined with Divisors, Lemma 31.13.2 shows we may assume that $S = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 38.30.5 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \setminus U$. Since S_i is quasi-compact we can replace S_i by a finite collection $W_1, \ldots, W_n$ of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \ldots \mathcal{I}_n$ (this blowup is U-admissible by construction). The base change of $S' \to S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i' \mathcal{I}_1 \ldots \hat{\mathcal{I}}_i \ldots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \to S_i$ (Divisors, Lemma 31.32.12). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^*\mathcal{F}$ is flat over S_i. Hence $j_i^*\mathcal{F}$ is flat over S, see Lemma 38.2.3. By Morphisms, Lemma 29.25.13 we see that $\mathcal{F}$ is flat over S. □

0815 **Theorem 38.30.7.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *X is quasi-compact,*
(2) *X is locally of finite presentation over S,*
(3) *$\mathcal{F}$ is a module of finite type,*
(4) *$\mathcal{F}_U$ is of finite presentation, and*
(5) *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_S S'}$-module of finite presentation and flat over S'.

Proof. We first prove that we can find a U-admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 38.30.6 for a precise statement. In particular, we may assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of $\mathcal{F}$ to the fibre). As $X \to S$ is of finite type $d = \max_{s \in S} \dim(\mathrm{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d.

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 38.3.2 to get $g : X' \to X$, $e : S' \to S$, $i : Z' \to X'$, and $\pi : Z' \to Y'$. Observe that $Y' \to S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d. Because the problem is étale local it suffices to prove the theorem for $g^*\mathcal{F}/X'/S'$. Because $i : Z' \to X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 31.33.4) it suffices to prove the flattening result for $\mathcal{G}$. Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_*\mathcal{G}/Y'/S'$. Thus we may assume that $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d.

Next, we apply a blowup as in Lemma 38.30.3. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s. Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a map $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$. Because $\mathcal{F}$ is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \to S$ is surjective. Say $\mathcal{F}$ is the quasi-coherent module associated to the A-module N. Since $\mathcal{F}$ is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_{\mathfrak{p}} : A_{\mathfrak{p}}^{\oplus r} \to N_{\mathfrak{p}}$$

is universally injective for all primes $\mathfrak{p}$ of R, see Lemma 38.10.1. Hence α is universally injective, see Algebra, Lemma 10.82.12. Set $\mathcal{H} = \mathrm{Coker}(\alpha)$. By Divisors, Lemma 31.33.7 we see that, given a U-admissible blowup $S' \to S$ the strict transforms of $\mathcal{F}'$ and $\mathcal{H}'$ fit into an exact sequence

$$0 \to \mathcal{O}_{X \times_S S'}^{\oplus r} \to \mathcal{F}' \to \mathcal{H}' \to 0$$

Hence Lemma 38.10.1 also shows that $\mathcal{F}'$ is flat at a point x' if and only if $\mathcal{H}'$ is flat at that point. In particular $\mathcal{H}_U$ is flat over U and $\mathcal{H}_U$ is a module of finite presentation. We may apply the induction hypothesis to $\mathcal{H}$ to see that there exists a U-admissible blowup such that the strict transform $\mathcal{H}'$ is flat as desired.

To finish the proof of the theorem we still have to show that $\mathcal{F}'$ is a module of finite presentation (after possibly another U-admissible blowup). This follows from Lemma 38.11.1 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 38.30.1). This finishes the proof of the theorem. □

38.31. Applications

081Q In this section we apply some of the results above.

081R **Lemma 38.31.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is of finite type and quasi-separated, and*
(2) *$X_U \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is flat and of finite presentation over S'.

Proof. Since $X \to S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \to S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U-admissible blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \ldots \cup W_n$ be a finite affine open covering. If we can find a U-admissible blowup $S_i \to S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U-admissible blowing up $S' \to S$ dominating all $S_i \to S$ which does the job (see Divisors, Lemma 31.34.4; see also Remark 38.30.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 29.39.2 we can choose an immersion $j : X \to \mathbf{A}^n_S$ over S. Let $V \subset \mathbf{A}^n_S$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \to V$ over S. Apply Theorem 38.30.7 to $V \to S$ and the quasi-coherent module $i_*\mathcal{O}_X$ to obtain a U-admissible blowup $S' \to S$ such that the strict transform of $i_*\mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \to S$. Let $i' : X' \to V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors, Lemma 31.33.4), we see that $i'_*\mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V \times_S S'}$-module. This implies the lemma. □

0B49 **Lemma 38.31.2.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is proper, and*
(2) *$X_U \to U$ is finite locally free.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is finite locally free over S'.

Proof. By Lemma 38.31.1 we may assume that $X \to S$ is flat and of finite presentation. After replacing S by a U-admissible blowup if necessary, we may assume that $U \subset S$ is scheme theoretically dense. Then f is finite by Lemma 38.11.4. Hence f is finite locally free by Morphisms, Lemma 29.48.2. □

081S **Lemma 38.31.3.** *Let $\varphi : X \to S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ such that the strict transform X' of X is isomorphic to an open subscheme of S'.*

Proof. The discussion in Remark 38.30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Next, we do

another U-admissible blowup to get to the situation where $X \to S$ is flat and of finite presentation, see Lemma 38.31.1. In this case the result follows from Lemma 38.11.5. □

The following lemma says that a proper modification can be dominated by a blowup.

081T **Lemma 38.31.4.** *Let $\varphi : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ which dominates X, i.e., such that there exists a factorization $S' \to X \to S$ of the blowup morphism.*

Proof. The discussion in Remark 38.30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Choose another U-admissible blowup $S' \to S$ such that the strict transform X' of X is an open subscheme of S', see Lemma 38.31.3. Since $X' \to S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. □

0CP1 **Lemma 38.31.5.** *Let S be a scheme. Let $U \subset W \subset S$ be open subschemes. Let $f : X \to W$ be a morphism and let $s : U \to X$ be a morphism such that $f \circ s = id_U$. Assume*

(1) *f is proper,*
(2) *S is quasi-compact and quasi-separated, and*
(3) *U and W are quasi-compact.*

Then there exists a U-admissible blowup $b : S' \to S$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms, Lemma 29.6.8. By Lemma 38.31.4 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors, Lemma 31.34.3 to extend $W' \to W$ to a U-admissible blowup of S. □

38.32. Compactifications

0ATT Let S be a quasi-compact and quasi-separated scheme. We will say a scheme X over S *has a compactification over S* or *is compactifyable over S* if there exists a quasi-compact open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over S. If X has a compactification over S, then $X \to S$ is separated and of finite type. It is a theorem of Nagata, see [**Lüt93**], [**Con07b**], [**Nag56**], [**Nag57a**], [**Nag62a**], and [**Nag63**], that the converse is true as well. We will prove this theorem in the next section, see Theorem 38.33.8.

Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated finite type morphism of schemes. The category of *compactifications of X over S* is the category defined as follows:

(1) Objects are open immersions $j : X \to \overline{X}$ over S with $\overline{X} \to S$ proper.
(2) Morphisms $(j' : X \to \overline{X}') \to (j : X \to \overline{X})$ are morphisms $f : \overline{X}' \to \overline{X}$ of schemes over S such that $f \circ j' = j$.

If $j : X \to \overline{X}$ is a compactification, then j is a quasi-compact open immersion, see Schemes, Remark 26.21.18.

Warning. We do *not* assume compactifications $j : X \to \overline{X}$ to have dense image. Consequently, if $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications, it may not be the case that $f^{-1}(j(X)) = j'(X)$.

0ATU **Lemma 38.32.1.** *Let S be a quasi-compact and quasi-separated scheme. Let X be a compactifyable scheme over S.*

(a) *The category of compactifications of X over S is cofiltered.*
(b) *The full subcategory consisting of compactifications $j : X \to \overline{X}$ such that $j(X)$ is dense and scheme theoretically dense in $\overline{X}$ is initial (Categories, Definition 4.17.3).*
(c) *If $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications of X such that $j'(X)$ is dense in $\overline{X}'$, then $f^{-1}(j(X)) = j'(X)$.*

Proof. To prove part (a) we have to check conditions (1), (2), (3) of Categories, Definition 4.20.1. Condition (1) holds exactly because we assumed that X is compactifyable. Let $j_i : X \to \overline{X}_i$, $i = 1, 2$ be two compactifications. Then we can consider the scheme theoretic image $\overline{X}$ of $(j_1, j_2) : X \to \overline{X}_1 \times_S \overline{X}_2$. This determines a third compactification $j : X \to \overline{X}$ which dominates both j_i:

$$(X, \overline{X}_1) \longleftarrow (X, \overline{X}) \longrightarrow (X, \overline{X}_2)$$

Thus (2) holds. Let $f_1, f_2 : \overline{X}_1 \to \overline{X}_2$ be two morphisms between compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$. Let $\overline{X} \subset \overline{X}_1$ be the equalizer of f_1 and f_2. As $\overline{X}_2 \to S$ is separated, we see that $\overline{X}$ is a closed subscheme of $\overline{X}_1$ and hence proper over S. Moreover, we obtain an open immersion $X \to \overline{X}$ because $f_1|_X = f_2|_X = \mathrm{id}_X$. The morphism $(X \to \overline{X}) \to (j_1 : X \to \overline{X}_1)$ given by the closed immersion $\overline{X} \to \overline{X}_1$ equalizes f_1 and f_2 which proves condition (3).

Proof of (b). Let $j : X \to \overline{X}$ be a compactification. If $\overline{X}'$ denotes the scheme theoretic closure of X in $\overline{X}$, then X is dense and scheme theoretically dense in $\overline{X}'$ by Morphisms, Lemma 29.7.7. This proves the first condition of Categories, Definition 4.17.3. Since we have already shown the category of compactifications of X is cofiltered, the second condition of Categories, Definition 4.17.3 follows from the first (we omit the solution to this categorical exercise).

Proof of (c). After replacing $\overline{X}'$ with the scheme theoretic closure of $j'(X)$ (which doesn't change the underlying topological space) this follows from Morphisms, Lemma 29.6.8. □

We can also consider the category of all compactifications (for varying X). It turns out that this category, localized at the set of morphisms which induce an isomorphism on the interior is equivalent to the category of compactifyable schemes over S.

0A9Z **Lemma 38.32.2.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with Y separated and of finite type over S and X compactifyable over S. Then X has a compactification over Y.*

Proof. Let $j : X \to \overline{X}$ be a compactification of X over S. Then we let $\overline{X}'$ be the scheme theoretic image of $(j, f) : X \to \overline{X} \times_S Y$. The morphism $\overline{X}' \to Y$ is proper because $\overline{X} \times_S Y \to Y$ is proper as a base change of $\overline{X} \to S$. On the other hand, since Y is separated over S, the morphism $(1, f) : X \to X \times_S Y$ is a closed

immersion (Schemes, Lemma 26.21.10) and hence $X \to \overline{X}'$ is an open immersion by Morphisms, Lemma 29.6.8 applied to the "partial section" $s = (j, f)$ to the projection $\overline{X} \times_S Y \to \overline{X}$. □

Let S be a quasi-compact and quasi-separated scheme. We define the *category of compactifications* to be the category whose objects are pairs $(X, \overline{X})$ where $\overline{X}$ is a scheme proper over S and $X \subset \overline{X}$ is a quasi-compact open and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \end{array}$$

of morphisms of schemes over S.

0ATV **Lemma 38.32.3.** *Let S be a quasi-compact and quasi-separated scheme. The collection of morphisms $(u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X})$ such that u is an isomorphism forms a right multiplicative system (Categories, Definition 4.27.1) of arrows in the category of compactifications.*

Proof. Axiom RMS1 is trivial to verify. Let us check RMS2 holds. Suppose given a diagram

$$\begin{array}{ccc} & & (X', \overline{X}') \\ & & \downarrow{\scriptstyle (u,\overline{u})} \\ (Y, \overline{Y}) & \xrightarrow{(f,\overline{f})} & (X, \overline{X}) \end{array}$$

with $u : X' \to X$ an isomorphism. Then we let $Y' = Y \times_X X'$ with the projection map $v : Y' \to Y$ (an isomorphism). We also set $\overline{Y}' = \overline{Y} \times_{\overline{X}} \overline{X}'$ with the projection map $\overline{v} : \overline{Y}' \to \overline{Y}$ It is clear that $Y' \to \overline{Y}'$ is an open immersion. The diagram

$$\begin{array}{ccc} (Y', \overline{Y}') & \xrightarrow[(g,\overline{g})]{} & (X', \overline{X}') \\ {\scriptstyle (v,\overline{v})}\downarrow & & \downarrow{\scriptstyle (u,\overline{u})} \\ (Y, \overline{Y}) & \xrightarrow{(f,\overline{f})} & (X, \overline{X}) \end{array}$$

shows that axiom RMS2 holds.

Let us check RMS3 holds. Suppose given a pair of morphisms $(f, \overline{f}), (g, \overline{g}) : (X, \overline{X}) \to (Y, \overline{Y})$ of compactifications and a morphism $(v, \overline{v}) : (Y, \overline{Y}) \to (Y', \overline{Y}')$ such that v is an isomorphism and such that $(v, \overline{v}) \circ (f, \overline{f}) = (v, \overline{v}) \circ (g, \overline{g})$. Then $f = g$. Hence if we let $\overline{X}' \subset \overline{X}$ be the equalizer of $\overline{f}$ and $\overline{g}$, then $(u, \overline{u}) : (X, \overline{X}') \to (X, \overline{X})$ will be a morphism of the category of compactifications such that $(f, \overline{f}) \circ (u, \overline{u}) = (g, \overline{g}) \circ (u, \overline{u})$ as desired. □

0ATW **Lemma 38.32.4.** *Let S be a quasi-compact and quasi-separated scheme. The functor $(X, \overline{X}) \mapsto X$ defines an equivalence from the category of compactifications localized (Categories, Lemma 4.27.11) at the right multiplicative system of Lemma 38.32.3 to the category of compactifyable schemes over S.*

Proof. Denote $\mathcal{C}$ the category of compactifications and denote $Q : \mathcal{C} \to \mathcal{C}'$ the localization functor of Categories, Lemma 4.27.16. Denote $\mathcal{D}$ the category of compactifyable schemes over S. It is clear from the lemma just cited and our choice of multiplicative system that we obtain a functor $\mathcal{C}' \to \mathcal{D}$. This functor is clearly essentially surjective. If $f : X \to Y$ is a morphism of compactifyable schemes, then we choose an open immersion $Y \to \overline{Y}$ into a scheme proper over S, and then we choose an embedding $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $\overline{Y}$ (possible by Lemma 38.32.2 applied to $X \to \overline{Y}$). This gives a morphism $(X, \overline{X}) \to (Y, \overline{Y})$ of compactifications which produces our given morphism $X \to Y$. Finally, suppose given a pair of morphisms in the localized category with the same source and target: say

$$a = ((f, \overline{f}) : (X', \overline{X}') \to (Y, \overline{Y}), (u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X}))$$

and

$$b = ((g, \overline{g}) : (X'', \overline{X}'') \to (Y, \overline{Y}), (v, \overline{v}) : (X'', \overline{X}'') \to (X, \overline{X}))$$

which produce the same morphism $X \to Y$ over S, in other words $f \circ u^{-1} = g \circ v^{-1}$. By Categories, Lemma 4.27.13 we may assume that $(X', \overline{X}') = (X'', \overline{X}'')$ and $(u, \overline{u}) = (v, \overline{v})$. In this case we can consider the equalizer $\overline{X}''' \subset \overline{X}'$ of $\overline{f}$ and $\overline{g}$. The morphism $(w, \overline{w}) : (X', \overline{X}''') \to (X', \overline{X}')$ is in the multiplicative subset and we see that $a = b$ in the localized category by precomposing with $(w, \overline{w})$. □

38.33. Nagata compactification

0F3T In this section we prove the theorem announced in Section 38.32.

0F3U **Lemma 38.33.1.** *Let $X \to S$ be a morphism of schemes. If $X = U \cup V$ is an open cover such that $U \to S$ and $V \to S$ are separated and $U \cap V \to U \times_S V$ is closed, then $X \to S$ is separated.*

Proof. Omitted. Hint: check that $\Delta : X \to X \times_S X$ is closed by using the open covering of $X \times_S X$ given by $U \times_S U$, $U \times_S V$, $V \times_S U$, and $V \times_S V$. □

0F3V **Lemma 38.33.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open.*

1. *If $Z_1, Z_2 \subset X$ are closed subschemes of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.*
2. *If $T_1, T_2 \subset U$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.*

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms, Lemma 29.21.7. Denote $Z \subset X$ the closed subscheme cut out by the product $\mathcal{I}_1\mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors, Lemma 31.34.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors, Definition 31.34.1 and Morphisms, Lemma 29.21.3). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as

the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors, Lemma 31.33.2. Thus (1) is proved.

Proof of (2). By Properties, Lemma 28.24.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Properties, Lemma 28.22.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subschemes $Z_i = V(\mathcal{I}_i)$ to conclude. □

0F3W **Lemma 38.33.3.** *Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset V$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in Y'.*

Proof. Let $T' \subset V$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in V and $T \cap T' = \emptyset$. Since V is a spectral topological space, we can find constructible closed subsets T_c, T'_c with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of V containing T' not meeting T and set $T_c = V \setminus W$, then choose a quasi-compact open W' of V containing T_c not meeting T' and set $T'_c = V \setminus W'$). By Lemma 38.33.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in Y. Set $Y_0 = Y \setminus \overline{T}'_c$, $V_0 = V \setminus T'_c$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see Lemma 38.31.3. By Divisors, Lemma 31.34.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. □

0F3X **Lemma 38.33.4.** *Let S be a quasi-compact and quasi-separated scheme. Let $U \to X_1$ and $U \to X_2$ be open immersions of schemes over S and assume U, X_1, X_2 of finite type and separated over S. Then there exists a commutative diagram*

$$\begin{array}{ccccc} X'_1 & \longrightarrow & X & \longleftarrow & X'_2 \\ \downarrow & \searrow & \uparrow & \swarrow & \downarrow \\ X_1 & \longleftarrow & U & \longrightarrow & X_2 \end{array}$$

of schemes over S where $X'_i \to X_i$ is a U-admissible blowup, $X'_i \to X$ is an open immersion, and X is separated and finite type over S.

Proof. Throughout the proof all schemes will be separated of finite type over S. This in particular implies these schemes are quasi-compact and quasi-separated and the morphisms between them are quasi-compact and separated. See Schemes, Sections 26.19 and 26.21. We will use that if $U \to W$ is an immersion of such schemes over S, then the scheme theoretic image Z of U in W is a closed subscheme of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $U \subset Z$ is dense topologically. See Morphisms, Lemma 29.7.7.

Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $U \to X_1 \times_S X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms, Lemma

29.6.8. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subscheme of X_i^i, see Lemma 38.31.3. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors, Lemma 31.33.2. Let X'_{12} be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors, Lemma 31.32.12 for what this entails. We obtain in particular a commutative diagram

$$\begin{array}{ccc} X'_{12} & \longrightarrow & X_{12}^2 \\ \downarrow & & \downarrow \\ X_{12}^1 & \longrightarrow & X_{12} \end{array}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X_i' \to X_i^i$ restricting to $X'_{12} \to X_{12}^i$, see Divisors, Lemma 31.34.3. Then $X'_{12} \subset X_i'$ is an open subscheme and the diagram

$$\begin{array}{ccc} X'_{12} & \longrightarrow & X_i' \\ \downarrow & & \downarrow \\ X_{12}^i & \longrightarrow & X_i^i \end{array}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X'_{12} \to X_1' \times_S X_2'$ is an immersion and proper (use that $X'_{12} \to X_{12}$ is proper and $X_{12} \to X_1 \times_S X_2$ is closed and $X_1' \times_S X_2' \to X_1 \times_S X_2$ is separated and apply Morphisms, Lemma 29.41.7). Thus $X'_{12} \to X_1' \times_S X_2'$ is a closed immersion. It follows that if we define X by glueing X_1' and X_2' along the common open subscheme X'_{12}, then $X \to S$ is of finite type and separated (Lemma 38.33.1). As compositions of U-admissible blowups are U-admissible blowups (Divisors, Lemma 31.34.2) the lemma is proved. □

0F3Y **Lemma 38.33.5.** *Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $U \subset X$ be an open subscheme. Let $V \to X \times_S Y$ be a quasi-compact morphism whose composition with the first projection maps into U. Let $Z \subset X \times_S Y$ be the scheme theoretic image of $V \to X \times_S Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_S Y$ is the strict transform of Z with respect to the blowing up.*

Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_S Y$ which is a closed immersion (as Z' is a closed subscheme of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subscheme such that $V \to Z'$ factors through Z''. Since both $V \to X \times_S Y$ and $V \to X' \times_S Y$ are quasi-compact (for the latter this follows from Schemes, Lemma 26.21.14 and the fact that $X' \times_S Y \to X \times_S Y$ is separated as a base change of a proper morphism), by Morphisms, Lemma 29.6.3 we see that $Z \cap (U \times_S Y) = Z'' \cap (U \times_S Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. □

0F3Z **Lemma 38.33.6.** *Let S be a quasi-compact and quasi-separated scheme. Let U be a scheme of finite type and separated over S. Let $V \subset U$ be a quasi-compact open. If V has a compactification $V \subset Y$ over S, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.*

Proof. Consider the scheme theoretic image $Z \subset Y \times_S U$ of the "diagonal" morphism $V \to Y \times_S U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 38.33.5. Hence by Lemma 38.31.3 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_S U \to U$ is proper and $V' \cong Z$ is a closed subscheme of $Y \times_S U$. □

The following lemma is formulated in the Noetherian case only. The version for quasi-compact and quasi-separated schemes is true as well, but will be trivially implied by the main theorem in this section.

0F40 **Lemma 38.33.7.** *Let S be a Noetherian scheme. Let U be a scheme of finite type and separated over S. Let $U = U_1 \cup U_2$ be opens such that U_1 and U_2 have compactifications over S and such that $U_1 \cap U_2$ is dense in U. Then U has a compactification over S.*

Proof. Choose a compactification $U_i \subset X_i$ for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and a proper morphism $\psi_i : V_i \to U$ extending $\mathrm{id} : U_i \to U_i$, see Lemma 38.33.6. Picture

$$\begin{array}{ccccc} U_i & \longrightarrow & V_i & \longrightarrow & X_i \\ \downarrow & \swarrow_{\psi_i} & & & \\ U & & & & \end{array}$$

If $\{i, j\} = \{1, 2\}$ denote $Z_i = U \setminus U_j = U_i \setminus (U_1 \cap U_2)$ and $Z_j = U \setminus U_i = U_j \setminus (U_1 \cap U_2)$. Thus we have

$$U = U_1 \amalg Z_2 = Z_1 \amalg U_2 = Z_1 \amalg (U_1 \cap U_2) \amalg Z_2$$

Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_i is an isomorphism over an open neighbourhood of Z_i. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subsets of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subsets of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the closures of $Z_{i,i}$ and $Z_{i,j}$ in X_i. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 38.33.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_S X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_S V_2$ (Schemes, Lemma 26.21.9) and the open immersion $V_1 \times_S V_2 \to X_1 \times_S X_2$. Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_S X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over S. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 38.33.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subscheme

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}(Z_{1,2}) = (p_2|_{V_{12}})^{-1}(Z_{2,2}) = \psi^{-1}(Z_2) \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 38.33.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict transform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $Z_{1,2}$ in X_1'. After replacing X_1 by X_1' and X_{12} by X_{12}' we may assume that p_1 is an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$.

The reduction of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_S X_2$. Namely, the inverse image $p_1^{-1}(\overline{Z}_{1,2})$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $Z_{12,2}$ is dense in $p_1^{-1}(\overline{Z}_{1,2})$. Thus p_2 maps $p_1^{-1}(\overline{Z}_{1,2})$ into $\overline{Z}_{2,2}$. Since $\overline{Z}_{2,2} \cap \overline{Z}_{2,1} = \emptyset$ we conclude.

Consider the schemes

$$W_i = U \coprod_{U_i} (X_i \setminus \overline{Z}_{i,j}), \quad i = 1, 2$$

obtained by glueing. Let us apply Lemma 38.33.1 to see that $W_i \to S$ is separated. First, $U \to S$ and $X_i \to S$ are separated. The immersion $U_i \to U \times_S (X_i \setminus \overline{Z}_{i,j})$ is closed because any specialization $u_i \rightsquigarrow u$ with $u_i \in U_i$ and $u \in U \setminus U_i$ can be lifted uniquely to a specialization $u_i \rightsquigarrow v_i$ in V_i along the proper morphism $\psi_i : V_i \to U$ and then v_i must be in $Z_{i,j}$. Thus the image of the immersion is closed, whence the immersion is a closed immersion.

On the other hand, for any valuation ring A over S with fraction field K and any morphism $\gamma : \mathrm{Spec}(K) \to (U_1 \cap U_2)$ over S, there is an i and an extension of γ to a morphism $h_i : \mathrm{Spec}(A) \to W_i$. Namely, for both $i = 1, 2$ there is a morphism $g_i : \mathrm{Spec}(A) \to X_i$ extending γ by the valuative criterion of properness for X_i over S, see Morphisms, Lemma 29.42.1. Thus we only are in trouble if $g_i(\mathfrak{m}_A) \in \overline{Z}_{i,j}$ for $i = 1, 2$. This is impossible by the emptyness of the intersection of X_{12} and $\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}$ we proved above.

Consider a diagram

$$\begin{array}{ccccc} W_1' & \longrightarrow & W & \longleftarrow & W_2' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ W_1 & \longleftarrow & U & \longrightarrow & W_2 \end{array}$$

as in Lemma 38.33.4. By the previous paragraph for every solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\gamma} & W \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where $\mathrm{Im}(\gamma) \subset U_1 \cap U_2$ there is an i and an extension $h_i : \mathrm{Spec}(A) \to W_i$ of γ. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \mathrm{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists. Since W is separated over S, we see that the arrow is unique as well. This implies that $W \to S$ is universally closed by Morphisms, Lemma 29.42.2. As $W \to S$ is already of finite type and separated, we win. □

0F41 **Theorem 38.33.8.** *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated, finite type morphism. Then X has a compactification over S.*

See [**Lüt93**], [**Con07b**], [**Nag56**], [**Nag57a**], [**Nag62a**], and [**Nag63**]

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. There exists a closed immersion $X \to X'$ with $X' \to S$ of finite presentation and separated. See Limits, Proposition 32.9.6. If we find a compactification of X' over S, then taking the scheme theoretic image of X in this will give a compactification of X over S. Thus we may assume $X \to S$ is separated and of finite presentation. We may write $S = \lim S_i$ as a directed limit of a system of Noetherian schemes with affine transition morphisms. See Limits, Proposition 32.5.4. We can choose an i and a morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $X_i \to S_i$ is separated, see Limits, Lemma 32.8.6. If we can find a compactification of X_i over S_i, then the base change of this to S will be a compactification of X over S. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. We can choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ such that $U_1 \cap \ldots \cap U_n$ is dense in X. This follows from Properties, Lemma 28.29.4 and the fact that X is quasi-compact with finitely many irreducible components. For each i we can choose an $n_i \geq 0$ and an immersion $U_i \to \mathbf{A}_S^{n_i}$ by Morphisms, Lemma 29.39.2. Hence U_i has a compactification over S for $i = 1, \ldots, n$ by taking the scheme theoretic image in $\mathbf{P}_S^{n_i}$. Applying Lemma 38.33.7 $(n-1)$ times we conclude that the theorem is true. □

38.34. The h topology

0ETQ For us, loosely speaking, an h sheaf is a sheaf for the Zariski topology which satisfies the sheaf property for surjective proper morphisms of finite presentation, see Lemma 38.34.17. However, it may be worth pointing out that the definition of the h topology on the category of schemes depends on the reference.

Voevodsky initially defined an h covering to be a finite collection of finite type morphisms which are jointly universally submersive (Morphisms, Definition 29.24.1). See [**Voe96**, Definition 3.1.2]. This definition works best if the underlying category of schemes is restricted to all schemes of finite type over a fixed Noetherian base scheme. In this setting, Voevodsky relates h coverings to ph coverings. The ph

topology is generated by Zariski coverings and proper surjective morphisms. See Topologies, Section 34.8 for more information.

In Topologies, Section 34.10 we study the V topology. A quasi-compact morphism $X \to Y$ defines a V covering, if any specialization of points of Y is the image of a specialization of points in X and the same is true after any base change (Topologies, Lemma 34.10.13). In this case $X \to Y$ is universally submersive (Topologies, Lemma 34.10.14). It turns out the notion of a V covering is a good replacement for "families of morphisms with fixed target which are jointly universally submersive" when working with non-Noetherian schemes.

Our approach will be to first prove the equivalence between ph covers and V coverings for (possibly infinite) families of morphisms which are locally of finite presentation. We will then use these families as our notion of h coverings in the Stacks project. For Noetherian schemes and finite families these coverings match those in Voevodsky's definition, see Lemma 38.34.3. On the category of schemes of finite presentation over a fixed quasi-compact and quasi-separated scheme S these coverings determine the same topology as the one in [**BS17**, Definition 2.7].

0ETR **Lemma 38.34.1.** *Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes with fixed target with f_i locally of finite presentation for all i. The following are equivalent*

(1) *$\{X_i \to X\}$ is a ph covering, and*
(2) *$\{X_i \to X\}$ is a V covering.*

Proof. Let $U \subset X$ be affine open. Looking at Topologies, Definitions 34.8.4 and 34.10.7 it suffices to show that the base change $\{X_i \times_X U \to U\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Thus we may assume X is affine and we have to show $\{X_i \to X\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Since a standard ph covering is a standard V covering, see Topologies, Lemma 34.10.3 it suffices to prove the other implication.

Assume X is affine and assume $\{f_i : X_i \to X\}_{i \in I}$ can be refined by a standard V covering $\{g_j : Y_j \to X\}_{j=1,\ldots,m}$. For each j choose an i_j and a morphism $h_j : Y_j \to X_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since Y_j is affine hence quasi-compact, for each j we can find finitely many affine opens $U_{j,k} \subset X_{i_j}$ such that $\mathrm{Im}(h_j) \subset \bigcup U_{j,k}$. Then $\{U_{j,k} \to X\}_{j,k}$ refines $\{X_i \to X\}$ and is a standard V covering (as it is a finite family of morphisms of affines and it inherits the lifting property for valuation rings from the corresponding property of $\{Y_j \to X\}$). Thus we reduce to the case discussed in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation. We have to show that $\{X_i \to X\}$ can be refined by a standard ph covering. Choose a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for the finitely presented morphism

$$\coprod\nolimits_{i=1,\ldots,n} f_i : \coprod\nolimits_{i=1,\ldots,n} X_i \longrightarrow X$$

of affines. We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is

flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (see Properties, Lemma 28.24.1), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 29.6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a standard ph covering and hence a standard V covering (see above). By the transitivity property of standard V coverings (Topologies, Lemma 34.10.5) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a standard V covering. This reduces us to the case described in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation and there is a dense quasi-compact open $U \subset X$ such that $X_i \times_X U \to U$ is flat. By Theorem 38.30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f_i' : X_i' \to X'$ of f_i is flat. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is scheme theoretically dense (see Remark 38.30.1). Hence for every point $x \in X'$ there is a valuation ring V and a morphism $g : \mathrm{Spec}(V) \to X'$ such that the generic point of $\mathrm{Spec}(V)$ maps into U and the closed point of $\mathrm{Spec}(V)$ maps to x, see Morphisms, Lemma 29.6.5. Since $\{X_i \to X\}$ is a standard V covering, we can choose an extension of valuation rings $V \subset W$, an index i, and a morphism $\mathrm{Spec}(W) \to X_i$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(W) & & \longrightarrow & & X_i \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & X' & \longrightarrow & X \end{array}$$

is commutative. Since $X_i' \subset X' \times_X X_i$ is a closed subscheme containing the open $U \times_X X_i$, since $\mathrm{Spec}(W)$ is an integral scheme, and since the induced morphism $h : \mathrm{Spec}(W) \to X' \times_X X_i$ maps the generic point of $\mathrm{Spec}(W)$ into $U \times_X X_i$, we conclude that h factors through the closed subscheme $X_i' \subset X' \times_X X_i$. We conclude that $\{f_i' : X_i' \to X'\}$ is a V covering. In particular, $\coprod f_i'$ is surjective. In particular $\{X_i' \to X'\}$ is an fppf covering. Since an fppf covering is a ph covering (More on Morphisms, Lemma 37.48.7), we can find a standard ph covering $\{Y_j \to X'\}$ refining $\{X_i' \to X\}$. Say this covering is given by a proper surjective morphism $Y \to X'$ and a finite affine open covering $Y = \bigcup Y_j$. Then the composition $Y \to X$ is proper surjective and we conclude that $\{Y_j \to X\}$ is a standard ph covering. This finishes the proof. □

Here is our definition.

0ETS **Definition 38.34.2.** Let T be a scheme. A *h covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ such that each f_i is locally of finite presentation and one of the equivalent conditions of Lemma 38.34.1 is satisfied.

For Noetherian schemes this is the same thing as a ph covering (we record this in Lemma 38.34.4 below) and we recover Voevodsky's notion.

0ETT **Lemma 38.34.3.** *Let X be a Noetherian scheme. Let $\{X_i \to X\}_{i \in I}$ be a finite family of finite type morphisms. The following are equivalent*

(1) $\coprod_{i\in I} X_i \to X$ *is universally submersive (Morphisms, Definition 29.24.1), and*
(2) $\{X_i \to X\}_{i\in I}$ *is an h covering.*

Proof. The implication (2) $\Rightarrow$ (1) follows from the more general Topologies, Lemma 34.10.14 and our definition of h covers. Assume $\coprod X_i \to X$ is universally submersive. We will show that $\{X_i \to X\}$ can be refined by a ph covering; this will suffice by Topologies, Lemma 34.8.7 and our definition of h coverings. The argument will be the same as the one used in the proof of Lemma 38.34.1.

Choose a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for the finitely presented morphism

$$\coprod\nolimits_{i=1,\ldots,n} f_i : \coprod\nolimits_{i=1,\ldots,n} X_i \longrightarrow X$$

We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (all schemes in this paragraph are Noetherian), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 29.6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a ph covering. By the transitivity property of ph coverings (Topologies, Lemma 34.8.8) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a ph covering. This reduces us to the case described in the next paragraph.

Assume $\coprod X_i \to X$ is universally submersive and there is a dense open $U \subset X$ such that $X_i \times_X U \to U$ is flat for all i. By Theorem 38.30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f_i' : X_i' \to X'$ of f_i is flat for all i. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is dense (see Remark 38.30.1). Hence for every point $x \in X'$ there is a discrete valuation ring A and a morphism $g : \mathrm{Spec}(A) \to X'$ such that the generic point of $\mathrm{Spec}(A)$ maps into U and the closed point of $\mathrm{Spec}(A)$ maps to x, see Limits, Lemma 32.15.1. Set

$$W = \mathrm{Spec}(A) \times_X \coprod X_i = \coprod \mathrm{Spec}(A) \times_X X_i$$

Since $\coprod X_i \to X$ is universally submersive, there is a specialization $w' \rightsquigarrow w$ in W such that w' maps to the generic point of $\mathrm{Spec}(A)$ and w maps to the closed point of $\mathrm{Spec}(A)$. (If not, then the closed fibre of $W \to \mathrm{Spec}(A)$ is stable under generalizations, hence open, which contradicts the fact that $W \to \mathrm{Spec}(A)$ is submersive.) Say $w' \in \mathrm{Spec}(A) \times_X X_i$ so of course $w \in \mathrm{Spec}(A) \times_X X_i$ as well. Let $x_i' \rightsquigarrow x_i$ be the image of $w' \rightsquigarrow w$ in $X' \times_X X_i$. Since $x_i' \in X_i'$ and since $X_i' \subset X' \times_X X_i$ is a closed subscheme we see that $x_i \in X_i'$. Since x_i maps to $x \in X'$ we conclude that $\coprod X_i' \to X'$ is surjective! In particular $\{X_i' \to X'\}$ is an fppf covering. But an fppf covering is a ph covering (More on Morphisms, Lemma 37.48.7). Since $X' \to X$ is proper surjective, we conclude that $\{X_i' \to X\}$ is a ph covering and the proof is complete. $\square$

0H45 **Lemma 38.34.4.** *Let X be a locally Noetherian scheme. A family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ with target X is an h covering if and only if it is a ph covering.*

Proof. By Definition 38.34.2 a h covering is a ph covering. Conversely, if $\{f_i : X_i \to X\}$ is a ph covering, then the morphisms f_i are locally of finite type (Topologies, Definition 34.8.4). Since X is locally Noetherian, each f_i is locally of finite presentation and we see that we have a h covering by definition. □

The following lemma and [**Ryd07b**, Theorem 8.4] shows our definition agrees with (or at least is closely related to) the definition in the paper [**Ryd07b**] by David Rydh. We restrict to affine base for simplicity.

0ETU **Lemma 38.34.5.** *Let X be an affine scheme. Let $\{X_i \to X\}_{i\in I}$ be an h covering. Then there exists a surjective proper morphism*

$$Y \longrightarrow X$$

of finite presentation (!) and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i\in I}$.

Proof. By assumption there exists a proper surjective morphism $Y \to X$ and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i\in I}$. This means that for each j there is an index $i_j \in I$ and a morphism $h_j : Y_j \to X_{i_j}$ over X. See Definition 38.34.2 and Topologies, Definition 34.8.4. The problem is that we don't know that $Y \to X$ is of finite presentation. By Limits, Lemma 32.13.2 we can write

$$Y = \lim Y_\lambda$$

as a directed limit of schemes Y_λ proper and of finite presentation over X such that the morphisms $Y \to Y_\lambda$ and the the transition morphisms are closed immersions. Observe that each $Y_\lambda \to X$ is surjective. By Limits, Lemma 32.4.11 we can find a λ and quasi-compact opens $Y_{\lambda,j} \subset Y_\lambda$, $j = 1, \ldots, m$ covering Y_λ and restricting to Y_j in Y. Then $Y_j = \lim Y_{\lambda,j}$. After increasing λ we may assume $Y_{\lambda,j}$ is affine for all j, see Limits, Lemma 32.4.13. Finally, since $X_i \to X$ is locally of finite presentation we can use the functorial characterization of morphisms which are locally of finite presentation (Limits, Proposition 32.6.1) to find a λ such that for each j there is a morphism $h_{\lambda,j} : Y_{\lambda,j} \to X_{i_j}$ whose restriction to Y_j is the morphism h_j chosen above. Thus $\{Y_{\lambda,j} \to X\}$ refines $\{X_i \to X\}$ and the proof is complete. □

We return to the development of the general theory of h coverings.

0ETV **Lemma 38.34.6.** *An fppf covering is a h covering. Hence syntomic, smooth, étale, and Zariski coverings are h coverings as well.*

Proof. This is true because in an fppf covering the morphisms are required to be locally of finite presentation and because fppf coverings are ph covering, see More on Morphisms, Lemma 37.48.7. The second statement follows from the first and Topologies, Lemma 34.7.2. □

0ETW **Lemma 38.34.7.** *Let $f : Y \to X$ be a surjective proper morphism of schemes which is of finite presentation. Then $\{Y \to X\}$ is an h covering.*

Proof. Combine Topologies, Lemmas 34.10.10 and 34.8.6. □

0ETX **Lemma 38.34.8.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms such that f_i is locally of finite presentation for all i. The following are equivalent*

(1) *$\{T_i \to T\}_{i\in I}$ is an h covering,*
(2) *there is an h covering which refines $\{T_i \to T\}_{i\in I}$, and*
(3) *$\{\coprod_{i\in I} T_i \to T\}$ is an h covering.*

Proof. This follows from the analogous statement for ph coverings (Topologies, Lemma 34.8.7) or from the analogous statement for V coverings (Topologies, Lemma 34.10.8). □

Next, we show that our notion of an h covering satisfies the conditions of Sites, Definition 7.6.2.

0ETY **Lemma 38.34.9.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an h covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is an h covering and for each i we have an h covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an h covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is an h covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an h covering.*

Proof. Follows immediately from the corresponding statement for either ph or V coverings (Topologies, Lemma 34.8.8 or 34.10.9) and the fact that the class of morphisms which are locally of finite presentation is preserved under base change and composition. □

Next, we define the big h sites we will work with in the Stacks project. It makes sense to read the general discussion in Topologies, Section 34.2 before proceeding.

0ETZ **Definition 38.34.10.** A *big h site* is any site Sch_h as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of h coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of h coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

0EUY **Definition 38.34.11.** Let T be an affine scheme. A *standard h covering* of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ with each T_i affine, with f_i of finite presentation satisfying either of the following equivalent conditions: (1) $\{T_i \to T\}$ can be refined by a standard ph covering or (2) $\{T_i \to T\}$ is a V covering.

The equivalence of the conditions follows from Lemma 38.34.1, Topologies, Definition 34.8.4, and Lemma 34.8.7.

Before we continue with the introduction of the big h site of a scheme S, let us point out that the topology on a big h site Sch_h is in some sense induced from the h topology on the category of all schemes.

0EU0 **Lemma 38.34.12.** *Let Sch_h be a big h site as in Definition 38.34.10. Let $T \in \mathrm{Ob}(Sch_h)$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary h covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_h which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard h covering, then it is tautologically equivalent to a covering of Sch_h.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_h.*

Proof. Omitted. Hint: this is exactly the same as the proof of Topologies, Lemma 34.8.10. □

0EU1 **Definition 38.34.13.** Let S be a scheme. Let Sch_h be a big h site containing S.

(1) The *big h site of S*, denoted $(Sch/S)_h$, is the site Sch_h/S introduced in Sites, Section 7.25.
(2) The *big affine h site of S*, denoted $(Aff/S)_h$, is the full subcategory of $(Sch/S)_h$ whose objects are affine U/S. A covering of $(Aff/S)_h$ is any covering $\{U_i \to U\}$ of $(Sch/S)_h$ which is a standard h covering.

We explicitly state that the big affine h site is a site.

0EU2 **Lemma 38.34.14.** *Let S be a scheme. Let Sch_h be a big h site containing S. Then $(Aff/S)_h$ is a site.*

Proof. Reasoning as in the proof of Topologies, Lemma 34.4.9 it suffices to show that the collection of standard h coverings satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard h covering $\{T_i \to T\}_{i\in I}$ and for each i a standard h covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a h covering (Lemma 38.34.9), $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. Thus $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard h covering. □

0EU3 **Lemma 38.34.15.** *Let S be a scheme. Let Sch_h be a big h site containing S. The underlying categories of the sites Sch_h, $(Sch/S)_h$, and $(Aff/S)_h$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_h$ has a final object, namely S/S.*

Proof. For Sch_h it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_h)$. The fibre product $V \times_U W$ in Sch_h is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_h$. This proves the result for $(Sch/S)_h$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_h$. □

Next, we check that the big affine site defines the same topos as the big site.

0EU4 **Lemma 38.34.16.** *Let S be a scheme. Let Sch_h be a big h site containing S. The functor $(Aff/S)_h \to (Sch/S)_h$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_h)$ to $Sh((Sch/S)_h)$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_h \to (Sch/S)_h$. Being cocontinuous follows because any h covering of T/S, T affine, can be refined by a standard h covering

for example by Lemma 38.34.5. Hence (1) holds. We see u is continuous simply because a standard h covering is a h covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a h covering). □

0EU5 **Lemma 38.34.17.** *Let $\mathcal{F}$ be a presheaf on $(Sch/S)_h$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and*
(2) *if $f : V \to U$ is proper, surjective, and of finite presentation, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ as in (2) with U affine.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a covering in $(Sch/S)_h$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ of finite presentation and a finite affine open covering $V_j = \bigcup V_{jk}$ such that $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. See Lemma 38.34.5. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_h$ which is proper, surjective, and of finite presentation. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. □

Next, we establish some relationships between the topoi associated to these sites.

0EU6 **Lemma 38.34.18.** *Let Sch_h be a big h site. Let $f : T \to S$ be a morphism in Sch_h. The functor*

$$u : (Sch/T)_h \longrightarrow (Sch/S)_h, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_h \longrightarrow (Sch/T)_h, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_h) \longrightarrow Sh((Sch/S)_h)$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *We have* $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. *Also,* f_{big}^{-1} *has a left adjoint* $f_{big!}$ *which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0EU7 **Lemma 38.34.19.** *Given schemes* X, Y, Y *in* $(Sch/S)_h$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *we have* $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 38.34.18. □

38.35. More on the h topology

0EUZ In this section we prove a few more results on the h topology. First, some non-examples.

0EV0 **Example 38.35.1.** The "structure sheaf" $\mathcal{O}$ is not a sheaf in the h topology. For example, consider a surjective closed immersion of finite presentation $X \to Y$. Then $\{X \to Y\}$ is an h covering for example by Lemma 38.34.7. Moreover, note that $X \times_Y X = X$. Thus if $\mathcal{O}$ where a sheaf in the h topology, then $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ would be bijective. This is not the case as soon as X, Y are affine and the morphism $X \to Y$ is not an isomorphism.

0EV1 **Example 38.35.2.** On any of the sites $(Sch/S)_h$ the topology is not subcanonical, in other words, representable sheaves are not sheaves. Namely, the "structure sheaf" $\mathcal{O}$ is representable because $\mathcal{O}(X) = \mathrm{Mor}_S(X, \mathbf{A}^1_S)$ in $(Sch/S)_h$ and we saw in Example 38.35.1 that $\mathcal{O}$ is not a sheaf.

0EV2 **Lemma 38.35.3.** *Let* T *be an affine scheme which is written as a limit* $T = \lim_{i \in I} T_i$ *of a directed inverse system of affine schemes.*

(1) *Let* $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ *be a standard h covering of* T, *see Definition 38.34.11. Then there exists an index* i *and a standard h covering* $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ *whose base change* $T \times_{T_i} \mathcal{V}_i$ *to* T *is isomorphic to* $\mathcal{V}$.
(2) *Let* $\mathcal{V}_i$, $\mathcal{V}'_i$ *be a pair of standard h coverings of* T_i. *If* $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ *is a morphism of coverings of* T, *then there exists an index* $i' \geq i$ *and a morphism* $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ *whose base change to* T *is* f.
(3) *If* $f, g : \mathcal{V} \to \mathcal{V}'_i$ *are morphisms of standard h coverings of* T_i *whose base changes* f_T, g_T *to* T *are equal then there exists an index* $i' \geq i$ *such that* $f_{T_{i'}} = g_{T_{i'}}$.

In other words, the category of standard h coverings of T *is the colimit over* I *of the categories of standard h coverings of* T_i.

Proof. By Limits, Lemma 32.10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemma 32.8.2 the same is true for category of schemes which are affine and

of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of finitely presented morphisms with $V_{j,i}$ affine, and the base change family $\{T \times_{T_i} V_{j,i} \to T\}$ is an h covering, then for some $i' \geq i$ the family $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$ is an h covering. To see this we use Lemma 38.34.5 to choose a finitely presented, proper, surjective morphism $Y \to T$ and a finite affine open covering $Y = \bigcup_{k=1,\ldots,n} Y_k$ such that $\{Y_k \to T\}_{k=1,\ldots,n}$ refines $\{T \times_{T_i} V_{j,i} \to T\}$. Using the arguments above and Limits, Lemmas 32.13.1, 32.8.15, and 32.4.11 we can find an $i' \geq i$ and a finitely presented, surjective, proper morphism $Y_{i'} \to T_{i'}$ and an affine open covering $Y_{i'} = \bigcup_{k=1,\ldots,n} Y_{i',k}$ such that moreover $\{Y_{i',k} \to Y_{i'}\}$ refines $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$. It follows that this last mentioned family is a h covering and the proof is complete. □

0EV3 **Lemma 38.35.4.** *Let S be a scheme contained in a big site Sch_h. Let $F : (Sch/S)_h^{opp} \to Sets$ be an h sheaf satisfying property (b) of Topologies, Lemma 34.13.1 with $\mathcal{C} = (Sch/S)_h$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all h coverings and is limit preserving (Limits, Remark 32.6.2).*

Proof. This is proven by the arguments given in the proofs of Topologies, Lemmas 34.13.3 and 34.13.4 using Lemmas 38.35.3 and 38.34.12. Details omitted. □

38.36. Blow up squares and the ph topology

0EW0 Let X be a scheme. Let $Z \subset X$ be a closed subscheme such that the inclusion morphism is of finite presentation, i.e., the quasi-coherent sheaf of ideals corresponding to Z is of finite type. Let $b : X' \to X$ be the blowup of X in Z and let $E = b^{-1}(Z)$ be the exceptional divisor. See Divisors, Section 31.32. In this situation and in this section, let us say

0EV5 (38.36.0.1)
$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

is a *blow up square*.

0EW1 **Lemma 38.36.1.** *Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$, see Topologies, Definition 34.8.11. Then for any blow up square (38.36.0.1) in the category $(Sch/S)_{ph}$ the diagram*
$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$
is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism we see that $\{Z \amalg X' \to X\}$ is a ph covering (Topologies, Lemma 34.8.6). We have
$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$
Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is

in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. □

0EW2 **Lemma 38.36.2.** *Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$ as in Topologies, Definition 34.8.11. Let $X \to X'$ be a morphism of $(Sch/S)_{ph}$ which is a thickening. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.*

Proof. Observe that $X \to X'$ is a proper surjective morphism of and $X \times_{X'} X = X$. By the sheaf property for the ph covering $\{X \to X'\}$ (Topologies, Lemma 34.8.6) we conclude. □

38.37. Almost blow up squares and the h topology

0EV4 Consider a blow up square (38.36.0.1). Although the morphism $b : X' \to X$ is projective (Divisors, Lemma 31.32.13) in general there is no simple way to guarantee that b is of finite presentation. Since h coverings are constructed using morphisms of finite presentation, we need a variant. Namely, we will say a commutative diagram

0EV6 (38.37.0.1)
$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

of schemes is an *almost blow up square* if the following conditions are satisfied

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 28.24.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$. For some very simple examples of almost blow up squares, see Examples 38.37.10 and 38.37.11.

The base change of a blow up usually isn't a blow up, but almost blow ups are compatible with base change.

0EV7 **Lemma 38.37.1.** *Consider an almost blow up square (38.37.0.1). Let $Y \to X$ be any morphism. Then the base change*

$$\begin{array}{ccc} Y \times_X E & \longrightarrow & Y \times_X X' \\ \downarrow & & \downarrow \\ Y \times_X Z & \longrightarrow & Y \end{array}$$

is an almost blow up square too.

Proof. The morphism $Y \times_X X' \to Y$ is proper and of finite presentation by Morphisms, Lemmas 29.41.5 and 29.21.4. The morphism $Y \times_X Z \to Y$ is a closed immersion (Morphisms, Lemma 29.2.4) of finite presentation. The inverse image of $Y \times_X Z$ in $Y \times_X X'$ is equal to the inverse image of E in $Y \times_X X'$ and hence is locally principal (Divisors, Lemma 31.13.11). Let $X'' \subset X'$, resp. $Y'' \subset Y \times_X X'$ be the closed subscheme corresponding to the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$, resp. $\mathcal{O}_{Y \times_Y X'}$ supported on E, resp. $Y \times_X E$. Clearly, $Y'' \subset Y \times_X X''$ is the closed subscheme corresponding to the quasi-coherent ideal of sections of $\mathcal{O}_{Y \times_Y X''}$ supported on $Y \times_X (E \cap X'')$. Thus Y'' is the strict transform of Y relative to the blowing up $X'' \to X$, see Divisors, Definition 31.33.1. Thus by Divisors, Lemma 31.33.2 we see that Y'' is the blow up of $Y \times_X Z$ on Y. □

One can shrink almost blow up squares.

0EV8 **Lemma 38.37.2.** *Consider an almost blow up square (38.37.0.1). Let $W \to X'$ be a closed immersion of finite presentation. The following are equivalent*

(1) *$X' \setminus E$ is scheme theoretically contained in W,*
(2) *the blowup X'' of X in Z is scheme theoretically contained in W,*
(3) *the diagram*

$$\begin{array}{ccc} E \cap W & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is an almost blow up square. Here $E \cap W$ is the scheme theoretic intersection.

Proof. Assume (1). Then the surjection $\mathcal{O}_{X'} \to \mathcal{O}_W$ is an isomorphism over the open $X' \subset E$. Since the ideal sheaf of $X'' \subset X'$ is the sections of $\mathcal{O}_{X'}$ supported on E (by our definition of almost blow up squares) we conclude (2) is true. If (2) is true, then (3) holds. If (3) holds, then (1) holds because $X'' \cap (X' \setminus E)$ is isomorphic to $X \setminus Z$ which in turn is isomorphic to $X' \setminus E$. □

The actual blowup is the limit of shrinkings of any given almost blowup.

0EV9 **Lemma 38.37.3.** *Consider an almost blow up square (38.37.0.1) with X quasi-compact and quasi-separated. Then the blowup X'' of X in Z can be written as*

$$X'' = \lim X'_i$$

where the limit is over the directed system of closed subschemes $X'_i \subset X'$ of finite presentation satisfying the equivalent conditions of Lemma 38.37.2.

Proof. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals corresponding to X''. By Properties, Lemma 28.22.3 we can write $\mathcal{I}$ as the filtered colimit $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ of its quasi-coherent submodules of finite type. Since these modules correspond 1-to-1 to the closed subschemes X'_i the proof is complete. □

Almost blow up squares exist.

0EVA **Lemma 38.37.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Then there exists an almost blow up square as in (38.37.0.1).*

Proof. We may write $X = \lim X_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 32.5.4. We can find an index i and a closed immersion $Z_i \to X_i$ whose base change to X is the closed immersion $Z \to X$. See Limits, Lemmas 32.10.1 and 32.8.5. Let $b_i : X_i' \to X_i$ be the blowing up with center Z_i. This produces a blow up square

$$\begin{array}{ccc} E_i & \longrightarrow & X_i' \\ \downarrow & & \downarrow{\scriptstyle b_i} \\ Z_i & \longrightarrow & X_i \end{array}$$

where all the morphisms are finite type morphisms of Noetherian schemes and hence of finite presentation. Thus this is an almost blow up square. By Lemma 38.37.1 the base change of this diagram to X produces the desired almost blow up square. □

Almost blow up squares are unique up to shrinking as in Lemma 38.37.2.

0EVB **Lemma 38.37.5.** *Let X be a quasi-compact and quasi-separated scheme and let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Suppose given almost blow up squares (38.37.0.1)*

$$\begin{array}{ccc} E_k & \longrightarrow & X_k' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

for $k = 1, 2$, then there exists an almost blow up square

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

and closed immersions $i_k : X' \to X_k'$ over X with $E = i_k^{-1}(E_k)$.

Proof. Denote $X'' \to X$ the blowing up of Z in X. We view X'' as a closed subscheme of both X_1' and X_2'. Write $X'' = \lim X_{1,i}'$ as in Lemma 38.37.3. By Limits, Proposition 32.6.1 there exists an i and a morphism $h : X_{1,i}' \to X_2'$ agreeing with the inclusions $X'' \subset X_{1,i}'$ and $X'' \subset X_2'$. By Limits, Lemma 32.4.20 the restriction of h to $X_{1,i'}'$ is a closed immersion for some $i' \geq i$. This finishes the proof. □

Our flattening techniques for blowing up are inherited by almost blowups in favorable situations.

0EVC **Lemma 38.37.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let X be a scheme of finite presentation over Y. Let $V \subset Y$ be a quasi-compact open such*

that $X_V \to V$ is flat. Then there exist a commutative diagram

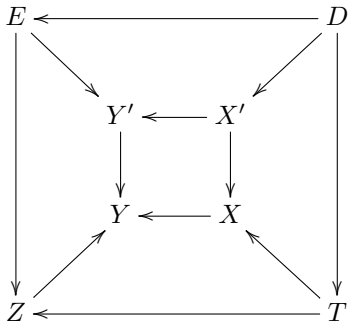

whose right and left hand squares are almost blow up squares, whose lower and top squares are cartesian, such that $Z \cap V = \emptyset$, and such that $X' \to Y'$ is flat (and of finite presentation).

Proof. If Y is a Noetherian scheme, then this lemma follows immediately from Lemma 38.31.1 because in this case blow up squares are almost blow up squares (we also use that strict transforms are blow ups). The general case is reduced to the Noetherian case by absolute Noetherian approximation.

We may write $Y = \lim Y_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 32.5.4. We can find an index i and a morphism $X_i \to Y_i$ of finite presentation whose base change to Y is $X \to Y$. See Limits, Lemmas 32.10.1. After increasing i we may assume V is the inverse image of an open subscheme $V_i \subset Y_i$, see Limits, Lemma 32.4.11. Finally, after increasing i we may assume that $X_{i,V_i} \to V_i$ is flat, see Limits, Lemma 32.8.7. By the Noetherian case, we may construct a diagram as in the lemma for $X_i \to Y_i \supset V_i$. The base change of this diagram by $Y \to Y_i$ provides the solution. Use that base change preserves properties of morphisms, see Morphisms, Lemmas 29.41.5, 29.21.4, 29.2.4, and 29.25.8 and that base change of an almost blow up square is an almost blow up square, see Lemma 38.37.1. □

0EVD **Lemma 38.37.7.** *Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then for any almost blow up square (38.37.0.1) in the category $(Sch/S)_h$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism of finite presentation we see that $\{Z \amalg X' \to X\}$ is an h covering (Lemma 38.34.7). We have

$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$

Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is

in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism of finite presentation as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. □

0EVE **Lemma 38.37.8.** *Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Let $X \to X'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.*

Proof. First proof. Observe that $X \to X'$ is a proper surjective morphism of finite presentation and $X \times_{X'} X = X$. By the sheaf property for the h covering $\{X \to X'\}$ (Lemma 38.34.7) we conclude.

Second proof (silly). The blow up of X' in X is the empty scheme. The reason is that the affine blowup algebra $A[\frac{I}{a}]$ (Algebra, Section 10.70) is zero if a is a nilpotent element of A. Details omitted. Hence we get an almost blow up square of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

Since $\mathcal{F}$ is a sheaf we have that $\mathcal{F}(\emptyset)$ is a singleton. Applying Lemma 38.37.7 we get the conclusion. □

0EVF **Proposition 38.37.9.** *Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied*

1. *$\mathcal{F}$ is a sheaf for the Zariski topology,*
2. *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,*
3. *given an almost blow up square (38.37.0.1) with X affine in the category $(Sch/S)_h$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets.

Proof. Assume $\mathcal{F}$ is a sheaf. Condition (1) holds because a Zariski covering is a h covering, see Lemma 38.34.6. Condition (2) holds because for f as in (2) we have that $\{X \to Y\}$ is an fppf covering (this is clear) and hence an h covering, see Lemma 38.34.6. Condition (3) holds by Lemma 38.37.7.

Conversely, assume $\mathcal{F}$ satisfies (1), (2), and (3). We will prove $\mathcal{F}$ is a sheaf by applying Lemma 38.34.17. Consider a surjective, finitely presented, proper morphism $f : X \to Y$ in $(Sch/S)_h$ with Y affine. It suffices to show that $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$.

First, assume that $f : X \to Y$ is in addition a closed immersion (in other words, f is a thickening). Then the blow up of Y in X is the empty scheme and this produces an almost blow up square consisting with $\emptyset, \emptyset, X, Y$ at the vertices (compare with the second proof of Lemma 38.37.8). Hence we see that condition (3) tells us that

$$\begin{array}{ccc} \mathcal{F}(\emptyset) & \longleftarrow & \mathcal{F}(\emptyset) \\ \uparrow & & \uparrow \\ \mathcal{F}(X) & \longleftarrow & \mathcal{F}(Y) \end{array}$$

is cartesian in the category of sets. Since $\mathcal{F}$ is a sheaf for the Zariski topology, we see that $\mathcal{F}(\emptyset)$ is a singleton. Hence we see that $\mathcal{F}(X) = \mathcal{F}(Y)$.

Interlude A: let $T \to T'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(T') \to \mathcal{F}(T)$ is bijective. Namely, choose an affine open covering $T' = \bigcup T'_i$ and let $T_i = T \times_{T'} T'_i$ be the corresponding affine opens of T. Then we have $\mathcal{F}(T'_i) \to \mathcal{F}(T_i)$ is bijective for all i by the result of the previous paragraph. Using the Zariski sheaf property we see that $\mathcal{F}(T') \to \mathcal{F}(T)$ is injective. Repeating the argument we find that it is bijective. Minor details omitted.

Interlude B: consider an almost blow up square (38.37.0.1) in the category $(Sch/S)_h$. Then we claim the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets. This is a consequence of condition (3) as follows by choosing an affine open covering of X and arguing as in Interlude A. We omit the details.

Next, let $f : X \to Y$ be a surjective, finitely presented, proper morphism in $(Sch/S)_h$ with Y affine. Choose a generic flatness stratification

$$Y \supset Y_0 \supset Y_1 \supset \ldots \supset Y_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for $f : X \to Y$. We are going to use all the properties of the stratification without further mention. Set $X_0 = X \times_Y Y_0$. By the Interlude B we have $\mathcal{F}(Y_0) = \mathcal{F}(Y)$, $\mathcal{F}(X_0) = \mathcal{F}(X)$, and $\mathcal{F}(X_0 \times_{Y_0} X_0) = \mathcal{F}(X \times_Y X)$.

We are going to prove the result by induction on t. If $t = 1$ then $X_0 \to Y_0$ is surjective, proper, flat, and of finite presentation and we see that the result holds by property (2). For $t > 1$ we may replace Y by Y_0 and X by X_0 (see above) and assume $Y = Y_0$.

Consider the quasi-compact open subscheme $V = Y \setminus Y_1 = Y_0 \setminus Y_1$. Choose a diagram

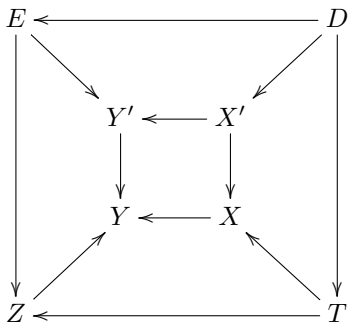

as in Lemma 38.37.6 for $f : X \to Y \supset V$. Then $f' : X' \to Y'$ is flat and of finite presentation. Also f' is proper (use Morphisms, Lemmas 29.41.4 and 29.41.7 to see this). Thus the image $W = f'(X') \subset Y'$ is an open (Morphisms, Lemma 29.25.10) and closed subscheme of Y'. Observe that $Y' \setminus E$ is contained in W. By Lemma 38.37.2 this means we may replace Y' by W in the above diagram. In other words, we may and do assume f' is surjective. At this point we know that

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(Y') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(D) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(T) & \longleftarrow & \mathcal{F}(X) \end{array}$$

are cartesian by Interlude B. Note that $Z \cap Y_1 \to Z$ is a thickening of finite presentation (as Z is set theoretically contained in Y_1 as a closed subscheme of Y disjoint from V). Thus we obtain a filtration

$$Z \supset Z \cap Y_1 \supset Z \cap Y_2 \subset \ldots \subset Z \cap Y_t = \emptyset$$

as above for the restriction $T = Z \times_Y X \to Z$ of f to T. Thus by induction hypothesis we find that $\mathcal{F}(Z) \to \mathcal{F}(T)$ is an injective map of sets whose image is the equalizer of the two maps $\mathcal{F}(T) \to \mathcal{F}(T \times_Z T)$.

Let $s \in \mathcal{F}(X)$ be in the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$. By the above we see that the restriction $s|_T$ comes from a unique element $t \in \mathcal{F}(Z)$ and similarly that the restriction $s|_{X'}$ comes from a unique element $t' \in \mathcal{F}(Y')$. Chasing sections using the restriction maps for $\mathcal{F}$ corresponding to the arrows in the huge commutative diagram above the reader finds that t and t' restrict to the same element of $\mathcal{F}(E)$ because they restrict to the same element of $\mathcal{F}(D)$ and we have (2); here we use that $D \to E$ is surjective, flat, proper, and of finite presentation as the restriction of $X' \to Y'$. Thus by the first of the two cartesian squares displayed above we get a unique section $u \in \mathcal{F}(Y)$ restricting to t and t' on Z and Y'. To see that u restrict to s on X use the second diagram. $\square$

0EVG **Example 38.37.10.** Let A be a ring. Let $f \in A$ be an element. Let $J \subset A$ be a finitely generated ideal annihilated by a power of f. Then

$$\begin{array}{ccc} E = \mathrm{Spec}(A/fA + J) & \longrightarrow & \mathrm{Spec}(A/J) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/fA) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVH **Example 38.37.11.** Let A be a ring. Let $f_1, f_2 \in A$ be elements.

$$\begin{array}{ccc} E = \mathrm{Proj}(A/(f_1, f_2)[T_0, T_1]) & \longrightarrow & \mathrm{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/(f_1, f_2)) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVI **Lemma 38.37.12.** *Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied*

1. *$\mathcal{F}$ is a sheaf for the Zariski topology,*
2. *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,*
3. *$\mathcal{F}$ turns an almost blow up square as in Example 38.37.10 in the category $(Sch/S)_h$ into a cartesian diagram of sets, and*
4. *$\mathcal{F}$ turns an almost blow up square as in Example 38.37.11 in the category $(Sch/S)_h$ into a cartesian diagram of sets.*

Proof. By Proposition 38.37.9 it suffices to show that given an almost blow up square (38.37.0.1) with X affine in the category $(Sch/S)_h$ the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets. The rough idea of the proof is to dominate the morphism by other almost blowup squares to which we can apply assumptions (3) and (4) locally.

Suppose we have an almost blow up square (38.37.0.1) in the category $(Sch/S)_h$, an open covering $X = \bigcup U_i$, and open coverings $U_i \cap U_j = \bigcup U_{ijk}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_i)) & \longleftarrow & \mathcal{F}(b^{-1}(U_i)) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_i) & \longleftarrow & \mathcal{F}(U_i) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_{ijk})) & \longleftarrow & \mathcal{F}(b^{-1}(U_{ijk})) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_{ijk}) & \longleftarrow & \mathcal{F}(U_{ijk}) \end{array}$$

are cartesian, then the same is true for

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

This follows as $\mathcal{F}$ is a sheaf in the Zariski topology.

In particular, if we have a blow up square (38.37.0.1) such that $b : X' \to X$ is a closed immersion and Z is a locally principal closed subscheme, then we see that $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$. Namely, affine locally on X we obtain an almost blow up square as in (3).

Let $Z \subset X$, $E_k \subset X'_k \to X$, $E \subset X' \to X$, and $i_k : X' \to X'_k$ be as in the statement of Lemma 38.37.5. Then

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & X'_k \end{array}$$

is an almost blow up square of the kind discussed in the previous paragraph. Thus

$$\mathcal{F}(X'_k) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(E_k)$$

for $k = 1, 2$ by the result of the previous paragraph. It follows that

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(X'_k) \times_{\mathcal{F}(E_k)} \mathcal{F}(Z)$$

is bijective for $k = 1$ if and only if it is bijective for $k = 2$. Thus given a closed immersion $Z \to X$ of finite presentation with X quasi-compact and quasi-separated, whether or not $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$ is independent of the choice of the almost blow up square (38.37.0.1) one chooses. (Moreover, by Lemma 38.37.4 there does indeed exist an almost blow up square for $Z \subset X$.)

Finally, consider an affine object X of $(Sch/S)_h$ and a closed immersion $Z \to X$ of finite presentation. We will prove the desired property for the pair (X, Z) by induction on the number of generators r for the ideal defining Z in X. If the number of generators is ≤ 2, then we can choose our almost blow up square as in Example 38.37.11 and we conclude by assumption (4).

Induction step. Suppose $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$ with $r > 2$. Choose a blow up square (38.37.0.1) for the pair (X, Z). Set $Z_1 = \mathrm{Spec}(A/(f_1, f_2))$ and let

$$\begin{array}{ccc} E_1 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_1 & \longrightarrow & X \end{array}$$

be the almost blow up square constructed in Example 38.37.11. By Lemma 38.37.1 the base changes

$$(I) \quad \begin{array}{ccc} Y \times_X E & \longrightarrow & Y \times_X X' \\ \downarrow & & \downarrow \\ Y \times_X Z & \longrightarrow & Y \end{array} \qquad \text{and} \qquad (II) \quad \begin{array}{ccc} E & \longrightarrow & Z_1 \times_X X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z_1 \end{array}$$

are almost blow up squares. The ideal of Z in Z_1 is generated by $r-2$ elements. The ideal of $Y \times_X Z$ is generated by the pullbacks of $f_1, \ldots, f_r$ to Y. Locally on Y the ideal generated by f_1, f_2 can be generated by one element, thus $Y \times_X Z$ is affine locally on Y cut out by at most $r-1$ elements. By induction hypotheses and the discussion above

$$\mathcal{F}(Y) = \mathcal{F}(Y \times_X X') \times_{\mathcal{F}(Y \times_X E)} \mathcal{F}(Y \times_X Z)$$

and

$$\mathcal{F}(Z_1) = \mathcal{F}(Z_1 \times_X X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$$

By assumption (4) we have

$$\mathcal{F}(X) = \mathcal{F}(Y) \times_{\mathcal{F}(E_1)} \mathcal{F}(Z_1)$$

Now suppose we have a pair (s', t) with $s' \in \mathcal{F}(X')$ and $t \in \mathcal{F}(Z)$ with same restriction in $\mathcal{F}(E)$. Then $(s'|Z_1 \times_X X', t)$ are the image of a unique element $t_1 \in \mathcal{F}(Z_1)$. Similarly, $(s'|_{Y \times_X X'}, t|_{Y \times_X Z})$ are the image of a unique element $s_Y \in \mathcal{F}(Y)$. We claim that s_Y and t_1 restrict to the same element of $\mathcal{F}(E_1)$. This is true because the almost blow up square

$$\begin{array}{ccc} E_1 \times_X E & \longrightarrow & E_1 \times_X X' \\ \downarrow & & \downarrow \\ E_1 \times_X Z & \longrightarrow & E_1 \end{array}$$

is the base change of almost blow up square (I) via $E_1 \to Y$ and the base change of almost blow up square (II) via $E_1 \to Z_1$ and because the pairs of sections used to construct s_Y and t_1 match. Thus by the third fibre product equality we see that there is a unique $s \in \mathcal{F}(X)$ mapping to s_Y in $\mathcal{F}(Y)$ and to t_1 in $\mathcal{F}(Z)$. We omit the verification that s maps to s' in $\mathcal{F}(X')$ and to t in $\mathcal{F}(Z)$; hint: use uniqueness of s just constructed and work affine locally. □

0EX9 **Lemma 38.37.13.** *Let $p : \mathcal{S} \to (Sch/S)_h$ be a category fibred in groupoids. Then $\mathcal{S}$ is a stack in groupoids if and only if the following conditions are satisfied*

(1) *$\mathcal{S}$ is a stack in groupoids for the Zariski topology,*
(2) *given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then*
$$\mathcal{S}_Y \longrightarrow \mathcal{S}_X \times_{\mathcal{S}_{X \times_Y X}} \mathcal{S}_X$$
is an equivalence of categories,
(3) *for an almost blow up square as in Example 38.37.10 or 38.37.11 in the category $(Sch/S)_h$ the functor*
$$\mathcal{S}_X \longrightarrow \mathcal{S}_Z \times_{\mathcal{S}_E} \mathcal{S}_{X'}$$
is an equivalence of categories.

Proof. This lemma is a formal consequence of Lemma 38.37.12 and our definition of stacks in groupoids. For example, assume (1), (2), (3). To show that $\mathcal{S}$ is a stack, we have to prove descent for morphisms and objects, see Stacks, Definition 8.5.1.

If x, y are objects of $\mathcal{S}$ over an object U of $(Sch/S)_h$, then our assumptions imply $Isom(x, y)$ is a presheaf on $(Sch/U)_h$ which satisfies (1), (2), (3), and (4) of Lemma 38.37.12 and therefore is a sheaf. Some details omitted.

Let $\{U_i \to U\}_{i\in I}$ be a covering of $(Sch/S)_h$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{S}$ relative to the family $\{U_i \to U\}_{i\in I}$, see Stacks, Definition 8.3.1. Consider the rule F which to V/U in $(Sch/U)_h$ associates the set of pairs (y, ψ_i) where y is an object of $\mathcal{S}_V$ and $\psi_i : y|_{U_i\times_U V} \to x_i|_{U_i\times_U V}$ is a morphism of $\mathcal{S}$ over $U_i \times_U V$ such that

$$\varphi_{ij}|_{U_i\times_U U_j\times_U V} \circ \psi_i|_{U_i\times_U U_j\times_U V} = \psi_j|_{U_i\times_U U_j\times_U V}$$

up to isomorphism. Since we already have descent for morphisms, it is clear that $F(V/U)$ is either empty or a singleton set. On the other hand, we have $F(U_{i_0}/U)$ is nonempty because it contains $(x_{i_0}, \varphi_{i_0 i})$. Since our goal is to prove that $F(U/U)$ is nonempty, it suffices to show that F is a sheaf on $(Sch/U)_h$. To do this we may use the criterion of Lemma 38.37.12. However, our assumptions (1), (2), (3) imply (by drawing some commutative diagrams which we omit), that properties (1), (2), (3), and (4) of Lemma 38.37.12 hold for F.

We omit the verification that if $\mathcal{S}$ is a stack in groupoids, then (1), (2), and (3) are satisfied. □

38.38. Absolute weak normalization and h coverings

0EVS In this section we use the criteria found in Section 38.37 to exhibit some h sheaves and we relate h sheafification of the structure sheaf to absolute weak normalization. We will need the following elementary lemma to do this.

0EVJ **Lemma 38.38.1.** *Let Z, X, X', E be an almost blow up square as in Example 38.37.11. Then $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$ and $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is a surjective map of rings whose kernel is an ideal of square zero.*

Proof. First assume that $A = \mathbf{Z}[f_1, f_2]$ is the polynomial ring. In this case our almost blow up square is the blowing up of $X = \mathrm{Spec}(A)$ in the closed subscheme Z and in fact $X' \subset \mathbf{P}^1_X$ is an effective Cartier divisor cut out by the global section $f_2T_0 - f_1T_1$ of $\mathcal{O}_{\mathbf{P}^1_X}(1)$. Thus we have a resolution

$$0 \to \mathcal{O}_{\mathbf{P}^1_X}(-1) \to \mathcal{O}_{\mathbf{P}^1_X} \to \mathcal{O}_{X'} \to 0$$

Using the description of the cohomology given in Cohomology of Schemes, Section 30.8 it follows that in this case $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is an isomorphism and $H^1(X', \mathcal{O}_{X'}) = 0$.

Next, we observe that any diagram as in Example 38.37.11 is the base change of the diagram in the previous paragraph by the ring map $\mathbf{Z}[f_1, f_2] \to A$. Hence by More on Morphisms, Lemmas 37.72.1, 37.72.2, and 37.72.4 we conclude that $H^1(X', \mathcal{O}_{X'})$ is zero in general and the surjectivity of the map $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ in general.

Next, in the general case, let us study the kernel. If $a \in A$ maps to zero, then looking on affine charts we see that

$$a = (f_1x - f_2)(a_0 + a_1x + \ldots + a_rx^r) \text{ in } A[x]$$

for some $r \geq 0$ and $a_0, \ldots, a_r \in A$ and similarly

$$a = (f_1 - f_2y)(b_0 + b_1y + \ldots + b_sy^s) \text{ in } A[y]$$

for some $s \geq 0$ and $b_0, \ldots, b_s \in A$. This means we have

$$a = f_2a_0,\ f_1a_0 = f_2a_1,\ \ldots,\ f_1a_r = 0,\ a = f_1b_0,\ f_2b_0 = f_1b_1,\ \ldots,\ f_2b_s = 0$$

If (a', r', a'_i, s', b'_j) is a second such system, then we have

$$aa' = f_1 f_2 a_0 b'_0 = f_1 f_2 a_1 b'_1 = f_1 f_2 a_2 b'_2 = \ldots = 0$$

as desired. □

For an $\mathbf{F}_p$-algebra A we set $\operatorname{colim}_F A$ equal to the colimit of the system

$$A \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} \ldots$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVK **Lemma 38.38.2.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 38.34.13. There is a unique sheaf $\mathcal{F}$ on $(Sch/S)_h$ such that*

$$\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$$

for any quasi-compact and quasi-separated object X of $(Sch/S)_h$.

Proof. Denote $\mathcal{F}$ the Zariski sheafification of the functor

$$X \longrightarrow \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$$

For quasi-compact and quasi-separated schemes X we have $\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$. by Sheaves, Lemma 6.29.1 and the fact that $\mathcal{O}$ is a sheaf for the Zariski topology. Thus it suffices to show that $\mathcal{F}$ is a h sheaf. To prove this we check conditions (1), (2), (3), and (4) of Lemma 38.37.12. Condition (1) holds because we performed an (almost unnecessary) Zariski sheafification. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 35.8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\operatorname{colim}_F A$ is the equalizer of the two maps $\operatorname{colim}_F B \to \operatorname{colim}_F C$.

We check condition (3). Let A, f, J be as in Example 38.37.10. We have to show that

$$\operatorname{colim}_F A = \operatorname{colim}_F A/J \times_{\operatorname{colim}_F A/fA+J} \operatorname{colim}_F A/fA$$

This reduces to the following algebra question: suppose $a', a'' \in A$ are such that $F^n(a' - a'') \in fA + J$. Find $a \in A$ and $m \geq 0$ such that $a - F^m(a') \in J$ and $a - F^m(a'') \in fA$ and show that the pair (a, m) is uniquely determined up to a replacement of the form $(a, m) \mapsto (F(a), m+1)$. To do this just write $F^n(a'-a'') = fh+g$ with $h \in A$ and $g \in J$ and set $a = F^n(a') - g = F^n(a'') + fh$ and set $m = n$. To see uniqueness, suppose (a_1, m_1) is a second solution. By a replacement of the form given above we may assume $m = m_1$. Then we see that $a - a_1 \in J$ and $a - a_1 \in fA$. Since J is annihilated by a power of f we see that $a - a_1$ is a nilpotent element. Hence $F^k(a - a_1)$ is zero for some large k. Thus after doing more replacements we get $a = a_1$.

We check condition (4). Let X, X', Z, E be as in Example 38.37.11. By Lemma 38.38.1 we see that

$$\mathcal{F}(X) = \operatorname{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \operatorname{colim}_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

Let p be a prime number. For an $\mathbf{F}_p$-algebra A we set $\lim_F A$ equal to the limit of the inverse system

$$\ldots \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} A$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVL **Lemma 38.38.3.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 38.34.13. The rule*

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X)$$

defines a sheaf on $(Sch/S)_h$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1), (2), (3), and (4) of Lemma 38.37.12. Condition (1) holds because limits of sheaves are sheaves and $\mathcal{O}$ is a Zariski sheaf. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 35.8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\lim_F A$ is the equalizer of the two maps $\lim_F B \to \lim_F C$.

We check condition (3). Let A, f, J be as in Example 38.37.10. We have to show that

$$\begin{aligned} \lim_F A &\to \lim_F A/J \times_{\lim_F A/fA+J} \lim_F A/fA \\ &= \lim_F (A/J \times_{A/fA+J} A/fA) \\ &= \lim_F A/(fA \cap J) \end{aligned}$$

is bijective. Since J is annihilated by a power of f we see that $\mathfrak{a} = fA \cap J$ is a nilpotent ideal, i.e., there exists an n such that $\mathfrak{a}^n = 0$. It is straightforward to verify that in this case $\lim_F A \to \lim_F A/\mathfrak{a}$ is bijective.

We check condition (4). Let X, X', Z, E be as in Example 38.37.11. By Lemma 38.38.1 and the same argument as above we see that

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X) \longrightarrow \lim_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

In the following lemma we use the absolute weak normalization X^{awn} of a scheme X, see Morphisms, Section 29.47.

0EVT **Lemma 38.38.4.** *Let $(Sch/S)_{ph}$ be a site as in Topologies, Definition 34.8.11. The rule*

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is a sheaf on $(Sch/S)_{ph}$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1) and (2) of Topologies, Lemma 34.8.15. Condition (1) holds because formation of X^{awn} commutes with open coverings, see Morphisms, Lemma 29.47.7 and its proof.

Let $\pi : Y \to X$ be a surjective proper morphism. We have to show that the equalizer of the two maps

$$\Gamma(Y^{awn}, \mathcal{O}_{Y^{awn}}) \to \Gamma((Y \times_X Y)^{awn}, \mathcal{O}_{(Y \times_X Y)^{awn}})$$

is equal to $\Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$. Let f be an element of this equalizer. Then we consider the morphism

$$f : Y^{awn} \longrightarrow \mathbf{A}^1_X$$

Since $Y^{awn} \to X$ is universally closed, the scheme theoretic image Z of f is a closed subscheme of $\mathbf{A}^1_X$ proper over X and $f : Y^{awn} \to Z$ is surjective. See Morphisms, Lemma 29.41.10. Thus $Z \to X$ is finite (Morphisms, Lemma 29.44.11) and surjective.

Let k be a field and let $z_1, z_2 : \mathrm{Spec}(k) \to Z$ be two morphisms equalized by $Z \to X$. We claim that $z_1 = z_2$. It suffices to show the images $\lambda_i = z_i^* f \in k$ agree (as the structure sheaf of Z is generated by f over the structure sheaf of X). To see this we choose a field extension K/k and morphisms $y_1, y_2 : \mathrm{Spec}(K) \to Y^{awn}$ such that $z_i \circ (\mathrm{Spec}(K) \to \mathrm{Spec}(k)) = f \circ y_i$. This is possible by the surjectivity of the map $Y^{awn} \to Z$. Choose an algebraically closed extension Ω/k of very large cardinality. For any k-algebra maps $\sigma_i : K \to \Omega$ we obtain

$$\mathrm{Spec}(\Omega) \xrightarrow{\sigma_1, \sigma_2} \mathrm{Spec}(K \otimes_k K) \xrightarrow{y_1, y_2} Y^{awn} \times_X Y^{awn}$$

Since the canonical morphism $(Y \times_X Y)^{awn} \to Y^{awn} \times_X Y^{awn}$ is a universal homeomorphism and since Ω is algebraically closed, we can lift the composition above uniquely to a morphism $\mathrm{Spec}(\Omega) \to (Y \times_X Y)^{awn}$. Since f is in the equalizer above, this proves that $\sigma_1(\lambda_1) = \sigma_2(\lambda_2)$. An easy lemma about field extensions shows that this implies $\lambda_1 = \lambda_2$; details omitted.

We conclude that $Z \to X$ is universally injective, i.e., $Z \to X$ is injective on points and induces purely inseparated residue field extensions (Morphisms, Lemma 29.10.2). All in all we conclude that $Z \to X$ is a universal homeomorphism, see Morphisms, Lemma 29.45.5.

Let $g : X^{awn} \to Z$ be the map obtained from the universal property of X^{awn}. Then $Y^{awn} \to X^{awn} \to Z$ and $f : Y^{awn} \to Z$ are two morphisms over X. By the universal property of $Y^{awn} \to Y$ the two corresponding morphisms $Y^{awn} \to Y \times_X Z$ over Y have to be equal. This implies that $g \circ \pi^{wan} = f$ as morphisms into $\mathbf{A}^1_X$ and we conclude that $g \in \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$ is the element we were looking for. $\square$

0EVU **Lemma 38.38.5.** *Let S be a scheme. Choose a site $(Sch/S)_h$ as in Definition 38.34.13. The rule*

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is the sheafification of the "structure sheaf" $\mathcal{O}$ on $(Sch/S)_h$. Similarly for the ph topology.

Proof. In Lemma 38.38.4 we have seen that the rule $\mathcal{F}$ of the lemma defines a sheaf in the ph topology and hence a fortiori a sheaf for the h topology. Clearly, there is a canonical map of presheaves of rings $\mathcal{O} \to \mathcal{F}$. To finish the proof, it suffices to show

(1) if $f \in \mathcal{O}(X)$ maps to zero in $\mathcal{F}(X)$, then there is a h covering $\{X_i \to X\}$ such that $f|_{X_i} = 0$, and
(2) given $f \in \mathcal{F}(X)$ there is a h covering $\{X_i \to X\}$ such that $f|_{X_i}$ is the image of $f_i \in \mathcal{O}(X_i)$.

Let f be as in (1). Then $f|_{X^{awn}} = 0$. This means that f is locally nilpotent. Thus if $X' \subset X$ is the closed subscheme cut out by f, then $X' \to X$ is a surjective closed immersion of finite presentation. Hence $\{X' \to X\}$ is the desired h covering. Let f be as in (2). After replacing X by the members of an affine open covering we may assume $X = \mathrm{Spec}(A)$ is affine. Then $f \in A^{awn}$, see Morphisms, Lemma 29.47.6. By Morphisms, Lemma 29.46.11 we can find a ring map $A \to B$ of finite presentation such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism and such that f is the image of an element $b \in B$ under the canonical map $B \to A^{awn}$. Then $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an h covering and we conclude. The statement about the

ph topology follows in the same manner (or it can be deduced from the statement for the h topology). □

Let p be a prime number. An $\mathbf{F}_p$-algebra A is called *perfect* if the map $F : A \to A$, $x \mapsto x^p$ is an automorphism of A.

0EVV **Lemma 38.38.6.** *Let p be a prime number. An $\mathbf{F}_p$-algebra A is absolutely weakly normal if and only if it is perfect.*

Proof. It is immediate from condition (2)(b) in Morphisms, Definition 29.47.1 that if A is absolutely weakly normal, then it is perfect.

Assume A is perfect. Suppose $x, y \in A$ with $x^3 = y^2$. If $p > 3$ then we can write $p = 2n + 3m$ for some $n, m > 0$. Choose $a, b \in A$ with $a^p = x$ and $b^p = y$. Setting $c = a^n b^m$ we have

$$c^{2p} = x^{2n} y^{2m} = x^{2n+3m} = x^p$$

and hence $c^2 = x$. Similarly $c^3 = y$. If $p = 2$, then write $x = a^2$ to get $a^6 = y^2$ which implies $a^3 = y$. If $p = 3$, then write $y = a^3$ to get $x^3 = a^6$ which implies $x = a^2$.

Suppose $x, y \in A$ with $\ell^\ell x = y^\ell$ for some prime number ℓ. If $\ell \neq p$, then $a = y/\ell$ satisfies $a^\ell = x$ and $\ell a = y$. If $\ell = p$, then $y = 0$ and $x = a^p$ for some a. □

0EVW **Lemma 38.38.7.** *Let p be a prime number.*

(1) *If A is an $\mathbf{F}_p$-algebra, then* $\operatorname{colim}_F A = A^{awn}$.
(2) *If S is a scheme over $\mathbf{F}_p$, then the h sheafification of $\mathcal{O}$ sends a quasi-compact and quasi-separated X to* $\operatorname{colim}_F \Gamma(X, \mathcal{O}_X)$.

Proof. Proof of (1). Observe that $A \to \operatorname{colim}_F A$ induces a universal homeomorphism on spectra by Algebra, Lemma 10.46.7. Thus it suffices to show that $B = \operatorname{colim}_F A$ is absolutely weakly normal, see Morphisms, Lemma 29.47.6. Note that the ring map $F : B \to B$ is an automorphism, in other words, B is a perfect ring. Hence Lemma 38.38.6 applies.

Proof of (2). This follows from (1) and Lemmas 38.38.2 and 38.38.5 by looking affine locally. □

38.39. Descent vector bundles in positive characteristic

0EXA A reference for this section is [**BS17**].

For a scheme S let us denote $Vect(S)$ the category of finite locally free $\mathcal{O}_S$-modules. Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. In this section we will work with the category

$$\operatorname{colim}_F Vect(S) = \operatorname{colim} \Big(Vect(S) \xrightarrow{F^*} Vect(S) \xrightarrow{F^*} Vect(S) \xrightarrow{F^*} \ldots \Big)$$

where $F : S \to S$ is the absolute Frobenius morphism. In down to earth terms an object of this category is a pair $(\mathcal{E}, n)$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_S$-module and $n \geq 0$ is an integer. For morphisms we take

$$\operatorname{Hom}_{\operatorname{colim}_F Vect(S)}((\mathcal{E}, n), (\mathcal{G}, m)) = \operatorname{colim}_N \operatorname{Hom}_S(F^{N-n,*}\mathcal{E}, F^{N-m,*}\mathcal{G})$$

where $F : S \to S$ is the absolute Frobenius morphism of S. Thus the object $(\mathcal{E}, n)$ is isomorphic to the object $(F^*\mathcal{E}, n + 1)$.

0EXB **Lemma 38.39.1.** *Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. The category $\operatorname{colim}_F Vect(S)$ is equivalent to the category of finite locally free modules over the sheaf of rings $\operatorname{colim}_F \mathcal{O}_S$ on S.*

Proof. Omitted. □

0EXC **Lemma 38.39.2.** *Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 38.37.10. Then the functor*

$$\operatorname{colim}_F Vect(X) \longrightarrow \operatorname{colim}_F Vect(Z) \times_{\operatorname{colim}_F Vect(E)} \operatorname{colim}_F Vect(X')$$

is an equivalence.

Proof. Let A, f, J be as in Example 38.37.10. Since all our schemes are affine and since we have internal Hom in the category of vector bundles, the fully faithfulness of the functor follows if we can show that

$$\operatorname{colim} P \otimes_{A,F^N} A = \operatorname{colim} P \otimes_{A,F^N} A/J \times_{\operatorname{colim} P \otimes_{A,F^N} A/fA+J} \operatorname{colim} P \otimes_{A,F^N} A/fA$$

for a finite projective A-module P. After writing P as a summand of a finite free module, this follows from the case where P is finite free. This case immediately reduces to the case $P = A$. The case $P = A$ follows from Lemma 38.38.2 (in fact we proved this case directly in the proof of this lemma).

Essential surjectivity. Here we obtain the following algebra problem. Suppose P_1 is a finite projective A/J-module, P_2 is a finite projective A/fA-module, and

$$\varphi : P_1 \otimes_{A/J} A/fA + J \longrightarrow P_2 \otimes_{A/fA} A/fA + J$$

is an isomorphism. Goal: show that there exists an N, a finite projective A-module P, an isomorphism $\varphi_1 : P \otimes_A A/J \to P_1 \otimes_{A/J,F^N} A/J$, and an isomorphism $\varphi_2 : P \otimes_A A/fA \to P_2 \otimes_{A/fA,F^N} A/fA$ compatible with φ in an obvious manner. This can be seen as follows. First, observe that

$$A/(J \cap fA) = A/J \times_{A/fA+J} A/fA$$

Hence by More on Algebra, Lemma 15.6.9 there is a finite projective module P' over $A/(J \cap fA)$ which comes with isomorphisms $\varphi'_1 : P' \otimes_A A/J \to P_1$ and $\varphi_2 : P' \otimes_A A/fA \to P_2$ compatible with φ. Since J is a finitely generated ideal and f-power torsion we see that $J \cap fA$ is a nilpotent ideal. Hence for some N there is a factorization

$$A \xrightarrow{\alpha} A/(J \cap fA) \xrightarrow{\beta} A$$

of F^N. Setting $P = P' \otimes_{A/(J\cap fA),\beta} A$ we conclude. □

0EXD **Lemma 38.39.3.** *Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 38.37.11. Then the functor*

$$G : \operatorname{colim}_F Vect(X) \longrightarrow \operatorname{colim}_F Vect(Z) \times_{\operatorname{colim}_F Vect(E)} \operatorname{colim}_F Vect(X')$$

is an equivalence.

Proof. Fully faithfulness. Suppose that $(\mathcal{E}, n)$ and $(\mathcal{F}, m)$ are objects of $\operatorname{colim}_F Vect(X)$. Let $(a, b) : G(\mathcal{E}, n) \to G(\mathcal{F}, m)$ be a morphism in the RHS. We may choose $N \gg 0$ and think of a as a map $a : F^{N-n,*}\mathcal{E}|_Z \to F^{N-m,*}\mathcal{F}|_Z$ and b as a map

$b : F^{N-n,*}\mathcal{E}|_{X'} \to F^{N-m,*}\mathcal{F}|_{X'}$ agreeing over E. Choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{E}|_{X_i}$ and $\mathcal{F}|_{X_i}$ are finite free $\mathcal{O}_{X_i}$-modules. For each i the base change

$$\begin{array}{ccc} E_i & \longrightarrow & X_i' \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & X_i \end{array}$$

is another almost blow up square as in Example 38.37.11. For these squares we know that

$$\operatorname{colim}_F H^0(X_i, \mathcal{O}_{X_i}) = \operatorname{colim}_F H^0(Z_i, \mathcal{O}_{Z_i}) \times_{\operatorname{colim}_F H^0(E_i, \mathcal{O}_{E_i})} \operatorname{colim}_F H^0(X_i', \mathcal{O}_{X_i'})$$

by Lemma 38.38.2 (see proof of the lemma). Hence after increasing N we may assume the maps $a|_{Z_i}$ and $b|_{X_i'}$ come from maps $c_i : F^{N-n,*}\mathcal{E}|_{X_i} \to F^{N-m,*}\mathcal{F}|_{X_i}$. After possibly increasing N we may assume c_i and c_j agree on $X_i \cap X_j$. Thus these maps glue to give the desired morphism $(\mathcal{E}, n) \to (\mathcal{F}, m)$ in the LHS.

Essential surjectivity. Let $(\mathcal{F}, \mathcal{G}, \varphi)$ be a triple consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{F}$, a finite locally free $\mathcal{O}_{X'}$-module $\mathcal{G}$, and an isomorphism $\varphi : \mathcal{F}|_E \to \mathcal{G}|_E$. We have to show that after replacing this triple by a Frobenius power pullback, it comes from a finite locally free $\mathcal{O}_X$-module.

Noetherian reduction; we urge the reader to skip this paragraph. Recall that $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/(f_1, f_2))$, $X' = \operatorname{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E = \mathbf{P}^1_Z$. By Limits, Lemma 32.10.3 we can find a finitely generated $\mathbf{F}_p$-subalgebra $A_0 \subset A$ containing f_1 and f_2 such that the triple $(\mathcal{F}, \mathcal{G}, \varphi)$ descends to $X_0 = \operatorname{Spec}(A_0)$ and $Z_0 = \operatorname{Spec}(A_0/(f_1, f_2))$, $X_0' = \operatorname{Proj}(A_0[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E_0 = \mathbf{P}^1_{Z_0}$. Thus we may assume our schemes are Noetherian.

Assume X is Noetherian. We may choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{F}|_{Z \cap X_i}$ is free. Since we can glue objects of $\operatorname{colim}_F \mathit{Vect}(X)$ in the Zariski topology (Lemma 38.39.1), and since we already know fully faithfulness over X_i and $X_i \cap X_j$ (see first paragraph of the proof), it suffices to prove the existence over each X_i. This reduces us to the case discussed in the next paragraph.

Assume X is Noetherian and $\mathcal{F} = \mathcal{O}_Z^{\oplus r}$. Using φ we get an isomorphism $\mathcal{O}_E^{\oplus r} \to \mathcal{G}|_E$. Let $I = (f_1, f_2) \subset A$. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the ideal sheaf of E; it is globally generated by f_1 and f_2. For any n there is a surjection

$$(\mathcal{I}^n/\mathcal{I}^{n+1})^{\oplus r} = \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_E} \mathcal{G}|_E \longrightarrow \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G}$$

Hence the first cohomology group of this module is zero. Here we use that $E = \mathbf{P}^1_Z$ and hence its structure sheaf and in fact any globally generated quasi-coherent module has vanishing H^1. Compare with More on Morphisms, Lemma 37.72.3. Then using the short exact sequences

$$0 \to \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^n\mathcal{G} \to 0$$

and induction, we see that

$$\lim H^0(X', \mathcal{G}/\mathcal{I}^n\mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$

is surjective. By the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5) this implies that

$$H^0(X', \mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$

is surjective. Thus we can choose a map $\alpha : \mathcal{O}_{X'}^{\oplus r} \to \mathcal{G}$ which is compatible with the given trivialization of $\mathcal{G}|_E$. Thus α is an isomorphism over an open neighbourhood of E in X'. Thus every point of Z has an affine open neighbourhood where we can solve the problem. Since $X' \setminus E \to X \setminus Z$ is an isomorphism, the same holds for points of X not in Z. Thus another Zariski glueing argument finishes the proof. □

0EXE **Proposition 38.39.4.** *Let p be a prime number. Let S be a scheme in characteristic p. Then the category fibred in groupoids*

$$p : \mathcal{S} \longrightarrow (Sch/S)_h$$

whose fibre category over U is the category of finite locally free $\operatorname{colim}_F \mathcal{O}_U$-modules over U is a stack in groupoids. Moreover, if U is quasi-compact and quasi-separated, then $\mathcal{S}_U$ is $\operatorname{colim}_F Vect(U)$.

Proof. The final assertion is the content of Lemma 38.39.1. To prove the proposition we will check conditions (1), (2), and (3) of Lemma 38.37.13.

Condition (1) holds because by definition we have glueing for the Zariski topology.

To see condition (2), suppose that $f : X \to Y$ is a surjective, flat, proper morphism of finite presentation over S with Y affine. Since $Y, X, X \times_Y X$ are quasi-compact and quasi-separated, we can use the description of fibre categories given in the statement of the proposition. Then it is clearly enough to show that

$$Vect(Y) \longrightarrow Vect(X) \times_{Vect(X \times_Y X)} Vect(X)$$

is an equivalence (as this will imply the same for the colimits). This follows immediately from fppf descent of finite locally free modules, see Descent, Proposition 35.5.2 and Lemma 35.7.6.

Condition (3) is the content of Lemmas 38.39.2 and 38.39.3. □

0EXF **Lemma 38.39.5.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres where S is the spectrum of a discrete valuation ring. Denote $\eta \in S$ the generic point and denote $X_n \subset X$ the closed subscheme cutout by the nth power of a uniformizer on S. Then there exists an integer n such that the following is true: any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ such that $\mathcal{E}|_{X_\eta}$ and $\mathcal{E}|_{X_n}$ are free, is free.*

Proof. We first reduce to the case where $X \to S$ has a section. Say $S = \operatorname{Spec}(A)$. Choose a closed point ξ of X_η. Choose an extension of discrete valuation rings $A \subset B$ such that the fraction field of B is $\kappa(\xi)$. This is possible by Krull-Akizuki (Algebra, Lemma 10.120.18) and the fact that $\kappa(\xi)$ is a finite extension of the fraction field of A. By the valuative criterion of properness (Morphisms, Lemma 29.42.1) we get a B-valued point $\tau : \operatorname{Spec}(B) \to X$ which induces a section $\sigma : \operatorname{Spec}(B) \to X_B$. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let $\mathcal{E}_B$ be the pullback to the base change X_B. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that $H^0(X_B, \mathcal{E}_B) = H^0(X, \mathcal{E}) \otimes_A B$. Thus if $\mathcal{E}_B$ is free of rank r, then the sections in $H^0(X, \mathcal{E})$ generate the free B-module $\tau^*\mathcal{E} = \sigma^*\mathcal{E}_B$. In particular, we can find r global sections $s_1, \ldots, s_r$ of $\mathcal{E}$ which generate $\tau^*\mathcal{E}$. Then

$$s_1, \ldots, s_r : \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{E}$$

is a map of finite locally free $\mathcal{O}_X$-modules of rank r and the pullback to X_B is a map of free $\mathcal{O}_{X_B}$-modules which restricts to an isomorphism in one point of each fibre. Taking the determinant we get a function $g \in \Gamma(X_\eta, \mathcal{O}_{X_B})$ which is invertible

in one point of each fibre. As the fibres are proper and connected, we see that g must be invertible (details omitted; hint: use Varieties, Lemma 33.9.3). Thus it suffices to prove the lemma for the base change $X_B \to \mathrm{Spec}(B)$.

Assume we have a section $\sigma : S \to X$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module which is assumed free on the generic fibre and on X_n (we will choose n later). Choose an isomorphism $\sigma^*\mathcal{E} = \mathcal{O}_S^{\oplus r}$. Consider the map

$$K = R\Gamma(X, \mathcal{E}) \longrightarrow R\Gamma(S, \sigma^*\mathcal{E}) = A^{\oplus r}$$

in $D(A)$. Arguing as above, we see $\mathcal{E}$ is free if (and only if) the induced map $H^0(K) = H^0(X, \mathcal{E}) \to A^{\oplus r}$ is surjective.

Set $L = R\Gamma(X, \mathcal{O}_X^{\oplus r})$ and observe that the corresponding map $L \to A^{\oplus r}$ has the desired property. Observe that $K \otimes_A Q(A) \cong L \otimes_A Q(A)$ by flat base change and the assumption that $\mathcal{E}$ is free on the generic fibre. Let $\pi \in A$ be a uniformizer. Observe that

$$K \otimes^{\mathbf{L}}_A A/\pi^m A = R\Gamma(X, \mathcal{E} \xrightarrow{\pi^m} \mathcal{E})$$

and similarly for L. Denote $\mathcal{E}_{tors} \subset \mathcal{E}$ the coherent subsheaf of sections supported on the special fibre and similarly for other $\mathcal{O}_X$-modules. Choose $k > 0$ such that $(\mathcal{O}_X)_{tors} \to \mathcal{O}_X/\pi^k\mathcal{O}_X$ is injective (Cohomology of Schemes, Lemma 30.10.3). Since $\mathcal{E}$ is locally free, we see that $\mathcal{E}_{tors} \subset \mathcal{E}/\pi^k\mathcal{E}$. Then for $n \geq m + k$ we have isomorphisms

$$\begin{aligned}
(\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}) &\cong (\mathcal{E}/\pi^k\mathcal{E} \xrightarrow{\pi^m} \mathcal{E}/\pi^{k+m}\mathcal{E}) \\
&\cong (\mathcal{O}_X^{\oplus r}/\pi^k\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r}/\pi^{k+m}\mathcal{O}_X^{\oplus r}) \\
&\cong (\mathcal{O}_X^{\oplus r} \xrightarrow{\pi^m} \mathcal{O}_X^{\oplus r})
\end{aligned}$$

in $D(\mathcal{O}_X)$. This determines an isomorphism

$$K \otimes^{\mathbf{L}}_A A/\pi^m A \cong L \otimes^{\mathbf{L}}_A A/\pi^m A$$

in $D(A)$ (holds when $n \geq m+k$). Observe that these isomorphisms are compatible with pulling back by σ hence in particular we conclude that $K \otimes^{\mathbf{L}}_A A/\pi^m A \to (A/\pi^m A)^{\oplus r}$ defines an surjection on degree 0 cohomology modules (as this is true for L). Since A is a discrete valuation ring, we have

$$K \cong \bigoplus H^i(K)[-i] \quad \text{and} \quad L \cong \bigoplus H^i(L)[-i]$$

in $D(A)$. See More on Algebra, Example 15.70.3. The cohomology groups $H^i(K) = H^i(X, \mathcal{E})$ and $H^i(L) = H^i(X, \mathcal{O}_X)^{\oplus r}$ are finite A-modules by Cohomology of Schemes, Lemma 30.19.2. By More on Algebra, Lemma 15.125.3 these modules are direct sums of cyclic modules. We have seen above that the rank β_i of the free part of $H^i(K)$ and $H^i(L)$ are the same. Next, observe that

$$H^i(L \otimes^{\mathbf{L}}_A A/\pi^m A) = H^i(L)/\pi^m H^i(L) \oplus H^{i+1}(L)[\pi^m]$$

and similarly for K. Let e be the largest integer such that $A/\pi^e A$ occurs as a summand of $H^i(X, \mathcal{O}_X)$, or equivalently $H^i(L)$, for some i. Then taking $m = e+1$ we see that $H^i(L \otimes^{\mathbf{L}}_A A/\pi^m A)$ is a direct sum of β_i copies of $A/\pi^m A$ and some other cyclic modules each annihilated by π^e. By the same reasoning for K and the isomorphism $K \otimes^{\mathbf{L}}_A A/\pi^m A \cong L \otimes^{\mathbf{L}}_A A/\pi^m A$ it follows that $H^i(K)$ cannot have any cyclic summands of the form $A/\pi^l A$ with $l > e$. (It also follows that K is

isomorphic to L as an object of $D(A)$, but we won't need this.) Then the only way the map

$$H^0(K \otimes^{\mathbf{L}}_A A/\pi^{e+1}A) = H^0(K)/\pi^{e+1}H^0(K) \oplus H^1(K)[\pi^{e+1}] \longrightarrow (A/\pi^{e+1}A)^{\oplus r}$$

is surjective, is if it is surjective on the first summand. This is what we wanted to show. (To be precise, the integer n in the statement of the lemma, if there is a section σ, should be equal to $k+e+1$ where k and e are as above and depend only on X.) □

0EXG **Lemma 38.39.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume*

(1) *f is flat and proper and $\mathcal{O}_S = f_*\mathcal{O}_X$,*
(2) *S is a normal Noetherian scheme,*
(3) *the pullback of $\mathcal{E}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is free for every codimension 1 point $s \in S$.*

Then $\mathcal{E}$ is isomorphic to the pullback of a finite locally free $\mathcal{O}_S$-module.

Proof. We will prove the canonical map

$$\Phi : f^*f_*\mathcal{E} \longrightarrow \mathcal{E}$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 30.5.2) and assumptions (1) and (3) we see that the pullback of this to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is an isomorphism for every codimension 1 point $s \in S$. By Divisors, Lemma 31.2.11 it suffices to prove that $\mathrm{depth}((f^*f_*\mathcal{E})_x) \geq 2$ for any point $x \in X$ mapping to a point $s \in S$ of codimension ≥ 2. Since f is flat and $(f^*f_*\mathcal{E})_x = (f_*\mathcal{E})_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$, it suffices to prove that $\mathrm{depth}((f_*\mathcal{E})_s) \geq 2$, see Algebra, Lemma 10.163.2. Since S is a normal Noetherian scheme and $\dim(\mathcal{O}_{S,s}) \geq 2$ we have $\mathrm{depth}(\mathcal{O}_{S,s}) \geq 2$, see Properties, Lemma 28.12.5. Thus we get what we want from Divisors, Lemma 31.6.6. □

We can use the results above to prove the following miraculous statement.

0EXH **Theorem 38.39.7.** *Let p be a prime number. Let Y be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Let $f : X \to Y$ be a proper, surjective morphism of finite presentation with geometrically connected fibres. Then the functor*

$$\mathrm{colim}_F \mathit{Vect}(Y) \longrightarrow \mathrm{colim}_F \mathit{Vect}(X)$$

is fully faithful with essential image described as follows. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume for all $y \in Y$ there exists integers $n_y, r_y \geq 0$ such that

$$F^{n_y,*}\mathcal{E}|_{X_{y,red}} \cong \mathcal{O}_{X_{y,red}}^{\oplus r_y}$$

Then for some $n \geq 0$ the nth Frobenius power pullback $F^{n,*}\mathcal{E}$ is the pullback of a finite locally free $\mathcal{O}_Y$-module.

Proof. Proof of fully faithfulness. Since vectorbundles on Y are locally trivial, this reduces to the statement that

$$\mathrm{colim}_F \Gamma(Y, \mathcal{O}_Y) \longrightarrow \mathrm{colim}_F \Gamma(X, \mathcal{O}_X)$$

is bijective. Since $\{X \to Y\}$ is an h covering, this will follow from Lemma 38.38.2 if we can show that the two maps

$$\mathrm{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \mathrm{colim}_F \Gamma(X \times_Y X, \mathcal{O}_{X \times_Y X})$$

are equal. Let $g \in \Gamma(X, \mathcal{O}_X)$ and denote g_1 and g_2 the two pullbacks of g to $X \times_Y X$. Since $X_{y,red}$ is geometrically connected, we see that $H^0(X_{y,red}, \mathcal{O}_{X_{y,red}})$ is a purely inseparable extension of $\kappa(y)$, see Varieties, Lemma 33.9.3. Thus $g^q|_{X_{y,red}}$ comes from an element of $\kappa(y)$ for some p-power q (which may depend on y). It follows that g_1^q and g_2^q map to the same element of the residue field at any point of $(X \times_Y X)_y = X_y \times_y X_y$. Hence $g_1 - g_2$ restricts to zero on $(X \times_Y X)_{red}$. Hence $(g_1 - g_2)^n = 0$ for some n which we may take to be a p-power as desired.

Description of essential image. Let $\mathcal{E}$ be as in the statement of the proposition. We first reduce to the Noetherian case.

Let $y \in Y$ be a point and view it as a morphism $y \to Y$ from the spectrum of the residue field into Y. We can write $y \to Y$ as a filtered limit of morphisms $Y_i \to Y$ of finite presentation with Y_i affine. (It is best to prove this yourself, but it also follows formally from Limits, Lemma 32.7.2 and 32.4.13.) For each i set $Z_i = Y_i \times_Y X$. Then $X_y = \lim Z_i$ and $X_{y,red} = \lim Z_{i,red}$. By Limits, Lemma 32.10.2 we can find an i such that $F^{n_y,*}\mathcal{E}|_{Z_{i,red}} \cong \mathcal{O}_{Z_{i,red}}^{\oplus r_y}$. Fix i. We have $Z_{i,red} = \lim Z_{i,j}$ where $Z_{i,j} \to Z_i$ is a thickening of finite presentation (Limits, Lemma 32.9.4). Using the same lemma as before we can find a j such that $F^{n_y,*}\mathcal{E}|_{Z_{i,j}} \cong \mathcal{O}_{Z_{i,j}}^{\oplus r_y}$. We conclude that for each $y \in Y$ there exists a morphism $Y_y \to Y$ of finite presentation whose image contains y and a thickening $Z_y \to Y_y \times_Y X$ such that $F^{n_y,*}\mathcal{E}|_{Z_y} \cong \mathcal{O}_{Z_y}^{\oplus r_y}$. Observe that the image of $Y_y \to Y$ is constructible (Morphisms, Lemma 29.22.2). Since Y is quasi-compact in the constructible topology (Topology, Lemma 5.23.2 and Properties, Lemma 28.2.4) we conclude that there are a finite number of morphisms

$$Y_1 \to Y,\ Y_2 \to Y,\ \ldots,\ Y_N \to Y$$

of finite presentation such that $Y = \bigcup \mathrm{Im}(Y_a \to Y)$ set theoretically and such that for each $a \in \{1, \ldots, N\}$ there exist integers $n_a, r_a \geq 0$ and there is a thickening $Z_a \subset Y_a \times_Y X$ of finite presentation such that $F^{n_a,*}\mathcal{E}|_{Z_a} \cong \mathcal{O}_{Z_a}^{\oplus r_a}$.

Formulated in this way, the condition descends to an absolute Noetherian approximation. We strongly urge the reader to skip this paragraph. First write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of schemes of finite type over $\mathbf{F}_p$ with affine transition morphisms (Limits, Lemma 32.7.2). Next, we can assume we have proper morphisms $f_i : X_i \to Y_i$ whose base change to Y recovers $f : X \to Y$, see Limits, Lemma 32.10.1. After increasing i we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is isomorphic to $\mathcal{E}$, see Limits, Lemma 32.10.3. Pick $0 \in I$ and denote $E \subset Y_0$ the constructible subset where the geometric fibres of f_0 are connected, see More on Morphisms, Lemma 37.28.6. Then $Y \to Y_0$ maps into E, see More on Morphisms, Lemma 37.28.2. Thus $Y_i \to Y_0$ maps into E for $i \gg 0$, see Limits, Lemma 32.4.10. Hence we see that the fibres of f_i are geometrically connected for $i \gg 0$. By Limits, Lemma 32.10.1 for large enough i we can find morphisms $Y_{i,a} \to Y_i$ of finite type whose base change to Y recovers $Y_a \to Y$, $a \in \{1, \ldots, N\}$. After possibly increasing i we can find thickenings $Z_{i,a} \subset Y_{i,a} \times_{Y_i} X_i$ whose base change to $Y_a \times_Y X$ recovers Z_a (same reference as before combined with Limits, Lemmas 32.8.5 and 32.8.15). Since $Z_a = \lim Z_{i,a}$ we find that after increasing i we may assume $F^{n_a,*}\mathcal{E}_i|_{Z_{i,a}} \cong \mathcal{O}_{Z_{i,a}}^{\oplus r_a}$, see Limits, Lemma 32.10.2. Finally, after increasing i one more time we may assume $\coprod Y_{i,a} \to Y_i$ is

surjective by Limits, Lemma 32.8.15. At this point all the assumptions hold for $X_i \to Y_i$ and $\mathcal{E}_i$ and we see that it suffices to prove result for $X_i \to Y_i$ and $\mathcal{E}_i$.

Assume Y is of finite type over $\mathbf{F}_p$. To prove the result we will use induction on $\dim(Y)$. We are trying to find an object of $\mathrm{colim}_F \mathit{Vect}(Y)$ which pulls back to the object of $\mathrm{colim}_F \mathit{Vect}(X)$ determined by $\mathcal{E}$. By the fully faithfulness already proven and because of Proposition 38.39.4 it suffices to construct a descent of $\mathcal{E}$ after replacing Y by the members of a h covering and X by the corresponding base change. This means that we may replace Y by a scheme proper and surjective over Y provided this does not increase the dimension of Y. If $T \subset T'$ is a thickening of schemes of finite type over $\mathbf{F}_p$ then $\mathrm{colim}_F \mathit{Vect}(T) = \mathrm{colim}_F \mathit{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that $T \times_{T'} T = T$. If $T' \to T$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$, then $\mathrm{colim}_F \mathit{Vect}(T) = \mathrm{colim}_F \mathit{Vect}(T')$ as $\{T \to T'\}$ is a h covering such that the diagonal $T \subset T \times_{T'} T$ is a thickening.

Using the general remarks made above, we may and do replace X by its reduction and we may assume X is reduced. Consider the Stein factorization $X \to Y' \to Y$, see More on Morphisms, Theorem 37.53.4. Then $Y' \to Y$ is a universal homeomorphism of schemes of finite type over $\mathbf{F}_p$. By the above we may replace Y by Y'. Thus we may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that Y is reduced. This reduces us to the case discussed in the next paragraph.

Assume Y is reduced and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. Then $X \to Y$ is flat over a dense open subscheme $V \subset Y$, see Morphisms, Proposition 29.27.2. By Lemma 38.31.1 there is a V-admissible blowing up $Y' \to Y$ such that the strict transform X' of X is flat over Y'. Observe that $\dim(Y') = \dim(Y)$ as Y and Y' have a common dense open subscheme. By More on Morphisms, Lemma 37.53.7 and the fact that $V \subset Y'$ is dense all fibres of $f' : X' \to Y'$ are geometrically connected. We still have $(f'_*\mathcal{O}_{X'})|_V = \mathcal{O}_V$. Write

$$Y' \times_Y X = X' \cup E \times_Y X$$

where $E \subset Y'$ is the exceptional divisor of the blowing up. By the general remarks above, it suffices to prove existence for $Y' \times_Y X \to Y'$ and the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Suppose that we find some object ξ' in $\mathrm{colim}_F \mathit{Vect}(Y')$ pulling back to the restriction of $\mathcal{E}$ to X' (viewed as an object of the colimit category). By induction on $\dim(Y)$ we can find an object ξ'' in $\mathrm{colim}_F \mathit{Vect}(E)$ pulling back to the restriction of $\mathcal{E}$ to $E \times_Y X$. Then the fully faithfullness determines a unique isomorphism $\xi'|_E \to \xi''$ compatible with the given identifications with the restriction of $\mathcal{E}$ to $E \times_{Y'} X'$. Since

$$\{E \times_Y X \to Y' \times_Y X, X' \to Y' \times_Y X\}$$

is a h covering given by a pair of closed immersions with

$$(E \times_Y X) \times_{(Y' \times_Y X)} X' = E \times_{Y'} X'$$

we conclude that ξ' pulls back to the restriction of $\mathcal{E}$ to $Y' \times_Y X$. Thus it suffices to find ξ' and we reduce to the case discussed in the next paragraph.

Assume Y is reduced, f is flat, and $f_*\mathcal{O}_X = \mathcal{O}_Y$ over a dense open subscheme of Y. In this case we consider the normalization $Y^\nu \to Y$ (Morphisms, Section 29.54). This is a finite surjective morphism (Morphisms, Lemma 29.54.11 and 29.18.2) which is an isomorphism over a dense open. Hence by our general remarks we may replace Y by Y^ν and X by $Y^\nu \times_Y X$. After this replacement we see that

$\mathcal{O}_Y = f_*\mathcal{O}_X$ (because the Stein factorization has to be an isomorphism in this case; small detail omitted).

Assume Y is a normal Noetherian scheme, that f is flat, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is trivial on the scheme theoretic fibres of f at the generic points of the irreducible components of Y (because colim_F *Vect*$(-)$ is an equivalence on universal homeomorphisms, see above). Similarly to the arguments above (in the reduction to the Noetherian case) we conclude there is a dense open subscheme $V \subset Y$ such that $\mathcal{E}|_{f^{-1}(V)}$ is free. Let $Z \subset Y$ be a closed subscheme such that $Y = V \amalg Z$ set theoretically. Let $z_1, \ldots, z_t \in Z$ be the generic points of the irreducible components of Z of codimension 1. Then $A_i = \mathcal{O}_{Y,z_i}$ is a discrete valuation ring. Let n_i be the integer found in Lemma 38.39.5 for the scheme X_{A_i} over A_i. After replacing $\mathcal{E}$ by a suitable Frobenius power pullback, we may assume $\mathcal{E}$ is free over $X_{A_i/\mathfrak{m}_i^{n_i}}$ (again because the colimit category is invariant under universal homeomorphisms, see above). Then Lemma 38.39.5 tells us that $\mathcal{E}$ is free on X_{A_i}. Thus finally we conclude by applying Lemma 38.39.6. □

38.40. Blowing up complexes

0ESM This section finds normal forms for perfect objects of the derived category after blowups.

0ESP **Lemma 38.40.1.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. For every $p, k \in \mathbf{Z}$ there is an finite type quasi-coherent sheaf of ideals $Fit_{p,k}(E) \subset \mathcal{O}_X$ with the following property: for $U \subset X$ open such that $E|_U$ is isomorphic to*

$$\ldots \to \mathcal{O}_U^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} \mathcal{O}_U^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} \mathcal{O}_U^{\oplus n_b} \to 0 \to \ldots$$

the restriction $Fit_{p,k}(E)|_U$ is generated by the minors of the matrix of d_p of size

$$-k + n_{p+1} - n_{p+2} + \ldots + (-1)^{b-p+1} n_b$$

Convention: the ideal generated by $r \times r$-minors is $\mathcal{O}_U$ if $r \leq 0$ and the ideal generated by $r \times r$-minors where $r > \min(n_p, n_{p+1})$ is zero.

Proof. Observe that E locally on X has the shape as stated in the lemma, see More on Algebra, Section 15.65, Cohomology, Section 20.47, and Derived Categories of Schemes, Section 36.10. Thus it suffices to prove that the ideal of minors is independent of the chosen representative. To do this, it suffices to check in local rings. Over a local ring $(R, \mathfrak{m}, \kappa)$ consider a bounded above complex

$$F^\bullet : \ldots \to R^{\oplus n_{b-2}} \xrightarrow{d_{b-2}} R^{\oplus n_{b-1}} \xrightarrow{d_{b-1}} R^{\oplus n_b} \to 0 \to \ldots$$

Denote $\mathrm{Fit}_{k,p}(F^\bullet) \subset R$ the ideal generated by the minors of size $k - n_{p+1} + n_{p+2} - \ldots + (-1)^{b-p} n_b$ in the matrix of d_p. Suppose some matrix coefficient of some differential of $F^\bullet$ is invertible. Then we pick a largest integer i such that d_i has an invertible matrix coefficient. By Algebra, Lemma 10.102.2 the complex $F^\bullet$ is isomorphic to a direct sum of a trivial complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ with nonzero terms in degrees i and $i+1$ and a complex $(F')^\bullet$. We leave it to the reader to see that $\mathrm{Fit}_{p,k}(F^\bullet) = \mathrm{Fit}_{p,k}((F')^\bullet)$; this is where the formula for the size of the minors is used. If $(F')^\bullet$ has another differential with an invertible matrix coefficient, we do it again, etc. Continuing in this manner, we eventually reach a complex $(F^\infty)^\bullet$ all of whose differentials have matrices with coefficients in

$\mathfrak{m}$. Here you may have to do an infinite number of steps, but for any cutoff only a finite number of these steps affect the complex in degrees $\geq$ the cutoff. Thus the "limit" $(F^\infty)^\bullet$ is a well defined bounded above complex of finite free modules, comes equipped with a quasi-isomorphism $(F^\infty)^\bullet \to F^\bullet$ into the complex we started with, and $\mathrm{Fit}_{p,k}(F^\bullet) = \mathrm{Fit}_{p,k}((F^\infty)^\bullet)$. Since the complex $(F^\infty)^\bullet$ is unique up to isomorphism by More on Algebra, Lemma 15.76.6 the proof is complete. □

0ESQ **Lemma 38.40.2.** *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $U \subset X$ be a scheme theoretically dense open subscheme such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$. Then there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.*

Proof. We will construct and study the blowup affine locally. Namely, suppose that $V \subset X$ is an affine open subscheme such that $E|_V$ can be represented by the complex

$$\mathcal{O}_V^{\oplus n_a} \xrightarrow{d_a} \ldots \xrightarrow{d_{b-1}} \mathcal{O}_V^{\oplus n_b}$$

Set $k_i = r_{i+1} - r_{i+2} + \ldots + (-1)^{b-i+1} r_b$. A computation which we omit show that over $U \cap V$ the rank of d_i is

$$\rho_i = -k_i + n_{i+1} - n_{i+2} + \ldots + (-1)^{b-i+1} n_b$$

in the sense that the cokernel of d_i is finite locally free of rank $n_{i+1} - \rho_i$. Let $\mathcal{I}_i \subset \mathcal{O}_V$ be the ideal generated by the minors of size $\rho_i \times \rho_i$ in the matrix of d_i.

On the one hand, comparing with Lemma 38.40.1 we see the ideal $\mathcal{I}_i$ corresponds to the global ideal $\mathrm{Fit}_{i,k_i}(E)$ which was shown to be independent of the choice of the complex representing $E|_V$. On the other hand, $\mathcal{I}_i$ is the $(n_{i+1} - \rho_i)$th Fitting ideal of $\mathrm{Coker}(d_i)$. Please keep this in mind.

We let $b : X' \to X$ be the blowing up in the product of the ideals $\mathrm{Fit}_{i,k_i}(E)$; this makes sense as locally on X almost all of these ideals are equal to the unit ideal (see above). This blowup dominates the blowups $b_i : X_i' \to X$ in the ideals $\mathrm{Fit}_{i,k_i}(E)$, see Divisors, Lemma 31.32.12. By Divisors, Lemma 31.35.3 each b_i is a U-admissible blowup. It follows that b is a U-admissible blowup (tiny detail omitted; compare with the proof of Divisors, Lemma 31.34.4). Finally, U is still a scheme theoretically dense open subscheme of X'. Thus after replacing X by X' we end up in the situation discussed in the next paragraph.

Assume $\mathrm{Fit}_{i,k_i}(E)$ is an invertible ideal for all i. Choose an affine open V and a complex of finite free modules representing $E|_V$ as above. It follows from Divisors, Lemma 31.35.3 that $\mathrm{Coker}(d_i)$ has tor dimension ≤ 1. Thus $\mathrm{Im}(d_i)$ is finite locally free as the kernel of a map from a finite locally free module to a finitely presented module of tor dimension ≤ 1. Hence $\mathrm{Ker}(d_i)$ is finite locally free as well (same argument). Thus the short exact sequence

$$0 \to \mathrm{Im}(d_{i-1}) \to \mathrm{Ker}(d_i) \to H^i(E)|_V \to 0$$

shows what we want and the proof is complete. □

0ESR **Lemma 38.40.3.** *Let X be an integral scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Then there exists a nonempty open $U \subset X$ such that $H^i(E|_U)$ is finite locally free of constant rank r_i for all $i \in \mathbf{Z}$ and there exists a U-admissible blowup $b : X' \to X$ such that $H^i(Lb^*E)$ is a perfect $\mathcal{O}_{X'}$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$.*

Proof. We strongly urge the reader to find their own proof of the existence of U. Let $\eta \in X$ be the generic point. The restriction of E to η is isomorphic in $D(\kappa(\eta))$ to a finite complex $V^\bullet$ of finite dimensional vector spaces with zero differentials. Set $r_i = \dim_{\kappa(\eta)} V^i$. Then the perfect object E' in $D(\mathcal{O}_X)$ represented by the complex with terms $\mathcal{O}_X^{\oplus r_i}$ and zero differentials becomes isomorphic to E after pulling back to η. Hence by Derived Categories of Schemes, Lemma 36.35.9 there is an open neighbourhood U of η such that $E|_U$ and $E'|_U$ are isomorphic. This proves the first assertion. The second follows from the first and Lemma 38.40.2 as any nonempty open is scheme theoretically dense in the integral scheme X. □

0F8J **Remark 38.40.4.** Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be a perfect object such that $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This property sometimes allows one to reduce questions about E to questions about $H^i(E)$. For example, suppose

$$\mathcal{E}^a \xrightarrow{d^a} \ldots \xrightarrow{d^{b-2}} \mathcal{E}^{b-1} \xrightarrow{d^{b-1}} \mathcal{E}^b$$

is a bounded complex of finite locally free $\mathcal{O}_X$-modules representing E. Then $\mathrm{Im}(d^i)$ and $\mathrm{Ker}(d^i)$ are finite locally free $\mathcal{O}_X$-modules for all i. Namely, suppose by induction we know this for all indices bigger than i. Then we can first use the short exact sequence

$$0 \to \mathrm{Im}(d^i) \to \mathrm{Ker}(d^{i+1}) \to H^{i+1}(E) \to 0$$

and the assumption that $H^{i+1}(E)$ is perfect of tor dimension ≤ 1 to conclude that $\mathrm{Im}(d^i)$ is finite locally free. The same argument used again for the short exact sequence

$$0 \to \mathrm{Ker}(d^i) \to \mathcal{E}^i \to \mathrm{Im}(d^i) \to 0$$

then gives that $\mathrm{Ker}(d^i)$ is finite locally free. It follows that the distinguished triangles

$$\tau_{\leq k-1}E \to \tau_{\leq k}E \to H^k(E)[-k] \to (\tau_{\leq k-1}E)[1]$$

are represented by the following short exact sequences of bounded complexes of finite locally free modules

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathrm{Ker}(d^{k-1}) & & \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{E}^a & \to & \ldots & \to & \mathcal{E}^{k-2} & \to & \mathcal{E}^{k-1} & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \mathrm{Im}(d^{k-1}) & \to & \mathrm{Ker}(d^k) \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Here the complexes are the rows and the "obvious" zeros are omitted from the display.

38.41. Blowing up perfect modules

0F8K This section tries to find normal forms for perfect modules of tor dimension ≤ 1 after blowups. We are only partially successful.

0ESS **Lemma 38.41.1.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. For any blowup $b : X' \to X$ we have $Lb^*\mathcal{F} = b^*\mathcal{F}$ and $b^*\mathcal{F}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1.*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine and we may assume the A-module M corresponding to $\mathcal{F}$ has a presentation

$$0 \to A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

Suppose $I \subset A$ is an ideal and $a \in I$. Recall that the affine blowup algebra $A[\frac{I}{a}]$ is a subring of A_a. Since localization is exact we see that $A_a^{\oplus m} \to A_a^{\oplus n}$ is injective. Hence $A[\frac{I}{a}]^{\oplus m} \to A[\frac{I}{a}]^{\oplus n}$ is injective too. This proves the lemma. □

0EST **Lemma 38.41.2.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then there exists a U-admissible blowup $b : X' \to X$ such that there is a canonical short exact sequence*

$$0 \to \mathcal{K} \to b^*\mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is finite locally free of rank r and $\mathcal{K}$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 whose restriction to U is zero.

Proof. Combine Divisors, Lemma 31.35.3 and Lemma 38.41.1. □

0ESU **Lemma 38.41.3.** *Let X be a scheme. Let $\mathcal{F}$ be a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1. Let $U \subset X$ be an open such that $\mathcal{F}|_U = 0$. Then there is a U-admissible blowup*

$$b : X' \to X$$

such that $\mathcal{F}' = b^*\mathcal{F}$ is equipped with two canonical locally finite filtrations

$$0 = F^0 \subset F^1 \subset F^2 \subset \ldots \subset \mathcal{F}' \quad \text{and} \quad \mathcal{F}' = F_1 \supset F_2 \supset F_3 \supset \ldots \supset 0$$

such that for each $n \geq 1$ there is an effective Cartier divisor $D_n \subset X'$ with the property that

$$F^i/F^{i-1} \quad \text{and} \quad F_i/F_{i+1}$$

are finite locally free of rank i on D_i.

Proof. Choose an affine open $V \subset X$ such that there exists a presentation

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus n} \to \mathcal{F} \to 0$$

for some n and some matrix A. The ideal we are going to blowup in is the product of the Fitting ideals $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$. This makes sense because in the affine situation above we see that $\mathrm{Fit}_k(\mathcal{F})|_V = \mathcal{O}_V$ for $k > n$. It is clear that this is a U-admissible blowing up. By Divisors, Lemma 31.32.12 we see that on X' the ideals $\mathrm{Fit}_k(\mathcal{F})$ are invertible. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathrm{Fit}_k(\mathcal{F})$ is an invertible ideal for $k \geq 0$. If $E_k \subset X$ is the effective Cartier divisor defined by $\mathrm{Fit}_k(\mathcal{F})$ for $k \geq 0$, then the effective Cartier divisors D_k in the statement of the lemma will satisfy

$$E_k = D_{k+1} + 2D_{k+2} + 3D_{k+3} + \ldots$$

This makes sense as the collection D_k will be locally finite. Moreover, it uniquely determines the effective Cartier divisors D_k hence it suffices to construct D_k locally.

Choose an affine open $V \subset X$ and presentation of $\mathcal{F}|_V$ as above. We will construct the divisors and filtrations by induction on the integer n in the presentation. We set $D_k|_V = \emptyset$ for $k > n$ and we set $D_n|V = E_{n-1}|_V$. After shrinking V we may assume that $\mathrm{Fit}_{n-1}(\mathcal{F})|_V$ is generated by a single nonzerodivisor $f \in \Gamma(V, \mathcal{O}_V)$. Since $\mathrm{Fit}_{n-1}(\mathcal{F})|_V$ is the ideal generated by the entries of A, we see that there is a matrix A' in $\Gamma(V, \mathcal{O}_V)$ such that $A = fA'$. Define $\mathcal{F}'$ on V by the short exact sequence

$$0 \to \mathcal{O}_V^{\oplus n} \xrightarrow{A'} \mathcal{O}_V^{\oplus n} \to \mathcal{F}' \to 0$$

Since the entries of A' generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$ we see that $\mathcal{F}'$ locally on V has a presentation with n decreased by 1, see Algebra, Lemma 10.102.2. Further note that $f^{n-k}\mathrm{Fit}_k(\mathcal{F}') = \mathrm{Fit}_k(\mathcal{F})|_V$ for $k = 0, \ldots, n$. Hence $\mathrm{Fit}_k(\mathcal{F}')$ is an invertible ideal for all k. We conclude by induction that there exist effective Cartier divisors $D'_k \subset V$ such that $\mathcal{F}'$ has two canonical filtrations as in the statement of the lemma. Then we set $D_k|_V = D'_k$ for $k = 1, \ldots, n-1$. Observe that the equalities between effective Cartier divisors displayed above hold with these choices. Finally, we come to the construction of the filtrations. Namely, we have short exact sequences

$$0 \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to \mathcal{F} \to \mathcal{F}' \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_{D_n \cap V}^{\oplus n} \to 0$$

coming from the two factorizations $A = A'f = fA'$ of A. These sequences are canonical because in the first one the submodule is $\mathrm{Ker}(f : \mathcal{F} \to \mathcal{F})$ and in the second one the quotient module is $\mathrm{Coker}(f : \mathcal{F} \to \mathcal{F})$. □

0ESV **Lemma 38.41.4.** *Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homorphism of perfect $\mathcal{O}_X$-modules of tor dimension ≤ 1. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U = 0$ and $\mathcal{G}|_U = 0$. Then there is a U-admissible blowup $b : X' \to X$ such that the kernel, image, and cokernel of $b^*\varphi$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.*

Proof. The assumptions tell us that the object $(\mathcal{F} \to \mathcal{G})$ of $D(\mathcal{O}_X)$ is perfect. Thus we get a U-admissible blowup that works for the cokernel and kernel by Lemmas 38.40.2 and 38.41.1 (to see what the complex looks like after pullback). The image is the kernel of the cokernel and hence is going to be perfect of tor dimension ≤ 1 as well. □

38.42. An operator introduced by Berthelot and Ogus

0F8L Please read Cohomology, Section 20.55 first.

Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{I} = \mathcal{I}_D \subset \mathcal{O}_X$ be the ideal sheaf of D, see Divisors, Section 31.14. Clearly we can apply the discussion in Cohomology, Section 20.55 to X and $\mathcal{I}$.

0F8M **Lemma 38.42.1.** *Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $\mathcal{I}$-torsion free for all i. Then $\eta_{\mathcal{I}}\mathcal{F}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Moreover, if $U = \mathrm{Spec}(A) \subset X$ is affine open and $D \cap U = V(f)$, then $\eta_f(\mathcal{F}^\bullet(U))$ is canonically isomorphic to $(\eta_{\mathcal{I}}\mathcal{F}^\bullet)(U)$.*

Proof. Omitted. □

0GTV **Lemma 38.42.2.** *Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The functor $L\eta_{\mathcal{I}} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ of Cohomology, Lemma 20.55.7 sends $D_{QCoh}(\mathcal{O}_X)$ into itself. Moreover, if $X = \mathrm{Spec}(A)$ is affine and $D = V(f)$, then the functor $L\eta_f$ on $D(A)$ defined in More on Algebra, Lemma 15.96.4 and the functor $L\eta_{\mathcal{I}}$ on $D_{QCoh}(\mathcal{O}_X)$ correspond via the equivalence of Derived Categories of Schemes, Lemma 36.3.5.*

Proof. Omitted. □

38.43. Blowing up complexes, II

0F8R The material in this section will be used to construct a version of Macpherson's graph construction in Section 38.44.

0F8S **Situation 38.43.1.** Here X is a scheme, $D \subset X$ is an effective Cartier divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and M is a perfect object of $D(\mathcal{O}_X)$.

Let (X, D, M) be a triple as in Situation 38.43.1. Consider an affine open $U = \mathrm{Spec}(A) \subset X$ such that

(1) $D \cap U = V(f)$ for some nonzerodivisor $f \in A$, and
(2) there exists a bounded complex $M^\bullet$ of finite free A-modules representing $M|_U$ (via the equivalence of Derived Categories of Schemes, Lemma 36.3.5).

We will say that $(U, A, f, M^\bullet)$ is an *affine chart* for (X, D, M). Consider the ideals $I_i(M^\bullet, f) \subset A$ defined in More on Algebra, Section 15.97. Let us say (X, S, M) is a *good triple* if for every $x \in D$ there exists an affine chart $(U, A, f, M^\bullet)$ with $x \in U$ and $I_i(M^\bullet, f)$ principal ideals for all $i \in \mathbf{Z}$.

0F8T **Lemma 38.43.2.** *In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ of D is defined (Divisors, Definition 31.13.12). Let $(U, A, f, M^\bullet)$ is an affine chart for (X, D, M). Let $V = \mathrm{Spec}(B) \subset Y$ is an affine open with $h(V) \subset U$. Denote $g \in B$ the image of $f \in A$. Then*

(1) *$(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for (Y, E, Lh^*M),*
(2) *$I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$ in B, and*
(3) *if (X, D, M) is a good triple, then (Y, E, Lh^*M) is a good triple.*

Proof. The first statement follows from the following observations: g is a nonzerodivisor in B which defines $E \cap V \subset V$ and $M^\bullet \otimes_A B$ represents $M^\bullet \otimes^{\mathbf{L}}_A B$ and hence represents the pullback of M to V by Derived Categories of Schemes, Lemma 36.3.8. Part (2) follows from part (1) and More on Algebra, Lemma 15.97.3. Combined with More on Algebra, Lemma 15.97.3 we conclude that the second statement of the lemma holds. □

0GTW **Lemma 38.43.3.** *Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. If (X, D, M) is a good triple, then $L\eta_{\mathcal{I}}M$ is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Translation of More on Algebra, Lemma 15.97.5. To do the translation use Lemma 38.42.2. □

0GTX **Lemma 38.43.4.** *Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. Assume (X, D, M) is a good triple. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing $L\eta_{\mathcal{I}}M$.*

Proof. By Cohomology, Lemma 20.55.7 the complex $\mathcal{Q}^\bullet = \eta_{\mathcal{I}}\mathcal{M}^\bullet$ represents $L\eta_{\mathcal{I}}M$. To check that this complex is locally bounded and consists of finite locally free, we may work affine locally. Then the boundedness is clear. Choose an affine chart $(U, A, f, M^\bullet)$ for (X, D, M) such that the ideals $I_i(M^\bullet, f)$ are principal and such that $\mathcal{M}^i|_U$ is finite free for each i. By our assumption that (X, D, M) is a good triple we can do this. Writing $N^i = \Gamma(U, \mathcal{M}^i|_U)$ we get a bounded complex $N^\bullet$ of finite free A-modules representing the same object in $D(A)$ as the complex $M^\bullet$ (by Derived Categories of Schemes, Lemma 36.3.5). Then $I_i(N^\bullet, f)$ is a principal ideal for all i by More on Algebra, Lemma 15.97.1. Hence the complex $\eta_f N^\bullet$ is a bounded complex of finite locally free A-modules. Since $\mathcal{Q}^i|_U$ is the quasi-coherent $\mathcal{O}_U$-module corresponding to $\eta_f N^i$ by Lemma 38.42.1 we conclude. □

0F9X **Lemma 38.43.5.** *In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. If (X, D, M) is a good triple, then*

$$Lh^*(L\eta_{\mathcal{I}}M) = L\eta_{\mathcal{J}}(Lh^*M)$$

in $D(\mathcal{O}_Y)$ where $\mathcal{J}$ is the ideal sheaf of E.

Proof. Translation of More on Algebra, Lemma 15.97.6. Use Lemmas 38.42.1 and 38.42.2 to do the translation. □

0GTY **Lemma 38.43.6.** *In Situation 38.43.1 there is a unique morphism $b : X' \to X$ such that*

1. *the pullback $D' = b^{-1}D$ is defined and (X', D', M') is a good triple where $M' = Lb^*M$, and*
2. *for any morphism of schemes $h : Y \to X$ such that the pullback $E = h^{-1}D$ is defined and (Y, E, Lh^*M) is a good triple, there is a unique factorization of h through b.*

Moreover, for any affine chart $(U, A, f, M^\bullet)$ the restriction $b^{-1}(U) \to U$ is the blowing up in the product of the ideals $I_i(M^\bullet, f)$ and for any quasi-compact open $W \subset X$ the restriction $b|_{b^{-1}(W)} : b^{-1}(W) \to W$ is a $W \setminus D$-admissible blowing up.

Proof. The proof is just that we will locally blow up X in the product ideals $I_i(M^\bullet, f)$ for any affine chart $(U, A, f, M^\bullet)$. The first few lemmas in More on Algebra, Section 15.97 show that this is well defined. The universal property (2) then follows from the universal property of blowing up. The details can be found below.

Let $U, A, f, M^\bullet$ be an affine chart for (X, D, M). All but a finite number of the ideals $I_i(M^\bullet, f)$ are equal to A hence it makes sense to look at

$$I = \prod\nolimits_i I_i(M^\bullet, f)$$

and this is a finitely generated ideal of A. Denote

$$b_U : U' \to U$$

the blowing up of U in I. Then $b_U^{-1}(U \cap D)$ is defined by Divisors, Lemma 31.32.11. Recall that $f^{r_i} \in I_i(M^\bullet, f)$ and hence b_U is a $(U \setminus D)$-admissible blowing up. By Divisors, Lemma 31.32.12 for each i the morphism b_U factors as $U' \to U_i' \to U$ where $U_i' \to U$ is the blowing up in $I_i(M^\bullet, f)$ and $U' \to U_i'$ is another blowing up. It follows that the pullback $I_i(M^\bullet, f)\mathcal{O}_{U'}$ of $I_i(M^\bullet, f)$ to U' is an invertible ideal sheaf, see Divisors, Lemmas 31.32.11 and 31.32.4. It follows that $(U', b^{-1}D, Lb^*M|_U)$ is a

good triple, see Lemma 38.43.2 for the behaviour of the ideals $I_i(-,-)$ under pullback. Finally, we claim that $b_U : U' \to U$ has the universal property mentioned in part (2) of the statement of the lemma. Namely, suppose $h : Y \to U$ is a morphism of schemes such that the pullback $E = h^{-1}(D \cap U)$ is defined and (Y, E, Lh^*M) is a good triple. Then Y is covered by affine charts $(V, B, g, N^\bullet)$ such that $I_i(N^\bullet, g)$ is an invertible ideal for each i. Then g and the image of f in B differ by a unit as they both cut out the effective Cartier divisor $E \cap V$. Hence we may assume g is the image of f by More on Algebra, Lemma 15.97.2. Then $I_i(N^\bullet, g)$ is isomorphic to $I_i(M^\bullet \otimes_A B, g)$ as a B-module by More on Algebra, Lemma 15.97.1. Thus $I_i(M^\bullet \otimes_A B, g) = I_i(M^\bullet, f)B$ (Lemma 38.43.2) is an invertible B-module. Hence the ideal IB is invertible. It follows that $I\mathcal{O}_Y$ is invertible. Hence we obtain a unique factorization of h through b_U by Divisors, Lemma 31.32.5.

Let $\mathcal{B}$ be the set of affine opens $U \subset X$ such that there exists an affine chart $(U, A, f, M^\bullet)$ for (X, D, M). Then $\mathcal{B}$ is a basis for the topology on X; details omitted. For $U \in \mathcal{B}$ we have the morphism $b_U : U' \to U$ constructed above which satisfies the universal property over U. If $U_1 \subset U_2 \subset X$ are both in $\mathcal{B}$, then $b_{U_1} : U_1' \to U_1$ is canonically isomorphic to

$$b_{U_2}|_{b_{U_2}^{-1}(U_1)} : b_{U_2}^{-1}(U_1) \longrightarrow U_1$$

by the universal property. In other words, we get an isomorphism $U_1' \to b_{U_2}^{-1}(U_1)$ over U_1. These isomorphisms satisfy the cocycle condition (again by the universal property) and hence by Constructions, Lemma 27.2.1 we get a morphism $b : X' \to X$ whose restriction to each U in $\mathcal{B}$ is isomorphic to $U' \to U$. Then the morphism $b : X' \to X$ satisfies properties (1) and (2) of the statement of the lemma as these properties may be checked locally (details omitted).

We still have to prove the final assertion of the lemma. Let $W \subset X$ be a quasi-compact open. Choose a finite covering $W = U_1 \cup \ldots \cup U_T$ such that for each $1 \leq t \leq T$ there exists an affine chart $(U_t, A_t, f_t, M_t^\bullet)$. We will use below that for any affine open $V = \mathrm{Spec}(B) \subset U_t \cap U_{t'}$ we have (a) the images of f_t and $f_{t'}$ in B differ by a unit, and (b) the complexes $M_t^\bullet \otimes_A B$ and $M_{t'} \otimes_A B$ define isomorphic objects of $D(B)$. For $i \in \mathbf{Z}$, set

$$N_i = \max\nolimits_{t=1,\ldots,T} \left(\sum\nolimits_{j \geq i} (-1)^{j-i} rk(M_t^j) \right)$$

Then $N_t - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j) \geq 0$ and we can consider the ideals

$$I_{t,i} = f_t^{N_i - \sum_{j \geq i} (-1)^{j-i} rk(M_t^j)} I_i(M_t^\bullet, f_t) \subset A_t$$

It follows from More on Algebra, Lemmas 15.97.2 and 15.97.1 that the ideals $I_{t,i}$ glue to a quasi-coherent, finite type ideal $\mathcal{I}_i \subset \mathcal{O}_W$. Moreover, all but a finite number of these ideals are equal to $\mathcal{O}_W$. Clearly, the morphism $X' \to X$ constructed above restricts to the blowing up of W in the product of the ideals $\mathcal{I}_i$. This finishes the proof. □

0F8U **Lemma 38.43.7.** *In Situation 38.43.1 let $b : X' \to X$ be the morphism of Lemma 38.43.6. Consider the effective Cartier divisor $D' = b^{-1}D$ with ideal sheaf $\mathcal{I}' \subset \mathcal{O}_{X'}$. Then $Q = L\eta_{\mathcal{I}'} Lb^* M$ is a perfect object of $D(\mathcal{O}_{X'})$.*

Proof. Follows from Lemmas 38.43.6 and 38.43.3. □

0F8V **Lemma 38.43.8.** *In Situation 38.43.1 let $h : Y \to X$ be a morphism of schemes such that the pullback $E = h^{-1}D$ is defined. Let $b : X' \to X$, resp. $c : Y' \to Y$ be as constructed in Lemma 38.43.6 for $D \subset X$ and M, resp. $E \subset Y$ and Lh^*M. Then Y' is the strict transform of Y with respect to $b : X' \to X$ (see proof for a precise formulation of this) and*

$$L\eta_{\mathcal{J}'}L(h \circ c)^*M = L(Y' \to X')^*Q$$

where $Q = L\eta_{\mathcal{I}'}Lb^*M$ as in Lemma 38.43.7. In particular, if (Y, E, Lh^*M) is a good triple and $k : Y \to X'$ is the unique morphism such that $h = b \circ k$, then $L\eta_{\mathcal{J}}Lh^*M = Lk^*Q$.

Proof. Denote $E' = c^{-1}E$. Then $(Y', E', L(h \circ c)^*M)$ is a good triple. Hence by the universal property of Lemma 38.43.6 there is a unique morphism

$$h' : Y' \longrightarrow X'$$

such that $b \circ h' = h \circ c$. In particular, there is a morphism $(h', c) : Y' \to X' \times_X Y$. We claim that given $W \subset X$ quasi-compact open, such that $b^{-1}(W) \to W$ is a blowing up, this morphism identifies $Y'|_W$ with the strict transform of Y_W with respect to $b^{-1}(W) \to W$. In turn, to see this is true is a local question on W, and we may therefore prove the statement over an affine chart. We do this in the next paragraph.

Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M). Recall from the proof of Lemma 38.43.7 that the restriction of $b : X' \to X$ to U is the blowing up of $U = \mathrm{Spec}(A)$ in the product of the ideals $I_i(M^\bullet, f)$. Now if $V = \mathrm{Spec}(B) \subset Y$ is any affine open with $h(V) \subset U$, then $(V, B, g, M^\bullet \otimes_A B)$ is an affine chart for (Y, E, Lh^*M) where $g \in B$ is the image of f, see Lemma 38.43.2. Hence the restriction of $c : Y' \to Y$ to V is the blowing up in the product of the ideals $I_i(M^\bullet, f)B$, i.e., the morphism $c : Y' \to Y$ over $h^{-1}(U)$ is the blowing up of $h^{-1}(U)$ in the ideal $\prod I_i(M^\bullet, f)\mathcal{O}_{h^{-1}(U)}$. Since this is also true for the strict transform, we see that our claim on strict transforms is true.

Having said this the equality $L\eta_{\mathcal{J}'}L(h \circ c)^*M = L(Y' \to X')^*Q$ follows from Lemma 38.43.5. The final statement is a special case of this (namely, the case where $c = \mathrm{id}_Y$ and $k = h'$). □

0F8W **Lemma 38.43.9.** *In Situation 38.43.1 let $W \subset X$ be the maximal open subscheme over which the cohomology sheaves of M are locally free. Then the morphism $b : X' \to X$ of Lemma 38.43.6 is an isomorphism over W.*

Proof. This is true because for any affine chart $(U, A, f, M^\bullet)$ with $U \subset W$ we have that $I_i(M^\bullet, f)$ are locally generated by a power of f by More on Algebra, Lemma 15.97.4. Since f is a nonzerodivisor, the blowing up $b^{-1}(U) \to U$ is an isomorphism. □

0GTZ **Lemma 38.43.10.** *Let $X, D, \mathcal{I}, M$ be as in Situation 38.43.1. If (X, D, M) is a good triple, then there exists a closed immersion*

$$i : T \longrightarrow D$$

of finite presentation with the following properties

(1) *T scheme theoretically contains $D \cap W$ where $W \subset X$ is the maximal open over which the cohomology sheaves of M are locally free,*

(2) *the cohomology sheaves of $Li^*L\eta_{\mathcal{I}}M$ are locally free, and*
(3) *for any point $t \in T$ with image $x = i(t) \in W$ the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$ and the rank of $H^i(Li^*L\eta_{\mathcal{I}}M)_t$ over $\mathcal{O}_{T,t}$ agree.*

Proof. Let $(U, A, f, M^\bullet)$ be an affine chart for (X, D, M) such that $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then we define $T \cap U \subset D \cap U$ as the closed subscheme defined by the ideal

$$J(M^\bullet, f) = \sum J_i(M^\bullet, f) \subset A/fA$$

studied in More on Algebra, Lemmas 15.97.8 and 15.97.9; in terms of the second lemma we see that $T \cap U \to D \cap U$ is given by the ring map $A/fA \to C$ studied there. Since (X, D, M) is a good triple we can cover X by affine charts of this form and by the first of the two lemmas, this construction glues. Hence we obtain a closed subscheme $T \subset D$ which on good affine charts as above is given by the ideal $J(M^\bullet, f)$. Then properties (1) and (2) follow from the second lemma. Details omitted. Small observation to help the reader: since $\eta_f M^\bullet$ is a complex of locally free modules by More on Algebra, Lemma 15.97.5 we see that $Li^*L\eta_{\mathcal{I}}M|_{T\cap U}$ is represented by the complex $\eta_f M^\bullet \otimes_A C$ of C-modules. The statement (3) on ranks follows from Cohomology, Lemma 20.55.10. □

0F8X **Lemma 38.43.11.** *In Situation 38.43.1. Let $b : X' \to X$ and D' be as in Lemma 38.43.6. Let $Q = L\eta_{\mathcal{I}'}Lb^*M$ be as in Lemma 38.43.7. Let $W \subset X$ be the maximal open where M has locally free cohomology modules. Then there exists a closed immersion $i : T \to D'$ of finite presentation such that*

(1) *$D' \cap b^{-1}(W) \subset T$ scheme theoretically,*
(2) *Li^*Q has locally free cohomology sheaves, and*
(3) *for $t \in T$ mapping to $w \in W$ the rank of $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$.*

Proof. Lemma 38.43.9 tells us that b is an isomorphism over W. Hence $b^{-1}(W) \subset X'$ is contained in the maximal open $W' \subset X'$ where Lb^*M has locally free cohomology sheaves. Then the actual statements in the lemma are an immediate consequence of Lemma 38.43.10 applied to (X', D', Lb^*M) and the other lemmas mentioned in the statement. □

0F8Y **Lemma 38.43.12.** *In Situation 38.43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 38.43.7. Let $\rho = (\rho_i)_{i\in\mathbf{Z}}$ be integers. Let $W(\rho) \subset X$ be the maximal open subscheme where $H^i(M)$ is locally free of rank ρ_i for all i. Let $i : T \to D'$ be as in Lemma 38.43.11. Then there exists an open and closed subscheme $T(\rho) \subset T$ containing $D' \cap b^{-1}(W(\rho))$ scheme theoretically such that $H^i(Li^*Q|_{T(\rho)})$ is locally free of rank ρ_i for all i.*

Proof. Let $T(\rho) \subset T$ be the open and closed subscheme where $H^i(Li^*Q)$ has rank ρ_i for all i. Then the statement is immediate from the assertion in Lemma 38.43.11 on ranks of the cohomology modules. □

0GU0 **Lemma 38.43.13.** *In Situation 38.43.1. Let $b : X' \to X$, $D' \subset X'$, and Q be as in Lemma 38.43.7. If there exists a locally bounded complex $\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing M, then there exists a locally bounded complex $\mathcal{Q}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules representing Q.*

Proof. Recall that $Q = L\eta_{\mathcal{I}'}Lb^*M$ where $\mathcal{I}'$ is the ideal sheaf of the effective Cartier divisor D'. The locally bounded complex $(\mathcal{M}')^\bullet = b^*\mathcal{M}^\bullet$ of finite locally free $\mathcal{O}_{X'}$-modules represents Lb^*M. Thus the lemma follows from Lemma 38.43.4. □

0F9Y **Lemma 38.43.14.** *Let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Let $M \in D(\mathcal{O}_X)$ be a perfect object. Let $W \subset X$ be the maximal open over which the cohomology sheaves $H^i(M)$ are locally free. There exists a proper morphism $b : X' \longrightarrow X$ and an object Q in $D(\mathcal{O}_{X'})$ with the following properties*

1. *$b : X' \to X$ is an isomorphism over $X \setminus D$,*
2. *$b : X' \to X$ is an isomorphism over W,*
3. *$D' = b^{-1}D$ is an effective Cartier divisor,*
4. *$Q = L\eta_{\mathcal{I}'}Lb^*M$ where $\mathcal{I}'$ is the ideal sheaf of D',*
5. *Q is a perfect object of $D(\mathcal{O}_{X'})$,*
6. *there exists a closed immersion $i : T \to D'$ of finite presentation such that*
   (a) *$D' \cap b^{-1}(W) \subset T$ scheme theoretically,*
   (b) *Li^*Q has finite locally free cohomology sheaves,*
   (c) *for $t \in T$ with image $w \in W$ the rank of $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_x$ over $\mathcal{O}_{X,x}$,*
7. *for any affine chart $(U, A, f, M^\bullet)$ for (X, D, M) the restriction of b to U is the blowing up of $U = \mathrm{Spec}(A)$ in the ideal $I = \prod I_i(M^\bullet, f)$, and*
8. *for any affine chart $(V, B, g, N^\bullet)$ for (X', D', Lb^*N) such that $I_i(N^\bullet, g)$ is principal, we have*
   (a) *$Q|_V$ corresponds to $\eta_g N^\bullet$,*
   (b) *$T \subset V \cap D'$ corresponds to the ideal $J(N^\bullet, g) = \sum J_i(N^\bullet, g) \subset B/gB$ studied in More on Algebra, Lemma 15.97.9.*
9. *If M can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules.*

Proof. This statement collects the information obtained in Lemmas 38.43.2, 38.43.3, 38.43.5, 38.43.6, 38.43.7, 38.43.8, 38.43.9, 38.43.10, 38.43.11, and 38.43.13. □

38.44. Blowing up complexes, III

0F8Z In this section we give an "algebra version" of the version of Macpherson's graph construction given in [**Ful98**, Section 18.1].

Let X be a scheme. Let E be a perfect object of $D(\mathcal{O}_X)$. Let $U \subset X$ be the maximal open subscheme such that $E|_U$ has locally free cohomology sheaves.

Consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{A}^1_X & \longrightarrow & \mathbf{P}^1_X & \longleftarrow & (\mathbf{P}^1_X)_\infty \\ & \searrow & \downarrow p & \swarrow \nearrow_{\infty} & \\ & & X & & \end{array}$$

Here we recall that $\mathbf{A}^1 = D_+(T_0)$ is the first standard affine open of $\mathbf{P}^1$ and that $\infty = V_+(T_0)$ is the complementary effective Cartier divisor and the diagram above

is the pullback of these schemes to X. Observe that $\infty : X \to (\mathbf{P}^1_X)_\infty$ is an isomorphism. Then

$$(\mathbf{P}^1_X, (\mathbf{P}^1_X)_\infty, Lp^*E)$$

is a triple as in Situation 38.43.1 in Section 38.43. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad W_\infty = b^{-1}((\mathbf{P}^1_X)_\infty)$$

be the blowing up and effective Cartier divisor constructed starting with this triple in Lemma 38.43.6. We also denote

$$Q = L\eta_{\mathcal{I}} Lb^*M = L\eta_{\mathcal{I}} L(p \circ b)^* E$$

the perfect object of $D(\mathcal{O}_W)$ considered in Lemma 38.43.7. Here $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of W_∞.

0F90 **Lemma 38.44.1.** *The construction above has the following properties:*

(1) *b is an isomorphism over $\mathbf{P}^1_U \cup \mathbf{A}^1_X$,*
(2) *the restriction of Q to $\mathbf{A}^1_X$ is equal to the pullback of E,*
(3) *there exists a closed immersion $i : T \to W_\infty$ of finite presentation such that $(W_\infty \to X)^{-1}U \subset T$ scheme theoretically and such that Li^*Q has locally free cohomology sheaves,*
(4) *for $t \in T$ with image $u \in U$ we have that the rank $H^i(Li^*Q)_t$ over $\mathcal{O}_{T,t}$ is equal to the rank of $H^i(M)_u$ over $\mathcal{O}_{U,u}$,*
(5) *if E can be represented by a locally bounded complex of finite locally free $\mathcal{O}_X$-modules, then Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules.*

Proof. This follows immediately from the results in Section 38.43; for a statement collecting everything needed, see Lemma 38.43.14. □

38.45. Other chapters

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CHAPTER 39

Groupoid Schemes

022L

39.1. Introduction

022M This chapter is devoted to generalities concerning groupoid schemes. See for example the beautiful paper [**KM97**] by Keel and Mori.

39.2. Notation

022N Let S be a scheme. If U, T are schemes over S we denote $U(T)$ for the set of T-valued points of U *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$. We try to reserve the letter T to denote a "test scheme" over S, as in the discussion that follows. Suppose we are given schemes X, Y over S and a morphism of schemes $f : X \to Y$ over S. For any scheme T over S we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which as indicated we denote by f also. In fact this construction is functorial in the scheme T/S. Yoneda's Lemma, see Categories, Lemma 4.3.5, says that f determines and is determined by this transformation of functors $f : h_X \to h_Y$. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are schemes over S, and if $m : X \times_S Y \to Z \times_S Z$ is a morphism of schemes over S, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

39.3. Equivalence relations

022O Recall that a *relation* R on a set A is just a subset of $R \subset A \times A$. We usually write aRb to indicate $(a, b) \in R$. We say the relation is *transitive* if $aRb, bRc \Rightarrow aRc$. We say the relation is *reflexive* if aRa for all $a \in A$. We say the relation is *symmetric* if $aRb \Rightarrow bRa$. A relation is called an *equivalence relation* if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require $R \to A \times A$ to be a map. Here is the definition.

022P **Definition 39.3.1.** Let S be a scheme. Let U be a scheme over S.

(1) A *pre-relation* on U over S is any morphism of schemes $j : R \to U \times_S U$. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A *relation* on U over S is a monomorphism of schemes $j : R \to U \times_S U$.

(3) A *pre-equivalence relation* is a pre-relation $j : R \to U \times_S U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all T/S.
(4) We say a morphism $R \to U \times_S U$ of schemes is an *equivalence relation on U over S* if and only if for every scheme T over S the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

02V8 **Lemma 39.3.2.** *Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. Finally, set*

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then j' is a pre-relation on U' over S. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

02V9 **Definition 39.3.3.** Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. The pre-relation $j' : R' \to U' \times_S U'$ is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

022Q **Lemma 39.3.4.** *Let $j : R \to U \times_S U$ be a pre-relation. Consider the relation on points of the scheme U defined by the rule*

$$x \sim y \Leftrightarrow \exists\ r \in R : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in R$ with $t(r) = x$, $s(r) = y$ and pick $r' \in R$ with $t(r') = y$, $s(r') = z$. Pick a field K fitting into the following commutative diagram

$$\begin{array}{ccc} \kappa(r) & \longrightarrow & K \\ \uparrow & & \uparrow \\ \kappa(y) & \longrightarrow & \kappa(r') \end{array}$$

Denote $x_K, y_K, z_K : \mathrm{Spec}(K) \to U$ the morphisms

$$\begin{array}{c} \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(x)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(y)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r')) \to \mathrm{Spec}(\kappa(z)) \to U \end{array}$$

By construction $(x_K, y_K) \in j(R(K))$ and $(y_K, z_K) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x_K, z_K) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

0DT7 **Lemma 39.3.5.** *Let $j : R \to U \times_S U$ be a pre-relation. Assume*

(1) *s, t are unramified,*
(2) *for any algebraically closed field k over S the map $R(k) \to U(k) \times U(k)$ is an equivalence relation,*

(3) *there are morphisms* $e : U \to R$, $i : R \to R$, $c : R \times_{s,U,t} R \to R$ *such that*

$$\begin{array}{ccc} U & \xrightarrow[e]{} & R \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle j} \\ U \times_S U & \longrightarrow & U \times_S U \end{array} \qquad \begin{array}{ccc} R & \xrightarrow[i]{} & R \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ U \times_S U & \xrightarrow{flip} & U \times_S U \end{array} \qquad \begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow[c]{} & R \\ {\scriptstyle j\times j}\downarrow & & \downarrow{\scriptstyle j} \\ U \times_S U \times_S U & \xrightarrow{pr_{02}} & U \times_S U \end{array}$$

are commutative.

Then j is an equivalence relation.

Proof. By condition (1) and Morphisms, Lemma 29.35.16 we see that j is unramified. Then $\Delta_j : R \to R \times_{U \times_S U} R$ is an open immersion by Morphisms, Lemma 29.35.13. However, then condition (2) says Δ_j is bijective on k-valued points, hence Δ_j is an isomorphism, hence j is a monomorphism. Then it easily follows from the commutative diagrams that $R(T) \subset U(T) \times U(T)$ is an equivalence relation for all schemes T over S. □

39.4. Group schemes

022R Let us recall that a *group* is a pair (G, m) where G is a set, and $m : G \times G \to G$ is a map of sets with the following properties:

(1) (associativity) $m(g, m(g', g'')) = m(m(g, g'), g'')$ for all $g, g', g'' \in G$,
(2) (identity) there exists a unique element $e \in G$ (called the *identity*, *unit*, or 1 of G) such that $m(g, e) = m(e, g) = g$ for all $g \in G$, and
(3) (inverse) for all $g \in G$ there exists a $i(g) \in G$ such that $m(g, i(g)) = m(i(g), g) = e$, where e is the identity.

Thus we obtain a map $e : \{*\} \to G$ and a map $i : G \to G$ so that the quadruple (G, m, e, i) satisfies the axioms listed above.

A *homomorphism of groups* $\psi : (G, m) \to (G', m')$ is a map of sets $\psi : G \to G'$ such that $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$. This automatically insures that $\psi(e) = e'$ and $i'(\psi(g)) = \psi(i(g))$. (Obvious notation.) We will use this below.

022S **Definition 39.4.1.** Let S be a scheme.

(1) A *group scheme over* S is a pair (G, m), where G is a scheme over S and $m : G \times_S G \to G$ is a morphism of schemes over S with the following property: For every scheme T over S the pair $(G(T), m)$ is a group.
(2) A *morphism* $\psi : (G, m) \to (G', m')$ *of group schemes over* S is a morphism $\psi : G \to G'$ of schemes over S such that for every T/S the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group scheme over the scheme S. By the discussion above (and the discussion in Section 39.2) we obtain morphisms of schemes over S: (identity) $e : S \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group listed above.

Let (G, m), (G', m') be group schemes over S. Let $f : G \to G'$ be a morphism of schemes over S. It follows from the definition that f is a morphism of group

schemes over S if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & G' \times_S G' \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m'} \\ G & \xrightarrow{f} & G' \end{array}$$

022T **Lemma 39.4.2.** *Let (G, m) be a group scheme over S. Let $S' \to S$ be a morphism of schemes. The pullback $(G_{S'}, m_{S'})$ is a group scheme over S'.*

Proof. Omitted. □

047D **Definition 39.4.3.** Let S be a scheme. Let (G, m) be a group scheme over S.

(1) A *closed subgroup scheme* of G is a closed subscheme $H \subset G$ such that $m|_{H \times_S H}$ factors through H and induces a group scheme structure on H over S.
(2) An *open subgroup scheme* of G is an open subscheme $G' \subset G$ such that $m|_{G' \times_S G'}$ factors through G' and induces a group scheme structure on G' over S.

Alternatively, we could say that H is a closed subgroup scheme of G if it is a group scheme over S endowed with a morphism of group schemes $i : H \to G$ over S which identifies H with a closed subscheme of G.

0G8L **Lemma 39.4.4.** *Let S be a scheme. Let (G, m, e, i) be a group scheme over S.*

(1) *A closed subscheme $H \subset G$ is a closed subgroup scheme if and only if $e : S \to G$, $m|_{H \times_S H} : H \times_S H \to G$, and $i|_H : H \to G$ factor through H.*
(2) *An open subscheme $H \subset G$ is an open subgroup scheme if and only if $e : S \to G$, $m|_{H \times_S H} : H \times_S H \to G$, and $i|_H : H \to G$ factor through H.*

Proof. Looking at T-valued points this translates into the well known conditions characterizing subsets of groups as subgroups. □

047E **Definition 39.4.5.** Let S be a scheme. Let (G, m) be a group scheme over S.

(1) We say G is a *smooth group scheme* if the structure morphism $G \to S$ is smooth.
(2) We say G is a *flat group scheme* if the structure morphism $G \to S$ is flat.
(3) We say G is a *separated group scheme* if the structure morphism $G \to S$ is separated.

Add more as needed.

39.5. Examples of group schemes

047F

022U **Example 39.5.1** (Multiplicative group scheme)**.** Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_m \times \mathbf{G}_m &\to \mathbf{G}_m \\ \mathrm{Spec}(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) &\to \mathrm{Spec}(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] &\leftarrow \mathbf{Z}[x, x^{-1}] \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that $\mathbf{G}_m$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{m,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

040M **Example 39.5.2** (Roots of unity)**.** Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the scheme

$$\mu_n = \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mu_n \times \mu_n &\to \mu_n \\ \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) &\to \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) &\leftarrow \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that μ_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mu_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

022V **Example 39.5.3** (Additive group scheme)**.** Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = \mathrm{Spec}(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_a \times \mathbf{G}_a &\to \mathbf{G}_a \\ \mathrm{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) &\to \mathrm{Spec}(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] &\leftarrow \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x &\leftarrow x \end{aligned}$$

Hence we see that $\mathbf{G}_a$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{a,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

022W **Example 39.5.4** (General linear group scheme)**.** Let $n \geq 1$. Consider the functor which associates to any scheme T the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\mathrm{GL}_n = \mathrm{Spec}(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where $d = \det((x_{ij}))$ with (x_{ij}) the $n \times n$ matrix with entry x_{ij} in the (i,j)-spot. The morphism giving the group structure is the morphism

$$\begin{aligned} \mathrm{GL}_n \times \mathrm{GL}_n &\to \mathrm{GL}_n \\ \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) &\to \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] &\leftarrow \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} &\leftarrow x_{ij} \end{aligned}$$

Hence we see that GL_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathrm{GL}_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathrm{GL}_{n,S}(T) = \mathrm{GL}_n(T) = \mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

022X **Example 39.5.5.** The determinant defines a morphism of group schemes

$$\det : \mathrm{GL}_n \longrightarrow \mathbf{G}_m$$

over $\mathbf{Z}$. By base change it gives a morphism of group schemes $\mathrm{GL}_{n,S} \to \mathbf{G}_{m,S}$ over any base scheme S.

03YW **Example 39.5.6** (Constant group)**.** Let G be an abstract group. Consider the functor which associates to any scheme T the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the scheme

$$G_{\mathrm{Spec}(\mathbf{Z})} = \coprod\nolimits_{g \in G} \mathrm{Spec}(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{\mathrm{Spec}(\mathbf{Z})} \times_{\mathrm{Spec}(\mathbf{Z})} G_{\mathrm{Spec}(\mathbf{Z})} \longrightarrow G_{\mathrm{Spec}(\mathbf{Z})}$$

which maps the component corresponding to the pair (g, g') to the component corresponding to gg'. For any scheme S the base change G_S is a group scheme over S whose functor of points is

$$T/S \longmapsto G_S(T) = \{f : T \to G \text{ locally constant}\}$$

as before.

39.6. Properties of group schemes

045W In this section we collect some simple properties of group schemes which hold over any base.

047G **Lemma 39.6.1.** *Let S be a scheme. Let G be a group scheme over S. Then $G \to S$ is separated (resp. quasi-separated) if and only if the identity morphism $e : S \to G$ is a closed immersion (resp. quasi-compact).*

Proof. We recall that by Schemes, Lemma 26.21.11 we have that e is an immersion which is a closed immersion (resp. quasi-compact) if $G \to S$ is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{G/S}} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle (g,g') \mapsto m(i(g),g')} \\ S & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/S}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact) so is $\Delta_{G/S}$, see Schemes, Lemma 26.18.2 (resp. Schemes, Lemma 26.19.3). □

047H **Lemma 39.6.2.** *Let S be a scheme. Let G be a group scheme over S. Let T be a scheme over S and let $\psi : T \to G$ be a morphism over S. If T is flat over S, then the morphism*

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

is flat. In particular, if G is flat over S, then $m : G \times_S G \to G$ is flat.

Proof. Consider the diagram

$$\begin{array}{ccccc} T \times_S G & \xrightarrow[(t,g) \mapsto (t, m(\psi(t),g))]{} & T \times_S G & \xrightarrow{\text{pr}} & G \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 29.25.8. □

047I **Lemma 39.6.3.** *Let (G, m, e, i) be a group scheme over the scheme S. Denote $f : G \to S$ the structure morphism. Then there exist canonical isomorphisms* [**MvdGE**, Proposition 3.15]

$$\Omega_{G/S} \cong f^*\mathcal{C}_{S/G} \cong f^*e^*\Omega_{G/S}$$

where $\mathcal{C}_{S/G}$ denotes the conormal sheaf of the immersion e. In particular, if S is the spectrum of a field, then $\Omega_{G/S}$ is a free $\mathcal{O}_G$-module.

Proof. By Morphisms, Lemma 29.32.10 we have

$$\Omega_{G \times_S G/G} = \text{pr}_0^*\Omega_{G/S}$$

where on the left hand side we view $G \times_S G$ as a scheme over G using pr_1. Let $\tau : G \times_S G \to G \times_S G$ be the "shearing map" given by $(g, h) \mapsto (m(g, h), h)$ on points. This map is an automorphism of $G \times_S G$ viewed as a scheme over G via the projection pr_1. Combining these two remarks we obtain an isomorphism

$$\tau^*\text{pr}_0^*\Omega_{G/S} \to \text{pr}_0^*\Omega_{G/S}$$

Since $\text{pr}_0 \circ \tau = m$ this can be rewritten as an isomorphism

$$m^*\Omega_{G/S} \to \text{pr}_0^*\Omega_{G/S}$$

Pulling back this isomorphism by $(e \circ f, \text{id}_G) : G \to G \times_S G$ and using that $m \circ (e \circ f, \text{id}_G) = \text{id}_G$ and $\text{pr}_0 \circ (e \circ f, \text{id}_G) = e \circ f$ we obtain an isomorphism

$$\Omega_{G/S} \to f^*e^*\Omega_{G/S}$$

as desired. By Morphisms, Lemma 29.32.16 we have $\mathcal{C}_{S/G} \cong e^*\Omega_{G/S}$. If S is the spectrum of a field, then any $\mathcal{O}_S$-module on S is free and the final statement follows. □

0BF5 **Lemma 39.6.4.** *Let S be a scheme. Let G be a group scheme over S. Let $s \in S$. Then the composition*

$$T_{G/S,e(s)} \oplus T_{G/S,e(s)} = T_{G\times_S G/S,(e(s),e(s))} \to T_{G/S,e(s)}$$

is addition of tangent vectors. Here the $=$ comes from Varieties, Lemma 33.16.7 and the right arrow is induced from $m : G \times_S G \to G$ via Varieties, Lemma 33.16.6.

Proof. We will use Varieties, Equation (33.16.3.1) and work with tangent vectors in fibres. An element θ in the first factor $T_{G_s/s,e(s)}$ is the image of θ via the map $T_{G_s/s,e(s)} \to T_{G_s\times G_s/s,(e(s),e(s))}$ coming from $(1, e) : G_s \to G_s \times G_s$. Since $m \circ (1, e) = 1$ we see that θ maps to θ by functoriality. Since the map is linear we see that (θ_1, θ_2) maps to $\theta_1 + \theta_2$. □

39.7. Properties of group schemes over a field

047J In this section we collect some properties of group schemes over a field. In the case of group schemes which are (locally) algebraic over a field we can say a lot more, see Section 39.8.

047K **Lemma 39.7.1.** *If (G, m) is a group scheme over a field k, then the multiplication map $m : G \times_k G \to G$ is open.*

Proof. The multiplication map is isomorphic to the projection map $\mathrm{pr}_0 : G \times_k G \to G$ because the diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow{(g,g')\mapsto(m(g,g'),g')} & G \times_k G \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle (g,g')\mapsto g} \\ G & \xrightarrow{\mathrm{id}} & G \end{array}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 29.23.4. □

0B7N **Lemma 39.7.2.** *If (G, m) is a group scheme over a field k. Let $U \subset G$ open and $T \to G$ a morphism of schemes. Then the image of the composition $T \times_k U \to G \times_k G \to G$ is open.*

Proof. For any field extension K/k the morphism $G_K \to G$ is open (Morphisms, Lemma 29.23.4). Every point ξ of $T \times_k U$ is the image of a morphism $(t, u) : \mathrm{Spec}(K) \to T \times_k U$ for some K. Then the image of $T_K \times_K U_K = (T \times_k U)_K \to G_K$ contains the translate $t \cdot U_K$ which is open. Combining these facts we see that the image of $T \times_k U \to G$ contains an open neighbourhood of the image of ξ. Since ξ was arbitrary we win. □

047L **Lemma 39.7.3.** *Let G be a group scheme over a field. Then G is a separated scheme.*

Proof. Say $S = \mathrm{Spec}(k)$ with k a field, and let G be a group scheme over S. By Lemma 39.6.1 we have to show that $e : S \to G$ is a closed immersion. By Morphisms, Lemma 29.20.2 the image of $e : S \to G$ is a closed point of G. It is clear that $\mathcal{O}_G \to e_*\mathcal{O}_S$ is surjective, since $e_*\mathcal{O}_S$ is a skyscraper sheaf supported at

the neutral element of G with value k. We conclude that e is a closed immersion by Schemes, Lemma 26.24.2. □

047M **Lemma 39.7.4.** *Let G be a group scheme over a field k. Then*

(1) *every local ring $\mathcal{O}_{G,g}$ of G has a unique minimal prime ideal,*
(2) *there is exactly one irreducible component Z of G passing through e, and*
(3) *Z is geometrically irreducible over k.*

Proof. For any point $g \in G$ there exists a field extension K/k and a K-valued point $g' \in G(K)$ mapping to g. If we think of g' as a K-rational point of the group scheme G_K, then we see that $\mathcal{O}_{G,g} \to \mathcal{O}_{G_K,g'}$ is a faithfully flat local ring map (as $G_K \to G$ is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 10.39.17). The result for $\mathcal{O}_{G_K,g'}$ implies the result for $\mathcal{O}_{G,g}$, see Algebra, Lemma 10.30.5. Hence in order to prove (1) it suffices to prove it for k-rational points g of G. In this case translation by g defines an automorphism $G \to G$ which maps e to g. Hence $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$. In this way we see that (2) implies (1), since irreducible components passing through e correspond one to one with minimal prime ideals of $\mathcal{O}_{G,e}$.

In order to prove (2) and (3) it suffices to prove (2) when k is algebraically closed. In this case, let Z_1, Z_2 be two irreducible components of G passing through e. Since k is algebraically closed the closed subscheme $Z_1 \times_k Z_2 \subset G \times_k G$ is irreducible too, see Varieties, Lemma 33.8.4. Hence $m(Z_1 \times_k Z_2)$ is contained in an irreducible component of G. On the other hand it contains Z_1 and Z_2 since $m|_{e\times G} = \mathrm{id}_G$ and $m|_{G\times e} = \mathrm{id}_G$. We conclude $Z_1 = Z_2$ as desired. □

04L9 **Remark 39.7.5.** Warning: The result of Lemma 39.7.4 does not mean that every irreducible component of G/k is geometrically irreducible. For example the group scheme $\mu_{3,\mathbf{Q}} = \mathrm{Spec}(\mathbf{Q}[x]/(x^3 - 1))$ over $\mathbf{Q}$ has two irreducible components corresponding to the factorization $x^3 - 1 = (x - 1)(x^2 + x + 1)$. The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over $\mathrm{Spec}(\mathbf{Q})$.

047R **Lemma 39.7.6.** *Let G be a group scheme over a perfect field k. Then the reduction G_{red} of G is a closed subgroup scheme of G.*

Proof. Omitted. Hint: Use that $G_{red} \times_k G_{red}$ is reduced by Varieties, Lemmas 33.6.3 and 33.6.7. □

047S **Lemma 39.7.7.** *Let k be a field. Let $\psi : G' \to G$ be a morphism of group schemes over k. If $\psi(G')$ is open in G, then $\psi(G')$ is closed in G.*

Proof. Let $U = \psi(G') \subset G$. Let $Z = G \setminus \psi(G') = G \setminus U$ with the reduced induced closed subscheme structure. By Lemma 39.7.2 the image of

$$Z \times_k G' \longrightarrow Z \times_k U \longrightarrow G$$

is open (the first arrow is surjective). On the other hand, since ψ is a homomorphism of group schemes, the image of $Z \times_k G' \to G$ is contained in Z (because translation by $\psi(g')$ preserves U for all points g' of G'; small detail omitted). Hence $Z \subset G$ is an open subset (although not necessarily an open subscheme). Thus $U = \psi(G')$ is closed. □

047T **Lemma 39.7.8.** *Let $i : G' \to G$ be an immersion of group schemes over a field k. Then i is a closed immersion, i.e., $i(G')$ is a closed subgroup scheme of G.*

Proof. To show that i is a closed immersion it suffices to show that $i(G')$ is a closed subset of G. Let $k \subset k'$ be a perfect extension of k. If $i(G'_{k'}) \subset G_{k'}$ is closed, then $i(G') \subset G$ is closed by Morphisms, Lemma 29.25.12 (as $G_{k'} \to G$ is flat, quasi-compact and surjective). Hence we may and do assume k is perfect. We will use without further mention that products of reduced schemes over k are reduced. We may replace G' and G by their reductions, see Lemma 39.7.6. Let $\overline{G'} \subset G$ be the closure of $i(G')$ viewed as a reduced closed subscheme. By Varieties, Lemma 33.24.1 we conclude that $\overline{G'} \times_k \overline{G'}$ is the closure of the image of $G' \times_k G' \to G \times_k G$. Hence

$$m\Big(\overline{G'} \times_k \overline{G'}\Big) \subset \overline{G'}$$

as m is continuous. It follows that $\overline{G'} \subset G$ is a (reduced) closed subgroup scheme. By Lemma 39.7.7 we see that $i(G') \subset \overline{G'}$ is also closed which implies that $i(G') = \overline{G'}$ as desired. □

0B7P **Lemma 39.7.9.** *Let G be a group scheme over a field k. If G is irreducible, then G is quasi-compact.*

Proof. Suppose that K/k is a field extension. If G_K is quasi-compact, then G is too as $G_K \to G$ is surjective. By Lemma 39.7.4 we see that G_K is irreducible. Hence it suffices to prove the lemma after replacing k by some extension. Choose K to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 33.14.2, we see that G_K is a Jacobson scheme all of whose closed points have residue field equal to K. In other words we may assume G is a Jacobson scheme all of whose closed points have residue field k.

Let $U \subset G$ be a nonempty affine open. Let $g \in G(k)$. Then $gU \cap U \neq \emptyset$. Hence we see that g is in the image of the morphism

$$U \times_{\mathrm{Spec}(k)} U \longrightarrow G, \quad (u_1, u_2) \longmapsto u_1 u_2^{-1}$$

Since the image of this morphism is open (Lemma 39.7.1) we see that the image is all of G (because G is Jacobson and closed points are k-rational). Since U is affine, so is $U \times_{\mathrm{Spec}(k)} U$. Hence G is the image of a quasi-compact scheme, hence quasi-compact. □

0B7Q **Lemma 39.7.10.** *Let G be a group scheme over a field k. If G is connected, then G is irreducible.*

Proof. By Varieties, Lemma 33.7.14 we see that G is geometrically connected. If we show that G_K is irreducible for some field extension K/k, then the lemma follows. Hence we may apply Varieties, Lemma 33.14.2 to reduce to the case where k is algebraically closed, G is a Jacobson scheme, and all the closed points are k-rational.

Let $Z \subset G$ be the unique irreducible component of G passing through the neutral element, see Lemma 39.7.4. Endowing Z with the reduced induced closed subscheme structure, we see that $Z \times_k Z$ is reduced and irreducible (Varieties, Lemmas 33.6.7 and 33.8.4). We conclude that $m|_{Z \times_k Z} : Z \times_k Z \to G$ factors through Z. Hence Z becomes a closed subgroup scheme of G.

To get a contradiction, assume there exists another irreducible component $Z' \subset G$. Then $Z \cap Z' = \emptyset$ by Lemma 39.7.4. By Lemma 39.7.9 we see that Z is quasi-compact. Thus we may choose a quasi-compact open $U \subset G$ with $Z \subset U$ and

$U \cap Z' = \emptyset$. The image W of $Z \times_k U \to G$ is open in G by Lemma 39.7.2. On the other hand, W is quasi-compact as the image of a quasi-compact space. We claim that W is closed.

Proof of the claim. Since W is quasi-compact, we see that points in the closure of W are specializations of points of W (Morphisms, Lemma 29.6.5). Thus we have to show that any irreducible component $Z'' \subset G$ of G which meets W is contained in W. As G is Jacobson and closed points are rational, $Z'' \cap W$ has a rational point $g \in Z''(k) \cap W(k)$ and hence $Z'' = Zg$. But $W = m(Z \times_k W)$ by construction, so $Z'' \cap W \neq \emptyset$ implies $Z'' \subset W$.

By the claim $W \subset G$ is an open and closed subset of G. Now $W \cap Z' = \emptyset$ since otherwise by the argument given in the preceding paragraph we would get $Z' = Zg$ for some $g \in W(k)$. Then as Z is a subgroup we could even pick $g \in U(k)$ which would contradict $Z' \cap U = \emptyset$. Hence $W \subset G$ is a proper open and closed subset which contradicts the assumption that G is connected. □

0B7R **Proposition 39.7.11.** *Let G be a group scheme over a field k. There exists a canonical closed subgroup scheme $G^0 \subset G$ with the following properties*

(1) *$G^0 \to G$ is a flat closed immersion,*
(2) *$G^0 \subset G$ is the connected component of the identity,*
(3) *G^0 is geometrically irreducible, and*
(4) *G^0 is quasi-compact.*

Proof. Let G^0 be the connected component of the identity with its canonical scheme structure (Morphisms, Definition 29.26.3). To show that G^0 is a closed subsgroup scheme we will use the criterion of Lemma 39.4.4. The morphism $e : \mathrm{Spec}(k) \to G$ factors through G^0 as we chose G^0 to be the connected component of G containing e. Since $i : G \to G$ is an automorphism fixing e, we see that i sends G^0 into itself. By Varieties, Lemma 33.7.13 the scheme G^0 is geometrically connected over k. Thus $G^0 \times_k G^0$ is connected (Varieties, Lemma 33.7.4). Thus $m(G^0 \times_k G^0) \subset G^0$ set theoretically. Thus $m|_{G^0 \times_k G^0} : G^0 \times_k G^0 \to G$ factors through G^0 by Morphisms, Lemma 29.26.1. Hence G^0 is a closed subgroup scheme of G. By Lemma 39.7.10 we see that G^0 is irreducible. By Lemma 39.7.4 we see that G^0 is geometrically irreducible. By Lemma 39.7.9 we see that G^0 is quasi-compact. □

0B7T **Lemma 39.7.12.** *Let k be a field. Let $T = \mathrm{Spec}(A)$ where A is a directed colimit of algebras which are finite products of copies of k. For any scheme X over k we have $|T \times_k X| = |T| \times |X|$ as topological spaces.*

Proof. By taking an affine open covering we reduce to the case of an affine X. Say $X = \mathrm{Spec}(B)$. Write $A = \mathrm{colim}\, A_i$ with $A_i = \prod_{t \in T_i} k$ and T_i finite. Then $T_i = |\mathrm{Spec}(A_i)|$ with the discrete topology and the transition morphisms $A_i \to A_{i'}$ are given by set maps $T_{i'} \to T_i$. Thus $|T| = \lim T_i$ as a topological space, see

Limits, Lemma 32.4.6. Similarly we have

$$\begin{aligned} |T \times_k X| &= |\operatorname{Spec}(A \otimes_k B)| \\ &= |\operatorname{Spec}(\operatorname{colim} A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(\prod\nolimits_{t \in T_i} B)| \\ &= \lim T_i \times |X| \\ &= (\lim T_i) \times |X| \\ &= |T| \times |X| \end{aligned}$$

by the lemma above and the fact that limits commute with limits. □

The following lemma says that in fact we can put a "algebraic profinite family of points" in an affine open. We urge the reader to read Lemma 39.8.6 first.

0B7U **Lemma 39.7.13.** *Let k be an algebraically closed field. Let G be a group scheme over k. Assume that G is Jacobson and that all closed points are k-rational. Let $T = \operatorname{Spec}(A)$ where A is a directed colimit of algebras which are finite products of copies of k. For any morphism $f : T \to G$ there exists an affine open $U \subset G$ containing $f(T)$.*

Proof. Let $G^0 \subset G$ be the closed subgroup scheme found in Proposition 39.7.11. The first two paragraphs serve to reduce to the case $G = G^0$.

Observe that T is a directed inverse limit of finite topological spaces (Limits, Lemma 32.4.6), hence profinite as a topological space (Topology, Definition 5.22.1). Let $W \subset G$ be a quasi-compact open containing the image of $T \to G$. After replacing W by the image of $G^0 \times W \to G \times G \to G$ we may assume that W is invariant under the action of left translation by G^0, see Lemma 39.7.2. Consider the composition

$$\psi = \pi \circ f : T \xrightarrow{f} W \xrightarrow{\pi} \pi_0(W)$$

The space $\pi_0(W)$ is profinite (Topology, Lemma 5.23.9 and Properties, Lemma 28.2.4). Let $F_\xi \subset T$ be the fibre of $T \to \pi_0(W)$ over $\xi \in \pi_0(W)$. Assume that for all ξ we can find an affine open $U_\xi \subset W$ with $F \subset U$. Since $\psi : T \to \pi_0(W)$ is universally closed as a map of topological spaces (Topology, Lemma 5.17.7), we can find a quasi-compact open $V_\xi \subset \pi_0(W)$ such that $\psi^{-1}(V_\xi) \subset f^{-1}(U_\xi)$ (easy topological argument omitted). After replacing U_ξ by $U_\xi \cap \pi^{-1}(V_\xi)$, which is open and closed in U_ξ hence affine, we see that $U_\xi \subset \pi^{-1}(V_\xi)$ and $U_\xi \cap T = \psi^{-1}(V_\xi)$. By Topology, Lemma 5.22.4 we can find a finite disjoint union decomposition $\pi_0(W) = \bigcup_{i=1,\ldots,n} V_i$ by quasi-compact opens such that $V_i \subset V_{\xi_i}$ for some i. Then we see that

$$f(T) \subset \bigcup\nolimits_{i=1,\ldots,n} U_{\xi_i} \cap \pi^{-1}(V_i)$$

the right hand side of which is a finite disjoint union of affines, therefore affine.

Let Z be a connected component of G which meets $f(T)$. Then Z has a k-rational point z (because all residue fields of the scheme T are isomorphic to k). Hence $Z = G^0 z$. By our choice of W, we see that $Z \subset W$. The argument in the preceding paragraph reduces us to the problem of finding an affine open neighbourhood of $f(T) \cap Z$ in W. After translation by a rational point we may assume that $Z = G^0$ (details omitted). Observe that the scheme theoretic inverse image $T' = f^{-1}(G^0) \subset$

T is a closed subscheme, which has the same type. After replacing T by T' we may assume that $f(T) \subset G^0$. Choose an affine open neighbourhood $U \subset G$ of $e \in G$, so that in particular $U \cap G^0$ is nonempty. We will show there exists a $g \in G^0(k)$ such that $f(T) \subset g^{-1}U$. This will finish the proof as $g^{-1}U \subset W$ by the left G^0-invariance of W.

The arguments in the preceding two paragraphs allow us to pass to G^0 and reduce the problem to the following: Assume G is irreducible and $U \subset G$ an affine open neighbourhood of e. Show that $f(T) \subset g^{-1}U$ for some $g \in G(k)$. Consider the morphism

$$U \times_k T \longrightarrow G \times_k T, \quad (t, u) \longrightarrow (uf(t)^{-1}, t)$$

which is an open immersion (because the extension of this morphism to $G \times_k T \to G \times_k T$ is an isomorphism). By our assumption on T we see that we have $|U \times_k T| = |U| \times |T|$ and similarly for $G \times_k T$, see Lemma 39.7.12. Hence the image of the displayed open immersion is a finite union of boxes $\bigcup_{i=1,\ldots,n} U_i \times V_i$ with $V_i \subset T$ and $U_i \subset G$ quasi-compact open. This means that the possible opens $Uf(t)^{-1}$, $t \in T$ are finite in number, say $Uf(t_1)^{-1}, \ldots, Uf(t_r)^{-1}$. Since G is irreducible the intersection

$$Uf(t_1)^{-1} \cap \ldots \cap Uf(t_r)^{-1}$$

is nonempty and since G is Jacobson with closed points k-rational, we can choose a k-valued point $g \in G(k)$ of this intersection. Then we see that $g \in Uf(t)^{-1}$ for all $t \in T$ which means that $f(t) \in g^{-1}U$ as desired. □

047V **Remark 39.7.14.** If G is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? By Proposition 39.7.11 this question is only interesting if G has infinitely many connected components (geometrically).

047U **Lemma 39.7.15.** *Let G be a group scheme over a field. There exists an open and closed subscheme $G' \subset G$ which is a countable union of affines.*

Proof. Let $e \in U(k)$ be a quasi-compact open neighbourhood of the identity element. By replacing U by $U \cap i(U)$ we may assume that U is invariant under the inverse map. As G is separated this is still a quasi-compact set. Set

$$G' = \bigcup\nolimits_{n \geq 1} m_n(U \times_k \ldots \times_k U)$$

where $m_n : G \times_k \ldots \times_k G \to G$ is the n-slot multiplication map $(g_1, \ldots, g_n) \mapsto m(m(\ldots(m(g_1, g_2), g_3), \ldots), g_n)$. Each of these maps are open (see Lemma 39.7.1) hence G' is an open subgroup scheme. By Lemma 39.7.7 it is also a closed subgroup scheme. □

39.8. Properties of algebraic group schemes

0BF6 Recall that a scheme over a field k is (locally) algebraic if it is (locally) of finite type over $\mathrm{Spec}(k)$, see Varieties, Definition 33.20.1. This is the sense of algebraic we are using in the title of this section.

045X **Lemma 39.8.1.** *Let k be a field. Let G be a locally algebraic group scheme over k. Then G is equidimensional and $\dim(G) = \dim_g(G)$ for all $g \in G$. For any closed point $g \in G$ we have $\dim(G) = \dim(\mathcal{O}_{G,g})$.*

Proof. Let us first prove that $\dim_g(G) = \dim_{g'}(G)$ for any pair of points $g, g' \in G$. By Morphisms, Lemma 29.28.3 we may extend the ground field at will. Hence we may assume that both g and g' are defined over k. Hence there exists an automorphism of G mapping g to g', whence the equality. By Morphisms, Lemma 29.28.1 we have $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \mathrm{trdeg}_k(\kappa(g))$. On the other hand, the dimension of G (or any open subset of G) is the supremum of the dimensions of the local rings of G, see Properties, Lemma 28.10.3. Clearly this is maximal for closed points g in which case $\mathrm{trdeg}_k(\kappa(g)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 29.16). Hence the lemma follows. □

The following result is sometimes referred to as Cartier's theorem.

047N **Lemma 39.8.2.** *Let k be a field of characteristic 0. Let G be a locally algebraic group scheme over k. Then the structure morphism $G \to \mathrm{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 39.6.3 the module of differentials of G over k is free. Hence smoothness follows from Varieties, Lemma 33.25.1. □

047O **Remark 39.8.3.** Any group scheme over a field of characteristic 0 is reduced, see [**Per75**, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [**Per76**, Proposition 4.2.8]. This was a question raised in [**Oor66**, page 80]. We have seen in Lemma 39.8.2 that this holds when the group scheme is locally of finite type.

047P **Lemma 39.8.4.** *Let k be a perfect field of characteristic $p > 0$ (see Lemma 39.8.2 for the characteristic zero case). Let G be a locally algebraic group scheme over k. If G is reduced then the structure morphism $G \to \mathrm{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 39.6.3 the sheaf $\Omega_{G/k}$ is free. Hence the lemma follows from Varieties, Lemma 33.25.2. □

047Q **Remark 39.8.5.** Let k be a field of characteristic $p > 0$. Let $\alpha \in k$ be an element which is not a pth power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}^2_{a,k}$$

is reduced and irreducible but not smooth (not even normal).

The following lemma is a special case of Lemma 39.7.13 with a somewhat easier proof.

0B7S **Lemma 39.8.6.** *Let k be an algebraically closed field. Let G be a locally algebraic group scheme over k. Let $g_1, \ldots, g_n \in G(k)$ be k-rational points. Then there exists an affine open $U \subset G$ containing $g_1, \ldots, g_n$.*

Proof. We first argue by induction on n that we may assume all g_i are on the same connected component of G. Namely, if not, then we can find a decomposition $G = W_1 \amalg W_2$ with W_i open in G and (after possibly renumbering) $g_1, \ldots, g_r \in W_1$ and $g_{r+1}, \ldots, g_n \in W_2$ for some $0 < r < n$. By induction we can find affine opens U_1 and U_2 of G with $g_1, \ldots, g_r \in U_1$ and $g_{r+1}, \ldots, g_n \in U_2$. Then

$$g_1, \ldots, g_n \in (U_1 \cap W_1) \cup (U_2 \cap W_2)$$

is a solution to the problem. Thus we may assume $g_1, \ldots, g_n$ are all on the same connected component of G. Translating by g_1^{-1} we may assume $g_1, \ldots, g_n \in G^0$

where $G^0 \subset G$ is as in Proposition 39.7.11. Choose an affine open neighbourhood U of e, in particular $U \cap G^0$ is nonempty. Since G^0 is irreducible we see that

$$G^0 \cap (Ug_1^{-1} \cap \ldots \cap Ug_n^{-1})$$

is nonempty. Since $G \to \mathrm{Spec}(k)$ is locally of finite type, also $G^0 \to \mathrm{Spec}(k)$ is locally of finite type, hence any nonempty open has a k-rational point. Thus we can pick $g \in G^0(k)$ with $g \in Ug_i^{-1}$ for all i. Then $g_i \in g^{-1}U$ for all i and $g^{-1}U$ is the affine open we were looking for. □

0BF7 **Lemma 39.8.7.** *Let k be a field. Let G be an algebraic group scheme over k. Then G is quasi-projective over k.*

Proof. By Varieties, Lemma 33.15.1 we may assume that k is algebraically closed. Let $G^0 \subset G$ be the connected component of G as in Proposition 39.7.11. Then every other connected component of G has a k-rational point and hence is isomorphic to G^0 as a scheme. Since G is quasi-compact and Noetherian, there are finitely many of these connected components. Thus we reduce to the case discussed in the next paragraph.

Let G be a connected algebraic group scheme over an algebraically closed field k. If the characteristic of k is zero, then G is smooth over k by Lemma 39.8.2. If the characteristic of k is $p > 0$, then we let $H = G_{red}$ be the reduction of G. By Divisors, Proposition 31.17.9 it suffices to show that H has an ample invertible sheaf. (For an algebraic scheme over k having an ample invertible sheaf is equivalent to being quasi-projective over k, see for example the very general More on Morphisms, Lemma 37.49.1.) By Lemma 39.7.6 we see that H is a group scheme over k. By Lemma 39.8.4 we see that H is smooth over k. This reduces us to the situation discussed in the next paragraph.

Let G be a quasi-compact irreducible smooth group scheme over an algebraically closed field k. Observe that the local rings of G are regular and hence UFDs (Varieties, Lemma 33.25.3 and More on Algebra, Lemma 15.122.2). The complement of a nonempty affine open of G is the support of an effective Cartier divisor D. This follows from Divisors, Lemma 31.16.6. (Observe that G is separated by Lemma 39.7.3.) We conclude there exists an effective Cartier divisor $D \subset G$ such that $G \setminus D$ is affine. We will use below that for any $n \geq 1$ and $g_1, \ldots, g_n \in G(k)$ the complement $G \setminus \bigcup Dg_i$ is affine. Namely, it is the intersection of the affine opens $G \setminus Dg_i \cong G \setminus D$ in the separated scheme G.

We may choose the top row of the diagram

$$\begin{array}{ccccc} G & \xleftarrow{j} & U & \xrightarrow{\pi} & \mathbf{A}^d_k \\ & & \uparrow & & \uparrow \\ & & W & \xrightarrow{\pi'} & V \end{array}$$

such that $U \neq \emptyset$, $j : U \to G$ is an open immersion, and π is étale, see Morphisms, Lemma 29.36.20. There is a nonempty affine open $V \subset \mathbf{A}^d_k$ such that with $W = \pi^{-1}(V)$ the morphism $\pi' = \pi|_W : W \to V$ is finite étale. In particular π' is finite locally free, say of degree n. Consider the effective Cartier divisor

$$\mathcal{D} = \{(g, w) \mid m(g, j(w)) \in D\} \subset G \times W$$

(This is the restriction to $G \times W$ of the pullback of $D \subset G$ under the flat morphism $m : G \times G \to G$.) Consider the closed subset[1] $T = (1 \times \pi')(\mathcal{D}) \subset G \times V$. Since π' is finite locally free, every irreducible component of T has codimension 1 in $G \times V$. Since $G \times V$ is smooth over k we conclude these components are effective Cartier divisors (Divisors, Lemma 31.15.7 and lemmas cited above) and hence T is the support of an effective Cartier divisor E in $G \times V$. If $v \in V(k)$, then $(\pi')^{-1}(v) = \{w_1, \ldots, w_n\} \subset W(k)$ and we see that

$$E_v = \bigcup\nolimits_{i=1,\ldots,n} Dj(w_i)^{-1}$$

in G set theoretically. In particular we see that $G \setminus E_v$ is affine open (see above). Moreover, if $g \in G(k)$, then there exists a $v \in V$ such that $g \notin E_v$. Namely, the set W' of $w \in W$ such that $g \notin Dj(w)^{-1}$ is nonempty open and it suffices to pick v such that the fibre of $W' \to V$ over v has n elements.

Consider the invertible sheaf $\mathcal{M} = \mathcal{O}_{G \times V}(E)$ on $G \times V$. By Varieties, Lemma 33.30.5 the isomorphism class $\mathcal{L}$ of the restriction $\mathcal{M}_v = \mathcal{O}_G(E_v)$ is independent of $v \in V(k)$. On the other hand, for every $g \in G(k)$ we can find a v such that $g \notin E_v$ and such that $G \setminus E_v$ is affine. Thus the canonical section (Divisors, Definition 31.14.1) of $\mathcal{O}_G(E_v)$ corresponds to a section s_v of $\mathcal{L}$ which does not vanish at g and such that G_{s_v} is affine. This means that $\mathcal{L}$ is ample by definition (Properties, Definition 28.26.1). □

0BF8 **Lemma 39.8.8.** *Let k be a field. Let G be a locally algebraic group scheme over k. Then the center of G is a closed subgroup scheme of G.*

Proof. Let $\mathrm{Aut}(G)$ denote the contravariant functor on the category of schemes over k which associates to S/k the set of automorphisms of the base change G_S as a group scheme over S. There is a natural transformation

$$G \longrightarrow \mathrm{Aut}(G), \quad g \longmapsto \mathrm{inn}_g$$

sending an S-valued point g of G to the inner automorphism of G determined by g. The center C of G is by definition the kernel of this transformation, i.e., the functor which to S associates those $g \in G(S)$ whose associated inner automorphism is trivial. The statement of the lemma is that this functor is representable by a closed subgroup scheme of G.

Choose an integer $n \geq 1$. Let $G_n \subset G$ be the nth infinitesimal neighbourhood of the identity element e of G. For every scheme S/k the base change $G_{n,S}$ is the nth infinitesimal neighbourhood of $e_S : S \to G_S$. Thus we see that there is a natural transformation $\mathrm{Aut}(G) \to \mathrm{Aut}(G_n)$ where the right hand side is the functor of automorphisms of G_n as a scheme (G_n isn't in general a group scheme). Observe that G_n is the spectrum of an artinian local ring A_n with residue field k which has finite dimension as a k-vector space (Varieties, Lemma 33.20.2). Since every automorphism of G_n induces in particular an invertible linear map $A_n \to A_n$, we obtain transformations of functors

$$G \to \mathrm{Aut}(G) \to \mathrm{Aut}(G_n) \to \mathrm{GL}(A_n)$$

[1] Using the material in Divisors, Section 31.17 we could take as effective Cartier divisor E the norm of the effective Cartier divisor $\mathcal{D}$ along the finite locally free morphism $1 \times \pi'$ bypassing some of the arguments.

The final group valued functor is representable, see Example 39.5.4, and the last arrow is visibly injective. Thus for every n we obtain a closed subgroup scheme

$$H_n = \mathrm{Ker}(G \to \mathrm{Aut}(G_n)) = \mathrm{Ker}(G \to \mathrm{GL}(A_n)).$$

As a first approximation we set $H = \bigcap_{n \geq 1} H_n$ (scheme theoretic intersection). This is a closed subgroup scheme which contains the center C.

Let h be an S-valued point of H with S locally Noetherian. Then the automorphism inn_h induces the identity on all the closed subschemes $G_{n,S}$. Consider the kernel $K = \mathrm{Ker}(\mathrm{inn}_h : G_S \to G_S)$. This is a closed subgroup scheme of G_S over S containing the closed subschemes $G_{n,S}$ for $n \geq 1$. This implies that K contains an open neighbourhood of $e(S) \subset G_S$, see Algebra, Remark 10.51.6. Let $G^0 \subset G$ be as in Proposition 39.7.11. Since G^0 is geometrically irreducible, we conclude that K contains G^0_S (for any nonempty open $U \subset G^0_{k'}$ and any field extension k'/k we have $U \cdot U^{-1} = G^0_{k'}$, see proof of Lemma 39.7.9). Applying this with $S = H$ we find that G^0 and H are subgroup schemes of G whose points commute: for any scheme S and any S-valued points $g \in G^0(S)$, $h \in H(S)$ we have $gh = hg$ in $G(S)$.

Assume that k is algebraically closed. Then we can pick a k-valued point g_i in each irreducible component G_i of G. Observe that in this case the connected components of G are the irreducible components of G are the translates of G^0 by our g_i. We claim that

$$C = H \cap \bigcap\nolimits_i \mathrm{Ker}(\mathrm{inn}_{g_i} : G \to G) \quad \text{(scheme theoretic intersection)}$$

Namely, C is contained in the right hand side. On the other hand, every S-valued point h of the right hand side commutes with G^0 and with g_i hence with everything in $G = \bigcup G^0 g_i$.

The case of a general base field k follows from the result for the algebraic closure $\overline{k}$ by descent. Namely, let $A \subset G_{\overline{k}}$ the closed subgroup scheme representing the center of $G_{\overline{k}}$. Then we have

$$A \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} A$$

as closed subschemes of $G_{\overline{k} \otimes_k \overline{k}}$ by the functorial nature of the center. Hence we see that A descends to a closed subgroup scheme $Z \subset G$ by Descent, Lemma 35.37.2 (and Descent, Lemma 35.23.21). Then Z represents C (small argument omitted) and the proof is complete. □

39.9. Abelian varieties

0BF9 An excellent reference for this material is Mumford's book on abelian varieties, see [**Mum70**]. We encourage the reader to look there. There are many equivalent definitions; here is one.

03RO **Definition 39.9.1.** Let k be a field. An *abelian variety* is a group scheme over k which is also a proper, geometrically integral variety over k[2].

We prove a few lemmas about this notion and then we collect all the results together in Proposition 39.9.11.

0BFA **Lemma 39.9.2.** *Let k be a field. Let A be an abelian variety over k. Then A is projective.*

[2]For equivalent definitions see Remark 39.9.12.

Proof. This follows from Lemma 39.8.7 and More on Morphisms, Lemma 37.50.1. □

0BFB **Lemma 39.9.3.** *Let k be a field. Let A be an abelian variety over k. For any field extension K/k the base change A_K is an abelian variety over K.*

Proof. Omitted. Note that this is why we insisted on A being geometrically integral; without that condition this lemma (and many others below) would be wrong. □

0BFC **Lemma 39.9.4.** *Let k be a field. Let A be an abelian variety over k. Then A is smooth over k.*

Proof. If k is perfect then this follows from Lemma 39.8.2 (characteristic zero) and Lemma 39.8.4 (positive characteristic). We can reduce the general case to this case by descent for smoothness (Descent, Lemma 35.23.29) and going to the perfect closure using Lemma 39.9.3. □

0BFD **Lemma 39.9.5.** *An abelian variety is an abelian group scheme, i.e., the group law is commutative.*

Proof. Let k be a field. Let A be an abelian variety over k. By Lemma 39.9.3 we may replace k by its algebraic closure. Consider the morphism

$$h : A \times_k A \longrightarrow A \times_k A, \quad (x, y) \longmapsto (x, xyx^{-1}y^{-1})$$

This is a morphism over A via the first projection on either side. Let $e \in A(k)$ be the unit. Then we see that $h|_{e\times A}$ is constant with value (e, e). By More on Morphisms, Lemma 37.44.3 there exists an open neighbourhood $U \subset A$ of e such that $h|_{U\times A}$ factors through some $Z \subset U \times A$ finite over U. This means that for $x \in U(k)$ the morphism $A \to A$, $y \mapsto xyx^{-1}y^{-1}$ takes finitely many values. Of course this means it is constant with value e. Thus $(x, y) \mapsto xyx^{-1}y^{-1}$ is constant with value e on $U \times A$ which implies that the group law on A is abelian. □

0BFE **Lemma 39.9.6.** *Let k be a field. Let A be an abelian variety over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module. Then there is an isomorphism*

$$m_{1,2,3}^*\mathcal{L} \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L} \cong m_{1,2}^*\mathcal{L} \otimes m_{1,3}^*\mathcal{L} \otimes m_{2,3}^*\mathcal{L}$$

of invertible modules on $A \times_k A \times_k A$ where $m_{i_1,\ldots,i_t} : A \times_k A \times_k A \to A$ is the morphism $(x_1, x_2, x_3) \mapsto \sum x_{i_j}$.

Proof. Apply the theorem of the cube (More on Morphisms, Theorem 37.33.8) to the difference

$$\mathcal{M} = m_{1,2,3}^*\mathcal{L} \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L} \otimes m_{1,2}^*\mathcal{L}^{\otimes -1} \otimes m_{1,3}^*\mathcal{L}^{\otimes -1} \otimes m_{2,3}^*\mathcal{L}^{\otimes -1}$$

This works because the restriction of $\mathcal{M}$ to $A \times A \times e = A \times A$ is equal to

$$n_{1,2}^*\mathcal{L} \otimes n_1^*\mathcal{L} \otimes n_2^*\mathcal{L} \otimes n_{1,2}^*\mathcal{L}^{\otimes -1} \otimes n_1^*\mathcal{L}^{\otimes -1} \otimes n_2^*\mathcal{L}^{\otimes -1} \cong \mathcal{O}_{A\times_k A}$$

where $n_{i_1,\ldots,i_t} : A \times_k A \to A$ is the morphism $(x_1, x_2) \mapsto \sum x_{i_j}$. Similarly for $A \times e \times A$ and $e \times A \times A$. □

0BFF **Lemma 39.9.7.** *Let k be a field. Let A be an abelian variety over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module. Then*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n(n+1)/2} \otimes ([-1]^*\mathcal{L})^{\otimes n(n-1)/2}$$

where $[n] : A \to A$ sends x to $x + x + \ldots + x$ with n summands and where $[-1] : A \to A$ is the inverse of A.

Proof. Consider the morphism $A \to A \times_k A \times_k A$, $x \mapsto (x, x, -x)$ where $-x = [-1](x)$. Pulling back the relation of Lemma 39.9.6 we obtain

$$\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [-1]^*\mathcal{L} \cong [2]^*\mathcal{L}$$

which proves the result for $n = 2$. By induction assume the result holds for $1, 2, \ldots, n$. Then consider the morphism $A \to A \times_k A \times_k A$, $x \mapsto (x, x, [n-1]x)$. Pulling back the relation of Lemma 39.9.6 we obtain

$$[n+1]^*\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [n-1]^*\mathcal{L} \cong [2]^*\mathcal{L} \otimes [n]^*\mathcal{L} \otimes [n]^*\mathcal{L}$$

and the result follows by elementary arithmetic. □

0BFG **Lemma 39.9.8.** *Let k be a field. Let A be an abelian variety over k. Let $[d] : A \to A$ be the multiplication by d. Then $[d]$ is finite locally free of degree $d^{2\dim(A)}$.*

Proof. By Lemma 39.9.2 (and More on Morphisms, Lemma 37.50.1) we see that A has an ample invertible module $\mathcal{L}$. Since $[-1] : A \to A$ is an automorphism, we see that $[-1]^*\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module as well. Thus $\mathcal{N} = \mathcal{L} \otimes [-1]^*\mathcal{L}$ is ample, see Properties, Lemma 28.26.5. Since $\mathcal{N} \cong [-1]^*\mathcal{N}$ we see that $[d]^*\mathcal{N} \cong \mathcal{N}^{\otimes d^2}$ by Lemma 39.9.7.

To get a contradiction, let $C \subset X$ be a proper curve contained in a fibre of $[d]$. Then $\mathcal{N}^{\otimes d^2}|_C \cong \mathcal{O}_C$ is an ample invertible $\mathcal{O}_C$-module of degree 0 which contradicts Varieties, Lemma 33.44.14 for example. (You can also use Varieties, Lemma 33.45.9.) Thus every fibre of $[d]$ has dimension 0 and hence $[d]$ is finite for example by Cohomology of Schemes, Lemma 30.21.1. Moreover, since A is smooth over k by Lemma 39.9.4 we see that $[d] : A \to A$ is flat by Algebra, Lemma 10.128.1 (we also use that schemes smooth over fields are regular and that regular rings are Cohen-Macaulay, see Varieties, Lemma 33.25.3 and Algebra, Lemma 10.106.3). Thus $[d]$ is finite flat hence finite locally free by Morphisms, Lemma 29.48.2.

Finally, we come to the formula for the degree. By Varieties, Lemma 33.45.11 we see that

$$\deg_{\mathcal{N}^{\otimes d^2}}(A) = \deg([d]) \deg_{\mathcal{N}}(A)$$

Since the degree of A with respect to $\mathcal{N}^{\otimes d^2}$, respectively $\mathcal{N}$ is the coefficient of $n^{\dim(A)}$ in the polynomial

$$n \longmapsto \chi(A, \mathcal{N}^{\otimes nd^2}), \quad \text{respectively} \quad n \longmapsto \chi(A, \mathcal{N}^{\otimes n})$$

we see that $\deg([d]) = d^{2\dim(A)}$. □

0BFH **Lemma 39.9.9.** *Let k be a field. Let A be a nonzero abelian variety over k. Then $[d] : A \to A$ is étale if and only if d is invertible in k.*

Proof. Observe that $[d](x+y) = [d](x) + [d](y)$. Since translation by a point is an automorphism of A, we see that the set of points where $[d] : A \to A$ is étale is either empty or equal to A (some details omitted). Thus it suffices to check whether $[d]$ is étale at the unit $e \in A(k)$. Since we know that $[d]$ is finite locally free (Lemma 39.9.8) to see that it is étale at e is equivalent to proving that $\mathrm{d}[d] : T_{A/k,e} \to T_{A/k,e}$ is injective. See Varieties, Lemma 33.16.8 and Morphisms, Lemma 29.36.16. By Lemma 39.6.4 we see that $\mathrm{d}[d]$ is given by multiplication by d on $T_{A/k,e}$. □

0C0Y **Lemma 39.9.10.** *Let k be a field of characteristic $p > 0$. Let A be an abelian variety of dimension g over k. The fibre of $[p] : A \to A$ over 0 has at most p^g distinct points.*

Proof. To prove this, we may and do replace k by the algebraic closure. By Lemma 39.6.4 the derivative of $[p]$ is multiplication by p as a map $T_{A/k,e} \to T_{A/k,e}$ and hence is zero (compare with proof of Lemma 39.9.9). Since $[p]$ commutes with translation we conclude that the derivative of $[p]$ is everywhere zero, i.e., that the induced map $[p]^*\Omega_{A/k} \to \Omega_{A/k}$ is zero. Looking at generic points, we find that the corresponding map $[p]^* : k(A) \to k(A)$ of function fields induces the zero map on $\Omega_{k(A)/k}$. Let $t_1, \ldots, t_g$ be a p-basis of $k(A)$ over k (More on Algebra, Definition 15.46.1 and Lemma 15.46.2). Then $[p]^*(t_i)$ has a pth root by Algebra, Lemma 10.158.2. We conclude that $k(A)[x_1, \ldots, x_g]/(x_1^p - t_1, \ldots, x_g^p - t_g)$ is a subextension of $[p]^* : k(A) \to k(A)$. Thus we can find an affine open $U \subset A$ such that $t_i \in \mathcal{O}_A(U)$ and $x_i \in \mathcal{O}_A([p]^{-1}(U))$. We obtain a factorization

$$[p]^{-1}(U) \xrightarrow{\pi_1} \mathrm{Spec}(\mathcal{O}(U)[x_1, \ldots, x_g]/(x_1^p - t_1, \ldots, x_g^p - t_g)) \xrightarrow{\pi_2} U$$

of $[p]$ over U. After shrinking U we may assume that π_1 is finite locally free (for example by generic flatness – actually it is already finite locally free in our case). By Lemma 39.9.8 we see that $[p]$ has degree p^{2g}. Since π_2 has degree p^g we see that π_1 has degree p^g as well. The morphism π_2 is a universal homeomorphism hence the fibres are singletons. We conclude that the (set theoretic) fibres of $[p]^{-1}(U) \to U$ are the fibres of π_1. Hence they have at most p^g elements. Since $[p]$ is a homomorphism of group schemes over k, the fibre of $[p] : A(k) \to A(k)$ has the same cardinality for every $a \in A(k)$ and the proof is complete. □

03RP **Proposition 39.9.11.** *Let A be an abelian variety over a field k. Then*

Wonderfully explained in [**Mum70**].

1. *A is projective over k,*
2. *A is a commutative group scheme,*
3. *the morphism $[n] : A \to A$ is surjective for all $n \geq 1$,*
4. *if k is algebraically closed, then $A(k)$ is a divisible abelian group,*
5. *$A[n] = \mathrm{Ker}([n] : A \to A)$ is a finite group scheme of degree $n^{2\dim A}$ over k,*
6. *$A[n]$ is étale over k if and only if $n \in k^*$,*
7. *if $n \in k^*$ and k is algebraically closed, then $A(k)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2\dim(A)}$,*
8. *if k is algebraically closed of characteristic $p > 0$, then there exists an integer $0 \leq f \leq \dim(A)$ such that $A(k)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ for all $m \geq 1$.*

Proof. Part (1) follows from Lemma 39.9.2. Part (2) follows from Lemma 39.9.5. Part (3) follows from Lemma 39.9.8. If k is algebraically closed then surjective morphisms of varieties over k induce surjective maps on k-rational points, hence (4) follows from (3). Part (5) follows from Lemma 39.9.8 and the fact that a base change of a finite locally free morphism of degree N is a finite locally free morphism of degree N. Part (6) follows from Lemma 39.9.9. Namely, if n is invertible in k, then $[n]$ is étale and hence $A[n]$ is étale over k. On the other hand, if n is not invertible in k, then $[n]$ is not étale at e and it follows that $A[n]$ is not étale over k at e (use Morphisms, Lemmas 29.36.16 and 29.35.15).

Assume k is algebraically closed. Set $g = \dim(A)$. Proof of (7). Let ℓ be a prime number which is invertible in k. Then we see that

$$A[\ell](k) = A(k)[\ell]$$

is a finite abelian group, annihilated by ℓ, of order ℓ^{2g}. It follows that it is isomorphic to $(\mathbf{Z}/\ell\mathbf{Z})^{2g}$ by the structure theory for finite abelian groups. Next, we consider the short exact sequence

$$0 \to A(k)[\ell] \to A(k)[\ell^2] \xrightarrow{\ell} A(k)[\ell] \to 0$$

Arguing similarly as above we conclude that $A(k)[\ell^2] \cong (\mathbf{Z}/\ell^2\mathbf{Z})^{2g}$. By induction on the exponent we find that $A(k)[\ell^m] \cong (\mathbf{Z}/\ell^m\mathbf{Z})^{2g}$. For composite integers n prime to the characteristic of k we take primary parts and we find the correct shape of the n-torsion in $A(k)$. The proof of (8) proceeds in exactly the same way, using that Lemma 39.9.10 gives $A(k)[p] \cong (\mathbf{Z}/p\mathbf{Z})^{\oplus f}$ for some $0 \leq f \leq g$. □

0H2U **Remark 39.9.12.** Let k be a field. There are $2 \times 4 \times 2 = 16$ equivalent definitions of abelian varieties. Let

- projective, proper,
- geometrically irreducible, irreducible, geometrically connected, connected,
- smooth, geometrically reduced

be three sets of properties, pick one from each of them, and let A be a group scheme over k with the chosen properties over k. Then A is an abelian variety. If we pick the options "proper, geometrically irreducible, geometrically reduced", then we recover Definition 39.9.1 (use Varieties, Lemma 33.9.2). The weakest possible options would be "proper, connected, and geometrically reduced", see for example Morphisms, Lemma 29.43.5 and Varieties, Lemma 33.25.4. So say A is a proper, connected, and geometrically reduced group scheme over k. Then A is geometrically irreducible by Lemmas 39.7.10 and 39.7.4 and hence an abelian variety. Finally, if A/k is an abelian variety, then it is projective and smooth over k (Proposition 39.9.11), whence satisfies the strongest possible options "projective, geometrically irreducible, smooth".

39.10. Actions of group schemes

022Y Let (G, m) be a group and let V be a set. Recall that a *(left) action* of G on V is given by a map $a : G \times V \to V$ such that

(1) (associativity) $a(m(g, g'), v) = a(g, a(g', v))$ for all $g, g' \in G$ and $v \in V$, and
(2) (identity) $a(e, v) = v$ for all $v \in V$.

We also say that V is a *G-set* (this usually means we drop the a from the notation – which is abuse of notation). A *map of G-sets* $\psi : V \to V'$ is any set map such that $\psi(a(g, v)) = a(g, \psi(v))$ for all $v \in V$.

022Z **Definition 39.10.1.** Let S be a scheme. Let (G, m) be a group scheme over S.

(1) An *action of G on the scheme X/S* is a morphism $a : G \times_S X \to X$ over S such that for every T/S the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.

(2) Suppose that X, Y are schemes over S each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of schemes over S such that for every T/S the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

03LD (39.10.1.1)
$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times a} & G \times_S X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ G \times_S X & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} G \times_S X & \xrightarrow{a} & X \\ {\scriptstyle e \times 1_X}\uparrow & \nearrow{\scriptstyle 1_X} & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\mathrm{id} \times \psi} & G \times_S Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{\psi} & Y \end{array}$$

commutes.

07S1 **Definition 39.10.2.** Let S, $G \to S$, and $X \to S$ as in Definition 39.10.1. Let $a : G \times_S X \to X$ be an action of G on X/S. We say the action is *free* if for every scheme T over S the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

07S2 **Lemma 39.10.3.** *Situation as in Definition 39.10.2, The action a is free if and only if*
$$G \times_S X \to X \times_S X, \quad (g, x) \mapsto (a(g, x), x)$$
is a monomorphism.

Proof. Immediate from the definitions. □

39.11. Principal homogeneous spaces

0497 In Cohomology on Sites, Definition 21.4.1 we have defined a torsor for a sheaf of groups on a site. Suppose $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ is a topology and (G, m) is a group scheme over S. Since τ is stronger than the canonical topology (see Descent, Lemma 35.13.7) we see that $\underline{G}$ (see Sites, Definition 7.12.3) is a sheaf of groups on $(Sch/S)_\tau$. Hence we already know what it means to have a torsor for $\underline{G}$ on $(Sch/S)_\tau$. A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a G-torsor.

0498 **Definition 39.11.1.** Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism of schemes $G \times_S X \to X \times_S X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism of schemes over S.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over S where G acts on G by left multiplication.

It is clear that if $S' \to S$ is a morphism of schemes then the pullback $X_{S'}$ of a pseudo G-torsor over S is a pseudo $G_{S'}$-torsor over S'.

0499 **Lemma 39.11.2.** *In the situation of Definition 39.11.1.*

(1) *The scheme X is a pseudo G-torsor if and only if for every scheme T over S the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to S$ has a section.*

Proof. Omitted. □

049A **Definition 39.11.3.** Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a pseudo G-torsor over S.

(1) We say X is a *principal homogeneous space* or a *G-torsor* if there exists a fpqc covering[3] $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section (i.e., is a trivial pseudo G_{S_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section.
(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-torsor over S" to indicate that X is a scheme over S equipped with an action of G which turns it into a principal homogeneous space over S. Next we show that this agrees with the notation introduced earlier when both apply.

049B **Lemma 39.11.4.** *Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then X is a G-torsor in the τ-topology if and only if $\underline{X}$ is a $\underline{G}$-torsor on $(Sch/S)_\tau$.*

Proof. Omitted. □

049C **Remark 39.11.5.** Let (G, m) be a group scheme over the scheme S. In this situation we have the following natural types of questions:

(1) If $X \to S$ is a pseudo G-torsor and $X \to S$ is surjective, then is X necessarily a G-torsor?
(2) Is every $\underline{G}$-torsor on $(Sch/S)_{fppf}$ representable? In other words, does every $\underline{G}$-torsor come from a fppf G-torsor?
(3) Is every G-torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on $G \to S$. For example: If S is the spectrum of a field, then the answer to (1) is yes because then $\{X \to S\}$ is a fpqc covering trivializing X. If $G \to S$ is affine, then the answer to (2) is yes (this follows from Descent, Lemma 35.37.1). If $G = \mathrm{GL}_{n,S}$ then the answer to (3) is yes and in fact any $\mathrm{GL}_{n,S}$-torsor is locally trivial (this follows from Descent, Lemma 35.7.6).

[3] This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [**ABD+66**, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

39.12. Equivariant quasi-coherent sheaves

03LE We think of "functions" as dual to "space". Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf of modules as "functions". This leads us naturally to the direction of the arrows chosen in the following definition.

03LF **Definition 39.12.1.** Let S be a scheme, let (G, m) be a group scheme over S, and let $a : G \times_S X \to X$ be an action of the group scheme G on X/S. A *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply an *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G\times_S X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_S X \to X$ is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^*\mathrm{pr}_1^*\mathcal{F} & \xrightarrow{\mathrm{pr}_{12}^*\alpha} & \mathrm{pr}_2^*\mathcal{F} \\ {\scriptstyle (1_G\times a)^*\alpha}\uparrow & & \uparrow{\scriptstyle (m\times 1_X)^*\alpha} \\ (1_G \times a)^*a^*\mathcal{F} & = & (m \times 1_X)^*a^*\mathcal{F} \end{array}$$

is commutative in the category of $\mathcal{O}_{G\times_S G\times_S X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (39.10.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

03LG **Lemma 39.12.2.** *Let S be a scheme. Let G be a group scheme over S. Let $f : Y \to X$ be a G-equivariant morphism between S-schemes endowed with G-actions. The rule $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules to the category of G-equivariant quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Omitted. □

Let us give an example.

0EKJ **Example 39.12.3.** Let A be a $\mathbf{Z}$-graded ring, i.e., A comes with a direct sum decomposition $A = \bigoplus_{n\in\mathbf{Z}} A_n$ and $A_n \cdot A_m \subset A_{n+m}$. Set $X = \mathrm{Spec}(A)$. Then we obtain a $\mathbf{G}_m$-action

$$a : \mathbf{G}_m \times X \longrightarrow X$$

by the ring map $\mu : A \to A \otimes \mathbf{Z}[x, x^{-1}]$, $f \mapsto f \otimes x^{\deg(f)}$. Namely, to check this we have to verify that

$$\begin{array}{ccc} A & \xrightarrow{\mu} & A \otimes \mathbf{Z}[x, x^{-1}] \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu\otimes 1} \\ A \otimes \mathbf{Z}[x, x^{-1}] & \xrightarrow{1\otimes m} & A \otimes \mathbf{Z}[x, x^{-1}] \otimes \mathbf{Z}[x, x^{-1}] \end{array}$$

where $m(x) = x \otimes x$, see Example 39.5.1. This is immediately clear when evaluating on a homogeneous element. Suppose that M is a graded A-module. Then we obtain a $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-module $\mathcal{F} = \widetilde{M}$ by using α as in Definition 39.12.1 corresponding to the $A \otimes \mathbf{Z}[x, x^{-1}]$-module map

$$M \otimes_{A,\mu} (A \otimes \mathbf{Z}[x, x^{-1}]) \longrightarrow M \otimes_{A,\mathrm{id}_A \otimes 1} (A \otimes \mathbf{Z}[x, x^{-1}])$$

sending $m \otimes 1 \otimes 1$ to $m \otimes 1 \otimes x^{\deg(m)}$ for $m \in M$ homogeneous.

0EKK **Lemma 39.12.4.** *Let $a : \mathbf{G}_m \times X \to X$ be an action on an affine scheme. Then X is the spectrum of a $\mathbf{Z}$-graded ring and the action is as in Example 39.12.3.*

Proof. Let $f \in A = \Gamma(X, \mathcal{O}_X)$. Then we can write

$$a^\sharp(f) = \sum\nolimits_{n \in \mathbf{Z}} f_n \otimes x^n \quad \text{in} \quad A \otimes \mathbf{Z}[x, x^{-1}] = \Gamma(\mathbf{G}_m \times X, \mathcal{O}_{\mathbf{G}_m \times X})$$

as a finite sum with f_n in A uniquely determined. Thus we obtain maps $A \to A$, $f \mapsto f_n$. Since a is an action, if we evaluate at $x = 1$, we see $f = \sum f_n$. Since a is an action we find that

$$\sum (f_n)_m \otimes x^m \otimes x^n = \sum f_n x^n \otimes x^n$$

(compare with computation in Example 39.12.3). Thus $(f_n)_m = 0$ if $n \neq m$ and $(f_n)_n = f_n$. Thus if we set

$$A_n = \{f \in A \mid f_n = f\}$$

then we get $A = \sum A_n$. On the other hand, the sum has to be direct since $f = 0$ implies $f_n = 0$ in the situation above. □

0EKL **Lemma 39.12.5.** *Let A be a graded ring. Let $X = \mathrm{Spec}(A)$ with action $a : \mathbf{G}_m \times X \to X$ as in Example 39.12.3. Let $\mathcal{F}$ be a $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-module. Then $M = \Gamma(X, \mathcal{F})$ has a canonical grading such that it is a graded A-module and such that the isomorphism $\widetilde{M} \to \mathcal{F}$ (Schemes, Lemma 26.7.4) is an isomorphism of $\mathbf{G}_m$-equivariant modules where the $\mathbf{G}_m$-equivariant structure on $\widetilde{M}$ is the one from Example 39.12.3.*

Proof. You can either prove this by repeating the arguments of Lemma 39.12.4 for the module M. Alternatively, you can consider the scheme $(X', \mathcal{O}_{X'}) = (X, \mathcal{O}_X \oplus \mathcal{F})$ where $\mathcal{F}$ is viewed as an ideal of square zero. There is a natural action $a' : \mathbf{G}_m \times X' \to X'$ defined using the action on X and on $\mathcal{F}$. Then apply Lemma 39.12.4 to X' and conclude. (The nice thing about this argument is that it immediately shows that the grading on A and M are compatible, i.e., that M is a graded A-module.) Details omitted. □

39.13. Groupoids

0230 Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 4.2.5. Hence a groupoid has a set of objects Ob, a set of arrows Arrows, a *source* and *target* map $s, t : \mathrm{Arrows} \to \mathrm{Ob}$, and a *composition law* $c : \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \to \mathrm{Arrows}$. These maps satisfy exactly the following axioms

(1) (associativity) $c \circ (1, c) = c \circ (c, 1)$ as maps $\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} \to \mathrm{Arrows}$,
(2) (identity) there exists a map $e : \mathrm{Ob} \to \mathrm{Arrows}$ such that

(a) $s \circ e = t \circ e = \mathrm{id}$ as maps $\mathrm{Ob} \to \mathrm{Ob}$,
(b) $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$ as maps $\mathrm{Arrows} \to \mathrm{Arrows}$,

(3) (inverse) there exists a map $i : \mathrm{Arrows} \to \mathrm{Arrows}$ such that
(a) $s \circ i = t$, $t \circ i = s$ as maps $\mathrm{Arrows} \to \mathrm{Ob}$, and
(b) $c \circ (1, i) = e \circ t$ and $c \circ (i, 1) = e \circ s$ as maps $\mathrm{Arrows} \to \mathrm{Arrows}$.

If this is the case the maps e and i are uniquely determined and i is a bijection. Note that if $(\mathrm{Ob}', \mathrm{Arrows}', s', t', c')$ is a second groupoid category, then a functor $f : (\mathrm{Ob}, \mathrm{Arrows}, s, t, c) \to (\mathrm{Ob}', \mathrm{Arrows}', s', t', c')$ is given by a pair of set maps $f : \mathrm{Ob} \to \mathrm{Ob}'$ and $f : \mathrm{Arrows} \to \mathrm{Arrows}'$ such that $s' \circ f = f \circ s$, $t' \circ f = f \circ t$, and $c' \circ (f, f) = f \circ c$. The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

0231 **Definition 39.13.1.** Let S be a scheme.

(1) A *groupoid scheme over* S, or simply a *groupoid over* S is a quintuple (U, R, s, t, c) where U and R are schemes over S, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of schemes over S with the following property: For any scheme T over S the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category in the sense described above.
(2) A *morphism* $f : (U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoid schemes over* S is given by morphisms of schemes $f : U \to U'$ and $f : R \to R'$ with the following property: For any scheme T over S the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid over S. Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes $e : U \to R$ and $i : R \to R$ over S such that for every scheme T over S the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid scheme. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid scheme over S we denote
$$j = (t, s) : R \longrightarrow U \times_S U$$
which is compatible with our conventions in Section 39.3 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid over S" to stress the existence of identity and inverse.

0232 **Lemma 39.13.2.** *Given a groupoid scheme (U, R, s, t, c) over S the morphism $j : R \to U \times_S U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. □

0233 **Lemma 39.13.3.** *Given an equivalence relation $j : R \to U \times_S U$ over S there is a unique way to extend it to a groupoid (U, R, s, t, c) over S.*

Proof. Omitted. This is a nice exercise in the definitions. □

02YE **Lemma 39.13.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. In the commutative diagram*

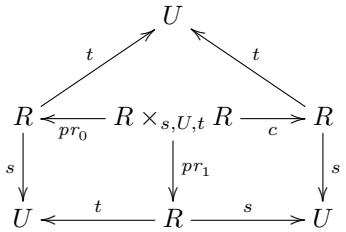

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

03C6 **Lemma 39.13.5.** *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. The diagram*

03C7 (39.13.5.1)

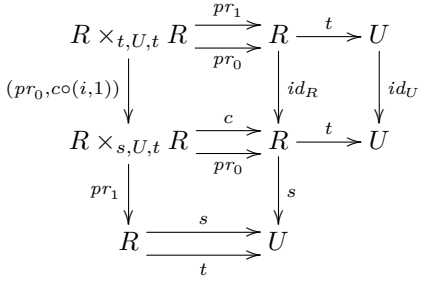

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 39.13.4. □

0DT8 **Lemma 39.13.6.** *Let (U, R, s, t, c) be a groupoid over a scheme S. Let $S' \to S$ be a morphism. Then the base changes $U' = S' \times_S U$, $R' = S' \times_S R$ endowed with the base changes s', t', c' of the morphisms s, t, c form a groupoid scheme (U', R', s', t', c') over S' and the projections determine a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes over S.*

Proof. Omitted. Hint: $R' \times_{s',U',t'} R' = S' \times_S (R \times_{s,U,t} R)$. □

39.14. Quasi-coherent sheaves on groupoids

03LH See the introduction of Section 39.12 for our choices in direction of arrows.

03LI **Definition 39.14.1.** Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

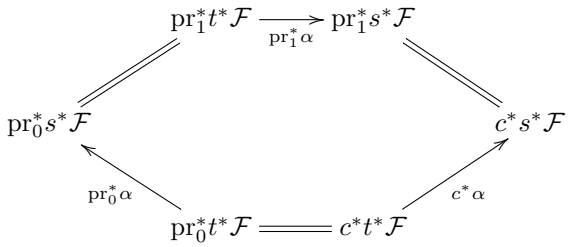

is a commutative in the category of $\mathcal{O}_{R\times_{s,U,t}R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 39.13.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

077Q **Lemma 39.14.2.** *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 39.14.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. □

03LJ **Lemma 39.14.3.** *Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. □

09VH **Lemma 39.14.4.** *Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Assume that*

(1) *$f : U \to U'$ is quasi-compact and quasi-separated,*

(2) *the square*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian, and

(3) *s' and t' are flat.*

Then pushforward f_* given by

$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U, R, s, t, c) to the category of quasi-coherent sheaves on (U', R', s', t', c') which is right adjoint to pullback as defined in Lemma 39.14.3.

Proof. Since $U \to U'$ is quasi-compact and quasi-separated we see that f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Schemes, Lemma 26.24.1). Moreover, since the squares

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

are cartesian we find that $(t')^* f_* \mathcal{F} = f_* t^* \mathcal{F}$ and $(s')^* f_* \mathcal{F} = f_* s^* \mathcal{F}$, see Cohomology of Schemes, Lemma 30.5.2. Thus it makes sense to think of $f_*\alpha$ as a map $(t')^* f_* \mathcal{F} \to (s')^* f_* \mathcal{F}$. A similar argument shows that $f_*\alpha$ satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and pushforward on modules on ringed spaces are adjoint. Some details omitted. □

077R **Lemma 39.14.5.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Schemes, Section 26.24. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

077S **Lemma 39.14.6.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^* \operatorname{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^* \operatorname{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^* \operatorname{Ker}(\varphi) \to s^* \operatorname{Ker}(\varphi)$ and $\lambda : t^* \operatorname{Coker}(\varphi) \to s^* \operatorname{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\operatorname{Ker}(\varphi), \kappa)$ and $(\operatorname{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\operatorname{Coim}(\varphi) = \operatorname{Im}(\varphi)$ follows because it holds over U. □

39.15. Colimits of quasi-coherent modules

07TS In this section we prove some technical results saying that under suitable assumptions every quasi-coherent module on a groupoid is a filtered colimit of "small" quasi-coherent modules.

07TR **Lemma 39.15.1.** *Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are flat, quasi-compact, and quasi-separated. For any quasi-coherent module $\mathcal{G}$ on U, there exists a canonical isomorphism $\alpha : t^*s_*t^*\mathcal{G} \to s^*s_*t^*\mathcal{G}$ which turns $(s_*t^*\mathcal{G}, \alpha)$ into a quasi-coherent module on (U, R, s, t, c). This construction defines a functor*

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$.

Proof. The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Schemes, Lemma 26.24.1. Hence $s_*t^*\mathcal{G}$ is quasi-coherent. With notation as in Lemma 39.13.4 we have

$$t^*s_*t^*\mathcal{G} = \mathrm{pr}_{1,*}\mathrm{pr}_0^*t^*\mathcal{G} = \mathrm{pr}_{1,*}c^*t^*\mathcal{G} = s^*s_*t^*\mathcal{G}$$

The middle equality because $t \circ c = t \circ \mathrm{pr}_0$ as morphisms $R \times_{s,U,t} R \to U$, and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Schemes, Lemma 30.5.2.

To verify the cocycle condition of Definition 39.14.1 for α and the adjointness property we describe the construction $\mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$ in another way. Consider the groupoid scheme $(R, R \times_{t,U,t} R, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ associated to the equivalence relation $R \times_{t,U,t} R$ on R, see Lemma 39.13.3. There is a morphism

$$f : (R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by $s : R \to U$ and $R \times_{t,U,t} R \to R$ given by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$; we omit the verification of the commutativity of the required diagrams. Since $t, s : R \to U$ are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{array}{ccc} R \times_{t,U,t} R & \xrightarrow[(r_0,r_1)\mapsto r_0^{-1}\circ r_1]{} & R \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow{\quad s \quad} & U \end{array}$$

by Lemma 39.13.5 it follows that Lemma 39.14.4 applies to f. Thus pushforward and pullback of quasi-coherent modules along f are adjoint functors. To finish the proof we will identify these functors with the functors described above. To do this, note that

$$t^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is an equivalence by the theory of descent of quasi-coherent sheaves as $\{t : R \to U\}$ is an fpqc covering, see Descent, Proposition 35.5.2.

Pushforward along f precomposed with the equivalence t^* sends $\mathcal{G}$ to $(s_*t^*\mathcal{G}, \alpha)$; we omit the verification that the isomorphism α obtained in this fashion is the same as the one constructed above.

Pullback along f postcomposed with the inverse of the equivalence t^* sends $(\mathcal{F}, \beta)$ to the descent relative to $\{t : R \to U\}$ of the module $s^*\mathcal{F}$ endowed with the descent datum γ on $R \times_{t,U,t} R$ which is the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. Consider the isomorphism $\beta : t^*\mathcal{F} \to s^*\mathcal{F}$. The canonical descent datum (Descent, Definition 35.2.3) on $t^*\mathcal{F}$ relative to $\{t : R \to U\}$ translates via β into the map

$$\mathrm{pr}_0^*s^*\mathcal{F} \xrightarrow{\mathrm{pr}_0^*\beta^{-1}} \mathrm{pr}_0^*t^*\mathcal{F} \xrightarrow{can} \mathrm{pr}_1^*t^*\mathcal{F} \xrightarrow{\mathrm{pr}_1^*\beta} \mathrm{pr}_1^*s^*\mathcal{F}$$

Since β satisfies the cocycle condition, this is equal to the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. To see this take the actual cocycle relation in Definition 39.14.1 and pull it back by the morphism $(\mathrm{pr}_0, c \circ (i, 1)) : R \times_{t,U,t} R \to R \times_{s,U,t} R$ which also plays a role in the commutative diagram of Lemma 39.13.5. It follows that $(s^*\mathcal{F}, \gamma)$ is isomorphic to $(t^*\mathcal{F}, can)$. All in all, we conclude that pullback by f postcomposed with the inverse of the equivalence t^* is isomorphic to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$. □

0GNF **Remark 39.15.2.** In the situation of Lemma 39.15.1 denote

$$F : QCoh(U, R, s, t, c) \to QCoh(\mathcal{O}_U), \quad (\mathcal{F}, \beta) \mapsto \mathcal{F}$$

the forgetful functor and denote

$$G : QCoh(\mathcal{O}_U) \to QCoh(U, R, s, t, c), \quad \mathcal{G} \mapsto (s_* t^* \mathcal{G}, \alpha)$$

the right adjoint constructed in the lemma. Then the unit $\eta : \mathrm{id} \to G \circ F$ of the adjunction evaluated on $(\mathcal{F}, \beta)$ is given by the map

$$\mathcal{F} \to s_* s^* \mathcal{F} \xrightarrow{\beta^{-1}} s_* t^* \mathcal{F}$$

We omit the verification.

07TT **Lemma 39.15.3.** *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module, and let $\varphi : \mathcal{G} \to f^*\mathcal{F}$ be a module map. Assume*

(1) *φ is injective,*
(2) *f is quasi-compact, quasi-separated, flat, and surjective,*
(3) *X, Y are locally Noetherian, and*
(4) *$\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.*

Then $\mathcal{F} \cap f_*\mathcal{G}$ defined as the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f_* f^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F} \cap f_*\mathcal{G} & \longrightarrow & f_*\mathcal{G} \end{array}$$

is a coherent $\mathcal{O}_X$-module.

Proof. We will freely use the characterization of coherent modules of Cohomology of Schemes, Lemma 30.9.1 as well as the fact that coherent modules form a Serre subcategory of $QCoh(\mathcal{O}_X)$, see Cohomology of Schemes, Lemma 30.9.3. If f has a section σ, then we see that $\mathcal{F} \cap f_*\mathcal{G}$ is contained in the image of $\sigma^*\mathcal{G} \to \sigma^* f^* \mathcal{F} = \mathcal{F}$, hence coherent. In general, to show that $\mathcal{F} \cap f_*\mathcal{G}$ is coherent, it suffices the show that $f^*(\mathcal{F} \cap f_*\mathcal{G})$ is coherent (see Descent, Lemma 35.7.1). Since f is flat this is equal to $f^*\mathcal{F} \cap f^* f_*\mathcal{G}$. Since f is flat, quasi-compact, and quasi-separated we see $f^* f_*\mathcal{G} = p_* q^* \mathcal{G}$ where $p, q : Y \times_X Y \to Y$ are the projections, see Cohomology of Schemes, Lemma 30.5.2. Since p has a section we win. □

Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Assume that U is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module.

07TU **Lemma 39.15.4.** *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that*

(1) *U, R are Noetherian,*

(2) s, t *are flat, quasi-compact, and quasi-separated.*

Then every quasi-coherent module $(\mathcal{F}, \beta)$ *on* (U, R, s, t, c) *is a filtered colimit of coherent modules.*

Proof. We will use the characterization of Cohomology of Schemes, Lemma 30.9.1 of coherent modules on locally Noetherian scheme without further mention. We can write $\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ as the filtered colimit of coherent submodules $\mathcal{H}_i \subset \mathcal{F}$, see Cohomology of Schemes, Lemma 30.10.4. Given a quasi-coherent sheaf $\mathcal{H}$ on U we denote $(s_*t^*\mathcal{H}, \alpha)$ the quasi-coherent sheaf on (U, R, s, t, c) of Lemma 39.15.1. Consider the adjunction map $(\mathcal{F}, \beta) \to (s_*t^*\mathcal{F}, \alpha)$ in $QCoh(U, R, s, t, c)$, see Remark 39.15.2. Set

$$(\mathcal{F}_i, \beta_i) = (\mathcal{F}, \beta) \times_{(s_*t^*\mathcal{F}, \alpha)} (s_*t^*\mathcal{H}_i, \alpha)$$

in $QCoh(U, R, s, t, c)$. Since restriction to U is an exact functor on $QCoh(U, R, s, t, c)$ by the proof of Lemma 39.14.6 we obtain a pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_*t^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_*t^*\mathcal{H}_i \end{array}$$

in other words $\mathcal{F}_i = \mathcal{F} \cap s_*t^*\mathcal{H}_i$. By the description of the adjunction map in Remark 39.15.2 this diagram is isomorphic to the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_*s^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_*t^*\mathcal{H}_i \end{array}$$

where the right vertical arrow is the result of applying s_* to the map

$$t^*\mathcal{H}_i \to t^*\mathcal{F} \xrightarrow{\beta} s^*\mathcal{F}$$

This arrow is injective as t is a flat morphism. It follows that $\mathcal{F}_i$ is coherent by Lemma 39.15.3. Finally, because s is quasi-compact and quasi-separated we see that s_* commutes with colimits (see Cohomology of Schemes, Lemma 30.6.1). Hence $s_*t^*\mathcal{F} = \operatorname{colim} s_*t^*\mathcal{H}_i$ and hence $(\mathcal{F}, \beta) = \operatorname{colim}(\mathcal{F}_i, \beta_i)$ as desired. □

Here is a curious lemma that is useful when working with groupoids on fields. In fact, this is the standard argument to prove that any representation of an algebraic group is a colimit of finite dimensional representations.

07TV **Lemma 39.15.5.** *Let* (U, R, s, t, c) *be a groupoid scheme over* S*. Assume that*

(1) U*,* R *are affine,*
(2) *there exist* $e_i \in \mathcal{O}_R(R)$ *such that every element* $g \in \mathcal{O}_R(R)$ *can be uniquely written as* $\sum s^*(f_i)e_i$ *for some* $f_i \in \mathcal{O}_U(U)$*.*

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ *on* (U, R, s, t, c) *is a filtered colimit of finite type quasi-coherent modules.*

Proof. The assumption means that $\mathcal{O}_R(R)$ is a free $\mathcal{O}_U(U)$-module via s with basis e_i. Hence for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$ we see that $s^*\mathcal{G}(R) = \bigoplus_i \mathcal{G}(U)e_i$. We will write $s(-)$ to indicate pullback of sections by s and similarly for other

morphisms. Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Let $\sigma \in \mathcal{F}(U)$. By the above we can write

$$\alpha(t(\sigma)) = \sum s(\sigma_i) e_i$$

for some unique $\sigma_i \in \mathcal{F}(U)$ (all but finitely many are zero of course). We can also write

$$c(e_i) = \sum \mathrm{pr}_1(f_{ij}) \mathrm{pr}_0(e_j)$$

as functions on $R \times_{s,U,t} R$. Then the commutativity of the diagram in Definition 39.14.1 means that

$$\sum \mathrm{pr}_1(\alpha(t(\sigma_i))) \mathrm{pr}_0(e_i) = \sum \mathrm{pr}_1(s(\sigma_i) f_{ij}) \mathrm{pr}_0(e_j)$$

(calculation omitted). Picking off the coefficients of $\mathrm{pr}_0(e_l)$ we see that $\alpha(t(\sigma_l)) = \sum s(\sigma_i) f_{il}$. Hence the submodule $\mathcal{G} \subset \mathcal{F}$ generated by the elements σ_i defines a finite type quasi-coherent module preserved by α. Hence it is a subobject of $\mathcal{F}$ in $QCoh(U, R, s, t, c)$. This submodule contains σ (as one sees by pulling back the first relation by e). Hence we win. □

We suggest the reader skip the rest of this section. Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Let κ be a cardinal. In the following we will say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is κ-generated if $\mathcal{F}$ is a κ-generated $\mathcal{O}_U$-module, see Properties, Definition 28.23.1.

077T **Lemma 39.15.6.** *Let (U, R, s, t, c) be a groupoid scheme over S. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) such that every κ-generated quasi-coherent module on (U, R, s, t, c) is isomorphic to one of the $(\mathcal{F}_t, \alpha_t)$.*

Proof. For each quasi-coherent module $\mathcal{F}$ on U there is a (possibly empty) set of maps $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ such that $(\mathcal{F}, \alpha)$ is a quasi-coherent modules on (U, R, s, t, c). By Properties, Lemma 28.23.2 there exists a set of isomorphism classes of κ-generated quasi-coherent $\mathcal{O}_U$-modules. □

077U **Lemma 39.15.7.** *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that s, t are flat. There exists a cardinal κ such that every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is the directed colimit of its κ-generated quasi-coherent submodules.*

Proof. In the statement of the lemma and in this proof a *submodule* of a quasi-coherent module $(\mathcal{F}, \alpha)$ is a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as subsheaves of $s^*\mathcal{F}$. This makes sense because since s, t are flat the pullbacks s^* and t^* are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 28.23.3. We urge the reader to read that proof first.

Choose an affine open covering $U = \bigcup_{i \in I} U_i$. For each pair i, j choose affine open coverings

$$U_i \cap U_j = \bigcup\nolimits_{k \in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup\nolimits_{k \in J_{ij}} W_{ijk}.$$

Write $U_i = \mathrm{Spec}(A_i)$, $U_{ijk} = \mathrm{Spec}(A_{ijk})$, $W_{ijk} = \mathrm{Spec}(B_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}, J_{ij}.

Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Set $M_i = \mathcal{F}(U_i)$, $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that α gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i,t} B_{ijk} \longrightarrow M_j \otimes_{A_j,s} B_{ijk}$$

see Schemes, Lemma 26.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ or $k \in J_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes b_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$ or $b_{m'} \in B_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ when $k \in I_{ij}$. Fix such a collection $S(i, j, k, m)$

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i,t} B_{ijk}) = N_j \otimes_{A_j,s} B_{ijk}$$

as submodules of M_{ijk} or $M_j \otimes_{A_j,s} B_{ijk}$. Thus there exists a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as submodules of $s^*\mathcal{F}$. In other words, $(\mathcal{G}, \alpha|_{t^*\mathcal{G}})$ is a submodule of $(\mathcal{F}, \alpha)$. Moreover, by construction $\mathcal{G}$ is κ-generated.

Let $\{(\mathcal{G}_t, \alpha_t)\}_{t \in T}$ be the set of κ-generated quasi-coherent submodules of $(\mathcal{F}, \alpha)$. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent submodule as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\operatorname{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

39.16. Groupoids and group schemes

03LK There are many ways to construct a groupoid out of an action a of a group G on a set V. We choose the one where we think of an element $g \in G$ as an arrow with source v and target $a(g, v)$. This leads to the following construction for group actions of schemes.

0234 **Lemma 39.16.1.** *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y with identity e_G and inverse i_G. Let X/Y be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X/Y. Then we get a groupoid scheme (U, R, s, t, c, e, i) over S in the following manner:*

(1) *We set $U = X$, and $R = G \times_Y X$.*

(2) *We set* $s : R \to U$ *equal to* $(g, x) \mapsto x$.
(3) *We set* $t : R \to U$ *equal to* $(g, x) \mapsto a(g, x)$.
(4) *We set* $c : R \times_{s,U,t} R \to R$ *equal to* $((g, x), (g', x')) \mapsto (m(g, g'), x')$.
(5) *We set* $e : U \to R$ *equal to* $x \mapsto (e_G(x), x)$.
(6) *We set* $i : R \to R$ *equal to* $(g, x) \mapsto (i_G(g), a(g, x))$.

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

03LL **Lemma 39.16.2.** *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y. Let X be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X over Y. Let (U, R, s, t, c) be the groupoid scheme constructed in Lemma 39.16.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_Y X \to X$, see Definitions 39.12.1 and 39.14.1. Using the translation in Lemma 39.16.1 the commutativity requirements of the two definitions match up exactly. □

39.17. The stabilizer group scheme

03LM Given a groupoid scheme we get a group scheme as follows.

0235 **Lemma 39.17.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The scheme G defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle j=(t,s)} \\ U & \xrightarrow{\Delta} & U \times_S U \end{array}$$

is a group scheme over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is an immersion we see that $G = j^{-1}(\Delta_{U/S})$ is a locally closed subscheme of R. Thinking of it in this way, the structure morphism $j^{-1}(\Delta_{U/S}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

0236 **Definition 39.17.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The group scheme $j^{-1}(\Delta_{U/S}) \to U$ is called the *stabilizer of the groupoid scheme* (U, R, s, t, c).

In the literature the stabilizer group scheme is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

0237 **Lemma 39.17.3.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S, and let G/U be its stabilizer. Denote R_t/U the scheme R seen as a scheme over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/S we define $a(g, r) = c(g, r)$. □

04Q2 **Lemma 39.17.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let G be the stabilizer group scheme of R. Let*

$$G_0 = G \times_{U, pr_0} (U \times_S U) = G \times_S U$$

as a group scheme over $U \times_S U$. The action of G on R of Lemma 39.17.3 induces an action of G_0 on R over $U \times_S U$ which turns R into a pseudo G_0-torsor over $U \times_S U$.

Proof. This is true because in a groupoid category $\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is a principal homogeneous set under the group $\mathrm{Mor}_{\mathcal{C}}(y, y)$. □

04Q3 **Lemma 39.17.5.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in U \times_S U$ be a point. Denote R_p the scheme theoretic fibre of $j = (t, s) : R \to U \times_S U$. If $R_p \neq \emptyset$, then the action*

$$G_{0,\kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

(see Lemma 39.17.4) which turns R_p into a $G_{\kappa(p)}$-torsor over $\kappa(p)$.

Proof. The action is a pseudo-torsor by the lemma cited in the statement. And if R_p is not the empty scheme, then $\{R_p \to p\}$ is an fpqc covering which trivializes the pseudo-torsor. □

39.18. Restricting groupoids

02VA Consider a (usual) groupoid $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we have a map of sets $g : \mathrm{Ob}' \to \mathrm{Ob}$. Then we can construct a groupoid $\mathcal{C}' = (\mathrm{Ob}', \mathrm{Arrows}', s', t', c')$ by thinking of a morphism between elements x', y' of Ob' as a morphism in $\mathcal{C}$ between $g(x'), g(y')$. In other words we set

$$\mathrm{Arrows}' = \mathrm{Ob}' \times_{g,\mathrm{Ob},t} \mathrm{Arrows} \times_{s,\mathrm{Ob},g} \mathrm{Ob}'.$$

with obvious choices for s', t', and c'. There is a canonical functor $\mathcal{C}' \to \mathcal{C}$ which is fully faithful, but not necessarily essentially surjective. This groupoid $\mathcal{C}'$ endowed with the functor $\mathcal{C}' \to \mathcal{C}$ is called the *restriction* of the groupoid $\mathcal{C}$ to Ob'.

02VB **Lemma 39.18.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Consider the following diagram*

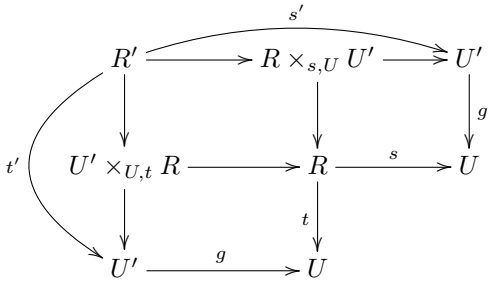

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s', U', t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid scheme over S and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to$

(U, R, s, t, c) of groupoid schemes over S. Moreover, for any scheme T over S the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see above) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. □

02VC **Definition 39.18.2.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. The morphism of groupoids $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 39.18.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

02VD **Lemma 39.18.3.** *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 39.18.2 and 39.3.3 agree via the constructions of Lemmas 39.13.2 and 39.13.3.*

Proof. What we are saying here is that R' of Lemma 39.18.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma. □

04ML **Lemma 39.18.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g. Let G be the stabilizer of (U, R, s, t, c) and let G' be the stabilizer of (U', R', s', t', c'). Then G' is the base change of G by g, i.e., there is a canonical identification $G' = U' \times_{g,U} G$.*

Proof. Omitted. □

39.19. Invariant subschemes

03LN In this section we discuss briefly the notion of an invariant subscheme.

03BC **Definition 39.19.1.** Let (U, R, s, t, c) be a groupoid scheme over the base scheme S.

(1) A subset $W \subset U$ is *set-theoretically R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) An open $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(3) A closed subscheme $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$. Here we use the scheme theoretic inverse image, see Schemes, Definition 26.17.7.
(4) A monomorphism of schemes $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as schemes over R.

For subsets and open subschemes $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$ as subsets of R. If $W \subset U$ is an R-equivariant open subscheme then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant closed subscheme, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

03LO **Lemma 39.19.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S.*

(1) *For any subset $W \subset U$ the subset $t(s^{-1}(W))$ is set-theoretically R-invariant.*

(2) *If s and t are open, then for every open $W \subset U$ the open $t(s^{-1}(W))$ is an R-invariant open subscheme.*
(3) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subschemes.*

Proof. Part (1) follows from Lemmas 39.3.4 and 39.13.2, namely, $t(s^{-1}(W))$ is the set of points of U equivalent to a point of W. Next, assume s and t open and $W \subset U$ open. Since s is open the set $W' = t(s^{-1}(W))$ is an open subset of U. Finally, assume that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = t(s^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over all affine opens of U we see (3). □

0APA **Lemma 39.19.3.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s and t quasi-compact and flat and U quasi-separated. Let $W \subset U$ be quasi-compact open. Then $t(s^{-1}(W))$ is an intersection of a nonempty family of quasi-compact open subsets of U.*

Proof. Note that $s^{-1}(W)$ is quasi-compact open in R. As a continuous map t maps the quasi-compact subset $s^{-1}(W)$ to a quasi-compact subset $t(s^{-1}(W))$. As t is flat and $s^{-1}(W)$ is closed under generalization, so is $t(s^{-1}(W))$, see (Morphisms, Lemma 29.25.9 and Topology, Lemma 5.19.6). Pick a quasi-compact open $W' \subset U$ containing $t(s^{-1}(W))$. By Properties, Lemma 28.2.4 we see that W' is a spectral space (here we use that U is quasi-separated). Then the lemma follows from Topology, Lemma 5.24.7 applied to $t(s^{-1}(W)) \subset W'$. □

0APB **Lemma 39.19.4.** *Assumptions and notation as in Lemma 39.19.3. There exists an R-invariant open $V \subset U$ and a quasi-compact open W' such that $W \subset V \subset W' \subset U$.*

Proof. Set $E = t(s^{-1}(W))$. Recall that E is set-theoretically R-invariant (Lemma 39.19.2). By Lemma 39.19.3 there exists a quasi-compact open W' containing E. Let $Z = U \setminus W'$ and consider $T = t(s^{-1}(Z))$. Observe that $Z \subset T$ and that $E \cap T = \emptyset$ because $s^{-1}(E) = t^{-1}(E)$ is disjoint from $s^{-1}(Z)$. Since T is the image of the closed subset $s^{-1}(Z) \subset R$ under the quasi-compact morphism $t : R \to U$ we see that any point ξ in the closure $\overline{T}$ is the specialization of a point of T, see Morphisms, Lemma 29.6.5 (and Morphisms, Lemma 29.6.3 to see that the scheme theoretic image is the closure of the image). Say $\xi' \rightsquigarrow \xi$ with $\xi' \in T$. Suppose that $r \in R$ and $s(r) = \xi$. Since s is flat we can find a specialization $r' \rightsquigarrow r$ in R such that $s(r') = \xi'$ (Morphisms, Lemma 29.25.9). Then $t(r') \rightsquigarrow t(r)$. We conclude that $t(r') \in T$ as T is set-theoretically invariant by Lemma 39.19.2. Thus $\overline{T}$ is a set-theoretically R-invariant closed subset and $V = U \setminus \overline{T}$ is the open we are looking for. It is contained in W' which finishes the proof. □

39.20. Quotient sheaves

02VE Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-relation over S. Say U, R, S are objects of a τ-site Sch_τ (see Topologies, Section 34.2). Then we can consider the functors

$$h_U, h_R : (Sch/S)_\tau^{opp} \longrightarrow Sets.$$

These are sheaves, see Descent, Lemma 35.13.7. The morphism j induces a map $j : h_R \to h_U \times h_U$. For each object $T \in \mathrm{Ob}((Sch/S)_\tau)$ we can take the equivalence relation $\sim_T$ generated by $j(T) : R(T) \to U(T) \times U(T)$ and consider the quotient. Hence we get a presheaf

02VF (39.20.0.1) $$(Sch/S)_\tau^{opp} \longrightarrow Sets, \quad T \longmapsto U(T)/\sim_T$$

02VG **Definition 39.20.1.** Let τ, S, and the pre-relation $j : R \to U \times_S U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (39.20.0.1) in the τ-topology. If $j : R \to U \times_S U$ comes from the action of a group scheme G/S on U as in Lemma 39.16.1 then we sometimes denote the quotient sheaf U/G.

This means exactly that the diagram

$$h_R \rightrightarrows h_U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_\tau$. Using the Yoneda embedding we may view $(Sch/S)_\tau$ as a full subcategory of sheaves on $(Sch/S)_\tau$ and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$R \overset{s}{\underset{t}{\rightrightarrows}} U \longrightarrow U/R$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

03BD **Definition 39.20.2.** In the situation of Definition 39.20.1. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable. We will say a groupoid (U, R, s, t, c) has a *representable quotient* if the quotient U/R with $j = (t, s)$ is representable.

The following lemma characterizes schemes M representing the quotient. It applies for example if $\tau = fppf$, $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

03C5 **Lemma 39.20.3.** *In the situation of Definition 39.20.1. Assume there is a scheme M, and a morphism $U \to M$ such that*

1. *the morphism $U \to M$ equalizes s, t,*
2. *the morphism $U \to M$ induces a surjection of sheaves $h_U \to h_M$ in the τ-topology, and*
3. *the induced map $(t, s) : R \to U \times_M U$ induces a surjection of sheaves $h_R \to h_{U \times_M U}$ in the τ-topology.*

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $h_U \to h_M$ factors through U/R. Condition (2) says that $U/R \to h_M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to h_M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

045Y **Lemma 39.20.4.** *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S. Assume U, R, S are objects of a τ-site Sch_τ. For $T \in \mathrm{Ob}((Sch/S)_\tau)$ and $a, b \in U(T)$ the following are equivalent:*

(1) *a and b map to the same element of $(U/R)(T)$, and*
(2) *there exists a τ-covering $\{f_i : T_i \to T\}$ of T and morphisms $r_i : T_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.*

In other words, in this case the map of τ-sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (39.20.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 39.3.1. □

045Z **Lemma 39.20.5.** *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S and $g : U' \to U$ a morphism of schemes over S. Let $j' : R' \to U' \times_S U'$ be the restriction of j to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If g defines a surjection $h_{U'} \to h_U$ of sheaves in the τ-topology (for example if $\{g : U' \to U\}$ is a τ-covering), then $U'/R' \to U/R$ is an isomorphism.

Proof. Suppose $\xi, \xi' \in (U'/R')(T)$ are sections which map to the same section of U/R. Then we can find a τ-covering $\mathcal{T} = \{T_i \to T\}$ of T such that $\xi|_{T_i}, \xi'|_{T_i}$ are given by $a_i, a_i' \in U'(T_i)$. By Lemma 39.20.4 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : T_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(T_i)$ and this shows that a_i and a_i' define the same section of U'/R' over T_i. By the sheaf condition this implies $\xi = \xi'$.

If $h_{U'} \to h_U$ is a surjection of sheaves, then of course $U'/R' \to U/R$ is surjective also. If $\{g : U' \to U\}$ is a τ-covering, then the map of sheaves $h_{U'} \to h_U$ is surjective, see Sites, Lemma 7.12.4. Hence $U'/R' \to U/R$ is surjective also in this case. □

02VH **Lemma 39.20.6.** *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ a morphism of schemes over S. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g,U,t} R \xrightarrow{pr_1} R \xrightarrow{s} U, \quad U' \times_{g,U,t} R \xrightarrow{h} U$$

defines a surjection of sheaves in the τ-topology then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is a τ-covering, or if $U' \to U$ defines

a surjection of sheaves in the τ-topology, or if $\{g : U' \to U\}$ is a covering in the τ-topology.

Proof. Injectivity follows on combining Lemmas 39.13.2 and 39.20.5. To see surjectivity (see Sites, Section 7.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ is the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find a τ-covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f'_i = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f'_i \in U'(T_i)$ maps to $g \circ f'_i \in U(T_i)$ and that $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (39.20.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is a τ-covering, then it induces a surjection of sheaves, see Sites, Lemma 7.12.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). □

07S3 **Lemma 39.20.7.** *Let S be a scheme. Let $f : (U, R, j) \to (U', R', j')$ be a morphism between equivalence relations over S. Assume that*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian. For any $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$ the diagram

$$\begin{array}{ccc} U & \longrightarrow & U/R \\ \downarrow & & \downarrow{\scriptstyle f} \\ U' & \longrightarrow & U'/R' \end{array}$$

is a fibre product square of τ-sheaves.

Proof. By Lemma 39.20.4 the quotient sheaves have a simple description which we will use below without further mention. We first show that

$$U \longrightarrow U' \times_{U'/R'} U/R$$

is injective. Namely, assume $a, b \in U(T)$ map to the same element on the right hand side. Then $f(a) = f(b)$. After replacing T by the members of a τ-covering we may assume that there exists an $r \in R(T)$ such that $a = s(r)$ and $b = t(r)$. Then $r' = f(r)$ is a T-valued point of R' with $s'(r') = t'(r')$. Hence $r' = e'(f(a))$ (where e' is the identity of the groupoid scheme associated to j', see Lemma 39.13.3). Because the first diagram of the lemma is cartesian this implies that r has to equal $e(a)$. Thus $a = b$.

Finally, we show that the displayed arrow is surjective. Let T be a scheme over S and let $(a', \bar{b})$ be a section of the sheaf $U' \times_{U'/R'} U/R$ over T. After replacing T by the members of a τ-covering we may assume that $\bar{b}$ is the class of an element $b \in U(T)$. After replacing T by the members of a τ-covering we may assume that there exists an $r' \in R'(T)$ such that $a' = t(r')$ and $s'(r') = f(b)$. Because the

first diagram of the lemma is cartesian we can find $r \in R(T)$ such that $s(r) = b$ and $f(r) = r'$. Then it is clear that $a = t(r) \in U(T)$ is a section which maps to $(a', \overline{b})$. □

39.21. Descent in terms of groupoids

0APC Cartesian morphisms are defined as follows.

0APD **Definition 39.21.1.** Let S be a scheme. Let $f : (U', R', s', t', c') \to (U, R, s, t, c)$ be a morphism of groupoid schemes over S. We say f is *cartesian*, or that (U', R', s', t', c') *is cartesian over* (U, R, s, t, c), if the diagram

$$\begin{array}{ccc} R' & \xrightarrow{f} & R \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ U' & \xrightarrow{f} & U \end{array}$$

is a fibre square in the category of schemes. A *morphism of groupoid schemes cartesian over* (U, R, s, t, c) is a morphism of groupoid schemes compatible with the structure morphisms towards (U, R, s, t, c).

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian groupoid schemes over a fixed groupoid scheme.

0APE **Lemma 39.21.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of groupoid schemes cartesian over (U, R, s, t, c) is equivalent to the category of pairs (V, φ) where V is a scheme over U and*

$$\varphi : V \times_{U,t} R \longrightarrow R \times_{s,U} V$$

is an isomorphism over R such that $e^*\varphi = id_V$ and such that

$$c^*\varphi = pr_1^*\varphi \circ pr_0^*\varphi$$

as morphisms of schemes over $R \times_{s,U,t} R$.

Proof. The pullback notation in the lemma signifies base change. The displayed formula makes sense because

$$(R \times_{s,U,t} R) \times_{\mathrm{pr}_1,R,\mathrm{pr}_1} (V \times_{U,t} R) = (R \times_{s,U,t} R) \times_{\mathrm{pr}_0,R,\mathrm{pr}_0} (R \times_{s,U} V)$$

as schemes over $R \times_{s,U,t} R$.

Given (V, φ) we set $U' = V$ and $R' = V \times_{U,t} R$. We set $t' : R' \to U'$ equal to the projection $V \times_{U,t} R \to V$. We set s' equal to φ followed by the projection $R \times_{s,U} V \to V$. We set c' equal to the composition

$$\begin{aligned} R' \times_{s',U',t'} R' &\xrightarrow{\varphi,1} (R \times_{s,U} V) \times_V (V \times_{U,t} R) \\ &\to R \times_{s,U} V \times_{U,t} R \\ &\xrightarrow{\varphi^{-1},1} V \times_{U,t} (R \times_{s,U,t} R) \\ &\xrightarrow{1,c} V \times_{U,t} R = R' \end{aligned}$$

A computation, which we omit shows that we obtain a groupoid scheme over (U, R, s, t, c). It is clear that this groupoid scheme is cartesian over (U, R, s, t, c).

Conversely, given $f : (U', R', s', t', c') \to (U, R, s, t, c)$ cartesian then the morphisms

$$U' \times_{U,t} R \xleftarrow{t', f} R' \xrightarrow{f, s'} R \times_{s,U} U'$$

are isomorphisms and we can set $V = U'$ and φ equal to the composition $(f, s') \circ (t', f)^{-1}$. We omit the proof that φ satisfies the conditions in the lemma. We omit the proof that these constructions are mutually inverse. □

Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Then we obtain a groupoid scheme $(X, X \times_Y X, \mathrm{pr}_1, \mathrm{pr}_0, c)$ over S. Namely, $j : X \times_Y X \to X \times_S X$ is an equivalence relation and we can take the associated groupoid, see Lemma 39.13.3.

0APF **Lemma 39.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. The construction of Lemma 39.21.2 determines an equivalence*

$$\begin{array}{c} \textit{category of groupoid schemes} \\ \textit{cartesian over } (X, X \times_Y X, \ldots) \end{array} \longrightarrow \begin{array}{c} \textit{category of descent data} \\ \textit{relative to } X/Y \end{array}$$

Proof. This is clear from Lemma 39.21.2 and the definition of descent data on schemes in Descent, Definition 35.34.1. □

39.22. Separation conditions

02YG This really means conditions on the morphism $j : R \to U \times_S U$ when given a groupoid (U, R, s, t, c) over S. As in the previous section we first formulate the corresponding diagram.

02YH **Lemma 39.22.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow[f \mapsto (f, s(f))]{} & R \times_{s,U} U & \longrightarrow & U \\ \downarrow{\scriptstyle \Delta_{R/U \times_S U}} & & \downarrow & & \downarrow \\ R \times_{(U \times_S U)} R & \xrightarrow{(f,g) \mapsto (f, f^{-1} \circ g)} & R \times_{s,U} G & \longrightarrow & G \end{array}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

02YI **Lemma 39.22.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is separated,*
   (b) *$G \to U$ is separated, and*
   (c) *$e : U \to G$ is a closed immersion.*
(2) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is quasi-separated,*
   (b) *$G \to U$ is quasi-separated, and*
   (c) *$e : U \to G$ is quasi-compact.*

Proof. The group scheme $G \to U$ is the base change of $R \to U \times_S U$ by the diagonal morphism $U \to U \times_S U$, see Lemma 39.17.1. Hence if j is separated (resp. quasi-separated), then $G \to U$ is separated (resp. quasi-separated). (See Schemes, Lemma 26.21.12). Thus (a) $\Rightarrow$ (b) in both (1) and (2).

If $G \to U$ is separated (resp. quasi-separated), then the morphism $U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. quasi-compact), see Schemes, Lemma 26.21.11. Thus (b) $\Rightarrow$ (a) in both (1) and (2).

By the result of Lemma 39.22.1 (and Schemes, Lemmas 26.18.2 and 26.19.3) we see that if e is a closed immersion (resp. quasi-compact) $\Delta_{R/U \times_S U}$ is a closed immersion (resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in both (1) and (2). □

39.23. Finite flat groupoids, affine case

03BE Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine. In this case we get two ring maps $s^\sharp, t^\sharp : A \longrightarrow B$. Let C be the equalizer of $s^\sharp$ and $t^\sharp$. In a formula

03BF (39.23.0.1) $$C = \{a \in A \mid t^\sharp(a) = s^\sharp(a)\}.$$

We will sometimes call this the *ring of R-invariant functions on U*. What properties does $M = \mathrm{Spec}(C)$ have? The first observation is that the diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

is commutative, i.e., the morphism $U \to M$ equalizes s, t. Moreover, if T is any affine scheme, and if $U \to T$ is a morphism which equalizes s, t, then $U \to T$ factors through $U \to M$. In other words, $U \to M$ is a coequalizer in the category of affine schemes.

We would like to find conditions that guarantee the morphism $U \to M$ is really a "quotient" in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where s, t are finite locally free.

03BG **Example 39.23.1.** Let k be a field. Let $U = \mathrm{GL}_{2,k}$. Let $B \subset \mathrm{GL}_2$ be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf $\mathrm{GL}_{2,k}/B$ (in the Zariski, étale or fppf topology, see Definition 39.20.1) is representable by the projective line: $\mathbf{P}^1 = \mathrm{GL}_{2,k}/B$. (Details omitted.) On the other hand, the ring of invariant functions in this case is just k. Note that in this case the morphisms $s, t : R = \mathrm{GL}_{2,k} \times_k B \to \mathrm{GL}_{2,k} = U$ are smooth of relative dimension 3.

Recall that in Exercises, Exercises 111.22.6 and 111.22.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If $\varphi : A \to B$ is a finite locally free ring map, then we will denote $\mathrm{Norm}_\varphi(b) \in A$ the norm of $b \in B$. In the case of a finite locally free morphism of schemes, the norm was constructed in Divisors, Lemma 31.17.6.

03BH **Lemma 39.23.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let C be as in (39.23.0.1). Let $f \in A$. Then $\mathrm{Norm}_{s^\sharp}(t^\sharp(f)) \in C$.*

Proof. Consider the commutative diagram

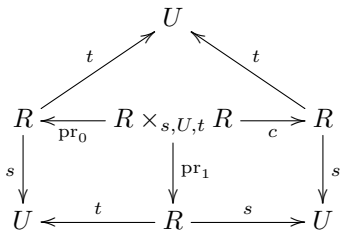

of Lemma 39.13.4. Think of $f \in \Gamma(U, \mathcal{O}_U)$. The commutativity of the top part of the diagram shows that $\mathrm{pr}_0^\sharp(t^\sharp(f)) = c^\sharp(t^\sharp(f))$ as elements of $\Gamma(R \times_{S,U,t} R, \mathcal{O})$. Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1^\sharp}(c^\sharp(t^\sharp(f)))$. Similarly we get $t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1^\sharp}(\mathrm{pr}_0^\sharp(t^\sharp(f)))$. Hence by the first equality of this proof we see that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f)))$ as desired. □

03BI **Lemma 39.23.3.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $s, t : R \to U$ finite locally free. Then*

$$U = \coprod\nolimits_{r \geq 1} U_r$$

is a disjoint union of R-invariant opens such that the restriction R_r of R to U_r has the property that $s, t : R_r \to U_r$ are finite locally free of rank r.

Proof. By Morphisms, Lemma 29.48.5 there exists a decomposition $U = \coprod_{r \geq 0} U_r$ such that $s : s^{-1}(U_r) \to U_r$ is finite locally free of rank r. As s is surjective we see that $U_0 = \emptyset$. Note that $u \in U_r \Leftrightarrow$ if and only if the scheme theoretic fibre $s^{-1}(u)$ has degree r over $\kappa(u)$. Now, if $z \in R$ with $s(z) = u$ and $t(z) = u'$ then using notation as in Lemma 39.13.4

$$\mathrm{pr}_1^{-1}(z) \to \mathrm{Spec}(\kappa(z))$$

is the base change of both $s^{-1}(u) \to \mathrm{Spec}(\kappa(u))$ and $s^{-1}(u') \to \mathrm{Spec}(\kappa(u'))$ by the lemma cited. Hence $u \in U_r \Leftrightarrow u' \in U_r$, in other words, the open subsets U_r are R-invariant. In particular the restriction of R to U_r is just $s^{-1}(U_r)$ and $s : R_r \to U_r$ is finite locally free of rank r. As $t : R_r \to U_r$ is isomorphic to s by the inverse of R_r we see that it has also rank r. □

03BJ **Lemma 39.23.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Then A is integral over C.*

Proof. First, by Lemma 39.23.3 we know that (U, R, s, t, c) is a disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that each $s, t : R_r \to U_r$ has constant rank r. As U is quasi-compact, we have $U_r = \emptyset$ for almost all r. It suffices to prove the lemma for each (U_r, R_r, s, t, c) and hence we may assume that s, t are finite locally free of rank r.

Assume that s, t are finite locally free of rank r. Let $f \in A$. Consider the element $x - f \in A[x]$, where we think of x as the coordinate on $\mathbf{A}^1$. Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \mathrm{id}_{\mathbf{A}^1}, t \times \mathrm{id}_{\mathbf{A}^1}, c \times \mathrm{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 39.23.2 to it and we see that $P(x) = \mathrm{Norm}_{s^\sharp}(t^\sharp(x - f))$ is an element of $C[x]$. Because $s^\sharp : A \to B$ is finite locally free of rank r we see that P is monic of degree r. Moreover $P(f) = 0$ by Cayley-Hamilton (Algebra, Lemma 10.16.1). More precisely, Cayley-Hamilton says the polynomial $s^\sharp(P) \in B[x]$ annihilates the element $t^\sharp(f)$. Since the coefficients of P are in C we get that $t^\sharp(P(f)) = 0$ in B. Since t is faithfully flat, we get $P(f) = 0$. □

03BK **Lemma 39.23.5.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Let $C \to C'$ be a ring map, and set $U' = \mathrm{Spec}(A \otimes_C C')$, $R' = \mathrm{Spec}(B \otimes_C C')$. Then*

(1) *The maps s, t, c induce maps s', t', c' such that (U', R', s', t', c') is a groupoid scheme. Let $C^1 \subset A'$ be the R'-invariant functions on U'.*
(2) *The canonical map $\varphi : C' \to C^1$ satisfies*
   (a) *for every $f \in C^1$ there exists an $n > 0$ and a polynomial $P \in C'[x]$ whose image in $C^1[x]$ is $(x - f)^n$, and*
   (b) *for every $f \in \mathrm{Ker}(\varphi)$ there exists an $n > 0$ such that $f^n = 0$.*
(3) *If $C \to C'$ is flat then φ is an isomorphism.*

Proof. The proof of part (1) is omitted. Let us denote $A' = A \otimes_C C'$ and $B' = B \otimes_C C'$. Then we have

$$C^1 = \{a \in A' \mid (t')^\sharp(a) = (s')^\sharp(a)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(a) = s^\sharp \otimes 1(a)\}.$$

In other words, C^1 is the kernel of the difference map $(t^\sharp - s^\sharp) \otimes 1$ which is just the base change of the C-linear map $t^\sharp - s^\sharp : A \to B$ by $C \to C'$. Hence (3) follows.

Proof of part (2)(b). Since $C \to A$ is integral (Lemma 39.23.4) and injective we see that $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$ is surjective, see Algebra, Lemma 10.36.17. Thus also $\mathrm{Spec}(A') \to \mathrm{Spec}(C')$ is surjective as a base change of a surjective morphism (Morphisms, Lemma 29.9.4). Hence $\mathrm{Spec}(C^1) \to \mathrm{Spec}(C')$ is surjective also. This implies (2)(b) holds for example by Algebra, Lemma 10.30.6.

Proof of part (2)(a). By Lemma 39.23.3 our groupoid scheme (U, R, s, t, c) decomposes as a finite disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that $s, t : R_r \to U_r$ are finite locally free of rank r. Pulling back by $U' = \mathrm{Spec}(C') \to U$ we obtain a similar decomposition of U' and $U^1 = \mathrm{Spec}(C^1)$. We will show in the next paragraph that (2)(a) holds for the corresponding system of rings $A_r, B_r, C_r, C'_r, C^1_r$ with $n = r$. Then given $f \in C^1$ let $P_r \in C_r[x]$ be the polynomial whose image in $C^1_r[x]$ is the image of $(x - f)^r$. Choosing a sufficiently divisible integer n we see that there is a polynomial $P \in C'[x]$ whose image in $C^1[x]$ is $(x-f)^n$; namely, we take P to be the unique element of $C'[x]$ whose image in $C'_r[x]$ is $P_r^{n/r}$.

In this paragraph we prove (2)(a) in case the ring maps $s^\sharp, t^\sharp : A \to B$ are finite locally free of a fixed rank r. Let $f \in C^1 \subset A' = A \otimes_C C'$. Choose a flat C-algebra D and a surjection $D \to C'$. Choose a lift $g \in A \otimes_C D$ of f. Consider the polynomial

$$P = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(x - g))$$

in $(A \otimes_C D)[x]$. By Lemma 39.23.2 and part (3) of the current lemma the coefficients of P are in D (compare with the proof of Lemma 39.23.4). On the other hand, the

image of P in $(A \otimes_C C')[x]$ is $(x-f)^r$ because $t^\sharp \otimes 1(x-f) = s^\sharp(x-f)$ and $s^\sharp$ is finite locally free of rank r. This proves what we want with P as in the statement (2)(a) given by the image of our P under the map $D[x] \to C'[x]$. □

03BL **Lemma 39.23.6.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$ and $R = \mathrm{Spec}(B)$ are affine and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (39.23.0.1). Then $U \to M = \mathrm{Spec}(C)$ has the following properties:*

(1) *the map on points $|U| \to |M|$ is surjective and $u_0, u_1 \in |U|$ map to the same point if and only if there exists a $r \in |R|$ with $t(r) = u_0$ and $s(r) = u_1$, in a formula*
$$|M| = |U|/|R|$$
(2) *for any algebraically closed field k we have*
$$M(k) = U(k)/R(k)$$

Proof. Since $C \to A$ is integral (Lemma 39.23.4) and injective we see that $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$ is surjective, see Algebra, Lemma 10.36.17. Thus $|U| \to |M|$ is surjective.

Let k be an algebraically closed field and let $C \to k$ be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 29.9.4) we see that $A \otimes_C k$ is not zero. Now $k \subset A \otimes_C k$ is a nonzero integral extension. Hence any residue field of $A \otimes_C k$ is an algebraic extension of k, hence equal to k. Thus we see that $U(k) \to M(k)$ is surjective.

Let $a_0, a_1 : A \to k$ be two ring maps. If there exists a ring map $b : B \to k$ such that $a_0 = b \circ t^\sharp$ and $a_1 = b \circ s^\sharp$ then we see that $a_0|_C = a_1|_C$ by definition. Thus the map $U(k) \to M(k)$ equalizes the two maps $R(k) \to U(k)$. Conversely, suppose that $a_0|_C = a_1|_C$. Let us name this algebra map $c : C \to k$. Consider the diagram

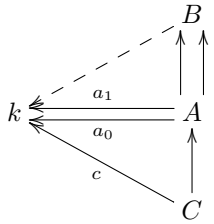

If we can construct a dotted arrow making the diagram commute, then the proof of part (2) of the lemma is complete. Since $s : A \to B$ is finite there exist finitely many ring maps $b_1, \ldots, b_n : B \to k$ such that $b_i \circ s^\sharp = a_1$. If the dotted arrow does not exist, then we see that none of the $a_i' = b_i \circ t^\sharp$, $i = 1, \ldots, n$ is equal to a_0. Hence the maximal ideals
$$\mathfrak{m}_i' = \mathrm{Ker}(a_i' \otimes 1 : A \otimes_C k \to k)$$
of $A \otimes_C k$ are distinct from $\mathfrak{m} = \mathrm{Ker}(a_0 \otimes 1 : A \otimes_C k \to k)$. By Algebra, Lemma 10.15.2 we would get an element $f \in A \otimes_C k$ with $f \in \mathfrak{m}$, but $f \notin \mathfrak{m}_i'$ for $i = 1, \ldots, n$. Consider the norm
$$g = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$
By Lemma 39.23.2 this lies in the invariants $C^1 \subset A \otimes_C k$ of the base change groupoid (base change via the map $c : C \to k$). On the one hand, $a_1(g) \in k^*$ since the value of $t^\sharp(f)$ at all the points (which correspond to $b_1, \ldots, b_n$) lying over a_1 is invertible (insert future reference on property determinant here). On the other

hand, since $f \in \mathfrak{m}$, we see that f is not a unit, hence $t^\sharp(f)$ is not a unit (as $t^\sharp \otimes 1$ is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that C^1 contains an element which is not nilpotent and not a unit. We will now show that this leads to a contradiction. Namely, apply Lemma 39.23.5 to the map $c : C \to C' = k$, then we see that the map of k into the invariants C^1 is injective and moreover, that for any element $x \in C^1$ there exists an integer $n > 0$ such that $x^n \in k$. Hence every element of C^1 is either a unit or nilpotent.

We still have to finish the proof of (1). We already know that $|U| \to |M|$ is surjective. It is clear that $|U| \to |M|$ is $|R|$-invariant. Finally, suppose $u_0, u_1 \in U$ maps to the same point $m \in M$. Then the induced field extensions $\kappa(u_0)/\kappa(m)$ and $\kappa(u_1)/\kappa(m)$ are algebraic (as A is integral over C as used above). Hence if k is an algebraic closure of $\kappa(m)$ then we can find $\kappa(m)$-embeddings $\overline{u}_0 : \kappa(u_0) \to k$ and $\overline{u}_1 : \kappa(u_1) \to k$. These determine k-valued points $\overline{u}_0, \overline{u}_1 \in U(k)$ mapping to the same point of $M(k)$. By part (2) we see that there exists a point $\overline{r} \in R(k)$ with $s(\overline{r}) = \overline{u}_0$ and $t(\overline{r}) = \overline{u}_1$. The image $r \in R$ of $\overline{r}$ is a point with $s(r) = u_0$ and $t(r) = u_1$ as desired. □

0DT9 **Lemma 39.23.7.** *Let S be a scheme. Let $f : (U', R', s', t') \to (U, R, s, t, c)$ be a morphism of groupoid schemes over S.*

(1) *U, R, U', R' are affine,*
(2) *s, t, s', t' are finite locally free,*
(3) *the diagrams*

$$\begin{array}{ccc} R' & \xrightarrow{f} & R \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ U' & \xrightarrow{f} & U \end{array} \qquad \begin{array}{ccc} R' & \xrightarrow{f} & R \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ U' & \xrightarrow{f} & U \end{array} \qquad \begin{array}{ccc} G' & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ U' & \xrightarrow{f} & U \end{array}$$

are cartesian where G and G' are the stabilizer group schemes, and
(4) *$f : U' \to U$ is étale.*

Then the map $C \to C'$ from the R-invariant functions on U to the R'-invariant functions on U' is étale and $U' = \mathrm{Spec}(C') \times_{\mathrm{Spec}(C)} U$.

Proof. Set $M = \mathrm{Spec}(C)$ and $M' = \mathrm{Spec}(C')$. Write $U = \mathrm{Spec}(A)$, $U' = \mathrm{Spec}(A')$, $R = \mathrm{Spec}(B)$, and $R' = \mathrm{Spec}(B')$. We will use the results of Lemmas 39.23.4, 39.23.5, and 39.23.6 without further mention.

Assume C is a strictly henselian local ring. Let $p \in M$ be the closed point and let $p' \in M'$ map to p. Claim: in this case there is a disjoint union decomposition $(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$ over (U, R, s, t, c) such that for the corresponding disjoint union decomposition $M' = M \amalg M''$ over M the point p' corresponds to $p \in M$.

The claim implies the lemma. Suppose that $M_1 \to M$ is a flat morphism of affine schemes. Then we can base change everything to M_1 without affecting the hypotheses (1) – (4). From Lemma 39.23.5 we see M_1, resp. M_1' is the spectrum of the R_1-invariant functions on U_1, resp. the R_1'-invariant functions on U_1'. Suppose that $p' \in M'$ maps to $p \in M$. Let M_1 be the spectrum of the strict henselization of $\mathcal{O}_{M,p}$ with closed point $p_1 \in M_1$. Choose a point $p_1' \in M_1'$ mapping to p_1 and p'.

From the claim we get

$$(U'_1, R'_1, s'_1, t'_1, c'_1) = (U_1, R_1, s_1, t_1, c_1) \amalg (U''_1, R''_1, s''_1, t''_1, c''_1)$$

and correspondingly $M'_1 = M_1 \amalg M''_1$ as a scheme over M_1. Write $M_1 = \mathrm{Spec}(C_1)$ and write $C_1 = \mathrm{colim}\, C_i$ as a filtered colimit of étale C-algebras. Set $M_i = \mathrm{Spec}(C_i)$. The $M_1 = \lim M_i$ and similarly for the other schemes. By Limits, Lemmas 32.4.11 and 32.8.11 we can find an i such that

$$(U'_i, R'_i, s'_i, t'_i, c'_i) = (U_i, R_i, s_i, t_i, c_i) \amalg (U''_i, R''_i, s''_i, t''_i, c''_i)$$

We conclude that $M'_i = M_i \amalg M''_i$. In particular $M' \to M$ becomes étale at a point over p' after an étale base change. This implies that $M' \to M$ is étale at p' (for example by Morphisms, Lemma 29.36.17). We will prove $U' \cong M' \times_M U$ after we prove the claim.

Proof of the claim. Observe that U_p and $U'_{p'}$ have finitely many points. For $u \in U_p$ we have $\kappa(u)/\kappa(p)$ is algebraic, hence $\kappa(u)$ is separably closed. As $U' \to U$ is étale, we conclude the morphism $U'_{p'} \to U_p$ induces isomorphisms on residue field extensions. Let $u' \in U'_{p'}$ with image $u \in U_p$. By assumption (3) the morphism of scheme theoretic fibres $(s')^{-1}(u') \to s^{-1}(u)$, $(t')^{-1}(u') \to t^{-1}(u)$, and $G'_{u'} \to G_u$ are isomorphisms. Observing that $U_p = t(s^{-1}(u))$ (set theoretically) we conclude that the points of $U'_{p'}$ surject onto the points of U_p. Suppose that u'_1 and u'_2 are points of $U'_{p'}$ mapping to the same point u of U_p. Then there exists a point $r' \in R'_{p'}$ with $s'(r') = u'_1$ and $t'(r') = u'_2$. Consider the two towers of fields

$$\kappa(r')/\kappa(u'_1)/\kappa(u)/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(u)/\kappa(p)$$

whose "ends" are the same as the two "ends" of the two towers

$$\kappa(r')/\kappa(u'_1)/\kappa(p')/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(p')/\kappa(p)$$

These two induce the same maps $\kappa(p') \to \kappa(r')$ as $(U'_{p'}, R'_{p'}, s', t', c')$ is a groupoid over p'. Since $\kappa(u)/\kappa(p)$ is purely inseparable, we conclude that the two induced maps $\kappa(u) \to \kappa(r')$ are the same. Therefore r' maps to a point of the fibre G_u. By assumption (3) we conclude that $r' \in (G')_{u'_1}$. Namely, we may think of G as a closed subscheme of R viewed as a scheme over U via s and use that the base change to U' gives $G' \subset R'$. In particular we have $u'_1 = u'_2$. We conclude that $U'_{p'} \to U_p$ is a bijective map on points inducing isomorphisms on residue fields. It follows that $U'_{p'}$ is a finite set of closed points (Algebra, Lemma 10.35.9) and hence $U'_{p'}$ is closed in U'. Let $J' \subset A'$ be the radical ideal cutting out $U'_{p'}$ set theoretically.

Second part proof of the claim. Let $\mathfrak{m} \subset C$ be the maximal ideal. Observe that $(A, \mathfrak{m}A)$ is a henselian pair by More on Algebra, Lemma 15.11.8. Let $J = \sqrt{\mathfrak{m}A}$. Then (A, J) is a henselian pair (More on Algebra, Lemma 15.11.7) and the étale ring map $A \to A'$ induces an isomorphism $A/J \to A'/J'$ by our deliberations above. We conclude that $A' = A \times A''$ by More on Algebra, Lemma 15.11.6. Consider the corresponding disjoint union decomposition $U' = U \amalg U''$. The open $(s')^{-1}(U)$ is the set of points of R' specializing to a point of $R'_{p'}$. Similarly for $(t')^{-1}(U)$. Similarly we have $(s')^{-1}(U'') = (t')^{-1}(U'')$ as this is the set of points which do not specialize to $R'_{p'}$. Hence we obtain a disjoint union decomposition

$$(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$$

This immediately gives $M' = M \amalg M''$ and the proof of the claim is complete.

We still have to prove that the canonical map $U' \to M' \times_M U$ is an isomorphism. It is an étale morphism (Morphisms, Lemma 29.36.18). On the other hand, by base changing to strictly henselian local rings (as in the third paragraph of the proof) and using the bijectivity $U'_{p'} \to U_p$ established in the course of the proof of the claim, we see that $U' \to M' \times_M U$ is universally bijective (some details omitted). However, a universally bijective étale morphism is an isomorphism (Descent, Lemma 35.25.2) and the proof is complete. □

03C8 **Lemma 39.23.8.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and*
(2) *there exist elements $x_i \in A$, $i \in I$ such that $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$.*

Then $A = \bigoplus_{i \in I} Cx_i$, and $B \cong A \otimes_C A$ where $C \subset A$ is the R-invariant functions on U as in (39.23.0.1).

Proof. During this proof we will write $s, t : A \to B$ instead of $s^\sharp, t^\sharp$, and similarly $c : B \to B \otimes_{s,A,t} B$. We write $p_0 : B \to B \otimes_{s,A,t} B$, $b \mapsto b \otimes 1$ and $p_1 : B \to B \otimes_{s,A,t} B$, $b \mapsto 1 \otimes b$. By Lemma 39.13.5 and the definition of C we have the following commutative diagram

$$\begin{array}{ccccc} B \otimes_{s,A,t} B & \overset{c}{\underset{p_0}{\leftleftarrows}} & B & \xleftarrow{t} & A \\ {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle s} & & \uparrow \\ B & \overset{s}{\underset{t}{\leftleftarrows}} & A & \longleftarrow & C \end{array}$$

Moreover the tow left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$B \otimes_{t,A,t} B \overset{p_1}{\underset{p_0}{\leftleftarrows}} B \xleftarrow{t} A$$

which is an equalizer diagram according to Descent, Lemma 35.3.6 because condition (2) implies in particular that s (and hence also then isomorphic arrow t) is faithfully flat. The lower row is an equalizer diagram by definition of C. We can use the x_i and get a commutative diagram

$$\begin{array}{ccccc} B \otimes_{s,A,t} B & \overset{c}{\underset{p_0}{\leftleftarrows}} & B & \xleftarrow{t} & A \\ {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle s} & & \uparrow \\ \bigoplus_{i \in I} Bx_i & \overset{s}{\underset{t}{\leftleftarrows}} & \bigoplus_{i \in I} Ax_i & \longleftarrow & \bigoplus_{i \in I} Cx_i \end{array}$$

where in the right vertical arrow we map x_i to x_i, in the middle vertical arrow we map x_i to $t(x_i)$ and in the left vertical arrow we map x_i to $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$ (equality by the commutativity of the top part of the diagram in Lemma 39.13.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism. □

03BM **Proposition 39.23.9.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine,*
(2) *$s, t : R \to U$ finite locally free, and*
(3) *$j = (t, s)$ is an equivalence relation.*

In this case, let $C \subset A$ be as in (39.23.0.1). Then $U \to M = \mathrm{Spec}(C)$ is finite locally free and $R = U \times_M U$. Moreover, M represents the quotient sheaf U/R in the fppf topology (see Definition 39.20.1).

Proof. During this proof we use the notation $s, t : A \to B$ instead of the notation $s^\sharp, t^\sharp$. By Lemma 39.20.3 it suffices to show that $C \to A$ is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that j is a monomorphism, and also finite (since already s and t are finite). Hence we see that j is a closed immersion by Morphisms, Lemma 29.44.15. Hence $A \otimes_C A \to B$ is surjective.

We will perform base change by flat ring maps $C \to C'$ as in Lemma 39.23.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime $\mathfrak{p} \subset C$, there exists a local flat ring map $C_\mathfrak{p} \to C'_\mathfrak{p}$ such that the result holds after a base change to $C'_\mathfrak{p}$. This implies immediately that $A \otimes_C A \to B$ is injective (use Algebra, Lemma 10.23.1). It also implies that $C \to A$ is flat, by combining Algebra, Lemmas 10.39.17, 10.39.18, and 10.39.8. Then since $U \to \mathrm{Spec}(C)$ is surjective also (Lemma 39.23.6) we conclude that $C \to A$ is faithfully flat. Then the isomorphism $B \cong A \otimes_C A$ implies that A is a finitely presented C-module, see Algebra, Lemma 10.83.2. Hence A is finite locally free over C, see Algebra, Lemma 10.78.2.

By Lemma 39.23.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 39.23.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s, t : A \to B$ are finite locally free of a fixed rank r.

The local ring maps $C_\mathfrak{p} \to C'_\mathfrak{p}$ we are going to use are any local flat ring maps such that the residue field of $C'_\mathfrak{p}$ is infinite. By Algebra, Lemma 10.159.1 such local ring maps exist.

Assume C is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field, and assume that $s, t : A \to B$ is finite locally free of constant rank $r > 0$. Since $C \subset A$ is integral (Lemma 39.23.4) all primes lying over $\mathfrak{m}$ are maximal, and all maximal ideals of A lie over $\mathfrak{m}$. Similarly for $C \subset B$. Pick a maximal ideal $\mathfrak{m}'$ of A lying over $\mathfrak{m}$ (exists by Lemma 39.23.6). Since $t : A \to B$ is finite locally free there exist at most finitely many maximal ideals of B lying over $\mathfrak{m}'$. Hence we conclude (by Lemma 39.23.6 again) that A has finitely many maximal ideals, i.e., A is semi-local. This in turn implies that B is semi-local as well. OK, and now, because $t \otimes s : A \otimes_C A \to B$ is surjective, we can apply Algebra, Lemma 10.78.8 to the ring map $C \to A$, the A-module $M = B$ (seen as an A-module via t) and the C-submodule $s(A) \subset B$. This lemma implies that there exist $x_1, \ldots, x_r \in A$ such that M is free over A on the basis $s(x_1), \ldots, s(x_r)$. Hence we conclude that $C \to A$ is finite free and $B \cong A \otimes_C A$ by applying Lemma 39.23.8. $\square$

39.24. Finite flat groupoids

03JD In this section we prove a lemma that will help to show that the quotient of a scheme by a finite flat equivalence relation is a scheme, provided that each equivalence class is contained in an affine. See Properties of Spaces, Proposition 66.14.1.

03JE **Lemma 39.24.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are finite locally free. Let $u \in U$ be a point such that $t(s^{-1}(\{u\}))$ is contained in an affine open of U. Then there exists an R-invariant affine open neighbourhood of u in U.*

Proof. Since s is finite locally free it has finite fibres. Hence $t(s^{-1}(\{u\})) = \{u_1, \ldots, u_n\}$ is a finite set. Note that $u \in \{u_1, \ldots, u_n\}$. Let $W \subset U$ be an affine open containing $\{u_1, \ldots, u_n\}$, in particular $u \in W$. Consider $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of U which can be loosely described as the set of points of U which are R-equivalent to a point of $U \setminus W$. Hence $W' = U \setminus t(Z)$ is an R-invariant, open subscheme of U contained in W, and $\{u_1, \ldots, u_n\} \subset W'$. Picture

$$\{u_1, \ldots, u_n\} \subset W' \subset W \subset U.$$

Let $f \in \Gamma(W, \mathcal{O}_W)$ be an element such that $\{u_1, \ldots, u_n\} \subset D(f) \subset W'$. Such an f exists by Algebra, Lemma 10.15.2. By our choice of W' we have $s^{-1}(W') \subset t^{-1}(W)$, and hence we get a diagram

$$\begin{array}{ccc} s^{-1}(W') & \xrightarrow{t} & W \\ {\scriptstyle s}\downarrow & & \\ W' & & \end{array}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \mathrm{Norm}_s(t^\sharp f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction g is a function on W' which is nonzero in u, as $t^\sharp(f)$ is nonzero in each of the points of R lying over u, since f is nonzero in $u_1, \ldots, u_n$. Similarly, $D(g) \subset W'$ is equal to the set of points w such that f is not zero in any of the points equivalent to w. This means that $D(g)$ is an R-invariant affine open of W'. The final picture is

$$\{u_1, \ldots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win. □

39.25. Descending quasi-projective schemes

0CCH We can use Lemma 39.24.1 to show that a certain type of descent datum is effective.

0H8K **Lemma 39.25.1.** *Let $X \to Y$ be a surjective finite locally free morphism. Let d be a positive integer. Assume that for every geometric point $\bar{y} : \mathrm{Spec}(k) \to Y$ the fiber $X_{\bar{y}}$ has at most d points. Let V be a scheme over X such that for all $(y, v_1, \ldots, v_d)$ where $y \in Y$ and $v_1, \ldots, v_d \in V_y$ there exists an affine open $U \subset V$ with $v_1, \ldots, v_d \in U$. Then any descent datum on $V/X/Y$ is effective.*

Proof. Let φ be a descent datum as in Descent, Definition 35.34.1. Recall that the functor from schemes over Y to descent data relative to $\{X \to Y\}$ is fully faithful, see Descent, Lemma 35.35.11. Thus using Constructions, Lemma 27.2.1 it suffices to prove the lemma in the case that Y is affine. Some details omitted (this argument can be avoided if Y is separated or has affine diagonal, because then every morphism from an affine scheme to X is affine).

Assume Y is affine. If V is also affine, then we have effectivity by Descent, Lemma 35.37.1. Hence by Descent, Lemma 35.35.13 it suffices to prove that every point v of V has a φ-invariant affine open neighbourhood. Consider the groupoid $(X, X \times_Y X, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$. By Lemma 39.21.3 the descent datum φ determines and is determined by a cartesian morphism of groupoid schemes

$$(V, R, s, t, c) \longrightarrow (X, X \times_Y X, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

over $\mathrm{Spec}(\mathbf{Z})$. Since $X \to Y$ is finite locally free, we see that $\mathrm{pr}_i : X \times_Y X \to X$ and hence s and t are finite locally free, and their geometric fibers have at most d points. In particular the R-orbit $t(s^{-1}(\{v\}))$ of our point $v \in V$ has at most d points. Using the equivalence of categories of Lemma 39.21.3 once more we see that φ-invariant opens of V are the same thing as R-invariant opens of V. Our assumption shows there exists an affine open of V containing the orbit $t(s^{-1}(\{v\}))$ as all the points in this orbit map to the same point of Y. Thus Lemma 39.24.1 provides an R-invariant affine open containing v. □

0CCI **Lemma 39.25.2.** *Let $X \to Y$ be a surjective finite locally free morphism. Let V be a scheme over X such that for all $(y, v_1, \ldots, v_d)$ where $y \in Y$ and $v_1, \ldots, v_d \in V_y$ there exists an affine open $U \subset V$ with $v_1, \ldots, v_d \in U$. Then any descent datum on $V/X/Y$ is effective.*

Proof. As in the proof of Lemma 39.25.1, we may assume that Y is affine, in particular quasi-compact. Thus there exists an integer d such that all geometric fibers of $X \to Y$ have at most d points, see the discussion in Morphisms, Section 29.57. Our assumption on V trivially implies that we can apply Lemma 39.25.1, which completes the proof. □

0CCJ **Lemma 39.25.3.** *Let $X \to Y$ be a surjective finite locally free morphism. Let V be a scheme over X such that one of the following holds*

(1) *$V \to X$ is projective,*
(2) *$V \to X$ is quasi-projective,*
(3) *there exists an ample invertible sheaf on V,*
(4) *there exists an X-ample invertible sheaf on V,*
(5) *there exists an X-very ample invertible sheaf on V.*

Then any descent datum on $V/X/Y$ is effective.

Proof. We check the condition in Lemma 39.25.2. Let $y \in Y$ and $v_1, \ldots, v_d \in V$ points over y. Case (1) is a special case of (2), see Morphisms, Lemma 29.43.10. Case (2) is a special case of (4), see Morphisms, Definition 29.40.1. If there exists an ample invertible sheaf on V, then there exists an affine open containing $v_1, \ldots, v_d$ by Properties, Lemma 28.29.5. Thus (3) is true. In cases (4) and (5) it is harmless to replace Y by an affine open neighbourhood of y. Then X is affine too. In case (4) we see that V has an ample invertible sheaf by Morphisms, Definition 29.37.1 and the

result follows from case (3). In case (5) we can replace V by a quasi-compact open containing $v_1, \ldots, v_d$ and we reduce to case (4) by Morphisms, Lemma 29.38.2. □

0H8L **Lemma 39.25.4.** *Let $X \to Y$ be a surjective finite locally free morphism which is radicial. Let V be a scheme over X. Then any descent datum on $V/X/Y$ is effective.*

Proof. Apply Lemma 39.25.1 with $d = 1$. □

39.26. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
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CHAPTER 40

More on Groupoid Schemes

40.1. Introduction

04LB This chapter is devoted to advanced topics on groupoid schemes. Even though the results are stated in terms of groupoid schemes, the reader should keep in mind the 2-cartesian diagram

04LC (40.1.0.1)
$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 78.20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97**] by Keel and Mori.

40.2. Notation

04LD We continue to abide by the conventions and notation introduced in Groupoids, Section 39.2.

40.3. Useful diagrams

04LE We briefly restate the results of Groupoids, Lemmas 39.13.4 and 39.13.5 for easy reference in this chapter. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. In the commutative diagram

04LF (40.3.0.1)
$$\begin{array}{ccccc} & & U & & \\ & \overset{t}{\nearrow} & & \overset{t}{\nwarrow} & \\ R & \xleftarrow[\mathrm{pr}_0]{} & R \times_{s,U,t} R & \xrightarrow[c]{} & R \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} \\ U & \xleftarrow{t} & R & \xrightarrow{s} & U \end{array}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

04LG (40.3.0.2)

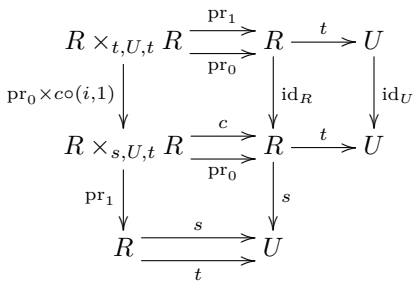

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

40.4. Sheaf of differentials

04R8 The following lemma is the analogue of Groupoids, Lemma 39.6.3.

04R9 **Lemma 40.4.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The sheaf of differentials of R seen as a scheme over U via t is a quotient of the pullback via t of the conormal sheaf of the immersion $e : U \to R$. In a formula: there is a canonical surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$. If s is flat, then this map is an isomorphism.*

Proof. Note that $e : U \to R$ is an immersion as it is a section of the morphism s, see Schemes, Lemma 26.21.11. Consider the following diagram

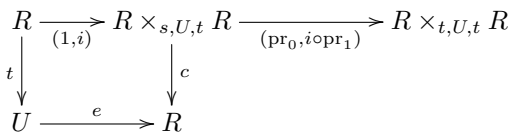

The square on the left is cartesian, because if $a \circ b = e$, then $b = i(a)$. The composition of the horizontal maps is the diagonal morphism of $t : R \to U$. The right top horizontal arrow is an isomorphism. Hence since $\Omega_{R/U}$ is the conormal sheaf of the composition it is isomorphic to the conormal sheaf of $(1, i)$. By Morphisms, Lemma 29.31.4 we get the surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$ and if c is flat, then this is an isomorphism. Since c is a base change of s by the properties of Diagram (40.3.0.2) we conclude that if s is flat, then c is flat, see Morphisms, Lemma 29.25.8. □

40.5. Local structure

0CK3 Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $u \in U$ be a point. In this section we explain what kind of structure we obtain on the local rings

$$A = \mathcal{O}_{U,u} \quad \text{and} \quad B = \mathcal{O}_{R,e(u)}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a

commutative diagram

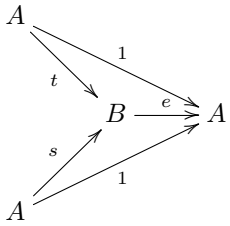

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s,U,t} R,(e(u),e(u))}$$

Then we have

$$C = (B \otimes_{s,A,t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e, e) : C \longrightarrow A$$

The composition law $c : R \times_{s,U,t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

0CK4 **Lemma 40.5.1.** *The map $I/I^2 \to J/J^2$ induced by c is the composition*

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B,\ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B,\ b_1 \otimes b_2 \mapsto t(e(b_1))b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

The statement on i follows from the statement on c and the fact that $c \circ (1, i) = e \circ t$. Some details omitted. □

40.6. Properties of groupoids

02YD Let (U, R, s, t, c) be a groupoid scheme. The idea behind the results in this section is that $s : R \to U$ is a base change of the morphism $U \to [U/R]$ (see Diagram (40.1.0.1). Hence the local properties of $s : R \to U$ should reflect local properties of the morphism $U \to [U/R]$. This doesn't work, because $[U/R]$ is not always an algebraic stack, and hence we cannot speak of geometric or algebraic properties

of $U \to [U/R]$. But it turns out that we can make some of it work without even referring to the quotient stack at all.

Here is a first example of such a result. The open $W \subset U'$ found in the lemma is roughly speaking the locus where the morphism $U' \to [U/R]$ has property $\mathcal{P}$.

04LH **Lemma 40.6.1.** *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. Let $g : U' \to U$ be a morphism of schemes. Denote h the composition*

$$h : U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U.$$

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of schemes. Assume

(1) *$\mathcal{R} \Rightarrow \mathcal{Q}$,*
(2) *$\mathcal{Q}$ is preserved under base change and composition,*
(3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

If s, t have $\mathcal{R}$ and g has $\mathcal{Q}$, then there exists an open subscheme $W \subset U'$ such that $W \times_{g,U,t} R = W(\mathcal{P}, h)$.

Proof. Note that the following diagram is commutative

$$\begin{array}{ccc} U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow{\mathrm{pr}_{12}} & R \times_{t,U,t} R \\ \downarrow{\scriptstyle \mathrm{pr}_{01}} \downarrow{\scriptstyle \mathrm{pr}_{02}} & & \downarrow{\scriptstyle \mathrm{pr}_0} \downarrow{\scriptstyle \mathrm{pr}_1} \\ U' \times_{g,U,t} R & \xrightarrow{\mathrm{pr}_1} & R \end{array}$$

with both squares cartesian (this uses that the two maps $t \circ \mathrm{pr}_i : R \times_{t,U,t} R \to U$ are equal). Combining this with the properties of diagram (40.3.0.2) we get a commutative diagram

$$\begin{array}{ccc} U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow{c \circ (i,1)} & R \\ \downarrow{\scriptstyle \mathrm{pr}_{01}} \downarrow{\scriptstyle \mathrm{pr}_{02}} & & \downarrow{\scriptstyle t} \downarrow{\scriptstyle s} \\ U' \times_{g,U,t} R & \xrightarrow{h} & U \end{array}$$

where both squares are cartesian.

Assume s, t have $\mathcal{R}$ and g has $\mathcal{Q}$. Then h has $\mathcal{Q}$ as a composition of s (which has $\mathcal{R}$ hence $\mathcal{Q}$) and a base change of g (which has $\mathcal{Q}$). Thus $W(\mathcal{P}, h) \subset U' \times_{g,U,t} R$ exists. By our assumptions we have $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ since both are the largest open on which $c \circ (i, 1)$ has $\mathcal{P}$. Note that the projection $U' \times_{g,U,t} R \to U'$ has a section, namely $\sigma : U' \to U' \times_{g,U,t} R$, $u' \mapsto (u', e(g(u')))$. Also via the isomorphism

$$(U' \times_{g,U,t} R) \times_{U'} (U' \times_{g,U,t} R) = U' \times_{g,U,t} R \times_{t,U,t} R$$

the two projections of the left hand side to $U' \times_{g,U,t} R$ agree with the morphisms pr_{01} and pr_{02} on the right hand side. Since $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ we conclude that $W(\mathcal{P}, h)$ is the inverse image of a subset of U, which is necessarily the open set $W = \sigma^{-1}(W(\mathcal{P}, h))$. □

04LI **Remark 40.6.2.** Warning: Lemma 40.6.1 should be used with care. For example, it applies to $\mathcal{P}$ ="flat", $\mathcal{Q}$ ="empty", and $\mathcal{R}$ ="flat and locally of finite presentation". But given a morphism of schemes $f : X \to Y$ the largest open $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

047W **Remark 40.6.3.** Notwithstanding the warning in Remark 40.6.2 there are some cases where Lemma 40.6.1 can be used without causing too much ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} = \mathcal{R}$ ="locally of finite type", and $\mathcal{P}$ ="relative dimension $\leq d$". See Morphisms, Definition 29.29.1 and Morphisms, Lemmas 29.28.4 and 29.28.3.
(2) $\mathcal{Q} = \mathcal{R}$ ="locally of finite type", and $\mathcal{P}$ ="locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms, Lemma 29.29.5.
(3) $\mathcal{Q} = \mathcal{R}$ ="locally of finite type", and $\mathcal{P}$ ="unramified". See Morphisms, Lemmas 29.35.3 and 29.35.15.

What is interesting about the cases listed above is that we do not need to assume that s, t are flat to get a conclusion about the locus where the morphism h has property $\mathcal{P}$. We continue the list:

(4) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="flat". See More on Morphisms, Theorem 37.15.1 and Lemma 37.15.2.
(5) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="Cohen-Macaulay". See More on Morphisms, Definition 37.22.1 and More on Morphisms, Lemmas 37.22.6 and 37.22.7.
(6) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="syntomic" use Morphisms, Lemma 29.30.12 (the locus is automatically open).
(7) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="smooth". See Morphisms, Lemma 29.34.15 (the locus is automatically open).
(8) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="étale". See Morphisms, Lemma 29.36.17 (the locus is automatically open).

Here is the second result. The R-invariant open $W \subset U$ should be thought of as the inverse image of the largest open of $[U/R]$ over which the morphism $U \to [U/R]$ has property $\mathcal{P}$.

03JC **Lemma 40.6.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$[1]. Let $\mathcal{P}$ be a property of morphisms of schemes which is τ-local on the target (Descent, Definition 35.22.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $s|_{s^{-1}(W)} : s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant, see Groupoids, Definition 39.19.1.*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 35.22.3. In Diagram (40.3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from

[1]The fact that *fpqc* is missing is not a typo.

the aforementioned Descent, Lemma 35.22.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

06QQ **Lemma 40.6.5.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be its stabilizer group scheme. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids, Definition 39.19.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 35.22.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent, Lemma 35.22.3. □

40.7. Comparing fibres

04LJ Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Diagram (40.3.0.1) gives us a way to compare the fibres of the map $s : R \to U$ in a groupoid. For a point $u \in U$ we will denote $F_u = s^{-1}(u)$ the scheme theoretic fibre of $s : R \to U$ over u. For example the diagram implies that if $u, u' \in U$ are points such that $s(r) = u$ and $t(r) = u'$, then $(F_u)_{\kappa(r)} \cong (F_{u'})_{\kappa(r)}$. This is a special case of the more general and more precise Lemma 40.7.1 below. To see this take $r' = i(r)$.

A pair (X, x) consisting of a scheme X and a point $x \in X$ is sometimes called the *germ of X at x*. A *morphism of germs* $f : (X, x) \to (S, s)$ is a morphism $f : U \to S$ defined on an open neighbourhood of x with $f(x) = s$. Two such f, f' are said to give the same morphism of germs if and only if f and f' agree in some open neighbourhood of x. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We temporarily introduce the following concept: We say that two morphisms of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *isomorphic locally on the base in the τ-topology*, if there exists a pointed scheme (S'', s'') and morphisms of germs $g : (S'', s'') \to (S, s)$, and $g' : (S'', s'') \to (S', s')$ such that

(1) g and g' are an open immersion (resp. étale, smooth, syntomic, flat and locally of finite presentation) at s'',
(2) there exists an isomorphism
$$(S'' \times_{g,S,f} X, \tilde{x}) \cong (S'' \times_{g',S',f'} X', \tilde{x}')$$
of germs over the germ (S'', s'') for some choice of points $\tilde{x}$ and $\tilde{x}'$ lying over (s'', x) and (s'', x').

Finally, we simply say that the maps of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *flat locally on the base isomorphic* if there exist S'', s'', g, g' as above but with (1) replaced by the condition that g and g' are flat at s'' (this is much weaker than any of the τ conditions above as a flat morphism need not be open).

02YF **Lemma 40.7.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $r, r' \in R$ with $t(r) = t(r')$ in U. Set $u = s(r)$, $u' = s(r')$. Denote $F_u = s^{-1}(u)$ and $F_{u'} = s^{-1}(u')$ the scheme theoretic fibres.*

(1) *There exists a common field extension $\kappa(u) \subset k$, $\kappa(u') \subset k$ and an isomorphism $(F_u)_k \cong (F_{u'})_k$.*
(2) *We may choose the isomorphism of (1) such that a point lying over r maps to a point lying over r'.*
(3) *If the morphisms s, t are flat then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are flat locally on the base isomorphic.*
(4) *If the morphisms s, t are étale (resp. smooth, syntomic, or flat and locally of finite presentation) then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally on the base isomorphic in the étale (resp. smooth, syntomic, or fppf) topology.*

Proof. We repeatedly use the properties and the existence of diagram (40.3.0.1). By the properties of the diagram (and Schemes, Lemma 26.17.5) there exists a point ξ of $R \times_{s,U,t} R$ with $\mathrm{pr}_0(\xi) = r$ and $c(\xi) = r'$. Let $\tilde{r} = \mathrm{pr}_1(\xi) \in R$.

Proof of (1). Set $k = \kappa(\tilde{r})$. Since $t(\tilde{r}) = u$ and $s(\tilde{r}) = u'$ we see that k is a common extension of both $\kappa(u)$ and $\kappa(u')$ and in fact that both $(F_u)_k$ and $(F_{u'})_k$ are isomorphic to the fibre of $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ over $\tilde{r}$. Hence (1) is proved.

Part (2) follows since the point ξ maps to r, resp. r'.

Part (3) is clear from the above (using the point ξ for $\tilde{u}$ and $\tilde{u}'$) and the definitions.

If s and t are flat and of finite presentation, then they are open morphisms (Morphisms, Lemma 29.25.10). Hence the image of some affine open neighbourhood V'' of $\tilde{r}$ will cover an open neighbourhood V of u, resp. V' of u'. These can be used to show that properties (1) and (2) of the definition of "locally on the base isomorphic in the τ-topology". □

40.8. Cohen-Macaulay presentations

04LK Given any groupoid (U, R, s, t, c) with s, t flat and locally of finite presentation there exists an "equivalent" groupoid (U', R', s', t', c') such that s' and t' are Cohen-Macaulay morphisms (and locally of finite presentation). See More on Morphisms, Section 37.22 for more information on Cohen-Macaulay morphisms. Here "equivalent" can be taken to mean that the quotient stacks $[U/R]$ and $[U'/R']$ are equivalent stacks, see Groupoids in Spaces, Section 78.20 and Section 78.25.

0460 **Lemma 40.8.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Assume s and t are flat and locally of finite presentation. Then there exists an open $U' \subset U$ such that*

(1) *$t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay,*
(2) *$s^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism t is Cohen-Macaulay,*
(3) *the morphism $t|_{s^{-1}(U')} : s^{-1}(U') \to U$ is surjective,*
(4) *the morphism $s|_{t^{-1}(U')} : t^{-1}(U') \to U$ is surjective, and*
(5) *the restriction $R' = s^{-1}(U') \cap t^{-1}(U')$ of R to U' defines a groupoid (U', R', s', t', c') which has the property that the morphisms s' and t' are Cohen-Macaulay and locally of finite presentation.*

Proof. Apply Lemma 40.6.1 with $g = \mathrm{id}$ and $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="Cohen-Macaulay", see Remark 40.6.3. This gives us an open $U' \subset U$ such that Let $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay. This proves (1). Let $i : R \to R$ be the inverse of the groupoid. Since i is an isomorphism, and $s \circ i = t$ and $t \circ i = s$ we see that $s^{-1}(U')$ is also the largest open of R on which t is Cohen-Macaulay. This proves (2). By More on Morphisms, Lemma 37.22.7 the open subset $t^{-1}(U')$ is dense in every fibre of $s : R \to U$. This proves (3). Same argument for (4). Part (5) is a formal consequence of (1) and (2) and the discussion of restrictions in Groupoids, Section 39.18. □

40.9. Restricting groupoids

04MM In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

04MN (40.9.0.1)

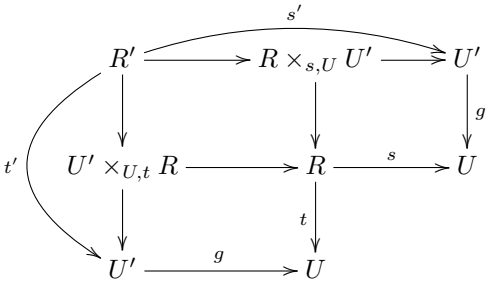

of a restriction. See Groupoids, Lemma 39.18.1.

04MP **Lemma 40.9.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

1. *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
2. *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
3. *If s, t are flat and g is flat, then s', t' are flat.*
4. *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms, Lemmas 29.15.3 and 29.15.4. Hence (1) is clear from Diagram (40.9.0.1). For the other cases, see Morphisms, Lemmas 29.21.3, 29.21.4, 29.25.6, and 29.25.8. □

The following lemma could have been used to prove the results of the preceding lemma in a more uniform way.

04MV **Lemma 40.9.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g, and let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$. If $\mathcal{P}$ is a property of morphisms of schemes such that*

1. *h has property $\mathcal{P}$, and*

(2) $\mathcal{P}$ *is preserved under base change,*

then s', t' *have property* $\mathcal{P}$*.*

Proof. This is clear as s' is the base change of h by Diagram (40.9.0.1) and t' is isomorphic to s' as a morphism of schemes. □

04MW **Lemma 40.9.3.** *Let* S *be a scheme. Let* (U, R, s, t, c) *be a groupoid scheme over* S*. Let* $g : U' \to U$ *and* $g' : U'' \to U'$ *be morphisms of schemes. Set* $g'' = g \circ g'$*. Let* (U', R', s', t', c') *be the restriction of* R *to* U'*. Let* $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$*, let* $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$*, and let* $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$*. The following diagram is commutative*

$$\begin{array}{ccccc} U'' \times_{g',U',t} R' & \longleftarrow & (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g'',U,t} R \\ \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle h''} \\ U' & \xleftarrow{pr_0} & U' \times_{g,U,t} R & \xrightarrow{h} & U \end{array}$$

with both squares cartesian where the left upper horizontal arrow is given by the rule

$$\begin{array}{ccc} (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g',U',t} R' \\ ((u', r_0), (u'', r_1)) & \longmapsto & (u'', (c(r_1, i(r_0)), (g'(u''), u'))) \end{array}$$

with notation as explained in the proof.

Proof. We work this out by exploiting the functorial point of view and reducing the lemma to a statement on arrows in restrictions of a groupoid category. In the last formula of the lemma the notation $((u', r_0), (u'', r_1))$ indicates a T-valued point of $(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R)$. This means that u', u'', r_0, r_1 are T-valued points of U', U'', R, R and that $g(u') = t(r_0)$, $g(g'(u'')) = g''(u'') = t(r_1)$, and $s(r_0) = s(r_1)$. It would be more correct here to write $g \circ u' = t \circ r_0$ and so on but this makes the notation even more unreadable. If we think of r_1 and r_0 as arrows in a groupoid category then we can represent this by the picture

$$t(r_0) = g(u') \xleftarrow{r_0} s(r_0) = s(r_1) \xrightarrow{r_1} t(r_1) = g(g'(u''))$$

This diagram in particular demonstrates that the composition $c(r_1, i(r_0))$ makes sense. Recall that

$$R' = R \times_{(t,s),U \times_S U, g \times g} U' \times_S U'$$

hence a T-valued point of R' looks like $(r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. In particular given $((u', r_0), (u'', r_1))$ as above we get the T-valued point $(c(r_1, i(r_0)), (g'(u''), u'))$ of R' because we have $t(c(r_1, i(r_0))) = t(r_1) = g(g'(u''))$ and $s(c(r_1, i(r_0))) = s(i(r_0)) = t(r_0) = g(u')$. We leave it to the reader to show that the left square commutes with this definition.

To show that the left square is cartesian, suppose we are given (v'', p') and (v', p) which are T-valued points of $U'' \times_{g',U',t} R'$ and $U' \times_{g,U,t} R$ with $v' = s'(p')$. This also means that $g'(v'') = t'(p')$ and $g(v') = t(p)$. By the discussion above we know that we can write $p' = (r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. Using this notation we see that $v' = s'(p') = u'_1$ and $g'(v'') = t'(p') = u'_0$. Here is a picture

$$s(p) \xrightarrow{p} g(v') = g(u'_1) \xrightarrow{r} g(u'_0) = g(g'(v''))$$

What we have to show is that there exists a unique T-valued point $((u', r_0), (u'', r_1))$ as above such that $v' = u'$, $p = r_0$, $v'' = u''$ and $p' = (c(r_1, i(r_0)), (g'(u''), u'))$. Comparing the two diagrams above it is clear that we have no choice but to take

$$((u', r_0), (u'', r_1)) = ((v', p), (v'', c(r, p))$$

Some details omitted. □

04MX **Lemma 40.9.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf, fpqc\}$. Let $\mathcal{P}$ be a property of morphisms of schemes which is preserved under base change, and which is local on the target for the τ-topology. If*

(1) *$h(U' \times_U R)$ is open in U,*
(2) *$\{h : U' \times_U R \to h(U' \times_U R)\}$ is a τ-covering,*
(3) *h' has property $\mathcal{P}$,*

then h'' has property $\mathcal{P}$. Conversely, if

(a) *$\{t : R \to U\}$ is a τ-covering,*
(d) *h'' has property $\mathcal{P}$,*

then h' has property $\mathcal{P}$.

Proof. This follows formally from the properties of the diagram of Lemma 40.9.3. In the first case, note that the image of the morphism h'' is contained in the image of h, as $g'' = g \circ g'$. Hence we may replace the U in the lower right corner of the diagram by $h(U' \times_U R)$. This explains the significance of conditions (1) and (2) in the lemma. In the second case, note that $\{\mathrm{pr}_0 : U' \times_{g,U,t} R \to U'\}$ is a τ-covering as a base change of τ and condition (a). □

40.10. Properties of groupoids on fields

04LL A "groupoid on a field" indicates a groupoid scheme (U, R, s, t, c) where U is the spectrum of a field. It does **not** mean that (U, R, s, t, c) is defined over a field, more precisely, it does **not** mean that the morphisms $s, t : R \to U$ are equal. Given any field k, an abstract group G and a group homomorphism $\varphi : G \to \mathrm{Aut}(k)$ we obtain a groupoid scheme (U, R, s, t, c) over $\mathbf{Z}$ by setting

$$\begin{aligned} U &= \mathrm{Spec}(k) \\ R &= \coprod_{g \in G} \mathrm{Spec}(k) \\ s &= \coprod_{g \in G} \mathrm{Spec}(\mathrm{id}_k) \\ t &= \coprod_{g \in G} \mathrm{Spec}(\varphi(g)) \\ c &= \text{composition in } G \end{aligned}$$

This example still is a groupoid scheme over $\mathrm{Spec}(k^G)$. Hence, if G is finite, then $U = \mathrm{Spec}(k)$ is finite over $\mathrm{Spec}(k^G)$. In some sense our goal in this section is to show that suitable finiteness conditions on s, t force any groupoid on a field to be defined over a finite index subfield $k' \subset k$.

If k is a field and (G, m) is a group scheme over k with structure morphism $p : G \to \mathrm{Spec}(k)$, then $(\mathrm{Spec}(k), G, p, p, m)$ is an example of a groupoid on a field (and in this case of course the whole structure is defined over a field). Hence this section can be viewed as the analogue of Groupoids, Section 39.7.

04LM **Lemma 40.10.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then the composition morphism $c : R \times_{s,U,t} R \to R$ is open.*

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (40.3.0.2). The projection is open by Morphisms, Lemma 29.23.4. □

04LN **Lemma 40.10.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then R is a separated scheme.*

Proof. By Groupoids, Lemma 39.7.3 the stabilizer group scheme $G \to U$ is separated. By Groupoids, Lemma 39.22.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 26.17). Hence R is a separated scheme, see Schemes, Lemma 26.21.12. □

04LP **Lemma 40.10.3.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any points $r, r' \in R$ there exists a field extension k'/k and points $r_1, r_2 \in R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k')$ and a diagram*

$$R \xleftarrow{pr_0} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\varphi} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{pr_0} R$$

such that φ is an isomorphism of schemes over $\mathrm{Spec}(k')$, we have $\varphi(r_1) = r_2$, $pr_0(r_1) = r$, and $pr_0(r_2) = r'$.

Proof. This is a special case of Lemma 40.7.1 parts (1) and (2). □

04LQ **Lemma 40.10.4.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

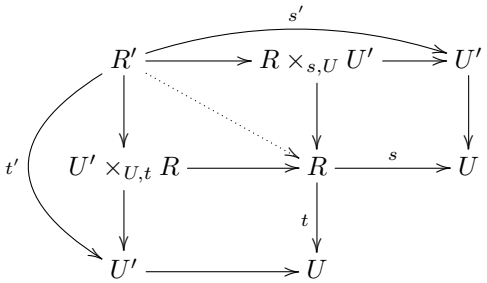

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 29.23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then

you conclude by using Morphisms, Lemmas 29.9.4, 29.9.2, 29.23.3, 29.11.9, 29.11.8, 29.25.8, and 29.25.6. □

04LR **Lemma 40.10.5.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any point $r \in R$ there exist*

(1) *a field extension k'/k with k' algebraically closed,*
(2) *a point $r' \in R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s', t' : R' \to \mathrm{Spec}(k')$ induce isomorphisms $k' \to \kappa(r')$.*

Proof. Translating the geometric statement into a statement on fields, this means that we can find a diagram

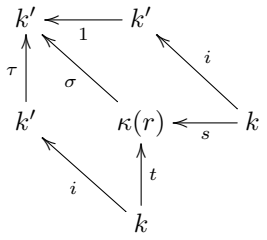

where $i : k \to k'$ is the embedding of k into k', the maps $s, t : k \to \kappa(r)$ are induced by $s, t : R \to U$, and the map $\tau : k' \to k'$ is an automorphism. To produce such a diagram we may proceed in the following way:

(1) Pick $i : k \to k'$ a field map with k' algebraically closed of very large transcendence degree over k.
(2) Pick an embedding $\sigma : \kappa(r) \to k'$ such that $\sigma \circ s = i$. Such a σ exists because we can just choose a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of $\kappa(r)$ over k and find $y_\alpha \in k'$, $\alpha \in A$ which are algebraically independent over $i(k)$, and map $s(k)(\{x_\alpha\})$ into k' by the rules $s(\lambda) \mapsto i(\lambda)$ for $\lambda \in k$ and $x_\alpha \mapsto y_\alpha$ for $\alpha \in A$. Then extend to $\tau : \kappa(\alpha) \to k'$ using that k' is algebraically closed.
(3) Pick an automorphism $\tau : k' \to k'$ such that $\tau \circ i = \sigma \circ t$. To do this pick a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of k over its prime field. On the one hand, extend $\{i(x_\alpha)\}$ to a transcendence basis of k' by adding $\{y_\beta\}_{\beta \in B}$ and extend $\{\sigma(t(x_\alpha))\}$ to a transcendence basis of k' by adding $\{z_\gamma\}_{\gamma \in C}$. As k' is algebraically closed we can extend the isomorphism $\sigma \circ t \circ i^{-1} : i(k) \to \sigma(t(k))$ to an isomorphism $\tau' : \overline{i(k)} \to \overline{\sigma(t(k))}$ of their algebraic closures in k'. As k' has large transcendence degree we see that the sets B and C have the same cardinality. Thus we can use a bijection $B \to C$ to extend τ' to an isomorphism

$$\overline{i(k)}(\{y_\beta\}) \longrightarrow \overline{\sigma(t(k))}(\{z_\gamma\})$$

and then since k' is the algebraic closure of both sides we see that this extends to an automorphism $\tau : k' \to k'$ as desired.

This proves the lemma. □

04LS **Lemma 40.10.6.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. If $r \in R$ is a point such that s, t induce isomorphisms $k \to \kappa(r)$, then the map*

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

(see proof for precise notation) is an automorphism $R \to R$ which maps e to r.

Proof. This is completely obvious if you think about groupoids in a functorial way. But we will also spell it out completely. Denote $a : U \to R$ the morphism with image r such that $s \circ a = \mathrm{id}_U$ which exists by the hypothesis that $s : k \to \kappa(r)$ is an isomorphism. Similarly, denote $b : U \to R$ the morphism with image r such that $t \circ b = \mathrm{id}_U$. Note that $b = a \circ (t \circ a)^{-1}$, in particular $a \circ s \circ b = b$.

Consider the morphism $\Psi : R \to R$ given on T-valued points by

$$(f : T \to R) \longmapsto (c(a \circ t \circ f, f) : T \to R)$$

To see this is defined we have to check that $s \circ a \circ t \circ f = t \circ f$ which is obvious as $s \circ a = 1$. Note that $\Phi(e) = a$, so that in order to prove the lemma it suffices to show that Φ is an automorphism of R. Let $\Phi : R \to R$ be the morphism given on T-valued points by

$$(g : T \to R) \longmapsto (c(i \circ b \circ t \circ g, g) : T \to R).$$

This is defined because $s \circ i \circ b \circ t \circ g = t \circ b \circ t \circ g = t \circ g$. We claim that Φ and Ψ are inverse to each other. To see this we compute

$$\begin{aligned}
& c(a \circ t \circ c(i \circ b \circ t \circ g, g), c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ t \circ i \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ s \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(c(b \circ t \circ g, i \circ b \circ t \circ g), g)) \\
&= c(e, g) \\
&= g
\end{aligned}$$

where we have used the relation $a \circ s \circ b = b$ shown above. In the other direction we have

$$\begin{aligned}
& c(i \circ b \circ t \circ c(a \circ t \circ f, f), c(a \circ t \circ f, f)) \\
&= c(i \circ b \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ (t \circ a)^{-1} \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(c(i \circ a \circ t \circ f, a \circ t \circ f), f) \\
&= c(e, f) \\
&= f
\end{aligned}$$

The lemma is proved. □

0B7V **Lemma 40.10.7.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, $W \subset R$ is open, and $Z \to R$ is a morphism of schemes, then the image of the composition $Z \times_{s,U,t} W \to R \times_{s,U,t} R \to R$ is open.*

Proof. Write $U = \mathrm{Spec}(k)$. Consider a field extension k'/k. Denote $U' = \mathrm{Spec}(k')$. Let R' be the restriction of R via $U' \to U$. Set $Z' = Z \times_R R'$ and $W' = R' \times_R W$. Consider a point $\xi = (z, w)$ of $Z \times_{s,U,t} W$. Let $r \in R$ be the image of z under $Z \to R$. Pick $k' \supset k$ and $r' \in R'$ as in Lemma 40.10.5. We can choose $z' \in Z'$ mapping to z and r'. Then we can find $\xi' \in Z' \times_{s',U',t'} W'$ mapping to z' and ξ. The open $c(r', W')$ (Lemma 40.10.6) is contained in the image of $Z' \times_{s',U',t'} W' \to R'$. Observe that $Z' \times_{s',U',t'} W' = (Z \times_{s,U,t} W) \times_{R \times_{s,U,t} R} (R' \times_{s',U',t'} R')$. Hence the image of $Z' \times_{s',U',t'} W' \to R' \to R$ is contained in the image of $Z \times_{s,U,t} W \to R$. As $R' \to R$ is open (Lemma 40.10.4) we conclude the image contains an open neighbourhood of the image of ξ as desired. □

04LT **Lemma 40.10.8.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. By abuse of notation denote $e \in R$ the image of the identity morphism $e : U \to R$. Then*

(1) *every local ring $\mathcal{O}_{R,r}$ of R has a unique minimal prime ideal,*
(2) *there is exactly one irreducible component Z of R passing through e, and*
(3) *Z is geometrically irreducible over k via either s or t.*

Proof. Let $r \in R$ be a point. In this proof we will use the correspondence between irreducible components of R passing through a point r and minimal primes of the local ring $\mathcal{O}_{R,r}$ without further mention. Choose $k \subset k'$ and $r' \in R'$ as in Lemma 40.10.5. Note that $\mathcal{O}_{R,r} \to \mathcal{O}_{R',r'}$ is faithfully flat and local, see Lemma 40.10.4. Hence the result for $r' \in R'$ implies the result for $r \in R$. In other words we may assume that $s, t : k \to \kappa(r)$ are isomorphisms. By Lemma 40.10.6 there exists an automorphism moving e to r. Hence we may assume $r = e$, i.e., part (1) follows from part (2).

We first prove (2) in case k is separably algebraically closed. Namely, let $X, Y \subset R$ be irreducible components passing through e. Then by Varieties, Lemma 33.8.4 and 33.8.3 the scheme $X \times_{s,U,t} Y$ is irreducible as well. Hence $c(X \times_{s,U,t} Y) \subset R$ is an irreducible subset. We claim it contains both X and Y (as subsets of R). Namely, let T be the spectrum of a field. If $x : T \to X$ is a T-valued point of X, then $c(x, e \circ s \circ x) = x$ and $e \circ s \circ x$ factors through Y as $e \in Y$. Similarly for points of Y. This clearly implies that $X = Y$, i.e., there is a unique irreducible component of R passing through e.

Proof of (2) and (3) in general. Let $k \subset k'$ be a separable algebraic closure, and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. By the previous paragraph there is exactly one irreducible component Z' of R' passing through e'. Denote $e'' \in R \times_{s,U} U'$ the base change of e. As $R' \to R \times_{s,U} U'$ is faithfully flat, see Lemma 40.10.4, and $e' \mapsto e''$ we see that there is exactly one irreducible component Z'' of $R \times_{s,k} k'$ passing through e''. This implies, as $R \times_k k' \to R$ is faithfully flat, that there is exactly one irreducible component Z of R passing through e. This proves (2).

To prove (3) let $Z''' \subset R \times_k k'$ be an arbitrary irreducible component of $Z \times_k k'$. By Varieties, Lemma 33.8.14 we see that $Z''' = \sigma(Z'')$ for some $\sigma \in \mathrm{Gal}(k'/k)$. Since $\sigma(e'') = e''$ we see that $e'' \in Z'''$ and hence $Z''' = Z''$. This means that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism s. The same argument implies that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism t. □

04LU **Lemma 40.10.9.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then*

(1) *R is equidimensional,*
(2) *$\dim(R) = \dim_r(R)$ for all $r \in R$,*
(3) *for any $r \in R$ we have $trdeg_{s(k)}(\kappa(r)) = trdeg_{t(k)}(\kappa(r))$, and*
(4) *for any closed point $r \in R$ we have $\dim(R) = \dim(\mathcal{O}_{R,r})$.*

Proof. Let $r, r' \in R$. Then $\dim_r(R) = \dim_{r'}(R)$ by Lemma 40.10.3 and Morphisms, Lemma 29.28.3. By Morphisms, Lemma 29.28.1 we have

$$\dim_r(R) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{s(k)}(\kappa(r)) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{t(k)}(\kappa(r)).$$

On the other hand, the dimension of R (or any open subset of R) is the supremum of the dimensions of the local rings of R, see Properties, Lemma 28.10.3. Clearly this is maximal for closed points r in which case $\mathrm{trdeg}_k(\kappa(r)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 29.16). Hence the lemma follows. □

04MQ **Lemma 40.10.10.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then $\dim(R) = \dim(G)$ where G is the stabilizer group scheme of R.*

Proof. Let $Z \subset R$ be the irreducible component passing through e (see Lemma 40.10.8) thought of as an integral closed subscheme of R. Let k'_s, resp. k'_t be the integral closure of $s(k)$, resp. $t(k)$ in $\Gamma(Z, \mathcal{O}_Z)$. Recall that k'_s and k'_t are fields, see Varieties, Lemma 33.28.4. By Varieties, Proposition 33.31.1 we have $k'_s = k'_t$ as subrings of $\Gamma(Z, \mathcal{O}_Z)$. As e factors through Z we obtain a commutative diagram

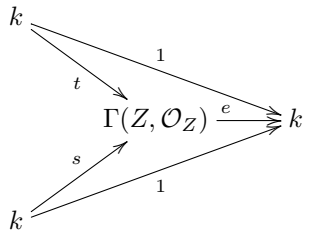

This on the one hand shows that $k'_s = s(k)$, $k'_t = t(k)$, so $s(k) = t(k)$, which combined with the diagram above implies that $s = t$! In other words, we conclude that Z is a closed subscheme of $G = R \times_{(t,s),U \times_S U, \Delta} U$. The lemma follows as both G and R are equidimensional, see Lemma 40.10.9 and Groupoids, Lemma 39.8.1. □

04MR **Remark 40.10.11.** Warning: Lemma 40.10.10 is wrong without the condition that s and t are locally of finite type. An easy example is to start with the action

$$\mathbf{G}_{m,\mathbf{Q}} \times_{\mathbf{Q}} \mathbf{A}^1_{\mathbf{Q}} \to \mathbf{A}^1_{\mathbf{Q}}$$

and restrict the corresponding groupoid scheme to the generic point of $\mathbf{A}^1_{\mathbf{Q}}$. In other words restrict via the morphism $\mathrm{Spec}(\mathbf{Q}(x)) \to \mathrm{Spec}(\mathbf{Q}[x]) = \mathbf{A}^1_{\mathbf{Q}}$. Then you get a groupoid scheme (U, R, s, t, c) with $U = \mathrm{Spec}(\mathbf{Q}(x))$ and

$$R = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(xy)}, P \in \mathbf{Q}[T], P \neq 0\right]\right)$$

In this case $\dim(R) = 1$ and $\dim(G) = 0$.

04RA **Lemma 40.10.12.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(k)$ with k a field,*
(2) *s, t are locally of finite type, and*
(3) *the characteristic of k is zero.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 40.4.1 the sheaf of differentials of $R \to U$ is free. Hence smoothness follows from Varieties, Lemma 33.25.1. □

04RB **Lemma 40.10.13.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(k)$ with k a field,*
(2) *s, t are locally of finite type,*
(3) *R is reduced, and*
(4) *k is perfect.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 40.4.1 the sheaf $\Omega_{R/U}$ is free. Hence the lemma follows from Varieties, Lemma 33.25.2. □

40.11. Morphisms of groupoids on fields

04Q4 This section studies morphisms between groupoids on fields. This is slightly more general, but very akin to, studying morphisms of groupschemes over a field.

04Q5 **Situation 40.11.1.** Let S be a scheme. Let $U = \mathrm{Spec}(k)$ be a scheme over S with k a field. Let (U, R_1, s_1, t_1, c_1), (U, R_2, s_2, t_2, c_2) be groupoid schemes over S with identical first component. Let $a : R_1 \to R_2$ be a morphism such that (id_U, a) defines a morphism of groupoid schemes over S, see Groupoids, Definition 39.13.1. In particular, the following diagrams commute

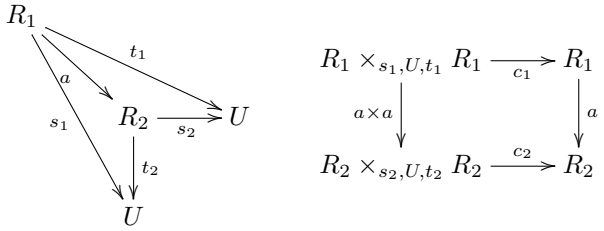

The following lemma is a generalization of Groupoids, Lemma 39.7.7.

04Q6 **Lemma 40.11.2.** *Notation and assumptions as in Situation 40.11.1. If $a(R_1)$ is open in R_2, then $a(R_1)$ is closed in R_2.*

Proof. Let $r_2 \in R_2$ be a point in the closure of $a(R_1)$. We want to show $r_2 \in a(R_1)$. Pick $k \subset k'$ and $r_2' \in R_2'$ adapted to (U, R_2, s_2, t_2, c_2) and r_2 as in Lemma 40.10.5. Let R_i' be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to U = \mathrm{Spec}(k)$. Let $a' : R_1' \to R_2'$ be the base change of a. The diagram

$$\begin{array}{ccc} R_1' & \xrightarrow{a'} & R_2' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ R_1 & \xrightarrow{a} & R_2 \end{array}$$

is a fibre square. Hence the image of a' is the inverse image of the image of a via the morphism $p_2 : R'_2 \to R_2$. By Lemma 40.10.4 the map p_2 is surjective and open. Hence by Topology, Lemma 5.6.4 we see that r'_2 is in the closure of $a'(R'_1)$. This means that we may assume that $r_2 \in R_2$ has the property that the maps $k \to \kappa(r_2)$ induced by s_2 and t_2 are isomorphisms.

In this case we can use Lemma 40.10.6. This lemma implies $c(r_2, a(R_1))$ is an open neighbourhood of r_2. Hence $a(R_1) \cap c(r_2, a(R_1)) \neq \emptyset$ as we assumed that r_2 was a point of the closure of $a(R_1)$. Using the inverse of R_2 and R_1 we see this means $c_2(a(R_1), a(R_1))$ contains r_2. As $c_2(a(R_1), a(R_1)) \subset a(c_1(R_1, R_1)) = a(R_1)$ we conclude $r_2 \in a(R_1)$ as desired. □

04Q7 **Lemma 40.11.3.** *Notation and assumptions as in Situation 40.11.1. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then $c_2(Z \times_{s_2,U,t_2} Z) \subset Z$ set theoretically.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} R_1 \times_{s_1,U,t_1} R_1 & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 \times_{s_2,U,t_2} R_2 & \longrightarrow & R_2 \end{array}$$

By Varieties, Lemma 33.24.2 the closure of the image of the left vertical arrow is (set theoretically) $Z \times_{s_2,U,t_2} Z$. Hence the result follows. □

04Q8 **Lemma 40.11.4.** *Notation and assumptions as in Situation 40.11.1. Assume that k is perfect. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 33.6.3. Hence the scheme $Z \times_{s_2,U,t_2} Z \subset Z$ is reduced, see Varieties, Lemma 33.6.7. Hence by Lemma 40.11.3 we see that c induces a morphism $Z \times_{s_2,U,t_2} Z \to Z$. Finally, it is clear that e_2 factors through Z and that the map $i_2 : R_2 \to R_2$ preserves Z. Since the morphisms of the septuple $(U, R_2, s_2, t_2, c_2, e_2, i_2)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

04Q9 **Lemma 40.11.5.** *Notation and assumptions as in Situation 40.11.1. If the image $a(R_1)$ is a locally closed subset of R_2 then it is a closed subset.*

Proof. Let $k \subset k'$ be a perfect closure of the field k. Let R'_i be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Note that the morphisms $R'_i \to R_i$ are universal homeomorphisms as compositions of base changes of the universal homeomorphism $U' \to U$ (see diagram in statement of Lemma 40.10.4). Hence it suffices to prove that $a'(R'_1)$ is closed in R'_2. In other words, we may assume that k is perfect.

If k is perfect, then the closure of the image is a groupoid scheme $Z \subset R_2$, by Lemma 40.11.4. By the same lemma applied to $\mathrm{id}_{R_1} : R_1 \to R_1$ we see that $(R_2)_{red}$ is a groupoid scheme. Thus we may apply Lemma 40.11.2 to the morphism $a|_{(R_2)_{red}} : (R_2)_{red} \to Z$ to conclude that Z equals the image of a. □

04QA **Lemma 40.11.6.** *Notation and assumptions as in Situation 40.11.1. Assume that $a : R_1 \to R_2$ is a quasi-compact morphism. Let $Z \subset R_2$ be the scheme theoretic image (see Morphisms, Definition 29.6.2) of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. The main difficulty is to show that $c_2|_{Z \times_{s_2,U,t_2} Z}$ maps into Z. Consider the commutative diagram

$$\begin{array}{ccc} R_1 \times_{s_1,U,t_1} R_1 & \longrightarrow & R_1 \\ \downarrow{\scriptstyle a \times a} & & \downarrow \\ R_2 \times_{s_2,U,t_2} R_2 & \longrightarrow & R_2 \end{array}$$

By Varieties, Lemma 33.24.3 we see that the scheme theoretic image of $a \times a$ is $Z \times_{s_2,U,t_2} Z$. By the commutativity of the diagram we conclude that $Z \times_{s_2,U,t_2} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 40.11.4 it is also true that $i_2(Z) \subset Z$ and that e_2 factors through Z. Hence we conclude as in the proof of that lemma. □

04QB **Lemma 40.11.7.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then $pr_{02}(Z \times_{pr_1,U,pr_0} Z) \subset Z$ set theoretically.*

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.3. But we can also prove it directly as follows.

Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{array}{ccc} R_s \times_k {}_tR & \xrightarrow{c} & R \\ \downarrow{\scriptstyle j_s \times {}_tj} & & \downarrow{\scriptstyle j} \\ U_s^2 \times_k {}_tU^2 & \longrightarrow & U \times_S U \end{array}$$

By Varieties, Lemma 33.24.2 we see that the closure of the image of $j_s \times {}_tj$ is $Z_s \times_k {}_tZ$. By the commutativity of the diagram we conclude that $Z_s \times_k {}_tZ$ maps into Z by the bottom horizontal arrow. □

04QC **Lemma 40.11.8.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a perfect field. Let $Z \subset U \times_S U$ be the reduced*

closed subscheme (see Schemes, Definition 26.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.4. But we can also prove it directly as follows.

We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 33.6.3. Hence the scheme $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \subset Z$ is reduced, see Varieties, Lemma 33.6.7. Hence by Lemma 40.11.7 we see that pr_{02} induces a morphism $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \to Z$. Finally, it is clear that $\Delta_{U/S}$ factors through Z and that the map $\sigma : U \times_S U \to U \times_S U$, $(x, y) \mapsto (y, x)$ preserves Z. Since $(U, U \times_S U, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}, \Delta_{U/S}, \sigma)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

04QD **Lemma 40.11.9.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field and assume R is quasi-compact (equivalently s, t are quasi-compact). Let $Z \subset U \times_S U$ be the scheme theoretic image (see Morphisms, Definition 29.6.2) of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 40.11.6. But we can also prove it directly as follows.

The main difficulty is to show that $\mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z}$ maps into Z. Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{array}{ccc} R_s \times_{k\ {}_tR} & \xrightarrow{c} & R \\ \downarrow{\scriptstyle j_s \times_t j} & & \downarrow{\scriptstyle j} \\ U_s^2 \times_{k\ {}_tU^2} & \longrightarrow & U \times_S U \end{array}$$

By Varieties, Lemma 33.24.3 we see that the scheme theoretic image of $j_s \times {}_tj$ is $Z_s \times_{k\ {}_tZ} Z$. By the commutativity of the diagram we conclude that $Z_s \times_{k\ {}_tZ} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 40.11.8 it is also true that $\sigma(Z) \subset Z$ and that $\Delta_{U/S}$ factors through Z. Hence we conclude as in the proof of that lemma. □

40.12. Slicing groupoids

04LV The following lemma shows that we may slice a Cohen-Macaulay groupoid scheme in order to reduce the dimension of the fibres, provided that the dimension of the stabilizer is small. This is an essential step in the process of improving a given presentation of a quotient stack.

04MY **Situation 40.12.1.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let $u \in U$ be a point, and let $u' \in U'$ be a point such that $g(u') = u$. Given these data, denote (U', R', s', t', c') the restriction of (U, R, s, t, c) via the morphism g. Denote $G \to U$ the stabilizer group scheme of R, which is a locally closed subscheme of R. Denote h the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U.$$

Denote $F_u = s^{-1}(u)$ (scheme theoretic fibre), and G_u the scheme theoretic fibre of G over u. Similarly for R' we denote $F'_{u'} = (s')^{-1}(u')$. Because $g(u') = u$ we have

$$F'_{u'} = h^{-1}(u) \times_{\mathrm{Spec}(\kappa(u))} \mathrm{Spec}(\kappa(u')).$$

The point $e(u) \in R$ may be viewed as a point on G_u and F_u also, and $e'(u')$ is a point of R' (resp. $G'_{u'}$, resp. $F'_{u'}$) which maps to $e(u)$ in R (resp. G_u, resp. F_u).

0461 **Lemma 40.12.2.** *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. With notation as in Situation 40.12.1, set*

$$d_1 = \dim(G_u), \quad d_2 = \dim_{e(u)}(F_u).$$

If $d_2 > d_1$, then there exist an affine scheme U' and a morphism $g : U' \to U$ such that (with notation as in Situation 40.12.1)

1. *g is an immersion*
2. *$u \in U'$,*
3. *g is locally of finite presentation,*
4. *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay at $(u, e(u))$, and*
5. *we have $\dim_{e'(u)}(F'_u) = d_2 - 1$.*

Proof. Let $\mathrm{Spec}(A) \subset U$ be an affine neighbourhood of u such that u corresponds to a closed point of U, see Morphisms, Lemma 29.16.4. Let $\mathrm{Spec}(B) \subset R$ be an affine neighbourhood of $e(u)$ which maps via j into the open $\mathrm{Spec}(A) \times_S \mathrm{Spec}(A) \subset U \times_S U$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to u. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $e(u)$. Pictures:

$$\begin{array}{ccc} B & \xleftarrow{s} & A \\ {\scriptstyle t}\uparrow & & \\ A & & \end{array} \quad \text{and} \quad \begin{array}{ccc} B_{\mathfrak{q}} & \xleftarrow{s} & A_{\mathfrak{m}} \\ {\scriptstyle t}\uparrow & & \\ A_{\mathfrak{m}} & & \end{array}$$

Note that the two induced maps $s, t : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{q})$ are equal and isomorphisms as $s \circ e = t \circ e = \mathrm{id}_U$. In particular we see that $\mathfrak{q}$ is a maximal ideal as well. The ring maps $s, t : A \to B$ are of finite presentation and flat. By assumption the ring

$$\mathcal{O}_{F_u, e(u)} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$$

is Cohen-Macaulay of dimension d_2. The equality of dimension holds by Morphisms, Lemma 29.28.1.

Let R'' be the restriction of R to $u = \mathrm{Spec}(\kappa(u))$ via the morphism $\mathrm{Spec}(\kappa(u)) \to U$. As $u \to U$ is locally of finite type, we see that $(\mathrm{Spec}(\kappa(u)), R'', s'', t'', c'')$ is a groupoid scheme with s'', t'' locally of finite type, see Lemma 40.9.1. By Lemma 40.10.10 this implies that $\dim(G'') = \dim(R'')$. We also have $\dim(R'') =$

$\dim_{e''}(R'') = \dim(\mathcal{O}_{R'',e''})$, see Lemma 40.10.9. By Groupoids, Lemma 39.18.4 we have $G'' = G_u$. Hence we conclude that $\dim(\mathcal{O}_{R'',e''}) = d_1$.

As a scheme R'' is

$$R'' = R \times_{(U \times_S U)} \Big(\mathrm{Spec}(\kappa(\mathfrak{m})) \times_S \mathrm{Spec}(\kappa(\mathfrak{m})) \Big)$$

Hence an affine open neighbourhood of e'' is the spectrum of the ring

$$B \otimes_{(A \otimes A)} (\kappa(\mathfrak{m}) \otimes \kappa(\mathfrak{m})) = B/s(\mathfrak{m})B + t(\mathfrak{m})B$$

We conclude that

$$\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$$

and so now we know that this ring has dimension d_1.

We claim this implies we can find an element $f \in \mathfrak{m}$ such that

$$\dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + fB_{\mathfrak{q}}) < d_2$$

Namely, suppose $\mathfrak{n}_j \supset s(\mathfrak{m})B_{\mathfrak{q}}$, $j = 1, \ldots, m$ correspond to the minimal primes of the local ring $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$. There are finitely many as this ring is Noetherian (since it is essentially of finite type over a field – but also because a Cohen-Macaulay ring is Noetherian). By the Cohen-Macaulay condition we have $\dim(B_{\mathfrak{q}}/\mathfrak{n}_j) = d_2$, for example by Algebra, Lemma 10.104.4. Note that $\dim(B_{\mathfrak{q}}/(\mathfrak{n}_j + t(\mathfrak{m})B_{\mathfrak{q}})) \leq d_1$ as it is a quotient of the ring $\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$ which has dimension d_1. As $d_1 < d_2$ this implies that $\mathfrak{m} \not\subset t^{-1}(\mathfrak{n}_i)$. By prime avoidance, see Algebra, Lemma 10.15.2, we can find $f \in \mathfrak{m}$ with $t(f) \notin \mathfrak{n}_j$ for $j = 1, \ldots, m$. For this choice of f we have the displayed inequality above, see Algebra, Lemma 10.60.13.

Set $A' = A/fA$ and $U' = \mathrm{Spec}(A')$. Then it is clear that $U' \to U$ is an immersion, locally of finite presentation and that $u \in U'$. Thus (1), (2) and (3) of the lemma hold. The morphism

$$U' \times_{g,U,t} R \longrightarrow U$$

factors through $\mathrm{Spec}(A)$ and corresponds to the ring map

$$B/t(f)B = A/(f) \otimes_{A,t} B \xleftarrow{s} A$$

Now, we see $t(f)$ is not a zerodivisor on $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$ as this is a Cohen-Macaulay ring of positive dimension and f is not contained in any minimal prime, see for example Algebra, Lemma 10.104.2. Hence by Algebra, Lemma 10.128.5 we conclude that $s : A_{\mathfrak{m}} \to B_{\mathfrak{q}}/t(f)B_{\mathfrak{q}}$ is flat with fibre ring $B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}})$ which is Cohen-Macaulay by Algebra, Lemma 10.104.2 again. This implies part (4) of the lemma. To see part (5) note that by Diagram (40.9.0.1) the fibre F'_u is equal to the fibre of h over u. Hence $\dim_{e'(u)}(F'_u) = \dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}}))$ by Morphisms, Lemma 29.28.1 and the dimension of this ring is $d_2 - 1$ by Algebra, Lemma 10.104.2 once more. This proves the final assertion of the lemma and we win. □

Now that we know how to slice we can combine it with the preceding material to get the following "optimal" result. It is optimal in the sense that since G_u is a locally closed subscheme of F_u one always has the inequality $\dim(G_u) = \dim_{e(u)}(G_u) \leq \dim_{e(u)}(F_u)$ so it is not possible to slice more than in the lemma.

04MZ **Lemma 40.12.3.** *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. With notation as in Situation 40.12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay and locally of finite presentation,*
(5) *the morphisms $s', t' : R' \to U'$ are Cohen-Macaulay and locally of finite presentation, and*
(6) *$\dim_{e(u)}(F'_u) = \dim(G'_u)$.*

Proof. As s is locally of finite presentation the scheme F_u is locally of finite type over $\kappa(u)$. Hence $\dim_{e(u)}(F_u) < \infty$ and we may argue by induction on $\dim_{e(u)}(F_u)$.

If $\dim_{e(u)}(F_u) = \dim(G_u)$ there is nothing to prove. Assume $\dim_{e(u)}(F_u) > \dim(G_u)$. This means that Lemma 40.12.2 applies and we find a morphism $g : U' \to U$ which has properties (1), (2), (3), instead of (6) we have $\dim_{e(u)}(F'_u) < \dim_{e(u)}(F_u)$, and instead of (4) and (5) we have that the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U$$

is Cohen-Macaulay at the point $(u, e(u))$. We apply Remark 40.6.3 and we obtain an open subscheme $U'' \subset U'$ such that $U'' \times_{g,U,t} R \subset U' \times_{g,U,t} R$ is the largest open subscheme on which h is Cohen-Macaulay. Since $(u, e(u)) \in U'' \times_{g,U,t} R$ we see that $u \in U''$. Hence we may replace U' by U'' and assume that in fact h is Cohen-Macaulay everywhere! By Lemma 40.9.2 we conclude that s', t' are locally of finite presentation and Cohen-Macaulay (use Morphisms, Lemma 29.21.4 and More on Morphisms, Lemma 37.22.6).

By construction $\dim_{e'(u)}(F'_u) < \dim_{e(u)}(F_u)$, so we may apply the induction hypothesis to (U', R', s', t', c') and the point $u \in U'$. Note that u is also a finite type point of U' (for example you can see this using the characterization of finite type points from Morphisms, Lemma 29.16.4). Let $g' : U'' \to U'$ and $(U'', R'', s'', t'', c'')$ be the solution of the corresponding problem starting with (U', R', s', t', c') and the point $u \in U'$. We claim that the composition

$$g'' = g \circ g' : U'' \longrightarrow U$$

is a solution for the original problem. Properties (1), (2), (3), (5), and (6) are immediate. To see (4) note that the morphism

$$h'' = s \circ \mathrm{pr}_1 : U'' \times_{g'',U,t} R \longrightarrow U$$

is locally of finite presentation and Cohen-Macaulay by an application of Lemma 40.9.4 (use More on Morphisms, Lemma 37.22.11 to see that Cohen-Macaulay morphisms are fppf local on the target). □

In case the stabilizer group scheme has fibres of dimension 0 this leads to the following slicing lemma.

04N0 **Lemma 40.12.4.** *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 29.16.3. Assume that $G \to U$ is locally quasi-finite. With notation as in Situation 40.12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is flat, locally of finite presentation, and locally quasi-finite, and*
(5) *the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation, and locally quasi-finite.*

Proof. Take $g : U' \to U$ as in Lemma 40.12.3. Since $h^{-1}(u) = F'_u$ we see that h has relative dimension ≤ 0 at $(u, e(u))$. Hence, by Remark 40.6.3, we obtain an open subscheme $U'' \subset U'$ such that $u \in U''$ and $U'' \times_{g,U,t} R$ is the maximal open subscheme of $U' \times_{g,U,t} R$ on which h has relative dimension ≤ 0. After replacing U' by U'' we see that h has relative dimension ≤ 0. This implies that h is locally quasi-finite by Morphisms, Lemma 29.29.5. Since it is still locally of finite presentation and Cohen-Macaulay we see that it is flat, locally of finite presentation and locally quasi-finite, i.e., (4) above holds. This implies that s' is flat, locally of finite presentation and locally quasi-finite as a base change of h, see Lemma 40.9.2. □

40.13. Étale localization of groupoids

03FK In this section we begin applying the étale localization techniques of More on Morphisms, Section 37.41 to groupoid schemes. More advanced material of this kind can be found in More on Groupoids in Spaces, Section 79.15. Lemma 40.13.2 will be used to prove results on algebraic spaces separated and quasi-finite over a scheme, namely Morphisms of Spaces, Proposition 67.50.2 and its corollary Morphisms of Spaces, Lemma 67.51.1.

03FL **Lemma 40.13.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume that*

(1) *$U \to S$ is locally of finite type,*
(2) *$U \to S$ is quasi-finite at u,*
(3) *$U \to S$ is separated,*
(4) *$R \to S$ is separated,*
(5) *s, t are flat and locally of finite presentation, and*
(6) *$s^{-1}(\{u\})$ is finite.*

Then there exists an étale neighbourhood $(S', p') \to (S, p)$ with $\kappa(p) = \kappa(p')$ and a base change diagram

$$\begin{array}{ccccc} R' \amalg W' & = & S' \times_S R & \longrightarrow & R \\ & & \downarrow{\scriptstyle t'} \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle t} \downarrow{\scriptstyle s} \\ U' \amalg W & = & S' \times_S U & \longrightarrow & U \\ & & \downarrow & & \downarrow \\ & & S' & \longrightarrow & S \end{array}$$

where the equal signs are decompositions into open and closed subschemes such that

(a) *there exists a point u' of U' mapping to u in U,*
(b) *the fibre $(U')_{p'}$ equals $t'\big((s')^{-1}(\{u'\})\big)$ set theoretically,*
(c) *the fibre $(R')_{p'}$ equals $(s')^{-1}\big((U')_{p'}\big)$ set theoretically,*
(d) *the schemes U' and R' are finite over S',*
(e) *we have $s'(R') \subset U'$ and $t'(R') \subset U'$,*
(f) *we have $c'(R' \times_{s', U', t'} R') \subset R'$ where c' is the base change of c, and*
(g) *the morphisms s', t', c' determine a groupoid structure by taking the system $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{s', U', t'} R'})$.*

Proof. Let us denote $f : U \to S$ the structure morphism of U. By assumption (6) we can write $s^{-1}(\{u\}) = \{r_1, \ldots, r_n\}$. Since this set is finite, we see that s is quasi-finite at each of these finitely many inverse images, see Morphisms, Lemma 29.20.7. Hence we see that $f \circ s : R \to S$ is quasi-finite at each r_i (Morphisms, Lemma 29.20.12). Hence r_i is isolated in the fibre R_p, see Morphisms, Lemma 29.20.6. Write $t(\{r_1, \ldots, r_n\}) = \{u_1, \ldots, u_m\}$. Note that it may happen that $m < n$ and note that $u \in \{u_1, \ldots, u_m\}$. Since t is flat and locally of finite presentation, the morphism of fibres $t_p : R_p \to U_p$ is flat and locally of finite presentation (Morphisms, Lemmas 29.25.8 and 29.21.4), hence open (Morphisms, Lemma 29.25.10). The fact that each r_i is isolated in R_p implies that each $u_j = t(r_i)$ is isolated in U_p. Using Morphisms, Lemma 29.20.6 again, we see that f is quasi-finite at $u_1, \ldots, u_m$.

Denote $F_u = s^{-1}(u)$ and $F_{u_j} = s^{-1}(u_j)$ the scheme theoretic fibres. Note that F_u is finite over $\kappa(u)$ as it is locally of finite type over $\kappa(u)$ with finitely many points (for example it follows from the much more general Morphisms, Lemma 29.57.9). By Lemma 40.7.1 we see that F_u and F_{u_j} become isomorphic over a common field extension of $\kappa(u)$ and $\kappa(u_j)$. Hence we see that F_{u_j} is finite over $\kappa(u_j)$. In particular we see $s^{-1}(\{u_j\})$ is a finite set for each $j = 1, \ldots, m$. Thus we see that assumptions (2) and (6) hold for each u_j also (above we saw that $U \to S$ is quasi-finite at u_j). Hence the argument of the first paragraph applies to each u_j and we see that $R \to U$ is quasi-finite at each of the points of

$$\{r_1, \ldots, r_N\} = s^{-1}(\{u_1, \ldots, u_m\})$$

Note that $t(\{r_1, \ldots, r_N\}) = \{u_1, \ldots, u_m\}$ and $t^{-1}(\{u_1, \ldots, u_m\}) = \{r_1, \ldots, r_N\}$ since R is a groupoid[2]. Moreover, we have $\mathrm{pr}_0(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$

[2]Explanation in groupoid language: The original set $\{r_1, \ldots, r_n\}$ was the set of arrows with source u. The set $\{u_1, \ldots, u_m\}$ was the set of objects isomorphic to u. And $\{r_1, \ldots, r_N\}$ is the set of all arrows between all the objects equivalent to u.

and $\mathrm{pr}_1(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$. Similarly we get $e(\{u_1, \ldots, u_m\}) \subset \{r_1, \ldots, r_N\}$ and $i(\{r_1, \ldots, r_N\}) = \{r_1, \ldots, r_N\}$.

We may apply More on Morphisms, Lemma 37.41.4 to the pairs $(U \to S, \{u_1, \ldots, u_m\})$ and $(R \to S, \{r_1, \ldots, r_N\})$ to get an étale neighbourhood $(S', p') \to (S, p)$ which induces an identification $\kappa(p) = \kappa(p')$ such that $S' \times_S U$ and $S' \times_S R$ decompose as

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

with $U' \to S'$ finite and $(U')_{p'}$ mapping bijectively to $\{u_1, \ldots, u_m\}$, and $R' \to S'$ finite and $(R')_{p'}$ mapping bijectively to $\{r_1, \ldots, r_N\}$. Moreover, no point of $W_{p'}$ (resp. $(W')_{p'}$) maps to any of the points u_j (resp. r_i). At this point (a), (b), (c), and (d) of the lemma are satisfied. Moreover, the inclusions of (e) and (f) hold on fibres over p', i.e., $s'((R')_{p'}) \subset (U')_{p'}$, $t'((R')_{p'}) \subset (U')_{p'}$, and $c'((R' \times_{s',U',t'} R')_{p'}) \subset (R')_{p'}$.

We claim that we can replace S' by a Zariski open neighbourhood of p' so that the inclusions of (e) and (f) hold. For example, consider the set $E = (s'|_{R'})^{-1}(W)$. This is open and closed in R' and does not contain any points of R' lying over p'. Since $R' \to S'$ is closed, after replacing S' by $S' \setminus (R' \to S')(E)$ we reach a situation where E is empty. In other words s' maps R' into U'. Note that this property is preserved under further shrinking S'. Similarly, we can arrange it so that t' maps R' into U'. At this point (e) holds. In the same manner, consider the set $E = (c'|_{R' \times_{s',U',t'} R'})^{-1}(W')$. It is open and closed in the scheme $R' \times_{s',U',t'} R'$ which is finite over S', and does not contain any points lying over p'. Hence after replacing S' by $S' \setminus (R' \times_{s',U',t'} R' \to S')(E)$ we reach a situation where E is empty. In other words we obtain the inclusion in (f). We may repeat the argument also with the identity $e' : S' \times_S U \to S' \times_S R$ and the inverse $i' : S' \times_S R \to S' \times_S R$ so that we may assume (after shrinking S' some more) that $(e'|_{U'})^{-1}(W') = \emptyset$ and $(i'|_{R'})^{-1}(W') = \emptyset$.

At this point we see that we may consider the structure

$$(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t',U',s'} R'}, e'|_{U'}, i'|_{R'}).$$

The axioms of a groupoid scheme over S' hold because they hold for the groupoid scheme $(S' \times_S U, S' \times_S R, s', t', c', e', i')$. □

03X5 **Lemma 40.13.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume assumptions (1) – (6) of Lemma 40.13.1 hold as well as*

(7) *$j : R \to U \times_S U$ is universally closed*[3].

Then we can choose $(S', p') \to (S, p)$ and decompositions $S' \times_S U = U' \amalg W$ and $S' \times_S R = R' \amalg W'$ and $u' \in U'$ such that (a) – (g) of Lemma 40.13.1 hold as well as

(h) *R' is the restriction of $S' \times_S R$ to U'.*

Proof. We apply Lemma 40.13.1 for the groupoid (U, R, s, t, c) over the scheme S with points p and u. Hence we get an étale neighbourhood $(S', p') \to (S, p)$ and disjoint union decompositions

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

[3]In view of the other conditions this is equivalent to requiring j to be proper.

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), and (g). We may shrink S' to a smaller neighbourhood of p' without affecting the conclusions (a) – (g). We will show that for a suitable shrinking conclusion (h) holds as well. Let us denote j' the base change of j to S'. By conclusion (e) it is clear that

$$j'^{-1}(U' \times_{S'} U') = R' \amalg Rest$$

for some open and closed $Rest$ piece. Since $U' \to S'$ is finite by conclusion (d) we see that $U' \times_{S'} U'$ is finite over S'. Since j is universally closed, also j' is universally closed, and hence $j'|_{Rest}$ is universally closed too. By conclusions (b) and (c) we see that the fibre of

$$(U' \times_{S'} U' \to S') \circ j'|_{Rest} : Rest \longrightarrow S'$$

over p' is empty. Hence, since $Rest \to S'$ is closed as a composition of closed morphisms, after replacing S' by $S' \setminus \mathrm{Im}(Rest \to S')$, we may assume that $Rest = \emptyset$. And this is exactly the condition that R' is the restriction of $S' \times_S R$ to the open subscheme $U' \subset S' \times_S U$, see Groupoids, Lemma 39.18.3 and its proof. □

40.14. Finite groupoids

0AB8 A groupoid scheme (U, R, s, t, c) is sometimes called *finite* if the morphisms s and t are finite. This is potentially confusing as it doesn't imply that U or R or the quotient sheaf U/R are finite over anything.

0AB9 **Lemma 40.14.1.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists a sequence of R-invariant closed subschemes*

$$U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$$

such that $\bigcap Z_r = \emptyset$ and such that $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r.

Proof. Let $\{Z_r\}$ be the stratification of U given by the Fitting ideals of the finite type quasi-coherent modules $s_*\mathcal{O}_R$. See Divisors, Lemma 31.9.6. Since the identity $e : U \to R$ is a section to s we see that $s_*\mathcal{O}_R$ contains $\mathcal{O}_S$ as a direct summand. Hence $U = Z_{-1} = Z_0$ (details omitted). Since formation of Fitting ideals commutes with base change (More on Algebra, Lemma 15.8.4) we find that $s^{-1}(Z_r)$ corresponds to the rth Fitting ideal of $\mathrm{pr}_{1,*}\mathcal{O}_{R \times_{s,U,t} R}$ because the lower right square of diagram (40.3.0.1) is cartesian. Using the fact that the lower left square is also cartesian we conclude that $s^{-1}(Z_r) = t^{-1}(Z_r)$, in other words Z_r is R-invariant. The morphism $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r because the module $s_*\mathcal{O}_R$ pulls back to a finite locally free module of rank r on $Z_{r-1} \setminus Z_r$ by Divisors, Lemma 31.9.6. □

0ABA **Lemma 40.14.2.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists an open subscheme $W \subset U$ and a closed subscheme $W' \subset W$ such that*

1. *W and W' are R-invariant,*
2. *$U = t(s^{-1}(\overline{W}))$ set theoretically,*
3. *W is a thickening of W', and*
4. *the maps s', t' of the restriction (W', R', s', t', c') are finite locally free.*

Proof. Consider the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 40.14.1.

We will construct disjoint unions $W = \coprod_{r \geq 1} W_r$ and $W' = \coprod_{r \geq 1} W'_r$ with each $W'_r \to W_r$ a thickening of R-invariant subschemes of U such that the morphisms s'_r, t'_r of the restrictions $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r. To begin we set $W_1 = W'_1 = U \setminus Z_1$. This is an R-invariant open subscheme of U, it is true that W_0 is a thickening of W'_0, and the maps s'_1, t'_1 of the restriction $(W'_1, R'_1, s'_1, t'_1, c'_1)$ are isomorphisms, i.e., finite locally free of rank 1. Moreover, every point of $U \setminus Z_1$ is in $t(s^{-1}(\overline{W_1}))$.

Assume we have found subschemes $W'_r \subset W_r \subset U$ for $r \leq n$ such that

(1) $W_1, \ldots, W_n$ are disjoint,
(2) W_r and W'_r are R-invariant,
(3) $U \setminus Z_n \subset \bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ set theoretically,
(4) W_r is a thickening of W'_r,
(5) the maps s'_r, t'_r of the restriction $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r.

Then we set

$$W_{n+1} = Z_n \setminus \Big(Z_{n+1} \cup \bigcup\nolimits_{r \leq n} t(s^{-1}(\overline{W_r}))\Big)$$

set theoretically and

$$W'_{n+1} = Z_n \setminus \Big(Z_{n+1} \cup \bigcup\nolimits_{r \leq n} t(s^{-1}(\overline{W_r}))\Big)$$

scheme theoretically. Then W_{n+1} is an R-invariant open subscheme of U because $Z_{n+1} \setminus \overline{U \setminus Z_{n+1}}$ is open in U and $\overline{U \setminus Z_{n+1}}$ is contained in the closed subset $\bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ we are removing by property (3) and the fact that t is a closed morphism. It is clear that W'_{n+1} is a closed subscheme of W_{n+1} with the same underlying topological space. Finally, properties (1), (2) and (3) are clear and property (5) follows from Lemma 40.14.1.

By Lemma 40.14.1 we have $\bigcap Z_r = \emptyset$. Hence every point of U is contained in $U \setminus Z_n$ for some n. Thus we see that $U = \bigcup_{r \geq 1} t(s^{-1}(\overline{W_r}))$ set theoretically and we see that (2) holds. Thus $W' \subset W$ satisfy (1), (2), (3), and (4). □

Let (U, R, s, t, c) be a groupoid scheme. Given a point $u \in U$ the *R-orbit* of u is the subset $t(s^{-1}(\{u\}))$ of U.

0ABB **Lemma 40.14.3.** *In Lemma 40.14.2 assume in addition that s and t are of finite presentation. Then*

(1) *the morphism $W' \to W$ is of finite presentation, and*
(2) *if $u \in U$ is a point whose R-orbit consists of generic points of irreducible components of U, then $u \in W$.*

Proof. In this case the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 40.14.1 is given by closed immersions $Z_k \to U$ of finite presentation, see Divisors, Lemma 31.9.6. Part (1) follows immediately from this as $W' \to W$ is locally given by intersecting the open W by Z_r. To see part (2) let $\{u_1, \ldots, u_n\}$ be the orbit of u. Since the closed subschemes Z_k are R-invariant and $\bigcap Z_k = \emptyset$, we find an k such that $u_i \in Z_k$ and $u_i \notin Z_{k+1}$ for all i. The image of $Z_k \to U$ and $Z_{k+1} \to U$ is locally constructible (Morphisms, Theorem 29.22.3). Since $u_i \in U$ is a generic point of an irreducible component of U, there exists an open neighbourhood U_i of

u_i which is contained in $Z_k \setminus Z_{k+1}$ set theoretically (Properties, Lemma 28.2.2). In the proof of Lemma 40.14.2 we have constructed W as a disjoint union $\coprod W_r$ with $W_r \subset Z_{r-1} \setminus Z_r$ such that $U = \bigcup t(s^{-1}(\overline{W_r}))$. As $\{u_1, \ldots, u_n\}$ is an R-orbit we see that $u \in t(s^{-1}(\overline{W_r}))$ implies $u_i \in \overline{W_r}$ for some i which implies $U_i \cap W_r \neq \emptyset$ which implies $r = k$. Thus we conclude that u is in

$$W_{k+1} = Z_k \setminus \Big(Z_{k+1} \cup \bigcup\nolimits_{r \leq k} t(s^{-1}(\overline{W_r}))\Big)$$

as desired. □

0ABC **Lemma 40.14.4.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite and of finite presentation and U quasi-separated. Let $u_1, \ldots, u_m \in U$ be points whose orbits consist of generic points of irreducible components of U. Then there exist R-invariant subschemes $V' \subset V \subset U$ such that*

(1) *$u_1, \ldots, u_m \in V'$,*
(2) *V is open in U,*
(3) *V' and V are affine,*
(4) *$V' \subset V$ is a thickening of finite presentation,*
(5) *the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.*

Proof. Let $W' \subset W \subset U$ be as in Lemma 40.14.2. By Lemma 40.14.3 we get $u_j \in W$ and that $W' \to W$ is a thickening of finite presentation. By Limits, Lemma 32.11.3 it suffices to find an R-invariant affine open subscheme V' of W' containing u_j (because then we can let $V \subset W$ be the corresponding open subscheme which will be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c') to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c) whose morphisms s and t are finite locally free. By Properties, Lemma 28.29.1 we can find an affine open containing the union of the orbits of $u_1, \ldots, u_m$. Finally, we can apply Groupoids, Lemma 39.24.1 to conclude. □

The following lemma is a special case of Lemma 40.14.4 but we redo the argument as it is slightly easier in this case (it avoids using Lemma 40.14.3).

0ABD **Lemma 40.14.5.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t finite, U is locally Noetherian, and $u_1, \ldots, u_m \in U$ points whose orbits consist of generic points of irreducible components of U. Then there exist R-invariant subschemes $V' \subset V \subset U$ such that*

(1) *$u_1, \ldots, u_m \in V'$,*
(2) *V is open in U,*
(3) *V' and V are affine,*
(4) *$V' \subset V$ is a thickening,*
(5) *the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.*

Proof. Let $\{u_{j1}, \ldots, u_{jn_j}\}$ be the orbit of u_j. Let $W' \subset W \subset U$ be as in Lemma 40.14.2. Since $U = t(s^{-1}(\overline{W}))$ we see that at least one $u_{ji} \in \overline{W}$. Since u_{ji} is a generic point of an irreducible component and U locally Noetherian, this implies that $u_{ji} \in W$. Since W is R-invariant, we conclude that $u_j \in W$ and in fact the whole orbit is contained in W. By Cohomology of Schemes, Lemma 30.13.3 it suffices to find an R-invariant affine open subscheme V' of W' containing $u_1, \ldots, u_m$

(because then we can let $V \subset W$ be the corresponding open subscheme which will be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c') to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c) whose morphisms s and t are finite locally free. By Properties, Lemma 28.29.1 we can find an affine open containing $\{u_{ij}\}$ (a locally Noetherian scheme is quasi-separated by Properties, Lemma 28.5.4). Finally, we can apply Groupoids, Lemma 39.24.1 to conclude. □

0ABE **Lemma 40.14.6.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S with s, t integral. Let $g : U' \to U$ be an integral morphism such that every R-orbit in U meets $g(U')$. Let (U', R', s', t', c') be the restriction of R to U'. If $u' \in U'$ is contained in an R'-invariant affine open, then the image $u \in U$ is contained in an R-invariant affine open of U.*

Proof. Let $W' \subset U'$ be an R'-invariant affine open. Set $\tilde{R} = U' \times_{g,U,t} R$ with maps $\mathrm{pr}_0 : \tilde{R} \to U'$ and $h = s \circ \mathrm{pr}_1 : \tilde{R} \to U$. Observe that pr_0 and h are integral. It follows that $\tilde{W} = \mathrm{pr}_0^{-1}(W')$ is affine. Since W' is R'-invariant, the image $W = h(\tilde{W})$ is set theoretically R-invariant and $\tilde{W} = h^{-1}(W)$ set theoretically (details omitted). Thus, if we can show that W is open, then W is a scheme and the morphism $\tilde{W} \to W$ is integral surjective which implies that W is affine by Limits, Proposition 32.11.2. However, our assumption on orbits meeting U' implies that $h : \tilde{R} \to U$ is surjective. Since an integral surjective morphism is submersive (Topology, Lemma 5.6.5 and Morphisms, Lemma 29.44.7) it follows that W is open. □

The following technical lemma produces "almost" invariant functions in the situation of a finite groupoid on a quasi-affine scheme.

0ABF **Lemma 40.14.7.** *Let (U, R, s, t, c) be a groupoid scheme with s, t finite and of finite presentation. Let $u_1, \ldots, u_m \in U$ be points whose R-orbits consist of generic points of irreducible components of U. Let $j : U \to \mathrm{Spec}(A)$ be an immersion. Let $I \subset A$ be an ideal such that $j(U) \cap V(I) = \emptyset$ and $V(I) \cup j(U)$ is closed in $\mathrm{Spec}(A)$. Then there exists an $h \in I$ such that $j^{-1}D(h)$ is an R-invariant affine open subscheme of U containing $u_1, \ldots, u_m$.*

Proof. Let $u_1, \ldots, u_m \in V' \subset V \subset U$ be as in Lemma 40.14.4. Since $U \setminus V$ is closed in U, j an immersion, and $V(I) \cup j(U)$ is closed in $\mathrm{Spec}(A)$, we can find an ideal $J \subset I$ such that $V(J) = V(I) \cup j(U \setminus V)$. For example we can take the ideal of elements of I which vanish on $j(U \setminus V)$. Thus we can replace (U, R, s, t, c), $j : U \to \mathrm{Spec}(A)$, and I by (V', R', s', t', c'), $j|_{V'} : V' \to \mathrm{Spec}(A)$, and J. In other words, we may assume that U is affine and that s and t are finite locally free. Take any $f \in I$ which does not vanish at all the points in the R-orbits of $u_1, \ldots, u_m$ (Algebra, Lemma 10.15.2). Consider

$$g = \mathrm{Norm}_s(t^\sharp(j^\sharp(f))) \in \Gamma(U, \mathcal{O}_U)$$

Since $f \in I$ and since $V(I) \cup j(U)$ is closed we see that $U \cap D(f) \to D(f)$ is a closed immersion. Hence $f^n g$ is the image of an element $h \in I$ for some $n > 0$. We claim that h works. Namely, we have seen in Groupoids, Lemma 39.23.2 that g is an R-invariant function, hence $D(g) \subset U$ is R-invariant. Since f does not vanish on the orbit of u_j, the function g does not vanish at u_j. Moreover, we have $V(g) \supset V(j^\sharp(f))$ and hence $j^{-1}D(h) = D(g)$. □

0ABG **Lemma 40.14.8.** *Let (U, R, s, t, c) be a groupoid scheme. If s, t are finite, and $u, u' \in R$ are distinct points in the same orbit, then u' is not a specialization of u.*

Proof. Let $r \in R$ with $s(r) = u$ and $t(r) = u'$. If $u \rightsquigarrow u'$ then we can find a nontrivial specialization $r \rightsquigarrow r'$ with $s(r') = u'$, see Schemes, Lemma 26.19.8. Set $u'' = t(r')$. Note that $u'' \neq u'$ as there are no specializations in the fibres of a finite morphism. Hence we can continue and find a nontrivial specialization $r' \rightsquigarrow r''$ with $s(r'') = u''$, etc. This shows that the orbit of u contains an infinite sequence $u \rightsquigarrow u' \rightsquigarrow u'' \rightsquigarrow \ldots$ of specializations which is nonsense as the orbit $t(s^{-1}(\{u\}))$ is finite. □

0ABH **Lemma 40.14.9.** *Let $j : V \to \mathrm{Spec}(A)$ be a quasi-compact immersion of schemes. Let $f \in A$ be such that $j^{-1}D(f)$ is affine and $j(V) \cap V(f)$ is closed. Then V is affine.*

Proof. This follows from Morphisms, Lemma 29.11.15 but we will also give a direct proof. Let $A' = \Gamma(V, \mathcal{O}_V)$. Then $j' : V \to \mathrm{Spec}(A')$ is a quasi-compact open immersion, see Properties, Lemma 28.18.4. Let $f' \in A'$ be the image of f. Then $(j')^{-1}D(f') = j^{-1}D(f)$ is affine. On the other hand, $j'(V) \cap V(f')$ is a subscheme of $\mathrm{Spec}(A')$ which maps isomorphically to the closed subscheme $j(V) \cap V(f)$ of $\mathrm{Spec}(A)$. Hence it is closed in $\mathrm{Spec}(A')$ for example by Schemes, Lemma 26.21.11. Thus we may replace A by A' and assume that j is an open immersion and $A = \Gamma(V, \mathcal{O}_V)$.

In this case we claim that $j(V) = \mathrm{Spec}(A)$ which finishes the proof. If not, then we can find a principal affine open $D(g) \subset \mathrm{Spec}(A)$ which meets the complement and avoids the closed subset $j(V) \cap V(f)$. Note that j maps $j^{-1}D(f)$ isomorphically onto $D(f)$, see Properties, Lemma 28.18.3. Hence $D(g)$ meets $V(f)$. On the other hand, $j^{-1}D(g)$ is a principal open of the affine open $j^{-1}D(f)$ hence affine. Hence by Properties, Lemma 28.18.3 again we see that $D(g)$ is isomorphic to $j^{-1}D(g) \subset j^{-1}D(f)$ which implies that $D(g) \subset D(f)$. This contradiction finishes the proof. □

0ABI **Lemma 40.14.10.** *Let (U, R, s, t, c) be a groupoid scheme. Let $u \in U$. Assume*

(1) *s, t are finite morphisms,*
(2) *U is separated and locally Noetherian,*
(3) *$\dim(\mathcal{O}_{U,u'}) \leq 1$ for every point u' in the orbit of u.*

Then u is contained in an R-invariant affine open of U.

Proof. The R-orbit of u is finite. By conditions (2) and (3) it is contained in an affine open U' of U, see Varieties, Proposition 33.42.7. Then $t(s^{-1}(U \setminus U'))$ is an R-invariant closed subset of U which does not contain u. Thus $U \setminus t(s^{-1}(U \setminus U'))$ is an R-invariant open of U' containing u. Replacing U by this open we may assume U is quasi-affine.

By Lemma 40.14.6 we may replace U by its reduction and assume U is reduced. This means R-invariant subschemes $W' \subset W \subset U$ of Lemma 40.14.2 are equal $W' = W$. As $U = t(s^{-1}(\overline{W}))$ some point u' of the R-orbit of u is contained in $\overline{W}$ and by Lemma 40.14.6 we may replace U by $\overline{W}$ and u by u'. Hence we may assume there is a dense open R-invariant subscheme $W \subset U$ such that the morphisms s_W, t_W of the restriction (W, R_W, s_W, t_W, c_W) are finite locally free.

If $u \in W$ then we are done by Groupoids, Lemma 39.24.1 (because W is quasi-affine so any finite set of points of W is contained in an affine open, see Properties,

Lemma 28.29.5). Thus we assume $u \notin W$ and hence none of the points of the orbit of u is in W. Let $\xi \in U$ be a point with a nontrivial specialization to a point u' in the orbit of u. Since there are no specializations among the points in the orbit of u (Lemma 40.14.8) we see that ξ is not in the orbit. By assumption (3) we see that ξ is a generic point of U and hence $\xi \in W$. As U is Noetherian there are finitely many of these points $\xi_1, \ldots, \xi_m \in W$. Because s_W, t_W are flat the orbit of each ξ_j consists of generic points of irreducible components of W (and hence U).

Let $j : U \to \mathrm{Spec}(A)$ be an immersion of U into an affine scheme (this is possible as U is quasi-affine). Let $J \subset A$ be an ideal such that $V(J) \cap j(W) = \emptyset$ and $V(J) \cup j(W)$ is closed. Apply Lemma 40.14.7 to the groupoid scheme (W, R_W, s_W, t_W, c_W), the morphism $j|_W : W \to \mathrm{Spec}(A)$, the points ξ_j, and the ideal J to find an $f \in J$ such that $(j|_W)^{-1}D(f)$ is an R_W-invariant affine open containing ξ_j for all j. Since $f \in J$ we see that $j^{-1}D(f) \subset W$, i.e., $j^{-1}D(f)$ is an R-invariant affine open of U contained in W containing all ξ_j.

Let Z be the reduced induced closed subscheme structure on

$$U \setminus j^{-1}D(f) = j^{-1}V(f).$$

Then Z is set theoretically R-invariant (but it may not be scheme theoretically R-invariant). Let (Z, R_Z, s_Z, t_Z, c_Z) be the restriction of R to Z. Since $Z \to U$ is finite, it follows that s_Z and t_Z are finite. Since $u \in Z$ the orbit of u is in Z and agrees with the R_Z-orbit of u viewed as a point of Z. Since $\dim(\mathcal{O}_{U,u'}) \leq 1$ and since $\xi_j \notin Z$ for all j, we see that $\dim(\mathcal{O}_{Z,u'}) \leq 0$ for all u' in the orbit of u. In other words, the R_Z-orbit of u consists of generic points of irreducible components of Z.

Let $I \subset A$ be an ideal such that $V(I) \cap j(U) = \emptyset$ and $V(I) \cup j(U)$ is closed. Apply Lemma 40.14.7 to the groupoid scheme (Z, R_Z, s_Z, t_Z, c_Z), the restriction $j|_Z$, the ideal I, and the point $u \in Z$ to obtain $h \in I$ such that $j^{-1}D(h) \cap Z$ is an R_Z-invariant open affine containing u.

Consider the R_W-invariant (Groupoids, Lemma 39.23.2) function

$$g = \mathrm{Norm}_{s_W}(t_W^\sharp(j^\sharp(h)|_W)) \in \Gamma(W, \mathcal{O}_W)$$

(In the following we only need the restriction of g to $j^{-1}D(f)$ and in this case the norm is along a finite locally free morphism of affines.) We claim that

$$V = (W_g \cap j^{-1}D(f)) \cup (j^{-1}D(h) \cap Z)$$

is an R-invariant affine open of U which finishes the proof of the lemma. It is set theoretically R-invariant by construction. As V is a constructible set, to see that it is open it suffices to show it is closed under generalization in U (Topology, Lemma 5.19.10 or the more general Topology, Lemma 5.23.6). Since $W_g \cap j^{-1}D(f)$ is open in U, it suffices to consider a specialization $u_1 \rightsquigarrow u_2$ of U with $u_2 \in j^{-1}D(h) \cap Z$. This means that h is nonzero in $j(u_2)$ and $u_2 \in Z$. If $u_1 \in Z$, then $j(u_1) \rightsquigarrow j(u_2)$ and since h is nonzero in $j(u_2)$ it is nonzero in $j(u_1)$ which implies $u_1 \in V$. If $u_1 \notin Z$ and also not in $W_g \cap j^{-1}D(f)$, then $u_1 \in W$, $u_1 \notin W_g$ because the complement of $Z = j^{-1}V(f)$ is contained in $W \cap j^{-1}D(f)$. Hence there exists a point $r_1 \in R$ with $s(r_1) = u_1$ such that h is zero in $t(r_1)$. Since s is finite we can find a specialization $r_1 \rightsquigarrow r_2$ with $s(r_2) = u_2$. However, then we conclude that h is zero in $u_2' = t(r_2)$ which contradicts the fact that $j^{-1}D(h) \cap Z$ is R-invariant and u_2 is in it. Thus V is open.

Observe that $V \subset j^{-1}D(h)$ for our function $h \in I$. Thus we obtain an immersion

$$j' : V \longrightarrow \mathrm{Spec}(A_h)$$

Let $f' \in A_h$ be the image of f. Then $(j')^{-1}D(f')$ is the principal open determined by g in the affine open $j^{-1}D(f)$ of U. Hence $(j')^{-1}D(f)$ is affine. Finally, $j'(V) \cap V(f') = j'(j^{-1}D(h) \cap Z)$ is closed in $\mathrm{Spec}(A_h/(f')) = \mathrm{Spec}((A/f)_h) = D(h) \cap V(f)$ by our choice of $h \in I$ and the ideal I. Hence we can apply Lemma 40.14.9 to conclude that V is affine as claimed above. □

40.15. Descending ind-quasi-affine morphisms

0APG Ind-quasi-affine morphisms were defined in More on Morphisms, Section 37.66. This section is the analogue of Descent, Section 35.38 for ind-quasi-affine-morphisms.

Let X be a quasi-separated scheme. Let $E \subset X$ be a subset which is an intersection of a nonempty family of quasi-compact opens of X. Say $E = \bigcap_{i \in I} U_i$ with $U_i \subset X$ quasi-compact open and I nonempty. By adding finite intersections we may assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. In this situation we have

0APH (40.15.0.1) $$\Gamma(E, \mathcal{F}|_E) = \mathrm{colim}\, \Gamma(U_i, \mathcal{F}|_{U_i})$$

for any sheaf $\mathcal{F}$ defined on X. Namely, fix $i_0 \in I$ and replace X by U_{i_0} and I by $\{i \in I \mid U_i \subset U_{i_0}\}$. Then X is quasi-compact and quasi-separated, hence a spectral space, see Properties, Lemma 28.2.4. Then we see the equality holds by Topology, Lemma 5.24.7 and Sheaves, Lemma 6.29.4. (In fact, the formula holds for higher cohomology groups as well if $\mathcal{F}$ is abelian, see Cohomology, Lemma 20.19.3.)

0API **Lemma 40.15.1.** *Let X be an ind-quasi-affine scheme. Let $E \subset X$ be an intersection of a nonempty family of quasi-compact opens of X. Set $A = \Gamma(E, \mathcal{O}_X|_E)$ and $Y = \mathrm{Spec}(A)$. Then the canonical morphism*

$$j : (E, \mathcal{O}_X|_E) \longrightarrow (Y, \mathcal{O}_Y)$$

of Schemes, Lemma 26.6.4 determines an isomorphism $(E, \mathcal{O}_X|_E) \to (E', \mathcal{O}_Y|_{E'})$ where $E' \subset Y$ is an intersection of quasi-compact opens. If $W \subset E$ is open in X, then $j(W)$ is open in Y.

Proof. Note that $(E, \mathcal{O}_X|_E)$ is a locally ringed space so that Schemes, Lemma 26.6.4 applies to $A \to \Gamma(E, \mathcal{O}_X|_E)$. Write $E = \bigcap_{i \in I} U_i$ with $I \neq \emptyset$ and $U_i \subset X$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. Set $A_i = \Gamma(U_i, \mathcal{O}_{U_i})$. We obtain commutative diagrams

$$\begin{array}{ccc} (E, \mathcal{O}_X|_E) & \longrightarrow & (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \\ \downarrow & & \downarrow \\ (U_i, \mathcal{O}_{U_i}) & \longrightarrow & (\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)}) \end{array}$$

Since U_i is quasi-affine, we see that $U_i \to \mathrm{Spec}(A_i)$ is a quasi-compact open immersion. On the other hand $A = \mathrm{colim}\, A_i$. Hence $\mathrm{Spec}(A) = \lim \mathrm{Spec}(A_i)$ as topological spaces (Limits, Lemma 32.4.6). Since $E = \lim U_i$ (by Topology, Lemma 5.24.7) we see that $E \to \mathrm{Spec}(A)$ is a homeomorphism onto its image E' and that E' is the intersection of the inverse images of the opens $U_i \subset \mathrm{Spec}(A_i)$ in $\mathrm{Spec}(A)$. For any $e \in E$ the local ring $\mathcal{O}_{X,e}$ is the value of $\mathcal{O}_{U_i,e}$ which is the same as the value on $\mathrm{Spec}(A)$.

To prove the final assertion of the lemma we argue as follows. Pick $i, j \in I$ with $U_i \subset U_j$. Consider the following commutative diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & \mathrm{Spec}(A_i) \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & \mathrm{Spec}(A_j) \end{array} \quad \begin{array}{ccc} W & \longrightarrow & \mathrm{Spec}(A_i) \\ \downarrow & & \downarrow \\ W & \longrightarrow & \mathrm{Spec}(A_j) \end{array} \quad \begin{array}{ccc} W & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ W & \longrightarrow & \mathrm{Spec}(A_j) \end{array}$$

By Properties, Lemma 28.18.5 the first diagram is cartesian. Hence the second is cartesian as well. Passing to the limit we find that the third diagram is cartesian, so the top horizontal arrow of this diagram is an open immersion. □

0APJ **Lemma 40.15.2.** *Suppose given a cartesian diagram*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

of schemes. Let $E \subset Y$ be an intersection of a nonempty family of quasi-compact opens of Y. Then

$$\Gamma(f^{-1}(E), \mathcal{O}_X|_{f^{-1}(E)}) = \Gamma(E, \mathcal{O}_Y|_E) \otimes_A B$$

provided Y is quasi-separated and $A \to B$ is flat.

Proof. Write $E = \bigcap_{i \in I} V_i$ with $V_i \subset Y$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $V_k \subset V_i \cap V_j$. Then we have similarly that $f^{-1}(E) = \bigcap_{i \in I} f^{-1}(V_i)$ in X. Thus the result follows from equation (40.15.0.1) and the corresponding result for V_i and $f^{-1}(V_i)$ which is Cohomology of Schemes, Lemma 30.5.2. □

0APK **Lemma 40.15.3** (Gabber)**.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$, see Descent, Definition 35.34.3. If each morphism $V_i \to X_i$ is ind-quasi-affine, then the descent datum is effective.*

Proof. Being ind-quasi-affine is a property of morphisms of schemes which is preserved under any base change, see More on Morphisms, Lemma 37.66.6. Hence Descent, Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is ind-quasi-affine, in other words, V is ind-quasi-affine.

Let (U, R, s, t, c) be the groupoid scheme over S with $U = X$ and $R = X \times_S X$ and s, t, c as usual. By Groupoids, Lemma 39.21.3 the pair (V, φ) corresponds to a cartesian morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes. Let $u' \in U'$ be any point. By Groupoids, Lemmas 39.19.2, 39.19.3, and 39.19.4 we can choose $u' \in W \subset E \subset U'$ where W is open and R'-invariant, and E is set-theoretically R'-invariant and an intersection of a nonempty family of quasi-compact opens.

Translating back to (V, φ), for any $v \in V$ we can find $v \in W \subset E \subset V$ with the following properties: (a) W is open and $\varphi(W \times_S X) = X \times_S W$ and (b) E an

intersection of quasi-compact opens and $\varphi(E \times_S X) = X \times_S E$ set-theoretically. Here we use the notation $E \times_S X$ to mean the inverse image of E in $V \times_S X$ by the projection morphism and similarly for $X \times_S E$. By Lemma 40.15.2 this implies that φ defines an isomorphism

$$\begin{aligned}\Gamma(E, \mathcal{O}_V|_E) \otimes_R A &= \Gamma(E \times_S X, \mathcal{O}_{V \times_S X}|_{E \times_S X}) \\ &\to \Gamma(X \times_S E, \mathcal{O}_{X \times_S V}|_{X \times_S E}) \\ &= A \otimes_R \Gamma(E, \mathcal{O}_V|_E)\end{aligned}$$

of $A \otimes_R A$-algebras which we will call ψ. The cocycle condition for φ translates into the cocycle condition for ψ as in Descent, Definition 35.3.1 (details omitted). By Descent, Proposition 35.3.9 we find an R-algebra R' and an isomorphism $\chi : R' \otimes_R A \to \Gamma(E, \mathcal{O}_V|_E)$ of A-algebras, compatible with ψ and the canonical descent datum on $R' \otimes_R A$.

By Lemma 40.15.1 we obtain a canonical "embedding"

$$j : (E, \mathcal{O}_V|_E) \longrightarrow \mathrm{Spec}(\Gamma(E, \mathcal{O}_V|_E)) = \mathrm{Spec}(R' \otimes_R A)$$

of locally ringed spaces. The construction of this map is canonical and we get a commutative diagram

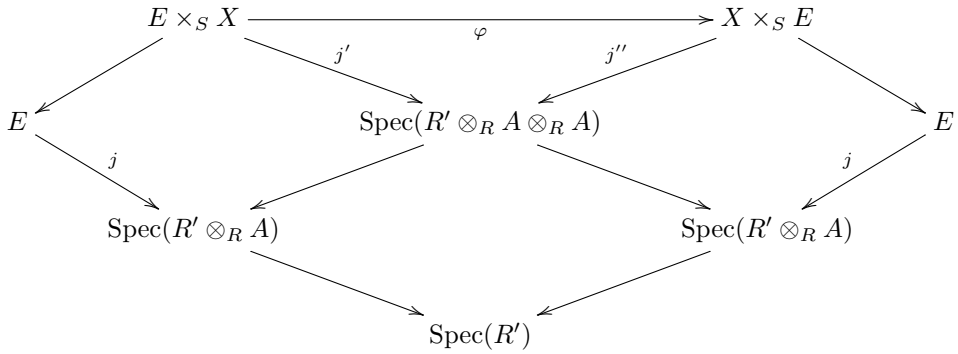

where j' and j'' come from the same construction applied to $E \times_S X \subset V \times_S X$ and $X \times_S E \subset X \times_S V$ via χ and the identifications used to construct ψ. It follows that $j(W)$ is an open subscheme of $\mathrm{Spec}(R' \otimes_R A)$ whose inverse image under the two projections $\mathrm{Spec}(R' \otimes_R A \otimes_R A) \to \mathrm{Spec}(R' \otimes_R A)$ are equal. By Descent, Lemma 35.13.6 we find an open $W_0 \subset \mathrm{Spec}(R')$ whose base change to $\mathrm{Spec}(A)$ is $j(W)$. Contemplating the diagram above we see that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. By Descent, Lemma 35.35.13 we see that our descent datum is effective. □

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CHAPTER 41

Étale Morphisms of Schemes

024J

41.1. Introduction

024K In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

We will refer to the other chapters of the Stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

41.2. Conventions

039F In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 10.31.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 29.21.9.
(3) Add more like this here.

41.3. Unramified morphisms

024L We first define "unramified homomorphisms of local rings" for Noetherian local rings. We cannot use the term "unramified" as there already is a notion of an unramified ring map (Algebra, Section 10.151) and it is different. After discussing the notion a bit we globalize it to describe unramified morphisms of locally Noetherian schemes.

024M **Definition 41.3.1.** Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be *unramified homomorphism of local rings* if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 10.151. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and $\Omega_{S/R} = 0$. We say $R \to S$ is unramified at a prime $\mathfrak{q} \subset S$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is an unramified ring map. It is shown in Algebra, Lemmas 10.151.5 and 10.151.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak{q}$ of S lying over $\mathfrak{p} \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak{q} \Leftrightarrow \mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}} \text{ and } \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

039G **Lemma 41.3.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is unramified at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an unramified homomorphism of local rings.*

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 10.97.6.

039H **Lemma 41.3.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.*

(1) *if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,*
(2) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$, then $A^\wedge \to B^\wedge$ is surjective,*
(3) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,*
(4) *if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only if the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).*

Proof. Part (1) is a special case of Algebra, Lemma 10.97.7. For part (2), note that the $\kappa(\mathfrak{m}_A)$-vector space $B^\wedge/\mathfrak{m}_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

039I **Lemma 41.3.4.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an unramified homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is unramified.*

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak{m}_A B^\wedge = (\mathfrak{m}_A B)B^\wedge = \mathfrak{m}_B B^\wedge$ and hence $\mathfrak{m}_A B = \mathfrak{m}_B$.

Assume the equivalent conditions (1) and (2). By Lemma 41.3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma 10.151.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 10.151.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

024N **Definition 41.3.5.** (See Morphisms, Definition 29.35.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is *unramified at* x if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.
(2) The morphism $f : X \to Y$ is said to be *unramified* if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is unramified is open.

039J **Lemma 41.3.6.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 41.3.5 if and only if it is unramified in the sense of Morphisms, Definition 29.35.1.*

Proof. This follows from Lemma 41.3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 29.35.5 and 29.35.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 29.35.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 29.29.1 and Morphisms, Lemma 29.29.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 29.35.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 29.35.16.

41.4. Three other characterizations of unramified morphisms

024O The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 29.35.14 for (part of) the statement for general schemes.

024P **Theorem 41.4.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,*
(3) *there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{i} & \mathbf{A}^n_V \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials dg for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and
(4) *the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 29.35.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 41.3.5 of this chapter but it does follow from Morphisms, Definition 29.35.1 which we proved to be equivalent in Lemma 41.3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 29.35.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by Morphisms, Lemma 29.32.7. Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 29.32.15 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 29.32.5 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 28.5.2 and Morphisms, Lemma 29.15.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \notin \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 10.131.8). This means that $\mathrm{d}(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1,\ldots,x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. □

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

see Morphisms, Lemma 29.32.9. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 29.35.16.

(2) Since $\Omega_{X/Y}$ is isomorphic to the conormal sheaf of the diagonal morphism (Morphisms, Lemma 29.32.7) we see that if $X \to Y$ is a monomorphism of locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 29.35.7 and 29.35.8.
(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the "ramification locus".

41.5. The functorial characterization of unramified morphisms

024Q In basic algebraic geometry we learn that some classes of morphisms can be characterized functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterization.

024R **Theorem 41.5.1.** *Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:*

(1) *f is unramified,*
(2) *the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map*

$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T_0, X)$$

is injective.

Proof. See More on Morphisms, Lemma 37.6.8 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Say $T = \mathrm{Spec}(C)$. Let J be the square-zero ideal of C with $T_0 = \mathrm{Spec}(C/J)$. Assume that we are given the diagram

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \downarrow{\scriptstyle \phi} & \searrow{\scriptstyle \bar{\phi}} & \\ R & \longrightarrow & C & \longrightarrow & C/J \end{array}$$

Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 41.4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. □

41.6. Topological properties of unramified morphisms

024S The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections. The material in this section does not require any Noetherian hypotheses.

024T **Proposition 41.6.1.** *Sections of unramified morphisms.*

(1) *Any section of an unramified morphism is an open immersion.*
(2) *Any section of a separated morphism is a closed immersion.*
(3) *Any section of an unramified separated morphism is open and closed.*

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 26.21.3 (resp. Morphisms, Lemma 29.35.13). Hence so is the graph as a base change (by Schemes, Lemma 26.18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). □

We can now explicitly describe the sections of unramified morphisms.

024U **Theorem 41.6.2.** *Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence*

$$\text{sections of } f \leftrightarrow \left\{ \begin{array}{c} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{array} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 41.6.1 part (3). □

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [**Gro71**, Exposé V]). See also the more general Morphisms, Lemma 29.35.17.

024V **Proposition 41.6.3.** *Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume*

(1) *Y is connected*
(2) *$x = f(y) = g(y)$, and*
(3) *the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.*

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 29.35.5 and Schemes, Lemma 26.21.12). Thus according to Theorem 41.6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) guarantee, namely $f'(y) = g'(y) \in X_Y$. □

0AKI **Lemma 41.6.4.** *Let S be a Noetherian scheme. Let $X \to S$ be a quasi-compact unramified morphism. Let $Y \to S$ be a morphism with Y Noetherian. Then $\mathrm{Mor}_S(Y, X)$ is a finite set.*

Proof. Assume first $X \to S$ is separated (which is often the case in practice). Since Y is Noetherian it has finitely many connected components. Thus we may assume Y is connected. Choose a point $y \in Y$ with image $s \in S$. Since $X \to S$ is unramified and quasi-compact then fibre X_s is finite, say $X_s = \{x_1, \ldots, x_n\}$ and $\kappa(x_i)/\kappa(s)$ is a finite field extension. See Morphisms, Lemma 29.35.10, 29.20.5, and 29.20.10. For each i there are at most finitely many $\kappa(s)$-algebra maps $\kappa(x_i) \to \kappa(y)$ (by elementary field theory). Thus $\mathrm{Mor}_S(Y, X)$ is finite by Proposition 41.6.3.

General case. There exists a nonempty open $U \subset S$ such that $X_U \to U$ is finite (in particular separated), see Morphisms, Lemma 29.51.1 (the lemma applies since we've already seen above that a quasi-compact unramified morphism is quasi-finite and since $X \to S$ is quasi-separated by Morphisms, Lemma 29.15.7). Let $Z \subset S$ be the reduced closed subscheme supported on the complement of U. By Noetherian induction, we see that $\mathrm{Mor}_Z(Y_Z, X_Z)$ is finite (details omitted). By the result of the first paragraph the set $\mathrm{Mor}_U(Y_U, X_U)$ is finite. Thus it suffices to show that

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_Z(Y_Z, X_Z) \times \mathrm{Mor}_U(Y_U, X_U)$$

is injective. This follows from the fact that the set of points where two morphisms $a, b : Y \to X$ agree is open in Y, due to the fact that $\Delta : X \to X \times_S X$ is open, see Morphisms, Lemma 29.35.13. □

41.7. Universally injective, unramified morphisms

06ND Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 29.10.1. Universally injective and unramified morphisms can be characterized as follows.

05VH **Lemma 41.7.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is unramified and a monomorphism,*
2. *f is unramified and universally injective,*
3. *f is locally of finite type and a monomorphism,*
4. *f is universally injective, locally of finite type, and formally unramified,*
5. *f is locally of finite type and X_s is either empty or $X_s \to s$ is an isomorphism for all $s \in S$.*

Proof. We have seen in More on Morphisms, Lemma 37.6.8 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 29.35.13) and surjective (Morphisms, Lemma 29.10.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 26.23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 26.23.11. Condition (5) implies (4) by Morphisms, Lemmas 29.10.2 and 29.35.12. □

This leads to the following useful characterization of closed immersions.

04XV **Lemma 41.7.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is a proper monomorphism,*
(3) *f is proper, unramified, and universally injective,*
(4) *f is universally closed, unramified, and a monomorphism,*
(5) *f is universally closed, unramified, and universally injective,*
(6) *f is universally closed, locally of finite type, and a monomorphism,*
(7) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (4) – (7) follows immediately from Lemma 41.7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 26.23.3 and has finite fibres. Hence More on Morphisms, Lemma 37.44.1 shows f is finite. Then Morphisms, Lemma 29.44.15 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 29.41.6 and Schemes, Lemma 26.23.8). By Lemma 41.7.1 we see that (2) implies (3). It is clear that (3) implies (5). □

Here is another result of a similar flavor.

04DG **Lemma 41.7.3.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is unramified,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[1].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = \mathrm{Spec}(A)$. By definition of a finite morphism this implies $X = \mathrm{Spec}(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. Conditions (2), (3) and (4) imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{p})$. By Algebra, Lemma 10.41.11 we have $B_\mathfrak{q} = B_\mathfrak{p}$ (note that a finite ring map satisfies going up, see Algebra, Section 10.41.) Hence we see that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 10.20.1) that $B_\mathfrak{p} = \varphi(A_\mathfrak{p})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. □

The topological results presented above will be used to give a functorial characterization of étale morphisms similar to Theorem 41.5.1.

[1] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

41.8. Examples of unramified morphisms

024W Here are a few examples.

024X **Example 41.8.1.** Let k be a field. Unramified quasi-compact morphisms $X \to \mathrm{Spec}(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalization forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $\mathrm{Spec}(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to \mathrm{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

024Y **Example 41.8.2.** Property (3) in Theorem 41.4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \ldots, g_m)$ be an ideal in $R[x_1, \ldots, x_n]$. Let $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak{q}$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak{q} \quad \in \quad \mathrm{Mat}(n \times m, \kappa(\mathfrak{q}))$$

has rank n. Then the morphism $f : Z = \mathrm{Spec}(R[x_1, \ldots, x_n]/I) \to \mathrm{Spec}(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak{q}$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 10.137.5, Lemma 10.137.6 and Example 10.137.7).

024Z **Example 41.8.3.** Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\mathrm{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a morphism $X \to \mathrm{Spec}(\mathcal{O}_K)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is finite Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $\mathrm{Spec}(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

41.9. Flat morphisms

0250 This section simply exists to summarize the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for "hyperplane sections" of certain modules to be flat.

0251 **Definition 41.9.1.** Flatness of modules and rings.

(1) A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is *faithfully flat* over A.
(3) A morphism of rings $f : A \to B$ is said to be *flat (resp. faithfully flat)* if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. See Algebra, Lemma 10.39.18.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 10.39.7.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 10.78.5.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ are flat for all $\mathfrak{q} \in \mathrm{Spec}(B)$. See Algebra, Lemma 10.39.18
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 10.39.17.
(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 10.39.16.
(8) If A is a Noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 10.97.3.
(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 10.39.8.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

0252 **Theorem 41.9.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak{m}_B$ is an element such that multiplication by t is injective on $M/\mathfrak{m}_A M$, then M/tM is also A-flat.*

Proof. See Algebra, Lemma 10.99.1. See also [**Mat70a**, Section 20]. □

0253 **Definition 41.9.3.** (See Morphisms, Definition 29.25.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) Let $x \in X$. We say $\mathcal{F}$ is *flat over Y at $x \in X$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. This uses the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ to think of $\mathcal{F}_x$ as a $\mathcal{O}_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is *flat at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.
(3) We say f is *flat* if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be *faithfully flat*.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)
(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 29.25.8 and 29.25.6.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 28.22.1 to see that Y has "enough" ideal sheaves and use the characterization of flatness in Algebra, Lemma 10.39.5.)

41.10. Topological properties of flat morphisms

0254 We "recall" below some openness properties that flat morphisms enjoy.

0255 **Theorem 41.10.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over Y is an open set. In particular the set of points where f is flat is open in X.*

Proof. See More on Morphisms, Theorem 37.15.1. □

039K **Theorem 41.10.2.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.*

Proof. See Morphisms, Lemma 29.25.10. □

0256 **Theorem 41.10.3.** *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. See Morphisms, Lemma 29.25.12. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrized by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

41.11. Étale morphisms

0257 In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterization presented in Theorem 41.16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the

basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

We first define "étale homomorphisms of local rings" for Noetherian local rings. We cannot use the term "étale", as there already is a notion of an étale ring map (Algebra, Section 10.143) and it is different.

0258 **Definition 41.11.1.** Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an *étale homomorphism of local rings* if it is flat and an unramified homomorphism of local rings (please see Definition 41.3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 10.143. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 10.143.2) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

039L **Lemma 41.11.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_\mathfrak{p} \to B_\mathfrak{q}$ is an étale homomorphism of local rings.*

Proof. See Algebra, Lemmas 10.143.3 (flatness of étale maps), 10.143.5 (étale maps are unramified) and 10.143.7 (flat and unramified maps are étale). □

039M **Lemma 41.11.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an étale homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is étale.*

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 41.3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 41.3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 41.9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

0259 **Definition 41.11.4.** (See Morphisms, Definition 29.36.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

(1) Let $x \in X$. We say f is *étale at* $x \in X$ if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
(2) The morphism is said to be *étale* if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

039N **Lemma 41.11.5.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 41.11.4 if and only if it is étale at x in the sense of Morphisms, Definition 29.36.1.*

Proof. This follows from Lemma 41.11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) An étale morphism is unramified. (Clear from our definitions.)
(2) Étaleness is local on the source and the target in the Zariski topology.
(3) Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 29.36.4 and 29.36.3.
(4) Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
(5) Étale morphisms have relative dimension 0. See Morphisms, Definition 29.29.1 and Morphisms, Lemma 29.29.5.
(6) A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 29.36.8.
(7) Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 41.10.2.
(8) Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 29.36.18.

41.12. The structure theorem

025A

We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a *standard étale ring map*, see Algebra, Definition 10.144.1. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g/(f)$.

Then the map

$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$

is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map

$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$

with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

025B **Theorem 41.12.1.** *Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 41.11.2 we see that $A \to B'$ is étale at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 10.144.4 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

039O **Theorem 41.12.2.** *Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a quotient of a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 41.3.2 we see that $A \to B'$ is unramified at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 10.152.1 to see that a principal localization of B' is a quotient of a standard étale A-algebra. □

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

025C **Theorem 41.12.3.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is étale at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the*

following diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[j]{} & \mathrm{Spec}(R[t]_{f'}/(f)) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) \end{array}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 29.36.14 although the statements differ slightly. See also, Varieties, Lemma 33.18.3 for a variant for unramified morphisms. □

41.13. Étale and smooth morphisms

039P An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

039Q **Theorem 41.13.1.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist an integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[\pi]{} & \mathbf{A}^n_R & = & \mathrm{Spec}(R[x_1, \ldots, x_n]) \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) & & \end{array}$$

where π is étale.

Proof. See Morphisms, Lemma 29.36.20. □

41.14. Topological properties of étale morphisms

025F We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the *fundamental property of étale morphisms*, see [**Gro71**, Exposé I.5].

025G **Theorem 41.14.1.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is an open immersion,*
(2) *f is universally injective and étale, and*
(3) *f is a flat monomorphism, locally of finite presentation.*

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 29.36.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 29.36.12 and 29.36.11. By Lemma 41.7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 29.25.10. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism.

Hence f is quasi-compact. Hence Descent, Lemma 35.25.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

04DH **Lemma 41.14.2.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is étale,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 41.7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 41.14.1). Hence after shrinking U we obtain an isomorphism. □

0EBS **Lemma 41.14.3.** *Let $U \to X$ be an étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.*

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 33.36.6. The morphism $F_{U/X}$ is étale as a morphism between schemes étale over X (Morphisms, Lemma 29.36.18). Hence $F_{U/X}$ is an isomorphism by Theorem 41.14.1. □

41.15. Topological invariance of the étale topology

06NE Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

025H **Theorem 41.15.1.** *Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S_0}(Y_0, X_0)$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $f : X \to S$.

Uniqueness. Suppose we have two sections σ, σ' through which σ_0 factors. Because $X \to S$ is étale we see that $\Delta : X \to X \times_S X$ is an open immersion (Morphisms, Lemma 29.35.13). The morphism $(\sigma, \sigma') : S \to X \times_S X$ factors through this open because for any $s \in S$ we have $(\sigma, \sigma')(s) = (\sigma_0(s), \sigma_0(s))$. Thus $\sigma = \sigma'$.

To prove existence we first reduce to the affine case (we suggest the reader skip this step). Let $X = \bigcup X_i$ be an affine open covering such that each X_i maps into an affine open S_i of S. For every $s \in S$ we can choose an i such that $\sigma_0(s) \in X_i$.

[2] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Choose an affine open neighbourhood $U \subset S_i$ of s such that $\sigma_0(U_0) \subset X_{i,0}$. Note that $X' = X_i \times_S U = X_i \times_{S_i} U$ is affine. If we can lift $\sigma_0|_{U_0} : U_0 \to X'_0$ to $U \to X'$, then by uniqueness these local lifts will glue to a global morphism $S \to X$. Thus we may assume S and X are affine.

Existence when S and X are affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Then $A \to B$ is étale and in particular smooth (of relative dimension 0). As $|S_0| = |S|$ we see that $S_0 = \mathrm{Spec}(A/I)$ with $I \subset A$ locally nilpotent. Thus existence follows from Algebra, Lemma 10.138.17. □

From the proof of preceding theorem, we also obtain one direction of the promised functorial characterization of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 59.45.2.

039R **Theorem 41.15.2** (Une equivalence remarquable de catégories)**.** *Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme with the same underlying topological space (for example if the ideal sheaf of S_0 in S has square zero). The functor* [**DG67**, IV, Theorem 18.1.2]

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\text{schemes } X \text{ étale over } S\} \leftrightarrow \{\text{schemes } X_0 \text{ étale over } S_0\}$$

Proof. By Theorem 41.15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma 26.14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$ with I locally nilpotent. By Algebra, Lemma 10.143.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial\overline{f}_1/\partial x_1 & \partial\overline{f}_2/\partial x_1 & \ldots & \partial\overline{f}_n/\partial x_1 \\ \partial\overline{f}_1/\partial x_2 & \partial\overline{f}_2/\partial x_2 & \ldots & \partial\overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\overline{f}_1/\partial x_n & \partial\overline{f}_2/\partial x_n & \ldots & \partial\overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in $\overline{A}$. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Set

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Since I is locally nilpotent the ideal IA is locally nilpotent (Algebra, Lemma 10.32.3). Observe that $\overline{A} = A/IA$. It follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A by Algebra, Lemma 10.32.4. Hence $R \to A$ is étale and the proof is complete. □

41.16. The functorial characterization

025J We finally present the promised functorial characterization. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterization.

025K **Theorem 41.16.1.** *Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent*

(1) *f is étale,*
(2) *for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_S(Y_0, X)$$

is bijective.

Proof. This is More on Morphisms, Lemma 37.8.9. □

This characterization says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

41.17. Étale local structure of unramified morphisms

04HG In the chapter More on Morphisms, Section 37.41 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is

$$\begin{array}{ccccc} V & \longrightarrow & X_U & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & U & \longrightarrow & S \end{array}$$

see More on Morphisms, Equation (37.41.0.1). We start with a very general case.

04HH **Lemma 41.17.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. By Morphisms, Definition 29.35.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms,

Lemma 37.41.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 41.7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

04HI **Lemma 41.17.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod_{i,j} V_{i,j}$$

such that

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 41.17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 29.41.7. Suppose $(i, j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite unramified morphism is étale locally on the base a closed immersion.

04HJ **Lemma 41.17.3.** *Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = \coprod_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over s is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 41.17.2 to this set (a finite morphism is separated, see Morphisms, Section 29.44). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

41.18. Étale local structure of étale morphisms

04HK This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 41.17 and change "closed immersion" into "isomorphism".

04HL **Lemma 41.18.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 41.14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

04HM **Lemma 41.18.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod_{i,j} V_{i,j}$$

of schemes such that

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.2. As in the proof of Lemma 41.18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a "topological covering space", i.e., a finite product of copies of the base.

04HN **Lemma 41.18.3.** *Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = \coprod_j V_j$$

of schemes such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 41.17.3. As in the proof of Lemma 41.18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

41.19. Permanence properties

025L In what follows, we present a few "permanence" properties of étale homomorphisms of Noetherian local rings (as defined in Definition 41.11.1). See More on Algebra, Sections 15.43 and 15.45 for the analogue of this material for the completion and henselization of a Noetherian local ring.

039S **Lemma 41.19.1.** *Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then* $\dim(A) = \dim(B)$.

Proof. See for example Algebra, Lemma 10.112.7. □

039T **Proposition 41.19.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then $depth(A) = depth(B)$*

Proof. See Algebra, Lemma 10.163.2. □

025Q **Proposition 41.19.3.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. □

025N **Proposition 41.19.4.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.*

Proof. If B is regular, then A is regular by Algebra, Lemma 10.110.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(B)$ (see Lemma 41.19.1). On the other hand, $\mathfrak{m}B$ is the maximal ideal of B and hence $\mathfrak{m}_B/\mathfrak{m}_B = \mathfrak{m}B/\mathfrak{m}^2B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 10.112.8.) □

025O **Proposition 41.19.5.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.*

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 10.164.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 41.11.2. Then we see that Algebra, Lemma 10.163.7 applies to $A \to B'$ and B' is reduced. Hence B is reduced. □

039U **Remark 41.19.6.** The result on "reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 110.17. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

025P **Proposition 41.19.7.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so.* [**Gro71**, Expose I, Theorem 9.5 part (i)]

Proof. See Algebra, Lemma 10.164.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 41.11.2. Then we see that Algebra, Lemma 10.163.9 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. □

The preceding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for $\mathbf{C}$-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the $\mathbf{C}$-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

41.20. Descending étale morphisms

0BTH In order to understand the language used in this section we encourage the reader to take a look at Descent, Section 35.34. Let $f : X \to S$ be a morphism of schemes. Consider the pullback functor

0BTI (41.20.0.1) $$\text{schemes } U \text{ étale over } S \longrightarrow \begin{matrix}\text{descent data } (V, \varphi) \text{ relative to } X/S \\ \text{with } V \text{ étale over } X\end{matrix}$$

sending U to the canonical descent datum $(X \times_S U, can)$.

0BTJ **Lemma 41.20.1.** *If $f : X \to S$ is surjective, then the functor (41.20.0.1) is faithful.*

Proof. Let $a, b : U_1 \to U_2$ be two morphisms between schemes étale over S. Assume the base changes of a and b to X agree. We have to show that $a = b$. By Proposition 41.6.3 it suffices to show that a and b agree on points and residue fields. This is clear because for every $u \in U_1$ we can find a point $v \in X \times_S U_1$ mapping to u. □

0BTK **Lemma 41.20.2.** *Assume $f : X \to S$ is submersive and any étale base change of f is submersive. Then the functor (41.20.0.1) is fully faithful.*

Proof. By Lemma 41.20.1 the functor is faithful. Let $U_1 \to S$ and $U_2 \to S$ be étale morphisms and let $a : X \times_S U_1 \to X \times_S U_2$ be a morphism compatible with canonical descent data. We will prove that a is the base change of a morphism $U_1 \to U_2$.

Let $U_2' \subset U_2$ be an open subscheme. Consider $W = a^{-1}(X \times_S U_2')$. This is an open subscheme of $X \times_S U_1$ which is compatible with the canonical descent datum on $V_1 = X \times_S U_1$. This means that the two inverse images of W by the projections $V_1 \times_{U_1} V_1 \to V_1$ agree. Since $V_1 \to U_1$ is surjective (as the base change of $X \to S$) we conclude that W is the inverse image of some subset $U_1' \subset U_1$. Since W is open, our assumption on f implies that $U_1' \subset U_1$ is open.

Let $U_2 = \bigcup U_{2,i}$ be an affine open covering. By the result of the preceding paragraph we obtain an open covering $U_1 = \bigcup U_{1,i}$ such that $X \times_S U_{1,i} = a^{-1}(X \times_S U_{2,i})$. If we can prove there exists a morphism $U_{1,i} \to U_{2,i}$ whose base change is the morphism $a_i : X \times_S U_{1,i} \to X \times_S U_{2,i}$ then we can glue these morphisms to a morphism $U_1 \to U_2$ (using faithfulness). In this way we reduce to the case that U_2 is affine. In particular $U_2 \to S$ is separated (Schemes, Lemma 26.21.13).

Assume $U_2 \to S$ is separated. Then the graph Γ_a of a is a closed subscheme of

$$V = (X \times_S U_1) \times_X (X \times_S U_2) = X \times_S U_1 \times_S U_2$$

by Schemes, Lemma 26.21.10. On the other hand the graph is open for example because it is a section of an étale morphism (Proposition 41.6.1). Since a is a morphism of descent data, the two inverse images of $\Gamma_a \subset V$ under the projections $V \times_{U_1 \times_S U_2} V \to V$ are the same. Hence arguing as in the second paragraph of the proof we find an open and closed subscheme $\Gamma \subset U_1 \times_S U_2$ whose base change to X gives Γ_a. Then $\Gamma \to U_1$ is an étale morphism whose base change to X is an isomorphism. This means that $\Gamma \to U_1$ is universally bijective, hence an isomorphism by Theorem 41.14.1. Thus Γ is the graph of a morphism $U_1 \to U_2$ and the base change of this morphism is a as desired. □

0BTL **Lemma 41.20.3.** *Let $f : X \to S$ be a morphism of schemes. In the following cases the functor (41.20.0.1) is fully faithful:*

(1) *f is surjective and universally closed (e.g., finite, integral, or proper),*
(2) *f is surjective and universally open (e.g., locally of finite presentation and flat, smooth, or etale),*
(3) *f is surjective, quasi-compact, and flat.*

Proof. This follows from Lemma 41.20.2. For example a closed surjective map of topological spaces is submersive (Topology, Lemma 5.6.5). Finite, integral, and proper morphisms are universally closed, see Morphisms, Lemmas 29.44.7 and 29.44.11 and Definition 29.41.1. On the other hand an open surjective map of topological spaces is submersive (Topology, Lemma 5.6.4). Flat locally finitely presented, smooth, and étale morphisms are universally open, see Morphisms, Lemmas 29.25.10, 29.34.10, and 29.36.13. The case of surjective, quasi-compact, flat morphisms follows from Morphisms, Lemma 29.25.12. □

0BTM **Lemma 41.20.4.** *Let $f : X \to S$ be a morphism of schemes. Let (V, φ) be a descent datum relative to X/S with $V \to X$ étale. Let $S = \bigcup S_i$ be an open covering. Assume that*

(1) *the pullback of the descent datum (V, φ) to $X \times_S S_i/S_i$ is effective,*
(2) *the functor (41.20.0.1) for $X \times_S (S_i \cap S_j) \to (S_i \cap S_j)$ is fully faithful, and*
(3) *the functor (41.20.0.1) for $X \times_S (S_i \cap S_j \cap S_k) \to (S_i \cap S_j \cap S_k)$ is faithful.*

Then (V, φ) is effective.

Proof. (Recall that pullbacks of descent data are defined in Descent, Definition 35.34.7.) Set $X_i = X \times_S S_i$. Denote (V_i, φ_i) the pullback of (V, φ) to X_i/S_i. By assumption (1) we can find an étale morphism $U_i \to S_i$ which comes with an isomorphism $X_i \times_{S_i} U_i \to V_i$ compatible with *can* and φ_i. By assumption (2) we obtain isomorphisms $\psi_{ij} : U_i \times_{S_i} (S_i \cap S_j) \to U_j \times_{S_j} (S_i \cap S_j)$. By assumption (3) these isomorphisms satisfy the cocycle condition so that (U_i, ψ_{ij}) is a descend datum for the Zariski covering $\{S_i \to S\}$. Then Descent, Lemma 35.35.10 (which is essentially just a reformulation of Schemes, Section 26.14) tells us that there exists a morphism of schemes $U \to S$ and isomorphisms $U \times_S S_i \to U_i$ compatible with ψ_{ij}. The isomorphisms $U \times_S S_i \to U_i$ determine corresponding isomorphisms $X_i \times_S U \to V_i$ which glue to a morphism $X \times_S U \to V$ compatible with the canonical descent datum and φ. □

0BTN **Lemma 41.20.5.** *Let (A, I) be a henselian pair. Let $U \to \mathrm{Spec}(A)$ be a quasi-compact, separated, étale morphism such that $U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite. Then*

$$U = U_{fin} \amalg U_{away}$$

where $U_{fin} \to \mathrm{Spec}(A)$ is finite and U_{away} has no points lying over Z.

Proof. By Zariski's main theorem, the scheme U is quasi-affine. In fact, we can find an open immersion $U \to T$ with T affine and $T \to \mathrm{Spec}(A)$ finite, see More on Morphisms, Lemma 37.43.3. Write $Z = \mathrm{Spec}(A/I)$ and denote $U_Z \to T_Z$ the base change. Since $U_Z \to Z$ is finite, we see that $U_Z \to T_Z$ is closed as well as open. Hence by More on Algebra, Lemma 15.11.6 we obtain a unique decomposition $T = T' \amalg T''$ with $T'_Z = U_Z$. Set $U_{fin} = U \cap T'$ and $U_{away} = U \cap T''$. Since $T'_Z \subset U_Z$

we see that all closed points of T' are in U hence $T' \subset U$, hence $U_{fin} = T'$, hence $U_{fin} \to \mathrm{Spec}(A)$ is finite. We omit the proof of uniqueness of the decomposition. □

0BTP **Proposition 41.20.6.** *Let $f : X \to S$ be a surjective integral morphism. The functor (41.20.0.1) induces an equivalence*

$$\begin{array}{c}\textit{schemes quasi-compact,}\\ \textit{separated, étale over } S\end{array} \longrightarrow \begin{array}{c}\textit{descent data } (V, \varphi) \textit{ relative to } X/S \textit{ with}\\ V \textit{ quasi-compact, separated, étale over } X\end{array}$$

Proof. By Lemma 41.20.3 the functor (41.20.0.1) is fully faithful and the same remains the case after any base change $S \to S'$. Let (V, φ) be a descent data relative to X/S with $V \to X$ quasi-compact, separated, and étale. We can use Lemma 41.20.4 to see that it suffices to prove the effectivity Zariski locally on S. In particular we may and do assume that S is affine.

If S is affine we can find a directed set Λ and an inverse system $X_\lambda \to S_\lambda$ of finite morphisms of affine schemes of finite type over $\mathrm{Spec}(\mathbf{Z})$ such that $(X \to S) = \lim(X_\lambda \to S_\lambda)$. See Algebra, Lemma 10.127.15. Since limits commute with limits we deduce that $X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda$ and $X \times_S X \times_S X = \lim X_\lambda \times_{S_\lambda} X_\lambda \times_{S_\lambda} X_\lambda$. Observe that $V \to X$ is a morphism of finite presentation. Using Limits, Lemmas 32.10.1 we can find an λ and a descent datum $(V_\lambda, \varphi_\lambda)$ relative to X_λ/S_λ whose pullback to X/S is (V, φ). Of course it is enough to show that $(V_\lambda, \varphi_\lambda)$ is effective. Note that V_λ is quasi-compact by construction. After possibly increasing λ we may assume that $V_\lambda \to X_\lambda$ is separated and étale, see Limits, Lemma 32.8.6 and 32.8.10. Thus we may assume that f is finite surjective and S affine of finite type over $\mathbf{Z}$.

Consider an open $S' \subset S$ such that the pullback (V', φ') of (V, φ) to $X' = X \times_S S'$ is effective. Below we will prove, that $S' \neq S$ implies there is a strictly larger open over which the descent datum is effective. Since S is Noetherian (and hence has a Noetherian underlying topological space) this will finish the proof. Let $\xi \in S$ be a generic point of an irreducible component of the closed subset $Z = S \setminus S'$. If $\xi \in S'' \subset S$ is an open over which the descent datum is effective, then the descent datum is effective over $S' \cup S''$ by the glueing argument of the first paragraph. Thus in the rest of the proof we may replace S by an affine open neighbourhood of ξ.

After a first such replacement we may assume that Z is irreducible with generic point Z. Let us endow Z with the reduced induced closed subscheme structure. After another shrinking we may assume $X_Z = X \times_S Z = f^{-1}(Z) \to Z$ is flat, see Morphisms, Proposition 29.27.1. Let (V_Z, φ_Z) be the pullback of the descent datum to X_Z/Z. By More on Morphisms, Lemma 37.57.1 this descent datum is effective and we obtain an étale morphism $U_Z \to Z$ whose base change is isomorphic to V_Z in a manner compatible with descent data. Of course $U_Z \to Z$ is quasi-compact and separated (Descent, Lemmas 35.23.1 and 35.23.6). Thus after shrinking once more we may assume that $U_Z \to Z$ is finite, see Morphisms, Lemma 29.51.1.

Let $S = \mathrm{Spec}(A)$ and let $I \subset A$ be the prime ideal corresponding to $Z \subset S$. Let (A^h, IA^h) be the henselization of the pair (A, I). Denote $S^h = \mathrm{Spec}(A^h)$ and $Z^h = V(IA^h) \cong Z$. We claim that it suffices to show effectivity after base change to S^h. Namely, $\{S^h \to S, S' \to S\}$ is an fpqc covering ($A \to A^h$ is flat by More on Algebra, Lemma 15.12.2) and by More on Morphisms, Lemma 37.57.1 we have fpqc descent for separated étale morphisms. Namely, if $U^h \to S^h$ and $U' \to S'$ are the objects corresponding to the pullbacks (V^h, φ^h) and (V', φ'), then the required

isomorphisms

$$U^h \times_S S^h \to S^h \times_S V^h \quad \text{and} \quad U^h \times_S S' \to S^h \times_S U'$$

are obtained by the fully faithfulness pointed out in the first paragraph. In this way we reduce to the situation described in the next paragraph.

Here $S = \mathrm{Spec}(A)$, $Z = V(I)$, $S' = S \setminus Z$ where (A, I) is a henselian pair, we have $U' \to S'$ corresponding to the descent datum (V', φ') and we have a finite étale morphism $U_Z \to Z$ corresponding to the descent datum (V_Z, φ_Z). We no longer have that A is of finite type over $\mathbf{Z}$; but the rest of the argument will not even use that A is Noetherian. By More on Algebra, Lemma 15.13.2 we can find a finite étale morphism $U_{fin} \to S$ whose restriction to Z is isomorphic to $U_Z \to Z$. Write $X = \mathrm{Spec}(B)$ and $Y = V(IB)$. Since (B, IB) is a henselian pair (More on Algebra, Lemma 15.11.8) and since the restriction $V \to X$ to Y is finite (as base change of $U_Z \to Z$) we see that there is a canonical disjoint union decomposition

$$V = V_{fin} \amalg V_{away}$$

were $V_{fin} \to X$ is finite and where V_{away} has no points lying over Y. See Lemma 41.20.5. Using the uniqueness of this decomposition over $X \times_S X$ we see that φ preserves it and we obtain

$$(V, \varphi) = (V_{fin}, \varphi_{fin}) \amalg (V_{away}, \varphi_{away})$$

in the category of descent data. By More on Algebra, Lemma 15.13.2 there is a unique isomorphism

$$X \times_S U_{fin} \longrightarrow V_{fin}$$

compatible with the given isomorphism $Y \times_Z U_Z \to V \times_X Y$ over Y. By the uniqueness we see that this isomorphism is compatible with descent data, i.e., $(X \times_S U_{fin}, can) \cong (V_{fin}, \varphi_{fin})$. Denote $U'_{fin} = U_{fin} \times_S S'$. By fully faithfulness we obtain a morphism $U'_{fin} \to U'$ which is the inclusion of an open (and closed) subscheme. Then we set $U = U_{fin} \amalg_{U'_{fin}} U'$ (glueing of schemes as in Schemes, Section 26.14). The morphisms $X \times_S U_{fin} \to V$ and $X \times_S U' \to V$ glue to a morphism $X \times_S U \to V$ which is the desired isomorphism. □

41.21. Normal crossings divisors

0CBN Here is the definition.

0BI9 **Definition 41.21.1.** Let X be a locally Noetherian scheme. A *strict normal crossings divisor* on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{X,p}$.

We often encounter effective Cartier divisors E on locally Noetherian schemes X such that there exists a strict normal crossings divisor D with $E \subset D$ set theoretically. In this case we have $E = \sum a_i D_i$ with $a_i \geq 0$ where $D = \bigcup_{i \in I} D_i$ is the decomposition of D into its irreducible components. Observe that $D' = \bigcup_{a_i > 0} D_i$ is a strict normal crossings divisor with $E = D'$ set theoretically. When the above happens we will say that E is *supported on a strict normal crossings divisor*.

0BIA **Lemma 41.21.2.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $D_i \subset D$, $i \in I$ be its irreducible components viewed as reduced closed subschemes of X. The following are equivalent*

(1) *D is a strict normal crossings divisor, and*
(2) *D is reduced, each D_i is an effective Cartier divisor, and for $J \subset I$ finite the scheme theoretic intersection $D_J = \bigcap_{j \in J} D_j$ is a regular scheme each of whose irreducible components has codimension $|J|$ in X.*

Proof. Assume D is a strict normal crossings divisor. Pick $p \in D$ and choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 41.21.1. Since $\mathcal{O}_{X,p}/(x_i)$ is a regular local ring (and in particular a domain) we see that the irreducible components $D_1, \ldots, D_r$ of D passing through p correspond 1-to-1 to the height one primes $(x_1), \ldots, (x_r)$ of $\mathcal{O}_{X,p}$. By Algebra, Lemma 10.106.3 we find that the intersections $D_{i_1} \cap \ldots \cap D_{i_s}$ have codimension s in an open neighbourhood of p and that this intersection has a regular local ring at p. Since this holds for all $p \in D$ we conclude that (2) holds.

Assume (2). Let $p \in D$. Since $\mathcal{O}_{X,p}$ is finite dimensional we see that p can be contained in at most $\dim(\mathcal{O}_{X,p})$ of the components D_i. Say $p \in D_1, \ldots, D_r$ for some $r \geq 1$. Let $x_1, \ldots, x_r \in \mathfrak{m}_p$ be local equations for $D_1, \ldots, D_r$. Then x_1 is a nonzerodivisor in $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X,p}/(x_1) = \mathcal{O}_{D_1,p}$ is regular. Hence $\mathcal{O}_{X,p}$ is regular, see Algebra, Lemma 10.106.7. Since $D_1 \cap \ldots \cap D_r$ is a regular (hence normal) scheme it is a disjoint union of its irreducible components (Properties, Lemma 28.7.6). Let $Z \subset D_1 \cap \ldots \cap D_r$ be the irreducible component containing p. Then $\mathcal{O}_{Z,p} = \mathcal{O}_{X,p}/(x_1, \ldots, x_r)$ is regular of codimension r (note that since we already know that $\mathcal{O}_{X,p}$ is regular and hence Cohen-Macaulay, there is no ambiguity about codimension as the ring is catenary, see Algebra, Lemmas 10.106.3 and 10.104.4). Hence $\dim(\mathcal{O}_{Z,p}) = \dim(\mathcal{O}_{X,p}) - r$. Choose additional $x_{r+1}, \ldots, x_n \in \mathfrak{m}_p$ which map to a minimal system of generators of $\mathfrak{m}_{Z,p}$. Then $\mathfrak{m}_p = (x_1, \ldots, x_n)$ by Nakayama's lemma and we see that D is a normal crossings divisor. □

0CBP **Lemma 41.21.3.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a strict normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a strict normal crossings divisor on Y.*

Proof. As f is flat the pullback is defined by Divisors, Lemma 31.13.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ as in Definition 41.21.1. Since f is smooth the local ring homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ is flat and the fibre ring

$$\mathcal{O}_{Y,q}/\mathfrak{m}_p\mathcal{O}_{Y,q} = \mathcal{O}_{Y_p,q}$$

is a regular local ring (see for example Algebra, Lemma 10.140.3). Pick $y_1, \ldots, y_n \in \mathfrak{m}_q$ which map to a regular system of parameters in $\mathcal{O}_{Y_p,q}$. Then $x_1, \ldots, x_d, y_1, \ldots, y_n$ generate the maximal ideal $\mathfrak{m}_q$. Hence $\mathcal{O}_{Y,q}$ is a regular local ring of dimension $d+n$ by Algebra, Lemma 10.112.7 and $x_1, \ldots, x_d, y_1, \ldots, y_n$ is a regular system of parameters. Since f^*D is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{Y,q}$ we conclude that the lemma is true. □

Here is the definition of a normal crossings divisor.

0BSF **Definition 41.21.4.** Let X be a locally Noetherian scheme. A *normal crossings divisor* on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ there exists an étale morphism $U \to X$ with p in the image and $D \times_X U$ a strict normal crossings divisor on U.

For example $D = V(x^2 + y^2)$ is a normal crossings divisor (but not a strict one) on $\mathrm{Spec}(\mathbf{R}[x, y])$ because after pulling back to the étale cover $\mathrm{Spec}(\mathbf{C}[x, y])$ we obtain $(x - iy)(x + iy) = 0$.

0CBQ **Lemma 41.21.5.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a normal crossings divisor. If $f : Y \to X$ is a smooth morphism of schemes, then the pullback f^*D is a normal crossings divisor on Y.*

Proof. As f is flat the pullback is defined by Divisors, Lemma 31.13.13 hence the statement makes sense. Let $q \in f^*D$ map to $p \in D$. Choose an étale morphism $U \to X$ whose image contains p such that $D \times_X U \subset U$ is a strict normal crossings divisor as in Definition 41.21.4. Set $V = Y \times_X U$. Then $V \to Y$ is étale as a base change of $U \to X$ (Morphisms, Lemma 29.36.4) and the pullback $D \times_X V$ is a strict normal crossings divisor on V by Lemma 41.21.3. Thus we have checked the condition of Definition 41.21.4 for $q \in f^*D$ and we conclude. □

0CBR **Lemma 41.21.6.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. The following are equivalent*

(1) *D is a normal crossings divisor in X,*
(2) *D is reduced, the normalization $\nu : D^\nu \to D$ is unramified, and for any $n \geq 1$ the scheme*

$$Z_n = D^\nu \times_D \ldots \times_D D^\nu \setminus \{(p_1, \ldots, p_n) \mid p_i = p_j \text{ for some } i \neq j\}$$

is regular, the morphism $Z_n \to X$ is a local complete intersection morphism whose conormal sheaf is locally free of rank n.

Proof. First we explain how to think about condition (2). The diagonal of an unramified morphism is open (Morphisms, Lemma 29.35.13). On the other hand $D^\nu \to D$ is separated, hence the diagonal $D^\nu \to D^\nu \times_D D^\nu$ is closed. Thus Z_n is an open and closed subscheme of $D^\nu \times_D \ldots \times_D D^\nu$. On the other hand, $Z_n \to X$ is unramified as it is the composition

$$Z_n \to D^\nu \times_D \ldots \times_D D^\nu \to \ldots \to D^\nu \times_D D^\nu \to D^\nu \to D \to X$$

and each of the arrows is unramified. Since an unramified morphism is formally unramified (More on Morphisms, Lemma 37.6.8) we have a conormal sheaf $\mathcal{C}_n = \mathcal{C}_{Z_n/X}$ of $Z_n \to X$, see More on Morphisms, Definition 37.7.2.

Formation of normalization commutes with étale localization by More on Morphisms, Lemma 37.19.3. Checking that local rings are regular, or that a morphism is unramified, or that a morphism is a local complete intersection or that a morphism is unramified and has a conormal sheaf which is locally free of a given rank, may be done étale locally (see More on Algebra, Lemma 15.44.3, Descent, Lemma 35.23.30, More on Morphisms, Lemma 37.62.19 and Descent, Lemma 35.7.6).

By the remark of the preceding paragraph and the definition of normal crossings divisor it suffices to prove that a strict normal crossings divisor $D = \bigcup_{i \in I} D_i$ satisfies (2). In this case $D^\nu = \coprod D_i$ and $D^\nu \to D$ is unramified (being unramified is local on the source and $D_i \to D$ is a closed immersion which is unramified). Similarly, $Z_1 = D^\nu \to X$ is a local complete intersection morphism because we may check this locally on the source and each morphism $D_i \to X$ is a regular immersion as it is the inclusion of a Cartier divisor (see Lemma 41.21.2 and More on Morphisms, Lemma 37.62.9). Since an effective Cartier divisor has an invertible

conormal sheaf, we conclude that the requirement on the conormal sheaf is satisfied. Similarly, the scheme Z_n for $n \geq 2$ is the disjoint union of the schemes $D_J = \bigcap_{j \in J} D_j$ where $J \subset I$ runs over the subsets of order n. Since $D_J \to X$ is a regular immersion of codimension n (by the definition of strict normal crossings and the fact that we may check this on stalks by Divisors, Lemma 31.20.8) it follows in the same manner that $Z_n \to X$ has the required properties. Some details omitted.

Assume (2). Let $p \in D$. Since $D^\nu \to D$ is unramified, it is finite (by Morphisms, Lemma 29.44.4). Hence $D^\nu \to X$ is finite unramified. By Lemma 41.17.3 and étale localization (permissible by the discussion in the second paragraph and the definition of normal crossings divisors) we reduce to the case where $D^\nu = \coprod_{i \in I} D_i$ with I finite and $D_i \to U$ a closed immersion. After shrinking X if necessary, we may assume $p \in D_i$ for all $i \in I$. The condition that $Z_1 = D^\nu \to X$ is an unramified local complete intersection morphism with conormal sheaf locally free of rank 1 implies that $D_i \subset X$ is an effective Cartier divisor, see More on Morphisms, Lemma 37.62.3 and Divisors, Lemma 31.21.3. To finish the proof we may assume $X = \mathrm{Spec}(A)$ is affine and $D_i = V(f_i)$ with $f_i \in A$ a nonzerodivisor. If $I = \{1, \ldots, r\}$, then $p \in Z_r = V(f_1, \ldots, f_r)$. The same reference as above implies that $(f_1, \ldots, f_r)$ is a Koszul regular ideal in A. Since the conormal sheaf has rank r, we see that $f_1, \ldots, f_r$ is a minimal set of generators of the ideal defining Z_r in $\mathcal{O}_{X,p}$. This implies that $f_1, \ldots, f_r$ is a regular sequence in $\mathcal{O}_{X,p}$ such that $\mathcal{O}_{X,p}/(f_1, \ldots, f_r)$ is regular. Thus we conclude by Algebra, Lemma 10.106.7 that $f_1, \ldots, f_r$ can be extended to a regular system of parameters in $\mathcal{O}_{X,p}$ and this finishes the proof. □

0CBS **Lemma 41.21.7.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme. If X is J-2 or Nagata, then following are equivalent*

(1) *D is a normal crossings divisor in X,*
(2) *for every $p \in D$ the pullback of D to the spectrum of the strict henselization $\mathcal{O}^{sh}_{X,p}$ is a strict normal crossings divisor.*

Proof. The implication (1) ⇒ (2) is straightforward and does not need the assumption that X is J-2 or Nagata. Namely, let $p \in D$ and choose an étale neighbourhood $(U, u) \to (X, p)$ such that the pullback of D is a strict normal crossings divisor on U. Then $\mathcal{O}^{sh}_{X,p} = \mathcal{O}^{sh}_{U,u}$ and we see that the trace of D on $\mathrm{Spec}(\mathcal{O}^{sh}_{U,u})$ is cut out by part of a regular system of parameters as this is already the case in $\mathcal{O}_{U,u}$.

To prove the implication in the other direction we will use the criterion of Lemma 41.21.6. Observe that formation of the normalization $D^\nu \to D$ commutes with strict henselization, see More on Morphisms, Lemma 37.19.4. If we can show that $D^\nu \to D$ is finite, then we see that $D^\nu \to D$ and the schemes Z_n satisfy all desired properties because these can all be checked on the level of local rings (but the finiteness of the morphism $D^\nu \to D$ is not something we can check on local rings). We omit the detailed verifications.

If X is Nagata, then $D^\nu \to D$ is finite by Morphisms, Lemma 29.54.11.

Assume X is J-2. Choose a point $p \in D$. We will show that $D^\nu \to D$ is finite over a neighbourhood of p. By assumption there exists a regular system of parameters $f_1, \ldots, f_d$ of $\mathcal{O}^{sh}_{X,p}$ and $1 \leq r \leq d$ such that the trace of D on $\mathrm{Spec}(\mathcal{O}^{sh}_{X,p})$ is cut out by $f_1 \ldots f_r$. Then

$$D^\nu \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,p}) = \coprod\nolimits_{i=1,\ldots,r} V(f_i)$$

Choose an affine étale neighbourhood $(U, u) \to (X, p)$ such that f_i comes from $f_i \in \mathcal{O}_U(U)$. Set $D_i = V(f_i) \subset U$. The strict henselization of $\mathcal{O}_{D_i,u}$ is $\mathcal{O}^{sh}_{X,p}/(f_i)$ which is regular. Hence $\mathcal{O}_{D_i,u}$ is regular (for example by More on Algebra, Lemma 15.45.10). Because X is J-2 the regular locus is open in D_i. Thus after replacing U by a Zariski open we may assume that D_i is regular for each i. It follows that

$$\coprod\nolimits_{i=1,\ldots,r} D_i = D^\nu \times_X U \longrightarrow D \times_X U$$

is the normalization morphism and it is clearly finite. In other words, we have found an étale neighbourhood (U, u) of (X, p) such that the base change of $D^\nu \to D$ to this neighbourhood is finite. This implies $D^\nu \to D$ is finite by descent (Descent, Lemma 35.23.25) and the proof is complete. □

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Part 3

Topics in Scheme Theory

CHAPTER 42

Chow Homology and Chern Classes

02P3

42.1. Introduction

02P4 In this chapter we discuss Chow homology groups and the construction of Chern classes of vector bundles as elements of operational Chow cohomology groups (everything with $\mathbf{Z}$-coefficients).

We start this chapter by giving the shortest possible algebraic proof of the Key Lemma 42.6.3. We first define the Herbrand quotient (Section 42.2) and we compute it in some cases (Section 42.3). Next, we prove some simple algebra lemmas on existence of suitable factorizations after modifications (Section 42.4). Using these we construct/define the tame symbol in Section 42.5. Only the most basic properties of the tame symbol are needed to prove the Key Lemma, which we do in Section 42.6.

Next, we introduce the basic setup we work with in the rest of this chapter in Section 42.7. To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and is endowed with a dimension function. These assumptions suffice to be able to define the Chow homology groups $\mathrm{CH}_*(X)$ and the action of capping with Chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

Next, we follow the first few chapters of [**Ful98**] in order to define cycles, flat pullback, proper pushforward, and rational equivalence, except that we have been less precise about the supports of the cycles involved.

We diverge from the presentation given in [**Ful98**] by using the Key lemma mentioned above to prove a basic commutativity relation in Section 42.27. Using this we prove that the operation of intersecting with an invertible sheaf passes through rational equivalence and is commutative, see Section 42.28. One more application of the Key lemma proves that the Gysin map of an effective Cartier divisor passes through rational equivalence, see Section 42.30. Having proved this, it is straightforward to define Chern classes of vector bundles, prove additivity, prove the splitting principle, introduce Chern characters, Todd classes, and state the Grothendieck-Riemann-Roch theorem.

There are two appendices. In Appendix A (Section 42.68) we discuss an alternative (longer) construction of the tame symbol and corresponding proof of the Key Lemma. Finally, in Appendix B (Section 42.69) we briefly discuss the relationship with K-theory of coherent sheaves and we discuss some blowup lemmas. We suggest the reader look at their introductions for more information.

We will return to the Chow groups $\mathrm{CH}_*(X)$ for smooth projective varieties over algebraically closed fields in the next chapter. Using a moving lemma as in [**Sam56**], [**Che58a**], and [**Che58b**] and Serre's Tor-formula (see [**Ser00**] or [**Ser65**]) we will define a ring structure on $\mathrm{CH}_*(X)$. See Intersection Theory, Section 43.1 ff.

42.2. Periodic complexes and Herbrand quotients

02PF Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

02PG **Definition 42.2.1.** Let R be a ring.

(1) A 2-*periodic complex* over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi) \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

(2) A $(2,1)$-*periodic complex* over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules* $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2,1)$-periodic complexes. It is clear that the collection of 2-periodic complexes forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = g \circ \varphi$ and $\psi' \circ g = f \circ \psi$. We obtain an abelian category, with kernels and cokernels as in Homology, Lemma 12.13.3.

02PH **Definition 42.2.2.** Let (M, N, φ, ψ) be a 2-periodic complex over a ring R whose cohomology modules have finite length. In this case we define the *multiplicity* of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \mathrm{length}_R(H^0(M, N, \varphi, \psi)) - \mathrm{length}_R(H^1(M, N, \varphi, \psi))$$

In the case of a $(2,1)$-periodic complex (M, φ, ψ), we denote this by $e_R(M, \varphi, \psi)$ and we will sometimes call this the *(additive) Herbrand quotient*.

If the cohomology groups of (M, φ, ψ) are finite abelian groups, then it is customary to call the *(multiplicative) Herbrand quotient*

$$q(M, \varphi, \psi) = \frac{\#H^0(M, \varphi, \psi)}{\#H^1(M, \varphi, \psi)}$$

In words: the multiplicative Herbrand quotient is the number of elements of H^0 divided by the number of elements of H^1. If R is local and if the residue field of R is finite with q elements, then we see that

$$q(M, \varphi, \psi) = q^{e_R(M,\varphi,\psi)}$$

An example of a $(2,1)$-periodic complex over a ring R is any triple of the form $(M, 0, \psi)$ where M is an R-module and ψ is an R-linear map. If the kernel and cokernel of ψ have finite length, then we obtain

0EA6 (42.2.2.1) $$e_R(M, 0, \psi) = \text{length}_R(\text{Coker}(\psi)) - \text{length}_R(\text{Ker}(\psi))$$

We state and prove the obligatory lemmas on these notations.

0EA7 **Lemma 42.2.3.** *Let R be a ring. Suppose that we have a short exact sequence of 2-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. We abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 10.52.3) we see the result. □

0EA8 **Lemma 42.2.4.** *Let R be a ring. If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length, then $e_R(M, N, \varphi, \psi) = length_R(M) - length_R(N)$. In particular, if (M, φ, ψ) is a $(2,1)$-periodic complex such that M has finite length, then $e_R(M, \varphi, \psi) = 0$.*

Proof. Observe that on the category of 2-periodic complexes with M, N of finite length the quantity "$\text{length}_R(M) - \text{length}_R(N)$" is additive in short exact sequences (precise statement left to the reader). Consider the short exact sequence

$$0 \to (M, \text{Im}(\varphi), \varphi, 0) \to (M, N, \varphi, \psi) \to (0, N/\text{Im}(\varphi), 0, 0) \to 0$$

The initial remark combined with the additivity of Lemma 42.2.3 reduces us to the cases (a) $M = 0$ and (b) φ is surjective. We leave those cases to the reader. □

0EA9 **Lemma 42.2.5.** *Let R be a ring. Let $f : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a map of $(2,1)$-periodic complexes whose cohomology modules have finite length. If $\text{Ker}(f)$ and $\text{Coker}(f)$ have finite length, then $e_R(M, \varphi, \psi) = e_R(M', \varphi', \psi')$.*

Proof. Apply the additivity of Lemma 42.2.3 and observe that $(\text{Ker}(f), \varphi, \psi)$ and $(\text{Coker}(f), \varphi', \psi')$ have vanishing multiplicity by Lemma 42.2.4. □

42.3. Calculation of some multiplicities

0EAA To prove equality of certain cycles later on we need to compute some multiplicities. Our main tool, besides the elementary lemmas on multiplicities given in the previous section, will be Algebra, Lemma 10.121.7.

02QF **Lemma 42.3.1.** *Let R be a Noetherian local ring. Let M be a finite R-module. Let $x \in R$. Assume that*

(1) $\dim(\mathit{Supp}(M)) \leq 1$, *and*
(2) $\dim(\mathit{Supp}(M/xM)) \leq 0$.

Write $\mathit{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Then

$$e_R(M, 0, x) = \sum\nolimits_{i=1,\ldots,t} \mathit{ord}_{R/\mathfrak{q}_i}(x)\mathit{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

Proof. We first make some preparatory remarks. The result of the lemma holds if M has finite length, i.e., if $t = 0$, because both the left hand side and the right hand side are zero in this case, see Lemma 42.2.4. Also, if we have a short exact sequence $0 \to M \to M' \to M'' \to 0$ of modules satisfying (1) and (2), then lemma for 2 out of 3 of these implies the lemma for the third by the additivity of length (Algebra, Lemma 10.52.3) and additivty of multiplicities (Lemma 42.2.3).

Denote M_i the image of M in $M_{\mathfrak{q}_i}$, so $\mathrm{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The kernel and cokernel of the map $M \to \bigoplus M_i$ have support $\{\mathfrak{m}\}$ and hence have finite length. By our preparatory remarks, it follows that it suffices to prove the lemma for each M_i. Thus we may assume that $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$. In this case we have a finite filtration $M \supset \mathfrak{q}M \supset \mathfrak{q}^2M \supset \ldots \supset \mathfrak{q}^nM = 0$ by Algebra, Lemma 10.62.4. Again additivity shows that it suffices to prove the lemma in the case M is annihilated by $\mathfrak{q}$. In this case we can view M as a $R/\mathfrak{q}$-module, i.e., we may assume that R is a Noetherian local domain of dimension 1 with fraction field K. Dividing by the torsion submodule, i.e., by the kernel of $M \to M \otimes_R K = V$ (the torsion has finite length hence is handled by our preliminary remarks) we may assume that $M \subset V$ is a lattice (Algebra, Definition 10.121.3). Then $x : M \to M$ is injective and $\mathrm{length}_R(M/xM) = d(M, xM)$ (Algebra, Definition 10.121.5). Since $\mathrm{length}_K(V) = \dim_K(V)$ we see that $\det(x : V \to V) = x^{\dim_K(V)}$ and $\mathrm{ord}_R(\det(x : V \to V)) = \dim_K(V)\mathrm{ord}_R(x)$. Thus the desired equality follows from Algebra, Lemma 10.121.7 in this case. □

02QG **Lemma 42.3.2.** *Let R be a Noetherian local ring. Let $x \in R$. If M is a finite Cohen-Macaulay module over R with $\dim(\mathit{Supp}(M)) = 1$ and $\dim(\mathit{Supp}(M/xM)) = 0$, then*

$$\mathit{length}_R(M/xM) = \sum\nolimits_i \mathit{length}_R(R/(x, \mathfrak{q}_i))\mathit{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M. If $I \subset R$ is an ideal such that x is a nonzerodivisor on R/I and $\dim(R/I) = 1$, then

$$\mathit{length}_R(R/(x, I)) = \sum\nolimits_i \mathit{length}_R(R/(x, \mathfrak{q}_i))\mathit{length}_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I.

Proof. These are special cases of Lemma 42.3.1. □

Here is another case where we can determine the value of a multiplicity.

0EAB **Lemma 42.3.3.** *Let R be a ring. Let M be an R-module. Let $\varphi : M \to M$ be an endomorphism and $n > 0$ such that $\varphi^n = 0$ and such that $\mathrm{Ker}(\varphi)/\mathrm{Im}(\varphi^{n-1})$ has finite length as an R-module. Then*

$$e_R(M, \varphi^i, \varphi^{n-i}) = 0$$

for $i = 0, \ldots, n$.

Proof. The cases $i = 0, n$ are trivial as $\varphi^0 = \mathrm{id}_M$ by convention. Let us think of M as an $R[t]$-module where multiplication by t is given by φ. Let us write $K_i = \mathrm{Ker}(t^i : M \to M)$ and

$$a_i = \mathrm{length}_R(K_i/t^{n-i}M), \quad b_i = \mathrm{length}_R(K_i/tK_{i+1}), \quad c_i = \mathrm{length}_R(K_1/t^iK_{i+1})$$

Boundary values are $a_0 = a_n = b_0 = c_0 = 0$. The c_i are integers for $i < n$ as K_1/t^iK_{i+1} is a quotient of $K_1/t^{n-1}M$ which is assumed to have finite length. We will use frequently that $K_i \cap t^jM = t^jK_{i+j}$. For $0 < i < n-1$ we have an exact sequence

$$0 \to K_1/t^{n-i-1}K_{n-i} \to K_{i+1}/t^{n-i-1}M \xrightarrow{t} K_i/t^{n-i}M \to K_i/tK_{i+1} \to 0$$

By induction on i we conclude that a_i and b_i are integers for $i < n$ and that

$$c_{n-i-1} - a_{i+1} + a_i - b_i = 0$$

For $0 < i < n-1$ there is a short exact sequence

$$0 \to K_i/tK_{i+1} \to K_{i+1}/tK_{i+2} \xrightarrow{t^i} K_1/t^{i+1}K_{i+2} \to K_1/t^iK_{i+1} \to 0$$

which gives

$$b_i - b_{i+1} + c_{i+1} - c_i = 0$$

Since $b_0 = c_0$ we conclude that $b_i = c_i$ for $i < n$. Then we see that

$$a_2 = a_1 + b_{n-2} - b_1, \quad a_3 = a_2 + b_{n-3} - b_2, \quad \ldots$$

It is straightforward to see that this implies $a_i = a_{n-i}$ as desired. □

0EAC **Lemma 42.3.4.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R with M finite and with cohomology groups of finite length over R. Let $x \in R$ be such that $\dim(\mathit{Supp}(M/xM)) \leq 0$. Then*

$$e_R(M, x\varphi, \psi) = e_R(M, \varphi, \psi) - e_R(\mathrm{Im}(\varphi), 0, x)$$

and

$$e_R(M, \varphi, x\psi) = e_R(M, \varphi, \psi) + e_R(\mathrm{Im}(\psi), 0, x)$$

Proof. We will only prove the first formula as the second is proved in exactly the same manner. Let $M' = M[x^\infty]$ be the x-power torsion submodule of M. Consider the short exact sequence $0 \to M' \to M \to M'' \to 0$. Then M'' is x-power torsion free (More on Algebra, Lemma 15.89.4). Since φ, ψ map M' into M' we obtain a short exact sequence

$$0 \to (M', \varphi', \psi') \to (M, \varphi, \psi) \to (M'', \varphi'', \psi'') \to 0$$

of $(2,1)$-periodic complexes. Also, we get a short exact sequence $0 \to M' \cap \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi'') \to 0$. We have $e_R(M', \varphi, \psi) = e_R(M', x\varphi, \psi) = e_R(M' \cap \mathrm{Im}(\varphi), 0, x) = 0$ by Lemma 42.2.5. By additivity (Lemma 42.2.3) we see that it suffices to prove the lemma for (M'', φ'', ψ''). This reduces us to the case discussed in the next paragraph.

Assume $x : M \to M$ is injective. In this case $\mathrm{Ker}(x\varphi) = \mathrm{Ker}(\varphi)$. On the other hand we have a short exact sequence

$$0 \to \mathrm{Im}(\varphi)/x\,\mathrm{Im}(\varphi) \to \mathrm{Ker}(\psi)/\,\mathrm{Im}(x\varphi) \to \mathrm{Ker}(\psi)/\,\mathrm{Im}(\varphi) \to 0$$

This together with (42.2.2.1) proves the formula. □

42.4. Preparation for tame symbols

0EAD In this section we put some lemma that will help us define the tame symbol in the next section.

0EAE **Lemma 42.4.1.** *Let A be a Noetherian ring. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be pairwise distinct maximal ideals of A. For $i = 1, \ldots, r$ let $\varphi_i : A_{\mathfrak{m}_i} \to B_i$ be a ring map whose kernel and cokernel are annihilated by a power of $\mathfrak{m}_i$. Then there exists a ring map $\varphi : A \to B$ such that*

(1) *the localization of φ at $\mathfrak{m}_i$ is isomorphic to φ_i, and*
(2) *$\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by a power of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$.*

Moreover, if each φ_i is finite, injective, or surjective then so is φ.

Proof. Set $I = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$. Set $A_i = A_{\mathfrak{m}_i}$ and $A' = \prod A_i$. Then $IA' = \prod \mathfrak{m}_i A_i$ and $A \to A'$ is a flat ring map such that $A/I \cong A'/IA'$. Thus we may use More on Algebra, Lemma 15.90.17 to see that there exists an A-module map $\varphi : A \to B$ with φ_i isomorphic to the localization of φ at $\mathfrak{m}_i$. Then we can use the discussion in More on Algebra, Remark 15.90.20 to endow B with an A-algebra structure matching the given A-algebra structure on B_i. The final statement of the lemma follows easily from the fact that $\mathrm{Ker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Ker}(\varphi_i)$ and $\mathrm{Coker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Coker}(\varphi_i)$. □

The following lemma is very similar to Algebra, Lemma 10.119.3.

02Q7 **Lemma 42.4.2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors. There exists a finite ring extension $R \subset R'$ with R'/R annihilated by a power of $\mathfrak{m}$ and nonzerodivisors $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.*

Proof. If a or b is a unit, then the lemma is true with $R = R'$. Thus we may assume $a, b \in \mathfrak{m}$. Set $I = (a, b)$. The idea is to blow up R in I. Instead of doing the algebraic argument we work geometrically. Let $X = \mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$. By Divisors, Lemma 31.32.4 the morphism $X \to \mathrm{Spec}(R)$ is an isomorphism over the punctured spectrum $U = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. Thus we may and do view U as an open subscheme of X. The morphism $X \to \mathrm{Spec}(R)$ is projective by Divisors, Lemma 31.32.13. Also, every generic point of X lies in U, for example by Divisors, Lemma 31.32.10. It follows from Varieties, Lemma 33.17.2 that $X \to \mathrm{Spec}(R)$ is finite. Thus $X = \mathrm{Spec}(R')$ is affine and $R \to R'$ is finite. We have $R_a \cong R'_a$ as $U = D(a)$. Hence a power of a annihilates the finite R-module R'/R. As $\mathfrak{m} = \sqrt{(a)}$ we see that R'/R is annihilated by a power of $\mathfrak{m}$. By Divisors, Lemma 31.32.4 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 10.78.7, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. □

0EAF **Lemma 42.4.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors with $a \in \mathfrak{m}$. There exists an integer $n = n(R, a, b)$ such that for a finite ring extension $R \subset R'$ if $b = a^m c$ for some $c \in R'$, then $m \leq n$.*

Proof. Choose a minimal prime $\mathfrak{q} \subset R$. Observe that $\dim(R/\mathfrak{q}) = 1$, in particular $R/\mathfrak{q}$ is not a field. We can choose a discrete valuation ring A dominating $R/\mathfrak{q}$ with the same fraction field, see Algebra, Lemma 10.119.1. Observe that a and b map to nonzero elements of A as nonzerodivisors in R are not contained in $\mathfrak{q}$. Let v be the discrete valuation on A. Then $v(a) > 0$ as $a \in \mathfrak{m}$. We claim $n = v(b)/v(a)$ works.

Let $R \subset R'$ be given. Set $A' = A \otimes_R R'$. Since $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective (Algebra, Lemma 10.36.17) also $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective (Algebra, Lemma 10.30.3). Pick a prime $\mathfrak{q}' \subset A'$ lying over $(0) \subset A$. Then $A \subset A'' = A'/\mathfrak{q}'$ is a finite extension of rings (again inducing a surjection on spectra). Pick a maximal ideal $\mathfrak{m}'' \subset A''$ lying over the maximal ideal of A and a discrete valuation ring A''' dominating $A''_{\mathfrak{m}''}$ (see lemma cited above). Then $A \to A'''$ is an extension of discrete valuation rings and we have $b = a^m c$ in A'''. Thus $v'''(b) \geq mv'''(a)$. Since $v''' = ev$ where e is the ramification index of A'''/A, we find that $m \leq n$ as desired. □

0EAG **Lemma 42.4.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $r \geq 2$ and let $a_1, \ldots, a_r \in A$ be nonzerodivisors not all units. Then there exist*

1. *a finite ring extension $A \subset B$ with B/A annihilated by a power of $\mathfrak{m}$,*
2. *for each maximal ideal $\mathfrak{m}_j \subset B$ a nonzerodivisor $\pi_j \in B_j = B_{\mathfrak{m}_j}$, and*
3. *factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ in B_j with $u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$.*

Proof. Since at least one a_i is not a unit we find that $\mathfrak{m}$ is not an associated prime of A. Moreover, for any $A \subset B$ as in the statement $\mathfrak{m}$ is not an associated prime of B and $\mathfrak{m}_j$ is not an associate prime of B_j. Keeping this in mind will help check the arguments below.

First, we claim that it suffices to prove the lemma for $r = 2$. We will argue this by induction on r; we suggest the reader skip the proof. Suppose we are given $A \subset B$ and π_j in $B_j = B_{\mathfrak{m}_j}$ and factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ for $i = 1, \ldots, r-1$ in B_j with $u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$. Then by the case $r = 2$ for π_j and a_r in B_j we can find extensions $B_j \subset C_j$ and for every maximal ideal $\mathfrak{m}_{j,k}$ of C_j a nonzerodivisor $\pi_{j,k} \in C_{j,k} = (C_j)_{\mathfrak{m}_{j,k}}$ and factorizations

$$\pi_j = v_{j,k}\pi_{j,k}^{f_{j,k}} \quad \text{and} \quad a_r = w_{j,k}\pi_{j,k}^{g_{j,k}}$$

as in the lemma. There exists a unique finite extension $B \subset C$ with C/B annihilated by a power of $\mathfrak{m}$ such that $C_j \cong C_{\mathfrak{m}_j}$ for all j, see Lemma 42.4.1. The maximal ideals of C correspond 1-to-1 to the maximal ideals $\mathfrak{m}_{j,k}$ in the localizations and in these localizations we have

$$a_i = u_{i,j}\pi_j^{e_{i,j}} = u_{i,j}v_{j,k}^{e_{i,j}}\pi_{j,k}^{e_{i,j}f_{j,k}}$$

for $i \leq r-1$. Since a_r factors correctly too the proof of the induction step is complete.

Proof of the case $r = 2$. We will use induction on

$$\ell = \min(\mathrm{length}_A(A/a_1A),\ \mathrm{length}_A(A/a_2A)).$$

If $\ell = 0$, then either a_1 or a_2 is a unit and the lemma holds with $A = B$. Thus we may and do assume $\ell > 0$.

Suppose we have a finite extension of rings $A \subset A'$ such that A'/A is annihilated by a power of $\mathfrak{m}$ and such that $\mathfrak{m}$ is not an associated prime of A'. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \subset A'$ be the maximal ideals and set $A'_i = A'_{\mathfrak{m}_i}$. If we can solve the problem for a_1, a_2

in each A'_i, then we can apply Lemma 42.4.1 to produce a solution for a_1, a_2 in A. Choose $x \in \{a_1, a_2\}$ such that $\ell = \text{length}_A(A/xA)$. By Lemma 42.2.5 and (42.2.2.1) we have $\text{length}_A(A/xA) = \text{length}_A(A'/xA')$. On the other hand, we have

$$\text{length}_A(A'/xA') = \sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \text{length}_{A'_i}(A'_i/xA'_i)$$

by Algebra, Lemma 10.52.12. Since $x \in \mathfrak{m}$ we see that each term on the right hand side is positive. We conclude that the induction hypothesis applies to a_1, a_2 in each A'_i if $r > 1$ or if $r = 1$ and $[\kappa(\mathfrak{m}_1) : \kappa(\mathfrak{m})] > 1$. We conclude that we may assume each A' as above is local with the same residue field as A.

Applying the discussion of the previous paragraph, we may replace A by the ring constructed in Lemma 42.4.2 for $a_1, a_2 \in A$. Then since A is local we find, after possibly switching a_1 and a_2, that $a_2 \in (a_1)$. Write $a_2 = a_1^m c$ with $m > 0$ maximal. In fact, by Lemma 42.4.3 we may assume m is maximal even after replacing A by any finite extension $A \subset A'$ as in the previous paragraph. If c is a unit, then we are done. If not, then we replace A by the ring constructed in Lemma 42.4.2 for $a_1, c \in A$. Then either (1) $c = a_1 c'$ or (2) $a_1 = ca'_1$. The first case cannot happen since it would give $a_2 = a_1^{m+1} c'$ contradicting the maximality of m. In the second case we get $a_1 = ca'_1$ and $a_2 = c^{m+1}(a'_1)^m$. Then it suffices to prove the lemma for A and c, a'_1. If a'_1 is a unit we're done and if not, then $\text{length}_A(A/cA) < \ell$ because cA is a strictly bigger ideal than $a_1 A$. Thus we win by induction hypothesis. □

42.5. Tame symbols

0EAH Consider a Noetherian local ring $(A, \mathfrak{m})$ of dimension 1. We denote $Q(A)$ the total ring of fractions of A, see Algebra, Example 10.9.8. The *tame symbol* will be a map

$$\partial_A(-,-) : Q(A)^* \times Q(A)^* \longrightarrow \kappa(\mathfrak{m})^*$$

satisfying the following properties:

0EAI (1) $\partial_A(f, gh) = \partial_A(f, g)\partial_A(f, h)$ for $f, g, h \in Q(A)^*$,

0EAJ (2) $\partial_A(f, g)\partial_A(g, f) = 1$ for $f, g \in Q(A)^*$,

0EAK (3) $\partial_A(f, 1 - f) = 1$ for $f \in Q(A)^*$ such that $1 - f \in Q(A)^*$,

0EAL (4) $\partial_A(aa', b) = \partial_A(a, b)\partial_A(a', b)$ and $\partial_A(a, bb') = \partial_A(a, b)\partial_A(a, b')$ for $a, a', b, b' \in A$ nonzerodivisors,

0EAM (5) $\partial_A(b, b) = (-1)^m$ with $m = \text{length}_A(A/bA)$ for $b \in A$ a nonzerodivisor,

0EAN (6) $\partial_A(u, b) = u^m \bmod \mathfrak{m}$ with $m = \text{length}_A(A/bA)$ for $u \in A$ a unit and $b \in A$ a nonzerodivisor, and

0EAP (7) $\partial_A(a, b - a)\partial_A(b, b) = \partial_A(b, b - a)\partial_A(a, b)$ for $a, b \in A$ such that $a, b, b - a$ are nonzerodivisors.

Since it is easier to work with elements of A we will often think of ∂_A as a map defined on pairs of nonzerodivisors of A satisfying (4), (5), (6), (7). It is an exercise to see that setting

$$\partial_A(\frac{a}{b}, \frac{c}{d}) = \partial_A(a, c)\partial_A(a, d)^{-1}\partial_A(b, c)^{-1}\partial_A(b, d)$$

we get a well defined map $Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3) as well as the other properties.

We do not claim there is a unique map with these properties. Instead, we will give a recipe for constructing such a map. Namely, given $a_1, a_2 \in A$ nonzerodivisors, we

choose a ring extension $A \subset B$ and local factorizations as in Lemma 42.4.4. Then we define

0EAQ (42.5.0.1) $$\partial_A(a_1, a_2) = \prod\nolimits_j \mathrm{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j}$$

where $m_j = \mathrm{length}_{B_j}(B_j/\pi_j B_j)$ and the product is taken over the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of B.

0EAR **Lemma 42.5.1.** *The formula (42.5.0.1) determines a well defined element of $\kappa(\mathfrak{m})^*$. In other words, the right hand side does not depend on the choice of the local factorizations or the choice of B.*

Proof. Independence of choice of factorizations. Suppose we have a Noetherian 1-dimensional local ring B, elements $a_1, a_2 \in B$, and nonzerodivisors π, θ such that we can write

$$a_1 = u_1\pi^{e_1} = v_1\theta^{f_1}, \quad a_2 = u_2\pi^{e_2} = v_2\theta^{f_2}$$

with $e_i, f_i \geq 0$ integers and u_i, v_i units in B. Observe that this implies

$$a_1^{e_2} = u_1^{e_2}u_2^{-e_1}a_2^{e_1}, \quad a_1^{f_2} = v_1^{f_2}v_2^{-f_1}a_2^{f_1}$$

On the other hand, setting $m = \mathrm{length}_B(B/\pi B)$ and $k = \mathrm{length}_B(B/\theta B)$ we find $e_2 m = \mathrm{length}_B(B/a_2B) = f_2 k$. Expanding $a_1^{e_2 m} = a_1^{f_2 k}$ using the above we find

$$(u_1^{e_2}u_2^{-e_1})^m = (v_1^{f_2}v_2^{-f_1})^k$$

This proves the desired equality up to signs. To see the signs work out we have to show me_1e_2 is even if and only if kf_1f_2 is even. This follows as both $me_2 = kf_2$ and $me_1 = kf_1$ (same argument as above).

Independence of choice of B. Suppose given two extensions $A \subset B$ and $A \subset B'$ as in Lemma 42.4.4. Then

$$C = (B \otimes_A B')/(\mathfrak{m}\text{-power torsion})$$

will be a third one. Thus we may assume we have $A \subset B \subset C$ and factorizations over the local rings of B and we have to show that using the same factorizations over the local rings of C gives the same element of $\kappa(\mathfrak{m})$. By transitivity of norms (Fields, Lemma 9.20.5) this comes down to the following problem: if B is a Noetherian local ring of dimension 1 and $\pi \in B$ is a nonzerodivisor, then

$$\lambda^m = \prod \mathrm{Norm}_{\kappa_k/\kappa}(\lambda)^{m_k}$$

Here we have used the following notation: (1) κ is the residue field of B, (2) λ is an element of κ, (3) $\mathfrak{m}_k \subset C$ are the maximal ideals of C, (4) $\kappa_k = \kappa(\mathfrak{m}_k)$ is the residue field of $C_k = C_{\mathfrak{m}_k}$, (5) $m = \mathrm{length}_B(B/\pi B)$, and (6) $m_k = \mathrm{length}_{C_k}(C_k/\pi C_k)$. The displayed equality holds because $\mathrm{Norm}_{\kappa_k/\kappa}(\lambda) = \lambda^{[\kappa_k:\kappa]}$ as $\lambda \in \kappa$ and because $m = \sum m_k[\kappa_k : \kappa]$. First, we have $m = \mathrm{length}_B(B/xB) = \mathrm{length}_B(C/\pi C)$ by Lemma 42.2.5 and (42.2.2.1). Finally, we have $\mathrm{length}_B(C/\pi C) = \sum m_k[\kappa_k : \kappa]$ by Algebra, Lemma 10.52.12. □

0EAS **Lemma 42.5.2.** *The tame symbol (42.5.0.1) satisfies (4), (5), (6), (7) and hence gives a map $\partial_A : Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3).*

Proof. Let us prove (4). Let $a_1, a_2, a_3 \in A$ be nonzerodivisors. Choose $A \subset B$ as in Lemma 42.4.4 for a_1, a_2, a_3. Then the equality

$$\partial_A(a_1a_2, a_3) = \partial_A(a_1, a_3)\partial_A(a_2, a_3)$$

follows from the equality

$$(-1)^{(e_{1,j}+e_{2,j})e_{3,j}}(u_{1,j}u_{2,j})^{e_{3,j}}u_{3,j}^{-e_{1,j}-e_{2,j}} = (-1)^{e_{1,j}e_{3,j}}u_{1,j}^{e_{3,j}}u_{3,j}^{-e_{1,j}}(-1)^{e_{2,j}e_{3,j}}u_{2,j}^{e_{3,j}}u_{3,j}^{-e_{2,j}}$$

in B_j. Properties (5) and (6) are equally immediate.

Let us prove (7). Let $a_1, a_2, a_1 - a_2 \in A$ be nonzerodivisors and set $a_3 = a_1 - a_2$. Choose $A \subset B$ as in Lemma 42.4.4 for a_1, a_2, a_3. Then it suffices to show

$$(-1)^{e_{1,j}e_{2,j}+e_{1,j}e_{3,j}+e_{2,j}e_{3,j}+e_{2,j}}u_{1,j}^{e_{2,j}-e_{3,j}}u_{2,j}^{e_{3,j}-e_{1,j}}u_{3,j}^{e_{1,j}-e_{2,j}} \bmod \mathfrak{m}_j = 1$$

This is clear if $e_{1,j} = e_{2,j} = e_{3,j}$. Say $e_{1,j} > e_{2,j}$. Then we see that $e_{3,j} = e_{2,j}$ because $a_3 = a_1 - a_2$ and we see that $u_{3,j}$ has the same residue class as $-u_{2,j}$. Hence the formula is true – the signs work out as well and this verification is the reason for the choice of signs in (42.5.0.1). The other cases are handled in exactly the same manner. □

0EAT **Lemma 42.5.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $A \subset B$ be a finite ring extension with B/A annihilated by a power of $\mathfrak{m}$ and $\mathfrak{m}$ not an associated prime of B. For $a, b \in A$ nonzerodivisors we have*

$$\partial_A(a, b) = \prod Norm_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}(\partial_{B_j}(a, b))$$

where the product is over the maximal ideals $\mathfrak{m}_j$ of B and $B_j = B_{\mathfrak{m}_j}$.

Proof. Choose $B_j \subset C_j$ as in Lemma 42.4.4 for a, b. By Lemma 42.4.1 we can choose a finite ring extension $B \subset C$ with $C_j \cong C_{\mathfrak{m}_j}$ for all j. Let $\mathfrak{m}_{j,k} \subset C$ be the maximal ideals of C lying over $\mathfrak{m}_j$. Let

$$a = u_{j,k}\pi_{j,k}^{f_{j,k}}, \quad b = v_{j,k}\pi_{j,k}^{g_{j,k}}$$

be the local factorizations which exist by our choice of $C_j \cong C_{\mathfrak{m}_j}$. By definition we have

$$\partial_A(a, b) = \prod\nolimits_{j,k} \mathrm{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m})}((-1)^{f_{j,k}g_{j,k}}u_{j,k}^{g_{j,k}}v_{j,k}^{-f_{j,k}} \bmod \mathfrak{m}_{j,k})^{m_{j,k}}$$

and

$$\partial_{B_j}(a, b) = \prod\nolimits_{k} \mathrm{Norm}_{\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)}((-1)^{f_{j,k}g_{j,k}}u_{j,k}^{g_{j,k}}v_{j,k}^{-f_{j,k}} \bmod \mathfrak{m}_{j,k})^{m_{j,k}}$$

The result follows by transitivity of norms for $\kappa(\mathfrak{m}_{j,k})/\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})$, see Fields, Lemma 9.20.5. □

0EPG **Lemma 42.5.4.** *Let $(A, \mathfrak{m}, \kappa) \to (A', \mathfrak{m}', \kappa')$ be a local homomorphism of Noetherian local rings. Assume $A \to A'$ is flat and $\dim(A) = \dim(A') = 1$. Set $m = length_{A'}(A'/\mathfrak{m}A')$. For $a_1, a_2 \in A$ nonzerodivisors $\partial_A(a_1, a_2)^m$ maps to $\partial_{A'}(a_1, a_2)$ via $\kappa \to \kappa'$.*

Proof. If a_1, a_2 are both units, then $\partial_A(a_1, a_2) = 1$ and $\partial_{A'}(a_1, a_2) = 1$ and the result is true. If not, then we can choose a ring extension $A \subset B$ and local factorizations as in Lemma 42.4.4. Denote $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be the maximal ideals of B. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be the maximal ideals of B with residue fields $\kappa_1, \ldots, \kappa_m$. For

each $j \in \{1, \ldots, m\}$ denote $\pi_j \in B_j = B_{\mathfrak{m}_j}$ a nonzerodivisor such that we have factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ as in the lemma. By definition we have

$$\partial_A(a_1, a_2) = \prod\nolimits_j \mathrm{Norm}_{\kappa_j/\kappa}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j}$$

where $m_j = \mathrm{length}_{B_j}(B_j/\pi_j B_j)$.

Set $B' = A' \otimes_A B$. Since A' is flat over A we see that $A' \subset B'$ is a ring extension with B'/A' annihilated by a power of $\mathfrak{m}'$. Let

$$\mathfrak{m}'_{j,l}, \quad l = 1, \ldots, n_j$$

be the maximal ideals of B' lying over $\mathfrak{m}_j$. Denote $\kappa'_{j,l}$ the residue field of $\mathfrak{m}'_{j,l}$. Denote $B'_{j,l}$ the localization of B' at $\mathfrak{m}'_{j,l}$. As factorizations of a_1 and a_2 in $B'_{j,l}$ we use the image of the factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ given to us in B_j. By definition we have

$$\partial_{A'}(a_1, a_2) = \prod\nolimits_{j,l} \mathrm{Norm}_{\kappa'_{j,l}/\kappa'}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}'_{j,l})^{m'_{j,l}}$$

where $m'_{j,l} = \mathrm{length}_{B'_{j,l}}(B'_{j,l}/\pi_j B'_{j,l})$.

Comparing the formulae above we see that it suffices to show that for each j and for any unit $u \in B_j$ we have

0GU1 (42.5.4.1) $$\left(\mathrm{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak{m}_j)^{m_j}\right)^m = \prod\nolimits_l \mathrm{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak{m}'_{j,l})^{m'_{j,l}}$$

in κ'. We are going to use the construction of determinants of endomorphisms of finite length modules in More on Algebra, Section 15.121 to prove this. Set $M = B_j/\pi_j B_j$. By More on Algebra, Lemma 15.121.2 we have

$$\mathrm{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak{m}_j)^{m_j} = \det_\kappa(u : M \to M)$$

Thus, by More on Algebra, Lemma 15.121.3, the left hand side of (42.5.4.1) is equal to $\det_{\kappa'}(u : M \otimes_A A' \to M \otimes_A A')$. We have an isomorphism

$$M \otimes_A A' = (B_j/\pi_j B_j) \otimes_A A' = \bigoplus\nolimits_l B'_{j,l}/\pi_j B'_{j,l}$$

of A'-modules. Setting $M'_l = B'_{j,l}/\pi_j B'_{j,l}$ we see that $\mathrm{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak{m}'_{j,l})^{m'_{j,l}} = \det_{\kappa'}(u_j : M'_l \to M'_l)$ by More on Algebra, Lemma 15.121.2 again. Hence (42.5.4.1) holds by multiplicativity of the determinant construction, see More on Algebra, Lemma 15.121.1. □

42.6. A key lemma

0EAU In this section we apply the results above to prove Lemma 42.6.3. This lemma is a low degree case of the statement that there is a complex for Milnor K-theory similar to the Gersten-Quillen complex in Quillen's K-theory. See Remark 42.6.4.

0EAV **Lemma 42.6.1.** *Let $(A, \mathfrak{m})$ be a 2-dimensional Noetherian local ring. Let $t \in \mathfrak{m}$ be a nonzerodivisor. Say $V(t) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. Let $A_{\mathfrak{q}_i} \subset B_i$ be a finite ring extension with $B_i/A_{\mathfrak{q}_i}$ annihilated by a power of t. Then there exists a finite extension $A \subset B$ of local rings identifying residue fields with $B_i \cong B_{\mathfrak{q}_i}$ and B/A annihilated by a power of t.*

Proof. Choose $n > 0$ such that $B_i \subset t^{-n}A_{\mathfrak{q}_i}$. Let $M \subset t^{-n}A$, resp. $M' \subset t^{-2n}A$ be the A-submodule consisting of elements mapping to B_i in $t^{-n}A_{\mathfrak{q}_i}$, resp. $t^{-2n}A_{\mathfrak{q}_i}$. Then $M \subset M'$ are finite A-modules as A is Noetherian and $M_{\mathfrak{q}_i} = M'_{\mathfrak{q}_i} = B_i$ as localization is exact. Thus M'/M is annihilated by $\mathfrak{m}^c$ for some $c > 0$. Observe that $M \cdot M \subset M'$ under the multiplication $t^{-n}A \times t^{-n}A \to t^{-2n}A$. Hence $B = A + \mathfrak{m}^{c+1}M$ is a finite A-algebra with the correct localizations. We omit the verification that B is local with maximal ideal $\mathfrak{m} + \mathfrak{m}^{c+1}M$. □

0EAW **Lemma 42.6.2.** *Let $(A, \mathfrak{m})$ be a 2-dimensional Noetherian local ring. Let $a, b \in A$ be nonzerodivisors. Then we have*

$$\sum ord_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(a, b)) = 0$$

where the sum is over the height 1 primes $\mathfrak{q}$ of A.

Proof. If $\mathfrak{q}$ is a height 1 prime of A such that a, b map to a unit of $A_\mathfrak{q}$, then $\partial_{A_\mathfrak{q}}(a, b) = 1$. Thus the sum is finite. In fact, if $V(ab) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$ then the sum is over $i = 1, \ldots, r$. For each i we pick an extension $A_{\mathfrak{q}_i} \subset B_i$ as in Lemma 42.4.4 for a, b. By Lemma 42.6.1 with $t = ab$ and the given list of primes we may assume we have a finite local extension $A \subset B$ with B/A annihilated by a power of ab and such that for each i the $B_{\mathfrak{q}_i} \cong B_i$. Observe that if $\mathfrak{q}_{i,j}$ are the primes of B lying over $\mathfrak{q}_i$ then we have

$$\mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) = \sum\nolimits_j \mathrm{ord}_{B/\mathfrak{q}_{i,j}}(\partial_{B_{\mathfrak{q}_{i,j}}}(a, b))$$

by Lemma 42.5.3 and Algebra, Lemma 10.121.8. Thus we may replace A by B and reduce to the case discussed in the next paragraph.

Assume for each i there is a nonzerodivisor $\pi_i \in A_{\mathfrak{q}_i}$ and units $u_i, v_i \in A_{\mathfrak{q}_i}$ such that for some integers $e_i, f_i \geq 0$ we have

$$a = u_i\pi_i^{e_i}, \quad b = v_i\pi_i^{f_i}$$

in $A_{\mathfrak{q}_i}$. Setting $m_i = \mathrm{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}/\pi_i)$ we have $\partial_{A_{\mathfrak{q}_i}}(a, b) = ((-1)^{e_i f_i} u_i^{f_i} v_i^{-e_i})^{m_i}$ by definition. Since a, b are nonzerodivisors the $(2, 1)$-periodic complex $(A/(ab), a, b)$ has vanishing cohomology. Denote M_i the image of $A/(ab)$ in $A_{\mathfrak{q}_i}/(ab)$. Then we have a map

$$A/(ab) \longrightarrow \bigoplus M_i$$

whose kernel and cokernel are supported in $\{\mathfrak{m}\}$ and hence have finite length. Thus we see that

$$\sum e_A(M_i, a, b) = 0$$

by Lemma 42.2.5. Hence it suffices to show $e_A(M_i, a, b) = -\mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b))$.

Let us prove this first, in case π_i, u_i, v_i are the images of elements $\pi_i, u_i, v_i \in A$ (using the same symbols should not cause any confusion). In this case we get

$$\begin{aligned} e_A(M_i, a, b) &= e_A(M_i, u_i\pi_i^{e_i}, v_i\pi_i^{f_i}) \\ &= e_A(M_i, \pi_i^{e_i}, \pi_i^{f_i}) - e_A(\pi_i^{e_i}M_i, 0, u_i) + e_A(\pi_i^{f_i}M_i, 0, v_i) \\ &= 0 - f_i m_i \mathrm{ord}_{A/\mathfrak{q}_i}(u_i) + e_i m_i \mathrm{ord}_{A/\mathfrak{q}_i}(v_i) \\ &= -m_i \mathrm{ord}_{A/\mathfrak{q}_i}(u_i^{f_i} v_i^{-e_i}) = -\mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) \end{aligned}$$

The second equality holds by Lemma 42.3.4. Observe that $M_i \subset (M_i)_{\mathfrak{q}_i} = A_{\mathfrak{q}_i}/(\pi_i^{e_i+f_i})$ and $(\pi_i^{e_i}M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{f_i}$ and $(\pi_i^{f_i}M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{e_i}$. The 0 in the third equality

comes from Lemma 42.3.3 and the other two terms come from Lemma 42.3.1. The last two equalities follow from multiplicativity of the order function and from the definition of our tame symbol.

In general, we may first choose $c \in A$, $c \notin \mathfrak{q}_i$ such that $c\pi_i \in A$. After replacing π_i by $c\pi_i$ and u_i by $c^{-e_i}u_i$ and v_i by $c^{-f_i}v_i$ we may and do assume π_i is in A. Next, choose an $c \in A$, $c \notin \mathfrak{q}_i$ with $cu_i, cv_i \in A$. Then we observe that

$$e_A(M_i, ca, cb) = e_A(M_i, a, b) - e_A(aM_i, 0, c) + e_A(bM_i, 0, c)$$

by Lemma 42.3.1. On the other hand, we have

$$\partial_{A_{\mathfrak{q}_i}}(ca, cb) = c^{m_i(f_i - e_i)}\partial_{A_{\mathfrak{q}_i}}(a, b)$$

in $\kappa(\mathfrak{q}_i)^*$ because c is a unit in $A_{\mathfrak{q}_i}$. The arguments in the previous paragraph show that $e_A(M_i, ca, cb) = -\mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(ca, cb))$. Thus it suffices to prove

$$e_A(aM_i, 0, c) = \mathrm{ord}_{A/\mathfrak{q}_i}(c^{m_i f_i}) \quad \text{and} \quad e_A(bM_i, 0, c) = \mathrm{ord}_{A/\mathfrak{q}_i}(c^{m_i e_i})$$

and this follows from Lemma 42.3.1 by the description (see above) of what happens when we localize at $\mathfrak{q}_i$. □

0EAX **Lemma 42.6.3** (Key Lemma)**.** *Let A be a 2-dimensional Noetherian local domain with fraction field K. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A^*_{\mathfrak{q}}$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(\partial_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height 1 prime $\mathfrak{q}$ of A where $f, g \in A^*_{\mathfrak{q}}$ we have $\partial_{A_{\mathfrak{q}}}(f, g) = 1$.

When A is an excellent ring this is [**Kat86**, Proposition 1].

Proof. Since the tame symbols $\partial_{A_{\mathfrak{q}}}(f, g)$ are bilinear and the order functions $\mathrm{ord}_{A/\mathfrak{q}}$ are additive it suffices to prove the formula when f and g are elements of A. This case is proven in Lemma 42.6.2. □

0EAY **Remark 42.6.4** (Milnor K-theory)**.** For a field k let us denote $K^M_*(k)$ the quotient of the tensor algebra on k^* divided by the two-sided ideal generated by the elements $x \otimes 1 - x$ for $x \in k \setminus \{0, 1\}$. Thus $K^M_0(k) = \mathbf{Z}$, $K^M_1(k) = k^*$, and

$$K^M_2(k) = k^* \otimes_{\mathbf{Z}} k^*/\langle x \otimes 1 - x \rangle$$

If A is a discrete valuation ring with fraction field $F = \mathrm{Frac}(A)$ and residue field κ, there is a tame symbol

$$\partial_A : K^M_{i+1}(F) \to K^M_i(\kappa)$$

defined as in Section 42.5; see [**Kat86**]. More generally, this map can be extended to the case where A is an excellent local domain of dimension 1 using normalization and norm maps on K^M_i, see [**Kat86**]; presumably the method in Section 42.5 can be used to extend the construction of the tame symbol ∂_A to arbitrary Noetherian local domains A of dimension 1. Next, let X be a Noetherian scheme with a dimension function δ. Then we can use these tame symbols to get the arrows in the following:

$$\bigoplus\nolimits_{\delta(x)=j+1} K^M_{i+1}(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j} K^M_i(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j-1} K^M_{i-1}(\kappa(x))$$

However, it is not clear, that the composition is zero, i.e., that we obtain a complex of abelian groups. For excellent X this is shown in [**Kat86**]. When $i = 1$ and j arbitrary, this follows from Lemma 42.6.3.

42.7. Setup

02QK We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. It is exactly the generality discussed in [**Tho90**]. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

02QL **Situation 42.7.1.** Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 29.17.1 for the notion of a universally catenary scheme, and see Topology, Definition 5.20.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 28.11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 29.17.5.

Let (S, δ) be as in Situation 42.7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \mathrm{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 29.52.3. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \mathrm{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

02QM **Example 42.7.2.** Here $S = \mathrm{Spec}(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 42.7.1 by Morphisms, Lemma 29.17.5.

02QN **Example 42.7.3.** Here $S = \mathrm{Spec}(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a maximal ideal and $\delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = (0)$ corresponds to the generic point. This is an example of Situation 42.7.1 by Morphisms, Lemma 29.17.5.

0F91 **Example 42.7.4.** Here S is a Cohen-Macaulay scheme. Then S is universally catenary by Morphisms, Lemma 29.17.5. We set $\delta(s) = -\dim(\mathcal{O}_{S,s})$. If $s' \rightsquigarrow s$ is a nontrivial specialization of points of S, then $\mathcal{O}_{S,s'}$ is the localization of $\mathcal{O}_{S,s}$ at a nonmaximal prime ideal $\mathfrak{p} \subset \mathcal{O}_{S,s}$, see Schemes, Lemma 26.13.2. Thus $\dim(\mathcal{O}_{S,s}) = \dim(\mathcal{O}_{S,s'}) + \dim(\mathcal{O}_{S,s}/\mathfrak{p}) > \dim(\mathcal{O}_{S,s'})$ by Algebra, Lemma 10.104.4. Hence $\delta(s') > \delta(s)$. If $s' \rightsquigarrow s$ is an immediate specialization, then there is no prime

ideal strictly between $\mathfrak{p}$ and $\mathfrak{m}_s$ and we find $\delta(s') = \delta(s) + 1$. Thus δ is a dimension function. In other words, the pair (S, δ) is an example of Situation 42.7.1.

If S is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

02QO **Lemma 42.7.5.** *Let (S, δ) be as in Situation 42.7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:*

(1) $\delta_{X/S}(\xi)$,
(2) $\dim(Z)$, *and*
(3) $\dim(\mathcal{O}_{Z,z})$ *where z is a closed point of Z.*

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 29.16.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 28.10.3) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z)$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z)/\kappa(f(z))$ is finite, see Morphisms, Lemma 29.16.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = \mathrm{Spec}(\mathbf{C}[[t]])$ and $X = \mathrm{Spec}(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

02QP **Definition 42.7.6.** Let (S, δ) as in Situation 42.7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the *δ-dimension of Z*. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

42.8. Cycles

02QQ Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

02QR **Definition 42.8.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A *cycle on X* is a formal sum

$$\alpha = \sum n_Z [Z]$$

where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite (Topology, Definition 5.28.4).

(2) A *k-cycle* on X is a cycle
$$\alpha = \sum n_Z [Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(3) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

0GU2 **Remark 42.8.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Then we can write
$$Z_k(X) = \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \quad \subset \quad \bigoplus\nolimits_{\delta(x)=k} K_0^M(\kappa(x))$$
with the following notation and conventions:

(1) $K_0^M(\kappa(x)) = \mathbf{Z}$ is the degree 0 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4), and

(2) the direct sum on the right is over all points $x \in X$ with $\delta(x) = k$,

(3) the notation $\bigoplus'_x$ signifies that we consider the subgroup consisting of locally finite elements; namely, elements $\sum_x n_x$ such that for every quasi-compact open $U \subset X$ the set of $x \in U$ with $n_x \neq 0$ is finite.

0H46 **Definition 42.8.3.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The *support* of a cycle $\alpha = \sum n_Z[Z]$ on X is
$$\mathrm{Supp}(\alpha) = \bigcup\nolimits_{n_Z \neq 0} Z \subset X$$

Since the collection $\{Z; n_Z \neq 0\}$ is locally finite we see that $\mathrm{Supp}(\alpha)$ is a closed subset of X. If α is a k-cycle, then every irreducible component Z of $\mathrm{Supp}(\alpha)$ has δ-dimension k.

0H47 **Definition 42.8.4.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. A cycle α on X is *effective* if it can be written as $\alpha = \sum n_Z[Z]$ with $n_Z \geq 0$ for all Z.

The set of all effective cycles is a monoid because the sum of two effective cycles is effective, but it is not a group (unless $X = \emptyset$).

42.9. Cycle associated to a closed subscheme

02QS

02QT **Lemma 42.9.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.*

(1) *Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$\mathit{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$

(2) *If* $\dim_\delta(Z) \leq k$ *and* $\xi \in Z$ *with* $\delta(\xi) = k$*, then* ξ *is a generic point of an irreducible component of* Z*.*

Proof. Let $Z' \subset Z$, $\xi \in Z'$ be as in (1). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 28.10.3). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 10.60.7). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 10.52.5.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (2) holds. □

02QU **Definition 42.9.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 42.9.1) is called the *multiplicity of* Z' *in* Z.

(2) Assume $\dim_\delta(Z) \leq k$. The *k-cycle associated to Z* is

$$[Z]_k = \sum m_{Z',Z}[Z']$$

where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Divisors, Lemma 31.26.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

42.10. Cycle associated to a coherent sheaf

02QV

02QW **Lemma 42.10.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.*

(2) *Let $Z' \subset Support(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*

$$length_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$

(3) *If $\dim_\delta(Support(\mathcal{F})) \leq k$ and $\xi \in Support(\mathcal{F})$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $Support(\mathcal{F})$.*

Proof. By Cohomology of Schemes, Lemma 30.9.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 26.12.4). Hence (1) follows from Divisors, Lemma 31.26.1 applied to Z and (3) follows from Lemma 42.9.1 applied to Z.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 26.13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 10.62.3. □

02QX **Definition 42.10.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For any irreducible component $Z' \subset \text{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 42.10.1) is called the *multiplicity of Z' in $\mathcal{F}$*.
(2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is
$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$
where the sum is over the irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 42.10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

02QY **Lemma 42.10.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.*

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. □

02QZ **Lemma 42.10.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 10.52.3. □

42.11. Preparation for proper pushforward

02R0

02R1 **Lemma 42.11.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.*

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible (Topology, Lemmas 5.8.2 and 5.8.3). If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 29.8.7 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence $R(Y) \subset R(X)$ is a finite extension by Morphisms, Lemma 29.51.7 (which applies by Morphisms, Lemma 29.15.8). □

02R2 **Lemma 42.11.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set
$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$
the lemma is proved. □

42.12. Proper pushforward

02R3

02R4 **Definition 42.12.1.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if} \quad \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 42.11.1.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 42.11.2 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

02R5 **Lemma 42.12.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(W) \subset Y$ and $W'' = g(f(W)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (g \circ f)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 42.11.1. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (g \circ f)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 29.51.9 to conclude. □

A closed immersion is proper. If $i : Z \to X$ is a closed immersion then the maps

$$i_* : Z_k(Z) \longrightarrow Z_k(X)$$

are all *injective*.

0F92 **Lemma 42.12.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups*

$$Z_k(X_1 \cap X_2) \longrightarrow Z_k(X_1) \oplus Z_k(X_2) \longrightarrow Z_k(X) \longrightarrow 0$$

is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. The groups $Z_k(X_1)$, $Z_k(X_2)$, $Z_k(X_1 \cap X_2)$ are free on the subset of these Z such that $Z \subset X_1$, $Z \subset X_2$, $Z \subset X_1 \cap X_2$. This immediately proves the lemma in this case. The general case is similar and the proof is omitted. □

02R6 **Lemma 42.12.4.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a proper morphism of schemes which are locally of finite type over S.*

(1) *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then*
$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$
(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*
$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Schemes, Proposition 30.19.1.

Proof. Part (1) follows from (2) and Lemma 42.10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Cohomology of Schemes, Lemma 30.9.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$. Consider the commutative diagram of schemes
$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f|_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{i'} & Y \end{array}$$
We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that
$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$
as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 29.6.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions
$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$
whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 42.9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 29.51.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak{p} \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\mathrm{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak{p}$. Then $A_\mathfrak{p} = \prod A_{\mathfrak{q}_i}$ since $A_\mathfrak{p}$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak{p}$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 10.52.12. □

42.13. Preparation for flat pullback

02R7 Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 29.29.1.

02R8 **Lemma 42.13.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

provided $f^{-1}(Z)$ is nonempty. If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 29.25.10 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 29.28.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

02R9 **Lemma 42.13.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of X.*

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

42.14. Flat pullback

02RA In the following we use $f^{-1}(Z)$ to denote the *scheme theoretic inverse image* of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 26.4 and Lemma 26.17.6 and Definition 26.17.7.)

02RB **Definition 42.14.1.** Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k + r)$-cycle on X to the scheme theoretic inverse image
$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$
This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 42.13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum
$$f^*\alpha = \sum n_i f^*[Z_i]$$
where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 42.13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of X (see Schemes, Lemma 26.12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

02RC **Lemma 42.14.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. Such a basis element maps either to the basis element $[Z \cap U]$ or to zero if $Z \subset Y$.

Hence the lemma is clear in this case. The general case is similar and the proof is omitted. □

02RD **Lemma 42.14.3.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 29.29.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k + s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k + s + r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 42.13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 10.52.14. □

02RE **Lemma 42.14.4.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. The statements on dimensions follow immediately from Lemma 42.13.1. Part (1) follows from part (2) by Lemma 42.10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Schemes, Lemma 30.9.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 17.9.2), hence $f^*\mathcal{F}$ is coherent (Cohomology of Schemes, Lemma 30.9.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 42.10.1.) We have $f^*\mathcal{F}_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 10.52.13. □

42.15. Push and pull

02RF In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

02RG **Lemma 42.15.1.** *Let (S, δ) be as in Situation 42.7.1. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_* (g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 29.41.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 29.29.2 and 29.25.8. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 42.10.3. By Lemmas 42.12.4 and 42.14.4 it therefore suffices to show that $f'_*(g')^* \mathcal{O}_W$ is isomorphic to $g^* f_* \mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Schemes, Lemma 30.5.2. □

02RH **Lemma 42.15.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 29.48.1). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 29.48.2, and a finite morphism is proper by Morphisms, Lemma 29.44.11. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 29.48.4) we see that $f_* f^* \mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_* f^* [Z] = f_* f^* [\mathcal{O}_Z]_k = [f_* f^* \mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 42.14.4 and 42.12.4. □

42.16. Preparation for principal divisors

02RI Some of the material in this section partially overlaps with the discussion in Divisors, Section 31.26.

02RK **Lemma 42.16.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral.*

(1) *If $Z \subset X$ is an integral closed subscheme, then the following are equivalent:*
   (a) *Z is a prime divisor,*
   (b) *Z has codimension 1 in X, and*
   (c) $\dim_\delta(Z) = \dim_\delta(X) - 1$.
(2) *If Z is an irreducible component of an effective Cartier divisor on X, then* $\dim_\delta(Z) = \dim_\delta(X) - 1$.

Proof. Part (1) follows from the definition of a prime divisor (Divisors, Definition 31.26.2) and the definition of a dimension function (Topology, Definition 5.20.1). Let $\xi \in Z$ be the generic point of an irreducible component Z of an effective Cartier divisor $D \subset X$. Then $\dim(\mathcal{O}_{D,\xi}) = 0$ and $\mathcal{O}_{D,\xi} = \mathcal{O}_{X,\xi}/(f)$ for some nonzerodivisor $f \in \mathcal{O}_{X,\xi}$ (Divisors, Lemma 31.15.2). Then $\dim(\mathcal{O}_{X,\xi}) = 1$ by Algebra, Lemma 10.60.13. Hence Z is as in (1) by Properties, Lemma 28.10.3 and the proof is complete. □

02RM **Lemma 42.16.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that*

(1) *X, Y are integral,*
(2) *Y is locally Noetherian*
(3) *f is proper, dominant and $R(Y) \subset R(X)$ is finite, and*
(4) $\dim(\mathcal{O}_{Y,\xi}) = 1$.

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. This lemma is a special case of Varieties, Lemma 33.17.2. Here is a direct argument in this case. By Cohomology of Schemes, Lemma 30.21.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = \mathrm{Spec}(R)$. Note that R is Noetherian, as Y is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = \mathrm{Spec}(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains such that the induced extension of fraction fields is finite. Thus the lemma follows from Algebra, Lemma 10.113.2. □

42.17. Principal divisors

02RN The following definition is the analogue of Divisors, Definition 31.26.5 in our current setup.

02RO **Definition 42.17.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$-cycle

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

defined in Divisors, Definition 31.26.5. This makes sense because prime divisors have δ-dimension $n - 1$ by Lemma 42.16.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\mathrm{div}_X(fg) = \mathrm{div}_X(f) + \mathrm{div}_X(g)$$

in $Z_{n-1}(X)$. See Divisors, Lemma 31.26.6. The following lemma will be superseded by the more general Lemma 42.20.2.

02RR **Lemma 42.17.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then*

$$f^*(div_Y(g)) = div_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n + r - 1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n - 1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an integral closed subscheme of Y of δ-dimension $n - 1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have g in the fraction field of A. What we have to show is that

$$\mathrm{ord}_A(g)\mathrm{length}_B(B/\mathfrak{m}_A B) = \mathrm{ord}_B(g).$$

This follows from Algebra, Lemma 10.52.13 (details omitted). □

42.18. Principal divisors and pushforward

02RS The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

02RT **Lemma 42.18.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_* div(f) = div(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n - 1$. We want to show that the coefficient of $[Z]$ in $p_*\mathrm{div}(f)$ and $\mathrm{div}(g)$ are equal. We may apply Lemma 42.16.2 to the morphism $p : X \to Y$ and the generic point $\xi \in Z$. Hence we may replace Y by an affine open neighbourhood of ξ and assume that $p : X \to Y$ is finite. Write $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension L/K of fraction fields. Now we have $f \in L$, $g = \mathrm{Nm}(f) \in K$, and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\mathrm{div}_Y(g)$ is $\mathrm{ord}_{R_\mathfrak{p}}(g)$. The coefficient of $[Z]$ in $p_*\mathrm{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\mathrm{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 10.121.8. □

An important role in the discussion of principal divisors is played by the "universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

0F93 (42.18.1.1) $$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, T_1])$$

the closed subscheme cut out by the section T_1, resp. T_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 31.13.1 and Lemma 31.14.10. The following lemma says that loosely speaking we have "$\mathrm{div}(T_1/T_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

02RQ **Lemma 42.18.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then*

1. *the projection morphism $p : Y \to X$ is proper,*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,*
3. *the pullbacks $Y_0 = q^{-1}D_0$ and $Y_\infty = q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are defined (Divisors, Definition 31.13.12),*
4. *we have*
$$div_Y(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$
5. *we have*
$$div_X(f) = p_* div_Y(f)$$
6. *if we view Y_0 and Y_∞ as closed subschemes of X via the morphism p then we have*
$$div_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 29.41.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence (3) by Divisors, Lemma 31.13.13. We obtain $\dim_\delta(Y_0) = n - 1$ and $\dim_\delta(Y_\infty) = n - 1$ from Lemma 42.16.1.

Consider the effective Cartier divisor Y_0. At every point $\xi \in Y_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for Y_0 is given by f. In particular, if $\delta(\xi) = n - 1$ so ξ is the generic point of an integral closed subscheme Z of δ-dimension $n - 1$, then we see that the coefficient of $[Z]$ in $\mathrm{div}_Y(f)$ is

$$\mathrm{ord}_Z(f) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y_0,\xi})$$

which is the coefficient of $[Z]$ in $[Y_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[Y_\infty]$ agrees with the terms with negative coefficients in the expression for $\mathrm{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $Y_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. It follows that

$$p_*\mathcal{O}_{Y_0} = \mathcal{O}_{Y'_0}$$

where $Y'_0 \subset X$ is the image of $Y_0 \to X$. By Lemma 42.12.4 we have $p_*[Y_0]_{n-1} = [Y'_0]_{n-1}$. The same is true for D_∞ and Y_∞. Hence (6) is a consequence of (5). Finally, (5) follows immediately from Lemma 42.18.1. □

The following lemma says that the degree of a principal divisor on a proper curve is zero.

02RU **Lemma 42.18.3.** *Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to \mathrm{Spec}(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then*

$$\sum\nolimits_{x \in X \text{ closed}} [\kappa(x) : K] ord_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 10.121.2. In other words, $c_* div(f) = 0$.

Proof. Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbf{P}^1_K & \xrightarrow{c'} & \mathrm{Spec}(K) \end{array}$$

that we constructed in Lemma 42.18.2 starting with X and the rational function f over $S = \mathrm{Spec}(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\mathrm{div}_X(f) = c_*p_*\mathrm{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\mathrm{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\mathrm{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 29.48.1). Since $\mathrm{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\mathrm{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Then by Lemma 42.15.2 we get $q_*\mathrm{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 42.16.2. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 10.114.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K, q(y)}$ as soon as it is torsion free (More on Algebra, Lemma 15.22.11). This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 29.48.2. □

42.19. Rational equivalence

02RV In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 42.19.5. However, if we do not allow these

then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

02RW **Definition 42.19.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_* \mathrm{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.
(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.
(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
(4) We define
$$\mathrm{CH}_k(X) = Z_k(X)/\sim_{rat}$$
to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles modulo rational equivalence on X*.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all.

0GU3 **Remark 42.19.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Let us show that we have a presentation
$$\bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \to \mathrm{CH}_k(X) \to 0$$
Here we use the notation and conventions introduced in Remark 42.8.2 and in addition

(1) $K_1^M(\kappa(x)) = \kappa(x)^*$ is the degree 1 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4), and
(2) the differential ∂ is defined as follows: given an element $\xi = \sum_x f_x$ we denote $W_x = \overline{x}$ the integral closed subscheme of X with generic point x and we set
$$\partial(\xi) = \sum (W_x \to X)_* \mathrm{div}(f_x)$$
in $Z_k(X)$ which makes sense as we have seen that the second term of the complex is equal to $Z_k(X)$ by Remark 42.8.2.

The fact that we obtain a presentation of $\mathrm{CH}_k(X)$ follows immediately by comparing with Definition 42.19.1.

A very simple but important lemma is the following.

02RX **Lemma 42.19.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*
$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j\in J}$ be a locally finite collection of integral closed subschemes of U of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha-\beta)|_U = \sum (i_j)_*\mathrm{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that

$$\alpha - \beta - \sum (i_j')_*\mathrm{div}(f_j)$$

is a cycle supported on Y and the lemma follows (see Lemma 42.14.2). □

0F94 **Lemma 42.19.4.** *Let (S,δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups*

$$\mathrm{CH}_k(X_1 \cap X_2) \longrightarrow \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \longrightarrow \mathrm{CH}_k(X) \longrightarrow 0$$

is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. By Lemma 42.12.3 the arrow $\mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \to \mathrm{CH}_k(X)$ is surjective. Suppose that (α_1, α_2) maps to zero under this map. Write $\alpha_1 = \sum n_{1,i}[W_{1,i}]$ and $\alpha_2 = \sum n_{2,i}[W_{2,i}]$. Then we obtain a locally finite collection $\{W_j\}_{j\in J}$ of integral closed subschemes of X of δ-dimension $k+1$ and $f_j \in R(W_j)^*$ such that

$$\sum n_{1,i}[W_{1,i}] + \sum n_{2,i}[W_{2,i}] = \sum (i_j)_*\mathrm{div}(f_j)$$

as cycles on X where $i_j : W_j \to X$ is the inclusion morphism. Choose a disjoint union decomposition $J = J_1 \amalg J_2$ such that $W_j \subset X_1$ if $j \in J_1$ and $W_j \subset X_2$ if $j \in J_2$. (This is possible because the W_j are integral.) Then we can write the equation above as

$$\sum n_{1,i}[W_{1,i}] - \sum\nolimits_{j\in J_1} (i_j)_*\mathrm{div}(f_j) = -\sum n_{2,i}[W_{2,i}] + \sum\nolimits_{j\in J_2} (i_j)_*\mathrm{div}(f_j)$$

Hence this expression is a cycle (!) on $X_1 \cap X_2$. In other words the element (α_1, α_2) is in the image of the first arrow and the proof is complete. □

02RY **Example 42.19.5.** Here is a "strange" example. Suppose that S is the spectrum of a field k with δ as in Example 42.7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$. This gives a zero cycle $[0] \in Z_0(X)$. The "strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_*\mathrm{div}(f_j)$ is exactly the 0-cycle $[0]$. In fact it turns out that $\mathrm{CH}_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

02RZ **Remark 42.19.6.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $\mathrm{CH}_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

42.20. Rational equivalence and push and pull

02S0 In this section we show that flat pullback and proper pushforward commute with rational equivalence.

0EPH **Lemma 42.20.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then for $g \in R(Y)^*$ we have*

$$f^* div_Y(g) = \sum n_j i_{j,*} div_{X_j}(g \circ f|_{X_j})$$

as $(k + r - 1)$-cycles on X where the sum is over the irreducible components X_j of X and n_j is the multiplicity of X_j in X.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k + r - 1$. We have to show that the coefficient n of $[Z]$ in $f^*\mathrm{div}(g)$ is equal to the coefficient m of $[Z]$ in $\sum i_{j,*}\mathrm{div}(g \circ f|_{X_j})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 42.13.1 we have $\dim_\delta(Z') \geq k - 1$. Thus either $Z' = Y$ or Z' is a prime divisor on Y. If $Z' = Y$, then the coefficients n and m are both zero: this is clear for n by definition of f^* and follows for m because $g \circ f|_{X_j}$ is a unit in any point of X_j mapping to the generic point of Y. Hence we may assume that $Z' \subset Y$ is a prime divisor.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, A is a domain, and $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. The rational function g is an element of the fraction field $Q(A)$ of A. By construction, the closed subschemes X_j which meet ξ correspond 1-to-1 with minimal primes

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset B$$

The integers n_j are the corresponding lengths

$$n_i = \mathrm{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

The rational functions $g \circ f|_{X_j}$ correspond to the image $g_i \in \kappa(\mathfrak{q}_i)^*$ of $g \in Q(A)$. Putting everything together we see that

$$n = \mathrm{ord}_A(g)\mathrm{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \mathrm{ord}_{B/\mathfrak{q}_i}(g_i)\mathrm{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

Writing $g = x/y$ for some nonzero $x, y \in A$ we see that it suffices to prove

$$\text{length}_A(A/(x))\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/xB)$$

(equality uses Algebra, Lemma 10.52.13) equals

$$\sum\nolimits_{i=1,\ldots,s} \text{length}_{B/\mathfrak{q}_i}(B/(x,\mathfrak{q}_i))\text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

and similarly for y. As $A \to B$ is flat it follows that x is a nonzerodivisor in B. Hence the desired equality follows from Lemma 42.3.2. □

02S1 **Lemma 42.20.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k+r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j\in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(g_j)) = \sum f^* i_{j,*}\text{div}(g_j)$$

is rationally equivalent to zero on X. The sum on the right makes sense as $\{W_j\}$ is locally finite in X by Lemma 42.13.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \to W_j$ the first projection. By Lemma 42.15.1 we can write the sum above as

$$\sum i'_{j,*}(f_j^*\text{div}(g_j))$$

By Lemma 42.20.1 we see that each $f_j^*\text{div}(g_j)$ is rationally equivalent to zero on W'_j. Hence each $i'_{j,*}(f_j^*\text{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 42.19.6. □

02S2 **Lemma 42.20.3.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j\in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\text{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\text{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 42.11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*} \mathrm{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\mathrm{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a single integral closed subscheme $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ W' & \xrightarrow{i'} & Y \end{array}$$

Note that $p_* i_* \mathrm{div}(f) = i'_*(p')_*\mathrm{div}(f)$ by Lemma 42.12.2. As explained above we have to show that $(p')_*\mathrm{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\mathrm{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\mathrm{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to \mathrm{Spec}(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram

$$\begin{array}{ccc} W_\eta & \longrightarrow & W \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathrm{Spec}(K) & \longrightarrow & W' \end{array}$$

Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $p'(Z) = W'$. Note that the multiplicity of Z_ξ in $\mathrm{div}(f)$ is equal to $\mathrm{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_*\mathrm{div}(f)$ is equal to the multiplicity of $[\mathrm{Spec}(K)]$ in $c_*\mathrm{div}(f_\eta)$. By Lemma 42.18.3 this is zero.

The case $\dim_\delta(W') = k+1$. In this case Lemma 42.18.1 applies, and we see that indeed $p'_*\mathrm{div}(f) = \mathrm{div}(g)$ for some $g \in R(W')^*$ as desired. □

42.21. Rational equivalence and the projective line

02S3 Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \mathrm{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \mathrm{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as in (42.18.1.1).

02S4 **Lemma 42.21.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k+1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then*

(1) *W_0, W_∞ are effective Cartier divisors of W,*

(2) W_0, W_∞ *can be viewed as closed subschemes of* X *and*
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
(3) *for any locally finite family of integral closed subschemes* $W_i \subset X \times_S \mathbf{P}^1_S$ *of* δ*-dimension* $k+1$ *with* $W_i \neq (W_i)_0$ *and* $W_i \neq (W_i)_\infty$ *we have* $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ *on* X*, and*
(4) *for any* $\alpha \in Z_k(X)$ *with* $\alpha \sim_{rat} 0$ *there exists a locally finite family of integral closed subschemes* $W_i \subset X \times_S \mathbf{P}^1_S$ *as above such that* $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.

Proof. Part (1) follows from Divisors, Lemma 31.13.13 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumption.

Since $X \times_S D_0 \to X$ is a closed immersion, we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. The morphism $p : W \to X$ is proper as a composition of the closed immersion $W \to X \times_S \mathbf{P}^1_S$ and the proper morphism $X \times_S \mathbf{P}^1_S \to X$. By Lemma 42.18.2 we have $[W_0]_k \sim_{rat} [W_\infty]_k$ as cycles on W. Hence part (2) follows from Lemma 42.20.3 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k+1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i \in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_* \mathrm{div}(f_i)$. Let
$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$
be the closure of the graph of the rational map f_i as in Lemma 42.18.2. Then we have that $(V_i \to X)_* \mathrm{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. □

02S5 **Lemma 42.21.2.** *Let* (S, δ) *be as in Situation 42.7.1. Let* X *be a scheme locally of finite type over* S*. Let* Z *be a closed subscheme of* $X \times \mathbf{P}^1$*. Assume*
(1) $\dim_\delta(Z) \leq k+1$,
(2) $\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$, *and*
(3) *for any embedded point* ξ *(Divisors, Definition 31.4.1) of* Z *either* $\xi \notin Z_0 \cup Z_\infty$ *or* $\delta(\xi) < k$.

Then $[Z_0]_k \sim_{rat} [Z_\infty]_k$ *as* k*-cycles on* X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of Z which have δ-dimension $k+1$. Write
$$[Z]_{k+1} = \sum n_i [W_i]$$
with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Divisors, Lemma 31.26.1. We claim that
$$[Z_0]_k = \sum n_i [(W_i)_0]_k$$
and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 42.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same

as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by assumption (3) we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzerodivisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 42.3.2. □

02S6 **Lemma 42.21.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume*

(1) $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k + 1$,
(2) $\dim_\delta(\text{Supp}(\mathcal{F}_0)) \leq k$, $\dim_\delta(\text{Supp}(\mathcal{F}_\infty)) \leq k$, *and*
(3) *for any embedded associated point ξ of $\mathcal{F}$ either $\xi \notin (X \times \mathbf{P}^1)_0 \cup (X \times \mathbf{P}^1)_\infty$ or $\delta(\xi) < k$.*

Then $[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$ as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 42.10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 42.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(\text{Supp}(M)) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by assumption (3) we see that M is a Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(\text{Supp}(M/tM)) = 0$ which implies that t is a nonzerodivisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal

primes $\mathfrak{q}_i$ of $\mathrm{Ass}(M)$. The multiplicities n_i correspond to the lengths $\mathrm{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \mathrm{length}_A(M/tM)$$

and

$$m = \sum \mathrm{length}_A(A/(t, \mathfrak{q}_i)A)\mathrm{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 42.3.2. □

42.22. Chow groups and envelopes

0GU4 Here is the definition.

0GU5 **Definition 42.22.1.** Let X be a scheme. An *envelope* is a proper morphism $f : Y \to X$ which is completely decomposed (More on Morphisms, Definition 37.78.1). [**Ful98**, Definition 18.3]

The exact sequence of Lemma 42.22.4 is the main motivation for the definition.

0GU6 **Lemma 42.22.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. If $f : Y \to X$ and $g : Z \to Y$ are envelopes, then $f \circ g$ is an envelope.*

Proof. Follows from Morphisms, Lemma 29.41.4 and More on Morphisms, Lemma 37.78.2. □

0GU7 **Lemma 42.22.3.** *Let (S, δ) be as in Situation 42.7.1. Let $X' \to X$ be a morphism of schemes locally of finite type over S. If $f : Y \to X$ is an envelope, then the base change $f' : Y' \to X'$ of f is an envelope too.*

Proof. Follows from Morphisms, Lemma 29.41.5 and More on Morphisms, Lemma 37.78.3. □

0GU8 **Lemma 42.22.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $f : Y \to X$ be an envelope. Then we have an exact sequence*

$$\mathrm{CH}_k(Y \times_X Y) \xrightarrow{p_* - q_*} \mathrm{CH}_k(Y) \xrightarrow{f_*} \mathrm{CH}_k(X) \to 0$$

for all $k \in \mathbf{Z}$. Here $p, q : Y \times_X Y \to Y$ are the projections.

Proof. Since f is an envelope, f is proper and hence pushforward on cycles and cycle classes is defined, see Sections 42.12 and 42.15. Similarly, the morphisms p and q are proper as base changes of f. The composition of the arrows is zero as $f_* \circ p_* = (p \circ f)_* = (q \circ f)_* = f_* \circ q_*$, see Lemma 42.12.2.

Let us show that $f_* : Z_k(Y) \to Z_k(X)$ is surjective. Namely, suppose that we have $\alpha = \sum n_i[Z_i] \in Z_k(X)$ where $Z_i \subset X$ is a locally finite family of integral closed subschemes. Let $x_i \in Z_i$ be the generic point. Since f is an envelope and hence completely decomposed, there exists a point $y_i \in Y$ with $f(y_i) = x_i$ and with $\kappa(y_i)/\kappa(x_i)$ trivial. Let $W_i \subset Y$ be the integral closed subscheme with generic point y_i. Since f is closed, we see that $f(W_i) = Z_i$. It follows that the family of closed subschemes W_i is locally finite on Y. Since $\kappa(y_i)/\kappa(x_i)$ is trivial we see that $\dim_\delta(W_i) = \dim_\delta(Z_i) = k$. Hence $\beta = \sum n_i[W_i]$ is in $Z_k(Y)$. Finally, since $\kappa(y_i)/\kappa(x_i)$ is trivial, the degree of the dominant morphism $f|_{W_i} : W_i \to Z_i$ is 1 and we conclude that $f_*\beta = \alpha$.

Since $f_* : Z_k(Y) \to Z_k(X)$ is surjective, a fortiori the map $f_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X)$ is surjective.

Let $\beta \in Z_k(Y)$ be an element such that $f_*\beta$ is zero in $\mathrm{CH}_k(X)$. This means we can find a locally finite family of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+1$ and $f_j \in R(Z_j)^*$ such that

$$f_*\beta = \sum (Z_j \to X)_* \mathrm{div}(f_j)$$

as cycles where $i_j : Z_j \to X$ is the given closed immersion. Arguing exactly as above, we can find a locally finite family of integral closed subschemes $W_j \subset Y$ with $f(W_j) = Z_j$ and such that $W_j \to Z_j$ is birational, i.e., induces an isomorphism $R(Z_j) = R(W_j)$. Denote $g_j \in R(W_j)^*$ the element corresponding to f_j. Observe that $W_j \to Z_j$ is proper and that $(W_j \to Z_j)_*\mathrm{div}(g_j) = \mathrm{div}(f_j)$ as cycles on Z_j. It follows from this that if we replace β by the rationally equivalent cycle

$$\beta' = \beta - \sum (W_j \to Y)_* \mathrm{div}(g_j)$$

then we find that $f_*\beta' = 0$. (This uses Lemma 42.12.2.) Thus to finish the proof of the lemma it suffices to show the claim in the following paragraph.

Claim: if $\beta \in Z_k(Y)$ and $f_*\beta = 0$, then $\beta = \delta + p_*\gamma - q_*\gamma$ in $Z_k(Y)$ for some $\gamma \in Z_k(Y \times_X Y)$. Namely, write $\beta = \sum_{j \in J} n_j[W_j]$ with $\{W_j\}_{j\in J}$ a locally finite family of integral closed subschemes of Y with $\dim_\delta(W_j) = k$. Fix an integral closed subscheme $Z \subset X$. Consider the subset $J_Z = \{j \in J : f(W_j) = Z\}$. This is a finite set. There are three cases:

(1) $J_Z = \emptyset$. In this case we set $\gamma_Z = 0$.
(2) $J_Z \neq \emptyset$ and $\dim_\delta(Z) = k$. The condition $f_*\beta = 0$ implies by looking at the coefficient of Z that $\sum_{j\in J_Z} n_j \deg(W_j/Z) = 0$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W_j' \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W_j' \to W$ is dominant and $\deg(W_j'/W) = \deg(W_j/W)$. Thus if we set $\gamma_Z = \sum_{j\in J_Z} n_j[W_j']$ then we see that $p_*\gamma_Z = \sum_{j\in J_Z} n_j[W_j]$ and $q_*\gamma_Z = \sum_{j\in J_Z} n_j \deg(W_j'/W)[W] = 0$.
(3) $J_Z \neq \emptyset$ and $\dim_\delta(Z) < k$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W_j' \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W_j' \to W$ is dominant and $k = \dim_\delta(W_j') > \dim_\delta(W) = \dim_\delta(Z)$. Thus if we set $\gamma_Z = \sum_{j\in J_Z} n_j[W_j']$ then we see that $p_*\gamma_Z = \sum_{j\in J_Z} n_j[W_j]$ and $q_*\gamma_Z = 0$.

Since the family of integral closed subschemes $\{f(W_j)\}$ is locally finite on X (Lemma 42.11.2) we see that the k-cycle

$$\gamma = \sum\nolimits_{Z \subset X \text{ integral closed}} \gamma_Z$$

on $Y \times_X Y$ is well defined. By our computations above it follows that $p_*\gamma_Z = \beta$ and $q_*\gamma_Z = 0$ which implies what we wanted to prove. $\square$

42.23. Chow groups and K-groups

0FDQ In this section we are going to compare K_0 of the category of coherent sheaves to the chow groups.

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. We denote $Coh(X) = Coh(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Cohomology of Schemes, Lemma 30.9.2. For any $k \in \mathbf{Z}$ we let $Coh_{\leq k}(X)$ be the full subcategory of $Coh(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

02S8 **Lemma 42.23.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The categories $Coh_{\leq k}(X)$ are Serre subcategories of the abelian category $Coh(X)$.*

Proof. The definition of a Serre subcategory is Homology, Definition 12.10.1. The proof of the lemma is straightforward and omitted. □

02S9 **Lemma 42.23.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The maps*

$$Z_k(X) \longrightarrow K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)), \quad \sum n_Z[Z] \mapsto \Big[\bigoplus\nolimits_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}\Big] - \Big[\bigoplus\nolimits_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}\Big]$$

and

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X), \quad \mathcal{F} \longmapsto [\mathcal{F}]_k$$

are mutually inverse isomorphisms.

Proof. Note that if $\sum n_Z[Z]$ is in $Z_k(X)$, then the direct sums $\bigoplus_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}$ and $\bigoplus_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}$ are coherent sheaves on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $Coh_{\leq k}(X)$, see Lemma 42.10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in Coh_{\leq k-1}(X)$. By part (1) of Homology, Lemma 12.11.3 this implies that the second map is well defined too. It is clear that the composition of the first map with the second map is the identity.

Conversely, say we start with a coherent sheaf $\mathcal{F}$ on X. Write $[\mathcal{F}]_k = \sum_{i \in I} n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X$, $i \in I$ pairwise distinct integral closed subschemes of δ-dimension k. We have to show that

$$[\mathcal{F}] = [\bigoplus\nolimits_{i \in I} \mathcal{O}_{Z_i}^{\oplus n_i}]$$

in $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$. Denote $\xi_i \in Z_i$ the generic point. If we set

$$\mathcal{F}' = \text{Ker}(\mathcal{F} \to \bigoplus \xi_{i,*}\mathcal{F}_{\xi_i})$$

then $\mathcal{F}'$ is the maximal coherent submodule of $\mathcal{F}$ whose support has dimension $\leq k-1$. In particular $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}'$ have the same class in $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$. Thus after replacing $\mathcal{F}$ by $\mathcal{F}/\mathcal{F}'$ we may and do assume that the kernel $\mathcal{F}'$ displayed above is zero.

For each $i \in I$ we choose a filtration

$$\mathcal{F}_{\xi_i} = \mathcal{F}_i^0 \supset \mathcal{F}_i^1 \supset \ldots \supset \mathcal{F}_i^{n_i} = 0$$

such that the successive quotients are of dimension 1 over the residue field at ξ_i. This is possible as the length of $\mathcal{F}_{\xi_i}$ over $\mathcal{O}_{X,\xi_i}$ is n_i. For $p > n_i$ set $\mathcal{F}_i^p = 0$. For $p \geq 0$ we denote

$$\mathcal{F}^p = \text{Ker}\left(\mathcal{F} \longrightarrow \bigoplus \xi_{i,*}(\mathcal{F}_{\xi_i}/\mathcal{F}_i^p)\right)$$

Then $\mathcal{F}^p$ is coherent, $\mathcal{F}^0 = \mathcal{F}$, and $\mathcal{F}^p/\mathcal{F}^{p+1}$ is isomorphic to a free $\mathcal{O}_{Z_i}$-module of rank 1 (if $n_i > p$) or 0 (if $n_i \leq p$) in an open neighbourhood of ξ_i. Moreover, $\mathcal{F}' = \bigcap \mathcal{F}^p = 0$. Since every quasi-compact open $U \subset X$ contains only a finite number of ξ_i we conclude that $\mathcal{F}^p|_U$ is zero for $p \gg 0$. Hence $\bigoplus_{p\geq 0} \mathcal{F}^p$ is a coherent $\mathcal{O}_X$-module. Consider the short exact sequences

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \bigoplus\nolimits_{p>0} \mathcal{F}^p/\mathcal{F}^{p+1} \to 0$$

and

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. This already shows that

$$[\mathcal{F}] = [\bigoplus \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$. Next, for every $p \geq 0$ and $i \in I$ such that $n_i > p$ we choose a nonzero ideal sheaf $\mathcal{I}_{i,p} \subset \mathcal{O}_{Z_i}$ and a map $\mathcal{I}_{i,p} \to \mathcal{F}^p/\mathcal{F}^{p+1}$ on X which is an isomorphism over the open neighbourhood of ξ_i mentioned above. This is possible by Cohomology of Schemes, Lemma 30.10.6. Then we consider the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{Q} \to 0$$

and the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{O}_{Z_i} \to \mathcal{Q}' \to 0$$

Observe that both $\mathcal{Q}$ and $\mathcal{Q}'$ are zero in a neighbourhood of the points ξ_i and that they are supported on $\bigcup Z_i$. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ are in $Coh_{\leq k-1}(X)$. Since

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_{Z_i}^{\oplus n_i} \cong \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{O}_{Z_i}$$

this concludes the proof. □

0FDR **Lemma 42.23.3.** *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over (S, δ) as in Situation 42.7.1. Then $\pi_* : Coh(X) \to Coh(Y)$ is an exact functor which sends $Coh_{\leq k}(X)$ into $Coh_{\leq k}(Y)$ and induces homomorphisms on K_0 of these categories and their quotients. The maps of Lemma 42.23.2 fit into a commutative diagram*

$$\begin{array}{ccccc} Z_k(X) & \longrightarrow & K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) & \longrightarrow & Z_k(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ Z_k(Y) & \longrightarrow & K_0(Coh_{\leq k}(Y)/Coh_{\leq k-1}(Y)) & \longrightarrow & Z_k(Y) \end{array}$$

Proof. A finite morphism is affine, hence pushforward of quasi-coherent modules along π is an exact functor by Cohomology of Schemes, Lemma 30.2.3. A finite morphism is proper, hence π_* sends coherent sheaves to coherent sheaves, see Cohomology of Schemes, Proposition 30.19.1. The statement on dimensions of supports is clear. Commutativity on the right follows immediately from Lemma 42.12.4. Since the horizontal arrows are bijections, we find that we have commutativity on the left as well. □

0FDS **Lemma 42.23.4.** *Let X be a scheme locally of finite type over (S, δ) as in Situation 42.7.1. There is a canonical map*

$$\mathrm{CH}_k(X) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$$

induced by the map $Z_k(X) \to K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ from Lemma 42.23.2.

Proof. We have to show that an element α of $Z_k(X)$ which is rationally equivalent to zero, is mapped to zero in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. Write $\alpha = \sum (i_j)_* \mathrm{div}(f_j)$ as in Definition 42.19.1. Observe that

$$\pi = \coprod i_j : W = \coprod W_j \longrightarrow X$$

is a finite morphism as each $i_j : W_j \to X$ is a closed immersion and the family of W_j is locally finite in X. Hence we may use Lemma 42.23.3 to reduce to the case of W. Since W is a disjoint union of integral scheme, we reduce to the case discussed in the next paragraph.

Assume X is integral of δ-dimension $k+1$. Let f be a nonzero rational function on X. Let $\alpha = \mathrm{div}(f)$. We have to show that α is mapped to zero in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of f, see Divisors, Definition 31.23.10. Then we have short exact sequences

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

and

$$0 \to \mathcal{I} \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_X/f\mathcal{I} \to 0$$

See Divisors, Lemma 31.23.9. We claim that

$$[\mathcal{O}_X/\mathcal{I}]_k - [\mathcal{O}_X/f\mathcal{I}]_k = \mathrm{div}(f)$$

The claim implies the element $\alpha = \mathrm{div}(f)$ is represented by $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ in $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$. Then the short exact sequences show that this element maps to zero in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$.

To prove the claim, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and let $\xi \in Z$ be its generic point. Then $I = \mathcal{I}_\xi \subset A = \mathcal{O}_{X,\xi}$ is an ideal such that $fI \subset A$. Now the coefficient of $[Z]$ in $\mathrm{div}(f)$ is $\mathrm{ord}_A(f)$. (Of course as usual we identify the function field of X with the fraction field of A.) On the other hand, the coefficient of $[Z]$ in $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ is

$$\mathrm{length}_A(A/I) - \mathrm{length}_A(A/fI)$$

Using the distance function of Algebra, Definition 10.121.5 we can rewrite this as

$$d(A, I) - d(A, fI) = d(I, fI) = \mathrm{ord}_A(f)$$

The equalities hold by Algebra, Lemmas 10.121.6 and 10.121.7. (Using these lemmas isn't necessary, but convenient.) □

02SD **Remark 42.23.5.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. We will see later (in Lemma 42.69.3) that the map

$$\mathrm{CH}_k(X) \longrightarrow K_0(Coh_{k+1}(X)/Coh_{\leq k-1}(X))$$

of Lemma 42.23.4 is injective. Composing with the canonical map

$$K_0(Coh_{k+1}(X)/Coh_{\leq k-1}(X)) \longrightarrow K_0(Coh(X)/Coh_{\leq k-1}(X))$$

we obtain a canonical map

$$\mathrm{CH}_k(X) \longrightarrow K_0(Coh(X)/Coh_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map should be injective or not. It seems reasonable to expect the kernel of this map to be torsion. We will return to this question (insert future reference).

0FDT **Lemma 42.23.6.** *Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Denote $Coh_Z(X) \subset Coh(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Then the exact inclusion functor $Coh(Z) \to Coh_Z(X)$ induces an isomorphism*

$$K'_0(Z) = K_0(Coh(Z)) \longrightarrow K_0(Coh_Z(X))$$

Proof. Let $\mathcal{F}$ be an object of $Coh_Z(X)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Consider the descending filtration

$$\ldots \subset \mathcal{F}^p = \mathcal{I}^p\mathcal{F} \subset \mathcal{F}^{p-1} \subset \ldots \subset \mathcal{F}^0 = \mathcal{F}$$

Exactly as in the proof of Lemma 42.23.4 this filtration is locally finite and hence $\bigoplus_{p\geq 0} \mathcal{F}^p$, $\bigoplus_{p\geq 1} \mathcal{F}^p$, and $\bigoplus_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ are coherent $\mathcal{O}_X$-modules supported on Z. Hence we get

$$[\mathcal{F}] = [\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(Coh_Z(X))$ exactly as in the proof of Lemma 42.23.4. Since the coherent module $\bigoplus_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ is annihilated by $\mathcal{I}$ we conclude that $[\mathcal{F}]$ is in the image. Actually, we claim that the map

$$\mathcal{F} \longmapsto c(\mathcal{F}) = [\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

factors through $K_0(Coh_Z(X))$ and is an inverse to the map in the statement of the lemma. To see this all we have to show is that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence in $Coh_Z(X)$, then we get $c(\mathcal{G}) = c(\mathcal{F}) + c(\mathcal{H})$. Observe that for all $q \geq 0$ we have a short exact sequence

$$0 \to (\mathcal{F} \cap \mathcal{I}^q\mathcal{G})/(\mathcal{F} \cap \mathcal{I}^{q+1}\mathcal{G}) \to \mathcal{G}^q/\mathcal{G}^{q+1} \to \mathcal{H}^q/\mathcal{H}^{q+1} \to 0$$

For $p, q \geq 0$ consider the coherent submodule

$$\mathcal{F}^{p,q} = \mathcal{I}^p\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$$

Arguing exactly as above and using that the filtrations $\mathcal{F}^p = \mathcal{I}^p\mathcal{F}$ and $\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$ are locally finite, we find that

$$[\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}] = [\bigoplus\nolimits_{p,q\geq 0} \mathcal{F}^{p,q}/(\mathcal{F}^{p+1,q} + \mathcal{F}^{p,q+1})] = [\bigoplus\nolimits_{q\geq 0} (\mathcal{F} \cap \mathcal{I}^q\mathcal{G})/(\mathcal{F} \cap \mathcal{I}^{q+1}\mathcal{G})]$$

in $K_0(Coh(Z))$. Combined with the exact sequences above we obtain the desired result. Some details omitted. □

42.24. The divisor associated to an invertible sheaf

02SI The following definition is the analogue of Divisors, Definition 31.27.4 in our current setup.

02SJ **Definition 42.24.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to* s is the $(n-1)$-cycle
$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z]$$
defined in Divisors, Definition 31.27.4. This makes sense because Weil divisors have δ-dimension $n-1$ by Lemma 42.16.1.

(2) We define *Weil divisor associated to* $\mathcal{L}$ as
$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_{\mathcal{L}}(s) \in \mathrm{CH}_{n-1}(X)$$
where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Divisors, Lemma 31.27.3.

Let X and S be as in Definition 42.24.1 above. Set $n = \dim_\delta(X)$. It is clear from the definitions that $Cl(X) = \mathrm{CH}_{n-1}(X)$ where $Cl(X)$ is the Weil divisor class group of X as defined in Divisors, Definition 31.26.7. The map
$$\mathrm{Pic}(X) \longrightarrow \mathrm{CH}_{n-1}(X), \quad \mathcal{L} \longmapsto c_1(\mathcal{L}) \cap [X]$$
is the same as the map $\mathrm{Pic}(X) \to Cl(X)$ constructed in Divisors, Equation (31.27.5.1) for arbitrary locally Noetherian integral schemes. In particular, this map is a homomorphism of abelian groups, it is injective if X is a normal scheme, and an isomorphism if all local rings of X are UFDs. See Divisors, Lemmas 31.27.6 and 31.27.7. There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

02SK **Lemma 42.24.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*
$$div_{\mathcal{L}}(s) = [Z(s)]_{n-1}$$
in $Z_{n-1}(X)$ and
$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$
in $\mathrm{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = f s_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 31.14.8 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \mathrm{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. □

The following lemma will be superseded by the more general Lemma 42.26.2.

02SM **Lemma 42.24.3.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then*
$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$

in $\mathrm{CH}_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\mathrm{div}_{\mathcal{L}}(s) = \mathrm{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = gs_\xi$ with $g \in R(Y)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that

$$\mathrm{div}_{\mathcal{L}}(s)|_V = \mathrm{div}_Y(g)|_V.$$

In exactly the same way, since f^*s_ξ generates $f^*\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = gf^*s_\xi$ we also have

$$\mathrm{div}_{\mathcal{L}}(f^*s)|_{f^{-1}(V)} = \mathrm{div}_X(g)|_{f^{-1}(V)}.$$

Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pullbacks of principal divisors, see Lemma 42.17.2. □

42.25. Intersecting with an invertible sheaf

02SN In this section we study the following construction.

02SO **Definition 42.25.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to \mathrm{CH}_k(X)$$

called *intersection with the first Chern class of* $\mathcal{L}$.

(1) Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define

$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$

where the right hand side is defined in Definition 42.24.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set

$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

02SP **Lemma 42.25.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ *for every* $\alpha \in Z_{k+1}(X)$. *Moreover,* $c_1(\mathcal{O}_X) \cap \alpha = 0$ *for all* α.

Proof. The additivity follows directly from Divisors, Lemma 31.27.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. □

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

0EPI **Lemma 42.25.3.** *Let (S, δ) be as in Situation 42.7.1. Let Y be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $s \in \Gamma(Y, \mathcal{L})$. Assume*

(1) $\dim_\delta(Y) \leq k + 1$,
(2) $\dim_\delta(Z(s)) \leq k$, *and*
(3) *for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to \mathcal{L}_\xi$.*

Write $[Y]_{k+1} = \sum n_i[Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k + 1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then

02SR (42.25.3.1) $$[Z(s)]_k = \sum n_i[Z(s_i)]_k$$

as k-cycles on Y.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. We want to compare the coefficient n of $[Z]$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z]$ in the expression $[Z(s)]_k$. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $A = \mathcal{O}_{Y,\xi}$, $L = \mathcal{L}_\xi$. Then $L = As_\xi$. Write $s = fs_\xi$ for some (unique) $f \in A$. Hypothesis (3) means that $f : A \to A$ is injective. Since $\dim_\delta(Y) \leq k + 1$ and $\dim_\delta(Z) = k$ we have $\dim(A) = 0$ or 1. We have

$$m = \text{length}_A(A/(f))$$

which is finite in either case.

If $\dim(A) = 0$, then $f : A \to A$ being injective implies that $f \in A^*$. Hence in this case m is zero. Moreover, the condition $\dim(A) = 0$ means that ξ does not lie on any irreducible component of δ-dimension $k + 1$, i.e., $n = 0$ as well.

Now, let $\dim(A) = 1$. Since A is a Noetherian local ring it has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. These correspond 1-1 with the Y_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i})$. Also, the multiplicity of $[Z]$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak{q}_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f)) = \sum \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i})\text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 42.3.2. □

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

02SQ **Lemma 42.25.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y \subset X$ be a closed subscheme. Let $s \in \Gamma(Y, \mathcal{L}|_Y)$. Assume*

(1) $\dim_\delta(Y) \leq k + 1$,
(2) $\dim_\delta(Z(s)) \leq k$, *and*
(3) *for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to (\mathcal{L}|_Y)_\xi$*[1].

[1]For example, this holds if s is a regular section of $\mathcal{L}|_Y$.

Then

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\mathrm{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 42.24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 42.25.3. □

42.26. Intersecting with an invertible sheaf and push and pull

0AYA In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

0EPJ **Lemma 42.26.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of $\mathcal{L}$. Then we have*

$$f^* div_{\mathcal{L}}(s) = \sum n_i div_{f^*\mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ-dimension $n + r$, the section $s_i = f|^*_{X_i}(s)$ is the pullback of s, and $n_i = m_{X_i,X}$ is the multiplicity of X_i in X.

Proof. To prove this equality of cycles, we may work locally on Y. Hence we may assume Y is affine and $s = p/q$ for some nonzero sections $p \in \Gamma(Y, \mathcal{L})$ and $q \in \Gamma(Y, \mathcal{O})$. If we can show both

$$f^*\mathrm{div}_{\mathcal{L}}(p) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(p_i) \quad \text{and} \quad f^*\mathrm{div}_{\mathcal{O}}(q) = \sum n_i \mathrm{div}_{\mathcal{O}_{X_i}}(q_i)$$

(with obvious notations) then we win by the additivity, see Divisors, Lemma 31.27.5. Thus we may assume that $s \in \Gamma(Y, \mathcal{L})$. In this case we may apply the equality (42.25.3.1) to see that

$$[Z(f^*(s))]_{k+r-1} = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

where $f^*(s) \in f^*\mathcal{L}$ denotes the pullback of s to X. On the other hand we have

$$f^*\mathrm{div}_{\mathcal{L}}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 42.24.2 and 42.14.4. Since $Z(f^*(s)) = f^{-1}(Z(s))$ we win. □

02SS **Lemma 42.26.2.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $\mathrm{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\mathrm{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subscheme $f^{-1}(W_i)$ of X. By the discussion in Remark 42.19.6 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. Consider the closed subscheme $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{array}{ccc} W' & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & Y \end{array}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ-dimension $k + r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i[W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. By Lemma 42.26.1 we have the following equality of cycles

$$h^*\mathrm{div}_{\mathcal{L}|_W}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\mathrm{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\mathrm{CH}_{k+r-1}(X)$. □

02ST **Lemma 42.26.3.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Then*

$$f_*\left(div_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)] div_{\mathcal{L}}(s).$$

as cycles on Y. In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = [R(X) : R(Y)]c_1(\mathcal{L}) \cap [Y] = c_1(\mathcal{L}) \cap f_*[X]$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 42.18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_*\left(\mathrm{div}_X(g)\right) = [R(X) : R(Y)]\mathrm{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 42.18.1 we see this true since $\mathrm{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

02SU **Lemma 42.26.4.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\mathrm{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an integral closed subscheme). Consider the diagram

$$\begin{array}{ccc} X' = \coprod W_i & \xrightarrow{q} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' = \coprod W_i' & \xrightarrow{q'} & Y. \end{array}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 42.26.3. □

42.27. The key formula

0AYB Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible sheaves on X. Let s be a nonzero meromorphic section of $\mathcal{L}$ and let t be a nonzero meromorphic section of $\mathcal{N}$. Let $Z_i \subset X$, $i \in I$ be a locally finite set of irreducible closed subsets of codimension 1 with the following property: If $Z \notin \{Z_i\}$ with generic point ξ, then s is a generator for $\mathcal{L}_\xi$ and t is a generator for $\mathcal{N}_\xi$. Such a set exists by Divisors, Lemma 31.27.2. Then

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z_i,\mathcal{L}}(s)[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i. Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

Set $B_i = \mathcal{O}_{X,\xi_i}$. Then by definition

$$\mathrm{ord}_{Z_i,\mathcal{L}}(s) = \mathrm{ord}_{B_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$$

Since t_i is a generator of $\mathcal{N}_{\xi_i}$ we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{B_i}(f_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\mathrm{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^*$$

see Section 42.5.

0AYC **Lemma 42.27.1** (Key formula)**.** *In the situation above the cycle*

$$\sum (Z_i \to X)_* \left(ord_{B_i}(f_i) div_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - ord_{B_i}(g_i) div_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right)$$

is equal to the cycle

$$\sum (Z_i \to X)_* div(\partial_{B_i}(f_i, g_i))$$

Proof. First, let us examine what happens if we replace s_i by us_i for some unit u in B_i. Then f_i gets replaced by $u^{-1}f_i$. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$-\mathrm{ord}_{B_i}(g_i)\mathrm{div}(u|_{Z_i})$$

(where $u|_{Z_i}$ is the image of u in the residue field) by Divisors, Lemma 31.27.3 and in the second expression we add

$$\mathrm{div}(\partial_{B_i}(u^{-1}, g_i))$$

by bi-linearity of the tame symbol. These terms agree by property (6) of the tame symbol.

Let $Z \subset X$ be an irreducible closed with $\dim_\delta(Z) = n - 2$. To show that the coefficients of Z of the two cycles of the lemma is the same, we may do a replacement $s_i \mapsto us_i$ as in the previous paragraph. In exactly the same way one shows that we may do a replacement $t_i \mapsto vt_i$ for some unit v of B_i.

Since we are proving the equality of cycles we may argue one coefficient at a time. Thus we choose an irreducible closed $Z \subset X$ with $\dim_\delta(Z) = n - 2$ and compare coefficients. Let $\xi \in Z$ be the generic point and set $A = \mathcal{O}_{X,\xi}$. This is a Noetherian local domain of dimension 2. Choose generators σ and τ for $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$. After shrinking X, we may and do assume σ and τ define trivializations of the invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ over all of X. Because Z_i is locally finite after shrinking X we may assume $Z \subset Z_i$ for all $i \in I$ and that I is finite. Then ξ_i corresponds to a prime $\mathfrak{q}_i \subset A$ of height 1. We may write $s_i = a_i\sigma$ and $t_i = b_i\tau$ for some a_i and b_i units in $A_{\mathfrak{q}_i}$. By the remarks above, it suffices to prove the lemma when $a_i = b_i = 1$ for all i.

Assume $a_i = b_i = 1$ for all i. Then the first expression of the lemma is zero, because we choose σ and τ to be trivializing sections. Write $s = f\sigma$ and $t = g\tau$ with f and g in the fraction field of A. By the previous paragraph we have reduced to the case $f_i = f$ and $g_i = g$ for all i. Moreover, for a height 1 prime $\mathfrak{q}$ of A which is not in $\{\mathfrak{q}_i\}$ we have that both f and g are units in $A_\mathfrak{q}$ (by our choice of the family $\{Z_i\}$ in the discussion preceding the lemma). Thus the coefficient of Z in the second expression of the lemma is

$$\sum\nolimits_i \mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{B_i}(f, g))$$

which is zero by the key Lemma 42.6.3. □

0GU9 **Remark 42.27.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. We claim that there is a complex

$$\bigoplus\nolimits'_{\delta(x)=k+2} K_2^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x))$$

Here we use notation and conventions introduced in Remark 42.19.2 and in addition

(1) $K_2^M(\kappa(x))$ is the degree 2 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 42.6.4) which is the quotient of $\kappa(x)^* \otimes_{\mathbf{Z}} \kappa(x)^*$ by the subgroup generated by elements of the form $\lambda \otimes (1-\lambda)$ for $\lambda \in \kappa(x) \setminus \{0, 1\}$, and
(2) the first differential ∂ is defined as follows: given an element $\xi = \sum_x \alpha_x$ in the first term we set

$$\partial(\xi) = \sum\nolimits_{x \rightsquigarrow x',\ \delta(x')=k+1} \partial_{\mathcal{O}_{W_x, x'}}(\alpha_x)$$

where $\partial_{\mathcal{O}_{W_x,x'}} : K_2^M(\kappa(x)) \to K_1^M(\kappa(x))$ is the tame symbol constructed in Section 42.5.

We claim that we get a complex, i.e., that $\partial \circ \partial = 0$. To see this it suffices to take an element ξ as above and a point $x'' \in X$ with $\delta(x'') = k$ and check that the coefficient of x'' in the element $\partial(\partial(\xi))$ is zero. Because $\xi = \sum \alpha_x$ is a locally finite sum, we may in fact assume by additivity that $\xi = \alpha_x$ for some $x \in X$ with $\delta(x) = k+2$ and $\alpha_x \in K_2^M(\kappa(x))$. By linearity again we may assume that $\alpha_x = f \otimes g$ for some $f, g \in \kappa(x)^*$. Denote $W \subset X$ the integral closed subscheme with generic point x. If $x'' \notin W$, then it is immediately clear that the coefficient of x in $\partial(\partial(\xi))$ is zero. If $x'' \in W$, then we see that the coefficient of x'' in $\partial(\partial(x))$ is equal to

$$\sum\nolimits_{x \rightsquigarrow x' \rightsquigarrow x'',\ \delta(x')=k+1} \mathrm{ord}_{\mathcal{O}_{\overline{\{x'\}}, x''}}(\partial_{\mathcal{O}_{W,x'}}(f, g))$$

The key algebraic Lemma 42.6.3 says exactly that this is zero.

0GUA **Remark 42.27.3.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. The complex in Remark 42.27.2 and the presentation of $\mathrm{CH}_k(X)$ in Remark 42.19.2 suggests that we can define a first higher Chow group

$$\mathrm{CH}_k^M(X, 1) = H_1(\text{the complex of Remark 42.27.2})$$

We use the supscript ^{M} to distinguish our notation from the higher chow groups defined in the literature, e.g., in the papers by Spencer Bloch ([**Blo86**] and [**Blo94**]). Let $U \subset X$ be open with complement $Y \subset X$ (viewed as reduced closed subscheme). Then we find a split short exact sequence

$$0 \to \bigoplus\nolimits'_{y \in Y, \delta(y)=k+i} K_i^M(\kappa(y)) \to \bigoplus\nolimits'_{x \in X, \delta(x)=k+i} K_i^M(\kappa(x)) \to \bigoplus\nolimits'_{u \in U, \delta(u)=k+i} K_i^M(\kappa(u)) \to 0$$

for $i = 2, 1, 0$ compatible with the boundary maps in the complexes of Remark 42.27.2. Applying the snake lemma (see Homology, Lemma 12.13.6) we obtain a six term exact sequence

$$\mathrm{CH}_k^M(Y, 1) \to \mathrm{CH}_k^M(X, 1) \to \mathrm{CH}_k^M(U, 1) \to \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(U) \to 0$$

extending the canonical exact sequence of Lemma 42.19.3. With some work, one may also define flat pullback and proper pushforward for the first higher chow group $\mathrm{CH}_k^M(X, 1)$. We will return to this later (insert future reference here).

42.28. Intersecting with an invertible sheaf and rational equivalence

02TG Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

02TH **Lemma 42.28.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_{\mathcal{L}}(s)$ and $\beta = div_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\mathrm{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 42.27.1 and the discussion preceding it. □

02TI **Lemma 42.28.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 42.25.1 is zero. By Definition 42.19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By Lemma 42.26.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 42.28.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 42.25.2 we see that the right hand side is zero in $\mathrm{CH}_k(X)$ as desired. □

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. We will denote

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 42.28.2. We will denote $c_1(\mathcal{L})^s \cap -$ the s-fold iterate of this operation for all $s \geq 0$.

02TJ **Lemma 42.28.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in \mathrm{CH}_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\mathrm{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 42.26.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 42.28.1 and the definitions. □

42.29. Gysin homomorphisms

02T7 In this section we define the Gysin map for the zero locus D of a section of an invertible sheaf. An interesting case occurs when D is an effective Cartier divisor, but the generalization to arbitrary D allows us a flexibility to formulate various compatibilities, see Remark 42.29.7 and Lemmas 42.29.8, 42.29.9, and 42.30.5. These results can be generalized to locally principal closed subschemes endowed with a virtual normal bundle (Remark 42.29.2) or to pseudo-divisors (Remark 42.29.3).

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a regular global section, see Divisors, Lemma 31.14.10. If D corresponds to $(\mathcal{L}, s)$, then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

02T8 **Definition 42.29.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s)$ be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the zero scheme of s, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D).$$

by the following rules:

(1) Given an integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k+1$ we define
   (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
   (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set
$$i^*\alpha = \sum n_j i^*[W_j]$$

(3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class $i^*\alpha$ to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

0B70 **Remark 42.29.2.** Let X be a scheme locally of finite type over S as in Situation 42.7.1. Let $(D, \mathcal{N}, \sigma)$ be a triple consisting of a locally principal (Divisors, Definition 31.13.1) closed subscheme $i : D \to X$, an invertible $\mathcal{O}_D$-module $\mathcal{N}$, and a surjection $\sigma : \mathcal{N}^{\otimes -1} \to i^*\mathcal{I}_D$ of $\mathcal{O}_D$-modules[2]. Here $\mathcal{N}$ should be thought of as a *virtual normal bundle of D in X*. The construction of $i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D)$ in Definition 42.29.1 generalizes to such triples, see Section 42.54.

[2]This condition assures us that if D is an effective Cartier divisor, then $\mathcal{N} = \mathcal{O}_X(D)|_D$.

0B7D **Remark 42.29.3.** Let X be a scheme locally of finite type over S as in Situation 42.7.1. In [**Ful98**] a *pseudo-divisor* on X is defined as a triple $D = (\mathcal{L}, Z, s)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, $Z \subset X$ is a closed subset, and $s \in \Gamma(X \setminus Z, \mathcal{L})$ is a nowhere vanishing section. Similarly to the above, one can define for every α in $\mathrm{CH}_{k+1}(X)$ a product $D \cdot \alpha$ in $\mathrm{CH}_k(Z \cap |\alpha|)$ where $|\alpha|$ is the support of α.

02T9 **Lemma 42.29.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Let α be a $(k+1)$-cycle on X. Then $i_* i^* \alpha = c_1(\mathcal{L}) \cap \alpha$ in $\mathrm{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.*

Proof. Write $\alpha = \sum n_j [W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of s we see that $D \cap W_j$ is the zero scheme of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 42.25.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j])$$

in $\mathrm{CH}_k(X)$ by Definition 42.25.1. The right hand side matches (termwise) the pushforward of the class $i^*\alpha$ on D from Definition 42.29.1. Hence we win. □

02TB **Lemma 42.29.5.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1.*

(1) *Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k+1$ and $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Then*

$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$

in $\mathrm{CH}_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}]$ and $[D \cap Z]_k = [\mathcal{O}_{D \cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 42.10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j [W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum m_j [D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(\mathrm{Supp}(M)) = 1$, the map $f : M \to M$ is injective, and $\mathrm{length}_A(M/fM) < \infty$. Moreover, $\mathrm{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 42.3.2. $\square$

0B6Z **Remark 42.29.6.** Let $X \to S$, $\mathcal{L}$, s, $i : D \to X$ be as in Definition 42.29.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \to Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$ is an integral closed subscheme. The possibility to do this will be useful later on.

0B6Y **Remark 42.29.7.** Let $f : X' \to X$ be a morphism of schemes locally of finite type over S as in Situation 42.7.1. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow[i']{} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

02TA **Lemma 42.29.8.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow[i']{} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 42.29.7. For any $(k+1)$-cycle α' on X' we have $i^*f_*\alpha' = g_*(i')^*\alpha'$ in $\mathrm{CH}_k(D)$ (this makes sense as f_* is defined on the level of cycles).

Proof. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 42.26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\mathrm{CH}_k(W)$ by the second assertion of Lemma 42.26.3. By Remark 42.19.6 the result follows for general α'. $\square$

0B71 **Lemma 42.29.9.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X' \to X$ be a flat morphism of relative dimension r of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow[i']{} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 42.29.7. For any $(k+1)$-cycle α on X we have $(i')^*f^*\alpha = g^*i^*\alpha$ in $\mathrm{CH}_{k+r}(D')$ (this makes sense as f^* is defined on the level of cycles).

Proof. Suppose $\alpha = [W]$ for some integral closed subscheme $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subschemes of D'. Hence the equality holds as cycles, see Lemma 42.14.4. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 42.26.2. By Remark 42.19.6 the result follows for general α'. □

42.30. Gysin homomorphisms and rational equivalence

02TK In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence. We also prove an important commutativity property.

02TM **Lemma 42.30.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module and let t be a nonzero meromorphic section of $\mathcal{N}$. Then $i^* div_{\mathcal{N}}(t) = c_1(\mathcal{N}|_D) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$.*

Proof. Write $\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$ for some integral closed subschemes $Z_i \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i, see Divisors, Lemmas 31.26.1, 31.26.4, and 31.27.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 42.27 to our current situation. For each i let $\xi_i \in Z_i$ be its generic point. Let $B_i = \mathcal{O}_{X,\xi_i}$. For each i we pick generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of s_i. We claim that

$$i^*\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\mathrm{div}_{\mathcal{N}}(t)$ and the fact that $\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 42.24.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 42.27.1) gives the equality

$$\sum \Big(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\Big) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^*\mathrm{div}_{\mathcal{N}}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D. This finishes the proof. □

02TO **Lemma 42.30.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\mathrm{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ D & \xrightarrow{i} & X \end{array}$$

of Remark 42.29.7. Since $X' \to X$ is proper we see that $i^*p_* = q_*(i')^*$ by Lemma 42.29.8. As we know that q_* factors through rational equivalence (Lemma 42.20.3), it suffices to prove the result for $\alpha' = \sum \mathrm{div}_{W_j}(f_j)$ on X'. Clearly this reduces us to the case where X is integral and $\alpha = \mathrm{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^*\alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 42.28.2. If $D \neq X$, then we see that $i^*\mathrm{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$ by Lemma 42.30.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map (Lemma 42.25.2). □

0F95 **Lemma 42.30.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Then $i^*i_* : \mathrm{CH}_k(D) \to \mathrm{CH}_{k-1}(D)$ sends α to $c_1(\mathcal{L}|_D) \cap \alpha$.*

Proof. This is immediate from the definition of i_* on cycles and the definition of i^* given in Definition 42.29.1. □

0B72 **Lemma 42.30.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$ in $\mathrm{CH}_{k-2}(D)$ for all $\alpha \in \mathrm{CH}_k(X)$.*

Proof. With exactly the same proof as in Lemma 42.30.2 this follows from Lemmas 42.26.4, 42.28.3, and 42.30.1. □

0B73 **Lemma 42.30.5.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ and $(\mathcal{L}', s', i' : D' \to X)$ be two triples as in Definition 42.29.1. Then the diagram*

$$\begin{array}{ccc} \mathrm{CH}_k(X) & \xrightarrow{i^*} & \mathrm{CH}_{k-1}(D) \\ {\scriptstyle (i')^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathrm{CH}_{k-1}(D') & \xrightarrow{(j')^*} & \mathrm{CH}_{k-2}(D \cap D') \end{array}$$

commutes where each of the maps is a Gysin map.

Proof. Denote $j : D \cap D' \to D$ and $j' : D \cap D' \to D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'}$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^*i^*\alpha = j^*(i')^*\alpha$ for all $\alpha \in \mathrm{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k. Let us prove the equality in case $\alpha = [W]$. We will deduce it from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we

require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\mathrm{div}_{\mathcal{L}|_W}(\sigma) = \sum \mathrm{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{L}'|_W}(\sigma') = \sum \mathrm{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 42.27. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i, let $\xi_i \in Z_i$ be the generic point. As in Section 42.27 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}_{W,\xi_i}$ and which is equal to the image of s, resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i\sigma_i$ and $\sigma' = f'_i\sigma'_i$ so that $n_i = \mathrm{ord}_{B_i}(f_i)$ and $n'_i = \mathrm{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^*i^*[W] = \sum \mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 42.27.1) now gives the equality

$$\sum \Big(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})\Big) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^*i^*[W]$ and $j^*(i')^*[W]$ on $D \cap D' \cap W$. By Remark 42.19.6 the result follows for general α. □

42.31. Relative effective Cartier divisors

02TP Relative effective Cartier divisors are defined and studied in Divisors, Section 31.18. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

02TR **Lemma 42.31.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be a relative effective Cartier divisor (Divisors, Definition 31.18.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \mathrm{CH}_{k+1}(Y)$ we have*

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $\mathrm{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $\mathrm{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension $k+1$. By Divisors, Lemma 31.18.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 42.29.5 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as schemes. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^*p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j[W_j]$ is a general $k+1$ cycle, then we get $i^*\alpha = \sum m_j i^*p^*[W_j] = \sum m_j(p|_D)^*[W_j]$ as

cycles. This proves then first equality. To deduce the second from the first apply Lemma 42.29.4. □

42.32. Affine bundles

02TS For an affine bundle the pullback map is surjective on Chow groups.

02TT **Lemma 42.32.1.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{array}{ccc} f^{-1}(Z_j) & \longrightarrow & X \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f} \\ Z_j & \longrightarrow & Y \end{array}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 42.19.6). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 42.14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(f^{-1}(V))$ of Lemma 42.19.3 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(W)/R(Y)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. □

0B74 **Lemma 42.32.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let*

$$p : L = \underline{\mathrm{Spec}}(Sym^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 42.32.1. Let $o : X \to L$ be the zero section of $L \to X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \to \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier divisor on L. Hence by Lemma 42.31.1 we see that $o^* \circ p^* = \mathrm{id}$ and we conclude that p^* is injective too. □

02TU **Remark 42.32.3.** We will see later (Lemma 42.36.3) that if X is a vector bundle of rank r over Y then the pullback map $\mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. This is true whenever $X \to Y$ satisfies the assumptions of Lemma 42.32.1, see [**Tot14**, Lemma 2.2]. We will sketch a proof in Remark 42.32.8 using higher chow groups.

0F96 **Lemma 42.32.4.** *In the situation of Lemma 42.32.2 denote $o : X \to L$ the zero section (see proof of the lemma). Then we have*

(1) *$o(X)$ is the zero scheme of a regular global section of $p^*\mathcal{L}^{\otimes -1}$,*
(2) *$o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(L)$ as o is a closed immersion,*
(3) *$o^* : \mathrm{CH}_{k+1}(L) \to \mathrm{CH}_k(X)$ as $o(X)$ is an effective Cartier divisor,*
(4) *$o^*p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(X)$ is the identity map,*
(5) *$o_*\alpha = -p^*(c_1(\mathcal{L}) \cap \alpha)$ for any $\alpha \in \mathrm{CH}_k(X)$, and*
(6) *$o^*o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(X)$ is equal to the map $\alpha \mapsto -c_1(\mathcal{L}) \cap \alpha$.*

Proof. Since $p_*\mathcal{O}_L = \mathrm{Sym}^*(\mathcal{L})$ we have $p_*(p^*\mathcal{L}^{\otimes -1}) = \mathrm{Sym}^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ by the projection formula (Cohomology, Lemma 20.54.2) and the section mentioned in (1) is the canonical trivialization $\mathcal{O}_X \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$. We omit the proof that the vanishing locus of this section is precisely $o(X)$. This proves (1).

Parts (2), (3), and (4) we've seen in the course of the proof of Lemma 42.32.2. Of course (4) is the first formula in Lemma 42.31.1.

Part (5) follows from the second formula in Lemma 42.31.1, additivity of capping with c_1 (Lemma 42.25.2), and the fact that capping with c_1 commutes with flat pullback (Lemma 42.26.2).

Part (6) follows from Lemma 42.30.3 and the fact that $o^*p^*\mathcal{L} = \mathcal{L}$. □

0F97 **Lemma 42.32.5.** *Let Y be a scheme. Let $\mathcal{L}_i$, $i = 1, 2$ be invertible $\mathcal{O}_Y$-modules. Let s be a global section of $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Denote $i : D \to X$ the zero scheme of s. Then there exists a commutative diagram*

$$\begin{array}{ccccc} D_1 & \xrightarrow{i_1} & L & \xleftarrow{i_2} & D_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_2} \\ D & \xrightarrow{i} & Y & \xleftarrow{i} & D \end{array}$$

and sections s_i of $p^*\mathcal{L}_i$ such that the following hold:

(1) *$p^*s = s_1 \otimes s_2$,*
(2) *p is of finite type and flat of relative dimension 1,*

(3) D_i *is the zero scheme of* s_i,
(4) $D_i \cong \underline{\mathrm{Spec}}(Sym^*(\mathcal{L}_{1-i}^{\otimes -1})|_D))$ *over* D *for* $i = 1, 2$,
(5) $p^{-1}D = D_1 \cup D_2$ *(scheme theoretic union),*
(6) $D_1 \cap D_2$ *(scheme theoretic intersection) maps isomorphically to* D*, and*
(7) $D_1 \cap D_2 \to D_i$ *is the zero section of the line bundle* $D_i \to D$ *for* $i = 1, 2$.

Moreover, the formation of this diagram and the sections s_i *commutes with arbitrary base change.*

Proof. Let $p : X \to Y$ be the relative spectrum of the quasi-coherent sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A} = \left(\bigoplus\nolimits_{a_1,a_2 \geq 0} \mathcal{L}_1^{\otimes -a_1} \otimes_{\mathcal{O}_Y} \mathcal{L}_2^{\otimes -a_2}\right) / \mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by local sections of the form $st - t$ for t a local section of any summand $\mathcal{L}_1^{\otimes -a_1} \otimes \mathcal{L}_2^{\otimes -a_2}$ with $a_1, a_2 > 0$. The sections s_i viewed as maps $p^*\mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$ are defined as the adjoints of the maps $\mathcal{L}_i^{\otimes -1} \to \mathcal{A} = p_*\mathcal{O}_X$. For any $y \in Y$ we can choose an affine open $V \subset Y$, say $V = \mathrm{Spec}(B)$, containing y and trivializations $z_i : \mathcal{O}_V \to \mathcal{L}_i^{\otimes -1}|_V$. Observe that $f = s(z_1 z_2) \in A$ cuts out the closed subscheme D. Then clearly

$$p^{-1}(V) = \mathrm{Spec}(B[z_1, z_2]/(z_1 z_2 - f))$$

Since D_i is cut out by z_i everything is clear. □

0F98 **Lemma 42.32.6.** *In the situation of Lemma 42.32.5 assume* Y *is locally of finite type over* (S, δ) *as in Situation 42.7.1. Then we have* $i_1^* p^* \alpha = p_1^* i^* \alpha$ *in* $\mathrm{CH}_k(D_1)$ *for all* $\alpha \in \mathrm{CH}_k(Y)$.

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We distinguish two cases.

Assume $W \subset D$. Then $i^*[W] = c_1(\mathcal{L}_1) \cap [W] + c_1(\mathcal{L}_2) \cap [W]$ in $\mathrm{CH}_{k-1}(D)$ by our definition of Gysin homomorphisms and the additivity of Lemma 42.25.2. Hence $p_1^* i^*[W] = p_1^*(c_1(\mathcal{L}_1) \cap [W]) + p_1^*(c_1(\mathcal{L}_2) \cap [W])$. On the other hand, we have $p^*[W] = [p^{-1}(W)]_{k+1}$ by construction of flat pullback. And $p^{-1}(W) = W_1 \cup W_2$ (scheme theoretically) where $W_i = p_i^{-1}(W)$ is a line bundle over W by the lemma (since formation of the diagram commutes with base change). Then $[p^{-1}(W)]_{k+1} = [W_1] + [W_2]$ as W_i are integral closed subschemes of L of δ-dimension $k + 1$. Hence

$$\begin{aligned} i_1^* p^*[W] &= i_1^*[p^{-1}(W)]_{k+1} \\ &= i_1^*([W_1] + [W_2]) \\ &= c_1(p_1^*\mathcal{L}_1) \cap [W_1] + [W_1 \cap W_2]_k \\ &= c_1(p_1^*\mathcal{L}_1) \cap p_1^*[W] + [W_1 \cap W_2]_k \\ &= p_1^*(c_1(\mathcal{L}_1) \cap [W]) + [W_1 \cap W_2]_k \end{aligned}$$

by construction of Gysin homomorphisms, the definition of flat pullback (for the second equality), and compatibility of $c_1 \cap -$ with flat pullback (Lemma 42.26.2). Since $W_1 \cap W_2$ is the zero section of the line bundle $W_1 \to W$ we see from Lemma 42.32.4 that $[W_1 \cap W_2]_k = p_1^*(c_1(\mathcal{L}_2) \cap [W])$. Note that here we use the fact that D_1 is the line bundle which is the relative spectrum of the inverse of $\mathcal{L}_2$. Thus we get the same thing as before.

Assume $W \not\subset D$. In this case, both $i_1^* p^*[W]$ and $p_1^* i^*[W]$ are represented by the $k - 1$ cycle associated to the scheme theoretic inverse image of W in D_1. □

0F99 **Lemma 42.32.7.** *In Situation 42.7.1 let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. There exists a commutative diagram*

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow{i} & X \end{array}$$

such that

(1) *p and g are of finite type and flat of relative dimension 1,*
(2) *$p^* : \mathrm{CH}_k(D) \to \mathrm{CH}_{k+1}(D')$ is injective for all k,*
(3) *$D' \subset X'$ is the zero scheme of a global section $s' \in \Gamma(X', \mathcal{O}_{X'})$,*
(4) *$p^* i^* = (i')^* g^*$ as maps $\mathrm{CH}_k(X) \to \mathrm{CH}_k(D')$.*

Moreover, these properties remain true after arbitrary base change by morphisms $Y \to X$ which are locally of finite type.

Proof. Observe that $(i')^*$ is defined because we have the triple $(\mathcal{O}_{X'}, s', i' : D' \to X')$ as in Definition 42.29.1. Thus the statement makes sense.

Set $\mathcal{L}_1 = \mathcal{O}_X$, $\mathcal{L}_2 = \mathcal{L}$ and apply Lemma 42.32.5 with the section s of $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Take $D' = D_1$. The results now follow from the lemma, from Lemma 42.32.6 and injectivity by Lemma 42.32.2. □

0GUB **Remark 42.32.8.** Let (S, δ) be as in Situation 42.7.1. Let Y be locally of finite type over S. Let $r \geq 0$. Let $f : X \to Y$ be a morphism of schemes. Assume every $y \in Y$ is contained in an open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. In this remark we sketch a proof of the fact that $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. First, by Lemma 42.32.1 the map is surjective. Let $\alpha \in \mathrm{CH}_k(Y)$ with $f^*\alpha = 0$. We will prove that $\alpha = 0$.

Step 1. We may assume that $\dim_\delta(Y) < \infty$. (This is immediate in all cases in practice so we suggest the reader skip this step.) Namely, any rational equivalence witnessing that $f^*\alpha = 0$ on X, will use a locally finite collection of integral closed subschemes of dimension $k + r + 1$. Taking the union of the closures of the images of these in Y we get a closed subscheme $Y' \subset Y$ of $\dim_\delta(Y') \leq k + r + 1$ such that α is the image of some $\alpha' \in \mathrm{CH}_k(Y')$ and such that $(f')^*\alpha = 0$ where f' is the base change of f to Y'.

Step 2. Assume $d = \dim_\delta(Y) < \infty$. Then we can use induction on d. If $d < k$, then $\alpha = 0$ and we are done; this is the base case of the induction. In general, our assumption on f shows we can choose a dense open $V \subset Y$ such that $U = f^{-1}(V) = \mathbf{A}^r_V$. Denote $Y' \subset Y$ the complement of V as a reduced closed subscheme and set $X' = f^{-1}(Y')$. Consider

$$\begin{array}{ccccccccc} \mathrm{CH}^M_{k+r}(U, 1) & \longrightarrow & \mathrm{CH}_{k+r}(X') & \longrightarrow & \mathrm{CH}_{k+r}(X) & \longrightarrow & \mathrm{CH}_{k+r}(U) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{CH}^M_k(V, 1) & \longrightarrow & \mathrm{CH}_k(Y') & \longrightarrow & \mathrm{CH}_k(Y) & \longrightarrow & \mathrm{CH}_k(V) & \longrightarrow & 0 \end{array}$$

Here we use the first higher Chow groups of V and U and the six term exact sequences constructed in Remark 42.27.3, as well as flat pullback for these higher chow groups and compatibility of flat pullback with these six term exact

sequences. Since $U = \mathbf{A}^r_V$ the vertical map on the right is an isomorphism. The map $\mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ is bijective by induction on d. Hence to finish the argument is suffices to show that

$$\mathrm{CH}^M_k(V, 1) \longrightarrow \mathrm{CH}^M_{k+r}(U, 1)$$

is surjective. Arguing as in the proof of Lemma 42.32.1 this reduces to Step 3 below.

Step 3. Let F be a field. Then $\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = 0$. (In the proof of the lemma cited above we proved analogously that $\mathrm{CH}_0(\mathbf{A}^1_F) = 0$.) We have

$$\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = \mathrm{Coker}\left(\partial : K^M_2(F(T)) \longrightarrow \bigoplus\nolimits_{\mathfrak{p} \subset F[T] \text{ maximal}} \kappa(\mathfrak{p})^*\right)$$

The classical argument for the vanishing of the cokernel is to show by induction on the degree of $\kappa(\mathfrak{p})/F$ that the summand corresponding to $\mathfrak{p}$ is in the image. If $\mathfrak{p}$ is generated by the irreducible monic polynomial $P(T) \in F[T]$ and if $u \in \kappa(x)^*$ is the residue class of some $Q(T) \in F[T]$ with $\deg(Q) < \deg(P)$ then one shows that $\partial(Q, P)$ produces the element u at $\mathfrak{p}$ and perhaps some other units at primes dividing Q which have lower degree. This finishes the sketch of the proof.

42.33. Bivariant intersection theory

0B75 In order to intelligently talk about higher Chern classes of vector bundles we introduce bivariant chow classes as in [**Ful98**]. Our definition differs from [**Ful98**] in two respects: (1) we work in a different setting, and (2) we only require our bivariant classes commute with the Gysin homomorphisms for zero schemes of sections of invertible modules (Section 42.29). We will see later, in Lemma 42.54.8, that our bivariant classes commute with all higher codimension Gysin homomorphisms and hence satisfy all properties required of them in [**Ful98**]; see also [**Ful98**, Theorem 17.1].

0B76 **Definition 42.33.1.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. A *bivariant class c of degree p for f* is given by a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map Similar to [**Ful98**, Definition 17.1]

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

(1) if $Y'' \to Y'$ is a proper, then $c \cap (Y'' \to Y')_* \alpha'' = (X'' \to X')_*(c \cap \alpha'')$ for all α'' on Y'' where $X'' = Y'' \times_Y X$,
(2) if $Y'' \to Y'$ is flat locally of finite type of fixed relative dimension, then $c \cap (Y'' \to Y')^* \alpha' = (X'' \to X')^*(c \cap \alpha')$ for all α' on Y', and
(3) if $(\mathcal{L}', s', i' : D' \to Y')$ is as in Definition 42.29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X', then we have $c \cap (i')^* \alpha' = (j')^*(c \cap \alpha')$ for all α' on Y'.

The collection of all bivariant classes of degree p for f is denoted $A^p(X \to Y)$.

Let (S, δ) be as in Situation 42.7.1. Let $X \to Y$ and $Y \to Z$ be morphisms of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. It is clear that $A^p(X \to Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \to Y) \times A^q(Y \to Z) \to A^{p+q}(X \to Z)$$

which is associative.

0B78 **Lemma 42.33.2.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a flat morphism of relative dimension r between schemes locally of finite type over S. Then the rule that to $Y' \to Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.*

Proof. This follows from Lemmas 42.20.2, 42.14.3, 42.15.1, and 42.29.9. □

0B79 **Lemma 42.33.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. Then the rule that to $f : X' \to X$ assigns $(i')^* : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 42.30.2, 42.29.8, 42.29.9, and 42.30.5. □

0EPK **Lemma 42.33.4.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. Let $c \in A^p(X \to Z)$ and assume f is proper. Then the rule that to $Z' \to Z$ assigns $\alpha \longmapsto f'_*(c \cap \alpha)$ is a bivariant class denoted $f_* \circ c \in A^p(Y \to Z)$.*

Proof. This follows from Lemmas 42.12.2, 42.15.1, and 42.29.8. □

0F9Z **Remark 42.33.5.** Let (S, δ) be as in Situation 42.7.1. Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes locally of finite type over S. Let $X' = Y' \times_Y X$. Then there is an obvious restriction map

$$A^p(X \to Y) \longrightarrow A^p(X' \to Y'), \quad c \longmapsto res(c)$$

obtained by viewing a scheme Y'' locally of finite type over Y' as a scheme locally of finite type over Y and setting $res(c) \cap \alpha'' = c \cap \alpha''$ for any $\alpha'' \in \mathrm{CH}_k(Y'')$. This restriction operation is compatible with compositions in an obvious manner.

0FA2 **Remark 42.33.6.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. For $i = 1, 2$ let $Z_i \to X$ be a morphism of schemes locally of finite type. Let $c_i \in A^{p_i}(Z_i \to X)$, $i = 1, 2$ be bivariant classes. For any $\alpha \in \mathrm{CH}_k(X)$ we can ask whether

$$c_1 \cap c_2 \cap \alpha = c_2 \cap c_1 \cap \alpha$$

in $\mathrm{CH}_{k-p_1-p_2}(Z_1 \times_X Z_2)$. If this is true and if it holds after any base change by $X' \to X$ locally of finite type, then we say c_1 and c_2 *commute*. Of course this is the same thing as saying that

$$res(c_1) \circ c_2 = res(c_2) \circ c_1$$

in $A^{p_1+p_2}(Z_1 \times_X Z_2 \to X)$. Here $res(c_1) \in A^{p_1}(Z_1 \times_X Z_2 \to Z_2)$ is the restriction of c_1 as in Remark 42.33.5; similarly for $res(c_2)$.

0FA3 **Example 42.33.7.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ a triple as in Definition 42.29.1. Let $Z \to X$ be a morphism of schemes locally of finite type and let $c \in A^p(Z \to X)$ be a bivariant class. Then the bivariant Gysin class $c' \in A^1(D \to X)$ of Lemma 42.33.3 commutes with c in the sense of Remark 42.33.6. Namely, this is a restatement of condition (3) of Definition 42.33.1.

0FDU **Remark 42.33.8.** There is a more general type of bivariant class that doesn't seem to be considered in the literature. Namely, suppose we are given a diagram

$$X \longrightarrow Z \longleftarrow Y$$

of schemes locally of finite type over (S, δ) as in Situation 42.7.1. Let $p \in \mathbf{Z}$. Then we can consider a rule c which assigns to every $Z' \to Z$ locally of finite type maps

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

for all $k \in \mathbf{Z}$ where $X' = X \times_Z Z'$ and $Y' = Z' \times_Z Y$ compatible with

(1) proper pushforward if given $Z'' \to Z'$ proper,
(2) flat pullback if given $Z'' \to Z'$ flat of fixed relative dimension, and
(3) Gysin maps if given $D' \subset Z'$ as in Definition 42.29.1.

We omit the detailed formulations. Suppose we denote the collection of all such operations $A^p(X \to Z \leftarrow Y)$. A simple example of the utility of this concept is when we have a proper morphism $f : X_2 \to X_1$. Then f_* isn't a bivariant operation in the sense of Definition 42.33.1 but it is in the above generalized sense, namely, $f_* \in A^0(X_1 \to X_1 \leftarrow X_2)$.

42.34. Chow cohomology and the first Chern class

0FDV We will be most interested in $A^p(X) = A^p(X \to X)$, which will always mean the bivariant cohomology classes for id_X. Namely, that is where Chern classes will live.

0B7E **Definition 42.34.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The *Chow cohomology* of X is the graded $\mathbf{Z}$-algebra $A^*(X)$ whose degree p component is $A^p(X \to X)$.

Warning: It is not clear that the $\mathbf{Z}$-algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

0B7F **Remark 42.34.2.** Let (S, δ) be as in Situation 42.7.1. Let $f : Y' \to Y$ be a morphism of schemes locally of finite type over S. As a special case of Remark 42.33.5 there is a canonical $\mathbf{Z}$-algebra map $res : A^*(Y) \to A^*(Y')$. This map is often denoted f^* in the literature.

0B77 **Lemma 42.34.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the rule that to $f : X' \to X$ assigns $c_1(f^*\mathcal{L}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(X')$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 42.28.2, 42.26.4, 42.26.2, and 42.30.4. □

The lemma above finally allows us to make the following definition.

0FDW **Definition 42.34.4.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The *first Chern class* $c_1(\mathcal{L}) \in A^1(X)$ of $\mathcal{L}$ is the bivariant class of Lemma 42.34.3.

For finite locally free modules we construct the Chern classes in Section 42.38. Let us prove that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

0B7B **Lemma 42.34.5.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

(1) *$c_1(\mathcal{L}) \in A^1(X)$ is in the center of $A^*(X)$ and*
(2) *if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_1(\mathcal{L}) = c_1(f^*\mathcal{L}) \circ c$.*

Proof. Of course (2) implies (1). Let $p : L \to X$ be as in Lemma 42.32.2 and let $o : X \to L$ be the zero section. Denote $p' : L' \to X'$ and $o' : X' \to L'$ their base changes. By Lemma 42.32.4 we have

$$p^*(c_1(\mathcal{L}) \cap \alpha) = -o_*\alpha \quad \text{and} \quad (p')^*(c_1(f^*\mathcal{L}) \cap \alpha') = -o'_*\alpha'$$

Since c is a bivariant class we have

$$\begin{aligned}(p')^*(c \cap c_1(\mathcal{L}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}) \cap \alpha) \\ &= -c \cap o_*\alpha \\ &= -o'_*(c \cap \alpha) \\ &= (p')^*(c_1(f^*\mathcal{L}) \cap c \cap \alpha)\end{aligned}$$

Since $(p')^*$ is injective by one of the lemmas cited above we obtain $c \cap c_1(\mathcal{L}) \cap \alpha = c_1(f^*\mathcal{L}) \cap c \cap \alpha$. The same is true after any base change by $Y \to X$ locally of finite type and hence we have the equality of bivariant classes stated in (2). □

0FDX **Lemma 42.34.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be a finite type scheme over S which has an ample invertible sheaf. Assume $d = \dim(X) < \infty$ (here we really mean dimension and not δ-dimension). Then for any invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ on X we have $c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_{d+1}) = 0$ in $A^{d+1}(X)$.*

Proof. We prove this by induction on d. The base case $d = 0$ is true because in this case X is a finite set of closed points and hence every invertible module is trivial. Assume $d > 0$. By Divisors, Lemma 31.15.12 we can write $\mathcal{L}_{d+1} \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$. Then $c_1(\mathcal{L}_{d+1})$ is the difference of $c_1(\mathcal{O}_X(D))$ and $c_1(\mathcal{O}_X(D'))$ and hence we may assume $\mathcal{L}_{d+1} = \mathcal{O}_X(D)$ for some effective Cartier divisor.

Denote $i : D \to X$ the inclusion morphism and denote $i^* \in A^1(D \to X)$ the bivariant class given by the Gysin hommomorphism as in Lemma 42.33.3. We have $i_* \circ i^* = c_1(\mathcal{L}_{d+1})$ in $A^1(X)$ by Lemma 42.29.4 (and Lemma 42.33.4 to make sense of the left hand side). Since $c_1(\mathcal{L}_i)$ commutes with both i_* and i^* (by definition of bivariant classes) we conclude that

$$c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_{d+1}) = i_* \circ c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_d) \circ i^* = i_* \circ c_1(\mathcal{L}_1|_D) \circ \ldots \circ c_1(\mathcal{L}_d|_D) \circ i^*$$

Thus we conclude by induction on d. Namely, we have $\dim(D) < d$ as none of the generic points of X are in D. □

0FA0 **Remark 42.34.7.** Let (S, δ) be as in Situation 42.7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S and let $p \geq 0$. In this setting we define

$$A^{(p)}(Z \to X) = \prod\nolimits_{i \leq p-1} A^i(X) \times \prod\nolimits_{i \geq p} A^i(Z \to X).$$

Then $A^{(p)}(Z \to X)$ canonically comes equipped with the structure of a graded algebra. In fact, more generally there is a multiplication

$$A^{(p)}(Z \to X) \times A^{(q)}(Z \to X) \longrightarrow A^{(\max(p,q))}(Z \to X)$$

In order to define these we define maps

$$\begin{aligned}A^i(Z \to X) \times A^j(X) &\to A^{i+j}(Z \to X) \\ A^i(X) \times A^j(Z \to X) &\to A^{i+j}(Z \to X) \\ A^i(Z \to X) \times A^j(Z \to X) &\to A^{i+j}(Z \to X)\end{aligned}$$

For the first we use composition of bivariant classes. For the second we use restriction $A^i(X) \to A^i(Z)$ (Remark 42.33.5) and composition $A^i(Z) \times A^j(Z \to X) \to A^{i+j}(Z \to X)$. For the third, we send (c, c') to $res(c) \circ c'$ where $res : A^i(Z \to X) \to A^i(Z)$ is the restriction map (see Remark 42.33.5). We omit the verification that these multiplications are associative in a suitable sense.

0FA1 **Remark 42.34.8.** Let (S, δ) be as in Situation 42.7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S. Denote $res : A^p(Z \to X) \to A^p(Z)$ the restriction map of Remark 42.33.5. For $c \in A^p(Z \to X)$ we have $res(c) \cap \alpha = c \cap i_*\alpha$ for $\alpha \in \mathrm{CH}_*(Z)$. Namely $res(c) \cap \alpha = c \cap \alpha$ and compatibility of c with proper pushforward gives $(Z \to Z)_*(c \cap \alpha) = c \cap (Z \to X)_*\alpha$.

42.35. Lemmas on bivariant classes

0FDY In this section we prove some elementary results on bivariant classes. Here is a criterion to see that an operation passes through rational equivalence.

0B7A **Lemma 42.35.1.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map*

$$c \cap - : Z_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying condition (3) of Definition 42.33.1 whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ factors through rational equivalence.

Very weak form of [**Ful98**, Theorem 17.1]

Proof. The statement makes sense because given a triple $(\mathcal{L}, s, i : D \to X)$ as in Definition 42.29.1 such that $\mathcal{L}|_D \cong \mathcal{O}_D$, then the operation i^* is defined on the level of cycles, see Remark 42.29.6. Let $\alpha \in Z_k(X')$ be a cycle which is rationally equivalent to zero. We have to show that $c \cap \alpha = 0$. By Lemma 42.21.1 there exists a cycle $\beta \in Z_{k+1}(X' \times \mathbf{P}^1)$ such that $\alpha = i_0^*\beta - i_\infty^*\beta$ where $i_0, i_\infty : X' \to X' \times \mathbf{P}^1$ are the closed immersions of X' over $0, \infty$. Since these are examples of effective Cartier divisors with trivial normal bundles, we see that $c \cap i_0^*\beta = j_0^*(c \cap \beta)$ and $c \cap i_\infty^*\beta = j_\infty^*(c \cap \beta)$ where $j_0, j_\infty : Y' \to Y' \times \mathbf{P}^1$ are closed immersions as before. Since $j_0^*(c \cap \beta) \sim_{rat} j_\infty^*(c \cap \beta)$ (follows from Lemma 42.21.1) we conclude. □

0F9A **Lemma 42.35.2.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map*

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying conditions (1), (2) of Definition 42.33.1 and condition (3) whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ is a bivariant class.

Weak form of [**Ful98**, Theorem 17.1]

Proof. Let $Y' \to Y$ be a morphism of schemes which is locally of finite type. Let $(\mathcal{L}', s', i' : D' \to Y')$ be as in Definition 42.29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X'. We have to show that $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all $\alpha' \in \mathrm{CH}_k(Y')$.

Denote $g : Y'' \to Y'$ the smooth morphism of relative dimension 1 with $i'' : D'' \to Y''$ and $p : D'' \to D'$ constructed in Lemma 42.32.7. (Warning: D'' isn't the full inverse image of D'.) Denote $f : X'' \to X'$ and $E'' \subset X''$ their base changes by

$X' \to Y'$. Picture

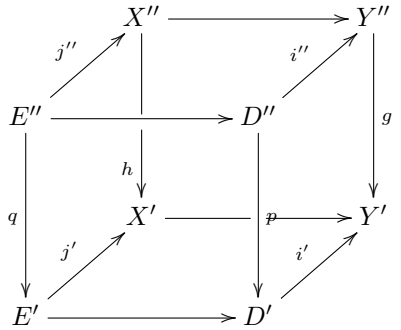

By the properties given in the lemma we know that $\beta' = (i')^*\alpha'$ is the unique element of $\mathrm{CH}_{k-1}(D')$ such that $p^*\beta' = (i'')^*g^*\alpha'$. Similarly, we know that $\gamma' = (j')^*(c \cap \alpha')$ is the unique element of $\mathrm{CH}_{k-1-p}(E')$ such that $q^*\gamma' = (j'')^*h^*(c \cap \alpha')$. Since we know that

$$(j'')^*h^*(c \cap \alpha') = (j'')^*(c \cap g^*\alpha') = c \cap (i'')^*g^*\alpha'$$

by our assumptions on c; note that the modified version of (3) assumed in the statement of the lemma applies to i'' and its base change j''. We similarly know that

$$q^*(c \cap \beta') = c \cap p^*\beta'$$

We conclude that $\gamma' = c \cap \beta'$ by the uniqueness pointed out above. □

Here a criterion for when a bivariant class is zero.

02UC **Lemma 42.35.3.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $c \in A^p(X \to Y)$. For $Y'' \to Y' \to Y$ set $X'' = Y'' \times_Y X$ and $X' = Y' \times_Y X$. The following are equivalent*

(1) *c is zero,*
(2) *$c \cap [Y'] = 0$ in $\mathrm{CH}_*(X')$ for every integral scheme Y' locally of finite type over Y, and*
(3) *for every integral scheme Y' locally of finite type over Y, there exists a proper birational morphism $Y'' \to Y'$ such that $c \cap [Y''] = 0$ in $\mathrm{CH}_*(X'')$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. Assumption (3) implies (2) because $(Y'' \to Y')_*[Y''] = [Y']$ and hence $c \cap [Y'] = (X'' \to X')_*(c \cap [Y''])$ as c is a bivariant class. Assume (2). Let $Y' \to Y$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(Y')$. Write $\alpha = \sum n_i[Y'_i]$ with $Y'_i \subset Y'$ a locally finite collection of integral closed subschemes of δ-dimension k. Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y'_i]$ on $Y'' = \coprod Y'_i$ under the proper morphism $Y'' \to Y'$. By the properties of bivariant classes it suffices to prove that $c \cap \alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(X'_i)$ where $X'_i = Y'_i \times_Y X$. This follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \to \mathrm{CH}_{k-p}(X'_i)$ are given by flat pullback. Since capping with c commutes with flat pullback, we see that it suffices to show that $c \cap [Y'_i]$ is zero in $\mathrm{CH}_{k-p}(X'_i)$ which is true by assumption. □

0FDZ **Lemma 42.35.4.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Assume we have disjoint union*

decompositions $X = \coprod_{i\in I} X_i$ and $Y = \coprod_{j\in J} Y_j$ by open and closed subschemes and a map $a : I \to J$ of sets such that $f(X_i) \subset Y_{a(i)}$. Then

$$A^p(X \to Y) = \prod\nolimits_{i\in I} A^p(X_i \to Y_{a(i)})$$

Proof. Suppose given an element $(c_i) \in \prod_i A^p(X_i \to Y_{a(i)})$. Then given $\beta \in \mathrm{CH}_k(Y)$ we can map this to the element of $\mathrm{CH}_{k-p}(X)$ whose restriction to X_i is $c_i \cap \beta|_{Y_{a(i)}}$. This works because $\mathrm{CH}_{k-p}(X) = \prod_i \mathrm{CH}_{k-p}(X_i)$. The same construction works after base change by any $Y' \to Y$ locally of finite type and we get $c \in A^p(X \to Y)$. Thus we obtain a map Ψ from the right hand side of the formula to the left hand side of the formula. Conversely, given $c \in A^p(X \to Y)$ and an element $\beta_i \in \mathrm{CH}_k(Y_{a(i)})$ we can consider the element $(c \cap (Y_{a(i)} \to Y)_*\beta_i)|_{X_i}$ in $\mathrm{CH}_{k-p}(X_i)$. The same thing works after base change by any $Y' \to Y$ locally of finite type and we get $c_i \in A^p(X_i \to Y_{a(i)})$. Thus we obtain a map Φ from the left hand side of the formula to the right hand side of the formula. It is immediate that $\Phi \circ \Psi = \mathrm{id}$. For the converse, suppose that $c \in A^p(X \to Y)$ and $\beta \in \mathrm{CH}_k(Y)$. Say $\Phi(c) = (c_i)$. Let $j \in J$. Because c commutes with flat pullback we get

$$(c \cap \beta)|_{\coprod_{a(i)=j} X_i} = c \cap \beta|_{Y_j}$$

Because c commutes with proper pushforward we get

$$(\coprod\nolimits_{a(i)=j} X_i \to X)_*((c \cap \beta)|_{\coprod_{a(i)=j} X_i}) = c \cap (Y_j \to Y)_*\beta|_{Y_j}$$

The left hand side is the cycle on X restricting to $(c \cap \beta)|_{X_i}$ on X_i for $i \in I$ with $a(i) = j$ and 0 else. The right hand side is a cycle on X whose restriction to X_i is $c_i \cap \beta|_{Y_j}$ for $i \in I$ with $a(i) = j$. Thus $c \cap \beta = \Psi((c_i))$ as desired. □

0FE0 **Remark 42.35.5.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $X = \coprod_{i\in I} X_i$ and $Y = \coprod_{j\in J} Y_j$ be the decomposition of X and Y into their connected components (the connected components are open as X and Y are locally Noetherian, see Topology, Lemma 5.9.6 and Properties, Lemma 28.5.5). Let $a(i) \in J$ be the index such that $f(X_i) \subset Y_{a(i)}$. Then $A^p(X \to Y) = \prod A^p(X_i \to Y_{a(i)})$ by Lemma 42.35.4. In this setting it is convenient to set

$$A^*(X \to Y)^\wedge = \prod\nolimits_i A^*(X_i \to Y_{a(i)})$$

This "completed" bivariant group is the subset

$$A^*(X \to Y)^\wedge \quad \subset \quad \prod\nolimits_{p\geq 0} A^p(X)$$

consisting of elements $c = (c_0, c_1, c_2, \ldots)$ such that for each connected component X_i the image of c_p in $A^p(X_i \to Y_{a(i)})$ is zero for almost all p. If $Y \to Z$ is a second morphism, then the composition $A^*(X \to Y) \times A^*(Y \to Z) \to A^*(X \to Z)$ extends to a composition $A^*(X \to Y)^\wedge \times A^*(Y \to Z)^\wedge \to A^*(X \to Z)^\wedge$ of completions. We sometimes call $A^*(X)^\wedge = A^*(X \to X)^\wedge$ the *completed bivariant cohomology ring* of X.

0GUC **Lemma 42.35.6.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $g : Y' \to Y$ be an envelope*

(Definition 42.22.1) and denote $X' = Y' \times_Y X$. Let $p \in \mathbf{Z}$ and let $c' \in A^p(X' \to Y')$. If the two restrictions

$$res_1(c') = res_2(c') \in A^p(X' \times_X X' \to Y' \times_Y Y')$$

are equal (see proof), then there exists a unique $c \in A^p(X \to Y)$ whose restriction $res(c) = c'$ in $A^p(X' \to Y')$.

Proof. We have a commutative diagram

$$\begin{array}{ccccc} X' \times_X X' & \xrightarrow[b]{a} & X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' \times_Y Y' & \xrightarrow[q]{p} & Y' & \xrightarrow{g} & Y \end{array}$$

The element $res_1(c')$ is the restriction (see Remark 42.33.5) of c' for the cartesian square with morphisms a, f', p, f'' and the element $res_2(c')$ is the restriction of c' for the cartesian square with morphisms b, f', q, f''. Assume $res_1(c') = res_2(c')$ and let $\beta \in \mathrm{CH}_k(Y)$. By Lemma 42.22.4 we can find a $\beta' \in \mathrm{CH}_k(Y')$ with $g_*\beta' = \beta$. Then we set

$$c \cap \beta = h_*(c' \cap \beta')$$

To see that this is independent of the choice of β' it suffices to show that $h_*(c' \cap (p_*\gamma - q_*\gamma))$ is zero for $\gamma \in \mathrm{CH}_k(Y' \times_Y Y')$. Since c' is a bivariant class we have

$$h_*(c' \cap (p_*\gamma - q_*\gamma)) = h_*(a_*(c' \cap \gamma) - b_*(c' \cap \gamma)) = 0$$

the last equality since $h_* \circ a_* = h_* \circ b_*$ as $h \circ a = h \circ b$.

Observe that our choice for $c \cap \beta$ is forced by the requirement that $res(c) = c'$ and the compatibility of bivariant classes with proper pushforward.

Of course, in order to define the bivariant class c we need to construct maps $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(Y_1 \times_Y X)$ for any morphism $Y_1 \to Y$ locally of finite type satisfying the conditions listed in Definition 42.33.1. Denote $Y_1' = Y' \times_Y Y_1$, $X_1 = X \times_Y Y_1$. The morphism $Y_1' \to Y_1$ is an envelope by Lemma 42.22.3. Hence we can use the base changed diagram

$$\begin{array}{ccccc} X_1' \times_{X_1} X_1' & \xrightarrow[b_1]{a_1} & X_1' & \xrightarrow{h_1} & X_1 \\ \downarrow{\scriptstyle f_1''} & & \downarrow{\scriptstyle f_1'} & & \downarrow{\scriptstyle f_1} \\ Y_1' \times_{Y_1} Y_1' & \xrightarrow[q_1]{p_1} & Y_1' & \xrightarrow{g_1} & Y_1 \end{array}$$

and the same arguments to get a well defined map $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(X_1)$ as before.

Next, we have to check conditions (1), (2), and (3) of Definition 42.33.1 for c. For example, suppose that $t : Y_2 \to Y_1$ is a proper morphism of schemes locally of finite type over Y. Denote as above the base changes of the first diagram to Y_1, resp. Y_2, by subscripts $_1$, resp. $_2$. Denote $t' : Y_2' \to Y_1'$, $s : X_2 \to X_1$, and $s' : X_2' \to X_1'$ the base changes of t to Y', X, and X'. We have to show that

$$s_*(c \cap \beta_2) = c \cap t_*\beta_2$$

for $\beta_2 \in \mathrm{CH}_k(Y_2)$. Choose $\beta_2' \in \mathrm{CH}_k(Y_2')$ with $g_{2,*}\beta_2' = \beta_2$. Since c' is a bivariant class and the diagrams

$$\begin{array}{ccc} X_2' & \xrightarrow[h_2]{} & X_2 \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ X_1' & \xrightarrow{h_1} & X_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_2' & \xrightarrow[f_2']{} & Y_2' \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle t'} \\ X_2' & \xrightarrow{f_1'} & Y_1' \end{array}$$

are cartesian we have

$$s_*(c \cap \beta_2) = s_*(h_{2,*}(c' \cap \beta_2')) = h_{1,*}s'_*(c' \cap \beta_2') = h_{1,*}(c' \cap (t'_*\beta_2'))$$

and the final expression computes $c \cap t_*\beta_2$ by construction: $t'_*\beta_2' \in \mathrm{CH}_k(Y_1')$ is a class whose image by $g_{1,*}$ is $t_*\beta_2$. This proves condition (1). The other conditions are proved in the same manner and we omit the detailed arguments. □

42.36. Projective space bundle formula

02TV Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

02TW **Lemma 42.36.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in \mathrm{CH}_k(X)$ the element*

$$\pi_*\left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in \mathrm{CH}_{k+r-1-s}(X)$$

is 0 if $s < r-1$ and is equal to α when $s = r-1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r-1$. If $s < r-1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k+r-1-s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section

$Z \to \pi^{-1}(Z)$. Hence we see successively that

$$\begin{aligned} \pi^*[Z] &= [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] &= [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] &= [Z(s_1) \cap Z(s_2)] \\ \ldots &= \ldots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] &= [Z(s_1) \cap \ldots \cap Z(s_{r-1})] \end{aligned}$$

by repeated applications of Lemma 42.25.4. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. □

02TX **Lemma 42.36.2** (Projective space bundle formula)**.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} \mathrm{CH}_{k+i}(X) \longrightarrow \mathrm{CH}_{k+r-1}(P),$$
$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 42.36.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. Consider the commutative diagram

$$\begin{array}{ccc} \coprod P_i & \xrightarrow{p} & P \\ {\scriptstyle \coprod \pi_i}\downarrow & & \downarrow{\scriptstyle \pi} \\ \coprod X_i & \xrightarrow{q} & X \end{array}$$

Observe that p_* is surjective. If $\beta \in \mathrm{CH}_k(\coprod X_i)$ then $\pi^*q_*\beta = p_*(\coprod \pi_i)^*\beta$, see Lemma 42.15.1. Similarly for capping with $c_1(\mathcal{O}(1))$ by Lemma 42.26.4. Hence, if the map of the lemma is surjective for each of the morphisms $\pi_i : P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus \mathrm{CH}_{k+i}(T) \to \mathrm{CH}_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha_i' \in \mathrm{CH}_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha_0', \ldots, \alpha_{r-1}')$

is by Lemma 42.19.3 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$ as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \amalg \mathbf{A}^{r-2} \amalg \ldots \amalg \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \amalg U^{r-2} \amalg \ldots \amalg U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. By Lemma 42.32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in \mathrm{CH}_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in \mathrm{CH}_{k+r-1}(P^{r-2})$. By Lemma 42.32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in \mathrm{CH}_{k+1}(X)$. By Lemma 42.31.1 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in \mathrm{CH}_{k+r-1}(P^{r-3})$. And so on. □

02TY **Lemma 42.36.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{\mathrm{Spec}}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is an isomorphism for all k.

Proof. (For the case of linebundles, see Lemma 42.32.2.) For surjectivity see Lemma 42.32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 31.14.10). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 42.19.3. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 42.36.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 42.31.1 and 42.26.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 42.26.4 unless all α and all β_i are zero. □

42.37. The Chern classes of a vector bundle

02TZ We can use the projective space bundle formula to define the Chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (42.37.1.1). The reason for the signs will be explained later.

02U0 **Definition 42.37.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 42.36.2 there are elements $c_i \in \mathrm{CH}_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

02U1 (42.37.1.1) $$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0.$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $\mathrm{CH}_{n-i}(X)$. We call these the *Chern classes of $\mathcal{E}$ on X*.

(3) The *total Chern class of $\mathcal{E}$ on X* is the combination
$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$
which is an element of $\mathrm{CH}_*(X) = \bigoplus_{k \in \mathbf{Z}} \mathrm{CH}_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

02U2 **Lemma 42.37.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class of $\mathcal{L}$ on X of Definition 42.37.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 42.24.1.*

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 42.24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (42.37.1.1) reads
$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$
Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. □

02U3 **Remark 42.37.3.** We could also rewrite equation 42.37.1.1 as

05M8 (42.37.3.1) $$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0.$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

42.38. Intersecting with Chern classes

02U4 In this section we define Chern classes of vector bundles on X as bivariant classes on X, see Lemma 42.38.7 and the discussion following this lemma. Our construction follows the familiar pattern of first defining the operation on prime cycles and then summing. In Lemma 42.38.2 we show that the result is determined by the usual formula on the associated projective bundle. Next, we show that capping with Chern classes passes through rational equivalence, commutes with proper pushforward, commutes with flat pullback, and commutes with the Gysin maps for inclusions of effective Cartier divisors. These lemmas could have been avoided by directly

using the characterization in Lemma 42.38.2 and using Lemma 42.33.4; the reader who wishes to see this worked out should consult Chow Groups of Spaces, Lemma 82.28.1.

02U5 **Definition 42.38.1.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to \mathrm{CH}_{k-j}(X)$$

called *intersection with the jth Chern class of $\mathcal{E}$.*

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in \mathrm{CH}_{k-j}(X)$$

where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 42.37.1.

(2) For a general k-cycle $\alpha = \sum n_i[W_i]$ we set

$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

If $\mathcal{E}$ has rank 1 then this agrees with our previous definition (Definition 42.25.1) by Lemma 42.37.2.

02U6 **Lemma 42.38.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $\mathrm{CH}_{k-j}(X)$ such that $\alpha_0 = \alpha$ and*

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 42.36.2. The identity holds by definition for $\alpha = [W]$ where W is an integral closed subscheme of X. For a general k-cycle α on X write $\alpha = \sum n_a[W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram

$$\begin{array}{ccccc} P' & = & \coprod P_a & \xrightarrow{\coprod i'_a} & P \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \coprod \pi_a} & & \downarrow{\scriptstyle \pi} \\ X' & = & \coprod W_a & \xrightarrow{\coprod i_a} & X \end{array}$$

The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $\mathrm{CH}_{k-j}(W_a)$. We have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each a by definition. Thus clearly we have

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$

with $\beta_j = \sum n_a \beta_{a,j} \in \mathrm{CH}_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^* \beta_j$ by Lemma 42.15.1. By the projection formula of Lemma 42.26.4 we conclude that

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\beta_j) = 0$$

Since $p_*\beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. □

We will consistently use this characterization of Chern classes to prove many more properties.

02U7 **Lemma 42.38.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $\mathrm{CH}_{k-j}(X)$.*

Proof. By Lemma 42.38.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the *same* equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 42.20.2.) Hence they are equal. □

In other words capping with Chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-j}(X).$$

Our next task is to show that Chern classes are bivariant classes, see Definition 42.33.1.

02U9 **Lemma 42.38.4.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then*

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*\alpha$$

Proof. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Then $P_X = X \times_Y P$ is the projective bundle associated to $p^*\mathcal{E}$ and $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of P_X. Consider the fibre product diagram

$$\begin{array}{ccc} P_X & \xrightarrow{p'} & P \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{p} & Y \end{array}$$

Apply proper pushforward p'_* (Lemma 42.20.3) to the displayed equality above. Using Lemmas 42.26.4 and 42.15.1 we obtain

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\alpha_{r-i}) = 0$$

in the chow group of P. By the characterization of Lemma 42.38.2 we conclude. □

02U8 **Lemma 42.38.5.** *Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*\alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{array}{ccc} P_X = \mathbf{P}(f^*\mathcal{E}) & \xrightarrow{f'} & P \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

Note that $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply flat pullback $(f')^*$ (Lemma 42.20.2) to the displayed equation above. By Lemmas 42.26.2 and 42.14.3 we see that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. By the characterization of Lemma 42.38.2 we conclude. □

0B7G **Lemma 42.38.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 42.29.1. Then $c_j(\mathcal{E}|_D) \cap i^*\alpha = i^*(c_j(\mathcal{E}) \cap \alpha)$ for all $\alpha \in \mathrm{CH}_k(X)$.*

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to X, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{array}{ccc} P_D = \mathbf{P}(\mathcal{E}|_D) & \xrightarrow{i'} & P \\ \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{i} & X \end{array}$$

Note that $\mathcal{O}_{P_D}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply the Gysin map $(i')^*$ (Lemma 42.30.2) to the displayed equation above. Applying Lemmas 42.30.4 and 42.29.9 we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_D}(1))^i \cap \pi_D^*(i^*\alpha_{r-i}) = 0$$

in the chow group of P_D. By the characterization of Lemma 42.38.2 we conclude. □

At this point we have enough material to be able to prove that capping with Chern classes defines a bivariant class.

0B7H **Lemma 42.38.7.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Let $0 \leq p \leq r$. Then the rule that to $f : X' \to X$ assigns $c_p(f^*\mathcal{E}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-p}(X')$ is a bivariant class of degree p.*

Proof. Immediate from Lemmas 42.38.3, 42.38.4, 42.38.5, and 42.38.6 and Definition 42.33.1. □

This lemma allows us to define the Chern classes of a finite locally free module as follows.

0FE1 **Definition 42.38.8.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. For $i = 0, \ldots, r$ the *ith Chern class* of $\mathcal{E}$ is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 42.38.7. The *total Chern class* of $\mathcal{E}$ is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X.

By the remark following Definition 42.38.1 if $\mathcal{E}$ is invertible, then this definition agrees with Definition 42.34.4. Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

02UA **Lemma 42.38.9.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then*

(1) *$c_j(\mathcal{E}) \in A^j(X)$ is in the center of $A^*(X)$ and*
(2) *if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_j(\mathcal{E}) = c_j(f^*\mathcal{E}) \circ c$.*

In particular, if $\mathcal{F}$ is a second locally free $\mathcal{O}_X$-module on X of rank s, then

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. It is immediate that (2) implies (1). Let $\alpha \in \mathrm{CH}_k(X)$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 42.38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Denote $\pi' : P' \to X'$ the base change of π by f. Using Lemma 42.34.5 and the properties of bivariant classes we obtain

$$\begin{aligned} 0 &= c \cap \left(\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) \right) \\ &= \sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(c \cap \alpha_{r-i}) \end{aligned}$$

in the Chow group of P' (calculation omitted). Hence we see that $c \cap \alpha_j$ is equal to $c_j(f^*\mathcal{E}) \cap (c \cap \alpha)$ by the characterization of Lemma 42.38.2. This proves the lemma. □

0ESW **Remark 42.38.10.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. If the rank of $\mathcal{E}$ is not constant then we can still define the Chern classes of $\mathcal{E}$. Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

where $X_r \subset X$ is the open and closed subspace where the rank of $\mathcal{E}$ is r. By Lemma 42.35.4 we have $A^p(X) = \prod A^p(X_r)$. Hence we can define $c_p(\mathcal{E})$ to be the product of the classes $c_p(\mathcal{E}|_{X_r})$ in $A^p(X_r)$. Explicitly, if $X' \to X$ is a morphism locally of

finite type, then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod\nolimits_{r\geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then $c_p(\mathcal{E}) \in A^p(X)$ is the bivariant class which preserves these direct product decompositions and acts by the already defined operations $c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_p(\mathcal{E})$ is nonzero for infinitely many p. It follows that the total chern class is an element

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots \in A^*(X)^\wedge$$

of the completed bivariant cohomology ring, see Remark 42.35.5. In this setting we define the "rank" of $\mathcal{E}$ to be the element $r(\mathcal{E}) \in A^0(X)$ as the bivariant operation which sends $(\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$ to $(r\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$. Note that it is still true that $c_p(\mathcal{E})$ and $r(\mathcal{E})$ are in the center of $A^*(X)$.

0FA4 **Remark 42.38.11.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. In general we write $X = \coprod X_r$ as in Remark 42.38.10. If only a finite number of the X_r are nonempty, then we can set

$$c_{top}(\mathcal{E}) = \sum\nolimits_r c_r(\mathcal{E}|_{X_r}) \in A^*(X) = \bigoplus A^*(X_r)$$

where the equality is Lemma 42.35.4. If infinitely many X_r are nonempty, we will use the same notation to denote

$$c_{top}(\mathcal{E}) = \prod c_r(\mathcal{E}|_{X_r}) \in \prod A^r(X_r) \subset A^*(X)^\wedge$$

see Remark 42.35.5 for notation.

42.39. Polynomial relations among Chern classes

02UB Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. By Lemma 42.38.9 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) $\mathbf{Z}$-subalgebra of the Chow cohomology algebra $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \to X$ which are locally of finite type. By Lemma 42.35.3 this is equivalent to the requirement that given any morphism $f : Y \to X$ where Y is an integral scheme locally of finite type over S the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 42.25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

02UD **Lemma 42.39.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have*

02UE (42.39.1.1) $$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in $A^*(X)$.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral and by Lemma 42.35.3 it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphism

$$\begin{array}{ccc} P & \xrightarrow{g} & P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

see Constructions, Lemma 27.20.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $\mathrm{CH}_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (42.37.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. □

Some example cases of (42.39.1.1) are

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + r c_1(\mathcal{L})$$

$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1) c_1(\mathcal{E}) c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2$$

$$c_3(\mathcal{E} \otimes \mathcal{L}) = c_3(\mathcal{E}) + (r-2) c_2(\mathcal{E}) c_1(\mathcal{L}) + \binom{r-1}{2} c_1(\mathcal{E}) c_1(\mathcal{L})^2 + \binom{r}{3} c_1(\mathcal{L})^3$$

42.40. Additivity of Chern classes

02UF All of the preliminary lemmas follow trivially from the final result.

02UG **Lemma 42.40.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

in $A^*(X)$.

Proof. By Lemma 42.35.3 it suffices to show that if X is integral then $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 42.31.1. By Lemma 42.38.2 we see that $\alpha_j = c_j(\mathcal{F}) \cap [X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 42.26.4 we get

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 42.38.2 we conclude that $c_r(\mathcal{E}) \cap [X] = 0$ and $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. □

02UH **Lemma 42.40.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^*(X)$.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 42.40.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ were we set $c_r(\mathcal{F} \otimes \mathcal{L}^{-1}) = 0$ by convention. Applying Lemma 42.39.1 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a "solution" of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. □

02UI **Lemma 42.40.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^*(X)$.

Proof. By Lemma 42.35.3 we may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 42.40.2. Assume $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 42.38.5. □

02UJ **Lemma 42.40.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a locally free rank $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r} (1 + x_i)$$

in $A^*(X)$.

Proof. Apply Lemma 42.40.2 and induction. □

42.41. Degrees of zero cycles

0AZ0 We start defining the degree of a zero cycle on a proper scheme over a field. One approach is to define it directly as in Lemma 42.41.2 and then show it is well defined by Lemma 42.18.3. Instead we define it as follows.

0AZ1 **Definition 42.41.1.** Let k be a field (Example 42.7.2). Let $p : X \to \mathrm{Spec}(k)$ be proper. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}(k))$$

(Lemma 42.20.3) combined with the natural isomorphism $\mathrm{CH}_0(\mathrm{Spec}(k)) = \mathbf{Z}$ which maps $[\mathrm{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

0AZ2 **Lemma 42.41.2.** *Let k be a field. Let X be proper over k. Let $\alpha = \sum n_i[Z_i]$ be in $Z_0(X)$. Then*

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \to \mathrm{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 42.12.1). □

Next, we make the connection with degrees of vector bundles over 1-dimensional proper schemes over fields as defined in Varieties, Section 33.44.

0AZ3 **Lemma 42.41.3.** *Let k be a field. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of constant rank. Then*

$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

where the left hand side is defined in Varieties, Definition 33.44.1.

Proof. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1 with reduced induced scheme structure and let m_i be the multiplicity of C_i in X. Then $[X]_1 = \sum m_i[C_i]$ and $c_1(\mathcal{E}) \cap [X]_1$ is the sum of the pushforwards of the cycles $m_i c_1(\mathcal{E}|_{C_i}) \cap [C_i]$. Since we have a similar decomposition of the degree of $\mathcal{E}$ by Varieties, Lemma 33.44.6 it suffices to prove the lemma in case X is a proper curve over k.

Assume X is a proper curve over k. By Divisors, Lemma 31.36.1 there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules. Since $f_*[X']_1 = [X]_1$ we conclude from Lemma 42.38.4 that

$$\deg(c_1(\mathcal{E}) \cap [X]_1) = \deg(c_1(f^*\mathcal{E}) \cap [X']_1)$$

Since we have a similar relationship for the degree by Varieties, Lemma 33.44.4 we reduce to the case where $\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_X$-modules. In this case, we may use additivity of the degree (Varieties, Lemma 33.44.3) and of first Chern classes (Lemma 42.40.3) to reduce to the case discussed in the next paragraph.

Assume X is a proper curve over k and $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. By Divisors, Lemma 31.15.12 we see that $\mathcal{E}$ is isomorphic to $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors D, D' on X (this also uses that X is projective, see Varieties, Lemma 33.43.4 for example). By additivity of degree under tensor product of invertible sheaves (Varieties, Lemma 33.44.7) and additivity of c_1 under tensor product of invertible sheaves (Lemma 42.25.2 or 42.39.1) we reduce to the case $\mathcal{E} = \mathcal{O}_X(D)$. In this case the left hand side gives $\deg(D)$ (Varieties, Lemma 33.44.9) and the right hand side gives $\deg([D]_0)$ by Lemma 42.25.4. Since

$$[D]_0 = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x})[x] = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[x]$$

by definition, we see

$$\deg([D]_0) = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[\kappa(x) : k] = \dim_k \Gamma(D, \mathcal{O}_D) = \deg(D)$$

The penultimate equality by Algebra, Lemma 10.52.12 using that D is affine. □

Finally, we can tie everything up with the numerical intersections defined in Varieties, Section 33.45.

0BFI **Lemma 42.41.4.** *Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d)$$

where the left hand side is defined in Varieties, Definition 33.45.3. In particular,

$$\deg_{\mathcal{L}}(Z) = \deg(c_1(\mathcal{L})^d \cap [Z]_d)$$

if $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module.

Proof. We will prove this by induction on d. If $d = 0$, then the result is true by Varieties, Lemma 33.33.3. Assume $d > 0$.

Let $Z_i \subset Z$, $i = 1, \ldots, t$ be the irreducible components of dimension d with reduced induced scheme structure and let m_i be the multiplicity of Z_i in Z. Then $[Z]_d = \sum m_i[Z_i]$ and $c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by Varieties, Lemma 33.45.2 it suffices to prove the lemma in case $Z = X$ is a proper variety of dimension d over k.

By Chow's lemma there exists a birational proper morphism $f : Y \to X$ with Y H-projective over k. See Cohomology of Schemes, Lemma 30.18.1 and Remark 30.18.2. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Varieties, Lemma 33.45.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \ldots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 42.26.4. Thus we may replace X by Y and assume that X is projective over k.

If X is a proper d-dimensional projective variety, then we can write $\mathcal{L}_1 = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$ by Divisors, Lemma 31.15.12. By additivity for both sides of the equation (Varieties, Lemma 33.45.5 and Lemma 42.25.2) we reduce to the case $\mathcal{L}_1 = \mathcal{O}_X(D)$ for some effective Cartier divisor D. By Varieties, Lemma 33.45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

and by Lemma 42.25.4 we have

$$c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X] = c_1(\mathcal{L}_2) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [D]_{d-1}$$

Thus we obtain the result from our induction hypothesis. □

42.42. Cycles of given codimension

0FE2 In some cases there is a second grading on the abelian group of all cycles given by codimension.

0FE3 **Lemma 42.42.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Write $\delta = \delta_{X/S}$ as in Section 42.7. The following are equivalent*

1. *There exists a decomposition $X = \coprod_{n \in \mathbf{Z}} X_n$ into open and closed subschemes such that $\delta(\xi) = n$ whenever $\xi \in X_n$ is a generic point of an irreducible component of X_n.*
2. *For all $x \in X$ there exists an open neighbourhood $U \subset X$ of x and an integer n such that $\delta(\xi) = n$ whenever $\xi \in U$ is a generic point of an irreducible component of U.*
3. *For all $x \in X$ there exists an integer n_x such that $\delta(\xi) = n_x$ for any generic point ξ of an irreducible component of X containing x.*

The conditions are satisfied if X is either normal or Cohen-Macaulay[3].

[3] In fact, it suffices if X is (S_2). Compare with Local Cohomology, Lemma 51.3.2.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Conversely, if (3) holds, then we set $X_n = \{x \in X \mid n_x = n\}$ and we get a decomposition as in (1). Namely, X_n is open because given x the union of the irreducible components of X passing through x minus the union of the irreducible components of X not passing through x is an open neighbourhood of x. If X is normal, then X is a disjoint union of integral schemes (Properties, Lemma 28.7.7) and hence the properties hold. If X is Cohen-Macaulay, then $\delta' : X \to \mathbf{Z}$, $x \mapsto -\dim(\mathcal{O}_{X,x})$ is a dimension function on X (see Example 42.7.4). Since $\delta - \delta'$ is locally constant (Topology, Lemma 5.20.3) and since $\delta'(\xi) = 0$ for every generic point ξ of X we see that (2) holds. $\square$

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1. For an integral closed subscheme $Z \subset X$ we have the codimension $\mathrm{codim}(Z, X)$ of Z in X, see Topology, Definition 5.11.1. We define a *codimension p-cycle* to be a cycle $\alpha = \sum n_Z[Z]$ on X such that $n_Z \neq 0 \Rightarrow \mathrm{codim}(Z, X) = p$. The abelian group of all codimension p-cycles is denoted $Z^p(X)$. Let $X = \coprod X_n$ be the decomposition given in Lemma 42.42.1 part (1). Recalling that our cycles are defined as locally finite sums, it is clear that

$$Z^p(X) = \prod\nolimits_n Z_{n-p}(X_n)$$

Moreover, we see that $\prod_p Z^p(X) = \prod_k Z_k(X)$. We could now define rational equivalence of codimension p cycles on X in exactly the same manner as before and in fact we could redevelop the whole theory from scratch for cycles of a given codimension for X as in Lemma 42.42.1. However, instead we simply define the *Chow group of codimension p-cycles* as

$$\mathrm{CH}^p(X) = \prod\nolimits_n \mathrm{CH}_{n-p}(X_n)$$

As before we have $\prod_p \mathrm{CH}^p(X) = \prod_k \mathrm{CH}_k(X)$. If X is quasi-compact, then the product in the formula is finite (and hence is a direct sum) and we have $\bigoplus_p \mathrm{CH}^p(X) = \bigoplus_k \mathrm{CH}_k(X)$. If X is quasi-compact and finite dimensional, then only a finite number of these groups is nonzero.

Many of the constructions and results for Chow groups proved above have natural counterparts for the Chow groups $\mathrm{CH}^*(X)$. Each of these is shown by decomposing the relevant schemes into "equidimensional" pieces as in Lemma 42.42.1 and applying the results already proved for the factors in the product decomposition given above. Let us list some of them.

(1) If $f : X \to Y$ is a flat morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 then flat pullback determines a map
$$f^* : \mathrm{CH}^p(Y) \to \mathrm{CH}^p(X)$$
(2) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 let us say f has *codimension* $r \in \mathbf{Z}$ if for all pairs of irreducible components $Z \subset X$, $W \subset Y$ with $f(Z) \subset W$ we have $\dim_\delta(W) - \dim_\delta(Z) = r$.
(3) If $f : X \to Y$ is a proper morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 and f has codimension r, then proper pushforward is a map
$$f_* : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+r}(Y)$$

(4) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.42.1 and f has codimension r and $c \in A^q(X \to Y)$, then c induces maps
$$c \cap - : \mathrm{CH}^p(Y) \to \mathrm{CH}^{p+q-r}(X)$$
(5) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1 and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then
$$c_1(\mathcal{L}) \cap - : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+1}(X)$$
(6) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1 and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then
$$c_i(\mathcal{E}) \cap - : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+i}(X)$$
Warning: the property for a morphism to have codimension r is not preserved by base change.

0FE4 **Remark 42.42.2.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.42.1. Let $X = \coprod X_n$ be the decomposition into open and closed subschemes such that every irreducible component of X_n has δ-dimension n. In this situation we sometimes set
$$[X] = \sum\nolimits_n [X_n]_n \in \mathrm{CH}^0(X)$$
This class is a kind of "fundamental class" of X in Chow theory.

42.43. The splitting principle

02UK In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

02UL **Lemma 42.43.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a projective flat morphism $\pi : P \to X$ of relative dimension d such that*

(1) *for any morphism $f : Y \to X$ the map $\pi_Y^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
(2) *each $\pi^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.*

Moreover, when (1) holds the restriction map $A^*(X) \to A^*(P)$ (Remark 42.34.2) is injective.

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum(r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \mathrm{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the morphism $\pi_i : P_i = \mathbf{P}(\mathcal{E}_i) \to X$. We have a short exact sequence
$$0 \to \mathcal{F} \to \pi_i^*\mathcal{E}_i \to \mathcal{O}_{P_i}(1) \to 0$$
of finite locally free $\mathcal{O}_{P_i}$-modules of ranks $r_i - 1$, r_i, and 1. Observe that π_i^* is injective on chow groups after any base change by the projective bundle formula (Lemma 42.36.2). By the induction hypothesis applied to the finite locally free $\mathcal{O}_{P_i}$-modules $\mathcal{F}$ and $\pi_{i'}^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $\pi : P \to P_i$ with properties

stated as in the lemma. Then the composition $\pi_i \circ \pi : P \to X$ does the job. Some details omitted. □

0FVE **Remark 42.43.2.** The proof of Lemma 42.43.1 shows that the morphism $\pi : P \to X$ has the following additional properties:

(1) π is a finite composition of projective space bundles associated to locally free modules of finite constant rank, and
(2) for every $\alpha \in \mathrm{CH}_k(X)$ we have $\alpha = \pi_*(\xi_1 \cap \ldots \cap \xi_d \cap \pi^*\alpha)$ where ξ_i is the first Chern class of some invertible $\mathcal{O}_P$-module.

The second observation follows from the first and Lemma 42.36.1. We will add more observations here as needed.

Let (S, δ), X, and $\mathcal{E}_i$ be as in Lemma 42.43.1. The *splitting principle* refers to the practice of symbolically writing

$$c(\mathcal{E}_i) = \prod(1 + x_{i,j})$$

The symbols $x_{i,1}, \ldots, x_{i,r_i}$ are called the *Chern roots* of $\mathcal{E}_i$. In other words, the pth Chern class of $\mathcal{E}_i$ is the pth elementary symmetric function in the Chern roots. The usefulness of the splitting principle comes from the assertion that in order to prove a polynomial relation among Chern classes of the $\mathcal{E}_i$ it is enough to prove the corresponding relation among the Chern roots.

Namely, let $\pi : P \to X$ be as in Lemma 42.43.1. Recall that there is a canonical $\mathbf{Z}$-algebra map $\pi^* : A^*(X) \to A^*(P)$, see Remark 42.34.2. The injectivity of π_Y^* on Chow groups for every Y over X, implies that the map $\pi^* : A^*(X) \to A^*(P)$ is injective (details omitted). We have

$$\pi^* c(\mathcal{E}_i) = \prod(1 + c_1(\mathcal{L}_{i,j}))$$

by Lemma 42.40.4. Thus we may think of the Chern roots $x_{i,j}$ as the elements $c_1(\mathcal{L}_{i,j}) \in A^*(P)$ and the displayed equation as taking place in $A^*(P)$ after applying the injective map $\pi^* : A^*(X) \to A^*(P)$ to the left hand side of the equation.

To see how this works, it is best to give some examples.

0FA5 **Lemma 42.43.3.** *In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^i(X)$.

Proof. Choose a morphism $\pi : P \to X$ as in Lemma 42.43.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of $\mathcal{E}$ and $\mathcal{E}^\vee$ after pulling back to P. Thus we may assume there exist invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_1^\perp \subset \mathcal{E}_2^\perp \subset \ldots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 42.25.2. By Lemma 42.40.4 we have

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod\nolimits_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0FA6 **Lemma 42.43.4.** *In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ and $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-modules of ranks r and s. Then we have*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = rc_2(\mathcal{F}) + sc_2(\mathcal{E}) + \binom{r}{2} c_1(\mathcal{F})^2 + (rs-1)c_1(\mathcal{F})c_1(\mathcal{E}) + \binom{s}{2} c_1(\mathcal{E})^2$$

and so on in $A^*(X)$.

Proof. Arguing exactly as in the proof of Lemma 42.43.3 we may assume we have invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ $\mathcal{N}_i$, $i = 1, \ldots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \ldots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \ldots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \ldots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 42.40.4 we have

$$c(\mathcal{E}) = \prod (1 + x_i), \quad c(\mathcal{F}) = \prod (1 + y_j), \quad \text{and} \quad c(\mathcal{E} \otimes \mathcal{F}) = \prod (1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0FA7 **Remark 42.43.5.** The equalities proven above remain true even when we work with finite locally free $\mathcal{O}_X$-modules whose rank is allowed to be nonconstant. In fact, we can work with polynomials in the rank and the Chern classes as follows. Consider the graded polynomial ring $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ where r has degree 0 and c_i has degree i. Let

$$P \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$$

be a homogeneous polynomial of degree p. Then for any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ on X we can consider

$$P(\mathcal{E}) = P(r(\mathcal{E}), c_1(\mathcal{E}), c_2(\mathcal{E}), c_3(\mathcal{E}), \ldots) \in A^p(X)$$

see Remark 42.38.10 for notation and conventions. To prove relations among these polynomials (for multiple finite locally free modules) we can work locally on X and use the splitting principle as above. For example, we claim that

$$c_2(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = P(\mathcal{E})$$

where $P = 2rc_2 - (r-1)c_1^2$. Namely, since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^\vee$ this follows easily from Lemmas 42.43.3 and 42.43.4 above by decomposing X into parts where the rank of $\mathcal{E}$ is constant as in Remark 42.38.10.

0F9B **Example 42.43.6.** For every $p \geq 1$ there is a unique homogeneous polynomial $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ of degree p such that, for any $n \geq p$ we have

$$P_p(s_1, s_2, \ldots, s_p) = \sum x_i^p$$

in $\mathbf{Z}[x_1, \ldots, x_n]$ where $s_1, \ldots, s_p$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$, so

$$s_i = \sum\nolimits_{1 \leq j_1 < \ldots < j_i \leq n} x_{j_1} x_{j_2} \ldots x_{j_i}$$

The existence of P_p comes from the well known fact that the elementary symmetric functions generate the ring of all symmetric functions over the integers. Another way to characterize $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ is that we have

$$\log(1 + c_1 + c_2 + c_3 + \ldots) = \sum\nolimits_{p \geq 1} (-1)^{p-1} \frac{P_p}{p}$$

as formal power series. This is clear by writing $1 + c_1 + c_2 + \ldots = \prod(1 + x_i)$ and applying the power series for the logarithm function. Expanding the left hand side we get

$$\begin{aligned} &(c_1 + c_2 + \ldots) - (1/2)(c_1 + c_2 + \ldots)^2 + (1/3)(c_1 + c_2 + \ldots)^3 - \ldots \\ &= c_1 + (c_2 - (1/2)c_1^2) + (c_3 - c_1c_2 + (1/3)c_1^3) + \ldots \end{aligned}$$

In this way we find that

$$\begin{aligned} P_1 &= c_1, \\ P_2 &= c_1^2 - 2c_2, \\ P_3 &= c_1^3 - 3c_1c_2 + 3c_3, \\ P_4 &= c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4, \end{aligned}$$

and so on. Since the Chern classes of a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ are the elementary symmetric polynomials in the Chern roots x_i, we see that

$$P_p(\mathcal{E}) = \sum x_i^p$$

For convenience we set $P_0 = r$ in $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ so that $P_0(\mathcal{E}) = r(\mathcal{E})$ as a bivariant class (as in Remarks 42.38.10 and 42.43.5).

42.44. Chern classes and sections

0FA8 A brief section whose main result is that we may compute the top Chern class of a finite locally free module using the vanishing locus of a "regular section.

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $f : X' \to X$ be locally of finite type. Let

$$s \in \Gamma(X', f^*\mathcal{E})$$

be a global section of the pullback of $\mathcal{E}$ to X'. Let $Z(s) \subset X'$ be the zero scheme of s. More precisely, we define $Z(s)$ to be the closed subscheme whose quasi-coherent sheaf of ideals is the image of the map $s : f^*\mathcal{E}^\vee \to \mathcal{O}_{X'}$.

0FA9 **Lemma 42.44.1.** *In the situation described just above assume* $\dim_\delta(X') = n$*, that* $f^*\mathcal{E}$ *has constant rank* r*, that* $\dim_\delta(Z(s)) \leq n - r$*, and that for every generic point* $\xi \in Z(s)$ *with* $\delta(\xi) = n - r$ *the ideal of* $Z(s)$ *in* $\mathcal{O}_{X',\xi}$ *is generated by a regular sequence of length* r*. Then*

$$c_r(\mathcal{E}) \cap [X']_n = [Z(s)]_{n-r}$$

in $\mathrm{CH}_*(X')$*.*

Proof. Since $c_r(\mathcal{E})$ is a bivariant class (Lemma 42.38.7) we may assume $X = X'$ and we have to show that $c_r(\mathcal{E}) \cap [X]_n = [Z(s)]_{n-r}$ in $\mathrm{CH}_{n-r}(X)$. We will prove the lemma by induction on $r \geq 0$. (The case $r = 0$ is trivial.) The case $r = 1$ is handled by Lemma 42.25.4. Assume $r > 1$.

Let $\pi : P \to X$ be the projective space bundle associated to $\mathcal{E}$ and consider the short exact sequence

$$0 \to \mathcal{E}' \to \pi^*\mathcal{E} \to \mathcal{O}_P(1) \to 0$$

By the projective space bundle formula (Lemma 42.36.2) it suffices to prove the equality after pulling back by π. Observe that $\pi^{-1}Z(s) = Z(\pi^*s)$ has δ-dimension $\leq n-1$ and that the assumption on regular sequences at generic points of δ-dimension $n-1$ holds by flat pullback, see Algebra, Lemma 10.68.5. Let $t \in \Gamma(P, \mathcal{O}_P(1))$ be the image of π^*s. We claim

$$[Z(t)]_{n+r-2} = c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1}$$

Assuming the claim we finish the proof as follows. The restriction $\pi^*s|_{Z(t)}$ maps to zero in $\mathcal{O}_P(1)|_{Z(t)}$ hence comes from a unique element $s' \in \Gamma(Z(t), \mathcal{E}'|_{Z(t)})$. Note that $Z(s') = Z(\pi^*s)$ as closed subschemes of P. If $\xi \in Z(s')$ is a generic point with $\delta(\xi) = n-1$, then the ideal of $Z(s')$ in $\mathcal{O}_{Z(t),\xi}$ can be generated by a regular sequence of length $r-1$: it is generated by $r-1$ elements which are the images of $r-1$ elements in $\mathcal{O}_{P,\xi}$ which together with a generator of the ideal of $Z(t)$ in $\mathcal{O}_{P,\xi}$ form a regular sequence of length r in $\mathcal{O}_{P,\xi}$. Hence we can apply the induction hypothesis to s' on $Z(t)$ to get $c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} = [Z(s')]_{n-1}$. Combining all of the above we obtain

$$\begin{aligned} c_r(\pi^*\mathcal{E}) \cap [P]_{n+r-1} &= c_{r-1}(\mathcal{E}') \cap c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1} \\ &= c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} \\ &= [Z(s')]_{n-1} \\ &= [Z(\pi^*s)]_{n-1} \end{aligned}$$

which is what we had to show.

Proof of the claim. This will follow from an application of the already used Lemma 42.25.4. We have $\pi^{-1}(Z(s)) = Z(\pi^*s) \subset Z(t)$. On the other hand, for $x \in X$ if $P_x \subset Z(t)$, then $t|_{P_x} = 0$ which implies that s is zero in the fibre $\mathcal{E} \otimes \kappa(x)$, which implies $x \in Z(s)$. It follows that $\dim_\delta(Z(t)) \leq n + (r-1) - 1$. Finally, let $\xi \in Z(t)$ be a generic point with $\delta(\xi) = n + r - 2$. If ξ is not the generic point of the fibre of $P \to X$ it is immediate that a local equation of $Z(t)$ is a nonzerodivisor in $\mathcal{O}_{P,\xi}$ (because we can check this on the fibre by Algebra, Lemma 10.99.2). If ξ is the generic point of a fibre, then $x = \pi(\xi) \in Z(s)$ and $\delta(x) = n + r - 2 - (r-1) = n-1$. This is a contradiction with $\dim_\delta(Z(s)) \leq n - r$ because $r > 1$ so this case doesn't happen. □

0FAA **Lemma 42.44.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let*

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$$

be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Consider the closed embedding

$$i : N' = \underline{\mathrm{Spec}}_X(Sym((\mathcal{N}')^\vee)) \longrightarrow N = \underline{\mathrm{Spec}}_X(Sym(\mathcal{N}^\vee))$$

For $\alpha \in \mathrm{CH}_k(X)$ we have

$$i_*(p')^*\alpha = p^*(c_{top}(\mathcal{E}) \cap \alpha)$$

where $p' : N' \to X$ and $p : N \to X$ are the structure morphisms.

Proof. Here $c_{top}(\mathcal{E})$ is the bivariant class defined in Remark 42.38.11. By its very definition, in order to verify the formula, we may assume that $\mathcal{E}$ has constant rank. We may similarly assume $\mathcal{N}'$ and $\mathcal{N}$ have constant ranks, say r' and r, so $\mathcal{E}$ has rank $r - r'$ and $c_{top}(\mathcal{E}) = c_{r-r'}(\mathcal{E})$. Observe that $p^*\mathcal{E}$ has a canonical section

$$s \in \Gamma(N, p^*\mathcal{E}) = \Gamma(X, p_*p^*\mathcal{E}) = \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}(\mathcal{N}^\vee) \supset \Gamma(X, \mathcal{H}om(\mathcal{N}, \mathcal{E}))$$

corresponding to the surjection $\mathcal{N} \to \mathcal{E}$ given in the statement of the lemma. The vanishing scheme of this section is exactly $N' \subset N$. Let $Y \subset X$ be an integral closed subscheme of δ-dimension n. Then we have

(1) $p^*[Y] = [p^{-1}(Y)]$ since $p^{-1}(Y)$ is integral of δ-dimension $n + r$,
(2) $(p')^*[Y] = [(p')^{-1}(Y)]$ since $(p')^{-1}(Y)$ is integral of δ-dimension $n + r'$,
(3) the restriction of s to $p^{-1}Y$ has vanishing scheme $(p')^{-1}Y$ and the closed immersion $(p')^{-1}Y \to p^{-1}Y$ is a regular immersion (locally cut out by a regular sequence).

We conclude that

$$(p')^*[Y] = c_{r-r'}(p^*\mathcal{E}) \cap p^*[Y] \quad \text{in} \quad \mathrm{CH}_*(N)$$

by Lemma 42.44.1. This proves the lemma. □

42.45. The Chern character and tensor products

02UM Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. We define the *Chern character* of a finite locally free $\mathcal{O}_X$-module to be the formal expression

$$ch(\mathcal{E}) = \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the Chern roots of $\mathcal{E}$. Writing this as a polynomial in the Chern classes we obtain

$$\begin{aligned} ch(\mathcal{E}) &= r(\mathcal{E}) + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) \\ &\quad + \frac{1}{24}(c_1(\mathcal{E})^4 - 4c_1(\mathcal{E})^2c_2(\mathcal{E}) + 4c_1(\mathcal{E})c_3(\mathcal{E}) + 2c_2(\mathcal{E})^2 - 4c_4(\mathcal{E})) + \ldots \\ &= \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(\mathcal{E})}{p!} \end{aligned}$$

with P_p polynomials in the Chern classes as in Example 42.43.6. The degree p component of the above is

$$ch_p(\mathcal{E}) = \frac{P_p(\mathcal{E})}{p!} \in A^p(X) \otimes \mathbf{Q}$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of $ch_p(\mathcal{E})$ as an element of $A^p(X) \otimes \mathbf{Q}$.

0ESX **Remark 42.45.1.** In the discussion above we have defined the components of the Chern character $ch_p(\mathcal{E}) \in A^p(X) \otimes \mathbf{Q}$ of $\mathcal{E}$ even if the rank of $\mathcal{E}$ is not constant. See Remarks 42.38.10 and 42.43.5. Thus the full Chern character of $\mathcal{E}$ is an element of $\prod_{p \geq 0}(A^p(X) \otimes \mathbf{Q})$. If X is quasi-compact and $\dim(X) < \infty$ (usual dimension), then one can show using Lemma 42.34.6 and the splitting principle that $ch(\mathcal{E}) \in A^*(X) \otimes \mathbf{Q}$.

0F9C **Lemma 42.45.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then we have the equality*

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

More precisely, we have $P_p(\mathcal{E}) = P_p(\mathcal{E}_1) + P_p(\mathcal{E}_2)$ in $A^p(X)$ where P_p is as in Example 42.43.6.

Proof. It suffices to prove the more precise statement. By Section 42.43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,1}, \ldots, x_{1,r_1}, x_{2,1}, \ldots, x_{2,r_2}$ are the Chern roots of $\mathcal{E}$. Hence we get the result from our choice of P_p in Example 42.43.6. □

0F9D **Lemma 42.45.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be finite locally free $\mathcal{O}_X$-modules. Then we have the equality*

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

More precisely, we have

$$P_p(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(\mathcal{E}_1)P_{p_2}(\mathcal{E}_2)$$

in $A^p(X)$ where P_p is as in Example 42.43.6.

Proof. It suffices to prove the more precise statement. By Section 42.43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,i} + x_{2,j}$, $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ are the Chern roots of $\mathcal{E}_1 \otimes \mathcal{E}_2$. Hence we get the result from the binomial formula for $(x_{1,i} + x_{2,j})^p$ and the shape of our polynomials P_p in Example 42.43.6. □

0FAB **Lemma 42.45.4.** *In Situation 42.7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then $ch_i(\mathcal{E}^\vee) = (-1)^i ch_i(\mathcal{E})$ in $A^i(X) \otimes \mathbf{Q}$.*

Proof. Follows from the corresponding result for Chern classes (Lemma 42.43.3). □

42.46. Chern classes and the derived category

0ESY In this section we define the total Chern class of a perfect object E of the derived category of a scheme X, under the assumption that E may be represented by a finite complex of finite locally free modules on an envelope of X.

Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let

$$\mathcal{E}^a \to \mathcal{E}^{a+1} \to \ldots \to \mathcal{E}^b$$

be a bounded complex of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we define the *total Chern class of the complex* by the formula

$$c(\mathcal{E}^\bullet) = \prod\nolimits_{n=a,\ldots,b} c(\mathcal{E}^n)^{(-1)^n} \in \prod\nolimits_{p \geq 0} A^p(X)$$

Here the inverse is the formal inverse, so

$$(1 + c_1 + c_2 + c_3 + \ldots)^{-1} = 1 - c_1 + c_1^2 - c_2 - c_1^3 + 2c_1c_2 - c_3 + \ldots$$

We will denote $c_p(\mathcal{E}^\bullet) \in A^p(X)$ the degree p part of $c(\mathcal{E}^\bullet)$. We similarly define the *Chern character of the complex* by the formula

$$ch(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n ch(\mathcal{E}^n) \in \prod\nolimits_{p\geq 0} (A^p(X) \otimes \mathbf{Q})$$

We will denote $ch_p(\mathcal{E}^\bullet) \in A^p(X) \otimes \mathbf{Q}$ the degree p part of $ch(\mathcal{E}^\bullet)$. Finally, for $P_p \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ as in Example 42.43.6 we define

$$P_p(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n P_p(\mathcal{E}^n)$$

in $A^p(X)$. Then we have $ch_p(\mathcal{E}^\bullet) = (1/p!)P_p(\mathcal{E}^\bullet)$ as usual. The next lemma shows that these constructions only depends on the image of the complex in the derived category.

0ESZ **Lemma 42.46.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be an object such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Then a slight generalization of the above constructions*

$$c(\mathcal{E}^\bullet) \in \prod\nolimits_{p\geq 0} A^p(X), \quad ch(\mathcal{E}^\bullet) \in \prod\nolimits_{p\geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(\mathcal{E}^\bullet) \in A^p(X)$$

are independent of the choice of the complex $\mathcal{E}^\bullet$.

Proof. We prove this for the total Chern class; the other two cases follow by the same arguments using Lemma 42.45.2 instead of Lemma 42.40.3.

As in Remark 42.38.10 in order to define the total chern class $c(\mathcal{E}^\bullet)$ we decompose X into open and closed subschemes

$$X = \coprod\nolimits_{i\in I} X_i$$

such that the rank $\mathcal{E}^n$ is constant on X_i for all n and i. (Since these ranks are locally constant functions on X we can do this.) Since $\mathcal{E}^\bullet$ is locally bounded, we see that only a finite number of the sheaves $\mathcal{E}^n|_{X_i}$ are nonzero for a fixed i. Hence we can define

$$c(\mathcal{E}^\bullet|_{X_i}) = \prod\nolimits_n c(\mathcal{E}^n|_{X_i})^{(-1)^n} \in \prod\nolimits_{p\geq 0} A^p(X_i)$$

as above. By Lemma 42.35.4 we have $A^p(X) = \prod_i A^p(X_i)$. Hence for each $p \in \mathbf{Z}$ we have a unique element $c_p(\mathcal{E}^\bullet) \in A^p(X)$ restricting to $c_p(\mathcal{E}^\bullet|_{X_i})$ on X_i for all i.

Suppose we have a second locally bounded complex $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Let $g : Y \to X$ be a morphism locally of finite type with Y integral. By Lemma 42.35.3 it suffices to show that with $c(g^*\mathcal{E}^\bullet) \cap [Y]$ is the same as $c(g^*\mathcal{F}^\bullet) \cap [Y]$ and it even suffices to prove this after replacing Y by an integral scheme proper and birational over Y. Then first we conclude that $g^*\mathcal{E}^\bullet$ and $g^*\mathcal{F}^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of constant rank. Next, by More on Flatness, Lemma 38.40.3 we may assume that $H^i(Lg^*E)$ is perfect of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This reduces us to the case discussed in the next paragraph.

Assume X is integral, $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes of finite locally free modules of constant rank, and $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. We have to show that $c(\mathcal{E}^\bullet) \cap [X]$ is the same as $c(\mathcal{F}^\bullet) \cap [X]$. Denote $d^i_\mathcal{E} : \mathcal{E}^i \to \mathcal{E}^{i+1}$ and $d^i_\mathcal{F} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ the differentials of our complexes. By More on

Flatness, Remark 38.40.4 we know that $\mathrm{Im}(d^i_{\mathcal{E}})$, $\mathrm{Ker}(d^i_{\mathcal{E}})$, $\mathrm{Im}(d^i_{\mathcal{F}})$, and $\mathrm{Ker}(d^i_{\mathcal{F}})$ are finite locally free $\mathcal{O}_X$-modules for all i. By additivity (Lemma 42.40.3) we see that

$$c(\mathcal{E}^\bullet) = \prod\nolimits_i c(\mathrm{Ker}(d^i_{\mathcal{E}}))^{(-1)^i} c(\mathrm{Im}(d^i_{\mathcal{E}}))^{(-1)^i}$$

and similarly for $\mathcal{F}^\bullet$. Since we have the short exact sequences

$$0 \to \mathrm{Im}(d^i_{\mathcal{E}}) \to \mathrm{Ker}(d^i_{\mathcal{E}}) \to H^i(E) \to 0 \quad \text{and} \quad 0 \to \mathrm{Im}(d^i_{\mathcal{F}}) \to \mathrm{Ker}(d^i_{\mathcal{F}}) \to H^i(E) \to 0$$

we reduce to the problem stated and solved in the next paragraph.

Assume X is integral and we have two short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

with $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{F}$, $\mathcal{F}'$ finite locally free. Problem: show that $c(\mathcal{E})c(\mathcal{E}')^{-1} \cap [X] = c(\mathcal{F})c(\mathcal{F}')^{-1} \cap [X]$. To do this, consider the short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \oplus \mathcal{F} \to \mathcal{Q} \to 0$$

defining $\mathcal{G}$. Since $\mathcal{Q}$ has tor dimension ≤ 1 we see that $\mathcal{G}$ is finite locally free. A diagram chase shows that the kernel of the surjection $\mathcal{G} \to \mathcal{F}$ maps isomorphically to $\mathcal{E}'$ in $\mathcal{E}$ and the kernel of the surjection $\mathcal{G} \to \mathcal{E}$ maps isomorphically to $\mathcal{F}'$ in $\mathcal{F}$. (Working affine locally this follows from or is equivalent to Schanuel's lemma, see Algebra, Lemma 10.109.1.) We conclude that

$$c(\mathcal{E})c(\mathcal{F}') = c(\mathcal{G}) = c(\mathcal{F})c(\mathcal{E}')$$

as desired. □

0GUD **Lemma 42.46.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume there exists an envelope $f : Y \to X$ (Definition 42.22.1) such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. Then there exists unique bivariant classes $c(E) \in \prod_{p \geq 0} A^p(X)$, $ch(E) \in \prod_{p \geq 0} A^p(X) \otimes \mathbf{Q}$, and $P_p(E) \in A^p(X)$, independent of the choice of $f : Y \to X$ and $\mathcal{E}^\bullet$, such that the restriction of these classes to Y are equal to $c(\mathcal{E}^\bullet)$, $ch(\mathcal{E}^\bullet)$, and $P_p(\mathcal{E}^\bullet)$.*

Proof. Fix $p \in \mathbf{Z}$. We will prove the lemma for the chern class $c_p(E) \in A^p(X)$ and omit the arguments for the other cases.

Let $g : T \to X$ be a morphism locally of finite type such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_T$-modules representing Lg^*E in $D(\mathcal{O}_T)$. The bivariant class $c_p(\mathcal{E}^\bullet) \in A^p(T)$ is independent of the choice of $\mathcal{E}^\bullet$ by Lemma 42.46.1. Let us write $c_p(Lg^*E) \in A^p(T)$ for this class. For any further morphism $h : T' \to T$ which is locally of finite type, setting $g' = g \circ h$ we see that $L(g')^*E = L(g \circ h)^*E = Lh^*Lg^*E$ is represented by $h^*\mathcal{E}^\bullet$ in $D(\mathcal{O}_{T'})$. We conclude that $c_p(L(g')^*E)$ makes sense and is equal to the restriction (Remark 42.33.5) of $c_p(Lg^*E)$ to T' (strictly speaking this requires an application of Lemma 42.38.7).

Let $f : Y \to X$ and $\mathcal{E}^\bullet$ be as in the statement of the lemma. We obtain a bivariant class $c_p(E) \in A^p(X)$ from an application of Lemma 42.35.6 to $f : Y \to X$ and the class $c' = c_p(Lf^*E)$ we constructed in the previous paragraph. The assumption in the lemma is satisfied because by the discussion in the previous paragraph we have $res_1(c') = c_p(Lg^*E) = res_2(c')$ where $g = f \circ p = f \circ q : Y \times_X Y \to X$.

Finally, suppose that $f' : Y' \to X$ is a second envelope such that $L(f')^*E$ is represented by a bounded complex of finite locally free $\mathcal{O}_{Y'}$-modules. Then it

follows that the restrictions of $c_p(Lf^*E)$ and $c_p(L(f')^*E)$ to $Y \times_X Y'$ are equal. Since $Y \times_X Y' \to X$ is an envelope (Lemmas 42.22.3 and 42.22.2), we see that our two candidates for $c_p(E)$ agree by the unicity in Lemma 42.35.6. □

0F9E **Definition 42.46.3.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object.

(1) We say the *Chern classes of E are defined*[4] if there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules.
(2) If the Chern classes of E are defined, then we define

$$c(E) \in \prod\nolimits_{p\geq 0} A^p(X), \quad ch(E) \in \prod\nolimits_{p\geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(E) \in A^p(X)$$

by an application of Lemma 42.46.2.

This definition applies in many but not all situations envisioned in this chapter, see Lemma 42.46.4. Perhaps an elementary construction of these bivariant classes for general $E/X/(S, \delta)$ as in the definition exists; we don't know.

0GUE **Lemma 42.46.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. If one of the following conditions hold, then the Chern classes of E are defined:*

(1) *there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules,*
(2) *E can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules,*
(3) *the irreducible components of X are quasi-compact,*
(4) *X is quasi-compact,*
(5) *there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' whose chern classes are defined, or*
(6) *add more here.*

Proof. Condition (1) is just Definition 42.46.3 part (1). Condition (2) implies (1).

As in (3) assume the irreducible components X_i of X are quasi-compact. We view X_i as a reduced integral closed subscheme over X. The morphism $\coprod X_i \to X$ is an envelope. For each i there exists an envelope $X_i' \to X_i$ such that X_i' has an ample family of invertible modules, see More on Morphisms, Proposition 37.80.3. Observe that $f : Y = \coprod X_i' \to X$ is an envelope; small detail omitted. By Derived Categories of Schemes, Lemma 36.36.7 each X_i' has the resolution property. Thus the perfect object $L(f|_{X_i'})^*E$ of $D(\mathcal{O}_{X_i'})$ can be represented by a bounded complex of finite locally free $\mathcal{O}_{X_i'}$-modules, see Derived Categories of Schemes, Lemma 36.37.2. This proves (3) implies (1).

Part (4) implies (3).

Let $g : X \to X'$ and E' be as in part (5). Then there exists an envelope $f' : Y' \to X'$ such that $L(f')^*E'$ is represented by a locally bounded complex $(\mathcal{E}')^\bullet$ of $\mathcal{O}_{Y'}$-modules. Then the base change $f : Y \to X$ is an envelope by Lemma 42.22.3. Moreover, the pulllback $\mathcal{E}^\bullet = g^*(\mathcal{E}')^\bullet$ represents Lf^*E and we see that the chern classes of E are defined. □

[4] See Lemma 42.46.4 for some criteria.

0GUF **Lemma 42.46.5.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume the Chern classes of E are defined. For $g : W \to X$ locally of finite type with W integral, there exists a commutative diagram*

$$\begin{array}{ccc} W' & \xrightarrow{b} & W \\ & {\scriptstyle g'} \searrow \quad \swarrow {\scriptstyle g} & \\ & X & \end{array}$$

with W' integral and $b : W' \to W$ proper birational such that $L(g')^*E$ is represented by a bounded complex $\mathcal{E}^\bullet$ of locally free $\mathcal{O}_{W'}$-modules of constant rank and we have $res(c_p(E)) = c_p(\mathcal{E}^\bullet)$ in $A^p(W')$.

Proof. Choose an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. The base change $Y \times_X W \to W$ of f is an envelope by Lemma 42.22.3. Choose a point $\xi \in Y \times_X W$ mapping to the generic point of W with the same residue field. Consider the integral closed subscheme $W' \subset Y \times_X W$ with generic point ξ. The restriction of the projection $Y \times_X W \to W$ to W' is a proper birational morphism $b : W' \to W$. Set $g' = g \circ b$. Finally, consider the pullback $(W' \to Y)^*\mathcal{E}^\bullet$. This is a locally bounded complex of finite locally free modules on W'. Since W' is integral it follows that it is bounded and that the terms have constant rank. Finally, by construction $(W' \to Y)^*\mathcal{E}^\bullet$ represents $L(g')^*E$ and by construction its pth chern class gives the restriction of $c_p(E)$ by $W' \to X$. This finishes the proof. □

0FAC **Lemma 42.46.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be perfect. If the Chern classes of E are defined then*

1. *$c_p(E)$ is in the center of the algebra $A^*(X)$, and*
2. *if $g : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_p(E) = c_p(Lg^*E) \circ c$.*

Proof. Part (1) follows immediately from part (2). Let $g : X' \to X$ and $c \in A^*(X' \to X)$ be as in (2). To show that $c \circ c_p(E) - c_p(Lg^*E) \circ c = 0$ we use the criterion of Lemma 42.35.3. Thus we may assume that X is integral and by Lemma 42.46.5 we may even assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we have to show that

$$c \cap c_p(\mathcal{E}^\bullet) \cap [X] = c_p(\mathcal{E}^\bullet) \cap c \cap [X]$$

in $\mathrm{CH}_*(X')$. This is immediate from Lemma 42.38.9 and the construction of $c_p(\mathcal{E}^\bullet)$ as a polynomial in the chern classes of the locally free modules $\mathcal{E}^n$. □

0F9F **Lemma 42.46.7.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. If one of the following conditions holds

1. *there exists an envelope $f : Y \to X$ such that $Lf^*E_1 \to Lf^*E_2$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules,*
2. *$E_1 \to E_2$ can be represented be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules,*

(3) *the irreducible components of X are quasi-compact,*
(4) *X is quasi-compact, or*
(5) *add more here,*

then the Chern classes of E_1, E_2, E_3 are defined and we have $c(E_2) = c(E_1)c(E_3)$, $ch(E_2) = ch(E_1) + ch(E_3)$, and $P_p(E_2) = P_p(E_1) + P_p(E_3)$.

Proof. Let $f : Y \to X$ be an envelope and let $\alpha^\bullet : \mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules representing $Lf^*E_1 \to Lf^*E_2$. Then the cone $C(\alpha)^\bullet$ represents Lf^*E_3. Since $C(\alpha)^n = \mathcal{E}_2^n \oplus \mathcal{E}_1^{n+1}$ we see that $C(\alpha)^\bullet$ is a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules. We conclude that the Chern classes of E_1, E_2, E_3 are defined. Moreover, recall that $c_p(E_1)$ is defined as the unique element of $A^p(X)$ which restricts to $c_p(\mathcal{E}_1^\bullet)$ in $A^p(Y)$. Similarly for E_2 and E_3. Hence it suffices to prove $c(\mathcal{E}_2^\bullet) = c(\mathcal{E}_1^\bullet)c(C(\alpha)^\bullet)$ in $\prod_{p\geq 0} A^p(Y)$. In turn, it suffices to prove this after restricting to a connected component of Y. Hence we may assume the complexes $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of fixed rank. In this case the desired equality follows from the multiplicativity of Lemma 42.40.3. In the case of ch or P_p we use Lemmas 42.45.2.

In the previous paragraph we have seen that the lemma holds if condition (1) is satisfied. Since (2) implies (1) this deals with the second case. Assume (3). Arguing exactly as in the proof of Lemma 42.46.4 we find an envelope $f : Y \to X$ such that Y is a disjoint union $Y = \coprod Y_i$ of quasi-compact (and quasi-separated) schemes each having the resolution property. Then we may represent the restriction of $Lf^*E_1 \to Lf^*E_2$ to Y_i by a map of bounded complexes of finite locally free modules, see Derived Categories of Schemes, Proposition 36.37.5. In this way we see that condition (3) implies condition (1). Of course condition (4) implies condition (3) and the proof is complete. □

0FAD **Remark 42.46.8.** The Chern classes of a perfect complex, when defined, satisfy a kind of splitting principle. Namely, suppose that $(S, \delta), X, E$ are as in Definition 42.46.3 such that the Chern classes of E are defined. Say we want to prove a relation between the bivariant classes $c_p(E)$, $P_p(E)$, and $ch_p(E)$. To do this, we may choose an envelope $f : Y \to X$ and a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. By the uniqueness in Lemma 42.46.2 it suffices to prove the desired relation between the bivariant classes $c_p(\mathcal{E}^\bullet)$, $P_p(\mathcal{E}^\bullet)$, and $ch_p(\mathcal{E}^\bullet)$. Thus we may replace X by a connected component of Y and assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free modules of fixed rank. Using the splitting principle (Lemma 42.43.1) we may assume each $\mathcal{E}^i$ has a filtration whose successive quotients $\mathcal{L}_{i,j}$ are invertible modules. Setting $x_{i,j} = c_1(\mathcal{L}_{i,j})$ we see that

$$c(E) = \prod\nolimits_{i \text{ even}} (1 + x_{i,j}) \prod\nolimits_{i \text{ odd}} (1 + x_{i,j})^{-1}$$

and

$$P_p(E) = \sum\nolimits_{i \text{ even}} (x_{i,j})^p - \sum\nolimits_{i \text{ odd}} (x_{i,j})^p$$

Formally taking the logarithm for the expression for $c(E)$ above we find that

$$\log(c(E)) = \sum (-1)^{p-1} \frac{P_p(E)}{p}$$

Looking at the construction of the polynomials P_p in Example 42.43.6 it follows that $P_p(E)$ is the exact same expression in the Chern classes of E as in the case of

vector bundles, in other words, we have

$$\begin{aligned}
P_1(E) &= c_1(E), \\
P_2(E) &= c_1(E)^2 - 2c_2(E), \\
P_3(E) &= c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E), \\
P_4(E) &= c_1(E)^4 - 4c_1(E)^2c_2(E) + 4c_1(E)c_3(E) + 2c_2(E)^2 - 4c_4(E),
\end{aligned}$$

and so on. On the other hand, the bivariant class $P_0(E) = r(E) = ch_0(E)$ cannot be recovered from the Chern class $c(E)$ of E; the chern class doesn't know about the rank of the complex.

0FAE **Lemma 42.46.9.** *In Situation 42.7.1 let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object whose Chern classes are defined. Then $c_i(E^\vee) = (-1)^i c_i(E)$, $P_i(E^\vee) = (-1)^i P_i(E)$, and $ch_i(E^\vee) = (-1)^i ch_i(E)$ in $A^i(X)$.*

Proof. First proof: argue as in the proof of Lemma 42.46.6 to reduce to the case where E is represented by a bounded complex of finite locally free modules of fixed rank and apply Lemma 42.43.3. Second proof: use the splitting principle discussed in Remark 42.46.8 and use that the chern roots of $E^\vee$ are the negatives of the chern roots of E. □

0FAF **Lemma 42.46.10.** *In Situation 42.7.1 let X be locally of finite type over S. Let E be a perfect object of $D(\mathcal{O}_X)$ whose Chern classes are defined. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$c_i(E \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) c_1(\mathcal{L})^j$$

provided E has constant rank $r \in \mathbf{Z}$.

Proof. In the case where E is locally free of rank r this is Lemma 42.39.1. The reader can deduce the lemma from this special case by a formal computation. An alternative is to use the splitting principle of Remark 42.46.8. In this case one ends up having to prove the following algebra fact: if we write formally

$$\frac{\prod_{a=1,\ldots,n}(1+x_a)}{\prod_{n=1,\ldots,m}(1+y_b)} = 1 + c_1 + c_2 + c_3 + \ldots$$

with c_i homogeneous of degree i in $\mathbf{Z}[x_i, y_j]$ then we have

$$\frac{\prod_{a=1,\ldots,n}(1+x_a+t)}{\prod_{b=1,\ldots,m}(1+y_b+t)} = \sum\nolimits_{i\geq 0} \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j} t^j$$

where $r = n - m$. We omit the details. □

0FAG **Lemma 42.46.11.** *In Situation 42.7.1 let X be locally of finite type over S. Let E and F be perfect objects of $D(\mathcal{O}_X)$ whose Chern classes are defined. Then we have*

$$c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = r(E)c_1(\mathcal{F}) + r(F)c_1(\mathcal{E})$$

and for $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we have the expression

$$r(E)c_2(F) + r(F)c_2(E) + \binom{r(E)}{2} c_1(F)^2 + (r(E)r(F) - 1)c_1(F)c_1(E) + \binom{r(F)}{2} c_1(E)^2$$

and so on for higher Chern classes in $A^*(X)$. Similarly, we have $ch(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = ch(E)ch(F)$ in $A^*(X) \otimes \mathbf{Q}$. More precisely, we have

$$P_p(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(E) P_{p_2}(F)$$

in $A^p(X)$.

Proof. After choosing an envelope $f : Y \to X$ such that Lf^*E and Lf^*F can be represented by locally bounded complexes of finite locally free $\mathcal{O}_X$-modules this follows by a computation from the corresponding result for vector bundles in Lemmas 42.43.4 and 42.45.3. A better proof is probably to use the splitting principle as in Remark 42.46.8 and reduce the lemma to computations in polynomial rings which we describe in the next paragraph.

Let A be a commutative ring (for us this will be the subring of the bivariant chow ring of X generated by Chern classes). Let S be a finite set together with maps $\epsilon : S \to \{\pm 1\}$ and $f : S \to A$. Define

$$P_p(S, f, \epsilon) = \sum\nolimits_{s \in S} \epsilon(s) f(s)^p$$

in A. Given a second triple (S', ϵ', f') the equality that has to be shown for P_p is the equality

$$P_p(S \times S', f + f', \epsilon\epsilon') = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(S, f, \epsilon) P_{p_2}(S', f', \epsilon')$$

To see this is true, one reduces to the polynomial ring on variables $S \amalg S'$ and one shows that each term $f(s)^i f'(s')^j$ occurs on the left and right hand side with the same coefficient. To verify the formulas for $c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ and $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we use the splitting principle to reduce to checking these formulae in a torsion free ring. Then we use the relationship between $P_j(E)$ and $c_i(E)$ proved in Remark 42.46.8. For example

$$c_1(E \otimes F) = P_1(E \otimes F) = r(F)P_1(E) + r(E)P_1(F) = r(F)c_1(E) + r(E)c_1(F)$$

the middle equation because $r(E) = P_0(E)$ by definition. Similarly, we have

$$\begin{aligned}
&2c_2(E \otimes F) \\
&= c_1(E \otimes F)^2 - P_2(E \otimes F) \\
&= (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)P_2(E) - P_1(E)P_1(F) - r(E)P_2(F) \\
&= (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)(c_1(E)^2 - 2c_2(E)) - c_1(E)c_1(F) - \\
&\quad r(E)(c_1(F)^2 - 2c_2(F))
\end{aligned}$$

which the reader can verify agrees with the formula in the statement of the lemma up to a factor of 2. □

42.47. A baby case of localized Chern classes

0F9G In this section we discuss some properties of the bivariant classes constructed in the following lemma; most of these properties follow immediately from the characterization given in the lemma. We urge the reader to skip the rest of the section.

0F9H **Lemma 42.47.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E_2 \in D(\mathcal{O}_{X_2})$ be a perfect object. Assume*

(1) *Chern classes of E_2 are defined,*
(2) *the restriction $E_2|_{X_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0.*

Then there is a canonical bivariant class

$$P'_p(E_2), \text{ resp. } c'_p(E_2) \in A^p(X_2 \to X)$$

characterized by the property

$$P'_p(E_2) \cap i_{2,*}\alpha_2 = P_p(E_2) \cap \alpha_2 \quad \text{and} \quad P'_p(E_2) \cap i_{1,*}\alpha_1 = 0,$$

respectively

$$c'_p(E_2) \cap i_{2,*}\alpha_2 = c_p(E_2) \cap \alpha_2 \quad \text{and} \quad c'_p(E_2) \cap i_{1,*}\alpha_1 = 0$$

for $\alpha_i \in \mathrm{CH}_k(X_i)$ and similarly after any base change $X' \to X$ locally of finite type.

Proof. We are going to use the material of Section 42.46 without further mention.

Assume $E_2|_{X_1 \cap X_2}$ is zero. Consider a morphism of schemes $X' \to X$ which is locally of finite type and denote $i'_j : X'_j \to X'$ the base change of i_j. By Lemma 42.19.4 we can write any element $\alpha' \in \mathrm{CH}_k(X')$ as $i'_{1,*}\alpha'_1 + i'_{2,*}\alpha'_2$ where $\alpha'_2 \in \mathrm{CH}_k(X'_2)$ is well defined up to an element in the image of pushforward by $X'_1 \cap X'_2 \to X'_2$. Then we can set $P'_p(E_2) \cap \alpha' = P_p(E_2) \cap \alpha'_2 \in \mathrm{CH}_{k-p}(X'_2)$. This is well defined by our assumption that E_2 restricts to zero on $X_1 \cap X_2$.

If $E_2|_{X_1 \cap X_2}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0, then $c_p(E_2|_{X_1 \cap X_2}) = 0$ by rank considerations and we can argue in exactly the same manner. □

0FAH **Lemma 42.47.2.** *In Lemma 42.47.1 the bivariant class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$ does not depend on the choice of X_1.*

Proof. Suppose that $X'_1 \subset X$ is another closed subscheme such that $X = X'_1 \cup X_2$ set theoretically and the restriction $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X'_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0. Then $X = (X_1 \cap X'_1) \cup X_2$. Hence we can write any element $\alpha \in \mathrm{CH}_k(X)$ as $i_*\beta + i_{2,*}\alpha_2$ with $\alpha_2 \in \mathrm{CH}_k(X'_2)$ and $\beta \in \mathrm{CH}_k(X_1 \cap X'_1)$. Thus it is clear that $P'_p(E_2) \cap \alpha = P_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, resp. $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, is independent of whether we use X_1 or X'_1. Similarly after any base change. □

0GUG **Lemma 42.47.3.** *In Lemma 42.47.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $X' = X'_1 \cup X'_2$ and $E'_2 \in D(\mathcal{O}_{X'_2})$ the pullbacks to X'. Then the class $P'_p(E'_2)$, resp. $c'_p(E'_2)$ in $A^p(X'_2 \to X')$ constructed in Lemma 42.47.1 using $X' = X'_1 \cup X'_2$ and E'_2 is the restriction (Remark 42.33.5) of the class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$.*

Proof. Immediate from the characterization of these classes in Lemma 42.47.1. □

0F9I **Lemma 42.47.4.** *In Lemma 42.47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ such that $E|_{X_1}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. If Chern classes of E are defined, then $i_{2,*} \circ P'_p(E_2) = P_p(E)$, resp. $i_{2,*} \circ c'_p(E_2) = c_p(E)$ (with $\circ$ as in Lemma 42.33.4).*

Proof. First, assume $E|_{X_1}$ is zero. With notations as in the proof of Lemma 42.47.1 the lemma in this case follows from

$$\begin{aligned} P_p(E) \cap \alpha' &= i'_{1,*}(P_p(E) \cap \alpha'_1) + i'_{2,*}(P_p(E) \cap \alpha'_2) \\ &= i'_{1,*}(P_p(E|_{X_1}) \cap \alpha'_1) + i'_{2,*}(P'_p(E_2) \cap \alpha') \\ &= i'_{2,*}(P'_p(E_2) \cap \alpha') \end{aligned}$$

The case where $E|_{X_1}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0 is similar. □

0FAI **Lemma 42.47.5.** *In Lemma 42.47.1 suppose we have closed subschemes $X'_2 \subset X_2$ and $X_1 \subset X'_1 \subset X$ such that $X = X'_1 \cup X'_2$ set theoretically. Assume $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. Then we have $(X'_2 \to X_2)_* \circ P'_p(E_2|_{X'_2}) = P'_p(E_2)$, resp. $(X'_2 \to X_2)_* \circ c'_p(E_2|_{X'_2}) = c_p(E_2)$ (with $\circ$ as in Lemma 42.33.4).*

Proof. This follows immediately from the characterization of these classes in Lemma 42.47.1. □

0FAJ **Lemma 42.47.6.** *In Lemma 42.47.1 let $f : Y \to X$ be locally of finite type and say $c \in A^*(Y \to X)$. Then*

$$c \circ P'_p(E_2) = P'_p(Lf_2^* E_2) \circ c \quad \text{resp.} \quad c \circ c'_p(E_2) = c'_p(Lf_2^* E_2) \circ c$$

in $A^*(Y_2 \to Y)$ where $f_2 : Y_2 \to X_2$ is the base change of f.

Proof. Let $\alpha \in \mathrm{CH}_k(X)$. We may write

$$\alpha = \alpha_1 + \alpha_2$$

with $\alpha_i \in \mathrm{CH}_k(X_i)$; we are omitting the pushforwards by the closed immersions $X_i \to X$. The reader then checks that $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2$, $c \cap c'_p(E_2) \cap \alpha = c \cap c_p(E_2) \cap \alpha_2$, $c \cap \alpha = c \cap \alpha_1 + c \cap \alpha_2$, and $c'_p(Lf_2^* E_2) \cap c \cap \alpha = c_p(Lf_2^* E_2) \cap c \cap \alpha_2$. We conclude by Lemma 42.46.6. □

0FAK **Lemma 42.47.7.** *In Lemma 42.47.1 assume $E_2|_{X_1 \cap X_2}$ is zero. Then*

$$\begin{aligned} P'_1(E_2) &= c'_1(E_2), \\ P'_2(E_2) &= c'_1(E_2)^2 - 2c'_2(E_2), \\ P'_3(E_2) &= c'_1(E_2)^3 - 3c'_1(E_2)c'_2(E_2) + 3c'_3(E_2), \\ P'_4(E_2) &= c'_1(E_2)^4 - 4c'_1(E_2)^2 c'_2(E_2) + 4c'_1(E_2)c'_3(E_2) + 2c'_2(E_2)^2 - 4c'_4(E_2), \end{aligned}$$

and so on with multiplication as in Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(E_2)$ are defined for all $p \geq 1$. The equalities follow immediately from the characterization of the classes produced by Lemma 42.47.1 and the corresponding result for capping with the Chern classes of E_2 given in Remark 42.46.8. □

0FAL **Lemma 42.47.8.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E, F \in D(\mathcal{O}_X)$ be perfect objects. Assume*

1. *Chern classes of E and F are defined,*
2. *the restrictions $E|_{X_1 \cap X_2}$ and $F|_{X_1 \cap X_2}$ are isomorphic to a finite locally free $\mathcal{O}_{X_1}$-modules of rank $< p$ and $< q$ sitting in cohomological degree 0.*

With notation as in Remark 42.34.7 set

$$c^{(p)}(E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c'_p(E|_{X_2}) + c'_{p+1}(E|_{X_2}) + \ldots \in A^{(p)}(X_2 \to X)$$

with $c'_p(E|_{X_2})$ as in Lemma 42.47.1. Similarly for $c^{(q)}(F)$ and $c^{(p+q)}(E \oplus F)$. Then $c^{(p+q)}(E \oplus F) = c^{(p)}(E)c^{(q)}(F)$ in $A^{(p+q)}(X_2 \to X)$.

Proof. Immediate from the characterization of the classes in Lemma 42.47.1 and the additivity in Lemma 42.46.7. □

0FAM **Lemma 42.47.9.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E, F \in D(\mathcal{O}_{X_2})$ be perfect objects. Assume*

(1) *Chern classes of E and F are defined,*
(2) *the restrictions $E|_{X_1 \cap X_2}$ and $F|_{X_1 \cap X_2}$ are zero,*

Denote $P'_p(E), P'_p(F), P'_p(E \oplus F) \in A^p(X_2 \to X)$ for $p \geq 0$ the classes constructed in Lemma 42.47.1. Then $P'_p(E \oplus F) = P'_p(E) + P'_p(F)$.

Proof. Immediate from the characterization of the classes in Lemma 42.47.1 and the additivity in Lemma 42.46.7. □

0FAN **Lemma 42.47.10.** *In Lemma 42.47.1 assume E_2 has constant rank 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$c'_i(E_2 \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{-i+j}{j} c'_{i-j}(E_2) c_1(\mathcal{L})^j$$

Proof. The assumption on rank implies that $E_2|_{X_1 \cap X_2}$ is zero. Hence $c'_i(E_2)$ is defined for all $i \geq 1$ and the statement makes sense. The actual equality follows immediately from Lemma 42.46.10 and the characterization of c'_i in Lemma 42.47.1. □

0FE5 **Lemma 42.47.11.** *In Situation 42.7.1 let X be locally of finite type over S. Let*

$$X = X_1 \cup X_2 = X'_1 \cup X'_2$$

be two ways of writing X as a set theoretic union of closed subschemes. Let E, E' be perfect objects of $D(\mathcal{O}_X)$ whose Chern classes are defined. Assume that $E|_{X_1}$ and $E'|_{X'_1}$ are zero[5] for $i = 1, 2$. Denote

(1) *$r = P'_0(E) \in A^0(X_2 \to X)$ and $r' = P'_0(E') \in A^0(X'_2 \to X)$,*
(2) *$\gamma_p = c'_p(E|_{X_2}) \in A^p(X_2 \to X)$ and $\gamma'_p = c'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$,*
(3) *$\chi_p = P'_p(E|_{X_2}) \in A^p(X_2 \to X)$ and $\chi'_p = P'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$*

the classes constructed in Lemma 42.47.1. Then we have

$$c'_1((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_1 + r'\gamma_1$$

in $A^1(X_2 \cap X'_2 \to X)$ and

$$c'_2((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_2 + r'\gamma_2 + \binom{r}{2}(\gamma'_1)^2 + (rr' - 1)\gamma'_1\gamma_1 + \binom{r'}{2}\gamma_1^2$$

in $A^2(X_2 \cap X'_2 \to X)$ and so on for higher Chern classes. Similarly, we have

$$P'_p((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} \chi_{p_1}\chi'_{p_2}$$

[5] Presumably there is a variant of this lemma where we only assume these restrictions are isomorphic to a finite locally free modules of rank $< p$ and $< p'$.

in $A^p(X_2 \cap X_2' \to X)$.

Proof. First we observe that the statement makes sense. Namely, we have $X = (X_2 \cap X_2') \cup Y$ where $Y = (X_1 \cap X_1') \cup (X_1 \cap X_2') \cup (X_2 \cap X_1')$ and the object $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E'$ restricts to zero on Y. The actual equalities follow from the characterization of our classes in Lemma 42.47.1 and the equalities of Lemma 42.46.11. We omit the details. □

42.48. Gysin at infinity

0FAP This section is about the bivariant class constructed in the next lemma. We urge the reader to skip the rest of the section.

0F9J **Lemma 42.48.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $b : W \to \mathbf{P}^1_X$ be a proper morphism of schemes which is an isomorphism over $\mathbf{A}^1_X$. Denote $i_\infty : W_\infty \to W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. Then there is a canonical bivariant class*

$$C \in A^0(W_\infty \to X)$$

with the property that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ for $\alpha \in \mathrm{CH}_k(X)$ and similarly after any base change by $X' \to X$ locally of finite type.

Proof. Given $\alpha \in \mathrm{CH}_k(X)$ there exists a $\beta \in \mathrm{CH}_{k+1}(W)$ restricting to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, see Lemma 42.14.2. A second choice of β differs from β by a cycle supported on W_∞, see Lemma 42.19.3. Since the normal bundle of the effective Cartier divisor $D_\infty \subset \mathbf{P}^1_X$ of (42.18.1.1) is trivial, the Gysin homomorphism i_∞^* kills cycle classes supported on W_∞, see Remark 42.29.6. Hence setting $C \cap \alpha = i_\infty^* \beta$ is well defined.

Since W_∞ and $W_0 = X \times \{0\}$ are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_X$, we see that $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_X}(1)) \cap \beta$ by Lemma 42.29.4. By our choice of β we have $i_0^*\beta = \alpha$ as cycles on $W_0 = X \times \{0\}$, see for example Lemma 42.31.1. Thus we see that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ as stated in the lemma.

Observe that the assumptions on b are preserved by any base change by $X' \to X$ locally of finite type. Hence we get an operation $C \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_k(W'_\infty)$ by the same construction as above. To see that this family of operations defines a bivariant class, we consider the diagram

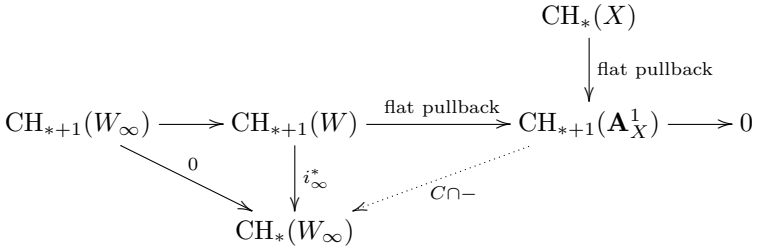

for X as indicated and the base change of this diagram for any $X' \to X$. We know that flat pullback and i_∞^* are bivariant operations, see Lemmas 42.33.2 and 42.33.3. Then a formal argument (involving huge diagrams of schemes and their chow groups) shows that the dotted arrow is a bivariant operation. □

0GUH **Lemma 42.48.2.** *In Lemma 42.48.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $b' : W' \to \mathbf{P}^1_{X'}$ and $i'_\infty : W'_\infty \to W'$ the base changes of b and i_∞. Then the class $C' \in A^0(W'_\infty \to X')$ constructed as in Lemma 42.48.1 using b' is the restriction (Remark 42.33.5) of C.*

Proof. Immediate from the construction and the fact that a similar statement holds for flat pullback and i_∞^*. □

0FAQ **Lemma 42.48.3.** *In Lemma 42.48.1 let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Let $C' \in A^0(W'_\infty \to X)$ and $C \in A^0(W_\infty \to X)$ be the classes constructed in Lemma 42.48.1. Then $g_{\infty,*} \circ C' = C$ in $A^0(W_\infty \to X)$.*

Proof. Set $b' = b \circ g : W' \to \mathbf{P}^1_X$. Denote $i'_\infty : W'_\infty \to W'$ the inclusion morphism. Denote $g_\infty : W'_\infty \to W_\infty$ the restriction of g. Given $\alpha \in \mathrm{CH}_k(X)$ choose $\beta' \in \mathrm{CH}_{k+1}(W')$ restricting to the flat pullback of α on $(b')^{-1}\mathbf{A}^1_X$. Then $\beta = g_*\beta' \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}\mathbf{A}^1_X$. Then $i_\infty^*\beta = g_{\infty,*}(i'_\infty)^*\beta'$ by Lemma 42.29.8. This and the corresponding fact after base change by morphisms $X' \to X$ locally of finite type, corresponds to the assertion made in the lemma. □

0FAR **Lemma 42.48.4.** *In Lemma 42.48.1 we have $C \circ (W_\infty \to X)_* \circ i_\infty^* = i_\infty^*$.*

Proof. Let $\beta \in \mathrm{CH}_{k+1}(W)$. Denote $i_0 : X = X \times \{0\} \to W$ the closed immersion of the fibre over 0 in $\mathbf{P}^1$. Then $(W_\infty \to X)_* i_\infty^*\beta = i_0^*\beta$ in $\mathrm{CH}_k(X)$ because $i_{\infty,*}i_\infty^*\beta$ and $i_{0,*}i_0^*\beta$ represent the same class on W (for example by Lemma 42.29.4) and hence pushforward to the same class on X. The restriction of β to $b^{-1}(\mathbf{A}^1_X)$ restricts to the flat pullback of $i_0^*\beta = (W_\infty \to X)_* i_\infty^*\beta$ because we can check this after pullback by i_0, see Lemmas 42.32.2 and 42.32.4. Hence we may use β when computing the image of $(W_\infty \to X)_* i_\infty^*\beta$ under C and we get the desired result. □

0FAS **Lemma 42.48.5.** *In Lemma 42.48.1 let $f : Y \to X$ be a morphism locally of finite type and $c \in A^*(Y \to X)$. Then $C \circ c = c \circ C$ in $A^*(W_\infty \times_X Y \to X)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} W_\infty \times_X Y & = & W_{Y,\infty} & \xrightarrow{i_{Y,\infty}} & W_Y & \xrightarrow{b_Y} & \mathbf{P}^1_Y & \xrightarrow{p_Y} & Y \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow f \\ & & W_\infty & \xrightarrow{i_\infty} & W & \xrightarrow{b} & \mathbf{P}^1_X & \xrightarrow{p} & X \end{array}$$

with cartesian squares. For an elemnent $\alpha \in \mathrm{CH}_k(X)$ choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α. Then $c \cap \beta$ is a class in $\mathrm{CH}_*(W_Y)$ whose restriction to $b_Y^{-1}(\mathbf{A}^1_Y)$ is the flat pullback of $c \cap \alpha$. Next, we have

$$i_{Y,\infty}^*(c \cap \beta) = c \cap i_\infty^*\beta$$

because c is a bivariant class. This exactly says that $C \cap c \cap \alpha = c \cap C \cap \alpha$. The same argument works after any base change by $X' \to X$ locally of finite type. This proves the lemma. □

42.49. Preparation for localized Chern classes

0FAT In this section we discuss some properties of the bivariant classes constructed in the following lemma. We urge the reader to skip the rest of the section.

0F9K **Lemma 42.49.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. Let*

$$b : W \longrightarrow \mathbf{P}^1_X$$

be a proper morphism of schemes. Let $Q \in D(\mathcal{O}_W)$ be a perfect object. Denote $W_\infty \subset W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. We assume

(A0) *Chern classes of Q are defined (Section 42.46),*
(A1) *b is an isomorphism over $\mathbf{A}^1_X$,*
(A2) *there exists a closed subscheme $T \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z$ such that $Q|_T$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$ sitting in cohomological degree 0.*

Then there exists a canonical bivariant class

$$P'_p(Q), \text{ resp. } c'_p(Q) \in A^p(Z \to X)$$

with $(Z \to X)_* \circ P'_p(Q) = P_p(Q|_{X \times \{0\}})$, resp. $(Z \to X)_* \circ c'_p(Q) = c_p(Q|_{X \times \{0\}})$.

Proof. Denote $E \subset W_\infty$ the inverse image of Z. Then $W_\infty = T \cup E$ and b induces a proper morphism $E \to Z$. Denote $C \in A^0(W_\infty \to X)$ the bivariant class constructed in Lemma 42.48.1. Denote $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ in $A^p(E \to W_\infty)$ the bivariant class constructed in Lemma 42.47.1. This makes sense because $(Q|_E)|_{T \cap E}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{E \cap T}$-module of rank $< p$ sitting in cohomological degree 0 by assumption (A2). Then we define

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

This is a bivariant class, see Lemma 42.33.4. Since $E \to Z \to X$ is equal to $E \to W_\infty \to W \to X$ we see that

$$\begin{aligned}
(Z \to X)_* \circ c'_p(Q) &= (W \to X)_* \circ i_{\infty,*} \circ (E \to W_\infty)_* \circ c'_p(Q|_E) \circ C \\
&= (W \to X)_* \circ i_{\infty,*} \circ c_p(Q|_{W_\infty}) \circ C \\
&= (W \to X)_* \circ c_p(Q) \circ i_{\infty,*} \circ C \\
&= (W \to X)_* \circ c_p(Q) \circ i_{0,*} \\
&= (W \to X)_* \circ i_{0,*} \circ c_p(Q|_{X \times \{0\}}) \\
&= c_p(Q|_{X \times \{0\}})
\end{aligned}$$

The second equality holds by Lemma 42.47.4. The third equality because $c_p(Q)$ is a bivariant class. The fourth equality by Lemma 42.48.1. The fifth equality because $c_p(Q)$ is a bivariant class. The final equality because $(W_0 \to W) \circ (W \to X)$ is the identity on X if we identify W_0 with X as we've done above. The exact same sequence of equations works to prove the property for $P'_p(Q)$. □

0GUI **Lemma 42.49.2.** *In Lemma 42.49.1 let $X' \to X$ be a morphism which is locally of finite type. Denote Z', $b' : W' \to \mathbf{P}^1_{X'}$, and $T' \subset W'_\infty$ the base changes of Z, $b : W \to \mathbf{P}^1_X$, and $T \subset W_\infty$. Set $Q' = (W' \to W)^* Q$. Then the class $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z' \to X')$ constructed as in Lemma 42.49.1 using b', Q', and T' is the restriction (Remark 42.33.5) of the class $P'_p(Q)$, resp. $c'_p(Q)$ in $A^p(Z \to X)$.*

Proof. Recall that the construction is as follows

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

Thus the lemma follows from the corresponding base change property for C (Lemma 42.48.2) and the fact that the same base change property holds for the classes constructed in Lemma 42.47.1 (small detail omitted). □

0FAU **Lemma 42.49.3.** *In Lemma 42.49.1 the bivariant class $P'_p(Q)$, resp. $c'_p(Q)$ is independent of the choice of the closed subscheme T. Moreover, given a proper morphism $g : W' \to W$ which is an isomorphism over $\mathbf{A}^1_X$, then setting $Q' = g^*Q$ we have $P'_p(Q) = P'_p(Q')$, resp. $c'_p(Q) = c'_p(Q')$.*

Proof. The independence of T follows immediately from Lemma 42.47.2.

Let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Observe that taking $T' = g^{-1}(T) \subset W'_\infty$ is a closed subscheme satisfying (A2) hence the operator $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z \to X)$ corresponding to $b' = b \circ g : W' \to \mathbf{P}^1_X$ and Q' is defined. Denote $E' \subset W'_\infty$ the inverse image of Z in W'_∞. Recall that

$$c'_p(Q') = (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C'$$

with $C' \in A^0(W'_\infty \to X)$ and $c'_p(Q'|_{E'}) \in A^p(E' \to W'_\infty)$. By Lemma 42.48.3 we have $g_{\infty,*} \circ C' = C$. Observe that E' is also the inverse image of E in W'_∞ by g_∞. Since moreover $Q' = g^*Q$ we find that $c'_p(Q'|_{E'})$ is simply the restriction of $c'_p(Q|_E)$ to schemes lying over W'_∞, see Remark 42.33.5. Thus we obtain

$$\begin{aligned} c'_p(Q') &= (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C' \\ &= (E \to Z)_* \circ (E' \to E)_* \circ c'_p(Q|_E) \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ g_{\infty,*} \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ C \\ &= c'_p(Q) \end{aligned}$$

In the third equality we used that $c'_p(Q|_E)$ commutes with proper pushforward as it is a bivariant class. The equality $P'_p(Q) = P'_p(Q')$ is proved in exactly the same way. □

0FAV **Lemma 42.49.4.** *In Lemma 42.49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Denote $C \in A^0(W_\infty \to X)$ the class of Lemma 42.48.1. Then*

$$C \circ c_p(Q|_{X \times \{0\}}) = C \circ (Z \to X)_* \circ c'_p(Q) = c_p(Q|_{W_\infty}) \circ C$$

Proof. The first equality holds because $c_p(Q|_{X\times\{0\}}) = (Z \to X)_* \circ c'_p(Q)$ by Lemma 42.49.1. We may prove the second equality one cycle class at a time (see Lemma 42.35.3). Since the construction of the bivariant classes in the lemma is compatible with base change, we may assume we have some $\alpha \in \mathrm{CH}_k(X)$ and we

have to show that $C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) = c_p(Q|_{W_\infty}) \cap C \cap \alpha$. Observe that

$$\begin{aligned}
C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) &= C \cap (Z \to X)_*(E \to Z)_*(c'_p(Q|_E) \cap C \cap \alpha) \\
&= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap C \cap \alpha) \\
&= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap i_\infty^*\beta) \\
&= C \cap (W_\infty \to X)_*(c_p(Q|_{W_\infty}) \cap i_\infty^*\beta) \\
&= C \cap (W_\infty \to X)_* i_\infty^*(c_p(Q) \cap \beta) \\
&= i_\infty^*(c_p(Q) \cap \beta) \\
&= c_p(Q|_{W_\infty}) \cap i_\infty^*\beta \\
&= c_p(Q|_{W_\infty}) \cap C \cap \alpha
\end{aligned}$$

as desired. For the first equality we used that $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where $E \subset W_\infty$ is the inverse image of Z and $c'_p(Q|_E)$ is the class constructed in Lemma 42.47.1. The second equality is just the statement that $E \to Z \to X$ is equal to $E \to W_\infty \to X$. For the third equality we choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α so that $C \cap \alpha = i_\infty^*\beta$ by construction. The fourth equality is Lemma 42.47.4. The fifth equality is the fact that $c_p(Q)$ is a bivariant class and hence commutes with i_∞^*. The sixth equality is Lemma 42.48.4. The seventh uses again that $c_p(Q)$ is a bivariant class. The final holds as $C \cap \alpha = i_\infty^*\beta$. □

0FAW **Lemma 42.49.5.** *In Lemma 42.49.1 let $Y \to X$ be a morphism locally of finite type and let $c \in A^*(Y \to X)$ be a bivariant class. Then*

$$P'_p(Q) \circ c = c \circ P'_p(Q) \quad \textit{resp.} \quad c'_p(Q) \circ c = c \circ c'_p(Q)$$

in $A^*(Y \times_X Z \to X)$.

Proof. Let $E \subset W_\infty$ be the inverse image of Z. Recall that $P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C$, resp. $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where C is as in Lemma 42.48.1 and $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ are as in Lemma 42.47.1. By Lemma 42.48.5 we see that C commutes with c and by Lemma 42.47.6 we see that $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ commutes with c. Since c is a bivariant class it commutes with proper pushforward by $E \to Z$ by definition. This finishes the proof. □

0FAX **Lemma 42.49.6.** *In Lemma 42.49.1 assume $Q|_T$ is zero. In $A^*(Z \to X)$ we have*

$$\begin{aligned}
P'_1(Q) &= c'_1(Q), \\
P'_2(Q) &= c'_1(Q)^2 - 2c'_2(Q), \\
P'_3(Q) &= c'_1(Q)^3 - 3c'_1(Q)c'_2(Q) + 3c'_3(Q), \\
P'_4(Q) &= c'_1(Q)^4 - 4c'_1(Q)^2c'_2(Q) + 4c'_1(Q)c'_3(Q) + 2c'_2(Q)^2 - 4c'_4(Q),
\end{aligned}$$

and so on with multiplication as in Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(Q)$ are defined for all $p \geq 1$. In the proof of Lemma 42.49.1 we have constructed the classes $P'_p(Q)$ and $c'_p(Q)$ using the bivariant class $C \in A^0(W_\infty \to X)$ of Lemma 42.48.1 and the bivariant classes $P'_p(Q|_E)$ and $c'_p(Q|_E)$ of Lemma 42.47.1 for the restriction $Q|_E$ of Q to the inverse image E of Z in W_∞. Observe

that by Lemma 42.47.7 we have the desired relationship between $P'_p(Q|_E)$ and $c'_p(Q|_E)$. Recall that

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C \quad \text{and} \quad c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

To finish the proof it suffices to show the multiplications defined in Remark 42.34.7 on the classes $a_p = c'_p(Q)$ and on the classes $b_p = c'_p(Q|_E)$ agree:

$$a_{p_1} a_{p_2} \ldots a_{p_r} = (E \to Z)_* \circ b_{p_1} b_{p_2} \ldots b_{p_r} \circ C$$

Some details omitted. If $r = 1$, then this is true. For $r > 1$ note that by Remark 42.34.8 the multiplication in Remark 42.34.7 proceeds by inserting $(Z \to X)_*$, resp. $(E \to W_\infty)_*$ in between the factors of the product $a_{p_1} a_{p_2} \ldots a_{p_r}$, resp. $b_{p_1} b_{p_2} \ldots b_{p_r}$ and taking compositions as bivariant classes. Now by Lemma 42.47.1 we have

$$(E \to W_\infty)_* \circ b_{p_i} = c_{p_i}(Q|_{W_\infty})$$

and by Lemma 42.49.4 we have

$$C \circ (Z \to X)_* \circ a_{p_i} = c_{p_i}(Q|_{W_\infty}) \circ C$$

for $i = 2, \ldots, r$. A calculation shows that the left and right hand side of the desired equality both simplify to

$$(E \to Z)_* \circ c'_{p_1}(Q|_E) \circ c_{p_2}(Q|_{W_\infty}) \circ \ldots \circ c_{p_r}(Q|_{W_\infty}) \circ C$$

and the proof is complete. □

0FAY **Lemma 42.49.7.** *In Lemma 42.49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined with $Q'|_T$ isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p'$ placed in cohomological degree 0. With notation as in Remark 42.34.7 set*

$$c^{(p)}(Q) = 1 + c_1(Q|_{X\times\{0\}}) + \ldots + c_{p-1}(Q|_{X\times\{0\}}) + c'_p(Q) + c'_{p+1}(Q) + \ldots$$

in $A^{(p)}(Z \to X)$ with $c'_i(Q)$ for $i \geq p$ as in Lemma 42.49.1. Similarly for $c^{(p')}(Q')$ and $c^{(p+p')}(Q \oplus Q')$. Then $c^{(p+p')}(Q \oplus Q') = c^{(p)}(Q)c^{(p')}(Q')$ in $A^{(p+p')}(Z \to X)$.

Proof. Recall that the image of $c'_i(Q)$ in $A^p(X)$ is equal to $c_i(Q|_{X\times\{0\}})$ for $i \geq p$ and similarly for Q' and $Q \oplus Q'$, see Lemma 42.49.1. Hence the equality in degrees $< p + p'$ follows from the additivity of Lemma 42.46.7.

Let's take $n \geq p + p'$. As in the proof of Lemma 42.49.1 let $E \subset W_\infty$ denote the inverse image of Z. Observe that we have the equality

$$c^{(p+p')}(Q|_E \oplus Q'|_E) = c^{(p)}(Q|_E)c^{(p')}(Q'|_E)$$

in $A^{(p+p')}(E \to W_\infty)$ by Lemma 42.47.8. Since by construction

$$c'_p(Q \oplus Q') = (E \to Z)_* \circ c'_p(Q|_E \oplus Q'|_E) \circ C$$

we conclude that suffices to show for all $i + j = n$ we have

$$(E \to Z)_* \circ c_i^{(p)}(Q|_E) c_j^{(p')}(Q'|_E) \circ C = c_i^{(p)}(Q) c_j^{(p')}(Q')$$

in $A^n(Z \to X)$ where the multiplication is the one from Remark 42.34.7 on both sides. There are three cases, depending on whether $i \geq p$, $j \geq p'$, or both.

Assume $i \geq p$ and $j \geq p'$. In this case the products are defined by inserting $(E \to W_\infty)_*$, resp. $(Z \to X)_*$ in between the two factors and taking compositions as bivariant classes, see Remark 42.34.8. In other words, we have to show

$$(E \to Z)_* \circ c_i'(Q|_E) \circ (E \to W_\infty)_* \circ c_j'(Q'|_E) \circ C = c_i'(Q) \circ (Z \to X)_* \circ c_j'(Q')$$

By Lemma 42.47.1 the left hand side is equal to

$$(E \to Z)_* \circ c_i'(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C$$

Since $c_i'(Q) = (E \to Z)_* \circ c_i'(Q|_E) \circ C$ the right hand side is equal to

$$(E \to Z)_* \circ c_i'(Q|_E) \circ C \circ (Z \to X)_* \circ c_j'(Q')$$

which is immediately seen to be equal to the above by Lemma 42.49.4.

Assume $i \geq p$ and $j < p$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c_i'(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C = c_i'(Q) \circ c_j(Q'|_{X \times \{0\}})$$

Again using that $c_i'(Q) = (E \to Z)_* \circ c_i'(Q|_E) \circ C$ we see that it suffices to show $c_j(Q'|_{W_\infty}) \circ C = C \circ c_j(Q'|_{X \times \{0\}})$ which is part of Lemma 42.49.4.

Assume $i < p$ and $j \geq p'$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c_i(Q|_E) \circ c_j'(Q'|_E) \circ C = c_i(Q|_{Z \times \{0\}}) \circ c_j'(Q')$$

However, since $c_j'(Q|_E)$ and $c_j'(Q')$ are bivariant classes, they commute with capping with Chern classes (Lemma 42.38.9). Hence it suffices to prove

$$(E \to Z)_* \circ c_j'(Q'|_E) \circ c_i(Q|_{W_\infty}) \circ C = c_j'(Q') \circ c_i(Q|_{X \times \{0\}})$$

which we reduces us to the case discussed in the preceding paragraph. □

0FAZ **Lemma 42.49.8.** *In Lemma 42.49.1 assume $Q|_T$ is zero. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined such that the restriction $Q'|_T$ is zero. In this case the classes $P_p'(Q), P_p'(Q'), P_p'(Q \oplus Q') \in A^p(Z \to X)$ constructed in Lemma 42.49.1 satisfy $P_p'(Q \oplus Q') = P_p'(Q) + P_p'(Q')$.*

Proof. This follows immediately from the construction of these classes and Lemma 42.47.9. □

42.50. Localized Chern classes

0FB0 Outline of the construction. Let F be a field, let X be a variety over F, let E be a perfect object of $D(\mathcal{O}_X)$, and let $Z \subset X$ be a closed subscheme such that $E|_{X \setminus Z} = 0$. Then we want to construct elements

$$c_p(Z \to X, E) \in A^p(Z \to X)$$

We will do this by constructing a diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & X \\ {\scriptstyle f}\downarrow & & \\ \mathbf{P}^1_F & & \end{array}$$

and a perfect object Q of $D(\mathcal{O}_W)$ such that

(1) f is flat, and f, q are proper; for $t \in \mathbf{P}^1_F$ denote W_t the fibre of f, $q_t : W_t \to X$ the restriction of q, and $Q_t = Q|_{W_t}$,
(2) $q_t : W_t \to X$ is an isomorphism and $Q_t = q_t^* E$ for $t \in \mathbf{A}^1_F$,

(3) $q_\infty : W_\infty \to X$ is an isomorphism over $X \setminus Z$,
(4) if $T \subset W_\infty$ is the closure of $q_\infty^{-1}(X \setminus Z)$ then $Q_\infty|_T$ is zero.

The idea is to think of this as a family $\{(W_t, Q_t)\}$ parametrized by $t \in \mathbf{P}^1$. For $t \neq \infty$ we see that $c_p(Q_t)$ is just $c_p(E)$ on the chow groups of $Q_t = X$. But for $t = \infty$ we see that $c_p(Q_\infty)$ sends classes on Q_∞ to classes supported on $E = q_\infty^{-1}(Z)$ since $Q_\infty|_T = 0$. We think of E as the exceptional locus of $q_\infty : W_\infty \to X$. Since any $\alpha \in \mathrm{CH}_*(X)$ gives rise to a "family" of cycles $\alpha_t \in \mathrm{CH}_*(W_t)$ it makes sense to define $c_p(Z \to X, E) \cap \alpha$ as the pushforward $(E \to Z)_*(c_p(Q_\infty) \cap \alpha_\infty)$.

To make this work there are two main ingredients: (1) the construction of W and Q is a sort of algebraic Macpherson's graph construction; it is done in More on Flatness, Section 38.44. (2) the construction of the actual class given W and Q is done in Section 42.49 relying on Sections 42.48 and 42.47.

0GUJ **Situation 42.50.1.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be a closed immersion. Let $E \in D(\mathcal{O}_X)$ be an object. Let $p \geq 0$. Assume

(1) E is a perfect object of $D(\mathcal{O}_X)$,
(2) the restriction $E|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, and
(3) at least one[6] of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) there exists a locally bounded complex of finite locally free $\mathcal{O}_X$-modules representing E, or (d) there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object on X' and the irreducible components of X' are quasi-compact.

0FB2 **Lemma 42.50.2.** *In Situation 42.50.1 there exists a canonical bivariant class*

$$P_p(Z \to X, E) \in A^p(Z \to X), \quad \textit{resp.} \quad c_p(Z \to X, E) \in A^p(Z \to X)$$

with the property that

0FB1 (42.50.2.1) $$i_* \circ P_p(Z \to X, E) = P_p(E), \quad \textit{resp.} \quad i_* \circ c_p(Z \to X, E) = c_p(E)$$

as bivariant classes on X (with $\circ$ as in Lemma 42.33.4).

Proof. The construction of these bivariant classes is as follows. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad T \longrightarrow W_\infty \quad \text{and} \quad Q$$

be the blowing up, the perfect object Q in $D(\mathcal{O}_W)$, and the closed immersion constructed in More on Flatness, Section 38.44 and Lemma 38.44.1. Let $T' \subset T$ be the open and closed subscheme such that $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{T'}$-module of rank $< p$ sitting in cohomological degree 0. By condition (2) of Situation 42.50.1 the morphisms

$$T' \to T \to W_\infty \to X$$

are all isomorphisms of schemes over the open subscheme $X \setminus Z$ of X. Below we check the chern classes of Q are defined. Recalling that $Q|_{X \times \{0\}} \cong E$ by construction, we conclude that the bivariant class constructed in Lemma 42.49.1 using W, b, Q, T' gives us classes

$$P_p(Z \to X, E) = P'_p(Q) \in A^p(Z \to X)$$

[6] Please ignore this technical condition on a first reading; see discussion in Remark 42.50.5.

and

$$c_p(Z \to X, E) = c'_p(Q) \in A^p(Z \to X)$$

satisfying (42.50.2.1).

In this paragraph we prove that the chern classes of Q are defined (Definition 42.46.3); we suggest the reader skip this. If assumption (3)(a) or (3)(b) of Situation 42.50.1 holds, i.e., if X has quasi-compact irreducible components, then the same is true for W (because $W \to X$ is proper). Hence we conclude that the chern classes of any perfect object of $D(\mathcal{O}_W)$ are defined by Lemma 42.46.4. If (3)(c) hold, i.e., if E can be represented by a locally bounded complex of finite locally free modules, then the object Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules by part (5) of More on Flatness, Lemma 38.44.1. Hence the chern classes of Q are defined. Finally, assume (3)(d) holds, i.e., assume we have a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' and the irreducible components of X' are quasi-compact. Let $b' : W' \to \mathbf{P}^1_{X'}$ and $Q' \in D(\mathcal{O}_{W'})$ be the morphism and perfect object constructed as in More on Flatness, Section 38.44 starting with the triple $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})_\infty, L(p')^*E')$. By the discussion above we see that the chern classes of Q' are defined. Since b and b' were constructed via an application of More on Flatness, Lemma 38.43.6 it follows from More on Flatness, Lemma 38.43.8 that there exists a morphism $W \to W'$ such that $Q = L(W \to W')^*Q'$. Then it follows from Lemma 42.46.4 that the chern classes of Q are defined. □

0FB5 **Definition 42.50.3.** With (S, δ), X, $E \in D(\mathcal{O}_X)$, and $i : Z \to X$ as in Situation 42.50.1.

(1) If the restriction $E|_{X\setminus Z}$ is zero, then for all $p \geq 0$ we define

$$P_p(Z \to X, E) \in A^p(Z \to X)$$

by the construction in Lemma 42.50.2 and we define the *localized Chern character* by the formula

$$ch(Z \to X, E) = \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(Z \to X, E)}{p!} \quad \text{in} \quad \prod\nolimits_{p\geq 0} A^p(Z \to X) \otimes \mathbf{Q}$$

(2) If the restriction $E|_{X\setminus Z}$ is isomorphic to a finite locally free $\mathcal{O}_{X\setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, then we define the *localized pth Chern class* $c_p(Z \to X, E)$ by the construction in Lemma 42.50.2.

In the situation of the definition assume $E|_{X\setminus Z}$ is zero. Then, to be sure, we have the equality

$$i_* \circ ch(Z \to X, E) = ch(E)$$

in $A^*(X) \otimes \mathbf{Q}$ because we have shown the equality (42.50.2.1) above.

Here is an important sanity check.

0FB3 **Lemma 42.50.4.** *In Situation 42.50.1 let $f : X' \to X$ be a morphism of schemes which is locally of finite type. Denote $E' = f^*E$ and $Z' = f^{-1}(Z)$. Then the bivariant class of Definition 42.50.3*

$$P_p(Z' \to X', E') \in A^p(Z' \to X'), \quad \textit{resp.} \quad c_p(Z' \to X', E') \in A^p(Z' \to X')$$

constructed as in Lemma 42.50.2 using X', Z', E' is the restriction (Remark 42.33.5) of the bivariant class $P_p(Z \to X, E) \in A^p(Z \to X)$, resp. $c_p(Z \to X, E) \in A^p(Z \to X)$.

Proof. Denote $p : \mathbf{P}^1_X \to X$ and $p' : \mathbf{P}^1_{X'} \to X'$ the structure morphisms. Recall that $b : W \to \mathbf{P}^1_X$ and $b' : W' \to \mathbf{P}^1_{X'}$ are the morphism constructed from the triples $(\mathbf{P}^1_X, (\mathbf{P}^1_X)\infty, p^*E)$ and $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})\infty, (p')^*E')$ in More on Flatness, Lemma 38.43.6. Furthermore $Q = L\eta_{\mathcal{I}_\infty} p^*E$ and $Q = L\eta_{\mathcal{I}'_\infty}(p')^*E'$ where $\mathcal{I}_\infty \subset \mathcal{O}_W$ is the ideal sheaf of W_∞ and $\mathcal{I}'_\infty \subset \mathcal{O}_{W'}$ is the ideal sheaf of W'_∞. Next, $h : \mathbf{P}^1_{X'} \to \mathbf{P}^1_X$ is a morphism of schemes such that the pullback of the effective Cartier divisor $(\mathbf{P}^1_X)_\infty$ is the effective Cartier divisor $(\mathbf{P}^1_{X'})_\infty$ and such that $h^*p^*E = (p')^*E'$. By More on Flatness, Lemma 38.43.8 we obtain a commutative diagram

$$\begin{array}{ccccc} W' & \xrightarrow{g} & \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W & \xrightarrow{q} & W \\ & \searrow^{b'} & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle b} \\ & & \mathbf{P}^1_{X'} & \longrightarrow & \mathbf{P}^1_X \end{array}$$

such that W' is the "strict transform" of $\mathbf{P}^1_{X'}$ with respect to b and such that $Q' = (q \circ g)^*Q$. Now recall that $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c'_p(Q)$ where $P'_p(Q)$, resp. $c'_p(Q)$ are constructed in Lemma 42.49.1 using b, Q, T' where T' is a closed subscheme $T' \subset W_\infty$ with the following two properties: (a) T' contains all points of W_∞ lying over $X \setminus Z$, and (b) $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. In the construction of Lemma 42.49.1 we chose a particular closed subscheme T' with properties (a) and (b) but the precise choice of T' is immaterial, see Lemma 42.49.3.

Next, by Lemma 42.49.2 the restriction of the bivariant class $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c_p(Q')$ to X' corresponds to the class $P'_p(q^*Q)$, resp. $c'_p(q^*Q)$ constructed as in Lemma 42.49.1 using $r : \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W \to \mathbf{P}^1_{X'}$, the complex q^*Q, and the inverse image $q^{-1}(T')$.

Now by the second statement of Lemma 42.49.3 we have $P'_p(Q') = P'_p(q^*Q)$, resp. $c'_p(q^*Q) = c'_p(Q')$. Since $P_p(Z' \to X', E') = P'_p(Q')$, resp. $c_p(Z' \to X', E') = c'_p(Q')$ we conclude that the lemma is true. □

0GUK **Remark 42.50.5.** In Situation 42.50.1 it would have been more natural to replace assumption (3) with the assumption: "the chern classes of E are defined". In fact, combining Lemmas 42.50.2 and 42.50.4 with Lemma 42.35.6 it is easy to extend the definition to this (slightly) more general case. If we ever need this we will do so here.

0FB4 **Lemma 42.50.6.** *In Situation 42.50.1 we have*

$$P_p(Z \to X, E) \cap i_*\alpha = P_p(E|_Z) \cap \alpha, \quad \textit{resp.} \quad c_p(Z \to X, E) \cap i_*\alpha = c_p(E|_Z) \cap \alpha$$

in $\mathrm{CH}_*(Z)$ *for any* $\alpha \in \mathrm{CH}_*(Z)$.

Proof. We only prove the second equality and we omit the proof of the first. Since $c_p(Z \to X, E)$ is a bivariant class and since the base change of $Z \to X$ by $Z \to X$ is $\mathrm{id} : Z \to Z$ we have $c_p(Z \to X, E) \cap i_*\alpha = c_p(Z \to X, E) \cap \alpha$. By Lemma 42.50.4 the restriction of $c_p(Z \to X, E)$ to Z (!) is the localized Chern class for $\mathrm{id} : Z \to Z$ and $E|_Z$. Thus the result follows from (42.50.2.1) with $X = Z$. □

0FB6 **Lemma 42.50.7.** *In Situation 42.50.1 if* $\alpha \in \mathrm{CH}_k(X)$ *has support disjoint from* Z, *then* $P_p(Z \to X, E) \cap \alpha = 0$, *resp.* $c_p(Z \to X, E) \cap \alpha = 0$.

Proof. This is immediate from the construction of the localized Chern classes. It also follows from the fact that we can compute $c_p(Z \to X, E) \cap \alpha$ by first restricting $c_p(Z \to X, E)$ to the support of α, and then using Lemma 42.50.4 to see that this restriction is zero. □

0FB7 **Lemma 42.50.8.** *In Situation 42.50.1 assume $Z \subset Z' \subset X$ where Z' is a closed subscheme of X. Then $P_p(Z' \to X, E) = (Z \to Z')_* \circ P_p(Z \to X, E)$, resp. $c_p(Z' \to X, E) = (Z \to Z')_* \circ c_p(Z \to X, E)$ (with $\circ$ as in Lemma 42.33.4).*

Proof. The construction of $P_p(Z' \to X, E)$, resp. $c_p(Z' \to X, E)$ in Lemma 42.50.2 uses the exact same morphism $b : W \to \mathbf{P}^1_X$ and perfect object Q of $D(\mathcal{O}_W)$. Then we can use Lemma 42.47.5 to conclude. Some details omitted. □

0FB8 **Lemma 42.50.9.** *In Lemma 42.47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ whose restriction to X_1 is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. Then the class $P'_p(E_2)$, resp. $c'_p(E_2)$ of Lemma 42.47.1 agrees with $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ of Definition 42.50.3 provided E satisfies assumption (3) of Situation 42.50.1.*

Proof. The assumptions on E imply that there is an open $U \subset X$ containing X_1 such that $E|_U$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_U$-module of rank $< p$. See More on Algebra, Lemma 15.76.7. Let $Z \subset X$ be the complement of U in X endowed with the reduced induced closed subscheme structure. Then $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ by Lemma 42.50.8. Now we can prove that $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ satisfies the characterization of $P'_p(E_2)$, resp. $c'_p(E_2)$ given in Lemma 42.47.1. Namely, by the relation $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ just proven and the fact that $X_1 \cap Z = \emptyset$, the composition $P_p(X_2 \to X, E) \circ i_{1,*}$, resp. $c_p(X_2 \to X, E) \circ i_{1,*}$ is zero by Lemma 42.50.7. On the other hand, $P_p(X_2 \to X, E) \circ i_{2,*} = P_p(E_2)$, resp. $c_p(X_2 \to X, E) \circ i_{2,*} = c_p(E_2)$ by Lemma 42.50.6. □

42.51. Two technical lemmas

0FE6 In this section we develop some additional tools to allow us to work more comfortably with localized Chern classes. The following lemma is a more precise version of something we've already encountered in the proofs of Lemmas 42.49.6 and 42.49.7.

0FE7 **Lemma 42.51.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $b : W \longrightarrow \mathbf{P}^1_X$ be a proper morphism of schemes. Let $n \geq 1$. For $i = 1, \ldots, n$ let $Z_i \subset X$ be a closed subscheme, let $Q_i \in D(\mathcal{O}_W)$ be a perfect object, let $p_i \geq 0$ be an integer, and let $T_i \subset W_\infty$, $i = 1, \ldots, n$ be closed. Denote $W_i = b^{-1}(\mathbf{P}^1_{Z_i})$. Assume*

(1) *for $i = 1, \ldots, n$ the assumption of Lemma 42.49.1 hold for b, Z_i, Q_i, T_i, p_i,*
(2) *$Q_i|_{W \setminus W_i}$ is zero, resp. isomorphic to a finite locally free module of rank $< p_i$ placed in cohomological degree 0,*
(3) *Q_i on W satisfies assumption (3) of Situation 42.50.1.*

Then $P'_{p_n}(Q_n) \circ \ldots \circ P'_{p_1}(Q_1)$ is equal to

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ P'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ P'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n+\ldots+p_1}(Z_n \cap \ldots \cap Z_1 \to X)$*, resp.* $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ *is equal to*

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n+\ldots+p_1}(Z_n \cap \ldots \cap Z_1 \to X)$.

Proof. Let us prove the statement on Chern classes by induction on n; the statement on $P_p(-)$ is proved in the exact same manner. The case $n = 1$ is the construction of $c'_{p_1}(Q_1)$ because $W_{1,\infty}$ is the inverse image of Z_1 in W_∞. For $n > 1$ we have by induction that $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ is equal to

$$c'_{p_n}(Q_n) \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_* \circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

By Lemma 42.49.2 the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is computed by the closed subset $Z_n \cap \ldots \cap Z_1$, the morphism $b' : W_{n-1} \cap \ldots \cap W_1 \to \mathbf{P}^1_{Z_{n-1} \cap \ldots \cap Z_1}$ and the restriction of Q_n to $W_{n-1} \cap \ldots \cap W_1$. Observe that $(b')^{-1}(Z_n) = W_n \cap \ldots \cap W_1$ and that $(W_n \cap \ldots \cap W_1)_\infty = W_{n,\infty} \cap \ldots \cap W_{1,\infty}$. Denote $C_{n-1} \in A^0(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)$ the class of Lemma 42.48.1. We conclude the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is

$$\begin{aligned} &(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{(W_n \cap \ldots \cap W_1)_\infty}) \circ C_{n-1} \\ &= (W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ C_{n-1} \end{aligned}$$

where the equality follows from Lemma 42.47.3 (we omit writing the restriction on the right). Hence the above becomes

$$\begin{aligned} (W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \\ C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_* \\ \circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C \end{aligned}$$

By Lemma 42.48.4 we know that the composition $C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_*$ is the identity on elements in the image of the Gysin map

$$(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^*$$

Thus it suffices to show that any element in the image of $c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$ is in the image of the Gysin map. We may write

$$c'_{p_i}(Q_i|_{W_{i,\infty}}) = \text{restriction of } c_{p_i}(W_i \to W, Q_i) \text{ to } W_{i,\infty}$$

by Lemma 42.50.9 and assumptions (2) and (3) on Q_i in the statement of the lemma. Thus, if $\beta \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, then

$$\begin{aligned} &c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap C \cap \alpha \\ &= c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1,\infty}}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap i_\infty^* \beta \\ &= c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_{n-1}}(W_1 \to W, Q_1) \cap i_\infty^* \beta \\ &= (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^* \left(c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_1}(W_1 \to W, Q_1) \cap \beta\right) \end{aligned}$$

as desired. Namely, for the last equality we use that $c_{p_i}(W_i \to W, Q_i)$ is a bivariant class and hence commutes with i_∞^* by definition. □

The following lemma gives us a tremendous amount of flexibility if we want to compute the localized Chern classes of a complex.

0FE8 **Lemma 42.51.2.** *Assume $(S, \delta), X, Z, b : W \to \mathbf{P}^1_X, Q, T, p$ satisfy the assumptions of Lemma 42.49.1. Let $F \in D(\mathcal{O}_X)$ be a perfect object such that*

(1) *the restriction of Q to $b^{-1}(\mathbf{A}^1_X)$ is isomorphic to the pullback of F,*
(2) *$F|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X\setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, and*
(3) *Q on W and F on X satisfy assumption (3) of Situation 42.50.1.*

Then the class $P'_p(Q)$, resp. $c'_p(Q)$ in $A^p(Z \to X)$ constructed in Lemma 42.49.1 is equal to $P_p(Z \to X, F)$, resp. $c_p(Z \to X, F)$ from Definition 42.50.3.

Proof. The assumptions are preserved by base change with a morphism $X' \to X$ locally of finite type. Hence it suffices to show that $P_p(Z \to X, F) \cap \alpha = P'_p(Q) \cap \alpha$, resp. $c_p(Z \to X, F) \cap \alpha = c'_p(Q) \cap \alpha$ for any $\alpha \in \mathrm{CH}_k(X)$. Choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is equal to the flat pullback of α as in the construction of C in Lemma 42.48.1. Denote $W' = b^{-1}(Z)$ and denote $E = W'_\infty \subset W_\infty$ the inverse image of Z by $W_\infty \to X$. The lemma follows from the following sequence of equalities (the case of P_p is similar)

$$\begin{aligned}
c'_p(Q) \cap \alpha &= (E \to Z)_*(c'_p(Q|_E) \cap i_\infty^* \beta) \\
&= (E \to Z)_*(c_p(E \to W_\infty, Q|_{W_\infty}) \cap i_\infty^* \beta) \\
&= (W'_\infty \to Z)_*(c_p(W' \to W, Q) \cap i_\infty^* \beta) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(W' \to W, Q) \cap \beta)) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*((i'_0)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*(c_p(Z' \to X, F) \cap i_0^* \beta)) \\
&= c_p(Z \to X, F) \cap \alpha
\end{aligned}$$

The first equality is the construction of $c'_p(Q)$ in Lemma 42.49.1. The second is Lemma 42.50.9. The base change of $W' \to W$ by $W_\infty \to W$ is the morphism $E = W'_\infty \to W_\infty$. Hence the third equality holds by Lemma 42.50.4. The fourth equality, in which $i'_\infty : W'_\infty \to W'$ is the inclusion morphism, follows from the fact that $c_p(W' \to W, Q)$ is a bivariant class. For the fifth equality, observe that $c_p(W' \to W, Q)$ and $c_p(Z' \to X, F)$ restrict to the same bivariant class in $A^p((b')^{-1} \to b^{-1}(\mathbf{A}^1_X))$ by assumption (1) of the lemma which says that Q and F restrict to the same object of $D(\mathcal{O}_{b^{-1}(\mathbf{A}^1_X)})$; use Lemma 42.50.4. Since $(i'_\infty)^*$ annihilates cycles supported on W'_∞ (see Remark 42.29.6) we conclude the fifth equality is true. The sixth equality holds because W'_∞ and W'_0 are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_Z$ and hence $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W'; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_Z}(1)) \cap c_p(Z \to X, F) \cap \beta$ by Lemma 42.29.4. The seventh equality holds because $c_p(Z \to X, F)$ is a bivariant class. By construction $W'_0 = Z$ and $i_0^*\beta = \alpha$ which explains why the final equality holds. □

42.52. Properties of localized Chern classes

0FB9 The main results in this section are additivity and multiplicativity for localized Chern classes.

0FBA **Lemma 42.52.1.** *In Situation 42.50.1 assume $E|_{X\setminus Z}$ is zero. Then*

$$P_1(Z \to X, E) = c_1(Z \to X, E),$$
$$P_2(Z \to X, E) = c_1(Z \to X, E)^2 - 2c_2(Z \to X, E),$$
$$P_3(Z \to X, E) = c_1(Z \to X, E)^3 - 3c_1(Z \to X, E)c_2(Z \to X, E) + 3c_3(Z \to X, E),$$

and so on where the products are taken in the algebra $A^{(1)}(Z \to X)$ of Remark 42.34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c_p(Z \to X, E)$ are defined for all $p \geq 1$. The result itself follows immediately from the more general Lemma 42.49.6 as the localized Chern classes where defined using the procedure of Lemma 42.49.1 in Section 42.50. □

0FBB **Lemma 42.52.2.** *In Situation 42.50.1 let $Y \to X$ be locally of finite type and $c \in A^*(Y \to X)$. Then*

$$P_p(Z \to X, E) \circ c = c \circ P_p(Z \to X, E),$$

respectively

$$c_p(Z \to X, E) \circ c = c \circ c_p(Z \to X, E)$$

in $A^*(Y \times_X Z \to X)$.

Proof. This follows from Lemma 42.49.5. More precisely, let

$$b : W \to \mathbf{P}^1_X \quad \text{and} \quad Q \quad \text{and} \quad T' \subset T \subset W_\infty$$

be as in the proof of Lemma 42.50.2. By definition $c_p(Z \to X, E) = c'_p(Q)$ as bivariant operations where the right hand side is the bivariant class constructed in Lemma 42.49.1 using W, b, Q, T'. By Lemma 42.49.5 we have $P'_p(Q) \circ c = c \circ P'_p(Q)$, resp. $c'_p(Q) \circ c = c \circ c'_p(Q)$ in $A^*(Y \times_X Z \to X)$ and we conclude. □

0FBC **Remark 42.52.3.** In Situation 42.50.1 it is convenient to define

$$c^{(p)}(Z \to X, E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c_p(Z \to X, E) + c_{p+1}(Z \to X, E) + \ldots$$

as an element of the algebra $A^{(p)}(Z \to X)$ considered in Remark 42.34.7.

0FBD **Lemma 42.52.4.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

1. *the restrictions $E_1|_{X\setminus Z}$ and $E_3|_{X\setminus Z}$ are isomorphic to finite locally free $\mathcal{O}_{X\setminus Z}$-modules of rank $< p_1$ and $< p_3$ placed in degree 0, and*
2. *at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.*

With notation as in Remark 42.52.3 we have

$$c^{(p_1+p_3)}(Z \to X, E_2) = c^{(p_1)}(Z \to X, E_1)c^{(p_3)}(Z \to X, E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$.

Proof. Observe that the assumptions imply that $E_2|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p_1 + p_3$. Thus the statement makes sense.

Let $f : Y \to X$ be an envelope. Expanding the left and right hand sides of the formula in the statement of the lemma we see that we have to prove some equalities of classes in $A^*(X)$ and in $A^*(Z \to X)$. By the uniqueness in Lemma 42.35.6 it suffices to prove the corresponding relations in $A^*(Y)$ and $A^*(Z \to Y)$. Since moreover the construction of the classes involved is compatible with base change (Lemma 42.50.4) we may replace X by Y and the distinguished triangle by its pullback.

In the proof of Lemma 42.46.7 we have seen that conditions (2)(a), (2)(b), and (2)(c) imply condition (2)(d). Combined with the discussion in the previous paragraph we reduce to the case discussed in the next paragraph.

Let $\varphi^\bullet : \mathcal{E}_3^\bullet[-1] \to \mathcal{E}_1^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules representing the map $E_3[-1] \to E_1$ in the derived category. Consider the scheme $X' = \mathbf{A}^1 \times X$ with projection $g : X' \to X$. Let $Z' = g^{-1}(Z) = \mathbf{A}^1 \times Z$. Denote t the coordinate on $\mathbf{A}^1$. Consider the cone $\mathcal{C}^\bullet$ of the map of complexes

$$tg^*\varphi^\bullet : g^*\mathcal{E}_3^\bullet[-1] \longrightarrow g^*\mathcal{E}_1^\bullet$$

over X'. We obtain a distinguished triangle

$$g^*\mathcal{E}_1^\bullet \to \mathcal{C}^\bullet \to g^*\mathcal{E}_3^\bullet \to g^*\mathcal{E}_1^\bullet[1]$$

where the first three terms form a termwise split short exact sequence of complexes. Clearly $\mathcal{C}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules whose restriction to $X' \setminus Z'$ is isomorphic to a finite locally free $\mathcal{O}_{X' \setminus Z'}$-module of rank $< p_1 + p_3$ placed in degree 0. Thus we have the localized Chern classes

$$c_p(Z' \to X', \mathcal{C}^\bullet) \in A^p(Z' \to X')$$

for $p \geq p_1 + p_3$. For any $\alpha \in \mathrm{CH}_k(X)$ consider

$$c_p(Z' \to X', \mathcal{C}^\bullet) \cap g^*\alpha \in \mathrm{CH}_{k+1-p}(\mathbf{A}^1 \times X)$$

If we restrict to $t = 0$, then the map $tg^*\varphi^\bullet$ restricts to zero and $\mathcal{C}^\bullet|_{t=0}$ is the direct sum of $\mathcal{E}_1^\bullet$ and $\mathcal{E}_3^\bullet$. By compatibility of localized Chern classes with base change (Lemma 42.50.4) we conclude that

$$i_0^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$. On the other hand, if we restrict to $t = 1$, then the map $tg^*\varphi^\bullet$ restricts to φ and $\mathcal{C}^\bullet|_{t=1}$ is a bounded complex of finite locally free modules representing E_2. We conclude that

$$i_1^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_2)$$

in $A^{(p_1+p_3)}(Z \to X)$. Since $i_0^* = i_1^*$ by definition of rational equivalence (more precisely this follows from the formulae in Lemma 42.32.4) we conclude that

$$c^{(p_1+p_2)}(Z \to X, E_2) = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

This reduces us to the case discussed in the next paragraph.

Assume $E_2 = E_1 \oplus E_3$ and the triples (X, Z, E_i) are as in Situation 42.50.1. For $i = 1, 3$ let

$$b_i : W_i \to \mathbf{P}^1_X \quad \text{and} \quad Q_i \quad \text{and} \quad T_i' \subset T_i \subset W_{i,\infty}$$

be as in the proof of Lemma 42.50.2. By definition

$$c_p(Z \to X, E_i) = c'_p(Q_i)$$

where the right hand side is the bivariant class constructed in Lemma 42.49.1 using W_i, b_i, Q_i, T'_i. Set $W = W_1 \times_{b_1, \mathbf{P}^1_X, b_2} W_2$ and consider the cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g_3} & W_3 \\ {\scriptstyle g_1}\downarrow & \searrow^{b} & \downarrow{\scriptstyle b_3} \\ W_1 & \xrightarrow{b_1} & \mathbf{P}^1_X \end{array}$$

Of course $b^{-1}(\mathbf{A}^1)$ maps isomorphically to $\mathbf{A}^1_X$. Observe that $T' = g_1^{-1}(T'_1) \cap g_2^{-1}(T'_2)$ still contains all the points of W_∞ lying over $X \setminus Z$. By Lemma 42.49.3 we may use W, b, $g_i^*\mathcal{Q}_i$, and T' to construct $c_p(Z \to X, E_i)$ for $i = 1, 3$. Also, by the stronger independence given in Lemma 42.51.2 we may use W, b, $g_1^*Q_1 \oplus g_3^*Q_3$, and T' to compute the classes $c_p(Z \to X, E_2)$. Thus the desired equality follows from Lemma 42.49.7. □

0FBE **Lemma 42.52.5.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

1. *the restrictions $E_1|_{X\setminus Z}$ and $E_3|_{X\setminus Z}$ are zero, and*
2. *at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.*

Then we have

$$P_p(Z \to X, E_2) = P_p(Z \to X, E_1) + P_p(Z \to X, E_3)$$

for all $p \in \mathbf{Z}$ and consequently $ch(Z \to X, E_2) = ch(Z \to X, E_1) + ch(Z \to X, E_3)$.

Proof. The proof is exactly the same as the proof of Lemma 42.52.4 except it uses Lemma 42.49.8 at the very end. For $p > 0$ we can deduce this lemma from Lemma 42.52.4 with $p_1 = p_3 = 1$ and the relationship between $P_p(Z \to X, E)$ and $c_p(Z \to X, E)$ given in Lemma 42.52.1. The case $p = 0$ can be shown directly (it is only interesting if X has a connected component entirely contained in Z). □

0FBF **Lemma 42.52.6.** *In Situation 42.7.1 let X be locally of finite type over S. Let $Z_i \subset X$, $i = 1, 2$ be closed subschemes. Let F_i, $i = 1, 2$ be perfect objects of $D(\mathcal{O}_X)$. Assume for $i = 1, 2$ that $F_i|_{X\setminus Z_i}$ is zero[7] and that F_i on X satisfies assumption (3) of Situation 42.50.1. Denote $r_i = P_0(Z_i \to X, F_i) \in A^0(Z_i \to X)$. Then we have*

$$c_1(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = r_1 c_1(Z_2 \to X, F_2) + r_2 c_1(Z_1 \to X, F_1)$$

[7] Presumably there is a variant of this lemma where we only assume $F_i|_{X\setminus Z_i}$ is isomorphic to a finite locally free $\mathcal{O}_{X\setminus Z_i}$-module of rank $< p_i$.

in $A^1(Z_1 \cap Z_2 \to X)$ *and*

$$\begin{aligned} c_2(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = r_1c_2(Z_2 \to X, F_2) + r_2c_2(Z_1 \to X, F_1) + \\ \binom{r_1}{2} c_1(Z_2 \to X, F_2)^2 + \\ (r_1r_2 - 1)c_1(Z_2 \to X, F_2)c_1(Z_1 \to X, F_1) + \\ \binom{r_2}{2} c_1(Z_1 \to X, F_1)^2 \end{aligned}$$

in $A^2(Z_1 \cap Z_2 \to X)$ *and so on for higher Chern classes. Similarly, we have*

$$ch(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = ch(Z_1 \to X, F_1)ch(Z_2 \to X, F_2)$$

in $\prod_{p \geq 0} A^p(Z_1 \cap Z_2 \to X) \otimes \mathbf{Q}$. *More precisely, we have*

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(Z_1 \to X, F_1)P_{p_2}(Z_2 \to X, F_2)$$

in $A^p(Z_1 \cap Z_2 \to X)$.

Proof. Choose proper morphisms $b_i : W_i \to \mathbf{P}^1_X$ and $Q_i \in D(\mathcal{O}_{W_i})$ as well as closed subschemes $T_i \subset W_{i,\infty}$ as in the construction of the localized Chern classes for F_i or more generally as in Lemma 42.51.2. Choose a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g_2} & W_2 \\ \downarrow{\scriptstyle g_1} & \searrow^{b} & \downarrow{\scriptstyle b_2} \\ W_1 & \xrightarrow{b_1} & \mathbf{P}^1_X \end{array}$$

where all morphisms are proper and isomorphisms over $\mathbf{A}^1_X$. For example, we can take W to be the closure of the graph of the isomorphism between $b_1^{-1}(\mathbf{A}^1_X)$ and $b_2^{-1}(\mathbf{A}^1_X)$. By Lemma 42.51.2 we may work with W, $b = b_i \circ g_i$, $Lg_i^*Q_i$, and $g_i^{-1}(T_i)$ to construct the localized Chern classes $c_p(Z_i \to X, F_i)$. Thus we reduce to the situation described in the next paragraph.

Assume we have

(1) a proper morphism $b : W \to \mathbf{P}^1_X$ which is an isomorphism over $\mathbf{A}^1_X$,
(2) $E_i \subset W_\infty$ is the inverse image of Z_i,
(3) perfect objects $Q_i \in D(\mathcal{O}_W)$ whose Chern classes are defined, such that
   (a) the restriction of Q_i to $b^{-1}(\mathbf{A}^1_X)$ is the pullback of F_i, and
   (b) there exists a closed subscheme $T_i \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z_i$ such that $Q_i|_{T_i}$ is zero.

By Lemma 42.51.2 we have

$$c_p(Z_i \to X, F_i) = c'_p(Q_i) = (E_i \to Z_i)_* \circ c'_p(Q_i|_{E_i}) \circ C$$

and

$$P_p(Z_i \to X, F_i) = P'_p(Q_i) = (E_i \to Z_i)_* \circ P'_p(Q_i|_{E_i}) \circ C$$

for $i = 1, 2$. Next, we observe that $Q = Q_1 \otimes^{\mathbf{L}}_{\mathcal{O}_W} Q_2$ satisfies (3)(a) and (3)(b) for $F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2$ and $T_1 \cup T_2$. Hence we see that

$$c_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ c'_p(Q|_{E_1 \cap E_2}) \circ C$$

and

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ P'_p(Q|_{E_1 \cap E_2}) \circ C$$

by the same lemma. By Lemma 42.47.11 the classes $c'_p(Q|_{E_1 \cap E_2})$ and $P'_p(Q|_{E_1 \cap E_2})$ can be expanded in the correct manner in terms of the classes $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$. Then finally Lemma 42.51.1 tells us that polynomials in $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$ agree with the corresponding polynomials in $c'_p(Q_i)$ and $P'_p(Q_i)$ as desired. □

42.53. Blowing up at infinity

0FBG Let X be a scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Denote $X' \to X$ the blowing up with center Z. Let $b : W \to \mathbf{P}^1_X$ be the blowing up with center $\infty(Z)$. Denote $E \subset W$ the exceptional divisor. There is a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & W \\ \downarrow & & \downarrow b \\ X & \xrightarrow{\infty} & \mathbf{P}^1_X \end{array}$$

whose horizontal arrows are closed immersion (Divisors, Lemma 31.33.2). Denote $E \subset W$ the exceptional divisor and $W_\infty \subset W$ the inverse image of $(\mathbf{P}^1_X)_\infty$. Then the following are true

(1) b is an isomorphism over $\mathbf{A}^1_X \cup \mathbf{P}^1_{X \setminus Z}$,
(2) X' is an effective Cartier divisor on W,
(3) $X' \cap E$ is the exceptional divisor of $X' \to X$,
(4) $W_\infty = X' + E$ as effective Cartier divisors on W,
(5) $E = \underline{\mathrm{Proj}}_Z(\mathcal{C}_{Z/X,*}[S])$ where S is a variable placed in degree 1,
(6) $X' \cap E = \underline{\mathrm{Proj}}_Z(\mathcal{C}_{Z/X,*})$,
0FBH (7) $E \setminus X' = E \setminus (X' \cap E) = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*}) = C_Z X$,
0FE9 (8) there is a closed immersion $\mathbf{P}^1_Z \to W$ whose composition with b is the inclusion morphism $\mathbf{P}^1_Z \to \mathbf{P}^1_X$ and whose base change by ∞ is the composition $Z \to C_Z X \to E \to W_\infty$ where the first arrow is the vertex of the cone.

We recall that $\mathcal{C}_{Z/X,*}$ is the conormal algebra of Z in X, see Divisors, Definition 31.19.1 and that $C_Z X$ is the normal cone of Z in X, see Divisors, Definition 31.19.5.

We now give the proof of the numbered assertions above. We strongly urge the reader to work through some examples instead of reading the proofs.

Part (1) follows from the corresponding assertion of Divisors, Lemma 31.32.4. Observe that $E \subset W$ is an effective Cartier divisor by the same lemma.

Observe that W_∞ is an effective Cartier divisor by Divisors, Lemma 31.32.11. Since $E \subset W_\infty$ we can write $W_\infty = D + E$ for some effective Cartier divisor D, see Divisors, Lemma 31.13.8. We will see below that $D = X'$ which will prove (2) and (4).

Since X' is the strict transform of the closed immersion $\infty : X \to \mathbf{P}^1_X$ (see above) it follows that the exceptional divisor of $X' \to X$ is equal to the intersection $X' \cap E$

(for example because both are cut out by the pullback of the ideal sheaf of Z to X'). This proves (3).

The intersection of $\infty(Z)$ with $\mathbf{P}^1_Z$ is the effective Cartier divisor $(\mathbf{P}^1_Z)_\infty$ hence the strict transform of $\mathbf{P}^1_Z$ by the blowing up b maps isomorphically to $\mathbf{P}^1_Z$ (see Divisors, Lemmas 31.33.2 and 31.32.7). This gives us the morphism $\mathbf{P}^1_Z \to W$ mentioned in (8). It is a closed immersion as b is separated, see Schemes, Lemma 26.21.11.

Suppose that $\mathrm{Spec}(A) \subset X$ is an affine open and that $Z \cap \mathrm{Spec}(A)$ corresponds to the finitely generated ideal $I \subset A$. An affine neighbourhood of $\infty(Z \cap \mathrm{Spec}(A))$ is the affine space over A with coordinate $s = T_0/T_1$. Denote $J = (I, s) \subset A[s]$ the ideal generated by I and s. Let $B = A[s] \oplus J \oplus J^2 \oplus \ldots$ be the Rees algebra of $(A[s], J)$. Observe that

$$J^n = I^n \oplus sI^{n-1} \oplus s^2 I^{n-2} \ldots \oplus s^n A \oplus s^{n+1} A \oplus \ldots$$

as an A-submodule of $A[s]$ for all $n \geq 0$. Consider the open subscheme

$$\mathrm{Proj}(B) = \mathrm{Proj}(A[s] \oplus J \oplus J^2 \oplus \ldots) \subset W$$

Finally, denote S the element $s \in J$ viewed as a degree 1 element of B.

Since formation of Proj commutes with base change (Constructions, Lemma 27.11.6) we see that

$$E = \mathrm{Proj}(B \otimes_{A[s]} A/I) = \mathrm{Proj}((A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S])$$

The verification that $B \otimes_{A[s]} A/I = \bigoplus J^n/J^{n+1}$ is as given follows immediately from our description of the powers J^n above. This proves (5) because the conormal algebra of $Z \cap \mathrm{Spec}(A)$ in $\mathrm{Spec}(A)$ corresponds to the graded A-algebra $A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$ by Divisors, Lemma 31.19.2.

Recall that $\mathrm{Proj}(B)$ is covered by the affine opens $D_+(S)$ and $D_+(f^{(1)})$ for $f \in I$ which are the spectra of affine blowup algebras $A[s][\frac{J}{s}]$ and $A[s][\frac{J}{f}]$, see Divisors, Lemma 31.32.2 and Algebra, Definition 10.70.1. We will describe each of these affine opens and this will finish the proof.

The open $D_+(S)$, i.e., the spectrum of $A[s][\frac{J}{s}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{s}] = \textstyle\sum s^{-n} I^n[s] \subset A[s, s^{-1}]$$

The element s is a nonzerodivisor in this ring, defines the exceptional divisor E as well as W_∞. Hence $D \cap D_+(S) = \emptyset$. Finally, the quotient of $A[s][\frac{J}{s}]$ by s is the conormal algebra

$$A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

This proves (7).

The open $D_+(f^{(1)})$, i.e., the spectrum of $A[s][\frac{J}{f}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{f}] = A[\tfrac{I}{f}][\tfrac{s}{f}]$$

where $\frac{s}{f}$ is a variable. The element f is a nonzerodivisor in this ring whose zero scheme defines the exceptional divisor E. Since s defines W_∞ and $s = f \cdot \frac{s}{f}$ we conclude that $\frac{s}{f}$ defines the divisor D constructed above. Then we see that

$$D \cap D_+(f^{(1)}) = \mathrm{Spec}(A[\tfrac{I}{f}])$$

which is the corresponding open of the blowup X' over $\mathrm{Spec}(A)$. Namely, the surjective graded $A[s]$-algebra map $B \to A \oplus I \oplus I^2 \oplus \ldots$ to the Rees algebra of (A, I) corresponds to the closed immersion $X' \to W$ over $\mathrm{Spec}(A[s])$. This proves $D = X'$ as desired.

Let us prove (6). Observe that the zero scheme of $\frac{s}{f}$ in the previous paragraph is the restriction of the zero scheme of S on the affine open $D_+(f^{(1)})$. Hence we see that $S = 0$ defines $X' \cap E$ on E. Thus (6) follows from (5).

Finally, we have to prove the last part of (8). This is clear because the map $\mathbf{P}^1_Z \to W$ is affine locally given by the surjection

$$B \to B \otimes_{A[s]} A/I = (A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S] \to A/I[S]$$

and the identification $\mathrm{Proj}(A/I[S]) = \mathrm{Spec}(A/I)$. Some details omitted.

42.54. Higher codimension gysin homomorphisms

0FBI Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. In this section we are going to consider triples

$$(Z \to X, \mathcal{N}, \sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X})$$

consisting of a closed immersion $Z \to X$ and a locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ from the dual of $\mathcal{N}$ to the conormal sheaf of Z in X, see Morphisms, Section 29.31. We will say $\mathcal{N}$ is a *virtual normal sheaf for Z in X*.

0FBJ **Lemma 42.54.1.** *Let (S, δ) be as in Situation 42.7.1. Let*

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & X \end{array}$$

be a cartesian diagram of schemes locally of finite type over S whose horizontal arrows are closed immersions. If $\mathcal{N}$ is a virtual normal sheaf for Z in X, then $\mathcal{N}' = g^*\mathcal{N}$ is a virtual normal sheaf for Z' in X'.

Proof. This follows from the surjectivity of the map $g^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$ proved in Morphisms, Lemma 29.31.4. □

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal bundle for a closed immersion $Z \to X$. In this situation we set

$$p : N = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{N}^\vee)) \longrightarrow Z$$

equal to the vector bundle over Z whose sections correspond to sections of $\mathcal{N}$. In this situation we have canonical closed immersions

$$C_Z X \longrightarrow N_Z X \longrightarrow N$$

The first closed immersion is Divisors, Equation (31.19.5.1) and the second closed immersion corresponds to the surjection $\mathrm{Sym}(\mathcal{N}^\vee) \to \mathrm{Sym}(\mathcal{C}_{Z/X})$ induced by σ. Let

$$b : W \longrightarrow \mathbf{P}^1_X$$

be the blowing up in $\infty(Z)$ constructed in Section 42.53. By Lemma 42.48.1 we have a canonical bivariant class in

$$C \in A^0(W_\infty \to X)$$

Consider the open immersion $j : C_Z X \to W_\infty$ of (7) and the closed immersion $i : C_Z X \to N$ constructed above. By Lemma 42.36.3 for every $\alpha \in \mathrm{CH}_k(X)$ there exists a unique $\beta \in \mathrm{CH}_*(Z)$ such that

$$i_* j^*(C \cap \alpha) = p^* \beta$$

We set $c(Z \to X, \mathcal{N}) \cap \alpha = \beta$.

0FBK **Lemma 42.54.2.** *The construction above defines a bivariant class*[8]

$$c(Z \to X, \mathcal{N}) \in A^*(Z \to X)^\wedge$$

and moreover the construction is compatible with base change as in Lemma 42.54.1. If $\mathcal{N}$ has constant rank r, then $c(Z \to X, \mathcal{N}) \in A^r(Z \to X)$.

Proof. Since both $i_* \circ j^* \circ C$ and p^* are bivariant classes (see Lemmas 42.33.2 and 42.33.4) we can use the equation

$$i_* \circ j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

(suitably interpreted) to define $c(Z \to X, \mathcal{N})$ as a bivariant class. This works because p^* is always bijective on chow groups by Lemma 42.36.3.

Let $X' \to X$, $Z' \to X'$, and $\mathcal{N}'$ be as in Lemma 42.54.1. Write $c = c(Z \to X, \mathcal{N})$ and $c' = c(Z' \to X', \mathcal{N}')$. The second statement of the lemma means that c' is the restriction of c as in Remark 42.33.5. Since we claim this is true for all X'/X locally of finite type, a formal argument shows that it suffices to check that $c' \cap \alpha' = c \cap \alpha'$ for $\alpha' \in \mathrm{CH}_k(X')$. To see this, note that we have a commutative diagram

$$\begin{array}{ccccccc} C_{Z'}X' & \longrightarrow & W'_\infty & \longrightarrow & W' & \longrightarrow & \mathbf{P}^1_{X'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_Z X & \longrightarrow & W_\infty & \longrightarrow & W & \longrightarrow & \mathbf{P}^1_X \end{array}$$

which induces closed immersions:

$$W' \to W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'}, \quad W'_\infty \to W_\infty \times_X X', \quad C_{Z'}X' \to C_Z X \times_Z Z'$$

To get $c \cap \alpha'$ we use the class $C \cap \alpha'$ defined using the morphism $W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'} \to \mathbf{P}^1_{X'}$ in Lemma 42.48.1. To get $c' \cap \alpha'$ on the other hand, we use the class $C' \cap \alpha'$ defined using the morphism $W' \to \mathbf{P}^1_{X'}$. By Lemma 42.48.3 the pushforward of $C' \cap \alpha'$ by the closed immersion $W'_\infty \to (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$, is equal to $C \cap \alpha'$. Hence the same is true for the pullbacks to the opens

$$C_{Z'}X' \subset W'_\infty, \quad C_Z X \times_Z Z' \subset (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$$

[8]The notation $A^*(Z \to X)^\wedge$ is discussed in Remark 42.35.5. If X is quasi-compact, then $A^*(Z \to X)^\wedge = A^*(Z \to X)$.

by Lemma 42.15.1. Since we have a commutative diagram

$$\begin{array}{ccc} C_{Z'}X' & \longrightarrow & N' \\ \downarrow & & \| \\ C_ZX \times_Z Z' & \longrightarrow & N \times_Z Z' \end{array}$$

these classes pushforward to the same class on N' which proves that we obtain the same element $c \cap \alpha' = c' \cap \alpha'$ in $\mathrm{CH}_*(Z')$. □

0FBL **Lemma 42.54.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Suppose that we have a short exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$ of finite locally free $\mathcal{O}_Z$-modules such that the given surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through a map $\sigma' : (\mathcal{N}')^\vee \to \mathcal{C}_{Z/X}$. Then*

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ c(Z \to X, \mathcal{N}')$$

as bivariant classes.

Proof. Denote $N' \to N$ the closed immersion of vector bundles corresponding to the surjection $\mathcal{N}^\vee \to (\mathcal{N}')^\vee$. Then we have closed immersions

$$C_ZX \to N' \to N$$

Thus the desired relationship between the bivariant classes follows immediately from Lemma 42.44.2. □

0FV7 **Lemma 42.54.4.** *Let (S, δ) be as in Situation 42.7.1. Consider a cartesian diagram*

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & X \end{array}$$

of schemes locally of finite type over S whose horizontal arrows are closed immersions. Let $\mathcal{N}$, resp. $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset X$, resp. $Z' \to X'$. Assume given a short exact sequence $0 \to \mathcal{N}' \to g^*\mathcal{N} \to \mathcal{E} \to 0$ of finite locally free modules on Z' such that the diagram

$$\begin{array}{ccc} g^*\mathcal{N}^\vee & \longrightarrow & (\mathcal{N}')^\vee \\ \downarrow & & \downarrow \\ g^*\mathcal{C}_{Z/X} & \longrightarrow & \mathcal{C}_{Z'/X'} \end{array}$$

commutes. Then we have

$$res(c(Z \to X, \mathcal{N})) = c_{top}(\mathcal{E}) \circ c(Z' \to X', \mathcal{N}')$$

in $A^*(Z' \to X')^\wedge$.

Proof. By Lemma 42.54.2 we have $res(c(Z \to X, \mathcal{N})) = c(Z' \to X', g^*\mathcal{N})$ and the equality follows from Lemma 42.54.3. □

Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Assume $Z \times_X Y \to Y$ is a regular closed immersion, see Divisors, Section 31.21. In this case the conormal sheaf $\mathcal{C}_{Z \times_X Y/Y}$ is a finite locally free $\mathcal{O}_{Z \times_X Y}$-module and we obtain a short exact sequence

$$0 \to \mathcal{E}^\vee \to \mathcal{N}^\vee|_{Z \times_X Y} \to \mathcal{C}_{Z \times_X Y/Y} \to 0$$

The quotient $\mathcal{N}|_{Y \times_X Z} \to \mathcal{E}$ is called the *excess normal sheaf* of the situation.

0FBM **Lemma 42.54.5.** *In the situation described just above assume* $\dim_\delta(Y) = n$ *and that* $\mathcal{C}_{Y \times_X Z/Z}$ *has constant rank* r*. Then*

$$c(Z \to X, \mathcal{N}) \cap [Y]_n = c_{top}(\mathcal{E}) \cap [Z \times_X Y]_{n-r}$$

in $\mathrm{CH}_*(Z \times_X Y)$*.*

Proof. The bivariant class $c_{top}(\mathcal{E}) \in A^*(Z \times_X Y)$ was defined in Remark 42.38.11. By Lemma 42.54.2 we may replace X by Y. Thus we may assume $Z \to X$ is a regular closed immersion of codimension r, we have $\dim_\delta(X) = n$, and we have to show that $c(Z \to X, \mathcal{N}) \cap [X]_n = c_{top}(\mathcal{E}) \cap [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$. By Lemma 42.54.3 we may even assume $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is an isomorphism. In other words, we have to show $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n = [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$.

Let us trace through the steps in the definition of $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n$. Let $b : W \to \mathbf{P}^1_X$ be the blowing up of $\infty(Z)$. We first have to compute $C \cap [X]_n$ where $C \in A^0(W_\infty \to X)$ is the class of Lemma 42.48.1. To do this, note that $[W]_{n+1}$ is a cycle on W whose restriction to $\mathbf{A}^1_X$ is equal to the flat pullback of $[X]_n$. Hence $C \cap [X]_n$ is equal to $i_\infty^*[W]_{n+1}$. Since W_∞ is an effective Cartier divisor on W we have $i_\infty^*[W]_{n+1} = [W_\infty]_n$, see Lemma 42.29.5. The restriction of this class to the open $C_Z X \subset W_\infty$ is of course just $[C_Z X]_n$. Because $Z \subset X$ is regularly embedded we have

$$\mathcal{C}_{Z/X,*} = \mathrm{Sym}(\mathcal{C}_{Z/X})$$

as graded $\mathcal{O}_Z$-algebras, see Divisors, Lemma 31.21.5. Hence $p : N = C_Z X \to Z$ is the structure morphism of the vector bundle associated to the finite locally free module $\mathcal{C}_{Z/X}$ of rank r. Then it is clear that $p^*[Z]_{n-r} = [C_Z X]_n$ and the proof is complete. □

0FEA **Lemma 42.54.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Given integers r, n assume*

(1) *$\mathcal{N}$ is locally free of rank r,*
(2) *every irreducible component of Y has δ-dimension n,*
(3) *$\dim_\delta(Z \times_X Y) \leq n - r$, and*
(4) *for $\xi \in Z \times_X Y$ with $\delta(\xi) = n - r$ the local ring $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay.*

Then $c(Z \to X, \mathcal{N}) \cap [Y]_n = [Z \times_X Y]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X Y)$.

Proof. The statement makes sense as $Z \times_X Y$ is a closed subscheme of Y. Because $\mathcal{N}$ has rank r we know that $c(Z \to X, \mathcal{N}) \cap [Y]_n$ is in $\mathrm{CH}_{n-r}(Z \times_X Y)$. Since $\dim_\delta(Z \cap Y) \leq n - r$ the chow group $\mathrm{CH}_{n-r}(Z \times_X Y)$ is freely generated by the cycle classes of the irreducible components $W \subset Z \times_X Y$ of δ-dimension $n - r$. Let $\xi \in W$ be the generic point. By assumption (2) we see that $\dim(\mathcal{O}_{Y,\xi}) = r$. On the

other hand, since $\mathcal{N}$ has rank r and since $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is surjective, we see that the ideal sheaf of Z is locally cut out by r equations. Hence the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ of $Z \times_X Y$ in Y is locally generated by r elements. Since $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay of dimension r and since $\mathcal{I}_\xi$ is an ideal of definition (as ξ is a generic point of $Z \times_X Y$) it follows that $\mathcal{I}_\xi$ is generated by a regular sequence (Algebra, Lemma 10.104.2). By Divisors, Lemma 31.20.8 we see that $\mathcal{I}$ is generated by a regular sequence over an open neighbourhood $V \subset Y$ of ξ. By our description of $\mathrm{CH}_{n-r}(Z \times_X Y)$ it suffices to show that $c(Z \to X, \mathcal{N}) \cap [V]_n = [Z \times_X V]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X V)$. This follows from Lemma 42.54.5 because the excess normal sheaf is 0 over V. □

0FBN **Lemma 42.54.7.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 42.29.1. The Gysin homomorphism i^* viewed as an element of $A^1(D \to X)$ (see Lemma 42.33.3) is the same as the bivariant class $c(D \to X, \mathcal{N}) \in A^1(D \to X)$ constructed using $\mathcal{N} = i^*\mathcal{L}$ viewed as a virtual normal sheaf for D in X.*

Proof. We will use the criterion of Lemma 42.35.3. Thus we may assume that X is an integral scheme and we have to show that $i^*[X]$ is equal to $c \cap [X]$. Let $n = \dim_\delta(X)$. As usual, there are two cases.

If $X = D$, then we see that both classes are represented by $c_1(\mathcal{N}) \cap [X]_n$. See Lemma 42.54.5 and Definition 42.29.1.

If $D \neq X$, then $D \to X$ is an effective Cartier divisor and in particular a regular closed immersion of codimension 1. Again by Lemma 42.54.5 we conclude $c(D \to X, \mathcal{N}) \cap [X]_n = [D]_{n-1}$. The same is true by definition for the Gysin homomorphism and we conclude once again. □

0FBP **Lemma 42.54.8.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $Z \subset X$ be a closed subscheme with virtual normal sheaf $\mathcal{N}$. Let $Y \to X$ be locally of finite type and $c \in A^*(Y \to X)$. Then c and $c(Z \to X, \mathcal{N})$ commute (Remark 42.33.6).*

Proof. To check this we may use Lemma 42.35.3. Thus we may assume X is an integral scheme and we have to show $c \cap c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to X, \mathcal{N}) \cap c \cap [X]$ in $\mathrm{CH}_*(Z \times_X Y)$.

If $Z = X$, then $c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{N})$ by Lemma 42.54.5 which commutes with the bivariant class c, see Lemma 42.38.9.

Assume that Z is not equal to X. By Lemma 42.35.3 it even suffices to prove the result after blowing up X (in a nonzero ideal). Let us blowup X in the ideal sheaf of Z. This reduces us to the case where Z is an effective Cartier divisor, see Divisors, Lemma 31.32.4,

If Z is an effective Cartier divisor, then we have

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ i^*$$

where $i^* \in A^1(Z \to X)$ is the Gysin homomorphism associated to $i : Z \to X$ (Lemma 42.33.3) and $\mathcal{E}$ is the dual of the kernel of $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$, see Lemmas 42.54.3 and 42.54.7. Then we conclude because Chern classes are in the center of the bivariant ring (in the strong sense formulated in Lemma 42.38.9) and c commutes with the Gysin homomorphism i^* by definition of bivariant classes. □

Let (S, δ) be as in Situation 42.7.1. Let X be an integral scheme locally of finite type over S of δ-dimension n. Let $Z \subset Y \subset X$ be closed subschemes which are both effective Cartier divisors in X. Denote $o : Y \to C_Y X$ the zero section of the normal line cone of Y in X. As $C_Y X$ is a line bundle over Y we obtain a bivariant class $o^* \in A^1(Y \to C_Y X)$, see Lemma 42.33.3.

0FEB **Lemma 42.54.9.** *With notation as above we have*

$$o^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X)$.

Proof. Denote $W \to \mathbf{P}^1_X$ the blowing up of $\infty(Z)$ as in Section 42.53. Similarly, denote $W' \to \mathbf{P}^1_X$ the blowing up of $\infty(Y)$. Since $\infty(Z) \subset \infty(Y)$ we get an opposite inclusion of ideal sheaves and hence a map of the graded algebras defining these blowups. This produces a rational morphism from W to W' which in fact has a canonical representative

$$W \supset U \longrightarrow W'$$

See Constructions, Lemma 27.18.1. A local calculation (omitted) shows that U contains at least all points of W not lying over ∞ and the open subscheme $C_Z X$ of the special fibre. After shrinking U we may assume $U_\infty = C_Z X$ and $\mathbf{A}^1_X \subset U$. Another local calculation (omitted) shows that the morphism $U_\infty \to W'_\infty$ induces the canonical morphism $C_Z X \to C_Y X \subset W'_\infty$ of normal cones induced by the inclusion of ideals sheaves coming from $Z \subset Y$. Denote $W'' \subset W$ the strict transform of $\mathbf{P}^1_Y \subset \mathbf{P}^1_X$ in W. Then W'' is the blowing up of $\mathbf{P}^1_Y$ in $\infty(Z)$ by Divisors, Lemma 31.33.2 and hence $(W'' \cap U)_\infty = C_Z Y$.

Consider the effective Cartier divisor $i : \mathbf{P}^1_Y \to W'$ from (8) and its associated bivariant class $i^* \in A^1(\mathbf{P}^1_Y \to W')$ from Lemma 42.33.3. We similarly denote $(i'_\infty)^* \in A^1(W'_\infty \to W')$ the Gysin map at infinity. Observe that the restriction of i'_∞ (Remark 42.33.5) to U is the restriction of $i^*_\infty \in A^1(W_\infty \to W)$ to U. On the one hand we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^*_\infty i^*[U]_{n+1} = i^*_\infty[(W'' \cap U)_\infty]_{n+1} = [C_Z Y]_n$$

because i^*_∞ kills all classes supported over ∞, because $i^*[U]$ and $[W'']$ agree as cycles over $\mathbf{A}^1$, and because $C_Z Y$ is the fibre of $W'' \cap U$ over ∞. On the other hand, we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^* i^*_\infty[U]_{n+1} = i^*[U_\infty] = o^*[C_Y X]_n$$

because $(i'_\infty)^*$ and i^* commute (Lemma 42.30.5) and because the fibre of $i : \mathbf{P}^1_Y \to W'$ over ∞ factors as $o : Y \to C_Y X$ and the open immersion $C_Y X \to W'_\infty$. The lemma follows. □

0FEC **Lemma 42.54.10.** *Let (S, δ) be as in Situation 42.7.1. Let $Z \subset Y \subset X$ be closed subschemes of a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for $Z \subset X$. Let $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset Y$. Let $\mathcal{N}''$ be a virtual normal sheaf for $Y \subset X$. Assume there is a commutative diagram*

$$\begin{array}{ccccc} (\mathcal{N}'')^\vee|_Z & \longrightarrow & \mathcal{N}^\vee & \longrightarrow & (\mathcal{N}')^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_{Y/X}|_Z & \longrightarrow & \mathcal{C}_{Z/X} & \longrightarrow & \mathcal{C}_{Z/Y} \end{array}$$

where the sequence at the bottom is from More on Morphisms, Lemma 37.7.12 and the top sequence is a short exact sequence. Then

$$c(Z \to X, \mathcal{N}) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{N}'')$$

in $A^*(Z \to X)^\wedge$.

Proof. Observe that the assumptions remain satisfied after any base change by a morphism $X' \to X$ which is locally of finite type (the short exact sequence of virtual normal sheaves is locally split hence remains exact after any base change). Thus to check the equality of bivariant classes we may use Lemma 42.35.3. Thus we may assume X is an integral scheme and we have to show $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X]$.

If $Y = X$, then we have

$$\begin{aligned} c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X] &= c(Z \to Y, \mathcal{N}') \cap c_{top}(\mathcal{N}'') \cap [Y] \\ &= c_{top}(\mathcal{N}''|_Z) \cap c(Z \to Y, \mathcal{N}') \cap [Y] \\ &= c(Z \to X, \mathcal{N}) \cap [X] \end{aligned}$$

The first equality by Lemma 42.54.3. The second because Chern classes commute with bivariant classes (Lemma 42.38.9). The third equality by Lemma 42.54.3.

Assume $Y \neq X$. By Lemma 42.35.3 it even suffices to prove the result after blowing up X in a nonzero ideal. Let us blowup X in the product of the ideal sheaf of Y and the ideal sheaf of Z. This reduces us to the case where both Y and Z are effective Cartier divisors on X, see Divisors, Lemmas 31.32.4 and 31.32.12.

Denote $\mathcal{N}'' \to \mathcal{E}$ the surjection of finite locally free $\mathcal{O}_Z$-modules such that $0 \to \mathcal{E}^\vee \to (\mathcal{N}'')^\vee \to \mathcal{C}_{Y/X} \to 0$ is a short exact sequence. Then $\mathcal{N} \to \mathcal{E}|_Z$ is a surjection as well. Denote $\mathcal{N}_1$ the finite locally free kernel of this map and observe that $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through $\mathcal{N}_1$. By Lemma 42.54.3 we have

$$c(Y \to X, \mathcal{N}'') = c_{top}(\mathcal{E}) \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

and

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}|_Z) \circ c(Z \to X, \mathcal{N}_1)$$

Since Chern classes of bundles commute with bivariant classes (Lemma 42.38.9) it suffices to prove

$$c(Z \to X, \mathcal{N}_1) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

in $A^*(Z \to X)$. This we may assume that $\mathcal{N}'' = \mathcal{C}_{Y/X}$. This reduces us to the case discussed in the next paragraph.

In this paragraph Z and Y are effective Cartier divisors on X integral of dimension n, we have $\mathcal{N}'' = \mathcal{C}_{Y/X}$. In this case $c(Y \to X, \mathcal{C}_{Y/X}^\vee) \cap [X] = [Y]_{n-1}$ by Lemma 42.54.5. Thus we have to prove that $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$. Denote N and N' the vector bundles over Z associated to $\mathcal{N}$ and $\mathcal{N}'$. Consider the commutative diagram

$$\begin{array}{ccccc} N' & \xrightarrow[i]{} & N & \longrightarrow & (C_Y X) \times_Y Z \\ \uparrow & & \uparrow & & \\ C_Z Y & \longrightarrow & C_Z X & & \end{array}$$

of cones and vector bundles over Z. Observe that N' is a relative effective Cartier divisor in N over Z and that

$$\begin{array}{ccc} N' & \xrightarrow{i} & N \\ \downarrow & & \downarrow \\ Z & \xrightarrow{o} & (C_Y X) \times_Y Z \end{array}$$

is cartesian where o is the zero section of the line bundle $C_Y X$ over Y. By Lemma 42.54.9 we have $o^*[C_Z X]_n = [C_Z Y]_{n-1}$ in

$$\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X) = \mathrm{CH}_{n-1}(Z \times_{o,(C_Y X)\times_Y Z} C_Z X)$$

By the cartesian property of the square above this implies that

$$i^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(N')$. Now observe that $\gamma = c(Z \to X, \mathcal{N}) \cap [X]$ and $\gamma' = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$ are characterized by $p^*\gamma = [C_Z X]_n$ in $\mathrm{CH}_n(N)$ and by $(p')^*\gamma' = [C_Z Y]_{n-1}$ in $\mathrm{CH}_{n-1}(N')$. Hence the proof is finished as $i^* \circ p^* = (p')^*$ by Lemma 42.31.1. □

0FBQ **Remark 42.54.11** (Variant for immersions)**.** Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be an immersion of schemes. In this situation

(1) the conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X is defined (Morphisms, Definition 29.31.1),
(2) we say a pair consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is a virtual normal bundle for the immersion $Z \to X$,
(3) choose an open subscheme $U \subset X$ such that $Z \to X$ factors through a closed immersion $Z \to U$ and set $c(Z \to X, \mathcal{N}) = c(Z \to U, \mathcal{N}) \circ (U \to X)^*$.

The bivariant class $c(Z \to X, \mathcal{N})$ does not depend on the choice of the open subscheme U. All of the lemmas have immediate counterparts for this slightly more general construction. We omit the details.

42.55. Calculating some classes

0FED To get further we need to compute the values of some of the classes we've constructed above.

0FEE **Lemma 42.55.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then*

$$\prod\nolimits_{n=0,\ldots,r} c(\wedge^n \mathcal{E})^{(-1)^n} = 1 - (r-1)! c_r(\mathcal{E}) + \ldots$$

Proof. By the splitting principle we can turn this into a calculation in the polynomial ring on the Chern roots $x_1, \ldots, x_r$ of $\mathcal{E}$. See Section 42.43. Observe that

$$c(\wedge^n \mathcal{E}) = \prod\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} (1 + x_{i_1} + \ldots + x_{i_n})$$

Thus the logarithm of the left hand side of the equation in the lemma is

$$-\sum\nolimits_{p \geq 1} \sum\nolimits_{n=0}^{r} \sum\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} \frac{(-1)^{p+n}}{p} (x_{i_1} + \ldots + x_{i_n})^p$$

Please notice the minus sign in front. However, we have

$$\sum\nolimits_{p\geq 0}\sum\nolimits_{n=0}^{r}\sum\nolimits_{1\leq i_1<\ldots<i_n\leq r}\frac{(-1)^{p+n}}{p!}(x_{i_1}+\ldots+x_{i_n})^p=\prod(1-e^{-x_i})$$

Hence we see that the first nonzero term in our Chern class is in degree r and equal to the predicted value. □

0FEF **Lemma 42.55.2.** *Let (S,δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{C}$ be a locally free $\mathcal{O}_X$-module of rank r. Consider the morphisms*

$$X=\underline{Proj}_X(\mathcal{O}_X[T])\xrightarrow{i}E=\underline{Proj}_X(Sym^*(\mathcal{C})[T])\xrightarrow{\pi}X$$

Then $c_t(i_*\mathcal{O}_X)=0$ for $t=1,\ldots,r-1$ and in $A^0(C\to E)$ we have

$$p^*\circ\pi_*\circ c_r(i_*\mathcal{O}_X)=(-1)^{r-1}(r-1)!j^*$$

where $j:C\to E$ and $p:C\to X$ are the inclusion and structure morphism of the vector bundle $C=\underline{\mathrm{Spec}}(Sym^*(\mathcal{C}))$.

Proof. The canonical map $\pi^*\mathcal{C}\to\mathcal{O}_E(1)$ vanishes exactly along $i(X)$. Hence the Koszul complex on the map

$$\pi^*\mathcal{C}\otimes\mathcal{O}_E(-1)\to\mathcal{O}_E$$

is a resolution of $i_*\mathcal{O}_X$. In particular we see that $i_*\mathcal{O}_X$ is a perfect object of $D(\mathcal{O}_E)$ whose Chern classes are defined. The vanishing of $c_t(i_*\mathcal{O}_X)$ for $t=1,\ldots,t-1$ follows from Lemma 42.55.1. This lemma also gives

$$c_r(i_*\mathcal{O}_X)=-(r-1)!c_r(\pi^*\mathcal{C}\otimes\mathcal{O}_E(-1))$$

On the other hand, by Lemma 42.43.3 we have

$$c_r(\pi^*\mathcal{C}\otimes\mathcal{O}_E(-1))=(-1)^rc_r(\pi^*\mathcal{C}^\vee\otimes\mathcal{O}_E(1))$$

and $\pi^*\mathcal{C}^\vee\otimes\mathcal{O}_E(1)$ has a section s vanishing exactly along $i(X)$.

After replacing X by a scheme locally of finite type over X, it suffices to prove that both sides of the equality have the same effect on an element $\alpha\in\mathrm{CH}_*(E)$. Since $C\to X$ is a vector bundle, every cycle class on C is of the form $p^*\beta$ for some $\beta\in\mathrm{CH}_*(X)$ (Lemma 42.36.3). Hence by Lemma 42.19.3 we can write $\alpha=\pi^*\beta+\gamma$ where γ is supported on $E\setminus C$. Using the equalities above it suffices to show that

$$p^*(\pi_*(c_r(\pi^*\mathcal{C}^\vee\otimes\mathcal{O}_E(1))\cap[W]))=j^*[W]$$

when $W\subset E$ is an integral closed subscheme which is either (a) disjoint from C or (b) is of the form $W=\pi^{-1}Y$ for some integral closed subscheme $Y\subset X$. Using the section s and Lemma 42.44.1 we find in case (a) $c_r(\pi^*\mathcal{C}^\vee\otimes\mathcal{O}_E(1))\cap[W]=0$ and in case (b) $c_r(\pi^*\mathcal{C}^\vee\otimes\mathcal{O}_E(1))\cap[W]=[i(Y)]$. The result follows easily from this; details omitted. □

0FEG **Lemma 42.55.3.** *Let (S,δ) be as in Situation 42.7.1. Let $i:Z\to X$ be a regular closed immersion of codimension r between schemes locally of finite type over S. Let $\mathcal{N}=\mathcal{C}_{Z/X}^\vee$ be the normal sheaf. If X is quasi-compact (or has quasi-compact irreducible components), then $c_t(Z\to X,i_*\mathcal{O}_Z)=0$ for $t=1,\ldots,r-1$ and*

$$c_r(Z\to X,i_*\mathcal{O}_Z)=(-1)^{r-1}(r-1)!c(Z\to X,\mathcal{N})\quad in\quad A^r(Z\to X)$$

where $c_t(Z\to X,i_*\mathcal{O}_Z)$ is the localized Chern class of Definition 42.50.3.

Proof. For any $x \in Z$ we can choose an affine open neighbourhood $\mathrm{Spec}(A) \subset X$ such that $Z \cap \mathrm{Spec}(A) = V(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r \in A$ is a regular sequence. See Divisors, Definition 31.21.1 and Lemma 31.20.8. Then we see that the Koszul complex on $f_1, \ldots, f_r$ is a resolution of $A/(f_1, \ldots, f_r)$ for example by More on Algebra, Lemma 15.30.2. Hence $A/(f_1, \ldots, f_r)$ is perfect as an A-module. It follows that $F = i_*\mathcal{O}_Z$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to $X \setminus Z$ is zero. The assumption that X is quasi-compact (or has quasi-compact irreducible components) means that the localized Chern classes $c_t(Z \to X, i_*\mathcal{O}_Z)$ are defined, see Situation 42.50.1 and Definition 42.50.3. All in all we conclude that the statement makes sense.

Denote $b : W \to \mathbf{P}^1_X$ the blowing up in $\infty(Z)$ as in Section 42.53. By (8) we have a closed immersion

$$i' : \mathbf{P}^1_Z \longrightarrow W$$

We claim that $Q = i'_*\mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$ and that F and Q satisfy the assumptions of Lemma 42.51.2.

Assume the claim. The output of Lemma 42.51.2 is that we have

$$c_p(Z \to X, F) = c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

for all $p \geq 1$. Observe that $Q|_E$ is equal to the pushforward of the structure sheaf of Z via the morphism $Z \to E$ which is the base change of i' by ∞. Thus the vanishing of $c_t(Z \to X, F)$ for $1 \leq t \leq r-1$ by Lemma 42.55.2 applied to $E \to Z$. Because $\mathcal{C}_{Z/X} = \mathcal{N}^\vee$ is locally free the bivariant class $c(Z \to X, \mathcal{N})$ is characterized by the relation

$$j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

where $j : C_Z X \to W_\infty$ and $p : C_Z X \to Z$ are the given maps. (Recall $C \in A^0(W_\infty \to X)$ is the class of Lemma 42.48.1.) Thus the displayed equation in the statement of the lemma follows from the corresponding equation in Lemma 42.55.2.

Proof of the claim. Let A and $f_1, \ldots, f_r$ be as above. Consider the affine open $\mathrm{Spec}(A[s]) \subset \mathbf{P}^1_X$ as in Section 42.53. Recall that $s = 0$ defines $(\mathbf{P}^1_X)_\infty$ over this open. Hence over $\mathrm{Spec}(A[s])$ we are blowing up in the ideal generated by the regular sequence $s, f_1, \ldots, f_r$. By More on Algebra, Lemma 15.31.2 the $r+1$ affine charts are global complete intersections over $A[s]$. The chart corresponding to the affine blowup algebra

$$A[s][f_1/s, \ldots, f_r/s] = A[s, y_1, \ldots, y_r]/(sy_i - f_i)$$

contains $i'(Z \cap \mathrm{Spec}(A))$ as the closed subscheme cut out by $y_1, \ldots, y_r$. Since $y_1, \ldots, y_r, sy_1 - f_1, \ldots, sy_r - f_r$ is a regular sequence in the polynomial ring $A[s, y_1, \ldots, y_r]$ we find that i' is a regular immersion. Some details omitted. As above we conclude that $Q = i'_*\mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$. All the other assumptions on F and Q in Lemma 42.51.2 (and Lemma 42.49.1) are immediately verified. □

0FEH **Lemma 42.55.4.** *In the situation of Lemma 42.55.3 say* $\dim_\delta(X) = n$. *Then we have*

(1) $c_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ *for* $t = 1, \ldots, r-1$,
(2) $c_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = (-1)^{r-1}(r-1)![Z]_{n-r}$,
(3) $ch_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ *for* $t = 0, \ldots, r-1$, *and*
(4) $ch_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = [Z]_{n-r}$.

Proof. Parts (1) and (2) follow immediately from Lemma 42.55.3 combined with Lemma 42.54.5. Then we deduce parts (3) and (4) using the relationship between $ch_p = (1/p!)P_p$ and c_p given in Lemma 42.52.1. (Namely, $(-1)^{r-1}(r-1)!ch_r = c_r$ provided $c_1 = c_2 = \ldots = c_{r-1} = 0$.) □

42.56. An Adams operator

0FEI We do the minimal amount of work to define the second adams operator. Let X be a scheme. Recall that $Vect(X)$ denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(Vect(X))$ associated to this category in Derived Categories of Schemes, Section 36.38. Finally, $K_0(Vect(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FEJ **Lemma 42.56.1.** *Let X be a scheme. There is a ring map*

$$\psi^2 : K_0(Vect(X)) \longrightarrow K_0(Vect(X))$$

which sends $[\mathcal{L}]$ to $[\mathcal{L}^{\otimes 2}]$ when $\mathcal{L}$ is invertible and is compatible with pullbacks.

Proof. Let X be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We will consider the element

$$\psi^2(\mathcal{E}) = [\mathrm{Sym}^2(\mathcal{E})] - [\wedge^2(\mathcal{E})]$$

of $K_0(Vect(X))$.

Let X be a scheme and consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Let us think of this as a filtration on $\mathcal{F}$ with 2 steps. The induced filtration on $\mathrm{Sym}^2(\mathcal{F})$ has 3 steps with graded pieces $\mathrm{Sym}^2(\mathcal{E})$, $\mathcal{E} \otimes \mathcal{F}$, and $\mathrm{Sym}^2(\mathcal{G})$. Hence

$$[\mathrm{Sym}^2(\mathcal{F})] = [\mathrm{Sym}^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\mathrm{Sym}^2(\mathcal{G})]$$

In exactly the same manner one shows that

$$[\wedge^2(\mathcal{F})] = [\wedge^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\wedge^2(\mathcal{G})]$$

Thus we see that $\psi^2(\mathcal{F}) = \psi^2(\mathcal{E}) + \psi^2(\mathcal{G})$. We conclude that we obtain a well defined additive map $\psi^2 : K_0(Vect(X)) \to K_0(Vect(X))$.

It is clear that this map commutes with pullbacks.

We still have to show that ψ^2 is a ring map. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be finite locally free $\mathcal{O}_X$-modules. Observe that there is a short exact sequence

$$0 \to \wedge^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \mathrm{Sym}^2(\mathcal{E} \otimes \mathcal{F}) \to \mathrm{Sym}^2(\mathcal{E}) \otimes \mathrm{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $(e \wedge e') \otimes (f \wedge f')$ to $(e \otimes f)(e' \otimes f') - (e' \otimes f)(e \otimes f')$ and the second map sends $(e \otimes f)(e' \otimes f')$ to $ee' \otimes ff'$. Similarly, there is a short exact sequence

$$0 \to \mathrm{Sym}^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \wedge^2(\mathcal{E} \otimes \mathcal{F}) \to \wedge^2(\mathcal{E}) \otimes \mathrm{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $ee' \otimes f \wedge f'$ to $(e \otimes f) \wedge (e' \otimes f') + (e' \otimes f) \wedge (e \otimes f')$ and the second map sends $(e \otimes f) \wedge (e' \otimes f')$ to $(e \wedge e') \otimes (ff')$. As above this proves the map ψ^2 is multiplicative. Since it is clear that $\psi^2(1) = 1$ this concludes the proof. □

0FEK **Remark 42.56.2.** Let X be a scheme such that 2 is invertible on X. Then the Adams operator ψ^2 can be defined on the K-group $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ (Derived Categories of Schemes, Definition 36.38.2) in a straightforward manner. Namely, given a perfect complex L on X we get an action of the group $\{\pm 1\}$ on $L \otimes^{\mathbf{L}} L$ by switching the factors. Then we can set

$$\psi^2(L) = [(L \otimes^{\mathbf{L}} L)^+] - [(L \otimes^{\mathbf{L}} L)^-]$$

where $(-)^+$ denotes taking invariants and $(-)^-$ denotes taking anti-invariants (suitably defined). Using exactness of taking invariants and anti-invariants one can argue similarly to the proof of Lemma 42.56.1 to show that this is well defined. When 2 is not invertible on X the situation is a good deal more complicated and another approach has to be used.

0FV8 **Lemma 42.56.3.** *Let X be a scheme. There is a ring map $\psi^{-1} : K_0(\mathit{Vect}(X)) \to K_0(\mathit{Vect}(X))$ which sends $[\mathcal{E}]$ to $[\mathcal{E}^\vee]$ when $\mathcal{E}$ is finite locally free and is compatible with pullbacks.*

Proof. The only thing to check is that taking duals is compatible with short exact sequences and with pullbacks. This is clear. □

0FEL **Remark 42.56.4.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The Chern class map defines a canonical map

$$c : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots$ and extending multiplicatively. Thus $-[\mathcal{E}]$ is sent to the formal inverse $c(\mathcal{E})^{-1}$ which is why we have the infinite product on the right hand side. This is well defined by Lemma 42.40.3.

0FEM **Remark 42.56.5.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The Chern character map defines a canonical ring map

$$ch : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X) \otimes \mathbf{Q}$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $ch(\mathcal{E})$ and extending additively. This is well defined by Lemma 42.45.2 and a ring homomorphism by Lemma 42.45.3.

0FEN **Lemma 42.56.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. If ψ^2 is as in Lemma 42.56.1 and c and ch are as in Remarks 42.56.4 and 42.56.5 then we have $c_i(\psi^2(\alpha)) = 2^i c_i(\alpha)$ and $ch_i(\psi^2(\alpha)) = 2^i ch_i(\alpha)$ for all $\alpha \in K_0(\mathit{Vect}(X))$.*

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(\mathit{Vect}(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r, see Remark 42.38.10. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 42.43.1.

Then $[p^*\mathcal{E}] = \sum[\mathcal{L}_j]$ and hence $\psi^2([p^{\mathcal{E}}]) = \sum[\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1(\mathcal{L}_j)$ we have

$$c(\alpha) = \prod(1 + x_j) \quad \text{and} \quad c(\psi^2(\alpha)) = \prod(1 + 2x_j)$$

in $\prod A^i(P)$ and we have

$$ch(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. □

0FEP **Remark 42.56.7.** Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are settheoretically supported on Z. Denote $Coh_Z(X) \subset Coh(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Observe that given $E \in D_{Z,perf}(\mathcal{O}_X)$ Zariski locally on X only a finite number of the cohomology sheaves $H^i(E)$ are nonzero (and they are all settheoretically supported on Z). Hence we can define

$$K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(Coh_Z(X)) = K'_0(Z)$$

(equality by Lemma 42.23.6) by the rule

$$E \longmapsto [\bigoplus\nolimits_{i\in\mathbf{Z}} H^{2i}(E)] - [\bigoplus\nolimits_{i\in\mathbf{Z}} H^{2i+1}(E)]$$

This works because given a distinguished triangle in $D_{Z,perf}(\mathcal{O}_X)$ we have a long exact sequence of cohomology sheaves.

0FEQ **Remark 42.56.8.** Let X, Z, $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. Assume X is regular. Then there is a canonical map

$$K_0(Coh(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X))$$

defined as follows. For any coherent $\mathcal{O}_Z$-module $\mathcal{F}$ denote $\mathcal{F}[0]$ the object of $D(\mathcal{O}_X)$ which has $\mathcal{F}$ in degree 0 and is zero in other degrees. Then $\mathcal{F}[0]$ is a perfect complex on X by Derived Categories of Schemes, Lemma 36.11.8. Hence $\mathcal{F}[0]$ is an object of $D_{Z,perf}(\mathcal{O}_X)$. On the other hand, given a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_Z$-modules we obtain a distinguished triangle $\mathcal{F}[0] \to \mathcal{F}'[0] \to \mathcal{F}''[0] \to \mathcal{F}[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(Coh(Z)) \to K_0(D_{Z,perf}(\mathcal{O}_X))$ by sending $[\mathcal{F}]$ to $[\mathcal{F}[0]]$ with apologies for the horrendous notation.

0FER **Lemma 42.56.9.** *Let X be a Noetherian regular scheme. Let $Z \subset X$ be a closed subscheme. The maps constructed in Remarks 42.56.7 and 42.56.8 are mutually inverse and we get $K'_0(Z) = K_0(D_{Z,perf}(\mathcal{O}_X))$.*

Proof. Clearly the composition

$$K_0(Coh(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(Coh(Z))$$

is the identity map. Thus it suffices to show the first arrow is surjective. Let E be an object of $D_{Z,perf}(\mathcal{O}_X)$. Recall that $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.8. Hence the cohomologies $H^i(E)$ are coherent, can be

viewed as objects of $D_{Z,perf}(\mathcal{O}_X)$, and only a finite number are nonzero. Using the distinguished triangles of canonical truncations the reader sees that

$$[E] = \sum (-1)^i [H^i(E)[0]]$$

in $K_0(D_{Z,perf}(\mathcal{O}_X))$. Then it suffices to show that $[\mathcal{F}[0]]$ is in the image of the map for any coherent $\mathcal{O}_X$-module set theoretically supported on Z. Since we can find a finite filtration on $\mathcal{F}$ whose subquotients are $\mathcal{O}_Z$-modules, the proof is complete. □

0FES **Remark 42.56.10.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. If X is quasi-compact (or more generally the irreducible components of X are quasi-compact), then the localized Chern classes define a canonical map

$$c(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow A^0(X) \times \prod\nolimits_{i \geq 1} A^i(Z \to X)$$

by sending a generator $[E]$ on the left hand side to

$$c(Z \to X, E) = 1 + c_1(Z \to X, E) + c_2(Z \to X, E) + \ldots$$

and extending multiplicatively (with product on the right hand side as in Remark 42.34.7). The quasi-compactness condition on X guarantees that the localized chern classes are defined (Situation 42.50.1 and Definition 42.50.3) and that these localized chern classes convert distinguished triangles into the corresponding products in the bivariant chow rings (Lemma 42.52.4).

0FET **Remark 42.56.11.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 42.56.7. If the irreducible components of X are quasi-compact, then the localized Chern character defines a canonical additive and multiplicative map

$$ch(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(Z \to X) \otimes \mathbf{Q}$$

by sending a generator $[E]$ on the left hand side to $ch(Z \to X, E)$ and extending additively. Namely, the condition on the irreducible components of X guarantees that the localized chern character is defined (Situation 42.50.1 and Definition 42.50.3) and that these localized chern characters convert distinguished triangles into the corresponding sums in the bivariant chow rings (Lemma 42.52.5). The multiplication on $K_0(D_{Z,perf}(X))$ is defined using derived tensor product (Derived Categories of Schemes, Remark 36.38.9) hence $ch(Z \to X, \alpha\beta) = ch(Z \to X, \alpha)ch(Z \to X, \beta)$ by Lemma 42.52.6. If X is quasi-compact, then the map $ch(Z \to X, -)$ has image contained in $A^*(Z \to X) \otimes \mathbf{Q}$; we omit the details.

0FEU **Remark 42.56.12.** Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S and assume X is quasi-compact (or more generally the irreducible components of X are quasi-compact). With $Z = X$ and notation as in Remarks 42.56.10 and 42.56.11 we have $D_{Z,perf}(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$ and we see that

$$K_0(D_{Z,perf}(\mathcal{O}_X)) = K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$$

see Derived Categories of Schemes, Definition 36.38.2. Hence we get

$$c : K_0(X) \to \prod A^i(X) \quad \text{and} \quad ch : K_0(X) \to \prod A^i(X) \otimes \mathbf{Q}$$

as a special case of Remarks 42.56.10 and 42.56.11. Of course, instead we could have just directly used Definition 42.46.3 and Lemmas 42.46.7 and 42.46.11 to construct

these maps (as this immediately seen to produce the same classes). Recall that there is a canonical map $K_0(\mathit{Vect}(X)) \to K_0(X)$ which sends a finite locally free module to itself viewed as a perfect complex (placed in degree 0), see Derived Categories of Schemes, Section 36.38. Then the diagram

$$\begin{array}{ccc} K_0((\mathit{Vect}(X)) & \longrightarrow & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \\ & {\scriptstyle c}\searrow \quad \swarrow{\scriptstyle c} & \\ & \prod A^i(X) & \end{array}$$

commutes where the south-east arrow is the one constructed in Remark 42.56.4. Similarly, the diagram

$$\begin{array}{ccc} K_0((\mathit{Vect}(X)) & \longrightarrow & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \\ & {\scriptstyle ch}\searrow \quad \swarrow{\scriptstyle ch} & \\ & \prod A^i(X) \otimes \mathbf{Q} & \end{array}$$

commutes where the south-east arrow is the one constructed in Remark 42.56.5.

42.57. Chow groups and K-groups revisited

0FEV This section is the continuation of Section 42.23. Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. The K-group $K'_0(X) = K_0(\mathit{Coh}(X))$ of coherent sheaves on X has a canonical increasing filtration

$$F_k K'_0(X) = \mathrm{Im}\left(K_0(\mathit{Coh}_{\leq k}(X)) \to K_0(\mathit{Coh}(X)\right)$$

This is called the filtration by dimension of supports. Observe that

$$\mathrm{gr}_k K'_0(X) \subset K'_0(X)/F_{k-1}K'_0(X) = K_0(\mathit{Coh}(X)/\mathit{Coh}_{\leq k-1}(X))$$

where the equality holds by Homology, Lemma 12.11.3. The discussion in Remark 42.23.5 shows that there are canonical maps

$$\mathrm{CH}_k(X) \longrightarrow \mathrm{gr}_k K'_0(X)$$

defined by sending the class of an integral closed subscheme $Z \subset X$ of δ-dimension k to the class of $[\mathcal{O}_Z]$ on the right hand side.

0FEW **Proposition 42.57.1.** *Let (S, δ) be as in Situation 42.7.1. Assume given a closed immersion $X \to Y$ of schemes locally of finite type over S with Y regular and quasi-compact. Then the composition*

$$K'_0(X) \to K_0(D_{X,perf}(\mathcal{O}_Y)) \to A^*(X \to Y) \otimes \mathbf{Q} \to \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

of the map $\mathcal{F} \mapsto \mathcal{F}[0]$ from Remark 42.56.8, the map $ch(X \to Y, -)$ from Remark 42.56.11, and the map $c \mapsto c \cap [Y]$ induces an isomorphism

$$K'_0(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

which depends on the choice of Y. Moreover, the canonical map

$$\mathrm{CH}_k(X) \otimes \mathbf{Q} \longrightarrow gr_k K'_0(X) \otimes \mathbf{Q}$$

(see above) is an isomorphism of $\mathbf{Q}$-vector spaces for all $k \in \mathbf{Z}$.

Proof. Since Y is regular, the construction in Remark 42.56.8 applies. Since Y is quasi-compact, the construction in Remark 42.56.11 applies. We have that Y is locally equidimensional (Lemma 42.42.1) and thus the "fundamental cycle" $[Y]$ is defined as an element of $\mathrm{CH}_*(Y)$, see Remark 42.42.2. Combining this with the map $\mathrm{CH}_k(X) \to \mathrm{gr}_k K'_0(X)$ constructed above we see that it suffices to prove

(1) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module whose support has δ-dimension $\leq k$, then the composition above sends $[\mathcal{F}]$ into $\bigoplus_{k' \leq k} \mathrm{CH}_{k'}(X) \otimes \mathbf{Q}$.
(2) If $Z \subset X$ is an integral closed subscheme of δ-dimension k, then the composition above sends $[\mathcal{O}_Z]$ to an element whose degree k part is the class of $[Z]$ in $\mathrm{CH}_k(X) \otimes \mathbf{Q}$.

Namely, if this holds, then our maps induce maps $\mathrm{gr}_k K'_0(X) \otimes \mathbf{Q} \to CH_k(X) \otimes \mathbf{Q}$ which are inverse to the canonical maps $\mathrm{CH}_k(X) \otimes \mathbf{Q} \to \mathrm{gr}_k K'_0(X) \otimes \mathbf{Q}$ given above the proposition.

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ the composition above sends $[\mathcal{F}]$ to

$$ch(X \to Y, \mathcal{F}[0]) \cap [Y] \in \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

If $\mathcal{F}$ is (set theoretically) supported on a closed subscheme $Z \subset X$, then we have

$$ch(X \to Y, \mathcal{F}[0]) = (Z \to X)_* \circ ch(Z \to Y, \mathcal{F}[0])$$

by Lemma 42.50.8. We conclude that in this case we end up in the image of $\mathrm{CH}_*(Z) \to \mathrm{CH}_*(X)$. Hence we get condition (1).

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. The composition above sends $[\mathcal{O}_Z]$ to the element

$$ch(X \to Y, \mathcal{O}_Z[0]) \cap [Y] = (Z \to X)_* ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y]$$

by the same argument as above. Thus it suffices to prove that the degree k part of $ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y] \in \mathrm{CH}_*(Z) \otimes \mathbf{Q}$ is $[Z]$. Since $\mathrm{CH}_k(Z) = \mathbf{Z}$, in order to prove this we may replace Y by an open neighbourhood of the generic point ξ of Z. Since the maximal ideal of the regular local ring $\mathcal{O}_{X,\xi}$ is generated by a regular sequence (Algebra, Lemma 10.106.3) we may assume the ideal of Z is generated by a regular sequence, see Divisors, Lemma 31.20.8. Thus we deduce the result from Lemma 42.55.4. □

42.58. Rational intersection products on regular schemes

0FEX We will show that $\mathrm{CH}_*(X) \otimes \mathbf{Q}$ has an intersection product if X is Noetherian, regular, finite dimensional, with affine diagonal. The basis for the construction is the following result (which is a corollary of the proposition in the previous section).

0FEY **Lemma 42.58.1.** *Let (S, δ) be as in Situation 42.7.1. Let X be a quasi-compact regular scheme of finite type over S with affine diagonal and $\delta_{X/S} : X \to \mathbf{Z}$ bounded. Then the composition*

$$K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \longrightarrow A^*(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

of the map ch from Remark 42.56.5 and the map $c \mapsto c \cap [X]$ is an isomorphism.

Proof. We have $K'_0(X) = K_0(X) = K_0(\mathit{Vect}(X))$ by Derived Categories of Schemes, Lemmas 36.38.4, 36.36.8, and 36.38.5. By Remark 42.56.12 the composition given agrees with the map of Proposition 42.57.1 for $X = Y$. Thus the result follows from the proposition. □

Let X, S, δ be as in Lemma 42.58.1. For simplicity let us work with cycles of a given codimension, see Section 42.42. Let $[X]$ be the fundamental cycle of X, see Remark 42.42.2. Pick $\alpha \in CH^i(X)$ and $\beta \in CH^j(X)$. By the lemma we can find a unique $\alpha' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ with $ch(\alpha') \cap [X] = \alpha$. Of course this means that $ch_{i'}(\alpha') \cap [X] = 0$ if $i' \neq i$ and $ch_i(\alpha') \cap [X] = \alpha$. By Lemma 42.56.6 we see that $\alpha'' = 2^{-i}\psi^2(\alpha')$ is another solution. By uniqueness we get $\alpha'' = \alpha'$ and we conclude that $ch_{i'}(\alpha) = 0$ in $A^{i'}(X) \otimes \mathbf{Q}$ for $i' \neq i$. Then we can define

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch_i(\alpha') \cap \beta$$

in $\mathrm{CH}^{i+j}(X) \otimes \mathbf{Q}$ by the property of α' we observed above. This is a symmetric pairing: namely, if we pick $\beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ lifting β, then we get

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch(\alpha') \cap ch(\beta') \cap [X]$$

and we know that Chern classes commute. The intersection product is associative for the same reason

$$(\alpha \cdot \beta) \cdot \gamma = ch(\alpha') \cap ch(\beta') \cap ch(\gamma') \cap [X]$$

because we know composition of bivariant classes is associative. Perhaps a better way to formulate this is as follows: there is a unique commutative, associative intersection product on $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ compatible with grading such that the isomorphism $K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \to \mathrm{CH}^*(X) \otimes \mathbf{Q}$ is an isomorphism of rings.

42.59. Gysin maps for local complete intersection morphisms

0FEZ Before reading this section, we suggest the reader read up on regular immersions (Divisors, Section 31.21) and local complete intersection morphisms (More on Morphisms, Section 37.62).

Let (S, δ) be as in Situation 42.7.1. Let $i : X \to Y$ be a regular immersion[9] of schemes locally of finite type over S. In particular, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free (see Divisors, Lemma 31.21.5). Hence the normal sheaf

$$\mathcal{N}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$$

is finite locally free as well and we have a surjection $\mathcal{N}_{X/Y}^\vee \to \mathcal{C}_{X/Y}$ (because an isomorphism is also a surjection). The construction in Section 42.54 gives us a canonical bivariant class

$$i^! = c(X \to Y, \mathcal{N}_{X/Y}) \in A^*(X \to Y)^\wedge$$

We need a couple of lemmas about this notion.

0FF0 **Lemma 42.59.1.** *Let (S, δ) be as in Situation 42.7.1. Let $i : X \to Y$ and $j : Y \to Z$ be regular immersions of schemes locally of finite type over S. Then $j \circ i$ is a regular immersion and $(j \circ i)^! = i^! \circ j^!$.*

Proof. The first statement is Divisors, Lemma 31.21.7. By Divisors, Lemma 31.21.6 there is a short exact sequence

$$0 \to i^*(\mathcal{C}_{Y/Z}) \to \mathcal{C}_{X/Z} \to \mathcal{C}_{X/Y} \to 0$$

Thus the result by the more general Lemma 42.54.10. □

[9] See Divisors, Definition 31.21.1. Observe that regular immersions are the same thing as Koszul-regular immersions or quasi-regular immersions for locally Noetherian schemes, see Divisors, Lemma 31.21.3. We will use this without further mention in this section.

0FF1 **Lemma 42.59.2.** *Let (S, δ) be as in Situation 42.7.1. Let $p : P \to X$ be a smooth morphism of schemes locally of finite type over S and let $s : X \to P$ be a section. Then s is a regular immersion and $1 = s^! \circ p^*$ in $A^*(X)^\wedge$ where $p^* \in A^*(P \to X)^\wedge$ is the bivariant class of Lemma 42.33.2.*

Proof. The first statement is Divisors, Lemma 31.22.8. It suffices to show that $s^! \cap p^*[Z] = [Z]$ in $\mathrm{CH}_*(X)$ for any integral closed subscheme $Z \subset X$ as the assumptions are preserved by base change by $X' \to X$ locally of finite type. After replacing P by an open neighbourhood of $s(Z)$ we may assume $P \to X$ is smooth of fixed relative dimension r. Say $\dim_\delta(Z) = n$. Then every irreducible component of $p^{-1}(Z)$ has dimension $r+n$ and $p^*[Z]$ is given by $[p^{-1}(Z)]_{n+r}$. Observe that $s(X) \cap p^{-1}(Z) = s(Z)$ scheme theoretically. Hence by the same reference as used above $s(X) \cap p^{-1}(Z)$ is a closed subscheme regularly embedded in $\overline{p}^{-1}(Z)$ of codimension r. We conclude by Lemma 42.54.5. □

Let (S, δ) be as in Situation 42.7.1. Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

of schemes locally of finite type over S such that g is smooth and i is a regular immersion. Combining the bivariant class $i^!$ discussed above with the bivariant class $g^* \in A^*(P \to Y)^\wedge$ of Lemma 42.33.2 we obtain

$$f^! = i^! \circ g^* \in A^*(X \to Y)$$

Observe that the morphism f is a local complete intersection morphism, see More on Morphisms, Definition 37.62.2. Conversely, if $f : X \to Y$ is a local complete intersection morphism of locally Noetherian schemes and $f = g \circ i$ with g smooth, then i is a regular immersion. We claim that our construction of $f^!$ only depends on the morphism f and not on the choice of factorization $f = g \circ i$.

0FF2 **Lemma 42.59.3.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. The bivariant class $f^!$ is independent of the choice of the factorization $f = g \circ i$ with g smooth (provided one exists).*

Proof. Given a second such factorization $f = g' \circ i'$ we can consider the smooth morphism $g'' : P \times_Y P' \to Y$, the immersion $i'' : X \to P \times_Y P'$ and the factorization $f = g'' \circ i''$. Thus we may assume that we have a diagram

$$\begin{array}{ccccc} & & P' & & \\ & {\scriptstyle i'}\nearrow & \downarrow{\scriptstyle p} & \searrow{\scriptstyle g'} & \\ X & \xrightarrow{i} & P & \xrightarrow{g} & Y \end{array}$$

where p is a smooth morphism. Then $(g')^* = p^* \circ g^*$ (Lemma 42.14.3) and hence it suffices to show that $i^! = (i')^! \circ p^*$ in $A^*(X \to P)$. Consider the commutative

diagram

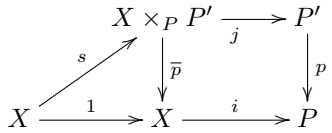

where $s = (1, i')$. Then s and j are regular immersions (by Divisors, Lemma 31.22.8 and Divisors, Lemma 31.21.4) and $i' = j \circ s$. By Lemma 42.59.1 we have $(i')^! = s^! \circ j^!$. Since the square is cartesian, the bivariant class $j^!$ is the restriction (Remark 42.33.5) of $i^!$ to P', see Lemma 42.54.2. Since bivariant classes commute with flat pullbacks we find $j^! \circ p^* = \overline{p}^* \circ i^!$. Thus it suffices to show that $s^! \circ \overline{p}^* = \mathrm{id}$ which is done in Lemma 42.59.2. □

0FF3 **Definition 42.59.4.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. We say *the Gysin map for f exists* if we can write $f = g \circ i$ with g smooth and i an immersion. In this case we define the *Gysin map* $f^! = i^! \circ g^* \in A^*(X \to Y)$ as above.

It follows from the definition that for a regular immersion this agrees with the construction earlier and for a smooth morphism this agrees with flat pullback. In fact, this agreement holds for all syntomic morphisms.

0FF4 **Lemma 42.59.5.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. If the Gysin map exists for f and f is flat, then $f^!$ is equal to the bivariant class of Lemma 42.33.2.*

Proof. Choose a factorization $f = g \circ i$ with $i : X \to P$ an immersion and $g : P \to Y$ smooth. Observe that for any morphism $Y' \to Y$ which is locally of finite type, the base changes of f', g', i' satisfy the same assumptions (see Morphisms, Lemmas 29.34.5 and 29.30.4 and More on Morphisms, Lemma 37.62.8). Thus we reduce to proving that $f^*[Y] = i^!(g^*[Y])$ in case Y is integral, see Lemma 42.35.3. Set $n = \dim_\delta(Y)$. After decomposing X and P into connected components we may assume f is flat of relative dimension r and g is smooth of relative dimension t. Then $f^*[Y] = [X]_{n+s}$ and $g^*[Y] = [P]_{n+t}$. On the other hand i is a regular immersion of codimension $t - s$. Thus $i^![P]_{n+t} = [X]_{n+s}$ (Lemma 42.54.5) and the proof is complete. □

0FF5 **Lemma 42.59.6.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be local complete intersection morphisms of schemes locally of finite type over S. Assume the Gysin map exists for $g \circ f$ and g. Then the Gysin map exists for f and $(g \circ f)^! = f^! \circ g^!$.*

Proof. Observe that $g \circ f$ is a local complete intersection morphism by More on Morphisms, Lemma 37.62.7 and hence the statement of the lemma makes sense. If $X \to P$ is an immersion of X into a scheme P smooth over Z then $X \to P \times_Z Y$ is an immersion of X into a scheme smooth over Y. This prove the first assertion of the lemma. Let $Y \to P'$ be an immersion of Y into a scheme P' smooth over Z.

Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & P \times_Z Y & \xrightarrow{a} & P \times_Z P' \\ \downarrow & & \swarrow{\scriptstyle p} & & \swarrow{\scriptstyle q} \\ Y & & \xrightarrow[b]{} & & P' \\ \downarrow & & & & \\ Z & & & & \end{array}$$

Here the horizontal arrows are regular immersions, the south-west arrows are smooth, and the square is cartesian. Whence $a^! \circ q^* = p^* \circ b^!$ as bivariant classes commute with flat pullback. Combining this fact with Lemmas 42.59.1 and 42.14.3 the reader finds the statement of the lemma holds true. Small detail omitted. □

0FF6 **Lemma 42.59.7.** *Let (S, δ) be as in Situation 42.7.1. Consider a commutative diagram*

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

of schemes locally of finite type over S with both square cartesian. Assume $f : X \to Y$ is a local complete intersection morphism such that the Gysin map exists for f. Let $c \in A^*(Y'' \to Y')$. Denote $res(f^!) \in A^*(X' \to Y')$ the restriction of $f^!$ to Y' (Remark 42.33.5). Then c and $res(f^!)$ commute (Remark 42.33.6).

Proof. Choose a factorization $f = g \circ i$ with g smooth and i an immersion. Since $f^! = i^! \circ g^!$ it suffices to prove the lemma for $g^!$ (which is given by flat pullback) and for $i^!$. The result for flat pullback is part of the definition of a bivariant class. The case of $i^!$ follows immediately from Lemma 42.54.8. □

0FF7 **Lemma 42.59.8.** *Let (S, δ) be as in Situation 42.7.1. Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

of schemes locally of finite type over S. Assume

1. *f is a local complete intersection morphism and the Gysin map exists for f,*
2. *X, X', Y, Y' satisfy the equivalent conditions of Lemma 42.42.1,*
3. *for $x' \in X'$ with images x, y', and y in X, Y', and Y we have $n_{x'} - n_{y'} = n_x - n_y$ where $n_{x'}$, n_x, $n_{y'}$, and n_y are as in the lemma, and*
4. *for every generic point $\xi \in X'$ the local ring $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay.*

Then $f^![Y'] = [X']$ where $[Y']$ and $[X']$ are as in Remark 42.42.2.

Proof. Recall that $n_{x'}$ is the common value of $\delta(\xi)$ where ξ is the generic point of an irreducible component passing through x'. Moreover, the functions $x' \mapsto n_{x'}$, $x \mapsto n_x$, $y' \mapsto n_{y'}$, and $y \mapsto n_y$ are locally constant. Let X'_n, X_n, Y'_n, and Y_n be the open and closed subscheme of X', X, Y', and Y where the function has value

n. Recall that $[X'] = \sum [X'_n]_n$ and $[Y'] = \sum [Y'_n]_n$. Having said this, it is clear that to prove the lemma we may replace X' by one of its connected components and X, Y', Y by the connected component that it maps into. Then we know that X', X, Y', and Y are δ-equidimensional in the sense that each irreducible component has the same δ-dimension. Say n', n, m', and m is this common value for X', X, Y', and Y. The last assumption means that $n' - m' = n - m$.

Choose a factorization $f = g \circ i$ where $i : X \to P$ is an immersion and $g : P \to Y$ is smooth. As X is connected, we see that the relative dimension of $P \to Y$ at points of $i(X)$ is constant. Hence after replacing P by an open neighbourhood of $i(X)$, we may assume that $P \to Y$ has constant relative dimension and $i : X \to P$ is a closed immersion. Denote $g' : Y' \times_Y P \to Y'$ the base change of g and denote $i' : X' \to Y' \times_Y P$ the base change of i. It is clear that $g^*[Y] = [P]$ and $(g')^*[Y'] = [Y' \times_Y P]$. Finally, if $\xi' \in X'$ is a generic point, then $\mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is Cohen-Macaulay. Namely, the local ring map $\mathcal{O}_{Y', f'(\xi)} \to \mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is flat with regular fibre (see Algebra, Section 10.142), a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3), $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay by assumption (4) and we get what we want from Algebra, Lemma 10.163.3. Thus we reduce to the case discussed in the next paragraph.

Assume f is a regular closed immersion and X', X, Y', and Y are δ-equidimensional of δ-dimensions n', n, m', and m and $m' - n' = m - n$. In this case we obtain the result immediately from Lemma 42.54.6. □

0FF8 **Remark 42.59.9.** Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. Assume the Gysin map exists for f. Then $f^! \circ c_i(\mathcal{E}) = c_i(f^*\mathcal{E}) \circ f^!$ and similarly for the Chern character, see Lemma 42.59.7. If X and Y satisfy the equivalent conditions of Lemma 42.42.1 and Y is Cohen-Macaulay (for example), then $f^![Y] = [X]$ by Lemma 42.59.8. In this case we also get $f^!(c_i(\mathcal{E}) \cap [Y]) = c_i(f^*\mathcal{E}) \cap [X]$ and similarly for the Chern character.

0FV9 **Lemma 42.59.10.** *Let (S, δ) be as in Situation 42.7.1. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes locally of finite type over S. Assume

(1) *both f and f' are local complete intersection morphisms, and*
(2) *the Gysin map exists for f*

Then $\mathcal{C} = \mathrm{Ker}(H^{-1}((g')^* NL_{X/Y}) \to H^{-1}(NL_{X'/Y'}))$ is a finite locally free $\mathcal{O}_{X'}$-module, the Gysin map exists for f', and we have

$$res(f^!) = c_{top}(\mathcal{C}^\vee) \circ (f')^!$$

in $A^*(X' \to Y')$.

Proof. The fact that $\mathcal{C}$ is finite locally free follows immediately from More on Algebra, Lemma 15.86.5. Choose a factorization $f = h \circ i$ with $h : P \to Y$ smooth

and i an immersion. Then we can factor $f' = h' \circ i'$ where $h' : P' \to Y'$ and $i' : X' \to P'$ the base changes. Picture

$$\begin{array}{ccccc} X' & \longrightarrow & P' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i} & P & \xrightarrow{h} & Y \end{array}$$

In particular, we see that the Gysin map exists for f'. By More on Morphisms, Lemmas 37.13.13 we have

$$NL_{X/Y} = \big(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\big)$$

where $\mathcal{C}_{X/P}$ is the conormal sheaf of the embedding i. Similarly for the primed version. We have $(g')^*i^*\Omega_{P/Y} = (i')^*\Omega_{P'/Y'}$ because $\Omega_{P/Y}$ pulls back to $\Omega_{P'/Y'}$ by Morphisms, Lemma 29.32.10. Also, recall that $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ is surjective, see Morphisms, Lemma 29.31.4. We deduce that the sheaf $\mathcal{C}$ is canonicallly isomorphic to the kernel of the map $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ of finite locally free modules. Recall that $i^!$ is defined using $\mathcal{N} = \mathcal{C}_{Z/X}^\vee$ and similarly for $(i')^!$. Thus we have

$$res(i^!) = c_{top}(\mathcal{C}^\vee) \circ (i')^!$$

in $A^*(X' \to P')$ by an application of Lemma 42.54.4. Since finally we have $f^! = i^! \circ h^*$, $(f')^! = (i')^! \circ (h')^*$, and $(h')^* = res(h^*)$ we conclude. □

0FVA **Lemma 42.59.11** (Blow up formula)**.** *Let (S, δ) be as in Situation 42.7.1. Let $i : Z \to X$ be a regular closed immersion of schemes locally of finite type over S. Let $b : X' \to X$ be the blowing up with center Z. Picture*

$$\begin{array}{ccc} E & \xrightarrow[j]{} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Assume that the Gysin map exists for b. Then we have

$$res(b^!) = c_{top}(\mathcal{F}^\vee) \circ \pi^*$$

in $A^*(E \to Z)$ where $\mathcal{F}$ is the kernel of the canonical map $\pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$.

Proof. Observe that the morphism b is a local complete intersection morphism by More on Algebra, Lemma 15.31.2 and hence the statement makes sense. Since $Z \to X$ is a regular immersion (and hence a fortiori quasi-regular) we see that $\mathcal{C}_{Z/X}$ is finite locally free and the map $\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \to \mathcal{C}_{Z/X,*}$ is an isomorphism, see Divisors, Lemma 31.21.5. Since $E = \mathrm{Proj}(\mathcal{C}_{Z/X,*})$ we conclude that $E = \mathbf{P}(\mathcal{C}_{Z/X})$ is a projective space bundle over Z. Thus $E \to Z$ is smooth and certainly a local complete intersection morphism. Thus Lemma 42.59.10 applies and we see that

$$res(b^!) = c_{top}(\mathcal{C}^\vee) \circ \pi^!$$

with $\mathcal{C}$ as in the statement there. Of course $\pi^* = \pi^!$ by Lemma 42.59.5. It remains to show that $\mathcal{F}$ is equal to the kernel $\mathcal{C}$ of the map $H^{-1}(j^*\, NL_{X'/X}) \to H^{-1}(NL_{E/Z})$.

Since $E \to Z$ is smooth we have $H^{-1}(NL_{E/Z}) = 0$, see More on Morphisms, Lemma 37.13.7. Hence it suffices to show that $\mathcal{F}$ can be identified with $H^{-1}(j^*\, NL_{X'/X})$. By More on Morphisms, Lemmas 37.13.11 and 37.13.9 we have an exact sequence

$$0 \to H^{-1}(j^*\, NL_{X'/X}) \to H^{-1}(NL_{E/X}) \to \mathcal{C}_{E/X'} \to \ldots$$

By the same lemmas applied to $E \to Z \to X$ we obtain an isomorphism $\pi^*\mathcal{C}_{Z/X} = H^{-1}(\pi^* NL_{Z/X}) \to H^{-1}(NL_{E/X})$. Thus we conclude. □

0FF9 **Lemma 42.59.12.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the Gysin map exists for f. Then*

$$f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$$

in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ where the intersection product is as in Section 42.58.

Proof. The first statement follows from More on Morphisms, Lemma 37.62.11. Observe that $f^![Y] = [X]$, see Lemma 42.59.8. Write $\alpha = ch(\alpha') \cap [Y]$ and $\beta = ch(\beta') \cap [Y]$ where $\alpha', \beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ as in Section 42.58. Setting $c = ch(\alpha')$ and $c' = ch(\beta')$ we find $\alpha \cdot \beta = c \cap c' \cap [Y]$ by construction. By Lemma 42.59.7 we know that $f^!$ commutes with both c and c'. Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]) \\ &= c \cap c' \cap f^![Y] \\ &= c \cap c' \cap [X] \\ &= (c \cap [X]) \cdot (c' \cap [X]) \\ &= (c \cap f^![Y]) \cdot (c' \cap f^![Y]) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired. □

0FFA **Lemma 42.59.13.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the Gysin map exists for f and that f is proper. Then*

$$f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$$

in $\mathrm{CH}^*(Y) \otimes \mathbf{Q}$ where the intersection product is as in Section 42.58.

Proof. The first statement follows from More on Morphisms, Lemma 37.62.11. Observe that $f^![Y] = [X]$, see Lemma 42.59.8. Write $\alpha = ch(\alpha') \cap [X]$ and $\beta = ch(\beta') \cap [Y]$ $\alpha' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ and $\beta' \in K_0(\mathit{Vect}(Y)) \otimes \mathbf{Q}$ as in Section 42.58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the construction of the intersection product. By Lemma 42.59.7 we know that $f^!$ commutes with c'. The fact that Chern classes are in the

center of the bivariant ring justifies switching the order of capping $[X]$ with c and c'. Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the construction of the intersection product. □

42.60. Gysin maps for diagonals

0FBR Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ be a smooth morphism of schemes locally of finite type over S. Then the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$ is a regular immersion, see More on Morphisms, Lemma 37.62.18. Thus we have the Gysin map

$$\Delta^! \in A^*(X \to X \times_Y X)^\wedge$$

constructed in Section 42.59. If $X \to Y$ has constant relative dimension d, then $\Delta^! \in A^d(X \to X \times_Y X)$.

0FBS **Lemma 42.60.1.** *In the situation above we have $\Delta^! \circ pr_i^! = 1$ in $A^0(X)$.*

Proof. Observe that the projections $\mathrm{pr}_i : X \times_Y X \to X$ are smooth and hence we have Gysin maps for these projections as well. Thus the lemma makes sense and is a special case of Lemma 42.59.6. □

0FBT **Proposition 42.60.2.** *Let (S, δ) be as in Situation 42.7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. If g is smooth of relative dimension d, then $A^p(X \to Y) = A^{p-d}(X \to Z)$.* [**Ful98**, Proposition 17.4.2]

Proof. We will use that smooth morphisms are local complete intersection morphisms whose Gysin maps exist (see Section 42.59). In particular we have $g^! \in A^{-d}(Y \to Z)$. Then we can send $c \in A^p(X \to Y)$ to $c \circ g^! \in A^{p-d}(X \to Z)$.

Conversely, let $c' \in A^{p-d}(X \to Z)$. Denote $res(c')$ the restriction (Remark 42.33.5) of c' by the morphism $Y \to Z$. Since the diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\mathrm{pr}_2} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

is cartesian we find $res(c') \in A^{p-d}(X \times_Z Y \to Y)$. Let $\Delta : Y \to Y \times_Z Y$ be the diagonal and denote $res(\Delta^!)$ the restriction of $\Delta^!$ to $X \times_Z Y$ by the morphism $X \times_Z Y \to Y \times_Z Y$. Since the diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times_Z Y \end{array}$$

is cartesian we see that $res(\Delta^!) \in A^d(X \to X \times_Z Y)$. Combining these two restrictions we obtain

$$res(\Delta^!) \circ res(c') \in A^p(X \to Y)$$

Thus we have produced maps $A^p(X \to Y) \to A^{p-d}(X \to Z)$ and $A^{p-d}(X \to Z) \to A^p(X \to Y)$. To finish the proof we will show these maps are mutually inverse.

Let us start with $c \in A^p(X \to Y)$. Consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & & \\ \downarrow & & \downarrow & & \\ X \times_Z Y & \longrightarrow & Y \times_Z Y & \xrightarrow[p_2]{} & Y \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

whose squares are carteisan. The lower two square of this diagram show that $res(c \circ g^!) = res(c) \cap p_2^!$ where in this formula $res(c)$ means the restriction of c via p_1. Looking at the upper square of the diagram and using Lemma 42.59.7 we get $c \circ \Delta^! = res(\Delta^!) \circ res(c)$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c \circ g^!) &= res(\Delta^!) \circ res(c) \circ p_2^! \\ &= c \circ \Delta^! \circ p_2^! \\ &= c \end{aligned}$$

The final equality by Lemma 42.60.1.

Conversely, let us start with $c' \in A^{p-d}(X \to Z)$. Looking at the lower rectangle of the diagram above we find $res(c') \circ g^! = \mathrm{pr}_1^! \circ c'$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c') \circ g^! &= res(\Delta^!) \circ \mathrm{pr}_1^! \circ c' \\ &= c' \end{aligned}$$

The final equality holds because the left two squares of the diagram show that $\mathrm{id} = res(\Delta^! \circ p_1^!) = res(\Delta^!) \circ \mathrm{pr}_1^!$. This finishes the proof. □

42.61. Exterior product

0FBU Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 42.7.2.

Consider a cartesian square

$$\begin{array}{ccc} X \times_k Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(k) = S \end{array}$$

of schemes locally of finite type over k. Then there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m}(X \times_k Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of dimensions n and m we have

$$[X'] \times [Y'] = [X' \times_k Y']_{n+m}$$

in $\mathrm{CH}_{n+m}(X \times_k Y)$.

0FBV **Lemma 42.61.1.** *The map $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$ is well defined.*

Proof. A first remark is that if $\alpha = \sum n_i[X_i]$ and $\beta = \sum m_j[Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of dimensions n and m, then $X_i \times_k Y_j$ is a locally finite collection of closed subschemes of $X \times_k Y$ of dimensions $n + m$ and we can indeed consider

$$\alpha \times \beta = \sum n_i m_j [X_i \times_k Y_j]_{n+m}$$

as a $(n+m)$-cycle on $X \times_k Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_k Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of dimension n. Then flat pullback along the morphism $p' : X' \to \mathrm{Spec}(k)$ is an element $(p')^* \in A^{-n}(X' \to \mathrm{Spec}(k))$ by Lemma 42.33.2 and hence $c' = i_* \circ (p')^* \in A^{-n}(X \to \mathrm{Spec}(k))$ by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n}(X \times_k Y)$$

which the reader easily sends $[Y']$ to $[X' \times_k Y']_{n+m}$ for any integral closed subscheme $Y' \subset Y$ of dimension m. Hence the construction $([X'], [Y']) \mapsto [X' \times_k Y']_{n+m}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$. By symmetry the same is true for the other variable and we conclude. □

0FBW **Lemma 42.61.2.** *Let k be a field. Let X be a scheme locally of finite type over k. Then we have a canonical identification*

$$A^p(X \to \mathrm{Spec}(k)) = \mathrm{CH}_{-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[\mathrm{Spec}(k)] \in \mathrm{CH}_0(\mathrm{Spec}(k))$. We get a map $A^p(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{-p}(X)$ by sending c to $c \cap [\mathrm{Spec}(k)]$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{-p}(X)$. Then we can define $c_\alpha \in A^p(X \to \mathrm{Spec}(k))$ as follows: given $X' \to \mathrm{Spec}(k)$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_k X')$. To show that this is a bivariant class we write $\alpha = \sum n_i[X_i]$ as in Definition 42.8.1. Consider the composition

$$\coprod X_i \xrightarrow{g} X \to \mathrm{Spec}(k)$$

and denote $f : \coprod X_i \to \mathrm{Spec}(k)$ the composition. Then g is proper and f is flat of relative dimension $-p$. Pullback along f is a bivariant class $f^* \in A^p(\coprod X_i \to \mathrm{Spec}(k))$ by Lemma 42.33.2. Denote $\nu \in A^0(\coprod X_i)$ the bivariant class which multiplies a cycle by n_i on the ith component. Thus $\nu \circ f^* \in A^p(\coprod X_i \to X)$. Finally, we have a bivariant class

$$g_* \circ \nu \circ f^*$$

by Lemma 42.33.4. The reader easily verifies that c_α is equal to this class and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [\mathrm{Spec}(k)] = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to \mathrm{Spec}(k))$ and set $\alpha = c \cap [\mathrm{Spec}(k)]$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over $\mathrm{Spec}(k)$, see Lemma 42.35.3. However, then $p' : X' \to \mathrm{Spec}(k)$ is flat of relative dimension $\dim(X')$ and hence $[X'] = (p')^*[\mathrm{Spec}(k)]$. Thus the fact that the bivariant classes c and c_α agree on $[\mathrm{Spec}(k)]$ implies they agree when capped against $[X']$ and the proof is complete. □

0FBX **Lemma 42.61.3.** *Let k be a field. Let X be a scheme locally of finite type over k. Let $c \in A^p(X \to \mathrm{Spec}(k))$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_k Y \to Z)$.*

Proof. In the proof of Lemma 42.61.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, and flat pullback. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. □

0FBY **Remark 42.61.4.** The upshot of Lemmas 42.61.2 and 42.61.3 is the following. Let k be a field. Let X be a scheme locally of finite type over k. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_k Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to \mathrm{Spec}(k))$ the bivariant class corresponding to α, see proof of Lemma 42.61.2.

0FBZ **Lemma 42.61.5.** *Exterior product is associative. More precisely, let k be a field, let X, Y, Z be schemes locally of finite type over k, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_k Y \times_k Z)$.*

Proof. Omitted. Hint: associativity of fibre product of schemes. □

42.62. Intersection products

0FC0 Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 42.7.2.

Let X be a smooth scheme over k. The bivariant class $\Delta^!$ of Section 42.60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_k Z) \times_{X \times_k X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Z) \xrightarrow{\times} \mathrm{CH}_{n+m}(Y \times_k Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 42.61 and the second arrow is the Gysin map for the diagonal studied in Section 42.60. If X is equidimensional of dimension d, then we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$ and in general we can decompose into the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the *intersection product*. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

0FC1 **Lemma 42.62.1.** *The product defined above is associative. More precisely, let k be a field, let X be smooth over k, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.*

Proof. By Lemma 42.61.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_k Z \times_k W)$. Consider the closed immersions

$$\Delta_{12} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 42.33.5) of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 42.54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\
&= \Delta^!(\Delta_{12}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!(\alpha \times (\beta \times \gamma)) \\
&= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\
&= \alpha \cdot (\beta \cdot \gamma)
\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta_{23}^!(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 42.61.4. Similarly for the second equation. □

0FC2 **Lemma 42.62.2.** *Let k be a field. Let X be a smooth scheme over k, equidimensional of dimension d. The map*

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to \mathrm{Spec}(k)$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d}(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 42.60.2 and the second is the isomorphism $c \mapsto c \cap [\mathrm{Spec}(k)]$ of Lemma 42.61.2. From the proof of Lemma 42.61.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 42.60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$

for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. □

0FFB **Lemma 42.62.3.** *Let k be a field. Let $f : X \to Y$ be a morphism of schemes smooth over k. Then the Gysin map exists for f and $f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$.*

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the Gysin map exists for f (Definition 42.59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 42.59.8). Write $\alpha = c \cap [Y]_e$ and $\beta = c' \cap [Y]_e$ and hence $\alpha \cdot \beta = c \cap c' \cap [Y]_e$, see Lemma 42.62.2. By Lemma 42.59.7 we know that $f^!$ commutes with both c and c'. Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]_e) \\ &= c \cap c' \cap f^![Y]_e \\ &= c \cap c' \cap [X]_d \\ &= (c \cap [X]_d) \cdot (c' \cap [X]_d) \\ &= (c \cap f^![Y]_e) \cdot (c' \cap f^![Y]_e) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired where we have used Lemma 42.62.2 for X as well.

An alternative proof can be given by proving that $(f \times f)^!(\alpha \times \beta) = f^!\alpha \times f^!\beta$ and using Lemma 42.59.6. □

0FFC **Lemma 42.62.4.** *Let k be a field. Let $f : X \to Y$ be a proper morphism of schemes smooth over k. Then the Gysin map exists for f and $f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$.*

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the Gysin map exists for f (Definition 42.59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 42.59.8). Write $\alpha = c \cap [X]_d$ and $\beta = c' \cap [Y]_e$, see Lemma 42.62.2. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the result of Lemma 42.62.2 for X. By Lemma 42.59.7 we know that $f^!$ commutes with c'. The commutativity of the intersection product justifies switching the order of capping $[X]_d$ with c and c' (via the lemma). Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the lemma. □

0FFD **Lemma 42.62.5.** *Let k be a field. Let X be an integral scheme smooth over k. Let $Y, Z \subset X$ be integral closed subschemes. Set $d = \dim(Y) + \dim(Z) - \dim(X)$. Assume*

(1) $\dim(Y \cap Z) \leq d$, *and*
(2) $\mathcal{O}_{Y,\xi}$ *and* $\mathcal{O}_{Z,\xi}$ *are Cohen-Macaulay for every* $\xi \in Y \cap Z$ *with* $\delta(\xi) = d$.

Then $[Y] \cdot [Z] = [Y \cap Z]_d$ in $\mathrm{CH}_d(X)$.

Proof. Recall that $[Y] \cdot [Z] = \Delta^!([Y \times Z])$ where $\Delta^! = c(\Delta : X \to X \times X, \mathcal{T}_{X/k})$ is a higher codimension Gysin map (Section 42.54) with $\mathcal{T}_{X/k} = \mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$ locally free of rank $\dim(X)$. We have the equality of schemes

$$Y \cap Z = X \times_{\Delta,(X\times X)} (Y \times Z)$$

and $\dim(Y \times Z) = \dim(Y) + \dim(Z)$ and hence conditions (1), (2), and (3) of Lemma 42.54.6 hold. Finally, if $\xi \in Y \cap Z$, then we have a flat local homomorphism

$$\mathcal{O}_{Y,\xi} \longrightarrow \mathcal{O}_{Y\times Z,\xi}$$

whose "fibre" is $\mathcal{O}_{Z,\xi}$. It follows that if both $\mathcal{O}_{Y,\xi}$ and $\mathcal{O}_{Z,\xi}$ are Cohen-Macaulay, then so is $\mathcal{O}_{Y\times Z,\xi}$, see Algebra, Lemma 10.163.3. In this way we see that all the hypotheses of Lemma 42.54.6 are satisfied and we conclude. □

0FFE **Lemma 42.62.6.** *Let k be a field. Let X be a scheme smooth over k. Let $i : Y \to X$ be a regular closed immersion. Let $\alpha \in \mathrm{CH}_*(X)$. If Y is equidimensional of dimension e, then $\alpha \cdot [Y]_e = i_*(i^!(\alpha))$ in $\mathrm{CH}_*(X)$.*

Proof. After decomposing X into connected components we may and do assume X is equidimensional of dimension d. Write $\alpha = c \cap [X]_n$ with $x \in A^*(X)$, see Lemma 42.62.2. Then

$$i_*(i^!(\alpha)) = i_*(i^!(c \cap [X]_n)) = i_*(c \cap i^![X]_n) = i_*(c \cap [Y]_e) = c \cap i_*[Y]_e = \alpha \cdot [Y]_e$$

The first equality by choice of c. Then second equality by Lemma 42.59.7. The third because $i^![X]_d = [Y]_e$ in $\mathrm{CH}_*(Y)$ (Lemma 42.59.8). The fourth because bivariant classes commute with proper pushforward. The last equality by Lemma 42.62.2. □

0FFF **Lemma 42.62.7.** *Let k be a field. Let X be a smooth scheme over k which is quasi-compact and has affine diagonal. Then the intersection product on $\mathrm{CH}^*(X)$ constructed in this section agrees after tensoring with $\mathbf{Q}$ with the intersection product constructed in Section 42.58.*

Proof. Let $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$. Write $\alpha = ch(\alpha') \cap [X]$ and $\beta = ch(\beta') \cap [X]$ $\alpha', \beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ as in Section 42.58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. Then the intersection product in Section 42.58 produces $c \cap c' \cap [X]$. This is the same as $\alpha \cdot \beta$ by Lemma 42.62.2 (or rather the generalization that $A^i(X) \to \mathrm{CH}^i(X)$, $c \mapsto c \cap [X]$ is an isomorphism for any smooth scheme X over k). □

42.63. Exterior product over Dedekind domains

0FC3 Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 42.7.1; see Example 42.7.3. Observe that S is normal (Algebra, Lemma 10.120.17) and hence a disjoint union of normal

integral schemes (Properties, Lemma 28.7.7). Thus all of the arguments below reduce to the case where S is irreducible. On the other hand, we allow S to be nonseparated (so S could be the affine line with 0 doubled for example).

Consider a cartesian square

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of schemes locally of finite type over S. We claim there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m-1}(X \times_S Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of δ-dimensions n and m we set

(1) $[X'] \times [Y'] = [X' \times_S Y']_{n+m-1}$ if X' or Y' dominates an irreducible component of S,
(2) $[X'] \times [Y'] = 0$ if neither X' nor Y' dominates an irreducible component of S.

0FC4 **Lemma 42.63.1.** *The map $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$ is well defined.*

Proof. Consider n and m cycles $\alpha = \sum_{i \in I} n_i [X_i]$ and $\beta = \sum_{j \in J} m_j [Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of δ-dimensions n and m. Let $K \subset I \times J$ be the set of pairs $(i, j) \in I \times J$ such that X_i or Y_j dominates an irreducible component of S. Then $\{X_i \times_S Y_j\}_{(i,j) \in K}$ is a locally finite collection of closed subschemes of $X \times_S Y$ of δ-dimension $n + m - 1$. This means we can indeed consider

$$\alpha \times \beta = \sum\nolimits_{(i,j) \in K} n_i m_j [X_i \times_S Y_j]_{n+m-1}$$

as a $(n + m - 1)$-cycle on $X \times_S Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_S Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n which dominates an irreducible component of S. Then $p' : X' \to S$ is flat of relative dimension $n - 1$, see More on Algebra, Lemma 15.22.11. Hence flat pullback along p' is an element $(p')^* \in A^{-n+1}(X' \to S)$ by Lemma 42.33.2 and hence $c' = i_* \circ (p')^* \in A^{-n+1}(X \to S)$ by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which sends $[Y']$ to $[X' \times_S Y']_{n+m-1}$ for any integral closed subscheme $Y' \subset Y$ of δ-dimension m.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n such that the composition $X' \to X \to S$ factors through a closed point $s \in S$. Since s is a closed point of the spectrum of a Dedekind domain, we see that s is an effective Cartier divisor on S whose normal bundle is trivial. Denote $c \in A^1(s \to S)$ the Gysin homomorphism, see Lemma 42.33.3. The morphism

$p' : X' \to s$ is flat of relative dimension n. Hence flat pullback along p' is an element $(p')^* \in A^{-n}(X' \to S)$ by Lemma 42.33.2. Thus

$$c' = i_* \circ (p')^* \circ c \in A^{-n}(X \to S)$$

by Lemma 42.33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which for any integral closed subscheme $Y' \subset Y$ of δ-dimension m sends $[Y']$ to either $[X' \times_S Y']_{n+m-1}$ if Y' dominates an irreducible component of S or to 0 if not.

From the previous two paragraphs we conclude the construction $([X'], [Y']) \mapsto [X' \times_S Y']_{n+m-1}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$. By symmetry the same is true for the other variable and we conclude. □

0FC5 **Lemma 42.63.2.** *Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Then we have a canonical identification*

$$A^p(X \to S) = \mathrm{CH}_{1-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[S]_1 \in \mathrm{CH}_1(S)$. We get a map $A^p(X \to S) \to \mathrm{CH}_{1-p}(X)$ by sending c to $c \cap [S]_1$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{1-p}(X)$. Then we can define $c_\alpha \in A^p(X \to S)$ as follows: given $X' \to S$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_S X')$. To show that this is a bivariant class we write $\alpha = \sum_{i \in I} n_i [X_i]$ as in Definition 42.8.1. In particular the morphism

$$g : \coprod\nolimits_{i \in I} X_i \longrightarrow X$$

is proper. Pick $i \in I$. If X_i dominates an irreducible component of S, then the structure morphism $p_i : X_i \to S$ is flat and we have $\xi_i = p_i^* \in A^p(X_i \to S)$. On the other hand, if p_i factors as $p_i' : X_i \to s_i$ followed by the inclusion $s_i \to S$ of a closed point, then we have $\xi_i = (p_i')^* \circ c_i \in A^p(X_i \to S)$ where $c_i \in A^1(s_i \to S)$ is the Gysin homomorphism and $(p_i')^*$ is flat pullback. Observe that

$$A^p(\coprod\nolimits_{i \in I} X_i \to S) = \prod\nolimits_{i \in I} A^p(X_i \to S)$$

Thus we have

$$\xi = \sum n_i \xi_i \in A^p(\coprod\nolimits_{i \in I} X_i \to S)$$

Finally, since g is proper we have a bivariant class

$$g_* \circ \xi \in A^p(X \to S)$$

by Lemma 42.33.4. The reader easily verifies that c_α is equal to this class (please compare with the proof of Lemma 42.63.1) and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [S]_1 = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to S)$ and set $\alpha = c \cap [S]_1$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over S, see Lemma 42.35.3. However, either $p' : X' \to S$ is flat of relative dimension $\dim_\delta(X') - 1$ and hence $[X'] = (p')^*[S]_1$ or $X' \to S$ factors as $X' \to s \to S$ and hence $[X'] = (p')^*(s \to S)^*[S]_1$. Thus the fact that the bivariant classes c and c_α agree on $[S]_1$ implies they agree when capped against $[X']$ (since bivariant classes commute with flat pullback and Gysin maps) and the proof is complete. □

0FC6 **Lemma 42.63.3.** *Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $c \in A^p(X \to S)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_S Y \to X \times_S Z)$.*

Proof. In the proof of Lemma 42.63.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, flat pullback, and Gysin maps. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. □

0FC7 **Remark 42.63.4.** The upshot of Lemmas 42.63.2 and 42.63.3 is the following. Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_S Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to S)$ the bivariant class corresponding to α, see proof of Lemma 42.63.2.

0FC8 **Lemma 42.63.5.** *Exterior product is associative. More precisely, let (S, δ) be as above, let X, Y, Z be schemes locally of finite type over S, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_S Y \times_S Z)$.*

Proof. Omitted. Hint: associativity of fibre product of schemes. □

42.64. Intersection products over Dedekind domains

0FC9 Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 42.7.1; see Example 42.7.3 and discussion in Section 42.63.

Let X be a smooth scheme over S. The bivariant class $\Delta^!$ of Section 42.60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_S Z) \times_{X \times_S X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \xrightarrow{\times} \mathrm{CH}_{n+m-1}(Y \times_S Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 42.63 and the second arrow is the Gysin map for the diagonal studied in Section 42.60. If X is equidimensional of dimension d, then $X \to S$ is smooth of relative dimension $d - 1$ and hence we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$. In general we can decompose into

the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the *intersection product*. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

0FCA **Lemma 42.64.1.** *The product defined above is associative. More precisely, with (S, δ) as above, let X be smooth over S, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.*

Proof. By Lemma 42.63.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_S Z \times_S W)$. Consider the closed immersions

$$\Delta_{12} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 42.33.5) of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 42.54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\
&= \Delta^!(\Delta_{12}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!(\alpha \times (\beta \times \gamma)) \\
&= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\
&= \alpha \cdot (\beta \cdot \gamma)
\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta_{23}^!(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 42.61.4. Similarly for the second equation. □

0FCB **Lemma 42.64.2.** *Let (S, δ) be as above. Let X be a smooth scheme over S, equidimensional of dimension d. The map*

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to S$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d+1}(X \to S) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 42.60.2 and the second is the isomorphism $c \mapsto c \cap [S]_1$ of Lemma 42.63.2. From the proof of Lemma 42.63.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 42.60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$ for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. □

42.65. Todd classes

02UN A final class associated to a vector bundle $\mathcal{E}$ of rank r is its *Todd class* $Todd(\mathcal{E})$. In terms of the Chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}$$

In terms of the Chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

42.66. Grothendieck-Riemann-Roch

02UO Let (S, δ) be as in Situation 42.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^if_*\mathcal{E}$ are locally free sheaves on Y of finite rank. The Grothendieck-Riemann-Roch theorem say in this case that

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum (-1)^i ch(R^if_*\mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. If $Y = \mathrm{Spec}(k)$ where k is a field, then we can restate this as

$$\chi(X, \mathcal{E}) = \deg(Todd(T_{X/k})ch(\mathcal{E}))$$

The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

42.67. Change of base scheme

0FVF In this section we explain how to compare theories for different base schemes.

0FVG **Situation 42.67.1.** Here (S, δ) and (S', δ') are as in Situation 42.7.1. Furthermore $g : S' \to S$ is a flat morphism of schemes and $c \in \mathbf{Z}$ is an integer such that: for all $s \in S$ and $s' \in S'$ a generic point of an irreducible component of $g^{-1}(\{s\})$ we have $\delta'(s') = \delta(s) + c$.

We will see that for a scheme X locally of finite type over S there is a well defined map $\mathrm{CH}_k(X) \to \mathrm{CH}_{k+c}(X \times_S S')$ of Chow groups which (by and large) commutes with the operations we have defined in this chapter.

0FVH **Lemma 42.67.2.** *In Situation 42.67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $S' \to S$. If X is integral with $\dim_\delta(X) = k$, then every irreducible component Z' of X' has $\dim_{\delta'}(Z') = k + c$,*

Proof. The projection $X' \to X$ is flat as a base change of the flat morphism $S' \to S$ (Morphisms, Lemma 29.25.8). Hence every generic point x' of an irreducible component of X' maps to the generic point $x \in X$ (because generalizations lift along $X' \to X$ by Morphisms, Lemma 29.25.9). Let $s \in S$ be the image of x. Recall that the scheme $S'_s = S' \times_S s$ has the same underlying topological space as $g^{-1}(\{s\})$ (Schemes, Lemma 26.18.5). We may view x' as a point of the scheme $S'_s \times_s x$ which comes equipped with a monomorphism $S'_s \times_s x \to S' \times_S X$. Of course, x' is a generic point of an irreducible component of $S'_s \times_s x$ as well. Using the flatness of $\mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\kappa(s)) = s$ and arguing as above, we see that x' maps to a generic point s' of an irreducible component of $g^{-1}(\{s\})$. Hence $\delta'(s') = \delta(s) + c$ by assumption. We have $\dim_x(X_s) = \dim_{x'}(X_{s'})$ by Morphisms, Lemma 29.28.3. Since x is a generic point of an irreducible component X_s (this is an irreducible scheme but we don't need this) and x' is a generic point of an irreducible component of $X'_{s'}$ we conclude that $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \mathrm{trdeg}_{\kappa(s')}(\kappa(x'))$ by Morphisms, Lemma 29.28.1. Then

$$\delta_{X'/S'}(x') = \delta(s') + \mathrm{trdeg}_{\kappa(s')}(\kappa(x')) = \delta(s) + c + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \delta_{X/S}(x) + c$$

This proves what we want by Definition 42.7.6. □

In Situation 42.67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $g : S' \to S$. There is a unique homomorphism

$$g^* : Z_k(X) \longrightarrow Z_{k+c}(X')$$

which given an integral closed subscheme $Z \subset X$ of δ-dimension k sends $[Z]$ to $[Z \times_S S']_{k+c}$. This makes sense by Lemma 42.67.2.

0FVI **Lemma 42.67.3.** *In Situation 42.67.1 let $X \to S$ locally of finite type and let $X' \to S$ be the base change by $S' \to S$.*

1. *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$ and base change $Z' \subset X'$. Then we have $\dim_{\delta'}(Z')) \leq k + c$ and $[Z']_{k+c} = g^*[Z]_k$ in $Z_{k+c}(X')$.*
2. *Let $\mathcal{F}$ be a coherent sheaf on X with $\dim_\delta(\mathit{Supp}(\mathcal{F})) \leq k$ and base change $\mathcal{F}'$ on X'. Then we have $\dim_\delta(\mathit{Supp}(\mathcal{F}')) \leq k + c$ and $g^*[\mathcal{F}]_k = [\mathcal{F}']_{k+c}$ in $Z_{k+c}(X')$.*

Proof. The proof is exactly the same is the proof of Lemma 42.14.4 and we suggest the reader skip it.

The statements on dimensions follow from Lemma 42.67.2. Part (1) follows from part (2) by Lemma 42.10.3 and the fact that the base change of the coherent module $\mathcal{O}_Z$ is $\mathcal{O}_{Z'}$.

Proof of (2). As X, X' are locally Noetherian we may apply Cohomology of Schemes, Lemma 30.9.1 to see that $\mathcal{F}$ is of finite type, hence $\mathcal{F}'$ is of finite type (Modules, Lemma 17.9.2), hence $\mathcal{F}'$ is coherent (Cohomology of Schemes, Lemma 30.9.1 again). Thus the lemma makes sense. Let $W \subset X$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X'$ be an integral closed subscheme of δ'-dimension $k + c$ mapping into W under $X' \to X$. We have to show that the coefficient n of $[W']$ in $g^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[\mathcal{F}']_{k+c}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X',\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 42.10.1.) We have $\mathcal{F}'_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 10.52.13. □

0FVJ **Lemma 42.67.4.** *In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. The map $g^* : Z_k(X) \to Z_{k+c}(X')$ above factors through rational equivalence to give a map*

$$g^* : \text{CH}_k(X) \longrightarrow \text{CH}_{k+c}(X')$$

of chow groups.

Proof. Suppose that $\alpha \in Z_k(X)$ is a k-cycle which is rationally equivalent to zero. By Lemma 42.21.1 there exists a locally finite family of integral closed subschemes $W_i \subset X \times \mathbf{P}^1$ of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$. Thus it suffices to prove for $W \subset X \times \mathbf{P}^1$ integral closed of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ we have

(1) the base change $W' \subset X' \times \mathbf{P}^1$ satisfies the assumptions of Lemma 42.21.2 with k replaced by $k + c$, and
(2) $g^*[W_0]_k = [(W')_0]_{k+c}$ and $g^*[W_\infty]_k = [(W')_\infty]_{k+c}$.

Part (2) follows immediately from Lemma 42.67.3 and the fact that $(W')_0$ is the base change of W_0 (by associativity of fibre products). For part (1), first the statement on dimensions follows from Lemma 42.67.2. Then let $w' \in (W')_0$ with image $w \in W_0$ and $z \in \mathbf{P}^1_S$. Denote $t \in \mathcal{O}_{\mathbf{P}^1_S,z}$ the usual equation for $0 : S \to \mathbf{P}^1_S$. Since $\mathcal{O}_{W,w} \to \mathcal{O}_{W',w'}$ is flat and since t is a nonzerodivisor on $\mathcal{O}_{W,w}$ (as W is integral and $W \neq W_0$) we see that also t is a nonzerodivisor in $\mathcal{O}_{W',w'}$. Hence W' has no associated points lying on W'_0. □

0FVK **Lemma 42.67.5.** *In Situation 42.67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is flat of relative dimension r. Then $f' : Y' \to X'$ is flat of relative dimension r and the*

diagrams

$$\begin{array}{ccc} Z_{k+r}(Y) & \xrightarrow[g^*]{} & Z_{k+c+r}(Y') \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (f')^*} \\ Z_k(X) & \xrightarrow{g^*} & Z_{k+c}(X') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CH}_{k+r}(Y) & \xrightarrow[g^*]{} & \mathrm{CH}_{k+c+r}(Y') \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (f')^*} \\ \mathrm{CH}_k(X) & \xrightarrow{g^*} & \mathrm{CH}_{k+c}(X') \end{array}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.14.4 we have $(f')^*g^*[Z] = [Z' \times_{X'} Y']_{k+c+r}$. Conversely, we have $f^*[Z] = [Z \times_X Y]_{k+r}$ by Definition 42.14.1. By Lemma 42.67.3 we have $g^*f^*[Z] = [(Z \times_X Y)']_{k+r+c}$. Since $(Z \times_X Y)' = Z' \times_{X'} Y'$ by associativity of fibre product we conclude. □

0FVL **Lemma 42.67.6.** *In Situation 42.67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is proper. Then $f' : Y' \to X'$ is proper and the diagram*

$$\begin{array}{ccc} Z_k(Y) & \xrightarrow[g^*]{} & Z_{k+c}(Y') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ Z_k(X) & \xrightarrow{g^*} & Z_{k+c}(X') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CH}_k(Y) & \xrightarrow[g^*]{} & \mathrm{CH}_{k+c}(Y') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ \mathrm{CH}_k(X) & \xrightarrow{g^*} & \mathrm{CH}_{k+c}(X') \end{array}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.12.4 we have $(f')_*g^*[Z] = [f'_*\mathcal{O}_{Z'}]_{k+c}$. By the same lemma we have $f_*[Z] = [f_*\mathcal{O}_Z]_k$. By Lemma 42.67.3 we have $g^*f_*[Z] = [(X' \to X)^*f_*\mathcal{O}_Z]_{k+r}$. Thus it suffices to show that

$$(X' \to X)^*f_*\mathcal{O}_Z \cong f'_*\mathcal{O}_{Z'}$$

as coherent modules on X'. As $X' \to X$ is flat and as $\mathcal{O}_{Z'} = (Y' \to Y)^*\mathcal{O}_Z$, this follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

0FVM **Lemma 42.67.7.** *In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with base change $\mathcal{L}'$ on X'. Then the diagram*

$$\begin{array}{ccc} \mathrm{CH}_k(X) & \xrightarrow[g^*]{} & \mathrm{CH}_{k+c}(X') \\ {\scriptstyle c_1(\mathcal{L})\cap -}\downarrow & & \downarrow{\scriptstyle c_1(\mathcal{L}')\cap -} \\ \mathrm{CH}_{k-1}(X) & \xrightarrow{g^*} & \mathrm{CH}_{k+c-1}(X') \end{array}$$

of chow groups commutes.

Proof. Let $p : L \to X$ be the line bundle associated to $\mathcal{L}$ with zero section $o : X \to L$. For $\alpha \in CH_k(X)$ we know that $\beta = c_1(\mathcal{L})\cap\alpha$ is the unique element of $\mathrm{CH}_{k-1}(X)$ such that $o_*\alpha = -p^*\beta$, see Lemmas 42.32.2 and 42.32.4. The same characterization holds after pullback. Hence the lemma follows from Lemmas 42.67.5 and 42.67.6. □

0FVN **Lemma 42.67.8.** *In Situation 42.67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank r with base change $\mathcal{E}'$ on X'. Then the diagram*

$$\begin{array}{ccc} \mathrm{CH}_k(X) & \xrightarrow{g^*} & \mathrm{CH}_{k+c}(X') \\ \downarrow{\scriptstyle c_i(\mathcal{E})\cap -} & & \downarrow{\scriptstyle c_i(\mathcal{E}')\cap -} \\ \mathrm{CH}_{k-i}(X) & \xrightarrow{g^*} & \mathrm{CH}_{k+c-i}(X') \end{array}$$

of chow groups commutes for all i.

Proof. Set $P = \mathbf{P}(\mathcal{E})$. The base change P' of P is equal to $\mathbf{P}(\mathcal{E}')$. Since we already know that flat pullback and cupping with c_1 of an invertible module commute with base change (Lemmas 42.67.5 and 42.67.7) the lemma follows from the characterization of capping with $c_i(\mathcal{E})$ given in Lemma 42.38.2. □

0FVP **Lemma 42.67.9.** *Let (S, δ), (S', δ'), (S'', δ'') be as in Situation 42.7.1. Let $g : S' \to S$ and $g' : S'' \to S'$ be flat morphisms of schemes and let $c, c' \in \mathbf{Z}$ be integers such that $S, \delta, S', \delta', g, c$ and S', δ', S'', g', c' are as in Situation 42.67.1. Let $X \to S$ be locally of finite type and denote $X' \to S'$ and $X'' \to S''$ the base changes by $S' \to S$ and $S'' \to S$. Then*

1. *$S, \delta, S'', \delta'', g \circ g', c + c'$ is as in Situation 42.67.1,*
2. *the maps $g^* : Z_k(X) \to Z_{k+c}(X')$ and $(g')^* : Z_{k+c}(X') \to Z_{k+c+c'}(X'')$ of compose to give the map $(g \circ g')^* : Z_k(X) \to Z_{k+c+c'}(X'')$, and*
3. *the maps $g^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c}(X')$ and $(g')^* : \mathrm{CH}_{k+c}(X') \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 42.67.4 compose to give the map $(g \circ g')^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 42.67.4.*

Proof. Let $s \in S$ and let $s'' \in S''$ be a generic point of an irreducible component of $(g \circ g')^{-1}(\{s\})$. Set $s' = g'(s'')$. Clearly, s'' is a generic point of an irreducible component of $(g')^{-1}(\{s'\})$. Moreover, since g' is flat and hence generalizations lift along g' (Morphisms, Lemma 29.25.8) we see that also s' is a generic point of an irreducible component of $g^{-1}(\{s\})$. Thus by assumption $\delta'(s') = \delta(s) + c$ and $\delta''(s'') = \delta'(s') + c'$. We conclude $\delta''(s'') = \delta(s) + c + c'$ and the first part of the statement is true.

For the second part, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Denote $Z' \subset X'$ and $Z'' \subset X''$ the base changes. By definition we have $g^*[Z] = [Z']_{k+c}$. By Lemma 42.67.3 we have $(g')^*[Z']_{k+c} = [Z'']_{k+c+c'}$. This proves the final statement. □

0FVQ **Lemma 42.67.10.** *In Situation 42.67.1 assume $c = 0$ and assume that $S' = \lim_{i \in I} S_i$ is a filtered limit of schemes S_i affine over S such that*

1. *with δ_i equal to $S_i \to S \xrightarrow{\delta} \mathbf{Z}$ the pair (S_i, δ_i) is as in Situation 42.7.1,*
2. *$S_i, \delta_i, S, \delta, S \to S_i, c = 0$ is as in Situation 42.67.1,*
3. *$S_i, \delta_i, S_{i'}, \delta_{i'}, S_i \to S_{i'}, c = 0$ for $i \geq i'$ is as in Situation 42.67.1.*

Then for a quasi-compact scheme X of finite type over S with base change X' and X_i by $S' \to S$ and $S_i \to S$ we have $Z_k(X') = \operatorname{colim} Z_k(X_i)$ and $\mathrm{CH}_k(X') = \operatorname{colim} \mathrm{CH}_k(X_i)$.

Proof. By the result of Lemma 42.67.9 we obtain a system of cycle groups $Z_k(X_i)$ and a system of chow groups $\mathrm{CH}_k(X_i)$ as well as maps $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ and

$\operatorname{colim} \mathrm{CH}_i(X_i) \to \mathrm{CH}_k(X')$. We may replace S by a quasi-compact open through which $X \to S$ factors, hence we may and do assume all the schemes occurring in this proof are Noetherian (and hence quasi-compact and quasi-separated).

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is surjective. Namely, let $Z' \subset X'$ be an integral closed subscheme of δ'-dimension k. By Limits, Lemma 32.10.1 we can find an i and a morphism $Z_i \to X_i$ of finite presentation whose base change is Z'. After increasing i we may assume Z_i is a closed subscheme of X_i, see Limits, Lemma 32.8.5. Then $Z' \to X_i$ factors through Z_i and we may replace Z_i by the scheme theoretic image of $Z' \to X_i$. In this way we see that we may assume Z_i is an integral closed subscheme of X_i. By Lemma 42.67.2 we conclude that $\dim_{\delta_i}(Z_i) = \dim_{\delta'}(Z') = k$. Thus $Z_k(X_i) \to Z_k(X')$ maps $[Z_i]$ to $[Z']$ and we conclude surjectivity holds.

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is injective. Let $\alpha_i = \sum n_j[Z_j] \in Z_k(X_i)$ be a cycle whose image in $Z_k(X')$ is zero. We may and do assume $Z_j \neq Z_{j'}$ if $j \neq j'$ and $n_j \neq 0$ for all j. Denote $Z'_j \subset X'$ the base change of Z_j. By Lemma 42.67.2 each irreducible component of Z'_j has δ'-dimension k. Moreover, as Z_j is irreducible and $Z'_j \to Z_j$ is flat (as the base change of $S' \to S$) we see that $Z'_j \to Z_j$ is dominant. Hence if Z'_j is nonempty, then some irreducible component, say Z', of Z'_j dominates Z_j. It follows that Z' cannot be an irreducible component of $Z'_{j'}$ for $j' \neq j$. Hence if Z'_j is nonempty, then we see that $(S' \to S_i)^*\alpha_i = \sum [Z'_j]_r$ is nonzero (as the coefficient of Z' would be nonzero). Thus we see that $Z'_j = \emptyset$ for all j. However, this means that the base change of Z_j by some transition map $S_{i'} \to S_i$ is empty by Limits, Lemma 32.4.3. Thus α_i dies in the colimit as desired.

The surjectivity of $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ implies that $\operatorname{colim} \mathrm{CH}_k(X_i) \to \mathrm{CH}_k(X')$ is surjective. To finish the proof we show that this map is injective. Let $\alpha_i \in \mathrm{CH}_k(X_i)$ be a cycle whose image $\alpha' \in \mathrm{CH}_k(X')$ is zero. Then there exist integral closed subschemes $W'_l \subset X'$, $l = 1, \ldots, r$ of δ"-dimension $k+1$ and nonzero rational functions f'_l on W'_l such that $\alpha' = \sum_{l=1,\ldots,r} \operatorname{div}_{W'_l}(f'_l)$. Arguing as above we can find an i and integral closed subschemes $W_{i,l} \subset X_i$ of δ_i-dimension $k+1$ whose base change is W'_l. After increasing i we may assume we have rational functions $f_{i,l}$ on $W_{i,l}$. Namely, we may think of f'_l as a section of the structure sheaf over a nonempty open $U'_l \subset W'_l$, we can descend these opens by Limits, Lemma 32.4.11 and after increasing i we may descend f'_l by Limits, Lemma 32.4.7. We claim that

$$\alpha_i = \sum\nolimits_{l=1,\ldots,r} \operatorname{div}_{W_{i,l}}(f_{i,l})$$

after possibly increasing i.

To prove the claim, let $Z'_{l,j} \subset W'_l$ be a finite collection of integral closed subschemes of δ'-dimension k such that f'_l is an invertible regular function outside $\bigcup_j Y'_{l,j}$. After increasing i (by the arguments above) we may assume there exist integral closed subschemes $Z_{i,l,j} \subset W_i$ of δ_i-dimension k such that $f_{i,l}$ is an invertible regular function outside $\bigcup_j Z_{i,l,j}$. Then we may write

$$\operatorname{div}_{W'_l}(f'_l) = \sum n_{l,j}[Z'_{l,j}]$$

and

$$\operatorname{div}_{W_{i,l}}(f_{i,l}) = \sum n_{i,l,j}[Z_{i,l,j}]$$

To prove the claim it suffices to show that $n_{l,i} = n_{i,l,j}$. Namely, this will imply that $\beta_i = \alpha_i - \sum_{l=1,\ldots,r} \mathrm{div}_{W_{i,l}}(f_{i,l})$ is a cycle on X_i whose pullback to X' is zero as a cycle! It follows that β_i pulls back to zero as a cycle on $X_{i'}$ for some $i' \geq i$ by an easy argument we omit.

To prove the equality $n_{l,i} = n_{i,l,j}$ we choose a generic point $\xi' \in Z'_{l,j}$ and we denote $\xi \in Z_{i,l,j}$ the image which is a generic point also. Then the local ring map

$$\mathcal{O}_{W_{i,l},\xi} \longrightarrow \mathcal{O}_{W'_l,\xi'}$$

is flat as $W'_l \to W_{i,l}$ is the base change of the flat morphism $S' \to S_i$. We also have $\mathfrak{m}_\xi \mathcal{O}_{W'_l,\xi'} = \mathfrak{m}_{\xi'}$ because $Z_{i,l,j}$ pulls back to $Z'_{l,j}$! Thus the equality of

$$n_{l,j} = \mathrm{ord}_{Z'_{l,j}}(f'_l) = \mathrm{ord}_{\mathcal{O}_{W'_l,\xi'}}(f'_l) \quad \text{and} \quad n_{i,l,j} = \mathrm{ord}_{Z_{i,l,j}}(f_{i,l}) = \mathrm{ord}_{\mathcal{O}_{W_{i,l},\xi}}(f_{i,l})$$

follows from Algebra, Lemma 10.52.13 and the construction of ord in Algebra, Section 10.121. □

42.68. Appendix A: Alternative approach to key lemma

0EAZ In this appendix we first define determinants $\det_\kappa(M)$ of finite length modules M over local rings $(R, \mathfrak{m}, \kappa)$, see Subsection 42.68.1. The determinant $\det_\kappa(M)$ is a 1-dimensional κ-vector space. We use this in Subsection 42.68.12 to define the determinant $\det_\kappa(M, \varphi, \psi) \in \kappa^*$ of an exact $(2,1)$-periodic complex (M, φ, ψ) with M of finite length. In Subsection 42.68.26 we use these determinants to construct a tame symbol $d_R(a,b) = \det_\kappa(R/ab, a, b)$ for a pair of nonzerodivisors $a, b \in R$ when R is Noetherian of dimension 1. Although there is no doubt that

$$d_R(a,b) = \partial_R(a,b)$$

where ∂_R is as in Section 42.5, we have not (yet) added the verification. The advantage of the tame symbol as constructed in this appendix is that it extends (for example) to pairs of injective endomorphisms φ, ψ of a finite R-module M of dimension 1 such that $\varphi(\psi(M)) = \psi(\varphi(M))$. In Subsection 42.68.40 we relate Herbrand quotients and determinants. An easy to state version of the main result (Proposition 42.68.43) is the formula

$$-e_R(M, \varphi, \psi) = \mathrm{ord}_R(\det_K(M_K, \varphi, \psi))$$

when (M, φ, ψ) is a $(2,1)$-periodic complex whose Herbrand quotient e_R (Definition 42.2.2) is defined over a 1-dimensonal Noetherian local domain R with fraction field K. We use this proposition to give an alternative proof of the key lemma (Lemma 42.6.3) for the tame symbol constructed in this appendix, see Lemma 42.68.46.

02P5 **42.68.1. Determinants of finite length modules.** The material in this section is related to the material in the paper [**KM76**] and to the material in the thesis [**Ros09**].

Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\varphi : M \to M$ be an R-linear endomorphism of a finite length R-module M. In More on Algebra, Section 15.121 we have already defined the determinant $\det_\kappa(\varphi)$ (and the trace and the characteristic polynomial) of φ relative to κ. In this section, we will construct a canonical 1-dimensional κ-vector space $\det_\kappa(M)$ such that $\det_\kappa(\varphi : M \to M) : \det_\kappa(M) \to \det_\kappa(M)$ is equal to multiplication by $\det_\kappa(\varphi)$. If M is annihilated by $\mathfrak{m}$, then M can be viewed as a finite dimension κ-vector space and then we have $\det_\kappa(M) = \wedge^n_\kappa(M)$ where $n = \dim_\kappa(M)$. Our construction will generalize this to all finite length modules

over R and if R contains its residue field, then the determinant $\det_\kappa(M)$ will be given by the usual determinant in a suitable sense, see Remark 42.68.9.

02P6 **Definition 42.68.2.** Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

(1) Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r \rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of M generated by $x_1, \ldots, x_r$.
(2) We will say an l-tuple of elements $(e_1, \ldots, e_l)$ of M is *admissible* if $\mathfrak{m}e_i \subset \langle e_1, \ldots, e_{i-1} \rangle$ for $i = 1, \ldots, l$.
(3) A *symbol* $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
(4) An *admissible relation* between symbols is one of the following:
   (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$, then $[e_1, \ldots, e_l] = 0$,
   (b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1} \rangle$, then
   $$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$
   where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and
   (c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2} \rangle$ then
   $$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$
(5) We define the *determinant of the finite length R-module M* to be
$$\det_\kappa(M) = \left\{ \frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}} \right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

02P7 **Lemma 42.68.3.** *With notations as above we have* $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that
$$\text{length}_R(\langle f_1, \ldots, f_i \rangle) = i$$
for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l]$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l \rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1} \rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l \rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i \rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1} \rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak{m}f_1 = 0$. Hence by applying the third admissible relation $i - 1$ times we see that
$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$

Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l \rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$

for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l \rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i \rangle$ and we go through the same process as above to see that

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$

Continuing in this vein we obtain the desired result. □

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over κ.

02P8 **Lemma 42.68.4.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak{m}$. Let $l = \dim_\kappa(M)$. Then the map*

$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$

is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 42.68.3 the left hand side has dimension at most one we see that the map is an isomorphism. □

02P9 **Lemma 42.68.5.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l \rangle = M$.*

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 42.68.3 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 42.68.4. Choose a nonzero element $f \in M$ with $\mathfrak{m}f = 0$. Set $\overline{M} = M/\langle f \rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \notin \langle e_1, \ldots, e_l \rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l \rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i \rangle$. We may write $e_i = \lambda f + x$ for some unit $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1} \rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = (-1)^i \overline{\lambda} [\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$

with i minimal. Then $i \neq a$ and $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}}_i, \ldots, \overline{e}_{a-1} \rangle$ if $i < a$ or $\overline{e}_a \in \langle \overline{e}_1, \ldots, \overline{e}_{a-1} \rangle$ if $i > a$. Thus the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1} \rangle$. If $i < a$ then the desired equality is

$$(-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda \overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$(-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda \overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = (-1)^i \overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m} e_a \subset \langle e_1, \ldots, e_{a-2} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1} \rangle$. We distinguish 4 cases:

Case 1: $i < a - 1$. The desired equality is

$$\begin{aligned} &(-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] \\ &= (-1)^{i+1} \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l] \end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 2: $i > a$. The desired equality is

$$\begin{aligned} &(-1)^i \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] \\ &= (-1)^{i+1} \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] \end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 3: $i = a$. We write $e_a = \lambda f + \mu e_{a-1} + y$ with $y \in \langle e_1, \ldots, e_{a-2} \rangle$. Then

$$\psi([e_1, \ldots, e_l]) = (-1)^a \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. If $\overline{\mu}$ is nonzero, then we have $e_{a-1} = -\mu^{-1}\lambda f + \mu^{-1} e_a - \mu^{-1} y$ and we obtain

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a \overline{\mu^{-1}\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. Since in $\overline{M}$ we have $\overline{e}_a = \mu \overline{e}_{a-1} + \overline{y}$ we see the two outcomes are equal by relation (a) for $\det_\kappa(\overline{M})$. If on the other hand $\overline{\mu}$ is zero, then we can write $e_a = \lambda f + y$ with $y \in \langle e_1, \ldots, e_{a-2} \rangle$ and we have

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is equal to $\psi([e_1, \ldots, e_l])$.

Case 4: $i = a - 1$. Here we have

$$\psi([e_1, \ldots, e_l]) = (-1)^{a-1} \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

by definition. If $f \notin \langle e_1, \ldots, e_{a-2}, e_a \rangle$ then

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^{a+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

Since $(-1)^{a-1} = (-1)^{a+1}$ the two expressions are the same. Finally, assume $f \in \langle e_1, \ldots, e_{a-2}, e_a \rangle$. In this case we see that $e_{a-1} = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{a-2} \rangle$ and $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2} \rangle$ for units $\lambda, \mu \in R$. We conclude that both $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a \rangle$. In this case a relation of type (a) applies to both $[e_1, \ldots, e_l]$ and $[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]$ and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

05M7 (42.68.5.1)
$$\left\{\begin{array}{c}\text{finite length } R\text{-modules} \\ \text{with isomorphisms}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{1-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms}\end{array}\right\}$$

This is typical for a "determinant functor" (see [**Knu02**]), as is the following additivity property.

02PA **Lemma 42.68.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence*

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K \to L \to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K' & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K' \to L' \to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \to L \to M},$$

(2) *for every commutative square of finite length R-modules with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D & \to & E & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G & \to & H & \to & I & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the following diagram is commutative

$$\begin{array}{ccc}
\det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) & \xrightarrow{\gamma_{A\to B\to C} \otimes \gamma_{G\to H\to I}} & \det_\kappa(B) \otimes \det_\kappa(H) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \gamma_{B\to E\to H}} \\
 & & \det_\kappa(E) \\
 & & \uparrow{\scriptstyle \gamma_{D\to E\to F}} \\
\det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) & \xrightarrow{\gamma_{A\to D\to G} \otimes \gamma_{C\to F\to I}} & \det_\kappa(D) \otimes \det_\kappa(F)
\end{array}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = length_R(C)$ and $g = length_R(G)$, and

(3) *the map $\gamma_{K\to L\to M}$ agrees with the usual isomorphism if $0 \to K \to L \to M \to 0$ is actually a short exact sequence of κ-vector spaces.*

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K\to L\to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not guaranteed that $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k \rangle$ (see proof of Lemma 42.68.3), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 42.68.5 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m} \rangle$. Hence it is clear that the map $\gamma_{K\to L\to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \; & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \; & \gamma_{K' \to L' \to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \\ = \; & (-1)^{cg} [\alpha_1, \ldots, \alpha_a, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \end{aligned}$$

(for suitable lifts $\tilde{x}$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde{\zeta}_1$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_1 \subset A$), then move $\tilde{\zeta}_2$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. □

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 12.19.1)

$$0 = F^m \subset F^{m-1} \subset \ldots \subset F^{n+1} \subset F^n = M$$

then there is a well defined and canonical isomorphism

$$\gamma_{(M,F)} : \det_\kappa(F^{m-1}/F^m) \otimes_\kappa \ldots \otimes_k \det_\kappa(F^n/F^{n+1}) \longrightarrow \det_\kappa(M)$$

To construct it we use isomorphisms of Lemma 42.68.6 coming from the short exact sequences $0 \to F^{i-1}/F^i \to M/F^i \to M/F^{i-1} \to 0$. Part (2) of Lemma 42.68.6 with $G = 0$ shows we obtain the same isomorphism if we use the short exact sequences $0 \to F^i \to F^{i-1} \to F^{i-1}/F^i \to 0$.

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

02PB **Lemma 42.68.7.** *Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor*

$$\det_\kappa : \left\{ \begin{array}{c} \textit{finite length } R\textit{-modules} \\ \textit{with isomorphisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} 1\textit{-dimensional } \kappa\textit{-vector spaces} \\ \textit{with isomorphisms} \end{array} \right\}$$

endowed with the maps $\gamma_{K \to L \to M}$ is characterized by the following properties

1. *its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 42.68.4), and*
2. *(1), (2) and (3) of Lemma 42.68.6 hold.*

Proof. Omitted. □

02PC **Lemma 42.68.8.** *Let $(R', \mathfrak{m}') \to (R, \mathfrak{m})$ be a local ring homomorphism which induces an isomorphism on residue fields κ. Then for every finite length R-module the restriction $M_{R'}$ is a finite length R'-module and there is a canonical isomorphism*

$$\det_{R,\kappa}(M) \longrightarrow \det_{R',\kappa}(M_{R'})$$

This isomorphism is functorial in M and compatible with the isomorphisms $\gamma_{K \to L \to M}$ of Lemma 42.68.6 defined for $\det_{R,\kappa}$ and $\det_{R',\kappa}$.

Proof. If the length of M as an R-module is l, then the length of M as an R'-module (i.e., $M_{R'}$) is l as well, see Algebra, Lemma 10.52.12. Note that an admissible sequence $x_1, \ldots, x_l$ of M over R is an admissible sequence of M over R' as $\mathfrak{m}'$ maps into $\mathfrak{m}$. The isomorphism is obtained by mapping the symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$. It is immediate to verify that this is functorial for isomorphisms and compatible with the isomorphisms γ of Lemma 42.68.6. □

0BDQ **Remark 42.68.9.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring and assume either the characteristic of κ is zero or it is p and $pR = 0$. Let $M_1, \ldots, M_n$ be finite length R-modules. We will show below that there exists an ideal $I \subset \mathfrak{m}$ annihilating M_i for $i = 1, \ldots, n$ and a section $\sigma : \kappa \to R/I$ of the canonical surjection $R/I \to \kappa$. The restriction $M_{i,\kappa}$ of M_i via σ is a κ-vector space of dimension $l_i = \text{length}_R(M_i)$ and using Lemma 42.68.8 we see that

$$\det_\kappa(M_i) = \wedge_\kappa^{l_i}(M_{i,\kappa})$$

These isomorphisms are compatible with the isomorphisms $\gamma_{K\to M\to L}$ of Lemma 42.68.6 for short exact sequences of finite length R-modules annihilated by I. The conclusion is that verifying a property of $\det_\kappa$ often reduces to verifying corresponding properties of the usual determinant on the category finite dimensional vector spaces.

For I we can take the annihilator (Algebra, Definition 10.40.3) of the module $M = \bigoplus M_i$. In this case we see that $R/I \subset \text{End}_R(M)$ hence has finite length. Thus R/I is an Artinian local ring with residue field κ. Since an Artinian local ring is complete we see that R/I has a coefficient ring by the Cohen structure theorem (Algebra, Theorem 10.160.8) which is a field by our assumption on R.

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 42.68.11 for a random example.

02PD **Lemma 42.68.10.** *Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then*

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = \text{length}_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \text{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 42.68.6 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. □

02PE **Example 42.68.11.** Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

0BDR **42.68.12. Periodic complexes and determinants.** Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Subsection 42.68.1. Let us abbreviate $K_\varphi = \mathrm{Ker}(\varphi)$, $I_\varphi = \mathrm{Im}(\varphi)$, $K_\psi = \mathrm{Ker}(\psi)$, and $I_\psi = \mathrm{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 42.68.6. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

02PJ **Definition 42.68.13.** Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The *determinant of* (M, φ, ψ) is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by $(-1)^{\mathrm{length}_R(I_\varphi)\mathrm{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

02PK **Remark 42.68.14.** Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \mathrm{Im}(\varphi)$, $I_\psi = \mathrm{Im}(\psi)$ as above. Assume that $\mathrm{length}_R(I_\varphi) = a$ and $\mathrm{length}_R(I_\psi) = b$, so that $a + b = \mathrm{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$. Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

02PL **Lemma 42.68.15.** *Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. □

02PM **Lemma 42.68.16.** *Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, φ) is exact. Then $\mathrm{length}_R(M) = 2\mathrm{length}_R(\mathrm{Im}(\varphi))$ and*

$$\det_\kappa(M, \varphi, \varphi) = (-1)^{\mathrm{length}_R(\mathrm{Im}(\varphi))} = (-1)^{\frac{1}{2}\mathrm{length}_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. □

02PN **Lemma 42.68.17.** *Let R be a local ring with residue field κ. Let M be a finite length R-module.*

(1) *if $\varphi : M \to M$ is an isomorphism then $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.*
(2) *if $\psi : M \to M$ is an isomorphism then $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.*

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 42.68.13, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). □

02PO **Lemma 42.68.18.** *Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{\varphi,1} & \longrightarrow & K_{\varphi,2} & \longrightarrow & K_{\varphi,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{\varphi,1} & \longrightarrow & I_{\varphi,2} & \longrightarrow & I_{\varphi,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition $\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the

composition of the left vertical maps in the following diagram

$$\begin{array}{ccc}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) & \xrightarrow{\gamma} & \det_\kappa(M_2) \\
\downarrow{\scriptstyle \gamma^{-1}\otimes\gamma^{-1}} & & \downarrow{\scriptstyle \gamma^{-1}} \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) & \xrightarrow{\gamma\otimes\gamma} & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \\
\downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) & \xrightarrow{\gamma\otimes\gamma} & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \\
\downarrow{\scriptstyle \gamma\otimes\gamma} & & \downarrow{\scriptstyle \gamma} \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) & \xrightarrow{\gamma} & \det_\kappa(M_2)
\end{array}$$

The top and bottom squares are commutative up to sign by applying Lemma 42.68.6 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\text{length}(I_{\varphi,1})\text{length}(K_{\varphi,3})+\text{length}(I_{\psi,1})\text{length}(K_{\psi,3})} \\
&= (-1)^{\text{length}(I_{\varphi,1})\text{length}(I_{\psi,3})+\text{length}(I_{\psi,1})\text{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. □

02PP **Example 42.68.19.** Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

02PQ **Example 42.68.20.** Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 42.68.19 shows that

$$\begin{aligned}
\det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \bmod p \\
&= (-1)^{\text{ord}_p(\alpha)\text{ord}_p(\beta)} \frac{\alpha^{\text{ord}_p(\beta)}}{\beta^{\text{ord}_p(\alpha)}} \bmod p
\end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 42.68.37 for a more general case (and a proof).

02PR **Example 42.68.21.** Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a + b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \text{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \text{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 42.68.18.

02PS **Example 42.68.22.** Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \mathrm{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 42.68.19.

02PT **Example 42.68.23.** Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \mathrm{Im}(\varphi)$, and $K_\varphi = \mathrm{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

02PU **Lemma 42.68.24.** *Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that*

(1) *$I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,*
(2) *$K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.*

Then

(1) *The triple $(M, \alpha, \beta\gamma)$ is an exact $(2,1)$-periodic complex.*
(2) *The triple $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex.*
(3) *The triple $(M/K_\beta, \alpha, \gamma)$ is an exact $(2,1)$-periodic complex.*
(4) *We have*

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 42.68.14 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned}&[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r] \\ &= (-1)^{r(s+t)} D[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]\end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. □

02PV **Lemma 42.68.25.** *Let R be a local ring with residue field κ. Let $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a morphism of $(2,1)$-periodic complexes over R. Assume*

(1) *M, M' have finite length,*
(2) *(M, φ, ψ), (M', φ', ψ') are exact,*
(3) *the maps φ, ψ induce the zero map on $K = \text{Ker}(\alpha)$, and*
(4) *the maps φ, ψ induce the zero map on $Q = \text{Coker}(\alpha)$.*

Denote $N = \alpha(M) \subset M'$. We obtain two short exact sequences of $(2,1)$-periodic complexes

$$0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0$$
$$0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. Then

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, then $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \mathrm{length}_R(K)$, $a = \mathrm{length}_R(I_\varphi)$, $b = \mathrm{length}_R(I_\psi)$. Then we see that $\mathrm{length}_R(M) = a + b$, and $\mathrm{length}_R(N) = a + b - k$, $\mathrm{length}_R(Q) = k$ and $\mathrm{length}_R(M') = a+b$. The exact sequences below will show that also $\mathrm{length}_R(I_{\varphi'}) = a$ and $\mathrm{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by Greek letters and elements by Roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z_i' \in M$ such that $\varphi z_i' = z_i$,
(3) elements $z_i'' \in M$ such that $\psi z_i'' = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and

(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 42.68.14 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z_1'', \ldots, z_k'', \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \; & (-1)^{ab} D [z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z_1', \ldots, z_k', \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 42.68.14 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b, w_1, \ldots, w_k] \\ = \; & (-1)^{ab} D' [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \; & (-1)^{ab} D [\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \; & (-1)^{ab} D' [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z_1'', \ldots, z_k''$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0 [\alpha z_1'', \ldots, \alpha z_k'']$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1 [\alpha z_1', \ldots, \alpha z_k']$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i [z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0 [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \; & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1 [\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \; & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. □

02PW **42.68.26. Symbols.** The correct generality for this construction is perhaps the situation of the following lemma.

02PX **Lemma 42.68.27.** *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_A(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2,1)$-periodic complex.*

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_\mathfrak{q}$ is a finite length $A_\mathfrak{q}$-module, see Algebra, Lemma 10.62.3. Hence both φ and ψ induce isomorphisms $M_\mathfrak{q} \to M_\mathfrak{q}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. □

02PY **Lemma 42.68.28.** *Let A be a Noetherian local ring. Let $a, b \in A$.*

(1) *If M is a finite A-module of dimension 1 such that a, b are nonzerodivisors on M, then $\text{length}_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2,1)$-periodic exact complex.*
(2) *If a, b are nonzerodivisors and $\dim(A) = 1$ then $\text{length}_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2,1)$-periodic exact complex.*

In particular, in these cases $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 42.68.27. □

02PZ **Definition 42.68.29.** Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzerodivisors on M. We define the *symbol associated to* M, a, b to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

02Q0 **Lemma 42.68.30.** *Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(\text{Supp}(M)) = 1$. Assume a, b, c are nonzerodivisors on M. Then*

$$d_M(a, bc) = d_M(a, b) d_M(a, c)$$

and $d_M(a, b) d_M(b, a) = 1$.

Proof. The first statement follows from Lemma 42.68.24 applied to $M/abcM$ and endomorphisms α, β, γ given by multiplication by a, b, c. The second comes from Lemma 42.68.15. □

02Q1 **Definition 42.68.31.** Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the *tame symbol* of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 42.68.30.

It is clear that we may extend more generally $d_M(-, -)$ to certain rings of fractions of A (even if A is not a domain).

0AY9 **Lemma 42.68.32.** *Let A be a Noetherian local ring and M a finite A-module of dimension 1. Let $a \in A$ be a nonzerodivisor on M. Then $d_M(a, a) = (-1)^{\text{length}_A(M/aM)}$.*

Proof. Immediate from Lemma 42.68.16. □

02Q2 **Lemma 42.68.33.** *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzerodivisor on M, and let $u \in A^*$. Then*

$$d_M(u, b) = u^{length_A(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{ord_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 42.68.17. The lemma then follows from Lemma 42.68.10. □

02Q3 **Lemma 42.68.34.** *Let A be a Noetherian local ring. Let $a, b \in A$. Let*

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzerodivisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b)d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 42.68.18. □

02Q4 **Lemma 42.68.35.** *Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume*

(1) *a, b are nonzerodivisors on both M and M', and*
(2) *$\dim(\mathrm{Ker}(\alpha)), \dim(\mathrm{Coker}(\alpha)) \leq 0$.*

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\mathrm{length}(M/bM) = \mathrm{length}(M'/bM')$ and Lemma 42.68.33. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 42.68.30). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\mathrm{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trivially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 42.68.25. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \mathrm{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. □

02Q5 **Lemma 42.68.36.** *Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(Supp(M)) = 1$. Let $a, b \in A$ nonzerodivisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then*

$$d_M(a,b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a,b)^{length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 10.62.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$. Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 10.62.5. Hence $d_{M_j}(a,b)$ is defined for each j and

$$d_{M_j}(a,b) = \begin{cases} d_{M_{j-1}}(a,b) d_{A/\mathfrak{q}_i}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 42.68.34 in the first instance and Lemma 42.68.35 in the second. Hence the lemma. □

02Q6 **Lemma 42.68.37.** *Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have*

$$d_A(x,y) = (-1)^{ord_A(x) ord_A(y)} \frac{x^{ord_A(y)}}{y^{ord_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 42.68.14 we see that $d_A(x,y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1}t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x,y) [t^{l-1}, \ldots, t^a, u^{-1}t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x,y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. □

02Q8 **Lemma 42.68.38.** *Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b - a$ are nonzerodivisors. Then*

$$d_M(a, b-a) d_M(b,b) = d_M(b, b-a) d_M(a,b)$$

in κ^*.

Proof. By Lemma 42.68.36 it suffices to show the relation when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q} \subset A$ with $\dim(A/\mathfrak{q}) = 1$.

In case $M = A/\mathfrak{q}$ we may replace A by $A/\mathfrak{q}$ and a, b by their images in $A/\mathfrak{q}$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak{q}$ are the same and that for any $A/\mathfrak{q}$-module M the determinant taken over A or over $A/\mathfrak{q}$ are canonically identified. See Lemma 42.68.8.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemmas 42.68.33 and 42.68.32.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 42.4.2. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 42.68.27. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 42.68.35 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b' - a')d^A_{A'}(b', b') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{ccc} A'/(a'(b' - a')) & \cong & A'/(a') \oplus A'/(b' - a') \\ A'/(b'(b' - a')) & \cong & A'/(b') \oplus A'/(b' - a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 42.68.17 and 42.68.18 we conclude

$$\begin{array}{ccc} d^A_{A'}(a', b' - a') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b' - a')}) \\ d^A_{A'}(b', b' - a') & = & \det_\kappa(-a'|_{A'/(b')})^{-1} \det_\kappa(b'|_{A'/(b' - a')}) \\ d^A_{A'}(a', b') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b')}) \end{array}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))} d^A_{A'}(a', b' - a') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 42.68.16 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$ and the lemma is proved. □

The tame symbol is a Steinberg symbol.

02Q9 **Lemma 42.68.39.** *Let A be a Noetherian local domain of dimension 1 with fraction field K. For $x \in K \setminus \{0, 1\}$ we have*

$$d_A(x, 1 - x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1 - x = (b - a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1 - x) = d_A(a, b - a)d_A(a, b)^{-1}d_A(b, b - a)^{-1}d_A(b, b)$$

Thus we have to show that $d_A(a, b - a)d_A(b, b) = d_A(b, b - a)d_A(a, b)$. This is Lemma 42.68.38. □

02QA **42.68.40. Lengths and determinants.** In this section we use the determinant to compare lattices. The key lemma is the following.

02QB **Lemma 42.68.41.** *Let R be a Noetherian local ring. Let $\mathfrak{q} \subset R$ be a prime with $\dim(R/\mathfrak{q}) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties*

(1) $M = \langle x_1, \ldots, x_l \rangle$,
(2) $\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \cong R/\mathfrak{q}$ *for* $i = 1, \ldots, l$,
(3) $N = \langle y_1, \ldots, y_l \rangle$, *and*
(4) $\langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle \cong R/\mathfrak{q}$ *for* $i = 1, \ldots, l$.

Then φ is injective if and only if $\varphi_\mathfrak{q}$ is an isomorphism, and in this case we have

$$\mathit{length}_R(\mathrm{Coker}(\varphi)) = \mathit{ord}_{R/\mathfrak{q}}(f)$$

where $f \in \kappa(\mathfrak{q})$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak{q}$ and y_1 is a basis of N over $R/\mathfrak{q}$ and we have $\varphi(x_1) = fy_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak{q}$. Moreover, $\mathrm{Coker}(\varphi) = R/(f, \mathfrak{q})$ and hence the lemma holds by definition of $\mathrm{ord}_{R/\mathfrak{q}}(f)$ (see Algebra, Definition 10.121.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \notin \mathfrak{q}$. Then the induced maps

$$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \longrightarrow \langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak{q})$. Hence we see that

$$\mathrm{length}_R(\mathrm{Coker}(\varphi)) = \sum \mathrm{ord}_{R/\mathfrak{q}}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l [y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i \rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle x_1 \rangle & \longrightarrow & \langle x_1, \ldots, x_l \rangle & \longrightarrow & \langle x_1, \ldots, x_l \rangle / \langle x_1 \rangle & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle y_1 \rangle & \longrightarrow & \langle y_1, \ldots, y_l \rangle & \longrightarrow & \langle y_1, \ldots, y_l \rangle / \langle y_1 \rangle & \longrightarrow & 0 \end{array}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1 y_1 + \ldots + a_{i-1} y_{i-1} + a y_i$ with $a_j, a \in R$ and $a \notin \mathfrak{q}$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ a x_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ a y_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l \rangle$ and $N' = \langle y'_1, \ldots, y'_l \rangle$. Since $\varphi(x'_1) = a_1 y'_1 + \ldots + a_{i-1} y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l \rangle$ we get a commutative diagram of R-modules and maps

$$\begin{array}{ccc} M' & \xrightarrow{\varphi'} & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak{q}}(a)$ and similarly $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak{q}}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i \ \text{ord}_{R/\mathfrak{q}}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^i f$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots + a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak{q} x_1 = 0$ which implies also $\mathfrak{q} y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

02QC **Lemma 42.68.42.** *Let R be a local Noetherian ring. Let $\mathfrak{q} \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak{q}$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that*

(1) *the support of $M/\langle x_1, \ldots, x_l \rangle$ does not contain $\mathfrak{q}$, and*
(2) *$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Moreover, in this case $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$.

Proof. The condition that $\mathfrak{q}$ is a minimal prime in the support of M implies that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is finite (see Algebra, Lemma 10.62.3). Hence we can find $y_1, \ldots, y_l \in M_\mathfrak{q}$ such that $\langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \notin \mathfrak{q}$ such that $f_i y_i$ is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak{q} = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_j y_i \in \langle y_1, \ldots, y_{i-1} \rangle$ inside the module $M_\mathfrak{q}$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \notin \mathfrak{q}$ such that $f_{ij} g_j z_i \in \langle z_1, \ldots, z_{i-1} \rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11} f_{12} \ldots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \ldots f_{1t} f_{21} f_{22} \ldots f_{2t} z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_\mathfrak{q}$ we still have that $R_\mathfrak{q}x_1+\ldots+R_\mathfrak{q}x_i/R_\mathfrak{q}x_1+\ldots+R_\mathfrak{q}x_{i-1} \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. By construction, $\mathfrak{q}x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak{q}$ such that localizing at $\mathfrak{q}$ gives a q-dimensional vector space over $\kappa(\mathfrak{q})$. Hence it is isomorphic to $R/\mathfrak{q}$. □

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 42.68.44 his/herself.

02QD **Proposition 42.68.43.** *Let R be a local Noetherian ring with residue field κ. Suppose that (M, φ, ψ) is a $(2,1)$-periodic complex over R. Assume*

(1) *M is a finite R-module,*
(2) *the cohomology modules of (M, φ, ψ) are of finite length, and*
(3) $\dim(\mathit{Supp}(M)) = 1$.

Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the minimal primes of the support of M. Then we have[10]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}\left(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i})\right)$$

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, where $\mathfrak{m} \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak{q}_i}$, so $\mathrm{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2,1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t}(M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support contained in $\{\mathfrak{m}\}$. Hence by Lemma 42.2.5 we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak{q}_i} = M_{i,\mathfrak{q}_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\mathrm{ord}_{R/\mathfrak{q}_i}\left(\det_{\kappa(\mathfrak{q}_i)}(M_{i,\mathfrak{q}_i}, \varphi_{i,\mathfrak{q}_i}, \psi_{i,\mathfrak{q}_i})\right)$$

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$, and finite length cohomology modules. The proof in this case follows from Lemma 42.68.41 and careful bookkeeping. Denote $K_\varphi = \mathrm{Ker}(\varphi)$, $I_\varphi = \mathrm{Im}(\varphi)$, $K_\psi = \mathrm{Ker}(\psi)$, and $I_\psi = \mathrm{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \mathrm{length}_{R_\mathfrak{q}}(I_{\varphi,\mathfrak{q}}) = \mathrm{length}_{R_\mathfrak{q}}(K_{\psi,\mathfrak{q}}), \quad b = \mathrm{length}_{R_\mathfrak{q}}(I_{\psi,\mathfrak{q}}) = \mathrm{length}_{R_\mathfrak{q}}(K_{\varphi,\mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak{q}$. Note that $l = \mathrm{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is equal to $l = a + b$.

We are going to use Lemma 42.68.42 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak{m}, \mathfrak{q}\}$. In this case $N_\mathfrak{q}$ has finite length, say $n \in \mathbf{N}$. Let us call a sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2)

[10] Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 42.68.13.

of Lemma 42.68.42 a "good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n\rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak{m}\}$ and hence has finite length (Algebra, Lemma 10.62.3). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$ is a generator, see Lemma 42.68.5.

Having said this we choose good sequences

$$\begin{array}{llllll} x_1, \ldots, x_b & \text{in} & K_\varphi, & t_1, \ldots, t_a & \text{in} & K_\psi, \\ y_1, \ldots, y_a & \text{in} & I_\varphi \cap \langle t_1, \ldots t_a\rangle, & s_1, \ldots, s_b & \text{in} & I_\psi \cap \langle x_1, \ldots, x_b\rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde{y}_i \in M$ of $y_i \in I_\varphi$ and $\tilde{s}_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak{q}\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and it may not be the case that $\mathfrak{q}\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. However, using that $\mathfrak{q}$ is finitely generated (as in the proof of Lemma 42.68.42) we can find a $d \in R$, $d \notin \mathfrak{q}$ such that $\mathfrak{q}d\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and $\mathfrak{q}d\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. Thus after replacing y_i by dy_i, $\tilde{y}_i$ by $d\tilde{y}_i$, s_i by ds_i and $\tilde{s}_i$ by $d\tilde{s}_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_b$ and $t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle \cap \langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak{q}} = K_{\psi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\psi,\mathfrak{q}})$. Similarly, as $I_{\psi,\mathfrak{q}} = K_{\varphi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\varphi,\mathfrak{q}})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned} [y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\ [s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\ [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\ [z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak{q})$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak{q})}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 42.68.13 we have

$$\begin{aligned} [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\ &\mapsto f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\ &\mapsto f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\ &\mapsto f_\varphi h/g[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \\ &= f_\varphi h/f_\psi g[z_1, \ldots, z_l] \end{aligned}$$

This means that $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{aligned}
k_\varphi &= \text{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\
k_\psi &= \text{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\
i_\varphi &= \text{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\
i_\psi &= \text{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\
m_\varphi &= \text{length}_R(M/\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle) \\
m_\psi &= \text{length}_R(M/\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle) \\
\delta_\varphi &= \text{length}_R(\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle\langle z_1, \ldots, z_l\rangle) \\
\delta_\psi &= \text{length}_R(\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle\langle z_1, \ldots, z_l\rangle)
\end{aligned}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 42.68.41.

Next, let us compute the multiplicity of the periodic complex

$$\begin{aligned}
e_R(M, \varphi, \psi) &= \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\
&= \text{length}_R(\langle x_1, \ldots, x_b\rangle/\langle s_1, \ldots, s_b\rangle) + k_\varphi - i_\psi \\
&\quad -\text{length}_R(\langle t_1, \ldots, t_a\rangle/\langle y_1, \ldots, y_a\rangle) - k_\psi + i_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + m_\varphi - m_\psi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + \delta_\psi - \delta_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(f_\psi g/f_\varphi h)
\end{aligned}$$

where we used the key Lemma 42.68.41 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ this proves the proposition. □

In most applications the following lemma suffices.

02QE **Lemma 42.68.44.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that*

(1) $\text{Ker}(\psi)$ *and* $\text{Coker}(\psi)$ *have finite length, and*
(2) $\dim(\mathit{Supp}(M)) \leq 1$.

Write $\mathit{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ and denote $f_i \in \kappa(\mathfrak{q}_i)^*$ the element such that $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \to \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$ is multiplication by f_i. Then we have

$$\mathit{length}_R(\text{Coker}(\psi)) - \mathit{length}_R(\text{Ker}(\psi)) = \sum\nolimits_{i=1,\ldots,t} \mathit{ord}_{R/\mathfrak{q}_i}(f_i).$$

Proof. Recall that $H^0(M, 0, \psi) = \text{Coker}(\psi)$ and $H^1(M, 0, \psi) = \text{Ker}(\psi)$, see remarks above Definition 42.2.2. The lemma follows by combining Proposition 42.68.43 with Lemma 42.68.17.

Alternative proof. Reduce to the case $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ as in the proof of Proposition 42.68.43. Then directly combine Lemmas 42.68.41 and 42.68.42 to prove this specific case of Proposition 42.68.43. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

02QI **42.68.45. Application to the key lemma.** In this section we apply the results above to show the analogue of the key lemma (Lemma 42.6.3) with the tame symbol d_A constructed above. Please see Remark 42.6.4 for the relationship with Milnor K-theory.

02QJ **Lemma 42.68.46** (Key Lemma)**.** *Let A be a 2-dimensional Noetherian local domain with fraction field K. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A^*_{\mathfrak{q}}$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f,g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f,g)) = 0$$

since at any height one prime $\mathfrak{q}$ of A where $f, g \in A^*_{\mathfrak{q}}$ we have $d_{A_{\mathfrak{q}}}(f,g) = 1$ by Lemma 42.68.33.

When A is an excellent ring this is [**Kat86**, Proposition 1].

Proof. Since the tame symbols $d_{A_{\mathfrak{q}}}(f,g)$ are additive (Lemma 42.68.30) and the order functions $\mathrm{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 10.121.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\mathrm{height}(\mathfrak{q})=1} \mathrm{ord}_{A/\mathfrak{q}}(\det_\kappa(A_{\mathfrak{q}}/(ab), a, b)) = 0$$

By Proposition 42.68.43 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. □

42.69. Appendix B: Alternative approaches

0AYD In this appendix we first briefly try to connect the material in the main text with K-theory of coherent sheaves. In particular we describe how cupping with c_1 of an invertible module is related to tensoring by this invertible module, see Lemma 42.69.7. This material is obviously very interesting and deserves a much more detailed and expansive exposition.

02S7 **42.69.1. Rational equivalence and K-groups.** This section is a continuation of Section 42.23. The motivation for the following lemma is Homology, Lemma 12.11.3.

02SB **Lemma 42.69.2.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let*

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (12.11.2.1). Assume that

1. $\dim_\delta(Supp(\mathcal{F})) \leq k + 1$.
2. $\dim_\delta(Supp(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ *for* $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j\in J}$ be the collection of irreducible components of $\mathrm{Supp}(\mathcal{F})$ which have δ-dimension $k+1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 42.10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 42.68.13 for notation. We claim that

$$-[H^0(\mathcal{F},\varphi,\psi)]_k + [H^1(\mathcal{F},\varphi,\psi)]_k = \sum (W_j \to X)_* \mathrm{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F},\varphi,\psi)]_k - [H^1(\mathcal{F},\varphi,\psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_*\mathrm{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(\mathrm{Supp}(M)) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\mathrm{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \mathrm{ord}_{A/\mathfrak{q}_i}(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 42.68.43. □

02SC **Lemma 42.69.3.** *Let (S, δ) be as in Situation 42.7.1. Let X be a scheme locally of finite type over S. The map*

$$\mathrm{CH}_k(X) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$$

from Lemma 42.23.4 induces a bijection from $\mathrm{CH}_k(X)$ onto the image $B_k(X)$ of the map

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X)).$$

Proof. By Lemma 42.23.2 we have $Z_k(X) = K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ compatible with the map of Lemma 42.23.4. Thus, suppose we have an element $[A]-[B]$ of $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., maps to zero in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. We have to show that $[A]-[B]$ corresponds to a cycle rationally equivalent to zero on X. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A]-[B]$ maps to zero in the group $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A}\oplus\mathcal{A}'] - [\mathcal{B}\oplus\mathcal{B}']$ is zero in $K_0(Coh_{k+1}(X))$ (use part (1) of Homology, Lemma 12.11.3). By part (2) of Homology, Lemma 12.11.3 this means there exists a $(2,1)$-periodic complex $(\mathcal{F},\varphi,\psi)$ in the category $Coh_{\leq k+1}(X)$ such that $\mathcal{A}\oplus\mathcal{A}' = H^0(\mathcal{F},\varphi,\psi)$ and $\mathcal{B}\oplus\mathcal{B}' = H^1(\mathcal{F},\varphi,\psi)$. By Lemma 42.69.2 this implies that

$$[\mathcal{A}\oplus\mathcal{A}']_k \sim_{rat} [\mathcal{B}\oplus\mathcal{B}']_k$$

This proves that $[A]-[B]$ maps to a cycle rationally equivalent to zero by the map

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

of Lemma 42.23.2. This is what we had to prove and the proof is complete. □

02SV **42.69.4. Cartier divisors and K-groups.** In this section we describe how the intersection with the first Chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

02QH **Lemma 42.69.5.** *Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume*

(1) $\dim(A) = 1$,
(2) *both a and b are nonzerodivisors in A,*
(3) *A has no embedded primes,*
(4) *M has no embedded associated primes,*
(5) *$Supp(M) = \mathrm{Spec}(A)$.*

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$length_A(M/(a/b)IM) - length_A(M/IM) = \sum\nolimits_i length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) ord_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $\mathrm{Spec}(A)$ we see that $\mathrm{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzerodivisors on M. Note that

$$\begin{aligned}
&\mathrm{length}_A(M/(a/b)IM) \\
&= \mathrm{length}_A(bM/aIM) \\
&= \mathrm{length}_A(M/aIM) - \mathrm{length}_A(M/bM) \\
&= \mathrm{length}_A(M/aM) + \mathrm{length}_A(aM/aIM) - \mathrm{length}_A(M/bM) \\
&= \mathrm{length}_A(M/aM) + \mathrm{length}_A(M/IM) - \mathrm{length}_A(M/bM)
\end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\mathrm{length}_A(M/aM) - \mathrm{length}_A(M/bM).$$

Applying the second formula of Lemma 42.3.2 with $x = a, b$ respectively and using Algebra, Definition 10.121.2 of the ord-functions we get the result. □

02SW **Lemma 42.69.6.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume*

(1) $\dim_\delta(X) \leq k + 1$,
(2) *X has no embedded points,*
(3) *$\mathcal{F}$ has no embedded associated points,*
(4) *the support of $\mathcal{F}$ is X, and*
(5) *the section s is regular meromorphic.*

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 31.23.10. Then we have the following:

(1) *there are short exact sequences*

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\
0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0
\end{array}$$

(2) *the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,*

(3) *the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k+1$, and*
(4) *writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have*

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* div_{\mathcal{L}|_{X_i}}(s_i)$$

in $Z_k(X)$, in particular

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$

in $\mathrm{CH}_k(X)$.

Proof. Recall from Divisors, Lemma 31.24.5 the existence of injective maps $1 : \mathcal{IF} \to \mathcal{F}$ and $s : \mathcal{IF} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 31.23.5 and 31.23.8 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \mathrm{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzerodivisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is

$$\mathrm{length}_A(M/(a/b)IM) - \mathrm{length}_A(M/IM)$$

and the coefficient of $[Z]$ in the right hand side is

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 42.69.5. □

02SX **Lemma 42.69.7.** *Let (S, δ) be as in Situation 42.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\dim_\delta(Supp(\mathcal{F})) \leq k+1$. Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 42.69.3 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to \mathrm{CH}_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 31.4.6. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\mathrm{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 31.4.6 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\mathrm{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 42.10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes\mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 31.4.7. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\mathrm{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 31.25.4 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 42.69.6 there exist short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

such that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 20.54.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. This already shows that $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 42.26.4 and 42.12.4. And this agree with the image of the element under $B_k(X) \to \mathrm{CH}_k(X)$ by definition. Hence the lemma is proved. □

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CHAPTER 43

Intersection Theory

0AZ6

43.1. Introduction

0AZ7 In this chapter we construct the intersection product on the Chow groups modulo rational equivalence on a nonsingular projective variety over an algebraically closed field. Our tools are Serre's Tor formula (see [**Ser65**, Chapter V]), reduction to the diagonal, and the moving lemma.

We first recall cycles and how to construct proper pushforward and flat pullback of cycles. Next, we introduce rational equivalence of cycles which gives us the Chow groups $\mathrm{CH}_*(X)$. Proper pushforward and flat pullback factor through rational equivalence to give operations on Chow groups. This takes up Sections 43.3, 43.4, 43.5, 43.6, 43.7, 43.8, 43.9, 43.10, and 43.11. For proofs we mostly refer to the chapter on Chow homology where these results have been proven in the setting of schemes locally of finite type over a universally catenary Noetherian base, see Chow Homology, Section 42.7 ff.

Since we work on a nonsingular projective X any irreducible component of the intersection $V \cap W$ of two irreducible closed subvarieties has dimension at least $\dim(V)+\dim(W)-\dim(X)$. We say V and W intersect properly if equality holds for every irreducible component Z. In this case we define the intersection multiplicity $e_Z = e(X, V \cdot W, Z)$ by the formula

$$e_Z = \sum\nolimits_i (-1)^i \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

We need to do a little bit of commutative algebra to show that these intersection multiplicities agree with intuition in simple cases, namely, that sometimes

$$e_Z = \mathrm{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{V \cap W, Z},$$

in other words, only Tor_0 contributes. This happens when V and W are Cohen-Macaulay in the generic point of Z or when W is cut out by a regular sequence in $\mathcal{O}_{X,Z}$ which also defines a regular sequence on $\mathcal{O}_{V,Z}$. However, Example 43.14.4 shows that higher tors are necessary in general. Moreover, there is a relationship with the Samuel multiplicity. These matters are discussed in Sections 43.13, 43.14, 43.15, 43.16, and 43.17.

Reduction to the diagonal is the statement that we can intersect V and W by intersecting $V \times W$ with the diagonal in $X \times X$. This innocuous statement, which is clear on the level of scheme theoretic intersections, reduces an intersection of a general pair of closed subschemes, to the case where one of the two is locally cut out by a regular sequence. We use this, following Serre, to obtain positivity of intersection multiplicities. Moreover, reduction to the diagonal leads to additivity of intersection multiplicities, associativity, and a projection formula. This can be found in Sections 43.18, 43.19, 43.20, 43.21, and 43.22.

Finally, we come to the moving lemmas and applications. There are two parts to the moving lemma. The first is that given closed subvarieties

$$Z \subset X \subset \mathbf{P}^N$$

with X nonsingular, we can find a subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that the other components Z_j are "more general" than Z. The second part is that one can move $C \subset \mathbf{P}^N$ over a rational curve to a subvariety in general position with respect to any given list of subvarieties. Combined these results imply that it suffices to define the intersection product of cycles on X which intersect properly which was done above. Of course this only leads to an intersection product on $\mathrm{CH}_*(X)$ if one can show, as we do in the text, that these products pass through rational equivalence. This and some applications are discussed in Sections 43.23, 43.24, 43.25, 43.26, 43.27, and 43.28.

43.2. Conventions

0AZ8 We fix an algebraically closed ground field $\mathbf{C}$ of any characteristic. All schemes and varieties are over $\mathbf{C}$ and all morphisms are over $\mathbf{C}$. A variety X is *nonsingular* if X is a regular scheme (see Properties, Definition 28.9.1). In our case this means that the morphism $X \to \mathrm{Spec}(\mathbf{C})$ is smooth (see Varieties, Lemma 33.12.6).

43.3. Cycles

0AZ9 Let X be a variety. A *closed subvariety* of X is an integral closed subscheme $Z \subset X$. A *k-cycle* on X is a finite formal sum $\sum n_i [Z_i]$ where each Z_i is a closed subvariety of dimension k. Whenever we use the notation $\alpha = \sum n_i [Z_i]$ for a k-cycle we always assume the subvarieties Z_i are pairwise distinct and $n_i \neq 0$ for all i. In this case the *support* of α is the closed subset

$$\mathrm{Supp}(\alpha) = \bigcup Z_i \subset X$$

of dimension k. The group of k-cycles is denoted $Z_k(X)$. See Chow Homology, Section 42.8.

43.4. Cycle associated to closed subscheme

0AZA Suppose that X is a variety and that $Z \subset X$ be a closed subscheme with $\dim(Z) \leq k$. Let Z_i be the irreducible components of Z of dimension k and let n_i be the *multiplicity of Z_i in Z* defined as

$$n_i = \mathrm{length}_{\mathcal{O}_{X,Z_i}} \mathcal{O}_{Z,Z_i}$$

where $\mathcal{O}_{X,Z_i}$, resp. $\mathcal{O}_{Z,Z_i}$ is the local ring of X, resp. Z at the generic point of Z_i. We define the k-cycle associated to Z to be the k-cycle

$$[Z]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 42.9.

43.5. Cycle associated to a coherent sheaf

0AZB Suppose that X is a variety and that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module with $\dim(\text{Supp}(\mathcal{F})) \leq k$. Let Z_i be the irreducible components of $\text{Supp}(\mathcal{F})$ of dimension k and let n_i be the *multiplicity of Z_i in $\mathcal{F}$* defined as

$$n_i = \text{length}_{\mathcal{O}_{X,Z_i}} \mathcal{F}_{\xi_i}$$

where $\mathcal{O}_{X,Z_i}$ is the local ring of X at the generic point ξ_i of Z_i and $\mathcal{F}_{\xi_i}$ is the stalk of $\mathcal{F}$ at this point. We define the k-cycle associated to $\mathcal{F}$ to be the k-cycle

$$[\mathcal{F}]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 42.10. Note that, if $Z \subset X$ is a closed subscheme with $\dim(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$ by definition.

43.6. Proper pushforward

0AZC Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f_*[Z]$ to be 0 if $\dim(f(Z)) < k$ and $d \cdot [f(Z)]$ if $\dim(f(Z)) = k$ where

$$d = [\mathbf{C}(Z) : \mathbf{C}(f(Z))] = \deg(Z/f(Z))$$

is the degree of the dominant morphism $Z \to f(Z)$, see Morphisms, Definition 29.51.8. Let $\alpha = \sum n_i [Z_i]$ be a k-cycle on X. The *pushforward* of α is the sum $f_*\alpha = \sum n_i f_*[Z_i]$ where each $f_*[Z_i]$ is defined as above. This defines a homomorphism

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

See Chow Homology, Section 42.12.

0AZD **Lemma 43.6.1.** *Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\text{Supp}(\mathcal{F})) \leq k$, then $f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k$. In particular, if $Z \subset X$ is a closed subscheme of dimension $\leq k$, then $f_*[Z]_k = [f_*\mathcal{O}_Z]_k$.* See [**Ser65**, Chapter V].

Proof. See Chow Homology, Lemma 42.12.4. □

0B0N **Lemma 43.6.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of varieties. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Special case of Chow Homology, Lemma 42.12.2. □

43.7. Flat pullback

0AZE Suppose that $f : X \to Y$ is a flat morphism of varieties. By Morphisms, Lemma 29.28.2 every fibre of f has dimension $r = \dim(X) - \dim(Y)$[1]. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f^*[Z]$ to be the $(k+r)$-cycle associated to the scheme theoretic inverse image: $f^*[Z] = [f^{-1}(Z)]_{k+r}$. Let $\alpha = \sum n_i [Z_i]$ be a k-cycle on Y. The *pullback* of α is the sum $f^*\alpha = \sum n_i f^*[Z_i]$ where each $f^*[Z_i]$ is defined as above. This defines a homomorphism

$$f^* : Z_k(Y) \longrightarrow Z_{k+r}(X)$$

See Chow Homology, Section 42.14.

[1] Conversely, if $f : X \to Y$ is a dominant morphism of varieties, X is Cohen-Macaulay, Y is nonsingular, and all fibres have the same dimension r, then f is flat. This follows from Algebra, Lemma 10.128.1 and Varieties, Lemma 33.20.4 showing $\dim(X) = \dim(Y) + r$.

0AZF **Lemma 43.7.1.** *Let $f : X \to Y$ be a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Then $f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$ if $\mathcal{F}$ is a coherent sheaf on Y and the dimension of the support of $\mathcal{F}$ is at most k.*

Proof. See Chow Homology, Lemma 42.14.4. □

0B0P **Lemma 43.7.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of varieties. Then $g \circ f$ is flat and $f^* \circ g^* = (g \circ f)^*$ as maps $Z_k(Z) \to Z_{k+\dim(X)-\dim(Z)}(X)$.*

Proof. Special case of Chow Homology, Lemma 42.14.3. □

43.8. Rational Equivalence

0AZG We are going to define rational equivalence in a way which at first glance may seem different from what you are used to, or from what is in [**Ful98**, Chapter I] or Chow Homology, Section 42.19. However, in Section 43.9 we will show that the two notions agree.

Let X be a variety. Let $W \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k+1$. Let a, b be distinct closed points of $\mathbf{P}^1$. Assume that $X \times a$, $X \times b$ and W intersect properly:

$$\dim(W \cap X \times a) \leq k, \quad \dim(W \cap X \times b) \leq k.$$

This is true as soon as $W \to \mathbf{P}^1$ is dominant or if W is contained in a fibre of the projection over a closed point different from a or b (this is an uninteresting case which we will discard). In this situation the scheme theoretic fibre W_a of the morphism $W \to \mathbf{P}^1$ is equal to the scheme theoretic intersection $W \cap X \times a$ in $X \times \mathbf{P}^1$. Identifying $X \times a$ and $X \times b$ with X we may think of the fibres W_a and W_b as closed subschemes of X of dimension $\leq k$[2]. A basic example of a rational equivalence is

$$[W_a]_k \sim_{rat} [W_b]_k$$

The cycles $[W_a]_k$ and $[W_b]_k$ are easy to compute in practice (given W) because they are obtained by proper intersection with a Cartier divisor (we will see this in Section 43.17). Since the automorphism group of $\mathbf{P}^1$ is 2-transitive we may move the pair of closed points a, b to any pair we like. A traditional choice is to choose $a = 0$ and $b = \infty$.

More generally, let $\alpha = \sum n_i[W_i]$ be a $(k+1)$-cycle on $X \times \mathbf{P}^1$. Let a_i, b_i be pairs of distinct closed points of $\mathbf{P}^1$. Assume that $X \times a_i$, $X \times b_i$ and W_i intersect properly, in other words, each W_i, a_i, b_i satisfies the condition discussed above. A *cycle rationally equivalent to zero* is any cycle of the form

$$\sum n_i([W_{i,a_i}]_k - [W_{i,b_i}]_k).$$

This is indeed a k-cycle. The collection of k-cycles rationally equivalent to zero is an additive subgroup of the group of k-cycles. We say two k-cycles are *rationally equivalent*, notation $\alpha \sim_{rat} \alpha'$, if $\alpha - \alpha'$ is a cycle rationally equivalent to zero.

We define

$$\mathrm{CH}_k(X) = Z_k(X)/\sim_{rat}$$

to be the *Chow group of k-cycles on X*. We will see in Lemma 43.9.1 that this agrees with the Chow group as defined in Chow Homology, Definition 42.19.1.

[2]We will sometimes think of W_a as a closed subscheme of $X \times \mathbf{P}^1$ and sometimes as a closed subscheme of X. It should always be clear from context which point of view is taken.

43.9. Rational equivalence and rational functions

0AZH Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k+1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W. For every subvariety $Z \subset W$ of dimension k one can define the order of vanishing $\mathrm{ord}_{W,Z}(f)$ of f at Z. If f is an element of the local ring $\mathcal{O}_{W,Z}$, then one has

$$\mathrm{ord}_{W,Z}(f) = \mathrm{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{W,Z}/f\mathcal{O}_{W,Z}$$

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$ is the local ring of X, resp. W at the generic point of Z. In general one extends the definition by multiplicativity. The *principal divisor associated to* f is

$$\mathrm{div}_W(f) = \sum \mathrm{ord}_{W,Z}(f)[Z]$$

in $Z_k(W)$. Since $W \subset X$ is a closed subvariety we may think of $\mathrm{div}_W(f)$ as a cycle on X. See Chow Homology, Section 42.17.

0AZI **Lemma 43.9.1.** *Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k+1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W. Then $div_W(f)$ is rationally equivalent to zero on X. Conversely, these principal divisors generate the abelian group of cycles rationally equivalent to zero on X.*

Proof. The first assertion follows from Chow Homology, Lemma 42.18.2. More precisely, let $W' \subset X \times \mathbf{P}^1$ be the closure of the graph of f. Then $\mathrm{div}_W(f) = [W'_0]_k - [W'_\infty]$ in $Z_k(W) \subset Z_k(X)$, see part (6) of Chow Homology, Lemma 42.18.2.

For the second, let $W' \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k+1$ which dominates $\mathbf{P}^1$. We will show that $[W'_0]_k - [W'_\infty]_k$ is a principal divisor which will finish the proof. Let $W \subset X$ be the image of W' under the projection to X. Then $W \subset X$ is a closed subvariety and $W' \to W$ is proper and dominant with fibres of dimension 0 or 1. If $\dim(W) = k$, then $W' = W \times \mathbf{P}^1$ and we see that $[W'_0]_k - [W'_\infty]_k = [W] - [W] = 0$. If $\dim(W) = k+1$, then $W' \to W$ is generically finite[3]. Let f denote the projection $W' \to \mathbf{P}^1$ viewed as an element of $\mathbf{C}(W')^*$. Let $g = \mathrm{Nm}(f) \in \mathbf{C}(W)^*$ be the norm. By Chow Homology, Lemma 42.18.1 we have

$$\mathrm{div}_W(g) = \mathrm{pr}_{X,*}\mathrm{div}_{W'}(f)$$

Since $\mathrm{div}_{W'}(f) = [W'_0]_k - [W'_\infty]_k$ by Chow Homology, Lemma 42.18.2 the proof is complete. □

43.10. Proper pushforward and rational equivalence

0AZJ Suppose that $f : X \to Y$ is a proper morphism of varieties. Let $\alpha \sim_{rat} 0$ be a k-cycle on X rationally equivalent to 0. Then the pushforward of α is rationally equivalent to zero: $f_*\alpha \sim_{rat} 0$. See Chapter I of [**Ful98**] or Chow Homology, Lemma 42.20.3.

[3] If $W' \to W$ is birational, then the result follows from Chow Homology, Lemma 42.18.2. Our task is to show that even if $W' \to W$ has degree > 1 the basic rational equivalence $[W'_0]_k \sim_{rat} [W'_\infty]_k$ comes from a principal divisor on a subvariety of X.

Therefore we obtain a commutative diagram

$$\begin{array}{ccc} Z_k(X) & \longrightarrow & \mathrm{CH}_k(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ Z_k(Y) & \longrightarrow & \mathrm{CH}_k(Y) \end{array}$$

of groups of k-cycles.

43.11. Flat pullback and rational equivalence

0AZK Suppose that $f : X \to Y$ is a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Let $\alpha \sim_{rat} 0$ be a k-cycle on Y rationally equivalent to 0. Then the pullback of α is rationally equivalent to zero: $f^*\alpha \sim_{rat} 0$. See Chapter I of [**Ful98**] or Chow Homology, Lemma 42.20.2.

Therefore we obtain a commutative diagram

$$\begin{array}{ccc} Z_{k+r}(X) & \longrightarrow & \mathrm{CH}_{k+r}(X) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ Z_k(Y) & \longrightarrow & \mathrm{CH}_k(Y) \end{array}$$

of groups of k-cycles.

43.12. The short exact sequence for an open

0B5Z Let X be a variety and let $U \subset X$ be an open subvariety. Let $X \setminus U = \bigcup Z_i$ be the decomposition into irreducible components[4]. Then for each $k \geq 0$ there exists a commutative diagram

$$\begin{array}{ccccccc} \bigoplus Z_k(Z_i) & \longrightarrow & Z_k(X) & \longrightarrow & Z_k(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus \mathrm{CH}_k(Z_i) & \longrightarrow & \mathrm{CH}_k(X) & \longrightarrow & \mathrm{CH}_k(U) & \longrightarrow & 0 \end{array}$$

with exact rows. Here the vertical arrows are the canonical quotient maps. The left horizontal arrows are given by proper pushforward along the closed immersions $Z_i \to X$. The right horizontal arrows are given by flat pullback along the open immersion $j : U \to X$. Since we have seen that these maps factor through rational equivalence we obtain the commutativity of the squares. The top row is exact simply because every subvariety of X is either contained in some Z_i or has irreducible intersection with U. The bottom row is exact because every principal divisor $\mathrm{div}_W(f)$ on U is the restriction of a principal divisor on X. More precisely, if $W \subset U$ is a $(k+1)$-dimensional closed subvariety and $f \in \mathbf{C}(W)^*$, then denote $\overline{W}$ the closure of W in X. Then $W \subset \overline{W}$ is an open immersion, so $\mathbf{C}(W) = \mathbf{C}(\overline{W})$ and we may think of f as a nonconstant rational function on $\overline{W}$. Then clearly

$$j^*\mathrm{div}_{\overline{W}}(f) = \mathrm{div}_W(f)$$

in $Z_k(X)$. The exactness of the lower row follows easily from this. For details see Chow Homology, Lemma 42.19.3.

[4]Since in this chapter we only consider Chow groups of varieties, we are prohibited from taking $Z_k(X \setminus U)$ and $\mathrm{CH}_k(X \setminus U)$, hence the approach using the varieties Z_i.

43.13. Proper intersections

0AZL First a few lemmas to get dimension estimates.

0AZM **Lemma 43.13.1.** *Let X and Y be varieties. Then $X \times Y$ is a variety and* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Proof. The scheme $X \times Y = X \times_{\mathrm{Spec}(\mathbf{C})} Y$ is a variety by Varieties, Lemma 33.3.3. The statement on dimension is Varieties, Lemma 33.20.5. □

Recall that a regular immersion $i : X \to Y$ of schemes is a closed immersion whose corresponding sheaf of ideals is locally generated by a regular sequence, see Divisors, Section 31.21. Moreover, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free of rank equal to the length of the regular sequence. Let us say i is a *regular immersion of codimension c* if $\mathcal{C}_{X/Y}$ is locally free of rank c.

More generally, recall (More on Morphisms, Section 37.62) that $f : X \to Y$ is a local complete intersection morphism if we can cover X by opens U such that we can factor $f|_U$ as

$$\begin{array}{ccc} U & \xrightarrow{i} & \mathbf{A}^n_Y \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

where i is a Koszul regular immersion (if Y is locally Noetherian this is the same as asking i to be a regular immersion, see Divisors, Lemma 31.21.3). Let us say that f is a *local complete intersection morphism of relative dimension r* if for any factorization as above, the closed immersion i has conormal sheaf of rank $n - r$ (in other words if i is a Koszul-regular immersion of codimension $n - r$ which in the Noetherian case just means it is regular immersion of codimension $n - r$).

0AZN **Lemma 43.13.2.** *Let $f : X \to Y$ be a morphism of varieties.*

(1) *If $Z \subset Y$ is a subvariety dimension d and f is a regular immersion of codimension c, then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d - c$.*
(2) *If $Z \subset Y$ is a subvariety of dimension d and f is a local complete intersection morphism of relative dimension r, then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d + r$.*

Proof. Proof of (1). We may work locally, hence we may assume that $Y = \mathrm{Spec}(A)$ and $X = V(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in A. If $Z = \mathrm{Spec}(A/\mathfrak{p})$, then we see that $f^{-1}(Z) = \mathrm{Spec}(A/\mathfrak{p} + (f_1, \ldots, f_c))$. If V is an irreducible component of $f^{-1}(Z)$, then we can choose a closed point $v \in V$ not contained in any other irreducible component of $f^{-1}(Z)$. Then

$$\dim(Z) = \dim \mathcal{O}_{Z,v} \quad \text{and} \quad \dim(V) = \dim \mathcal{O}_{V,v} = \dim \mathcal{O}_{Z,v}/(f_1, \ldots, f_c)$$

The first equality for example by Algebra, Lemma 10.116.1 and the second equality by our choice of closed point. The result now follows from the fact that dividing by one element in the maximal ideal decreases the dimension by at most 1, see Algebra, Lemma 10.60.13.

Proof of (2). Choose a factorization as in the definition of a local complete intersection and apply (1). Some details omitted. □

0B0Q **Lemma 43.13.3.** *Let X be a nonsingular variety. Then the diagonal $\Delta : X \to X \times X$ is a regular immersion of codimension $\dim(X)$.*

Proof. In fact, any closed immersion between nonsingular projective varieties is a regular immersion, see Divisors, Lemma 31.22.11. □

The following lemma demonstrates how reduction to the diagonal works.

0AZP **Lemma 43.13.4.** *Let X be a nonsingular variety and let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Then every irreducible component Z of $V \cap W$ has dimension $\geq r + s - \dim(X)$.*

Proof. Since $V \cap W = \Delta^{-1}(V \times W)$ (scheme theoretically) we conclude by Lemmas 43.13.3 and 43.13.2. □

This lemma suggests the following definition.

0AZQ **Definition 43.13.5.** Let X be a nonsingular variety.

(1) Let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. We say that W and V *intersect properly* if $\dim(V \cap W) \leq r + s - \dim(X)$.
(2) Let $\alpha = \sum n_i[W_i]$ be an s-cycle, and $\beta = \sum_j m_j[V_j]$ be an r-cycle on X. We say that α and β *intersect properly* if W_i and V_j intersect properly for all i and j.

43.14. Intersection multiplicities using Tor formula

0AZR A basic fact we will use frequently is that given sheaves of modules $\mathcal{F}$, $\mathcal{G}$ on a ringed space $(X, \mathcal{O}_X)$ and a point $x \in X$ we have

$$\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

as $\mathcal{O}_{X,x}$-modules. This can be seen in several ways from our construction of derived tensor products in Cohomology, Section 20.26, for example it follows from Cohomology, Lemma 20.26.4. Moreover, if X is a scheme and $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, then the modules $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are quasi-coherent too, see Derived Categories of Schemes, Lemma 36.3.9. More important for our purposes is the following result.

0AZS **Lemma 43.14.1.** *Let X be a locally Noetherian scheme.*

(1) *If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $Tor_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is too.*
(2) *If L and K are in $D^-_{Coh}(\mathcal{O}_X)$, then so is $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$.*

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using previously developed general theory.

Proof of (1). Since formation of Tor commutes with localization we may assume X is affine. Hence $X = \mathrm{Spec}(A)$ for some Noetherian ring A and $\mathcal{F}$, $\mathcal{G}$ correspond to finite A-modules M and N (Cohomology of Schemes, Lemma 30.9.1). By Derived Categories of Schemes, Lemma 36.3.9 we may compute the Tor's by first computing the Tor's of M and N over A, and then taking the associated $\mathcal{O}_X$-module. Since the modules $\mathrm{Tor}_p^A(M, N)$ are finite by Algebra, Lemma 10.75.7 we conclude.

By Derived Categories of Schemes, Lemma 36.10.3 the assumption is equivalent to asking L and K to be (locally) pseudo-coherent. Then $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$ is pseudo-coherent by Cohomology, Lemma 20.47.5. □

0AZT **Lemma 43.14.2.** *Let X be a nonsingular variety. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-module $Tor_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent, has stalk at x equal to $Tor_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$, is supported on $Supp(\mathcal{F}) \cap Supp(\mathcal{G})$, and is nonzero only for $p \in \{0, \ldots, \dim(X)\}$.*

Proof. The result on stalks was discussed above and it implies the support condition. The Tor's are coherent by Lemma 43.14.1. The vanishing of negative Tor's is immediate from the construction. The vanishing of Tor_p for $p > \dim(X)$ can be seen as follows: the local rings $\mathcal{O}_{X,x}$ are regular (as X is nonsingular) of dimension $\leq \dim(X)$ (Algebra, Lemma 10.116.1), hence $\mathcal{O}_{X,x}$ has finite global dimension $\leq \dim(X)$ (Algebra, Lemma 10.110.8) which implies that Tor-groups of modules vanish beyond the dimension (More on Algebra, Lemma 15.67.19). □

Let X be a nonsingular variety and $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Assume V and W intersect properly. In this case Lemma 43.13.4 tells us all irreducible components of $V \cap W$ have dimension equal to $r+s-\dim(X)$. The sheaves $\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)$ are coherent, supported on $V \cap W$, and zero if $j < 0$ or $j > \dim(X)$ (Lemma 43.14.2). We define the *intersection product* as

$$W \cdot V = \sum\nolimits_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)]_{r+s-\dim(X)} \cdot$$

We stress that this makes sense only because of our assumption that V and W intersect properly. This fact will necessitate a moving lemma in order to define the intersection product in general.

With this notation, the cycle $V \cdot W$ is a formal linear combination $\sum e_Z Z$ of the irreducible components Z of the intersection $V \cap W$. The integers e_Z are called the *intersection multiplicities*

$$e_Z = e(X, V \cdot W, Z) = \sum\nolimits_i (-1)^i \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$, resp. $\mathcal{O}_{V,Z}$ denotes the local ring of X, resp. W, resp. V at the generic point of Z. These alternating sums of lengths of Tor's satisfy many good properties, as we will see later on.

In the case of transversal intersections, the intersection number is 1.

0B1I **Lemma 43.14.3.** *Let X be a nonsingular variety. Let $V, W \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ and assume that the multiplicity (in the sense of Section 43.4) of Z in the closed subscheme $V \cap W$ is 1. Then $e(X, V \cdot W, Z) = 1$ and V and W are smooth in a general point of Z.*

Proof. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in Z$ is the generic point. Then $\dim(A) = \dim(X) - \dim(Z)$, see Varieties, Lemma 33.20.3. Let $I, J \subset A$ cut out the trace of V and W in $\mathrm{Spec}(A)$. Set $\overline{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $\dim_\kappa \overline{I} \leq \dim(X) - \dim(V)$ with equality if and only if A/I is regular (this follows from the lemma cited above and the definition of regular rings, see Algebra, Definition 10.60.10 and the discussion preceding it). Similarly for $\overline{J}$. If the multiplicity is 1, then $\mathrm{length}_A(A/I + J) = 1$, hence $I + J = \mathfrak{m}$, hence $\overline{I} + \overline{J} = \mathfrak{m}/\mathfrak{m}^2$. Then we get equality everywhere (because the intersection is proper). Hence we find $f_1, \ldots, f_a \in I$ and $g_1, \ldots g_b \in J$ such that $\overline{f}_1, \ldots, \overline{g}_b$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$. Then $f_1, \ldots, g_b$ is a regular system of parameters and a regular sequence (Algebra, Lemma 10.106.3). The same lemma shows $A/(f_1, \ldots, f_a)$ is a regular local ring of dimension

$\dim(X) - \dim(V)$, hence $A/(f_1, \ldots, f_a) \to A/I$ is an isomorphism (if the kernel is nonzero, then the dimension of A/I is strictly less, see Algebra, Lemmas 10.106.2 and 10.60.13). We conclude $I = (f_1, \ldots, f_a)$ and $J = (g_1, \ldots, g_b)$ by symmetry. Thus the Koszul complex $K_\bullet(A, f_1, \ldots, f_a)$ on $f_1, \ldots, f_a$ is a resolution of A/I, see More on Algebra, Lemma 15.30.2. Hence

$$\begin{aligned} \mathrm{Tor}_p^A(A/I, A/J) &= H_p(K_\bullet(A, f_1, \ldots, f_a) \otimes_A A/J) \\ &= H_p(K_\bullet(A/J, f_1 \bmod J, \ldots, f_a \bmod J)) \end{aligned}$$

Since we've seen above that $f_1 \bmod J, \ldots, f_a \bmod J$ is a regular system of parameters in the regular local ring A/J we conclude that there is only one cohomology group, namely $H_0 = A/(I + J) = \kappa$. This finishes the proof. □

0B2S **Example 43.14.4.** In this example we show that it is necessary to use the higher tors in the formula for the intersection multiplicities above. Let X be a nonsingular variety of dimension 4. Let $p \in X$ be a closed point. Let $V, W \subset X$ be closed subvarieties in X. Assume that there is an isomorphism

$$\mathcal{O}_{X,p}^\wedge \cong \mathbf{C}[[x, y, z, w]]$$

such that the ideal of V is (xz, xw, yz, yw) and the ideal of W is $(x - z, y - w)$. Then a computation shows that

$$\text{length } \mathbf{C}[[x, y, z, w]]/(xz, xw, yz, yw, x - z, y - w) = 3$$

On the other hand, the multiplicity $e(X, V \cdot W, p) = 2$ as can be seen from the fact that formal locally V is the union of two smooth planes $x = y = 0$ and $z = w = 0$ at p, each of which has intersection multiplicity 1 with the plane $x - z = y - w = 0$ (Lemma 43.14.3). To make an actual example, take a general morphism $f : \mathbf{P}^2 \to \mathbf{P}^4$ given by 5 homogeneous polynomials of degree > 1. The image $V \subset \mathbf{P}^4 = X$ will have singularities of the type described above, because there will be $p_1, p_2 \in \mathbf{P}^2$ with $f(p_1) = f(p_2)$. To find W take a general plane passing through such a point.

43.15. Algebraic multiplicities

0AZU Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A-module and let $I \subset A$ be an ideal of definition (Algebra, Definition 10.59.1). Recall that the function

$$\chi_{I,M}(n) = \text{length}_A(M/I^n M) = \sum\nolimits_{p=0,\ldots,n-1} \text{length}_A(I^p M/I^{p+1} M)$$

is a numerical polynomial (Algebra, Proposition 10.59.5). The degree of this polynomial is equal to $\dim(\mathrm{Supp}(M))$ by Algebra, Lemma 10.62.6.

0AZV **Definition 43.15.1.** In the situation above, assume $d \geq \dim(\mathrm{Supp}(M))$. In this case, if $d > \dim(\mathrm{Supp}(M))$, then we set $e_I(M, d) = 0$ and if $d = \dim(\mathrm{Supp}(M))$, then we set $e_I(M, d)$ equal to $d!$ times the leading coefficient of the numerical polynomial $\chi_{I,M}$. Thus in both cases we have

$$\chi_{I,M}(n) \sim e_I(M, d)\frac{n^d}{d!} + \text{lower order terms}$$

The *multiplicity of M for the ideal of definition I* is $e_I(M) = e_I(M, \dim(\mathrm{Supp}(M)))$.

We have the following properties of these multiplicities.

0AZW **Lemma 43.15.2.** *Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite A-modules. Let $d \geq \dim(Supp(M))$. Then*

$$e_I(M, d) = e_I(M', d) + e_I(M'', d)$$

Proof. Immediate from the definitions and Algebra, Lemma 10.59.10. □

0AZX **Lemma 43.15.3.** *Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let M be a finite A-module. Let $d \geq \dim(Supp(M))$. Then*

$$e_I(M, d) = \sum length_{A_\mathfrak{p}}(M_\mathfrak{p}) e_I(A/\mathfrak{p}, d)$$

where the sum is over primes $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) = d$.

Proof. Both the left and side and the right hand side are additive in short exact sequences of modules of dimension $\leq d$, see Lemma 43.15.2 and Algebra, Lemma 10.52.3. Hence by Algebra, Lemma 10.62.1 it suffices to prove this when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q}$ of A with $\dim(A/\mathfrak{q}) \leq d$. This case is obvious. □

0AZY **Lemma 43.15.4.** *Let P be a polynomial of degree r with leading coefficient a. Then*

$$r!a = \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} P(t - i)$$

for any t.

Proof. Let us write Δ the operator which to a polynomial P associates the polynomial $\Delta(P) = P(t) - P(t-1)$. We claim that

$$\Delta^r(P) = \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} P(t - i)$$

This is true for $r = 0, 1$ by inspection. Assume it is true for r. Then we compute

$$\begin{aligned} \Delta^{r+1}(P) &= \sum\nolimits_{i=0,\ldots,r} (-1)^i \binom{r}{i} \Delta(P)(t - i) \\ &= \sum\nolimits_{n=-r,\ldots,0} (-1)^i \binom{r}{i} (P(t - i) - P(t - i - 1)) \end{aligned}$$

Thus the claim follows from the equality

$$\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1}$$

The lemma follows from the fact that $\Delta(P)$ is of degree $r-1$ with leading coefficient ra if the degree of P is r. □

An important fact is that one can compute the multiplicity in terms of the Koszul complex. Recall that if R is a ring and $f_1, \ldots, f_r \in R$, then $K_\bullet(f_1, \ldots, f_r)$ denotes the Koszul complex, see More on Algebra, Section 15.28.

0AZZ **Theorem 43.15.5.** *Let A be a Noetherian local ring. Let $I = (f_1, \ldots, f_r) \subset A$ be an ideal of definition. Let M be a finite A-module. Then*

$$e_I(M, r) = \sum (-1)^i length_A H_i(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$$

[**Ser65**, Theorem 1 in part B of Chapter IV]

Proof. Let us change the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ into a cochain complex $K^\bullet$ by setting $K^n = K_{-n}(f_1, \ldots, f_r)$. Then $K^\bullet$ is sitting in degrees $-r, \ldots, 0$ and $H^i(K^\bullet \otimes_A M) = H_{-i}(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$. The statement of the theorem makes sense as the modules $H^i(K^\bullet \otimes M)$ are annihilated by $f_1, \ldots, f_r$ (More on Algebra, Lemma 15.28.6) hence have finite length. Define a filtration on the complex $K^\bullet$ by setting

$$F^p(K^n \otimes_A M) = I^{\max(0,p+n)}(K^n \otimes_A M), \quad p \in \mathbf{Z}$$

Since $f_i I^p \subset I^{p+1}$ this is a filtration by subcomplexes. Thus we have a filtered complex and we obtain a spectral sequence, see Homology, Section 12.24. We have

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q} = \bigoplus\nolimits_{p,q} \mathrm{gr}^p(K^{p+q} \otimes_A M) = \mathrm{Gr}_I(K^\bullet \otimes_A M)$$

Since K^n is finite free we have

$$\mathrm{Gr}_I(K^\bullet \otimes_A M) = \mathrm{Gr}_I(K^\bullet) \otimes_{\mathrm{Gr}_I(A)} \mathrm{Gr}_I(M)$$

Note that $\mathrm{Gr}_I(K^\bullet)$ is the Koszul complex over $\mathrm{Gr}_I(A)$ on the elements $\overline{f}_1, \ldots, \overline{f}_r \in I/I^2$. A simple calculation (omitted) shows that the differential d_0 on E_0 agrees with the differential coming from the Koszul complex. Since $\mathrm{Gr}_I(M)$ is a finite $\mathrm{Gr}_I(A)$-module and since $\mathrm{Gr}_I(A)$ is Noetherian (as a quotient of $A/I[x_1, \ldots, x_r]$ with $x_i \mapsto \overline{f}_i$), the cohomology module $E_1 = \bigoplus E_1^{p,q}$ is a finite $\mathrm{Gr}_I(A)$-module. However, as above E_1 is annihilated by $\overline{f}_1, \ldots, \overline{f}_r$. We conclude E_1 has finite length. In particular we find that $\mathrm{Gr}_F^p(K^\bullet \otimes M)$ is acyclic for $p \gg 0$.

Next, we check that the spectral sequence above converges using Homology, Lemma 12.24.10. The required equalities follow easily from the Artin-Rees lemma in the form stated in Algebra, Lemma 10.51.3. Thus we see that

$$\begin{aligned} \sum (-1)^i \mathrm{length}_A(H^i(K^\bullet \otimes_A M)) &= \sum (-1)^{p+q} \mathrm{length}_A(E_\infty^{p,q}) \\ &= \sum (-1)^{p+q} \mathrm{length}_A(E_1^{p,q}) \end{aligned}$$

because as we've seen above the length of E_1 is finite (of course this uses additivity of lengths). Pick t so large that $\mathrm{Gr}_F^p(K^\bullet \otimes M)$ is acyclic for $p \geq t$ (see above). Using additivity again we see that

$$\sum (-1)^{p+q} \mathrm{length}_A(E_1^{p,q}) = \sum\nolimits_n \sum\nolimits_{p \leq t} (-1)^n \mathrm{length}_A(\mathrm{gr}^p(K^n \otimes_A M))$$

This is equal to

$$\sum\nolimits_{n=-r,\ldots,0} (-1)^n \binom{r}{|n|} \chi_{I,M}(t+n)$$

by our choice of filtration above and the definition of $\chi_{I,M}$ in Algebra, Section 10.59. The lemma follows from Lemma 43.15.4 and the definition of $e_I(M, r)$. □

0B00 **Remark 43.15.6** (Trivial generalization)**.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A-module. Let $I \subset A$ be an ideal. The following are equivalent

(1) $I' = I + \mathrm{Ann}(M)$ is an ideal of definition (Algebra, Definition 10.59.1),
(2) the image $\overline{I}$ of I in $\overline{A} = A/\mathrm{Ann}(M)$ is an ideal of definition,
(3) $\mathrm{Supp}(M/IM) \subset \{\mathfrak{m}\}$,
(4) $\dim(\mathrm{Supp}(M/IM)) \leq 0$, and
(5) $\mathrm{length}_A(M/IM) < \infty$.

This follows from Algebra, Lemma 10.62.3 (details omitted). If this is the case we have $M/I^nM = M/(I')^nM$ for all n and $M/I^nM = M/\overline{I}^nM$ for all n if M is viewed as an $\overline{A}$-module. Thus we can define

$$\chi_{I,M}(n) = \text{length}_A(M/I^nM) = \sum\nolimits_{p=0,\ldots,n-1} \text{length}_A(I^pM/I^{p+1}M)$$

and we get

$$\chi_{I,M}(n) = \chi_{I',M}(n) = \chi_{\overline{I},M}(n)$$

for all n by the equalities above. All the results of Algebra, Section 10.59 and all the results in this section, have analogues in this setting. In particular we can define multiplicities $e_I(M, d)$ for $d \geq \dim(\text{Supp}(M))$ and we have

$$\chi_{I,M}(n) \sim e_I(M, d)\frac{n^d}{d!} + \text{lower order terms}$$

as in the case where I is an ideal of definition.

43.16. Computing intersection multiplicities

0B01 In this section we discuss some cases where the intersection multiplicities can be computed by different means. Here is a first example.

0B02 **Lemma 43.16.1.** *Let X be a nonsingular variety and $W, V \subset X$ closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ. Assume that $\mathcal{O}_{W,\xi}$ and $\mathcal{O}_{V,\xi}$ are Cohen-Macaulay. Then*

$$e(X, V \cdot W, Z) = \mathit{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{V\cap W,\xi})$$

where $V \cap W$ is the scheme theoretic intersection. In particular, if both V and W are Cohen-Macaulay, then $V \cdot W = [V \cap W]_{\dim(V)+\dim(W)-\dim(X)}$.

Proof. Set $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{V,\xi}$, and $C = \mathcal{O}_{W,\xi}$. By Auslander-Buchsbaum (Algebra, Proposition 10.111.1) we can find a finite free resolution $F_\bullet \to B$ of length

$$\text{depth}(A) - \text{depth}(B) = \dim(A) - \dim(B) = \dim(C)$$

First equality as A and B are Cohen-Macaulay and the second as V and W intersect properly. Then $F_\bullet \otimes_A C$ is a complex of finite free modules representing $B \otimes^{\mathbf{L}}_A C$ hence has cohomology modules with support in $\{\mathfrak{m}_A\}$. By the Acyclicity lemma (Algebra, Lemma 10.102.8) which applies as C is Cohen-Macaulay we conclude that $F_\bullet \otimes_A C$ has nonzero cohomology only in degree 0. This finishes the proof. □

0B03 **Lemma 43.16.2.** *Let A be a Noetherian local ring. Let $I = (f_1, \ldots, f_r)$ be an ideal generated by a regular sequence. Let M be a finite A-module. Assume that $\dim(\mathit{Supp}(M/IM)) = 0$. Then*

$$e_I(M, r) = \sum(-1)^i \mathit{length}_A(\mathit{Tor}_i^A(A/I, M))$$

Here $e_I(M, r)$ is as in Remark 43.15.6.

Proof. Since $f_1, \ldots, f_r$ is a regular sequence the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ is a resolution of A/I over A, see More on Algebra, Lemma 15.30.7. Thus the right hand side is equal to

$$\sum(-1)^i \text{length}_A H_i(K_\bullet(f_1, \ldots, f_r) \otimes_A M)$$

Now the result follows immediately from Theorem 43.15.5 if I is an ideal of definition. In general, we replace A by $\overline{A} = A/\text{Ann}(M)$ and $f_1, \ldots, f_r$ by $\overline{f}_1, \ldots, \overline{f}_r$ which is allowed because

$$K_\bullet(f_1, \ldots, f_r) \otimes_A M = K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \otimes_{\overline{A}} M$$

Since $e_I(M, r) = e_{\overline{I}}(M, r)$ where $\overline{I} = (\overline{f}_1, \ldots, \overline{f}_r) \subset \overline{A}$ is an ideal of definition the result follows from Theorem 43.15.5 in this case as well. □

0B04 **Lemma 43.16.3.** *Let X be a nonsingular variety. Let $W, V \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ. Suppose the ideal of V in $\mathcal{O}_{X,\xi}$ is cut out by a regular sequence $f_1, \ldots, f_c \in \mathcal{O}_{X,\xi}$. Then $e(X, V \cdot W, Z)$ is equal to $c!$ times the leading coefficient in the Hilbert polynomial*

$$t \mapsto \mathit{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{W,\xi}/(f_1, \ldots, f_c)^t, \quad t \gg 0.$$

In particular, this coefficient is > 0.

Proof. The equality

$$e(X, V \cdot W, Z) = e_{(f_1,\ldots,f_c)}(\mathcal{O}_{W,\xi}, c)$$

follows from the more general Lemma 43.16.2. To see that $e_{(f_1,\ldots,f_c)}(\mathcal{O}_{W,\xi}, c)$ is > 0 or equivalently that $e_{(f_1,\ldots,f_c)}(\mathcal{O}_{W,\xi}, c)$ is the leading coefficient of the Hilbert polynomial it suffices to show that the dimension of $\mathcal{O}_{W,\xi}$ is c, because the degree of the Hilbert polynomial is equal to the dimension by Algebra, Proposition 10.60.9. Say $\dim(V) = r$, $\dim(W) = s$, and $\dim(X) = n$. Then $\dim(Z) = r + s - n$ as the intersection is proper. Thus the transcendence degree of $\kappa(\xi)$ over $\mathbf{C}$ is $r + s - n$, see Algebra, Lemma 10.116.1. We have $r + c = n$ because V is cut out by a regular sequence in a neighbourhood of ξ, see Divisors, Lemma 31.20.8 and then Lemma 43.13.2 applies (for example). Thus

$$\dim(\mathcal{O}_{W,\xi}) = s - (r + s - n) = s - ((n - c) + s - n) = c$$

the first equality by Algebra, Lemma 10.116.3. □

0B05 **Lemma 43.16.4.** *In Lemma 43.16.3 assume that $c = 1$, i.e., V is an effective Cartier divisor. Then*

$$e(X, V \cdot W, Z) = \mathit{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi}).$$

Proof. In this case the image of f_1 in $\mathcal{O}_{W,\xi}$ is nonzero by properness of intersection, hence a nonzerodivisor divisor. Moreover, $\mathcal{O}_{W,\xi}$ is a Noetherian local domain of dimension 1. Thus

$$\text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1^t\mathcal{O}_{W,\xi}) = t\text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi})$$

for all $t \geq 1$, see Algebra, Lemma 10.121.1. This proves the lemma. □

0B06 **Lemma 43.16.5.** *In Lemma 43.16.3 assume that the local ring $\mathcal{O}_{W,\xi}$ is Cohen-Macaulay. Then we have*

$$e(X, V \cdot W, Z) = \mathit{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi} + \ldots + f_c\mathcal{O}_{W,\xi}).$$

Proof. This follows immediately from Lemma 43.16.1. Alternatively, we can deduce it from Lemma 43.16.3. Namely, by Algebra, Lemma 10.104.2 we see that $f_1, \ldots, f_c$ is a regular sequence in $\mathcal{O}_{W,\xi}$. Then Algebra, Lemma 10.69.2 shows that $f_1, \ldots, f_c$ is a quasi-regular sequence. This easily implies the length of $\mathcal{O}_{W,\xi}/(f_1, \ldots, f_c)^t$ is

$$\binom{c+t}{c} \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{W,\xi}/f_1\mathcal{O}_{W,\xi} + \ldots + f_c\mathcal{O}_{W,\xi}).$$

Looking at the leading coefficient we conclude. □

43.17. Intersection product using Tor formula

0B08 Let X be a nonsingular variety. Let $\alpha = \sum n_i[W_i]$ be an r-cycle and $\beta = \sum_j m_j[V_j]$ be an s-cycle on X. Assume that α and β intersect properly, see Definition 43.13.5. In this case we define

$$\alpha \cdot \beta = \sum\nolimits_{i,j} n_i m_j W_i \cdot V_j.$$

where $W_i \cdot V_j$ is as defined in Section 43.14. If $\beta = [V]$ where V is a closed subvariety of dimension s, then we sometimes write $\alpha \cdot \beta = \alpha \cdot V$.

0B07 **Lemma 43.17.1.** *Let X be a nonsingular variety. Let $a, b \in \mathbf{P}^1$ be distinct closed points. Let $k \geq 0$.*

1. *If $W \subset X \times \mathbf{P}^1$ is a closed subvariety of dimension $k+1$ which intersects $X \times a$ properly, then*
   (a) *$[W_a]_k = W \cdot X \times a$ as cycles on $X \times \mathbf{P}^1$, and*
   (b) *$[W_a]_k = pr_{X,*}(W \cdot X \times a)$ as cycles on X.*
2. *Let α be a $(k+1)$-cycle on $X \times \mathbf{P}^1$ which intersects $X \times a$ and $X \times b$ properly. Then $pr_{X,*}(\alpha \cdot X \times a - \alpha \cdot X \times b)$ is rationally equivalent to zero.*
3. *Conversely, any k-cycle which is rationally equivalent to 0 is of this form.*

Proof. First we observe that $X \times a$ is an effective Cartier divisor in $X \times \mathbf{P}^1$ and that W_a is the scheme theoretic intersection of W with $X \times a$. Hence the equality in (1)(a) is immediate from the definitions and the calculation of intersection multiplicity in case of a Cartier divisor given in Lemma 43.16.4. Part (1)(b) holds because $W_a \to X \times \mathbf{P}^1 \to X$ maps isomorphically onto its image which is how we viewed W_a as a closed subscheme of X in Section 43.8. Parts (2) and (3) are formal consequences of part (1) and the definitions. □

For transversal intersections of closed subschemes the intersection multiplicity is 1.

0B1J **Lemma 43.17.2.** *Let X be a nonsingular variety. Let $r, s \geq 0$ and let $Y, Z \subset X$ be closed subschemes with $\dim(Y) \leq r$ and $\dim(Z) \leq s$. Assume $[Y]_r = \sum n_i[Y_i]$ and $[Z]_s = \sum m_j[Z_j]$ intersect properly. Let T be an irreducible component of $Y_{i_0} \cap Z_{j_0}$ for some i_0 and j_0 and assume that the multiplicity (in the sense of Section 43.4) of T in the closed subscheme $Y \cap Z$ is 1. Then*

1. *the coefficient of T in $[Y]_r \cdot [Z]_s$ is 1,*
2. *Y and Z are nonsingular at the generic point of T,*
3. *$n_{i_0} = 1$, $m_{j_0} = 1$, and*
4. *T is not contained in Y_i or Z_j for $i \neq i_0$ and $j \neq j_0$.*

Proof. Set $n = \dim(X)$, $a = n - r$, $b = n - s$. Observe that $\dim(T) = r + s - n = n - a - b$ by the assumption that the intersections are proper. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in T$ is the generic point. Then $\dim(A) = a + b$, see

Varieties, Lemma 33.20.3. Let $I_0, I, J_0, J \subset A$ cut out the trace of Y_{i_0}, Y, Z_{j_0}, Z in $\mathrm{Spec}(A)$. Then $\dim(A/I) = \dim(A/I_0) = b$ and $\dim(A/J) = \dim(A/J_0) = a$ by the same reference. Set $\overline{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $I \subset I_0 \subset \mathfrak{m}$ and $J \subset J_0 \subset \mathfrak{m}$ and $I + J = \mathfrak{m}$. By Lemma 43.14.3 and its proof we see that $I_0 = (f_1, \ldots, f_a)$ and $J_0 = (g_1, \ldots, g_b)$ where $f_1, \ldots, g_b$ is a regular system of parameters for the regular local ring A. Since $I + J = \mathfrak{m}$, the map

$$I \oplus J \to \mathfrak{m}/\mathfrak{m}^2 = \kappa f_1 \oplus \ldots \oplus \kappa f_a \oplus \kappa g_1 \oplus \ldots \oplus \kappa g_b$$

is surjective. We conclude that we can find $f_1', \ldots, f_a' \in I$ and $g_1', \ldots, g_b' \in J$ whose residue classes in $\mathfrak{m}/\mathfrak{m}^2$ are equal to the residue classes of $f_1, \ldots, f_a$ and $g_1, \ldots, g_b$. Then $f_1', \ldots, g_b'$ is a regular system of parameters of A. By Algebra, Lemma 10.106.3 we find that $A/(f_1', \ldots, f_a')$ is a regular local ring of dimension b. Thus any nontrivial quotient of $A/(f_1', \ldots, f_a')$ has strictly smaller dimension (Algebra, Lemmas 10.106.2 and 10.60.13). Hence $I = (f_1', \ldots, f_a') = I_0$. By symmetry $J = J_0$. This proves (2), (3), and (4). Finally, the coefficient of T in $[Y]_r \cdot [Z]_s$ is the coefficient of T in $Y_{i_0} \cdot Z_{j_0}$ which is 1 by Lemma 43.14.3. □

43.18. Exterior product

0B09 Let X and Y be varieties. Let V, resp. W be a closed subvariety of X, resp. Y. The product $V \times W$ is a closed subvariety of $X \times Y$ (Lemma 43.13.1). For a k-cycle $\alpha = \sum n_i[V_i]$ and a l-cycle $\beta = \sum m_j[V_j]$ on Y we define the *exterior product* of α and β to be the cycle $\alpha \times \beta = \sum n_i m_j [V_i \times W_j]$. Exterior product defines a $\mathbf{Z}$-linear map

$$Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) \longrightarrow Z_{r+s}(X \times Y)$$

Let us prove that exterior product factors through rational equivalence.

0B0S **Lemma 43.18.1.** *Let X and Y be varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. If $\alpha \sim_{rat} 0$ or $\beta \sim_{rat} 0$, then $\alpha \times \beta \sim_{rat} 0$.*

Proof. By linearity and symmetry in X and Y, it suffices to prove this when $\alpha = [V]$ for some subvariety $V \subset X$ of dimension r and $\beta = [W_a]_s - [W_b]_s$ for some closed subvariety $W \subset Y \times \mathbf{P}^1$ of dimension $s+1$ which intersects $Y \times a$ and $Y \times b$ properly. In this case the lemma follows if we can prove

$$[(V \times W)_a]_{r+s} = [V] \times [W_a]_s$$

and similarly with a replaced by b. Namely, then we see that $\alpha \times \beta = [(V \times W)_a]_{r+s} - [(V \times W)_b]_{r+s}$ as desired. To see the displayed equality we note the equality

$$V \times W_a = (V \times W)_a$$

of schemes. The projection $V \times W_a \to W_a$ induces a bijection of irreducible components (see for example Varieties, Lemma 33.8.4). Let $W' \subset W_a$ be an irreducible component with generic point ζ. Then $V \times W'$ is the corresponding irreducible component of $V \times W_a$ (see Lemma 43.13.1). Let ξ be the generic point of $V \times W'$. We have to show that

$$\mathrm{length}_{\mathcal{O}_{Y,\zeta}}(\mathcal{O}_{W_a,\zeta}) = \mathrm{length}_{\mathcal{O}_{X \times Y,\xi}}(\mathcal{O}_{V \times W_a,\xi})$$

In this formula we may replace $\mathcal{O}_{Y,\zeta}$ by $\mathcal{O}_{W_a,\zeta}$ and we may replace $\mathcal{O}_{X \times Y,\zeta}$ by $\mathcal{O}_{V \times W_a,\zeta}$ (see Algebra, Lemma 10.52.5). As $\mathcal{O}_{W_a,\zeta} \to \mathcal{O}_{V \times W_a,\xi}$ is flat, by Algebra, Lemma 10.52.13 it suffices to show that

$$\mathrm{length}_{\mathcal{O}_{V \times W_a,\xi}}(\mathcal{O}_{V \times W_a,\xi}/\mathfrak{m}_\zeta \mathcal{O}_{V \times W_a,\xi}) = 1$$

This is true because the quotient on the right is the local ring $\mathcal{O}_{V \times W', \xi}$ of a variety at a generic point hence equal to $\kappa(\xi)$. $\square$

We conclude that exterior product defines a commutative diagram

$$\begin{array}{ccc} Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) & \longrightarrow & Z_{r+s}(X \times Y) \\ \downarrow & & \downarrow \\ \mathrm{CH}_r(X) \otimes_{\mathbf{Z}} \mathrm{CH}_s(Y) & \longrightarrow & \mathrm{CH}_{r+s}(X \times Y) \end{array}$$

for any pair of varieties X and Y. For nonsingular varieties we can think of the exterior product as an intersection product of pullbacks.

0B0R **Lemma 43.18.2.** *Let X and Y be nonsingular varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. Then*

(1) *$pr_Y^*(\beta) = [X] \times \beta$ and $pr_X^*(\alpha) = \alpha \times [Y]$,*
(2) *$\alpha \times [Y]$ and $[X] \times \beta$ intersect properly on $X \times Y$, and*
(3) *we have $\alpha \times \beta = (\alpha \times [Y]) \cdot ([X] \times \beta) = pr_Y^*(\alpha) \cdot pr_X^*(\beta)$ in $Z_{r+s}(X \times Y)$.*

Proof. By linearity we may assume $\alpha = [V]$ and $\beta = [W]$. Then (1) says that $\mathrm{pr}_Y^{-1}(W) = X \times W$ and $\mathrm{pr}_X^{-1}(V) = V \times Y$. This is clear. Part (2) holds because $X \times W \cap V \times Y = V \times W$ and $\dim(V \times W) = r + s$ by Lemma 43.13.1.

Proof of (3). Let ξ be the generic point of $V \times W$. Since the projections $X \times W \to W$ is smooth as a base change of $X \to \mathrm{Spec}(\mathbf{C})$, we see that $X \times W$ is nonsingular at every point lying over the generic point of W, in particular at ξ. Similarly for $V \times Y$. Hence $\mathcal{O}_{X \times W, \xi}$ and $\mathcal{O}_{V \times Y, \xi}$ are Cohen-Macaulay local rings and Lemma 43.16.1 applies. Since $V \times Y \cap X \times W = V \times W$ scheme theoretically the proof is complete. $\square$

43.19. Reduction to the diagonal

0B0A Let X be a nonsingular variety. We will use Δ to denote either the diagonal morphism $\Delta : X \to X \times X$ or the image $\Delta \subset X \times X$. Reduction to the diagonal is the statement that intersection products on X can be reduced to intersection products of exterior products with the diagonal on $X \times X$.

0B0T **Lemma 43.19.1.** *Let X be a nonsingular variety.*

(1) *If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then there are canonical isomorphisms*

$$Tor_i^{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times X}} pr_2^*\mathcal{G}) = \Delta_* Tor_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

(2) *If K and M are in $D_{QCoh}(\mathcal{O}_X)$, then there is a canonical isomorphism*

$$L\Delta^* \left(Lpr_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} Lpr_2^* M \right) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

in $D_{QCoh}(\mathcal{O}_X)$ and a canonical isomorphism

$$\mathcal{O}_\Delta \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} Lpr_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times X}} Lpr_2^* M = \Delta_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D_{QCoh}(X \times X)$.

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using more general theory. As (2) implies (1) the reader can skip the proof of (1).

Proof of (1). Choose an affine open $\mathrm{Spec}(A) \subset X$. Then A is a Noetherian $\mathbf{C}$-algebra and $\mathcal{F}$, $\mathcal{G}$ correspond to finite A-modules M and N (Cohomology of Schemes, Lemma 30.9.1). By Derived Categories of Schemes, Lemma 36.3.9 we may compute Tor_i over $\mathcal{O}_X$ by first computing the Tor's of M and N over A, and then taking the associated $\mathcal{O}_X$-module. For the Tor_i over $\mathcal{O}_{X\times X}$ we compute the tor of A and $M \otimes_{\mathbf{C}} N$ over $A \otimes_{\mathbf{C}} A$ and then take the associated $\mathcal{O}_{X\times X}$-module. Hence on this affine patch we have to prove that

$$\mathrm{Tor}_i^{A\otimes_{\mathbf{C}} A}(A, M \otimes_{\mathbf{C}} N) = \mathrm{Tor}_i^A(M, N)$$

To see this choose resolutions $F_\bullet \to M$ and $G_\bullet \to M$ by finite free A-modules (Algebra, Lemma 10.71.1). Note that $\mathrm{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet)$ is a resolution of $M \otimes_{\mathbf{C}} N$ as it computes Tor groups over $\mathbf{C}$! Of course the terms of $F_\bullet \otimes_{\mathbf{C}} G_\bullet$ are finite free $A \otimes_{\mathbf{C}} A$-modules. Hence the left hand side of the displayed equation is the module

$$H_i(A \otimes_{A\otimes_{\mathbf{C}} A} \mathrm{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet))$$

and the right hand side is the module

$$H_i(\mathrm{Tot}(F_\bullet \otimes_A G_\bullet))$$

Since $A \otimes_{A\otimes_{\mathbf{C}} A} (F_p \otimes_{\mathbf{C}} G_q) = F_p \otimes_A G_q$ we see that these modules are equal. This defines an isomorphism over the affine open $\mathrm{Spec}(A) \times \mathrm{Spec}(A)$ (which is good enough for the application to equality of intersection numbers). We omit the proof that these isomorphisms glue.

Proof of (2). The second statement follows from the first by the projection formula as stated in Derived Categories of Schemes, Lemma 36.22.1. To see the first, represent K and M by K-flat complexes $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$. Since pullback and tensor product preserve K-flat complexes (Cohomology, Lemmas 20.26.5 and 20.26.8) we see that it suffices to show

$$\Delta^*\mathrm{Tot}(\mathrm{pr}_1^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^*\mathcal{M}^\bullet) = \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

Thus it suffices to see that there are canonical isomorphisms

$$\Delta^*(\mathrm{pr}_1^*\mathcal{K} \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^*\mathcal{M}) \longrightarrow \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$$

whenever $\mathcal{K}$ and $\mathcal{M}$ are $\mathcal{O}_X$-modules (not necessarily quasi-coherent or flat). We omit the details. □

0B0U **Lemma 43.19.2.** *Let X be a nonsingular variety. Let α, resp. β be an r-cycle, resp. s-cycle on X. Assume α and β intersect properly. Then*

1. *$\alpha \times \beta$ and $[\Delta]$ intersect properly*
2. *we have $\Delta_*(\alpha \cdot \beta) = [\Delta] \cdot \alpha \times \beta$ as cycles on $X \times X$,*
3. *if X is proper, then $pr_{1,*}([\Delta] \cdot \alpha \times \beta) = \alpha \cdot \beta$, where $pr_1 : X \times X \to X$ is the projection.*

Proof. By linearity it suffices to prove this when $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V \subset X$ and $W \subset Y$ which intersect properly. Recall that $V \times W$ is a closed subvariety of dimension $r+s$. Observe that scheme theoretically we have $V \cap W = \Delta^{-1}(V \times W)$ as well as $\Delta(V \cap W) = \Delta \cap V \times W$. This proves (1).

Proof of (2). Let $Z \subset V \cap W$ be an irreducible component with generic point ξ. We have to show that the coefficient of Z in $\alpha \cdot \beta$ is the same as the coefficient of $\Delta(Z)$ in $[\Delta] \cdot \alpha \times \beta$. The first is given by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X,\xi}} \text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi$$

and the second by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X\times Y,\Delta(\xi)}} \text{Tor}_i^{\mathcal{O}_{X\times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V\times W})_{\Delta(\xi)}$$

However, by Lemma 43.19.1 we have

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi \cong \text{Tor}_i^{\mathcal{O}_{X\times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V\times W})_{\Delta(\xi)}$$

as $\mathcal{O}_{X\times X,\Delta(\xi)}$-modules. Thus equality of lengths (by Algebra, Lemma 10.52.5 to be precise).

Part (2) implies (3) because $\text{pr}_{1,*} \circ \Delta_* = \text{id}$ by Lemma 43.6.2. □

0B0V **Proposition 43.19.3.** *Let X be a nonsingular variety. Let $V \subset X$ and $W \subset Y$ be closed subvarieties which intersect properly. Let $Z \subset V \cap W$ be an irreducible component. Then $e(X, V \cdot W, Z) > 0$.*

This is one of the main results of [**Ser65**].

Proof. By Lemma 43.19.2 we have

$$e(X, V \cdot W, Z) = e(X \times X, \Delta \cdot V \times W, \Delta(Z))$$

Since $\Delta : X \to X \times X$ is a regular immersion (see Lemma 43.13.3), we see that $e(X \times X, \Delta \cdot V \times W, \Delta(Z))$ is a positive integer by Lemma 43.16.3. □

The following is a key lemma in the development of the theory as is done in this chapter. Essentially, this lemma tells us that the intersection numbers have a suitable additivity property.

0B0W **Lemma 43.19.4.** *Let X be a nonsingular variety. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X with $\dim(\mathit{Supp}(\mathcal{F})) \leq r$, $\dim(\mathit{Supp}(\mathcal{G})) \leq s$, and $\dim(\mathit{Supp}(\mathcal{F}) \cap \mathit{Supp}(\mathcal{G})) \leq r + s - \dim X$. In this case $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly and*

[**Ser65**, Chapter V]

$$[\mathcal{F}]_r \cdot [\mathcal{G}]_s = \sum (-1)^p [\mathit{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}.$$

Proof. The statement that $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly is immediate. Since we are proving an equality of cycles we may work locally on X. (Observe that the formation of the intersection product of cycles, the formation of Tor-sheaves, and forming the cycle associated to a coherent sheaf, each commute with restriction to open subschemes.) Thus we may and do assume that X is affine.

Denote

$$RHS(\mathcal{F}, \mathcal{G}) = [\mathcal{F}]_r \cdot [\mathcal{G}]_s \quad \text{and} \quad LHS(\mathcal{F}, \mathcal{G}) = \sum (-1)^p [\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}$$

Consider a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of coherent sheaves on X with $\text{Supp}(\mathcal{F}_i) \subset \text{Supp}(\mathcal{F})$, then both $LHS(\mathcal{F}_i, \mathcal{G})$ and $RHS(\mathcal{F}_i, \mathcal{G})$ are defined for $i = 1, 2, 3$ and we have

$$RHS(\mathcal{F}_2, \mathcal{G}) = RHS(\mathcal{F}_1, \mathcal{G}) + RHS(\mathcal{F}_3, \mathcal{G})$$

and similarly for LHS. Namely, the support condition guarantees that everything is defined, the short exact sequence and additivity of lengths gives

$$[\mathcal{F}_2]_r = [\mathcal{F}_1]_r + [\mathcal{F}_3]_r$$

(Chow Homology, Lemma 42.10.4) which implies additivity for RHS. The long exact sequence of Tors

$$\ldots \to \mathrm{Tor}_1(\mathcal{F}_3, \mathcal{G}) \to \mathrm{Tor}_0(\mathcal{F}_1, \mathcal{G}) \to \mathrm{Tor}_0(\mathcal{F}_2, \mathcal{G}) \to \mathrm{Tor}_0(\mathcal{F}_3, \mathcal{G}) \to 0$$

and additivity of lengths as before implies additivity for LHS.

By Algebra, Lemma 10.62.1 and the fact that X is affine, we can find a filtration of $\mathcal{F}$ whose graded pieces are structure sheaves of closed subvarieties of $\mathrm{Supp}(\mathcal{F})$. The additivity shown in the previous paragraph, implies that it suffices to prove $LHS = RHS$ with $\mathcal{F}$ replaced by $\mathcal{O}_V$ where $V \subset \mathrm{Supp}(\mathcal{F})$. By symmetry we can do the same for $\mathcal{G}$. This reduces us to proving that

$$LHS(\mathcal{O}_V, \mathcal{O}_W) = RHS(\mathcal{O}_V, \mathcal{O}_W)$$

where $W \subset \mathrm{Supp}(\mathcal{G})$ is a closed subvariety. If $\dim(V) = r$ and $\dim(W) = s$, then this equality is the **definition** of $V \cdot W$. On the other hand, if $\dim(V) < r$ or $\dim(W) < s$, i.e., $[V]_r = 0$ or $[W]_s = 0$, then we have to prove that $RHS(\mathcal{O}_V, \mathcal{O}_W) = 0$ [5].

Let $Z \subset V \cap W$ be an irreducible component of dimension $r+s-\dim(X)$. This is the maximal dimension of a component and it suffices to show that the coefficient of Z in RHS is zero. Let $\xi \in Z$ be the generic point. Write $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X\times X,\Delta(\xi)}$, and $C = \mathcal{O}_{V\times W,\Delta(\xi)}$. By Lemma 43.19.1 we have

$$\text{coeff of } Z \text{ in } RHS(\mathcal{O}_V, \mathcal{O}_W) = \sum (-1)^i \mathrm{length}_B \mathrm{Tor}_i^B(A, C)$$

Since $\dim(V) < r$ or $\dim(W) < s$ we have $\dim(V \times W) < r + s$ which implies $\dim(C) < \dim(X)$ (small detail omitted). Moreover, the kernel I of $B \to A$ is generated by a regular sequence of length $\dim(X)$ (Lemma 43.13.3). Hence vanishing by Lemma 43.16.2 because the Hilbert function of C with respect to I has degree $\dim(C) < n$ by Algebra, Proposition 10.60.9. □

0B0X **Remark 43.19.5.** Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring. Let M and N be nonzero finite A-modules such that $M \otimes_A N$ is supported in $\{\mathfrak{m}\}$. Then

$$\chi(M, N) = \sum (-1)^i \mathrm{length}_A \mathrm{Tor}_i^A(M, N)$$

is finite. Let $r = \dim(\mathrm{Supp}(M))$ and $s = \dim(\mathrm{Supp}(N))$. In [**Ser65**] it is shown that $r + s \leq \dim(A)$ and the following conjectures are made:

(1) if $r + s < \dim(A)$, then $\chi(M, N) = 0$, and
(2) if $r + s = \dim(A)$, then $\chi(M, N) > 0$.

The arguments that prove Lemma 43.19.4 and Proposition 43.19.3 can be leveraged (as is done in Serre's text) to show that (1) and (2) are true if A contains a field. Currently, conjecture (1) is known in general and it is known that $\chi(M, N) \geq 0$ in general (Gabber). Positivity is, as far as we know, still an open problem.

[5] The reader can see that this is not a triviality by taking $r = s = 1$ and X a nonsingular surface and $V = W$ a closed point x of X. In this case there are 3 nonzero Tors of lengths $1, 2, 1$ at x.

43.20. Associativity of intersections

0B1K It is clear that proper intersections as defined above are commutative. Using the key Lemma 43.19.4 we can prove that (proper) intersection products are associative.

0B1L **Lemma 43.20.1.** *Let X be a nonsingular variety. Let U, V, W be closed subvarieties. Assume that U, V, W intersect properly pairwise and that $\dim(U \cap V \cap W) \leq \dim(U) + \dim(V) + \dim(W) - 2\dim(X)$. Then*

$$U \cdot (V \cdot W) = (U \cdot V) \cdot W$$

as cycles on X.

Proof. We are going to use Lemma 43.19.4 without further mention. This implies that

$$\begin{aligned} V \cdot W &= \sum (-1)^i [\mathrm{Tor}_i(\mathcal{O}_V, \mathcal{O}_W)]_{b+c-n} \\ U \cdot (V \cdot W) &= \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathcal{O}_U, \mathrm{Tor}_i(\mathcal{O}_V, \mathcal{O}_W))]_{a+b+c-2n} \\ U \cdot V &= \sum (-1)^i [\mathrm{Tor}_i(\mathcal{O}_U, \mathcal{O}_V)]_{a+b-n} \\ (U \cdot V) \cdot W &= \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathrm{Tor}_i(\mathcal{O}_U, \mathcal{O}_V), \mathcal{O}_W))]_{a+b+c-2n} \end{aligned}$$

where $\dim(U) = a$, $\dim(V) = b$, $\dim(W) = c$, $\dim(X) = n$. The assumptions in the lemma guarantee that the coherent sheaves in the formulae above satisfy the required bounds on dimensions of supports in order to make sense of these. Now consider the object

$$K = \mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_W$$

of the derived category $D_{Coh}(\mathcal{O}_X)$. We claim that the expressions obtained above for $U \cdot (V \cdot W)$ and $(U \cdot V) \cdot W$ are equal to

$$\sum (-1)^k [H^k(K)]_{a+b+c-2n}$$

This will prove the lemma. By symmetry it suffices to prove one of these equalities. To do this we represent $\mathcal{O}_U$ and $\mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_W$ by K-flat complexes $M^\bullet$ and $L^\bullet$ and use the spectral sequence associated to the double complex $M^\bullet \otimes_{\mathcal{O}_X} L^\bullet$ in Homology, Section 12.25. This is a spectral sequence with E_2 page

$$E_2^{p,q} = \mathrm{Tor}_{-p}(\mathcal{O}_U, \mathrm{Tor}_{-q}(\mathcal{O}_V, \mathcal{O}_W))$$

converging to $H^{p+q}(K)$ (details omitted; compare with More on Algebra, Example 15.62.4). Since lengths are additive in short exact sequences we see that the result is true. □

43.21. Flat pullback and intersection products

0B0B Short discussion of the interaction between intersections and flat pullback.

0B0Y **Lemma 43.21.1.** *Let $f : X \to Y$ be a flat morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on Y with $\dim(Supp(\mathcal{F})) \leq r$, $\dim(Supp(\mathcal{G})) \leq s$, and $\dim(Supp(\mathcal{F}) \cap Supp(\mathcal{G})) \leq r + s - \dim(Y)$. In this case the cycles $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly and*

$$f^*([\mathcal{F}]_r \cdot [\mathcal{G}]_s) = [f^*\mathcal{F}]_{r+e} \cdot [f^*\mathcal{G}]_{s+e}$$

Proof. The statement that $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly is immediate from the assumption that f has relative dimension e. By Lemmas 43.19.4 and 43.7.1 it suffices to show that

$$f^*\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) = \mathrm{Tor}_i^{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

as $\mathcal{O}_X$-modules. This follows from Cohomology, Lemma 20.27.3 and the fact that f^* is exact, so $Lf^*\mathcal{F} = f^*\mathcal{F}$ and similarly for $\mathcal{G}$. □

0B0Z **Lemma 43.21.2.** *Let $f : X \to Y$ be a flat morphism of nonsingular varieties. Let α be a r-cycle on Y and β an s-cycle on Y. Assume that α and β intersect properly. Then $f^*\alpha$ and $f^*\beta$ intersect properly and $f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta$.*

Proof. By linearity we may assume that $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V, W \subset Y$ of dimension r, s. Say f has relative dimension e. Then the lemma is a special case of Lemma 43.21.1 because $[V] = [\mathcal{O}_V]_r$, $[W] = [\mathcal{O}_W]_r$, $f^*[V] = [f^{-1}(V)]_{r+e} = [f^*\mathcal{O}_V]_{r+e}$, and $f^*[W] = [f^{-1}(W)]_{s+e} = [f^*\mathcal{O}_W]_{s+e}$. □

43.22. Projection formula for flat proper morphisms

0B0C Short discussion of the projection formula for flat proper morphisms.

0B10 **Lemma 43.22.1.** *Let $f : X \to Y$ be a flat proper morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let α be an r-cycle on X and let β be a s-cycle on Y. Assume that α and $f^*(\beta)$ intersect properly. Then $f_*(\alpha)$ and β intersect properly and*

$$f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$$

See [**Ser65**, Chapter V, C), Section 7, formula (10)] for a more general formula.

Proof. By linearity we reduce to the case where $\alpha = [V]$ and $\beta = [W]$ for some closed subvariety $V \subset X$ and $W \subset Y$ of dimension r and s. Then $f^{-1}(W)$ has pure dimension $s + e$. We assume the cycles $[V]$ and $f^*[W]$ intersect properly. We will use without further mention the fact that $V \cap f^{-1}(W) \to f(V) \cap W$ is surjective.

Let a be the dimension of the generic fibre of $V \to f(V)$. If $a > 0$, then $f_*[V] = 0$. In particular $f_*\alpha$ and β intersect properly. To finish this case we have to show that $f_*([V] \cdot f^*[W]) = 0$. However, since every fibre of $V \to f(V)$ has dimension $\geq a$ (see Morphisms, Lemma 29.28.4) we conclude that every irreducible component Z of $V \cap f^{-1}(W)$ has fibres of dimension $\geq a$ over $f(Z)$. This certainly implies what we want.

Assume that $V \to f(V)$ is generically finite. Let $Z \subset f(V) \cap W$ be an irreducible component. Let $Z_i \subset V \cap f^{-1}(W)$, $i = 1, \ldots, t$ be the irreducible components of $V \cap f^{-1}(W)$ dominating Z. By assumption each Z_i has dimension $r + s + e - \dim(X) = r + s - \dim(Y)$. Hence $\dim(Z) \leq r + s - \dim(Y)$. Thus we see that $f(V)$ and W intersect properly, $\dim(Z) = r + s - \dim(Y)$, and each $Z_i \to Z$ is generically finite. In particular, it follows that $V \to f(V)$ has finite fibre over the generic point ξ of Z. Thus $V \to Y$ is finite in an open neighbourhood of ξ, see Cohomology of Schemes, Lemma 30.21.2. Using a very general projection formula for derived tensor products, we get

$$Rf_*(\mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{O}_W) = Rf_*\mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{O}_W$$

see Derived Categories of Schemes, Lemma 36.22.1. Since f is flat, we see that $Lf^*\mathcal{O}_W = f^*\mathcal{O}_W$. Since $f|_V$ is finite in an open neighbourhood of ξ we have

$$(Rf_*\mathcal{F})_\xi = (f_*\mathcal{F})_\xi$$

for any coherent sheaf on X whose support is contained in V (see Cohomology of Schemes, Lemma 30.20.8). Thus we conclude that

0B11 (43.22.1.1)
$$\Big(f_*\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)\Big)_\xi = \Big(\mathrm{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{O}_V, \mathcal{O}_W)\Big)_\xi$$

for all i. Since $f^*[W] = [f^*\mathcal{O}_W]_{s+e}$ by Lemma 43.7.1 we have

$$[V] \cdot f^*[W] = \sum (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

by Lemma 43.19.4. Applying Lemma 43.6.1 we find

$$f_*([V] \cdot f^*[W]) = \sum (-1)^i [f_*\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

Since $f_*[V] = [f_*\mathcal{O}_V]_r$ by Lemma 43.6.1 we have

$$[f_*V] \cdot [W] = \sum (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(f_*\mathcal{O}_V, \mathcal{O}_W)]_{r+s-\dim(Y)}$$

again by Lemma 43.19.4. Comparing the formula for $f_*([V] \cdot f^*[W])$ with the formula for $f_*[V] \cdot [W]$ and looking at the coefficient of Z by taking lengths of stalks at ξ, we see that (43.22.1.1) finishes the proof. □

0B1M **Lemma 43.22.2.** *Let $X \to P$ be a closed immersion of nonsingular varieties. Let $C' \subset P \times \mathbf{P}^1$ be a closed subvariety of dimension $r+1$. Assume*

(1) *the fibre $C = C'_0$ has dimension r, i.e., $C' \to \mathbf{P}^1$ is dominant,*
(2) *C' intersects $X \times \mathbf{P}^1$ properly,*
(3) *$[C]_r$ intersects X properly.*

Then setting $\alpha = [C]_r \cdot X$ viewed as cycle on X and $\beta = C' \cdot X \times \mathbf{P}^1$ viewed as cycle on $X \times \mathbf{P}^1$, we have

$$\alpha = pr_{X,*}(\beta \cdot X \times 0)$$

as cycles on X where $pr_X : X \times \mathbf{P}^1 \to X$ is the projection.

Proof. Let $\mathrm{pr} : P \times \mathbf{P}^1 \to P$ be the projection. Since we are proving an equality of cycles it suffices to think of α, resp. β as a cycle on P, resp. $P \times \mathbf{P}^1$ and prove the result for pushing forward by pr. Because $\mathrm{pr}^*X = X \times \mathbf{P}^1$ and pr defines an isomorphism of C'_0 onto C the projection formula (Lemma 43.22.1) gives

$$\mathrm{pr}_*([C'_0]_r \cdot X \times \mathbf{P}^1) = [C]_r \cdot X = \alpha$$

On the other hand, we have $[C'_0]_r = C' \cdot P \times 0$ as cycles on $P \times \mathbf{P}^1$ by Lemma 43.17.1. Hence

$$[C'_0]_r \cdot X \times \mathbf{P}^1 = (C' \cdot P \times 0) \cdot X \times \mathbf{P}^1 = (C' \cdot X \times \mathbf{P}^1) \cdot P \times 0$$

by associativity (Lemma 43.20.1) and commutativity of the intersection product. It remains to show that the intersection product of $C' \cdot X \times \mathbf{P}^1$ with $P \times 0$ on $P \times \mathbf{P}^1$ is equal (as a cycle) to the intersection product of β with $X \times 0$ on $X \times \mathbf{P}^1$. Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k[E_k]$ and hence $\beta = \sum n_k[E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1 \subset P \times \mathbf{P}^1$. Then both intersections are equal to $\sum m_k[E_{k,0}]$ by Lemma 43.17.1 applied twice. This finishes the proof. □

43.23. Projections

0B1N Recall that we are working over a fixed algebraically closed ground field $\mathbf{C}$. If V is a finite dimensional vector space over $\mathbf{C}$ then we set

$$\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$$

where $\text{Sym}(V)$ is the symmetric algebra on V over $\mathbf{C}$. See Constructions, Example 27.21.2. The normalization is chosen such that $V = \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1))$. Of course we have $\mathbf{P}(V) \cong \mathbf{P}^n_{\mathbf{C}}$ if $\dim(V) = n + 1$. We note that $\mathbf{P}(V)$ is a nonsingular projective variety.

Let $p \in \mathbf{P}(V)$ be a closed point. The point p corresponds to a surjection $V \to L_p$ of vector spaces where $\dim(L_p) = 1$, see Constructions, Lemma 27.12.3. Let us denote $W_p = \text{Ker}(V \to L_p)$. *Projection from p* is the morphism

$$r_p : \mathbf{P}(V) \setminus \{p\} \longrightarrow \mathbf{P}(W_p)$$

of Constructions, Lemma 27.11.1. Here is a lemma to warm up.

0B1P **Lemma 43.23.1.** *Let V be a vector space of dimension $n+1$. Let $X \subset \mathbf{P}(V)$ be a closed subscheme. If $X \neq \mathbf{P}(V)$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the restriction of the projection r_p defines a finite morphism $r_p|_X : X \to \mathbf{P}(W_p)$.*

Proof. We claim the lemma holds with $U = \mathbf{P}(V) \setminus X$. For a closed point p of U we indeed obtain a morphism $r_p|_X : X \to \mathbf{P}(W_p)$. This morphism is proper because X is a proper scheme (Morphisms, Lemmas 29.43.5 and 29.41.7). On the other hand, the fibres of r_p are affine lines as can be seen by a direct calculation. Hence the fibres of $r_p|X$ are proper and affine, whence finite (Morphisms, Lemma 29.44.11). Finally, a proper morphism with finite fibres is finite (Cohomology of Schemes, Lemma 30.21.1). □

0B1Q **Lemma 43.23.2.** *Let V be a vector space of dimension $n + 1$. Let $X \subset \mathbf{P}(V)$ be a closed subvariety. Let $x \in X$ be a nonsingular point.*

1. *If $\dim(X) < n-1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \to r_p(X)$ is an isomorphism over an open neighbourhood of $r_p(x)$.*
2. *If $\dim(X) = n-1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \to \mathbf{P}(W_p)$ is étale at x.*

Proof. Proof of (1). Note that if $x, y \in X$ have the same image under r_p then p is on the line $\overline{xy}$. Consider the finite type scheme

$$T = \{(y, p) \mid y \in X \setminus \{x\},\ p \in \mathbf{P}(V),\ p \in \overline{xy}\}$$

and the morphisms $T \to X$ and $T \to \mathbf{P}(V)$ given by $(y, p) \mapsto y$ and $(y, p) \mapsto p$. Since each fibre of $T \to X$ is a line, we see that the dimension of T is $\dim(X)+1 < \dim(\mathbf{P}(V))$. Hence $T \to \mathbf{P}(V)$ is not surjective. On the other hand, consider the finite type scheme

$$T' = \{p \mid p \in \mathbf{P}(V) \setminus \{x\},\ \overline{xp} \text{ tangent to } X \text{ at } x\}$$

Then the dimension of T' is $\dim(X) < \dim(\mathbf{P}(V))$. Thus the morphism $T' \to \mathbf{P}(V)$ is not surjective either. Let $U \subset \mathbf{P}(V) \setminus X$ be nonempty open and disjoint from these images; such a U exists because the images of T and T' in $\mathbf{P}(V)$ are

constructible by Morphisms, Lemma 29.22.2. Then for $p \in U$ closed the projection $r_p|_X : X \to \mathbf{P}(W_p)$ is injective on the tangent space at x and $r_p^{-1}(\{r_p(x)\}) = \{x\}$. This means that r_p is unramified at x (Varieties, Lemma 33.16.8), finite by Lemma 43.23.1, and $r_p^{-1}(\{r_p(x)\}) = \{x\}$ thus Étale Morphisms, Lemma 41.7.3 applies and there is an open neighbourhood R of $r_p(x)$ in $\mathbf{P}(W_p)$ such that $(r_p|_X)^{-1}(R) \to R$ is a closed immersion which proves (1).

Proof of (2). In this case we still conclude that the morphism $T' \to \mathbf{P}(V)$ is not surjective. Arguing as above we conclude that for $U \subset \mathbf{P}(V)$ avoiding X and the image of T', the projection $r_p|_X : X \to \mathbf{P}(W_p)$ is étale at x and finite. □

0H8A **Lemma 43.23.3.** *Let V be a vector space. Let $Z \subset X \subset \mathbf{P}(V)$ be closed subvarieties with $Z \neq X \neq \mathbf{P}(V)$. Let $x \in Z$ be a point which is nonsingular on X. Then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_Z : Z \to r_p(Z)$ is an isomorphism over an open neighbourhood of $r_p(x)$.*

Proof. Arguing as in the proof of Lemma 43.23.2 we find a nonempty Zariski open U such that for $p \in U$ the map $r_p|_X : X \to r_p(X)$ is étale at x and $r_p^{-1}(r_p(\{x\})) \cap Z = \{x\}$. Details omitted. Set $Z' = r_p(Z)$ viewed as a closed subvariety of $\mathbf{P}(W_p)$. Then the morphism $\pi : Z \to Z'$ is finite (Lemma 43.23.1), unramified at x (small detail omitted) and $\pi^{-1}(\pi(\{x\})) = \{x\}$. The result follows from Étale Morphisms, Lemma 41.7.3. □

0B1R **Lemma 43.23.4.** *Let V be a vector space of dimension $n + 1$. Let $Y, Z \subset \mathbf{P}(V)$ be closed subvarieties. There is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ we have*

$$Y \cap r_p^{-1}(r_p(Z)) = (Y \cap Z) \cup E$$

with $E \subset Y$ closed and $\dim(E) \leq \dim(Y) + \dim(Z) + 1 - n$.

Proof. Set $Y' = Y \setminus Y \cap Z$. Let $y \in Y'$, $z \in Z$ be closed points with $r_p(y) = r_p(z)$. Then p is on the line $\overline{yz}$ passing through y and z. Consider the finite type scheme

$$T = \{(y, z, p) \mid y \in Y', z \in Z, p \in \overline{yz}\}$$

and the morphism $T \to \mathbf{P}(V)$ given by $(y, z, p) \mapsto p$. Observe that T is irreducible and that $\dim(T) = \dim(Y) + \dim(Z) + 1$. Hence the general fibre of $T \to \mathbf{P}(V)$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$, more precisely, there exists a nonempty open $U \subset \mathbf{P}(V) \setminus (Y \cup Z)$ over which the fibre has dimension at most $\dim(Y) + \dim(Z) + 1 - n$ (Varieties, Lemma 33.20.4). Let $p \in U$ be a closed point and let $F \subset T$ be the fibre of $T \to \mathbf{P}(V)$ over p. Then

$$(Y \cap r_p^{-1}(r_p(Z))) \setminus (Y \cap Z)$$

is the image of $F \to Y$, $(y, z, p) \mapsto y$. Again by Varieties, Lemma 33.20.4 the closure of the image of $F \to Y$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$. □

0B2T **Lemma 43.23.5.** *Let V be a vector space. Let $B \subset \mathbf{P}(V)$ be a closed subvariety of codimension ≥ 2. Let $p \in \mathbf{P}(V)$ be a closed point, $p \notin B$. Then there exists a line $\ell \subset \mathbf{P}(V)$ with $p \in \ell$ and $\ell \cap B = \emptyset$. Moreover, these lines sweep out an open subset of $\mathbf{P}(V)$.*

Proof. Consider the image of B under the projection $r_p : \mathbf{P}(V) \to \mathbf{P}(W_p)$. Since $\dim(W_p) = \dim(V) - 1$, we see that $r_p(B)$ has codimension ≥ 1 in $\mathbf{P}(W_p)$. For any $q \in \mathbf{P}(V)$ with $r_p(q) \notin r_p(B)$ we see that the line $\ell = \overline{pq}$ connecting p and q works. □

0B2U **Lemma 43.23.6.** *Let V be a vector space. Let $G = PGL(V)$. Then $G \times \mathbf{P}(V) \to \mathbf{P}(V)$ is doubly transitive.*

Proof. Omitted. Hint: This follows from the fact that $\mathrm{GL}(V)$ acts doubly transitive on pairs of linearly independent vectors. □

0B2V **Lemma 43.23.7.** *Let k be a field. Let $n \geq 1$ be an integer and let $x_{ij}, 1 \leq i, j \leq n$ be variables. Then*

$$\det \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & \ldots & \ldots & x_{nn} \end{pmatrix}$$

is an irreducible element of the polynomial ring $k[x_{ij}]$.

Proof. Let V be an n dimensional vector space. Translating into geometry the lemma signifies that the locus C of non-invertible linear maps $V \to V$ is irreducible (note that since the determinant has degree 1 in each variable, it is squarefree). Let W be a vector space of dimension $n - 1$. By elementary linear algebra, the morphism

$$\mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, V), \quad (\psi, \varphi) \longmapsto \psi \circ \varphi$$

has image C. Since the source is irreducible, so is the image. Details omitted. □

Let V be a vector space of dimension $n+1$. Set $E = \mathrm{End}(V)$. Let $E^\vee = \mathrm{Hom}(E, \mathbf{C})$ be the dual vector space. Write $\mathbf{P} = \mathbf{P}(E^\vee)$. There is a canonical linear map

$$V \longrightarrow V \otimes_{\mathbf{C}} E^\vee = \mathrm{Hom}(E, V)$$

sending $v \in V$ to the map $g \mapsto g(v)$ in $\mathrm{Hom}(E, V)$. Recall that we have a canonical map $E^\vee \to \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ which is an isomorphism. Hence we obtain a canonical map

$$\psi : V \otimes \mathcal{O}_{\mathbf{P}} \to V \otimes \mathcal{O}_{\mathbf{P}}(1)$$

of sheaves of modules on $\mathbf{P}$ which on global sections recovers the given map. Recall that a projective bundle $\mathbf{P}(\mathcal{E})$ is defined as the relative Proj of the symmetric algebra on $\mathcal{E}$, see Constructions, Definition 27.21.1. We are going to study the rational map between $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1))$ and $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}})$ associated to ψ. By Constructions, Lemma 27.16.10 we have a canonical isomorphism

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

By Constructions, Lemma 27.20.1 we see that

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1)) = \mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

Combining this with Constructions, Lemma 27.18.1 we obtain

0B2W (43.23.7.1) $$\mathbf{P} \times \mathbf{P}(V) \supset U(\psi) \xrightarrow{r_\psi} \mathbf{P} \times \mathbf{P}(V)$$

To understand this better we work out what happens on fibres over $\mathbf{P}$. Let $g \in E$ be nonzero. This defines a nonzero map $E^\vee \to \mathbf{C}$, hence a point $[g] \in \mathbf{P}$. On the other

hand, g defines a $\mathbf{C}$-linear map $g : V \to V$. Hence we obtain, by Constructions, Lemma 27.11.1 a map

$$\mathbf{P}(V) \supset U(g) \xrightarrow{r_g} \mathbf{P}(V)$$

What we will use below is that $U(g)$ is the fibre $U(\psi)_{[g]}$ and that r_g is the fibre of r_ψ over the point $[g]$. Another observation we will use is that the complement of $U(g)$ in $\mathbf{P}(V)$ is the image of the closed immersion

$$\mathbf{P}(\mathrm{Coker}(g)) \longrightarrow \mathbf{P}(V)$$

and the image of r_g is the image of the closed immersion

$$\mathbf{P}(\mathrm{Im}(g)) \longrightarrow \mathbf{P}(V)$$

0B1S **Lemma 43.23.8.** *With notation as above. Let X, Y be closed subvarieties of $\mathbf{P}(V)$ which intersect properly such that $X \neq \mathbf{P}(V)$ and $X \cap Y \neq \emptyset$. For a general line $\ell \subset \mathbf{P}$ with $[id_V] \in \ell$ we have*

(1) *$X \subset U_g$ for all $[g] \in \ell$,*
(2) *$g(X)$ intersects Y properly for all $[g] \in \ell$.*

Proof. Let $B \subset \mathbf{P}$ be the set of "bad" points, i.e., those points $[g]$ that violate either (1) or (2). Note that $[\mathrm{id}_V] \notin B$ by assumption. Moreover, B is closed. Hence it suffices to prove that $\dim(B) \leq \dim(\mathbf{P}) - 2$ (Lemma 43.23.5).

First, consider the open $G = \mathrm{PGL}(V) \subset \mathbf{P}$ consisting of points $[g]$ such that $g : V \to V$ is invertible. Since G acts doubly transitively on $\mathbf{P}(V)$ (Lemma 43.23.6) we see that

$$T = \{(x, y, [g]) \mid x \in X, y \in Y, [g] \in G, r_g(x) = y\}$$

is a locally trivial fibration over $X \times Y$ with fibre equal to the stabilizer of a point in G. Hence T is a variety. Observe that the fibre of $T \to G$ over $[g]$ is $r_g(X) \cap Y$. The morphism $T \to G$ is surjective, because any translate of X intersects Y (note that by the assumption that X and Y intersect properly and that $X \cap Y \neq \emptyset$ we see that $\dim(X) + \dim(Y) \geq \dim(\mathbf{P}(V))$ and then Varieties, Lemma 33.34.3 implies all translates of X intersect Y). Since the dimensions of fibres of a dominant morphism of varieties do not jump in codimension 1 (Varieties, Lemma 33.20.4) we conclude that $B \cap G$ has codimension ≥ 2.

Next we look at the complement $Z = \mathbf{P} \setminus G$. This is an irreducible variety because the determinant is an irreducible polynomial (Lemma 43.23.7). Thus it suffices to prove that B does not contain the generic point of Z. For a general point $[g] \in Z$ the cokernel $V \to \mathrm{Coker}(g)$ has dimension 1, hence $U(g)$ is the complement of a point. Since $X \neq \mathbf{P}(V)$ we see that for a general $[g] \in Z$ we have $X \subset U(g)$. Moreover, the morphism $r_g|_X : X \to r_g(X)$ is finite, hence $\dim(r_g(X)) = \dim(X)$. On the other hand, for such a g the image of r_g is the closed subspace $H = \mathbf{P}(\mathrm{Im}(g)) \subset \mathbf{P}(V)$ which has codimension 1. For general point of Z we see that $H \cap Y$ has dimension 1 less than Y (compare with Varieties, Lemma 33.35.3). Thus we see that we have to show that $r_g(X)$ and $H \cap Y$ intersect properly in H. For a fixed choice of H, we can by postcomposing g by an automorphism, move $r_g(X)$ by an arbitrary automorphism of $H = \mathbf{P}(\mathrm{Im}(g))$. Thus we can argue as above to conclude that the intersection of $H \cap Y$ with $r_g(X)$ is proper for general g with given $H = \mathbf{P}(\mathrm{Im}(g))$. Some details omitted. □

43.24. Moving Lemma

0B0D The moving lemma states that given an r-cycle α and an s-cycle β there exists α', $\alpha' \sim_{rat} \alpha$ such that α' and β intersect properly (Lemma 43.24.3). See [**Sam56**], [**Che58a**], [**Che58b**]. The key to this is Lemma 43.24.1; the reader may find this lemma in the form stated in [**Ful98**, Example 11.4.1] and find a proof in [**Rob72**].

0B0E **Lemma 43.24.1.** *Let $X \subset \mathbf{P}^N$ be a nonsingular closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $Z \subset X$ be a closed subvariety of dimension d and $T_i \subset X$, $i \in I$ be a finite collection of closed subvarieties of dimension d'. Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that* See [**Rob72**].

$$C \cdot X = Z + \sum\nolimits_{j \in J} m_j Z_j$$

where $Z_j \subset X$ are irreducible of dimension d, distinct from Z, and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X.

Proof. Write $\mathbf{P}^N = \mathbf{P}(V_N)$ so $\dim(V_N) = N+1$. We are going to choose a sequence of projections from points

$$\begin{aligned} & r_N : \mathbf{P}(V_N) \setminus \{p_N\} \to \mathbf{P}(V_{N-1}), \\ & r_{N-1} : \mathbf{P}(V_{N-1}) \setminus \{p_{N-1}\} \to \mathbf{P}(V_{N-2}), \\ & \ldots, \\ & r_{n+1} : \mathbf{P}(V_{n+1}) \setminus \{p_{n+1}\} \to \mathbf{P}(V_n) \end{aligned}$$

as in Section 43.23. At each step we will choose $p_N, p_{N-1}, \ldots, p_{n+1}$ in a suitable Zariski open set. Pick a closed point $x \in Z \subset X$. For every i pick closed points $x_{it} \in T_i \cap Z$, at least one in each irreducible component of $T_i \cap Z$. Taking the composition we obtain a morphism

$$\pi = (r_{n+1} \circ \ldots \circ r_N)|_X : X \longrightarrow \mathbf{P}(V_n)$$

which has the following properties

(1) π is finite,
(2) π is étale at x and all x_{it},
(3) $\pi|_Z : Z \to \pi(Z)$ is an isomorphism over an open neighbourhood of $\pi(x_{it})$,
(4) $T_i \cap \pi^{-1}(\pi(Z)) = (T_i \cap Z) \cup E_i$ with $E_i \subset T_i$ closed and $\dim(E_i) \leq d + d' + 1 - (n+1) = d + d' - n$.

It follows in a straightforward manner from Lemmas 43.23.1, 43.23.2, 43.23.3, and 43.23.4 and induction that we can do this; observe that the last projection is from $\mathbf{P}(V_{n+1})$ and that $\dim(V_{n+1}) = n+2$ which explains the inequality in (4). In a bit more detail, let $X_j, Z_j, T_{i,j} \subset \mathbf{P}(V_j)$ be the images of X, Z, T_i for $j = N, \ldots, n$. Then $T_{i,j+1} \cap r_{j+1}^{-1}(Z_j) = (T_{i,j+1} \cap Z_{j+1}) \cup E_{i,j+1}$ where we have a dimension bound on $E_{i,j+1}$ from Lemma 43.23.4. Then note that $T_i \cap \pi^{-1}(\pi(Z))$ is the union of $T_i \cap Z$ and the inverse images of the $E_{i,j}$. Since the transition morphisms between the X_j are finite we obtain the desired dimension bound.

Let $C \subset \mathbf{P}(V_N)$ be the scheme theoretic closure of $(r_{n+1} \circ \ldots \circ r_N)^{-1}(\pi(Z))$. Because π is étale at the point x of Z, we see that the closed subscheme $C \cap X = \pi^{-1}(\pi(Z))$

contains Z with multiplicity 1 (local calculation omitted). Hence by Lemma 43.17.2 we conclude that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

for some subvarieties $Z_j \subset X$ of dimension d. Note that

$$C \cap X = \pi^{-1}(\pi(Z))$$

set theoretically. Hence $T_i \cap Z_j \subset T_i \cap \pi^{-1}(\pi(Z)) \subset (T_i \cap Z) \cup E_i$. For any irreducible component of $T_i \cap Z_j$ contained in E_i we have the desired dimension bound. Finally, let V be an irreducible component of $T_i \cap Z_j$ which is contained in $T_i \cap Z$. To finish the proof it suffices to show that V does not contain any of the points x_{it}, because then $\dim(V) < \dim(Z \cap T_i)$. To show this it suffices to show that $x_{it} \notin Z_j$ for all i, t, j.

Set $Z' = \pi(Z)$ and $Z'' = \pi^{-1}(Z')$, scheme theoretically. By condition (3) we can find an open $U \subset \mathbf{P}(V_n)$ containing $\pi(x_{it})$ such that $\pi^{-1}(U) \cap Z \to U \cap Z'$ is an isomorphism. In particular, $Z \to Z'$ is a local isomorphism at x_{it}. On the other hand, $Z'' \to Z'$ is étale at x_{it} by condition (2). Hence the closed immersion $Z \to Z''$ is étale at x_{it} (Morphisms, Lemma 29.36.18). Thus $Z = Z''$ in a Zariski neighbourhood of x_{it} which proves the assertion. □

The actual moving is done using the following lemma.

0B1T **Lemma 43.24.2.** *Let $C \subset \mathbf{P}^N$ be a closed subvariety. Let $X \subset \mathbf{P}^N$ be subvariety and let $T_i \subset X$ be a finite collection of closed subvarieties. Assume that C and X intersect properly. Then there exists a closed subvariety $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ such that*

(1) *$C' \to \mathbf{P}^1$ is dominant,*
(2) *$C'_0 = C$ scheme theoretically,*
(3) *C' and $X \times \mathbf{P}^1$ intersect properly,*
(4) *C'_∞ properly intersects each of the given T_i.*

Proof. If $C \cap X = \emptyset$, then we take the constant family $C' = C \times \mathbf{P}^1$. Thus we may and do assume $C \cap X \neq \emptyset$.

Write $\mathbf{P}^N = \mathbf{P}(V)$ so $\dim(V) = N + 1$. Let $E = \mathrm{End}(V)$. Let $E^\vee = \mathrm{Hom}(E, \mathbf{C})$. Set $\mathbf{P} = \mathbf{P}(E^\vee)$ as in Lemma 43.23.8. Choose a general line $\ell \subset \mathbf{P}$ passing through id_V. Set $C' \subset \ell \times \mathbf{P}(V)$ equal to the closed subscheme having fibre $r_g(C)$ over $[g] \in \ell$. More precisely, C' is the image of

$$\ell \times C \subset \mathbf{P} \times \mathbf{P}(V)$$

under the morphism (43.23.7.1). By Lemma 43.23.8 this makes sense, i.e., $\ell \times C \subset U(\psi)$. The morphism $\ell \times C \to C'$ is finite and $C'_{[g]} = r_g(C)$ set theoretically for all $[g] \in \ell$. Parts (1) and (2) are clear with $0 = [\mathrm{id}_V] \in \ell$. Part (3) follows from the fact that $r_g(C)$ and X intersect properly for all $[g] \in \ell$. Part (4) follows from the fact that a general point $\infty = [g] \in \ell$ is a general point of $\mathbf{P}$ and for such as point $r_g(C) \cap T$ is proper for any closed subvariety T of $\mathbf{P}(V)$. Details omitted. □

0B1U **Lemma 43.24.3.** *Let X be a nonsingular projective variety. Let α be an r-cycle and β be an s-cycle on X. Then there exists an r-cycle α' such that $\alpha' \sim_{rat} \alpha$ and such that α' and β intersect properly.*

Proof. Write $\beta = \sum n_i[T_i]$ for some subvarieties $T_i \subset X$ of dimension s. By linearity we may assume that $\alpha = [Z]$ for some irreducible closed subvariety $Z \subset X$ of dimension r. We will prove the lemma by induction on the maximum e of the integers

$$\dim(Z \cap T_i)$$

The base case is $e = r + s - \dim(X)$. In this case Z intersects β properly and the lemma is trivial.

Induction step. Assume that $e > r + s - \dim(X)$. Choose an embedding $X \subset \mathbf{P}^N$ and apply Lemma 43.24.1 to find a closed subvariety $C \subset \mathbf{P}^N$ such that $C \cdot X = [Z] + \sum m_j[Z_j]$ and such that the induction hypothesis applies to each Z_j. Next, apply Lemma 43.24.2 to C, X, T_i to find $C' \subset \mathbf{P}^N \times \mathbf{P}^1$. Let $\gamma = C' \cdot X \times \mathbf{P}^1$ viewed as a cycle on $X \times \mathbf{P}^1$. By Lemma 43.22.2 we have

$$[Z] + \sum m_j[Z_j] = \mathrm{pr}_{X,*}(\gamma \cdot X \times 0)$$

On the other hand the cycle $\gamma_\infty = \mathrm{pr}_{X,*}(\gamma \cdot X \times \infty)$ is supported on $C'_\infty \cap X$ hence intersects β properly. Thus we see that $[Z] \sim_{rat} -\sum m_j[Z_j] + \gamma_\infty$ by Lemma 43.17.1. Since by induction each $[Z_j]$ is rationally equivalent to a cycle which properly intersects β this finishes the proof. □

43.25. Intersection products and rational equivalence

0B0F With definitions as above we show that the intersection product is well defined modulo rational equivalence. We first deal with a special case.

0B60 **Lemma 43.25.1.** *Let X be a nonsingular variety. Let $W \subset X \times \mathbf{P}^1$ be an $(s+1)$-dimensional subvariety dominating $\mathbf{P}^1$. Let W_a, resp. W_b be the fibre of $W \to \mathbf{P}^1$ over a, resp. b. Let V be a r-dimensional subvariety of X such that V intersects both W_a and W_b properly. Then $[V] \cdot [W_a]_r \sim_{rat} [V] \cdot [W_b]_r$.*

Proof. We have $[W_a]_r = \mathrm{pr}_{X,*}(W \cdot X \times a)$ and similarly for $[W_b]_r$, see Lemma 43.17.1. Thus we reduce to showing

$$V \cdot \mathrm{pr}_{X,*}(W \cdot X \times a) \sim_{rat} V \cdot \mathrm{pr}_{X,*}(W \cdot X \times b).$$

Applying the projection formula Lemma 43.22.1 we get

$$V \cdot \mathrm{pr}_{X,*}(W \cdot X \times a) = \mathrm{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a))$$

and similarly for b. Thus we reduce to showing

$$\mathrm{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \sim_{rat} \mathrm{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b))$$

If $V \times \mathbf{P}^1$ intersects W properly, then associativity for the intersection multiplicities (Lemma 43.20.1) gives $V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = (V \times \mathbf{P}^1 \cdot W) \cdot X \times a$ and similarly for b. Thus we reduce to showing

$$\mathrm{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times a) \sim_{rat} \mathrm{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times b)$$

which is true by Lemma 43.17.1.

The argument above does not quite work. The obstruction is that we do not know that $V \times \mathbf{P}^1$ and W intersect properly. We only know that V and W_a and V and W_b intersect properly. Let Z_i, $i \in I$ be the irreducible components of $V \times \mathbf{P}^1 \cap W$. Then we know that $\dim(Z_i) \geq r+1+s+1-n-1 = r+s+1-n$ where $n = \dim(X)$, see Lemma 43.13.4. Since we have assumed that V and W_a intersect properly, we

see that $\dim(Z_{i,a}) = r + s - n$ or $Z_{i,a} = \emptyset$. On the other hand, if $Z_{i,a} \neq \emptyset$, then $\dim(Z_{i,a}) \geq \dim(Z_i) - 1 = r + s - n$. It follows that $\dim(Z_i) = r + s + 1 - n$ if Z_i meets $X \times a$ and in this case $Z_i \to \mathbf{P}^1$ is surjective. Thus we may write $I = I' \amalg I''$ where I' is the set of $i \in I$ such that $Z_i \to \mathbf{P}^1$ is surjective and I'' is the set of $i \in I$ such that Z_i lies over a closed point $t_i \in \mathbf{P}^1$ with $t_i \neq a$ and $t_i \neq b$. Consider the cycle

$$\gamma = \sum\nolimits_{i \in I'} e_i [Z_i]$$

where we take

$$e_i = \sum\nolimits_p (-1)^p \text{length}_{\mathcal{O}_{X \times \mathbf{P}^1, Z_i}} \text{Tor}_p^{\mathcal{O}_{X \times \mathbf{P}^1, Z_i}}(\mathcal{O}_{V \times \mathbf{P}^1, Z_i}, \mathcal{O}_{W, Z_i})$$

We will show that γ can be used as a replacement for the intersection product of $V \times \mathbf{P}^1$ and W.

We will show this using associativity of intersection products in exactly the same way as above. Let $U = \mathbf{P}^1 \setminus \{t_i, i \in I''\}$. Note that $X \times a$ and $X \times b$ are contained in $X \times U$. The subvarieties

$$V \times U, \quad W_U, \quad X \times a \quad \text{of} \quad X \times U$$

intersect transversally pairwise by our choice of U and moreover $\dim(V \times U \cap W_U \cap X \times a) = \dim(V \cap W_a)$ has the expected dimension. Thus we see that

$$V \times U \cdot (W_U \cdot X \times a) = (V \times U \cdot W_U) \cdot X \times a$$

as cycles on $X \times U$ by Lemma 43.20.1. By construction γ restricts to the cycle $V \times U \cdot W_U$ on $X \times U$. Trivially, $V \times \mathbf{P}^1 \cdot (W \times X \times a)$ restricts to $V \times U \cdot (W_U \cdot X \times a)$ on $X \times U$. Hence

$$V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = \gamma \cdot X \times a$$

as cycles on $X \times \mathbf{P}^1$ (because both sides are contained in $X \times U$ and are equal after restricting to $X \times U$ by what was said before). Since we have the same for b we conclude

$$\begin{aligned} V \cdot [W_a] &= \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \\ &= \text{pr}_{X,*}(\gamma \cdot X \times a) \\ &\sim_{rat} \text{pr}_{X,*}(\gamma \cdot X \times b) \\ &= \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b)) \\ &= V \cdot [W_b] \end{aligned}$$

The first and the last equality by the first paragraph of the proof, the second and penultimate equalities were shown in this paragraph, and the middle equivalence is Lemma 43.17.1. □

0B1V **Theorem 43.25.2.** *Let X be a nonsingular projective variety. Let α, resp. β be an r, resp. s cycle on X. Assume that α and β intersect properly so that $\alpha \cdot \beta$ is defined. Finally, assume that $\alpha \sim_{rat} 0$. Then $\alpha \cdot \beta \sim_{rat} 0$.*

Proof. Pick a closed immersion $X \subset \mathbf{P}^N$. By linearity it suffices to prove the result when $\beta = [Z]$ for some s-dimensional closed subvariety $Z \subset X$ which intersects α properly. The condition $\alpha \sim_{rat} 0$ means there are finitely many $(r+1)$-dimensional closed subvarieties $W_i \subset X \times \mathbf{P}^1$ such that

$$\alpha = \sum [W_{i,a_i}]_r - [W_{i,b_i}]_r$$

for some pairs of points a_i, b_i of $\mathbf{P}^1$. Let W^t_{i,a_i} and W^t_{i,b_i} be the irreducible components of W_{i,a_i} and W_{i,b_i}. We will use induction on the maximum d of the integers

$$\dim(Z \cap W^t_{i,a_i}), \quad \dim(Z \cap W^t_{i,b_i})$$

The main problem in the rest of the proof is that although we know that Z intersects α properly, it may not be the case that Z intersects the "intermediate" varieties W^t_{i,a_i} and W^t_{i,b_i} properly, i.e., it may happen that $d > r + s - \dim(X)$.

Base case: $d = r + s - \dim(X)$. In this case all the intersections of Z with the W^t_{i,a_i} and W^t_{i,b_i} are proper and the desired result follows from Lemma 43.25.1, because it applies to show that $[Z] \cdot [W_{i,a_i}]_r \sim_{rat} [Z] \cdot [W_{i,b_i}]_r$ for each i.

Induction step: $d > r + s - \dim(X)$. Apply Lemma 43.24.1 to $Z \subset X$ and the family of subvarieties $\{W^t_{i,a_i}, W^t_{i,b_i}\}$. Then we find a closed subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that

$$\dim(Z_j \cap W^t_{i,a_i}) \leq \dim(Z \cap W^t_{i,a_i}), \quad \dim(Z_j \cap W^t_{i,b_i}) \leq \dim(Z \cap W^t_{i,b_i})$$

with strict inequality if the right hand side is $> r + s - \dim(X)$. This implies two things: (a) the induction hypothesis applies to each Z_j, and (b) $C \cdot X$ and α intersect properly (because α is a linear combination of those $[W^t_{i,a_i}]$ and $[W^t_{i,a_i}]$ which intersect Z properly). Next, pick $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ as in Lemma 43.24.2 with respect to C, X, and W^t_{i,a_i}, W^t_{i,b_i}. Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k [E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1$ of dimension $s + 1$. Note that $n_k > 0$ for all k by Proposition 43.19.3. By Lemma 43.22.2 we have

$$[Z] + \sum m_j [Z_j] = \sum n_k [E_{k,0}]_s$$

Since $E_{k,0} \subset C \cap X$ we see that $[E_{k,0}]_s$ and α intersect properly. On the other hand, the cycle

$$\gamma = \sum n_k [E_{k,\infty}]_s$$

is supported on $C'_\infty \cap X$ and hence properly intersects each W^t_{i,a_i}, W^t_{i,b_i}. Thus by the base case and linearity, we see that

$$\gamma \cdot \alpha \sim_{rat} 0$$

As we have seen that $E_{k,0}$ and $E_{k,\infty}$ intersect α properly Lemma 43.25.1 applied to $E_k \subset X \times \mathbf{P}^1$ and α gives

$$[E_{k,0}] \cdot \alpha \sim_{rat} [E_{k,\infty}] \cdot \alpha$$

Putting everything together we have

$$\begin{aligned}
[Z] \cdot \alpha &= (\sum n_k [E_{k,0}]_r - \sum m_j [Z_j]) \cdot \alpha \\
&\sim_{rat} \sum n_k [E_{k,0}] \cdot \alpha \quad \text{(by induction hypothesis)} \\
&\sim_{rat} \sum n_k [E_{k,\infty}] \cdot \alpha \quad \text{(by the lemma)} \\
&= \gamma \cdot \alpha \\
&\sim_{rat} 0 \quad \text{(by base case)}
\end{aligned}$$

This finishes the proof. □

0B61 **Remark 43.25.3.** Lemma 43.24.3 and Theorem 43.25.2 also hold for nonsingular quasi-projective varieties with the same proof. The only change is that one needs to prove the following version of the moving Lemma 43.24.1: Let $X \subset \mathbf{P}^N$ be a closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $X^{reg} \subset X$ be the open subset of nonsingular points. Let $Z \subset X^{reg}$ be a closed subvariety of dimension d and $T_i \subset X^{reg}$, $i \in I$ be a finite collection of closed subvarieties of dimension d'. Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that

$$(C \cdot X)|_{X^{reg}} = Z + \sum\nolimits_{j \in J} m_j Z_j$$

where $Z_j \subset X^{reg}$ are irreducible of dimension d, distinct from Z, and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X^{reg}.

43.26. Chow rings

0B0G Let X be a nonsingular projective variety. We define the intersection product

$$\mathrm{CH}_r(X) \times \mathrm{CH}_s(X) \longrightarrow \mathrm{CH}_{r+s-\dim(X)}(X), \quad (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

as follows. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(X)$. If α and β intersect properly, we use the definition given in Section 43.17. If not, then we choose $\alpha \sim_{rat} \alpha'$ as in Lemma 43.24.3 and we set

$$\alpha \cdot \beta = \text{class of } \alpha' \cdot \beta \in \mathrm{CH}_{r+s-\dim(X)}(X)$$

This is well defined and passes through rational equivalence by Theorem 43.25.2. The intersection product on $\mathrm{CH}_*(X)$ is commutative (this is clear), associative (Lemma 43.20.1) and has a unit $[X] \in \mathrm{CH}_{\dim(X)}(X)$.

We often use $\mathrm{CH}^c(X) = \mathrm{CH}_{\dim X - c}(X)$ to denote the Chow group of cycles of codimension c, see Chow Homology, Section 42.42. The intersection product defines a product

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \longrightarrow \mathrm{CH}^{k+l}(X)$$

which is commutative, associative, and has a unit $1 = [X] \in \mathrm{CH}^0(X)$.

43.27. Pullback for a general morphism

0B0H Let $f : X \to Y$ be a morphism of nonsingular projective varieties. We define

$$f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+\dim X - \dim Y}(X)$$

by the rule

$$f^*(\alpha) = pr_{X,*}(\Gamma_f \cdot pr_Y^*(\alpha))$$

where $\Gamma_f \subset X \times Y$ is the graph of f. Note that in this generality, it is defined only on cycle classes and not on cycles. With the notation CH^* introduced in Section 43.26 we may think of pullback as a map

$$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$$

in other words, it is a map of graded abelian groups.

0B2X **Lemma 43.27.1.** *Let $f : X \to Y$ be a morphism of nonsingular projective varieties. The pullback map on chow groups satisfies:*

(1) *$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ is a ring map,*
(2) *$(g \circ f)^* = f^* \circ g^*$ for a composable pair f, g,*

(3) *the projection formula holds:* $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$*, and*
(4) *if* f *is flat then it agrees with the previous definition.*

Proof. All of these follow readily from the results above.

For (1) it suffices to show that $\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha) \cdot \mathrm{pr}_{X,*}(\Gamma_f \cdot \beta)$ for cycles α, β on $X \times Y$. If α is a cycle on $X \times Y$ which intersects Γ_f properly, then it is easy to see that

$$\Gamma_f \cdot \alpha = \Gamma_f \cdot \mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha))$$

as cycles because Γ_f is a graph. Thus we get the first equality in

$$\begin{aligned} \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) &= \mathrm{pr}_{X,*}(\Gamma_f \cdot \mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha)) \cdot \beta) \\ &= \mathrm{pr}_{X,*}(\mathrm{pr}_X^*(\mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha)) \cdot (\Gamma_f \cdot \beta)) \\ &= \mathrm{pr}_{X,*}(\Gamma_f \cdot \alpha) \cdot \mathrm{pr}_{X,*}(\Gamma_f \cdot \beta) \end{aligned}$$

the last step by the projection formula in the flat case (Lemma 43.22.1).

If $g : Y \to Z$ then property (2) follows formally from the observation that

$$\Gamma = \mathrm{pr}_{X\times Y}^*\Gamma_f \cdot \mathrm{pr}_{Y\times Z}^*\Gamma_g$$

in $Z_*(X \times Y \times Z)$ where $\Gamma = \{(x, f(x), g(f(x))\}$ and maps isomorphically to $\Gamma_{g\circ f}$ in $X \times Z$. The equality follows from the scheme theoretic equality and Lemma 43.14.3.

For (3) we use the projection formula for flat maps twice

$$\begin{aligned} f_*(\alpha \cdot pr_{X,*}(\Gamma_f \cdot pr_Y^*(\beta))) &= f_*(pr_{X,*}(pr_X^*\alpha \cdot \Gamma_f \cdot pr_Y^*(\beta))) \\ &= pr_{Y,*}(pr_X^*\alpha \cdot \Gamma_f \cdot pr_Y^*(\beta))) \\ &= pt_{Y,*}(pr_X^*\alpha \cdot \Gamma_f) \cdot \beta \\ &= f_*(\alpha) \cdot \beta \end{aligned}$$

where in the last equality we use the remark on graphs made above. This proves (3).

Property (4) rests on identifying the intersection product $\Gamma_f \cdot pr_Y^*\alpha$ in the case f is flat. Namely, in this case if $V \subset Y$ is a closed subvariety, then every generic point ξ of the scheme $f^{-1}(V) \cong \Gamma_f \cap pr_Y^{-1}(V)$ lies over the generic point of V. Hence the local ring of $pr_Y^{-1}(V) = X \times V$ at ξ is Cohen-Macaulay. Since $\Gamma_f \subset X \times Y$ is a regular immersion (as a morphism of smooth projective varieties) we find that

$$\Gamma_f \cdot pr_Y^*[V] = [\Gamma_f \cap pr_Y^{-1}(V)]_d$$

with d the dimension of $\Gamma_f \cap pr_Y^{-1}(V)$, see Lemma 43.16.5. Since $\Gamma_f \cap pr_Y^{-1}(V)$ maps isomorphically to $f^{-1}(V)$ we conclude. □

43.28. Pullback of cycles

0B0I Suppose that X and Y be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. Suppose that $Z \subset Y$ is a closed subvariety. Let $f^{-1}(Z)$ be the scheme theoretic inverse image:

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a fibre product diagram of schemes. In particular $f^{-1}(Z) \subset X$ is a closed subscheme of X. In this case we always have

$$\dim f^{-1}(Z) \geq \dim Z + \dim X - \dim Y.$$

If equality holds in the formula above, then $f^*[Z] = [f^{-1}(Z)]_{\dim Z + \dim X - \dim Y}$ provided that the scheme Z is Cohen-Macaulay at the images of the generic points of $f^{-1}(Z)$. This follows by identifying $f^{-1}(Z)$ with the scheme theoretic intersection of Γ_f and $X \times Z$ and using Lemma 43.16.5. Details are similar to the proof of part (4) of Lemma 43.27.1 above.

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CHAPTER 44

Picard Schemes of Curves

0B92 **44.1. Introduction**

0B93 In this chapter we do just enough work to construct the Picard scheme of a projective nonsingular curve over an algebraically closed field. See [**Kle05**] for a more thorough discussion as well as historical background.

Later in the Stacks project we will discuss Hilbert and Quot functors in much greater generality.

44.2. Hilbert scheme of points

0B94 Let $X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. For a scheme T over S we let

$$\mathrm{Hilb}^d_{X/S}(T) = \left\{ \begin{array}{l} Z \subset X_T \text{ closed subscheme such that} \\ Z \to T \text{ is finite locally free of degree } d \end{array} \right\}$$

If $T' \to T$ is a morphism of schemes over S and if $Z \in \mathrm{Hilb}^d_{X/S}(T)$, then the base change $Z_{T'} \subset X_{T'}$ is an element of $\mathrm{Hilb}^d_{X/S}(T')$. In this way we obtain a functor

$$\mathrm{Hilb}^d_{X/S} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hilb}^d_{X/S}(T)$$

In general $\mathrm{Hilb}^d_{X/S}$ is an algebraic space (insert future reference here). In this section we will show that $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme if any finite number of points in a fibre of $X \to S$ are contained in an affine open. If $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme, we often denote this scheme by $\underline{\mathrm{Hilb}}^d_{X/S}$.

0B95 **Lemma 44.2.1.** *Let $X \to S$ be a morphism of schemes. The functor $\mathrm{Hilb}^d_{X/S}$ satisfies the sheaf property for the fpqc topology (Topologies, Definition 34.9.13).*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies, Lemma 34.9.8) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ is a collection of elements such that Z_i and $Z_{i'}$ map to the same element of $\mathrm{Hilb}^d_{X/S}(T_i \times_T T_{i'})$. By effective descent for closed immersions (Descent, Lemma 35.37.2) there is a closed immersion $Z \to X_T$ whose base change by $X_i \to X_T$ is equal to $Z_i \to X_i$. The morphism $Z \to T$ then has the property that its base change to T_i is the morphism $Z_i \to T_i$. Hence $Z \to T$ is finite locally free of degree d by Descent, Lemma 35.23.32. □

0B96 **Lemma 44.2.2.** *Let $X \to S$ be a morphism of schemes. If $X \to S$ is of finite presentation, then the functor $\mathrm{Hilb}^d_{X/S}$ is limit preserving (Limits, Remark 32.6.2).*

Proof. Let $T = \lim T_i$ be a limit of affine schemes over S. We have to show that $\mathrm{Hilb}^d_{X/S}(T) = \mathrm{colim}\, \mathrm{Hilb}^d_{X/S}(T_i)$. Observe that if $Z \to X_T$ is an element of $\mathrm{Hilb}^d_{X/S}(T)$, then $Z \to T$ is of finite presentation. Hence by Limits, Lemma 32.10.1 there exists an i, a scheme Z_i of finite presentation over T_i, and a morphism $Z_i \to X_{T_i}$ over T_i whose base change to T gives $Z \to X_T$. We apply Limits, Lemma 32.8.5 to see that we may assume $Z_i \to X_{T_i}$ is a closed immersion after increasing i. We apply Limits, Lemma 32.8.8 to see that $Z_i \to T_i$ is finite locally free of degree d after possibly increasing i. Then $Z_i \in \mathrm{Hilb}^d_{X/S}(T_i)$ as desired. □

Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes over S. Then there is a transformation of functors

$$\mathrm{Hilb}^d_{X/S} \longrightarrow \mathrm{Hilb}^d_{Y/S}$$

which maps an element $Z \in \mathrm{Hilb}^d_{X/S}(T)$ to $i_T(Z) \subset Y_T$ in $\mathrm{Hilb}^d_{Y/S}$. Here $i_T : X_T \to Y_T$ is the base change of i.

0B97 **Lemma 44.2.3.** *Let S be a scheme. Let $i : X \to Y$ be a closed immersion of schemes. If $\mathrm{Hilb}^d_{Y/S}$ is representable by a scheme, so is $\mathrm{Hilb}^d_{X/S}$ and the corresponding morphism of schemes $\underline{\mathrm{Hilb}}^d_{X/S} \to \underline{\mathrm{Hilb}}^d_{Y/S}$ is a closed immersion.*

Proof. Let T be a scheme over S and let $Z \in \mathrm{Hilb}^d_{Y/S}(T)$. Claim: there is a closed subscheme $T_X \subset T$ such that a morphism of schemes $T' \to T$ factors through T_X if and only if $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$. Applying this to a scheme T_{univ} representing $\mathrm{Hilb}^d_{Y/S}$ and the universal object[1] $Z_{univ} \in \mathrm{Hilb}^d_{Y/S}(T_{univ})$ we get a closed subscheme $T_{univ,X} \subset T_{univ}$ such that $Z_{univ,X} = Z_{univ} \times_{T_{univ}} T_{univ,X}$ is a closed subscheme of $X \times_S T_{univ,X}$ and hence defines an element of $\mathrm{Hilb}^d_{X/S}(T_{univ,X})$. A formal argument then shows that $T_{univ,X}$ is a scheme representing $\mathrm{Hilb}^d_{X/S}$ with universal object $Z_{univ,X}$.

Proof of the claim. Consider $Z' = X_T \times_{Y_T} Z$. Given $T' \to T$ we see that $Z_{T'} \to Y_{T'}$ factors through $X_{T'}$ if and only if $Z'_{T'} \to Z_{T'}$ is an isomorphism. Thus the claim follows from the very general More on Flatness, Lemma 38.23.4. However, in this special case one can prove the statement directly as follows: first reduce to the case $T = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$. After shrinking T further we may assume there is an isomorphism $\varphi : B \to A^{\oplus d}$ as A-modules. Then $Z' = \mathrm{Spec}(B/J)$ for some ideal $J \subset B$. Let $g_\beta \in J$ be a collection of generators and write $\varphi(g_\beta) = (g^1_\beta, \ldots, g^d_\beta)$. Then it is clear that T_X is given by $\mathrm{Spec}(A/(g^j_\beta))$. □

0B98 **Lemma 44.2.4.** *Let $X \to S$ be a morphism of schemes. If $X \to S$ is separated and $\mathrm{Hilb}^d_{X/S}$ is representable, then $\underline{\mathrm{Hilb}}^d_{X/S} \to S$ is separated.*

Proof. In this proof all unadorned products are over S. Let $H = \underline{\mathrm{Hilb}}^d_{X/S}$ and let $Z \in \mathrm{Hilb}^d_{X/S}(H)$ be the universal object. Consider the two objects $Z_1, Z_2 \in \mathrm{Hilb}^d_{X/S}(H \times H)$ we get by pulling back Z by the two projections $H \times H \to H$. Then $Z_1 = Z \times H \subset X_{H \times H}$ and $Z_2 = H \times Z \subset X_{H \times H}$. Since H represents the functor $\mathrm{Hilb}^d_{X/S}$, the diagonal morphism $\Delta : H \to H \times H$ has the following universal property: A morphism of schemes $T \to H \times H$ factors through Δ if and only if $Z_{1,T} = Z_{2,T}$ as elements of $\mathrm{Hilb}^d_{X/S}(T)$. Set $Z = Z_1 \times_{X_{H \times H}} Z_2$. Then we see

[1] See Categories, Section 4.3

that $T \to H \times H$ factors through Δ if and only if the morphisms $Z_T \to Z_{1,T}$ and $Z_T \to Z_{2,T}$ are isomorphisms. It follows from the very general More on Flatness, Lemma 38.23.4 that Δ is a closed immersion. In the proof of Lemma 44.2.3 the reader finds an alternative easier proof of the needed result in our special case. □

0B99 **Lemma 44.2.5.** *Let $X \to S$ be a morphism of affine schemes. Let $d \geq 0$. Then $\mathrm{Hilb}^d_{X/S}$ is representable.*

Proof. Say $S = \mathrm{Spec}(R)$. Then we can choose a closed immersion of X into the spectrum of $R[x_i; i \in I]$ for some set I (of sufficiently large cardinality. Hence by Lemma 44.2.3 we may assume that $X = \mathrm{Spec}(A)$ where $A = R[x_i; i \in I]$. We will use Schemes, Lemma 26.15.4 to prove the lemma in this case.

Condition (1) of the lemma follows from Lemma 44.2.1.

For every subset $W \subset A$ of cardinality d we will construct a subfunctor F_W of $\mathrm{Hilb}^d_{X/S}$. (It would be enough to consider the case where W consists of a collection of monomials in the x_i but we do not need this.) Namely, we will say that $Z \in \mathrm{Hilb}^d_{X/S}(T)$ is in $F_W(T)$ if and only if the $\mathcal{O}_T$-linear map

$$\bigoplus\nolimits_{f \in W} \mathcal{O}_T \longrightarrow (Z \to T)_*\mathcal{O}_Z, \quad (g_f) \longmapsto \sum g_f f|_Z$$

is surjective (equivalently an isomorphism). Here for $f \in A$ and $Z \in \mathrm{Hilb}^d_{X/S}(T)$ we denote $f|_Z$ the pullback of f by the morphism $Z \to X_T \to X$.

Openness, i.e., condition (2)(b) of the lemma. This follows from Algebra, Lemma 10.79.4.

Covering, i.e., condition (2)(c) of the lemma. Since

$$A \otimes_R \mathcal{O}_T = (X_T \to T)_*\mathcal{O}_{X_T} \to (Z \to T)_*\mathcal{O}_Z$$

is surjective and since $(Z \to T)_*\mathcal{O}_Z$ is finite locally free of rank d, for every point $t \in T$ we can find a finite subset $W \subset A$ of cardinality d whose images form a basis of the d-dimensional $\kappa(t)$-vector space $((Z \to T)_*\mathcal{O}_Z)_t \otimes_{\mathcal{O}_{T,t}} \kappa(t)$. By Nakayama's lemma there is an open neighbourhood $V \subset T$ of t such that $Z_V \in F_W(V)$.

Representable, i.e., condition (2)(a) of the lemma. Let $W \subset A$ have cardinality d. We claim that F_W is representable by an affine scheme over R. We will construct this affine scheme here, but we encourage the reader to think it through for themselves. Choose a numbering $f_1, \ldots, f_d$ of the elements of W. We will construct a universal element $Z_{univ} = \mathrm{Spec}(B_{univ})$ of F_W over $T_{univ} = \mathrm{Spec}(R_{univ})$ as follows. First, we set

$$R_{univ} = R[c^m_{kl}, b^l, b^l_i]/\mathfrak{a}_{univ}$$

for some ideal $\mathfrak{a}_{univ}$ which we describe below. Here the indices k, l, m, i are all ranging through $\{1, \ldots, d\}$. Then we set

$$B_{univ} = R_{univ}e_1 \oplus R_{univ}e_2 \oplus \ldots \oplus R_{univ}e_d$$

as an R_{univ}-module with algebra structure given by the rule

$$e_k e_l = \sum c^m_{kl} e_m$$

The closed immersion $Z_{univ} \to X_{T_{univ}}$ is given by the R_{univ}-algebra map

$$\Psi : A \otimes_R R_{univ} \longrightarrow B_{univ}$$

mapping $1 \otimes 1$ to $\sum b^l e_l$ and x_i to $\sum b_i^l e_l$ (for existence see below). As for the ideal $\mathfrak{a}_{univ}$ we have the following prescription:

(1) multiplication on B_{univ} is commutative, i.e., we should have $c_{lk}^m - c_{kl}^m \in \mathfrak{a}_{univ}$,
(2) multiplication on B_{univ} is associative, i.e., we should have $c_{lk}^m c_{mn}^p - c_{lq}^p c_{kn}^q \in \mathfrak{a}_{univ}$,
(3) $\sum b^l e_l$ is a multiplicative 1 in B_{univ}, in other words, we should have $(\sum b^l e_l)e_k = e_k$ for all k, which means $\sum b^l c_{lk}^m - \delta_{km} \in \mathfrak{a}_{univ}$ (Kronecker delta).

Since we've guaranteed that B_{univ} is a commutative R_{univ}-algebra with 1 given by $\sum b^l e_l$ the homomorphism Ψ exists. The final condition is that

(5) f_l to map to e_l in B_{univ}. Write $\Psi(f_l) - e_l \equiv \sum h_l^m e_m$ for some $h_l^m \in R[c_{kl}^m, b^l, b_i^l]$ then we should have $h_l^m \in \mathfrak{a}_{univ}$.

With $\mathfrak{a}_{univ} \subset R[c_{kl}^m, b^l, b_i^l]$ generated by the elements listed in (1) – (5) it is clear that F_W is represented by $\mathrm{Spec}(R_{univ})$. □

0B9A **Proposition 44.2.6.** *Let $X \to S$ be a morphism of schemes. Let $d \geq 0$. Assume for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$. Then $\mathrm{Hilb}^d_{X/S}$ is representable by a scheme.*

Proof. Either using relative glueing (Constructions, Section 27.2) or using the functorial point of view (Schemes, Lemma 26.15.4) we reduce to the case where S is affine. Details omitted.

Assume S is affine. For $U \subset X$ affine open, denote $F_U \subset \mathrm{Hilb}^d_{X/S}$ the subfunctor such that for a scheme T/S an element $Z \in \mathrm{Hilb}^d_{X/S}(T)$ is in $F_U(T)$ if and only if $Z \subset U_T$. We will use Schemes, Lemma 26.15.4 and the subfunctors F_U to conclude.

Condition (1) is Lemma 44.2.1.

Condition (2)(a) follows from the fact that $F_U = \mathrm{Hilb}^d_{U/S}$ and that this is representable by Lemma 44.2.5. Namely, if $Z \in F_U(T)$, then Z can be viewed as a closed subscheme of U_T which is finite locally free of degree d over T and hence $Z \in \mathrm{Hilb}^d_{U/S}(T)$. Conversely, if $Z \in \mathrm{Hilb}^d_{U/S}(T)$ then $Z \to U_T \to X_T$ is a closed immersion[2] and we may view Z as an element of $F_U(T)$.

Let $Z \in \mathrm{Hilb}^d_{X/S}(T)$ for some scheme T over S. Let

$$B = (Z \to T)\big((Z \to X_T \to X)^{-1}(X \setminus U)\big)$$

This is a closed subset of T and it is clear that over the open $T_{Z,U} = T \setminus B$ the restriction $Z_{t'}$ maps into $U_{T'}$. On the other hand, for any $b \in B$ the fibre Z_b does not map into U. Thus we see that given a morphism $T' \to T$ we have $Z_{T'} \in F_U(T')$ $\Leftrightarrow$ $T' \to T$ factors through the open $T_{Z,U}$. This proves condition (2)(b).

[2] This is clear if $X \to S$ is separated as in this case Morphisms, Lemma 29.41.7 tells us that the immersion $\varphi : Z \to X_T$ has closed image and hence is a closed immersion by Schemes, Lemma 26.10.4. We suggest the reader skip the rest of this footnote as we don't know of any instance where the assumptions on $X \to S$ hold but $X \to S$ is not separated. In the general case, let $x \in X_T$ be a point in the closure of $\varphi(Z)$. We have to show that $x \in \varphi(Z)$. Let $t \in T$ be the image of x. By assumption on $X \to S$ we can choose an affine open $W \subset X_T$ containing x and $\varphi(Z_t)$. Then $\varphi^{-1}(W)$ is an open containing the whole fibre Z_t and since $Z \to T$ is closed, we may after replacing T by an open neighbourhood of t assume that $Z = \varphi^{-1}(W)$. Then $\varphi(Z) \subset W$ is closed by the separated case (as $W \to T$ is separated) and we conclude $x \in \varphi(Z)$.

Condition (2)(c) follows from our assumption on X/S. All we have to do is show the following: If T is the spectrum of a field and $Z \subset X_T$ is a closed subscheme, finite flat of degree d over T, then $Z \to X_T \to X$ factors through an affine open U of X. This is clear because Z will have at most d points and these will all map into the fibre of X over the image point of $T \to S$. □

0B9B **Remark 44.2.7.** Let $f : X \to S$ be a morphism of schemes. The assumption of Proposition 44.2.6 and hence the conclusion holds in each of the following cases:

(1) X is quasi-affine,
(2) f is quasi-affine,
(3) f is quasi-projective,
(4) f is locally projective,
(5) there exists an ample invertible sheaf on X,
(6) there exists an f-ample invertible sheaf on X, and
(7) there exists an f-very ample invertible sheaf on X.

Namely, in each of these cases, every finite set of points of a fibre X_s is contained in a quasi-compact open U of X which comes with an ample invertible sheaf, is isomorphic to an open of an affine scheme, or is isomorphic to an open of Proj of a graded ring (in each case this follows by unwinding the definitions). Thus the existence of suitable affine opens by Properties, Lemma 28.29.5.

44.3. Moduli of divisors on smooth curves

0B9C For a smooth morphism $X \to S$ of relative dimension 1 the functor $\mathrm{Hilb}^d_{X/S}$ parametrizes relative effective Cartier divisors as defined in Divisors, Section 31.18.

0B9D **Lemma 44.3.1.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D \subset X$ be a closed subscheme. Consider the following conditions*

(1) *$D \to S$ is finite locally free,*
(2) *D is a relative effective Cartier divisor on X/S,*
(3) *$D \to S$ is locally quasi-finite, flat, and locally of finite presentation, and*
(4) *$D \to S$ is locally quasi-finite and flat.*

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$$

If S is locally Noetherian, then the last arrow is an if and only if. If $X \to S$ is proper (and S arbitrary), then the first arrow is an if and only if.

Proof. Equivalence of (2) and (3). This follows from Divisors, Lemma 31.18.9 if we can show the equivalence of (2) and (3) when S is the spectrum of a field k. Let $x \in X$ be a closed point. As X is smooth of relative dimension 1 over k and we see that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 1 (see Varieties, Lemma 33.25.3). Thus $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and hence a PID. It follows that every sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is nonvanishing at all the generic points of X is invertible (Divisors, Lemma 31.15.2). In other words, every closed subscheme of X which does not contain a generic point is an effective Cartier divisor. It follows that (2) and (3) are equivalent.

If S is Noetherian, then any locally quasi-finite morphism $D \to S$ is locally of finite presentation (Morphisms, Lemma 29.21.9), whence (3) is equivalent to (4).

If $X \to S$ is proper (and S is arbitrary), then $D \to S$ is proper as well. Since a proper locally quasi-finite morphism is finite (More on Morphisms, Lemma 37.44.1) and a finite, flat, and finitely presented morphism is finite locally free (Morphisms, Lemma 29.48.2), we see that (1) is equivalent to (2). □

0B9E **Lemma 44.3.2.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. Then $D_1 + D_2$ is finite locally free of degree $d_1 + d_2$ over S.*

Proof. By Lemma 44.3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. Thus $D = D_1 + D_2$ is a relative effective Cartier divisor on X/S by Divisors, Lemma 31.18.3. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 44.3.1. Applying Morphisms, Lemma 29.41.11 the surjective integral morphism $D_1 \amalg D_2 \to D$ we find that $D \to S$ is separated. Then Morphisms, Lemma 29.41.9 implies that $D \to S$ is proper. This implies that $D \to S$ is finite (More on Morphisms, Lemma 37.44.1) and in turn we see that $D \to S$ is finite locally free (Morphisms, Lemma 29.48.2). Thus it suffice to show that the degree of $D \to S$ is $d_1 + d_2$. To do this we may base change to a fibre of $X \to S$, hence we may assume that $S = \mathrm{Spec}(k)$ for some field k. In this case, there exists a finite set of closed points $x_1, \ldots, x_n \in X$ such that D_1 and D_2 are supported on $\{x_1, \ldots, x_n\}$. In fact, there are nonzerodivisors $f_{i,j} \in \mathcal{O}_{X,x_i}$ such that

$$D_1 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1})) \quad \text{and} \quad D_2 = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,2}))$$

Then we see that

$$D = \coprod \mathrm{Spec}(\mathcal{O}_{X,x_i}/(f_{i,1}f_{i,2}))$$

From this one sees easily that D has degree $d_1 + d_2$ over k (if need be, use Algebra, Lemma 10.121.1). □

0B9F **Lemma 44.3.3.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. Let $D_1, D_2 \subset X$ be closed subschemes finite locally free of degrees d_1, d_2 over S. If $D_1 \subset D_2$ (as closed subschemes) then there is a closed subscheme $D \subset X$ finite locally free of degree $d_2 - d_1$ over S such that $D_2 = D_1 + D$.*

Proof. This proof is almost exactly the same as the proof of Lemma 44.3.2. By Lemma 44.3.1 we see that D_1 and D_2 are relative effective Cartier divisors on X/S. By Divisors, Lemma 31.18.4 there is a relative effective Cartier divisor $D \subset X$ such that $D_2 = D_1 + D$. Hence $D \to S$ is locally quasi-finite, flat, and locally of finite presentation by Lemma 44.3.1. Since D is a closed subscheme of D_2, we see that $D \to S$ is finite. It follows that $D \to S$ is finite locally free (Morphisms, Lemma 29.48.2). Thus it suffice to show that the degree of $D \to S$ is $d_2 - d_1$. This follows from Lemma 44.3.2. □

Let $X \to S$ be a smooth morphism of schemes of relative dimension 1. By Lemma 44.3.1 for a scheme T over S and $D \in \mathrm{Hilb}^d_{X/S}(T)$, we can view D as a relative effective Cartier divisor on X_T/T such that $D \to T$ is finite locally free of degree d. Hence, by Lemma 44.3.2 we obtain a transformation of functors

$$\mathrm{Hilb}^{d_1}_{X/S} \times \mathrm{Hilb}^{d_2}_{X/S} \longrightarrow \mathrm{Hilb}^{d_1+d_2}_{X/S}, \quad (D_1, D_2) \longmapsto D_1 + D_2$$

If $\mathrm{Hilb}^d_{X/S}$ is representable for all degrees d, then this transformation of functors corresponds to a morphism of schemes

$$\underline{\mathrm{Hilb}}^{d_1}_{X/S} \times_S \underline{\mathrm{Hilb}}^{d_2}_{X/S} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/S}$$

over S. Observe that $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$. A special case of the morphism above is the morphism

$$\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \longrightarrow \underline{\mathrm{Hilb}}^{d+1}_{X/S}, \quad (D, x) \longmapsto D + x$$

0B9G **Lemma 44.3.4.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1 such that the functors $\mathrm{Hilb}^d_{X/S}$ are representable. The morphism $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ is finite locally free of degree $d + 1$.*

Proof. Let $D_{univ} \subset X \times_S \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ be the universal object. There is a commutative diagram

$$\begin{array}{ccccc} \underline{\mathrm{Hilb}}^d_{X/S} \times_S X & \longrightarrow & D_{univ} & \hookrightarrow & \underline{\mathrm{Hilb}}^{d+1}_{X/S} \times_S X \\ & \searrow & \downarrow & \swarrow & \\ & & \underline{\mathrm{Hilb}}^{d+1}_{X/S} & & \end{array}$$

where the top horizontal arrow maps (D', x) to $(D' + x, x)$. We claim this morphism is an isomorphism which certainly proves the lemma. Namely, given a scheme T over S, a T-valued point ξ of D_{univ} is given by a pair $\xi = (D, x)$ where $D \subset X_T$ is a closed subscheme finite locally free of degree $d + 1$ over T and $x : T \to X$ is a morphism whose graph $x : T \to X_T$ factors through D. Then by Lemma 44.3.3 we can write $D = D' + x$ for some $D' \subset X_T$ finite locally free of degree d over T. Sending $\xi = (D, x)$ to the pair (D', x) is the desired inverse. □

0B9H **Lemma 44.3.5.** *Let $X \to S$ be a smooth morphism of schemes of relative dimension 1 such that the functors $\mathrm{Hilb}^d_{X/S}$ are representable. The schemes $\underline{\mathrm{Hilb}}^d_{X/S}$ are smooth over S of relative dimension d.*

Proof. We have $\underline{\mathrm{Hilb}}^0_{X/S} = S$ and $\underline{\mathrm{Hilb}}^1_{X/S} = X$ thus the result is true for $d = 0, 1$. Assuming the result for d, we see that $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X$ is smooth over S (Morphisms, Lemma 29.34.5 and 29.34.4). Since $\underline{\mathrm{Hilb}}^d_{X/S} \times_S X \to \underline{\mathrm{Hilb}}^{d+1}_{X/S}$ is finite locally free of degree $d + 1$ by Lemma 44.3.4 the result follows from Descent, Lemma 35.14.5. We omit the verification that the relative dimension is as claimed (you can do this by looking at fibres, or by keeping track of the dimensions in the argument above). □

We collect all the information obtained sofar in the case of a proper smooth curve over a field.

0B9I **Proposition 44.3.6.** *Let X be a geometrically irreducible smooth proper curve over a field k.*

1. *The functors $\mathrm{Hilb}^d_{X/k}$ are representable by smooth proper varieties $\underline{\mathrm{Hilb}}^d_{X/k}$ of dimension d over k.*
2. *For a field extension k'/k the k'-rational points of $\underline{\mathrm{Hilb}}^d_{X/k}$ are in 1-to-1 bijection with effective Cartier divisors of degree d on $X_{k'}$.*

(3) *For $d_1, d_2 \geq 0$ there is a morphism*

$$\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} \longrightarrow \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k}$$

which is finite locally free of degree $\binom{d_1+d_2}{d_1}$.

Proof. The functors $\mathrm{Hilb}^d_{X/k}$ are representable by Proposition 44.2.6 (see also Remark 44.2.7) and the fact that X is projective (Varieties, Lemma 33.43.4). The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are separated over k by Lemma 44.2.4. The schemes $\underline{\mathrm{Hilb}}^d_{X/k}$ are smooth over k by Lemma 44.3.5. Starting with $X = \underline{\mathrm{Hilb}}^1_{X/k}$, the morphisms of Lemma 44.3.4, and induction we find a morphism

$$X^d = X \times_k X \times_k \ldots \times_k X \longrightarrow \underline{\mathrm{Hilb}}^d_{X/k}, \quad (x_1, \ldots, x_d) \longrightarrow x_1 + \ldots + x_d$$

which is finite locally free of degree $d!$. Since X is proper over k, so is X^d, hence $\underline{\mathrm{Hilb}}^d_{X/k}$ is proper over k by Morphisms, Lemma 29.41.9. Since X is geometrically irreducible over k, the product X^d is irreducible (Varieties, Lemma 33.8.4) hence the image is irreducible (in fact geometrically irreducible). This proves (1). Part (2) follows from the definitions. Part (3) follows from the commutative diagram

$$\begin{array}{ccc} X^{d_1} \times_k X^{d_2} & = & X^{d_1+d_2} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_k \underline{\mathrm{Hilb}}^{d_2}_{X/k} & \longrightarrow & \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k} \end{array}$$

and multiplicativity of degrees of finite locally free morphisms. □

0B9J **Remark 44.3.7.** Let X be a geometrically irreducible smooth proper curve over a field k as in Proposition 44.3.6. Let $d \geq 0$. The universal closed object is a relatively effective divisor

$$D_{univ} \subset \underline{\mathrm{Hilb}}^{d+1}_{X/k} \times_k X$$

over $\underline{\mathrm{Hilb}}^{d+1}_{X/k}$ by Lemma 44.3.1. In fact, D_{univ} is isomorphic as a scheme to $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$, see proof of Lemma 44.3.4. In particular, D_{univ} is an effective Cartier divisor and we obtain an invertible module $\mathcal{O}(D_{univ})$. If $[D] \in \underline{\mathrm{Hilb}}^{d+1}_{X/k}$ denotes the k-rational point corresponding to the effective Cartier divisor $D \subset X$ of degree $d+1$, then the restriction of $\mathcal{O}(D_{univ})$ to the fibre $[D] \times X$ is $\mathcal{O}_X(D)$.

44.4. The Picard functor

0B9K Given any scheme X we denote $\mathrm{Pic}(X)$ the set of isomorphism classes of invertible $\mathcal{O}_X$-modules. See Modules, Definition 17.25.9. Given a morphism $f : X \to Y$ of schemes, pullback defines a group homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. The assignment $X \rightsquigarrow \mathrm{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

Let us define the Picard functor for a morphism of schemes $f : X \to S$. The idea behind our construction is that we'll take it to be the sheaf $R^1 f_* \mathbf{G}_m$ where we use the fppf topology to compute the higher direct image. Unwinding the definitions this leads to the following more direct definition.

0B9L **Definition 44.4.1.** Let Sch_{fppf} be a big site as in Topologies, Definition 34.7.8. Let $f : X \to S$ be a morphism of this site. The *Picard functor* $\mathrm{Pic}_{X/S}$ is the fppf sheafification of the functor

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad T \longmapsto \mathrm{Pic}(X_T)$$

If this functor is representable, then we denote $\underline{\mathrm{Pic}}_{X/S}$ a scheme representing it.

An often used remark is that if $T \in \mathrm{Ob}((Sch/S)_{fppf})$, then $\mathrm{Pic}_{X_T/T}$ is the restriction of $\mathrm{Pic}_{X/S}$ to $(Sch/T)_{fppf}$. It turns out to be nontrivial to see what the value of $\mathrm{Pic}_{X/S}$ is on schemes T over S. Here is a lemma that helps with this task.

0B9M **Lemma 44.4.2.** *Let $f : X \to S$ be as in Definition 44.4.1. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \mathrm{Ob}((Sch/S)_{fppf})$, then*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T)$$

is an exact sequence for all T.

Proof. We may replace S by T and X by X_T and assume that $S = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_S$-module. If $f^*\mathcal{N} \cong \mathcal{O}_X$, then we see that $f_*f^*\mathcal{N} \cong f_*\mathcal{O}_X \cong \mathcal{O}_S$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \to f_*f^*\mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_S \to f_*f^*\mathcal{O}_S$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_S$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is in the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/S}(S)$. Then there exists an fppf covering $\{S_i \to S\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{S_i}. Choose a trivializing section s_i. Then $\mathrm{pr}_0^*s_i$ and $\mathrm{pr}_1^*s_j$ are both trivialising sections of $\mathcal{L}$ over $X_{S_i \times_S S_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_S S_j}, \mathcal{O}^*_{X_{S_i \times_S S_j}}) = \Gamma(S_i \times_S S_j, \mathcal{O}^*_{S_i \times_S S_j})$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $S_i \times_S S_j \times_S S_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{S_i \to S\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_S$-module $\mathcal{N}$ with trivializations over S_i whose associated descent datum is $\{f_{ij}\}$. Then $f^*\mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful (see proposition cited above). □

0B9N **Lemma 44.4.3.** *Let $f : X \to S$ be as in Definition 44.4.1. Assume f has a section σ and that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Then*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T) \to 0$$

is a split exact sequence with splitting given by $\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T)$.

Proof. Denote $K(T) = \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$. Since σ is a section of f we see that $\mathrm{Pic}(X_T)$ is the direct sum of $\mathrm{Pic}(T)$ and $K(T)$. Thus by Lemma 44.4.2 we see that $K(T) \subset \mathrm{Pic}_{X/S}(T)$ for all T. Moreover, it is clear from the construction that $\mathrm{Pic}_{X/S}$ is the sheafification of the presheaf K. To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \to T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j. Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i$ for all i. Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma_{T_i \times_T T_j}^* \varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \to \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij}, then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some regular function f_{ijk} on the scheme $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X, we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \to X\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ and an isomorphism $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. □

44.5. A representability criterion

0B9P To prove the Picard functor is representable we will use the following criterion.

0B9Q **Lemma 44.5.1.** *Let k be a field. Let $G : (Sch/k)^{opp} \to Groups$ be a functor. With terminology as in Schemes, Definition 26.15.3, assume that*

(1) *G satisfies the sheaf property for the Zariski topology,*
(2) *there exists a subfunctor $F \subset G$ such that*
   (a) *F is representable,*
   (b) *$F \subset G$ is representable by open immersion,*
   (c) *for every field extension K of k and $g \in G(K)$ there exists a $g' \in G(k)$ such that $g'g \in F(K)$.*

Then G is representable by a group scheme over k.

Proof. This follows from Schemes, Lemma 26.15.4. Namely, take $I = G(k)$ and for $i = g' \in I$ take $F_i \subset G$ the subfunctor which associates to T over k the set of elements $g \in G(T)$ with $g'g \in F(T)$. Then $F_i \cong F$ by multiplication by g'. The map $F_i \to G$ is isomorphic to the map $F \to G$ by multiplication by g', hence is representable by open immersions. Finally, the collection $(F_i)_{i \in I}$ covers G by assumption (2)(c). Thus the lemma mentioned above applies and the proof is complete. □

44.6. The Picard scheme of a curve

0B9R In this section we will apply Lemma 44.5.1 to show that $\mathrm{Pic}_{X/k}$ is representable, when k is an algebraically closed field and X is a smooth projective curve over k. To make this work we use a bit of cohomology and base change developed in the chapter on derived categories of schemes.

0B9U **Lemma 44.6.1.** *Let k be a field. Let X be a smooth projective curve over k which has a k-rational point. Then the hypotheses of Lemma 44.4.3 are satisfied.*

Proof. The meaning of the phrase "has a k-rational point" is exactly that the structure morphism $f : X \to \mathrm{Spec}(k)$ has a section, which verifies the first condition. By Varieties, Lemma 33.26.2 we see that $k' = H^0(X, \mathcal{O}_X)$ is a field extension of k. Since X has a k-rational point there is a k-algebra homomorphism $k' \to k$ and we conclude $k' = k$. Since k is a field, any morphism $T \to \mathrm{Spec}(k)$ is flat. Hence we see by cohomology and base change (Cohomology of Schemes, Lemma 30.5.2) that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. This finishes the proof. □

Let X be a smooth projective curve over a field k with a k-rational point σ. Then the functor

$$\mathrm{Pic}_{X/k,\sigma} : (Sch/k)^{opp} \longrightarrow Ab, \quad T \longmapsto \mathrm{Ker}(\mathrm{Pic}(X_T) \xrightarrow{\sigma_T^*} \mathrm{Pic}(T))$$

is isomorphic to $\mathrm{Pic}_{X/k}$ on $(Sch/k)_{fppf}$ by Lemmas 44.6.1 and 44.4.3. Hence it will suffice to prove that $\mathrm{Pic}_{X/k,\sigma}$ is representable. We will use the notation "$\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$" to signify that T is a scheme over k and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$-module whose restriction to T via σ_T is isomorphic to $\mathcal{O}_T$.

0B9V **Lemma 44.6.2.** *Let k be a field. Let X be a smooth projective curve over k with a k-rational point σ. For a scheme T over k, consider the subset $F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$ consisting of $\mathcal{L}$ such that $Rf_{T,*}\mathcal{L}$ is isomorphic to an invertible $\mathcal{O}_T$-module placed in degree 0. Then $F \subset \mathrm{Pic}_{X/k,\sigma}$ is a subfunctor and the inclusion is representable by open immersions.*

Proof. Immediate from Derived Categories of Schemes, Lemma 36.32.3 applied with $i = 0$ and $r = 1$ and Schemes, Definition 26.15.3. □

To continue it is convenient to make the following definition.

0B9W **Definition 44.6.3.** Let k be a field. Let X be a smooth projective geometrically irreducible curve over k. The *genus* of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

0B9X **Lemma 44.6.4.** *Let k be a field. Let X be a smooth projective curve of genus g over k with a k-rational point σ. The open subfunctor F defined in Lemma 44.6.2 is representable by an open subscheme of $\underline{\mathrm{Hilb}}^g_{X/k}$.*

Proof. In this proof unadorned products are over $\mathrm{Spec}(k)$. By Proposition 44.3.6 the scheme $H = \underline{\mathrm{Hilb}}^g_{X/k}$ exists. Consider the universal divisor $D_{univ} \subset H \times X$ and the associated invertible sheaf $\mathcal{O}(D_{univ})$, see Remark 44.3.7. We adjust by tensoring with the pullback via $\sigma_H : H \to H \times X$ to get

$$\mathcal{L}_H = \mathcal{O}(D_{univ}) \otimes_{\mathcal{O}_{H\times X}} \mathrm{pr}_H^*\sigma_H^*\mathcal{O}(D_{univ})^{\otimes -1} \in \mathrm{Pic}_{X/k,\sigma}(H)$$

By the Yoneda lemma (Categories, Lemma 4.3.5) the invertible sheaf $\mathcal{L}_H$ defines a natural transformation

$$h_H \longrightarrow \mathrm{Pic}_{X/k,\sigma}$$

Because F is an open subfuctor, there exists a maximal open $W \subset H$ such that $\mathcal{L}_H|_{W\times X}$ is in $F(W)$. Of course, this open is nothing else than the open subscheme constructed in Derived Categories of Schemes, Lemma 36.32.3 with $i = 0$ and $r = 1$ for the morphism $H \times X \to H$ and the sheaf $\mathcal{F} = \mathcal{O}(D_{univ})$. Applying the Yoneda

lemma again we obtain a commutative diagram

$$\begin{array}{ccc} h_W & \longrightarrow & F \\ \downarrow & & \downarrow \\ h_H & \longrightarrow & \mathrm{Pic}_{X/k,\sigma} \end{array}$$

To finish the proof we will show that the top horizontal arrow is an isomorphism.

Let $\mathcal{L} \in F(T) \subset \mathrm{Pic}_{X/k,\sigma}(T)$. Let $\mathcal{N}$ be the invertible $\mathcal{O}_T$-module such that $Rf_{T,*}\mathcal{L} \cong \mathcal{N}[0]$. The adjunction map

$$f_T^*\mathcal{N} \longrightarrow \mathcal{L} \quad \text{corresponds to a section } s \text{ of } \quad \mathcal{L} \otimes f_T^*\mathcal{N}^{\otimes -1}$$

on X_T. Claim: The zero scheme of s is a relative effective Cartier divisor D on $(T \times X)/T$ finite locally free of degree g over T.

Let us finish the proof of the lemma admitting the claim. Namely, D defines a morphism $m : T \to H$ such that D is the pullback of D_{univ}. Then

$$(m \times \mathrm{id}_X)^*\mathcal{O}(D_{univ}) \cong \mathcal{O}_{T\times X}(D)$$

Hence $(m \times \mathrm{id}_X)^*\mathcal{L}_H$ and $\mathcal{O}(D)$ differ by the pullback of an invertible sheaf on H. This in particular shows that $m : T \to H$ factors through the open $W \subset H$ above. Moreover, it follows that these invertible modules define, after adjusting by pullback via σ_T as above, the same element of $\mathrm{Pic}_{X/k,\sigma}(T)$. Chasing diagrams using Yoneda's lemma we see that $m \in h_W(T)$ maps to $\mathcal{L} \in F(T)$. We omit the verification that the rule $F(T) \to h_W(T)$, $\mathcal{L} \mapsto m$ defines an inverse of the transformation of functors above.

Proof of the claim. Since D is a locally principal closed subscheme of $T \times X$, it suffices to show that the fibres of D over T are effective Cartier divisors, see Lemma 44.3.1 and Divisors, Lemma 31.18.9. Because taking cohomology of $\mathcal{L}$ commutes with base change (Derived Categories of Schemes, Lemma 36.30.4) we reduce to $T = \mathrm{Spec}(K)$ where K/k is a field extension. Then $\mathcal{L}$ is an invertible sheaf on X_K with $H^0(X_K, \mathcal{L}) = K$ and $H^1(X_K, \mathcal{L}) = 0$. Thus

$$\deg(\mathcal{L}) = \chi(X_K, \mathcal{L}) - \chi(X_K, \mathcal{O}_{X_K}) = 1 - (1 - g) = g$$

See Varieties, Definition 33.44.1. To finish the proof we have to show a nonzero section of $\mathcal{L}$ defines an effective Cartier divisor on X_K. This is clear. □

0B9Y **Lemma 44.6.5.** *Let k be a separably closed field. Let X be a smooth projective curve of genus g over k. Let K/k be a field extension and let $\mathcal{L}$ be an invertible sheaf on X_K. Then there exists an invertible sheaf $\mathcal{L}_0$ on X such that $\dim_K H^0(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 1$ and $\dim_K H^1(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 0$.*

Proof. This proof is a variant of the proof of Varieties, Lemma 33.44.16. We encourage the reader to read that proof first.

First we pick an ample invertible sheaf $\mathcal{L}_0$ and we replace $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0^{\otimes n}|_{X_K}$ for some $n \gg 0$. The result will be that we may assume that $H^0(X_K, \mathcal{L}) \neq 0$ and $H^1(X_K, \mathcal{L}) = 0$. Namely, we will get the vanishing by Cohomology of Schemes, Lemma 30.17.1 and the nonvanishing because the degree of the tensor product is $\gg 0$. We will finish the proof by descending induction on $t = \dim_K H^0(X_K, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Observe that for a k-rational point x of X, the inverse image x_K is a K-rational point of X_K. Moreover, there are infinitely many k-rational points by Varieties, Lemma 33.25.6. Therefore the points x_K form a Zariski dense collection of points of X_K.

Let $s \in H^0(X_K, \mathcal{L})$ be nonzero. From the previous paragraph we deduce there exists a k-rational point x such that s does not vanish in x_K. Let $\mathcal{I}$ be the ideal sheaf of $i : x_K \to X_K$ as in Varieties, Lemma 33.43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L} \to \mathcal{L} \to i_* i^* \mathcal{L} \to 0$$

Observe that $H^0(X_K, i_* i^* \mathcal{L}) = H^0(x_K, i^* \mathcal{L})$ has dimension 1 over K. Since s does not vanish at x we conclude that

$$H^0(X_K, \mathcal{L}) \longrightarrow H^0(X, i_* i^* \mathcal{L})$$

is surjective. Hence $\dim_K H^0(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = 0$ thereby finishing the proof of the induction step. □

0B9Z **Proposition 44.6.6.** *Let k be a separably closed field. Let X be a smooth projective curve over k. The Picard functor $\mathrm{Pic}_{X/k}$ is representable.*

Proof. Since k is separably closed there exists a k-rational point σ of X, see Varieties, Lemma 33.25.6. As discussed above, it suffices to show that the functor $\mathrm{Pic}_{X/k,\sigma}$ classifying invertible modules trivial along σ is representable. To do this we will check conditions (1), (2)(a), (2)(b), and (2)(c) of Lemma 44.5.1.

The functor $\mathrm{Pic}_{X/k,\sigma}$ satisfies the sheaf condition for the fppf topology because it is isomorphic to $\mathrm{Pic}_{X/k}$. It would be more correct to say that we've shown the sheaf condition for $\mathrm{Pic}_{X/k,\sigma}$ in the proof of Lemma 44.4.3 which applies by Lemma 44.6.1. This proves condition (1)

As our subfunctor we use F as defined in Lemma 44.6.2. Condition (2)(b) follows. Condition (2)(a) is Lemma 44.6.4. Condition (2)(c) is Lemma 44.6.5. □

In fact, the proof given above produces more information which we collect here.

0BA0 **Lemma 44.6.7.** *Let k be a separably closed field. Let X be a smooth projective curve of genus g over k.*

1. *$\underline{\mathrm{Pic}}_{X/k}$ is a disjoint union of g-dimensional smooth proper varieties $\underline{\mathrm{Pic}}^d_{X/k}$,*
2. *k-points of $\underline{\mathrm{Pic}}^d_{X/k}$ correspond to invertible $\mathcal{O}_X$-modules of degree d,*
3. *$\underline{\mathrm{Pic}}^0_{X/k}$ is an open and closed subgroup scheme,*
4. *for $d \geq 0$ there is a canonical morphism $\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$*
5. *the morphisms γ_d are surjective for $d \geq g$ and smooth for $d \geq 2g - 1$,*
6. *the morphism $\underline{\mathrm{Hilb}}^g_{X/k} \to \underline{\mathrm{Pic}}^g_{X/k}$ is birational.*

Proof. Pick a k-rational point σ of X. Recall that $\mathrm{Pic}_{X/k}$ is isomorphic to the functor $\mathrm{Pic}_{X/k,\sigma}$. By Derived Categories of Schemes, Lemma 36.32.2 for every $d \in \mathbf{Z}$ there is an open subfunctor

$$\mathrm{Pic}^d_{X/k,\sigma} \subset \mathrm{Pic}_{X/k,\sigma}$$

whose value on a scheme T over k consists of those $\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$ such that $\chi(X_t, \mathcal{L}_t) = d + 1 - g$ and moreover we have

$$\mathrm{Pic}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \mathrm{Pic}^d_{X/k,\sigma}$$

as fppf sheaves. It follows that the scheme $\underline{\mathrm{Pic}}_{X/k}$ (which exists by Proposition 44.6.6) has a corresponding decomposition

$$\underline{\mathrm{Pic}}_{X/k,\sigma} = \coprod\nolimits_{d \in \mathbf{Z}} \underline{\mathrm{Pic}}^d_{X/k,\sigma}$$

where the points of $\underline{\mathrm{Pic}}^d_{X/k,\sigma}$ correspond to isomorphism classes of invertible modules of degree d on X.

Fix $d \geq 0$. There is a morphism

$$\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \longrightarrow \underline{\mathrm{Pic}}^d_{X/k}$$

coming from the invertible sheaf $\mathcal{O}(D_{univ})$ on $\underline{\mathrm{Hilb}}^d_{X/k} \times_k X$ (Remark 44.3.7) by the Yoneda lemma (Categories, Lemma 4.3.5). Our proof of the representability of the Picard functor of X/k in Proposition 44.6.6 and Lemma 44.6.4 shows that γ_g induces an open immersion on a nonempty open of $\underline{\mathrm{Hilb}}^g_{X/k}$. Moreover, the proof shows that the translates of this open by k-rational points of the group scheme $\underline{\mathrm{Pic}}_{X/k}$ define an open covering. Since $\underline{\mathrm{Hilb}}^g_{X/K}$ is smooth of dimension g (Proposition 44.3.6) over k, we conclude that the group scheme $\underline{\mathrm{Pic}}_{X/k}$ is smooth of dimension g over k.

By Groupoids, Lemma 39.7.3 we see that $\underline{\mathrm{Pic}}_{X/k}$ is separated. Hence, for every $d \geq 0$, the image of γ_d is a proper variety over k (Morphisms, Lemma 29.41.10).

Let $d \geq g$. Then for any field extension K/k and any invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}$ of degree d, we see that $\chi(X_K, \mathcal{L}) = d + 1 - g > 0$. Hence $\mathcal{L}$ has a nonzero section and we conclude that $\mathcal{L} = \mathcal{O}_{X_K}(D)$ for some divisor $D \subset X_K$ of degree d. It follows that γ_d is surjective.

Combining the facts mentioned above we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$. This finishes the proof of (2) because now we see that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper for $d \geq g$ but then all $\underline{\mathrm{Pic}}^d_{X/k}$ are proper by translation.

It remains to prove that γ_d is smooth for $d \geq 2g - 1$. Consider an invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree d. Then the fibre of the point corresponding to $\mathcal{L}$ is

$$Z = \{D \subset X \mid \mathcal{O}_X(D) \cong \mathcal{L}\} \subset \underline{\mathrm{Hilb}}^d_{X/k}$$

with its natural scheme structure. Since any isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ is well defined up to multiplying by a nonzero scalar, we see that the canonical section $1 \in \mathcal{O}_X(D)$ is mapped to a section $s \in \Gamma(X, \mathcal{L})$ well defined up to multiplication by a nonzero scalar. In this way we obtain a morphism

$$Z \longrightarrow \mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*))$$

(dual because of our conventions). This morphism is an isomorphism, because given an section of $\mathcal{L}$ we can take the associated effective Cartier divisor, in other words we can construct an inverse of the displayed morphism; we omit the precise formulation and proof. Since $\dim H^0(X, \mathcal{L}) = d + 1 - g$ for every $\mathcal{L}$ of degree $d \geq 2g - 1$ by Varieties, Lemma 33.44.17 we see that $\mathrm{Proj}(\mathrm{Sym}(\Gamma(X, \mathcal{L})^*)) \cong \mathbf{P}^{d-g}_k$. We conclude that $\dim(Z) = \dim(\mathbf{P}^{d-g}_k) = d - g$. We conclude that the fibres of the morphism γ_d

all have dimension equal to the difference of the dimensions of $\underline{\mathrm{Hilb}}^d_{X/k}$ and $\underline{\mathrm{Pic}}^d_{X/k}$. It follows that γ_d is flat, see Algebra, Lemma 10.128.1. As moreover the fibres are smooth, we conclude that γ_d is smooth by Morphisms, Lemma 29.34.3. □

44.7. Some remarks on Picard groups

0CDS This section continues the discussion in Varieties, Section 33.30 and will be continued in Algebraic Curves, Section 53.17.

0CDT **Lemma 44.7.1.** *Let k be a field. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. If X has a k-rational point, then for any Galois extension k'/k we have*

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{k'})^{Gal(k'/k)}$$

Moreover the action of $Gal(k'/k)$ on $\mathrm{Pic}(X_{k'})$ is continuous.

Proof. Since $\mathrm{Gal}(k'/k) = \mathrm{Aut}(k'/k)$ it acts (from the right) on $\mathrm{Spec}(k')$, hence it acts (from the right) on $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$, and since $\mathrm{Pic}(-)$ is a contravariant functor, it acts (from the left) on $\mathrm{Pic}(X_{k'})$. If k'/k is an infinite Galois extension, then we write $k' = \mathrm{colim}\, k'_\lambda$ as a filtered colimit of finite Galois extensions, see Fields, Lemma 9.22.3. Then $X_{k'} = \lim X_{k_\lambda}$ (as in Limits, Section 32.2) and we obtain

$$\mathrm{Pic}(X_{k'}) = \mathrm{colim}\, \mathrm{Pic}(X_{k_\lambda})$$

by Limits, Lemma 32.10.3. Moreover, the transition maps in this system of abelian groups are injective by Varieties, Lemma 33.30.3. It follows that every element of $\mathrm{Pic}(X_{k'})$ is fixed by one of the open subgroups $\mathrm{Gal}(k'/k_\lambda)$, which exactly means that the action is continuous. Injectivity of the transition maps implies that it suffices to prove the statement on fixed points in the case that k'/k is finite Galois.

Assume k'/k is finite Galois with Galois group $G = \mathrm{Gal}(k'/k)$. Let $\mathcal{L}$ be an element of $\mathrm{Pic}(X_{k'})$ fixed by G. We will use Galois descent (Descent, Lemma 35.6.1) to prove that $\mathcal{L}$ is the pullback of an invertible sheaf on X. Recall that $f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \to X_{k'}$ and that σ acts on $\mathrm{Pic}(X_{k'})$ by pulling back by f_σ. Hence for each $\sigma \in G$ we can choose an isomorphism $\varphi_\sigma : \mathcal{L} \to f_\sigma^*\mathcal{L}$ because $\mathcal{L}$ is a fixed by the G-action. The trouble is that we don't know if we can choose φ_σ such that the cocycle condition $\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ holds. To see that this is possible we use that X has a k-rational point $x \in X(k)$. Of course, x similarly determines a k'-rational point $x' \in X_{k'}$ which is fixed by f_σ for all σ. Pick a nonzero element s in the fibre of $\mathcal{L}$ at x'; the fibre is the 1-dimensional $k' = \kappa(x')$-vector space

$$\mathcal{L}_{x'} \otimes_{\mathcal{O}_{X_{k'}, x'}} \kappa(x').$$

Then $f_\sigma^* s$ is a nonzero element of the fibre of $f_\sigma^*\mathcal{L}$ at x'. Since we can multiply φ_σ by an element of $(k')^*$ we may assume that φ_σ sends s to $f_\sigma^* s$. Then we see that both $\varphi_{\sigma\tau}$ and $f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ send s to $f_{\sigma\tau}^* s = f_\tau^* f_\sigma^* s$. Since $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ these two isomorphisms have to be the same (as one is a global unit times the other and they agree in x') and the proof is complete. □

0CD5 **Lemma 44.7.2.** *Let k be a field of characteristic $p > 0$. Let X be a quasi-compact and quasi-separated scheme over k with $H^0(X, \mathcal{O}_X) = k$. Let n be an integer prime to p. Then the map*

$$\mathrm{Pic}(X)[n] \longrightarrow \mathrm{Pic}(X_{k'})[n]$$

is bijective for any purely inseparable extension k'/k.

Proof. First we observe that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k'})$ is injective by Varieties, Lemma 33.30.3. Hence we have to show the map in the lemma is surjective. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X_{k'}}$-module which has order dividing n in $\mathrm{Pic}(X_{k'})$. Choose an isomorphism $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_{X_{k'}}$ of invertible modules. We will prove that we can descend the pair $(\mathcal{L}, \alpha)$ to X.

Set $A = k' \otimes_k k'$. Since k'/k is purely inseparable, the kernel of the multiplication map $A \to k'$ is a locally nilpotent ideal I of A. Observe that

$$X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A) = X_{k'} \times_X X_{k'}$$

comes with two projections $\mathrm{pr}_i : X_A \to X_{k'}$, $i = 0, 1$ which agree over A/I. Hence the invertible modules $\mathcal{L}_i = \mathrm{pr}_i^*\mathcal{L}$ agree over the closed subscheme $X_{A/I} = X_{k'}$. Since $X_{A/I} \to X_A$ is a thickening and since $\mathcal{L}_i$ are n-torsion, we see that there exists an isomorphism $\varphi : \mathcal{L}_0 \to \mathcal{L}_1$ by More on Morphisms, Lemma 37.4.2. We may pick φ to reduce to the identity modulo I. Namely, $H^0(X, \mathcal{O}_X) = k$ implies $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ by Cohomology of Schemes, Lemma 30.5.2 and $A \to k'$ is surjective hence we can adjust φ by multiplying by a suitable element of A. Consider the map

$$\lambda : \mathcal{O}_{X_A} \xrightarrow{\mathrm{pr}_0^*\alpha^{-1}} \mathcal{L}_0^{\otimes n} \xrightarrow{\varphi^{\otimes n}} \mathcal{L}_1^{\otimes n} \xrightarrow{\mathrm{pr}_0^*\alpha} \mathcal{O}_{X_A}$$

We can view λ as an element of A because $H^0(X_A, \mathcal{O}_{X_A}) = A$ (same reference as above). Since φ reduces to the identity modulo I we see that $\lambda = 1 \bmod I$. Then there is a unique nth root of λ in $1 + I$ (Algebra, Lemma 10.32.8) and after multiplying φ by its inverse we get $\lambda = 1$. We claim that $(\mathcal{L}, \varphi)$ is a descent datum for the fpqc covering $\{X_{k'} \to X\}$ (Descent, Definition 35.2.1). If true, then $\mathcal{L}$ is the pullback of an invertible $\mathcal{O}_X$-module $\mathcal{N}$ by Descent, Proposition 35.5.2. Injectivity of the map on Picard groups shows that $\mathcal{N}$ is a torsion element of $\mathrm{Pic}(X)$ of the same order as $\mathcal{L}$.

Proof of the claim. To see this we have to verify that

$$\mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi = \mathrm{pr}_{02}^*\varphi \quad \text{on} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

As before the diagonal morphism $\Delta : X_{k'} \to X_{k' \otimes_k k' \otimes_k k'}$ is a thickening. The left and right hand sides of the equality signs are maps $a, b : p_0^*\mathcal{L} \to p_2^*\mathcal{L}$ compatible with $p_0^*\alpha$ and $p_2^*\alpha$ where $p_i : X_{k' \otimes_k k' \otimes_k k'} \to X_{k'}$ are the projection morphisms. Finally, a, b pull back to the same map under Δ. Affine locally (in local trivializations) this means that a, b are given by multiplication by invertible functions which reduce to the same function modulo a locally nilpotent ideal and which have the same nth powers. Then it follows from Algebra, Lemma 10.32.8 that these functions are the same. □

44.8. Other chapters

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Schemes

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CHAPTER 45

Weil Cohomology Theories

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45.1. Introduction

0FFH In this chapter we discuss Weil cohomology theories for smooth projective schemes over a base field. Briefly, for us such a cohomology theory H^* is one which has Künneth, Poincaré duality, and cycle classes (with suitable compatibilities). We warn the reader that there is no universal agreement in the literature as to what constitutes a "Weil cohomology theory".

Before reading this chapter the reader should take a look at Categories, Section 4.43 and Homology, Section 12.17 where we define (symmetric) monoidal categories and we develop just enough basic language concerning these categories for the needs of this chapter. Equipped with this language we construct in Section 45.3 the symmetric monoidal graded category whose objects are smooth projective schemes and whose morphisms are correspondences. In Section 45.4 we add images of projectors and invert the Lefschetz motive in order to obtain the symmetric monoidal Karoubian category M_k of Chow motives. This category comes equipped with a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

As we will see below, a key property of a Weil cohomology theory is that it factors over h.

First, in the case of an algebraically closed base field, we define what we call a "classical Weil cohomology theory", see Section 45.7. This notion is the same as the notion introduced in [**Kle68**, Section 1.2] and agrees with the notion introduced in [**Kle72**, page 65]. However, our notion does not a priori agree with the notion introduced in [**Kle94**, page 10] because there the author adds two Lefschetz type axioms and it isn't known whether any classical Weil cohomology theory as defined in this chapter satisfies those axioms. At the end of Section 45.7 we show that a classical Weil cohomology theory is of the form $H^* = G \circ h$ where G is a symmetric monoidal functor from M_k to the category of graded vector spaces over the coefficient field of H^*.

In Section 45.8 we prove a couple of lemmas on cycle groups over non-closed fields which will be used in discussing Weil cohomology theories on smooth projective schemes over arbitrary fields.

Our motivation for our axioms of a Weil cohomology theory H^* over a general base field k are the following

(1) $H^* = G \circ h$ for a symmetric monoidal functor G from M_k to the category of graded vector spaces over the coefficient field F of H^*,

(2) G should send the Tate motive (inverse of the Lefschetz motive) to a 1-dimensional vector space $F(1)$ sitting in degree -2,
(3) when k is algebraically closed we should recover the notion discussion in Section 45.7 up to choosing a basis element of $F(1)$.

First, in Section 45.9 we analyze the first two conditions. After developing a few more results in Section 45.10 in Section 45.11 we add the necessary axioms to obtain property (3).

In the final Section 45.14 we detail an alternative approach to Weil cohomology theories, using a first Chern class map instead of cycle classes. It is this approach that will be most suited for proving that certain cohomology theories are Weil cohomology theories in later chapters, see de Rham Cohomology, Section 50.22.

45.2. Conventions and notation

0FFI Let F be a field. In this chapter we view the category of F-graded vector spaces as an F-linear symmetric monoidal category with associativity constraint as usual and with commutativity constraint involving signs. See Homology, Example 12.17.4.

Let R be a ring. In this chapter a *graded commutative R-algebra A* is a commutative differential graded R-algebra (Differential Graded Algebra, Definitions 22.3.1 and 22.3.3) whose differential is zero. Thus A is an R-module endowed with a grading $A = \bigoplus_{n\in\mathbf{Z}} A^n$ by R-submodules. The R-bilinear multiplication

$$A^n \times A^m \longrightarrow A^{n+m}, \quad \alpha \times \beta \longmapsto \alpha \cup \beta$$

will be called the *cup product* in this chapter. The commutativity constraint is $\alpha \cup \beta = (-1)^{nm}\beta \cup \alpha$ if $\alpha \in A^n$ and $\beta \in A^m$. Finally, there is a multiplicative unit $1 \in A^0$, or equivalently, there is an additive and multiplicative map $R \to A^0$ which is compatible the R-module structure on A.

Let k be a field. Let X be a scheme of finite type over k. The Chow groups $\mathrm{CH}_k(X)$ of X of cycles of dimension k modulo rational equivalence have been defined in Chow Homology, Definition 42.19.1. If X is normal or Cohen-Macaulay, then we can also consider the Chow groups $\mathrm{CH}^p(X)$ of cycles of codimension p (Chow Homology, Section 42.42) and then $[X] \in \mathrm{CH}^0(X)$ denotes the "fundamental class" of X, see Chow Homology, Remark 42.42.2. If X is smooth and α and β are cycles on X, then $\alpha \cdot \beta$ denotes the intersection product of α and β, see Chow Homology, Section 42.62.

45.3. Correspondences

0FFZ Let k be a field. For schemes X and Y over k we denote $X \times Y$ the product of X and Y in the category of schemes over k. In this section we construct the graded category over $\mathbf{Q}$ whose objects are smooth projective schemes over k and whose morphisms are correspondences.

Let X and Y be smooth projective schemes over k. Let $X = \coprod X_d$ be the decomposition of X into the open and closed subschemes which are equidimensional with $\dim(X_d) = d$. We define the $\mathbf{Q}$-vector space *of correspondences of degree r from X to Y* by the formula:

$$\mathrm{Corr}^r(X, Y) = \bigoplus\nolimits_d \mathrm{CH}^{d+r}(X_d \times Y) \otimes \mathbf{Q} \subset \mathrm{CH}^*(X \times Y) \otimes \mathbf{Q}$$

Given $c \in \mathrm{Corr}^r(X, Y)$ and $\beta \in \mathrm{CH}_k(Y) \otimes \mathbf{Q}$ we can define the *pullback* of β by c using the formula

$$c^*(\beta) = \mathrm{pr}_{1,*}(c \cdot \mathrm{pr}_2^*\beta) \quad \text{in} \quad \mathrm{CH}_{k-r}(X) \otimes \mathbf{Q}$$

This makes sense because pr_2 is flat of relative dimension d on $X_d \times Y$, hence $\mathrm{pr}_2^*\beta$ is a cycle of dimension $d + k$ on $X_d \times Y$, hence $c \cdot \mathrm{pr}_2^*\alpha$ is a cycle of dimension $k - r$ on $X_d \times Y$ whose pushforward by the proper morphism pr_1 is a cycle of the same dimension. Similarly, switching to grading by codimension, given $\alpha \in \mathrm{CH}^i(X) \otimes \mathbf{Q}$ we can define the *pushforward* of α by c using the formula

$$c_*(\alpha) = \mathrm{pr}_{2,*}(c \cdot \mathrm{pr}_1^*\alpha) \quad \text{in} \quad \mathrm{CH}^{i+r}(Y) \otimes \mathbf{Q}$$

This makes sense because $\mathrm{pr}_1^*\alpha$ is a cycle of codimension i on $X \times Y$, hence $c \cdot \mathrm{pr}_1^*\alpha$ is a cycle of codimension $i + d + r$ on $X_d \times Y$, which pushes forward to a cycle of codimension $i + r$ on Y.

Given a three smooth projective schemes X, Y, Z over k we define a *composition of correspondences*

$$\mathrm{Corr}^s(Y, Z) \times \mathrm{Corr}^r(X, Y) \longrightarrow \mathrm{Corr}^{r+s}(X, Z)$$

by the rule

$$(c', c) \longmapsto c' \circ c = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*c \cdot \mathrm{pr}_{23}^*c')$$

where $\mathrm{pr}_{12} : X \times Y \times Z \to X \times Y$ is the projection and similarly for pr_{13} and pr_{23}.

0FG0 **Lemma 45.3.1.** *We have the following for correspondences:*

(1) *composition of correspondences is* $\mathbf{Q}$*-bilinear and associative,*
(2) *there is a canonical isomorphism*

$$\mathrm{CH}_{-r}(X) \otimes \mathbf{Q} = \mathit{Corr}^r(X, \mathrm{Spec}(k))$$

such that pullback by correspondences corresponds to composition,
(3) *there is a canonical isomorphism*

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathit{Corr}^r(\mathrm{Spec}(k), X)$$

such that pushforward by correspondences corresponds to composition,
(4) *composition of correspondences is compatible with pushforward and pullback of cycles.*

Proof. Bilinearity follows immediately from the linearity of pushforward and pullback and the bilinearity of the intersection product. To prove associativity, say we have X, Y, Z, W and $c \in \mathrm{Corr}(X, Y)$, $c' \in \mathrm{Corr}(Y, Z)$, and $c'' \in \mathrm{Corr}(Z, W)$. Then we have

$$\begin{aligned}
c'' \circ (c' \circ c) &= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{12}^{123,*}c \cdot \mathrm{pr}_{23}^{123,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}(\mathrm{pr}_{12}^{123,*}c \cdot \mathrm{pr}_{23}^{123,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{134,*}^{1234}(\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}\mathrm{pr}_{134,*}^{1234}((\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{1234,*}c'') \\
&= \mathrm{pr}_{14,*}^{1234}((\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{1234,*}c'')
\end{aligned}$$

Here we use the notation

$$p_{134}^{1234} : X \times Y \times Z \times W \to X \times Z \times W \quad \text{and} \quad p_{14}^{134} : X \times Z \times W \to X \times W$$

the projections and similarly for other indices. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123} = \mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}$ by Chow Homology, Lemma 42.15.1. The third equality holds because intersection product commutes with the Gysin map for p_{123}^{1234} (which is given by flat pullback), see Chow Homology, Lemma 42.62.3. The fourth equality follows from the projection formula for p_{134}^{1234}, see Chow Homology, Lemma 42.62.4. The fourth equality is that proper pushforward is compatible with composition, see Chow Homology, Lemma 42.12.2. Since intersection product is associative by Chow Homology, Lemma 42.62.1 this concludes the proof of associativity of composition of correspondences.

We omit the proofs of (2) and (3) as these are essentially proved by carefully bookkeeping where various cycles live and in what (co)dimension.

The statement on pushforward and pullback of cycles means that $(c' \circ c)^*(\alpha) = c^*((c')^*(\alpha))$ and $(c' \circ c)_*(\alpha) = (c')_*(c_*(\alpha))$. This follows on combining (1), (2), and (3). □

0FG1 **Example 45.3.2.** Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Denote $\Gamma_f \subset X \times Y$ the graph of f. More precisely, Γ_f is the image of the closed immersion

$$(f, \mathrm{id}_Y) : Y \longrightarrow X \times Y$$

Let $X = \coprod X_d$ be the decomposition of X into its open and closed parts X_d which are equidimensional of dimension d. Then $\Gamma_f \cap (X_d \times Y)$ has pure codimension d. Hence $[\Gamma_f] \in \mathrm{CH}^*(X \times Y) \otimes \mathbf{Q}$ is contained in $\mathrm{Corr}^0(X \times Y)$, i.e., $[\Gamma_f]$ is a correspondence of degree 0 from X to Y.

0FG2 **Lemma 45.3.3.** *Smooth projective schemes over k with correspondences and composition of correspondences as defined above form a graded category over $\mathbf{Q}$ (Differential Graded Algebra, Definition 22.25.1).*

Proof. Everything is clear from the construction and Lemma 45.3.1 except for the existence of identity morphisms. Given a smooth projective scheme X consider the class $[\Delta]$ of the diagonal $\Delta \subset X \times X$ in $\mathrm{Corr}^0(X, X)$. We note that Δ is equal to the graph of the identity $\mathrm{id}_X : X \to X$ which is a fact we will use below.

To prove that $[\Delta]$ can serve as an identity we have to show that $[\Delta] \circ c = c$ and $c' \circ [\Delta] = c'$ for any correspondences $c \in \mathrm{Corr}^r(Y, X)$ and $c' \in \mathrm{Corr}^s(X, Y)$. For the second case we have to show that

$$c' = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^* c')$$

where $\mathrm{pr}_{12} : X \times X \times Y \to X \times X$ is the projection and similarly for pr_{13} and pr_{23}. We may write $c' = \sum a_i[Z_i]$ for some integral closed subschemes $Z_i \subset X \times Y$ and rational numbers a_i. Thus it clearly suffices to show that

$$[Z] = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^*[Z])$$

in the chow group of $X \times Y$ for any integral closed subscheme Z of $X \times Y$. After replacing X and Y by the irreducible component containing the image of Z under the two projections we may assume X and Y are integral as well. Then we have to show

$$[Z] = \mathrm{pr}_{13,*}([\Delta \times Y] \cdot [X \times Z])$$

Denote $Z' \subset X \times X \times Y$ the image of Z by the morphism $(\Delta, 1) : X \times Y \to X \times X \times Y$. Then Z' is a closed subscheme of $X \times X \times Y$ isomorphic to Z and $Z' = \Delta \times Y \cap X \times Z$ scheme theoretically. By Chow Homology, Lemma 42.62.5[1] we conclude that

$$[Z'] = [\Delta \times Y] \cdot [X \times Z]$$

Since Z' maps isomorphically to Z by pr_{13} also we conclude. The verification that $[\Delta] \circ c = c$ is similar and we omit it. □

0FG3 **Lemma 45.3.4.** *There is a contravariant functor from the category of smooth projective schemes over k to the category of correspondences which is the identity on objects and sends $f : Y \to X$ to the element $[\Gamma_f] \in Corr^0(X, Y)$.*

Proof. In the proof of Lemma 45.3.3 we have seen that this construction sends identities to identities. To finish the proof we have to show if $g : Z \to Y$ is another morphism of smooth projective schemes over k, then we have $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{f \circ g}]$ in $\mathrm{Corr}^0(X, Z)$. Arguing as in the proof of Lemma 45.3.3 we see that it suffices to show

$$[\Gamma_{f \circ g}] = \mathrm{pr}_{13,*}([\Gamma_f \times Z] \cdot [X \times \Gamma_g])$$

in $\mathrm{CH}^*(X \times Z)$ when X, Y, Z are integral. Denote $Z' \subset X \times Y \times Z$ the image of the closed immersion $(f \circ g, g, 1) : Z \to X \times Y \times Z$. Then $Z' = \Gamma_f \times Z \cap X \times \Gamma_g$ scheme theoretically and we conclude using Chow Homology, Lemma 42.62.5 that

$$[Z'] = [\Gamma_f \times Z] \cdot [X \times \Gamma_g]$$

Since it is clear that $\mathrm{pr}_{13,*}([Z']) = [\Gamma_{f \circ g}]$ the proof is complete. □

0FG4 **Remark 45.3.5.** Let X and Y be smooth projective schemes over k. Assume X is equidimensional of dimension d and Y is equidimensional of dimension e. Then the isomorphism $X \times Y \to Y \times X$ switching the factors determines an isomorphism

$$\mathrm{Corr}^r(X, Y) \longrightarrow \mathrm{Corr}^{d-e+r}(Y, X), \quad c \longmapsto c^t$$

called the *transpose*. It acts on cycles as well as cycle classes. An example which is sometimes useful, is the transpose $[\Gamma_f]^t = [\Gamma_f^t]$ of the graph of a morphism $f : Y \to X$.

0FG5 **Lemma 45.3.6.** *Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Let $[\Gamma_f] \in Corr^0(X, Y)$ be as in Example 45.3.2. Then*

1. *pushforward of cycles by the correspondence $[\Gamma_f]$ agrees with the Gysin map $f^! : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$,*
2. *pullback of cycles by the correspondence $[\Gamma_f]$ agrees with the pushforward map $f_* : \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X)$,*
3. *if X and Y are equidimensional of dimensions d and e, then*
   (a) *pushforward of cycles by the correspondence $[\Gamma_f^t]$ of Remark 45.3.5 corresponds to pushforward of cycles by f, and*
   (b) *pullback of cycles by the correspondence $[\Gamma_f^t]$ of Remark 45.3.5 corresponds to the Gysin map $f^!$.*

[1] The reader verifies that $\dim(Z') = \dim(\Delta \times Y) + \dim(X \times Z) - \dim(X \times X \times Y)$ and that Z' has a unique generic point mapping to the generic point of Z (where the local ring is CM) and to some point of X (where the local ring is CM). Thus all the hypothese of the lemma are indeed verified.

Proof. Proof of (1). Recall that $[\Gamma_f]_*(\alpha) = \mathrm{pr}_{2,*}([\Gamma_f] \cdot \mathrm{pr}_1^*\alpha)$. We have

$$[\Gamma_f] \cdot \mathrm{pr}_1^*\alpha = (f,1)_*((f,1)^!\mathrm{pr}_1^*\alpha) = (f,1)_*((f,1)^!\mathrm{pr}_1^!\alpha) = (f,1)_*(f^!\alpha)$$

The first equality by Chow Homology, Lemma 42.62.6. The second by Chow Homology, Lemma 42.59.5. The third because $\mathrm{pr}_1 \circ (f,1) = f$ and Chow Homology, Lemma 42.59.6. Then we coclude because $\mathrm{pr}_{2,*} \circ (f,1)_* = 1_*$ by Chow Homology, Lemma 42.12.2.

Proof of (2). Recall that $[\Gamma_f]_*(\beta) = \mathrm{pr}_{1,*}([\Gamma_f] \cdot \mathrm{pr}_2^*\beta)$. Arguing exactly as above we have

$$[\Gamma_f] \cdot \mathrm{pr}_2^*\beta = (f,1)_*\beta$$

Thus the result follows as before.

Proof of (3). Proved in exactly the same manner as above. □

0FG6 **Example 45.3.7.** Let $X = \mathbf{P}^1_k$. Then we have

$$\mathrm{Corr}^0(X,X) = \mathrm{CH}^1(X \times X) \otimes \mathbf{Q} = \mathrm{CH}_1(X \times X) \otimes \mathbf{Q}$$

Choose a k-rational point $x \in X$ and consider the cycles $c_0 = [x \times X]$ and $c_2 = [X \times x]$. A computation shows that $1 = [\Delta] = c_0 + c_2$ in $\mathrm{Corr}^0(X,X)$ and that we have the following rules for composition $c_0 \circ c_0 = c_0$, $c_0 \circ c_2 = 0$, $c_2 \circ c_0 = 0$, and $c_2 \circ c_2 = c_2$. In other words, c_0 and c_2 are orthogonal idempotents in the algebra $\mathrm{Corr}^0(X,X)$ and in fact we get

$$\mathrm{Corr}^0(X,X) = \mathbf{Q} \times \mathbf{Q}$$

as a $\mathbf{Q}$-algebra.

The category of correspondences is a symmetric monoidal category. Given smooth projective schemes X and Y over k, we define $X \otimes Y = X \times Y$. Given four smooth projective schemes X, X', Y, Y' over k we define a tensor product

$$\otimes : \mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^{r'}(X',Y') \longrightarrow \mathrm{Corr}^{r+r'}(X \times X', Y \times Y')$$

by the rule

$$(c, c') \longmapsto c \otimes c' = \mathrm{pr}_{13}^*c \cdot \mathrm{pr}_{24}^*c'$$

where $\mathrm{pr}_{13} : X \times X' \times Y \times Y' \to X \times Y$ and $\mathrm{pr}_{24} : X \times X' \times Y \times Y' \to X' \times Y'$ are the projections. As associativity constraint

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

we use the usual associativity constraint on products of schemes. The commutativity constraint will be given by the isomorphism $X \times Y \to Y \times X$ switching the factors.

0FG7 **Lemma 45.3.8.** *The tensor product of correspondences defined above turns the category of correspondences into a symmetric monoidal category with unit* $\mathrm{Spec}(k)$.

Proof. Omitted. □

0FG8 **Lemma 45.3.9.** *Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Assume X and Y equidimensional of dimensions d and e. Denote $a = [\Gamma_f] \in Corr^0(X,Y)$ and $a^t = [\Gamma_f^t] \in Corr^{d-e}(Y,X)$. Set $\eta_X = [\Gamma_{X \to X \times X}] \in$*

$Corr^0(X \times X, X)$, $\eta_Y = [\Gamma_{Y \to Y \times Y}] \in Corr^0(Y \times Y, Y)$, $[X] \in Corr^{-d}(X, \mathrm{Spec}(k))$, and $[Y] \in Corr^{-e}(Y, \mathrm{Spec}(k))$. The diagram

$$\begin{array}{ccccc} X \otimes Y & \xrightarrow[a \otimes id]{} & Y \otimes Y & \xrightarrow[\eta_Y]{} & Y \\ {\scriptstyle id \otimes a^t}\downarrow & & & & \downarrow{\scriptstyle [Y]} \\ X \otimes X & \xrightarrow{\eta_X} & X & \xrightarrow{[X]} & \mathrm{Spec}(k) \end{array}$$

is commutative in the category of correspondences.

Proof. Recall that $\mathrm{Corr}^r(W, \mathrm{Spec}(k)) = \mathrm{CH}_{-r}(W)$ for any smooth projective scheme W over k and given $c \in \mathrm{Corr}^s(W', W)$ the composition with c agrees with pullback by c as a map $\mathrm{CH}_{-r}(W) \to \mathrm{CH}_{-r-s}(W')$ (Lemma 45.3.1). Finally, we have Lemma 45.3.6 which tells us how to convert this into usual pushforward and pullback of cycles. We have

$$(a \otimes \mathrm{id})^* \eta_Y^* [Y] = (a \otimes \mathrm{id})^* [\Delta_Y] = (f \times \mathrm{id})_* \Delta_Y = [\Gamma_f]$$

and the other way around we get

$$(\mathrm{id} \otimes a^t)^* \eta_X^* [X] = (\mathrm{id} \otimes a^t)^* [\Delta_X] = (\mathrm{id} \times f)^! [\Delta_X] = [\Gamma_f]$$

The last equality follows from Chow Homology, Lemma 42.59.8. In other words, going either way around the diagram we obtain the element of $\mathrm{Corr}^d(X \times Y, \mathrm{Spec}(k))$ corresponding to the cycle $\Gamma_f \subset X \times Y$. □

45.4. Chow motives

0FG9 We fix a base field k. In this section we construct an additive Karoubian $\mathbf{Q}$-linear category M_k endowed with a symmetric monoidal structure and a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

which maps products to tensor products and disjoint unions to direct sums. Our construction will be characterized by the fact that h factors through the symmetric monoidal category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0 such that the image of the projector c_2 on $h(\mathbf{P}^1_k)$ from Example 45.3.7 is invertible in M_k, see Lemma 45.4.8. At the end of the section we will show that every motive, i.e., every object of M_k to has a (left) dual, see Lemma 45.4.10.

A *motive* or a *Chow motive* over k will be a triple (X, p, m) where

(1) X is a smooth projective scheme over k,
(2) $p \in \mathrm{Corr}^0(X, X)$ satisfies $p \circ p = p$,
(3) $m \in \mathbf{Z}$.

Given a second motive (Y, q, n) we define a *morphism of motives* or a *morphism of Chow motives* to be an element of

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \circ \mathrm{Corr}^{n-m}(X, Y) \circ p \subset \mathrm{Corr}^{n-m}(X, Y)$$

Composition of morphisms of motives is defined using the composition of correspondences defined above.

0FGA **Lemma 45.4.1.** *The category M_k whose objects are motives over k and morphisms are morphisms of motives over k is a $\mathbf{Q}$-linear category. There is a contravariant functor*

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

defined by $h(X) = (X, 1, 0)$ and $h(f) = [\Gamma_f]$.

Proof. Follows immediately from Lemma 45.3.4. □

0FGB **Lemma 45.4.2.** *The category M_k is Karoubian.*

Proof. Let $M = (X, p, m)$ be a motive and let $a \in \mathrm{Mor}(M, M)$ be a projector. Then $a = a \circ a$ both in $\mathrm{Mor}(M, M)$ as well as in $\mathrm{Corr}^0(X, X)$. Set $N = (X, a, m)$. Since we have $a = p \circ a \circ a$ in $\mathrm{Corr}^0(X, X)$ we see that $a : N \to M$ is a morphism of M_k. Next, suppose that $b : (Y, q, n) \to M$ is a morphism such that $(1 - a) \circ b = 0$. Then $b = a \circ b$ as well as $b = b \circ q$. Hence b is a morphism $b : (Y, q, n) \to N$. Thus we see that the projector $1 - a$ has a kernel, namely N and we find that M_k is Karoubian, see Homology, Definition 12.4.1. □

We define a functor

$$\otimes : M_k \times M_k \longrightarrow M_k$$

On objects we use the formula

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

On morphisms, we use

$$\begin{array}{c} \mathrm{Mor}((X, p, m), (Y, q, n)) \times \mathrm{Mor}((X', p', m'), (Y', q', n')) \\ \downarrow \\ \mathrm{Mor}((X \times X', p \otimes p', m + m'), (Y \times Y', q \otimes q', n + n')) \end{array}$$

given by the rule $(a, a') \longmapsto a \otimes a'$ where $\otimes$ on correspondences is as in Section 45.3. This makes sense: by definition of morphisms of motives we can write $a = q \circ c \circ p$ and $a' = q' \circ c' \circ p'$ with $c \in \mathrm{Corr}^{n-m}(X, Y)$ and $c' \in \mathrm{Corr}^{n'-m'}(X', Y')$ and then we obtain

$$a \otimes a' = (q \circ c \circ p) \otimes (q' \circ c' \circ p') = (q \otimes q') \circ (c \otimes c') \circ (p \otimes p')$$

which is indeed a morphism of motives from $(X \times X', p \otimes p', m + m')$ to $(Y \times Y', q \otimes q', n + n')$.

0FGC **Lemma 45.4.3.** *The category M_k with tensor product defined as above is symmetric monoidal with the obvious associativity and commutativity constraints and with unit $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$.*

Proof. Follows readily from Lemma 45.3.8. Details omitted. □

The motives $\mathbf{1}(n) = (\mathrm{Spec}(k), 1, n)$ are useful. Observe that

$$\mathbf{1} = \mathbf{1}(0) \quad \text{and} \quad \mathbf{1}(n + m) = \mathbf{1}(n) \otimes \mathbf{1}(m)$$

Thus tensoring with $\mathbf{1}(1)$ is an autoequivalence of the category of motives. Given a motive M we sometimes write $M(n) = M \otimes \mathbf{1}(n)$. Observe that if $M = (X, p, m)$, then $M(n) = (X, p, m + n)$.

0FGD **Lemma 45.4.4.** *With notation as in Example 45.3.7*

(1) *the motive* $(X, c_0, 0)$ *is isomorphic to the motive* $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$.
(2) *the motive* $(X, c_2, 0)$ *is isomorphic to the motive* $\mathbf{1}(-1) = (\mathrm{Spec}(k), 1, -1)$.

Proof. We will use Lemma 45.3.4 without further mention. The structure morphism $X \to \mathrm{Spec}(k)$ gives a correspondence $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), X)$. On the other hand, the rational point x is a morphism $\mathrm{Spec}(k) \to X$ which gives a correspondence $b \in \mathrm{Corr}^0(X, \mathrm{Spec}(k))$. We have $b \circ a = 1$ as a correspondence on $\mathrm{Spec}(k)$. The composition $a \circ b$ corresponds to the graph of the composition $X \to x \to X$ which is $c_0 = [x \times X]$. Thus $a = a \circ b \circ a = c_0 \circ a$ and $b = b \circ a \circ b = b \circ c_0$. Hence, unwinding the definitions, we see that a and b are mutually inverse morphisms $a : (\mathrm{Spec}(k), 1, 0) \to (X, c_0, 0)$ and $b : (X, c_0, 0) \to (\mathrm{Spec}(k), 1, 0)$.

We will proceed exactly as above to prove the second statement. Denote

$$a' \in \mathrm{Corr}^1(\mathrm{Spec}(k), X) = \mathrm{CH}^1(X)$$

the class of the point x. Denote

$$b' \in \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) = \mathrm{CH}_1(X)$$

the class of $[X]$. We have $b' \circ a' = 1$ as a correspondence on $\mathrm{Spec}(k)$ because $[x] \cdot [X] = [x]$ on $X = \mathrm{Spec}(k) \times X \times \mathrm{Spec}(k)$. Computing the intersection product $\mathrm{pr}_{12}^* b' \cdot \mathrm{pr}_{23}^* a'$ on $X \times \mathrm{Spec}(k) \times X$ gives the cycle $X \times \mathrm{Spec}(k) \times x$. Hence the composition $a' \circ b'$ is equal to c_2 as a correspondence on X. Thus $a' = a' \circ b \circ a' = c_2 \circ a'$ and $b' = b' \circ a' \circ b' = b' \circ c_2$. Recall that

$$\mathrm{Mor}((\mathrm{Spec}(k), 1, -1), (X, c_2, 0)) = c_2 \circ \mathrm{Corr}^1(\mathrm{Spec}(k), X) \subset \mathrm{Corr}^1(\mathrm{Spec}(k), X)$$

and

$$\mathrm{Mor}((X, c_2, 0), (\mathrm{Spec}(k), 1, -1)) = \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) \circ c_2 \subset \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k))$$

Hence, we see that a' and b' are mutually inverse morphisms $a' : (\mathrm{Spec}(k), 1, -1) \to (X, c_2, 0)$ and $b' : (X, c_2, 0) \to (\mathrm{Spec}(k), 1, -1)$. □

0FGE **Remark 45.4.5** (Lefschetz and Tate motive)**.** Let $X = \mathbf{P}^1_k$ and c_2 be as in Example 45.3.7. In the literature the motive $(X, c_2, 0)$ is sometimes called the *Lefschetz motive* and depending on the reference the notation L, $\mathbf{L}$, $\mathbf{Q}(-1)$, or $h^2(\mathbf{P}^1_k)$ may be used to denote it. By Lemma 45.4.4 the Lefschetz motive is isomorphic to $\mathbf{1}(-1)$. Hence the Lefschetz motive is invertible (Categories, Definition 4.43.4) with inverse $\mathbf{1}(1)$. The motive $\mathbf{1}(1)$ is sometimes called the *Tate motive* and depending on the reference the notation L^{-1}, $\mathbf{L}^{-1}$, $\mathbf{T}$, or $\mathbf{Q}(1)$ may be used to denote it.

0FGF **Lemma 45.4.6.** *The category* M_k *is additive.*

Proof. Let (Y, p, m) and (Z, q, n) be motives. If $n = m$, then a direct sum is given by $(Y \amalg Z, p + q, m)$, with obvious notation. Details omitted.

Suppose that $n < m$. Let X, c_2 be as in Example 45.3.7. Then we consider

$$\begin{aligned}(Z, q, n) &= (Z, q, m) \otimes (\mathrm{Spec}(k), 1, -1) \otimes \ldots \otimes (\mathrm{Spec}(k), 1, -1) \\ &\cong (Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0) \\ &\cong (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)\end{aligned}$$

where we have used Lemma 45.4.4. This reduces us to the case discussed in the first paragraph. □

0FGG **Lemma 45.4.7.** *In* M_k *we have* $h(\mathbf{P}^1_k) \cong \mathbf{1} \oplus \mathbf{1}(-1)$.

Proof. This follows from Example 45.3.7 and Lemma 45.4.4. □

0FGH **Lemma 45.4.8.** *Let X, c_2 be as in Example 45.3.7. Let $\mathcal{C}$ be a $\mathbf{Q}$-linear Karoubian symmetric monoidal category. Any $\mathbf{Q}$-linear functor*

$$F : \left\{ \begin{array}{c} \text{smooth projective schemes over } k \\ \text{morphisms are correspondences of degree } 0 \end{array} \right\} \longrightarrow \mathcal{C}$$

of symmetric monoidal categories such that the image of $F(c_2)$ on $F(X)$ is an invertible object, factors uniquely through a functor $F : M_k \to \mathcal{C}$ of symmetric monoidal categories.

Proof. Denote U in $\mathcal{C}$ the invertible object which is assumed to exist in the statement of the lemma. We extend F to motives by setting

$$F(X, p, m) = (\text{the image of the projector } F(p) \text{ in } F(X)) \otimes U^{\otimes -m}$$

which makes sense because U is invertible and because $\mathcal{C}$ is Karoubian. An important feature of this choice is that $F(X, c_2, 0) = U$. Observe that

$$\begin{aligned} F((X, p, m) \otimes (Y, q, n)) &= F(X \times Y, p \otimes q, m + n) \\ &= (\text{the image of } F(p \otimes q) \text{ in } F(X \times Y)) \otimes U^{\otimes -m-n} \\ &= F(X, p, m) \otimes F(Y, q, n) \end{aligned}$$

Thus we see that our rule is compatible with tensor products on the level of objects (details omitted).

Next, we extend F to morphisms of motives. Suppose that

$$a \in \mathrm{Hom}((Y, p, m), (Z, q, n)) = q \circ \mathrm{Corr}^{n-m}(Y, Z) \circ p \subset \mathrm{Corr}^{n-m}(Y, Z)$$

is a morphism. If $n = m$, then a is a correspondence of degree 0 and we can use $F(a) : F(Y) \to F(Z)$ to get the desired map $F(Y, p, m) \to F(Z, q, n)$. If $n < m$ we get canonical identifications

$$\begin{aligned} s : F((Z, q, n)) &\to F(Z, q, m) \otimes U^{m-n} \\ &\to F(Z, q, m) \otimes F(X, c_2, 0) \otimes \ldots \otimes F(X, c_2, 0) \\ &\to F((Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0)) \\ &\to F((Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)) \end{aligned}$$

Namely, for the first isomorphism we use the definition of F on motives above. For the second, we use the choice of U. For the third we use the compatibility of F on tensor products of motives. The fourth is the definition of tensor products on motives. On the other hand, since we similarly have an isomorphism

$$\sigma : (Z, q, n) \to (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)$$

(see proof of Lemma 45.4.6). Composing a with this isomorphism gives

$$\sigma \circ a \in \mathrm{Hom}((Y, p, m), (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m))$$

Putting everything together we obtain

$$s^{-1} \circ F(\sigma \circ a) : F(Y, p, m) \to F(Z, q, n)$$

If $n > m$ we similarly define isomorphisms

$$t : F((Y, p, m)) \to F((Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n))$$

and

$$\tau : (Y, p, m)) \to (Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n)$$

and we set $F(a) = F(a \circ \tau^{-1}) \circ t$. We omit the verification that this construction defines a functor of symmetric monoidal categories. □

0FGI **Lemma 45.4.9.** *Let X be a smooth projective scheme over k which is equidimensional of dimension d. Then $h(X)(d)$ is a left dual to $h(X)$ in M_k.*

Proof. We will use Lemma 45.3.1 without further mention. We compute

$$\mathrm{Hom}(\mathbf{1}, h(X) \otimes h(X)(d)) = \mathrm{Corr}^d(\mathrm{Spec}(k), X \times X) = \mathrm{CH}^d(X \times X)$$

Here we have $\eta = [\Delta]$. On the other hand, we have

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

and here we have the class $\epsilon = [\Delta]$ of the diagonal as well. The composition of the correspondence $[\Delta] \otimes 1$ with $1 \otimes [\Delta]$ either way is the correspondence $[\Delta] = 1$ in $\mathrm{Corr}^0(X, X)$ which proves the required diagrams of Categories, Definition 4.43.5 commute. Namely, observe that

$$[\Delta] \otimes 1 \in \mathrm{Corr}^d(X, X \times X \times X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$ with obvious notation. Similarly, the class

$$1 \otimes [\Delta] \in \mathrm{Corr}^{-d}(X \times X \times X, X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$. The composition $(1 \otimes [\Delta]) \circ ([\Delta] \otimes 1)$ is by definition the pushforward $\mathrm{pr}_{15,*}^{12345}$ of the intersection product

$$[\mathrm{pr}_{23}^{12345,-1}(\Delta) \cap \mathrm{pr}_{14}^{12345,-1}(\Delta)] \cdot [\mathrm{pr}_{34}^{12345,-1}(\Delta) \cap \mathrm{pr}_{15}^{12345,-1}(\Delta)] = [\text{small diagonal in } X^5]$$

which is equal to Δ as desired. We omit the proof of the formula for the composition in the other order. □

0FGJ **Lemma 45.4.10.** *Every object of M_k has a left dual.*

Proof. Let $M = (X, p, m)$ be an object of M_k. Then M is a summand of $(X, 0, m) = h(X)(m)$. By Homology, Lemma 12.17.3 it suffices to show that $h(X)(m) = h(X) \otimes \mathbf{1}(m)$ has a dual. By construction $\mathbf{1}(-m)$ is a left dual of $\mathbf{1}(m)$. Hence it suffices to show that $h(X)$ has a left dual, see Categories, Lemma 4.43.8. Let $X = \coprod X_i$ be the decomposition of X into irreducible components. Then $h(X) = \bigoplus h(X_i)$ and it suffices to show that $h(X_i)$ has a left dual, see Homology, Lemma 12.17.2. This follows from Lemma 45.4.9. □

45.5. Chow groups of motives

0FGK We define the Chow groups of a motive as follows.

0FGL **Definition 45.5.1.** Let k be a base field. Let $M = (X, p, m)$ be a Chow motive over k. For $i \in \mathbf{Z}$ we define the *ith Chow group of M* by the formula

$$\mathrm{CH}^i(M) = p\left(\mathrm{CH}^{i+m}(X) \otimes \mathbf{Q}\right)$$

We have $\mathrm{CH}^i(h(X)) = \mathrm{CH}^i(X) \otimes \mathbf{Q}$ if X is a smooth projective scheme over k.

Observe that $\mathrm{CH}^i(-)$ is a functor from M_k to $\mathbf{Q}$-vector spaces. Indeed, if $c : M \to N$ is a morphism of motives $M = (X, p, m)$ and $N = (Y, q, n)$, then c is a correspondence of degree $n - m$ from X to Y and hence pushforward along c (Section 45.3) is a family of maps

$$c_* : \mathrm{CH}^{i+m}(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^{i+n}(Y) \otimes \mathbf{Q}$$

Since $c = q \circ c \circ p$ by definition of morphisms of motives, we see that indeed we obtain

$$c_* : \mathrm{CH}^i(M) \to \mathrm{CH}^i(N)$$

for all $i \in \mathbf{Z}$. This is compatible with compositions of morphisms of motives by Lemma 45.3.1. This functoriality of Chow groups can also be deduced from the following lemma.

0FGM **Lemma 45.5.2.** *Let k be a base field. The functor $\mathrm{CH}^i(-)$ on the category of motives M_k is representable by $\mathbf{1}(-i)$, i.e., we have*

$$\mathrm{CH}^i(M) = \mathrm{Hom}_{M_k}(\mathbf{1}(-i), M)$$

functorially in M in M_k.

Proof. Immediate from the definitions and Lemma 45.3.1. □

The reader can imagine that we can use Lemma 45.5.2, the Yoneda lemma, and the duality in Lemma 45.4.9 to obtain the following.

0FGN **Lemma 45.5.3** (Manin)**.** *Let k be a base field. Let $c : M \to N$ be a morphism of motives. If for every smooth projective scheme X over k the map $c \otimes 1 : M \otimes h(X) \to N \otimes h(X)$ induces an isomorphism on Chow groups, then c is an isomorphism.*

Proof. Any object L of M_k is a summand of $h(X)(m)$ for some smooth projective scheme X over k and some $m \in \mathbf{Z}$. Observe that the Chow groups of $M \otimes h(X)(m)$ are the same as the Chow groups of $M \otimes h(X)$ up to a shift in degrees. Hence our assumption implies that $c \otimes 1 : M \otimes L \to N \otimes L$ induces an isomorphism on Chow groups for every object L of M_k. By Lemma 45.5.2 we see that

$$\mathrm{Hom}_{M_k}(\mathbf{1}, M \otimes L) \to \mathrm{Hom}_{M_k}(\mathbf{1}, N \otimes L)$$

is an isomorphism for every L. Since every object of M_k has a left dual (Lemma 45.4.10) we conclude that

$$\mathrm{Hom}_{M_k}(K, M) \to \mathrm{Hom}_{M_k}(K, N)$$

is an isomorphism for every object K of M_k, see Categories, Lemma 4.43.6. We conclude by the Yoneda lemma (Categories, Lemma 4.3.5). □

45.6. Projective space bundle formula

0FGP Let k be a base field. Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Our convention is that the *projective bundle associated to $\mathcal{E}$* is the morphism

$$P = \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{p} X$$

over X with $\mathcal{O}_P(1)$ normalized so that $p_*(\mathcal{O}_P(1)) = \mathcal{E}$. Recall that

$$[\Gamma_p] \in \mathrm{Corr}^0(X, P) \subset \mathrm{CH}^*(X \times P) \otimes \mathbf{Q}$$

See Example 45.3.2. For $i = 0, \ldots, r-1$ consider the correspondences

$$c_i = c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^i \cap [\Gamma_p] \in \mathrm{Corr}^i(X, P)$$

We may and do think of c_i as a morphism $h(X)(-i) \to h(P)$.

0FGQ **Lemma 45.6.1** (Projective space bundle formula)**.** *In the situation above, the map*

$$\sum\nolimits_{i=0,\ldots,r-1} c_i : \bigoplus\nolimits_{i=0,\ldots,r-1} h(X)(-i) \longrightarrow h(P)$$

is an isomorphism in the category of motives.

Proof. By Lemma 45.5.3 it suffices to show that our map defines an isomorphism on Chow groups of motives after taking the product with any smooth projective scheme Z. Observe that $P \times Z \to X \times Z$ is the projective bundle associated to the pullback of $\mathcal{E}$ to $X \times Z$. Hence the statement on Chow groups is true by the projective space bundle formula given in Chow Homology, Lemma 42.36.2. Namely, pushforward of cycles along $[\Gamma_p]$ is given by pullback of cycles by p according to Lemma 45.3.6 and Chow Homology, Lemma 42.59.5. Hence pushforward along c_i sends α to $c_1(\mathcal{O}_P(1))^i \cap p^*\alpha$. Some details omitted. □

In the situation above, for $j = 0, \ldots, r-1$ consider the correspondences

$$c'_j = c_1(\mathrm{pr}_1^*\mathcal{O}_P(1))^{r-1-j} \cap [\Gamma_p^t] \in \mathrm{Corr}^{-j}(P, X)$$

For $i, j \in \{0, \ldots, r-1\}$ we have

$$c'_j \circ c_i = \mathrm{pr}_{13,*}\left(c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^{i+r-1-j} \cap (\mathrm{pr}_{12}^*[\Gamma_p] \cdot \mathrm{pr}_{23}^*[\Gamma_p^t])\right)$$

The cycles $\mathrm{pr}_{12}^{-1}\Gamma_p$ and $\mathrm{pr}_{23}^{-1}\Gamma_p^t$ intersect transversally and with intersection equal to the image of $(p, 1, p) : P \to X \times P \times X$. Observe that the fibres of $(p, p) = \mathrm{pr}_{13} \circ (p, 1, p) : P \to X \times X$ have dimension $r-1$. We immediately conclude $c'_j \circ c_i = 0$ for $i + r - 1 - j < r - 1$, in other words when $i < j$. On the other hand, by the projective space bundle formula (Chow Homology, Lemma 42.36.2) the cycle $c_1(\mathcal{O}_P(1))^{r-1} \cap [P]$ maps to $[X]$ in X. Hence for $i = j$ the pushforward above gives the class of the diagonal and hence we see that

$$c'_i \circ c_i = 1 \in \mathrm{Corr}^0(X, X)$$

for all $i \in \{0, \ldots, r-1\}$. Thus we see that the matrix of the composition

$$\bigoplus h(X)(-i) \xrightarrow{\bigoplus c_i} h(P) \xrightarrow{\bigoplus c'_j} \bigoplus h(X)(-j)$$

is invertible (upper triangular with 1s on the diagonal). We conclude from the projective space bundle formula (Lemma 45.6.1) that also the composition the other way around is invertible, but it seems a bit harder to prove this directly.

0FGR **Lemma 45.6.2.** *Let $p : P \to X$ be as in Lemma 45.6.1. The class $[\Delta_P]$ of the diagonal of P in $\mathrm{CH}^*(P \times P)$ can be written as*

$$[\Delta_P] = \left(\sum\nolimits_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(pr_1^*\mathcal{S}^\vee) \cap c_1(pr_2^*\mathcal{O}_P(1))^i\right) \cap (p \times p)^*[\Delta_X]$$

where $\mathcal{S}$ is the kernel of the canonical surjection $p^*\mathcal{E} \to \mathcal{O}_P(1)$.

Proof. Observe that $(p \times p)^*[\Delta_X] = [P \times_X P]$. Since $\Delta_P \subset P \times_X P \subset P \times P$ and since capping with Chern classes commutes with proper pushforward (Chow Homology, Lemma 42.38.4) it suffices to show that the class of $\Delta_P \subset P \times_X P$ in $\mathrm{CH}^*(P \times_X P)$ is equal to

$$\left(\sum\nolimits_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(q_1^*\mathcal{S}^\vee) \cap c_1(q_2^*\mathcal{O}_P(1))^i\right) \cap [P \times_X P]$$

where $q_i : P \times_X P \to P$, $i = 1, 2$ are the projections. Set $q = p \circ q_1 = p \circ q_2 : P \times_X P \to X$. Consider the maps

$$q_1^*\mathcal{S} \otimes q_2^*\mathcal{O}_P(-1) \to q^*\mathcal{E} \otimes q^*\mathcal{E}^\vee \to \mathcal{O}_{P \times_X P}$$

where the final arrow is the pullback by q of the evaluation map $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{O}_X$. The source of the composition is a module locally free of rank $r - 1$ and a local calculation shows that this map vanishes exactly along Δ_P. By Chow Homology, Lemma 42.44.1 the class $[\Delta_P]$ is the top Chern class of the dual

$$q_1^*\mathcal{S}^\vee \otimes q_2^*\mathcal{O}_P(1)$$

The desired result follows from Chow Homology, Lemma 42.39.1. □

45.7. Classical Weil cohomology theories

0FGS In this section we define what we will call a classical Weil cohomology theory. This is exactly what is called a Weil cohomology theory in [**Kle68**, Section 1.2].

We fix an algebraically closed field k (the base field). In this section *variety* will mean a variety over k, see Varieties, Section 33.3. We fix a field F of characteristic 0 (the coefficient field). A Weil cohomology theory is given by data (D1), (D2), and (D3) subject to axioms (A), (B), and (C).

The data is given by:

(D1) A contravariant functor H^* from the category of smooth projective varieties to the category of graded commutative F-algebras.
(D2) For every smooth projective variety X a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)$.
(D3) For every smooth projective variety X of dimension d a map $\int_X : H^{2d}(X) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D1). Given a smooth projective variety X we say that $H^*(X)$ is the *cohomology* of X. Given a morphism $f : X \to Y$ of smooth projective varieties we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the *pullback map*.

Remarks on (D2). The map γ is called the *cycle class map*. We say that $\gamma(\alpha)$ is the *cohomology class* of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective varieties and Z integral, then $[Z]$ could mean the class of the cycle $[Z]$ in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the *trace map* and is sometimes denoted Tr_X.

The first axiom is often called *Poincaré duality*

(A) Let X be a smooth projective variety of dimension d. Then
  (a) $\dim_F H^i(X) < \infty$ for all i,
  (b) $H^i(X) \times H^{2d-i}(X) \to H^{2d}(X) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$,
  (c) $H^i(X) = 0$ unless $i \in [0, 2d]$, and
  (d) $\int_X : H^{2d}(X) \to F$ is an isomorphism.

Let $f : X \to Y$ be a morphism of smooth projective varieties with $\dim(X) = d$ and $\dim(Y) = e$. Using Poincaré duality we can define a *pushforward*

$$f_* : H^{2d-i}(X) \longrightarrow H^{2e-i}(Y)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)$, the element $f_*a \in H^{2e-i}(Y)$ is characterized by

$$\int_X f^*b \cup a = \int_Y b \cup f_*a$$

for all $b \in H^i(Y)$.

0FGT **Lemma 45.7.1.** *Assume given (D1) and (D3) satisfying (A). For $f : X \to Y$ a morphism of smooth projective varieties we have $f_*(f^*b \cup a) = b \cup f_*a$. If $g : Y \to Z$ is a second morphism of smooth projective varieties, then $g_* \circ f_* = (g \circ f)_*$.*

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_*a = \int_X f^*c \cup f^*b \cup a = \int_Y c \cup f_*(f^*b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_*a = \int_X (g \circ f)^*c \cup a = \int_X f^*g^*c \cup a = \int_Y g^*c \cup f_*a = \int_Z c \cup g_*f_*a$$

This ends the proof. □

The second axiom says that H^* respects the monoidal structure given by products via the *Künneth formula*

(B) Let X and Y be smooth projective varieties. The map

$$H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y), \quad a \otimes b \mapsto \mathrm{pr}_1^*a \cup \mathrm{pr}_2^*b$$

is an isomorphism.

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
  (a) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f^!\beta) = f^*\gamma(\beta)$ for $\beta \in \mathrm{CH}^*(Y)$,
  (b) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ for $\alpha \in \mathrm{CH}^*(X)$,
  (c) for any smooth projective variety X we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \mathrm{CH}^*(X)$, and
  (d) $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$.

0FGU **Remark 45.7.2.** Let X be a smooth projective variety. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \mathrm{pr}_1^*\alpha \cdot \mathrm{pr}_2^*\beta$ in $\mathrm{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\mathrm{pr}_1^*\gamma(\alpha) \cup \mathrm{pr}_2^*\gamma(\beta)$. This is how axiom (C)(c) is formulated in [**Kle68**].

0FGV **Definition 45.7.3.** Let k be an algebraically closed field. Let F be a field of characteristic 0. A *classical Weil cohomology theory* over k with coefficients in F is given by data (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying (A), (B), and (C).

We do a tiny bit of work.

0FGW **Lemma 45.7.4.** *Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety of dimension d. The diagram*

$$\begin{array}{ccc} \mathrm{CH}^d(X) & \xrightarrow{\gamma} & H^{2d}(X) \\ \| & & \downarrow \int_X \\ \mathrm{CH}_0(X) & \xrightarrow{\deg} & F \end{array}$$

commutes where $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ *is the degree of zero cycles discussed in Chow Homology, Section 42.41.*

Proof. The result holds for $\mathrm{Spec}(k)$ by axiom (C)(d). Let $x : \mathrm{Spec}(k) \to X$ be a closed point of X. Then we have $\gamma([x]) = x_*\gamma([\mathrm{Spec}(k)])$ in $H^{2d}(X)$ by axiom (C)(b). Hence $\int_X \gamma([x]) = 1$ by the definition of x_*. □

0FGX **Lemma 45.7.5.** *Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X and Y be smooth projective varieties. Then $\int_{X\times Y} = \int_X \otimes \int_Y$.*

Proof. Say $\dim(X) = d$ and $\dim(Y) = e$. By axiom (B) we have $H^{2d+2e}(X \times Y) = H^{2d}(X) \otimes H^{2e}(Y)$ and by axiom (A)(d) this is 1-dimensional. By Lemma 45.7.4 this 1-dimensional vector space generated by the class $\gamma([x \times y])$ of a closed point (x, y) and $\int_{X\times Y} \gamma([x \times y]) = 1$. Since $\gamma([x \times y]) = \gamma([x]) \otimes \gamma([y])$ by axioms (C)(a) and (C)(c) and since $\int_X \gamma([x]) = 1$ and $\int_Y \gamma([y]) = 1$ we conclude. □

0FGY **Lemma 45.7.6.** *Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X and Y be smooth projective varieties. Then $pr_{2,*} : H^*(X \times Y) \to H^*(Y)$ sends $a \otimes b$ to $(\int_X a)b$.*

Proof. This is equivalent to the result of Lemma 45.7.5. □

0FGZ **Lemma 45.7.7.** *Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety of dimension d. Choose a basis $e_{i,j}, j =$*

$1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write

$$\gamma([\Delta]) = \sum\nolimits_{i=0,\ldots,2d} \sum\nolimits_j e_{i,j} \otimes e'_{2d-i,j} \quad \text{in} \quad \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i \delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 45.7.2. On the other hand we have $\gamma([\Delta]) = \Delta_* \gamma([X]) = \Delta_* 1$ by axiom (C)(b) and the fact that $\gamma([X]) = 1$. Namely, $[X] \cdot [X] = [X]$ hence by axiom (C)(c) the cohomology class $\gamma([X])$ is 0 or 1 in the 1-dimensional F-algebra $H^0(X)$; here we have also used axioms (A)(d) and (A)(b). But $\gamma([X])$ cannot be zero as $[X] \cdot [x] = [x]$ for a closed point x of X and we have the nonvanishing of $\gamma([x])$ by Lemma 45.7.4. Hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_* 1 \cup a \otimes b = \int_X a \cup b$$

by the definition of Δ_*. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by Lemma 45.7.5; note that we made two switches of order so that the sign is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. □

0FH0 **Lemma 45.7.8.** *Let H^* be a classical Weil cohomology theory (Definition 45.7.3). Let X be a smooth projective variety. We have*

$$\sum\nolimits_{i=0,\ldots,2\dim(X)} (-1)^i \dim_F H^i(X) = \deg([\Delta] \cdot [\Delta]) = \deg(c_d(\mathcal{T}_X) \cap [X])$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 42.62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 42.54.5). Since the conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 29.32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 45.7.4 we have

$$\begin{aligned} \deg([\Delta] \cdot [\Delta]) &= \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_*(\Delta^* \gamma([\Delta])) \\ &= \int_X \Delta^* \gamma([\Delta]) \end{aligned}$$

Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 45.7.7. Recalling that Δ^* is given by cup product we obtain

$$\int_X \sum\nolimits_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} (-1)^i = \sum (-1)^i \beta_i$$

as desired. □

We will now tie classical Weil cohomology theories in with motives as follows.

0FH1 **Lemma 45.7.9.** *Let k be an algebraically closed field. Let F be a field of characteristic 0. Consider a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D1), (D2), (D3) satisfying all of (A), (B), (C) except for possibly (A)(c) and (A)(d).

Proof. We obtain a contravariant functor from the category of smooth projective varieties to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X \times Y) = G(h(X \times Y)) = G(h(X) \otimes h(Y)) = G(h(X)) \otimes G(h(Y)) = H^*(X) \otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X) \otimes h(X) \to h(X) \otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 12.17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2 and we may choose an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces. Here and below $F[n]$ means the graded F-vector space which has F in degree $-n$ and zero elsewhere. Using compatibility with tensor products, we find for all $n \in \mathbf{Z}$ an isomorphism $F[2n] \to G(\mathbf{1}(n))$ compatible with tensor products.

Let X be a smooth projective variety. By Lemma 45.3.1 we have

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X) = \mathrm{Hom}(\mathbf{1}(-r), h(X))$$

Applying the functor G we obtain

$$\gamma : \mathrm{CH}^r(X) \otimes \mathbf{Q} \longrightarrow \mathrm{Hom}(G(\mathbf{1}(-r)), H^*(X)) = H^{2r}(X)$$

This is the datum (D2).

Let X be a smooth projective variety of dimension d. By Lemma 45.3.1 we have

$$\mathrm{Mor}(h(X)(d), \mathbf{1}) = \mathrm{Mor}((X, 1, d), (\mathrm{Spec}(k), 1, 0)) = \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G we obtain

$$H^*(X) \otimes F[-2d] = G(h(X)(d)) \longrightarrow G(\mathbf{1}) = F$$

This map is zero except in degree 0 where we obtain $\int_X : H^{2d}(X) \to F$. This is the datum (D3).

Let X be a smooth projective variety of dimension d. By Lemma 45.4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes F[-2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. By Homology, Lemma 12.17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$ produces nondegenerate pairings $H^{2d-i}(X) \otimes_F H^i(X) \to F$. In the proof of Lemma 45.4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A) parts (a) and (b).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it as a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of smooth projective varieties, then $f^!\alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 45.3.6. Hence $f^!\alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 45.3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of smooth projective varieties and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 45.3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 45.3.6 and 45.3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 45.3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 45.7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be smooth projective varieties and let α, β be cycles on them. Denote a, b the corresponding correspondences from $\mathrm{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\mathrm{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\mathrm{Spec}(k)]$ corresponds to the identity morphism on $h(\mathrm{Spec}(k))$. This finishes the proof of the lemma. □

0FH2 **Lemma 45.7.10.** *Let k be an algebraically closed field. Let F be a field of characteristic 0. Let H^* be a classical Weil cohomology theory. Then we can construct a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \textit{graded } F\textit{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. By Lemma 45.4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space. Since every smooth projective scheme is canonically a disjoint union of smooth projective varieties, it suffices to construct G on the category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given a smooth projective variety X we set $G(X) = H^*(X)$.

Given a correspondence $c \in \mathrm{Corr}^0(X, Y)$ between smooth projective varieties we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \mathrm{pr}_{2,*}(\gamma(c) \cup \mathrm{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b), and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \mathrm{Corr}^0(Y, Z)$. Namely, for $a \in H^*(X)$ we have

$$\begin{aligned}
(G(c') \circ G(c))(a) &= \mathrm{pr}_{3,*}^{23}(\gamma(c') \cup \mathrm{pr}_2^{23,*}(\mathrm{pr}_{2,*}^{12}(\gamma(c) \cup \mathrm{pr}_1^{12,*} a))) \\
&= \mathrm{pr}_{3,*}^{23}(\gamma(c') \cup \mathrm{pr}_{23,*}^{123}(\mathrm{pr}_{12}^{123,*}(\gamma(c) \cup \mathrm{pr}_1^{12,*} a))) \\
&= \mathrm{pr}_{3,*}^{23}\mathrm{pr}_{23,*}^{123}(\mathrm{pr}_{23}^{123,*}\gamma(c') \cup \mathrm{pr}_{12}^{123,*}\gamma(c) \cup \mathrm{pr}_1^{123,*} a) \\
&= \mathrm{pr}_{3,*}^{23}\mathrm{pr}_{23,*}^{123}(\gamma(\mathrm{pr}_{23}^{123,*} c') \cup \gamma(\mathrm{pr}_{12}^{123,*} c) \cup \mathrm{pr}_1^{123,*} a) \\
&= \mathrm{pr}_{3,*}^{13}\mathrm{pr}_{13,*}^{123}(\gamma(\mathrm{pr}_{23}^{123,*} c' \cdot \mathrm{pr}_{12}^{123,*} c) \cup \mathrm{pr}_1^{123,*} a) \\
&= \mathrm{pr}_{3,*}^{13}(\gamma(\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{23}^{123,*} c' \cdot \mathrm{pr}_{12}^{123,*} c)) \cup \mathrm{pr}_1^{13,*} a) \\
&= G(c' \circ c)(a)
\end{aligned}$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\mathrm{pr}_2^{23,*} \circ \mathrm{pr}_{2,*}^{12} = \mathrm{pr}_{23,*}^{123} \circ \mathrm{pr}_{12}^{123,*}$ as follows immediately from the description of pushforward along projections given in Lemma 45.7.6. The third equality holds by Lemma 45.7.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the Gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 42.59.5). The fifth equality uses axiom (C)(c) as well as Lemma 45.7.1 to see that $\mathrm{pr}_{3,*}^{23} \circ \mathrm{pr}_{23,*}^{123} = \mathrm{pr}_{3,*}^{13} \circ \mathrm{pr}_{13,*}^{123}$. The sixth equality uses the projection formula from Lemma 45.7.1 as well as axiom (C)(b) to see that $\mathrm{pr}_{13,*}^{123}\gamma(\mathrm{pr}_{23}^{123,*} c' \cdot \mathrm{pr}_{12}^{123,*} c) = \gamma(\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{23}^{123,*} c' \cdot \mathrm{pr}_{12}^{123,*} c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \mathrm{Corr}^0(X, X)$ is the identity in the category of correspondences (see Lemma 45.3.3 and its proof), then we have to show that $G([\Delta]) = \mathrm{id}$. This follows from the determination of $\gamma([\Delta])$ in Lemma 45.7.7 and Lemma 45.7.6. This finishes the construction of G as a functor on smooth projective varieties and correspondences of degree 0.

It follows from axioms (A)(c) and (A)(d) that $G(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B) shows our functor

is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By axiom (A)(d) the map $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ is an isomorphism. By axiom (A)(b) we see that $\dim_F H^0(\mathbf{P}^1) = 1$. By Lemma 45.7.8 and axiom (A)(c) we obtain $2 - \dim_F H^1(\mathbf{P}^1) = c_1(T_{\mathbf{P}^1}) = 2$. Hence $H^1(\mathbf{P}^1) = 0$. Thus

$$G(\mathbf{P}^1) = H^0(\mathbf{P}^1) \oplus H^2(\mathbf{P}^1)$$

Recall that $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1, \mathbf{P}^1)$, see Example 45.3.7. We have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 45.4.4. Since $F = G(\mathrm{Spec}(k))$, it follows from functoriality that $G(c_0)$ is the projector onto the summand $H^0(\mathbf{P}^1) \subset G(\mathbf{P}^1)$. Hence $G(c_2)$ must necessarily be the projection onto $H^2(\mathbf{P}^1)$ and the proof is complete. □

0FH3 **Proposition 45.7.11.** *Let k be an algebraically closed field. Let F be a field of characteristic 0. A classical Weil cohomology theory is the same thing as a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \textit{graded } F\textit{-vector spaces}$$

of symmetric monoidal categories together with an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces such that in addition

(1) *$G(h(X))$ lives in nonnegative degrees, and*
(2) *$\dim_F G^0(h(X)) = 1$*

for any smooth projective variety X.

Proof. Given G and $F[2] \to G(\mathbf{1}(1))$ by setting $H^*(X) = G(h(X))$ we obtain data (D1), (D2), and (D3) satisfying all of (A), (B), and (C) except for possibly (A)(c) and (A)(d), see Lemma 45.7.9 and its proof. Observe that assumptions (1) and (2) imply axioms (A)(c) and (A)(d) in the presence of the known axioms (A)(a) and (A)(b).

Conversely, given H^* we get a functor G by the construction of Lemma 45.7.10. Let $X = \mathbf{P}^1, c_0, c_2$ be as in Example 45.3.7. We have constructed an isomorphism $1(-1) \to (X, c_2, 0)$ of motives in Lemma 45.4.4. In the proof of Lemma 45.7.10 we have seen that $G(1(-1)) = G(X, c_2, 0) = H^2(\mathbf{P}^1)[-2]$. Hence the isomorphism $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ of axiom (A)(d) gives an isomorphism $G(1(-1)) \to F[-2]$ which determines an isomorphism $F[2] \to G(\mathbf{1}(1))$. Finally, since $G(h(X)) = H^*(X)$ assumptions (1) and (2) follow from axiom (A). □

45.8. Cycles over non-closed fields

0FH4 Some lemmas which will help us in our study of motives over base fields which are not algebraically closed.

0FH5 **Lemma 45.8.1.** *Let k be a field. Let X be a smooth projective scheme over k. Then $\mathrm{CH}_0(X)$ is generated by classes of closed points whose residue fields are separable over k.*

Proof. The lemma is immediate if k has characteristic 0 or is perfect. Thus we may assume k is an infinite field of characteristic $p > 0$.

We may assume X is irreducible of dimension d. Then $k' = H^0(X, \mathcal{O}_X)$ is a finite separable field extension of k and that X is geometrically integral over k'. See Varieties, Lemmas 33.25.4, 33.9.3, and 33.9.4. We may and do replace k by k' and assume that X is geometrically integral.

Let $x \in X$ be a closed point. To prove the lemma we are going to show that $[x] \in \mathrm{CH}_0(X)$ is rationally equivalent to an integer linear combination of classes of closed points whose residue fields are separable over k. Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0\}$$

After replacing $\mathcal{L}$ by a power we may assume (a) $\mathcal{L}$ is very ample, (b) V generates $\mathcal{L}$ over $X \setminus x$, (c) the morphism $X \setminus x \to \mathbf{P}(V)$ is an immersion, (d) the map $V \to \mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ is surjective, see Morphisms, Lemma 29.39.5, Varieties, Lemma 33.47.1, and Properties, Proposition 28.26.13. Consider the set

$$V^d \supset U = \{(s_1, \ldots, s_d) \in V^d \mid s_1, \ldots, s_d \text{ generate } \mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x \text{ over } \kappa(x)\}$$

Since $\mathcal{O}_{X,x}$ is a regular local ring of dimension d we have $\dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) = d$ and hence we see that U is a nonempty (Zariski) open of V^d. For $(s_1, \ldots, s_d) \in U$ set $H_i = Z(s_i)$. Since $s_1, \ldots, s_d$ generate $\mathfrak{m}_x\mathcal{L}_x$ we see that

$$H_1 \cap \ldots \cap H_d = x \amalg Z$$

scheme theoretically for some closed subscheme $Z \subset X$. By Bertini (in the form of Varieties, Lemma 33.47.3) for a general element $s_1 \in V$ the scheme $H_1 \cap (X \setminus x)$ is smooth over k of dimension $d - 1$. Having chosen s_1, for a general element $s_2 \in V$ the scheme $H_1 \cap H_2 \cap (X \setminus x)$ is smooth over k of dimension $d - 2$. And so on. We conclude that for sufficiently general $(s_1, \ldots, s_d) \in U$ the scheme Z is étale over $\mathrm{Spec}(k)$. In particular $H_1 \cap \ldots \cap H_d$ has dimension 0 and hence

$$[H_1] \cdot \ldots \cdot [H_d] = [x] + [Z]$$

in $\mathrm{CH}_0(X)$ by repeated application of Chow Homology, Lemma 42.62.5 (details omitted). This finishes the proof as it shows that $[x] \sim_{rat} -[Z] + [Z']$ where $Z' = H_1' \cap \ldots \cap H_d'$ is a general complete intersection of vanishing loci of sufficiently general sections of $\mathcal{L}$ which will be étale over k by the same argument as before. □

0FH6 **Lemma 45.8.2.** *Let K/k be an algebraic field extension. Let X be a finite type scheme over k. Then $\mathrm{CH}_i(X_K) = \mathrm{colim}\, \mathrm{CH}_i(X_{k'})$ where the colimit is over the subextensions $K/k'/k$ with k'/k finite.*

Proof. This is a special case of Chow Homology, Lemma 42.67.10. □

0FH7 **Lemma 45.8.3.** *Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. Let n be an integer invertible in k. Then there exists a finite separable extension k'/k such that the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$.*

Proof. Let k' be a separable algebraic closure of k. Suppose that we can show the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$. Then we conclude by Lemma 45.8.2. Thus we may and do assume k is separably algebraically closed.

Suppose $\dim(X) > 1$. Let $\mathcal{L}$ be an ample invertible sheaf on X. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0 \text{ and } s(x') = 0\}$$

After replacing $\mathcal{L}$ by a power we see that for a general $v \in V$ the corresponding divisor $H_v \subset X$ is smooth away from x and x', see Varieties, Lemmas 33.47.1 and 33.47.3. To find v we use that k is infinite (being separably algebraically closed). If we choose s general, then the image of s in $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ will be nonzero, which implies that H_v is smooth at x (details omitted). Similarly for x'. Thus H_v is smooth. By Varieties, Lemma 33.48.3 (applied to the base change of everything to the algebraic closure of k) we see that H_v is geometrically connected. It suffices to prove the result for $[x]-[x']$ seen as an element of $\mathrm{CH}_0(H_v)$. In this way we reduce to the case of a curve.

Assume X is a curve. Then we see that $\mathcal{O}_X(x - x')$ defines a k-rational point g of $J = \underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. Recall that J is a proper smooth variety over k which is also a group scheme over k (same reference). Hence J is geometrically integral (see Varieties, Lemma 33.7.13 and 33.25.4). In other words, J is an abelian variety, see Groupoids, Definition 39.9.1. Thus $[n] : J \to J$ is finite étale by Groupoids, Proposition 39.9.11 (this is where we use n is invertible in k). Since k is separably closed we conclude that $g = [n](g')$ for some $g' \in J(k)$. If $\mathcal{L}$ is the degree 0 invertible module on X corresponding to g', then we conclude that $\mathcal{O}_X(x - x') \cong \mathcal{L}^{\otimes n}$ as desired. □

0FH8 **Lemma 45.8.4.** *Let K/k be an algebraic extension of fields. Let X be a finite type scheme over k. The kernel of the map $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_K)$ constructed in Lemma 45.8.2 is torsion.*

Proof. It clearly suffices to show that the kernel of flat pullback $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_{k'})$ by $\pi : X_{k'} \to X$ is torsion for any finite extension k'/k. This is clear because $\pi_*\pi^*\alpha = [k' : k]\alpha$ by Chow Homology, Lemma 42.15.2. □

0FH9 **Lemma 45.8.5** (Voevodsky)**.** *Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. For n large enough the class of the zero cycle* [**Voe95**]

$$([x] - [x']) \times \ldots \times ([x] - [x']) \in \mathrm{CH}_0(X^n)$$

is torsion.

Proof. If we can show this after base change to the algebraic closure of k, then the result follows over k because the kernel of pullback is torsion by Lemma 45.8.4. Hence we may and do assume k is algebraically closed.

Using Bertini we can choose a smooth curve $C \subset X$ passing through x and x'. See proof of Lemma 45.8.3. Hence we may assume X is a curve.

Assume X is a curve and k is algebraically closed. Write $S^n(X) = \underline{\mathrm{Hilb}}^n_{X/k}$ with notation as in Picard Schemes of Curves, Sections 44.2 and 44.3. There is a canonical morphism

$$\pi : X^n \longrightarrow S^n(X)$$

which sends the k-rational point $(x_1, \ldots, x_n)$ to the k-rational point corresponding to the divisor $[x_1]+\ldots+[x_n]$ on X. There is a faithful action of the symmetric group S_n on X^n. The morphism π is S_n-invariant and the fibres of π are S_n-orbits (set

theoretically). Finally, π is finite flat of degree $n!$, see Picard Schemes of Curves, Lemma 44.3.4.

Let α_n be the zero cycle on X^n given by the formula in the statement of the lemma. Let $\mathcal{L} = \mathcal{O}_X(x - x')$. Then $c_1(\mathcal{L}) \cap [X] = [x] - [x']$. Thus

$$\alpha_n = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n]$$

where $\mathcal{L}_i = \mathrm{pr}_i^*\mathcal{L}$ and $\mathrm{pr}_i : X^n \to X$ is the ith projection. By either Divisors, Lemma 31.17.6 or Divisors, Lemma 31.17.7 there is a norm for π. Set $\mathcal{N} = \mathrm{Norm}_\pi(\mathcal{L}_1)$, see Divisors, Lemma 31.17.2. We have

$$\pi^*\mathcal{N} = (\mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n)^{\otimes (n-1)!}$$

in $\mathrm{Pic}(X^n)$ by a calculation. Deails omitted; hint: this follows from the fact that $\mathrm{Norm}_\pi : \pi_*\mathcal{O}_{X^n} \to \mathcal{O}_{S^n(X)}$ composed with the natural map $\pi_*\mathcal{O}_{S^n(X)} \to \mathcal{O}_{X^n}$ is equal to the product over all $\sigma \in S_n$ of the action of σ on $\pi_*\mathcal{O}_{X^n}$. Consider

$$\beta_n = c_1(\mathcal{N})^n \cap [S^n(X)]$$

in $\mathrm{CH}_0(S^n(X))$. Observe that $c_1(\mathcal{L}_i) \cap c_1(\mathcal{L}_i) = 0$ because $\mathcal{L}_i$ is pulled back from a curve, see Chow Homology, Lemma 42.34.6. Thus we see that

$$\begin{aligned} \pi^*\beta_n &= ((n-1)!)^n(\textstyle\sum_{i=1,\ldots,n} c_1(\mathcal{L}_i))^n \cap [X^n] \\ &= ((n-1)!)^n n^n c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n] \\ &= (n!)^n \alpha_n \end{aligned}$$

Thus it suffices to show that β_n is torsion.

There is a canonical morphism

$$f : S^n(X) \longrightarrow \underline{\mathrm{Pic}}^n_{X/k}$$

See Picard Schemes of Curves, Lemma 44.6.7. For $n \geq 2g - 1$ this morphism is a projective space bundle (details omitted; compare with the proof of Picard Schemes of Curves, Lemma 44.6.7). The invertible sheaf $\mathcal{N}$ is trivial on the fibres of f, see below. Thus by the projective space bundle formula (Chow Homology, Lemma 42.36.2) we see that $\mathcal{N} = f^*\mathcal{M}$ for some invertible module $\mathcal{M}$ on $\underline{\mathrm{Pic}}^n_{X/k}$. Of course, then we see that

$$c_1(\mathcal{N})^n = f^*(c_1(\mathcal{M})^n)$$

is zero because $n > g = \dim(\underline{\mathrm{Pic}}^n_{X/k})$ and we can use Chow Homology, Lemma 42.34.6 as before.

We still have to show that $\mathcal{N}$ is trivial on a fibre F of f. Since the fibres of f are projective spaces and since $\mathrm{Pic}(\mathbf{P}^m_k) = \mathbf{Z}$ (Divisors, Lemma 31.28.5), this can be shown by computing the degree of $\mathcal{N}$ on a line contained in the fibre. Instead we will prove it by proving that $\mathcal{N}$ is algebraically equivalent to zero. First we claim there is a connected finite type scheme T over k, an invertible module $\mathcal{L}'$ on $T \times X$ and k-rational points $p, q \in T$ such that $\mathcal{M}_p \cong \mathcal{O}_X$ and $\mathcal{M}_q = \mathcal{L}$. Namely, since $\mathcal{L} = \mathcal{O}_X(x - x')$ we can take $T = X$, $p = x'$, $q = x$, and $\mathcal{L}' = \mathcal{O}_{X \times X}(\Delta) \otimes \mathrm{pr}_2^*\mathcal{O}_X(-x')$. Then we let $\mathcal{L}'_i$ on $T \times X^n$ for $i = 1, \ldots, n$ be the pullback of $\mathcal{L}'$ by $\mathrm{id}_T \times \mathrm{pr}_i : T \times X^n \to T \times X$. Finally, we let $\mathcal{N}' = \mathrm{Norm}_{\mathrm{id}_T \times \pi}(\mathcal{L}'_1)$ on $T \times S^n(X)$. By construction we have $\mathcal{N}'_p = \mathcal{O}_{S^n(X)}$ and $\mathcal{N}'_q = \mathcal{N}$. We conclude that

$$\mathcal{N}'|_{T \times F}$$

is an invertible module on $T \times F \cong T \times \mathbf{P}^m_k$ whose fibre over p is the trivial invertible module and whose fibre over q is $\mathcal{N}|_F$. Since the euler characteristic of the trivial bundle is 1 and since this euler characteristic is locally constant in families (Derived Categories of Schemes, Lemma 36.32.2) we conclude $\chi(F, \mathcal{N}^{\otimes s}|_F) = 1$ for all $s \in \mathbf{Z}$. This can happen only if $\mathcal{N}|_F \cong \mathcal{O}_F$ (see Cohomology of Schemes, Lemma 30.8.1) and the proof is complete. Some details omitted. □

45.9. Weil cohomology theories, I

0FHA This section is the analogue of Section 45.7 over arbitrary fields. In other words, we work out what data and axioms correspond to functors G of symmetric monoidal categories from the category of motives to the category of graded vector spaces such that $G(\mathbf{1}(1))$ sits in degree -2. In Section 45.11 we will define a Weil cohomology theory by adding a single supplementary condition.

We fix a field k (the base field). We fix a field F of characteristic 0 (the coefficient field). The data is given by:

(D0) A 1-dimensional F-vector space $F(1)$.
(D1) A contravariant functor H^* from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
(D2) For every smooth projective scheme X over k a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)(i)$.
(D3) For every nonempty smooth projective scheme X over k which is equidimensional of dimension d a map $\int_X : H^{2d}(X)(d) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D0). The vector space $F(1)$ gives rise to *Tate twists* on the category of F-vector spaces. Namely, for $n \in \mathbf{Z}$ we set $F(n) = F(1)^{\otimes n}$ if $n \geq 0$, we set $F(-1) = \mathrm{Hom}_F(F(1), F)$, and we set $F(n) = F(-1)^{\otimes -n}$ if $n < 0$. Please compare with More on Algebra, Section 15.118. For an F-vector space V we define the *nth Tate twist*

$$V(n) = V \otimes_F F(n)$$

We will use obvious notation, e.g., given F-vector spaces U, V and W and a linear map $U \otimes_F V \to W$ we obtain a linear map $U(n) \otimes_F V(m) \to W(n+m)$ for $n, m \in \mathbf{Z}$.

Remarks on (D1). Given a smooth projective scheme X over k we say that $H^*(X)$ is the *cohomology* of X. Given a morphism $f : X \to Y$ of smooth projective schemes over k we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the *pullback map*.

Remarks on (D2). The map γ is called the *cycle class map*. We say that $\gamma(\alpha)$ is the *cohomology class* of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective over k and Z integral, then $[Z]$ could mean the class of the cycle $[Z]$ in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the *trace map* and is sometimes denoted Tr_X.

The first axiom is often called *Poincaré duality*

(A) Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Then
  (a) $\dim_F H^i(X) < \infty$ for all i,
  (b) $H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$.

Let $f : X \to Y$ be a morphism of nonempty smooth projective schemes with X equidimensional of dimension d and Y is equidimensional of dimension e. Using Poincaré duality we can define a *pushforward*

$$f_* : H^{2d-i}(X)(d) \longrightarrow H^{2e-i}(Y)(e)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)(d)$, the element $f_*a \in H^{2e-i}(Y)(e)$ is characterized by

$$\int_X f^*b \cup a = \int_Y b \cup f_*a$$

for all $b \in H^i(Y)$.

0FHB **Lemma 45.9.1.** *Assume given (D0), (D1), and (D3) satisfying (A). For $f : X \to Y$ a morphism of nonempty equidimensional smooth projective schemes over k we have $f_*(f^*b \cup a) = b \cup f_*a$. If $g : Y \to Z$ is a second morphism with Z nonempty smooth projective and equidimensional, then $g_* \circ f_* = (g \circ f)_*$.*

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_*a = \int_X f^*c \cup f^*b \cup a = \int_Y c \cup f_*(f^*b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_*a = \int_X (g \circ f)^*c \cup a = \int_X f^*g^*c \cup a = \int_Y g^*c \cup f_*a = \int_Z c \cup g_*f_*a$$

This ends the proof. □

The second axiom says that H^* respects the monoidal structure given by products via the *Künneth formula*

(B) Let X and Y be smooth projective schemes over k.
  (a) $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $\alpha \otimes \beta \mapsto \mathrm{pr}_1^*\alpha \cup \mathrm{pr}_2^*\beta$ is an isomorphism,
  (b) if X and Y are nonempty and equidimensional, then $\int_{X\times Y} = \int_X \otimes \int_Y$ via (a).

Using axiom (B)(b) we can compute pushforwards along projections.

0FHC **Lemma 45.9.2.** *Assume given (D0), (D1), and (D3) satisfying (A) and (B). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. Then $pr_{2,*} : H^*(X \times Y)(d+e) \to H^*(Y)(e)$ sends $a \otimes b$ to $(\int_X a)b$.*

Proof. This follows from axioms (B)(a) and (B)(b). □

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
  (a) for a morphism $f : X \to Y$ of smooth projective schemes over k we have $\gamma(f^!\beta) = f^*\gamma(\beta)$ for $\beta \in \mathrm{CH}^*(Y)$,

(b) for a morphism $f : X \to Y$ of nonempty equidimensional smooth projective schemes over k we have $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ for $\alpha \in \mathrm{CH}^*(X)$,
(c) for any smooth projective scheme X over k we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \mathrm{CH}^*(X)$, and
(d) $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$.

Let us elucidate axiom (C)(b). Namely, say $f : X \to Y$ is as in (C)(b) with $\dim(X) = d$ and $\dim(Y) = e$. Then we see that pushforward on Chow groups gives

$$f_* : \mathrm{CH}^{d-i}(X) = \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y) = \mathrm{CH}^{e-i}(Y)$$

Say $\alpha \in \mathrm{CH}^{d-i}(X)$. On the one hand, we have $f_*\alpha \in \mathrm{CH}^{e-i}(Y)$ and hence $\gamma(f_*\alpha) \in H^{2e-2i}(Y)(e-i)$. On the other hand, we have $\gamma(\alpha) \in H^{2d-2i}(X)(d-i)$ and hence $f_*\gamma(\alpha) \in H^{2e-2i}(Y)(e-i)$ as well. Thus the condition $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ makes sense.

0FHD **Remark 45.9.3.** Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C)(a). Let X be a smooth projective scheme over k. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \mathrm{pr}_1^*\alpha \cdot \mathrm{pr}_2^*\beta$ in $\mathrm{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\mathrm{pr}_1^*\gamma(\alpha) \cup \mathrm{pr}_2^*\gamma(\beta)$.

0FHE **Lemma 45.9.4.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^i(\mathrm{Spec}(k)) = 0$ for $i \neq 0$ and there is a unique F-algebra isomorphism $F = H^0(\mathrm{Spec}(k))$. We have $\gamma([\mathrm{Spec}(k)]) = 1$ and $\int_{\mathrm{Spec}(k)} 1 = 1$.*

Proof. By axiom (C)(d) we see that $H^0(\mathrm{Spec}(k))$ is nonzero and even $\gamma([\mathrm{Spec}(k)])$ is nonzero. Since $\mathrm{Spec}(k) \times \mathrm{Spec}(k) = \mathrm{Spec}(k)$ we get

$$H^*(\mathrm{Spec}(k)) \otimes_F H^*(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$$

by axiom (B)(a) which implies (look at dimensions) that only H^0 is nonzero and moreover has dimension 1. Thus $F = H^0(\mathrm{Spec}(k))$ via the unique F-algebra isomorphism given by mapping $1 \in F$ to $1 \in H^0(\mathrm{Spec}(k))$. Since $[\mathrm{Spec}(k)] \cdot [\mathrm{Spec}(k)] = [\mathrm{Spec}(k)]$ in the Chow ring of $\mathrm{Spec}(k)$ we conclude that $\gamma([\mathrm{Spec}(k)]) \cup \gamma([\mathrm{Spec}(k)]) = \gamma([\mathrm{Spec}(k)])$ by axiom (C)(c). Since we already know that $\gamma([\mathrm{Spec}(k)])$ is nonzero we conclude that it has to be equal to 1. Finally, we have $\int_{\mathrm{Spec}(k)} 1 = 1$ since $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$ by axiom (C)(d). □

0FHF **Lemma 45.9.5.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. If $X = \emptyset$, then $H^*(X) = 0$. If X is nonempty, then $\gamma([X]) = 1$ and $1 \neq 0$ in $H^0(X)$.*

Proof. First assume X is nonempty. Observe that $[X]$ is the pullback of $[\mathrm{Spec}(k)]$ by the structure morphism $p : X \to \mathrm{Spec}(k)$. Hence we get $\gamma([X]) = 1$ by axiom (C)(a) and Lemma 45.9.4. Let $X' \subset X$ be an irreducible component. By functoriality it suffices to show $1 \neq 0$ in $H^0(X')$. Thus we may and do assume X is irreducible, and in particular nonempty and equidimensional, say of dimension d. To see that $1 \neq 0$ it suffices to show that $H^*(X)$ is nonzero.

Let $x \in X$ be a closed point whose residue field k' is separable over k, see Varieties, Lemma 33.25.6. Let $i : \mathrm{Spec}(k') \to X$ be the inclusion morphism. Denote $p : X \to \mathrm{Spec}(k)$ is the structure morphism. Observe that $p_* i_* [\mathrm{Spec}(k')] = [k' : k][\mathrm{Spec}(k)]$ in $\mathrm{CH}_0(\mathrm{Spec}(k))$. Using axiom (C)(b) twice and Lemma 45.9.4 we conclude that

$$p_* i_* \gamma([\mathrm{Spec}(k')]) = \gamma([k' : k][\mathrm{Spec}(k)]) = [k' : k] \in F = H^0(\mathrm{Spec}(k))$$

is nonzero. Thus $i_*\gamma([\mathrm{Spec}(k)]) \in H^{2d}(X)(d)$ is nonzero (because it maps to something nonzero via p_*). This concludes the proof in case X is nonempty.

Finally, we consider the case of the empty scheme. Axiom (B)(a) gives $H^*(\emptyset) \otimes H^*(\emptyset) = H^*(\emptyset)$ and we get that $H^*(\emptyset)$ is either zero or 1-dimensional in degree 0. Then axiom (B)(a) again shows that $H^*(\emptyset) \otimes H^*(X) = H^*(\emptyset)$ for all smooth projective schemes X over k. Using axiom (A)(b) and the nonvanishing of $H^0(X)$ we've seen above we find that $H^*(X)$ is nonzero in at least two degrees if $\dim(X) > 0$. This then forces $H^*(\emptyset)$ to be zero. □

0FHG **Lemma 45.9.6.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $i : X \to Y$ be a closed immersion of nonempty smooth projective equidimensional schemes over k. Then $\gamma([X]) = i_*1$ in $H^{2c}(Y)(c)$ where $c = \dim(Y) - \dim(X)$.*

Proof. This is true because $1 = \gamma([X])$ in $H^0(X)$ by Lemma 45.9.5 and then we can apply axiom (C)(b). □

0FHH **Lemma 45.9.7.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Choose a basis $e_{i,j}, j = 1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write*

$$\gamma([\Delta]) = \sum\nolimits_i \sum\nolimits_j e_{i,j} \otimes e'_{2d-i,j} \quad \text{in} \quad \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)(d)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i \delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 45.9.3. On the other hand, recall that $\gamma([\Delta]) = \Delta_* 1$ (Lemma 45.9.6) and hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_* 1 \cup a \otimes b = \int_X a \cup b$$

by Lemma 45.9.1. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by axiom (B)(b); note that we made two switches of order so that the sign for each term is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. □

0FHI **Lemma 45.9.8.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^*(\mathbf{P}^1_k)$ is 1-dimensional in dimensions 0 and 2 and zero in other degrees.*

Proof. Let $x \in \mathbf{P}^1_k$ be a k-rational point. Observe that $\Delta = \mathrm{pr}_1^* x + \mathrm{pr}_2^* x$ as divisors on $\mathbf{P}^1_k \times \mathbf{P}^1_k$. Using axiom (C)(a) and additivity of γ we see that

$$\gamma([\Delta]) = \mathrm{pr}_1^*\gamma([x]) + \mathrm{pr}_2^*\gamma([x]) = \gamma([x]) \otimes 1 + 1 \otimes \gamma([x])$$

in $H^*(\mathbf{P}^1_k \times \mathbf{P}^1_k) = H^*(\mathbf{P}^1_k) \otimes_F H^*(\mathbf{P}^1_k)$. However, by Lemma 45.9.7 we know that $\gamma([\Delta])$ cannot be written as a sum of fewer than $\sum \beta_i$ pure tensors where $\beta_i = \dim_F H^i(\mathbf{P}^1_k)$. Thus we see that $\sum \beta_i \leq 2$. By Lemma 45.9.5 we have $H^0(\mathbf{P}^1_k) \neq 0$. By Poincaré duality, more precisely axiom (A)(b), we have $\beta_0 = \beta_2$. Therefore the lemma holds. □

0FHJ **Lemma 45.9.9.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If X and Y are smooth projective schemes over k, then $H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$, $a \mapsto (i^*a, j^*a)$ is an isomorphism where i, j are the coprojections.*

Proof. If X or Y is empty, then this is true because $H^*(\emptyset) = 0$ by Lemma 45.9.5. Thus we may assume both X and Y are nonempty.

We first show that the map is injective. First, observe that we can find morphisms $X' \to X$ and $Y' \to Y$ of smooth projective schemes so that X' and Y' are equidimensional of the same dimension and such that $X' \to X$ and $Y' \to Y$ each have a section. Namely, decompose $X = \coprod X_d$ and $Y = \coprod Y_e$ into open and closed subschemes equidimensional of dimension d and e. Then take $X' = \coprod X_d \times \mathbf{P}^{n-d}$ and $Y' = \coprod Y_e \times \mathbf{P}^{n-e}$ for some n sufficiently large. Thus pullback by $X' \amalg Y' \to X \amalg Y$ is injective (because there is a section) and it suffices to show the injectivity for X', Y' as we do in the next parapgrah.

Let us show the map is injective when X and Y are equidimensional of the same dimension d. Observe that $[X \amalg Y] = [X] + [Y]$ in $\mathrm{CH}^0(X \amalg Y)$ and that $[X]$ and $[Y]$ are orthogonal idempotents in $\mathrm{CH}^0(X \amalg Y)$. Thus

$$1 = \gamma([X \amalg Y] = \gamma([X]) + \gamma([Y]) = i_*1 + j_*1$$

is a decomposition into orthogonal idempotents. Here we have used Lemmas 45.9.5 and 45.9.6 and axiom (C)(c). Then we see that

$$a = a \cup 1 = a \cup i_*1 + a \cup j_*1 = i_*(i^*a) + j_*(j^*a)$$

by the projection formula (Lemma 45.9.1) and hence the map is injective.

We show the map is surjective. Write $e = \gamma([X])$ and $f = \gamma([Y])$ viewed as elements in $H^0(X \amalg Y)$. We have $i^*e = 1$, $i^*f = 0$, $j^*e = 0$, and $j^*f = 1$ by axiom (C)(a). Hence if $i^* : H^*(X \amalg Y) \to H^*(X)$ and $j^* : H^*(X \amalg Y) \to H^*(Y)$ are surjective, then so is (i^*, j^*). Namely, for $a, a' \in H^*(X \amalg Y)$ we have

$$(i^*a, j^*a') = (i^*(a \cup e + a' \cup f), j^*(a \cup e + a' \cup f))$$

By symmetry it suffices to show $i^* : H^*(X \amalg Y) \to H^*(X)$ is surjective. If there is a morphism $Y \to X$, then there is a morphism $g : X \amalg Y \to X$ with $g \circ i = \mathrm{id}_X$ and we conclude. To finish the proof, observe that in order to prove i^* is surjective, it suffices to do so after tensoring by a nonzero graded F-vector space. Hence by axiom (B)(b) and nonvanishing of cohomology (Lemma 45.9.5) it suffices to prove i^* is surjective after replacing X and Y by $X \times \mathrm{Spec}(k')$ and $Y \times \mathrm{Spec}(k')$ for some

finite separable extension k'/k. If we choose k' such that there exists a closed point $x \in X$ with $\kappa(x) = k'$ (and this is possible by Varieties, Lemma 33.25.6) then there is a morphism $Y \times \mathrm{Spec}(k') \to X \times \mathrm{Spec}(k')$ and we find that the proof is complete. □

0FHK **Lemma 45.9.10.** *Let k be a field. Let F be a field of characteristic 0. Assume given a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D0), (D1), (D2), and (D3) satisfying all of (A), (B), and (C) above.

Proof. This proof is the same as the proof of Lemma 45.7.9; we urge the reader to read the proof of that lemma instead.

We obtain a contravariant functor from the category of smooth projective schemes over k to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X \times Y) = G(h(X \times Y)) = G(h(X) \otimes h(Y)) = G(h(X)) \otimes G(h(Y)) = H^*(X) \otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X) \otimes h(X) \to h(X) \otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 12.17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2. Our datum (D0) is the vector space $F(1) = G^{-2}(\mathbf{1}(1))$. Since G is a symmetric monoidal functor we see that $F(n) = G^{-2n}(\mathbf{1}(n))$ for all $n \in \mathbf{Z}$. It follows that

$$H^{2r}(X)(r) = G^{2r}(h(X)) \otimes G^{-2r}(\mathbf{1}(r)) = G^0(h(X)(r))$$

a formula we will frequently use below.

Let X be a smooth projective scheme over k. By Lemma 45.3.1 we have

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X) = \mathrm{Hom}(\mathbf{1}(-r), h(X)) = \mathrm{Hom}(\mathbf{1}, h(X)(r))$$

Applying the functor G this maps into $\mathrm{Hom}(G(\mathbf{1}), G(h(X)(r)))$. By taking the image of 1 in $G^0(\mathbf{1}) = F$ into $G^0(h(X)(r)) = H^{2r}(X)(r)$ we obtain

$$\gamma : \mathrm{CH}^r(X) \otimes \mathbf{Q} \longrightarrow H^{2r}(X)(r)$$

This is the datum (D2).

Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 45.3.1 we have

$$\mathrm{Mor}(h(X)(d), \mathbf{1}) = \mathrm{Mor}((X, 1, d), (\mathrm{Spec}(k), 1, 0)) = \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G and taking degree 0 parts we obtain

$$H^{2d}(X)(d) = G^0(h(X)(d)) \longrightarrow G^0(\mathbf{1}) = F$$

This map $\int_X : H^{2d}(X)(d) \to F$ is the datum (D3).

Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. By Lemma 45.4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes_F F(d)[2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. Here $[n]$ is the shift functor on graded vector spaces. By Homology, Lemma 12.17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$ produces nondegenerate pairings $H^{2d-i}(X)(d) \otimes_F H^i(X) \to F$. In the proof of Lemma 45.4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of nonempty equidimensional smooth projective schemes over k, then $f^!\alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 45.3.6. Hence $f^!\alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 45.3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of nonempty equidimensional smooth projective schemes over k and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 45.3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 45.3.6 and 45.3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 45.3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 45.7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be nonempty smooth projective schemes over k and let α, β be cycles on them. Denote a, b the corresponding correspondences from $\mathrm{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\mathrm{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\mathrm{Spec}(k)]$ corresponds to the identity morphism on $h(\mathrm{Spec}(k))$. This finishes the proof of the lemma. □

0FHL **Lemma 45.9.11.** *Let k be a field. Let F be a field of characteristic 0. Given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we can construct a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. The proof of this lemma is the same as the proof of Lemma 45.7.10; we urge the reader to read the proof of that lemma instead.

By Lemma 45.4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1_k)$ is an invertible graded F-vector space.

Let X be a smooth projective scheme over k. There is a canonical decomposition

$$X = \coprod\nolimits_{0 \leq d \leq \dim(X)} X_d$$

into open and closed subschemes such that X_d is equidimensional of dimension d. By Lemma 45.9.9 we have correspondingly

$$H^*(X) \longrightarrow \prod\nolimits_{0 \leq d \leq \dim(X)} H^*(X_d)$$

If Y is a second smooth projective scheme over k and we similarly decompose $Y = \coprod Y_e$, then

$$\mathrm{Corr}^0(X, Y) = \bigoplus \mathrm{Corr}^0(X_d, Y_e)$$

As well we have $X \otimes Y = \coprod X_d \otimes Y_e$ in the category of correspondences. From these observations it follows that it suffices to construct G on the category whose objects are equidimensional smooth projective schemes over k and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given an equdimensional smooth projective scheme X over k we set $G(X) = H^*(X)$. Observe that $G(X) = 0$ if $X = \emptyset$ (Lemma 45.9.5). Thus maps from and to $G(\emptyset)$ are zero and we may and do assume our schemes are nonempty in the discussions below.

Given a correspondence $c \in \mathrm{Corr}^0(X, Y)$ between nonempty equidmensional smooth projective schemes over k we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \mathrm{pr}_{2,*}(\gamma(c) \cup \mathrm{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b), and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \mathrm{Corr}^0(Y, Z)$. Namely,

for $a \in H^*(X)$ we have

$$\begin{aligned}
(G(c') \circ G(c))(a) &= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}_2^{23,*}(\mathrm{pr}^{12}_{2,*}(\gamma(c) \cup \mathrm{pr}_1^{12,*}a))) \\
&= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}^{123}_{23,*}(\mathrm{pr}_{12}^{123,*}(\gamma(c) \cup \mathrm{pr}_1^{12,*}a))) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\mathrm{pr}_{23}^{123,*}\gamma(c') \cup \mathrm{pr}_{12}^{123,*}\gamma(c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\gamma(\mathrm{pr}_{23}^{123,*}c') \cup \gamma(\mathrm{pr}_{12}^{123,*}c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}^{13}_{3,*}\mathrm{pr}^{123}_{13,*}(\gamma(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}^{13}_{3,*}(\gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c)) \cup \mathrm{pr}_1^{13,*}a) \\
&= G(c' \circ c)(a)
\end{aligned}$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\mathrm{pr}_2^{23,*} \circ \mathrm{pr}^{12}_{2,*} = \mathrm{pr}^{123}_{23,*} \circ \mathrm{pr}_{12}^{123,*}$ as follows immediately from the description of pushforward along projections given in Lemma 45.9.2. The third equality holds by Lemma 45.9.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the Gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 42.59.5). The fifth equality uses axiom (C)(c) as well as Lemma 45.9.1 to see that $\mathrm{pr}^{23}_{3,*} \circ \mathrm{pr}^{123}_{23,*} = \mathrm{pr}^{13}_{3,*} \circ \mathrm{pr}^{123}_{13,*}$. The sixth equality uses the projection formula from Lemma 45.9.1 as well as axiom (C)(b) to see that $\mathrm{pr}^{123}_{13,*}\gamma(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c) = \gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \mathrm{Corr}^0(X, X)$ is the identity in the category of correspondences (Lemma 45.3.3), then we have to show that $G([\Delta]) = \mathrm{id}$. This follows from the determination of $\gamma([\Delta])$ in Lemma 45.9.7 and Lemma 45.9.2. This finishes the construction of G as a functor on smooth projective schemes over k and correspondences of degree 0.

By Lemma 45.9.4 we have that $G(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B)(a) shows our functor is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1_k) = H^*(\mathbf{P}^1_k)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By Lemma 45.9.8 we only have nonzero cohomology in degrees 0 and 2 both of dimension 1. We have $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1_k, \mathbf{P}^1_k)$, see Example 45.3.7. Further we have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1_k)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1_k, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 45.4.4. Thus $G(c_0)$ is the projector onto the degree 0 part. It follows that $G(c_2)$ must be the projector onto the degree 2 part and the proof is complete. □

0FHM **Proposition 45.9.12.** *Let k be a field. Let F be a field of characteristic 0. There is a 1-to-1 correspondence between the following*

(1) *data (D0), (D1), (D2), and (D3) satisfying (A), (B), and(C), and*
(2) *$\mathbf{Q}$-linear symmetric monoidal functors*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

such that $G(\mathbf{1}(1))$ is nonzero only in degree -2.

Proof. Given G as in (2) by setting $H^*(X) = G(h(X))$ we obtain data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C), see Lemma 45.9.10 and its proof.

Conversely, given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we get a functor G as in (2) by the construction of the proof of Lemma 45.9.11.

We omit the detailed proof that these constructions are inverse to each other. □

45.10. Further properties

0FHN In this section we prove a few more results one obtains if given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) as in Section 45.9.

0FHP **Lemma 45.10.1.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X, Y be nonempty smooth projective schemes both equidimensional of dimension d over k. Then $\int_{X \amalg Y} = \int_X + \int_Y$.*

Proof. Denote $i : X \to X \amalg Y$ and $j : Y \to X \amalg Y$ be the coprojections. By Lemma 45.9.9 the map $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism. The statement of the lemma means that under the isomorphism $(i^*, j^*) : H^{2d}(X \amalg Y)(d) \to H^{2d}(X)(d) \oplus H^{2d}(Y)(d)$ the map $\int_X + \int_Y$ is transformed into $\int_{X \amalg Y}$. This is true because

$$\int_{X \amalg Y} a = \int_{X \amalg Y} i_*(i^*a) + j_*(j^*a) = \int_X i^*a + \int_Y j^*a$$

where the equality $a = i_*(i^*a) + j_*(j^*a)$ was shown in the proof of Lemma 45.9.9. □

0FHQ **Lemma 45.10.2.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme of dimension zero over k. Then*

1. *$H^i(X) = 0$ for $i \neq 0$,*
2. *$H^0(X)$ is a finite separable algebra over F,*
3. *$\dim_F H^0(X) = \deg(X \to \mathrm{Spec}(F))$,*
4. *$\int_X : H^0(X) \to F$ is the trace map,*
5. *$\gamma([X]) = 1$, and*
6. *$\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$.*

Proof. We can write $X = \mathrm{Spec}(k')$ where k' is a finite separable algebra over k. Observe that $\deg(X \to \mathrm{Spec}(k)) = [k' : k]$. Choose a finite Galois extension k''/k containing each of the factors of k'. (Recall that a finite separable k-algebra is a product of finite separable field extension of k.) Set $\Sigma = \mathrm{Hom}_k(k', k'')$. Then we get

$$k' \otimes_k k'' = \prod\nolimits_{\sigma \in \Sigma} k''$$

Setting $Y = \mathrm{Spec}(k'')$ axioms (B)(a) and Lemma 45.9.9 give

$$H^*(X) \otimes_F H^*(Y) = \prod\nolimits_{\sigma \in \Sigma} H^*(Y)$$

as graded commutative F-algebras. By Lemma 45.9.5 the F-algebra $H^*(Y)$ is nonzero. Comparing dimensions on either side of the displayed equation we conclude that $H^*(X)$ sits only in degree 0 and $\dim_F H^0(X) = [k' : k]$. Applying this to Y we get $H^*(Y) = H^0(Y)$. Since

$$H^0(X) \otimes_F H^0(Y) = H^0(Y) \times \ldots \times H^0(Y)$$

as F-algebras, it follows that $H^0(X)$ is a separable F-algebra because we may check this after the faithfully flat base change $F \to H^0(Y)$.

The displayed isomorphism above is given by the map

$$H^0(X) \otimes_F H^0(Y) \longrightarrow \prod\nolimits_{\sigma \in \Sigma} H^0(Y), \quad a \otimes b \longmapsto \prod\nolimits_\sigma \mathrm{Spec}(\sigma)^* a \cup b$$

Via this isomorphism we have $\int_{X \times Y} = \sum_\sigma \int_Y$ by Lemma 45.10.1. Thus

$$\int_X a = \mathrm{pr}_{1,*}(a \otimes 1) = \sum \mathrm{Spec}(\sigma)^* a$$

in $H^0(Y)$; the first equality by Lemma 45.9.2 and the second by the observation we just made. Choose an algebraic closure $\overline{F}$ and a F-algebra map $\tau : H^0(Y) \to \overline{F}$. The isomorphism above base changes to the isomorphism

$$H^0(X) \otimes_F \overline{F} \longrightarrow \prod\nolimits_{\sigma \in \Sigma} \overline{F}, \quad a \otimes b \longmapsto \prod\nolimits_\sigma \tau(\mathrm{Spec}(\sigma)^* a) b$$

It follows that $a \mapsto \tau(\mathrm{Spec}(\sigma)^* a)$ is a full set of embeddings of $H^0(X)$ into $\overline{F}$. Applying τ to the formula for $\int_X a$ obtained above we conclude that $\int_X$ is the trace map. By Lemma 45.9.5 we have $\gamma([X]) = 1$. Finally, we have $\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$ because $\gamma([X]) = 1$ and the trace of 1 is equal to $[k' : k]$ □

0FHR **Lemma 45.10.3.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme equidimensional of dimension d over k. The diagram*

$$\begin{array}{ccc} \mathrm{CH}^d(X) & \xrightarrow{\gamma} & H^{2d}(X)(d) \\ \| & & \downarrow \int_X \\ \mathrm{CH}_0(X) & \xrightarrow{\deg} & F \end{array}$$

commutes where $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ *is the degree of zero cycles discussed in Chow Homology, Section 42.41.*

Proof. Let x be a closed point of X whose residue field is separable over k. View x as a scheme and denote $i : x \to X$ the inclusion morphism. To avoid confusion denote $\gamma' : \mathrm{CH}_0(x) \to H^0(x)$ the cycle class map for x. Then we have

$$\int_X \gamma([x]) = \int_X \gamma(i_*[x]) = \int_X i_* \gamma'([x]) = \int_x \gamma'([x]) = \deg(x \to \mathrm{Spec}(k))$$

The second equality is axiom (C)(b) and the third equality is the definition of i_* on cohomology. The final equality is Lemma 45.10.2. This proves the lemma because $\mathrm{CH}_0(X)$ is generated by the classes of points x as above by Lemma 45.8.1. □

0FHS **Lemma 45.10.4.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We have*

$$\sum\nolimits_i (-1)^i \dim_F H^i(X) = \deg(\Delta \cdot \Delta) = \deg(c_d(\mathcal{T}_{X/k}))$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 42.62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 42.54.5). Since the

conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 29.32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_{X/k} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 45.10.3 we have

$$\begin{aligned}\deg([\Delta] \cdot [\Delta]) &= \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta]) \\ &= \int_{X \times X} \Delta_*(\Delta^* \gamma([\Delta])) \\ &= \int_X \Delta^* \gamma([\Delta])\end{aligned}$$

We have used Lemmas 45.9.6 and 45.9.1. Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 45.9.7. Recalling that Δ^* is given by cup product (Remark 45.9.3) we obtain

$$\int_X \sum\nolimits_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum\nolimits_{i,j} (-1)^i = \sum (-1)^i \beta_i$$

as desired. □

0FHT **Lemma 45.10.5.** *Let F be a field of characteristic 0. Let F' and F_i, $i = 1, \ldots, r$ be finite separable F-algebras. Let A be a finite F-algebra. Let $\sigma, \sigma' : A \to F'$ and $\sigma_i : A \to F_i$ be F-algebra maps. Assume σ and σ' surjective. If there is a relation*

$$Tr_{F'/F} \circ \sigma - Tr_{F'/F} \circ \sigma' = n(\sum m_i \, Tr_{F_i/F} \circ \sigma_i)$$

where $n > 1$ and m_i are integers, then $\sigma = \sigma'$.

Proof. We may write $A = \prod A_j$ as a finite product of local Artinian F-algebras $(A_j, \mathfrak{m}_j, \kappa_j)$, see Algebra, Lemma 10.53.2 and Proposition 10.60.7. Denote $A' = \prod \kappa_j$ where the product is over those j such that κ_j/k is separable. Then each of the maps $\sigma, \sigma', \sigma_i$ factors over the map $A \to A'$. After replacing A by A' we may assume A is a finite separable F-algebra.

Choose an algebraic closure $\overline{F}$. Set $\overline{A} = A \otimes_F \overline{F}$, $\overline{F}' = F' \otimes_F \overline{F}$, and $\overline{F}_i = F_i \otimes_F \overline{F}$. We can base change σ, σ', σ_i to get $\overline{F}$ algebra maps $\overline{A} \to \overline{F}'$ and $\overline{A} \to \overline{F}_i$. Moreover $\mathrm{Tr}_{\overline{F}'/\overline{F}}$ is the base change of $\mathrm{Tr}_{F'/F}$ and similarly for $\mathrm{Tr}_{F_i/F}$. Thus we may replace F by $\overline{F}$ and we reduce to the case discussed in the next paragraph.

Assume F is algebraically closed and A a finite separable F-algebra. Then each of A, F', F_i is a product of copies of F. Let us say an element e of a product $F \times \ldots \times F$ of copies of F is a minimal idempotent if it generates one of the factors, i.e., if $e = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $e \in A$ be a minimal idempotent. Since σ and σ' are surjective, we see that $\sigma(e)$ and $\sigma'(e)$ are minimal idempotents or zero. If $\sigma \neq \sigma'$, then we can choose a minimal idempotent $e \in A$ such that $\sigma(e) = 0$ and $\sigma'(e) \neq 0$ or vice versa. Then $\mathrm{Tr}_{F'/F}(\sigma(e)) = 0$ and $\mathrm{Tr}_{F'/F}(\sigma'(e)) = 1$ or vice versa. On the other hand, $\sigma_i(e)$ is an idempotent and hence $\mathrm{Tr}_{F_i/F}(\sigma_i(e)) = r_i$ is an integer. We conclude that

$$-1 = \sum nm_i r_i = n(\sum m_i r_i) \quad \text{or} \quad 1 = \sum nm_i r_i = n(\sum m_i r_i)$$

which is impossible. □

0FHU **Lemma 45.10.6.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a smooth projective scheme over k'. Let $x, x' \in X$ be k'-rational points. If $\gamma(x) \neq \gamma(x')$, then $[x] - [x']$ is not divisible by any integer $n > 1$ in $\mathrm{CH}_0(X)$.*

Proof. If x and x' lie on distinct irreducible components of X, then the result is obvious. Thus we may X irreducible of dimension d. Say $[x] - [x']$ is divisible by $n > 1$ in $\mathrm{CH}_0(X)$. We may write $[x] - [x'] = n(\sum m_i[x_i])$ in $\mathrm{CH}_0(X)$ for some $x_i \in X$ closed points whose residue fields are separable over k by Lemma 45.8.1. Then

$$\gamma([x]) - \gamma([x']) = n(\sum m_i \gamma([x_i]))$$

in $H^{2d}(X)(d)$. Denote $i^*, (i')^*, i_i^*$ the pullback maps $H^0(X) \to H^0(x)$, $H^0(X) \to H^0(x')$, $H^0(X) \to H^0(x_i)$. Recall that $H^0(x)$ is a finite separable F-algebra and that $\int_x : H^0(x) \to F$ is the trace map (Lemma 45.10.2) which we will denote Tr_x. Similarly for x' and x_i. Then by Poincaré duality in the form of axiom (A)(b) the equation above is dual to

$$\mathrm{Tr}_x \circ i^* - \mathrm{Tr}_{x'} \circ (i')^* = n(\sum m_i \mathrm{Tr}_{x_i} \circ i_i^*)$$

which takes place in $\mathrm{Hom}_F(H^0(X), F)$. Finally, observe that i^* and $(i')^*$ are surjective as x and x' are k'-rational points and hence the compositions $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x)$ and $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x')$ are isomorphisms. By Lemma 45.10.5 we conclude that $i^* = (i')^*$ which contradicts the assumption that $\gamma([x]) \neq \gamma([x'])$. □

0FHV **Lemma 45.10.7.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a geometrically irreducible smooth projective scheme over k' of dimension d. Then $\gamma : \mathrm{CH}_0(X) \to H^{2d}(X)(d)$ factors through $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$.*

Proof. By Lemma 45.8.1 it suffices to show: given closed points $x, x' \in X$ whose residue fields are separable over k we have $\deg(x')\gamma([x]) = \deg(x)\gamma([x'])$.

We first reduce to the case of k'-rational points. Let k''/k' be a Galois extension such that $\kappa(x)$ and $\kappa(x')$ embed into k'' over k. Set $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$ and denote $p : Y \to X$ the projection. By our choice of k''/k' there exists a k''-rational point y, resp. y' on Y mapping to x, resp. x'. Then $p_*[y] = [k'' : \kappa(x)][x]$ and $p_*[y'] = [k'' : \kappa(x')][x']$ in $\mathrm{CH}_0(X)$. By compatibility with pushforwards given in axiom (C)(b) it suffices to prove $\gamma([y]) = \gamma([y'])$ in $\mathrm{CH}^{2d}(Y)(d)$. This reduces us to the discussion in the next paragraph.

Assume x and x' are k'-rational points. By Lemma 45.8.3 there exists a finite separable extension k''/k' of fields such that the pullback $[y] - [y']$ of the difference $[x] - [x']$ becomes divisible by an integer $n > 1$ on $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$. (Note that $y, y' \in Y$ are k''-rational points.) By Lemma 45.10.6 we have $\gamma([y]) = \gamma([y'])$ in $H^{2d}(Y)(d)$. By compatibility with pushforward in axiom (C)(b) we conclude the same for x and x'. □

0FHW **Lemma 45.10.8.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $f : X \to Y$ be a dominant morphism of irreducible smooth projective schemes over k. Then $H^*(Y) \to H^*(X)$ is injective.*

Proof. There exists an integral closed subscheme $Z \subset X$ of the same dimension as Y mapping onto Y. Thus $f_*[Z] = m[Y]$ for some $m > 0$. Then $f_*\gamma([Z]) = m\gamma([Y]) = m$ in $H^*(Y)$ because of Lemma 45.9.5. Hence by the projection formula (Lemma 45.9.1) we have $f_*(f^*a \cup \gamma([Z])) = ma$ and we conclude. □

0FHX **Lemma 45.10.9.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $k''/k'/k$ be finite separable algebras and let X be a smooth projective scheme over k'. Then*

$$H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Proof. We will use the results of Lemma 45.10.2 without further mention. Write

$$k' \otimes_k k'' = k'' \times l$$

for some finite separable k'-algebra l. Write $F' = H^0(\mathrm{Spec}(k'))$, $F'' = H^0(\mathrm{Spec}(k''))$, and $G = H^0(\mathrm{Spec}(l))$. Since $\mathrm{Spec}(k') \times \mathrm{Spec}(k'') = \mathrm{Spec}(k'') \amalg \mathrm{Spec}(l)$ we deduce from axiom (B)(a) and Lemma 45.9.9 that we have

$$F' \otimes_F F'' = F'' \times G$$

The map from left to right identifies F'' with $F' \otimes_{F'} F''$. By the same token we have

$$H^*(X) \otimes_F F'' = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')) \times H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(l))$$

as modules over $F' \otimes_F F'' = F'' \times G$. This proves the lemma. □

45.11. Weil cohomology theories, II

0FHY For us a Weil cohomology theory will be the analogue of a classical Weil cohomology theory (Section 45.7) when the ground field k is not algebraically closed. In Section 45.9 we listed axioms which guarantee our cohomology theory comes from a symmetric monoidal functor on the category of motives over k. Missing from our axioms so far are the condition $H^i(X) = 0$ for $i < 0$ and a condition on $H^{2d}(X)(d)$ for X equidimensional of dimension d corresponding to the classical axioms (A)(c) and (A)(d). Let us first convince the reader that it is necessary to impose such conditions.

0FHZ **Example 45.11.1.** Let $k = \mathbf{C}$ and $F = \mathbf{C}$ both be equal to the field of complex numbers. For X smooth projective over k denote $H^{p,q}(X) = H^q(X, \Omega^p_{X/k})$. Let $(H')^*$ be the functor which sends X to $(H')^*(X) = \bigoplus H^{p,q}(X)$ with the usual cup product. This is a classical Weil cohomology theory (insert future reference here). By Proposition 45.7.11 we obtain a $\mathbf{Q}$-linear symmetric monoidal functor G' from M_k to the category of graded F-vector spaces. Of course, in this case for every M in M_k the value $G'(M)$ is naturally bigraded, i.e., we have

$$(G')(M) = \bigoplus (G')^{p,q}(M), \quad (G')^n = \bigoplus\nolimits_{n=p+q} (G')^{p,q}(M)$$

with $(G')^{p,q}$ sitting in total degree $p+q$ as indicated. Now we are going to construct a $\mathbf{Q}$-linear symmetric monoidal functor G to the category of graded F-vector spaces by setting

$$G^n(M) = \bigoplus\nolimits_{n=3p-q} (G')^{p,q}(M)$$

We omit the verification that this defines a symmetric monoidal functor (a technical point is that because we chose odd numbers 3 and -1 above the functor G

is compatible with the commutativity constraints). Observe that $G(\mathbf{1}(1))$ is still sitting in degree -2! Hence by Lemma 45.7.9 we obtain a functor H^*, cycle classes γ, and trace maps satisfying all classical axioms (A), (B), (C), except for possibly the classical axioms (A)(a) and (A)(d). However, if E is an elliptic curve over k, then we find $\dim H^{-1}(E) = 1$, i.e., axiom (A)(a) is indeed violated.

0FI0 **Lemma 45.11.2.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. Set $k' = \Gamma(X, \mathcal{O}_X)$. The following are equivalent*

(1) *there exist finitely many closed points $x_1, \ldots, x_r \in X$ whose residue fields are separable over k such that $H^0(X) \to H^0(x_1) \oplus \ldots \oplus H^0(x_r)$ is injective,*
(2) *the map $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism.*

If this is true, then $H^0(X)$ is a finite separable algebra over F. If X is equidimensional of dimension d, then (1) and (2) are also equivalent to

(3) *the classes of closed points generate $H^{2d}(X)(d)$ as a module over $H^0(X)$.*

Proof. We observe that the statement makes sense because k' is a finite separable algebra over k (Varieties, Lemma 33.9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. The compatibility of H^* with direct sums (Lemmas 45.9.9 and 45.10.1) shows that it suffices to prove the lemma when X is connected. Hence we may assume X is irreducible and we have to show the equivalence of (1), (2), and (3). Set $d = \dim(X)$. This implies that k' is a field finite separable over k and that X is geometrically irreducible over k', see Varieties, Lemmas 33.9.3 and 33.9.4.

By Lemma 45.8.1 we see that the closed points in (3) may be assumed to have separable residue fields over k. By axioms (A)(a) and (A)(b) we see that conditions (1) and (3) are equivalent.

If (2) holds, then pick any closed point $x \in X$ whose residue field is finite separable over k'. Then $H^0(\mathrm{Spec}(k')) = H^0(X) \to H^0(x)$ is injective for example by Lemma 45.10.8.

Assume the equivalent conditions (1) and (3) hold. Choose $x_1, \ldots, x_r \in X$ as in (1). Choose a finite separable extension k''/k'. By Lemma 45.10.9 we have

$$H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^0(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Thus in order to show that $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism we may replace k' by k''. Thus we may assume $x_1, \ldots, x_r$ are k'-rational points (this replaces each x_i with multiple points, so r is increased in this step). By Lemma 45.10.7 $\gamma(x_1) = \gamma(x_2) = \ldots = \gamma(x_r)$. By axiom (A)(b) all the maps $H^0(X) \to H^0(x_i)$ are the same. This means (2) holds.

Finally, Lemma 45.10.2 implies $H^0(X)$ is a separable F-algebra if (1) holds. □

0FI1 **Lemma 45.11.3.** *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If there exists a smooth projective scheme Y over k such that $H^i(Y)$ is nonzero for some $i < 0$, then there exists an equidimensional smooth projective scheme X over k such that the equivalent conditions of Lemma 45.11.2 fail for X.*

Proof. By Lemma 45.9.9 we may assume Y is irreducible and a fortiori equidimensional. If i is odd, then after replacing Y by $Y \times Y$ we find an example where

Y is equidimensional and $i = -2l$ for some $l > 0$. Set $X = Y \times (\mathbf{P}^1_k)^l$. Using axiom (B)(a) we obtain

$$H^0(X) \supset H^0(Y) \oplus H^i(Y) \otimes_F H^2(\mathbf{P}^1_k)^{\otimes_F l}$$

with both summands nonzero. Thus it is clear that $H^0(X)$ cannot be isomorphic to H^0 of the spectrum of $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$ as this falls into the first summand. □

Thus it makes sense to finally make the following definition.

0FI2 **Definition 45.11.4.** Let k be a field. Let F be a field of characteristic 0. A *Weil cohomology theory* over k with coefficients in F is given by data (D0), (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying axioms (A), (B), and (C) of Section 45.9 and in addition such that the equivalent conditions (1) and (2) of Lemma 45.11.2 hold for every smooth projective X over k.

By Lemma 45.11.3 this means also that there are no nonzero negative cohomology groups. In particular, if k is algebraically closed, then a Weil cohomology theory as above together with an isomorphism $F \to F(1)$ is the same thing as a classical Weil cohomology theory.

0FI3 **Remark 45.11.5.** Let H^* be a Weil cohomology theory (Definition 45.11.4). Let X be a geometrically irreducible smooth projective scheme of dimension d over k' with k'/k a finite separable extension of fields. Suppose that

$$H^0(\mathrm{Spec}(k')) = F_1 \times \ldots \times F_r$$

for some fields F_i. Then we accordingly can write

$$H^*(X) = \prod\nolimits_{i=1,\ldots,r} H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

Now, our final assumption in Definition 45.11.4 tells us that $H^0(X)$ is free of rank 1 over $\prod F_i$. In other words, each of the factors $H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$ has dimension 1 over F_i. Poincaré duality then tells us that the same is true for cohomology in degree $2d$. What isn't clear however is that the same holds in other degrees. Namely, we don't know that given $0 < n < \dim(X)$ the integers

$$\dim_{F_i} H^n(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

are independent of i! This question is closely related to the following open question: given an algebraically closed base field $\overline{k}$, a field of characteristic zero F, a classical Weil cohomology theory H^* over $\overline{k}$ with coefficient field F, and a smooth projective variety X over $\overline{k}$ is it true that the betti numbers of X

$$\beta_i = \dim_F H^i(X)$$

are independent of F and the Weil cohomology theory H^*?

0GIJ **Proposition 45.11.6.** *Let k be a field. Let F be a field of characteristic 0. A Weil cohomology theory is the same thing as a $\mathbf{Q}$-linear symmetric monoidal functor*

$$G : M_k \longrightarrow \textit{graded } F\textit{-vector spaces}$$

such that

(1) *$G(\mathbf{1}(1))$ is nonzero only in degree -2, and*

(2) *for every smooth projective scheme X over k with $k' = \Gamma(X, \mathcal{O}_X)$ the homomorphism $G(h(\mathrm{Spec}(k'))) \to G(h(X))$ of graded F-vector spaces is an isomorphism in degree* 0.

Proof. Immediate consequence of Proposition 45.9.12 and Definition 45.11.4. Of course we could replace (2) by the condition that $G(h(X)) \to \bigoplus G(h(x_i))$ is injective in degree 0 for some choice of closed points $x_1, \ldots, x_r \in X$ whose residue fields are separable over k. □

45.12. Chern classes

0FI4 In this section we discuss how given a first Chern class and a projective space bundle formula we can get all Chern classes. A reference for this section is [**Gro58**] although our axioms are slightly different.

Let $\mathcal{C}$ be a category of schemes with the following properties

(1) Every $X \in \mathrm{Ob}(\mathcal{C})$ is quasi-compact and quasi-separated.
(2) If $X \in \mathrm{Ob}(\mathcal{C})$ and $U \subset X$ is open and closed, then $U \to X$ is a morphism of $\mathcal{C}$. If $X' \to X$ is a morphism of $\mathcal{C}$ factoring through U, then $X' \to U$ is a morphism of $\mathcal{C}$.
(3) If $X \in \mathrm{Ob}(\mathcal{C})$ and if $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then
   (a) $p : \mathbf{P}(\mathcal{E}) \to X$ is a morphism of $\mathcal{C}$,
   (b) for a morphism $f : X' \to X$ in $\mathcal{C}$ the induced morphism $\mathbf{P}(f^*\mathcal{E}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$,
   (c) if $\mathcal{E} \to \mathcal{F}$ is a surjection onto another finite locally free $\mathcal{O}_X$-module then the closed immersion $\mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$.

Next, assume given a contravariant functor A from the category $\mathcal{C}$ to the category of graded algebras. Here a graded algebra A is a unital, associative, not necessarily commutative $\mathbf{Z}$-algebra A endowed with a grading $A = \bigoplus_{i \geq 0} A^i$. Given a morphism $f : X' \to X$ of $\mathcal{C}$ we denote $f^* : A(X) \to A(X')$ the induced algebra map. We will denote the product of $a, b \in A(X)$ by $a \cup b$. Finally, we assume given for every object X of $\mathcal{C}$ an additive map

$$c_1^A : \mathrm{Pic}(X) \longrightarrow A^1(X)$$

We assume the following axioms are satisfied

(1) Given $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{L} \in \mathrm{Pic}(X)$ the element $c_1^A(\mathcal{L})$ is in the center of the algebra $A(X)$.
(2) If $X \in \mathrm{Ob}(\mathcal{C})$ and $X = U \amalg V$ with U and V open and closed, then $A(X) = A(U) \times A(V)$ via the induced maps $A(X) \to A(U)$ and $A(X) \to A(V)$.
(3) If $f : X' \to X$ is a morphism of $\mathcal{C}$ and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $f^* c_1^A(\mathcal{L}) = c_1^A(f^*\mathcal{L})$.
(4) Given $X \in \mathrm{Ob}(\mathcal{C})$ and locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank r consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$ of $\mathcal{C}$. Then the map
$$\bigoplus\nolimits_{i=0,\ldots,r-1} A(X) \longrightarrow A(P), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum c_1^A(\mathcal{O}_P(1))^i \cup p^*(a_i)$$
is bijective.
(5) Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r + 1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of $\mathcal{C}$ which exhibits

P' as an effective Cartier divisor on P. Then for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$.

To formulate our result recall that $\mathit{Vect}(X)$ denotes the (exact) category of finite locally free $\mathcal{O}_X$-modules. In Derived Categories of Schemes, Section 36.38 we have defined the zeroth K-group $K_0(\mathit{Vect}(X))$ of this category. Moreover, we have seen that $K_0(\mathit{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FI5 **Proposition 45.12.1.** *In the situation above there is a unique rule which assigns to every $X \in \mathrm{Ob}(\mathcal{C})$ a "total Chern class"*

$$c^A : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

with the following properties

(1) *For $X \in \mathrm{Ob}(\mathcal{C})$ we have $c^A(\alpha + \beta) = c^A(\alpha)c^A(\beta)$ and $c^A(0) = 1$.*
(2) *If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* \circ c^A = c^A \circ f^*$.*
(3) *Given $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{L} \in \mathrm{Pic}(X)$ we have $c^A([\mathcal{L}]) = 1 + c_1^A(\mathcal{L})$.*

Proof. Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define an element $c^A(\mathcal{E}) \in A(X)$.

As a first step, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite. Hence $A(X) = \prod A(X_r)$. Thus it suffices to define $c^A(\mathcal{E})$ when $\mathcal{E}$ has constant rank r. In this case let $p : P \to X$ be the projective bundle of $\mathcal{E}$. We can uniquely define elements $c_i^A(\mathcal{E}) \in A^i(X)$ for $i \geq 0$ such that $c_0^A(\mathcal{E}) = 1$ and the equation

0FI6 (45.12.1.1) $$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cup p^* c_{r-i}^A(\mathcal{E}) = 0$$

is true. As usual we set $c^A(\mathcal{E}) = c_0^A(\mathcal{E}) + c_1^A(\mathcal{E}) + \ldots + c_r^A(\mathcal{E})$ in $A(X)$.

If $\mathcal{E}$ is invertible, then $c^A(\mathcal{E}) = 1 + c_1^A(\mathcal{L})$. This follows immediately from the construction above.

The elements $c_i^A(\mathcal{E})$ are in the center of $A(X)$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ be the corresponding projective bundle. if $a \in A(X)$ then $p^*a \cup (-1)^r c_1(\mathcal{O}_P(1))^r = (-1)^r c_1(\mathcal{O}_P(1))^r \cup p^*a$ and hence we must have the same for all the other terms in the expression defining $c_i^A(\mathcal{E})$ as well and we conclude.

If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* c_i^A(\mathcal{E}) = c_i^A(f^*\mathcal{E})$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X'$ be the projective bundles corresponding to $\mathcal{E}$ and $f^*\mathcal{E}$. The induced morphism $g : P' \to P$ is a morphism of $\mathcal{C}$. The pullback by g of the equality defining $c_i^A(\mathcal{E})$ is the corresponding equation for $f^*\mathcal{E}$ and we conclude.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

of finite locally free $\mathcal{O}_X$-modules with $\mathcal{L}$ invertible. Then

$$c^A(\mathcal{E}) = c^A(\mathcal{L})c^A(\mathcal{F})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$ has constant rank $r+1$ and $\mathcal{F}$ has constant rank r. The inclusion

$$i : P' = \mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}(\mathcal{E}) = P$$

is a morphism of $\mathcal{C}$ and it is the zero scheme of a regular section of the invertible module $\mathcal{L}^{\otimes -1} \otimes \mathcal{O}_P(1)$. The element

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})$$

pulls back to zero on P' by definition. Hence we see that

$$\big(c_1^A(\mathcal{O}_P(1)) - c_1^A(\mathcal{L})\big) \cup \Big(\sum\nolimits_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})\Big) = 0$$

in $A^*(P)$ by assumption (5) on our cohomology A. By definition of $c_1^A(\mathcal{E})$ this gives the desired equality.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Then

$$c^A(\mathcal{F}) = c^A(\mathcal{E}) c^A(\mathcal{G})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ have constant ranks r, s, and t. We prove it by induction on r. The case $r = 1$ was done above. If $r > 1$, then it suffices to check this after pulling back by the morphism $\mathbf{P}(\mathcal{E}^\vee) \to X$. Thus we may assume we have an invertible submodule $\mathcal{L} \subset \mathcal{E}$ such that both $\mathcal{E}' = \mathcal{E}/\mathcal{L}$ and $\mathcal{F}' = \mathcal{E}/\mathcal{L}$ are finite locally free (of ranks $s-1$ and $t-1$). Then we have

$$c^A(\mathcal{E}) = c^A(\mathcal{L}) c^A(\mathcal{E}') \quad \text{and} \quad c^A(\mathcal{F}) = c^A(\mathcal{L}) c^A(\mathcal{F}')$$

Since we have the short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{F}' \to \mathcal{G} \to 0$$

we see by induction hypothesis that

$$c^A(\mathcal{F}') = c^A(\mathcal{E}') c^A(\mathcal{G})$$

Thus the result follows from a formal calculation.

At this point for $X \in \mathrm{Ob}(\mathcal{C})$ we can define $c^A : K_0(\mathit{Vect}(X)) \to A(X)$. Namely, we send a generator $[\mathcal{E}]$ to $c^A(\mathcal{E})$ and we extend multiplicatively. Thus for example $c^A(-[\mathcal{E}]) = c^A(\mathcal{E})^{-1}$ is the formal inverse of $a^A([\mathcal{E}])$. The multiplicativity in short exact sequences shown above guarantees that this works.

Uniqueness. Suppose $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We want to show that conditions (1), (2), and (3) of the lemma uniquely determine $c^A([\mathcal{E}])$. To prove this we may assume $\mathcal{E}$ has constant rank r; this already uses (2). Then we may use induction on r. If $r = 1$, then uniqueness follows from (3). If $r > 1$ we pullback using (2) to the projective bundle $p : P \to X$ and we see that we may assume we have a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ with $\mathcal{E}'$ and $\mathcal{E}''$ having lower rank. By induction hypothesis $c^A(\mathcal{E}')$ and $c^A(\mathcal{E}'')$ are uniquely determined. Thus uniqueness for $\mathcal{E}$ by the axiom (1). □

0FI7 **Lemma 45.12.2.** *In the situation above. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a morphism $p : P \to X$ in $\mathcal{C}$ such that*

(1) $p^* : A(X) \to A(P)$ *is injective,*
(2) *each* $p^*\mathcal{E}_i$ *has a filtration whose successive quotients* $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ *are invertible* $\mathcal{O}_P$*-modules.*

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum(r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \mathrm{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the projective bundle $p : P \to X$ associated to $\mathcal{E}_i$. We have a short exact sequence

$$0 \to \mathcal{F} \to p^*\mathcal{E}_i \to \mathcal{O}_P(1) \to 0$$

of finite locally free $\mathcal{O}_P$-modules of ranks $r_i - 1$, r_i, and 1. Observe that $p^* : A(X) \to A(P)$ is injective by assumption. By the induction hypothesis applied to the finite locally free $\mathcal{O}_P$-modules $\mathcal{F}$ and $p^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $p' : P' \to P$ with properties stated as in the lemma. Then the composition $p \circ p' : P' \to X$ does the job. □

0FI8 **Lemma 45.12.3.** *Let* $X \in \mathrm{Ob}(\mathcal{C})$*. Let* $\mathcal{E}$ *be a finite locally free* $\mathcal{O}_X$*-module. Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Then*

$$c_i^A(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}^A(\mathcal{E}) \cup c_1^A(\mathcal{L})^j$$

Proof. By the construction of c_i^A we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X$ be the projective bundle associated to $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$. Then there is an isomorphism $g : P \to P'$ such that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes p^*\mathcal{L}$. See Constructions, Lemma 27.20.1. Thus

$$g^* c_1^A(\mathcal{O}_{P'}(1)) = c_1^A(\mathcal{O}_P(1)) + p^* c_1^A(\mathcal{L})$$

The desired equality follows formally from this and the definition of Chern classes using equation (45.12.1.1). □

0FI9 **Proposition 45.12.4.** *In the situation above assume* $A(X)$ *is a* $\mathbf{Q}$*-algebra for all* $X \in \mathrm{Ob}(\mathcal{C})$*. Then there is a unique rule which assigns to every* $X \in \mathrm{Ob}(\mathcal{C})$ *a "chern character"*

$$ch^A : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

with the following properties

(1) ch^A *is a ring map for all* $X \in \mathrm{Ob}(\mathcal{C})$*.*
(2) *If* $f : X' \to X$ *is a morphism of* $\mathcal{C}$*, then* $f^* \circ ch^A = ch^A \circ f^*$*.*
(3) *Given* $X \in \mathrm{Ob}(\mathcal{C})$ *and* $\mathcal{L} \in \mathrm{Pic}(X)$ *we have* $ch^A([\mathcal{L}]) = \exp(c_1^A(\mathcal{L}))$*.*

Proof. Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define the rank $r^A(\mathcal{E}) \in A^0(X)$. Namely, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite, say $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$. Then $A(X) = A(X_0) \times A(X_1) \times \ldots \times A(X_n)$. Thus we can define $r^A(\mathcal{E}) = (0, 1, \ldots, n) \in A^0(X)$.

Let $P_p(c_1, \ldots, c_p)$ be the polynomials constructed in Chow Homology, Example 42.43.6. Then we can define

$$ch^A(\mathcal{E}) = r^A(\mathcal{E}) + \sum\nolimits_{i \geq 1} (1/i!) P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) \in \prod\nolimits_{i \geq 0} A^i(X)$$

where ci^A are the Chern classes of Proposition 45.12.1. It follows immediately that we have property (2) and (3) of the lemma.

We still have to show the following three statements

(1) If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}) = ch^A(\mathcal{E}_1) + ch^A(\mathcal{E}_2)$.
(2) If $\mathcal{E}_1$ and $\mathcal{E}_2 \to 0$ are finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = ch^A(\mathcal{E}_1)ch^A(\mathcal{E}_2)$.

Namely, the first will prove that ch^A factors through $K_0(\mathit{Vect}(X))$ and the first and the second will combined show that ch^A is a ring map.

To prove these statements we can reduce to the case where $\mathcal{E}_1$ and $\mathcal{E}_2$ have constant ranks r_1 and r_2. In this case the equalities in $A^0(X)$ are immediate. To prove the equalities in higher degrees, by Lemma 45.12.2 we may assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ have filtrations whose graded pieces are invertible modules $\mathcal{L}_{1,j}$, $j = 1, \ldots, r_1$ and $\mathcal{L}_{2,j}$, $j = 1, \ldots, r_2$. Using the multiplicativity of Chern classes we get

$$c_i^A(\mathcal{E}_1) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}))$$

where s_i is the ith elementary symmetric function as in Chow Homology, Example 42.43.6. Similarly for $c_i^A(\mathcal{E}_2)$. In case (1) we get

$$c_i^A(\mathcal{E}) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}), c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{2,r_2}))$$

and for case (2) we get

$$c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = s_i(c_1^A(\mathcal{L}_{1,1}) + c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}) + c_1^A(\mathcal{L}_{2,r_2}))$$

By the definition of the polynomials P_i we see that this means

$$P_i(c_1^A(\mathcal{E}_1), \ldots, c_i^A(\mathcal{E}_1)) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i$$

and similarly for $\mathcal{E}_2$. In case (1) we have also

$$P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i + \sum\nolimits_{j=1,\ldots,r_2} c_1^A(\mathcal{L}_{2,j})^i$$

In case (2) we get accordingly

$$P_i(c_1^A(\mathcal{E}_1 \otimes \mathcal{E}_2), \ldots, c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2)) = \sum\nolimits_{j=1,\ldots,r_1} \sum\nolimits_{j'=1,\ldots,r_2} (c_1^A(\mathcal{L}_{1,j}) + c_1^A(\mathcal{L}_{2,j'}))^i$$

Thus the desired equalities are now consequences of elementary identities between symmetric polynomials.

We omit the proof of uniqueness. □

0FIA **Lemma 45.12.5.** *In the situation above let $X \in \mathrm{Ob}(\mathcal{C})$. If ψ^2 is as in Chow Homology, Lemma 42.56.1 and c^A and ch^A are as in Propositions 45.12.1 and 45.12.4 then we have $c_i^A(\psi^2(\alpha)) = 2^i c_i^A(\alpha)$ and $ch_i^A(\psi^2(\alpha)) = 2^i ch_i^A(\alpha)$ for all $\alpha \in K_0(\mathit{Vect}(X))$.*

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(\mathit{Vect}(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded

parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 45.12.2. Then $[p^*\mathcal{E}] = \sum[\mathcal{L}_j]$ and hence $\psi^2([p^*\mathcal{E}]) = \sum[\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1^A(\mathcal{L}_j)$ we have

$$c^A(\alpha) = \prod(1 + x_j) \quad \text{and} \quad c^A(\psi^2(\alpha)) = \prod(1 + 2x_j)$$

in $\prod A^i(P)$ and we have

$$ch^A(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch^A(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. □

45.13. Exterior powers and K-groups

0FIB We do the minimal amount of work to define the lambda operators. Let X be a scheme. Recall that $\mathit{Vect}(X)$ denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(\mathit{Vect}(X))$ associated to this category in Derived Categories of Schemes, Section 36.38. Finally, $K_0(\mathit{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 36.38.6.

0FIC **Lemma 45.13.1.** *Let X be a scheme. There are maps*

$$\lambda^r : K_0(\mathit{Vect}(X)) \longrightarrow K_0(\mathit{Vect}(X))$$

which sends $[\mathcal{E}]$ to $[\wedge^r(\mathcal{E})]$ when $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and which are compatible with pullbacks.

Proof. Consider the ring $R = K_0(\mathit{Vect}(X))[[t]]$ where t is a variable. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we set

$$c(\mathcal{E}) = \sum\nolimits_{i=0}^{\infty} [\wedge^i(\mathcal{E})]t^i$$

in R. We claim that given a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of finite locally free $\mathcal{O}_X$-modules we have $c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'')$. The claim implies that c extends to a map

$$c : K_0(\mathit{Vect}(X)) \longrightarrow R$$

which converts addition in $K_0(\mathit{Vect}(X))$ to multiplication in R. Writing $c(\alpha) = \sum \lambda^i(\alpha)t^i$ we obtain the desired operators λ^i.

To see the claim, we consider the short exact sequence as a filtration on $\mathcal{E}$ with 2 steps. We obtain an induced filtration on $\wedge^r(\mathcal{E})$ with $r + 1$ steps and subquotients

$$\wedge^r(\mathcal{E}'), \wedge^{r-1}(\mathcal{E}') \otimes \mathcal{E}'', \wedge^{r-2}(\mathcal{E}') \otimes \wedge^2(\mathcal{E}''), \ldots \wedge^r(\mathcal{E}'')$$

Thus we see that $[\wedge^r(\mathcal{E})]$ is equal to

$$\sum\nolimits_{i=0}^{r} [\wedge^{r-i}(\mathcal{E}')][\wedge^i(\mathcal{E}'')]$$

and the result follows easily from this and elementary algebra. □

45.14. Weil cohomology theories, III

0FID Let k be a field. Let F be a field of characteristic zero. Suppose we are given the following data

(D0) A 1-dimensional F-vector space $F(1)$.
(D1) A contravariant functor $H^*(-)$ from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
(D2') For every smooth projective scheme X over k a homomorphism $c_1^H : \mathrm{Pic}(X) \to H^2(X)(1)$ of abelian groups.

We will use the terminology, notation, and conventions regarding (D0) and (D1) as discussed in Section 45.9. Given a smooth projective scheme X over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ the cohomology class $c_1^H(\mathcal{L}) \in H^2(X)(1)$ of (D2') is sometimes called the *first Chern class of $\mathcal{L}$ in cohomology*.

Here is the list of axioms.

(A1) H^* is compatible with finite coproducts
(A2) c_1^H is compatible with pullbacks
(A3) Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r \geq 1$. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map

$$\bigoplus\nolimits_{i=0,\ldots,r-1} H^*(X)(-i) \longrightarrow H^*(P), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum c_1^H(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is an isomorphism of F-vector spaces.
(A4) Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$.
(A5) H^* is compatible with finite products
(A6) Let X be a nonempty smooth, projective scheme over k equidimensional of dimension d. Then there exists an F-linear map $\lambda : H^{2d}(X)(d) \to F$ such that $(\mathrm{id} \otimes \lambda)\gamma([\Delta]) = 1$ in $H^*(X)$.
(A7) If $b : X' \to X$ is the blowing up of a smooth center in a smooth projective scheme X over k[2], then $b^* : H^*(X) \to H^*(X')$ is injective.
(A8) If X is a smooth projective scheme over k and $k' = \Gamma(X, \mathcal{O}_X)$, then the map $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism.
(A9) Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. For $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y))$ where λ_Y and λ_X are as in axiom (A6) for X and Y.

Let us explain more precisely what we mean by each of these axioms. Axioms (A3), (A4), and (A7) are clear as stated.

Ad (A1). This means that $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections.

Ad (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^*c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$.

[2]Then X' is smooth and projective over k as well, see More on Morphisms, Lemma 37.17.3.

Ad (A5). This means that $H^*(\mathrm{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections.

Ad (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 45.14.2 if we have axioms (A1) – (A4) we can consider the class of the diagonal

$$\gamma([\Delta]) \in H^{2d}(X \times X)(d) = \bigoplus\nolimits_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

where the tensor decomposition comes from axiom (A5). Given an F-linear map $\lambda : H^{2d}(X)(d) \to F$ we may also view λ as an F-linear map $\lambda : H^*(X)(d) \to F$ by precomposing with the projection onto $H^{2d}(X)(d)$. Having said this axiom (A6) makes sense.

Ad (A8). Let X be a smooth projective scheme over k. Then $k' = \Gamma(X, \mathcal{O}_X)$ is a finite separable k-algebra (Varieties, Lemma 33.9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. Thus we may apply H^* to $\mathrm{Spec}(k')$ and axiom (A8) makes sense.

Ad (A9). We will see in Remark 45.14.6 that if we have axioms (A1) – (A7) then the map λ of axiom (A6) is unique.

0FIE **Lemma 45.14.1.** *Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a ring homomorphism*

$$ch^H : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\mathcal{L}) = \exp(c_1^H(\mathcal{L}))$ for any invertible $\mathcal{O}_X$-module $\mathcal{L}$.

Proof. Immediate from Proposition 45.12.4 applied to the category of smooth projective schemes over k, the functor $A : X \mapsto \bigoplus_{i \geq 0} H^{2i}(X)(i)$, and the map c_1^H. □

0FIF **Lemma 45.14.2.** *Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a graded ring homomorphism*

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus\nolimits_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\alpha) = \gamma(ch(\alpha))$ for α in $K_0(\mathit{Vect}(X))$.

Proof. Recall that we have an isomorphism

$$K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^*(X) \otimes \mathbf{Q}, \quad \alpha \longmapsto ch(\alpha) \cap [X]$$

see Chow Homology, Lemma 42.58.1. It is an isomorphism of rings by Chow Homology, Remark 42.56.5. We define γ by the formula $\gamma(\alpha) = ch^H(\alpha')$ where ch^H is as in Lemma 45.14.1 and $\alpha' \in K_0(\mathit{Vect}(X))$ is such that $ch(\alpha') \cap [X] = \alpha$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$.

The construction $\alpha \mapsto \gamma(\alpha)$ is compatible with pullbacks because both ch^H and taking Chern classes is compatible with pullbacks, see Lemma 45.14.1 and Chow Homology, Remark 42.59.9.

We still have to see that γ is graded. Let $\psi^2 : K_0(\mathit{Vect}(X)) \to K_0(\mathit{Vect}(X))$ be the second Adams operator, see Chow Homology, Lemma 42.56.1. If $\alpha \in \mathrm{CH}^i(X)$ and $\alpha' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ is the unique element with $ch(\alpha') \cap [X] = \alpha$, then we have seen in Chow Homology, Section 42.58 that $\psi^2(\alpha') = 2^i\alpha'$. Hence we conclude that $ch^H(\alpha') \in H^{2i}(X)(i)$ by Lemma 45.12.5 as desired. □

0FIH **Lemma 45.14.3.** *Let $b : X' \to X$ be the blowing up of a smooth projective scheme over k in a smooth closed subscheme $Z \subset X$. Picture*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Assume every irreducible component of Z has codimension r in X. Then there exists a cycle $\theta \in \mathrm{CH}^{r-1}(X')$ such that $b^![Z] = [E] \cdot \theta$ in $\mathrm{CH}^r(X')$ and $\pi_* j^!(\theta) = [Z]$ in $\mathrm{CH}^r(Z)$.

Proof. The scheme X is smooth and projective over k and hence we have $K_0(X) = K_0(\mathit{Vect}(X))$. See Derived Categories of Schemes, Lemmas 36.36.2 and 36.38.5. Let $\alpha \in K_0(\mathrm{Vect}(X))$ be an element whose restriction to Z is $\mathcal{C}_{Z/X}$. By Chow Homology, Lemma 42.56.3 there exists an element $\alpha^\vee$ which restricts to $\mathcal{C}^\vee_{Z/X}$. By the blow up formula (Chow Homology, Lemma 42.59.11) we have

$$b^![Z] = b^!i_*[Z] = j_*res(b^!)([Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap \pi^*[Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap [E])$$

where $\mathcal{F}$ is the kernel of the surjection $\pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$. Observe that $b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]$ is an element of $K_0(\mathrm{Vect}(X'))$ which restricts to $[\pi^*\mathcal{C}^\vee_{Z/X}] - [\mathcal{C}^\vee_{E/X'}] = [\mathcal{F}^\vee]$ on E. Since capping with Chern classes commutes with j_* we conclude that the above is equal to

$$c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [E]$$

in the chow group of X'. Hence we see that setting

$$\theta = c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']$$

we get the first relation $\theta \cdot [E] = b^![Z]$ for example by Chow Homology, Lemma 42.62.2. For the second relation observe that

$$j^!\theta = j^!(c_{r-1}(b^*\alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']) = c_{r-1}(\mathcal{F}^\vee) \cap j^![X'] = c_{r-1}(\mathcal{F}^\vee) \cap [E]$$

in the chow groups of E. To prove that π_* of this is equal to $[Z]$ it suffices to prove that the degree of the codimension $r-1$ cycle $(-1)^{r-1}c_{r-1}(\mathcal{F}) \cap [E]$ on the fibres of π is 1. This is a computation we omit. □

0FII **Lemma 45.14.4.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A4) and (A7). Let X be a smooth projective scheme over k. Let $Z \subset X$ be a smooth closed subscheme such that every irreducible component of Z has codimension r in X. Assume the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the restriction of an element of $K_0(X)$. If $a \in H^*(X)$ and $a|_Z = 0$ in $H^*(Z)$, then $\gamma([Z]) \cup a = 0$.*

Proof. Let $b : X' \to X$ be the blowing up. By (A7) it suffices to show that

$$b^*(\gamma([Z]) \cup a) = b^*\gamma([Z]) \cup b^*a = 0$$

By Lemma 45.14.3 we have

$$b^*\gamma([Z]) = \gamma(b^*[Z]) = \gamma([E] \cdot \theta) = \gamma([E]) \cup \gamma(\theta)$$

Hence because b^*a restricts to zero on E and since $\gamma([E]) = c_1^H(\mathcal{O}_{X'}(E))$ we get what we want from (A4). □

0FIJ **Lemma 45.14.5.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (A) of Section 45.9 holds with $\int_X = \lambda$ as in axiom (A6).*

Proof. Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We will show that the graded F-vector space $H^*(X)(d)[2d]$ is a left dual to $H^*(X)$. This will prove what we want by Homology, Lemma 12.17.5. We are going to use axiom (A5) which in particular says that

$$H^*(X \times X)(d) = \bigoplus H^i(X) \otimes H^j(X)(d) = \bigoplus H^i(X)(d) \otimes H^j(X)$$

Define a map

$$\eta : F \longrightarrow H^*(X \times X)(d)$$

by multiplying by $\gamma([\Delta]) \in H^{2d}(X \times X)(d)$. On the other hand, define a map

$$\epsilon : H^*(X \times X)(d) \longrightarrow H^*(X)(d) \xrightarrow{\lambda} F$$

by first using pullback Δ^* by the diagonal morphism $\Delta : X \to X \times X$ and then using the F-linear map $\lambda : H^{2d}(X)(d) \to F$ of axiom (A6) precomposed by the projection $H^*(X)(d) \to H^{2d}(X)(d)$. In order to show that $H^*(X)(d)$ is a left dual to $H^*(X)$ we have to show that the composition of the maps

$$\eta \otimes 1 : H^*(X) \longrightarrow H^*(X \times X \times X)(d)$$

and

$$1 \otimes \epsilon : H^*(X \times X \times X)(d) \longrightarrow H^*(X)$$

is the identity. If $a \in H^*(X)$ then we see that the composition maps a to

$$(1 \otimes \lambda)(\Delta_{23}^*(q_{12}^*\gamma([\Delta]) \cup q_3^*a)) = (1 \otimes \lambda)(\gamma([\Delta]) \cup p_2^*a)$$

where $q_i : X \times X \times X \to X$ and $q_{ij} : X \times X \times X \to X \times X$ are the projections, $\Delta_{23} : X \times X \to X \times X \times X$ is the diagonal, and $p_i : X \times X \to X$ are the projections. The equality holds because $\Delta_{23}^*(q_{12}^*\gamma([\Delta]) = \Delta_{23}^*\gamma([\Delta \times X]) = \gamma([\Delta])$ and because $\Delta_{23}^*q_3^*a = p_2^*a$. Since $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ (see below) the above simplifies to

$$(1 \otimes \lambda)(\gamma([\Delta]) \cup p_1^*a) = a$$

by our choice of λ as desired. The second condition $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}$ of Categories, Definition 4.43.5 is proved in exactly the same manner.

Note that p_1^*a and pr_2^*a restrict to the same cohomology class on $\Delta \subset X \times X$. Moreover we have $\mathcal{C}_{\Delta/X\times X} = \Omega^1_\Delta$ which is the restriction of $p_1^*\Omega^1_X$. Hence Lemma 45.14.4 implies $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ and the proof is complete. □

0FIK **Remark 45.14.6** (Uniqueness of trace maps)**.** Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. Combining what was said in the proofs of Lemma 45.14.5 and Homology, Lemma 12.17.5 we see that

$$\gamma([\Delta]) \in \bigoplus\nolimits_i H^i(X) \otimes H^{2d-i}(X)(d)$$

defines a perfect duality between $H^i(X)$ and $H^{2d-i}(X)(d)$ for all i. In particular, the linear map $\int_X = \lambda : H^{2d}(X)(d) \to F$ of axiom (A6) is unique! We will call the linear map $\int_X$ the trace map of X from now on.

0FIL **Lemma 45.14.7.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (B) of Section 45.9 holds.*

Proof. Axiom (B)(a) is immediate from axiom (A5). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. To see that axiom (B)(b) holds, observe that the diagonal $\Delta_{X\times Y}$ of $X \times Y$ is the intersection product of the pullbacks of the diagonals Δ_X of X and Δ_Y of Y by the projections $p : X \times Y \times X \times Y \to X \times X$ and $q : X \times Y \times X \times Y \to Y \times Y$. Compatibility of γ with intersection products then gives that

$$\gamma([\Delta_{X\times Y}]) \in H^{2d+2e}(X \times Y \times X \times Y)(d + e)$$

is the cup product of the pullbacks of $\gamma([\Delta_X])$ and $\gamma([\Delta_Y])$ by p and q. Write

$$\gamma([\Delta_{X\times Y}]) = \sum \eta_{X\times Y,i} \text{ with } \eta_{X\times Y,i} \in H^i(X \times Y) \otimes H^{2d+2e-i}(X \times Y)(d + e)$$

and similarly $\gamma([\Delta_X]) = \sum \eta_{X,i}$ and $\gamma([\Delta_Y]) = \sum \eta_{Y,i}$. The observation above implies we have

$$\eta_{X\times Y,0} = \sum\nolimits_{i\in\mathbf{Z}} p^*\eta_{X,i} \cup q^*\eta_{Y,-i}$$

(If our cohomology theory vanishes in negative degrees, which will be true in almost all cases, then only the term for $i = 0$ contributes and $\eta_{X\times Y,0}$ lies in $H^0(X) \otimes H^0(Y) \otimes H^{2d}(X)(d) \otimes H^{2e}(Y)(e)$ as expected, but we don't need this.) Since $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2e}(Y)(e) \to F$ send $\eta_{X,0}$ and $\eta_{Y,0}$ to 1 in $H^*(X)$ and $H^*(Y)$, we see that $\lambda_X \otimes \lambda_Y$ sends $\eta_{X\times Y,0}$ to 1 in $H^*(X) \otimes H^*(Y) = H^*(X \times Y)$ and the proof is complete. □

0FIM **Lemma 45.14.8.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (C)(d) of Section 45.9 holds.*

Proof. We have $\gamma([\mathrm{Spec}(k)]) = 1 \in H^*(\mathrm{Spec}(k))$ by construction. Since

$$H^0(\mathrm{Spec}(k)) = F, \quad H^0(\mathrm{Spec}(k) \times \mathrm{Spec}(k)) = H^0(\mathrm{Spec}(k)) \otimes_F H^0(\mathrm{Spec}(k))$$

the map $\int_{\mathrm{Spec}(k)} = \lambda$ of axiom (A6) must send 1 to 1 because we have seen that $\int_{\mathrm{Spec}(k)\times\mathrm{Spec}(k)} = \int_{\mathrm{Spec}(k)} \int_{\mathrm{Spec}(k)}$ in Lemma 45.14.7. □

Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then we obtain data (D0), (D1), (D2), and (D3) of Section 45.9 satisfying axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 45.9, see Lemmas 45.14.5, 45.14.7, and 45.14.8. Moreover, we have the pushforwards $f_* : H^*(X) \to H^*(Y)$ as constructed in Section 45.9. The only axiom of Section 45.9 which isn't clear yet is axiom (C)(b).

0FIN **Lemma 45.14.9.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let $p : P \to X$ be as in axiom (A3) with X nonempty equidimensional. Then γ commutes with pushforward along p.*

Proof. It suffices to prove this on generators for $\mathrm{CH}_*(P)$. Thus it suffices to prove this for a cycle class of the form $\xi^i \cdot p^*\alpha$ where $0 \leq i \leq r - 1$ and $\alpha \in \mathrm{CH}_*(X)$. Note that $p_*(\xi^i \cdot p^*\alpha) = 0$ if $i < r - 1$ and $p_*(\xi^{r-1} \cdot p^*\alpha) = \alpha$. On the other hand,

we have $\gamma(\xi^i \cdot p^*\alpha) = c^i \cup p^*\gamma(\alpha)$ and by the projection formula (Lemma 45.9.1) we have

$$p_*\gamma(\xi^i \cdot p^*\alpha) = p_*(c^i) \cup \gamma(\alpha)$$

Thus it suffices to show that $p_*c^i = 0$ for $i < r-1$ and $p_*c^{r-1} = 1$. Equivalently, it suffices to prove that $\lambda_P : H^{2d+2r-2}(P)(d+r-1) \to F$ defined by the rules

$$\lambda_P(c^i \cup p^*(a)) = \begin{cases} 0 & \text{if} \quad i < r-1 \\ \int_X(a) & \text{if} \quad i = r-1 \end{cases}$$

satisfies the condition of axiom (A5). This follows from the computation of the class of the diagonal of P in Lemma 45.6.2. □

0FVR **Lemma 45.14.10.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If k'/k is a Galois extension, then we have $\int_{\mathrm{Spec}(k')} 1 = [k' : k]$.*

Proof. We observe that

$$\mathrm{Spec}(k') \times \mathrm{Spec}(k') = \coprod\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} (\mathrm{Spec}(\sigma) \times \mathrm{id})^{-1}\Delta$$

as cycles on $\mathrm{Spec}(k') \times \mathrm{Spec}(k')$. Our construction of γ always sends $[X]$ to 1 in $H^0(X)$. Thus $1 \otimes 1 = 1 = \sum(\mathrm{Spec}(\sigma) \times \mathrm{id})^*\gamma([\Delta])$. Denote $\lambda : H^0(\mathrm{Spec}(k')) \to F$ the map from axiom (A6), in other words $(\mathrm{id} \otimes \lambda)(\gamma(\Delta)) = 1$ in $H^0(\mathrm{Spec}(k'))$. We obtain

$$\begin{aligned} \lambda(1)1 &= (\mathrm{id} \otimes \lambda)(1 \otimes 1) \\ &= (\mathrm{id} \otimes \lambda)(\sum(\mathrm{Spec}(\sigma) \times \mathrm{id})^*\gamma([\Delta])) \\ &= \sum(\mathrm{Spec}(\sigma) \times \mathrm{id})^*((\mathrm{id} \otimes \lambda)(\gamma([\Delta])) \\ &= \sum(\mathrm{Spec}(\sigma) \times \mathrm{id})^*(1) \\ &= [k' : k] \end{aligned}$$

Since λ is another name for $\int_{\mathrm{Spec}(k')}$ (Remark 45.14.6) the proof is complete. □

0FIP **Lemma 45.14.11.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k.*

Proof. We will use without further mention that we've constructed our cycle class map γ in Lemma 45.14.2 compatible with intersection products and pullbacks and that we've already shown axioms (A), (B), (C)(a), (C)(c), and (C)(d) of Section 45.9, see Lemma 45.14.5, Remark 45.14.6, and Lemmas 45.14.7 and 45.14.8. In particular, we may use (for example) Lemma 45.9.1 to see that pushforward on H^* is compatible with composition and satisfies the projection formula.

Let $f : X \to Y$ be a morphism of nonempty equidimensional smooth projective schemes over k. We are trying to show $f_*\gamma(\alpha) = \gamma(f_*\alpha)$ for any cycle class α on X. We can write α as a $\mathbf{Q}$-linear combination of products of Chern classes of locally free $\mathcal{O}_X$-modules (Chow Homology, Lemma 42.58.1). Thus we may assume α is a product of Chern classes of finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_1, \ldots, \mathcal{E}_r$. Pick $p : P \to X$ as in the splitting principle (Chow Homology, Lemma 42.43.1). By Chow Homology, Remark 42.43.2 we see that p is a composition of projective space bundles and that $\alpha = p_*(\xi_1 \cap \ldots \cap \xi_d \cap \cdot p^*\alpha)$ where ξ_i are first Chern classes

of invertible modules. By Lemma 45.14.9 we know that p_* commutes with cycle classes. Thus it suffices to prove the property for the composition $f \circ p$. Since $p^*\mathcal{E}_1, \ldots, p^*\mathcal{E}_r$ have filtrations whose successive quotients are invertible modules, this reduces us to the case where α is of the form $\xi_1 \cap \ldots \cap \xi_t \cap [X]$ for some first Chern classes ξ_i of invertible modules $\mathcal{L}_i$.

Assume $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some invertible modules $\mathcal{L}_i$ on X. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For $n \gg 0$ the invertible $\mathcal{O}_X$-modules $\mathcal{L}^{\otimes n}$ and $\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}$ are both very ample on X over k, see Morphisms, Lemma 29.39.8. Since $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}) - c_1(\mathcal{L}^{\otimes n})$ this reduces us to the case where $\mathcal{L}_1$ is very ample. Repeating this with $\mathcal{L}_i$ for $i = 2, \ldots, t$ we reduce to the case where $\mathcal{L}_i$ is very ample on X over k for all $i = 1, \ldots, t$.

Assume k is infinite and $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some very ample invertible modules $\mathcal{L}_i$ on X over k. By Bertini in the form of Varieties, Lemma 33.47.3 we can successively find regular sections s_i of $\mathcal{L}_i$ such that the schemes $Z(s_1) \cap \ldots \cap Z(s_i)$ are smooth over k and of codimension i in X. By the construction of capping with the first class of an invertible module (going back to Chow Homology, Definition 42.24.1), this reduces us to the case where $\alpha = [Z]$ for some nonempty smooth closed subscheme $Z \subset X$ which is equidimensional.

Assume $\alpha = [Z]$ where $Z \subset X$ is a smooth closed subscheme. Choose a closed embedding $X \to \mathbf{P}^n$. We can factor f as

$$X \to Y \times \mathbf{P}^n \to Y$$

Since we know the result for the second morphism by Lemma 45.14.9 it suffices to prove the result when $\alpha = [Z]$ where $i : Z \to X$ is a closed immersion and f is a closed immersion. Then $j = f \circ i$ is a closed embedding too. Using the hypothesis for i and j we win.

We still have to prove the lemma in case k is finite. We urge the reader to skip the rest of the proof. Everything we said above continues to work, except that we do not know we can choose the sections s_i cutting out our Z over k as k is finite. However, we do know that we can find s_i over the algebraic closure $\overline{k}$ of k (by the same lemma). This means that we can find a finite extension k'/k such that our sections s_i are defined over k'. Denote $\pi : X_{k'} \to X$ the projection. The arguments above shows that we get the desired conclusion (from the assumption in the lemma) for the cycle $\pi^*\alpha$ and the morphism $f \circ \pi : X_{k'} \to Y$. We have $\pi_*\pi^*\alpha = [k' : k]\alpha$, see Chow Homology, Lemma 42.15.2. On the other hand, we have

$$\pi_*\gamma(\pi^*\alpha) = \pi_*\pi^*\gamma(\alpha) = \gamma(\alpha)\pi_*1$$

by the projection formula for our cohomology theory. Observe that π is a projection (!) and hence we have $\pi_*(1) = \int_{\mathrm{Spec}(k')}(1)1$ by Lemma 45.9.2. Thus to finish the proof in the finite field case, it suffices to prove that $\int_{\mathrm{Spec}(k')}(1) = [k' : k]$ which we do in Lemma 45.14.10. □

In the lemmas below we use the Grassmannians defined and constructed in Constructions, Section 27.22.

0FIQ **Lemma 45.14.12.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Given integers $0 < l < n$ and a nonempty equidimensional smooth*

projective scheme X over k consider the projection morphism $p : X \times \mathbf{G}(l, n) \to X$. Then γ commutes with pushforward along p.

Proof. If $l = 1$ or $l = n - 1$ then p is a projective bundle and the result follows from Lemma 45.14.9. In general there exists a morphism

$$h : Y \to X \times \mathbf{G}(l, n)$$

such that both h and $p \circ h$ are compositions of projective space bundles. Namely, denote $\mathbf{G}(1, 2, \ldots, l; n)$ the partial flag variety. Then the morphism

$$\mathbf{G}(1, 2, \ldots, l; n) \to \mathbf{G}(l, n)$$

is a compostion of projective space bundles and similarly the structure morphism $\mathbf{G}(1, 2, \ldots, l; n) \to \mathrm{Spec}(k)$ is of this form. Thus we may set $Y = X \times \mathbf{G}(1, 2, \ldots, l; n)$. Since every cycle on $X \times \mathbf{G}(l, n)$ is the pushforward of a cycle on Y, the result for $Y \to X$ and the result for $Y \to X \times \mathbf{G}(l, n)$ imply the result for p. □

0FIR **Lemma 45.14.13.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k such that the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the pullback of a class in $K_0(X)$.*

Proof. By Lemma 45.14.11 it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k. Say Z has codimension r in X. Let $\mathcal{L}$ be a sufficiently ample invertible module on X. Choose $n > 0$ and a surjection

$$\mathcal{O}_Z^{\oplus n} \to \mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$$

This gives a morphism $g : Z \to \mathbf{G}(n - r, n)$ to the Grassmannian over k, see Constructions, Section 27.22. Consider the composition

$$Z \to X \times \mathbf{G}(n - r, n) \to X$$

Pushforward along the second morphism is compatible with classes of cycles by Lemma 45.14.12. The conormal sheaf $\mathcal{C}$ of the closed immersion $Z \to X \times \mathbf{G}(n-r, n)$ sits in a short exact sequence

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{C} \to g^*\Omega_{\mathbf{G}(n-r,n)} \to 0$$

See More on Morphisms, Lemma 37.11.13. Since $\mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$ is the pull back of a finite locally free sheaf on $\mathbf{G}(n - r, n)$ we conclude that the class of $\mathcal{C}$ in $K_0(Z)$ is the pullback of a class in $K_0(X \times \mathbf{G}(n - r, n))$. Hence we have the property for $Z \to X \times \mathbf{G}(n - r, n)$ and we conclude. □

0FVS **Lemma 45.14.14.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If $k''/k'/k$ are finite separable field extensions, then $H^0(\mathrm{Spec}(k')) \to H^0(\mathrm{Spec}(k''))$ is injective.*

Proof. We may replace k'' by its normal closure over k which is Galois over k, see Fields, Lemma 9.21.5. Then k'' is Galois over k' as well, see Fields, Lemma 9.21.4. We deduce we have an isomorphism

$$k' \otimes_k k'' \longrightarrow \prod\nolimits_{\sigma \in \mathrm{Gal}(k''/k')} k'', \quad \eta \otimes \zeta \longmapsto (\eta\sigma(\zeta))_\sigma$$

This produces an isomorphism $\coprod_\sigma \mathrm{Spec}(k'') \to \mathrm{Spec}(k') \times \mathrm{Spec}(k'')$ which on cohomology produces the isomorphism

$$H^*(\mathrm{Spec}(k')) \otimes_F H^*(\mathrm{Spec}(k'')) \to \prod\nolimits_\sigma H^*(\mathrm{Spec}(k'')), \quad a' \otimes a'' \longmapsto (\pi^* a' \cup \mathrm{Spec}(\sigma)^* a'')_\sigma$$

where $\pi : \mathrm{Spec}(k'') \to \mathrm{Spec}(k')$ is the morphism corresponding to the inclusion of k' in k''. We conclude the lemma is true by taking $a'' = 1$. □

0FIS **Lemma 45.14.15.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Let $b : X' \to X$ be a blowing up of a smooth projective scheme X over k which is nonempty equidimensional of dimension d in a nonwhere dense smooth center Z. Then $b_*(1) = 1$.*

Proof. We may replace X by a connected component of X (some details omitted). Thus we may assume X is connected and hence irreducible. Set $k' = \Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'})$; we omit the proof of the equality. Choose a closed point $x' \in X'$ which isn't contained in the exceptional divisor and whose residue field k'' is separable over k; this is possible by Varieties, Lemma 33.25.6. Denote $x \in X$ the image (whose residue field is equal to k'' as well of course). Consider the diagram

$$\begin{array}{ccc} x' \times X' & \longrightarrow & X' \times X' \\ \downarrow & & \downarrow \\ x \times X & \longrightarrow & X \times X \end{array}$$

The class of the diagonal $\Delta = \Delta_X$ pulls back to the class of the "diagonal point" $\delta_x : x \to x \times X$ and similarly for the class of the diagonal Δ'. On the other hand, the diagonal point δ_x pulls back to the diagonal point $\delta_{x'}$ by the left vertical arrow. Write $\gamma([\Delta]) = \sum \eta_i$ with $\eta_i \in H^i(X) \otimes H^{2d-i}(X)(d)$ and $\gamma([\Delta']) = \sum \eta'_i$ with $\eta'_i \in H^i(X') \otimes H^{2d-i}(X')(d)$. The arguments above show that η_0 and η'_0 map to the same class in

$$H^0(x') \otimes_F H^{2d}(X')(d)$$

We have $H^0(\mathrm{Spec}(k')) = H^0(X) = H^0(X')$ by axiom (A8). By Lemma 45.14.14 this common value maps injectively into $H^0(x')$. We conclude that η_0 maps to η'_0 by the map

$$H^0(X) \otimes_F H^{2d}(X)(d) \longrightarrow H^0(X') \otimes_F H^{2d}(X')(d)$$

This means that $\int_X$ is equal to $\int_{X'}$ composed with the pullback map. This proves the lemma. □

0FIT **Lemma 45.14.16.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then the cycle class map γ commutes with pushforward.*

Proof. Let $i : Z \to X$ be as in Lemma 45.14.13. Consider the diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Let $\theta \in \mathrm{CH}^{r-1}(X')$ be as in Lemma 45.14.3. Then $\pi_* j^! \theta = [Z]$ in $\mathrm{CH}_*(Z)$ implies that $\pi_* \gamma(j^! \theta) = 1$ by Lemma 45.14.9 because π is a projective space bundle. Hence

we see that

$$i_*(1) = i_*(\pi_*(\gamma(j^!\theta))) = b_*j_*(j^*\gamma(\theta)) = b_*(j_*(1) \cup \gamma(\theta))$$

We have $j_*(1) = \gamma([E])$ by (A9). Thus this is equal to

$$b_*(\gamma([E]) \cup \gamma(\theta)) = b_*(\gamma([E] \cdot \theta)) = b_*(\gamma(b^*[Z])) = b_*b^*\gamma([Z]) = b_*(1) \cup \gamma([Z])$$

Since $b_*(1) = 1$ by Lemma 45.14.15 the proof is complete. □

0FIU **Proposition 45.14.17.** *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then we have a Weil cohomology theory.*

Proof. We have axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 45.9 by Lemmas 45.14.5, 45.14.7, and 45.14.8. We have axiom (C)(b) by Lemma 45.14.16. Finally, the additional condition of Definition 45.11.4 holds because it is the same as our axiom (A8). □

The following lemma is sometimes useful to show that we get a Weil cohomology theory over a nonclosed field by reducing to a closed one.

0FVT **Lemma 45.14.18.** *Let k'/k be an extension of fields. Let F'/F be an extension of fields of characteristic 0. Assume given*

1. *data (D0), (D1), (D2') for k and F denoted $F(1), H^*, c_1^H$,*
2. *data (D0), (D1), (D2') for k' and F' denoted $F'(1), (H')^*, c_1^{H'}$, and*
3. *an isomorphism $F(1) \otimes_F F' \to F'(1)$, functorial isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ on the category of smooth projective schemes X over k such that the diagrams*

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow[c_1^H]{} & H^2(X)(1) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X_{k'}) & \xrightarrow{c_1^{H'}} & (H')^2(X_{k'})(1) \end{array}$$

commute.

In this case, if $F'(1), (H')^*, c_1^{H'}$ satisfy axioms (A1) – (A9), then the same is true for $F(1), H^*, c_1^H$.

Proof. We go by the axioms one by one.

Axiom (A1). We have to show $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections. By the functorial nature of the isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ this follows from linear algebra: if $\varphi : V \to W$ is an F-linear map of F-vector spaces, then φ is an isomorphism if and only if $\varphi_{F'} : V \otimes_F F' \to W \otimes_F F'$ is an isomorphism.

Axiom (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^*c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$. This is immediately clear from the corresponding property for $c_1^{H'}$, the commutative diagrams in the lemma, and the fact that the canonical map $V \to V \otimes_F F'$ is injective for any F-vector space V.

Axiom (A3). This follows from the principle stated in the proof of axiom (A1) and compatibility of c_1^H and $c_1^{H'}$.

Axiom (A4). Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have to show $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$. Denote $a' \in (H')^*(X_{k'})$ the image of a. The assumption implies that $(i')^*a' = 0$ where $i' : Y_{k'} \to X_{k'}$ is the base change of i. Hence we get $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'})) = 0$ by the axiom for $(H')^*$. Since $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'}))$ is the image of $a \cup c_1^H(\mathcal{O}_X(Y))$ we conclude by the princple stated in the proof of axiom (A2).

Axiom (A5). This means that $H^*(\mathrm{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections. This follows from the principle stated in the proof of axiom (A1).

We interrupt the flow of the arguments to show that for every smooth projective scheme X over k the diagram

$$\begin{array}{ccc} \mathrm{CH}^*(X) & \xrightarrow{\gamma} & \bigoplus H^{2i}(X)(i) \\ {\scriptstyle g^*}\downarrow & & \downarrow \\ \mathrm{CH}^*(X_{k'}) & \xrightarrow{\gamma'} & \bigoplus (H')^{2i}(X_{k'})(i) \end{array}$$

commutes. Observe that we have γ as we know axioms (A1) – (A4) already; see Lemma 45.14.2. Also, the left vertical arrow is the one discussed in Chow Homology, Section 42.67 for the morphism of base schemes $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. More precisely, it is the map given in Chow Homology, Lemma 42.67.4. Pick $\alpha \in \mathrm{CH}^*(X)$. Write $\alpha = ch(\beta) \cap [X]$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ for some $\beta \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ so that $\gamma(\alpha) = ch^H(\beta)$; this is our construction of γ. Since the map of Chow Homology, Lemma 42.67.4 is compatible with capping with Chern classes by Chow Homology, Lemma 42.67.8 we see that $g^*\alpha = ch((X_{k'} \to X)^*\beta) \cap [X_{k'}]$. Hence $\gamma'(g^*\alpha) = ch^{H'}((X_{k'} \to X)^*\beta)$. Thus commutativity of the diagram will hold if for any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r and $0 \leq i \leq r$ the element $c_i^H(\mathcal{E})$ of $H^{2i}(X)(i)$ maps to the element $c_i^{H'}(\mathcal{E}_{k'})$ in $(H')^{2i}(X_{k'})(i)$. Because we have the projective space bundle formula for both X and X' we may replace X by a projective space bundle over X finitely many times to show this. Thus we may assume $\mathcal{E}$ has a filtration whose graded pieces are invertible $\mathcal{O}_X$-modules $\mathcal{L}_1, \ldots, \mathcal{L}_r$. See Chow Homology, Lemma 42.43.1 and Remark 42.43.2. Then $c_i^H(\mathcal{E}$ is the ith elementary symmetric polynomial in $c_1^H(\mathcal{L}_1), \ldots, c_1^H(\mathcal{L}_r)$ and we conclude by our assumption that we have agreement for first Chern classes.

Axiom (A6). Suppose given F-vector spaces V, W, an element $v \in V$, and a tensor $\xi \in V \otimes_F W$. Denote $V' = V \otimes_F F'$, $W' = W \otimes_F F'$ and v', ξ' the images of v, ξ in V', $V' \otimes_{F'} W'$. The linear algebra principle we will use in the proof of axiom (A6) is the following: there exists an F-linear map $\lambda : W \to F$ such that $(1 \otimes \lambda)\xi = v$ if and only if there exists an F'-linear map $\lambda' : W \otimes_F F' \to F'$ such that $(1 \otimes \lambda')\xi' = v'$.

Let X be a nonempty equidimensional smooth projective scheme over k of dimension d. Denote $\gamma = \gamma([\Delta])$ in $H^{2d}(X \times X)(d)$ (unadorned fibre products will be over k). Observe/recall that this makes sense as we know axioms (A1) – (A4) already; see Lemma 45.14.2. We may decompose

$$\gamma = \sum \gamma_i, \quad \gamma_i \in H^i(X) \otimes_F H^{2d-i}(X)(d)$$

in the Künneth decomposition. Similarly, denote $\gamma' = \gamma([\Delta']) = \sum \gamma'_i$ in $(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d)$. By the linear algebra princple mentioned above, it suffices to show that γ_0 maps to γ'_0 in $(H')^0(X) \otimes_{F'} (H')^{2d}(X')(d)$. By the compatibility of Künneth decompositions we see that it suffice to show that γ maps to γ' in

$$(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d) = (H')^{2d}((X \times X)_{k'})(d)$$

Since $\Delta_{k'} = \Delta'$ this follows from the discussion above.

Axiom (A7). This follows from the linear algebra fact: a linear map $V \to W$ of F-vector spaces is injective if and only if $V \otimes_F F' \to W \otimes_F F'$ is injective.

Axiom (A8). Follows from the linear algebra fact used in the proof of axiom (A1).

Axiom (A9). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. Let λ_Y and λ_X be as in axiom (A6) for X and Y. We have to show: for $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y))$. By Remark 45.14.6 we know that $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2d-2}(Y)(d-1)$ are uniquely determined by the requirement in axiom (A6). Having said this, it follows from our proof of axiom (A6) for H^* above that $\lambda_X \otimes \mathrm{id}_{F'}$ corresponds to $\lambda_{X_{k'}}$ via the given identification $H^{2d}(X)(d) \otimes_F F' = H^{2d}(X_{k'})(d)$. Thus the fact that we know axiom (A9) for $F'(1), (H')^*, c_1^{H'}$ implies the axiom for $F(1), H^*, c_1^H$ by a diagram chase. This completes the proof of the lemma. □

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CHAPTER 46

Adequate Modules

06Z1

46.1. Introduction

06Z2 For any scheme X the category $QCoh(\mathcal{O}_X)$ of quasi-coherent modules is abelian and a weak Serre subcategory of the abelian category of all $\mathcal{O}_X$-modules. The same thing works for the category of quasi-coherent modules on an algebraic space X viewed as a subcategory of the category of all $\mathcal{O}_X$-modules on the small étale site of X. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the pushforward f_* and higher direct images preserve quasi-coherence.

Next, let X be a scheme and let $\mathcal{O}$ be the structure sheaf on one of the big sites of X, say, the big fppf site. The category of quasi-coherent $\mathcal{O}$-modules is abelian (in fact it is equivalent to the category of usual quasi-coherent $\mathcal{O}_X$-modules on the scheme X we mentioned above) but its imbedding into $Mod(\mathcal{O})$ is not exact. An example is the map of quasi-coherent modules

$$\mathcal{O}_{\mathbf{A}^1_k} \longrightarrow \mathcal{O}_{\mathbf{A}^1_k}$$

on $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ given by multiplication by x. In the abelian category of quasi-coherent sheaves this map is injective, whereas in the abelian category of all $\mathcal{O}$-modules on the big site of $\mathbf{A}^1_k$ this map has a nontrivial kernel as we see by evaluating on sections over $\mathrm{Spec}(k[x]/(x)) = \mathrm{Spec}(k)$. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the functor $f_{big,*}$ does not preserve quasi-coherence.

In this chapter we introduce the category of what we will call adequate modules, closely related to quasi-coherent modules, which "fixes" the two problems mentioned above. Another solution, which we will implement when we talk about quasi-coherent modules on algebraic stacks, is to consider $\mathcal{O}$-modules which are locally quasi-coherent and satisfy the flat base change property. See Cohomology of Stacks, Section 103.8, Cohomology of Stacks, Remark 103.10.7, and Derived Categories of Stacks, Section 104.5.

46.2. Conventions

06Z3 In this chapter we fix $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ and we fix a big τ-site Sch_τ as in Topologies, Section 34.2. All schemes will be objects of Sch_τ. In particular, given a scheme S we obtain sites $(Aff/S)_\tau \subset (Sch/S)_\tau$. The structure sheaf $\mathcal{O}$ on these sites is defined by the rule $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$.

All rings A will be such that $\mathrm{Spec}(A)$ is (isomorphic to) an object of Sch_τ. Given a ring A we denote Alg_A the category of A-algebras whose objects are the A-algebras B of the form $B = \Gamma(U, \mathcal{O}_U)$ where U is an affine object of Sch_τ. Thus given an

affine scheme $S = \mathrm{Spec}(A)$ the functor

$$(Aff/S)_\tau \longrightarrow Alg_A, \quad U \longmapsto \mathcal{O}(U)$$

is an equivalence.

46.3. Adequate functors

06US In this section we discuss a topic closely related to direct images of quasi-coherent sheaves. Most of this material was taken from the paper [**Jaf97**].

06Z4 **Definition 46.3.1.** Let A be a ring. A *module-valued functor* is a functor $F : Alg_A \to Ab$ such that

(1) for every object B of Alg_A the group $F(B)$ is endowed with the structure of a B-module, and
(2) for any morphism $B \to B'$ of Alg_A the map $F(B) \to F(B')$ is B-linear.

A *morphism of module-valued functors* is a transformation of functors $\varphi : F \to G$ such that $F(B) \to G(B)$ is B-linear for all $B \in \mathrm{Ob}(Alg_A)$.

Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of module-valued functors on Alg_A is equivalent to the category $PMod((Aff/S)_\tau, \mathcal{O})$ of presheaves of $\mathcal{O}$-modules. The equivalence is given by the rule which assigns to the module-valued functor F the presheaf $\mathcal{F}$ defined by the rule $\mathcal{F}(U) = F(\mathcal{O}(U))$. This is clear from the equivalence $(Aff/S)_\tau \to Alg_A$, $U \mapsto \mathcal{O}(U)$ given in Section 46.2. The quasi-inverse sets $F(B) = \mathcal{F}(\mathrm{Spec}(B))$.

An important special case of a module-valued functor comes about as follows. Let M be an A-module. Then we will denote $\underline{M}$ the module-valued functor $B \mapsto M \otimes_A B$ (with obvious B-module structure). Note that if $M \to N$ is a map of A-modules then there is an associated morphism $\underline{M} \to \underline{N}$ of module-valued functors. Conversely, any morphism of module-valued functors $\underline{M} \to \underline{N}$ comes from an A-module map $M \to N$ as the reader can see by evaluating on $B = A$. In other words Mod_A is a full subcategory of the category of module-valued functors on Alg_A.

Given and A-module map $\varphi : M \to N$ then $\mathrm{Coker}(\underline{M} \to \underline{N}) = \underline{Q}$ where $Q = \mathrm{Coker}(M \to N)$ because $\otimes$ is right exact. But this isn't the case for the kernel in general: for example an injective map of A-modules need not be injective after base change. Thus the following definition makes sense.

06UT **Definition 46.3.2.** Let A be a ring. A module-valued functor F on Alg_A is called

(1) *adequate* if there exists a map of A-modules $M \to N$ such that F is isomorphic to $\mathrm{Ker}(\underline{M} \to \underline{N})$.
(2) *linearly adequate* if F is isomorphic to the kernel of a map $\underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Note that F is adequate if and only if there exists an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and F is linearly adequate if and only if there exists an exact sequence $0 \to F \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Let A be a ring. In this section we will show the category of adequate functors on Alg_A is abelian (Lemmas 46.3.10 and 46.3.11) and has a set of generators (Lemma 46.3.6). We will also see that it is a weak Serre subcategory of the category of all module-valued functors on Alg_A (Lemma 46.3.16) and that it has arbitrary colimits (Lemma 46.3.12).

06UU **Lemma 46.3.3.** *Let A be a ring. Let F be an adequate functor on Alg_A. If $B = \operatorname{colim} B_i$ is a filtered colimit of A-algebras, then $F(B) = \operatorname{colim} F(B_i)$.*

Proof. This holds because for any A-module M we have $M \otimes_A B = \operatorname{colim} M \otimes_A B_i$ (see Algebra, Lemma 10.12.9) and because filtered colimits commute with exact sequences, see Algebra, Lemma 10.8.8. □

06UV **Remark 46.3.4.** Consider the category $Alg_{fp,A}$ whose objects are A-algebras B of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and whose morphisms are A-algebra maps. Every A-algebra B is a filtered colimit of finitely presented A-algebra, i.e., a filtered colimit of objects of $Alg_{fp,A}$. By Lemma 46.3.3 we conclude every adequate functor F is determined by its restriction to $Alg_{fp,A}$. For some questions we can therefore restrict to functors on $Alg_{fp,A}$. For example, the category of adequate functors does not depend on the choice of the big τ-site chosen in Section 46.2.

06UW **Lemma 46.3.5.** *Let A be a ring. Let F be an adequate functor on Alg_A. If $B \to B'$ is flat, then $F(B) \otimes_B B' \to F(B')$ is an isomorphism.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. This gives the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(B) \otimes_B B' & \longrightarrow & (M \otimes_A B) \otimes_B B' & \longrightarrow & (N \otimes_A B) \otimes_B B' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(B') & \longrightarrow & M \otimes_A B' & \longrightarrow & N \otimes_A B' \end{array}$$

where the rows are exact (the top one because $B \to B'$ is flat). Since the right two vertical arrows are isomorphisms, so is the left one. □

06UX **Lemma 46.3.6.** *Let A be a ring. Let F be an adequate functor on Alg_A. Then there exists a surjection $L \to F$ with L a direct sum of linearly adequate functors.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ where $\underline{M} \to \underline{N}$ is given by $\varphi : M \to N$. By Lemma 46.3.3 it suffices to construct $L \to F$ such that $L(B) \to F(B)$ is surjective for every finitely presented A-algebra B. Hence it suffices to construct, given a finitely presented A-algebra B and an element $\xi \in F(B)$ a map $L \to F$ with L linearly adequate such that ξ is in the image of $L(B) \to F(B)$. (Because there is a set worth of such pairs (B, ξ) up to isomorphism.)

To do this write $\sum_{i=1,\ldots,n} m_i \otimes b_i$ the image of ξ in $\underline{M}(B) = M \otimes_A B$. We know that $\sum \varphi(m_i) \otimes b_i = 0$ in $N \otimes_A B$. As N is a filtered colimit of finitely presented A-modules, we can find a finitely presented A-module N', a commutative diagram of A-modules

$$\begin{array}{ccc} A^{\oplus n} & \longrightarrow & N' \\ {\scriptstyle m_1,\ldots,m_n}\downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

such that $(b_1, \ldots, b_n)$ maps to zero in $N' \otimes_A B$. Choose a presentation $A^{\oplus l} \to A^{\oplus k} \to N' \to 0$. Choose a lift $A^{\oplus n} \to A^{\oplus k}$ of the map $A^{\oplus n} \to N'$ of the diagram. Then we see that there exist $(c_1, \ldots, c_l) \in B^{\oplus l}$ such that $(b_1, \ldots, b_n, c_1, \ldots, c_l)$ maps to zero in $B^{\oplus k}$ under the map $B^{\oplus n} \oplus B^{\oplus l} \to B^{\oplus k}$. Consider the commutative

diagram

$$\begin{array}{ccc} A^{\oplus n} \oplus A^{\oplus l} & \longrightarrow & A^{\oplus k} \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

where the left vertical arrow is zero on the summand $A^{\oplus l}$. Then we see that L equal to the kernel of $\underline{A^{\oplus n+l}} \to \underline{A^{\oplus k}}$ works because the element $(b_1, \ldots, b_n, c_1, \ldots, c_l) \in L(B)$ maps to ξ. □

Consider a graded A-algebra $B = \bigoplus_{d \geq 0} B_d$. Then there are two A-algebra maps $p, a : B \to B[t, t^{-1}]$, namely $p : b \mapsto b$ and $a : b \mapsto t^{\deg(b)} b$ where b is homogeneous. If F is a module-valued functor on Alg_A, then we define

06UY (46.3.6.1) $$F(B)^{(k)} = \{\xi \in F(B) \mid t^k F(p)(\xi) = F(a)(\xi)\}.$$

For functors which behave well with respect to flat ring extensions this gives a direct sum decomposition. This amounts to the fact that representations of $\mathbf{G}_m$ are completely reducible.

06UZ **Lemma 46.3.7.** *Let A be a ring. Let F be a module-valued functor on Alg_A. Assume that for $B \to B'$ flat the map $F(B) \otimes_B B' \to F(B')$ is an isomorphism. Let B be a graded A-algebra. Then*

1. *$F(B) = \bigoplus_{k \in \mathbf{Z}} F(B)^{(k)}$, and*
2. *the map $B \to B_0 \to B$ induces map $F(B) \to F(B)$ whose image is contained in $F(B)^{(0)}$.*

Proof. Let $x \in F(B)$. The map $p : B \to B[t, t^{-1}]$ is free hence we know that

$$F(B[t, t^{-1}]) = \bigoplus\nolimits_{k \in \mathbf{Z}} F(p)(F(B)) \cdot t^k = \bigoplus\nolimits_{k \in \mathbf{Z}} F(B) \cdot t^k$$

as indicated we drop the $F(p)$ in the rest of the proof. Write $F(a)(x) = \sum t^k x_k$ for some $x_k \in F(B)$. Denote $\epsilon : B[t, t^{-1}] \to B$ the B-algebra map $t \mapsto 1$. Note that the compositions $\epsilon \circ p, \epsilon \circ a : B \to B[t, t^{-1}] \to B$ are the identity. Hence we see that

$$x = F(\epsilon)(F(a)(x)) = F(\epsilon)(\sum t^k x_k) = \sum x_k.$$

On the other hand, we claim that $x_k \in F(B)^{(k)}$. Namely, consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow[a]{} & B[t, t^{-1}] \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle f} \\ B[s, s^{-1}] & \xrightarrow{g} & B[t, s, t^{-1}, s^{-1}] \end{array}$$

where $a'(b) = s^{\deg(b)} b$, $f(b) = b$, $f(t) = st$ and $g(b) = t^{\deg(b)} b$ and $g(s) = s$. Then

$$F(g)(F(a'))(x) = F(g)(\sum s^k x_k) = \sum s^k F(a)(x_k)$$

and going the other way we see

$$F(f)(F(a))(x) = F(f)(\sum t^k x_k) = \sum (st)^k x_k.$$

Since $B \to B[s, t, s^{-1}, t^{-1}]$ is free we see that $F(B[t, s, t^{-1}, s^{-1}]) = \bigoplus_{k,l \in \mathbf{Z}} F(B) \cdot t^k s^l$ and comparing coefficients in the expressions above we find $F(a)(x_k) = t^k x_k$ as desired.

Finally, the image of $F(B_0) \to F(B)$ is contained in $F(B)^{(0)}$ because $B_0 \to B \xrightarrow{a} B[t, t^{-1}]$ is equal to $B_0 \to B \xrightarrow{p} B[t, t^{-1}]$. □

As a particular case of Lemma 46.3.7 note that

$$\underline{M}(B)^{(k)} = M \otimes_A B_k$$

where B_k is the degree k part of the graded A-algebra B.

06V0 **Lemma 46.3.8.** *Let A be a ring. Given a solid diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & \underline{A^{\oplus n}} & \longrightarrow & \underline{A^{\oplus m}} \\ & & \downarrow{\scriptstyle \varphi} & \swarrow & & & \\ & & \underline{M} & & & & \end{array}$$

of module-valued functors on Alg_A with exact row there exists a dotted arrow making the diagram commute.

Proof. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the ring $B = A[x_1, \ldots, x_n]/(\sum a_{ij} x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Note that ξ has the following universal property: for any A-algebra C and any $\xi' \in L(C)$ there exists an A-algebra map $B \to C$ such that ξ maps to ξ' via the map $L(B) \to L(C)$.

Note that B is a graded A-algebra, hence we can use Lemmas 46.3.7 and 46.3.5 to decompose the values of our functors on B into graded pieces. Note that $\xi \in L(B)^{(1)}$ as $(x_1, \ldots, x_n)$ is an element of degree one in $\underline{A^{\oplus n}}(B)$. Hence we see that $\varphi(\xi) \in \underline{M}(B)^{(1)} = M \otimes_A B_1$. Since B_1 is generated by $x_1, \ldots, x_n$ as an A-module we can write $\varphi(\xi) = \sum m_i \otimes x_i$. Consider the map $A^{\oplus n} \to M$ which maps the ith basis vector to m_i. By construction the associated map $\underline{A^{\oplus n}} \to \underline{M}$ maps the element ξ to $\varphi(\xi)$. It follows from the universal property mentioned above that the diagram commutes. □

06V1 **Lemma 46.3.9.** *Let A be a ring. Let $\varphi : F \to \underline{M}$ be a map of module-valued functors on Alg_A with F adequate. Then* $\mathrm{Coker}(\varphi)$ *is adequate.*

Proof. By Lemma 46.3.6 we may assume that $F = \bigoplus L_i$ is a direct sum of linearly adequate functors. Choose exact sequences $0 \to L_i \to \underline{A^{\oplus n_i}} \to \underline{A^{\oplus m_i}}$. For each i choose a map $A^{\oplus n_i} \to M$ as in Lemma 46.3.8. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus L_i & \longrightarrow & \bigoplus \underline{A^{\oplus n_i}} & \longrightarrow & \bigoplus \underline{A^{\oplus m_i}} \\ & & \downarrow & \swarrow & & & \\ & & \underline{M} & & & & \end{array}$$

Consider the A-modules

$$Q = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to M \oplus \bigoplus A^{\oplus m_i}) \quad \text{and} \quad P = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to \bigoplus A^{\oplus m_i}).$$

Then we see that $\mathrm{Coker}(\varphi)$ is isomorphic to the kernel of $\underline{Q} \to \underline{P}$. □

06V2 **Lemma 46.3.10.** *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then* $\mathrm{Coker}(\varphi)$ *is adequate.*

Proof. Choose an injection $G \to \underline{M}$. Then we have an injection $G/F \to \underline{M}/F$. By Lemma 46.3.9 we see that $\underline{M}/F$ is adequate, hence we can find an injection $\underline{M}/F \to \underline{N}$. Composing we obtain an injection $G/F \to \underline{N}$. By Lemma 46.3.9 the cokernel of the induced map $G \to \underline{N}$ is adequate hence we can find an injection $\underline{N}/G \to \underline{K}$. Then $0 \to G/F \to \underline{N} \to \underline{K}$ is exact and we win. □

06V3 **Lemma 46.3.11.** *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then* $\mathrm{Ker}(\varphi)$ *is adequate.*

Proof. Choose an injection $F \to \underline{M}$ and an injection $G \to \underline{N}$. Denote $F \to \underline{M \oplus N}$ the diagonal map so that

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \underline{M \oplus N} & \longrightarrow & \underline{N} \end{array}$$

commutes. By Lemma 46.3.10 we can find a module map $M \oplus N \to K$ such that F is the kernel of $\underline{M \oplus N} \to \underline{K}$. Then $\mathrm{Ker}(\varphi)$ is the kernel of $\underline{M \oplus N} \to \underline{K \oplus N}$. □

06V4 **Lemma 46.3.12.** *Let A be a ring. An arbitrary direct sum of adequate functors on Alg_A is adequate. A colimit of adequate functors is adequate.*

Proof. The statement on direct sums is immediate. A general colimit can be written as a kernel of a map between direct sums, see Categories, Lemma 4.14.12. Hence this follows from Lemma 46.3.11. □

06V5 **Lemma 46.3.13.** *Let A be a ring. Let F, G be module-valued functors on Alg_A. Let $\varphi : F \to G$ be a transformation of functors. Assume*

(1) *φ is additive,*
(2) *for every A-algebra B and $\xi \in F(B)$ and unit $u \in B^*$ we have $\varphi(u\xi) = u\varphi(\xi)$ in $G(B)$, and*
(3) *for any flat ring map $B \to B'$ we have $G(B) \otimes_B B' = G(B')$.*

Then φ is a morphism of module-valued functors.

Proof. Let B be an A-algebra, $\xi \in F(B)$, and $b \in B$. We have to show that $\varphi(b\xi) = b\varphi(\xi)$. Consider the ring map

$$B \to B' = B[x, y, x^{-1}, y^{-1}]/(x + y - b).$$

This ring map is faithfully flat, hence $G(B) \subset G(B')$. On the other hand

$$\varphi(b\xi) = \varphi((x + y)\xi) = \varphi(x\xi) + \varphi(y\xi) = x\varphi(\xi) + y\varphi(\xi) = (x + y)\varphi(\xi) = b\varphi(\xi)$$

because x, y are units in B'. Hence we win. □

06V6 **Lemma 46.3.14.** *Let A be a ring. Let $0 \to \underline{M} \to G \to L \to 0$ be a short exact sequence of module-valued functors on Alg_A with L linearly adequate. Then G is adequate.*

Proof. We first point out that for any flat A-algebra map $B \to B'$ the map $G(B) \otimes_B B' \to G(B')$ is an isomorphism. Namely, this holds for $\underline{M}$ and L, see Lemma 46.3.5 and hence follows for G by the five lemma. In particular, by Lemma 46.3.7 we see that $G(B) = \bigoplus_{k \in \mathbf{Z}} G(B)^{(k)}$ for any graded A-algebra B.

Choose an exact sequence $0 \to L \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the graded A-algebra

$B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Observe that $\xi \in L(B)^{(1)}$. The map

$$\mathrm{Hom}_A(B, C) \longrightarrow L(C), \quad f \longmapsto L(f)(\xi)$$

defines an isomorphism of functors. The reason is that f is determined by the images $c_i = f(x_i) \in C$ which have to satisfy the relations $\sum a_{ij}c_j = 0$. And $L(C)$ is the set of n-tuples $(c_1, \ldots, c_n)$ satisfying the relations $\sum a_{ij}c_j = 0$.

Since the value of each of the functors $\underline{M}$, G, L on B is a direct sum of its weight spaces (by the lemma mentioned above) exactness of $0 \to \underline{M} \to G \to L \to 0$ implies the sequence $0 \to \underline{M}(B)^{(1)} \to G(B)^{(1)} \to L(B)^{(1)} \to 0$ is exact. Thus we may choose an element $\theta \in G(B)^{(1)}$ mapping to ξ.

Consider the graded A-algebra

$$C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$$

There are three graded A-algebra homomorphisms $p_1, p_2, m : B \to C$ defined by the rules

$$p_1(x_i) = x_i, \quad p_1(x_i) = y_i, \quad m(x_i) = x_i + y_i.$$

We will show that the element

$$\tau = G(m)(\theta) - G(p_1)(\theta) - G(p_2)(\theta) \in G(C)$$

is zero. First, τ maps to zero in $L(C)$ by a direct calculation. Hence τ is an element of $\underline{M}(C)$. Moreover, since m, p_1, p_2 are graded algebra maps we see that $\tau \in G(C)^{(1)}$ and since $\underline{M} \subset G$ we conclude

$$\tau \in \underline{M}(C)^{(1)} = M \otimes_A C_1.$$

We may write uniquely $\tau = \underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)$ with $\tau_i \in M \otimes_A B_1 = \underline{M}(B)^{(1)}$ because $C_1 = p_1(B_1) \oplus p_2(B_1)$. Consider the ring map $q_1 : C \to B$ defined by $x_i \mapsto x_i$ and $y_i \mapsto 0$. Then $\underline{M}(q_1)(\tau) = \underline{M}(q_1)(\underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)) = \tau_1$. On the other hand, because $q_1 \circ m = q_1 \circ p_1$ we see that $G(q_1)(\tau) = -G(q_1 \circ p_2)(\tau)$. Since $q_1 \circ p_2$ factors as $B \to A \to B$ we see that $G(q_1 \circ p_2)(\tau)$ is in $G(B)^{(0)}$, see Lemma 46.3.7. Hence $\tau_1 = 0$ because it is in $G(B)^{(0)} \cap \underline{M}(B)^{(1)} \subset G(B)^{(0)} \cap G(B)^{(1)} = 0$. Similarly $\tau_2 = 0$, whence $\tau = 0$.

Since $\theta \in G(B)$ we obtain a transformation of functors

$$\psi : L(-) = \mathrm{Hom}_A(B, -) \longrightarrow G(-)$$

by mapping $f : B \to C$ to $G(f)(\theta)$. Since θ is a lift of ξ the map ψ is a right inverse of $G \to L$. In terms of ψ the statements proved above have the following meaning: $\tau = 0$ means that ψ is additive and $\theta \in G(B)^{(1)}$ implies that for any A-algebra D we have $\psi(ul) = u\psi(l)$ in $G(D)$ for $l \in L(D)$ and $u \in D^*$ a unit. This implies that ψ is a morphism of module-valued functors, see Lemma 46.3.13. Clearly this implies that $G \cong \underline{M} \oplus L$ and we win. □

06V7 **Remark 46.3.15.** Let A be a ring. The proof of Lemma 46.3.14 shows that any extension $0 \to \underline{M} \to E \to L \to 0$ of module-valued functors on Alg_A with L linearly adequate splits. It uses only the following properties of the module-valued functor $F = \underline{M}$:

(1) $F(B) \otimes_B B' \to F(B')$ is an isomorphism for a flat ring map $B \to B'$, and

(2) $F(C)^{(1)} = F(p_1)(F(B)^{(1)}) \oplus F(p_2)(F(B)^{(1)})$ where $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$ and $C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$.

These two properties hold for any adequate functor F; details omitted. Hence we see that L is a projective object of the abelian category of adequate functors.

06V8 **Lemma 46.3.16.** *Let A be a ring. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A. If F and H are adequate, so is G.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. If we can show that $(\underline{M} \oplus G)/F$ is adequate, then G is the kernel of the map of adequate functors $(\underline{M} \oplus G)/F \to \underline{N}$, hence adequate by Lemma 46.3.11. Thus we may assume $F = \underline{M}$.

We can choose a surjection $L \to H$ where L is a direct sum of linearly adequate functors, see Lemma 46.3.6. If we can show that the pullback $G \times_H L$ is adequate, then G is the cokernel of the map $\mathrm{Ker}(L \to H) \to G \times_H L$ hence adequate by Lemma 46.3.10. Thus we may assume that $H = \bigoplus L_i$ is a direct sum of linearly adequate functors. By Lemma 46.3.14 each of the pullbacks $G \times_H L_i$ is adequate. By Lemma 46.3.12 we see that $\bigoplus G \times_H L_i$ is adequate. Then G is the cokernel of

$$\bigoplus\nolimits_{i \neq i'} F \longrightarrow \bigoplus G \times_H L_i$$

where ξ in the summand (i, i') maps to $(0, \ldots, 0, \xi, 0, \ldots, 0, -\xi, 0, \ldots, 0)$ with nonzero entries in the summands i and i'. Thus G is adequate by Lemma 46.3.10. □

06V9 **Lemma 46.3.17.** *Let $A \to A'$ be a ring map. If F is an adequate functor on Alg_A, then its restriction F' to $Alg_{A'}$ is adequate too.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. Then $F'(B') = F(B') = \mathrm{Ker}(M \otimes_A B' \to N \otimes_A B')$. Since $M \otimes_A B' = M \otimes_A A' \otimes_{A'} B'$ and similarly for N we see that F' is the kernel of $\underline{M \otimes_A A'} \to \underline{N \otimes_A A'}$. □

06VA **Lemma 46.3.18.** *Let $A \to A'$ be a ring map. If F' is an adequate functor on $Alg_{A'}$, then the module-valued functor $F : B \mapsto F'(A' \otimes_A B)$ on Alg_A is adequate too.*

Proof. Choose an exact sequence $0 \to F' \to \underline{M'} \to \underline{N'}$. Then

$$\begin{aligned} F(B) &= F'(A' \otimes_A B) \\ &= \mathrm{Ker}(M' \otimes_{A'} (A' \otimes_A B) \to N' \otimes_{A'} (A' \otimes_A B)) \\ &= \mathrm{Ker}(M' \otimes_A B \to N' \otimes_A B) \end{aligned}$$

Thus F is the kernel of $\underline{M} \to \underline{N}$ where $M = M'$ and $N = N'$ viewed as A-modules. □

06VB **Lemma 46.3.19.** *Let $A = A_1 \times \ldots \times A_n$ be a product of rings. An adequate functor over A is the same thing as a sequence $F_1, \ldots, F_n$ of adequate functors F_i over A_i.*

Proof. This is true because an A-algebra B is canonically a product $B_1 \times \ldots \times B_n$ and the same thing holds for A-modules. Setting $F(B) = \coprod F_i(B_i)$ gives the correspondence. Details omitted. □

06VH **Lemma 46.3.20.** *Let $A \to A'$ be a ring map and let F be a module-valued functor on Alg_A such that*

(1) *the restriction F' of F to the category of A'-algebras is adequate, and*

(2) *for any A-algebra B the sequence*

$$0 \to F(B) \to F(B \otimes_A A') \to F(B \otimes_A A' \otimes_A A')$$

is exact.

Then F is adequate.

Proof. The functors $B \to F(B \otimes_A A')$ and $B \mapsto F(B \otimes_A A' \otimes_A A')$ are adequate, see Lemmas 46.3.18 and 46.3.17. Hence F as a kernel of a map of adequate functors is adequate, see Lemma 46.3.11. □

46.4. Higher exts of adequate functors

06Z5 Let A be a ring. In Lemma 46.3.16 we have seen that any extension of adequate functors in the category of module-valued functors on Alg_A is adequate. In this section we show that the same remains true for higher ext groups.

06Z6 **Lemma 46.4.1.** *Let A be a ring. For every module-valued functor F on Alg_A there exists a morphism $Q(F) \to F$ of module-valued functors on Alg_A such that (1) $Q(F)$ is adequate and (2) for every adequate functor G the map* $\mathrm{Hom}(G, Q(F)) \to \mathrm{Hom}(G, F)$ *is a bijection.*

Proof. Choose a set $\{L_i\}_{i \in I}$ of linearly adequate functors such that every linearly adequate functor is isomorphic to one of the L_i. This is possible. Suppose that we can find $Q(F) \to F$ with (1) and (2)' or every $i \in I$ the map $\mathrm{Hom}(L_i, Q(F)) \to \mathrm{Hom}(L_i, F)$ is a bijection. Then (2) holds. Namely, combining Lemmas 46.3.6 and 46.3.11 we see that every adequate functor G sits in an exact sequence

$$K \to L \to G \to 0$$

with K and L direct sums of linearly adequate functors. Hence (2)' implies that $\mathrm{Hom}(L, Q(F)) \to \mathrm{Hom}(L, F)$ and $\mathrm{Hom}(K, Q(F)) \to \mathrm{Hom}(K, F)$ are bijections, whence the same thing for G.

Consider the category $\mathcal{I}$ whose objects are pairs (i, φ) where $i \in I$ and $\varphi : L_i \to F$ is a morphism. A morphism $(i, \varphi) \to (i', \varphi')$ is a map $\psi : L_i \to L_{i'}$ such that $\varphi' \circ \psi = \varphi$. Set

$$Q(F) = \mathrm{colim}_{(i,\varphi) \in \mathrm{Ob}(\mathcal{I})} L_i$$

There is a natural map $Q(F) \to F$, by Lemma 46.3.12 it is adequate, and by construction it has property (2)'. □

06Z7 **Lemma 46.4.2.** *Let A be a ring. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 46.4.1. Then*

(1) *i is fully faithful, exact, and its image is a weak Serre subcategory,*
(2) *$\mathcal{P}$ has enough injectives,*
(3) *the functor Q is a right adjoint to i hence left exact,*
(4) *Q transforms injectives into injectives,*
(5) *$\mathcal{A}$ has enough injectives.*

Proof. This lemma just collects some facts we have already seen so far. Part (1) is clear from the definitions, the characterization of weak Serre subcategories (see Homology, Lemma 12.10.3), and Lemmas 46.3.10, 46.3.11, and 46.3.16. Recall that $\mathcal{P}$ is equivalent to the category $PMod((Aff/\operatorname{Spec}(A))_\tau, \mathcal{O})$. Hence (2) by Injectives,

Proposition 19.8.5. Part (3) follows from Lemma 46.4.1 and Categories, Lemma 4.24.5. Parts (4) and (5) follow from Homology, Lemmas 12.29.1 and 12.29.3. □

Let A be a ring. As in Formal Deformation Theory, Section 90.11 given an A-algebra B and an B-module N we set $B[N]$ equal to the R-algebra with underlying B-module $B \oplus N$ with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. Note that this construction is functorial in the pair (B, N) where morphism $(B, N) \to (B', N')$ is given by an A-algebra map $B \to B'$ and an B-module map $N \to N'$. In some sense the functor TF of pairs defined in the following lemma is the tangent space of F. Below we will only consider pairs (B, N) such that $B[N]$ is an object of Alg_A.

06Z8 **Lemma 46.4.3.** *Let A be a ring. Let F be a module valued functor. For every $B \in \mathrm{Ob}(Alg_A)$ and B-module N there is a canonical decomposition*

$$F(B[N]) = F(B) \oplus TF(B, N)$$

characterized by the following properties

(1) $TF(B, N) = \mathrm{Ker}(F(B[N]) \to F(B))$,
(2) *there is a B-module structure $TF(B, N)$ compatible with $B[N]$-module structure on $F(B[N])$,*
(3) *TF is a functor from the category of pairs (B, N),*
06Z9 (4) *there are canonical maps $N \otimes_B F(B) \to TF(B, N)$ inducing a transformation between functors defined on the category of pairs (B, N),*
(5) *$TF(B, 0) = 0$ and the map $TF(B, N) \to TF(B, N')$ is zero when $N \to N'$ is the zero map.*

Proof. Since $B \to B[N] \to B$ is the identity we see that $F(B) \to F(B[N])$ is a direct summand whose complement is $TF(N, B)$ as defined in (1). This construction is functorial in the pair (B, N) simply because given a morphism of pairs $(B, N) \to (B', N')$ we obtain a commutative diagram

$$\begin{array}{ccccc} B' & \longrightarrow & B'[N'] & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ B & \longrightarrow & B[N] & \longrightarrow & B \end{array}$$

in Alg_A. The B-module structure comes from the $B[N]$-module structure and the ring map $B \to B[N]$. The map in (4) is the composition

$$N \otimes_B F(B) \longrightarrow B[N] \otimes_{B[N]} F(B[N]) \longrightarrow F(B[N])$$

whose image is contained in $TF(B, N)$. (The first arrow uses the inclusions $N \to B[N]$ and $F(B) \to F(B[N])$ and the second arrow is the multiplication map.) If $N = 0$, then $B = B[N]$ hence $TF(B, 0) = 0$. If $N \to N'$ is zero then it factors as $N \to 0 \to N'$ hence the induced map is zero since $TF(B, 0) = 0$. □

Let A be a ring. Let M be an A-module. Then the module-valued functor $\underline{M}$ has tangent space $T\underline{M}$ given by the rule $T\underline{M}(B, N) = N \otimes_A M$. In particular, for B given, the functor $N \mapsto T\underline{M}(B, N)$ is additive and right exact. It turns out this also holds for injective module-valued functors.

06ZA **Lemma 46.4.4.** *Let A be a ring. Let I be an injective object of the category of module-valued functors. Then for any $B \in \mathrm{Ob}(Alg_A)$ and short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of B-modules the sequence*

$$TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$$

is exact.

Proof. We will use the results of Lemma 46.4.3 without further mention. Denote $h : Alg_A \to Sets$ the functor given by $h(C) = \mathrm{Mor}_A(B[N], C)$. Similarly for h_1 and h_2. The map $B[N] \to B[N_2]$ corresponding to the surjection $N \to N_2$ is surjective. It corresponds to a map $h_2 \to h$ such that $h_2(C) \to h(C)$ is injective for all A-algebras C. On the other hand, there are two maps $p, q : h \to h_1$, corresponding to the zero map $N_1 \to N$ and the injection $N_1 \to N$. Note that

$$h_2 \longrightarrow h \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} h_1$$

is an equalizer diagram. Denote $\mathcal{O}_h$ the module-valued functor $C \mapsto \bigoplus_{h(C)} C$. Similarly for $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$. Note that

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{O}_h, F) = F(B[N])$$

where $\mathcal{P}$ is the category of module-valued functors on Alg_A. We claim there is an equalizer diagram

$$\mathcal{O}_{h_2} \longrightarrow \mathcal{O}_h \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} \mathcal{O}_{h_1}$$

in $\mathcal{P}$. Namely, suppose that $C \in \mathrm{Ob}(Alg_A)$ and $\xi = \sum_{i=1,\ldots,n} c_i \cdot f_i$ where $c_i \in C$ and $f_i : B[N] \to C$ is an element of $\mathcal{O}_h(C)$. If $p(\xi) = q(\xi)$, then we see that

$$\sum c_i \cdot f_i \circ z = \sum c_i \cdot f_i \circ y$$

where $z, y : B[N_1] \to B[N]$ are the maps $z : (b, m_1) \mapsto (b, 0)$ and $y : (b, m_1) \mapsto (b, m_1)$. This means that for every i there exists a j such that $f_j \circ z = f_i \circ y$. Clearly, this implies that $f_i(N_1) = 0$, i.e., f_i factors through a unique map $\overline{f}_i : B[N_2] \to C$. Hence ξ is the image of $\overline{\xi} = \sum c_i \cdot \overline{f}_i$. Since I is injective, it transforms this equalizer diagram into a coequalizer diagram

$$I(B[N_1]) \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} I(B[N]) \longrightarrow I(B[N_2])$$

This diagram is compatible with the direct sum decompositions $I(B[N]) = I(B) \oplus TI(B, N)$ and $I(B[N_i]) = I(B) \oplus TI(B, N_i)$. The zero map $N \to N_1$ induces the zero map $TI(B, N) \to TI(B, N_1)$. Thus we see that the coequalizer property above means we have an exact sequence $TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$ as desired. □

06ZB **Lemma 46.4.5.** *Let A be a ring. Let F be a module-valued functor such that for any $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Then*

(1) *$TF(B, N_1 \oplus N_2) = TF(B, N_1) \oplus TF(B, N_2)$,*
(2) *there is a second functorial B-module structure on $TF(B, N)$ defined by setting $x \cdot b = TF(B, b \cdot 1_N)(x)$ for $x \in TF(B, N)$ and $b \in B$,*

06ZC (3) *the canonical map $N \otimes_B F(B) \to TF(B, N)$ of Lemma 46.4.3 is B-linear also with respect to the second B-module structure,*

06ZD (4) *given a finitely presented B-module N there is a canonical isomorphism $TF(B,B) \otimes_B N \to TF(B,N)$ where the tensor product uses the second B-module structure on $TF(B,B)$.*

Proof. We will use the results of Lemma 46.4.3 without further mention. The maps $N_1 \to N_1 \oplus N_2$ and $N_2 \to N_1 \oplus N_2$ give a map $TF(B,N_1) \oplus TF(B,N_2) \to TF(B, N_1 \oplus N_2)$ which is injective since the maps $N_1 \oplus N_2 \to N_1$ and $N_1 \oplus N_2 \to N_2$ induce an inverse. Since TF is right exact we see that $TF(B,N_1) \to TF(B, N_1 \oplus N_2) \to TF(B,N_2) \to 0$ is exact. Hence $TF(B,N_1) \oplus TF(B,N_2) \to TF(B, N_1 \oplus N_2)$ is an isomorphism. This proves (1).

To see (2) the only thing we need to show is that $x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$. (Associativity and additivity are clear.) To see this consider

$$N \xrightarrow{(b_1, b_2)} N \oplus N \xrightarrow{+} N$$

and apply $TF(B, -)$.

Part (3) follows immediately from the fact that $N \otimes_B F(B) \to TF(B,N)$ is functorial in the pair (B, N).

Suppose N is a finitely presented B-module. Choose a presentation $B^{\oplus m} \to B^{\oplus n} \to N \to 0$. This gives an exact sequence

$$TF(B, B^{\oplus m}) \to TF(B, B^{\oplus n}) \to TF(B,N) \to 0$$

by right exactness of $TF(B,-)$. By part (1) we can write $TF(B, B^{\oplus m}) = TF(B,B)^{\oplus m}$ and $TF(B, B^{\oplus n}) = TF(B,B)^{\oplus n}$. Next, suppose that $B^{\oplus m} \to B^{\oplus n}$ is given by the matrix $T = (b_{ij})$. Then the induced map $TF(B,B)^{\oplus m} \to TF(B,B)^{\oplus n}$ is given by the matrix with entries $TF(B, b_{ij} \cdot 1_B)$. This combined with right exactness of $\otimes$ proves (4). □

06ZE **Example 46.4.6.** Let F be a module-valued functor as in Lemma 46.4.5. It is not always the case that the two module structures on $TF(B,N)$ agree. Here is an example. Suppose $A = \mathbf{F}_p$ where p is a prime. Set $F(B) = B$ but with B-module structure given by $b \cdot x = b^p x$. Then $TF(B,N) = N$ with B-module structure given by $b \cdot x = b^p x$ for $x \in N$. However, the second B-module structure is given by $x \cdot b = bx$. Note that in this case the canonical map $N \otimes_B F(B) \to TF(B,N)$ is zero as raising an element $n \in B[N]$ to the pth power is zero.

In the following lemma we will frequently use the observation that if $0 \to F \to G \to H \to 0$ is an exact sequence of module-valued functors on Alg_A, then for any pair (B,N) the sequence $0 \to TF(B,N) \to TG(B,N) \to TH(B,N) \to 0$ is exact. This follows from the fact that $0 \to F(B[N]) \to G(B[N]) \to H(B[N]) \to 0$ is exact.

06ZF **Lemma 46.4.7.** *Let A be a ring. For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B,-)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A.*

1. *If $(*)$ holds for F, G then $(*)$ holds for H.*
2. *If $(*)$ holds for F, H then $(*)$ holds for G.*
3. *If $H' \to H$ is morphism of module-valued functors on Alg_A and $(*)$ holds for F, G, H, and H', then $(*)$ holds for $G \times_H H'$.*

Proof. Let B be given. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of B-modules. Part (1) follows from a diagram chase in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact horizontal rows and exact columns involving TF and TG. To prove part (2) we do a diagram chase in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

with exact horizontal rows and exact columns involving TF and TH. Part (3) follows from part (2) as $G \times_H H'$ sits in the exact sequence $0 \to F \to G \times_H H' \to H' \to 0$. □

Most of the work in this section was done in order to prove the following key vanishing result.

06ZG **Lemma 46.4.8.** *Let A be a ring. Let M, P be A-modules with P of finite presentation. Then $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Choose an injective resolution $\underline{M} \to I^\bullet$ in $\mathcal{P}$, see Lemma 46.4.2. By Derived Categories, Lemma 13.27.2 any element of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ comes from a morphism $\varphi : \underline{P} \to I^i$ with $d^i \circ \varphi = 0$. We will prove that the Yoneda extension

$$E : 0 \to \underline{M} \to I^0 \to \ldots \to I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P} \to \underline{P} \to 0$$

of $\underline{P}$ by $\underline{M}$ associated to φ is trivial, which will prove the lemma by Derived Categories, Lemma 13.27.5.

For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Recall that the module-valued functors $\underline{M}, I^n, \underline{P}$ each have property $(*)$, see Lemma 46.4.4 and the remarks preceding it. By splitting $0 \to$

$\underline{M} \to I^\bullet$ into short exact sequences we find that each of the functors $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n) \subset I^n$ has property $(*)$ by Lemma 46.4.7 and also that $I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P}$ has property $(*)$.

Thus we may assume the Yoneda extension is given as

$$E : 0 \to \underline{M} \to F_{i-1} \to \ldots \to F_0 \to \underline{P} \to 0$$

where each of the module-valued functors F_j has property $(*)$. Set $G_j(B) = TF_j(B, B)$ viewed as a B-module via the *second* B-module structure defined in Lemma 46.4.5. Since TF_j is a functor on pairs we see that G_j is a module-valued functor on Alg_A. Moreover, since E is an exact sequence the sequence $G_{j+1} \to G_j \to G_{j-1}$ is exact (see remark preceding Lemma 46.4.7). Observe that $T\underline{M}(B, B) = M \otimes_A B = \underline{M}(B)$ and that the two B-module structures agree on this. Thus we obtain a Yoneda extension

$$E' : 0 \to \underline{M} \to G_{i-1} \to \ldots \to G_0 \to \underline{P} \to 0$$

Moreover, the canonical maps

$$F_j(B) = B \otimes_B F_j(B) \longrightarrow TF_j(B, B) = G_j(B)$$

of Lemma 46.4.3 (4) are B-linear by Lemma 46.4.5 (3) and functorial in B. Hence a map

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \underline{M} & \longrightarrow & F_{i-1} & \longrightarrow & \ldots & \longrightarrow & F_0 & \longrightarrow & \underline{P} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \underline{M} & \longrightarrow & G_{i-1} & \longrightarrow & \ldots & \longrightarrow & G_0 & \longrightarrow & \underline{P} & \longrightarrow & 0
\end{array}$$

of Yoneda extensions. In particular we see that E and E' have the same class in $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ by the lemma on Yoneda Exts mentioned above. Finally, let N be a A-module of finite presentation. Then we see that

$$0 \to T\underline{M}(A, N) \to TF_{i-1}(A, N) \to \ldots \to TF_0(A, N) \to T\underline{P}(A, N) \to 0$$

is exact. By Lemma 46.4.5 (4) with $B = A$ this translates into the exactness of the sequence of A-modules

$$0 \to M \otimes_A N \to G_{i-1}(A) \otimes_A N \to \ldots \to G_0(A) \otimes_A N \to P \otimes_A N \to 0$$

Hence the sequence of A-modules $0 \to M \to G_{i-1}(A) \to \ldots \to G_0(A) \to P \to 0$ is universally exact, in the sense that it remains exact on tensoring with any finitely presented A-module N. Let $K = \mathrm{Ker}(G_0(A) \to P)$ so that we have exact sequences

$$0 \to K \to G_0(A) \to P \to 0 \quad \text{and} \quad G_2(A) \to G_1(A) \to K \to 0$$

Tensoring the second sequence with N we obtain that $K \otimes_A N = \mathrm{Coker}(G_2(A) \otimes_A N \to G_1(A) \otimes_A N)$. Exactness of $G_2(A) \otimes_A N \to G_1(A) \otimes_A N \to G_0(A) \otimes_A N$ then implies that $K \otimes_A N \to G_0(A) \otimes_A N$ is injective. By Algebra, Theorem 10.82.3 this means that the A-module extension $0 \to K \to G_0(A) \to P \to 0$ is exact, and because P is assumed of finite presentation this means the sequence is split, see Algebra, Lemma 10.82.4. Any splitting $P \to G_0(A)$ defines a map $\underline{P} \to G_0$ which splits the surjection $G_0 \to \underline{P}$. Thus the Yoneda extension E' is equivalent to the trivial Yoneda extension and we win. $\square$

06ZH **Lemma 46.4.9.** *Let A be a ring. Let M be an A-module. Let L be a linearly adequate functor on Alg_A. Then $\mathrm{Ext}^i_{\mathcal{P}}(L, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Since L is linearly adequate there exists an exact sequence

$$0 \to L \to \underline{A^{\oplus m}} \to \underline{A^{\oplus n}} \to \underline{P} \to 0$$

Here $P = \mathrm{Coker}(A^{\oplus m} \to A^{\oplus n})$ is the cokernel of the map of finite free A-modules which is given by the definition of linearly adequate functors. By Lemma 46.4.8 we have the vanishing of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ and $\mathrm{Ext}^i_{\mathcal{P}}(\underline{A}, \underline{M})$ for $i > 0$. Let $K = \mathrm{Ker}(\underline{A^{\oplus n}} \to \underline{P})$. By the long exact sequence of Ext groups associated to the exact sequence $0 \to K \to \underline{A^{\oplus n}} \to \underline{P} \to 0$ we conclude that $\mathrm{Ext}^i_{\mathcal{P}}(K, \underline{M}) = 0$ for $i > 0$. Repeating with the sequence $0 \to L \to \underline{A^{\oplus m}} \to K \to 0$ we win. □

06ZI **Lemma 46.4.10.** *With notation as in Lemma 46.4.2 we have $R^pQ(F) = 0$ for all $p > 0$ and any adequate functor F.*

Proof. Choose an exact sequence $0 \to F \to \underline{M^0} \to \underline{M^1}$. Set $M^2 = \mathrm{Coker}(M^0 \to M^1)$ so that $0 \to F \to \underline{M^0} \to \underline{M^1} \to \underline{M^2} \to 0$ is a resolution. By Derived Categories, Lemma 13.21.3 we obtain a spectral sequence

$$R^pQ(\underline{M^q}) \Rightarrow R^{p+q}Q(F)$$

Since $Q(\underline{M^q}) = \underline{M^q}$ it suffices to prove $R^pQ(\underline{M}) = 0$, $p > 0$ for any A-module M.

Choose an injective resolution $\underline{M} \to I^\bullet$ in the category $\mathcal{P}$. Suppose that $R^iQ(\underline{M})$ is nonzero. Then $\mathrm{Ker}(Q(I^i) \to Q(I^{i+1}))$ is strictly bigger than the image of $Q(I^{i-1}) \to Q(I^i)$. Hence by Lemma 46.3.6 there exists a linearly adequate functor L and a map $\varphi : L \to Q(I^i)$ mapping into the kernel of $Q(I^i) \to Q(I^{i+1})$ which does not factor through the image of $Q(I^{i-1}) \to Q(I^i)$. Because Q is a left adjoint to the inclusion functor the map φ corresponds to a map $\varphi' : L \to I^i$ with the same properties. Thus φ' gives a nonzero element of $\mathrm{Ext}^i_{\mathcal{P}}(L, \underline{M})$ contradicting Lemma 46.4.9. □

46.5. Adequate modules

06VF In Descent, Section 35.8 we have seen that quasi-coherent modules on a scheme S are the same as quasi-coherent modules on any of the big sites $(Sch/S)_\tau$ associated to S. We have seen that there are two issues with this identification:

(1) $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is not exact in general (Descent, Lemma 35.10.2), and
(2) given a quasi-compact and quasi-separated morphism $f : X \to S$ the functor f_* does not preserve quasi-coherent sheaves on the big sites in general (Descent, Proposition 35.9.4).

Part (1) means that we cannot define a triangulated subcategory of $D(\mathcal{O})$ consisting of complexes whose cohomology sheaves are quasi-coherent. Part (2) means that $Rf_*\mathcal{F}$ isn't a complex with quasi-coherent cohomology sheaves even when $\mathcal{F}$ is quasi-coherent and f is quasi-compact and quasi-separated. Moreover, the examples given in the proofs of Descent, Lemma 35.10.2 and Descent, Proposition 35.9.4 are not of a pathological nature.

In this section we discuss a slightly larger category of $\mathcal{O}$-modules on $(Sch/S)_\tau$ with contains the quasi-coherent modules, is abelian, and is preserved under f_* when f

is quasi-compact and quasi-separated. To do this, suppose that S is a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. For any affine object $U = \mathrm{Spec}(A)$ of $(Sch/S)_\tau$ we can restrict $\mathcal{F}$ to $(Aff/U)_\tau$ to get a presheaf of $\mathcal{O}$-modules on this site. The corresponding module-valued functor, see Section 46.3, will be denoted

$$F = F_{\mathcal{F},A} : Alg_A \longrightarrow Ab, \quad B \longmapsto \mathcal{F}(\mathrm{Spec}(B))$$

The assignment $\mathcal{F} \mapsto F_{\mathcal{F},A}$ is an exact functor of abelian categories.

06VG **Definition 46.5.1.** A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $(Sch/S)_\tau$ is *adequate* if there exists a τ-covering $\{\mathrm{Spec}(A_i) \to S\}_{i\in I}$ such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$.

We will see below that the category of adequate $\mathcal{O}$-modules is independent of the chosen topology τ.

06VI **Lemma 46.5.2.** *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate.*

Proof. Let $\{\mathrm{Spec}(A_i) \to S\}_{i\in I}$ be a τ-covering such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$. We can find a standard affine τ-covering $\{\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A)\}_{j=1,\ldots,m}$ such that $\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A) \to S$ factors through $\mathrm{Spec}(A_{i(j)})$ for some $i(j) \in I$. Then we see that $F_{\mathcal{F},A'_j}$ is the restriction of $F_{\mathcal{F},A_{i(j)}}$ to the category of A'_j-algebras. Hence $F_{\mathcal{F},A'_j}$ is adequate by Lemma 46.3.17. By Lemma 46.3.19 the sequence $F_{\mathcal{F},A'_j}$ corresponds to an adequate "product" functor F' over $A' = A'_1 \times \ldots \times A'_m$. As $\mathcal{F}$ is a sheaf (for the Zariski topology) this product functor F' is equal to $F_{\mathcal{F},A'}$, i.e., is the restriction of F to A'-algebras. Finally, $\{\mathrm{Spec}(A') \to \mathrm{Spec}(A)\}$ is a τ-covering. It follows from Lemma 46.3.20 that $F_{\mathcal{F},A}$ is adequate. □

06ZJ **Lemma 46.5.3.** *Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is equivalent to the category of adequate module-valued functors on Alg_A.*

Proof. Given an adequate module $\mathcal{F}$ the functor $F_{\mathcal{F},A}$ is adequate by Lemma 46.5.2. Given an adequate functor F we choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and we consider the $\mathcal{O}$-module $\mathcal{F} = \mathrm{Ker}(M^a \to N^a)$ where M^a denotes the quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on S. Note that $F = F_{\mathcal{F},A}$, in particular the module $\mathcal{F}$ is adequate by definition. We omit the proof that the constructions define mutually inverse equivalences of categories. □

06VJ **Lemma 46.5.4.** *Let $f : T \to S$ be a morphism of schemes. The pullback $f^*\mathcal{F}$ of an adequate $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_\tau$ is an adequate $\mathcal{O}$-module on $(Sch/T)_\tau$.*

Proof. The pullback map $f^* : Mod((Sch/S)_\tau, \mathcal{O}) \to Mod((Sch/T)_\tau, \mathcal{O})$ is given by restriction, i.e., $f^*\mathcal{F}(V) = \mathcal{F}(V)$ for any scheme V over T. Hence this lemma follows immediately from Lemma 46.5.2 and the definition. □

Here is a characterization of the category of adequate $\mathcal{O}$-modules. To understand the significance, consider a map $\mathcal{G} \to \mathcal{H}$ of quasi-coherent $\mathcal{O}_S$-modules on a scheme S. The cokernel of the associated map $\mathcal{G}^a \to \mathcal{H}^a$ of $\mathcal{O}$-modules is quasi-coherent because it is equal to $(\mathcal{H}/\mathcal{G})^a$. But the kernel of $\mathcal{G}^a \to \mathcal{H}^a$ in general isn't quasi-coherent. However, it is adequate.

06VK **Lemma 46.5.5.** *Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent*

(1) $\mathcal{F}$ *is adequate,*
(2) *there exists an affine open covering* $S = \bigcup S_i$ *and maps of quasi-coherent* $\mathcal{O}_{S_i}$*-modules* $\mathcal{G}_i \to \mathcal{H}_i$ *such that* $\mathcal{F}|_{(Sch/S_i)_\tau}$ *is the kernel of* $\mathcal{G}_i^a \to \mathcal{H}_i^a$
(3) *there exists a* τ*-covering* $\{S_i \to S\}_{i\in I}$ *and maps of* $\mathcal{O}_{S_i}$*-quasi-coherent modules* $\mathcal{G}_i \to \mathcal{H}_i$ *such that* $\mathcal{F}|_{(Sch/S_i)_\tau}$ *is the kernel of* $\mathcal{G}_i^a \to \mathcal{H}_i^a$*,*
(4) *there exists a* τ*-covering* $\{f_i : S_i \to S\}_{i\in I}$ *such that each* $f_i^*\mathcal{F}$ *is adequate,*
(5) *for any affine scheme* U *over* S *the restriction* $\mathcal{F}|_{(Sch/U)_\tau}$ *is the kernel of a map* $\mathcal{G}^a \to \mathcal{H}^a$ *of quasi-coherent* $\mathcal{O}_U$*-modules.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. By definition, the functor F is adequate if and only if there exists a map of A-modules $M \to N$ such that $F = \mathrm{Ker}(\underline{M} \to \underline{N})$. Combining with Lemmas 46.5.2 and 46.5.3 we see that (1) and (5) are equivalent.

It is clear that (5) implies (2) and (2) implies (3). If (3) holds then we can refine the covering $\{S_i \to S\}$ such that each $S_i = \mathrm{Spec}(A_i)$ is affine. Then we see, by the preliminary remarks of the proof, that $F_{\mathcal{F},A_i}$ is adequate. Thus $\mathcal{F}$ is adequate by definition. Hence (3) implies (1).

Finally, (4) is equivalent to (1) using Lemma 46.5.4 for one direction and that a composition of τ-coverings is a τ-covering for the other. □

Just like is true for quasi-coherent sheaves the category of adequate modules is independent of the topology.

06VL **Lemma 46.5.6.** *Let* $\mathcal{F}$ *be an adequate* $\mathcal{O}$*-module on* $(Sch/S)_\tau$*. For any surjective flat morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affines over* S *the extended Čech complex*

$$0 \to \mathcal{F}(\mathrm{Spec}(A)) \to \mathcal{F}(\mathrm{Spec}(B)) \to \mathcal{F}(\mathrm{Spec}(B \otimes_A B)) \to \ldots$$

is exact. In particular $\mathcal{F}$ *satisfies the sheaf condition for fpqc coverings, and is a sheaf of* $\mathcal{O}$*-modules on* $(Sch/S)_{fppf}$*.*

Proof. With $A \to B$ as in the lemma let $F = F_{\mathcal{F},A}$. This functor is adequate by Lemma 46.5.2. By Lemma 46.3.5 since $A \to B$, $A \to B \otimes_A B$, etc are flat we see that $F(B) = F(A) \otimes_A B$, $F(B \otimes_A B) = F(A) \otimes_A B \otimes_A B$, etc. Exactness follows from Descent, Lemma 35.3.6.

Thus $\mathcal{F}$ satisfies the sheaf condition for τ-coverings (in particular Zariski coverings) and any faithfully flat covering of an affine by an affine. Arguing as in the proofs of Descent, Lemma 35.5.1 and Descent, Proposition 35.5.2 we conclude that $\mathcal{F}$ satisfies the sheaf condition for all fpqc coverings (made out of objects of $(Sch/S)_\tau$). Details omitted. □

Lemma 46.5.6 shows in particular that for any pair of topologies τ, τ' the collection of adequate modules for the τ-topology and the τ'-topology are identical (as presheaves of modules on the underlying category Sch/S).

07AH **Definition 46.5.7.** Let S be a scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is denoted $Adeq(\mathcal{O})$ or $Adeq((Sch/S)_\tau, \mathcal{O})$. If we want to think just about the abelian category of adequate modules without choosing a topology we simply write $Adeq(S)$.

06VM **Lemma 46.5.8.** *Let* S *be a scheme. Let* $\mathcal{F}$ *be an adequate* $\mathcal{O}$*-module on* $(Sch/S)_\tau$*.*

(1) *The restriction* $\mathcal{F}|_{S_{Zar}}$ *is a quasi-coherent* $\mathcal{O}_S$*-module on the scheme* S*.*

(2) *The restriction $\mathcal{F}|_{S_{étale}}$ is the quasi-coherent module associated to $\mathcal{F}|_{S_{Zar}}$.*
(3) *For any affine scheme U over S we have $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.*
(4) *There is a canonical isomorphism*

$$H^q(S, \mathcal{F}|_{S_{Zar}}) = H^q((Sch/S)_\tau, \mathcal{F}).$$

Proof. By Lemma 46.3.5 and Lemma 46.5.2 we see that for any flat morphism of affines $U \to V$ over S we have $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. This works in particular if $U \subset V \subset S$ are affine opens of S, hence $\mathcal{F}|_{S_{Zar}}$ is quasi-coherent. Thus (1) holds.

Let $S' \to S$ be an étale morphism of schemes. Then for $U \subset S'$ affine open mapping into an affine open $V \subset S$ we see that $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ because $U \to V$ is étale, hence flat. Therefore $\mathcal{F}|_{S'_{Zar}}$ is the pullback of $\mathcal{F}|_{S_{Zar}}$. This proves (2).

We are going to apply Cohomology on Sites, Lemma 21.10.9 to the site $(Sch/S)_\tau$ with $\mathcal{B}$ the set of affine schemes over S and Cov the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Descent, Lemma 35.9.1 and Lemma 46.5.6 for the case of a covering by a single affine. Hence we conclude that $H^p(U, \mathcal{F}) = 0$ for every affine scheme U over S. This proves (3). In exactly the same way as in the proof of Descent, Proposition 35.9.3 this implies the equality of cohomologies (4). □

06VN **Remark 46.5.9.** Let S be a scheme. We have functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$. Namely, the functor $u : \mathcal{F} \mapsto \mathcal{F}^a$ comes from taking the associated $\mathcal{O}$-module which is adequate by Lemma 46.5.5. Conversely, the functor v comes from restriction $v : \mathcal{G} \mapsto \mathcal{G}|_{S_{Zar}}$, see Lemma 46.5.8. Since $\mathcal{F}^a$ can be described as the pullback of $\mathcal{F}$ under a morphism of ringed topoi $((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Descent, Remark 35.8.6 and since restriction is the pushforward we see that u and v are adjoint as follows

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(u\mathcal{F}, \mathcal{G})$$

where $\mathcal{O}$ denotes the structure sheaf on the big site. It is immediate from the description that the adjunction mapping $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for all quasi-coherent sheaves.

06VP **Lemma 46.5.10.** *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. If for every affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate, then the sheafification of $\mathcal{F}$ is an adequate $\mathcal{O}$-module.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. The sheafification $\mathcal{F}^\# = (\mathcal{F}^+)^+$, see Sites, Section 7.10. By construction

$$(\mathcal{F})^+(U) = \mathrm{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F})$$

where the colimit is over coverings in the site $(Sch/S)_\tau$. Since U is affine it suffices to take the limit over standard affine τ-coverings $\mathcal{U} = \{U_i \to U\}_{i\in I} = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_{i\in I}$ of U. Since each $A \to A_i$ and $A \to A_i \otimes_A A_j$ is flat we see that

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(\prod F(A) \otimes_A A_i \to \prod F(A) \otimes_A A_i \otimes_A A_j)$$

by Lemma 46.3.5. Since $A \to \prod A_i$ is faithfully flat we see that this always is canonically isomorphic to $F(A)$ by Descent, Lemma 35.3.6. Thus the presheaf $(\mathcal{F})^+$ has the same value as $\mathcal{F}$ on all affine schemes over S. Repeating the argument once more we deduce the same thing for $\mathcal{F}^\# = ((\mathcal{F})^+)^+$. Thus $F_{\mathcal{F},A} = F_{\mathcal{F}^\#,A}$ and we conclude that $\mathcal{F}^\#$ is adequate. □

06VQ **Lemma 46.5.11.** *Let S be a scheme.*

(1) *The category $Adeq(\mathcal{O})$ is abelian.*
(2) *The functor $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ is exact.*
(3) *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate, then $\mathcal{F}_2$ is adequate.*
(4) *The category $Adeq(\mathcal{O})$ has colimits and $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ commutes with them.*

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of adequate $\mathcal{O}$-modules. To prove (1) and (2) it suffices to show that $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ computed in $Mod((Sch/S)_\tau, \mathcal{O})$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F = F_{\mathcal{F},A}$ and $G = F_{\mathcal{G},A}$. By Lemmas 46.3.11 and 46.3.10 the kernel K and cokernel Q of the induced map $F \to G$ are adequate functors. Because the kernel is computed on the level of presheaves, we see that $K = F_{\mathcal{K},A}$ and we conclude $\mathcal{K}$ is adequate. To prove the result for the cokernel, denote $\mathcal{Q}'$ the presheaf cokernel of φ. Then $Q = F_{\mathcal{Q}',A}$ and $\mathcal{Q} = (\mathcal{Q}')^\#$. Hence $\mathcal{Q}$ is adequate by Lemma 46.5.10.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. The sequence of functors

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact, because for $V = \mathrm{Spec}(B)$ affine over U we have $H^1(V, \mathcal{F}_1) = 0$ by Lemma 46.5.8. Since F_1 and F_3 are adequate functors by Lemma 46.5.2 we see that F_2 is adequate by Lemma 46.3.16. Thus $\mathcal{F}_2$ is adequate.

Let $\mathcal{I} \to Adeq(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Denote $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ the colimit computed in $Mod((Sch/S)_\tau, \mathcal{O})$. To prove (4) it suffices to show that $\mathcal{F}$ is adequate. Let $\mathcal{F}' = \mathrm{colim}_i \mathcal{F}_i$ be the colimit computed in presheaves of $\mathcal{O}$-modules. Then $\mathcal{F} = (\mathcal{F}')^\#$. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. By Lemma 46.3.12 the functor $\mathrm{colim}_i F_i = F_{\mathcal{F}',A}$ is adequate. Lemma 46.5.10 shows that $\mathcal{F}$ is adequate. □

The following lemma tells us that the total direct image $Rf_*\mathcal{F}$ of an adequate module under a quasi-compact and quasi-separated morphism is a complex whose cohomology sheaves are adequate.

06VR **Lemma 46.5.12.** *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. First we explain how to compute the higher direct images. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then $R^i f_*\mathcal{F}$ is the ith cohomology sheaf of the complex $f_*\mathcal{I}^\bullet$. Hence $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to an object U/S of $(Sch/S)_\tau$ the module

$$\begin{aligned}
\frac{\mathrm{Ker}(f_*\mathcal{I}^i(U) \to f_*\mathcal{I}^{i+1}(U))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(U) \to f_*\mathcal{I}^i(U))} &= \frac{\mathrm{Ker}(\mathcal{I}^i(U \times_S T) \to \mathcal{I}^{i+1}(U \times_S T))}{\mathrm{Im}(\mathcal{I}^{i-1}(U \times_S T) \to \mathcal{I}^i(U \times_S T))} \\
&= H^i(U \times_S T, \mathcal{F}) \\
&= H^i((Sch/U \times_S T)_\tau, \mathcal{F}|_{(Sch/U \times_S T)_\tau}) \\
&= H^i(U \times_S T, \mathcal{F}|_{(U \times_S T)_{Zar}})
\end{aligned}$$

The first equality by Topologies, Lemma 34.7.12 (and its analogues for other topologies), the second equality by definition of cohomology of $\mathcal{F}$ over an object of $(Sch/T)_\tau$, the third equality by Cohomology on Sites, Lemma 21.7.1, and the last equality by Lemma 46.5.8. Thus by Lemma 46.5.10 it suffices to prove the claim stated in the following paragraph.

Let A be a ring. Let T be a scheme quasi-compact and quasi-separated over A. Let $\mathcal{F}$ be an adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$. For an A-algebra B set $T_B = T \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B = \mathcal{F}|_{(T_B)_{Zar}}$ the restriction of $\mathcal{F}$ to the small Zariski site of T_B. (Recall that this is a "usual" quasi-coherent sheaf on the scheme T_B, see Lemma 46.5.8.) Claim: The functor

$$B \longmapsto H^q(T_B, \mathcal{F}_B)$$

is adequate. We will prove the lemma by the usual procedure of cutting T into pieces.

Case I: T is affine. In this case the schemes T_B are all affine and $H^q(T_B, \mathcal{F}_B) = 0$ for all $q \geq 1$. The functor $B \mapsto H^0(T_B, \mathcal{F}_B)$ is adequate by Lemma 46.3.18.

Case II: T is separated. Let n be the minimal number of affines needed to cover T. We argue by induction on n. The base case is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Observe that

$$U \cap V = (V_1 \cap V_n) \cup \ldots \cup (V_{n-1} \cap V_n)$$

is also a union of $n-1$ affine opens as T is separated, see Schemes, Lemma 26.21.7. Note that for each B the base changes U_B, V_B and $(U \cap V)_B = U_B \cap V_B$ behave in the same way. Hence we see that for each B we have a long exact sequence

$$0 \to H^0(T_B, \mathcal{F}_B) \to H^0(U_B, \mathcal{F}_B) \oplus H^0(V_B, \mathcal{F}_B) \to H^0((U \cap V)_B, \mathcal{F}_B) \to H^1(T_B, \mathcal{F}_B) \to \ldots$$

functorial in B, see Cohomology, Lemma 20.8.2. By induction hypothesis the functors $B \mapsto H^q(U_B, \mathcal{F}_B)$, $B \mapsto H^q(V_B, \mathcal{F}_B)$, and $B \mapsto H^q((U \cap V)_B, \mathcal{F}_B)$ are adequate. Using Lemmas 46.3.11 and 46.3.10 we see that our functor $B \mapsto H^q(T_B, \mathcal{F}_B)$ sits in the middle of a short exact sequence whose outer terms are adequate. Thus the claim follows from Lemma 46.3.16.

Case III: General quasi-compact and quasi-separated case. The proof is again by induction on the number n of affines needed to cover T. The base case $n = 1$ is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Note that since T is quasi-separated $U \cap V$ is a quasi-compact open of an affine scheme, hence Case II applies to it. The rest of the argument proceeds in exactly the same manner as in the paragraph above and is omitted. □

46.6. Parasitic adequate modules

06ZK In this section we start comparing adequate modules and quasi-coherent modules on a scheme S. Recall that there are functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$ satisfying the adjunction

$$\mathcal{H}om_{QCoh(\mathcal{O}_S)}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{Adeq(\mathcal{O})}(u\mathcal{F}, \mathcal{G})$$

and such that $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for every quasi-coherent sheaf $\mathcal{F}$, see Remark 46.5.9. Hence u is a fully faithful embedding and we can identify $QCoh(\mathcal{O}_S)$ with a full subcategory of $Adeq(\mathcal{O})$. The functor v is exact but u is not left exact in general. The kernel of v is the subcategory of parasitic adequate modules.

In Descent, Definition 35.12.1 we give the definition of a parasitic module. For adequate modules the notion does not depend on the chosen topology.

06ZM **Lemma 46.6.1.** *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent:*

1. *$v\mathcal{F} = 0$,*
2. *$\mathcal{F}$ is parasitic,*
3. *$\mathcal{F}$ is parasitic for the τ-topology,*
4. *$\mathcal{F}(U) = 0$ for all $U \subset S$ open, and*
5. *there exists an affine open covering $S = \bigcup U_i$ such that $\mathcal{F}(U_i) = 0$ for all i.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate from the definitions. Assume (5). Suppose that $S = \bigcup U_i$ is an affine open covering such that $\mathcal{F}(U_i) = 0$ for all i. Let $V \to S$ be a flat morphism. There exists an affine open covering $V = \bigcup V_j$ such that each V_j maps into some U_i. As the morphism $V_j \to S$ is flat, also $V_j \to U_i$ is flat. Hence the corresponding ring map $A_i = \mathcal{O}(U_i) \to \mathcal{O}(V_j) = B_j$ is flat. Thus by Lemma 46.5.2 and Lemma 46.3.5 we see that $\mathcal{F}(U_i) \otimes_{A_i} B_j \to \mathcal{F}(V_j)$ is an isomorphism. Hence $\mathcal{F}(V_j) = 0$. Since $\mathcal{F}$ is a sheaf for the Zariski topology we conclude that $\mathcal{F}(V) = 0$. In this way we see that (5) implies (2).

This proves the equivalence of (2), (3), (4), and (5). As (1) is equivalent to (3) (see Remark 46.5.9) we conclude that all five conditions are equivalent. □

Let S be a scheme. The subcategory of parasitic adequate modules is a Serre subcategory of $Adeq(\mathcal{O})$. The quotient is the category of quasi-coherent modules.

06ZN **Lemma 46.6.2.** *Let S be a scheme. The subcategory $\mathcal{C} \subset Adeq(\mathcal{O})$ of parasitic adequate modules is a Serre subcategory. Moreover, the functor v induces an equivalence of categories*

$$Adeq(\mathcal{O})/\mathcal{C} = QCoh(\mathcal{O}_S).$$

Proof. The category $\mathcal{C}$ is the kernel of the exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, see Lemma 46.6.1. Hence it is a Serre subcategory by Homology, Lemma 12.10.4. By Homology, Lemma 12.10.6 we obtain an induced exact functor $\overline{v} : Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$. Because u is a right inverse to v we see right away that $\overline{v}$ is essentially surjective. We see that $\overline{v}$ is faithful by Homology, Lemma 12.10.7. Because u is a right inverse to v we finally conclude that $\overline{v}$ is fully faithful. □

06ZP **Lemma 46.6.3.** *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any parasitic adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. We have already seen in Lemma 46.5.12 that these higher direct images are adequate. Hence it suffices to show that $(R^i f_*\mathcal{F})(U_i) = 0$ for any τ-covering $\{U_i \to S\}$ open. And $R^i f_*\mathcal{F}$ is parasitic by Descent, Lemma 35.12.3. □

46.7. Derived categories of adequate modules, I

06VS Let S be a scheme. We continue the discussion started in Section 46.6. The exact functor v induces a functor

$$D(Adeq(\mathcal{O})) \longrightarrow D(QCoh(\mathcal{O}_S))$$

and similarly for bounded versions.

06ZQ **Lemma 46.7.1.** *Let S be a scheme. Let $\mathcal{C} \subset Adeq(\mathcal{O})$ denote the full subcategory consisting of parasitic adequate modules. Then*

$$D(Adeq(\mathcal{O}))/D_{\mathcal{C}}(Adeq(\mathcal{O})) = D(QCoh(\mathcal{O}_S))$$

and similarly for the bounded versions.

Proof. Follows immediately from Derived Categories, Lemma 13.17.3. □

Next, we look for a description the other way around by looking at the functors

$$K^+(QCoh(\mathcal{O}_S)) \longrightarrow K^+(Adeq(\mathcal{O})) \longrightarrow D^+(Adeq(\mathcal{O})) \longrightarrow D^+(QCoh(\mathcal{O}_S)).$$

In some cases the derived category of adequate modules is a localization of the homotopy category of complexes of quasi-coherent modules at universal quasi-isomorphisms. Let S be a scheme. A map of complexes $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules is said to be a *universal quasi-isomorphism* if for every morphism of schemes $f : T \to S$ the pullback $f^*\varphi$ is a quasi-isomorphism.

06ZR **Lemma 46.7.2.** *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The bounded below derived category $D^+(Adeq(\mathcal{O}))$ is the localization of $K^+(QCoh(\mathcal{O}_U))$ at the multiplicative subset of universal quasi-isomorphisms.*

Proof. If $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of quasi-coherent $\mathcal{O}_U$-modules, then $u\varphi : u\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ is a quasi-isomorphism if and only if φ is a universal quasi-isomorphism. Hence the collection S of universal quasi-isomorphisms is a saturated multiplicative system compatible with the triangulated structure by Derived Categories, Lemma 13.5.4. Hence $S^{-1}K^+(QCoh(\mathcal{O}_U))$ exists and is a triangulated category, see Derived Categories, Proposition 13.5.6. We obtain a canonical functor $can : S^{-1}K^+(QCoh(\mathcal{O}_U)) \to D^+(Adeq(\mathcal{O}))$ by Derived Categories, Lemma 13.5.7.

Note that, almost by definition, every adequate module on U has an embedding into a quasi-coherent sheaf, see Lemma 46.5.5. Hence by Derived Categories, Lemma 13.15.5 given $\mathcal{F}^\bullet \in \mathrm{Ob}(K^+(Adeq(\mathcal{O})))$ there exists a quasi-isomorphism $\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet \in \mathrm{Ob}(K^+(QCoh(\mathcal{O}_U)))$. This proves that *can* is essentially surjective.

Similarly, suppose that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below complexes of quasi-coherent $\mathcal{O}_U$-modules. A morphism in $D^+(Adeq(\mathcal{O}))$ between these consists of a pair $f : u\mathcal{F}^\bullet \to \mathcal{H}^\bullet$ and $s : u\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ where s is a quasi-isomorphism. Pick a quasi-isomorphism $s' : \mathcal{H}^\bullet \to u\mathcal{E}^\bullet$. Then we see that $s' \circ f : \mathcal{F} \to \mathcal{E}^\bullet$ and the universal quasi-isomorphism $s' \circ s : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ give a morphism in $S^{-1}K^+(QCoh(\mathcal{O}_U))$ mapping to the given morphism. This proves the "fully" part of full faithfulness. Faithfulness is proved similarly. □

06ZS **Lemma 46.7.3.** *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The inclusion functor*

$$Adeq(\mathcal{O}) \to Mod((Sch/U)_\tau, \mathcal{O})$$

has a right adjoint A[1]. Moreover, the adjunction mapping $A(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every adequate module $\mathcal{F}$.

[1]This is the "adequator".

Proof. By Topologies, Lemma 34.7.11 (and similarly for the other topologies) we may work with $\mathcal{O}$-modules on $(Aff/U)_\tau$. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 46.4.1. We have the commutative diagram

06ZT (46.7.3.1)
$$\begin{array}{ccccc} Adeq(\mathcal{O}) & \xrightarrow[k]{} & Mod((Aff/U)_\tau, \mathcal{O}) & \xrightarrow[j]{} & PMod((Aff/U)_\tau, \mathcal{O}) \\ \| & & & & \| \\ \mathcal{A} & & \xrightarrow{\quad i \quad} & & \mathcal{P} \end{array}$$

The left vertical equality is Lemma 46.5.3 and the right vertical equality was explained in Section 46.3. Define $A(\mathcal{F}) = Q(j(\mathcal{F}))$. Since j is fully faithful it follows immediately that A is a right adjoint of the inclusion functor k. Also, since k is fully faithful too, the final assertion follows formally. □

The functor A is a right adjoint hence left exact. Since the inclusion functor is exact, see Lemma 46.5.11 we conclude that A transforms injectives into injectives, and that the category $Adeq(\mathcal{O})$ has enough injectives, see Homology, Lemma 12.29.3 and Injectives, Theorem 19.8.4. This also follows from the equivalence in (46.7.3.1) and Lemma 46.4.2.

06ZU **Lemma 46.7.4.** *Let $U = \mathrm{Spec}(A)$ be an affine scheme. For any object $\mathcal{F}$ of $Adeq(\mathcal{O})$ we have $R^pA(\mathcal{F}) = 0$ for all $p > 0$ where A is as in Lemma 46.7.3.*

Proof. With notation as in the proof of Lemma 46.7.3 choose an injective resolution $k(\mathcal{F}) \to \mathcal{I}^\bullet$ in the category of $\mathcal{O}$-modules on $(Aff/U)_\tau$. By Cohomology on Sites, Lemmas 21.12.2 and Lemma 46.5.8 the complex $j(\mathcal{I}^\bullet)$ is exact. On the other hand, each $j(\mathcal{I}^n)$ is an injective object of the category of presheaves of modules by Cohomology on Sites, Lemma 21.12.1. It follows that $R^pA(\mathcal{F}) = R^pQ(j(k(\mathcal{F})))$. Hence the result now follows from Lemma 46.4.10. □

Let S be a scheme. By the discussion in Section 46.5 the embedding $Adeq(\mathcal{O}) \subset Mod((Sch/S)_\tau, \mathcal{O})$ exhibits $Adeq(\mathcal{O})$ as a weak Serre subcategory of the category of all $\mathcal{O}$-modules. Denote

$$D_{Adeq}(\mathcal{O}) \subset D(\mathcal{O}) = D(Mod((Sch/S)_\tau, \mathcal{O}))$$

the triangulated subcategory of complexes whose cohomology sheaves are adequate, see Derived Categories, Section 13.17. We obtain a canonical functor

$$D(Adeq(\mathcal{O})) \longrightarrow D_{Adeq}(\mathcal{O})$$

see Derived Categories, Equation (13.17.1.1).

06ZV **Lemma 46.7.5.** *If $U = \mathrm{Spec}(A)$ is an affine scheme, then the bounded below version*

06VV (46.7.5.1)
$$D^+(Adeq(\mathcal{O})) \longrightarrow D^+_{Adeq}(\mathcal{O})$$

of the functor above is an equivalence.

Proof. Let $A : Mod(\mathcal{O}) \to Adeq(\mathcal{O})$ be the right adjoint to the inclusion functor constructed in Lemma 46.7.3. Since A is left exact and since $Mod(\mathcal{O})$ has enough injectives, A has a right derived functor $RA : D^+_{Adeq}(\mathcal{O}) \to D^+(Adeq(\mathcal{O}))$. We claim that RA is a quasi-inverse to (46.7.5.1). To see this the key fact is that if $\mathcal{F}$ is an

adequate module, then the adjunction map $\mathcal{F} \to RA(\mathcal{F})$ is a quasi-isomorphism by Lemma 46.7.4.

Namely, to prove the lemma in full it suffices to show:

(1) Given $\mathcal{F}^\bullet \in K^+(Adeq(\mathcal{O}))$ the canonical map $\mathcal{F}^\bullet \to RA(\mathcal{F}^\bullet)$ is a quasi-isomorphism, and
(2) given $\mathcal{G}^\bullet \in K^+(Mod(\mathcal{O}))$ the canonical map $RA(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Both (1) and (2) follow from the key fact via a spectral sequence argument using one of the spectral sequences of Derived Categories, Lemma 13.21.3. Some details omitted. □

06ZW **Lemma 46.7.6.** *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be adequate $\mathcal{O}$-modules. For any $i \geq 0$ the natural map*

$$\mathrm{Ext}^i_{Adeq(\mathcal{O})}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^i_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. By definition these ext groups are computed as hom sets in the derived category. Hence this follows immediately from Lemma 46.7.5. □

46.8. Pure extensions

06ZX We want to characterize extensions of quasi-coherent sheaves on the big site of an affine schemes in terms of algebra. To do this we introduce the following notion.

06ZY **Definition 46.8.1.** Let A be a ring.

(1) An A-module P is said to be *pure projective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(P, K) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$ is exact.
(2) An A-module I is said to be *pure injective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(N, I) \to \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(K, I) \to 0$ is exact.

Let's characterize pure projectives.

06ZZ **Lemma 46.8.2.** *Let A be a ring.*

(1) *A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented A-modules.*
(2) *For any module M there exists a universally exact sequence $0 \to N \to P \to M \to 0$ with P pure projective.*

Proof. First note that a finitely presented A-module is pure projective by Algebra, Theorem 10.82.3. Hence a direct summand of a direct sum of finitely presented A-modules is indeed pure projective. Let M be any A-module. Write $M = \mathrm{colim}_{i \in I} P_i$ as a filtered colimit of finitely presented A-modules. Consider the sequence

$$0 \to N \to \bigoplus P_i \to M \to 0.$$

For any finitely presented A-module P the map $\mathrm{Hom}_A(P, \bigoplus P_i) \to \mathrm{Hom}_A(P, M)$ is surjective, as any map $P \to M$ factors through some P_i. Hence by Algebra, Theorem 10.82.3 this sequence is universally exact. This proves (2). If now M is pure projective, then the sequence is split and we see that M is a direct summand of $\bigoplus P_i$. □

Let's characterize pure injectives.

0700 **Lemma 46.8.3.** *Let A be a ring. For any A-module M set $M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$.*

(1) *For any A-module M the A-module $M^\vee$ is pure injective.*
(2) *An A-module I is pure injective if and only if the map $I \to (I^\vee)^\vee$ splits.*
(3) *For any module M there exists a universally exact sequence $0 \to M \to I \to N \to 0$ with I pure injective.*

Proof. We will use the properties of the functor $M \mapsto M^\vee$ found in More on Algebra, Section 15.55 without further mention. Part (1) holds because $\mathrm{Hom}_A(N, M^\vee) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_A M, \mathbf{Q}/\mathbf{Z})$ and because $\mathbf{Q}/\mathbf{Z}$ is injective in the category of abelian groups. Hence if $I \to (I^\vee)^\vee$ is split, then I is pure injective. We claim that for any A-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is universally injective. To see this note that $ev^\vee : ((M^\vee)^\vee)^\vee \to M^\vee$ has a right inverse, namely $ev' : M^\vee \to ((M^\vee)^\vee)^\vee$. Then for any A-module N applying the exact faithful functor $^\vee$ to the map $N \otimes_A M \to N \otimes_A (M^\vee)^\vee$ gives

$$\mathrm{Hom}_A(N, ((M^\vee)^\vee)^\vee) = \left(N \otimes_A (M^\vee)^\vee\right)^\vee \to \left(N \otimes_A M\right)^\vee = \mathrm{Hom}_A(N, M^\vee)$$

which is surjective by the existence of the right inverse. The claim follows. The claim implies (3) and the necessity of the condition in (2). □

Before we continue we make the following observation which we will use frequently in the rest of this section.

0701 **Lemma 46.8.4.** *Let A be a ring.*

(1) *Let $L \to M \to N$ be a universally exact sequence of A-modules. Let $K = \mathrm{Im}(M \to N)$. Then $K \to N$ is universally injective.*
(2) *Any universally exact complex can be split into universally exact short exact sequences.*

Proof. Proof of (1). For any A-module T the sequence $L \otimes_A T \to M \otimes_A T \to K \otimes_A T \to 0$ is exact by right exactness of $\otimes$. By assumption the sequence $L \otimes_A T \to M \otimes_A T \to N \otimes_A T$ is exact. Combined this shows that $K \otimes_A T \to N \otimes_A T$ is injective.

Part (2) means the following: Suppose that $M^\bullet$ is a universally exact complex of A-modules. Set $K^i = \mathrm{Ker}(d^i) \subset M^i$. Then the short exact sequences $0 \to K^i \to M^i \to K^{i+1} \to 0$ are universally exact. This follows immediately from part (1). □

0702 **Definition 46.8.5.** Let A be a ring. Let M be an A-module.

(1) A *pure projective resolution* $P_\bullet \to M$ is a universally exact sequence
$$\ldots \to P_1 \to P_0 \to M \to 0$$
with each P_i pure projective.
(2) A *pure injective resolution* $M \to I^\bullet$ is a universally exact sequence
$$0 \to M \to I^0 \to I^1 \to \ldots$$
with each I^i pure injective.

These resolutions satisfy the usual uniqueness properties among the class of all universally exact left or right resolutions.

0703 **Lemma 46.8.6.** *Let A be a ring.*

(1) *Any A-module has a pure projective resolution.*

Let $M \to N$ be a map of A-modules. Let $P_\bullet \to M$ be a pure projective resolution and let $N_\bullet \to N$ be a universally exact resolution.

(2) *There exists a map of complexes $P_\bullet \to N_\bullet$ inducing the given map*

$$M = \mathrm{Coker}(P_1 \to P_0) \to \mathrm{Coker}(N_1 \to N_0) = N$$

(3) *two maps $\alpha, \beta : P_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Part (1) follows immediately from Lemma 46.8.2. Before we prove (2) and (3) note that by Lemma 46.8.4 we can split the universally exact complex $N_\bullet \to N \to 0$ into universally exact short exact sequences $0 \to K_0 \to N_0 \to N \to 0$ and $0 \to K_i \to N_i \to K_{i-1} \to 0$.

Proof of (2). Because P_0 is pure projective we can find a map $P_0 \to N_0$ lifting the map $P_0 \to M \to N$. We obtain an induced map $P_1 \to F_0 \to N_0$ which ends up in K_0. Since P_1 is pure projective we may lift this to a map $P_1 \to N_1$. This in turn induces a map $P_2 \to P_1 \to N_1$ which maps to zero into N_0, i.e., into K_1. Hence we may lift to get a map $P_2 \to N_2$. Repeat.

Proof of (3). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : P_0 \to N_0$ is contained in K_0. Hence we may lift γ_0 to a map $h_0 : P_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{P,1} : P_1 \to N_1$. By our choice of h_0 we see that the image of γ_1' is contained in K_1. Since P_1 is pure projective may lift γ_1' to a map $h_1 : P_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

0704 **Lemma 46.8.7.** *Let A be a ring.*

(1) *Any A-module has a pure injective resolution.*

Let $M \to N$ be a map of A-modules. Let $M \to M^\bullet$ be a universally exact resolution and let $N \to I^\bullet$ be a pure injective resolution.

(2) *There exists a map of complexes $M^\bullet \to I^\bullet$ inducing the given map*

$$M = \mathrm{Ker}(M^0 \to M^1) \to \mathrm{Ker}(I^0 \to I^1) = N$$

(3) *two maps $\alpha, \beta : M^\bullet \to I^\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. This lemma is dual to Lemma 46.8.6. The proof is identical, except one has to reverse all the arrows. □

Using the material above we can define pure extension groups as follows. Let A be a ring and let M, N be A-modules. Choose a pure injective resolution $N \to I^\bullet$. By Lemma 46.8.7 the complex

$$\mathrm{Hom}_A(M, I^\bullet)$$

is well defined up to homotopy. Hence its ith cohomology module is a well defined invariant of M and N.

0705 **Definition 46.8.8.** Let A be a ring and let M, N be A-modules. The ith *pure extension module* $\mathrm{Pext}^i_A(M, N)$ is the ith cohomology module of the complex $\mathrm{Hom}_A(M, I^\bullet)$ where $I^\bullet$ is a pure injective resolution of N.

Warning: It is not true that an exact sequence of A-modules gives rise to a long exact sequence of pure extensions groups. (You need a universally exact sequence for this.) We collect some facts which are obvious from the material above.

0706 **Lemma 46.8.9.** *Let A be a ring.*

(1) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever N is pure injective,*
(2) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever M is pure projective, in particular if M is an A-module of finite presentation,*
(3) *$Pext^i_A(M, N)$ is also the ith cohomology module of the complex $\mathrm{Hom}_A(P_\bullet, N)$ where $P_\bullet$ is a pure projective resolution of M.*

Proof. To see (3) consider the double complex

$$A^{\bullet,\bullet} = \mathrm{Hom}_A(P_\bullet, I^\bullet)$$

Each of its rows is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(M, I^q)$. Each of its columns is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(P_p, N)$. Hence the two spectral sequences associated to this complex in Homology, Section 12.25 degenerate, giving the equality. □

46.9. Higher exts of quasi-coherent sheaves on the big site

0707 It turns out that the module-valued functor $\underline{I}$ associated to a pure injective module I gives rise to an injective object in the category of adequate functors on Alg_A. Warning: It is not true that a pure projective module gives rise to a projective object in the category of adequate functors. We do have plenty of projective objects, namely, the linearly adequate functors.

0708 **Lemma 46.9.1.** *Let A be a ring. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. The injective objects of $\mathcal{A}$ are exactly the functors $\underline{I}$ where I is a pure injective A-module.*

Proof. Let I be an injective object of $\mathcal{A}$. Choose an embedding $I \to \underline{M}$ for some A-module M. As I is injective we see that $\underline{M} = I \oplus F$ for some module-valued functor F. Then $M = I(A) \oplus F(A)$ and it follows that $I = \underline{I(A)}$. Thus we see that any injective object is of the form $\underline{I}$ for some A-module I. It is clear that the module I has to be pure injective since any universally exact sequence $0 \to M \to N \to L \to 0$ gives rise to an exact sequence $0 \to \underline{M} \to \underline{N} \to \underline{L} \to 0$ of $\mathcal{A}$.

Finally, suppose that I is a pure injective A-module. Choose an embedding $\underline{I} \to J$ into an injective object of $\mathcal{A}$ (see Lemma 46.4.2). We have seen above that $J = \underline{I'}$ for some A-module I' which is pure injective. As $\underline{I} \to \underline{I'}$ is injective the map $I \to I'$ is universally injective. By assumption on I it splits. Hence $\underline{I}$ is a summand of $J = \underline{I'}$ whence an injective object of the category $\mathcal{A}$. □

Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M be an A-module. We will use the notation M^a to denote the quasi-coherent sheaf of $\mathcal{O}$-modules on $(Sch/U)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on U. Now we have all the notation in place to formulate the following lemma.

0709 **Lemma 46.9.2.** *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M, N be A-modules. For all i we have a canonical isomorphism*

$$\mathrm{Ext}^i_{Mod(\mathcal{O})}(M^a, N^a) = Pext^i_A(M, N)$$

functorial in M and N.

Proof. Let us construct a canonical arrow from right to left. Namely, if $N \to I^\bullet$ is a pure injective resolution, then $M^a \to (I^\bullet)^a$ is an exact complex of (adequate) $\mathcal{O}$-modules. Hence any element of $\mathrm{Pext}^i_A(M, N)$ gives rise to a map $N^a \to M^a[i]$ in $D(\mathcal{O})$, i.e., an element of the group on the left.

To prove this map is an isomorphism, note that we may replace $\mathrm{Ext}^i_{Mod(\mathcal{O})}(M^a, N^a)$ by $\mathrm{Ext}^i_{Adeq(\mathcal{O})}(M^a, N^a)$, see Lemma 46.7.6. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. We have seen that $\mathcal{A}$ is equivalent to $Adeq(\mathcal{O})$, see Lemma 46.5.3; see also the proof of Lemma 46.7.3. Hence now it suffices to prove that

$$\mathrm{Ext}^i_{\mathcal{A}}(\underline{M}, \underline{N}) = \mathrm{Pext}^i_A(M, N)$$

However, this is clear from Lemma 46.9.1 as a pure injective resolution $N \to I^\bullet$ exactly corresponds to an injective resolution of $\underline{N}$ in $\mathcal{A}$. □

46.10. Derived categories of adequate modules, II

070T Let S be a scheme. Denote $\mathcal{O}_S$ the structure sheaf of S and $\mathcal{O}$ the structure sheaf of the big site $(Sch/S)_\tau$. In Descent, Remark 35.8.4 we constructed a morphism of ringed sites

070U (46.10.0.1) $$f : ((Sch/S)_\tau, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_S).$$

In the previous sections have seen that the functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_S)$ transforms adequate sheaves into quasi-coherent sheaves, and induces an exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, and in fact that $f_* = v$ induces an equivalence $Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$ where $\mathcal{C}$ is the subcategory of parasitic adequate modules. Moreover, the functor f^* transforms quasi-coherent modules into adequate modules, and induces a functor $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ which is a left adjoint to v.

There is a very similar relationship between $D_{Adeq}(\mathcal{O})$ and $D_{QCoh}(S)$. First we explain why the category $D_{Adeq}(\mathcal{O})$ is independent of the chosen topology.

070V **Remark 46.10.1.** Let S be a scheme. Let $\tau, \tau' \in \{Zar, étale, smooth, syntomic, fppf\}$. Denote $\mathcal{O}_\tau$, resp. $\mathcal{O}_{\tau'}$ the structure sheaf $\mathcal{O}$ viewed as a sheaf on $(Sch/S)_\tau$, resp. $(Sch/S)_{\tau'}$. Then $D_{Adeq}(\mathcal{O}_\tau)$ and $D_{Adeq}(\mathcal{O}_{\tau'})$ are canonically isomorphic. This follows from Cohomology on Sites, Lemma 21.29.1. Namely, assume τ is stronger than the topology τ', let $\mathcal{C} = (Sch/S)_{fppf}$, and let $\mathcal{B}$ the collection of affine schemes over S. Assumptions (1) and (2) we've seen above. Assumption (3) is clear and assumption (4) follows from Lemma 46.5.8.

070W **Remark 46.10.2.** Let S be a scheme. The morphism f see (46.10.0.1) induces adjoint functors $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$ and $Lf^* : D_{QCoh}(S) \to D_{Adeq}(\mathcal{O})$. Moreover $Rf_*Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$.

We sketch the proof. By Remark 46.10.1 we may assume the topology τ is the Zariski topology. We will use the existence of the unbounded total derived functors Lf^* and Rf_* on $\mathcal{O}$-modules and their adjointness, see Cohomology on Sites, Lemma 21.19.1. In this case f_* is just the restriction to the subcategory S_{Zar} of $(Sch/S)_{Zar}$. Hence it is clear that $Rf_* = f_*$ induces $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$. Suppose that $\mathcal{G}^\bullet$ is an object of $D_{QCoh}(S)$. We may choose a system $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ of

bounded above complexes of flat $\mathcal{O}_S$-modules whose transition maps are termwise split injectives and a diagram

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the properties (1), (2), (3) listed in Derived Categories, Lemma 13.29.1 where $\mathcal{P}$ is the collection of flat $\mathcal{O}_S$-modules. Then $Lf^*\mathcal{G}^\bullet$ is computed by colim $f^*\mathcal{K}_n^\bullet$, see Cohomology on Sites, Lemmas 21.18.1 and 21.18.2 (note that our sites have enough points by Étale Cohomology, Lemma 59.30.1). We have to see that $H^i(Lf^*\mathcal{G}^\bullet) = \operatorname{colim} H^i(f^*\mathcal{K}_n^\bullet)$ is adequate for each i. By Lemma 46.5.11 we conclude that it suffices to show that each $H^i(f^*\mathcal{K}_n^\bullet)$ is adequate.

The adequacy of $H^i(f^*\mathcal{K}_n^\bullet)$ is local on S, hence we may assume that $S = \operatorname{Spec}(A)$ is affine. Because S is affine $D_{QCoh}(S) = D(QCoh(\mathcal{O}_S))$, see the discussion in Derived Categories of Schemes, Section 36.3. Hence there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}_n^\bullet$ where $\mathcal{F}^\bullet$ is a bounded above complex of flat quasi-coherent modules. Then $f^*\mathcal{F}^\bullet \to f^*\mathcal{K}_n^\bullet$ is a quasi-isomorphism, and the cohomology sheaves of $f^*\mathcal{F}^\bullet$ are adequate.

The final assertion $Rf_*Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$ follows from the explicit description of the functors above. (In plain English: if $\mathcal{F}$ is quasi-coherent and $p > 0$, then $L_pf^*\mathcal{F}$ is a parasitic adequate module.)

070X **Remark 46.10.3.** Remark 46.10.2 above implies we have an equivalence of derived categories

$$D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O}) \longrightarrow D_{QCoh}(S)$$

where $\mathcal{C}$ is the category of parasitic adequate modules. Namely, it is clear that $D_{\mathcal{C}}(\mathcal{O})$ is the kernel of Rf_*, hence a functor as indicated. For any object X of $D_{Adeq}(\mathcal{O})$ the map $Lf^*Rf_*X \to X$ maps to a quasi-isomorphism in $D_{QCoh}(S)$, hence $Lf^*Rf_*X \to X$ is an isomorphism in $D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})$. Finally, for X, Y objects of $D_{Adeq}(\mathcal{O})$ the map

$$Rf_* : \mathrm{Hom}_{D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})}(X, Y) \to \mathrm{Hom}_{D_{QCoh}(S)}(Rf_*X, Rf_*Y)$$

is bijective as Lf^* gives an inverse (by the remarks above).

46.11. Other chapters

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CHAPTER 47

Dualizing Complexes

08XG

47.1. Introduction

08XH In this chapter we discuss dualizing complexes in commutative algebra. A reference is [**Har66**].

We begin with a discussion of essential surjections and essential injections, projective covers, injective hulls, duality for Artinian rings, and study injective hulls of residue fields, leading quickly to a proof of Matlis duality. See Sections 47.2, 47.3, 47.4, 47.5, 47.6, and 47.7 and Proposition 47.7.8.

This is followed by three sections discussing local cohomology in great generality, see Sections 47.8, 47.9, and 47.10. We apply some of this to a discussion of depth in Section 47.11. In another application we show how, given a finitely generated ideal I of a ring A, the "I-complete" and "I-torsion" objects of the derived category of A are equivalent, see Section 47.12. To learn more about local cohomology, for example the finiteness theorem (which relies on local duality – see below) please visit Local Cohomology, Section 51.1.

The bulk of this chapter is devoted to duality for a ring map and dualizing complexes. See Sections 47.13, 47.14, 47.15, 47.16, 47.17, 47.18, 47.19, 47.20, 47.21, 47.22, and 47.23. The key definition is that of a dualizing complex $\omega_A^\bullet$ over a Noetherian ring A as an object $\omega_A^\bullet \in D^+(A)$ whose cohomology modules $H^i(\omega_A^\bullet)$ are finite A-modules, which has finite injective dimension, and is such that the map

$$A \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$$

is a quasi-isomorphism. After establishing some elementary properties of dualizing complexes, we show a dualizing complex gives rise to a dimension function. Next, we prove Grothendieck's local duality theorem. After briefly discussing dualizing modules and Cohen-Macaulay rings, we introduce Gorenstein rings and we show many familiar Noetherian rings have dualizing complexes. In a last section we apply the material to show there is a good theory of Noetherian local rings whose formal fibres are Gorenstein or local complete intersections.

In the last few sections, we describe an algebraic construction of the "upper shriek functors" used in algebraic geometry, for example in the book [**Har66**]. This topic is continued in the chapter on duality for schemes. See Duality for Schemes, Section 48.1.

47.2. Essential surjections and injections

08XI We will mostly work in categories of modules, but we may as well make the definition in general.

08XJ **Definition 47.2.1.** Let $\mathcal{A}$ be an abelian category.

(1) An injection $A \subset B$ of $\mathcal{A}$ is *essential*, or we say that B is an *essential extension of* A, if every nonzero subobject $B' \subset B$ has nonzero intersection with A.
(2) A surjection $f : A \to B$ of $\mathcal{A}$ is *essential* if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

08XK **Lemma 47.2.2.** *Let $\mathcal{A}$ be an abelian category.*

(1) *If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.*
(2) *If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.*
(3) *If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.*
(4) *Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel K, the morphism $C \to B/f(K)$ is an essential surjection.*

Proof. Omitted. □

08XL **Lemma 47.2.3.** *Let R be a ring. Let M be an R-module. Let $E = \operatorname{colim} E_i$ be a filtered colimit of R-modules. Suppose given a compatible system of essential injections $M \to E_i$ of R-modules. Then $M \to E$ is an essential injection.*

Proof. Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 10.8.8). □

08XM **Lemma 47.2.4.** *Let R be a ring. Let $M \subset N$ be R-modules. The following are equivalent*

(1) *$M \subset N$ is an essential extension,*
(2) *for all $x \in N$ nonzero there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.*

Proof. Assume (1) and let $x \in N$ be a nonzero element. By (1) we have $Rx \cap M \neq 0$. This implies (2).

Assume (2). Let $N' \subset N$ be a nonzero submodule. Pick $x \in N'$ nonzero. By (2) we can find $f \in R$ with $fx \in M$ and $fx \neq 0$. Thus $N' \cap M \neq 0$. □

47.3. Injective modules

08XN Some results about injective modules over rings.

08XP **Lemma 47.3.1.** *Let R be a ring. Any product of injective R-modules is injective.*

Proof. Special case of Homology, Lemma 12.27.3. □

08XQ **Lemma 47.3.2.** *Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.*

Proof. This is true because $\operatorname{Hom}_R(M, E) = \operatorname{Hom}_S(M \otimes_R S, E)$ by Algebra, Lemma 10.14.3 and the fact that tensoring with S is exact. □

08YV **Lemma 47.3.3.** *Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.*

Proof. This is true because $\operatorname{Hom}_R(N, E) = \operatorname{Hom}_S(N, E)$ for any S-module N, see Algebra, Lemma 10.107.14. □

08XR **Lemma 47.3.4.** *Let $R \to S$ be a ring map. If E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.*

Proof. This is true because $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) = \mathrm{Hom}_R(N, E)$ by Algebra, Lemma 10.14.4. □

08XS **Lemma 47.3.5.** *Let R be a ring. Let I be an injective R-module. Let $E \subset I$ be a submodule. The following are equivalent*

(1) *E is injective, and*
(2) *for all $E \subset E' \subset I$ with $E \subset E'$ essential we have $E = E'$.*

In particular, an R-module is injective if and only if every essential extension is trivial.

Proof. The final assertion follows from the first and the fact that the category of R-modules has enough injectives (More on Algebra, Section 15.55).

Assume (1). Let $E \subset E' \subset I$ as in (2). Then the map $\mathrm{id}_E : E \to E$ can be extended to a map $\alpha : E' \to E$. The kernel of α has to be zero because it intersects E trivially and E' is an essential extension. Hence $E = E'$.

Assume (2). Let $M \subset N$ be R-modules and let $\varphi : M \to E$ be an R-module map. In order to prove (1) we have to show that φ extends to a morphism $N \to E$. Consider the set $\mathcal{S}$ of pairs (M', φ') where $M \subset M' \subset N$ and $\varphi' : M' \to E$ is an R-module map agreeing with φ on M. We define an ordering on $\mathcal{S}$ by the rule $(M', \varphi') \leq (M'', \varphi'')$ if and only if $M' \subset M''$ and $\varphi''|_{M'} = \varphi'$. It is clear that we can take the maximum of a totally ordered subset of $\mathcal{S}$. Hence by Zorn's lemma we may assume (M, φ) is a maximal element.

Choose an extension $\psi : N \to I$ of φ composed with the inclusion $E \to I$. This is possible as I is injective. If $\psi(N) \subset E$, then ψ is the desired extension. If $\psi(N)$ is not contained in E, then by (2) the inclusion $E \subset E + \psi(N)$ is not essential. hence we can find a nonzero submodule $K \subset E + \psi(N)$ meeting E in 0. This means that $M' = \psi^{-1}(E + K)$ strictly contains M. Thus we can extend φ to M' using

$$M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E$$

This contradicts the maximality of (M, φ). □

08XT **Example 47.3.6.** Let R be a reduced ring. Let $\mathfrak{p} \subset R$ be a minimal prime so that $K = R_{\mathfrak{p}}$ is a field (Algebra, Lemma 10.25.1). Then K is an injective R-module. Namely, we have $\mathrm{Hom}_R(M, K) = \mathrm{Hom}_K(M_{\mathfrak{p}}, K)$ for any R-module M. Since localization is an exact functor and taking duals is an exact functor on K-vector spaces we conclude $\mathrm{Hom}_R(-, K)$ is an exact functor, i.e., K is an injective R-module.

08XV **Lemma 47.3.7.** *Let R be a Noetherian ring. A direct sum of injective modules is injective.*

Proof. Let E_i be a family of injective modules parametrized by a set I. Set $E = \bigoplus E_i$. To show that E is injective we use Injectives, Lemma 19.2.6. Thus let $\varphi : J \to E$ be a module map from an ideal of R into E. As J is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigoplus_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigoplus_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}. □

0A6I **Lemma 47.3.8.** *Let R be a Noetherian ring. Let $S \subset R$ be a multiplicative subset. If E is an injective R-module, then $S^{-1}E$ is an injective $S^{-1}R$-module.*

Proof. Since $R \to S^{-1}R$ is an epimorphism of rings, it suffices to show that $S^{-1}E$ is injective as an R-module, see Lemma 47.3.3. To show this we use Injectives, Lemma 19.2.6. Thus let $I \subset R$ be an ideal and let $\varphi : I \to S^{-1}E$ be an R-module map. As I is a finitely presented R-module (because R is Noetherian) we can find an $f \in S$ and an R-module map $I \to E$ such that $f\varphi$ is the composition $I \to E \to S^{-1}E$ (Algebra, Lemma 10.10.2). Then we can extend $I \to E$ to a homomorphism $R \to E$. Then the composition

$$R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E$$

is the desired extension of φ to R. □

08XW **Lemma 47.3.9.** *Let R be a Noetherian ring. Let I be an injective R-module.*

(1) *Let $f \in R$. Then $E = \bigcup I[f^n] = I[f^\infty]$ is an injective submodule of I.*
(2) *Let $J \subset R$ be an ideal. Then the J-power torsion submodule $I[J^\infty]$ is an injective submodule of I.*

Proof. We will use Lemma 47.3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that E' is an essential extension of E. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E\}$. Since R is Noetherian, we may write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. By definition E is the set of elements of I annihilated by powers of f, so we may choose integers n_i so that $f^{n_i} g_i x = 0$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of E' not in E and is annihilated by J. Set $J' = \{a \in R \mid ax' \in E\}$ so $J \subset J'$. Conversely, we have $a \in J'$ if and only if $ax' \in E$ if and only if $f^m ax' = 0$ for some $m \geq 0$. But then $f^m ax' = f^{m+n} ax$ implies $ax \in E$, i.e., $a \in J$. Hence $J = J'$. Thus $J = J' = \mathrm{Ann}(x')$, so $Rx' \cap E = 0$. Hence E' is not an essential extension of E, a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^\infty]$ is equal to

$$(\ldots((I[f_1^\infty])[f_2^\infty])\ldots)[f_t^\infty]$$

and the result follows from (1) and induction. □

0A6J **Lemma 47.3.10.** *Let A be a Noetherian ring. Let E be an injective A-module. Then $E \otimes_A A[x]$ has injective-amplitude $[0, 1]$ as an object of $D(A[x])$. In particular, $E \otimes_A A[x]$ has finite injective dimension as an $A[x]$-module.*

Proof. Let us write $E[x] = E \otimes_A A[x]$. Consider the short exact sequence of $A[x]$-modules

$$0 \to E[x] \to \mathrm{Hom}_A(A[x], E[x]) \to \mathrm{Hom}_A(A[x], E[x]) \to 0$$

where the first map sends $p \in E[x]$ to $f \mapsto fp$ and the second map sends φ to $f \mapsto \varphi(xf) - x\varphi(f)$. The second map is surjective because $\mathrm{Hom}_A(A[x], E[x]) = \prod_{n \geq 0} E[x]$ as an abelian group and the map sends (e_n) to $(e_{n+1} - xe_n)$ which is surjective. As an A-module we have $E[x] \cong \bigoplus_{n \geq 0} E$ which is injective by Lemma 47.3.7. Hence the $A[x]$-module $\mathrm{Hom}_A(A[x], E[x])$ is injective by Lemma 47.3.4 and the proof is complete. □

47.4. Projective covers

08XX In this section we briefly discuss projective covers.

08XY **Definition 47.4.1.** Let R be a ring. A surjection $P \to M$ of R-modules is said to be a *projective cover*, or sometimes a *projective envelope*, if P is a projective R-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if k is a field and $R = k[x]$ is the polynomial ring over k, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with P projective over R, the proper submodule $(x-1)P$ surjects onto M. Hence f is not essential.

08XZ **Lemma 47.4.2.** *Let R be a ring and let M be an R-module. If a projective cover of M exists, then it is unique up to isomorphism.*

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because P is a projective R-module and $P' \to M$ is surjective, we can find an R-module map $\alpha : P \to P'$ compatible with the maps to M. Since $P' \to M$ is essential, we see that α is surjective. As P' is a projective R-module we can choose a direct sum decomposition $P = \mathrm{Ker}(\alpha) \oplus P'$. Since $P' \to M$ is surjective and since $P \to M$ is essential we conclude that $\mathrm{Ker}(\alpha)$ is zero as desired. □

Here is an example where projective covers exist.

08Y0 **Lemma 47.4.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Any finite R-module has a projective cover.*

Proof. Let M be a finite R-module. Let $r = \dim_\kappa(M/\mathfrak{m}M)$. Choose $x_1, \ldots, x_r \in M$ mapping to a basis of $M/\mathfrak{m}M$. Consider the map $f : R^{\oplus r} \to M$. By Nakayama's lemma this is a surjection (Algebra, Lemma 10.20.1). If $N \subset R^{\oplus r}$ is a proper submodule, then $N/\mathfrak{m}N \to \kappa^{\oplus r}$ is not surjective (by Nakayama's lemma again) hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is not surjective. Thus f is an essential surjection. □

47.5. Injective hulls

08Y1 In this section we briefly discuss injective hulls.

08Y2 **Definition 47.5.1.** Let R be a ring. An injection $M \to I$ of R-modules is said to be an *injective hull* if I is an injective R-module and $M \to I$ is an essential injection.

Injective hulls always exist.

08Y3 **Lemma 47.5.2.** *Let R be a ring. Any R-module has an injective hull.*

Proof. Let M be an R-module. By More on Algebra, Section 15.55 the category of R-modules has enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. We order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset of $\mathcal{S}$, then $\bigcup E_\alpha$ is an essential extension of M too (Lemma 47.2.3). Thus we can apply Zorn's lemma and find a maximal element $E \in \mathcal{S}$. We claim $M \subset E$ is an injective hull, i.e., E is an injective R-module. This follows from Lemma 47.3.5. □

08Y4 **Lemma 47.5.3.** *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then*

(1) *for any R-module map $\varphi : M \to N$ there exists an R-module map $\psi : E \to E'$ such that*

$$\begin{array}{ccc} M & \longrightarrow & E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ N & \longrightarrow & E' \end{array}$$

commutes,
(2) *if φ is injective, then ψ is injective,*
(3) *if φ is an essential injection, then ψ is an isomorphism,*
(4) *if φ is an isomorphism, then ψ is an isomorphism,*
(5) *if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \oplus I'$ compatible with the embeddings of M,*

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (1) follows from the fact that E' is an injective R-module. Part (2) follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential injection. Then $E \cong \psi(E) \subset E'$ by (2) which implies $E' = \psi(E) \oplus E''$ because E is injective. Since E' is an essential extension of M (Lemma 47.2.2) we get $E'' = 0$. Part (4) is a special case of (3). Assume $M \to I$ as in (5). Choose a map $\alpha : E \to I$ extending the map $M \to I$. Arguing as before we see that α is injective. Thus as before $\alpha(E)$ splits off from I. This proves (5). □

08Y5 **Example 47.5.4.** Let R be a domain with fraction field K. Then $R \subset K$ is an injective hull of R. Namely, by Example 47.3.6 we see that K is an injective R-module and by Lemma 47.2.4 we see that $R \subset K$ is an essential extension.

08Y6 **Definition 47.5.5.** An object X of an additive category is called *indecomposable* if it is nonzero and if $X = Y \oplus Z$, then either $Y = 0$ or $Z = 0$.

08Y7 **Lemma 47.5.6.** *Let R be a ring. Let E be an indecomposable injective R-module. Then*

(1) *E is the injective hull of any nonzero submodule of E,*
(2) *the intersection of any two nonzero submodules of E is nonzero,*
(3) *$End_R(E, E)$ is a noncommutative local ring with maximal ideal those $\varphi : E \to E$ whose kernel is nonzero, and*
(4) *the set of zerodivisors on E is a prime ideal $\mathfrak{p}$ of R and E is an injective $R_\mathfrak{p}$-module.*

Proof. Part (1) follows from Lemma 47.5.3. Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set $A = \mathrm{End}_R(E, E)$ and $I = \{\varphi \in A \mid \mathrm{Ker}(\varphi) \neq 0\}$. The statement means that I is a two sided ideal and that any $\varphi \in A$, $\varphi \notin I$ is invertible. Suppose φ and ψ are not injective. Then $\mathrm{Ker}(\varphi) \cap \mathrm{Ker}(\psi)$ is nonzero by (2). Hence $\varphi + \psi \in I$. It follows that I is a two sided ideal. If $\varphi \in A$, $\varphi \notin I$, then $E \cong \varphi(E) \subset E$ is an injective submodule, hence $E = \varphi(E)$ because E is indecomposable.

Proof of (4). Consider the ring map $R \to A$ and let $\mathfrak{p} \subset R$ be the inverse image of the maximal ideal I. Then it is clear that $\mathfrak{p}$ is a prime ideal and that $R \to A$ extends to $R_\mathfrak{p} \to A$. Thus E is an $R_\mathfrak{p}$-module. It follows from Lemma 47.3.3 that E is injective as an $R_\mathfrak{p}$-module. □

08Y8 **Lemma 47.5.7.** *Let $\mathfrak{p} \subset R$ be a prime of a ring R. Let E be the injective hull of $R/\mathfrak{p}$. Then*

(1) *E is indecomposable,*
(2) *E is the injective hull of $\kappa(\mathfrak{p})$,*
(3) *E is the injective hull of $\kappa(\mathfrak{p})$ over the ring $R_\mathfrak{p}$.*

Proof. By Lemma 47.2.4 the inclusion $R/\mathfrak{p} \subset \kappa(\mathfrak{p})$ is an essential extension. Then Lemma 47.5.3 shows (2) holds. For $f \in R$, $f \notin \mathfrak{p}$ the map $f : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ is an isomorphism hence the map $f : E \to E$ is an isomorphism, see Lemma 47.5.3. Thus E is an $R_\mathfrak{p}$-module. It is injective as an $R_\mathfrak{p}$-module by Lemma 47.3.3. Finally, let $E' \subset E$ be a nonzero injective R-submodule. Then $J = (R/\mathfrak{p}) \cap E'$ is nonzero. After shrinking E' we may assume that E' is the injective hull of J (see Lemma 47.5.3 for example). Observe that $R/\mathfrak{p}$ is an essential extension of J for example by Lemma 47.2.4. Hence $E' \to E$ is an isomorphism by Lemma 47.5.3 part (3). Hence E is indecomposable. □

08Y9 **Lemma 47.5.8.** *Let R be a Noetherian ring. Let E be an indecomposable injective R-module. Then there exists a prime ideal $\mathfrak{p}$ of R such that E is the injective hull of $\kappa(\mathfrak{p})$.*

Proof. Let $\mathfrak{p}$ be the prime ideal found in Lemma 47.5.6. Say $\mathfrak{p} = (f_1, \ldots, f_r)$. Pick a nonzero element $x \in \bigcap \mathrm{Ker}(f_i : E \to E)$, see Lemma 47.5.6. Then $(R_\mathfrak{p})x$ is a module isomorphic to $\kappa(\mathfrak{p})$ inside E. We conclude by Lemma 47.5.6. □

08YA **Proposition 47.5.9** (Structure of injective modules over Noetherian rings)**.** *Let R be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.*

Proof. The second statement is Lemma 47.5.8. For the first statement, let I be an injective R-module. We will use transfinite recursion to construct $I_\alpha \subset I$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$ for $\beta < \alpha$. For $\alpha = 0$ we let $I_0 = 0$. Suppose given an ordinal α such that I_α has been constructed. Then I_α is an injective R-module by Lemma 47.3.7. Hence $I \cong I_\alpha \oplus I'$. If $I' = 0$ we are done. If not, then I' has an associated prime by Algebra, Lemma 10.63.7. Thus I' contains a copy of $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence I' contains an indecomposable submodule E by Lemmas 47.5.3 and 47.5.7. Set $I_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and I_β has been constructed for $\beta < \alpha$, then we set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Observe that $I_\alpha = \bigoplus_{\beta<\alpha} E_{\beta+1}$. This concludes the proof. □

47.6. Duality over Artinian local rings

08YW Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Recall that this implies R is Noetherian and that R has finite length as an R-module. Moreover an R-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 10.52 and 10.53 and Algebra, Proposition 10.60.7 for more details.

08YX **Lemma 47.6.1.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For every finite R-module M we have*

$$\mathit{length}_R(M) = \mathit{length}_R(\mathrm{Hom}_R(M, E))$$

In particular, the injective hull E of κ is a finite R-module.

Proof. Because E is an essential extension of κ we have $\kappa = E[\mathfrak{m}]$ where $E[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion in E (notation as in More on Algebra, Section 15.89). Hence $\mathrm{Hom}_R(\kappa, E) \cong \kappa$ and the equality of lengths holds for $M = \kappa$. We prove the displayed equality of the lemma by induction on the length of M. If M is nonzero there exists a surjection $M \to \kappa$ with kernel M'. Since the functor $M \mapsto \mathrm{Hom}_R(M, E)$ is exact we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\kappa, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0.$$

Additivity of length for this sequence and the sequence $0 \to M' \to M \to \kappa \to 0$ and the equality for M' (induction hypothesis) and κ implies the equality for M. The final statement of the lemma follows as $E = \mathrm{Hom}_R(R, E)$. □

08YY **Lemma 47.6.2.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For any finite R-module M the evaluation map*

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$$

is an isomorphism. In particular $R = \mathrm{Hom}_R(E, E)$.

Proof. Observe that the displayed arrow is injective. Namely, if $x \in M$ is a nonzero element, then there is a nonzero map $Rx \to \kappa$ which we can extend to a map $\varphi : M \to E$ that doesn't vanish on x. Since the source and target of the arrow have the same length by Lemma 47.6.1 we conclude it is an isomorphism. The final statement follows on taking $M = R$. □

To state the next lemma, denote Mod_R^{fg} the category of finite R-modules over a ring R.

08YZ **Lemma 47.6.3.** *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an exact anti-equivalence $Mod_R^{fg} \to Mod_R^{fg}$ and $D \circ D \cong id$.*

Proof. We have seen that $D \circ D = \mathrm{id}$ on Mod_R^{fg} in Lemma 47.6.2. It follows immediately that D is an anti-equivalence. □

08Z0 **Lemma 47.6.4.** *Assumptions and notation as in Lemma 47.6.3. Let $I \subset R$ be an ideal and M a finite R-module. Then*

$$D(M[I]) = D(M)/ID(M) \quad \text{and} \quad D(M/IM) = D(M)[I]$$

Proof. Say $I = (f_1, \ldots, f_t)$. Consider the map

$$M^{\oplus t} \xrightarrow{f_1, \ldots, f_t} M$$

with cokernel M/IM. Applying the exact functor D we conclude that $D(M/IM)$ is $D(M)[I]$. The other case is proved in the same way. □

47.7. Injective hull of the residue field

08Z1 Most of our results will be for Noetherian local rings in this section.

08Z2 **Lemma 47.7.1.** *Let $R \to S$ be a surjective map of local rings with kernel I. Let E be the injective hull of the residue field of R over R. Then $E[I]$ is the injective hull of the residue field of S over S.*

Proof. Observe that $E[I] = \mathrm{Hom}_R(S, E)$ as $S = R/I$. Hence $E[I]$ is an injective S-module by Lemma 47.3.4. Since E is an essential extension of $\kappa = R/\mathfrak{m}_R$ it follows that $E[I]$ is an essential extension of κ as well. The result follows. □

08Z3 **Lemma 47.7.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let E be the injective hull of κ. Let M be a $\mathfrak{m}$-power torsion R-module with $n = \dim_\kappa(M[\mathfrak{m}]) < \infty$. Then M is isomorphic to a submodule of $E^{\oplus n}$.*

Proof. Observe that $E^{\oplus n}$ is the injective hull of $\kappa^{\oplus n} = M[\mathfrak{m}]$. Thus there is an R-module map $M \to E^{\oplus n}$ which is injective on $M[\mathfrak{m}]$. Since M is $\mathfrak{m}$-power torsion the inclusion $M[\mathfrak{m}] \subset M$ is an essential extension (for example by Lemma 47.2.4) we conclude that the kernel of $M \to E^{\oplus n}$ is zero. □

08Z4 **Lemma 47.7.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Let E_n be an injective hull of κ over $R/\mathfrak{m}^n$. Then $E = \bigcup E_n$ and $E_n = E[\mathfrak{m}^n]$.*

Proof. We have $E_n = E[\mathfrak{m}^n]$ by Lemma 47.7.1. We have $E = \bigcup E_n$ because $\bigcup E_n = E[\mathfrak{m}^\infty]$ is an injective R-submodule which contains κ, see Lemma 47.3.9. □

The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

08Z5 **Lemma 47.7.4.** *Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/\mathfrak{m}_R \cong S/\mathfrak{m}_R S$. Then the injective hull of the residue field of R is the injective hull of the residue field of S.*

Proof. Note that $\mathfrak{m}_R S = \mathfrak{m}_S$ as the quotient by the former is a field. Set $\kappa = R/\mathfrak{m}_R = S/\mathfrak{m}_S$. Let E_R be the injective hull of κ over R. Let E_S be the injective hull of κ over S. Observe that E_S is an injective R-module by Lemma 47.3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 47.7.3 because $R \to S$ induces an isomorphism $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all n. □

08Z6 **Lemma 47.7.5.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then $\mathrm{Hom}_R(E, E)$ is canonically isomorphic to the completion of R.*

Proof. Write $E = \bigcup E_n$ with $E_n = E[\mathfrak{m}^n]$ as in Lemma 47.7.3. Any endomorphism of E preserves this filtration. Hence

$$\mathrm{Hom}_R(E, E) = \lim \mathrm{Hom}_R(E_n, E_n)$$

The lemma follows as $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = R/\mathfrak{m}^n$ by Lemma 47.6.2. □

08Z7 **Lemma 47.7.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then E satisfies the descending chain condition.*

Proof. If $E \supset M_1 \supset M_2 \supset \ldots$ is a sequence of submodules, then

$$\mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(M_1, E) \to \mathrm{Hom}_R(M_2, E) \to \ldots$$

is a sequence of surjections. By Lemma 47.7.5 each of these is a module over the completion $R^\wedge = \mathrm{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 10.97.6) the sequence stabilizes: $\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) = \ldots$. Since

E is injective, this can only happen if $\mathrm{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if M_n/M_{n+1} is nonzero, then it contains a nonzero element annihilated by $\mathfrak{m}$, because E is $\mathfrak{m}$-power torsion by Lemma 47.7.3. In this case M_n/M_{n+1} has a nonzero map into E, contradicting the assumed vanishing. This finishes the proof. □

08Z8 **Lemma 47.7.7.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ.*

(1) *For an R-module M the following are equivalent:*
   (a) *M satisfies the ascending chain condition,*
   (b) *M is a finite R-module, and*
   (c) *there exist n, m and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.*

(2) *For an R-module M the following are equivalent:*
   (a) *M satisfies the descending chain condition,*
   (b) *M is $\mathfrak{m}$-power torsion and $\dim_\kappa(M[\mathfrak{m}]) < \infty$, and*
   (c) *there exist n, m and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.*

Proof. We omit the proof of (1).

Let M be an R-module with the descending chain condition. Let $x \in M$. Then $\mathfrak{m}^n x$ is a descending chain of submodules, hence stabilizes. Thus $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$ for some n. By Nakayama's lemma (Algebra, Lemma 10.20.1) this implies $\mathfrak{m}^n x = 0$, i.e., x is $\mathfrak{m}$-power torsion. Since $M[\mathfrak{m}]$ is a vector space over κ it has to be finite dimensional in order to have the descending chain condition.

Assume that M is $\mathfrak{m}$-power torsion and has a finite dimensional $\mathfrak{m}$-torsion submodule $M[\mathfrak{m}]$. By Lemma 47.7.2 we see that M is a submodule of $E^{\oplus n}$ for some n. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 47.7.6 the module E has the descending chain condition hence so do $E^{\oplus n}$ and N. Therefore N satisfies (2)(a) which implies N satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 47.7.2 again we see that N is a submodule of $E^{\oplus m}$ for some m. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since E satisfies the descending chain condition by Lemma 47.7.6 so does M. □

08Z9 **Proposition 47.7.8** (Matlis duality)**.** *Let $(R, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Let E be an injective hull of κ over R. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an anti-equivalence*

$$\left\{\begin{array}{c}\text{R-modules with the}\\ \text{descending chain condition}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{R-modules with the}\\ \text{ascending chain condition}\end{array}\right\}$$

and we have $D \circ D = id$ on either side of the equivalence.

Proof. By Lemma 47.7.5 we have $R = \mathrm{Hom}_R(E, E) = D(E)$. Of course we have $E = \mathrm{Hom}_R(R, E) = D(R)$. Since E is injective the functor D is exact. The result now follows immediately from the description of the categories in Lemma 47.7.7. □

0EGL **Remark 47.7.9.** Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Here is an addendum to Matlis duality: If N is an $\mathfrak{m}$-power torsion module and $M = \mathrm{Hom}_R(N, E)$ is a finite module over the completion of R, then N satisfies the descending chain condition. Namely, for any submodules $N'' \subset N' \subset N$ with $N'' \neq N'$, we can find an embedding $\kappa \subset N''/N'$ and hence a nonzero map $N' \to E$ annihilating N'' which we can extend to a map $N \to E$ annihilating N''.

Thus $N \supset N' \mapsto M' = \mathrm{Hom}_R(N/N', E) \subset M$ is an inclusion preserving map from submodules of N to submodules of M, whence the conclusion.

47.8. Deriving torsion

0BJA Let A be a ring and let $I \subset A$ be a finitely generated ideal (if I is not finitely generated perhaps a different definition should be used). Let $Z = V(I) \subset \mathrm{Spec}(A)$. Recall that the category I^∞-torsion of I-power torsion modules only depends on the closed subset Z and not on the choice of the finitely generated ideal I such that $Z = V(I)$, see More on Algebra, Lemma 15.89.6. In this section we will consider the functor

$$H^0_I : \mathrm{Mod}_A \longrightarrow I^\infty\text{-torsion}, \quad M \longmapsto M[I^\infty] = \bigcup M[I^n]$$

which sends M to the submodule of I-power torsion.

Let A be a ring and let I be a finitely generated ideal. Note that I^∞-torsion is a Grothendieck abelian category (direct sums exist, filtered colimits are exact, and $\bigoplus A/I^n$ is a generator by More on Algebra, Lemma 15.89.2). Hence the derived category $D(I^\infty\text{-torsion})$ exists, see Injectives, Remark 19.13.3. Our functor H^0_I is left exact and has a derived extension which we will denote

$$R\Gamma_I : D(A) \longrightarrow D(I^\infty\text{-torsion}).$$

Warning: this functor does not deserve the name local cohomology unless the ring A is Noetherian. The functors H^0_I, $R\Gamma_I$, and the satellites H^p_I only depend on the closed subset $Z \subset \mathrm{Spec}(A)$ and not on the choice of the finitely generated ideal I such that $V(I) = Z$. However, we insist on using the subscript I for the functors above as the notation $R\Gamma_Z$ is going to be used for a different functor, see (47.9.0.1), which agrees with the functor $R\Gamma_I$ only (as far as we know) in case A is Noetherian (see Lemma 47.10.1).

0A6L **Lemma 47.8.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor $R\Gamma_I$ is right adjoint to the functor $D(I^\infty\text{-torsion}) \to D(A)$.*

Proof. This follows from the fact that taking I-power torsion submodules is the right adjoint to the inclusion functor I^∞-torsion $\to \mathrm{Mod}_A$. See Derived Categories, Lemma 13.30.3. □

0954 **Lemma 47.8.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For any object K of $D(A)$ we have*

$$R\Gamma_I(K) = \mathit{hocolim}\, R\mathrm{Hom}_A(A/I^n, K)$$

in $D(A)$ and

$$R^q\Gamma_I(K) = \mathrm{colim}_n \mathrm{Ext}^q_A(A/I^n, K)$$

as modules for all $q \in \mathbf{Z}$.

Proof. Let $J^\bullet$ be a K-injective complex representing K. Then

$$R\Gamma_I(K) = J^\bullet[I^\infty] = \mathrm{colim}\, J^\bullet[I^n] = \mathrm{colim}\, \mathrm{Hom}_A(A/I^n, J^\bullet)$$

where the first equality is the definition of $R\Gamma_I(K)$. By Derived Categories, Lemma 13.33.7 we obtain the first displayed equality in the statement of the lemma. The second displayed equality in the statement of the lemma then follows because $H^q(\mathrm{Hom}_A(A/I^n, J^\bullet)) = \mathrm{Ext}^q_A(A/I^n, K)$ and because filtered colimits are exact in the category of abelian groups. □

0A6M **Lemma 47.8.3.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_I(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_I(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

Let A be a ring and $I \subset A$ a finitely generated ideal. By More on Algebra, Lemma 15.89.5 the category of I-power torsion modules is a Serre subcategory of the category of all A-modules. Hence there is a functor

0A6N (47.8.3.1) $$D(I^\infty\text{-torsion}) \longrightarrow D_{I^\infty\text{-torsion}}(A)$$

where the right hand side is the full subcategory of $D(A)$ consising of objects whose cohomologies are I-power torsion modules. Both the functor and the category are discussed in Derived Categories, Section 13.17.

0A6P **Lemma 47.8.4.** *Let A be a ring and let I be a finitely generated ideal. Let M and N be I-power torsion modules.*

(1) $\mathrm{Hom}_{D(A)}(M, N) = \mathrm{Hom}_{D(I^\infty\text{-torsion})}(M, N)$,
(2) $\mathrm{Ext}^1_{D(A)}(M, N) = \mathrm{Ext}^1_{D(I^\infty\text{-torsion})}(M, N)$,
(3) $\mathrm{Ext}^2_{D(I^\infty\text{-torsion})}(M, N) \to \mathrm{Ext}^2_{D(A)}(M, N)$ *is not surjective in general,*
(4) *(47.8.3.1) is not an equivalence in general.*

Proof. Parts (1) and (2) follow immediately from the fact that I-power torsion forms a Serre subcategory of Mod_A. Part (4) follows from part (3).

For part (3) let A be a ring with an element $f \in A$ such that $A[f]$ contains a nonzero element x annihilated by f and A contains elements x_n with $f^n x_n = x$. Such a ring A exists because we can take

$$A = \mathbf{Z}[f, x, x_n]/(fx, f^n x_n - x)$$

Given A set $I = (f)$. Then the exact sequence

$$0 \to A[f] \to A \xrightarrow{f} A \to A/fA \to 0$$

defines an element in $\mathrm{Ext}^2_A(A/fA, A[f])$. We claim this element does not come from an element of $\mathrm{Ext}^2_{D(f^\infty\text{-torsion})}(A/fA, A[f])$. Namely, if it did, then there would be an exact sequence

$$0 \to A[f] \to M \to N \to A/fA \to 0$$

where M and N are f-power torsion modules defining the same 2 extension class. Since $A \to A$ is a complex of free modules and since the 2 extension classes are the same we would be able to find a map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A[f] & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A/fA & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\
0 & \longrightarrow & A[f] & \longrightarrow & M & \longrightarrow & N & \longrightarrow & A/fA & \longrightarrow & 0
\end{array}$$

(some details omitted). Then we could replace M by the image of φ and N by the image of ψ. Then M would be a cyclic module, hence $f^n M = 0$ for some n. Considering $\varphi(x_{n+1})$ we get a contradiction with the fact that $f^{n+1} x_n = x$ is nonzero in $A[f]$. □

47.9. Local cohomology

0952 Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. We will construct a functor

0A6Q (47.9.0.1) $$R\Gamma_Z : D(A) \longrightarrow D_{I^\infty\text{-torsion}}(A).$$

which is right adjoint to the inclusion functor. For notation see Section 47.8. The cohomology modules of $R\Gamma_Z(K)$ are the *local cohomology groups of K with respect to Z*. By Lemma 47.8.4 this functor will in general **not** be equal to $R\Gamma_I(-)$ even viewed as functors into $D(A)$. In Section 47.10 we will show that if A is Noetherian, then the two agree.

We will continue the discussion of local cohomology in the chapter on local cohomology, see Local Cohomology, Section 51.1. For example, there we will show that $R\Gamma_Z$ computes cohomology with support in Z for the associated complex of quasi-coherent sheaves on $\mathrm{Spec}(A)$. See Local Cohomology, Lemma 51.2.1.

0A6R **Lemma 47.9.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. There exists a right adjoint $R\Gamma_Z$ (47.9.0.1) to the inclusion functor $D_{I^\infty\text{-torsion}}(A) \to D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have*

$$R\Gamma_Z(K) = (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \otimes^{\mathbf{L}}_A K$$

functorially in $K \in D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$ is an ideal. Let $K^\bullet$ be a complex of A-modules. There is a canonical map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \longrightarrow A.$$

from the extended Čech complex to A. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right) \longrightarrow K^\bullet$$

in $D(A)$. We claim the cohomology modules of the complex on the left are I-power torsion, i.e., the LHS is an object of $D_{I^\infty\text{-torsion}}(A)$. Namely, we have

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) = \mathrm{colim}\, K(A, f_1^n, \ldots, f_r^n)$$

by More on Algebra, Lemma 15.29.6. Moreover, multiplication by f_i^n on the complex $K(A, f_1^n, \ldots, f_r^n)$ is homotopic to zero by More on Algebra, Lemma 15.28.6. Since

$$H^q(LHS) = \mathrm{colim}\, H^q(\mathrm{Tot}(K^\bullet \otimes_A K(A, f_1^n, \ldots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{I^\infty\text{-torsion}}(A)$, then the complexes $K^\bullet \otimes_A A_{f_{i_0}\ldots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $LHS \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$R\Gamma_Z(K^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right)$$

is functorial in $K^\bullet$ and defines an exact functor $D(A) \to D_{I^\infty\text{-torsion}}(A)$ between triangulated categories. It follows formally from the existence of the natural transformation $R\Gamma_Z \to \mathrm{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $R\Gamma_Z$ is the desired right adjoint. □

0BJB **Lemma 47.9.2.** *Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Set $Z = V(I)$ and $Y = V(J)$. Then*

$$R\Gamma_Z(M_A) = R\Gamma_Y(M)_A$$

functorially in $M \in D(B)$. Here $(-)_A$ denotes the restriction functors $D(B) \to D(A)$ and $D_{J^\infty\text{-}torsion}(B) \to D_{I^\infty\text{-}torsion}(A)$.

Proof. This follows from uniqueness of adjoint functors as both $R\Gamma_Z((-)_A)$ and $R\Gamma_Y(-)_A$ are right adjoint to the functor $D_{I^\infty\text{-torsion}}(A) \to D(B)$, $K \mapsto K \otimes^{\mathbf{L}}_A B$. Alternatively, one can use the description of $R\Gamma_Z$ and $R\Gamma_Y$ in terms of alternating Čech complexes (Lemma 47.9.1). Namely, if $I = (f_1, \ldots, f_r)$ then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then the statement of the lemma follows from the existence of a canonical isomorphism

$$\begin{aligned} & M_A \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \\ & = M \otimes_B (B \to \prod\nolimits_{i_0} B_{g_{i_0}} \to \prod\nolimits_{i_0<i_1} B_{g_{i_0}g_{i_1}} \to \ldots \to B_{g_1\ldots g_r}) \end{aligned}$$

for any B-module M. □

0ALZ **Lemma 47.9.3.** *Let $A \to B$ be a ring homomorphism and let $I \subset A$ be a finitely generated ideal. Set $J = IB$. Let $Z = V(I)$ and $Y = V(J)$. Then*

$$R\Gamma_Z(K) \otimes^{\mathbf{L}}_A B = R\Gamma_Y(K \otimes^{\mathbf{L}}_A B)$$

functorially in $K \in D(A)$.

Proof. Write $I = (f_1, \ldots, f_r)$. Then J is generated by the images $g_1, \ldots, g_r \in B$ of $f_1, \ldots, f_r$. Then we have

$$(A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1\ldots f_r}) \otimes_A B = (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1\ldots g_r})$$

as complexes of B-modules. Represent K by a K-flat complex $K^\bullet$ of A-modules. Since the total complexes associated to

$$K^\bullet \otimes_A (A \to \prod A_{f_{i_0}} \to \ldots \to A_{f_1\ldots f_r}) \otimes_A B$$

and

$$K^\bullet \otimes_A B \otimes_B (B \to \prod B_{g_{i_0}} \to \ldots \to B_{g_1\ldots g_r})$$

represent the left and right hand side of the displayed formula of the lemma (see Lemma 47.9.1) we conclude. □

0A6S **Lemma 47.9.4.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_Z(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_Z(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

0ALY **Lemma 47.9.5.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For $K, L \in D(A)$ we have*

$$R\Gamma_Z(K \otimes^{\mathbf{L}}_A L) = K \otimes^{\mathbf{L}}_A R\Gamma_Z(L) = R\Gamma_Z(K) \otimes^{\mathbf{L}}_A L = R\Gamma_Z(K) \otimes^{\mathbf{L}}_A R\Gamma_Z(L)$$

If K or L is in $D_{I^\infty\text{-}torsion}(A)$ then so is $K \otimes^{\mathbf{L}}_A L$.

Proof. By Lemma 47.9.1 we know that $R\Gamma_Z$ is given by $C \otimes^{\mathbf{L}} -$ for some $C \in D(A)$. Hence, for $K, L \in D(A)$ general we have

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes^{\mathbf{L}} L \otimes_A^{\mathbf{L}} C = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

The other equalities follow formally from this one. This also implies the last statement of the lemma. □

0BJC **Lemma 47.9.6.** *Let A be a ring and let $I, J \subset A$ be finitely generated ideals. Set $Z = V(I)$ and $Y = V(J)$. Then $Z \cap Y = V(I + J)$ and $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y \cap Z}$ as functors $D(A) \to D_{(I+J)^\infty\text{-torsion}}(A)$. For $K \in D^+(A)$ there is a spectral sequence*

$$E_2^{p,q} = H_Y^p(H_Z^q(K)) \Rightarrow H_{Y \cap Z}^{p+q}(K)$$

as in Derived Categories, Lemma 13.22.2.

Proof. There is a bit of abuse of notation in the lemma as strictly speaking we cannot compose $R\Gamma_Y$ and $R\Gamma_Z$. The meaning of the statement is simply that we are composing $R\Gamma_Z$ with the inclusion $D_{I^\infty\text{-torsion}}(A) \to D(A)$ and then with $R\Gamma_Y$. Then the equality $R\Gamma_Y \circ R\Gamma_Z = R\Gamma_{Y \cap Z}$ follows from the fact that

$$D_{I^\infty\text{-torsion}}(A) \to D(A) \xrightarrow{R\Gamma_Y} D_{(I+J)^\infty\text{-torsion}}(A)$$

is right adjoint to the inclusion $D_{(I+J)^\infty\text{-torsion}}(A) \to D_{I^\infty\text{-torsion}}(A)$. Alternatively one can prove the formula using Lemma 47.9.1 and the fact that the tensor product of extended Čech complexes on $f_1, \ldots, f_r$ and $g_1, \ldots, g_m$ is the extended Čech complex on $f_1, \ldots, f_n.g_1, \ldots, g_m$. The final assertion follows from this and the cited lemma. □

The following lemma is the analogue of More on Algebra, Lemma 15.92.25 for complexes with torsion cohomologies.

0AM0 **Lemma 47.9.7.** *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I = B/IB$. Then base change and restriction induce quasi-inverse equivalences $D_{I^\infty\text{-torsion}}(A) = D_{(IB)^\infty\text{-torsion}}(B)$.*

Proof. More precisely the functors are $K \mapsto K \otimes_A^{\mathbf{L}} B$ for K in $D_{I^\infty\text{-torsion}}(A)$ and $M \mapsto M_A$ for M in $D_{(IB)^\infty\text{-torsion}}(B)$. The reason this works is that $H^i(K \otimes_A^{\mathbf{L}} B) = H^i(K) \otimes_A B = H^i(K)$. The first equality holds as $A \to B$ is flat and the second by More on Algebra, Lemma 15.90.2. □

The following lemma was shown for Hom and Ext^1 of modules in More on Algebra, Lemmas 15.90.3 and 15.90.8.

05EH **Lemma 47.9.8.** *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I \to B/IB$ is an isomorphism. For $K \in D_{I^\infty\text{-torsion}}(A)$ and $L \in D(A)$ the map*

$$R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

is a quasi-isomorphism. In particular, if M, N are A-modules and M is I-power torsion, then the canonical map

$$\operatorname{Ext}_A^i(M, N) \longrightarrow \operatorname{Ext}_B^i(M \otimes_A B, N \otimes_A B)$$

is an isomorphism for all i.

Proof. Let $Z = V(I) \subset \mathrm{Spec}(A)$ and $Y = V(IB) \subset \mathrm{Spec}(B)$. Since the cohomology modules of K are I power torsion, the canonical map $R\Gamma_Z(L) \to L$ induces an isomorphism

$$R\mathrm{Hom}_A(K, R\Gamma_Z(L)) \to R\mathrm{Hom}_A(K, L)$$

in $D(A)$. Similarly, the cohomology modules of $K \otimes_A B$ are IB power torsion and we have an isomorphism

$$R\mathrm{Hom}_B(K \otimes_A B, R\Gamma_Y(L \otimes_A B)) \to R\mathrm{Hom}_B(K \otimes_A B, L \otimes_A B)$$

in $D(B)$. By Lemma 47.9.3 we have $R\Gamma_Z(L) \otimes_A B = R\Gamma_Y(L \otimes_A B)$. Hence it suffices to show that the map

$$R\mathrm{Hom}_A(K, R\Gamma_Z(L)) \to R\mathrm{Hom}_B(K \otimes_A B, R\Gamma_Z(L) \otimes_A B)$$

is a quasi-isomorphism. This follows from Lemma 47.9.7. □

47.10. Local cohomology for Noetherian rings

0BJD Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. Recall that (47.8.3.1) is the functor

$$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$$

In fact, there is a natural transformation of functors

0A6U (47.10.0.1) $$(47.8.3.1) \circ R\Gamma_I(-) \longrightarrow R\Gamma_Z(-)$$

Namely, given a complex of A-modules $K^\bullet$ the canonical map $R\Gamma_I(K^\bullet) \to K^\bullet$ in $D(A)$ factors (uniquely) through $R\Gamma_Z(K^\bullet)$ as $R\Gamma_I(K^\bullet)$ has I-power torsion cohomology modules (see Lemma 47.8.1). In general this map is not an isomorphism (we've seen this in Lemma 47.8.4).

0955 **Lemma 47.10.1.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal.*

1. *the adjunction $R\Gamma_I(K) \to K$ is an isomorphism for $K \in D_{I^\infty\text{-torsion}}(A)$,*
2. *the functor (47.8.3.1) $D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$ is an equivalence,*
3. *the transformation of functors (47.10.0.1) is an isomorphism, in other words $R\Gamma_I(K) = R\Gamma_Z(K)$ for $K \in D(A)$.*

Proof. A formal argument, which we omit, shows that it suffices to prove (1).

Let M be an I-power torsion A-module. Choose an embedding $M \to J$ into an injective A-module. Then $J[I^\infty]$ is an injective A-module, see Lemma 47.3.9, and we obtain an embedding $M \to J[I^\infty]$. Thus every I-power torsion module has an injective resolution $M \to J^\bullet$ with J^n also I-power torsion. It follows that $R\Gamma_I(M) = M$ (this is not a triviality and this is not true in general if A is not Noetherian). Next, suppose that $K \in D^+_{I^\infty\text{-torsion}}(A)$. Then the spectral sequence

$$R^q\Gamma_I(H^p(K)) \Rightarrow R^{p+q}\Gamma_I(K)$$

(Derived Categories, Lemma 13.21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $R\Gamma_I(K) \to K$ is an isomorphism.

Suppose K is an arbitrary object of $D_{I^\infty\text{-torsion}}(A)$. We have

$$R^q\Gamma_I(K) = \mathrm{colim}\, \mathrm{Ext}^q_A(A/I^n, K)$$

by Lemma 47.8.2. Choose $f_1, \ldots, f_r \in A$ generating I. Let $K^\bullet_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex with terms in degrees $-r, \ldots, 0$. Since the pro-objects

$\{A/I^n\}$ and $\{K_n^\bullet\}$ in $D(A)$ are the same by More on Algebra, Lemma 15.95.1, we see that

$$R^q\Gamma_I(K) = \operatorname{colim} \operatorname{Ext}^q_A(K_n^\bullet, K)$$

Pick any complex $K^\bullet$ of A-modules representing K. Since $K_n^\bullet$ is a finite complex of finite free modules we see that

$$\operatorname{Ext}^q_A(K_n, K) = H^q(\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet))$$

where $(K_n^\bullet)^\vee$ is the dual of the complex $K_n^\bullet$. See More on Algebra, Lemma 15.74.2. As $(K_n^\bullet)^\vee$ is a complex of finite free A-modules sitting in degrees $0, \ldots, r$ we see that the terms of the complex $\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet)$ are the same as the terms of the complex $\operatorname{Tot}((K_n^\bullet)^\vee \otimes_A \tau_{\geq q-r-2}K^\bullet)$ in degrees $q-1$ and higher. Hence we see that

$$\operatorname{Ext}^q_A(K_n, K) = \operatorname{Ext}^q_A(K_n, \tau_{\geq q-r-2}K)$$

for all n. It follows that

$$R^q\Gamma_I(K) = R^q\Gamma_I(\tau_{\geq q-r-2}K) = H^q(\tau_{\geq q-r-2}K) = H^q(K)$$

Thus we see that the map $R\Gamma_I(K) \to K$ is an isomorphism. □

0956 **Lemma 47.10.2.** *Let A be a Noetherian ring and let $I = (f_1, \ldots, f_r)$ be an ideal of A. Set $Z = V(I) \subset \operatorname{Spec}(A)$. There are canonical isomorphisms*

$$R\Gamma_I(A) \to (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \to R\Gamma_Z(A)$$

in $D(A)$. If M is an A-module, then we have similarly

$$R\Gamma_I(M) \cong (M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1\ldots f_r}) \cong R\Gamma_Z(M)$$

in $D(A)$.

Proof. This follows from Lemma 47.10.1 and the computation of the functor $R\Gamma_Z$ in Lemma 47.9.1. □

0957 **Lemma 47.10.3.** *If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then in $D(B)$ we have*

$$R\Gamma_I(A) \otimes^{\mathbf{L}}_A B = R\Gamma_Z(A) \otimes^{\mathbf{L}}_A B = R\Gamma_Y(B) = R\Gamma_{IB}(B)$$

where $Y = V(IB) \subset \operatorname{Spec}(B)$.

Proof. Combine Lemmas 47.10.2 and 47.9.3. □

47.11. Depth

0AVY In this section we revisit the notion of depth introduced in Algebra, Section 10.72.

0AVZ **Lemma 47.11.1.** *Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M be a finite A-module such that $IM \neq M$. Then the following integers are equal:*

1. *$\operatorname{depth}_I(M)$,*
2. *the smallest integer i such that $\operatorname{Ext}^i_A(A/I, M)$ is nonzero, and*
3. *the smallest integer i such that $H^i_I(M)$ is nonzero.*

Moreover, we have $\operatorname{Ext}^i_A(N, M) = 0$ for $i < \operatorname{depth}_I(M)$ for any finite A-module N annihilated by a power of I.

Proof. We prove the equality of (1) and (2) by induction on $\text{depth}_I(M)$ which is allowed by Algebra, Lemma 10.72.4.

Base case. If $\text{depth}_I(M) = 0$, then I is contained in the union of the associated primes of M (Algebra, Lemma 10.63.9). By prime avoidance (Algebra, Lemma 10.15.2) we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\text{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\text{depth}_I(M) > 0$. Let $f \in I$ be a nonzerodivisor on M such that $\text{depth}_I(M/fM) = \text{depth}_I(M) - 1$. Consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

and the associated long exact sequence for $\text{Ext}^*_A(A/I, -)$. Note that $\text{Ext}^i_A(A/I, M)$ is a finite A/I-module (Algebra, Lemmas 10.71.9 and 10.71.8). Hence we obtain

$$\text{Hom}_A(A/I, M/fM) = \text{Ext}^1_A(A/I, M)$$

and short exact sequences

$$0 \to \text{Ext}^i_A(A/I, M) \to \text{Ext}^i_A(A/I, M/fM) \to \text{Ext}^{i+1}_A(A/I, M) \to 0$$

Thus the equality of (1) and (2) by induction.

Observe that $\text{depth}_I(M) = \text{depth}_{I^n}(M)$ for all $n \geq 1$ for example by Algebra, Lemma 10.68.9. Hence by the equality of (1) and (2) we see that $\text{Ext}^i_A(A/I^n, M) = 0$ for all n and $i < \text{depth}_I(M)$. Let N be a finite A-module annihilated by a power of I. Then we can choose a short exact sequence

$$0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0$$

for some $n, m \geq 0$. Then $\text{Hom}_A(N, M) \subset \text{Hom}_A((A/I^n)^{\oplus m}, M)$ and $\text{Ext}^i_A(N, M) \subset \text{Ext}^{i-1}_A(N', M)$ for $i < \text{depth}_I(M)$. Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have $H^p_I(M) = \text{colim}\, \text{Ext}^p_A(A/I^n, M)$ by Lemma 47.8.2. Thus we see that $H^i_I(M) = 0$ for $i < \text{depth}_I(M)$. For $i = \text{depth}_I(M)$, using the vanishing of $\text{Ext}^{i-1}_A(I/I^n, M)$ we see that the map $\text{Ext}^i_A(A/I, M) \to H^i_I(M)$ is injective which proves nonvanishing in the correct degree. □

0BUV **Lemma 47.11.2.** *Let A be a Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finite A-modules. Let $I \subset A$ be an ideal.*

(1) $\textit{depth}_I(N) \geq \min\{\textit{depth}_I(N'), \textit{depth}_I(N'')\}$
(2) $\textit{depth}_I(N'') \geq \min\{\textit{depth}_I(N), \textit{depth}_I(N') - 1\}$
(3) $\textit{depth}_I(N') \geq \min\{\textit{depth}_I(N), \textit{depth}_I(N'') + 1\}$

Proof. Assume $IN \neq N$, $IN' \neq N'$, and $IN'' \neq N''$. Then we can use the characterization of depth using the Ext groups $\text{Ext}^i(A/I, N)$, see Lemma 47.11.1, and use the long exact cohomology sequence

$$\begin{aligned} 0 \to \text{Hom}_A(A/I, N') \to \text{Hom}_A(A/I, N) \to \text{Hom}_A(A/I, N'') \\ \to \text{Ext}^1_A(A/I, N') \to \text{Ext}^1_A(A/I, N) \to \text{Ext}^1_A(A/I, N'') \to \ldots \end{aligned}$$

from Algebra, Lemma 10.71.6. This argument also works if $IN = N$ because in this case $\text{Ext}^i_A(A/I, N) = 0$ for all i. Similarly in case $IN' \neq N'$ and/or $IN'' \neq N''$. □

0BUW **Lemma 47.11.3.** *Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let M a finite A-module with $IM \neq M$.*

(1) *If $x \in I$ is a nonzerodivisor on M, then $depth_I(M/xM) = depth_I(M) - 1$.*
(2) *Any M-regular sequence $x_1, \ldots, x_r$ in I can be extended to an M-regular sequence in I of length $depth_I(M)$.*

Proof. Part (2) is a formal consequence of part (1). Let $x \in I$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 47.11.2 we see that $\text{depth}_I(M/xM) \geq \text{depth}_I(M) - 1$. On the other hand, if $x_1, \ldots, x_r \in I$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence (1) holds. □

0BUX **Lemma 47.11.4.** *Let R be a Noetherian local ring. If M is a finite Cohen-Macaulay R-module and $I \subset R$ a nontrivial ideal. Then*

$$depth_I(M) = \dim(Supp(M)) - \dim(Supp(M/IM)).$$

Proof. We will prove this by induction on $\text{depth}_I(M)$.

If $\text{depth}_I(M) = 0$, then I is contained in one of the associated primes $\mathfrak{p}$ of M (Algebra, Lemma 10.63.18). Then $\mathfrak{p} \in \text{Supp}(M/IM)$, hence $\dim(\text{Supp}(M/IM)) \geq \dim(R/\mathfrak{p}) = \dim(\text{Supp}(M))$ where equality holds by Algebra, Lemma 10.103.7. Thus the lemma holds in this case.

If $\text{depth}_I(M) > 0$, we pick $x \in I$ which is a nonzerodivisor on M. Note that $(M/xM)/I(M/xM) = M/IM$. On the other hand we have $\text{depth}_I(M/xM) = \text{depth}_I(M) - 1$ by Lemma 47.11.3 and $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$ by Algebra, Lemma 10.63.10. Thus the result by induction hypothesis. □

0BUY **Lemma 47.11.5.** *Let $R \to S$ be a flat local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m} \subset R$ the maximal ideal. Let $I \subset S$ be an ideal. If $S/\mathfrak{m}S$ is Cohen-Macaulay, then*

$$depth_I(S) \geq \dim(S/\mathfrak{m}S) - \dim(S/\mathfrak{m}S + I)$$

Proof. By Algebra, Lemma 10.99.3 any sequence in S which maps to a regular sequence in $S/\mathfrak{m}S$ is a regular sequence in S. Thus it suffices to prove the lemma in case R is a field. This is a special case of Lemma 47.11.4. □

0AW0 **Lemma 47.11.6.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Let $Z = V(I)$. Then $H^0_I(M) = H^0_Z(M)$. Let N be the common value and set $M' = M/N$. Then*

(1) *$H^0_I(M') = 0$ and $H^p_I(M) = H^p_I(M')$ and $H^p_I(N) = 0$ for all $p > 0$,*
(2) *$H^0_Z(M') = 0$ and $H^p_Z(M) = H^p_Z(M')$ and $H^p_Z(N) = 0$ for all $p > 0$.*

Proof. By definition $H^0_I(M) = M[I^\infty]$ is I-power torsion. By Lemma 47.9.1 we see that

$$H^0_Z(M) = \text{Ker}(M \longrightarrow M_{f_1} \times \ldots \times M_{f_r})$$

if $I = (f_1, \ldots, f_r)$. Thus $H^0_I(M) \subset H^0_Z(M)$ and conversely, if $x \in H^0_Z(M)$, then it is annihilated by a $f_i^{e_i}$ for some $e_i \geq 1$ hence annihilated by some power of I. This proves the first equality and moreover N is I-power torsion. By Lemma 47.8.1 we see that $R\Gamma_I(N) = N$. By Lemma 47.9.1 we see that $R\Gamma_Z(N) = N$. This proves the higher vanishing of $H^p_I(N)$ and $H^p_Z(N)$ in (1) and (2). The vanishing of $H^0_I(M')$ and $H^0_Z(M')$ follow from the preceding remarks and the fact that M' is I-power torsion free by More on Algebra, Lemma 15.89.4. The equality of higher cohomologies for M and M' follow immediately from the long exact cohomology sequence. □

47.12. Torsion versus complete modules

0A6V Let A be a ring and let I be a finitely generated ideal. In this case we can consider the derived category $D_{I^\infty\text{-torsion}}(A)$ of complexes with I-power torsion cohomology modules (Section 47.9) and the derived category $D_{comp}(A, I)$ of derived complete complexes (More on Algebra, Section 15.92). In this section we show these categories are equivalent. A more general statement can be found in [**DG02**].

0A6W **Lemma 47.12.1.** *Let A be a ring and let I be a finitely generated ideal. Let $R\Gamma_Z$ be as in Lemma 47.9.1. Let $^\wedge$ denote derived completion as in More on Algebra, Lemma 15.92.10. For an object K in $D(A)$ we have*

$$R\Gamma_Z(K^\wedge) = R\Gamma_Z(K) \quad \text{and} \quad (R\Gamma_Z(K))^\wedge = K^\wedge$$

in $D(A)$.

Proof. Choose $f_1, \ldots, f_r \in A$ generating I. Recall that

$$K^\wedge = R\operatorname{Hom}_A\left((A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

by More on Algebra, Lemma 15.92.10. Hence the cone $C = \operatorname{Cone}(K \to K^\wedge)$ is given by

$$R\operatorname{Hom}_A\left((\prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

which can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$R\operatorname{Hom}_A(A_{f_{i_0 \ldots f_{i_p}}}, K), \quad p > 0$$

These complexes vanish on applying $R\Gamma_Z$, see Lemma 47.9.4. Applying $R\Gamma_Z$ to the distinguished triangle $K \to K^\wedge \to C \to K[1]$ we see that the first formula of the lemma is correct.

Recall that

$$R\Gamma_Z(K) = K \otimes^{\mathbf{L}} (A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r})$$

by Lemma 47.9.1. Hence the cone $C = \operatorname{Cone}(R\Gamma_Z(K) \to K)$ can be represented by a complex endowed with a finite filtration whose successive quotients are isomorphic to

$$K \otimes_A A_{f_{i_0 \ldots f_{i_p}}}, \quad p > 0$$

These complexes vanish on applying $^\wedge$, see More on Algebra, Lemma 15.92.12. Applying derived completion to the distinguished triangle $R\Gamma_Z(K) \to K \to C \to R\Gamma_Z(K)[1]$ we see that the second formula of the lemma is correct. □

The following result is a special case of a very general phenomenon concerning admissible subcategories of a triangulated category.

0A6X **Proposition 47.12.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functors $R\Gamma_Z$ and $^\wedge$ define quasi-inverse equivalences of categories*

$$D_{I^\infty\text{-}torsion}(A) \leftrightarrow D_{comp}(A, I)$$

This is a special case of [**PSY14b**, Theorem 1.1].

Proof. Follows immediately from Lemma 47.12.1. □

The following addendum of the proposition above makes the correspondence on morphisms more precise.

0A6Y **Lemma 47.12.3.** *With notation as in Lemma 47.12.1. For objects K, L in $D(A)$ there is a canonical isomorphism*

$$R\operatorname{Hom}_A(K^\wedge, L^\wedge) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

in $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Denote $C = (A \to \prod A_{f_i} \to \ldots \to A_{f_1 \ldots f_r})$ the alternating Čech complex. Then derived completion is given by $R\operatorname{Hom}_A(C, -)$ (More on Algebra, Lemma 15.92.10) and local cohomology by $C \otimes^{\mathbf{L}} -$ (Lemma 47.9.1). Combining the isomorphism

$$R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C) = R\operatorname{Hom}_A(K, R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C))$$

(More on Algebra, Lemma 15.74.1) and the map

$$L \to R\operatorname{Hom}_A(C, L \otimes^{\mathbf{L}} C)$$

(More on Algebra, Lemma 15.74.6) we obtain a map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C)$$

On the other hand, the right hand side is derived complete as it is equal to

$$R\operatorname{Hom}_A(C, R\operatorname{Hom}_A(K, L \otimes^{\mathbf{L}} C)).$$

Thus γ factors through the derived completion of $R\operatorname{Hom}_A(K, L)$ by the universal property of derived completion. However, the derived completion goes inside the $R\operatorname{Hom}_A$ by More on Algebra, Lemma 15.92.13 and we obtain the desired map.

To show that the map of the lemma is an isomorphism we may assume that K and L are derived complete, i.e., $K = K^\wedge$ and $L = L^\wedge$. In this case we are looking at the map

$$\gamma : R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L))$$

By Proposition 47.12.2 we know that the cohomology groups of the left and the right hand side coincide. In other words, we have to check that the map γ sends a morphism $\alpha : K \to L$ in $D(A)$ to the morphism $R\Gamma_Z(\alpha) : R\Gamma_Z(K) \to R\Gamma_Z(L)$. We omit the verification (hint: note that $R\Gamma_Z(\alpha)$ is just the map $\alpha \otimes \mathrm{id}_C : K \otimes^{\mathbf{L}} C \to L \otimes^{\mathbf{L}} C$ which is almost the same as the construction of the map in More on Algebra, Lemma 15.74.6). □

0EEW **Lemma 47.12.4.** *Let I and J be ideals in a Noetherian ring A. Let M be a finite A-module. Set $Z = V(J)$. Consider the derived I-adic completion $R\Gamma_Z(M)^\wedge$ of local cohomology. Then*

1. *we have $R\Gamma_Z(M)^\wedge = R\lim R\Gamma_Z(M/I^nM)$, and*
2. *there are short exact sequences*

$$0 \to R^1\lim H^{i-1}_Z(M/I^nM) \to H^i(R\Gamma_Z(M)^\wedge) \to \lim H^i_Z(M/I^nM) \to 0$$

In particular $R\Gamma_Z(M)^\wedge$ has vanishing cohomology in negative degrees.

Proof. Suppose that $J = (g_1, \ldots, g_m)$. Then $R\Gamma_Z(M)$ is computed by the complex

$$M \to \prod M_{g_{j_0}} \to \prod M_{g_{j_0} g_{j_1}} \to \ldots \to M_{g_1 g_2 \ldots g_m}$$

by Lemma 47.9.1. By More on Algebra, Lemma 15.95.6 the derived I-adic completion of this complex is given by the complex

$$\lim M/I^nM \to \prod \lim (M/I^nM)_{g_{j_0}} \to \ldots \to \lim (M/I^nM)_{g_1 g_2 \ldots g_m}$$

of usual completions. Since $R\Gamma_Z(M/I^nM)$ is computed by the complex $M/I^nM \to \prod (M/I^nM)_{g_{j_0}} \to \ldots \to (M/I^nM)_{g_1g_2\ldots g_m}$ and since the transition maps between these complexes are surjective, we conclude that (1) holds by More on Algebra, Lemma 15.88.1. Part (2) then follows from More on Algebra, Lemma 15.88.4. □

0EEX **Lemma 47.12.5.** *With notation and hypotheses as in Lemma 47.12.4 assume A is I-adically complete. Then*

$$H^0(R\Gamma_Z(M)^\wedge) = \operatorname{colim} H^0_{V(J')}(M)$$

where the filtered colimit is over $J' \subset J$ such that $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof. Since M is a finite A-module, we have that M is I-adically complete. The proof of Lemma 47.12.4 shows that

$$H^0(R\Gamma_Z(M)^\wedge) = \operatorname{Ker}(M^\wedge \to \prod M^\wedge_{g_j}) = \operatorname{Ker}(M \to \prod M^\wedge_{g_j})$$

where on the right hand side we have usual I-adic completion. The kernel K_j of $M_{g_j} \to M^\wedge_{g_j}$ is $\bigcap I^n M_{g_j}$. By Algebra, Lemma 10.51.5 for every $\mathfrak{p} \in V(IA_{g_j})$ we find an $f \in A_{g_j}$, $f \notin \mathfrak{p}$ such that $(K_j)_f = 0$.

Let $s \in H^0(R\Gamma_Z(M)^\wedge)$. By the above we may think of s as an element of M. The support Z' of s intersected with $D(g_j)$ is disjoint from $D(g_j) \cap V(I)$ by the arguments above. Thus Z' is a closed subset of $\operatorname{Spec}(A)$ with $Z' \cap V(I) \subset V(J)$. Then $Z' \cup V(J) = V(J')$ for some ideal $J' \subset J$ with $V(J') \cap V(I) \subset V(J)$ and we have $s \in H^0_{V(J')}(M)$. Conversely, any $s \in H^0_{V(J')}(M)$ with $J' \subset J$ and $V(J') \cap V(I) \subset V(J)$ maps to zero in $M^\wedge_{g_j}$ for all j. This proves the lemma. □

47.13. Trivial duality for a ring map

0A6Z Let $A \to B$ be a ring homomorphism. Consider the functor

$$\operatorname{Hom}_A(B, -) : \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_B, \quad M \longmapsto \operatorname{Hom}_A(B, M)$$

This functor is left exact and has a derived extension $R\operatorname{Hom}(B, -) : D(A) \to D(B)$.

0A70 **Lemma 47.13.1.** *Let $A \to B$ be a ring homomorphism. The functor $R\operatorname{Hom}(B, -)$ constructed above is right adjoint to the restriction functor $D(B) \to D(A)$.*

Proof. This is a consequence of the fact that restriction and $\operatorname{Hom}_A(B, -)$ are adjoint functors by Algebra, Lemma 10.14.4. See Derived Categories, Lemma 13.30.3. □

0C0F **Lemma 47.13.2.** *Let $A \to B \to C$ be ring maps. Then $R\operatorname{Hom}(C, -) \circ R\operatorname{Hom}(B, -) : D(A) \to D(C)$ is the functor $R\operatorname{Hom}(C, -) : D(A) \to D(C)$.*

Proof. Follows from uniqueness of right adjoints and Lemma 47.13.1. □

0A71 **Lemma 47.13.3.** *Let $\varphi : A \to B$ be a ring homomorphism. For K in $D(A)$ we have*

$$\varphi_* R\operatorname{Hom}(B, K) = R\operatorname{Hom}_A(B, K)$$

where $\varphi_* : D(B) \to D(A)$ is restriction. In particular $R^q \operatorname{Hom}(B, K) = \operatorname{Ext}^q_A(B, K)$.

Proof. Choose a K-injective complex $I^\bullet$ representing K. Then $R\operatorname{Hom}(B, K)$ is represented by the complex $\operatorname{Hom}_A(B, I^\bullet)$ of B-modules. Since this complex, as a complex of A-modules, represents $R\operatorname{Hom}_A(B, K)$ we see that the lemma is true. □

Let A be a Noetherian ring. We will denote

$$D_{Coh}(A) \subset D(A)$$

the full subcategory consisting of those objects K of $D(A)$ whose cohomology modules are all finite A-modules. This makes sense by Derived Categories, Section 13.17 because as A is Noetherian, the subcategory of finite A-modules is a Serre subcategory of Mod_A.

0A72 **Lemma 47.13.4.** *With notation as above, assume $A \to B$ is a finite ring map of Noetherian rings. Then $R\mathrm{Hom}(B, -)$ maps $D^+_{Coh}(A)$ into $D^+_{Coh}(B)$.*

Proof. We have to show: if $K \in D^+(A)$ has finite cohomology modules, then the complex $R\mathrm{Hom}(B, K)$ has finite cohomology modules too. This follows for example from Lemma 47.13.3 if we can show the ext modules $\mathrm{Ext}^i_A(B, K)$ are finite A-modules. Since K is bounded below there is a convergent spectral sequence

$$\mathrm{Ext}^p_A(B, H^q(K)) \Rightarrow \mathrm{Ext}^{p+q}_A(B, K)$$

This finishes the proof as the modules $\mathrm{Ext}^p_A(B, H^q(K))$ are finite by Algebra, Lemma 10.71.9. □

0A73 **Remark 47.13.5.** Let A be a ring and let $I \subset A$ be an ideal. Set $B = A/I$. In this case the functor $\mathrm{Hom}_A(B, -)$ is equal to the functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M[I]$$

which sends M to the submodule of I-torsion.

0BZB **Situation 47.13.6.** Let $R \to A$ be a ring map. We will give an alternative construction of $R\mathrm{Hom}(A, -)$ which will stand us in good stead later in this chapter. Namely, suppose we have a differential graded algebra (E, d) over R and a quasi-isomorphism $E \to A$ where we view A as a differential graded algebra over R with zero differential. Then we have commutative diagrams

$$\begin{array}{ccc} D(E, \mathrm{d}) & \longleftarrow & D(A) \\ & \searrow \quad \swarrow & \\ & D(R) & \end{array} \quad \text{and} \quad \begin{array}{ccc} D(E, \mathrm{d}) & \xrightarrow[-\otimes^{\mathbf{L}}_E A]{} & D(A) \\ & \nwarrow_{-\otimes^{\mathbf{L}}_R E} \quad \nearrow_{-\otimes^{\mathbf{L}}_R A} & \\ & D(R) & \end{array}$$

where the horizontal arrows are equivalences of categories (Differential Graded Algebra, Lemma 22.37.1). It is clear that the first diagram commutes. The second diagram commutes because the first one does and our functors are their left adjoints (Differential Graded Algebra, Example 22.33.6) or because we have $E \otimes^{\mathbf{L}}_E A = E \otimes_E A$ and we can use Differential Graded Algebra, Lemma 22.34.1.

0BZC **Lemma 47.13.7.** *In Situation 47.13.6 the functor $R\mathrm{Hom}(A, -)$ is equal to the composition of $R\mathrm{Hom}(E, -) : D(R) \to D(E, d)$ and the equivalence $- \otimes^{\mathbf{L}}_E A : D(E, d) \to D(A)$.*

Proof. This is true because $R\mathrm{Hom}(E, -)$ is the right adjoint to $- \otimes^{\mathbf{L}}_R E$, see Differential Graded Algebra, Lemma 22.33.5. Hence this functor plays the same role as the functor $R\mathrm{Hom}(A, -)$ for the map $R \to A$ (Lemma 47.13.1), whence these functors must correspond via the equivalence $- \otimes^{\mathbf{L}}_E A : D(E, \mathrm{d}) \to D(A)$. □

0BZD **Lemma 47.13.8.** *In Situation 47.13.6 assume that*

(1) *E viewed as an object of $D(R)$ is compact, and*
(2) *$N = \mathrm{Hom}_R^\bullet(E^\bullet, R)$ computes $R\mathrm{Hom}(E, R)$.*

Then $R\mathrm{Hom}(E, -) : D(R) \to D(E)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$.

Proof. Special case of Differential Graded Algebra, Lemma 22.33.9. □

0BZE **Lemma 47.13.9.** *In Situation 47.13.6 assume A is a perfect R-module. Then*

$$R\mathrm{Hom}(A, -) : D(R) \to D(A)$$

is given by $K \mapsto K \otimes_R^{\mathbf{L}} M$ where $M = R\mathrm{Hom}(A, R) \in D(A)$.

Proof. We apply Divided Power Algebra, Lemma 23.6.10 to choose a Tate resolution (E, d) of A over R. Note that $E^i = 0$ for $i > 0$, $E^0 = R[x_1, \ldots, x_n]$ is a polynomial algebra, and E^i is a finite free E^0-module for $i < 0$. It follows that E viewed as a complex of R-modules is a bounded above complex of free R-modules. We check the assumptions of Lemma 47.13.8. The first holds because A is perfect (hence compact by More on Algebra, Proposition 15.79.3) and the second by More on Algebra, Lemma 15.74.2. From the lemma conclude that $K \mapsto R\mathrm{Hom}(E, K)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N$ for some differential graded E-module N. Observe that

$$(R \otimes_R E) \otimes_E^{\mathbf{L}} A = R \otimes_E E \otimes_E A$$

in $D(A)$. Hence by Differential Graded Algebra, Lemma 22.34.2 we conclude that the composition of $- \otimes_R^{\mathbf{L}} N$ and $- \otimes_R^{\mathbf{L}} A$ is of the form $- \otimes_R M$ for some $M \in D(A)$. To finish the proof we apply Lemma 47.13.7. □

0BZH **Lemma 47.13.10.** *Let $R \to A$ be a surjective ring map whose kernel I is an invertible R-module. The functor $R\mathrm{Hom}(A, -) : D(R) \to D(A)$ is isomorphic to $K \mapsto K \otimes_R^{\mathbf{L}} N[-1]$ where N is inverse of the invertible A-module $I \otimes_R A$.*

Proof. Since A has the finite projective resolution

$$0 \to I \to R \to A \to 0$$

we see that A is a perfect R-module. By Lemma 47.13.9 it suffices to prove that $R\mathrm{Hom}(A, R)$ is represented by $N[-1]$ in $D(A)$. This means $R\mathrm{Hom}(A, R)$ has a unique nonzero cohomology module, namely N in degree 1. As $\mathrm{Mod}_A \to \mathrm{Mod}_R$ is fully faithful it suffice to prove this after applying the restriction functor $i_* : D(A) \to D(R)$. By Lemma 47.13.3 we have

$$i_* R\mathrm{Hom}(A, R) = R\mathrm{Hom}_R(A, R)$$

Using the finite projective resolution above we find that the latter is represented by the complex $R \to I^{\otimes -1}$ with R in degree 0. The map $R \to I^{\otimes -1}$ is injective and the cokernel is N. □

47.14. Base change for trivial duality

0E28 In this section we consider a cocartesian square of rings

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ R & \xrightarrow{\rho} & R' \end{array}$$

In other words, we have $A' = A \otimes_R R'$. If A and R' are **tor independent over** R then there is a canonical base change map

0E29 (47.14.0.1) $$R\mathrm{Hom}(A, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}(A', K \otimes^{\mathbf{L}}_R R')$$

in $D(A')$ functorial for K in $D(R)$. Namely, by the adjointness of Lemma 47.13.1 such an arrow is the same thing as a map

$$\varphi'_* \left(R\mathrm{Hom}(A, K) \otimes^{\mathbf{L}}_A A'\right) \longrightarrow K \otimes^{\mathbf{L}}_R R'$$

in $D(R')$ where $\varphi'_* : D(A') \to D(R')$ is the restriction functor. We may apply More on Algebra, Lemma 15.61.2 to the left hand side to get that this is the same thing as a map

$$\varphi_*(R\mathrm{Hom}(A, K)) \otimes^{\mathbf{L}}_R R' \longrightarrow K \otimes^{\mathbf{L}}_R R'$$

in $D(R')$ where $\varphi_* : D(A) \to D(R)$ is the restriction functor. For this we can choose $can \otimes^{\mathbf{L}} \mathrm{id}_{R'}$ where $can : \varphi_*(R\mathrm{Hom}(A, K)) \to K$ is the counit of the adjunction between $R\mathrm{Hom}(A, -)$ and φ_*.

0E2A **Lemma 47.14.1.** *In the situation above, the map (47.14.0.1) is an isomorphism if and only if the map*

$$R\mathrm{Hom}_R(A, K) \otimes^{\mathbf{L}}_R R' \longrightarrow R\mathrm{Hom}_R(A, K \otimes^{\mathbf{L}}_R R')$$

of More on Algebra, Lemma 15.74.5 is an isomorphism.

Proof. To see that the map is an isomorphism, it suffices to prove it is an isomorphism after applying φ'_*. Applying the functor φ'_* to (47.14.0.1) and using that $A' = A \otimes^{\mathbf{L}}_R R'$ we obtain the base change map $R\mathrm{Hom}_R(A, K) \otimes^{\mathbf{L}}_R R' \to R\mathrm{Hom}_{R'}(A \otimes^{\mathbf{L}}_R R', K \otimes^{\mathbf{L}}_R R')$ for derived hom of More on Algebra, Equation (15.100.1.1). Unwinding the left and right hand side exactly as in the proof of More on Algebra, Lemma 15.100.2 and in particular using More on Algebra, Lemma 15.100.1 gives the desired result. □

0BZM **Lemma 47.14.2.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

1. *A is pseudo-coherent as an R-module,*
2. *R' has finite tor dimension as an R-module (for example $R \to R'$ is flat),*
3. *A and R' are tor independent over R.*

Then (47.14.0.1) is an isomorphism for $K \in D^+(R)$.

Proof. Follows from Lemma 47.14.1 and More on Algebra, Lemma 15.99.3 part (4). □

0BZP **Lemma 47.14.3.** *Let $R \to A$ and $R \to R'$ be ring maps and $A' = A \otimes_R R'$. Assume*

1. *A is perfect as an R-module,*
2. *A and R' are tor independent over R.*

Then (47.14.0.1) is an isomorphism for all $K \in D(R)$.

Proof. Follows from Lemma 47.14.1 and More on Algebra, Lemma 15.99.3 part (1). □

47.15. Dualizing complexes

0A7A In this section we define dualizing complexes for Noetherian rings.

0A7B **Definition 47.15.1.** Let A be a Noetherian ring. A *dualizing complex* is a complex of A-modules $\omega_A^\bullet$ such that

(1) $\omega_A^\bullet$ has finite injective dimension,
(2) $H^i(\omega_A^\bullet)$ is a finite A-module for all i, and
(3) $A \to R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.

This definition takes some time getting used to. It is perhaps a good idea to prove some of the following lemmas yourself without reading the proofs.

0G4H **Lemma 47.15.2.** *Let A be a Noetherian ring. Let $K, L \in D_{Coh}(A)$ and assume L has finite injective dimension. Then $R\mathrm{Hom}_A(K, L)$ is in $D_{Coh}(A)$.*

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 13.12.4. Since L has finite injective dimension we see that $R\mathrm{Hom}_A(\tau_{\geq n+1}K, L)$ has vanishing cohomology in degrees $\geq c - n$ for some constant c. Hence, given i, we see that $\mathrm{Ext}^i_A(K, L) \to \mathrm{Ext}^i_A(\tau_{\leq n}K, L)$ is an isomorphism for some $n \gg -i$. By Derived Categories of Schemes, Lemma 36.11.5 applied to $\tau_{\leq n}K$ and L we see conclude that $\mathrm{Ext}^i_A(K, L)$ is a finite A-module for all i. Hence $R\mathrm{Hom}_A(K, L)$ is indeed an object of $D_{Coh}(A)$. □

0A7C **Lemma 47.15.3.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then the functor*

$$D : K \longmapsto R\mathrm{Hom}_A(K, \omega_A^\bullet)$$

is an anti-equivalence $D_{Coh}(A) \to D_{Coh}(A)$ which exchanges $D^+_{Coh}(A)$ and $D^-_{Coh}(A)$ and induces an anti-equivalence $D^b_{Coh}(A) \to D^b_{Coh}(A)$. Moreover $D \circ D$ is isomorphic to the identity functor.

Proof. Let K be an object of $D_{Coh}(A)$. From Lemma 47.15.2 we see $R\mathrm{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. By More on Algebra, Lemma 15.99.2 and the assumptions on the dualizing complex we obtain a canonical isomorphism

$$K = R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes^{\mathbf{L}}_A K \longrightarrow R\mathrm{Hom}_A(R\mathrm{Hom}_A(K, \omega_A^\bullet), \omega_A^\bullet)$$

Thus our functor has a quasi-inverse and the proof is complete. □

Let R be a ring. Recall that an object L of $D(R)$ is *invertible* if it is an invertible object for the symmetric monoidal structure on $D(R)$ given by derived tensor product. In More on Algebra, Lemma 15.127.4 we have seen this means L is perfect, $L = \bigoplus H^n(L)[-n]$, this is a finite sum, each $H^n(L)$ is finite projective, and there is an open covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that $L \otimes_R R_{f_i} \cong R_{f_i}[-n_i]$ for some integers n_i.

0A7E **Lemma 47.15.4.** *Let A be a Noetherian ring. Let $F : D^b_{Coh}(A) \to D^b_{Coh}(A)$ be an A-linear equivalence of categories. Then $F(A)$ is an invertible object of $D(A)$.*

Proof. Let $\mathfrak{m} \subset A$ be a maximal ideal with residue field κ. Consider the object $F(\kappa)$. Since $\kappa = \mathrm{Hom}_{D(A)}(\kappa, \kappa)$ we find that all cohomology groups of $F(\kappa)$ are annihilated by $\mathfrak{m}$. We also see that

$$\mathrm{Ext}^i_A(\kappa, \kappa) = \mathrm{Ext}^i_A(F(\kappa), F(\kappa)) = \mathrm{Hom}_{D(A)}(F(\kappa), F(\kappa)[i])$$

is zero for $i < 0$. Say $H^a(F(\kappa)) \neq 0$ and $H^b(F(\kappa)) \neq 0$ with a minimal and b maximal (so in particular $a \leq b$). Then there is a nonzero map

$$F(\kappa) \to H^b(F(\kappa))[-b] \to H^a(F(\kappa))[-b] \to F(\kappa)[a-b]$$

in $D(A)$ (nonzero because it induces a nonzero map on cohomology). This proves that $b = a$. We conclude that $F(\kappa) = \kappa[-a]$.

Let G be a quasi-inverse to our functor F. Arguing as above we find an integer b such that $G(\kappa) = \kappa[-b]$. On composing we find $a + b = 0$. Let E be a finite A-module which is annihilated by a power of $\mathfrak{m}$. Arguing by induction on the length of E we find that $G(E) = E'[-b]$ for some finite A-module E' annihilated by a power of $\mathfrak{m}$. Then $E[-a] = F(E')$. Next, we consider the groups

$$\mathrm{Ext}^i_A(A, E') = \mathrm{Ext}^i_A(F(A), F(E')) = \mathrm{Hom}_{D(A)}(F(A), E[-a+i])$$

The left hand side is nonzero if and only if $i = 0$ and then we get E'. Applying this with $E = E' = \kappa$ and using Nakayama's lemma this implies that $H^j(F(A))_\mathfrak{m}$ is zero for $j > a$ and generated by 1 element for $j = a$. On the other hand, if $H^j(F(A))_\mathfrak{m}$ is not zero for some $j < a$, then there is a map $F(A) \to E[-a+i]$ for some $i < 0$ and some E (More on Algebra, Lemma 15.66.7) which is a contradiction. Thus we see that $F(A)_\mathfrak{m} = M[-a]$ for some $A_\mathfrak{m}$-module M generated by 1 element. However, since

$$A_\mathfrak{m} = \mathrm{Hom}_{D(A)}(A, A)_\mathfrak{m} = \mathrm{Hom}_{D(A)}(F(A), F(A))_\mathfrak{m} = \mathrm{Hom}_{A_\mathfrak{m}}(M, M)$$

we see that $M \cong A_\mathfrak{m}$. We conclude that there exists an element $f \in A$, $f \notin \mathfrak{m}$ such that $F(A)_f$ is isomorphic to $A_f[-a]$. This finishes the proof. □

0A7F **Lemma 47.15.5.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ and $(\omega'_A)^\bullet$ are dualizing complexes, then $(\omega'_A)^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes^{\mathbf{L}}_A L$ for some invertible object L of $D(A)$.*

Proof. By Lemmas 47.15.3 and 47.15.4 the functor $K \mapsto R\mathrm{Hom}_A(R\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$ maps A to an invertible object L. In other words, there is an isomorphism

$$L \longrightarrow R\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet)$$

Since L has finite tor dimension, this means that we can apply More on Algebra, Lemma 15.99.2 to see that

$$R\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet) \otimes^{\mathbf{L}}_A K \longrightarrow R\mathrm{Hom}_A(R\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$$

is an isomorphism for K in $D^b_{Coh}(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof. □

0A7G **Lemma 47.15.6.** *Let A be a Noetherian ring. Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules (Lemma 47.3.8) representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\mathrm{Hom}_A(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 15.100.2. □

0A7H **Lemma 47.15.7.** *Let A be a Noetherian ring. Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of A-modules such that $(\omega_A^\bullet)_{f_i}$ is a dualizing complex for A_{f_i} for all i, then $\omega_A^\bullet$ is a dualizing complex for A.*

Proof. Consider the double complex

$$\prod\nolimits_{i_0} (\omega_A^\bullet)_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \to \ldots$$

The associated total complex is quasi-isomorphic to $\omega_A^\bullet$ for example by Descent, Remark 35.3.10 or by Derived Categories of Schemes, Lemma 36.9.4. By assumption the complexes $(\omega_A^\bullet)_{f_i}$ have finite injective dimension as complexes of A_{f_i}-modules. This implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over $A_{f_{i_0} \ldots f_{i_p}}$, see Lemma 47.3.8. This in turn implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over A, see Lemma 47.3.2. Hence $\omega_A^\bullet$ has finite injective dimension as a complex of A-modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since $H^n(\omega_A^\bullet)_{f_i}$ is a finite A_{f_i} module for each i we see that $H^i(\omega_A^\bullet)$ is a finite A-module, see Algebra, Lemma 10.23.2. Finally, the (derived) base change of the map $A \to R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ to A_{f_i} is the map $A_{f_i} \to R\mathrm{Hom}_A((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$ by More on Algebra, Lemma 15.100.2. Hence we deduce that $A \to R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is an isomorphism and the proof is complete. □

0AX0 **Lemma 47.15.8.** *Let $A \to B$ be a finite ring map of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\mathrm{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\mathrm{Hom}_A(B, I^\bullet)$ is a bounded complex of injective B-modules (Lemma 47.3.4) representing $R\mathrm{Hom}(B, \omega_A^\bullet)$. Thus $R\mathrm{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By Lemma 47.13.4 it is an object of $D_{Coh}(B)$. Finally, we compute

$$\mathrm{Hom}_{D(B)}(R\mathrm{Hom}(B, \omega_A^\bullet), R\mathrm{Hom}(B, \omega_A^\bullet)) = \mathrm{Hom}_{D(A)}(R\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for $n \neq 0$ we compute

$$\mathrm{Hom}_{D(B)}(R\mathrm{Hom}(B, \omega_A^\bullet), R\mathrm{Hom}(B, \omega_A^\bullet)[n]) = \mathrm{Hom}_{D(A)}(R\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the first equality holds by Lemma 47.13.1 and the second equality holds by Lemma 47.15.3. □

0A7I **Lemma 47.15.9.** *Let $A \to B$ be a surjective homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\mathrm{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Special case of Lemma 47.15.8. □

0A7J **Lemma 47.15.10.** *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.*

Proof. Set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. It follows from Lemma 47.3.10 and More on Algebra, Lemma 15.70.5 that $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\mathrm{Hom}_B(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 15.100.2. □

0A7K **Proposition 47.15.11.** *Let A be a Noetherian ring which has a dualizing complex. Then any A-algebra essentially of finite type over A has a dualizing complex.*

Proof. This follows from a combination of Lemmas 47.15.6, 47.15.9, and 47.15.10. □

0A7L **Lemma 47.15.12.** *Let A be a Noetherian ring. Let $\omega_A^\bullet$ be a dualizing complex. Let $\mathfrak{m} \subset A$ be a maximal ideal and set $\kappa = A/\mathfrak{m}$. Then $R\mathrm{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbf{Z}$.*

Proof. This is true because $R\mathrm{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over κ (Lemma 47.15.9), because dualizing complexes over κ are unique up to shifts (Lemma 47.15.5), and because κ is a dualizing complex over κ. □

47.16. Dualizing complexes over local rings

0A7M In this section $(A, \mathfrak{m}, \kappa)$ will be a Noetherian local ring endowed with a dualizing complex $\omega_A^\bullet$ such that the integer n of Lemma 47.15.12 is zero. More precisely, we assume that $R\mathrm{Hom}_A(\kappa, \omega_A^\bullet) = \kappa[0]$. In this case we will say that the dualizing complex is *normalized*. Observe that a normalized dualizing complex is unique up to isomorphism and that any other dualizing complex for A is isomorphic to a shift of a normalized one (Lemma 47.15.5).

0AX1 **Lemma 47.16.1.** *Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{m}', \kappa')$ be a finite local map of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\omega_B^\bullet = R\mathrm{Hom}(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.*

Proof. By Lemma 47.15.8 the complex $\omega_B^\bullet$ is dualizing for B. We have

$$R\mathrm{Hom}_B(\kappa', \omega_B^\bullet) = R\mathrm{Hom}_B(\kappa', R\mathrm{Hom}(B, \omega_A^\bullet)) = R\mathrm{Hom}_A(\kappa', \omega_A^\bullet)$$

by Lemma 47.13.1. Since κ' is isomorphic to a finite direct sum of copies of κ as an A-module and since $\omega_A^\bullet$ is normalized, we see that this complex only has cohomology placed in degree 0. Thus $\omega_B^\bullet$ is a normalized dualizing complex as well. □

0A7N **Lemma 47.16.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $A \to B$ be surjective. Then $\omega_B^\bullet = R\mathrm{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.*

Proof. Special case of Lemma 47.16.1. □

0A7P **Lemma 47.16.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let F be an A-linear self-equivalence of the category of finite length A-modules. Then F is isomorphic to the identity functor.*

Proof. Since κ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that F is exact. Hence $F(E)$ has the same length as E for all finite length modules E. Since $\mathrm{Hom}(E, \kappa) = \mathrm{Hom}(F(E), F(\kappa)) \cong \mathrm{Hom}(F(E), \kappa)$ we conclude from Nakayama's lemma that E and $F(E)$ have the same number of generators. Hence $F(A/\mathfrak{m}^n)$ is a cyclic A-module. Pick a generator $e \in F(A/\mathfrak{m}^n)$. Since F is A-linear we conclude that $\mathfrak{m}^n e = 0$. The map $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ has to be an isomorphism as the lengths are equal. Pick an element

$$e \in \lim F(A/\mathfrak{m}^n)$$

which maps to a generator for all n (small argument omitted). Then we obtain a system of isomorphisms $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ compatible with all A-module maps $A/\mathfrak{m}^n \to A/\mathfrak{m}^{n'}$ (by A-linearity of F again). Since any finite length module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma. □

0A7Q **Lemma 47.16.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let E be an injective hull of κ. Then there exists a functorial isomorphism*

$$R\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)[0]$$

for N running through the finite length A-modules.

Proof. By induction on the length of N we see that $R\operatorname{Hom}_A(N, \omega_A^\bullet)$ is a module of finite length sitting in degree 0. Thus $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ induces an anti-equivalence on the category of finite length modules. Since the same is true for $\operatorname{Hom}_A(-, E)$ by Proposition 47.7.8 we see that

$$N \longmapsto \operatorname{Hom}_A(R\operatorname{Hom}_A(N, \omega_A^\bullet), E)$$

is an equivalence as in Lemma 47.16.3. Hence it is isomorphic to the identity functor. Since $\operatorname{Hom}_A(-, E)$ applied twice is the identity (Proposition 47.7.8) we obtain the statement of the lemma. □

0A7U **Lemma 47.16.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module and let $d = \dim(\mathit{Supp}(M))$. Then*

1. *if $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero, then $i \in \{-d, \ldots, 0\}$,*
2. *the dimension of the support of $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is at most $-i$,*
3. *$\mathit{depth}(M)$ is the smallest integer $\delta \geq 0$ such that $\operatorname{Ext}_A^{-\delta}(M, \omega_A^\bullet) \neq 0$.*

Proof. We prove this by induction on d. If $d = 0$, this follows from Lemma 47.16.4 and Matlis duality (Proposition 47.7.8) which guarantees that $\operatorname{Hom}_A(M, E)$ is nonzero if M is nonzero.

Assume the result holds for modules with support of dimension $< d$ and that M has depth > 0. Choose an $f \in \mathfrak{m}$ which is a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

Since $\dim(\operatorname{Supp}(M/fM)) = d - 1$ (Algebra, Lemma 10.63.10) we may apply the induction hypothesis. Writing $E^i = \operatorname{Ext}_A^i(M, \omega_A^\bullet)$ and $F^i = \operatorname{Ext}_A^i(M/fM, \omega_A^\bullet)$ we obtain a long exact sequence

$$\ldots \to F^i \to E^i \xrightarrow{f} E^i \to F^{i+1} \to \ldots$$

By induction $E^i/fE^i = 0$ for $i+1 \notin \{-\dim(\operatorname{Supp}(M/fM)), \ldots, -\operatorname{depth}(M/fM)\}$. By Nakayama's lemma (Algebra, Lemma 10.20.1) and Algebra, Lemma 10.72.7 we conclude $E^i = 0$ for $i \notin \{-\dim(\operatorname{Supp}(M)), \ldots, -\operatorname{depth}(M)\}$. Moreover, in the boundary case $i = -\operatorname{depth}(M)$ we deduce that E^i is nonzero as F^{i+1} is nonzero by induction. Since $E^i/fE^i \subset F^{i+1}$ we get

$$\dim(\operatorname{Supp}(F^{i+1})) \geq \dim(\operatorname{Supp}(E^i/fE^i)) \geq \dim(\operatorname{Supp}(E^i)) - 1$$

(see lemma used above) we also obtain the dimension estimate (2).

If M has depth 0 and $d > 0$ we let $N = M[\mathfrak{m}^\infty]$ and set $M' = M/N$ (compare with Lemma 47.11.6). Then M' has depth > 0 and $\dim(\operatorname{Supp}(M')) = d$. Thus we know

the result for M' and since $R\mathrm{Hom}_A(N, \omega_A^\bullet) = \mathrm{Hom}_A(N, E)$ (Lemma 47.16.4) the long exact cohomology sequence of Ext's implies the result for M. □

0BUJ **Remark 47.16.6.** Let $(A, \mathfrak{m})$ and $\omega_A^\bullet$ be as in Lemma 47.16.5. By More on Algebra, Lemma 15.70.2 we see that $\omega_A^\bullet$ has injective-amplitude in $[-d, 0]$ because part (3) of that lemma applies. In particular, for any A-module M (not necessarily finite) we have $\mathrm{Ext}_A^i(M, \omega_A^\bullet) = 0$ for $i \notin \{-d, \ldots, 0\}$.

0B5A **Lemma 47.16.7.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. The following are equivalent*

(1) *M is Cohen-Macaulay,*
(2) *$\mathrm{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero for at most one i,*
(3) *$\mathrm{Ext}_A^{-i}(M, \omega_A^\bullet)$ is zero for $i \neq \dim(\mathit{Supp}(M))$.*

Denote CM_d the category of finite Cohen-Macaulay A-modules of depth d. Then $M \mapsto \mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ defines an anti-auto-equivalence of CM_d.

Proof. We will use the results of Lemma 47.16.5 without further mention. Fix a finite module M. If M is Cohen-Macaulay, then only $\mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ can be nonzero, hence (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) is immediate. Assume (2) and let $N = \mathrm{Ext}_A^{-\delta}(M, \omega_A^\bullet)$ be the nonzero Ext where $\delta = \mathrm{depth}(M)$. Then, since

$$M[0] = R\mathrm{Hom}_A(R\mathrm{Hom}_A(M, \omega_A^\bullet), \omega_A^\bullet) = R\mathrm{Hom}_A(N[\delta], \omega_A^\bullet)$$

(Lemma 47.15.3) we conclude that $M = \mathrm{Ext}_A^{-\delta}(N, \omega_A^\bullet)$. Thus $\delta \geq \dim(\mathrm{Supp}(M))$. However, since we also know that $\delta \leq \dim(\mathrm{Supp}(M))$ (Algebra, Lemma 10.72.3) we conclude that M is Cohen-Macaulay.

To prove the final statement, it suffices to show that $N = \mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ is in CM_d for M in CM_d. Above we have seen that $M[0] = R\mathrm{Hom}_A(N[d], \omega_A^\bullet)$ and this proves the desired result by the equivalence of (1) and (3). □

0A7R **Lemma 47.16.8.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. If $\dim(A) = 0$, then $\omega_A^\bullet \cong E[0]$ where E is an injective hull of the residue field.*

Proof. Immediate from Lemma 47.16.4. □

0A7S **Lemma 47.16.9.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subset \mathfrak{m}$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \mathrm{Hom}_A(I, E)[0] \to \omega_B^\bullet[1]$$

in $D(A)$ where E is an injective hull of κ and $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 47.16.4 and 47.16.2. □

0A7T **Lemma 47.16.10.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \omega_A^\bullet \to \omega_B^\bullet[1]$$

in $D(A)$ where $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 47.16.2. □

0A7V **Lemma 47.16.11.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = e$. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero if and only if $i = -e$.*

Proof. Since $A_\mathfrak{p}$ has dimension zero, there exists an integer $n > 0$ such that $\mathfrak{p}^n A_\mathfrak{p}$ is zero. Set $B = A/\mathfrak{p}^n$ and $\omega_B^\bullet = R\mathrm{Hom}_A(B, \omega_A^\bullet)$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ we see that

$$(\omega_B^\bullet)_\mathfrak{p} = R\mathrm{Hom}_A(B, \omega_A^\bullet) \otimes_A^{\mathbf{L}} A_\mathfrak{p} = R\mathrm{Hom}_{A_\mathfrak{p}}(B_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}) = (\omega_A^\bullet)_\mathfrak{p}$$

The second equality holds by More on Algebra, Lemma 15.100.2. By Lemma 47.16.2 we may replace A by B. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A^\bullet)_\mathfrak{p}$ can only be nonzero if $i = -e$ by Lemma 47.16.5 parts (1) and (2). On the other hand, since $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex for the nonzero ring $A_\mathfrak{p}$ (Lemma 47.15.6) we see that the remaining module has to be nonzero. □

47.17. Dualizing complexes and dimension functions

0A7W Our results in the local setting have the following consequence: a Noetherian ring which has a dualizing complex is a universally catenary ring of finite dimension.

0A7X **Lemma 47.17.1.** *Let A be a Noetherian ring. Let $\mathfrak{p}$ be a minimal prime of A. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero for exactly one i.*

Proof. The complex $\omega_A^\bullet \otimes_A A_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (Lemma 47.15.6). The dimension of $A_\mathfrak{p}$ is zero as $\mathfrak{p}$ is minimal. Hence the result follows from Lemma 47.16.8. □

Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Lemma 47.15.12 allows us to define a function

$$\delta = \delta_{\omega_A^\bullet} : \mathrm{Spec}(A) \longrightarrow \mathbf{Z}$$

by mapping $\mathfrak{p}$ to the integer of Lemma 47.15.12 for the dualizing complex $(\omega_A^\bullet)_\mathfrak{p}$ over $A_\mathfrak{p}$ (Lemma 47.15.6) and the residue field $\kappa(\mathfrak{p})$. To be precise, we define $\delta(\mathfrak{p})$ to be the unique integer such that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})]$$

is a normalized dualizing complex over the Noetherian local ring $A_\mathfrak{p}$.

0A7Y **Lemma 47.17.2.** *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B^\bullet = R\mathrm{Hom}(B, \omega_A^\bullet)$ be the dualizing complex for B of Lemma 47.15.9. Then we have*

$$\delta_{\omega_B^\bullet} = \delta_{\omega_A^\bullet}|_{\mathrm{Spec}(B)}$$

Proof. This follows from the definition of the functions and Lemma 47.16.2. □

0A7Z **Lemma 47.17.3.** *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. The function $\delta = \delta_{\omega_A^\bullet}$ defined above is a dimension function (Topology, Definition 5.20.1).*

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be an immediate specialization. We have to show that $\delta(\mathfrak{p}) = \delta(\mathfrak{q}) + 1$. We may replace A by $A/\mathfrak{p}$, the complex $\omega_A^\bullet$ by $\omega_{A/\mathfrak{p}}^\bullet = R\mathrm{Hom}(A/\mathfrak{p}, \omega_A^\bullet)$, the prime $\mathfrak{p}$ by (0), and the prime $\mathfrak{q}$ by $\mathfrak{q}/\mathfrak{p}$, see Lemma 47.17.2. Thus we may assume that A is a domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is a prime ideal of height 1.

Then $H^i(\omega_A^\bullet)_{(0)}$ is nonzero for exactly one i, say i_0, by Lemma 47.17.1. In fact $i_0 = -\delta((0))$ because $(\omega_A^\bullet)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex for $A_\mathfrak{q}$. By Lemma 47.16.11 we see that

$$H^e((\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})])_{(0)} = H^{e-\delta(\mathfrak{q})}(\omega_A^\bullet)_{(0)}$$

is nonzero only for $e = -\dim(A_\mathfrak{q}) = -1$. We conclude

$$-\delta((0)) = -1 - \delta(\mathfrak{q})$$

as desired. □

0A80 **Lemma 47.17.4.** *Let A be a Noetherian ring which has a dualizing complex. Then A is universally catenary of finite dimension.*

Proof. Because $\mathrm{Spec}(A)$ has a dimension function by Lemma 47.17.3 it is catenary, see Topology, Lemma 5.20.2. Hence A is catenary, see Algebra, Lemma 10.105.2. It follows from Proposition 47.15.11 that A is universally catenary.

Because any dualizing complex $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ the values of the function $\delta_{\omega_A^\bullet}$ in minimal primes are bounded by Lemma 47.17.1. On the other hand, for a maximal ideal $\mathfrak{m}$ with residue field κ the integer $i = -\delta(\mathfrak{m})$ is the unique integer such that $\mathrm{Ext}^i_A(\kappa, \omega_A^\bullet)$ is nonzero (Lemma 47.15.12). Since $\omega_A^\bullet$ has finite injective dimension these values are bounded too. Since the dimension of A is the maximal value of $\delta(\mathfrak{p}) - \delta(\mathfrak{m})$ where $\mathfrak{p} \subset \mathfrak{m}$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of $\mathrm{Spec}(A)$ is bounded. □

0AWE **Lemma 47.17.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $d = \dim(A)$ and $\omega_A = H^{-d}(\omega_A^\bullet)$. Then*

1. *the support of ω_A is the union of the irreducible components of* $\mathrm{Spec}(A)$ *of dimension d,*
2. *ω_A satisfies (S_2), see Algebra, Definition 10.157.1.*

Proof. We will use Lemma 47.16.5 without further mention. By Lemma 47.16.11 the support of ω_A contains the irreducible components of dimension d. Let $\mathfrak{p} \subset A$ be a prime. By Lemma 47.17.3 the complex $(\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]$ is a normalized dualizing complex for $A_\mathfrak{p}$. Hence if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$, then $(\omega_A)_\mathfrak{p} = 0$. This proves the support of ω_A is the union of the irreducible components of dimension d, because the complement of this union is exactly the primes $\mathfrak{p}$ of A for which $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$ as A is catenary (Lemma 47.17.4). On the other hand, if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) = d$, then

$$(\omega_A)_\mathfrak{p} = H^{-\dim(A_\mathfrak{p})}((\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})])$$

Hence in order to prove ω_A has (S_2) it suffices to show that the depth of ω_A is at least $\min(\dim(A), 2)$. We prove this by induction on $\dim(A)$. The case $\dim(A) = 0$ is trivial.

Assume $\text{depth}(A) > 0$. Choose a nonzerodivisor $f \in \mathfrak{m}$ and set $B = A/fA$. Then $\dim(B) = \dim(A) - 1$ and we may apply the induction hypothesis to B. By Lemma 47.16.10 we see that multiplication by f is injective on ω_A and we get $\omega_A/f\omega_A \subset \omega_B$. This proves the depth of ω_A is at least 1. If $\dim(A) > 1$, then $\dim(B) > 0$ and ω_B has depth > 0. Hence ω_A has depth > 1 and we conclude in this case.

Assume $\dim(A) > 0$ and $\text{depth}(A) = 0$. Let $I = A[\mathfrak{m}^\infty]$ and set $B = A/I$. Then B has depth ≥ 1 and $\omega_A = \omega_B$ by Lemma 47.16.9. Since we proved the result for ω_B above the proof is done. □

47.18. The local duality theorem

0A81 The main result in this section is due to Grothendieck.

0A82 **Lemma 47.18.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let $Z = V(\mathfrak{m}) \subset \text{Spec}(A)$. Then $E = R^0\Gamma_Z(\omega_A^\bullet)$ is an injective hull of κ and $R\Gamma_Z(\omega_A^\bullet) = E[0]$.*

Proof. By Lemma 47.10.1 we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$. Thus

$$R\Gamma_Z(\omega_A^\bullet) = R\Gamma_\mathfrak{m}(\omega_A^\bullet) = \text{hocolim } R\text{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet)$$

by Lemma 47.8.2. Let E' be an injective hull of the residue field. By Lemma 47.16.4 we can find isomorphisms

$$R\text{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet) \cong \text{Hom}_A(A/\mathfrak{m}^n, E')[0]$$

compatible with transition maps. Since $E' = \bigcup E'[\mathfrak{m}^n] = \text{colim}\, \text{Hom}_A(A/\mathfrak{m}^n, E')$ by Lemma 47.7.3 we conclude that $E \cong E'$ and that all other cohomology groups of the complex $R\Gamma_Z(\omega_A^\bullet)$ are zero. □

0A83 **Remark 47.18.2.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with a normalized dualizing complex $\omega_A^\bullet$. By Lemma 47.18.1 above we see that $R\Gamma_Z(\omega_A^\bullet)$ is an injective hull of the residue field placed in degree 0. In fact, this gives a "construction" or "realization" of the injective hull which is slightly more canonical than just picking any old injective hull. Namely, a normalized dualizing complex is unique up to isomorphism, with group of automorphisms the group of units of A, whereas an injective hull of κ is unique up to isomorphism, with group of automorphisms the group of units of the completion $A^\wedge$ of A with respect to $\mathfrak{m}$.

Here is the main result of this section.

0A84 **Theorem 47.18.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $Z = V(\mathfrak{m}) \subset \text{Spec}(A)$. Denote $^\wedge$ derived completion with respect to $\mathfrak{m}$. Then*

$$R\text{Hom}_A(K, \omega_A^\bullet)^\wedge \cong R\text{Hom}_A(R\Gamma_Z(K), E[0])$$

for K in $D(A)$.

Proof. Observe that $E[0] \cong R\Gamma_Z(\omega_A^\bullet)$ by Lemma 47.18.1. By More on Algebra, Lemma 15.92.13 completion on the left hand side goes inside. Thus we have to prove

$$R\text{Hom}_A(K^\wedge, (\omega_A^\bullet)^\wedge) = R\text{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(\omega_A^\bullet))$$

This follows from the equivalence between $D_{comp}(A, \mathfrak{m})$ and $D_{\mathfrak{m}^\infty\text{-torsion}}(A)$ given in Proposition 47.12.2. More precisely, it is a special case of Lemma 47.12.3. □

Here is a special case of the theorem above.

0AAK **Lemma 47.18.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $K \in D_{Coh}(A)$. Then*

$$\mathrm{Ext}_A^{-i}(K, \omega_A^\bullet)^\wedge = \mathrm{Hom}_A(H^i_\mathfrak{m}(K), E)$$

where $^\wedge$ denotes $\mathfrak{m}$-adic completion.

Proof. By Lemma 47.15.3 we see that $R\mathrm{Hom}_A(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. It follows that the cohomology modules of the derived completion of $R\mathrm{Hom}_A(K, \omega_A^\bullet)$ are equal to the usual completions $\mathrm{Ext}^i_A(K, \omega_A^\bullet)^\wedge$ by More on Algebra, Lemma 15.95.4. On the other hand, we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$ for $Z = V(\mathfrak{m})$ by Lemma 47.10.1. Moreover, the functor $\mathrm{Hom}_A(-, E)$ is exact hence factors through cohomology. Hence the lemma is consequence of Theorem 47.18.3. □

47.19. Dualizing modules

0DW3 If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring and $\omega_A^\bullet$ is a normalized dualizing complex, then we say the module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$, described in Lemma 47.17.5, is a *dualizing module* for A. This module is a canonical module of A. It seems generally agreed upon to define a *canonical module* for a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ to be a finite A-module K such that

$$\mathrm{Hom}_A(K, E) \cong H^{\dim(A)}_\mathfrak{m}(A)$$

where E is an injective hull of the residue field. A dualizing module is canonical because

$$\mathrm{Hom}_A(H^{\dim(A)}_\mathfrak{m}(A), E) = (\omega_A)^\wedge$$

by Lemma 47.18.4 and hence applying $\mathrm{Hom}_A(-, E)$ we get

$$\begin{aligned} \mathrm{Hom}_A(\omega_A, E) &= \mathrm{Hom}_A((\omega_A)^\wedge, E) \\ &= \mathrm{Hom}_A(\mathrm{Hom}_A(H^{\dim(A)}_\mathfrak{m}(A), E), E) \\ &= H^{\dim(A)}_\mathfrak{m}(A) \end{aligned}$$

the first equality because E is $\mathfrak{m}$-power torsion, the second by the above, and the third by Matlis duality (Proposition 47.7.8). The utility of the definition of a canonical module given above lies in the fact that it makes sense even if A does not have a dualizing complex.

47.20. Cohen-Macaulay rings

0DW4 Cohen-Macaulay modules and rings were studied in Algebra, Sections 10.103 and 10.104.

0AWR **Lemma 47.20.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Then $\mathrm{depth}(A)$ is equal to the smallest integer $\delta \geq 0$ such that $H^{-\delta}(\omega_A^\bullet) \neq 0$.*

Proof. This follows immediately from Lemma 47.16.5. Here are two other ways to see that it is true.

First alternative. By Nakayama's lemma we see that δ is the smallest integer such that $\mathrm{Hom}_A(H^{-\delta}(\omega_A^\bullet), \kappa) \neq 0$. In other words, it is the smallest integer such that $\mathrm{Ext}_A^{-\delta}(\omega_A^\bullet, \kappa)$ is nonzero. Using Lemma 47.15.3 and the fact that $\omega_A^\bullet$ is normalized

this is equal to the smallest integer such that $\mathrm{Ext}^\delta_A(\kappa, A)$ is nonzero. This is equal to the depth of A by Algebra, Lemma 10.72.5.

Second alternative. By the local duality theorem (in the form of Lemma 47.18.4) δ is the smallest integer such that $H^\delta_{\mathfrak{m}}(A)$ is nonzero. This is equal to the depth of A by Lemma 47.11.1. □

0AWS **Lemma 47.20.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$. The following are equivalent*

(1) *A is Cohen-Macaulay,*
(2) *$\omega_A^\bullet$ is concentrated in a single degree, and*
(3) *$\omega_A^\bullet = \omega_A[\dim(A)]$.*

In this case ω_A is a maximal Cohen-Macaulay module.

Proof. Follows immediately from Lemma 47.16.7. □

0DW5 **Lemma 47.20.3.** *Let A be a Noetherian ring. If there exists a finite A-module ω_A such that $\omega_A[0]$ is a dualizing complex, then A is Cohen-Macaulay.*

Proof. We may replace A by the localization at a prime (Lemma 47.15.6 and Algebra, Definition 10.104.6). In this case the result follows immediately from Lemma 47.20.2. □

0EHS **Lemma 47.20.4.** *Let A be a Noetherian ring with dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Then*

$$U = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\mathrm{Spec}(A)$ *whose intersection with $\mathrm{Supp}(M)$ is dense.*

Proof. If $\mathfrak{p}$ is a generic point of $\mathrm{Supp}(M)$, then $\mathrm{depth}(M_\mathfrak{p}) = \dim(M_\mathfrak{p}) = 0$ and hence $\mathfrak{p} \in U$. This proves denseness. If $\mathfrak{p} \in U$, then we see that

$$R\mathrm{Hom}_A(M, \omega_A^\bullet)_\mathfrak{p} = R\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p})$$

has a unique nonzero cohomology module, say in degree i_0, by Lemma 47.16.7. Since $R\mathrm{Hom}_A(M, \omega_A^\bullet)$ has only a finite number of nonzero cohomology modules H^i and since each of these is a finite A-module, we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $(H^i)_f = 0$ for $i \neq i_0$. Then $R\mathrm{Hom}_A(M, \omega_A^\bullet)_f$ has a unique nonzero cohomology module and reversing the arguments just given we find that $D(f) \subset U$. □

0EHT **Lemma 47.20.5.** *Let A be a Noetherian ring. If A has a dualizing complex $\omega_A^\bullet$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p}$ is Cohen-Macaulay$\}$ is a dense open subset of* $\mathrm{Spec}(A)$.

Proof. Immediate consequence of Lemma 47.20.4 and the definitions. □

47.21. Gorenstein rings

0DW6 So far, the only explicit dualizing complex we've seen is κ on κ for a field κ, see proof of Lemma 47.15.12. By Proposition 47.15.11 this means that any finite type algebra over a field has a dualizing complex. However, it turns out that there are Noetherian (local) rings which do not have a dualizing complex. Namely, we have seen that a ring which has a dualizing complex is universally catenary (Lemma 47.17.4) but there are examples of Noetherian local rings which are not catenary, see Examples, Section 110.19.

Nonetheless many rings in algebraic geometry have dualizing complexes simply because they are quotients of Gorenstein rings. This condition is in fact both necessary and sufficient. That is: a Noetherian ring has a dualizing complex if and only if it is a quotient of a finite dimensional Gorenstein ring. This is Sharp's conjecture ([**Sha79**]) which can be found as [**Kaw02**, Corollary 1.4] in the literature. Returning to our current topic, here is the definition of Gorenstein rings.

0DW7 **Definition 47.21.1.** Gorenstein rings.

(1) Let A be a Noetherian local ring. We say A is *Gorenstein* if $A[0]$ is a dualizing complex for A.
(2) Let A be a Noetherian ring. We say A is *Gorenstein* if $A_\mathfrak{p}$ is Gorenstein for every prime $\mathfrak{p}$ of A.

This definition makes sense, because if $A[0]$ is a dualizing complex for A, then $S^{-1}A[0]$ is a dualizing complex for $S^{-1}A$ by Lemma 47.15.6. We will see later that a finite dimensional Noetherian ring is Gorenstein if it has finite injective dimension as a module over itself.

0DW8 **Lemma 47.21.2.** *A Gorenstein ring is Cohen-Macaulay.*

Proof. Follows from Lemma 47.20.2. □

An example of a Gorenstein ring is a regular ring.

0AWX **Lemma 47.21.3.** *A regular local ring is Gorenstein. A regular ring is Gorenstein.*

Proof. Let A be a regular ring of finite dimension d. Then A has finite global dimension d, see Algebra, Lemma 10.110.8. Hence $\mathrm{Ext}^{d+1}_A(M, A) = 0$ for all A-modules M, see Algebra, Lemma 10.109.8. Thus A has finite injective dimension as an A-module by More on Algebra, Lemma 15.70.2. It follows that $A[0]$ is a dualizing complex, hence A is Gorenstein by the remark following the definition. □

0DW9 **Lemma 47.21.4.** *Let A be a Noetherian ring.*

(1) *If A has a dualizing complex $\omega_A^\bullet$, then*
   (a) *A is Gorenstein $\Leftrightarrow$ $\omega_A^\bullet$ is an invertible object of $D(A)$,*
   (b) *$A_\mathfrak{p}$ is Gorenstein $\Leftrightarrow$ $(\omega_A^\bullet)_\mathfrak{p}$ is an invertible object of $D(A_\mathfrak{p})$,*
   (c) *$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p}$ is Gorenstein$\}$ is an open subset.*
(2) *If A is Gorenstein, then A has a dualizing complex if and only if $A[0]$ is a dualizing complex.*

Proof. For invertible objects of $D(A)$, see More on Algebra, Lemma 15.127.4 and the discussion in Section 47.15.

By Lemma 47.15.6 for every $\mathfrak{p}$ the complex $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex over $A_\mathfrak{p}$. By definition and uniqueness of dualizing complexes (Lemma 47.15.5) we see that (1)(b) holds.

To see (1)(c) assume that $A_\mathfrak{p}$ is Gorenstein. Let n_x be the unique integer such that $H^{n_x}((\omega_A^\bullet)_\mathfrak{p})$ is nonzero and isomorphic to $A_\mathfrak{p}$. Since $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ there are finitely many nonzero finite A-modules $H^i(\omega_A^\bullet)$. Thus there exists some $f \in A$, $f \notin \mathfrak{p}$ such that only $H^{n_x}((\omega_A^\bullet)_f)$ is nonzero and generated by 1 element over A_f. Since dualizing complexes are faithful (by definition) we conclude that $A_f \cong H^{n_x}((\omega_A^\bullet)_f)$. In this way we see that $A_\mathfrak{q}$ is Gorenstein for every $\mathfrak{q} \in D(f)$. This proves that the set in (1)(c) is open.

Proof of (1)(a). The implication $\Leftarrow$ follows from (1)(b). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $A_{\mathfrak{p}}$ is Gorenstein, then for some $f \in A$, $f \notin \mathfrak{p}$ the complex $(\omega_A^\bullet)_f$ has only one nonzero cohomology module which is invertible.

If $A[0]$ is a dualizing complex then A is Gorenstein by part (1). Conversely, we see that part (1) shows that $\omega_A^\bullet$ is locally isomorphic to a shift of A. Since being a dualizing complex is local (Lemma 47.15.7) the result is clear. □

0BJI **Lemma 47.21.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then A is Gorenstein if and only if $\mathrm{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$.*

Proof. Observe that $A[0]$ is a dualizing complex for A if and only if A has finite injective dimension as an A-module (follows immediately from Definition 47.15.1). Thus the lemma follows from More on Algebra, Lemma 15.70.7. □

0BJJ **Lemma 47.21.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then A is Gorenstein if and only if B is Gorenstein.*

Proof. If A is Gorenstein, then B is Gorenstein by Lemma 47.16.10. Conversely, suppose that B is Gorenstein. Then $\mathrm{Ext}^i_B(\kappa, B)$ is zero for $i \gg 0$ (Lemma 47.21.5). Recall that $R\mathrm{Hom}(B, -) : D(A) \to D(B)$ is a right adjoint to restriction (Lemma 47.13.1). Hence

$$R\mathrm{Hom}_A(\kappa, A) = R\mathrm{Hom}_B(\kappa, R\mathrm{Hom}(B, A)) = R\mathrm{Hom}_B(\kappa, B[1])$$

The final equality by direct computation or by Lemma 47.13.10. Thus we see that $\mathrm{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$ and A is Gorenstein (Lemma 47.21.5). □

0DWA **Lemma 47.21.7.** *If $A \to B$ is a local complete intersection homomorphism of rings and A is a Noetherian Gorenstein ring, then B is a Gorenstein ring.*

Proof. By More on Algebra, Definition 15.33.2 we can write $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. Observe that a polynomial ring over a Gorenstein ring A is Gorenstein: reduce to A local and then use Lemmas 47.15.10 and 47.21.4. A Koszul-regular ideal is by definition locally generated by a Koszul-regular sequence, see More on Algebra, Section 15.32. Looking at local rings of $A[x_1, \ldots, x_n]$ we see it suffices to show: if R is a Noetherian local Gorenstein ring and $f_1, \ldots, f_c \in \mathfrak{m}_R$ is a Koszul regular sequence, then $R/(f_1, \ldots, f_c)$ is Gorenstein. This follows from Lemma 47.21.6 and the fact that a Koszul regular sequence in R is just a regular sequence (More on Algebra, Lemma 15.30.7). □

0BJL **Lemma 47.21.8.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. The following are equivalent*

(1) *B is Gorenstein, and*
(2) *A and $B/\mathfrak{m}_A B$ are Gorenstein.*

Proof. Below we will use without further mention that a local Gorenstein ring has finite injective dimension as well as Lemma 47.21.5. By More on Algebra, Lemma 15.66.4 we have

$$\mathrm{Ext}^i_A(\kappa_A, A) \otimes_A B = \mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$$

for all i.

Assume (2). Using that $R\mathrm{Hom}(B/\mathfrak{m}_A B, -) : D(B) \to D(B/\mathfrak{m}_A B)$ is a right adjoint to restriction (Lemma 47.13.1) we obtain

$$R\mathrm{Hom}_B(\kappa_B, B) = R\mathrm{Hom}_{B/\mathfrak{m}_A B}(\kappa_B, R\mathrm{Hom}(B/\mathfrak{m}_A B, B))$$

The cohomology modules of $R\mathrm{Hom}(B/\mathfrak{m}_A B, B)$ are the modules $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B) = \mathrm{Ext}^i_A(\kappa_A, A) \otimes_A B$. Since A is Gorenstein, we conclude only a finite number of these are nonzero and each is isomorphic to a direct sum of copies of $B/\mathfrak{m}_A B$. Hence since $B/\mathfrak{m}_A B$ is Gorenstein we conclude that $R\mathrm{Hom}_B(B/\mathfrak{m}_B, B)$ has only a finite number of nonzero cohomology modules. Hence B is Gorenstein.

Assume (1). Since B has finite injective dimension, $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is 0 for $i \gg 0$. Since $A \to B$ is faithfully flat we conclude that $\mathrm{Ext}^i_A(\kappa_A, A)$ is 0 for $i \gg 0$. We conclude that A is Gorenstein. This implies that $\mathrm{Ext}^i_A(\kappa_A, A)$ is nonzero for exactly one i, namely for $i = \dim(A)$, and $\mathrm{Ext}^{\dim(A)}_A(\kappa_A, A) \cong \kappa_A$ (see Lemmas 47.16.1, 47.20.2, and 47.21.2). Thus we see that $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is zero except for one i, namely $i = \dim(A)$ and $\mathrm{Ext}^{\dim(A)}_B(B/\mathfrak{m}_A B, B) \cong B/\mathfrak{m}_A B$. Thus $B/\mathfrak{m}_A B$ is Gorenstein by Lemma 47.16.1. □

0EBT **Lemma 47.21.9.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local Gorenstein ring of dimension d. Let E be the injective hull of κ. Then $Tor^A_i(E, \kappa)$ is zero for $i \neq d$ and $Tor^A_d(E, \kappa) = \kappa$.*

Proof. Since A is Gorenstein $\omega_A^\bullet = A[d]$ is a normalized dualizing complex for A. Also E is the only nonzero cohomology module of $R\Gamma_\mathfrak{m}(\omega_A^\bullet)$ sitting in degree 0, see Lemma 47.18.1. By Lemma 47.9.5 we have

$$E \otimes^{\mathbf{L}}_A \kappa = R\Gamma_\mathfrak{m}(\omega_A^\bullet) \otimes^{\mathbf{L}}_A \kappa = R\Gamma_\mathfrak{m}(\omega_A^\bullet \otimes^{\mathbf{L}}_A \kappa) = R\Gamma_\mathfrak{m}(\kappa[d]) = \kappa[d]$$

and the lemma follows. □

47.22. The ubiquity of dualizing complexes

0DWB Many Noetherian rings have dualizing complexes.

0AWD **Lemma 47.22.1.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. If $A \to B$ is flat and $\mathfrak{m}_A B = \mathfrak{m}_B$, then $\omega_A^\bullet \otimes_A B$ is a normalized dualizing complex for B.*

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D^b_{Coh}(B)$. Let κ_A and κ_B be the residue fields of A and B. By More on Algebra, Lemma 15.100.2 we see that

$$R\mathrm{Hom}_B(\kappa_B, \omega_A^\bullet \otimes_A B) = R\mathrm{Hom}_A(\kappa_A, \omega_A^\bullet) \otimes_A B = \kappa_A[0] \otimes_A B = \kappa_B[0]$$

Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 15.70.7. Finally, we can use the same arguments to see that

$$R\mathrm{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. □

0DWC **Lemma 47.22.2.** *Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal such that $A/I = B/IB$ and such that IB is contained in the Jacobson radical of B. Let $\omega_A^\bullet$ be a dualizing complex. Then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.*

Proof. It is clear that $\omega_A^\bullet \otimes_A B$ is in $D^b_{Coh}(B)$. By More on Algebra, Lemma 15.100.2 we see that

$$R\mathrm{Hom}_B(K \otimes_A B, \omega_A^\bullet \otimes_A B) = R\mathrm{Hom}_A(K, \omega_A^\bullet) \otimes_A B$$

for any $K \in D^b_{Coh}(A)$. For any ideal $IB \subset J \subset B$ there is a unique ideal $I \subset J' \subset A$ such that $A/J' \otimes_A B = B/J$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension by More on Algebra, Lemma 15.70.6. Finally, we also have

$$R\mathrm{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = R\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. □

0DWD **Lemma 47.22.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $\omega_A^\bullet$ be a dualizing complex.*

(1) *$\omega_A^\bullet \otimes_A A^h$ is a dualizing complex on the henselization (A^h, I^h) of the pair (A, I),*
(2) *$\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex on the I-adic completion $A^\wedge$, and*
(3) *if A is local, then $\omega_A^\bullet \otimes_A A^h$, resp. $\omega_A^\bullet \otimes_A A^{sh}$ is a dualzing complex on the henselization, resp. strict henselization of A.*

Proof. Immediate from Lemmas 47.22.1 and 47.22.2. See More on Algebra, Sections 15.11, 15.43, and 15.45 and Algebra, Sections 10.96 and 10.97 for information on completions and henselizations. □

0BFR **Lemma 47.22.4.** *The following types of rings have a dualizing complex:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains,*
(5) *any ring which is obtained from one of the rings above by taking an algebra essentially of finite type, or by taking an ideal-adic completion, or by taking a henselization, or by taking a strict henselization.*

Proof. Part (5) follows from Proposition 47.15.11 and Lemma 47.22.3. By Lemma 47.21.3 a regular local ring has a dualizing complex. A complete Noetherian local ring is the quotient of a regular local ring by the Cohen structure theorem (Algebra, Theorem 10.160.8). Let A be a Dedekind domain. Then every ideal I is a finite projective A-module (follows from Algebra, Lemma 10.78.2 and the fact that the local rings of A are discrete valuation ring and hence PIDs). Thus every A-module has finite injective dimension at most 1 by More on Algebra, Lemma 15.70.2. It follows easily that $A[0]$ is a dualizing complex. □

47.23. Formal fibres

0BJM This section is a continuation of More on Algebra, Section 15.51. There we saw there is a (fairly) good theory of Noetherian rings A whose local rings have Cohen-Macaulay formal fibres. Namely, we proved (1) it suffices to check the formal fibres of localizations at maximal ideals are Cohen-Macaulay, (2) the property is inherited by rings of finite type over A, (3) the fibres of $A \to A^\wedge$ are Cohen-Macaulay for any completion $A^\wedge$ of A, and (4) the property is inherited by henselizations of A. See More on Algebra, Lemma 15.51.4, Proposition 15.51.5, Lemma 15.51.6, and Lemma 15.51.7. Similarly, for Noetherian rings whose local rings have formal fibres which

are geometrically reduced, geometrically normal, (S_n), and geometrically (R_n). In this section we will see that the same is true for Noetherian rings whose local rings have formal fibres which are Gorenstein or local complete intersections. This is relevant to this chapter because a Noetherian ring which has a dualizing complex is an example.

0BJN **Lemma 47.23.1.** *Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 15.51 hold for $P(k \to R) =$"R is a Gorenstein ring".*

Proof. Since we already know the result holds for Cohen-Macaulay instead of Gorenstein, we may in each step assume the ring we have is Cohen-Macaulay. This is not particularly helpful for the proof, but psychologically may be useful.

Part (A). Let K/k be a finitely generated field extension. Let R be a Gorenstein k-algebra. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. Hence $R \otimes_k A$ is Gorenstein by Lemma 47.21.7. Thus $R \otimes_k K$ is too as a localization.

Proof of (B). This is clear because a ring is Gorenstein if and only if all of its local rings are Gorenstein.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are Gorenstein and $B \to C$ is regular. We have to show the fibres of $A \to C$ are Gorenstein. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is Gorenstein and $C/\mathfrak{m}_B C$ is regular, in particular Gorenstein (Lemma 47.21.3). Then C is Gorenstein by Lemma 47.21.8.

Part (D). This follows from Lemma 47.21.8. Part (E) is immediate as the condition does not refer to the ground field. □

0AWY **Lemma 47.23.2.** *Let A be a Noetherian local ring. If A has a dualizing complex, then the formal fibres of A are Gorenstein.*

Proof. Let $\mathfrak{p}$ be a prime of A. The formal fibre of A at $\mathfrak{p}$ is isomorphic to the formal fibre of $A/\mathfrak{p}$ at (0). The quotient $A/\mathfrak{p}$ has a dualizing complex (Lemma 47.15.9). Thus it suffices to check the statement when A is a local domain and $\mathfrak{p} = (0)$. Let $\omega_A^\bullet$ be a dualizing complex for A. Then $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex for the completion $A^\wedge$ (Lemma 47.22.1). Then $\omega_A^\bullet \otimes_A K$ is a dualizing complex for the fraction field K of A (Lemma 47.15.6). Hence $\omega_A^\bullet \otimes_A K$ is isomorphic to $K[n]$ for some $n \in \mathbf{Z}$. Similarly, we conclude a dualizing complex for the formal fibre $A^\wedge \otimes_A K$ is

$$\omega_A^\bullet \otimes_A A^\wedge \otimes_{A^\wedge} (A^\wedge \otimes_A K) = (\omega_A^\bullet \otimes_A K) \otimes_K (A^\wedge \otimes_A K) \cong (A^\wedge \otimes_A K)[n]$$

as desired. □

Here is the verification promised in Divided Power Algebra, Remark 23.9.3.

0BJP **Lemma 47.23.3.** *Properties (A), (B), (C), (D), and (E) of More on Algebra, Section 15.51 hold for $P(k \to R) =$"R is a local complete intersection". See Divided Power Algebra, Definition 23.8.5.*

Proof. Part (A). Let K/k be a finitely generated field extension. Let R be a k-algebra which is a local complete intersection. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11. Then $R \to R \otimes_k A$ is a relative global complete intersection. It follows that $R \otimes_k A$ is a local complete intersection by Divided Power Algebra, Lemma 23.8.9.

Proof of (B). This is clear because a ring is a local complete intersection if and only if all of its local rings are complete intersections.

Part (C). Let $A \to B \to C$ be flat maps of Noetherian rings. Assume the fibres of $A \to B$ are local complete intersections and $B \to C$ is regular. We have to show the fibres of $A \to C$ are local complete intersections. Clearly, we may assume $A = k$ is a field. Then we may assume that $B \to C$ is a regular local homomorphism of Noetherian local rings. Then B is a complete intersection and $C/\mathfrak{m}_B C$ is regular, in particular a complete intersection (by definition). Then C is a complete intersection by Divided Power Algebra, Lemma 23.8.9.

Part (D). This follows by the same arguments as in (C) from the other implication in Divided Power Algebra, Lemma 23.8.9. Part (E) is immediate as the condition does not refer to the ground field. □

47.24. Upper shriek algebraically

0BZI For a finite type homomorphism $R \to A$ of Noetherian rings we will construct a functor $\varphi^! : D(R) \to D(A)$ well defined up to nonunique isomorphism which as we will see in Duality for Schemes, Remark 48.17.5 agrees up to isomorphism with the upper shriek functors one encounters in the duality theory for schemes. To motivate the construction we mention two additional properties:

(1) $\varphi^!$ sends a dualizing complex for R (if it exists) to a dualizing complex for A, and
(2) $\omega^\bullet_{A/R} = \varphi^!(R)$ is a kind of relative dualizing complex: it lies in $D^b_{Coh}(A)$ and restricts to a dualizing complex on the fibres provided $R \to A$ is flat.

These statements are Lemmas 47.24.3 and 47.25.2.

Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. We will define a functor $\varphi^! : D(R) \to D(A)$ in the following way

(1) If $\varphi : R \to A$ is surjective we set $\varphi^!(K) = R\mathrm{Hom}(A, K)$. Here we use the functor $R\mathrm{Hom}(A, -) : D(R) \to D(A)$ of Section 47.13, and
(2) in general we choose a surjection $\psi : P \to A$ with $P = R[x_1, \ldots, x_n]$ and we set $\varphi^!(K) = \psi^!(K \otimes^{\mathbf{L}}_R P)[n]$. Here we use the functor $- \otimes^{\mathbf{L}}_R P : D(R) \to D(P)$ of More on Algebra, Section 15.60.

Note the shift $[n]$ by the number of variables in the polynomial ring. This construction is **not** canonical and the functor $\varphi^!$ will only be well defined up to a (nonunique) isomorphism of functors[1].

0BZJ **Lemma 47.24.1.** *Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. The functor $\varphi^!$ is well defined up to isomorphism.*

[1] It is possible to make the construction canonical: use $\Omega^n_{P/R}[n]$ instead of $P[n]$ in the construction and use this in Lemma 47.24.1. The material in this section becomes a lot more involved if one wants to do this.

Proof. Suppose that $\psi_1 : P_1 = R[x_1, \ldots, x_n] \to A$ and $\psi_2 : P_2 = R[y_1, \ldots, y_m] \to A$ are two surjections from polynomial rings onto A. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

where f_j and g_i are chosen such that $\psi_1(f_j) = \psi_2(y_j)$ and $\psi_2(g_i) = \psi_1(x_i)$. By symmetry it suffices to prove the functors defined using $P \to A$ and $P[y_1, \ldots, y_m] \to A$ are isomorphic. By induction we may assume $m = 1$. This reduces us to the case discussed in the next paragraph.

Here $\psi : P \to A$ is given and $\chi : P[y] \to A$ induces ψ on P. Write $Q = P[y]$. Choose $g \in P$ with $\psi(g) = \chi(y)$. Denote $\pi : Q \to P$ the P-algebra map with $\pi(y) = g$. Then $\chi = \psi \circ \pi$ and hence $\chi^! = \psi^! \circ \pi^!$ as both are adjoint to the restriction functor $D(A) \to D(Q)$ by the material in Section 47.13. Thus

$$\chi^! \left(K \otimes^{\mathbf{L}}_R Q\right)[n+1] = \psi^! \left(\pi^! \left(K \otimes^{\mathbf{L}}_R Q\right)[1]\right)[n]$$

Hence it suffices to show that $\pi^!(K \otimes^{\mathbf{L}}_R Q[1]) = K \otimes^{\mathbf{L}}_R P$ Thus it suffices to show that the functor $\pi^!(-) : D(Q) \to D(P)$ is isomorphic to $K \mapsto K \otimes^{\mathbf{L}}_Q P[-1]$. This follows from Lemma 47.13.10. □

0BZK **Lemma 47.24.2.** *Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings.*

(1) *$\varphi^!$ maps $D^+(R)$ into $D^+(A)$ and $D^+_{Coh}(R)$ into $D^+_{Coh}(A)$.*
(2) *if φ is perfect, then $\varphi^!$ maps $D^-(R)$ into $D^-(A)$, $D^-_{Coh}(R)$ into $D^-_{Coh}(A)$, and $D^b_{Coh}(R)$ into $D^b_{Coh}(A)$.*

Proof. Choose a factorization $R \to P \to A$ as in the definition of $\varphi^!$. The functor $-\otimes^{\mathbf{L}}_R : D(R) \to D(P)$ preserves the subcategories $D^+, D^+_{Coh}, D^-, D^-_{Coh}, D^b_{Coh}$. The functor $R\mathrm{Hom}(A, -) : D(P) \to D(A)$ preserves D^+ and D^+_{Coh} by Lemma 47.13.4. If $R \to A$ is perfect, then A is perfect as a P-module, see More on Algebra, Lemma 15.83.2. Recall that the restriction of $R\mathrm{Hom}(A, K)$ to $D(P)$ is $R\mathrm{Hom}_P(A, K)$. By More on Algebra, Lemma 15.75.15 we have $R\mathrm{Hom}_P(A, K) = E \otimes^{\mathbf{L}}_P K$ for some perfect $E \in D(P)$. Since we can represent E by a finite complex of finite projective P-modules it is clear that $R\mathrm{Hom}_P(A, K)$ is in $D^-(P), D^-_{Coh}(P), D^b_{Coh}(P)$ as soon as K is. Since the restriction functor $D(A) \to D(P)$ reflects these subcategories, the proof is complete. □

0BZL **Lemma 47.24.3.** *Let φ be a finite type homomorphism of Noetherian rings. If $\omega^\bullet_R$ is a dualizing complex for R, then $\varphi^!(\omega^\bullet_R)$ is a dualizing complex for A.*

Proof. Follows from Lemmas 47.15.10 and 47.15.9, □

0BZN **Lemma 47.24.4.** *Let $R \to R'$ be a flat homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a finite type ring map. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial maps*

$$\varphi^!(K) \otimes^{\mathbf{L}}_A A' \longrightarrow (\varphi')^!(K \otimes^{\mathbf{L}}_R R')$$

for K in $D(R)$ which are isomorphisms for $K \in D^+(R)$.

Proof. Choose a factorization $R \to P \to A$ where P is a polynomial ring over R. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 15.60.5 it suffices to construct maps

$$R\mathrm{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R\mathrm{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. For this we use the map (47.14.0.1) constructed in Section 47.14 for P, A, P', A'. The map is an isomorphism for $K \in D^+(R)$ by Lemma 47.14.2. □

0BZR **Lemma 47.24.5.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be a perfect ring map (More on Algebra, Definition 15.83.1) such that R' and A are tor independent over R. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism*

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. We may choose a factorization $R \to P \to A$ where P is a polynomial ring over R such that A is a perfect P-module, see More on Algebra, Lemma 15.83.2. This gives a corresponding factorization $R' \to P' \to A'$ by base change. Since we have $(K \otimes_R^{\mathbf{L}} P) \otimes_P^{\mathbf{L}} P' = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} P'$ by More on Algebra, Lemma 15.60.5 it suffices to construct maps

$$R\mathrm{Hom}(A, K \otimes_R^{\mathbf{L}} P[n]) \otimes_A^{\mathbf{L}} A' \longrightarrow R\mathrm{Hom}(A', (K \otimes_R^{\mathbf{L}} P[n]) \otimes_P^{\mathbf{L}} P')$$

functorial in K. We have

$$A \otimes_P^{\mathbf{L}} P' = A \otimes_R^{\mathbf{L}} R' = A'$$

The first equality by More on Algebra, Lemma 15.61.2 applied to R, R', P, P'. The second equality because A and R' are tor independent over R. Hence A and P' are tor independent over P and we can use the map (47.14.0.1) constructed in Section 47.14 for P, A, P', A' get the desired arrow. By Lemma 47.14.3 to finish the proof it suffices to prove that A is a perfect P-module which we saw above. □

0BZS **Lemma 47.24.6.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $\varphi : R \to A$ be flat of finite type. Let $\varphi' : R' \to A' = A \otimes_R R'$ be the map induced by φ. Then we have a functorial isomorphism*

$$\varphi^!(K) \otimes_A^{\mathbf{L}} A' = (\varphi')^!(K \otimes_R^{\mathbf{L}} R')$$

for K in $D(R)$.

Proof. Special case of Lemma 47.24.5 by More on Algebra, Lemma 15.83.4. □

0BZT **Lemma 47.24.7.** *Let $A \xrightarrow{a} B \xrightarrow{b} C$ be finite type homomorphisms of Noetherian rings. Then there is a transformation of functors $b^! \circ a^! \to (b \circ a)^!$ which is an isomorphism on $D^+(A)$.*

Proof. Choose a polynomial ring $P = A[x_1, \ldots, x_n]$ over A and a surjection $P \to B$. Choose elements $c_1, \ldots, c_m \in C$ generating C over B. Set $Q = P[y_1, \ldots, y_m]$

and denote $Q' = Q \otimes_P B = B[y_1, \ldots, y_m]$. Let $\chi : Q' \to C$ be the surjection sending y_j to c_j. Picture

$$\begin{array}{ccccc} & & Q & \xrightarrow{\psi'} & Q' & \xrightarrow{\chi} & C \\ & & \uparrow & & \uparrow & & \\ A & \longrightarrow & P & \xrightarrow{\psi} & B & & \end{array}$$

By Lemma 47.14.2 for $M \in D(P)$ we have an arrow $\psi^!(M) \otimes^{\mathbf{L}}_B Q' \to (\psi')^!(M \otimes^{\mathbf{L}}_P Q)$ which is an isomorphism whenever M is bounded below. Also we have $\chi^! \circ (\psi')^! = (\chi \circ \psi')^!$ as both functors are adjoint to the restriction functor $D(C) \to D(Q)$ by Section 47.13. Then we see

$$\begin{aligned} b^!(a^!(K)) &= \chi^!(\psi^!(K \otimes^{\mathbf{L}}_A P)[n] \otimes^{\mathbf{L}}_B Q)[m] \\ &\to \chi^!((\psi')^!(K \otimes^{\mathbf{L}}_A P \otimes^{\mathbf{L}}_P Q))[n+m] \\ &= (\chi \circ \psi')^!(K \otimes^{\mathbf{L}}_A Q)[n+m] \\ &= (b \circ a)^!(K) \end{aligned}$$

where we have used in addition to the above More on Algebra, Lemma 15.60.5. □

0C0G **Lemma 47.24.8.** *Let $\varphi : R \to A$ be a finite map of Noetherian rings. Then $\varphi^!$ is isomorphic to the functor $R\mathrm{Hom}(A, -) : D(R) \to D(A)$ from Section 47.13.*

Proof. Suppose that A is generated by $n > 1$ elements over R. Then can factor $R \to A$ as a composition of two finite ring maps where in both steps the number of generators is $< n$. Since we have Lemma 47.24.7 and Lemma 47.13.2 we conclude that it suffices to prove the lemma when A is generated by one element over R. Since A is finite over R, it follows that A is a quotient of $B = R[x]/(f)$ where f is a monic polynomial in x (Algebra, Lemma 10.36.3). Again using the lemmas on composition and the fact that we have agreement for surjections by definition, we conclude that it suffices to prove the lemma for $R \to B = R[x]/(f)$. In this case, the functor $\varphi^!$ is isomorphic to $K \mapsto K \otimes^{\mathbf{L}}_R B$; you prove this by using Lemma 47.13.10 for the map $R[x] \to B$ (note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero). For the functor $R\mathrm{Hom}(B, -) : D(R) \to D(B)$ we can use Lemma 47.13.9 to see that it suffices to show $\mathrm{Hom}_R(B, R) \cong B$ as B-modules. Suppose that f has degree d. Then an R-basis for B is given by $1, x, \ldots, x^{d-1}$. Let $\delta_i : B \to R$, $i = 0, \ldots, d-1$ be the R-linear map which picks off the coefficient of x^i with respect to the given basis. Then $\delta_0, \ldots, \delta_{d-1}$ is a basis for $\mathrm{Hom}_R(B, R)$. Finally, for $0 \leq i \leq d-1$ a computation shows that

$$x^i \delta_{d-1} = \delta_{d-1-i} + b_1 \delta_{d-i} + \ldots + b_i \delta_{d-1}$$

for some $c_1, \ldots, c_d \in R$[2]. Hence $\mathrm{Hom}_R(B, R)$ is a principal B-module with generator δ_{d-1}. By looking at ranks we conclude that it is a rank 1 free B-module. □

0C0H **Lemma 47.24.9.** *Let R be a Noetherian ring and let $f \in R$. If φ denotes the map $R \to R_f$, then $\varphi^!$ is isomorphic to $- \otimes^{\mathbf{L}}_R R_f$. More generally, if $\varphi : R \to R'$ is a map such that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an open immersion, then $\varphi^!$ is isomorphic to $- \otimes^{\mathbf{L}}_R R'$.*

[2] If $f = x^d + a_1 x^{d-1} + \ldots + a_d$, then $c_1 = -a_1$, $c_2 = a_1^2 - a_2$, $c_3 = -a_1^3 + 2a_1 a_2 - a_3$, etc.

Proof. Choose the presentation $R \to R[x] \to R[x]/(fx-1) = R_f$ and observe that $fx-1$ is a nonzerodivisor in $R[x]$. Thus we can apply using Lemma 47.13.10 to compute the functor $\varphi^!$. Details omitted; note that the shift in the definition of $\varphi^!$ and in the lemma add up to zero.

In the general case note that $R' \otimes_R R' = R'$. Hence the result follows from the base change results above. Either Lemma 47.24.4 or Lemma 47.24.5 will do. □

0BZU **Lemma 47.24.10.** *Let $\varphi : R \to A$ be a perfect homomorphism of Noetherian rings (for example φ is flat of finite type). Then $\varphi^!(K) = K \otimes^{\mathbf{L}}_R \varphi^!(R)$ for $K \in D(R)$.*

Proof. (The parenthetical statement follows from More on Algebra, Lemma 15.83.4.) We can choose a factorization $R \to P \to A$ where P is a polynomial ring in n variables over R and then A is a perfect P-module, see More on Algebra, Lemma 15.83.2. Recall that $\varphi^!(K) = R\mathrm{Hom}(A, K \otimes^{\mathbf{L}}_R P[n])$. Thus the result follows from Lemma 47.13.9 and More on Algebra, Lemma 15.60.5. □

0E9L **Lemma 47.24.11.** *Let $\varphi : A \to B$ be a finite type homomorphism of Noetherian rings. Let $\omega^\bullet_A$ be a dualizing complex for A. Set $\omega^\bullet_B = \varphi^!(\omega^\bullet_A)$. Denote $D_A(K) = R\mathrm{Hom}_A(K, \omega^\bullet_A)$ for $K \in D_{Coh}(A)$ and $D_B(L) = R\mathrm{Hom}_B(L, \omega^\bullet_B)$ for $L \in D_{Coh}(B)$. Then there is a functorial isomorphism*

$$\varphi^!(K) = D_B(D_A(K) \otimes^{\mathbf{L}}_A B)$$

for $K \in D_{Coh}(A)$.

Proof. Observe that $\omega^\bullet_B$ is a dualizing complex for B by Lemma 47.24.3. Let $A \to B \to C$ be finite type homomorphisms of Noetherian rings. If the lemma holds for $A \to B$ and $B \to C$, then the lemma holds for $A \to C$. This follows from Lemma 47.24.7 and the fact that $D_B \circ D_B \cong \mathrm{id}$ by Lemma 47.15.3. Thus it suffices to prove the lemma in case $A \to B$ is a surjection and in the case where B is a polynomial ring over A.

Assume $B = A[x_1, \ldots, x_n]$. Since $D_A \circ D_A \cong \mathrm{id}$, it suffices to prove $D_B(K \otimes_A B) \cong D_A(K) \otimes_A B[n]$ for K in $D_{Coh}(A)$. Choose a bounded complex $I^\bullet$ of injectives representing $\omega^\bullet_A$. Choose a quasi-isomorphism $I^\bullet \otimes_A B \to J^\bullet$ where $J^\bullet$ is a bounded complex of B-modules. Given a complex $K^\bullet$ of A-modules, consider the obvious map of complexes

$$\mathrm{Hom}^\bullet(K^\bullet, I^\bullet) \otimes_A B[n] \longrightarrow \mathrm{Hom}^\bullet(K^\bullet \otimes_A B, J^\bullet[n])$$

The left hand side represents $D_A(K) \otimes_A B[n]$ and the right hand side represents $D_B(K \otimes_A B)$. Thus it suffices to prove this map is a quasi-isomorphism if the cohomology modules of $K^\bullet$ are finite A-modules. Observe that the cohomology of the complex in degree r (on either side) only depends on finitely many of the K^i. Thus we may replace $K^\bullet$ by a truncation, i.e., we may assume $K^\bullet$ represents an object of $D^-_{Coh}(A)$. Then $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite free A-modules. Therefore we may assume $K^\bullet$ is a bounded above complex of finite free A-modules. In this case it is easy to that the displayed map is an isomorphism of complexes which finishes the proof in this case.

Assume that $A \to B$ is surjective. Denote $i_* : D(B) \to D(A)$ the restriction functor and recall that $\varphi^!(-) = R\mathrm{Hom}(A, -)$ is a right adjoint to i_* (Lemma 47.13.1). For

$F \in D(B)$ we have

$$\begin{aligned}\mathrm{Hom}_B(F, D_B(D_A(K) \otimes^{\mathbf{L}}_A B)) &= \mathrm{Hom}_B((D_A(K) \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_B F, \omega^\bullet_B) \\ &= \mathrm{Hom}_A(D_A(K) \otimes^{\mathbf{L}}_A i_*F, \omega^\bullet_A) \\ &= \mathrm{Hom}_A(i_*F, D_A(D_A(K))) \\ &= \mathrm{Hom}_A(i_*F, K) \\ &= \mathrm{Hom}_B(F, \varphi^!(K))\end{aligned}$$

The first equality follows from More on Algebra, Lemma 15.74.1 and the definition of D_B. The second equality by the adjointness mentioned above and the equality $i_*((D_A(K) \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_B F) = D_A(K) \otimes^{\mathbf{L}}_A i_*F$ (More on Algebra, Lemma 15.60.1). The third equality follows from More on Algebra, Lemma 15.74.1. The fourth because $D_A \circ D_A = \mathrm{id}$. The final equality by adjointness again. Thus the result holds by the Yoneda lemma. □

47.25. Relative dualizing complexes in the Noetherian case

0E9M Let $\varphi : R \to A$ be a finite type homomorphism of Noetherian rings. Then we define the *relative dualizing complex of A over R* as the object

$$\omega^\bullet_{A/R} = \varphi^!(R)$$

of $D(A)$. Here $\varphi^!$ is as in Section 47.24. From the material in that section we see that $\omega^\bullet_{A/R}$ is well defined up to (non-unique) isomorphism.

0BZV **Lemma 47.25.1.** *Let $R \to R'$ be a homomorphism of Noetherian rings. Let $R \to A$ be of finite type. Set $A' = A \otimes_R R'$. If*

(1) *$R \to R'$ is flat, or*
(2) *$R \to A$ is flat, or*
(3) *$R \to A$ is perfect and R' and A are tor independent over R,*

then there is an isomorphism $\omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A A' \to \omega^\bullet_{A'/R'}$ in $D(A')$.

Proof. Follows from Lemmas 47.24.4, 47.24.6, and 47.24.5 and the definitions. □

0BZW **Lemma 47.25.2.** *Let $\varphi : R \to A$ be a finite type map of Noetherian rings.*

(1) *If φ is perfect, then*
   (a) *$\omega^\bullet_{A/R}$ is in $D^b_{Coh}(A)$,*
   (b) *$\omega^\bullet_{A/R}$ has finite tor dimension over R, and*
   (c) *$A \to R\mathrm{Hom}_A(\omega^\bullet_{A/R}, \omega^\bullet_{A/R})$ is an isomorphism.*
(2) *If φ is flat, then (1)(a), (1)(b), and (1)(c) hold and*
   (a) *$\omega^\bullet_{A/R}$ is R-perfect (More on Algebra, Definition 15.84.1), and*
   (b) *for every map $R \to k$ to a field the base change $\omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R k)$ is a dualizing complex for $A \otimes_R k$.*

Proof. Assume $R \to A$ is perfect. Choose $R \to P \to A$ as in the definition of $\varphi^!$. Then A is perfect as a P-modue (More on Algebra, Lemma 15.83.2). This shows that $\omega^\bullet_{A/R}$ is in $D^b_{Coh}(A)$ by Lemma 47.24.2. This proves (1)(a). To show $\omega^\bullet_{A/R}$ has finite tor dimension as a complex of R-modules, observe that $\omega^\bullet_{A/R} = \varphi^!(R) = R\mathrm{Hom}(A, P)[n]$ maps to $R\mathrm{Hom}_P(A, P)[n]$ in $D(P)$, which is perfect in $D(P)$ (More on Algebra, Lemma 15.75.15), hence has finite tor dimension in $D(R)$

as $R \to P$ is flat. This proves (1)(b). The object $R\mathrm{Hom}_A(\omega^\bullet_{A/R}, \omega^\bullet_{A/R})$ of $D(A)$ maps in $D(P)$ to

$$\begin{aligned} R\mathrm{Hom}_P(\omega^\bullet_{A/R}, R\mathrm{Hom}(A,P)[n]) &= R\mathrm{Hom}_P(R\mathrm{Hom}_P(A,P)[n],P)[n] \\ &= R\mathrm{Hom}_P(R\mathrm{Hom}_P(A,P),P) \end{aligned}$$

This is equal to A by the already used More on Algebra, Lemma 15.75.15. This proves (1)(c).

Assume φ is flat. Then $R \to A$ is a perfect ring map (More on Algebra, Lemma 15.83.4) and we see that (1)(a), (1)(b), and (1)(c) hold. Of course, then $\omega^\bullet_{A/R}$ is R-perfect by (1)(a) and (1)(b) and the definitions. Let $R \to k$ be as in (2)(b). By Lemma 47.25.1 there is an isomorphism

$$\omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R k) \cong \omega^\bullet_{A \otimes_R k/k}$$

and the right hand side is a dualizing complex by Lemma 47.24.3. This finishes the proof. □

0E0P **Lemma 47.25.3.** *Let K/k be an extension of fields. Let A be a finite type k-algebra. Let $A_K = A \otimes_k K$. If $\omega^\bullet_A$ is a dualizing complex for A, then $\omega^\bullet_A \otimes_A A_K$ is a dualizing complex for A_K.*

Proof. By the uniqueness of dualizing complexes, it doesn't matter which dualizing complex we pick for A; we omit the detailed proof. Denote $\varphi : k \to A$ the algebra structure. We may take $\omega^\bullet_A = \varphi^!(k[0])$ by Lemma 47.24.3. We conclude by Lemma 47.25.2. □

0E4B **Lemma 47.25.4.** *Let $\varphi : R \to A$ be a local complete intersection homomorphism of Noetherian rings. Then $\omega^\bullet_{A/R}$ is an invertible object of $D(A)$ and $\varphi^!(K) = K \otimes^{\mathbf{L}}_R \omega^\bullet_{A/R}$ for all $K \in D(R)$.*

Proof. Recall that a local complete intersection homomorphism is a perfect ring map by More on Algebra, Lemma 15.83.6. Hence the final statement holds by Lemma 47.24.10. By More on Algebra, Definition 15.33.2 we can write $A = R[x_1, \ldots, x_n]/I$ where I is a Koszul-regular ideal. The construction of $\varphi^!$ in Section 47.24 shows that it suffices to show the lemma in case $A = R/I$ where $I \subset R$ is a Koszul-regular ideal. Checking $\omega^\bullet_{A/R}$ is invertible in $D(A)$ is local on $\mathrm{Spec}(A)$ by More on Algebra, Lemma 15.127.4. Moreover, formation of $\omega^\bullet_{A/R}$ commutes with localization on R by Lemma 47.24.4. Combining More on Algebra, Definition 15.32.1 and Lemma 15.30.7 and Algebra, Lemma 10.68.6 we can find $g_1, \ldots, g_r \in R$ generating the unit ideal in A such that $I_{g_j} \subset R_{g_j}$ is generated by a regular sequence. Thus we may assume $A = R/(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in R. Then we consider the ring maps

$$R \to R/(f_1) \to R/(f_1, f_2) \to \ldots \to R/(f_1, \ldots, f_c) = A$$

and we use Lemma 47.24.7 (and the final statement already proven) to see that it suffices to prove the lemma for each step. Finally, in case $A = R/(f)$ for some nonzerodivisor f we see that the lemma is true since $\varphi^!(R) = R\mathrm{Hom}(A, R)$ is invertible by Lemma 47.13.10. □

0E4C **Lemma 47.25.5.** *Let $\varphi : R \to A$ be a flat finite type homomorphism of Noetherian rings. The following are equivalent*

(1) *the fibres* $A \otimes_R \kappa(\mathfrak{p})$ *are Gorenstein for all primes* $\mathfrak{p} \subset R$*, and*
(2) $\omega^\bullet_{A/R}$ *is an invertible object of* $D(A)$*, see More on Algebra, Lemma 15.127.4.*

Proof. If (2) holds, then the fibre rings $A \otimes_R \kappa(\mathfrak{p})$ have invertible dualizing complexes, and hence are Gorenstein. See Lemmas 47.25.2 and 47.21.4.

For the converse, assume (1). Observe that $\omega^\bullet_{A/R}$ is in $D^b_{Coh}(A)$ by Lemma 47.24.2 (since flat finite type homomorphisms of Noetherian rings are perfect, see More on Algebra, Lemma 15.83.4). Take a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Then

$$\omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A \kappa(\mathfrak{q}) = \omega^\bullet_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R \kappa(\mathfrak{p})) \otimes^{\mathbf{L}}_{(A \otimes_R \kappa(\mathfrak{p}))} \kappa(\mathfrak{q})$$

Applying Lemmas 47.25.2 and 47.21.4 and assumption (1) we find that this complex has 1 nonzero cohomology group which is a 1-dimensional $\kappa(\mathfrak{q})$-vector space. By More on Algebra, Lemma 15.78.1 we conclude that $(\omega^\bullet_{A/R})_f$ is an invertible object of $D(A_f)$ for some $f \in A$, $f \notin \mathfrak{q}$. This proves (2) holds. □

The following lemma is useful to see how dimension functions change when passing to a finite type algebra over a Noetherian ring.

0E9N **Lemma 47.25.6.** *Let* $\varphi : R \to A$ *be a finite type homomorphism of Noetherian rings. Assume* R *local and let* $\mathfrak{m} \subset A$ *be a maximal ideal lying over the maximal ideal of* R*. If* $\omega^\bullet_R$ *is a normalized dualizing complex for* R*, then* $\varphi^!(\omega^\bullet_R)_\mathfrak{m}$ *is a normalized dualizing complex for* $A_\mathfrak{m}$*.*

Proof. We already know that $\varphi^!(\omega^\bullet_R)$ is a dualizing complex for A, see Lemma 47.24.3. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in the construction of $\varphi^!$. If we can prove the lemma for $R \to P$ and the maximal ideal $\mathfrak{m}'$ of P corresponding to $\mathfrak{m}$, then we obtain the result for $R \to A$ by applying Lemma 47.16.2 to $P_{\mathfrak{m}'} \to A_\mathfrak{m}$ or by applying Lemma 47.17.2 to $P \to A$. In the case $A = R[x_1, \ldots, x_n]$ we see that $\dim(A_\mathfrak{m}) = \dim(R) + n$ for example by Algebra, Lemma 10.112.7 (combined with Algebra, Lemma 10.114.1 to compute the dimension of the fibre). The fact that $\omega^\bullet_R$ is normalized means that $i = -\dim(R)$ is the smallest index such that $H^i(\omega^\bullet_R)$ is nonzero (follows from Lemmas 47.16.5 and 47.16.11). Then $\varphi^!(\omega^\bullet_R)_\mathfrak{m} = \omega^\bullet_R \otimes_R A_\mathfrak{m}[n]$ has its first nonzero cohomology module in degree $-\dim(R) - n$ and therefore is the normalized dualizing complex for $A_\mathfrak{m}$. □

0E9P **Lemma 47.25.7.** *Let* $R \to A$ *be a finite type homomorphism of Noetherian rings. Let* $\mathfrak{q} \subset A$ *be a prime ideal lying over* $\mathfrak{p} \subset R$*. Then*

$$H^i(\omega^\bullet_{A/R})_\mathfrak{q} \neq 0 \Rightarrow -d \leq i$$

where d *is the dimension of the fibre of* $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ *over* $\mathfrak{p}$ *at the point* $\mathfrak{q}$*.*

Proof. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 47.24 so that $\omega^\bullet_{A/R} = R\operatorname{Hom}(A, P)[n]$. We have to show that $R\operatorname{Hom}(A, P)_\mathfrak{q}$ has vanishing cohomology in degrees $< n - d$. By Lemma 47.13.3 this means we have to show that $\operatorname{Ext}^i_P(P/I, P)_\mathfrak{r} = 0$ for $i < n - d$ where $\mathfrak{r} \subset P$ is the prime corresponding to $\mathfrak{q}$ and I is the kernel of $P \to A$. We may rewrite this as $\operatorname{Ext}^i_{P_\mathfrak{r}}(P_\mathfrak{r}/IP_\mathfrak{r}, P_\mathfrak{r})$ by More on Algebra, Lemma 15.66.4. Thus we have to show

$$\operatorname{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq n - d$$

by Lemma 47.11.1. By Lemma 47.11.5 we have

$$\operatorname{depth}_{IP_\mathfrak{r}}(P_\mathfrak{r}) \geq \dim((P \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r}) - \dim((P/I \otimes_R \kappa(\mathfrak{p}))_\mathfrak{r})$$

The two expressions on the right hand side agree by Algebra, Lemma 10.116.4. □

0E9Q **Lemma 47.25.8.** *Let $R \to A$ be a flat finite type homomorphism of Noetherian rings. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Then*

$$H^i(\omega^\bullet_{A/R})_\mathfrak{q} \neq 0 \Rightarrow -d \leq i \leq 0$$

where d is the dimension of the fibre of $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ over $\mathfrak{p}$ at the point $\mathfrak{q}$. If all fibres of $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ have dimension $\leq d$, then $\omega^\bullet_{A/R}$ has tor amplitude in $[-d, 0]$ as a complex of R-modules.

Proof. The lower bound has been shown in Lemma 47.25.7. Choose a factorization $R \to P \to A$ with $P = R[x_1, \ldots, x_n]$ as in Section 47.24 so that $\omega^\bullet_{A/R} = R\mathrm{Hom}(A, P)[n]$. The upper bound means that $\mathrm{Ext}^i_P(A, P)$ is zero for $i > n$. This follows from More on Algebra, Lemma 15.78.5 which shows that A is a perfect P-module with tor amplitude in $[-n, 0]$.

Proof of the final statement. Let $R \to R'$ be a ring homomorphism of Noetherian rings. Set $A' = A \otimes_R R'$. Then

$$\omega^\bullet_{A'/R'} = \omega^\bullet_{A/R} \otimes^\mathbf{L}_A A' = \omega^\bullet_{A/R} \otimes^\mathbf{L}_R R'$$

The first isomorphism by Lemma 47.25.1 and the second, which takes place in $D(R')$, by More on Algebra, Lemma 15.61.2. By the first part of the proof (note that the fibres of $\mathrm{Spec}(A') \to \mathrm{Spec}(R')$ have dimension $\leq d$) we conclude that $\omega^\bullet_{A/R} \otimes^\mathbf{L}_R R'$ has cohomology only in degrees $[-d, 0]$. Taking $R' = R \oplus M$ to be the square zero thickening of R by a finite R-module M, we see that $R\mathrm{Hom}(A, P) \otimes^\mathbf{L}_R M$ has cohomology only in the interval $[-d, 0]$ for any finite R-module M. Since any R-module is a filtered colimit of finite R-modules and since tensor products commute with colimits we conclude. □

0E9R **Lemma 47.25.9.** *Let $R \to A$ be a finite type homomorphism of Noetherian rings. Let $\mathfrak{p} \subset R$ be a prime ideal. Assume*

(1) *$R_\mathfrak{p}$ is Cohen-Macaulay, and*
(2) *for any minimal prime $\mathfrak{q} \subset A$ we have $trdeg_{\kappa(R \cap \mathfrak{q})}\kappa(\mathfrak{q}) \leq r$.*

Then

$$H^i(\omega^\bullet_{A/R})_\mathfrak{p} \neq 0 \Rightarrow -r \leq i$$

and $H^{-r}(\omega^\bullet_{A/R})_\mathfrak{p}$ is (S_2) as an $A_\mathfrak{p}$-module.

Proof. We may replace R by $R_\mathfrak{p}$ by Lemma 47.25.1. Thus we may assume R is a Cohen-Macaulay local ring and we have to show the assertions of the lemma for the A-modules $H^i(\omega^\bullet_{A/R})$.

Let $R^\wedge$ be the completion of R. The map $R \to R^\wedge$ is flat and $R^\wedge$ is Cohen-Macaulay (More on Algebra, Lemma 15.43.3). Observe that the minimal primes of $A \otimes_R R^\wedge$ lie over minimal primes of A by the flatness of $A \to A \otimes_R R^\wedge$ (and going down for flatness, see Algebra, Lemma 10.39.19). Thus condition (2) holds for the finite type ring map $R^\wedge \to A \otimes_R R^\wedge$ by Morphisms, Lemma 29.28.3. Appealing to Lemma 47.25.1 once again it suffices to prove the lemma for $R^\wedge \to A \otimes_R R^\wedge$. In this way, using Lemma 47.22.4, we may assume R is a Noetherian local Cohen-Macaulay ring which has a dualizing complex $\omega^\bullet_R$.

Let $\mathfrak{m} \subset A$ be a maximal ideal. It suffices to show that the assertions of the lemma hold for $H^i(\omega^\bullet_{A/R})_\mathfrak{m}$. If $\mathfrak{m}$ does not lie over the maximal ideal of R, then we replace R by a localization to reduce to this case (small detail omitted).

We may assume $\omega^\bullet_R$ is normalized. Setting $d = \dim(R)$ we see that $\omega^\bullet_R = \omega_R[d]$ for some R-module ω_R, see Lemma 47.20.2. Set $\omega^\bullet_A = \varphi^!(\omega^\bullet_R)$. By Lemma 47.24.11 we have

$$\omega^\bullet_{A/R} = R\operatorname{Hom}_A(\omega_R[d] \otimes^{\mathbf{L}}_R A, \omega^\bullet_A)$$

By the dimension formula we have $\dim(A_\mathfrak{m}) \leq d+r$, see Morphisms, Lemma 29.52.2 and use that $\kappa(\mathfrak{m})$ is finite over the residue field of R by the Hilbert Nullstellensatz. By Lemma 47.25.6 we see that $(\omega^\bullet_A)_\mathfrak{m}$ is a normalized dualizing complex for $A_\mathfrak{m}$. Hence $H^i((\omega^\bullet_A)_\mathfrak{m})$ is nonzero only for $-d-r \leq i \leq 0$, see Lemma 47.16.5. Since $\omega_R[d] \otimes^{\mathbf{L}}_R A$ lives in degrees $\leq -d$ we conclude the vanishing holds. Finally, we also see that

$$H^{-r}(\omega^\bullet_{A/R})_\mathfrak{m} = \operatorname{Hom}_A(\omega_R \otimes_R A, H^{-d-r}(\omega^\bullet_A))_\mathfrak{m}$$

Since $H^{-d-r}(\omega^\bullet_A)_\mathfrak{m}$ is (S_2) by Lemma 47.17.5 we find that the final statement is true by More on Algebra, Lemma 15.23.11. □

47.26. More on dualizing complexes

0E49 Some lemmas which don't fit anywhere else very well.

0E4A **Lemma 47.26.1.** *Let $A \to B$ be a faithfully flat map of Noetherian rings. If $K \in D(A)$ and $K \otimes^{\mathbf{L}}_A B$ is a dualizing complex for B, then K is a dualizing complex for A.*

Proof. Since $A \to B$ is flat we have $H^i(K) \otimes_A B = H^i(K \otimes^{\mathbf{L}}_A B)$. Since $K \otimes^{\mathbf{L}}_A B$ is in $D^b_{Coh}(B)$ we first find that K is in $D^b(A)$ and then we see that $H^i(K)$ is a finite A-module by Algebra, Lemma 10.83.2. Let M be a finite A-module. Then

$$R\operatorname{Hom}_A(M, K) \otimes_A B = R\operatorname{Hom}_B(M \otimes_A B, K \otimes^{\mathbf{L}}_A B)$$

by More on Algebra, Lemma 15.100.2. Since $K \otimes^{\mathbf{L}}_A B$ has finite injective dimension, say injective-amplitude in $[a, b]$, we see that the right hand side has vanishing cohomology in degrees $> b$. Since $A \to B$ is faithfully flat, we find that $R\operatorname{Hom}_A(M, K)$ has vanishing cohomology in degrees $> b$. Thus K has finite injective dimension by More on Algebra, Lemma 15.70.2. To finish the proof we have to show that the map $A \to R\operatorname{Hom}_A(K, K)$ is an isomorphism. For this we again use More on Algebra, Lemma 15.100.2 and the fact that $B \to R\operatorname{Hom}_B(K \otimes^{\mathbf{L}}_A B, K \otimes^{\mathbf{L}}_A B)$ is an isomorphism. □

0E4D **Lemma 47.26.2.** *Let $\varphi : A \to B$ be a homomorphism of Noetherian rings. Assume*

(1) *$A \to B$ is syntomic and induces a surjective map on spectra, or*
(2) *$A \to B$ is a faithfully flat local complete intersection, or*
(3) *$A \to B$ is faithfully flat of finite type with Gorenstein fibres.*

Then $K \in D(A)$ is a dualizing complex for A if and only if $K \otimes^{\mathbf{L}}_A B$ is a dualizing complex for B.

Proof. Observe that $A \to B$ satisfies (1) if and only if $A \to B$ satisfies (2) by More on Algebra, Lemma 15.33.5. Observe that in both (2) and (3) the relative dualzing complex $\varphi^!(A) = \omega^\bullet_{B/A}$ is an invertible object of $D(B)$, see Lemmas 47.25.4 and

47.25.5. Moreover we have $\varphi^!(K) = K \otimes^{\mathbf{L}}_A \omega^\bullet_{B/A}$ in both cases, see Lemma 47.24.10 for case (3). Thus $\varphi^!(K)$ is the same as $K \otimes^{\mathbf{L}}_A B$ up to tensoring with an invertible object of $D(B)$. Hence $\varphi^!(K)$ is a dualizing complex for B if and only if $K \otimes^{\mathbf{L}}_A B$ is (as being a dualizing complex is local and invariant under shifts). Thus we see that if K is dualizing for A, then $K \otimes^{\mathbf{L}}_A B$ is dualizing for B by Lemma 47.24.3. To descend the property, see Lemma 47.26.1. □

0E4E **Lemma 47.26.3.** *Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{n}, l)$ be a flat local homorphism of Noetherian rings such that $\mathfrak{n} = \mathfrak{m}B$. If E is the injective hull of κ, then $E \otimes_A B$ is the injective hull of l.*

Proof. Write $E = \bigcup E_n$ as in Lemma 47.7.3. It suffices to show that $E_n \otimes_{A/\mathfrak{m}^n} B/\mathfrak{n}^n$ is the injective hull of l over $B/\mathfrak{n}$. This reduces us to the case where A and B are Artinian local. Observe that $\text{length}_A(A) = \text{length}_B(B)$ and $\text{length}_A(E) = \text{length}_B(E \otimes_A B)$ by Algebra, Lemma 10.52.13. By Lemma 47.6.1 we have $\text{length}_A(E) = \text{length}_A(A)$ and $\text{length}_B(E') = \text{length}_B(B)$ where E' is the injective hull of l over B. We conclude $\text{length}_B(E') = \text{length}_B(E \otimes_A B)$. Observe that

$$\dim_l((E \otimes_A B)[\mathfrak{n}]) = \dim_l(E[\mathfrak{m}] \otimes_A B) = \dim_\kappa(E[\mathfrak{m}]) = 1$$

where we have used flatness of $A \to B$ and $\mathfrak{n} = \mathfrak{m}B$. Thus there is an injective B-module map $E \otimes_A B \to E'$ by Lemma 47.7.2. By equality of lengths shown above this is an isomorphism. □

0E4F **Lemma 47.26.4.** *Let $\varphi : A \to B$ be a flat homorphism of Noetherian rings such that for all primes $\mathfrak{q} \subset B$ we have $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, for example if φ is étale. If I is an injective A-module, then $I \otimes_A B$ is an injective B-module.*

Proof. Étale maps satisfy the assumption by Algebra, Lemma 10.143.5. By Lemma 47.3.7 and Proposition 47.5.9 we may assume I is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$. Then I is a module over $A_\mathfrak{p}$. It suffices to prove $I \otimes_A B = I \otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ is injective as a $B_\mathfrak{p}$-module, see Lemma 47.3.2. Thus we may assume $(A, \mathfrak{m}, \kappa)$ is local Noetherian and $I = E$ is the injective hull of the residue field κ. Our assumption implies that the Noetherian ring $B/\mathfrak{m}B$ is a product of fields (details omitted). Thus there are finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ in B lying over $\mathfrak{m}$ and they are all maximal ideals. Write $E = \bigcup E_n$ as in Lemma 47.7.3. Then $E \otimes_A B = \bigcup E_n \otimes_A B$ and $E_n \otimes_A B$ is a finite B-module with support $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ hence decomposes as a product over the localizations at $\mathfrak{m}_i$. Thus $E \otimes_A B = \prod (E \otimes_A B)_{\mathfrak{m}_i}$. Since $(E \otimes_A B)_{\mathfrak{m}_i} = E \otimes_A B_{\mathfrak{m}_i}$ is the injective hull of the residue field of $\mathfrak{m}_i$ by Lemma 47.26.3 we conclude. □

47.27. Relative dualizing complexes

0E2B For a finite type ring map $\varphi : R \to A$ of Noetherian rings we have the relative dualizing complex $\omega^\bullet_{A/R} = \varphi^!(R)$ considered in Section 47.25. If R is not Noetherian, a similarly constructed complex will in general not have good properties. In this section, we give a definition of a relative dualizing complex for a flat and finitely presented ring maps $R \to A$ of non-Noetherian rings. The definition is chosen to globalize to flat and finitely presented morphisms of schemes, see Duality for Schemes, Section 48.28. We will show that relative dualizing complexes exist (when the definition applies), are unique up to (noncanonical) isomorphism, and that in the Noetherian case we recover the complex of Section 47.25.

The Noetherian reader may safely skip this section!

0E2C **Definition 47.27.1.** Let $R \to A$ be a flat ring map of finite presentation. A *relative dualizing complex* is an object $K \in D(A)$ such that

(1) K is R-perfect (More on Algebra, Definition 15.84.1), and
(2) $R\mathrm{Hom}_{A\otimes_R A}(A, K \otimes^{\mathbf{L}}_A (A \otimes_R A))$ is isomorphic to A.

To understand this definition you may have to read and understand some of the following lemmas. Lemmas 47.27.3 and 47.27.2 show this definition does not clash with the definition in Section 47.25.

0E2D **Lemma 47.27.2.** *Let $R \to A$ be a flat ring map of finite presentation. Any two relative dualizing complexes for $R \to A$ are isomorphic.*

Proof. Let K and L be two relative dualizing complexes for $R \to A$. Denote $K_1 = K \otimes^{\mathbf{L}}_A (A \otimes_R A)$ and $L_2 = (A \otimes_R A) \otimes^{\mathbf{L}}_A L$ the derived base changes via the first and second coprojections $A \to A \otimes_R A$. By symmetry the assumption on L_2 implies that $R\mathrm{Hom}_{A\otimes_R A}(A, L_2)$ is isomorphic to A. By More on Algebra, Lemma 15.99.3 part (3) applied twice we have

$$A \otimes^{\mathbf{L}}_{A\otimes_R A} L_2 \cong R\mathrm{Hom}_{A\otimes_R A}(A, K_1 \otimes^{\mathbf{L}}_{A\otimes_R A} L_2) \cong A \otimes^{\mathbf{L}}_{A\otimes_R A} K_1$$

Applying the restriction functor $D(A \otimes_R A) \to D(A)$ for either coprojection we obtain the desired result. □

0E2E **Lemma 47.27.3.** *Let $\varphi : R \to A$ be a flat finite type ring map of Noetherian rings. Then the relative dualizing complex $\omega^\bullet_{A/R} = \varphi^!(R)$ of Section 47.25 is a relative dualizing complex in the sense of Definition 47.27.1.*

Proof. From Lemma 47.25.2 we see that $\varphi^!(R)$ is R-perfect. Denote $\delta : A \otimes_R A \to A$ the multiplication map and $p_1, p_2 : A \to A \otimes_R A$ the coprojections. Then

$$\varphi^!(R) \otimes^{\mathbf{L}}_A (A \otimes_R A) = \varphi^!(R) \otimes^{\mathbf{L}}_{A,p_1} (A \otimes_R A) = p_2^!(A)$$

by Lemma 47.24.4. Recall that $R\mathrm{Hom}_{A\otimes_R A}(A, \varphi^!(R) \otimes^{\mathbf{L}}_A (A \otimes_R A))$ is the image of $\delta^!(\varphi^!(R) \otimes^{\mathbf{L}}_A (A \otimes_R A))$ under the restriction map $\delta_* : D(A) \to D(A \otimes_R A)$. Use the definition of $\delta^!$ from Section 47.24 and Lemma 47.13.3. Since $\delta^!(p_2^!(A)) \cong A$ by Lemma 47.24.7 we conclude. □

0E2F **Lemma 47.27.4.** *Let $R \to A$ be a flat ring map of finite presentation. Then*

(1) *there exists a relative dualizing complex K in $D(A)$, and*
(2) *for any ring map $R \to R'$ setting $A' = A \otimes_R R'$ and $K' = K \otimes^{\mathbf{L}}_A A'$, then K' is a relative dualizing complex for $R' \to A'$.*

Moreover, if

$$\xi : A \longrightarrow K \otimes^{\mathbf{L}}_A (A \otimes_R A)$$

is a generator for the cyclic module $\mathrm{Hom}_{D(A\otimes_R A)}(A, K \otimes^{\mathbf{L}}_A (A \otimes_R A))$ *then in (2) the derived base change of ξ by $A \otimes_R A \to A' \otimes_{R'} A'$ is a generator for the cyclic module* $\mathrm{Hom}_{D(A'\otimes_{R'} A')}(A', K' \otimes^{\mathbf{L}}_{A'} (A' \otimes_{R'} A'))$

Proof. We first reduce to the Noetherian case. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemma 47.27.3 there exists a relative dualizing complex $K_0 \in D(A_0)$. Thus if we show (2) for K_0, then we find that $K_0 \otimes^{\mathbf{L}}_{A_0} A$ is a dualizing complex for $R \to A$ and that it also satisfies (2) by transitivity of derived

base change. The uniqueness of relative dualizing complexes (Lemma 47.27.2) then shows that this holds for any relative dualizing complex.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes^{\mathbf{L}}_A A'$. To finish the proof we have to show that K' is a relative dualizing complex for $R' \to A'$. By More on Algebra, Lemma 15.84.5 we see that K' is R'-perfect in all cases. By Lemmas 47.25.1 and 47.27.3 if R' is Noetherian, then K' is a relative dualizing complex for $R' \to A'$ (in either sense). Transitivity of derived tensor product shows that $K \otimes^{\mathbf{L}}_A (A \otimes_R A) \otimes^{\mathbf{L}}_{A \otimes_R A} (A' \otimes_{R'} A') = K' \otimes^{\mathbf{L}}_{A'} (A' \otimes_{R'} A')$. Flatness of $R \to A$ guarantees that $A \otimes^{\mathbf{L}}_{A \otimes_R A} (A' \otimes_{R'} A') = A'$; namely $A \otimes_R A$ and R' are tor independent over R so we can apply More on Algebra, Lemma 15.61.2. Finally, A is pseudo-coherent as an $A \otimes_R A$-module by More on Algebra, Lemma 15.83.8. Thus we have checked all the assumptions of More on Algebra, Lemma 15.84.6. We find there exists a bounded below complex $E^\bullet$ of R-flat finitely presented $A \otimes_R A$-modules such that $E^\bullet \otimes_R R'$ represents $R\mathrm{Hom}_{A' \otimes_{R'} A'}(A', K' \otimes^{\mathbf{L}}_{A'} (A' \otimes_{R'} A'))$ and these identifications are compatible with derived base change. Let $n \in \mathbf{Z}$, $n \neq 0$. Define Q^n by the sequence

$$E^{n-1} \to E^n \to Q^n \to 0$$

Since $\kappa(\mathfrak{p})$ is a Noetherian ring, we know that $H^n(E^\bullet \otimes_R \kappa(\mathfrak{p})) = 0$, see remarks above. Chasing diagrams this means that

$$Q^n \otimes_R \kappa(\mathfrak{p}) \to E^{n+1} \otimes_R \kappa(\mathfrak{p})$$

is injective. Hence for a prime $\mathfrak{q}$ of $A \otimes_R A$ lying over $\mathfrak{p}$ we have $Q^n_\mathfrak{q}$ is $R_\mathfrak{p}$-flat and $Q^n_\mathfrak{p} \to E^{n+1}_\mathfrak{q}$ is $R_\mathfrak{p}$-universally injective, see Algebra, Lemma 10.99.1. Since this holds for all primes, we conclude that Q^n is R-flat and $Q^n \to E^{n+1}$ is R-universally injective. In particular $H^n(E^\bullet \otimes_R R') = 0$ for any ring map $R \to R'$. Let $Z^0 = \mathrm{Ker}(E^0 \to E^1)$. Since there is an exact sequence $0 \to Z^0 \to E^0 \to E^1 \to Q^1 \to 0$ we see that Z^0 is R-flat and that $Z^0 \otimes_R R' = \mathrm{Ker}(E^0 \otimes_R R' \to E^1 \otimes_R R')$ for all $R \to R'$. Then the short exact sequence $0 \to Q^{-1} \to Z^0 \to H^0(E^\bullet) \to 0$ shows that

$$H^0(E^\bullet \otimes_R R') = H^0(E^\bullet) \otimes_R R' = A \otimes_R R' = A'$$

as desired. This equality furthermore gives the final assertion of the lemma. □

0E2G **Lemma 47.27.5.** *Let $R \to A$ be a flat ring map of finite presentation. Let K be a relative dualizing complex. Then $A \to R\mathrm{Hom}_A(K, K)$ is an isomorphism.*

Proof. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. By Lemmas 47.27.2, 47.27.3, and 47.27.4 there exists a relative dualizing complex $K_0 \in D(A_0)$ and its derived base change is K. This reduces us to the situation discussed in the next paragraph.

Assume R Noetherian and let K be a relative dualizing complex for $R \to A$. Given a ring map $R \to R'$ set $A' = A \otimes_R R'$ and $K' = K \otimes^{\mathbf{L}}_A A'$. To finish the proof we show $R\mathrm{Hom}_{A'}(K', K') = A'$. By Lemma 47.25.2 we know this is true whenever R' is Noetherian. Since a general R' is a filtered colimit of Noetherian R-algebras, we find the result holds by More on Algebra, Lemma 15.84.7. □

0E2H **Lemma 47.27.6.** *Let $R \to A \to B$ be a ring maps which are flat and of finite presentation. Let $K_{A/R}$ and $K_{B/A}$ be relative dualizing complexes for $R \to A$ and $A \to B$. Then $K = K_{A/R} \otimes^{\mathbf{L}}_A K_{B/A}$ is a relative dualizing complex for $R \to B$.*

Proof. We will use reduction to the Noetherian case. Namely, by Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$ subalgebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ such that $A = A_0 \otimes_{R_0} R$. After increasing R_0 and correspondingly replacing A_0 we may assume there is a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{R_0} R$ (use the same lemma). If we prove the lemma for $R_0 \to A_0 \to B_0$, then the lemma follows by Lemmas 47.27.2, 47.27.3, and 47.27.4. This reduces us to the situation discussed in the next paragraph.

Assume R is Noetherian and denote $\varphi : R \to A$ and $\psi : A \to B$ the given ring maps. Then $K_{A/R} \cong \varphi^!(R)$ and $K_{B/A} \cong \psi^!(A)$, see references given above. Then

$$K = K_{A/R} \otimes^{\mathbf{L}}_A K_{B/A} \cong \varphi^!(R) \otimes^{\mathbf{L}}_A \psi^!(A) \cong \psi^!(\varphi^!(R)) \cong (\psi \circ \varphi)^!(R)$$

by Lemmas 47.24.10 and 47.24.7. Thus K is a relative dualizing complex for $R \to B$. □

47.28. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
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CHAPTER 48

Duality for Schemes

0DWE

48.1. Introduction

0DWF This chapter studies relative duality for morphisms of schemes and the dualizing complex on a scheme. A reference is [**Har66**].

Dualizing complexes for Noetherian rings were defined and studied in Dualizing Complexes, Section 47.15 ff. In this chapter we continue this by studying dualizing complexes on schemes, see Section 48.2.

The bulk of this chapter is devoted to studying the right adjoint of pushforward in the setting of derived categories of sheaves of modules with quasi-coherent cohomology sheaves. See Sections 48.3, 48.4, 48.5, 48.6, 48.7, 48.8, 48.9, 48.11, 48.13, 48.14, and 48.15. Here we follow the papers [**Nee96**], [**LN07**], [**Lip09**], and [**Nee14**].

We discuss the important and useful upper shriek functors $f^!$ for separated morphisms of finite type between Noetherian schemes in Sections 48.16, 48.17, and 48.18 culminating in the overview Section 48.19.

In Section 48.20 we explain alternative theory of duality and dualizing complexes when working over a fixed locally Noetherian base endowed with a dualizing complex (this section corresponds to a remark in Hartshorne's book).

In the remaining sections we give a few applications.

This chapter is continued by the chapter on duality on algebraic spaces, see Duality for Spaces, Section 86.1.

48.2. Dualizing complexes on schemes

0A85 We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

0A86 **Lemma 48.2.1.** *Let X be a locally Noetherian scheme. Let K be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *For every affine open $U = \mathrm{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for A such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\widetilde{\ }: D(A) \to D(\mathcal{O}_U)$.*
2. *There is an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that for each i there exists a dualizing complex $\omega_i^\bullet$ for A_i such that $K|_{U_i}$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\widetilde{\ }: D(A_i) \to D(\mathcal{O}_{U_i})$.*

Proof. Assume (2) and let $U = \mathrm{Spec}(A)$ be an affine open of X. Since condition (2) implies that K is in $D_{QCoh}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated

complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 36.3.5. We will show that $\omega_A^\bullet$ is a dualizing complex for A which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens U_i, see Schemes, Lemma 26.11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 36.3.5. By Dualizing Complexes, Lemma 47.15.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over A_{f_j} for $j = 1, \ldots, m$. This implies that $\omega_A^\bullet$ is dualizing by Dualizing Complexes, Lemma 47.15.7. □

0A87 **Definition 48.2.2.** Let X be a locally Noetherian scheme. An object K of $D(\mathcal{O}_X)$ is called a *dualizing complex* if K satisfies the equivalent conditions of Lemma 48.2.1.

Please see remarks made at the beginning of this section.

0A88 **Lemma 48.2.3.** *Let A be a Noetherian ring and let $X = \mathrm{Spec}(A)$. Let K, L be objects of $D(A)$. If $K \in D_{Coh}(A)$ and L has finite injective dimension, then*

$$R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) = R\widetilde{\mathrm{Hom}_A(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. In this case, Derived Categories of Schemes, Lemma 36.10.8, tells us that the nth cohomology sheaf of $R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f)$$

Since A is Noetherian, the A_f-module $I \otimes_A A_f$ is injective (Dualizing Complexes, Lemma 47.3.8). Hence we see that

$$\begin{aligned}
\mathrm{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f) &= \mathrm{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f) \\
&= \mathrm{Hom}_{A_f}(H^{-n}(K) \otimes_A A_f, I \otimes_A A_f) \\
&= \mathrm{Hom}_A(H^{-n}(K), I) \otimes_A A_f
\end{aligned}$$

The last equality because $H^{-n}(K)$ is a finite A-module, see Algebra, Lemma 10.10.2. This proves that the canonical map

$$R\widetilde{\mathrm{Hom}_A(K, L)} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

is a quasi-isomorphism in this case and the proof is done. □

0G4I **Lemma 48.2.4.** *Let X be a Noetherian scheme. Let $K, L, M \in D_{QCoh}(\mathcal{O}_X)$. Then the map*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

of Cohomology, Lemma 20.42.9 is an isomorphism in the following two cases

(1) *$K \in D^-_{Coh}(\mathcal{O}_X)$, $L \in D^+_{Coh}(\mathcal{O}_X)$, and M affine locally has finite injective dimension (see proof), or*

(2) *K and L are in $D_{Coh}(\mathcal{O}_X)$, the object $R\mathcal{H}om(L, M)$ has finite tor dimension, and L and M affine locally have finite injective dimension (in particular L and M are bounded).*

Proof. Proof of (1). We say M has affine locally finite injective dimension if X has an open covering by affines $U = \mathrm{Spec}(A)$ such that the object of $D(A)$ corresponding to $M|_U$ (Derived Categories of Schemes, Lemma 36.3.5) has finite injective dimension[1]. To prove the lemma we may replace X by U, i.e., we may assume $X = \mathrm{Spec}(A)$ for some Noetherian ring A. Observe that $R\mathcal{H}om(K, L)$ is in $D^+_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.5. Moreover, the formation of the left and right hand side of the arrow commutes with the functor $D(A) \to D_{QCoh}(\mathcal{O}_X)$ by Lemma 48.2.3 and Derived Categories of Schemes, Lemma 36.10.8 (to be sure this uses the assumptions on K, L, M and what we just proved about $R\mathcal{H}om(K, L)$). Then finally the arrow is an isomorphism by More on Algebra, Lemmas 15.99.1 part (2).

Proof of (2). We argue as above. A small change is that here we get $R\mathcal{H}om(K, L)$ in $D_{Coh}(\mathcal{O}_X)$ because affine locally (which is allowable by Lemma 48.2.3) we may appeal to Dualizing Complexes, Lemma 47.15.2. Then we finally conclude by More on Algebra, Lemma 15.99.2. □

0A89 **Lemma 48.2.5.** *Let K be a dualizing complex on a locally Noetherian scheme X. Then K is an object of $D_{Coh}(\mathcal{O}_X)$ and $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $\mathrm{id} \to D \circ D$. If X is quasi-compact, then D exchanges $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and induces an anti-equivalence $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$.

Proof. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 48.2.1. By Lemma 48.2.3 the diagram

$$\begin{array}{ccc} D_{Coh}(A) & \longrightarrow & D_{Coh}(\mathcal{O}_U) \\ {\scriptstyle R\mathrm{Hom}_A(-,\omega_A^\bullet)}\downarrow & & \downarrow{\scriptstyle R\mathcal{H}om_{\mathcal{O}_X}(-,K|_U)} \\ D_{Coh}(A) & \longrightarrow & D(\mathcal{O}_U) \end{array}$$

commutes. We conclude that D sends $D_{Coh}(\mathcal{O}_X)$ into $D_{Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(K, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(using Cohomology, Lemma 20.42.9 for the second arrow) is an isomorphism for all L because this is true on affines by Dualizing Complexes, Lemma 47.15.3[2] and we have already seen on affines that we recover what happens in algebra. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 47.15.3. □

[1] This condition is independent of the choice of the affine open cover of the Noetherian scheme X. Details omitted.

[2] An alternative is to first show that $R\mathcal{H}om_{\mathcal{O}_X}(K, K) = \mathcal{O}_X$ by working affine locally and then use Lemma 48.2.4 part (2) to see the map is an isomorphism.

Let X be a locally ringed space. Recall that an object L of $D(\mathcal{O}_X)$ is *invertible* if it is an invertible object for the symmetric monoidal structure on $D(\mathcal{O}_X)$ given by derived tensor product. In Cohomology, Lemma 20.52.2 we have seen this means L is perfect and there is an open covering $X = \bigcup U_i$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i. In this case, the function

$$x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^{n_x}(L_x) \neq 0$$

is locally constant on X. In particular, we have $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing L.

0ATP **Lemma 48.2.6.** *Let X be a locally Noetherian scheme. If K and K' are dualizing complexes on X, then K' is isomorphic to $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for some invertible object L of $D(\mathcal{O}_X)$.*

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true, see Dualizing Complexes, Lemma 47.15.5 and its proof. The evaluation map $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K'$ is an isomorphism for the same reason. □

0AWF **Lemma 48.2.7.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X. Then X is universally catenary and the function $X \to \mathbf{Z}$ defined by*

$$x \longmapsto \delta(x) \text{ such that } \omega^\bullet_{X,x}[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,x}$$

is a dimension function.

Proof. Immediate from the affine case Dualizing Complexes, Lemma 47.17.3 and the definitions. □

0ECM **Lemma 48.2.8.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X with associated dimension function δ. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $\mathcal{E}^i = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)$. Then $\mathcal{E}^i$ is a coherent $\mathcal{O}_X$-module and for $x \in X$ we have*

1. *$\mathcal{E}^i_x$ is nonzero only for $\delta(x) \leq i \leq \delta(x) + \dim(\mathrm{Supp}(\mathcal{F}_x))$,*
2. *$\dim(\mathrm{Supp}(\mathcal{E}_x^{i+\delta(x)})) \leq i$,*
3. *$\mathrm{depth}(\mathcal{F}_x)$ is the smallest integer $i \geq 0$ such that $\mathcal{E}_x^{i+\delta(x)} \neq 0$, and*
4. *we have $x \in \mathrm{Supp}(\bigoplus_{j \leq i} \mathcal{E}^j) \Leftrightarrow \mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \delta(x) \leq i$.*

Proof. Lemma 48.2.5 tells us that $\mathcal{E}^i$ is coherent. Choosing an affine neighbourhood of x and using Derived Categories of Schemes, Lemma 36.10.8 and More on Algebra, Lemma 15.100.2 part (3) we have

$$\mathcal{E}^i_x = \mathcal{E}xt^{-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)_x = \mathrm{Ext}^{-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega^\bullet_{X,x}) = \mathrm{Ext}^{\delta(x)-i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \omega^\bullet_{X,x}[-\delta(x)])$$

By construction of δ in Lemma 48.2.7 this reduces parts (1), (2), and (3) to Dualizing Complexes, Lemma 47.16.5. Part (4) is a formal consequence of (3) and (1). □

48.3. Right adjoint of pushforward

0A9D References for this section and the following are [**Nee96**], [**LN07**], [**Lip09**], and [**Nee14**].

Let $f : X \to Y$ be a morphism of schemes. In this section we consider the right adjoint to the functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. In the literature, if this functor exists, then it is sometimes denoted $f^\times$. This notation is not universally accepted and we refrain from using it. We will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms f) with the notation in [**Har66**].

0A9E **Lemma 48.3.1.** *Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ has a right adjoint.*

This is almost the same as [**Nee96**, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 13.38.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma 36.3.1. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem 36.15.3. Since X and Y are quasi-compact and quasi-separated, so is f, see Schemes, Lemmas 26.21.13 and 26.21.14. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Schemes, Lemma 36.4.5. This finishes the proof. □

0A9F **Example 48.3.2.** Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ corresponds to restriction $D(B) \to D(A)$ via the equivalences $D(B) \to D_{QCoh}(\mathcal{O}_X)$ and $D(A) \to D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \longmapsto R\mathrm{Hom}(B, K)$ of Dualizing Complexes, Section 47.13.

0A9G **Example 48.3.3.** If $f : X \to Y$ is a separated finite type morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{Coh}(\mathcal{O}_Y)$ into $D_{Coh}(\mathcal{O}_X)$. Namely, let k be a field and consider the morphism $f : \mathbf{A}^1_k \to \mathrm{Spec}(k)$. By Example 48.3.2 this corresponds to the question of whether $R\mathrm{Hom}(B, -)$ maps $D_{Coh}(A)$ into $D_{Coh}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\mathrm{Hom}(k[x], k) = \left(\prod\nolimits_{n \geq 0} k\right)[0]$$

which is not a finite $k[x]$-module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

0A9H **Example 48.3.4.** If $f : X \to Y$ is a proper or even finite morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ does not map $D^-_{QCoh}(\mathcal{O}_Y)$ into $D^-_{QCoh}(\mathcal{O}_X)$. Namely, let k be a field, let $k[\epsilon]$ be the dual numbers over k, let $X = \mathrm{Spec}(k)$, and let $Y = \mathrm{Spec}(k[\epsilon])$. Then $\mathrm{Ext}^i_{k[\epsilon]}(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example 48.3.2.

0A9I **Lemma 48.3.5.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_* of Lemma 48.3.1. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.*

Proof. By Derived Categories of Schemes, Lemma 36.4.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that

$H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. □

Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0B6H (48.3.5.1) $$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology, Remark 20.42.11 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

0A9Q **Lemma 48.3.6.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$. Then the map (48.3.5.1)*

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

becomes an isomorphism after applying the functor $DQ_Y : D(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Y)$ discussed in Derived Categories of Schemes, Section 36.21.

Proof. The statement makes sense as DQ_Y exists by Derived Categories of Schemes, Lemma 36.21.1. Since DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$ to prove the lemma we have to show that for any $M \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.3.5.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

To see this we use the following string of equalities

$$\begin{aligned}
\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\
&= \mathrm{Hom}_X(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, a(K)) \\
&= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L), K) \\
&= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L, K) \\
&= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))
\end{aligned}$$

The first equality holds by Cohomology, Lemma 20.28.1. The second equality by Cohomology, Lemma 20.42.2. The third equality by construction of a. The fourth equality by Derived Categories of Schemes, Lemma 36.22.1 (this is the important step). The fifth by Cohomology, Lemma 20.42.2. □

0GEU **Example 48.3.7.** The statement of Lemma 48.3.6 is not true without applying the "coherator" DQ_Y. Indeed, suppose $Y = \mathrm{Spec}(R)$ and $X = \mathbf{A}^1_R$. Take $L = \mathcal{O}_X$ and $K = \mathcal{O}_Y$. The left hand side of the arrow is in $D_{QCoh}(\mathcal{O}_Y)$ but the right hand side of the arrow is isomorphic to $\prod_{n\geq 0}\mathcal{O}_Y$ which is not quasi-coherent.

0GEV **Remark 48.3.8.** In the situation of Lemma 48.3.6 we have

$$DQ_Y(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = Rf_*DQ_X(R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

by Derived Categories of Schemes, Lemma 36.21.2. Thus if $R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \in D_{QCoh}(\mathcal{O}_X)$, then we can "erase" the DQ_Y on the left hand side of the arrow. On the other hand, if we know that $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \in D_{QCoh}(\mathcal{O}_Y)$, then we can "erase" the DQ_Y from the right hand side of the arrow. If both are true then we see that (48.3.5.1) is an isomorphism. Combining this with Derived Categories of Schemes, Lemma 36.10.8 we see that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism if

(1) L and Rf_*L are perfect, or
(2) K is bounded below and L and Rf_*L are pseudo-coherent.

For (2) we use that $a(K)$ is bounded below if K is bounded below, see Lemma 48.3.5.

0GEW **Example 48.3.9.** Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $L \in D^-_{Coh}(X)$ and $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Then the map $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism. Namely, the complexes L and Rf_*L are pseudo-coherent by Derived Categories of Schemes, Lemmas 36.10.3 and 36.11.3 and the discussion in Remark 48.3.8 applies.

0B6I **Lemma 48.3.10.** *Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact schemes. For all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ (48.3.5.1) induces an isomorphism $R\mathrm{Hom}_X(L, a(K)) \to R\mathrm{Hom}_Y(Rf_*L, K)$ of global derived homs.*

Proof. By the construction in Cohomology, Section 20.44 we have

$$R\mathrm{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\mathrm{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

Thus the lemma is a consequence of Lemma 48.3.6. Namely, a map $E \to E'$ in $D(\mathcal{O}_Y)$ which induces an isomorphism $DQ_Y(E) \to DQ_Y(E')$ induces a quasi-isomorphism $R\Gamma(Y, E) \to R\Gamma(Y, E')$. Indeed we have $H^i(Y, E) = \mathrm{Ext}^i_Y(\mathcal{O}_Y, E) = \mathrm{Hom}(\mathcal{O}_Y[-i], E) = \mathrm{Hom}(\mathcal{O}_Y[-i], DQ_Y(E))$ because $\mathcal{O}_Y[-i]$ is in $D_{QCoh}(\mathcal{O}_Y)$ and DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$. □

48.4. Right adjoint of pushforward and restriction to opens

0E4G In this section we study the question to what extent the right adjoint of pushforward commutes with restriction to open subschemes. This is a base change question, so let's first discuss this more generally.

We often want to know whether the right adjoints to pushforward commutes with base change. Thus we consider a cartesian square

0A9J (48.4.0.1)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact and quasi-separated schemes. Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 48.3.1). Consider the base change map of Cohomology, Remark 20.28.3. It induces a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0A9K **Lemma 48.4.1.** *In diagram (48.4.0.1) assume that g is flat or more generally that f and g are Tor independent. Then $a \circ Rg_* \leftarrow Rg'_* \circ a'$ is an isomorphism.*

Proof. In this case the base change map $Lg^* \circ Rf_*K \longrightarrow Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.22.5. Thus the corresponding transformation between adjoint functors is an isomorphism as well. □

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$\begin{array}{ccc} U & \xrightarrow{j'} & X \\ {\scriptstyle f|_U}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & Y \end{array}$$

as in (48.4.0.1). By Lemma 48.4.1 the map $\xi : a \circ Rj_* \leftarrow Rj'_* \circ a'$ is an isomorphism where a and a' are the right adjoints to Rf_* and $R(f|_U)_*$. We obtain a transformation of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_U)$

0A9L (48.4.1.1) $$(j')^* \circ a \to (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \to a' \circ j^*$$

where the first arrow comes from $\mathrm{id} \to Rj_* \circ j^*$ and the final arrow from the isomorphism $(j')^* \circ Rj'_* \to \mathrm{id}$. In particular, we see that (48.4.1.1) is an isomorphism when evaluated on K if and only if $a(K)|_U \to a(Rj_*(K|_V))|_U$ is an isomorphism.

0A9M **Example 48.4.2.** There is a finite morphism $f : X \to Y$ of Noetherian schemes such that (48.4.1.1) is not an isomorphism when evaluated on some $K \in D_{Coh}(\mathcal{O}_Y)$. Namely, let $X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$ with $A = k[x, \epsilon]$ where k is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbf{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\mathrm{Ext}^i_A(B, M_n) = M_n, \quad i \geq 0$$

because B has the free periodic resolution $\ldots \to A \to A \to A$ with maps given by multiplication by ϵ. Consider the object $K = \bigoplus M_n[n] = \prod M_n[n]$ of $D_{Coh}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 13.33.5 and 13.34.2). Then we see that $a(K)$ corresponds to $R\,\mathrm{Hom}(B, K)$ by Example 48.3.2 and

$$H^0(R\,\mathrm{Hom}(B, K)) = \mathrm{Ext}^0_A(B, K) = \prod\nolimits_{n\geq 1} \mathrm{Ext}^n_A(B.M_n) = \prod\nolimits_{n\geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of x, whereas the complex K does have every element of its cohomology annihilated by a power of x. In other words, for the map (48.4.1.1) with $V = D(x)$ and $U = D(x)$ and the complex K cannot be an isomorphism because $(j')^*(a(K))$ is nonzero and $a'(j^*K)$ is zero.

0A9N **Lemma 48.4.3.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$.*

(1) *For every $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (48.4.1.1) is an isomorphism on all K in $D^+_{QCoh}(\mathcal{O}_Y)$.*
(2) *For every $Q \in D_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (48.4.1.1) is an isomorphism on all K in $D_{QCoh}(\mathcal{O}_Y)$.*
(3) *If a commutes with direct sums, then the equivalent conditions of (1) imply the equivalent conditions of (2).*

Proof. Proof of (1). Let $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Choose a distinguished triangle

$$K \to Rj_*K|_V \to Q \to K[1]$$

Observe that Q is in $D^+_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 36.4.1) and is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 36.6.1). Applying a we obtain a distinguished triangle

$$a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]$$

on X. If $a(Q)$ is supported on $X \setminus U$, then restricting to U the map $a(K)|_U \to a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (48.4.1.1) is an isomorphism on K. The converse is immediate.

The proof of (2) is exactly the same as the proof of (1).

Proof of (3). Assume the equivalent conditions of (1) hold. Set $T = Y \setminus V$. We will use the notation $D_{QCoh,T}(\mathcal{O}_Y)$ and $D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)$ to denote complexes whose cohomology sheaves are supported on T and $f^{-1}(T)$. Since a commutes with direct sums, the strictly full, saturated, triangulated subcategory $\mathcal{D}$ with objects

$$\{Q \in D_{QCoh,T}(\mathcal{O}_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)\}$$

is preserved by direct sums and hence derived colimits. On the other hand, the category $D_{QCoh,T}(\mathcal{O}_Y)$ is generated by a perfect object E (see Derived Categories of Schemes, Lemma 36.15.4). By assumption we see that $E \in \mathcal{D}$. By Derived Categories, Lemma 13.37.3 every object Q of $D_{QCoh,T}(\mathcal{O}_Y)$ is a derived colimit of a system $Q_1 \to Q_2 \to Q_3 \to \ldots$ such that the cones of the transition maps are direct sums of shifts of E. Arguing by induction we see that $Q_n \in \mathcal{D}$ for all n and finally that Q is in $\mathcal{D}$. Thus the equivalent conditions of (2) hold. □

0A9P **Lemma 48.4.4.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper morphism. If*[3]

(1) *f is flat and of finite presentation, or*
(2) *Y is Noetherian*

then the equivalent conditions of Lemma 48.4.3 part (1) hold for all quasi-compact opens V of Y.

[3]This proof works for those morphisms of quasi-compact and quasi-separated schemes such that Rf_*P is pseudo-coherent for all P perfect on X. It follows easily from a theorem of Kiehl [**Kie72**] that this holds if f is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this chapter.

Proof. Let $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex P_U on U and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 36.15.3). Then using Derived Categories of Schemes, Lemma 36.13.11 we may assume there is a perfect complex P on X and a map $P \to a(Q)$ whose restriction to U is nonzero. By definition of a this map is adjoint to a map $Rf_*P \to Q$.

The complex Rf_*P is pseudo-coherent. In case (1) this follows from Derived Categories of Schemes, Lemma 36.30.5. In case (2) this follows from Derived Categories of Schemes, Lemmas 36.11.3 and 36.10.3. Thus we may apply Derived Categories of Schemes, Lemma 36.17.5 and get a map $I \to \mathcal{O}_Y$ of perfect complexes whose restriction to V is an isomorphism such that the composition $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P \to Rf_*P \to Q$ is zero. By Derived Categories of Schemes, Lemma 36.22.1 we have $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P = Rf_*(Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)$. We conclude that the composition

$$Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P \to P \to a(Q)$$

is zero. However, the restriction to U is the map $P|_U \to a(Q)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. □

48.5. Right adjoint of pushforward and base change, I

0AA5 The map (48.4.1.1) is a special case of a base change map. Namely, suppose that we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact and quasi-separated schemes, i.e., a diagram as in (48.4.0.1). Assume f and g are **Tor independent**. Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0AA6 (48.5.0.1) $$L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$$

The first arrow comes from the adjunction map $\mathrm{id} \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g')^*Rg'_* \to \mathrm{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 48.4.1. Alternatively, we can think of (48.5.0.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned} \mathrm{Hom}_X(M, a(K)) &= \mathrm{Hom}_Y(Rf_*M, K) \\ &\to \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\ &\leftarrow \mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\ &= \mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\ &= \mathrm{Hom}_X(M, Rg'_*a'(Lg^*K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma 36.22.5) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks 20.28.4 and 20.28.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0ATQ **Lemma 48.5.1.** *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{l} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{m} & Z \end{array}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (48.5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_*. Let a, b, and c be the right adjoints of Lemma 48.3.1 for f, g, and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \to k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \to a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \to a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $l^* \circ f_* \to f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \to l_* \circ l^*$ and $k^* \circ k_* \to 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \to l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \to b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \to k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \to c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

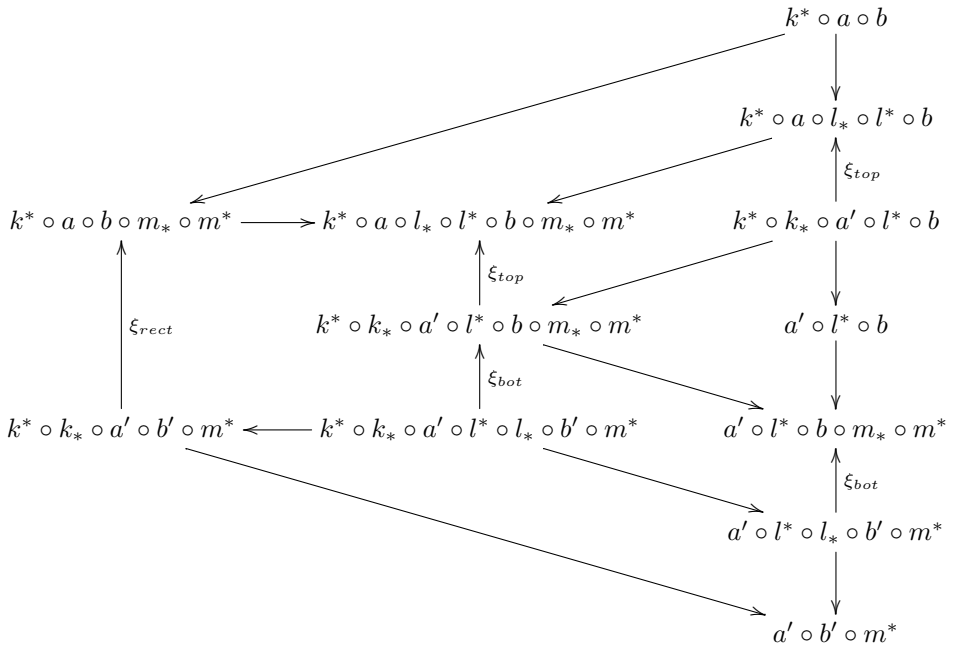

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show the diagram

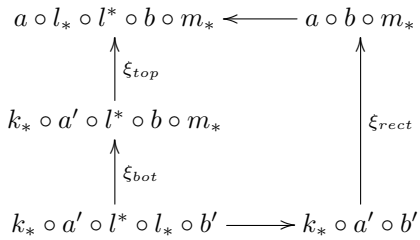

becomes commutative if we invert the arrows ξ_{top}, ξ_{bot}, and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

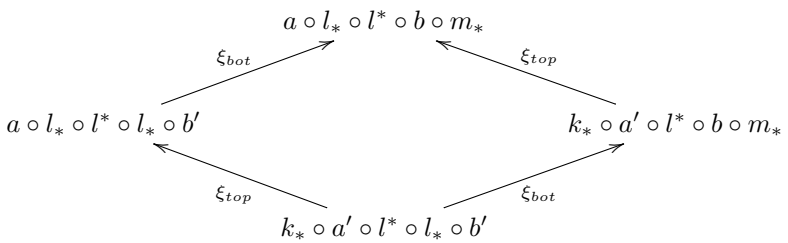

commutes by Categories, Lemma 4.28.2. Since the diagrams

$$\begin{array}{ccc} a \circ l_* \circ l^* \circ b \circ m_* & \longleftarrow & a \circ b \circ m \\ \uparrow & & \uparrow \\ a \circ l_* \circ l^* \circ l_* \circ b' & \longleftarrow & a \circ l_* \circ b' \end{array} \quad \text{and} \quad \begin{array}{ccc} a \circ l_* \circ l^* \circ l_* \circ b' & \longrightarrow & a \circ l_* \circ b' \\ \uparrow & & \uparrow \\ k_* \circ a' \circ l^* \circ l_* \circ b' & \longrightarrow & k_* \circ a' \circ b' \end{array}$$

commute (see references cited) and since the composition of $l_* \to l_* \circ l^* \circ l_* \to l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology, Remark 20.28.4 and the proof is complete. □

0ATR **Lemma 48.5.2.** *Consider a commutative diagram*

$$\begin{array}{ccccc} X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h'} & Y' & \xrightarrow{h} & Y \end{array}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (48.5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_*. Let a, a', and a'' be the right adjoints of Lemma 48.3.1 for f, f', and f''. The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \to g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \to a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \to a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $h^* \circ f_* \to f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \to h_* \circ h^*$ and $g^* \circ g_* \to 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \to a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \to k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \to a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

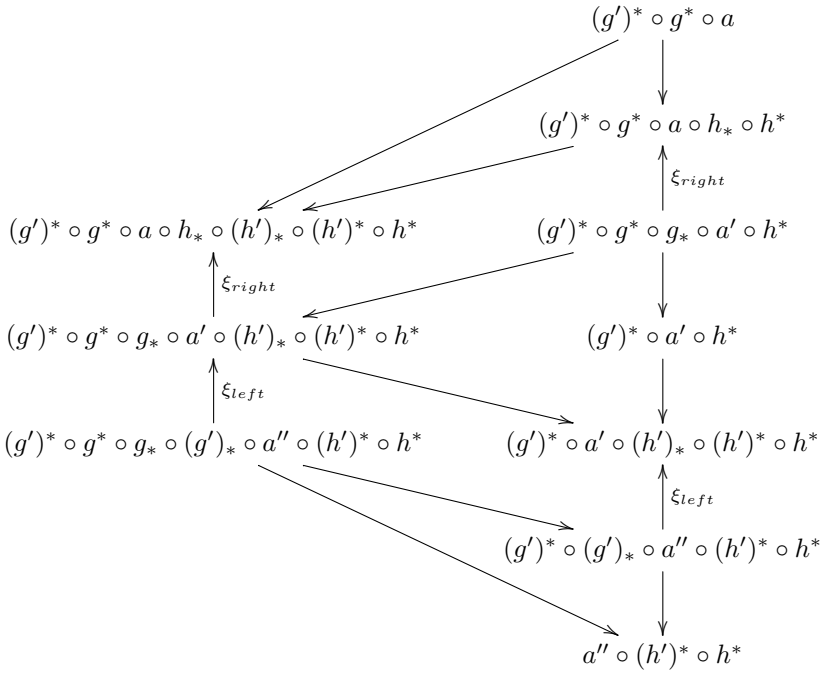

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect}. This is the statement dual to Cohomology, Remark 20.28.5 and the proof is complete. □

0ATS **Remark 48.5.3.** Consider a commutative diagram

$$\begin{array}{ccccc} X'' & \xrightarrow[k']{} & X' & \xrightarrow[k]{} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\ {\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z \end{array}$$

of quasi-compact and quasi-separated schemes where all squares are cartesian and where (f, l), (g, m), (f', l'), (g', m') are Tor independent pairs of maps. Let a, a', a'', b, b', b'' be the right adjoints of Lemma 48.3.1 for f, f', f'', g, g', g''. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{array}{cc} A & B \\ C & D \end{array}$$

Then the maps (48.5.0.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\gamma_A : (k')^* \circ a' \to a'' \circ (l')^* \quad \gamma_B : k^* \circ a \to a' \circ l^*$$
$$\gamma_C : (l')^* \circ b' \to b'' \circ (m')^* \quad \gamma_D : l^* \circ b \to b' \circ m^*$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\gamma_{A+B} : (k \circ k')^* \circ a \to a'' \circ (l \circ l')^*$$
$$\gamma_{C+D} : (l \circ l')^* \circ b \to b'' \circ (m \circ m')^*$$
$$\gamma_{A+C} : (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^*$$
$$\gamma_{B+D} : k^* \circ (a \circ b) \to (a' \circ b') \circ m^*$$

By Lemma 48.5.2 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 48.5.1 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \mathrm{id}_{l^*}) \circ (\mathrm{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 4.28 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 48.5.1 and 48.5.2 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{array}{ccc} (k')^* \circ a' \circ l^* \circ b & \xrightarrow[\gamma_D]{} & (k')^* \circ a' \circ b' \circ m^* \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_A} \\ a'' \circ (l')^* \circ l^* \circ b & \xrightarrow{\gamma_D} & a'' \circ (l')^* \circ b' \circ m^* \end{array}$$

commutes. This is true by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

48.6. Right adjoint of pushforward and base change, II

0BZF In this section we prove that the base change map of Section 48.5 is an isomorphism in some cases. We first observe that it suffices to check over affine opens, provided formation of the right adjoint of pushforward commutes with restriction to opens.

0E9S **Remark 48.6.1.** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow[g']{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact and quasi-separated schemes with (g, f) Tor independent. Let $V \subset Y$ and $V' \subset Y'$ be affine opens with $g(V') \subset V$. Form the cartesian diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & Y' \end{array}$$

Assume (48.4.1.1) with respect to K and the first diagram and (48.4.1.1) with respect to Lg^*K and the second diagram are isomorphisms. Then the restriction of the base change map (48.5.0.1)

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

to U' is isomorphic to the base change map (48.5.0.1) for $K|_V$ and the cartesian diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

This follows from the fact that (48.4.1.1) is a special case of the base change map (48.5.0.1) and that the base change maps compose correctly if we stack squares horizontally, see Lemma 48.5.2. Thus in order to check the base change map restricted to U' is an isomorphism it suffices to work with the last diagram.

0AA8 **Lemma 48.6.2.** *In diagram (48.4.0.1) assume*

(1) *$g : Y' \to Y$ is a morphism of affine schemes,*
(2) *$f : X \to Y$ is proper, and*
(3) *f and g are Tor independent.*

Then the base change map (48.5.0.1) induces an isomorphism

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

in the following cases

(1) *for all $K \in D_{QCoh}(\mathcal{O}_Y)$ if f is flat of finite presentation,*
(2) *for all $K \in D_{QCoh}(\mathcal{O}_Y)$ if f is perfect and Y Noetherian,*
(3) *for $K \in D^+_{QCoh}(\mathcal{O}_Y)$ if g has finite Tor dimension and Y Noetherian.*

Proof. Write $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 36.15.3. Then $L(g')^*M$ is a generator for $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Remark 36.16.4. Hence it suffices to show that (48.5.0.1) induces an isomorphism

0E45 (48.6.2.1) $$R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology, Section 20.44, as this will imply the cone of $L(g')^*a(K) \to a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (48.6.2.1).

The left hand side. Because M is perfect, the canonical map

$$R\mathrm{Hom}_X(M, a(K)) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Schemes, Lemma 36.22.6. We can combine this with the isomorphism $R\mathrm{Hom}_Y(Rf_*M, K) = R\mathrm{Hom}_X(M, a(K))$ of Lemma 48.3.10 to get that the left hand side equals $R\mathrm{Hom}_Y(Rf_*M, K) \otimes^{\mathbf{L}}_A A'$.

The right hand side. Here we first use the isomorphism

$$R\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 48.3.10. Then we use the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Schemes, Lemma 36.22.5. Hence we may rewrite this as $R\mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 36.4.1) we have a canonical map

$$\beta : R\mathrm{Hom}_Y(Rf_*M, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (15.100.1.1) where we have used Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.8 to translate back and forth into algebra.

(1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Schemes, Lemma 36.30.4 and β is an isomorphism by More on Algebra, Lemma 15.100.2 part (1).
(2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms, Lemma 37.61.13 and β is an isomorphism as before.
(3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Schemes, Lemmas 36.11.3 and 36.10.3) and β is an isomorphism by More on Algebra, Lemma 15.100.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (48.6.2.1) given in the second and third paragraph are in fact given by (48.6.2.1). To make our formulas manageable we will use $(-,-)_X = R\mathrm{Hom}_X(-,-)$, use $-\otimes A'$ in stead of $-\otimes^{\mathbf{L}}_A A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

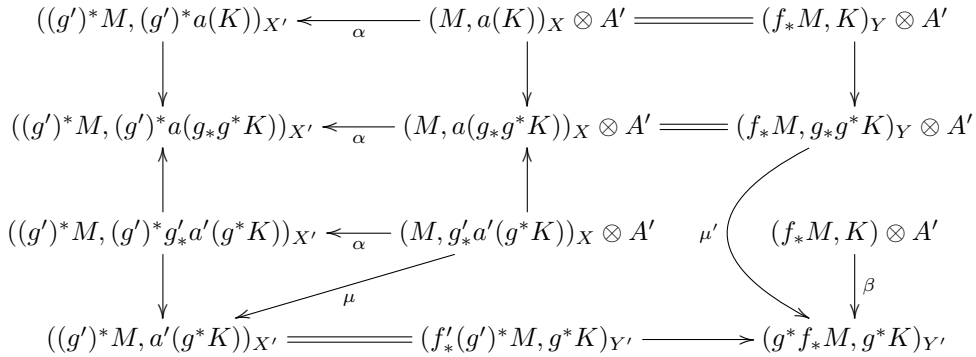

The arrows labeled α are the maps from Derived Categories of Schemes, Lemma 36.22.6 for the diagram with corners X', X, Y', Y. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \to a \circ g_*$ of Lemma 48.4.1 and therefore the middle square is commutative. Going down the left hand

side is (48.6.2.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \to E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \to E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_* a(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Schemes, Remark 36.22.7.

Observe that

$$\begin{array}{ccccc} (M, a(g_*g^*K))_X & = & (f_*M, g_*g^*K)_Y & = & (g^*f_*M, g^*K)_{Y'} \\ \uparrow & & & & \uparrow \\ (M, g'_*a'(g^*K))_X & = & ((g')^*M, a'(g^*K))_{X'} & = & (f'_*(g')^*M, g^*K)_{Y'} \end{array}$$

is commutative by the very definition of the transformation $g'_* \circ a' \to a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \to (f_*M, g_*g^*K)_X \otimes A'$ and μ'. To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \to (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{array}{ccc} R\operatorname{Hom}_A(E, K) & \xrightarrow[\text{induced by } \beta]{} & R\operatorname{Hom}_{A'}(E \otimes^{\mathbf{L}}_A A', K \otimes^{\mathbf{L}}_A A') \\ & \searrow \quad \nearrow & \\ & R\operatorname{Hom}_A(E, K \otimes^{\mathbf{L}}_A A') & \end{array}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 15.100 the proof is complete. □

48.7. Right adjoint of pushforward and trace maps

0AWG Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 48.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\operatorname{Tr}_f : Rf_* \circ a \longrightarrow \mathrm{id}$$

The corresponding map $\operatorname{Tr}_{f,K} : Rf_*a(K) \longrightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the *trace map*. This is the map which has the property that the bijection

$$\operatorname{Hom}_X(L, a(K)) \longrightarrow \operatorname{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \operatorname{Tr}_{f,K} \circ Rf_*\varphi$$

The map (48.3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

comes about by composition with $\operatorname{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0B6J **Lemma 48.7.1** (Trace map and base change)**.** *Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Then the maps $1 \star Tr_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $Tr_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 20.28.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (48.5.0.1). More precisely, the diagram*

$$\begin{array}{ccc} Lg^* \circ Rf_* \circ a & \xrightarrow{1 \star Tr_f} & Lg^* \\ \downarrow{\scriptstyle \beta \star 1} & & \uparrow{\scriptstyle Tr_{f'} \star 1} \\ Rf'_* \circ L(g')^* \circ a & \xrightarrow{1 \star \alpha} & Rf'_* \circ a' \circ Lg^* \end{array}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 48.4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^*$$

where the first arrow is the unit for (g^*, g_*), the second arrow is Tr_f, and the third arrow is the counit for (g^*, g_*). This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \circ g^* \to g^*$ of unit and counit is the identity. Consider the diagram

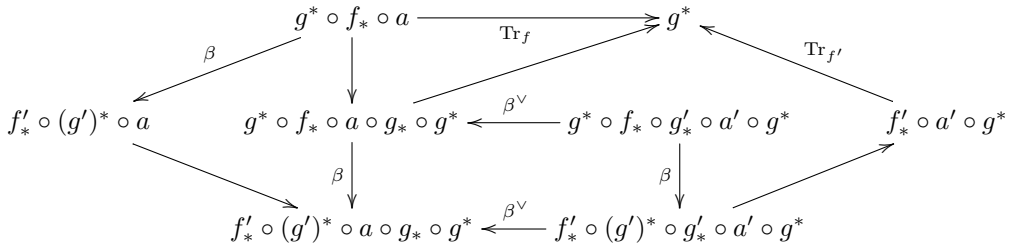

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By Categories, Lemma 4.24.8 the square

$$\begin{array}{ccc} g^* \circ f_* \circ g'_* \circ a' & \xrightarrow{\beta} & f'_* \circ (g')^* \circ g'_* \circ a' \\ \downarrow{\scriptstyle \beta^\vee} & & \downarrow \\ g^* \circ f_* \circ a \circ g_* & \longrightarrow & \mathrm{id} \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\mathrm{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. □

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 48.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\eta_f : \mathrm{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

0B6K **Lemma 48.7.2.** *Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Then the maps $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ and $\eta_{f'} \star 1 : L(g')^* \to a' \circ Rf'_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology, Remark 20.28.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (48.5.0.1). More precisely, the diagram*

$$\begin{array}{ccc} L(g')^* & \xrightarrow{1\star\eta_f} & L(g')^* \circ a \circ Rf_* \\ {\scriptstyle \eta_{f'}\star 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ a' \circ Rf'_* \circ L(g')^* & \xleftarrow{\beta} & a' \circ Lg^* \circ Rf_* \end{array}$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 48.7.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 48.4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

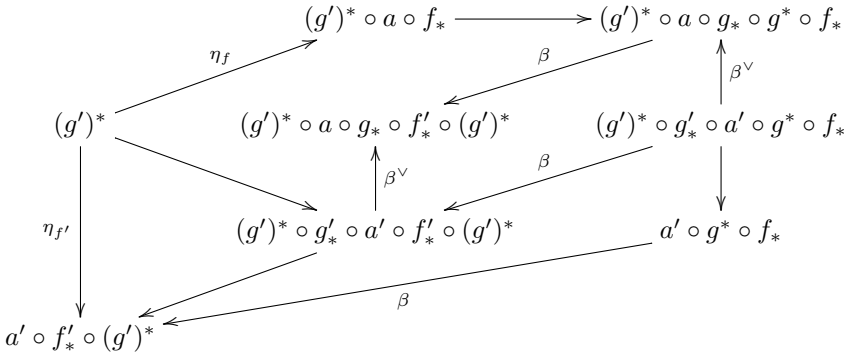

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 4.24.8 the square

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & g'_* \circ a' \circ g^* \circ f_* \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ g'_* \circ a' \circ g^* \circ f_* & \xrightarrow{\beta^\vee} & a \circ g_* \circ f'_* \circ (g')^* \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. □

0B6L **Example 48.7.3.** Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. As seen in Example 48.3.2 the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ sends an object K of $D(A) = D_{QCoh}(\mathcal{O}_Y)$ to $R\mathrm{Hom}(B, K)$ in $D(B) = D_{QCoh}(\mathcal{O}_X)$. The trace map is the map

$$\mathrm{Tr}_{f,K} : R\mathrm{Hom}(B, K) \longrightarrow R\mathrm{Hom}(A, K) = K$$

induced by the A-module map $A \to B$.

48.8. Right adjoint of pushforward and pullback

0B6N Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint of pushforward as in Lemma 48.3.1. For $K, L \in D_{QCoh}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \longrightarrow a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(a(L)) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L$$

(equality by Derived Categories of Schemes, Lemma 36.22.1) for which we use the trace map $Rf_*a(L) \to L$. When $L = \mathcal{O}_Y$ we obtain a map

0A9S (48.8.0.1) $$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0A9T **Lemma 48.8.1.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. The map $Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \to a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (48.8.0.1) is an isomorphism if K is perfect.*

Proof. Let $K^\vee$ be the "dual" to K, see Cohomology, Lemma 20.50.5. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\vee, L) \\
&= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\vee, a(L)) \\
&= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L))
\end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology, Lemma 20.54.3) or the more general Derived Categories of Schemes, Lemma 36.22.1. Hence the result by the Yoneda lemma. □

0B6P **Lemma 48.8.2.** *Suppose we have a diagram (48.4.0.1) where f and g are tor independent. Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram*

$$\begin{array}{ccc}
L(g')^*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) & \longrightarrow & L(g')^*a(K) \\
\downarrow & & \downarrow \\
L(f')^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} a'(\mathcal{O}_{Y'}) & \longrightarrow & a'(Lg^*K)
\end{array}$$

commutes where the horizontal arrows are the maps (48.8.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology, Remark 20.28.3 and (48.5.0.1).

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. Let us write (48.8.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\to a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\to a(K \otimes \mathcal{O}_Y) \\ &\to a(K) \end{aligned}$$

Here the first arrow is the unit η_f, the second arrow is a applied to Cohomology, Equation (20.54.2.1) which is an isomorphism by Derived Categories of Schemes, Lemma 36.22.1, the third arrow is a applied to $\mathrm{id}_K \otimes \mathrm{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 48.7.2 and 48.7.1. For the arrow using Cohomology, Equation (20.54.2.1) this is Cohomology, Remark 20.54.5. For the multiplication map it is clear. This finishes the proof. □

0B6Q **Lemma 48.8.3.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open such that $f^{-1}(V) \to V$ is an isomorphism. Then for $K \in D^+_{QCoh}(\mathcal{O}_Y)$ the map (48.8.0.1) restricts to an isomorphism over $f^{-1}(V)$.*

Proof. By Lemma 48.4.4 the map (48.4.1.1) is an isomorphism for objects of $D^+_{QCoh}(\mathcal{O}_Y)$. Hence Lemma 48.8.2 tells us the restriction of (48.8.0.1) for K to $f^{-1}(V)$ is the map (48.8.0.1) for $K|_V$ and $f^{-1}(V) \to V$. Thus it suffices to show that the map is an isomorphism when f is the identity morphism. This is clear. □

0B6R **Lemma 48.8.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of quasi-compact and quasi-separated schemes and set $h = g \circ f$. Let a, b, c be the adjoints of Lemma 48.3.1 for f, g, h. For any $K \in D_{QCoh}(\mathcal{O}_Z)$ the diagram*

$$\begin{array}{ccccc} Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) & \longrightarrow & a(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} b(\mathcal{O}_Z)) & \longrightarrow & a(b(K)) \\ \| & & & & \| \\ Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*b(\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) & \longrightarrow & Lh^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} c(\mathcal{O}_Z) & \longrightarrow & c(K) \end{array}$$

is commutative where the arrows are (48.8.0.1) and we have used $Lh^* = Lf^* \circ Lg^*$ and $c = a \circ b$.

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. The composition of the top arrows is adjoint to a map

$$g_*f_*(f^*(g^*K \otimes b(\mathcal{O}_Z)) \otimes a(\mathcal{O}_Y)) \to K$$

The left hand side is equal to $K \otimes g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))$ by Derived Categories of Schemes, Lemma 36.22.1 and inspection of the definitions shows the map comes from the map

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{g_*\epsilon} g_*(b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y)) \xrightarrow{g_*\alpha} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

tensored with id_K. Here ϵ is the isomorphism from Derived Categories of Schemes, Lemma 36.22.1 and β comes from the counit map $g_*b \to \mathrm{id}$. Similarly, the composition of the lower horizontal arrows is adjoint to id_K tensored with the composition

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xrightarrow{g_*f_*\delta} g_*f_*(ab(\mathcal{O}_Z)) \xrightarrow{g_*\gamma} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

where γ comes from the counit map $f_*a \to \mathrm{id}$ and δ is the map whose adjoint is the composition

$$f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{\epsilon} b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y) \xrightarrow{\alpha} b(\mathcal{O}_Z)$$

By general properties of adjoint functors, adjoint maps, and counits (see Categories, Section 4.24) we have $\gamma \circ f_*\delta = \alpha \circ \epsilon^{-1}$ as desired. □

48.9. Right adjoint of pushforward for closed immersions

0A74 Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that i is a homeomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

a sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 17.13.4 there is a sheaf of $\mathcal{O}_Z$-modules, which we will denote $\mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$, such that

$$i_*\,\mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

as $\mathcal{O}_X$-modules. We spell out what this means.

0A75 **Lemma 48.9.1.** *With notation as above. The functor $\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : Mod(\mathcal{O}_Z) \to Mod(\mathcal{O}_X)$. For $V \subset Z$ open we have*

$$\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0\}$$

where $U \subset X$ is an open whose intersection with Z is V.

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$-modules. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{i_*\mathcal{O}_Z}(i_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))$$

The first equality by Modules, Lemma 17.22.3 and the second by the fully faithfulness of i_*, see Modules, Lemma 17.13.4. The description of sections is left to the reader. □

The functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Z), \quad \mathcal{F} \longmapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$$

is left exact and has a derived extension

$$R\mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z).$$

0A76 **Lemma 48.9.2.** *With notation as above. The functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.*

Proof. This is a consequence of the fact that i_* and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 48.9.1. See Derived Categories, Lemma 13.30.3. □

0A77 **Lemma 48.9.3.** *With notation as above. We have*

$$Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$$

in $D(\mathcal{O}_X)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$. □

0E2I **Lemma 48.9.4.** *With notation as above. For $M \in D(\mathcal{O}_Z)$ we have*

$$R\mathcal{H}om_{\mathcal{O}_X}(Ri_*M, K) = Ri_*R\mathcal{H}om_{\mathcal{O}_Z}(M, R\mathcal{H}om(\mathcal{O}_Z, K))$$

in $D(\mathcal{O}_Z)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ and the fact that if $\mathcal{K}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}om(\mathcal{O}_Z, \mathcal{K}^\bullet)$ is a K-injective complex of $\mathcal{O}_Z$-modules, see Derived Categories, Lemma 13.31.9. □

0A78 **Lemma 48.9.5.** *Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes (any closed immersion if X is locally Noetherian). Then*

(1) *$R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$, and*
(2) *if $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, then the diagram*

$$\begin{array}{ccc} D^+(B) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_Z) \\ {\scriptstyle R\mathrm{Hom}(B,-)}\uparrow & & \uparrow{\scriptstyle R\mathcal{H}om(\mathcal{O}_Z,-)} \\ D^+(A) & \longrightarrow & D^+_{QCoh}(\mathcal{O}_X) \end{array}$$

is commutative.

Proof. To explain the parenthetical remark, if X is locally Noetherian, then i is pseudo-coherent by More on Morphisms, Lemma 37.60.9.

Let K be an object of $D^+_{QCoh}(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 29.4.1 it suffices to show that i_* applied to $H^n(R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on X. By Lemma 48.9.3 this means we have to show that $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$ is in $D_{QCoh}(\mathcal{O}_X)$. Since i is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 36.10.8.

Assume $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective A-modules representing an object K of $D^+(A)$. Then we know that $R\mathrm{Hom}(B, K) = \mathrm{Hom}_A(B, I^\bullet)$ viewed as a complex of B-modules. Choose a quasi-isomorphism

$$\widetilde{I^\bullet} \longrightarrow \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$-modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 48.9.1 that there is a map

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet))$$

Observe that $\mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet)$ represents $R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$. Applying the universal property of the $\widetilde{\ }$ functor we obtain a map

$$\widetilde{\mathrm{Hom}_A(B, I^\bullet)} \longrightarrow R\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$$

in $D(\mathcal{O}_Z)$. We may check that this map is an isomorphism in $D(\mathcal{O}_Z)$ after applying i_*. However, once we apply i_* we obtain the isomorphism of Derived Categories of Schemes, Lemma 36.10.8 via the identification of Lemma 48.9.3. □

0A79 **Lemma 48.9.6.** *Let $i : Z \to X$ be a closed immersion of schemes. Assume X is a locally Noetherian. Then $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{Coh}(\mathcal{O}_X)$ into $D^+_{Coh}(\mathcal{O}_Z)$.*

Proof. The question is local on X, hence we may assume that X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ with A Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 48.9.5 to translate the question into algebra. The corresponding algebra result is a consequence of Dualizing Complexes, Lemma 47.13.4. □

0A9X **Lemma 48.9.7.** *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a pseudo-coherent closed immersion (if X is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Z)$ be the right adjoint to Ri_*. Then there is a functorial isomorphism*

$$a(K) = R\mathcal{H}om(\mathcal{O}_Z, K)$$

for $K \in D^+_{QCoh}(\mathcal{O}_X)$.

Proof. (The parenthetical statement follows from More on Morphisms, Lemma 37.60.9.) By Lemma 48.9.2 the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. Moreover, by Lemma 48.9.5 and Lemma 48.3.5 both $R\mathcal{H}om(\mathcal{O}_Z, -)$ and a map $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. □

0B6M **Example 48.9.8.** If $i : Z \to X$ is closed immersion of Noetherian schemes, then the diagram

$$\begin{tikzcd}
i_*a(K) \arrow[rr, "\mathrm{Tr}_{i,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\
i_*R\mathcal{H}om(\mathcal{O}_Z, K) \arrow[r, equal] & R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K) \arrow[r] & K
\end{tikzcd}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. Here the horizontal equality sign is Lemma 48.9.3 and the lower horizontal arrow is induced by the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$. The commutativity of the diagram is a consequence of Lemma 48.9.7.

48.10. Right adjoint of pushforward for closed immersions and base change

0E2J Consider a cartesian diagram of schemes

$$\begin{tikzcd}
Z' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

where i is a closed immersion. If Z and X' are tor independent over X, then there is a canonical base change map

0E2K (48.10.0.1) $$Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

in $D(\mathcal{O}_{Z'})$ functorial for K in $D(\mathcal{O}_X)$. Namely, by adjointness of Lemma 48.9.2 such an arrow is the same thing as a map

$$Ri'_*Lg^*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

in $D(\mathcal{O}_{X'})$. By tor independence we have $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ (see Derived Categories of Schemes, Lemma 36.22.9). Thus this is the same thing as a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^*K$$

For this we can use $Lf^*(can)$ where $can : Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ is the counit of the adjunction.

0E2L **Lemma 48.10.1.** *In the situation above, the map (48.10.0.1) is an isomorphism if and only if the base change map*

$$Lf^*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, Lf^*K)$$

of Cohomology, Remark 20.42.13 is an isomorphism.

Proof. The statement makes sense because $\mathcal{O}_{Z'} = Lf^*\mathcal{O}_Z$ by the assumed tor independence. Since i'_* is exact and faithful we see that it suffices to show the map (48.10.0.1) is an isomorphism after applying Ri'_*. Since $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ by the assumed tor independence and Derived Categories of Schemes, Lemma 36.22.9 we obtain a map

$$Lf^*Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Ri'_*R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^*K)$$

whose source and target are as in the statement of the lemma by Lemma 48.9.3. We omit the verification that this is the same map as the one constructed in Cohomology, Remark 20.42.13. □

0E2M **Lemma 48.10.2.** *In the situation above, assume f is flat and i pseudo-coherent. Then (48.10.0.1) is an isomorphism for K in $D^+_{QCoh}(\mathcal{O}_X)$.*

Proof. First proof. To prove this map is an isomorphism, we may work locally. Hence we may assume X, X', Z, Z' are affine, say corresponding to the rings A, A', B, B'. Then B and A' are tor independent over A. By Lemma 48.10.1 it suffices to check that

$$R\operatorname{Hom}_A(B, K) \otimes^{\mathbf{L}}_A A' = R\operatorname{Hom}_{A'}(B', K \otimes^{\mathbf{L}}_A A')$$

in $D(A')$ for all $K \in D^+(A)$. Here we use Derived Categories of Schemes, Lemma 36.10.8 and the fact that B, resp. B' is pseudo-coherent as an A-module, resp. A'-module to compare derived hom on the level of rings and schemes. The displayed equality follows from More on Algebra, Lemma 15.99.3 part (3). See also the discussion in Dualizing Complexes, Section 47.14.

Second proof[4]. Let $z' \in Z'$ with image $z \in Z$. First show that (48.10.0.1) on stalks at z' induces the map

$$R\operatorname{Hom}(\mathcal{O}_{Z,z}, K_z) \otimes^{\mathbf{L}}_{\mathcal{O}_{Z,x}} \mathcal{O}_{Z',z'} \longrightarrow R\operatorname{Hom}(\mathcal{O}_{Z',z'}, K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X,z}} \mathcal{O}_{X',z'})$$

from Dualizing Complexes, Equation (47.14.0.1). Namely, the constructions of these maps are identical. Then apply Dualizing Complexes, Lemma 47.14.2. □

0E2N **Lemma 48.10.3.** *Let $i : Z \to X$ be a pseudo-coherent closed immersion of schemes. Let $M \in D_{QCoh}(\mathcal{O}_X)$ locally have tor-amplitude in $[a, \infty)$. Let $K \in D^+_{QCoh}(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Li^*M = R\mathcal{H}om(\mathcal{O}_Z, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(\mathcal{O}_Z)$.

Proof. A map from LHS to RHS is the same thing as a map

$$Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

by Lemmas 48.9.2 and 48.9.3. For this map we take the counit $Ri_*R\mathcal{H}om(\mathcal{O}_Z, K) \to K$ tensored with id_M. To see this map is an isomorphism under the hypotheses

[4]This proof shows it suffices to assume K is in $D^+(\mathcal{O}_X)$.

given, translate into algebra using Lemma 48.9.5 and then for example use More on Algebra, Lemma 15.99.3 part (3). Instead of using Lemma 48.9.5 you can look at stalks as in the second proof of Lemma 48.10.2. □

48.11. Right adjoint of pushforward for finite morphisms

0AWZ If $i : Z \to X$ is a closed immersion of schemes, then there is a right adjoint $\mathcal{H}om(\mathcal{O}_Z, -)$ to the functor $i_* : Mod(\mathcal{O}_Z) \to Mod(\mathcal{O}_X)$ whose derived extension $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. See Section 48.9. In the case of a finite morphism $f : Y \to X$ this strategy cannot work, as the functor $f_* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is not exact in general and hence does not have a right adjoint. A replacement is to consider the exact functor $Mod(f_*\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ and consider the corresponding right adjoint and its derived extension.

Let $f : Y \to X$ be an affine morphism of schemes. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F})$$

is a sheaf of $f_*\mathcal{O}_Y$-modules. We obtain a functor $Mod(\mathcal{O}_X) \to Mod(f_*\mathcal{O}_Y)$ which we will denote $\mathcal{H}om(f_*\mathcal{O}_Y, -)$.

0BUZ **Lemma 48.11.1.** *With notation as above. The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is a right adjoint to the restriction functor $Mod(f_*\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$. For an affine open $U \subset X$ we have*

$$\Gamma(U, \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{F})) = \mathrm{Hom}_A(B, \mathcal{F}(U))$$

where $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(f^{-1}(U))$.

Proof. Adjointness follows from Modules, Lemma 17.22.3. As f is affine we see that $f_*\mathcal{O}_Y$ is the quasi-coherent sheaf corresponding to B viewed as an A-module. Hence the description of sections over U follows from Schemes, Lemma 26.7.1. □

The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is left exact. Let

$$R\mathcal{H}om(f_*\mathcal{O}_Y, -) : D(\mathcal{O}_X) \longrightarrow D(f_*\mathcal{O}_Y)$$

be its derived extension.

0BV0 **Lemma 48.11.2.** *With notation as above. The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is the right adjoint of the functor $D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. Follows from Lemma 48.11.1 and Derived Categories, Lemma 13.30.3. □

0BV1 **Lemma 48.11.3.** *With notation as above. The composition*

$$D(\mathcal{O}_X) \xrightarrow{R\mathcal{H}om(f_*\mathcal{O}_Y, -)} D(f_*\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

is the functor $K \mapsto R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K)$.

Proof. This is immediate from the construction. □

0AX2 **Lemma 48.11.4.** *Let $f : Y \to X$ be a finite pseudo-coherent morphism of schemes (a finite morphism of Noetherian schemes is pseudo-coherent). The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$. If X is quasi-compact*

and quasi-separated, then the diagram

$$\begin{array}{ccc} D^+_{QCoh}(\mathcal{O}_X) & \xrightarrow{\quad a \quad} & D^+_{QCoh}(\mathcal{O}_Y) \\ & {\scriptstyle R\mathcal{H}om(f_*\mathcal{O}_Y,-)} \searrow \quad \swarrow {\scriptstyle \Phi} & \\ & D^+_{QCoh}(f_*\mathcal{O}_Y) & \end{array}$$

is commutative, where a is the right adjoint of Lemma 48.3.1 for f and Φ is the equivalence of Derived Categories of Schemes, Lemma 36.5.4.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 37.60.9.) Since f is pseudo-coherent, the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ is pseudo-coherent, see More on Morphisms, Lemma 37.60.8. Thus $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(f_*\mathcal{O}_Y)$, see Derived Categories of Schemes, Lemma 36.10.8. Then $\Phi \circ a$ and $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ agree on $D^+_{QCoh}(\mathcal{O}_X)$ because these functors are both right adjoint to the restriction functor $D^+_{QCoh}(f_*\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$. To see this use Lemmas 48.3.5 and 48.11.2. □

0AX3 **Remark 48.11.5.** If $f : Y \to X$ is a finite morphism of Noetherian schemes, then the diagram

$$\begin{array}{ccc} Rf_*a(K) & \xrightarrow{\mathrm{Tr}_{f,K}} & K \\ \| & & \| \\ R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) & \longrightarrow & K \end{array}$$

is commutative for $K \in D^+_{QCoh}(\mathcal{O}_X)$. This follows from Lemma 48.11.4. The lower horizontal arrow is induced by the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ and the upper horizontal arrow is the trace map discussed in Section 48.7.

48.12. Right adjoint of pushforward for proper flat morphisms

0E4H For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated schemes the right adjoint of pushforward enjoys some remarkable properties.

0E4I **Lemma 48.12.1.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Lemma 36.30.4 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 36.17.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes,

Theorem 36.15.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

0E4J **Lemma 48.12.2.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then*

(1) *for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,*
(2) *for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 48.4.3, 48.4.4, and 48.12.1. □

0E4K **Lemma 48.12.3.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation. The map (48.8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 48.12.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (48.8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 36.18.2) generated by a single perfect object (Derived Categories of Schemes, Theorem 36.15.3) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (48.8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 48.8.1. □

The following lemma shows that the base change map (48.5.0.1) is an isomorphism for proper, flat morphisms of finite presentation. We will see in Example 48.15.2 that this does not remain true for perfect proper morphisms; in that case one has to make a tor independence condition.

0AAB **Lemma 48.12.4.** *Let $g : Y' \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Then the base change map (48.5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 48.12.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 48.6.1. In this case the statement follows from Lemma 48.6.2. □

0B6S **Remark 48.12.5.** Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation. Let a be the adjoint of Lemma 48.3.1 for f. In this situation, $\omega^\bullet_{X/Y} = a(\mathcal{O}_Y)$ is sometimes called the *relative dualizing complex*. By Lemma 48.12.3 there is a functorial isomorphism $a(K) = Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}$ for $K \in D_{QCoh}(\mathcal{O}_Y)$. Moreover, the trace map

$$\mathrm{Tr}_{f,\mathcal{O}_Y} : Rf_*\omega^\bullet_{X/Y} \to \mathcal{O}_Y$$

of Section 48.7 induces the trace map for all K in $D_{QCoh}(\mathcal{O}_Y)$. More precisely the diagram

$$\begin{tikzcd}
Rf_*a(K) \arrow[rr, "\mathrm{Tr}_{f,K}"'] \arrow[d, equal] & & K \arrow[d, equal] \\
Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) \arrow[r, equal] & K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\omega^\bullet_{X/Y} \arrow[r, "\mathrm{id}_K \otimes \mathrm{Tr}_{f,\mathcal{O}_Y}"] & K
\end{tikzcd}$$

where the equality on the lower right is Derived Categories of Schemes, Lemma 36.22.1. If $g : Y' \to Y$ is a morphism of quasi-compact and quasi-separated schemes and $X' = Y' \times_Y X$, then by Lemma 48.12.4 we have $\omega^\bullet_{X'/Y'} = L(g')^*\omega^\bullet_{X/Y}$ where $g' : X' \to X$ is the projection and by Lemma 48.7.1 the trace map

$$\mathrm{Tr}_{f',\mathcal{O}_{Y'}} : Rf'_*\omega^\bullet_{X'/Y'} \to \mathcal{O}_{Y'}$$

for $f' : X' \to Y'$ is the base change of $\mathrm{Tr}_{f,\mathcal{O}_Y}$ via the base change isomorphism.

0G81 **Remark 48.12.6.** Let $f : X \to Y$, $\omega^\bullet_{X/Y}$, and $\mathrm{Tr}_{f,\mathcal{O}_Y}$ be as in Remark 48.12.5. Let K and M be in $D_{QCoh}(\mathcal{O}_X)$ with M pseudo-coherent (for example perfect). Suppose given a map $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to \omega^\bullet_{X/Y}$ which corresponds to an isomorphism $K \to R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/Y})$ via Cohomology, Equation (20.42.0.1). Then the relative cup product (Cohomology, Remark 20.28.7)

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \to Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \to Rf_*\omega^\bullet_{X/Y} \xrightarrow{\mathrm{Tr}_{f,\mathcal{O}_Y}} \mathcal{O}_Y$$

determines an isomorphism $Rf_*K \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$. Namely, since $\omega^\bullet_{X/Y} = a(\mathcal{O}_Y)$ the canonical map (48.3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/Y}) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$$

is an isomorphism by Lemma 48.3.6 and Remark 48.3.8 and the fact that M and Rf_*M are pseudo-coherent, see Derived Categories of Schemes, Lemma 36.30.5. To see that the relative cup product induces this isomorphism use the commutativity of the diagram in Cohomology, Remark 20.42.12.

0E4L **Lemma 48.12.7.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes which is proper, flat, and of finite presentation with relative dualizing complex $\omega^\bullet_{X/Y}$ (Remark 48.12.5). Then*

1. *$\omega^\bullet_{X/Y}$ is a Y-perfect object of $D(\mathcal{O}_X)$,*
2. *$Rf_*\omega^\bullet_{X/Y}$ has vanishing cohomology sheaves in positive degrees,*
3. *$\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ is an isomorphism.*

Proof. In view of the fact that formation of $\omega^\bullet_{X/Y}$ commutes with base change (see Remark 48.12.5), we may and do assume that Y is affine. For a perfect object E of $D(\mathcal{O}_X)$ we have

$$\begin{aligned}
Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) &= Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) \\
&= R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y) \\
&= (Rf_*E^\vee)^\vee
\end{aligned}$$

For the first equality, see Cohomology, Lemma 20.50.5. For the second equality, see Lemma 48.3.6, Remark 48.3.8, and Derived Categories of Schemes, Lemma 36.30.4. The third equality is the definition of the dual. In particular these references also

show that the outcome is a perfect object of $D(\mathcal{O}_Y)$. We conclude that $\omega^\bullet_{X/Y}$ is Y-perfect by More on Morphisms, Lemma 37.69.6. This proves (1).

Let M be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_Y(M, Rf_*\omega^\bullet_{X/Y}) &= \mathrm{Hom}_X(Lf^*M, \omega^\bullet_{X/Y}) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\mathcal{O}_X, \mathcal{O}_Y)\end{aligned}$$

The first equality holds by Cohomology, Lemma 20.28.1. The second equality by construction of a. The third equality by Derived Categories of Schemes, Lemma 36.22.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N, see Derived Categories of Schemes, Lemma 36.30.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 15.75.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*\omega^\bullet_{X/Y}) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y}) &= \mathrm{Hom}_X(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}, \omega^\bullet_{X/Y}) \\ &= \mathrm{Hom}_Y(Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y})), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\ &= R\Gamma(Y, Rf_*E^\vee) \\ &= \mathrm{Hom}_X(E, \mathcal{O}_X)\end{aligned}$$

The first equality holds by Cohomology, Lemma 20.42.2. The second equality is the definition of $\omega^\bullet_{X/Y}$. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology, Lemma 20.50.5. The fourth equality follows from the equality $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ shown in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology, Lemma 20.50.5) and the fact that Rf_*E is perfect by Derived Categories of Schemes, Lemma 36.30.4. The last equality is Leray for f. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.10.8. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E. Since the perfect objects generate (Derived Categories of Schemes, Theorem 36.15.3) we conclude that α is an isomorphism. □

0E2P **Lemma 48.12.8** (Rigidity)**.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a proper, flat morphism of finite presentation with relative dualizing complex $\omega^\bullet_{X/Y}$ (Remark 48.12.5). There is a canonical isomorphism*

0E2Q (48.12.8.1) $$\mathcal{O}_X = c(Lpr_1^*\omega^\bullet_{X/Y}) = c(Lpr_2^*\omega^\bullet_{X/Y})$$

and a canonical isomorphism

0E2R (48.12.8.2)
$$\omega^\bullet_{X/Y} = c\left(Lpr_1^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} Lpr_2^*\omega^\bullet_{X/Y}\right)$$

where c is the right adjoint of Lemma 48.3.1 for the diagonal $\Delta : X \to X \times_Y X$.

Proof. Let a be the right adjoint to Rf_* as in Lemma 48.3.1. Consider the cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

Let b be the right adjoint for p as in Lemma 48.3.1. Then

$$\begin{aligned} \omega^\bullet_{X/Y} &= c(b(\omega^\bullet_{X/Y})) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} b(\mathcal{O}_X)) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} Lq^*a(\mathcal{O}_Y)) \\ &= c(Lp^*\omega^\bullet_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_Y X}} Lq^*\omega^\bullet_{X/Y}) \end{aligned}$$

as in (48.12.8.2). Explanation as follows:

(1) The first equality holds as $\mathrm{id} = c \circ b$ because $\mathrm{id}_X = p \circ \Delta$.
(2) The second equality holds by Lemma 48.12.3.
(3) The third holds by Lemma 48.12.4 and the fact that $\mathcal{O}_X = Lf^*\mathcal{O}_Y$.
(4) The fourth holds because $\omega^\bullet_{X/Y} = a(\mathcal{O}_Y)$.

Equation (48.12.8.1) is proved in exactly the same way. □

0BRU **Remark 48.12.9.** Lemma 48.12.8 means our relative dualizing complex is *rigid* in a sense analogous to the notion introduced in [**vdB97**]. Namely, since the functor on the right of (48.12.8.2) is "quadratic" in $\omega^\bullet_{X/Y}$ and the functor on the left of (48.12.8.2) is "linear" this "pins down" the complex $\omega^\bullet_{X/Y}$ to some extent. There is an approach to duality theory using "rigid" (relative) dualizing complexes, see for example [**Nee11**], [**Yek10**], and [**YZ09**]. We will return to this in Section 48.28.

48.13. Right adjoint of pushforward for perfect proper morphisms

0AA9 The correct generality for this section would be to consider perfect proper morphisms of quasi-compact and quasi-separated schemes, see [**LN07**].

0A9R **Lemma 48.13.1.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By More on Morphisms, Lemma 37.61.13 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 36.17.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 36.15.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

0AAA **Lemma 48.13.2.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then*

(1) *for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,*
(2) *for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (48.4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 48.4.3, 48.4.4, and 48.13.1. □

0A9U **Lemma 48.13.3.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. The map (48.8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 48.13.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (48.8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 36.18.2) generated by a single perfect object (Derived Categories of Schemes, Theorem 36.15.3) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (48.8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 48.8.1. □

0BZG **Lemma 48.13.4.** *Let $f : X \to Y$ be a perfect proper morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism with Y' Noetherian. If X and Y' are tor independent over Y, then the base change map (48.5.0.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 48.13.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes, see Remark 48.6.1. In this case the statement follows from Lemma 48.6.2. □

48.14. Right adjoint of pushforward for effective Cartier divisors

0B4A Let X be a scheme and let $i : D \to X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of i, see Morphisms, Section 29.31 and Divisors, Section 31.13. Recall that $R\mathcal{H}om(\mathcal{O}_D, -)$ denotes the right adjoint to $i_* : D(\mathcal{O}_D) \to D(\mathcal{O}_X)$ and has the property $i_*R\mathcal{H}om(\mathcal{O}_D, -) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, -)$, see Section 48.9.

0B4B **Lemma 48.14.1.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$.*

Proof. Equivalently, we are saying that $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ has a unique nonzero cohomology sheaf in degree 1 and that this sheaf is isomorphic to $\mathcal{N}$. Since i_* is exact and fully faithful, it suffices to prove that $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ is isomorphic to

$i_*\mathcal{N}[-1]$. We have $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X)$ by Lemma 48.9.3. We have a resolution

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where $\mathcal{I}$ is the ideal sheaf of D which we can use to compute. Since $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) = \mathcal{O}_X(D)$ by a local computation, we see that

$$R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

where on the right hand side we have $\mathcal{O}_X$ in degree 0 and $\mathcal{O}_X(D)$ in degree 1. The result follows from the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$$

coming from the fact that D is the zero scheme of the canonical section of $\mathcal{O}_X(D)$ and from the fact that $\mathcal{N} = i^*\mathcal{O}_X(D)$. □

For every object K of $D(\mathcal{O}_X)$ there is a canonical map

0B4C (48.14.1.1) $$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

in $D(\mathcal{O}_D)$ functorial in K and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$i_*(Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \longrightarrow K$$

where the equality is Cohomology, Lemma 20.54.4 and the arrow comes from the canonical map $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{O}_X$ induced by $\mathcal{O}_X \to i_*\mathcal{O}_D$.

If $K \in D_{QCoh}(\mathcal{O}_X)$, then (48.14.1.1) is equal to (48.8.0.1) via the identification $a(K) = R\mathcal{H}om(\mathcal{O}_D, K)$ of Lemma 48.9.7. If $K \in D_{QCoh}(\mathcal{O}_X)$ and X is Noetherian, then the following lemma is a special case of Lemma 48.13.3.

0AA4 **Lemma 48.14.2.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (48.14.1.1) combined with Lemma 48.14.1 defines an isomorphism*

$$Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_D} \mathcal{N}[-1] \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

functorial in K in $D(\mathcal{O}_X)$.

Proof. Since i_* is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying i_*. We will use the short exact sequences $0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$ and $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*\mathcal{N} \to 0$ used in the proof of Lemma 48.14.1 without further mention. By Cohomology, Lemma 20.54.4 which was used to define the map (48.14.1.1) the left hand side becomes

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} i_*\mathcal{N}[-1] = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{O}_X \to \mathcal{O}_X(D))$$

The right hand side becomes

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, K) &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), K) \\ &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \end{aligned}$$

the final equality by Cohomology, Lemma 20.50.5. Since the map comes from the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \to \mathcal{O}_X), \mathcal{O}_X) = (\mathcal{O}_X \to \mathcal{O}_X(D))$$

the lemma is clear. □

48.15. Right adjoint of pushforward in examples

0BQV In this section we compute the right adjoint to pushforward in some examples. The isomorphisms are canonical but only in the weakest possible sense, i.e., we do not prove or claim that these isomorphisms are compatible with various operations such as base change and compositions of morphisms. There is a huge literature on these types of issues; the reader can start with the material in [**Har66**], [**Con00**] (these citations use a different starting point for duality but address the issue of constructing canonical representatives for relative dualizing complexes) and then continue looking at works by Joseph Lipman and collaborators.

0A9W **Lemma 48.15.1.** *Let Y be a quasi-compact and quasi-separated scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module of rank $n+1$ with determinant $\mathcal{L} = \wedge^{n+1}(\mathcal{E})$. Let $f : X = \mathbf{P}(\mathcal{E}) \to Y$ be the projection. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then there is an isomorphism*

$$c : f^*\mathcal{L}(-n-1)[n] \longrightarrow a(\mathcal{O}_Y)$$

In particular, if $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$, then $X = \mathbf{P}^n_Y$ and we obtain $a(\mathcal{O}_Y) = \mathcal{O}_X(-n-1)[n]$.

Proof. In (the proof of) Cohomology of Schemes, Lemma 30.8.4 we constructed a canonical isomorphism

$$R^n f_*(f^*\mathcal{L}(-n-1)) \longrightarrow \mathcal{O}_Y$$

Moreover, $Rf_*(f^*\mathcal{L}(-n-1))[n] = R^n f_*(f^*\mathcal{L}(-n-1))$, i.e., the other higher direct images are zero. Thus we find an isomorphism

$$Rf_*(f^*\mathcal{L}(-n-1)[n]) \longrightarrow \mathcal{O}_Y$$

This isomorphism determines c as in the statement of the lemma because a is the right adjoint of Rf_*. By Lemma 48.4.4 construction of the a is local on the base. In particular, to check that c is an isomorphism, we may work locally on Y. In other words, we may assume Y is affine and $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$. In this case the sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \ldots, \mathcal{O}_X(-n)$ generate $D_{QCoh}(X)$, see Derived Categories of Schemes, Lemma 36.16.3. Hence it suffices to show that $c : \mathcal{O}_X(-n-1)[n] \to a(\mathcal{O}_Y)$ is transformed into an isomorphism under the functors

$$F_{i,p}(-) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), (-)[p])$$

for $i \in \{-n, \ldots, 0\}$ and $p \in \mathbf{Z}$. For $F_{0,p}$ this holds by construction of the arrow c! For $i \in \{-n, \ldots, -1\}$ we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), \mathcal{O}_X(-n-1)[n+p]) = H^p(X, \mathcal{O}_X(-n-1-i)) = 0$$

by the computation of cohomology of projective space (Cohomology of Schemes, Lemma 30.8.1) and we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), a(\mathcal{O}_Y)[p]) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\mathcal{O}_X(i), \mathcal{O}_Y[p]) = 0$$

because $Rf_*\mathcal{O}_X(i) = 0$ by the same lemma. Hence the source and the target of $F_{i,p}(c)$ vanish and $F_{i,p}(c)$ is necessarily an isomorphism. This finishes the proof. □

0AAC **Example 48.15.2.** The base change map (48.5.0.1) is not an isomorphism if f is perfect proper and g is perfect. Let k be a field. Let $Y = \mathbf{A}^2_k$ and let $f : X \to Y$ be the blowup of Y in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor f as

$$X \xrightarrow{i} \mathbf{P}^1_Y \xrightarrow{p} Y$$

This gives a factorization $a = c \circ b$ where a, b, and c are the right adjoints of Lemma 48.3.1 of Rf_*, Rp_*, and Ri_*. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbf{P}^1_Y$ and denote $\mathcal{O}_X(n)$ its restriction to X. Note that $X \subset \mathbf{P}^1_Y$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 48.15.1 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 48.9.7 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 48.14.2. Let $Y' = \mathrm{Spec}(k)$ be the origin in Y. The restriction of $a(\mathcal{O}_Y)$ to $X' = E = \mathbf{P}^1_k$ is an invertible sheaf of degree -1 placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\mathrm{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible sheaf of degree -2 placed in cohomological degree -1, so different. In this example the hypothesis of Tor independence in Lemma 48.6.2 is violated.

0BQW **Lemma 48.15.3.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. There is a canonical isomorphism $c : \mathcal{N} \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof. Consider the canonical short exact sequence

0BQX (48.15.3.1) $$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_Y/\mathcal{I}^2 \to \mathcal{O}_X \to 0$$

Let $U \subset X$ be open and let $s \in \mathcal{N}(U)$. Then we can pushout (48.15.3.1) via s to get an extension E_s of $\mathcal{O}_X|_U$ by $\mathcal{O}_X|_U$. This in turn defines a section $c(s)$ of $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ over U. See Cohomology, Lemma 20.42.1 and Derived Categories, Lemma 13.27.6. Conversely, given an extension

$$0 \to \mathcal{O}_X|_U \to \mathcal{E} \to \mathcal{O}_X|_U \to 0$$

of $\mathcal{O}_U$-modules, we can find an open covering $U = \bigcup U_i$ and sections $e_i \in \mathcal{E}(U_i)$ mapping to $1 \in \mathcal{O}_X(U_i)$. Then e_i defines a map $\mathcal{O}_Y|_{U_i} \to \mathcal{E}|_{U_i}$ whose kernel contains $\mathcal{I}^2$. In this way we see that $\mathcal{E}|_{U_i}$ comes from a pushout as above. This shows that c is surjective. We omit the proof of injectivity. □

0BQY **Lemma 48.15.4.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 31.20.2) then composition on $R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ defines isomorphisms*

$$\wedge^i(\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

for all i.

Proof. By composition we mean the map

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

of Cohomology, Lemma 20.42.5. This induces multiplication maps

$$\mathcal{E}xt^a_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{E}xt^b_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^{a+b}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

Please compare with More on Algebra, Equation (15.63.0.1). The statement of the lemma means that the induced map

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes \ldots \otimes \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

factors through the wedge product and then induces an isomorphism. To see this is true we may work locally on Y. Hence we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Denote

$$\mathcal{A} = \mathcal{O}_Y\langle \xi_1, \ldots, \xi_r \rangle$$

the sheaf of strictly commutative differential graded $\mathcal{O}_Y$-algebras which is a (divided power) polynomial algebra on $\xi_1, \ldots, \xi_r$ in degree -1 over $\mathcal{O}_Y$ with differential d given by the rule $\mathrm{d}\xi_i = f_i$. Let us denote $\mathcal{A}^\bullet$ the underlying complex of $\mathcal{O}_Y$-modules which is the Koszul complex mentioned above. Thus the canonical map $\mathcal{A}^\bullet \to \mathcal{O}_X$ is a quasi-isomorphism. We obtain quasi-isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{A}^\bullet) \to \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

by Cohomology, Lemma 20.46.9. The differentials of the latter complex are zero, and hence

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$$

For $j \in \{1, \ldots, r\}$ let $\delta_j : \mathcal{A} \to \mathcal{A}$ be the derivation of degree 1 with $\delta_j(\xi_i) = \delta_{ij}$ (Kronecker delta). A computation shows that $\delta_j \circ \mathrm{d} = -\mathrm{d} \circ \delta_j$ which shows that we get a morphism of complexes

$$\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1].$$

Whence δ_j defines a section of the corresponding $\mathcal{E}xt$-sheaf. Another computation shows that $\delta_1, \ldots, \delta_r$ map to a basis for $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-1}, \mathcal{O}_X)$ over $\mathcal{O}_X$. Since it is clear that $\delta_j \circ \delta_j = 0$ and $\delta_j \circ \delta_{j'} = -\delta_{j'} \circ \delta_j$ as endomorphisms of $\mathcal{A}$ and hence in the $\mathcal{E}xt$-sheaves we obtain the statement that our map above factors through the exterior power. To see we get the desired isomorphism the reader checks that the elements

$$\delta_{j_1} \circ \ldots \circ \delta_{j_i}$$

for $j_1 < \ldots < j_i$ map to a basis of the sheaf $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$ over $\mathcal{O}_X$. □

0BQZ **Lemma 48.15.5.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 31.20.2) then*

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r\mathcal{N}[-r]$$

where $r : Y \to \{1, 2, 3, \ldots\}$ sends y to the minimal number of generators of $\mathcal{I}$ needed in a neighbourhood of y.

Proof. We can use Lemmas 48.15.3 and 48.15.4 to see that we have isomorphisms $\wedge^i\mathcal{N} \to \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ for $i \geq 0$. Thus it suffices to show that the map $\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^r_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

and that $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$ is zero for $i \neq r$. These statements are local on Y. Thus we may assume that we have global sections $f_1, \ldots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Let $\mathcal{A}^\bullet$ be the Koszul complex on $f_1, \ldots, f_r$ as introduced in the proof of Lemma 48.15.4. Then

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by Cohomology, Lemma 20.46.9. Denote $1 : \mathcal{A}^\bullet \to \mathcal{O}_Y$ the map of differential graded $\mathcal{O}_Y$-algebras given by the identity map of $\mathcal{A}^0 = \mathcal{O}_Y \to \mathcal{O}_Y$ in degree 0. With δ_j as in the proof of Lemma 48.15.4 we get an isomorphism of graded $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y\langle\delta_1, \ldots, \delta_r\rangle \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by mapping $\delta_{j_1} \ldots \delta_{j_i}$ to $1 \circ \delta_{j_1} \circ \ldots \circ \delta_{j_i}$ in degree i. Via this isomorphism the differential on the right hand side induces a differential d on the left hand side. By

our sign rules we have $\mathrm{d}(1) = -\sum f_j\delta_j$. Since $\delta_j : \mathcal{A}^\bullet \to \mathcal{A}^\bullet[1]$ is a morphism of complexes, it follows that

$$\mathrm{d}(\delta_{j_1} \ldots \delta_{j_i}) = (-\sum f_j\delta_j)\delta_{j_1} \ldots \delta_{j_i}$$

Observe that we have $\mathrm{d} = \sum f_j\delta_j$ on the differential graded algebra $\mathcal{A}$. Therefore the map defined by the rule

$$1 \circ \delta_{j_1} \ldots \delta_{j_i} \longmapsto (\delta_{j_1} \circ \ldots \circ \delta_{j_i})(\xi_1 \ldots \xi_r)$$

will define an isomorphism of complexes

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{A}^\bullet[-r]$$

if r is odd and commuting with differentials up to sign if r is even. In any case these complexes have isomorphic cohomology, which shows the desired vanishing. The isomorphism on cohomology in degree r under the map

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

also follows in a straightforward manner from this. (We observe that our choice of conventions regarding Koszul complexes does intervene in the definition of the isomorphism $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r\mathcal{N}[-r]$.) □

0BR0 **Lemma 48.15.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let $i : X \to Y$ be a Koszul-regular closed immersion. Let a be the right adjoint of $Ri_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 48.3.1. Then there is an isomorphism*

$$\wedge^r\mathcal{N}[-r] \longrightarrow a(\mathcal{O}_Y)$$

where $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$ is the normal sheaf of i (Morphisms, Section 29.31) and r is its rank viewed as a locally constant function on X.

Proof. Recall, from Lemmas 48.9.7 and 48.9.3, that $a(\mathcal{O}_Y)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ whose pushforward to Y is $R\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \mathcal{O}_Y)$. Thus the result follows from Lemma 48.15.5. □

0BRT **Lemma 48.15.7.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a smooth proper morphism of relative dimension d. Let a be the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ as in Lemma 48.3.1. Then there is an isomorphism*

$$\wedge^d\Omega_{X/S}[d] \longrightarrow a(\mathcal{O}_S)$$

in $D(\mathcal{O}_X)$.

Proof. Set $\omega^\bullet_{X/S} = a(\mathcal{O}_S)$ as in Remark 48.12.5. Let c be the right adjoint of Lemma 48.3.1 for $\Delta : X \to X \times_S X$. Because Δ is the diagonal of a smooth morphism it is a Koszul-regular immersion, see Divisors, Lemma 31.22.11. In particular, Δ is a perfect proper morphism (More on Morphisms, Lemma 37.61.7) and we obtain

$$\begin{aligned}
\mathcal{O}_X &= c(L\mathrm{pr}_1^*\omega^\bullet_{X/S}) \\
&= L\Delta^*(L\mathrm{pr}_1^*\omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} c(\mathcal{O}_{X\times_S X}) \\
&= \omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_X} c(\mathcal{O}_{X\times_S X}) \\
&= \omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \wedge^d(\mathcal{N}_\Delta)[-d]
\end{aligned}$$

The first equality is (48.12.8.1) because $\omega^\bullet_{X/S} = a(\mathcal{O}_S)$. The second equality by Lemma 48.13.3. The third equality because $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$. The fourth equality

by Lemma 48.15.6. Observe that $\wedge^d(\mathcal{N}_\Delta)$ is an invertible $\mathcal{O}_X$-module. Hence $\wedge^d(\mathcal{N}_\Delta)[-d]$ is an invertible object of $D(\mathcal{O}_X)$ and we conclude that $a(\mathcal{O}_S) = \omega^\bullet_{X/S} = \wedge^d(\mathcal{C}_\Delta)[d]$. Since the conormal sheaf $\mathcal{C}_\Delta$ of Δ is $\Omega_{X/S}$ by Morphisms, Lemma 29.32.7 the proof is complete. □

48.16. Upper shriek functors

0A9Y In this section, we construct the functors $f^!$ for morphisms between schemes which are of finite type and separated over a fixed Noetherian base using compactifications. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

0F42 **Situation 48.16.1.** Here S is a Noetherian scheme and FTS_S is the category whose

(1) objects are schemes X over S such that the structure morphism $X \to S$ is both separated and of finite type, and
(2) morphisms $f : X \to Y$ between objects are morphisms of schemes over S.

In Situation 48.16.1 given a morphism $f : X \to Y$ in FTS_S, we will define an exact functor

$$f^! : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$$

of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over Y which is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $\overline{a} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{\overline{X}})$ be the right adjoint of $R\overline{f}_*$ constructed in Lemma 48.3.1. Then we set

$$f^!K = \overline{a}(K)|_X$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$. The result is an object of $D^+_{QCoh}(\mathcal{O}_X)$ by Lemma 48.3.5.

0AA0 **Lemma 48.16.2.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. The category of compactifications of X over Y is defined in More on Flatness, Section 38.32. By More on Flatness, Theorem 38.33.8 and Lemma 38.32.2 it is nonempty. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $j^* \circ \overline{a} : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$ where $\overline{a}$ is the right adjoint of $R\overline{f}_*$ constructed in Lemma 48.3.1.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$[5]. Let $\overline{c}$ be the right adjoint of Lemma 48.3.1 for g. Then $\overline{c} \circ \overline{a}_2 = \overline{a}_1$ because these functors are adjoint to $R\overline{f}_{2,*} \circ Rg_* = R(\overline{f}_2 \circ g)_*$. By (48.4.1.1) we have a canonical transformation

$$j_1^* \circ \overline{c} \longrightarrow j_2^*$$

[5]This may fail with our definition of compactification. See More on Flatness, Section 38.32.

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_2}) \to D^+_{QCoh}(\mathcal{O}_X)$ which is an isomorphism by Lemma 48.4.4. The composition

$$j_1^* \circ \overline{a}_1 \longrightarrow j_1^* \circ \overline{c} \circ \overline{a}_2 \longrightarrow j_2^* \circ \overline{a}_2$$

is an isomorphism of functors which we will denote by α_g.

Consider two compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$ of X over Y. By More on Flatness, Lemma 38.32.1 part (b) we can find a compactification $j : X \to \overline{X}$ with dense image and morphisms $g_i : \overline{X} \to \overline{X}_i$ of compactifications. By More on Flatness, Lemma 38.32.1 part (c) we have $g_i^{-1}(j_i(X)) = j(X)$. Hence we get isomorpisms

$$\alpha_{g_i} : j^* \circ \overline{a} \longrightarrow j_i^* \circ \overline{a}_i$$

by the previous paragraph. We obtain an isomorphism

$$\alpha_{g_2} \circ \alpha_{g_1}^{-1} : j_1^* \circ \overline{a}_1 \to j_2^* \circ \overline{a}_2$$

To finish the proof we have to show that these isomorphisms are well defined. We claim it suffices to show the composition of isomorphisms constructed in the previous paragraph is another (for a precise statement see the next paragraph). We suggest the reader check this is true on a napkin, but we will also completely spell it out in the rest of this paragraph. Namely, consider a second choice of a compactification $j' : X \to \overline{X}'$ with dense image and morphisms of compactifications $g'_i : \overline{X}' \to \overline{X}_i$. By More on Flatness, Lemma 38.32.1 we can find a compactification $j'' : X \to \overline{X}''$ with dense image and morphisms of compactifications $h : \overline{X}'' \to \overline{X}$ and $h' : \overline{X}'' \to \overline{X}'$. We may even assume $g_1 \circ h = g'_1 \circ h'$ and $g_2 \circ h = g'_2 \circ h'$. The result of the next paragraph gives

$$\alpha_{g_i} \circ \alpha_h = \alpha_{g_i \circ h} = \alpha_{g'_i \circ h'} = \alpha_{g'_i} \circ \alpha_{h'}$$

for $i = 1, 2$. Since these are all isomorphisms of functors we conclude that $\alpha_{g_2} \circ \alpha_{g_1}^{-1} = \alpha_{g'_2} \circ \alpha_{g'_1}^{-1}$ as desired.

Suppose given compactifications $j_i : X \to \overline{X}_i$ for $i = 1, 2, 3$. Suppose given morphisms $g : \overline{X}_1 \to \overline{X}_2$ and $h : \overline{X}_2 \to \overline{X}_3$ of compactifications such that $g^{-1}(j_2(X)) = j_1(X)$ and $h^{-1}(j_2(X)) = j_3(X)$. Let $\overline{a}_i$ be as above. The claim above means that

$$\alpha_g \circ \alpha_h = \alpha_{g \circ h} : j_1^* \circ \overline{a}_1 \to j_3^* \circ \overline{a}_3$$

Let $\overline{c}$, resp. $\overline{d}$ be the right adjoint of Lemma 48.3.1 for g, resp. h. Then $\overline{c} \circ \overline{a}_2 = \overline{a}_1$ and $\overline{d} \circ \overline{a}_3 = \overline{a}_2$ and there are canonical transformations

$$j_1^* \circ \overline{c} \longrightarrow j_2^* \quad \text{and} \quad j_2^* \circ \overline{d} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_2}) \to D^+_{QCoh}(\mathcal{O}_X)$ and $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ for the same reasons as above. Denote $\overline{e}$ the right adjoint of Lemma 48.3.1 for $h \circ g$. There is a canonical transformation

$$j_1^* \circ \overline{e} \longrightarrow j_3^*$$

of functors $D^+_{QCoh}(\mathcal{O}_{\overline{X}_3}) \to D^+_{QCoh}(\mathcal{O}_X)$ given by (48.4.1.1). Spelling things out we have to show that the composition

$$\alpha_h \circ \alpha_g : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{c} \circ \overline{a}_2 \to j_2^* \circ \overline{a}_2 \to j_2^* \circ \overline{d} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

is the same as the composition

$$\alpha_{h \circ g} : j_1^* \circ \overline{a}_1 \to j_1^* \circ \overline{e} \circ \overline{a}_3 \to j_3^* \circ \overline{a}_3$$

We split this into two parts. The first is to show that the diagram

$$\begin{array}{ccc} \overline{a}_1 & \longrightarrow & \overline{c} \circ \overline{a}_2 \\ \downarrow & & \downarrow \\ \overline{e} \circ \overline{a}_3 & \longrightarrow & \overline{c} \circ \overline{d} \circ \overline{a}_3 \end{array}$$

commutes where the lower horizontal arrow comes from the identification $\overline{e} = \overline{c} \circ \overline{d}$. This is true because the corresponding diagram of total direct image functors

$$\begin{array}{ccc} R\overline{f}_{1,*} & \longrightarrow & Rg_* \circ R\overline{f}_{2,*} \\ \downarrow & & \downarrow \\ R(h \circ g)_* \circ R\overline{f}_{3,*} & \longrightarrow & Rg_* \circ Rh_* \circ R\overline{f}_{3,*} \end{array}$$

is commutative (insert future reference here). The second part is to show that the composition

$$j_1^* \circ \overline{c} \circ \overline{d} \to j_2^* \circ \overline{d} \to j_3^*$$

is equal to the map

$$j_1^* \circ \overline{e} \to j_3^*$$

via the identification $\overline{e} = \overline{c} \circ \overline{d}$. This was proven in Lemma 48.5.1 (note that in the current case the morphisms f', g' of that lemma are equal to id_X). □

0ATX **Lemma 48.16.3.** *In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. Then there is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$.*

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2 twice. Let $\overline{a}$ be the right adjoint of Lemma 48.3.1 for $\overline{X} \to \overline{Y}$ and let $\overline{b}$ be the right adjoint of Lemma 48.3.1 for $\overline{Y} \to Z$. Then $\overline{a} \circ \overline{b}$ is the right adjoint of Lemma 48.3.1 for the composition $\overline{X} \to Z$. Hence $g^! = i^* \circ \overline{b}$ and $(g \circ f)^! = (X \to \overline{X})^* \circ \overline{a} \circ \overline{b}$. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & U & \xrightarrow{j'} & \overline{X} \\ \downarrow & \swarrow & & & \swarrow \\ Y & & \xrightarrow{i} & & \overline{Y} \\ \downarrow & & & \swarrow & \\ Z & & & & \end{array}$$

Let $\overline{a}'$ be the right adjoint of Lemma 48.3.1 for $U \to Y$. Then $f^! = j^* \circ \overline{a}'$. We obtain

$$\gamma : (j')^* \circ \overline{a} \to \overline{a}' \circ i^*$$

by (48.4.1.1) and we can use it to define

$$(g \circ f)^! = (j' \circ j)^* \circ \overline{a} \circ \overline{b} = j^* \circ (j')^* \circ \overline{a} \circ \overline{b} \to j^* \circ \overline{a}' \circ i^* \circ \overline{b} = f^! \circ g^!$$

which is an isomorphism on objects of $D^+_{QCoh}(\mathcal{O}_Z)$ by Lemma 48.4.4. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

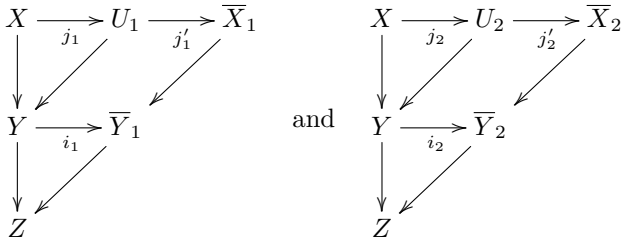

We can first choose a compactification $i : Y \to \overline{Y}$ with dense image of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 38.32.1. By More on Flatness, Lemma 38.32.3 and Categories, Lemmas 4.27.13 and 4.27.14 we can choose a compactification $X \to \overline{X}$ with dense image of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

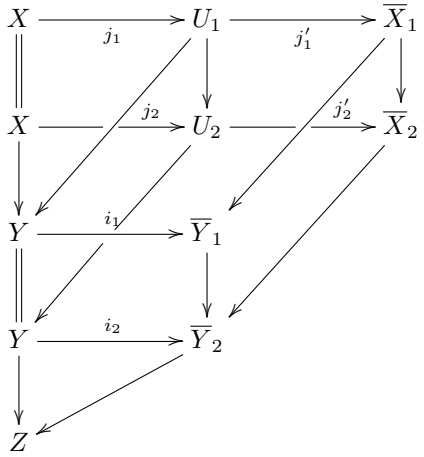

and moreover the compactifications $X \to \overline{X}_1$ and $Y \to \overline{Y}_2$ have dense image. We use $\overline{a}_i$, $\overline{a}'_i$, $\overline{c}$, and $\overline{c}'$ for the right adjoint of Lemma 48.3.1 for $\overline{X}_i \to \overline{Y}_i$, $U_i \to Y$, $\overline{X}_1 \to \overline{X}_2$, and $U_1 \to U_2$. Each of the squares

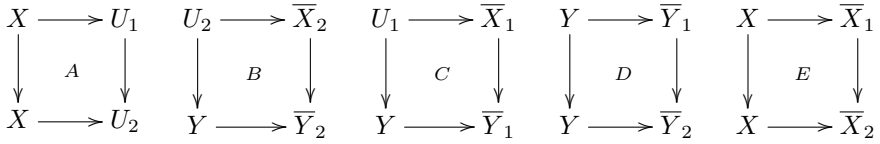

is cartesian (see More on Flatness, Lemma 38.32.1 part (c) for A, D, E and recall that U_i is the inverse image of Y by $\overline{X}_i \to \overline{Y}_i$ for B, C) and hence gives rise to a base change map (48.4.1.1) as follows

$$\gamma_A : j_1^* \circ \overline{c}' \to j_2^* \qquad \gamma_B : (j'_2)^* \circ \overline{a}_2 \to \overline{a}'_2 \circ i_2^* \qquad \gamma_C : (j'_1)^* \circ \overline{a}_1 \to \overline{a}'_1 \circ i_1^*$$
$$\gamma_D : i_1^* \circ \overline{d} \to i_2^* \quad \gamma_E : (j'_1 \circ j_1)^* \circ \overline{c} \to (j'_2 \circ j_2)^*$$

Denote $f_1^! = j_1^* \circ \overline{a}'_1$, $f_2^! = j_2^* \circ \overline{a}'_2$, $g_1^! = i_1^* \circ \overline{b}_1$, $g_2^! = i_2^* \circ \overline{b}_2$, $(g \circ f)_1^! = (j'_1 \circ j_1)^* \circ \overline{a}_1 \circ \overline{b}_1$, and $(g \circ f)_2^! = (j'_2 \circ j_2)^* \circ \overline{a}_2 \circ \overline{b}_2$. The construction given in the first paragraph of the proof and in Lemma 48.16.2 uses

(1) γ_C for the map $(g \circ f)_1^! \to f_1^! \circ g_1^!$,
(2) γ_B for the map $(g \circ f)_2^! \to f_2^! \circ g_2^!$,
(3) γ_A for the map $f_1^! \to f_2^!$,
(4) γ_D for the map $g_1^! \to g_2^!$, and
(5) γ_E for the map $(g \circ f)_1^! \to (g \circ f)_2^!$.

We have to show that the diagram

$$\begin{array}{ccc} (g \circ f)_1^! & \xrightarrow{\gamma_E} & (g \circ f)_2^! \\ {\scriptstyle \gamma_C}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\ f_1^! \circ g_1^! & \xrightarrow{\gamma_A \circ \gamma_D} & f_2^! \circ g_2^! \end{array}$$

is commutative. We will use Lemmas 48.5.1 and 48.5.2 and with (abuse of) notation as in Remark 48.5.3 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_A \circ \gamma_F$ where

$$\begin{array}{ccc} U_1 & \longrightarrow & \overline{X}_1 \\ \downarrow & F & \downarrow \\ U_2 & \longrightarrow & \overline{X}_2 \end{array}$$

Thus we see that

$$\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 48.5.3). Thus it suffices to prove that $\gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F$. Since both of these are equal to the map (48.4.1.1) for the square

$$\begin{array}{ccc} U_1 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \overline{Y}_2 \end{array}$$

we conclude. □

0ATY **Lemma 48.16.4.** *In Situation 48.16.1 the constructions of Lemmas 48.16.2 and 48.16.3 define a pseudo functor from the category FTS_S into the 2-category of categories (see Categories, Definition 4.29.5).*

Proof. To show this we have to prove given morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ that

$$\begin{array}{ccc} (h \circ g \circ f)^! & \xrightarrow{\gamma_{A+B}} & f^! \circ (h \circ g)^! \\ {\scriptstyle \gamma_{B+C}}\downarrow & & \downarrow{\scriptstyle \gamma_C} \\ (g \circ f)^! \circ h^! & \xrightarrow{\gamma_A} & f^! \circ g^! \circ h^! \end{array}$$

is commutative (for the meaning of the γ's, see below). To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then

a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \overline{X} \\
{\scriptstyle f}\downarrow & & \downarrow & A & \downarrow & B & \downarrow \\
Y & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \overline{Y} \\
{\scriptstyle g}\downarrow & & \downarrow & & \downarrow & C & \downarrow \\
Z & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & \overline{Z} \\
{\scriptstyle h}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & T & \longrightarrow & T & \longrightarrow & T
\end{array}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 48.16.3 we see that the maps in the first displayed diagram use the maps (48.4.1.1) for the rectangles $A + B$, $B + C$, A, and C as indicated. Since by Lemmas 48.5.1 and 48.5.2 we have $\gamma_{A+B} = \gamma_A \circ \gamma_B$ and $\gamma_{B+C} = \gamma_C \circ \gamma_B$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 48.5.3). □

0B6T **Lemma 48.16.5.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. There are canonical maps*

$$\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \longrightarrow f^!K$$

functorial in K in $D^+_{QCoh}(\mathcal{O}_Y)$. If $g : Y \to Z$ is another morphism of FTS_S, then the diagram

$$\begin{array}{ccccc}
Lf^*(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y & \xrightarrow[\mu_f]{} & f^!(Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g^!\mathcal{O}_Z) & \xrightarrow[f^!\mu_g]{} & f^!g^!K \\
\| & & & & \| \\
Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y & \xrightarrow{\mu_f} & Lf^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!g^!\mathcal{O}_Z & \xrightarrow{\mu_{g\circ f}} & f^!g^!K
\end{array}$$

commutes for all $K \in D^+_{QCoh}(\mathcal{O}_Z)$.

Proof. If f is proper, then $f^! = a$ and we can use (48.8.0.1) and if g is also proper, then Lemma 48.8.4 proves the commutativity of the diagram (in greater generality).

Let us define the map $\mu_{f,K}$. Choose a compactification $j : X \to \overline{X}$ of X over Y. Since $f^!$ is defined as $j^* \circ \overline{a}$ we obtain $\mu_{f,K}$ as the restriction of the map (48.8.0.1)

$$L\overline{f}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}(\mathcal{O}_Y) \longrightarrow \overline{a}(K)$$

to X. To see this is independent of the choice of the compactification we argue as in the proof of Lemma 48.16.2. We urge the reader to read the proof of that lemma first.

Assume given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$. Denote $\overline{c}$ the right adjoint for pushforward

of Lemma 48.3.1 for the morphism g. The maps

$$L\overline{f}_1^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_1(\mathcal{O}_Y) \longrightarrow \overline{a}_1(K) \quad \text{and} \quad L\overline{f}_2^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{\overline{X}}} \overline{a}_2(\mathcal{O}_Y) \longrightarrow \overline{a}_2(K)$$

fit into the commutative diagram

$$\begin{array}{ccccc} Lg^*(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) & \xrightarrow[\sigma]{} & \overline{c}(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) & \longrightarrow & \overline{c}(\overline{a}_2(K)) \\ \| & & & & \| \\ L\overline{f}_1^*K \otimes^{\mathbf{L}} Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) & \xrightarrow{1\otimes\tau} & L\overline{f}_1^*K \otimes^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y) & \longrightarrow & \overline{a}_1(K) \end{array}$$

by Lemma 48.8.4. By Lemma 48.8.3 the maps σ and τ restrict to an isomorphism over X. In fact, we can say more. Recall that in the proof of Lemma 48.16.2 we used the map (48.4.1.1) $\gamma : j_1^* \circ \overline{c} \to j_2^*$ to construct our isomorphism $\alpha_g : j_1^* \circ \overline{a}_1 \to j_2^* \circ \overline{a}_2$. Pulling back to map σ by j_1 we obtain the identity map on $j_2^*\left(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)\right)$ if we identify $j_1^*\overline{c}(\mathcal{O}_{\overline{X}_2})$ with $\mathcal{O}_X$ via $j_1^* \circ \overline{c} \to j_2^*$, see Lemma 48.8.2. Similarly, the map $\tau : Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) \to \overline{a}_1(\mathcal{O}_Y) = \overline{c}(\overline{a}_2(\mathcal{O}_Y))$ pulls back to the identity map on $j_2^*\overline{a}_2(\mathcal{O}_Y)$. We conclude that pulling back by j_1 and applying γ wherever we can we obtain a commutative diagram

$$\begin{array}{ccccc} j_2^*\left(L\overline{f}_2^*K \otimes^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)\right) & & \longrightarrow & & j_2^*\overline{a}_2(K) \\ \downarrow & & & \nwarrow^{\alpha_g} & \\ j_1^*L\overline{f}_1^*K \otimes^{\mathbf{L}} j_2^*\overline{a}_2(\mathcal{O}_Y) & \xleftarrow{1\otimes\alpha_g} & j_1^*(L\overline{f}_1^*K \otimes^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y)) & \longrightarrow & j_1^*\overline{a}_1(K) \end{array}$$

The commutativity of this diagram exactly tells us that the map $\mu_{f,K}$ constructed using the compactification $\overline{X}_1$ is the same as the map $\mu_{f,K}$ constructed using the compactification $\overline{X}_2$ via the identification α_g used in the proof of Lemma 48.16.2. Some categorical arguments exactly as in the proof of Lemma 48.16.2 now show that $\mu_{f,K}$ is well defined (small detail omitted).

Having said this, the commutativity of the diagram in the statement of our lemma follows from the construction of the isomorphism $(g \circ f)^! \to f^! \circ g^!$ (first part of the proof of Lemma 48.16.3 using $\overline{X} \to \overline{Y} \to Z$) and the result of Lemma 48.8.4 for $\overline{X} \to \overline{Y} \to Z$. □

48.17. Properties of upper shriek functors

0ATZ Here are some properties of the upper shriek functors.

0AU0 **Lemma 48.17.1.** *In Situation 48.16.1 let Y be an object of FTS_S and let $j : X \to Y$ be an open immersion. Then there is a canonical isomorphism $j^! = j^*$ of functors.*

For an étale morphism $f : X \to Y$ of FTS_S we also have $f^* \cong f^!$, see Lemma 48.18.2.

Proof. In this case we may choose $\overline{X} = Y$ as our compactification. Then the right adjoint of Lemma 48.3.1 for $\mathrm{id} : Y \to Y$ is the identity functor and hence $j^! = j^*$ by definition. □

0G4J **Lemma 48.17.2.** *In Situation 48.16.1 let*

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j'} & Y \end{array}$$

be a commutative diagram of FTS_S where j and j' are open immersions. Then $j^* \circ f^! = g^! \circ (j')^*$ as functors $D^+_{QCoh}(\mathcal{O}_Y) \to D^+(\mathcal{O}_U)$.

Proof. Let $h = f \circ j = j' \circ g$. By Lemma 48.16.3 we have $h^! = j^! \circ f^! = g^! \circ (j')^!$. By Lemma 48.17.1 we have $j^! = j^*$ and $(j')^! = (j')^*$. □

0AA1 **Lemma 48.17.3.** *In Situation 48.16.1 let Y be an object of FTS_S and let $f : X = \mathbf{A}^1_Y \to Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.*

Proof. Since $X = \mathbf{A}^1_Y \subset \mathbf{P}^1_Y$ and since $\mathcal{O}_{\mathbf{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 48.15.1 and 48.13.3. □

0AA2 **Lemma 48.17.4.** *In Situation 48.16.1 let Y be an object of FTS_S and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$ of functors.*

Proof. This is a restatement of Lemma 48.9.7. □

0BV2 **Remark 48.17.5** (Local description upper shriek)**.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Using the lemmas above we can compute $f^!$ locally as follows. Suppose that we are given affine opens

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{i} & Y \end{array}$$

Since $j^! \circ f^! = g^! \circ i^!$ (Lemma 48.16.3) and since $j^!$ and $i^!$ are given by restriction (Lemma 48.17.1) we see that

$$(f^!E)|_U = g^!(E|_V)$$

for any $E \in D^+_{QCoh}(\mathcal{O}_X)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ and let $\varphi : R \to A$ be the finite type ring map corresponding to g. Choose a presentation $A = P/I$ where $P = R[x_1, \ldots, x_n]$ is a polynomial algebra in n variables over R. Choose an object $K \in D^+(R)$ corresponding to $E|_V$ (Derived Categories of Schemes, Lemma 36.3.5). Then we claim that $f^!E|_U$ corresponds to

$$\varphi^!(K) = R\,\mathrm{Hom}(A, K \otimes^{\mathbf{L}}_R P)[n]$$

where $R\,\mathrm{Hom}(A, -) : D(P) \to D(A)$ is the functor of Dualizing Complexes, Section 47.13 and where $\varphi^! : D(R) \to D(A)$ is the functor of Dualizing Complexes, Section 47.24. Namely, the choice of presentation gives a factorization

$$U \to \mathbf{A}^n_V \to \mathbf{A}^{n-1}_V \to \ldots \to \mathbf{A}^1_V \to V$$

Applying Lemma 48.17.3 exactly n times we see that $(\mathbf{A}^n_V \to V)^!(E|_V)$ corresponds to $K \otimes^{\mathbf{L}}_R P[n]$. By Lemmas 48.9.5 and 48.17.4 the last step corresponds to applying $R\,\mathrm{Hom}(A, -)$.

0AU1 **Lemma 48.17.6.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Then $f^!$ maps $D^+_{Coh}(\mathcal{O}_Y)$ into $D^+_{Coh}(\mathcal{O}_X)$.*

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. The lemma follows from By Lemmas 48.17.3 and 48.9.6 and Dualizing Complexes, Lemma 47.15.10 and induction. □

0AA3 **Lemma 48.17.7.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If K is a dualizing complex for Y, then $f^!K$ is a dualizing complex for X.*

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. By Lemma 48.17.3 and Dualizing Complexes, Lemma 47.15.10 and induction we see that the $p^!K$ is a dualizing complex on $\mathbf{A}^n_Y$ where $p : \mathbf{A}^n_Y \to Y$ is the projection. Similarly, by Dualizing Complexes, Lemma 47.15.9 and Lemmas 48.9.5 and 48.17.4 we see that $i^!$ transforms dualizing complexes into dualizing complexes. □

0AU2 **Lemma 48.17.8.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let K be a dualizing complex on Y. Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{Coh}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^!K)$ for $E \in D_{Coh}(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$f^!M \longrightarrow D_X(Lf^*D_Y(M))$$

for $M \in D^+_{Coh}(\mathcal{O}_Y)$.

Proof. Choose compactification $j : X \subset \overline{X}$ of X over Y (More on Flatness, Theorem 38.33.8 and Lemma 38.32.2). Let a be the right adjoint of Lemma 48.3.1 for $\overline{X} \to Y$. Set $D_{\overline{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(E, a(K))$ for $E \in D_{Coh}(\mathcal{O}_{\overline{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \longrightarrow D_{\overline{X}}(L\overline{f}^*D_Y(M))$$

for $M \in D_{Coh}(\mathcal{O}_Y)$. For $F \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned}
\mathrm{Hom}_{\overline{X}}(F, D_{\overline{X}}(L\overline{f}^*D_Y(M))) &= \mathrm{Hom}_{\overline{X}}(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M), a(K)) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} L\overline{f}^*D_Y(M)), K) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} D_Y(M), K) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F), D_Y(D_Y(M))) \\
&= \mathrm{Hom}_Y(R\overline{f}_*(F), M) \\
&= \mathrm{Hom}_{\overline{X}}(F, a(M))
\end{aligned}$$

The first equality by Cohomology, Lemma 20.42.2. The second by definition of a. The third by Derived Categories of Schemes, Lemma 36.22.1. The fourth equality by Cohomology, Lemma 20.42.2 and the definition of D_Y. The fifth equality by

Lemma 48.2.5. The final equality by definition of a. Hence we see that $a(M) = D_{\overline{X}}(L\overline{f}^* D_Y(M))$ by Yoneda's lemma. □

0B6U **Lemma 48.17.9.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume f is perfect (e.g., flat). Then*

(a) *$f^!\mathcal{O}_Y$ is in $D^b_{Coh}(\mathcal{O}_X)$,*
(b) *$f^!\mathcal{O}_Y$ has finite tor dimension in $D(f^{-1}\mathcal{O}_Y)$,*
(c) *$\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism,*
(d) *$f^!$ maps $D^b_{Coh}(\mathcal{O}_Y)$ into $D^b_{Coh}(\mathcal{O}_X)$,*
(e) *the map $\mu_{f,K} : Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y \to f^!K$ of Lemma 48.16.5 is an isomorphism for all $K \in D^+_{QCoh}(\mathcal{O}_Y)$.*

Proof. (A flat morphism of finite presentation is perfect, see More on Morphisms, Lemma 37.61.5.) Assertions (a), (b), and (c) are local on X. Thus we may assume X and Y are affine. Then Remark 48.17.5 turns (a), (b), and (c) into (1)(a), (1)(b), and (1)(c) of Dualizing Complexes, Lemma 47.25.2. (Use Derived Categories of Schemes, Lemmas 36.10.5, 36.10.1, 36.10.3 36.10.8 to match the assertions.)

To prove (d) and (e) we begin with a series of preliminary remarks.

(1) We already know that $f^!$ sends $D^+_{Coh}(\mathcal{O}_Y)$ into $D^+_{Coh}(\mathcal{O}_X)$, see Lemma 48.17.6.
(2) If f is an open immersion, then $f^! = f^*$ and (d) and (e) are true because we can take $\overline{X} = Y$ in the construction of $f^!$ and μ_f, see Lemma 48.17.1.
(3) If f is a perfect proper morphism, then (e) is true by Lemma 48.13.3.
(4) If there exists an open covering $X = \bigcup U_i$ and (d) is true for $U_i \to Y$, then (d) is true for $X \to Y$. Same for (e). This holds because the construction of $f^!$ and μ_f commutes with passing to open subschemes.
(5) If $g : Y \to Z$ is a second perfect morphism in FTS_S and (e) holds for f and g, then $f^!g^!\mathcal{O}_Z = Lf^*g^!\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^!\mathcal{O}_Y$ and (e) holds for $g \circ f$ by the commutative diagram of Lemma 48.16.5.
(6) If (d) and (e) hold for both f and g, then (d) and (e) hold for $g \circ f$. Namely, then $f^!g^!\mathcal{O}_Z$ is bounded above (by the previous point) and $L(g \circ f)^*$ has finite cohomological dimension and (d) follows from (e) which we saw above.

From these points we see it suffices to prove (d) and (e) in case X is affine. Choose an immersion $X \to \mathbf{A}^n_Y$ (Morphisms, Lemma 29.39.2) which we factor as $X \to U \to \mathbf{A}^n_Y \to Y$ where $X \to U$ is a closed immersion and $U \subset \mathbf{A}^n_Y$ is open. Note that $X \to U$ is a perfect closed immersion by More on Morphisms, Lemma 37.61.8. Thus it suffices to prove the lemma for a perfect closed immersion and for the projection $\mathbf{A}^n_Y \to Y$.

Let $f : X \to Y$ be a perfect closed immersion. We already know (d) holds. Let $K \in D^b_{Coh}(\mathcal{O}_Y)$. Then $f^!K = R\mathcal{H}om(\mathcal{O}_X, K)$ (Lemma 48.17.4) and $f_*f^!K = R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$. Since f is perfect, the complex $f_*\mathcal{O}_X$ is perfect and hence $R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is bounded above. This proves that (d) holds. Some details omitted.

Let $f : \mathbf{A}^n_Y \to Y$ be the projection. Then (d) holds by repeated application of Lemma 48.17.3. Finally, (e) is true because it holds for $\mathbf{P}^n_Y \to Y$ (flat and proper) and because $\mathbf{A}^n_Y \subset \mathbf{P}^n_Y$ is an open. □

0E9T **Lemma 48.17.10.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is flat, then $f^!\mathcal{O}_Y$ is a Y-perfect object of $D(\mathcal{O}_X)$ and $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism.*

Proof. Both assertions are local on X. Thus we may assume X and Y are affine. Then Remark 48.17.5 turns the lemma into an algebra lemma, namely Dualizing Complexes, Lemma 47.25.2. (Use Derived Categories of Schemes, Lemma 36.35.3 to match the languages.) □

0B6V **Lemma 48.17.11.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume $f : X \to Y$ is a local complete intersection morphism. Then*

(1) *$f^!\mathcal{O}_Y$ is an invertible object of $D(\mathcal{O}_X)$, and*
(2) *$f^!$ maps perfect complexes to perfect complexes.*

Proof. Recall that a local complete intersection morphism is perfect, see More on Morphisms, Lemma 37.62.4. By Lemma 48.17.9 it suffices to show that $f^!\mathcal{O}_Y$ is an invertible object in $D(\mathcal{O}_X)$. This question is local on X and Y. Hence we may assume that $X \to Y$ factors as $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is a Koszul regular immersion. See More on Morphisms, Section 37.62. The result holds for $\mathbf{A}^n_Y \to Y$ by Lemma 48.17.3. Thus it suffices to prove the lemma when f is a Koszul regular immersion. Working locally once again we reduce to the case $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, where $A = B/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r \in B$ (use that for Noetherian local rings the notion of Koszul regular and regular are the same, see More on Algebra, Lemma 15.30.7). Thus $X \to Y$ is a composition

$$X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = Y$$

where each arrow is the inclusion of an effective Cartier divisor. In this way we reduce to the case of an inclusion of an effective Cartier divisor $i : D \to X$. In this case $i^!\mathcal{O}_X = \mathcal{N}[1]$ by Lemma 48.14.1 and the proof is complete. □

48.18. Base change for upper shriek

0BZX In Situation 48.16.1 let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram in FTS_S such that X and Y' are Tor independent over Y. Our setup is currently not sufficient to construct a base change map $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ in this generality. The reason is that in general it will not be possible to choose a compactification $j : X \to \overline{X}$ over Y such that $\overline{X}$ and Y' are tor independent over Y and hence our construction of the base change map in Section 48.5 does not apply[6].

A partial remedy will be found in Section 48.28. Namely, if the morphism f is flat, then there is a good notion of a relative dualizing complex and using Lemmas

[6]The reader who is well versed with derived algebraic geometry will realize this is not a "real" problem. Namely, taking $\overline{X}'$ to be the derived fibre product of $\overline{X}$ and Y' over Y, one can argue exactly as in the proof of Lemma 48.18.1 to define this map. After all, the Tor independence of X and Y' guarantees that X' will be an open subscheme of the derived scheme $\overline{X}'$.

48.28.9 48.28.6, and 48.17.9 we may construct a canonical base change isomorphism. If we ever need to use this, we will add precise statements and proofs later in this chapter.

0E9U **Lemma 48.18.1.** *In Situation 48.16.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of FTS_S with g flat. Then there is an isomorphism $L(g')^* \circ f^! \to (f')^! \circ Lg^*$ on $D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. Namely, because g is flat, for every choice of compactification $j : X \to \overline{X}$ of X over Y the scheme $\overline{X}$ is Tor independent of Y'. Denote $j' : X' \to \overline{X}'$ the base change of j and $\overline{g}' : \overline{X}' \to \overline{X}$ the projection. We define the base change map as the composition

$$L(g')^* \circ f^! = L(g')^* \circ j^* \circ a = (j')^* \circ L(\overline{g}')^* \circ a \longrightarrow (j')^* \circ a' \circ Lg^* = (f')^! \circ Lg^*$$

where the middle arrow is the base change map (48.5.0.1) and a and a' are the right adjoints to pushforward of Lemma 48.3.1 for $\overline{X} \to Y$ and $\overline{X}' \to Y'$. This construction is independent of the choice of compactification (we will formulate a precise lemma and prove it, if we ever need this result).

To finish the proof it suffices to show that the base change map $L(g')^* \circ a \to a' \circ Lg^*$ is an isomorphism on $D^+_{QCoh}(\mathcal{O}_Y)$. By Lemma 48.4.4 formation of a and a' commutes with restriction to affine opens of Y and Y'. Thus by Remark 48.6.1 we may assume that Y and Y' are affine. Thus the result by Lemma 48.6.2. □

0FWI **Lemma 48.18.2.** *In Situation 48.16.1 let $f : X \to Y$ be an étale morphism of FTS_S. Then $f^! \cong f^*$ as functors on $D^+_{QCoh}(\mathcal{O}_Y)$.*

Proof. We are going to use that an étale morphism is flat, syntomic, and a local complete intersection morphism (Morphisms, Lemma 29.36.10 and 29.36.12 and More on Morphisms, Lemma 37.62.8). By Lemma 48.17.9 it suffices to show $f^!\mathcal{O}_Y = \mathcal{O}_X$. By Lemma 48.17.11 we know that $f^!\mathcal{O}_Y$ is an invertible module. Consider the commutative diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

and the diagonal $\Delta : X \to X \times_Y X$. Since Δ is an open immersion (by Morphisms, Lemmas 29.35.13 and 29.36.5), by Lemma 48.17.1 we have $\Delta^! = \Delta^*$. By Lemma 48.16.3 we have $\Delta^! \circ p_1^! \circ f^! = f^!$. By Lemma 48.18.1 applied to the diagram we have $p_1^!\mathcal{O}_X = p_2^* f^!\mathcal{O}_Y$. Hence we conclude

$$f^!\mathcal{O}_Y = \Delta^! p_1^! f^!\mathcal{O}_Y = \Delta^*(p_1^* f^!\mathcal{O}_Y \otimes p_1^!\mathcal{O}_X) = \Delta^*(p_2^* f^!\mathcal{O}_Y \otimes p_1^* f^!\mathcal{O}_Y) = (f^!\mathcal{O}_Y)^{\otimes 2}$$

where in the second step we have used Lemma 48.17.9 once more. Thus $f^!\mathcal{O}_Y = \mathcal{O}_X$ as desired. □

In the rest of this section, we formulate some easy to prove results which would be consequences of a good theory of the base change map.

0BZY **Lemma 48.18.3** (Makeshift base change)**.** *In Situation 48.16.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of FTS_S. Let $E \in D^+_{QCoh}(\mathcal{O}_Y)$ be an object such that Lg^*E is in $D^+(\mathcal{O}_Y)$. If f is flat, then $L(g')^*f^!E$ and $(f')^!Lg^*E$ restrict to isomorphic objects of $D(\mathcal{O}_{U'})$ for $U' \subset X'$ affine open mapping into affine opens of Y, Y', and X.

Proof. By our assumptions we immediately reduce to the case where X, Y, Y', and X' are affine. Say $Y = \mathrm{Spec}(R)$, $Y' = \mathrm{Spec}(R')$, $X = \mathrm{Spec}(A)$, and $X' = \mathrm{Spec}(A')$. Then $A' = A \otimes_R R'$. Let E correspond to $K \in D^+(R)$. Denoting $\varphi : R \to A$ and $\varphi' : R' \to A'$ the given maps we see from Remark 48.17.5 that $L(g')^*f^!E$ and $(f')^!Lg^*E$ correspond to $\varphi^!(K) \otimes^{\mathbf{L}}_A A'$ and $(\varphi')^!(K \otimes^{\mathbf{L}}_R R')$ where $\varphi^!$ and $(\varphi')^!$ are the functors from Dualizing Complexes, Section 47.24. The result follows from Dualizing Complexes, Lemma 47.24.6. □

0BZZ **Lemma 48.18.4.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume f is flat. Set $\omega^\bullet_{X/Y} = f^!\mathcal{O}_Y$ in $D^b_{Coh}(X)$. Let $y \in Y$ and $h : X_y \to X$ the projection. Then $Lh^*\omega^\bullet_{X/Y}$ is a dualizing complex on X_y.*

Proof. The complex $\omega^\bullet_{X/Y}$ is in D^b_{Coh} by Lemma 48.17.9. Being a dualizing complex is a local property. Hence by Lemma 48.18.3 it suffices to show that $(X_y \to y)^!\mathcal{O}_y$ is a dualizing complex on X_y. This follows from Lemma 48.17.7. □

48.19. A duality theory

0AU3 In this section we spell out what kind of a duality theory our very general results above give for finite type separated schemes over a fixed Noetherian base scheme.

Recall that a dualizing complex on a Noetherian scheme X, is an object of $D(\mathcal{O}_X)$ which affine locally gives a dualizing complex for the corresponding rings, see Definition 48.2.2.

Given a Noetherian scheme S denote FTS_S the category of schemes which are of finite type and separated over S. Then:

(1) the functors $f^!$ turn D^+_{QCoh} into a pseudo functor on FTS_S,
(2) if $f : X \to Y$ is a proper morphism in FTS_S, then $f^!$ is the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ to $D^+_{QCoh}(\mathcal{O}_Y)$ and there is a canonical isomorphism
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, f^!M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, M)$$
for all $K \in D^-_{Coh}(\mathcal{O}_X)$ and $M \in D^+_{QCoh}(\mathcal{O}_Y)$,
(3) if an object X of FTS_S has a dualizing complex $\omega^\bullet_X$, then the functor $D_X = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega^\bullet_X)$ defines an involution of $D_{Coh}(\mathcal{O}_X)$ switching $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and fixing $D^b_{Coh}(\mathcal{O}_X)$,

(4) if $f : X \to Y$ is a morphism of FTS_S and $\omega_Y^\bullet$ is a dualizing complex on Y, then
   (a) $\omega_X^\bullet = f^!\omega_Y^\bullet$ is a dualizing complex for X,
   (b) $f^!M = D_X(Lf^*D_Y(M))$ canonically for $M \in D^+_{Coh}(\mathcal{O}_Y)$, and
   (c) if in addition f is proper then
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, \omega_Y^\bullet)$$
   for K in $D^-_{Coh}(\mathcal{O}_X)$,
(5) if $f : X \to Y$ is a closed immersion in FTS_S, then $f^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$,
(6) if $f : Y \to X$ is a finite morphism in FTS_S, then $f_*f^!(-) = R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -)$,
(7) if $f : X \to Y$ is the inclusion of an effective Cartier divisor into an object of FTS_S, then $f^!(-) = Lf^*(-) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(X)[-1]$,
(8) if $f : X \to Y$ is a Koszul regular immersion of codimension c into an object of FTS_S, then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \wedge^c\mathcal{N}[-c]$, and
(9) if $f : X \to Y$ is a smooth proper morphism of relative dimension d in FTS_S, then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}[d]$.

This follows from Lemmas 48.2.5, 48.3.6, 48.9.7, 48.11.4, 48.14.2, 48.15.6, 48.15.7, 48.16.3, 48.16.4, 48.17.4, 48.17.7, 48.17.8, and 48.17.9 and Example 48.3.9. We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 13.38.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

48.20. Glueing dualizing complexes

0AU5 We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over S given a pair $(S, \omega_S^\bullet)$ as in Situation 48.20.1. This is similar to [**Har66**, Remark on page 310].

0AU4 **Situation 48.20.1.** Here S is a Noetherian scheme and $\omega_S^\bullet$ is a dualizing complex.

In Situation 48.20.1 let X be a scheme of finite type over S. Let $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$ be a finite open covering of X by objects of FTS_S, see Situation 48.16.1. All this means is that the morphisms $U_i \to S$ are separated (as they are already of finite type). Every affine scheme of finite type over S is an object of FTS_S by Schemes, Lemma 26.21.13 hence such open coverings certainly exist. Then for each $i, j, k \in \{1, \ldots, n\}$ the morphisms $p_i : U_i \to S$, $p_{ij} : U_i \cap U_j \to S$, and $p_{ijk} : U_i \cap U_j \cap U_k \to S$ are separated and each of these schemes is an object of FTS_S. From such an open covering we obtain

(1) $\omega_i^\bullet = p_i^!\omega_S^\bullet$ a dualizing complex on U_i, see Section 48.19,
(2) for each i, j a canonical isomorphism $\varphi_{ij} : \omega_i^\bullet|_{U_i \cap U_j} \to \omega_j^\bullet|_{U_i \cap U_j}$, and
0AU6 (3) for each i, j, k we have
$$\varphi_{ik}|_{U_i \cap U_j \cap U_k} = \varphi_{jk}|_{U_i \cap U_j \cap U_k} \circ \varphi_{ij}|_{U_i \cap U_j \cap U_k}$$
   in $D(\mathcal{O}_{U_i \cap U_j \cap U_k})$.

Here, in (2) we use that $(U_i \cap U_j \to U_i)^!$ is given by restriction (Lemma 48.17.1) and that we have canonical isomorphisms
$$(U_i \cap U_j \to U_i)^! \circ p_i^! = p_{ij}^! = (U_i \cap U_j \to U_j)^! \circ p_j^!$$

by Lemma 48.16.3 and to get (3) we use that the upper shriek functors form a pseudo functor by Lemma 48.16.4.

In the situation just described a *dualizing complex normalized relative to* $\omega_S^\bullet$ *and* $\mathcal{U}$ is a pair (K, α_i) where $K \in D(\mathcal{O}_X)$ and $\alpha_i : K|_{U_i} \to \omega_i^\bullet$ are isomorphisms such that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. Since being a dualizing complex on a scheme is a local property we see that dualizing complexes normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ are indeed dualizing complexes.

0AU7 **Lemma 48.20.2.** *In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then it is unique up to unique isomorphism.*

Proof. If (K, α_i) and (K', α_i') are two, then we consider $L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$. By Lemma 48.2.6 and its proof, this is an invertible object of $D(\mathcal{O}_X)$. Using α_i and α_i' we obtain an isomorphism

$$\alpha_i^t \otimes \alpha_i' : L|_{U_i} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_i^\bullet, \omega_i^\bullet) = \mathcal{O}_{U_i}[0]$$

This already implies that $L = H^0(L)[0]$ in $D(\mathcal{O}_X)$. Moreover, $H^0(L)$ is an invertible sheaf with given trivializations on the opens U_i of X. Finally, the condition that $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$ and $\alpha_j'|_{U_i \cap U_j} \circ (\alpha_i')^{-1}|_{U_i \cap U_j}$ both give φ_{ij} implies that the transition maps are 1 and we get an isomorphism $H^0(L) = \mathcal{O}_X$. □

0AU8 **Lemma 48.20.3.** *In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$, $\mathcal{V}$ be two finite open coverings of X by schemes separated over S. If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ and these complexes are canonically isomorphic.*

Proof. It suffices to prove this when $\mathcal{U}$ is given by the opens $U_1, \ldots, U_n$ and $\mathcal{V}$ by the opens $U_1, \ldots, U_{n+m}$. In fact, we may and do even assume $m = 1$. To go from a dualizing complex (K, α_i) normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ to a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ is achieved by forgetting about α_i for $i = n + 1$. Conversely, let (K, α_i) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. To finish the proof we need to construct a map $\alpha_{n+1} : K|_{U_{n+1}} \to \omega_{n+1}^\bullet$ satisfying the desired conditions. To do this we observe that $U_{n+1} = \bigcup U_i \cap U_{n+1}$ is an open covering. It is clear that $(K|_{U_{n+1}}, \alpha_i|_{U_i \cap U_{n+1}})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. On the other hand, by condition (3) the pair $(\omega_{n+1}^\bullet|_{U_{n+1}}, \varphi_{n+1i})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. By Lemma 48.20.2 we obtain a unique isomorphism

$$\alpha_{n+1} : K|_{U_{n+1}} \longrightarrow \omega_{n+1}^\bullet$$

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. □

0AU9 **Lemma 48.20.4.** *In Situation 48.20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S. Then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$.*

Proof. Say $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$. We prove the lemma by induction on n. The base case $n = 1$ is immediate. Assume $n > 1$. Set $X' = U_1 \cup \ldots \cup U_{n-1}$ and

let $(K', \{\alpha'_i\}_{i=1,\ldots,n-1})$ be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}' : X' = \bigcup_{i=1,\ldots,n-1} U_i$. It is clear that $(K'|_{X'\cap U_n}, \alpha'_i|_{U_i\cap U_n})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. On the other hand, by condition (3) the pair $(\omega_n^\bullet|_{X'\cap U_n}, \varphi_{ni})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1,\ldots,n-1} U_i \cap U_n$. By Lemma 48.20.2 we obtain a unique isomorphism

$$\epsilon : K'|_{X'\cap U_n} \longrightarrow \omega_i^\bullet|_{X'\cap U_n}$$

compatible with the given local isomorphisms. By Cohomology, Lemma 20.45.1 we obtain $K \in D(\mathcal{O}_X)$ together with isomorphisms $\beta : K|_{X'} \to K'$ and $\gamma : K|_{U_n} \to \omega_n^\bullet$ such that $\epsilon = \gamma|_{X'\cap U_n} \circ \beta|^{-1}_{X'\cap U_n}$. Then we define

$$\alpha_i = \alpha'_i \circ \beta|_{U_i}, i = 1, \ldots, n-1, \text{ and } \alpha_n = \gamma$$

We still need to verify that φ_{ij} is given by $\alpha_j|_{U_i\cap U_j} \circ \alpha_i^{-1}|_{U_i\cap U_j}$. For $i, j \leq n-1$ this follows from the corresponding condition for α'_i. For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i\cap U_n} \circ \alpha_i^{-1}|_{U_i\cap U_n} = \gamma|_{U_i\cap U_n} \circ \beta^{-1}|_{U_i\cap U_n} \circ (\alpha'_i)^{-1}|_{U_i\cap U_n} = \epsilon|_{U_i\cap U_n} \circ (\alpha'_i)^{-1}|_{U_i\cap U_n}$$

This is equal to α_{in} exactly because ϵ is the unique map compatible with the maps α'_i and α_{ni}. □

Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. The upshot of the lemmas above is that given any scheme X of finite type over S, there is a pair (K, α_U) given up to unique isomorphism, consisting of an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\alpha_U : K|_U \to \omega_U^\bullet$ for every open subscheme $U \subset X$ which is separated over S. Here $\omega_U^\bullet = (U \to S)^!\omega_S^\bullet$ is a dualizing complex on U, see Section 48.19. Moreover, if $\mathcal{U} : X = \bigcup U_i$ is a finite open covering by opens which are separated over S, then (K, α_{U_i}) is a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. Namely, uniqueness up to unique isomorphism by Lemma 48.20.2, existence for one open covering by Lemma 48.20.4, and the fact that K then works for all open coverings is Lemma 48.20.3.

0AUA **Definition 48.20.5.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex on S. Let X be a scheme of finite type over S. The complex K constructed above is called the *dualizing complex normalized relative to* $\omega_S^\bullet$ and is denoted $\omega_X^\bullet$.

As the terminology suggest, a dualizing complex normalized relative to $\omega_S^\bullet$ is not just an object of the derived category of X but comes equipped with the local isomorphisms described above. This does not conflict with setting $\omega_X^\bullet = p^!\omega_S^\bullet$ where $p : X \to S$ is the structure morphism if X is separated over S. More generally we have the following sanity check.

0AUB **Lemma 48.20.6.** *Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $f : X \to Y$ be a morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then $\omega_X^\bullet$ is a dualizing complex normalized relative to $\omega_Y^\bullet$.*

Proof. This is just a matter of bookkeeping. Choose a finite affine open covering $\mathcal{V} : Y = \bigcup V_j$. For each j choose a finite affine open covering $f^{-1}(V_j) = U_{ji}$. Set $\mathcal{U} : X = \bigcup U_{ji}$. The schemes V_j and U_{ji} are separated over S, hence we have the upper shriek functors for $q_j : V_j \to S$, $p_{ji} : U_{ji} \to S$ and $f_{ji} : U_{ji} \to V_j$ and $f'_{ji} : U_{ji} \to Y$. Let (L, β_j) be a dualizing complex normalized relative to $\omega_S^\bullet$ and

$\mathcal{V}$. Let (K, γ_{ji}) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. (In other words, $L = \omega_Y^\bullet$ and $K = \omega_X^\bullet$.) We can define

$$\alpha_{ji} : K|_{U_{ji}} \xrightarrow{\gamma_{ji}} p_{ji}^! \omega_S^\bullet = f_{ji}^! q_j^! \omega_S^\bullet \xrightarrow{f_{ji}^! \beta_j^{-1}} f_{ji}^! (L|_{V_j}) = (f'_{ji})^!(L)$$

To finish the proof we have to show that $\alpha_{ji}|_{U_{ji} \cap U_{j'i'}} \circ \alpha_{j'i'}^{-1}|_{U_{ji} \cap U_{j'i'}}$ is the canonical isomorphism $(f'_{ji})^!(L)|_{U_{ji} \cap U_{j'i'}} \to (f'_{j'i'})^!(L)|_{U_{ji} \cap U_{j'i'}}$. This is formal and we omit the details. □

0AUC **Lemma 48.20.7.** *Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $j : X \to Y$ be an open immersion of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then there is a canonical isomorphism $\omega_X^\bullet = \omega_Y^\bullet|_X$.*

Proof. Immediate from the construction of normalized dualizing complexes given just above Definition 48.20.5. □

0AUD **Lemma 48.20.8.** *Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. Let $f : X \to Y$ be a proper morphism of schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Let a be the right adjoint of Lemma 48.3.1 for f. Then there is a canonical isomorphism $a(\omega_Y^\bullet) = \omega_X^\bullet$.*

Proof. Let $p : X \to S$ and $q : Y \to S$ be the structure morphisms. If X and Y are separated over S, then this follows from the fact that $\omega_X^\bullet = p^!\omega_S^\bullet$, $\omega_Y^\bullet = q^!\omega_S^\bullet$, $f^! = a$, and $f^! \circ q^! = p^!$ (Lemma 48.16.3). In the general case we first use Lemma 48.20.6 to reduce to the case $Y = S$. In this case X and Y are separated over S and we've just seen the result. □

Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. For a scheme X of finite type over S denote $\omega_X^\bullet$ the dualizing complex for X normalized relative to $\omega_S^\bullet$. Define $D_X(-) = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ as in Lemma 48.2.5. Let $f : X \to Y$ be a morphism of finite type schemes over S. Define

$$f_{new}^! = D_X \circ Lf^* \circ D_Y : D_{Coh}^+(\mathcal{O}_Y) \to D_{Coh}^+(\mathcal{O}_X)$$

If $f : X \to Y$ and $g : Y \to Z$ are composable morphisms between schemes of finite type over S, define

$$\begin{aligned} (g \circ f)_{new}^! &= D_X \circ L(g \circ f)^* \circ D_Z \\ &= D_X \circ Lf^* \circ Lg^* \circ D_Z \\ &\to D_X \circ Lf^* \circ D_Y \circ D_Y \circ Lg^* \circ D_Z \\ &= f_{new}^! \circ g_{new}^! \end{aligned}$$

where the arrow is defined in Lemma 48.2.5. We collect the results together in the following lemma.

0AUE **Lemma 48.20.9.** *Let $(S, \omega_S^\bullet)$ be as in Situation 48.20.1. With $f_{new}^!$ and $\omega_X^\bullet$ defined for all (morphisms of) schemes of finite type over S as above:*

1. *the functors $f_{new}^!$ and the arrows $(g \circ f)_{new}^! \to f_{new}^! \circ g_{new}^!$ turn D_{Coh}^+ into a pseudo functor from the category of schemes of finite type over S into the 2-category of categories,*
2. *$\omega_X^\bullet = (X \to S)_{new}^! \omega_S^\bullet$,*
3. *the functor D_X defines an involution of $D_{Coh}(\mathcal{O}_X)$ switching $D_{Coh}^+(\mathcal{O}_X)$ and $D_{Coh}^-(\mathcal{O}_X)$ and fixing $D_{Coh}^b(\mathcal{O}_X)$,*

(4) $\omega_X^\bullet = f_{new}^! \omega_Y^\bullet$ *for* $f : X \to Y$ *a morphism of finite type schemes over* S,
(5) $f_{new}^! M = D_X(Lf^* D_Y(M))$ *for* $M \in D^+_{Coh}(\mathcal{O}_Y)$, *and*
(6) *if in addition* f *is proper, then* $f_{new}^!$ *is isomorphic to the restriction of the right adjoint of* $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ *to* $D^+_{Coh}(\mathcal{O}_Y)$ *and there is a canonical isomorphism*

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f_{new}^! M) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, M)$$

for $K \in D^-_{Coh}(\mathcal{O}_X)$ *and* $M \in D^+_{Coh}(\mathcal{O}_Y)$, *and*

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, \omega_Y^\bullet)$$

for $K \in D^-_{Coh}(\mathcal{O}_X)$ *and*

If X *is separated over* S, *then* $\omega_X^\bullet$ *is canonically isomorphic to* $(X \to S)^! \omega_S^\bullet$ *and if* f *is a morphism between schemes separated over* S, *then there is a canonical isomorphism*[7] $f_{new}^! K = f^! K$ *for* K *in* D^+_{Coh}.

Proof. Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be morphisms of schemes of finite type over S. We have to show that

$$\begin{array}{ccc} (h \circ g \circ f)^!_{new} & \longrightarrow & f^!_{new} \circ (h \circ g)^!_{new} \\ \downarrow & & \downarrow \\ (g \circ f)^!_{new} \circ h^!_{new} & \longrightarrow & f^!_{new} \circ g^!_{new} \circ h^!_{new} \end{array}$$

is commutative. Let $\eta_Y : \mathrm{id} \to D_Y^2$ and $\eta_Z : \mathrm{id} \to D_Z^2$ be the canonical isomorphisms of Lemma 48.2.5. Then, using Categories, Lemma 4.28.2, a computation (omitted) shows that both arrows $(h \circ g \circ f)^!_{new} \to f^!_{new} \circ g^!_{new} \circ h^!_{new}$ are given by

$$1 \star \eta_Y \star 1 \star \eta_Z \star 1 : D_X \circ Lf^* \circ Lg^* \circ Lh^* \circ D_T \longrightarrow D_X \circ Lf^* \circ D_Y^2 \circ Lg^* \circ D_Z^2 \circ Lh^* \circ D_T$$

This proves (1). Part (2) is immediate from the definition of $(X \to S)^!_{new}$ and the fact that $D_S(\omega_S^\bullet) = \mathcal{O}_S$. Part (3) is Lemma 48.2.5. Part (4) follows by the same argument as part (2). Part (5) is the definition of $f^!_{new}$.

Proof of (6). Let a be the right adjoint of Lemma 48.3.1 for the proper morphism $f : X \to Y$ of schemes of finite type over S. The issue is that we do not know X or Y is separated over S (and in general this won't be true) hence we cannot immediately apply Lemma 48.17.8 to f over S. To get around this we use the canonical identification $\omega_X^\bullet = a(\omega_Y^\bullet)$ of Lemma 48.20.8. Hence $f^!_{new}$ is the restriction of a to $D^+_{Coh}(\mathcal{O}_Y)$ by Lemma 48.17.8 applied to $f : X \to Y$ over the base scheme Y! The displayed equalities hold by Example 48.3.9.

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma 48.17.8. □

0BV3 **Remark 48.20.10.** Let S be a Noetherian scheme which has a dualizing complex. Let $f : X \to Y$ be a morphism of schemes of finite type over S. Then the functor

$$f^!_{new} : D^+_{Coh}(\mathcal{O}_Y) \to D^+_{Coh}(\mathcal{O}_X)$$

is independent of the choice of the dualizing complex $\omega_S^\bullet$ up to canonical isomorphism. We sketch the proof. Any second dualizing complex is of the form $\omega_S^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{L}$ where $\mathcal{L}$ is an invertible object of $D(\mathcal{O}_S)$, see Lemma 48.2.6. For any separated

[7] We haven't checked that these are compatible with the isomorphisms $(g \circ f)^! \to f^! \circ g^!$ and $(g \circ f)^!_{new} \to f^!_{new} \circ g^!_{new}$. We will do this here if we need this later.

morphism $p : U \to S$ of finite type we have $p^!(\omega_S^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{L}) = p^!(\omega_S^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_U} Lp^*\mathcal{L}$ by Lemma 48.8.1. Hence, if $\omega_X^\bullet$ and $\omega_Y^\bullet$ are the dualizing complexes normalized relative to $\omega_S^\bullet$ we see that $\omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*\mathcal{L}$ and $\omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lb^*\mathcal{L}$ are the dualizing complexes normalized relative to $\omega_S^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{L}$ (where $a : X \to S$ and $b : Y \to S$ are the structure morphisms). Then the result follows as

$$\begin{aligned}
& R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lb^*\mathcal{L}), \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R(\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lb^*\mathcal{L}), \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*\mathcal{L}, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*\mathcal{L}) \\
& = R\mathcal{H}om_{\mathcal{O}_X}(Lf^*R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet), \omega_X^\bullet)
\end{aligned}$$

for $K \in D^+_{Coh}(\mathcal{O}_Y)$. The last equality because $La^*\mathcal{L}$ is invertible in $D(\mathcal{O}_X)$.

0B6X **Example 48.20.11.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a proper morphism of finite type schemes over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. In this situation we have $a(\omega_Y^\bullet) = \omega_X^\bullet$ (Lemma 48.20.8) and hence the trace map (Section 48.7) is a canonical arrow

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet \longrightarrow \omega_Y^\bullet$$

which produces the isomorphisms (Lemma 48.20.9)

$$\mathrm{Hom}_X(L, \omega_X^\bullet) = \mathrm{Hom}_Y(Rf_*L, \omega_Y^\bullet)$$

and

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, \omega_Y^\bullet)$$

for L in $D_{QCoh}(\mathcal{O}_X)$.

0AX4 **Remark 48.20.12.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to Y$ be a finite morphism between schemes of finite type over S. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then we have

$$f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet)$$

in $D^+_{QCoh}(f_*\mathcal{O}_X)$ by Lemmas 48.11.4 and 48.20.8 and the trace map of Example 48.20.11 is the map

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet = f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet) \longrightarrow \omega_Y^\bullet$$

which often goes under the name "evaluation at 1".

0B6W **Remark 48.20.13.** Let $f : X \to Y$ be a flat proper morphism of finite type schemes over a pair $(S, \omega_S^\bullet)$ as in Situation 48.20.1. The relative dualizing complex (Remark 48.12.5) is $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$. By Lemma 48.20.8 we have the first canonical isomorphism in

$$\omega_X^\bullet = a(\omega_Y^\bullet) = Lf^*\omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}^\bullet$$

in $D(\mathcal{O}_X)$. The second canonical isomorphism follows from the discussion in Remark 48.12.5.

48.21. Dimension functions

0BV4 We need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.

0AWL **Lemma 48.21.1.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let X be a scheme of finite type over S and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. If $x \in X$ is a closed point lying over a closed point s of S, then $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$ provided that $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$.*

Proof. We may replace X by an affine neighbourhood of x, hence we may and do assume that $f : X \to S$ is separated. Then $\omega_X^\bullet = f^!\omega_S^\bullet$. We have to show that $R\mathrm{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ is sitting in degree 0. Let $i_x : x \to X$ denote the inclusion morphism which is a closed immersion as x is a closed point. Hence $R\mathrm{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ represents $i_x^!\omega_X^\bullet$ by Lemma 48.17.4. Consider the commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{i_x} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f} \\ s & \xrightarrow{i_s} & S \end{array}$$

By Morphisms, Lemma 29.20.3 the extension $\kappa(x)/\kappa(s)$ is finite and hence π is a finite morphism. We conclude that

$$i_x^!\omega_X^\bullet = i_x^! f^!\omega_S^\bullet = \pi^! i_s^!\omega_S^\bullet$$

Thus if $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$, then $i_s^!\omega_S^\bullet = \kappa(s)[0]$ by the same reasoning as above. We have

$$R\pi_*(\pi^!(\kappa(s)[0])) = R\mathcal{H}om_{\mathcal{O}_s}(R\pi_*(\kappa(x)[0]), \kappa(s)[0]) = \widetilde{\mathrm{Hom}_{\kappa(s)}(\kappa(x), \kappa(s))}$$

The first equality by Example 48.3.9 applied with $L = \kappa(x)[0]$. The second equality holds because π_* is exact. Thus $\pi^!(\kappa(s)[0])$ is supported in degree 0 and we win. □

0AWM **Lemma 48.21.2.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \to S$ be of finite type and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. For all $x \in X$ we have*

$$\delta_X(x) - \delta_S(f(x)) = trdeg_{\kappa(f(x))}(\kappa(x))$$

where δ_S, resp. δ_X is the dimension function of $\omega_S^\bullet$, resp. $\omega_X^\bullet$, see Lemma 48.2.7.

Proof. We may replace X by an affine neighbourhood of x. Hence we may and do assume there is a compactification $X \subset \overline{X}$ over S. Then we may replace X by $\overline{X}$ and assume that X is proper over S. We may also assume X is connected by replacing X by the connected component of X containing x. Next, recall that both δ_X and the function $x \mapsto \delta_S(f(x)) + \mathrm{trdeg}_{\kappa(f(x))}(\kappa(x))$ are dimension functions on X, see Morphisms, Lemma 29.52.3 (and the fact that S is universally catenary by Lemma 48.2.7). By Topology, Lemma 5.20.3 we see that the difference is locally constant, hence constant as X is connected. Thus it suffices to prove equality in any point of X. By Properties, Lemma 28.5.9 the scheme X has a closed point x. Since $X \to S$ is proper the image s of x is closed in S. Thus we may apply Lemma 48.21.1 to conclude. □

0BV5 **Lemma 48.21.3.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$ with image $y \in Y$. Then*

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i.$$

Proof. Since the statement is local on X we may assume X and Y are affine schemes. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$. Then $f^!\mathcal{O}_Y$ corresponds to the relative dualizing complex $\omega^\bullet_{A/R}$ of Dualizing Complexes, Section 47.25 by Remark 48.17.5. Thus the lemma follows from Dualizing Complexes, Lemma 47.25.7. □

0BV6 **Lemma 48.21.4.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$ with image $y \in Y$. If f is flat, then*

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i \leq 0.$$

In fact, if all fibres of f have dimension $\leq d$, then $f^!\mathcal{O}_Y$ has tor-amplitude in $[-d, 0]$ as an object of $D(X, f^{-1}\mathcal{O}_Y)$.

Proof. Arguing exactly as in the proof of Lemma 48.21.3 this follows from Dualizing Complexes, Lemma 47.25.8. □

0E9V **Lemma 48.21.5.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$ with image $y \in Y$. Assume*

(1) *$\mathcal{O}_{Y,y}$ is Cohen-Macaulay, and*
(2) *$\mathrm{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) \leq r$ for any generic point ξ of an irreducible component of X containing x.*

Then

$$H^i(f^!\mathcal{O}_Y)_x \neq 0 \Rightarrow -r \leq i$$

and the stalk $H^{-r}(f^!\mathcal{O}_Y)_x$ is (S_2) as an $\mathcal{O}_{X,x}$-module.

Proof. After replacing X by an open neighbourhood of x, we may assume every irreducible component of X passes through x. Then arguing exactly as in the proof of Lemma 48.21.3 this follows from Dualizing Complexes, Lemma 47.25.9. □

0BV7 **Lemma 48.21.6.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is flat and quasi-finite, then*

$$f^!\mathcal{O}_Y = \omega_{X/Y}[0]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y.

Proof. Consequence of Lemma 48.21.4 and the fact that the cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 48.17.6. □

0BV8 **Lemma 48.21.7.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is Cohen-Macaulay (More on Morphisms, Definition 37.22.1), then*

$$f^!\mathcal{O}_Y = \omega_{X/Y}[d]$$

for some coherent $\mathcal{O}_X$-module $\omega_{X/Y}$ flat over Y where d is the locally constant function on X which gives the relative dimension of X over Y.

Proof. The relative dimension d is well defined and locally constant by Morphisms, Lemma 29.29.4. The cohomology sheaves of $f^!\mathcal{O}_Y$ are coherent by Lemma 48.17.6. We will get flatness of $\omega_{X/Y}$ from Lemma 48.21.4 if we can show the other cohomology sheaves of $f^!\mathcal{O}_Y$ are zero.

The question is local on X, hence we may assume X and Y are affine and the morphism has relative dimension d. If $d = 0$, then the result follows directly from Lemma 48.21.6. If $d > 0$, then we may assume there is a factorization

$$X \xrightarrow{g} \mathbf{A}^d_Y \xrightarrow{p} Y$$

with g quasi-finite and flat, see More on Morphisms, Lemma 37.22.8. Then $f^! = g^! \circ p^!$. By Lemma 48.17.3 we see that $p^!\mathcal{O}_Y \cong \mathcal{O}_{\mathbf{A}^d_Y}[-d]$. We conclude by the case $d = 0$. □

0BV9 **Remark 48.21.8.** Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. In this case Lemmas 48.21.3, 48.21.4, 48.21.6, and 48.21.7 are true for any morphism $f : X \to Y$ of finite type schemes over S but with $f^!$ replaced by $f^!_{new}$. This is clear because in each case the proof reduces immediately to the affine case and then $f^! = f^!_{new}$ by Lemma 48.20.9.

48.22. Dualizing modules

0AWH This section is a continuation of Dualizing Complexes, Section 47.19.

Let X be a Noetherian scheme and let $\omega_X^\bullet$ be a dualizing complex. Let $n \in \mathbf{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7). Sometimes $H^n(\omega_X^\bullet)$ is called a *dualizing module* or *dualizing sheaf* for X and then it is often denoted by ω_X. We will say "let ω_X be a dualizing module" to indicate the above.

Care has to be taken when using dualizing modules ω_X on Noetherian schemes X:

(1) the integer n may change when passing from X to an open U of X and then it won't be true that $\omega_X|_U = \omega_U$,
(2) the dualizing complex isn't unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with X of finite type over a base change S which is endowed with a fixed dualizing complex $\omega_S^\bullet$ and $\omega_X^\bullet$ will be the dualizing complex normalized relative to $\omega_S^\bullet$. The first problem will not occur if X is equidimensional, more precisely, if the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7) maps every generic point of X to the same integer.

0AWI **Example 48.22.1.** Say $S = \mathrm{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring, and $\omega_S^\bullet$ corresponds to a normalized dualizing complex $\omega_A^\bullet$. Then if $f : X \to S$ is proper over S and $\omega_X^\bullet = f^!\omega_S^\bullet$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

is a dualizing module and is often called the dualizing module of X (with S and $\omega_S^\bullet$ being understood). We will see that this has good properties.

0AWJ **Example 48.22.2.** Say X is an equidimensional scheme of finite type over a field k. Then it is customary to take $\omega_X^\bullet$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

as the dualizing module of X. If X is separated over k, then $\omega_X^\bullet = f^!\mathcal{O}_{\mathrm{Spec}(k)}$ where $f : X \to \mathrm{Spec}(k)$ is the structure morphism by Lemma 48.20.9. If X is proper over k, then this is a special case of Example 48.22.1.

0AWK **Lemma 48.22.3.** *Let X be a connected Noetherian scheme and let ω_X be a dualizing module on X. The support of ω_X is the union of the irreducible components of maximal dimension with respect to any dimension function and ω_X is a coherent $\mathcal{O}_X$-module having property (S_2).*

Proof. By our conventions discussed above there exists a dualizing complex $\omega_X^\bullet$ such that ω_X is the leftmost nonvanishing cohomology sheaf. Since X is connected, any two dimension functions differ by a constant (Topology, Lemma 5.20.3). Hence we may use the dimension function associated to $\omega_X^\bullet$ (Lemma 48.2.7). With these remarks in place, the lemma now follows from Dualizing Complexes, Lemma 47.17.5 and the definitions (in particular Cohomology of Schemes, Definition 30.11.1). □

0AWN **Lemma 48.22.4.** *Let X/A with $\omega_X^\bullet$ and ω_X be as in Example 48.22.1. Then*

(1) *$H^i(\omega_X^\bullet) \neq 0 \Rightarrow i \in \{-\dim(X), \ldots, 0\}$,*
(2) *the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,*
(3) *$Supp(\omega_X)$ is the union of the components of dimension $\dim(X)$, and*
(4) *ω_X has property (S_2).*

Proof. Let δ_X and δ_S be the dimension functions associated to $\omega_X^\bullet$ and $\omega_S^\bullet$ as in Lemma 48.21.2. As X is proper over A, every closed subscheme of X contains a closed point x which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\overline{\{\xi\}})$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\mathrm{Spec}(\mathcal{O}_{X,x})$ in which case the result follows from Dualizing Complexes, Lemmas 47.16.5 and 47.17.5. Some details omitted. The last two statements can also be deduced from Lemma 48.22.3. □

0AWP **Lemma 48.22.5.** *Let X/A with dualizing module ω_X be as in Example 48.22.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module ω_X represents the functor*

$$\mathcal{F} \longmapsto \mathrm{Hom}_A(H^d(X, \mathcal{F}), \omega_A)$$

on the category of coherent $\mathcal{O}_X$-modules.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_X(\mathcal{F}, \omega_X) &= \mathrm{Ext}_X^{-\dim(X)}(\mathcal{F}, \omega_X^\bullet) \\
&= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], \omega_X^\bullet) \\
&= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], f^!(\omega_A^\bullet)) \\
&= \mathrm{Hom}_S(Rf_*\mathcal{F}[\dim(X)], \omega_A^\bullet) \\
&= \mathrm{Hom}_A(H^d(X, \mathcal{F}), \omega_A)
\end{aligned}$$

The first equality because $H^i(\omega_X^\bullet) = 0$ for $i < -\dim(X)$, see Lemma 48.22.4 and Derived Categories, Lemma 13.27.3. The second equality is follows from the definition of Ext groups. The third equality is our choice of $\omega_X^\bullet$. The fourth equality holds because $f^!$ is the right adjoint of Lemma 48.3.1 for f, see Section 48.19. The final equality holds because $R^i f_*\mathcal{F}$ is zero for $i > d$ (Cohomology of Schemes, Lemma 30.20.9) and $H^j(\omega_A^\bullet)$ is zero for $j < -\dim(A)$. □

48.23. Cohen-Macaulay schemes

0AWQ This section is the continuation of Dualizing Complexes, Section 47.20. Duality takes a particularly simple form for Cohen-Macaulay schemes.

0AWT **Lemma 48.23.1.** *Let X be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$.*

(1) *X is Cohen-Macaulay $\Leftrightarrow$ $\omega_X^\bullet$ locally has a unique nonzero cohomology sheaf,*
(2) *$\mathcal{O}_{X,x}$ is Cohen-Macaulay $\Leftrightarrow$ $\omega_{X,x}^\bullet$ has a unique nonzero cohomology,*
(3) *$U = \{x \in X \mid \mathcal{O}_{X,x}$ is Cohen-Macaulay$\}$ is open and Cohen-Macaulay.*

If X is connected and Cohen-Macaulay, then there is an integer n and a coherent Cohen-Macaulay $\mathcal{O}_X$-module ω_X such that $\omega_X^\bullet = \omega_X[-n]$.

Proof. By definition and Dualizing Complexes, Lemma 47.15.6 for every $x \in X$ the complex $\omega_{X,x}^\bullet$ is a dualizing complex over $\mathcal{O}_{X,x}$. By Dualizing Complexes, Lemma 47.20.2 we see that (2) holds.

To see (3) assume that $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Let n_x be the unique integer such that $H^{n_x}(\omega_{X,x}^\bullet)$ is nonzero. For an affine neighbourhood $V \subset X$ of x we have $\omega_X^\bullet|_V$ is in $D^b_{Coh}(\mathcal{O}_V)$ hence there are finitely many nonzero coherent modules $H^i(\omega_X^\bullet)|_V$. Thus after shrinking V we may assume only H^{n_x} is nonzero, see Modules, Lemma 17.9.5. In this way we see that $\mathcal{O}_{X,v}$ is Cohen-Macaulay for every $v \in V$. This proves that U is open as well as a Cohen-Macaulay scheme.

Proof of (1). The implication $\Leftarrow$ follows from (2). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then in a neighbourhood of x the complex $\omega_X^\bullet$ has only one nonzero cohomology sheaf.

Assume X is connected and Cohen-Macaulay. The above shows that the map $x \mapsto n_x$ is locally constant. Since X is connected it is constant, say equal to n. Setting $\omega_X = H^n(\omega_X^\bullet)$ we see that the lemma holds because ω_X is Cohen-Macaulay by Dualizing Complexes, Lemma 47.20.2 (and Cohomology of Schemes, Definition 30.11.4). □

0AWU **Lemma 48.23.2.** *Let X be a locally Noetherian scheme. If there exists a coherent sheaf ω_X such that $\omega_X[0]$ is a dualizing complex on X, then X is a Cohen-Macaulay scheme.*

Proof. This follows immediately from Dualizing Complexes, Lemma 47.20.3 and our definitions. □

0C0Z **Lemma 48.23.3.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$. If f is flat, then the following are equivalent*

(1) *f is Cohen-Macaulay at x,*
(2) *$f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf in a neighbourhood of x.*

Proof. One direction of the lemma follows from Lemma 48.21.7. To prove the converse, we may assume $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf. Let $y = f(x)$. Let $\xi_1, \ldots, \xi_n \in X_y$ be the generic points of the fibre X_y specializing to x. Let $d_1, \ldots, d_n$ be the dimensions of the corresponding irreducible components of X_y. The morphism $f : X \to Y$ is Cohen-Macaulay at η_i by More on Morphisms,

Lemma 37.22.7. Hence by Lemma 48.21.7 we see that $d_1 = \ldots = d_n$. If d denotes the common value, then $d = \dim_x(X_y)$. After shrinking X we may assume all fibres have dimension at most d (Morphisms, Lemma 29.28.4). Then the only nonzero cohomology sheaf $\omega = H^{-d}(f^!\mathcal{O}_Y)$ is flat over Y by Lemma 48.21.4. Hence, if $h : X_y \to X$ denotes the canonical morphism, then $Lh^*(f^!\mathcal{O}_Y) = Lh^*(\omega[d]) = (h^*\omega)[d]$ by Derived Categories of Schemes, Lemma 36.22.8. Thus $h^*\omega[d]$ is the dualizing complex of X_y by Lemma 48.18.4. Hence X_y is Cohen-Macaulay by Lemma 48.23.1. This proves f is Cohen-Macaulay at x as desired. □

0C10 **Remark 48.23.4.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Assume f is a Cohen-Macaulay morphism of relative dimension d. Let $\omega_{X/Y} = H^{-d}(f^!\mathcal{O}_Y)$ be the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 48.21.7. Then there is a canonical isomorphism

$$f^!K = Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$, see Lemma 48.17.9. In particular, if S has a dualizing complex $\omega_S^\bullet$, $\omega_Y^\bullet = (Y \to S)^!\omega_S^\bullet$, and $\omega_X^\bullet = (X \to S)^!\omega_S^\bullet$ then we have

$$\omega_X^\bullet = Lf^*\omega_Y^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_{X/Y}[d]$$

Thus if further X and Y are connected and Cohen-Macaulay and if ω_Y and ω_X denote the unique nonzero cohomology sheaves of $\omega_Y^\bullet$ and $\omega_X^\bullet$, then we have

$$\omega_X = f^*\omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}.$$

Similar results hold for X and Y arbitrary finite type schemes over S (i.e., not necessarily separated over S) with dualizing complexes normalized with respect to $\omega_S^\bullet$ as in Section 48.20.

48.24. Gorenstein schemes

0AWV This section is the continuation of Dualizing Complexes, Section 47.21.

0AWW **Definition 48.24.1.** Let X be a scheme. We say X is *Gorenstein* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

This definition makes sense because a Noetherian ring is said to be Gorenstein if and only if all of its local rings are Gorenstein, see Dualizing Complexes, Definition 47.21.1.

0C00 **Lemma 48.24.2.** *A Gorenstein scheme is Cohen-Macaulay.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.2. □

0DWG **Lemma 48.24.3.** *A regular scheme is Gorenstein.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.3. □

0BFQ **Lemma 48.24.4.** *Let X be a locally Noetherian scheme.*

(1) *If X has a dualizing complex $\omega_X^\bullet$, then*
   (a) *X is Gorenstein $\Leftrightarrow$ $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$,*
   (b) *$\mathcal{O}_{X,x}$ is Gorenstein $\Leftrightarrow$ $\omega_{X,x}^\bullet$ is an invertible object of $D(\mathcal{O}_{X,x})$,*
   (c) *$U = \{x \in X \mid \mathcal{O}_{X,x}$ is Gorenstein$\}$ is an open Gorenstein subscheme.*

(2) *If X is Gorenstein, then X has a dualizing complex if and only if $\mathcal{O}_X[0]$ is a dualizing complex.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 47.21.4. □

0BVA **Lemma 48.24.5.** *If $f : Y \to X$ is a local complete intersection morphism with X a Gorenstein scheme, then Y is Gorenstein.*

Proof. By More on Morphisms, Lemma 37.62.5 it suffices to prove the corresponding statement about ring maps. This is Dualizing Complexes, Lemma 47.21.7. □

0C01 **Lemma 48.24.6.** *The property $\mathcal{P}(S) =$"S is Gorenstein" is local in the syntomic topology.*

Proof. Let $\{S_i \to S\}$ be a syntomic covering. The scheme S is locally Noetherian if and only if each S_i is Noetherian, see Descent, Lemma 35.16.1. Thus we may now assume S and S_i are locally Noetherian. If S is Gorenstein, then each S_i is Gorenstein by Lemma 48.24.5. Conversely, if each S_i is Gorenstein, then for each point $s \in S$ we can pick i and $t \in S_i$ mapping to s. Then $\mathcal{O}_{S,s} \to \mathcal{O}_{S_i,t}$ is a flat local ring homomorphism with $\mathcal{O}_{S_i,t}$ Gorenstein. Hence $\mathcal{O}_{S,s}$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. □

48.25. Gorenstein morphisms

0C02 This section is one in a series. The corresponding sections for normal morphisms, regular morphisms, and Cohen-Macaulay morphisms can be found in More on Morphisms, Sections 37.20, 37.21, and 37.22.

The following lemma says that it does not make sense to define geometrically Gorenstein schemes, since these would be the same as Gorenstein schemes.

0C03 **Lemma 48.25.1.** *Let X be a locally Noetherian scheme over the field k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Gorenstein} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Gorenstein}$$

If X is locally of finite type over k, the same holds for any field extension k'/k.

Proof. In both cases the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{k'},x'}$ is a faithfully flat local homomorphism of Noetherian local rings. Thus if $\mathcal{O}_{X_{k'},x'}$ is Gorenstein, then so is $\mathcal{O}_{X,x}$ by Dualizing Complexes, Lemma 47.21.8. To go up, we use Dualizing Complexes, Lemma 47.21.8 as well. Thus we have to show that

$$\mathcal{O}_{X_{k'},x'}/\mathfrak{m}_x\mathcal{O}_{X_{k'},x'} = \kappa(x) \otimes_k k'$$

is Gorenstein. Note that in the first case $k \to k'$ is finitely generated and in the second case $k \to \kappa(x)$ is finitely generated. Hence this follows as property (A) holds for Gorenstein, see Dualizing Complexes, Lemma 47.23.1. □

The lemma above guarantees that the following is the correct definition of Gorenstein morphisms.

0C04 **Definition 48.25.2.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Gorenstein at* x if f is flat at x, and the local ring of the scheme X_y at x is Gorenstein.
(2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X.

Here is a translation.

0C05 **Lemma 48.25.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Gorenstein, and*
(2) *f is flat and its fibres are Gorenstein schemes.*

Proof. This follows directly from the definitions. □

0C06 **Lemma 48.25.4.** *A Gorenstein morphism is Cohen-Macaulay.*

Proof. Follows from Lemma 48.24.2 and the definitions. □

0C15 **Lemma 48.25.5.** *A syntomic morphism is Gorenstein. Equivalently a flat local complete intersection morphism is Gorenstein.*

Proof. Recall that a syntomic morphism is flat and its fibres are local complete intersections over fields, see Morphisms, Lemma 29.30.11. Since a local complete intersection over a field is a Gorenstein scheme by Lemma 48.24.5 we conclude. The properties "syntomic" and "flat and local complete intersection morphism" are equivalent by More on Morphisms, Lemma 37.62.8. □

0C11 **Lemma 48.25.6.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Assume that the fibres X_y, Y_z and X_z of f, g, and $g \circ f$ are locally Noetherian.*

(1) *If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.*
(2) *If f and g are Gorenstein, then $g \circ f$ is Gorenstein.*
(3) *If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.*
(4) *If $g \circ f$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.*

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 47.21.8. □

0C12 **Lemma 48.25.7.** *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(x)}$ is Gorenstein for all $x \in X$.*

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 47.21.8. □

0C07 **Lemma 48.25.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

(1) *If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.*
(2) *If f is flat at x and f' is Gorenstein at x', then f is Gorenstein at x.*
(3) *If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.*

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 33.11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y',y'} & \longleftarrow & \mathcal{O}_{Y,y} \end{array}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Gorenstein at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f' is Gorenstein at x'.

Assume f is flat at x and f' is Gorenstein at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f is Gorenstein at x.

Assume $Y' \to Y$ is flat at y' and f' is Gorenstein at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Gorenstein, see Lemma 48.25.1. Hence we see that f is Gorenstein at x. □

0E0Q **Lemma 48.25.9.** *Let $f : X \to Y$ be a morphism of schemes which is flat and locally of finite type. Then formation of the set $\{x \in X \mid f \text{ is Gorenstein at } x\}$ commutes with arbitrary base change.*

Proof. The assumption implies any fibre of f is locally of finite type over a field and hence locally Noetherian and the same is true for any base change. Thus the statement makes sense. Looking at fibres we reduce to the following problem: let X be a scheme locally of finite type over a field k, let K/k be a field extension, and let $x_K \in X_K$ be a point with image $x \in X$. Problem: show that $\mathcal{O}_{X_K,x_K}$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ is Gorenstein.

The problem can be solved using a bit of algebra as follows. Choose an affine open $\mathrm{Spec}(A) \subset X$ containing x. Say x corresponds to $\mathfrak{p} \subset A$. With $A_K = A \otimes_k K$ we see that $\mathrm{Spec}(A_K) \subset X_K$ contains x_K. Say x_K corresponds to $\mathfrak{p}_K \subset A_K$. Let $\omega_A^\bullet$ be a dualizing complex for A. By Dualizing Complexes, Lemma 47.25.3 $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K. Now we are done because $A_\mathfrak{p} \to (A_K)_{\mathfrak{p}_K}$ is a flat local homomorphism of Noetherian rings and hence $(\omega_A^\bullet)_\mathfrak{p}$ is an invertible object of $D(A_\mathfrak{p})$ if and only if $(\omega_A^\bullet)_\mathfrak{p} \otimes_{A_\mathfrak{p}} (A_K)_{\mathfrak{p}_K}$ is an invertible object of $D((A_K)_{\mathfrak{p}_K})$. Some details omitted; hint: look at cohomology modules. □

0C08 **Lemma 48.25.10.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let $x \in X$. If f is flat, then the following are equivalent*

(1) *f is Gorenstein at x,*
(2) *$f^!\mathcal{O}_Y$ is isomorphic to an invertible object in a neighbourhood of x.*

In particular, the set of points where f is Gorenstein is open in X.

Proof. Set $\omega^\bullet = f^!\mathcal{O}_Y$. By Lemma 48.18.4 this is a bounded complex with coherent cohomology sheaves whose derived restriction $Lh^*\omega^\bullet$ to the fibre X_y is a dualizing complex on X_y. Denote $i : x \to X_y$ the inclusion of a point. Then the following are equivalent

(1) f is Gorenstein at x,
(2) $\mathcal{O}_{X_y,x}$ is Gorenstein,
(3) $Lh^*\omega^\bullet$ is invertible in a neighbourhood of x,
(4) $Li^*Lh^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(5) $L(h \circ i)^*\omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
(6) $\omega^\bullet$ is invertible in a neighbourhood of x.

The equivalence of (1) and (2) is by definition (as f is flat). The equivalence of (2) and (3) follows from Lemma 48.24.4. The equivalence of (3) and (4) follows from More on Algebra, Lemma 15.78.1. The equivalence of (4) and (5) holds because $Li^*Lh^* = L(h \circ i)^*$. The equivalence of (5) and (6) holds by More on Algebra, Lemma 15.78.1. Thus the lemma is clear. □

0C09 **Lemma 48.25.11.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent*

(1) *f is Gorenstein at x,*
(2) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is Gorenstein at x,*
(3) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite Gorenstein morphism $U \to \mathbf{A}^d_S$ over S,*
(4) *for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is Gorenstein at x.*

In particular, the set of points where f is Gorenstein is open in X.

Proof. Choose affine open $U = \mathrm{Spec}(A) \subset X$ with $x \in U$ and $V = \mathrm{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 10.125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[8] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Having said this, the lemma translates into the following algebra problem (translation omitted). Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ the following are equivalent

(a) $\mathrm{Spec}(\varphi)$ is Gorenstein at $\mathfrak{p}$, and
(b) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein at $\mathfrak{p}$.
(c) $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is Gorenstein in an open neighbourhood of $\mathfrak{p}$.

In each case $R[x_1, \ldots, x_n] \to A$ is flat at $\mathfrak{p}$ hence by openness of flatness (Algebra, Theorem 10.129.4), we may assume $R[x_1, \ldots, x_n] \to A$ is flat (replace A by a suitable principal localization). By Algebra, Lemma 10.168.1 there exists $R_0 \subset R$ and $R_0[x_1, \ldots, x_n] \to A_0$ such that R_0 is of finite type over $\mathbf{Z}$ and $R_0 \to A_0$ is of finite type and $R_0[x_1, \ldots, x_n] \to A_0$ is flat. Note that the set of points where a flat finite type morphism is Gorenstein commutes with base change by Lemma 48.25.8. In this way we reduce to the case where R is Noetherian.

[8] If S is quasi-separated, then g will be quasi-finite.

Thus we may assume X and S affine and that we have a factorization of f of the form

$$X \xrightarrow{g} \mathbf{A}^n_S \xrightarrow{p} S$$

with g flat and quasi-finite and S Noetherian. Then X and $\mathbf{A}^n_S$ are separated over S and we have

$$f^!\mathcal{O}_S = g^!p^!\mathcal{O}_S = g^!\mathcal{O}_{\mathbf{A}^n_S}[n]$$

by know properties of upper shriek functors (Lemmas 48.16.3 and 48.17.3). Hence the equivalence of (a), (b), and (c) by Lemma 48.25.10. □

0C0A **Lemma 48.25.12.** *The property $\mathcal{P}(f)$ = "the fibres of f are locally Noetherian and f is Gorenstein" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f)$ = "f is flat", and $\mathcal{P}_2(f)$ = "the fibres of f are locally Noetherian and Gorenstein". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 35.23.17 and 35.27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. And if in addition X_y is Gorenstein, then X_{i,y_i} is Gorenstein, see Lemma 48.25.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Lemma 48.24.6. □

48.26. More on dualizing complexes

0E4M Some lemmas which don't fit anywhere else very well.

0E4N **Lemma 48.26.1.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Assume*

(1) *f is syntomic and surjective, or*
(2) *f is a surjective flat local complete intersection morphism, or*
(3) *f is a surjective Gorenstein morphism of finite type.*

Then $K \in D_{QCoh}(\mathcal{O}_Y)$ is a dualizing complex on Y if and only if Lf^*K is a dualizing complex on X.

Proof. Taking affine opens and using Derived Categories of Schemes, Lemma 36.3.5 this translates into Dualizing Complexes, Lemma 47.26.2. □

48.27. Duality for proper schemes over fields

0FVU In this section we work out the consequences of the very general material above on dualizing complexes and duality for proper schemes over fields.

0FVV **Lemma 48.27.1.** *Let X be a proper scheme over a field k. There exists a dualizing complex $\omega_X^\bullet$ with the following properties*

(1) *$H^i(\omega_X^\bullet)$ is nonzero only for $i \in [-\dim(X), 0]$,*
(2) *$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$ is a coherent (S_2)-module whose support is the irreducible components of dimension $\dim(X)$,*
(3) *the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,*
(4) *for $x \in X$ closed the module $H^i(\omega_{X,x}^\bullet) \oplus \ldots \oplus H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if $\text{depth}(\mathcal{O}_{X,x}) \leq -i$,*
(5) *for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms*[9]
$$\text{Ext}^i_X(K, \omega_X^\bullet) = \text{Hom}_k(H^{-i}(X, K), k)$$
compatible with shifts and distinguished triangles,
(6) *there are functorial isomorphisms $\text{Hom}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^{\dim(X)}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X, and*
(7) *if $X \to \text{Spec}(k)$ is smooth of relative dimension d, then $\omega_X^\bullet \cong \wedge^d \Omega_{X/k}[d]$ and $\omega_X \cong \wedge^d \Omega_{X/k}$.*

Proof. Denote $f : X \to \text{Spec}(k)$ the structure morphism. Let a be the right adjoint of pushforward of this morphism, see Lemma 48.3.1. Consider the relative dualizing complex
$$\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$$
Compare with Remark 48.12.5. Since f is proper we have $f^!(\mathcal{O}_{\text{Spec}(k)}) = a(\mathcal{O}_{\text{Spec}(k)})$ by definition, see Section 48.16. Applying Lemma 48.17.7 we find that $\omega_X^\bullet$ is a dualizing complex. Moreover, we see that $\omega_X^\bullet$ and ω_X are as in Example 48.22.1 and as in Example 48.22.2.

Parts (1), (2), and (3) follow from Lemma 48.22.4.

For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Lemma 48.21.1. Part (4) then follows from Dualizing Complexes, Lemma 47.20.1.

Part (5) holds by construction as a is the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_{\text{Spec}(k)}) = D(k)$ which we can identify with $K \mapsto R\Gamma(X, K)$. We also use that the derived category $D(k)$ of k-modules is the same as the category of graded k-vector spaces.

Part (6) follows from Lemma 48.22.5 for coherent $\mathcal{F}$ and in general by unwinding (5) for $K = \mathcal{F}[0]$ and $i = -\dim(X)$.

Part (7) follows from Lemma 48.15.7. □

0FVW **Remark 48.27.2.** Let k, X, and $\omega_X^\bullet$ be as in Lemma 48.27.1. The identity on the complex $\omega_X^\bullet$ corresponds, via the functorial isomorphism in part (5), to a map
$$t : H^0(X, \omega_X^\bullet) \longrightarrow k$$

[9] This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$ in fact it suffices to consider $K \in D^b_{Coh}(\mathcal{O}_X)$.

For an arbitrary K in $D_{QCoh}(\mathcal{O}_X)$ the identification $\mathrm{Hom}(K, \omega_X^\bullet)$ with $H^0(X, K)^\vee$ in part (5) corresponds to the pairing

$$\mathrm{Hom}_X(K, \omega_X^\bullet) \times H^0(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

This follows from the functoriality of the isomorphisms in (5). Similarly for any $i \in \mathbf{Z}$ we get the pairing

$$\mathrm{Ext}^i_X(K, \omega_X^\bullet) \times H^{-i}(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

Here we think of α as a morphism $K[-i] \to \omega_X^\bullet$ and β as an element of $H^0(X, K[-i])$ in order to define $\alpha(\beta)$. Observe that if K is general, then we only know that this pairing is nondegenerate on one side: the pairing induces an isomorphism of $\mathrm{Hom}_X(K, \omega_X^\bullet)$, resp. $\mathrm{Ext}^i_X(K, \omega_X^\bullet)$ with the k-linear dual of $H^0(X, K)$, resp. $H^{-i}(X, K)$ but in general not vice versa. If K is in $D^b_{Coh}(\mathcal{O}_X)$, then $\mathrm{Hom}_X(K, \omega_X^\bullet)$, $\mathrm{Ext}_X(K, \omega_X^\bullet)$, $H^0(X, K)$, and $H^i(X, K)$ are finite dimensional k-vector spaces (by Derived Categories of Schemes, Lemmas 36.11.5 and 36.11.4) and the pairings are perfect in the usual sense.

0FVX **Remark 48.27.3.** We continue the discussion in Remark 48.27.2 and we use the same notation k, X, $\omega_X^\bullet$, and t. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we obtain perfect pairings

$$\langle -, - \rangle : \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\bullet) \times H^{-i}(X, \mathcal{F}) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

of finite dimensional k-vector spaces. These pairings satisfy the following (obvious) functoriality: if $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism of coherent $\mathcal{O}_X$-modules, then we have

$$\langle \alpha \circ \varphi, \beta \rangle = \langle \alpha, \varphi(\beta) \rangle$$

for $\alpha \in \mathrm{Ext}^i_X(\mathcal{G}, \omega_X^\bullet)$ and $\beta \in H^{-i}(X, \mathcal{F})$. In other words, the k-linear map $\mathrm{Ext}^i_X(\mathcal{G}, \omega_X^\bullet) \to \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\bullet)$ induced by φ is, via the pairings, the k-linear dual of the k-linear map $H^{-i}(X, \mathcal{F}) \to H^{-i}(X, \mathcal{G})$ induced by φ. Formulated in this manner, this still works if φ is a homomorphism of quasi-coherent $\mathcal{O}_X$-modules.

0FVY **Lemma 48.27.4.** *Let k, X, and $\omega_X^\bullet$ be as in Lemma 48.27.1. Let $t : H^0(X, \omega_X^\bullet) \to k$ be as in Remark 48.27.2. Let $E \in D(\mathcal{O}_X)$ be perfect. Then the pairings*

$$H^i(X, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee) \times H^{-i}(X, E) \longrightarrow k, \quad (\xi, \eta) \longmapsto t((1_{\omega_X^\bullet} \otimes \epsilon)(\xi \cup \eta))$$

are perfect for all i. Here $\cup$ denotes the cup product of Cohomology, Section 20.31 and $\epsilon : E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to \mathcal{O}_X$ is as in Cohomology, Example 20.50.7.

Proof. By replacing E with $E[-i]$ this reduces to the case $i = 0$. By Cohomology, Lemma 20.51.2 we see that the pairing is the same as the one discussed in Remark 48.27.2 whence the result by the discussion in that remark. □

0FVZ **Lemma 48.27.5.** *Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension d. The module ω_X of Lemma 48.27.1 has the following properties*

1. *ω_X is a dualizing module on X (Section 48.22),*
2. *ω_X is a coherent Cohen-Macaulay module whose support is X,*
3. *there are functorial isomorphisms $\mathrm{Ext}^i_X(K, \omega_X[d]) = \mathrm{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts and distinguished triangles for $K \in D_{QCoh}(X)$,*
4. *there are functorial isomorphisms $\mathrm{Ext}^{d-i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^i(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X.*

Proof. It is clear from Lemma 48.27.1 that ω_X is a dualizing module (as it is the left most nonvanishing cohomology sheaf of a dualizing complex). We have $\omega_X^\bullet = \omega_X[d]$ and ω_X is Cohen-Macaulay as X is Cohen-Macualay, see Lemma 48.23.1. The other statements follow from this combined with the corresponding statements of Lemma 48.27.1. □

0FW0 **Remark 48.27.6.** Let X be a proper Cohen-Macaulay scheme over a field k which is equidimensional of dimension d. Let $\omega_X^\bullet$ and ω_X be as in Lemma 48.27.1. By Lemma 48.27.5 we have $\omega_X^\bullet = \omega_X[d]$. Let $t : H^d(X, \omega_X) \to k$ be the map of Remark 48.27.2. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then we have perfect pairings

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{E}^\vee) \times H^{d-i}(X, \mathcal{E}) \longrightarrow k, \quad (\xi, \eta) \longmapsto t(1 \otimes \epsilon)(\xi \cup \eta))$$

where $\cup$ is the cup-product and $\epsilon : \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_X$ is the evaluation map. This is a special case of Lemma 48.27.4.

Here is a sanity check for the dualizing complex.

0FW1 **Lemma 48.27.7.** *Let X be a proper scheme over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 48.27.1.*

(1) *If $X \to \mathrm{Spec}(k)$ factors as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for some field k', then $\omega_X^\bullet$ and ω_X are as in Lemma 48.27.1 for the morphism $X \to \mathrm{Spec}(k')$.*
(2) *If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 48.27.1 for the morphism $X_K \to \mathrm{Spec}(K)$.*

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism and denote $f' : X \to \mathrm{Spec}(k')$ the given factorization. In the proof of Lemma 48.27.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$ where a be is the right adjoint of Lemma 48.3.1 for f. Thus we have to show $a(\mathcal{O}_{\mathrm{Spec}(k)}) \cong a'(\mathcal{O}_{\mathrm{Spec}(k)})$ where a' be is the right adjoint of Lemma 48.3.1 for f'. Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 30.19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Lemma 48.3.1 for $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that $\mathrm{Hom}_k(k', k) \cong k'$ as k'-modules, see Example 48.3.2. This holds because these are k'-vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Lemma 48.6.2. See discussion in Remark 48.12.5. □

48.28. Relative dualizing complexes

0E2S For a proper, flat morphism of finite presentation we have a rigid relative dualizing complex, see Remark 48.12.5 and Lemma 48.12.8. For a separated and finite type morphism $f : X \to Y$ of Noetherian schemes, we can consider $f^!\mathcal{O}_Y$. In this section we define relative dualizing complexes for morphisms which are flat and locally of finite presentation (but not necessarily quasi-separated or quasi-compact) between schemes (not necessarily locally Noetherian). We show such complexes exist, are unique up to unique isomorphism, and agree with the cases mentioned above. Before reading this section, please read Dualizing Complexes, Section 47.27.

0E2T **Definition 48.28.1.** Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $W \subset X \times_S X$ be any open such that the diagonal $\Delta_{X/S} : X \to X \times_S X$ factors through a closed immersion $\Delta : X \to W$. A *relative dualizing complex* is a pair (K, ξ) consisting of an object $K \in D(\mathcal{O}_X)$ and a map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\mathrm{pr}_1^* K|_W$$

in $D(\mathcal{O}_W)$ such that

(1) K is S-perfect (Derived Categories of Schemes, Definition 36.35.1), and
(2) ξ defines an isomorphism of $\Delta_*\mathcal{O}_X$ with $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$.

By Lemma 48.9.3 condition (2) is equivalent to the existence of an isomorphism

$$\mathcal{O}_X \longrightarrow R\mathcal{H}om(\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$$

in $D(\mathcal{O}_X)$ whose pushforward via Δ is equal to ξ. Since $R\mathcal{H}om(\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ is independent of the choice of the open W, so is the category of pairs (K, ξ). If $X \to S$ is separated, then we can choose $W = X \times_S X$. We will reduce many of the arguments to the case of rings using the following lemma.

0E2U **Lemma 48.28.2.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then for any commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array}$$

whose horizontal arrows are open immersions, the restriction of K to $\mathrm{Spec}(A)$ corresponds via Derived Categories of Schemes, Lemma 36.3.5 to a relative dualizing complex for $R \to A$ in the sense of Dualizing Complexes, Definition 47.27.1.

Proof. Since formation of $R\mathcal{H}om$ commutes with restrictions to opens we may as well assume $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. Observe that relatively perfect objects of $D(\mathcal{O}_X)$ are pseudo-coherent and hence are in $D_{QCoh}(\mathcal{O}_X)$ (Derived Categories of Schemes, Lemma 36.10.1). Thus the statement makes sense. Observe that taking Δ_*, $L\mathrm{pr}_1^*$, and $R\mathcal{H}om$ is compatible with what happens on the algebraic side by Derived Categories of Schemes, Lemmas 36.3.7, 36.3.8, 36.10.8. For the last one we observe that $L\mathrm{pr}_1^* K$ is S-perfect (hence bounded below) and that $\Delta_*\mathcal{O}_X$ is a pseudo-coherent object of $D(\mathcal{O}_W)$; translated into algebra this means that A is pseudo-coherent as an $A \otimes_R A$-module which follows from More on Algebra, Lemma 15.83.8 applied to $R \to A \otimes_R A \to A$. Thus we recover exactly the conditions in Dualizing Complexes, Definition 47.27.1. □

0E2V **Lemma 48.28.3.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(K, K)$ is an isomorphism.*

Proof. Looking affine locally this reduces using Lemma 48.28.2 to the algebraic case which is Dualizing Complexes, Lemma 47.27.5. □

0E2W **Lemma 48.28.4.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. If (K, ξ) and (L, η) are two relative dualizing complexes on X/S, then there is a unique isomorphism $K \to L$ sending ξ to η.*

Proof. Let $U \subset X$ be an affine open mapping into an affine open of S. Then there is an isomorphism $K|_U \to L|_U$ by Lemma 48.28.2 and Dualizing Complexes, Lemma 47.27.2. The reader can reuse the argument of that lemma in the schemes case to obtain a proof in this case. We will instead use a glueing argument.

Suppose we have an isomorphism $\alpha : K \to L$. Then $\alpha(\xi) = u\eta$ for some invertible section $u \in H^0(W, \Delta_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. (Because both η and $\alpha(\xi)$ are generators of an invertible $\Delta_*\mathcal{O}_X$-module by assumption.) Hence after replacing α by $u^{-1}\alpha$ we see that $\alpha(\xi) = \eta$. Since the automorphism group of K is $H^0(X, \mathcal{O}_X^*)$ by Lemma 48.28.3 there is at most one such α.

Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each $U \in \mathcal{B}$ we have a unique isomorphism $\alpha_U : K|_U \to L|_U$ mapping ξ to η by the discussion in the previous two paragraphs. Observe that $\mathrm{Ext}^i(K|_U, K|_U) = 0$ for $i < 0$ and any open U of X by Lemma 48.28.3. By Cohomology, Lemma 20.45.2 applied to $\mathrm{id} : X \to X$ we get a unique morphism $\alpha : K \to L$ agreeing with α_U for all $U \in \mathcal{B}$. Then α sends ξ to η as this is true locally. □

0E2X **Lemma 48.28.5.** *Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. There exists a relative dualizing complex (K, ξ).*

Proof. Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S. For each U we have a relative dualizing complex (K_U, ξ_U) for U over S. Namely, choose an affine open $V \subset S$ such that $U \to X \to S$ factors through V. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$. By Dualizing Complexes, Lemma 47.27.4 there exists a relative dualizing complex $K_A \in D(A)$ for $R \to A$. Arguing backwards through the proof of Lemma 48.28.2 this determines an V-perfect object $K_U \in D(\mathcal{O}_U)$ and a map

$$\xi : \Delta_*\mathcal{O}_U \to L\mathrm{pr}_1^* K_U$$

in $D(\mathcal{O}_{U \times_V U})$. Since being V-perfect is the same as being S-perfect and since $U \times_V U = U \times_S U$ we find that (K_U, ξ_U) is as desired.

If $U' \subset U \subset X$ with $U', U \in \mathcal{B}$, then we have a unique isomorphism $\rho^U_{U'} : K_U|_{U'} \to K_{U'}$ in $D(\mathcal{O}_{U'})$ sending $\xi_U|_{U' \times_S U'}$ to $\xi_{U'}$ by Lemma 48.28.4 (note that trivially the restriction of a relative dualizing complex to an open is a relative dualizing complex). The uniqueness guarantees that $\rho^U_{U''} = \rho^V_{U''} \circ \rho^U_{U'}|_{U''}$ for $U'' \subset U' \subset U$ in $\mathcal{B}$. Observe that $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$ for $U \in \mathcal{B}$ by Lemma 48.28.3 applied to U/S and K_U. Thus the BBD glueing lemma (Cohomology, Theorem 20.45.8) tells us there is a unique solution, namely, an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ such that we have $\rho^U_{U'} \circ \rho_U|_{U'} = \rho_{U'}$ for all $U' \subset U$, $U, U' \in \mathcal{B}$.

To finish the proof we have to construct the map

$$\xi : \Delta_*\mathcal{O}_X \longrightarrow L\mathrm{pr}_1^* K|_W$$

in $D(\mathcal{O}_W)$ inducing an isomorphism from $\Delta_*\mathcal{O}_X$ to $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$. Since we may change W, we choose $W = \bigcup_{U \in \mathcal{B}} U \times_S U$. We can use ρ_U to get isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)|_{U \times_S U} \xrightarrow{\rho_U} R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_*\mathcal{O}_U, L\mathrm{pr}_1^* K_U)$$

As W is covered by the opens $U \times_S U$ we conclude that the cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ are zero except in degree 0. Moreover, we obtain

isomorphisms

$$H^0\left(U \times_S U, R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)\right) \xrightarrow{\rho_U} H^0\left((R\mathcal{H}om_{\mathcal{O}_{U\times_S U}}(\Delta_*\mathcal{O}_U, L\mathrm{pr}_1^* K_U)\right)$$

Let τ_U in the LHS be an element mapping to ξ_U under this map. The compatibilities between $\rho^U_{U'}$, ξ_U, $\xi_{U'}$, ρ_U, and $\rho_{U'}$ for $U' \subset U \subset X$ open $U', U \in \mathcal{B}$ imply that $\tau_U|_{U'\times_S U'} = \tau_{U'}$. Thus we get a global section τ of the 0th cohomology sheaf $H^0(R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W))$. Since the other cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ are zero, this global section τ determines a morphism ξ as desired. Since the restriction of ξ to $U \times_S U$ gives ξ_U, we see that it satisfies the final condition of Definition 48.28.1. □

0E2Y **Lemma 48.28.6.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes. Assume $X \to S$ is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex for f. Set $K' = L(g')^* K$. Let ξ' be the derived base change of ξ (see proof). Then (K', ξ') is a relative dualizing complex for f'.

Proof. Consider the cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \Delta_{X'/S'}}\downarrow & & \downarrow{\scriptstyle \Delta_{X/S}} \\ X' \times_{S'} X' & \xrightarrow{g'\times g'} & X \times_S X \end{array}$$

Choose $W \subset X \times_S X$ open such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$. Choose $W' \subset X' \times_{S'} X'$ open such that $\Delta_{X'/S'}$ factors through a closed immersion $\Delta' : X \to W'$ and such that $(g' \times g')(W') \subset W$. Let us still denote $g' \times g' : W' \to W$ the induced morphism. We have

$$L(g' \times g')^* \Delta_* \mathcal{O}_X = \Delta'_* \mathcal{O}_{X'} \quad \text{and} \quad L(g' \times g')^* L\mathrm{pr}_1^* K|_W = L\mathrm{pr}_1^* K'|_{W'}$$

The first equality holds because X and $X' \times_{S'} X'$ are tor independent over $X \times_S X$ (see for example More on Morphisms, Lemma 37.69.1). The second holds by transitivity of derived pullback (Cohomology, Lemma 20.27.2). Thus $\xi' = L(g' \times g')^* \xi$ can be viewed as a map

$$\xi' : \Delta'_* \mathcal{O}_{X'} \longrightarrow L\mathrm{pr}_1^* K'|_{W'}$$

Having said this the proof of the lemma is straightforward. First, K' is S'-perfect by Derived Categories of Schemes, Lemma 36.35.6. To check that ξ' induces an isomorphism of $\Delta'_* \mathcal{O}_{X'}$ to $R\mathcal{H}om_{\mathcal{O}_{W'}}(\Delta'_* \mathcal{O}_{X'}, L\mathrm{pr}_1^* K'|_{W'})$ we may work affine locally. By Lemma 48.28.2 we reduce to the corresponding statement in algebra which is proven in Dualizing Complexes, Lemma 47.27.4. □

0E2Z **Lemma 48.28.7.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a proper, flat morphism of finite presentation. The relative dualizing complex $\omega^\bullet_{X/S}$ of Remark 48.12.5 together with (48.12.8.1) is a relative dualizing complex in the sense of Definition 48.28.1.*

Proof. In Lemma 48.12.7 we proved that $\omega^\bullet_{X/S}$ is S-perfect. Let c be the right adjoint of Lemma 48.3.1 for the diagonal $\Delta : X \to X \times_S X$. Then we can apply Δ_* to (48.12.8.1) to get an isomorphism

$$\Delta_*\mathcal{O}_X \to \Delta_*(c(L\mathrm{pr}_1^*\omega^\bullet_{X/S})) = R\mathcal{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X, L\mathrm{pr}_1^*\omega^\bullet_{X/S})$$

The equality holds by Lemmas 48.9.7 and 48.9.3. This finishes the proof. □

0E4P **Remark 48.28.8.** Let $X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. By Lemma 48.28.5 there exists a relative dualizing complex $(\omega^\bullet_{X/S}, \xi)$ in the sense of Definition 48.28.1. Consider any morphism $g : S' \to S$ where S' is quasi-compact and quasi-separated (for example an affine open of S). By Lemma 48.28.6 we see that $(L(g')^*\omega^\bullet_{X/S}, L(g')^*\xi)$ is a relative dualizing complex for the base change $f' : X' \to S'$ in the sense of Definition 48.28.1. Let $\omega^\bullet_{X'/S'}$ be the relative dualizing complex for $X' \to S'$ in the sense of Remark 48.12.5. Combining Lemmas 48.28.7 and 48.28.4 we see that there is a unique isomorphism

$$\omega^\bullet_{X'/S'} \longrightarrow L(g')^*\omega^\bullet_{X/S}$$

compatible with (48.12.8.1) and $L(g')^*\xi$. These isomorphisms are compatible with morphisms between quasi-compact and quasi-separated schemes over S and the base change isomorphisms of Lemma 48.12.4 (if we ever need this compatibility we will carefully state and prove it here).

0E9W **Lemma 48.28.9.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. If f is flat, then $f^!\mathcal{O}_Y$ is (the first component of) a relative dualizing complex for X over Y in the sense of Definition 48.28.1.*

Proof. By Lemma 48.17.10 we have that $f^!\mathcal{O}_Y$ is Y-perfect. As f is separated the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion and $\Delta_*\Delta^!(-) = R\mathcal{H}om_{\mathcal{O}_{X\times_Y X}}(\mathcal{O}_X, -)$, see Lemmas 48.9.7 and 48.9.3. Hence to finish the proof it suffices to show $\Delta^!(L\mathrm{pr}_1^* f^!(\mathcal{O}_Y)) \cong \mathcal{O}_X$ where $\mathrm{pr}_1 : X \times_Y X \to X$ is the first projection. We have

$$\mathcal{O}_X = \Delta^!\mathrm{pr}_1^!\mathcal{O}_X = \Delta^!\mathrm{pr}_1^! L\mathrm{pr}_2^*\mathcal{O}_Y = \Delta^!(L\mathrm{pr}_1^* f^!\mathcal{O}_Y)$$

where $\mathrm{pr}_2 : X \times_Y X \to X$ is the second projection and where we have used the base change isomorphism $\mathrm{pr}_1^! \circ L\mathrm{pr}_2^* = L\mathrm{pr}_1^* \circ f^!$ of Lemma 48.18.1. □

0E30 **Lemma 48.28.10.** *Let $f : Y \to X$ and $X \to S$ be morphisms of schemes which are flat and of finite presentation. Let (K, ξ) and (M, η) be a relative dualizing complex for $X \to S$ and $Y \to X$. Set $E = M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*K$. Then (E, ζ) is a relative dualizing complex for $Y \to S$ for a suitable ζ.*

Proof. Using Lemma 48.28.2 and the algebraic version of this lemma (Dualizing Complexes, Lemma 47.27.6) we see that E is affine locally the first component of a relative dualizing complex. In particular we see that E is S-perfect since this may be checked affine locally, see Derived Categories of Schemes, Lemma 36.35.3.

Let us first prove the existence of ζ in case the morphisms $X \to S$ and $Y \to X$ are separated so that $\Delta_{X/S}$, $\Delta_{Y/X}$, and $\Delta_{Y/S}$ are closed immersions. Consider the

following diagram

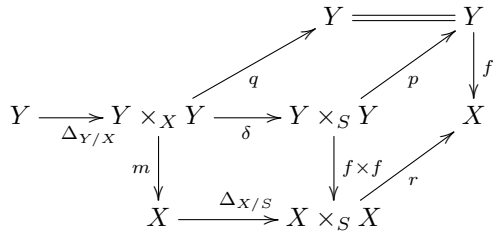

where p, q, r are the first projections. By Lemma 48.9.4 we have

$$R\mathcal{H}om_{\mathcal{O}_{Y\times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_*\left(R\mathcal{H}om_{\mathcal{O}_{Y\times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, R\mathcal{H}om(\mathcal{O}_{Y\times_X Y}, Lp^*E))\right)$$

By Lemma 48.10.3 we have

$$R\mathcal{H}om(\mathcal{O}_{Y\times_X Y}, Lp^*E) = R\mathcal{H}om(\mathcal{O}_{Y\times_X Y}, L(f\times f)^*Lr^*K) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y\times_S Y}} Lq^*M$$

By Lemma 48.10.2 we have

$$R\mathcal{H}om(\mathcal{O}_{Y\times_X Y}, L(f\times f)^*Lr^*K) = Lm^*R\mathcal{H}om(\mathcal{O}_X, Lr^*K)$$

The last expression is isomorphic (via ξ) to $Lm^*\mathcal{O}_X = \mathcal{O}_{Y\times_X Y}$. Hence the expression preceding is isomorphic to Lq^*M. Hence

$$R\mathcal{H}om_{\mathcal{O}_{Y\times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_*\left(R\mathcal{H}om_{\mathcal{O}_{Y\times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, Lq^*M)\right)$$

The material inside the parentheses is isomorphic to $\Delta_{Y/X,*} * \mathcal{O}_X$ via η. This finishes the proof in the separated case.

In the general case we choose an open $W \subset X \times_S X$ such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \to W$ and we choose an open $V \subset Y \times_X Y$ such that $\Delta_{Y/X}$ factors through a closed immersion $\Delta' : Y \to V$. Finally, choose an open $W' \subset Y \times_S Y$ whose intersection with $Y \times_X Y$ gives V and which maps into W. Then we consider the diagram

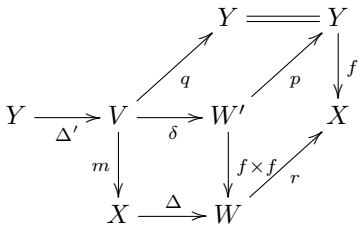

and we use exactly the same argument as before. □

48.29. The fundamental class of an lci morphism

0E9X In this section we will use the computations made in Section 48.15. Thus our result will suffer from the same kind of non-uniqueness as we have in that section.

0E9Y **Lemma 48.29.1.** *Let X be a locally ringed space. Let*

$$\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{E}_1$ and $\mathcal{E}_0$ are locally free of ranks r_1, r_0. Then there is a canonical map

$$\wedge^{r_0-r_1}\mathcal{F} \longrightarrow \wedge^{r_1}(\mathcal{E}_1^\vee) \otimes \wedge^{r_0}\mathcal{E}_0$$

which is an isomorphism on the stalk at $x \in X$ if and only if $\mathcal{F}$ is locally free of rank $r_0 - r_1$ in an open neighbourhood of x.

Proof. If $r_1 > r_0$ then $\wedge^{r_0-r_1}\mathcal{F} = 0$ by convention and the unique map cannot be an isomorphism. Thus we may assume $r = r_0 - r_1 \geq 0$. Define the map by the formula

$$s_1 \wedge \ldots \wedge s_r \mapsto t_1^\vee \wedge \ldots \wedge t_{r_1}^\vee \otimes \alpha(t_1) \wedge \ldots \wedge \alpha(t_{r_1}) \wedge \tilde{s}_1 \wedge \ldots \wedge \tilde{s}_r$$

where $t_1, \ldots, t_{r_1}$ is a local basis for $\mathcal{E}_1$, correspondingly $t_1^\vee, \ldots, t_{r_1}^\vee$ is the dual basis for $\mathcal{E}_1^\vee$, and s_i' is a local lift of s_i to a section of $\mathcal{E}_0$. We omit the proof that this is well defined.

If $\mathcal{F}$ is locally free of rank r, then it is straightforward to verify that the map is an isomorphism. Conversely, assume the map is an isomorphism on stalks at x. Then $\wedge^r\mathcal{F}_x$ is invertible. This implies that $\mathcal{F}_x$ is generated by at most r elements. This can only happen if α has rank r modulo $\mathfrak{m}_x$, i.e., α has maximal rank modulo $\mathfrak{m}_x$. This implies that α has maximal rank in a neighbourhood of x and hence $\mathcal{F}$ is locally free of rank r in a neighbourhood as desired. □

0E9Z **Lemma 48.29.2.** *Let Y be a Noetherian scheme. Let $f : X \to Y$ be a local complete intersection morphism which factors as an immersion $X \to P$ followed by a proper smooth morphism $P \to Y$. Let r be the locally constant function on X such that $\omega_{X/Y} = H^{-r}(f^!\mathcal{O}_Y)$ is the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 48.17.11. Then there is a map*

$$\wedge^r\Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the stalk at a point x if and only if f is smooth at x.

Proof. The assumption implies that X is compactifiable over Y hence $f^!$ is defined, see Section 48.16. Let $j : W \to P$ be an open subscheme such that $X \to P$ factors through a closed immersion $i : X \to W$. Moreover, we have $f^! = i^! \circ j^! \circ g^!$ where $g : P \to Y$ is the given morphism. We have $g^!\mathcal{O}_Y = \wedge^d\Omega_{P/Y}[d]$ by Lemma 48.15.7 where d is the locally constant function giving the relative dimension of P over Y. We have $j^! = j^*$. We have $i^!\mathcal{O}_W = \wedge^c\mathcal{N}[-c]$ where c is the codimension of X in W (a locally constant function on X) and where $\mathcal{N}$ is the normal sheaf of the Koszul-regular immersion i, see Lemma 48.15.6. Combining the above we find

$$f^!\mathcal{O}_Y = \left(\wedge^c\mathcal{N} \otimes_{\mathcal{O}_X} \wedge^d\Omega_{P/Y}|_X\right)[d-c]$$

where we have also used Lemma 48.17.9. Thus $r = d|_X - c$ as locally constant functions on X. The conormal sheaf of $X \to P$ is the module $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of i, see Morphisms, Section 29.31. Consider the canonical exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{P/Y}|_X \to \Omega_{X/Y} \to 0$$

of Morphisms, Lemma 29.32.15. We obtain our map by an application of Lemma 48.29.1.

If f is smooth at x, then the map is an isomorphism by an application of Lemma 48.29.1 and the fact that $\Omega_{X/Y}$ is locally free at x of rank r. Conversely, assume

that our map is an isomorphism on stalks at x. Then the lemma shows that $\Omega_{X/Y}$ is free of rank r after replacing X by an open neighbourhood of x. On the other hand, we may also assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$ where $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and where $f_1, \ldots, f_m$ is a Koszul regular sequence (this follows from the definition of local complete intersection morphisms). Clearly this implies $r = n - m$. We conclude that the rank of the matrix of partials $\partial f_j/\partial x_i$ in the residue field at x is m. Thus after reordering the variables we may assume the determinant of $(\partial f_j/\partial x_i)_{1 \leq i,j \leq m}$ is invertible in an open neighbourhood of x. It follows that $R \to A$ is smooth at this point, see for example Algebra, Example 10.137.7. □

0EA0 **Lemma 48.29.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $r \geq 0$. Assume*

(1) *Y is Cohen-Macaulay (Properties, Definition 28.8.1),*
(2) *f factors as $X \to P \to Y$ where the first morphism is an immersion and the second is smooth and proper,*
(3) *if $x \in X$ and $\dim(\mathcal{O}_{X,x}) \leq 1$, then f is Koszul at x (More on Morphisms, Definition 37.62.2), and*
(4) *if ξ is a generic point of an irreducible component of X, then we have $trdeg_{\kappa(f(\xi))}\kappa(\xi) = r$.*

Then with $\omega_{X/Y} = H^{-r}(f^!\mathcal{O}_Y)$ there is a map

$$\wedge^r \Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the locus where f is smooth.

Proof. Let $U \subset X$ be the open subscheme over which f is a local complete intersection morphism. Since f has relative dimension r at all generic points by assumption (4) we see that the locally constant function of Lemma 48.29.2 is constant with value r and we obtain a map

$$\wedge^r \Omega_{X/Y}|_U = \wedge^r \Omega_{U/Y} \longrightarrow \omega_{U/Y} = \omega_{X/Y}|_U$$

which is an isomorphism in the smooth points of f (this locus is contained in U because a smooth morphism is a local complete intersection morphism). By Lemma 48.21.5 and the assumption that Y is Cohen-Macaulay the module $\omega_{X/Y}$ is (S_2). Since U contains all the points of codimension 1 by condition (3) and using Divisors, Lemma 31.5.11 we see that $j_*\omega_{U/Y} = \omega_{X/Y}$. Hence the map over U extends to X and the proof is complete. □

48.30. Extension by zero for coherent modules

0G2G The material in this section and the next few can be found in the appendix by Deligne of [**Har66**].

In this section $j : U \to X$ will be an open immersion of Noetherian schemes. We are going to consider inverse systems (K_n) in $D^b_{Coh}(\mathcal{O}_X)$ constructed as follows. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Then we can set

$$K_n = \mathcal{I}^n \mathcal{F}^\bullet$$

More precisely, K_n is the object of $D^b_{Coh}(\mathcal{O}_X)$ represented by the complex whose term in degree q is the coherent submodule $\mathcal{I}^n\mathcal{F}^q$ of $\mathcal{F}^q$. Observe that the maps

$\ldots \to K_3 \to K_2 \to K_1$ induce isomorphisms on restriction to U. Let us call such a system a *Deligne system*.

0G2H **Lemma 48.30.1.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let (K_n) be a Deligne system and denote $K \in D^b_{Coh}(\mathcal{O}_U)$ the value of the constant system $(K_n|_U)$. Let L be an object of $D^b_{Coh}(\mathcal{O}_X)$. Then $\operatorname{colim} \operatorname{Hom}_X(K_n, L) = \operatorname{Hom}_U(K, L|_U)$.*

Proof. Let $L \to M \to N \to L[1]$ be a distinguished triangle in $D^b_{Coh}(\mathcal{O}_X)$. Then we obtain a commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, L) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, M) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, N) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \operatorname{Hom}_U(K, L|_U) & \longrightarrow & \operatorname{Hom}_U(K, M|_U) & \longrightarrow & \operatorname{Hom}_U(K, N|_U) & \longrightarrow & \ldots \end{array}$$

whose rows are exact by Derived Categories, Lemma 13.4.2 and Algebra, Lemma 10.8.8. Hence if the statement of the lemma holds for $N[-1]$, L, N, and $L[1]$ then it holds for M by the 5-lemma. Thus, using the distinguished triangles for the canonical truncations of L (see Derived Categories, Remark 13.12.4) we reduce to the case that L has only one nonzero cohomology sheaf.

Choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules and a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n\mathcal{F}^\bullet$. Using "stupid" truncations we obtain compatible termwise split short exact sequences of complexes

$$0 \to \sigma_{\geq a+1}\mathcal{I}^n\mathcal{F}^\bullet \to \mathcal{I}^n\mathcal{F}^\bullet \to \sigma_{\leq a}\mathcal{I}^n\mathcal{F}^\bullet \to 0$$

which in turn correspond to compatible systems of distinguished triangles in $D^b_{Coh}(\mathcal{O}_X)$. Arguing as above we reduce to the case where $\mathcal{F}^\bullet$ has only one nonzero term. This reduces us to the case discussed in the next paragraph.

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a coherent $\mathcal{O}_X$-module $\mathcal{G}$ we have to show that the canonical map

$$\operatorname{colim} \operatorname{Ext}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G}) \longrightarrow \operatorname{Ext}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$$

is an isomorphism for all $i \geq 0$. For $i = 0$ this is Cohomology of Schemes, Lemma 30.10.5. Assume $i > 0$.

Injectivity. Let $\xi \in \operatorname{Ext}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G})$ be an element whose restriction to U is zero. We have to show there exists an $m \geq n$ such that the restriction of ξ to $\mathcal{I}^m\mathcal{F} = \mathcal{I}^{m-n}\mathcal{I}^n\mathcal{F}$ is zero. After replacing $\mathcal{F}$ by $\mathcal{I}^n\mathcal{F}$ we may assume $n = 0$, i.e., we have $\xi \in \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G})$ whose restriction to U is zero. By Derived Categories of Schemes, Proposition 36.11.2 we have $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$. Hence we can compute the Ext group in the abelian category of coherent $\mathcal{O}_X$-modules. This implies there exists an surjection $\alpha : \mathcal{F}'' \to \mathcal{F}$ such that $\xi \circ \alpha = 0$ (this is where we use that $i > 0$). Set $\mathcal{F}' = \operatorname{Ker}(\alpha)$ so that we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F}'' \to \mathcal{F} \to 0$$

It follows that ξ is the image of an element $\xi' \in \operatorname{Ext}^{i-1}_X(\mathcal{F}', \mathcal{G})$ whose restriction to U is in the image of $\operatorname{Ext}^{i-1}_U(\mathcal{F}''|_U, \mathcal{G}|_U) \to \operatorname{Ext}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U)$. By Artin-Rees the inverse systems $(\mathcal{I}^n\mathcal{F}')$ and $(\mathcal{I}^n\mathcal{F}'' \cap \mathcal{F}')$ are pro-isomorphic, see Cohomology

of Schemes, Lemma 30.10.3. Since we have the compatible system of short exact sequences

$$0 \to \mathcal{F}' \cap \mathcal{I}^n\mathcal{F}'' \to \mathcal{I}^n\mathcal{F}'' \to \mathcal{I}^n\mathcal{F} \to 0$$

we obtain a commutative diagram

$$\begin{array}{ccccc}
\operatorname{colim} \operatorname{Ext}^{i-1}_X(\mathcal{I}^n\mathcal{F}'', \mathcal{G}) & \longrightarrow & \operatorname{colim} \operatorname{Ext}^{i-1}_X(\mathcal{F}' \cap \mathcal{I}^n\mathcal{F}'', \mathcal{G}) & \longrightarrow & \operatorname{colim} \operatorname{Ext}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ext}^{i-1}_U(\mathcal{F}''|_U, \mathcal{G}|_U) & \longrightarrow & \operatorname{Ext}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U) & \longrightarrow & \operatorname{Ext}^{i-1}_U(\mathcal{F}|_U, \mathcal{G}|_U)
\end{array}$$

with exact rows. By induction on i and the comment on inverse systems above we find that the left two vertical arrows are isomorphisms. Now ξ gives an element in the top right group which is the image of ξ' in the middle top group, which in turn maps to an element of the bottom middle group coming from some element in the left bottom group. We conclude that ξ maps to zero in $\operatorname{Ext}^i_X(\mathcal{I}^n\mathcal{F}, \mathcal{G})$ for some n as desired.

Surjectivity. Let $\xi \in \operatorname{Ext}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$. Arguing as above using that $i > 0$ we can find an surjection $\mathcal{H} \to \mathcal{F}|_U$ of coherent $\mathcal{O}_U$-modules such that ξ maps to zero in $\operatorname{Ext}^i_U(\mathcal{H}, \mathcal{G}|_U)$. Then we can find a map $\varphi : \mathcal{F}'' \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to U is $\mathcal{H} \to \mathcal{F}|_U$, see Properties, Lemma 28.22.4. Observe that the lemma doesn't guarantee φ is surjective but this won't matter (it is possible to pick a surjective φ with a little bit of additional work). Denote $\mathcal{F}' = \operatorname{Ker}(\varphi)$. The short exact sequence

$$0 \to \mathcal{F}'|_U \to \mathcal{F}''|_U \to \mathcal{F}|_U \to 0$$

shows that ξ is the image of ξ' in $\operatorname{Ext}^{i-1}_U(\mathcal{F}'|_U, \mathcal{G}|_U)$. By induction on i we can find an n such that ξ' is the image of some ξ'_n in $\operatorname{Ext}^{i-1}_X(\mathcal{I}^n\mathcal{F}', \mathcal{G})$. By Artin-Rees we can find an $m \geq n$ such that $\mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'' \subset \mathcal{I}^n\mathcal{F}'$. Using the short exact sequence

$$0 \to \mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'' \to \mathcal{I}^m\mathcal{F}'' \to \mathcal{I}^m \operatorname{Im}(\varphi) \to 0$$

the image of ξ'_n in $\operatorname{Ext}^{i-1}_X(\mathcal{F}' \cap \mathcal{I}^m\mathcal{F}'', \mathcal{G})$ maps by the boundary map to an element ξ_m of $\operatorname{Ext}^i_X(\mathcal{I}^m \operatorname{Im}(\varphi), \mathcal{G})$ which maps to ξ. Since $\operatorname{Im}(\varphi)$ and $\mathcal{F}$ agree over U we see that $\mathcal{F}/\mathcal{I}^m \operatorname{Im}(\varphi)$ is supported on $X \setminus U$. Hence there exists an $l \geq m$ such that $\mathcal{I}^l\mathcal{F} \subset \mathcal{I}^m \operatorname{Im}(\varphi)$, see Cohomology of Schemes, Lemma 30.10.2. Taking the image of ξ_m in $\operatorname{Ext}^i_X(\mathcal{I}^l\mathcal{F}, \mathcal{G})$ we win. □

0G4K **Lemma 48.30.2.** *The result of Lemma 48.30.1 holds even for $L \in D^+_{Coh}(\mathcal{O}_X)$.*

Proof. Namely, if (K_n) is a Deligne system then there exists a $b \in \mathbf{Z}$ such that $H^i(K_n) = 0$ for $i > b$. Then $\operatorname{Hom}(K_n, L) = \operatorname{Hom}(K_n, \tau_{\leq b}L)$ and $\operatorname{Hom}(K, L) = \operatorname{Hom}(K, \tau_{\leq b}L)$. Hence using the result of the lemma for $\tau_{\leq b}L$ we win. □

0G4L **Lemma 48.30.3.** *Let $j : U \to X$ be an open immersion of Noetherian schemes.*

(1) *Let (K_n) and (L_n) be Deligne systems. Let K and L be the values of the constant systems $(K_n|_U)$ and $(L_n|_U)$. Given a morphism $\alpha : K \to L$ of $D(\mathcal{O}_U)$ there is a unique morphism of pro-systems $(K_n) \to (L_n)$ of $D^b_{Coh}(\mathcal{O}_X)$ whose restriction to U is α.*
(2) *Given $K \in D^b_{Coh}(\mathcal{O}_U)$ there exists a Deligne system (K_n) such that $(K_n|_U)$ is constant with value K.*

(3) *The pro-object (K_n) of $D^b_{Coh}(\mathcal{O}_X)$ of (2) is unique up to unique isomorphism (as a pro-object).*

Proof. Part (1) is an immediate consequence of Lemma 48.30.1 and the fact that morphisms between pro-systems are the same as morphisms between the functors they corepresent, see Categories, Remark 4.22.7.

Let K be as in (2). We can choose $K' \in D^b_{Coh}(\mathcal{O}_X)$ whose restriction to U is isomorphic to K, see Derived Categories of Schemes, Lemma 36.13.2. By Derived Categories of Schemes, Proposition 36.11.2 we can represent K' by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$ (for example choose $\mathcal{I}$ to correspond to the reduced induced subscheme structure on $X \setminus U$). Then we can set K_n equal to the object represented by the complex $\mathcal{I}^n\mathcal{F}^\bullet$ as in the introduction to this section.

Part (3) is immediate from parts (1) and (2). □

0G4M **Lemma 48.30.4.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let*

$$K \to L \to M \to K[1]$$

be a distinguished triangle of $D^b_{Coh}(\mathcal{O}_U)$. Then there exists an inverse system of distinguished triangles

$$K_n \to L_n \to M_n \to K_n[1]$$

in $D^b_{Coh}(\mathcal{O}_X)$ such that (K_n), (L_n), (M_n) are Deligne systems and such that the restriction of these distinguished triangles to U is isomorphic to the distinguished triangle we started out with.

Proof. Let (K_n) be as in Lemma 48.30.3 part (2). Choose an object L' of $D^b_{Coh}(\mathcal{O}_X)$ whose restriction to U is L (we can do this as the lemma shows). By Lemma 48.30.1 we can find an n and a morphism $K_n \to L'$ on X whose restriction to U is the given arrow $K \to L$. We conclude there is a morphism $K' \to L'$ of $D^b_{Coh}(\mathcal{O}_X)$ whose restriction to U is the given arrow $K \to L$.

By Derived Categories of Schemes, Proposition 36.11.2 we can find a morphism $\alpha^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of bounded complexes of coherent $\mathcal{O}_X$-modules representing $K' \to L'$. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$. Then we let $K_n = \mathcal{I}^n\mathcal{F}^\bullet$ and $L_n = \mathcal{I}^n\mathcal{G}^\bullet$. Observe that $\alpha^\bullet$ induces a morphism of complexes $\alpha^\bullet_n : \mathcal{I}^n\mathcal{F}^\bullet \to \mathcal{I}^n\mathcal{G}^\bullet$. From the construction of cones in Derived Categories, Section 13.9 it is clear that

$$C(\alpha_n)^\bullet = \mathcal{I}^n C(\alpha^\bullet)$$

and hence we can set $M_n = C(\alpha_n)^\bullet$. Namely, we have a compatible system of distinguished triangles (see discussion in Derived Categories, Section 13.12)

$$K_n \to L_n \to M_n \to K_n[1]$$

whose restriction to U is isomorphic to the distinguished triangle we started out with by axiom TR3 and Derived Categories, Lemma 13.4.3. □

0G4N **Remark 48.30.5.** Let $j : U \to X$ be an open immersion of Noetherian schemes. Sending $K \in D^b_{Coh}(\mathcal{O}_U)$ to a Deligne system whose restriction to U is K determines a functor

$$Rj_! : D^b_{Coh}(\mathcal{O}_U) \longrightarrow \text{Pro-}D^b_{Coh}(\mathcal{O}_X)$$

which is "exact" by Lemma 48.30.4 and which is "left adjoint" to the functor $j^* : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_U)$ by Lemma 48.30.1.

0G4P **Remark 48.30.6.** Let (A_n) and (B_n) be inverse systems of a category $\mathcal{C}$. Let us say a linear-pro-morphism from (A_n) to (B_n) is given by a compatible family of morphisms $\varphi_n : A_{cn+d} \to B_n$ for all $n \geq 1$ for some fixed integers $c, d \geq 1$. We'll say $(\varphi_n : A_{cn+d} \to B_n)$ and $(\psi_n : A_{c'n+d'} \to B_n)$ determine the same morphism if there exist $c'' \geq \max(c, c')$ and $d'' \geq \max(d, d')$ such that the two induced morphisms $A_{c''n+d''} \to B_n$ are the same for all n. It seems likely that Deligne systems (K_n) with given value on U are well defined up to linear-pro-isomorphisms. If we ever need this we will carefully formulate and prove this here.

0G4Q **Lemma 48.30.7.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let*

$$K_n \to L_n \to M_n \to K_n[1]$$

be an inverse system of distinguished triangles in $D^b_{Coh}(\mathcal{O}_X)$. If (K_n) and (M_n) are pro-isomorphic to Deligne systems, then so is (L_n).

Proof. Observe that the systems $(K_n|_U)$ and $(M_n|_U)$ are essentially constant as they are pro-isomorphic to constant systems. Denote K and M their values. By Derived Categories, Lemma 13.42.2 we see that the inverse system $L_n|_U$ is essentially constant as well. Denote L its value. Let $N \in D^b_{Coh}(\mathcal{O}_X)$. Consider the commutative diagram

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(M_n, N) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(L_n, N) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, N) \longrightarrow \ldots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & \operatorname{Hom}_U(M, N|_U) & \longrightarrow & \operatorname{Hom}_U(L, N|_U) & \longrightarrow & \operatorname{Hom}_U(K, N|_U) \longrightarrow \ldots
\end{array}$$

By Lemma 48.30.1 and the fact that isomorphic ind-systems have the same colimit, we see that the vertical arrows two to the right and two to the left of the middle one are isomorphisms. By the 5-lemma we conclude that the middle vertical arrow is an isomorphism. Now, if (L'_n) is a Deligne system whose restriction to U has constant value L (which exists by Lemma 48.30.3), then we have $\operatorname{colim} \operatorname{Hom}_X(L'_n, N) = \operatorname{Hom}_U(L, N|_U)$ as well. Hence the pro-systems (L_n) and (L'_n) are pro-isomorphic by Categories, Remark 4.22.7. □

0G4R **Lemma 48.30.8.** *Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}^\bullet$ be a complex of coherent $\mathcal{O}_X$-modules. Let $p \in \mathbf{Z}$. Set $\mathcal{H} = H^p(\mathcal{F}^\bullet)$ and $\mathcal{H}_n = H^p(\mathcal{I}^n\mathcal{F}^\bullet)$. Then there are canonical $\mathcal{O}_X$-module maps*

$$\ldots \to \mathcal{H}_3 \to \mathcal{H}_2 \to \mathcal{H}_1 \to \mathcal{H}$$

There exists a $c > 0$ such that for $n \geq c$ the image of $\mathcal{H}_n \to \mathcal{H}$ is contained in $\mathcal{I}^{n-c}\mathcal{H}$ and there is a canonical $\mathcal{O}_X$-module map $\mathcal{I}^n\mathcal{H} \to \mathcal{H}_{n-c}$ such that the compositions

$$\mathcal{I}^n\mathcal{H} \to \mathcal{H}_{n-c} \to \mathcal{I}^{n-2c}\mathcal{H} \quad \text{and} \quad \mathcal{H}_n \to \mathcal{I}^{n-c}\mathcal{H} \to \mathcal{H}_{n-2c}$$

are the canonical ones. In particular, the inverse systems $(\mathcal{H}_n)$ and $(\mathcal{I}^n\mathcal{H})$ are isomorphic as pro-objects of $Mod(\mathcal{O}_X)$.

Proof. If X is affine, translated into algebra this is More on Algebra, Lemma 15.102.1. In the general case, argue exactly as in the proof of that lemma replacing the reference to Artin-Rees in algebra with a reference to Cohomology of Schemes, Lemma 30.10.3. Details omitted. □

0G4S **Lemma 48.30.9.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $a \leq b$ be integers. Let (K_n) be an inverse system of $D^b_{Coh}(\mathcal{O}_X)$ such that $H^i(K_n) = 0$ for $i \notin [a, b]$. The following are equivalent*

(1) *(K_n) is pro-isomorphic to a Deligne system,*
(2) *for every $p \in \mathbf{Z}$ there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that the pro-systems $(H^p(K_n))$ and $(\mathcal{I}^n\mathcal{F})$ are pro-isomorphic.*

Proof. Assume (1). To prove (2) holds we may assume (K_n) is a Deligne system. By definition we may choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$-modules and a quasi-coherent sheaf of ideals cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n\mathcal{F}^\bullet$. Thus the result follows from Lemma 48.30.8.

Assume (2). We will prove that (K_n) is as in (1) by induction on $b - a$. If $a = b$ then (1) holds essentially by assumption. If $a < b$ then we consider the compatible system of distinguished triangles

$$\tau_{\leq a}K_n \to K_n \to \tau_{\geq a+1}K_n \to (\tau_{\leq a}K_n)[1]$$

See Derived Categories, Remark 13.12.4. By induction on $b-a$ we know that $\tau_{\leq a}K_n$ and $\tau_{\geq a+1}K_n$ are pro-isomorphic to Deligne systems. We conclude by Lemma 48.30.7. □

0G4T **Lemma 48.30.10.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let (K_n) be an inverse system in $D^b_{Coh}(\mathcal{O}_X)$. Let $X = W_1 \cup \ldots \cup W_r$ be an open covering. The following are equivalent*

(1) *(K_n) is pro-isomorphic to a Deligne system,*
(2) *for each i the restriction $(K_n|_{W_i})$ is pro-isomorphic to a Deligne system with respect to the open immersion $U \cap W_i \to W_i$.*

Proof. By induction on r. If $r = 1$ then the result is clear. Assume $r > 1$. Set $V = W_1 \cup \ldots \cup W_{r-1}$. By induction we see that $(K_n|_V)$ is a Deligne system. This reduces us to the discussion in the next paragraph.

Assume $X = V \cup W$ is an open covering and $(K_n|_W)$ and $(K_n|_V)$ are pro-isomorphic to Deligne systems. We have to show that (K_n) is pro-isomorphic to a Deligne system. Observe that $(K_n|_{V\cap W})$ is pro-isomorphic to a Deligne system (it follows immediately from the construction of Deligne systems that restrictions to opens preserves them). In particular the pro-systems $(K_n|_{U\cap V})$, $(K_n|_{U\cap W})$, and $(K_n|_{U\cap V\cap W})$ are essentially constant. It follows from the distinguished triangles in Cohomology, Lemma 20.33.2 and Derived Categories, Lemma 13.42.2 that $(K_n|_U)$ is essentially constant. Denote $K \in D^b_{Coh}(\mathcal{O}_U)$ the value of this system. Let L be

an object of $D^b_{Coh}(\mathcal{O}_X)$. Consider the diagram

$$\begin{array}{ccc}
\operatorname{colim}\operatorname{Ext}^{-1}(K_n|_V, L|_V) \oplus \operatorname{colim}\operatorname{Ext}^{-1}(K_n|_W, L|_W) & \longrightarrow & \operatorname{Ext}^{-1}(K|_{U\cap V}, L|_{U\cap V}) \oplus \operatorname{Ext}^{-1}(K|_{U\cap W}, L|_{U\cap W}) \\
\downarrow & & \downarrow \\
\operatorname{colim}\operatorname{Ext}^{-1}(K_n|_{V\cap W}, L|_{V\cap W}) & \longrightarrow & \operatorname{Ext}^{-1}(K|_{U\cap V\cap W}, L|_{U\cap V\cap W}) \\
\downarrow & & \downarrow \\
\operatorname{colim}\operatorname{Hom}(K_n, L) & \longrightarrow & \operatorname{Hom}(K|_U, L|_U) \\
\downarrow & & \downarrow \\
\operatorname{colim}\operatorname{Hom}(K_n|_V, L|_V) \oplus \operatorname{colim}\operatorname{Hom}(K_n|_W, L|_W) & \longrightarrow & \operatorname{Hom}(K|_{U\cap V}, L|_{U\cap V}) \oplus \operatorname{Hom}(K|_{U\cap W}, L|_{U\cap W}) \\
\downarrow & & \downarrow \\
\operatorname{colim}\operatorname{Hom}(K_n|_{V\cap W}, L|_{V\cap W}) & \longrightarrow & \operatorname{Hom}(K|_{U\cap V\cap W}, L|_{U\cap V\cap W})
\end{array}$$

The vertical sequences are exact by Cohomology, Lemma 20.33.3 and the fact that filtered colimits are exact. All horizontal arrows except for the middle one are isomorphisms by Lemma 48.30.1 and the fact that pro-isomorphic systems have the same colimits. Hence the middle one is an isomorphism too by the 5-lemma. It follows that (K_n) is pro-isomorphic to a Deligne system for K. Namey, if (K'_n) is a Deligne system whose restriction to U has constant value K (which exists by Lemma 48.30.3), then we have $\operatorname{colim}\operatorname{Hom}_X(K'_n, L) = \operatorname{Hom}_U(K, L|_U)$ as well. Hence the pro-systems (K_n) and (K'_n) are pro-isomorphic by Categories, Remark 4.22.7. □

0G4U **Lemma 48.30.11.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Let K be in $D^b_{Coh}(\mathcal{O}_X)$. Then*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n$$

is pro-isomorphic to a Deligne system with constant value $K|_U$ over U.

Proof. By Lemma 48.30.10 the question is local on X. Thus we may assume X is the spectrum of a Noetherian ring. In this case the statement follows from the algebra version which is More on Algebra, Lemma 15.102.6. □

48.31. Preliminaries to compactly supported cohomology

0G4V In Situation 48.16.1 let $f : X \to Y$ be a morphism in the category FTS_S. Using the constructions in the previous section, we will construct a functor

$$Rf_! : D^b_{Coh}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{Coh}(\mathcal{O}_Y)$$

which reduces to the functor of Remark 48.30.5 if f is an open immersion and in general is constructed using a compactification of f. Before we do this, we need the following lemmas to prove our construction is well defined.

0G4W **Lemma 48.31.1.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open subscheme and set $U = f^{-1}(V)$. Picture*

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j'} & Y \end{array}$$

Then we have a canonical isomorphism $Rj'_! \circ Rg_* \to Rf_* \circ Rj_!$ of functors $D^b_{Coh}(\mathcal{O}_U) \to$ Pro-$D^b_{Coh}(\mathcal{O}_Y)$ where $Rj_!$ and $Rj'_!$ are as in Remark 48.30.5.

First proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_U)$. Let (K_n) be a Deligne system for $U \to X$ whose restriction to U is constant with value K. Of course this means that (K_n) represents $Rj_!K$ in Pro-$D^b_{Coh}(\mathcal{O}_X)$. Observe that both $Rj'_!Rg_*K$ and $Rf_*Rj_!K$ restrict to the constant pro-object with value Rg_*K on V. This is immediate for the first one and for the second one it follows from the fact that $(Rf_*K_n)|_V = Rg_*(K_n|_U) = Rg_*K$. By the uniqueness of Deligne systems in Lemma 48.30.3 it suffices to show that (Rf_*K_n) is pro-isomorphic to a Deligne system. The lemma referenced will also show that the isomorphism we obtain is functorial.

Proof that (Rf_*K_n) is pro-isomorphic to a Deligne system. First, we observe that the question is independent of the choice of the Deligne system (K_n) corresponding to K (by the aforementioned uniqueness). By Lemmas 48.30.4 and 48.30.7 if we have a distinguished triangle

$$K \to L \to M \to K[1]$$

in $D^b_{Coh}(\mathcal{O}_U)$ and the result holds for K and M, then the result holds for L. Using the distinguished triangles of canonical truncations (Derived Categories, Remark 13.12.4) we reduce to the problem studied in the next paragraph.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals cutting out $Y \setminus V$. Denote $\mathcal{J}^n\mathcal{F}$ the image of $f^*\mathcal{J}^n \otimes \mathcal{F} \to \mathcal{F}$. We have to show that $(Rf_*(\mathcal{J}^n\mathcal{F}))$ is a Deligne system. By Lemma 48.30.10 the question is local on Y. Thus we may assume $Y = \mathrm{Spec}(A)$ is affine and $\mathcal{J}$ corresponds to an ideal $I \subset A$. By Lemma 48.30.9 it suffices to show that the inverse system of cohomology modules $(H^p(X, I^n\mathcal{F}))$ is pro-isomorphic to the inverse system (I^nM) for some finite A-module M. This is shown in Cohomology of Schemes, Lemma 30.20.3. □

Second proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_U)$. Let L be an object of $D^b_{Coh}(\mathcal{O}_Y)$. We will construct a bijection

$$\mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) \longrightarrow \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rf_*Rj_!K, L)$$

functorial in K and L. Fixing K this will determine an isomorphism of pro-objects $Rf_*Rj_!K \to Rj'_!Rg_*K$ by Categories, Remark 4.22.7 and varying K we obtain that this determines an isomorphism of functors. To actually produce the isomorphism

we use the sequence of functorial equalities

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) &= \mathrm{Hom}_V(Rg_*K, L|_V) \\ &= \mathrm{Hom}_U(K, g^!(L|_V)) \\ &= \mathrm{Hom}_U(K, f^!L|_U)) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_X)}(Rj_!K, f^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rf_*Rj_!K, L)\end{aligned}$$

The first equality is true by Lemma 48.30.1. The second equality is true because g is proper (as the base change of f to V) and hence $g^!$ is the right adjoint of pushforward by construction, see Section 48.16. The third equality holds as $g^!(L|_V) = f^!L|_U$ by Lemma 48.17.2. Since $f^!L$ is in $D^+_{Coh}(\mathcal{O}_X)$ by Lemma 48.17.6 the fourth equality follows from Lemma 48.30.2. The fifth equality holds again because $f^!$ is the right adjoint to Rf_* as f is proper. □

0G4X **Lemma 48.31.2.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $j' : U \to X'$ be a compactification of U over X (see proof) and denote $f : X' \to X$ the structure morphism. Then we have a canonical isomorphism $Rj_! \to Rf_* \circ R(j')_!$ of functors $D^b_{Coh}(\mathcal{O}_U) \to Pro\text{-}D^b_{Coh}(\mathcal{O}_X)$ where $Rj_!$ and $Rj'_!$ are as in Remark 48.30.5.*

Proof. The fact that X' is a compactification of U over X means precisely that $f : X' \to X$ is proper, that j' is an open immersion, and $j = f \circ j'$. See More on Flatness, Section 38.32. If $j'(U) = f^{-1}(j(U))$, then the lemma follows immediately from Lemma 48.31.1. If $j'(U) \neq f^{-1}(j(U))$, then denote $X'' \subset X'$ the scheme theoretic closure of $j' : U \to X'$ and denote $j'' : U \to X''$ the corresponding open immersion. Picture

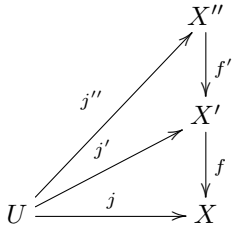

By More on Flatness, Lemma 38.32.1 part (c) and the discussion above we have isomorphisms $Rf'_* \circ Rj''_! = Rj'_!$ and $R(f \circ f')_* \circ Rj''_! = Rj_!$. Since $R(f \circ f')_* = Rf_* \circ Rf'_*$ we conclude. □

0G4Y **Remark 48.31.3.** Let $X \supset U \supset U'$ be open subschemes of a Noetherian scheme X. Denote $j : U \to X$ and $j' : U' \to X$ the inclusion morphisms. We claim there is a canonical map

$$Rj'_!(K|_{U'}) \longrightarrow Rj_!K$$

functorial for K in $D^b_{Coh}(\mathcal{O}_U)$. Namely, by Lemma 48.30.1 we have for any L in $D^b_{Coh}(\mathcal{O}_X)$ the map

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_X)}(Rj_!K, L) &= \mathrm{Hom}_U(K, L|_U) \\ &\to \mathrm{Hom}_{U'}(K|_{U'}, L|_{U'}) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_X)}(Rj'_!(K|_{U'}), L)\end{aligned}$$

functorial in L and K'. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rj'_!(K|_{U'}) \to Rj_!K$ which is functorial in K by the functoriality of the arrow above in K.

Here is an explicit construction of this arrow. Namely, suppose that $\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules whose restriction to U represents K in the derived category. We have seen in the proof of Lemma 48.30.3 that such a complex always exists. Let $\mathcal{I}$, resp. $\mathcal{I}'$ be a quasi-coherent sheaf of ideals on X with $V(\mathcal{I}) = X \setminus U$, resp. $V(\mathcal{I}') = X \setminus U'$. After replacing $\mathcal{I}$ by $\mathcal{I} + \mathcal{I}'$ we may assume $\mathcal{I}' \subset \mathcal{I}$. By construction $Rj_!K$, resp. $Rj'_!(K|_{U'})$ is represented by the inverse system (K_n), resp. (K'_n) of $D^b_{Coh}(\mathcal{O}_X)$ with

$$K_n = \mathcal{I}^n\mathcal{F}^\bullet \quad \text{resp.} \quad K'_n = (\mathcal{I}')^n\mathcal{F}^\bullet$$

Clearly the map constructed above is given by the maps $K'_n \to K_n$ coming from the inclusions $(\mathcal{I}')^n \subset \mathcal{I}^n$.

48.32. Compactly supported cohomology for coherent modules

0G4Z In Situation 48.16.1 given a morphism $f : X \to Y$ in FTS_S, we will define a functor

$$Rf_! : D^b_{Coh}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{Coh}(\mathcal{O}_Y)$$

Namely, we choose a compactification $j : X \to \overline{X}$ over Y which is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Then we set

$$Rf_!K = R\overline{f}_*Rj_!K$$

for $K \in D^b_{Coh}(\mathcal{O}_X)$ where $Rj_!$ is as in Remark 48.30.5.

0G50 **Lemma 48.32.1.** *The functor $Rf_!$ is, up to isomorphism, independent of the choice of the compactification.*

In fact, the functor $Rf_!$ will be characterized as a "left adjoint" to $f^!$ which will determine it up to unique isomorphism.

Proof. Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 38.33.8 and Lemmas 38.32.1 and 38.32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ Rj_!$. Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ Rj_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ Rj_{1,!}$$

using Lemma 48.31.2 in the first equality. In this way we see our functor is independent of the choice of compactification up to isomorphism. □

0G51 **Proposition 48.32.2.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Then the functors $Rf_!$ and $f^!$ are adjoint in the following sense: for all $K \in D^b_{Coh}(\mathcal{O}_X)$ and $L \in D^+_{Coh}(\mathcal{O}_Y)$ we have*

$$\text{Hom}_X(K, f^!L) = \text{Hom}_{Pro\text{-}D^+_{Coh}(\mathcal{O}_Y)}(Rf_!K, L)$$

bifunctorially in K and L.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Then we have

$$\begin{aligned}\mathrm{Hom}_X(K, f^!L) &= \mathrm{Hom}_X(K, j^*\overline{f}^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_{\overline{X}})}(Rj_!K, \overline{f}^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_Y)}(Rf_*Rj_!K, L) \\ &= \mathrm{Hom}_{\text{Pro-}D^+_{Coh}(\mathcal{O}_Y)}(Rf_!K, L)\end{aligned}$$

The first equality follows immediately from the construction of $f^!$ in Section 48.16. By Lemma 48.17.6 we have $\overline{f}^!L$ in $D^+_{Coh}(\mathcal{O}_{\overline{X}})$ hence the second equality follows from Lemma 48.30.2. Since $\overline{f}$ is proper the functor $\overline{f}^!$ is the right adjoint of pushforward by construction. This is why we have the third equality. The fourth equality holds because $Rf_! = Rf_*Rj_!$. □

0G52 **Lemma 48.32.3.** *In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S. Let*

$$K \to L \to M \to K[1]$$

be a distinguished triangle of $D^b_{Coh}(\mathcal{O}_X)$. Then there exists an inverse system of distinguished triangles

$$K_n \to L_n \to M_n \to K_n[1]$$

in $D^b_{Coh}(\mathcal{O}_Y)$ such that the pro-systems (K_n), (L_n), and (M_n) give $Rf_!K$, $Rf_!L$, and $Rf_!M$.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Choose an inverse system of distinguished triangles

$$\overline{K}_n \to \overline{L}_n \to \overline{M}_n \to \overline{K}_n[1]$$

in $D^b_{Coh}(\mathcal{O}_{\overline{X}})$ as in Lemma 48.30.4 corresponding to the open immersion j and the given distinguished triangle. Take $K_n = R\overline{f}_*\overline{K}_n$ and similarly for L_n and M_n. This works by the very definition of $Rf_!$. □

0G53 **Remark 48.32.4.** Let $\mathcal{C}$ be a category. Suppose given an inverse system

$$\ldots \xrightarrow{\alpha_4} (M_{3,n}) \xrightarrow{\alpha_3} (M_{2,n}) \xrightarrow{\alpha_2} (M_{1,n})$$

of inverse systems in the category of pro-objects of $\mathcal{C}$. In other words, the arrows α_i are morphisms of pro-objects. By Categories, Example 4.22.6 we can represent each α_i by a pair (m_i, a_i) where $m_i : \mathbf{N} \to \mathbf{N}$ is an increasing function and $a_{i,n} : M_{i,m_i(n)} \to M_{i-1,n}$ is a morphism of $\mathcal{C}$ making the diagrams

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & M_{i,m_i(3)} & \longrightarrow & M_{i,m_i(2)} & \longrightarrow & M_{i,m_i(1)} \\
 & & \downarrow{\scriptstyle a_{i,3}} & & \downarrow{\scriptstyle a_{i,2}} & & \downarrow{\scriptstyle a_{i,1}} \\
\ldots & \longrightarrow & M_{i-1,3} & \longrightarrow & M_{i-1,2} & \longrightarrow & M_{i-1,1}
\end{array}$$

commute. By replacing $m_i(n)$ by $\max(n, m_i(n))$ and adjusting the morphisms $a_i(n)$ accordingly (as in the example referenced) we may assume that $m_i(n) \geq n$. In this situation consider the inverse system

$$\ldots \to M_{4,m_4(m_3(m_2(4)))} \to M_{3,m_3(m_2(3))} \to M_{2,m_2(2)} \to M_{1,1}$$

with general term

$$M_k = M_{k,m_k(m_{k-1}(\ldots(m_2(k))\ldots))}$$

For any object N of $\mathcal{C}$ we have

$$\operatorname{colim}_i \operatorname{colim}_n \operatorname{Mor}_{\mathcal{C}}(M_{i,n}, N) = \operatorname{colim}_k \operatorname{Mor}_{\mathcal{C}}(M_k, N)$$

We omit the details. In other words, we see that the inverse system (M_k) has the property

$$\operatorname{colim}_i \operatorname{Mor}_{\text{Pro-}\mathcal{C}}((M_{i,n}), N) = \operatorname{Mor}_{\text{Pro-}\mathcal{C}}((M_k), N)$$

This property determines the inverse system (M_k) up to pro-isomorphism by the discussion in Categories, Remark 4.22.7. In this way we can turn certain inverse systems in Pro-$\mathcal{C}$ into pro-objects with countable index categories.

0G54 **Remark 48.32.5.** In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. Let us define the composition

$$Rg_! \circ Rf_! : D^b_{Coh}(\mathcal{O}_X) \longrightarrow \text{Pro-}D^b_{Coh}(\mathcal{O}_Z)$$

Namely, by the very construction of $Rf_!$ for K in $D^b_{Coh}(\mathcal{O}_X)$ the output $Rf_!K$ is the pro-isomorphism class of an inverse system (M_n) in $D^b_{Coh}(\mathcal{O}_Y)$. Then, since $Rg_!$ is constructed similarly, we see that

$$\ldots \to Rg_!M_3 \to Rg_!M_2 \to Rg_!M_1$$

is an inverse system of Pro-$D^b_{Coh}(\mathcal{O}_Y)$. By the discussion in Remark 48.32.4 there is a unique pro-isomorphism class, which we will denote $Rg_!Rf_!K$, of inverse systems in $D^b_{Coh}(\mathcal{O}_Z)$ such that

$$\operatorname{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Z)}(Rg_!Rf_!K, L) = \operatorname{colim}_n \operatorname{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Z)}(Rg_!M_n, L)$$

We omit the discussion necessary to see that this construction is functorial in K as it will immediately follow from the next lemma.

0G55 **Lemma 48.32.6.** *In Situation 48.16.1 let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of FTS_S. With notation as in Remark 48.32.5 we have $Rg_! \circ Rf_! = R(g \circ f)_!$.*

Proof. By the discussion in Categories, Remark 4.22.7 it suffices to show that we obtain the same answer if we compute Hom into L in $D^b_{Coh}(\mathcal{O}_Z)$. To do this we compute, using the notation in Remark 48.32.5, as follows

$$\begin{aligned}
\operatorname{Hom}_Z(Rg_!Rf_!K, L) &= \operatorname{colim}_n \operatorname{Hom}_Z(Rg_!M_n, L) \\
&= \operatorname{colim}_n \operatorname{Hom}_Y(M_n, g^!L) \\
&= \operatorname{Hom}_Y(Rf_!K, g^!L) \\
&= \operatorname{Hom}_X(K, f^!g^!L) \\
&= \operatorname{Hom}_X(K, (g \circ f)^!L) \\
&= \operatorname{Hom}_Z(R(g \circ f)_!K, L)
\end{aligned}$$

The first equality is the definition of $Rg_!Rf_!K$. The second equality is Proposition 48.32.2 for g. The third equality is the fact that $Rf_!K$ is given by (M_n). The fourth equality is Proposition 48.32.2 for f. The fifth equality is Lemma 48.16.3. The sixth is Proposition 48.32.2 for $g \circ f$. □

0G56 **Remark 48.32.7.** In Situation 48.16.1 let $f : X \to Y$ be a morphism of FTS_S and let $U \subset X$ be an open. Set $g = f|_U : U \to Y$. Then there is a canonical morphism

$$Rg_!(K|_U) \longrightarrow Rf_!K$$

functorial in K in $D^b_{Coh}(\mathcal{O}_X)$ which can be defined in at least 3 ways.

(1) Denote $i : U \to X$ the inclusion morphism. We have $Rg_! = Rf_! \circ Ri_!$ by Lemma 48.32.6 and we can use $Rf_!$ applied to the map $Ri_!(K|_U) \to K$ which is a special case of Remark 48.31.3.
(2) Choose a compactification $j : X \to \overline{X}$ of X over Y with structure morphism $\overline{f} : \overline{X} \to Y$. Set $j' = j \circ i : U \to \overline{X}$. We can use that $Rf_! = R\overline{f}_* \circ Rj_!$ and $Rg_! = R\overline{f}_* \circ Rj'_!$ and we can use $R\overline{f}_*$ applied to the map $Rj'_!(K|_U) \to Rj_!K$ of Remark 48.31.3.
(3) We can use

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\ &\to \mathrm{Hom}_U(K|_U, f^!L|_U) \\ &= \mathrm{Hom}_U(K|_U, g^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rg_!(K|_U), L)\end{aligned}$$

functorial in L and K. Here we have used Proposition 48.32.2 twice and the construction of upper shriek functors which shows that $g^! = i^* \circ f^!$. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rg_!(K|_U) \to Rf_!K$ in Pro-$D^b_{Coh}(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K.

Each of these three constructions gives the same arrow; we omit the details.

0G57 **Remark 48.32.8.** Let us generalize the covariance of compactly supported cohomology given in Remark 48.32.7 to étale morphisms. Namely, in Situation 48.16.1 suppose given a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

of FTS_S with h étale. Then there is a canonical morphism

$$Rg_!(h^*K) \longrightarrow Rf_!K$$

functorial in K in $D^b_{Coh}(\mathcal{O}_X)$. We define this transformation using the sequence of maps

$$\begin{aligned}\mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\ &\to \mathrm{Hom}_U(h^*K, h^*(f^!L)) \\ &= \mathrm{Hom}_U(h^*K, h^!f^!L) \\ &= \mathrm{Hom}_U(h^*K, g^!L) \\ &= \mathrm{Hom}_{\text{Pro-}D^b_{Coh}(\mathcal{O}_Y)}(Rg_!(h^*K), L)\end{aligned}$$

functorial in L and K. Here we have used Proposition 48.32.2 twice, we have used the equality $h^* = h^!$ of Lemma 48.18.2, and we have used the equality $h^! \circ f^! = g^!$

of Lemma 48.16.3. The functoriality in L shows by Categories, Remark 4.22.7 that we obtain a canonical map $Rg_!(h^*K) \to Rf_!K$ in Pro-$D^b_{Coh}(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K.

0G58 **Remark 48.32.9.** In Remarks 48.32.7 and 48.32.8 we have seen that the construction of compactly supported cohomology is covariant with respect to open immersions and étale morphisms. In fact, the correct generality is that given a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

of FTS_S with h flat and quasi-finite there exists a canonical transformation

$$Rg_! \circ h^* \longrightarrow Rf_!$$

As in Remark 48.32.8 this map can be constructed using a transformation of functors $h^* \to h^!$ on $D^+_{Coh}(\mathcal{O}_X)$. Recall that $h^!K = h^*K \otimes \omega_{U/X}$ where $\omega_{U/X} = h^!\mathcal{O}_X$ is the relative dualizing sheaf of the flat quasi-finite morphism h (see Lemmas 48.17.9 and 48.21.6). Recall that $\omega_{U/X}$ is the same as the relative dualizing module which will be constructed in Discriminants, Remark 49.2.11 by Discriminants, Lemma 49.15.1. Thus we can use the trace element $\tau_{U/X} : \mathcal{O}_U \to \omega_{U/X}$ which will be constructed in Discriminants, Remark 49.4.7 to define our transformation. If we ever need this, we will precisely formulate and prove the result here.

48.33. Duality for compactly supported cohomology

0G59 Let k be a field. Let U be a separated scheme of finite type over k. Let K be an object of $D^b_{Coh}(\mathcal{O}_U)$. Let us define the compactly supported cohomology $H^i_c(U, K)$ of K as follows. Choose an open immersion $j : U \to X$ into a scheme proper over k and a Deligne system (K_n) for $j : U \to X$ whose restriction to U is constant with value K. Then we set

$$H^i_c(U, K) = \lim H^i(X, K_n)$$

We view this as a topological k-vector space using the limit topology (see More on Algebra, Section 15.36). There are several points to make here.

First, this definition is independent of the choice of X and (K_n). Namely, if $p : U \to \mathrm{Spec}(k)$ denotes the structure morphism, then we already know that $Rp_!K = (R\Gamma(X, K_n))$ is well defined up to pro-isomorphism in $D(k)$ hence so is the limit defining $H^i_c(U, K)$.

Second, it may seem more natural to use the expression

$$H^i(R\lim R\Gamma(X, K_n)) = R\Gamma(X, R\lim K_n)$$

but this would give the same answer: since the k-vector spaces $H^j(X, K_n)$ are finite dimensional, these inverse systems satisfy Mittag-Leffler and hence $R^1\lim$ terms of Cohomology, Lemma 20.37.1 vanish.

If $U' \subset U$ is an open subscheme, then there is a canonical map

$$H^i_c(U', K|_{U'}) \longrightarrow H^i_c(U, K)$$

functorial for K in $D^b_{Coh}(\mathcal{O}_U)$. See for example Remark 48.32.7. In fact, using Remark 48.32.8 we see that more generally such a map exists for an étale morphism $U' \to U$ of separated schemes of finite type over k.

If V is a k-vector space then we put a topology on $\mathrm{Hom}_k(V, k)$ as follows: write $V = \bigcup V_i$ as the filtered union of its finite dimensional k-subvector spaces and use the limit topology on $\mathrm{Hom}_k(V, k) = \lim \mathrm{Hom}_k(V_i, k)$. If $\dim_k V < \infty$ then the topology on $\mathrm{Hom}_k(V, k)$ is discrete. More generally, if $V = \mathrm{colim}_n V_n$ is written as a directed colimit of finite dimensional vector spaces, then $\mathrm{Hom}_k(V, k) = \lim \mathrm{Hom}_k(V_n, k)$ as topological vector spaces.

0G5A **Lemma 48.33.1.** *Let $p : U \to \mathrm{Spec}(k)$ be separated of finite type where k is a field. Let $\omega^\bullet_{U/k} = p^!\mathcal{O}_{\mathrm{Spec}(k)}$. There are canonical isomorphisms*

$$\mathrm{Hom}_k(H^i(U, K), k) = H^{-i}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k}))$$

of topological k-vector spaces functorial for K in $D^b_{Coh}(\mathcal{O}_U)$.

Proof. Choose a compactification $j : U \to X$ over k. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent ideal sheaf with $V(\mathcal{I}) = X \setminus U$. By Derived Categories of Schemes, Proposition 36.11.2 we may choose $M \in D^b_{Coh}(\mathcal{O}_X)$ with $K = M|_U$. We have

$$H^i(U, K) = \mathrm{Ext}^i_U(\mathcal{O}_U, M|_U) = \mathrm{colim}\, \mathrm{Ext}^i_X(\mathcal{I}^n, M) = \mathrm{colim}\, H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M))$$

by Lemma 48.30.1. Since $\mathcal{I}^n$ is a coherent $\mathcal{O}_X$-module, we have $\mathcal{I}^n$ in $D^-_{Coh}(\mathcal{O}_X)$, hence $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)$ is in $D^+_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.5.

Let $\omega^\bullet_{X/k} = q^!\mathcal{O}_{\mathrm{Spec}(k)}$ where $q : X \to \mathrm{Spec}(k)$ is the structure morphism, see Section 48.27. We find that

$$\begin{aligned}
&\mathrm{Hom}_k(H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)), k) \\
&\quad = \mathrm{Ext}^{-i}_X(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k}) \\
&\quad = H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k}))
\end{aligned}$$

by Lemma 48.27.1. By Lemma 48.2.4 part (1) the canonical map

$$R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega^\bullet_{X/k})$$

is an isomorphism. Observe that $\omega^\bullet_{U/k} = \omega^\bullet_{X/k}|_U$ because $p^!$ is constructed as $q^!$ composed with restriction to U. Hence $R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k})$ is an object of $D^b_{Coh}(\mathcal{O}_X)$ which restricts to $R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k})$ on U. Hence by Lemma 48.30.11 we conclude that

$$\lim H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}^n)$$

is an avatar for the right hand side of the equality of the lemma. Combining all the isomorphisms obtained in this manner we get the isomorphism of the lemma. □

0G5B **Lemma 48.33.2.** *With notation as in Lemma 48.33.1 suppose $U' \subset U$ is an open subscheme. Then the diagram*

$$\begin{array}{ccc}
\mathrm{Hom}_k(H^i(U, K), k) & \longrightarrow & H^{-i}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega^\bullet_{U/k})) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_k(H^i(U', K|_{U'}), k) & \longrightarrow & H^{-i}_c(U', R\mathcal{H}om_{\mathcal{O}_{U'}}(K, \omega^\bullet_{U'/k}))
\end{array}$$

is commutative. Here the horizontal arrows are the isomorphisms of Lemma 48.33.1, the vertical arrow on the left is the contragredient to the restriction map $H^i(U, K) \to H^i(U', K|_{U'})$, and the right vertical arrow is Remark 48.32.7 (see discussion before the lemma).

Proof. We strongly urge the reader to skip this proof. Choose X and M as in the proof of Lemma 48.33.1. We are going to drop the subscript $\mathcal{O}_X$ from $R\mathcal{H}om$ and $\otimes^{\mathbf{L}}$. We write

$$H^i(U, K) = \operatorname{colim} H^i(X, R\mathcal{H}om(\mathcal{I}^n, M))$$

and

$$H^i(U', K|_{U'}) = \operatorname{colim} H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M))$$

as in the proof of Lemma 48.33.1 where we choose $\mathcal{I}' \subset \mathcal{I}$ as in the discussion in Remark 48.31.3 so that the map $H^i(U, K) \to H^i(U', K|_{U'})$ is induced by the maps $(\mathcal{I}')^n \to \mathcal{I}^n$. We similarly write

$$H^i_c(U, R\mathcal{H}om(K, \omega^\bullet_{U/k})) = \lim H^i(X, R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} \mathcal{I}^n)$$

and

$$H^i_c(U', R\mathcal{H}om(K|_{U'}, \omega^\bullet_{U'/k})) = \lim H^i(X, R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} (\mathcal{I}')^n)$$

so that the arrow $H^i_c(U', R\mathcal{H}om(K|_{U'}, \omega^\bullet_{U'/k})) \to H^i_c(U, R\mathcal{H}om(K, \omega^\bullet_{U/k}))$ is similarly deduced from the maps $(\mathcal{I}')^n \to \mathcal{I}^n$. The diagrams

$$\begin{array}{ccc} R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} \mathcal{I}^n & \longrightarrow & R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega^\bullet_{X/k}) \\ \uparrow & & \uparrow \\ R\mathcal{H}om(M, \omega^\bullet_{X/k}) \otimes^{\mathbf{L}} (\mathcal{I}')^n & \longrightarrow & R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega^\bullet_{X/k}) \end{array}$$

commute because the construction of the horizontal arrows in Cohomology, Lemma 20.42.9 is functorial in all three entries. Hence we finally come down to the assertion that the diagrams

$$\begin{array}{ccc} \operatorname{Hom}_k(H^i(X, R\mathcal{H}om(\mathcal{I}^n, M)), k) & \longrightarrow & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega^\bullet_{X/k})) \\ \uparrow & & \uparrow \\ \operatorname{Hom}_k(H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M)), k) & \longrightarrow & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega^\bullet_{X/k})) \end{array}$$

commute. This is true because the duality isomorphism

$$\operatorname{Hom}_k(H^i(X, L), k) = \operatorname{Ext}^{-i}_X(L, \omega^\bullet_{X/k}) = H^{-i}(X, R\mathcal{H}om(L, \omega^\bullet_{X/k}))$$

is functorial for L in $D_{QCoh}(\mathcal{O}_X)$. □

0G5C **Lemma 48.33.3.** *Let X be a proper scheme over a field k. Let $K \in D^b_{Coh}(\mathcal{O}_X)$ with $H^i(K) = 0$ for $i < 0$. Set $\mathcal{F} = H^0(K)$. Let $Z \subset X$ be closed with complement $U = X \setminus U$. Then*

$$H^0_c(U, K|_U) \subset H^0(X, \mathcal{F})$$

is given by those global sections of $\mathcal{F}$ which vanish in an open neighbourhood of Z.

Proof. Consider the map $H^0_c(U, K|_U) \to H^0_X(X, K) = H^0(X, K) = H^0(X, \mathcal{F})$ of Remark 48.32.7. To study this we represent K by a bounded complex $\mathcal{F}^\bullet$ with $\mathcal{F}^i = 0$ for $i < 0$. Then we have by definition

$$H^0_c(U, K|_U) = \lim H^0(X, \mathcal{I}^n\mathcal{F}^\bullet) = \lim \mathrm{Ker}(H^0(X, \mathcal{I}^n\mathcal{F}^0) \to H^0(X, \mathcal{I}^n\mathcal{F}^1))$$

By Artin-Rees (Cohomology of Schemes, Lemma 30.10.3) this is the same as $\lim H^0(X, \mathcal{I}^n\mathcal{F})$. Thus the arrow $H^0_c(U, K|_U) \to H^0(X, \mathcal{F})$ is injective and the image consists of those global sections of $\mathcal{F}$ which are contained in the subsheaf $\mathcal{I}^n\mathcal{F}$ for any n. The characterization of these as the sections which vanish in a neighbourhood of Z comes from Krull's intersection theorem (Algebra, Lemma 10.51.4) by looking at stalks of $\mathcal{F}$. See discussion in Algebra, Remark 10.51.6 for the case of functions. □

48.34. Lichtenbaum's theorem

0G5D The theorem below was conjectured by Lichtenbaum and proved by Grothendieck (see [**Har67**]). There is a very nice proof of the theorem by Kleiman in [**Kle67**]. A generalization of the theorem to the case of cohomology with supports can be found in [**Lyu91**]. The most interesting part of the argument is contained in the proof of the following lemma.

0G5E **Lemma 48.34.1.** *Let U be a variety. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If $H^d(U, \mathcal{F})$ is nonzero, then $\dim(U) \geq d$ and if equality holds, then U is proper.*

Proof. By the Grothendieck's vanishing result in Cohomology, Proposition 20.20.7 we conclude that $\dim(U) \geq d$. Assume $\dim(U) = d$. Choose a compactification $U \to X$ such that U is dense in X. (This is possible by More on Flatness, Theorem 38.33.8 and Lemma 38.32.2.) After replacing X by its reduction we find that X is a proper variety of dimension d and we see that U is proper if and only if $U = X$. Set $Z = X \setminus U$. We will show that $H^d(U, \mathcal{F})$ is zero if Z is nonempty.

Choose a coherent $\mathcal{O}_X$-module $\mathcal{G}$ whose restriction to U is $\mathcal{F}$, see Properties, Lemma 28.22.5. Let $\omega_X^\bullet$ denote the dualizing complex of X as in Section 48.27. Set $\omega_U^\bullet = \omega_X^\bullet|_U$. Then $H^d(U, \mathcal{F})$ is dual to

$$H^{-d}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$$

by Lemma 48.33.1. By Lemma 48.27.1 we see that the cohomology sheaves of $\omega_X^\bullet$ vanish in degrees $< -d$ and $H^{-d}(\omega_X^\bullet) = \omega_X$ is a coherent $\mathcal{O}_X$-module which is (S_2) and whose support is X. In particular, ω_X is torsion free, see Divisors, Lemma 31.11.10. Thus we see that the cohomology sheaf

$$H^{-d}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \omega_X^\bullet)) = \mathcal{H}om(\mathcal{G}, \omega_X)$$

is torsion free, see Divisors, Lemma 31.11.12. Consequently this sheaf has no nonzero sections vanishing on any nonempty open of X (those would be torsion sections). Thus it follows from Lemma 48.33.3 that $H^{-d}_c(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$ is zero, and hence $H^d(U, \mathcal{F})$ is zero as desired. □

0G5F **Theorem 48.34.2.** *Let X be a nonempty separated scheme of finite type over a field k. Let $d = \dim(X)$. The following are equivalent*

1. *$H^d(X, \mathcal{F}) = 0$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X,*
2. *$H^d(X, \mathcal{F}) = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X, and*
3. *no irreducible component $X' \subset X$ of dimension d is proper over k.*

Proof. Assume there exists an irreducible component $X' \subset X$ (which we view as an integral closed subscheme) which is proper and has dimension d. Let $\omega_{X'}$ be a dualizing module of X' over k, see Lemma 48.27.1. Then $H^d(X', \omega_{X'})$ is nonzero as it is dual to $H^0(X', \mathcal{O}_{X'})$ by the lemma. Hence we see that $H^d(X, \omega_{X'}) = H^d(X', \omega_{X'})$ is nonzero and we conclude that (1) does not hold. In this way we see that (1) implies (3).

Let us prove that (3) implies (1). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $H^d(X, \mathcal{F})$ is nonzero. Choose a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

as in Cohomology of Schemes, Lemma 30.12.3. We obtain exact sequences

$$H^d(X, \mathcal{F}_i) \to H^d(X, \mathcal{F}_{i+1}) \to H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$$

Thus for some $i \in \{1, \ldots, m\}$ we find that $H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$ is nonzero. By our choice of the filtration this means that there exists an integral closed subscheme $Z \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ such that $H^d(Z, \mathcal{I})$ is nonzero. By Lemma 48.34.1 we conclude $\dim(Z) = d$ and Z is proper over k contradicting (3). Hence (3) implies (1).

Finally, let us show that (1) and (2) are equivalent for any Noetherian scheme X. Namely, (2) trivially implies (1). On the other hand, assume (1) and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as the filtered colimit of its coherent submodules, see Properties, Lemma 28.22.3. Then we have $H^d(X, \mathcal{F}) = \operatorname{colim} H^d(X, \mathcal{F}_i) = 0$ by Cohomology, Lemma 20.19.1. Thus (2) is true. □

48.35. Other chapters

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CHAPTER 49

Discriminants and Differents

0DWH

49.1. Introduction

0DWI In this chapter we study the different and discriminant of locally quasi-finite morphisms of schemes. A good reference for some of this material is [**Kun86**].

Given a quasi-finite morphism $f : Y \to X$ of Noetherian schemes there is a relative dualizing module $\omega_{Y/X}$. In Section 49.2 we construct this module from scratch, using Zariski's main theorem and étale localization methods. The key property is that given a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

with $g : X' \to X$ flat, $Y' \subset X' \times_X Y$ open, and $f' : Y' \to X'$ finite, then there is a canonical isomorphism

$$f'_*(g')^*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_{X'}}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_{X'})$$

as sheaves of $f'_*\mathcal{O}_{Y'}$-modules. In Section 49.4 we prove that if f is flat, then there is a canonical global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which for every commutative diagram as above maps $(g')^*\tau_{Y/X}$ to the trace map of Section 49.3 for the finite locally free morphism f'. In Section 49.9 we define the different for a flat quasi-finite morphism of Noetherian schemes as the annihilator of the cokernel of $\tau_{Y/X} : \mathcal{O}_X \to \omega_{Y/X}$.

The main goal of this chapter is to prove that for quasi-finite syntomic[1] f the different agrees with the Kähler different. The Kähler different is the zeroth fitting ideal of $\Omega_{Y/X}$, see Section 49.7. This agreement is not obvious; we use a slick argument due to Tate, see Section 49.12. On the way we also discuss the Noether different and the Dedekind different.

Only in the end of this chapter, see Sections 49.15 and 49.16, do we make the link with the more advanced material on duality for schemes.

49.2. Dualizing modules for quasi-finite ring maps

0BUK Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. By Zariski's main theorem (Algebra, Lemma 10.123.14) there exists a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. We set

0BSZ (49.2.0.1) $$\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$$

in this situation. The reader can think of this as a kind of relative dualizing module, see Lemmas 49.15.1 and 49.2.12. In this section we will show by elementary

[1] AKA flat and lci.

commutative algebra methods that $\omega_{B/A}$ is independent of the choice of the factorization and that formation of $\omega_{B/A}$ commutes with flat base change. To help prove the independence of factorizations we compare two given factorizations.

0BT0 **Lemma 49.2.1.** *Let $A \to B$ be a quasi-finite ring map. Given two factorizations $A \to B' \to B$ and $A \to B'' \to B$ with $A \to B'$ and $A \to B''$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(B'')$ open immersions, there exists an A-subalgebra $B''' \subset B$ finite over A such that $\mathrm{Spec}(B) \to \mathrm{Spec}(B''')$ an open immersion and $B' \to B$ and $B'' \to B$ factor through B'''.*

Proof. Let $B''' \subset B$ be the A-subalgebra generated by the images of $B' \to B$ and $B'' \to B$. As B' and B'' are each generated by finitely many elements integral over A, we see that B''' is generated by finitely many elements integral over A and we conclude that B''' is finite over A (Algebra, Lemma 10.36.5). Consider the maps

$$B = B' \otimes_{B'} B \to B''' \otimes_{B'} B \to B \otimes_{B'} B = B$$

The final equality holds because $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion (and hence a monomorphism). The second arrow is injective as $B' \to B$ is flat. Hence both arrows are isomorphisms. This means that

$$\begin{array}{ccc} \mathrm{Spec}(B''') & \longleftarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B') & \longleftarrow & \mathrm{Spec}(B) \end{array}$$

is cartesian. Since the base change of an open immersion is an open immersion we conclude. □

0BT1 **Lemma 49.2.2.** *The module (49.2.0.1) is well defined, i.e., independent of the choice of the factorization.*

Proof. Let B', B'', B''' be as in Lemma 49.2.1. We obtain a canonical map

$$\omega''' = \mathrm{Hom}_A(B''', A) \otimes_{B'''} B \longrightarrow \mathrm{Hom}_A(B', A) \otimes_{B'} B = \omega'$$

and a similar one involving B''. If we show these maps are isomorphisms then the lemma is proved. Let $g \in B'$ be an element such that $B'_g \to B_g$ is an isomorphism and hence $B'_g \to (B''')_g \to B_g$ are isomorphisms. It suffices to show that $(\omega''')_g \to \omega'_g$ is an isomorphism. The kernel and cokernel of the ring map $B' \to B'''$ are finite A-modules and g-power torsion. Hence they are annihilated by a power of g. This easily implies the result. □

0BT2 **Lemma 49.2.3.** *Let $A \to B$ be a quasi-finite map of Noetherian rings.*

(1) *If $A \to B$ factors as $A \to A_f \to B$ for some $f \in A$, then $\omega_{B/A} = \omega_{B/A_f}$.*
(2) *If $g \in B$, then $(\omega_{B/A})_g = \omega_{B_g/A}$.*
(3) *If $f \in A$, then $\omega_{B_f/A_f} = (\omega_{B/A})_f$.*

Proof. Say $A \to B' \to B$ is a factorization with $A \to B'$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ an open immersion. In case (1) we may use the factorization $A_f \to B'_f \to B$ to compute ω_{B/A_f} and use Algebra, Lemma 10.10.2. In case (2) use the factorization $A \to B' \to B_g$ to see the result. Part (3) follows from a combination of (1) and (2). □

Let $A \to B$ be a quasi-finite ring map of Noetherian rings, let $A \to A_1$ be an arbitrary ring map of Noetherian rings, and set $B_1 = B \otimes_A A_1$. We obtain a cocartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_1 \end{array}$$

Observe that $A_1 \to B_1$ is quasi-finite as well (Algebra, Lemma 10.122.9). In this situation we will define a canonical B-linear base change map

0BVB (49.2.3.1) $$\omega_{B/A} \longrightarrow \omega_{B_1/A_1}$$

Namely, we choose a factorization $A \to B' \to B$ as in the construction of $\omega_{B/A}$. Then $B'_1 = B' \otimes_A A_1$ is finite over A_1 and we can use the factorization $A_1 \to B'_1 \to B_1$ in the construction of ω_{B_1/A_1}. Thus we have to construct a map

$$\mathrm{Hom}_A(B', A) \otimes_{B'} B \longrightarrow \mathrm{Hom}_{A_1}(B' \otimes_A A_1, A_1) \otimes_{B'_1} B_1$$

Thus it suffices to construct a B'-linear map $\mathrm{Hom}_A(B', A) \to \mathrm{Hom}_{A_1}(B' \otimes_A A_1, A_1)$ which we will denote $\varphi \mapsto \varphi_1$. Namely, given an A-linear map $\varphi : B' \to A$ we let φ_1 be the map such that $\varphi_1(b' \otimes a_1) = \varphi(b')a_1$. This is clearly A_1-linear and the construction is complete.

0BVC **Lemma 49.2.4.** *The base change map (49.2.3.1) is independent of the choice of the factorization $A \to B' \to B$. Given ring maps $A \to A_1 \to A_2$ the composition of the base change maps for $A \to A_1$ and $A_1 \to A_2$ is the base change map for $A \to A_2$.*

Proof. Omitted. Hint: argue in exactly the same way as in Lemma 49.2.2 using Lemma 49.2.1. □

0BT3 **Lemma 49.2.5.** *If $A \to A_1$ is flat, then the base change map (49.2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.*

Proof. Assume that $A \to A_1$ is flat. By construction of $\omega_{B/A}$ we may assume that $A \to B$ is finite. Then $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ and $\omega_{B_1/A_1} = \mathrm{Hom}_{A_1}(B_1, A_1)$. Since $B_1 = B \otimes_A A_1$ the result follows from More on Algebra, Lemma 15.66.4. □

0BT4 **Lemma 49.2.6.** *Let $A \to B \to C$ be quasi-finite homomorphisms of Noetherian rings. There is a canonical map $\omega_{B/A} \otimes_B \omega_{C/B} \to \omega_{C/A}$.*

Proof. Choose $A \to B' \to B$ with $A \to B'$ finite such that $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion. Then $B' \to C$ is quasi-finite too. Choose $B' \to C' \to C$ with $B' \to C'$ finite and $\mathrm{Spec}(C) \to \mathrm{Spec}(C')$ an open immersion. Then the source of the arrow is

$$\mathrm{Hom}_A(B', A) \otimes_{B'} B \otimes_B \mathrm{Hom}_B(B \otimes_{B'} C', B) \otimes_{B \otimes_{B'} C'} C$$

which is equal to

$$\mathrm{Hom}_A(B', A) \otimes_{B'} \mathrm{Hom}_{B'}(C', B) \otimes_{C'} C$$

This indeed comes with a canonical map to $\mathrm{Hom}_A(C', A) \otimes_{C'} C = \omega_{C/A}$ coming from composition $\mathrm{Hom}_A(B', A) \times \mathrm{Hom}_{B'}(C', B) \to \mathrm{Hom}_A(C', A)$. □

0BT5 **Lemma 49.2.7.** *Let $A \to B$ and $A \to C$ be quasi-finite maps of Noetherian rings. Then $\omega_{B \times C/A} = \omega_{B/A} \times \omega_{C/A}$ as modules over $B \times C$.*

Proof. Choose factorizations $A \to B' \to B$ and $A \to C' \to C$ such that $A \to B'$ and $A \to C'$ are finite and such that $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ and $\mathrm{Spec}(C) \to \mathrm{Spec}(C')$ are open immersions. Then $A \to B' \times C' \to B \times C$ is a similar factorization. Using this factorization to compute $\omega_{B \times C/A}$ gives the lemma. □

0BVD **Lemma 49.2.8.** *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Then $Ass_B(\omega_{B/A})$ is the set of primes of B lying over associated primes of A.*

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion on spectra. As $\omega_{B/A} = \omega_{B'/A} \otimes_{B'} B$ it suffices to prove the statement for $\omega_{B'/A}$. Thus we may assume $A \to B$ is finite.

Assume $\mathfrak{p} \in \mathrm{Ass}(A)$ and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p}$. Let $x \in A$ be an element whose annihilator is $\mathfrak{p}$. Choose a nonzero $\kappa(\mathfrak{p})$ linear map $\lambda : \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$. Since $A/\mathfrak{p} \subset B/\mathfrak{q}$ is a finite extension of rings, there is an $f \in A$, $f \notin \mathfrak{p}$ such that $f\lambda$ maps $B/\mathfrak{q}$ into $A/\mathfrak{p}$. Hence we obtain a nonzero A-linear map

$$B \to B/\mathfrak{q} \to A/\mathfrak{p} \to A, \quad b \mapsto f\lambda(b)x$$

An easy computation shows that this element of $\omega_{B/A}$ has annihilator $\mathfrak{q}$, whence $\mathfrak{q} \in \mathrm{Ass}(\omega_{B/A})$.

Conversely, suppose that $\mathfrak{q} \subset B$ is a prime ideal lying over a prime $\mathfrak{p} \subset A$ which is not an associated prime of A. We have to show that $\mathfrak{q} \notin \mathrm{Ass}_B(\omega_{B/A})$. After replacing A by $A_\mathfrak{p}$ and B by $B_\mathfrak{p}$ we may assume that $\mathfrak{p}$ is a maximal ideal of A. This is allowed by Lemma 49.2.5 and Algebra, Lemma 10.63.16. Then there exists an $f \in \mathfrak{m}$ which is a nonzerodivisor on A. Then f is a nonzerodivisor on $\omega_{B/A}$ and hence $\mathfrak{q}$ is not an associated prime of this module. □

0BVE **Lemma 49.2.9.** *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. Then $\omega_{B/A}$ is a flat A-module.*

Proof. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. We will show that the localization $\omega_{B/A,\mathfrak{q}}$ is flat over $A_\mathfrak{p}$. This suffices by Algebra, Lemma 10.39.18. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. By Lemma 49.2.5 and Algebra, Lemma 10.100.1 it suffices to show $\omega_{B'/A',\mathfrak{q}'}$ is flat over $A'_{\mathfrak{p}'}$. Since $\omega_{B'/A'} = \omega_{C/A'} \times \omega_{D/A'}$ by Lemma 49.2.7 this reduces us to the case where B is finite flat over A. In this case B is finite locally free as an A-module and $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ is the dual finite locally free A-module. □

0BVF **Lemma 49.2.10.** *If $A \to B$ is flat, then the base change map (49.2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.*

Proof. If $A \to B$ is finite flat, then B is finite locally free as an A-module. In this case $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ is the dual finite locally free A-module and formation of this module commutes with arbitrary base change which proves the lemma in this case. In the next paragraph we reduce the general (quasi-finite flat) case to the finite flat case just discussed.

Let $\mathfrak{q}_1 \subset B_1$ be a prime. We will show that the localization of the map at the prime $\mathfrak{q}_1$ is an isomorphism, which suffices by Algebra, Lemma 10.23.1. Let $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$ be the prime ideals lying under $\mathfrak{q}_1$. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. Set $A'_1 = A' \otimes_A A_1$ and consider the base change maps (49.2.3.1) for the ring maps $A \to A' \to A'_1$ and $A \to A_1 \to A'_1$ as in the diagram

$$\begin{array}{ccc} \omega_{B'/A'} \otimes_{B'} B'_1 & \longrightarrow & \omega_{B'_1/A'_1} \\ \uparrow & & \uparrow \\ \omega_{B/A} \otimes_B B'_1 & \longrightarrow & \omega_{B_1/A_1} \otimes_{B_1} B'_1 \end{array}$$

where $B' = B \otimes_A A'$, $B_1 = B \otimes_A A_1$, and $B'_1 = B \otimes_A (A' \otimes_A A_1)$. By Lemma 49.2.4 the diagram commutes. By Lemma 49.2.5 the vertical arrows are isomorphisms. As $B_1 \to B'_1$ is étale and hence flat it suffices to prove the top horizontal arrow is an isomorphism after localizing at a prime $\mathfrak{q}'_1$ of B'_1 lying over $\mathfrak{q}$ (there is such a prime and use Algebra, Lemma 10.39.17). Thus we may assume that $B = C \times D$ with $A \to C$ finite and $\mathfrak{q}$ corresponding to a prime of C. In this case the dualizing module $\omega_{B/A}$ decomposes in a similar fashion (Lemma 49.2.7) which reduces the question to the finite flat case $A \to C$ handled above. □

0BVG **Remark 49.2.11.** Let $f : Y \to X$ be a locally quasi-finite morphism of locally Noetherian schemes. It is clear from Lemma 49.2.3 that there is a unique coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ on Y such that for every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is a canonical isomorphism

$$H^0(V, \omega_{Y/X}) = \omega_{B/A}$$

and where these isomorphisms are compatible with restriction maps.

0C0I **Lemma 49.2.12.** *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let $\omega^\bullet_{B/A} \in D(B)$ be the algebraic relative dualizing complex discussed in Dualizing Complexes, Section 47.25. Then there is a (nonunique) isomorphism $\omega_{B/A} = H^0(\omega^\bullet_{B/A})$.*

Proof. Choose a factorization $A \to B' \to B$ where $A \to B'$ is finite and $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is an open immersion. Then $\omega^\bullet_{B/A} = \omega^\bullet_{B'/A} \otimes^{\mathbf{L}}_B B'$ by Dualizing Complexes, Lemmas 47.24.7 and 47.24.9 and the definition of $\omega^\bullet_{B/A}$. Hence it suffices to show there is an isomorphism when $A \to B$ is finite. In this case we can use Dualizing Complexes, Lemma 47.24.8 to see that $\omega^\bullet_{B/A} = R\mathrm{Hom}(B, A)$ and hence $H^0(\omega^\bullet_{B/A}) = \mathrm{Hom}_A(B, A)$ as desired. □

49.3. Discriminant of a finite locally free morphism

0BVH Let X be a scheme and let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module. Then there is a canonical *trace* map

$$\mathrm{Trace} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{O}_X$$

See Exercises, Exercise 111.22.6. This map has the property that Trace(id) is the locally constant function on $\mathcal{O}_X$ corresponding to the rank of $\mathcal{F}$.

Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then there exists a canonical *trace for* π which is an $\mathcal{O}_Y$-linear map

$$\text{Trace}_\pi : \pi_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

sending a local section f of $\pi_*\mathcal{O}_X$ to the trace of multiplication by f on $\pi_*\mathcal{O}_X$. Over affine opens this recovers the construction in Exercises, Exercise 111.22.7. The composition

$$\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_*\mathcal{O}_X \xrightarrow{\text{Trace}_\pi} \mathcal{O}_Y$$

equals multiplication by the degree of π (which is a locally constant function on Y). In analogy with Fields, Section 9.20 we can define the trace pairing

$$Q_\pi : \pi_*\mathcal{O}_X \times \pi_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

by the rule $(f, g) \mapsto \text{Trace}_\pi(fg)$. We can think of Q_π as a linear map $\pi_*\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X, \mathcal{O}_Y)$ between locally free modules of the same rank, and hence obtain a determinant

$$\det(Q_\pi) : \wedge^{top}(\pi_*\mathcal{O}_X) \longrightarrow \wedge^{top}(\pi_*\mathcal{O}_X)^{\otimes -1}$$

or in other words a global section

$$\det(Q_\pi) \in \Gamma(Y, \wedge^{top}(\pi_*\mathcal{O}_X)^{\otimes -2})$$

The *discriminant of* π is by definition the closed subscheme $D_\pi \subset Y$ cut out by this global section. Clearly, D_π is a locally principal closed subscheme of Y.

0BJF **Lemma 49.3.1.** *Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then π is étale if and only if its discriminant is empty.*

Proof. By Morphisms, Lemma 29.36.8 it suffices to check that the fibres of π are étale. Since the construction of the trace pairing commutes with base change we reduce to the following question: Let k be a field and let A be a finite dimensional k-algebra. Show that A is étale over k if and only if the trace pairing $Q_{A/k} : A \times A \to k$, $(a, b) \mapsto \text{Trace}_{A/k}(ab)$ is nondegenerate.

Assume $Q_{A/k}$ is nondegenerate. If $a \in A$ is a nilpotent element, then ab is nilpotent for all $b \in A$ and we conclude that $Q_{A/k}(a, -)$ is identically zero. Hence A is reduced. Then we can write $A = K_1 \times \ldots \times K_n$ as a product where each K_i is a field (see Algebra, Lemmas 10.53.2, 10.53.6, and 10.25.1). In this case the quadratic space $(A, Q_{A/k})$ is the orthogonal direct sum of the spaces $(K_i, Q_{K_i/k})$. It follows from Fields, Lemma 9.20.7 that each K_i is separable over k. This means that A is étale over k by Algebra, Lemma 10.143.4. The converse is proved by reading the argument backwards. □

49.4. Traces for flat quasi-finite ring maps

0BSY The trace referred to in the title of this section is of a completely different nature than the trace discussed in Duality for Schemes, Section 48.7. Namely, it is the trace as discussed in Fields, Section 9.20 and generalized in Exercises, Exercises 111.22.6 and 111.22.7.

Let $A \to B$ be a finite flat map of Noetherian rings. Then B is finite flat as an A-module and hence finite locally free (Algebra, Lemma 10.78.2). Given $b \in B$ we can

consider the *trace* $\mathrm{Trace}_{B/A}(b)$ of the A-linear map $B \to B$ given by multiplication by b on B. By the references above this defines an A-linear map $\mathrm{Trace}_{B/A} : B \to A$. Since $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as $A \to B$ is finite, we see that $\mathrm{Trace}_{B/A} \in \omega_{B/A}$.

For a general flat quasi-finite ring map we define the notion of a trace as follows.

0BT6 **Definition 49.4.1.** Let $A \to B$ be a flat quasi-finite map of Noetherian rings. The *trace element* is the unique[2] element $\tau_{B/A} \in \omega_{B/A}$ with the following property: for any Noetherian A-algebra A_1 such that $B_1 = B \otimes_A A_1$ comes with a product decomposition $B_1 = C \times D$ with $A_1 \to C$ finite the image of $\tau_{B/A}$ in ω_{C/A_1} is Trace_{C/A_1}. Here we use the base change map (49.2.3.1) and Lemma 49.2.7 to get $\omega_{B/A} \to \omega_{B_1/A_1} \to \omega_{C/A_1}$.

We first prove that trace elements are unique and then we prove that they exist.

0BT7 **Lemma 49.4.2.** *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Then there is at most one trace element in $\omega_{B/A}$.*

Proof. Let $\mathfrak{q} \subset B$ be a prime ideal lying over the prime $\mathfrak{p} \subset A$. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A_1$ and a prime ideal $\mathfrak{p}_1 \subset A_1$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}_1) = \kappa(\mathfrak{p})$ and such that

$$B_1 = B \otimes_A A_1 = C \times D$$

with $A_1 \to C$ finite and such that the unique prime $\mathfrak{q}_1$ of $B \otimes_A A_1$ lying over $\mathfrak{q}$ and $\mathfrak{p}_1$ corresponds to a prime of C. Observe that $\omega_{C/A_1} = \omega_{B/A} \otimes_B C$ (combine Lemmas 49.2.5 and 49.2.7). Since the collection of ring maps $B \to C$ obtained in this manner is a jointly injective family of flat maps and since the image of $\tau_{B/A}$ in ω_{C/A_1} is prescribed the uniqueness follows. □

Here is a sanity check.

0BT8 **Lemma 49.4.3.** *Let $A \to B$ be a finite flat map of Noetherian rings. Then $\mathrm{Trace}_{B/A} \in \omega_{B/A}$ is the trace element.*

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Of course in this case $A_1 \to D$ is also finite. Set $B_1 = B \otimes_A A_1$. Since the construction of traces commutes with base change we see that $\mathrm{Trace}_{B/A}$ maps to Trace_{B_1/A_1}. Thus the proof is finished by noticing that $\mathrm{Trace}_{B_1/A_1} = (\mathrm{Trace}_{C/A_1}, \mathrm{Trace}_{D/A_1})$ under the isomorphism $\omega_{B_1/A_1} = \omega_{C/A_1} \times \omega_{D/A_1}$ of Lemma 49.2.7. □

0BT9 **Lemma 49.4.4.** *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $\tau \in \omega_{B/A}$ be a trace element.*

1. *If $A \to A_1$ is a map with A_1 Noetherian, then with $B_1 = A_1 \otimes_A B$ the image of τ in ω_{B_1/A_1} is a trace element.*
2. *If $A = R_f$ for some ring R and $f \in R$, then τ is a trace element in $\omega_{B/R}$.*
3. *If $g \in B$, then the image of τ in $\omega_{B_g/A}$ is a trace element.*
4. *If $B = B_1 \times B_2$, then τ maps to a trace element in both $\omega_{B_1/A}$ and $\omega_{B_2/A}$.*

Proof. Part (1) is a formal consequence of the definition.

Statement (2) makes sense because $\omega_{B/R} = \omega_{B/A}$ by Lemma 49.2.3. Denote τ' the element τ but viewed as an element of $\omega_{B/R}$. To see that (2) is true suppose that we

[2] Uniqueness and existence will be justified in Lemmas 49.4.2 and 49.4.6.

have $R \to R_1$ with R_1 Noetherian and a product decomposition $B \otimes_R R_1 = C \times D$ with $R_1 \to C$ finite. Then with $A_1 = (R_1)_f$ we see that $B \otimes_A A_1 = C \times D$. Since $R_1 \to C$ is finite, a fortiori $A_1 \to C$ is finite. Hence we can use the defining property of τ to get the corresponding property of τ'.

Statement (3) makes sense because $\omega_{B_g/A} = (\omega_{B/A})_g$ by Lemma 49.2.3. The proof is similar to the proof of (2). Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B_1 = B \otimes_A A_1$. Then $\mathrm{Spec}(C) \to \mathrm{Spec}(B_1)$ is an open immersion as $B_g \otimes_A A_1 = (B_1)_g$ and the image is closed because $B_1 \to C$ is finite (as $A_1 \to C$ is finite). Thus we see that $B_1 = C \times D_1$ and $D = (D_1)_g$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_g/A}$.

Statement (4) makes sense because $\omega_{B/A} = \omega_{B_1/A} \times \omega_{B_2/A}$ by Lemma 49.2.7. Suppose we have $A \to A'$ with A' Noetherian and a product decomposition $B \otimes_A A' = C \times D$ with $A' \to C$ finite. Then it is clear that we can refine this product decomposition into $B \otimes_A A' = C_1 \times C_2 \times D_1 \times D_2$ with $A' \to C_i$ finite such that $B_i \otimes_A A' = C_i \times D_i$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_i/A}$. This uses the obvious fact that $\mathrm{Trace}_{C/A'} = (\mathrm{Trace}_{C_1/A'}, \mathrm{Trace}_{C_2/A'})$ under the decomposition $\omega_{C/A'} = \omega_{C_1/A'} \times \omega_{C_2/A'}$. □

0BTA **Lemma 49.4.5.** *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $g_1, \ldots, g_m \in B$ be elements generating the unit ideal. Let $\tau \in \omega_{B/A}$ be an element whose image in $\omega_{B_{g_i}/A}$ is a trace element for $A \to B_{g_i}$. Then τ is a trace element.*

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. We have to show that the image of τ in ω_{C/A_1} is Trace_{C/A_1}. Observe that $g_1, \ldots, g_m$ generate the unit ideal in $B_1 = B \otimes_A A_1$ and that τ maps to a trace element in $\omega_{(B_1)_{g_i}/A_1}$ by Lemma 49.4.4. Hence we may replace A by A_1 and B by B_1 to get to the situation as described in the next paragraph.

Here we assume that $B = C \times D$ with $A \to C$ is finite. Let τ_C be the image of τ in $\omega_{C/A}$. We have to prove that $\tau_C = \mathrm{Trace}_{C/A}$ in $\omega_{C/A}$. By the compatibility of trace elements with products (Lemma 49.4.4) we see that τ_C maps to a trace element in $\omega_{C_{g_i}/A}$. Hence, after replacing B by C we may assume that $A \to B$ is finite flat.

Assume $A \to B$ is finite flat. In this case $\mathrm{Trace}_{B/A}$ is a trace element by Lemma 49.4.3. Hence $\mathrm{Trace}_{B/A}$ maps to a trace element in $\omega_{B_{g_i}/A}$ by Lemma 49.4.4. Since trace elements are unique (Lemma 49.4.2) we find that $\mathrm{Trace}_{B/A}$ and τ map to the same elements in $\omega_{B_{g_i}/A} = (\omega_{B/A})_{g_i}$. As $g_1, \ldots, g_m$ generate the unit ideal of B the map $\omega_{B/A} \to \prod \omega_{B_{g_i}/A}$ is injective and we conclude that $\tau_C = \mathrm{Trace}_{B/A}$ as desired. □

0BTB **Lemma 49.4.6.** *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. There exists a trace element $\tau \in \omega_{B/A}$.*

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ an open immersion. Let $g_1, \ldots, g_n \in B'$ be elements such that $\mathrm{Spec}(B) = \bigcup D(g_i)$ as opens of $\mathrm{Spec}(B')$. Suppose that we can prove the existence of trace elements τ_i for the quasi-finite flat ring maps $A \to B_{g_i}$. Then for all i, j the elements

τ_i and τ_j map to trace elements of $\omega_{B_{g_ig_j}/A}$ by Lemma 49.4.4. By uniqueness of trace elements (Lemma 49.4.2) they map to the same element. Hence the sheaf condition for the quasi-coherent module associated to $\omega_{B/A}$ (see Algebra, Lemma 10.24.1) produces an element $\tau \in \omega_{B/A}$. Then τ is a trace element by Lemma 49.4.5. In this way we reduce to the case treated in the next paragraph.

Assume we have $A \to B'$ finite and $g \in B'$ with $B = B'_g$ flat over A. It is our task to construct a trace element in $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$. Choose a resolution $F_1 \to F_0 \to B' \to 0$ of B' by finite free A-modules F_0 and F_1. Then we have an exact sequence

$$0 \to \mathrm{Hom}_A(B', A) \to F_0^\vee \to F_1^\vee$$

where $F_i^\vee = \mathrm{Hom}_A(F_i, A)$ is the dual finite free module. Similarly we have the exact sequence

$$0 \to \mathrm{Hom}_A(B', B') \to F_0^\vee \otimes_A B' \to F_1^\vee \otimes_A B'$$

The idea of the construction of τ is to use the diagram

$$B' \xrightarrow{\mu} \mathrm{Hom}_A(B', B') \leftarrow \mathrm{Hom}_A(B', A) \otimes_A B' \xrightarrow{ev} A$$

where the first arrow sends $b' \in B'$ to the A-linear operator given by multiplication by b' and the last arrow is the evaluation map. The problem is that the middle arrow, which sends $\lambda' \otimes b'$ to the map $b'' \mapsto \lambda'(b'')b'$, is not an isomorphism. If B' is flat over A, the exact sequences above show that it is an isomorphism and the composition from left to right is the usual trace $\mathrm{Trace}_{B'/A}$. In the general case, we consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Hom}_A(B', A) \otimes_A B' & \longrightarrow & \mathrm{Hom}_A(B', A) \otimes_A B'_g \\ & \overset{\psi}{\nearrow} & \downarrow & \nearrow & \downarrow \\ B' & \xrightarrow[\mu]{} & \mathrm{Hom}_A(B', B') & \longrightarrow & \mathrm{Ker}(F_0^\vee \otimes_A B'_g \to F_1^\vee \otimes_A B'_g) \end{array}$$

By flatness of $A \to B'_g$ we see that the right vertical arrow is an isomorphism. Hence we obtain the unadorned dotted arrow. Since $B'_g = \mathrm{colim} \frac{1}{g^n} B'$, since colimits commute with tensor products, and since B' is a finitely presented A-module we can find an $n \geq 0$ and a B'-linear (for right B'-module structure) map $\psi : B' \to \mathrm{Hom}_A(B', A) \otimes_A B'$ whose composition with the left vertical arrow is $g^n\mu$. Composing with ev we obtain an element $ev \circ \psi \in \mathrm{Hom}_A(B', A)$. Then we set

$$\tau = (ev \circ \psi) \otimes g^{-n} \in \mathrm{Hom}_A(B', A) \otimes_{B'} B'_g = \omega_{B'_g/A} = \omega_{B/A}$$

We omit the easy verification that this element does not depend on the choice of n and ψ above.

Let us prove that τ as constructed in the previous paragraph has the desired property in a special case. Namely, say $B' = C' \times D'$ and $g = (f, h)$ where $A \to C'$ flat, D'_h is flat, and f is a unit in C'. To show: τ maps to $\mathrm{Trace}_{C'/A}$ in $\omega_{C'/A}$. In this case we first choose n_D and $\psi_D : D' \to \mathrm{Hom}_A(D', A) \otimes_A D'$ as above for the pair (D', h) and we can let $\psi_C : C' \to \mathrm{Hom}_A(C', A) \otimes_A C' = \mathrm{Hom}_A(C', C')$ be the map seconding $c' \in C'$ to multiplication by c'. Then we take $n = n_D$ and $\psi = (f^{n_D}\psi_C, \psi_D)$ and the desired compatibility is clear because $\mathrm{Trace}_{C'/A} = ev \circ \psi_C$ as remarked above.

To prove the desired property in general, suppose given $A \to A_1$ with A_1 Noetherian and a product decomposition $B'_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B'_1 = B' \otimes_A A_1$. Then $\mathrm{Spec}(C) \to \mathrm{Spec}(B'_1)$ is an open immersion as $B'_g \otimes_A A_1 = (B'_1)_g$ and the image is closed as $B'_1 \to C$ is finite (since $A_1 \to C$ is finite). Thus $B'_1 = C \times D'$ and $D'_g = D$. We conclude that $B'_1 = C \times D'$ and g over A_1 are as in the previous paragraph. Since formation of the displayed diagram above commutes with base change, the formation of τ commutes with the base change $A \to A_1$ (details omitted; use the resolution $F_1 \otimes_A A_1 \to F_0 \otimes_A A_1 \to B'_1 \to 0$ to see this). Thus the desired compatibility follows from the result of the previous paragraph. □

0BVJ **Remark 49.4.7.** Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 49.2.11. It is clear from the uniqueness, existence, and compatibility with localization of trace elements (Lemmas 49.4.2, 49.4.6, and 49.4.4) that there exists a global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

such that for every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ that element $\tau_{Y/X}$ maps to $\tau_{B/A}$ under the canonical isomorphism $H^0(V, \omega_{Y/X}) = \omega_{B/A}$.

0C13 **Lemma 49.4.8.** *Let k be a field and let A be a finite k-algebra. Assume A is local with residue field k'. The following are equivalent*

1. *$\mathrm{Trace}_{A/k}$ is nonzero,*
2. *$\tau_{A/k} \in \omega_{A/k}$ is nonzero, and*
3. *k'/k is separable and $\mathrm{length}_A(A)$ is prime to the characteristic of k.*

Proof. Conditions (1) and (2) are equivalent by Lemma 49.4.3. Let $\mathfrak{m} \subset A$. Since $\dim_k(A) < \infty$ it is clear that A has finite length over A. Choose a filtration

$$A = I_0 \supset \mathfrak{m} = I_1 \supset I_2 \supset \ldots I_n = 0$$

by ideals such that $I_i/I_{i+1} \cong k'$ as A-modules. See Algebra, Lemma 10.52.11 which also shows that $n = \mathrm{length}_A(A)$. If $a \in \mathfrak{m}$ then $aI_i \subset I_{i+1}$ and it is immediate that $\mathrm{Trace}_{A/k}(a) = 0$. If $a \notin \mathfrak{m}$ with image $\lambda \in k'$, then we conclude

$$\mathrm{Trace}_{A/k}(a) = \sum\nolimits_{i=0,\ldots,n-1} \mathrm{Trace}_k(a : I_i/I_{i-1} \to I_i/I_{i-1}) = n\mathrm{Trace}_{k'/k}(\lambda)$$

The proof of the lemma is finished by applying Fields, Lemma 9.20.7. □

49.5. Finite morphisms

0FKW In this section we collect some observations about the constructions in the previous sections for finite morphisms. Let $f : Y \to X$ be a finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 49.2.11.

The first remark is that

$$f_*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

as sheaves of $f_*\mathcal{O}_Y$-modules. Since f is affine, this formula uniquely characterizes $\omega_{Y/X}$, see Morphisms, Lemma 29.11.7. The formula holds because for $\mathrm{Spec}(A) =$

$U \subset X$ affine open, the inverse image $V = f^{-1}(U)$ is the spectrum of a finite A-algebra B and hence

$$H^0(U, f_*\omega_{Y/X}) = H^0(V, \omega_{Y/X}) = \omega_{B/A} = \mathrm{Hom}_A(B, A) = H^0(U, \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X))$$

by construction. In particular, we obtain a canonical evaluation map

$$f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

which is given by evaluation at 1 if we think of $f_*\omega_{Y/X}$ as the sheaf $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$.

The second remark is that using the evaluation map we obtain canonical identifications

$$\mathrm{Hom}_Y(\mathcal{F}, f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathrm{Hom}_X(f_*\mathcal{F}, \mathcal{G})$$

functorially in the quasi-coherent module $\mathcal{F}$ on Y and the finite locally free module $\mathcal{G}$ on X. If $\mathcal{G} = \mathcal{O}_X$ this follows immediately from the above and Algebra, Lemma 10.14.4. For general $\mathcal{G}$ we can use the same lemma and the isomorphisms

$$f_*(f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{G})$$

of $f_*\mathcal{O}_Y$-modules where the first equality is the projection formula (Cohomology, Lemma 20.54.2). An alternative is to prove the formula affine locally by direct computation.

The third remark is that if f is in addition flat, then the composition

$$f_*\mathcal{O}_Y \xrightarrow{f_*\tau_{Y/X}} f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

is equal to the trace map Trace_f discussed in Section 49.3. This follows immediately by looking over affine opens.

The fourth remark is that if f is flat and X Noetherian, then we obtain

$$\mathrm{Hom}_Y(K, Lf^*M \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathrm{Hom}_X(Rf_*K, M)$$

for any K in $D_{QCoh}(\mathcal{O}_Y)$ and M in $D_{QCoh}(\mathcal{O}_X)$. This follows from the material in Duality for Schemes, Section 48.12, but can be proven directly in this case as follows. First, if X is affine, then it holds by Dualizing Complexes, Lemmas 47.13.1 and 47.13.9[3] and Derived Categories of Schemes, Lemma 36.3.5. Then we can use the induction principle (Cohomology of Schemes, Lemma 30.4.1) and Mayer-Vietoris (in the form of Cohomology, Lemma 20.33.3) to finish the proof.

49.6. The Noether different

0BVK There are many different differents available in the literature. We list some of them in this and the next sections; for more information we suggest the reader consult [**Kun86**].

Let $A \to B$ be a ring map. Denote

$$\mu : B \otimes_A B \longrightarrow B, \quad b \otimes b' \longmapsto bb'$$

the multiplication map. Let $I = \mathrm{Ker}(\mu)$. It is clear that I is generated by the elements $b \otimes 1 - 1 \otimes b$ for $b \in B$. Hence the annihilator $J \subset B \otimes_A B$ of I is a B-module in a canonical manner. The *Noether different* of B over A is the image

[3] There is a simpler proof of this lemma in our case.

of J under the map $\mu : B \otimes_A B \to B$. Equivalently, the Noether different is the image of the map

$$J = \mathrm{Hom}_{B \otimes_A B}(B, B \otimes_A B) \longrightarrow B, \quad \varphi \longmapsto \mu(\varphi(1))$$

We begin with some obligatory lemmas.

0BVL **Lemma 49.6.1.** *Let $A \to B_i$, $i = 1, 2$ be ring maps. Set $B = B_1 \times B_2$.*

(1) *The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J_1 \times J_2$ where J_i is the annihilator of $\mathrm{Ker}(B_i \otimes_A B_i \to B_i)$.*
(2) *The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_i$ is the Noether different of B_i over A.*

Proof. Omitted. □

0BVM **Lemma 49.6.2.** *Let $A \to B$ be a finite type ring map. Let $A \to A'$ be a flat ring map. Set $B' = B \otimes_A A'$.*

(1) *The annihilator J' of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$ is $J \otimes_A A'$ where J is the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$.*
(2) *The Noether different $\mathfrak{D}'$ of B' over A' is $\mathfrak{D}B'$, where $\mathfrak{D}$ is the Noether different of B over A.*

Proof. Choose generators $b_1, \ldots, b_n$ of B as an A-algebra. Then

$$J = \mathrm{Ker}(B \otimes_A B \xrightarrow{b_i \otimes 1 - 1 \otimes b_i} (B \otimes_A B)^{\oplus n})$$

Hence we see that the formation of J commutes with flat base change. The result on the Noether different follows immediately from this. □

0BVN **Lemma 49.6.3.** *Let $A \to B' \to B$ be ring maps with $A \to B'$ of finite type and $B' \to B$ inducing an open immersion of spectra.*

(1) *The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J' \otimes_{B'} B$ where J' is the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$.*
(2) *The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}'B$, where $\mathfrak{D}'$ is the Noether different of B' over A.*

Proof. Write $I = \mathrm{Ker}(B \otimes_A B \to B)$ and $I' = \mathrm{Ker}(B' \otimes_A B' \to B')$. As $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion, it follows that $B = (B \otimes_A B) \otimes_{B' \otimes_A B'} B'$. Thus we see that $I = I'(B \otimes_A B)$. Since I' is finitely generated and $B' \otimes_A B' \to B \otimes_A B$ is flat, we conclude that $J = J'(B \otimes_A B)$, see Algebra, Lemma 10.40.4. Since the $B' \otimes_A B'$-module structure of J' factors through $B' \otimes_A B' \to B'$ we conclude that (1) is true. Part (2) is a consequence of (1). □

0BVP **Remark 49.6.4.** Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let J be the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$. There is a canonical B-bilinear pairing

0BVQ (49.6.4.1) $$\omega_{B/A} \times J \longrightarrow B$$

defined as follows. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$. We first define

$$\mathrm{Hom}_A(B', A) \times J' \longrightarrow B', \quad (\lambda, \sum b_i \otimes c_i) \longmapsto \sum \lambda(b_i) c_i$$

This is B'-bilinear exactly because for $\xi \in J'$ and $b \in B'$ we have $(b \otimes 1)\xi = (1 \otimes b)\xi$. By Lemma 49.6.3 and the fact that $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ we can extend this to a B-bilinear pairing as displayed above.

0BVR **Lemma 49.6.5.** *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings.*

(1) *If $A \to A'$ is a flat map of Noetherian rings, then*

$$\begin{array}{ccc} \omega_{B/A} \times J & \longrightarrow & B \\ \downarrow & & \downarrow \\ \omega_{B'/A'} \times J' & \longrightarrow & B' \end{array}$$

is commutative where notation as in Lemma 49.6.2 and horizontal arrows are given by (49.6.4.1).

(2) *If $B = B_1 \times B_2$, then*

$$\begin{array}{ccc} \omega_{B/A} \times J & \longrightarrow & B \\ \downarrow & & \downarrow \\ \omega_{B_i/A} \times J_i & \longrightarrow & B_i \end{array}$$

is commutative for $i = 1, 2$ where notation as in Lemma 49.6.1 and horizontal arrows are given by (49.6.4.1).

Proof. Because of the construction of the pairing in Remark 49.6.4 both (1) and (2) reduce to the case where $A \to B$ is finite. Then (1) follows from the fact that the contraction map $\mathrm{Hom}_A(M, A) \otimes_A M \otimes_A M \to M$, $\lambda \otimes m \otimes m' \mapsto \lambda(m)m'$ commuted with base change. To see (2) use that $J = J_1 \times J_2$ is contained in the summands $B_1 \otimes_A B_1$ and $B_2 \otimes_A B_2$ of $B \otimes_A B$. □

0BVS **Lemma 49.6.6.** *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The pairing of Remark 49.6.4 induces an isomorphism $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$.*

Proof. We first prove this when $A \to B$ is finite and flat. In this case we can localize on A and assume B is finite free as an A-module. Let $b_1, \ldots, b_n$ be a basis of B as an A-module and denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A}$. Note that $\sum b_i \otimes c_i \in J$ maps to the element of $\mathrm{Hom}_B(\omega_{B/A}, B)$ which sends $b_i^\vee$ to c_i. Suppose $\varphi : \omega_{B/A} \to B$ is B-linear. Then we claim that $\xi = \sum b_i \otimes \varphi(b_i^\vee)$ is an element of J. Namely, the B-linearity of φ exactly implies that $(b \otimes 1)\xi = (1 \otimes b)\xi$ for all $b \in B$. Thus our map has an inverse and it is an isomorphism.

Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. We will show that the localization

$$J_{\mathfrak{q}} \longrightarrow \mathrm{Hom}_B(\omega_B/A, B)_{\mathfrak{q}}$$

is an isomorphism. This suffices by Algebra, Lemma 10.23.1. By Algebra, Lemma 10.145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$. By Lemmas 49.2.5, 49.6.2, and 49.6.5 the map $J' \to \mathrm{Hom}_{B'}(\omega_{B'/A'}, B')$ is gotten by applying the functor $- \otimes_B B'$ to the map $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$. Since $B_{\mathfrak{q}} \to B'_{\mathfrak{q}'}$

is faithfully flat it suffices to prove the result for $(A' \to B', \mathfrak{q}')$. By Lemmas 49.2.7, 49.6.1, and 49.6.5 this reduces us to the case proved in the first paragraph of the proof. □

0BVT **Lemma 49.6.7.** *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The diagram*

$$\begin{array}{ccc} J & \longrightarrow & \mathrm{Hom}_B(\omega_{B/A}, B) \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\varphi\mapsto\varphi(\tau_{B/A})} & \\ & B & \end{array}$$

commutes where the horizontal arrow is the isomorphism of Lemma 49.6.6. Hence the Noether different of B over A is the image of the map $\mathrm{Hom}_B(\omega_{B/A}, B) \to B$.

Proof. Exactly as in the proof of Lemma 49.6.6 this reduces to the case of a finite free map $A \to B$. In this case $\tau_{B/A} = \mathrm{Trace}_{B/A}$. Choose a basis $b_1, \ldots, b_n$ of B as an A-module. Let $\xi = \sum b_i \otimes c_i \in J$. Then $\mu(\xi) = \sum b_i c_i$. On the other hand, the image of ξ in $\mathrm{Hom}_B(\omega_{B/A}, B)$ sends $\mathrm{Trace}_{B/A}$ to $\sum \mathrm{Trace}_{B/A}(b_i)c_i$. Thus we have to show

$$\sum b_i c_i = \sum \mathrm{Trace}_{B/A}(b_i) c_i$$

when $\xi = \sum b_i \otimes c_i \in J$. Write $b_i b_j = \sum_k a_{ij}^k b_k$ for some $a_{ij}^k \in A$. Then the right hand side is $\sum_{i,j} a_{ij}^j c_i$. On the other hand, $\xi \in J$ implies

$$(b_j \otimes 1)(\sum\nolimits_i b_i \otimes c_i) = (1 \otimes b_j)(\sum\nolimits_i b_i \otimes c_i)$$

which implies that $b_j c_i = \sum_k a_{jk}^i c_k$. Thus the left hand side is $\sum_{i,j} a_{ij}^i c_j$. Since $a_{ij}^k = a_{ji}^k$ the equality holds. □

0BVU **Lemma 49.6.8.** *Let $A \to B$ be a finite type ring map. Let $\mathfrak{D} \subset B$ be the Noether different. Then $V(\mathfrak{D})$ is the set of primes $\mathfrak{q} \subset B$ such that $A \to B$ is not unramified at $\mathfrak{q}$.*

Proof. Assume $A \to B$ is unramified at $\mathfrak{q}$. After replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ we may assume $A \to B$ is unramified (Algebra, Definition 10.151.1 and Lemma 49.6.3). In this case $\Omega_{B/A} = 0$. Hence if $I = \mathrm{Ker}(B \otimes_A B \to B)$, then $I/I^2 = 0$ by Algebra, Lemma 10.131.13. Since $A \to B$ is of finite type, we see that I is finitely generated. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) there exists an element of the form $1 + i$ annihilating I. It follows that $\mathfrak{D} = B$.

Conversely, assume that $\mathfrak{D} \not\subset \mathfrak{q}$. Then after replacing B by a principal localization as above we may assume $\mathfrak{D} = B$. This means there exists an element of the form $1 + i$ in the annihilator of I. Conversely this implies that $I/I^2 = \Omega_{B/A}$ is zero and we conclude. □

49.7. The Kähler different

0BVV Let $A \to B$ be a finite type ring map. The *Kähler different* is the zeroth fitting ideal of $\Omega_{B/A}$ as a B-module. We globalize the definition as follows.

0BVW **Definition 49.7.1.** Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. The *Kähler different* is the 0th fitting ideal of $\Omega_{Y/X}$.

The Kähler different is a quasi-coherent sheaf of ideals on Y.

0BVX **Lemma 49.7.2.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

with f locally of finite type. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the Kähler different of f, resp. f'. Then $Y' \to Y$ induces an isomorphism $R' \to R \times_Y Y'$.

Proof. This is true because $\Omega_{Y'/X'}$ is the pullback of $\Omega_{Y/X}$ (Morphisms, Lemma 29.32.10) and then we can apply More on Algebra, Lemma 15.8.4. □

0BVY **Lemma 49.7.3.** *Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. Let $R \subset Y$ be the closed subscheme defined by the Kähler different. Then $R \subset Y$ is exactly the set of points where f is not unramified.*

Proof. This is a copy of Divisors, Lemma 31.10.2. □

0BVZ **Lemma 49.7.4.** *Let A be a ring. Let $n \geq 1$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. The Kähler different of B over A is the ideal of B generated by $\det(\partial f_i/\partial x_j)$.*

Proof. This is true because $\Omega_{B/A}$ has a presentation

$$\bigoplus\nolimits_{i=1,\ldots,n} Bf_i \xrightarrow{\mathrm{d}} \bigoplus\nolimits_{j=1,\ldots,n} B\mathrm{d}x_j \to \Omega_{B/A} \to 0$$

by Algebra, Lemma 10.131.9. □

49.8. The Dedekind different

0BW0 Let $A \to B$ be a ring map. We say *the Dedekind different is defined* if A is Noetherian, $A \to B$ is finite, any nonzerodivisor on A is a nonzerodivisor on B, and $K \to L$ is étale where $K = Q(A)$ and $L = B \otimes_A K$. Then $K \subset L$ is finite étale and

$$\mathcal{L}_{B/A} = \{x \in L \mid \mathrm{Trace}_{L/K}(bx) \in A \text{ for all } b \in B\}$$

is the Dedekind complementary module. In this situation the *Dedekind different* is

$$\mathfrak{D}_{B/A} = \{x \in L \mid x\mathcal{L}_{B/A} \subset B\}$$

viewed as a B-submodule of L. By Lemma 49.8.1 the Dedekind different is an ideal of B either if A is normal or if B is flat over A.

0BW1 **Lemma 49.8.1.** *Assume the Dedekind different of $A \to B$ is defined. Consider the statements*

1. *$A \to B$ is flat,*
2. *A is a normal ring,*
3. *$\mathit{Trace}_{L/K}(B) \subset A$,*
4. *$1 \in \mathcal{L}_{B/A}$, and*
5. *the Dedekind different $\mathfrak{D}_{B/A}$ is an ideal of B.*

Then we have (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4), and (4) $\Rightarrow$ (5).

Proof. The equivalence of (3) and (4) and the implication (4) $\Rightarrow$ (5) are immediate.

If $A \to B$ is flat, then we see that $\mathrm{Trace}_{B/A} : B \to A$ is defined and that $\mathrm{Trace}_{L/K}$ is the base change. Hence (3) holds.

If A is normal, then A is a finite product of normal domains, hence we reduce to the case of a normal domain. Then K is the fraction field of A and $L = \prod L_i$ is a finite product of finite separable field extensions of K. Then $\mathrm{Trace}_{L/K}(b) = \sum \mathrm{Trace}_{L_i/K}(b_i)$ where $b_i \in L_i$ is the image of b. Since b is integral over A as B is finite over A, these traces are in A. This is true because the minimal polynomial of b_i over K has coefficients in A (Algebra, Lemma 10.38.6) and because $\mathrm{Trace}_{L_i/K}(b_i)$ is an integer multiple of one of these coefficients (Fields, Lemma 9.20.3). □

0BW2 **Lemma 49.8.2.** *If the Dedekind different of $A \to B$ is defined, then there is a canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$.*

Proof. Recall that $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as $A \to B$ is finite. We send $x \in \mathcal{L}_{B/A}$ to the map $b \mapsto \mathrm{Trace}_{L/K}(bx)$. Conversely, given an A-linear map $\varphi : B \to A$ we obtain a K-linear map $\varphi_K : L \to K$. Since $K \to L$ is finite étale, we see that the trace pairing is nondegenerate (Lemma 49.3.1) and hence there exists a $x \in L$ such that $\varphi_K(y) = \mathrm{Trace}_{L/K}(xy)$ for all $y \in L$. Then $x \in \mathcal{L}_{B/A}$ maps to φ in $\omega_{B/A}$. □

0BW3 **Lemma 49.8.3.** *If the Dedekind different of $A \to B$ is defined and $A \to B$ is flat, then*

(1) *the canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$ sends $1 \in \mathcal{L}_{B/A}$ to the trace element $\tau_{B/A} \in \omega_{B/A}$, and*
(2) *the Dedekind different is $\mathfrak{D}_{B/A} = \{b \in B \mid b\omega_{B/A} \subset B\tau_{B/A}\}$.*

Proof. The first assertion follows from the proof of Lemma 49.8.1 and Lemma 49.4.3. The second assertion is immediate from the first and the definitions. □

49.9. The different

0BTC The motivation for the following definition is that it recovers the Dedekind different in the finite flat case as we will see below.

0BW4 **Definition 49.9.1.** Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be the relative dualizing module and let $\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$ be the trace element (Remarks 49.2.11 and 49.4.7). The annihilator of

$$\mathrm{Coker}(\mathcal{O}_Y \xrightarrow{\tau_{Y/X}} \omega_{Y/X})$$

is the *different* of Y/X. It is a coherent ideal $\mathfrak{D}_f \subset \mathcal{O}_Y$.

We will generalize this in Remark 49.14.2 below. Observe that $\mathfrak{D}_f$ is locally generated by one element if $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module. We first state the agreement with the Dedekind different.

0BW5 **Lemma 49.9.2.** *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If the Dedekind different of $A \to B$ is defined, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on V.

Proof. This is clear from Lemmas 49.8.1 and 49.8.3. □

0BW6 **Lemma 49.9.3.** *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If $\omega_{Y/X}|_V$ is invertible, i.e., if $\omega_{B/A}$ is an invertible B-module, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}}$$

as coherent ideal sheaves on V where $\mathfrak{D} \subset B$ is the Noether different of B over A.

Proof. Consider the map

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/X}, \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y, \quad \varphi \longmapsto \varphi(\tau_{Y/X})$$

The image of this map corresponds to the Noether different on affine opens, see Lemma 49.6.7. Hence the result follows from the elementary fact that given an invertible module ω and a global section τ the image of $\tau : \mathcal{H}om(\omega, \mathcal{O}) = \omega^{\otimes -1} \to \mathcal{O}$ is the same as the annihilator of $\mathrm{Coker}(\tau : \mathcal{O} \to \omega)$. □

0BW7 **Lemma 49.9.4.** *Consider a cartesian diagram of Noetherian schemes*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

with f flat and quasi-finite. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the different $\mathfrak{D}_f$, resp. $\mathfrak{D}_{f'}$. Then $Y' \to Y$ induces a bijective closed immersion $R' \to R \times_Y Y'$. If g is flat or if $\omega_{Y/X}$ is invertible, then $R' = R \times_Y Y'$.

Proof. There is an immediate reduction to the case where X, X', Y, Y' are affine. In other words, we have a cocartesian diagram of Noetherian rings

$$\begin{array}{ccc} B' & \longleftarrow & B \\ \uparrow & & \uparrow \\ A' & \longleftarrow & A \end{array}$$

with $A \to B$ flat and quasi-finite. The base change map $\omega_{B/A} \otimes_B B' \to \omega_{B'/A'}$ is an isomorphism (Lemma 49.2.10) and maps the trace element $\tau_{B/A}$ to the trace element $\tau_{B'/A'}$ (Lemma 49.4.4). Hence the finite B-module $Q = \mathrm{Coker}(\tau_{B/A} : B \to \omega_{B/A})$ satisfies $Q \otimes_B B' = \mathrm{Coker}(\tau_{B'/A'} : B' \to \omega_{B'/A'})$. Thus $\mathfrak{D}_{B/A}B' \subset \mathfrak{D}_{B'/A'}$ which means we obtain the closed immersion $R' \to R \times_Y Y'$. Since $R = \mathrm{Supp}(Q)$ and $R' = \mathrm{Supp}(Q \otimes_B B')$ (Algebra, Lemma 10.40.5) we see that $R' \to R \times_Y Y'$ is bijective by Algebra, Lemma 10.40.6. The equality $\mathfrak{D}_{B/A}B' = \mathfrak{D}_{B'/A'}$ holds if $B \to B'$ is flat, e.g., if $A \to A'$ is flat, see Algebra, Lemma 10.40.4. Finally, if $\omega_{B/A}$ is invertible, then we can localize and assume $\omega_{B/A} = B\lambda$. Writing $\tau_{B/A} = b\lambda$ we see that $Q = B/bB$ and $\mathfrak{D}_{B/A} = bB$. The same reasoning over B' gives $\mathfrak{D}_{B'/A'} = bB'$ and the lemma is proved. □

0BW8 **Lemma 49.9.5.** *Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. Then $\mathrm{Norm}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ maps $f_*\mathfrak{D}_f$ into the ideal sheaf of the discriminant D_f.*

Proof. The norm map is constructed in Divisors, Lemma 31.17.6 and the discriminant of f in Section 49.3. The question is affine local, hence we may assume $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and f given by a finite locally free ring map $A \to B$. Localizing further we may assume B is finite free as an A-module. Choose a basis $b_1, \ldots, b_n \in B$ for B as an A-module. Denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as an A-module. Since the norm of b is the determinant of $b : B \to B$ as an A-linear map, we see that $\mathrm{Norm}_{B/A}(b) = \det(b_i^\vee(bb_j))$. The discriminant is the principal closed subscheme of $\mathrm{Spec}(A)$ defined by $\det(\mathrm{Trace}_{B/A}(b_i b_j))$. If $b \in \mathfrak{D}_{B/A}$ then there exist $c_i \in B$ such that $b \cdot b_i^\vee = c_i \cdot \mathrm{Trace}_{B/A}$ where we use a dot to indicate the B-module structure on $\omega_{B/A}$. Write $c_i = \sum a_{il} b_l$. We have

$$\begin{aligned}
\mathrm{Norm}_{B/A}(b) &= \det(b_i^\vee(bb_j)) \\
&= \det((b \cdot b_i^\vee)(b_j)) \\
&= \det((c_i \cdot \mathrm{Trace}_{B/A})(b_j)) \\
&= \det(\mathrm{Trace}_{B/A}(c_i b_j)) \\
&= \det(a_{il}) \det(\mathrm{Trace}_{B/A}(b_l b_j))
\end{aligned}$$

which proves the lemma. □

0BW9 **Lemma 49.9.6.** *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. The closed subscheme $R \subset Y$ defined by the different $\mathfrak{D}_f$ is exactly the set of points where f is not étale (equivalently not unramified).*

Proof. Since f is of finite presentation and flat, we see that it is étale at a point if and only if it is unramified at that point. Moreover, the formation of the locus of ramified points commutes with base change. See Morphisms, Section 29.36 and especially Morphisms, Lemma 29.36.17. By Lemma 49.9.4 the formation of R commutes set theoretically with base change. Hence it suffices to prove the lemma when X is the spectrum of a field. On the other hand, the construction of $(\omega_{Y/X}, \tau_{Y/X})$ is local on Y. Since Y is a finite discrete space (being quasi-finite over a field), we may assume Y has a unique point.

Say $X = \mathrm{Spec}(k)$ and $Y = \mathrm{Spec}(B)$ where k is a field and B is a finite local k-algebra. If $Y \to X$ is étale, then B is a finite separable extension of k, and the trace element $\mathrm{Trace}_{B/k}$ is a basis element of $\omega_{B/k}$ by Fields, Lemma 9.20.7. Thus $\mathfrak{D}_{B/k} = B$ in this case. Conversely, if $\mathfrak{D}_{B/k} = B$, then we see from Lemma 49.9.5 and the fact that the norm of 1 equals 1 that the discriminant is empty. Hence $Y \to X$ is étale by Lemma 49.3.1. □

0BWA **Lemma 49.9.7.** *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $R \subset Y$ be the closed subscheme defined by $\mathfrak{D}_f$.*

1. *If $\omega_{Y/X}$ is invertible, then R is a locally principal closed subscheme of Y.*
2. *If $\omega_{Y/X}$ is invertible and f is finite, then the norm of R is the discriminant D_f of f.*
3. *If $\omega_{Y/X}$ is invertible and f is étale at the associated points of Y, then R is an effective Cartier divisor and there is an isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$.*

Proof. Proof of (1). We may work locally on Y, hence we may assume $\omega_{Y/X}$ is free of rank 1. Say $\omega_{Y/X} = \mathcal{O}_Y \lambda$. Then we can write $\tau_{Y/X} = h\lambda$ and then we see that R is defined by h, i.e., R is locally principal.

Proof of (2). We may assume $Y \to X$ is given by a finite free ring map $A \to B$ and that $\omega_{B/A}$ is free of rank 1 as B-module. Choose a B-basis element λ for $\omega_{B/A}$ and write $\text{Trace}_{B/A} = b \cdot \lambda$ for some $b \in B$. Then $\mathfrak{D}_{B/A} = (b)$ and D_f is cut out by $\det(\text{Trace}_{B/A}(b_i b_j))$ where $b_1, \ldots, b_n$ is a basis of B as an A-module. Let $b_1^\vee, \ldots, b_n^\vee$ be the dual basis. Writing $b_i^\vee = c_i \cdot \lambda$ we see that $c_1, \ldots, c_n$ is a basis of B as well. Hence with $c_i = \sum a_{il} b_l$ we see that $\det(a_{il})$ is a unit in A. Clearly, $b \cdot b_i^\vee = c_i \cdot \text{Trace}_{B/A}$ hence we conclude from the computation in the proof of Lemma 49.9.5 that $\text{Norm}_{B/A}(b)$ is a unit times $\det(\text{Trace}_{B/A}(b_i b_j))$.

Proof of (3). In the notation above we see from Lemma 49.9.6 and the assumption that h does not vanish in the associated points of Y, which implies that h is a nonzerodivisor. The canonical isomorphism sends 1 to $\tau_{Y/X}$, see Divisors, Lemma 31.14.10. □

49.10. Quasi-finite syntomic morphisms

0DWJ This section discusses the fact that a quasi-finite syntomic morphism has an invertible relative dualizing module.

0BWE **Lemma 49.10.1.** *Let $f : Y \to X$ be a morphism of schemes. The following are equivalent*

(1) *f is locally quasi-finite and syntomic,*
(2) *f is locally quasi-finite, flat, and a local complete intersection morphism,*
(3) *f is locally quasi-finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,*
(4) *f is locally quasi-finite and for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ an integer n and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$,*
(5) *for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,*
(6) *f is locally quasi-finite, flat, locally of finite presentation, and $NL_{Y/X}$ has tor-amplitude in $[-1, 0]$, and*
(7) *f is flat, locally of finite presentation, $NL_{Y/X}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,*

Proof. The equivalence of (1) and (2) is More on Morphisms, Lemma 37.62.8. The equivalence of (1) and (3) is Morphisms, Lemma 29.30.11.

If $A \to B$ is as in (4), then $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n)$ is a relative global complete intersection by see Algebra, Definition 10.136.5. Thus (4) implies (5). It is clear that (5) implies (4).

Condition (5) implies (1): by Algebra, Lemma 10.136.13 a relative global complete intersection is syntomic and the definition of a relative global complete intersection guarantees that a relative global complete intersection on n variables with n equations is quasi-finite, see Algebra, Definition 10.136.5 and Lemma 10.122.2.

Either Algebra, Lemma 10.136.15 or Morphisms, Lemma 29.30.10 shows that (1) implies (5).

More on Morphisms, Lemma 37.62.17 shows that (6) is equivalent to (1). If the equivalent conditions (1) – (6) hold, then we see that affine locally $Y \to X$ is given

by a relative global complete intersection $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with the same number of variables as the number of equations. Using this presentation we see that

$$NL_{B/A} = \Big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B\mathrm{d}x_i\Big)$$

By Algebra, Lemma 10.136.12 the module $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the congruence classes of the elements $f_1, \ldots, f_n$. Thus $NL_{B/A}$ has rank 0 and so does $NL_{Y/X}$. In this way we see that (1) – (6) imply (7).

Finally, assume (7). By More on Morphisms, Lemma 37.62.17 we see that f is syntomic. Thus on suitable affine opens f is given by a relative global complete intersection $A \to B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Morphisms, Lemma 29.30.10. Exactly as above we see that $NL_{B/A}$ is a perfect complex of rank $n - m$. Thus $n = m$ and we see that (5) holds. This finishes the proof. □

0DWK **Lemma 49.10.2.** *Invertibility of the relative dualizing module.*

(1) *If $A \to B$ is a quasi-finite flat homomorphism of Noetherian rings, then $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B\otimes_A \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$ is an invertible $B \otimes_A \kappa(\mathfrak{p})$-module for all primes $\mathfrak{p} \subset A$.*

(2) *If $Y \to X$ is a quasi-finite flat morphism of Noetherian schemes, then $\omega_{Y/X}$ is invertible if and only if $\omega_{Y_x/x}$ is invertible for all $x \in X$.*

Proof. Proof of (1). As $A \to B$ is flat, the module $\omega_{B/A}$ is A-flat, see Lemma 49.2.9. Thus $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B/A} \otimes_A \kappa(\mathfrak{p})$ is an invertible $B\otimes_A\kappa(\mathfrak{p})$-module for every prime $\mathfrak{p} \subset A$, see More on Morphisms, Lemma 37.16.7. Still using that $A \to B$ is flat, we have that formation of $\omega_{B/A}$ commutes with base change, see Lemma 49.2.10. Thus we see that invertibility of the relative dualizing module, in the presence of flatness, is equivalent to invertibility of the relative dualizing module for the maps $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$.

Part (2) follows from (1) and the fact that affine locally the dualizing modules are given by their algebraic counterparts, see Remark 49.2.11. □

0DWL **Lemma 49.10.3.** *Let k be a field. Let $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a global complete intersection over k of dimension 0. Then $\omega_{B/k}$ is invertible.*

Proof. By Noether normalization, see Algebra, Lemma 10.115.4 we see that there exists a finite injection $k \to B$, i.e., $\dim_k(B) < \infty$. Hence $\omega_{B/k} = \mathrm{Hom}_k(B, k)$ as a B-module. By Dualizing Complexes, Lemma 47.15.8 we see that $R\,\mathrm{Hom}(B, k)$ is a dualizing complex for B and by Dualizing Complexes, Lemma 47.13.3 we see that $R\,\mathrm{Hom}(B, k)$ is equal to $\omega_{B/k}$ placed in degree 0. Thus it suffices to show that B is Gorenstein (Dualizing Complexes, Lemma 47.21.4). This is true by Dualizing Complexes, Lemma 47.21.7. □

0BWF **Lemma 49.10.4.** *Let $f : Y \to X$ be a morphism of locally Noetherian schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module.*

Proof. We may assume $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ and we have to show $\omega_{B/A}$ is invertible. This follows in combining Lemmas 49.10.2 and 49.10.3. □

0FK8 **Example 49.10.5.** Let $n \geq 1$ and $d \geq 1$ be integers. Let T be the set of multi-indices $E = (e_1, \ldots, e_n)$ with $e_i \geq 0$ and $\sum e_i \leq d$. Consider the ring

$$A = \mathbf{Z}[a_{i,E}; 1 \leq i \leq n, E \in T]$$

In $A[x_1, \ldots, x_n]$ consider the elements $f_i = \sum_{E \in T} a_{i,E} x^E$ where $x^E = x_1^{e_1} \ldots x_n^{e_n}$ as is customary. Consider the A-algebra

$$B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Denote $X_{n,d} = \mathrm{Spec}(A)$ and let $Y_{n,d} \subset \mathrm{Spec}(B)$ be the maximal open subscheme such that the restriction of the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A) = X_{n,d}$ is quasi-finite, see Algebra, Lemma 10.123.13.

0FK9 **Lemma 49.10.6.** *With notation as in Example 49.10.5 the schemes $X_{n,d}$ and $Y_{n,d}$ are regular and irreducible, the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic, and there is a dense open subscheme $V \subset Y_{n,d}$ such that $Y_{n,d} \to X_{n,d}$ restricts to an étale morphism $V \to X_{n,d}$.*

Proof. The scheme $X_{n,d}$ is the spectrum of the polynomial ring A. Hence $X_{n,d}$ is regular and irreducible. Since we can write

$$f_i = a_{i,(0,\ldots,0)} + \sum\nolimits_{E \in T, E \neq (0,\ldots,0)} a_{i,E} x^E$$

we see that the ring B is isomorphic to the polynomial ring on $x_1, \ldots, x_n$ and the elements $a_{i,E}$ with $E \neq (0, \ldots, 0)$. Hence $\mathrm{Spec}(B)$ is an irreducible and regular scheme and so is the open $Y_{n,d}$. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic by Lemma 49.10.1. To find V it suffices to find a single point where $Y_{n,d} \to X_{n,d}$ is étale (the locus of points where a morphism is étale is open by definition). Thus it suffices to find a point of $X_{n,d}$ where the fibre of $Y_{n,d} \to X_{n,d}$ is nonempty and étale, see Morphisms, Lemma 29.36.15. We choose the point corresponding to the ring map $\chi : A \to \mathbf{Q}$ sending f_i to $x_i^d - 1$. Then

$$B \otimes_{A,\chi} \mathbf{Q} = \mathbf{Q}[x_1, \ldots, x_n]/(x_1^d - 1, \ldots, x_n^d - 1)$$

which is a nonzero étale algebra over $\mathbf{Q}$. □

0FKA **Lemma 49.10.7.** *Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then for every $y \in Y$ there exist n, d and a commutative diagram*

$$\begin{array}{ccccc} Y & \longleftarrow & V & \longrightarrow & Y_{n,d} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & U & \longrightarrow & X_{n,d} \end{array}$$

where $U \subset X$ and $V \subset Y$ are open with $y \in V$, where $Y_{n,d} \to X_{n,d}$ is as in Example 49.10.5, and where the square on the right hand side is cartesian.

Proof. By Lemma 49.10.1 we can choose U and V affine so that $U = \mathrm{Spec}(R)$ and $V = \mathrm{Spec}(S)$ with $S = R[y_1, \ldots, y_n]/(g_1, \ldots, g_n)$. With notation as in Example 49.10.5 if we pick d large enough, then we can write each g_i as $g_i = \sum_{E \in T} g_{i,E} y^E$ with $g_{i,E} \in R$. Then the map $A \to R$ sending $a_{i,E}$ to $g_{i,E}$ and the map $B \to S$

sending $x_i \to y_i$ give a cocartesian diagram of rings

$$\begin{array}{ccc} S & \longleftarrow & B \\ \uparrow & & \uparrow \\ R & \longleftarrow & A \end{array}$$

which proves the lemma. □

49.11. Finite syntomic morphisms

0FKX This section is the analogue of Section 49.10 for finite syntomic morphisms.

0FKY **Lemma 49.11.1.** *Let $f : Y \to X$ be a morphism of schemes. The following are equivalent*

(1) *f is finite and syntomic,*
(2) *f is finite, flat, and a local complete intersection morphism,*
(3) *f is finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,*
(4) *f is finite and for every $x \in X$ there is an affine open $x \in U = \mathrm{Spec}(A) \subset X$ an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $f^{-1}(U)$ is isomorphic to the spectrum of $A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,*
(5) *f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(6) *f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,*

Proof. The equivalence of (1), (2), (3), (5), and (6) and the implication (4) $\Rightarrow$ (1) follow immediately from Lemma 49.10.1. Assume the equivalent conditions (1), (2), (3), (5), (6) hold. Choose a point $x \in X$ and an affine open $U = \mathrm{Spec}(A)$ of x in X and say x corresponds to the prime ideal $\mathfrak{p} \subset A$. Write $f^{-1}(U) = \mathrm{Spec}(B)$. Write $B = A[x_1, \ldots, x_n]/I$. Since $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$ by (6) we see that I/I^2 is a finite locally free B-module of rank n. Since $B_{\mathfrak{p}}$ is semi-local we see that $(I/I^2)_{\mathfrak{p}}$ is free of rank n, see Algebra, Lemma 10.78.7. Thus after replacing A by a principal localization at an element not in $\mathfrak{p}$ we may assume I/I^2 is a free B-module of rank n. Thus by Algebra, Lemma 10.136.6 we can find a presentation of B over A with the same number of variables as equations. In other words, we may assume $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. This proves (4). □

0FKZ **Example 49.11.2.** Let $d \geq 1$ be an integer. Consider variables a_{ij}^l for $1 \leq i, j, l \leq d$ and denote

$$A_d = \mathbf{Z}[a_{ij}^k]/J$$

where J is the ideal generated by the elements

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m - \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k - a_{ji}^k & \forall i, j, k \\ a_{i1}^j - \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} indices the Kronecker delta function. We define an A_d-algebra B_d as follows: as an A_d-module we set

$$B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$$

The algebra structure is given by $A_d \to B_d$ mapping 1 to e_1. The multiplication on B_d is the A_d-bilinar map

$$m : B_d \times B_d \longrightarrow B_d, \quad m(e_i, e_j) = \sum a_{ij}^k e_k$$

It is straightforward to check that the relations given above exactly force this to be an A_d-algebra structure. The morphism

$$\pi_d : Y_d = \mathrm{Spec}(B_d) \longrightarrow \mathrm{Spec}(A_d) = X_d$$

is the "universal" finite free morphism of rank d.

0FL0 **Lemma 49.11.3.** *With notation as in Example 49.11.2 there is an open subscheme $U_d \subset X_d$ with the following property: a morphism of schemes $X \to X_d$ factors through U_d if and only if $Y_d \times_{X_d} X \to X$ is syntomic.*

Proof. Recall that being syntomic is the same thing as being flat and a local complete intersection morphism, see More on Morphisms, Lemma 37.62.8. The set $W_d \subset Y_d$ of points where π_d is Koszul is open in Y_d and its formation commutes with arbitrary base change, see More on Morphisms, Lemma 37.62.21. Since π_d is finite and hence closed, we see that $Z = \pi_d(Y_d \setminus W_d)$ is closed. Since clearly $U_d = X_d \setminus Z$ and since its formation commutes with base change we find that the lemma is true. □

0FL1 **Lemma 49.11.4.** *With notation as in Example 49.11.2 and U_d as in Lemma 49.11.3 then U_d is smooth over* $\mathrm{Spec}(\mathbf{Z})$.

Proof. Let us use More on Morphisms, Lemma 37.12.1 to show that $U_d \to \mathrm{Spec}(\mathbf{Z})$ is smooth. Namely, suppose that $\mathrm{Spec}(A) \to U_d$ is a morphism and $A' \to A$ is a small extension. Then $B = A \otimes_{A_d} B_d$ is a finite free A-algebra which is syntomic over A (by construction of U_d). By Smoothing Ring Maps, Proposition 16.3.2 there exists a syntomic ring map $A' \to B'$ such that $B \cong B' \otimes_{A'} A$. Set $e'_1 = 1 \in B'$. For $1 < i \leq d$ choose lifts $e'_i \in B'$ of the elements $1 \otimes e_i \in A \otimes_{A_d} B_d = B$. Then $e'_1, \ldots, e'_d$ is a basis for B' over A' (for example see Algebra, Lemma 10.101.1). Thus we can write $e'_i e'_j = \sum \alpha_{ij}^l e'_l$ for unique elements $\alpha_{ij}^l \in A'$ which satisfy the relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j = \delta_{ij}$ in A'. This determines a morphism $\mathrm{Spec}(A') \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A'$. This morphism agrees with the given morphism $\mathrm{Spec}(A) \to U_d$. Since $\mathrm{Spec}(A')$ and $\mathrm{Spec}(A)$ have the same underlying topological space, we see that we obtain the desired lift $\mathrm{Spec}(A') \to U_d$ and we conclude that U_d is smooth over $\mathbf{Z}$. □

0FL2 **Lemma 49.11.5.** *With notation as in Example 49.11.2 consider the open subscheme $U'_d \subset X_d$ over which π_d is étale. Then U'_d is a dense subset of the open U_d of Lemma 49.11.3.*

Proof. By exactly the same reasoning as in the proof of Lemma 49.11.3, using Morphisms, Lemma 29.36.17, there is a maximal open $U'_d \subset X_d$ over which π_d is étale. Moreover, since an étale morphism is syntomic, we see that $U'_d \subset U_d$. To finish the proof we have to show that $U'_d \subset U_d$ is dense. Let $u : \mathrm{Spec}(k) \to U_d$ be a morphism where k is a field. Let $B = k \otimes_{A_d} B_d$ as in the proof of Lemma 49.11.4. We will show there is a local domain A' with residue field k and a finite syntomic A' algebra B' with $B = k \otimes_{A'} B'$ whose generic fibre is étale. Exactly as in the previous paragraph this will determine a morphism $\mathrm{Spec}(A') \to U_d$ which will map the generic point into U'_d and the closed point to u, thereby finishing the proof.

By Lemma 49.11.1 part (4) we can choose a presentation $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Let d' be the maximum total degree of the polynomials $f_1, \ldots, f_n$. Let $Y_{n,d'} \to X_{n,d'}$ be as in Example 49.10.5. By construction there is a morphism $u' : \mathrm{Spec}(k) \to X_{n,d'}$ such that

$$\mathrm{Spec}(B) \cong Y_{n,d'} \times_{X_{n,d'},u'} \mathrm{Spec}(k)$$

Denote $A = \mathcal{O}^h_{X_{n,d'},u'}$ the henselization of the local ring of $X_{n,d'}$ at the image of u'. Then we can write

$$Y_{n,d'} \times_{X_{n,d'}} \mathrm{Spec}(A) = Z \amalg W$$

with $Z \to \mathrm{Spec}(A)$ finite and $W \to \mathrm{Spec}(A)$ having empty closed fibre, see Algebra, Lemma 10.153.3 part (13) or the discussion in More on Morphisms, Section 37.41. By Lemma 49.10.6 the local ring A is regular (here we also use More on Algebra, Lemma 15.45.10) and the morphism $Z \to \mathrm{Spec}(A)$ is étale over the generic point of $\mathrm{Spec}(A)$ (because it is mapped to the generic point of $X_{d,n'}$). By construction $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong \mathrm{Spec}(B)$. This proves what we want except that the map from residue field of A to k may not be an isomorphism. By Algebra, Lemma 10.159.1 there exists a flat local ring map $A \to A'$ such that the residue field of A' is k. If A' isn't a domain, then we choose a minimal prime $\mathfrak{p} \subset A'$ (which lies over the unique minimal prime of A by flatness) and we replace A' by $A'/\mathfrak{p}$. Set B' equal to the unique A'-algebra such that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') = \mathrm{Spec}(B')$. This finishes the proof. □

0FL3 **Remark 49.11.6.** Let $\pi_d : Y_d \to X_d$ be as in Example 49.11.2. Let $U_d \subset X_d$ be the maximal open over which $V_d = \pi_d^{-1}(U_d)$ is finite syntomic as in Lemma 49.11.3. Then it is also true that V_d is smooth over $\mathbf{Z}$. (Of course the morphism $V_d \to U_d$ is not smooth when $d \geq 2$.) Arguing as in the proof of Lemma 49.11.4 this corresponds to the following deformation problem: given a small extension $C' \to C$ and a finite syntomic C-algebra B with a section $B \to C$, find a finite syntomic C'-algebra B' and a section $B' \to C'$ whose tensor product with C recovers $B \to C$. By Lemma 49.11.1 we may write $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ as a relative global complete intersection. After a change of coordinates we may assume $x_1, \ldots, x_n$ are in the kernel of $B \to C$. Then the polynomials f_i have vanishing constant terms. Choose any lifts $f_i' \in C'[x_1, \ldots, x_n]$ of f_i with vanishing constant terms. Then $B' = C'[x_1, \ldots, x_n]/(f_1', \ldots, f_n')$ with section $B' \to C'$ sending x_i to zero works.

0FL4 **Lemma 49.11.7.** *Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 49.11.1 then for every $x \in X$ there exist a d and a commutative diagram*

$$\begin{array}{ccccccc} Y & \longleftarrow & V & \longrightarrow & V_d & \longrightarrow & Y_d \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_d} \\ X & \longleftarrow & U & \longrightarrow & U_d & \longrightarrow & X_d \end{array}$$

with the following properties

(1) *$U \subset X$ is open, $x \in U$, and $V = f^{-1}(U)$,*
(2) *$\pi_d : Y_d \to X_d$ is as in Example 49.11.2,*
(3) *$U_d \subset X_d$ is as in Lemma 49.11.3 and $V_d = \pi_d^{-1}(U_d) \subset Y_d$,*
(4) *where the middle square is cartesian.*

Proof. Choose an affine open neighbourhood $U = \mathrm{Spec}(A) \subset X$ of x. Write $V = f^{-1}(U) = \mathrm{Spec}(B)$. Then B is a finite locally free A-module and the inclusion $A \subset B$ is a locally direct summand. Thus after shrinking U we can choose a basis $1 = e_1, e_2, \ldots, e_d$ of B as an A-module. Write $e_i e_j = \sum \alpha_{ij}^l e_l$ for unique elements $\alpha_{ij}^l \in A$ which satisfy the relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j = \delta_{ij}$ in A. This determines a morphism $\mathrm{Spec}(A) \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A$. By construction $V \cong \mathrm{Spec}(A) \times_{X_d} Y_d$. By the definition of U_d we see that $\mathrm{Spec}(A) \to X_d$ factors through U_d. This finishes the proof. □

49.12. A formula for the different

0BWB In this section we discuss the material in [**MR70**, Appendix A] due to Tate. In our language, this will show that the different is equal to the Kähler different in the case of a flat, quasi-finite, local complete intersection morphism. First we compute the Noether different in a special case.

0BWC **Lemma 49.12.1.** *Let $A \to P$ be a ring map. Let $f_1, \ldots, f_n \in P$ be a Koszul regular sequence. Assume $B = P/(f_1, \ldots, f_n)$ is flat over A. Let $g_1, \ldots, g_n \in P \otimes_A B$ be a Koszul regular sequence generating the kernel of the multiplication map $P \otimes_A B \to B$. Write $f_i \otimes 1 = \sum g_{ij} g_j$. Then the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$ is a principal ideal generated by the image of $\det(g_{ij})$.* [**MR70**, Appendix]

Proof. The Koszul complex $K_\bullet = K(P, f_1, \ldots, f_n)$ is a resolution of B by finite free P-modules. The Koszul complex $M_\bullet = K(P \otimes_A B, g_1, \ldots, g_n)$ is a resolution of B by finite free $P \otimes_A B$-modules. There is a map of complexes

$$K_\bullet \longrightarrow M_\bullet$$

which in degree 1 is given by the matrix (g_{ij}) and in degree n by $\det(g_{ij})$. See More on Algebra, Lemma 15.28.3. As B is a flat A-module, we can view $M_\bullet$ as a complex of flat P-modules (via $P \to P \otimes_A B$, $p \mapsto p \otimes 1$). Thus we may use both complexes to compute $\mathrm{Tor}_*^P(B, B)$ and it follows that the displayed map defines a quasi-isomorphism after tensoring with B. It is clear that $H_n(K_\bullet \otimes_P B) = B$. On the other hand, $H_n(M_\bullet \otimes_P B)$ is the kernel of

$$B \otimes_A B \xrightarrow{g_1, \ldots, g_n} (B \otimes_A B)^{\oplus n}$$

Since $g_1, \ldots, g_n$ generate the kernel of $B \otimes_A B \to B$ this proves the lemma. □

0BWD **Lemma 49.12.2.** *Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then*

1. *B is flat over A and $A \to B$ is a relative local complete intersection,*
2. *the annihilator J of $I = \mathrm{Ker}(B \otimes_A B \to B)$ is free of rank 1 over B,*
3. *the Noether different of B over A is generated by $\det(\partial f_i / \partial x_j)$ in B.*

Proof. Note that $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n)$ is a relative global complete intersection over A, see Algebra, Definition 10.136.5. By Algebra, Lemma 10.136.13 we see that B is flat over A.

Write $P' = A[x, x_1, \ldots, x_n]$ and $P = P'/(xh - 1) = A[x_1, \ldots, x_n, 1/h]$. Then we have $P' \to P \to B$. By More on Algebra, Lemma 15.33.4 we see that $xh - 1, f_1, \ldots, f_n$ is a Koszul regular sequence in P'. Since $xh - 1$ is a Koszul regular sequence of length one in P' (by the same lemma for example) we conclude that $f_1, \ldots, f_n$ is a Koszul regular sequence in P by More on Algebra, Lemma 15.30.14.

Let $g_i \in P \otimes_A B$ be the image of $x_i \otimes 1 - 1 \otimes x_i$. Let us use the short hand $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$ in $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n]$ so that g_i is the image of $y_i - z_i$. For a polynomial $f \in A[x_1, \ldots, x_n]$ we write $f(y) = f \otimes 1$ and $f(z) = 1 \otimes f$ in the above tensor product. Then we have

$$P \otimes_A B/(g_1, \ldots, g_n) = \frac{A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]}{(f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n)}$$

which is clearly isomorphic to B. Hence by the same arguments as above we find that $f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n$ is a Koszul regular sequence in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$. The sequence $f_1(z), \ldots, f_n(z)$ is a Koszul regular in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$ by flatness of the map

$$P \longrightarrow A[y_1, \ldots, y_n, z_1, \ldots, z_n, \tfrac{1}{h(y)h(z)}], \quad x_i \longmapsto z_i$$

and More on Algebra, Lemma 15.30.5. By More on Algebra, Lemma 15.30.14 we conclude that $g_1, \ldots, g_n$ is a regular sequence in $P \otimes_A B$.

At this point we have verified all the assumptions of Lemma 49.12.1 above with P, $f_1, \ldots, f_n$, and $g_i \in P \otimes_A B$ as above. In particular the annihilator J of I is freely generated by one element δ over B. Set $f_{ij} = \partial f_i/\partial x_j \in A[x_1, \ldots, x_n]$. An elementary computation shows that we can write

$$f_i(y) = f_i(z_1 + g_1, \ldots, z_n + g_n) = f_i(z) + \sum\nolimits_j f_{ij}(z) g_j + \sum\nolimits_{j,j'} F_{ijj'} g_j g_{j'}$$

for some $F_{ijj'} \in A[y_1, \ldots, y_n, z_1, \ldots, z_n]$. Taking the image in $P \otimes_A B$ the terms $f_i(z)$ map to zero and we obtain

$$f_i \otimes 1 = \sum\nolimits_j \left(1 \otimes f_{ij} + \sum\nolimits_{j'} F_{ijj'} g_{j'}\right) g_j$$

Thus we conclude from Lemma 49.12.1 that $\delta = \det(g_{ij})$ with $g_{ij} = 1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'}$. Since $g_{j'}$ maps to zero in B, we conclude that the image of $\det(\partial f_i/\partial x_j)$ in B generates the Noether different of B over A. □

0BWG **Lemma 49.12.3.** *Let $f : Y \to X$ be a morphism of Noetherian schemes. If f satisfies the equivalent conditions of Lemma 49.10.1 then the different $\mathfrak{D}_f$ of f is the Kähler different of f.*

Proof. By Lemmas 49.9.3 and 49.10.4 the different of f affine locally is the same as the Noether different. Then the lemma follows from the computation of the Noether different and the Kähler different on standard affine pieces done in Lemmas 49.7.4 and 49.12.2. □

0BWH **Lemma 49.12.4.** *Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then there is an isomorphism $B \to \omega_{B/A}$ mapping $\det(\partial f_i/\partial x_j)$ to $\tau_{B/A}$.*

Proof. Let J be the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$. By Lemma 49.12.2 the map $A \to B$ is flat and J is a free B-module with generator ξ mapping to $\det(\partial f_i/\partial x_j)$ in B. Thus the lemma follows from Lemma 49.6.7 and the fact (Lemma 49.10.4) that $\omega_{B/A}$ is an invertible B-module. (Warning: it is necessary to prove $\omega_{B/A}$ is invertible because a finite B-module M such that $\mathrm{Hom}_B(M, B) \cong B$ need not be free.) □

0BWI **Example 49.12.5.** Let A be a Noetherian ring. Let $f, h \in A[x]$ such that

$$B = (A[x]/(f))_h = A[x, 1/h]/(f)$$

is quasi-finite over A. Let $f' \in A[x]$ be the derivative of f with respect to x. The ideal $\mathfrak{D} = (f') \subset B$ is the Noether different of B over A, is the Kähler different of B over A, and is the ideal whose associated quasi-coherent sheaf of ideals is the different of $\mathrm{Spec}(B)$ over $\mathrm{Spec}(A)$.

0BWJ **Lemma 49.12.6.** *Let S be a Noetherian scheme. Let X, Y be smooth schemes of relative dimension n over S. Let $f : Y \to X$ be a locally quasi-finite morphism over S. Then f is flat and the closed subscheme $R \subset Y$ cut out by the different of f is the locally principal closed subscheme cut out by*

$$\wedge^n(df) \in \Gamma(Y, (f^*\Omega^n_{X/S})^{\otimes -1} \otimes_{\mathcal{O}_Y} \Omega^n_{Y/S})$$

If f is étale at the associated points of Y, then R is an effective Cartier divisor and

$$f^*\Omega^n_{X/S} \otimes_{\mathcal{O}_Y} \mathcal{O}(R) = \Omega^n_{Y/S}$$

as invertible sheaves on Y.

Proof. To prove that f is flat, it suffices to prove $Y_s \to X_s$ is flat for all $s \in S$ (More on Morphisms, Lemma 37.16.3). Flatness of $Y_s \to X_s$ follows from Algebra, Lemma 10.128.1. By More on Morphisms, Lemma 37.62.10 the morphism f is a local complete intersection morphism. Thus the statement on the different follows from the corresponding statement on the Kähler different by Lemma 49.12.3. Finally, since we have the exact sequence

$$f^*\Omega_{X/S} \xrightarrow{\mathrm{d}f} \Omega_{Y/S} \to \Omega_{Y/X} \to 0$$

by Morphisms, Lemma 29.32.9 and since $\Omega_{X/S}$ and $\Omega_{Y/S}$ are finite locally free of rank n (Morphisms, Lemma 29.34.12), the statement for the Kähler different is clear from the definition of the zeroth fitting ideal. If f is étale at the associated points of Y, then $\wedge^n \mathrm{d}f$ does not vanish in the associated points of Y, which implies that the local equation of R is a nonzerodivisor. Hence R is an effective Cartier divisor. The canonical isomorphism sends 1 to $\wedge^n \mathrm{d}f$, see Divisors, Lemma 31.14.10. □

49.13. The Tate map

0FKB In this section we produce an isomorphism between the determinant of the relative cotangent complex and the relative dualizing module for a locally quasi-finite syntomic morphism of locally Noetherian schemes. Following [**Gar84**, 1.4.4] we dub the isomorphism the Tate map. Our approach is to avoid doing local calculations as much as is possible.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. We will use all the equivalent conditions for this notion given in Lemma 49.10.1 without further mention in this section. In particular, we see that $NL_{Y/X}$ is a perfect object of $D(\mathcal{O}_Y)$ with tor-amplitude in $[-1, 0]$. Thus we have a canonical invertible module $\det(NL_{Y/X})$ on Y and a global section

$$\delta(NL_{Y/X}) \in \Gamma(Y, \det(NL_{Y/X}))$$

See Derived Categories of Schemes, Lemma 36.39.1. Suppose given a commutative diagram of schemes

$$\begin{array}{ccc} Y' & \xrightarrow[b]{} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then the canonical map

$$Lb^* NL_{Y/X} \longrightarrow NL_{Y'/X'}$$

is a quasi-isomorphism by More on Morphisms, Lemma 37.13.16. Thus we get a canonical isomorphism $b^* \det(NL_{Y/X}) \to \det(NL_{Y'/X'})$ which sends the canonical section $\delta(NL_{Y/X})$ to $\delta(NL_{Y'/X'})$, see Derived Categories of Schemes, Remark 36.39.2.

0FKC **Remark 49.13.1.** Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. What does the pair $(\det(NL_{Y/X}), \delta(NL_{Y/X}))$ look like locally? Choose affine opens $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ with $f(V) \subset U$ and an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Then

$$NL_{B/A} = \Big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B\mathrm{d}x_i \Big)$$

and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$. See proof of Lemma 49.10.1. Thus $\det(L_{B/A})$ is free on the generator

$$\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

and the section $\delta(NL_{B/A})$ is the element

$$\delta(NL_{B/A}) = \det(\partial f_j/\partial x_i) \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

by definition.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of locally Noetherian schemes. By Remarks 49.2.11 and 49.4.7 we have a coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ and a canonical global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

which affine locally recovers the pair $\omega_{B/A}, \tau_{B/A}$. By Lemma 49.10.4 the module $\omega_{Y/X}$ is invertible. Suppose given a commutative diagram of locally Noetherian schemes

$$\begin{array}{ccc} Y' & \xrightarrow[b]{} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then there is a canonical base change map

$$b^* \omega_{Y/X} \longrightarrow \omega_{Y'/X'}$$

which is an isomorphism mapping $\tau_{Y/X}$ to $\tau_{Y'/X'}$. Namely, the base change map in the affine setting is (49.2.3.1), it is an isomorphism by Lemma 49.2.10, and it maps $\tau_{Y/X}$ to $\tau_{Y'/X'}$ by Lemma 49.4.4 part (1).

0FKD **Proposition 49.13.2.** *There exists a unique rule that to every locally quasi-finite syntomic morphism of locally Noetherian schemes $Y \to X$ assigns an isomorphism*

$$c_{Y/X} : \det(NL_{Y/X}) \longrightarrow \omega_{Y/X}$$

satisfying the following two properties

(1) *the section $\delta(NL_{Y/X})$ is mapped to $\tau_{Y/X}$, and*
(2) *the rule is compatible with restriction to opens and with base change.*

Proof. Let us reformulate the statement of the proposition. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $Y \to X$ of locally Noetherian schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{array}{ccc} Y' & \xrightarrow{b} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{a} & X \end{array}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The proposition means that for every object Y/X of $\mathcal{C}$ we have an isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$ and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^* c_{Y/X} = c_{Y'/X'}$ via the identifications $b^* \det(NL_{Y/X}) = \det(NL_{Y'/X'})$ and $b^*\omega_{Y/X} = \omega_{Y'/X'}$ described above.

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine opens $V \subset Y$ and $U \subset X$ with $y \in V$ and $f(V) \subset U$ such that there exists some isomorphism

$$\det(NL_{Y/X})|_V \longrightarrow \omega_{Y/X}|_V$$

mapping $\delta(NL_{Y/X})|_V$ to $\tau_{Y/X}|_V$. This follows from picking affine opens as in Lemma 49.10.1 part (5), the affine local description of $\delta(NL_{Y/X})$ in Remark 49.13.1, and Lemma 49.12.4. If the annihilator of the section $\tau_{Y/X}$ is zero, then these local maps are unique and automatically glue. Hence if the annihilator of $\tau_{Y/X}$ is zero, then there is a unique isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$ and the annihilator of $\tau_{Y'/X'}$ is zero as well, then $b^*c_{Y/X}$ is the unique isomorphism $c_{Y'/X'} : \det(NL_{Y'/X'}) \to \omega_{Y'/X'}$ with $c_{Y'/X'}(\delta(NL_{Y'/X'})) = \tau_{Y'/X'}$. This follows formally from the fact that $b^*\delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ and $b^*\tau_{Y/X} = \tau_{Y'/X'}$.

We can summarize the results of the previous paragraph as follows. Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that the annihilator of $\tau_{Y/X}$ is zero. Then we have solved the problem on $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c_{Y/X}$ the solution we've just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. **Claim.** $b_1^* c_{Y_1/X_1} = b_2^* c_{Y_2/X_2}$. We will first show that the claim implies the proposition and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Example 49.10.5. Then $Y_{n,d}$ is an irreducible regular scheme and the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Lemma 49.10.6. Thus $\tau_{Y_{n,d}/X_{n,d}}$ is nonzero for example by Lemma 49.9.6. Now

a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Lemma 49.10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c_{Y_{n,d}/X_{n,d}}$ constructed above by b to V. The claim guarantees these local isomorphisms glue! Thus we get a well defined global isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c_{Y'/X'}$ is the pullback by b of the locally constructed map $c_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Therefore we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine, say they are the spectra of rings B, A, B_1, A_1, B_2, A_2. Picture

$$\begin{array}{ccccc} B_1 & \longrightarrow & B & \longleftarrow & B_2 \\ \uparrow & & \uparrow & & \uparrow \\ A_1 & \longrightarrow & A & \longleftarrow & A_2 \end{array}$$

By assumption the spectrum of B is an affine open of both the spectrum of $A \otimes_{A_1} B_1$ and $A \otimes_{A_2} B_2$. Shrinking more we may assume there exist elements $g_i \in A \otimes_{A_i} B_i$ such that our maps give isomorphisms $(A \otimes_{A_i} B_i)_{g_i} = B$, see Properties, Lemma 28.29.9. Let x_α, y_β be a sufficiently large collection of variables such that we may choose surjections

$$A_1' = A_1[x_\alpha] \to A \quad \text{and} \quad A_2' = A_2[y_\beta] \to A$$

of A_1 and A_2-algebras. Then we can choose lifts $h_i \in A_i' \otimes_{A_i} B_i$ of $g_i \in A \otimes_{A_i} B_i$ and we consider the diagram

$$\begin{array}{ccccc} (A_1' \otimes_{A_1} B_1)_{h_1} & \longrightarrow & B & \longleftarrow & (A_2' \otimes_{A_1} B_2)_{h_2} \\ \uparrow & & \uparrow & & \uparrow \\ A_1' & \longrightarrow & A & \longleftarrow & A_2' \end{array}$$

By construction the two squares are cocartesian. Next, we consider the ring map

$$A' = A_1' \times_A A_2' \longrightarrow B' = (A_1' \otimes_{A_1} B_1)_{h_1} \times_B (A_2' \otimes_{A_1} B_2)_{h_2}$$

By More on Algebra, Lemma 15.6.4 we have $A_1' \otimes_{A'} B' = (A_1' \otimes_{A_1} B_1)_{h_1}$ and $A_2' \otimes_{A'} B' = (A_2' \otimes_{A_2} B_2)_{h_2}$. In particular the fibres of the morphism $Y' = \mathrm{Spec}(B') \to \mathrm{Spec}(A') = X'$ are open subschemes of base changes of the fibres of the maps $Y_i \to X_i$, hence finite and local complete intersections (see Lemma 49.10.1). By More on Algebra, Lemma 15.7.7 the ring map $A' \to B'$ is flat and of finite presentation. Thus by Lemma 49.10.1 part (3) we conclude that $Y' \to X'$ is locally quasi-finite

and syntomic. Thus we obtain the commutative diagram

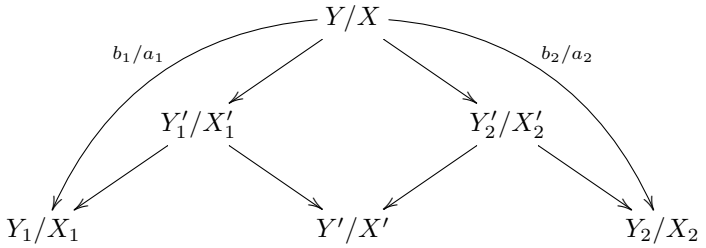

where Y_i'/X_i' corresponds to $A_i' \to (A_i' \otimes_{A_i} B_i)_{h_i}$.

In this paragraph, we fix the issue that A' and A_i' may not be Noetherian by a limit argument; we suggest the reader skip this. We can find a finitely generated $\mathbf{Z}$-subalgebra $A'' \subset A'$ and a quasi-finite and syntomic ring map $A'' \to B''$ such that $B' = A' \otimes_{A''} B''$. This follows from Limits, Lemmas 32.10.1, 32.8.16, and 32.18.2. Then the ring maps $A'' \to A_1' = A_1[x_\alpha]$ will factor through a finite polynomial subalgebra $A_i[x_1, \ldots, x_n$ and the ring map $A'' \to A_2' = A_2[y_\beta]$ will factor through $A_2[y_1, \ldots, y_m]$. After increasing the sets of variables, we may also assume that $h_1 \in A_1[x_1, \ldots, x_n] \otimes_{A_1} B_1$ and $h_2 \in A_2[y_1, \ldots, y_m] \otimes_{A_2} B_2$. Increasing the sets of variables one more time, we can assume that the maps $B'' \to (A_i' \otimes_{A_i} B_i)_{h_i}$ factor through $(A_1[x_1, \ldots, x_n] \otimes_{A_1} B_1)_{h_1}$ and $(A_2[y_1, \ldots, y_m] \otimes_{A_2} B_2)_{h_2}$. Replacing Y', X', Y_1', X_1', Y_2', X_2' by the spectra of B'', A'', $(A_1[x_1, \ldots, x_n] \otimes_{A_1} B_1)_{h_1}$, $A_1[x_1, \ldots, x_n]$, $(A_2[y_1, \ldots, y_m] \otimes_{A_2} B_2)_{h_2}$, $A_2[y_1, \ldots, y_m]$, we see that we get a diagram as above where all schemes are Noetherian.

Note that Y_i'/X_i' is an object of $\mathcal{C}_{nice}$ as it obtained by taking an open subscheme of the product of an affine space with Y_i/X_i. In particular, the pullback of c_{Y_i/X_i} via (b_i/a_i) is the same as the pullback of $c_{Y_i'/X_i'}$ via $Y/X \to Y_i'/X_i'$. Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$ (but this is likely not the case). Namely, then pulling back $c_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. To get around the problem that Y'/X' is not in $\mathcal{C}_{nice}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1', Y_1', X_2', Y_2', X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

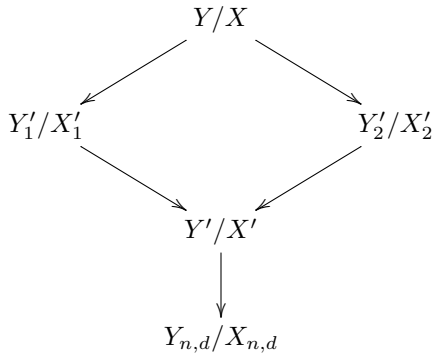

and then we can use the already given argument by pulling back from $c_{Y_{n,d}/X_{n,d}}$. This finishes the proof. $\square$

49.14. A generalization of the different

0BWK In this section we generalize Definition 49.9.1 to take into account all cases of ring maps $A \to B$ where the Dedekind different is defined and $1 \in \mathcal{L}_{B/A}$. First we explain the condition "$A \to B$ maps nonzerodivisors to nonzerodivisors and induces a flat map $Q(A) \to Q(A) \otimes_A B$".

0BWL **Lemma 49.14.1.** *Let $A \to B$ be a map of Noetherian rings. Consider the conditions*

(1) *nonzerodivisors of A map to nonzerodivisors of B,*
(2) *(1) holds and $Q(A) \to Q(A) \otimes_A B$ is flat,*
(3) *$A \to B_{\mathfrak{q}}$ is flat for every $\mathfrak{q} \in Ass(B)$,*
(4) *(3) holds and $A \to B_{\mathfrak{q}}$ is flat for every $\mathfrak{q}$ lying over an element in $Ass(A)$.*

Then we have the following implications

$$\begin{array}{ccc} (1) & \Longleftarrow & (2) \\ \Uparrow & & \Downarrow \\ (3) & \Longleftarrow & (4) \end{array}$$

If going up holds for $A \to B$ then (2) and (4) are equivalent.

Proof. The horizontal implications in the diagram are trivial. Let $S \subset A$ be the set of nonzerodivisors so that $Q(A) = S^{-1}A$ and $Q(A) \otimes_A B = S^{-1}B$. Recall that $S = A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$ by Algebra, Lemma 10.63.9. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$.

Assume (2). If $\mathfrak{q} \in \mathrm{Ass}(B)$ then $\mathfrak{q}$ consists of zerodivisors, hence (1) implies the same is true for $\mathfrak{p}$. Hence $\mathfrak{p}$ corresponds to a prime of $S^{-1}A$. Hence $A \to B_{\mathfrak{q}}$ is flat by our assumption (2). If $\mathfrak{q}$ lies over an associated prime $\mathfrak{p}$ of A, then certainly $\mathfrak{p} \in \mathrm{Spec}(S^{-1}A)$ and the same argument works.

Assume (3). Let $f \in A$ be a nonzerodivisor. If f were a zerodivisor on B, then f is contained in an associated prime $\mathfrak{q}$ of B. Since $A \to B_{\mathfrak{q}}$ is flat by assumption, we conclude that $\mathfrak{p}$ is an associated prime of A by Algebra, Lemma 10.65.3. This would imply that f is a zerodivisor on A, a contradiction.

Assume (4) and going up for $A \to B$. We already know (1) holds. If $\mathfrak{q}$ corresponds to a prime of $S^{-1}B$ then $\mathfrak{p}$ is contained in an associated prime $\mathfrak{p}'$ of A. By going up there exists a prime $\mathfrak{q}'$ containing $\mathfrak{q}$ and lying over $\mathfrak{p}$. Then $A \to B_{\mathfrak{q}'}$ is flat by (4). Hence $A \to B_{\mathfrak{q}}$ is flat as a localization. Thus $A \to S^{-1}B$ is flat and so is $S^{-1}A \to S^{-1}B$, see Algebra, Lemma 10.39.18. □

0BWM **Remark 49.14.2.** We can generalize Definition 49.9.1. Suppose that $f : Y \to X$ is a quasi-finite morphism of Noetherian schemes with the following properties

(1) the open $V \subset Y$ where f is flat contains $\mathrm{Ass}(\mathcal{O}_Y)$ and $f^{-1}(\mathrm{Ass}(\mathcal{O}_X))$,
(2) the trace element $\tau_{V/X}$ comes from a section $\tau \in \Gamma(Y, \omega_{Y/X})$.

Condition (1) implies that V contains the associated points of $\omega_{Y/X}$ by Lemma 49.2.8. In particular, τ is unique if it exists (Divisors, Lemma 31.2.8). Given τ we can define the different $\mathfrak{D}_f$ as the annihilator of $\mathrm{Coker}(\tau : \mathcal{O}_Y \to \omega_{Y/X})$. This agrees with the Dedekind different in many cases (Lemma 49.14.3). However, for non-flat maps between non-normal rings, this generalization no longer measures ramification of the morphism, see Example 49.14.4.

0BWN **Lemma 49.14.3.** *Assume the Dedekind different is defined for $A \to B$. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. The generalization of Remark 49.14.2 applies to the morphism $f : Y \to X$ if and only if $1 \in \mathcal{L}_{B/A}$ (e.g., if A is normal, see Lemma 49.8.1). In this case $\mathfrak{D}_{B/A}$ is an ideal of B and we have*

$$\mathfrak{D}_f = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on Y.

Proof. As the Dedekind different for $A \to B$ is defined we can apply Lemma 49.14.1 to see that $Y \to X$ satisfies condition (1) of Remark 49.14.2. Recall that there is a canonical isomorphism $c : \mathcal{L}_{B/A} \to \omega_{B/A}$, see Lemma 49.8.2. Let $K = Q(A)$ and $L = K \otimes_A B$ as above. By construction the map c fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{B/A} & \longrightarrow & L \\ {\scriptstyle c}\downarrow & & \downarrow \\ \omega_{B/A} & \longrightarrow & \mathrm{Hom}_K(L, K) \end{array}$$

where the right vertical arrow sends $x \in L$ to the map $y \mapsto \mathrm{Trace}_{L/K}(xy)$ and the lower horizontal arrow is the base change map (49.2.3.1) for $\omega_{B/A}$. We can factor the lower horizontal map as

$$\omega_{B/A} = \Gamma(Y, \omega_{Y/X}) \to \Gamma(V, \omega_{V/X}) \to \mathrm{Hom}_K(L, K)$$

Since all associated points of $\omega_{V/X}$ map to associated primes of A (Lemma 49.2.8) we see that the second map is injective. The element $\tau_{V/X}$ maps to $\mathrm{Trace}_{L/K}$ in $\mathrm{Hom}_K(L, K)$ by the very definition of trace elements (Definition 49.4.1). Thus τ as in condition (2) of Remark 49.14.2 exists if and only if $1 \in \mathcal{L}_{B/A}$ and then $\tau = c(1)$. In this case, by Lemma 49.8.1 we see that $\mathfrak{D}_{B/A} \subset B$. Finally, the agreement of $\mathfrak{D}_f$ with $\mathfrak{D}_{B/A}$ is immediate from the definitions and the fact $\tau = c(1)$ seen above. □

0BWP **Example 49.14.4.** Let k be a field. Let $A = k[x, y]/(xy)$ and $B = k[u, v]/(uv)$ and let $A \to B$ be given by $x \mapsto u^n$ and $y \mapsto v^m$ for some $n, m \in \mathbf{N}$ prime to the characteristic of k. Then $A_{x+y} \to B_{x+y}$ is (finite) étale hence we are in the situation where the Dedekind different is defined. A computation shows that

$$\mathrm{Trace}_{L/K}(1) = (nx + my)/(x + y), \quad \mathrm{Trace}_{L/K}(u^i) = 0, \quad \mathrm{Trace}_{L/K}(v^j) = 0$$

for $1 \leq i < n$ and $1 \leq j < m$. We conclude that $1 \in \mathcal{L}_{B/A}$ if and only if $n = m$. Moreover, a computation shows that if $n = m$, then $\mathcal{L}_{B/A} = B$ and the Dedekind different is B as well. In other words, we find that the different of Remark 49.14.2 is defined for $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ if and only if $n = m$, and in this case the different is the unit ideal. Thus we see that in nonflat cases the nonvanishing of the different does not guarantee the morphism is étale or unramified.

49.15. Comparison with duality theory

0DWM In this section we compare the elementary algebraic constructions above with the constructions in the chapter on duality theory for schemes.

0BUL **Lemma 49.15.1.** *Let $f : Y \to X$ be a quasi-finite separated morphism of Noetherian schemes. For every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is an isomorphism*

$$H^0(V, f^!\mathcal{O}_X) = \omega_{B/A}$$

where $f^!$ is as in Duality for Schemes, Section 48.16. These isomorphisms are compatible with restriction maps and define a canonical isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{Y/X}$ with $\omega_{Y/X}$ as in Remark 49.2.11. Similarly, if $f : Y \to X$ is a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$, then $H^0(f^!_{new}\mathcal{O}_X) = \omega_{Y/X}$.

Proof. By Zariski's main theorem we can choose a factorization $f = f' \circ j$ where $j : Y \to Y'$ is an open immersion and $f' : Y' \to X$ is a finite morphism, see More on Morphisms, Lemma 37.43.3. By our construction in Duality for Schemes, Lemma 48.16.2 we have $f^! = j^* \circ a'$ where $a' : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{Y'})$ is the right adjoint to Rf'_* of Duality for Schemes, Lemma 48.3.1. By Duality for Schemes, Lemma 48.11.4 we see that $\Phi(a'(\mathcal{O}_X)) = R\mathcal{H}om(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$ in $D^+_{QCoh}(f'_*\mathcal{O}_{Y'})$. In particular $a'(\mathcal{O}_X)$ has vanishing cohomology sheaves in degrees < 0. The zeroth cohomology sheaf is determined by the isomorphism

$$f'_*H^0(a'(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$$

as $f'_*\mathcal{O}_{Y'}$-modules via the equivalence of Morphisms, Lemma 29.11.7. Writing $(f')^{-1}U = V' = \mathrm{Spec}(B')$, we obtain

$$H^0(V', a'(\mathcal{O}_X)) = \mathrm{Hom}_A(B', A).$$

As the zeroth cohomology sheaf of $a'(\mathcal{O}_X)$ is a quasi-coherent module we find that the restriction to V is given by $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ as desired.

The statement about restriction maps signifies that the restriction mappings of the quasi-coherent $\mathcal{O}_{Y'}$-module $H^0(a'(\mathcal{O}_X))$ for opens in Y' agrees with the maps defined in Lemma 49.2.3 for the modules $\omega_{B/A}$ via the isomorphisms given above. This is clear.

Let $f : Y \to X$ be a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$. Consider opens $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ and V and U separated over S. Denote $f|_V : V \to U$ the restriction of f. By the discussion above and Duality for Schemes, Lemma 48.20.9 there are canonical isomorphisms

$$H^0(f^!_{new}\mathcal{O}_X)|_V = H^0((f|_V)^!\mathcal{O}_U) = \omega_{V/U} = \omega_{Y/X}|_V$$

We omit the verification that these isomorphisms glue to a global isomorphism $H^0(f^!_{new}\mathcal{O}_X) \to \omega_{Y/X}$. □

0BVI **Lemma 49.15.2.** *Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. The map*

$$\mathit{Trace}_f : f_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

of Section 49.3 corresponds to a map $\mathcal{O}_Y \to f^!\mathcal{O}_X$ (see proof). Denote $\tau_{Y/X} \in H^0(Y, f^!\mathcal{O}_X)$ the image of 1. Via the isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{X/Y}$ of Lemma 49.15.1 this agrees with the construction in Remark 49.4.7.

Proof. The functor $f^!$ is defined in Duality for Schemes, Section 48.16. Since f is finite (and hence proper), we see that $f^!$ is given by the right adjoint to pushforward for f. In Duality for Schemes, Section 48.11 we have made this adjoint explicit. In particular, the object $f^!\mathcal{O}_X$ consists of a single cohomology sheaf placed in degree 0 and for this sheaf we have

$$f_*f^!\mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

To see this we use also that $f_*\mathcal{O}_Y$ is finite locally free as f is a finite flat morphism of Noetherian schemes and hence all higher Ext sheaves are zero. Some details omitted. Thus finally

$$\text{Trace}_f \in \text{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \Gamma(X, f_*f^!\mathcal{O}_X) = \Gamma(Y, f^!\mathcal{O}_X)$$

On the other hand, we have $f^!\mathcal{O}_X = \omega_{Y/X}$ by the identification of Lemma 49.15.1. Thus we now have two elements, namely Trace_f and $\tau_{Y/X}$ from Remark 49.4.7 in

$$\Gamma(Y, f^!\mathcal{O}_X) = \Gamma(Y, \omega_{Y/X})$$

and the lemma says these elements are the same.

Let $U = \text{Spec}(A) \subset X$ be an affine open with inverse image $V = \text{Spec}(B) \subset Y$. Since f is finite, we see that $A \to B$ is finite and hence the $\omega_{Y/X}(V) = \text{Hom}_A(B, A)$ by construction and this isomorphism agrees with the identification of $f_*f^!\mathcal{O}_Y$ with $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ discussed above. Hence the agreement of Trace_f and $\tau_{Y/X}$ follows from the fact that $\tau_{B/A} = \text{Trace}_{B/A}$ by Lemma 49.4.3. □

49.16. Quasi-finite Gorenstein morphisms

0C14 This section discusses quasi-finite Gorenstein morphisms.

0C16 **Lemma 49.16.1.** *Let $f : Y \to X$ be a quasi-finite morphism of Noetherian schemes. The following are equivalent*

(1) *f is Gorenstein,*
(2) *f is flat and the fibres of f are Gorenstein,*
(3) *f is flat and $\omega_{Y/X}$ is invertible (Remark 49.2.11),*
(4) *for every $y \in Y$ there are affine opens $y \in V = \text{Spec}(B) \subset Y$, $U = \text{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is flat and $\omega_{B/A}$ is an invertible B-module.*

Proof. Parts (1) and (2) are equivalent by definition. Parts (3) and (4) are equivalent by the construction of $\omega_{Y/X}$ in Remark 49.2.11. Thus we have to show that (1)-(2) is equivalent to (3)-(4).

First proof. Working affine locally we can assume f is a separated morphism and apply Lemma 49.15.1 to see that $\omega_{Y/X}$ is the zeroth cohomology sheaf of $f^!\mathcal{O}_X$. Under both assumptions f is flat and quasi-finite, hence $f^!\mathcal{O}_X$ is isomorphic to $\omega_{Y/X}[0]$, see Duality for Schemes, Lemma 48.21.6. Hence the equivalence follows from Duality for Schemes, Lemma 48.25.10.

Second proof. By Lemma 49.10.2, we see that it suffices to prove the equivalence of (2) and (3) when X is the spectrum of a field k. Then $Y = \text{Spec}(B)$ where B is a finite k-algebra. In this case $\omega_{B/A} = \omega_{B/k} = \text{Hom}_k(B, k)$ placed in degree 0 is a dualizing complex for B, see Dualizing Complexes, Lemma 47.15.8. Thus the equivalence follows from Dualizing Complexes, Lemma 47.21.4. □

0C17 **Remark 49.16.2.** Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of Noetherian schemes. Let $\mathfrak{D}_f \subset \mathcal{O}_Y$ be the different and let $R \subset Y$ be the closed subscheme cut out by $\mathfrak{D}_f$. Then we have

(1) $\mathfrak{D}_f$ is a locally principal ideal,
(2) R is a locally principal closed subscheme,
(3) $\mathfrak{D}_f$ is affine locally the same as the Noether different,
(4) formation of R commutes with base change,

(5) if f is finite, then the norm of R is the discriminant of f, and
(6) if f is étale in the associated points of Y, then R is an effective Cartier divisor and $\omega_{Y/X} = \mathcal{O}_Y(R)$.

This follows from Lemmas 49.9.3, 49.9.4, and 49.9.7.

0C18 **Remark 49.16.3.** Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of compactifyable schemes over S. Assume moreover Y and X Cohen-Macaulay and f étale at the generic points of Y. Then we can combine Duality for Schemes, Remark 48.23.4 and Remark 49.16.2 to see that we have a canonical isomorphism

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \omega_{Y/X} = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R)$$

of $\mathcal{O}_Y$-modules. If further f is finite, then the isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$ comes from the global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which corresponds via duality to the map $\mathrm{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Lemma 49.15.2.

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CHAPTER 50

de Rham Cohomology

0FK4

50.1. Introduction

0FK5 In this chapter we start with a discussion of the de Rham complex of a morphism of schemes and we end with a proof that de Rham cohomology defines a Weil cohomology theory when the base field has characteristic zero.

50.2. The de Rham complex

07HX Let $p : X \to S$ be a morphism of schemes. There is a complex

$$\Omega^\bullet_{X/S} = \mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

of $p^{-1}\mathcal{O}_S$-modules with $\Omega^i_{X/S} = \wedge^i(\Omega_{X/S})$ placed in degree i and differential determined by the rule $\mathrm{d}(g_0 \mathrm{d}g_1 \wedge \ldots \wedge \mathrm{d}g_p) = \mathrm{d}g_0 \wedge \mathrm{d}g_1 \wedge \ldots \wedge \mathrm{d}g_p$ on local sections. See Modules, Section 17.30.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes, there are canonical maps of complexes $f^{-1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X'/S'}$ and $\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$. See Modules, Section 17.30. Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

In particular, if $f : Y \to X$ is a morphism of schemes over a base scheme S, then there is a map of complexes

$$\Omega^\bullet_{X/S} \longrightarrow f_*\Omega^\bullet_{Y/S}$$

Linearizing, we see that for every $p \geq 0$ we obtain a canonical map

$$\Omega^p_{X/S} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \longrightarrow f_*\Omega^p_{Y/S}$$

0FL5 **Lemma 50.2.1.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

be a cartesian diagram of schemes. Then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. Combine Morphisms, Lemma 29.32.10 with the fact that formation of exterior power commutes with base change. □

0FLV **Lemma 50.2.2.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

If $X' \to X$ and $S' \to S$ are étale, then the maps discussed above induce isomorphisms $f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}$.

Proof. We have $\Omega_{S'/S} = 0$ and $\Omega_{X'/X} = 0$, see for example Morphisms, Lemma 29.36.15. Then by the short exact sequences of Morphisms, Lemmas 29.32.9 and 29.34.16 we see that $\Omega_{X'/S'} = \Omega_{X'/S} = f^*\Omega_{X/S}$. Taking exterior powers we conclude. □

50.3. de Rham cohomology

0FL6 Let $p : X \to S$ be a morphism of schemes. We define the *de Rham cohomology of X over S* to be the cohomology groups

$$H^i_{dR}(X/S) = H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$$

Since $\Omega^\bullet_{X/S}$ is a complex of $p^{-1}\mathcal{O}_S$-modules, these cohomology groups are naturally modules over $H^0(S, \mathcal{O}_S)$.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes, using the canonical maps of Section 50.2 we obtain pullback maps

$$f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \longrightarrow R\Gamma(X', \Omega^\bullet_{X'/S'})$$

and

$$f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$$

These pullbacks satisfy an obvious composition law. In particular, if we work over a fixed base scheme S, then de Rham cohomology is a contravariant functor on the category of schemes over S.

0FLW **Lemma 50.3.1.** *Let $X \to S$ be a morphism of affine schemes given by the ring map $R \to A$. Then $R\Gamma(X, \Omega^\bullet_{X/S}) = \Omega^\bullet_{A/R}$ in $D(R)$ and $H^i_{dR}(X/S) = H^i(\Omega^\bullet_{A/R})$.*

Proof. This follows from Cohomology of Schemes, Lemma 30.2.2 and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7). □

0FLX **Lemma 50.3.2.** *Let $p : X \to S$ be a morphism of schemes. If p is quasi-compact and quasi-separated, then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{QCoh}(\mathcal{O}_S)$.*

Proof. There is a spectral sequence with first page $E_1^{a,b} = R^b p_*\Omega^a_{X/S}$ converging to the cohomology of $Rp_*\Omega^\bullet_{X/S}$ (see Derived Categories, Lemma 13.21.3). Hence by Homology, Lemma 12.25.3 it suffices to show that $R^b p_*\Omega^a_{X/S}$ is quasi-coherent. This follows from Cohomology of Schemes, Lemma 30.4.5. □

0FLY **Lemma 50.3.3.** *Let $p : X \to S$ be a proper morphism of schemes with S locally Noetherian. Then $Rp_*\Omega^\bullet_{X/S}$ is an object of $D_{Coh}(\mathcal{O}_S)$.*

Proof. In this case by Morphisms, Lemma 29.32.12 the modules $\Omega^i_{X/S}$ are coherent. Hence we can use exactly the same argument as in the proof of Lemma 50.3.2 using Cohomology of Schemes, Proposition 30.19.1. □

0FLZ **Lemma 50.3.4.** *Let A be a Noetherian ring. Let X be a proper scheme over $S = \mathrm{Spec}(A)$. Then $H^i_{dR}(X/S)$ is a finite A-module for all i.*

Proof. This is a special case of Lemma 50.3.3. □

0FM0 **Lemma 50.3.5.** *Let $f : X \to S$ be a proper smooth morphism of schemes. Then $Rf_*\Omega^p_{X/S}$, $p \geq 0$ and $Rf_*\Omega^\bullet_{X/S}$ are perfect objects of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base.*

Proof. Since f is smooth the modules $\Omega^p_{X/S}$ are finite locally free $\mathcal{O}_X$-modules, see Morphisms, Lemma 29.34.12. Their formation commutes with arbitrary change of base by Lemma 50.2.1. Hence $Rf_*\Omega^p_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary base change, see Derived Categories of Schemes, Lemma 36.30.4. This proves the first assertion of the lemma.

To prove that $Rf_*\Omega^\bullet_{X/S}$ is perfect on S we may work locally on S. Thus we may assume S is quasi-compact. This means we may assume that $\Omega^n_{X/S}$ is zero for n large enough. For every $p \geq 0$ we claim that $Rf_*\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$ whose formation commutes with arbitrary change of base. By the above we see that this is true for $p \gg 0$. Suppose the claim holds for p and consider the distinguished triangle

$$\sigma_{\geq p}\Omega^\bullet_{X/S} \to \sigma_{\geq p-1}\Omega^\bullet_{X/S} \to \Omega^{p-1}_{X/S}[-(p-1)] \to (\sigma_{\geq p}\Omega^\bullet_{X/S})[1]$$

in $D(f^{-1}\mathcal{O}_S)$. Applying the exact functor Rf_* we obtain a distinguished triangle in $D(\mathcal{O}_S)$. Since we have the 2-out-of-3 property for being perfect (Cohomology, Lemma 20.49.7) we conclude $Rf_*\sigma_{\geq p-1}\Omega^\bullet_{X/S}$ is a perfect object of $D(\mathcal{O}_S)$. Similarly for the commutation with arbitrary base change. □

50.4. Cup product

0FM1 Consider the maps $\Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ given by $(\omega, \eta) \longmapsto \omega \wedge \eta$. Using the formula for d given in Section 50.2 and the Leibniz rule for $\mathrm{d} : \mathcal{O}_X \to \Omega_{X/S}$ we see that $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \mathrm{d}(\eta)$. This means that $\wedge$ defines a morphism

0FM2 (50.4.0.1) $$\wedge : \mathrm{Tot}(\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}$$

of complexes of $p^{-1}\mathcal{O}_S$-modules.

Combining the cup product of Cohomology, Section 20.31 with (50.4.0.1) we find a $H^0(S, \mathcal{O}_S)$-bilinear cup product map

$$\cup : H^i_{dR}(X/S) \times H^j_{dR}(X/S) \longrightarrow H^{i+j}_{dR}(X/S)$$

For example, if $\omega \in \Gamma(X, \Omega^i_{X/S})$ and $\eta \in \Gamma(X, \Omega^j_{X/S})$ are closed, then the cup product of the de Rham cohomology classes of ω and η is the de Rham cohomology class of $\omega \wedge \eta$, see discussion in Cohomology, Section 20.31.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow[f]{} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes, the pullback maps $f^* : R\Gamma(X, \Omega^\bullet_{X/S}) \to R\Gamma(X', \Omega^\bullet_{X'/S'})$ and $f^* : H^i_{dR}(X/S) \longrightarrow H^i_{dR}(X'/S')$ are compatible with the cup product defined above.

0FM3 **Lemma 50.4.1.** *Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{dR}(X/S)$ is associative and graded commutative.*

Proof. This follows from Cohomology, Lemmas 20.31.5 and 20.31.6 and the fact that $\wedge$ is associative and graded commutative. □

0FU6 **Remark 50.4.2.** Let $p : X \to S$ be a morphism of schemes. Then we can think of $\Omega^\bullet_{X/S}$ as a sheaf of differential graded $p^{-1}\mathcal{O}_S$-algebras, see Differential Graded Sheaves, Definition 24.12.1. In particular, the discussion in Differential Graded Sheaves, Section 24.32 applies. For example, this means that for any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[f]{} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow{h} & T \end{array}$$

of schemes there is a canonical relative cup product

$$\mu : Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

in $D(Y, q^{-1}\mathcal{O}_T)$ which is associative and which on cohomology reproduces the cup product discussed above.

0FU7 **Remark 50.4.3.** Let $f : X \to S$ be a morphism of schemes. Let $\xi \in H^n_{dR}(X/S)$. According to the discussion Differential Graded Sheaves, Section 24.32 there exists a canonical morphism

$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n]$$

in $D(f^{-1}\mathcal{O}_S)$ uniquely characterized by (1) and (2) of the following list of properties:

(1) ξ' can be lifted to a map in the derived category of right differential graded $\Omega^\bullet_{X/S}$-modules, and
(2) $\xi'(1) = \xi$ in $H^0(X, \Omega^\bullet_{X/S}[n]) = H^n_{dR}(X/S)$,
(3) the map ξ' sends $\eta \in H^m_{dR}(X/S)$ to $\xi \cup \eta$ in $H^{n+m}_{dR}(X/S)$,
(4) the construction of ξ' commutes with restrictions to opens: for $U \subset X$ open the restriction $\xi'|_U$ is the map corresponding to the image $\xi|_U \in H^n_{dR}(U/S)$,
(5) for any diagram as in Remark 50.4.2 we obtain a commutative diagram

$$\begin{array}{ccc} Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} & \xrightarrow{\mu} & Rf_*\Omega^\bullet_{X/S} \\ {\scriptstyle \xi'\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \xi'} \\ Rf_*\Omega^\bullet_{X/S}[n] \otimes^{\mathbf{L}}_{q^{-1}\mathcal{O}_T} Rf_*\Omega^\bullet_{X/S} & \xrightarrow{\mu} & Rf_*\Omega^\bullet_{X/S}[n] \end{array}$$

in $D(Y, q^{-1}\mathcal{O}_T)$.

50.5. Hodge cohomology

0FM4 Let $p : X \to S$ be a morphism of schemes. We define the *Hodge cohomology of X over S* to be the cohomology groups

$$H^n_{Hodge}(X/S) = \bigoplus\nolimits_{n=p+q} H^q(X, \Omega^p_{X/S})$$

viewed as a graded $H^0(X, \mathcal{O}_X)$-module. The wedge product of forms combined with the cup product of Cohomology, Section 20.31 defines a $H^0(X, \mathcal{O}_X)$-bilinear cup product

$$\cup : H^i_{Hodge}(X/S) \times H^j_{Hodge}(X/S) \longrightarrow H^{i+j}_{Hodge}(X/S)$$

Of course if $\xi \in H^q(X, \Omega^p_{X/S})$ and $\xi' \in H^{q'}(X, \Omega^{p'}_{X/S})$ then $\xi \cup \xi' \in H^{q+q'}(X, \Omega^{p+p'}_{X/S})$.

0FM5 **Lemma 50.5.1.** *Let $p : X \to S$ be a morphism of schemes. The cup product on $H^*_{Hodge}(X/S)$ is associative and graded commutative.*

Proof. The proof is identical to the proof of Lemma 50.4.1. □

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes, there are pullback maps $f^* : H^i_{Hodge}(X/S) \longrightarrow H^i_{Hodge}(X'/S')$ compatible with gradings and with the cup product defined above.

50.6. Two spectral sequences

0FM6 Let $p : X \to S$ be a morphism of schemes. Since the category of $p^{-1}\mathcal{O}_S$-modules on X has enough injectives there exist a Cartan-Eilenberg resolution for $\Omega^\bullet_{X/S}$. See Derived Categories, Lemma 13.21.2. Hence we can apply Derived Categories, Lemma 13.21.3 to get two spectral sequences both converging to the de Rham cohomology of X over S.

The first is customarily called *the Hodge-to-de Rham spectral sequence*. The first page of this spectral sequence has

$$E_1^{p,q} = H^q(X, \Omega^p_{X/S})$$

which are the Hodge cohomology groups of X/S (whence the name). The differential d_1 on this page is given by the maps $d_1^{p,q} : H^q(X, \Omega^p_{X/S}) \to H^q(X.\Omega^{p+1}_{X/S})$ induced by the differential $\mathrm{d} : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$. Here is a picture

$$\begin{array}{ccccccc} H^2(X, \mathcal{O}_X) & \longrightarrow & H^2(X, \Omega^1_{X/S}) & \longrightarrow & H^2(X, \Omega^2_{X/S}) & \longrightarrow & H^2(X, \Omega^3_{X/S}) \\ & & & \searrow & & \searrow & \\ H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \Omega^1_{X/S}) & \longrightarrow & H^1(X, \Omega^2_{X/S}) & \longrightarrow & H^1(X, \Omega^3_{X/S}) \\ & & & \searrow & & \searrow & \\ H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, \Omega^1_{X/S}) & \longrightarrow & H^0(X, \Omega^2_{X/S}) & \longrightarrow & H^0(X, \Omega^3_{X/S}) \end{array}$$

where we have drawn striped arrows to indicate the source and target of the differentials on the E_2 page and a dotted arrow for a differential on the E_3 page. Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = \mathrm{Ker}(\mathrm{d} : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/S}))$$

Of course, this is also immediately clear from the fact that the de Rham complex starts in degree 0 with $\mathcal{O}_X \to \Omega^1_{X/S}$.

The second spectral sequence is usually called *the conjugate spectral sequence*. The second page of this spectral sequence has

$$E_2^{p,q} = H^p(X, H^q(\Omega^\bullet_{X/S})) = H^p(X, \mathcal{H}^q)$$

where $\mathcal{H}^q = H^q(\Omega^\bullet_{X/S})$ is the qth cohomology sheaf of the de Rham complex of X/S. The differentials on this page are given by $E_2^{p,q} \to E_2^{p+2,q-1}$. Here is a picture

$$\begin{array}{cccc}
H^0(X, \mathcal{H}^2) & H^1(X, \mathcal{H}^2) & H^2(X, \mathcal{H}^2) & H^3(X, \mathcal{H}^2) \\
H^0(X, \mathcal{H}^1) & H^1(X, \mathcal{H}^1) & H^2(X, \mathcal{H}^1) & H^3(X, \mathcal{H}^1) \\
H^0(X, \mathcal{H}^0) & H^1(X, \mathcal{H}^0) & H^2(X, \mathcal{H}^0) & H^3(X, \mathcal{H}^0)
\end{array}$$

Looking in degree 0 we conclude that

$$H^0_{dR}(X/S) = H^0(X, \mathcal{H}^0)$$

which is obvious if you think about it. In degree 1 we get an exact sequence

$$0 \to H^1(X, \mathcal{H}^0) \to H^1_{dR}(X/S) \to H^0(X, \mathcal{H}^1) \to H^2(X, \mathcal{H}^0) \to H^2_{dR}(X/S)$$

It turns out that if $X \to S$ is smooth and S lives in characteristic p, then the sheaves $\mathcal{H}^q$ are computable (in terms of a certain sheaves of differentials) and the conjugate spectral sequence is a valuable tool (insert future reference here).

50.7. The Hodge filtration

0FM7 Let $X \to S$ be a morphism of schemes. The Hodge filtration on $H^n_{dR}(X/S)$ is the filtration induced by the Hodge-to-de Rham spectral sequence (Homology, Definition 12.24.5). To avoid misunderstanding, we explicitly define it as follows.

0FM8 **Definition 50.7.1.** Let $X \to S$ be a morphism of schemes. The *Hodge filtration* on $H^n_{dR}(X/S)$ is the filtration with terms

$$F^p H^n_{dR}(X/S) = \mathrm{Im}\left(H^n(X, \sigma_{\geq p}\Omega^\bullet_{X/S}) \longrightarrow H^n_{dR}(X/S)\right)$$

where $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is as in Homology, Section 12.15.

Of course $\sigma_{\geq p}\Omega^\bullet_{X/S}$ is a subcomplex of the relative de Rham complex and we obtain a filtration

$$\Omega^\bullet_{X/S} = \sigma_{\geq 0}\Omega^\bullet_{X/S} \supset \sigma_{\geq 1}\Omega^\bullet_{X/S} \supset \sigma_{\geq 2}\Omega^\bullet_{X/S} \supset \sigma_{\geq 3}\Omega^\bullet_{X/S} \supset \ldots$$

of the relative de Rham complex with $\mathrm{gr}^p(\Omega^\bullet_{X/S}) = \Omega^p_{X/S}[-p]$. The spectral sequence constructed in Cohomology, Lemma 20.29.1 for $\Omega^\bullet_{X/S}$ viewed as a filtered

complex of sheaves is the same as the Hodge-to-de Rham spectral sequence constructed in Section 50.6 by Cohomology, Example 20.29.4. Further the wedge product (50.4.0.1) sends $\mathrm{Tot}(\sigma_{\geq i}\Omega^\bullet_{X/S} \otimes_{p^{-1}\mathcal{O}_S} \sigma_{\geq j}\Omega^\bullet_{X/S})$ into $\sigma_{\geq i+j}\Omega^\bullet_{X/S}$. Hence we get commutative diagrams

$$\begin{array}{ccc} H^n(X, \sigma_{\geq i}\Omega^\bullet_{X/S})) \times H^m(X, \sigma_{\geq j}\Omega^\bullet_{X/S})) & \longrightarrow & H^{n+m}(X, \sigma_{\geq i+j}\Omega^\bullet_{X/S})) \\ \downarrow & & \downarrow \\ H^n_{dR}(X/S) \times H^m_{dR}(X/S) & \xrightarrow{\cup} & H^{n+m}_{dR}(X/S) \end{array}$$

In particular we find that

$$F^i H^n_{dR}(X/S) \cup F^j H^m_{dR}(X/S) \subset F^{i+j} H^{n+m}_{dR}(X/S)$$

50.8. Künneth formula

0FM9 An important feature of de Rham cohomology is that there is a Künneth formula.

Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms and $f = a \circ p = b \circ q$. Here is a picture

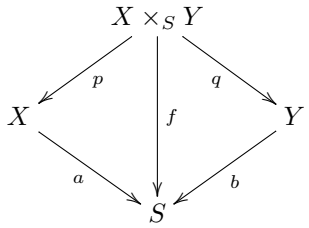

In this section, given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\mathcal{G}$$

The bifunctor $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}$ on quasi-coherent modules extends to a bifunctor on quasi-coherent modules and differential operators of finite over over S, see Morphisms, Remark 29.33.3. The differentials of the de Rham complexes $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Y/S}$ are differential operators of order 1 over S by Modules, Lemma 17.30.5. Thus it makes sense to consider the complex

$$\mathrm{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S})$$

Please see the discussion in Derived Categories of Schemes, Section 36.24.

0FMA **Lemma 50.8.1.** *In the situation above there is a canonical isomorphism*

$$\mathit{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S}) \longrightarrow \Omega^\bullet_{X\times_S Y/S}$$

of complexes of $f^{-1}\mathcal{O}_S$-modules.

Proof. We know that $\Omega_{X\times_S Y/S} = p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S}$ by Morphisms, Lemma 29.32.11. Taking exterior powers we obtain

$$\Omega^n_{X\times_S Y/S} = \bigoplus\nolimits_{i+j=n} p^*\Omega^i_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\Omega^j_{Y/S} = \bigoplus\nolimits_{i+j=n} \Omega^i_{X/S} \boxtimes \Omega^j_{Y/S}$$

by elementary properties of exterior powers. These identifications determine isomorphisms between the terms of the complexes on the left and the right of the

arrow in the lemma. We omit the verification that these maps are compatible with differentials. □

Set $A = \Gamma(S, \mathcal{O}_S)$. Combining the result of Lemma 50.8.1 with the map Derived Categories of Schemes, Equation (36.24.0.2) we obtain a cup product

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

On the level of cohomology, using the discussion in More on Algebra, Section 15.63, we obtain a canonical map

$$H^i_{dR}(X/S) \otimes_A H^j_{dR}(Y/S) \longrightarrow H^{i+j}_{dR}(X \times_S Y/S), \quad (\xi, \zeta) \longmapsto p^*\xi \cup q^*\zeta$$

We note that the construction above indeed proceeds by first pulling back and then taking the cup product.

0FMB **Lemma 50.8.2.** *Assume X and Y are smooth, quasi-compact, with affine diagonal over $S = \mathrm{Spec}(A)$. Then the map*

$$R\Gamma(X, \Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \Omega^\bullet_{Y/S}) \longrightarrow R\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S})$$

is an isomorphism in $D(A)$.

Proof. By Morphisms, Lemma 29.34.12 the sheaves $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are finite locally free $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules. On the other hand, X and Y are flat over S (Morphisms, Lemma 29.34.9) and hence we find that $\Omega^n_{X/S}$ and $\Omega^m_{Y/S}$ are flat over S. Also, observe that $\Omega^\bullet_{X/S}$ is a locally bounded. Thus the result by Lemma 50.8.1 and Derived Categories of Schemes, Lemma 36.24.1. □

There is a relative version of the cup product, namely a map

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X \times_S Y/S}$$

in $D(\mathcal{O}_S)$. The construction combines Lemma 50.8.1 with the map Derived Categories of Schemes, Equation (36.24.0.1). The construction shows that this map is given by the diagram

$$\begin{array}{ccc}
Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} & & \\
\downarrow{\scriptstyle \text{units of adjunction}} & & \\
Rf_*(p^{-1}\Omega^\bullet_{X/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(q^{-1}\Omega^\bullet_{Y/S}) & \longrightarrow & Rf_*(\Omega^\bullet_{X \times_S Y/S}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*(\Omega^\bullet_{X \times_S Y/S}) \\
\downarrow{\scriptstyle \text{relative cup product}} & & \downarrow{\scriptstyle \text{relative cup product}} \\
Rf_*(p^{-1}\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) & \longrightarrow & Rf_*(\Omega^\bullet_{X \times_S Y/S} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \\
\downarrow{\scriptstyle \text{from derived to usual}} & & \downarrow{\scriptstyle \text{from derived to usual}} \\
Rf_*\mathrm{Tot}(p^{-1}\Omega^\bullet_{X/S} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\Omega^\bullet_{Y/S}) & \longrightarrow & Rf_*\mathrm{Tot}(\Omega^\bullet_{X \times_S Y/S} \otimes_{f^{-1}\mathcal{O}_S} \Omega^\bullet_{X \times_S Y/S}) \\
\downarrow{\scriptstyle \text{canonical map}} & & \downarrow{\scriptstyle \eta \otimes \omega \mapsto \eta \wedge \omega} \\
Rf_*\mathrm{Tot}(\Omega^\bullet_{X/S} \boxtimes \Omega^\bullet_{Y/S}) & = & Rf_*\Omega^\bullet_{X \times_S Y/S}
\end{array}$$

Here the first arrow uses the units $\mathrm{id} \to Rp_*p^{-1}$ and $\mathrm{id} \to Rq_*q^{-1}$ of adjunction as well as the identifications $Rf_*p^{-1} = Ra_*Rp_*p^{-1}$ and $Rf_*q^{-1} = Rb_*Rq_*q^{-1}$. The

second arrow is the relative cup product of Cohomology, Remark 20.28.7. The third arrow is the map sending a derived tensor product of complexes to the totalization of the tensor product of complexes. The final equality is Lemma 50.8.1. This construction recovers on global section the construction given earlier.

0FMC **Lemma 50.8.3.** *Assume $X \to S$ and $Y \to S$ are smooth and quasi-compact and the morphisms $X \to X \times_S X$ and $Y \to Y \times_S Y$ are affine. Then the relative cup product*

$$Ra_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*\Omega^\bullet_{Y/S} \longrightarrow Rf_*\Omega^\bullet_{X\times_S Y/S}$$

is an isomorphism in $D(\mathcal{O}_S)$.

Proof. Immediate consequence of Lemma 50.8.2. □

50.9. First Chern class in de Rham cohomology

0FLE Let $X \to S$ be a morphism of schemes. There is a map of complexes

$$\mathrm{d}\log : \mathcal{O}_X^*[-1] \longrightarrow \Omega^\bullet_{X/S}$$

which sends the section $g \in \mathcal{O}_X^*(U)$ to the section $\mathrm{d}\log(g) = g^{-1}\mathrm{d}g$ of $\Omega^1_{X/S}(U)$. Thus we can consider the map

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) = H^2(X, \mathcal{O}_X^*[-1]) \longrightarrow H^2_{dR}(X/S)$$

where the first equality is Cohomology, Lemma 20.6.1. The image of the isomorphism class of the invertible module $\mathcal{L}$ is denoted $c_1^{dR}(\mathcal{L}) \in H^2_{dR}(X/S)$.

We can also use the map $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S}$ to define a Chern class in Hodge cohomology

$$c_1^{Hodge} : \mathrm{Pic}(X) \longrightarrow H^1(X, \Omega^1_{X/S}) \subset H^2_{Hodge}(X/S)$$

These constructions are compatible with pullbacks.

0FMD **Lemma 50.9.1.** *Given a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes the diagrams

$$\begin{array}{ccc} \mathrm{Pic}(X') & \xleftarrow{f^*} & \mathrm{Pic}(X) \\ {\scriptstyle c_1^{dR}}\downarrow & & \downarrow{\scriptstyle c_1^{dR}} \\ H^2_{dR}(X'/S') & \xleftarrow{f^*} & H^2_{dR}(X/S) \end{array} \qquad \begin{array}{ccc} \mathrm{Pic}(X') & \xleftarrow{f^*} & \mathrm{Pic}(X) \\ {\scriptstyle c_1^{Hodge}}\downarrow & & \downarrow{\scriptstyle c_1^{Hodge}} \\ H^1(X', \Omega^1_{X'/S'}) & \xleftarrow{f^*} & H^1(X, \Omega^1_{X/S}) \end{array}$$

commute.

Proof. Omitted. □

Let us "compute" the element $c_1^{dR}(\mathcal{L})$ in Čech cohomology (with sign rules for Čech differentials as in Cohomology, Section 20.25). Namely, choose an open covering $\mathcal{U} : X = \bigcup_{i\in I} U_i$ such that we have a trivializing section s_i of $\mathcal{L}|_{U_i}$ for all i.

On the overlaps $U_{i_0i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0i_1}$ such that $f_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} s_{i_0}|^{-1}_{U_{i_0i_1}}$ [1]. Of course we have

$$f_{i_1i_2}|_{U_{i_0i_1i_2}} f^{-1}_{i_0i_2}|_{U_{i_0i_1i_2}} f_{i_0i_1}|_{U_{i_0i_1i_2}} = 1$$

The cohomology class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$ is the image of the Čech cohomology class of the cocycle $\{f_{i_0i_1}\}$ in $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}_X^*)$. Therefore we see that $c_1^{dR}(\mathcal{L})$ is the image of the cohomology class associated to the Čech cocycle $\{\alpha_{i_0\ldots i_p}\}$ in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$ of degree 2 given by

(1) $\alpha_{i_0} = 0$ in $\Omega^2_{X/S}(U_{i_0})$,
(2) $\alpha_{i_0i_1} = f^{-1}_{i_0i_1} \mathrm{d} f_{i_0i_1}$ in $\Omega^1_{X/S}(U_{i_0i_1})$, and
(3) $\alpha_{i_0i_1i_2} = 0$ in $\mathcal{O}_{X/S}(U_{i_0i_1i_2})$.

Suppose we have invertible modules $\mathcal{L}_k$, $k = 1, \ldots, a$ each trivialized over U_i for all $i \in I$ giving rise to cocycles f_{k,i_0i_1} and $\alpha_k = \{\alpha_{k,i_0\ldots i_p}\}$ as above. Using the rule in Cohomology, Section 20.25 we can compute

$$\beta = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_a$$

to be given by the cocycle $\beta = \{\beta_{i_0\ldots i_p}\}$ described as follows

(1) $\beta_{i_0\ldots i_p} = 0$ in $\Omega^{2a-p}_{X/S}(U_{i_0\ldots i_p})$ unless $p = a$, and
(2) $\beta_{i_0\ldots i_a} = (-1)^{a(a-1)/2} \alpha_{1,i_0i_1} \wedge \alpha_{2,i_1i_2} \wedge \ldots \wedge \alpha_{a,i_{a-1}i_a}$ in $\Omega^a_{X/S}(U_{i_0\ldots i_a})$.

Thus this is a cocycle representing $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ Of course, the same computation shows that the cocycle $\{\beta_{i_0\ldots i_a}\}$ in $\check{\mathcal{C}}^a(\mathcal{U}, \Omega^a_{X/S}))$ represents the cohomology class $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$

0FME **Remark 50.9.2.** Here is a reformulation of the calculations above in more abstract terms. Let $p : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If we view $\mathrm{d}\log$ as a map

$$\mathcal{O}_X^*[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$$

then using $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ as above we find a cohomology class

$$\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$$

The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$ recovers $c_1^{dR}(\mathcal{L})$. In particular we see that $c_1^{dR}(\mathcal{L}) \in F^1H^2_{dR}(X/S)$, see Section 50.7. The image of $\gamma_1(\mathcal{L})$ under the map $\sigma_{\geq 1}\Omega^\bullet_{X/S} \to \Omega^1_{X/S}[-1]$ recovers $c_1^{Hodge}(\mathcal{L})$. Taking the cup product (see Section 50.7) we obtain

$$\xi = \gamma_1(\mathcal{L}_1) \cup \ldots \cup \gamma_1(\mathcal{L}_a) \in H^{2a}(X, \sigma_{\geq a}\Omega^\bullet_{X/S})$$

The commutative diagrams in Section 50.7 show that ξ is mapped to $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ in $H^{2a}_{dR}(X/S)$ by the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$. Also, it follows $c_1^{dR}(\mathcal{L}_1) \cup \ldots \cup c_1^{dR}(\mathcal{L}_a)$ is contained in $F^aH^{2a}_{dR}(X/S)$. Similarly, the map $\sigma_{\geq a}\Omega^\bullet_{X/S} \to \Omega^a_{X/S}[-a]$ sends ξ to $c_1^{Hodge}(\mathcal{L}_1) \cup \ldots \cup c_1^{Hodge}(\mathcal{L}_a)$ in $H^a(X, \Omega^a_{X/S})$.

[1] The Čech differential of a 0-cycle $\{a_{i_0}\}$ has $a_{i_1} - a_{i_0}$ over $U_{i_0i_1}$.

0FMF **Remark 50.9.3.** Let $p : X \to S$ be a morphism of schemes. For $i > 0$ denote $\Omega^i_{X/S,log} \subset \Omega^i_{X/S}$ the abelian subsheaf generated by local sections of the form

$$\mathrm{d}\log(u_1) \wedge \ldots \wedge \mathrm{d}\log(u_i)$$

where $u_1, \ldots, u_n$ are invertible local sections of $\mathcal{O}_X$. For $i = 0$ the subsheaf $\Omega^0_{X/S,log} \subset \mathcal{O}_X$ is the image of $\mathbf{Z} \to \mathcal{O}_X$. For every $i \geq 0$ we have a map of complexes

$$\Omega^i_{X/S,log}[-i] \longrightarrow \Omega^\bullet_{X/S}$$

because the derivative of a logarithmic form is zero. Moreover, wedging logarithmic forms gives another, hence we find bilinear maps

$$\wedge : \Omega^i_{X/S,log} \times \Omega^j_{X/S,log} \longrightarrow \Omega^{i+j}_{X/S,log}$$

compatible with (50.4.0.1) and the maps above. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Using the map of abelian sheaves $\mathrm{d}\log : \mathcal{O}_X^* \to \Omega^1_{X/S,log}$ and the identification $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ we find a canonical cohomology class

$$\tilde{\gamma}_1(\mathcal{L}) \in H^1(X, \Omega^1_{X/S,log})$$

These classes have the following properties

(1) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \sigma_{\geq 1}\Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to the class $\gamma_1(\mathcal{L}) \in H^2(X, \sigma_{\geq 1}\Omega^\bullet_{X/S})$ of Remark 50.9.2,
(2) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log}[-1] \to \Omega^\bullet_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{dR}(\mathcal{L})$ in $H^2_{dR}(X/S)$,
(3) the image of $\tilde{\gamma}_1(\mathcal{L})$ under the canonical map $\Omega^1_{X/S,log} \to \Omega^1_{X/S}$ sends $\tilde{\gamma}_1(\mathcal{L})$ to $c_1^{Hodge}(\mathcal{L})$ in $H^1(X, \Omega^1_{X/S})$,
(4) the construction of these classes is compatible with pullbacks,
(5) add more here.

50.10. de Rham cohomology of a line bundle

0FU8 A line bundle is a special case of a vector bundle, which in turn is a cone endowed with some extra structure. To intelligently talk about the de Rham complex of these, it makes sense to discuss the de Rham complex of a graded ring.

0FU9 **Remark 50.10.1** (de Rham complex of a graded ring)**.** Let G be an abelian monoid written additively with neutral element 0. Let $R \to A$ be a ring map and assume A comes with a grading $A = \bigoplus_{g \in G} A_g$ by R-modules such that R maps into A_0 and $A_g \cdot A_{g'} \subset A_{g+g'}$. Then the module of differentials comes with a grading

$$\Omega_{A/R} = \bigoplus\nolimits_{g \in G} \Omega_{A/R,g}$$

where $\Omega_{A/R,g}$ is the R-submodule of $\Omega_{A/R}$ generated by $a_0\mathrm{d}a_1$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1$. Similarly, we obtain

$$\Omega^p_{A/R} = \bigoplus\nolimits_{g \in G} \Omega^p_{A/R,g}$$

where $\Omega^p_{A/R,g}$ is the R-submodule of $\Omega^p_{A/R}$ generated by $a_0\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_p$ with $a_i \in A_{g_i}$ such that $g = g_0 + g_1 + \ldots + g_p$. Of course the differentials preserve the grading and the wedge product is compatible with the gradings in the obvious manner.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : C \to X$ be a cone, see Constructions, Definition 27.7.2. Recall that this means π is affine and we have a grading $\pi_*\mathcal{O}_C = \bigoplus_{n\geq 0} \mathcal{A}_n$ with $\mathcal{A}_0 = \mathcal{O}_X$. Using the discussion in Remark 50.10.1 over affine opens we find that[2]

$$\pi_*(\Omega^\bullet_{C/S}) = \bigoplus\nolimits_{n\geq 0} \Omega^\bullet_{C/S,n}$$

is canonically a direct sum of subcomplexes. Moreover, we have a factorization

$$\Omega^\bullet_{X/S} \to \Omega^\bullet_{C/S,0} \to \pi_*(\Omega^\bullet_{C/S})$$

and we know that $\omega \wedge \eta \in \Omega^{p+q}_{C/S,n+m}$ if $\omega \in \Omega^p_{C/S,n}$ and $\eta \in \Omega^q_{C/S,m}$.

Let $f : X \to S$ be a morphism of schemes. Let $\pi : L \to X$ be the line bundle associated to the invertible $\mathcal{O}_X$-module $\mathcal{L}$. This means that π is the unique affine morphism such that

$$\pi_*\mathcal{O}_L = \bigoplus\nolimits_{n\geq 0} \mathcal{L}^{\otimes n}$$

as $\mathcal{O}_X$-algebras. Thus L is a cone over X. By the discussion above we find a canonical direct sum decomposition

$$\pi_*(\Omega^\bullet_{L/S}) = \bigoplus\nolimits_{n\geq 0} \Omega^\bullet_{L/S,n}$$

compatible with wedge product, compatible with the decomposition of $\pi_*\mathcal{O}_L$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L/S,0}$ of degree 0.

There is another case which will be useful to us. Namely, consider the complement[3] $L^\star \subset L$ of the zero section $o : X \to L$ in our line bundle L. A local computation shows we have a canonical isomorphism

$$(L^\star \to X)_*\mathcal{O}_{L^\star} = \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$$

of $\mathcal{O}_X$-algebras. The right hand side is a $\mathbf{Z}$-graded quasi-coherent $\mathcal{O}_X$-algebra. Using the discussion in Remark 50.10.1 over affine opens we find that

$$(L^\star \to X)_*(\Omega^\bullet_{L^\star/S}) = \bigoplus\nolimits_{n\in\mathbf{Z}} \Omega^\bullet_{L^\star/S,n}$$

compatible with wedge product, compatible with the decomposition of $(L^\star \to X)_*\mathcal{O}_{L^\star}$ above, and such that $\Omega_{X/S}$ maps into the part $\Omega_{L^\star/S,0}$ of degree 0. The complex $\Omega^\bullet_{L^\star/S,0}$ will be of particular interest to us.

0FUF **Lemma 50.10.2.** *With notation as above, there is a short exact sequence of complexes*

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1] \to 0$$

Proof. We have constructed the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ above.

Construction of $\mathrm{Res} : \Omega^\bullet_{L^\star/S,0} \to \Omega^\bullet_{X/S}[-1]$. Let $U \subset X$ be an open and let $s \in \mathcal{L}(U)$ and $s' \in \mathcal{L}^{\otimes -1}(U)$ be sections such that $s's = 1$. Then s gives an invertible section of the sheaf of algebras $(L^\star \to X)_*\mathcal{O}_{L^\star}$ over U with inverse $s' = s^{-1}$. Then we can consider the 1-form $\mathrm{d}\log(s) = s'\mathrm{d}(s)$ which is an element of $\Omega^1_{L^\star/S,0}(U)$ by our construction of the grading on $\Omega^1_{L^\star/S}$. Our computations on affines given below will

[2] With excuses for the notation!

[3] The scheme $L^\star$ is the $\mathbf{G}_m$-torsor over X associated to L. This is why the grading we get below is a $\mathbf{Z}$-grading, compare with Groupoids, Example 39.12.3 and Lemmas 39.12.4 and 39.12.5.

show that 1 and $\mathrm{d}\log(s)$ freely generate $\Omega^\bullet_{L^\star/S,0}|_U$ as a right module over $\Omega^\bullet_{X/S}|_U$. Thus we can define Res over U by the rule

$$\mathrm{Res}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \omega$$

for all $\omega', \omega \in \Omega^\bullet_{X/S}(U)$. This map is independent of the choice of local generator s and hence glues to give a global map. Namely, another choice of s would be of the form gs for some invertible $g \in \mathcal{O}_X(U)$ and we would get $\mathrm{d}\log(gs) = g^{-1}\mathrm{d}(g) + \mathrm{d}\log(s)$ from which the independence easily follows. Finally, observe that our rule for Res is compatible with differentials as $\mathrm{d}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \mathrm{d}(\omega') - \mathrm{d}\log(s) \wedge \mathrm{d}(\omega)$ and because the differential on $\Omega^\bullet_{X/S}[-1]$ sends ω' to $-\mathrm{d}(\omega')$ by our sign convention in Homology, Definition 12.14.7.

Local computation. We can cover X by affine opens $U \subset X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$ which moreover map into an affine open $V \subset S$. Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$ and choose a generator s of $\mathcal{L}$. We find that we have

$$L^\star \times_X U = \mathrm{Spec}(A[s, s^{-1}])$$

Computing differentials we see that

$$\Omega^1_{A[s,s^{-1}]/R} = A[s, s^{-1}] \otimes_A \Omega^1_{A/R} \oplus A[s, s^{-1}]\mathrm{d}\log(s)$$

and therefore taking exterior powers we obtain

$$\Omega^p_{A[s,s^{-1}]/R} = A[s, s^{-1}] \otimes_A \Omega^p_{A/R} \oplus A[s, s^{-1}]\mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

Taking degree 0 parts we find

$$\Omega^p_{A[s,s^{-1}]/R,0} = \Omega^p_{A/R} \oplus \mathrm{d}\log(s) \otimes_A \Omega^{p-1}_{A/R}$$

and the proof of the lemma is complete. □

0FUG **Lemma 50.10.3.** *The "boundary" map $\delta : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in $D(X, f^{-1}\mathcal{O}_S)$ coming from the short exact sequence in Lemma 50.10.2 is the map of Remark 50.4.3 for $\xi = c_1^{dR}(\mathcal{L})$.*

Proof. To be precise we consider the shift

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S} \to 0$$

of the short exact sequence of Lemma 50.10.2. As the degree zero part of a grading on $(L^\star \to X)_*\Omega^\bullet_{L^\star/S}$ we see that $\Omega^\bullet_{L^\star/S,0}$ is a differential graded $\mathcal{O}_X$-algebra and that the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{L^\star/S,0}$ is a homomorphism of differential graded $\mathcal{O}_X$-algebras. Hence we may view $\Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{L^\star/S,0}[1]$ as a map of right differential graded $\Omega^\bullet_{X/S}$-modules on X. The map $\mathrm{Res} : \Omega^\bullet_{L^\star/S,0}[1] \to \Omega^\bullet_{X/S}$ is a map of right differential graded $\Omega^\bullet_{X/S}$-modules since it is locally defined by the rule $\mathrm{Res}(\omega' + \mathrm{d}\log(s) \wedge \omega) = \omega$, see proof of Lemma 50.10.2. Thus by the discussion in Differential Graded Sheaves, Section 24.32 we see that δ comes from a map $\delta' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$ in the derived category $D(\Omega^\bullet_{X/S}, \mathrm{d})$ of right differential graded modules over the de Rham complex. The uniqueness averted in Remark 50.4.3 shows it suffices to prove that $\delta(1) = c_1^{dR}(\mathcal{L})$.

We claim that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & E & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{d}\log} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{X/S}[1] & \longrightarrow & \Omega^\bullet_{L^\star/S,0}[1] & \longrightarrow & \Omega^\bullet_{X/S} & \longrightarrow & 0 \end{array}$$

where the top row is a short exact sequence of abelian sheaves whose boundary map sends 1 to the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$. It suffices to prove the claim by the compatibility of boundary maps with maps between short exact sequences. We define E as the sheafification of the rule

$$U \longmapsto \{(s, n) \mid n \in \mathbf{Z},\ s \in \mathcal{L}^{\otimes n}(U) \text{ generator}\}$$

with group structure given by $(s, n) \cdot (t, m) = (s \otimes t, n + m)$. The middle vertical map sends (s, n) to $\mathrm{d}\log(s)$. This produces a map of short exact sequences because the map $Res : \Omega^1_{L^\star/S,0} \to \mathcal{O}_X$ constructed in the proof of Lemma 50.10.2 sends $\mathrm{d}\log(s)$ to 1 if s is a local generator of $\mathcal{L}$. To calculate the boundary of 1 in the top row, choose local trivializations s_i of $\mathcal{L}$ over opens U_i as in Section 50.9. On the overlaps $U_{i_0 i_1} = U_{i_0} \cap U_{i_1}$ we have an invertible function $f_{i_0 i_1}$ such that $f_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} s_{i_0}|^{-1}_{U_{i_0 i_1}}$ and the cohomology class of $\mathcal{L}$ is given by the Čech cocycle $\{f_{i_0 i_1}\}$. Then of course we have

$$(f_{i_0 i_1}, 0) = (s_{i_1}, 1)|_{U_{i_0 i_1}} \cdot (s_{i_0}, 1)|^{-1}_{U_{i_0 i_1}}$$

as sections of E which finishes the proof. □

0FUH **Lemma 50.10.4.** *With notation as above we have*

(1) $\Omega^p_{L^\star/S,n} = \Omega^p_{L^\star/S,0} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ *for all* $n \in \mathbf{Z}$ *as quasi-coherent* $\mathcal{O}_X$*-modules,*
(2) $\Omega^\bullet_{X/S} = \Omega^\bullet_{L/X,0}$ *as complexes, and*
(3) *for* $n > 0$ *and* $p \geq 0$ *we have* $\Omega^p_{L/X,n} = \Omega^p_{L^\star/S,n}$.

Proof. In each case there is a globally defined canonical map which is an isomorphism by local calculations which we omit. □

0FUI **Lemma 50.10.5.** *In the situation above, assume there is a morphism* $S \to \mathrm{Spec}(\mathbf{Q})$. *Then* $\Omega^\bullet_{X/S} \to \pi_*\Omega^\bullet_{L/S}$ *is a quasi-isomorphism and* $H^*_{dR}(X/S) = H^*_{dR}(L/S)$.

Proof. Let R be a $\mathbf{Q}$-algebra. Let A be an R-algebra. The affine local statement is that the map

$$\Omega^\bullet_{A/R} \longrightarrow \Omega^\bullet_{A[t]/R}$$

is a quasi-isomorphism of complexes of R-modules. In fact it is a homotopy equivalence with homotopy inverse given by the map sending $g\omega + g'\mathrm{d}t \wedge \omega'$ to $g(0)\omega$ for $g, g' \in A[t]$ and $\omega, \omega' \in \Omega^\bullet_{A/R}$. The homotopy sends $g\omega + g'\mathrm{d}t \wedge \omega'$ to $(\int g')\omega'$ were $\int g' \in A[t]$ is the polynomial with vanishing constant term whose derivative with respect to t is g'. Of course, here we use that R contains $\mathbf{Q}$ as $\int t^n = (1/n)t^{n+1}$. □

0FUJ **Example 50.10.6.** Lemma 50.10.5 is false in positive characteristic. The de Rham complex of $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ over a field k looks like a direct sum

$$k \oplus \bigoplus\nolimits_{n \geq 1} (k \cdot t^n \xrightarrow{n} k \cdot t^{n-1}\mathrm{d}t)$$

Hence if the characteristic of k is $p > 0$, then we see that both $H^0_{dR}(\mathbf{A}^1_k/k)$ and $H^1_{dR}(\mathbf{A}^1_k/k)$ are infinite dimensional over k.

50.11. de Rham cohomology of projective space

0FMG Let A be a ring. Let $n \geq 1$. The structure morphism $\mathbf{P}^n_A \to \mathrm{Spec}(A)$ is a proper smooth of relative dimension n. It is smooth of relative dimension n and of finite type as $\mathbf{P}^n_A$ has a finite affine open covering by schemes each isomorphic to $\mathbf{A}^n_A$, see Constructions, Lemma 27.13.3. It is proper because it is also separated and universally closed by Constructions, Lemma 27.13.4. Let us denote $\mathcal{O}$ and $\mathcal{O}(d)$ the structure sheaf $\mathcal{O}_{\mathbf{P}^n_A}$ and the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$. Let us denote $\Omega = \Omega_{\mathbf{P}^n_A/A}$ the sheaf of relative differentials and Ω^p its exterior powers.

0FMH **Lemma 50.11.1.** *There exists a short exact sequence*

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

Proof. To explain this, we recall that $\mathbf{P}^n_A = \mathrm{Proj}(A[T_0, \ldots, T_n])$, and we write symbolically

$$\mathcal{O}(-1)^{\oplus n+1} = \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

The first arrow

$$\Omega \to \bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$$

in the short exact sequence above is given on each of the standard opens $D_+(T_i) = \mathrm{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$ mentioned above by the rule

$$\sum\nolimits_{j\neq i} g_j \mathrm{d}(T_j/T_i) \longmapsto \sum\nolimits_{j\neq i} g_j/T_i \mathrm{d}T_j - (\sum\nolimits_{j\neq i} g_j T_j/T_i^2)\mathrm{d}T_i$$

This makes sense because $1/T_i$ is a section of $\mathcal{O}(-1)$ over $D_+(T_i)$. The map

$$\bigoplus\nolimits_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j \to \mathcal{O}$$

is given by sending $\mathrm{d}T_j$ to T_j, more precisely, on $D_+(T_i)$ we send the section $\sum g_j \mathrm{d}T_j$ to $\sum T_j g_j$. We omit the verification that this produces a short exact sequence. □

Given an integer $k \in \mathbf{Z}$ and a quasi-coherent $\mathcal{O}_{\mathbf{P}^n_A}$-module $\mathcal{F}$ denote as usual $\mathcal{F}(k)$ the kth Serre twist of $\mathcal{F}$. See Constructions, Definition 27.10.1.

0FUK **Lemma 50.11.2.** *In the situation above we have the following cohomology groups*

1. $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ *unless* $0 \leq p = q \leq n$,
2. *for* $0 \leq p \leq n$ *the* A*-module* $H^p(\mathbf{P}^n_A, \Omega^p)$ *free of rank* 1.
3. *for* $q > 0$, $k > 0$, *and* p *arbitrary we have* $H^q(\mathbf{P}^n_A, \Omega^p(k)) = 0$, *and*
4. *add more here.*

Proof. We are going to use the results of Cohomology of Schemes, Lemma 30.8.1 without further mention. In particular, the statements are true for $H^q(\mathbf{P}^n_A, \mathcal{O}(k))$.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 50.11.1. Since $\mathcal{O}(-1)$ has vanishing cohomology in all degrees, this gives that $H^q(\mathbf{P}^n_A, \Omega)$ is zero except in degree 1 where it is freely generated by the boundary of 1 in $H^0(\mathbf{P}^n_A, \mathcal{O})$.

Assume $p > 1$. Let us think of the short exact sequence above as defining a 2 step filtration on $\mathcal{O}(-1)^{\oplus n+1}$. The induced filtration on $\wedge^p\mathcal{O}(-1)^{\oplus n+1}$ looks like this

$$0 \to \Omega^p \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

Observe that $\wedge^p\mathcal{O}(-1)^{\oplus n+1}$ is isomorphic to a direct sum of $n+1$ choose p copies of $\mathcal{O}(-p)$ and hence has vanishing cohomology in all degrees. By induction hypothesis, this shows that $H^q(\mathbf{P}^n_A, \Omega^p)$ is zero unless $q = p$ and $H^p(\mathbf{P}^n_A, \Omega^p)$ is free of rank 1 with generator the boundary of the generator in $H^{p-1}(\mathbf{P}^n_A, \Omega^{p-1})$.

Let $k > 0$. Observe that $\Omega^n = \mathcal{O}(-n-1)$ for example by the short exact sequence above for $p = n+1$. Hence $\Omega^n(k)$ has vanishing cohomology in positive degrees. Using the short exact sequences

$$0 \to \Omega^p(k) \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right)(k) \to \Omega^{p-1}(k) \to 0$$

and *descending* induction on p we get the vanishing of cohomology of $\Omega^p(k)$ in positive degrees for all p. □

0FMI **Lemma 50.11.3.** *We have $H^q(\mathbf{P}^n_A, \Omega^p) = 0$ unless $0 \leq p = q \leq n$. For $0 \leq p \leq n$ the A-module $H^p(\mathbf{P}^n_A, \Omega^p)$ free of rank 1 with basis element $c_1^{Hodge}(\mathcal{O}(1))^p$.*

Proof. We have the vanishing and freeness by Lemma 50.11.2. For $p = 0$ it is certainly true that $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$ is a generator.

Proof for $p = 1$. Consider the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

of Lemma 50.11.1. In the proof of Lemma 50.11.2 we have seen that the generator of $H^1(\mathbf{P}^n_A, \Omega)$ is the boundary ξ of $1 \in H^0(\mathbf{P}^n_A, \mathcal{O})$. As in the proof of Lemma 50.11.1 we will identify $\mathcal{O}(-1)^{\oplus n+1}$ with $\bigoplus_{j=0,\ldots,n} \mathcal{O}(-1)\mathrm{d}T_j$. Consider the open covering

$$\mathcal{U} : \mathbf{P}^n_A = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

We can lift the restriction of the global section 1 of $\mathcal{O}$ to $U_i = D_+(T_i)$ by the section $T_i^{-1}\mathrm{d}T_i$ of $\bigoplus \mathcal{O}(-1)\mathrm{d}T_j$ over U_i. Thus the cocyle representing ξ is given by

$$T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \in \Omega(U_{i_0 i_1})$$

On the other hand, for each i the section T_i is a trivializing section of $\mathcal{O}(1)$ over U_i. Hence we see that $f_{i_0 i_1} = T_{i_1}/T_{i_0} \in \mathcal{O}^*(U_{i_0 i_1})$ is the cocycle representing $\mathcal{O}(1)$ in $\mathrm{Pic}(\mathbf{P}^n_A)$, see Section 50.9. Hence $c_1^{Hodge}(\mathcal{O}(1))$ is given by the cocycle $\mathrm{d}\log(T_{i_1}/T_{i_0})$ which agrees with what we got for ξ above.

Proof for general p by induction. The base cases $p = 0, 1$ were handled above. Assume $p > 1$. In the proof of Lemma 50.11.2 we have seen that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is the boundary of $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ in the long exact cohomology sequence associated to

$$0 \to \Omega^p \to \wedge^p \left(\mathcal{O}(-1)^{\oplus n+1}\right) \to \Omega^{p-1} \to 0$$

By the calculation in Section 50.9 the cohomology class $c_1^{Hodge}(\mathcal{O}(1))^{p-1}$ is, up to a sign, represented by the cocycle with terms

$$\beta_{i_0 \ldots i_{p-1}} = \mathrm{d}\log(T_{i_1}/T_{i_0}) \wedge \mathrm{d}\log(T_{i_2}/T_{i_1}) \wedge \ldots \wedge \mathrm{d}\log(T_{i_{p-1}}/T_{i_{p-2}})$$

in $\Omega^{p-1}(U_{i_0\ldots i_{p-1}})$. These $\beta_{i_0\ldots i_{p-1}}$ can be lifted to the sections $\tilde{\beta}_{i_0\ldots i_{p-1}} = T_{i_0}^{-1}\mathrm{d}T_{i_0}\wedge \beta_{i_0\ldots i_{p-1}}$ of $\wedge^p(\bigoplus \mathcal{O}(-1)\mathrm{d}T_j)$ over $U_{i_0\ldots i_{p-1}}$. We conclude that the generator of $H^p(\mathbf{P}^n_A, \Omega^p)$ is given by the cocycle whose components are

$$\begin{aligned}\sum\nolimits_{a=0}^{p}(-1)^a\tilde{\beta}_{i_0\ldots\hat{i_a}\ldots i_p} &= T_{i_1}^{-1}\mathrm{d}T_{i_1}\wedge\beta_{i_1\ldots i_p} + \sum\nolimits_{a=1}^{p}(-1)^a T_{i_0}^{-1}\mathrm{d}T_{i_0}\wedge\beta_{i_0\ldots\hat{i_a}\ldots i_p}\\ &= (T_{i_1}^{-1}\mathrm{d}T_{i_1} - T_{i_0}^{-1}\mathrm{d}T_{i_0})\wedge\beta_{i_1\ldots i_p} + T_{i_0}^{-1}\mathrm{d}T_{i_0}\wedge\mathrm{d}(\beta)_{i_0\ldots i_p}\\ &= \mathrm{d}\log(T_{i_1}/T_{i_0})\wedge\beta_{i_1\ldots i_p}\end{aligned}$$

viewed as a section of Ω^p over $U_{i_0\ldots i_p}$. This is up to sign the same as the cocycle representing $c_1^{Hodge}(\mathcal{O}(1))^p$ and the proof is complete. □

0FMJ **Lemma 50.11.4.** *For $0 \leq i \leq n$ the de Rham cohomology $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 with basis element $c_1^{dR}(\mathcal{O}(1))^i$. In all other degrees the de Rham cohomology of $\mathbf{P}^n_A$ over A is zero.*

Proof. Consider the Hodge-to-de Rham spectral sequence of Section 50.6. By the computation of the Hodge cohomology of $\mathbf{P}^n_A$ over A done in Lemma 50.11.3 we see that the spectral sequence degenerates on the E_1 page. In this way we see that $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ is a free A-module of rank 1 for $0 \leq i \leq n$ and zero else. Observe that $c_1^{dR}(\mathcal{O}(1))^i \in H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $i = 0, \ldots, n$ and that for $i = n$ this element is the image of $c_1^{Hodge}(\mathcal{L})^n$ by the map of complexes

$$\Omega^n_{\mathbf{P}^n_A/A}[-n] \longrightarrow \Omega^\bullet_{\mathbf{P}^n_A/A}$$

This follows for example from the discussion in Remark 50.9.2 or from the explicit description of cocycles representing these classes in Section 50.9. The spectral sequence shows that the induced map

$$H^n(\mathbf{P}^n_A, \Omega^n_{\mathbf{P}^n_A/A}) \longrightarrow H^{2n}_{dR}(\mathbf{P}^n_A/A)$$

is an isomorphism and since $c_1^{Hodge}(\mathcal{L})^n$ is a generator of of the source (Lemma 50.11.3), we conclude that $c_1^{dR}(\mathcal{L})^n$ is a generator of the target. By the A-bilinearity of the cup products, it follows that also $c_1^{dR}(\mathcal{L})^i$ is a generator of $H^{2i}_{dR}(\mathbf{P}^n_A/A)$ for $0 \leq i \leq n$. □

50.12. The spectral sequence for a smooth morphism

0FMK Consider a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

where f is a smooth morphism. Then we obtain a locally split short exact sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

by Morphisms, Lemma 29.34.16. Let us think of this as a descending filtration F on $\Omega_{X/S}$ with $F^0\Omega_{X/S} = \Omega_{X/S}$, $F^1\Omega_{X/S} = f^*\Omega_{Y/S}$, and $F^2\Omega_{X/S} = 0$. Applying the functor $\wedge^p$ we obtain for every p an induced filtration

$$\Omega^p_{X/S} = F^0\Omega^p_{X/S} \supset F^1\Omega^p_{X/S} \supset F^2\Omega^p_{X/S} \supset \ldots \supset F^{p+1}\Omega^p_{X/S} = 0$$

whose successive quotients are

$$\mathrm{gr}^k\Omega^p_{X/S} = F^k\Omega^p_{X/S}/F^{k+1}\Omega^p_{X/S} = f^*\Omega^k_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{p-k}_{X/Y} = f^{-1}\Omega^k_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{p-k}_{X/Y}$$

for $k = 0, \ldots, p$. In fact, the reader can check using the Leibniz rule that $F^k\Omega^\bullet_{X/S}$ is a subcomplex of $\Omega^\bullet_{X/S}$. In this way $\Omega^\bullet_{X/S}$ has the structure of a filtered complex. We can also see this by observing that

$$F^k\Omega^\bullet_{X/S} = \mathrm{Im}\left(\wedge : \mathrm{Tot}(f^{-1}\sigma_{\geq k}\Omega^\bullet_{Y/S} \otimes_{p^{-1}\mathcal{O}_S} \Omega^\bullet_{X/S}) \longrightarrow \Omega^\bullet_{X/S}\right)$$

is the image of a map of complexes on X. The filtered complex

$$\Omega^\bullet_{X/S} = F^0\Omega^\bullet_{X/S} \supset F^1\Omega^\bullet_{X/S} \supset F^2\Omega^\bullet_{X/S} \supset \ldots$$

has the following associated graded parts

$$\mathrm{gr}^k\Omega^\bullet_{X/S} = f^{-1}\Omega^k_{Y/S}[-k] \otimes_{f^{-1}\mathcal{O}_Y} \Omega^\bullet_{X/Y}$$

by what was said above.

0FMM **Lemma 50.12.1.** *Let $f : X \to Y$ be a quasi-compact, quasi-separated, and smooth morphism of schemes over a base scheme S. There is a bounded spectral sequence with first page*

$$E_1^{p,q} = H^q(\Omega^p_{Y/S} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y})$$

converging to $R^{p+q}f_*\Omega^\bullet_{X/S}$.

Proof. Consider $\Omega^\bullet_{X/S}$ as a filtered complex with the filtration introduced above. The spectral sequence is the spectral sequence of Cohomology, Lemma 20.29.5. By Derived Categories of Schemes, Lemma 36.23.2 we have

$$Rf_*\mathrm{gr}^k\Omega^\bullet_{X/S} = \Omega^k_{Y/S}[-k] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\Omega^\bullet_{X/Y}$$

and thus we conclude. □

0FMN **Remark 50.12.2.** In Lemma 50.12.1 consider the cohomology sheaves

$$\mathcal{H}^q_{dR}(X/Y) = H^q(Rf_*\Omega^\bullet_{X/Y})$$

If f is proper in addition to being smooth and S is a scheme over $\mathbf{Q}$ then $\mathcal{H}^q_{dR}(X/Y)$ is finite locally free (insert future reference here). If we only assume $\mathcal{H}^q_{dR}(X/Y)$ are flat $\mathcal{O}_Y$-modules, then we obtain (tiny argument omitted)

$$E_1^{p,q} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

and the differentials in the spectral sequence are maps

$$d_1^{p,q} : \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \longrightarrow \Omega^{p+1}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$$

In particular, for $p = 0$ we obtain a map $d_1^{0,q} : \mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y)$ which turns out to be an integrable connection ∇ (insert future reference here) and the complex

$$\mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \Omega^2_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y) \to \ldots$$

with differentials given by $d_1^{\bullet,q}$ is the de Rham complex of ∇. The connection ∇ is known as the *Gauss-Manin connection*.

50.13. Leray-Hirsch type theorems

0FUL In this section we prove that for a smooth proper morphism one can sometimes express the de Rham cohomology upstairs in terms of the de Rham cohomology downstairs.

0FMP **Lemma 50.13.1.** *Let $f : X \to Y$ be a smooth proper morphism of schemes. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/Y)$, $1 \leq i \leq N$. Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. Then the map*

$$\bigoplus\nolimits_{i=1}^{N} \mathcal{O}_Y[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/Y}$$

associated to $\xi_1, \ldots, \xi_N$ is an isomorphism.

Proof. By Lemma 50.3.5 $Rf_*\Omega^\bullet_{X/Y}$ is a perfect object of $D(\mathcal{O}_Y)$ whose formation commutes with arbitrary base change. Thus the map of the lemma is a map $a : K \to L$ between perfect objects of $D(\mathcal{O}_Y)$ whose derived restriction to any point is an isomorphism by our assumption on fibres. Then the cone C on a is a perfect object of $D(\mathcal{O}_Y)$ (Cohomology, Lemma 20.49.7) whose derived restriction to any point is zero. It follows that C is zero by More on Algebra, Lemma 15.76.7 and a is an isomorphism. (This also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7 to translate into algebra.) □

We first prove the main result of this section in the following special case.

0FUM **Lemma 50.13.2.** *Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Assume*

1. *Y and S are affine, and*
2. *there exist integers N and $n_1, \ldots, n_N \geq 0$ and $\xi_i \in H^{n_i}_{dR}(X/S)$, $1 \leq i \leq N$ such that for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$.*

Then the map

$$\bigoplus\nolimits_{i=1}^{N} H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^*a_i$$

is an isomorphism.

Proof. Say $Y = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. In this case $\Omega^\bullet_{A/R}$ computes $R\Gamma(Y, \Omega^\bullet_{Y/S})$ by Lemma 50.3.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Consider the complex

$$K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S}))$$

as in Cohomology, Section 20.25. Let us collect some facts about this complex most of which can be found in the reference just given:

1. $K^\bullet$ is a complex of R-modules whose terms are A-modules,
2. $K^\bullet$ represents $R\Gamma(X, \Omega^\bullet_{X/S})$ in $D(R)$ (Cohomology of Schemes, Lemma 30.2.2 and Cohomology, Lemma 20.25.2),
3. there is a natural map $\Omega^\bullet_{A/R} \to K^\bullet$ of complexes of R-modules which is A-linear on terms and induces the pullback map $H^*_{dR}(Y/S) \to H^*_{dR}(X/S)$ on cohomology,
4. $K^\bullet$ has a multiplication denoted $\wedge$ which turns it into a differential graded R-algebra,

(5) the multiplication on $K^\bullet$ induces the cup product on $H^*_{dR}(X/S)$ (Cohomology, Section 20.31),
(6) the filtration F on $\Omega^*_{X/S}$ induces a filtration

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \ldots$$

by subcomplexes on $K^\bullet$ such that
(a) $F^kK^n \subset K^n$ is an A-submmodule,
(b) $F^kK^\bullet \wedge F^lK^\bullet \subset F^{k+l}K^\bullet$,
(c) $\mathrm{gr}^kK^\bullet$ is a complex of A-modules,
(d) $\mathrm{gr}^0K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ and represents $R\Gamma(X, \Omega^\bullet_{X/Y})$ in $D(A)$,
(e) multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0K^\bullet \to \mathrm{gr}^kK^\bullet$

We omit the detailed proofs of these statements; please see discussion leading up to the construction of the spectral sequence in Lemma 50.12.1.

For every $i = 1, \ldots, N$ we choose a cocycle $x_i \in K^{n_i}$ representing ξ_i. Next, we look at the map of complexes

$$\tilde{x} : M^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} \Omega^\bullet_{A/R}[-n_i] \longrightarrow K^\bullet$$

which sends ω in the ith summand to $x_i \wedge \omega$. All that remains is to show that this map is a quasi-isomorphism. We endow $M^\bullet$ with the structure of a filtered complex by the rule

$$F^kM^\bullet = \bigoplus\nolimits_{i=1,\ldots,N} (\sigma_{\geq k}\Omega^\bullet_{A/R})[-n_i]$$

With this choice the map $\tilde{x}$ is a morphism of filtered complexes. Observe that $\mathrm{gr}^0M^\bullet = \bigoplus A[-n_i]$ and multiplication induces an isomorphism $\Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0M^\bullet \to \mathrm{gr}^kM^\bullet$. By construction and Lemma 50.13.1 we see that

$$\mathrm{gr}^0\tilde{x} : \mathrm{gr}^0M^\bullet \longrightarrow \mathrm{gr}^0K^\bullet$$

is an isomorphism in $D(A)$. It follows that for all $k \geq 0$ we obtain isomorphisms

$$\mathrm{gr}^k\tilde{x} : \mathrm{gr}^kM^\bullet = \Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0M^\bullet \longrightarrow \Omega^k_{A/R}[-k] \otimes_A \mathrm{gr}^0K^\bullet = \mathrm{gr}^kK^\bullet$$

in $D(A)$. Namely, the complex $\mathrm{gr}^0K^\bullet = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/Y}))$ is K-flat as a complex of A-modules by Derived Categories of Schemes, Lemma 36.23.3. Hence the tensor product on the right hand side is the derived tensor product as is true by inspection on the left hand side. Finally, taking the derived tensor product $\Omega^k_{A/R}[-k] \otimes^{\mathbf{L}}_A -$ is a functor on $D(A)$ and therefore sends isomorphisms to isomorphisms. Arguing by induction on k we deduce that

$$\tilde{x} : M^\bullet/F^kM^\bullet \to K^\bullet/F^kK^\bullet$$

is an isomorphism in $D(R)$ since we have the short exact sequences

$$0 \to F^kM^\bullet/F^{k+1}M^\bullet \to M^\bullet/F^{k+1}M^\bullet \to \mathrm{gr}^kM^\bullet \to 0$$

and similarly for $K^\bullet$. This proves that $\tilde{x}$ is a quasi-isomorphism as the filtrations are finite in any given degree. □

0FMR **Proposition 50.13.3.** *Let $f : X \to Y$ be a smooth proper morphism of schemes over a base S. Let N and $n_1, \ldots, n_N \geq 0$ be integers and let $\xi_i \in H^{n_i}_{dR}(X/S)$,*

$1 \leq i \leq N$. Assume for all points $y \in Y$ the images of $\xi_1, \ldots, \xi_N$ in $H^*_{dR}(X_y/y)$ form a basis over $\kappa(y)$. The map

$$\tilde{\xi} = \bigoplus \tilde{\xi}_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

(see proof) is an isomorphism in $D(Y, (Y \to S)^{-1}\mathcal{O}_S)$ and correspondingly the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y/S) \longrightarrow H^*_{dR}(X/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i \cup f^*a_i$$

is an isomorphism.

Proof. Denote $p : X \to S$ and $q : Y \to S$ be the structure morphisms. Let $\xi'_i : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[n_i]$ be the map of Remark 50.4.3 corresponding to ξ_i. Denote

$$\tilde{\xi}_i : \Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}[n_i]$$

the composition of ξ'_i with the canonical map $\Omega^\bullet_{Y/S} \to Rf_*\Omega^\bullet_{X/S}$. Using

$$R\Gamma(Y, Rf_*\Omega^\bullet_{X/S}) = R\Gamma(X, \Omega^\bullet_{X/S})$$

on cohomology $\tilde{\xi}_i$ is the map $\eta \mapsto \xi_i \cup f^*\eta$ from $H^m_{dR}(Y/S)$ to $H^{m+n}_{dR}(X/S)$. Further, since the formation of ξ'_i commutes with restrictions to opens, so does the formation of $\tilde{\xi}_i$ commute with restriction to opens.

Thus we can consider the map

$$\tilde{\xi} = \bigoplus \tilde{\xi}_i[-n_i] : \bigoplus \Omega^\bullet_{Y/S}[-n_i] \longrightarrow Rf_*\Omega^\bullet_{X/S}$$

To prove the lemma it suffices to show that this is an isomorphism in $D(Y, q^{-1}\mathcal{O}_S)$. If we could show $\tilde{\xi}$ comes from a map of filtered complexes (with suitable filtrations), then we could appeal to the spectral sequence of Lemma 50.12.1 to finish the proof. This takes more work than is necessary and instead our approach will be to reduce to the affine case (whose proof does in some sense use the spectral sequence).

Indeed, if $Y' \subset Y$ is is any open with inverse image $X' \subset X$, then $\tilde{\xi}|_{X'}$ induces the map

$$\bigoplus\nolimits_{i=1}^N H^*_{dR}(Y'/S) \longrightarrow H^*_{dR}(X'/S), \quad (a_1, \ldots, a_N) \longmapsto \sum \xi_i|_{X'} \cup f^*a_i$$

on cohomology over Y', see discussion above. Thus it suffices to find a basis for the topology on Y such that the proposition holds for the members of the basis (in particular we can forget about the map $\tilde{\xi}$ when we do this). This reduces us to the case where Y and S are affine which is handled by Lemma 50.13.2 and the proof is complete. □

50.14. Projective space bundle formula

0FMS The title says it all.

0FMT **Proposition 50.14.1.** *Let $X \to S$ be a morphism of schemes. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank r. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map*

$$\bigoplus\nolimits_{i=0,\ldots,r-1} H^*_{dR}(X/S) \longrightarrow H^*_{dR}(P/S)$$

given by the rule

$$(a_0, \ldots, a_{r-1}) \longmapsto \sum\nolimits_{i=0,\ldots,r-1} c_1^{dR}(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is an isomorphism.

Proof. Choose an affine open $\mathrm{Spec}(A) \subset X$ such that $\mathcal{E}$ restricts to the trivial locally free module $\mathcal{O}_{\mathrm{Spec}(A)}^{\oplus r}$. Then $P \times_X \mathrm{Spec}(A) = \mathbf{P}_A^{r-1}$. Thus we see that p is proper and smooth, see Section 50.11. Moreover, the classes $c_1^{dR}(\mathcal{O}_P(1))^i$, $i = 0, 1, \ldots, r-1$ restricted to a fibre $X_y = \mathbf{P}_y^{r-1}$ freely generate the de Rham cohomology $H^*_{dR}(X_y/y)$ over $\kappa(y)$, see Lemma 50.11.4. Thus we've verified the conditions of Proposition 50.13.3 and we win. □

0FUN **Remark 50.14.2.** In the situation of Proposition 50.14.1 we get moreover that the map

$$\tilde{\xi} : \bigoplus\nolimits_{t=0,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism in $D(X, (X \to S)^{-1}\mathcal{O}_X)$ as follows immediately from the application of Proposition 50.13.3. Note that the arrow for $t = 0$ is simply the canonical map $c_{P/X} : \Omega^\bullet_{X/S} \to Rp_*\Omega^\bullet_{P/S}$ of Section 50.2. In fact, we can pin down this map further in this particular case. Namely, consider the canonical map

$$\xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2]$$

of Remark 50.4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))$. Then

$$\xi'[2(t-1)] \circ \ldots \circ \xi'[2] \circ \xi' : \Omega^\bullet_{P/S} \to \Omega^\bullet_{P/S}[2t]$$

is the map of Remark 50.4.3 corresponding to $c_1^{dR}(\mathcal{O}_P(1))^t$. Tracing through the choices made in the proof of Proposition 50.13.3 we find the value

$$\tilde{\xi}|_{\Omega^\bullet_{X/S}[-2t]} = Rp_*\xi'[-2] \circ \ldots \circ Rp_*\xi'[-2(t-1)] \circ Rp_*\xi'[-2t] \circ c_{P/X}[-2t]$$

for the restriction of our isomorphism to the summand $\Omega^\bullet_{X/S}[-2t]$. This has the following simple consequence we will use below: let

$$M = \bigoplus\nolimits_{t=1,\ldots,r-1} \Omega^\bullet_{X/S}[-2t] \quad \text{and} \quad K = \bigoplus\nolimits_{t=0,\ldots,r-2} \Omega^\bullet_{X/S}[-2t]$$

viewed as subcomplexes of the source of the arrow $\tilde{\xi}$. It follows formally from the discussion above that

$$c_{P/X} \oplus \tilde{\xi}|_M : \Omega^\bullet_{X/S} \oplus M \longrightarrow Rp_*\Omega^\bullet_{P/S}$$

is an isomorphism and that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{id}} & M[2] \\ {\scriptstyle \tilde{\xi}|_K}\downarrow & & \downarrow{\scriptstyle (\tilde{\xi}|_M)[2]} \\ Rp_*\Omega^\bullet_{P/S} & \xrightarrow{Rp_*\xi'} & Rp_*\Omega^\bullet_{P/S}[2] \end{array}$$

commutes where $\mathrm{id} : K \to M[2]$ identifies the summand corresponding to t in the deomposition of K to the summand corresponding to $t+1$ in the decomposition of M.

50.15. Log poles along a divisor

0FMU Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. If X étale locally along Y looks like $Y \times \mathbf{A}^1$, then there is a canonical short exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

having many good properties we will discuss in this section. There is a variant of this construction where one starts with a normal crossings divisor (Étale Morphisms, Definition 41.21.1) which we will discuss elsewhere (insert future reference here).

0FMV **Definition 50.15.1.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. We say the *de Rham complex of log poles is defined for $Y \subset X$ over S* if for all $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

(1) $\mathcal{O}_{X,y} \to \Omega_{X/S,y}$, $g \mapsto g\mathrm{d}f$ is a split injection, and
(2) $\Omega^p_{X/S,y}$ is f-torsion free for all p.

An easy local calculation shows that it suffices for every $y \in Y$ to find one local equation f for which conditions (1) and (2) hold.

0FMW **Lemma 50.15.2.** *Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. There is a canonical short exact sequence of complexes*

$$0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1] \to 0$$

Proof. Our assumption is that for every $y \in Y$ and local equation $f \in \mathcal{O}_{X,y}$ of Y we have

$$\Omega_{X/S,y} = \mathcal{O}_{X,y}\mathrm{d}f \oplus M \quad \text{and} \quad \Omega^p_{X/S,y} = \wedge^{p-1}(M)\mathrm{d}f \oplus \wedge^p(M)$$

for some module M with f-torsion free exterior powers $\wedge^p(M)$. It follows that

$$\Omega^p_{Y/S,y} = \wedge^p(M/fM) = \wedge^p(M)/f \wedge^p (M)$$

Below we will tacitly use these facts. In particular the sheaves $\Omega^p_{X/S}$ have no nonzero local sections supported on Y and we have a canonical inclusion

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(Y)$$

see More on Flatness, Section 38.42. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme such that $Y \cap U = V(f)$ for some nonzerodivisor $f \in A$. Let us consider the $\mathcal{O}_U$-submodule of $\Omega^p_{X/S}(Y)|_U$ generated by $\Omega^p_{X/S}|_U$ and $\mathrm{d}\log(f) \wedge \Omega^{p-1}_{X/S}$ where $\mathrm{d}\log(f) = f^{-1}\mathrm{d}(f)$. This is independent of the choice of f as another generator of the ideal of Y on U is equal to uf for a unit $u \in A$ and we get

$$\mathrm{d}\log(uf) - \mathrm{d}\log(f) = \mathrm{d}\log(u) = u^{-1}\mathrm{d}u$$

which is a section of $\Omega_{X/S}$ over U. These local sheaves glue to give a quasi-coherent submodule

$$\Omega^p_{X/S} \subset \Omega^p_{X/S}(\log Y) \subset \Omega^p_{X/S}(Y)$$

Let us agree to think of $\Omega^p_{Y/S}$ as a quasi-coherent $\mathcal{O}_X$-module. There is a unique surjective $\mathcal{O}_X$-linear map

$$\mathrm{Res} : \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y/S}$$

defined by the rule

$$\mathrm{Res}(\eta' + \mathrm{d}\log(f) \wedge \eta) = \eta|_{Y \cap U}$$

for all opens U as above and all $\eta' \in \Omega^p_{X/S}(U)$ and $\eta \in \Omega^{p-1}_{X/S}(U)$. If a form η over U restricts to zero on $Y \cap U$, then $\eta = \mathrm{d}f \wedge \eta' + f\eta''$ for some forms η' and η'' over U. We conclude that we have a short exact sequence

$$0 \to \Omega^p_{X/S} \to \Omega^p_{X/S}(\log Y) \to \Omega^{p-1}_{Y/S} \to 0$$

for all p. We still have to define the differentials $\Omega^p_{X/S}(\log Y) \to \Omega^{p+1}_{X/S}(\log Y)$. On the subsheaf $\Omega^p_{X/S}$ we use the differential of the de Rham complex of X over S. Finally, we define $\mathrm{d}(\mathrm{d}\log(f) \wedge \eta) = -\mathrm{d}\log(f) \wedge \mathrm{d}\eta$. The sign is forced on us by the Leibniz rule (on $\Omega^\bullet_{X/S}$) and it is compatible with the differential on $\Omega^\bullet_{Y/S}[-1]$ which is after all $-\mathrm{d}_{Y/S}$ by our sign convention in Homology, Definition 12.14.7. In this way we obtain a short exact sequence of complexes as stated in the lemma. □

0FUA **Definition 50.15.3.** Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Then the complex

$$\Omega^\bullet_{X/S}(\log Y)$$

constructed in Lemma 50.15.2 is the *de Rham complex of log poles for $Y \subset X$ over S*.

This complex has many good properties.

0FUP **Lemma 50.15.4.** *Let $p : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S.*

(1) *The maps $\wedge : \Omega^p_{X/S} \times \Omega^q_{X/S} \to \Omega^{p+q}_{X/S}$ extend uniquely to $\mathcal{O}_X$-bilinear maps*

$$\wedge : \Omega^p_{X/S}(\log Y) \times \Omega^q_{X/S}(\log Y) \to \Omega^{p+q}_{X/S}(\log Y)$$

satisfying the Leibniz rule $d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d(\eta)$,

(2) *with multiplication as in (1) the map $\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}(\log(Y)$ is a homomorphism of differential graded $\mathcal{O}_S$-algebras,*

(3) *via the maps in (1) we have $\Omega^p_{X/S}(\log Y) = \wedge^p(\Omega^1_{X/S}(\log Y))$, and*

(4) *the map $Res : \Omega^\bullet_{X/S}(\log Y) \to \Omega^\bullet_{Y/S}[-1]$ satisfies*

$$Res(\omega \wedge \eta) = Res(\omega) \wedge \eta|_Y$$

for ω a local section of $\Omega^p_{X/S}(\log Y)$ and η a local section of $\Omega^q_{X/S}$.

Proof. This follows by direct calculation from the local construction of the complex in the proof of Lemma 50.15.2. Details omitted. □

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of schemes. Let $Y \subset X$ be an effective Cartier divisor whose pullback $Y' = f^*Y$ is defined (Divisors, Definition 31.13.12). Assume the de Rham complex of log poles

is defined for $Y \subset X$ over S and the de Rham complex of log poles is defined for $Y' \subset X'$ over S'. In this case we obtain a map of short exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}\Omega^\bullet_{X/S} & \longrightarrow & f^{-1}\Omega^\bullet_{X/S}(\log Y) & \longrightarrow & f^{-1}\Omega^\bullet_{Y/S}[-1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{X'/S'} & \longrightarrow & \Omega^\bullet_{X'/S'}(\log Y') & \longrightarrow & \Omega^\bullet_{Y'/S'}[-1] & \longrightarrow & 0 \end{array}$$

Linearizing, for every p we obtain a linear map $f^*\Omega^p_{X/S}(\log Y) \to \Omega^p_{X'/S'}(\log Y')$.

0FUQ **Lemma 50.15.5.** *Let $f : X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Denote*

$$\delta : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}[2]$$

in $D(X, f^{-1}\mathcal{O}_S)$ the "boundary" map coming from the short exact sequence in Lemma 50.15.2. Denote

$$\xi' : \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}[2]$$

in $D(X, f^{-1}\mathcal{O}_S)$ the map of Remark 50.4.3 corresponding to $\xi = c_1^{dR}(\mathcal{O}_X(-Y))$. Denote

$$\zeta' : \Omega^\bullet_{Y/S} \to \Omega^\bullet_{Y/S}[2]$$

in $D(Y, f|_Y^{-1}\mathcal{O}_S)$ the map of Remark 50.4.3 corresponding to $\zeta = c_1^{dR}(\mathcal{O}_X(-Y)|_Y)$. Then the diagram

$$\begin{array}{ccc} \Omega^\bullet_{X/S} & \longrightarrow & \Omega^\bullet_{Y/S} \\ {\scriptstyle \xi'}\downarrow & \swarrow{\scriptstyle \delta} & \downarrow{\scriptstyle \zeta'} \\ \Omega^\bullet_{X/S}[2] & \longrightarrow & \Omega^\bullet_{Y/S}[2] \end{array}$$

is commutative in $D(X, f^{-1}\mathcal{O}_S)$.

Proof. More precisely, we define δ as the boundary map corresponding to the shifted short exact sequence

$$0 \to \Omega^\bullet_{X/S}[1] \to \Omega^\bullet_{X/S}(\log Y)[1] \to \Omega^\bullet_{Y/S} \to 0$$

It suffices to prove each triangle commutes. Set $\mathcal{L} = \mathcal{O}_X(-Y)$. Denote $\pi : L \to X$ the line bundle with $\pi_*\mathcal{O}_L = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$.

Commutativity of the upper left triangle. By Lemma 50.10.3 the map ξ' is the boundary map of the triangle given in Lemma 50.10.2. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_{X/S}[1] & \longrightarrow & \Omega^\bullet_{L^*/S,0}[1] & \longrightarrow & \Omega^\bullet_{X/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{X/S}[1] & \longrightarrow & \Omega^\bullet_{X/S}(\log Y)[1] & \longrightarrow & \Omega^\bullet_{Y/S} & \longrightarrow & 0 \end{array}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(s) \wedge \omega \longmapsto \omega' + \mathrm{d}\log(f) \wedge \omega$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where s is a local generator of $\mathcal{L}$ and $f \in \mathcal{O}_X(-Y)$ is the corresponding section of the ideal sheaf of Y in X. Since the constructions of the maps in Lemmas 50.10.2 and 50.15.2 match exactly, this works.

Commutativity of the lower right triangle. Denote $\overline{L}$ the restriction of L to Y. By Lemma 50.10.3 the map ζ' is the boundary map of the triangle given in Lemma 50.10.2 using the line bundle $\overline{L}$ on Y. By functoriality it suffices to prove there exists a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_{X/S}[1] & \longrightarrow & \Omega^\bullet_{X/S}(\log Y)[1] & \longrightarrow & \Omega^\bullet_{Y/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{Y/S}[1] & \longrightarrow & \Omega^\bullet_{\overline{L}^\star/S,0}[1] & \longrightarrow & \Omega^\bullet_{Y/S} & \longrightarrow & 0 \end{array}$$

where the left and right vertical arrows are the obvious ones. We can define the middle vertical arrow by the rule

$$\omega' + \mathrm{d}\log(f) \wedge \omega \longmapsto \omega'|_Y + \mathrm{d}\log(\overline{s}) \wedge \omega|_Y$$

where ω', ω are local sections of $\Omega^\bullet_{X/S}$ and where f is a local generator of $\mathcal{O}_X(-Y)$ viewed as a function on X and where $\overline{s}$ is $f|_Y$ viewed as a section of $\mathcal{L}|_Y = \mathcal{O}_X(-Y)|_Y$. Since the constructions of the maps in Lemmas 50.10.2 and 50.15.2 match exactly, this works. □

0FMX **Lemma 50.15.6.** *Let $X \to S$ be a morphism of schemes. Let $Y \subset X$ be an effective Cartier divisor. Assume the de Rham complex of log poles is defined for $Y \subset X$ over S. Let $b \in H^m_{dR}(X/S)$ be a de Rham cohomology class whose restriction to Y is zero. Then $c_1^{dR}(\mathcal{O}_X(Y)) \cup b = 0$ in $H^{m+2}_{dR}(X/S)$.*

Proof. This follows immediately from Lemma 50.15.5. Namely, we have

$$c_1^{dR}(\mathcal{O}_X(Y)) \cup b = -c_1^{dR}(\mathcal{O}_X(-Y)) \cup b = -\xi'(b) = -\delta(b|_Y) = 0$$

as desired. For the second equality, see Remark 50.4.3. □

0FMY **Lemma 50.15.7.** *Let $X \to T \to S$ be morphisms of schemes. Let $Y \subset X$ be an effective Cartier divisor. If both $X \to T$ and $Y \to T$ are smooth, then the de Rham complex of log poles is defined for $Y \subset X$ over S.*

Proof. Let $y \in Y$ be a point. By More on Morphisms, Lemma 37.17.1 there exists an integer $0 \geq m$ and a commutative diagram

$$\begin{array}{ccccc} Y & \longleftarrow & V & \longrightarrow & \mathbf{A}^m_T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0)} \\ X & \longleftarrow & U & \xrightarrow{\pi} & \mathbf{A}^{m+1}_T \end{array}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_T)$, and $y \in V$. Denote $z \in \mathbf{A}^m_T$ the image of y. Then we have

$$\Omega^p_{X/S,y} = \Omega^p_{\mathbf{A}^{m+1}_T/S,z} \otimes_{\mathcal{O}_{\mathbf{A}^{m+1}_T,z}} \mathcal{O}_{X,x}$$

by Lemma 50.2.2. Denote $x_1, \ldots, x_{m+1}$ the coordinate functions on $\mathbf{A}^{m+1}_T$. Since the conditions (1) and (2) in Definition 50.15.1 do not depend on the choice of the local coordinate, it suffices to check the conditions (1) and (2) when f is the

image of x_{m+1} by the flat local ring homomorphism $\mathcal{O}_{\mathbf{A}^{m+1}_T,z} \to \mathcal{O}_{X,x}$. In this way we see that it suffices to check conditions (1) and (2) for $\mathbf{A}^m_T \subset \mathbf{A}^{m+1}_T$ and the point z. To prove this case we may assume $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ are affine. Let $A \to B$ be the ring map corresponding to the morphism $T \to S$ and set $P = B[x_1, \ldots, x_{m+1}]$ so that $\mathbf{A}^{m+1}_T = \mathrm{Spec}(P)$. We have

$$\Omega_{P/A} = \Omega_{B/A} \otimes_B P \oplus \bigoplus\nolimits_{j=1,\ldots,m} P\mathrm{d}x_j \oplus P\mathrm{d}x_{m+1}$$

Hence the map $P \to \Omega_{P/A}$, $g \mapsto g\mathrm{d}x_{m+1}$ is a split injection and x_{m+1} is a nonzerodivisor on $\Omega^p_{P/A}$ for all $p \geq 0$. Localizing at the prime ideal corresponding to z finishes the proof. □

0FMZ **Remark 50.15.8.** Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If the map

$$\mathcal{O}^\wedge_{X,y} \longrightarrow \mathcal{O}^\wedge_{Y,y}$$

has a section for all $y \in Y$, then the de Rham complex of log poles is defined for $Y \subset X$ over S. If we ever need this result we will formulate a precise statement and add a proof here.

0FN0 **Remark 50.15.9.** Let S be a locally Noetherian scheme. Let X be locally of finite type over S. Let $Y \subset X$ be an effective Cartier divisor. If for every $y \in Y$ we can find a diagram of schemes over S

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} V$$

with φ étale and $\psi|_{\varphi^{-1}(Y)} : \varphi^{-1}(Y) \to V$ étale, then the de Rham complex of log poles is defined for $Y \subset X$ over S. A special case is when the pair (X, Y) étale locally looks like $(V \times \mathbf{A}^1, V \times \{0\})$. If we ever need this result we will formulate a precise statement and add a proof here.

50.16. Calculations

0FUB In this section we calculate some Hodge and de Rham cohomology groups for a standard blowing up.

We fix a ring R and we set $S = \mathrm{Spec}(R)$. Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion

$$Z = \mathbf{A}^m_S \longrightarrow \mathbf{A}^{m+n}_S = X, \quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0).$$

We are going to consider the blowing up L of X along the closed subscheme Z. Write

$$X = \mathbf{A}^{m+n}_S = \mathrm{Spec}(A) \quad \text{with} \quad A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

We will consider $A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$ as a graded R-algebra by setting $\deg(x_i) = 0$ and $\deg(y_j) = 1$. With this grading we have

$$P = \mathrm{Proj}(A) = \mathbf{A}^m_S \times_S \mathbf{P}^{n-1}_S = Z \times_S \mathbf{P}^{n-1}_S = \mathbf{P}^{n-1}_Z$$

Observe that the ideal cutting out Z in X is the ideal A_+. Hence L is the Proj of the Rees algebra

$$A \oplus A_+ \oplus (A_+)^2 \oplus \ldots = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

Hence L is an example of the phenomenon studied in more generality in More on Morphisms, Section 37.51; we will use the observations we made there without further mention. In particular, we have a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow[0]{} & L & \xrightarrow[\pi]{} & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X & \longrightarrow & Z \end{array}$$

such that $\pi : L \to P$ is a line bundle over $P = Z \times_S \mathbf{P}^{n-1}_S$ with zero section 0 whose image $E = 0(P) \subset L$ is the exceptional divisor of the blowup b.

0FUR **Lemma 50.16.1.** *For $a \geq 0$ we have*

(1) *the map $\Omega^a_{X/S} \to b_*\Omega^a_{L/S}$ is an isomorphism,*
(2) *the map $\Omega^a_{Z/S} \to p_*\Omega^a_{P/S}$ is an isomorphism, and*
(3) *the map $Rb_*\Omega^a_{L/S} \to i_*Rp_*\Omega^a_{P/S}$ is an isomorphism on cohomology sheaves in degree ≥ 1.*

Proof. Let us first prove part (2). Since $P = Z \times_S \mathbf{P}^{n-1}_S$ we see that

$$\Omega^a_{P/S} = \bigoplus\nolimits_{a=r+s} \mathrm{pr}_1^*\Omega^r_{Z/S} \otimes \mathrm{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S}$$

Recalling that $p = \mathrm{pr}_1$ by the projection formula (Cohomology, Lemma 20.54.2) we obtain

$$p_*\Omega^a_{P/S} = \bigoplus\nolimits_{a=r+s} \Omega^r_{Z/S} \otimes \mathrm{pr}_{1,*}\mathrm{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S}$$

By the calculations in Section 50.11 and in particular in the proof of Lemma 50.11.3 we have $\mathrm{pr}_{1,*}\mathrm{pr}_2^*\Omega^s_{\mathbf{P}^{n-1}_S/S} = 0$ except if $s = 0$ in which case we get $\mathrm{pr}_{1,*}\mathcal{O}_P = \mathcal{O}_Z$. This proves (2).

By the material in Section 50.10 and in particular Lemma 50.10.4 we have $\pi_*\Omega^a_{L/S} = \Omega^a_{P/S} \oplus \bigoplus_{k\geq 1} \Omega^a_{L/S,k}$. Since the composition $\pi \circ 0$ in the diagram above is the identity morphism on P to prove part (3) it suffices to show that $\Omega^a_{L/S,k}$ has vanishing higher cohomology for $k > 0$. By Lemmas 50.10.2 and 50.10.4 there are short exact sequences

$$0 \to \Omega^a_{P/S} \otimes \mathcal{O}_P(k) \to \Omega^a_{L/S,k} \to \Omega^{a-1}_{P/S} \otimes \mathcal{O}_P(k) \to 0$$

where $\Omega^{a-1}_{P/S} = 0$ if $a = 0$. Since $P = Z \times_S \mathbf{P}^{n-1}_S$ we have

$$\Omega^a_{P/S} = \bigoplus\nolimits_{i+j=a} \Omega^i_{Z/S} \boxtimes \Omega^j_{\mathbf{P}^{n-1}_S/S}$$

by Lemma 50.8.1. Since $\Omega^i_{Z/S}$ is free of finite rank we see that it suffices to show that the higher cohomology of $\mathcal{O}_Z \boxtimes \Omega^j_{\mathbf{P}^{n-1}_S/S}(k)$ is zero for $k > 0$. This follows from Lemma 50.11.2 applied to $P = Z \times_S \mathbf{P}^{n-1}_S = \mathbf{P}^{n-1}_Z$ and the proof of (3) is complete.

We still have to prove (1). If $n = 1$, then we are blowing up an effective Cartier divisor and b is an isomorphism and we have (1). If $n > 1$, then the composition

$$\Gamma(X, \Omega^a_{X/S}) \to \Gamma(L, \Omega^a_{L/S}) \to \Gamma(L \setminus E, \Omega^a_{L/S}) = \Gamma(X \setminus Z, \Omega^a_{X/S})$$

is an isomorphism as $\Omega^a_{X/S}$ is finite free (small detail omitted). Thus the only way (1) can fail is if there are nonzero elements of $\Gamma(L, \Omega^a_{L/S})$ which vanish outside of

$E = 0(P)$. Since L is a line bundle over P with zero section $0 : P \to L$, it suffices to show that on a line bundle there are no nonzero sections of a sheaf of differentials which vanish identically outside the zero section. The reader sees this is true either (preferably) by a local calculation or by using that $\Omega_{L/S,k} \subset \Omega_{L^\star/S,k}$ (see references above). □

We suggest the reader skip to the next section at this point.

0G5G **Lemma 50.16.2.** *For $a \geq 0$ there are canonical maps*

$$b^*\Omega^a_{X/S} \longrightarrow \Omega^a_{L/S} \longrightarrow b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_L} \mathcal{O}_L((n-1)E)$$

whose composition is induced by the inclusion $\mathcal{O}_L \subset \mathcal{O}_L((n-1)E)$.

Proof. The first arrow in the displayed formula is discussed in Section 50.2. To get the second arrow we have to show that if we view a local section of $\Omega^a_{L/S}$ as a "meromorphic section" of $b^*\Omega^a_{X/S}$, then it has a pole of order at most $n-1$ along E. To see this we work on affine local charts on L. Namely, recall that L is covered by the spectra of the affine blowup algebras $A[\frac{I}{y_i}]$ where $I = A_+$ is the ideal generated by $y_1, \ldots, y_n$. See Algebra, Section 10.70 and Divisors, Lemma 31.32.2. By symmetry it is enough to work on the chart corresponding to $i = 1$. Then

$$A[\frac{I}{y_1}] = R[x_1, \ldots, x_m, y_1, t_2, \ldots, t_n]$$

where $t_i = y_i/y_1$, see More on Algebra, Lemma 15.31.2. Thus the module $\Omega^1_{L/S}$ is over the corresponding affine open freely generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$, $\mathrm{d}y_1$, and $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$. Of course, the first $m+1$ of these generators come from $b^*\Omega^1_{X/S}$ and for the remaining $n-1$ we have

$$\mathrm{d}t_j = \mathrm{d}\frac{y_j}{y_1} = \frac{1}{y_1}\mathrm{d}y_j - \frac{y_j}{y_1^2}\mathrm{d}y_1 = \frac{\mathrm{d}y_j - t_j\mathrm{d}y_1}{y_1}$$

which has a pole of order 1 along E since E is cut out by y_1 on this chart. Since the wedges of a of these elements give a basis of $\Omega^a_{L/S}$ over this chart, and since there are at most $n-1$ of the $\mathrm{d}t_j$ involved this finishes the proof. □

0G5H **Lemma 50.16.3.** *Let $E = 0(P)$ be the exceptional divisor of the blowing up b. For any locally free $\mathcal{O}_X$-module $\mathcal{E}$ and $0 \leq i \leq n-1$ the map*

$$\mathcal{E} \longrightarrow Rb_*(b^*\mathcal{E} \otimes_{\mathcal{O}_L} \mathcal{O}_L(iE))$$

is an isomorphism in $D(\mathcal{O}_X)$.

Proof. By the projection formula it is enough to show this for $\mathcal{E} = \mathcal{O}_X$, see Cohomology, Lemma 20.54.2. Since X is affine it suffices to show that the maps

$$H^0(X, \mathcal{O}_X) \to H^0(L, \mathcal{O}_L) \to H^0(L, \mathcal{O}_L(iE))$$

are isomorphisms and that $H^j(X, \mathcal{O}_L(iE)) = 0$ for $j > 0$ and $0 \leq i \leq n-1$, see Cohomology of Schemes, Lemma 30.4.6. Since π is affine, we can compute global sections and cohomology after taking π_*, see Cohomology of Schemes, Lemma 30.2.4. If $n = 1$, then $L \to X$ is an isomorphism and $i = 0$ hence the first statement holds. If $n > 1$, then we consider the composition

$$H^0(X, \mathcal{O}_X) \to H^0(L, \mathcal{O}_L) \to H^0(L, \mathcal{O}_L(iE)) \to H^0(L \setminus E, \mathcal{O}_L) = H^0(X \setminus Z, \mathcal{O}_X)$$

Since $H^0(X \setminus Z, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ in this case as Z has codimension $n \geq 2$ in X (details omitted) we conclude the first statement holds. For the second, recall that $\mathcal{O}_L(E) = \mathcal{O}_L(-1)$, see Divisors, Lemma 31.32.4. Hence we have

$$\pi_*\mathcal{O}_L(iE) = \pi_*\mathcal{O}_L(-i) = \bigoplus\nolimits_{k \geq -i} \mathcal{O}_P(k)$$

as discussed in More on Morphisms, Section 37.51. Thus we conclude by the vanishing of the cohomology of twists of the structure sheaf on $P = \mathbf{P}^{n-1}_Z$ shown in Cohomology of Schemes, Lemma 30.8.1. □

50.17. Blowing up and de Rham cohomology

0FUC Fix a base scheme S, a smooth morphism $X \to S$, and a closed subscheme $Z \subset X$ which is also smooth over S. Denote $b : X' \to X$ the blowing up of X along Z. Denote $E \subset X'$ the exceptional divisor. Picture

0FUS (50.17.0.1)
$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Our goal in this section is to prove that the map $b^* : H^*_{dR}(X/S) \longrightarrow H^*_{dR}(X'/S)$ is injective (although a lot more can be said).

0FUU **Lemma 50.17.1.** *With notation as in More on Morphisms, Lemma 37.17.3 for $a \geq 0$ we have*

(1) *the map $\Omega^a_{X/S} \to b_*\Omega^a_{X'/S}$ is an isomorphism,*
(2) *the map $\Omega^a_{Z/S} \to p_*\Omega^a_{E/S}$ is an isomorphism,*
(3) *the map $Rb_*\Omega^a_{X'/S} \to i_*Rp_*\Omega^a_{E/S}$ is an isomorphism on cohomology sheaves in degree ≥ 1.*

Proof. Let $\epsilon : X_1 \to X$ be a surjective étale morphism. Denote $i_1 : Z_1 \to X_1$, $b_1 : X'_1 \to X_1$, $E_1 \subset X'_1$, and $p_1 : E_1 \to Z_1$ the base changes of the objects considered in More on Morphisms, Lemma 37.17.3. Observe that i_1 is a closed immersion of schemes smooth over S and that b_1 is the blowing up with center Z_1 by Divisors, Lemma 31.32.3. Suppose that we can prove (1), (2), and (3) for the morphisms b_1, p_1, and i_1. Then by Lemma 50.2.2 we obtain that the pullback by ϵ of the maps in (1), (2), and (3) are isomorphisms. As ϵ is a surjective flat morphism we conclude. Thus working étale locally, by More on Morphisms, Lemma 37.17.1, we may assume we are in the situation discussed in Section 50.16. In this case the lemma is the same as Lemma 50.16.1. □

0FUV **Lemma 50.17.2.** *With notation as in More on Morphisms, Lemma 37.17.3 and denoting $f : X \to S$ the structure morphism there is a canonical distinguished triangle*

$$\Omega^\bullet_{X/S} \to Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S}) \to \Omega^\bullet_{X/S}[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$ where the four maps

$$\begin{array}{ccc} \Omega^\bullet_{X/S} & \to & Rb_*(\Omega^\bullet_{X'/S}), \\ \Omega^\bullet_{X/S} & \to & i_*\Omega^\bullet_{Z/S}, \\ Rb_*(\Omega^\bullet_{X'/S}) & \to & i_*Rp_*(\Omega^\bullet_{E/S}), \\ i_*\Omega^\bullet_{Z/S} & \to & i_*Rp_*(\Omega^\bullet_{E/S}) \end{array}$$

are the canonical ones (Section 50.2), except with sign reversed for one of them.

Proof. Choose a distinguished triangle

$$C \to Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*\Omega^\bullet_{E/S} \to C[1]$$

in $D(X, f^{-1}\mathcal{O}_S)$. It suffices to show that $\Omega^\bullet_{X/S}$ is isomorphic to C in a manner compatible with the canonical maps. By the axioms of triangulated categories there exists a map of distinguished triangles

$$\begin{array}{ccccccc} C' & \longrightarrow & b_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} & \longrightarrow & i_*p_*\Omega^\bullet_{E/S} & \longrightarrow & C'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & Rb_*\Omega^\bullet_{X'/S} \oplus i_*\Omega^\bullet_{Z/S} & \longrightarrow & i_*Rp_*\Omega^\bullet_{E/S} & \longrightarrow & C[1] \end{array}$$

By Lemma 50.17.1 part (3) and Derived Categories, Proposition 13.4.23 we conclude that $C' \to C$ is an isomorphism. By Lemma 50.17.1 part (2) the map $i_*\Omega^\bullet_{Z/S} \to i_*p_*\Omega^\bullet_{E/S}$ is an isomorphism. Thus $C' = b_*\Omega^\bullet_{X'/S}$ in the derived category. Finally we use Lemma 50.17.1 part (1) tells us this is equal to $\Omega^\bullet_{X/S}$. We omit the verification this is compatible with the canonical maps. □

0FUW **Proposition 50.17.3.** *With notation as in More on Morphisms, Lemma 37.17.3 the map $\Omega^\bullet_{X/S} \to Rb_*\Omega^\bullet_{X'/S}$ has a splitting in $D(X, (X \to S)^{-1}\mathcal{O}_S)$.*

Proof. Consider the triangle constructed in Lemma 50.17.2. We claim that the map

$$Rb_*(\Omega^\bullet_{X'/S}) \oplus i_*\Omega^\bullet_{Z/S} \to i_*Rp_*(\Omega^\bullet_{E/S})$$

has a splitting whose image contains the summand $i_*\Omega^\bullet_{Z/S}$. By Derived Categories, Lemma 13.4.11 this will show that the first arrow of the triangle has a splitting which vanishes on the summand $i_*\Omega^\bullet_{Z/S}$ which proves the lemma. We will prove the claim by decomposing $Rp_*\Omega^\bullet_{E/S}$ into a direct sum where the first piece corresponds to $\Omega^\bullet_{Z/S}$ and the second piece can be lifted through $Rb_*\Omega^\bullet_{X'/S}$.

Proof of the claim. We may decompose X into open and closed subschemes having fixed relative dimension to S, see Morphisms, Lemma 29.34.12. Since the derived category $D(X, f^{-1}\mathcal{O})_S)$ correspondingly decomposes as a product of categories, we may assume X has fixed relative dimension N over S. We may decompose $Z = \coprod Z_m$ into open and closed subschemes of relative dimension $m \geq 0$ over S. The restriction $i_m : Z_m \to X$ of i to Z_m is a regular immersion of codimension $N - m$, see Divisors, Lemma 31.22.11. Let $E = \coprod E_m$ be the corresponding decomposition, i.e., we set $E_m = p^{-1}(Z_m)$. If $p_m : E_m \to Z_m$ denotes the restriction of p to E_m, then we have a canonical isomorphism

$$\tilde{\xi}_m : \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_{m,*}\Omega^\bullet_{E_m/S}$$

in $D(Z_m, (Z_m \to S)^{-1}\mathcal{O}_S)$ where in degree 0 we have the canonical map $\Omega^\bullet_{Z_m/S} \to Rp_{m,*}\Omega^\bullet_{E_m/S}$. See Remark 50.14.2. Thus we have an isomorphism

$$\tilde{\xi} : \bigoplus\nolimits_m \bigoplus\nolimits_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t] \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

in $D(Z, (Z \to S)^{-1}\mathcal{O}_S)$ whose restriction to the summand $\Omega^\bullet_{Z/S} = \bigoplus \Omega^\bullet_{Z_m/S}$ of the source is the canonical map $\Omega^\bullet_{Z/S} \to Rp_*(\Omega^\bullet_{E/S})$. Consider the subcomplexes M_m

and K_m of the complex $\bigoplus_{t=0,\ldots,N-m-1} \Omega^\bullet_{Z_m/S}[-2t]$ introduced in Remark 50.14.2. We set

$$M = \bigoplus M_m \quad \text{and} \quad K = \bigoplus K_m$$

We have $M = K[-2]$ and by construction the map

$$c_{E/Z} \oplus \tilde{\xi}|_M : \Omega^\bullet_{Z/S} \oplus M \longrightarrow Rp_*(\Omega^\bullet_{E/S})$$

is an isomorphism (see remark referenced above).

Consider the map

$$\delta : \Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S}$$

in $D(X', (X' \to S)^{-1}\mathcal{O}_S)$ of Lemma 50.15.5 with the property that the composition

$$\Omega^\bullet_{E/S}[-2] \longrightarrow \Omega^\bullet_{X'/S} \longrightarrow \Omega^\bullet_{E/S}$$

is the map θ' of Remark 50.4.3 for $c_1^{dR}(\mathcal{O}_{X'}(-E))|_E) = c_1^{dR}(\mathcal{O}_E(1))$. The final assertion of Remark 50.14.2 tells us that the diagram

$$\begin{array}{ccc} K[-2] & \xrightarrow{\text{id}} & M \\ {\scriptstyle (\tilde{\xi}|_K)[-2]}\downarrow & & \downarrow{\scriptstyle \tilde{x}|_M} \\ Rp_*\Omega^\bullet_{E/S}[-2] & \xrightarrow{Rp_*\theta'} & Rp_*\Omega^\bullet_{E/S} \end{array}$$

commutes. Thus we see that we can obtain the desired splitting of the claim as the map

$$\begin{aligned} Rp_*(\Omega^\bullet_{E/S}) &\xrightarrow{(c_{E/Z}\oplus\tilde{\xi}|_M)^{-1}} \Omega^\bullet_{Z/S} \oplus M \\ &\xrightarrow{\text{id}\oplus\text{id}^{-1}} \Omega^\bullet_{Z/S} \oplus K[-2] \\ &\xrightarrow{\text{id}\oplus(\tilde{\xi}|_K)[-2]} \Omega^\bullet_{Z/S} \oplus Rp_*\Omega^\bullet_{E/S}[-2] \\ &\xrightarrow{\text{id}\oplus Rb_*\delta} \Omega^\bullet_{Z/S} \oplus Rb_*\Omega^\bullet_{X'/S} \end{aligned}$$

The relationship between θ' and δ stated above together with the commutative diagram involving θ', $\tilde{\xi}|_K$, and $\tilde{\xi}|_M$ above are exactly what's needed to show that this is a section to the canonical map $\Omega^\bullet_{Z/S} \oplus Rb_*(\Omega^\bullet_{X'/S}) \to Rp_*(\Omega^\bullet_{E/S})$ and the proof of the claim is complete. □

Lemma 50.17.5 shows that producing the splitting on Hodge cohomology is a good deal easier than the result of Proposition 50.17.3. We urge the reader to skip ahead to the next section.

0G5I **Lemma 50.17.4.** *Let $i : Z \to X$ be a closed immersion of schemes which is regular of codimension c. Then $\text{Ext}^q_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}) = 0$ for $q < c$ for $\mathcal{E}$ locally free on X and $\mathcal{F}$ any $\mathcal{O}_Z$-module.*

Proof. By the local to global spectral sequence of Ext it suffices to prove this affine locally on X. See Cohomology, Section 20.43. Thus we may assume $X = \text{Spec}(A)$ and there exists a regular sequence $f_1, \ldots, f_c$ in A such that $Z = \text{Spec}(A/(f_1, \ldots, f_c))$. We may assume $c \geq 1$. Then we see that $f_1 : \mathcal{E} \to \mathcal{E}$ is injective. Since $i_*\mathcal{F}$ is annihilated by f_1 this shows that the lemma holds for $i = 0$ and that we have a surjection

$$\text{Ext}^{q-1}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E}) \longrightarrow \text{Ext}^q_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E})$$

Thus it suffices to show that the source of this arrow is zero. Next we repeat this argument: if $c \geq 2$ the map $f_2 : \mathcal{E}/f_1\mathcal{E} \to \mathcal{E}/f_1\mathcal{E}$ is injective. Since $i_*\mathcal{F}$ is annihilated by f_2 this shows that the lemma holds for $q = 1$ and that we have a surjection

$$\mathrm{Ext}^{q-2}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E} + f_2\mathcal{E}) \longrightarrow \mathrm{Ext}^{q-1}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}/f_1\mathcal{E})$$

Continuing in this fashion the lemma is proved. □

0G5J **Lemma 50.17.5.** *With notation as in More on Morphisms, Lemma 37.17.3 for $a \geq 0$ there is a unique arrow $Rb_*\Omega^a_{X'/S} \to \Omega^a_{X/S}$ in $D(\mathcal{O}_X)$ whose composition with $\Omega^a_{X/S} \to Rb_*\Omega^a_{X'/S}$ is the identity on $\Omega^a_{X/S}$.*

Proof. We may decompose X into open and closed subschemes having fixed relative dimension to S, see Morphisms, Lemma 29.34.12. Since the derived category $D(X, f^{-1}\mathcal{O})_S)$ correspondingly decomposes as a product of categories, we may assume X has fixed relative dimension N over S. We may decompose $Z = \coprod Z_m$ into open and closed subschemes of relative dimension $m \geq 0$ over S. The restriction $i_m : Z_m \to X$ of i to Z_m is a regular immersion of codimension $N - m$, see Divisors, Lemma 31.22.11. Let $E = \coprod E_m$ be the corresponding decomposition, i.e., we set $E_m = p^{-1}(Z_m)$. We claim that there are natural maps

$$b^*\Omega^a_{X/S} \to \Omega^a_{X'/S} \to b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(\sum (N - m - 1)E_m)$$

whose composition is induced by the inclusion $\mathcal{O}_{X'} \to \mathcal{O}_{X'}(\sum(N - m - 1)E_m)$. Namely, in order to prove this, it suffices to show that the cokernel of the first arrow is locally on X' annihilated by a local equation of the effective Cartier divisor $\sum(N - m - 1)E_m$. To see this in turn we can work étale locally on X as in the proof of Lemma 50.17.1 and apply Lemma 50.16.2. Computing étale locally using Lemma 50.16.3 we see that the induced composition

$$\Omega^a_{X/S} \to Rb_*\Omega^a_{X'/S} \to Rb_*\left(b^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(\sum (N - m - 1)E_m)\right)$$

is an isomorphism in $D(\mathcal{O}_X)$ which is how we obtain the existence of the map in the lemma.

For uniqueness, it suffices to show that there are no nonzero maps from $\tau_{\geq 1}Rb_*\Omega_{X'/S}$ to $\Omega^a_{X/S}$ in $D(\mathcal{O}_X)$. For this it suffices in turn to show that there are no nonzero maps from $R^qb_*\Omega_{X'/s}[-q]$ to $\Omega^a_{X/S}$ in $D(\mathcal{O}_X)$ for $q \geq 1$ (details omitted). By Lemma 50.17.1 we see that $R^qb_*\Omega_{X'/s} \cong i_*R^qp_*\Omega^a_{E/S}$ is the pushforward of a module on $Z = \coprod Z_m$. Moreover, observe that the restriction of $R^qp_*\Omega^a_{E/S}$ to Z_m is nonzero only for $q < N - m$. Namely, the fibres of $E_m \to Z_m$ have dimension $N - m - 1$ and we can apply Limits, Lemma 32.19.2. Thus the desired vanishing follows from Lemma 50.17.4. □

50.18. Comparing sheaves of differential forms

0FL7 The goal of this section is to construct for every locally quasi-finite syntomic morphism of schemes $f : Y \to X$ maps

$$c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

for all $p \geq 0$ satisfying the following properties

0H9C (1) $c^0_{Y/X}(1) = \delta(NL_{Y/X})$ see Discriminants, Section 49.13,

0H9D (2) $c^{q+p}_{Y/X}(\omega \wedge \eta) = \omega \wedge c^p_{Y/X}(\eta)$ for local sections ω of $f^*\Omega^q_{X/\mathbf{Z}}$ and η of $\Omega^p_{Y/\mathbf{Z}}$,

0H9E (3) the formation of $c^p_{Y/X}$ is compatible with restriction to opens and with base change (as formulated in Lemma 50.18.4).

Note that these conditions imply that the maps $c^p_{Y/X}$ are $\mathcal{O}_Y$-linear and that the composition with $f^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{Y/\mathbf{Z}}$ is multiplication by $\delta(NL_{Y/X})$. We will in fact show that there exists a unique collection of operators $c^p_{Y/X}$ with properties (1), (2), (3). This result will be applied in Section 50.19 to the construction of the trace map on the de Rham complexes if f is finite.

0FL8 **Lemma 50.18.1.** *Let R be a ring and consider a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0 & \longrightarrow & L^0 & \longrightarrow & M^0 & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \partial} & & \uparrow & & \\ & & & & L^{-1} & = & M^{-1} & & \end{array}$$

of R-modules with exact top row and M^0 and M^{-1} finite free of the same rank. For $p \geq 0$ there are canonical maps

$$c^p : \wedge^p(H^0(L^\bullet)) \longrightarrow \wedge^p(K^0) \otimes_R \det(M^\bullet)$$

with the following properties:

(1) *$c^0(1) = \delta(M^\bullet)$, and*
(2) *$c^{q+p}(\omega \wedge \eta) = \omega \wedge c^p(\eta)$ for $\omega \in \wedge^q(K^0)$ and $\eta \in \wedge^p(H^0(L^\bullet))$.*

In particular, the composition of c^p with the map $\wedge^p(K^0) \to \wedge^p(H^0(L^\bullet))$ is multiplication with $\delta(M^\bullet)$.

Proof. Say M^0 and M^{-1} are free of rank n. For every $p \geq 0$ there is a unique surjection

$$\pi_p : \wedge^{p+n}(L^0) \longrightarrow \wedge^p(K^0) \otimes \wedge^n(M^0)$$

with the following two properties; (i) we have $\pi_p(k_1 \wedge \ldots \wedge k_p \wedge l_1 \wedge \ldots \wedge l_n) = k_1 \wedge \ldots \wedge k_p \otimes m_1 \wedge \ldots \wedge m_n$ if $k_1, \ldots, k_p \in K^0$ and $l_1, \ldots, l_n \in L^0$ map to $m_1, \ldots, m_n \in M^0$ and (ii) the kernel of π_p is the submodule generated by wedges $l_1 \wedge \ldots \wedge l_{p+n}$ such that $> p$ of the l_j are in K^0. Pick a basis $e_1, \ldots, e_n$ for $L^{-1} = M^{-1}$. Then we define our map by the rule

$$\eta \mapsto \pi_p(\tilde{\eta} \wedge \partial(e_1) \wedge \ldots \wedge \partial(e_n)) \otimes (e_1 \wedge \ldots \wedge e_n)^{\otimes -1}$$

where $\eta \in \wedge^p(H^0(L^\bullet))$ and $\tilde{\eta} \in \wedge^p(L^0)$ is a lift of η. The expression on the right hand side makes sense because $\det(M^\bullet) = \wedge^n(M^0) \otimes \wedge^n(M^{-1})^{\otimes -1}$. The expression on the right is independent of the choice of $\tilde{\eta}$ since $\partial(\xi) \wedge \partial(e_1) \wedge \ldots \wedge \partial(e_n) = 0$ for any $\xi \in L^{-1}$. A straightforward calculation shows that the map is independent of the choice of the basis of $L^{-1} = M^{-1}$. It is immediate from the definition of $\delta(M^\bullet)$ in More on Algebra, Section 15.123 that $c^0(1) = \delta(M^\bullet)$. Finally, the rule in statement (2) can be directly verified from the definition. □

0H9F **Lemma 50.18.2.** *Let $R_1 \to R_2$ be a ring homomorphism. For $i = 1, 2$ consider commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i^0 & \longrightarrow & L_i^0 & \longrightarrow & M_i^0 & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \partial_i} & & \uparrow & & \\ & & & & L_i^{-1} & = & M_i^{-1} & & \end{array}$$

of R_i-modules as in Lemma 50.18.1. Assume we have maps $K_1^0 \to K_2^0$, $L_1^j \to L_2^j$, $M_1^j \to M_2^j$ compatible with the given ring map $R_1 \to R_2$ and compatible with the maps in the displayed diagrams. If the maps $M_1^j \otimes_{R_1} R_2 \to M_2^j$ are isomorphisms, then the diagrams

$$\begin{array}{ccc} \wedge^p(H^0(L_1^\bullet)) & \xrightarrow[c^p]{} & \wedge^p(K_1^0) \otimes_{R_1} \det(M_1^\bullet) \\ \downarrow & & \downarrow \\ \wedge^p(H^0(L_2^\bullet)) & \xrightarrow{c^p} & \wedge^p(K_2^0) \otimes_{R_2} \det(M_2^\bullet) \end{array}$$

commute.

Proof. This follows from the explicit description of the map given in the proof of Lemma 50.18.1. Note that we need the assumption that $M_1^j \otimes_{R_1} R_2 \to M_2^j$ are isomorphisms to see that the basis we use for M_1^{-1} maps to a basis for M_2^{-1} (and hence in checking the compatibility in both constructions we can use the "same" basis). □

0FL9 **Remark 50.18.3.** Let A be a ring. Let $P = A[x_1, \ldots, x_n]$. Let $f_1, \ldots, f_n \in P$ and set $B = P/(f_1, \ldots, f_n)$. Assume $A \to B$ is quasi-finite. Then B is a relative global complete intersection over A (Algebra, Definition 10.136.5) and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$ by Algebra, Lemma 10.136.12. Consider the following diagram

$$\begin{array}{ccccc} \Omega_{A/\mathbf{Z}} \otimes_A B & \longrightarrow & \Omega_{P/\mathbf{Z}} \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \\ & & \uparrow & & \uparrow \\ & & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 & = & (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \end{array}$$

The right column represents $NL_{B/A}$ in $D(B)$ hence has cohomology $\Omega_{B/A}$ in degree 0. The top row is the split short exact sequence $0 \to \Omega_{A/\mathbf{Z}} \otimes_A B \to \Omega_{P/\mathbf{Z}} \otimes_P B \to \Omega_{P/A} \otimes_P B \to 0$. The middle column has cohomology $\Omega_{B/\mathbf{Z}}$ in degree 0 by Algebra, Lemma 10.131.9. Thus by Lemma 50.18.1 we obtain canonical B-module maps

$$\Omega^p_{B/\mathbf{Z}} \longrightarrow \Omega^p_{A/\mathbf{Z}} \otimes_A \det(NL_{B/A})$$

for $p \geq 0$ satisfying properties (1) and (2) and in particular satisfy the property that the composition with $\Omega^p_{A/\mathbf{Z}} \to \Omega^p_{B/\mathbf{Z}}$ is multiplication by $\delta(NL_{B/A})$.

0H9G **Lemma 50.18.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow[b]{} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{a} & X \end{array}$$

of schemes which induces an isomorphism of Y' with an open subscheme of $X' \times_X Y$. Assume f is locally quasi-finite and syntomic and assume given maps $c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \to f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$ for $p \geq 0$ satisfying (1) and (2). Then there is at most one collection of maps $c^p_{Y'/X'} : \Omega^p_{Y'/\mathbf{Z}} \to (f')^*\Omega^p_{X'/\mathbf{Z}} \otimes_{\mathcal{O}_{Y'}} \det(NL_{Y'/X'})$

for $p \geq 0$ satisfying (1) and (2) such that the diagrams

$$\begin{array}{ccccc} b^*\Omega^p_{Y/\mathbf{Z}} & \xrightarrow[b^*c^p_{Y/X}]{} & b^*(f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})) & = & (f')^*a^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} b^*\det(NL_{Y/X})) \\ \downarrow & & & & \downarrow \\ \Omega^p_{Y'/\mathbf{Z}} & & \xrightarrow{c^p_{Y'/X'}} & & (f')^*\Omega^p_{X'/\mathbf{Z}} \otimes_{\mathcal{O}_{Y'}} \det(NL_{Y'/X'}) \end{array}$$

commute for all $p \geq 0$. Here the vertical arrows use the maps $b^*\Omega^p_{Y/\mathbf{Z}} \to \Omega^p_{Y'/\mathbf{Z}}$ and $a^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{X'/\mathbf{Z}}$ of Section 50.2 as well as the identification $b^*\det(NL_{Y/X}) = \det(NL_{Y'/X'})$ of Discriminants, Section 49.13.

Proof. The map

$$(f')^*\Omega_{X'/\mathbf{Z}} \oplus b^*\Omega_{Y/\mathbf{Z}} \longrightarrow \Omega_{Y'/\mathbf{Z}}$$

is surjective because Y' is an open subscheme of the fibre product; the corresponding algebra statement is that $\Omega_{B\otimes_A C/\mathbf{Z}}$ is generated as a module by the images of $\Omega_{B/\mathbf{Z}}$ and $\Omega_{C/\mathbf{Z}}$. Thus for all p the map

$$\bigoplus\nolimits_{i=0,\ldots,p} (f')^*\Omega^i_{X'/\mathbf{Z}} \otimes b^*\Omega^{p-i}_{Y/\mathbf{Z}} \longrightarrow \Omega^p_{Y'/\mathbf{Z}}$$

is surjective. Conditions (1) and (2) combined with the commutativity of the diagram in the lemma, implies that the map $c^p_{Y'/X'}$ is determined (but existence is not immediate). □

0H9H **Lemma 50.18.5.** *Consider a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow[b]{} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{a} & X \end{array}$$

of schemes which induces an isomorphism of Y' with an open subscheme of $X' \times_X Y$. Assume f is locally quasi-finite and syntomic. Then for every $y' \in Y'$ we can find opens $V' \subset Y'$, $V \subset Y$, $U' \subset X$, $U \subset X$ with $y' \in V'$, with $f'(V') \subset U'$, $b(V') \subset V$, $a(U') \subset U$, $f(V) \subset U$, and such that for $p \geq 0$ there are maps $c^p_{V/U}$ and $c^p_{V'/U'}$ satisfying (1) and (2) which are compatible with the diagram

$$\begin{array}{ccc} V' & \xrightarrow[b]{} & V \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

in the sense explained in Lemma 50.18.4.

Proof. It is clear that we may replace Y' by $X' \times_X Y$ in order to prove this. Pick affine opens $V \subset Y$ and $U \subset X$ with V containing the image of y' as in Discriminants, Lemma 49.10.1 part (5). Pick an affine open $U' \subset X'$ containing the image of y' and mapping into U. After replacing X, Y, X', Y' by $U, V, U', U' \times_U V$ we may assume we are in the situation described in the next paragraph.

Assume that $X' = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Y' = \mathrm{Spec}(A' \otimes_A B)$ with $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ a relative global complete intersection over A. Then also $B' = A'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_n)$ is a relative global complete intersection over A' where f'_i is the image of f_i in $A'[x_1, \ldots, x_n]$, see Algebra, Lemma

10.136.9. The construction in Remark 50.18.3 provides us with the maps $c^p_{V/U}$ and $c^p_{V'/U'}$. These maps are compatible by Lemma 50.18.2 and the compatibility of the presentations of B and B' over A and A'. Some details omitted. □

0FLA **Lemma 50.18.6.** *There exists a unique rule that to every locally quasi-finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_Y$-module maps*

$$c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

for $p \geq 0$ satisfying (1), (2), and (3). In particular, the composition of $c^p_{Y/X}$ with $f^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{Y/\mathbf{Z}}$ is multiplication by $\delta(NL_{Y/X})$.

Proof. This proof is very similar to the proof of Discriminants, Proposition 49.13.2 and we suggest the reader look at that proof first.

Let us reformulate the statement. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $f : Y \to X$ of schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{array}{ccc} Y' & \xrightarrow{b} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{a} & X \end{array}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The lemma means that for every object Y/X of $\mathcal{C}$ we have maps $c^p_{Y/X}$, $p \geq 0$ with properties (1) and (2), and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have that $c^p_{Y/X}$ and $c^p_{Y'/X'}$ are compatible as in Lemma 50.18.4.

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some maps

$$\Omega^p_{Y/\mathbf{Z}}|_V \longrightarrow \left(f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})\right)|_V$$

with properties (1) and (2). This follows from picking affine opens as in Discriminants, Lemma 49.10.1 part (5) and then using the construction in Remark 50.18.3.

Note that the étale locus of f is exactly the set of points where $\delta(NL_{Y/X})$ does not vanish, see discussion in Discriminants, Section 49.13. In particular $f^*\Omega^p_{X/\mathbf{Z}} \to \Omega^p_{Y/\mathbf{Z}}$ becomes an isomorphism after inverting $\delta(NL_{Y/X})$. We conclude that if $\Omega^p_{X/\mathbf{Z}}$ is finite locally free and the annihilator of $\delta(NL_{Y/X})$ in $\mathcal{O}_Y$ is zero, then the local maps constructed in the previous paragraph are unique and automatically glue!

Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that

(1) $\Omega_{X/\mathbf{Z}}$ is locally free, and
(2) the annihilator of $\delta(NL_{Y/X})$ in $\mathcal{O}_Y$ is zero.

By the remarks in the previous paragraph, we see that for any object Y/X of $\mathcal{C}_{nice}$ we have unique maps $c^p_{Y/X}$, $p \geq 0$ satisfying conditions (1) and (2). If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}_{nice}$, then the maps $c^p_{Y/X}$ and $c^p_{Y'/X'}$ are compatible as in Lemma 50.18.4: namely, locally there do exist compatible maps by Lemma 50.18.5 and the uniqueness just mentioned shows these maps agree with $c^p_{Y/X}$ and $c^p_{Y'/X'}$. In other words, we have solved the problem on the full subcategory $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c^p_{Y/X}$ the solution we've just found.

In fact, more generally, suppose we have a morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ such that Y/X is an object of $\mathcal{C}_{nice}$. The same argument, but this time using the uniqueness in Lemma 50.18.4, tells us there are maps $c^p_{Y'/X'}$, $p \geq 0$ satisfying conditions (1) and (2) compatible with the already constructed maps $c^p_{Y/X}$. Let us call this the base change $(b/a)^* c^p_{Y/X}$.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. **Claim.** The two base changes $(b_i/a_i)^* c^p_{Y_i/X_i}$, $i = 1, 2$ are equal. We will first show that the claim implies the lemma and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Discriminants, Example 49.10.5. Then $Y_{n,d}$ and $Y_{n,d}$ are irreducible schemes of finite type and smooth over $\mathbf{Z}$. Namely, $X_{n,d}$ is a spectrum of a polynomial ring over $\mathbf{Z}$ and $Y_{n,d}$ is an open subscheme of such. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Discriminants, Lemma 49.10.6. Thus $\delta(NL_{Y_{n,d}/X_{n,d}})$ is nonzero: for example we have the local description of $\delta(NL_{Y/X})$ in Discriminants, Remark 49.13.1 and we have the local description of étale morphisms in Morphisms, Lemma 29.36.15 part (8). Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Discriminants, Lemma 49.10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Then we have the base change $c^p_{V/U} = (b/a)^* c^p_{Y_{n,d}/X_{n,d}}$. The claim guarantees these locally constructed maps $c^p_{V/U}$ glue! Thus we get a well defined global maps $c^p_{Y/X}$ with properties (1) and (2). Finally, let $b/a : Y'/X' \to Y/X$ be an arbitray morphism of $\mathcal{C}$. We have the maps $c^p_{Y'/X'}$, $p \geq 0$ and $c^p_{Y/X}$, $p \geq 0$ constructed in this paragraph. To check they are compatible as in Lemma 50.18.4 we may work locally on Y'/X' and Y/X. Thus we may assume that there exists a morphism $Y/X \to Y_{n,d}/X_{n,d}$. By construction, both the family $c^p_{Y'/X'}$ and the family $c^p_{X/Y}$ are compatible with the map towards $Y_{n,d}/X_{n,d}$. It follows easily from this (and the uniqueness in Lemma 50.18.4) that $c^p_{Y'/X'}$ is compatible with $c^p_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Therefore we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine, say they are the spectra of rings B, A, B_1, A_1, B_2, A_2. Picture

$$\begin{array}{ccccc} B_1 & \longrightarrow & B & \longleftarrow & B_2 \\ \uparrow & & \uparrow & & \uparrow \\ A_1 & \longrightarrow & A & \longleftarrow & A_2 \end{array}$$

By assumption the spectrum of B is an affine open of both the spectrum of $A \otimes_{A_1} B_1$ and $A \otimes_{A_2} B_2$. Shrinking more we may assume there exist elements $g_i \in A \otimes_{A_i} B_i$

such that our maps give isomorphisms $(A \otimes_{A_i} B_i)_{g_i} = B$, see Properties, Lemma 28.29.9. Let x_α, y_β be a sufficiently large collection of variables such that we may choose surjections

$$A'_1 = A_1[x_\alpha] \to A \quad \text{and} \quad A'_2 = A_2[y_\beta] \to A$$

of A_1 and A_2-algebras. Then we can choose lifts $h_i \in A'_i \otimes_{A_i} B_i$ of $g_i \in A \otimes_{A_i} B_i$ and we consider the diagram

$$\begin{array}{ccccc} (A'_1 \otimes_{A_1} B_1)_{h_1} & \longrightarrow & B & \longleftarrow & (A'_2 \otimes_{A_1} B_2)_{h_2} \\ \uparrow & & \uparrow & & \uparrow \\ A'_1 & \longrightarrow & A & \longleftarrow & A'_2 \end{array}$$

By construction the two squares are cocartesian. Next, we consider the ring map

$$A' = A'_1 \times_A A'_2 \longrightarrow B' = (A'_1 \otimes_{A_1} B_1)_{h_1} \times_B (A'_2 \otimes_{A_1} B_2)_{h_2}$$

By More on Algebra, Lemma 15.6.4 we have $A'_1 \otimes_{A'} B' = (A'_1 \otimes_{A_1} B_1)_{h_1}$ and $A'_2 \otimes_{A'} B' = (A'_2 \otimes_{A_2} B_2)_{h_2}$. In particular the fibres of the morphism $Y' = \mathrm{Spec}(B') \to \mathrm{Spec}(A') = X'$ are open subschemes of base changes of the fibres of the maps $Y_i \to X_i$, hence finite and local complete intersections (see Discriminants, Lemma 49.10.1). By More on Algebra, Lemma 15.7.7 the ring map $A' \to B'$ is flat and of finite presentation. Thus by Discriminants, Lemma 49.10.1 part (3) we conclude that Y'/X' is an object of $\mathcal{C}$. Consider now the commutative diagram

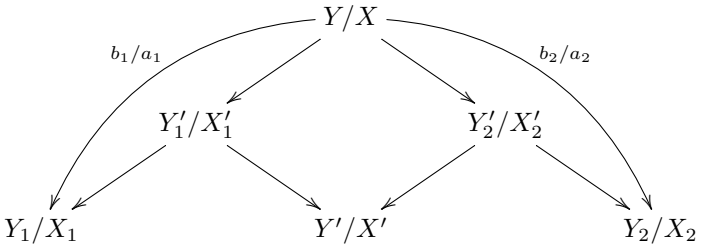

where Y'_i/X'_i corresponds to $A'_i \to (A'_i \otimes_{A_i} B_i)_{h_i}$. Note that Y'_i/X'_i is an object of $\mathcal{C}_{nice}$ as it obtained by taking an open subscheme of the product of an infinite dimensional affine space with Y_i/X_i; small detail omitted. In particular, the pullback of $c^p_{Y_i/X_i}$ via (b_i/a_i) is the same as the pullback of $c^p_{Y'_i/X'_i}$ via $Y/X \to Y'_i/X'_i$. Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$, but this is almost never the case. Namely, then pulling back $c^p_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. To get around the problem that Y'/X' is not in $\mathcal{C}_{nice}$ we note the arguments above show that, after possibly shrinking all of the schemes $X, Y, X'_1, Y'_1, X'_2, Y'_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like

so:

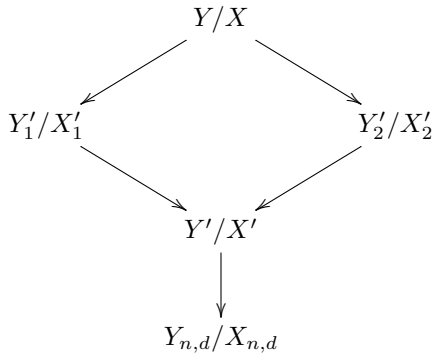

and then we can use the already given argument by pulling back from $c^p_{Y_{n,d}/X_{n,d}}$. This finishes the proof. □

50.19. Trace maps on de Rham complexes

0FK6 A reference for some of the material in this section is [**Gar84**]. Let S be a scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes over S. Then there is a trace map $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 49.3. In this situation a trace map on de Rham complexes is a map of complexes

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/S} \longrightarrow \Omega^\bullet_{X/S}$$

such that $\Theta_{Y/X}$ is equal to Trace_f in degree 0 and satisfies

$$\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$$

for local sections ω of $\Omega^\bullet_{X/S}$ and η of $f_*\Omega^\bullet_{Y/S}$. It is not clear to us whether such a trace map $\Theta_{Y/X}$ exists for every finite locally free morphism $Y \to X$; please email stacks.project@gmail.com if you have a counterexample or a proof.

0FK7 **Example 50.19.1.** Here is an example where we do not have a trace map on de Rham complexes. For example, consider the $\mathbf{C}$-algebra $B = \mathbf{C}[x, y]$ with action of $G = \{\pm 1\}$ given by $x \mapsto -x$ and $y \mapsto -y$. The invariants $A = B^G$ form a normal domain of finite type over $\mathbf{C}$ generated by x^2, xy, y^2. We claim that for the inclusion $A \subset B$ there is no reasonable trace map $\Omega_{B/\mathbf{C}} \to \Omega_{A/\mathbf{C}}$ on 1-forms. Namely, consider the element $\omega = x\mathrm{d}y \in \Omega_{B/\mathbf{C}}$. Since ω is invariant under the action of G if a "reasonable" trace map exists, then 2ω should be in the image of $\Omega_{A/\mathbf{C}} \to \Omega_{B/\mathbf{C}}$. This is not the case: there is no way to write 2ω as a linear combination of $\mathrm{d}(x^2)$, $\mathrm{d}(xy)$, and $\mathrm{d}(y^2)$ even with coefficients in B. This example contradicts the main theorem in [**Zan99**].

0FLB **Lemma 50.19.2.** *There exists a unique rule that to every finite syntomic morphism of schemes $f : Y \to X$ assigns $\mathcal{O}_X$-module maps*

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbf{Z}} \longrightarrow \Omega^p_{X/\mathbf{Z}}$$

satisfying the following properties

1. *the composition with $\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}}$ is equal to $id \otimes \text{Trace}_f$ where $\text{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ is the map from Discriminants, Section 49.3,*
2. *the rule is compatible with base change.*

Proof. First, assume that X is locally Noetherian. By Lemma 50.18.6 we have a canonical map

$$c^p_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

By Discriminants, Proposition 49.13.2 we have a canonical isomorphism

$$c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$$

mapping $\delta(NL_{Y/X})$ to $\tau_{Y/X}$. Combined these maps give

$$c^p_{Y/X} \otimes c_{Y/X} : \Omega^p_{Y/\mathbf{Z}} \longrightarrow f^*\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$$

By Discriminants, Section 49.5 this is the same thing as a map

$$\Theta^p_{Y/X} : f_*\Omega^p_{Y/\mathbf{Z}} \longrightarrow \Omega^p_{X/\mathbf{Z}}$$

Recall that the relationship between $c^p_{Y/X} \otimes c_{Y/X}$ and $\Theta^p_{Y/X}$ uses the evaluation map $f_*\omega_{Y/X} \to \mathcal{O}_X$ which sends $\tau_{Y/X}$ to $\mathrm{Trace}_f(1)$, see Discriminants, Section 49.5. Hence property (1) holds. Property (2) holds for base changes by $X' \to X$ with X' locally Noetherian because both $c^p_{Y/X}$ and $c_{Y/X}$ are compatible with such base changes. For $f : Y \to X$ finite syntomic and X locally Noetherian, we will continue to denote $\Theta^p_{Y/X}$ the solution we've just found.

Uniqueness. Suppose that we have a finite syntomic morphism $f : Y \to X$ such that X is smooth over $\mathrm{Spec}(\mathbf{Z})$ and f is étale over a dense open of X. We claim that in this case $\Theta^p_{Y/X}$ is uniquely determined by property (1). Namely, consider the maps

$$\Omega^p_{X/\mathbf{Z}} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\Omega^p_{Y/\mathbf{Z}} \to \Omega^p_{X/\mathbf{Z}}$$

The sheaf $\Omega^p_{X/\mathbf{Z}}$ is torsion free (by the assumed smoothness), hence it suffices to check that the restriction of $\Theta^p_{Y/X}$ is uniquely determined over the dense open over which f is étale, i.e., we may assume f is étale. However, if f is étale, then $f^*\Omega_{X/\mathbf{Z}} = \Omega_{Y/\mathbf{Z}}$ hence the first arrow in the displayed equation is an isomorphism. Since we've pinned down the composition, this guarantees uniqueness.

Let $f : Y \to X$ be a finite syntomic morphism of locally Noetherian schemes. Let $x \in X$. By Discriminants, Lemma 49.11.7 we can find $d \geq 1$ and a commutative diagram

$$\begin{array}{ccccc} Y & \longleftarrow & V & \longrightarrow & V_d \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & U & \longrightarrow & U_d \end{array}$$

such that $x \in U \subset X$ is open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Thus $\Theta^p_{Y/X}|_V$ is the pullback of the map $\Theta^p_{V_d/U_d}$. However, by the discussion on uniqueness above and Discriminants, Lemmas 49.11.4 and 49.11.5 the map $\Theta^p_{V_d/U_d}$ is uniquely determined by the requirement (1). Hence uniqueness holds.

At this point we know that we have existence and uniqueness for all finite syntomic morphisms $Y \to X$ with X locally Noetherian. We could now give an argument similar to the proof of Lemma 50.18.6 to extend to general X. However, instead it is possible to directly use absolute Noetherian approximation to finish the proof. Namely, to construct $\Theta^p_{Y/X}$ it suffices to do so Zariski locally on X (provided we also show the uniqueness). Hence we may assume X is affine (small detail omitted).

Then we can write $X = \lim_{i \in I} X_i$ as the limit over a directed set I of Noetherian affine schemes. By Algebra, Lemma 10.127.8 we can find $0 \in I$ and a finitely presented morphism of affines $f_0 : Y_0 \to X_0$ whose base change to X is $Y \to X$. After increasing 0 we may assume $Y_0 \to X_0$ is finite and syntomic, see Algebra, Lemma 10.168.9 and 10.168.3. For $i \geq 0$ also the base change $f_i : Y_i = Y_0 \times_{X_0} X_i \to X_i$ is finite syntomic. Then

$$\Gamma(X, f_*\Omega^p_{Y/\mathbf{Z}}) = \Gamma(Y, \Omega^p_{Y/\mathbf{Z}}) = \operatorname{colim}_{i \geq 0} \Gamma(Y_i, \Omega^p_{Y_i/\mathbf{Z}}) = \operatorname{colim}_{i \geq 0} \Gamma(X_i, f_{i,*}\Omega^p_{Y_i/\mathbf{Z}})$$

Hence we can (and are forced to) define $\Theta^p_{Y/X}$ as the colimit of the maps $\Theta^p_{Y_i/X_i}$. This map is compatible with any cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with X' affine as we know this for the case of Noetherian affine schemes by the arguments given above (small detail omitted; hint: if we also write $X' = \lim_{j \in J} X'_j$ then for every $i \in I$ there is a $j \in J$ and a morphism $X'_j \to X_i$ compatible with the morphism $X' \to X$). This finishes the proof. □

0FLC **Proposition 50.19.3.** *Let $f : Y \to X$ be a finite syntomic morphism of schemes.* [**Gar84**] *The maps $\Theta^p_{Y/X}$ of Lemma 50.19.2 define a map of complexes*

$$\Theta_{Y/X} : f_*\Omega^\bullet_{Y/\mathbf{Z}} \longrightarrow \Omega^\bullet_{X/\mathbf{Z}}$$

with the following properties

1. *in degree 0 we get $\mathrm{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Discriminants, Section 49.3,*
2. *we have $\Theta_{Y/X}(\omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$ for ω in $\Omega^\bullet_{X/\mathbf{Z}}$ and η in $f_*\Omega^\bullet_{Y/\mathbf{Z}}$,*
3. *if f is a morphism over a base scheme S, then $\Theta_{Y/X}$ induces a map of complexes $f_*\Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}$.*

Proof. By Discriminants, Lemma 49.11.7 for every $x \in X$ we can find $d \geq 1$ and a commutative diagram

$$\begin{array}{ccccccc} Y & \longleftarrow & V & \longrightarrow & V_d & \longrightarrow & Y_d = \operatorname{Spec}(B_d) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & U & \longrightarrow & U_d & \longrightarrow & X_d = \operatorname{Spec}(A_d) \end{array}$$

such that $x \in U \subset X$ is affine open, $V = f^{-1}(U)$ and $V = U \times_{U_d} V_d$. Write $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ and observe that $B = A \otimes_{A_d} B_d$ and recall that $B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$. Suppose we have $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_s \in B$. We may write $b_j = \sum a_{j,l} e_l$ with $a_{j,l} \in A$. Set $N = r + sd$ and consider the factorizations

$$\begin{array}{ccccc} V & \longrightarrow & V' = \mathbf{A}^N \times V_d & \longrightarrow & V_d \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U' = \mathbf{A}^N \times U_d & \longrightarrow & U_d \end{array}$$

Here the horizontal lower right arrow is given by the morphism $U \to U_d$ (from the earlier diagram) and the morphism $U \to \mathbf{A}^N$ given by $a_1, \ldots, a_r, a_{1,1}, \ldots, a_{s,d}$.

Then we see that the functions $a_1, \ldots, a_r$ are in the image of $\Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ and the functions $b_1, \ldots, b_s$ are in the image of $\Gamma(V', \mathcal{O}_{V'}) \to \Gamma(V, \mathcal{O}_V)$. In this way we see that for any finite collection of elements[4] of the groups

$$\Gamma(V, \Omega^i_{Y/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U, \Omega^j_{X/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

we can find a factorizations $V \to V' \to V_d$ and $U \to U' \to U_d$ with $V' = \mathbf{A}^N \times V_d$ and $U' = \mathbf{A}^N \times U_d$ as above such that these sections are the pullbacks of sections from

$$\Gamma(V', \Omega^i_{V'/\mathbf{Z}}), \quad i = 0, 1, 2, \ldots \quad \text{and} \quad \Gamma(U', \Omega^j_{U'/\mathbf{Z}}), \quad j = 0, 1, 2, \ldots$$

The upshot of this is that to check $\mathrm{d} \circ \Theta_{Y/X} = \Theta_{Y/X} \circ \mathrm{d}$ it suffices to check this is true for $\Theta_{V'/U'}$. Similarly, for property (2) of the lemma.

By Discriminants, Lemmas 49.11.4 and 49.11.5 the scheme U_d is smooth and the morphism $V_d \to U_d$ is étale over a dense open of U_d. Hence the same is true for the morphism $V' \to U'$. Since $\Omega_{U'/\mathbf{Z}}$ is locally free and hence $\Omega^p_{U'/\mathbf{Z}}$ is torsion free, it suffices to check the desired relations after restricting to the open over which V' is finite étale. Then we may check the relations after a surjective étale base change. Hence we may split the finite étale cover and assume we are looking at a morphism of the form

$$\coprod\nolimits_{i=1,\ldots,d} W \longrightarrow W$$

with W smooth over $\mathbf{Z}$. In this case any local properties of our construction are trivial to check (provided they are true). This finishes the proof of (1) and (2).

Finally, we observe that (3) follows from (2) because $\Omega_{Y/S}$ is the quotient of $\Omega_{Y/\mathbf{Z}}$ by the submodule generated by pullbacks of local sections of $\Omega_{S/\mathbf{Z}}$. □

0FLD **Example 50.19.4.** Let A be a ring. Let $f = x^d + \sum_{0 \leq i < d} a_{d-i} x^i \in A[x]$. Let $B = A[x]/(f)$. By Proposition 50.19.3 we have a morphism of complexes

$$\Theta_{B/A} : \Omega^\bullet_B \longrightarrow \Omega^\bullet_A$$

In particular, if $t \in B$ denotes the image of $x \in A[x]$ we can consider the elements

$$\Theta_{B/A}(t^i \mathrm{d}t) \in \Omega^1_A, \quad i = 0, \ldots, d-1$$

What are these elements? By the same principle as used in the proof of Proposition 50.19.3 it suffices to compute this in the universal case, i.e., when $A = \mathbf{Z}[a_1, \ldots, a_d]$ or even when A is replaced by the fraction field $\mathbf{Q}(a_1, \ldots, a_d)$. Writing symbolically

$$f = \prod\nolimits_{i=1,\ldots,d} (x - \alpha_i)$$

we see that over $\mathbf{Q}(\alpha_1, \ldots, \alpha_d)$ the algebra B becomes split:

$$\mathbf{Q}(a_0, \ldots, a_{d-1})[x]/(f) \longrightarrow \prod\nolimits_{i=1,\ldots,d} \mathbf{Q}(\alpha_1, \ldots, \alpha_d), \quad t \longmapsto (\alpha_1, \ldots, \alpha_d)$$

Thus for example

$$\Theta(\mathrm{d}t) = \sum \mathrm{d}\alpha_i = -\mathrm{d}a_1$$

Next, we have

$$\Theta(t\mathrm{d}t) = \sum \alpha_i \mathrm{d}\alpha_i = a_1 \mathrm{d}a_1 - \mathrm{d}a_2$$

[4] After all these elements will be finite sums of elements of the form $a_0 \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_i$ with $a_0, \ldots, a_i \in A$ or finite sums of elements of the form $b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_j$ with $b_0, \ldots, b_j \in B$.

Next, we have

$$\Theta(t^2\mathrm{d}t) = \sum \alpha_i^2 \mathrm{d}\alpha_i = -a_1^2\mathrm{d}a_1 + a_1\mathrm{d}a_2 + a_2\mathrm{d}a_1 - \mathrm{d}a_3$$

(modulo calculation error), and so on. This suggests that if $f(x) = x^d - a$ then

$$\Theta_{B/A}(t^i\mathrm{d}t) = \begin{cases} 0 & \text{if} \quad i = 0, \ldots, d-2 \\ \mathrm{d}a & \text{if} \quad i = d-1 \end{cases}$$

in Ω_A. This is true for in this particular case one can do the calculation for the extension $\mathbf{Q}(a)[x]/(x^d - a)$ to verify this directly.

0FW2 **Lemma 50.19.5.** *Let p be a prime number. Let $X \to S$ be a smooth morphism of relative dimension d of schemes in characteristic p. The relative Frobenius $F_{X/S} : X \to X^{(p)}$ of X/S (Varieties, Definition 33.36.4) is finite syntomic and the corresponding map*

$$\Theta_{X/X^{(p)}} : F_{X/S,*}\Omega^\bullet_{X/S} \to \Omega^\bullet_{X^{(p)}/S}$$

is zero in all degrees except in degree d where it defines a surjection.

Proof. Observe that $F_{X/S}$ is a finite morphism by Varieties, Lemma 33.36.8. To prove that $F_{X/S}$ is flat, it suffices to show that the morphism $F_{X/S,s} : X_s \to X_s^{(p)}$ between fibres is flat for all $s \in S$, see More on Morphisms, Theorem 37.16.2. Flatness of $X_s \to X_s^{(p)}$ follows from Algebra, Lemma 10.128.1 (and the finiteness already shown). By More on Morphisms, Lemma 37.62.10 the morphism $F_{X/S}$ is a local complete intersection morphism. Hence $F_{X/S}$ is finite syntomic (see More on Morphisms, Lemma 37.62.8).

For every point $x \in X$ we may choose a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \longleftarrow & \mathbf{A}^d_S \end{array}$$

where π is étale and $x \in U$ is open in X, see Morphisms, Lemma 29.36.20. Observe that $\mathbf{A}^d_S \to \mathbf{A}^d_S$, $(x_1, \ldots, x_d) \mapsto (x_1^p, \ldots, x_d^p)$ is the relative Frobenius for $\mathcal{A}^d_S$ over S. The commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[F_{X/S}]{} & U^{(p)} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi^{(p)}} \\ \mathbf{A}^d_S & \xrightarrow{x_i \mapsto x_i^p} & \mathbf{A}^d_S \end{array}$$

of Varieties, Lemma 33.36.5 for $\pi : U \to \mathbf{A}^d_S$ is cartesian by Étale Morphisms, Lemma 41.14.3. Since the construction of Θ is compatible with base change and since $\Omega_{U/S} = \pi^*\Omega_{\mathbf{A}^d_S/S}$ (Lemma 50.2.2) we conclude that it suffices to show the lemma for $\mathbf{A}^d_S$.

Let A be a ring of characteristic p. Consider the unique A-algebra homomorphism $A[y_1, \ldots, y_d] \to A[x_1, \ldots, x_d]$ sending y_i to x_i^p. The arguments above reduce us to computing the map

$$\Theta^i : \Omega^i_{A[x_1,\ldots,x_d]/A} \to \Omega^i_{A[y_1,\ldots,y_d]/A}$$

We urge the reader to do the computation in this case for themselves. As in Example 50.19.4 we may reduce this to computing a formula for Θ^i in the universal case

$$\mathbf{Z}[y_1, \ldots, y_d] \to \mathbf{Z}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

In turn, we can find the formula for Θ^i by computing in the complex case, i.e., for the $\mathbf{C}$-algebra map

$$\mathbf{C}[y_1, \ldots, y_d] \to \mathbf{C}[x_1, \ldots, x_d], \quad y_i \mapsto x_i^p$$

We may even invert $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$. In this case, we have $\mathrm{d}x_i = p^{-1}x_i^{-p+1}\mathrm{d}y_i$. Hence we see that

$$\begin{aligned}\Theta^i(x_1^{e_1} \ldots x_d^{e_d}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_i) &= p^{-i}\Theta^i(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i) \\ &= p^{-i}\mathrm{Trace}(x_1^{e_1-p+1} \ldots x_i^{e_i-p+1}x_{i+1}^{e_{i+1}} \ldots x_d^{e_d})\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i\end{aligned}$$

by the properties of Θ^i. An elementary computation shows that the trace in the expression above is zero unless $e_1, \ldots, e_i$ are congruent to -1 modulo p and $e_{i+1}, \ldots, e_d$ are divisible by p. Moreover, in this case we obtain

$$p^{d-i}y_1^{(e_1-p+1)/p} \ldots y_i^{(e_i-p+1)/p}y_{i+1}^{e_{i+1}/p} \ldots y_d^{e_d/p}\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_i$$

We conclude that we get zero in characteristic p unless $d = i$ and in this case we get every possible d-form. □

50.20. Poincaré duality

0FW3 In this section we prove Poincar'e duality for the de Rham cohomology of a proper smooth scheme over a field. Let us first explain how this works for Hodge cohomology.

0FW4 **Lemma 50.20.1.** *Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map*

$$t : H^d(X, \Omega^d_{X/k}) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all p, q the pairing

$$H^q(X, \Omega^p_{X/k}) \times H^{d-q}(X, \Omega^{d-p}_{X/k}) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By Duality for Schemes, Lemma 48.27.1 we have $\omega_X^\bullet = \Omega^d_{X/k}[d]$. Since $\Omega_{X/k}$ is locally free of rank d (Morphisms, Lemma 29.34.12) we have

$$\Omega^d_{X/k} \otimes_{\mathcal{O}_X} (\Omega^p_{X/k})^\vee \cong \Omega^{d-p}_{X/k}$$

Thus we obtain a k-linear map $t : H^d(X, \Omega^d_{X/k}) \to k$ such that the statement is true by Duality for Schemes, Lemma 48.27.4. In particular the pairing $H^0(X, \mathcal{O}_X) \times H^d(X, \Omega^d_{X/k}) \to k$ is perfect, which implies that any k-linear map $t' : H^d(X, \Omega^d_{X/k}) \to k$ is of the form $\xi \mapsto t(g\xi)$ for some $g \in H^0(X, \mathcal{O}_X)$. Of course, in order for t' to still produce a duality between $H^0(X, \mathcal{O}_X)$ and $H^d(X, \Omega^d_{X/k})$ we need g to be a unit. Denote $\langle -, - \rangle_{p,q}$ the pairing constructed using t and denote $\langle -, - \rangle'_{p,q}$ the pairing constructed using t'. Clearly we have

$$\langle \xi, \xi' \rangle'_{p,q} = \langle g\xi, \xi' \rangle_{p,q}$$

for $\xi \in H^q(X, \Omega^p_{X/k})$ and $\xi' \in H^{d-q}(X, \Omega^{d-p}_{X/k})$. Since g is a unit, i.e., invertible, we see that using t' instead of t we still get perfect pairings for all p, q. □

0FW5 **Lemma 50.20.2.** *Let k be a field. Let X be a smooth proper scheme over k. The map*

$$d : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_{X/k})$$

is zero.

Proof. Since X is smooth over k it is geometrically reduced over k, see Varieties, Lemma 33.25.4. Hence $H^0(X, \mathcal{O}_X) = \prod k_i$ is a finite product of finite separable field extensions k_i/k, see Varieties, Lemma 33.9.3. It follows that $\Omega_{H^0(X,\mathcal{O}_X)/k} = \prod \Omega_{k_i/k} = 0$ (see for example Algebra, Lemma 10.158.1). Since the map of the lemma factors as

$$H^0(X, \mathcal{O}_X) \to \Omega_{H^0(X,\mathcal{O}_X)/k} \to H^0(X, \Omega_{X/k})$$

by functoriality of the de Rham complex (see Section 50.2), we conclude. □

0FW6 **Lemma 50.20.3.** *Let k be a field. Let X be a smooth proper scheme over k equidimensional of dimension d. The map*

$$d : H^d(X, \Omega^{d-1}_{X/k}) \to H^d(X, \Omega^d_{X/k})$$

is zero.

Proof. It is tempting to think this follows from a combination of Lemmas 50.20.2 and 50.20.1. However this doesn't work because the maps $\mathcal{O}_X \to \Omega^1_{X/k}$ and $\Omega^{d-1}_{X/k} \to \Omega^d_{X/k}$ are not $\mathcal{O}_X$-linear and hence we cannot use the functoriality discussed in Duality for Schemes, Remark 48.27.3 to conclude the map in Lemma 50.20.2 is dual to the one in this lemma.

We may replace X by a connected component of X. Hence we may assume X is irreducible. By Varieties, Lemmas 33.25.4 and 33.9.3 we see that $k' = H^0(X, \mathcal{O}_X)$ is a finite separable extension k'/k. Since $\Omega_{k'/k} = 0$ (see for example Algebra, Lemma 10.158.1) we see that $\Omega_{X/k} = \Omega_{X/k'}$ (see Morphisms, Lemma 29.32.9). Thus we may replace k by k' and assume that $H^0(X, \mathcal{O}_X) = k$.

Assume $H^0(X, \mathcal{O}_X) = k$. We conclude that $\dim H^d(X, \Omega^d_{X/k}) = 1$ by Lemma 50.20.1. Assume first that the characteristic of k is a prime number p. Denote $F_{X/k} : X \to X^{(p)}$ the relative Frobenius of X over k; please keep in mind the facts proved about this morphism in Lemma 50.19.5. Consider the commutative diagram

$$\begin{array}{ccccc}
H^d(X, \Omega^{d-1}_{X/k}) & \longrightarrow & H^d(X^{(p)}, F_{X/k,*}\Omega^{d-1}_{X/k}) & \xrightarrow{\Theta^{d-1}} & H^d(X^{(p)}, \Omega^{d-1}_{X^{(p)}/k}) \\
\downarrow & & \downarrow & & \downarrow \\
H^d(X, \Omega^d_{X/k}) & \longrightarrow & H^d(X^{(p)}, F_{X/k,*}\Omega^d_{X/k}) & \xrightarrow{\Theta^d} & H^d(X^{(p)}, \Omega^d_{X^{(p)}/k})
\end{array}$$

The left two horizontal arrows are isomorphisms as $F_{X/k}$ is finite, see Cohomology of Schemes, Lemma 30.2.4. The right square commutes as $\Theta_{X^{(p)}/X}$ is a morphism of complexes and Θ^{d-1} is zero. Thus it suffices to show that Θ^d is nonzero (because the dimension of the source of the map Θ^d is 1 by the discussion above). However, we know that

$$\Theta^d : F_{X/k,*}\Omega^d_{X/k} \to \Omega^d_{X^{(p)}/k}$$

is surjective and hence surjective after applying the right exact functor $H^d(X^{(p)}, -)$ (right exactness by the vanishing of cohomology beyond d as follows from Cohomology, Proposition 20.20.7). Finally, $H^d(X^{(d)}, \Omega^d_{X^{(d)}/k})$ is nonzero for example because it is dual to $H^0(X^{(d)}, \mathcal{O}_{X^{(p)}})$ by Lemma 50.20.1 applied to $X^{(p)}$ over k. This finishes the proof in this case.

Finally, assume the characteristic of k is 0. We can write k as the filtered colimit of its finite type $\mathbf{Z}$-subalgebras R. For one of these we can find a cartesian diagram of schemes

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

such that $Y \to \mathrm{Spec}(R)$ is smooth of relative dimension d and proper. See Limits, Lemmas 32.10.1, 32.8.9, 32.18.4, and 32.13.1. The modules $M^{i,j} = H^j(Y, \Omega^i_{Y/R})$ are finite R-modules, see Cohomology of Schemes, Lemma 30.19.2. Thus after replacing R by a localization we may assume all of these modules are finite free. We have $M^{i,j} \otimes_R k = H^j(X, \Omega^i_{X/k})$ by flat base change (Cohomology of Schemes, Lemma 30.5.2). Thus it suffices to show that $M^{d-1,d} \to M^{d,d}$ is zero. This is a map of finite free modules over a domain, hence it suffices to find a dense set of primes $\mathfrak{p} \subset R$ such that after tensoring with $\kappa(\mathfrak{p})$ we get zero. Since R is of finite type over $\mathbf{Z}$, we can take the collection of primes $\mathfrak{p}$ whose residue field has positive characteristic (details omitted). Observe that

$$M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) = H^d(Y_{\kappa(\mathfrak{p})}, \Omega^{d-1}_{Y_{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p})})$$

for example by Limits, Lemma 32.19.2. Similarly for $M^{d,d}$. Thus we see that $M^{d-1,d} \otimes_R \kappa(\mathfrak{p}) \to M^{d,d} \otimes_R \kappa(\mathfrak{p})$ is zero by the case of positive characteristic handled above. □

0FW7 **Proposition 50.20.4.** *Let k be a field. Let X be a nonempty smooth proper scheme over k equidimensional of dimension d. There exists a k-linear map*

$$t : H^{2d}_{dR}(X/k) \longrightarrow k$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all i the pairing

$$H^i_{dR}(X/k) \times H^{2d-i}_{dR}(X/k) \longrightarrow k, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is perfect.

Proof. By the Hodge-to-de Rham spectral sequence (Section 50.6), the vanishing of $\Omega^i_{X/k}$ for $i > d$, the vanishing in Cohomology, Proposition 20.20.7 and the results of Lemmas 50.20.2 and 50.20.3 we see that $H^0_{dR}(X/k) = H^0(X, \mathcal{O}_X)$ and $H^d(X, \Omega^d_{X/k}) = H^{2d}_{dR}(X/k)$. More precisely, these identifications come from the maps of complexes

$$\Omega^\bullet_{X/k} \to \mathcal{O}_X[0] \quad \text{and} \quad \Omega^d_{X/k}[-d] \to \Omega^\bullet_{X/k}$$

Let us choose $t : H^{2d}_{dR}(X/k) \to k$ which via this identification corresponds to a t as in Lemma 50.20.1. Then in any case we see that the pairing displayed in the lemma is perfect for $i = 0$.

Denote $\underline{k}$ the constant sheaf with value k on X. Let us abbreviate $\Omega^\bullet = \Omega^\bullet_{X/k}$. Consider the map (50.4.0.1) which in our situation reads

$$\wedge : \mathrm{Tot}(\Omega^\bullet \otimes_{\underline{k}} \Omega^\bullet) \longrightarrow \Omega^\bullet$$

For every integer $p = 0, 1, \ldots, d$ this map annihilates the subcomplex $\mathrm{Tot}(\sigma_{>p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet)$ for degree reasons. Hence we find that the restriction of $\wedge$ to the subcomplex $\mathrm{Tot}(\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet)$ factors through a map of complexes

$$\gamma_p : \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow \Omega^\bullet$$

Using the same procedure as in Section 50.4 we obtain cup products

$$H^i(X, \sigma_{\leq p}\Omega^\bullet) \times H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \longrightarrow H^{2d}_{dR}(X, \Omega^\bullet)$$

We will prove by induction on p that these cup products via t induce perfect pairings between $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$. For $p = d$ this is the assertion of the proposition.

The base case is $p = 0$. In this case we simply obtain the pairing between $H^i(X, \mathcal{O}_X)$ and $H^{d-i}(X, \Omega^d)$ of Lemma 50.20.1 and the result is true.

Induction step. Say we know the result is true for p. Then we consider the distinguished triangle

$$\Omega^{p+1}[-p-1] \to \sigma_{\leq p+1}\Omega^\bullet \to \sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$$

and the distinguished triangle

$$\sigma_{\geq d-p}\Omega^\bullet \to \sigma_{\geq d-p-1}\Omega^\bullet \to \Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$$

Observe that both are distinguished triangles in the homotopy category of complexes of sheaves of $\underline{k}$-modules; in particular the maps $\sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$ and $\Omega^{d-p-1}[-d+p+1] \to (\sigma_{\geq d-p}\Omega^\bullet)[1]$ are given by actual maps of complexes, namely using the differential $\Omega^p \to \Omega^{p+1}$ and the differential $\Omega^{d-p-1} \to \Omega^{d-p}$. Consider the long exact cohomology sequences associated to these distinguished triangles

$$\begin{array}{ccc}
H^{i-1}(X, \sigma_{\leq p}\Omega^\bullet) & & H^{2d-i+1}(X, \sigma_{\geq d-p}\Omega^\bullet) \\
\downarrow{\scriptstyle a} & & \uparrow{\scriptstyle a'} \\
H^i(X, \Omega^{p+1}[-p-1]) & & H^{2d-i}(X, \Omega^{d-p-1}[-d+p+1]) \\
\downarrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} \\
H^i(X, \sigma_{\leq p+1}\Omega^\bullet) & & H^{2d-i}(X, \sigma_{\geq d-p-1}\Omega^\bullet) \\
\downarrow{\scriptstyle c} & & \uparrow{\scriptstyle c'} \\
H^i(X, \sigma_{\leq p}\Omega^\bullet) & & H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet) \\
\downarrow{\scriptstyle d} & & \uparrow{\scriptstyle d'} \\
H^{i+1}(X, \Omega^{p+1}[-p-1]) & & H^{2d-i-1}(X, \Omega^{d-p-1}[-d+p+1])
\end{array}$$

By induction and Lemma 50.20.1 we know that the pairings constructed above between the k-vectorspaces on the first, second, fourth, and fifth rows are perfect. By the 5-lemma, in order to show that the pairing between the cohomology groups

in the middle row is perfect, it suffices to show that the pairs (a, a'), (b, b'), (c, c'), and (d, d') are compatible with the given pairings (see below).

Let us prove this for the pair (c, c'). Here we observe simply that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \\ \downarrow{\scriptstyle \gamma_p} & & \downarrow \\ \Omega^\bullet & \xleftarrow{\gamma_{p+1}} & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \end{array}$$

Hence if we have $\alpha \in H^i(X, \sigma_{\leq p+1}\Omega^\bullet)$ and $\beta \in H^{2d-i}(X, \sigma_{\geq d-p}\Omega^\bullet)$ then we get $\gamma_p(\alpha \cup c'(\beta)) = \gamma_{p+1}(c(\alpha) \cup \beta)$ by functoriality of the cup product.

Similarly for the pair (b, b') we use the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) & \longleftarrow & \mathrm{Tot}(\Omega^{p+1}[-p-1] \otimes_{\underline{k}} \sigma_{\geq d-p-1}\Omega^\bullet) \\ \downarrow{\scriptstyle \gamma_{p+1}} & & \downarrow \\ \Omega^\bullet & \xleftarrow{\wedge} & \Omega^{p+1}[-p-1] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p+1] \end{array}$$

and argue in the same manner.

For the pair (d, d') we use the commutative diagram

$$\begin{array}{ccc} \Omega^{p+1}[-p] \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p] & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \Omega^{d-p-1}[-d+p]) \\ \downarrow & & \downarrow \\ \Omega^\bullet & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{\underline{k}} \sigma_{\geq d-p}\Omega^\bullet) \end{array}$$

and we look at cohomology classes in $H^i(X, \sigma_{\leq p}\Omega^\bullet)$ and $H^{2d-i}(X, \Omega^{d-p-1}[-d+p])$. Changing i to $i-1$ we get the result for the pair (a, a') thereby finishing the proof that our pairings are perfect.

We omit the argument showing the uniqueness of t up to precomposing by multiplication by a unit in $H^0(X, \mathcal{O}_X)$. □

50.21. Chern classes

0FW8 The results proved so far suffice to use the discussion in Weil Cohomology Theories, Section 45.12 to produce Chern classes in de Rham cohomology.

0FW9 **Lemma 50.21.1.** *There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X a total Chern class*

$$c^{dR} : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$$

with the following properties

1. *we have $c^{dR}(\alpha + \beta) = c^{dR}(\alpha)c^{dR}(\beta)$ for $\alpha, \beta \in K_0(\mathit{Vect}(X))$,*
2. *if $f : X \to X'$ is a morphism of quasi-compact and quasi-separated schemes, then $c^{dR}(f^*\alpha) = f^*c^{dR}(\alpha)$,*
3. *given $\mathcal{L} \in \mathrm{Pic}(X)$ we have $c^{dR}([\mathcal{L}]) = 1 + c_1^{dR}(\mathcal{L})$*

The construction can easily be extended to all schemes, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 45.12.

Proof. We will apply Weil Cohomology Theories, Proposition 45.12.1 to get this.

Let $\mathcal{C}$ be the category of all quasi-compact and quasi-separated schemes. This certainly satisfies conditions (1), (2), and (3) (a), (b), and (c) of Weil Cohomology Theories, Section 45.12.

As our contravariant functor A from $\mathcal{C}$ to the category of graded algebras will send X to $A(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Z})$ endowed with its cup product. Functoriality is discussed in Section 50.3 and the cup product in Section 50.4. For the additive maps c_1^A we take c_1^{dR} constructed in Section 50.9.

In fact, we obtain commutative algebras by Lemma 50.4.1 which shows we have axiom (1) for A.

To check axiom (2) for A it suffices to check that $H^*_{dR}(X \coprod Y/\mathbf{Z}) = H^*_{dR}(X/\mathbf{Z}) \times H^*_{dR}(Y/\mathbf{Z})$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (3) for A is just the statement that taking first Chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 50.9.1.

Axiom (4) for A is the projective space bundle formula which we proved in Proposition 50.14.1.

Axiom (5). Let X be a quasi-compact and quasi-separated scheme and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r+1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of smooth projective schemes over X which exhibits P' as an effective Cartier divisor on P. Thus by Lemma 50.15.7 the complex of log poles for $P' \subset P$ over $\mathbf{Z}$ is defined. Hence for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$ by Lemma 50.15.6. This finishes the proof. □

0FWA **Remark 50.21.2.** The analogues of Weil Cohomology Theories, Lemmas 45.12.2 (splitting principle) and 45.12.3 (chern classes of tensor products) hold for de Rham Chern classes on quasi-compact and quasi-separated schemes. This is clear as we've shown in the proof of Lemma 50.21.1 that all the axioms of Weil Cohomology Theories, Section 45.12 are satisfied.

Working with schemes over $\mathbf{Q}$ we can construct a Chern character.

0FWB **Lemma 50.21.3.** *There is a unique rule which assigns to every quasi-compact and quasi-separated scheme X over $\mathbf{Q}$ a "chern character"*

$$ch^{dR} : K_0(\mathit{Vect}(X)) \longrightarrow \prod_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$$

with the following properties

1. *ch^{dR} is a ring map for all X,*
2. *if $f : X' \to X$ is a morphism of quasi-compact and quasi-separated schemes over $\mathbf{Q}$, then $f^* \circ ch^{dR} = ch^{dR} \circ f^*$, and*
3. *given $\mathcal{L} \in \mathrm{Pic}(X)$ we have $ch^{dR}([\mathcal{L}]) = \exp(c_1^{dR}(\mathcal{L}))$.*

The construction can easily be extended to all schemes over $\mathbf{Q}$, but to do so one needs to slightly upgrade the discussion in Weil Cohomology Theories, Section 45.12.

Proof. Exactly as in the proof of Lemma 50.21.1 one shows that the category of quasi-compact and quasi-separated schemes over $\mathbf{Q}$ together with the functor $A^*(X) = \bigoplus_{i \geq 0} H^{2i}_{dR}(X/\mathbf{Q})$ satisfy the axioms of Weil Cohomology Theories, Section 45.12. Moreover, in this case $A(X)$ is a $\mathbf{Q}$-algebra for all X. Hence the lemma follows from Weil Cohomology Theories, Proposition 45.12.4. □

50.22. A Weil cohomology theory

0FWC Let k be a field of characteristic 0. In this section we prove that the functor

$$X \longmapsto H^*_{dR}(X/k)$$

defines a Weil cohomology theory over k with coefficients in k as defined in Weil Cohomology Theories, Definition 45.11.4. We will proceed by checking the constructions earlier in this chapter provide us with data (D0), (D1), and (D2') satisfying axioms (A1) – (A9) of Weil Cohomology Theories, Section 45.14.

Throughout the rest of this section we fix the field k of characteristic 0 and we set $F = k$. Next, we take the following data

(D0) For our 1-dimensional F vector space $F(1)$ we take $F(1) = F = k$.
(D1) For our functor H^* we take the functor sending a smooth projective scheme X over k to $H^*_{dR}(X/k)$. Functoriality is discussed in Section 50.3 and the cup product in Section 50.4. We obtain graded commutative F-algebras by Lemma 50.4.1.
(D2') For the maps $c_1^H : \mathrm{Pic}(X) \to H^2(X)(1)$ we use the de Rham first Chern class introduced in Section 50.9.

We are going to show axioms (A1) – (A9) hold.

In this paragraph, we are going to reduce the checking of the axioms to the case where k is algebraically closed by using Weil Cohomology Theories, Lemma 45.14.18. Denote k' the algebraic closure of k. Set $F' = k'$. We obtain data (D0), (D1), (D2') over k' with coefficient field F' in exactly the same way as above. By Lemma 50.3.5 there are functorial isomorphisms

$$H^{2d}_{dR}(X/k) \otimes_k k' \longrightarrow H^{2d}_{dR}(X_{k'}/k')$$

for X smooth and projective over k. Moreover, the diagrams

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow[c_1^{dR}]{} & H^2_{dR}(X/k) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X_{k'}) & \xrightarrow{c_1^{dR}} & H^2_{dR}(X_{k'}/k') \end{array}$$

commute by Lemma 50.9.1. This finishes the proof of the reduction.

Assume k is algebraically closed field of characteristic zero. We will show axioms (A1) – (A9) for the data (D0), (D1), and (D2') given above.

Axiom (A1). Here we have to check that $H^*_{dR}(X \coprod Y/k) = H^*_{dR}(X/k) \times H^*_{dR}(Y/k)$. This is a consequence of the fact that de Rham cohomology is constructed by taking the cohomology of a sheaf of differential graded algebras (in the Zariski topology).

Axiom (A2). This is just the statement that taking first Chern classes of invertible modules is compatible with pullbacks. This follows from the more general Lemma 50.9.1.

Axiom (A3). This follows from the more general Proposition 50.14.1.

Axiom (A4). This follows from the more general Lemma 50.15.6.

Already at this point, using Weil Cohomology Theories, Lemmas 45.14.1 and 45.14.2, we obtain a Chern character and cycle class maps

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus\nolimits_{i \geq 0} H^{2i}_{dR}(X/k)$$

for X smooth projective over k which are graded ring homomorphisms compatible with pullbacks between morphisms $f : X \to Y$ of smooth projective schemes over k.

Axiom (A5). We have $H^*_{dR}(\mathrm{Spec}(k)/k) = k = F$ in degree 0. We have the Künneth formula for the product of two smooth projective k-schemes by Lemma 50.8.2 (observe that the derived tensor products in the statement are harmless as we are tensoring over the field k).

Axiom (A7). This follows from Proposition 50.17.3.

Axiom (A8). Let X be a smooth projective scheme over k. By the explanatory text to this axiom in Weil Cohomology Theories, Section 45.14 we see that $k' = H^0(X, \mathcal{O}_X)$ is a finite separable k-algebra. It follows that $H^*_{dR}(\mathrm{Spec}(k')/k) = k'$ sitting in degree 0 because $\Omega_{k'/k} = 0$. By Lemma 50.20.2 we also have $H^0_{dR}(X, \mathcal{O}_X) = k'$ and we get the axiom.

Axiom (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Denote $\Delta : X \to X \times_{\mathrm{Spec}(k)} X$ the diagonal morphism of X over k. We have to show that there exists a k-linear map

$$\lambda : H^{2d}_{dR}(X/k) \longrightarrow k$$

such that $(1 \otimes \lambda)\gamma([\Delta]) = 1$ in $H^0_{dR}(X/k)$. Let us write

$$\gamma = \gamma([\Delta]) = \gamma_0 + \ldots + \gamma_{2d}$$

with $\gamma_i \in H^i_{dR}(X/k) \otimes_k H^{2d-i}_{dR}(X/k)$ the Künneth components. Our problem is to show that there is a linear map $\lambda : H^{2d}_{dR}(X/k) \to k$ such that $(1 \otimes \lambda)\gamma_0 = 1$ in $H^0_{dR}(X/k)$.

Let $X = \coprod X_i$ be the decomposition of X into connected and hence irreducible components. Then we have correspondingly $\Delta = \coprod \Delta_i$ with $\Delta_i \subset X_i \times X_i$. It follows that

$$\gamma([\Delta]) = \sum \gamma([\Delta_i])$$

and moreover $\gamma([\Delta_i])$ corresponds to the class of $\Delta_i \subset X_i \times X_i$ via the decomposition

$$H^*_{dR}(X \times X) = \prod\nolimits_{i,j} H^*_{dR}(X_i \times X_j)$$

We omit the details; one way to show this is to use that in $\mathrm{CH}^0(X \times X)$ we have idempotents $e_{i,j}$ corresponding to the open and closed subschemes $X_i \times X_j$ and to use that γ is a ring map which sends $e_{i,j}$ to the corresponding idempotent in the displayed product decomposition of cohomology. If we can find $\lambda_i : H^{2d}_{dR}(X_i/k) \to k$ with $(1 \otimes \lambda_i)\gamma([\Delta_i]) = 1$ in $H^0_{dR}(X_i/k)$ then taking $\lambda = \sum \lambda_i$ will solve the problem for X. Thus we may and do assume X is irreducible.

Proof of Axiom (A6) for X irreducible. Since k is algebraically closed we have $H^0_{dR}(X/k) = k$ because $H^0(X, \mathcal{O}_X) = k$ as X is a projective variety over an algebraically closed field (see Varieties, Lemma 33.9.3 for example). Let $x \in X$ be any closed point. Consider the cartesian diagram

$$\begin{array}{ccc} x & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{x \times \mathrm{id}} & X \times_{\mathrm{Spec}(k)} X \end{array}$$

Compatibility of γ with pullbacks implies that $\gamma([\Delta])$ maps to $\gamma([x])$ in $H^{2d}_{dR}(X/k)$, in other words, we have $\gamma_0 = 1 \otimes \gamma([x])$. We conclude two things from this: (a) the class $\gamma([x])$ is independent of x, (b) it suffices to show the class $\gamma([x])$ is nonzero, and hence (c) it suffices to find any zero cycle α on X such that $\gamma(\alpha) \neq 0$. To do this we choose a finite morphism

$$f : X \longrightarrow \mathbf{P}^d_k$$

To see such a morphism exist, see Intersection Theory, Section 43.23 and in particular Lemma 43.23.1. Observe that f is finite syntomic (local complete intersection morphism by More on Morphisms, Lemma 37.62.10 and flat by Algebra, Lemma 10.128.1). By Proposition 50.19.3 we have a trace map

$$\Theta_f : f_*\Omega^\bullet_{X/k} \longrightarrow \Omega^\bullet_{\mathbf{P}^d_k/k}$$

whose composition with the canonical map

$$\Omega^\bullet_{\mathbf{P}^d_k/k} \longrightarrow f_*\Omega^\bullet_{X/k}$$

is multiplication by the degree of f. Hence we see that we get a map

$$\Theta : H^{2d}_{dR}(X/k) \to H^{2d}_{dR}(\mathbf{P}^d_k/k)$$

such that $\Theta \circ f^*$ is multiplication by a positive integer. Hence if we can find a zero cycle on $\mathbf{P}^d_k$ whose class is nonzero, then we conclude by the compatibility of γ with pullbacks. This is true by Lemma 50.11.4 and this finishes the proof of axiom (A6).

Below we will use the following without further mention. First, by Weil Cohomology Theories, Remark 45.14.6 the map $\lambda_X : H^{2d}_{dR}(X/k) \to k$ is unique. Second, in the proof of axiom (A6) we have seen that $\lambda_X(\gamma([x])) = 1$ when X is irreducible, i.e., the composition of the cycle class map $\gamma : \mathrm{CH}^d(X) \to H^{2d}_{dR}(X/k)$ with λ_X is the degree map.

Axiom (A9). Let $Y \subset X$ be a nonempty smooth divisor on a nonempty smooth equidimensional projective scheme X over k of dimension d. We have to show that the diagram

$$\begin{array}{ccc} H^{2d-2}_{dR}(X/k) & \xrightarrow[c_1^{dR}(\mathcal{O}_X(Y))\cap -]{} & H^{2d}_{dR}(X) \\ {\scriptstyle restriction}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ H^{2d-2}_{dR}(Y/k) & \xrightarrow{\lambda_Y} & k \end{array}$$

commutes where λ_X and λ_Y are as in axiom (A6). Above we have seen that if we decompose $X = \coprod X_i$ into connected (equivalently irreducible) components, then we have correspondingly $\lambda_X = \sum \lambda_{X_i}$. Similarly, if we decompoese $Y = \coprod Y_j$

into connected (equivalently irreducible) components, then we have $\lambda_Y = \sum \lambda_{Y_j}$. Moreover, in this case we have $\mathcal{O}_X(Y) = \otimes_j \mathcal{O}_X(Y_j)$ and hence

$$c_1^{dR}(\mathcal{O}_X(Y)) = \sum\nolimits_j c_1^{dR}(\mathcal{O}_X(Y_j))$$

in $H^2_{dR}(X/k)$. A straightforward diagram chase shows that it suffices to prove the commutativity of the diagram in case X and Y are both irreducible. Then $H^{2d-2}_{dR}(Y/k)$ is 1-dimensional as we have Poincaré duality for Y by Weil Cohomology Theories, Lemma 45.14.5. By axiom (A4) the kernel of restriction (left vertical arrow) is contained in the kernel of cupping with $c_1^{dR}(\mathcal{O}_X(Y))$. This means it suffices to find one cohomology class $a \in H^{2d-2}_{dR}(X)$ whose restriction to Y is nonzero such that we have commutativity in the diagram for a. Take any ample invertible module $\mathcal{L}$ and set

$$a = c_1^{dR}(\mathcal{L})^{d-1}$$

Then we know that $a|_Y = c_1^{dR}(\mathcal{L}|_Y)^{d-1}$ and hence

$$\lambda_Y(a|_Y) = \deg(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y])$$

by our description of λ_Y above. This is a positive integer by Chow Homology, Lemma 42.41.4 combined with Varieties, Lemma 33.45.9. Similarly, we find

$$\lambda_X(c_1^{dR}(\mathcal{O}_X(Y)) \cap a) = \deg(c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X])$$

Since we know that $c_1(\mathcal{O}_X(Y)) \cap [X] = [Y]$ more or less by definition we have an equality of zero cycles

$$(Y \to X)_* \left(c_1(\mathcal{L}|_Y)^{d-1} \cap [Y]\right) = c_1(\mathcal{O}_X(Y)) \cap c_1(\mathcal{L})^{d-1} \cap [X]$$

on X. Thus these cycles have the same degree and the proof is complete.

0FWD **Proposition 50.22.1.** *Let k be a field of characteristic zero. The functor that sends a smooth projective scheme X over k to $H^*_{dR}(X/k)$ is a Weil cohomology theory in the sense of Weil Cohomology Theories, Definition 45.11.4.*

Proof. In the discussion above we showed that our data (D0), (D1), (D2') satisfies axioms (A1) – (A9) of Weil Cohomology Theories, Section 45.14. Hence we conclude by Weil Cohomology Theories, Proposition 45.14.17.

Please don't read what follows. In the proof of the assertions we also used Lemmas 50.3.5, 50.9.1, 50.15.6, 50.8.2, 50.20.2, and 50.11.4, Propositions 50.14.1, 50.17.3, and 50.19.3, Weil Cohomology Theories, Lemmas 45.14.18, 45.14.1, 45.14.2, and 45.14.5, Weil Cohomology Theories, Remark 45.14.6, Varieties, Lemmas 33.9.3 and 33.45.9, Intersection Theory, Section 43.23 and Lemma 43.23.1, More on Morphisms, Lemma 37.62.10, Algebra, Lemma 10.128.1, and Chow Homology, Lemma 42.41.4. □

0FWE **Remark 50.22.2.** In exactly the same manner as above one can show that Hodge cohomology $X \mapsto H^*_{Hodge}(X/k)$ equipped with c_1^{Hodge} determines a Weil cohomology theory. If we ever need this, we will precisely formulate and prove this here. This leads to the following amusing consequence: If the betti numbers of a Weil cohomology theory are independent of the chosen Weil cohomology theory (over our field k of characteristic 0), then the Hodge-to-de Rham spectral sequence degenerates at E_1! Of course, the degeneration of the Hodge-to-de Rham spectral sequence is known (see for example [**DI87**] for a marvelous algebraic proof), but it is by no means an easy result! This suggests that proving the independence of betti

numbers is a hard problem as well and as far as we know is still an open problem. See Weil Cohomology Theories, Remark 45.11.5 for a related question.

50.23. Gysin maps for closed immersions

0G82 In this section we define the Gysin map for closed immersions.

0G83 **Remark 50.23.1.** Let $X \to S$ be a morphism of schemes. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Let $Z \subset X$ be the closed subscheme cut out by $f_1, \ldots, f_c$. Below we will study the *Gysin map*

0G84 (50.23.1.1) $$\gamma^p_{f_1,\ldots,f_c} : \Omega^p_{Z/S} \longrightarrow \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

defined as follows. Given a local section ω of $\Omega^p_{Z/S}$ which is the restriction of a section $\tilde{\omega}$ of $\Omega^p_{X/S}$ we set

$$\gamma^p_{f_1,\ldots,f_c}(\omega) = c_{f_1,\ldots,f_c}(\tilde{\omega}|_Z) \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$$

where $c_{f_1,\ldots,f_c} : \Omega^p_{X/S} \otimes \mathcal{O}_Z \to \mathcal{H}^c_Z(\Omega^p_{X/S})$ is the map constructed in Derived Categories of Schemes, Remark 36.6.10. This is well defined: given ω we can change our choice of $\tilde{\omega}$ by elements of the form $\sum f_i\omega'_i + \sum \mathrm{d}(f_i) \wedge \omega''_i$ which are mapped to zero by the construction.

0G85 **Lemma 50.23.2.** *The Gysin map (50.23.1.1) is compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.*

Proof. This follows from an almost trivial calculation once we correctly interpret this. First, we recall that the functor $\mathcal{H}^c_Z$ computed on the category of $\mathcal{O}_X$-modules agrees with the similarly defined functor on the category of abelian sheaves on X, see Cohomology, Lemma 20.34.8. Hence, the differential $\mathrm{d} : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$ induces a map $\mathcal{H}^c_Z(\Omega^p_{X/S}) \to \mathcal{H}^c_Z(\Omega^{p+1}_{X/S})$. Moreover, the formation of the extended alternating Čech complex in Derived Categories of Schemes, Remark 36.6.4 works on the category of abelian sheaves. The map

$$\mathrm{Coker}\left(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c}\right) \longrightarrow i_*\mathcal{H}^c_Z(\mathcal{F})$$

used in the construction of $c_{f_1,\ldots,f_c}$ in Derived Categories of Schemes, Remark 36.6.10 is well defined and functorial on the category of all abelian sheaves on X. Hence we see that the lemma follows from the equality

$$\mathrm{d}\left(\frac{\tilde{\omega} \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c}{f_1 \ldots f_c}\right) = \frac{\mathrm{d}(\tilde{\omega}) \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c}{f_1 \ldots f_c}$$

which is clear. □

0G86 **Lemma 50.23.3.** *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a closed immersion of finite presentation whose conormal sheaf $\mathcal{C}_{Z/X}$ is locally free of rank c. Then there is a canonical map*

$$\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

which is locally given by the maps $\gamma^p_{f_1,\ldots,f_c}$ of Remark 50.23.1.

Proof. The assumptions imply that given $x \in Z \subset X$ there exists an open neighbourhood U of x such that Z is cut out by c elements $f_1, \ldots, f_c \in \mathcal{O}_X(U)$. Thus it suffices to show that given $f_1, \ldots, f_c$ and $g_1, \ldots, g_c$ in $\mathcal{O}_X(U)$ cutting out $Z \cap U$, the maps $\gamma^p_{f_1,\ldots,f_c}$ and $\gamma^p_{g_1,\ldots,g_c}$ are the same. To do this, after shrinking U we may assume $g_j = \sum a_{ji} f_i$ for some $a_{ji} \in \mathcal{O}_X(U)$. Then we have $c_{f_1,\ldots,f_c} = \det(a_{ji}) c_{g_1,\ldots,g_c}$ by Derived Categories of Schemes, Lemma 36.6.12. On the other hand we have

$$\mathrm{d}(g_1) \wedge \ldots \wedge \mathrm{d}(g_c) \equiv \det(a_{ji}) \mathrm{d}(f_1) \wedge \ldots \wedge \mathrm{d}(f_c) \bmod (f_1, \ldots, f_c)\Omega^c_{X/S}$$

Combining these relations, a straightforward calculation gives the desired equality. □

0G87 **Lemma 50.23.4.** *Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. The Gysin map γ^p is compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.*

Proof. We may check this locally and then it follows from Lemma 50.23.2. □

0G88 **Lemma 50.23.5.** *Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. Given $\alpha \in H^q(X, \Omega^p_{X/S})$ we have $\gamma^p(\alpha|_Z) = i^{-1}\alpha \wedge \gamma^0(1)$ in $H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S}))$. Please see proof for notation.*

Proof. The restriction $\alpha|_Z$ is the element of $H^q(Z, \Omega^p_{Z/S})$ given by functoriality for Hodge cohomology. Applying functoriality for cohomology using $\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$ we get get $\gamma^p(\alpha|_Z)$ in $H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S}))$. This explains the left hand side of the formula.

To explain the right hand side, we first pullback by the map of ringed spaces $i : (Z, i^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$ to get the element $i^{-1}\alpha \in H^q(Z, i^{-1}\Omega^p_{X/S})$. Let $\gamma^0(1) \in H^0(Z, \mathcal{H}^c_Z(\Omega^c_{X/S}))$ be the image of $1 \in H^0(Z, \mathcal{O}_Z) = H^0(Z, \Omega^0_{Z/S})$ by γ^0. Using cup product we obtain an element

$$i^{-1}\alpha \cup \gamma^0(1) \in H^{q+c}(Z, i^{-1}\Omega^p_{X/S} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{H}^c_Z(\Omega^c_{X/S}))$$

Using Cohomology, Remark 20.34.9 and wedge product there are canonical maps

$$i^{-1}\Omega^p_{X/S} \otimes^{\mathbf{L}}_{i^{-1}\mathcal{O}_X} R\mathcal{H}_Z(\Omega^c_{X/S}) \to R\mathcal{H}_Z(\Omega^p_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \Omega^c_{X/S}) \to R\mathcal{H}_Z(\Omega^{p+c}_{X/S})$$

By Derived Categories of Schemes, Lemma 36.6.8 the objects $R\mathcal{H}_Z(\Omega^j_{X/S})$ have vanishing cohomology sheaves in degrees $> c$. Hence on cohomology sheaves in degree c we obtain a map

$$i^{-1}\Omega^p_{X/S} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{H}^c_Z(\Omega^c_{X/S}) \longrightarrow \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$$

The expression $i^{-1}\alpha \wedge \gamma^0(1)$ is the image of the cup product $i^{-1}\alpha \cup \gamma^0(1)$ by the functoriality of cohomology.

Having explained the content of the formula in this manner, by general properties of cup products (Cohomology, Section 20.31), it now suffices to prove that the diagram

$$\begin{array}{ccccc} i^{-1}\Omega^p_X \otimes \Omega^0_Z & & \xrightarrow[\mathrm{id}\otimes\gamma^0]{} & & i^{-1}\Omega^p_X \otimes \mathcal{H}^c_Z(\Omega^c_X) \\ \downarrow & & & & \downarrow \wedge \\ \Omega^p_Z \otimes \Omega^0_Z & \xrightarrow{\wedge} & \Omega^p_Z & \xrightarrow{\gamma^p} & \mathcal{H}^c_Z(\Omega^{p+c}_X) \end{array}$$

is commutative in the category of sheaves on Z (with obvious abuse of notation). This boils down to a simple computation for the maps $\gamma^j_{f_1,\ldots,f_c}$ which we omit; in fact these maps are chosen exactly such that this works and such that 1 maps to $\frac{\mathrm{d}f_1\wedge\ldots\wedge\mathrm{d}f_c}{f_1\ldots f_c}$. □

0G89 **Lemma 50.23.6.** *Let $c \geq 0$ be an integer. Let*

$$\begin{array}{ccccc} Z' & \longrightarrow & X' & \longrightarrow & S' \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & X & \longrightarrow & S \end{array}$$

be a commutative diagram of schemes. Assume

(1) *$Z \to X$ and $Z' \to X'$ satisfy the assumptions of Lemma 50.23.3,*
(2) *the left square in the diagram is cartesian, and*
(3) *$h^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$ (Morphisms, Lemma 29.31.3) is an isomorphism.*

Then the diagram

$$\begin{array}{ccc} h^*\Omega^p_{Z/S} & \xrightarrow[h^{-1}\gamma^p]{} & \mathcal{O}_{X'}|_{Z'} \otimes_{h^{-1}\mathcal{O}_X|_Z} h^{-1}\mathcal{H}^c_Z(\Omega^{p+c}_{X/S}) \\ \downarrow & & \downarrow \\ \Omega^p_{Z'/S'} & \xrightarrow{\gamma^p} & \mathcal{H}^c_{Z'}(\Omega^{p+c}_{X'/S'}) \end{array}$$

is commutative. The left vertical arrow is functoriality of modules of differentials and the right vertical arrow uses Cohomology, Remark 20.34.12.

Proof. More precisely, consider the composition

$$\begin{aligned} \mathcal{O}_{X'}|_{Z'} \otimes^{\mathbf{L}}_{h^{-1}\mathcal{O}_X|_Z} h^{-1}R\mathcal{H}_Z(\Omega^{p+c}_{X/S}) &\to R\mathcal{H}_{Z'}(Lg^*\Omega^{p+c}_{X/S}) \\ &\to R\mathcal{H}_{Z'}(g^*\Omega^{p+c}_{X/S}) \\ &\to R\mathcal{H}_{Z'}(\Omega^{p+c}_{X'/S'}) \end{aligned}$$

where the first arrow is given by Cohomology, Remark 20.34.12 and the last one by functoriality of differentials. Since we have the vanishing of cohomology sheaves in degrees $> c$ by Derived Categories of Schemes, Lemma 36.6.8 this induces the right vertical arrow. We can check the commutativity locally. Thus we may assume Z is cut out by $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Then Z' is cut out by $f'_i = g^\sharp(f_i)$. The maps $c_{f_1,\ldots,f_c}$ and $c_{f'_1,\ldots,f'_c}$ fit into the commutative diagram

$$\begin{array}{ccc} h^*i^*\Omega^p_{X/S} & \xrightarrow[h^{-1}c_{f_1,\ldots,f_c}]{} & \mathcal{O}_{X'}|_{Z'} \otimes_{h^{-1}\mathcal{O}_X|_Z} h^{-1}\mathcal{H}^c_Z(\Omega^p_{X/S}) \\ \downarrow & & \downarrow \\ (i')^*\Omega^p_{X'/S'} & \xrightarrow{c_{f'_1,\ldots,f'_c}} & \mathcal{H}^c_{Z'}(\Omega^p_{X'/S'}) \end{array}$$

See Derived Categories of Schemes, Remark 36.6.14. Recall given a p-form ω on Z we define $\gamma^p(\omega)$ by choosing (locally on X and Z) a p-form $\tilde{\omega}$ on X lifting ω and taking $\gamma^p(\omega) = c_{f_1,\ldots,f_c}(\tilde{\omega}) \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$. Since the form $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_c$ pulls back to $\mathrm{d}f'_1 \wedge \ldots \wedge \mathrm{d}f'_c$ we conclude. □

0G8A **Remark 50.23.7.** Let $X \to S$, $i : Z \to X$, and $c \geq 0$ be as in Lemma 50.23.3. Let $p \geq 0$ and assume that $\mathcal{H}^i_Z(\Omega^{p+c}_{X/S}) = 0$ for $i = 0, \ldots, c-1$. This vanishing holds if $X \to S$ is smooth and $Z \to X$ is a Koszul regular immersion, see Derived Categories of Schemes, Lemma 36.6.9. Then we obtain a map

$$\gamma^{p,q} : H^q(Z, \Omega^p_{Z/S}) \longrightarrow H^{q+c}(X, \Omega^{p+c}_{X/S})$$

by first using $\gamma^p : \Omega^p_{Z/S} \to \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})$ to map into

$$H^q(Z, \mathcal{H}^c_Z(\Omega^{p+c}_{X/S})) = H^q(Z, R\mathcal{H}_Z(\Omega^{p+c}_{X/S})[c]) = H^q(X, i_* R\mathcal{H}_Z(\Omega^{p+c}_{X/S})[c])$$

and then using the adjunction map $i_* R\mathcal{H}_Z(\Omega^{p+c}_{X/S}) \to \Omega^{p+c}_{X/S}$ to continue on to the desired Hodge cohomology module.

0G8B **Lemma 50.23.8.** *Let $X \to S$ and $i : Z \to X$ be as in Lemma 50.23.3. Assume $X \to S$ is smooth and $Z \to X$ Koszul regular. The Gysin maps $\gamma^{p,q}$ are compatible with the de Rham differentials on $\Omega^\bullet_{X/S}$ and $\Omega^\bullet_{Z/S}$.*

Proof. This follows immediately from Lemma 50.23.4. □

0G8C **Lemma 50.23.9.** *Let $X \to S$, $i : Z \to X$, and $c \geq 0$ be as in Lemma 50.23.3. Assume $X \to S$ smooth and $Z \to X$ Koszul regular. Given $\alpha \in H^q(X, \Omega^p_{X/S})$ we have $\gamma^{p,q}(\alpha|_Z) = \alpha \cup \gamma^{0,0}(1)$ in $H^{q+c}(X, \Omega^{p+c}_{X/S})$ with $\gamma^{a,b}$ as in Remark 50.23.7.*

Proof. This lemma follows from Lemma 50.23.5 and Cohomology, Lemma 20.34.11. We suggest the reader skip over the more detailed discussion below.

We will use without further mention that $R\mathcal{H}_Z(\Omega^j_{X/S}) = \mathcal{H}^c_Z(\Omega^j_{X/S})[-c]$ for all j as pointed out in Remark 50.23.7. We will also silently use the identifications $H^{q+c}_Z(X, \Omega^j_{X/S}) = H^{q+c}(Z, R\mathcal{H}_Z(\Omega^j_{X/S}) = H^q(Z, \mathcal{H}^c_Z(\Omega^j_{X/S}))$, see Cohomology, Lemma 20.34.4 for the first one. With these identifications

(1) $\gamma^0(1) \in H^c_Z(X, \Omega^c_{X/S})$ maps to $\gamma^{0,0}(1)$ in $H^c(X, \Omega^c_{X/S})$,
(2) the right hand side $i^{-1}\alpha \wedge \gamma^0(1)$ of the equality in Lemma 50.23.5 is the (image by wedge product of the) cup product of Cohomology, Remark 20.34.10 of the elements α and $\gamma^0(1)$, in other words, the constructions in the proof of Lemma 50.23.5 and in Cohomology, Remark 20.34.10 match,
(3) by Cohomology, Lemma 20.34.11 this maps to $\alpha \cup \gamma^{0,0}(1)$ in $H^{q+c}(X, \Omega^p_{X/S} \otimes \Omega^c_{X/S})$, and
(4) the left hand side $\gamma^p(\alpha|_Z)$ of the equality in Lemma 50.23.5 maps to $\gamma^{p,q}(\alpha|_Z)$.

This finishes the proof. □

0G8D **Lemma 50.23.10.** *Let $c \geq 0$ and*

$$\begin{array}{ccccc} Z' & \longrightarrow & X' & \longrightarrow & S' \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & X & \longrightarrow & S \end{array}$$

satisfy the assumptions of Lemma 50.23.6 and assume in addition that $X \to S$ and $X' \to S'$ are smooth and that $Z \to X$ and $Z' \to X'$ are Koszul regular immersions.

Then the diagram

$$\begin{array}{ccc} H^q(Z, \Omega^p_{Z/S}) & \xrightarrow{\gamma^{p,q}} & H^{q+c}(X, \Omega^{p+c}_{X/S}) \\ \downarrow & & \downarrow \\ H^q(Z', \Omega^p_{Z'/S'}) & \xrightarrow{\gamma^{p,q}} & H^{q+c}(X', \Omega^{p+c}_{X'/S'}) \end{array}$$

is commutative where $\gamma^{p,q}$ is as in Remark 50.23.7.

Proof. This follows on combining Lemma 50.23.6 and Cohomology, Lemma 20.34.13. □

0G8E **Lemma 50.23.11.** *Let k be a field. Let X be an irreducible smooth proper scheme over k of dimension d. Let $Z \subset X$ be the reduced closed subscheme consisting of a single k-rational point x. Then the image of $1 \in k = H^0(Z, \mathcal{O}_Z) = H^0(Z, \Omega^0_{Z/k})$ by the map $H^0(Z, \Omega^0_{Z/k}) \to H^d(X, \Omega^d_{X/k})$ of Remark 50.23.7 is nonzero.*

Proof. The map $\gamma^0 : \mathcal{O}_Z \to \mathcal{H}^d_Z(\Omega^d_{X/k}) = R\mathcal{H}_Z(\Omega^d_{X/k})[d]$ is adjoint to a map

$$g^0 : i_*\mathcal{O}_Z \longrightarrow \Omega^d_{X/k}[d]$$

in $D(\mathcal{O}_X)$. Recall that $\Omega^d_{X/k} = \omega_X$ is a dualizing sheaf for X/k, see Duality for Schemes, Lemma 48.27.1. Hence the k-linear dual of the map in the statement of the lemma is the map

$$H^0(X, \mathcal{O}_X) \to \mathrm{Ext}^d_X(i_*\mathcal{O}_Z, \omega_X)$$

which sends 1 to g^0. Thus it suffices to show that g^0 is nonzero. This we may do in any neighbourhood U of the point x. Choose U such that there exist $f_1, \ldots, f_d \in \mathcal{O}_X(U)$ vanishing only at x and generating the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$. We may assume assume $U = \mathrm{Spec}(R)$ is affine. Looking over the construction of γ^0 we find that our extension is given by

$$k \to (R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r})[d] \to R[d]$$

where 1 maps to $1/f_1 \ldots f_c$ under the first map. This is nonzero because $1/f_1 \ldots f_c$ is a nonzero element of local cohomology group $H^d_{(f_1, \ldots, f_d)}(R)$ in this case, □

50.24. Relative Poincaré duality

0G8F In this section we prove Poincar'e duality for the relative de Rham cohomology of a proper smooth scheme over a base. We strongly urge the reader to look at Section 50.20 first.

0G8G **Situation 50.24.1.** Here S is a quasi-compact and quasi-separated scheme and $f : X \to S$ is a proper smooth morphism of schemes all of whose fibres are nonempty and equidimensional of dimension n.

0G8H **Lemma 50.24.2.** *In Situation 50.24.1 the pushforward $f_*\mathcal{O}_X$ is a finite étale $\mathcal{O}_S$-algebra and locally on S we have $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X \oplus P$ in $D(\mathcal{O}_S)$ with P perfect of tor amplitude in $[1, \infty)$. The map $d : f_*\mathcal{O}_X \to f_*\Omega_{X/S}$ is zero.*

Proof. The first part of the statement follows from Derived Categories of Schemes, Lemma 36.32.8. Setting $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$ we get a factorization $X \to S' \to S$ (this is the Stein factorization, see More on Morphisms, Section 37.53, although we don't need this) and we see that $\Omega_{X/S} = \Omega_{X/S'}$ for example by Morphisms, Lemma 29.32.9 and 29.36.15. This of course implies that $\mathrm{d} : f_*\mathcal{O}_X \to f_*\Omega_{X/S}$ is zero. □

0G8I **Lemma 50.24.3.** *In Situation 50.24.1 there exists an $\mathcal{O}_S$-module map*

$$t : Rf_*\Omega^n_{X/S}[n] \longrightarrow \mathcal{O}_S$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: for all p the pairing

$$Rf_*\Omega^p_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\Omega^{n-p}_{X/S}[n] \longrightarrow \mathcal{O}_S$$

given by the relative cup product composed with t is a perfect pairing of perfect complexes on S.

Proof. We first assume S is a Noetherian scheme. Let $\omega^\bullet_{X/S}$ be the relative dualizing complex of X over S as in Duality for Schemes, Remark 48.12.5 and let $Rf_*\omega^\bullet_{X/S} \to \mathcal{O}_S$ be its trace map. By Duality for Schemes, Lemma 48.15.7 (here we use that S is Noetherian) there exists an isomorphism $\omega^\bullet_{X/S} \cong \Omega^n_{X/S}[n]$ and using this isomorphism we obtain t. The complexes $Rf_*\Omega^p_{X/S}$ are perfect by Lemma 50.3.5. Since $\Omega^p_{X/S}$ is locally free and since $\Omega^p_{X/S} \otimes_{\mathcal{O}_X} \Omega^{n-p}_{X/S} \to \Omega^n_{X/S}$ exhibits an isomorphism $\Omega^p_{X/S} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^{n-p}_{X/S}, \Omega^n_{X/S})$ we see that the pairing induced by the relative cup product is perfect by Duality for Schemes, Remark 48.12.6.

Proof of existence in the general case. By absolute Noetherian approximation we can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ S & \xrightarrow{g} & S_0 \end{array}$$

with S_0 Noetherian and f_0 a proper and smooth morphism all of whose fibres are nonempty and equidimensional of dimension n. See Limits, Lemmas 32.10.1, 32.8.9, 32.18.4, 32.8.15, and 32.13.1. Choose a map

$$t_0 : Rf_{0,*}\Omega^n_{X_0/S_0}[n] \longrightarrow \mathcal{O}_{S_0}$$

as in the statement of the lemma. By cohomology and base change the complexes $Rf_*(\Omega^p_{X/S})$ are equal to $Lg^*(Rf_{0,*}\Omega^p_{X_0/S_0})$ for all $p \geq 0$, see Lemma 50.3.5. In particular the pullback of t_0 by g gives us a map t as in the statement of the lemmma. Namely, by Cohomology, Lemma 20.31.8, the cup product map

$$Rf_*\Omega^p_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\Omega^{n-p}_{X/S}[n] \longrightarrow Rf_*\Omega^n_{X/S}$$

is the pullback by g of the corresponding cup product map for f_0. Hence the fact that t_0 produces perfect pairings, pulls back to the same property for t on S.

Uniqueness of t. Choose a distinguished triangle $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X \to P \to f_*\mathcal{O}_X[1]$. By Lemma 50.24.2 the object P is perfect of tor amplitude in $[1, \infty)$ and the triangle is locally on S split. Thus $R\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X)$ is perfect of tor amplitude in $(-\infty, -1]$. Hence duality (above) shows that locally on S we have

$$Rf_*\Omega^n_{X/S}[n] \cong R\mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X, \mathcal{O}_S) \oplus R\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X)$$

This shows that $R^n f_* \Omega^n_{X/S}$ is finite locally free and that we obtain a perfect $\mathcal{O}_S$-bilinear pairing

$$f_*\mathcal{O}_X \times R^n f_* \Omega^n_{X/S} \longrightarrow \mathcal{O}_S$$

using t. This implies that any $\mathcal{O}_S$-linear map $t' : R^n f_* \Omega^n_{X/S} \to \mathcal{O}_S$ is of the form $t' = t \circ g$ for some $g \in \Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. In order for t' to still determine a perfect pairing g will have to be a unit. This finishes the proof. □

0G8J **Lemma 50.24.4.** *In Situation 50.24.1 the map $d : R^n f_* \Omega^{n-1}_{X/S} \to R^n f_* \Omega^n_{X/S}$ is zero.*

As we mentioned in the proof of Lemma 50.20.3 this lemma is not an easy consequence of Lemmas 50.24.3 and 50.24.2.

Proof in case S is reduced. Assume S is reduced. Observe that $\mathrm{d} : R^n f_* \Omega^{n-1}_{X/S} \to R^n f_* \Omega^n_{X/S}$ is an $\mathcal{O}_S$-linear map of (quasi-coherent) $\mathcal{O}_S$-modules. The $\mathcal{O}_S$-module $R^n f_* \Omega^n_{X/S}$ is finite locally free (as the dual of the finite locally free $\mathcal{O}_S$-module $f_*\mathcal{O}_X$ by Lemmas 50.24.3 and 50.24.2). Since S is reduced it suffices to show that the stalk of d in every generic point $\eta \in S$ is zero; this follows by looking at sections over affine opens, using that the target of d is locally free, and Algebra, Lemma 10.25.2 part (2). Since S is reduced we have $\mathcal{O}_{S,\eta} = \kappa(\eta)$, see Algebra, Lemma 10.25.1. Thus d_η is identified with the map

$$\mathrm{d} : H^n(X_\eta, \Omega^{n-1}_{X_\eta/\kappa(\eta)}) \longrightarrow H^n(X_\eta, \Omega^n_{X_\eta/\kappa(\eta)})$$

which is zero by Lemma 50.20.3. □

Proof in the general case. Observe that the question is flat local on S: if $S' \to S$ is a surjective flat morphism of schemes and the map is zero after pullback to S', then the map is zero. Also, formation of the map commutes with base change by flat morphisms by flat base change (Cohomology of Schemes, Lemma 30.5.2).

Consider the Stein factorization $X \to S' \to S$ as in More on Morphisms, Theorem 37.53.5. By Lemma 50.24.2 the morphism $\pi : S' \to S$ is finite étale. The morphism $f : X \to S'$ is proper (by the theorem), smooth (by More on Morphisms, Lemma 37.13.12) with geometrically connected fibres by the theorem on Stein factorization. In the proof of Lemma 50.24.2 we saw that $\Omega_{X/S} = \Omega_{X/S'}$ because $S' \to S$ is étale. Hence $\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S'}$. We have

$$R^q f_* \Omega^p_{X/S} = \pi_* R^q f'_* \Omega^p_{X/S'}$$

for all p, q by the Leray spectral sequence (Cohomology, Lemma 20.13.8), the fact that π is finite hence affine, and Cohomology of Schemes, Lemma 30.2.3 (of course we also use that $R^q f'_* \Omega^p_{X'/S}$ is quasi-coherent). Thus the map of the lemma is π_* applied to $\mathrm{d} : R^n f'_* \Omega^{n-1}_{X/S'} \to R^n f'_* \Omega^n_{X/S'}$. In other words, in order to prove the lemma we may replace $f : X \to S$ by $f' : X \to S'$ to reduce to the case discussed in the next pargraph.

Assume f has geometrically connected fibres and $f_*\mathcal{O}_X = \mathcal{O}_S$. For every $s \in S$ we can choose an étale neighbourhood $(S', s') \to (S, s)$ such that the base change $X' \to S'$ of S has a section. See More on Morphisms, Lemma 37.38.6. By the initial remarks of the proof this reduces us to the case discussed in the next paragraph.

Assume f has geometrically connected fibres, $f_*\mathcal{O}_X = \mathcal{O}_S$, and we have a section $s : S \to X$ of f. We may and do assume $S = \mathrm{Spec}(A)$ is affine. The map $s^* : R\Gamma(X, \mathcal{O}_X) \to R\Gamma(S, \mathcal{O}_S) = A$ is a splitting of the map $A \to R\Gamma(X, \mathcal{O}_X)$. Thus we can write

$$R\Gamma(X, \mathcal{O}_X) = A \oplus P$$

where P is the "kernel" of s^*. By Lemma 50.24.2 the object P of $D(A)$ is perfect of tor amplitude in $[1, n]$. As in the proof of Lemma 50.24.3 we see that $H^n(X, \Omega^n_{X/S})$ is a locally free A-module of rank 1 (and in fact dual to A so free of rank 1 – we will soon choose a generator but we don't want to check it is the same generator nor will it be necessary to do so).

Denote $Z \subset X$ the image of s which is a closed subscheme of X by Schemes, Lemma 26.21.11. Observe that $Z \to X$ is a regular (and a fortiori Koszul regular by Divisors, Lemma 31.21.2) closed immersion by Divisors, Lemma 31.22.8. Of course $Z \to X$ has codimension n. Thus by Remark 50.23.7 we can consider the map

$$\gamma^{0,0} : H^0(Z, \Omega^0_{Z/S}) \longrightarrow H^n(X, \Omega^n_{X/S})$$

and we set $\xi = \gamma^{0,0}(1) \in H^n(X, \Omega^n_{X/S})$.

We claim ξ is a basis element. Namely, since we have base change in top degree (see for example Limits, Lemma 32.19.2) we see that $H^n(X, \Omega^n_{X/S}) \otimes_A k = H^n(X_k, \Omega^n_{X_k/k})$ for any ring map $A \to k$. By the compatibility of the construction of ξ with base change, see Lemma 50.23.10, we see that the image of ξ in $H^n(X_k, \Omega^n_{X_k/k})$ is nonzero by Lemma 50.23.11 if k is a field. Thus ξ is a nowhere vanishing section of an invertible module and hence a generator.

Let $\theta \in H^n(X, \Omega^{n-1}_{X/S})$. We have to show that $\mathrm{d}(\theta)$ is zero in $H^n(X, \Omega^n_{X/S})$. We may write $\mathrm{d}(\theta) = a\xi$ for some $a \in A$ as ξ is a basis element. Then we have to show $a = 0$.

Consider the closed immersion

$$\Delta : X \to X \times_S X$$

This is also a section of a smooth morphism (namely either projection) and hence a regular and Koszul immersion of codimension n as well. Thus we can consider the maps

$$\gamma^{p,q} : H^q(X, \Omega^p_{X/S}) \longrightarrow H^{q+n}(X \times_S X, \Omega^{p+n}_{X \times_S X/S})$$

of Remark 50.23.7. Consider the image

$$\gamma^{n-1,n}(\theta) \in H^{2n}(X \times_S X, \Omega^{2n-1}_{X \times_S X})$$

By Lemma 50.8.1 we have

$$\Omega^{2n-1}_{X \times_S X} = \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S} \oplus \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S}$$

By the Künneth formula (either Derived Categories of Schemes, Lemma 36.23.1 or Derived Categories of Schemes, Lemma 36.23.4) we see that

$$H^{2n}(X \times_S X, \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S}) = H^n(X, \Omega^{n-1}_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$$

and

$$H^{2n}(X \times_S X, \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S}) = H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^{n-1}_{X/S})$$

Namely, since we are looking in top degree there no higher tor groups that intervene. Combined with the fact that ξ is a generator this means we can write

$$\gamma^{n-1,n}(\theta) = \theta_1 \otimes \xi + \xi \otimes \theta_2$$

with $\theta_1, \theta_2 \in H^n(X, \Omega^{n-1}_{X/S})$. Arguing in exactly the same manner we can write

$$\gamma^{n,n}(\xi) = b\xi \otimes \xi$$

in $H^{2n}(X \times_S X, \Omega^{2n}_{X\times_S X/S}) = H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$ for some $b \in H^0(S, \mathcal{O}_S)$.

Claim: $\theta_1 = \theta$, $\theta_2 = \theta$, and $b = 1$. Let us show that the claim implies the desired result $a = 0$. Namely, by Lemma 50.23.8 we have

$$\gamma^{n,n}(\mathrm{d}(\theta)) = \mathrm{d}(\gamma^{n-1,n}(\theta))$$

By our choices above this gives

$$a\xi \otimes \xi = \gamma^{n,n}(a\xi) = \mathrm{d}(\theta \otimes \xi + \xi \otimes \theta) = a\xi \otimes \xi + (-1)^n a\xi \otimes \xi$$

The right most equality comes from the fact that the map $\mathrm{d} : \Omega^{2n-1}_{X\otimes_S X/S} \to \Omega^{2n}_{X\times_S X/S}$ by Lemma 50.8.1 is the sum of the differential $\mathrm{d} \boxtimes 1 : \Omega^{n-1}_{X/S} \boxtimes \Omega^n_{X/S} \to \Omega^n_{X/S} \boxtimes \Omega^n_{X/S}$ and the differential $(-1)^n 1 \boxtimes \mathrm{d} : \Omega^n_{X/S} \boxtimes \Omega^{n-1}_{X/S} \to \Omega^n_{X/S} \boxtimes \Omega^n_{X/S}$. Please see discussion in Section 50.8 and Derived Categories of Schemes, Section 36.24 for more information. Since $\xi \otimes \xi$ is a basis for the rank 1 free A-module $H^n(X, \Omega^n_{X/S}) \otimes_A H^n(X, \Omega^n_{X/S})$ we conclude

$$a = a + (-1)^n a \Rightarrow a = 0$$

as desired.

In the rest of the proof we prove the claim above. Let us denote $\eta = \gamma^{0,0}(1) \in H^n(X \times_S X, \Omega^n_{X\times_S X/S})$. Since $\Omega^n_{X\times_S X/S} = \bigoplus_{p+p'=n} \Omega^p_{X/S} \boxtimes \Omega^{p'}_{X/S}$ we may write

$$\eta = \eta_0 + \eta_1 + \ldots + \eta_n$$

where η_p is in $H^n(X \times_S X, \Omega^p_{X/S} \boxtimes \Omega^{n-p}_{X/S})$. For $p = 0$ we can write

$$\begin{aligned} H^n(X \times_S X, \mathcal{O}_X \boxtimes \Omega^n_{X/S}) &= H^n(R\Gamma(X, \mathcal{O}_X) \otimes^{\mathbf{L}}_A R\Gamma(X, \Omega^n_{X/S})) \\ &= A \otimes_A H^n(X, \Omega^n_{X/S}) \oplus H^n(P \otimes^{\mathbf{L}}_A R\Gamma(X, \Omega^n_{X/S})) \end{aligned}$$

by our previously given decomposition $R\Gamma(X, \mathcal{O}_X) = A \oplus P$. Consider the morphism $(s, \mathrm{id}) : X \to X \times_S X$. Then $(s, \mathrm{id})^{-1}(\Delta) = Z$ scheme theoretically. Hence we see that $(s, \mathrm{id})^*\eta = \xi$ by Lemma 50.23.10. This means that

$$\xi = (s, \mathrm{id})^*\eta = (s^* \otimes \mathrm{id})(\eta_0)$$

This means exactly that the first component of η_0 in the direct sum decomposition above is ξ. In other words, we can write

$$\eta_0 = 1 \otimes \xi + \eta_0'$$

with $\eta_0' \in H^n(P \otimes^{\mathbf{L}}_A R\Gamma(X, \Omega^n_{X/S}))$. In exactly the same manner for $p = n$ we can write

$$\begin{aligned} H^n(X \times_S X, \Omega^n_{X/S} \boxtimes \mathcal{O}_X) &= H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes^{\mathbf{L}}_A R\Gamma(X, \mathcal{O}_X)) \\ &= H^n(X, \Omega^n_{X/S}) \otimes_A A \oplus H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes^{\mathbf{L}}_A P) \end{aligned}$$

and we can write

$$\eta_n = \xi \otimes 1 + \eta_n'$$

with $\eta'_n \in H^n(R\Gamma(X, \Omega^n_{X/S}) \otimes^{\mathbf{L}}_A P)$.

Observe that $\mathrm{pr}_1^*\theta = \theta \otimes 1$ and $\mathrm{pr}_2^*\theta = 1 \otimes \theta$ are Hodge cohomology classes on $X \times_S X$ which pull back to θ by Δ. Hence by Lemma 50.23.9 we have

$$\theta_1 \otimes \xi + \xi \otimes \theta_2 = \gamma^{n-1,n}(\theta) = (\theta \otimes 1) \cup \eta = (1 \otimes \theta) \cup \eta$$

in the Hodge cohomology ring of $X \times_S X$ over S. In terms of the direct sum decomposition on the modules of differentials of $X \times_S X/S$ we obtain

$$\theta_1 \otimes \xi = (\theta \otimes 1) \cup \eta_0 \quad \text{and} \quad \xi \otimes \theta_2 = (1 \otimes \theta) \cup \eta_n$$

Looking at the formula $\eta_0 = 1 \otimes \xi + \eta'_0$ we found above, we see that to show that $\theta_1 = \theta$ it suffices to prove that

$$(\theta \otimes 1) \cup \eta'_0 = 0$$

To do this, observe that cupping with $\theta \otimes 1$ is given by the action on cohomology of the map

$$(P \otimes^{\mathbf{L}}_A R\Gamma(X, \Omega^n_{X/S}))[-n] \xrightarrow{\theta \otimes 1} R\Gamma(X, \Omega^{n-1}_{X/S}) \otimes^{\mathbf{L}}_A R\Gamma(X, \Omega^n_{X/S})$$

in the derived category, see Cohomology, Remark 20.31.2. This map is the derived tensor product of the two maps

$$\theta : P[-n] \to R\Gamma(X, \Omega^{n-1}_{X/S}) \quad \text{and} \quad 1 : R\Gamma(X, \Omega^n_{X/S}) \to R\Gamma(X, \Omega^n_{X/S})$$

by Derived Categories of Schemes, Remark 36.23.5. However, the first of these is zero in $D(A)$ because it is a map from a perfect complex of tor amplitude in $[n+1, 2n]$ to a complex with cohomology only in degrees $0, 1, \ldots, n$, see More on Algebra, Lemma 15.77.1. A similar argument works to show the vanishing of $(1 \otimes \theta) \cup \eta'_n$. Finally, in exactly the same manner we obtain

$$b\xi \otimes \xi = \gamma^{n,n}(\xi) = (\xi \otimes 1) \cup \eta_0$$

and we conclude as before by showing that $(\xi \otimes 1) \cup \eta'_0 = 0$ in the same manner as above. This finishes the proof. □

0G8K **Proposition 50.24.5.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a proper smooth morphism of schemes all of whose fibres are nonempty and equidimensional of dimension n. There exists an $\mathcal{O}_S$-module map*

$$t : R^{2n} f_* \Omega^\bullet_{X/S} \longrightarrow \mathcal{O}_S$$

unique up to precomposing by multiplication by a unit of $H^0(X, \mathcal{O}_X)$ with the following property: the pairing

$$Rf_*\Omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\Omega^\bullet_{X/S}[2n] \longrightarrow \mathcal{O}_S, \quad (\xi, \xi') \longmapsto t(\xi \cup \xi')$$

is a perfect pairing of perfect complexes on S.

Proof. The proof is exactly the same as the proof of Proposition 50.20.4.

By the relative Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Rightarrow R^{p+q} f_* \Omega^\bullet_{X/S}$$

(Section 50.6), the vanishing of $\Omega^i_{X/S}$ for $i > n$, the vanishing in for example Limits, Lemma 32.19.2 and the results of Lemmas 50.24.2 and 50.24.4 we see that

$R^0 f_* \Omega_{X/S} = R^0 f_* \mathcal{O}_X$ and $R^n f_* \Omega^n_{X/S} = R^{2n} f_* \Omega^\bullet_{X/S}$. More precisely, these identifications come from the maps of complexes

$$\Omega^\bullet_{X/S} \to \mathcal{O}_X[0] \quad \text{and} \quad \Omega^n_{X/S}[-n] \to \Omega^\bullet_{X/S}$$

Let us choose $t : R^{2n} f_* \Omega_{X/S} \to \mathcal{O}_S$ which via this identification corresponds to a t as in Lemma 50.24.3.

Let us abbreviate $\Omega^\bullet = \Omega^\bullet_{X/S}$. Consider the map (50.4.0.1) which in our situation reads

$$\wedge : \mathrm{Tot}(\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \Omega^\bullet) \longrightarrow \Omega^\bullet$$

For every integer $p = 0, 1, \ldots, n$ this map annihilates the subcomplex $\mathrm{Tot}(\sigma_{>p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet)$ for degree reasons. Hence we find that the restriction of $\wedge$ to the subcomplex $\mathrm{Tot}(\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \geq_{n-p} \Omega^\bullet)$ factors through a map of complexes

$$\gamma_p : \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \longrightarrow \Omega^\bullet$$

Using the same procedure as in Section 50.4 we obtain relative cup products

$$Rf_*\sigma_{\leq p}\Omega^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rf_*\sigma_{\geq n-p}\Omega^\bullet \longrightarrow Rf_*\Omega^\bullet$$

We will prove by induction on p that these cup products via t induce perfect pairings between $Rf_*\sigma_{\leq p}\Omega^\bullet$ and $Rf_*\sigma_{\geq n-p}\Omega^\bullet[2n]$. For $p = n$ this is the assertion of the proposition.

The base case is $p = 0$. In this case we have

$$Rf_*\sigma_{\leq p}\Omega^\bullet = Rf_*\mathcal{O}_X \quad \text{and} \quad Rf_*\sigma_{\geq n-p}\Omega^\bullet[2n] = Rf_*(\Omega^n[-n])[2n] = Rf_*\Omega^n[n]$$

In this case we simply obtain the pairing between $Rf_*\mathcal{O}_X$ and $Rf_*\Omega^n[n]$ of Lemma 50.24.3 and the result is true.

Induction step. Say we know the result is true for p. Then we consider the distinguished triangle

$$\Omega^{p+1}[-p-1] \to \sigma_{\leq p+1}\Omega^\bullet \to \sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$$

and the distinguished triangle

$$\sigma_{\geq n-p}\Omega^\bullet \to \sigma_{\geq n-p-1}\Omega^\bullet \to \Omega^{n-p-1}[-n+p+1] \to (\sigma_{\geq n-p}\Omega^\bullet)[1]$$

Observe that both are distinguished triangles in the homotopy category of complexes of sheaves of $f^{-1}\mathcal{O}_S$-modules; in particular the maps $\sigma_{\leq p}\Omega^\bullet \to \Omega^{p+1}[-p]$ and $\Omega^{n-p-1}[-d+p+1] \to (\sigma_{\geq n-p}\Omega^\bullet)[1]$ are given by actual maps of complexes, namely using the differential $\Omega^p \to \Omega^{p+1}$ and the differential $\Omega^{n-p-1} \to \Omega^{n-p}$.

Consider the distinguished triangles associated gotten from these distinguished triangles by applying Rf_*

$$\begin{array}{ccc}
Rf_*\sigma_{\leq p}\Omega^\bullet & & Rf_*\sigma_{\geq n-p}\Omega^\bullet \\
\downarrow{\scriptstyle a} & & \uparrow{\scriptstyle a'} \\
Rf_*\Omega^{p+1}[-p-1] & & Rf_*\Omega^{n-p-1}[-n+p+1] \\
\downarrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} \\
Rf_*\sigma_{\leq p+1}\Omega^\bullet & & Rf_*\sigma_{\geq n-p-1}\Omega^\bullet \\
\downarrow{\scriptstyle c} & & \uparrow{\scriptstyle c'} \\
Rf_*\sigma_{\leq p}\Omega^\bullet & & Rf_*\sigma_{\geq n-p}\Omega^\bullet \\
\downarrow{\scriptstyle d} & & \uparrow{\scriptstyle d'} \\
Rf_*\Omega^{p+1}[-p-1] & & Rf_*\Omega^{n-p-1}[-n+p+1]
\end{array}$$

We will show below that the pairs (a, a'), (b, b'), (c, c'), and (d, d') are compatible with the given pairings. This means we obtain a map from the distinguished triangle on the left to the distuiguished triangle obtained by applying $R\mathcal{H}om(-, \mathcal{O}_S)$ to the distinguished triangle on the right. By induction and Lemma 50.20.1 we know that the pairings constructed above between the complexes on the first, second, fourth, and fifth rows are perfect, i.e., determine isomorphisms after taking duals. By Derived Categories, Lemma 13.4.3 we conclude the pairing between the complexes in the middle row is perfect as desired.

Let $e : K \to K'$ and $e' : M' \to M$ be maps of objects of $D(\mathcal{O}_S)$ and let $K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M \to \mathcal{O}_S$ and $K' \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' \to \mathcal{O}_S$ be pairings. Then we say these pairings are compatible if the diagram

$$\begin{array}{ccc}
K' \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' & \xleftarrow[e\otimes 1]{} & K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M' \\
\downarrow & & \downarrow{\scriptstyle 1\otimes e'} \\
\mathcal{O}_S & \longleftarrow & K \otimes^{\mathbf{L}}_{\mathcal{O}_S} M
\end{array}$$

commutes. This indeed means that the diagram

$$\begin{array}{ccc}
K & \longrightarrow & R\mathcal{H}om(M, \mathcal{O}_S) \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle R\mathcal{H}om(e', -)} \\
K' & \longrightarrow & R\mathcal{H}om(M', \mathcal{O}_S)
\end{array}$$

commutes and hence is sufficient for our purposes.

Let us prove this for the pair (c, c'). Here we observe simply that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \\
\downarrow{\scriptstyle \gamma_p} & & \downarrow \\
\Omega^\bullet & \xleftarrow{\gamma_{p+1}} & \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet)
\end{array}$$

By functoriality of the cup product we obtain commutativity of the desired diagram. Similarly for the pair (b, b') we use the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\sigma_{\leq p+1}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet) & \longleftarrow & \mathrm{Tot}(\Omega^{p+1}[-p-1] \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p-1}\Omega^\bullet) \\ \downarrow{\scriptstyle \gamma_{p+1}} & & \downarrow \\ \Omega^\bullet & \xleftarrow{\wedge} & \Omega^{p+1}[-p-1] \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p+1] \end{array}$$

For the pairs (d, d') and (a, a') we use the commutative diagram

$$\begin{array}{ccc} \Omega^{p+1}[-p] \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p] & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \Omega^{n-p-1}[-n+p]) \\ \downarrow & & \downarrow \\ \Omega^\bullet & \longleftarrow & \mathrm{Tot}(\sigma_{\leq p}\Omega^\bullet \otimes_{f^{-1}\mathcal{O}_S} \sigma_{\geq n-p}\Omega^\bullet) \end{array}$$

We omit the argument showing the uniqueness of t up to precomposing by multiplication by a unit in $H^0(X, \mathcal{O}_X)$. □

50.25. Other chapters

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CHAPTER 51

Local Cohomology

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51.1. Introduction

0DWP This chapter continues the study of local cohomology. A reference is [**Gro68**]. The definition of local cohomology can be found in Dualizing Complexes, Section 47.9. For Noetherian rings taking local cohomology is the same as deriving a suitable torsion functor as is shown in Dualizing Complexes, Section 47.10. The relationship with depth can be found in Dualizing Complexes, Section 47.11.

We discuss finiteness properties of local cohomology leading to a proof of a fairly general version of Grothendieck's finiteness theorem, see Theorem 51.11.6 and Lemma 51.12.1 (higher direct images of coherent modules under open immersions). Our methods incorporate a few very slick arguments the reader can find in papers of Faltings, see [**Fal78b**] and [**Fal81**].

As applications we offer a discussion of Hartshorne-Lichtenbaum vanishing. We also discuss the action of Frobenius and of differential operators on local cohomology.

51.2. Generalities

0DWQ The following lemma tells us that the functor $R\Gamma_Z$ is related to cohomology with supports.

0A6T **Lemma 51.2.1.** *Let A be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(A)$. For $K \in D(A)$ corresponding to $\widetilde{K} \in D_{QCoh}(\mathcal{O}_X)$ via Derived Categories of Schemes, Lemma 36.3.5 there is a functorial isomorphism*

$$R\Gamma_Z(K) = R\Gamma_Z(X, \widetilde{K})$$

where on the left we have Dualizing Complexes, Equation (47.9.0.1) and on the right we have the functor of Cohomology, Section 20.34.

Proof. By Cohomology, Lemma 20.34.5 there exists a distinguished triangle

$$R\Gamma_Z(X, \widetilde{K}) \to R\Gamma(X, \widetilde{K}) \to R\Gamma(U, \widetilde{K}) \to R\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $R\Gamma(X, \widetilde{K}) = K$ by Derived Categories of Schemes, Lemma 36.3.5. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. By Derived Categories of Schemes, Lemma 36.9.4 the alternating Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of A-modules representing K. Working through the definitions we find

$$R\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_A (\prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = R\Gamma(X, \widetilde{K^\bullet}) \to R\Gamma(U, \widetilde{K}^\bullet)$ is induced by the diagonal map from A into $\prod A_{f_i}$. Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right)$$

By Dualizing Complexes, Lemma 47.9.1 this complex computes $R\Gamma_Z(K)$ and we see the lemma holds. □

0DWR **Lemma 51.2.2.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $X = \text{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then*

(1) *there exists an A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
(2) *given M there is an exact sequence*

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
(3) *we may take $M = H^0(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. The existence of M follows from Properties, Lemma 28.22.1 and the fact that quasi-coherent sheaves on X correspond to A-modules (Schemes, Lemma 26.7.5). Then we look at the distinguished triangle

$$R\Gamma_Z(X, \widetilde{M}) \to R\Gamma(X, \widetilde{M}) \to R\Gamma(U, \widetilde{M}|_U) \to R\Gamma_Z(X, \widetilde{M})[1]$$

of Cohomology, Lemma 20.34.5. Since X is affine we have $R\Gamma(X, \widetilde{M}) = M$ by Cohomology of Schemes, Lemma 30.2.2. By our choice of M we have $\mathcal{F} = \widetilde{M}|_U$ and hence this produces an exact sequence

$$0 \to H^0_Z(X, \widetilde{M}) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(X, \widetilde{M}) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(X, \widetilde{M})$ for $p \geq 1$. By Lemma 51.2.1 we have $H^i_Z(M) = H^i_Z(X, \widetilde{M})$ for all i. Thus (1) and (2) do hold. Finally, setting $M' = H^0(U, \mathcal{F})$ we see that the kernel and cokernel of $M \to M'$ are I-power torsion. Therefore $\widetilde{M}|_U \to \widetilde{M'}|_U$ is an isomorphism and we can indeed use M' as predicted in (3). It goes without saying that we obtain zero for both $H^0_Z(M')$ and $H^0_Z(M')$. □

0DWS **Lemma 51.2.3.** *Let $I, J \subset A$ be finitely generated ideals of a ring A. If M is an I-power torsion module, then the canonical map*

$$H^i_{V(I)\cap V(J)}(M) \to H^i_{V(J)}(M)$$

is an isomorphism for all i.

Proof. Use the spectral sequence of Dualizing Complexes, Lemma 47.9.6 to reduce to the statement $R\Gamma_I(M) = M$ which is immediate from the construction of local cohomology in Dualizing Complexes, Section 47.9. □

0DWT **Lemma 51.2.4.** *Let $S \subset A$ be a multiplicative set of a ring A. Let M be an A-module with $S^{-1}M = 0$. Then $\text{colim}_{f\in S} H^0_{V(f)}(M) = M$ and $\text{colim}_{f\in S} H^1_{V(f)}(M) = 0$.*

Proof. The statement on H^0 follows directly from the definitions. To see the statement on H^1 observe that $R\Gamma_{V(f)}$ and $H^1_{V(f)}$ commute with colimits. Hence we may assume M is annihilated by some $f \in S$. Then $H^1_{V(ff')}(M) = 0$ for all $f' \in S$ (for example by Lemma 51.2.3). □

0DWU **Lemma 51.2.5.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Let $\mathfrak{p}$ be a prime ideal. Let M be an A-module. Let $i \geq 0$ be an integer and consider the map*

$$\Psi : \operatorname{colim}_{f \in A, f \notin \mathfrak{p}} H^i_{V((I,f))}(M) \longrightarrow H^i_{V(I)}(M)$$

Then

(1) $\operatorname{Im}(\Psi)$ *is the set of elements which map to zero in $H^i_{V(I)}(M)_\mathfrak{p}$,*
(2) *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is injective,*
(3) *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = H^i_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is an isomorphism.*

Proof. For $f \in A$, $f \notin \mathfrak{p}$ the spectral sequence of Dualizing Complexes, Lemma 47.9.6 degenerates to give short exact sequences

$$0 \to H^1_{V(f)}(H^{i-1}_{V(I)}(M)) \to H^i_{V((I,f))}(M) \to H^0_{V(f)}(H^i_{V(I)}(M)) \to 0$$

This proves (1) and part (2) follows from this and Lemma 51.2.4. Part (3) is a formal consequence. □

0DWV **Lemma 51.2.6.** *Let $I \subset I' \subset A$ be finitely generated ideals of a Noetherian ring A. Let M be an A-module. Let $i \geq 0$ be an integer. Consider the map*

$$\Psi : H^i_{V(I')}(M) \to H^i_{V(I)}(M)$$

The following are true:

(1) *if $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(I) \setminus V(I')$, then Ψ is surjective,*
(2) *if $H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(I) \setminus V(I')$, then Ψ is injective,*
(3) *if $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(I) \setminus V(I')$, then Ψ is an isomorphism.*

Proof. Proof of (1). Let $\xi \in H^i_{V(I)}(M)$. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ is the image of some $\xi'' \in H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^i_{V(I'')}(M)_\mathfrak{p} = H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 51.2.5 we can increase I'' which contradicts maximality.

Proof of (2). Let $\xi' \in H^i_{V(I')}(M)$ be in the kernel of Ψ. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ' maps to zero in $H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, then choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^{i-1}_{V(I'')}(M)_\mathfrak{p} = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 51.2.5 we can increase I'' which contradicts maximality.

Part (3) is formal from parts (1) and (2). □

51.3. Hartshorne's connectedness lemma

0FIV The title of this section refers to the following result.

0BLR **Lemma 51.3.1.** *Let A be a Noetherian local ring of depth ≥ 2. Then the punctured spectra of A, A^h, and A^{sh} are connected.* [**Har62**, Proposition 2.1]

Proof. Let U be the punctured spectrum of A. If U is disconnected then we see that $\Gamma(U, \mathcal{O}_U)$ has a nontrivial idempotent. But A, being local, does not have a nontrivial idempotent. Hence $A \to \Gamma(U, \mathcal{O}_U)$ is not an isomorphism. By Lemma 51.2.2 we conclude that either $H^0_{\mathfrak{m}}(A)$ or $H^1_{\mathfrak{m}}(A)$ is nonzero. Thus $\text{depth}(A) \leq 1$ by Dualizing Complexes, Lemma 47.11.1. To see the result for A^h and A^{sh} use More on Algebra, Lemma 15.45.8. □

0FIW **Lemma 51.3.2.** *Let A be a Noetherian local ring which is catenary and (S_2). Then $\text{Spec}(A)$ is equidimensional.* [**DG67**, IV Corollary 5.10.9]

Proof. Set $X = \text{Spec}(A)$. Say $d = \dim(A) = \dim(X)$. Inside X consider the union X_1 of the irreducible components of dimension d and the union X_2 of the irreducible components of dimension $< d$. Of course $X = X_1 \cup X_2$. If $X_2 = \emptyset$, then the lemma holds. If not, then $Z = X_1 \cap X_2$ is a nonempty closed subset of X because it contains at least the closed point of X. Hence we can choose a generic point $z \in Z$ of an irreducible component of Z. Recall that the spectrum of $\mathcal{O}_{Z,z}$ is the set of points of X specializing to z. Since z is both contained in an irreducible component of dimension d and in an irreducible component of dimension $< d$ we obtain nontrivial specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ such that the closures of x_1 and x_2 have different dimensions. Since X is catenary, this can only happen if at least one of the specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ is not immediate! Thus $\dim(\mathcal{O}_{Z,z}) \geq 2$. Therefore $\text{depth}(\mathcal{O}_{Z,z}) \geq 2$ because A is (S_2). However, the punctured spectrum U of $\mathcal{O}_{Z,z}$ is disconnected because the closed subsets $U \cap X_1$ and $U \cap X_2$ are disjoint (by our choice of z) and cover U. This is a contradiction with Lemma 51.3.1 and the proof is complete. □

51.4. Cohomological dimension

0DX6 A quick section about cohomological dimension.

0DX7 **Lemma 51.4.1.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Set $Y = V(I) \subset X = \text{Spec}(A)$. Let $d \geq -1$ be an integer. The following are equivalent*

1. *$H^i_Y(A) = 0$ for $i > d$,*
2. *$H^i_Y(M) = 0$ for $i > d$ for every A-module M, and*
3. *if $d = -1$, then $Y = \emptyset$, if $d = 0$, then Y is open and closed in X, and if $d > 0$ then $H^i(X \setminus Y, \mathcal{F}) = 0$ for $i \geq d$ for every quasi-coherent $\mathcal{O}_{X \setminus Y}$-module $\mathcal{F}$.*

Proof. Observe that $R\Gamma_Y(-)$ has finite cohomological dimension by Dualizing Complexes, Lemma 47.9.1 for example. Hence there exists an integer i_0 such that $H^i_Y(M) = 0$ for all A-modules M and $i \geq i_0$.

Let us prove that (1) and (2) are equivalent. It is immediate that (2) implies (1). Assume (1). By descending induction on $i > d$ we will show that $H^i_Y(M) = 0$ for all A-modules M. For $i \geq i_0$ we have seen this above. To do the induction step, let $i_0 > i > d$. Choose any A-module M and fit it into a short exact sequence

$0 \to N \to F \to M \to 0$ where F is a free A-module. Since $R\Gamma_Y$ is a right adjoint, we see that $H^i_Y(-)$ commutes with direct sums. Hence $H^i_Y(F) = 0$ as $i > d$ by assumption (1). Then we see that $H^i_Y(M) = H^{i+1}_Y(N) = 0$ as desired.

Assume $d = -1$ and (2) holds. Then $0 = H^0_Y(A/I) = A/I \Rightarrow A = I \Rightarrow Y = \emptyset$. Thus (3) holds. We omit the proof of the converse.

Assume $d = 0$ and (2) holds. Set $J = H^0_I(A) = \{x \in A \mid I^n x = 0 \text{ for some } n > 0\}$. Then

$$H^1_Y(A) = \text{Coker}(A \to \Gamma(X \backslash Y, \mathcal{O}_{X \backslash Y})) \quad \text{and} \quad H^1_Y(I) = \text{Coker}(I \to \Gamma(X \backslash Y, \mathcal{O}_{X \backslash Y}))$$

and the kernel of the first map is equal to J. See Lemma 51.2.2. We conclude from (2) that $I(A/J) = A/J$. Thus we may pick $f \in I$ mapping to 1 in A/J. Then $1 - f \in J$ so $I^n(1 - f) = 0$ for some $n > 0$. Hence $f^n = f^{n+1}$. Then $e = f^n \in I$ is an idempotent. Consider the complementary idempotent $e' = 1 - f^n \in J$. For any element $g \in I$ we have $g^m e' = 0$ for some $m > 0$. Thus I is contained in the radical of ideal $(e) \subset I$. This means $Y = V(I) = V(e)$ is open and closed in X as predicted in (3). Conversely, if $Y = V(I)$ is open and closed, then the functor $H^0_Y(-)$ is exact and has vanshing higher derived functors.

If $d > 0$, then we see immediately from Lemma 51.2.2 that (2) is equivalent to (3). □

0DX8 **Definition 51.4.2.** Let $I \subset A$ be a finitely generated ideal of a ring A. The smallest integer $d \geq -1$ satisfying the equivalent conditions of Lemma 51.4.1 is called the *cohomological dimension of I in A* and is denoted $\text{cd}(A, I)$.

Thus we have $\text{cd}(A, I) = -1$ if $I = A$ and $\text{cd}(A, I) = 0$ if I is locally nilpotent or generated by an idempotent. Observe that $\text{cd}(A, I)$ exists by the following lemma.

0DX9 **Lemma 51.4.3.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Then*

(1) *$cd(A, I)$ is at most equal to the number of generators of I,*
(2) *$cd(A, I) \leq r$ if there exist $f_1, \ldots, f_r \in A$ such that $V(f_1, \ldots, f_r) = V(I)$,*
(3) *$cd(A, I) \leq c$ if $\text{Spec}(A) \setminus V(I)$ can be covered by c affine opens.*

Proof. The explicit description for $R\Gamma_Y(-)$ given in Dualizing Complexes, Lemma 47.9.1 shows that (1) is true. We can deduce (2) from (1) using the fact that $R\Gamma_Z$ depends only on the closed subset Z and not on the choice of the finitely generated ideal $I \subset A$ with $V(I) = Z$. This follows either from the construction of local cohomology in Dualizing Complexes, Section 47.9 combined with More on Algebra, Lemma 15.89.6 or it follows from Lemma 51.2.1. To see (3) we use Lemma 51.4.1 and the vanishing result of Cohomology of Schemes, Lemma 30.4.2. □

0ECP **Lemma 51.4.4.** *Let $I, J \subset A$ be finitely generated ideals of a ring A. Then $cd(A, I + J) \leq cd(A, I) + cd(A, J)$.*

Proof. Use the definition and Dualizing Complexes, Lemma 47.9.6. □

0DXA **Lemma 51.4.5.** *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Then $cd(B, IB) \leq cd(A, I)$. If $A \to B$ is faithfully flat, then equality holds.*

Proof. Use the definition and Dualizing Complexes, Lemma 47.9.3. □

0DXB **Lemma 51.4.6.** *Let $I \subset A$ be a finitely generated ideal of a ring A. Then $cd(A, I) = \max cd(A_\mathfrak{p}, I_\mathfrak{p})$.*

Proof. Let $Y = V(I)$ and $Y' = V(I_\mathfrak{p}) \subset \mathrm{Spec}(A_\mathfrak{p})$. Recall that $R\Gamma_Y(A) \otimes_A A_\mathfrak{p} = R\Gamma_{Y'}(A_\mathfrak{p})$ by Dualizing Complexes, Lemma 47.9.3. Thus we conclude by Algebra, Lemma 10.23.1. □

0DXC **Lemma 51.4.7.** *Let $I \subset A$ be a finitely generated ideal of a ring A. If M is a finite A-module, then $H^i_{V(I)}(M) = 0$ for $i > \dim(\mathit{Supp}(M))$. In particular, we have $\mathit{cd}(A, I) \leq \dim(A)$.*

Proof. We first prove the second statement. Recall that $\dim(A)$ denotes the Krull dimension. By Lemma 51.4.6 we may assume A is local. If $V(I) = \emptyset$, then the result is true. If $V(I) \neq \emptyset$, then $\dim(\mathrm{Spec}(A) \setminus V(I)) < \dim(A)$ because the closed point is missing. Observe that $U = \mathrm{Spec}(A) \setminus V(I)$ is a quasi-compact open of the spectral space $\mathrm{Spec}(A)$, hence a spectral space itself. See Algebra, Lemma 10.26.2 and Topology, Lemma 5.23.5. Thus Cohomology, Proposition 20.22.4 implies $H^i(U, \mathcal{F}) = 0$ for $i \geq \dim(A)$ which implies what we want by Lemma 51.4.1. In the Noetherian case the reader may use Grothendieck's Cohomology, Proposition 20.20.7.

We will deduce the first statement from the second. Let $\mathfrak{a}$ be the annihilator of the finite A-module M. Set $B = A/\mathfrak{a}$. Recall that $\mathrm{Spec}(B) = \mathrm{Supp}(M)$, see Algebra, Lemma 10.40.5. Set $J = IB$. Then M is a B-module and $H^i_{V(I)}(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 47.9.2. Since $\mathrm{cd}(B, J) \leq \dim(B) = \dim(\mathrm{Supp}(M))$ by the first part we conclude. □

0DXD **Lemma 51.4.8.** *Let $I \subset A$ be a finitely generated ideal of a ring A. If $\mathit{cd}(A, I) = 1$ then $\mathrm{Spec}(A) \setminus V(I)$ is nonempty affine.*

Proof. This follows from Lemma 51.4.1 and Cohomology of Schemes, Lemma 30.3.1. □

0DXE **Lemma 51.4.9.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then $H^d_\mathfrak{m}(A)$ is nonzero and $\mathit{cd}(A, \mathfrak{m}) = d$.*

Proof. By one of the characterizations of dimension, there exists an ideal of definition for A generated by d elements, see Algebra, Proposition 10.60.9. Hence $\mathrm{cd}(A, \mathfrak{m}) \leq d$ by Lemma 51.4.3. Thus $H^d_\mathfrak{m}(A)$ is nonzero if and only if $\mathrm{cd}(A, \mathfrak{m}) = d$ if and only if $\mathrm{cd}(A, \mathfrak{m}) \geq d$.

Let $A \to A^\wedge$ be the map from A to its completion. Observe that $A^\wedge$ is a Noetherian local ring of the same dimension as A with maximal ideal $\mathfrak{m}A^\wedge$. See Algebra, Lemmas 10.97.6, 10.97.4, and 10.97.3 and More on Algebra, Lemma 15.43.1. By Lemma 51.4.5 it suffices to prove the lemma for $A^\wedge$.

By the previous paragraph we may assume that A is a complete local ring. Then A has a normalized dualizing complex $\omega_A^\bullet$ (Dualizing Complexes, Lemma 47.22.4). The local duality theorem (in the form of Dualizing Complexes, Lemma 47.18.4) tells us $H^d_\mathfrak{m}(A)$ is Matlis dual to $\mathrm{Ext}^{-d}(A, \omega_A^\bullet) = H^{-d}(\omega_A^\bullet)$ which is nonzero for example by Dualizing Complexes, Lemma 47.16.11. □

0DXF **Lemma 51.4.10.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be a proper ideal. Let $\mathfrak{p} \subset A$ be a prime ideal such that $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$. Then $\dim(A/\mathfrak{p}) \leq \mathit{cd}(A, I)$.*

Proof. By Lemma 51.4.5 we have $\mathrm{cd}(A, I) \geq \mathrm{cd}(A/\mathfrak{p}, I(A/\mathfrak{p}))$. Since $V(I) \cap V(\mathfrak{p}) = \{\mathfrak{m}\}$ we have $\mathrm{cd}(A/\mathfrak{p}, I(A/\mathfrak{p})) = \mathrm{cd}(A/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$. By Lemma 51.4.9 this is equal to $\dim(A/\mathfrak{p})$. □

0EHU **Lemma 51.4.11.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $b : X' \to X = \mathrm{Spec}(A)$ be the blowing up of I. If the fibres of b have dimension $\leq d - 1$, then $cd(A, I) \leq d$.*

Proof. Set $U = X \setminus V(I)$. Denote $j : U \to X'$ the canonical open immersion, see Divisors, Section 31.32. Since the exceptional divisor is an effective Cartier divisor (Divisors, Lemma 31.32.4) we see that j is affine, see Divisors, Lemma 31.13.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then $R^p j_* \mathcal{F} = 0$ for $p > 0$, see Cohomology of Schemes, Lemma 30.2.3. On the other hand, we have $R^q b_*(j_* \mathcal{F}) = 0$ for $q \geq d$ by Limits, Lemma 32.19.2. Thus by the Leray spectral sequence (Cohomology, Lemma 20.13.8) we conclude that $R^n(b \circ j)_* \mathcal{F} = 0$ for $n \geq d$. Thus $H^n(U, \mathcal{F}) = 0$ for $n \geq d$ (by Cohomology, Lemma 20.13.6). This means that $\mathrm{cd}(A, I) \leq d$ by definition. □

51.5. More general supports

0EEY Let A be a Noetherian ring. Let M be an A-module. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization (Topology, Definition 5.19.1). Let us define

$$H^0_T(M) = \mathrm{colim}_{Z \subset T} H^0_Z(M)$$

where the colimit is over the directed partially ordered set of closed subsets Z of $\mathrm{Spec}(A)$ contained in T[1]. In other words, an element m of M is in $H^0_T(M) \subset M$ if and only if the support $V(\mathrm{Ann}_R(m))$ of m is contained in T.

0EEZ **Lemma 51.5.1.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. For an A-module M the following are equivalent*

(1) *$H^0_T(M) = M$, and*
(2) *$Supp(M) \subset T$.*

The category of such A-modules is a Serre subcategory of the category A-modules closed under direct sums.

Proof. The equivalence holds because the support of an element of M is contained in the support of M and conversely the support of M is the union of the supports of its elements. The category of these modules is a Serre subcategory (Homology, Definition 12.10.1) of Mod_A by Algebra, Lemma 10.40.9. We omit the proof of the statement on direct sums. □

Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let us denote $\mathrm{Mod}_{A,T} \subset \mathrm{Mod}_A$ the Serre subcategory described in Lemma 51.5.1. Let us denote $D_T(A) \subset D(A)$ the strictly full saturated triangulated subcategory of $D(A)$ (Derived Categories, Lemma 13.17.1) consisting of complexes of A-modules whose cohomology modules are in $\mathrm{Mod}_{A,T}$. We obtain functors

$$D(\mathrm{Mod}_{A,T}) \to D_T(A) \to D(A)$$

See discussion in Derived Categories, Section 13.17. Denote $RH^0_T : D(A) \to D(\mathrm{Mod}_{A,T})$ the right derived extension of H^0_T. We will denote

$$R\Gamma_T : D^+(A) \to D^+_T(A),$$

[1] Since T is stable under specialization we have $T = \bigcup_{Z \subset T} Z$, see Topology, Lemma 5.19.3.

the composition of $RH^0_T : D^+(A) \to D^+(\mathrm{Mod}_{A,T})$ with $D^+(\mathrm{Mod}_{A,T}) \to D^+_T(A)$. If the dimension of A is finite[2], then we will denote

$$R\Gamma_T : D(A) \to D_T(A)$$

the composition of RH^0_T with $D(\mathrm{Mod}_{A,T}) \to D_T(A)$.

0EF0 **Lemma 51.5.2.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. The functor RH^0_T is the right adjoint to the functor $D(Mod_{A,T}) \to D(A)$.*

Proof. This follows from the fact that the functor $H^0_T(-)$ is the right adjoint to the inclusion functor $\mathrm{Mod}_{A,T} \to \mathrm{Mod}_A$, see Derived Categories, Lemma 13.30.3. □

0EF1 **Lemma 51.5.3.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. For any object K of $D(A)$ we have*

$$H^i(RH^0_T(K)) = \mathrm{colim}_{Z \subset T \text{ closed}} H^i_Z(K)$$

Proof. Let $J^\bullet$ be a K-injective complex representing K. By definition RH^0_T is represented by the complex

$$H^0_T(J^\bullet) = \mathrm{colim}\, H^0_Z(J^\bullet)$$

where the equality follows from our definition of H^0_T. Since filtered colimits are exact the cohomology of this complex in degree i is $\mathrm{colim}\, H^i(H^0_Z(J^\bullet)) = \mathrm{colim}\, H^i_Z(K)$ as desired. □

0EF2 **Lemma 51.5.4.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. The functor $D^+(Mod_{A,T}) \to D^+_T(A)$ is an equivalence.*

Proof. Let M be an object of $\mathrm{Mod}_{A,T}$. Choose an embedding $M \to J$ into an injective A-module. By Dualizing Complexes, Proposition 47.5.9 the module J is a direct sum of injective hulls of residue fields. Let E be an injective hull of the residue field of $\mathfrak{p}$. Since E is $\mathfrak{p}$-power torsion we see that $H^0_T(E) = 0$ if $\mathfrak{p} \notin T$ and $H^0_T(E) = E$ if $\mathfrak{p} \in T$. Thus $H^0_T(J)$ is injective as a direct sum of injective hulls (by the proposition) and we have an embedding $M \to H^0_T(J)$. Thus every object M of $\mathrm{Mod}_{A,T}$ has an injective resolution $M \to J^\bullet$ with J^n also in $\mathrm{Mod}_{A,T}$. It follows that $RH^0_T(M) = M$.

Next, suppose that $K \in D^+_T(A)$. Then the spectral sequence

$$R^q H^0_T(H^p(K)) \Rightarrow R^{p+q} H^0_T(K)$$

(Derived Categories, Lemma 13.21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $RH^0_T(K) \to K$ is an isomorphism. Thus the functor $D^+(\mathrm{Mod}_{A,T}) \to D^+_T(A)$ is an equivalence with quasi-inverse given by RH^0_T. □

0EF3 **Lemma 51.5.5.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. If $\dim(A) < \infty$, then functor $D(Mod_{A,T}) \to D_T(A)$ is an equivalence.*

[2] If $\dim(A) = \infty$ the construction may have unexpected properties on unbounded complexes.

Proof. Say $\dim(A) = d$. Then we see that $H^i_Z(M) = 0$ for $i > d$ for every closed subset Z of $\mathrm{Spec}(A)$, see Lemma 51.4.7. By Lemma 51.5.3 we find that H^0_T has bounded cohomological dimension.

Let $K \in D_T(A)$. We claim that $RH^0_T(K) \to K$ is an isomorphism. We know this is true when K is bounded below, see Lemma 51.5.4. However, since H^0_T has bounded cohomological dimension, we see that the ith cohomology of $RH^0_T(K)$ only depends on $\tau_{\geq -d+i}K$ and we conclude. Thus $D(\mathrm{Mod}_{A,T}) \to D_T(A)$ is an equivalence with quasi-inverse RH^0_T. □

0EF4 **Remark 51.5.6.** Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. The upshot of the discussion above is that $R\Gamma_T : D^+(A) \to D^+_T(A)$ is the right adjoint to the inclusion functor $D^+_T(A) \to D^+(A)$. If $\dim(A) < \infty$, then $R\Gamma_T : D(A) \to D_T(A)$ is the right adjoint to the inclusion functor $D_T(A) \to D(A)$. In both cases we have

$$H^i_T(K) = H^i(R\Gamma_T(K)) = R^iH^0_T(K) = \mathrm{colim}_{Z \subset T \text{ closed}} H^i_Z(K)$$

This follows by combining Lemmas 51.5.2, 51.5.3, 51.5.4, and 51.5.5.

0EF5 **Lemma 51.5.7.** *Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $T' \subset \mathrm{Spec}(B)$ be the inverse image of T. Then the canonical map*

$$R\Gamma_T(K) \otimes^{\mathbf{L}}_A B \longrightarrow R\Gamma_{T'}(K \otimes^{\mathbf{L}}_A B)$$

is an isomorphism for $K \in D^+(A)$. If A and B have finite dimension, then this is true for $K \in D(A)$.

Proof. From the map $R\Gamma_T(K) \to K$ we get a map $R\Gamma_T(K) \otimes^{\mathbf{L}}_A B \to K \otimes^{\mathbf{L}}_A B$. The cohomology modules of $R\Gamma_T(K) \otimes^{\mathbf{L}}_A B$ are supported on T' and hence we get the arrow of the lemma. This arrow is an isomorphism if T is a closed subset of $\mathrm{Spec}(A)$ by Dualizing Complexes, Lemma 47.9.3. Recall that $H^i_T(K)$ is the colimit of $H^i_Z(K)$ where Z runs over the (directed set of) closed subsets of T, see Lemma 51.5.3. Correspondingly $H^i_{T'}(K \otimes^{\mathbf{L}}_A B) = \mathrm{colim}\, H^i_{Z'}(K \otimes^{\mathbf{L}}_A B)$ where Z' is the inverse image of Z. Thus the result because $\otimes_A B$ commutes with filtered colimits and there are no higher Tors. □

0EF6 **Lemma 51.5.8.** *Let A be a ring and let $T, T' \subset \mathrm{Spec}(A)$ subsets stable under specialization. For $K \in D^+(A)$ there is a spectral sequence*

$$E_2^{p,q} = H^p_T(H^p_{T'}(K)) \Rightarrow H^{p+q}_{T \cap T'}(K)$$

as in Derived Categories, Lemma 13.22.2.

Proof. Let E be an object of $D_{T \cap T'}(A)$. Then we have

$$\mathrm{Hom}(E, R\Gamma_T(R\Gamma_{T'}(K))) = \mathrm{Hom}(E, R\Gamma_{T'}(K)) = \mathrm{Hom}(E, K)$$

The first equality by the adjointness property of $R\Gamma_T$ and the second by the adjointness property of $R\Gamma_{T'}$. On the other hand, if $J^\bullet$ is a bounded below complex of injectives representing K, then $H^0_{T'}(J^\bullet)$ is a complex of injective A-modules representing $R\Gamma_{T'}(K)$ and hence $H^0_T(H^0_{T'}(J^\bullet))$ is a complex representing $R\Gamma_T(R\Gamma_{T'}(K))$. Thus $R\Gamma_T(R\Gamma_{T'}(K))$ is an object of $D^+_{T \cap T'}(A)$. Combining these two facts we find that $R\Gamma_{T \cap T'} = R\Gamma_T \circ R\Gamma_{T'}$. This produces the spectral sequence by the lemma referenced in the statement. □

0EF7 **Lemma 51.5.9.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Assume A has finite dimension. Then*

$$R\Gamma_T(K) = R\Gamma_T(A) \otimes^{\mathbf{L}}_A K$$

for $K \in D(A)$. For $K, L \in D(A)$ we have

$$R\Gamma_T(K \otimes^{\mathbf{L}}_A L) = K \otimes^{\mathbf{L}}_A R\Gamma_T(L) = R\Gamma_T(K) \otimes^{\mathbf{L}}_A L = R\Gamma_T(K) \otimes^{\mathbf{L}}_A R\Gamma_T(L)$$

If K or L is in $D_T(A)$ then so is $K \otimes^{\mathbf{L}}_A L$.

Proof. By construction we may represent $R\Gamma_T(A)$ by a complex $J^\bullet$ in $\mathrm{Mod}_{A,T}$. Thus if we represent K by a K-flat complex $K^\bullet$ then we see that $R\Gamma_T(A) \otimes^{\mathbf{L}}_A K$ is represented by the complex $\mathrm{Tot}(J^\bullet \otimes_A K^\bullet)$ in $\mathrm{Mod}_{A,T}$. Using the map $R\Gamma_T(A) \to A$ we obtain a map $R\Gamma_T(A) \otimes^{\mathbf{L}}_A K \to K$. Thus by the adjointness property of $R\Gamma_T$ we obtain a canonical map

$$R\Gamma_T(A) \otimes^{\mathbf{L}}_A K \longrightarrow R\Gamma_T(K)$$

factoring the just constructed map. Observe that $R\Gamma_T$ commutes with direct sums in $D(A)$ for example by Lemma 51.5.3, the fact that directed colimits commute with direct sums, and the fact that usual local cohomology commutes with direct sums (for example by Dualizing Complexes, Lemma 47.9.1). Thus by More on Algebra, Remark 15.59.11 it suffices to check the map is an isomorphism for $K = A[k]$ where $k \in \mathbf{Z}$. This is clear.

The final statements follow from the result we've just shown by transitivity of derived tensor products. □

51.6. Filtrations on local cohomology

0EHV Some tricks related to the spectral sequence of Lemma 51.5.8.

0EF8 **Lemma 51.6.1.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and for every A-module M there is an exact sequence*

$$0 \to \mathrm{colim}_{Z,f} H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak{p} \in T \setminus T'} H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$$

where the colimit is over closed subsets $Z \subset T$ and $f \in A$ with $V(f) \cap Z \subset T'$.

Proof. For every Z and f the spectral sequence of Dualizing Complexes, Lemma 47.9.6 degenerates to give short exact sequences

$$0 \to H^1_f(H^{i-1}_Z(M)) \to H^i_{Z \cap V(f)}(M) \to H^0_f(H^i_Z(M)) \to 0$$

We will use this without further mention below.

Let $\xi \in H^i_T(M)$ map to zero in the direct sum. Then we first write ξ as the image of some $\xi' \in H^i_Z(M)$ for some closed subset $Z \subset T$, see Lemma 51.5.3. Then ξ' maps to zero in $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$ for every $\mathfrak{p} \in Z$, $\mathfrak{p} \notin T'$. Since there are finitely many of these primes, we may choose $f \in A$ not contained in any of these such that f annihilates ξ'. Then ξ' is the image of some $\xi'' \in H^i_{Z'}(M)$ where $Z' = Z \cap V(f)$. By our choice of f we have $Z' \subset T'$ and we get exactness at the penultimate spot.

Let $\xi \in H^i_{T'}(M)$ map to zero in $H^i_T(M)$. Choose closed subsets $Z' \subset Z$ with $Z' \subset T'$ and $Z \subset T$ such that ξ comes from $\xi' \in H^i_{Z'}(M)$ and maps to zero in $H^i_Z(M)$. Then we can find $f \in A$ with $V(f) \cap Z = Z'$ and we conclude. □

0EF9 **Lemma 51.6.2.** *Let A be a Noetherian ring of finite dimension. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n\geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ the induced map*

$$H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \longrightarrow H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$$

is zero for $k \geq m'(m)$. Let $m'' : \mathbf{N} \to \mathbf{N}$ be the $2^{\dim(T)}$-fold self-composition of m'. Then the map $H^i_T(M_k) \to H^i_T(M_m)$ is zero for all $k \geq m''(m)$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n)$$

of inverse systems of abelian groups. Suppose that for every m there exists an integer $m'(m) \geq m$ such that

$$A_k \to A_m \quad \text{and} \quad C_k \to C_m$$

are zero for $k \geq m'(m)$. Then for $k \geq m'(m'(m))$ the map $B_k \to B_m$ is zero.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak{p}\in T\setminus T'} H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$$

by Lemma 51.6.1. Thus we conclude by the initial remark of the proof. □

0EFA **Lemma 51.6.3.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n\geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume the dimension of A is finite and that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ we have*

1. *$H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ is zero for $k \geq m'(m)$, and*
2. *$H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ has image $G(\mathfrak{p}, m)$ independent of $k \geq m'(m)$ and moreover $G(\mathfrak{p}, m)$ maps injectively into $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{0,\mathfrak{p}})$.*

Then there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m''(m) \geq m$ such that for $k \geq m''(m)$ the image of $H^i_T(M_k) \to H^i_T(M_m)$ maps injectively into $H^i_T(M_{m_0})$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n) \to (D_n)$$

of inverse systems of abelian groups. Suppose that there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that the maps

$$\mathrm{Im}(B_k \to B_m) \longrightarrow B_{m_0} \quad \text{and} \quad \mathrm{Im}(D_k \to D_m) \longrightarrow D_{m_0}$$

are injective for $k \geq m'(m)$ and $A_k \to A_m$ is zero for $k \geq m'(m)$. Then for $m \geq m'(m_0)$ and $k \geq m'(m'(m))$ the map

$$\mathrm{Im}(C_k \to C_m) \to C_{m'(m_0)}$$

is injective. Namely, let $c_0 \in C_m$ be the image of $c_3 \in C_k$ and say c_0 maps to zero in $C_{m'(m_0)}$. Picture

$$C_k \to C_{m'(m'(m))} \to C_{m'(m)} \to C_m \to C_{m'(m_0)}, \quad c_3 \mapsto c_2 \mapsto c_1 \mapsto c_0 \mapsto 0$$

We have to show $c_0 = 0$. The image d_3 of c_3 maps to zero in C_{m_0} and hence we see that the image $d_1 \in D_{m'(m)}$ is zero. Thus we can choose $b_1 \in B_{m'(m)}$ mapping to the image c_1. Since c_3 maps to zero in $C_{m'(m_0)}$ we find an element $a_{-1} \in A_{m'(m_0)}$ which maps to the image $b_{-1} \in B_{m'(m_0)}$ of b_1. Since a_{-1} maps to zero in A_{m_0} we conclude that b_1 maps to zero in B_{m_0}. Thus the image $b_0 \in B_m$ is zero which of course implies $c_0 = 0$ as desired.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$0 \to \mathrm{colim}_{Z,f} H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak{p} \in T \setminus T'} H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$$

by Lemma 51.6.1. Thus we conclude by the initial remark of the proof and the fact that we've seen the system of groups

$$\left\{\mathrm{colim}_{Z,f} H^1_f(H^{i-1}_Z(M_n))\right\}_{n \geq 0}$$

is pro-zero in Lemma 51.6.2; this uses that the function $m''(m)$ in that lemma for $H^{i-1}_Z(M)$ is independent of Z. □

51.7. Finiteness of local cohomology, I

0AW7 We will follow Faltings approach to finiteness of local cohomology modules, see [**Fal78b**] and [**Fal81**]. Here is a lemma which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

0AW8 **Lemma 51.7.1.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let M be a finite A-module. Let $n \geq 0$. The following are equivalent* [**Fal78b**, Lemma 3]

(1) *$H^i_T(M)$ is finite for $i \leq n$,*
(2) *there exists an ideal $J \subset A$ with $V(J) \subset T$ such that J annihilates $H^i_T(M)$ for $i \leq n$.*

If $T = V(I) = Z$ for an ideal $I \subset A$, then these are also equivalent to

(3) *there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(M)$ for $i \leq n$.*

Proof. We prove the equivalence of (1) and (2) by induction on n. For $n = 0$ we have $H^0_T(M) \subset M$ is finite. Hence (1) is true. Since $H^0_T(M) = \mathrm{colim}\, H^0_{V(J)}(M)$ with J as in (2) we see that (2) is true. Assume that $n > 0$.

Assume (1) is true. Recall that $H^i_J(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 47.10.1. Thus $H^i_T(M) = \mathrm{colim}\, H^i_J(M)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 51.5.3. Since $H^i_T(M)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(M) \to H^i_T(M)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Assume we have J as in (2). Let $N = H^0_T(M)$ and $M' = M/N$. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 51.5.6. Thus we find that $H^0_T(M') = 0$ and $H^i_T(M') = H^i_T(M)$ for $i > 0$. We conclude that we may replace M by M'. Thus we may assume that $H^0_T(M) = 0$. This means that the finite set of associated primes of M are not in T. By prime avoidance (Algebra, Lemma 10.15.2) we can find $f \in J$ not contained in any of the associated primes of M. Then the long exact local cohomology sequence associated to the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

turns into short exact sequences

$$0 \to H^i_T(M) \to H^i_T(M/fM) \to H^{i+1}_T(M) \to 0$$

for $i < n$. We conclude that J^2 annihilates $H^i_T(M/fM)$ for $i < n$. By induction hypothesis we see that $H^i_T(M/fM)$ is finite for $i < n$. Using the short exact sequence once more we see that $H^{i+1}_T(M)$ is finite for $i < n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. □

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

0AW9 **Lemma 51.7.2.** *Let A be a Noetherian ring, $I \subset A$ an ideal, M a finite A-module, and $n \geq 0$ an integer. Let $Z = V(I)$. The following are equivalent*

(1) *the modules $H^i_Z(M)$ are finite for $i \leq n$, and*
(2) *for all $\mathfrak{p} \in \mathrm{Spec}(A)$ the modules $H^i_Z(M)_\mathfrak{p}$, $i \leq n$ are finite $A_\mathfrak{p}$-modules.*

This is a special case of [**Fal81**, Satz 1].

Proof. The implication (1) ⇒ (2) is immediate. We prove the converse by induction on n. The case $n = 0$ is clear because both (1) and (2) are always true in that case.

Assume $n > 0$ and that (2) is true. Let $N = H^0_Z(M)$ and $M' = M/N$. By Dualizing Complexes, Lemma 47.11.6 we may replace M by M'. Thus we may assume that $H^0_Z(M) = 0$. This means that $\mathrm{depth}_I(M) > 0$ (Dualizing Complexes, Lemma 47.11.1). Pick $f \in I$ a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

which produces a long exact sequence

$$0 \to H^0_Z(M/fM) \to H^1_Z(M) \to H^1_Z(M) \to H^1_Z(M/fM) \to H^2_Z(M) \to \ldots$$

and similarly after localization. Thus assumption (2) implies that the modules $H^i_Z(M/fM)_\mathfrak{p}$ are finite for $i < n$. Hence by induction assumption $H^i_Z(M/fM)$ are finite for $i < n$.

Let $\mathfrak{p}$ be a prime of A which is associated to $H^i_Z(M)$ for some $i \leq n$. Say $\mathfrak{p}$ is the annihilator of the element $x \in H^i_Z(M)$. Then $\mathfrak{p} \in Z$, hence $f \in \mathfrak{p}$. Thus $fx = 0$ and hence x comes from an element of $H^{i-1}_Z(M/fM)$ by the boundary map δ in the long exact sequence above. It follows that $\mathfrak{p}$ is an associated prime of the finite module $\mathrm{Im}(\delta)$. We conclude that $\mathrm{Ass}(H^i_Z(M))$ is finite for $i \leq n$, see Algebra, Lemma 10.63.5.

Recall that

$$H^i_Z(M) \subset \prod\nolimits_{\mathfrak{p} \in \mathrm{Ass}(H^i_Z(M))} H^i_Z(M)_\mathfrak{p}$$

by Algebra, Lemma 10.63.19. Since by assumption the modules on the right hand side are finite and I-power torsion, we can find integers $e_{\mathfrak{p},i} \geq 0$, $i \leq n$, $\mathfrak{p} \in \mathrm{Ass}(H^i_Z(M))$ such that $I^{e_{\mathfrak{p},i}}$ annihilates $H^i_Z(M)_\mathfrak{p}$. We conclude that I^e with $e = \max\{e_{\mathfrak{p},i}\}$ annihilates $H^i_Z(M)$ for $i \leq n$. By Lemma 51.7.1 we see that $H^i_Z(M)$ is finite for $i \leq n$. □

0BPX **Lemma 51.7.3.** *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $i \geq 0$ be an integer. Set $Z = V(I)$. If $H^i_Z(A)$ is annihilated by J^n for some n, then $H^i_Z(M)$ annihilated by J^m for some $m = m(M)$ for every finitely presented A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. Consider the annihilator $\mathfrak{a}$ of $H^i_Z(M)$. Let $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin Z$. By assumption there exists an $f \in I$, $f \notin \mathfrak{p}$ and an isomorphism $\varphi : A_f^{\oplus r} \to M_f$ of A_f-modules. Clearing denominators (and using that M is of finite presentation) we find maps

$$a : A^{\oplus r} \longrightarrow M \quad \text{and} \quad b : M \longrightarrow A^{\oplus r}$$

with $a_f = f^N\varphi$ and $b_f = f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. Thus we see that $H^i_Z(M)$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As $U = \mathrm{Spec}(A) \setminus Z$ is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_j)$ and $f_j^{2N_j}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. □

0BPY **Lemma 51.7.4.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $Z = V(I)$. Let $n \geq 0$ be an integer. If $H^i_Z(A)$ is finite for $0 \leq i \leq n$, then the same is true for $H^i_Z(M)$, $0 \leq i \leq n$ for any finite A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. The assumption that $H^i_Z(A)$ is finite for $0 \leq i \leq n$ implies there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(A)$ for $0 \leq i \leq n$, see Lemma 51.7.1. Then Lemma 51.7.3 implies that $H^i_Z(M)$, $0 \leq i \leq n$ is annihilated by I^m for some $m = m(M, i)$. We may take the same m for all $0 \leq i \leq n$. Then Lemma 51.7.1 implies that $H^i_Z(M)$ is finite for $0 \leq i \leq n$ as desired. □

51.8. Finiteness of pushforwards, I

0BL8 In this section we discuss the easiest nontrivial case of the finiteness theorem, namely, the finiteness of the first local cohomology or what is equivalent, finiteness of $j_*\mathcal{F}$ where $j : U \to X$ is an open immersion, X is locally Noetherian, and $\mathcal{F}$ is a coherent sheaf on U. Following a method of Kollár ([**Kol16b**] and [**Kol15**]) we find a necessary and sufficient condition, see Proposition 51.8.7. The reader who is interested in higher direct images or higher local cohomology groups should skip ahead to Section 51.12 or Section 51.11 (which are developed independently of the rest of this section).

0BJZ **Lemma 51.8.1.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. For $x \in U$ let $i_x : W_x \to U$ be the integral closed subscheme with generic point x. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent*

(1) *for all $x \in Ass(\mathcal{F})$ the $\mathcal{O}_X$-module $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent,*
(2) *$j_*\mathcal{F}$ is coherent.*

Proof. We first prove that (1) implies (2). Assume (1) holds. The statement is local on X, hence we may assume X is affine. Then U is quasi-compact, hence $\mathrm{Ass}(\mathcal{F})$ is finite (Divisors, Lemma 31.2.5). Thus we may argue by induction on the number of associated points. Let $x \in U$ be a generic point of an irreducible component of the support of $\mathcal{F}$. By Divisors, Lemma 31.2.5 we have $x \in \mathrm{Ass}(\mathcal{F})$. By our choice of x we have $\dim(\mathcal{F}_x) = 0$ as $\mathcal{O}_{X,x}$-module. Hence $\mathcal{F}_x$ has finite length as an $\mathcal{O}_{X,x}$-module (Algebra, Lemma 10.62.3). Thus we may use induction on this length.

Set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. This is a coherent $\mathcal{O}_X$-module by assumption. We have $\mathcal{G}_x = \kappa(x)$. Choose a nonzero map $\varphi_x : \mathcal{F}_x \to \kappa(x) = \mathcal{G}_x$. By Cohomology of Schemes, Lemma 30.9.6 there is an open $x \in V \subset U$ and a map $\varphi_V : \mathcal{F}|_V \to \mathcal{G}|_V$ whose stalk at x is φ_x. Choose $f \in \Gamma(X, \mathcal{O}_X)$ which does not vanish at x such that $D(f) \subset V$. By Cohomology of Schemes, Lemma 30.10.5 (for example) we see that φ_V extends to $f^n\mathcal{F} \to \mathcal{G}|_U$ for some n. Precomposing with multiplication by f^n we obtain a map $\mathcal{F} \to \mathcal{G}|_U$ whose stalk at x is nonzero. Let $\mathcal{F}' \subset \mathcal{F}$ be the kernel. Note that $\mathrm{Ass}(\mathcal{F}') \subset \mathrm{Ass}(\mathcal{F})$, see Divisors, Lemma 31.2.4. Since $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) - 1$ we may apply the induction hypothesis to conclude $j_*\mathcal{F}'$ is coherent. Since $\mathcal{G} = j_*(\mathcal{G}|_U) = j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent, we can consider the exact sequence

$$0 \to j_*\mathcal{F}' \to j_*\mathcal{F} \to \mathcal{G}$$

By Schemes, Lemma 26.24.1 the sheaf $j_*\mathcal{F}$ is quasi-coherent. Hence the image of $j_*\mathcal{F}$ in $j_*(\mathcal{G}|_U)$ is coherent by Cohomology of Schemes, Lemma 30.9.3. Finally, $j_*\mathcal{F}$ is coherent by Cohomology of Schemes, Lemma 30.9.2.

Assume (2) holds. Exactly in the same manner as above we reduce to the case X affine. We pick $x \in \mathrm{Ass}(\mathcal{F})$ and we set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. Then we choose a nonzero map $\varphi_x : \mathcal{G}_x = \kappa(x) \to \mathcal{F}_x$ which exists exactly because x is an associated point of $\mathcal{F}$. Arguing exactly as above we may assume φ_x extends to an $\mathcal{O}_U$-module map $\varphi : \mathcal{G}|_U \to \mathcal{F}$. Then φ is injective (for example by Divisors, Lemma 31.2.10) and we find an injective map $\mathcal{G} = j_*(\mathcal{G}|_V) \to j_*\mathcal{F}$. Thus (1) holds. □

0BK0 **Lemma 51.8.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \mathrm{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then*

(1) *there exists a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
(2) *given M there is an exact sequence*

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
(3) *given M and $p \geq 0$ the following are equivalent*
   (a) *$R^pj_*\mathcal{F}$ is coherent,*
   (b) *$H^p(U, \mathcal{F})$ is a finite A-module,*
   (c) *$H^{p+1}_Z(M)$ is a finite A-module,*
(4) *if the equivalent conditions in (3) hold for $p = 0$, we may take $M = \Gamma(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. By Properties, Lemma 28.22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ whose restriction to U is isomorphic to $\mathcal{F}$. Say $\mathcal{F}'$ corresponds to the finite A-module M as in (1). Note that $R^p j_*\mathcal{F}$ is quasi-coherent (Cohomology of Schemes, Lemma 30.4.5) and corresponds to the A-module $H^p(U, \mathcal{F})$. By Lemma 51.2.1 and the discussion in Cohomology, Sections 20.21 and 20.34 we obtain an exact sequence

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$. Here we use that $H^j(X, \mathcal{F}') = 0$ for $j > 0$ as X is affine and $\mathcal{F}'$ is quasi-coherent (Cohomology of Schemes, Lemma 30.2.2). This proves (2). Parts (3) and (4) are straightforward from (2); see also Lemma 51.2.2. □

0AWA **Lemma 51.8.3.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume*

(1) *X is Nagata,*
(2) *X is universally catenary, and*
(3) *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$.*

Then $j_*\mathcal{F}$ is coherent.

Proof. By Lemma 51.8.1 it suffices to prove $j_* i_{x,*}\mathcal{O}_{W_x}$ is coherent for $x \in \mathrm{Ass}(\mathcal{F})$. Let $\pi : Y \to X$ be the normalization of X in $\mathrm{Spec}(\kappa(x))$, see Morphisms, Section 29.54. By Morphisms, Lemma 29.53.14 the morphism π is finite. Since π is finite $\mathcal{G} = \pi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module by Cohomology of Schemes, Lemma 30.9.9. Observe that $W_x = U \cap \pi(Y)$. Thus $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ factors through $i_x : W_x \to U$ and we obtain a canonical map

$$i_{x,*}\mathcal{O}_{W_x} \longrightarrow (\pi|_{\pi^{-1}(U)})_*(\mathcal{O}_{\pi^{-1}(U)}) = (\pi_*\mathcal{O}_Y)|_U = \mathcal{G}|_U$$

This map is injective (for example by Divisors, Lemma 31.2.10). Hence $j_* i_{x,*}\mathcal{O}_{W_x} \subset j_*\mathcal{G}|_U$ and it suffices to show that $j_*\mathcal{G}|_U$ is coherent.

It remains to prove that $j_*(\mathcal{G}|_U)$ is coherent. We claim Divisors, Lemma 31.5.11 applies to

$$\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)$$

which finishes the proof. It suffices to show that $\mathrm{depth}(\mathcal{G}_z) \geq 2$ for $z \in Z$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to z. By Algebra, Lemma 10.72.11 it suffices to show that $\mathrm{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. If not, then by Properties, Lemma 28.12.5 we see that $\dim(\mathcal{O}_{Y,y_i}) = 1$ for some i. This is impossible by the dimension formula (Morphisms, Lemma 29.52.1) for $\pi : Y \to \overline{\{x\}}$ and assumption (3). □

0BK1 **Lemma 51.8.4.** *Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be the inclusion of a nonempty open subscheme with complement Z. Assume that for all $z \in Z$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}^\wedge_{X,z}$ we have $\dim(\mathcal{O}^\wedge_{X,z}/\mathfrak{p}) \geq 2$. Then $j_*\mathcal{O}_U$ is coherent.*

Proof. We may assume X is affine. Using Lemmas 51.7.2 and 51.8.2 we reduce to $X = \mathrm{Spec}(A)$ where $(A, \mathfrak{m})$ is a Noetherian local domain and $\mathfrak{m} \in Z$. Then we can use induction on $d = \dim(A)$. (The base case is $d = 0, 1$ which do not happen by our assumption on the local rings.) Set $V = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$. Observe that the local rings of V have dimension strictly smaller than d. Repeating the arguments

for $j' : U \to V$ we and using induction we conclude that $j'_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-module. Pick a nonzero $f \in A$ which vanishes on Z. Since $D(f) \cap V \subset U$ we find an n such that multiplication by f^n on U extends to a map $f^n : j'_*\mathcal{O}_U \to \mathcal{O}_V$ over V (for example by Cohomology of Schemes, Lemma 30.10.5). This map is injective hence there is an injective map

$$j_*\mathcal{O}_U = j''_*j'_*\mathcal{O}_U \to j''_*\mathcal{O}_V$$

on X where $j'' : V \to X$ is the inclusion morphism. Hence it suffices to show that $j''_*\mathcal{O}_V$ is coherent. In other words, we may assume that X is the spectrum of a local Noetherian domain and that Z consists of the closed point.

Assume $X = \mathrm{Spec}(A)$ with $(A, \mathfrak{m})$ local and $Z = \{\mathfrak{m}\}$. Let $A^\wedge$ be the completion of A. Set $X^\wedge = \mathrm{Spec}(A^\wedge)$, $Z^\wedge = \{\mathfrak{m}^\wedge\}$, $U^\wedge = X^\wedge \setminus Z^\wedge$, and $\mathcal{F}^\wedge = \mathcal{O}_{U^\wedge}$. The ring $A^\wedge$ is universally catenary and Nagata (Algebra, Remark 10.160.9 and Lemma 10.162.8). Moreover, condition (3) of Lemma 51.8.3 for $X^\wedge, Z^\wedge, U^\wedge, \mathcal{F}^\wedge$ holds by assumption! Thus we see that $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ is coherent. Since the morphism $c : X^\wedge \to X$ is flat we conclude that the pullback of $j_*\mathcal{O}_U$ is $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ (Cohomology of Schemes, Lemma 30.5.2). Finally, since c is faithfully flat we conclude that $j_*\mathcal{O}_U$ is coherent by Descent, Lemma 35.7.1. □

0BK2 **Remark 51.8.5.** Let $j : U \to X$ be an open immersion of locally Noetherian schemes. Let $x \in U$. Let $i_x : W_x \to U$ be the integral closed subscheme with generic point x and let $\overline{\{x\}}$ be the closure in X. Then we have a commutative diagram

$$\begin{array}{ccc} W_x & \xrightarrow{j'} & \overline{\{x\}} \\ {\scriptstyle i_x}\downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X \end{array}$$

We have $j_*i_{x,*}\mathcal{O}_{W_x} = i_*j'_*\mathcal{O}_{W_x}$. As the left vertical arrow is a closed immersion we see that $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent if and only if $j'_*\mathcal{O}_{W_x}$ is coherent.

0AWC **Remark 51.8.6.** Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If there exists an $x \in \mathrm{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ such that $\dim(\mathcal{O}_{\overline{\{x\}},z}) \leq 1$, then $j_*\mathcal{F}$ is not coherent. To prove this we can do a flat base change to the spectrum of $\mathcal{O}_{X,z}$. Let $X' = \overline{\{x\}}$. The assumption implies $\mathcal{O}_{X' \cap U} \subset \mathcal{F}$. Thus it suffices to see that $j_*\mathcal{O}_{X' \cap U}$ is not coherent. This is clear because $X' = \{x, z\}$, hence $j_*\mathcal{O}_{X' \cap U}$ corresponds to $\kappa(x)$ as an $\mathcal{O}_{X,z}$-module which cannot be finite as x is not a closed point.

In fact, the converse of Lemma 51.8.4 holds true: given an open immersion $j : U \to X$ of integral Noetherian schemes and there exists a $z \in X \setminus U$ and an associated prime $\mathfrak{p}$ of the completion $\mathcal{O}^\wedge_{X,z}$ with $\dim(\mathcal{O}^\wedge_{X,z}/\mathfrak{p}) = 1$, then $j_*\mathcal{O}_U$ is not coherent. Namely, you can pass to the local ring, you can enlarge U to the punctured spectrum, you can pass to the completion, and then the argument above gives the nonfiniteness.

0BK3 **Proposition 51.8.7** (Kollár)**.** *Let $j : U \to X$ be an open immersion of locally Noetherian schemes with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent*

See [**Kol17**] and see [**DG67**, IV, Proposition 7.2.2] for a special case.

(1) $j_*\mathcal{F}$ *is coherent,*
(2) *for* $x \in Ass(\mathcal{F})$ *and* $z \in Z \cap \overline{\{x\}}$ *and any associated prime* $\mathfrak{p}$ *of the completion* $\mathcal{O}^\wedge_{\overline{\{x\}},z}$ *we have* $\dim(\mathcal{O}^\wedge_{\overline{\{x\}},z}/\mathfrak{p}) \geq 2$.

Proof. If (2) holds we get (1) by a combination of Lemmas 51.8.1, Remark 51.8.5, and Lemma 51.8.4. If (2) does not hold, then $j_*i_{x,*}\mathcal{O}_{W_x}$ is not finite for some $x \in \mathrm{Ass}(\mathcal{F})$ by the discussion in Remark 51.8.6 (and Remark 51.8.5). Thus $j_*\mathcal{F}$ is not coherent by Lemma 51.8.1. □

0BL9 **Lemma 51.8.8.** *Let* A *be a Noetherian ring and let* $I \subset A$ *be an ideal. Set* $Z = V(I)$. *Let* M *be a finite* A*-module. The following are equivalent*

(1) $H^1_Z(M)$ *is a finite* A*-module, and*
(2) *for all* $\mathfrak{p} \in Ass(M)$, $\mathfrak{p} \notin Z$ *and all* $\mathfrak{q} \in V(\mathfrak{p} + I)$ *the completion of* $(A/\mathfrak{p})_\mathfrak{q}$ *does not have associated primes of dimension* 1.

Proof. Follows immediately from Proposition 51.8.7 via Lemma 51.8.2. □

The formulation in the following lemma has the advantage that conditions (1) and (2) are inherited by schemes of finite type over X. Moreover, this is the form of finiteness which we will generalize to higher direct images in Section 51.12.

0AWB **Lemma 51.8.9.** *Let* X *be a locally Noetherian scheme. Let* $j : U \to X$ *be the inclusion of an open subscheme with complement* Z. *Let* $\mathcal{F}$ *be a coherent* $\mathcal{O}_U$*-module. Assume*

(1) X *is universally catenary,*
(2) *for every* $z \in Z$ *the formal fibres of* $\mathcal{O}_{X,z}$ *are* (S_1).

In this situation the following are equivalent

(a) *for* $x \in Ass(\mathcal{F})$ *and* $z \in Z \cap \overline{\{x\}}$ *we have* $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$, *and*
(b) $j_*\mathcal{F}$ *is coherent.*

Proof. Let $x \in \mathrm{Ass}(\mathcal{F})$. By Proposition 51.8.7 it suffices to check that $A = \mathcal{O}_{\overline{\{x\}},z}$ satisfies the condition of the proposition on associated primes of its completion if and only if $\dim(A) \geq 2$. Observe that A is universally catenary (this is clear) and that its formal fibres are (S_1) as follows from More on Algebra, Lemma 15.51.10 and Proposition 15.51.5. Let $\mathfrak{p}' \subset A^\wedge$ be an associated prime. As $A \to A^\wedge$ is flat, by Algebra, Lemma 10.65.3, we find that $\mathfrak{p}'$ lies over $(0) \subset A$. The formal fibre $A^\wedge \otimes_A F$ is (S_1) where F is the fraction field of A. We conclude that $\mathfrak{p}'$ is a minimal prime, see Algebra, Lemma 10.157.2. Since A is universally catenary it is formally catenary by More on Algebra, Proposition 15.110.5. Hence $\dim(A^\wedge/\mathfrak{p}') = \dim(A)$ which proves the equivalence. □

51.9. Depth and dimension

0DWW Some helper lemmas.

0DWX **Lemma 51.9.1.** *Let* A *be a Noetherian ring. Let* $I \subset A$ *be an ideal. Let* M *be a finite* A*-module. Let* $\mathfrak{p} \in V(I)$ *be a prime ideal. Assume* $e = depth_{IA_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. *Then there exists a nonempty open* $U \subset V(\mathfrak{p})$ *such that* $depth_{IA_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ *for all* $\mathfrak{q} \in U$.

Proof. By definition of depth we have $IM_\mathfrak{p} \neq M_\mathfrak{p}$ and there exists an $M_\mathfrak{p}$-regular sequence $f_1, \ldots, f_e \in IA_\mathfrak{p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in I$ form an M-regular sequence, see Algebra, Lemma 10.68.6. Consider the module $M' = M/IM$. Since $\mathfrak{p} \in \mathrm{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak{p}) \subset \mathrm{Supp}(M')$. Thus for $\mathfrak{q} \in V(\mathfrak{p})$ we get $IM_\mathfrak{q} \neq M_\mathfrak{q}$. Hence, using that localization is exact, we see that $\mathrm{depth}_{IA_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for any $\mathfrak{q} \in V(I)$ by definition of depth. □

0DWY **Lemma 51.9.2.** *Let A be a Noetherian ring. Let M be a finite A-module. Let $\mathfrak{p}$ be a prime ideal. Assume $e = depth_{A_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak{p})$ such that $depth_{A_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for all $\mathfrak{q} \in U$ and for all but finitely many $\mathfrak{q} \in U$ we have $depth_{A_\mathfrak{q}}(M_\mathfrak{q}) > e$.*

Proof. By definition of depth we have $\mathfrak{p}M_\mathfrak{p} \neq M_\mathfrak{p}$ and there exists an $M_\mathfrak{p}$-regular sequence $f_1, \ldots, f_e \in \mathfrak{p}A_\mathfrak{p}$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in \mathfrak{p}$ form an M-regular sequence, see Algebra, Lemma 10.68.6. Consider the module $M' = M/(f_1, \ldots, f_e)M$. Since $\mathfrak{p} \in \mathrm{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak{p}) \subset \mathrm{Supp}(M')$. Thus for $\mathfrak{q} \in V(\mathfrak{p})$ we get $\mathfrak{q}M_\mathfrak{q} \neq M_\mathfrak{q}$. Hence, using that localization is exact, we see that $\mathrm{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \geq e$ for any $\mathfrak{q} \in V(I)$ by definition of depth. Moreover, as soon as $\mathfrak{q}$ is not an associated prime of the module M', then the depth goes up. Thus we see that the final statement holds by Algebra, Lemma 10.63.5. □

0ECN **Lemma 51.9.3.** *Let X be a Noetherian scheme with dualizing complex $\omega_X^\bullet$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer. Assume $\mathcal{F}$ is (S_k). Then there is a finite number of points $x \in X$ such that*

$$depth(\mathcal{F}_x) = k \quad and \quad \dim(Supp(\mathcal{F}_x)) > k$$

Proof. We will prove this lemma by induction on k. The base case $k = 0$ says that $\mathcal{F}$ has a finite number of embedded associated points, which follows from Divisors, Lemma 31.2.5.

Assume $k > 0$ and the result holds for all smaller k. We can cover X by finitely many affine opens, hence we may assume $X = \mathrm{Spec}(A)$ is affine. Then $\mathcal{F}$ is the coherent $\mathcal{O}_X$-module associated to a finite A-module M which satisfies (S_k). We will use Algebra, Lemmas 10.63.10 and 10.72.7 without further mention.

Let $f \in A$ be a nonzerodivisor on M. Then M/fM has (S_{k-1}). By induction we see that there are finitely many primes $\mathfrak{p} \in V(f)$ with $\mathrm{depth}((M/fM)_\mathfrak{p}) = k - 1$ and $\dim(\mathrm{Supp}((M/fM)_\mathfrak{p})) > k - 1$. These are exactly the primes $\mathfrak{p} \in V(f)$ with $\mathrm{depth}(M_\mathfrak{p}) = k$ and $\dim(\mathrm{Supp}(M_\mathfrak{p})) > k$. Thus we may replace A by A_f and M by M_f in trying to prove the finiteness statement.

Since M satisfies (S_k) and $k > 0$ we see that M has no embedded associated primes (Algebra, Lemma 10.157.2). Thus $\mathrm{Ass}(M)$ is the set of generic points of the support of M. Thus Dualizing Complexes, Lemma 47.20.4 shows the set $U = \{\mathfrak{q} \mid M_\mathfrak{q} \text{ is Cohen-Macaulay}\}$ is an open containing $\mathrm{Ass}(M)$. By prime avoidance (Algebra, Lemma 10.15.2) we can pick $f \in A$ with $f \notin \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $D(f) \subset U$. Then f is a nonzerodivisor on M (Algebra, Lemma 10.63.9). After replacing A by A_f and M by M_f (see above) we find that M is Cohen-Macaulay. Thus for all $\mathfrak{q} \subset A$ we have $\dim(M_\mathfrak{q}) = \mathrm{depth}(M_\mathfrak{q})$ and hence the set described in the lemma is empty and a fortiori finite. □

0DWZ **Lemma 51.9.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Set $E^i = Ext_A^{-i}(M, \omega_A^\bullet)$. Then*

(1) *E^i is a finite A-module nonzero only for $0 \leq i \leq \dim(Supp(M))$,*
(2) *$\dim(Supp(E^i)) \leq i$,*
(3) *$depth(M)$ is the smallest integer $\delta \geq 0$ such that $E^\delta \neq 0$,*
(4) *$\mathfrak{p} \in Supp(E^0 \oplus \ldots \oplus E^i) \Leftrightarrow depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) \leq i$,*
(5) *the annihilator of E^i is equal to the annihilator of $H^i_\mathfrak{m}(M)$.*

Proof. Parts (1), (2), and (3) are copies of the statements in Dualizing Complexes, Lemma 47.16.5. For a prime $\mathfrak{p}$ of A we have that $(\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]$ is a normalized dualzing complex for $A_\mathfrak{p}$. See Dualizing Complexes, Lemma 47.17.3. Thus

$$E^i_\mathfrak{p} = \mathrm{Ext}_A^{-i}(M, \omega_A^\bullet)_\mathfrak{p} = \mathrm{Ext}_{A_\mathfrak{p}}^{-i+\dim(A/\mathfrak{p})}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})])$$

is zero for $i - \dim(A/\mathfrak{p}) < \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ and nonzero for $i = \dim(A/\mathfrak{p}) + \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ by part (3) over $A_\mathfrak{p}$. This proves part (4). If E is an injective hull of the residue field of A, then we have

$$\mathrm{Hom}_A(H^i_\mathfrak{m}(M), E) = \mathrm{Ext}_A^{-i}(M, \omega_A^\bullet)^\wedge = (E^i)^\wedge = E^i \otimes_A A^\wedge$$

by the local duality theorem (in the form of Dualizing Complexes, Lemma 47.18.4). Since $A \to A^\wedge$ is faithfully flat, we find (5) is true by Matlis duality (Dualizing Complexes, Proposition 47.7.8). □

51.10. Annihilators of local cohomology, I

0EFB This section discusses a result due to Faltings, see [**Fal78b**].

0EFC **Proposition 51.10.1.** *Let A be a Noetherian ring which has a dualizing complex.* [**Fal78b**]. *Let $T \subset T' \subset \mathrm{Spec}(A)$ be subsets stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent*

(1) *there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(M)$ for $i \leq s$, and*
(2) *for all $\mathfrak{p} \notin T'$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have*

$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. Let $\omega_A^\bullet$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 47.17. An important role will be played by the finite A-modules

$$E^i = \mathrm{Ext}^i_A(M, \omega_A^\bullet)$$

For $\mathfrak{p} \subset A$ we will write $H^i_\mathfrak{p}$ to denote the local cohomology of an $A_\mathfrak{p}$-module with respect to $\mathfrak{p}A_\mathfrak{p}$. Then we see that the $\mathfrak{p}A_\mathfrak{p}$-adic completion of

$$(E^i)_\mathfrak{p} = \mathrm{Ext}^{\delta(\mathfrak{p})+i}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})])$$

is Matlis dual to

$$H^{-\delta(\mathfrak{p})-i}_\mathfrak{p}(M_\mathfrak{p})$$

by Dualizing Complexes, Lemma 47.18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak{p}$ if and only if J annihilates $H^{-\delta(\mathfrak{p})-i}_\mathfrak{p}(M_\mathfrak{p})$.

Set $T_n = \{\mathfrak{p} \in T \mid \delta(\mathfrak{p}) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(M)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $0 \leq a \leq i$. The case IH_0 is trivial because $H^0_T(M)$ is a submodule of M and hence finite and hence is annihilated by some ideal J with $V(J) \subset T$.

Induction step. Assume IH_{i-1} holds for some $0 < i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(M)$ for $0 \leq a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(M)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(M) = 0$ (Algebra, Lemma 10.63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(M)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak{q} \in T_n$. Let $T_\mathfrak{q} \subset \mathrm{Spec}(A_\mathfrak{q})$ be the inverse image of T. We have $H^j_T(M)_\mathfrak{q} = H^j_{T_\mathfrak{q}}(M_\mathfrak{q})$ by Lemma 51.5.7. Consider the spectral sequence

$$H^p_\mathfrak{q}(H^q_{T_\mathfrak{q}}(M_\mathfrak{q})) \Rightarrow H^{p+q}_\mathfrak{q}(M_\mathfrak{q})$$

of Lemma 51.5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_\mathfrak{q}(M_\mathfrak{q})$ is annihilated by J'' for all $\mathfrak{q} \in T_n \setminus T_{n-1}$. Claim: $J(J')^i J''$ will work for $n-1$. Namely, let $\mathfrak{q} \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E^{0,i}_\infty = E^{0,i}_{i+2} \subset \ldots \subset E^{0,i}_3 \subset E^{0,i}_2 = H^0_\mathfrak{q}(H^i_{T_\mathfrak{q}}(M_\mathfrak{q}))$$

The module $E^{0,i}_\infty$ is annihilated by J''. The subquotients $E^{0,i}_j/E^{0,i}_{j+1}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d^{0,i}_j$ is a subquotient of

$$H^j_\mathfrak{q}(H^{i-j+1}_{T_\mathfrak{q}}(M_\mathfrak{q})) = H^j_\mathfrak{q}(H^{i-j+1}_T(M)_\mathfrak{q})$$

and $H^{i-j+1}_T(M)_\mathfrak{q}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(M)_\mathfrak{q} \subset H^0_\mathfrak{q}(H^i_T(M)_\mathfrak{q})$ since the non-closed points of $\mathrm{Spec}(A_\mathfrak{q})$ have higher δ values. Thus $\mathfrak{q}$ cannot be an associated prime of $J(J')^i J'' H^i_T(M)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak{q})-i})_\mathfrak{q} = (E^{-n-i})_\mathfrak{q}$$

for all $\mathfrak{q} \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak{q}$, then it works for all $\mathfrak{q}$ in an open neighbourhood of $\mathfrak{q}$ as the modules E^{-n-i} are finite. Since every subset of $\mathrm{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 5.9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak{q}$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\mathrm{Supp}(E^{-n-i}) \cap \mathrm{Spec}(A_\mathfrak{q}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} \notin T'$ is zero. Using local duality over $A_\mathfrak{p}$ we find that we need to prove that

$$H^{i+n-\delta(\mathfrak{p})}_\mathfrak{p}(M_\mathfrak{p}) = H^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}_\mathfrak{p}(M_\mathfrak{p})$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and Dualizing Complexes, Lemma 47.11.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak{q} \in T$, $\mathfrak{p} \subset \mathfrak{q}$ with $\mathfrak{p} \notin T'$ such that

$$i = \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq s$$

is minimal. Then $H^{i-\dim((A/\mathfrak{p})_\mathfrak{q})}_\mathfrak{p}(M_\mathfrak{p})$ is nonzero by the nonvanishing in Dualizing Complexes, Lemma 47.11.1. Set $n = \delta(\mathfrak{q})$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_\mathfrak{q} = 0$. Thus $H^i_\mathfrak{q}(M_\mathfrak{q})$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(M)_\mathfrak{q}$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(M)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. □

0EFE **Lemma 51.10.2.** *Let I be an ideal of a Noetherian ring A. Let M be a finite A-module, let $\mathfrak{p} \subset A$ be a prime ideal, and let $s \geq 0$ be an integer. Assume*

(1) *A has a dualizing complex,*
(2) *$\mathfrak{p} \notin V(I)$, and*
(3) *for all primes $\mathfrak{p}' \subset \mathfrak{p}$ and $\mathfrak{q} \in V(I)$ with $\mathfrak{p}' \subset \mathfrak{q}$ we have*

$$\text{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_\mathfrak{q}) > s$$

Then there exists an $f \in A$, $f \notin \mathfrak{p}$ which annihilates $H^i_{V(I)}(M)$ for $i \leq s$.

Proof. Consider the sets

$$T = V(I) \quad \text{and} \quad T' = \bigcup\nolimits_{f \in A, f \notin \mathfrak{p}} V(f)$$

These are subsets of $\text{Spec}(A)$ stable under specialization. Observe that $T \subset T'$ and $\mathfrak{p} \notin T'$. Assumption (3) says that hypothesis (2) of Proposition 51.10.1 holds. Hence we can find $J \subset A$ with $V(J) \subset T'$ such that $JH^i_{V(I)}(M) = 0$ for $i \leq s$. Choose $f \in A$, $f \notin \mathfrak{p}$ with $V(J) \subset V(f)$. A power of f annihilates $H^i_{V(I)}(M)$ for $i \leq s$. □

51.11. Finiteness of local cohomology, II

0BJQ We continue the discussion of finiteness of local cohomology started in Section 51.7. Using Faltings Annihilator Theorem we easily prove the following fundamental result.

0EFD **Proposition 51.11.1.** *Let A be a Noetherian ring which has a dualizing complex.* [**Fal78b**]. *Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent*

(1) *$H^i_T(M)$ is a finite A-module for $i \leq s$, and*
(2) *for all $\mathfrak{p} \notin T$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have*

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. Formal consequence of Proposition 51.10.1 and Lemma 51.7.1. □

Besides some lemmas for later use, the rest of this section is concerned with the question to what extend the condition in Proposition 51.11.1 that A has a dualizing complex can be weakened. The answer is roughly that one has to assume the formal fibres of A are (S_n) for sufficiently large n.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \mathrm{Spec}(A)$ and $Z = V(I) \subset X$. Let M be a finite A-module. We define

0BJR (51.11.1.1) $$s_{A,I}(M) = \min\{\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \mid \mathfrak{p} \in X \setminus Z, \mathfrak{q} \in Z, \mathfrak{p} \subset \mathfrak{q}\}$$

Our conventions on depth are that the depth of 0 is ∞ thus we only need to consider primes $\mathfrak{p}$ in the support of M. It will turn out that $s_{A,I}(M)$ is an important invariant of the situation.

0BJS **Lemma 51.11.2.** *Let $A \to B$ be a finite homomorphism of Noetherian rings. Let $I \subset A$ be an ideal and set $J = IB$. Let M be a finite B-module. If A is universally catenary, then $s_{B,J}(M) = s_{A,I}(M)$.*

Proof. Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $I \subset \mathfrak{q}$ and $I \not\subset \mathfrak{p}$. Since $A \to B$ is finite there are finitely many primes $\mathfrak{p}_i$ lying over $\mathfrak{p}$. By Algebra, Lemma 10.72.11 we have

$$\mathrm{depth}(M_\mathfrak{p}) = \min \mathrm{depth}(M_{\mathfrak{p}_i})$$

Let $\mathfrak{p}_i \subset \mathfrak{q}_{ij}$ be primes lying over $\mathfrak{q}$. By going up for $A \to B$ (Algebra, Lemma 10.36.22) there is at least one $\mathfrak{q}_{ij}$ for each i. Then we see that

$$\dim((B/\mathfrak{p}_i)_{\mathfrak{q}_{ij}}) = \dim((A/\mathfrak{p})_\mathfrak{q})$$

by the dimension formula, see Algebra, Lemma 10.113.1. This implies that the minimum of the quantities used to define $s_{B,J}(M)$ for the pairs $(\mathfrak{p}_i, \mathfrak{q}_{ij})$ is equal to the quantity for the pair $(\mathfrak{p}, \mathfrak{q})$. This proves the lemma. □

0EHW **Lemma 51.11.3.** *Let A be a Noetherian ring which has a dualizing complex. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let A', M' be the I-adic completions of A, M. Let $\mathfrak{p}' \subset \mathfrak{q}'$ be prime ideals of A' with $\mathfrak{q}' \in V(IA')$ lying over $\mathfrak{p} \subset \mathfrak{q}$ in A. Then*

$$\mathit{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) \geq \mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$$

and

$$\mathit{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$$

Proof. We have

$$\mathrm{depth}(M'_{\mathfrak{p}'}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) \geq \mathrm{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$, see Algebra, Lemma 10.163.1. Since the fibres of $A \to A'$ are Cohen-Macaulay (Dualizing Complexes, Lemma 47.23.2 and More on Algebra, Section 15.51) we see that $\mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) = \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'})$. Thus we obtain

$$\begin{aligned}
\mathrm{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) &= \mathrm{depth}(M_\mathfrak{p}) + \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) \\
&= \mathrm{depth}(M_\mathfrak{p}) + \dim((A'/\mathfrak{p}A')_{\mathfrak{q}'}) \\
&= \mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})
\end{aligned}$$

Second equality because A' is catenary and third equality by More on Algebra, Lemma 15.43.1 as $(A/\mathfrak{p})_\mathfrak{q}$ and $(A'/\mathfrak{p}A')_{\mathfrak{q}'}$ have the same I-adic completions. □

0BJT **Lemma 51.11.4.** *Let A be a universally catenary Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then*

$$s_{A,I}(M) \geq s_{A^\wedge,I^\wedge}(M^\wedge)$$

If the formal fibres of A are (S_n), then $\min(n+1, s_{A,I}(M)) \leq s_{A^\wedge,I^\wedge}(M^\wedge)$.

Proof. Write $X = \mathrm{Spec}(A)$, $X^\wedge = \mathrm{Spec}(A^\wedge)$, $Z = V(I) \subset X$, and $Z^\wedge = V(I^\wedge)$. Let $\mathfrak{p}' \subset \mathfrak{q}' \subset A^\wedge$ be primes with $\mathfrak{p}' \notin Z^\wedge$ and $\mathfrak{q}' \in Z^\wedge$. Let $\mathfrak{p} \subset \mathfrak{q}$ be the corresponding primes of A. Then $\mathfrak{p} \notin Z$ and $\mathfrak{q} \in Z$. Picture

$$\begin{array}{ccccc} \mathfrak{p}' & \longrightarrow & \mathfrak{q}' & \longrightarrow & A^\wedge \\ \downarrow & & \downarrow & & \uparrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{q} & \longrightarrow & A \end{array}$$

Let us write

$$\begin{aligned} a &= \dim(A/\mathfrak{p}) = \dim(A^\wedge/\mathfrak{p}A^\wedge), \\ b &= \dim(A/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{q}A^\wedge), \\ a' &= \dim(A^\wedge/\mathfrak{p}'), \\ b' &= \dim(A^\wedge/\mathfrak{q}') \end{aligned}$$

Equalities by More on Algebra, Lemma 15.43.1. We also write

$$\begin{aligned} p &= \dim(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) = \dim((A^\wedge/\mathfrak{p}A^\wedge)_{\mathfrak{p}'}) \\ q &= \dim(A^\wedge_{\mathfrak{q}'}/\mathfrak{p}A^\wedge_{\mathfrak{q}'}) = \dim((A^\wedge/\mathfrak{q}A^\wedge)_{\mathfrak{q}'}) \end{aligned}$$

Since A is universally catenary we see that $A^\wedge/\mathfrak{p}A^\wedge = (A/\mathfrak{p})^\wedge$ is equidimensional of dimension a (More on Algebra, Proposition 15.110.5). Hence $a = a' + p$. Similarly $b = b' + q$. By Algebra, Lemma 10.163.1 applied to the flat local ring map $A_\mathfrak{p} \to A^\wedge_{\mathfrak{p}'}$ we have

$$\mathrm{depth}(M^\wedge_{\mathfrak{p}'}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'})$$

The quantity we are minimizing for $s_{A,I}(M)$ is

$$s(\mathfrak{p}, \mathfrak{q}) = \mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) = \mathrm{depth}(M_\mathfrak{p}) + a - b$$

(last equality as A is catenary). The quantity we are minimizing for $s_{A^\wedge,I^\wedge}(M^\wedge)$ is

$$s(\mathfrak{p}', \mathfrak{q}') = \mathrm{depth}(M^\wedge_{\mathfrak{p}'}) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) = \mathrm{depth}(M^\wedge_{\mathfrak{p}'}) + a' - b'$$

(last equality as $A^\wedge$ is catenary). Now we have enough notation in place to start the proof.

Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $\mathfrak{p} \notin Z$ and $\mathfrak{q} \in Z$ such that $s_{A,I}(M) = s(\mathfrak{p}, \mathfrak{q})$. Then we can pick $\mathfrak{q}'$ minimal over $\mathfrak{q}A^\wedge$ and $\mathfrak{p}' \subset \mathfrak{q}'$ minimal over $\mathfrak{p}A^\wedge$ (using going down for $A \to A^\wedge$). Then we have four primes as above with $p = 0$ and $q = 0$. Moreover, we have $\mathrm{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) = 0$ also because $p = 0$. This means that $s(\mathfrak{p}', \mathfrak{q}') = s(\mathfrak{p}, \mathfrak{q})$. Thus we get the first inequality.

Assume that the formal fibres of A are (S_n). Then $\mathrm{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) \geq \min(n, p)$. Hence

$$s(\mathfrak{p}', \mathfrak{q}') \geq s(\mathfrak{p}, \mathfrak{q}) + q + \min(n, p) - p \geq s_{A,I}(M) + q + \min(n, p) - p$$

Thus the only way we can get in trouble is if $p > n$. If this happens then

$$\begin{aligned} s(\mathfrak{p}', \mathfrak{q}') &= \mathrm{depth}(M^\wedge_{\mathfrak{p}'}) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) \\ &= \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) \\ &\geq 0 + n + 1 \end{aligned}$$

because $(A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}$ has at least two primes. This proves the second inequality. $\square$

The method of proof of the following lemma works more generally, but the stronger results one gets will be subsumed in Theorem 51.11.6 below.

0BJU **Lemma 51.11.5.** *Let A be a Gorenstein Noetherian local ring. Let $I \subset A$ be an ideal and set $Z = V(I) \subset \mathrm{Spec}(A)$. Let M be a finite A-module. Let $s = s_{A,I}(M)$ as in (51.11.1.1). Then $H^i_Z(M)$ is finite for $i < s$, but $H^s_Z(M)$ is not finite.* This is a special case of [**Fal78b**, Satz 1].

Proof. Since a Gorenstein local ring has a dualizing complex, this is a special case of Proposition 51.11.1. It would be helpful to have a short proof of this special case, which will be used in the proof of a general finiteness theorem below. □

Observe that the hypotheses of the following theorem are satisfied by excellent Noetherian rings (by definition), by Noetherian rings which have a dualizing complex (Dualizing Complexes, Lemma 47.17.4 and Dualizing Complexes, Lemma 47.23.2), and by quotients of regular Noetherian rings.

0BJV **Theorem 51.11.6.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. Let M be a finite A-module. Set $s = s_{A,I}(M)$ as in (51.11.1.1). Assume that* This is a special case of [**Fal81**, Satz 2].

(1) *A is universally catenary,*
(2) *the formal fibres of the local rings of A are Cohen-Macaulay.*

Then $H^i_Z(M)$ is finite for $0 \leq i < s$ and $H^s_Z(M)$ is not finite.

Proof. By Lemma 51.7.2 we may assume that A is a local ring.

If A is a Noetherian complete local ring, then we can write A as the quotient of a regular complete local ring B by Cohen's structure theorem (Algebra, Theorem 10.160.8). Using Lemma 51.11.2 and Dualizing Complexes, Lemma 47.9.2 we reduce to the case of a regular local ring which is a consequence of Lemma 51.11.5 because a regular local ring is Gorenstein (Dualizing Complexes, Lemma 47.21.3).

Let A be a Noetherian local ring. Let $\mathfrak{m}$ be the maximal ideal. We may assume $I \subset \mathfrak{m}$, otherwise the lemma is trivial. Let $A^\wedge$ be the completion of A, let $Z^\wedge = V(IA^\wedge)$, and let $M^\wedge = M \otimes_A A^\wedge$ be the completion of M (Algebra, Lemma 10.97.1). Then $H^i_Z(M) \otimes_A A^\wedge = H^i_{Z^\wedge}(M^\wedge)$ by Dualizing Complexes, Lemma 47.9.3 and flatness of $A \to A^\wedge$ (Algebra, Lemma 10.97.2). Hence it suffices to show that $H^i_{Z^\wedge}(M^\wedge)$ is finite for $i < s$ and not finite for $i = s$, see Algebra, Lemma 10.83.2. Since we know the result is true for $A^\wedge$ it suffices to show that $s_{A,I}(M) = s_{A^\wedge,I^\wedge}(M^\wedge)$. This follows from Lemma 51.11.4. □

0BJW **Remark 51.11.7.** The astute reader will have realized that we can get away with a slightly weaker condition on the formal fibres of the local rings of A. Namely, in the situation of Theorem 51.11.6 assume A is universally catenary but make no assumptions on the formal fibres. Suppose we have an n and we want to prove that $H^i_Z(M)$ are finite for $i \leq n$. Then the exact same proof shows that it suffices that $s_{A,I}(M) > n$ and that the formal fibres of local rings of A are (S_n). On the other hand, if we want to show that $H^s_Z(M)$ is not finite where $s = s_{A,I}(M)$, then our arguments prove this if the formal fibres are (S_{s-1}).

51.12. Finiteness of pushforwards, II

0BJX This section is the continuation of Section 51.8. In this section we reap the fruits of the labor done in Section 51.11.

0BJY **Lemma 51.12.1.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $n \geq 0$ be an integer. Assume*

(1) *X is universally catenary,*
(2) *for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_n).*

In this situation the following are equivalent

(a) *for $x \in Supp(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > n$,*
(b) *$R^p j_*\mathcal{F}$ is coherent for $0 \leq p < n$.*

Proof. The statement is local on X, hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = V(I)$. Let M be a finite A-module whose associated coherent $\mathcal{O}_X$-module restricts to $\mathcal{F}$ over U, see Lemma 51.8.2. This lemma also tells us that $R^p j_*\mathcal{F}$ is coherent if and only if $H^{p+1}_Z(M)$ is a finite A-module. Observe that the minimum of the expressions $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z})$ is the number $s_{A,I}(M)$ of (51.11.1.1). Having said this the lemma follows from Theorem 51.11.6 as elucidated by Remark 51.11.7. □

0BLT **Lemma 51.12.2.** *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $n \geq 0$ be an integer. If $R^p j_*\mathcal{O}_U$ is coherent for $0 \leq p < n$, then the same is true for $R^p j_*\mathcal{F}$, $0 \leq p < n$ for any finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. The question is local on X, hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = V(I)$. Via Lemma 51.8.2 our lemma follows from Lemma 51.7.4. □

0BM5 **Lemma 51.12.3.** *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $p \geq 0$ be an integer. Set $U = \mathrm{Spec}(A) \setminus V(I)$. If $H^p(U, \mathcal{O}_U)$ is annihilated by J^n for some n, then $H^p(U, \mathcal{F})$ annihilated by J^m for some $m = m(\mathcal{F})$ for every finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.* [**BdJ14**, Lemma 1.9]

Proof. Consider the annihilator $\mathfrak{a}$ of $H^p(U, \mathcal{F})$. Let $u \in U$. There exists an open neighbourhood $u \in U' \subset U$ and an isomorphism $\varphi : \mathcal{O}_{U'}^{\oplus r} \to \mathcal{F}|_{U'}$. Pick $f \in A$ such that $u \in D(f) \subset U'$. There exist maps

$$a : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F} \quad \text{and} \quad b : \mathcal{F} \longrightarrow \mathcal{O}_U^{\oplus r}$$

whose restriction to $D(f)$ are equal to $f^N\varphi$ and $f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. This follows from Properties, Lemma 28.17.3 since U is quasi-compact (I is finitely generated), separated, and $\mathcal{F}$ and $\mathcal{O}_U^{\oplus r}$ are finitely presented. Thus we see that $H^p(U, \mathcal{F})$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As U is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_i)$ and $f_i^{2N_i}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. □

51.13. Annihilators of local cohomology, II

0EHX We extend the discussion of annihilators of local cohomology in Section 51.10 to bounded below complexes with finite cohomology modules.

0EHY **Definition 51.13.1.** Let I be an ideal of a Noetherian ring A. Let $K \in D^+_{Coh}(A)$. We define the *I-depth* of K, denoted $\text{depth}_I(K)$, to be the maximal $m \in \mathbf{Z} \cup \{\infty\}$ such that $H^i_I(K) = 0$ for all $i < m$. If A is local with maximal ideal $\mathfrak{m}$ then we call $\text{depth}_{\mathfrak{m}}(K)$ simply the *depth* of K.

This definition does not conflict with Algebra, Definition 10.72.1 by Dualizing Complexes, Lemma 47.11.1.

0EHZ **Proposition 51.13.2.** *Let A be a Noetherian ring which has a dualizing complex. Let $T \subset T' \subset \text{Spec}(A)$ be subsets stable under specialization. Let $s \in \mathbf{Z}$. Let K be an object of $D^+_{Coh}(A)$. The following are equivalent*

(1) *there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(K)$ for $i \leq s$, and*
(2) *for all $\mathfrak{p} \notin T'$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have*

$$depth_{A_\mathfrak{p}}(K_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. This lemma is the natural generalization of Proposition 51.10.1 whose proof the reader should read first. Let $\omega^\bullet_A$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 47.17. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(K, \omega^\bullet_A)$$

For $\mathfrak{p} \subset A$ we will write $H^i_\mathfrak{p}$ to denote the local cohomology of an object of $D(A_\mathfrak{p})$ with respect to $\mathfrak{p}A_\mathfrak{p}$. Then we see that the $\mathfrak{p}A_\mathfrak{p}$-adic completion of

$$(E^i)_\mathfrak{p} = \text{Ext}^{\delta(\mathfrak{p})+i}_{A_\mathfrak{p}}(K_\mathfrak{p}, (\omega^\bullet_A)_\mathfrak{p}[-\delta(\mathfrak{p})])$$

is Matlis dual to

$$H^{-\delta(\mathfrak{p})-i}_\mathfrak{p}(K_\mathfrak{p})$$

by Dualizing Complexes, Lemma 47.18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak{p}$ if and only if J annihilates $H^{-\delta(\mathfrak{p})-i}_\mathfrak{p}(K_\mathfrak{p})$.

Set $T_n = \{\mathfrak{p} \in T \mid \delta(\mathfrak{p}) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(K)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $a \leq i$. The case IH_i is trivial for i small enough because K is bounded below.

Induction step. Assume IH_{i-1} holds for some $i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(K)$ for $a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(K)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(K) = 0$ (Algebra, Lemma 10.63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(K)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak{q} \in T_n$. Let $T_\mathfrak{q} \subset \text{Spec}(A_\mathfrak{q})$ be the inverse image of T. We have $H^j_T(K)_\mathfrak{q} = H^j_{T_\mathfrak{q}}(K_\mathfrak{q})$ by Lemma 51.5.7. Consider the spectral sequence

$$H^p_\mathfrak{q}(H^q_{T_\mathfrak{q}}(K_\mathfrak{q})) \Rightarrow H^{p+q}_\mathfrak{q}(K_\mathfrak{q})$$

of Lemma 51.5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_{\mathfrak{q}}(K_{\mathfrak{q}})$ is annihilated by J'' for all $\mathfrak{q} \in T_n \setminus T_{n-1}$. Claim: $J(J')^i J''$ will work for $n-1$. Namely, let $\mathfrak{q} \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E_\infty^{0,i} = E_{i+2}^{0,i} \subset \ldots \subset E_3^{0,i} \subset E_2^{0,i} = H^0_{\mathfrak{q}}(H^i_{T_{\mathfrak{q}}}(K_{\mathfrak{q}}))$$

The module $E_\infty^{0,i}$ is annihilated by J''. The subquotients $E_j^{0,i}/E_{j+1}^{0,i}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d_j^{0,i}$ is a subquotient of

$$H^j_{\mathfrak{q}}(H^{i-j+1}_{T_{\mathfrak{q}}}(K_{\mathfrak{q}})) = H^j_{\mathfrak{q}}(H^{i-j+1}_T(K)_{\mathfrak{q}})$$

and $H^{i-j+1}_T(K)_{\mathfrak{q}}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(K)_{\mathfrak{q}} \subset H^0_{\mathfrak{q}}(H^i_T(K)_{\mathfrak{q}})$ since the non-closed points of $\mathrm{Spec}(A_{\mathfrak{q}})$ have higher δ values. Thus $\mathfrak{q}$ cannot be an associated prime of $J(J')^i J'' H^i_T(K)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak{q})-i})_{\mathfrak{q}} = (E^{-n-i})_{\mathfrak{q}}$$

for all $\mathfrak{q} \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak{q}$, then it works for all $\mathfrak{q}$ in an open neighbourhood of $\mathfrak{q}$ as the modules E^{-n-i} are finite. Since every subset of $\mathrm{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 5.9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak{q}$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\mathrm{Supp}(E^{-n-i}) \cap \mathrm{Spec}(A_{\mathfrak{q}}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} \notin T'$ is zero. Using local duality over $A_{\mathfrak{p}}$ we find that we need to prove that

$$H^{i+n-\delta(\mathfrak{p})}_{\mathfrak{p}}(K_{\mathfrak{p}}) = H^{i-\dim((A/\mathfrak{p})_{\mathfrak{q}})}_{\mathfrak{p}}(K_{\mathfrak{p}})$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and our definition of depth in Definition 51.13.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak{q} \in T$, $\mathfrak{p} \subset \mathfrak{q}$ with $\mathfrak{p} \notin T'$ such that

$$i = \mathrm{depth}_{A_{\mathfrak{p}}}(K_{\mathfrak{p}}) + \dim((A/\mathfrak{p})_{\mathfrak{q}}) \leq s$$

is minimal. Then $H^{i-\dim((A/\mathfrak{p})_{\mathfrak{q}})}_{\mathfrak{p}}(K_{\mathfrak{p}})$ is nonzero by the our definition of depth in Definition 51.13.1. Set $n = \delta(\mathfrak{q})$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_{\mathfrak{q}} = 0$. Thus $H^i_{\mathfrak{q}}(K_{\mathfrak{q}})$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(K)_{\mathfrak{q}}$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(K)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. □

51.14. Finiteness of local cohomology, III

0EI0 We extend the discussion of finiteness of local cohomology in Sections 51.7 and 51.11 to bounded below complexes with finite cohomology modules.

0EI1 **Lemma 51.14.1.** *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let K be an object of $D^+_{Coh}(A)$. Let $n \in \mathbf{Z}$. The following are equivalent*

(1) $H^i_T(K)$ *is finite for* $i \leq n$,
(2) *there exists an ideal* $J \subset A$ *with* $V(J) \subset T$ *such that* J *annihilates* $H^i_T(K)$ *for* $i \leq n$.

If $T = V(I) = Z$ *for an ideal* $I \subset A$, *then these are also equivalent to*

(3) *there exists an* $e \geq 0$ *such that* I^e *annihilates* $H^i_Z(K)$ *for* $i \leq n$.

Proof. This lemma is the natural generalization of Lemma 51.7.1 whose proof the reader should read first. Assume (1) is true. Recall that $H^i_J(K) = H^i_{V(J)}(K)$, see Dualizing Complexes, Lemma 47.10.1. Thus $H^i_T(K) = \operatorname{colim} H^i_J(K)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 51.5.3. Since $H^i_T(K)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(K) \to H^i_T(K)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Let $a \in \mathbf{Z}$ be an integer such that $H^i(K) = 0$ for $i < a$. We prove (2) $\Rightarrow$ (1) by descending induction on a. If $a > n$, then we have $H^i_T(K) = 0$ for $i \leq n$ hence both (1) and (2) are true and there is nothing to prove.

Assume we have J as in (2). Observe that $N = H^a_T(K) = H^0_T(H^a(K))$ is finite as a submodule of the finite A-module $H^a(K)$. If $n = a$ we are done; so assume $a < n$ from now on. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 51.5.6. Choose a distinguished triangle

$$N[-a] \to K \to K' \to N[-a+1]$$

Then we see that $H^a_T(K') = 0$ and $H^i_T(K) = H^i_T(K')$ for $i > a$. We conclude that we may replace K by K'. Thus we may assume that $H^a_T(K) = 0$. This means that the finite set of associated primes of $H^a(K)$ are not in T. By prime avoidance (Algebra, Lemma 10.15.2) we can find $f \in J$ not contained in any of the associated primes of $H^a(K)$. Choose a distinguished triangle

$$L \to K \xrightarrow{f} K \to L[1]$$

By construction we see that $H^i(L) = 0$ for $i \leq a$. On the other hand we have a long exact cohomology sequence

$$0 \to H^{a+1}_T(L) \to H^{a+1}_T(K) \xrightarrow{f} H^{a+1}_T(K) \to H^{a+2}_T(L) \to H^{a+2}_T(K) \xrightarrow{f} \ldots$$

which breaks into the identification $H^{a+1}_T(L) = H^{a+1}_T(K)$ and short exact sequences

$$0 \to H^{i-1}_T(K) \to H^i_T(L) \to H^i_T(K) \to 0$$

for $i \leq n$ since $f \in J$. We conclude that J^2 annihilates $H^i_T(L)$ for $i \leq n$. By induction hypothesis applied to L we see that $H^i_T(L)$ is finite for $i \leq n$. Using the short exact sequence once more we see that $H^i_T(K)$ is finite for $i \leq n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. □

0EI2 **Proposition 51.14.2.** *Let* A *be a Noetherian ring which has a dualizing complex. Let* $T \subset \operatorname{Spec}(A)$ *be a subset stable under specialization. Let* $s \in \mathbf{Z}$. *Let* $K \in D^+_{Coh}(A)$. *The following are equivalent*

(1) $H^i_T(K)$ *is a finite* A*-module for* $i \leq s$, *and*
(2) *for all* $\mathfrak{p} \notin T$, $\mathfrak{q} \in T$ *with* $\mathfrak{p} \subset \mathfrak{q}$ *we have*

$$\mathit{depth}_{A_\mathfrak{p}}(K_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. Formal consequence of Proposition 51.13.2 and Lemma 51.14.1. □

51.15. Improving coherent modules

0DX2 Similar constructions can be found in [**DG67**] and more recently in [**Kol15**] and [**Kol16b**].

0DX3 **Lemma 51.15.1.** *Let X be a Noetherian scheme. Let $T \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is a unique map $\mathcal{F} \to \mathcal{F}'$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F} \to \mathcal{F}'$ is surjective,*
(2) *$\mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for $x \notin T$,*
(3) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 1$ for $x \in T$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding quotient for $f^{-1}(T) \subset Y$ and $f^*\mathcal{F}$.

Proof. Condition (3) just means that $\text{Ass}(\mathcal{F}') \cap T = \emptyset$. Thus $\mathcal{F} \to \mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections whose support is contained in T. This proves uniqueness. The statement on pullbacks follows from Divisors, Lemma 31.3.1 and the uniqueness.

Existence of $\mathcal{F} \to \mathcal{F}'$. By the uniqueness it suffices to prove the existence and uniqueness locally on X; small detail omitted. Thus we may assume $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ is the coherent module associated to the finite A-module M. Set $M' = M/H^0_T(M)$ with $H^0_T(M)$ as in Section 51.5. Then $M_\mathfrak{p} = M'_\mathfrak{p}$ for $\mathfrak{p} \notin T$ which proves (1). On the other hand, we have $H^0_T(M) = \text{colim}\, H^0_Z(M)$ where Z runs over the closed subsets of X contained in T. Thus by Dualizing Complexes, Lemmas 47.11.6 we have $H^0_T(M') = 0$, i.e., no associated prime of M' is in T. Therefore $\text{depth}(M'_\mathfrak{p}) \geq 1$ for $\mathfrak{p} \in T$. □

0DX4 **Lemma 51.15.2.** *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}' = j_*(\mathcal{F}|_U)$ is coherent. Then $\mathcal{F} \to \mathcal{F}'$ is the unique map of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}|_U \to \mathcal{F}'|_U$ is an isomorphism,*
(2) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ for $x \in X$, $x \notin U$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding map for $f^{-1}(U) \subset Y$.

Proof. We have $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ by Divisors, Lemma 31.6.6 part (3). The uniqueness of $\mathcal{F} \to \mathcal{F}'$ follows from Divisors, Lemma 31.5.11. The compatibility with flat pullbacks follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

0DX5 **Lemma 51.15.3.** *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume X is universally catenary and the formal fibres of local rings have (S_1). Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \in X \setminus Z$,*
(2) *$\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) = 1$ for $x \in Z$ such that there exists an immediate specialization $x' \rightsquigarrow x$ with $x' \notin Z$ and $x' \in Ass(\mathcal{F})$,*
(3) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for the remaining $x \in Z$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(Z) \subset Y$.

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be the map constructed in Lemma 51.15.1 for the subset Z of X. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on Z.

We first prove uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. We get a factorization $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ since $\text{Ass}(\mathcal{F}'') \cap Z = \emptyset$ by conditions (2) and (3). Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}'|_U \to \mathcal{F}''|_U$ is an isomorphism. We see that U contains all the points as in (2). Then by Divisors, Lemma 31.5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. Recall that $\text{Ass}(\mathcal{F}') = \{x_1, \ldots, x_n\}$ is finite and $x_i \notin Z$. Let Y_i be the closure of $\{x_i\}$. Let $Z_{i,j}$ be the irreducible components of $Z \cap Y_i$. Observe that $\text{Supp}(\mathcal{F}') \cap Z = \bigcup Z_{i,j}$. Let $z_{i,j} \in Z_{i,j}$ be the generic point. Let

$$d_{i,j} = \dim(\mathcal{O}_{\overline{\{x_i\}}, z_{i,j}})$$

If $d_{i,j} = 1$, then $z_{i,j}$ is one of the points as in (2). Thus we do not need to modify $\mathcal{F}'$ at these points. Furthermore, still assuming $d_{i,j} = 1$, using Lemma 51.9.2 we can find an open neighbourhood $z_{i,j} \in V_{i,j} \subset X$ such that $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$ for $z \in Z_{i,j} \cap V_{i,j}$, $z \neq z_{i,j}$. Set

$$Z' = X \setminus \left(X \setminus Z \cup \bigcup\nolimits_{d_{i,j}=1} V_{i,j}) \right)$$

Denote $j' : X \setminus Z' \to X$. By our choice of Z' the assumptions of Lemma 51.8.9 are satisfied. We conclude by setting $\mathcal{F}'' = j'_*(\mathcal{F}'|_{X \setminus Z'})$ and applying Lemma 51.15.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 10.163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

0EI3 **Lemma 51.15.4.** *Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that if $x \rightsquigarrow x'$ is an immediate specialization of points in X with $x' \in T'$ and $x \notin T'$, then $\text{depth}(\mathcal{F}_x) \geq 1$. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T'$,*
(2) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset Y$.

Proof. Proof of uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}|_U \to \mathcal{F}''|_U$ is an isomorphism. From (1) we see that U contains the set $X \setminus T'$. From (2) and Divisors, Lemma 31.5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}|_U)$ where $j : U \to X$ is the inclusion morphism. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. Let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 51.15.1 using T'. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on T'. We will define

$$\mathcal{F}'' = \text{colim}\, j_*(\mathcal{F}'|_V)$$

where $j : V \to X$ runs over the open subschemes such that $X \setminus V \subset T'$. Observe that the colimit is filtered as T' is stable under specialization. Each of the maps $\mathcal{F}' \to j_*(\mathcal{F}'|_V)$ is injective as $\text{Ass}(\mathcal{F}')$ is disjoint from T'. Thus $\mathcal{F}' \to \mathcal{F}''$ is injective.

Suppose $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ corresponds to the finite A-module M. Then $\mathcal{F}'$ corresponds to $M' = M/H^0_{T'}(M)$, see proof of Lemma 51.15.1. Applying Lemmas 51.2.2 and 51.5.3 we see that $\mathcal{F}''$ corresponds to an A-module M'' which fits into the short exact sequence

$$0 \to M' \to M'' \to H^1_{T'}(M') \to 0$$

By Proposition 51.11.1 and our condition on immediate specializations in the statement of the lemma we see that M'' is a finite A-module. In this way we see that $\mathcal{F}''$ is coherent.

From the coherence of $\mathcal{F}''$ and the injectivity of the transition maps in the displayed filtered colimit, we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_V)$ for any sufficiently small V as above. Then conditions (1) and (2) follow from Lemma 51.15.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 10.163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

0EI4 **Lemma 51.15.5.** *Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset T \subset X$ be subsets stable under specialization such that if $x \rightsquigarrow x'$ is an immediate specialization of points in X and $x' \in T'$, then $x \in T$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T$,*
(2) *$\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 1$ for $x \in T$, $x \notin T'$, and*
(3) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset f^{-1}(T) \subset Y$.

Proof. First, let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 51.15.1 using T. Second, let $\mathcal{F}' \to \mathcal{F}''$ be the unique map of coherent modules construction in Lemma 51.15.4 using T'. Then $\mathcal{F} \to \mathcal{F}''$ is as desired. □

51.16. Hartshorne-Lichtenbaum vanishing

0EB0 This vanishing result is the local analogue of Lichtenbaum's theorem that the reader can find in Duality for Schemes, Section 48.34. This and much else besides can be found in [**Har68**].

0EB1 **Lemma 51.16.1.** *Let A be a Noetherian ring of dimension d. Let $I \subset I' \subset A$ be ideals. If I' is contained in the Jacobson radical of A and $cd(A, I') < d$, then $cd(A, I) < d$.*

Proof. By Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$. We will use Lemma 51.2.6 to show

$$H^d_{V(I')}(A) \to H^d_{V(I)}(A)$$

is surjective which will finish the proof. Pick $\mathfrak{p} \in V(I) \setminus V(I')$. By our assumption on I' we see that $\mathfrak{p}$ is not a maximal ideal of A. Hence $\dim(A_\mathfrak{p}) < d$. Then $H^d_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) = 0$ by Lemma 51.4.7. □

0EB2 **Lemma 51.16.2.** *Let A be a Noetherian ring of dimension d. Let $I \subset A$ be an ideal. If $H^d_{V(I)}(M) = 0$ for some finite A-module whose support contains all the irreducible components of dimension d, then $\text{cd}(A, I) < d$.*

Proof. By Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$. Thus for any finite A-module N we have $H^i_{V(I)}(N) = 0$ for $i > d$. Let us say property $\mathcal{P}$ holds for the finite A-module N if $H^d_{V(I)}(N) = 0$. One of our assumptions is that $\mathcal{P}(M)$ holds. Observe that $\mathcal{P}(N_1 \oplus N_2) \Leftrightarrow (\mathcal{P}(N_1) \wedge \mathcal{P}(N_2))$. Observe that if $N \to N'$ is surjective, then $\mathcal{P}(N) \Rightarrow \mathcal{P}(N')$ as we have the vanishing of $H^{d+1}_{V(I)}$ (see above). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of A with $\dim(A/\mathfrak{p}_i) = d$. Observe that $\mathcal{P}(N)$ holds if the support of N is disjoint from $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for dimension reasons, see Lemma 51.4.7. For each i set $M_i = M/\mathfrak{p}_i M$. This is a finite A-module annihilated by $\mathfrak{p}_i$ whose support is equal to $V(\mathfrak{p}_i)$ (here we use the assumption on the support of M). Finally, if $J \subset A$ is an ideal, then we have $\mathcal{P}(JM_i)$ as JM_i is a quotient of a direct sum of copies of M. Thus it follows from Cohomology of Schemes, Lemma 30.12.8 that $\mathcal{P}$ holds for every finite A-module. □

0EB3 **Lemma 51.16.3.** *Let A be a Noetherian local ring of dimension d. Let $f \in A$ be an element which is not contained in any minimal prime of dimension d. Then $f : H^d_{V(I)}(M) \to H^d_{V(I)}(M)$ is surjective for any finite A-module M and any ideal $I \subset A$.*

Proof. The support of M/fM has dimension $< d$ by our assumption on f. Thus $H^d_{V(I)}(M/fM) = 0$ by Lemma 51.4.7. Thus $H^d_{V(I)}(fM) \to H^d_{V(I)}(M)$ is surjective. Since by Lemma 51.4.7 we know $\text{cd}(A, I) \leq d$ we also see that the surjection $M \to fM$, $x \mapsto fx$ induces a surjection $H^d_{V(I)}(M) \to H^d_{V(I)}(fM)$. □

0EB4 **Lemma 51.16.4.** *Let A be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $I \subset A$ be an ideal. If $H^0_{V(I)}(\omega_A^\bullet) = 0$, then $\text{cd}(A, I) < \dim(A)$.*

Proof. Set $d = \dim(A)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Recall that the finite A-module $H^{-i}(\omega_A^\bullet)$ is nonzero only for $i \in \{0, \ldots, d\}$ and that the support of $H^{-i}(\omega_A^\bullet)$ has dimension $\leq i$, see Lemma 51.9.4. Set $\omega_A = H^{-d}(\omega_A^\bullet)$. By prime avoidance (Algebra, Lemma 10.15.2) we can find $f \in A$, $f \notin \mathfrak{p}_i$ which annihilates $H^{-i}(\omega_A^\bullet)$ for $i < d$. Consider the distinguished triangle

$$\omega_A[d] \to \omega_A^\bullet \to \tau_{\geq -d+1}\omega_A^\bullet \to \omega_A[d+1]$$

See Derived Categories, Remark 13.12.4. By Derived Categories, Lemma 13.12.5 we see that f^d induces the zero endomorphism of $\tau_{\geq -d+1}\omega_A^\bullet$. Using the axioms of a triangulated category, we find a map

$$\omega_A^\bullet \to \omega_A[d]$$

whose composition with $\omega_A[d] \to \omega_A^\bullet$ is multiplication by f^d on $\omega_A[d]$. Thus we conclude that f^d annihilates $H^d_{V(I)}(\omega_A)$. By Lemma 51.16.3 we conlude $H^d_{V(I)}(\omega_A) = 0$. Then we conclude by Lemma 51.16.2 and the fact that $(\omega_A)_{\mathfrak{p}_i}$ is nonzero (see for example Dualizing Complexes, Lemma 47.16.11). □

0EB5 **Lemma 51.16.5.** *Let $(A, \mathfrak{m})$ be a complete Noetherian local domain. Let $\mathfrak{p} \subset A$ be a prime ideal of dimension 1. For every $n \geq 1$ there is an $m \geq n$ such that $\mathfrak{p}^{(m)} \subset \mathfrak{p}^n$.*

Proof. Recall that the symbolic power $\mathfrak{p}^{(m)}$ is defined as the kernel of $A \to A_\mathfrak{p}/\mathfrak{p}^m A_\mathfrak{p}$. Since localization is exact we conclude that in the short exact sequence

$$0 \to \mathfrak{a}_n \to A/\mathfrak{p}^n \to A/\mathfrak{p}^{(n)} \to 0$$

the support of $\mathfrak{a}_n$ is contained in $\{\mathfrak{m}\}$. In particular, the inverse system $(\mathfrak{a}_n)$ is Mittag-Leffler as each $\mathfrak{a}_n$ is an Artinian A-module. We conclude that the lemma is equivalent to the requirement that $\lim \mathfrak{a}_n = 0$. Let $f \in \lim \mathfrak{a}_n$. Then f is an element of $A = \lim A/\mathfrak{p}^n$ (here we use that A is complete) which maps to zero in the completion $A^\wedge_\mathfrak{p}$ of $A_\mathfrak{p}$. Since $A_\mathfrak{p} \to A^\wedge_\mathfrak{p}$ is faithfully flat, we see that f maps to zero in $A_\mathfrak{p}$. Since A is a domain we see that f is zero as desired. □

0EB6 **Proposition 51.16.6.** *Let A be a Noetherian local ring with completion $A^\wedge$. Let $I \subset A$ be an ideal such that* [**Har68**, Theorem 3.1]

$$\dim V(IA^\wedge + \mathfrak{p}) \geq 1$$

for every minimal prime $\mathfrak{p} \subset A^\wedge$ of dimension $\dim(A)$. Then $cd(A, I) < \dim(A)$.

Proof. Since $A \to A^\wedge$ is faithfully flat we have $H^d_{V(I)}(A) \otimes_A A^\wedge = H^d_{V(IA^\wedge)}(A^\wedge)$ by Dualizing Complexes, Lemma 47.9.3. Thus we may assume A is complete.

Assume A is complete. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Consider the complete local ring $A_i = A/\mathfrak{p}_i$. We have $H^d_{V(I)}(A_i) = H^d_{V(IA_i)}(A_i)$ by Dualizing Complexes, Lemma 47.9.2. By Lemma 51.16.2 it suffices to prove the lemma for (A_i, IA_i). Thus we may assume A is a complete local domain.

Assume A is a complete local domain. We can choose a prime ideal $\mathfrak{p} \supset I$ with $\dim(A/\mathfrak{p}) = 1$. By Lemma 51.16.1 it suffices to prove the lemma for $\mathfrak{p}$.

By Lemma 51.16.4 it suffices to show that $H^0_{V(\mathfrak{p})}(\omega^\bullet_A) = 0$. Recall that

$$H^0_{V(\mathfrak{p})}(\omega^\bullet_A) = \operatorname{colim} \operatorname{Ext}^0_A(A/\mathfrak{p}^n, \omega^\bullet_A)$$

By Lemma 51.16.5 we see that the colimit is the same as

$$\operatorname{colim} \operatorname{Ext}^0_A(A/\mathfrak{p}^{(n)}, \omega^\bullet_A)$$

Since $\operatorname{depth}(A/\mathfrak{p}^{(n)}) = 1$ we see that these ext groups are zero by Lemma 51.9.4 as desired. □

0EB7 **Lemma 51.16.7.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Assume A is excellent, normal, and $\dim V(I) \geq 1$. Then $cd(A, I) < \dim(A)$. In particular, if $\dim(A) = 2$, then $\operatorname{Spec}(A) \setminus V(I)$ is affine.*

Proof. By More on Algebra, Lemma 15.52.6 the completion $A^\wedge$ is normal and hence a domain. Thus the assumption of Proposition 51.16.6 holds and we conclude. The statement on affineness follows from Lemma 51.4.8. □

51.17. Frobenius action

0EBU Let p be a prime number. Let A be a ring with $p = 0$ in A. The *Frobenius endomorphism* of A is the map

$$F : A \longrightarrow A, \quad a \longmapsto a^p$$

In this section we prove lemmas on modules which have Frobenius actions.

0EBV **Lemma 51.17.1.** *Let p be a prime number. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with $p = 0$ in A. Let M be a finite A-module such that $M \otimes_{A,F} A \cong M$. Then M is finite free.*

Proof. Choose a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ which induces an isomorphism $\kappa^{\oplus n} \to M/\mathfrak{m}M$. Let $T = (a_{ij})$ be the matrix of the map $A^{\oplus m} \to A^{\oplus n}$. Observe that $a_{ij} \in \mathfrak{m}$. Applying base change by F, using right exactness of base change, we get a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ where the matrix is $T = (a_{ij}^p)$. Thus we have a presentation with $a_{ij} \in \mathfrak{m}^p$. Repeating this construction we find that for each $e \geq 1$ there exists a presentation with $a_{ij} \in \mathfrak{m}^e$. This implies the fitting ideals (More on Algebra, Definition 15.8.3) $\mathrm{Fit}_k(M)$ for $k < n$ are contained in $\bigcap_{e \geq 1} \mathfrak{m}^e$. Since this is zero by Krull's intersection theorem (Algebra, Lemma 10.51.4) we conclude that M is free of rank n by More on Algebra, Lemma 15.8.8. □

In this section, we say elements $f_1, \ldots, f_r$ of a ring A are *independent* if $\sum a_i f_i = 0$ implies $a_i \in (f_1, \ldots, f_r)$. In other words, with $I = (f_1, \ldots, f_r)$ we have I/I^2 is free over A/I with basis $f_1, \ldots, f_r$.

0EBW **Lemma 51.17.2.** *Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent, then $f_1, \ldots, f_r$ are independent.* See [**Lec64**] and [**Mat70a**, Lemma 1 page 299].

Proof. Say $\sum a_i f_i = 0$. Then $\sum a_i g_r f_i = 0$. Hence $a_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$. Write $a_r = \sum_{i<r} b_i f_i + b f_r g_r$. Then $0 = \sum_{i<r} (a_i + b_i f_r) f_i + b f_r^2 g_r$. Thus $a_i + b_i f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which implies $a_i \in (f_1, \ldots, f_r)$ as desired. □

0EBX **Lemma 51.17.3.** *Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent and if the A-module $A/(f_1, \ldots, f_{r-1}, f_r g_r)$ has finite length, then* See [**Lec64**] and [**Mat70a**, Lemma 2 page 300].

$$\begin{aligned} &\mathit{length}_A(A/(f_1, \ldots, f_{r-1}, f_r g_r)) \\ &= \mathit{length}_A(A/(f_1, \ldots, f_{r-1}, f_r)) + \mathit{length}_A(A/(f_1, \ldots, f_{r-1}, g_r)) \end{aligned}$$

Proof. We claim there is an exact sequence

$$0 \to A/(f_1, \ldots, f_{r-1}, g_r) \xrightarrow{f_r} A/(f_1, \ldots, f_{r-1}, f_r g_r) \to A/(f_1, \ldots, f_{r-1}, f_r) \to 0$$

Namely, if $a f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$, then $\sum_{i<r} a_i f_i + (a + b g_r) f_r = 0$ for some $b, a_i \in A$. Hence $\sum_{i<r} a_i g_r f_i + (a + b g_r) g_r f_r = 0$ which implies $a + b g_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which means that a maps to zero in $A/(f_1, \ldots, f_{r-1}, g_r)$. This proves the claim. To finish use additivity of lengths (Algebra, Lemma 10.52.3). □

0EBY **Lemma 51.17.4.** *Let $(A, \mathfrak{m})$ be a local ring. If $\mathfrak{m} = (x_1, \ldots, x_r)$ and $x_1^{e_1}, \ldots, x_r^{e_r}$ are independent for some $e_i > 0$, then $\mathit{length}_A(A/(x_1^{e_1}, \ldots, x_r^{e_r})) = e_1 \ldots e_r$.* See [**Lec64**] and [**Mat70a**, Lemma 3 page 300].

Proof. Use Lemmas 51.17.2 and 51.17.3 and induction. □

0EBZ **Lemma 51.17.5.** *Let $\varphi : A \to B$ be a flat ring map. If $f_1, \ldots, f_r \in A$ are independent, then $\varphi(f_1), \ldots, \varphi(f_r) \in B$ are independent.*

Proof. Let $I = (f_1, \ldots, f_r)$ and $J = \varphi(I)B$. By flatness we have $I/I^2 \otimes_A B = J/J^2$. Hence freeness of I/I^2 over A/I implies freeness of J/J^2 over B/J. □

0EC0 **Lemma 51.17.6** (Kunz)**.** *Let p be a prime number. Let A be a Noetherian ring with $p = 0$. The following are equivalent* [**Kun69**]

(1) *A is regular, and*
(2) *$F : A \to A$, $a \mapsto a^p$ is flat.*

Proof. Observe that $\mathrm{Spec}(F) : \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is the identity map. Being regular is defined in terms of the local rings and being flat is something about local rings, see Algebra, Lemma 10.39.18. Thus we may and do assume A is a Noetherian local ring with maximal ideal $\mathfrak{m}$.

Assume A is regular. Let $x_1, \ldots, x_d$ be a system of parameters for A. Applying F we find $F(x_1), \ldots, F(x_d) = x_1^p, \ldots, x_d^p$, which is a system of parameters for A. Hence F is flat, see Algebra, Lemmas 10.128.1 and 10.106.3.

Conversely, assume F is flat. Write $\mathfrak{m} = (x_1, \ldots, x_r)$ with r minimal. Then $x_1, \ldots, x_r$ are independent in the sense defined above. Since F is flat, we see that $x_1^p, \ldots, x_r^p$ are independent, see Lemma 51.17.5. Hence $\mathrm{length}_A(A/(x_1^p, \ldots, x_r^p)) = p^r$ by Lemma 51.17.4. Let $\chi(n) = \mathrm{length}_A(A/\mathfrak{m}^n)$ and recall that this is a numerical polynomial of degree $\dim(A)$, see Algebra, Proposition 10.60.9. Choose $n \gg 0$. Observe that

$$\mathfrak{m}^{pn+pr} \subset F(\mathfrak{m}^n)A \subset \mathfrak{m}^{pn}$$

as can be seen by looking at monomials in $x_1, \ldots, x_r$. We have

$$A/F(\mathfrak{m}^n)A = A/\mathfrak{m}^n \otimes_{A,F} A$$

By flatness of F this has length $\chi(n)\mathrm{length}_A(A/F(\mathfrak{m})A)$ (Algebra, Lemma 10.52.13) which is equal to $p^r\chi(n)$ by the above. We conclude

$$\chi(pn + pr) \geq p^r\chi(n) \geq \chi(pn)$$

Looking at the leading terms this implies $r = \dim(A)$, i.e., A is regular. □

51.18. Structure of certain modules

0EC1 Some results on the structure of certain types of modules over regular local rings. These types of results and much more can be found in [**HS93**], [**Lyu93**], [**Lyu97**].

0EC2 **Lemma 51.18.1.** *Let k be a field of characteristic 0. Let $d \geq 1$. Let $A = k[[x_1, \ldots, x_d]]$ with maximal ideal $\mathfrak{m}$. Let M be an $\mathfrak{m}$-power torsion A-module endowed with additive operators $D_1, \ldots, D_d$ satisfying the leibniz rule*

$$D_i(fz) = \partial_i(f)z + fD_i(z)$$

for $f \in A$ and $z \in M$. Here ∂_i is differentiation with respect to x_i. Then M is isomorphic to a direct sum of copies of the injective hull E of k.

Special case of [**Lyu93**, Theorem 2.4]

Proof. Choose a set J and an isomorphism $M[\mathfrak{m}] \to \bigoplus_{j\in J} k$. Since $\bigoplus_{j\in J} E$ is injective (Dualizing Complexes, Lemma 47.3.7) we can extend this isomorphism to an A-module homomorphism $\varphi : M \to \bigoplus_{j\in J} E$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Let $z \in M$. Then $x_1^n z = 0$ for some $n \geq 0$. We will prove that $z \in x_1 M$ by induction on n. If $n = 0$, then $z = 0$ and the result is true. If $n > 0$, then applying D_1 we find $0 = nx_1^{n-1}z + x_1^n D_1(z)$. Hence $x_1^{n-1}(nz + x_1 D_1(z)) = 0$. By induction we get $nz + x_1 D_1(z) \in x_1 M$. Since n is invertible, we conclude $z \in x_1 M$. Thus we see that M is x_1-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j\in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular

$x_1 e \in M$. There exists an element $z_1 \in M$ with $x_1 z_1 = x_1 e$. Hence $x_1(z_1 - e) = 0$. Replacing e by $e - z_1$ we may assume e is annihilated by x_1. Thus it suffices to prove that

$$\varphi[x_1] : M[x_1] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x_1] = \bigoplus\nolimits_{j \in J} E[x_1]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x_1]$ is the injective hull of the residue field of $k[[x_2, \ldots, x_d]]$, see Dualizing Complexes, Lemma 47.7.1. Observe that $M[x_1]$ as a module over $k[[x_2, \ldots, x_d]]$ is $\mathfrak{m}/(x_1)$-power torsion and comes equipped with operators $D_2, \ldots, D_d$ satisfying the displayed Leibniz rule. Thus by induction on d we conclude that $\varphi[x_1]$ is surjective as desired. □

0EC3 **Lemma 51.18.2.** *Let p be a prime number. Let $(A, \mathfrak{m}, k)$ be a regular local ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ be the Frobenius endomorphism. Let M be a $\mathfrak{m}$-power torsion module such that $M \otimes_{A,F} A \cong M$. Then M is isomorphic to a direct sum of copies of the injective hull E of k.*

Follows from [**HS93**, Corollary 3.6] with a little bit of work. Also follows directly from [**Lyu97**, Theorem 1.4].

Proof. Choose a set J and an A-module homorphism $\varphi : M \to \bigoplus_{j \in J} E$ which maps $M[\mathfrak{m}]$ isomorphically onto $(\bigoplus_{j \in J} E)[\mathfrak{m}] = \bigoplus_{j \in J} k$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Recall that F is flat, see Lemma 51.17.6. Let $x_1, \ldots, x_d$ be a minimal system of generators of $\mathfrak{m}$. Denote

$$M_n = M[x_1^{p^n}, \ldots, x_d^{p^n}]$$

the submodule of M consisting of elements killed by $x_1^{p^n}, \ldots, x_d^{p^n}$. So $M_0 = M[\mathfrak{m}]$ is a vector space over k. Also $M = \bigcup M_n$ by our assumption that M is $\mathfrak{m}$-power torsion. Since F^n is flat and $F^n(x_i) = x_i^{p^n}$ we have

$$M_n \cong (M \otimes_{A,F^n} A)[x_1^{p^n}, \ldots, x_d^{p^n}] = M[x_1, \ldots, x_d] \otimes_{A,F^n} A = M_0 \otimes_k A/(x_1^{p^n}, \ldots, x_d^{p^n})$$

Thus M_n is free over $A/(x_1^{p^n}, \ldots, x_d^{p^n})$. A computation shows that every element of $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ annihilated by $x_1^{p^n - 1}$ is divisible by x_1; for example you can use that $A/(x_1^{p^n}, \ldots, x_d^{p^n}) \cong k[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$ by Algebra, Lemma 10.160.10. Thus the same is true for every element of M_n. Since every element of M is in M_n for all $n \gg 0$ and since every element of M is killed by some power of x_1, we conclude that M is x_1-divisible.

Let $x = x_1$. Above we have seen that M is x-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j \in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular $xe \in M$. There exists an element $z_1 \in M$ with $xz_1 = xe$. Hence $x(z_1 - e) = 0$. Replacing e by $e - z_1$ we may assume e is annihilated by x. Thus it suffices to prove that

$$\varphi[x] : M[x] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x] = \bigoplus\nolimits_{j \in J} E[x]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x]$ is the injective hull of the residue field of the regular ring A/xA, see

Dualizing Complexes, Lemma 47.7.1. Observe that $M[x]$ as a module over A/xA is $\mathfrak{m}/(x)$-power torsion and we have

$$\begin{aligned} M[x] \otimes_{A/xA,F} A/xA &= M[x] \otimes_{A,F} A \otimes_A A/xA \\ &= (M \otimes_{A,F} A)[x^p] \otimes_A A/xA \\ &\cong M[x^p] \otimes_A A/xA \end{aligned}$$

Argue using flatness of F as before. We claim that $M[x^p] \otimes_A A/xA \to M[x]$, $z \otimes 1 \mapsto x^{p-1}z$ is an isomorphism. This can be seen by proving it for each of the modules M_n, $n > 0$ defined above where it follows by the same result for $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ and $x = x_1$. Thus by induction on $\dim(A)$ we conclude that $\varphi[x]$ is surjective as desired. □

51.19. Additional structure on local cohomology

0EC4 Here is a sample result.

0EC5 **Lemma 51.19.1.** *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on the local cohomology modules $H^i_Z(A)$ satisfying the Leibniz rule with respect to θ.*

Proof. Let $f_1, \ldots, f_r$ be elements generating I. Recall that $R\Gamma_Z(A)$ is computed by the complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}$$

See Dualizing Complexes, Lemma 47.9.1. Since θ extends uniquely to an additive operator on any localization of A satisfying the Leibniz rule with respect to θ, the lemma is clear. □

0EC6 **Lemma 51.19.2.** *Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. There exists an isomorphism $R\Gamma_Z(A) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma_Z(A)$.*

Proof. Follows from Dualizing Complexes, Lemma 47.9.3 and the fact that $Z = V(f_1^p, \ldots, f_r^p)$ if $I = (f_1, \ldots, f_r)$. □

0EC7 **Lemma 51.19.3.** *Let A be a ring. Let $V \to \mathrm{Spec}(A)$ be quasi-compact, quasi-separated, and étale. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to θ.*

Proof. If V is separated, then we can argue using an affine open covering $V = \bigcup_{j=1,\ldots m} V_j$. Namely, because V is separated we may write $V_{j_0\ldots j_p} = \mathrm{Spec}(B_{j_0\ldots j_p})$. See Schemes, Lemma 26.21.7. Then we find that the A-module $H^i(V, \mathcal{O}_V)$ is the ith cohomology group of the Čech complex

$$\prod B_{j_0} \to \prod B_{j_0 j_1} \to \prod B_{j_0 j_1 j_2} \to \ldots$$

See Cohomology of Schemes, Lemma 30.2.6. Each $B = B_{j_0\ldots j_p}$ is an étale A-algebra. Hence $\Omega_B = \Omega_A \otimes_A B$ and we conclude θ extends uniquely to a derivation $\theta_B : B \to B$. These maps define an endomorphism of the Čech complex and define the desired operators on the cohomology groups.

In the general case we use a hypercovering of V by affine opens, exactly as in the first part of the proof of Cohomology of Schemes, Lemma 30.7.3. We omit the details. □

0EC8 **Remark 51.19.4.** We can upgrade Lemmas 51.19.1 and 51.19.3 to include higher order differential operators, using Algebra, Lemma 10.150.8 and Remark 10.150.9. If we ever need this we will state and prove a precise lemma here.

0EC9 **Lemma 51.19.5.** *Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. If $V \to \mathrm{Spec}(A)$ is quasi-compact, quasi-separated, and étale, then there exists an isomorphism $R\Gamma(V, \mathcal{O}_V) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma(V, \mathcal{O}_V)$.*

Proof. Observe that the relative Frobenius morphism

$$V \longrightarrow V \times_{\mathrm{Spec}(A),\mathrm{Spec}(F)} \mathrm{Spec}(A)$$

of V over A is an isomorphism, see Étale Morphisms, Lemma 41.14.3. Thus the lemma follows from cohomology and base change, see Derived Categories of Schemes, Lemma 36.22.5. Observe that since V is étale over A, it is flat over A. □

51.20. A bit of uniformity, I

0G9S The main task of this section is to formulate and prove Lemma 51.20.2.

0G9T **Lemma 51.20.1.** *Let R be a ring. Let $M \to M'$ be a map of R-modules with M of finite presentation such that $Tor_1^R(M, N) \to Tor_1^R(M', N)$ is zero for all R-modules N. Then $M \to M'$ factors through a free R-module.*

Proof. We may choose a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^{\oplus r} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & \bigoplus_{i\in I} R & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

whose right vertical arrow is the given map. We can factor this map through the short exact sequence

0G9U (51.20.1.1) $$0 \to K' \to E \to M \to 0$$

which is the pushout of the first short exact sequence by $K \to K'$. By a diagram chase we see that the assumption in the lemma implies that the boundary map $\mathrm{Tor}_1^R(M, N) \to K' \otimes_R N$ induced by (51.20.1.1) is zero, i.e., the sequence (51.20.1.1) is universally exact. This implies by Algebra, Lemma 10.82.4 that (51.20.1.1) is split (this is where we use that M is of finite presentation). Hence the map $M \to M'$ factors through $\bigoplus_{i\in I} R$ and we win. □

0G9V **Lemma 51.20.2.** *Let R be a ring. Let $\alpha : M \to M'$ be a map of R-modules. Let $P_\bullet \to M$ and $P'_\bullet \to M'$ be resolutions by projective R-modules. Let $e \geq 0$ be an integer. Consider the following conditions*

(1) *We can find a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ inducing α on cohomology with $a_i = 0$ for $i > e$.*

(2) *We can find a map of complexes* $a_\bullet : P_\bullet \to P'_\bullet$ *inducing* α *on cohomology with* $a_{e+1} = 0$.
(3) *The map* $\mathrm{Ext}^i_R(M', N) \to \mathrm{Ext}^i_R(M, N)$ *is zero for all* R*-modules* N *and* $i > e$.
(4) *The map* $\mathrm{Ext}^{e+1}_R(M', N) \to \mathrm{Ext}^{e+1}_R(M, N)$ *is zero for all* R*-modules* N.
(5) *Let* $N = \mathrm{Im}(P'_{e+1} \to P'_e)$ *and denote* $\xi \in \mathrm{Ext}^{e+1}_R(M', N)$ *the canonical element (see proof). Then* ξ *maps to zero in* $\mathrm{Ext}^{e+1}_R(M, N)$.
(6) *The map* $Tor^R_i(M, N) \to Tor^R_i(M', N)$ *is zero for all* R*-modules* N *and* $i > e$.
(7) *The map* $Tor^R_{e+1}(M, N) \to Tor^R_{e+1}(M', N)$ *is zero for all* R*-modules* N.

Then we always have the implications

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Leftrightarrow (7)$$

If M *is* $(-e-1)$*-pseudo-coherent (for example if* R *is Noetherian and* M *is a finite* R*-module), then all conditions are equivalent.*

Proof. It is clear that (2) implies (1). If $a_\bullet$ is as in (1), then we can consider the map of complexes $a'_\bullet : P_\bullet \to P'_\bullet$ with $a'_i = a_i$ for $i \leq e+1$ and $a'_i = 0$ for $i \geq e+1$ to get a map of complexes as in (2). Thus (1) is equivalent to (2).

By the construction of the Ext and Tor functors using resolutions (Algebra, Sections 10.71 and 10.75) we see that (1) and (2) imply all of the other conditions.

It is clear that (3) implies (4) implies (5). Let N be as in (5). The canonical map $\tilde{\xi} : P'_{e+1} \to N$ precomposed with $P'_{e+2} \to P'_{e+1}$ is zero. Hence we may consider the class ξ of $\tilde{\xi}$ in

$$\mathrm{Ext}^{e+1}_R(M', N) = \frac{\mathrm{Ker}(\mathrm{Hom}(P'_{e+1}, N \to \mathrm{Hom}(P'_{e+2}, N)}{\mathrm{Im}(\mathrm{Hom}(P'_e, N \to \mathrm{Hom}(P'_{e+1}, N)}$$

Choose a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ lifting α, see Derived Categories, Lemma 13.19.6. If ξ maps to zero in $\mathrm{Ext}^{e+1}_R(M', N)$, then we find a map $\varphi : P_e \to N$ such that $\tilde{\xi} \circ a_{e+1} = \varphi \circ d$. Thus we obtain a map of complexes

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & P_{e+1} & \longrightarrow & P_e & \longrightarrow & P_{e-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle a_e - \varphi} & & \downarrow{\scriptstyle a_{e-1}} & & \\
\ldots & \longrightarrow & P'_{e+1} & \longrightarrow & P'_e & \longrightarrow & P'_{e-1} & \longrightarrow & \ldots
\end{array}$$

as in (2). Hence (1) – (5) are equivalent.

The equivalence of (6) and (7) follows from dimension shifting; we omit the details.

Assume M is $(-e-1)$-pseudo-coherent. (The parenthetical statement in the lemma follows from More on Algebra, Lemma 15.65.17.) We will show that (7) implies (4) which finishes the proof. We will use induction on e. The base case is $e = 0$. Then M is of finite presentation by More on Algebra, Lemma 15.65.4 and we conclude from Lemma 51.20.1 that $M \to M'$ factors through a free module. Of course if $M \to M'$ factors through a free module, then $\mathrm{Ext}^i_R(M', N) \to \mathrm{Ext}^i_R(M, N)$ is zero for all $i > 0$ as desired. Assume $e > 0$. We may choose a map of short exact

sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^{\oplus r} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & \bigoplus_{i\in I} R & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

whose right vertical arrow is the given map. We obtain $\mathrm{Tor}_{i+1}^R(M,N) = \mathrm{Tor}_i^R(K,N)$ and $\mathrm{Ext}_R^{i+1}(M,N) = \mathrm{Ext}_R^i(K,N)$ for $i \geq 1$ and all R-modules N and similarly for M', K'. Hence we see that $\mathrm{Tor}_e^R(K,N) \to \mathrm{Tor}_e^R(K',N)$ is zero for all R-modules N. By More on Algebra, Lemma 15.65.2 we see that K is $(-e)$-pseudo-coherent. By induction we conclude that $\mathrm{Ext}^e(K',N) \to \mathrm{Ext}^e(K,N)$ is zero for all R-modules N, which gives what we want. □

0EH1 **Lemma 51.20.3.** *Let I be an ideal of a Noetherian ring A. For all $n \geq 1$ there exists an $m > n$ such that the map $A/I^m \to A/I^n$ satisfies the equivalent conditions of Lemma 51.20.2 with $e = cd(A,I)$.*

Proof. Let $\xi \in \mathrm{Ext}_A^{e+1}(A/I^n, N)$ be the element constructed in Lemma 51.20.2 part (5). Since $e = \mathrm{cd}(A,I)$ we have $0 = H_Z^{e+1}(N) = H_I^{e+1}(N) = \mathrm{colim}\, \mathrm{Ext}^{e+1}(A/I^m, N)$ by Dualizing Complexes, Lemmas 47.10.1 and 47.8.2. Thus we may pick $m \geq n$ such that ξ maps to zero in $\mathrm{Ext}_A^{e+1}(A/I^m, N)$ as desired. □

51.21. A bit of uniformity, II

0G9W Let I be an ideal of a Noetherian ring A. Let M be a finite A-module. Let $i > 0$. By More on Algebra, Lemma 15.27.3 there exists a $c = c(A,I,M,i)$ such that $\mathrm{Tor}_i^A(M, A/I^n) \to \mathrm{Tor}_i^A(M, A/I^{n-c})$ is zero for all $n \geq c$. In this section, we discuss some results which show that one sometimes can choose a constant c which works for all A-modules M simultaneously (and for a range of indices i). This material is related to uniform Artin-Rees as discussed in [**Hun92**] and [**AHS15**].

In Remark 51.21.9 we will apply this to show that various pro-systems related to derived completion are (or are not) strictly pro-isomorphic.

The following lemma can be significantly strengthened.

0G9X **Lemma 51.21.1.** *Let I be an ideal of a Noetherian ring A. For every $m \geq 0$ and $i > 0$ there exist a $c = c(A,I,m,i) \geq 0$ such that for every A-module M annihilated by I^m the map*

$$Tor_i^A(M, A/I^n) \to Tor_i^A(M, A/I^{n-c})$$

is zero for all $n \geq c$.

Proof. By induction on i. Base case $i = 1$. The short exact sequence $0 \to I^n \to A \to A/I^n \to 0$ determines an injection $\mathrm{Tor}_1^A(M, A/I^n) \subset I^n \otimes_A M$, see Algebra, Remark 10.75.9. As M is annihilated by I^m we see that the map $I^n \otimes_A M \to I^{n-m} \otimes_A M$ is zero for $n \geq m$. Hence the result holds with $c = m$.

Induction step. Let $i > 1$ and assume c works for $i-1$. By More on Algebra, Lemma 15.27.3 applied to $M = A/I^m$ we can choose $c' \geq 0$ such that $\mathrm{Tor}_i(A/I^m, A/I^n) \to \mathrm{Tor}_i(A/I^m, A/I^{n-c'})$ is zero for $n \geq c'$. Let M be annihilated by I^m. Choose a short exact sequence

$$0 \to S \to \bigoplus\nolimits_{i\in I} A/I^m \to M \to 0$$

The corresponding long exact sequence of tors gives an exact sequence

$$\mathrm{Tor}_i^A(\bigoplus\nolimits_{i\in I} A/I^m, A/I^n) \to \mathrm{Tor}_i^A(M, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^n)$$

for all integers $n \geq 0$. If $n \geq c+c'$, then the map $\mathrm{Tor}_{i-1}^A(S, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^{n-c})$ is zero and the map $\mathrm{Tor}_i^A(A/I^m, A/I^{n-c}) \to \mathrm{Tor}_i^A(A/I^m, A/I^{n-c-c'})$ is zero. Combined with the short exact sequences this implies the result holds for i with constant $c + c'$. □

0G9Y **Lemma 51.21.2.** *Let $I = (a_1, \ldots, a_t)$ be an ideal of a Noetherian ring A. Set $a = a_1$ and denote $B = A[\frac{I}{a}]$ the affine blowup algebra. There exists a $c > 0$ such that $Tor_i^A(B, M)$ is annihilated by I^c for all A-modules M and $i \geq t$.*

Proof. Recall that B is the quotient of $A[x_2, \ldots, x_t]/(a_1x_2 - a_2, \ldots, a_1x_t - a_t)$ by its a_1-torsion, see Algebra, Lemma 10.70.6. Let

$$B_\bullet = \text{Koszul complex on } a_1x_2 - a_2, \ldots, a_1x_t - a_t \text{ over } A[x_2, \ldots, x_t]$$

viewed as a chain complex sitting in degrees $(t-1), \ldots, 0$. The complex $B_\bullet[1/a_1]$ is isomorphic to the Koszul complex on $x_2 - a_2/a_1, \ldots, x_t - a_t/a_1$ which is a regular sequence in $A[1/a_1][x_2, \ldots, x_t]$. Since regular sequences are Koszul regular, we conclude that the augmentation

$$\epsilon : B_\bullet \longrightarrow B$$

is a quasi-isomorphism after inverting a_1. Since the homology modules of the cone $C_\bullet$ on ϵ are finite $A[x_2, \ldots, x_n]$-modules and since $C_\bullet$ is bounded, we conclude that there exists a $c \geq 0$ such that a_1^c annihilates all of these. By Derived Categories, Lemma 13.12.5 this implies that, after possibly replacing c by a larger integer, that a_1^c is zero on $C_\bullet$ in $D(A)$. The proof is finished once the reader contemplates the distinguished triangle

$$B_\bullet \otimes_A^{\mathbf{L}} M \to B \otimes_A^{\mathbf{L}} M \to C_\bullet \otimes_A^{\mathbf{L}} M$$

Namely, the first term is represented by $B_\bullet \otimes_A M$ which is sitting in homological degrees $(t-1), \ldots, 0$ in view of the fact that the terms in the Koszul complex $B_\bullet$ are free (and hence flat) A-modules. Whence $\mathrm{Tor}_i^A(B, M) = H_i(C_\bullet \otimes_A^{\mathbf{L}} M)$ for $i > t-1$ and this is annihilated by a_1^c. Since $a_1^c B = I^c B$ and since the tor module is a module over B we conclude. □

For the rest of the discussion in this section we fix a Noetherian ring A and an ideal $I \subset A$. We denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n\geq 0} I^n$. By Cohomology of Schemes, Lemmas 30.14.2 and 30.14.3 we can choose an integer $q(A, I) \geq 0$ such that for all $q \geq q(A, I)$ we have $H^i(X, \mathcal{O}_X(q)) = 0$ for $i > 0$ and $H^0(X, \mathcal{O}_X(q)) = I^q$.

0G9Z **Lemma 51.21.3.** *In the situation above, for $q \geq q(A, I)$ and any A-module M we have*

$$R\Gamma(X, Lp^*\widetilde{M}(q)) \cong M \otimes_A^{\mathbf{L}} I^q$$

in $D(A)$.

Proof. Choose a free resolution $F_\bullet \to M$. Then $\widetilde{F}_\bullet$ is a flat resolution of $\widetilde{M}$. Hence $Lp^*\widetilde{M}$ is given by the complex $p^*\widetilde{F}_\bullet$. Thus $Lp^*\widetilde{M}(q)$ is given by the complex $p^*\widetilde{F}_\bullet(q)$. Since $p^*\widetilde{F}_i(q)$ are right acyclic for $\Gamma(X, -)$ by our choice of $q \geq q(A, I)$ and since we have $\Gamma(X, p^*\widetilde{F}_i(q)) = I^q F_i$ by our choice of $q \geq q(A, I)$, we get that $R\Gamma(X, Lp^*\widetilde{M}(q))$ is given by the complex with terms $I^q F_i$ by Derived Categories of Schemes, Lemma 36.4.3. The result follows as the complex $I^q F_\bullet$ computes $M \otimes^{\mathbf{L}}_A I^q$ by definition. □

0GA0 **Lemma 51.21.4.** *In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the cohomology sheaves $H^j(Lp^*\widetilde{M})$ are annihilated by I^c for $j \leq -t$.*

Proof. Say $I = (a_1, \ldots, a_t)$. The question is affine local on X. For $1 \leq i \leq t$ let $B_i = A[\frac{I}{a_i}]$ be the affine blowup algebra. Then X has an affine open covering by the spectra of the rings B_i, see Divisors, Lemma 31.32.2. By the description of derived pullback given in Derived Categories of Schemes, Lemma 36.3.8 we conclude it suffices to prove that for each i there exists a $c \geq 0$ such that

$$\mathrm{Tor}^A_j(B_i, M)$$

is annihilated by I^c for $j \geq t$. This is Lemma 51.21.2. □

0GA1 **Lemma 51.21.5.** *In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the tor modules $Tor^A_i(M, A/I^q)$ are annihilated by I^c for $i > t$ and all $q \geq 0$.*

Proof. Let $q(A, I)$ be as above. For $q \geq q(A, I)$ we have

$$R\Gamma(X, Lp^*\widetilde{M}(q)) = M \otimes^{\mathbf{L}}_A I^q$$

by Lemma 51.21.3. We have a bounded and convergent spectral sequence

$$H^a(X, H^b(Lp^*\widetilde{M}(q))) \Rightarrow \mathrm{Tor}^A_{-a-b}(M, I^q)$$

by Derived Categories of Schemes, Lemma 36.4.4. Let d be an integer as in Cohomology of Schemes, Lemma 30.4.4 (actually we can take $d = t$, see Cohomology of Schemes, Lemma 30.4.2). Then we see that $H^{-i}(X, Lp^*\widetilde{M}(q)) = \mathrm{Tor}^A_i(M, I^q)$ has a finite filtration with at most d steps whose graded are subquotients of the modules

$$H^a(X, H^{-i-a}(Lp^*\widetilde{M})(q)), \quad a = 0, 1, \ldots, d-1$$

If $i \geq t$ then all of these modules are annihilated by I^c where $c = c(A, I)$ is as in Lemma 51.21.4 because the cohomology sheaves $H^{-i-a}(Lp^*\widetilde{M})$ are all annihilated by I^c by the lemma. Hence we see that $\mathrm{Tor}^A_i(M, I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i \geq t$. Using the short exact sequence $0 \to I^q \to A \to A/I^q \to 0$ we find that $\mathrm{Tor}_i(M, A/I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i > t$. We conclude that I^m with $m = \max(dc, q(A, I) - 1)$ annihilates $\mathrm{Tor}^A_i(M, A/I^q)$ for all $q \geq 0$ and $i > t$ as desired. □

0GA2 **Lemma 51.21.6.** *Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that the maps*

$$Tor^A_{t+1}(M, A/I^n) \to Tor^A_{t+1}(M, A/I^{n-c})$$

are zero for any A-module M and all $n \geq N$.

Proof. Let c_1 be the constant found in Lemma 51.21.5. Please keep in mind that this constant c_1 works for Tor_i for all $i > t$ simultaneously.

Say $I = (a_1, \ldots, a_t)$. For an A-module M we set

$$\ell(M) = \#\{i \mid 1 \leq i \leq t,\ a_i^{c_1} \text{ is zero on } M\}$$

This is an element of $\{0, 1, \ldots, t\}$. We will prove by descending induction on $0 \leq s \leq t$ the following statement H_s: there exist $N, c \geq 0$ such that for every module M with $\ell(M) \geq s$ the maps

$$\mathrm{Tor}^A_{t+1+i}(M, A/I^n) \to \mathrm{Tor}^A_{t+1+i}(M, A/I^{n-c})$$

are zero for $i = 0, \ldots, s$ for all $n \geq N$.

Base case: $s = t$. If $\ell(M) = t$, then M is annihilated by $(a_1^{c_1}, \ldots, a_t^{c_1}\}$ and hence by $I^{t(c_1-1)+1}$. We conclude from Lemma 51.21.1 that H_t holds by taking $c = N$ to be the maximum of the integers $c(A, I, t(c_1-1)+1, t+1), \ldots, c(A, I, t(c_1-1)+1, 2t+1)$ found in the lemma.

Induction step. Say $0 \leq s < t$ we have N, c as in H_{s+1}. Consider a module M with $\ell(M) = s$. Then we can choose an i such that $a_i^{c_1}$ is nonzero on M. It follows that $\ell(M[a_i^c]) \geq s + 1$ and $\ell(M/a_i^{c_1}M) \geq s + 1$ and the induction hypothesis applies to them. Consider the exact sequence

$$0 \to M[a_i^{c_1}] \to M \xrightarrow{a_i^{c_1}} M \to M/a_i^{c_1}M \to 0$$

Denote $E \subset M$ the image of the middle arrow. Consider the corresponding diagram of Tor modules

$$\begin{array}{ccccc} & & & & \mathrm{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q) \\ & & & & \downarrow \\ \mathrm{Tor}_i(M[a_i^{c_1}], A/I^q) & \longrightarrow & \mathrm{Tor}_i(M, A/I^q) & \longrightarrow & \mathrm{Tor}_i(E, A/I^q) \\ & & & \searrow^{0} & \downarrow \\ & & & & \mathrm{Tor}_i(M, A/I^q) \end{array}$$

with exact rows and columns (for every q). The south-east arrow is zero by our choice of c_1. We conclude that the module $\mathrm{Tor}_i(M, A/I^q)$ is sandwiched between a quotient module of $\mathrm{Tor}_i(M[a_i^{c_1}], A/I^q)$ and a submodule of $\mathrm{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q)$. Hence we conclude H_s holds with N replaced by $N + c$ and c replaced by $2c$. Some details omitted. □

0GA3 **Proposition 51.21.7.** *Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that for $n \geq N$ the maps*

$$A/I^n \to A/I^{n-c}$$

satisfy the equivalent conditions of Lemma 51.20.2 with $e = t$.

Proof. Immediate consequence of Lemmas 51.21.6 and 51.20.2. □

0GA4 **Remark 51.21.8.** The paper [**AHS15**] shows, besides many other things, that if A is local, then Proposition 51.21.7 also holds with $e = t$ replaced by $e = \dim(A)$. Looking at Lemma 51.20.3 it is natural to ask whether Proposition 51.21.7 holds with $e = t$ replaced with $e = \mathrm{cd}(A, I)$. We don't know.

0GA5 **Remark 51.21.9.** Let I be an ideal of a Noetherian ring A. Say $I = (f_1, \ldots, f_r)$. Denote $K_n^\bullet$ the Koszul complex on $f_1^n, \ldots, f_r^n$ as in More on Algebra, Situation 15.92.15 and denote $K_n \in D(A)$ the corresponding object. Let $M^\bullet$ be a bounded complex of finite A-modules and denote $M \in D(A)$ the corresponding object. Consider the following inverse systems in $D(A)$:

(1) $M^\bullet/I^nM^\bullet$, i.e., the complex whose terms are M^i/I^nM^i,
(2) $M \otimes^{\mathbf{L}}_A A/I^n$,
(3) $M \otimes^{\mathbf{L}}_A K_n$, and
(4) $M \otimes^{\mathbf{L}}_P P/J^n$ (see below).

All of these inverse systems are isomorphic as pro-objects: the isomorphism between (2) and (3) follows from More on Algebra, Lemma 15.95.1. The isomorphism between (1) and (2) is given in More on Algebra, Lemma 15.101.3. For the last one, see below.

However, we can ask if these isomorphisms of pro-systems are "strict"; this terminology and question is related to the discussion in [**Qui**, pages 61, 62]. Namely, given a category $\mathcal{C}$ we can define a "strict pro-category" whose objects are inverse systems (X_n) and whose morphisms $(X_n) \to (Y_n)$ are given by tuples (c, φ_n) consisting of a $c \geq 0$ and morphisms $\varphi_n : X_n \to Y_{n-c}$ for all $n \geq c$ satisfying an obvious compatibility condition and up to a certain equivalence (given essentially by increasing c). Then we ask whether the above inverse systems are isomorphic in this strict pro-category.

This clearly cannot be the case for (1) and (3) even when $M = A[0]$. Namely, the system $H^0(K_n) = A/(f_1^n, \ldots, f_r^n)$ is not strictly pro-isomorphic in the category of modules to the system A/I^n in general. For example, if we take $A = \mathbf{Z}[x_1, \ldots, x_r]$ and $f_i = x_i$, then $H^0(K_n)$ is not annihilated by $I^{r(n-1)}$.[3]

It turns out that the results above show that the natural map from (2) to (1) discussed in More on Algebra, Lemma 15.101.3 is a strict pro-isomorphism. We will sketch the proof. Using standard arguments involving stupid truncations, we first reduce to the case where $M^\bullet$ is given by a single finite A-module M placed in degree 0. Pick $N, c \geq 0$ as in Proposition 51.21.7. The proposition implies that for $n \geq N$ we get factorizations

$$M \otimes^{\mathbf{L}}_A A/I^n \to \tau_{\geq -t}(M \otimes^{\mathbf{L}}_A A/I^n) \to M \otimes^{\mathbf{L}}_A A/I^{n-c}$$

of the transition maps in the system (2). On the other hand, by More on Algebra, Lemma 15.27.3, we can find another constant $c' = c'(M) \geq 0$ such that the maps $\mathrm{Tor}_i^A(M, A/I^{n'}) \to \mathrm{Tor}_i(M, A/I^{n'-c'})$ are zero for $i = 1, 2, \ldots, t$ and $n' \geq c'$. Then it follows from Derived Categories, Lemma 13.12.5 that the map

$$\tau_{\geq -t}(M \otimes^{\mathbf{L}}_A A/I^{n+tc'}) \to \tau_{\geq -t}(M \otimes^{\mathbf{L}}_A A/I^n)$$

factors through $M \otimes^{\mathbf{L}}_A A/I^{n+tc'} \to M/I^{n+tc'}M$. Combined with the previous result we obtain a factorization

$$M \otimes^{\mathbf{L}}_A A/I^{n+tc'} \to M/I^{n+tc'}M \to M \otimes^{\mathbf{L}}_A A/I^{n-c}$$

[3] Of course, we can ask whether these pro-systems are isomorphic in a category whose objects are inverse systems and where maps are given by tuples (r, c, φ_n) consisting of $r \geq 1$, $c \geq 0$ and maps $\varphi_n : X_{rn} \to Y_{n-c}$ for $n \geq c$.

which gives us what we want. If we ever need this result, we will carefully state it and provide a detailed proof.

For number (4) suppose we have a Noetherian ring P, a ring homomorphism $P \to A$, and an ideal $J \subset P$ such that $I = JA$. By More on Algebra, Section 15.60 we get a functor $M \otimes^{\mathbf{L}}_P - : D(P) \to D(A)$ and we get an inverse system $M \otimes^{\mathbf{L}}_P P/J^n$ in $D(A)$ as in (4). If P is Noetherian, then the system in (4) is pro-isomorphic to the system in (1) because we can compare with Koszul complexes. If $P \to A$ is finite, then the system (4) is strictly pro-isomorphic to the system (2) because the inverse system $A \otimes^{\mathbf{L}}_P P/J^n$ is strictly pro-isomorphic to the inverse system A/I^n (by the discussion above) and because we have

$$M \otimes^{\mathbf{L}}_P P/J^n = M \otimes^{\mathbf{L}}_A (A \otimes^{\mathbf{L}}_P P/J^n)$$

by More on Algebra, Lemma 15.60.1.

A standard example in (4) is to take $P = \mathbf{Z}[x_1, \ldots, x_r]$, the map $P \to A$ sending x_i to f_i, and $J = (x_1, \ldots, x_r)$. In this case one shows that

$$M \otimes^{\mathbf{L}}_P P/J^n = M \otimes^{\mathbf{L}}_{A[x_1,\ldots,x_r]} A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$$

and we reduce to one of the cases discussed above (although this case is strictly easier as $A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$ has tor dimension at most r for all n and hence the step using Proposition 51.21.7 can be avoided). This case is discussed in the proof of [**BS13**, Proposition 3.5.1].

51.22. A bit of uniformity, III

0GA6 In this section we fix a Noetherian ring A and an ideal $I \subset A$. Our goal is to prove Lemma 51.22.7 which we will use in a later chapter to solve a lifting problem, see Algebraization of Formal Spaces, Lemma 88.5.3.

Throughout this section we denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n\geq 0} I^n$. We also consider the fibre product

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \mathrm{Spec}(A/I) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Then Y is the exceptional divisor of the blowup and hence an effective Cartier divisor on X such that $\mathcal{O}_X(-1) = \mathcal{O}_X(Y)$. Since taking Proj commutes with base change we have

$$Y = \mathrm{Proj}(\bigoplus\nolimits_{n\geq 0} I^n/I^{n+1}) = \mathrm{Proj}(S)$$

where $S = \mathrm{Gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$.

We denote $d = d(S) = d(\mathrm{Gr}_I(A)) = d(\bigoplus_{n\geq 0} I^n/I^{n+1})$ the maximum of the dimensions of the fibres of p (and we set it equal to 0 if $X = \emptyset$). This is well defined. In fact, we have

(1) $d \leq t - 1$ if $I = (a_1, \ldots, a_t)$ since then $X \subset \mathbf{P}^{t-1}_A$, and

(2) d is also the maximal dimension of the fibres of $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ provided that Y is nonempty and $d = 0$ if $Y = \emptyset$ (equivalently $S = 0$, equivalently $I = A$).

Hence d only depends on the isomorphism class of $S = \mathrm{Gr}_I(A)$. Observe that $H^i(X, \mathcal{F}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i > d$ by Cohomology of Schemes, Lemmas 30.20.9 and 30.4.6. Of course the same is true for coherent modules on Y.

We denote $q = q(S) = q(\mathrm{Gr}_I(A)) = q(\bigoplus_{n\geq 0} I^n/I^{n+1})$ the integer defined as follows. Note that the algebra $S = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is a Noetherian graded ring generated in degree 1 over degree 0. Hence by Cohomology of Schemes, Lemmas 30.14.2 and 30.14.3 we can define $q(S)$ as the smallest integer $q(S) \geq 0$ such that for all $q \geq q(S)$ we have $H^i(Y, \mathcal{O}_Y(q)) = 0$ for $1 \leq i \leq d$ and $H^0(Y, \mathcal{O}_Y(q)) = I^q/I^{q+1}$. (If $S = 0$, then $q(S) = 0$.)

For $n \geq 1$ we may consider the effective Cartier divisor nY which we will denote Y_n.

0GA7 **Lemma 51.22.1.** *With $q_0 = q(S)$ and $d = d(S)$ as above, we have*

(1) *for $n \geq 1$, $q \geq q_0$, and $i > 0$ we have $H^i(X, \mathcal{O}_{Y_n}(q)) = 0$,*
(2) *for $n \geq 1$ and $q \geq q_0$ we have $H^0(X, \mathcal{O}_{Y_n}(q)) = I^q/I^{q+n}$,*
(3) *for $q \geq q_0$ and $i > 0$ we have $H^i(X, \mathcal{O}_X(q)) = 0$,*
(4) *for $q \geq q_0$ we have $H^0(X, \mathcal{O}_X(q)) = I^q$.*

Proof. If $I = A$, then X is affine and the statements are trivial. Hence we may and do assume $I \neq A$. Thus Y and X are nonempty schemes.

Let us prove (1) and (2) by induction on n. The base case $n = 1$ is our definition of q_0 as $Y_1 = Y$. Recall that $\mathcal{O}_X(1) = \mathcal{O}_X(-Y)$. Hence we have a short exact sequence

$$0 \to \mathcal{O}_{Y_n}(1) \to \mathcal{O}_{Y_{n+1}} \to \mathcal{O}_Y \to 0$$

Hence for $i > 0$ we find

$$H^i(X, \mathcal{O}_{Y_n}(q+1)) \to H^i(X, \mathcal{O}_{Y_{n+1}}(q)) \to H^i(X, \mathcal{O}_Y(q))$$

and we obtain the desired vanishing of the middle term from the given vanishing of the outer terms. For $i = 0$ we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{q+1}/I^{q+1+n} & \longrightarrow & I^q/I^{q+1+n} & \longrightarrow & I^q/I^{q+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{O}_{Y_n}(q+1)) & \longrightarrow & H^0(X, \mathcal{O}_{Y_{n+1}}(q)) & \longrightarrow & H^0(Y, \mathcal{O}_Y(q)) & \longrightarrow & 0 \end{array}$$

with exact rows for $q \geq q_0$ (for the bottom row observe that the next term in the long exact cohomology sequence vanishes for $q \geq q_0$). Since $q \geq q_0$ the left and right vertical arrows are isomorphisms and we conclude the middle one is too.

We omit the proofs of (3) and (4) which are similar. In fact, one can deduce (3) and (4) from (1) and (2) using the theorem on formal functors (but this would be overkill). □

Let us introduce a notation: given $n \geq c \geq 0$ *an (A, n, c)-module* is a finite A-module M which is annihilated by I^n and which as an A/I^n-module is I^c/I^n-projective, see More on Algebra, Section 15.71.

We will use the following abuse of notation: given an A-module M we denote p^*M the quasi-coherent module gotten by pulling back by p the quasi-coherent module $\widetilde{M}$ on $\mathrm{Spec}(A)$ associated to M. For example we have $\mathcal{O}_{Y_n} = p^*(A/I^n)$. For a short exact sequence $0 \to K \to L \to M \to 0$ of A-modules we obtain an exact sequence

$$p^*K \to p^*L \to p^*M \to 0$$

as $\widetilde{\ }$ is an exact functor and p^* is a right exact functor.

0GA8 **Lemma 51.22.2.** *Let $0 \to K \to L \to M \to 0$ be a short exact sequence of A-modules such that K and L are annihilated by I^n and M is an (A, n, c)-module. Then the kernel of $p^*K \to p^*L$ is scheme theoretically supported on Y_c.*

Proof. Let $\mathrm{Spec}(B) \subset X$ be an affine open. The restriction of the exact sequence over $\mathrm{Spec}(B)$ corresponds to the sequence of B-modules

$$K \otimes_A B \to L \otimes_A B \to M \otimes_A B \to 0$$

which is isomorphismic to the sequence

$$K \otimes_{A/I^n} B/I^nB \to L \otimes_{A/I^n} B/I^nB \to M \otimes_{A/I^n} B/I^nB \to 0$$

Hence the kernel of the first map is the image of the module $\mathrm{Tor}_1^{A/I^n}(M, B/I^nB)$. Recall that the exceptional divisor Y is cut out by $I\mathcal{O}_X$. Hence it suffices to show that $\mathrm{Tor}_1^{A/I^n}(M, B/I^nB)$ is annihilated by I^c. Since multiplication by $a \in I^c$ on M factors through a finite free A/I^n-module, this is clear. □

We have the canonical map $\mathcal{O}_X \to \mathcal{O}_X(1)$ which vanishes exactly along Y. Hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we always have canonical maps $\mathcal{F}(q) \to \mathcal{F}(q+n)$ for any $q \in \mathbf{Z}$ and $n \geq 0$.

0GA9 **Lemma 51.22.3.** *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is scheme theoretically supported on Y_c if and only if the canonical map $\mathcal{F} \to \mathcal{F}(c)$ is zero.*

Proof. This is true because $\mathcal{O}_X \to \mathcal{O}_X(1)$ vanishes exactly along Y. □

0GAA **Lemma 51.22.4.** *With $q_0 = q(S)$ and $d = d(S)$ as above, suppose we have integers $n \geq c \geq 0$, an (A, n, c)-module M, an index $i \in \{0, 1, \ldots, d\}$, and an integer q. Then we distinguish the following cases*

1. *In the case $i = d \geq 1$ and $q \geq q_0$ we have $H^d(X, p^*M(q)) = 0$.*
2. *In the case $i = d - 1 \geq 1$ and $q \geq q_0$ we have $H^{d-1}(X, p^*M(q)) = 0$.*
3. *In the case $d-1 > i > 0$ and $q \geq q_0+(d-1-i)c$ the map $H^i(X, p^*M(q)) \to H^i(X, p^*M(q-(d-1-i)c))$ is zero.*
4. *In the case $i = 0$, $d \in \{0, 1\}$, and $q \geq q_0$, there is a surjection*
$$I^qM \longrightarrow H^0(X, p^*M(q))$$
5. *In the case $i = 0$, $d > 1$, and $q \geq q_0 + (d-1)c$ the map*
$$H^0(X, p^*M(q)) \to H^0(X, p^*M(q-(d-1)c))$$
has image contained in the image of the canonical map $I^{q-(d-1)c}M \to H^0(X, p^*M(q-(d-1)c))$.

Proof. Let M be an (A, n, c)-module. Choose a short exact sequence

$$0 \to K \to (A/I^n)^{\oplus r} \to M \to 0$$

We will use below that K is an (A, n, c)-module, see More on Algebra, Lemma 15.71.6. Consider the corresponding exact sequence

$$p^*K \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$

We split this into short exact sequences

$$0 \to \mathcal{F} \to p^*K \to \mathcal{G} \to 0 \quad \text{and} \quad 0 \to \mathcal{G} \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$

By Lemma 51.22.2 the coherent module $\mathcal{F}$ is scheme theoretically supported on Y_c.

Proof of (1). Assume $d > 0$. We have to prove $H^d(X, p^*M(q)) = 0$ for $q \geq q_0$. By the vanishing of the cohomology of twists of $\mathcal{G}$ in degrees $> d$ and the long exact cohomology sequence associated to the second short exact sequence above, it suffices to prove that $H^d(X, \mathcal{O}_{Y_n}(q)) = 0$. This is true by Lemma 51.22.1.

Proof of (2). Assume $d > 1$. We have to prove $H^{d-1}(X, p^*M(q)) = 0$ for $q \geq q_0$. Arguing as in the previous paragraph, we see that it suffices to show that $H^d(X, \mathcal{G}(q)) = 0$. Using the first short exact sequence and the vanishing of the cohomology of twists of $\mathcal{F}$ in degrees $> d$ we see that it suffices to show $H^d(X, p^*K(q))$ is zero which is true by (1) and the fact that K is an (A, n, c)-module (see above).

Proof of (3). Let $0 < i < d-1$ and assume the statement holds for $i+1$ except in the case $i = d-2$ we have statement (2). Using the long exact sequence of cohomology associated to the second short exact sequence above we find an injection

$$H^i(X, p^*M(q - (d-1-i)c)) \subset H^{i+1}(X, \mathcal{G}(q - (d-1-i)c))$$

as $q-(d-1-i)c \geq q_0$ gives the vanishing of $H^i(X, \mathcal{O}_{Y_n}(q-(d-1-i)c))$ (see above). Thus it suffices to show that the map $H^{i+1}(X, \mathcal{G}(q)) \to H^{i+1}(X, \mathcal{G}(q-(d-1-i)c))$ is zero. To study this, we consider the maps of exact sequences

$$\begin{array}{ccccc}
H^{i+1}(X, p^*K(q)) & \longrightarrow & H^{i+1}(X, \mathcal{G}(q)) & \longrightarrow & H^{i+2}(X, \mathcal{F}(q)) \\
\downarrow & \swarrow & \downarrow & & \downarrow \\
H^{i+1}(X, p^*K(q-c)) & \longrightarrow & H^{i+1}(X, \mathcal{G}(q-c)) & \longrightarrow & H^{i+2}(X, \mathcal{F}(q-c)) \\
\downarrow & & \downarrow & & \\
H^{i+1}(X, p^*K(q-(d-1-i)c)) & \longrightarrow & H^{i+1}(X, \mathcal{G}(q-(d-1-i)c)) & &
\end{array}$$

Since $\mathcal{F}$ is scheme theoretically supported on Y_c we see that the canonical map $\mathcal{G}(q) \to \mathcal{G}(q-c)$ factors through $p^*K(q-c)$ by Lemma 51.22.3. This gives the dotted arrow in the diagram. (In fact, for the proof it suffices to observe that the vertical arrow on the extreme right is zero in order to get the dotted arrow as a map of sets.) Thus it suffices to show that $H^{i+1}(X, p^*K(q-c)) \to H^{i+1}(X, p^*K(q-(d-1-i)c))$ is zero. If $i = d-2$, then the source of this arrow is zero by (2) as $q-c \geq q_0$ and K is an (A, n, c)-module. If $i < d-2$, then as K is an (A, n, c)-module, we get from the induction hypothesis that the map is indeed zero since $q-c-(q-(d-1-i)c) = (d-2-i)c = (d-1-(i+1))c$ and since $q-c \geq q_0 + (d-1-(i+1))c$. In this way we conclude the proof of (3).

Proof of (4). Assume $d \in \{0, 1\}$ and $q \geq q_0$. Then the first short exact sequence gives a surjection $H^1(X, p^*K(q)) \to H^1(X, \mathcal{G}(q))$ and the source of this arrow is zero by case (1). Hence for all $q \in \mathbf{Z}$ we see that the map

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q)) \longrightarrow H^0(X, p^*M(q))$$

is surjective. For $q \geq q_0$ the source is equal to $(I^q/I^{q+n})^{\oplus r}$ by Lemma 51.22.1 and this easily proves the statement.

Proof of (5). Assume $d > 1$. Arguing as in the proof of (4) we see that it suffices to show that the image of

$$H^0(X, p^*M(q)) \longrightarrow H^0(X, p^*M(q-(d-1)c))$$

is contained in the image of

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q-(d-1)c)) \longrightarrow H^0(X, p^*M(q-(d-1)c))$$

To show the inclusion above, it suffices to show that for $\sigma \in H^0(X, p^*M(q))$ with boundary $\xi \in H^1(X, \mathcal{G}(q))$ the image of ξ in $H^1(X, \mathcal{G}(q-(d-1)c))$ is zero. This follows by the exact same arguments as in the proof of (3). □

0GAB **Remark 51.22.5.** Given a pair (M, n) consisting of an integer $n \geq 0$ and a finite A/I^n-module M we set $M^\vee = \mathrm{Hom}_{A/I^n}(M, A/I^n)$. Given a pair $(\mathcal{F}, n)$ consisting of an integer n and a coherent $\mathcal{O}_{Y_n}$-module $\mathcal{F}$ we set

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{Y_n}}(\mathcal{F}, \mathcal{O}_{Y_n})$$

Given (M, n) as above, there is a canonical map

$$can : p^*(M^\vee) \longrightarrow (p^*M)^\vee$$

Namely, if we choose a presentation $(A/I^n)^{\oplus s} \to (A/I^n)^{\oplus r} \to M \to 0$ then we obtain a presentation $\mathcal{O}_{Y_n}^{\oplus s} \to \mathcal{O}_{Y_n}^{\oplus r} \to p^*M \to 0$. Taking duals we obtain exact sequences

$$0 \to M^\vee \to (A/I^n)^{\oplus r} \to (A/I^n)^{\oplus s}$$

and

$$0 \to (p^*M)^\vee \to \mathcal{O}_{Y_n}^{\oplus r} \to \mathcal{O}_{Y_n}^{\oplus s}$$

Pulling back the first sequence by p we find the desired map *can*. The construction of this map is functorial in the finite A/I^n-module M. The kernel and cokernel of *can* are scheme theoretically supported on Y_c if M is an (A, n, c)-module. Namely, in that case for $a \in I^c$ the map $a : M \to M$ factors through a finite free A/I^n-module for which *can* is an isomorphism. Hence a annihilates the kernel and cokernel of *can*.

0GAC **Lemma 51.22.6.** *With $q_0 = q(S)$ and $d = d(S)$ as above, let M be an (A, n, c)-module and let $\varphi : M \to I^n/I^{2n}$ be an A-linear map. Assume $n \geq \max(q_0 + (1+d)c, (2+d)c)$ and if $d = 0$ assume $n \geq q_0 + 2c$. Then the composition*

$$M \xrightarrow{\varphi} I^n/I^{2n} \to I^{n-(1+d)c}/I^{2n-(1+d)c}$$

is of the form $\sum a_i\psi_i$ with $a_i \in I^c$ and $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$.

Proof. The case $d > 1$. Since we have a compatible system of maps $p^*(I^q) \to \mathcal{O}_X(q)$ for $q \geq 0$ there are canonical maps $p^*(I^q/I^{q+\nu}) \to \mathcal{O}_{Y_\nu}(q)$ for $\nu \geq 0$. Using this and pulling back φ we obtain a map

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n)$$

such that the composition $M \to H^0(X, p^*M) \to H^0(X, \mathcal{O}_{Y_n}(n))$ is the given homomorphism φ combined with the map $I^n/I^{2n} \to H^0(X, \mathcal{O}_{Y_n}(n))$. Since $\mathcal{O}_{Y_n}(n)$ is invertible on Y_n the linear map χ determines a section

$$\sigma \in \Gamma(X, (p^*M)^\vee(n))$$

with notation as in Remark 51.22.5. The discussion in Remark 51.22.5 shows the cokernel and kernel of $can : p^*(M^\vee) \to (p^*M)^\vee$ are scheme theoretically supported on Y_c. By Lemma 51.22.3 the map $(p^*M)^\vee(n) \to (p^*M)^\vee(n-2c)$ factors through $p^*(M^\vee)(n-2c)$; small detail omitted. Hence the image of σ in $\Gamma(X, (p^*M)^\vee(n-2c))$ comes from an element

$$\sigma' \in \Gamma(X, p^*(M^\vee)(n-2c))$$

By Lemma 51.22.4 part (5), the fact that $M^\vee$ is an (A, n, c)-module by More on Algebra, Lemma 15.71.7, and the fact that $n \geq q_0 + (1+d)c$ so $n - 2c \geq q_0 + (d-1)c$ we see that the image of σ' in $H^0(X, p^*M^\vee(n-(1+d)c))$ is the image of an element τ in $I^{n-(1+d)c}M^\vee$. Write $\tau = \sum a_i\tau_i$ with $\tau_i \in I^{n-(2+d)c}M^\vee$; this makes sense as $n - (2+d)c \geq 0$. Then τ_i determines a homomorphism of modules $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$ using the evaluation map $M \otimes M^\vee \to A/I^n$.

Let us prove that this works[4]. Pick $z \in M$ and let us show that $\varphi(z)$ and $\sum a_i\psi_i(z)$ have the same image in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. First, the element z determines a map $p^*z : \mathcal{O}_{Y_n} \to p^*M$ whose composition with χ is equal to the map $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_n}(n)$ corresponding to $\varphi(z)$ via the map $I^n/I^{2n} \to \Gamma(\mathcal{O}_{Y_n}(n))$. Next z and p^*z determine evaluation maps $e_z : M^\vee \to A/I^n$ and $e_{p^*z} : (p^*M)^\vee \to \mathcal{O}_{Y_n}$. Since $\chi(p^*z)$ is the section corresponding to $\varphi(z)$ we see that $e_{p^*z}(\sigma)$ is the section corresponding to $\varphi(z)$. Here and below we abuse notation: for a map $a : \mathcal{F} \to \mathcal{G}$ of modules on X we also denote $a : \mathcal{F}(t) \to \mathcal{F}(t)$ the corresponding map of twisted modules. The diagram

$$\begin{array}{ccc} p^*(M^\vee) & \xrightarrow[p^*e_z]{} & \mathcal{O}_{Y_n} \\ {\scriptstyle can}\downarrow & & \| \\ (p^*M)^\vee & \xrightarrow{e_{p^*z}} & \mathcal{O}_{Y_n} \end{array}$$

commutes by functoriality of the construction *can*. Hence $(p^*e_z)(\sigma')$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-2c))$ is the section corresponding to the image of $\varphi(z)$ in I^{n-2c}/I^{2n-2c}. The next step is that σ' maps to the image of $\sum a_i\tau_i$ in $H^0(X, p^*M^\vee(n-(1+d)c))$. This implies that $(p^*e_z)(\sum a_i\tau_i) = \sum a_i p^*e_z(\tau_i)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is the section corresponding to the image of $\varphi(z)$ in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. Recall that ψ_i is defined from τ_i using an evaluation map. Hence if we denote

$$\chi_i : p^*M \longrightarrow \mathcal{O}_{Y_n}(n-(2+d)c)$$

the map we get from ψ_i, then we see by the same reasoning as above that the section corresponding to $\psi_i(z)$ is $\chi_i(p^*z) = e_{p^*z}(\chi_i) = p^*e_z(\tau_i)$. Hence we conclude that the image of $\varphi(z)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is equal to the image of $\sum a_i\psi_i(z)$. Since $n-(1+d)c \geq q_0$ we have $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c)) = I^{n-(1+d)c}/I^{2n-(1+d)c}$ by Lemma 51.22.1 and we conclude the desired compatibility is true.

The case $d = 1$. Here we argue as above that we get

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n), \quad \sigma \in \Gamma(X, (p^*M)^\vee(n)), \quad \sigma' \in \Gamma(X, p^*(M^\vee)(n-2c)),$$

and then we use Lemma 51.22.4 part (4) to see that σ' is the image of some element $\tau \in I^{n-2c}M^\vee$. The rest of the argument is the same.

The case $d = 0$. Argument is exactly the same as in the case $d = 1$. □

0GAD **Lemma 51.22.7.** *With $d = d(S)$ and $q_0 = q(S)$ as above. Then*

[4]We hope some reader will suggest a less dirty proof of this fact.

(1) *for integers* $n \geq c \geq 0$ *with* $n \geq \max(q_0 + (1+d)c, (2+d)c)$,
(2) *for* K *of* $D(A/I^n)$ *with* $H^i(K) = 0$ *for* $i \neq -1, 0$ *and* $H^i(K)$ *finite for* $i = -1, 0$ *such that* $\mathrm{Ext}^1_{A/I^c}(K, N)$ *is annihilated by* I^c *for all finite* A/I^n*-modules* N

the map

$$\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n}) \longrightarrow \mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$$

is zero.

Proof. The case $d > 0$. Let $K^{-1} \to K^0$ be a complex representing K as in More on Algebra, Lemma 15.85.5 part (5) with respect to the ideal I^c/I^n in the ring A/I^n. In particular K^{-1} is I^c/I^n-projective as multiplication by elements of I^c/I^n even factor through K^0. By More on Algebra, Lemma 15.85.4 part (1) we have

$$\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n}) = \mathrm{Coker}(\mathrm{Hom}_{A/I^n}(K^0, I^n/I^{2n}) \to \mathrm{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n}))$$

and similarly for other Ext groups. Hence any class ξ in $\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n})$ comes from an element $\varphi \in \mathrm{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n})$. Denote φ' the image of φ in $\mathrm{Hom}_{A/I^n}(K^{-1}, I^{n-(1+d)c}/I^{2n-(1+d)c})$. By Lemma 51.22.6 we can write $\varphi' = \sum a_i\psi_i$ with $a_i \in I^c$ and $\psi_i \in \mathrm{Hom}_{A/I^n}(M, I^{n-(2+d)c}/I^{2n-(2+d)c})$. Choose $h_i : K^0 \to K^{-1}$ such that $a_i \mathrm{id}_{K^{-1}} = h_i \circ d_K^{-1}$. Set $\psi = \sum \psi_i \circ h_i : K^0 \to I^{n-(2+d)c}/I^{2n-(2+d)c}$. Then $\varphi' = \psi \circ \mathrm{d}_K^{-1}$ and we conclude that ξ already maps to zero in $\mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-(1+d)c})$ and a fortiori in $\mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$.

The case $d = 0$[5]. Let ξ and φ be as above. We consider the diagram

$$\begin{array}{ccccc} & & K^0 & & \\ & & \uparrow & & \\ K^{-1} & \xrightarrow{\varphi} & I^n/I^{2n} & \longrightarrow & I^{n-c}/I^{2n-c} \end{array}$$

Pulling back to X and using the map $p^*(I^n/I^{2n}) \to \mathcal{O}_{Y_n}(n)$ we find a solid diagram

$$\begin{array}{ccccc} p^*K^0 & & \cdots\cdots & & \\ \uparrow & & & \searrow & \\ p^*K^{-1} & \longrightarrow & \mathcal{O}_{Y_n}(n) & \longrightarrow & \mathcal{O}_{Y_n}(n-c) \end{array}$$

We can cover X by affine opens $U = \mathrm{Spec}(B)$ such that there exists an $a \in I$ with the following property: $IB = aB$ and a is a nonzerodivisor on B. Namely, we can cover X by spectra of affine blowup algebras, see Divisors, Lemma 31.32.2. The restriction of $\mathcal{O}_{Y_n}(n) \to \mathcal{O}_{Y_n}(n-c)$ to U is isomorphic to the map of quasi-coherent $\mathcal{O}_U$-modules corresponding to the B-module map $a^c : B/a^nB \to B/a^nB$. Since $a^c : K^{-1} \to K^{-1}$ factors through K^0 we see that the dotted arrow exists over U. In other words, locally on X we can find the dotted arrow! Now the sheaf of dotted arrows fitting into the diagram is principal homogeneous under

$$\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathrm{Coker}(p^*K^{-1} \to p^*K^0), \mathcal{O}_{Y_n}(n-c))$$

which is a coherent $\mathcal{O}_X$-module. Hence the obstruction for finding the dotted arrow is an element of $H^1(X, \mathcal{F})$. This cohomology group is zero as $1 > d = 0$,

[5]The argument given for $d > 0$ works but gives a slightly weaker result.

see discussion following the definition of $d = d(S)$. This proves that we can find a dotted arrow $\psi : p^*K^0 \to \mathcal{O}_{Y_n}(n-c)$ fitting into the diagram. Since $n - c \geq q_0$ we find that ψ induces a map $K^0 \to I^{n-c}/I^{2n-c}$. Chasing the diagram we conclude that $\varphi' = \psi \circ \mathrm{d}_K^{-1}$ and the proof is finished as before. □

51.23. Other chapters

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CHAPTER 52

Algebraic and Formal Geometry

0EI5

52.1. Introduction

0EI6 This chapter continues the study of formal algebraic geometry and in particular the question of whether a formal object is the completion of an algebraic one. A fundamental reference is [**Gro68**]. Here is a list of results we have already discussed in the Stacks project:

(1) The theorem on formal functions, see Cohomology of Schemes, Section 30.20.
(2) Coherent formal modules, see Cohomology of Schemes, Section 30.23.
(3) Grothendieck's existence theorem, see Cohomology of Schemes, Sections 30.24, 30.25, and 30.27.
(4) Grothendieck's algebraization theorem, see Cohomology of Schemes, Section 30.28.
(5) Grothendieck's existence theorem more generally, see More on Flatness, Sections 38.28 and 38.29.

Let us give an overview of the contents of this chapter.

Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals. Many questions in this chapter have to do with inverse systems $(\mathcal{F}_n)$ of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. An important special case is where X is a scheme over a Noetherian ring A and $\mathcal{I} = I\mathcal{O}_X$ for some ideal $I \subset A$. In Cohomology, Sections 20.35, 20.36, and 20.39 we have some general results. In this chapter, Sections 52.2 and 52.3 contain results specific to schemes and quasi-coherent modules. In Section 52.4 we prove that the limit topology on $\lim H^p(X, \mathcal{F}_n)$ is I-adic in case $\mathrm{cd}(A, I) = 1$. One of the themes of this chapter will be to show that results proven in the principal ideal case $I = (f)$ also hold when we only assume $\mathrm{cd}(A, I) = 1$.

In Section 52.6 we discuss derived completion of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ with respect to a finite type sheaf of ideals $\mathcal{I}$. This section is the natural continuation of the theory of derived completion in commutative algebra as described in More on Algebra, Section 15.92. The first main result is that derived completion exists. The second main result is that for a morphism f of ringed sites derived completion commutes with derived pushforward:

$$(Rf_*K)^\wedge = Rf_*(K^\wedge)$$

if the ideal sheaf upstairs is locally generated by sections coming from the ideal downstairs, see Lemma 52.6.19. We stress that both main results are very elementary in case the ideals in question are globally finitely generated which will be true for all applications of this theory in this chapter. The displayed equality is the "correct" version of the theorem on formal functions, see discussion in Section 52.7.

Let A be a Noetherian ring and let I, J be two ideals of A. Let M be a finite A-module. The next topic in this chapter is the map

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge$$

from local cohomology of M into the derived I-adic completion of the same. It turns out that if we impose suitable depth conditions this map becomes an isomorphism on cohomology in a range of degrees. In Section 52.8 we work essentially in the generality just mentioned. In Section 52.9 we assume A is a local ring and $J = \mathfrak{m}$ is a maximal ideal. We encourage the reader to read this section before the other two in this part of the chapter. Finally, in Section 52.10 we bootstrap the local case to obtain stronger results back in the general case.

In the next part of this chapter we use the results on completion of local cohomology to get a nonexhaustive list of results on cohomology of the completion of coherent modules. More precisely, let A be a Noetherian ring, let $I \subset A$ be an ideal, and let $U \subset \mathrm{Spec}(A)$ be an open subscheme. If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, then we may consider the maps

$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

and ask if we get an isomorphism in a certain range of degrees. In Section 52.11 we work out some examples where U is the punctured spectrum of a local ring. In Section 52.12 we discuss the general case. In Section 52.14 we apply some of the results obtained to questions of connectedness in algebraic geometry.

The remaining sections of this chapter are devoted to a discussion of algebraization of coherent formal modules. In other words, given an inverse system of coherent modules $(\mathcal{F}_n)$ on U as above with $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$ we ask whether there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $\mathcal{F}_n = \mathcal{F}/I^n\mathcal{F}$ for all n. We encourage the reader to read Section 52.16 for a precise statement of the question, a useful general result (Lemma 52.16.10), and a nontrivial application (Lemma 52.16.11). To prove a result going essentially beyond this case quite a bit more theory has to be developed. Please see Section 52.22 for the strongest results of this type obtained in this chapter.

52.2. Formal sections, I

0EH3 We suggest looking at Cohomology, Section 20.35 first.

0EI8 **Lemma 52.2.1.** *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then

1. *$\mathcal{F} = R\lim \mathcal{F}_n$,*
2. *for any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for $p > 0$, and*
3. *for each p there is a short exact sequence $0 \to R^1\lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$.*

If moreover $\mathcal{I}$ is of finite type, then

4. *$\mathcal{F}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$, and*
5. *$\mathcal{I}^n\mathcal{F} = \lim_{m \geq n} \mathcal{I}^n\mathcal{F}_m$.*

Proof. Parts (1), (2), and (3) are general facts about inverse systems of quasi-coherent modules with surjective transition maps, see Derived Categories of Schemes, Lemma 36.3.2 and Cohomology, Lemma 20.37.1. Next, assume $\mathcal{I}$ is of finite type. Let $U \subset X$ be affine open. Say $U = \mathrm{Spec}(A)$ and $\mathcal{I}|_U$ corresponds to $I \subset A$. Observe that I is a finitely generated ideal. By the equivalence of categories between quasi-coherent $\mathcal{O}_U$-modules and A-modules (Schemes, Lemma 26.7.5) we find that $M_n = \mathcal{F}_n(U)$ is an inverse system of A-modules with $M_n = M_{n+1}/I^n M_{n+1}$. Thus

$$M = \mathcal{F}(U) = \lim \mathcal{F}_n(U) = \lim M_n$$

is an I-adically complete module with $M/I^n M = M_n$ by Algebra, Lemma 10.98.2. This proves (4). Part (5) translates into the statement that $\lim_{m \geq n} I^n M/I^m M = I^n M$. Since $I^m M = I^{m-n} \cdot I^n M$ this is just the statement that $I^m M$ is I-adically complete. This follows from Algebra, Lemma 10.96.3 and the fact that M is complete. □

52.3. Formal sections, II

0BLA We suggest looking at Cohomology, Sections 20.36 and 20.39 first.

0EH9 **Lemma 52.3.1.** *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of quasi-coherent $\mathcal{O}_X$-modules. The following are equivalent

1. *for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,*
2. *for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$*
3. *there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is f-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$.*
4. *there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is f-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$.*

Proof. The equivalence of (1), (2), (3) and the implication (4) $\Rightarrow$ (1) are proven in Cohomology, Lemma 20.36.1. Assume (1) holds. Set $\mathcal{F} = \lim \mathcal{F}_n$. By Lemma 52.2.1 part (4) we have $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$. Let $U \subset X$ be open and $s = (s_n) \in \mathcal{F}(U) = \lim \mathcal{F}_n(U)$. Choose $n \geq 1$. If $fs = 0$, then s_{n+1} is in the kernel of $\mathcal{F}_{n+1} \to \mathcal{F}_n$ by condition (1). Hence $s_n = 0$. Since n was arbitrary, we see $s = 0$. Thus $\mathcal{F}$ is f-torsion free. □

0BLD **Lemma 52.3.2.** *Let A be a ring and $f \in A$. Let X be a scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathcal{F}[f^n] = \mathrm{Ker}(f^n : \mathcal{F} \to \mathcal{F})$ stabilizes. Then* Slightly improved version of [**BdJ14**, Lemma 1.6]

$$R\Gamma(X, \lim \mathcal{F}/f^n\mathcal{F}) = R\Gamma(X, \mathcal{F})^\wedge$$

where the right hand side indicates the derived completion with respect to the ideal $(f) \subset A$. Consequently, for $p \in \mathbf{Z}$ we obtain a commutative diagram

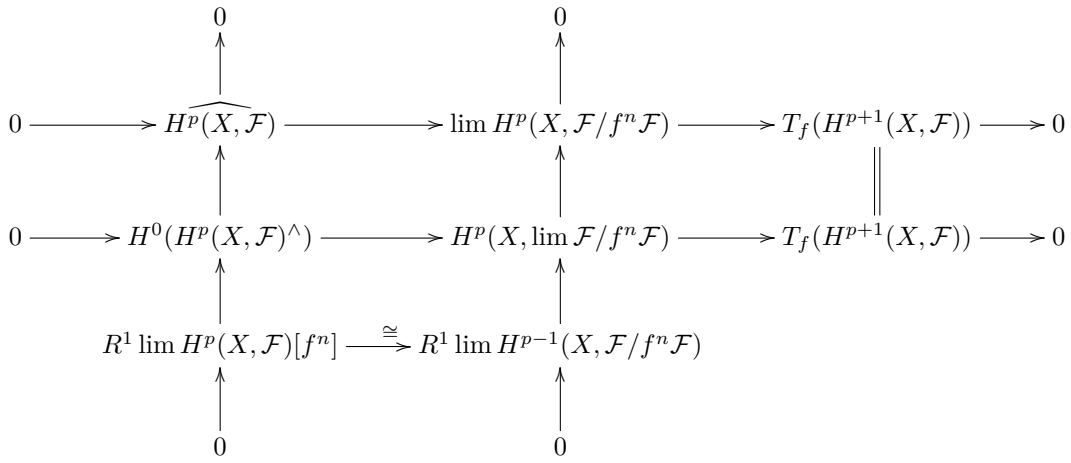

with exact rows and columns where $\widehat{H^p(X,\mathcal{F})} = \lim H^p(X,\mathcal{F})/f^n H^p(X,\mathcal{F})$ is the usual f-adic completion and $T_f(-)$ denotes the f-adic Tate module as in More on Algebra, Example 15.94.5.

Proof. By Lemma 52.2.1 we have $\lim \mathcal{F}/f^n\mathcal{F} = R\lim \mathcal{F}/f^n\mathcal{F}$. Everything else follows from Cohomology, Example 20.39.3. □

52.4. Formal sections, III

0EI9 In this section we prove Lemma 52.4.5 which (in the setting of Noetherian schemes and coherent modules) is the analogue of Cohomology, Lemma 20.36.2 in case the ideal I is not assumed principal but has the property that $\mathrm{cd}(A, I) = 1$.

0EIA **Lemma 52.4.1.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A. If $cd(A, I) = 1$, then there exist $c \geq 1$ and maps $\varphi_j : I^c \to A$ such that $\sum f_j\varphi_j : I^c \to I$ is the inclusion map.*

Proof. Since $\mathrm{cd}(A, I) = 1$ the complement $U = \mathrm{Spec}(A) \setminus V(I)$ is affine (Local Cohomology, Lemma 51.4.8). Say $U = \mathrm{Spec}(B)$. Then $IB = B$ and we can write $1 = \sum_{j=1,\ldots,r} f_j b_j$ for some $b_j \in B$. By Cohomology of Schemes, Lemma 30.10.5 we can represent b_j by maps $\varphi_j : I^c \to A$ for some $c \geq 0$. Then $\sum f_j\varphi_j : I^c \to I \subset A$ is the canonical embedding, after possibly replacing c by a larger integer, by the same lemma. □

0EIB **Lemma 52.4.2.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Then there is a unique graded A-algebra map*

$$\Phi : \bigoplus\nolimits_{n\geq 0} I^{nc} \to A[T_1, \ldots, T_r]$$

with $\Phi(g) = \sum \varphi_j(g)T_j$ for $g \in I^c$. Moreover, the composition of Φ with the map $A[T_1, \ldots, T_r] \to \bigoplus_{n\geq 0} I^n$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n\geq 0} I^{nc} \to \bigoplus_{n\geq 0} I^n$.

Proof. For each j and $m \geq c$ the restriction of φ_j to I^m is a map $\varphi_j : I^m \to I^{m-c}$. Given $j_1, \ldots, j_n \in \{1, \ldots, r\}$ we claim that the composition

$$\varphi_{j_1} \ldots \varphi_{j_n} : I^{nc} \to I^{(n-1)c} \to \ldots \to I^c \to A$$

is independent of the order of the indices $j_1, \ldots, j_n$. Namely, if $g = g_1 \ldots g_n$ with $g_i \in I^c$, then we see that

$$(\varphi_{j_1} \ldots \varphi_{j_n})(g) = \varphi_{j_1}(g_1) \ldots \varphi_{j_n}(g_n)$$

is independent of the ordering as multiplication in A is commutative. Thus we can define Φ by sending $g \in I^{nc}$ to

$$\Phi(g) = \sum\nolimits_{e_1+\ldots+e_r=n} (\varphi_1^{e_1} \circ \ldots \circ \varphi_r^{e_r})(g) T_1^{e_1} \ldots T_r^{e_r}$$

It is straightforward to prove that this is a graded A-algebra homomorphism with the desired property. Uniqueness is immediate as is the final property. This proves the lemma. □

0EIC **Lemma 52.4.3.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Let $A \to B$ be a ring map with B Noetherian and let N be a finite B-module. Then, after possibly increasing c and adjusting φ_j accordingly, there is a unique unique graded B-module map*

$$\Phi_N : \bigoplus\nolimits_{n \geq 0} I^{nc} N \to N[T_1, \ldots, T_r]$$

with $\Phi_N(gx) = \Phi(g)x$ for $g \in I^{nc}$ and $x \in N$ where Φ is as in Lemma 52.4.2. The composition of Φ_N with the map $N[T_1, \ldots, T_r] \to \bigoplus_{n \geq 0} I^n N$, $T_j \mapsto f_j$ is the inclusion map $\bigoplus_{n \geq 0} I^{nc} N \to \bigoplus_{n \geq 0} I^n N$.

Proof. The uniqueness is clear from the formula and the uniqueness of Φ in Lemma 52.4.2. Consider the Noetherian A-algebra $B' = B \oplus N$ where N is an ideal of square zero. To show the existence of Φ_N it is enough (via Lemma 52.4.1) to show that φ_j extends to a map $\varphi'_j : I^c B' \to B'$ after possibly increasing c to some c' (and replacing φ_j by the composition of the inclusion $I^{c'} \to I^c$ with φ_j). Recall that φ_j corresponds to a section

$$h_j \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \mathcal{O}_{\mathrm{Spec}(A)})$$

see Cohomology of Schemes, Lemma 30.10.5. (This is in fact how we chose our φ_j in the proof of Lemma 52.4.1.) Let us use the same lemma to represent the pullback

$$h'_j \in \Gamma(\mathrm{Spec}(B') \setminus V(IB'), \mathcal{O}_{\mathrm{Spec}(B')})$$

of h_j by a B'-linear map $\varphi'_j : I^{c'} B' \to B'$ for some $c' \geq c$. The agreement with φ_j will hold for c' sufficiently large by a further application of the lemma: namely we can test agreement on a finite list of generators of $I^{c'}$. Small detail omitted. □

0EH6 **Lemma 52.4.4.** *Let $I = (f_1, \ldots, f_r)$ be an ideal of a Noetherian ring A with $cd(A, I) = 1$. Let $c \geq 1$ and $\varphi_j : I^c \to A$, $j = 1, \ldots, r$ be as in Lemma 52.4.1. Let X be a Noetherian scheme over* $\mathrm{Spec}(A)$*. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Set $\mathcal{F} = \lim \mathcal{F}_n$. Then, after possibly increasing c and adjusting φ_j accordingly, there exists a unique graded $\mathcal{O}_X$-module map

$$\Phi_{\mathcal{F}} : \bigoplus\nolimits_{n\geq 0} I^{nc}\mathcal{F} \longrightarrow \mathcal{F}[T_1, \ldots, T_r]$$

with $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$ where Φ is as in Lemma 52.4.2. The composition of $\Phi_{\mathcal{F}}$ with the map $\mathcal{F}[T_1, \ldots, T_r] \to \bigoplus_{n\geq 0} I^n\mathcal{F}$, $T_j \mapsto f_j$ is the canonical inclusion $\bigoplus_{n\geq 0} I^{nc}\mathcal{F} \to \bigoplus_{n\geq 0} I^n\mathcal{F}$.

Proof. The uniqueness is immediate from the $\mathcal{O}_X$-linearity and the requirement that $\Phi_{\mathcal{F}}(gs) = \Phi(g)s$ for $g \in I^{nc}$ and s a local section of $\mathcal{F}$. Thus we may assume $X = \mathrm{Spec}(B)$ is affine. Observe that $(\mathcal{F}_n)$ is an object of the category $Coh(X, I\mathcal{O}_X)$ introduced in Cohomology of Schemes, Section 30.23. Let $B' = B^\wedge$ be the I-adic completion of B. By Cohomology of Schemes, Lemma 30.23.1 the object $(\mathcal{F}_n)$ corresponds to a finite B'-module N in the sense that $\mathcal{F}_n$ is the coherent module associated to the finite B-module N/I^nN. Applying Lemma 52.4.3 to $I \subset A \to B'$ and N we see that, after possibly increasing c and adjusting φ_j accordingly, we get unique maps

$$\Phi_N : \bigoplus\nolimits_{n\geq 0} I^{nc}N \to N[T_1, \ldots, T_r]$$

with the corresponding properties. Note that in degree n we obtain an inverse system of maps $N/I^mN \to \bigoplus_{e_1+\ldots+e_r=n} N/I^{m-nc}N \cdot T_1^{e_1} \ldots T_r^{e_r}$ for $m \geq nc$. Translating back into coherent sheaves we see that Φ_N corresponds to a system of maps

$$\Phi_m^n : I^{nc}\mathcal{F}_m \longrightarrow \bigoplus\nolimits_{e_1+\ldots+e_r=n} \mathcal{F}_{m-nc} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for varying $m \geq nc$ and $n \geq 1$. Taking the inverse limit of these maps over m we obtain $\Phi_{\mathcal{F}} = \bigoplus_n \lim_m \Phi_m^n$. Note that $\lim_m I^t\mathcal{F}_m = I^t\mathcal{F}$ as can be seen by evaluating on affines for example, but in fact we don't need this because it is clear there is a map $I^t\mathcal{F} \to \lim_m I^t\mathcal{F}_m$. □

0EH7 **Lemma 52.4.5.** *Let I be an ideal of a Noetherian ring A. Let X be a Noetherian scheme over* $\mathrm{Spec}(A)$. *Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. If $cd(A, I) = 1$, then for all $p \in \mathbf{Z}$ the limit topology on $\lim H^p(X, \mathcal{F}_n)$ is I-adic.

Proof. First it is clear that $I^t \lim H^p(X, \mathcal{F}_n)$ maps to zero in $H^p(X, \mathcal{F}_t)$. Thus the I-adic topology is finer than the limit topology. For the converse we set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_{\mathcal{F}}$ as in Lemma 52.4.4. We will use the results of Lemma 52.2.1 without further mention. In particular we have a short exact sequence

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}) \to \lim H^p(X, \mathcal{F}_n) \to 0$$

Thus we can lift any element ξ of $\lim H^p(X, \mathcal{F}_n)$ to an element $\xi' \in H^p(X, \mathcal{F})$. Suppose ξ maps to zero in $H^p(X, \mathcal{F}_{nc})$ for some n, in other words, suppose ξ is "small" in the limit topology. We have a short exact sequence

$$0 \to I^{nc}\mathcal{F} \to \mathcal{F} \to \mathcal{F}_{nc} \to 0$$

and hence the assumption means we can lift ξ' to an element $\xi'' \in H^p(X, I^{nc}\mathcal{F})$. Applying $\Phi_{\mathcal{F}}$ we get

$$\Phi_{\mathcal{F}}(\xi'') = \sum\nolimits_{e_1+\ldots+e_r=n} \xi'_{e_1,\ldots,e_r} \cdot T_1^{e_1} \ldots T_r^{e_r}$$

for some $\xi'_{e_1,\ldots,e_r} \in H^p(X, \mathcal{F})$. Letting $\xi_{e_1,\ldots,e_r} \in \lim H^p(X, \mathcal{F}_n)$ be the images and using the final assertion of Lemma 52.4.4 we conclude that

$$\xi = \sum f_1^{e_1} \ldots f_r^{e_r} \xi_{e_1,\ldots,e_r}$$

is in $I^n \lim H^p(X, \mathcal{F}_n)$ as desired. □

0EH8 **Example 52.4.6.** Let k be a field. Let $A = k[x, y][[s, t]]/(xs - yt)$. Let $I = (s, t)$ and $\mathfrak{a} = (x, y, s, t)$. Let $X = \mathrm{Spec}(A) - V(\mathfrak{a})$ and $\mathcal{F}_n = \mathcal{O}_X/I^n\mathcal{O}_X$. Observe that the rational function

$$g = \frac{t}{x} = \frac{s}{y}$$

is regular in an open neighbourhood $V \subset X$ of $V(I\mathcal{O}_X)$. Hence every power g^e determines a section $g^e \in M = \lim H^0(X, \mathcal{F}_n)$. Observe that $g^e \to 0$ as $e \to \infty$ in the limit topology on M since g^e maps to zero in $\mathcal{F}_e$. On the other hand, $g^e \notin IM$ for any e as the reader can see by computing $H^0(U, \mathcal{F}_n)$; computation omitted. Observe that $\mathrm{cd}(A, I) = 2$. Thus the result of Lemma 52.4.5 is sharp.

52.5. Mittag-Leffler conditions

0EFN When taking local cohomology with respect to the maximal ideal of a local Noetherian ring, we often get the Mittag-Leffler condition for free. This implies the same thing is true for higher cohomology groups of an inverse system of coherent sheaves with surjective transition maps on the puncture spectrum.

0DX0 **Lemma 52.5.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *Let M be a finite A-module. Then the A-module $H^i_{\mathfrak{m}}(M)$ satisfies the descending chain condition for any i.*
2. *Let $U = \mathrm{Spec}(A)\backslash\{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then the A-module $H^i(U, \mathcal{F})$ satisfies the descending chain condition for $i > 0$.*

Proof. We will prove part (1) by induction on the dimension of the support of M. The statement holds if $M = 0$, thus we may and do assume M is not zero.

Base case of the induction. If $\dim(\mathrm{Supp}(M)) = 0$, then the support of M is $\{\mathfrak{m}\}$ and we see that $H^0_{\mathfrak{m}}(M) = M$ and $H^i_{\mathfrak{m}}(M) = 0$ for $i > 0$ as is clear from the construction of local cohomology, see Dualizing Complexes, Section 47.9. Since M has finite length (Algebra, Lemma 10.52.8) it has the descending chain condition.

Induction step. Assume $\dim(\mathrm{Supp}(M)) > 0$. By the base case the finite module $H^0_{\mathfrak{m}}(M) \subset M$ has the descending chain condition. By Dualizing Complexes, Lemma 47.11.6 we may replace M by $M/H^0_{\mathfrak{m}}(M)$. Then $H^0_{\mathfrak{m}}(M) = 0$, i.e., M has depth ≥ 1, see Dualizing Complexes, Lemma 47.11.1. Choose $x \in \mathfrak{m}$ such that $x : M \to M$ is injective. By Algebra, Lemma 10.63.10 we have $\dim(\mathrm{Supp}(M/xM)) = \dim(\mathrm{Supp}(M)) - 1$ and the induction hypothesis applies. Pick an index i and consider the exact sequence

$$H^{i-1}_{\mathfrak{m}}(M/xM) \to H^i_{\mathfrak{m}}(M) \xrightarrow{x} H^i_{\mathfrak{m}}(M)$$

coming from the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. It follows that the x-torsion $H^i_{\mathfrak{m}}(M)[x]$ is a quotient of a module with the descending chain condition, and hence has the descending chain condition itself. Hence the $\mathfrak{m}$-torsion submodule $H^i_{\mathfrak{m}}(M)[\mathfrak{m}]$ has the descending chain condition (and hence is finite dimensional over $A/\mathfrak{m}$). Thus we conclude that the $\mathfrak{m}$-power torsion module $H^i_{\mathfrak{m}}(M)$ has the descending chain condition by Dualizing Complexes, Lemma 47.7.7.

Part (2) follows from (1) via Local Cohomology, Lemma 51.8.2. □

0DX1 **Lemma 52.5.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

(1) *Let (M_n) be an inverse system of finite A-modules. Then the inverse system $H^i_{\mathfrak{m}}(M_n)$ satisfies the Mittag-Leffler condition for any i.*
(2) *Let $U = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A. Let $\mathcal{F}_n$ be an inverse system of coherent $\mathcal{O}_U$-modules. Then the inverse system $H^i(U, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition for $i > 0$.*

Proof. Follows immediately from Lemma 52.5.1. □

0EHB **Lemma 52.5.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let (M_n) be an inverse system of finite A-modules. Let $M \to \lim M_n$ be a map where M is a finite A-module such that for some i the map $H^i_{\mathfrak{m}}(M) \to \lim H^i_{\mathfrak{m}}(M_n)$ is an isomorphism. Then the inverse system $H^i_{\mathfrak{m}}(M_n)$ is essentially constant with value $H^i_{\mathfrak{m}}(M)$.*

Proof. By Lemma 52.5.2 the inverse system $H^i_{\mathfrak{m}}(M_n)$ satisfies the Mittag-Leffler condition. Let $E_n \subset H^i_{\mathfrak{m}}(M_n)$ be the image of $H^i_{\mathfrak{m}}(M_{n'})$ for $n' \gg n$. Then (E_n) is an inverse system with surjective transition maps and $H^i_{\mathfrak{m}}(M) = \lim E_n$. Since $H^i_{\mathfrak{m}}(M)$ has the descending chain condition by Lemma 52.5.1 we find there can only be a finite number of nontrivial kernels of the surjections $H^i_{\mathfrak{m}}(M) \to E_n$. Thus $E_n \to E_{n-1}$ is an isomorphism for all $n \gg 0$ as desired. □

0DXJ **Lemma 52.5.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then*

$$H^i(R\Gamma_{\mathfrak{m}}(M)^\wedge) = \lim H^i_{\mathfrak{m}}(M/I^nM)$$

for all i where $R\Gamma_{\mathfrak{m}}(M)^\wedge$ denotes the derived I-adic completion.

Proof. Apply Dualizing Complexes, Lemma 47.12.4 and Lemma 52.5.2 to see the vanishing of the $R^1 \lim$ terms. □

52.6. Derived completion on a ringed site

0995 We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 15.92. Let $\mathcal{O}$ be a sheaf of rings on a site $\mathcal{C}$. Let f be a global section of $\mathcal{O}$. We denote $\mathcal{O}_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal{O}(U)_f$.

0996 **Lemma 52.6.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$.*

(1) *For $L, N \in D(\mathcal{O}_f)$ we have $R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}_f}(L, N)$. In particular the two $\mathcal{O}_f$-structures on $R\mathcal{H}om_{\mathcal{O}}(L, N)$ agree.*
(2) *For $K \in D(\mathcal{O})$ and $L \in D(\mathcal{O}_f)$ we have*

$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}_f}(L, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K))$$

In particular $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$.

(3) *If g is a second global section of $\mathcal{O}$, then*

$$R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 21.20.10 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

by Modules on Sites, Lemma 18.11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 21.20.11 and 21.20.10. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet))$$

by Modules on Sites, Lemma 18.27.8 and because $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. □

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 13.34.1) of the inverse system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{O})$.

0997 **Lemma 52.6.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent*

(1) $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_{\mathcal{O}}(L, K) = 0$ *for all L in $D(\mathcal{O}_f)$*,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 52.6.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(x_0, x_1 - fx_0, x_2 - fx_1, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_{\mathcal{O}}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 19.13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). □

0998 **Lemma 52.6.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.*

Proof. We will use the results of Lemma 52.6.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 10.24.2 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_{\mathcal{O}}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. □

We can make the following definition for any ringed site.

0999 **Definition 52.6.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is *derived complete with respect to* $\mathcal{I}$ if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

099A **Lemma 52.6.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_{\mathcal{O}}(K, L) \in D_{comp}(\mathcal{O})$.*

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 21.35.2. The right hand side is zero by Lemma 52.6.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 21.35.1. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 52.6.2 we conclude. □

099C **Lemma 52.6.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}, \mathcal{I})$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$.*

Proof. Using Lemma 52.6.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. □

099J **Lemma 52.6.7.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 18.7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. Namely, by Cohomology on Sites, Lemma 21.35.1 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g}, Rf_*K|_U)$ is zero, which implies that $T(Rf_*K|_U, g)$ is zero (Lemma 52.6.2) which is what we have to show (Definition 52.6.4). Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{V,g'}, K|_V) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. The second equality because K is derived complete and the first equality because the derived pullback of $\mathcal{O}_{U,g}$ is $\mathcal{O}'_{V,g'}$ and Cohomology on Sites, Lemma 21.19.1. □

The following lemma is the simplest case where one has derived completion.

099B **Lemma 52.6.8.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map*

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 52.6.1 the object $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_{\mathcal{O}}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Next we explain why derived completion is a completion.

0A0E **Lemma 52.6.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Let $K \in D(\mathcal{O})$. The derived completion $K^\wedge$ of Lemma 52.6.8 is given by the formula*

$$K^\wedge = R\lim K \otimes^{\mathbf{L}}_{\mathcal{O}} K_n$$

where $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ is the Koszul complex on $f_1^n, \ldots, f_r^n$ over $\mathcal{O}$.

Proof. In More on Algebra, Lemma 15.29.6 we have seen that the extended alternating Čech complex

$$\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free $\mathcal{O}$-modules with dual $\mathcal{H}om_{\mathcal{O}}(K^n, \mathcal{O}) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). By Lemma 52.6.8 the functor $E \mapsto E^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into E:

$$E^\wedge = R\mathcal{H}om(\operatorname{colim} K^n, E)$$

This is equal to $R\lim(E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n)$ by Cohomology on Sites, Lemma 21.48.8. □

099D **Lemma 52.6.10.** *There exist a way to construct*

(1) *for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,*
(2) *a map $K(A, I) \to A$ of complexes of A-modules,*
(3) *for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,*

such that

(a) *for $A \to B$ and $I \subset A$ finitely generated the diagram*

$$\begin{array}{ccc} K(A, I) & \longrightarrow & A \\ \downarrow & & \downarrow \\ K(B, IB) & \longrightarrow & B \end{array}$$

commutes,

(b) *for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A, I) \to K(B, IB) \to K(C, IC)$ is the map $K(A, I) \to K(C, IC)$.*
(c) *for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A, I) \otimes^{\mathbf{L}}_A B \to K(B, IB)$ is an isomorphism in $D(B)$, and*
(d) *if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram*

$$\begin{array}{ccc} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) & \longrightarrow & K(A, I) \\ \downarrow & & \downarrow \\ A & \xrightarrow{1} & A \end{array}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A, I) = \operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A_0', I_0') \to (A, I)$ are given by maps $(A_0, I_0) \to (A_0', I_0')$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a), (b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \mathrm{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing the image of $R\Gamma_I(A)$ in $D(A)$, see Dualizing Complexes, Section 47.9. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{J}^\bullet \\ \uparrow & & \uparrow \\ \mathcal{K}^\bullet & \longrightarrow & \mathcal{F}^\bullet \end{array}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 47.10.2 and 47.10.3. □

099E **Lemma 52.6.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism*

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U,f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{U,f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{U,f_1\ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 7.29.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 7.29.1. In other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathrm{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K^\bullet_{U,I} : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 52.6.10. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K^\bullet_{U,I}$ and $K^\bullet_{U,I'}$ are the *same* for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K^\bullet_U$ the common value.

The independence of choice of I also shows that $K^\bullet_U|_{\mathcal{C}/U'} = K^\bullet_{U'}$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K^\bullet_U$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 52.6.10. □

099F **Proposition 52.6.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.*

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 52.6.11. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 52.6.8. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. □

0CQH **Remark 52.6.13** (Comparison with completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 52.6.12. For any $n \geq 1$ the object $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ is derived complete as it is annihilated by powers of local sections of $\mathcal{I}$. Hence there is a canonical factorization

$$K \to K^\wedge \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$$

of the canonical map $K \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$. These maps are compatible for varying n and we obtain a comparison map

$$K^\wedge \longrightarrow R\lim \left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n\right)$$

The right hand side is more recognizable as a kind of completion. In general this comparison map is not an isomorphism.

0A0F **Remark 52.6.14** (Localization and derived completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 52.6.12. It follows from the construction in the proof of the proposition that $K^\wedge|_U$ is the derived completion of $K|_U$ for any $U \in \mathrm{Ob}(\mathcal{C})$. But we can also prove this as follows. From the definition of derived complete objects it follows that $K^\wedge|_U$ is derived complete. Thus we obtain a canonical map $a : (K|_U)^\wedge \to K^\wedge|_U$. On the other hand, if E is a derived complete object of $D(\mathcal{O}_U)$, then Rj_*E is a derived complete object of $D(\mathcal{O})$ by Lemma 52.6.7. Here j is the localization morphism (Modules on Sites, Section 18.19). Hence we also obtain a canonical map $b : K^\wedge \to Rj_*((K|_U)^\wedge)$. We omit the (formal) verification that the adjoint of b is the inverse of a.

099G **Remark 52.6.15** (Completed tensor product)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 52.6.12. Then we set

$$K \otimes^\wedge_{\mathcal{O}} L = (K \otimes^{\mathbf{L}}_{\mathcal{O}} L)^\wedge$$

This *completed tensor product* defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\mathrm{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_{\mathcal{O}}(L, M)) = \mathrm{Hom}_{D_{comp}(\mathcal{O})}(K \otimes^{\wedge}_{\mathcal{O}} L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_{\mathcal{O}}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 52.6.5.

099H **Lemma 52.6.16.** *Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)\mathcal{O}'$. Then the map of extended alternating Čech complexes*

$$\begin{array}{ccc} \mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r} \\ \downarrow \\ \mathcal{O}' \to \prod_{i_0} \mathcal{O}'_{f_{i_0}} \to \prod_{i_0<i_1} \mathcal{O}'_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}'_{f_1\ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 15.29.6. Hence it suffices to prove $K_n \to K_n \otimes_{\mathcal{O}} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism where H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 15.28.6. Hence these sheaves are annihilated by $(f_1, \ldots, f_r)^m$ for some $m \gg 0$. Thus $H^i \to H^i \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism by Modules on Sites, Lemma 18.28.16. □

099I **Lemma 52.6.17.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Lemma 52.6.7 implies restriction $r : D(\mathcal{O}') \to D(\mathcal{O})$ sends $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$. We will construct a quasi-inverse $E \mapsto E'$.

Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 52.6.11. Set $K' = K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}'$ in $D(\mathcal{O}')$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ which satisfies the conclusions of Lemma 52.6.11 with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to r(K')$ is a quasi-isomorphism by Lemma 52.6.16. Now, for $E \in D_{comp}(\mathcal{O}, \mathcal{I})$ we set

$$E' = R\mathcal{H}om_{\mathcal{O}}(r(K'), E)$$

viewed as an object in $D(\mathcal{O}')$ using the $\mathcal{O}'$-module structure on K'. Since E is derived complete we have $E = R\mathcal{H}om_{\mathcal{O}}(K, E)$, see proof of Proposition 52.6.12. On the other hand, since $K \to r(K')$ is an isomorphism in we see that there is an isomorphism $E \to r(E')$ in $D(\mathcal{O})$. To finish the proof we have to show that, if $E = r(M')$ for an object M' of $D_{comp}(\mathcal{O}', \mathcal{I}')$, then $E' \cong M'$. To get a map we use

$$M' = R\mathcal{H}om_{\mathcal{O}'}(\mathcal{O}', M') \to R\mathcal{H}om_{\mathcal{O}}(r(\mathcal{O}'), r(M')) \to R\mathcal{H}om_{\mathcal{O}}(r(K'), r(M')) = E'$$

where the second arrow uses the map $K' \to \mathcal{O}'$. To see that this is an isomorphism, one shows that r applied to this arrow is the same as the isomorphism $E \to r(E')$ above. Details omitted. □

099K **Lemma 52.6.18.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.*

Proof. The first statement we have seen in Lemma 52.6.7. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 52.6.12. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$\mathrm{Hom}(K, Rf_*M) = \mathrm{Hom}(Lf^*K, M) = \mathrm{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. □

0A0G **Lemma 52.6.19.** *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $\mathcal{I}' \subset \mathcal{O}'$ be the ideal generated by $f^\sharp(f^{-1}\mathcal{I})$. Then Rf_* commutes with derived completion, i.e., $Rf_*(K^\wedge) = (Rf_*K)^\wedge$.*

Generalization of [**BS13**, Lemma 6.5.9 (2)]. Compare with [**HLP14**, Theorem 6.5] in the setting of quasi-coherent modules and morphisms of (derived) algebraic stacks.

Proof. By Proposition 52.6.12 the derived completion functors exist. By Lemma 52.6.7 the object $Rf_*(K^\wedge)$ is derived complete, and hence we obtain a canonical map $(Rf_*K)^\wedge \to Rf_*(K^\wedge)$ by the universal property of derived completion. We may check this map is an isomorphism locally on $\mathcal{C}$. Thus, since derived completion commutes with localization (Remark 52.6.14) we may assume that $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$. Then $\mathcal{I}'$ is generated by $g_i = f^\sharp(f_i)$. By Lemma 52.6.9 we have to prove that

$$R\lim \left(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n)\right) = Rf_* \left(R\lim K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

Because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3) it suffices to prove that

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n) = Rf_* \left(K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

This follows from the projection formula (Cohomology on Sites, Lemma 21.50.1) and the fact that $Lf^*K(\mathcal{O}, f_1^n, \ldots, f_r^n) = K(\mathcal{O}', g_1^n, \ldots, g_r^n)$. □

0BLX **Lemma 52.6.20.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $\mathcal{C}$ be a site and let $\mathcal{O}$ be a sheaf of A-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then we have*

$$R\Gamma(\mathcal{C}, \mathcal{F})^\wedge = R\Gamma(\mathcal{C}, \mathcal{F}^\wedge)$$

in $D(A)$ where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $I\mathcal{O}$ and on the left hand wide we have the derived completion with respect to I. This produces two spectral sequences

$$E_2^{i,j} = H^i(H^j(\mathcal{C}, \mathcal{F})^\wedge) \quad \text{and} \quad E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{F}^\wedge))$$

both converging to $H^*(R\Gamma(\mathcal{C}, \mathcal{F})^\wedge) = H^*(\mathcal{C}, \mathcal{F}^\wedge)$

Proof. Apply Lemma 52.6.19 to the morphism of ringed topoi $(\mathcal{C}, \mathcal{O}) \to (pt, A)$ and take cohomology to get the first statement. The second spectral sequence is the second spectral sequence of Derived Categories, Lemma 13.21.3. The first spectral sequence is the spectral sequence of More on Algebra, Example 15.92.23 applied to $R\Gamma(\mathcal{C}, \mathcal{F})^\wedge$. □

0CQI **Remark 52.6.21.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion of Proposition 52.6.12. Let $U \in \mathrm{Ob}(\mathcal{C})$ be an object such that $\mathcal{I}$ is generated as an ideal sheaf by $f_1, \ldots, f_r \in \mathcal{I}(U)$. Set $A = \mathcal{O}(U)$ and $I = (f_1, \ldots, f_r) \subset A$. Warning: it may not be the case that $I = \mathcal{I}(U)$. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

where the right hand side is the derived completion of the object $R\Gamma(U, K)$ of $D(A)$ with respect to I. This is true because derived completion commutes with localization (Remark 52.6.14) and Lemma 52.6.20.

52.7. The theorem on formal functions

0A0H We interrupt the flow of the exposition to talk a little bit about derived completion in the setting of quasi-coherent modules on schemes and to use this to give a somewhat different proof of the theorem on formal functions. We give some pointers to the literature in Remark 52.7.4.

Lemma 52.6.19 is a (very formal) derived version of the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5). To make this more explicit, suppose $f : X \to S$ is a morphism of schemes, $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent sheaf of ideals of finite type, and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then the lemma says that

0A0I (52.7.0.1) $$Rf_*(\mathcal{F}^\wedge) = (Rf_*\mathcal{F})^\wedge$$

where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ and the right hand side is the derived completion of $Rf_*\mathcal{F}$ with respect to $\mathcal{I}$. To see that this gives back the theorem on formal functions we have to do a bit of work.

0A0L **Lemma 52.7.1.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be a pseudo-coherent object of $D(\mathcal{O}_X)$ with derived completion $K^\wedge$. Then*

$$H^p(U, K^\wedge) = \lim H^p(U, K)/I^n H^p(U, K) = H^p(U, K)^\wedge$$

for any affine open $U \subset X$ where $I = \mathcal{I}(U)$ and where on the right we have the derived completion with respect to I.

Proof. Write $U = \mathrm{Spec}(A)$. The ring A is Noetherian and hence $I \subset A$ is finitely generated. Then we have

$$R\Gamma(U, K^\wedge) = R\Gamma(U, K)^\wedge$$

by Remark 52.6.21. Now $R\Gamma(U, K)$ is a pseudo-coherent complex of A-modules (Derived Categories of Schemes, Lemma 36.10.2). By More on Algebra, Lemma 15.95.4 we conclude that the pth cohomology module of $R\Gamma(U, K^\wedge)$ is equal to the I-adic completion of $H^p(U, K)$. This proves the first equality. The second (less important) equality follows immediately from a second application of the lemma just used. □

0A0K **Lemma 52.7.2.** *Let X be a locally Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let K be an object of $D(\mathcal{O}_X)$. Then*

(1) *the derived completion $K^\wedge$ is equal to $R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$.*

Let K is a pseudo-coherent object of $D(\mathcal{O}_X)$. Then

(2) *the cohomology sheaf $H^q(K^\wedge)$ is equal to $\lim H^q(K)/\mathcal{I}^n H^q(K)$.*

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module[1]*. Then*

(3) *the derived completion $\mathcal{F}^\wedge$ is equal to $\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$,*
(4) *$\lim \mathcal{F}/\mathcal{I}^n\mathcal{F} = R\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$,*
(5) *$H^p(U, \mathcal{F}^\wedge) = 0$ for $p \neq 0$ for all affine opens $U \subset X$.*

Proof. Proof of (1). There is a canonical map

$$K \longrightarrow R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n),$$

see Remark 52.6.13. Derived completion commutes with passing to open subschemes (Remark 52.6.14). Formation of $R\lim$ commutes with passing to open subschemes. It follows that to check our map is an isomorphism, we may work locally. Thus we may assume $X = U = \mathrm{Spec}(A)$. Say $I = (f_1, \ldots, f_r)$. Let $K_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex. By More on Algebra, Lemma 15.95.1 we have seen that the pro-systems $\{K_n\}$ and $\{A/I^n\}$ of $D(A)$ are isomorphic. Using the equivalence $D(A) = D_{QCoh}(\mathcal{O}_X)$ of Derived Categories of Schemes, Lemma 36.3.5 we see that the pro-systems $\{K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\}$ and $\{\mathcal{O}_X/\mathcal{I}^n\}$ are isomorphic in $D(\mathcal{O}_X)$. This proves the second equality in

$$K^\wedge = R\lim \left(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} K(\mathcal{O}_X, f_1^n, \ldots, f_r^n)\right) = R\lim(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^n)$$

The first equality is Lemma 52.6.9.

Assume K is pseudo-coherent. For $U \subset X$ affine open we have $H^q(U, K^\wedge) = \lim H^q(U, K)/\mathcal{I}^n(U)H^q(U, K)$ by Lemma 52.7.1. As this is true for every U we see that $H^q(K^\wedge) = \lim H^q(K)/\mathcal{I}^n H^q(K)$ as sheaves. This proves (2).

Part (3) is a special case of (2). Parts (4) and (5) follow from Derived Categories of Schemes, Lemma 36.3.2. □

0A0M **Lemma 52.7.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let X be a Noetherian scheme over A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $H^p(X, \mathcal{F})$ is a finite A-module for all p. Then there are short exact sequences*

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \mathcal{F})^\wedge \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules where $H^p(X, \mathcal{F})^\wedge$ is the usual I-adic completion. If f is proper, then the $R^1 \lim$ term is zero.

Proof. Consider the two spectral sequences of Lemma 52.6.20. The first degenerates by More on Algebra, Lemma 15.95.4. We obtain $H^p(X, \mathcal{F})^\wedge$ in degree p. This is where we use the assumption that $H^p(X, \mathcal{F})$ is a finite A-module. The second degenerates because

$$\mathcal{F}^\wedge = \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$$

is a sheaf by Lemma 52.7.2. We obtain $H^p(X, \lim \mathcal{F}/I^n\mathcal{F})$ in degree p. Since $R\Gamma(X, -)$ commutes with derived limits (Injectives, Lemma 19.13.6) we also get

$$R\Gamma(X, \lim \mathcal{F}/I^n\mathcal{F}) = R\Gamma(X, R\lim \mathcal{F}/I^n\mathcal{F}) = R\lim R\Gamma(X, \mathcal{F}/I^n\mathcal{F})$$

By More on Algebra, Remark 15.88.6 we obtain exact sequences

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \lim \mathcal{F}/I^n\mathcal{F}) \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. Combining the above we get the first statement of the lemma. The vanishing of the $R^1 \lim$ term follows from Cohomology of Schemes, Lemma 30.20.4. □

[1] For example $H^q(K)$ for K pseudo-coherent on our locally Noetherian X.

0AKL **Remark 52.7.4.** Here are some references to discussions of related material the literature. It seems that a "derived formal functions theorem" for proper maps goes back to [**Lur04**, Theorem 6.3.1]. There is the discussion in [**Lur11**], especially Chapter 4 which discusses the affine story, see More on Algebra, Section 15.92. In [**GR13**, Section 2.9] one finds a discussion of proper base change and derived completion using (ind) coherent modules. An analogue of (52.7.0.1) for complexes of quasi-coherent modules can be found as [**HLP14**, Theorem 6.5]

52.8. Algebraization of local cohomology, I

0EFF Let A be a Noetherian ring and let I and J be two ideals of A. Let M be a finite A-module. In this section we study the cohomology groups of the object

$$R\Gamma_J(M)^\wedge \quad \text{of} \quad D(A)$$

where $^\wedge$ denotes derived I-adic completion. Observe that in Dualizing Complexes, Lemma 47.12.5 we have shown, if A is complete with respect to I, that there is an isomorphism

$$\operatorname{colim} H^0_Z(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

where the (directed) colimit is over the closed subsets $Z = V(J')$ with $J' \subset J$ and $V(J') \cap V(I) = V(J) \cap V(I)$. The union of these closed subsets is

0EFG (52.8.0.1) $$T = \{\mathfrak{p} \in \operatorname{Spec}(A) : V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)\}$$

This is a subset of $\operatorname{Spec}(A)$ stable under specialization. The result above becomes the statement that

$$H^0_T(M) \longrightarrow H^0(R\Gamma_J(M)^\wedge)$$

is an isomorphism provided A is complete with respect to I, see Local Cohomology, Lemma 51.5.3 and Remark 51.5.6. Our method to extend this isomorphism to higher cohomology groups rests on the following lemma.

0EFH **Lemma 52.8.1.** *Let I, J be ideals of a Noetherian ring A. Let M be a finite A-module. Let $\mathfrak{p} \subset A$ be a prime. Let s and d be integers. Assume*

1. *A has a dualizing complex,*
2. *$\mathfrak{p} \notin V(J) \cap V(I)$,*
3. *$cd(A, I) \leq d$, and*
4. *for all primes $\mathfrak{p}' \subset \mathfrak{p}$ we have $depth_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}}) > d + s$ for all $\mathfrak{q} \in V(\mathfrak{p}') \cap V(J) \cap V(I)$.*

Then there exists an $f \in A$, $f \notin \mathfrak{p}$ which annihilates $H^i(R\Gamma_J(M)^\wedge)$ for $i \leq s$ where $^\wedge$ indicates I-adic completion.

Proof. We will use that $R\Gamma_J = R\Gamma_{V(J)}$ and similarly for $I+J$, see Dualizing Complexes, Lemma 47.10.1. Observe that $R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge$, see Dualizing Complexes, Lemmas 47.12.1 and 47.9.6. Thus we may replace J by $I + J$ and assume $I \subset J$ and $\mathfrak{p} \notin V(J)$. Recall that

$$R\Gamma_J(M)^\wedge = R\operatorname{Hom}_A(R\Gamma_I(A), R\Gamma_J(M))$$

by the description of derived completion in More on Algebra, Lemma 15.92.10 combined with the description of local cohomology in Dualizing Complexes, Lemma

47.10.2. Assumption (3) means that $R\Gamma_I(A)$ has nonzero cohomology only in degrees $\leq d$. Using the canonical truncations of $R\Gamma_I(A)$ we find it suffices to show that

$$\mathrm{Ext}^i(N, R\Gamma_J(M))$$

is annihilated by an $f \in A$, $f \notin \mathfrak{p}$ for $i \leq s + d$ and any A-module N. In turn using the canonical truncations for $R\Gamma_J(M)$ we see that it suffices to show $H^i_J(M)$ is annihilated by an $f \in A$, $f \notin \mathfrak{p}$ for $i \leq s + d$. This follows from Local Cohomology, Lemma 51.10.2. □

0EFI **Lemma 52.8.2.** *Let I, J be ideals of a Noetherian ring. Let M be a finite A-module. Let s and d be integers. With T as in (52.8.0.1) assume*

(1) *A has a dualizing complex,*
(2) *if $\mathfrak{p} \in V(I)$, then no condition,*
(3) *if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \in T$, then $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ for some $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$,*
(4) *if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$, then*

$$\mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map

$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ and moreover these modules are annihilated by a power of $J_0 I$.

Proof. Let us consider the set

$$B = \{\mathfrak{p} \notin V(I),\ \mathfrak{p} \in T,\ \text{and } \mathrm{depth}(M_\mathfrak{p}) \leq s\}$$

Choose $J_0 \subset J$ such that $V(J_0)$ is the closure of $B \cup V(J)$.

Claim I: $V(J_0) \cap V(I) = V(J) \cap V(I)$.

Proof of Claim I. The inclusion $\supset$ holds by construction. Let $\mathfrak{p}$ be a minimal prime of $V(J_0)$. If $\mathfrak{p} \in B \cup V(J)$, then either $\mathfrak{p} \in T$ or $\mathfrak{p} \in V(J)$ and in both cases $V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)$ as desired. If $\mathfrak{p} \notin B \cup V(J)$, then $V(\mathfrak{p}) \cap B$ is dense, hence infinite, and we conclude that $\mathrm{depth}(M_\mathfrak{p}) < s$ by Local Cohomology, Lemma 51.9.2. In fact, let $V(\mathfrak{p}) \cap B = \{\mathfrak{p}_\lambda\}_{\lambda \in \Lambda}$. Pick $\mathfrak{q}_\lambda \in V(\mathfrak{p}_\lambda) \cap V(J) \cap V(I)$ as in (3). Let $\delta : \mathrm{Spec}(A) \to \mathbf{Z}$ be the dimension function associated to a dualizing complex $\omega_A^\bullet$ for A. Since Λ is infinite and δ is bounded, there exists an infinite subset $\Lambda' \subset \Lambda$ on which $\delta(\mathfrak{q}_\lambda)$ is constant. For $\lambda \in \Lambda'$ we have

$$\mathrm{depth}(M_{\mathfrak{p}_\lambda}) + \delta(\mathfrak{p}_\lambda) - \delta(\mathfrak{q}_\lambda) = \mathrm{depth}(M_{\mathfrak{p}_\lambda}) + \dim((A/\mathfrak{p}_\lambda)_{\mathfrak{q}_\lambda}) \leq d + s$$

by (3) and the definition of B. By the semi-continuity of the function depth + δ proved in Duality for Schemes, Lemma 48.2.8 we conclude that

$$\mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_{\mathfrak{q}_\lambda}) = \mathrm{depth}(M_\mathfrak{p}) + \delta(\mathfrak{p}) - \delta(\mathfrak{q}_\lambda) \leq d + s$$

Since also $\mathfrak{p} \notin V(I)$ we read off from (4) that $\mathfrak{p} \in T$, i.e., $V(\mathfrak{p}) \cap V(I) \subset V(J) \cap V(I)$. This finishes the proof of Claim I.

Claim II: $H^i_{J_0}(M) \to H^i_J(M)$ is an isomorphism for $i \leq s$ and $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$.

Proof of claim II. Choose $\mathfrak{p} \in V(J')$ not in $V(J_0)$. It suffices to show that $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for $i \leq s$, see Local Cohomology, Lemma 51.2.6. Observe that $\mathfrak{p} \in T$. Hence since $\mathfrak{p}$ is not in B we see that $\text{depth}(M_\mathfrak{p}) > s$ and the groups vanish by Dualizing Complexes, Lemma 47.11.1.

Claim III. The final statement of the lemma is true.

By Claim II for $i \leq s$ we have

$$H^i_T(M) = H^i_{J_0}(M) = H^i_{J'}(M)$$

for all ideals $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. See Local Cohomology, Lemma 51.5.3. Let us check the hypotheses of Local Cohomology, Proposition 51.10.1 for the subsets $T \subset T \cup V(I)$, the module M, and the integer s. We have to show that given $\mathfrak{p} \subset \mathfrak{q}$ with $\mathfrak{p} \notin T \cup V(I)$ and $\mathfrak{q} \in T$ we have

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

If $\text{depth}(M_\mathfrak{p}) \geq s$, then this is true because the dimension of $(A/\mathfrak{p})_\mathfrak{q}$ is at least 1. Thus we may assume $\text{depth}(M_\mathfrak{p}) < s$. If $\mathfrak{q} \in V(I)$, then $\mathfrak{q} \in V(J) \cap V(I)$ and the inequality holds by (4). If $\mathfrak{q} \notin V(I)$, then we can use (3) to pick $\mathfrak{q}' \in V(\mathfrak{q}) \cap V(J) \cap V(I)$ with $\dim((A/\mathfrak{q})_{\mathfrak{q}'}) \leq d$. Then assumption (4) gives

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_{\mathfrak{q}'}) > s + d$$

Since A is catenary this implies the inequality we want. Applying Local Cohomology, Proposition 51.10.1 we find $J'' \subset A$ with $V(J'') \subset T \cup V(I)$ such that J'' annihilates $H^i_T(M)$ for $i \leq s$. Then we can write $V(J'') \cup V(J_0) \cup V(I) = V(J'I)$ for some $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Replacing J_0 by J' the proof is complete. □

0EFJ **Lemma 52.8.3.** *In Lemma 52.8.2 if instead of the empty condition (2) we assume*

(2') *if $\mathfrak{p} \in V(I)$, $\mathfrak{p} \notin V(J) \cap V(I)$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$ for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$,*

then the conditions also imply that $H^i_{J_0}(M)$ is a finite A-module for $i \leq s$.

Proof. Recall that $H^i_{J_0}(M) = H^i_T(M)$, see proof of Lemma 52.8.2. Thus it suffices to check that for $\mathfrak{p} \notin T$ and $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$, see Local Cohomology, Proposition 51.11.1. Condition (2') tells us this is true for $\mathfrak{p} \in V(I)$. Since we know $H^i_T(M)$ is annihilated by a power of IJ_0 we know the condition holds if $\mathfrak{p} \notin V(IJ_0)$ by Local Cohomology, Proposition 51.10.1. This covers all cases and the proof is complete. □

0EFK **Lemma 52.8.4.** *If in Lemma 52.8.2 we additionally assume*

(6) *if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \in T$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$,*

then $H^i_{J_0}(M) = H^i_J(M) = H^i_{J+I}(M)$ for $i \leq s$ and these modules are annihilated by a power of I.

Proof. Choose $\mathfrak{p} \in V(J)$ or $\mathfrak{p} \in V(J_0)$ but $\mathfrak{p} \notin V(J + I) = V(J_0 + I)$. It suffices to show that $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for $i \leq s$, see Local Cohomology, Lemma 51.2.6. These groups vanish by condition (6) and Dualizing Complexes, Lemma 47.11.1. The final statement follows from Local Cohomology, Proposition 51.10.1. □

0EFL **Lemma 52.8.5.** *Let I, J be ideals of a Noetherian ring A. Let M be a finite A-module. Let s and d be integers. With T as in (52.8.0.1) assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $\mathfrak{p} \in V(I)$ no condition,*
(3) *$cd(A, I) \leq d$,*
(4) *if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$ then*
$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad or \quad depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$
for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$,
(5) *if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$, $V(\mathfrak{p}) \cap V(J) \cap V(I) \neq \emptyset$, and $depth(M_\mathfrak{p}) < s$, then one of the following holds[2]:*
   (a) *$\dim(Supp(M_\mathfrak{p})) < s + 2$[3], or*
   (b) *$\delta(\mathfrak{p}) > d + \delta_{max} - 1$ where δ is a dimension function and δ_{max} is the maximum of δ on $V(J) \cap V(I)$, or*
   (c) *$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s + \delta_{max} - \delta_{min} - 2$ for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.*

Then there exists an ideal $J_0 \subset J$ with $V(J_0) \cap V(I) = V(J) \cap V(I)$ such that for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$ the map
$$R\Gamma_{J'}(M) \longrightarrow R\Gamma_J(M)^\wedge$$
induces an isomorphism on cohomology in degrees $\leq s$. Here $^\wedge$ denotes derived I-adic completion.

We encourage the reader to read the proof in the local case first (Lemma 52.9.5) as it explains the structure of the proof without having to deal with all the inequalities.

Proof. For an ideal $\mathfrak{a} \subset A$ we have $R\Gamma_\mathfrak{a} = R\Gamma_{V(\mathfrak{a})}$, see Dualizing Complexes, Lemma 47.10.1. Next, we observe that
$$R\Gamma_J(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_{I+J}(M)^\wedge = R\Gamma_{I+J'}(M)^\wedge = R\Gamma_I(R\Gamma_{J'}(M))^\wedge = R\Gamma_{J'}(M)^\wedge$$
by Dualizing Complexes, Lemmas 47.9.6 and 47.12.1. This explains how we define the arrow in the statement of the lemma.

We claim that the hypotheses of Lemma 52.8.2 are implied by our current hypotheses on M. The only thing to verify is hypothesis (3). Thus let $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \in T$. Then $V(\mathfrak{p}) \cap V(I)$ is nonempty as I is contained in the Jacobson radical of A (Algebra, Lemma 10.96.6). Since $\mathfrak{p} \in T$ we have $V(\mathfrak{p}) \cap V(I) = V(\mathfrak{p}) \cap V(J) \cap V(I)$. Let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I)$ be the generic point of an irreducible component. We have $\mathrm{cd}(A_\mathfrak{q}, I_\mathfrak{q}) \leq d$ by Local Cohomology, Lemma 51.4.6. We have $V(\mathfrak{p}A_\mathfrak{q}) \cap V(I_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ by our choice of $\mathfrak{q}$ and we conclude $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ by Local Cohomology, Lemma 51.4.10.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^i(R\Gamma_J(M)^\wedge)$ is zero for $i < 0$. However, this vanishing was established in Dualizing Complexes, Lemma 47.12.4. We will prove the lemma by induction for $s \geq 0$.

The lemma for $s = 0$ follows immediately from the conclusion of Lemma 52.8.2 and Dualizing Complexes, Lemma 47.12.5.

Assume $s > 0$ and the lemma has been shown for smaller values of s. Let $M' \subset M$ be the maximal submodule whose support is contained in $V(I) \cup T$. Then M' is a

[2]Our method forces this additional condition. We will return to this (insert future reference).
[3]For example if M satisfies Serre's condition (S_s) on the complement of $V(I) \cup T$.

finite A-module whose support is contained in $V(J') \cup V(I)$ for some ideal $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. We claim that

$$R\Gamma_{J'}(M') \to R\Gamma_J(M')^\wedge$$

is an isomorphism for any choice of J'. Namely, we can choose a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J', with M_2 annihilated by a power of I, and with N annihilated by a power of $I + J'$. Thus it suffices to show that the claim holds for M_1, M_2, and N. In the case of M_1 we see that $R\Gamma_{J'}(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. This proves the claim for M_1 by the initial remarks of the proof. In the case of M_2 we see that $H^i_J(M_2) = H^i_{I+J}(M) = H^i_{I+J'}(M) = H^i_{J'}(M_2)$ are annihilated by a power of I and hence derived complete. Thus the claim in this case also. For N we can use either of the arguments just given. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. Thus we may assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$.

Let $\mathfrak{p} \in \text{Ass}(M)$ be such that $V(\mathfrak{p}) \cap V(J) \cap V(I) = \emptyset$. Since I is contained in the Jacobson radical of A this implies that $V(\mathfrak{p}) \cap V(J') = \emptyset$ for any $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus setting $N = H^0_{\mathfrak{p}}(M)$ we see that $R\Gamma_J(N) = R\Gamma_{J'}(N) = 0$ for all $J' \subset J$ with $V(J') \cap V(I) = V(J) \cap V(I)$. In particular $R\Gamma_J(N)^\wedge = 0$. Thus we may replace M by M/N as this changes the structure of M only in primes which do not play a role in conditions (4) or (5). Repeating we may assume that $V(\mathfrak{p}) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak{p} \in \text{Ass}(M)$.

Assume $\text{Ass}(M) \cap (V(I) \cup T) = \emptyset$ and that $V(\mathfrak{p}) \cap V(J) \cap V(I) \neq \emptyset$ for all $\mathfrak{p} \in \text{Ass}(M)$. Let $\mathfrak{p} \in \text{Ass}(M)$. We want to show that we may apply Lemma 52.8.1. It is in the verification of this that we will use the supplemental condition (5). Choose $\mathfrak{p}' \subset \mathfrak{p}$ and $\mathfrak{q}' \subset V(\mathfrak{p}) \cap V(J) \cap V(I)$.

(1) If $M_{\mathfrak{p}'} = 0$, then $\text{depth}(M_{\mathfrak{p}'}) = \infty$ and $\text{depth}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}'}) > d + s$.
(2) If $\text{depth}(M_{\mathfrak{p}'}) < s$, then $\text{depth}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}'}) > d + s$ by (4).

In the remaining cases we have $M_{\mathfrak{p}'} \neq 0$ and $\text{depth}(M_{\mathfrak{p}'}) \geq s$. In particular, we see that $\mathfrak{p}'$ is in the support of M and we can choose $\mathfrak{p}'' \subset \mathfrak{p}'$ with $\mathfrak{p}'' \in \text{Ass}(M)$.

(a) Observe that $\dim((A/\mathfrak{p}'')_{\mathfrak{p}'}) \geq \text{depth}(M_{\mathfrak{p}'})$ by Algebra, Lemma 10.72.9. If equality holds, then we have

$$\text{depth}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M_{\mathfrak{p}''}) + \dim((A/\mathfrak{p}'')_{\mathfrak{q}'}) > s + d$$

by (4) applied to $\mathfrak{p}''$ and we are done. This means we are only in trouble if $\dim((A/\mathfrak{p}'')_{\mathfrak{p}'}) > \text{depth}(M_{\mathfrak{p}'})$. This implies that $\dim(M_{\mathfrak{p}}) \geq s + 2$. Thus if (5)(a) holds, then this does not occur.
(b) If (5)(b) holds, then we get

$$\text{depth}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}'}) \geq s + \delta(\mathfrak{p}') - \delta(\mathfrak{q}') \geq s + 1 + \delta(\mathfrak{p}) - \delta_{max} > s + d$$

as desired.

(c) If (5)(c) holds, then we get

$$\begin{aligned}\operatorname{depth}(M_{\mathfrak{p}'}) + \dim((A/\mathfrak{p}')_{\mathfrak{q}'}) &\geq s + \delta(\mathfrak{p}') - \delta(\mathfrak{q}') \\ &\geq s + 1 + \delta(\mathfrak{p}) - \delta(\mathfrak{q}') \\ &= s + 1 + \delta(\mathfrak{p}) - \delta(\mathfrak{q}) + \delta(\mathfrak{q}) - \delta(\mathfrak{q}') \\ &> s + 1 + (s + d + \delta_{max} - \delta_{min} - 2) + \delta(\mathfrak{q}) - \delta(\mathfrak{q}') \\ &\geq 2s + d - 1 \geq s + d\end{aligned}$$

as desired. Observe that this argument works because we know that a prime $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$ exists.

Now we are ready to do the induction step.

Choose an ideal J_0 as in Lemma 52.8.2 and an integer $t > 0$ such that $(J_0 I)^t$ annihilates $H^s_J(M)$. The assumptions of Lemma 52.8.1 are satisfied for every $\mathfrak{p} \in \operatorname{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak{a} \subset A$ of $H^s(R\Gamma_J(M)^\wedge)$ is not contained in $\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Ass}(M)$. Thus we can find an $f \in \mathfrak{a}(J_0 I)^t$ not in any associated prime of M which is an annihilator of both $H^s(R\Gamma_J(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{array}{ccccccc} H^{s-1}_{J'}(M) & \longrightarrow & H^{s-1}_{J'}(M/fM) & \longrightarrow & H^s_{J'}(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{s-1}(R\Gamma_J(M)^\wedge) & \longrightarrow & H^{s-1}(R\Gamma_J(M/fM)^\wedge) & \longrightarrow & H^s(R\Gamma_J(M)^\wedge) & \longrightarrow & 0 \end{array}$$

for any $J' \subset J_0$ with $V(J') \cap V(I) = V(J) \cap V(I)$. Thus if we choose J' such that it works for M and M/fM and $s-1$ (possible by induction hypothesis – see next paragraph), then we conclude that the lemma is true.

To finish the proof we have to show that the module M/fM satisfies the hypotheses (4) and (5) for $s-1$. Thus we let $\mathfrak{p}$ be a prime in the support of M/fM with $\operatorname{depth}((M/fM)_{\mathfrak{p}}) < s-1$ and with $V(\mathfrak{p}) \cap V(J) \cap V(I)$ nonempty. Then $\dim(M_{\mathfrak{p}}) = \dim((M/fM)_{\mathfrak{p}})+1$ and $\operatorname{depth}(M_{\mathfrak{p}}) = \operatorname{depth}((M/fM)_{\mathfrak{p}})+1$. In particular, we know (4) and (5) hold for $\mathfrak{p}$ and M with the original value s. The desired inequalities then follow by inspection. □

0EFM **Example 52.8.6.** In Lemma 52.8.5 we do not know that the inverse systems $H^i_J(M/I^n M)$ satisfy the Mittag-Leffler condition. For example, suppose that $A = \mathbf{Z}_p[[x, y]]$, $I = (p)$, $J = (p, x)$, and $M = A/(xy - p)$. Then the image of $H^0_J(M/p^n M) \to H^0_J(M/pM)$ is the ideal generated by y^n in $M/pM = A/(p, xy)$.

52.9. Algebraization of local cohomology, II

0EFP In this section we redo the arguments of Section 52.8 when $(A, \mathfrak{m})$ is a local ring and we take local cohomology $R\Gamma_{\mathfrak{m}}$ with respect to $\mathfrak{m}$. As before our main tool is the following lemma.

0DXK **Lemma 52.9.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module and let $\mathfrak{p} \subset A$ be a prime. Let s and d be integers. Assume*

(1) *A has a dualizing complex,*

(2) $cd(A, I) \leq d$, *and*
(3) $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$.

Then there exists an $f \in A \setminus \mathfrak{p}$ *which annihilates* $H^i(R\Gamma_\mathfrak{m}(M)^\wedge)$ *for* $i \leq s$ *where* $^\wedge$ *indicates* I*-adic completion.*

Proof. According to Local Cohomology, Lemma 51.9.4 the function

$$\mathfrak{p}' \longmapsto \text{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) + \dim(A/\mathfrak{p}')$$

is lower semi-continuous on $\text{Spec}(A)$. Thus the value of this function on $\mathfrak{p}' \subset \mathfrak{p}$ is $> s + d$. Thus our lemma is a special case of Lemma 52.8.1 provided that $\mathfrak{p} \neq \mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}$, then we have $H^i_\mathfrak{m}(M) = 0$ for $i \leq s + d$ by the relationship between depth and local cohomology (Dualizing Complexes, Lemma 47.11.1). Thus the argument given in the proof of Lemma 52.8.1 shows that $H^i(R\Gamma_\mathfrak{m}(M)^\wedge) = 0$ for $i \leq s$ in this (degenerate) case. □

0DXM **Lemma 52.9.2.** *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring. Let* $I \subset A$ *be an ideal. Let* M *be a finite* A*-module. Let* s *and* d *be integers. Assume*

(1) A *has a dualizing complex,*
(2) *if* $\mathfrak{p} \in V(I)$*, then no condition,*
(3) *if* $\mathfrak{p} \notin V(I)$ *and* $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$*, then* $\dim(A/\mathfrak{p}) \leq d$*,*
(4) *if* $\mathfrak{p} \notin V(I)$ *and* $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$*, then*

$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad or \quad depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$$

Then there exists an ideal $J_0 \subset A$ *with* $V(J_0) \cap V(I) = \{\mathfrak{m}\}$ *such that for any* $J \subset J_0$ *with* $V(J) \cap V(I) = \{\mathfrak{m}\}$ *the map*

$$R\Gamma_J(M) \longrightarrow R\Gamma_{J_0}(M)$$

induces an isomorphism in cohomology in degrees $\leq s$ *and moreover these modules are annihilated by a power of* $J_0 I$*.*

Proof. This is a special case of Lemma 52.8.2. □

0DXN **Lemma 52.9.3.** *In Lemma 52.9.2 if instead of the empty condition (2) we assume*

(2') *if* $\mathfrak{p} \in V(I)$ *and* $\mathfrak{p} \neq \mathfrak{m}$*, then* $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > s$*,*

then the conditions also imply that $H^i_{J_0}(M)$ *is a finite* A*-module for* $i \leq s$*.*

Proof. This is a special case of Lemma 52.8.3. □

0EFQ **Lemma 52.9.4.** *If in Lemma 52.9.2 we additionally assume*

(6) *if* $\mathfrak{p} \notin V(I)$ *and* $V(\mathfrak{p}) \cap V(I) = \{\mathfrak{m}\}$*, then* $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$*,*

then $H^i_{J_0}(M) = H^i_J(M) = H^i_\mathfrak{m}(M)$ *for* $i \leq s$ *and these modules are annihilated by a power of* I*.*

Proof. This is a special case of Lemma 52.8.4. □

0DXP **Lemma 52.9.5.** *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring. Let* $I \subset A$ *be an ideal. Let* M *be a finite* A*-module. Let* s *and* d *be integers. Assume*

(1) A *is* I*-adically complete and has a dualizing complex,*
(2) *if* $\mathfrak{p} \in V(I)$*, no condition,*
(3) $cd(A, I) \leq d$*,*

(4) *if* $\mathfrak{p} \notin V(I)$ *and* $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$ *then*

$$depth_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad or \quad depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$$

Then there exists an ideal $J_0 \subset A$ *with* $V(J_0) \cap V(I) = \{\mathfrak{m}\}$ *such that for any* $J \subset J_0$ *with* $V(J) \cap V(I) = \{\mathfrak{m}\}$ *the map*

$$R\Gamma_J(M) \longrightarrow R\Gamma_J(M)^\wedge = R\Gamma_\mathfrak{m}(M)^\wedge$$

induces an isomorphism in cohomology in degrees $\leq s$. *Here* $^\wedge$ *denotes derived* I*-adic completion.*

Proof. This lemma is a special case of Lemma 52.8.5 since condition (5)(c) is implied by condition (4) as $\delta_{max} = \delta_{min} = \delta(\mathfrak{m})$. We will give the proof of this important special case as it is somewhat easier (fewer things to check).

There is no difference between $R\Gamma_\mathfrak{a}$ and $R\Gamma_{V(\mathfrak{a})}$ in our current situation, see Dualizing Complexes, Lemma 47.10.1. Next, we observe that

$$R\Gamma_\mathfrak{m}(M)^\wedge = R\Gamma_I(R\Gamma_J(M))^\wedge = R\Gamma_J(M)^\wedge$$

by Dualizing Complexes, Lemmas 47.9.6 and 47.12.1 which explains the equality sign in the statement of the lemma.

Observe that the lemma holds for $s < 0$. This is not a trivial case because it is not a priori clear that $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is zero for negative s. However, this vanishing was established in Lemma 52.5.4. We will prove the lemma by induction for $s \geq 0$.

The assumptions of Lemma 52.9.2 are satisfied by Local Cohomology, Lemma 51.4.10. The lemma for $s = 0$ follows from Lemma 52.9.2 and Dualizing Complexes, Lemma 47.12.5.

Assume $s > 0$ and the lemma holds for smaller values of s. Let $M' \subset M$ be the submodule of elements whose support is condained in $V(I) \cup V(J)$ for some ideal J with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Then M' is a finite A-module. We claim that

$$R\Gamma_J(M') \to R\Gamma_\mathfrak{m}(M')^\wedge$$

is an isomorphism for any choice of J. Namely, for any such module there is a short exact sequence $0 \to M_1 \oplus M_2 \to M' \to N \to 0$ with M_1 annihilated by a power of J, with M_2 annihilated by a power of I and with N annihilated by a power of $\mathfrak{m}$. In the case of M_1 we see that $R\Gamma_J(M_1) = M_1$ and since M_1 is a finite A-module and I-adically complete we have $M_1^\wedge = M_1$. Thus the claim holds for M_1. In the case of M_2 we see that $H^i_J(M_2)$ is annihilated by a power of I and hence derived complete. Thus the claim for M_2. By the same arguments the claim holds for N and we conclude that the claim holds. Considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$ we see that it suffices to prove the lemma for M/M'. This we may assume $\mathfrak{p} \in \mathrm{Ass}(M)$ implies $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, i.e., $\mathfrak{p}$ is a prime as in (4).

Choose an ideal J_0 as in Lemma 52.9.2 and an integer $t > 0$ such that $(J_0 I)^t$ annihilates $H^s_J(M)$. Here J denotes an arbitrary ideal $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. The assumptions of Lemma 52.9.1 are satisfied for every $\mathfrak{p} \in \mathrm{Ass}(M)$ (see previous paragraph). Thus the annihilator $\mathfrak{a} \subset A$ of $H^s(R\Gamma_\mathfrak{m}(M)^\wedge)$ is not contained in $\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Ass}(M)$. Thus we can find an $f \in \mathfrak{a}(J_0 I)^t$ not in any associated prime

of M which is an annihilator of both $H^s(R\Gamma_{\mathfrak{m}}(M)^\wedge)$ and $H^s_J(M)$. Then f is a nonzerodivisor on M and we can consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

Our choice of f shows that we obtain

$$\begin{array}{ccccccc} H^{s-1}_J(M) & \longrightarrow & H^{s-1}_J(M/fM) & \longrightarrow & H^s_J(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{s-1}(R\Gamma_{\mathfrak{m}}(M)^\wedge) & \longrightarrow & H^{s-1}(R\Gamma_{\mathfrak{m}}(M/fM)^\wedge) & \longrightarrow & H^s(R\Gamma_{\mathfrak{m}}(M)^\wedge) & \longrightarrow & 0 \end{array}$$

for any $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Thus if we choose J such that it works for M and M/fM and $s-1$ (possible by induction hypothesis), then we conclude that the lemma is true. □

52.10. Algebraization of local cohomology, III

0EFT In this section we bootstrap the material in Sections 52.8 and 52.9 to give a stronger result the following situation.

0EFU **Situation 52.10.1.** Here A is a Noetherian ring. We have an ideal $I \subset A$, a finite A-module M, and a subset $T \subset V(I)$ stable under specialization. We have integers s and d. We assume

(1) A has a dualizing complex,
(3) $\mathrm{cd}(A, I) \leq d$,
(4) given primes $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \notin V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$ we have
$$\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$
(6) given $\mathfrak{q} \in T$ denoting $A', \mathfrak{m}', I', M'$ are the usual I-adic completions of $A_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, I_\mathfrak{q}, M_\mathfrak{q}$ we have
$$\mathrm{depth}(M'_{\mathfrak{p}'}) > s$$
for all $\mathfrak{p}' \in \mathrm{Spec}(A') \setminus V(I')$ with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$.

The following lemma explains why in Situation 52.10.1 it suffices to look at triples $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ of primes in (4) even though the actual assumption only involves $\mathfrak{p}$ and $\mathfrak{q}$.

0EID **Lemma 52.10.2.** *In Situation 52.10.1 let $\mathfrak{p} \subset \mathfrak{q}$ be primes of A with $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in T$. If there does not exist an $\mathfrak{r} \in V(I) \setminus T$ with $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ then depth$(M_\mathfrak{p}) > s$.*

Proof. Choose $\mathfrak{q}' \in T$ with $\mathfrak{p} \subset \mathfrak{q}' \subset \mathfrak{q}$ such that there is no prime in T strictly in between $\mathfrak{p}$ and $\mathfrak{q}'$. To prove the lemma we may and do replace $\mathfrak{q}$ by $\mathfrak{q}'$. Next, let $\mathfrak{p}' \subset A_\mathfrak{q}$ be the prime corresponding to $\mathfrak{p}$. After doing this we obtain that $V(\mathfrak{p}') \cap V(IA_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ because of the nonexistence of a prime $\mathfrak{r}$ as in the lemma. Let $A', I', \mathfrak{m}', M'$ be the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Since $A_\mathfrak{q} \to A'$ is faithfully flat (Algebra, Lemma 10.97.3) we can choose $\mathfrak{p}'' \subset A'$ lying over $\mathfrak{p}'$ with $\dim(A'_{\mathfrak{p}''}/\mathfrak{p}'A'_{\mathfrak{p}''}) = 0$. Then we see that

$$\mathrm{depth}(M'_{\mathfrak{p}''}) = \mathrm{depth}((M_\mathfrak{q} \otimes_{A_\mathfrak{q}} A')_{\mathfrak{p}''}) = \mathrm{depth}(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A'_{\mathfrak{p}''}) = \mathrm{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$ and our choice of $\mathfrak{p}''$, see Algebra, Lemma 10.163.1. Since $\mathfrak{p}''$ lies over $\mathfrak{p}'$ we have $V(\mathfrak{p}'') \cap V(I') = \{\mathfrak{m}'\}$. Thus condition (6) in Situation 52.10.1 implies $\mathrm{depth}(M'_{\mathfrak{p}''}) > s$ which finishes the proof. □

The following tedious lemma explains the relationships between various collections of conditions one might impose.

0EFV **Lemma 52.10.3.** *In Situation 52.10.1 we have*

(E) *if $T' \subset T$ is a smaller specialization stable subset, then A, I, T', M satisfies the assumptions of Situation 52.10.1,*
(F) *if $S \subset A$ is a multiplicative subset, then $S^{-1}A, S^{-1}I, T', S^{-1}M$ satisfies the assumptions of Situation 52.10.1 where $T' \subset V(S^{-1}I)$ is the inverse image of T,*
(G) *the quadruple A', I', T', M' satisfies the assumptions of Situation 52.10.1 where A', I', M' are the usual I-adic completions of A, I, M and $T' \subset V(I')$ is the inverse image of T.*

Let $I \subset \mathfrak{a} \subset A$ be an ideal such that $V(\mathfrak{a}) \subset T$. Then

(A) *if I is contained in the Jacobson radical of A, then all hypotheses of Lemmas 52.8.2 and 52.8.4 are satisfied for $A, I, \mathfrak{a}, M$,*
(B) *if A is complete with respect to I, then all hypotheses except for possibly (5) of Lemma 52.8.5 are satisfied for $A, I, \mathfrak{a}, M$,*
(C) *if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$, then all hypotheses of Lemmas 52.9.2 and 52.9.4 hold for $A, \mathfrak{m}, I, M$,*
(D) *if A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$ and I-adically complete, then all hypotheses of Lemma 52.9.5 hold for $A, \mathfrak{m}, I, M$,*

Proof. Proof of (E). We have to prove assumptions (1), (3), (4), (6) of Situation 52.10.1 hold for A, I, T, M. Shrinking T to T' weakens assumption (6) and strengthens assumption (4). However, if we have $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{p} \notin V(I)$, $\mathfrak{r} \in V(I) \setminus T'$, $\mathfrak{q} \in T'$ as in assumption (4) for A, I, T', M, then either we can pick $\mathfrak{r} \in V(I) \setminus T$ and condition (4) for A, I, T, M kicks in or we cannot find such an $\mathfrak{r}$ in which case we get $\text{depth}(M_\mathfrak{p}) > s$ by Lemma 52.10.2. This proves (4) holds for A, I, T', M as desired.

Proof of (F). This is straightforward and we omit the details.

Proof of (G). We have to prove assumptions (1), (3), (4), (6) of Situation 52.10.1 hold for the I-adic completions A', I', T', M'. Please keep in mind that $\text{Spec}(A') \to \text{Spec}(A)$ induces an isomorphism $V(I') \to V(I)$.

Assumption (1): The ring A' has a dualizing complex, see Dualizing Complexes, Lemma 47.22.4.

Assumption (3): Since $I' = IA'$ this follows from Local Cohomology, Lemma 51.4.5.

Assumption (4): If we have primes $\mathfrak{p}' \subset \mathfrak{r}' \subset \mathfrak{q}'$ in A' with $\mathfrak{p}' \notin V(I')$, $\mathfrak{r}' \in V(I') \setminus T'$, $\mathfrak{q}' \in T'$ then their images $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ in the spectrum of A satisfy $\mathfrak{p} \notin V(I)$, $\mathfrak{r} \in V(I) \setminus T$, $\mathfrak{q} \in T$. Then we have

$$\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

by assumption (4) for A, I, T, M. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 51.11.3. Thus assumption (4) holds for A', I', T', M'.

Assumption (6): Let $\mathfrak{q}' \in T'$ lying over the prime $\mathfrak{q} \in T$. Then $A'_{\mathfrak{q}'}$ and $A_\mathfrak{q}$ have isomorphic I-adic completions and similarly for $M_\mathfrak{q}$ and $M'_{\mathfrak{q}'}$. Thus assumption (6) for A', I', T', M' is equivalent to assumption (6) for A, I, T, M.

Proof of (A). We have to check conditions (1), (2), (3), (4), and (6) of Lemmas 52.8.2 and 52.8.4 for $(A, I, \mathfrak{a}, M)$. Warning: the set T in the statement of these lemmas is not the same as the set T above.

Condition (1): This holds because we have assumed A has a dualizing complex in Situation 52.10.1.

Condition (2): This is empty.

Condition (3): Let $\mathfrak{p} \subset A$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I)$ be a generic point. Since $\mathrm{cd}(A_\mathfrak{q}, I_\mathfrak{q}) \leq d$ (Local Cohomology, Lemma 51.4.6) and since $V(\mathfrak{p}A_\mathfrak{q}) \cap V(I_\mathfrak{q}) = \{\mathfrak{q}A_\mathfrak{q}\}$ we get $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq d$ by Local Cohomology, Lemma 51.4.10 which proves (3).

Condition (4): Suppose $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. It suffices to show

$$\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq s \quad \text{or} \quad \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$$

If there exists a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$, then this follows immediately from assumption (4) in Situation 52.10.1. If not, then $\mathrm{depth}(M_\mathfrak{p}) > s$ by Lemma 52.10.2.

Condition (6): Let $\mathfrak{p} \notin V(I)$ with $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. Since I is contained in the Jacobson radical of A we see that $V(\mathfrak{p}) \cap V(I) \neq \emptyset$. Choose $\mathfrak{q} \in V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$. It is clear there does not exist a prime $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ with $\mathfrak{r} \in V(I) \setminus T$. By Lemma 52.10.2 we have $\mathrm{depth}(M_\mathfrak{p}) > s$ which proves (6).

Proof of (B). We have to check conditions (1), (2), (3), (4) of Lemma 52.8.5. Warning: the set T in the statement of this lemma is not the same as the set T above.

Condition (1): This holds because A is complete and has a dualizing complex.

Condition (2): This is empty.

Condition (3): This is the same as assumption (3) in Situation 52.10.1.

Condition (4): This is the same as assumption (4) in Lemma 52.8.2 which we proved in (A).

Proof of (C). This is true because the assumptions in Lemmas 52.9.2 and 52.9.4 are the same as the assumptions in Lemmas 52.8.2 and 52.8.4 in the local case and we proved these hold in (A).

Proof of (D). This is true because the assumptions in Lemma 52.9.5 are the same as the assumptions (1), (2), (3), (4) in Lemma 52.8.5 and we proved these hold in (B). □

0EFR **Lemma 52.10.4.** *In Situation 52.10.1 assume A is local with maximal ideal $\mathfrak{m}$ and $T = \{\mathfrak{m}\}$. Then $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M/I^nM)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I.*

Proof. Let $A', I', \mathfrak{m}', M'$ be the usual I-adic completions of $A, I, \mathfrak{m}, M$. Recall that we have $H^i_\mathfrak{m}(M) \otimes_A A' = H^i_{\mathfrak{m}'}(M')$ by flatness of $A \to A'$ and Dualizing Complexes, Lemma 47.9.3. Since $H^i_\mathfrak{m}(M)$ is $\mathfrak{m}$-power torsion we have $H^i_\mathfrak{m}(M) = H^i_\mathfrak{m}(M) \otimes_A A'$, see More on Algebra, Lemma 15.90.3. We conclude that $H^i_\mathfrak{m}(M) = H^i_{\mathfrak{m}'}(M')$. The exact same arguments will show that $H^i_\mathfrak{m}(M/I^nM) = H^i_{\mathfrak{m}'}(M'/(I')^nM')$ for all n and i.

Lemmas 52.9.5, 52.9.2, and 52.9.4 apply to $A', \mathfrak{m}', I', M'$ by Lemma 52.10.3 parts (C) and (D). Thus we get an isomorphism

$$H^i_{\mathfrak{m}'}(M') \longrightarrow H^i(R\Gamma_{\mathfrak{m}'}(M')^\wedge)$$

for $i \leq s$ where $^\wedge$ is derived I'-adic completion and these modules are annihilated by a power of I'. By Lemma 52.5.4 we obtain isomorphisms

$$H^i_{\mathfrak{m}'}(M') \longrightarrow \lim H^i_{\mathfrak{m}'}(M'/(I')^n M'))$$

for $i \leq s$. Combined with the already established comparison with local cohomology over A we conclude the lemma is true. □

0EFW **Lemma 52.10.5.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let M be a finite A-module. Let s and d be integers. If we assume*

(a) *A has a dualizing complex,*
(b) *$cd(A, I) \leq d$,*
(c) *if $\mathfrak{p} \notin V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ then $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ or $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > d + s$.*

Then $A, I, V(\mathfrak{a}), M, s, d$ are as in Situation 52.10.1.

Proof. We have to show that assumptions (1), (3), (4), and (6) of Situation 52.10.1 hold. It is clear that (a) $\Rightarrow$ (1), (b) $\Rightarrow$ (3), and (c) $\Rightarrow$ (4). To finish the proof in the next paragraph we show (6) holds.

Let $\mathfrak{q} \in V(\mathfrak{a})$. Denote $A', I', \mathfrak{m}', M'$ the I-adic completions of $A_\mathfrak{q}, I_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}$. Let $\mathfrak{p}' \subset A'$ be a nonmaximal prime with $V(\mathfrak{p}') \cap V(I') = \{\mathfrak{m}'\}$. Observe that this implies $\dim(A'/\mathfrak{p}') \leq d$ by Local Cohomology, Lemma 51.4.10. Denote $\mathfrak{p} \subset A$ the image of $\mathfrak{p}'$. We have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p})$ and $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$ by Local Cohomology, Lemma 51.11.3. By assumption (c) either we have $\text{depth}(M'_{\mathfrak{p}'}) \geq \text{depth}(M_\mathfrak{p}) > s$ and we're done or we have $\text{depth}(M'_{\mathfrak{p}'}) + \dim(A'/\mathfrak{p}') > s + d$ which implies $\text{depth}(M'_{\mathfrak{p}'}) > s$ because of the already shown inequality $\dim(A'/\mathfrak{p}') \leq d$. In both cases we obtain what we want. □

0EFX **Lemma 52.10.6.** *In Situation 52.10.1 the inverse systems $\{H^i_T(I^n M)\}_{n\geq 0}$ are pro-zero for $i \leq s$. Moreover, there exists an integer m_0 such that for all $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that for $k \geq m'(m)$ the image of $H^{s+1}_T(I^k M) \to H^{s+1}_T(I^m M)$ maps injectively to $H^{s+1}_T(I^{m_0} M)$.*

Proof. Fix m. Let $\mathfrak{q} \in T$. By Lemmas 52.10.3 and 52.10.4 we see that

$$H^i_\mathfrak{q}(M_\mathfrak{q}) \longrightarrow \lim H^i_\mathfrak{q}(M_\mathfrak{q}/I^n M_\mathfrak{q})$$

is an isomorphism for $i \leq s$. The inverse systems $\{H^i_\mathfrak{q}(I^n M_\mathfrak{q})\}_{n\geq 0}$ and $\{H^i_\mathfrak{q}(M/I^n M)\}_{n\geq 0}$ satisfy the Mittag-Leffler condition for all i, see Lemma 52.5.2. Thus looking at the inverse system of long exact sequences

$$0 \to H^0_\mathfrak{q}(I^n M_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}) \to H^0_\mathfrak{q}(M_\mathfrak{q}/I^n M_\mathfrak{q}) \to H^1_\mathfrak{q}(I^n M_\mathfrak{q}) \to H^1_\mathfrak{q}(M_\mathfrak{q}) \to \ldots$$

we conclude (some details omitted) that there exists an integer $m'(m, \mathfrak{q}) \geq m$ such that for all $k \geq m'(m, \mathfrak{q})$ the map $H^i_\mathfrak{q}(I^k M_\mathfrak{q}) \to H^i_\mathfrak{q}(I^m M_\mathfrak{q})$ is zero for $i \leq s$ and the image of $H^{s+1}_\mathfrak{q}(I^k M_\mathfrak{q}) \to H^{s+1}_\mathfrak{q}(I^m M_\mathfrak{q})$ is independent of $k \geq m'(m, \mathfrak{q})$ and maps injectively into $H^{s+1}_\mathfrak{q}(M_\mathfrak{q})$.

Suppose we can show that $m'(m, \mathfrak{q})$ can be chosen independently of $\mathfrak{q} \in T$. Then the lemma follows immediately from Local Cohomology, Lemmas 51.6.2 and 51.6.3.

Let $\omega_A^\bullet$ be a dualizing complex. Let $\delta : \mathrm{Spec}(A) \to \mathbf{Z}$ be the corresponding dimension function. Recall that δ attains only a finite number of values, see Dualizing Complexes, Lemma 47.17.4. Claim: for each $d \in \mathbf{Z}$ the integer $m'(m, \mathfrak{q})$ can be chosen independently of $\mathfrak{q} \in T$ with $\delta(\mathfrak{q}) = d$. Clearly the claim implies the lemma by what we said above.

Pick $\mathfrak{q} \in T$ with $\delta(\mathfrak{q}) = d$. Consider the ext modules

$$E(n, j) = \mathrm{Ext}^j_A(I^n M, \omega_A^\bullet)$$

A key feature we will use is that these are finite A-modules. Recall that $(\omega_A^\bullet)_\mathfrak{q}[-d]$ is a normalized dualizing complex for $A_\mathfrak{q}$ by definition of the dimension function associated to a dualizing complex, see Dualizing Complexes, Section 47.17. The local duality theorem (Dualizing Complexes, Lemma 47.18.4) tells us that the $\mathfrak{q}A_\mathfrak{q}$-adic completion of $E(n, -d - i)_\mathfrak{q}$ is Matlis dual to $H^i_\mathfrak{q}(I^n M_\mathfrak{q})$. Thus the choice of $m'(m, \mathfrak{q})$ for $i \leq s$ in the first paragraph tells us that for $k \geq m'(m, \mathfrak{q})$ and $j \geq -d - s$ the map

$$E(m, j)_\mathfrak{q} \to E(k, j)_\mathfrak{q}$$

is zero. Since these modules are finite and nonzero only for a finite number of possible j (small detail omitted), we can find an open neighbourhood $W \subset \mathrm{Spec}(A)$ of $\mathfrak{q}$ such that

$$E(m, j)_{\mathfrak{q}'} \to E(m'(m, \mathfrak{q}), j)_{\mathfrak{q}'}$$

is zero for $j \geq -d-s$ for all $\mathfrak{q}' \in W$. Then of course the maps $E(m, j)_{\mathfrak{q}'} \to E(k, j)_{\mathfrak{q}'}$ for $k \geq m'(m, \mathfrak{q})$ are zero as well.

For $i = s+1$ corresponding to $j = -d-s-1$ we obtain from local duality and the results of the first paragraph that

$$K_{k,\mathfrak{q}} = \mathrm{Ker}(E(m, -d - s - 1)_\mathfrak{q} \to E(k, -d - s - 1)_\mathfrak{q})$$

is independent of $k \geq m'(m, \mathfrak{q})$ and that

$$E(0, -d - s - 1)_\mathfrak{q} \to E(m, -d - s - 1)_\mathfrak{q}/K_{m'(m,\mathfrak{q}),\mathfrak{q}}$$

is surjective. For $k \geq m'(m, \mathfrak{q})$ set

$$K_k = \mathrm{Ker}(E(m, -d - s - 1) \to E(k, -d - s - 1))$$

Since K_k is an increasing sequence of submodules of the finite module $E(m, -d - s - 1)$ we see that, at the cost of increasing $m'(m, \mathfrak{q})$ a little bit, we may assume $K_{m'(m,\mathfrak{q})} = K_k$ for $k \geq m'(m, \mathfrak{q})$. After shrinking W further if necessary, we may also assume that

$$E(0, -d - s - 1)_{\mathfrak{q}'} \to E(m, -d - s - 1)_{\mathfrak{q}'}/K_{m'(m,\mathfrak{q}),\mathfrak{q}'}$$

is surjective for all $\mathfrak{q}' \in W$ (as before use that these modules are finite and that the map is surjective after localization at $\mathfrak{q}$).

Any subset, in particular $T_d = \{\mathfrak{q} \in T \text{ with } \delta(\mathfrak{q}) = d\}$, of the Noetherian topological space $\mathrm{Spec}(A)$ with the endowed topology is Noetherian and hence quasi-compact. Above we have seen that for every $\mathfrak{q} \in T_d$ there is an open neighbourhood W where $m'(m, \mathfrak{q})$ works for all $\mathfrak{q}' \in T_d \cap W$. We conclude that we can find an integer $m'(m, d)$ such that for all $\mathfrak{q} \in T_d$ we have

$$E(m, j)_\mathfrak{q} \to E(m'(m, d), j)_\mathfrak{q}$$

is zero for $j \geq -d-s$ and with $K_{m'(m,d)} = \mathrm{Ker}(E(m, -d-s-1) \to E(m'(m,d), -d-s-1))$ we have

$$K_{m'(m,d),\mathfrak{q}} = \mathrm{Ker}(E(m, -d-s-1)_\mathfrak{q} \to E(k, -d-s-1)_\mathfrak{q})$$

for all $k \geq m'(m,d)$ and the map

$$E(0, -d-s-1)_\mathfrak{q} \to E(m, -d-s-1)_\mathfrak{q}/K_{m'(m,d),\mathfrak{q}}$$

is surjective. Using the local duality theorem again (in the opposite direction) we conclude that the claim is correct. This finishes the proof. □

0EFY **Lemma 52.10.7.** *In Situation 52.10.1 there exists an integer $m_0 \geq 0$ such that*

(1) *$H^i_T(M) \to H^i_T(M/I^nM)$ is injective for all $i \leq s$ and $n \geq m_0$,*
(2) *$\{H^i_T(M/I^nM)\}_{n\geq 0}$ satisfies the Mittag-Leffler condition for $i < s$.*
(3) *$\{H^i_T(I^{m_0}M/I^nM)\}_{n\geq m_0}$ satisfies the Mittag-Leffler condition for $i \leq s$,*
(4) *$H^i_T(M) \to \lim H^i_T(M/I^nM)$ is an isomorphism for $i < s$,*
(5) *$H^s_T(I^{m_0}M) \to \lim H^s_T(I^{m_0}M/I^nM)$ is an isomorphism for $i \leq s$,*
(6) *$H^s_T(M) \to \lim H^s_T(M/I^nM)$ is injective with cokernel killed by I^{m_0}, and*
(7) *$R^1 \lim H^s_T(M/I^nM)$ is killed by I^{m_0}.*

Proof. To prove (1) let $0 \leq i \leq s$ and choose integers $m > n > 0$ such that $H^i_T(I^mM) \to H^i_T(I^nM)$ is zero; this is possibly by Lemma 52.10.6. The commutative diagram with exact rows

$$\begin{array}{ccccc} H^i_T(I^nM) & \longrightarrow & H^i_T(M) & \longrightarrow & H^i_T(M/I^nM) \\ {\scriptstyle 0}\uparrow & & {\scriptstyle 1}\uparrow & & \uparrow \\ H^i_T(I^mM) & \longrightarrow & H^i_T(M) & \longrightarrow & H^i_T(M/I^mM) \end{array}$$

shows that the map $H^i_T(M) \to H^i_T(M/I^mM)$ is injective. Whence (1) holds for any sufficiently large m_0.

Consider the long exact sequences

$$0 \to H^0_T(I^nM) \to H^0_T(M) \to H^0_T(M/I^nM) \to H^1_T(I^nM) \to H^1_T(M) \to \ldots$$

Parts (2) and (4) follow from this and Lemma 52.10.6.

Let m_0 and $m'(-)$ be as in Lemma 52.10.6. For $m \geq m_0$ consider the long exact sequence

$$H^s_T(I^mM) \to H^s_T(I^{m_0}M) \to H^s_T(I^{m_0}M/I^mM) \to H^{s+1}_T(I^mM) \to H^1_T(I^{m_0}M)$$

Then for $k \geq m'(m)$ the image of $H^{s+1}_T(I^kM) \to H^{s+1}_T(I^mM)$ maps injectively to $H^{s+1}_T(I^{m_0}M)$. Hence the image of $H^s_T(I^{m_0}M/I^kM) \to H^s_T(I^{m_0}M/I^mM)$ maps to zero in $H^{s+1}_T(I^mM)$ for all $k \geq m'(m)$. We conclude that (3) and (5) hold.

Consider the short exact sequences $0 \to I^{m_0}M \to M \to M/I^{m_0}M \to 0$ and $0 \to I^{m_0}M/I^nM \to M/I^nM \to M/I^{m_0}M \to 0$. We obtain a diagram

$$\begin{array}{ccccccc} H^{s-1}_T(M/I^{m_0}M) & \longrightarrow & \lim H^s_T(I^{m_0}M/I^nM) & \longrightarrow & \lim H^s_T(M/I^nM) & \longrightarrow & H^s_T(M/I^{m_0}M) \\ \| & & \uparrow{\scriptstyle \cong} & & \uparrow & & \| \\ H^{s-1}_T(M/I^{m_0}M) & \longrightarrow & H^s_T(I^{m_0}M) & \longrightarrow & H^s_T(M) & \longrightarrow & H^s_T(M/I^{m_0}M) \end{array}$$

whose lower row is exact. The top row is also exact (at the middle two spots) by Homology, Lemma 12.31.4. Part (6) follows.

Write $B_n = H^s_T(M/I^nM)$. Let $A_n \subset B_n$ be the image of $H^s_T(I^{m_0}M/I^nM) \to H^s_T(M/I^nM)$. Then (A_n) satisfies the Mittag-Leffler condition by (3) and Homology, Lemma 12.31.3. Also $C_n = B_n/A_n$ is killed by I^{m_0}. Thus $R^1 \lim B_n \cong R^1 \lim C_n$ is killed by I^{m_0} and we get (7). □

0EIE **Theorem 52.10.8.** *In Situation 52.10.1 for $i \leq s$ the inverse system $\{H^i_T(M/I^nM)\}_{n\geq 0}$ is essentially constant with value $H^i_T(M)$. In particular, $\{H^i_T(M/I^nM)\}_{n\geq 0}$ is Mittag-Leffler, $H^i_T(M)$ is annihilated by a power of I, and $H^i_T(M) = \lim H^i_T(M/I^nM)$.*

Proof. Let $0 \leq i \leq s$. If we show that $\{H^i_T(M/I^nM)\}_{n\geq 0}$ is Mittag-Leffler and $H^i_T(M) = \lim H^i_T(M/I^nM)$, then the injectivity of $H^i_T(M) \to H^i_T(M/I^nM)$ for $n \gg 0$ shown in Lemma 52.10.7 will imply that $\{H^i_T(M/I^nM)\}_{n\geq 0}$ is essentially constant with value $H^i_T(M)$. Small detail omitted.

We already know $\{H^i_T(M/I^nM)\}_{n\geq 0}$ is Mittag-Leffler and $H^i_T(M) = \lim H^i_T(M/I^nM)$ for $i < s$ by Lemma 52.10.7. We also know these statements hold for $i = s$ and the module $I^{m_0}M$ for some $m_0 > 0$. To finish of the proof we will show that in fact these assertions for $i = s$ holds for M.

Let $M' = H^0_I(M)$ and $M'' = M/M'$ so that we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and M'' has $H^0_I(M') = 0$ by Dualizing Complexes, Lemma 47.11.6. By Artin-Rees (Algebra, Lemma 10.51.2) we get short exact sequences

$$0 \to M' \to M/I^nM \to M''/I^nM'' \to 0$$

for n large enough. Consider the long exact sequences

$$H^s_T(M') \to H^s_T(M/I^nM) \to H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

Now it is a simple matter to see that if we have Mittag-Leffler for the inverse system $\{H^s_T(M''/I^nM'')\}_{n\geq 0}$ then we have Mittag-Leffler for the inverse system $\{H^s_T(M/I^nM)\}_{n\geq 0}$. (Note that the ML condition for an inverse system of groups G_n only depends on the values of the inverse system for sufficiently large n.) Moreover the sequence

$$H^s_T(M') \to \lim H^s_T(M/I^nM) \to \lim H^s_T(M''/I^nM'') \to H^{s+1}_T(M')$$

is exact because we have ML in the required spots, see Homology, Lemma 12.31.4. Hence, if $H^s_T(M'') \to \lim H^s_T(M''/I^nM'')$ is an isomorphism, then $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism too by the five lemma (Homology, Lemma 12.5.20). This reduces us to the case discussed in the next paragraph.

Assume that $H^0_I(M) = 0$. Choose generators $f_1, \ldots, f_r$ of I^{m_0} where m_0 is the integer found for M in Lemma 52.10.7. Then we consider the exact sequence

$$0 \to M \xrightarrow{f_1,\ldots,f_r} (I^{m_0}M)^{\oplus r} \to Q \to 0$$

defining Q. Some observations: the first map is injective exactly because $H^0_I(M) = 0$. The cokernel Q of this injection is a finite A-module such that for every $1 \leq j \leq r$ we have $Q_{f_j} \cong (M_{f_j})^{\oplus r-1}$. In particular, for a prime $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin V(I)$ we have $Q_\mathfrak{p} \cong (M_\mathfrak{p})^{\oplus r-1}$. Similarly, given $\mathfrak{q} \in T$ and $\mathfrak{p}' \subset A' = (A_\mathfrak{q})^\wedge$ not contained in $V(IA')$, we have $Q'_{\mathfrak{p}'} \cong (M'_{\mathfrak{p}'})^{\oplus r-1}$ where $Q' = (Q_\mathfrak{q})^\wedge$ and $M' = (M_\mathfrak{q})^\wedge$. Thus

the conditions in Situation 52.10.1 hold for A, I, T, Q. (Observe that Q may have nonvanishing $H^0_I(Q)$ but this won't matter.)

For any $n \geq 0$ we set $F^nM = M \cap I^n(I^{m_0}M)^{\oplus r}$ so that we get short exact sequences

$$0 \to F^nM \to I^n(I^{m_0}M)^{\oplus r} \to I^nQ \to 0$$

By Artin-Rees (Algebra, Lemma 10.51.2) there exists a $c \geq 0$ such that $I^nM \subset F^nM \subset I^{n-c}M$ for all $n \geq c$. Let m_0 be the integer and let $m'(m)$ be the function defined for $m \geq m_0$ found in Lemma 52.10.6 applied to M. Note that the integer m_0 is the same as our integer m_0 chosen above (you don't need to check this: you can just take the maximum of the two integers if you like). Finally, by Lemma 52.10.6 applied to Q for every integer m there exists an integer $m''(m) \geq m$ such that $H^s_T(I^kQ) \to H^s_T(I^mQ)$ is zero for all $k \geq m''(m)$.

Fix $m \geq m_0$. Choose $k \geq m'(m''(m+c))$. Choose $\xi \in H^{s+1}_T(I^kM)$ which maps to zero in $H^{s+1}_T(M)$. We want to show that ξ maps to zero in $H^{s+1}_T(I^mM)$. Namely, this will show that $\{H^s_T(M/I^nM)\}_{n\geq 0}$ is Mittag-Leffler exactly as in the proof of Lemma 52.10.7. Picture to help visualize the argument:

$$\begin{array}{ccccc}
 & & H^{s+1}_T(I^kM) & \longrightarrow & H^{s+1}_T(I^k(I^{m_0}M)^{\oplus r}) \\
 & & \downarrow & & \downarrow \\
H^s_T(I^{m''(m+c)}Q) & \xrightarrow{\delta} & H^{s+1}_T(F^{m''(m+c)}M) & \longrightarrow & H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r}) \\
\downarrow & & \downarrow & & \\
H^s_T(I^{m+c}Q) & \longrightarrow & H^{s+1}_T(F^{m+c}M) & & \\
 & & \downarrow & & \\
 & & H^{s+1}_T(I^mM) & &
\end{array}$$

The image of ξ in $H^{s+1}_T(I^k(I^{m_0}M)^{\oplus r})$ maps to zero in $H^{s+1}_T((I^{m_0}M)^{\oplus r})$ and hence maps to zero in $H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r})$ by choice of $m'(-)$. Thus the image $\xi' \in H^{s+1}_T(F^{m''(m+c)}M)$ maps to zero in $H^{s+1}_T(I^{m''(m+c)}(I^{m_0}M)^{\oplus r})$ and hence $\xi' = \delta(\eta)$ for some $\eta \in H^s_T(I^{m''(m+c)}Q)$. By our choice of $m''(-)$ we find that η maps to zero in $H^s_T(I^{m+c}Q)$. This in turn means that ξ' maps to zero in $H^{s+1}_T(F^{m+c}M)$. Since $F^{m+c}M \subset I^mM$ we conclude.

Finally, we prove the statement on limits. Consider the short exact sequences

$$0 \to M/F^nM \to (I^{m_0}M)^{\oplus r}/I^n(I^{m_0}M)^{\oplus r} \to Q/I^nQ \to 0$$

We have $\lim H^s_T(M/I^nM) = \lim H^s_T(M/F^nM)$ as these inverse systems are pro-isomorphic. We obtain a commutative diagram

$$\begin{array}{ccc}
H^{s-1}_T(Q) & \longrightarrow & \lim H^{s-1}_T(Q/I^nQ) \\
\downarrow & & \downarrow \\
H^s_T(M) & \longrightarrow & \lim H^s_T(M/I^nM) \\
\downarrow & & \downarrow \\
H^s_T((I^{m_0}M)^{\oplus r}) & \longrightarrow & \lim H^s_T((I^{m_0}M)^{\oplus r}/I^n(I^{m_0}M)^{\oplus r}) \\
\downarrow & & \downarrow \\
H^s_T(Q) & \longrightarrow & \lim H^s_T(Q/I^nQ)
\end{array}$$

The right column is exact because we have ML in the required spots, see Homology, Lemma 12.31.4. The lowest horizontal arrow is injective (!) by part (6) of Lemma 52.10.7. The horizontal arrow above it is bijective by part (5) of Lemma 52.10.7. The arrows in cohomological degrees $\leq s-1$ are isomorphisms. Thus we conclude $H^s_T(M) \to \lim H^s_T(M/I^nM)$ is an isomorphism by the five lemma (Homology, Lemma 12.5.20). This finishes the proof of the theorem. □

0EG0 **Lemma 52.10.9.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A and let M be a finite A-module. Let s and d be integers. Suppose that*

(1) *$A, I, V(\mathfrak{a}), M$ satisfy the assumptions of Situation 52.10.1 for s and d, and*
(2) *$A, I, \mathfrak{a}, M$ satisfy the conditions of Lemma 52.8.5 for $s+1$ and d with $J = \mathfrak{a}$.*

Then there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the map

$$H^{s+1}_J(M) \longrightarrow \lim H^{s+1}_{\mathfrak{a}}(M/I^nM)$$

is an isomorphism.

Proof. Namely, we have the existence of J_0 and the isomorphism $H^{s+1}_J(M) = H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge)$ by Lemma 52.8.5, we have a short exact sequence

$$0 \to R^1 \lim H^s_{\mathfrak{a}}(M/I^nM) \to H^{s+1}(R\Gamma_{\mathfrak{a}}(M)^\wedge) \to \lim H^{s+1}_{\mathfrak{a}}(M/I^nM) \to 0$$

by Dualizing Complexes, Lemma 47.12.4, and the module $R^1 \lim H^s_{\mathfrak{a}}(M/I^nM)$ is zero because $\{H^s_{\mathfrak{a}}(M/I^nM)\}_{n\geq 0}$ has Mittag-Leffler by Theorem 52.10.8. □

52.11. Algebraization of formal sections, I

0DXH In this section we study the problem of algebraization of formal sections in the local case. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let

$$X = \operatorname{Spec}(A) \supset U = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$$

and denote $Y = V(I)$ the closed subscheme corresponding to I. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. In this section we consider the limits

$$\lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$$

This is closely related to the cohomology of the pullback of $\mathcal{F}$ to the formal completion of U along Y; however, since we have not yet introduced formal schemes, we cannot use this terminology here.

0DXI **Lemma 52.11.1.** *Let U be the punctured spectrum of a Noetherian local ring A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $I \subset A$ be an ideal. Then*

$$H^i(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

for all i where $R\Gamma(U, \mathcal{F})^\wedge$ denotes the derived I-adic completion.

Proof. By Lemmas 52.6.20 and 52.7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

Thus we obtain short exact sequences

$$0 \to R^1 \lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) \to H^i(R\Gamma(U, \mathcal{F})^\wedge) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F}) \to 0$$

by Cohomology, Lemma 20.37.1. The $R^1 \lim$ terms vanish because the inverse systems of groups $H^i(U, \mathcal{F}/I^n\mathcal{F})$ satisfy the Mittag-Leffler condition by Lemma 52.5.2. □

0DXQ **Theorem 52.11.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume*

1. *$cd(A, I) \leq d$, i.e., $H^i(X \setminus Y, \mathcal{G}) = 0$ for $i \geq d$ and quasi-coherent $\mathcal{G}$ on X,*
2. *for any $x \in X \setminus Y$ whose closure $\overline{\{x\}}$ in X meets $U \cap Y$ we have*

$$depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq s \quad or \quad depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > d + s$$

Then there exists an open $V_0 \subset U$ containing $U \cap Y$ such that for any open $V \subset V_0$ containing $U \cap Y$ the map

$$H^i(V, \mathcal{F}) \to \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism for $i < s$. If in addition $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > s$ for all $x \in U \cap Y$, then these cohomology groups are finite A-modules.

The method of proof follows roughly the method of proof of [**Fal79**, Theorem 1] and [**Fal80b**, Satz 2]. The result is almost the same as [**Ray74**, Theorem 1.1] (affine complement case) and [**Ray75**, Theorem 3.9] (complement is union of few affines).

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Local Cohomology, Lemma 51.8.2. Then the assumptions of Lemma 52.9.5 are satisfied. Pick J_0 as in that lemma and set $V_0 = X \setminus V(J_0)$. Then opens $V \subset V_0$ containing $U \cap Y$ correspond 1-to-1 with ideals $J \subset J_0$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$. Moreover, for such a choice we have a distinguished triangle

$$R\Gamma_J(M) \to M \to R\Gamma(V, \mathcal{F}) \to R\Gamma_J(M)[1]$$

We similarly have a distinguished triangle

$$R\Gamma_\mathfrak{m}(M)^\wedge \to M \to R\Gamma(U, \mathcal{F})^\wedge \to R\Gamma_\mathfrak{m}(M)^\wedge[1]$$

involving derived I-adic completions. The cohomology groups of $R\Gamma(U, \mathcal{F})^\wedge$ are equal to the limits in the statement of the theorem by Lemma 52.11.1. The canonical map between these triangles and some easy arguments show that our theorem follows from the main Lemma 52.9.5 (note that we have $i < s$ here whereas we have $i \leq s$ in the lemma; this is because of the shift). The finiteness of the cohomology groups (under the additional assumption) follows from Lemma 52.9.3. □

0DXR **Lemma 52.11.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to an ideal I. Set $X = \mathrm{Spec}(A)$, $Y = V(I)$, and $U = X \setminus \{\mathfrak{m}\}$. Let $\mathcal{F}$ be a coherent sheaf on U. Assume for any associated point $x \in U$ of $\mathcal{F}$ we have $\dim(\overline{\{x\}}) > cd(A, I) + 1$ where $\overline{\{x\}}$ is the closure in X. Then the map*

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism of finite A-modules where the colimit is over opens $V \subset U$ containing $U \cap Y$.

Proof. Apply Theorem 52.11.2 with $s = 1$ (we get finiteness too). □

52.12. Algebraization of formal sections, II

0EG1 It is a bit difficult to succinctly state all possible consequences of the results in Sections 52.8 and 52.10 for cohomology of coherent sheaves on quasi-affine schemes and their completion with respect to an ideal. This section gives a nonexhaustive list of applications to H^0. The next section contains applications to higher cohomology.

0H48 **Lemma 52.12.1.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent modules on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes $V \subset U$ containing $U \cap V(I)$ ordered by reverse inclusion. Consider the commutative diagram*

$$\begin{array}{ccccc} \mathrm{colim}_{\mathcal{V}} H^0(V, \mathcal{F}') & \longrightarrow & \mathrm{colim}_{\mathcal{V}} H^0(V, \mathcal{F}) & \longrightarrow & \mathrm{colim}_{\mathcal{V}} H^0(V, \mathcal{F}'') \\ \downarrow & & \downarrow & & \downarrow \\ \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}') & \longrightarrow & \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}) & \longrightarrow & \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}'') \end{array}$$

If the left and right downarrows are isomorphisms so is the middle. If the middle and left downarrows are isomorphisms, so is the left.

Proof. The sequences in the diagram are exact in the middle and the first arrow is injective. Thus the final statement follows from an easy diagram chase. For the rest of the proof we assume the left and right downward arrows are isomorphisms. A diagram chase shows that the middle downward arrow is injective. All that remains is to show that it is surjective.

We may choose finite A-modules M and M' such that $\mathcal{F}$ and $\mathcal{F}'$ are the restriction of $\widetilde{M}$ and $\widetilde{M'}$ to U, see Local Cohomology, Lemma 51.8.2. After replacing M' by $\mathfrak{a}^n M'$ for some $n \geq 0$ we may assume that $\mathcal{F}' \to \mathcal{F}$ corresponds to a module map $M' \to M$, see Cohomology of Schemes, Lemma 30.10.5. After replacing M' by the image of $M' \to M$ and setting $M'' = M/M'$ we see that our short exact sequence corresponds to the restriction of the short exact sequence of coherent modules associated to the short exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules.

Let $\hat{s} \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ with image $\hat{s}'' \in \lim H^0(U, \mathcal{F}''/I^n\mathcal{F}'')$. By assumption we find $V \in \mathcal{V}$ and a section $s'' \in \mathcal{F}''(V)$ mapping to $\hat{s}''$. Let $J \subset A$ be an ideal such that $V(J) = \mathrm{Spec}(A) \setminus V$. By Cohomology of Schemes, Lemma 30.10.5 after replacing J by a power, we may assume there is an A-linear map $\varphi : J \to M''$ corresponding to s''. We fix this choice of J; in the rest of the proof we will replace V by a smaller V in $\mathcal{V}$, i.e, we will have $V \cap V(J) = \emptyset$.

Choose a presentation $A^{\oplus m} \to A^{\oplus n} \to J \to 0$. Denote $g_1, \ldots, g_n \in J$ the images of the basis vectors of $A^{\oplus n}$, so that $J = (g_1, \ldots, g_n)$. Let $A^{\oplus m} \to A^{\oplus n}$ be given by the matrix (a_{ji}) so that $\sum a_{ji} g_i = 0$, $j = 1, \ldots, m$. Since $M \to M''$ is surjective, for each i we can choose $m_i \in M$ mapping to $\varphi(g_i) \in M''$. Then the element $g_i \hat{s} - m_i$ of $\lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ lies in the submodule $\lim H^0(U, \mathcal{F}'/I^n\mathcal{F}')$. By assumption after shrinking V we may assume there are $s_i' \in \mathcal{F}'(V)$, $i = 1, \ldots, n$ with s_i' mapping to $g_i \hat{s} - m_i$. Set $s_i = s_i' + m_i$ in $\mathcal{F}(V)$. Note that $\sum a_{ji} s_i$ maps to $\sum a_{ji} g_i \hat{s} = 0$ by the map

$$\operatorname{colim}_{\mathcal{V}} \mathcal{F}(V') \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

Since this map is injective (see above), we may after shrinking V assume that $\sum a_{ji} s_i = 0$ in $\mathcal{F}(V)$ for all $j = 1, \ldots, m$. Then it follows that we obtain an A-module map $J \to \mathcal{F}(V)$ sending g_i to s_i. By the universal property of $\widetilde{J}$ this A-module map corresponds to an $\mathcal{O}_V$-module map $\widetilde{J}|_V \to \mathcal{F}$. However, since $V(J) \cap V = \emptyset$ we have $\widetilde{J}|_V = \mathcal{O}_V$. Thus we have produced a section $s \in \mathcal{F}(V)$. We omit the computation that shows that s maps to $\hat{s}$ by the map displayed above. $\square$

The following lemma will be superseded by Proposition 52.12.3.

0EIF **Lemma 52.12.2.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \operatorname{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + 1$,*
3. *one of the following holds:*
   (a) *the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_1)*
   (b) *the dimension of $V(\mathfrak{a})$ is at most 2[4].*

Then we obtain an isomorphism

$$\operatorname{colim} H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 51.8.2. Set $d = \operatorname{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\mathfrak{p}$ is not an associated prime of M and hence $\operatorname{depth}(M_\mathfrak{p}) \geq 1$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + 1$ by (2). Thus the hypotheses of Lemma 52.8.5 are satisfied for $s = 1$ and d; here we use condition (3). Thus we find there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_\mathfrak{a}(M)^\wedge)$$

[4] In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\operatorname{Spec}(A)$ is at most 2.

are isomorphisms for $i = 0, 1$. Consider the morphisms of exact triangles

$$\begin{array}{ccccccc}
R\Gamma_J(M) & \longrightarrow & M & \longrightarrow & R\Gamma(V, \mathcal{F}) & \longrightarrow & R\Gamma_J(M)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\Gamma_J(M)^\wedge & \longrightarrow & M & \longrightarrow & R\Gamma(V, \mathcal{F})^\wedge & \longrightarrow & R\Gamma_J(M)^\wedge[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R\Gamma_{\mathfrak{a}}(M)^\wedge & \longrightarrow & M & \longrightarrow & R\Gamma(U, \mathcal{F})^\wedge & \longrightarrow & R\Gamma_{\mathfrak{a}}(M)^\wedge[1]
\end{array}$$

where $V = \mathrm{Spec}(A) \setminus V(J)$. Recall that $R\Gamma_{\mathfrak{a}}(M)^\wedge \to R\Gamma_J(M)^\wedge$ is an isomorphism (because $\mathfrak{a}$, $\mathfrak{a}+I$, and $J+I$ cut out the same closed subscheme, for example see proof of Lemma 52.8.5). Hence $R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(V, \mathcal{F})^\wedge$. This produces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0_J(M) & \longrightarrow & M & \longrightarrow & \Gamma(V, \mathcal{F}) & \longrightarrow & H^1_J(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(R\Gamma_J(M)^\wedge) & \longrightarrow & M & \longrightarrow & H^0(R\Gamma(V, \mathcal{F})^\wedge) & \longrightarrow & H^1(R\Gamma_J(M)^\wedge) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^0(R\Gamma_{\mathfrak{a}}(M)^\wedge) & \longrightarrow & M & \longrightarrow & H^0(R\Gamma(U, \mathcal{F})^\wedge) & \longrightarrow & H^1(R\Gamma_{\mathfrak{a}}(M)^\wedge) & \longrightarrow & 0
\end{array}$$

with exact rows and isomorphisms for the lower vertical arrows. Hence we obtain an isomorphism $\Gamma(V, \mathcal{F}) \to H^0(R\Gamma(U, \mathcal{F})^\wedge)$. By Lemmas 52.6.20 and 52.7.2 we have

$$R\Gamma(U, \mathcal{F})^\wedge = R\Gamma(U, \mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F})$$

and we find $H^0(R\Gamma(U, \mathcal{F})^\wedge) = \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ by Cohomology, Lemma 20.37.1. □

Now we bootstrap the preceding lemma to get rid of condition (3).

0EG2 **Proposition 52.12.3.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > cd(A, I) + 1$.*

Then we obtain an isomorphism

$$\mathrm{colim}\, H^0(V, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. Let $T \subset U$ be the set of points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$. Let $\mathcal{F} \to \mathcal{F}'$ be the surjection of coherent modules on U constructed in Local Cohomology, Lemma 51.15.1. Since $\mathcal{F} \to \mathcal{F}'$ is an isomorphism over an open $V \subset U$ containing $U \cap V(I)$ it suffices to prove the lemma with $\mathcal{F}$ replaced by $\mathcal{F}'$. Hence we may and do assume for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $\mathrm{depth}(\mathcal{F}_x) \geq 1$.

Let $\mathcal{V}$ be the set of open subschemes $V \subset U$ containing $U \cap V(I)$ ordered by reverse inclusion. This is a directed set. We first claim that

$$\mathcal{F}(V) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

is injective for any $V \in \mathcal{F}$ (and in particular the map of the lemma is injective). Namely, an associated point x of $\mathcal{F}$ must have $\overline{\{x\}} \cap U \cap Y \neq \emptyset$ by the previous paragraph. If $y \in \overline{\{x\}} \cap U \cap Y$ then $\mathcal{F}_x$ is a localization of $\mathcal{F}_y$ and $\mathcal{F}_y \subset \lim \mathcal{F}_y/I^n\mathcal{F}_y$ by Krull's intersection theorem (Algebra, Lemma 10.51.4). This proves the claim as a section $s \in \mathcal{F}(V)$ in the kernel would have to have empty support, hence would have to be zero.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U, see Local Cohomology, Lemma 51.8.2. We may and do assume that $H^0_{\mathfrak{a}}(M) = 0$. Let $\mathrm{Ass}(M) \setminus V(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We will prove the lemma by induction on n. After reordering we may assume that $\mathfrak{p}_n$ is a minimal element of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect to inclusion, i.e, $\mathfrak{p}_n$ is a generic point of the support of M. Set

$$M' = H^0_{\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I}(M)$$

and $M'' = M/M'$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be the coherent $\mathcal{O}_U$-modules corresponding to M' and M''. Dualizing Complexes, Lemma 47.11.6 implies that M'' has only one associated prime, namely $\mathfrak{p}_n$. Hence $\mathcal{F}''$ has only one associated point and we see that condition (3)(a) of Lemma 52.12.2 holds; thus the map $\mathrm{colim}\, H^0(V, \mathcal{F}'') \to \lim H^0(U, \mathcal{F}''/I^n\mathcal{F}'')$ is an isomorphism. On the other hand, since $\mathfrak{p}_n \notin V(\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} I)$ we see that $\mathfrak{p}_n$ is not an associated prime of M'. Hence the induction hypothesis applies to M'; note that since $\mathcal{F}' \subset \mathcal{F}$ the condition $\mathrm{depth}(\mathcal{F}'_x) \geq 1$ at points x with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ holds, see Algebra, Lemma 10.72.6. Thus the map $\mathrm{colim}\, H^0(V, \mathcal{F}') \to \lim H^0(U, \mathcal{F}'/I^n\mathcal{F}')$ is an isomorphism too. We conclude by Lemma 52.12.1. □

0EIG **Lemma 52.12.4.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is I-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$, and $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + 1$,*
3. *for $x \in U$ with $\overline{\{x\}} \cap V(I) \subset V(\mathfrak{a})$ we have $depth(\mathcal{F}_x) \geq 2$,*

Then we obtain an isomorphism

$$H^0(U, \mathcal{F}) \longrightarrow \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$$

Proof. Let $\hat{s} \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$. By Proposition 52.12.3 we find that $\hat{s}$ is the image of an element $s \in \mathcal{F}(V)$ for some $V \subset U$ open containing $U \cap V(I)$. However, condition (3) shows that $\mathrm{depth}(\mathcal{F}_x) \geq 2$ for all $x \in U \setminus V$ and hence we find that $\mathcal{F}(V) = \mathcal{F}(U)$ by Divisors, Lemma 31.5.11 and the proof is complete. □

0EIH **Lemma 52.12.5.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor on M,*
3. *$H^1_{\mathfrak{a}}(M/fM)$ is a finite A-module.*

Then with $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\mathrm{colim}_V \Gamma(V, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap V(f)$.

Proof. Set $\mathcal{F} = \widetilde{M}|_U$. The finiteness of $H^1_{\mathfrak{a}}(M/fM)$ implies that $H^0(U, \mathcal{F}/f\mathcal{F})$ is finite, see Local Cohomology, Lemma 51.8.2. By Cohomology, Lemma 20.36.3 (which applies as f is a nonzerodivisor on $\mathcal{F}$) we see that $N = \lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is a finite A-module, is f-torsion free, and $N/fN \subset H^0(U, \mathcal{F}/f\mathcal{F})$. On the other hand, we have a map $M \to N$ and a compatible map

$$M/fM \longrightarrow H^0(U, \mathcal{F}/f\mathcal{F})$$

For $g \in \mathfrak{a}$ we see that $(M/fM)_g$ maps isomorphically to $H^0(U \cap D(f), \mathcal{F}/f\mathcal{F})$ since $\mathcal{F}/f\mathcal{F}$ is the restriction of $\widetilde{M/fM}$ to U. We conclude that $M_g \to N_g$ induces an isomorphism

$$M_g/fM_g = (M/fM)_g \to (N/fN)_g = N_g/fN_g$$

Since f is a nonzerodivisor on both N and M we conclude that $M_g \to N_g$ induces an isomorphism on f-adic completions which in turn implies $M_g \to N_g$ is an isomorphism in an open neighbourhood of $V(f) \cap D(g)$. Since $g \in \mathfrak{a}$ was arbitrary, we conclude that M and N determine isomorphic coherent modules over an open V as in the statement of the lemma. This finishes the proof. □

0H49 **Proposition 52.12.6.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $\mathcal{F}$ be a coherent module on $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

1. *A is f-adically complete and has a dualizing complex,*
2. *if $x \in Ass(\mathcal{F})$, $x \notin V(f)$, $\overline{\{x\}} \cap V(f) \not\subset V(\mathfrak{a})$, and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}},z}) > 2$.*

Then the map

$$\mathrm{colim}_V \Gamma(V, \mathcal{F}) \longrightarrow \lim \Gamma(U, \mathcal{F}/f^n\mathcal{F})$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap V(f)$.

First proof. Recall that A is universally catenary and with Gorenstein formal fibres, see Dualizing Complexes, Lemmas 47.23.2 and 47.17.4. Thus we may consider the map $\mathcal{F} \to \mathcal{F}'$ constructed in Local Cohomology, Lemma 51.15.3 for the closed subset $V(f) \cap U$ of U. Observe that

1. The kernel and cokernel of $\mathcal{F} \to \mathcal{F}'$ are supported on $V(f) \cap U$.
2. The module $\mathcal{F}'$ is f-torsion free as its stalks have depth ≥ 1 for all points of $V(f) \cap U$, i.e., $\mathcal{F}'$ has no associated points in $V(f) \cap U$.
3. If $y \in V(f) \cap U$ is an associated point of $\mathcal{F}'/f\mathcal{F}'$, then $\mathrm{depth}(\mathcal{F}'_y) = 1$ and hence (by the construction of $\mathcal{F}'$) there is an immediate specialization $x \rightsquigarrow y$ with $x \notin V(f)$ an associated point of $\mathcal{F}$. It follows that y cannot have an immediate specialization in $\mathrm{Spec}(A)$ to a point $z \in V(\mathfrak{a})$ by our assumption (2).
4. It follows from (3) that $H^0(U, \mathcal{F}'/f\mathcal{F}')$ is a finite A-module, see Local Cohomology, Lemma 51.12.1.

These observations will allow us to finish the proof.

First, we claim the lemma holds for $\mathcal{F}'$. Namely, choose a finite A-module M' such that $\mathcal{F}'$ is the restriction to U of the coherent module associated to M', see Local

Cohomology, Lemma 51.8.2. Since $\mathcal{F}'$ is f-torsion free, we may assume M' is f-torsion free as well. Observation (4) above shows that $H^1_{\mathfrak{a}}(M')$ is a finite A-module, see Local Cohomology, Lemma 51.8.2. Thus the claim by Lemma 52.12.5.

Second, we observe that the lemma holds trivially for any coherent $\mathcal{O}_U$-module supported on $V(f) \cap U$. Let $\mathcal{K}$, resp. $\mathcal{G}$, resp. $\mathcal{Q}$ be the kernel, resp. image, resp. cokernel of the map $\mathcal{F} \to \mathcal{F}'$. The short exact sequence $0 \to \mathcal{G} \to \mathcal{F}' \to \mathcal{Q} \to 0$ and Lemma 52.12.1 show that the result holds for $\mathcal{G}$. Then we do this again with the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0$ to finish the proof. □

Second proof. The proposition is a special case of Proposition 52.12.3. □

0EII **Lemma 52.12.7.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Assume*

(1) *A is f-adically complete,*
(2) *$H^1_{\mathfrak{a}}(M)$ and $H^2_{\mathfrak{a}}(M)$ are annihilated by a power of f.*

Then with $U = \operatorname{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$\Gamma(U, \widetilde{M}) \longrightarrow \lim \Gamma(U, \widetilde{M/f^nM})$$

is an isomorphism.

Proof. We may apply Lemma 52.3.2 to U and $\mathcal{F} = \widetilde{M}|_U$ because $\mathcal{F}$ is a Noetherian object in the category of coherent $\mathcal{O}_U$-modules. Since $H^1(U, \mathcal{F}) = H^2_{\mathfrak{a}}(M)$ (Local Cohomology, Lemma 51.8.2) is annihilated by a power of f, we see that its f-adic Tate module is zero. Hence the lemma shows $\lim H^0(U, \mathcal{F}/f^n\mathcal{F})$ is equal to the usual f-adic completion of $H^0(U, \mathcal{F})$. Consider the short exact sequence

$$0 \to M/H^0_{\mathfrak{a}}(M) \to H^0(U, \mathcal{F}) \to H^1_{\mathfrak{a}}(M) \to 0$$

of Local Cohomology, Lemma 51.8.2. Since $M/H^0_{\mathfrak{a}}(M)$ is a finite A-module, it is complete, see Algebra, Lemma 10.97.1. Since $H^1_{\mathfrak{a}}(M)$ is killed by a power of f, we conclude from Algebra, Lemma 10.96.4 that $H^0(U, \mathcal{F})$ is complete as well. This finishes the proof. □

52.13. Algebraization of formal sections, III

0EIJ The next section contains a nonexhaustive list of applications of the material on completion of local cohomology to higher cohomology of coherent modules on quasi-affine schemes and their completion with respect to an ideal.

0EG4 **Proposition 52.13.1.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $\mathcal{F}$ be a coherent module on $U = \operatorname{Spec}(A) \setminus V(\mathfrak{a})$. Let $s \geq 0$. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $x \in U \setminus V(I)$ then $\operatorname{depth}(\mathcal{F}_x) > s$ or*

$$\operatorname{depth}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > cd(A, I) + s + 1$$

for all $z \in V(\mathfrak{a}) \cap \overline{\{x\}}$,
(3) *one of the following conditions holds:*
   (a) *the restriction of $\mathcal{F}$ to $U \setminus V(I)$ is (S_{s+1}), or*
   (b) *the dimension of $V(\mathfrak{a})$ is at most 2[5].*

[5] In the sense that the difference of the maximal and minimal values on $V(\mathfrak{a})$ of a dimension function on $\operatorname{Spec}(A)$ is at most 2.

Then the maps

$$H^i(U, \mathcal{F}) \longrightarrow \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$$

are isomorphisms for $i < s$. Moreover we have an isomorphism

$$\operatorname{colim} H^s(V, \mathcal{F}) \longrightarrow \lim H^s(U, \mathcal{F}/I^n\mathcal{F})$$

where the colimit is over opens $V \subset U$ containing $U \cap V(I)$.

Proof. We may assume $s > 0$ as the case $s = 0$ was done in Proposition 52.12.3.

Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent module associated to M, see Local Cohomology, Lemma 51.8.2. Set $d = \operatorname{cd}(A, I)$. Let $\mathfrak{p}$ be a prime of A not contained in $V(I)$ and let $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$. Then either $\operatorname{depth}(M_\mathfrak{p}) \geq s + 1 > s$ or we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > d + s + 1$ by (2). By Lemma 52.10.5 we conclude that the assumptions of Situation 52.10.1 are satisfied for $A, I, V(\mathfrak{a}), M, s, d$. On the other hand, the hypotheses of Lemma 52.8.5 are satisfied for $s + 1$ and d; this is where condition (3) is used.

Applying Lemma 52.8.5 we find there exists an ideal $J_0 \subset \mathfrak{a}$ with $V(J_0) \cap V(I) = V(\mathfrak{a})$ such that for any $J \subset J_0$ with $V(J) \cap V(I) = V(\mathfrak{a})$ the maps

$$H^i_J(M) \longrightarrow H^i(R\Gamma_\mathfrak{a}(M)^\wedge)$$

is an isomorphism for $i \leq s + 1$.

For $i \leq s$ the map $H^i_\mathfrak{a}(M) \to H^i_J(M)$ is an isomorphism by Lemmas 52.10.3 and 52.8.4. Using the comparison of cohomology and local cohomology (Local Cohomology, Lemma 51.2.2) we deduce $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ is an isomorphism for $V = \operatorname{Spec}(A) \setminus V(J)$ and $i < s$.

By Theorem 52.10.8 we have $H^i_\mathfrak{a}(M) = \lim H^i_\mathfrak{a}(M/I^nM)$ for $i \leq s$. By Lemma 52.10.9 we have $H^{s+1}_\mathfrak{a}(M) = \lim H^{s+1}_\mathfrak{a}(M/I^nM)$.

The isomorphism $H^0(U, \mathcal{F}) = H^0(V, \mathcal{F}) = \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ follows from the above and Proposition 52.12.3. For $0 < i < s$ we get the desired isomorphisms $H^i(U, \mathcal{F}) = H^i(V, \mathcal{F}) = \lim H^i(U, \mathcal{F}/I^n\mathcal{F})$ in the same manner using the relation between local cohomology and cohomology; it is easier than the case $i = 0$ because for $i > 0$ we have

$$H^i(U, \mathcal{F}) = H^{i+1}_\mathfrak{a}(M), \quad H^i(V, \mathcal{F}) = H^{i+1}_J(M), \quad H^i(R\Gamma(U, \mathcal{F})^\wedge) = H^{i+1}(R\Gamma_\mathfrak{a}(M)^\wedge)$$

Similarly for the final statement. □

0EKM **Lemma 52.13.2.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let M be a finite A-module. Let $s \geq 0$. Assume*

1. *A is f-adically complete,*
2. *$H^i_\mathfrak{a}(M)$ is annihilated by a power of f for $i \leq s + 1$.*

Then with $U = \operatorname{Spec}(A) \setminus V(\mathfrak{a})$ the map

$$H^i(U, \widetilde{M}) \longrightarrow \lim H^i(U, \widetilde{M/f^nM})$$

is an isomorphism for $i < s$.

Proof. By induction on s. If $s = 0$, the assertion is empty. If $s = 1$, then the result is Lemma 52.12.7. Assume $s > 1$. By induction it suffices to prove the result for $i = s - 1 \geq 1$. We may apply Lemma 52.3.2 to U and $\mathcal{F} = \widetilde{M}|_U$ because $\mathcal{F}$ is a Noetherian object in the category of coherent $\mathcal{O}_U$-modules. Observe that

$H^j(U, \mathcal{F}) = H^{j+1}_{\mathfrak{a}}(M)$ for all j by Local Cohomology, Lemma 51.8.2. Thus for $j = s = (s-1)+1$ this is annihilated by a power of f by assumption. Thus it follows from Lemma 52.3.2 that $\lim H^{s-1}(U, \mathcal{F}/f^n\mathcal{F})$ is the usual f-adic completion of $H^{s-1}(U, \mathcal{F})$. Then again using that this module is killed by a power of f we see that the completion is simply equal to $H^{s-1}(U, \mathcal{F})$ as desired. □

52.14. Application to connectedness

0ECQ In this section we discuss Grothendieck's connectedness theorem and variants; the original version can be found as [**Gro68**, Exposee XIII, Theorem 2.1]. There is a version called Faltings' connectedness theorem in the literature; our guess is that this refers to [**Fal80a**, Theorem 6]. Let us state and prove the optimal version for complete local rings given in [**Var09**, Theorem 1.6].

0ECR **Lemma 52.14.1.** *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. Let I be a proper ideal of A. Set $X = \operatorname{Spec}(A)$ and $Y = V(I)$. Denote* [**Var09**, Theorem 1.6]

(1) *d the minimal dimension of an irreducible component of X, and*
(2) *c the minimal dimension of a closed subset $Z \subset X$ such that $X \setminus Z$ is disconnected.*

Then for $Z \subset Y$ closed we have $Y \setminus Z$ is connected if $\dim(Z) < \min(c, d-1) - cd(A, I)$. In particular, the punctured spectrum of A/I is connected if $cd(A, I) < \min(c, d-1)$.

Proof. Let us first prove the final assertion. As a first case, if the punctured spectrum of A/I is empty, then Local Cohomology, Lemma 51.4.10 shows every irreducible component of X has dimension $\leq \operatorname{cd}(A, I)$ and we get $\min(c, d-1) - \operatorname{cd}(A, I) < 0$ which implies the lemma holds in this case. Thus we may assume $U \cap Y$ is nonempty where $U = X \setminus \{\mathfrak{m}\}$ is the punctured spectrum of A. We may replace A by its reduction. Observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4) and that A is complete with respect to I (Algebra, Lemma 10.96.8). If we assume $d - 1 > \operatorname{cd}(A, I)$, then we may apply Lemma 52.11.3 to see that

$$\operatorname{colim} H^0(V, \mathcal{O}_V) \longrightarrow \lim H^0(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

is an isomorphism where the colimit is over opens $V \subset U$ containing $U \cap Y$. If $U \cap Y$ is disconnected, then its nth infinitesimal neighbourhood in U is disconnected for all n and we find the right hand side has a nontrivial idempotent (here we use that $U \cap Y$ is nonempty). Thus we can find a V which is disconnected. Set $Z = X \setminus V$. By Local Cohomology, Lemma 51.4.10 we see that every irreducible component of Z has dimension $\leq \operatorname{cd}(A, I)$. Hence $c \leq \operatorname{cd}(A, I)$ and this indeed proves the final statement.

We can deduce the statement of the lemma from what we just proved as follows. Suppose that $Z \subset Y$ closed and $Y \setminus Z$ is disconnected and $\dim(Z) = e$. Recall that a connected space is nonempty by convention. Hence we conclude either (a) $Y = Z$ or (b) $Y \setminus Z = W_1 \amalg W_2$ with W_i nonempty, open, and closed in $Y \setminus Z$. In case (b) we may pick points $w_i \in W_i$ which are closed in U, see Morphisms, Lemma 29.16.10. Then we can find $f_1, \ldots, f_e \in \mathfrak{m}$ such that $V(f_1, \ldots, f_e) \cap Z = \{\mathfrak{m}\}$ and in case (b) we may assume $w_i \in V(f_1, \ldots, f_e)$. Namely, we can inductively using prime avoidance choose f_i such that $\dim V(f_1, \ldots, f_i) \cap Z = e - i$ and such that in case (b) we have $w_1, w_2 \in V(f_i)$. It follows that the punctured spectrum of $A/I + (f_1, \ldots, f_e)$

is disconnected (small detail omitted). Since $\mathrm{cd}(A, I + (f_1, \ldots, f_e)) \leq \mathrm{cd}(A, I) + e$ by Local Cohomology, Lemmas 51.4.4 and 51.4.3 we conclude that

$$\mathrm{cd}(A, I) + e \geq \min(c, d - 1)$$

by the first part of the proof. This implies $e \geq \min(c, d-1) - \mathrm{cd}(A, I)$ which is what we had to show. □

0EG5 **Lemma 52.14.2.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $\mathfrak{p} \subset A$ is a minimal prime not contained in $V(I)$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$, then $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$,*
(3) *any nonempty open $V \subset \mathrm{Spec}(A)$ which contains $V(I) \backslash V(\mathfrak{a})$ is connected*[6].

Then $V(I) \setminus V(\mathfrak{a})$ is either empty or connected.

Proof. We may replace A by its reduction. Then we have the inequality in (2) for all associated primes of A. By Proposition 52.12.3 we see that

$$\mathrm{colim}\, H^0(V, \mathcal{O}_V) = \lim H^0(T_n, \mathcal{O}_{T_n})$$

where the colimit is over the opens V as in (3) and T_n is the nth infinitesimal neighbourhood of $T = V(I) \backslash V(\mathfrak{a})$ in $U = \mathrm{Spec}(A) \backslash V(\mathfrak{a})$. Thus T is either empty or connected, since if not, then the right hand side would have a nontrivial idempotent and we've assumed the left hand side does not. Some details omitted. □

0EG3 **Lemma 52.14.3.** *Let A be a Noetherian domain which has a dualizing complex and which is complete with respect to a nonzero $f \in A$. Let $f \in \mathfrak{a} \subset A$ be an ideal. Assume every irreducible component of $Z = V(\mathfrak{a})$ has codimension > 2 in $X = \mathrm{Spec}(A)$, i.e., assume every irreducible component of Z has codimension > 1 in $Y = V(f)$. Then $Y \setminus Z$ is connected.*

Proof. This is a special case of Lemma 52.14.2 (whose proof relies on Proposition 52.12.3). Below we prove it using the easier Proposition 52.12.6.

Set $U = X \setminus Z$. By Proposition 52.12.6 we have an isomorphism

$$\mathrm{colim}\, \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(U, \mathcal{O}_U/f^n\mathcal{O}_U)$$

where the colimit is over open $V \subset U$ containing $U \cap Y$. Hence if $U \cap Y$ is disconnected, then for some V there exists a nontrivial idempotent in $\Gamma(V, \mathcal{O}_V)$. This is impossible as V is an integral scheme as X is the spectrum of a domain. □

52.15. The completion functor

0EKN Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. In this section we consider inverse systems of coherent $\mathcal{O}_X$-modules $(\mathcal{F}_n)$ with $\mathcal{F}_n$ annihilated by I^n such that the transition maps induce isomorphisms $\mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1} \to \mathcal{F}_n$. The category of these inverse systems was denoted

$$Coh(X, \mathcal{I})$$

in Cohomology of Schemes, Section 30.23. This category is equivalent to the category of coherent modules on the formal completion of X along Y; however, since we

[6] For example if A is a domain.

have not yet introduced formal schemes or coherent modules on them, we cannot use this terminology here. We are particularly interested in the completion functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

See Cohomology of Schemes, Equation (30.23.3.1).

0EKP **Lemma 52.15.1.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Consider the following conditions*

1. *X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism,*
2. *X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$,*
3. *for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism, and*
4. *the completion functor $Coh(\mathcal{O}_X) \to Coh(X, \mathcal{I})$ is fully faithful on the full subcategory of finite locally free objects.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on X, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 30.23.5 we see that (3) implies (4).

Proof of (2) $\to$ (3). Namely, let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We claim we can find a universally exact sequence

$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r} \to (\mathcal{L}^{\otimes q})^{\oplus s}$$

for some $r, s \geq 0$ and $0 \ll p \ll q$. If this holds, then using the exact sequence

$$0 \to \lim \Gamma(\mathcal{E}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes p})^{\oplus r}|_{Y_n}) \to \lim \Gamma((\mathcal{L}^{\otimes q})^{\oplus s}|_{Y_n})$$

and the isomorphisms in (2) we get the isomorphism in (3). To prove the claim, consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ and apply Properties, Proposition 28.26.13 to find a surjection

$$(\mathcal{L}^{\otimes -p})^{\oplus r} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

Taking duals we obtain the first map in the exact sequence (it is universally injective because being a surjection is universal). Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 28.27.1.

We omit the proof of (4) $\Rightarrow$ (3). □

Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is *finite locally free* if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

0EK2 **Lemma 52.15.2.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion.*

(1) *X is quasi-affine and*
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$
is an isomorphism,
(2) *X has an ample invertible module $\mathcal{L}$ and*
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$
is an isomorphism for all $m \gg 0$,
(3) *for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map*
$$\mathrm{colim}_{V' \geq V} \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism, and
(4) *the completion functor*
$$\mathrm{colim}_{\mathcal{V}} \mathit{Coh}(\mathcal{O}_V) \longrightarrow \mathit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$
is fully faithful on the full subcategory of finite locally free objects (see explanation above).

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3).

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 52.15.1; we urge the reader to skip it.

Proof of (3) $\Rightarrow$ (4). If $\mathcal{F}$ and $\mathcal{G}$ are finite locally free on $V \in \mathcal{V}$, then considering $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{G}, \mathcal{F})$ and using Cohomology of Schemes, Lemma 30.23.5 we see that (3) implies (4).

Proof of (2) $\Rightarrow$ (3). Let $\mathcal{L}$ be ample on X and suppose that $\mathcal{E}$ is a finite locally free $\mathcal{O}_V$-module for some $V \in \mathcal{V}$. We claim we can find a universally exact sequence
$$0 \to \mathcal{E} \to (\mathcal{L}^{\otimes p})^{\oplus r}|_V \to (\mathcal{L}^{\otimes q})^{\oplus s}|_V$$
for some $r, s \geq 0$ and $0 \ll p \ll q$. If this is true, then the isomorphism in (2) will imply the isomorphism in (3). To prove the claim, recall that $\mathcal{L}|_V$ is ample, see Properties, Lemma 28.26.14. Consider the dual locally free module $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$ and apply Properties, Proposition 28.26.13 to find a surjection
$$(\mathcal{L}^{\otimes -p})^{\oplus r}|_V \longrightarrow \mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}, \mathcal{O}_V)$$
(it is universally injective because being a surjection is universal). Taking duals we obtain the first map in the exact sequence. Repeat with the cokernel to get the second. Some details omitted.

Proof of (1) $\Rightarrow$ (2). This is true because if X is quasi-affine then $\mathcal{O}_X$ is an ample invertible module, see Properties, Lemma 28.27.1.

We omit the proof of (4) $\Rightarrow$ (3). □

0EIQ **Lemma 52.15.3.** *Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The functor*
$$\mathit{Coh}(X, \mathcal{I}) \longrightarrow \mathit{Pro\text{-}QCoh}(\mathcal{O}_X)$$
is fully faithful, see Categories, Remark 4.22.5.

Proof. Let $(\mathcal{F}_n)$ and $(\mathcal{G}_n)$ be objects of $Coh(X, \mathcal{I})$. A morphism of pro-objects α from $(\mathcal{F}_n)$ to $(\mathcal{G}_n)$ is given by a system of maps $\alpha_n : \mathcal{F}_{m(n)} \to \mathcal{G}_n$ compatible with the transition maps where $\mathbf{N} \to \mathbf{N}$, $n \mapsto m(n)$ is an increasing function (in particular $m(n) \geq n$). Since $\mathcal{F}_n = \mathcal{F}_{m(n)}/\mathcal{I}^n\mathcal{F}_{m(n)}$ and since $\mathcal{G}_n$ is annihilated by $\mathcal{I}^n$ we see that α_n induces a map $\mathcal{F}_n \to \mathcal{G}_n$. □

Next we add some examples of the kind of fully faithfulness result we will be able to prove using the work done earlier in this chapter.

0EKQ **Lemma 52.15.4.** *Let $I \subset \mathfrak{a}$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *for any associated prime $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \not\subset V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$,*
(3) *for $\mathfrak{p} \subset A$ with $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \subset V(\mathfrak{a})$ we have $depth(A_\mathfrak{p}) \geq 2$.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.4. □

0EKS **Lemma 52.15.5.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A is f-adically complete,*
(2) *$H^1_\mathfrak{a}(A)$ and $H^2_\mathfrak{a}(A)$ are annihilated by a power of f.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.1 it suffices to show that

$$\Gamma(U, \mathcal{O}_U) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.7. □

0EKT **Lemma 52.15.6.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

(1) *A has a dualizing complex and is complete with respect to f,*
(2) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $depth(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > 2$.*

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. Follows from Lemma 52.15.5 and Local Cohomology, Proposition 51.10.1. □

0EKU **Lemma 52.15.7.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Assume*

(1) *A is I-adically complete and has a dualizing complex,*
(2) *for any associated prime $\mathfrak{p} \subset A$ with $I \not\subset \mathfrak{p}$ and $V(\mathfrak{p}) \cap V(I) \not\subset V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_\mathfrak{q}) > cd(A, I) + 1$.*

Then the completion functor

$$\mathrm{colim}_\mathcal{V} Coh(\mathcal{O}_V) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.2 it suffices to show that

$$\mathrm{colim}_\mathcal{V} \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Proposition 52.12.3. □

0EKV **Lemma 52.15.8.** *Let A be a Noetherian ring. Let $f \in \mathfrak{a} \subset A$ be an element of an ideal of A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(f)$ ordered by reverse inclusion. Assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{a}(A/fA)$ is a finite A-module.*

Then the completion functor

$$\mathrm{colim}_\mathcal{V} Coh(\mathcal{O}_V) \longrightarrow Coh(U, f\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. By Lemma 52.15.2 it suffices to show that

$$\mathrm{colim}_\mathcal{V} \Gamma(V, \mathcal{O}_V) = \lim \Gamma(U, \mathcal{O}_U/I^n\mathcal{O}_U)$$

This follows immediately from Lemma 52.12.5. □

0EIV **Lemma 52.15.9.** *Let $I \subset \mathfrak{a} \subset A$ be ideals of a Noetherian ring A. Let $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. Let $\mathcal{V}$ be the set of open subschemes of U containing $U \cap V(I)$ ordered by reverse inclusion. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_V$-modules for some $V \in \mathcal{V}$. The map*

$$\mathrm{colim}_{V' \geq V} \mathrm{Hom}_V(\mathcal{G}|_{V'}, \mathcal{F}|_{V'}) \longrightarrow \mathrm{Hom}_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective if the following assumptions hold:

(1) *A is I-adically complete and has a dualizing complex,*
(2) *if $x \in Ass(\mathcal{F})$, $x \notin V(I)$, $\overline{\{x\}} \cap V(I) \not\subset V(\mathfrak{a})$ and $z \in \overline{\{x\}} \cap V(\mathfrak{a})$, then $\dim(\mathcal{O}_{\overline{\{x\}}, z}) > cd(A, I) + 1$.*

Proof. We may choose coherent $\mathcal{O}_U$-modules $\mathcal{F}'$ and $\mathcal{G}'$ whose restriction to V is $\mathcal{F}$ and $\mathcal{G}$, see Properties, Lemma 28.22.5. We may modify our choice of $\mathcal{F}'$ to ensure that $\mathrm{Ass}(\mathcal{F}') \subset V$, see for example Local Cohomology, Lemma 51.15.1. Thus we may and do replace V by U and $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}'$ and $\mathcal{G}'$. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_U$-module. We have

$$\mathrm{Hom}_V(\mathcal{G}|_V, \mathcal{F}|_V) = H^0(V, \mathcal{H}) \quad \text{and} \quad \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 30.23.5. Thus if we can show that the assumptions of Proposition 52.12.3 hold for $\mathcal{H}$, then the proof is complete. This holds because $\text{Ass}(\mathcal{H}) \subset \text{Ass}(\mathcal{F})$. See Cohomology of Schemes, Lemma 30.11.2. □

52.16. Algebraization of coherent formal modules, I

0DXS The essential surjectivity of the completion functor (see below) was studied systematically in [**Gro68**], [**Ray75**], and [**Ray74**]. We work in the following affine situation.

0EHC **Situation 52.16.1.** Here A is a Noetherian ring and $I \subset \mathfrak{a} \subset A$ are ideals. We set $X = \text{Spec}(A)$, $Y = V(I) = \text{Spec}(A/I)$, and $Z = V(\mathfrak{a}) = \text{Spec}(A/\mathfrak{a})$. Furthermore $U = X \setminus Z$.

In this section we try to find conditions that guarantee an object of $Coh(U, I\mathcal{O}_U)$ is in the image of the completion functor $Coh(\mathcal{O}_U) \to Coh(U, I\mathcal{O}_U)$. See Cohomology of Schemes, Section 30.23 and Section 52.15.

0DXT **Lemma 52.16.2.** *In Situation 52.16.1. Consider an inverse system (M_n) of A-modules such that*

(1) *M_n is a finite A-module,*
(2) *M_n is annihilated by I^n,*
(3) *the kernel and cokernel of $M_{n+1}/I^nM_{n+1} \to M_n$ are $\mathfrak{a}$-power torsion.*

Then $(\widetilde{M_n}|_U)$ is in $Coh(U, I\mathcal{O}_U)$. Conversely, every object of $Coh(U, I\mathcal{O}_U)$ arises in this manner.

Proof. We omit the verification that $(\widetilde{M_n}|_U)$ is in $Coh(U, I\mathcal{O}_U)$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. By Local Cohomology, Lemma 51.8.2 we see that $\mathcal{F}_n = \widetilde{M_n}$ for some finite A/I^n-module M_n. After dividing M_n by $H^0_{\mathfrak{a}}(M_n)$ we may assume $M_n \subset H^0(U, \mathcal{F}_n)$, see Dualizing Complexes, Lemma 47.11.6 and the already referenced lemma. After replacing inductively M_{n+1} by the inverse image of M_n under the map $M_{n+1} \to H^0(U, \mathcal{F}_{n+1}) \to H^0(U, \mathcal{F}_n)$, we may assume M_{n+1} maps into M_n. This gives an inverse system (M_n) satisfying (1) and (2) such that $\mathcal{F}_n = \widetilde{M_n}$. To see that (3) holds, use that $M_{n+1}/I^nM_{n+1} \to M_n$ is a map of finite A-modules which induces an isomorphism after applying $\widetilde{\ }$ and restriction to U (here we use the first referenced lemma one more time). □

In Situation 52.16.1 we can study the completion functor Cohomology of Schemes, Equation (30.23.3.1)

0EIK
$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{52.16.2.1}$$

If A is I-adically complete, then this functor is fully faithful on suitable subcategories by our earlier work on algebraization of formal sections, see Section 52.15 and Lemma 52.19.6 for some sample results. Next, let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Still assuming A is I-adically complete, we can ask: When is $(\mathcal{F}_n)$ in the essential image of the completion functor displayed above?

0EIL **Lemma 52.16.3.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Consider the following conditions:*

(1) *$(\mathcal{F}_n)$ is in the essential image of the functor (52.16.2.1),*
(2) *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,*
(3) *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_V$-module for $U \cap Y \subset V \subset U$ open,*

(4) $(\mathcal{F}_n)$ *is the completion of the restriction to* U *of a coherent* $\mathcal{O}_X$*-module,*
(5) $(\mathcal{F}_n)$ *is the restriction to* U *of the completion of a coherent* $\mathcal{O}_X$*-module,*
(6) *there exists an object* $(\mathcal{G}_n)$ *of* $Coh(X, I\mathcal{O}_X)$ *whose restriction to* U *is* $(\mathcal{F}_n)$.

Then conditions (1), (2), (3), (4), and (5) are equivalent and imply (6). If A *is* I*-adically complete then condition (6) implies the others.*

Proof. Parts (1) and (2) are equivalent, because the completion of a coherent $\mathcal{O}_U$-module $\mathcal{F}$ is by definition the image of $\mathcal{F}$ under the functor (52.16.2.1). If $V \subset U$ is an open subscheme containing $U \cap Y$, then we have

$$Coh(V, I\mathcal{O}_V) = Coh(U, I\mathcal{O}_U)$$

since the category of coherent $\mathcal{O}_V$-modules supported on $V \cap Y$ is the same as the category of coherent $\mathcal{O}_U$-modules supported on $U \cap Y$. Thus the completion of a coherent $\mathcal{O}_V$-module is an object of $Coh(U, I\mathcal{O}_U)$. Having said this the equivalence of (2), (3), (4), and (5) holds because the functors $Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_U) \to Coh(\mathcal{O}_V)$ are essentially surjective. See Properties, Lemma 28.22.5.

It is always the case that (5) implies (6). Assume A is I-adically complete. Then any object of $Coh(X, I\mathcal{O}_X)$ corresponds to a finite A-module by Cohomology of Schemes, Lemma 30.23.1. Thus we see that (6) implies (5) in this case. □

0EHE **Example 52.16.4.** Let k be a field. Let $A = k[x, y][[t]]$ with $I = (t)$ and $\mathfrak{a} = (x, y, t)$. Let us use notation as in Situation 52.16.1. Observe that $U \cap Y = (D(x) \cap Y) \cup (D(y) \cap Y)$ is an affine open covering. For $n \geq 1$ consider the invertible module $\mathcal{L}_n$ of $\mathcal{O}_U/t^n\mathcal{O}_U$ given by glueing A_x/t^nA_x and A_y/t^nA_y via the invertible element of A_{xy}/t^nA_{xy} which is the image of any power series of the form

$$u = 1 + \frac{t}{xy} + \sum_{n \geq 2} a_n \frac{t^n}{(xy)^{\varphi(n)}}$$

with $a_n \in k[x, y]$ and $\varphi(n) \in \mathbf{N}$. Then $(\mathcal{L}_n)$ is an invertible object of $Coh(U, I\mathcal{O}_U)$ which is not the completion of a coherent $\mathcal{O}_U$-module $\mathcal{L}$. We only sketch the argument and we omit most of the details. Let $y \in U \cap Y$. Then the completion of the stalk $\mathcal{L}_y$ would be an invertible module hence $\mathcal{L}_y$ is invertible. Thus there would exist an open $V \subset U$ containing $U \cap Y$ such that $\mathcal{L}|_V$ is invertible. By Divisors, Lemma 31.28.3 we find an invertible A-module M with $\widetilde{M}|_V \cong \mathcal{L}|_V$. However the ring A is a UFD hence we see $M \cong A$ which would imply $\mathcal{L}_n \cong \mathcal{O}_U/I^n\mathcal{O}_U$. Since $\mathcal{L}_2 \ncong \mathcal{O}_U/I^2\mathcal{O}_U$ by construction we get a contradiction as desired.

Note that if we take $a_n = 0$ for $n \geq 2$, then we see that $\lim H^0(U, \mathcal{L}_n)$ is nonzero: in this case we the function x on $D(x)$ and the function $x + t/y$ on $D(y)$ glue. On the other hand, if we take $a_n = 1$ and $\varphi(n) = 2^n$ or even $\varphi(n) = n^2$ then the reader can show that $\lim H^0(U, \mathcal{L}_n)$ is zero; this gives another proof that $(\mathcal{L}_n)$ is not algebraizable in this case.

If in Situation 52.16.1 the ring A is not I-adically complete, then Lemma 52.16.3 suggests the correct thing is to ask whether $(\mathcal{F}_n)$ is in the essential image of the restriction functor

$$Coh(X, I\mathcal{O}_X) \longrightarrow Coh(U, I\mathcal{O}_U)$$

However, we can no longer say that this means $(\mathcal{F}_n)$ is algebraizable. Thus we introduce the following terminology.

0EIM **Definition 52.16.5.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ *extends to* X if there exists an object $(\mathcal{G}_n)$ of $Coh(X, I\mathcal{O}_X)$ whose restriction to U is isomorphic to $(\mathcal{F}_n)$.

This notion is equivalent to being algebraizable over the completion.

0EIN **Lemma 52.16.6.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let $A', I', \mathfrak{a}'$ be the I-adic completions of $A, I, \mathfrak{a}$. Set $X' = \mathrm{Spec}(A')$ and $U' = X' \setminus V(\mathfrak{a}')$. The following are equivalent*

(1) *$(\mathcal{F}_n)$ extends to X, and*
(2) *the pullback of $(\mathcal{F}_n)$ to U' is the completion of a coherent $\mathcal{O}_{U'}$-module.*

Proof. Recall that $A \to A'$ is a flat ring map which induces an isomorphism $A/I \to A'/I'$. See Algebra, Lemmas 10.97.2 and 10.97.4. Thus $X' \to X$ is a flat morphism inducing an isomorphism $Y' \to Y$. Thus $U' \to U$ is a flat morphism which induces an isomorphism $U' \cap Y' \to U \cap Y$. This implies that in the commutative diagram

$$\begin{array}{ccc} Coh(X', I\mathcal{O}_{X'}) & \longrightarrow & Coh(U', I\mathcal{O}_{U'}) \\ \uparrow & & \uparrow \\ Coh(X, I\mathcal{O}_X) & \longrightarrow & Coh(U, I\mathcal{O}_U) \end{array}$$

the vertical functors are equivalences. See Cohomology of Schemes, Lemma 30.23.10. The lemma follows formally from this and the results of Lemma 52.16.3. □

In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. To figure out if $(\mathcal{F}_n)$ extends to X it makes sense to look at the A-module

0EHD (52.16.6.1) $$M = \lim H^0(U, \mathcal{F}_n)$$

Observe that M has a limit topology which is (a priori) coarser than the I-adic topology since $M \to H^0(U, \mathcal{F}_n)$ annihilates $I^n M$. There are canonical maps

$$\widetilde{M}|_U \to \widetilde{M/I^nM}|_U \to \widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

One could hope that $\widetilde{M}$ restricts to a coherent module on U and that $(\mathcal{F}_n)$ is the completion of this module. This is naive because this has almost no chance of being true if A is not complete. But even if A is I-adically complete this notion is very difficult to work with. A less naive approach is to consider the following requirement.

0EIP **Definition 52.16.7.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. We say $(\mathcal{F}_n)$ *canonically extends to* X if the the inverse system

$$\{\widetilde{H^0(U, \mathcal{F}_n)}\}_{n \geq 1}$$

in $QCoh(\mathcal{O}_X)$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $Coh(X, I\mathcal{O}_X)$.

We will see in Lemma 52.16.8 that the condition in Definition 52.16.7 is stronger than the condition of Definition 52.16.5.

0EIR **Lemma 52.16.8.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. If $(\mathcal{F}_n)$ canonically extends to X, then*

(1) *$(\widetilde{H^0(U, \mathcal{F}_n)})$ is pro-isomorphic to an object $(\mathcal{G}_n)$ of $Coh(X, I\mathcal{O}_X)$ unique up to unique isomorphism,*

(2) *the restriction of* $(\mathcal{G}_n)$ *to* U *is isomorphic to* $(\mathcal{F}_n)$*, i.e.,* $(\mathcal{F}_n)$ *extends to* X*,*
(3) *the inverse system* $\{H^0(U, \mathcal{F}_n)\}$ *satisfies the Mittag-Leffler condition, and*
(4) *the module* M *in (52.16.6.1) is finite over the* I*-adic completion of* A *and the limit topology on* M *is the* I*-adic topology.*

Proof. The existence of $(\mathcal{G}_n)$ in (1) follows from Definition 52.16.7. The uniqueness of $(\mathcal{G}_n)$ in (1) follows from Lemma 52.15.3. Write $\mathcal{G}_n = \widetilde{M_n}$. Then $\{M_n\}$ is an inverse system of finite A-modules with $M_n = M_{n+1}/I^n M_{n+1}$. By Definition 52.16.7 the inverse system $\{H^0(U, \mathcal{F}_n)\}$ is pro-isomorphic to $\{M_n\}$. Hence we see that the inverse system $\{H^0(U, \mathcal{F}_n)\}$ satisfies the Mittag-Leffler condition and that $M = \lim M_n$ (as topological modules). Thus the properties of M in (4) follow from Algebra, Lemmas 10.98.2, 10.96.12, and 10.96.3. Since U is quasi-affine the canonical maps

$$\widetilde{H^0(U, \mathcal{F}_n)}|_U \to \mathcal{F}_n$$

are isomorphisms (Properties, Lemma 28.18.2). We conclude that $(\mathcal{G}_n|_U)$ and $(\mathcal{F}_n)$ are pro-isomorphic and hence isomorphic by Lemma 52.15.3. □

0EIS **Lemma 52.16.9.** *In Situation 52.16.1 let* $(\mathcal{F}_n)$ *be an object of* $Coh(U, I\mathcal{O}_U)$*. Let* $A \to A'$ *be a flat ring map. Set* $X' = \mathrm{Spec}(A')$*, let* $U' \subset X'$ *be the inverse image of* U*, and denote* $g : U' \to U$ *the induced morphism. Set* $(\mathcal{F}'_n) = (g^*\mathcal{F}_n)$*, see Cohomology of Schemes, Lemma 30.23.9. If* $(\mathcal{F}_n)$ *canonically extends to* X*, then* $(\mathcal{F}'_n)$ *canonically extends to* X'*. Moreover, the extension found in Lemma 52.16.8 for* $(\mathcal{F}_n)$ *pulls back to the extension for* $(\mathcal{F}'_n)$*.*

Proof. Let $f : X' \to X$ be the induced morphism. We have $H^0(U', \mathcal{F}'_n) = H^0(U, \mathcal{F}_n) \otimes_A A'$ by flat base change, see Cohomology of Schemes, Lemma 30.5.2. Thus if $(\mathcal{G}_n)$ in $Coh(X, I\mathcal{O}_X)$ is pro-isomorphic to $(\widetilde{H^0(U, \mathcal{F}_n)})$, then $(f^*\mathcal{G}_n)$ is pro-isomorphic to

$$(f^*\widetilde{H^0(U, \mathcal{F}_n)}) = (\widetilde{H^0(U, \mathcal{F}_n)} \otimes_A A') = (\widetilde{H^0(U', \mathcal{F}'_n)})$$

This finishes the proof. □

0EHH **Lemma 52.16.10.** *In Situation 52.16.1 let* $(\mathcal{F}_n)$ *be an object of* $Coh(U, I\mathcal{O}_U)$*. Let* M *be as in (52.16.6.1). Assume*

(a) *the inverse system* $H^0(U, \mathcal{F}_n)$ *has Mittag-Leffler,*
(b) *the limit topology on* M *agrees with the* I*-adic topology, and*
(c) *the image of* $M \to H^0(U, \mathcal{F}_n)$ *is a finite* A*-module for all* n*.*

Then $(\mathcal{F}_n)$ *extends canonically to* X*. In particular, if* A *is* I*-adically complete, then* $(\mathcal{F}_n)$ *is the completion of a coherent* $\mathcal{O}_U$*-module.*

Proof. Since $H^0(U, \mathcal{F}_n)$ has the Mittag-Leffler condition and since the limit topology on M is the I-adic topology we see that $\{M/I^nM\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic inverse systems of A-modules. Thus if we set

$$\mathcal{G}_n = \widetilde{M/I^nM}$$

then we see that to verify the condition in Definition 52.16.7 it suffices to show that M is a finite module over the I-adic completion of A. This follows from the fact that M/I^nM is finite by condition (c) and the above and Algebra, Lemma 10.96.12. □

The following is in some sense the most straightforward possible application Lemma 52.16.10 above.

0DXW **Lemma 52.16.11.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *$I = (f)$ is a principal ideal for a nonzerodivisor $f \in \mathfrak{a}$,*
(2) *$\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/f^n\mathcal{O}_U$-module,*
(3) *$H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$ are finite A-modules.*

Then $(\mathcal{F}_n)$ extends canonically to X. In particular, if A is complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Cohomology, Lemma 20.36.2.

As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and we have corresponding short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$. We will use Local Cohomology, Lemma 51.8.2 without further mention. Our assumptions imply that $H^0(U, \mathcal{O}_U/f\mathcal{O}_U)$ and $H^1(U, \mathcal{O}_U/f\mathcal{O}_U)$ are finite A-modules. Hence the same thing is true for $\mathcal{F}_1$, see Local Cohomology, Lemma 51.12.2. Using induction and the short exact sequences we find that $H^0(U, \mathcal{F}_n)$ are finite A-modules for all n. In this way we see hypothesis (c) is satisfied.

Finally, as $H^1(U, \mathcal{F}_1)$ is a finite A-module we can apply Cohomology, Lemma 20.36.4 to see hypothesis (a) holds. □

0EHI **Remark 52.16.12.** In Lemma 52.16.11 if A is universally catenary with Cohen-Macaulay formal fibres (for example if A has a dualizing complex), then the condition that $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$ are finite A-modules, is equivalent with

$$\operatorname{depth}((A/f)_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > 2$$

for all $\mathfrak{p} \in V(f) \setminus V(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ by Local Cohomology, Theorem 51.11.6.

For example, if A/fA is (S_2) and if every irreducible component of $Z = V(\mathfrak{a})$ has codimension ≥ 3 in $Y = \operatorname{Spec}(A/fA)$, then we get the finiteness of $H^1_\mathfrak{a}(A/fA)$ and $H^2_\mathfrak{a}(A/fA)$. This should be contrasted with the slightly weaker conditions found in Lemma 52.20.1 (see also Remark 52.20.2).

52.17. Algebraization of coherent formal modules, II

0EIT We continue the discussion started in Section 52.16. This section can be skipped on a first reading.

0EIU **Lemma 52.17.1.** *In Situation 52.16.1. Let $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ be a morphism of $Coh(U, I\mathcal{O}_U)$ whose kernel and cokernel are annihilated by a power of I. Then*

(1) *$(\mathcal{F}_n)$ extends to X if and only if $(\mathcal{F}'_n)$ extends to X, and*
(2) *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module if and only if $(\mathcal{F}'_n)$ is.*

Proof. Part (2) follows immediately from Cohomology of Schemes, Lemma 30.23.6. To see part (1), we first use Lemma 52.16.6 to reduce to the case where A is I-adically complete. However, in that case (1) reduces to (2) by Lemma 52.16.3. □

The following two lemmas where originally used in the proof of Lemma 52.16.10. We keep them here for the reader who is interested to know what intermediate results one can obtain.

0EHF **Lemma 52.17.2.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. If the inverse system $H^0(U, \mathcal{F}_n)$ has Mittag-Leffler, then the canonical maps*

$$\widetilde{M/I^nM}|_U \to \mathcal{F}_n$$

are surjective for all n where M is as in (52.16.6.1).

Proof. Surjectivity may be checked on the stalk at some point $y \in Y \setminus Z$. If y corresponds to the prime $\mathfrak{q} \subset A$, then we can choose $f \in \mathfrak{a}$, $f \notin \mathfrak{q}$. Then it suffices to show

$$M_f \longrightarrow H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

is surjective as $D(f)$ is affine (equality holds by Properties, Lemma 28.17.1). Since we have the Mittag-Leffler property, we find that

$$\mathrm{Im}(M \to H^0(U, \mathcal{F}_n)) = \mathrm{Im}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n))$$

for some $m \geq n$. Using the long exact sequence of cohomology we see that

$$\mathrm{Coker}(H^0(U, \mathcal{F}_m) \to H^0(U, \mathcal{F}_n)) \subset H^1(U, \mathrm{Ker}(\mathcal{F}_m \to \mathcal{F}_n))$$

Since $U = X \setminus V(\mathfrak{a})$ this H^1 is $\mathfrak{a}$-power torsion. Hence after inverting f the cokernel becomes zero. □

0EHG **Lemma 52.17.3.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let M be as in (52.16.6.1). Set*

$$\mathcal{G}_n = \widetilde{M/I^nM}.$$

If the limit topology on M agrees with the I-adic topology, then $\mathcal{G}_n|_U$ is a coherent $\mathcal{O}_U$-module and the map of inverse systems

$$(\mathcal{G}_n|_U) \longrightarrow (\mathcal{F}_n)$$

is injective in the abelian category $Coh(U, I\mathcal{O}_U)$.

Proof. Observe that $\mathcal{G}_n$ is a quasi-coherent $\mathcal{O}_X$-module annihilated by I^n and that $\mathcal{G}_{n+1}/I^n\mathcal{G}_{n+1} = \mathcal{G}_n$. Consider

$$M_n = \mathrm{Im}(M \longrightarrow H^0(U, \mathcal{F}_n))$$

The assumption says that the inverse systems (M_n) and (M/I^nM) are isomorphic as pro-objects of Mod_A. Pick $f \in \mathfrak{a}$ so $D(f) \subset U$ is an affine open. Then we have

$$(M_n)_f \subset H^0(U, \mathcal{F}_n)_f = H^0(D(f), \mathcal{F}_n)$$

Equality holds by Properties, Lemma 28.17.1. Thus $\widetilde{M_n}|_U \to \mathcal{F}_n$ is injective. It follows that $\widetilde{M_n}|_U$ is a coherent module (Cohomology of Schemes, Lemma 30.9.3). Since $M \to M/I^nM$ is surjective and factors as $M_{n'} \to M/I^nM$ for some $n' \geq n$ we find that $\mathcal{G}_n|_U$ is coherent as the quotient of a coherent module. Combined with

the initical remarks of the proof we conclude that $(\mathcal{G}_n|_U)$ indeed forms an object of $Coh(U, I\mathcal{O}_U)$. Finally, to show the injectivity of the map it suffices to show that

$$\lim(M/I^nM)_f = \lim H^0(D(f), \mathcal{G}_n) \to \lim H^0(D(f), \mathcal{F}_n)$$

is injective, see Cohomology of Schemes, Lemmas 30.23.2 and 30.23.1. The injectivity of $\lim(M_n)_f \to \lim H^0(D(f), \mathcal{F}_n)$ is clear (see above) and by our remark on pro-systems we have $\lim(M_n)_f = \lim(M/I^nM)_f$. This finishes the proof. □

52.18. A distance function

0EIW Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset. We define a function

0EIX (52.18.0.1) $$\delta_Z^Y = \delta_Z : Y \longrightarrow \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

which measures the "distance" of a point of Y from Z. For an informal discussion, please see Remark 52.18.3. Let $y \in Y$. We set $\delta_Z(y) = \infty$ if y is contained in a connected component of Y which does not meet Z. If y is contained in a connected component of Y which meets Z, then we can find $k \geq 0$ and a system

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \supset V_k \subset W_k$$

of integral closed subschemes of Y such that $V_0 \subset Z$ and $y \in W_k$ is the generic point. Set $c_i = \operatorname{codim}(V_i, W_i)$ for $i = 0, \ldots, k$ and $b_i = \operatorname{codim}(V_{i+1}, W_i)$ for $i = 0, \ldots, k-1$. For such a system we set

$$\delta(V_0, W_0, V_1, \ldots, W_k) = k + \max_{i=0,1,\ldots,k} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1})$$

This is $\geq k$ as we can take $i = k$ and we have $c_k \geq 0$. Finally, we set

$$\delta_Z(y) = \min \delta(V_0, W_0, V_1, \ldots, W_k)$$

where the minimum is over all systems of integral closed subschemes of Y as above.

0EIY **Lemma 52.18.1.** *Let Y be a Noetherian scheme and let $Z \subset Y$ be a closed subset.*

1. *For $y \in Y$ we have $\delta_Z(y) = 0 \Leftrightarrow y \in Z$.*
2. *The subsets $\{y \in Y \mid \delta_Z(y) \leq k\}$ are stable under specialization.*
3. *For $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we have $\dim(\mathcal{O}_{\overline{\{y\}}, z}) \geq \delta_Z(y)$.*
4. *If δ is a dimension function on Y, then $\delta_Z(y) \geq \delta(y) - \delta_{max}$ where δ_{max} is the maximum value of δ on Z.*
5. *If $Y = \operatorname{Spec}(A)$ is the spectrum of a catenary Noetherian local ring with maximal ideal $\mathfrak{m}$ and $Z = \{\mathfrak{m}\}$, then $\delta_Z(y) = \dim(\overline{\{y\}})$.*
6. *If $Y' \subset Y$ is an open subscheme, then $\delta^{Y'}_{Y' \cap Z}(y') \geq \delta^Y_Z(y')$ for $y' \in Y'$.*

Assume Y is catenary. Then

7. *Let $y' \rightsquigarrow y$ be an immediate specialization of points of Y. If Y is catenary, then $\delta_Z(y') \leq \delta_Z(y) + 1$.*
8. *Given a pattern of specializations*

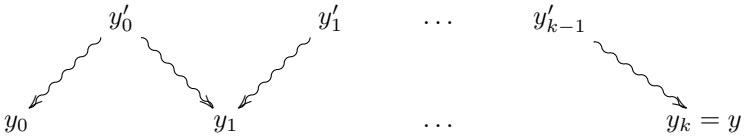

between points of Y with $y_0 \in Z$ and $y'_i \rightsquigarrow y_i$ an immediate specialization, then $\delta_Z(y_k) \leq k$.

Proof. Proof of (1). If $y \in Z$, then we can take $k = 0$ and $V_0 = W_0 = \overline{\{y\}}$ and we get $\delta(V_0, W_0) = 0$ so $\delta_Z(y) = 0$. If $y \notin Z$, then for every system $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ for y we either have $k = 0$ and $V_0 \neq W_0$ or $k > 0$. In both cases $\delta(V_0, W_0, \ldots, W_k) > 0$. Hence $\delta_Z(y) > 0$.

Proof of (2). Let $y \rightsquigarrow y'$ be a nontrivial specialization and let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ is a system for y. Here there are two cases. Case I: $V_k = W_k$, i.e., $c_k = 0$. In this case we can set $V_k' = W_k' = \overline{\{y'\}}$. An easy computation shows that $\delta(V_0, W_0, \ldots, V_k', W_k') \leq \delta(V_0, W_0, \ldots, V_k, W_k)$ because only b_{k-1} is changed into a bigger integer. Case II: $V_k \neq W_k$, i.e., $c_k > 0$. In this case, setting $V_{k+1} = W_{k+1} = \overline{\{y'\}}$ we see that $V_0 \subset W_0 \supset \ldots \subset W_k \supset V_{k+1} \subset W_{k+1}$ is a system for y'. Then $c_{k+1} = 0$ and $b_k > 0$ so we get

$$\begin{aligned}
&\delta(V_0, \ldots, W_{k+1}) \\
&= k + 1 + \max_{i=0,1,\ldots,k+1} (c_i + c_{i+1} + \ldots + c_k + c_{k+1} - b_i - b_{i+1} - \ldots - b_{k-1} - b_k) \\
&= k + 1 + \max_{i=0,1,\ldots,k+1} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1} - b_k) \\
&\leq k + \max_{i=0,1,\ldots,k} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) \\
&= \delta(V_0, \ldots, W_k)
\end{aligned}$$

The inequality holds because $c_k > 0$ and $b_k > 0$ which in particular implies that $\delta(V_0, \ldots, W_k) \geq k + c_k \geq k + 1$.

Proof of (3). Given $y \in Y$ and $z \in \overline{\{y\}} \cap Z$ we get the system

$$V_0 = \overline{\{z\}} \subset W_0 = \overline{\{y\}}$$

and $c_0 = \operatorname{codim}(V_0, W_0) = \dim(\mathcal{O}_{\overline{\{y\}}, z})$ by Properties, Lemma 28.10.3. Thus we see that $\delta(V_0, W_0) = 0 + c_0 = c_0$ which proves what we want.

Proof of (4). Let δ be a dimension function on Y. Let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ be a system for y. Let $y_i' \in W_i$ and $y_i \in V_i$ be the generic points, so $y_0 \in Z$ and $y_k = y$. Then we see that

$$\delta(y_i) - \delta(y_{i-1}) = \delta(y_{i-1}') - \delta(y_{i-1}) - \delta(y_{i-1}') + \delta(y_i) = c_{i-1} - b_{i-1}$$

Finally, we have $\delta(y_k') - \delta(y_{k-1}) = c_k$. Thus we see that

$$\delta(y) - \delta(y_0) = c_0 + \ldots + c_k - b_0 - \ldots - b_{k-1}$$

We conclude $\delta(V_0, W_0, \ldots, W_k) \geq k + \delta(y) - \delta(y_0)$ which proves what we want.

Proof of (5). The function $\delta(y) = \dim(\overline{\{y\}})$ is a dimension function. Hence $\delta(y) \leq \delta_Z(y)$ by part (4). By part (3) we have $\delta_Z(y) \leq \delta(y)$ and we are done.

Proof of (6). This is clear as their are fewer systems to consider in the computation of $\delta^{Y'}_{Y' \cap Z}$.

Proof of (7). Let $V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$ be a system for y. Set $W_k' = \overline{\{y'\}}$. Since Y is catenary, we see that $\operatorname{codim}(V_k, W_k') = \operatorname{codim}(V_k, W_k) + 1$. It follows easily that $\delta(V_0, \ldots, V_k, W_k') \leq \delta(V_0, \ldots, V_k, W_k) + 1$ which proves what we want.

Proof of (8): combine (7) and (2). $\square$

0EIZ **Lemma 52.18.2.** *Let Y be a universally catenary Noetherian scheme. Let $Z \subset Y$ be a closed subscheme. Let $f : Y' \to Y$ be a finite type morphism all of whose fibres have dimension $\leq e$. Set $Z' = f^{-1}(Z)$. Then*

$$\delta_Z(y) \leq \delta_{Z'}(y') + e - trdeg_{\kappa(y)}(\kappa(y'))$$

for $y' \in Y'$ with image $y \in Y$.

Proof. If $\delta_{Z'}(y') = \infty$, then there is nothing to prove. If $\delta_{Z'}(y') < \infty$, then we choose a system of integral closed subschemes

$$V'_0 \subset W'_0 \supset V'_1 \subset W'_1 \supset \ldots \subset W'_k$$

of Y' with $V'_0 \subset Z'$ and y' the generic point of W'_k such that $\delta_{Z'}(y') = \delta(V'_0, W'_0, \ldots, W'_k)$. Denote

$$V_0 \subset W_0 \supset V_1 \subset W_1 \supset \ldots \subset W_k$$

the scheme theoretic images of the above schemes in Y. Observe that y is the generic point of W_k and that $V_0 \subset Z$. For each i we look at the diagram

$$\begin{array}{ccccc} V'_i & \longrightarrow & W'_i & \longleftarrow & V'_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ V_i & \longrightarrow & W_i & \longleftarrow & V_{i+1} \end{array}$$

Denote n_i the relative dimension of V'_i/V_i and m_i the relative dimension of W'_i/W_i; more precisely these are the transcendence degrees of the corresponding extensions of the function fields. Set $c_i = \operatorname{codim}(V_i, W_i)$, $c'_i = \operatorname{codim}(V'_i, W'_i)$, $b_i = \operatorname{codim}(V_{i+1}, W_i)$, and $b'_i = \operatorname{codim}(V'_{i+1}, W'_i)$. By the dimension formula we have

$$c_i = c'_i + n_i - m_i \quad \text{and} \quad b_i = b'_i + n_{i+1} - m_i$$

See Morphisms, Lemma 29.52.1. Hence $c_i - b_i = c'_i - b'_i + n_i - n_{i+1}$. Thus we see that

$$\begin{aligned} & c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1} \\ & = c'_i + c'_{i+1} + \ldots + c'_k - b'_i - b'_{i+1} - \ldots - b'_{k-1} + n_i - n_k + c_k - c'_k \\ & = c'_i + c'_{i+1} + \ldots + c'_k - b'_i - b'_{i+1} - \ldots - b'_{k-1} + n_i - m_k \end{aligned}$$

Thus we see that

$$\begin{aligned} & \max_{i=0,\ldots,k} (c_i + c_{i+1} + \ldots + c_k - b_i - b_{i+1} - \ldots - b_{k-1}) \\ & \quad = \max_{i=0,\ldots,k} (c'_i + c'_{i+1} + \ldots + c'_k - b'_i - b'_{i+1} - \ldots - b'_{k-1} + n_i - m_k) \\ & \quad = \max_{i=0,\ldots,k} (c'_i + c'_{i+1} + \ldots + c'_k - b'_i - b'_{i+1} - \ldots - b'_{k-1} + n_i) - m_k \\ & \quad \leq \max_{i=0,\ldots,k} (c'_i + c'_{i+1} + \ldots + c'_k - b'_i - b'_{i+1} - \ldots - b'_{k-1}) + e - m_k \end{aligned}$$

Since $m_k = \operatorname{trdeg}_{\kappa(y)}(\kappa(y'))$ we conclude that

$$\delta(V_0, W_0, \ldots, W_k) \leq \delta(V'_0, W'_0, \ldots, W'_k) + e - \operatorname{trdeg}_{\kappa(y)}(\kappa(y'))$$

as desired. □

0EJ0 **Remark 52.18.3.** Let Y be a catenary Noetherian scheme and let $Z \subset Y$ be a closed subset. By Lemma 52.18.1 we have

$$\delta_Z(y) \leq \min \left\{ k \middle| \begin{array}{c} \text{there exist specializations in } Y \\ y_0 \leftarrow y_0' \rightarrow y_1 \leftarrow y_1' \rightarrow \ldots \leftarrow y_{k-1}' \rightarrow y_k = y \\ \text{with } y_0 \in Z \text{ and } y_i' \rightsquigarrow y_i \text{ immediate} \end{array} \right\}$$

We claim that if Y is of finite type over a field, then equality holds. If we ever need this result we will formulate a precise result and prove it here. However, in general if we define δ_Z by the right hand side of this inequality, then we don't know if Lemma 52.18.2 remains true.

0EJ1 **Example 52.18.4.** Let k be a field and $Y = \mathbf{A}^n_k$. Denote $\delta : Y \to \mathbf{Z}_{\geq 0}$ the usual dimension function.

(1) If $Z = \{z\}$ for some closed point z, then
   (a) $\delta_Z(y) = \delta(y)$ if $y \rightsquigarrow z$ and
   (b) $\delta_Z(y) = \delta(y) + 1$ if $y \not\rightsquigarrow z$.
(2) If Z is a closed subvariety and $W = \overline{\{y\}}$, then
   (a) $\delta_Z(y) = 0$ if $W \subset Z$,
   (b) $\delta_Z(y) = \dim(W) - \dim(Z)$ if Z is contained in W,
   (c) $\delta_Z(y) = 1$ if $\dim(W) \leq \dim(Z)$ and $W \not\subset Z$,
   (d) $\delta_Z(y) = \dim(W) - \dim(Z) + 1$ if $\dim(W) > \dim(Z)$ and $Z \not\subset W$.

A generalization of case (1) is if Y is of finite type over a field and $Z = \{z\}$ is a closed point. Then $\delta_Z(y) = \delta(y) + t$ where t is the minimum length of a chain of curves connecting z to a closed point of $\overline{\{y\}}$.

52.19. Algebraization of coherent formal modules, III

0EJ2 We continue the discussion started in Sections 52.16 and 52.17. We will use the distance function of Section 52.18 to formulate a some natural conditions on coherent formal modules in Situation 52.16.1.

In Situation 52.16.1 given a point $y \in U \cap Y$ we can consider the I-adic completion

$$\mathcal{O}^\wedge_{X,y} = \lim \mathcal{O}_{X,y}/I^n\mathcal{O}_{X,y}$$

This is a Noetherian local ring complete with respect to $I\mathcal{O}^\wedge_{X,y}$ with maximal ideal $\mathfrak{m}^\wedge_y$, see Algebra, Section 10.97. Let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let us define the "stalk" of $(\mathcal{F}_n)$ at y by the formula

$$\mathcal{F}^\wedge_y = \lim \mathcal{F}_{n,y}$$

This is a finite module over $\mathcal{O}^\wedge_{X,y}$. See Algebra, Lemmas 10.98.2 and 10.96.12.

0EJ3 **Definition 52.19.1.** In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let a, b be integers. Let δ^Y_Z be as in (52.18.0.1). We say $(\mathcal{F}_n)$ *satisfies the* (a, b)*-inequalities* if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$

(1) if $V(\mathfrak{p}) \cap V(I\mathcal{O}^\wedge_{X,y}) \neq \{\mathfrak{m}^\wedge_y\}$, then

$$\operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \delta^Y_Z(y) \geq a \quad \text{or} \quad \operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > b$$

(2) if $V(\mathfrak{p}) \cap V(I\mathcal{O}^\wedge_{X,y}) = \{\mathfrak{m}^\wedge_y\}$, then

$$\operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \delta^Y_Z(y) > a$$

We say $(\mathcal{F}_n)$ *satisfies the strict* (a, b)*-inequalities* if for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$ we have

$$\text{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \delta^Y_Z(y) > a \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > b$$

Here are some elementary observations.

0EJ4 **Lemma 52.19.2.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Let a, b be integers.*

(1) *If $(\mathcal{F}_n)$ is annihilated by a power of I, then $(\mathcal{F}_n)$ satisfies the (a, b)-inequalities for any a, b.*
(2) *If $(\mathcal{F}_n)$ satisfies the $(a + 1, b)$-inequalities, then $(\mathcal{F}_n)$ satisfies the strict (a, b)-inequalities.*

If $cd(A, I) \leq d$ and A has a dualizing complex, then

(3) *$(\mathcal{F}_n)$ satisfies the $(s, s + d)$-inequalities if and only if for all $y \in U \cap Y$ the tuple $\mathcal{O}^\wedge_{X,y}, I\mathcal{O}^\wedge_{X,y}, \{\mathfrak{m}^\wedge_y\}, \mathcal{F}^\wedge_y, s - \delta^Y_Z(y), d$ is as in Situation 52.10.1.*
(4) *If $(\mathcal{F}_n)$ satisfies the strict $(s, s + d)$-inequalities, then $(\mathcal{F}_n)$ satisfies the $(s, s + d)$-inequalities.*

Proof. Immediate except for part (4) which is a consequence of Lemma 52.10.5 and the translation in (3). □

0EKW **Lemma 52.19.3.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. If $cd(A, I) = 1$, then $\mathcal{F}$ satisfies the $(2, 3)$-inequalities if and only if*

$$depth((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > 3$$

for all $y \in U \cap Y$ and $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$.

Proof. Observe that for a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$, $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$ we have $V(\mathfrak{p}) \cap V(I\mathcal{O}^\wedge_{X,y}) = \{\mathfrak{m}^\wedge_y\} \Leftrightarrow \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = 1$ as $\text{cd}(A, I) = 1$. See Local Cohomology, Lemmas 51.4.5 and 51.4.10. OK, consider the three numbers $\alpha = \text{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) \geq 0$, $\beta = \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) \geq 1$, and $\gamma = \delta^Y_Z(y) \geq 1$. Then we see Definition 52.19.1 requires

(1) if $\beta > 1$, then $\alpha + \gamma \geq 2$ or $\alpha + \beta + \gamma > 3$, and
(2) if $\beta = 1$, then $\alpha + \gamma > 2$.

It is trivial to see that this is equivalent to $\alpha + \beta + \gamma > 3$. □

In the rest of this section, which we suggest the reader skip on a first reading, we will show that, when A is I-adically complete, the category of $(\mathcal{F}_n)$ of $Coh(U, I\mathcal{O}_U)$ which extend to X and satisfy the strict $(1, 1 + \text{cd}(A, I))$-inequalities is equivalent to a full subcategory of the category of coherent $\mathcal{O}_U$-modules.

0EJ5 **Lemma 52.19.4.** *In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities.*

Let $x \in X$ be a point. Let $W = \overline{\{x\}}$. If $W \cap Y$ has an irreducible component contained in Z and one which is not, then $depth(\mathcal{F}_x) \geq 1$.

Proof. Let $W \cap Y = W_1 \cup \ldots \cup W_n$ be the decomposition into irreducible components. By assumption, after renumbering, we can find $0 < m < n$ such that $W_1, \ldots, W_m \subset Z$ and $W_{m+1}, \ldots, W_n \not\subset Z$. We conclude that

$$W \cap Y \setminus ((W_1 \cup \ldots \cup W_m) \cap (W_{m+1} \cup \ldots \cup W_n))$$

is disconnected. By Lemma 52.14.2 we can find $1 \leq i \leq m < j \leq n$ and $z \in W_i \cap W_j$ such that $\dim(\mathcal{O}_{W,z}) \leq d + 1$. Choose an immediate specialization $y \rightsquigarrow z$ with $y \in W_j$, $y \notin Z$; existence of y follows from Properties, Lemma 28.6.4. Observe that $\delta_Z^Y(y) = 1$ and $\dim(\mathcal{O}_{W,y}) \leq d$. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}$ be the prime corresponding to x. Let $\mathfrak{p}' \subset \mathcal{O}_{X,y}^\wedge$ be a minimal prime over $\mathfrak{p}\mathcal{O}_{X,y}^\wedge$. Then we have

$$\text{depth}(\mathcal{F}_x) = \text{depth}((\mathcal{F}_y^\wedge)_{\mathfrak{p}'}) \quad \text{and} \quad \dim(\mathcal{O}_{W,y}) = \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}')$$

See Algebra, Lemma 10.163.1 and Local Cohomology, Lemma 51.11.3. Now we read off the conclusion from the inequalities given to us. □

0EJ6 **Lemma 52.19.5.** *In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and*
(3) *for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $depth(\mathcal{F}_x) \geq 2$.*

Then $H^0(U, \mathcal{F}) \to \lim H^0(U, \mathcal{F}/I^n\mathcal{F})$ is an isomorphism.

Proof. We will prove this by showing that Lemma 52.12.4 applies. Thus we let $x \in \text{Ass}(\mathcal{F})$ with $x \notin Y$. Set $W = \overline{\{x\}}$. By condition (3) we see that $W \cap Y \not\subset Z$. By Lemma 52.19.4 we see that no irreducible component of $W \cap Y$ is contained in Z. Thus if $z \in W \cap Z$, then there is an immediate specialization $y \rightsquigarrow z$, $y \in W \cap Y$, $y \notin Z$. For existence of y use Properties, Lemma 28.6.4. Then $\delta_Z^Y(y) = 1$ and the assumption implies that $\dim(\mathcal{O}_{W,y}) > d$. Hence $\dim(\mathcal{O}_{W,z}) > 1 + d$ and we win. □

0EJ7 **Lemma 52.19.6.** *In Situation 52.16.1 let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *the completion $\mathcal{F}^\wedge$ of $\mathcal{F}$ satisfies the strict $(1, 1 + d)$-inequalities, and*
(3) *for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $depth(\mathcal{F}_x) \geq 2$.*

Then the map

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective for every coherent $\mathcal{O}_U$-module $\mathcal{G}$.

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F})$. Using Cohomology of Schemes, Lemma 30.11.2 or More on Algebra, Lemma 15.23.10 we see that the completion of $\mathcal{H}$ satisfies the strict $(1, 1 + d)$-inequalities and that for $x \in U$ with $\overline{\{x\}} \cap Y \subset Z$ we have $\text{depth}(\mathcal{H}_x) \geq 2$. Details omitted. Thus by Lemma 52.19.5 we have

$$\text{Hom}_U(\mathcal{G}, \mathcal{F}) = H^0(U, \mathcal{H}) = \lim H^0(U, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \text{Mor}_{Coh(U, I\mathcal{O}_U)}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

See Cohomology of Schemes, Lemma 30.23.5 for the final equality. □

0EJ8 **Lemma 52.19.7.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$ and $d \geq 1$. Assume*

(1) *A is I-adically complete, has a dualizing complex, and $cd(A, I) \leq d$,*
(2) *$(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module,*

(3) $(\mathcal{F}_n)$ *satisfies the strict* $(1, 1+d)$*-inequalities.*

Then there exists a unique coherent $\mathcal{O}_U$*-module* $\mathcal{F}$ *whose completion is* $(\mathcal{F}_n)$ *such that for* $x \in U$ *with* $\overline{\{x\}} \cap Y \subset Z$ *we have* $\text{depth}(\mathcal{F}_x) \geq 2$*.*

Proof. Choose a coherent $\mathcal{O}_U$-module $\mathcal{F}$ whose completion is $(\mathcal{F}_n)$. Let $T = \{x \in U \mid \overline{\{x\}} \cap Y \subset Z\}$. We will construct $\mathcal{F}$ by applying Local Cohomology, Lemma 51.15.4 with $\mathcal{F}$ and T. Then uniqueness will follow from the mapping property of Lemma 52.19.6.

Since T is stable under specialization in U the only thing to check is the following. If $x' \rightsquigarrow x$ is an immediate specialization of points of U with $x \in T$ and $x' \notin T$, then $\text{depth}(\mathcal{F}_{x'}) \geq 1$. Set $W = \overline{\{x\}}$ and $W' = \overline{\{x'\}}$. Since $x' \notin T$ we see that $W' \cap Y$ is not contained in Z. If $W' \cap Y$ contains an irreducible component contained in Z, then we are done by Lemma 52.19.4. If not, we choose an irreducible component W_1 of $W \cap Y$ and an irreducible component W_1' of $W' \cap Y$ with $W_1 \subset W_1'$. Let $z \in W_1$ be the generic point. Let $y \rightsquigarrow z$, $y \in W_1'$ be an immediate specialization with $y \notin Z$; existence of y follows from $W_1' \not\subset Z$ (see above) and Properties, Lemma 28.6.4. Then we have the following $z \in Z$, $x \rightsquigarrow z$, $x' \rightsquigarrow y \rightsquigarrow z$, $y \in Y \setminus Z$, and $\delta_Z^Y(y) = 1$. By Local Cohomology, Lemma 51.4.10 and the fact that z is a generic point of $W \cap Y$ we have $\dim(\mathcal{O}_{W,z}) \leq d$. Since $x' \rightsquigarrow x$ is an immediate specialization we have $\dim(\mathcal{O}_{W',z}) \leq d+1$. Since $y \neq z$ we conclude $\dim(\mathcal{O}_{W',y}) \leq d$. If $\text{depth}(\mathcal{F}_{x'}) = 0$ then we would get a contradiction with assumption (3); details about passage from $\mathcal{O}_{X,y}$ to its completion omitted. This finishes the proof. □

52.20. Algebraization of coherent formal modules, IV

0EHJ In this section we prove two stronger versions of Lemma 52.16.11 in the local case, namely, Lemmas 52.20.1 and 52.20.4. Although these lemmas will be obsoleted by the more general Proposition 52.22.2, their proofs are significantly easier.

0DXU **Lemma 52.20.1.** *In Situation 52.16.1 let* $(\mathcal{F}_n)$ *be an object of* $Coh(U, I\mathcal{O}_U)$*. Assume*

(1) A *is local and* $\mathfrak{a} = \mathfrak{m}$ *is the maximal ideal,*
(2) A *has a dualizing complex,*
(3) $I = (f)$ *is a principal ideal for a nonzerodivisor* $f \in \mathfrak{m}$*,*
(4) $\mathcal{F}_n$ *is a finite locally free* $\mathcal{O}_U/f^n\mathcal{O}_U$*-module,*
(5) *if* $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$*, then* $\text{depth}((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$*, and*
(6) *if* $\mathfrak{p} \notin V(f)$ *and* $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$*, then* $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$*.*

Then $(\mathcal{F}_n)$ *extends canonically to* X*. In particular, if* A *is complete, then* $(\mathcal{F}_n)$ *is the completion of a coherent* $\mathcal{O}_U$*-module.*

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10.

Since $\mathcal{F}_n$ is locally free over $\mathcal{O}_U/f^n\mathcal{O}_U$ we see that we have short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$ for all n. Thus condition (b) holds by Cohomology, Lemma 20.36.2.

By induction on n and the short exact sequences $0 \to A/f^n \to A/f^{n+1} \to A/f \to 0$ we see that the associated primes of A/f^nA agree with the associated primes of A/fA. Since the associated points of $\mathcal{F}_n$ correspond to the associated primes of A/f^nA not equal to $\mathfrak{m}$ by assumption (3), we conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a

finite A-module by (5) and Local Cohomology, Proposition 51.8.7. Thus hypothesis (c) holds.

To finish the proof it suffices to show that there exists an $n > 1$ such that the image of

$$H^1(U, \mathcal{F}_n) \longrightarrow H^1(U, \mathcal{F}_1)$$

has finite length as an A-module. Namely, this will imply hypothesis (a) by Cohomology, Lemma 20.36.5. The image is independent of n for n large enough by Lemma 52.5.2. Let $\omega_A^\bullet$ be a normalized dualizing complex for A. By the local duality theorem and Matlis duality (Dualizing Complexes, Lemma 47.18.4 and Proposition 47.7.8) our claim is equivalent to: the image of

$$\mathrm{Ext}_A^{-2}(M_1, \omega_A^\bullet) \to \mathrm{Ext}_A^{-2}(M_n, \omega_A^\bullet)$$

has finite length for $n \gg 1$. The modules in question are finite A-modules supported at $V(f)$. Thus it suffices to show that this map is zero after localization at a prime $\mathfrak{q}$ containing f and different from $\mathfrak{m}$. Let $\omega_{A_\mathfrak{q}}^\bullet$ be a normalized dualizing complex on $A_\mathfrak{q}$ and recall that $\omega_{A_\mathfrak{q}}^\bullet = (\omega_A^\bullet)_\mathfrak{q}[\dim(A/\mathfrak{q})]$ by Dualizing Complexes, Lemma 47.17.3. Using the local structure of $\mathcal{F}_n$ given in (4) we find that it suffices to show the vanishing of

$$\mathrm{Ext}_{A_\mathfrak{q}}^{-2+\dim(A/\mathfrak{q})}(A_\mathfrak{q}/f, \omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}_{A_\mathfrak{q}}^{-2+\dim(A/\mathfrak{q})}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet)$$

for n large enough. If $\dim(A/\mathfrak{q}) > 3$, then this is immediate from Local Cohomology, Lemma 51.9.4. For the other cases we will use the long exact sequence

$$\ldots \xrightarrow{f^n} H^{-1}(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}_{A_\mathfrak{q}}^{-1}(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet) \to H^0(\omega_{A_\mathfrak{q}}^\bullet) \xrightarrow{f^n} H^0(\omega_{A_\mathfrak{q}}^\bullet) \to \mathrm{Ext}_{A_\mathfrak{q}}^0(A_\mathfrak{q}/f^n, \omega_{A_\mathfrak{q}}^\bullet) \to 0$$

If $\dim(A/\mathfrak{q}) = 2$, then $H^0(\omega_{A_\mathfrak{q}}^\bullet) = 0$ because $\mathrm{depth}(A_\mathfrak{q}) \geq 1$ as f is a nonzerodivisor. Thus the long exact sequence shows the condition is that

$$f^{n-1} : H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f \to H^{-1}(\omega_{A_\mathfrak{q}}^\bullet)/f^n$$

is zero. Now $H^{-1}(\omega_\mathfrak{q}^\bullet)$ is a finite module supported in the primes $\mathfrak{p} \subset A_\mathfrak{q}$ such that $\mathrm{depth}(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq 1$. Since $\dim((A/\mathfrak{p})_\mathfrak{q}) = \dim(A/\mathfrak{p}) - 2$ condition (6) tells us these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. Finally, if $\dim(A/\mathfrak{q}) = 1$, then condition (5) combined with the fact that f is a nonzerodivisor insures that $A_\mathfrak{q}$ has depth at least 2. Hence $H^0(\omega_{A_\mathfrak{q}}^\bullet) = H^{-1}(\omega_{A_\mathfrak{q}}^\bullet) = 0$ and the long exact sequence shows the claim is equivalent to the vanishing of

$$f^{n-1} : H^{-2}(\omega_{A_\mathfrak{q}}^\bullet)/f \to H^{-2}(\omega_{A_\mathfrak{q}}^\bullet)/f^n$$

Now $H^{-2}(\omega_\mathfrak{q}^\bullet)$ is a finite module supported in the primes $\mathfrak{p} \subset A_\mathfrak{q}$ such that $\mathrm{depth}(A_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \leq 2$. By condition (6) all of these primes are contained in $V(f)$. Thus the desired vanishing for n large enough. □

0DXV **Remark 52.20.2.** Let $(A, \mathfrak{m})$ be a complete Noetherian normal local domain of dimension ≥ 4 and let $f \in \mathfrak{m}$ be nonzero. Then assumptions (1), (2), (3), (5), and (6) of Lemma 52.20.1 are satisfied. Thus vectorbundles on the formal completion of U along $U \cap V(f)$ can be algebraized. In Lemma 52.20.4 we will generalize this to more general coherent formal modules; please also compare with Remark 52.20.7.

0EHK **Lemma 52.20.3.** *In Situation 52.16.1 let (M_n) be an inverse system of A-modules as in Lemma 52.16.2 and let $(\mathcal{F}_n)$ be the corresponding object of $Coh(U, I\mathcal{O}_U)$. Let $d \geq cd(A, I)$ and $s \geq 0$ be integers. With notation as above assume*

(1) *A is local with maximal ideal $\mathfrak{m} = \mathfrak{a}$,*
(2) *A has a dualizing complex, and*
(3) *$(\mathcal{F}_n)$ satisfies the $(s, s + d)$-inequalities (Definition 52.19.1).*

Let E be an injective hull of the residue field of A. Then for $i \leq s$ there exists a finite A-module N annihilated by a power of I and for $n \gg 0$ compatible maps

$$H^i_{\mathfrak{m}}(M_n) \to \mathrm{Hom}_A(N, E)$$

whose cokernels are finite length A-modules and whose kernels K_n form an inverse system such that $\mathrm{Im}(K_{n''} \to K_{n'})$ has finite length for $n'' \gg n' \gg 0$.

Proof. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\delta_Z^Y = \delta$ is the dimension function associated with this dualizing complex. Observe that $\mathrm{Ext}_A^{-i}(M_n, \omega_A^\bullet)$ is a finite A-module annihilated by I^n. Fix $0 \leq i \leq s$. Below we will find $n_1 > n_0 > 0$ such that if we set

$$N = \mathrm{Im}(\mathrm{Ext}_A^{-i}(M_{n_0}, \omega_A^\bullet) \to \mathrm{Ext}_A^{-i}(M_{n_1}, \omega_A^\bullet))$$

then the kernels of the maps

$$N \to \mathrm{Ext}_A^{-i}(M_n, \omega_A^\bullet), \quad n \geq n_1$$

are finite length A-modules and the cokernels Q_n form a system such that $\mathrm{Im}(Q_{n'} \to Q_{n''})$ has finite length for $n'' \gg n' \gg n_1$. This is equivalent to the statement that the system $\{\mathrm{Ext}_A^{-i}(M_n, \omega_A^\bullet)\}_{n \geq 1}$ is essentially constant in the quotient of the category of finite A-modules modulo the Serre subcategory of finite length A-modules. By the local duality theorem (Dualizing Complexes, Lemma 47.18.4) and Matlis duality (Dualizing Complexes, Proposition 47.7.8) we conclude that there are maps

$$H^i_{\mathfrak{m}}(M_n) \to \mathrm{Hom}_A(N, E), \quad n \geq n_1$$

as in the statement of the lemma.

Pick $f \in \mathfrak{m}$. Let $B = A_f^\wedge$ be the I-adic completion of the localization A_f. Recall that $\omega_{A_f}^\bullet = \omega_A^\bullet \otimes_A A_f$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$ are dualizing complexes (Dualizing Complexes, Lemma 47.15.6 and 47.22.3). Let M be the finite B-module $\lim M_{n,f}$ (compare with discussion in Cohomology of Schemes, Lemma 30.23.1). Then

$$\mathrm{Ext}_A^{-i}(M_n, \omega_A^\bullet)_f = \mathrm{Ext}_{A_f}^{-i}(M_{n,f}, \omega_{A_f}^\bullet) = \mathrm{Ext}_B^{-i}(M/I^nM, \omega_B^\bullet)$$

Since $\mathfrak{m}$ can be generated by finitely many $f \in \mathfrak{m}$ it suffices to show that for each f the system

$$\{\mathrm{Ext}_B^{-i}(M/I^nM, \omega_B^\bullet)\}_{n \geq 1}$$

is essentially constant. Some details omitted.

Let $\mathfrak{q} \subset IB$ be a prime ideal. Then $\mathfrak{q}$ corresponds to a point $y \in U \cap Y$. Observe that $\delta(\mathfrak{q}) = \dim(\overline{\{y\}})$ is also the value of the dimension function associated to $\omega_B^\bullet$ (we omit the details; use that $\omega_B^\bullet$ is gotten from $\omega_A^\bullet$ by tensoring up with B). Assumption (3) guarantees via Lemma 52.19.2 that Lemma 52.10.4 applies to $B_{\mathfrak{q}}, IB_{\mathfrak{q}}, \mathfrak{q}B_{\mathfrak{q}}, M_{\mathfrak{q}}$ with s replaced by $s - \delta(y)$. We obtain that

$$H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_{\mathfrak{q}}}(M_{\mathfrak{q}}) = \lim H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_{\mathfrak{q}}}((M/I^nM)_{\mathfrak{q}})$$

and this module is annihilated by a power of I. By Lemma 52.5.3 we find that the inverse systems $H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}((M/I^nM)_\mathfrak{q})$ are essentially constant with value $H^{i-\delta(\mathfrak{q})}_{\mathfrak{q}B_\mathfrak{q}}(M_\mathfrak{q})$. Since $(\omega_B^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex on $B_\mathfrak{q}$ the local duality theorem shows that the system

$$\mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet)_\mathfrak{q}$$

is essentially constant with value $\mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)_\mathfrak{q}$.

To finish the proof we globalize as in the proof of Lemma 52.10.6; the argument here is easier because we know the value of our system already. Namely, consider the maps

$$\alpha_n : \mathrm{Ext}^{-i}_B(M/I^nM, \omega_B^\bullet) \longrightarrow \mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)$$

for varying n. By the above, for every $\mathfrak{q}$ we can find an n such that α_n is surjective after localization at $\mathfrak{q}$. Since B is Noetherian and $\mathrm{Ext}^{-i}_B(M, \omega_B^\bullet)$ a finite module, we can find an n such that α_n is surjective. For any n such that α_n is surjective, given a prime $\mathfrak{q} \in V(IB)$ we can find an $n' > n$ such that $\mathrm{Ker}(\alpha_n)$ maps to zero in $\mathrm{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$ at least after localizing at $\mathfrak{q}$. Since $\mathrm{Ker}(\alpha_n)$ is a finite A-module and since supports of sections are quasi-compact, we can find an n' such that $\mathrm{Ker}(\alpha_n)$ maps to zero in $\mathrm{Ext}^{-i}(M/I^{n'}M, \omega_B^\bullet)$. In this way we see that $\mathrm{Ext}^{-i}(M/I^nM, \omega_B^\bullet)$ is essentially constant with value $\mathrm{Ext}^{-i}(M, \omega_B^\bullet)$. This finishes the proof. □

Here is a more general version of Lemma 52.20.1.

0EJ9 **Lemma 52.20.4.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is local and $\mathfrak{a} = \mathfrak{m}$ is the maximal ideal,*
(2) *A has a dualizing complex,*
(3) *$I = (f)$ is a principal ideal,*
(4) *$(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities.*

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. Recall that $Coh(U, I\mathcal{O}_U)$ is an abelian category, see Cohomology of Schemes, Lemma 30.23.2. Over affine opens of U the object $(\mathcal{F}_n)$ corresponds to a finite module over a Noetherian ring (Cohomology of Schemes, Lemma 30.23.1). Thus the kernels of the maps $f^N : (\mathcal{F}_n) \to (\mathcal{F}_n)$ stabilize for N large enough. By Lemmas 52.17.1 and 52.16.3 in order to prove the lemma we may replace $(\mathcal{F}_n)$ by the image of such a map. Thus we may assume f is injective on $(\mathcal{F}_n)$. After this replacement the equivalent conditions of Lemma 52.3.1 hold for the inverse system $(\mathcal{F}_n)$ on U. We will use this without further mention in the rest of the proof.

We will check hypotheses (a), (b), and (c) of Lemma 52.16.10. Hypothesis (b) holds by Cohomology, Lemma 20.36.2.

Pick an inverse system of modules $\{M_n\}$ as in Lemma 52.16.2. We may assume $H^0_\mathfrak{m}(M_n) = 0$ by replacing M_n by $M_n/H^0_\mathfrak{m}(M_n)$ if necessary. Then we obtain short exact sequences

$$0 \to M_n \to H^0(U, \mathcal{F}_n) \to H^1_\mathfrak{m}(M_n) \to 0$$

for all n. Let E be an injective hull of the residue field of A. By Lemma 52.20.3 and our current assumption (4) we can choose, an integer $m \geq 0$, finite A-modules N_1 and N_2 annihilated by f^c for some $c \geq 0$ and compatible systems of maps

$$H^i_{\mathfrak{m}}(M_n) \to \mathrm{Hom}_A(N_i, E), \quad i = 1, 2$$

for $n \geq m$ with the properties stated in the lemma.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the f-adic topology. Thus, given n, the module M/f^nM is a subquotient of $H^0(U, \mathcal{F}_N)$ for some $N \gg n$. Looking at the information obtained above we see that f^cM/f^nM is a finite A-module. Since f is a nonzerodivisor on M we conclude that $M/f^{n-c}M$ is a finite A-module. In this way we see that hypothesis (c) of Lemma 52.16.10 holds.

Next, we study the module

$$Ob = \lim H^1(U, \mathcal{F}_n) = \lim H^2_{\mathfrak{m}}(M_n)$$

For $n \geq m$ let K_n be the kernel of the map $H^2_{\mathfrak{m}}(M_n) \to \mathrm{Hom}_A(N_2, E)$. Set $K = \lim K_n$. We obtain an exact sequence

$$0 \to K \to Ob \to \mathrm{Hom}_A(N_2, E)$$

By the above the limit topology on $Ob = \lim H^2_{\mathfrak{m}}(M_n)$ is the f-adic topology. Since N_2 is annihilated by f^c we conclude the same is true for the limit topology on $K = \lim K_n$. Thus K/fK is a subquotient of K_n for $n \gg 1$. However, since $\{K_n\}$ is pro-isomorphic to an inverse system of finite length A-modules (by the conclusion of Lemma 52.20.3) we conclude that K/fK is a subquotient of a finite length A-module. It follows that K is a finite A-module, see Algebra, Lemma 10.96.12. (In fact, we even see that $\dim(\mathrm{Supp}(K)) = 1$ but we will not need this.)

Given $n \geq 1$ consider the boundary map

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow \lim_N H^1(U, f^n\mathcal{F}_N) \xrightarrow{f^{-n}} Ob$$

(the second map is an isomorphism) coming from the short exact sequences

$$0 \to f^n\mathcal{F}_N \to \mathcal{F}_N \to \mathcal{F}_n \to 0$$

For each n set

$$P_n = \mathrm{Im}(H^0(U, \mathcal{F}_{n+m}) \to H^0(U, \mathcal{F}_n))$$

where m is as above. Observe that $\{P_n\}$ is an inverse system and that the map $f : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps P_n into P_{n+1}. If $p \in P_n$, then $\delta_n(p) \in K \subset Ob$ because $\delta_n(p)$ maps to zero in $H^1(U, f^n\mathcal{F}_{n+m}) = H^2_{\mathfrak{m}}(M_m)$ and the composition of δ_n and $Ob \to \mathrm{Hom}_A(N_2, E)$ factors through $H^2_{\mathfrak{m}}(M_m)$ by our choice of m. Hence

$$\bigoplus\nolimits_{n \geq 0} \mathrm{Im}(P_n \to Ob)$$

is a finite graded $A[T]$-module where T acts via multiplication by f. Namely, it is a graded submodule of $K[T]$ and K is finite over A. Arguing as in the proof of Cohomology, Lemma 20.35.1[7] we find that the inverse system $\{P_n\}$ satisfies ML.

[7]Choose homogeneous generators of the form $\delta_{n_j}(p_j)$ for the displayed module. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $p \in P_n$ we can find $a_j \in A$ such that $p - \sum a_j f^{n-n_j} p_j$ is in the kernel of δ_n and hence in the image of $P_{n'}$ for all $n' \geq n$. Thus $\mathrm{Im}(P_n \to P_{n-k}) = \mathrm{Im}(P_{n'} \to P_{n-k})$ for all $n' \geq n$.

Since $\{P_n\}$ is pro-isomorphic to $\{H^0(U, \mathcal{F}_n)\}$ we conclude that $\{H^0(U, \mathcal{F}_n)\}$ has ML. Thus hypothesis (a) of Lemma 52.16.10 holds and the proof is complete. □

We can unwind condition of Lemma 52.20.4 as follows.

0EJA **Lemma 52.20.5.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,*
(2) *$cd(A, I) = 1$.*

Then $(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities if and only if for all $y \in U \cap Y$ with $\dim(\{y\}) = 1$ and every prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$, $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$ we have

$$depth((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) > 2$$

Proof. We will use Lemma 52.19.3 without further mention. In particular, we see the condition is necessary. Conversely, suppose the condition is true. Note that $\delta^Y_Z(y) = \dim(\overline{\{y\}})$ by Lemma 52.18.1. Let us write δ for this function. Let $y \in U \cap Y$. If $\delta(y) > 2$, then the inequality of Lemma 52.19.3 holds. Finally, suppose $\delta(y) = 2$. We have to show that

$$\operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) > 1$$

Choose a specialization $y \rightsquigarrow y'$ with $\delta(y') = 1$. Then there is a ring map $\mathcal{O}^\wedge_{X,y'} \to \mathcal{O}^\wedge_{X,y}$ which identifies the target with the completion of the localization of $\mathcal{O}^\wedge_{X,y'}$ at a prime $\mathfrak{q}$ with $\dim(\mathcal{O}^\wedge_{X,y'}/\mathfrak{q}) = 1$. Moreover, we then obtain

$$\mathcal{F}^\wedge_y = \mathcal{F}^\wedge_{y'} \otimes_{\mathcal{O}^\wedge_{X,y'}} \mathcal{O}^\wedge_{X,y}$$

Let $\mathfrak{p}' \subset \mathcal{O}^\wedge_{X,y'}$ be the image of $\mathfrak{p}$. By Local Cohomology, Lemma 51.11.3 we have

$$\begin{aligned}\operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) &= \operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \dim((\mathcal{O}^\wedge_{X,y}/\mathfrak{p})_{\mathfrak{p}'}) \\ &= \operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}') - 1\end{aligned}$$

the last equality because the specialization is immediate. Thus the lemma is prove by the assumed inequality for $y', \mathfrak{p}'$. □

0EJB **Lemma 52.20.6.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,*
(2) *A has a dualizing complex,*
(3) *$cd(A, I) = 1$,*
(4) *for $y \in U \cap Y$ the module $\mathcal{F}^\wedge_y$ is finite locally free outside $V(I\mathcal{O}^\wedge_{X,y})$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module, and*
(5) *one of the following is true*
   (a) *A_f is (S_2) and every irreducible component of X not contained in Y has dimension ≥ 4, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $depth(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then $(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities.

Proof. We will use the criterion of Lemma 52.20.5. Let $y \in U \cap Y$ with $\dim(\overline{\{y\}} = 1$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$. Condition (4) shows that $\operatorname{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) = \operatorname{depth}((\mathcal{O}^\wedge_{X,y})_\mathfrak{p})$. Thus we have to prove

$$\operatorname{depth}((\mathcal{O}^\wedge_{X,y})_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) > 2$$

Let $\mathfrak{p}_0 \subset A$ be the image of $\mathfrak{p}$. Let $\mathfrak{q} \subset A$ be the prime corresponding to y. By Local Cohomology, Lemma 51.11.3 we have

$$\begin{aligned}\text{depth}((\mathcal{O}^\wedge_{X,y})_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) &= \text{depth}(A_{\mathfrak{p}_0}) + \dim((A/\mathfrak{p}_0)_\mathfrak{q}) \\ &= \text{depth}(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1\end{aligned}$$

If (5)(a) holds, then we get that this is

$$\geq \min(2, \dim(A_{\mathfrak{p}_0})) + \dim(A/\mathfrak{p}_0) - 1$$

Note that in any case $\dim(A/\mathfrak{p}_0) \geq 2$. Hence if we get 2 for the minimum, then we are done. If not we get

$$\dim(A_{\mathfrak{p}_0}) + \dim(A/\mathfrak{p}_0) - 1 \geq 4 - 1$$

because every component of $\text{Spec}(A)$ passing through $\mathfrak{p}_0$ has dimension ≥ 4. If (5)(b) holds, then we win immediately. □

0EJC **Remark 52.20.7.** Let $(A, \mathfrak{m})$ be a Noetherian local ring which has a dualizing complex and is complete with respect to $f \in \mathfrak{m}$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, f\mathcal{O}_U)$ where U is the punctured spectrum of A. Set $Y = V(f) \subset X = \text{Spec}(A)$. If for $y \in U \cap V(f)$ closed in U, i.e., with $\dim(\overline{\{y\}}) = 1$, we assume the $\mathcal{O}^\wedge_{X,y}$-module $\mathcal{F}^\wedge_y$ satisfies the following two conditions

(1) $\mathcal{F}^\wedge_y[1/f]$ is (S_2) as a $\mathcal{O}^\wedge_{X,y}[1/f]$-module, and
(2) for $\mathfrak{p} \in \text{Ass}(\mathcal{F}^\wedge_y[1/f])$ we have $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) \geq 3$.

Then $(\mathcal{F}_n)$ is the completion of a coherent module on U. This follows from Lemmas 52.20.4 and 52.20.5.

52.21. Improving coherent formal modules

0EJD Let X be a Noetherian scheme. Let $Y \subset X$ be a closed subscheme with quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. In this section we construct maps $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ similar to the maps constructed in Local Cohomology, Section 51.15 for coherent modules. For a point $y \in Y$ we set

$$\mathcal{O}^\wedge_{X,y} = \lim \mathcal{O}_{X,y}/\mathcal{I}^n_y, \quad \mathcal{I}^\wedge_y = \lim \mathcal{I}_y/\mathcal{I}^n_y \quad \text{and} \quad \mathfrak{m}^\wedge_y = \lim \mathfrak{m}_y/\mathcal{I}^n_y$$

Then $\mathcal{O}^\wedge_{X,y}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}^\wedge_y$ complete with respect to $\mathcal{I}^\wedge_y = \mathcal{I}_y\mathcal{O}^\wedge_{X,y}$. We also set

$$\mathcal{F}^\wedge_y = \lim \mathcal{F}_{n,y}$$

Then $\mathcal{F}^\wedge_y$ is a finite module over $\mathcal{O}^\wedge_{X,y}$ with $\mathcal{F}^\wedge_y/(\mathcal{I}^\wedge_y)^n\mathcal{F}^\wedge_y = \mathcal{F}_{n,y}$ for all n, see Algebra, Lemmas 10.98.2 and 10.96.12.

0EJE **Lemma 52.21.1.** *In the situation above assume X locally has a dualizing complex. Let $T \subset Y$ be a subset stable under specialization. Assume for $y \in T$ and for a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have*

$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 0$$

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

(1) *for $y \in T$ we have $depth(\mathcal{F}'_{n,y}) \geq 1$,*
(2) *$(\mathcal{F}'_n)$ is isomorphic as a pro-system to an object $(\mathcal{G}_n)$ of $Coh(X, \mathcal{I})$,*
(3) *the induced morphism $(\mathcal{F}_n) \to (\mathcal{G}_n)$ of $Coh(X, \mathcal{I})$ is surjective with kernel annihilated by a power of $\mathcal{I}$.*

Proof. For every n we let $\mathcal{F}_n \to \mathcal{F}'_n$ be the surjection constructed in Local Cohomology, Lemma 51.15.1. Since this is the quotient of $\mathcal{F}_n$ by the subsheaf of sections supported on T we see that we get canonical maps $\mathcal{F}'_{n+1} \to \mathcal{F}'_n$ such that we obtain a map $(\mathcal{F}_n) \to (\mathcal{F}'_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules. Property (1) holds by construction.

To prove properties (2) and (3) we may assume that $X = \mathrm{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I_0-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 30.23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}'_n$ corresponds to the quotient $M'_n = M_n/H^0_T(M_n)$, see Local Cohomology, Lemma 51.15.1 and its proof.

Set $s = 0$ and $d = \mathrm{cd}(A, I)$. We claim that A, I, T, M, s, d satisfy assumptions (1), (3), (4), (6) of Situation 52.10.1. Namely, (1) and (3) are immediate from the above, (4) is the empty condition as $s = 0$, and (6) is the assumption we made in the statement of the lemma.

By Theorem 52.10.8 we see that $\{H^0_T(M_n)\}$ is essentially constant with value $H^0_T(M)$. Thus the limit of the short exact sequences $0 \to H^0_T(M_n) \to M_n \to M'_n \to 0$ is the short exact sequence

$$0 \to H^0_T(M) \to M \to \lim M'_n \to 0$$

Set $M' = \lim M'_n = M/H^0_T(M)$. This is a finite A-module. Let $(\mathcal{G}_n)$ be the object of $Coh(X, \mathcal{I})$ corresponding to M'. To finish the proof we have to show that the canonical map $\{M'/I^nM'\} \to \{M'_n\}$ is a pro-isomorphism. This is equivalent to saying that $\{H^0_T(M) + I^nM\} \to \{\mathrm{Ker}(M \to M'_n)\}$ is a pro-isomorphism. Dividing by I^nM it suffices to show that $\{H^0_T(M)/H^0_T(M) \cap I^nM\} \to \{H^0_T(M_n)\}$ is a pro-isomorphism. Since $H^0_T(M)$ is annihilated by a power of I, by Artin-Rees we see that $H^0_T(M) \cap I^nM = 0$ for all $n \gg 0$. Thus we have the desired pro-isomorphism as we've seen above that $\{H^0_T(M_n)\}$ is essentially constant with value $H^0_T(M)$. $\square$

0EJF **Lemma 52.21.2.** *In the situation above assume X locally has a dualizing complex. Let $T' \subset T \subset Y$ be subsets stable under specialization. Let $d \geq 0$ be an integer. Assume*

(a) *affine locally we have $X = \mathrm{Spec}(A_0)$ and $Y = V(I_0)$ and $cd(A_0, I_0) \leq d$,*
(b) *for $y \in T$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 0$$
(c) *for $y \in T'$ and for a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(\mathcal{I}^\wedge_y)$ and $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) \neq \{\mathfrak{m}^\wedge_y\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) \geq 1 \quad or \quad depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) > 1 + d$$
(d) *for $y \in T'$ and a nonmaximal prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\}$ we have*
$$depth_{(\mathcal{O}_{X,y})_\mathfrak{p}}((\mathcal{F}^\wedge_y)_\mathfrak{p}) > 1$$
(e) *if $y \rightsquigarrow y'$ is an immediate specialization and $y' \in T'$, then $y \in T$.*

Then there exists a canonical map $(\mathcal{F}_n) \to (\mathcal{F}''_n)$ of inverse systems of coherent $\mathcal{O}_X$-modules with the following properties

(1) *for $y \in T$ we have $depth(\mathcal{F}''_{n,y}) \geq 1$,*
(2) *for $y' \in T'$ we have $depth(\mathcal{F}''_{n,y'}) \geq 2$,*
(3) *$(\mathcal{F}''_n)$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $Coh(X, \mathcal{I})$,*
(4) *the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $Coh(X, \mathcal{I})$ has kernel and cokernel annihilated by a power of $\mathcal{I}$.*

Proof. As in Lemma 52.21.1 and its proof for every n we let $\mathcal{F}_n \to \mathcal{F}'_n$ be the surjection constructed in Local Cohomology, Lemma 51.15.1 using T. Next, we let $\mathcal{F}'_n \to \mathcal{F}''_n$ be the injection constructed in Local Cohomology, Lemma 51.15.5 and its proof. The constructions show that we get canonical maps $\mathcal{F}''_{n+1} \to \mathcal{F}''_n$ such that we obtain maps

$$(\mathcal{F}_n) \longrightarrow (\mathcal{F}'_n) \longrightarrow (\mathcal{F}''_n)$$

of inverse systems of coherent $\mathcal{O}_X$-modules. Properties (1) and (2) hold by construction.

To prove properties (3) and (4) we may assume that $X = \mathrm{Spec}(A_0)$ is affine and A_0 has a dualizing complex. Let $I_0 \subset A_0$ be the ideal corresponding to Y. Let A, I be the I_0-adic completions of A_0, I_0. For later use we observe that A has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Let M be the finite A-module corresponding to $(\mathcal{F}_n)$, see Cohomology of Schemes, Lemma 30.23.1. Then $\mathcal{F}_n$ corresponds to $M_n = M/I^nM$. Recall that $\mathcal{F}'_n$ corresponds to the quotient $M'_n = M_n/H^0_T(M_n)$. Also, recall that $M' = \lim M'_n$ is the quotient of M by $H^0_T(M)$, that $H^0_T(M)$ is I-power torsion, and that $\{M'_n\}$ and $\{M'/I^nM'\}$ are isomorphic as pro-systems. Finally, we see that $\mathcal{F}''_n$ corresponds to an extension

$$0 \to M'_n \to M''_n \to H^1_{T'}(M'_n) \to 0$$

see proof of Local Cohomology, Lemma 51.15.5.

Set $s = 1$. We claim that A, I, T', M', s, d satisfy assumptions (1), (3), (4), (6) of Situation 52.10.1. Namely, (1) and (3) are immediate, (4) is implied by (c), and (6) follows from (d). We omit the proofs of (c) $\Rightarrow$ (4) and (d) $\Rightarrow$ (6) .

By Theorem 52.10.8 we see that $\{H^1_{T'}(M'/I^nM')\}$ is essentially constant with value $H^1_{T'}(M')$. We deduce $\{H^1_{T'}(M'_n)\}$ is essentially constant with value $H^1_{T'}(M')$. Thus the limit of the short exact sequences displayed above is the short exact sequence

$$0 \to M' \to \lim M''_n \to H^1_{T'}(M') \to 0$$

Set $M'' = \lim M''_n$. It follows from Local Cohomology, Proposition 51.11.1 that $H^1_{T'}(M')$ and hence M'' are finite A-modules (in verifying the conditions use that M' has depth at least 1 at primes in $T - T'$). Let $(\mathcal{H}_n)$ be the object of $Coh(X, \mathcal{I})$ corresponding to the finite A-module M''. To finish the proof we have to show that the canonical map $\{M''/I^nM''\} \to \{M''_n\}$ is a pro-isomorphism. Since we already know that $\{M'/I^nM'\}$ is pro-isomorphic to $\{M'_n\}$ the reader verifies (omitted) this is equivalent to asking $\{H^1_{T'}(M')/I^nH^1_{T'}(M')\} \to \{H^1_{T'}(M'_n)\}$ to be a pro-isomorphism. Since $H^1_{T'}(M')/I^nH^1_{T'}(M') = H^1_{T'}(M')$ for n large enough, this follows because $\{H^1_{T'}(M'_n)\}$ is essentially constant with value $H^1_{T'}(M')$ as seen above. □

0EJG **Lemma 52.21.3.** *In Situation 52.16.1 assume that A has a dualizing complex. Let $d \geq cd(A, I)$. Let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume $(\mathcal{F}_n)$ satisfies the $(2, 2 + d)$-inequalities, see Definition 52.19.1. Then there exists a canonical*

map $(\mathcal{F}_n) \to (\mathcal{F}''_n)$ of inverse systems of coherent $\mathcal{O}_U$-modules with the following properties

(1) *$\text{depth}(\mathcal{F}''_{n,y}) + \delta_Z^Y(y) \geq 3$ for all $y \in U \cap Y$,*
(2) *$(\mathcal{F}''_n)$ is isomorphic as a pro-system to an object $(\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$,*
(3) *the induced morphism $(\mathcal{F}_n) \to (\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$ has kernel and cokernel annihilated by a power of I,*
(4) *the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n.*

Proof. The existence and properties (1), (2), (3) follow from Lemma 52.21.2 applied to U, $U \cap Y$, $T = \{y \in U \cap Y : \delta_Z^Y(y) \leq 2\}$, $T' = \{y \in U \cap Y : \delta_Z^Y(y) \leq 1\}$, and $(\mathcal{F}_n)$. The finiteness of the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ follows from Local Cohomology, Lemma 51.12.1 and the elementary properties of the function $\delta_Z^Y(-)$ proved in Lemma 52.18.1. □

52.22. Algebraization of coherent formal modules, V

0EJH In this section we prove our most general results on algebraization of coherent formal modules. We first prove it in case the ideal has cohomological dimension 1. Then we apply this to a blowup to prove a more general result.

0EJI **Lemma 52.22.1.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex and $cd(A, I) = 1$,*
(2) *$(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}''_n)$ of coherent $\mathcal{O}_U$-modules such that $\text{depth}(\mathcal{F}''_{n,y}) + \delta_Z^Y(y) \geq 3$ for all $y \in U \cap Y$.*

Then $(\mathcal{F}_n)$ extends canonically to X, see Definition 52.16.7.

Proof. We will check hypotheses (a), (b), and (c) of Lemma 52.16.10. Before we start, let us point out that the modules $H^0(U, \mathcal{F}''_n)$ and $H^1(U, \mathcal{F}''_n)$ are finite A-modules for all n by Local Cohomology, Lemma 51.12.1.

Observe that for each $p \geq 0$ the limit topology on $\lim H^p(U, \mathcal{F}_n)$ is the I-adic topology by Lemma 52.4.5. In particular, hypothesis (b) holds.

We know that $M = \lim H^0(U, \mathcal{F}_n)$ is an A-module whose limit topology is the I-adic topology. Thus, given n, the module M/I^nM is a subquotient of $H^0(U, \mathcal{F}_N)$ for some $N \gg n$. Since the inverse system $\{H^0(U, \mathcal{F}_N)\}$ is pro-isomorphic to an inverse system of finite A-modules, namely $\{H^0(U, \mathcal{F}''_N)\}$, we conclude that M/I^nM is finite. It follows that M is finite, see Algebra, Lemma 10.96.12. In particular hypothesis (c) holds.

For each $n \geq 0$ let us write $Ob_n = \lim_N H^1(U, I^n\mathcal{F}_N)$. A special case is $Ob = Ob_0 = \lim_N H^1(U, \mathcal{F}_N)$. Arguing exactly as in the previous paragraph we find that Ob is a finite A-module. (In fact, we also know that Ob/IOb is annihilated by a power of $\mathfrak{a}$, but it seems somewhat difficult to use this.)

We set $\mathcal{F} = \lim \mathcal{F}_n$, we pick generators $f_1, \ldots, f_r$ of I, we pick $c \geq 1$, and we choose $\Phi_{\mathcal{F}}$ as in Lemma 52.4.4. We will use the results of Lemma 52.2.1 without further mention. In particular, for each $n \geq 1$ there are maps

$$\delta_n : H^0(U, \mathcal{F}_n) \longrightarrow H^1(U, I^n\mathcal{F}) \longrightarrow Ob_n$$

The first comes from the short exact sequence $0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}_n \to 0$ and the second from $I^n\mathcal{F} = \lim I^n\mathcal{F}_N$. We will later use that if $\delta_n(s) = 0$ for $s \in H^0(U, \mathcal{F}_n)$

then we can for each $n' \geq n$ find $s' \in H^0(U, \mathcal{F}_{n'})$ mapping to s. Observe that there are commutative diagrams

$$\begin{array}{ccccc} H^0(U, \mathcal{F}_{nc}) & \longrightarrow & H^1(U, I^{nc}\mathcal{F}) & & \\ \downarrow & & \downarrow & \overset{\Phi_{\mathcal{F}}}{\searrow} & \\ & & & & \bigoplus_{e_1+\ldots+e_r=n} H^1(U, \mathcal{F}) \cdot T_1^{e_1} \ldots T_r^{e_r} \\ & & & \swarrow & \\ H^0(U, \mathcal{F}_n) & \longrightarrow & H^1(U, I^n\mathcal{F}) & & \end{array}$$

We conclude that the obstruction map $H^0(U, \mathcal{F}_n) \to Ob_n$ sends the image of $H^0(U, \mathcal{F}_{nc}) \to H^0(U, \mathcal{F}_n)$ into the submodule

$$Ob'_n = \mathrm{Im}\left(\bigoplus\nolimits_{e_1+\ldots+e_r=n} Ob \cdot T_1^{e_1} \ldots T_r^{e_r} \to Ob_n\right)$$

where on the summand $Ob \cdot T_1^{e_1} \ldots T_r^{e_r}$ we use the map on cohomology coming from the reductions modulo powers of I of the multiplication map $f_1^{e_1} \ldots f_r^{e_r} : \mathcal{F} \to I^n\mathcal{F}$. By construction

$$\bigoplus\nolimits_{n \geq 0} Ob'_n$$

is a finite graded module over the Rees algebra $\bigoplus_{n \geq 0} I^n$. For each n we set

$$M_n = \{s \in H^0(U, \mathcal{F}_n) \mid \delta_n(s) \in Ob'_n\}$$

Observe that $\{M_n\}$ is an inverse system and that $f_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$ on global sections maps M_n into M_{n+1}. By exactly the same argument as in the proof of Cohomology, Lemma 20.35.1 we find that $\{M_n\}$ is ML. Namely, because the Rees algebra is Noetherian we can choose a finite number of homogeneous generators of the form $\delta_{n_j}(z_j)$ with $z_j \in M_{n_j}$ for the graded submodule $\bigoplus_{n \geq 0} \mathrm{Im}(M_n \to Ob'_n)$. Then if $k = \max(n_j)$ we find that for $n \geq k$ and any $z \in M_n$ we can find $a_j \in I^{n-n_j}$ such that $z - \sum a_j z_j$ is in the kernel of δ_n and hence in the image of $M_{n'}$ for all $n' \geq n$ (because the vanishing of δ_n means that we can lift $z - \sum a_j z_j$ to an element $z' \in H^0(U, \mathcal{F}_{n'c})$ for all $n' \geq n$ and then the image of z' in $H^0(U, \mathcal{F}_{n'})$ is in $M_{n'}$ by what we proved above). Thus $\mathrm{Im}(M_n \to M_{n-k}) = \mathrm{Im}(M_{n'} \to M_{n-k})$ for all $n' \geq n$.

Choose n. By the Mittag-Leffler property of $\{M_n\}$ we just established we can find an $n' \geq n$ such that the image of $M_{n'} \to M_n$ is the same as the image of $M' \to M_n$. By the above we see that the image of $M' \to M_n$ contains the image of $H^0(U, \mathcal{F}_{n'c}) \to H^0(U, \mathcal{F}_n)$. Thus we see that $\{M_n\}$ and $\{H^0(U, \mathcal{F}_n)\}$ are pro-isomorphic. Therefore $\{H^0(U, \mathcal{F}_n)\}$ has ML and we finally conclude that hypothesis (a) holds. This concludes the proof. □

0EJJ **Proposition 52.22.2** (Algebraization in cohomological dimension 1)**.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex and $cd(A, I) = 1$,*
(2) *$(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities, see Definition 52.19.1.*

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

The local case of this result is [**Ray75**, IV Corollaire 2.9].

Proof. By Lemma 52.17.1 we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $Coh(U, I\mathcal{O}_U)$ found in Lemma 52.21.3. Thus we may assume that $(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}''_n)$ with the properties mentioned in Lemma 52.21.3. In Lemma 52.22.1 we proved that $(\mathcal{F}_n)$ canonically extends to X. The final statement follows from Lemma 52.16.8. □

0EJK **Lemma 52.22.3.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex,*
(2) *all fibres of the blowing up $b : X' \to X$ of I have dimension $\leq d-1$,*
(3) *one of the following is true*
   (a) *$(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities (Definition 52.19.1), or*
   (b) *for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$ we have*

$$depth((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X.

Proof. Let $Y' \subset X'$ be the exceptional divisor. Let $Z' \subset Y'$ be the inverse image of $Z \subset Y$. Then $U' = X' \setminus Z'$ is the inverse image of U. With $\delta^{Y'}_{Z'}$ as in (52.18.0.1) we set

$$T' = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1\} \subset T = \{y' \in Y' \mid \delta^{Y'}_{Z'}(y') = 1 \text{ or } 2\}$$

By Lemma 52.18.1 parts (1) and (2) these are specialization stable subsets of $U' \cap Y' = Y' \setminus Z'$. Consider the object $(b|^*_{U'}\mathcal{F}_n)$ of $Coh(U', I\mathcal{O}_{U'})$, see Cohomology of Schemes, Lemma 30.23.9. For $y' \in U' \cap Y'$ let us denote

$$\mathcal{F}^\wedge_{y'} = \lim (b|^*_{U'}\mathcal{F}_n)_{y'}$$

the "stalk" of this pullback at y'. We claim that conditions (a), (b), (c), (d), and (e) of Lemma 52.21.2 hold for the object $(b|^*_{U'}\mathcal{F}_n)$ on U' with d replaced by 1 and the subsets $T' \subset T \subset U' \cap Y'$. Condition (a) holds because Y' is an effective Cartier divisor and hence locally cut out by 1 equation. Condition (e) holds by Lemma 52.18.1 part (7). To prove (b), (c), and (d) we need some preparation.

Let $y' \in U' \cap Y'$ and let $\mathfrak{p}' \subset \mathcal{O}^\wedge_{X',y'}$ be a prime ideal not contained in $V(I\mathcal{O}^\wedge_{X',y'})$. Denote $y = b(y') \in U \cap Y$. Choose $f \in I$ such that y' is contained in the spectrum of the affine blowup algebra $A[\frac{I}{f}]$, see Divisors, Lemma 31.32.2. For any A-algebra B denote $B' = B[\frac{IB}{f}]$ the corresponding affine blowup algebra. Denote I-adic completion by $^\wedge$. By our choice of f we get a ring map $(\mathcal{O}^\wedge_{X,y})' \to \mathcal{O}^\wedge_{X',y'}$. If we let $\mathfrak{q}' \subset (\mathcal{O}^\wedge_{X,y})'$ be the inverse image of $\mathfrak{m}^\wedge_{y'}$, then we see that $((\mathcal{O}^\wedge_{X,y})'_{\mathfrak{q}'})^\wedge = \mathcal{O}^\wedge_{X',y'}$. Let $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ be the corresponding prime. At this point we have a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}^\wedge_{X,y} & \longrightarrow & (\mathcal{O}^\wedge_{X,y})' & \longrightarrow & (\mathcal{O}^\wedge_{X,y})'_{\mathfrak{q}'} & \xrightarrow{\beta} & \mathcal{O}^\wedge_{X',y'} \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow \\
\mathcal{O}^\wedge_{X,y}/\mathfrak{p} & \longrightarrow & (\mathcal{O}^\wedge_{X,y}/\mathfrak{p})' & \longrightarrow & (\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'} & \xrightarrow{\gamma} & ((\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'})^\wedge \\
 & & & & & & \downarrow \\
 & & & & & & \mathcal{O}^\wedge_{X',y'}/\mathfrak{p}'
\end{array}$$

whose vertical arrows are surjective. By More on Algebra, Lemma 15.43.1 and the dimension formula (Algebra, Lemma 10.113.1) we have

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'})^\wedge) = \dim((\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'}) = \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) - \operatorname{trdeg}(\kappa(y')/\kappa(y))$$

Tracing through the definitions of pullbacks, stalks, localizations, and completions we find

$$(\mathcal{F}^\wedge_y)_{\mathfrak{p}} \otimes_{(\mathcal{O}^\wedge_{X,y})_{\mathfrak{p}}} (\mathcal{O}^\wedge_{X',y'})_{\mathfrak{p}'} = (\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}$$

Details omitted. The ring maps β and γ in the diagram are flat with Gorenstein (hence Cohen-Macaulay) fibres, as these are completions of rings having a dualizing complex. See Dualizing Complexes, Lemmas 47.23.1 and 47.23.2 and the discussion in More on Algebra, Section 15.51. Observe that $(\mathcal{O}^\wedge_{X,y})_{\mathfrak{p}} = (\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak{p}}}$ where $\tilde{\mathfrak{p}}$ is the kernel of α in the diagram. On the other hand, $(\mathcal{O}^\wedge_{X,y})'_{\tilde{\mathfrak{p}}} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak{p}'}$ is flat with CM fibres by the above. Whence $(\mathcal{O}^\wedge_{X,y})_{\mathfrak{p}} \to (\mathcal{O}^\wedge_{X',y'})_{\mathfrak{p}'}$ is flat with CM fibres. Using Algebra, Lemma 10.163.1 we see that

$$\operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) = \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \dim(F_{\mathfrak{r}})$$

where F is the generic formal fibre of $(\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'}$ and $\mathfrak{r}$ is the prime corresponding to $\mathfrak{p}'$. Since $(\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'}$ is a universally catenary local domain, its I-adic completion is equidimensional and (universally) catenary by Ratliff's theorem (More on Algebra, Proposition 15.110.5). It then follows that

$$\dim(((\mathcal{O}^\wedge_{X,y}/\mathfrak{p})'_{\mathfrak{q}'})^\wedge) = \dim(F_{\mathfrak{r}}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}')$$

Combined with Lemma 52.18.2 we get

0EJL (52.22.3.1)
$$\begin{aligned}
&\operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \delta^{Y'}_{Z'}(y') \\
&= \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \dim(F_{\mathfrak{r}}) + \delta^{Y'}_{Z'}(y') \\
&\geq \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) + \dim(F_{\mathfrak{r}}) + \operatorname{trdeg}(\kappa(y')/\kappa(y)) - (d-1) \\
&= \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) - (d-1) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) - \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}')
\end{aligned}$$

Please keep in mind that $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) \geq \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}')$. Rewriting this we get

0EJM (52.22.3.2)
$$\begin{aligned}
&\operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}') + \delta^{Y'}_{Z'}(y') \\
&\geq \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) - (d-1)
\end{aligned}$$

This inequality will allow us to check the remaining conditions.

Conditions (b) and (d) of Lemma 52.21.2. Assume $V(\mathfrak{p}') \cap V(I\mathcal{O}^\wedge_{X',y'}) = \{\mathfrak{m}^\wedge_{y'}\}$. This implies that $\dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}') = 1$ because Z' is an effective Cartier divisor. The combination of (b) and (d) is equivalent with

$$\operatorname{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \delta^{Y'}_{Z'}(y') > 2$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we immediately conclude this is true by applying (52.22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then we see that in any case

$$\operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) \geq d+1 \quad \text{or} \quad \operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d+2$$

Looking at (52.22.3.1) and (52.22.3.2) above this gives what we want except possibly if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = 1$. However, if $\dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = 1$, then we have $V(\mathfrak{p}) \cap V(I\mathcal{O}^\wedge_{X,y}) = \{\mathfrak{m}^\wedge_y\}$ and we see that actually

$$\operatorname{depth}((\mathcal{F}^\wedge_y)_{\mathfrak{p}}) + \delta^Y_Z(y) > d+1$$

as $(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities and we conclude again.

Condition (c) of Lemma 52.21.2. Assume $V(\mathfrak{p}') \cap V(I\mathcal{O}^\wedge_{X',y'}) \neq \{\mathfrak{m}^\wedge_{y'}\}$. Then condition (c) is equivalent to

$$\text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \delta^{Y'}_{Z'}(y') \geq 2 \quad \text{or} \quad \text{depth}((\mathcal{F}^\wedge_{y'})_{\mathfrak{p}'}) + \dim(\mathcal{O}^\wedge_{X',y'}/\mathfrak{p}') + \delta^{Y'}_{Z'}(y') > 3$$

If $(\mathcal{F}_n)$ satisfies the inequalities in (3)(b) then we see the second of the two displayed inequalities holds true by applying (52.22.3.2). If $(\mathcal{F}_n)$ satisfies (3)(a), i.e., the $(d+1, d+2)$-inequalities, then this follows immediately from (52.22.3.1) and (52.22.3.2). This finishes the proof of our claim.

Choose $(b|^*_{U'}\mathcal{F}_n) \to (\mathcal{F}''_n)$ and $(\mathcal{H}_n)$ in $Coh(U', I\mathcal{O}_{U'})$ as in Lemma 52.21.2. For any affine open $W \subset X'$ observe that $\delta^{W\cap Y'}_{W\cap Z'}(y') \geq \delta^{Y'}_{Z'}(y')$ by Lemma 52.18.1 part (6). Hence we see that $(\mathcal{H}_n|_W)$ satisfies the assumptions of Lemma 52.22.1. Thus $(\mathcal{H}_n|_W)$ extends canonically to W. Let $(\mathcal{G}_{W,n})$ in $Coh(W, I\mathcal{O}_W)$ be the canonical extension as in Lemma 52.16.8. By Lemma 52.16.9 we see that for $W' \subset W$ there is a unique isomorphism

$$(\mathcal{G}_{W,n}|_{W'}) \longrightarrow (\mathcal{G}_{W',n})$$

compatible with the given isomorphisms $(\mathcal{G}_{W,n}|_{W\cap U}) \cong (\mathcal{H}_n|_{W\cap U})$. We conclude that there exists an object $(\mathcal{G}_n)$ of $Coh(X', I\mathcal{O}_{X'})$ whose restriction to U is isomorphic to $(\mathcal{H}_n)$.

If A is I-adically complete we can finish the proof as follows. By Grothendieck's existence theorem (Cohomology of Schemes, Lemma 30.24.3) we see that $(\mathcal{G}_n)$ is the completion of a coherent $\mathcal{O}_{X'}$-module. Then by Cohomology of Schemes, Lemma 30.23.6 we see that $(b|^*_{U'}\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_{U'}$-module $\mathcal{F}'$. By Cohomology of Schemes, Lemma 30.25.3 we see that there is a map

$$(\mathcal{F}_n) \longrightarrow ((b|_{U'})_*\mathcal{F}')^\wedge$$

whose kernel and cokernel is annihilated by a power of I. Then finally, we win by applying Lemma 52.17.1.

If A is not complete, then, before starting the proof, we may replace A by its completion, see Lemma 52.16.6. After completion the assumptions still hold: this is immediate for condition (3), follows from Dualizing Complexes, Lemma 47.22.4 for condition (1), and from Divisors, Lemma 31.32.3 for condition (2). Thus the complete case implies the general case. □

0EJN **Proposition 52.22.4** (Algebraization for ideals with few generators)**.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A has a dualizing complex,*
(2) *$V(I) = V(f_1, \ldots, f_d)$ for some $d \geq 1$ and $f_1, \ldots, f_d \in A$,*
(3) *one of the following is true*
   (a) *$(\mathcal{F}_n)$ satisfies the $(d+1, d+2)$-inequalities (Definition 52.19.1), or*
   (b) *for $y \in U \cap Y$ and a prime $\mathfrak{p} \subset \mathcal{O}^\wedge_{X,y}$ with $\mathfrak{p} \notin V(I\mathcal{O}^\wedge_{X,y})$ we have*

$$\textit{depth}((\mathcal{F}^\wedge_y)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) + \delta^Y_Z(y) > d + 2$$

Then $(\mathcal{F}_n)$ extends to X. In particular, if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We may assume $I = (f_1, \ldots, f_d)$, see Cohomology of Schemes, Lemma 30.23.11. Then we see that all fibres of the blowup of X in I have dimension at most $d - 1$. Thus we get the extension from Lemma 52.22.3. The final statement follows from Lemma 52.16.3. □

Please compare the next lemma with Remarks 52.16.12, 52.20.2, 52.20.7, and 52.23.2.

0EJP **Lemma 52.22.5.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is a local ring which has a dualizing complex,*
(2) *all irreducible components of X have the same dimension,*
(3) *the scheme $X \setminus Y$ is Cohen-Macaulay,*
(4) *I is generated by d elements,*
(5) *$\dim(X) - \dim(Z) > d + 2$, and*
(6) *for $y \in U \cap Y$ the module $\mathcal{F}_y^\wedge$ is finite locally free outside $V(I\mathcal{O}_{X,y}^\wedge)$, for example if $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_U/I^n\mathcal{O}_U$-module.*

Then $(\mathcal{F}_n)$ extends to X. In particular if A is I-adically complete, then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_U$-module.

Proof. We will show that the hypotheses (1), (2), (3)(b) of Proposition 52.22.4 are satisfied. This is clear for (1) and (2).

Let $y \in U \cap Y$ and let $\mathfrak{p}$ be a prime $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ with $\mathfrak{p} \notin V(I\mathcal{O}_{X,y}^\wedge)$. The last condition shows that $\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) = \text{depth}((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p})$. Since $X \setminus Y$ is Cohen-Macaulay we see that $(\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}$ is Cohen-Macaulay. Thus we see that

$$\begin{aligned}
&\text{depth}((\mathcal{F}_y^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\
&= \dim((\mathcal{O}_{X,y}^\wedge)_\mathfrak{p}) + \dim(\mathcal{O}_{X,y}^\wedge/\mathfrak{p}) + \delta_Z^Y(y) \\
&= \dim(\mathcal{O}_{X,y}^\wedge) + \delta_Z^Y(y)
\end{aligned}$$

The final equality because $\mathcal{O}_{X,y}$ is equidimensional by the second condition. Let $\delta(y) = \dim(\overline{\{y\}})$. This is a dimension function as A is a catenary local ring. By Lemma 52.18.1 we have $\delta_Z^Y(y) \geq \delta(y) - \dim(Z)$. Since X is equidimensional we get

$$\dim(\mathcal{O}_{X,y}^\wedge) + \delta_Z^Y(y) \geq \dim(\mathcal{O}_{X,y}^\wedge) + \delta(y) - \dim(Z) = \dim(X) - \dim(Z)$$

Thus we get the desired inequality and we win. □

0EJQ **Remark 52.22.6.** We are unable to prove or disprove the analogue of Proposition 52.22.4 where the assumption that I has d generators is replaced with the assumption $\text{cd}(A, I) \leq d$. If you know a proof or have a counter example, please email stacks.project@gmail.com. Another obvious question is to what extend the conditions in Proposition 52.22.4 are necessary.

52.23. Algebraization of coherent formal modules, VI

0EJR In this section we add a few more easier to prove cases.

0EJS **Proposition 52.23.1.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *there exist $f_1, \ldots, f_d \in I$ such that for $y \in U \cap Y$ the ideal $I\mathcal{O}_{X,y}$ is generated by $f_1, \ldots, f_d$ and $f_1, \ldots, f_d$ form a $\mathcal{F}_y^\wedge$-regular sequence,*

(2) $H^0(U, \mathcal{F}_1)$ *and* $H^1(U, \mathcal{F}_1)$ *are finite* A*-modules.*

Then $(\mathcal{F}_n)$ *extends canonically to* X*. In particular, if* A *is complete, then* $(\mathcal{F}_n)$ *is the completion of a coherent* $\mathcal{O}_U$*-module.*

Proof. We will prove this by verifying hypotheses (a), (b), and (c) of Lemma 52.16.10. For every n we have a short exact sequence

$$0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$$

Since $f_1, \ldots, f_d$ forms a regular sequence (and hence quasi-regular, see Algebra, Lemma 10.69.2) on each of the "stalks" $\mathcal{F}_y^\wedge$ and since we have $I\mathcal{F}_n = (f_1, \ldots, f_d)\mathcal{F}_n$ for all n, we find that

$$I^n\mathcal{F}_{n+1} = \bigoplus\nolimits_{e_1+\ldots+e_d=n} \mathcal{F}_1 \cdot f_1^{e_1} \ldots f_d^{e_d}$$

by checking on stalks. Using the assumption of finiteness of $H^0(U, \mathcal{F}_1)$ and induction, we first conclude that $M_n = H^0(U, \mathcal{F}_n)$ is a finite A-module for all n. In this way we see that condition (c) of Lemma 52.16.10 holds. We also see that

$$\bigoplus\nolimits_{n\geq 0} H^1(U, I^n\mathcal{F}_{n+1})$$

is a finite graded $R = \bigoplus I^n/I^{n+1}$-module. By Cohomology, Lemma 20.35.1 we conclude that condition (a) of Lemma 52.16.10 is satisfied. Finally, condition (b) of Lemma 52.16.10 is satisfied because $\bigoplus H^0(U, I^n\mathcal{F}_{n+1})$ is a finite graded R-module and we can apply Cohomology, Lemma 20.35.3. □

0EJT **Remark 52.23.2.** In the situation of Proposition 52.23.1 if we assume A has a dualizing complex, then the condition that $H^0(U, \mathcal{F}_1)$ and $H^1(U, \mathcal{F}_1)$ are finite is equivalent to

$$\text{depth}(\mathcal{F}_{1,y}) + \dim(\mathcal{O}_{\overline{\{y\}},z}) > 2$$

for all $y \in U \cap Y$ and $z \in Z \cap \overline{\{y\}}$. See Local Cohomology, Lemma 51.12.1. This holds for example if $\mathcal{F}_1$ is a finite locally free $\mathcal{O}_{U\cap Y}$-module, Y is (S_2), and $\text{codim}(Z', Y') \geq 3$ for every pair of irreducible components Y' of Y, Z' of Z with $Z' \subset Y'$.

0EJU **Proposition 52.23.3.** *In Situation 52.16.1 let* $(\mathcal{F}_n)$ *be an object of* $Coh(U, I\mathcal{O}_U)$*. Assume there is Noetherian local ring* $(R, \mathfrak{m})$ *and a ring map* $R \to A$ *such that*

(1) $I = \mathfrak{m}A$*,*
(2) *for* $y \in U \cap Y$ *the stalk* $\mathcal{F}_y^\wedge$ *is* R*-flat,*
(3) $H^0(U, \mathcal{F}_1)$ *and* $H^1(U, \mathcal{F}_1)$ *are finite* A*-modules.*

Then $(\mathcal{F}_n)$ *extends canonically to* X*. In particular, if* A *is complete, then* $(\mathcal{F}_n)$ *is the completion of a coherent* $\mathcal{O}_U$*-module.*

Proof. The proof is exactly the same as the proof of Proposition 52.23.1. Namely, if $\kappa = R/\mathfrak{m}$ then for $n \geq 0$ there is an isomorphism

$$I^n\mathcal{F}_{n+1} \cong \mathcal{F}_1 \otimes_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

and the right hand side is a finite direct sum of copies of $\mathcal{F}_1$. This can be checked by looking at stalks. Everything else is exactly the same. □

0EJV **Remark 52.23.4.** Proposition 52.23.3 is a local version of [**Bar10**, Theorem 2.10 (i)]. It is straightforward to deduce the global results from the local one; we will sketch the argument. Namely, suppose $(R, \mathfrak{m})$ is a complete Noetherian local ring and $X \to \mathrm{Spec}(R)$ is a proper morphism. For $n \geq 1$ set $X_n = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}^n)$. Let $Z \subset X_1$ be a closed subset of the special fibre. Set $U = X \setminus Z$ and denote $j : U \to X$ the inclusion morphism. Suppose given an object

$$(\mathcal{F}_n) \text{ of } Coh(U, \mathfrak{m}\mathcal{O}_U)$$

which is flat over R in the sense that $\mathcal{F}_n$ is flat over $R/\mathfrak{m}^n$ for all n. Assume that $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ are coherent modules. Then affine locally on X we get a canonical extension of $(\mathcal{F}_n)$ by Proposition 52.23.3 and formation of this extension commutes with localization (by Lemma 52.16.11). Thus we get a canonical global object $(\mathcal{G}_n)$ of $Coh(X, \mathfrak{m}\mathcal{O}_X)$ whose restriction of U is $(\mathcal{F}_n)$. By Grothendieck's existence theorem (Cohomology of Schemes, Proposition 30.25.4) we see there exists a coherent $\mathcal{O}_X$-module $\mathcal{G}$ whose completion is $(\mathcal{G}_n)$. In this way we see that $(\mathcal{F}_n)$ is algebraizable, i.e., it is the completion of a coherent $\mathcal{O}_U$-module.

We add that the coherence of $j_*\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1$ is a condition on the special fibre. Namely, if we denote $j_1 : U_1 \to X_1$ the special fibre of $j : U \to X$, then we can think of $\mathcal{F}_1$ as a coherent sheaf on U_1 and we have $j_*\mathcal{F}_1 = j_{1,*}\mathcal{F}_1$ and $R^1j_*\mathcal{F}_1 = R^1j_{1,*}\mathcal{F}_1$. Hence for example if X_1 is (S_2) and irreducible, we have $\dim(X_1) - \dim(Z) \geq 3$, and $\mathcal{F}_1$ is a locally free $\mathcal{O}_{U_1}$-module, then $j_{1,*}\mathcal{F}_1$ and $R^1j_{1,*}\mathcal{F}_1$ are coherent modules.

52.24. Application to the completion functor

0EKX In this section we just combine some already obtained results in order to conveniently reference them. There are many (stronger) results we could state here.

0EKY **Lemma 52.24.1.** *In Situation 52.16.1 assume*

(1) *A has a dualizing complex and is I-adically complete,*
(2) *$I = (f)$ generated by a single element,*
(3) *A is local with maximal ideal $\mathfrak{a} = \mathfrak{m}$,*
(4) *one of the following is true*
   (a) *A_f is (S_2) and for $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ minimal we have $\dim(A/\mathfrak{p}) \geq 4$, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $depth(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

Then with $U_0 = U \cap V(f)$ the completion functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} Coh(\mathcal{O}_{U'}) \longrightarrow Coh(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. It follows from Lemma 52.15.7 that the functor is fully faithful (details omitted). Let us prove essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $Coh(U, f\mathcal{O}_U)$. By either Lemma 52.20.4 or Proposition 52.22.2 there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ such that $(\mathcal{F}_n)$ is the completion of $\mathcal{F}$. Namely, for the application of either result the only thing to check is that $(\mathcal{F}_n)$ satisfies the $(2, 3)$-inequalities. This is done in Lemma 52.20.6. If $y \in U_0$, then the f-adic completion of the stalk $\mathcal{F}_y$ is isomorphic to a finite free module over the f-adic completion of $\mathcal{O}_{U,y}$. Hence $\mathcal{F}$ is finite locally free in an open neighbourhood U' of U_0. This finishes the proof. □

0EKZ **Lemma 52.24.2.** *In Situation 52.16.1 assume*

(1) $I = (f)$ *is principal,*
(2) A *is* f*-adically complete,*
(3) f *is a nonzerodivisor,*
(4) $H^1_{\mathfrak{a}}(A/fA)$ *and* $H^2_{\mathfrak{a}}(A/fA)$ *are finite* A*-modules.*

Then with $U_0 = U \cap V(f)$ *the completion functor*

$$\operatorname{colim}_{U_0 \subset U' \subset U \text{ open}} Coh(\mathcal{O}_{U'}) \longrightarrow Coh(U, f\mathcal{O}_U)$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. The functor is fully faithful by Lemma 52.15.8. Essential surjectivity follows from Lemma 52.16.11. □

52.25. Coherent triples

0F22 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $X = \operatorname{Spec}(A)$, $X_0 = \operatorname{Spec}(A/fA)$, $U = X \setminus V(\mathfrak{m})$, and $U_0 = U \cap X_0$. We say $(\mathcal{F}, \mathcal{F}_0, \alpha)$ is a *coherent triple* if we have

(1) $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module such that $f : \mathcal{F} \to \mathcal{F}$ is injective,
(2) $\mathcal{F}_0$ is a coherent $\mathcal{O}_{X_0}$-module,
(3) $\alpha : \mathcal{F}/f\mathcal{F} \to \mathcal{F}_0|_{U_0}$ is an isomorphism.

There is an obvious notion of a *morphism of coherent triples* which turns the collection of all coherent triples into a category.

The category of coherent triples is additive but not abelian. However, it is clear what a short exact sequence of coherent triples is.

Given two coherent triples $(\mathcal{F}, \mathcal{F}_0, \alpha)$ and $(\mathcal{G}, \mathcal{G}_0, \beta)$ it may not be the case that $(\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G}, \mathcal{F}_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{G}_0, \alpha \otimes \beta)$ is a coherent triple[8]. However, if the stalks $\mathcal{G}_x$ are free for all $x \in U_0$, then this does hold.

We will say the coherent triple $(\mathcal{G}, \mathcal{G}_0, \beta)$ is *locally free*, resp. *invertible* if $\mathcal{G}$ and $\mathcal{G}_0$ are locally free, resp. invertible modules. In this case tensoring with $(\mathcal{G}, \mathcal{G}_0, \beta)$ makes sense (see above) and turns short exact sequences of coherent triples into short exact sequences of coherent triples.

0F23 **Lemma 52.25.1.** *For any coherent triple* $(\mathcal{F}, \mathcal{F}_0, \alpha)$ *there exists a coherent* $\mathcal{O}_X$*-module* $\mathcal{F}'$ *such that* $f : \mathcal{F}' \to \mathcal{F}'$ *is injective, an isomorphism* $\alpha' : \mathcal{F}'|_U \to \mathcal{F}$*, and a map* $\alpha'_0 : \mathcal{F}'/f\mathcal{F}' \to \mathcal{F}_0$ *such that* $\alpha \circ (\alpha' \bmod f) = \alpha'_0|_{U_0}$.

Proof. Choose a finite A-module M such that $\mathcal{F}$ is the restriction to U of the coherent $\mathcal{O}_X$-module associated to M, see Local Cohomology, Lemma 51.8.2. Since $\mathcal{F}$ is f-torsion free, we may replace M by its quotient by f-power torsion. On the other hand, let $M_0 = \Gamma(X_0, \mathcal{F}_0)$ so that $\mathcal{F}_0$ is the coherent $\mathcal{O}_{X_0}$-module associated to the finite A/fA-module M_0. By Cohomology of Schemes, Lemma 30.10.5 there exists an n such that the isomorphism α_0 corresponds to an A/fA-module homomorphism $\mathfrak{m}^n M/fM \to M_0$ (whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$, but we don't need this). Thus if we take $M' = \mathfrak{m}^n M$ and we let $\mathcal{F}'$ be the coherent $\mathcal{O}_X$-module associated to M', then the lemma is clear. □

[8]Namely, it isn't necessarily the case that f is injective on $\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G}$.

Let $(\mathcal{F}, \mathcal{F}_0, \alpha)$ be a coherent triple. Choose $\mathcal{F}', \alpha', \alpha'_0$ as in Lemma 52.25.1. Set

0F24 (52.25.1.1) $$\chi(\mathcal{F}, \mathcal{F}_0, \alpha) = \text{length}_A(\text{Coker}(\alpha'_0)) - \text{length}_A(\text{Ker}(\alpha'_0))$$

The expression on the right makes sense as α'_0 is an isomorphism over U_0 and hence its kernel and coherent are coherent modules supported on $\{\mathfrak{m}\}$ which therefore have finite length (Algebra, Lemma 10.62.3).

0F25 **Lemma 52.25.2.** *The quantity $\chi(\mathcal{F}, \mathcal{F}_0, \alpha)$ in (52.25.1.1) does not depend on the choice of $\mathcal{F}', \alpha', \alpha'_0$ as in Lemma 52.25.1.*

Proof. Let $\mathcal{F}', \alpha', \alpha'_0$ and $\mathcal{F}'', \alpha'', \alpha''_0$ be two such choices. For $n > 0$ set $\mathcal{F}'_n = \mathfrak{m}^n\mathcal{F}'$. By Cohomology of Schemes, Lemma 30.10.5 for some n there exists an $\mathcal{O}_X$-module map $\mathcal{F}'_n \to \mathcal{F}''$ agreeing with the identification $\mathcal{F}''|_U = \mathcal{F}'|_U$ determined by α' and α''. Then the diagram

$$\begin{array}{ccc} \mathcal{F}'_n/f\mathcal{F}'_n & \longrightarrow & \mathcal{F}'/f\mathcal{F}' \\ \downarrow & & \downarrow{\scriptstyle \alpha'_0} \\ \mathcal{F}''/f\mathcal{F}'' & \xrightarrow{\alpha''_0} & \mathcal{F}_0 \end{array}$$

is commutative after restricting to U_0. Hence by Cohomology of Schemes, Lemma 30.10.5 it is commutative after restricting to $\mathfrak{m}^l(\mathcal{F}'_n/f\mathcal{F}'_n)$ for some $l > 0$. Since $\mathcal{F}'_{n+l}/f\mathcal{F}'_{n+l} \to \mathcal{F}'_n/f\mathcal{F}'_n$ factors through $\mathfrak{m}^l(\mathcal{F}'_n/f\mathcal{F}'_n)$ we see that after replacing n by $n+l$ the diagram is commutative. In other words, we have found a third choice $\mathcal{F}''', \alpha''', \alpha'''_0$ such that there are maps $\mathcal{F}''' \to \mathcal{F}''$ and $\mathcal{F}''' \to \mathcal{F}'$ over X compatible with the maps over U and X_0. This reduces us to the case discussed in the next paragraph.

Assume we have a map $\mathcal{F}'' \to \mathcal{F}'$ over X compatible with α', α'' over U and with α'_0, α''_0 over X_0. Observe that $\mathcal{F}'' \to \mathcal{F}'$ is injective as it is an isomorphism over U and since $f : \mathcal{F}'' \to \mathcal{F}''$ is injective. Clearly $\mathcal{F}'/\mathcal{F}''$ is supported on $\{\mathfrak{m}\}$ hence has finite length. We have the maps of coherent $\mathcal{O}_{X_0}$-modules

$$\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}' \xrightarrow{\alpha'_0} \mathcal{F}_0$$

whose composition is α''_0 and which are isomorphisms over U_0. Elementary homological algebra gives a 6-term exact sequence

$$\begin{aligned} 0 \to \text{Ker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}') \to \text{Ker}(\alpha''_0) \to \text{Ker}(\alpha'_0) \to \\ \text{Coker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}') \to \text{Coker}(\alpha''_0) \to \text{Coker}(\alpha'_0) \to 0 \end{aligned}$$

By additivity of lengths (Algebra, Lemma 10.52.3) we find that it suffices to show that

$$\text{length}_A(\text{Coker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}')) - \text{length}_A(\text{Ker}(\mathcal{F}''/f\mathcal{F}'' \to \mathcal{F}'/f\mathcal{F}')) = 0$$

This follows from applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'' & \xrightarrow{f} & \mathcal{F}'' & \longrightarrow & \mathcal{F}''/f\mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{f} & \mathcal{F}' & \longrightarrow & \mathcal{F}'/f\mathcal{F}' & \longrightarrow & 0 \end{array}$$

and the fact that $\mathcal{F}'/\mathcal{F}''$ has finite length. □

0F26 **Lemma 52.25.3.** *We have $\chi(\mathcal{G}, \mathcal{G}_0, \beta) = \chi(\mathcal{F}, \mathcal{F}_0, \alpha) + \chi(\mathcal{H}, \mathcal{H}_0, \gamma)$ if*

$$0 \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{G}, \mathcal{G}_0, \beta) \to (\mathcal{H}, \mathcal{H}_0, \gamma) \to 0$$

is a short exact sequence of coherent triples.

Proof. Choose $\mathcal{G}', \beta', \beta'_0$ as in Lemma 52.25.1 for the triple $(\mathcal{G}, \mathcal{G}_0, \beta)$. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}' \subset \mathcal{G}'$ be the kernel of the composition

$$\mathcal{G}' \xrightarrow{\beta'} j_*\mathcal{G} \to j_*\mathcal{H}$$

Observe that $\mathcal{H}' = \mathcal{G}'/\mathcal{F}'$ is a coherent subsheaf of $j_*\mathcal{H}$ and hence $f : \mathcal{H}' \to \mathcal{H}'$ is injective. Hence by the snake lemma we obtain a short exact sequence

$$0 \to \mathcal{F}'/f\mathcal{F}' \to \mathcal{G}'/f\mathcal{G}' \to \mathcal{H}'/f\mathcal{H}' \to 0$$

We have isomorphisms $\alpha' : \mathcal{F}'|_U \to \mathcal{F}$, $\beta' : \mathcal{G}'|_U \to \mathcal{G}$, and $\gamma' : \mathcal{H}'|_U \to \mathcal{H}$ by construction. To finish the proof we'll need to construct maps $\alpha'_0 : \mathcal{F}'/f\mathcal{F}' \to \mathcal{F}_0$ and $\gamma'_0 : \mathcal{H}'/f\mathcal{H}' \to \mathcal{H}_0$ as in Lemma 52.25.1 and fitting into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}'/f\mathcal{F}' & \longrightarrow & \mathcal{G}'/f\mathcal{G}' & \longrightarrow & \mathcal{H}'/f\mathcal{H}' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'_0} & & \downarrow{\scriptstyle \beta'_0} & & \downarrow{\scriptstyle \gamma'_0} & & \\
0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{H}_0 & \longrightarrow & 0
\end{array}$$

However, this may not be possible with our initial choice of $\mathcal{G}'$. From the displayed diagram we see the obstruction is exactly the composition

$$\delta : \mathcal{F}'/f\mathcal{F}' \to \mathcal{G}'/f\mathcal{G}' \xrightarrow{\beta'_0} \mathcal{G}_0 \to \mathcal{H}_0$$

Note that the restriction of δ to U_0 is zero by our choice of $\mathcal{F}'$ and $\mathcal{H}'$. Hence by Cohomology of Schemes, Lemma 30.10.5 there exists an $k > 0$ such that δ vanishes on $\mathfrak{m}^k \cdot (\mathcal{F}'/f\mathcal{F}')$. For $n > k$ set $\mathcal{G}'_n = \mathfrak{m}^n\mathcal{G}'$, $\mathcal{F}'_n = \mathcal{G}'_n \cap \mathcal{F}'$, and $\mathcal{H}'_n = \mathcal{G}'_n/\mathcal{F}'_n$. Observe that β'_0 can be composed with $\mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{G}'/f\mathcal{G}'$ to give a map $\beta'_{n,0} : \mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{G}_0$ as in Lemma 52.25.1. By Artin-Rees (Algebra, Lemma 10.51.2) we may choose n such that $\mathcal{F}'_n \subset \mathfrak{m}^k\mathcal{F}'$. As above the maps $f : \mathcal{F}'_n \to \mathcal{F}'_n$, $f : \mathcal{G}'_n \to \mathcal{G}'_n$, and $f : \mathcal{H}'_n \to \mathcal{H}'_n$ are injective and as above using the snake lemma we obtain a short exact sequence

$$0 \to \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{G}'_n/f\mathcal{G}'_n \to \mathcal{H}'_n/f\mathcal{H}'_n \to 0$$

As above we have isomorphisms $\alpha'_n : \mathcal{F}'_n|_U \to \mathcal{F}$, $\beta'_n : \mathcal{G}'_n|_U \to \mathcal{G}$, and $\gamma'_n : \mathcal{H}'_n|_U \to \mathcal{H}$. We consider the obstruction

$$\delta_n : \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{G}'_n/f\mathcal{G}'_n \xrightarrow{\beta'_{n,0}} \mathcal{G}_0 \to \mathcal{H}_0$$

as before. However, the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}'_n/f\mathcal{F}'_n & \longrightarrow & \mathcal{G}'_n/f\mathcal{G}'_n & \xrightarrow[\beta'_{n,0}]{} & \mathcal{G}_0 & \longrightarrow & \mathcal{H}_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}'/f\mathcal{F}' & \longrightarrow & \mathcal{G}'/f\mathcal{G}' & \xrightarrow{\beta'_0} & \mathcal{G}_0 & \longrightarrow & \mathcal{H}_0
\end{array}$$

our choice of n and our observation about δ show that $\delta_n = 0$. This produces the desired maps $\alpha'_{n,0} : \mathcal{F}'_n/f\mathcal{F}'_n \to \mathcal{F}_0$, and $\gamma'_{n,0} : \mathcal{H}'_n/f\mathcal{H}'_n \to \mathcal{H}_0$. OK, so we may use $\mathcal{F}'_n, \alpha'_n, \alpha'_{n,0}$, $\mathcal{G}'_n, \beta'_n, \beta'_{n,0}$, and $\mathcal{H}'_n, \gamma'_n, \gamma'_{n,0}$ to compute $\chi(\mathcal{F}, \mathcal{F}_0, \alpha)$, $\chi(\mathcal{G}, \mathcal{G}_0, \beta)$,

and $\chi(\mathcal{H}, \mathcal{H}_0, \gamma)$. Now finally the lemma follows from an application of the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'_n/f\mathcal{F}'_n & \longrightarrow & \mathcal{G}'_n/f\mathcal{G}'_n & \longrightarrow & \mathcal{H}'_n/f\mathcal{H}'_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{H}_0 & \longrightarrow & 0 \end{array}$$

and additivity of lengths (Algebra, Lemma 10.52.3). □

0F27 **Proposition 52.25.4.** *Let $(\mathcal{F}, \mathcal{F}_0, \alpha)$ be a coherent triple. Let $(\mathcal{L}, \mathcal{L}_0, \lambda)$ be an invertible coherent triple. Then the function*

$$\mathbf{Z} \longrightarrow \mathbf{Z}, \quad n \longmapsto \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial of degree $\leq \dim(\mathit{Supp}(\mathcal{F}))$.

More precisely, if $\mathcal{F} = 0$, then the function is constant. If $\mathcal{F}$ has finite support in U, then the function is constant. If the support of $\mathcal{F}$ in U has dimension 1, i.e., the closure of the support of $\mathcal{F}$ in X has dimension 2, then the function is linear, etc.

Proof. We will prove this by induction on the dimension of the support of $\mathcal{F}$.

The base case is when $\mathcal{F} = 0$. Then either $\mathcal{F}_0$ is zero or its support is $\{\mathfrak{m}\}$. In this case we have

$$(\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n} = (0, \mathcal{F}_0 \otimes \mathcal{L}_0^{\otimes n}, 0) \cong (0, \mathcal{F}_0, 0)$$

Thus the function of the lemma is constant with value equal to the length of $\mathcal{F}_0$.

Induction step. Assume the support of $\mathcal{F}$ is nonempty. Let $\mathcal{G}_0 \subset \mathcal{F}_0$ denote the submodule of sections supported on $\{\mathfrak{m}\}$. Then we get a short exact sequence

$$0 \to (0, \mathcal{G}_0, 0) \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{F}, \mathcal{F}_0/\mathcal{G}_0, \alpha) \to 0$$

This sequence remains exact if we tensor by the invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$, see discussion above. Thus by additivity of χ (Lemma 52.25.3) and the base case explained above, it suffices to prove the induction step for $(\mathcal{F}, \mathcal{F}_0/\mathcal{G}_0, \alpha)$. In this way we see that we may assume $\mathfrak{m}$ is not an associated point of $\mathcal{F}_0$.

Let $T = \mathrm{Ass}(\mathcal{F}) \cup \mathrm{Ass}(\mathcal{F}/f\mathcal{F})$. Since U is quasi-affine, we can find $s \in \Gamma(U, \mathcal{L})$ which does not vanish at any $u \in T$, see Properties, Lemma 28.29.7. After multiplying s by a suitable element of $\mathfrak{m}$ we may assume $\lambda(s \bmod f) = s_0|_{U_0}$ for some $s_0 \in \Gamma(X_0, \mathcal{L}_0)$; details omitted. We obtain a morphism

$$(s, s_0) : (\mathcal{O}_U, \mathcal{O}_{X_0}, 1) \longrightarrow (\mathcal{L}, \mathcal{L}_0, \lambda)$$

in the category of coherent triples. Let $\mathcal{G} = \mathrm{Coker}(s : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L})$ and $\mathcal{G}_0 = \mathrm{Coker}(s_0 : \mathcal{F}_0 \to \mathcal{F}_0 \otimes \mathcal{L}_0)$. Observe that $s_0 : \mathcal{F}_0 \to \mathcal{F}_0 \otimes \mathcal{L}_0$ is injective as it is injective on U_0 by our choice of s and as $\mathfrak{m}$ isn't an associated point of $\mathcal{F}_0$. It follows that there exists an isomorphism $\beta : \mathcal{G}/f\mathcal{G} \to \mathcal{G}_0|_{U_0}$ such that we obtain a short exact sequence

$$0 \to (\mathcal{F}, \mathcal{F}_0, \alpha) \to (\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda) \to (\mathcal{G}, \mathcal{G}_0, \beta) \to 0$$

By induction on the dimension of the support we know the proposition holds for the coherent triple $(\mathcal{G}, \mathcal{G}_0, \beta)$. Using the additivity of Lemma 52.25.3 we see that

$$n \longmapsto \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n+1}) - \chi((\mathcal{F}, \mathcal{F}_0, \alpha) \otimes (\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial. We conclude by a variant of Algebra, Lemma 10.58.5 for functions defined for all integers (details omitted). □

0F28 **Lemma 52.25.5.** *Assume depth$(A) \geq 3$ or equivalently depth$(A/fA) \geq 2$. Let $(\mathcal{L}, \mathcal{L}_0, \lambda)$ be an invertible coherent triple. Then*

$$\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = \mathit{length}_A \operatorname{Coker}(\Gamma(U, \mathcal{L}) \to \Gamma(U_0, \mathcal{L}_0))$$

and in particular this is ≥ 0. Moreover, $\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = 0$ if and only if $\mathcal{L} \cong \mathcal{O}_U$.

Proof. The equivalence of the depth conditions follows from Algebra, Lemma 10.72.7. By the depth condition we see that $\Gamma(U, \mathcal{O}_U) = A$ and $\Gamma(U_0, \mathcal{O}_{U_0}) = A/fA$, see Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.8.2. Using Local Cohomology, Lemma 51.12.2 we find that $M = \Gamma(U, \mathcal{L})$ is a finite A-module. This in turn implies $\operatorname{depth}(M) \geq 2$ for example by part (4) of Local Cohomology, Lemma 51.8.2 or by Divisors, Lemma 31.6.6. Also, we have $\mathcal{L}_0 \cong \mathcal{O}_{X_0}$ as X_0 is a local scheme. Hence we also see that $M_0 = \Gamma(X_0, \mathcal{L}_0) = \Gamma(U_0, \mathcal{L}_0|_{U_0})$ and that this module is isomorphic to A/fA.

By the above $\mathcal{F}' = \widetilde{M}$ is a coherent $\mathcal{O}_X$-module whose restriction to U is isomorphic to $\mathcal{L}$. The isomorphism $\lambda : \mathcal{L}/f\mathcal{L} \to \mathcal{L}_0|_{U_0}$ determines a map $M/fM \to M_0$ on global sections which is an isomorphism over U_0. Since $\operatorname{depth}(M) \geq 2$ we see that $H^0_{\mathfrak{m}}(M/fM) = 0$ and it follows that $M/fM \to M_0$ is injective. Thus by definition

$$\chi(\mathcal{L}, \mathcal{L}_0, \lambda) = \operatorname{length}_A \operatorname{Coker}(M/fM \to M_0)$$

which gives the first statement of the lemma.

Finally, if this length is 0, then $M \to M_0$ is surjective. Hence we can find $s \in M = \Gamma(U, \mathcal{L})$ mapping to a trivializing section of $\mathcal{L}_0$. Consider the finite A-modules K, Q defined by the exact sequence

$$0 \to K \to A \xrightarrow{s} M \to Q \to 0$$

The supports of K and Q do not meet U_0 because s is nonzero at points of U_0. Using Algebra, Lemma 10.72.6 we see that $\operatorname{depth}(K) \geq 2$ (observe that $As \subset M$ has depth ≥ 1 as a submodule of M). Thus the support of K if nonempty has dimension ≥ 2 by Algebra, Lemma 10.72.3. This contradicts $\operatorname{Supp}(M) \cap V(f) \subset \{\mathfrak{m}\}$ unless $K = 0$. When $K = 0$ we find that $\operatorname{depth}(Q) \geq 2$ and we conclude $Q = 0$ as before. Hence $A \cong M$ and $\mathcal{L}$ is trivial. □

52.26. Invertible modules on punctured spectra, I

0F29 In this section we prove some local Lefschetz theorems for the Picard group. Some of the ideas are taken from [**Kol13**], [**BdJ14**], and [**Kol16a**].

0F2A **Lemma 52.26.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor and assume that depth$(A/fA) \geq 2$, or equivalently depth$(A) \geq 3$. Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. The map*

$$\operatorname{Pic}(U) \to \operatorname{Pic}(U_0)$$

is injective on torsion.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. Observe that $\mathcal{L}$ maps to 0 in $\mathrm{Pic}(U_0)$ if and only if we can extend $\mathcal{L}$ to an invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$ as in Section 52.25. By Proposition 52.25.4 the function

$$n \longmapsto \chi((\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial. By Lemma 52.25.5 the value of this polynomial is zero if and only if $\mathcal{L}^{\otimes n}$ is trivial. Thus if $\mathcal{L}$ is torsion, then this polynomial has infinitely many zeros, hence is identically zero, hence $\mathcal{L}$ is trivial. □

0F2B **Proposition 52.26.2** (Kollár)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume* [**Kol16a**, Theorem 1.9]

(1) *A has a dualizing complex,*
(2) *f is a nonzerodivisor,*
(3) *$\mathrm{depth}(A/fA) \geq 2$, or equivalently $\mathrm{depth}(A) \geq 3$,*
(4) *if $f \in \mathfrak{p} \subset A$ is a prime ideal with $\dim(A/\mathfrak{p}) = 2$, then $\mathrm{depth}(A_\mathfrak{p}) \geq 2$.*

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. The map

$$\mathrm{Pic}(U) \to \mathrm{Pic}(U_0)$$

is injective. Finally, if (1), (2), (3), A is (S_2), and $\dim(A) \geq 4$, then (4) holds.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. Observe that $\mathcal{L}$ maps to 0 in $\mathrm{Pic}(U_0)$ if and only if we can extend $\mathcal{L}$ to an invertible coherent triple $(\mathcal{L}, \mathcal{L}_0, \lambda)$ as in Section 52.25. By Proposition 52.25.4 the function

$$n \longmapsto \chi((\mathcal{L}, \mathcal{L}_0, \lambda)^{\otimes n})$$

is a polynomial P. By Lemma 52.25.5 we have $P(n) \geq 0$ for all $n \in \mathbf{Z}$ with equality if and only if $\mathcal{L}^{\otimes n}$ is trivial. In particular $P(0) = 0$ and P is either identically zero and we win or P has even degree ≥ 2.

Set $M = \Gamma(U, \mathcal{L})$ and $M_0 = \Gamma(X_0, \mathcal{L}_0) = \Gamma(U_0, \mathcal{L}_0)$. Then M is a finite A-module of depth ≥ 2 and $M_0 \cong A/fA$, see proof of Lemma 52.25.5. Note that $H^2_\mathfrak{m}(M)$ is finite A-module by Local Cohomology, Lemma 51.7.4 and the fact that $H^i_\mathfrak{m}(A) = 0$ for $i = 0, 1, 2$ since $\mathrm{depth}(A) \geq 3$. Consider the short exact sequence

$$0 \to M/fM \to M_0 \to Q \to 0$$

Lemma 52.25.5 tells us Q has finite length equal to $\chi(\mathcal{L}, \mathcal{L}_0, \lambda)$. We obtain $Q = H^1_\mathfrak{m}(M/fM)$ and $H^i_\mathfrak{m}(M/fM) = H^i_\mathfrak{m}(M_0) \cong H^i_\mathfrak{m}(A/fA)$ for $i > 1$ from the long exact sequence of local cohomology associated to the displayed short exact sequence. Consider the long exact sequence of local cohomology associated to the sequence $0 \to M \to M \to M/fM \to 0$. It starts with

$$0 \to Q \to H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(A/fA)$$

Using additivity of lengths we see that $\chi(\mathcal{L}, \mathcal{L}_0, \lambda)$ is equal to the length of the image of $H^2_\mathfrak{m}(M) \to H^2_\mathfrak{m}(A/fA)$.

Let prove the lemma in a special case to elucidate the rest of the proof. Namely, assume for a moment that $H^2_\mathfrak{m}(A/fA)$ is a finite length module. Then we would have $P(1) \leq \mathrm{length}_A H^2_\mathfrak{m}(A/fA)$. The exact same argument applied to $\mathcal{L}^{\otimes n}$ shows that $P(n) \leq \mathrm{length}_A H^2_\mathfrak{m}(A/fA)$ for all n. Thus P cannot have positive degree and we win. In the rest of the proof we will modify this argument to give a linear upper bound for $P(n)$ which suffices.

Let us study the map $H^2_{\mathfrak{m}}(M) \to H^2_{\mathfrak{m}}(M_0) \cong H^2_{\mathfrak{m}}(A/fA)$. Choose a normalized dualizing complex $\omega_A^\bullet$ for A. By local duality (Dualizing Complexes, Lemma 47.18.4) this map is Matlis dual to the map

$$\mathrm{Ext}^{-2}_A(M, \omega_A^\bullet) \longleftarrow \mathrm{Ext}^{-2}_A(M_0, \omega_A^\bullet)$$

whose image therefore has the same (finite) length. The support (if nonempty) of the finite A-module $\mathrm{Ext}^{-2}_A(M_0, \omega_A^\bullet)$ consists of $\mathfrak{m}$ and a finite number of primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ containing f with $\dim(A/\mathfrak{p}_i) = 1$. Namely, by Local Cohomology, Lemma 51.9.4 the support is contained in the set of primes $\mathfrak{p} \subset A$ with $\mathrm{depth}_{A_\mathfrak{p}}(M_{0,\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq 2$. Thus it suffices to show there is no prime $\mathfrak{p}$ containing f with $\dim(A/\mathfrak{p}) = 2$ and $\mathrm{depth}_{A_\mathfrak{p}}(M_{0,\mathfrak{p}}) = 0$. However, because $M_{0,\mathfrak{p}} \cong (A/fA)_\mathfrak{p}$ this would give $\mathrm{depth}(A_\mathfrak{p}) = 1$ which contradicts assumption (4). Choose a section $t \in \Gamma(U, \mathcal{L}^{\otimes -1})$ which does not vanish in the points $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, see Properties, Lemma 28.29.7. Multiplication by t on global sections determines a map $t : M \to A$ which defines an isomorphism $M_{\mathfrak{p}_i} \to A_{\mathfrak{p}_i}$ for $i = 1, \ldots, r$. Denote $t_0 = t|_{U_0}$ the corresponding section of $\Gamma(U_0, \mathcal{L}_0^{\otimes -1})$ which similarly determines a map $t_0 : M_0 \to A/fA$ compatible with t. We conclude that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{-2}_A(M, \omega_A^\bullet) & \longleftarrow & \mathrm{Ext}^{-2}_A(M_0, \omega_A^\bullet) \\ \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle t_0} \\ \mathrm{Ext}^{-2}_A(A, \omega_A^\bullet) & \longleftarrow & \mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet) \end{array}$$

It follows that the length of the image of the top horizontal map is at most the length of $\mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ plus the length of the cokernel of t_0.

However, if we replace $\mathcal{L}$ by $\mathcal{L}^n$ for $n > 1$, then we can use

$$t^n : M_n = \Gamma(U, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(U, \mathcal{O}_U) = A$$

instead of t. This replaces t_0 by its nth power. Thus the length of the image of the map $\mathrm{Ext}^{-2}_A(M_n, \omega_A^\bullet) \leftarrow \mathrm{Ext}^{-2}_A(M_{n,0}, \omega_A^\bullet)$ is at most the length of $\mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ plus the length of the cokernel of

$$t_0^n : \mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet) \longrightarrow \mathrm{Ext}^{-2}_A(M_{n,0}, \omega_A^\bullet)$$

Via the isomorphism $M_0 \cong A/fA$ the map t_0 becomes $g : A/fA \to A/fA$ for some $g \in A/fA$ and via the corresponding isomorphisms $M_{n,0} \cong A/fA$ the map t_0^n becomes $g^n : A/fA \to A/fA$. Thus the length of the cokernel above is the length of the quotient of $\mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ by g^n. Since $\mathrm{Ext}^{-2}_A(A/fA, \omega_A^\bullet)$ is a finite A-module with support T of dimension 1 and since $V(g) \cap T$ consists of the closed point by our choice of t this length grows linearly in n by Algebra, Lemma 10.62.6.

To finish the proof we prove the final assertion. Assume $f \in \mathfrak{m} \subset A$ satisfies (1), (2), (3), A is (S_2), and $\dim(A) \geq 4$. Condition (1) implies A is catenary, see Dualizing Complexes, Lemma 47.17.4. Then $\mathrm{Spec}(A)$ is equidimensional by Local Cohomology, Lemma 51.3.2. Thus $\dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) \geq 4$ for every prime $\mathfrak{p}$ of A. Then $\mathrm{depth}(A_\mathfrak{p}) \geq \min(2, \dim(A_\mathfrak{p})) \geq \min(2, 4 - \dim(A/\mathfrak{p}))$ and hence (4) holds. □

0FIX **Remark 52.26.3.** In SGA2 we find the following result. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume A is a quotient of a regular ring, the element f is a nonzerodivisor, and

(a) if $\mathfrak{p} \subset A$ is a prime ideal with $\dim(A/\mathfrak{p}) = 1$, then $\text{depth}(A_\mathfrak{p}) \geq 2$, and
(b) $\text{depth}(A/fA) \geq 3$, or equivalently $\text{depth}(A) \geq 4$.

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then the map

$$\text{Pic}(U) \to \text{Pic}(U_0)$$

is injective. This is [**Gro68**, Exposee XI, Lemma 3.16][9]. This result from SGA2 follows from Proposition 52.26.2 because

(1) a quotient of a regular ring has a dualizing complex (see Dualizing Complexes, Lemma 47.21.3 and Proposition 47.15.11), and
(2) if $\text{depth}(A) \geq 4$ then $\text{depth}(A_\mathfrak{p}) \geq 2$ for all primes $\mathfrak{p}$ with $\dim(A/\mathfrak{p}) = 2$, see Algebra, Lemma 10.72.10.

52.27. Invertible modules on punctured spectra, II

0F2C Next we turn to surjectivity in local Lefschetz for the Picard group. First to extend an invertible module on U_0 to an open neighbourhood we have the following simple criterion.

0F2D **Lemma 52.27.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules, and*
(4) *$H^3_\mathfrak{m}(A/fA) = 0$[10].*

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then

$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{Pic}(U') \longrightarrow \text{Pic}(U_0)$$

is surjective.

Proof. Let $U_0 \subset U_n \subset U$ be the nth infinitesimal neighbourhood of U_0. Observe that the ideal sheaf of U_n in U_{n+1} is isomorphic to $\mathcal{O}_{U_0}$ as $U_0 \subset U$ is the principal closed subscheme cut out by the nonzerodivisor f. Hence we have an exact sequence of abelian groups

$$\text{Pic}(U_{n+1}) \to \text{Pic}(U_n) \to H^2(U_0, \mathcal{O}_{U_0}) = H^3_\mathfrak{m}(A/fA) = 0$$

see More on Morphisms, Lemma 37.4.1. Thus every invertible $\mathcal{O}_{U_0}$-module is the restriction of an invertible coherent formal module, i.e., an invertible object of $Coh(U, f\mathcal{O}_U)$. We conclude by applying Lemma 52.24.2. □

0F2E **Remark 52.27.2.** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. The conclusion of Lemma 52.27.1 holds if we assume

(1) A has a dualizing complex,
(2) A is f-adically complete,
(3) f is a nonzerodivisor,
(4) one of the following is true
  (a) A_f is (S_2) and for $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ minimal we have $\dim(A/\mathfrak{p}) \geq 4$, or
  (b) if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.
(5) $H^3_\mathfrak{m}(A/fA) = 0$.

[9] Condition (a) follows from condition (b), see Algebra, Lemma 10.72.10.
[10] Observe that (3) and (4) hold if $\text{depth}(A/fA) \geq 4$, or equivalently $\text{depth}(A) \geq 5$.

The proof is exactly the same as the proof of Lemma 52.27.1 using Lemma 52.24.1 instead of Lemma 52.24.2. Two points need to be made here: (a) it seems hard to find examples where one knows $H^3_{\mathfrak{m}}(A/fA) = 0$ without assuming $\text{depth}(A/fA) \geq 4$, and (b) the proof of Lemma 52.24.1 is a good deal harder than the proof of Lemma 52.24.2.

0F2F **Lemma 52.27.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Assume*

(1) *the conditions of Lemma 52.27.1 hold, and*
(2) *for every maximal ideal $\mathfrak{p} \subset A_f$ the punctured spectrum of $(A_f)_\mathfrak{p}$ has trivial Picard group.*

Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Then

$$\text{Pic}(U) \longrightarrow \text{Pic}(U_0)$$

is surjective.

Proof. Let $\mathcal{L}_0 \in \text{Pic}(U_0)$. By Lemma 52.27.1 there exists an open $U_0 \subset U' \subset U$ and $\mathcal{L}' \in \text{Pic}(U')$ whose restriction to U_0 is $\mathcal{L}_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as in (2). By assumption we can find invertible modules $\mathcal{L}'_i$ on $\text{Spec}(A_{\mathfrak{p}_i})$ agreeing with $\mathcal{L}'$ over the punctured spectrum $U' \times_U \text{Spec}(A_{\mathfrak{p}_i})$ since trivial invertible modules always extend. By Limits, Lemma 32.20.2 applied n times we see that $\mathcal{L}'$ extends to an invertible module on U. □

0F2G **Lemma 52.27.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $A^\wedge$ be its completion. Let U, resp. $U^\wedge$ be the punctured spectrum of A, resp. $A^\wedge$. Then $\text{Pic}(U) \to \text{Pic}(U^\wedge)$ is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module with pullback $\mathcal{L}^\wedge$ on $U^\wedge$. We have $H^0(U, \mathcal{O}_U) = A$ by our assumption on depth and Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.8.2. Thus $\mathcal{L}$ is trivial if and only if $M = H^0(U, \mathcal{L})$ is isomorphic to A as an A-module. (Details omitted.) Since $A \to A^\wedge$ is flat we have $M \otimes_A A^\wedge = \Gamma(U^\wedge, \mathcal{L}^\wedge)$ by flat base change, see Cohomology of Schemes, Lemma 30.5.2. Finally, it is easy to see that $M \cong A$ if and only if $M \otimes_A A^\wedge \cong A^\wedge$. □

0F2H **Lemma 52.27.5.** *Let $(A, \mathfrak{m})$ be a regular local ring. Then the Picard group of the punctured spectrum of A is trivial.*

Proof. Combine Divisors, Lemma 31.28.3 with More on Algebra, Lemma 15.122.2. □

Now we can bootstrap the earlier results to prove that Picard groups are trivial for punctured spectra of complete intersections of dimension ≥ 4. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 23.8.

0F2I **Proposition 52.27.6** (Grothendieck)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 4, then the Picard group of the punctured spectrum of A is trivial.*

Proof. By Lemma 52.27.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 52.27.5.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 52.27.3 to $f_r \in A'$. This is permissible:

(1) condition (1) of Lemma 52.27.1 holds because our local rings are complete,
(2) condition (2) of Lemma 52.27.1 holds holds as $f_1, \ldots, f_r$ is a regular sequence,
(3) condition (3) and (4) of Lemma 52.27.1 hold as $A = A'/f_rA'$ is Cohen-Macaulay of dimension $\dim(A) \geq 4$,
(4) condition (2) of Lemma 52.27.3 holds by induction hypothesis as $\dim((A'_{f_r})_{\mathfrak{p}}) \geq 4$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_{\mathfrak{p}} = B_{\mathfrak{q}}/(f_1, \ldots, f_{r-1})$ for some prime ideal $\mathfrak{q} \subset B$ and $B_{\mathfrak{q}}$ is regular.

This finishes the proof. □

0F2J **Example 52.27.7.** The dimension bound in Proposition 52.27.6 is sharp. For example the Picard group of the punctured spectrum of $A = k[[x, y, z, w]]/(xy - zw)$ is nontrivial. Namely, the ideal $I = (x, z)$ cuts out an effective Cartier divisor D on the punctured spectrum U of A as it is easy to see that I_x, I_y, I_z, I_w are invertible ideals in A_x, A_y, A_z, A_w. But on the other hand, A/I has depth ≥ 1 (in fact 2), hence I has depth ≥ 2 (in fact 3), hence $I = \Gamma(U, \mathcal{O}_U(-D))$. Thus if $\mathcal{O}_U(-D)$ were trivial, then we'd have $I \cong \Gamma(U, \mathcal{O}_U) = A$ which isn't true as I isn't generated by 1 element.

0F9L **Example 52.27.8.** Proposition 52.27.6 cannot be extended to quotients

$$A = B/(f_1, \ldots, f_r)$$

where B is regular and $\dim(B) - r \geq 4$. In other words, the condition that $f_1, \ldots, f_r$ be a regular sequence is (in general) needed for vanishing of the Picard group of the punctured spectrum of A. Namely, let k be a field and set

$$A = k[[a, b, x, y, z, u, v, w]]/(a^3, b^3, xa^2 + yab + zb^2, w^2)$$

Observe that $A = A_0[w]/(w^2)$ with $A_0 = k[[a, b, x, y, z, u, v]]/(a^3, b^3, xa^2 + yab + zb^2)$. We will show below that A_0 has depth 2. Denote U the punctured spectrum of A and U_0 the punctured spectrum of A_0. Observe there is a short exact sequence $0 \to A_0 \to A \to A_0 \to 0$ where the first arrow is given by multiplication by w. By More on Morphisms, Lemma 37.4.1 we find that there is an exact sequence

$$H^0(U, \mathcal{O}_U^*) \to H^0(U_0, \mathcal{O}_{U_0}^*) \to H^1(U_0, \mathcal{O}_{U_0}) \to \mathrm{Pic}(U)$$

Since the depth of A_0 and hence A is 2 we see that $H^0(U_0, \mathcal{O}_{U_0}) = A_0$ and $H^0(U, \mathcal{O}_U) = A$ and that $H^1(U_0, \mathcal{O}_{U_0})$ is nonzero, see Dualizing Complexes, Lemma 47.11.1 and Local Cohomology, Lemma 51.2.2. Thus the last arrow displayed above is nonzero and we conclude that $\mathrm{Pic}(U)$ is nonzero.

To show that A_0 has depth 2 it suffices to show that $A_1 = k[[a, b, x, y, z]]/(a^3, b^3, xa^2 + yab + zb^2)$ has depth 0. This is true because a^2b^2 maps to a nonzero element of A_1 which is annihilated by each of the variables a, b, x, y, z. For example $ya^2b^2 = (yab)(ab) = -(xa^2 + zb^2)(ab) = -xa^3b - yab^3 = 0$ in A_1. The other cases are similar.

52.28. Application to Lefschetz theorems

0EL0 In this section we discuss the relation between coherent sheaves on a projective scheme P and coherent modules on formal completion along an ample divisor Q.

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section[11] and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 31.14.8. For all $n \geq 1$ we denote $Q_n = Z(s^n)$ the nth infinitesimal neighbourhood of Q. If $\mathcal{F}$ is a coherent $\mathcal{O}_P$-module, then we denote $\mathcal{F}_n = \mathcal{F}|_{Q_n}$ the restriction, i.e., the pullback of $\mathcal{F}$ by the closed immersion $Q_n \to P$.

0EL1 **Proposition 52.28.1.** *In the situation above assume there exists an integer σ such that for all points $p \in P \setminus Q$ we have*

$$\mathit{depth}(\mathcal{F}_p) + \dim(\overline{\{p\}}) > \sigma$$

Then the map

$$H^i(P, \mathcal{F}) \longrightarrow \lim H^i(Q_n, \mathcal{F}_n)$$

is an isomorphism for $0 \leq i < \sigma$.

Proof. We will use More on Morphisms, Lemma 37.51.1 and we will use the notation used and results found More on Morphisms, Section 37.51 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 30.16.1.

We may and do think of s as an element $f \in A_1 \subset A$, i.e., a homogeneous element of degree 1 of A. Denote $Y = V(f) \subset X$ the closed subscheme defined by f. Then $U \cap Y = (\pi|_U)^{-1}(Q)$ scheme theoretically. Recall the notation $\mathcal{F}_U = \pi^*\mathcal{F}|_U = (\pi|_U)^*\mathcal{F}$. This is a coherent $\mathcal{O}_U$-module. Choose a finite A-module M such that $\mathcal{F}_U = \widetilde{M}|_U$ (for existence see Local Cohomology, Lemma 51.8.2). We claim that $H^i_Z(M)$ is annihilated by a power of f for $i \leq \sigma + 1$.

To prove the claim we will apply Local Cohomology, Proposition 51.10.1. Translating into geometry we see that it suffices to prove for $u \in U$, $u \notin Y$ and $z \in \overline{\{u\}} \cap Z$ that

$$\operatorname{depth}(\mathcal{F}_{U,u}) + \dim(\mathcal{O}_{\overline{\{u\}},z}) > \sigma + 1$$

This requires only a small amount of thought.

Observe that $Z = \operatorname{Spec}(A_0)$ is a finite set of closed points of X because A_0 is a finite dimensional k-algebra. (The reader who would like Z to be a singleton can replace the finite k-algebra A_0 by k; it won't affect anything else in the proof.)

The morphism $\pi : L \to P$ and its restriction $\pi|_U : U \to P$ are smooth of relative dimension 1. Let $u \in U$, $u \notin Y$ and $z \in \overline{\{u\}} \cap Z$. Let $p = \pi(u) \in P \setminus Q$ be its image. Then either u is a generic point of the fibre of π over p or a closed point of the fibre. If u is a generic point of the fibre, then $\operatorname{depth}(\mathcal{F}_{U,u}) = \operatorname{depth}(\mathcal{F}_p)$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}}) + 1$. If u is a closed point of the fibre, then $\operatorname{depth}(\mathcal{F}_{U,u}) = \operatorname{depth}(\mathcal{F}_p) + 1$ and $\dim(\overline{\{u\}}) = \dim(\overline{\{p\}})$. In both cases we have $\dim(\overline{\{u\}}) = \dim(\mathcal{O}_{\overline{\{u\}},z})$ because

[11] We do not require s to be a regular section. Correspondingly, Q is only a locally principal closed subscheme of P and not necessarily an effective Cartier divisor.

every point of Z is closed. Thus the desired inequality follows from the assumption in the statement of the lemma.

Let A' be the f-adic completion of A. So $A \to A'$ is flat by Algebra, Lemma 10.97.2. Denote $U' \subset X' = \mathrm{Spec}(A')$ the inverse image of U and similarly for Y' and Z'. Let $\mathcal{F}'$ on U' be the pullback of $\mathcal{F}_U$ and let $M' = M \otimes_A A'$. By flat base change for local cohomology (Local Cohomology, Lemma 51.5.7) we have

$$H^i_{Z'}(M') = H^i_Z(M) \otimes_A A'$$

and we find that for $i \leq \sigma + 1$ these are annihilated by a power of f. Consider the diagram

$$\begin{array}{ccccc} & & H^i(U, \mathcal{F}_U) & \longrightarrow & \lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) \\ & \swarrow & \downarrow & & \| \\ H^i(U, \mathcal{F}_U) \otimes_A A' & = & H^i(U', \mathcal{F}') & \longrightarrow & \lim H^i(U', \mathcal{F}'/f^n\mathcal{F}') \end{array}$$

The lower horizontal arrow is an isomorphism for $i < \sigma$ by Lemma 52.13.2 and the torsion property we just proved. The horizontal equal sign is flat base change (Cohomology of Schemes, Lemma 30.5.2) and the vertical equal sign is because $U \cap Y$ and $U' \cap Y'$ as well as their nth infinitesimal neighbourhoods are mapped isomorphically onto each other (as we are completing with respect to f).

Applying More on Morphisms, Equation (37.51.0.2) we have compatible direct sum decompositions

$$\lim H^i(U, \mathcal{F}_U/f^n\mathcal{F}_U) = \lim \left(\bigoplus\nolimits_{m \in \mathbf{Z}} H^i(Q_n, \mathcal{F}_n \otimes \mathcal{L}^{\otimes m}) \right)$$

and

$$H^i(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} H^i(P, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$$

Thus we conclude by Algebra, Lemma 10.98.4. □

0EL2 **Lemma 52.28.2.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s with nth infinitesimal neighbourhood $Y_n = Z(s^n)$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for all $x \in X \setminus Y$ we have*

$$\mathit{depth}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > 1$$

Then $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is an isomorphism for any open subscheme $V \subset X$ containing Y.

Proof. By Proposition 52.28.1 this is true for $V = X$. Thus it suffices to show that the map $\Gamma(V, \mathcal{F}) \to \lim \Gamma(Y_n, \mathcal{F}|_{Y_n})$ is injective. If $\sigma \in \Gamma(V, \mathcal{F})$ maps to zero, then its support is disjoint from Y (details omitted; hint: use Krull's intersection theorem). Then the closure $T \subset X$ of $\mathrm{Supp}(\sigma)$ is disjoint from Y. Whence T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 29.43.18) and hence finite over k (Morphisms, Lemma 29.44.11). Thus T is a finite set of closed points of X. Thus $\mathrm{depth}(\mathcal{F}_x) \geq 2$ is at least 1 for $x \in T$ by our assumption. We conclude that $\Gamma(V, \mathcal{F}) \to \Gamma(V \setminus T, \mathcal{F})$ is injective and $\sigma = 0$ as desired. □

0EL3 **Example 52.28.3.** Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote Y_n its nth infinitesimal neighbourhood. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Here are some special cases of Proposition 52.28.1.

(1) If X is a curve, we don't learn anything.
(2) If X is a Cohen-Macaulay (for example normal) surface, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism.
(3) If X is a Cohen-Macaulay threefold, then
$$H^0(X, \mathcal{E}) \to \lim H^0(Y_n, \mathcal{E}|_{Y_n}) \quad \text{and} \quad H^1(X, \mathcal{E}) \to \lim H^1(Y_n, \mathcal{E}|_{Y_n})$$
are isomorphisms.

Presumably the pattern is clear. If X is a normal threefold, then we can conclude the result for H^0 but not for H^1.

Before we prove the next main result, we need a lemma.

0EL4 **Lemma 52.28.4.** *In Situation 52.16.1 let $(\mathcal{F}_n)$ be an object of $Coh(U, I\mathcal{O}_U)$. Assume*

(1) *A is a graded ring, $\mathfrak{a} = A_+$, and I is a homogeneous ideal,*
(2) *$(\mathcal{F}_n) = (\widetilde{M_n}|_U)$ where (M_n) is an inverse system of graded A-modules, and*
(3) *$(\mathcal{F}_n)$ extends canonically to X.*

Then there is a finite graded A-module N such that

(a) *the inverse systems (N/I^nN) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules, and*
(b) *$(\mathcal{F}_n)$ is the completion of of the coherent module associated to N.*

Proof. Let $(\mathcal{G}_n)$ be the canonical extension as in Lemma 52.16.8. The grading on A and M_n determines an action
$$a : \mathbf{G}_m \times X \longrightarrow X$$
of the group scheme $\mathbf{G}_m$ on X such that $(\widetilde{M_n})$ becomes an inverse system of $\mathbf{G}_m$-equivariant quasi-coherent $\mathcal{O}_X$-modules, see Groupoids, Example 39.12.3. Since $\mathfrak{a}$ and I are homogeneous ideals the closed subschemes Z, Y and the open subscheme U are $\mathbf{G}_m$-invariant closed and open subschemes. The restriction $(\mathcal{F}_n)$ of $(\widetilde{M_n})$ is an inverse system of $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_U$-modules. In other words, $(\mathcal{F}_n)$ is a $\mathbf{G}_m$-equivariant coherent formal module, in the sense that there is an isomorphism
$$\alpha : (a^*\mathcal{F}_n) \longrightarrow (p^*\mathcal{F}_n)$$
over $\mathbf{G}_m \times U$ satisfying a suitable cocycle condition. Since a and p are flat morphisms of affine schemes, by Lemma 52.16.9 we conclude that there exists a unique isomorphism
$$\beta : (a^*\mathcal{G}_n) \longrightarrow (p^*\mathcal{G}_n)$$
over $\mathbf{G}_m \times X$ restricting to α on $\mathbf{G}_m \times U$. The uniqueness guarantees that β satisfies the corresponding cocycle condition. In this way each $\mathcal{G}_n$ becomes a $\mathbf{G}_m$-equivariant coherent $\mathcal{O}_X$-module in a manner compatible with transition maps.

By Groupoids, Lemma 39.12.5 we see that $\mathcal{G}_n$ with its $\mathbf{G}_m$-equivariant structure corresponds to a graded A-module N_n. The transition maps $N_{n+1} \to N_n$ are graded

module maps. Note that N_n is a finite A-module and $N_n = N_{n+1}/I^n N_{n+1}$ because $(\mathcal{G}_n)$ is an object of $Coh(X, I\mathcal{O}_X)$. Let N be the finite graded A-module foud in Algebra, Lemma 10.98.3. Then $N_n = N/I^n N$, whence $(\mathcal{G}_n)$ is the completion of the coherent module associated to N, and a fortiori we see that (b) is true.

To see (a) we have to unwind the situation described above a bit more. First, observe that the kernel and cokernel of $M_n \to H^0(U, \mathcal{F}_n)$ is A_+-power torsion (Local Cohomology, Lemma 51.8.2). Observe that $H^0(U, \mathcal{F}_n)$ comes with a natural grading such that these maps and the transition maps of the system are graded A-module map; for example we can use that $(U \to X)_*\mathcal{F}_n$ is a $\mathbf{G}_m$-equivariant module on X and use Groupoids, Lemma 39.12.5. Next, recall that (N_n) and $(H^0(U, \mathcal{F}_n))$ are pro-isomorphic by Definition 52.16.7 and Lemma 52.16.8. We omit the verification that the maps defining this pro-isomorphism are graded module maps. Thus (N_n) and (M_n) are pro-isomorphic in the category of graded A-modules modulo A_+-power torsion modules. □

Let k be a field. Let P be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Let $s \in \Gamma(P, \mathcal{L})$ be a section and let $Q = Z(s)$ be the zero scheme, see Divisors, Definition 31.14.8. Let $\mathcal{I} \subset \mathcal{O}_P$ be the ideal sheaf of Q. We will use $Coh(P, \mathcal{I})$ to denote the category of coherent formal modules introduced in Cohomology of Schemes, Section 30.23.

0EL5 **Proposition 52.28.5.** *In the situation above let $(\mathcal{F}_n)$ be an object of $Coh(P, \mathcal{I})$. Assume for all $q \in Q$ and for all primes $\mathfrak{p} \in \mathcal{O}^\wedge_{P,q}$, $\mathfrak{p} \notin V(\mathcal{I}^\wedge_q)$ we have*

$$depth((\mathcal{F}^\wedge_q)_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{P,q}/\mathfrak{p}) + \dim(\overline{\{q\}}) > 2$$

Then $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_P$-module.

Proof. By Cohomology of Schemes, Lemma 30.23.6 to prove the lemma, we may replace $(\mathcal{F}_n)$ by an object differing from it by $\mathcal{I}$-torsion (see below for more precision). Let $T' = \{q \in Q \mid \dim(\overline{\{q\}}) = 0\}$ and $T = \{q \in Q \mid \dim(\overline{\{q\}}) \leq 1\}$. The assumption in the proposition is exactly that $Q \subset P$, $(\mathcal{F}_n)$, and $T' \subset T \subset Q$ satisfy the conditions of Lemma 52.21.2 with $d = 1$; besides trivial manipulations of inequalities, use that $V(\mathfrak{p}) \cap V(\mathcal{I}^\wedge_y) = \{\mathfrak{m}^\wedge_y\} \Leftrightarrow \dim(\mathcal{O}^\wedge_{P,q}/\mathfrak{p}) = 1$ as $\mathcal{I}^\wedge_y$ is generated by 1 element. Combining these two remarks, we may replace $(\mathcal{F}_n)$ by the object $(\mathcal{H}_n)$ of $Coh(P, \mathcal{I})$ found in Lemma 52.21.2. Thus we may and do assume $(\mathcal{F}_n)$ is pro-isomorphic to an inverse system $(\mathcal{F}''_n)$ of coherent $\mathcal{O}_P$-modules such that $\text{depth}(\mathcal{F}''_{n,q}) + \dim(\overline{\{q\}}) \geq 2$ for all $q \in Q$.

We will use More on Morphisms, Lemma 37.51.1 and we will use the notation used and results found More on Morphisms, Section 37.51 without further mention; this proof will not make sense without at least understanding the statement of the lemma. Observe that in our case $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$ is a finite type k-algebra all of whose graded parts are finite dimensional k-vector spaces, see Cohomology of Schemes, Lemma 30.16.1.

By Cohomology of Schemes, Lemma 30.23.9 the pull back by $\pi|_U : U \to P$ is an object $(\pi|^*_U \mathcal{F}_n)$ of $Coh(U, f\mathcal{O}_U)$ which is pro-isomorphic to the inverse system $(\pi|^*_U \mathcal{F}''_n)$ of coherent $\mathcal{O}_U$-modules. We claim

$$\text{depth}(\pi|^*_U \mathcal{F}''_{n,y}) + \delta^Y_Z(y) \geq 3$$

for all $y \in U \cap Y$. Since all the points of Z are closed, we see that $\delta_Z^Y(y) \geq \dim(\overline{\{y\}})$ for all $y \in U \cap Y$, see Lemma 52.18.1. Let $q \in Q$ be the image of y. Since the morphism $\pi : U \to P$ is smooth of relative dimension 1 we see that either y is a closed point of a fibre of π or a generic point. Thus we see that

$$\text{depth}(\pi^*\mathcal{F}''_{n,y}) + \delta_Z^Y(y) \geq \text{depth}(\pi^*\mathcal{F}''_{n,y}) + \dim(\overline{\{y\}}) = \text{depth}(\mathcal{F}''_{n,q}) + \dim(\overline{\{q\}}) + 1$$

because either the depth goes up by 1 or the dimension. This proves the claim.

By Lemma 52.22.1 we conclude that $(\pi|_U^*\mathcal{F}_n)$ canonically extends to X. Observe that

$$M_n = \Gamma(U, \pi|_U^*\mathcal{F}_n) = \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(P, \mathcal{F}_n \otimes_{\mathcal{O}_P} \mathcal{L}^{\otimes m})$$

is canonically a graded A-module, see More on Morphisms, Equation (37.51.0.2). By Properties, Lemma 28.18.2 we have $\pi|_U^*\mathcal{F}_n = \widetilde{M_n}|_U$. Thus we may apply Lemma 52.28.4 to find a finite graded A-module N such that (M_n) and (N/I^nN) are pro-isomorphic in the category of graded A-modules modulo A_+-torsion modules. Let $\mathcal{F}$ be the coherent $\mathcal{O}_P$-module associated to N, see Cohomology of Schemes, Proposition 30.15.3. The same proposition tells us that $(\mathcal{F}/\mathcal{I}^n\mathcal{F})$ is pro-isomorphic to $(\mathcal{F}_n)$. Since both are objects of $Coh(P, \mathcal{I})$ we win by Lemma 52.15.3. □

0EL6 **Example 52.28.6.** Let k be a field and let X be a proper variety over k. Let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y)$ is ample and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $(\mathcal{E}_n)$ an object of $Coh(X, \mathcal{I})$ with $\mathcal{E}_n$ finite locally free. Here are some special cases of Proposition 52.28.5.

(1) If X is a curve or a surface, we don't learn anything.
(2) If X is a Cohen-Macaulay threefold, then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.
(3) More generally, if $\dim(X) \geq 3$ and X is (S_3), then $(\mathcal{E}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{E}$.

Of course, if $\mathcal{E}$ exists, then $\mathcal{E}$ is finite locally free in an open neighbourhood of Y.

0EL7 **Proposition 52.28.7.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s and denote $\mathcal{I} \subset \mathcal{O}_X$ the corresponding sheaf of ideals. Let $\mathcal{V}$ be the set of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have*

$$depth(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the completion functor

$$\text{colim}_{\mathcal{V}} Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I})$$

is an equivalence on the full subcategories of finite locally free objects.

Proof. To prove fully faithfulness it suffices to prove that

$$\text{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$

is an isomorphism for all m, see Lemma 52.15.2. This follows from Lemma 52.28.2.

Essential surjectivity. Let $(\mathcal{F}_n)$ be a finite locally free object of $Coh(X, \mathcal{I})$. Then for $y \in Y$ we have $\mathcal{F}_y^\wedge = \lim \mathcal{F}_{n,y}$ is is a finite free $\mathcal{O}_{X,y}^\wedge$-module. Let $\mathfrak{p} \subset \mathcal{O}_{X,y}^\wedge$ be a prime with $\mathfrak{p} \notin V(\mathcal{I}_y^\wedge)$. Then $\mathfrak{p}$ lies over a prime $\mathfrak{p}_0 \subset \mathcal{O}_{X,y}$ which corresponds

to a specialization $x \rightsquigarrow y$ with $x \notin Y$. By Local Cohomology, Lemma 51.11.3 and some dimension theory (see Varieties, Section 33.20) we have

$$\text{depth}((\mathcal{O}^\wedge_{X,y})_\mathfrak{p}) + \dim(\mathcal{O}^\wedge_{X,y}/\mathfrak{p}) = \text{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) - \dim(\overline{\{y\}})$$

Thus our assumptions imply the assumptions of Proposition 52.28.5 are satisfied and we find that $(\mathcal{F}_n)$ is the completion of a coherent $\mathcal{O}_X$-module $\mathcal{F}$. It then follows that $\mathcal{F}_y$ is finite free for all $y \in Y$ and hence $\mathcal{F}$ is finite locally free in an open neighbourhood V of Y. This finishes the proof. □

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CHAPTER 53

Algebraic Curves

0BRV

53.1. Introduction

0BRW In this chapter we develop some of the theory of algebraic curves. A reference covering algebraic curves over the complex numbers is the book [**ACGH85**].

What we already know. Besides general algebraic geometry, we have already proved some specific results on algebraic curves. Here is a list.

(1) We have discussed affine opens of and ample invertible sheaves on 1 dimensional Noetherian schemes in Varieties, Section 33.38.
(2) We have seen a curve is either affine or projective in Varieties, Section 33.43.
(3) We have discussed degrees of locally free modules on proper curves in Varieties, Section 33.44.
(4) We have discussed the Picard scheme of a nonsingular projective curve over an algebraically closed field in Picard Schemes of Curves, Section 44.1.

53.2. Curves and function fields

0BXX In this section we elaborate on the results of Varieties, Section 33.4 in the case of curves.

0BXY **Lemma 53.2.1.** *Let k be a field. Let X be a curve and Y a proper variety. Let $U \subset X$ be a nonempty open and let $f : U \to Y$ be a morphism. If $x \in X$ is a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring, then there exist an open $U \subset U' \subset X$ containing x and a morphism of varieties $f' : U' \to Y$ extending f.*

Proof. This is a special case of Morphisms, Lemma 29.42.5. □

0BXZ **Lemma 53.2.2.** *Let k be a field. Let X be a normal curve and Y a proper variety. The set of rational maps from X to Y is the same as the set of morphisms $X \to Y$.*

Proof. A rational map from X to Y can be extended to a morphism $X \to Y$ by Lemma 53.2.1 as every local ring is a discrete valuation ring (for example by Varieties, Lemma 33.43.8). Conversely, if two morphisms $f, g : X \to Y$ are equivalent as rational maps, then $f = g$ by Morphisms, Lemma 29.7.10. □

0CCK **Lemma 53.2.3.** *Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of curves over k. If Y is normal, then f is flat.*

Proof. Pick $x \in X$ mapping to $y \in Y$. Then $\mathcal{O}_{Y,y}$ is either a field or a discrete valuation ring (Varieties, Lemma 33.43.8). Since f is nonconstant it is dominant (as it must map the generic point of X to the generic point of Y). This implies that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective (Morphisms, Lemma 29.8.8). Hence $\mathcal{O}_{X,x}$ is torsion free

as a $\mathcal{O}_{Y,y}$-module and therefore $\mathcal{O}_{X,x}$ is flat as a $\mathcal{O}_{Y,y}$-module by More on Algebra, Lemma 15.22.10. □

0CCL **Lemma 53.2.4.** *Let k be a field. Let $f : X \to Y$ be a morphism of schemes over k. Assume*

(1) *Y is separated over k,*
(2) *X is proper of dimension ≤ 1 over k,*
(3) *$f(Z)$ has at least two points for every irreducible component $Z \subset X$ of dimension 1.*

Then f is finite.

Proof. The morphism f is proper by Morphisms, Lemma 29.41.7. Thus $f(X)$ is closed and images of closed points are closed. Let $y \in Y$ be the image of a closed point in X. Then $f^{-1}(\{y\})$ is a closed subset of X not containing any of the generic points of irreducible components of dimension 1 by condition (3). It follows that $f^{-1}(\{y\})$ is finite. Hence f is finite over an open neighbourhood of y by More on Morphisms, Lemma 37.44.2 (if Y is Noetherian, then you can use the easier Cohomology of Schemes, Lemma 30.21.2). Since we've seen above that there are enough of these points y, the proof is complete. □

0BY0 **Lemma 53.2.5.** *Let k be a field. Let $X \to Y$ be a morphism of varieties with Y proper and X a curve. There exists a factorization $X \to \overline{X} \to Y$ where $X \to \overline{X}$ is an open immersion and $\overline{X}$ is a projective curve.*

Proof. This is clear from Lemma 53.2.1 and Varieties, Lemma 33.43.6. □

Here is the main theorem of this section. We will say a morphism $f : X \to Y$ of varieties is *constant* if the image $f(X)$ consists of a single point y of Y. If this happens then y is a closed point of Y (since the image of a closed point of X will be a closed point of Y).

0BY1 **Theorem 53.2.6.** *Let k be a field. The following categories are canonically equivalent*

(1) *The category of finitely generated field extensions K/k of transcendence degree 1.*
(2) *The category of curves and dominant rational maps.*
(3) *The category of normal projective curves and nonconstant morphisms.*
(4) *The category of nonsingular projective curves and nonconstant morphisms.*
(5) *The category of regular projective curves and nonconstant morphisms.*
(6) *The category of normal proper curves and nonconstant morphisms.*

Proof. The equivalence between categories (1) and (2) is the restriction of the equivalence of Varieties, Theorem 33.4.1. Namely, a variety is a curve if and only if its function field has transcendence degree 1, see for example Varieties, Lemma 33.20.3.

The categories in (3), (4), (5), and (6) are the same. First of all, the terms "regular" and "nonsingular" are synonyms, see Properties, Definition 28.9.1. Being normal and regular are the same thing for Noetherian 1-dimensional schemes (Properties, Lemmas 28.9.4 and 28.12.6). See Varieties, Lemma 33.43.8 for the case of curves. Thus (3) is the same as (5). Finally, (6) is the same as (3) by Varieties, Lemma 33.43.4.

If $f : X \to Y$ is a nonconstant morphism of nonsingular projective curves, then f sends the generic point η of X to the generic point ξ of Y. Hence we obtain a morphism $k(Y) = \mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta} = k(X)$ in the category (1). If two morphisms $f, g : X \to Y$ gives the same morphism $k(Y) \to k(X)$, then by the equivalence between (1) and (2), f and g are equivalent as rational maps, so $f = g$ by Lemma 53.2.2. Conversely, suppose that we have a map $k(Y) \to k(X)$ in the category (1). Then we obtain a morphism $U \to Y$ for some nonempty open $U \subset X$. By Lemma 53.2.1 this extends to all of X and we obtain a morphism in the category (5). Thus we see that there is a fully faithful functor (5)$\to$(1).

To finish the proof we have to show that every K/k in (1) is the function field of a normal projective curve. We already know that $K = k(X)$ for some curve X. After replacing X by its normalization (which is a variety birational to X) we may assume X is normal (Varieties, Lemma 33.27.1). Then we choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ as in Varieties, Lemma 33.43.6. Since X is normal and since each of the local rings $\mathcal{O}_{\overline{X},x_i}$ is normal we conclude that $\overline{X}$ is a normal projective curve as desired. (Remark: We can also first compactify using Varieties, Lemma 33.43.5 and then normalize using Varieties, Lemma 33.27.1. Doing it this way we avoid using the somewhat tricky Morphisms, Lemma 29.53.16.) □

0BY2 **Definition 53.2.7.** Let k be a field. Let X be a curve. A *nonsingular projective model of* X is a pair (Y, φ) where Y is a nonsingular projective curve and $\varphi : k(X) \to k(Y)$ is an isomorphism of function fields.

A nonsingular projective model is determined up to unique isomorphism by Theorem 53.2.6. Thus we often say "the nonsingular projective model". We usually drop φ from the notation. Warning: it needn't be the case that Y is smooth over k but Lemma 53.2.8 shows this can only happen in positive characteristic.

0BY3 **Lemma 53.2.8.** *Let k be a field. Let X be a curve and let Y be the nonsingular projective model of X. If k is perfect, then Y is a smooth projective curve.*

Proof. See Varieties, Lemma 33.43.8 for example. □

0BY4 **Lemma 53.2.9.** *Let k be a field. Let X be a geometrically irreducible curve over k. For a field extension K/k denote Y_K a nonsingular projective model of $(X_K)_{red}$.*

(1) *If X is proper, then Y_K is the normalization of X_K.*
(2) *There exists K/k finite purely inseparable such that Y_K is smooth.*
(3) *Whenever Y_K is smooth*[1] *we have $H^0(Y_K, \mathcal{O}_{Y_K}) = K$.*
(4) *Given a commutative diagram*

$$\begin{array}{ccc} \Omega & \longleftarrow & K' \\ \uparrow & & \uparrow \\ K & \longleftarrow & k \end{array}$$

of fields such that Y_K and $Y_{K'}$ are smooth, then $Y_\Omega = (Y_K)_\Omega = (Y_{K'})_\Omega$.

Proof. Let X' be a nonsingular projective model of X. Then X' and X have isomorphic nonempty open subschemes. In particular X' is geometrically irreducible as X is (some details omitted). Thus we may assume that X is projective.

[1] Or even geometrically reduced.

Assume X is proper. Then X_K is proper and hence the normalization $(X_K)^\nu$ is proper as a scheme finite over a proper scheme (Varieties, Lemma 33.27.1 and Morphisms, Lemmas 29.44.11 and 29.41.4). On the other hand, X_K is irreducible as X is geometrically irreducible. Hence X_K^ν is proper, normal, irreducible, and birational to $(X_K)_{red}$. This proves (1) because a proper curve is projective (Varieties, Lemma 33.43.4).

Proof of (2). As X is proper and we have (1), we can apply Varieties, Lemma 33.27.4 to find K/k finite purely inseparable such that Y_K is geometrically normal. Then Y_K is geometrically regular as normal and regular are the same for curves (Properties, Lemma 28.12.6). Then Y is a smooth variety by Varieties, Lemma 33.12.6.

If Y_K is geometrically reduced, then Y_K is geometrically integral (Varieties, Lemma 33.9.2) and we see that $H^0(Y_K, \mathcal{O}_{Y_K}) = K$ by Varieties, Lemma 33.26.2. This proves (3) because a smooth variety is geometrically reduced (even geometrically regular, see Varieties, Lemma 33.12.6).

If Y_K is smooth, then for every extension Ω/K the base change $(Y_K)_\Omega$ is smooth over Ω (Morphisms, Lemma 29.34.5). Hence it is clear that $Y_\Omega = (Y_K)_\Omega$. This proves (4). □

53.3. Linear series

0CCM We deviate from the classical story (see Remark 53.3.6) by defining linear series in the following manner.

0CCN **Definition 53.3.1.** Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $d \geq 0$ and $r \geq 0$. A *linear series of degree d and dimension r* is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module of degree d (Varieties, Definition 33.44.1) and $V \subset H^0(X, \mathcal{L})$ is a k-subvector space of dimension $r + 1$. We will abbreviate this by saying $(\mathcal{L}, V)$ is a $\mathfrak{g}^r_d$ on X.

We will mostly use this when X is a nonsingular proper curve. In fact, the definition above is just one way to generalize the classical definition of a $\mathfrak{g}^r_d$. For example, if X is a proper curve, then one can generalize linear series by allowing $\mathcal{L}$ to be a torsion free coherent $\mathcal{O}_X$-module of rank 1. On a nonsingular curve every torsion free coherent module is locally free, so this agrees with our notion for nonsingular proper curves.

The following lemma explains the geometric meaning of linear series for proper nonsingular curves.

0CCP **Lemma 53.3.2.** *Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ on X. Then there exists a morphism*

$$\varphi : X \longrightarrow \mathbf{P}^r_k = Proj(k[T_0, \ldots, T_r])$$

of varieties over k and a map $\alpha : \varphi^*\mathcal{O}_{\mathbf{P}^r_k}(1) \to \mathcal{L}$ such that $\varphi^*T_0, \ldots, \varphi^*T_r$ are sent to a basis of V by α.

Proof. Let $s_0, \ldots, s_r \in V$ be a k-basis. Since X is nonsingular the image $\mathcal{L}' \subset \mathcal{L}$ of the map $s_0, \ldots, s_r : \mathcal{O}_X^{\oplus r+1} \to \mathcal{L}$ is an invertible $\mathcal{O}_X$-module for example by Divisors, Lemma 31.11.11. Then we use Constructions, Lemma 27.13.1 to get a morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k$$

as in the statement of the lemma. □

0CCQ **Lemma 53.3.3.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If X has a $\mathfrak{g}^r_d$, then X has a $\mathfrak{g}^s_d$ for all $0 \leq s \leq r$.*

Proof. This is true because a vector space V of dimension $r+1$ over k has a linear subspace of dimension $s+1$ for all $0 \leq s \leq r$. □

0CCR **Lemma 53.3.4.** *Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^1_d$ on X. Then the morphism $\varphi : X \to \mathbf{P}^1_k$ of Lemma 53.3.2 either*

(1) *is nonconstant and has degree $\leq d$, or*
(2) *factors through a closed point of $\mathbf{P}^1_k$ and in this case $H^0(X, \mathcal{O}_X) \neq k$.*

Proof. By Lemma 53.3.2 we see that $\mathcal{L}' = \varphi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ has a nonzero map $\mathcal{L}' \to \mathcal{L}$. Hence by Varieties, Lemma 33.44.12 we see that $0 \leq \deg(\mathcal{L}') \leq d$. If $\deg(\mathcal{L}') = 0$, then the same lemma tells us $\mathcal{L}' \cong \mathcal{O}_X$ and since we have two linearly independent sections we find we are in case (2). If $\deg(\mathcal{L}') > 0$ then φ is nonconstant (since the pullback of an invertible module by a constant morphism is trivial). Hence

$$\deg(\mathcal{L}') = \deg(X/\mathbf{P}^1_k)\deg(\mathcal{O}_{\mathbf{P}^1_k}(1))$$

by Varieties, Lemma 33.44.11. This finishes the proof as the degree of $\mathcal{O}_{\mathbf{P}^1_k}(1)$ is 1. □

0CCS **Lemma 53.3.5.** *Let k be a field. Let X be a proper curve over k with $H^0(X, \mathcal{O}_X) = k$. If X has a $\mathfrak{g}^r_d$, then $r \leq d$. If equality holds, then $H^1(X, \mathcal{O}_X) = 0$, i.e., the genus of X (Definition 53.8.1) is 0.*

Proof. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$. Since this will only increase r, we may assume $V = H^0(X, \mathcal{L})$. Choose a nonzero element $s \in V$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$, and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 31.14.10 and Remark 31.14.11. By Varieties, Lemma 33.44.9 we have $\deg(D) = \deg(\mathcal{L}) = d$. Since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[2]. Thus

$$\dim_k H^0(D, \mathcal{L}|_D) = \dim_k H^0(D, \mathcal{O}_D) = \deg(D) = d$$

On the other hand, by assumption $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim H^0(X, \mathcal{L}) = r+1$. We conclude that $r + 1 \leq 1 + d$, i.e., $r \leq d$ as in the lemma.

Assume equality holds. Then $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. If we knew that $H^1(X, \mathcal{L})$ was zero, then we would conclude that $H^1(X, \mathcal{O}_X)$ is zero by the long exact cohomology sequence and the proof would be complete. Our strategy will be to replace $\mathcal{L}$ by a large power which has vanishing. As $\mathcal{L}|_D$ is the trivial invertible module (see above), we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. Consider the multiplication map

$$\mu : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 2})$$

and consider the short exact sequence

$$0 \to \mathcal{L} \xrightarrow{s} \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}|_D \to 0$$

[2]In our case this follows from Divisors, Lemma 31.17.1 as $D \to \mathrm{Spec}(k)$ is finite.

Since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t maps to a trivialization of $\mathcal{L}|_D$ we see that $\mu(H^0(X, \mathcal{L}) \otimes t)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes 2})$ surjecting onto the global sections of $\mathcal{L}^{\otimes 2}|_D$. Thus we see that

$$\dim H^0(X, \mathcal{L}^{\otimes 2}) = r + 1 + d = 2r + 1 = \deg(\mathcal{L}^{\otimes 2}) + 1$$

Ok, so $\mathcal{L}^{\otimes 2}$ has the same property as $\mathcal{L}$, i.e., that the dimension of the space of global sections is equal to the degree plus one. Since $\mathcal{L}$ is ample (Varieties, Lemma 33.44.14) there exists some n_0 such that $\mathcal{L}^{\otimes n}$ has vanishing H^1 for all $n \geq n_0$ (Cohomology of Schemes, Lemma 30.16.1). Thus applying the argument above to $\mathcal{L}^{\otimes n}$ with $n = 2^m$ for some sufficiently large m we conclude the lemma is true. □

0CCT **Remark 53.3.6** (Classical definition)**.** Let X be a smooth projective curve over an algebraically closed field k. We say two effective Cartier divisors $D, D' \subset X$ are *linearly equivalent* if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules. Since $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ (Divisors, Lemma 31.27.7) we see that D and D' are linearly equivalent if and only if the Weil divisors associated to D and D' define the same element of $\mathrm{Cl}(X)$. Given an effective Cartier divisor $D \subset X$ of degree d the *complete linear system* or *complete linear series* $|D|$ of D is the set of effective Cartier divisors $E \subset X$ which are linearly equivalent to D. Another way to say it is that $|D|$ is the set of closed points of the fibre of the morphism

$$\gamma_d : \underline{\mathrm{Hilb}}^d_{X/k} \longrightarrow \underline{\mathrm{Pic}}^d_{X/k}$$

(Picard Schemes of Curves, Lemma 44.6.7) over the closed point corresponding to $\mathcal{O}_X(D)$. This gives $|D|$ a natural scheme structure and it turns out that $|D| \cong \mathbf{P}^m_k$ with $m + 1 = h^0(\mathcal{O}_X(D))$. In fact, more canonically we have

$$|D| = \mathbf{P}(H^0(X, \mathcal{O}_X(D))^\vee)$$

where $(-)^\vee$ indicates k-linear dual and $\mathbf{P}$ is as in Constructions, Example 27.21.2. In this language a *linear system* or a *linear series* on X is a closed subvariety $L \subset |D|$ which can be cut out by linear equations. If L has dimension r, then $L = \mathbf{P}(V^\vee)$ where $V \subset H^0(X, \mathcal{O}_X(D))$ is a linear subspace of dimension $r + 1$. Thus the classical linear series $L \subset |D|$ corresponds to the linear series $(\mathcal{O}_X(D), V)$ as defined above.

53.4. Duality

0E31 In this section we work out the consequences of the very general material on dualizing complexes and duality for proper 1-dimensional schemes over fields. If you are interested in the analogous discussion for higher dimension proper schemes over fields, see Duality for Schemes, Section 48.27.

0BS2 **Lemma 53.4.1.** *Let X be a proper scheme of dimension ≤ 1 over a field k. There exists a dualizing complex $\omega_X^\bullet$ with the following properties*

1. *$H^i(\omega_X^\bullet)$ is nonzero only for $i = -1, 0$,*
2. *$\omega_X = H^{-1}(\omega_X^\bullet)$ is a coherent Cohen-Macaulay module whose support is the irreducible components of dimension 1,*
3. *for $x \in X$ closed, the module $H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if either*
   (a) *$\dim(\mathcal{O}_{X,x}) = 0$ or*
   (b) *$\dim(\mathcal{O}_{X,x}) = 1$ and $\mathcal{O}_{X,x}$ is not Cohen-Macaulay,*

(4) *for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms*[3]
$$\mathrm{Ext}^i_X(K, \omega_X^\bullet) = \mathrm{Hom}_k(H^{-i}(X, K), k)$$
compatible with shifts and distinguished triangles,
(5) *there are functorial isomorphisms* $\mathrm{Hom}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ *for $\mathcal{F}$ quasi-coherent on X,*
(6) *if $X \to \mathrm{Spec}(k)$ is smooth of relative dimension 1, then $\omega_X \cong \Omega_{X/k}$.*

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism. We start with the relative dualizing complex
$$\omega_X^\bullet = \omega_{X/k}^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$$
as described in Duality for Schemes, Remark 48.12.5. Then property (4) holds by construction as a is the right adjoint for $f_* : D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_{\mathrm{Spec}(k)})$. Since f is proper we have $f^!(\mathcal{O}_{\mathrm{Spec}(k)}) = a(\mathcal{O}_{\mathrm{Spec}(k)})$ by definition, see Duality for Schemes, Section 48.16. Hence $\omega_X^\bullet$ and ω_X are as in Duality for Schemes, Example 48.22.1 and as in Duality for Schemes, Example 48.22.2. Parts (1) and (2) follow from Duality for Schemes, Lemma 48.22.4. For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Duality for Schemes, Lemma 48.21.1. Assertion (3) then follows from Dualizing Complexes, Lemma 47.20.2. Assertion (5) follows from Duality for Schemes, Lemma 48.22.5 for coherent $\mathcal{F}$ and in general by unwinding (4) for $K = \mathcal{F}[0]$ and $i = -1$. Assertion (6) follows from Duality for Schemes, Lemma 48.15.7. □

0BS3 **Lemma 53.4.2.** *Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension 1. The module ω_X of Lemma 53.4.1 has the following properties*

(1) *ω_X is a dualizing module on X (Duality for Schemes, Section 48.22),*
(2) *ω_X is a coherent Cohen-Macaulay module whose support is X,*
(3) *there are functorial isomorphisms* $\mathrm{Ext}^i_X(K, \omega_X[1]) = \mathrm{Hom}_k(H^{-i}(X, K), k)$ *compatible with shifts for $K \in D_{QCoh}(X)$,*
(4) *there are functorial isomorphisms* $\mathrm{Ext}^{1+i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^{-i}(X, \mathcal{F}), k)$ *for $\mathcal{F}$ quasi-coherent on X.*

Proof. Recall from the proof of Lemma 53.4.1 that ω_X is as in Duality for Schemes, Example 48.22.1 and hence is a dualizing module. The other statements follow from Lemma 53.4.1 and the fact that $\omega_X^\bullet = \omega_X[1]$ as X is Cohen-Macualay (Duality for Schemes, Lemma 48.23.1). □

0BS4 **Remark 53.4.3.** Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 53.4.1. If $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$ then we have canonical isomorphisms
$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^i(X, \mathcal{E}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_X^\bullet)$$
This follows from the lemma and Cohomology, Lemma 20.50.5. If X is Cohen-Macaulay and equidimensional of dimension 1, then we have canonical isomorphisms
$$\mathrm{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^{1+i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)$$

[3]This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$ in fact it suffices to consider $K \in D^b_{Coh}(\mathcal{O}_X)$.

by Lemma 53.4.2. In particular if $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we have

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

and

$$\dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

Here is a sanity check for the dualizing complex.

0E32 **Lemma 53.4.4.** *Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 53.4.1.*

(1) *If $X \to \mathrm{Spec}(k)$ factors as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for some field k', then $\omega_X^\bullet$ and ω_X satisfy properties (4), (5), (6) with k replaced with k'.*
(2) *If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 53.4.1 for the morphism $X_K \to \mathrm{Spec}(K)$.*

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism. Assertion (1) really means that $\omega_X^\bullet$ and ω_X are as in Lemma 53.4.1 for the morphism $f' : X \to \mathrm{Spec}(k')$. In the proof of Lemma 53.4.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$ where a be is the right adjoint of Duality for Schemes, Lemma 48.3.1 for f. Thus we have to show $a(\mathcal{O}_{\mathrm{Spec}(k)}) \cong a'(\mathcal{O}_{\mathrm{Spec}(k)})$ where a' be is the right adjoint of Duality for Schemes, Lemma 48.3.1 for f'. Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 30.19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Duality for Schemes, Lemma 48.3.1 for $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that $\mathrm{Hom}_k(k', k) \cong k'$ as k'-modules, see Duality for Schemes, Example 48.3.2. This holds because these are k'-vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Duality for Schemes, Lemma 48.6.2. See discussion in Duality for Schemes, Remark 48.12.5. □

0E33 **Lemma 53.4.5.** *Let X be a proper scheme of dimension ≤ 1 over a field k. Let $i : Y \to X$ be a closed immersion. Let $\omega_X^\bullet$, ω_X, $\omega_Y^\bullet$, ω_Y be as in Lemma 53.4.1. Then*

(1) $\omega_Y^\bullet = R\mathcal{H}om(\mathcal{O}_Y, \omega_X^\bullet)$,
(2) $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ *and* $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$.

Proof. Denote $g : Y \to \mathrm{Spec}(k)$ and $f : X \to \mathrm{Spec}(k)$ the structure morphisms. Then $g = f \circ i$. Denote a, b, c the right adjoint of Duality for Schemes, Lemma 48.3.1 for f, g, i. Then $b = c \circ a$ by uniqueness of right adjoints and because $Rg_* = Rf_* \circ Ri_*$. In the proof of Lemma 53.4.1 we set $\omega_X^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$ and $\omega_Y^\bullet = b(\mathcal{O}_{\mathrm{Spec}(k)})$. Hence $\omega_Y^\bullet = c(\omega_X^\bullet)$ which implies (1) by Duality for Schemes, Lemma 48.9.7. Since $\omega_X = H^{-1}(\omega_X^\bullet)$ and $\omega_Y = H^{-1}(\omega_Y^\bullet)$ we conclude that $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$. This implies $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Duality for Schemes, Lemma 48.9.3. □

0E34 **Lemma 53.4.6.** *Let X be a proper scheme over a field k which is Gorenstein, reduced, and equidimensional of dimension 1. Let $i : Y \to X$ be a reduced closed subscheme equidimensional of dimension 1. Let $j : Z \to X$ be the scheme theoretic closure of $X \setminus Y$. Then*

(1) *Y and Z are Cohen-Macaulay,*

(2) *if $\mathcal{I} \subset \mathcal{O}_X$, resp. $\mathcal{J} \subset \mathcal{O}_X$ is the ideal sheaf of Y, resp. Z in X, then*

$$\mathcal{I} = i_*\mathcal{I}' \quad \text{and} \quad \mathcal{J} = j_*\mathcal{J}'$$

where $\mathcal{I}' \subset \mathcal{O}_Z$, resp. $\mathcal{J}' \subset \mathcal{O}_Y$ is the ideal sheaf of $Y \cap Z$ in Z, resp. Y,

(3) *$\omega_Y = \mathcal{J}'(i^*\omega_X)$ and $i_*(\omega_Y) = \mathcal{J}\omega_X$,*

(4) *$\omega_Z = \mathcal{I}'(i^*\omega_X)$ and $i_*(\omega_Z) = \mathcal{I}\omega_X$,*

(5) *we have the following short exact sequences*

$$\begin{aligned}
0 \to \omega_X \to i_*i^*\omega_X \oplus j_*j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to i_*\omega_Y \to \omega_X \to j_*j^*\omega_X \to 0 \\
0 \to j_*\omega_Z \to \omega_X \to i_*i^*\omega_X \to 0 \\
0 \to i_*\omega_Y \oplus j_*\omega_Z \to \omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \omega_Y \to i^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \omega_Z \to j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0
\end{aligned}$$

Here ω_X, ω_Y, ω_Z are as in Lemma 53.4.1.

Proof. A reduced 1-dimensional Noetherian scheme is Cohen-Macaulay, so (1) is true. Since X is reduced, we see that $X = Y \cup Z$ scheme theoretically. With notation as in Morphisms, Lemma 29.4.6 and by the statement of that lemma we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0$$

Since $\mathcal{J} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Z)$, $\mathcal{J}' = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y\cap Z})$, $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$, and $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_Z \to \mathcal{O}_{Y\cap Z})$ a diagram chase implies (2). Observe that $\mathcal{I}+\mathcal{J}$ is the ideal sheaf of $Y \cap Z$ and that $\mathcal{I} \cap \mathcal{J} = 0$. Hence we have the following exact sequences

$$\begin{aligned}
0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \\
0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \\
0 \to \mathcal{J} \oplus \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{J}' \to \mathcal{O}_Y \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{I}' \to \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0
\end{aligned}$$

Since X is Gorenstein ω_X is an invertible $\mathcal{O}_X$-module (Duality for Schemes, Lemma 48.24.4). Since $Y \cap Z$ has dimension 0 we have $\omega_X|_{Y\cap Z} \cong \mathcal{O}_{Y\cap Z}$. Thus if we prove (3) and (4), then we obtain the short exact sequences of the lemma by tensoring the above short exact sequence with the invertible module ω_X. By symmetry it suffices to prove (3) and by (2) it suffices to prove $i_*(\omega_Y) = \mathcal{J}\omega_X$.

We have $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5. Again using that ω_X is invertible we finally conclude that it suffices to show $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X)$ maps isomorphically to $\mathcal{J}$ by evaluation at 1. In other words, that $\mathcal{J}$ is the annihilator of $\mathcal{I}$. This follows from the above. □

53.5. Riemann-Roch

0B5B

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. In Varieties, Section 33.44 we have defined the degree of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank by the formula

0BRX (53.5.0.1) $$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

see Varieties, Definition 33.44.1. In the chapter on Chow Homology we defined the first Chern class of $\mathcal{E}$ as an operation on cycles (Chow Homology, Section 42.38) and we proved that

0BRY (53.5.0.2) $$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

see Chow Homology, Lemma 42.41.3. Combining (53.5.0.1) and (53.5.0.2) we obtain our first version of the Riemann-Roch formula

0BRZ (53.5.0.3) $$\chi(X, \mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1) + \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X)$$

If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we can also consider the numerical intersection $(\mathcal{L} \cdot X)$ as defined in Varieties, Definition 33.45.3. However, this does not give anything new as

0BS0 (53.5.0.4) $$(\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

by Varieties, Lemma 33.45.12. If $\mathcal{L}$ is ample, then this integer is positive and is called the degree

0BS1 (53.5.0.5) $$\deg_{\mathcal{L}}(X) = (\mathcal{L} \cdot X) = \deg(\mathcal{L})$$

of X with respect to $\mathcal{L}$, see Varieties, Definition 33.45.10.

To obtain a true Riemann-Roch theorem we would like to write $\chi(X, \mathcal{O}_X)$ as the degree of a canonical zero cycle on X. We refer to [**Ful98**] for a fully general version of this. We will use duality to get a formula in the case where X is Gorenstein; however, in some sense this is a cheat (for example because this method cannot work in higher dimension).

We first use Lemmas 53.4.1 and 53.4.2 to get a relation between the euler characteristic of $\mathcal{O}_X$ and the euler characteristic of the dualizing complex or the dualizing module.

0BS5 **Lemma 53.5.1.** *Let X be a proper scheme of dimension ≤ 1 over a field k. With $\omega_X^\bullet$ and ω_X as in Lemma 53.4.1 we have*

$$\chi(X, \mathcal{O}_X) = \chi(X, \omega_X^\bullet)$$

If X is Cohen-Macaulay and equidimensional of dimension 1, then

$$\chi(X, \mathcal{O}_X) = -\chi(X, \omega_X)$$

Proof. We define the right hand side of the first formula as follows:

$$\chi(X, \omega_X^\bullet) = \sum\nolimits_{i \in \mathbf{Z}} (-1)^i \dim_k H^i(X, \omega_X^\bullet)$$

This is well defined because $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$, but also because

$$H^i(X, \omega_X^\bullet) = \text{Ext}^i(\mathcal{O}_X, \omega_X^\bullet) = H^{-i}(X, \mathcal{O}_X)$$

which is always finite dimensional and nonzero only if $i = 0, -1$. This of course also proves the first formula. The second is a consequence of the first because $\omega_X^\bullet = \omega_X[1]$ in the CM case, see Lemma 53.4.2. □

We will use Lemma 53.5.1 to get the desired formula for $\chi(X, \mathcal{O}_X)$ in the case that ω_X is invertible, i.e., that X is Gorenstein. The statement is that $-1/2$ of the first Chern class of ω_X capped with the cycle $[X]_1$ associated to X is a natural zero cycle on X with half-integer coefficients whose degree is $\chi(X, \mathcal{O}_X)$. The occurrence of fractions in the statement of Riemann-Roch cannot be avoided.

0BS6 **Lemma 53.5.2** (Riemann-Roch)**.** *Let X be a proper scheme over a field k which is Gorenstein and equidimensional of dimension 1. Let ω_X be as in Lemma 53.4.1. Then*

(1) *ω_X is an invertible $\mathcal{O}_X$-module,*
(2) *$\deg(\omega_X) = -2\chi(X, \mathcal{O}_X)$,*
(3) *for a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank we have*

$$\chi(X, \mathcal{E}) = \deg(\mathcal{E}) - \tfrac{1}{2} rank(\mathcal{E}) \deg(\omega_X)$$

and $\dim_k(H^i(X, \mathcal{E})) = \dim_k(H^{1-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X))$ for all $i \in \mathbf{Z}$.

Nonsingular (normal) curves are Gorenstein, see Duality for Schemes, Lemma 48.24.3.

Proof. Recall that Gorenstein schemes are Cohen-Macaulay (Duality for Schemes, Lemma 48.24.2) and hence ω_X is a dualizing module on X, see Lemma 53.4.2. It follows more or less from the definition of the Gorenstein property that the dualizing sheaf is invertible, see Duality for Schemes, Section 48.24. By (53.5.0.3) applied to ω_X we have

$$\chi(X, \omega_X) = \deg(c_1(\omega_X) \cap [X]_1) + \chi(X, \mathcal{O}_X)$$

Combined with Lemma 53.5.1 this gives

$$2\chi(X, \mathcal{O}_X) = -\deg(c_1(\omega_X) \cap [X]_1) = -\deg(\omega_X)$$

the second equality by (53.5.0.2). Putting this back into (53.5.0.3) for $\mathcal{E}$ gives the displayed formula of the lemma. The symmetry in dimensions is a consequence of duality for X, see Remark 53.4.3. □

53.6. Some vanishing results

0B5C This section contains some very weak vanishing results. Please see Section 53.22 for a few more and more interesting results.

0BY5 **Lemma 53.6.1.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then X is connected, Cohen-Macaulay, and equidimensional of dimension 1.*

Proof. Since $\Gamma(X, \mathcal{O}_X) = k$ has no nontrivial idempotents, we see that X is connected. This already shows that X is equidimensional of dimension 1 (any irreducible component of dimension 0 would be a connected component). Let $\mathcal{I} \subset \mathcal{O}_X$ be the maximal coherent submodule supported in closed points. Then $\mathcal{I}$ exists (Divisors, Lemma 31.4.6) and is globally generated (Varieties, Lemma 33.33.3). Since $1 \in \Gamma(X, \mathcal{O}_X)$ is not a section of $\mathcal{I}$ we conclude that $\mathcal{I} = 0$. Thus X does not have embedded points (Divisors, Lemma 31.4.6). Thus X has (S_1) by Divisors, Lemma 31.4.3. Hence X is Cohen-Macaulay. □

In this section we work in the following situation.

0B5D **Situation 53.6.2.** Here k is a field, X is a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$.

By Lemma 53.6.1 the scheme X is Cohen-Macaulay and equidimensional of dimension 1. The dualizing module ω_X discussed in Lemmas 53.4.1 and 53.4.2 has nonvanishing H^1 because in fact $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$. It turns out that anything slightly more "positive" than ω_X has vanishing H^1.

0B5E **Lemma 53.6.3.** *In Situation 53.6.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $H^1(X, \mathcal{Q}) = 0$ (for example if $\dim(Supp(\mathcal{Q})) = 0$), then either $H^1(X, \mathcal{F}) = 0$ or $\mathcal{F} = \omega_X \oplus \mathcal{Q}$.

Proof. (The parenthetical statement follows from Cohomology of Schemes, Lemma 30.9.10.) Since $H^0(X, \mathcal{O}_X) = k$ is dual to $H^1(X, \omega_X)$ (see Section 53.5) we see that $\dim H^1(X, \omega_X) = 1$. The sheaf ω_X represents the functor $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ on the category of coherent $\mathcal{O}_X$-modules (Duality for Schemes, Lemma 48.22.5). Consider an exact sequence as in the statement of the lemma and assume that $H^1(X, \mathcal{F}) \neq 0$. Since $H^1(X, \mathcal{Q}) = 0$ we see that $H^1(X, \omega_X) \to H^1(X, \mathcal{F})$ is an isomorphism. By the universal property of ω_X stated above, we conclude there is a map $\mathcal{F} \to \omega_X$ whose action on H^1 is the inverse of this isomorphism. The composition $\omega_X \to \mathcal{F} \to \omega_X$ is the identity (by the universal property) and the lemma is proved. □

0B62 **Lemma 53.6.4.** *In Situation 53.6.2. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is globally generated and not isomorphic to $\mathcal{O}_X$. Then $H^1(X, \omega_X \otimes \mathcal{L}) = 0$.*

Proof. By duality as discussed in Section 53.5 we have to show that $H^0(X, \mathcal{L}^{\otimes -1}) = 0$. If not, then we can choose a global section t of $\mathcal{L}^{\otimes -1}$ and a global section s of $\mathcal{L}$ such that $st \neq 0$. However, then st is a constant multiple of 1, by our assumption that $H^0(X, \mathcal{O}_X) = k$. It follows that $\mathcal{L} \cong \mathcal{O}_X$, which is a contradiction. □

0B5F **Lemma 53.6.5.** *In Situation 53.6.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $\dim(Supp(\mathcal{Q})) = 0$ and $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and such that there is no nonzero submodule $\mathcal{Q}' \subset \mathcal{F}$ such that $\mathcal{Q}' \to \mathcal{Q}$ is injective. Then the submodule of $\mathcal{F}$ generated by global sections surjects onto $\mathcal{Q}$.

Proof. Let $\mathcal{F}' \subset \mathcal{F}$ be the submodule generated by global sections and the image of $\omega_X \to \mathcal{F}$. Since $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{F}' \to \mathcal{Q}$ is not zero and $\omega_X \to \mathcal{F}'$ is not an isomorphism. Hence $H^1(X, \mathcal{F}') = 0$ by Lemma 53.6.3 and our assumption on $\mathcal{F}$. Consider the short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q}/\operatorname{Im}(\mathcal{F}' \to \mathcal{Q}) \to 0$$

If the quotient on the right is nonzero, then we obtain a contradiction because then $H^0(X, \mathcal{F})$ is bigger than $H^0(X, \mathcal{F}')$. □

Here is an example global generation statement.

0B5G **Lemma 53.6.6.** *In Situation 53.6.2 assume that X is integral. Let $0 \to \omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ torsion free, $\dim(Supp(\mathcal{Q})) = 0$, and $\dim_k H^0(X, \mathcal{Q}) \geq 2$. Then $\mathcal{F}$ is globally generated.*

Proof. Consider the submodule $\mathcal{F}'$ generated by the global sections. By Lemma 53.6.5 we see that $\mathcal{F}' \to \mathcal{Q}$ is surjective, in particular $\mathcal{F}' \neq 0$. Since X is a curve, we see that $\mathcal{F}' \subset \mathcal{F}$ is an inclusion of rank 1 sheaves, hence $\mathcal{Q}' = \mathcal{F}/\mathcal{F}'$ is supported in finitely many points. To get a contradiction, assume that $\mathcal{Q}'$ is nonzero. Then we see that $H^1(X, \mathcal{F}') \neq 0$. Then we get a nonzero map $\mathcal{F}' \to \omega_X$ by the universal property (Duality for Schemes, Lemma 48.22.5). The image of the composition $\mathcal{F}' \to \omega_X \to \mathcal{F}$ is generated by global sections, hence is inside of $\mathcal{F}'$. Thus we get a nonzero self map $\mathcal{F}' \to \mathcal{F}'$. Since $\mathcal{F}'$ is torsion free of rank 1 on a proper curve this has to be an automorphism (details omitted). But then this implies that $\mathcal{F}'$ is contained in $\omega_X \subset \mathcal{F}$ contradicting the surjectivity of $\mathcal{F}' \to \mathcal{Q}$. □

0B5H **Lemma 53.6.7.** *In Situation 53.6.2. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module with $\deg(\mathcal{L}) \geq 2$. Then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated.*

Proof. Assume k is algebraically closed. Let $x \in X$ be a closed point. Let $C_i \subset X$ be the irreducible components and for each i let $x_i \in C_i$ be the generic point. By Varieties, Lemma 33.22.2 we can choose a section $s \in H^0(X, \mathcal{L})$ such that s vanishes at x but not at x_i for all i. The corresponding module map $s : \mathcal{O}_X \to \mathcal{L}$ is injective with cokernel $\mathcal{Q}$ supported in finitely many points and with $H^0(X, \mathcal{Q}) \geq 2$. Consider the corresponding exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L} \to \omega_X \otimes \mathcal{Q} \to 0$$

By Lemma 53.6.5 we see that the module generated by global sections surjects onto $\omega_X \otimes \mathcal{Q}$. Since x was arbitrary this proves the lemma. Some details omitted.

We will reduce the case where k is not algebraically closed, to the algebraically closed field case. We suggest the reader skip the rest of the proof. Choose an algebraic closure $\overline{k}$ of k and consider the base change $X_{\overline{k}}$. Let us check that $X_{\overline{k}} \to \mathrm{Spec}(\overline{k})$ is an example of Situation 53.6.2. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that $H^0(X_{\overline{k}}, \mathcal{O}) = \overline{k}$. The scheme $X_{\overline{k}}$ is proper over $\overline{k}$ (Morphisms, Lemma 29.41.5) and equidimensional of dimension 1 (Morphisms, Lemma 29.28.3). The pullback of ω_X to $X_{\overline{k}}$ is the dualizing module of $X_{\overline{k}}$ by Lemma 53.4.4. The pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is very ample (Morphisms, Lemma 29.38.8). The degree of the pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is equal to the degree of $\mathcal{L}$ on X (Varieties, Lemma 33.44.2). Finally, we see that $\omega_X \otimes \mathcal{L}$ is globally generated if and only if its base change is so (Varieties, Lemma 33.22.1). In this way we see that the result follows from the result in the case of an algebraically closed ground field. □

53.7. Very ample invertible sheaves

0E8U An often used criterion for very ampleness of an invertible module $\mathcal{L}$ on a scheme X of finite type over an algebraically closed field is: sections of $\mathcal{L}$ separate points and tangent vectors (Varieties, Section 33.23). Here is another criterion for curves; please compare with Varieties, Subsection 33.35.6.

0E8V **Lemma 53.7.1.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

(1) *$\mathcal{L}$ has a regular global section,*
(2) *$H^1(X, \mathcal{L}) = 0$, and*
(3) *$\mathcal{L}$ is ample.*

Then $\mathcal{L}^{\otimes 6}$ is very ample on X over k.

Proof. Let s be a regular global section of $\mathcal{L}$. Let $i : Z = Z(s) \to X$ be the zero scheme of s, see Divisors, Section 31.14. By condition (3) we see that $Z \neq \emptyset$ (small detail omitted). Consider the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_Z) \to 0$$

Tensoring with $\mathcal{L}$ we obtain

$$0 \to \mathcal{L} \to \mathcal{L}^{\otimes 2} \to i_*(\mathcal{L}^{\otimes 2}|_Z) \to 0$$

Observe that Z has dimension 0 (Divisors, Lemma 31.13.5) and hence is the spectrum of an Artinian ring (Varieties, Lemma 33.20.2) hence $\mathcal{L}|_Z \cong \mathcal{O}_Z$ (Algebra, Lemma 10.78.7). The short exact sequence also shows that $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ (for example using Varieties, Lemma 33.33.3 to see vanishing in the spot on the right). Using induction on $n \geq 1$ and the sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

we see that $H^1(X, \mathcal{L}^{\otimes n}) = 0$ for $n > 0$ and that there exists a global section t_{n+1} of $\mathcal{L}^{\otimes n+1}$ which gives a trivialization of $\mathcal{L}^{\otimes n+1}|_Z \cong \mathcal{O}_Z$.

Consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \oplus H^0(X, \mathcal{L}^{\otimes 2}) \otimes_k H^0(X, \mathcal{L}^{\otimes n-1}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective for $n \geq 3$. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes 2}|_Z)$ is surjective (see above) and since t_{n-1} maps to a trivialization of $\mathcal{L}^{\otimes n-1}|_Z$ we see that $\mu_n(H^0(X, \mathcal{L}^{\otimes 2}) \otimes t_{n-1})$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_Z$. This proves the claim.

From the claim in the previous paragraph we conclude that the graded k-algebra

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2, 3$ over k. Recall that $X = \mathrm{Proj}(S)$, see Morphisms, Lemma 29.43.17. Thus $S^{(6)} = \bigoplus_n S_{6n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 6}$ is very ample as desired. □

0E8W **Lemma 53.7.2.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

(1) *$\mathcal{L}$ is globally generated,*
(2) *$H^1(X, \mathcal{L}) = 0$, and*
(3) *$\mathcal{L}$ is ample.*

Then $\mathcal{L}^{\otimes 2}$ is very ample on X over k.

Proof. Choose basis $s_0, \ldots, s_n$ of $H^0(X, \mathcal{L}^{\otimes 2})$ over k. By property (1) we see that $\mathcal{L}^{\otimes 2}$ is globally generated and we get a morphism

$$\varphi_{\mathcal{L}^{\otimes 2}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}^n_k$$

See Constructions, Section 27.13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 35.23.21. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. For each generic point $\eta_i \in X$ let $V_i \subset H^0(X, \mathcal{L})$ be the k-subvector space of sections vanishing at η_i. Since $\mathcal{L}$ is globally generated, we see that $V_i \neq H^0(X, \mathcal{L})$. Since X has only a finite number of irreducible components and k is infinite, we can find $s \in H^0(X, \mathcal{L})$ nonvanishing at η_i for all i. Then s is a regular section of $\mathcal{L}$ (because X is Cohen-Macaulay by Lemma 53.6.1 and hence $\mathcal{L}$ has no embedded associated points).

In particular, all of the statements given in the proof of Lemma 53.7.1 hold with this s. Moreover, as $\mathcal{L}$ is globally generated, we can find a global section $t \in H^0(X, \mathcal{L})$ such that $t|_Z$ is nonvanishing (argue as above using the finite number of points of Z). Then in the proof of Lemma 53.7.1 we can use t to see that additionally the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes 2}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 3})$$

is surjective. Thus

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2$ over k. Arguing as in the proof of Lemma 53.7.1 we find that $S^{(2)} = \bigoplus_n S_{2n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 2}$ is very ample as desired. Some details omitted. □

53.8. The genus of a curve

0BY6 If X is a smooth projective geometrically irreducible curve over a field k, then we've previously defined the genus of X as the dimension of $H^1(X, \mathcal{O}_X)$, see Picard Schemes of Curves, Definition 44.6.3. Observe that $H^0(X, \mathcal{O}_X) = k$ in this case, see Varieties, Lemma 33.26.2. Let us generalize this as follows.

0BY7 **Definition 53.8.1.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then the *genus* of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

This is sometimes called the *arithmetic genus* of X. In the literature the arithmetic genus of a proper curve X over k is sometimes defined as

$$p_a(X) = 1 - \chi(X, \mathcal{O}_X) = 1 - \dim_k H^0(X, \mathcal{O}_X) + \dim_k H^1(X, \mathcal{O}_X)$$

This agrees with our definition when it applies because we assume $H^0(X, \mathcal{O}_X) = k$. But note that

(1) $p_a(X)$ can be negative, and
(2) $p_a(X)$ depends on the base field k and should be written $p_a(X/k)$.

For example if $k = \mathbf{Q}$ and $X = \mathbf{P}^1_{\mathbf{Q}(i)}$ then $p_a(X/\mathbf{Q}) = -1$ and $p_a(X/\mathbf{Q}(i)) = 0$.

The assumption that $H^0(X, \mathcal{O}_X) = k$ in our definition has two consequences. On the one hand, it means there is no confusion about the base field. On the other hand, it implies the scheme X is Cohen-Macaulay and equidimensional of dimension 1 (Lemma 53.6.1). If ω_X denotes the dualizing module as in Lemmas 53.4.1 and 53.4.2 we see that

0BY8 (53.8.1.1) $$g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

by duality, see Remark 53.4.3.

If X is proper over k of dimension ≤ 1 and $H^0(X, \mathcal{O}_X)$ is not equal to the ground field k, instead of using the arithmetic genus $p_a(X)$ given by the displayed formula above we shall use the invariant $\chi(X, \mathcal{O}_X)$. In fact, it is advocated in [**Ser55b**,

page 276] and [**Hir95**, Introduction] that we should call $\chi(X, \mathcal{O}_X)$ the arithmetic genus.

0BY9 **Lemma 53.8.2.** *Let k'/k be a field extension. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then $X_{k'}$ is a proper scheme over k' having dimension 1 and $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$. Moreover the genus of $X_{k'}$ is equal to the genus of X.*

Proof. The dimension of $X_{k'}$ is 1 for example by Morphisms, Lemma 29.28.3. The morphism $X_{k'} \to \mathrm{Spec}(k')$ is proper by Morphisms, Lemma 29.41.5. The equality $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ follows from Cohomology of Schemes, Lemma 30.5.2. The equality of the genus follows from the same lemma. □

0C19 **Lemma 53.8.3.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein, then*

$$\deg(\omega_X) = 2g - 2$$

where g is the genus of X and ω_X is as in Lemma 53.4.1.

Proof. Immediate from Lemma 53.5.2. □

0C1A **Lemma 53.8.4.** *Let X be a smooth proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Then*

$$\dim_k H^0(X, \Omega_{X/k}) = g \quad \text{and} \quad \deg(\Omega_{X/k}) = 2g - 2$$

where g is the genus of X.

Proof. By Lemma 53.4.1 we have $\Omega_{X/k} = \omega_X$. Hence the formulas hold by (53.8.1.1) and Lemma 53.8.3. □

53.9. Plane curves

0BYA Let k be a field. A *plane curve* will be a curve X which is isomorphic to a closed subscheme of $\mathbf{P}^2_k$. Often the embedding $X \to \mathbf{P}^2_k$ will be considered given. By Divisors, Example 31.31.2 a curve is determined by the corresponding homogeneous ideal

$$I(X) = \mathrm{Ker}\left(k[T_0, T_2, T_2] \longrightarrow \bigoplus \Gamma(X, \mathcal{O}_X(n))\right)$$

Recall that in this situation we have

$$X = \mathrm{Proj}(k[T_0, T_2, T_2]/I)$$

as closed subschemes of $\mathbf{P}^2_k$. For more general information on these constructions we refer the reader to Divisors, Example 31.31.2 and the references therein. It turns out that $I(X) = (F)$ for some homogeneous polynomial $F \in k[T_0, T_1, T_2]$, see Lemma 53.9.1. Since X is irreducible, it follows that F is irreducible, see Lemma 53.9.2. Moreover, looking at the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \xrightarrow{F} \mathcal{O}_{\mathbf{P}^2_k} \to \mathcal{O}_X \to 0$$

where $d = \deg(F)$ we find that $H^0(X, \mathcal{O}_X) = k$ and that X has genus $(d-1)(d-2)/2$, see proof of Lemma 53.9.3.

To find smooth plane curves it is easiest to write explicit equations. Let p denote the characteristic of k. If p does not divide d, then we can take

$$F = T_0^d + T_1^d + T_2^d$$

The corresponding curve $X = V_+(F)$ is called the *Fermat curve* of degree d. It is smooth because on each standard affine piece $D_+(T_i)$ we obtain a curve isomorphic to the affine curve

$$\mathrm{Spec}(k[x, y]/(x^d + y^d + 1))$$

The ring map $k \to k[x, y]/(x^d + y^d + 1)$ is smooth by Algebra, Lemma 10.137.15 as dx^{d-1} and dy^{d-1} generate the unit ideal in $k[x, y]/(x^d + y^d + 1)$. If $p|d$ but $p \neq 3$ then you can use the equation

$$F = T_0^{d-1}T_1 + T_1^{d-1}T_2 + T_2^{d-1}T_0$$

Namely, on the affine pieces you get $x + x^{d-1}y + y^{d-1}$ with derivatives $1 - x^{d-2}y$ and $x^{d-1} - y^{d-2}$ whose common zero set (of all three) is empty[4]. We leave it to the reader to make examples in characteristic 3.

More generally for any field k and any n and d there exists a smooth hypersurface of degree d in $\mathbf{P}^n_k$, see for example [**Poo05**].

Of course, in this way we only find smooth curves whose genus is a triangular number. To get smooth curves of an arbitrary genus one can look for smooth curves lying on $\mathbf{P}^1 \times \mathbf{P}^1$ (insert future reference here).

0BYB **Lemma 53.9.1.** *Let $Z \subset \mathbf{P}^2_k$ be a closed subscheme which is equidimensional of dimension 1 and has no embedded points (equivalently Z is Cohen-Macaulay). Then the ideal $I(Z) \subset k[T_0, T_1, T_2]$ corresponding to Z is principal.*

Proof. This is a special case of Divisors, Lemma 31.31.3 (see also Varieties, Lemma 33.34.4). The parenthetical statement follows from the fact that a 1 dimensional Noetherian scheme is Cohen-Macaulay if and only if it has no embedded points, see Divisors, Lemma 31.4.4. □

0BYC **Lemma 53.9.2.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then Z is a curve if and only if F is irreducible.*

Proof. If F is reducible, say $F = F'F''$ then let Z' be the closed subscheme of $\mathbf{P}^2_k$ defined by F'. It is clear that $Z' \subset Z$ and that $Z' \neq Z$. Since Z' has dimension 1 as well, we conclude that either Z is not reduced, or that Z is not irreducible. Conversely, write $Z = \sum a_i D_i$ where D_i are the irreducible components of Z, see Divisors, Lemmas 31.15.8 and 31.15.9. Let $F_i \in k[T_0, T_1, T_2]$ be the homogeneous polynomial generating the ideal of D_i. Then it is clear that F and $\prod F_i^{a_i}$ cut out the same closed subscheme of $\mathbf{P}^2_k$. Hence $F = \lambda \prod F_i^{a_i}$ for some $\lambda \in k^*$ because both generate the ideal of Z. Thus we see that if F is irreducible, then Z is a prime divisor, i.e., a curve. □

0BYD **Lemma 53.9.3.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then $H^0(Z, \mathcal{O}_Z) = k$ and the genus of Z is $(d - 1)(d - 2)/2$ where $d = \deg(F)$.*

Proof. Let $S = k[T_0, T_1, T_2]$. There is an exact sequence of graded modules

$$0 \to S(-d) \xrightarrow{F} S \to S/(F) \to 0$$

[4] Namely, as $x^{d-1} = y^{d-2}$, then $0 = x + x^{d-1}y + y^{d-1} = x + 2x^{d-1}y$. Since $x \neq 0$ because $1 = x^{d-2}y$ we get $0 = 1 + 2x^{d-2}y = 3$ which is absurd unless $3 = 0$.

Denote $i : Z \to \mathbf{P}^2_k$ the given closed immersion. Applying the exact functor $\tilde{}$ (Constructions, Lemma 27.8.4) we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_Z \to 0$$

because F generates the ideal of Z. Note that the cohomology groups of $\mathcal{O}_{\mathbf{P}^2_k}(-d)$ and $\mathcal{O}_{\mathbf{P}^2_k}$ are given in Cohomology of Schemes, Lemma 30.8.1. On the other hand, we have $H^q(Z, \mathcal{O}_Z) = H^q(\mathbf{P}^2_k, i_*\mathcal{O}_Z)$ by Cohomology of Schemes, Lemma 30.2.4. Applying the long exact cohomology sequence we first obtain that

$$k = H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}) \longrightarrow H^0(Z, \mathcal{O}_Z)$$

is an isomorphism and next that the boundary map

$$H^1(Z, \mathcal{O}_Z) \longrightarrow H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-d)) \cong k[T_0, T_1, T_2]_{d-3}$$

is an isomorphism. Since it is easy to see that the dimension of this is $(d-1)(d-2)/2$ the proof is finished. □

0CCU **Lemma 53.9.4.** *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 53.9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. If $Z \to \mathrm{Spec}(k)$ is smooth in at least one point and k is infinite, then there exists a closed point $z \in Z$ contained in the smooth locus such that $\kappa(z)/k$ is finite separable of degree at most d.*

Proof. Suppose that $z' \in Z$ is a point where $Z \to \mathrm{Spec}(k)$ is smooth. After renumbering the coordinates if necessary we may assume z' is contained in $D_+(T_0)$. Set $f = F(1, x, y) \in k[x, y]$. Then $Z \cap D_+(X_0)$ is isomorphic to the spectrum of $k[x, y]/(f)$. Let f_x, f_y be the partial derivatives of f with respect to x, y. Since z' is a smooth point of Z/k we see that either f_x or f_y is nonzero in z' (see discussion in Algebra, Section 10.137). After renumbering the coordinates we may assume f_y is not zero at z'. Hence there is a nonempty open subscheme $V \subset Z \cap D_+(X_0)$ such that the projection

$$p : V \longrightarrow \mathrm{Spec}(k[x])$$

is étale. Because the degree of f as a polynomial in y is at most d, we see that the degrees of the fibres of the projection p are at most d (see discussion in Morphisms, Section 29.57). Moreover, as p is étale the image of p is an open $U \subset \mathrm{Spec}(k[x])$. Finally, since k is infinite, the set of k-rational points $U(k)$ of U is infinite, in particular not empty. Pick any $t \in U(k)$ and let $z \in V$ be a point mapping to t. Then z works. □

53.10. Curves of genus zero

0C6L Later we will need to know what a proper genus zero curve looks like. It turns out that a Gorenstein proper genus zero curve is a plane curve of degree 2, i.e., a conic, see Lemma 53.10.3. A general proper genus zero curve is obtained from a nonsingular one (over a bigger field) by a pushout procedure, see Lemma 53.10.5. Since a nonsingular curve is Gorenstein, these two results cover all possible cases.

0C6M **Lemma 53.10.1.** *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X has genus 0, then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree 0 is trivial.*

Proof. Namely, we have $\dim_k H^0(X, \mathcal{L}) \geq 0+1-0 = 1$ by Riemann-Roch (Lemma 53.5.2), hence $\mathcal{L}$ has a nonzero section, hence $\mathcal{L} \cong \mathcal{O}_X$ by Varieties, Lemma 33.44.12. □

0C6T **Lemma 53.10.2.** *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X has genus 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module of degree $d > 0$. Then we have*

(1) $\dim_k H^0(X, \mathcal{L}) = d + 1$ *and* $\dim_k H^1(X, \mathcal{L}) = 0$,
(2) *$\mathcal{L}$ is very ample and defines a closed immersion into $\mathbf{P}^d_k$.*

Proof. By definition of degree and genus we have

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = d + 1$$

Let s be a nonzero section of $\mathcal{L}$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$ and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 31.14.10 and Remark 31.14.11. Since $H^1(X, \mathcal{O}_X) = 0$ by assumption, we see that $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. As $\mathcal{L}|_D$ is generated by global sections (because $\dim(D) = 0$, see Varieties, Lemma 33.33.3) we conclude that the invertible module $\mathcal{L}$ is generated by global sections. In fact, since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[5] and hence we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. The short exact sequence also shows that $H^1(X, \mathcal{L}) = 0$.

For $n \geq 1$ consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to \mathcal{L}^{\otimes n+1}|_D \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t^n maps to a trivialization of $\mathcal{L}^{\otimes n}|_D$ we see that $\mu_n(H^0(X, \mathcal{L}) \otimes t^n)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_D$. This proves the claim.

Observe that $\mathcal{L}$ is ample by Varieties, Lemma 33.44.14. Hence Morphisms, Lemma 29.43.17 gives an isomorphism

$$X \longrightarrow \operatorname{Proj}\left(\bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})\right)$$

Since the maps μ_n are surjective for all $n \geq 1$ we see that the graded algebra on the right hand side is a quotient of the symmetric algebra on $H^0(X, \mathcal{L})$. Choosing a k-basis $s_0, \ldots, s_d$ of $H^0(X, \mathcal{L})$ we see that it is a quotient of a polynomial algebra in $d + 1$ variables. Since quotients of graded rings correspond to closed immersions of Proj (Constructions, Lemma 27.11.5) we find a closed immersion $X \to \mathbf{P}^d_k$. We omit the verification that this morphism is the morphism of Constructions, Lemma 27.13.1 associated to the sections $s_0, \ldots, s_d$ of $\mathcal{L}$. □

0C6N **Lemma 53.10.3.** *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein and has genus 0, then X is isomorphic to a plane curve of degree 2.*

[5] In our case this follows from Divisors, Lemma 31.17.1 as $D \to \operatorname{Spec}(k)$ is finite.

Proof. Consider the invertible sheaf $\mathcal{L} = \omega_X^{\otimes -1}$ where ω_X is as in Lemma 53.4.1. Then $\deg(\omega_X) = -2$ by Lemma 53.8.3 and hence $\deg(\mathcal{L}) = 2$. By Lemma 53.10.2 we conclude that choosing a basis s_0, s_1, s_2 of the k-vector space of global sections of $\mathcal{L}$ we obtain a closed immersion

$$\varphi_{(\mathcal{L},(s_0,s_1,s_2))} : X \longrightarrow \mathbf{P}^2_k$$

Thus X is a plane curve of some degree d. Let $F \in k[T_0, T_1, T_2]_d$ be its equation (Lemma 53.9.1). Because the genus of X is 0 we see that d is 1 or 2 (Lemma 53.9.3). Observe that F restricts to the zero section on $\varphi(X)$ and hence $F(s_0, s_1, s_2)$ is the zero section of $\mathcal{L}^{\otimes 2}$. Because s_0, s_1, s_2 are linearly independent we see that F cannot be linear, i.e., $d = \deg(F) \geq 2$. Thus $d = 2$ and the proof is complete. □

0C6U **Proposition 53.10.4** (Characterization of the projective line)**.** *Let k be a field. Let X be a proper curve over k. The following are equivalent*

1. *$X \cong \mathbf{P}^1_k$,*
2. *X is smooth and geometrically irreducible over k, X has genus 0, and X has an invertible module of odd degree,*
3. *X is geometrically integral over k, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,*
4. *$H^0(X, \mathcal{O}_X) = k$, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,*
5. *X is geometrically integral over k, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,*
6. *$H^0(X, \mathcal{O}_X) = k$, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,*
7. *$H^1(X, \mathcal{O}_X) = 0$ and X has an invertible $\mathcal{O}_X$-module of degree 1,*
8. *$H^1(X, \mathcal{O}_X) = 0$ and X has closed points $x_1, \ldots, x_n$ such that $\mathcal{O}_{X,x_i}$ is normal and $\gcd([\kappa(x_i) : k]) = 1$, and*
9. *add more here.*

Proof. We will prove that each condition (2) – (8) implies (1) and we omit the verification that (1) implies (2) – (8).

Assume (2). A smooth scheme over k is geometrically reduced (Varieties, Lemma 33.25.4) and regular (Varieties, Lemma 33.25.3). Hence X is Gorenstein (Duality for Schemes, Lemma 48.24.3). Thus we reduce to (3).

Assume (3). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 33.26.2. and we reduce to (4).

Assume (4). Since X is Gorenstein the dualizing module ω_X as in Lemma 53.4.1 has degree $\deg(\omega_X) = -2$ by Lemma 53.8.3. Combined with the assumed existence of an odd degree invertible module, we conclude there exists an invertible module of degree 1. In this way we reduce to (6).

Assume (5). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 33.26.2. and we reduce to (6).

Assume (6). Then $X \cong \mathbf{P}^1_k$ by Lemma 53.10.2.

Assume (7). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 33.26.2. If $d = [\kappa : k] > 1$, then every invertible sheaf has degree divisible

by d and there cannot be an invertible sheaf of degree 1. Hence $d = 1$ and we reduce to case (6).

Assume (8). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 33.26.2. Since $\kappa \subset \kappa(x_i)$ we see that $k = \kappa$ by the assumption on the gcd of the degrees. The same condition allows us to find integers a_i such that $1 = \sum a_i[\kappa(x_i) : k]$. Because x_i defines an effective Cartier divisor on X by Varieties, Lemma 33.43.8 we can consider the invertible module $\mathcal{O}_X(\sum a_i x_i)$. By our choice of a_i the degree of $\mathcal{L}$ is 1. Thus $X \cong \mathbf{P}^1_k$ by Lemma 53.10.2. □

0DJB **Lemma 53.10.5.** *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X is singular and has genus 0. Then there exists a diagram*

$$\begin{array}{ccccc} x' & \longrightarrow & X' & \longrightarrow & \mathrm{Spec}(k') \\ \downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow \\ x & \longrightarrow & X & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

where

(1) *k'/k is a nontrivial finite extension,*
(2) *$X' \cong \mathbf{P}^1_{k'}$,*
(3) *x' is a k'-rational point of X',*
(4) *x is a k-rational point of X,*
(5) *$X' \setminus \{x'\} \to X \setminus \{x\}$ is an isomorphism,*
(6) *$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to k'/k \to 0$ is a short exact sequence where $k'/k = \kappa(x')/\kappa(x)$ indicates the skyscraper sheaf on the point x.*

Proof. Let $\nu : X' \to X$ be the normalization of X, see Varieties, Sections 33.27 and 33.41. Since X is singular ν is not an isomorphism. Then $k' = H^0(X', \mathcal{O}_{X'})$ is a finite extension of k (Varieties, Lemma 33.26.2). The short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

and the fact that $\mathcal{Q}$ is supported in finitely many closed points give us that

(1) $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., X' has genus 0 as a curve over k',
(2) there is a short exact sequence $0 \to k \to k' \to H^0(X, \mathcal{Q}) \to 0$.

In particular k'/k is a nontrivial extension.

Next, we consider what is often called the *conductor ideal*

$$\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$$

This is a quasi-coherent $\mathcal{O}_X$-module. We view $\mathcal{I}$ as an ideal in $\mathcal{O}_X$ via the map $\varphi \mapsto \varphi(1)$. Thus $\mathcal{I}(U)$ is the set of $f \in \mathcal{O}_X(U)$ such that $f\,(\nu_*\mathcal{O}_{X'}(U)) \subset \mathcal{O}_X(U)$. In other words, the condition is that f annihilates $\mathcal{Q}$. In other words, there is a defining exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{Q})$$

Let $U \subset X$ be an affine open containing the support of $\mathcal{Q}$. Then $V = \mathcal{Q}(U) = H^0(X, \mathcal{Q})$ is a k-vector space of dimension $n-1$. The image of $\mathcal{O}_X(U) \to \mathrm{Hom}_k(V, V)$ is a commutative subalgebra, hence has dimension $\leq n-1$ over k (this is a property of commutative subalgebras of matrix algebras; details omitted). We conclude that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{A} \to 0$$

where $\text{Supp}(\mathcal{A}) = \text{Supp}(\mathcal{Q})$ and $\dim_k H^0(X, \mathcal{A}) \leq n - 1$. On the other hand, the description $\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$ provides $\mathcal{I}$ with a $\nu_*\mathcal{O}_{X'}$-module structure such that the inclusion map $\mathcal{I} \to \nu_*\mathcal{O}_{X'}$ is a $\nu_*\mathcal{O}_{X'}$-module map. We conclude that $\mathcal{I} = \nu_*\mathcal{I}'$ for some quasi-coherent sheaf of ideals $\mathcal{I}' \subset \mathcal{O}_{X'}$, see Morphisms, Lemma 29.11.7. Define $\mathcal{A}'$ as the cokernel:

$$0 \to \mathcal{I}' \to \mathcal{O}_{X'} \to \mathcal{A}' \to 0$$

Combining the exact sequences so far we obtain a short exact sequence $0 \to \mathcal{A} \to \nu_*\mathcal{A}' \to \mathcal{Q} \to 0$. Using the estimate above, combined with $\dim_k H^0(X, \mathcal{Q}) = n - 1$, gives

$$\dim_k H^0(X', \mathcal{A}') = \dim_k H^0(X, \mathcal{A}) + \dim_k H^0(X, \mathcal{Q}) \leq 2n - 2$$

However, since X' is a curve over k' we see that the left hand side is divisible by n (Varieties, Lemma 33.44.10). As $\mathcal{A}$ and $\mathcal{A}'$ cannot be zero, we conclude that $\dim_k H^0(X', \mathcal{A}') = n$ which means that $\mathcal{I}'$ is the ideal sheaf of a k'-rational point x'. By Proposition 53.10.4 we find $X' \cong \mathbf{P}^1_{k'}$. Going back to the equalities above, we conclude that $\dim_k H^0(X, \mathcal{A}) = 1$. This means that $\mathcal{I}$ is the ideal sheaf of a k-rational point x. Then $\mathcal{A} = \kappa(x) = k$ and $\mathcal{A}' = \kappa(x') = k'$ as skyscraper sheaves. Comparing the exact sequences given above, this immediately implies the result on structure sheaves as stated in the lemma. □

0DJC **Example 53.10.6.** In fact, the situation described in Lemma 53.10.5 occurs for any nontrivial finite extension k'/k. Namely, we can consider

$$A = \{f \in k'[x] \mid f(0) \in k\}$$

The spectrum of A is an affine curve, which we can glue to the spectrum of $B = k'[y]$ using the isomorphism $A_x \cong B_y$ sending x^{-1} to y. The result is a proper curve X with $H^0(X, \mathcal{O}_X) = k$ and singular point x corresponding to the maximal ideal $A \cap (x)$. The normalization of X is $\mathbf{P}^1_{k'}$ exactly as in the lemma.

53.11. Geometric genus

0BYE If X is a proper and **smooth** curve over k with $H^0(X, \mathcal{O}_X) = k$, then

$$p_g(X) = \dim_k H^0(X, \Omega_{X/k})$$

is called the *geometric genus* of X. By Lemma 53.8.4 the geometric genus of X agrees with the (arithmetic) genus. However, in higher dimensions there is a difference between the geometric genus and the arithmetic genus, see Remark 53.11.2.

For singular curves, we will define the geometric genus as follows.

0BYF **Definition 53.11.1.** Let k be a field. Let X be a geometrically irreducible curve over k. The *geometric genus* of X is the genus of a smooth projective model of X possibly defined over an extension field of k as in Lemma 53.2.9.

If k is perfect, then the nonsingular projective model Y of X is smooth (Lemma 53.2.8) and the geometric genus of X is just the genus of Y. But if k is not perfect, this may not be true. In this case we choose an extension K/k such that the nonsingular projective model Y_K of $(X_K)_{red}$ is a smooth projective curve and we define the geometric genus of X to be the genus of Y_K. This is well defined by Lemmas 53.2.9 and 53.8.2.

0BYG **Remark 53.11.2.** Suppose that X is a d-dimensional proper smooth variety over an algebraically closed field k. Then the *arithmetic genus* is often defined as $p_a(X) = (-1)^d(\chi(X, \mathcal{O}_X)-1)$ and the *geometric genus* as $p_g(X) = \dim_k H^0(X, \Omega^d_{X/k})$. In this situation the arithmetic genus and the geometric genus no longer agree even though it is still true that $\omega_X \cong \Omega^d_{X/k}$. For example, if $d = 2$, then we have

$$\begin{aligned} p_a(X) - p_g(X) &= h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - 1 - h^0(X, \Omega^2_{X/k}) \\ &= -h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^0(X, \omega_X) \\ &= -h^1(X, \mathcal{O}_X) \end{aligned}$$

where $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ and where the last equality follows from duality. Hence for a surface the difference $p_g(X) - p_a(X)$ is always nonnegative; it is sometimes called the irregularity of the surface. If $X = C_1 \times C_2$ is a product of smooth projective curves of genus g_1 and g_2, then the irregularity is $g_1 + g_2$.

53.12. Riemann-Hurwitz

0C1B Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. Then we obtain a canonical exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{\mathrm{d}f} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

by Morphisms, Lemma 29.32.9. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 29.34.12. Assume the first map is nonzero, i.e., assume f is generically étale, see Lemma 53.12.1. Let $R \subset X$ be the closed subscheme cut out by the different $\mathfrak{D}_f$ of f. By Discriminants, Lemma 49.12.6 this is the same as the vanishing locus of $\mathrm{d}f$, it is an effective Cartier divisor, and we get

$$f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R) = \Omega_{X/k}$$

In particular, if X, Y are projective with $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X, Y have genus g_X, g_Y, then we get the Riemann-Hurwitz formula

$$\begin{aligned} 2g_X - 2 &= \deg(\Omega_{X/k}) \\ &= \deg(f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)) \\ &= \deg(f)\deg(\Omega_{Y/k}) + \deg(R) \\ &= \deg(f)(2g_Y - 2) + \deg(R) \end{aligned}$$

The first and last equality by Lemma 53.8.4. The second equality by the isomorphism of invertible sheaves given above. The third equality by additivity of degrees (Varieties, Lemma 33.44.7), the formula for the degree of a pullback (Varieties, Lemma 33.44.11), and finally the formula for the degree of $\mathcal{O}_X(R)$ (Varieties, Lemma 33.44.9).

To use the Riemann-Hurwitz formula we need to compute $\deg(R) = \dim_k \Gamma(R, \mathcal{O}_R)$. By the structure of zero dimensional schemes over k (see for example Varieties, Lemma 33.20.2), we see that R is a finite disjoint union of spectra of Artinian local rings $R = \coprod_{x \in R} \mathrm{Spec}(\mathcal{O}_{R,x})$ with each $\mathcal{O}_{R,x}$ of finite dimension over k. Thus

$$\deg(R) = \sum\nolimits_{x \in R} \dim_k \mathcal{O}_{R,x} = \sum\nolimits_{x \in R} d_x[\kappa(x) : k]$$

with

$$d_x = \mathrm{length}_{\mathcal{O}_{R,x}} \mathcal{O}_{R,x} = \mathrm{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{R,x}$$

the multiplicity of x in R (see Algebra, Lemma 10.52.12). Let $x \in X$ be a closed point with image $y \in Y$. Looking at stalks we obtain an exact sequence

$$\Omega_{Y/k,y} \to \Omega_{X/k,x} \to \Omega_{X/Y,x} \to 0$$

Choosing local generators η_x and η_y of the (free rank 1) modules $\Omega_{X/k,x}$ and $\Omega_{Y/k,y}$ we see that $\eta_y \mapsto h\eta_x$ for some nonzero $h \in \mathcal{O}_{X,x}$. By the exact sequence we see that $\Omega_{X/Y,x} \cong \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules. Since the divisor R is cut out by h (see above) we have $\mathcal{O}_{R,x} = \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$. Thus we find the following equalities

$$\begin{aligned} d_x &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{R,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/h\mathcal{O}_{X,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x}) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(h) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(\text{“}\eta_y/\eta_x\text{”}) \end{aligned}$$

The first equality by our definition of d_x. The second and third we saw above. The fourth equality is the definition of ord, see Algebra, Definition 10.121.2. Note that since $\mathcal{O}_{X,x}$ is a discrete valuation ring, the integer $\text{ord}_{\mathcal{O}_{X,x}}(h)$ just the valuation of h. The fifth equality is a mnemonic.

Here is a case where one can “calculate” the multiplicity d_x in terms of other invariants. Namely, if $\kappa(x)$ is separable over k, then we may choose $\eta_x = \mathrm{d}s$ and $\eta_y = \mathrm{d}t$ where s and t are uniformizers in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ (Lemma 53.12.3). Then $t \mapsto us^{e_x}$ for some unit $u \in \mathcal{O}_{X,x}$ where e_x is the ramification index of the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$. Hence we get

$$\eta_y = \mathrm{d}t = \mathrm{d}(us^{e_x}) = es^{e_x-1}u\mathrm{d}s + s^{e_x}\mathrm{d}u$$

Writing $\mathrm{d}u = w\mathrm{d}s$ for some $w \in \mathcal{O}_{X,x}$ we see that

$$\text{“}\eta_y/\eta_x\text{”} = es^{e_x-1}u + s^{e_x}w = (e_xu + sw)s^{e_x-1}$$

We conclude that the order of vanishing of this is $e_x - 1$ unless the characteristic of $\kappa(x)$ is $p > 0$ and p divides e_x in which case the order of vanishing is $> e_x - 1$.

Combining all of the above we find that if k has characteristic zero, then

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum\nolimits_{x \in X}(e_x - 1)[\kappa(x) : k]$$

where e_x is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,f(x)}$. This precise formula will hold if and only if all the ramification is tame, i.e., when the residue field extensions $\kappa(x)/\kappa(y)$ are separable and e_x is prime to the characteristic of k, although the arguments above are insufficient to prove this. We refer the reader to Lemma 53.12.4 and its proof.

0C1C **Lemma 53.12.1.** *Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. The following are equivalent*

1. *$df : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero,*
2. *$\Omega_{X/Y}$ is supported on a proper closed subset of X,*
3. *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is unramified,*
4. *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is étale,*
5. *the extension $k(X)/k(Y)$ of function fields is finite separable.*

Proof. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 29.34.12. Using the exact sequence

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

of Morphisms, Lemma 29.32.9 we see that (1) and (2) are equivalent and equivalent to the condition that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero in the generic point. The equivalence of (2) and (3) follows from Morphisms, Lemma 29.35.2. The equivalence between (3) and (4) follows from Morphisms, Lemma 29.36.16 and the fact that flatness is automatic (Lemma 53.2.3). To see the equivalence of (5) and (4) use Algebra, Lemma 10.140.9. Some details omitted. □

0C1D **Lemma 53.12.2.** *Let $f : X \to Y$ be a morphism of smooth proper curves over a field k which satisfies the equivalent conditions of Lemma 53.12.1. If $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X and Y have genus g_X and g_Y, then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg(R)$$

where $R \subset X$ is the effective Cartier divisor cut out by the different of f.

Proof. See discussion above; we used Discriminants, Lemma 49.12.6, Lemma 53.8.4, and Varieties, Lemmas 33.44.7 and 33.44.11. □

0C1E **Lemma 53.12.3.** *Let $X \to \mathrm{Spec}(k)$ be smooth of relative dimension 1 at a closed point $x \in X$. If $\kappa(x)$ is separable over k, then for any uniformizer s in the discrete valuation ring $\mathcal{O}_{X,x}$ the element ds freely generates $\Omega_{X/k,x}$ over $\mathcal{O}_{X,x}$.*

Proof. The ring $\mathcal{O}_{X,x}$ is a discrete valuation ring by Algebra, Lemma 10.140.3. Since x is closed $\kappa(x)$ is finite over k. Hence if $\kappa(x)/k$ is separable, then any uniformizer s maps to a nonzero element of $\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ by Algebra, Lemma 10.140.4. Since $\Omega_{X/k,x}$ is free of rank 1 over $\mathcal{O}_{X,x}$ the result follows. □

0C1F **Lemma 53.12.4.** *Notation and assumptions as in Lemma 53.12.2. For a closed point $x \in X$ let d_x be the multiplicity of x in R. Then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum d_x[\kappa(x) : k]$$

Moreover, we have the following results

(1) *$d_x = \mathrm{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x})$,*
(2) *$d_x \geq e_x - 1$ where e_x is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$,*
(3) *$d_x = e_x - 1$ if and only if $\mathcal{O}_{X,x}$ is tamely ramified over $\mathcal{O}_{Y,y}$.*

Proof. By Lemma 53.12.2 and the discussion above (which used Varieties, Lemma 33.20.2 and Algebra, Lemma 10.52.12) it suffices to prove the results on the multiplicity d_x of x in R. Part (1) was proved in the discussion above. In the discussion above we proved (2) and (3) only in the case where $\kappa(x)$ is separable over k. In the rest of the proof we give a uniform treatment of (2) and (3) using material on differents of quasi-finite Gorenstein morphisms.

First, observe that f is a quasi-finite Gorenstein morphism. This is true for example because f is a flat quasi-finite morphism and X is Gorenstein (see Duality for Schemes, Lemma 48.25.7) or because it was shown in the proof of Discriminants, Lemma 49.12.6 (which we used above). Thus $\omega_{X/Y}$ is invertible by Discriminants, Lemma 49.16.1 and the same remains true after replacing X by opens and after performing a base change by some $Y' \to Y$. We will use this below without further mention.

Choose affine opens $U \subset X$ and $V \subset Y$ such that $x \in U$, $y \in V$, $f(U) \subset V$, and x is the only point of U lying over y. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Then $R \cap U$ is the different of $f|_U : U \to V$. By Discriminants, Lemma 49.9.4 formation of the different commutes with arbitrary base change in our case. By our choice of U and V we have

$$A \otimes_B \kappa(y) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{O}_{X,x}/(s^{e_x})$$

where e_x is the ramification index as in the statement of the lemma. Let $C = \mathcal{O}_{X,x}/(s^{e_x})$ viewed as a finite algebra over $\kappa(y)$. Let $\mathfrak{D}_{C/\kappa(y)}$ be the different of C over $\kappa(y)$ in the sense of Discriminants, Definition 49.9.1. It suffices to show: $\mathfrak{D}_{C/\kappa(y)}$ is nonzero if and only if the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$ is tamely ramified and in the tamely ramified case $\mathfrak{D}_{C/\kappa(y)}$ is equal to the ideal generated by s^{e_x-1} in C. Recall that tame ramification means exactly that $\kappa(x)/\kappa(y)$ is separable and that the characteristic of $\kappa(y)$ does not divide e_x. On the other hand, the different of $C/\kappa(y)$ is nonzero if and only if $\tau_{C/\kappa(y)} \in \omega_{C/\kappa(y)}$ is nonzero. Namely, since $\omega_{C/\kappa(y)}$ is an invertible C-module (as the base change of $\omega_{A/B}$) it is free of rank 1, say with generator λ. Write $\tau_{C/\kappa(y)} = h\lambda$ for some $h \in C$. Then $\mathfrak{D}_{C/\kappa(y)} = (h) \subset C$ whence the claim. By Discriminants, Lemma 49.4.8 we have $\tau_{C/\kappa(y)} \neq 0$ if and only if $\kappa(x)/\kappa(y)$ is separable and e_x is prime to the characteristic. Finally, even if $\tau_{C/\kappa(y)}$ is nonzero, then it is still the case that $s\tau_{C/\kappa(y)} = 0$ because $s\tau_{C/\kappa(y)} : C \to \kappa(y)$ sends c to the trace of the nilpotent operator sc which is zero. Hence $sh = 0$, hence $h \in (s^{e_x-1})$ which proves that $\mathfrak{D}_{C/\kappa(y)} \subset (s^{e_x-1})$ always. Since $(s^{e_x-1}) \subset C$ is the smallest nonzero ideal, we have proved the final assertion. □

53.13. Inseparable maps

0CCV Some remarks on the behaviour of the genus under inseparable maps.

0CCW **Lemma 53.13.1.** *Let k be a field. Let $f : X \to Y$ be a surjective morphism of curves over k. If X is smooth over k and Y is normal, then Y is smooth over k.*

Proof. Let $y \in Y$. Pick $x \in X$ mapping to y. By Varieties, Lemma 33.25.9 it suffices to show that f is flat at x. This follows from Lemma 53.2.3. □

0CCX **Lemma 53.13.2.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If the extension $k(X)/k(Y)$ of function fields is purely inseparable, then there exists a factorization*

$$X = X_0 \to X_1 \to \ldots \to X_n = Y$$

such that each X_i is a proper nonsingular curve and $X_i \to X_{i+1}$ is a degree p morphism with $k(X_{i+1}) \subset k(X_i)$ inseparable.

Proof. This follows from Theorem 53.2.6 and the fact that a finite purely inseparable extension of fields can always be gotten as a sequence of (inseparable) extensions of degree p, see Fields, Lemma 9.14.5. □

0CCY **Lemma 53.13.3.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is inseparable of degree p, then there is a unique isomorphism $Y = X^{(p)}$ such that f is $F_{X/k}$.*

Proof. The relative frobenius morphism $F_{X/k} : X \to X^{(p)}$ is constructed in Varieties, Section 33.36. Observe that $X^{(p)}$ is a smooth proper curve over k as a base change of X. The morphism $F_{X/k}$ has degree p by Varieties, Lemma 33.36.10. Thus $k(X^{(p)})$ and $k(Y)$ are both subfields of $k(X)$ with $[k(X) : k(Y)] = [k(X) : k(X^{(p)})] = p$. To prove the lemma it suffices to show that $k(Y) = k(X^{(p)})$ inside $k(X)$. See Theorem 53.2.6.

Write $K = k(X)$. Consider the map $\mathrm{d} : K \to \Omega_{K/k}$. It follows from Lemma 53.12.1 that both $k(Y)$ is contained in the kernel of d. By Varieties, Lemma 33.36.7 we see that $k(X^{(p)})$ is in the kernel of d. Since X is a smooth curve we know that $\Omega_{K/k}$ is a vector space of dimension 1 over K. Then More on Algebra, Lemma 15.46.2. implies that $\mathrm{Ker}(\mathrm{d}) = kK^p$ and that $[K : kK^p] = p$. Thus $k(Y) = kK^p = k(X^{(p)})$ for reasons of degree. □

0CCZ **Lemma 53.13.4.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is purely inseparable, then there is a unique $n \geq 0$ and a unique isomorphism $Y = X^{(p^n)}$ such that f is the n-fold relative Frobenius of X/k.*

Proof. The n-fold relative Frobenius of X/k is defined in Varieties, Remark 33.36.11. The lemma follows by combining Lemmas 53.13.3 and 53.13.2. □

0CD0 **Lemma 53.13.5.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. Assume*

(1) *X is smooth,*
(2) *$H^0(X, \mathcal{O}_X) = k$,*
(3) *$k(X)/k(Y)$ is purely inseparable.*

Then Y is smooth, $H^0(Y, \mathcal{O}_Y) = k$, and the genus of Y is equal to the genus of X.

Proof. By Lemma 53.13.4 we see that $Y = X^{(p^n)}$ is the base change of X by $F^n_{\mathrm{Spec}(k)}$. Thus Y is smooth and the result on the cohomology and genus follows from Lemma 53.8.2. □

0CD1 **Example 53.13.6.** This example will show that the genus can change under a purely inseparable morphism of nonsingular projective curves. Let k be a field of characteristic 3. Assume there exists an element $a \in k$ which is not a 3rd power. For example $k = \mathbf{F}_3(a)$ would work. Let X be the plane curve with homogeneous equation

$$F = T_1^2T_0 - T_2^3 + aT_0^3$$

as in Section 53.9. On the affine piece $D_+(T_0)$ using coordinates $x = T_1/T_0$ and $y = T_2/T_0$ we obtain $x^2 - y^3 + a = 0$ which defines a nonsingular affine curve. Moreover, the point at infinity $(0 : 1 : 0)$ is a smooth point. Hence X is a nonsingular projective curve of genus 1 (Lemma 53.9.3). On the other hand, consider the morphism $f : X \to \mathbf{P}^1_k$ which on $D_+(T_0)$ sends (x, y) to $x \in \mathbf{A}^1_k \subset \mathbf{P}^1_k$. Then f is a morphism of proper nonsingular curves over k inducing an inseparable function field extension of degree $p = 3$ but the genus of X is 1 and the genus of $\mathbf{P}^1_k$ is 0.

0CD2 **Proposition 53.13.7.** *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper smooth curves over k. Then we can factor f as*

$$X \longrightarrow X^{(p^n)} \longrightarrow Y$$

where $X^{(p^n)} \to Y$ is a nonconstant morphism of proper smooth curves inducing a separable field extension $k(X^{(p^n)})/k(Y)$, we have

$$X^{(p^n)} = X \times_{\mathrm{Spec}(k), F^n_{\mathrm{Spec}(k)}} \mathrm{Spec}(k),$$

and $X \to X^{(p^n)}$ is the n-fold relative frobenius of X.

Proof. By Fields, Lemma 9.14.6 there is a subextension $k(X)/E/k(Y)$ such that $k(X)/E$ is purely inseparable and $E/k(Y)$ is separable. By Theorem 53.2.6 this corresponds to a factorization $X \to Z \to Y$ of f with Z a nonsingular proper curve. Apply Lemma 53.13.4 to the morphism $X \to Z$ to conclude. □

0CD3 **Lemma 53.13.8.** *Let k be a field of characteristic $p > 0$. Let X be a smooth proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ with $r \geq 1$. Then one of the following two is true*

1. *there exists a $\mathfrak{g}^1_d$ whose corresponding morphism $X \to \mathbf{P}^1_k$ (Lemma 53.3.2) is generically étale (i.e., is as in Lemma 53.12.1), or*
2. *there exists a $\mathfrak{g}^r_{d'}$ on $X^{(p)}$ where $d' \leq d/p$.*

Proof. Pick two k-linearly independent elements $s, t \in V$. Then $f = s/t$ is the rational function defining the morphism $X \to \mathbf{P}^1_k$ corresponding to the linear series $(\mathcal{L}, ks + kt)$. If this morphism is not generically étale, then $f \in k(X^{(p)})$ by Proposition 53.13.7. Now choose a basis $s_0, \ldots, s_r$ of V and let $\mathcal{L}' \subset \mathcal{L}$ be the invertible sheaf generated by $s_0, \ldots, s_r$. Set $f_i = s_i/s_0$ in $k(X)$. If for each pair (s_0, s_i) we have $f_i \in k(X^{(p)})$, then the morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$$

factors through $X^{(p)}$ as this is true over the affine open $D_+(T_0)$ and we can extend the morphism over the affine part to the whole of the smooth curve $X^{(p)}$ by Lemma 53.2.2. Introducing notation, say we have the factorization

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{\psi} \mathbf{P}^r_k$$

of φ. Then $\mathcal{N} = \psi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ is an invertible $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{L}' = F^*_{X/k}\mathcal{N}$ and with $\psi^*T_0, \ldots, \psi^*T_r$ k-linearly independent (as they pullback to $s_0, \ldots, s_r$ on X). Finally, we have

$$d = \deg(\mathcal{L}) \geq \deg(\mathcal{L}') = \deg(F_{X/k}) \deg(\mathcal{N}) = p \deg(\mathcal{N})$$

as desired. Here we used Varieties, Lemmas 33.44.12, 33.44.11, and 33.36.10. □

0CD4 **Lemma 53.13.9.** *Let k be a field. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and genus $g \geq 2$. Then there exists a closed point $x \in X$ with $\kappa(x)/k$ separable of degree $\leq 2g - 2$.*

Proof. Set $\omega = \Omega_{X/k}$. By Lemma 53.8.4 this has degree $2g - 2$ and has g global sections. Thus we have a $\mathfrak{g}^{g-1}_{2g-2}$. By the trivial Lemma 53.3.3 there exists a $\mathfrak{g}^1_{2g-2}$ and by Lemma 53.3.4 we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

of some degree $d \leq 2g-2$. Since φ is flat (Lemma 53.2.3) and finite (Lemma 53.2.4) it is finite locally free of degree d (Morphisms, Lemma 29.48.2). Pick any rational point $t \in \mathbf{P}^1_k$ and any point $x \in X$ with $\varphi(x) = t$. Then

$$d \geq [\kappa(x) : \kappa(t)] = [\kappa(x) : k]$$

for example by Morphisms, Lemmas 29.57.3 and 29.57.2. Thus if k is perfect (for example has characteristic zero or is finite) then the lemma is proved. Thus we reduce to the case discussed in the next paragraph.

Assume that k is an infinite field of characteristic $p > 0$. As above we will use that X has a $\mathfrak{g}^{g-1}_{2g-2}$. The smooth proper curve $X^{(p)}$ has the same genus as X. Hence its genus is > 0. We conclude that $X^{(p)}$ does not have a $\mathfrak{g}^{g-1}_d$ for any $d \leq g-1$ by Lemma 53.3.5. Applying Lemma 53.13.8 to our $\mathfrak{g}^{g-1}_{2g-2}$ (and noting that $2g - 2/p \leq g - 1$) we conclude that possibility (2) does not occur. Hence we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

which is generically étale (in the sense of the lemma) and has degree $\leq 2g-2$. Let $U \subset X$ be the nonempty open subscheme where φ is étale. Then $\varphi(U) \subset \mathbf{P}^1_k$ is a nonempty Zariski open and we can pick a k-rational point $t \in \varphi(U)$ as k is infinite. Let $u \in U$ be a point with $\varphi(u) = t$. Then $\kappa(u)/\kappa(t)$ is separable (Morphisms, Lemma 29.36.7), $\kappa(t) = k$, and $[\kappa(u) : k] \leq 2g - 2$ as before. □

The following lemma does not really belong in this section but we don't know a good place for it elsewhere.

0C1G **Lemma 53.13.10.** *Let X be a smooth curve over a field k. Let $\overline{x} \in X_{\overline{k}}$ be a closed point with image $x \in X$. The ramification index of $\mathcal{O}_{X,x} \subset \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is the inseparable degree of $\kappa(x)/k$.*

Proof. After shrinking X we may assume there is an étale morphism $\pi : X \to \mathbf{A}^1_k$, see Morphisms, Lemma 29.36.20. Then we can consider the diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X_{\overline{k}},\overline{x}} & \longleftarrow & \mathcal{O}_{\mathbf{A}^1_{\overline{k}},\pi(\overline{x})} \\ \uparrow & & \uparrow \\ \mathcal{O}_{X,x} & \longleftarrow & \mathcal{O}_{\mathbf{A}^1_k,\pi(x)} \end{array}$$

The horizontal arrows have ramification index 1 as they correspond to étale morphisms. Moreover, the extension $\kappa(x)/\kappa(\pi(x))$ is separable hence $\kappa(x)$ and $\kappa(\pi(x))$ have the same inseparable degree over k. By multiplicativity of ramification indices it suffices to prove the result when x is a point of the affine line.

Assume $X = \mathbf{A}^1_k$. In this case, the local ring of X at x looks like

$$\mathcal{O}_{X,x} = k[t]_{(P)}$$

where P is an irreducible monic polynomial over k. Then $P(t) = Q(t^q)$ for some separable polynomial $Q \in k[t]$, see Fields, Lemma 9.12.1. Observe that $\kappa(x) = k[t]/(P)$ has inseparable degree q over k. On the other hand, over $\overline{k}$ we can factor $Q(t) = \prod(t - \alpha_i)$ with α_i pairwise distinct. Write $\alpha_i = \beta_i^q$ for some unique $\beta_i \in \overline{k}$. Then our point $\overline{x}$ corresponds to one of the β_i and we conclude because the ramification index of

$$k[t]_{(P)} \longrightarrow \overline{k}[t]_{(t-\beta_i)}$$

is indeed equal to q as the uniformizer P maps to $(t - \beta_i)^q$ times a unit. □

53.14. Pushouts

0E35 Let k be a field. Consider a solid diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow & & \vdots a \\ Z & \xrightarrow{i} & X \end{array}$$

of schemes over k satisfying

(a) X' is separated of finite type over k of dimension ≤ 1,
(b) $i : Z' \to X'$ is a closed immersion,
(c) Z' and Z are finite over $\mathrm{Spec}(k)$, and
(d) $Z' \to Z$ is surjective.

In this situation every finite set of points of X' are contained in an affine open, see Varieties, Proposition 33.42.7. Thus the assumptions of More on Morphisms, Proposition 37.67.3 are satisfied and we obtain the following

(1) the pushout $X = Z \amalg_{Z'} X'$ exists in the category of schemes,
(2) $i : Z \to X$ is a closed immersion,
(3) $a : X' \to X$ is integral surjective,
(4) $X \to \mathrm{Spec}(k)$ is separated by More on Morphisms, Lemma 37.67.4
(5) $X \to \mathrm{Spec}(k)$ is of finite type by More on Morphisms, Lemmas 37.67.5,
(6) thus $a : X' \to X$ is finite by Morphisms, Lemmas 29.44.4 and 29.15.8,
(7) if $X' \to \mathrm{Spec}(k)$ is proper, then $X \to \mathrm{Spec}(k)$ is proper by Morphisms, Lemma 29.41.9.

The following lemma can be generalized significantly.

0E36 **Lemma 53.14.1.** *In the situation above, let $Z = \mathrm{Spec}(k')$ where k' is a field and $Z' = \mathrm{Spec}(k'_1 \times \ldots \times k'_n)$ with k'_i/k' finite extensions of fields. Let $x \in X$ be the image of $Z \to X$ and $x'_i \in X'$ the image of $\mathrm{Spec}(k'_i) \to X'$. Then we have a fibre product diagram*

$$\begin{array}{ccc} \prod_{i=1,\ldots,n} k'_i & \longleftarrow & \prod_{i=1,\ldots,n} \mathcal{O}^\wedge_{X',x'_i} \\ \uparrow & & \uparrow \\ k' & \longleftarrow & \mathcal{O}^\wedge_{X,x} \end{array}$$

where the horizontal arrows are given by the maps to the residue fields.

Proof. Choose an affine open neighbourhood $\mathrm{Spec}(A)$ of x in X. Let $\mathrm{Spec}(A') \subset X'$ be the inverse image. By construction we have a fibre product diagram

$$\begin{array}{ccc} \prod_{i=1,\ldots,n} k'_i & \longleftarrow & A' \\ \uparrow & & \uparrow \\ k' & \longleftarrow & A \end{array}$$

Since everything is finite over A we see that the diagram remains a fibre product diagram after completion with respect to the maximal ideal $\mathfrak{m} \subset A$ corresponding to x (Algebra, Lemma 10.97.2). Finally, apply Algebra, Lemma 10.97.8 to identify the completion of A'. □

53.15. Glueing and squishing

0C1H Below we will indicate $k[\epsilon]$ the algebra of dual numbers over k as defined in Varieties, Definition 33.16.1.

0C1I **Lemma 53.15.1.** *Let k be an algebraically closed field. Let $k \subset A$ be a ring extension such that A has exactly two k-sub algebras, then either $A = k \times k$ or $A = k[\epsilon]$.*

Proof. The assumption means $k \neq A$ and any subring $k \subset C \subset A$ is equal to either k or A. Let $t \in A$, $t \notin k$. Then A is generated by t over k. Hence $A = k[x]/I$ for some ideal I. If $I = (0)$, then we have the subalgebra $k[x^2]$ which is not allowed. Otherwise I is generated by a monic polynomial P. Write $P = \prod_{i=1}^{d}(t - a_i)$. If $d > 2$, then the subalgebra generated by $(t-a_1)(t-a_2)$ gives a contradiction. Thus $d = 2$. If $a_1 \neq a_2$, then $A = k \times k$, if $a_1 = a_2$, then $A = k[\epsilon]$. □

0C1J **Example 53.15.2** (Glueing points)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by glueing a and b in X' if the following are true:

(1) $a, b \in X'(k)$ are distinct points which map to the same point $x \in X(k)$,
(2) f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{a-b} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the difference $h(a) - h(b) \in k$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to the multiplicative difference $h(a)h(b)^{-1} \in k^*$.

0C1K **Example 53.15.3** (Squishing a tangent vector)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by squishing the tangent vector ϑ in X' if the following are true:

(1) $\vartheta : \mathrm{Spec}(k[\epsilon]) \to X'$ is a closed immersion over k such that $f \circ \vartheta$ factors through a point $x \in X(k)$,
(2) f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the coefficient of ϵ in $\vartheta^\sharp(h) \in k[\epsilon]$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to $\mathrm{d}\log(\vartheta^\sharp(h))$ where $\mathrm{d}\log : k[\epsilon]^* \to k$ is the homomorphism of abelian groups sending $a + b\epsilon$ to $b/a \in k$.

0C1L **Lemma 53.15.4.** *Let k be an algebraically closed field. Let $f : X' \to X$ be a finite morphism of algebraic k-schemes such that $\mathcal{O}_X \subset f_*\mathcal{O}_{X'}$ and such that f is an isomorphism away from a finite set of points. Then there is a factorization*

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

such that each $X_i \to X_{i-1}$ is either the glueing of two points or the squishing of a tangent vector (see Examples 53.15.2 and 53.15.3).

Proof. Let $U \subset X$ be the maximal open set over which f is an isomorphism. Then $X \setminus U = \{x_1, \ldots, x_n\}$ with $x_i \in X(k)$. We will consider factorizations $X' \to Y \to X$ of f such that both morphisms are finite and

$$\mathcal{O}_X \subset g_*\mathcal{O}_Y \subset f_*\mathcal{O}_{X'}$$

where $g : Y \to X$ is the given morphism. By assumption $\mathcal{O}_{X,x} \to (f_*\mathcal{O}_{X'})_x$ is an isomorphism onless $x = x_i$ for some i. Hence the cokernel

$$f_*\mathcal{O}_{X'}/\mathcal{O}_X = \bigoplus \mathcal{Q}_i$$

is a direct sum of skyscraper sheaves $\mathcal{Q}_i$ supported at $x_1, \ldots, x_n$. Because the displayed quotient is a coherent $\mathcal{O}_X$-module, we conclude that $\mathcal{Q}_i$ has finite length over $\mathcal{O}_{X,x_i}$. Hence we can argue by induction on the sum of these lengths, i.e., the length of the whole cokernel.

If $n > 1$, then we can define an $\mathcal{O}_X$-subalgebra $\mathcal{A} \subset f_*\mathcal{O}_{X'}$ by taking the inverse image of $\mathcal{Q}_1$. This will give a nontrivial factorization and we win by induction.

Assume $n = 1$. We abbreviate $x = x_1$. Consider the finite k-algebra extension

$$A = \mathcal{O}_{X,x} \subset (f_*\mathcal{O}_{X'})_x = B$$

Note that $\mathcal{Q} = \mathcal{Q}_1$ is the skyscraper sheaf with value B/A. We have a k-subalgebra $A \subset A + \mathfrak{m}_A B \subset B$. If both inclusions are strict, then we obtain a nontrivial factorization and we win by induction as above. If $A + \mathfrak{m}_A B = B$, then $A = B$ by Nakayama, then f is an isomorphism and there is nothing to prove. We conclude that we may assume $B = A + \mathfrak{m}_A B$. Set $C = B/\mathfrak{m}_A B$. If C has more than 2 k-subalgebras, then we obtain a subalgebra between A and B by taking the inverse image in B. Thus we may assume C has exactly 2 k-subalgebras. Thus $C = k \times k$ or $C = k[\epsilon]$ by Lemma 53.15.1. In this case f is correspondingly the glueing two points or the squishing of a tangent vector. □

0C1M **Lemma 53.15.5.** *Let k be an algebraically closed field. If $f : X' \to X$ is the glueing of two points a, b as in Example 53.15.2, then there is an exact sequence*

$$k^* \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

The first map is zero if a and b are on different connected components of X' and injective if X' is proper and a and b are on the same connected component of X'.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 33.38.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{ab^{-1}} k^* \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}^*_{X'}) \subset H^1(X', \mathcal{O}^*_{X'})$ (for example by the Leray spectral sequence, see Cohomology, Lemma 20.13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of $ab^{-1} : H^0(X', \mathcal{O}^*_{X'}) \to k^*$. If a and b are on different connected components of X', then ab^{-1} is surjective. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 33.9.3. Thus ab^{-1} is zero if X' is proper and a and b are on the same connected component. □

0C1N **Lemma 53.15.6.** *Let k be an algebraically closed field. If $f : X' \to X$ is the squishing of a tangent vector ϑ as in Example 53.15.3, then there is an exact sequence*

$$(k, +) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

and the first map is injective if X' is proper and reduced.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 33.38.7. Using the short exact sequence

$$0 \to \mathcal{O}^*_X \to f_*\mathcal{O}^*_{X'} \xrightarrow{\vartheta} x_*k \to 0$$

of Example 53.15.3 we obtain

$$H^0(X', \mathcal{O}^*_{X'}) \xrightarrow{\vartheta} k \to H^1(X, \mathcal{O}^*_X) \to H^1(X, f_*\mathcal{O}^*_{X'})$$

We have $H^1(X, f_*\mathcal{O}^*_{X'}) \subset H^1(X', \mathcal{O}^*_{X'})$ (for example by the Leray spectral sequence, see Cohomology, Lemma 20.13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of the map $\vartheta : H^0(X', \mathcal{O}^*_{X'}) \to k$. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 33.9.3. Thus the final statement of the lemma. □

53.16. Multicross and nodal singularities

0C1P In this section we discuss the simplest possible curve singularities.

Let k be a field. Consider the complete local k-algebra

0C1U (53.16.0.1) $$A = \{(f_1, \ldots, f_n) \in k[[t]] \times \ldots \times k[[t]] \mid f_1(0) = \ldots = f_n(0)\}$$

In the language introduced in Varieties, Definition 33.40.4 we see that A is a wedge of n copies of the power series ring in 1 variable over k. Observe that $k[[t]] \times \ldots \times k[[t]]$ is the integral closure of A in its total ring of fractions. Hence the δ-invariant of A is $n - 1$. There is an isomorphism

$$k[[x_1, \ldots, x_n]]/(\{x_i x_j\}_{i \neq j}) \longrightarrow A$$

obtained by sending x_i to $(0, \ldots, 0, t, 0, \ldots, 0)$ in A. It follows that $\dim(A) = 1$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. In particular, A is regular if and only if $n = 1$.

0C1V **Lemma 53.16.1.** *Let k be a separably closed field. Let A be a 1-dimensional reduced Nagata local k-algebra with residue field k. Then*

$$\delta\text{-invariant } A \geq \text{number of branches of } A - 1$$

If equality holds, then $A^\wedge$ is as in (53.16.0.1).

Proof. Since the residue field of A is separably closed, the number of branches of A is equal to the number of geometric branches of A, see More on Algebra, Definition 15.107.6. The inequality holds by Varieties, Lemma 33.40.6. Assume equality holds. We may replace A by the completion of A; this does not change the number of branches or the δ-invariant, see More on Algebra, Lemma 15.109.7 and Varieties, Lemma 33.39.6. Then A is strictly henselian, see Algebra, Lemma 10.153.9. By Varieties, Lemma 33.40.5 we see that A is a wedge of complete discrete valuation rings. Each of these is isomorphic to $k[[t]]$ by Algebra, Lemma 10.160.10. Hence A is as in (53.16.0.1). □

0C1W **Definition 53.16.2.** Let k be an algebraically closed field. Let X be an algebraic 1-dimensional k-scheme. Let $x \in X$ be a closed point. We say x defines a *multicross singularity* if the completion $\mathcal{O}^\wedge_{X,x}$ is isomorphic to (53.16.0.1) for some $n \geq 2$. We say x is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if $n = 2$.

These singularities are in some sense the simplest kind of singularities one can have on a curve over an algebraically closed field.

0C1X **Lemma 53.16.3.** *Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$. The following are equivalent*

1. *x defines a multicross singularity,*
2. *the δ-invariant of X at x is the number of branches of X at x minus 1,*
3. *there is a sequence of morphisms $U_n \to U_{n-1} \to \ldots \to U_0 = U \subset X$ where U is an open neighbourhood of x, where U_n is nonsingular, and where each $U_i \to U_{i-1}$ is the glueing of two points as in Example 53.15.2.*

Proof. The equivalence of (1) and (2) is Lemma 53.16.1.

Assume (3). We will argue by descending induction on i that all singularities of U_i are multicross. This is true for U_n as U_n has no singular points. If U_i is gotten from U_{i+1} by glueing $a, b \in U_{i+1}$ to a point $c \in U_i$, then we see that

$$\mathcal{O}^\wedge_{U_i,c} \subset \mathcal{O}^\wedge_{U_{i+1},a} \times \mathcal{O}^\wedge_{U_{i+1},b}$$

is the set of elements having the same residue classes in k. Thus the number of branches at c is the sum of the number of branches at a and b, and the δ-invariant at c is the sum of the δ-invariants at a and b plus 1 (because the displayed inclusion has codimension 1). This proves that (2) holds as desired.

Assume the equivalent conditions (1) and (2). We may choose an open $U \subset X$ such that x is the only singular point of U. Then we apply Lemma 53.15.4 to the normalization morphism

$$U^\nu = U_n \to U_{n-1} \to \ldots \to U_1 \to U_0 = U$$

All we have to do is show that in none of the steps we are squishing a tangent vector. Suppose $U_{i+1} \to U_i$ is the smallest i such that this is the squishing of a tangent vector θ at $u' \in U_{i+1}$ lying over $u \in U_i$. Arguing as above, we see that u_i is a multicross singularity (because the maps $U_i \to \ldots \to U_0$ are glueing of pairs of points). But now the number of branches at u' and u is the same and the δ-invariant of U_i at u is 1 bigger than the δ-invariant of U_{i+1} at u'. By Lemma 53.16.1 this implies that u cannot be a multicross singularity which is a contradiction. □

0CDZ **Lemma 53.16.4.** *Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$ be a multicross singularity (Definition 53.16.2). If X is Gorenstein, then x is a node.*

Proof. The map $\mathcal{O}_{X,x} \to \mathcal{O}^\wedge_{X,x}$ is flat and unramified in the sense that $\kappa(x) = \mathcal{O}^\wedge_{X,x}/\mathfrak{m}_x\mathcal{O}^\wedge_{X,x}$. (See More on Algebra, Section 15.43.) Thus X is Gorenstein implies $\mathcal{O}_{X,x}$ is Gorenstein, implies $\mathcal{O}^\wedge_{X,x}$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. Thus it suffices to show that the ring A in (53.16.0.1) with $n \geq 2$ is Gorenstein if and only if $n = 2$.

If $n = 2$, then $A = k[[x, y]]/(xy)$ is a complete intersection and hence Gorenstein. For example this follows from Duality for Schemes, Lemma 48.24.5 applied to $k[[x, y]] \to A$ and the fact that the regular local ring $k[[x, y]]$ is Gorenstein by Dualizing Complexes, Lemma 47.21.3.

Assume $n > 2$. If A where Gorenstein, then A would be a dualizing complex over A (Duality for Schemes, Definition 48.24.1). Then $R\mathrm{Hom}(k, A)$ would be equal to $k[n]$ for some $n \in \mathbf{Z}$, see Dualizing Complexes, Lemma 47.15.12. It would follow that $\mathrm{Ext}^1_A(k, A) \cong k$ or $\mathrm{Ext}^1_A(k, A) = 0$ (depending on the value of n; in fact n has to be -1 but it doesn't matter to us here). Using the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

we find that

$$\mathrm{Ext}^1_A(k, A) = \mathrm{Hom}_A(\mathfrak{m}_A, A)/A$$

where $A \to \mathrm{Hom}_A(\mathfrak{m}_A, A)$ is given by $a \mapsto (a' \mapsto aa')$. Let $e_i \in \mathrm{Hom}_A(\mathfrak{m}_A, A)$ be the element that sends $(f_1, \ldots, f_n) \in \mathfrak{m}_A$ to $(0, \ldots, 0, f_i, 0, \ldots, 0)$. The reader verifies easily that $e_1, \ldots, e_{n-1}$ are k-linearly independent in $\mathrm{Hom}_A(\mathfrak{m}_A, A)/A$. Thus $\dim_k \mathrm{Ext}^1_A(k, A) \geq n - 1 \geq 2$ which finishes the proof. (Observe that $e_1 + \ldots + e_n$ is the image of 1 under the map $A \to \mathrm{Hom}_A(\mathfrak{m}_A, A)$.) □

53.17. Torsion in the Picard group

0C1Y In this section we bound the torsion in the Picard group of a 1-dimensional proper scheme over a field. We will use this in our study of semistable reduction for curves.

There does not seem to be an elementary way to obtain the result of Lemma 53.17.1. Analyzing the proof there are two key ingredients: (1) there is an abelian variety classifying degree zero invertible sheaves on a smooth projective curve and (2) the structure of torsion points on an abelian variety can be determined.

0C1Z **Lemma 53.17.1.** *Let k be an algebraically closed field. Let X be a smooth projective curve of genus g over k.*

1. *If $n \geq 1$ is invertible in k, then $\mathrm{Pic}(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.*
2. *If the characteristic of k is $p > 0$, then there exists an integer $0 \leq f \leq g$ such that $\mathrm{Pic}(X)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ for all $m \geq 1$.*

Proof. Let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ denote the subgroup of invertible sheaves of degree 0. In other words, there is a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The group $\mathrm{Pic}^0(X)$ is the k-points of the group scheme $\underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. The same lemma tells us that $\underline{\mathrm{Pic}}^0_{X/k}$ is a g-dimensional

abelian variety over k as defined in Groupoids, Definition 39.9.1. Thus we conclude by the results of Groupoids, Proposition 39.9.11. □

0CDU **Lemma 53.17.2.** *Let k be a field. Let n be prime to the characteristic of k. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and of genus g.*

(1) *If $g = 1$ then there exists a finite separable extension k'/k such that $X_{k'}$ has a k'-rational point and $\mathrm{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$.*
(2) *If $g \geq 2$ then there exists a finite separable extension k'/k with $[k' : k] \leq (2g-2)(n^{2g})!$ such that $X_{k'}$ has a k'-rational point and $\mathrm{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.*

Proof. Assume $g \geq 2$. First we may choose a finite separable extension of degree at most $2g-2$ such that X acquires a rational point, see Lemma 53.13.9. Thus we may assume X has a k-rational point $x \in X(k)$ but now we have to prove the lemma with $[k' : k] \leq (n^{2g})!$. Let $k \subset k^{sep} \subset \overline{k}$ be a separable algebraic closure inside an algebraic closure. By Lemma 53.17.1 we have

$$\mathrm{Pic}(X_{\overline{k}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 44.7.2 we conclude that

$$\mathrm{Pic}(X_{k^{sep}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 44.7.2 there is a continuous action

$$\mathrm{Gal}(k^{sep}/k) \longrightarrow \mathrm{Aut}(\mathrm{Pic}(X_{k^{sep}})[n]$$

and the lemma is true for the fixed field k' of the kernel of this map. The kernel is open because the action is continuous which implies that k'/k is finite. By Galois theory $\mathrm{Gal}(k'/k)$ is the image of the displayed arrow. Since the permutation group of a set of cardinality n^{2g} has cardinality $(n^{2g})!$ we conclude by Galois theory that $[k' : k] \leq (n^{2g})!$. (Of course this proves the lemma with the bound $|\mathrm{GL}_{2g}(\mathbf{Z}/n\mathbf{Z})|$, but all we want here is that there is some bound.)

If the genus is 1, then there is no upper bound on the degree of a finite separable field extension over which X acquires a rational point (details omitted). Still, there is such an extension for example by Varieties, Lemma 33.25.6. The rest of the proof is the same as in the case of $g \geq 2$. □

0C20 **Proposition 53.17.3.** *Let k be an algebraically closed field. Let X be a proper scheme over k which is reduced, connected, and has dimension 1. Let g be the genus of X and let g_{geom} be the sum of the geometric genera of the irreducible components of X. For any prime ℓ different from the characteristic of k we have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X)[\ell] \leq g + g_{geom}$$

and equality holds if and only if all the singularities of X are multicross.

Proof. Let $\nu : X^\nu \to X$ be the normalization (Varieties, Lemma 33.41.2). Choose a factorization

$$X^\nu = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

as in Lemma 53.15.4. Let us denote $h_i^0 = \dim_k H^0(X_i, \mathcal{O}_{X_i})$ and $h_i^1 = \dim_k H^1(X_i, \mathcal{O}_{X_i})$. By Lemmas 53.15.5 and 53.15.6 for each $n > i \geq 0$ we have one of the following there possibilities

(1) X_i is obtained by glueing $a, b \in X_{i+1}$ which are on different connected components: in this case $\mathrm{Pic}(X_i) = \mathrm{Pic}(X_{i+1})$, $h_{i+1}^0 = h_i^0 + 1$, $h_{i+1}^1 = h_i^1$,

(2) X_i is obtained by glueing $a, b \in X_{i+1}$ which are on the same connected component: in this case there is a short exact sequence

$$0 \to k^* \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$

and $h^0_{i+1} = h^0_i$, $h^1_{i+1} = h^1_i - 1$,

(3) X_i is obtained by squishing a tangent vector in X_{i+1}: in this case there is a short exact sequence

$$0 \to (k, +) \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$

and $h^0_{i+1} = h^0_i$, $h^1_{i+1} = h^1_i - 1$.

To prove the statements on dimensions of cohomology groups of the structure sheaf, use the exact sequences in Examples 53.15.2 and 53.15.3. Since k is algebraically closed of characteristic prime to ℓ we see that $(k, +)$ and k^* are ℓ-divisible and with ℓ-torsion $(k, +)[\ell] = 0$ and $k^*[\ell] \cong \mathbf{F}_\ell$. Hence

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_{i+1})[\ell] - \dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_i)[\ell]$$

is zero, except in case (2) where it is equal to -1. At the end of this process we get the normalization $X^\nu = X_n$ which is a disjoint union of smooth projective curves over k. Hence we have

(1) $h^1_n = g_{geom}$ and
(2) $\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_n)[\ell] = 2g_{geom}$.

The last equality by Lemma 53.17.1. Since $g = h^1_0$ we see that the number of steps of type (2) and (3) is at most $h^1_0 - h^1_n = g - g_{geom}$. By our comptation of the differences in ranks we conclude that

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X)[\ell] \leq g - g_{geom} + 2g_{geom} = g + g_{geom}$$

and equality holds if and only if no steps of type (3) occur. This indeed means that all singularities of X are multicross by Lemma 53.16.3. Conversely, if all the singularities are multicross, then Lemma 53.16.3 guarantees that we can find a sequence $X^\nu = X_n \to \ldots \to X_0 = X$ as above such that no steps of type (3) occur in the sequence and we find equality holds in the lemma (just glue the local sequences for each point to find one that works for all singular points of x; some details omitted). □

53.18. Genus versus geometric genus

0CE0 Let k be a field with algebraic closure $\overline{k}$. Let X be a proper scheme of dimension ≤ 1 over k. We define $g_{geom}(X/k)$ to be the sum of the geometric genera of the irreducible components of $X_{\overline{k}}$ which have dimension 1.

0CE1 **Lemma 53.18.1.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then*

$$g_{geom}(X/k) = \sum\nolimits_{C \subset X} g_{geom}(C/k)$$

where the sum is over irreducible components $C \subset X$ of dimension 1.

Proof. This is immediate from the definition and the fact that an irreducible component $\overline{Z}$ of $X_{\overline{k}}$ maps onto an irreducible component Z of X (Varieties, Lemma 33.8.11) of the same dimension (Morphisms, Lemma 29.28.3 applied to the generic point of $\overline{Z}$). □

0CE2 **Lemma 53.18.2.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then*

(1) *We have $g_{geom}(X/k) = g_{geom}(X_{red}/k)$.*
(2) *If $X' \to X$ is a birational proper morphism, then $g_{geom}(X'/k) = g_{geom}(X/k)$.*
(3) *If $X^\nu \to X$ is the normalization morphism, then $g_{geom}(X^\nu/k) = g_{geom}(X/k)$.*

Proof. Part (1) is immediate from Lemma 53.18.1. If $X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 33.17.2 and 33.17.3). Hence $X'_{\overline{k}} \to X_{\overline{k}}$ is an isomorphism over a dense open. Thus the irreducible components of $X'_{\overline{k}}$ and $X_{\overline{k}}$ are in bijective correspondence and the corresponding components have isomorphic function fields. In particular these components have isomorphic nonsingular projective models and hence have the same geometric genera. This proves (2). Part (3) follows from (1) and (2) and the fact that $X^\nu \to X_{red}$ is birational (Morphisms, Lemma 29.54.7). □

0CE3 **Lemma 53.18.3.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $f : Y \to X$ be a finite morphism such that there exists a dense open $U \subset X$ over which f is a closed immersion. Then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(Y, \mathcal{O}_Y)$$

Proof. Consider the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to \mathcal{F} \to 0$$

of coherent sheaves on X. By assumption $\mathcal{F}$ is supported in finitely many closed points and hence has vanishing higher cohomology (Varieties, Lemma 33.33.3). On the other hand, we have $H^2(X, \mathcal{G}) = 0$ by Cohomology, Proposition 20.20.7. It follows formally that the induced map $H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_Y)$ is surjective. Since $H^1(X, f_*\mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y)$ (Cohomology of Schemes, Lemma 30.2.4) we conclude the lemma holds. □

0CE4 **Lemma 53.18.4.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If $X' \to X$ is a birational proper morphism, then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(X', \mathcal{O}_{X'})$$

If X is reduced, $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ is surjective, and equality holds, then $X' = X$.

Proof. If $f : X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 33.17.2 and 33.17.3). Thus the inequality by Lemma 53.18.3. Assume X is reduced. Then $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective and we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{F} \to 0$$

Under the assumptions given in the second statement, we conclude from the long exact cohomology sequence that $H^0(X, \mathcal{F}) = 0$. Then $\mathcal{F} = 0$ because $\mathcal{F}$ is generated by global sections (Varieties, Lemma 33.33.3). and $\mathcal{O}_X = f_*\mathcal{O}_{X'}$. Since f is affine this implies $X = X'$. □

0CE5 **Lemma 53.18.5.** *Let k be a field. Let C be a proper curve over k. Set $\kappa = H^0(C, \mathcal{O}_C)$. Then*

$$[\kappa : k]_s \dim_\kappa H^1(C, \mathcal{O}_C) \geq g_{geom}(C/k)$$

Proof. Varieties, Lemma 33.26.2 implies κ is a field and a finite extension of k. By Fields, Lemma 9.14.8 we have $[\kappa : k]_s = |\operatorname{Mor}_k(\kappa, \overline{k})|$ and hence $\operatorname{Spec}(\kappa \otimes_k \overline{k})$ has $[\kappa : k]_s$ points each with residue field $\overline{k}$. Thus

$$C_{\overline{k}} = \bigcup\nolimits_{t \in \operatorname{Spec}(\kappa \otimes_k \overline{k})} C_t$$

(set theoretic union). Here $C_t = C \times_{\operatorname{Spec}(\kappa), t} \operatorname{Spec}(\overline{k})$ where we view t as a k-algebra map $t : \kappa \to \overline{k}$. The conclusion is that $g_{geom}(C/k) = \sum_t g_{geom}(C_t/\overline{k})$ and the sum is over an index set of size $[\kappa : k]_s$. We have

$$H^0(C_t, \mathcal{O}_{C_t}) = \overline{k} \quad \text{and} \quad \dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) = \dim_\kappa H^1(C, \mathcal{O}_C)$$

by cohomology and base change (Cohomology of Schemes, Lemma 30.5.2). Observe that the normalization C_t^ν is the disjoint union of the nonsingular projective models of the irreducible components of C_t (Morphisms, Lemma 29.54.6). Hence $\dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$ is equal to $g_{geom}(C_t/\overline{k})$. By Lemma 53.18.3 we have

$$\dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) \geq \dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$$

and this finishes the proof. □

0CE6 **Lemma 53.18.6.** *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let ℓ be a prime number invertible in k. Then*

$$\dim_{\mathbf{F}_\ell} \operatorname{Pic}(X)[\ell] \leq \dim_k H^1(X, \mathcal{O}_X) + g_{geom}(X/k)$$

where $g_{geom}(X/k)$ is as defined above.

Proof. The map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{k}})$ is injective by Varieties, Lemma 33.30.3. By Cohomology of Schemes, Lemma 30.5.2 $\dim_k H^1(X, \mathcal{O}_X)$ equals $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$. Hence we may assume k is algebraically closed.

Let X_{red} be the reduction of X. Then the surjection $\mathcal{O}_X \to \mathcal{O}_{X_{red}}$ induces a surjection $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_{X_{red}})$ because cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology, Proposition 20.20.7. Since $X_{red} \to X$ induces an isomorphism on irreducible components over $\overline{k}$ and an isomorphism on ℓ-torsion in Picard groups (Picard Schemes of Curves, Lemma 44.7.2) we may replace X by X_{red}. In this way we reduce to Proposition 53.17.3. □

53.19. Nodal curves

0C46 We have already defined ordinary double points over algebraically closed fields, see Definition 53.16.2. Namely, if $x \in X$ is a closed point of a 1-dimensional algebraic scheme over an algebraically closed field k, then x is an ordinary double point if and only if

$$\mathcal{O}_{X,x}^\wedge \cong k[[x, y]]/(xy)$$

See discussion following (53.16.0.1) in Section 53.16.

0C47 **Definition 53.19.1.** Let k be a field. Let X be a 1-dimensional locally algebraic k-scheme.

(1) We say a closed point $x \in X$ is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if there exists an ordinary double point $\overline{x} \in X_{\overline{k}}$ mapping to x.

(2) We say the *singularities of X are at-worst-nodal* if all closed points of X are either in the smooth locus of the structure morphism $X \to \mathrm{Spec}(k)$ or are ordinary double points.

Often a 1-dimensional algebraic scheme X is called a *nodal curve* if the singularities of X are at worst nodal. Sometimes a nodal curve is required to be proper. Since a nodal curve so defined need not be irreducible, this conflicts with our earlier definition of a curve as a variety of dimension 1.

0C48 **Lemma 53.19.2.** *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $I \subset \mathfrak{m}$ be an ideal.*

(1) *If A/I is reduced, then $I = (0)$, $I = \mathfrak{m}$, or $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*
(2) *If A/I has depth 1, then $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*

Proof. Assume $I \neq 0$. Write $I = (f_1, \ldots, f_r)$. As A is a UFD (More on Algebra, Lemma 15.122.2) we can write $f_i = fg_i$ where f is the gcd of $f_1, \ldots, f_r$. Thus the gcd of $g_1, \ldots, g_r$ is 1 which means that there is no height 1 prime ideal over $g_1, \ldots, g_r$. Then either $(g_1, \ldots, g_r) = A$ which implies $I = (f)$ or if not, then $\dim(A) = 2$ implies that $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$, i.e., $\mathfrak{m} = \sqrt{(g_1, \ldots, g_r)}$.

Assume A/I reduced, i.e., I radical. If f is a unit, then since I is radical we see that $I = \mathfrak{m}$. If $f \in \mathfrak{m}$, then we see that f^n maps to zero in A/I. Hence $f \in I$ by reducedness and we conclude $I = (f)$.

Assume A/I has depth 1. Then $\mathfrak{m}$ is not an associated prime of A/I. Since the class of f modulo I is annihilated by $g_1, \ldots, g_r$, this implies that the class of f is zero in A/I. Thus $I = (f)$ as desired. □

Let κ be a field and let V be a vector space over κ. We will say $q \in \mathrm{Sym}^2_\kappa(V)$ is *nondegenerate* if the induced κ-linear map $V^\vee \to V$ is an isomorphism. If $q = \sum_{i \leq j} a_{ij} x_i x_j$ for some κ-basis $x_1, \ldots, x_n$ of V, then this means that the determinant of the matrix

$$\begin{pmatrix} 2a_{11} & a_{12} & \ldots \\ a_{12} & 2a_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

is nonzero. This is equivalent to the condition that the partial derivatives of q with respect to the x_i cut out 0 scheme theoretically.

0C49 **Lemma 53.19.3.** *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. The following are equivalent*

(1) *κ/k is separable, A is reduced, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,*
(2) *κ/k is separable, $depth(A) = 1$, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,*
(3) *κ/k is separable, $A^\wedge \cong \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ as a k-algebra where $ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over κ.*

Proof. Assume (3). Then $A^\wedge$ is reduced because $ax^2+bxy+cy^2$ is either irreducible or a product of two nonassociated prime elements. Hence $A \subset A^\wedge$ is reduced. It follows that (1) is true.

Assume (1). Then A cannot be Artinian, since it would not be reduced because $\mathfrak{m} \neq (0)$. Hence $\dim(A) \geq 1$, hence $\text{depth}(A) \geq 1$ by Algebra, Lemma 10.157.3. On the other hand $\dim(A) = 2$ implies A is regular which contradicts the existence of q by Algebra, Lemma 10.106.1. Thus $\dim(A) \leq 1$ and we conclude $\text{depth}(A) = 1$ by Algebra, Lemma 10.72.3. It follows that (2) is true.

Assume (2). Since the depth of A is the same as the depth of $A^\wedge$ (More on Algebra, Lemma 15.43.2) and since the other conditions are insensitive to completion, we may assume that A is complete. Choose $\kappa \to A$ as in More on Algebra, Lemma 15.38.3. Since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$ we can choose $x_0, y_0 \in \mathfrak{m}$ which map to a basis. We obtain a continuous κ-algebra map

$$\kappa[[x, y]] \longrightarrow A$$

by the rules $x \mapsto x_0$ and $y \mapsto y_0$. Let q be the class of $ax_0^2 + bx_0y_0 + cy_0^2$ in $\text{Sym}^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Write $Q(x, y) = ax^2 + bxy + cy^2$ viewed as a polynomial in two variables. Then we see that

$$Q(x_0, y_0) = ax_0^2 + bx_0y_0 + cy_0^2 = \sum\nolimits_{i+j=3} a_{ij}x_0^i y_0^j$$

for some a_{ij} in A. We want to prove that we can increase the order of vanishing by changing our choice of x_0, y_0. Suppose that $x_1, y_1 \in \mathfrak{m}^2$. Then

$$Q(x_0 + x_1, y_0 + y_1) = Q(x_0, y_0) + (2ax_0 + by_0)x_1 + (bx_0 + 2cy_0)y_1 \bmod \mathfrak{m}^4$$

Nondegeneracy of Q means exactly that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are a κ-basis for $\mathfrak{m}/\mathfrak{m}^2$, see discussion preceding the lemma. Hence we can certainly choose $x_1, y_1 \in \mathfrak{m}^2$ such that $Q(x_0 + x_1, y_0 + y_1) \in \mathfrak{m}^4$. Continuing in this fashion by induction we can find $x_i, y_i \in \mathfrak{m}^{i+1}$ such that

$$Q(x_0 + x_1 + \ldots + x_n, y_0 + y_1 + \ldots + y_n) \in \mathfrak{m}^{n+3}$$

Since A is complete we can set $x_\infty = \sum x_i$ and $y_\infty = \sum y_i$ and we can consider the map $\kappa[[x, y]] \longrightarrow A$ sending x to x_∞ and y to y_∞. This map induces a surjection $\kappa[[x, y]]/(Q) \longrightarrow A$ by Algebra, Lemma 10.96.1. By Lemma 53.19.2 the kernel of $k[[x, y]] \to A$ is principal. But the kernel cannot contain a proper divisor of Q as such a divisor would have degree 1 in x, y and this would contradict $\dim(\mathfrak{m}/\mathfrak{m}^2) = 2$. Hence Q generates the kernel as desired. □

0C4A **Lemma 53.19.4.** *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Nagata local k-algebra. The following are equivalent*

1. *$k \to A$ is as in Lemma 53.19.3,*
2. *κ/k is separable, A is reduced of dimension 1, the δ-invariant of A is 1, and A has 2 geometric branches.*

If this holds, then the integral closure A' of A in its total ring of fractions has either 1 or 2 maximal ideals $\mathfrak{m}'$ and the extensions $\kappa(\mathfrak{m}')/k$ are separable.

Proof. In both cases A and $A^\wedge$ are reduced. In case (2) because the completion of a reduced local Nagata ring is reduced (More on Algebra, Lemma 15.43.6). In both cases A and $A^\wedge$ have dimension 1 (More on Algebra, Lemma 15.43.1). The δ-invariant and the number of geometric branches of A and $A^\wedge$ agree by Varieties, Lemma 33.39.6 and More on Algebra, Lemma 15.109.7. Let A' be the integral closure of A in its total ring of fractions as in Varieties, Lemma 33.39.2. By Varieties, Lemma 33.39.5 we see that $A' \otimes_A A^\wedge$ plays the same role for $A^\wedge$. Thus we may replace A by $A^\wedge$ and assume A is complete.

Assume (1) holds. It suffices to show that A has two geometric branches and δ-invariant 1. We may assume $A = \kappa[[x, y]]/(ax^2+bxy+cy^2)$ with $q = ax^2+bxy+cy^2$ nondegenerate. There are two cases.

Case I: q splits over κ. In this case we may after changing coordinates assume that $q = xy$. Then we see that

$$A' = \kappa[[x, y]]/(x) \times \kappa[[x, y]]/(y)$$

Case II: q does not split. In this case $c \neq 0$ and nondegenerate means $b^2 - 4ac \neq 0$. Hence $\kappa' = \kappa[t]/(a + bt + ct^2)$ is a degree 2 separable extension of κ. Then $t = y/x$ is integral over A and we conclude that

$$A' = \kappa'[[x]]$$

with y mapping to tx on the right hand side.

In both cases one verifies by hand that the δ-invariant is 1 and the number of geometric branches is 2. In this way we see that (1) implies (2). Moreover we conclude that the final statement of the lemma holds.

Assume (2) holds. More on Algebra, Lemma 15.107.7 implies A' either has two maximal ideals or A' has one maximal ideal and $[\kappa(\mathfrak{m}') : \kappa]_s = 2$.

Case I: A' has two maximal ideals $\mathfrak{m}'_1$, $\mathfrak{m}'_2$ with residue fields κ_1, κ_2. Since the δ-invariant is the length of A'/A and since there is a surjection $A'/A \to (\kappa_1 \times \kappa_2)/\kappa$ we see that $\kappa = \kappa_1 = \kappa_2$. Since A is complete (and henselian by Algebra, Lemma 10.153.9) and A' is finite over A we see that $A' = A_1 \times A_2$ (by Algebra, Lemma 10.153.4). Since A' is a normal ring it follows that A_1 and A_2 are discrete valuation rings. Hence A_1 and A_2 are isomorphic to $\kappa[[t]]$ (as k-algebras) by More on Algebra, Lemma 15.38.4. Since the δ-invariant is 1 we conclude that A is the wedge of A_1 and A_2 (Varieties, Definition 33.40.4). It follows easily that $A \cong \kappa[[x, y]]/(xy)$.

Case II: A' has a single maximal ideal $\mathfrak{m}'$ with residue field κ' and $[\kappa' : \kappa]_s = 2$. Arguing exactly as in Case I we see that $[\kappa' : \kappa] = 2$ and κ' is separable over κ. Since A' is normal we see that A' is isomorphic to $\kappa'[[t]]$ (see reference above). Since A'/A has length 1 we conclude that

$$A = \{f \in \kappa'[[t]] \mid f(0) \in \kappa\}$$

Then a simple computation shows that A as in case (1). □

0C4B **Lemma 53.19.5.** *Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset A$ be an ideal. For any $r \geq 0$ the ideal in A/I generated by the $r \times r$-minors of the matrix $(\partial f_j/\partial x_i)$ is independent of the choice of the generators of I or the regular system of parameters $x_1, \ldots, x_n$ of A.*

Proof. The "correct" proof of this lemma is to prove that this ideal is the $(n-r)$th Fitting ideal of a module of continuous differentials of A/I over k. Here is a direct proof. If $g_1, \ldots g_l$ is a second set of generators of I, then we can write $g_s = \sum a_{sj} f_j$ and we have the equality of matrices

$$(\partial g_s/\partial x_i) = (a_{sj})(\partial f_j/\partial x_i) + (\partial a_{sj}/\partial x_i f_j)$$

The final term is zero in A/I. By the Cauchy-Binet formula we see that the ideal of minors for the g_s is contained in the ideal for the f_j. By symmetry these ideals are the same. If $y_1, \ldots, y_n \in \mathfrak{m}_A$ is a second regular system of parameters, then

the matrix $(\partial y_j/\partial x_i)$ is invertible and we can use the chain rule for differentiation. Some details omitted. □

0C4C **Lemma 53.19.6.** *Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset \mathfrak{m}_A$ be an ideal. The following are equivalent*

(1) *$k \to A/I$ is as in Lemma 53.19.3,*
(2) *A/I is reduced and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$,*
(3) *$\text{depth}(A/I) = 1$ and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$.*

Proof. By Lemma 53.19.5 we may change our system of coordinates and the choice of generators during the proof.

If (1) holds, then we may change coordinates such that $x_1, \ldots, x_{n-2}$ map to zero in A/I and $A/I = k[[x_{n-1}, x_n]]/(ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2)$ for some nondegenerate quadric $ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2$. Then we can explicitly compute to show that both (2) and (3) are true.

Assume the $(n-1)\times(n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I+\mathfrak{m}_A$. Suppose that for some i and j the partial derivative $\partial f_j/\partial x_i$ is a unit in A. Then we may use the system of parameters $f_j, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n$ and the generators $f_j, f_1, \ldots, f_{j-1}, \hat{f}_j, f_{j+1}, \ldots, f_m$ of I. Then we get a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, f_2, \ldots, f_m$ of I. Next, we look for an $i \geq 2$ and $j \geq 2$ such that $\partial f_j/\partial x_i$ is a unit in A. If such a pair exists, then we can make a replacement as above and assume that we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, x_2, f_3, \ldots, f_m$ of I. Continuing, in finitely many steps we reach the situation where we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, \ldots, x_t, f_{t+1}, \ldots, f_m$ of I such that $\partial f_j/\partial x_i \in \mathfrak{m}_A$ for all $i, j \geq t+1$.

In this case the matrix of partial derivatives has the following block shape

$$\begin{pmatrix} I_{t\times t} & * \\ 0 & \mathfrak{m}_A \end{pmatrix}$$

Hence every $(n-1) \times (n-1)$-minor is in $\mathfrak{m}_A^{n-1-t}$. Note that $I \neq \mathfrak{m}_A$ otherwise the ideal of minors would contain 1. It follows that $n - 1 - t \leq 1$ because there is an element of $\mathfrak{m}_A \setminus \mathfrak{m}_A^2 + I$ (otherwise $I = \mathfrak{m}_A$ by Nakayama). Thus $t \geq n - 2$. We have seen that $t \neq n$ above and similarly if $t = n - 1$, then there is an invertible $(n-1) \times (n-1)$-minor which is disallowed as well. Hence $t = n - 2$. Then A/I is a quotient of $k[[x_{n-1}, x_n]]$ and Lemma 53.19.2 implies in both cases (2) and (3) that I is generated by $x_1, \ldots, x_{n-2}, f$ for some $f = f(x_{n-1}, x_n)$. In this case the condition on the minors exactly says that the quadratic term in f is nondegenerate, i.e., A/I is as in Lemma 53.19.3. □

0C4D **Lemma 53.19.7.** *Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent*

(1) *x is a node,*
(2) *$k \to \mathcal{O}_{X,x}$ is as in Lemma 53.19.3,*
(3) *any $\overline{x} \in X_{\overline{k}}$ mapping to x defines a nodal singularity,*
(4) *$\kappa(x)/k$ is separable, $\mathcal{O}_{X,x}$ is reduced, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,*

(5) $\kappa(x)/k$ *is separable,* $\text{depth}(\mathcal{O}_{X,x}) = 1$*, and the first Fitting ideal of* $\Omega_{X/k}$ *generates* $\mathfrak{m}_x$ *in* $\mathcal{O}_{X,x}$*,*
(6) $\kappa(x)/k$ *is separable and* $\mathcal{O}_{X,x}$ *is reduced, has* δ*-invariant* 1*, and has* 2 *geometric branches.*

Proof. First assume that k is algebraically closed. In this case the equivalence of (1) and (3) is trivial. The equivalence of (1) and (3) with (2) holds because the only nondegenerate quadric in two variables is xy up to change in coordinates. The equivalence of (1) and (6) is Lemma 53.16.1. After replacing X by an affine neighbourhood of x, we may assume there is a closed immersion $X \to \mathbf{A}^n_k$ mapping x to 0. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be generators for the ideal I of X in $\mathbf{A}^n_k$. Then $\Omega_{X/k}$ corresponds to the $R = k[x_1, \ldots, x_n]/I$-module $\Omega_{R/k}$ which has a presentation

$$R^{\oplus m} \xrightarrow{(\partial f_j/\partial x_i)} R^{\oplus n} \to \Omega_{R/k} \to 0$$

(See Algebra, Sections 10.131 and 10.134.) The first Fitting ideal of $\Omega_{R/k}$ is thus the ideal generated by the $(n-1) \times (n-1)$-minors of the matrix $(\partial f_j/\partial x_i)$. Hence (2), (4), (5) are equivalent by Lemma 53.19.6 applied to the completion of $k[x_1, \ldots, x_n] \to R$ at the maximal ideal $(x_1, \ldots, x_n)$.

Now assume k is an arbitrary field. In cases (2), (4), (5), (6) the residue field $\kappa(x)$ is separable over k. Let us show this holds as well in cases (1) and (3). Namely, let $Z \subset X$ be the closed subscheme of X defined by the first Fitting ideal of $\Omega_{X/k}$. The formation of Z commutes with field extension (Divisors, Lemma 31.10.1). If (1) or (3) is true, then there exists a point $\overline{x}$ of $X_{\overline{k}}$ such that $\overline{x}$ is an isolated point of multiplicity 1 of $Z_{\overline{k}}$ (as we have the equivalence of the conditions of the lemma over $\overline{k}$). In particular $Z_{\overline{x}}$ is geometrically reduced at $\overline{x}$ (because $\overline{k}$ is algebraically closed). Hence Z is geometrically reduced at x (Varieties, Lemma 33.6.6). In particular, Z is reduced at x, hence $Z = \text{Spec}(\kappa(x))$ in a neighbourhood of x and $\kappa(x)$ is geometrically reduced over k. This means that $\kappa(x)/k$ is separable (Algebra, Lemma 10.44.2).

The argument of the previous paragraph shows that if (1) or (3) holds, then the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat and since $\mathcal{O}_{X_{\overline{k}},\overline{x}}$ is reduced and has depth 1, we see that (4) and (5) hold (use Algebra, Lemmas 10.164.2 and 10.163.2). Conversely, (4) implies (5) by Algebra, Lemma 10.157.3. If (5) holds, then Z is geometrically reduced at x (because $\kappa(x)/k$ separable and Z is x in a neighbourhood). Hence $Z_{\overline{k}}$ is reduced at any point $\overline{x}$ of $X_{\overline{k}}$ lying over x. In other words, the first fitting ideal of $\Omega_{X_{\overline{k}}/\overline{k}}$ generates $\mathfrak{m}_{\overline{x}}$ in $\mathcal{O}_{X_{\overline{k}},\overline{x}}$. Moreover, since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat we see that $\text{depth}(\mathcal{O}_{X_{\overline{k}},\overline{x}}) = 1$ (see reference above). Hence (5) holds for $\overline{x} \in X_{\overline{k}}$ and we conclude that (3) holds (because of the equivalence over algebraically closed fields). In this way we see that (1), (3), (4), (5) are equivalent.

The equivalence of (2) and (6) follows from Lemma 53.19.4.

Finally, we prove the equivalence of (2) = (6) with (1) = (3) = (4) = (5). First we note that the geometric number of branches of X at x and the geometric number of branches of $X_{\overline{k}}$ at $\overline{x}$ are equal by Varieties, Lemma 33.40.2. We conclude from the information available to us at this point that in all cases this number is equal to 2. On the other hand, in case (1) it is clear that X is geometrically reduced at x, and hence

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } X_{\overline{k}} \text{ at } \overline{x}$$

by Varieties, Lemma 33.39.8. Since in case (1) the right hand side is 1, this forces the δ-invariant of X at x to be 1 (because if it were zero, then $\mathcal{O}_{X,x}$ would be a discrete valuation ring by Varieties, Lemma 33.39.4 which is unibranch, a contradiction). Thus (5) holds. Conversely, if (2) = (5) is true, then assumptions (a), (b), (c) of Varieties, Lemma 33.27.6 hold for $x \in X$ by Lemma 53.19.4. Thus Varieties, Lemma 33.39.9 applies and shows that we have equality in the above displayed inequality. We conclude that (5) holds for $\overline{x} \in X_{\overline{k}}$ and we are back in case (1) by the equivalence of the conditions over an algebraically closed field. □

0CBT **Remark 53.19.8** (The quadratic extension associated to a node)**.** Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. Assume that either $(A, \mathfrak{m}, \kappa)$ is as in Lemma 53.19.3, or A is Nagata as in Lemma 53.19.4, or A is complete and as in Lemma 53.19.6. Then A defines canonically a degree 2 separable κ-algebra κ' as follows

(1) let $q = ax^2 + bxy + cy^2$ be a nondegenerate quadric as in Lemma 53.19.3 with coordinates x, y chosen such that $a \neq 0$ and set $\kappa' = \kappa[x]/(ax^2 + bx + c)$,
(2) let $A' \supset A$ be the integral closure of A in its total ring of fractions and set $\kappa' = A'/\mathfrak{m}A'$, or
(3) let κ' be the κ-algebra such that $\mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}) = \mathrm{Spec}(\kappa')$.

The equivalence of (1) and (2) was shown in the proof of Lemma 53.19.4. We omit the equivalence of this with (3). If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then (3) shows that $\mathrm{Spec}(\kappa') = X^\nu \times_X \mathrm{Spec}(\kappa)$ where $\nu : X^\nu \to X$ is the normalization morphism.

0CBU **Remark 53.19.9** (Trivial quadratic extension)**.** Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be as in Remark 53.19.8 and let κ'/κ be the associated separable algebra of degree 2. Then the following are equivalent

(1) $\kappa' \cong \kappa \times \kappa$ as κ-algebra,
(2) the form q of Lemma 53.19.3 can be chosen to be xy,
(3) A has two branches,
(4) the extension A'/A of Lemma 53.19.4 has two maximal ideals, and
(5) $A^\wedge \cong \kappa[[x, y]]/(xy)$ as a k-algebra.

The equivalence between these conditions has been shown in the proof of Lemma 53.19.4. If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then this means exactly that there are two points x_1, x_2 of the normalization X^ν lying over x and that $\kappa(x) = \kappa(x_1) = \kappa(x_2)$.

0CBV **Definition 53.19.10.** Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. We say x is a *split node* if x is a node, $\kappa(x) = k$, and the equivalent assertions of Remark 53.19.9 hold for $A = \mathcal{O}_{X,x}$.

We formulate the obligatory lemma stating what we already know about this concept.

0CBW **Lemma 53.19.11.** *Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent*

(1) *x is a split node,*
(2) *x is a node and there are exactly two points x_1, x_2 of the normalization X^ν lying over x with $k = \kappa(x_1) = \kappa(x_2)$,*

(3) $\mathcal{O}^\wedge_{X,x} \cong k[[x,y]]/(xy)$ *as a* k*-algebra, and*
(4) *add more here.*

Proof. This follows from the discussion in Remark 53.19.9 and Lemma 53.19.7. □

0C56 **Lemma 53.19.12.** *Let* K/k *be an extension of fields. Let* X *be a locally algebraic* k*-scheme of dimension* 1*. Let* $y \in X_K$ *be a point with image* $x \in X$*. The following are equivalent*

(1) x *is a closed point of* X *and a node, and*
(2) y *is a closed point of* Y *and a node.*

Proof. If x is a closed point of X, then y is too (look at residue fields). But conversely, this need not be the case, i.e., it can happen that a closed point of Y maps to a nonclosed point of X. However, in this case y cannot be a node. Namely, then X would be geometrically unibranch at x (because x would be a generic point of X and $\mathcal{O}_{X,x}$ would be Artinian and any Artinian local ring is geometrically unibranch), hence Y is geometrically unibranch at y (Varieties, Lemma 33.40.3), which means that y cannot be a node by Lemma 53.19.7. Thus we may and do assume that both x and y are closed points.

Choose algebraic closures $\overline{k}$, $\overline{K}$ and a map $\overline{k} \to \overline{K}$ extending the given map $k \to K$. Using the equivalence of (1) and (3) in Lemma 53.19.7 we reduce to the case where k and K are algebraically closed. In this case we can argue as in the proof of Lemma 53.19.7 that the geometric number of branches and δ-invariants of X at x and Y at y are the same. Another argument can be given by choosing an isomorphism $k[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) \to \mathcal{O}^\wedge_{X,x}$ of k-algebras as in Varieties, Lemma 33.21.1. By Varieties, Lemma 33.21.2 this gives an isomorphism $K[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) \to \mathcal{O}^\wedge_{Y,y}$ of K-algebras. By definition we have to show that

$$k[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) \cong k[[s,t]]/(st)$$

if and only if

$$K[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) \cong K[[s,t]]/(st)$$

We encourage the reader to prove this for themselves. Since k and K are algebraically closed fields, this is the same as asking these rings to be as in Lemma 53.19.3. Via Lemma 53.19.6 this translates into a statement about the $(n-1)\times(n-1)$-minors of the matrix $(\partial g_j/\partial x_i)$ which is clearly independent of the field used. We omit the details. □

0C57 **Lemma 53.19.13.** *Let* k *be a field. Let* X *be a locally algebraic* k*-scheme of dimension* 1*. Let* $Y \to X$ *be an étale morphism. Let* $y \in Y$ *be a point with image* $x \in X$*. The following are equivalent*

(1) x *is a closed point of* X *and a node, and*
(2) y *is a closed point of* Y *and a node.*

Proof. By Lemma 53.19.12 we may base change to the algebraic closure of k. Then the residue fields of x and y are k. Hence the map $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$ is an isomorphism (for example by Étale Morphisms, Lemma 41.11.3 or More on Algebra, Lemma 15.43.9). Thus the lemma is clear. □

0CD6 **Lemma 53.19.14.** *Let* k'/k *be a finite separable field extension. Let* X *be a locally algebraic* k'*-scheme of dimension* 1*. Let* $x \in X$ *be a closed point. The following are equivalent*

(1) *x is a node, and*
(2) *x is a node when X viewed as a locally algebraic k-scheme.*

Proof. Follows immediately from the characterization of nodes in Lemma 53.19.7. □

0C4E **Lemma 53.19.15.** *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1. The following are equivalent*

(1) *the singularities of X are at-worst-nodal, and*
(2) *X is a local complete intersection over k and the closed subscheme $Z \subset X$ cut out by the first fitting ideal of $\Omega_{X/k}$ is unramified over k.*

Proof. We urge the reader to find their own proof of this lemma; what follows is just putting together earlier results and may hide what is really going on.

Assume (2). Since $Z \to \mathrm{Spec}(k)$ is quasi-finite (Morphisms, Lemma 29.35.10) we see that the residue fields of points $x \in Z$ are finite over k (as well as separable) by Morphisms, Lemma 29.20.5. Hence each $x \in Z$ is a closed point of X by Morphisms, Lemma 29.20.2. The local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay by Algebra, Lemma 10.135.3. Since $\dim(\mathcal{O}_{X,x}) = 1$ by dimension theory (Varieties, Section 33.20), we conclude that $\mathrm{depth}(\mathcal{O}_{X,x})) = 1$. Thus x is a node by Lemma 53.19.7. If $x \in X$, $x \notin Z$, then $X \to \mathrm{Spec}(k)$ is smooth at x by Divisors, Lemma 31.10.3.

Assume (1). Under this assumption X is geometrically reduced at every closed point (see Varieties, Lemma 33.6.6). Hence $X \to \mathrm{Spec}(k)$ is smooth on a dense open by Varieties, Lemma 33.25.7. Thus Z is closed and consists of closed points. By Divisors, Lemma 31.10.3 the morphism $X \setminus Z \to \mathrm{Spec}(k)$ is smooth. Hence $X \setminus Z$ is a local complete intersection by Morphisms, Lemma 29.34.7 and the definition of a local complete intersection in Morphisms, Definition 29.30.1. By Lemma 53.19.7 for every point $x \in Z$ the local ring $\mathcal{O}_{Z,x}$ is equal to $\kappa(x)$ and $\kappa(x)$ is separable over k. Thus $Z \to \mathrm{Spec}(k)$ is unramified (Morphisms, Lemma 29.35.11). Finally, Lemma 53.19.7 via part (3) of Lemma 53.19.3, shows that $\mathcal{O}_{X,x}$ is a complete intersection in the sense of Divided Power Algebra, Definition 23.8.5. However, Divided Power Algebra, Lemma 23.8.8 and Morphisms, Lemma 29.30.9 show that this agrees with the notion used to define a local complete intersection scheme over a field and the proof is complete. □

0E37 **Lemma 53.19.16.** *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. Then X is Gorenstein and geometrically reduced.*

Proof. The Gorenstein assertion follows from Lemma 53.19.15 and Duality for Schemes, Lemma 48.24.5. Or you can use that it suffices to check after passing to the algebraic closure (Duality for Schemes, Lemma 48.25.1), then use that a Noetherian local ring is Gorenstein if and only if its completion is so (by Dualizing Complexes, Lemma 47.21.8), and then prove that the local rings $k[[t]]$ and $k[[x,y]]/(xy)$ are Gorenstein by hand.

To see that X is geometrically reduced, it suffices to show that $X_{\overline{k}}$ is reduced (Varieties, Lemmas 33.6.3 and 33.6.4). But $X_{\overline{k}}$ is a nodal curve over an algebraically closed field. Thus the complete local rings of $X_{\overline{k}}$ are isomorphic to either $\overline{k}[[t]]$ or $\overline{k}[[x,y]]/(xy)$ which are reduced as desired. □

0E38 **Lemma 53.19.17.** *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. If $Y \subset X$ is a reduced closed subscheme equidimensional of dimension 1, then*

(1) *the singularities of Y are at-worst-nodal, and*
(2) *if $Z \subset X$ is the scheme theoretic closure of $X \setminus Y$, then*
   (a) *the scheme theoretic intersection $Y \cap Z$ is the disjoint union of spectra of finite separable extensions of k,*
   (b) *each point of $Y \cap Z$ is a node of X, and*
   (c) *$Y \to \mathrm{Spec}(k)$ is smooth at every point of $Y \cap Z$.*

Proof. Since X and Y are reduced and equidimensional of dimension 1, we see that Y is the scheme theoretic union of a subset of the irreducible components of X (in a reduced ring (0) is the intersection of the minimal primes). Let $y \in Y$ be a closed point. If y is in the smooth locus of $X \to \mathrm{Spec}(k)$, then y is on a unique irreducible component of X and we see that Y and X agree in an open neighbourhood of y. Hence $Y \to \mathrm{Spec}(k)$ is smooth at y. If y is a node of X but still lies on a unique irreducible component of X, then y is a node on Y by the same argument. Suppose that y lies on more than 1 irreducible component of X. Since the number of geometric branches of X at y is 2 by Lemma 53.19.7, there can be at most 2 irreducible components passing through y by Properties, Lemma 28.15.5. If Y contains both of these, then again $Y = X$ in an open neighbourhood of y and y is a node of Y. Finally, assume Y contains only one of the irreducible components. After replacing X by an open neighbourhood of x we may assume Y is one of the two irreducble components and Z is the other. By Properties, Lemma 28.15.5 again we see that X has two branches at y, i.e., the local ring $\mathcal{O}_{X,y}$ has two branches and that these branches come from $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,y}$. Write $\mathcal{O}^\wedge_{X,y} \cong \kappa(y)[[u,v]]/(uv)$ as in Remark 53.19.9. The field $\kappa(y)$ is finite separable over k by Lemma 53.19.7 for example. Thus, after possibly switching the roles of u and v, the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u,v]]/(uv) \to \kappa(y)[[u]]$ and the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u,v]]/(uv) \to \kappa(y)[[v]]$. The scheme theoretic intersection of $Y \cap Z$ is cut out by the sum of their ideas which in the completion is (u, v), i.e., the maximal ideal. Thus (2)(a) and (2)(b) are clear. Finally, (2)(c) holds: the completion of $\mathcal{O}_{Y,y}$ is regular, hence $\mathcal{O}_{Y,y}$ is regular (More on Algebra, Lemma 15.43.4) and $\kappa(y)/k$ is separable, hence smoothness in an open neighbourhood by Algebra, Lemma 10.140.5. □

53.20. Families of nodal curves

0C58 In the Stacks project curves are irreducible varieties of dimension 1, but in the literature a "semi-stable curve" or a "nodal curve" is usually not irreducible and often assumed to be proper, especially when used in a phrase such as "family of semistable curves" or "family of nodal curves", or "nodal family". Thus it is a bit difficult for us to choose a terminology which is consistent with the literature as well as internally consistent. Moreover, we really want to first study the notion introduced in the following lemma (which is local on the source).

0C59 **Lemma 53.20.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat, locally of finite presentation, every nonempty fibre X_s is equidimensional of dimension 1, and X_s has at-worst-nodal singularities, and*

(2) *f is syntomic of relative dimension 1 and the closed subscheme $\mathrm{Sing}(f) \subset X$ defined by the first Fitting ideal of $\Omega_{X/S}$ is unramified over S.*

Proof. Recall that the formation of $\mathrm{Sing}(f)$ commutes with base change, see Divisors, Lemma 31.10.1. Thus the lemma follows from Lemma 53.19.15, Morphisms, Lemma 29.30.11, and Morphisms, Lemma 29.35.12. (We also use the trivial Morphisms, Lemmas 29.30.6 and 29.30.7.) □

0C5A **Definition 53.20.2.** Let $f : X \to S$ be a morphism of schemes. We say f is *at-worst-nodal of relative dimension* 1 if f satisfies the equivalent conditions of Lemma 53.20.1.

Here are some reasons for the cumbersome terminology[6]. First, we want to make sure this notion is not confused with any of the other notions in the literature (see introduction to this section). Second, we can imagine several generalizations of this notion to morphisms of higher relative dimension (for example, one can ask for morphisms which are étale locally compositions of at-worst-nodal morphisms or one can ask for morphisms whose fibres are higher dimensional but have at worst ordinary double points).

0CD7 **Lemma 53.20.3.** *A smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1.*

Proof. Omitted. □

0C5B **Lemma 53.20.4.** *Let $f : X \to S$ be at-worst-nodal of relative dimension 1. Then the same is true for any base change of f.*

Proof. This is true because the base change of a syntomic morphism is syntomic (Morphisms, Lemma 29.30.4), the base change of a morphism of relative dimension 1 has relative dimension 1 (Morphisms, Lemma 29.29.2), the formation of $\mathrm{Sing}(f)$ commutes with base change (Divisors, Lemma 31.10.1), and the base change of an unramified morphism is unramified (Morphisms, Lemma 29.35.5). □

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

0DSC **Lemma 53.20.5.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then there is a maximal open subscheme $U \subset X$ such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of U commutes with arbitrary base change.*

Proof. By Morphisms, Lemma 29.30.12 we find that there is such an open where f is syntomic. Hence we may assume that f is a syntomic morphism. In particular f is a Cohen-Macaulay morphism (Duality for Schemes, Lemmas 48.25.5 and 48.25.4). Thus X is a disjoint union of open and closed subschemes on which f has given relative dimension, see Morphisms, Lemma 29.29.4. This decomposition is preserved by arbitrary base change, see Morphisms, Lemma 29.29.2. Discarding all but one piece we may assume f is syntomic of relative dimension 1. Let $\mathrm{Sing}(f) \subset X$ be the closed subscheem defined by the first fitting ideal of $\Omega_{X/S}$. There is a maximal open subscheme $W \subset \mathrm{Sing}(f)$ such that $W \to S$ is unramified and its formation

[6]But please email the maintainer of the Stacks project if you have a better suggestion.

commutes with base change (Morphisms, Lemma 29.35.15). Since also formation of $\mathrm{Sing}(f)$ commutes with base change (Divisors, Lemma 31.10.1), we see that

$$U = (X \setminus \mathrm{Sing}(f)) \cup W$$

is the maximal open subscheme of X such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1 and that formation of U commutes with base change. □

0C5C **Lemma 53.20.6.** *Let $f : X \to S$ be at-worst-nodal of relative dimension 1. If $Y \to X$ is an étale morphism, then the composition $g : Y \to S$ is at-worst-nodal of relative dimension 1.*

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 29.36.11, 29.36.12, 29.21.3, and 29.25.6). Thus it suffices to prove the fibres have at-worst-nodal singularities. This follows from Lemma 53.19.13 (and the fact that the composition of an étale morphism and a smooth morphism is smooth by Morphisms, Lemmas 29.36.5 and 29.34.4). □

0CD8 **Lemma 53.20.7.** *Let $S' \to S$ be an étale morphism of schemes. Let $f : X \to S'$ be at-worst-nodal of relative dimension 1. Then the composition $g : X \to S$ is at-worst-nodal of relative dimension 1.*

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 29.36.11, 29.36.12, 29.21.3, and 29.25.6). Thus it suffices to prove the fibres of g have at-worst-nodal singularities. This follows from Lemma 53.19.14 and the analogous result for smooth points. □

0C5D **Lemma 53.20.8.** *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to X\}$ be an étale covering. The following are equivalent*

(1) *f is at-worst-nodal of relative dimension 1,*
(2) *each $U_i \to S$ is at-worst-nodal of relative dimension 1.*

In other words, being at-worst-nodal of relative dimension 1 is étale local on the source.

Proof. One direction we have seen in Lemma 53.20.6. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is étale local on the source (Descent, Lemmas 35.28.1, 35.27.1, and 35.33.8). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 53.19.13 (and the fact that being smooth is étale local on the source by Descent, Lemma 35.30.1). □

0C5E **Lemma 53.20.9.** *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to S\}$ be an fpqc covering. The following are equivalent*

(1) *f is at-worst-nodal of relative dimension 1,*
(2) *each $X \times_S U_i \to U_i$ is at-worst-nodal of relative dimension 1.*

In other words, being at-worst-nodal of relative dimension 1 is fpqc local on the target.

Proof. One direction we have seen in Lemma 53.20.4. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is fpqc local on the target (Descent, Lemmas 35.23.13, 35.23.17, and Morphisms,

Lemma 29.28.3). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 53.19.12 (and the fact that being smooth is fpqc local on the target by Descent, Lemma 35.23.29). □

0C5F **Lemma 53.20.10.** *Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If*

(1) *f is at-worst-nodal of relative dimension 1, and*
(2) *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is at-worst-nodal of relative dimension 1.

Proof. By Limits, Lemma 32.8.16 there exists an i such that f_i is syntomic. Then $X_i = \coprod_{d \geq 0} X_{i,d}$ is a disjoint union of open and closed subschemes such that $X_{i,d} \to Y_i$ has relative dimension d, see Morphisms, Lemma 29.30.14. Because of the behaviour of dimensions of fibres under base change given in Morphisms, Lemma 29.28.3 we see that $X \to X_i$ maps into $X_{i,1}$. Then there exists an $i' \geq i$ such that $X_{i'} \to X_i$ maps into $X_{i,1}$, see Limits, Lemma 32.4.10. Thus $f_{i'} : X_{i'} \to Y_{i'}$ is syntomic of relative dimension 1 (by Morphisms, Lemma 29.28.3 again). Consider the morphism $\mathrm{Sing}(f_{i'}) \to Y_{i'}$. We know that the base change to Y is an unramified morphism. Hence by Limits, Lemma 32.8.4 we see that after increasing i' the morphism $\mathrm{Sing}(f_{i'}) \to Y_{i'}$ becomes unramified. This finishes the proof. □

0CBX **Lemma 53.20.11.** *Let $f : T \to S$ be a morphism of schemes. Let $t \in T$ with image $s \in S$. Assume*

(1) *f is flat at t,*
(2) *$\mathcal{O}_{S,s}$ is Noetherian,*
(3) *f is locally of finite type,*
(4) *t is a split node of the fibre T_s.*

Then there exists an $h \in \mathfrak{m}_s^\wedge$ and an isomorphism

$$\mathcal{O}_{T,t}^\wedge \cong \mathcal{O}_{S,s}^\wedge[[x, y]]/(xy - h)$$

of $\mathcal{O}_{S,s}^\wedge$-algebras.

Proof. We replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and T by the base change to $\mathrm{Spec}(\mathcal{O}_{S,s})$. Then T is locally Noetherian and hence $\mathcal{O}_{T,t}$ is Noetherian. Set $A = \mathcal{O}_{S,s}^\wedge$, $\mathfrak{m} = \mathfrak{m}_A$, and $B = \mathcal{O}_{T,t}^\wedge$. By More on Algebra, Lemma 15.43.8 we see that $A \to B$ is flat. Since $\mathcal{O}_{T,t}/\mathfrak{m}_s\mathcal{O}_{T,t} = \mathcal{O}_{T_s,t}$ we see that $B/\mathfrak{m}B = \mathcal{O}_{T_s,t}^\wedge$. By assumption (4) and Lemma 53.19.11 we conclude there exist $\overline{u}, \overline{v} \in B/\mathfrak{m}B$ such that the map

$$(A/\mathfrak{m})[[x, y]] \longrightarrow B/\mathfrak{m}B, \quad x \longmapsto \overline{u}, x \longmapsto \overline{v}$$

is surjective with kernel (xy).

Assume we have $n \geq 1$ and $u, v \in B$ mapping to $\overline{u}, \overline{v}$ such that

$$uv = h + \delta$$

for some $h \in A$ and $\delta \in \mathfrak{m}^n B$. We claim that there exist $u', v' \in B$ with $u - u', v - v' \in \mathfrak{m}^n B$ such that

$$u'v' = h' + \delta'$$

for some $h' \in A$ and $\delta' \in \mathfrak{m}^{n+1}B$. To see this, write $\delta = \sum f_i b_i$ with $f_i \in \mathfrak{m}^n$ and $b_i \in B$. Then write $b_i = a_i + ub_{i,1} + vb_{i,2} + \delta_i$ with $a_i \in A$, $b_{i,1}, b_{i,2} \in B$ and $\delta_i \in \mathfrak{m}B$. This is possible because the residue field of B agrees with the residue field of A and the images of u and v in $B/\mathfrak{m}B$ generate the maximal ideal. Then we set

$$u' = u - \sum b_{i,2} f_i, \quad v' = v - \sum b_{i,1} f_i$$

and we obtain

$$u'v' = h + \delta - \sum (b_{i,1}u + b_{i,2}v) f_i + \sum c_{ij} f_i f_j = h + \sum a_i f_i + \sum f_i \delta_i + \sum c_{ij} f_i f_j$$

for some $c_{i,j} \in B$. Thus we get a formula as above with $h' = h + \sum a_i f_i$ and $\delta' = \sum f_i \delta_i + \sum c_{ij} f_i f_j$.

Arguing by induction and starting with any lifts $u_1, v_1 \in B$ of $\overline{u}, \overline{v}$ the result of the previous paragraph shows that we find a sequence of elements $u_n, v_n \in B$ and $h_n \in A$ such that $u_n - u_{n+1} \in \mathfrak{m}^n B$, $v_n - v_{n+1} \in \mathfrak{m}^n B$, $h_n - h_{n+1} \in \mathfrak{m}^n$, and such that $u_n v_n - h_n \in \mathfrak{m}^n B$. Since A and B are complete we can set $u_\infty = \lim u_n$, $v_\infty = \lim v_n$, and $h_\infty = \lim h_n$, and then we obtain $u_\infty v_\infty = h_\infty$ in B. Thus we have an A-algebra map

$$A[[x, y]]/(xy - h_\infty) \longrightarrow B$$

sending x to u_∞ and v to v_∞. This is a map of flat A-algebras which is an isomorphism after dividing by $\mathfrak{m}$. It is surjective modulo $\mathfrak{m}$ and hence surjective by completeness and Algebra, Lemma 10.96.1. Then we can apply Algebra, Lemma 10.99.1 to conclude it is an isomorphism. □

0GKA **Lemma 53.20.12.** *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper,*
(2) *f is at-worst-nodal of relative dimension 1, and*
(3) *the geometric fibres of f are connected.*

Then (a) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change, (b) $R^1 f_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with any base change, and (c) $R^q f_* \mathcal{O}_X = 0$ for $q \geq 2$.

Proof. Part (a) follows from Derived Categories of Schemes, Lemma 36.32.6. By Derived Categories of Schemes, Lemma 36.32.5 locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Schemes, Lemma 36.30.4). Thus for $s \in S$ we have

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 15.76.7. Since everything is compatible with base change we conclude. □

53.21. Étale local structure of nodal families

0H8M Consider the morphism of schemes

$$\mathrm{Spec}(\mathbf{Z}[u, v, a]/(uv - a)) \longrightarrow \mathrm{Spec}(\mathbf{Z}[a])$$

The next lemma shows that this morphism is a model for the étale local structure of a nodal family of curves.

0CBY **Lemma 53.21.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is at-worst-nodal of relative dimension 1. Let $x \in X$ be a point which is a singular point of the fibre X_s. Then there exists a commutative diagram of schemes*

$$\begin{array}{ccccccc} X & \leftarrow & U & \longrightarrow & W & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ S & \leftarrow & & & V & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[a]) \end{array}$$

with $X \leftarrow U$, $S \leftarrow V$, and $U \to W$ étale morphisms, and with the right hand square cartesian, such that there exists a point $u \in U$ mapping to x in X.

First proof using Artin approximation. We first use absolute Noetherian approximation to reduce to the case of schemes of finite type over $\mathbf{Z}$. The question is local on X and S. Hence we may assume that X and S are affine. Then we can write $S = \mathrm{Spec}(R)$ and write R as a filtered colimit $R = \mathrm{colim}\, R_i$ of finite type $\mathbf{Z}$-algebras. Using Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(R_i)$ whose base change to S is f. After increasing i we may assume that f_i is at-worst-nodal of relative dimension 1, see Lemma 53.20.10. The image $x_i \in X_i$ of x will be a singular point of its fibre, for example because the formation of $\mathrm{Sing}(f)$ commutes with base change (Divisors, Lemma 31.10.1). If we can prove the lemma for $f_i : X_i \to S_i$ and x_i, then the lemma follows for $f : X \to S$ by base change. Thus we reduce to the case studied in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$. Let $s \in S$ be the image of x. Recall that $\kappa(x)$ is a finite separable extension of $\kappa(s)$, for example because $\mathrm{Sing}(f) \to S$ is unramified or because x is a node of the fibre X_s and we can apply Lemma 53.19.7. Furthermore, let $\kappa'/\kappa(x)$ be the degree 2 separable algebra associated to $\mathcal{O}_{X_s,x}$ in Remark 53.19.8. By More on Morphisms, Lemma 37.35.2 we can choose an étale neighbourhood $(V, v) \to (S, s)$ such that the extension $\kappa(v)/\kappa(s)$ realizes either the extension $\kappa(x)/\kappa(s)$ in case $\kappa' \cong \kappa(x) \times \kappa(x)$ or the extension $\kappa'/\kappa(s)$ if κ' is a field. After replacing X by $X \times_S V$ and S by V we reduce to the situation described in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$ and $x \in X_s$ is a split node, see Definition 53.19.10. By Lemma 53.20.11 we see that there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{S,s}[[s,t]]/(st-h)$$

for some $h \in \mathfrak{m}^\wedge_s \subset \mathcal{O}^\wedge_{S,s}$. In other words, if we consider the homomorphism

$$\sigma : \mathbf{Z}[a] \longrightarrow \mathcal{O}^\wedge_{S,s}$$

sending a to h, then there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma, y_\sigma}$$

where

$$Y_\sigma = \mathrm{Spec}(\mathbf{Z}[u,v,t]/(uv-a)) \times_{\mathrm{Spec}(\mathbf{Z}[a]),\sigma} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$$

and y_σ is the point of Y_σ lying over the closed point of $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ and having coordinates u, v equal to zero. Since $\mathcal{O}_{S,s}$ is a G-ring by More on Algebra, Proposition 15.50.12 we may apply More on Morphisms, Lemma 37.39.3 to conclude. □

Proof not using Artin approximation. We only sketch this proof; it was contributed by Mohan Swaminathan and it mimicks an argument of [**ACG11**, Proposition X.2.1]. Contrary to the previous proof, this proof uses the actual equations.

In exactly the same manner as in the first proof we reduce to the case where S is of finite type over $\mathbf{Z}$ and $x \in X_s$ is a split node, see Definition 53.19.10. By Lemma 53.19.11 we see that there exists an $\kappa(s)$-algebra isomorphism

$$\mathcal{O}^\wedge_{X_s,x} \cong \kappa(s)[[u,v]]/(uv)$$

Note that the tangent space of X_s at x has dimension 2. Thus by Varieties, Lemma 33.18.5 after Zariski shrinking we may assume there exists a closed immersion $X \to Y$ of schemes over S with $Y \to S$ smooth of relative dimension 2. Since X is syntomic of relative dimension 1 (by definition), we see that X is an effective Cartier divisor in Y (follows from Divisors, Lemma 31.22.10). Thus after shrinking Y and X we may assume $Y = \mathrm{Spec}(B)$ is affine and that there exists a nonzerodivisor $g \in B$ such that $X = \mathrm{Spec}(B/gB)$.

Write $S = \mathrm{Spec}(R)$ and denote $\mathfrak{p} \subset R$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x (viewed as a point of Y). We have maps

$$B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} \longrightarrow B_\mathfrak{q}/gB_\mathfrak{q} + \mathfrak{p}B_\mathfrak{q} \longrightarrow \kappa(s)[[u,v]]/(uv)$$

where the second arrow becomes an isomorphism after completion. After replacing B by a suitable princpal localization, we may assume that there exist $U, V \in B$ which map to u and v up to $(u,v)^2$. Then the completion of $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$ is $\kappa(s)[[U,V]]$. Furthermore, after shrinking Y and after multiplying g by a unit we may assume that g maps to UV plus higher order terms in $\kappa(s)[[U,V]]$. After shrinking Y we may further assume

(1) $\mathfrak{q} = \mathfrak{p}B + (U,V)$
(2) $\Omega_{B/R}$ is free on $\mathrm{d}U$ and $\mathrm{d}V$

Write $\mathrm{d}(g) = g_2\mathrm{d}U + g_1\mathrm{d}V$ with $g_1, g_2 \in B$. Note that the ideal $J = (g_1, g_2) \subset B$ is independent of the choice of the "coordinates" U and V (for fixed g). Looking modulo $\mathfrak{p}B_\mathfrak{q} + (U,V)^3B_\mathfrak{q}$ we see that g_1 and g_2 are congruent to U and V modulo $\mathfrak{p}B_\mathfrak{q} + (U,V)^2B_\mathfrak{q}$. Thus we may redo the argument above with U replaced by g_1 and V replaced by g_2. Then we find that $J = (U,V)$ which implies that $g_1, g_2 \in (U,V) = J$ (of course these are different from the g_1 and g_2 with our previous choice of coordinates).

Note that $\mathrm{Spec}(B/J) \to S = \mathrm{Spec}(R)$ is étale at x. Thus after replacing S by an étale neighbourhood and shrinking Y again, we may assume that $R \to B \to B/J$ is an isomorphism. Denote $r \in R$ the element whose image in B/J is the image of $-g$. Then we can write

$$g + r \equiv aU + bV \bmod (U,V)^2$$

for some $a, b \in R$. Taking derivatives using that $\mathrm{d}g \in J\Omega_{B/R}$ we find $a = b = 0$. Thus we can write

$$g + r = \alpha U^2 + \beta UV + \gamma V^2$$

for some $\alpha, \beta, \gamma \in B$. Then β maps to 1 in $\kappa(x)$ and α and γ map to 0 in $\kappa(x)$. Thus we may assume β and $\alpha + \beta + \gamma$ are invertible in B. Consider the cover Y' of Y given by

$$Y' = \mathrm{Spec}(B[t]/(t(1-t) - \alpha\gamma/\beta^2))$$

with point x' the unique point over x with $t = 1$. Note that $Y' \to Y$ is étale at x'. Consider the elements

$$T_1 = (\alpha + t\beta)U + ((1-t)\beta + \gamma)V \quad \text{and} \quad T_2 = \frac{(\alpha + (1-t)\beta)U + (t\beta + \gamma)V}{\alpha + \beta + \gamma}$$

of $\Gamma(Y', \mathcal{O}_{Y'})$. Then we see that

$$g = T_1 T_2 - r$$

Since T_1 and T_2 are also coordinates in the sense above, we see that the morphism $Y' \to \mathbf{A}^2_S$ given by T_1 and T_2 is étale in a neighbourhood of x' and that the pullback $X' \subset Y'$ of X is the inverse image of the closed subscheme of $\mathbf{A}^2_S = \operatorname{Spec}(R[x_1, x_2])$ given by the equation $x_1 x_2 - r = 0$ (again after possibly shrinking Y'). We leave it to the reader to put everything together to obtain the desired diagram. □

53.22. More vanishing results

0E39 Continuation of Section 53.6.

0E3A **Lemma 53.22.1.** *In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then*

$$\deg(\mathcal{L}) \leq 2g - 2 + \deg(Z)$$

with strict inequality unless $\mathcal{I}\mathcal{L} \cong \omega_X$.

Proof. Any curve, e.g. X, is Cohen-Macaulay. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 53.4.2. Since $\mathcal{I}\mathcal{L}$ is torsion free, this map is injective. Since a field is Gorenstein and X is reduced, we find that the Gorenstein locus $U \subset X$ of X is nonempty, see Duality for Schemes, Lemma 48.24.4. This lemma also tells us that $\omega_X|_U$ is invertible. In this way we see we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \omega_X \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is zero dimensional. Hence we have

$$\begin{aligned} 0 &\leq \dim \Gamma(X, \mathcal{Q}) \\ &= \chi(\mathcal{Q}) \\ &= \chi(\omega_X) - \chi(\mathcal{I}\mathcal{L}) \\ &= \chi(\omega_X) - \deg(\mathcal{L}) - \chi(\mathcal{I}) \\ &= 2g - 2 - \deg(\mathcal{L}) + \deg(Z) \end{aligned}$$

by Lemmas 53.5.1 and 53.5.2, by (53.8.1.1), and by Varieties, Lemmas 33.33.3 and 33.44.5. We have also used that $\deg(Z) = \dim_k \Gamma(Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z)$ and the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. The lemma follows. □

0E3B **Lemma 53.22.2.** *In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $\deg(\mathcal{L}) > 2g - 2 + \deg(Z)$, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ and one of the following possibilities occurs* [**Lee05**, Lemma 2]

(1) *$H^0(X, \mathcal{I}\mathcal{L}) \neq 0$, or*
(2) *$g = 0$ and $\deg(\mathcal{L}) = \deg(Z) - 1$.*

In case (2) if $Z = \emptyset$, then $X \cong \mathbf{P}^1_k$ and $\mathcal{L}$ corresponds to $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Proof. The vanishing of $H^1(X, \mathcal{I}\mathcal{L})$ follows from Lemma 53.22.1. If $H^0(X, \mathcal{I}\mathcal{L}) = 0$, then $\chi(\mathcal{I}\mathcal{L}) = 0$. From the short exact sequence $0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$ we conclude $\deg(\mathcal{L}) = g - 1 + \deg(Z)$. Thus $g - 1 + \deg(Z) > 2g - 2 + \deg(Z)$ which implies $g = 0$ hence (2) holds. If $Z = \emptyset$ in case (2), then $\mathcal{L}^{-1}$ is an invertible sheaf of degree 1. This implies there is an isomorphism $X \to \mathbf{P}^1_k$ and $\mathcal{L}^{-1}$ is the pullback of $\mathcal{O}_{\mathbf{P}^1}(1)$ by Lemma 53.10.2. □

0E3C **Lemma 53.22.3.** *In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}) \geq 2g$, then $\mathcal{L}$ is globally generated.* [**Lee05**, Lemma 3]

Proof. Let $Z \subset X$ be the closed subscheme cut out by the global sections of $\mathcal{L}$. By Lemma 53.22.2 we see that $Z \neq X$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf cutting out Z. Consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$$

If $Z \neq \emptyset$, then $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero as follows from the long exact sequence of cohomology. By Lemma 53.4.2 this gives a nonzero and hence injective map

$$\mathcal{I}\mathcal{L} \longrightarrow \omega_X$$

In particular, we find an injective map $H^0(X, \mathcal{L}) = H^0(X, \mathcal{I}\mathcal{L}) \to H^0(X, \omega_X)$. This is impossible as

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}) + \deg(\mathcal{L}) + 1 - g \geq g + 1$$

and $\dim H^0(X, \omega_X) = g$ by (53.8.1.1). □

0E3D **Lemma 53.22.4.** *In Situation 53.6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a nonempty 0-dimensional closed subscheme. If $\deg(\mathcal{L}) \geq 2g - 1 + \deg(Z)$, then $\mathcal{L}$ is globally generated and $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ is surjective.*

Proof. Global generation by Lemma 53.22.3. If $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ by Lemma 53.22.1. Hence surjectivity. □

0H2V **Lemma 53.22.5.** *In Situation 53.6.2, assume X is geometrically integral over k and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}) \geq 2g + 1$, then $\mathcal{L}$ is very ample.*

Proof. By Lemma 53.22.3, $\mathcal{L}$ is globally generated, and so it determines a morphism $f : X \to \mathbf{P}^n_k$ where $n = h^0(X, \mathcal{L}) - 1$. To show that $\mathcal{L}$ is very ample means to show that f is a closed immersion. It suffices to check that the base change of f to an algebraic closure $\overline{k}$ of k is a closed immersion (Descent, Lemma 35.23.21). So we may assume that k is algebraically closed; X remains integral, by assumption. Lemma 53.22.4 gives that for every 0-dimensional closed subscheme $Z \subset X$ of degree 2, the restriction map $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ is surjective. By Varieties, Lemma 33.23.2, $\mathcal{L}$ is very ample. □

0E3E **Lemma 53.22.6.** *Let k be a field. Let X be a proper scheme over k which is reduced, connected, and of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L}) \neq 0$, then there exists a reduced connected closed subscheme $Y \subset X$ of dimension 1 such that* Weak version of [**Lee05**, Lemma 4]

$$\deg(\mathcal{L}|_Y) \leq -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

where $Z \cap Y$ is the scheme theoretic intersection.

Proof. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 53.4.2. Let $Y \subset X$ be the union of the irreducible components C of X such that φ is nonzero in the generic point of C. Then Y is a reduced closed subscheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic points of the support of $\mathcal{J}$ (by choice of Y and as X is reduced), we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5 we find a map

$$\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_Y$$

of $\mathcal{O}_Y$-modules which is injective in the generic points of Y. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of $Z \cap Y$. There is a map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module, the map above factors through an injective map $\mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. We see that we get an exact sequence

$$0 \to \mathcal{I}'\mathcal{L}|_Y \to \omega_Y \to \mathcal{Q} \to 0$$

of coherent sheaves on Y where $\mathcal{Q}$ is supported in dimension 0 (this uses that ω_Y is an invertible module in the generic points of Y). We conclude that

$$0 \leq \dim \Gamma(Y, \mathcal{Q}) = \chi(\mathcal{Q}) = \chi(\omega_Y) - \chi(\mathcal{I}'\mathcal{L}) = -2\chi(\mathcal{O}_Y) - \deg(\mathcal{L}|_Y) + \deg(Z \cap Y)$$

by Lemma 53.5.1 and Varieties, Lemma 33.33.3. If Y is connected, then this proves the lemma. If not, then we repeat the last part of the argument for one of the connected components of Y. □

0E3F **Lemma 53.22.7.** *Let k be a field. Let X be a proper scheme over k which is reduced, connected, and of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every reduced connected closed subscheme $Y \subset X$ of dimension 1 we have*

$$\deg(\mathcal{L}|_Y) \geq 2 \dim_k H^1(Y, \mathcal{O}_Y)$$

Then $\mathcal{L}$ is globally generated.

Proof. By induction on the number of irreducible components of X. If X is irreducible, then the lemma holds by Lemma 53.22.3 applied to X viewed as a scheme over the field $k' = H^0(X, \mathcal{O}_X)$. Assume X is not irreducible. Before we continue, if k is finite, then we replace k by a purely transcendental extension K. This is allowed by Varieties, Lemmas 33.22.1, 33.44.2, 33.6.7, and 33.8.4, Cohomology of Schemes, Lemma 30.5.2, Lemma 53.4.4 and the elementary fact that K is geometrically integral over k.

Assume that $\mathcal{L}$ is not globally generated to get a contradiction. Then we may choose a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ and such that $\mathcal{O}_X/\mathcal{I}$ is nonzero with support of dimension 0. For example, take $\mathcal{I}$ the ideal sheaf of any closed point in the common vanishing locus of the global sections of $\mathcal{L}$. We consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{L}/\mathcal{I}\mathcal{L} \to 0$$

Since the support of $\mathcal{L}/\mathcal{I}\mathcal{L}$ has dimension 0 we see that $\mathcal{L}/\mathcal{I}\mathcal{L}$ is generated by global sections (Varieties, Lemma 33.33.3). From the short exact sequence, and the fact that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ we get an injection $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$.

Recall that the k-vector space $H^1(X, \mathcal{I}\mathcal{L})$ is dual to $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$. Choose $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$. By Lemma 53.22.6 we have $H^1(X, \mathcal{L}) = 0$. Hence

$$\dim_k H^0(X, \mathcal{I}\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

We conclude that φ is not injective on global sections, in particular φ is not injective. For every generic point $\eta \in X$ of an irreducible component of X denote $V_\eta \subset \mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$ the k-subvector space consisting of those φ which are zero at η. Since every associated point of $\mathcal{I}\mathcal{L}$ is a generic point of X, the above shows that $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = \bigcup V_\eta$. As X has finitely many generic points and k is infinite, we conclude $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = V_\eta$ for some η. Let $\eta \in C \subset X$ be the corresponding irreducible component. Let $Y \subset X$ be the union of the other irreducible components of X. Then Y is a nonempty reduced closed subscheme not equal to X. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Please keep in mind that the support of $\mathcal{J}$ is C.

Let $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$ be arbitrary. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic point η of the support of $\mathcal{J}$, we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 53.4.5 we find a factorization

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \xrightarrow{\varphi'} \omega_Y \to \omega_X$$

of φ. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the image of $\mathcal{I} \subset \mathcal{O}_X$. There is a surjective map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module on Y, the map φ' factors through a map $\varphi'' : \mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. Thus we have commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}\mathcal{L} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}/\mathcal{I}\mathcal{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}'\mathcal{L}|_Y & \longrightarrow & \mathcal{L}|_Y & \longrightarrow & \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}\mathcal{L} & \xrightarrow{\varphi} & \omega_X \\ \downarrow & & \uparrow \\ \mathcal{I}'\mathcal{L}|_Y & \xrightarrow{\varphi''} & \omega_Y \end{array}$$

Now we can finish the proof as follows: Since for every φ we have a φ'' and since $\omega_X \in Coh(\mathcal{O}_X)$ represents the functor $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$, we find that $H^1(X, \mathcal{I}\mathcal{L}) \to H^1(Y, \mathcal{I}'\mathcal{L}|_Y)$ is injective. Since the boundary $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$ is injective, we conclude the composition

$$H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

is injective. Since $\mathcal{L}/\mathcal{I}\mathcal{L} \to \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y$ is a surjective map of coherent modules whose supports have dimension 0, we see that the first map $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y)$ is surjective (and hence bijective). But by induction we have that $\mathcal{L}|_Y$ is globally generated (if Y is disconnected this still works of course) and hence the boundary map

$$H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

cannot be injective. This contradiction finishes the proof. □

53.23. Contracting rational tails

0E3G In this section we discuss the simplest possible case of contracting a scheme to improve positivity properties of its canonical sheaf.

0E3H **Example 53.23.1** (Contracting a rational tail)**.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A *rational tail* will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

(1) $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,
(2) the scheme theoretic intersection $C \cap X'$ is a single reduced point x,
(3) $H^0(C, \mathcal{O}_C)$ maps isomorphically to the residue field of x, and
(4) C has genus zero.

Since there are at least two irreducible components of X passing through x, we conclude that x is a node. Set $k' = H^0(C, \mathcal{O}_C) = \kappa(x)$. Then k'/k is a finite separable extension of fields (Lemma 53.19.7). There is a canonical morphism

$$c : X \longrightarrow X'$$

inducing the identity on X' and mapping C to $x \in X'$ via the canonical morphism $C \to \mathrm{Spec}(k') = x$. This follows from Morphisms, Lemma 29.4.6 since X is the scheme theoretic union of C and X' (as X is reduced). Moreover, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_{X'} \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this, denote $i_C : C \to X$, $i_{X'} : X' \to X$ and $i_x : x \to X$ the embeddings and use the exact sequence

$$0 \to \mathcal{O}_X \to i_{C,*}\mathcal{O}_C \oplus i_{X',*}\mathcal{O}_{X'} \to i_{x,*}\kappa(x) \to 0$$

of Morphisms, Lemma 29.4.6. Looking at the long exact sequence of higher direct images, it follows that it suffices to show $H^0(C, \mathcal{O}_C) = k'$ and $H^1(C, \mathcal{O}_C) = 0$ which follows from the assumptions. Observe that X' is also a proper scheme over k, of dimension 1 whose singularities are at-worst-nodal (Lemma 53.19.17) has $H^0(X', \mathcal{O}_{X'}) = k$, and X' has the same genus as X. We will say $c : X \to X'$ is the *contraction of a rational tail*.

0E63 **Lemma 53.23.2.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 53.23.1). Then* $\deg(\omega_X|_C) < 0$.

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemmas 53.4.6, 53.19.16, and 53.19.17. With k' as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as $C \cong \mathbf{P}^1_{k'}$ by Proposition 53.10.4 and $\deg(C \cap X') = [k' : k]$. Hence $\deg(\omega_X|_C) = -[k' : k]$ which is negative. □

0E3I **Lemma 53.23.3.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 53.23.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational tails.*

Proof. Let $x \in C$ and $k' = \kappa(x)$ be as in the example. Observe that $C \cong \mathbf{P}^1_{k'}$ by Proposition 53.10.4. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. Set $C_i = \mathbf{P}^1_{K'_i}$ and denote $x_i \in C_i$ the inverse image of x. Then $C_K = \coprod C_i$ and $X'_K \cap C_i = x_i$ as desired. □

0E3J **Lemma 53.23.4.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. If X does not have a rational tail (Example 53.23.1), then for every reduced connected closed subscheme $Y \subset X$, $Y \neq X$ of dimension 1 we have $\deg(\omega_X|_Y) \geq \dim_k H^1(Y, \mathcal{O}_Y)$.*

Proof. Let $Y \subset X$ be as in the statement. Then $k' = H^0(Y, \mathcal{O}_Y)$ is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to Y and coherent sheaves on Y, see Varieties, Lemma 33.44.10. Let $Z \subset X$ be as in Lemma 53.4.6. We will use the results of this lemma and of Lemmas 53.19.16 and 53.19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_Y \to \omega_X|_Y \to \mathcal{O}_{Y \cap Z} \to 0$$

See Lemma 53.4.6. We conclude that

$$\deg(\omega_X|_Y) = \deg(Y \cap Z) + \deg(\omega_Y) = \deg(Y \cap Z) - 2\chi(Y, \mathcal{O}_Y)$$

Hence, if the lemma is false, then

$$2[k' : k] > \deg(Y \cap Z) + \dim_k H^1(Y, \mathcal{O}_Y)$$

Since $Y \cap Z$ is nonempty and by the divisiblity mentioned above, this can happen only if $Y \cap Z$ is a single k'-rational point of the smooth locus of Y and $H^1(Y, \mathcal{O}_Y) = 0$. If Y is irreducible, then this implies Y is a rational tail. If Y is reducible, then since $\deg(\omega_X|_Y) = -[k' : k]$ we find there is some irreducible component C of Y such that $\deg(\omega_X|_C) < 0$, see Varieties, Lemma 33.44.6. Then the analysis above applied to C gives that C is a rational tail. □

0E3K **Lemma 53.23.5.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Assume X does not have a rational tail (Example 53.23.1). If*

1. *the genus of X is 0, then X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample,*
2. *the genus of X is 1, then $\omega_X \cong \mathcal{O}_X$,*
3. *the genus of X is ≥ 2, then $\omega_X^{\otimes m}$ is globally generated for $m \geq 2$.*

Proof. By Lemma 53.19.16 we find that X is Gorenstein, i.e., ω_X is an invertible $\mathcal{O}_X$-module.

If the genus of X is zero, then $\deg(\omega_X) < 0$, hence if X has more than one irreducible component, we get a contradiction with Lemma 53.23.4. In the irreducible case we see that X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample by Lemma 53.10.3.

If the genus of X is 1, then ω_X has a global section and $\deg(\omega_X|_C) = 0$ for all irreducible components. Namely, $\deg(\omega_X|_C) \geq 0$ for all irreducible components C by Lemma 53.23.4, the sum of these numbers is 0 by Lemma 53.8.3, and we can apply Varieties, Lemma 33.44.6. Then $\omega_X \cong \mathcal{O}_X$ by Varieties, Lemma 33.44.13.

Assume the genus g of X is greater than or equal to 2. If X is irreducible, then we are done by Lemma 53.22.3. Assume X reducible. By Lemma 53.23.4 the inequalities of Lemma 53.22.7 hold for every $Y \subset X$ as in the statement, except for $Y = X$. Analyzing the proof of Lemma 53.22.7 we see that (in the reducible case) the only inequality used for $Y = X$ are

$$\deg(\omega_X^{\otimes m}) > -2\chi(\mathcal{O}_X) \quad \text{and} \quad \deg(\omega_X^{\otimes m}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X)$$

Since these both hold under the assumption $g \geq 2$ and $m \geq 2$ we win. □

0E3L **Lemma 53.23.6.** *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Consider a sequence*

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational tails (Example 53.23.1) until none are left. Then

(1) *if the genus of X is 0, then X' is an irreducible plane conic,*
(2) *if the genus of X is 1, then $\omega_{X'} \cong \mathcal{O}_{X'}$,*
(3) *if the genus of X is > 1, then $\omega_{X'}^{\otimes m}$ is globally generated for $m \geq 2$.*

If the genus of X is ≥ 1, then the morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational tails until there are none left. Then we see that (1), (2), (3) hold by Lemma 53.23.5.

Uniqueness. To see that $f : X \to X'$ is independent of the choices made, it suffices to show: any rational tail $C \subset X$ is mapped to a point by $X \to X'$; some details omitted. If not, then we can find a section $s \in \Gamma(X', \omega_{X'}^{\otimes 2})$ which does not vanish in the generic point of the irreducible component $f(C)$. Since in each of the contractions $X_i \to X_{i+1}$ we have a section $X_{i+1} \to X_i$, there is a section $X' \to X$ of f. Then we have an exact sequence

$$0 \to \omega_{X'} \to \omega_X \to \omega_X|_{X''} \to 0$$

where $X'' \subset X$ is the union of the irreducible components contracted by f. See Lemma 53.4.6. Thus we get a map $\omega_{X'}^{\otimes 2} \to \omega_X^{\otimes 2}$ and we can take the image of s to get a section of $\omega_X^{\otimes 2}$ not vanishing in the generic point of C. This is a contradiction with the fact that the restriction of ω_X to a rational tail has negative degree (Lemma 53.23.2).

The statement on base field extensions follows from Lemma 53.23.3. Some details omitted. □

53.24. Contracting rational bridges

0E7M In this section we discuss the next simplest possible case (after the case discussed in Section 53.23) of contracting a scheme to improve positivity properties of its canonical sheaf.

0E3M **Example 53.24.1** (Contracting a rational bridge)**.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A *rational bridge* will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

(1) $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,

(2) the scheme theoretic intersection $C \cap X'$ has degree 2 over $H^0(C, \mathcal{O}_C)$, and
(3) C has genus zero.

Set $k' = H^0(C, \mathcal{O}_C)$ and $k'' = H^0(C \cap X', \mathcal{O}_{C \cap X'})$. Then k' is a field (Varieties, Lemma 33.9.3) and $\dim_{k'}(k'') = 2$. Since there are at least two irreducible components of X passing through each point of $C \cap X'$, we conclude these points are nodes of X and smooth points on both C and X' (Lemma 53.19.17). Hence k'/k is a finite separable extension of fields and k''/k' is either a degree 2 separable extension of fields or $k'' = k' \times k'$ (Lemma 53.19.7). By Section 53.14 there exists a pushout

$$\begin{array}{ccc} C \cap X' & \longrightarrow & X' \\ \downarrow & & \downarrow a \\ \mathrm{Spec}(k') & \longrightarrow & Y \end{array}$$

with many good properties (all of which we will use below without further mention). Let $y \in Y$ be the image of $\mathrm{Spec}(k') \to Y$. Then

$$\mathcal{O}^\wedge_{Y,y} \cong k'[[s,t]]/(st) \quad \text{or} \quad \mathcal{O}^\wedge_{Y,y} \cong \{f \in k''[[s]] : f(0) \in k'\}$$

depending on whether $C \cap X'$ has 2 or 1 points. This follows from Lemma 53.14.1 and the fact that $\mathcal{O}_{X',p} \cong \kappa(p)[[t]]$ for $p \in C \cap X'$ by More on Algebra, Lemma 15.38.4. Thus we see that $y \in Y$ is a node, see Lemmas 53.19.7 and 53.19.4 and in particular the discussion of Case II in the proof of (2) $\Rightarrow$ (1) in Lemma 53.19.4. Thus the singularities of Y are at-worst-nodal.

We can extend the commutative diagram above to a diagram

$$\begin{array}{ccccccc} C \cap X' & \longrightarrow & X' & \longrightarrow & X & \longleftarrow & C \\ \downarrow & & \downarrow a & \swarrow c & & & \\ \mathrm{Spec}(k') & \longrightarrow & Y & \longleftarrow & \mathrm{Spec}(k') & & \end{array}$$

where the two lower horizontal arrows are the same. Namely, X is the scheme theoretic union of X' and C (thus a pushout by Morphisms, Lemma 29.4.6) and the morphisms $C \to Y$ and $X' \to Y$ agree on $C \cap X'$. Finally, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_Y \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_C \oplus \mathcal{O}_{X'} \to \mathcal{O}_{C \cap X'} \to 0$$

of Morphisms, Lemma 29.4.6. The long exact sequence of higher direct images is

$$0 \to c_*\mathcal{O}_X \to c_*\mathcal{O}_C \oplus c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C \cap X'} \to R^1c_*\mathcal{O}_X \to R^1c_*\mathcal{O}_C \oplus R^1c_*\mathcal{O}_{X'}$$

Since $c|_{X'} = a$ is affine we see that $R^1c_*\mathcal{O}_{X'} = 0$. Since $c|_C$ factors as $C \to \mathrm{Spec}(k') \to X$ and since C has genus zero, we find that $R^1c_*\mathcal{O}_C = 0$. Since $\mathcal{O}_{X'} \to \mathcal{O}_{C \cap X'}$ is surjective and since $c|_{X'}$ is affine, we see that $c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C \cap X'}$ is surjective. This proves that $R^1c_*\mathcal{O}_X = 0$. Finally, we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ by the exact sequence and the description of the structure sheaf of the pushout in More on Morphisms, Proposition 37.67.3.

All of this means that Y is also a proper scheme over k having dimension 1 and $H^0(Y, \mathcal{O}_Y) = k$ whose singularities are at-worst-nodal (Lemma 53.19.17) and that

Y has the same genus as X. We will say $c : X \to Y$ is the *contraction of a rational bridge*.

0E64 **Lemma 53.24.2.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 53.24.1). Then* $\deg(\omega_X|_C) = 0$.

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemmas 53.4.6, 53.19.16, and 53.19.17. With $k''/k'/k$ as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as C has genus 0 (Lemma 53.5.2) and $\deg(C \cap X') = [k'' : k] = 2[k' : k]$. Hence $\deg(\omega_X|_C) = 0$. □

0E65 **Lemma 53.24.3.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 53.24.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational bridges.*

Proof. Let $k''/k'/k$ be as in the example. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. The corresponding product decomposition $k'' \otimes_k K = \prod K''_i$ gives degree 2 separable algebra extensions K''_i/K'_i. Set $C_i = C_{K'_i}$. Then $C_K = \coprod C_i$ and therefore each C_i has genus 0 (viewed as a curve over K'_i), because $H^1(C_K, \mathcal{O}_{C_K}) = 0$ by flat base change. Finally, we have $X'_K \cap C_i = \mathrm{Spec}(K''_i)$ has degree 2 over K'_i as desired. □

0E3N **Lemma 53.24.4.** *Let $c : X \to Y$ be the contraction of a rational bridge (Example 53.24.1). Then $c^*\omega_Y \cong \omega_X$.*

Proof. You can prove this by direct computation, but we prefer to use the characterization of ω_X as the coherent $\mathcal{O}_X$-module which represents the functor $Coh(\mathcal{O}_X) \to Sets$, $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k) = H^1(X, \mathcal{F})^\vee$, see Lemma 53.4.2 or Duality for Schemes, Lemma 48.22.5.

To be precise, denote $\mathcal{C}_Y$ the category whose objects are invertible $\mathcal{O}_Y$-modules and whose maps are $\mathcal{O}_Y$-module homomorphisms. Denote $\mathcal{C}_X$ the category whose objects are invertible $\mathcal{O}_X$-modules $\mathcal{L}$ with $\mathcal{L}|_C \cong \mathcal{O}_C$ and whose maps are $\mathcal{O}_Y$-module homomorphisms. We claim that the functor

$$c^* : \mathcal{C}_Y \to \mathcal{C}_X$$

is an equivalence of categories. Namely, by More on Morphisms, Lemma 37.72.8 it is essentially surjective. Then the projection formula (Cohomology, Lemma 20.54.2) shows $c_*c^*\mathcal{N} = \mathcal{N}$ and hence c^* is an equivalence with quasi-inverse given by c_*.

We claim ω_X is an object of $\mathcal{C}_X$. Namely, we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 53.4.6. Taking degrees we find $\deg(\omega_X|_C) = 0$ (small detail omitted). Thus $\omega_X|_C$ is trivial by Lemma 53.10.1 and ω_X is an object of $\mathcal{C}_X$.

Since $R^1c_*\mathcal{O}_X = 0$ the projection formula shows that $R^1c_*c^*\mathcal{N} = 0$ for $\mathcal{N} \in \mathrm{Ob}(\mathcal{C}_Y)$. Therefore the Leray spectral sequence (Cohomology, Lemma 20.13.6) the

diagram

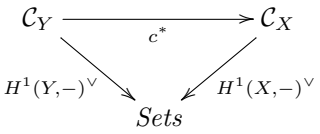

of categories and functors is commutative. Since $\omega_Y \in \mathrm{Ob}(\mathcal{C}_Y)$ represents the south-east arrow and $\omega_X \in \mathrm{Ob}(\mathcal{C}_X)$ represents the south-east arrow we conclude by the Yoneda lemma (Categories, Lemma 4.3.5). □

0E3P **Lemma 53.24.5.** *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume*

(1) *the singularities of X are at-worst-nodal,*
(2) *X does not have a rational tail (Example 53.23.1),*
(3) *X does not have a rational bridge (Example 53.24.1),*
(4) *the genus g of X is ≥ 2.*

Then ω_X is ample.

Proof. It suffices to show that $\deg(\omega_X|_C) > 0$ for every irreducible component C of X, see Varieties, Lemma 33.44.15. If $X = C$ is irreducible, this follows from $g \geq 2$ and Lemma 53.8.3. Otherwise, set $k' = H^0(C, \mathcal{O}_C)$. This is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to C and coherent sheaves on C, see Varieties, Lemma 33.44.10. Let $X' \subset X$ be the closure of $X \setminus C$ as in Lemma 53.4.6. We will use the results of this lemma and of Lemmas 53.19.16 and 53.19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 53.4.6. We conclude that

$$\deg(\omega_X|_C) = \deg(C \cap X') + \deg(\omega_C) = \deg(C \cap X') - 2\chi(C, \mathcal{O}_C)$$

Hence, if the lemma is false, then

$$2[k' : k] \geq \deg(C \cap X') + 2 \dim_k H^1(C, \mathcal{O}_C)$$

Since $C \cap X'$ is nonempty and by the divisiblity mentioned above, this can happen only if either

(a) $C \cap X'$ is a single k'-rational point of C and $H^1(C, \mathcal{O}_C) = 0$, and
(b) $C \cap X'$ has degree 2 over k' and $H^1(C, \mathcal{O}_C) = 0$.

The first possibility means C is a rational tail and the second that C is a rational bridge. Since both are excluded the proof is complete. □

0E3Q **Lemma 53.24.6.** *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and that X has no rational tails. Consider a sequence*

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational bridges (Example 53.24.1) until none are left. Then $\omega_{X'}$ ample. The morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational bridges until there are none left. Then $\omega_{X'}$ is ample by Lemma 53.24.5.

Denote $f : X \to X'$ the composition. By Lemma 53.24.4 and induction we see that $f^*\omega_{X'} = \omega_X$. We have $f_*\mathcal{O}_X = \mathcal{O}_{X'}$ because this is true for contraction of a rational bridge. Thus the projection formula says that $f_*f^*\mathcal{L} = \mathcal{L}$ for all invertible $\mathcal{O}_{X'}$-modules $\mathcal{L}$. Hence

$$\Gamma(X', \omega_{X'}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m})$$

for all m. Since X' is the Proj of the direct sum of these by Morphisms, Lemma 29.43.17 we conclude that the morphism $X \to X'$ is completely canonical.

Let K/k be an extension of fields, then ω_{X_K} is the pullback of ω_X (Lemma 53.4.4) and we have $\Gamma(X, \omega_X^{\otimes m}) \otimes_k K$ is equal to $\Gamma(X_K, \omega_{X_K}^{\otimes m})$ by Cohomology of Schemes, Lemma 30.5.2. Thus formation of $f : X \to X'$ commutes with base change by K/k by the arguments given above. Some details omitted. □

53.25. Contracting to a stable curve

0E7N In this section we combine the contraction morphisms found in Sections 53.23 and 53.24. Namely, suppose that k is a field and let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. Composing the morphism of Lemma 53.23.6 with the morphism of Lemma 53.24.6 we get a morphism

$$c : X \longrightarrow Y$$

such that Y also is a proper scheme over k of dimension 1 whose singularities are at worst nodal, with $k = H^0(Y, \mathcal{O}_Y)$ and having genus g, such that $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and such that ω_Y is ample on Y. Lemma 53.25.2 shows these conditions in fact characterize this morphism.

0E7P **Lemma 53.25.1.** *Let k be a field. Let $c : X \to Y$ be a morphism of proper schemes over k. Assume*

(1) *$\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$,*
(2) *X and Y are reduced, Gorenstein, and have dimension 1,*
(3) *$\exists\, m \in \mathbf{Z}$ with $H^1(X, \omega_X^{\otimes m}) = 0$ and $\omega_X^{\otimes m}$ generated by global sections.*

Then $c^*\omega_Y \cong \omega_X$.

Proof. The fibres of c are geometrically connected by More on Morphisms, Theorem 37.53.4. In particular c is surjective. There are finitely many closed points $y = y_1, \ldots, y_r$ of Y where X_y has dimension 1 and over $Y \setminus \{y_1, \ldots, y_r\}$ the morphism c is an isomorphism. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 30.21.1.

Let us carefully construct a map $b : c^*\omega_Y \to \omega_X$. Denote $f : X \to \mathrm{Spec}(k)$ and $g : Y \to \mathrm{Spec}(k)$ the structure morphisms. We have $f^!k = \omega_X[1]$ and $g^!k = \omega_Y[1]$, see Lemma 53.4.1 and its proof. Then $f^! = c^! \circ g^!$ and hence $c^!\omega_Y = \omega_X$. Thus there is a functorial isomorphism

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{F}, \omega_X) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rc_*\mathcal{F}, \omega_Y)$$

for coherent $\mathcal{O}_X$-modules $\mathcal{F}$ by definition of $c^!$[7]. This isomorphism is induced by a trace map $t : Rc_*\omega_X \to \omega_Y$ (the counit of the adjunction). By the projection

[7] As the restriction of the right adjoint of Duality for Schemes, Lemma 48.3.1 to $D^+_{QCoh}(\mathcal{O}_Y)$.

formula (Cohomology, Lemma 20.54.2) the canonical map $a : \omega_Y \to Rc_*c^*\omega_Y$ is an isomorphism. Combining the above we see there is a canonical map $b : c^*\omega_Y \to \omega_X$ such that

$$t \circ Rc_*(b) = a^{-1}$$

In particular, if we restrict b to $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ then it is an isomorphism (because it is a map between invertible modules whose composition with another gives the isomorphism a^{-1}).

Choose $m \in \mathbf{Z}$ as in (3) consider the map

$$b^{\otimes m} : \Gamma(Y, \omega_Y^{\otimes m}) \longrightarrow \Gamma(X, \omega_X^{\otimes m})$$

This map is injective because Y is reduced and by the last property of b mentioned in its construction. By Riemann-Roch (Lemma 53.5.2) we have $\chi(X, \omega_X^{\otimes m}) = \chi(Y, \omega_Y^{\otimes m})$. Thus

$$\dim_k \Gamma(Y, \omega_Y^{\otimes m}) \geq \dim_k \Gamma(X, \omega_X^{\otimes m}) = \chi(X, \omega_X^{\otimes m})$$

and we conclude $b^{\otimes m}$ induces an isomorphism on global sections. So $b^{\otimes m} : c^*\omega_Y^{\otimes m} \to \omega_X^{\otimes m}$ is surjective as generators of $\omega_X^{\otimes m}$ are in the image. Hence $b^{\otimes m}$ is an isomorphism. Thus b is an isomorphism. □

0E7Q **Lemma 53.25.2.** *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. There is a unique morphism (up to unique isomorphism)*

$$c : X \longrightarrow Y$$

of schemes over k having the following properties:

1. *Y is proper over k, $\dim(Y) = 1$, the singularities of Y are at-worst-nodal,*
2. *$\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and*
3. *ω_Y is ample on Y.*

Proof. Existence: A morphism with all the properties listed exists by combining Lemmas 53.23.6 and 53.24.6 as discussed in the introduction to this section. Moreover, we see that it can be written as a composition

$$X \to X_1 \to X_2 \ldots \to X_n \to X_{n+1} \to \ldots \to X_{n+n'}$$

where the first n morphisms are contractions of rational tails and the last n' morphisms are contractions of rational bridges. Note that property (2) holds for each contraction of a rational tail (Example 53.23.1) and contraction of a rational bridge (Example 53.24.1). It is easy to see that this property is inherited by compositions of morphisms.

Uniqueness: Let $c : X \to Y$ be a morphism satisfying conditions (1), (2), and (3). We will show that there is a unique isomorphism $X_{n+n'} \to Y$ compatible with the morphisms $X \to X_{n+n'}$ and c.

Before we start the proof we make some observations about c. We first observe that the fibres of c are geometrically connected by More on Morphisms, Theorem 37.53.4. In particular c is surjective. For a closed point $y \in Y$ the fibre X_y satisfies

$$H^1(X_y, \mathcal{O}_{X_y}) = 0 \quad \text{and} \quad H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$$

The first equality by More on Morphisms, Lemma 37.72.1 and the second by More on Morphisms, Lemma 37.72.4. Thus either $X_y = x$ where x is the unique point

of X mapping to y and has the same residue field as y, or X_y is a 1-dimensional proper scheme over $\kappa(y)$. Observe that in the second case X_y is Cohen-Macaulay (Lemma 53.6.1). However, since X is reduced, we see that X_y must be reduced at all of its generic points (details omitted), and hence X_y is reduced by Properties, Lemma 28.12.4. It follows that the singularities of X_y are at-worst-nodal (Lemma 53.19.17). Note that the genus of X_y is zero (see above). Finally, there are only a finite number of points y where the fibre X_y has dimension 1, say $\{y_1, \ldots, y_r\}$, and $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to $Y \setminus \{y_1, \ldots, y_r\}$ by c. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 30.21.1.

Let $C \subset X$ be a rational tail. We claim that c maps C to a point. Assume that this is not the case to get a contradiction. Then the image of C is an irreducible component $D \subset Y$. Recall that $H^0(C, \mathcal{O}_C) = k'$ is a finite separable extension of k and that C has a k'-rational point x which is also the unique intersection of C with the "rest" of X. We conclude from the general discussion above that $C \setminus \{x\} \subset c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to an open V of D. Let $y = c(x) \in D$. Observe that y is the only point of D meeting the "rest" of Y. If $y \notin \{y_1, \ldots, y_r\}$, then $C \cong D$ and it is clear that D is a rational tail of Y which is a contradiction with the ampleness of ω_Y (Lemma 53.23.2). Thus $y \in \{y_1, \ldots, y_r\}$ and $\dim(X_y) = 1$. Then $x \in X_y \cap C$ and x is a smooth point of X_y and C (Lemma 53.19.17). If $y \in D$ is a singular point of D, then y is a node and then $Y = D$ (because there cannot be another component of Y passing through y by Lemma 53.19.17). Then $X = X_y \cup C$ which means $g = 0$ because it is equal to the genus of X_y by the discussion in Example 53.23.1; a contradiction. If $y \in D$ is a smooth point of D, then $C \to D$ is an isomorphism (because the nonsingular projective model is unique and C and D are birational, see Section 53.2). Then D is a rational tail of Y which is a contradiction with ampleness of ω_Y.

Assume $n \geq 1$. If $C \subset X$ is the rational tail contracted by $X \to X_1$, then we see that C is mapped to a point of Y by the previous paragraph. Hence $c : X \to Y$ factors through $X \to X_1$ (because X is the pushout of C and X_1, see discussion in Example 53.23.1). After replacing X by X_1 we have decreased n. By induction we may assume $n = 0$, i.e., X does not have a rational tail.

Assume $n = 0$, i.e., X does not have any rational tails. Then $\omega_X^{\otimes 2}$ and $\omega_X^{\otimes 3}$ are globally generated by Lemma 53.23.5. It follows that $H^1(X, \omega_X^{\otimes 3}) = 0$ by Lemma 53.6.4. By Lemma 53.25.1 applied with $m = 3$ we find that $c^*\omega_Y \cong \omega_X$. We also have that $\omega_X = (X \to X_{n'})^*\omega_{X_{n'}}$ by Lemma 53.24.4 and induction. Applying the projection formula for both c and $X \to X_{n'}$ we conclude that

$$\Gamma(X_{n'}, \omega_{X_{n'}}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m}) = \Gamma(Y, \omega_Y^{\otimes m})$$

for all m. Since $X_{n'}$ and Y are the Proj of the direct sum of these by Morphisms, Lemma 29.43.17 we conclude that there is a canonical isomorphism $X_{n'} = Y$ as desired. We omit the verification that this is the unique isomorphism making the diagram commute. □

0E8X **Lemma 53.25.3.** *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and ω_X is ample. Then $\omega_X^{\otimes 3}$ is very ample and $H^1(X, \omega_X^{\otimes 3}) = 0$.*

Proof. Combining Varieties, Lemma 33.44.15 and Lemmas 53.23.2 and 53.24.2 we see that X contains no rational tails or bridges. Then we see that $\omega_X^{\otimes 3}$ is globally generated by Lemma 53.23.6. Choose a k-basis $s_0, \ldots, s_n$ of $H^0(X, \omega_X^{\otimes 3})$. We get a morphism

$$\varphi_{\omega_X^{\otimes 3}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}_k^n$$

See Constructions, Section 27.13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 35.23.21. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. We will use Varieties, Lemma 33.23.2 to prove the lemma. Let $Z \subset X$ be a closed subscheme of degree 2 over Z with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. We have to show that

$$H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}|_Z)$$

is surjective. Thus it suffices to show that $H^1(X, \mathcal{I}\mathcal{L}) = 0$. To do this we will use Lemma 53.22.6. Thus it suffices to show that

$$3 \deg(\omega_X|_Y) > -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

for every reduced connected closed subscheme $Y \subset X$. Since k is algebraically closed and Y connected and reduced we have $H^0(Y, \mathcal{O}_Y) = k$ (Varieties, Lemma 33.9.3). Hence $\chi(Y, \mathcal{O}_Y) = 1 - \dim H^1(Y, \mathcal{O}_Y)$. Thus we have to show

$$3 \deg(\omega_X|_Y) > -2 + 2 \dim H^1(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

which is true by Lemma 53.23.4 except possibly if $Y = X$ or if $\deg(\omega_X|_Y) = 0$. Since ω_X is ample the second possibility does not occur (see first lemma cited in this proof). Finally, if $Y = X$ we can use Riemann-Roch (Lemma 53.5.2) and the fact that $g \geq 2$ to see that the inquality holds. The same argument with $Z = \emptyset$ shows that $H^1(X, \omega_X^{\otimes 3}) = 0$. □

53.26. Vector fields

0E66 In this section we study the space of vector fields on a curve. Vector fields correspond to infinitesimal automorphisms, see More on Morphisms, Section 37.9, hence play an important role in moduli theory.

Let k be an algebraically closed field. Let X be a finite type scheme over k. Let $x \in X$ be a closed point. We will say an element $D \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ *fixes* x if $D(\mathcal{I}) \subset \mathcal{I}$ where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of x.

0E67 **Lemma 53.26.1.** *Let k be an algebraically closed field. Let X be a smooth, proper, connected curve over k. Let g be the genus of X.*

1. *If $g \geq 2$, then $Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is zero,*
2. *if $g = 1$ and $D \in Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D does not fix any closed point of X, and*
3. *if $g = 0$ and $D \in Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D fixes at most 2 closed points of X.*

Proof. Recall that we have a universal k-derivation $d : \mathcal{O}_X \to \Omega_{X/k}$ and hence $D = \theta \circ d$ for some $\mathcal{O}_X$-linear map $\theta : \Omega_{X/k} \to \mathcal{O}_X$. Recall that $\Omega_{X/k} \cong \omega_X$, see Lemma 53.4.1. By Riemann-Roch we have $\deg(\omega_X) = 2g - 2$ (Lemma 53.5.2). Thus we see that θ is forced to be zero if $g > 1$ by Varieties, Lemma 33.44.12. This proves part (1). If $g = 1$, then a nonzero θ does not vanish anywhere and if $g = 0$,

then a nonzero θ vanishes in a divisor of degree 2. Thus parts (2) and (3) follow if we show that vanishing of θ at a closed point $x \in X$ is equivalent to the statement that D fixes x (as defined above). Let $z \in \mathcal{O}_{X,x}$ be a uniformizer. Then dz is a basis element for $\Omega_{X,x}$, see Lemma 53.12.3. Since $D(z) = \theta(dz)$ we conclude. □

0E68 **Lemma 53.26.2.** *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Let $\nu : X^\nu \to X$ be the normalization. Let $S \subset X^\nu$ be the set of points where ν is not an isomorphism. Then*

$$Der_k(\mathcal{O}_X, \mathcal{O}_X) = \{D' \in Der_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu}) \mid D' \text{ fixes every } x^\nu \in S\}$$

Proof. Let $x \in X$ be a node. Let $x', x'' \in X^\nu$ be the inverse images of x. (Every node is a split node since k is algebriacally closed, see Definition 53.19.10 and Lemma 53.19.11.) Let $u \in \mathcal{O}_{X^\nu,x'}$ and $v \in \mathcal{O}_{X^\nu,x''}$ be uniformizers. Observe that we have an exact sequence

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu,x'} \times \mathcal{O}_{X^\nu,x''} \to k \to 0$$

This follows from Lemma 53.16.3. Thus we can view u and v as elements of $\mathcal{O}_{X,x}$ with $uv = 0$.

Let $D \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Then $0 = D(uv) = vD(u)+uD(v)$. Since (u) is annihilator of v in $\mathcal{O}_{X,x}$ and vice versa, we see that $D(u) \in (u)$ and $D(v) \in (v)$. As $\mathcal{O}_{X^\nu,x'} = k + (u)$ we conclude that we can extend D to $\mathcal{O}_{X^\nu,x'}$ and moreover the extension fixes x'. This produces a D' in the right hand side of the equality. Conversely, given a D' fixing x' and x'' we find that D' preserves the subring $\mathcal{O}_{X,x} \subset \mathcal{O}_{X^\nu,x'} \times \mathcal{O}_{X^\nu,x''}$ and this is how we go from right to left in the equality. □

0E69 **Lemma 53.26.3.** *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Assume the genus of X is at least 2 and that X has no rational tails or bridges. Then $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.*

Proof. Let $D \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Let X^ν be the normalization of X. Let $D' \in \mathrm{Der}_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu})$ be the element corresponding to D via Lemma 53.26.2. Let $C \subset X^\nu$ be an irreducible component. If the genus of C is > 1, then $D'|_{\mathcal{O}_C} = 0$ by Lemma 53.26.1 part (1). If the genus of C is 1, then there is at least one closed point c of C which maps to a node on X (since otherwise $X \cong C$ would have genus 1). By the correspondence this means that $D'|_{\mathcal{O}_C}$ fixes c hence is zero by Lemma 53.26.1 part (2). Finally, if the genus of C is zero, then there are at least 3 pairwise distinct closed points $c_1, c_2, c_3 \in C$ mapping to nodes in X, since otherwise either X is C with two points glued (two points of C mapping to the same node), or C is a rational bridge (two points mapping to different nodes of X), or C is a rational tail (one point mapping to a node of X). These three possibilities are not permitted since C has genus ≥ 2 and has no rational bridges, or rational tails. Whence $D'|_{\mathcal{O}_C}$ fixes c_1, c_2, c_3 hence is zero by Lemma 53.26.1 part (3). □

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CHAPTER 54

Resolution of Surfaces

0ADW

54.1. Introduction

0ADX This chapter discusses resolution of singularities of surfaces following Lipman [**Lip78**] and mostly following the exposition of Artin in [**Art86**]. The main result (Theorem 54.14.5) tells us that a Noetherian 2-dimensional scheme Y has a resolution of singularities when it has a finite normalization $Y^\nu \to Y$ with finitely many singular points $y_i \in Y^\nu$ and for each i the completion $\mathcal{O}^\wedge_{Y^\nu, y_i}$ is normal.

To be sure, if Y is a 2-dimensional scheme of finite type over a quasi-excellent base ring R (for example a field or a Dedekind domain with fraction field of characteristic 0 such as $\mathbf{Z}$) then the normalization of Y is finite, has finitely many singular points, and the completions of the local rings are normal. See the discussion in More on Algebra, Sections 15.47, 15.50, and 15.52 and More on Algebra, Lemma 15.42.2. Thus such a Y has a resolution of singularities.

A rough outline of the proof is as follows. Let A be a Noetherian local domain of dimension 2. The steps of the proof are as follows

- N replace A by its normalization,
- V prove Grauert-Riemenschneider,
- B show there is a maximum g of the lengths of $H^1(X, \mathcal{O}_X)$ over all normal modifications $X \to \mathrm{Spec}(A)$ and reduce to the case $g = 0$,
- R we say A defines a rational singularity if $g = 0$ and in this case after a finite number of blowups we may assume A is Gorenstein and $g = 0$,
- D we say A defines a rational double point if $g = 0$ and A is Gorenstein and in this case we explicitly resolve singularities.

Each of these steps needs assumptions on the ring A. We will discuss each of these in turn.

Ad N: Here we need to assume that A has a finite normalization (this is not automatic). Throughout most of the chapter we will assume that our scheme is Nagata if we need to know some normalization is finite. However, being Nagata is a slightly stronger condition than is given to us in the statement of the theorem. A solution to this (slight) problem would have been to use that our ring A is formally unramified (i.e., its completion is reduced) and to use Lemma 54.11.5. However, the way our proof works, it turns out it is easier to use Lemma 54.11.6 to lift finiteness of the normalization over the completion to finiteness of the normalization over A.

Ad V: This is Proposition 54.7.8 and it roughly states that for a normal modification $f : X \to \mathrm{Spec}(A)$ one has $R^1 f_* \omega_X = 0$ where ω_X is the dualizing module of X/A (Remark 54.7.7). In fact, by duality the result is equivalent to a statement (Lemma 54.7.6) about the object $Rf_*\mathcal{O}_X$ in the *derived category* $D(A)$. Having said this,

the proof uses the standard fact that components of the special fibre have positive conormal sheaves (Lemma 54.7.4).

Ad B: This is in some sense the most subtle part of the proof. In the end we only need to use the output of this step when A is a complete Noetherian local ring, although the writeup is a bit more general. The terminology is set in Definition 54.8.3. If g (as defined above) is bounded, then a straightforward argument shows that we can find a normal modification $X \to \mathrm{Spec}(A)$ such that all singular points of X are rational singularities, see Lemma 54.8.5. We show that given a finite extension $A \subset B$, then g is bounded for B if it is bounded for A in the following two cases: (1) if the fraction field extension is separable, see Lemma 54.8.5 and (2) if the fraction field extension has degree p, the characteristic is p, and A is regular and complete, see Lemma 54.8.10.

Ad R: Here we reduce the case $g = 0$ to the Gorenstein case. A marvellous fact, which makes everything work, is that the blowing up of a rational surface singularity is normal, see Lemma 54.9.4.

Ad D: The resolution of rational double points proceeds more or less by hand, see Section 54.12. A rational double point is a hypersurface singularity (this is true but we don't prove it as we don't need it). The local equation looks like

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

Using that the quadratic part cannot be zero because the multiplicity is 2 and remains 2 after any blowup and the fact that every blowup is normal one quickly achieves a resolution. One twist is that we do not have an invariant which decreases every blowup, but we rely on the material on formal arcs from Section 54.10 to demonstrate that the process stops.

To put everything together some additional work has to be done. The main kink is that we want to lift a resolution of the completion $A^\wedge$ to a resolution of $\mathrm{Spec}(A)$. In order to do this we first show that if a resolution exists, then there is a resolution by normalized blowups (Lemma 54.14.3). A sequence of normalized blowups can be lifted from the completion by Lemma 54.11.7. We then use this even in the proof of resolution of complete local rings A because our strategy works by induction on the degree of a finite inclusion $A_0 \subset A$ with A_0 regular, see Lemma 54.14.4. With a stronger result in B (such as is proved in Lipman's paper) this step could be avoided.

54.2. A trace map in positive characteristic

0ADY Some of the results in this section can be deduced from the much more general discussion on traces on differential forms in de Rham Cohomology, Section 50.19. See Remark 54.2.3 for a discussion.

We fix a prime number p. Let R be an $\mathbf{F}_p$-algebra. Given an $a \in R$ set $S = R[x]/(x^p - a)$. Define an R-linear map

$$\mathrm{Tr}_x : \Omega_{S/R} \longrightarrow \Omega_R$$

by the rule

$$x^i \mathrm{d}x \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \mathrm{d}a & \text{if} \quad i = p-1 \end{cases}$$

This makes sense as $\Omega_{S/R}$ is a free R-module with basis $x^i \mathrm{d}x$, $0 \leq i \leq p-1$. The following lemma implies that the trace map is well defined, i.e., independent of the choice of the coordinate x.

0ADZ **Lemma 54.2.1.** *Let $\varphi : R[x]/(x^p - a) \to R[y]/(y^p - b)$ be an R-algebra homomorphism. Then $Tr_x = Tr_y \circ \varphi$.*

Proof. Say $\varphi(x) = \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1}$ with $\lambda_i \in R$. The condition that mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1}$ induces an R-algebra homomorphism $R[x]/(x^p - a) \to R[y]/(y^p - b)$ is equivalent to the condition that

$$a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$$

in the ring R. Consider the polynomial ring

$$R_{univ} = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$$

with the element $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ Consider the universal algebra map $\varphi_{univ} : R_{univ}[x]/(x^p - a) \to R_{univ}[y]/(y^p - b)$ given by mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1}$. We obtain a canonical map

$$R_{univ} \longrightarrow R$$

sending b, λ_i to b, λ_i. By construction we get a commutative diagram

$$\begin{array}{ccc} R_{univ}[x]/(x^p - a) & \longrightarrow & R[x]/(x^p - a) \\ \downarrow{\scriptstyle \varphi_{univ}} & & \downarrow{\scriptstyle \varphi} \\ R_{univ}[y]/(y^p - b) & \longrightarrow & R[y]/(y^p - b) \end{array}$$

and the horizontal arrows are compatible with the trace maps. Hence it suffices to prove the lemma for the map φ_{univ}. Thus we may assume $R = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$ is a polynomial ring. We will check the lemma holds in this case by evaluating $\mathrm{Tr}_y(\varphi(x)^i \mathrm{d}\varphi(x))$ for $i = 0, \ldots, p-1$.

The case $0 \leq i \leq p-2$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^i (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. We have to show that the coefficient of y^{p-1} is zero. For this it suffices to show that the expression above as a polynomial in y has vanishing coefficients in front of the powers y^{pk-1}. Then we write our polynomial as

$$\frac{\mathrm{d}}{(i+1)\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{i+1}$$

and indeed the coefficients of y^{kp-1} are all zero.

The case $i = p-1$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{p-1} (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. To finish the proof we have to show that the coefficient of y^{p-1} times $\mathrm{d}b$ is $\mathrm{d}a$. Here we use that R is S/pS where $S = \mathbf{Z}[b, \lambda_0, \ldots, \lambda_{p-1}]$. Then the above, as a polynomial in y, is equal to

$$\frac{\mathrm{d}}{p\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p$$

Since $\frac{\mathrm{d}}{\mathrm{d}y}(y^{pk}) = pky^{pk-1}$ it suffices to understand the coefficients of y^{pk} in the polynomial $(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1})^p$ modulo p. The sum of these terms gives

$$\lambda_0^p + \lambda_1^p y^p + \ldots + \lambda_{p-1}^p y^{p(p-1)} \bmod p$$

Whence we see that we obtain after applying the operator $\frac{\mathrm{d}}{p\mathrm{d}y}$ and after reducing modulo $y^p - b$ the value

$$\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1} b^{p-2}$$

for the coefficient of y^{p-1} we wanted to compute. Now because $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ in R we obtain that

$$\mathrm{d}a = (\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2})\mathrm{d}b$$

in R. This proves that the coefficient of y^{p-1} is as desired. □

0AX5 **Lemma 54.2.2.** *Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ be ring extensions and assume that S is isomorphic to $R[x]/(x^p - a)$ for some $a \in R$. Then there are canonical R-linear maps*

$$Tr : \Omega^{t+1}_{S/\Lambda} \longrightarrow \Omega^{t+1}_{R/\Lambda}$$

for $t \geq 0$ such that

$$\eta_1 \wedge \ldots \wedge \eta_t \wedge x^i dx \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \eta_1 \wedge \ldots \wedge \eta_t \wedge da & \text{if} \quad i = p-1 \end{cases}$$

for $\eta_i \in \Omega_{R/\Lambda}$ and such that Tr annihilates the image of $S \otimes_R \Omega^{t+1}_{R/\Lambda} \to \Omega^{t+1}_{S/\Lambda}$.

Proof. For $t = 0$ we use the composition

$$\Omega_{S/\Lambda} \to \Omega_{S/R} \to \Omega_R \to \Omega_{R/\Lambda}$$

where the second map is Lemma 54.2.1. There is an exact sequence

$$H_1(L_{S/R}) \xrightarrow{\delta} \Omega_{R/\Lambda} \otimes_R S \to \Omega_{S/\Lambda} \to \Omega_{S/R} \to 0$$

(Algebra, Lemma 10.134.4). The module $\Omega_{S/R}$ is free over S with basis $\mathrm{d}x$ and the module $H_1(L_{S/R})$ is free over S with basis $x^p - a$ which δ maps to $-\mathrm{d}a \otimes 1$ in $\Omega_{R/\Lambda} \otimes_R S$. In particular, if we set

$$M = \mathrm{Coker}(R \to \Omega_{R/\Lambda}, 1 \mapsto -\mathrm{d}a)$$

then we see that $\mathrm{Coker}(\delta) = M \otimes_R S$. We obtain a canonical map

$$\Omega^{t+1}_{S/\Lambda} \to \wedge^t_S(\mathrm{Coker}(\delta)) \otimes_S \Omega_{S/R} = \wedge^t_R(M) \otimes_R \Omega_{S/R}$$

Now, since the image of the map $\mathrm{Tr} : \Omega_{S/R} \to \Omega_{R/\Lambda}$ of Lemma 54.2.1 is contained in $R\mathrm{d}a$ we see that wedging with an element in the image annihilates $\mathrm{d}a$. Hence there is a canonical map

$$\wedge^t_R(M) \otimes_R \Omega_{S/R} \to \Omega^{t+1}_{R/\Lambda}$$

mapping $\overline{\eta}_1 \wedge \ldots \wedge \overline{\eta}_t \wedge \omega$ to $\eta_1 \wedge \ldots \wedge \eta_t \wedge \mathrm{Tr}(\omega)$. □

0FLF **Remark 54.2.3.** Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ and Tr be as in Lemma 54.2.2. By de Rham Cohomology, Proposition 50.19.3 there is a canonical map of complexes

$$\Theta_{S/R} : \Omega^\bullet_{S/\Lambda} \longrightarrow \Omega^\bullet_{R/\Lambda}$$

The computation in de Rham Cohomology, Example 50.19.4 shows that $\Theta_{S/R}(x^i \mathrm{d}x) = \mathrm{Tr}_x(x^i \mathrm{d}x)$ for all i. Since $\mathrm{Trace}_{S/R} = \Theta^0_{S/R}$ is identically zero and since

$$\Theta_{S/R}(a \wedge b) = a \wedge \Theta_{S/R}(b)$$

for $a \in \Omega^i_{R/\Lambda}$ and $b \in \Omega^j_{S/\Lambda}$ it follows that $\mathrm{Tr} = \Theta_{S/R}$. The advantage of using Tr is that it is a good deal more elementary to construct.

0AX6 **Lemma 54.2.4.** *Let S be a scheme over $\mathbf{F}_p$. Let $f : Y \to X$ be a finite morphism of Noetherian normal integral schemes over S. Assume*

(1) *the extension of function fields is purely inseparable of degree p, and*
(2) *$\Omega_{X/S}$ is a coherent $\mathcal{O}_X$-module (for example if X is of finite type over S).*

For $i \geq 1$ there is a canonical map

$$Tr : f_*\Omega^i_{Y/S} \longrightarrow (\Omega^i_{X/S})^{**}$$

whose stalk in the generic point of X recovers the trace map of Lemma 54.2.2.

Proof. The exact sequence $f^*\Omega_{X/S} \to \Omega_{Y/S} \to \Omega_{Y/X} \to 0$ shows that $\Omega_{Y/S}$ and hence $f_*\Omega_{Y/S}$ are coherent modules as well. Thus it suffices to prove the trace map in the generic point extends to stalks at $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$, see Divisors, Lemma 31.12.14. Thus we reduce to the case discussed in the next paragraph.

Assume $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ with A a discrete valuation ring and B finite over A. Since the induced extension L/K of fraction fields is purely inseparable, we see that B is local too. Hence B is a discrete valuation ring too. Then either

(1) B/A has ramification index p and hence $B = A[x]/(x^p - a)$ where $a \in A$ is a uniformizer, or
(2) $\mathfrak{m}_B = \mathfrak{m}_A B$ and the residue field $B/\mathfrak{m}_A B$ is purely inseparable of degree p over $\kappa_A = A/\mathfrak{m}_A$. Choose any $x \in B$ whose residue class is not in κ_A and then we'll have $B = A[x]/(x^p - a)$ where $a \in A$ is a unit.

Let $\mathrm{Spec}(\Lambda) \subset S$ be an affine open such that X maps into $\mathrm{Spec}(\Lambda)$. Then we can apply Lemma 54.2.2 to see that the trace map extends to $\Omega^i_{B/\Lambda} \to \Omega^i_{A/\Lambda}$ for all $i \geq 1$. □

54.3. Quadratic transformations

0AGP In this section we study what happens when we blow up a nonsingular point on a surface. We hesitate the formally define such a morphism as a *quadratic transformation* as on the one hand often other names are used and on the other hand the phrase "quadratic transformation" is sometimes used with a different meaning.

0AGQ **Lemma 54.3.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$ wotj exceptional divisor E. There is a closed immersion*

$$r : X \longrightarrow \mathbf{P}^1_S$$

over S such that

(1) *$r|_E : E \to \mathbf{P}^1_\kappa$ is an isomorphism,*
(2) *$\mathcal{O}_X(E) = \mathcal{O}_X(-1) = r^*\mathcal{O}_{\mathbf{P}^1}(-1)$, and*
(3) *$\mathcal{C}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(1)$ and $\mathcal{N}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(-1)$.*

Proof. As A is regular of dimension 2 we can write $\mathfrak{m} = (x, y)$. Then x and y placed in degree 1 generate the Rees algebra $\bigoplus_{n\geq 0} \mathfrak{m}^n$ over A. Recall that $X = \text{Proj}(\bigoplus_{n\geq 0} \mathfrak{m}^n)$, see Divisors, Lemma 31.32.2. Thus the surjection

$$A[T_0, T_1] \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathfrak{m}^n, \quad T_0 \mapsto x,\ T_1 \mapsto y$$

of graded A-algebras induces a closed immersion $r : X \to \mathbf{P}^1_S = \text{Proj}(A[T_0, T_1])$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbf{P}^1_S}(1)$, see Constructions, Lemma 27.11.5. This proves (2) because $\mathcal{O}_X(E) = \mathcal{O}_X(-1)$ by Divisors, Lemma 31.32.4.

To prove (1) note that

$$\left(\bigoplus\nolimits_{n\geq 0} \mathfrak{m}^n\right) \otimes_A \kappa = \bigoplus\nolimits_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \kappa[\overline{x}, \overline{y}]$$

a polynomial algebra, see Algebra, Lemma 10.106.1. This proves that the fibre of $X \to S$ over $\text{Spec}(\kappa)$ is equal to $\text{Proj}(\kappa[\overline{x}, \overline{y}]) = \mathbf{P}^1_\kappa$, see Constructions, Lemma 27.11.6. Recall that E is the closed subscheme of X defined by $\mathfrak{m}\mathcal{O}_X$, i.e., $E = X_\kappa$. By our choice of the morphism r we see that $r|_E$ in fact produces the identification of $E = X_\kappa$ with the special fibre of $\mathbf{P}^1_S \to S$.

Part (3) follows from (1) and (2) and Divisors, Lemma 31.14.2. □

0AGR **Lemma 54.3.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then X is an irreducible regular scheme.*

Proof. Observe that X is integral by Divisors, Lemma 31.32.9 and Algebra, Lemma 10.106.2. To see X is regular it suffices to check that $\mathcal{O}_{X,x}$ is regular for closed points $x \in X$, see Properties, Lemma 28.9.2. Let $x \in X$ be a closed point. Since f is proper x maps to $\mathfrak{m}$, i.e., x is a point of the exceptional divisor E. Then E is an effective Cartier divisor and $E \cong \mathbf{P}^1_\kappa$. Thus if $g \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is a local equation for E, then $\mathcal{O}_{X,x}/(g) \cong \mathcal{O}_{\mathbf{P}^1_\kappa,x}$. Since $\mathbf{P}^1_\kappa$ is covered by two affine opens which are the spectrum of a polynomial ring over κ, we see that $\mathcal{O}_{\mathbf{P}^1_\kappa,x}$ is regular by Algebra, Lemma 10.114.1. We conclude by Algebra, Lemma 10.106.7. □

0C5G **Lemma 54.3.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then $\text{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(E)$.*

Proof. Recall that $E = \mathbf{P}^1_\kappa$ has Picard group $\mathbf{Z}$ with generator $\mathcal{O}(1)$, see Divisors, Lemma 31.28.5. By Lemma 54.3.1 the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(E)$ restricts to $\mathcal{O}(-1)$. Hence $\mathcal{O}_X(E)$ generates an infinite cyclic group in $\text{Pic}(X)$. Since A is regular it is a UFD, see More on Algebra, Lemma 15.122.2. Then the punctured spectrum $U = S \setminus \{\mathfrak{m}\} = X \setminus E$ has trivial Picard group, see Divisors, Lemma 31.28.4. Hence for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ there is an isomorphism $s : \mathcal{O}_U \to \mathcal{L}|_U$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_\mathcal{L}(s) = nE$ for some $n \in \mathbf{Z}$ (Divisors, Definition 31.27.4). By Divisors, Lemma 31.27.6 (and the fact that X is normal by Lemma 54.3.2) we conclude that $\mathcal{L} = \mathcal{O}_X(nE)$. □

0AGS **Lemma 54.3.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$,*
(2) *$H^1(X, \mathcal{O}_X(n)) = 0$ for $n \geq -1$,*
(3) *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ or $\mathcal{F}(1)$ is globally generated,*
(4) *$H^0(X, \mathcal{O}_X(n)) = \mathfrak{m}^{\max(0,n)}$,*

(5) $length_A H^1(X, \mathcal{O}_X(n)) = -n(-n-1)/2$ *if* $n < 0$.

Proof. If $\mathfrak{m} = (x, y)$, then X is covered by the spectra of the affine blowup algebras $A[\frac{\mathfrak{m}}{x}]$ and $A[\frac{\mathfrak{m}}{y}]$ because x and y placed in degree 1 generate the Rees algebra $\bigoplus \mathfrak{m}^n$ over A. See Divisors, Lemma 31.32.2 and Constructions, Lemma 27.8.9. Since X is separated by Constructions, Lemma 27.8.8 we see that cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology of Schemes, Lemma 30.4.2.

Let $i : E \to X$ be the exceptional divisor, see Divisors, Definition 31.32.1. Recall that $\mathcal{O}_X(-E) = \mathcal{O}_X(1)$ is f-relatively ample, see Divisors, Lemma 31.32.4. Hence we know that $H^1(X, \mathcal{O}_X(-nE)) = 0$ for some $n > 0$, see Cohomology of Schemes, Lemma 30.16.2. Consider the filtration

$$\mathcal{O}_X(-nE) \subset \mathcal{O}_X(-(n-1)E) \subset \ldots \subset \mathcal{O}_X(-E) \subset \mathcal{O}_X \subset \mathcal{O}_X(E)$$

The successive quotients are the sheaves

$$\mathcal{O}_X(-tE)/\mathcal{O}_X(-(t+1)E) = \mathcal{O}_X(t)/\mathcal{I}(t) = i_*\mathcal{O}_E(t)$$

where $\mathcal{I} = \mathcal{O}_X(-E)$ is the ideal sheaf of E. By Lemma 54.3.1 we have $E = \mathbf{P}^1_\kappa$ and $\mathcal{O}_E(1)$ indeed corresponds to the usual Serre twist of the structure sheaf on $\mathbf{P}^1$. Hence the cohomology of $\mathcal{O}_E(t)$ vanishes in degree 1 for $t \geq -1$, see Cohomology of Schemes, Lemma 30.8.1. Since this is equal to $H^1(X, i_*\mathcal{O}_E(t))$ (by Cohomology of Schemes, Lemma 30.2.4) we find that $H^1(X, \mathcal{O}_X(-(t+1)E)) \to H^1(X, \mathcal{O}_X(-tE))$ is surjective for $t \geq -1$. Hence

$$0 = H^1(X, \mathcal{O}_X(-nE)) \longrightarrow H^1(X, \mathcal{O}_X(-tE)) = H^1(X, \mathcal{O}_X(t))$$

is surjective for $t \geq -1$ which proves (2).

Let $\mathcal{F}$ be globally generated. This means there exists a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X \to \mathcal{F} \to 0$$

Note that $H^1(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} H^1(X, \mathcal{O}_X)$ by Cohomology, Lemma 20.19.1. By part (2) we have $H^1(X, \mathcal{O}_X) = 0$. If $\mathcal{F}(1)$ is globally generated, then we can find a surjection $\bigoplus_{i \in I} \mathcal{O}_X(-1) \to \mathcal{F}$ and argue in a similar fashion. In other words, part (3) follows from part (2).

For part (4) we note that for all n large enough we have $\Gamma(X, \mathcal{O}_X(n)) = \mathfrak{m}^n$, see Cohomology of Schemes, Lemma 30.14.3. If $n \geq 0$, then we can use the short exact sequence

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

and the vanishing of H^1 for the sheaf on the left to get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}^{\max(0,n)} & \longrightarrow & \mathfrak{m}^{\max(0,n-1)} & \longrightarrow & \mathfrak{m}^{\max(0,n)}/\mathfrak{m}^{\max(0,n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{O}_X(n)) & \longrightarrow & \Gamma(X, \mathcal{O}_X(n-1)) & \longrightarrow & \Gamma(E, \mathcal{O}_E(n-1)) & \longrightarrow & 0
\end{array}$$

with exact rows. In fact, the rows are exact also for $n < 0$ because in this case the groups on the right are zero. In the proof of Lemma 54.3.1 we have seen that the right vertical arrow is an isomorphism (details omitted). Hence if the left vertical arrow is an isomorphism, so is the middle one. In this way we see that (4) holds by descending induction on n.

Finally, we prove (5) by descending induction on n and the sequences

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

Namely, for $n \geq -1$ we already know $H^1(X, \mathcal{O}_X(n)) = 0$. Since

$$H^1(X, i_*\mathcal{O}_E(-2)) = H^1(E, \mathcal{O}_E(-2)) = H^1(\mathbf{P}^1_\kappa, \mathcal{O}(-2)) \cong \kappa$$

by Cohomology of Schemes, Lemma 30.8.1 which has length 1 as an A-module, we conclude from the long exact cohomology sequence that (5) holds for $n = -2$. And so on and so forth. □

0AGT **Lemma 54.3.5.** *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ be an ideal. Let $d \geq 0$ be the largest integer such that*

$$I\mathcal{O}_X \subset \mathcal{O}_X(-dE)$$

where E is the exceptional divisor. Set $\mathcal{I}' = I\mathcal{O}_X(dE) \subset \mathcal{O}_X$. Then $d > 0$, the sheaf $\mathcal{O}_X/\mathcal{I}'$ is supported in finitely many closed points $x_1, \ldots, x_r$ of X, and

$$\begin{aligned} \mathit{length}_A(A/I) &> \mathit{length}_A\Gamma(X, \mathcal{O}_X/\mathcal{I}') \\ &\geq \sum\nolimits_{i=1,\ldots,r} \mathit{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \end{aligned}$$

Proof. Since $I \subset \mathfrak{m}$ we see that every element of I vanishes on E. Thus we see that $d \geq 1$. On the other hand, since $\mathfrak{m}^n \subset I$ we see that $d \leq n$. Consider the short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to 0$$

Since $I\mathcal{O}_X$ is globally generated, we see that $H^1(X, I\mathcal{O}_X) = 0$ by Lemma 54.3.4. Hence we obtain a surjection $A/I \to \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-dE)/I\mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-dE) \to 0$$

By Divisors, Lemma 31.15.8 we see that $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ is supported in finitely many closed points of X. In particular, this coherent sheaf has vanishing higher cohomology groups (detail omitted). Thus in the following diagram

$$\begin{array}{ccccccccc} & & & & A/I & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) & \longrightarrow & \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X) & \longrightarrow & \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE)) & \longrightarrow & 0 \end{array}$$

the bottom row is exact and the vertical arrow surjective. We have

$$\mathrm{length}_A\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) < \mathrm{length}_A(A/I)$$

since $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE))$ is nonzero. Namely, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ is nonzero as $d > 0$.

To finish the proof we translate the results above into the statements of the lemma. Since $\mathcal{O}_X(dE)$ is invertible we have

$$\mathcal{O}_X/\mathcal{I}' = \mathcal{O}_X(-dE)/I\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(dE).$$

Thus $\mathcal{O}_X/\mathcal{I}'$ and $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ are supported in the same set of finitely many closed points, say $x_1, \ldots, x_r \in E \subset X$. Moreover we obtain

$$\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) = \bigoplus \mathcal{O}_X(-dE)_{x_i}/I\mathcal{O}_{X,x_i} \cong \bigoplus \mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i} = \Gamma(X, \mathcal{O}_X/\mathcal{I}')$$

because an invertible module over a local ring is trivial. Thus we obtain the strict inequality. We also get the second because

$$\text{length}_A(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \geq \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i})$$

as is immediate from the definition of length. □

0B4L **Lemma 54.3.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then $\Omega_{X/S} = i_*\Omega_{E/\kappa}$, where $i : E \to X$ is the immersion of the exceptional divisor.*

Proof. Writing $\mathbf{P}^1 = \mathbf{P}^1_S$, let $r : X \to \mathbf{P}^1$ be as in Lemma 54.3.1. Then we have an exact sequence

$$\mathcal{C}_{X/\mathbf{P}^1} \to r^*\Omega_{\mathbf{P}^1/S} \to \Omega_{X/S} \to 0$$

see Morphisms, Lemma 29.32.15. Since $\Omega_{\mathbf{P}^1/S}|_E = \Omega_{E/\kappa}$ by Morphisms, Lemma 29.32.10 it suffices to see that the first arrow defines a surjection onto the kernel of the canonical map $r^*\Omega_{\mathbf{P}^1/S} \to i_*\Omega_{E/\kappa}$. This we can do locally. With notation as in the proof of Lemma 54.3.1 on an affine open of X the morphism f corresponds to the ring map

$$A \to A[t]/(xt - y)$$

where $x, y \in \mathfrak{m}$ are generators. Thus $\text{d}(xt - y) = x\text{d}t$ and $y\text{d}t = t \cdot x\text{d}t$ which proves what we want. □

54.4. Dominating by quadratic transformations

0BFS Using the result above we can prove that blowups in points dominate any modification of a regular 2 dimensional scheme.

Let X be a scheme. Let $x \in X$ be a closed point. As usual, we view $i : x = \text{Spec}(\kappa(x)) \to X$ as a closed subscheme. The *blowing up $X' \to X$ of X at x* is the blowing up of X in the closed subscheme $x \subset X$. Observe that if X is locally Noetherian, then $X' \to X$ is projective (in particular proper) by Divisors, Lemma 31.32.13.

0AHH **Lemma 54.4.1.** *Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is regular of dimension 2 for $x \in T$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Denote I_i the stalk of $\mathcal{I}$ at x_i. Set

$$n_i = \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/I_i)$$

This is finite as $\mathcal{O}_X/\mathcal{I}$ is supported on T and hence $\mathcal{O}_{X,x_i}/I_i$ has support equal to $\{\mathfrak{m}_{x_i}\}$ (see Algebra, Lemma 10.62.3). We are going to use induction on $\sum n_i$. If $n_i = 0$ for all i, then $\mathcal{I} = \mathcal{O}_X$ and we are done.

Suppose $n_i > 0$. Let $X' \to X$ be the blowing up of X in x_i (see discussion above the lemma). Since $\text{Spec}(\mathcal{O}_{X,x_i}) \to X$ is flat we see that $X' \times_X \text{Spec}(\mathcal{O}_{X,x_i})$ is the

blowup of the ring $\mathcal{O}_{X,x_i}$ in the maximal ideal, see Divisors, Lemma 31.32.3. Hence the square in the commutative diagram

$$\begin{array}{ccc} \mathrm{Proj}(\bigoplus_{d\geq 0} \mathfrak{m}_{x_i}^d) & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{X,x_i}) & \longrightarrow & X \end{array}$$

is cartesian. Let $E \subset X'$ and $E' \subset \mathrm{Proj}(\bigoplus_{d\geq 0} \mathfrak{m}_{x_i}^d)$ be the exceptional divisors. Let $d \geq 1$ be the integer found in Lemma 54.3.5 for the ideal $\mathcal{I}_i \subset \mathcal{O}_{X,x_i}$. Since the horizontal arrows in the diagram are flat, since $E' \to E$ is surjective, and since E' is the pullback of E, we see that

$$\mathcal{I}\mathcal{O}_{X'} \subset \mathcal{O}_{X'}(-dE)$$

(some details omitted). Set $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}(dE) \subset \mathcal{O}_{X'}$. Then we see that $\mathcal{O}_{X'}/\mathcal{I}'$ is supported in finitely many closed points $T' \subset |X'|$ because this holds over $X \setminus \{x_i\}$ and for the pullback to $\mathrm{Proj}(\bigoplus_{d\geq 0} \mathfrak{m}_{x_i}^d)$. The final assertion of Lemma 54.3.5 tells us that the sum of the lengths of the stalks $\mathcal{O}_{X',x'}/\mathcal{I}'\mathcal{O}_{X',x'}$ for x' lying over x_i is $< n_i$. Hence the sum of the lengths has decreased.

By induction hypothesis, there exists a sequence

$$X'_n \to \ldots \to X'_1 \to X'$$

of blowups at closed points lying over T' such that $\mathcal{I}'\mathcal{O}_{X'_n}$ is invertible. Since $\mathcal{I}'\mathcal{O}_{X'}(-dE) = \mathcal{I}\mathcal{O}_{X'}$, we see that $\mathcal{I}\mathcal{O}_{X'_n} = \mathcal{I}'\mathcal{O}_{X'_n}(-d(f')^{-1}E)$ where $f' : X'_n \to X'$ is the composition. Note that $(f')^{-1}E$ is an effective Cartier divisor by Divisors, Lemma 31.32.11. Thus we are done by Divisors, Lemma 31.13.7. □

0AHI **Lemma 54.4.2.** *Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. Let $f : Y \to X$ be a proper morphism of schemes which is an isomorphism over $U = X \setminus T$. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Flatness, Lemma 38.31.4 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 54.4.1 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors, Lemma 31.32.5) we find the desired factorization. □

0C5H **Lemma 54.4.3.** *Let S be a scheme. Let X be a scheme over S which is regular and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point not lying over U and f_n and f agree.

Proof. We may assume U contains every point of codimension 1, see Morphisms, Lemma 29.42.5. Hence the complement $T \subset X$ of U is a finite set of closed points whose local rings are regular of dimension 2. Applying Divisors, Lemma 31.36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 54.4.2 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. □

54.5. Dominating by normalized blowups

0BBR In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

0BBS **Definition 54.5.1.** Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $x \in X$ be a closed point. The *normalized blowup of X at x* is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X in x and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the scheme X' has an open covering by opens which have finitely many irreducible components by Divisors, Lemma 31.32.10. See Morphisms, Definition 29.54.1 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that a scheme is Nagata if it has an open covering by affines which are spectra of Nagata rings (Properties, Definition 28.13.1).

0BFT **Lemma 54.5.2.** *In Definition 54.5.1 if X is Nagata, then the normalized blowing up of X at x is normal, Nagata, and proper over X.*

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors, Lemma 31.32.13). Thus X' is Nagata (Morphisms, Lemma 29.18.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms, Lemma 29.54.11) and we conclude that $X'' \to X$ is proper as well (Morphisms, Lemmas 29.44.11 and 29.41.4). It follows that the normalized blowing up is a normal (Morphisms, Lemma 29.54.5) Nagata algebraic space. □

In the following lemma we need to assume X is Noetherian in order to make sure that it has finitely many irreducible components. Then the properness of $f : Y \to X$ assures that Y has finitely many irreducible components too and it makes sense to require f to be birational (Morphisms, Definition 29.50.1).

0BBT **Lemma 54.5.3.** *Let X be a scheme which is Noetherian, Nagata, and has dimension 2. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} X_n & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_1 \longrightarrow X_0 \\ \downarrow & & & & & & \downarrow \\ Y & & & & & & \longrightarrow X \end{array}$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. We will use the results of Morphisms, Sections 29.18, 29.52, and 29.54 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Varieties, Lemma 33.17.3. Hence there is a finite set of closed points $T \subset X$ such that f is an isomorphism over $X \setminus T$. For each $x \in T$ the fibre Y_x is a proper geometrically connected scheme of dimension 1 over $\kappa(x)$, see More on Morphisms, Lemma 37.53.6. Thus

$$BadCurves(f) = \{C \subset Y \text{ closed} \mid \dim(C) = 1, f(C) = \text{a point}\}$$

is a finite set. We will prove the lemma by induction on the number of elements of $BadCurves(f)$. The base case is the case where $BadCurves(f)$ is empty, and in that case f is an isomorphism.

Fix $x \in T$. Let $X' \to X$ be the normalized blowup of X at x and let Y' be the normalization of $Y \times_X X'$. Picture

$$\begin{array}{ccc} Y' & \xrightarrow[f']{} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Let $x' \in X'$ be a closed point lying over x such that the fibre $Y'_{x'}$ has dimension ≥ 1. Let $C' \subset Y'$ be an irreducible component of $Y'_{x'}$, i.e., $C' \in BadCurves(f')$. Since $Y' \to Y \times_X X'$ is finite we see that C' must map to an irreducible component $C \subset Y_x$. If is clear that $C \in BadCurves(f)$. Since $Y' \to Y$ is birational and hence an isomorphism over points of codimension 1 in Y, we see that we obtain an injective map

$$BadCurves(f') \longrightarrow BadCurves(f)$$

Thus it suffices to show that after a finite number of these normalized blowups we get rid at of at least one of the bad curves, i.e., the displayed map is not surjective.

We will get rid of a bad curve using an argument due to Zariski. Pick $C \in BadCurves(f)$ lying over our x. Denote $\mathcal{O}_{Y,C}$ the local ring of Y at the generic point of C. Choose an element $u \in \mathcal{O}_{X,C}$ whose image in the residue field $R(C)$ is transcendental over $\kappa(x)$ (we can do this because $R(C)$ has transcendence degree 1 over $\kappa(x)$ by Varieties, Lemma 33.20.3). We can write $u = a/b$ with $a, b \in \mathcal{O}_{X,x}$ as $\mathcal{O}_{Y,C}$ and $\mathcal{O}_{X,x}$ have the same fraction fields. By our choice of u it must be the case that $a, b \in \mathfrak{m}_x$. Hence

$$N_{u,a,b} = \min\{\text{ord}_{\mathcal{O}_{Y,C}}(a), \text{ord}_{\mathcal{O}_{Y,C}}(b)\} > 0$$

Thus we can do descending induction on this integer. Let $X' \to X$ be the normalized blowing up of x and let Y' be the normalization of $X' \times_X Y$ as above. We will show that if C is the image of some bad curve $C' \subset Y'$ lying over $x' \in X'$, then there exists a choice of $a', b' \mathcal{O}_{X',x'}$ such that $N_{u,a',b'} < N_{u,a,b}$. This will finish the proof. Namely, since $X' \to X$ factors through the blowing up, we see that there exists a nonzero element $d \in \mathfrak{m}_{x'}$ such that $a = a'd$ and $b = b'd$ (namely, take d to be the local equation for the exceptional divisor of the blowup). Since $Y' \to Y$

is an isomorphism over an open containing the generic point of C (seen above) we see that $\mathcal{O}_{Y',C'} = \mathcal{O}_{Y,C}$. Hence

$$\mathrm{ord}_{\mathcal{O}_{Y,C}}(a) = \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a'd) = \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a') + \mathrm{ord}_{\mathcal{O}_{Y',C'}}(d) > \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a')$$

Similarly for b and the proof is complete. □

0C5I **Lemma 54.5.4.** *Let S be a scheme. Let X be a scheme over S which is Noetherian, Nagata, and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 \to X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_0 \to X$ is the normalization, $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point, and f_n and f agree.

Proof. Applying Divisors, Lemma 31.36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 54.5.3 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. □

54.6. Modifying over local rings

0AE1 Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points. Assume that the open embedding

$$U = S \setminus \{s_1, \ldots, s_n\} \longrightarrow S$$

is quasi-compact. Denote $FP_{S,\{s_1,\ldots,s_n\}}$ the category of morphisms $f : X \to S$ of finite presentation which induce an isomorphism $f^{-1}(U) \to U$. Morphisms are morphisms of schemes over S. For each i set $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and let $V_i = S_i \setminus \{s_i\}$. Denote FP_{S_i,s_i} the category of morphisms $g_i : Y_i \to S_i$ of finite presentation which induce an isomorphism $g_i^{-1}(V_i) \to V_i$. Morphisms are morphisms over S_i. Base change defines an functor

0BFU (54.6.0.1) $$F : FP_{S,\{s_1,\ldots,s_n\}} \longrightarrow FP_{S_1,s_1} \times \ldots \times FP_{S_n,s_n}$$

To reduce at least some of the problems in this chapter to the case of local rings we have the following lemma.

0BFV **Lemma 54.6.1.** *The functor F (54.6.0.1) is an equivalence.*

Proof. For $n = 1$ this is Limits, Lemma 32.21.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to S_i$ are objects of FP_{S_i,s_i}. Then by the case $n = 1$ we can find $f_i' : X_i' \to S$ of finite presentation which are isomorphisms over $S \setminus \{s_i\}$ and whose base change to S_i is g_i. Then we can set

$$f : X = X_1' \times_S \ldots \times_S X_n' \to S$$

This is an object of $FP_{S,\{s_1,\ldots,s_n\}}$ whose base change by $S_i \to S$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. □

0BFW **Lemma 54.6.2.** *Let S, s_i, S_i be as in (54.6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then f is separated, proper, finite, if and only if g_i is so for $i = 1, \ldots, n$.*

Proof. Follows from Limits, Lemma 32.21.2. □

0BFX **Lemma 54.6.3.** *Let S, s_i, S_i be as in (54.6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then $X_{s_i} \cong (Y_i)_{s_i}$ as schemes over $\kappa(s_i)$.*

Proof. This is clear. □

0BFY **Lemma 54.6.4.** *Let S, s_i, S_i be as in (54.6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Then there exists a factorization*

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{s_1, \ldots, s_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. Let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$. The first morphism $Z_1 \to S$ is given by blowing up one of the s_i, say s_1. Applying F to $Z_1 \to S$ we find a blowup $Z_{1,1} \to S_1$ at s_1 is the blowing up at s_1 and otherwise $Z_{i,0} = S_i$ for $i > 1$. In the next step, we either blow up one of the s_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to S$ over s_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{s_1} \cong (Z_{1,1})_{s_1}$ (Lemma 54.6.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$ we construct the sequence of blowing ups of S in exactly the same manner. □

Here is the analogue of Lemma 54.6.4 for normalized blowups.

0BFZ **Lemma 54.6.5.** *Let S, s_i, S_i be as in (54.6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Assume every quasi-compact open of S has finitely many irreducible components. Then there exists a factorization*

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. The assumption on S is used to assure us (successively) that the schemes we are normalizing have locally finitely many irreducible components so that the statement makes sense. Having said this the lemma follows by the exact same argument as used to prove Lemma 54.6.4. □

54.7. Vanishing

0AX7 In this section we will often work in the following setting. Recall that a modification is a proper birational morphism between integral schemes (Morphisms, Definition 29.51.11).

0AX8 **Situation 54.7.1.** Here $(A, \mathfrak{m}, \kappa)$ be a local Noetherian normal domain of dimension 2. Let s be the closed point of $S = \mathrm{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

By Varieties, Lemma 33.17.3 the morphism f defines an isomorphism $f^{-1}(U) \to U$. The special fibre X_s is proper over $\mathrm{Spec}(\kappa)$ and has dimension at most 1 by Varieties, Lemma 33.19.3. By Stein factorization (More on Morphisms, Lemma 37.53.6) we have $f_*\mathcal{O}_X = \mathcal{O}_S$ and the special fibre X_s is geometrically connected over κ. If X_s has dimension 0, then f is finite (More on Morphisms, Lemma 37.44.2) and hence an isomorphism (Morphisms, Lemma 29.54.8). We will discard this uninteresting case and we conclude that $\dim(C_i) = 1$ for $i = 1, \ldots, r$.

0B4M **Lemma 54.7.2.** *In Situation 54.7.1 there exists a U-admissible blowup $X' \to S$ which dominates X.*

Proof. This is a special case of More on Flatness, Lemma 38.31.4. □

0AX9 **Lemma 54.7.3.** *In Situation 54.7.1 there exists a nonzero $f \in \mathfrak{m}$ such that for every $i = 1, \ldots, r$ there exist*

(1) *a closed point $x_i \in C_i$ with $x_i \notin C_j$ for $j \neq i$,*
(2) *a factorization $f = g_i f_i$ of f in $\mathcal{O}_{X,x_i}$ such that $g_i \in \mathfrak{m}_{x_i}$ maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$.*

Proof. We will use the observations made following Situation 54.7.1 without further mention. Pick a closed point $x_i \in C_i$ which is not in C_j for $j \neq i$. Pick $g_i \in \mathfrak{m}_{x_i}$ which maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$. Since the fraction field of A is the fraction field of $\mathcal{O}_{X_i,x_i}$ we can write $g_i = a_i/b_i$ for some $a_i, b_i \in A$. Take $f = \prod a_i$. □

0AXA **Lemma 54.7.4.** *In Situation 54.7.1 assume X is normal. Let $Z \subset X$ be a nonempty effective Cartier divisor such that $Z \subset X_s$ set theoretically. Then the conormal sheaf of Z is not trivial. More precisely, there exists an i such that $C_i \subset Z$ and $\deg(\mathcal{C}_{Z/X}|_{C_i}) > 0$.*

Proof. We will use the observations made following Situation 54.7.1 without further mention. Let f be a function as in Lemma 54.7.3. Let $\xi_i \in C_i$ be the generic point. Let $\mathcal{O}_i$ be the local ring of X at ξ_i. Then $\mathcal{O}_i$ is a discrete valuation ring. Let e_i be the valuation of f in $\mathcal{O}_i$, so $e_i > 0$. Let $h_i \in \mathcal{O}_i$ be a local equation for Z and let d_i be its valuation. Then $d_i \geq 0$. Choose and fix i with d_i/e_i maximal (then $d_i > 0$ as Z is not empty). Replace f by f^{d_i} and Z by $e_i Z$. This is permissible, by the relation $\mathcal{O}_X(e_i Z) = \mathcal{O}_X(Z)^{\otimes e_i}$, the relation between the conormal sheaf and $\mathcal{O}_X(Z)$ (see Divisors, Lemmas 31.14.4 and 31.14.2, and since the degree gets multiplied by e_i, see Varieties, Lemma 33.44.7. Let $\mathcal{I}$ be the ideal sheaf of Z so that $\mathcal{C}_{Z/X} = \mathcal{I}|_Z$. Consider the image $\overline{f}$ of f in $\Gamma(Z, \mathcal{O}_Z)$. By our choices above we see that $\overline{f}$ vanishes in the generic points of irreducible components of Z (these are all generic points of C_j as Z is contained in the special fibre). On the other hand, Z is

(S_1) by Divisors, Lemma 31.15.6. Thus the scheme Z has no embedded associated points and we conclude that $\overline{f} = 0$ (Divisors, Lemmas 31.4.3 and 31.5.6). Hence f is a global section of $\mathcal{I}$ which generates $\mathcal{I}_{\xi_i}$ by construction. Thus the image s_i of f in $\Gamma(C_i, \mathcal{I}|_{C_i})$ is nonzero. However, our choice of f guarantees that s_i has a zero at x_i. Hence the degree of $\mathcal{I}|_{C_i}$ is > 0 by Varieties, Lemma 33.44.12. □

0AXB **Lemma 54.7.5.** *In Situation 54.7.1 assume X is normal and A Nagata. The map*

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(f^{-1}(U), \mathcal{O}_X)$$

is injective.

Proof. Let $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0$ be the extension corresponding to a nontrivial element ξ of $H^1(X, \mathcal{O}_X)$ (Cohomology, Lemma 20.5.1). Let $\pi : P = \mathbf{P}(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. The surjection $\mathcal{E} \to \mathcal{O}_X$ defines a section $\sigma : X \to P$ whose conormal sheaf is isomorphic to $\mathcal{O}_X$ (Divisors, Lemma 31.31.6). If the restriction of ξ to $f^{-1}(U)$ is trivial, then we get a map $\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$ splitting the injection $\mathcal{O}_X \to \mathcal{E}$. This defines a second section $\sigma' : f^{-1}(U) \to P$ disjoint from σ. Since ξ is nontrivial we conclude that σ' cannot extend to all of X and be disjoint from σ. Let $X' \subset P$ be the scheme theoretic image of σ' (Morphisms, Definition 29.6.2). Picture

$$\begin{array}{ccc} & & X' \longrightarrow P \\ & \overset{\sigma'}{\nearrow} & \overset{g}{\searrow} \quad \pi \downarrow \uparrow \sigma \\ f^{-1}(U) & \longrightarrow & X \end{array}$$

The morphism $P \backslash \sigma(X) \to X$ is affine. If $X' \cap \sigma(X) = \emptyset$, then $X' \to X$ is both affine and proper, hence finite (Morphisms, Lemma 29.44.11), hence an isomorphism (as X is normal, see Morphisms, Lemma 29.54.8). This is impossible as mentioned above.

Let X^ν be the normalization of X'. Since A is Nagata, we see that $X^\nu \to X'$ is finite (Morphisms, Lemmas 29.54.11 and 29.18.2). Let $Z \subset X^\nu$ be the pullback of the effective Cartier divisor $\sigma(X) \subset P$. By the above we see that Z is not empty and is contained in the closed fibre of $X^\nu \to S$. Since $P \to X$ is smooth, we see that $\sigma(X)$ is an effective Cartier divisor (Divisors, Lemma 31.22.8). Hence $Z \subset X^\nu$ is an effective Cartier divisor too. Since the conormal sheaf of $\sigma(X)$ in P is $\mathcal{O}_X$, the conormal sheaf of Z in X^ν (which is a priori invertible) is $\mathcal{O}_Z$ by Morphisms, Lemma 29.31.4. This is impossible by Lemma 54.7.4 and the proof is complete. □

0AXC **Lemma 54.7.6.** *In Situation 54.7.1 assume X is normal and A Nagata. Then*

$$\mathrm{Hom}_{D(A)}(\kappa[-1], Rf_*\mathcal{O}_X)$$

is zero. This uses $D(A) = D_{QCoh}(\mathcal{O}_S)$ to think of $Rf_*\mathcal{O}_X$ as an object of $D(A)$.

Proof. By adjointness of Rf_* and Lf^* such a map is the same thing as a map $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$. Note that

$$H^i(Lf^*\kappa[-1]) = \begin{cases} 0 & \text{if} \quad i > 1 \\ \mathcal{O}_{X_s} & \text{if} \quad i = 1 \\ \text{some } \mathcal{O}_{X_s}\text{-module} & \text{if} \quad i \leq 0 \end{cases}$$

Since $\mathrm{Hom}(H^0(Lf^*\kappa[-1]), \mathcal{O}_X) = 0$ as $\mathcal{O}_X$ is torsion free, the spectral sequence for Ext (Cohomology on Sites, Example 21.32.1) implies that $\mathrm{Hom}_{D(\mathcal{O}_X)}(Lf^*\kappa[-1], \mathcal{O}_X)$ is equal to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_s}, \mathcal{O}_X)$. We conclude that $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$ is given by an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_{X_s} \to 0$$

By Lemma 54.7.5 the pullback of this extension via the surjection $\mathcal{O}_X \to \mathcal{O}_{X_s}$ is zero (since this pullback is clearly split over $f^{-1}(U)$). Thus $1 \in \mathcal{O}_{X_s}$ lifts to a global section s of $\mathcal{E}$. Multiplying s by the ideal sheaf $\mathcal{I}$ of X_s we obtain an $\mathcal{O}_X$-module map $c_s : \mathcal{I} \to \mathcal{O}_X$. Applying f_* we obtain an A-linear map $f_*c_s : \mathfrak{m} \to A$. Since A is a Noetherian normal local domain this map is given by multiplication by an element $a \in A$. Changing s into $s - a$ we find that s is annihilated by $\mathcal{I}$ and the extension is trivial as desired. □

0B4R **Remark 54.7.7.** Let X be an integral Noetherian normal scheme of dimension 2. In this case the following are equivalent

(1) X has a dualizing complex $\omega_X^\bullet$,
(2) there is a coherent $\mathcal{O}_X$-module ω_X such that $\omega_X[n]$ is a dualizing complex, where n can be any integer.

This follows from the fact that X is Cohen-Macaulay (Properties, Lemma 28.12.7) and Duality for Schemes, Lemma 48.23.1. In this situation we will say that ω_X is a *dualizing module* in accordance with Duality for Schemes, Section 48.22. In particular, when A is a Noetherian normal local domain of dimension 2, then we say A *has a dualizing module* ω_A if the above is true. In this case, if $X \to \mathrm{Spec}(A)$ is a normal modification, then X has a dualizing module too, see Duality for Schemes, Example 48.22.1. In this situation we always denote ω_X the dualizing module normalized with respect to ω_A, i.e., such that $\omega_X[2]$ is the dualizing complex normalized relative to $\omega_A[2]$. See Duality for Schemes, Section 48.20.

The Grauert-Riemenschneider vanishing of the next proposition is a formal consequence of Lemma 54.7.6 and the general theory of duality.

0AXD **Proposition 54.7.8** (Grauert-Riemenschneider)**.** *In Situation 54.7.1 assume*

(1) *X is a normal scheme,*
(2) *A is Nagata and has a dualizing complex $\omega_A^\bullet$.*

Let ω_X be the dualizing module of X (Remark 54.7.7). Then $R^1 f_*\omega_X = 0$.

Proof. In this proof we will use the identification $D(A) = D_{QCoh}(\mathcal{O}_S)$ to identify quasi-coherent $\mathcal{O}_S$-modules with A-modules. Moreover, we may assume that $\omega_A^\bullet$ is normalized, see Dualizing Complexes, Section 47.16. Since X is a Noetherian normal 2-dimensional scheme it is Cohen-Macaulay (Properties, Lemma 28.12.7). Thus $\omega_X^\bullet = \omega_X[2]$ (Duality for Schemes, Lemma 48.23.1 and the normalization in Duality for Schemes, Example 48.22.1). If the proposition is false, then we can find a nonzero map $R^1 f_*\omega_X \to \kappa$. In other words we obtain a nonzero map $\alpha : Rf_*\omega_X^\bullet \to \kappa[1]$. Applying $R\,\mathrm{Hom}_A(-, \omega_A^\bullet)$ we get a nonzero map

$$\beta : \kappa[-1] \longrightarrow Rf_*\mathcal{O}_X$$

which is impossible by Lemma 54.7.6. To see that $R\,\mathrm{Hom}_A(-, \omega_A^\bullet)$ does what we said, first note that

$$R\,\mathrm{Hom}_A(\kappa[1], \omega_A^\bullet) = R\,\mathrm{Hom}_A(\kappa, \omega_A^\bullet)[-1] = \kappa[-1]$$

as $\omega_A^\bullet$ is normalized and we have

$$R\operatorname{Hom}_A(Rf_*\omega_X^\bullet, \omega_A^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\omega_X^\bullet, \omega_X^\bullet) = Rf_*\mathcal{O}_X$$

The first equality by Duality for Schemes, Example 48.3.9 and the fact that $\omega_X^\bullet = f^!\omega_A^\bullet$ by construction, and the second equality because $\omega_X^\bullet$ is a dualizing complex for X (which goes back to Duality for Schemes, Lemma 48.17.7). □

54.8. Boundedness

0AXE In this section we begin the discussion which will lead to a reduction to the case of rational singularities for 2-dimensional schemes.

0AXF **Lemma 54.8.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f} & \\ & \operatorname{Spec}(A) & \end{array}$$

where f and f' are modifications as in Situation 54.7.1 and X normal. Then we have a short exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}) \to H^0(X, R^1g_*\mathcal{O}_{X'}) \to 0$$

Also $\dim(\mathit{Supp}(R^1g_*\mathcal{O}_{X'})) = 0$ and $R^1g_*\mathcal{O}_{X'}$ is generated by global sections.

Proof. We will use the observations made following Situation 54.7.1 without further mention. As X is normal and g is dominant and birational, we have $g_*\mathcal{O}_{X'} = \mathcal{O}_X$, see for example More on Morphisms, Lemma 37.53.6. Since the fibres of g have dimension ≤ 1, we have $R^pg_*\mathcal{O}_{X'} = 0$ for $p > 1$, see for example Cohomology of Schemes, Lemma 30.20.9. The support of $R^1g_*\mathcal{O}_{X'}$ is contained in the set of points of X where the fibres of g' have dimension ≥ 1. Thus it is contained in the set of images of those irreducible components $C' \subset X'_s$ which map to points of X_s which is a finite set of closed points (recall that $X'_s \to X_s$ is a morphism of proper 1-dimensional schemes over κ). Then $R^1g_*\mathcal{O}_{X'}$ is globally generated by Cohomology of Schemes, Lemma 30.9.10. Using the morphism $f : X \to S$ and the references above we find that $H^p(X, \mathcal{F}) = 0$ for $p > 1$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence the short exact sequence of the lemma is a consequence of the Leray spectral sequence for g and $\mathcal{O}_{X'}$, see Cohomology, Lemma 20.13.4. □

0AXJ **Lemma 54.8.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. Let $a \in A$ be nonzero. There exists an integer N such that for every modification $f : X \to \operatorname{Spec}(A)$ with X normal the A-module*

$$M_{X,a} = \operatorname{Coker}(A \longrightarrow H^0(Z, \mathcal{O}_Z))$$

where $Z \subset X$ is cut out by a has length bounded by N.

Proof. By the short exact sequence $0 \to \mathcal{O}_X \xrightarrow{a} \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we see that

0AXK (54.8.2.1) $$M_{X,a} = H^1(X, \mathcal{O}_X)[a]$$

Here $N[a] = \{n \in N \mid an = 0\}$ for an A-module N. Thus if a divides b, then $M_{X,a} \subset M_{X,b}$. Suppose that for some $c \in A$ the modules $M_{X,c}$ have bounded length. Then for every X we have an exact sequence

$$0 \to M_{X,c} \to M_{X,c^2} \to M_{X,c}$$

where the second arrow is given by multiplication by c. Hence we see that M_{X,c^2} has bounded length as well. Thus it suffices to find a $c \in A$ for which the lemma is true such that a divides c^n for some $n > 0$. By More on Algebra, Lemma 15.126.6 we may assume $A/(a)$ is a reduced ring.

Assume that $A/(a)$ is reduced. Let $A/(a) \subset B$ be the normalization of $A/(a)$ in its quotient ring. Because A is Nagata, we see that $\mathrm{Coker}(A \to B)$ is finite. We claim the length of this finite module is a bound. To see this, consider $f : X \to \mathrm{Spec}(A)$ as in the lemma and let $Z' \subset Z$ be the scheme theoretic closure of $Z \cap f^{-1}(U)$. Then $Z' \to \mathrm{Spec}(A/(a))$ is finite for example by Varieties, Lemma 33.17.2. Hence $Z' = \mathrm{Spec}(B')$ with $A/(a) \subset B' \subset B$. On the other hand, we claim the map

$$H^0(Z, \mathcal{O}_Z) \to H^0(Z', \mathcal{O}_{Z'})$$

is injective. Namely, if $s \in H^0(Z, \mathcal{O}_Z)$ is in the kernel, then the restriction of s to $f^{-1}(U) \cap Z$ is zero. Hence the image of s in $H^1(X, \mathcal{O}_X)$ vanishes in $H^1(f^{-1}(U), \mathcal{O}_X)$. By Lemma 54.7.5 we see that s comes from an element $\tilde{s}$ of A. But by assumption $\tilde{s}$ maps to zero in B' which implies that $s = 0$. Putting everything together we see that $M_{X,a}$ is a subquotient of B'/A, namely not every element of B' extends to a global section of $\mathcal{O}_Z$, but in any case the length of $M_{X,a}$ is bounded by the length of B/A. □

In some cases, resolution of singularities reduces to the case of rational singularities.

0B4N **Definition 54.8.3.** Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2.

(1) We say A *defines a rational singularity* if for every normal modification $X \to \mathrm{Spec}(A)$ we have $H^1(X, \mathcal{O}_X) = 0$.
(2) We say that *reduction to rational singularities is possible for* A if the length of the A-modules

$$H^1(X, \mathcal{O}_X)$$

is bounded for all modifications $X \to \mathrm{Spec}(A)$ with X normal.

The meaning of the language in (2) is explained by Lemma 54.8.5. The following lemma says roughly speaking that local rings of modifications of $\mathrm{Spec}(A)$ with A defining a rational singularity also define rational singularities.

0BG0 **Lemma 54.8.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let $A \subset B$ be a local extension of domains with the same fraction field which is essentially of finite type such that $\dim(B) = 2$ and B normal. Then B defines a rational singularity.*

Proof. Choose a finite type A-algebra C such that $B = C_{\mathfrak{q}}$ for some prime $\mathfrak{q} \subset C$. After replacing C by the image of C in B we may assume that C is a domain with fraction field equal to the fraction field of A. Then we can choose a closed immersion $\mathrm{Spec}(C) \to \mathbf{A}^n_A$ and take the closure in $\mathbf{P}^n_A$ to conclude that B is isomorphic to $\mathcal{O}_{X,x}$ for some closed point $x \in X$ of a projective modification $X \to \mathrm{Spec}(A)$.

(Morphisms, Lemma 29.52.1, shows that $\kappa(x)$ is finite over κ and then Morphisms, Lemma 29.20.2 shows that x is a closed point.) Let $\nu : X^\nu \to X$ be the normalization. Since A is Nagata the morphism ν is finite (Morphisms, Lemma 29.54.11). Thus X^ν is projective over A by More on Morphisms, Lemma 37.50.2. Since $B = \mathcal{O}_{X,x}$ is normal, we see that $\mathcal{O}_{X,x} = (\nu_*\mathcal{O}_{X^\nu})_x$. Hence there is a unique point $x^\nu \in X^\nu$ lying over x and $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}$. Thus we may assume X is normal and projective over A. Let $Y \to \mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(B)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 32.21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 32.21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. Thus $H^1(X', \mathcal{O}_{X'}) = 0$ by our assumption on A. By Lemma 54.8.1 we have $R^1g_*\mathcal{O}_{X'} = 0$. Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired. □

0B4P **Lemma 54.8.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. If reduction to rational singularities is possible for A, then there exists a finite sequence of normalized blowups*

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

in closed points such that for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. In particular $X \to \mathrm{Spec}(A)$ is a modification and X is a normal scheme projective over A.

Proof. We choose a modification $X \to \mathrm{Spec}(A)$ with X normal which maximizes the length of $H^1(X, \mathcal{O}_X)$. By Lemma 54.8.1 for any further modification $g : X' \to X$ with X' normal we have $R^1g_*\mathcal{O}_{X'} = 0$ and $H^1(X, \mathcal{O}_X) = H^1(X', \mathcal{O}_{X'})$.

Let $x \in X$ be a closed point. We will show that $\mathcal{O}_{X,x}$ defines a rational singularity. Let $Y \to \mathrm{Spec}(\mathcal{O}_{X,x})$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 32.21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 32.21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. By maximality we have $R^1g_*\mathcal{O}_{X'} = 0$ (see first paragraph). Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired.

The conclusion is that we've found one normal modification X of $\mathrm{Spec}(A)$ such that the local rings of X at closed points all define rational singularities. Then we choose a sequence of normalized blowups $X_n \to \ldots \to X_1 \to \mathrm{Spec}(A)$ such that X_n dominates X, see Lemma 54.5.3. For a closed point $x' \in X_n$ mapping to $x \in X$ we can apply Lemma 54.8.4 to the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_n,x'}$ to see that $\mathcal{O}_{X_n,x'}$ defines a rational singularity. □

0AXL **Lemma 54.8.6.** *Let $A \to B$ be a finite injective local ring map of local normal Nagata domains of dimension 2. Assume that the induced extension of fraction fields is separable. If reduction to rational singularities is possible for A then it is possible for B.*

Proof. Let n be the degree of the fraction field extension L/K. Let $\mathrm{Trace}_{L/K} : L \to K$ be the trace. Since the extension is finite separable the trace pairing $(h, g) \mapsto \mathrm{Trace}_{L/K}(fg)$ is a nondegenerate bilinear form on L over K. See Fields, Lemma 9.20.7. Pick $b_1, \ldots, b_n \in B$ which form a basis of L over K. By the above $d = \det(\mathrm{Trace}_{L/K}(b_i b_j)) \in A$ is nonzero.

Let $Y \to \mathrm{Spec}(B)$ be a modification with Y normal. We can find a U-admissible blowup X' of $\mathrm{Spec}(A)$ such that the strict transform Y' of Y is finite over X', see More on Flatness, Lemma 38.31.2. Picture

$$\begin{array}{ccccc} Y' & \longrightarrow & Y & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & & & \downarrow \\ X' & & \longrightarrow & & \mathrm{Spec}(A) \end{array}$$

After replacing X' and Y' by their normalizations we may assume that X' and Y' are normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. In this way we reduce to the case where there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & \mathrm{Spec}(B) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathrm{Spec}(A) \end{array}$$

with X and Y normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ and π finite.

The trace map on L over K extends to a map of $\mathcal{O}_X$-modules $\mathrm{Trace} : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Consider the map

$$\Phi : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X^{\oplus n}, \quad s \longmapsto (\mathrm{Trace}(b_1 s), \ldots, \mathrm{Trace}(b_n s))$$

This map is injective (because it is injective in the generic point) and there is a map

$$\mathcal{O}_X^{\oplus n} \longrightarrow \pi_*\mathcal{O}_Y, \quad (s_1, \ldots, s_n) \longmapsto \sum b_i s_i$$

whose composition with Φ has matrix $\mathrm{Trace}(b_i b_j)$. Hence the cokernel of Φ is annihilated by d. Thus we see that we have an exact sequence

$$H^0(X, \mathrm{Coker}(\Phi)) \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)^{\oplus n}$$

Since the right hand side is bounded by assumption, it suffices to show that the d-torsion in $H^1(Y, \mathcal{O}_Y)$ is bounded. This is the content of Lemma 54.8.2 and (54.8.2.1). □

0B4Q **Lemma 54.8.7.** *Let A be a Nagata regular local ring of dimension 2. Then A defines a rational singularity.*

Proof. (The assumption that A be Nagata is not necessary for this proof, but we've only defined the notion of rational singularity in the case of Nagata 2-dimensional normal local domains.) Let $X \to \mathrm{Spec}(A)$ be a modification with X normal. By Lemma 54.4.2 we can dominate X by a scheme X_n which is the last in a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

of blowing ups in closed points. By Lemma 54.3.2 the schemes X_i are regular, in particular normal (Algebra, Lemma 10.157.5). By Lemma 54.8.1 we have

$H^1(X, \mathcal{O}_X) \subset H^1(X_n, \mathcal{O}_{X_n})$. Thus it suffices to prove $H^1(X_n, \mathcal{O}_{X_n}) = 0$. Using Lemma 54.8.1 again, we see that it suffices to prove $R^1(X_i \to X_{i-1})_*\mathcal{O}_{X_i} = 0$ for $i = 1, \ldots, n$. This follows from Lemma 54.3.4. □

0B4S **Lemma 54.8.8.** *Let A be a local normal Nagata domain of dimension 2 which has a dualizing complex $\omega_A^\bullet$. If there exists a nonzero $d \in A$ such that for all normal modifications $X \to \mathrm{Spec}(A)$ the cokernel of the trace map*

$$\Gamma(X, \omega_X) \to \omega_A$$

is annihilated by d, then reduction to rational singularities is possible for A.

Proof. For $X \to \mathrm{Spec}(A)$ as in the statement we have to bound $H^1(X, \mathcal{O}_X)$. Let ω_X be the dualizing module of X as in the statement of Grauert-Riemenschneider (Proposition 54.7.8). The trace map is the map $Rf_*\omega_X \to \omega_A$ described in Duality for Schemes, Section 48.7. By Grauert-Riemenschneider we have $Rf_*\omega_X = f_*\omega_X$ thus the trace map indeed produces a map $\Gamma(X, \omega_X) \to \omega_A$. By duality we have $Rf_*\omega_X = R\mathrm{Hom}_A(Rf_*\mathcal{O}_X, \omega_A)$ (this uses that $\omega_X[2]$ is the dualizing complex on X normalized relative to $\omega_A[2]$, see Duality for Schemes, Lemma 48.20.9 or more directly Section 48.19 or even more directly Example 48.3.9). The distinguished triangle

$$A \to Rf_*\mathcal{O}_X \to R^1f_*\mathcal{O}_X[-1] \to A[1]$$

is transformed by $R\mathrm{Hom}_A(-, \omega_A)$ into the short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) \to 0$$

(and $\mathrm{Ext}^i_A(R^1f_*\mathcal{O}_X, \omega_A) = 0$ for $i \neq 2$; this will follow from the discussion below as well). Since $R^1f_*\mathcal{O}_X$ is supported in $\{\mathfrak{m}\}$, the local duality theorem tells us that

$$\mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) = \mathrm{Ext}^0_A(R^1f_*\mathcal{O}_X, \omega_A[2]) = \mathrm{Hom}_A(R^1f_*\mathcal{O}_X, E)$$

is the Matlis dual of $R^1f_*\mathcal{O}_X$ (and the other ext groups are zero), see Dualizing Complexes, Lemma 47.18.4. By the equivalence of categories inherent in Matlis duality (Dualizing Complexes, Proposition 47.7.8), if $R^1f_*\mathcal{O}_X$ is not annihilated by d, then neither is the Ext^2 above. Hence we see that $H^1(X, \mathcal{O}_X)$ is annihilated by d. Thus the required boundedness follows from Lemma 54.8.2 and (54.8.2.1). □

0B4T **Lemma 54.8.9.** *Let p be a prime number. Let A be a regular local ring of dimension 2 and characteristic p. Let $A_0 \subset A$ be a subring such that Ω_{A/A_0} is free of rank $r < \infty$. Set $\omega_A = \Omega^r_{A/A_0}$. If $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowups in closed points, then there exists a map*

$$\varphi_X : (\Omega^r_{X/\mathrm{Spec}(A_0)})^{**} \longrightarrow \omega_X$$

extending the given identification in the generic point.

Proof. Observe that A is Gorenstein (Dualizing Complexes, Lemma 47.21.3) and hence the invertible module ω_A does indeed serve as a dualizing module. Moreover, any X as in the lemma has an invertible dualizing module ω_X as X is regular (hence Gorenstein) and proper over A, see Remark 54.7.7 and Lemma 54.3.2. Suppose we have constructed the map $\varphi_X : (\Omega^r_{X/A_0})^{**} \to \omega_X$ and suppose that $b : X' \to X$ is a blowup in a closed point. Set $\Omega^r_X = (\Omega^r_{X/A_0})^{**}$ and $\Omega^r_{X'} = (\Omega^r_{X'/A_0})^{**}$. Since $\omega_{X'} = b^!(\omega_X)$ a map $\Omega^r_{X'} \to \omega_{X'}$ is the same thing as a map $Rb_*(\Omega^r_{X'}) \to \omega_X$. See

discussion in Remark 54.7.7 and Duality for Schemes, Section 48.19. Thus in turn it suffices to produce a map

$$Rb_*(\Omega^r_{X'}) \longrightarrow \Omega^r_X$$

The sheaves $\Omega^r_{X'}$ and Ω^r_X are invertible, see Divisors, Lemma 31.12.15. Consider the exact sequence

$$b^*\Omega_{X/A_0} \to \Omega_{X'/A_0} \to \Omega_{X'/X} \to 0$$

A local calculation shows that $\Omega_{X'/X}$ is isomorphic to an invertible module on the exceptional divisor E, see Lemma 54.3.6. It follows that either

$$\Omega^r_{X'} \cong (b^*\Omega^r_X)(E) \quad \text{or} \quad \Omega^r_{X'} \cong b^*\Omega^r_X$$

see Divisors, Lemma 31.15.13. (The second possibility never happens in characteristic zero, but can happen in characteristic p.) In both cases we see that $R^1b_*(\Omega^r_{X'}) = 0$ and $b_*(\Omega^r_{X'}) = \Omega^r_X$ by Lemma 54.3.4. □

0B4U **Lemma 54.8.10.** *Let p be a prime number. Let A be a complete regular local ring of dimension 2 and characteristic p. Let L/K be a degree p inseparable extension of the fraction field K of A. Let $B \subset L$ be the integral closure of A. Then reduction to rational singularities is possible for B.*

Proof. We have $A = k[[x, y]]$. Write $L = K[x]/(x^p - f)$ for some $f \in A$ and denote $g \in B$ the congruence class of x, i.e., the element such that $g^p = f$. By Algebra, Lemma 10.158.2 we see that $\mathrm{d}f$ is nonzero in $\Omega_{K/\mathbf{F}_p}$. By More on Algebra, Lemma 15.46.5 there exists a subfield $k^p \subset k' \subset k$ with $p^e = [k : k'] < \infty$ such that $\mathrm{d}f$ is nonzero in Ω_{K/K_0} where K_0 is the fraction field of $A_0 = k'[[x^p, y^p]] \subset A$. Then

$$\Omega_{A/A_0} = A \otimes_k \Omega_{k/k'} \oplus A\mathrm{d}x \oplus A\mathrm{d}y$$

is finite free of rank $e + 2$. Set $\omega_A = \Omega^{e+2}_{A/A_0}$. Consider the canonical map

$$\mathrm{Tr} : \Omega^{e+2}_{B/A_0} \longrightarrow \Omega^{e+2}_{A/A_0} = \omega_A$$

of Lemma 54.2.4. By duality this determines a map

$$c : \Omega^{e+2}_{B/A_0} \to \omega_B = \mathrm{Hom}_A(B, \omega_A)$$

Claim: the cokernel of c is annihilated by a nonzero element of B.

Since $\mathrm{d}f$ is nonzero in Ω_{A/A_0} we can find $\eta_1, \ldots, \eta_{e+1} \in \Omega_{A/A_0}$ such that $\theta = \eta_1 \wedge \ldots \wedge \eta_{e+1} \wedge \mathrm{d}f$ is nonzero in $\omega_A = \Omega^{e+2}_{A/A_0}$. To prove the claim we will construct elements ω_i of Ω^{e+2}_{B/A_0}, $i = 0, \ldots, p-1$ which are mapped to $\varphi_i \in \omega_B = \mathrm{Hom}_A(B, \omega_A)$ with $\varphi_i(g^j) = \delta_{ij}\theta$ for $j = 0, \ldots, p-1$. Since $\{1, g, \ldots, g^{p-1}\}$ is a basis for L/K this proves the claim. We set $\eta = \eta_1 \wedge \ldots \wedge \eta_{e+1}$ so that $\theta = \eta \wedge \mathrm{d}f$. Set $\omega_i = \eta \wedge g^{p-1-i}\mathrm{d}g$. Then by construction we have

$$\varphi_i(g^j) = \mathrm{Tr}(g^j\eta \wedge g^{p-1-i}\mathrm{d}g) = \mathrm{Tr}(\eta \wedge g^{p-1-i+j}\mathrm{d}g) = \delta_{ij}\theta$$

by the explicit description of the trace map in Lemma 54.2.2.

Let $Y \to \mathrm{Spec}(B)$ be a normal modification. Exactly as in the proof of Lemma 54.8.6 we can reduce to the case where Y is finite over a modification X of $\mathrm{Spec}(A)$.

By Lemma 54.4.2 we may even assume $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowing ups in closed points. Picture:

$$\begin{array}{ccc} Y & \xrightarrow{g} & \mathrm{Spec}(B) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathrm{Spec}(A) \end{array}$$

We may apply Lemma 54.2.4 to π and we obtain the first arrow in

$$\pi_*(\Omega^{e+2}_{Y/A_0}) \xrightarrow{\mathrm{Tr}} (\Omega^{e+2}_{X/A_0})^{**} \xrightarrow{\varphi_X} \omega_X$$

and the second arrow is from Lemma 54.8.9 (because f is a sequence of blowups in closed points). By duality for the finite morphism π this corresponds to a map

$$c_Y : \Omega^{e+2}_{Y/A_0} \longrightarrow \omega_Y$$

extending the map c above. Hence we see that the image of $\Gamma(Y, \omega_Y) \to \omega_B$ contains the image of c. By our claim we see that the cokernel is annihilated by a fixed nonzero element of B. We conclude by Lemma 54.8.8. □

54.9. Rational singularities

0B4V In this section we reduce from rational singular points to Gorenstein rational singular points. See [**Lip69**] and [**Mat70b**].

0B4W **Situation 54.9.1.** Here $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let s be the closed point of $S = \mathrm{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification with X normal. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

0B4X **Lemma 54.9.2.** *In Situation 54.9.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

(1) *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and*
(2) *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.*

Proof. Part (1) follows from Cohomology of Schemes, Lemma 30.20.9. If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ as S has a rational singularity, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.19.1) we conclude. □

0B4Y **Lemma 54.9.3.** *In Situation 54.9.1 assume $E = X_s$ is an effective Cartier divisor. Let $\mathcal{I}$ be the ideal sheaf of E. Then $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ and $H^1(X, \mathcal{I}^n) = 0$.*

Proof. We have $H^0(X, \mathcal{O}_X) = A$, see discussion following Situation 54.7.1. Then $\mathfrak{m} \subset H^0(X, \mathcal{I}) \subset H^0(X, \mathcal{O}_X)$. The second inclusion is not an equality as $X_s \neq \emptyset$. Thus $H^0(X, \mathcal{I}) = \mathfrak{m}$. As $\mathcal{I}^n = \mathfrak{m}^n \mathcal{O}_X$ our Lemma 54.9.2 shows that $H^1(X, \mathcal{I}^n) = 0$.

Choose generators $x_1, \ldots, x_{\mu+1}$ of $\mathfrak{m}$. These define global sections of $\mathcal{I}$ which generate it. Hence a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus \mu+1} \to \mathcal{I} \to 0$$

Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank μ and $\mathcal{F} \otimes \mathcal{I}$ is globally generated by Constructions, Lemma 27.13.9. Hence $\mathcal{F} \otimes \mathcal{I}^n$ is globally generated for all $n \geq 1$. Thus for $n \geq 2$ we can consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{I}^{n-1} \to (\mathcal{I}^{n-1})^{\oplus \mu+1} \to \mathcal{I}^n \to 0$$

Applying the long exact sequence of cohomology using that $H^1(X, \mathcal{F} \otimes \mathcal{I}^{n-1}) = 0$ by Lemma 54.9.2 we obtain that every element of $H^0(X, \mathcal{I}^n)$ is of the form $\sum x_i a_i$ for some $a_i \in H^0(X, \mathcal{I}^{n-1})$. This shows that $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ by induction. □

0B4Z **Lemma 54.9.4.** *In Situation 54.9.1 the blowup of* $\mathrm{Spec}(A)$ *in* $\mathfrak{m}$ *is normal.*

Proof. Let $X' \to \mathrm{Spec}(A)$ be the blowup, in other words

$$X' = \mathrm{Proj}(A \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \ldots).$$

is the Proj of the Rees algebra. This in particular shows that X' is integral and that $X' \to \mathrm{Spec}(A)$ is a projective modification. Let X be the normalization of X'. Since A is Nagata, we see that $\nu : X \to X'$ is finite (Morphisms, Lemma 29.54.11). Let $E' \subset X'$ be the exceptional divisor and let $E \subset X$ be the inverse image. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be their ideal sheaves. Recall that $\mathcal{I}' = \mathcal{O}_{X'}(1)$ (Divisors, Lemma 31.32.13). Observe that $\mathcal{I} = \nu^*\mathcal{I}'$ and that E is an effective Cartier divisor (Divisors, Lemma 31.13.13). We are trying to show that ν is an isomorphism. As ν is finite, it suffices to show that $\mathcal{O}_{X'} \to \nu_*\mathcal{O}_X$ is an isomorphism. If not, then we can find an $n \geq 0$ such that

$$H^0(X', (\mathcal{I}')^n) \neq H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n)$$

for example because we can recover quasi-coherent $\mathcal{O}_{X'}$-modules from their associated graded modules, see Properties, Lemma 28.28.3. By the projection formula we have

$$H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n) = H^0(X, \nu^*(\mathcal{I}')^n) = H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$$

the last equality by Lemma 54.9.3. On the other hand, there is clearly an injection $\mathfrak{m}^n \to H^0(X', (\mathcal{I}')^n)$. Since $H^0(X', (\mathcal{I}')^n)$ is torsion free we conclude equality holds for all n, hence $X = X'$. □

0B63 **Lemma 54.9.5.** *In Situation 54.9.1. Let X be the blowup of* $\mathrm{Spec}(A)$ *in* $\mathfrak{m}$*. Let $E \subset X$ be the exceptional divisor. With $\mathcal{O}_X(1) = \mathcal{I}$ as usual and $\mathcal{O}_E(1) = \mathcal{O}_X(1)|_E$ we have*

(1) *E is a proper Cohen-Macaulay curve over κ.*
(2) *$\mathcal{O}_E(1)$ is very ample*
(3) *$\deg(\mathcal{O}_E(1)) \geq 1$ and equality holds only if A is a regular local ring,*
(4) *$H^1(E, \mathcal{O}_E(n)) = 0$ for $n \geq 0$, and*
(5) *$H^0(E, \mathcal{O}_E(n)) = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for $n \geq 0$.*

Proof. Since $\mathcal{O}_X(1)$ is very ample by construction, we see that its restriction to the special fibre E is very ample as well. By Lemma 54.9.4 the scheme X is normal. Then E is Cohen-Macaulay by Divisors, Lemma 31.15.6. Lemma 54.9.3 applies and we obtain (4) and (5) from the exact sequences

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to i_*\mathcal{O}_E(n) \to 0$$

and the long exact cohomology sequence. In particular, we see that

$$\deg(\mathcal{O}_E(1)) = \chi(E, \mathcal{O}_E(1)) - \chi(E, \mathcal{O}_E) = \dim(\mathfrak{m}/\mathfrak{m}^2) - 1$$

by Varieties, Definition 33.44.1. Thus (3) follows as well. □

0BBU **Lemma 54.9.6.** *In Situation 54.9.1 assume A has a dualizing complex $\omega_A^\bullet$. With ω_X the dualizing module of X, the trace map $H^0(X, \omega_X) \to \omega_A$ is an isomorphism and consequently there is a canonical map $f^*\omega_A \to \omega_X$.*

Proof. By Grauert-Riemenschneider (Proposition 54.7.8) we see that $Rf_*\omega_X = f_*\omega_X$. By duality we have a short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1 f_*\mathcal{O}_X, \omega_A) \to 0$$

(for example see proof of Lemma 54.8.8) and since A defines a rational singularity we obtain $f_*\omega_X = \omega_A$. □

0B64 **Lemma 54.9.7.** *In Situation 54.9.1 assume A has a dualizing complex $\omega_A^\bullet$ and is not regular. Let X be the blowup of $\mathrm{Spec}(A)$ in $\mathfrak{m}$ with exceptional divisor $E \subset X$. Let ω_X be the dualizing module of X. Then*

(1) *$\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$,*
(2) *$H^1(X, \omega_X(n)) = 0$ for $n \geq 0$,*
(3) *the map $f^*\omega_A \to \omega_X$ of Lemma 54.9.6 is surjective.*

Proof. We will use the results of Lemma 54.9.5 without further mention. Observe that $\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$ by Duality for Schemes, Lemmas 48.14.2 and 48.9.7. Thus $\omega_X|_E = \omega_E(1)$. Consider the short exact sequences

$$0 \to \omega_X(n+1) \to \omega_X(n) \to i_*\omega_E(n+1) \to 0$$

By Algebraic Curves, Lemma 53.6.4 we see that $H^1(E, \omega_E(n+1)) = 0$ for $n \geq 0$. Thus we see that the maps

$$\ldots \to H^1(X, \omega_X(2)) \to H^1(X, \omega_X(1)) \to H^1(X, \omega_X)$$

are surjective. Since $H^1(X, \omega_X(n))$ is zero for $n \gg 0$ (Cohomology of Schemes, Lemma 30.16.2) we conclude that (2) holds.

By Algebraic Curves, Lemma 53.6.7 we see that $\omega_X|_E = \omega_E \otimes \mathcal{O}_E(1)$ is globally generated. Since we seen above that $H^1(X, \omega_X(1)) = 0$ the map $H^0(X, \omega_X) \to H^0(E, \omega_X|_E)$ is surjective. We conclude that ω_X is globally generated hence (3) holds because $\Gamma(X, \omega_X) = \omega_A$ is used in Lemma 54.9.6 to define the map. □

0BBV **Lemma 54.9.8.** *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Assume A has a dualizing complex. Then there exists a finite sequence of blowups in singular closed points*

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_i is normal for each i and such that the dualizing sheaf ω_X of X is an invertible $\mathcal{O}_X$-module.

Proof. The dualizing module ω_A is a finite A-module whose stalk at the generic point is invertible. Namely, $\omega_A \otimes_A K$ is a dualizing module for the fraction field K of A, hence has rank 1. Thus there exists a blowup $b : Y \to \mathrm{Spec}(A)$ such that the strict transform of ω_A with respect to b is an invertible $\mathcal{O}_Y$-module, see Divisors, Lemma 31.35.3. By Lemma 54.5.3 we can choose a sequence of normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \mathrm{Spec}(A)$$

such that X_n dominates Y. By Lemma 54.9.4 and arguing by induction each $X_i \to X_{i-1}$ is simply a blowing up.

We claim that ω_{X_n} is invertible. Since ω_{X_n} is a coherent $\mathcal{O}_{X_n}$-module, it suffices to see its stalks are invertible modules. If $x \in X_n$ is a regular point, then this is clear from the fact that regular schemes are Gorenstein (Dualizing Complexes, Lemma 47.21.3). If x is a singular point of X_n, then each of the images $x_i \in X_i$ of x is a singular point (because the blowup of a regular point is regular by Lemma 54.3.2). Consider the canonical map $f_n^*\omega_A \to \omega_{X_n}$ of Lemma 54.9.6. For each i the morphism $X_{i+1} \to X_i$ is either a blowup of x_i or an isomorphism at x_i. Since x_i is always a singular point, it follows from Lemma 54.9.7 and induction that the maps $f_i^*\omega_A \to \omega_{X_i}$ is always surjective on stalks at x_i. Hence

$$(f_n^*\omega_A)_x \longrightarrow \omega_{X_n,x}$$

is surjective. On the other hand, by our choice of b the quotient of $f_n^*\omega_A$ by its torsion submodule is an invertible module $\mathcal{L}$. Moreover, the dualizing module is torsion free (Duality for Schemes, Lemma 48.22.3). It follows that $\mathcal{L}_x \cong \omega_{X_n,x}$ and the proof is complete. □

54.10. Formal arcs

0BG1 Let X be a locally Noetherian scheme. In this section we say that a *formal arc* in X is a morphism $a : T \to X$ where T is the spectrum of a complete discrete valuation ring R whose residue field κ is identified with the residue field of the image p of the closed point of $\mathrm{Spec}(R)$. Let us say that the formal arc a is *centered at* p in this case. We say the formal arc $T \to X$ is *nonsingular* if the induced map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2$ is surjective.

Let $a : T \to X$, $T = \mathrm{Spec}(R)$ be a nonsingular formal arc centered at a closed point p of X. Assume X is locally Noetherian. Let $b : X_1 \to X$ be the blowing up of X at x. Since a is nonsingular, we see that there is an element $f \in \mathfrak{m}_p$ which maps to a uniformizer in R. In particular, we find that the generic point of T maps to a point of X not equal to p. In other words, with K the fraction field of R, the restriction of a defines a morphism $\mathrm{Spec}(K) \to X \setminus \{p\}$. Since the morphism b is proper and an isomorphism over $X \setminus \{x\}$ we can apply the valuative criterion of properness to obtain a unique morphism a_1 making the following diagram commute

$$\begin{array}{ccc} T & \xrightarrow{a_1} & X_1 \\ & \searrow^{a} & \downarrow{\scriptstyle b} \\ & & X \end{array}$$

Let $p_1 \in X_1$ be the image of the closed point of T. Observe that p_1 is a closed point as it is a $\kappa = \kappa(p)$-rational point on the fibre of $X_1 \to X$ over x. Since we have a factorization

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X_1,p_1} \to R$$

we see that a_1 is a nonsingular formal arc as well.

We can repeat the process and obtain a sequence of blowing ups

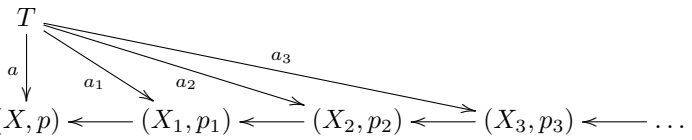

This kind of sequence of blowups can be characterized as follows.

0BG2 **Lemma 54.10.1.** *Let X be a locally Noetherian scheme. Let*

$$(X, p) = (X_0, p_0) \leftarrow (X_1, p_1) \leftarrow (X_2, p_2) \leftarrow (X_3, p_3) \leftarrow \ldots$$

be a sequence of blowups such that

(1) *p_i is closed, maps to p_{i-1}, and $\kappa(p_i) = \kappa(p_{i-1})$,*
(2) *there exists an $x_1 \in \mathfrak{m}_p$ whose image in $\mathfrak{m}_{p_i}$, $i > 0$ defines the exceptional divisor $E_i \subset X_i$.*

Then the sequence is obtained from a nonsingular arc $a : T \to X$ as above.

Proof. Let us write $\mathcal{O}_n = \mathcal{O}_{X_n, p_n}$ and $\mathcal{O} = \mathcal{O}_{X,p}$. Denote $\mathfrak{m} \subset \mathcal{O}$ and $\mathfrak{m}_n \subset \mathcal{O}_n$ the maximal ideals.

We claim that $x_1^t \notin \mathfrak{m}_n^{t+1}$. Namely, if this were the case, then in the local ring $\mathcal{O}_{n+1}$ the element x_1^t would be in the ideal of $(t+1)E_{n+1}$. This contradicts the assumption that x_1 defines E_{n+1}.

For every n choose generators $y_{n,1}, \ldots, y_{n,t_n}$ for $\mathfrak{m}_n$. As $\mathfrak{m}_n \mathcal{O}_{n+1} = x_1 \mathcal{O}_{n+1}$ by assumption (2), we can write $y_{n,i} = a_{n,i} x_1$ for some $a_{n,i} \in \mathcal{O}_{n+1}$. Since the map $\mathcal{O}_n \to \mathcal{O}_{n+1}$ defines an isomorphism on residue fields by (1) we can choose $c_{n,i} \in \mathcal{O}_n$ having the same residue class as $a_{n,i}$. Then we see that

$$\mathfrak{m}_n = (x_1, z_{n,1}, \ldots, z_{n,t_n}), \quad z_{n,i} = y_{n,i} - c_{n,i} x_1$$

and the elements $z_{n,i}$ map to elements of $\mathfrak{m}_{n+1}^2$ in $\mathcal{O}_{n+1}$.

Let us consider

$$J_n = \mathrm{Ker}(\mathcal{O} \to \mathcal{O}_n / \mathfrak{m}_n^{n+1})$$

We claim that $\mathcal{O}/J_n$ has length $n+1$ and that $\mathcal{O}/(x_1) + J_n$ equals the residue field. For $n = 0$ this is immediate. Assume the statement holds for n. Let $f \in J_n$. Then in $\mathcal{O}_n$ we have

$$f = a x_1^{n+1} + x_1^n A_1(z_{n,i}) + x_1^{n-1} A_2(z_{n,i}) + \ldots + A_{n+1}(z_{n,i})$$

for some $a \in \mathcal{O}_n$ and some A_i homogeneous of degree i with coefficients in $\mathcal{O}_n$. Since $\mathcal{O} \to \mathcal{O}_n$ identifies residue fields, we may choose $a \in \mathcal{O}$ (argue as in the construction of $z_{n,i}$ above). Taking the image in $\mathcal{O}_{n+1}$ we see that f and $a x_1^{n+1}$ have the same image modulo $\mathfrak{m}_{n+1}^{n+2}$. Since $x_n^{n+1} \notin \mathfrak{m}_{n+1}^{n+2}$ it follows that J_n / J_{n+1} has length 1 and the claim is true.

Consider $R = \lim \mathcal{O}/J_n$. This is a quotient of the $\mathfrak{m}$-adic completion of $\mathcal{O}$ hence it is a complete Noetherian local ring. On the other hand, it is not finite length and x_1 generates the maximal ideal. Thus R is a complete discrete valuation ring. The map $\mathcal{O} \to R$ lifts to a local homomorphism $\mathcal{O}_n \to R$ for every n. There are two ways to show this: (1) for every n one can use a similar procedure to construct $\mathcal{O}_n \to R_n$ and then one can show that $\mathcal{O} \to \mathcal{O}_n \to R_n$ factors through an isomorphism $R \to R_n$, or (2) one can use Divisors, Lemma 31.32.6 to show that $\mathcal{O}_n$

is a localization of a repeated affine blowup algebra to explicitly construct a map $\mathcal{O}_n \to R$. Having said this it is clear that our sequence of blowups comes from the nonsingular arc $a : T = \mathrm{Spec}(R) \to X$. □

The following lemma is a kind of Néron desingularization lemma.

0BG3 **Lemma 54.10.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension 2. Let $A \to R$ be a surjection onto a complete discrete valuation ring. This defines a nonsingular arc $a : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A)$. Let*

$$\mathrm{Spec}(A) = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

be the sequence of blowing ups constructed from a. If $A_\mathfrak{p}$ is a regular local ring where $\mathfrak{p} = \mathrm{Ker}(A \to R)$, then for some i the scheme X_i is regular at x_i.

Proof. Let $x_1 \in \mathfrak{m}$ map to a uniformizer of R. Observe that $\kappa(\mathfrak{p}) = K$ is the fraction field of R. Write $\mathfrak{p} = (x_2, \ldots, x_r)$ with r minimal. If $r = 2$, then $\mathfrak{m} = (x_1, x_2)$ and A is regular and the lemma is true. Assume $r > 2$. After renumbering if necessary, we may assume that x_2 maps to a uniformizer of $A_\mathfrak{p}$. Then $\mathfrak{p}/\mathfrak{p}^2 + (x_2)$ is annihilated by a power of x_1. For $i > 2$ we can find $n_i \geq 0$ and $a_i \in A$ such that

$$x_1^{n_i} x_i - a_i x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

for some $a_{jk} \in A$. If $n_i = 0$ for some i, then we can remove x_i from the list of generators of $\mathfrak{p}$ and we win by induction on r. If for some i the element a_i is a unit, then we can remove x_2 from the list of generators of $\mathfrak{p}$ and we win in the same manner. Thus either $a_i \in \mathfrak{p}$ or $a_i = u_i x_1^{m_1} \bmod \mathfrak{p}$ for some $m_1 > 0$ and unit $u_i \in A$. Thus we have either

$$x_1^{n_i} x_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k \quad \text{or} \quad x_1^{n_i} x_i - u_i x_1^{m_i} x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

We will prove that after blowing up the integers n_i, m_i decrease which will finish the proof.

Let us see what happens with these equations on the affine blowup algebra $A' = A[\mathfrak{m}/x_1]$. As $\mathfrak{m} = (x_1, \ldots, x_r)$ we see that A' is generated over R by $y_i = x_i/x_1$ for $i \geq 2$. Clearly $A \to R$ extends to $A' \to R$ with kernel $(y_2, \ldots, y_r)$. Then we see that either

$$x_1^{n_i - 1} y_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k \quad \text{or} \quad x_1^{n_i - 1} y_i - u_i x_1^{m_1 - 1} y_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k$$

and the proof is complete. □

54.11. Base change to the completion

0BG4 The following simple lemma will turn out to be a useful tool in what follows.

0BG5 **Lemma 54.11.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak{m}$. Let X be a scheme over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak{m}$-adic completion of A. For a point $q \in Y$ with image $p \in X$ lying over the closed point of $\mathrm{Spec}(A)$ the local ring map $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ induces an isomorphism on completions.*

Proof. We may assume X is affine. Then we may write $X = \mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B' = B \otimes_A A^\wedge$ be the prime corresponding to q and let $\mathfrak{p} \subset B$ be the prime ideal corresponding to p. By Algebra, Lemma 10.96.3 we have

$$B'/(\mathfrak{m}^\wedge)^n B' = A^\wedge/(\mathfrak{m}^\wedge)^n \otimes_A B = A/\mathfrak{m}^n \otimes_A B = B/\mathfrak{m}^n B$$

for all n. Since $\mathfrak{m}B \subset \mathfrak{p}$ and $\mathfrak{m}^\wedge B' \subset \mathfrak{q}$ we see that $B/\mathfrak{p}^n$ and $B'/\mathfrak{q}^n$ are both quotients of the ring displayed above by the nth power of the same prime ideal. The lemma follows. □

0BG6 **Lemma 54.11.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in Y$ with image $x \in X$. Then*

(1) *if $\mathcal{O}_{Y,y}$ is regular, then $\mathcal{O}_{X,x}$ is regular,*
(2) *if y is in the closed fibre, then $\mathcal{O}_{Y,y}$ is regular $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is regular, and*
(3) *If X is proper over A, then X is regular if and only if Y is regular.*

Proof. Since $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 10.97.3), we see that $Y \to X$ is flat. Hence (1) by Algebra, Lemma 10.164.4. Lemma 54.11.1 shows the morphism $Y \to X$ induces an isomorphism on complete local rings at points of the special fibres. Thus (2) by More on Algebra, Lemma 15.43.4. If X is proper over A, then Y is proper over $A^\wedge$ (Morphisms, Lemma 29.41.5) and we see every closed point of X and Y lies in the closed fibre. Thus we see that Y is a regular scheme if and only if X is so by Properties, Lemma 28.9.2. □

0AFK **Lemma 54.11.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with completion $A^\wedge$. Let $U \subset \mathrm{Spec}(A)$ and $U^\wedge \subset \mathrm{Spec}(A^\wedge)$ be the punctured spectra. If $Y \to \mathrm{Spec}(A^\wedge)$ is a $U^\wedge$-admissible blowup, then there exists a U-admissible blowup $X \to \mathrm{Spec}(A)$ such that $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$.*

Proof. By definition there exists an ideal $J \subset A^\wedge$ such that $V(J) = \{\mathfrak{m}A^\wedge\}$ and such that Y is the blowup of $S^\wedge$ in the closed subscheme defined by J, see Divisors, Definition 31.34.1. Since $A^\wedge$ is Noetherian this implies $\mathfrak{m}^n A^\wedge \subset J$ for some n. Since $A^\wedge/\mathfrak{m}^n A^\wedge = A/\mathfrak{m}^n$ we find an ideal $\mathfrak{m}^n \subset I \subset A$ such that $J = IA^\wedge$. Let $X \to S$ be the blowup in I. Since $A \to A^\wedge$ is flat we conclude that the base change of X is Y by Divisors, Lemma 31.32.3. □

0BG7 **Lemma 54.11.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Nagata local normal domain of dimension 2. Assume A defines a rational singularity and that the completion $A^\wedge$ of A is normal. Then*

(1) *$A^\wedge$ defines a rational singularity, and*
(2) *if $X \to \mathrm{Spec}(A)$ is the blowing up in $\mathfrak{m}$, then for a closed point $x \in X$ the completion $\mathcal{O}_{X,x}$ is normal.*

Proof. Let $Y \to \mathrm{Spec}(A^\wedge)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Varieties, Lemma 33.17.3 $Y \to \mathrm{Spec}(A^\wedge)$ is an isomorphism over the punctured spectrum $U^\wedge = \mathrm{Spec}(A^\wedge) \setminus \{\mathfrak{m}^\wedge\}$. By Lemma 54.7.2 there exists a $U^\wedge$-admissible blowup $Y' \to \mathrm{Spec}(A^\wedge)$ dominating Y. By Lemma 54.11.3 we find there exists a U-admissible blowup $X \to \mathrm{Spec}(A)$ whose base change to $A^\wedge$ dominates Y. Since A is Nagata, we can replace X by its normalization after which $X \to \mathrm{Spec}(A)$ is a normal modification (but possibly no longer a U-admissible blowup). Then $H^1(X, \mathcal{O}_X) = 0$ as A defines a rational singularity. It follows that $H^1(X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge), \mathcal{O}_{X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)}) = 0$ by flat base change (Cohomology of Schemes, Lemma 30.5.2 and flatness of $A \to A^\wedge$ by Algebra, Lemma 10.97.2). We find that $H^1(Y, \mathcal{O}_Y) = 0$ by Lemma 54.8.1.

Finally, let $X \to \mathrm{Spec}(A)$ be the blowing up of $\mathrm{Spec}(A)$ in $\mathfrak{m}$. Then $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is the blowing up of $\mathrm{Spec}(A^\wedge)$ in $\mathfrak{m}^\wedge$. By Lemma 54.9.4 we see that both

Y and X are normal. On the other hand, $A^\wedge$ is excellent (More on Algebra, Proposition 15.52.3) hence every affine open in Y is the spectrum of an excellent normal domain (More on Algebra, Lemma 15.52.2). Thus for $y \in Y$ the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}^\wedge_{Y,y}$ is regular and by More on Algebra, Lemma 15.42.2 we find that $\mathcal{O}^\wedge_{Y,y}$ is normal. If $x \in X$ is a closed point of the special fibre, then there is a unique closed point $y \in Y$ lying over x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces an isomorphism on completions (Lemma 54.11.1) we conclude. □

0BG8 **Lemma 54.11.5.** *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let X be a scheme over A. Assume*

(1) *A is analytically unramified (Algebra, Definition 10.162.9),*
(2) *X is locally of finite type over A, and*
(3) *$X \to \mathrm{Spec}(A)$ is étale at the generic points of irreducible components of X.*

Then the normalization of X is finite over X.

Proof. Since A is analytically unramified it is reduced by Algebra, Lemma 10.162.10. Since the normalization of X depends only on the reduction of X, we may replace X by its reduction X_{red}; note that $X_{red} \to X$ is an isomorphism over the open U where $X \to \mathrm{Spec}(A)$ is étale because U is reduced (Descent, Lemma 35.18.1) hence condition (3) remains true after this replacement. In addition we may and do assume that $X = \mathrm{Spec}(B)$ is affine.

The map

$$K = \prod\nolimits_{\mathfrak{p} \subset A \text{ minimal}} \kappa(\mathfrak{p}) \longrightarrow K^\wedge = \prod\nolimits_{\mathfrak{p}^\wedge \subset A^\wedge \text{ minimal}} \kappa(\mathfrak{p}^\wedge)$$

is injective because $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 10.97.3) hence induces a surjective map between sets of minimal primes (by going down for flat ring maps, see Algebra, Section 10.41). Both sides are finite products of fields as our rings are Noetherian. Let $L = \prod_{\mathfrak{q} \subset B \text{ minimal}} \kappa(\mathfrak{q})$. Our assumption (3) implies that $L = B \otimes_A K$ and that $K \to L$ is a finite étale ring map (this is true because $A \to B$ is generically finite, for example use Algebra, Lemma 10.122.11 or the more detailed results in Morphisms, Section 29.51). Since B is reduced we see that $B \subset L$. This implies that

$$C = B \otimes_A A^\wedge \subset L \otimes_A A^\wedge = L \otimes_K K^\wedge = M$$

Then M is the total ring of fractions of C and is a finite product of fields as a finite separable algebra over $K^\wedge$. It follows that C is reduced and that its normalization C' is the integral closure of C in M. The normalization B' of B is the integral closure of B in L. By flatness of $A \to A^\wedge$ we obtain an injective map $B' \otimes_A A^\wedge \to M$ whose image is contained in C'. Picture

$$B' \otimes_A A^\wedge \longrightarrow C'$$

As $A^\wedge$ is Nagata (by Algebra, Lemma 10.162.8), we see that C' is finite over $C = B \otimes_A A^\wedge$ (see Algebra, Lemmas 10.162.8 and 10.162.2). As C is Noetherian, we conclude that $B' \otimes_A A^\wedge$ is finite over $C = B \otimes_A A^\wedge$. Therefore by faithfully flat descent (Algebra, Lemma 10.83.2) we see that B' is finite over B which is what we had to show. □

0BG9 **Lemma 54.11.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. If the complement of the special fibre in Y is normal, then the normalization $X^\nu \to X$ is finite and the base change of X^ν to $\mathrm{Spec}(A^\wedge)$ recovers the normalization of Y.*

Proof. There is an immediate reduction to the case where $X = \mathrm{Spec}(B)$ is affine with B a finite type A-algebra. Set $C = B \otimes_A A^\wedge$ so that $Y = \mathrm{Spec}(C)$. Since $A \to A^\wedge$ is faithfully flat, for any prime $\mathfrak{q} \subset B$ there exists a prime $\mathfrak{r} \subset C$ lying over $\mathfrak{q}$. Then $B_\mathfrak{q} \to C_\mathfrak{r}$ is faithfully flat. Hence if $\mathfrak{q}$ does not lie over $\mathfrak{m}$, then $C_\mathfrak{r}$ is normal by assumption on Y and we conclude that $B_\mathfrak{q}$ is normal by Algebra, Lemma 10.164.3. In this way we see that X is normal away from the special fibre.

Recall that the complete Noetherian local ring $A^\wedge$ is Nagata (Algebra, Lemma 10.162.8). Hence the normalization $Y^\nu \to Y$ is finite (Morphisms, Lemma 29.54.11) and an isomorphism away from the special fibre. Say $Y^\nu = \mathrm{Spec}(C')$. Then $C \to C'$ is finite and an isomorphism away from $V(\mathfrak{m}C)$. Since $B \to C$ is flat and induces an isomorphism $B/\mathfrak{m}B \to C/\mathfrak{m}C$ there exists a finite ring map $B \to B'$ whose base change to C recovers $C \to C'$. See More on Algebra, Lemma 15.90.17 and Remark 15.90.20. Thus we find a finite morphism $X' \to X$ which is an isomorphism away from the special fibre and whose base change recovers $Y^\nu \to Y$. By the discussion in the first paragraph we see that X' is normal at points not on the special fibre. For a point $x \in X'$ on the special fibre we have a corresponding point $y \in Y^\nu$ and a flat map $\mathcal{O}_{X',x} \to \mathcal{O}_{Y^\nu,y}$. Since $\mathcal{O}_{Y^\nu,y}$ is normal, so is $\mathcal{O}_{X',x}$, see Algebra, Lemma 10.164.3. Thus X' is normal and it follows that it is the normalization of X. □

0BGA **Lemma 54.11.7.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain whose completion $A^\wedge$ is normal. Then given any sequence*

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \mathrm{Spec}(A^\wedge)$$

of normalized blowups, there exists a sequence of (proper) normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \mathrm{Spec}(A)$$

whose base change to $A^\wedge$ recovers the given sequence.

Proof. Given the sequence $Y_n \to \ldots \to Y_1 \to Y_0 = \mathrm{Spec}(A^\wedge)$ we inductively construct $X_n \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$. The base case is $i = 0$. Given X_i whose base change is Y_i, let $Y_i' \to Y_i$ be the blowing up in the closed point $y_i \in Y_i$ such that Y_{i+1} is the normalization of Y_i. Since the closed fibres of Y_i and X_i are isomorphic, the point y_i corresponds to a closed point x_i on the special fibre of X_i. Let $X_i' \to X_i$ be the blowup of X_i in x_i. Then the base change of X_i' to $\mathrm{Spec}(A^\wedge)$ is isomorphic to Y_i'. By Lemma 54.11.6 the normalization $X_{i+1} \to X_i'$ is finite and its base change to $\mathrm{Spec}(A^\wedge)$ is isomorphic to Y_{i+1}. □

54.12. Rational double points

0BGB In Section 54.9 we argued that resolution of 2-dimensional rational singularities reduces to the Gorenstein case. A Gorenstein rational surface singularity is a rational double point. We will resolve them by explicit computations.

According to the discussion in Examples, Section 110.20 there exists a normal Noetherian local domain A whose completion is isomorphic to $\mathbf{C}[[x, y, z]]/(z^2)$. In this case one could say that A has a rational double point singularity, but on

the other hand, $\mathrm{Spec}(A)$ does not have a resolution of singularities. This kind of behaviour cannot occur if A is a Nagata ring, see Algebra, Lemma 10.162.13.

However, it gets worse as there exists a local normal Nagata domain A whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and another whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$. This is Example 2.5 of [**Nis12**]. This is why we need to assume the completion of our ring is normal in this section.

0BGC **Situation 54.12.1.** Here $(A, \mathfrak{m}, \kappa)$ be a Nagata local normal domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. We assume A is not regular.

The arguments in this section will show that repeatedly blowing up singular points resolves $\mathrm{Spec}(A)$ in this situation. We will need the following lemma in the course of the proof.

0BGD **Lemma 54.12.2.** *Let κ be a field. Let $I \subset \kappa[x, y]$ be an ideal. Let*

$$a + bx + cy + dx^2 + exy + fy^2 \in I^2$$

for some $a, b, c, d, e, f \in k$ not all zero. If the colength of I in $\kappa[x, y]$ is > 1, then $a + bx + cy + dx^2 + exy + fy^2 = j(g + hx + iy)^2$ for some $j, g, h, i \in \kappa$.

Proof. Consider the partial derivatives $b + 2dx + ey$ and $c + ex + 2fy$. By the Leibniz rules these are contained in I. If one of these is nonzero, then after a linear change of coordinates, i.e., of the form $x \mapsto \alpha + \beta x + \gamma y$ and $y \mapsto \delta + \epsilon x + \zeta y$, we may assume that $x \in I$. Then we see that $I = (x)$ or $I = (x, F)$ with F a monic polynomial of degree ≥ 2 in y. In the first case the statement is clear. In the second case observe that we can write any element in I^2 in the form

$$A(x, y)x^2 + B(y)xF + C(y)F^2$$

for some $A(x, y) \in \kappa[x, y]$ and $B, C \in \kappa[y]$. Thus

$$a + bx + cy + dx^2 + exy + fy^2 = A(x, y)x^2 + B(y)xF + C(y)F^2$$

and by degree reasons we see that $B = C = 0$ and A is a constant.

To finish the proof we need to deal with the case that both partial derivatives are zero. This can only happen in characteristic 2 and then we get

$$a + dx^2 + fy^2 \in I^2$$

We may assume f is nonzero (if not, then switch the roles of x and y). After dividing by f we obtain the case where the characteristic of κ is 2 and

$$a + dx^2 + y^2 \in I^2$$

If a and d are squares in κ, then we are done. If not, then there exists a derivation $\theta : \kappa \to \kappa$ with $\theta(a) \neq 0$ or $\theta(d) \neq 0$, see Algebra, Lemma 10.158.2. We can extend this to a derivation of $\kappa[x, y]$ by setting $\theta(x) = \theta(y) = 0$. Then we find that

$$\theta(a) + \theta(d)x^2 \in I$$

The case $\theta(d) = 0$ is absurd. Thus we may assume that $\alpha + x^2 \in I$ for some $\alpha \in \kappa$. Combining with the above we find that $a + \alpha d + y^2 \in I$. Hence

$$J = (\alpha + x^2, a + \alpha d + y^2) \subset I$$

with codimension at most 2. Observe that J/J^2 is free over $\kappa[x, y]/J$ with basis $\alpha + x^2$ and $a + \alpha d + y^2$. Thus $a + dx^2 + y^2 = 1 \cdot (a + \alpha d + y^2) + d \cdot (\alpha + x^2) \in I^2$

implies that the inclusion $J \subset I$ is strict. Thus we find a nonzero element of the form $g + hx + iy + jxy$ in I. If $j = 0$, then I contains a linear form and we can conclude as in the first paragraph. Thus $j \neq 0$ and $\dim_\kappa(I/J) = 1$ (otherwise we could find an element as above in I with $j = 0$). We conclude that I has the form $(\alpha + x^2, \beta + y^2, g + hx + iy + jxy)$ with $j \neq 0$ and has colength 3. In this case $a + dx^2 + y^2 \in I^2$ is impossible. This can be shown by a direct computation, but we prefer to argue as follows. Namely, to prove this statement we may assume that κ is algebraically closed. Then we can do a coordinate change $x \mapsto \sqrt{\alpha} + x$ and $y \mapsto \sqrt{\beta} + y$ and assume that $I = (x^2, y^2, g' + h'x + i'y + jxy)$ with the same j. Then $g' = h' = i' = 0$ otherwise the colength of I is not 3. Thus we get $I = (x^2, y^2, xy)$ and the result is clear. □

Let $(A, \mathfrak{m}, \kappa)$ be as in Situation 54.12.1. Let $X \to \mathrm{Spec}(A)$ be the blowing up of $\mathfrak{m}$ in $\mathrm{Spec}(A)$. By Lemma 54.9.4 we see that X is normal. All singularities of X are rational singularities by Lemma 54.8.4. Since $\omega_A = A$ we see from Lemma 54.9.7 that $\omega_X \cong \mathcal{O}_X$ (see discussion in Remark 54.7.7 for conventions). Thus all singularities of X are Gorenstein. Moreover, the local rings of X at closed point have normal completions by Lemma 54.11.4. In other words, by blowing up $\mathrm{Spec}(A)$ we obtain a normal surface X whose singular points are as in Situation 54.12.1. We will use this below without further mention. (Note: we will see in the course of the discussion below that there are finitely many of these singular points.)

Let $E \subset X$ be the exceptional divisor. We have $\omega_E = \mathcal{O}_E(-1)$ by Lemma 54.9.7. By Lemma 54.9.5 we have $\kappa = H^0(E, \mathcal{O}_E)$. Thus E is a Gorenstein curve and by Riemann-Roch as discussed in Algebraic Curves, Section 53.5 we have

$$\chi(E, \mathcal{O}_E) = 1 - g = -(1/2)\deg(\omega_E) = (1/2)\deg(\mathcal{O}_E(1))$$

where $g = \dim_\kappa H^1(E, \mathcal{O}_E) \geq 0$. Since $\deg(\mathcal{O}_E(1))$ is positive by Varieties, Lemma 33.44.15 we find that $g = 0$ and $\deg(\mathcal{O}_E(1)) = 2$. It follows that we have

$$\dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 2n + 1$$

by Lemma 54.9.5 and Riemann-Roch on E.

Choose $x_1, x_2, x_3 \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Because $\dim_\kappa(\mathfrak{m}^2/\mathfrak{m}^3) = 5$ the images of $x_i x_j$, $i \geq j$ in this κ-vector space satisfy a relation. In other words, we can find $a_{ij} \in A$, $i \geq j$, not all contained in $\mathfrak{m}$, such that

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

for some $a_{ijk} \in A$ where $i \leq j \leq k$. Denote $a \mapsto \overline{a}$ the map $A \to \kappa$. The quadratic form $q = \sum \overline{a}_{ij}t_it_j \in \kappa[t_1, t_2, t_3]$ is well defined up to multiplication by an element of κ^* by our choices. If during the course of our arguments we find that $\overline{a}_{ij} = 0$ in κ, then we can subsume the term $a_{ij}x_ix_j$ in the right hand side and assume $a_{ij} = 0$; this operation changes the a_{ijk} but not the other $a_{i'j'}$.

The blowing up is covered by 3 affine charts corresponding to the "variables" x_1, x_2, x_3. By symmetry it suffices to study one of the charts. To do this let

$$A' = A[\mathfrak{m}/x_1]$$

be the affine blowup algebra (as in Algebra, Section 10.70). Since x_1, x_2, x_3 generate $\mathfrak{m}$ we see that A' is generated by $y_2 = x_2/x_1$ and $y_3 = x_3/x_1$ over A. We will

occasionally use $y_1 = 1$ to simplify formulas. Moreover, looking at our relation above we find that

$$a_{11} + a_{12}y_2 + a_{13}y_3 + a_{22}y_2^2 + a_{23}y_2y_3 + a_{33}y_3^2 = x_1(\sum a_{ijk}y_iy_jy_k)$$

in A'. Recall that $x_1 \in A'$ defines the exceptional divisor E on our affine open of X which is therefore scheme theoretically given by

$$\kappa[y_2, y_3]/(\overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2)$$

In other words, $E \subset \mathbf{P}^2_\kappa = \text{Proj}(\kappa[t_1, t_2, t_3])$ is the zero scheme of the quadratic form q introduced above.

The quadratic form q is an important invariant of the singularity defined by A. Let us say we are in **case II** if q is a square of a linear form times an element of κ^* and in **case I** otherwise. Observe that we are in case II exactly if, after changing our choice of x_1, x_2, x_3, we have

$$x_3^2 = \sum a_{ijk}x_ix_jx_k$$

in the local ring A.

Let $\mathfrak{m}' \subset A'$ be a maximal ideal lying over $\mathfrak{m}$ with residue field κ'. In other words, $\mathfrak{m}'$ corresponds to a closed point $p \in E$ of the exceptional divisor. Recall that the surjection

$$\kappa[y_2, y_3] \to \kappa'$$

has kernel generated by two elements $f_2, f_3 \in \kappa[y_2, y_3]$ (see for example Algebra, Example 10.27.3 or the proof of Algebra, Lemma 10.114.1). Let $z_2, z_3 \in A'$ map to f_2, f_3 in $\kappa[y_2, y_3]$. Then we see that $\mathfrak{m}' = (x_1, z_2, z_3)$ because x_2 and x_3 become divisible by x_1 in A'.

Claim. If X is singular at p, then $\kappa' = \kappa$ or we are in case II. Namely, if $A'_{\mathfrak{m}'}$ is singular, then $\dim_{\kappa'} \mathfrak{m}'/(\mathfrak{m}')^2 = 3$ which implies that $\dim_{\kappa'} \overline{\mathfrak{m}}'/(\overline{\mathfrak{m}}')^2 = 2$ where $\overline{m}'$ is the maximal ideal of $\mathcal{O}_{E,p} = \mathcal{O}_{X,p}/x_1\mathcal{O}_{X,p}$. This implies that

$$q(1, y_2, y_3) = \overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2 \in (f_2, f_3)^2$$

otherwise there would be a relation between the classes of z_2 and z_3 in $\overline{\mathfrak{m}}'/(\overline{\mathfrak{m}}')^2$. The claim now follows from Lemma 54.12.2.

Resolution in case I. By the claim any singular point of X is κ-rational. Pick such a singular point p. We may choose our $x_1, x_2, x_3 \in \mathfrak{m}$ such that p lies on the chart described above and has coordinates $y_2 = y_3 = 0$. Since it is a singular point arguing as in the proof of the claim we find that $q(1, y_2, y_3) \in (y_2, y_3)^2$. Thus we can choose $a_{11} = a_{12} = a_{13} = 0$ and $q(t_1, t_2, t_3) = q(t_2, t_3)$. It follows that

$$E = V(q) \subset \mathbf{P}^1_\kappa$$

either is the union of two distinct lines meeting at p or is a degree 2 curve with a unique κ-rational point (small detail omitted; use that q is not a square of a linear form up to a scalar). In both cases we conclude that X has a unique singular point p which is κ-rational. We need a bit more information in this case. First, looking at higher terms in the expression above, we find that $\overline{a}_{111} = 0$ because p is singular. Then we can write $a_{111} = b_{111}x_1 \bmod (x_2, x_3)$ for some $b_{111} \in A$. Then the quadratic form at p for the generators x_1, y_2, y_3 of $\mathfrak{m}'$ is

$$q' = \overline{b}_{111}t_1^2 + \overline{a}_{112}t_1t_2 + \overline{a}_{113}t_1t_3 + \overline{a}_{22}t_2^2 + \overline{a}_{23}t_2t_3 + \overline{a}_{33}t_3^2$$

We see that $E' = V(q')$ intersects the line $t_1 = 0$ in either two points or one point of degree 2. We conclude that p lies in case I.

Suppose that the blowing up $X' \to X$ of X at p again has a singular point p'. Then we see that p' is a κ-rational point and we can blow up to get $X'' \to X'$. If this process does not stop we get a sequence of blowings up

$$\mathrm{Spec}(A) \leftarrow X \leftarrow X' \leftarrow X'' \leftarrow \ldots$$

We want to show that Lemma 54.10.1 applies to this situation. To do this we have to say something about the choice of the element x_1 of $\mathfrak{m}$. Suppose that A is in case I and that X has a singular point. Then we will say that $x_1 \in \mathfrak{m}$ is a *good coordinate* if for any (equivalently some) choice of x_2, x_3 the quadratic form $q(t_1, t_2, t_3)$ has the property that $q(0, t_2, t_3)$ is not a scalar times a square. We have seen above that a good coordinate exists. If x_1 is a good coordinate, then the singular point $p \in E$ of X does not lie on the hypersurface $t_1 = 0$ because either this does not have a rational point or if it does, then it is not singular on X. Observe that this is equivalent to the statement that the image of x_1 in $\mathcal{O}_{X,p}$ cuts out the exceptional divisor E. Now the computations above show that if x_1 is a good coordinate for A, then $x_1 \in \mathfrak{m}'\mathcal{O}_{X,p}$ is a good coordinate for p. This of course uses that the notion of good coordinate does not depend on the choice of x_2, x_3 used to do the computation. Hence x_1 maps to a good coordinate at p', p'', etc. Thus Lemma 54.10.1 applies and our sequence of blowing ups comes from a nonsingular arc $A \to R$. Then the map $A^\wedge \to R$ is a surjection. Since the completion of A is normal, we conclude by Lemma 54.10.2 that after a finite number of blowups

$$\mathrm{Spec}(A^\wedge) \leftarrow X^\wedge \leftarrow (X')^\wedge \leftarrow \ldots$$

the resulting scheme $(X^{(n)})^\wedge$ is regular. Since $(X^{(n)})^\wedge \to X^{(n)}$ induces isomorphisms on complete local rings (Lemma 54.11.1) we conclude that the same is true for $X^{(n)}$.

Resolution in case II. Here we have

$$x_3^2 = \sum a_{ijk} x_i x_j x_k$$

in A for some choice of generators x_1, x_2, x_3 of $\mathfrak{m}$. Then $q = t_3^2$ and $E = 2C$ where C is a line. Recall that in A' we get

$$y_3^2 = x_1(\sum a_{ijk} y_i y_j y_k)$$

Since we know that X is normal, we get a discrete valuation ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of C. The element $y_3 \in A'$ maps to a uniformizer of $\mathcal{O}_{X,\xi}$. Since x_1 scheme theoretically cuts out E which is C with multiplicity 2, we see that x_1 is a unit times y_3^2 in $\mathcal{O}_{X,\xi}$. Looking at our equality above we conclude that

$$h(y_2) = \overline{a}_{111} + \overline{a}_{112} y_2 + \overline{a}_{122} y_2^2 + \overline{a}_{222} y_2^3$$

must be nonzero in the residue field of ξ. Now, suppose that $p \in C$ defines a singular point. Then y_3 is zero at p and p must correspond to a zero of h by the reasoning used in proving the claim above. If h does not have a double zero at p, then the quadratic form q' at p is not a square and we conclude that p falls in

case I which we have treated above[1]. Since the degree of h is 3 we get at most one singular point $p \in C$ falling into case II which is moreover κ-rational. After changing our choice of x_1, x_2, x_3 we may assume this is the point $y_2 = y_3 = 0$. Then $h = \overline{a}_{122}y_2^2 + \overline{a}_{222}y_2^3$. Moreover, it still has to be the case that $\overline{a}_{113} = 0$ for the quadratic form q' to have the right shape. Thus the local ring $\mathcal{O}_{X,p}$ defines a singularity as in the next paragraph.

The final case we treat is the case where we can choose our generators x_1, x_2, x_3 of $\mathfrak{m}$ such that

$$x_3^2 + x_1(ax_2^2 + bx_2x_3 + cx_3^2) \in \mathfrak{m}^4$$

for some $a, b, c \in A$. This is a subclass of case II. If $\overline{a} = 0$, then we can write $a = a_1x_1 + a_2x_2 + a_3x_3$ and we get after blowing up

$$y_3^2 + x_1(a_1x_1y_2^2 + a_2x_1y_2^3 + a_3x_1y_2^2y_3 + by_2y_3 + cy_3^2) = x_1^2(\sum a_{ijkl}y_iy_jy_ky_l)$$

This means that X is not normal[2] a contradiction. By the result of the previous paragraph, if the blowup X has a singular point p which falls in case II, then there is only one and it is κ-rational. Computing the affine blowup algebras $A[\frac{\mathfrak{m}}{x_2}]$ and $A[\frac{\mathfrak{m}}{x_3}]$ the reader easily sees that p cannot be contained the corresponding opens of X. Thus p is in the spectrum of $A[\frac{\mathfrak{m}}{x_1}]$. Doing the blowing up as before we see that p must be the point with coordinates $y_2 = y_3 = 0$ and the new equation looks like

$$y_3^2 + x_1(ay_2^2 + by_2y_3 + cy_3^2) \in (\mathfrak{m}')^4$$

which has the same shape as before and has the property that x_1 defines the exceptional divisor. Thus if the process does not stop we get an infinite sequence of blowups and on each of these x_1 defines the exceptional divisor in the local ring of the singular point. Thus we can finish the proof using Lemmas 54.10.1 and 54.10.2 and the same reasoning as before.

0BGE **Lemma 54.12.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. Then there exists a finite sequence of blowups in singular closed points*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_n is regular and such that each intervening schemes X_i is normal with finitely many singular points of the same type.

Proof. This is exactly what was proved in the discussion above. □

54.13. Implied properties

0BGF In this section we prove that for a Noetherian integral scheme the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian rings.

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[1]The maximal ideal at p in A' is generated by y_3, x_1 and a third element g whose image in $\kappa[y_2]$ is the prime divisor of h corresponding to p. If this prime divisor doesn't divide h twice, then we see that the quadratic form at p looks like

$$y_3^2 - x_1((something)x_1 + (something)y_3 + (unit)g)$$

and this can never be a square in $\kappa[y_3, x_1, g]$.

[2]Namely, the equation shows that you get something singular along the 1-dimensional locus $x_1 = y_3 = 0$ which cannot happen for a normal surface.

0BGG **Lemma 54.13.1.** *Let Y be a Noetherian integral scheme. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.*

Proof. It suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Schemes, Lemma 30.19.2 we see that C is a finite A-module. As X is normal (Properties, Lemma 28.9.4) we see that C is normal domain (Properties, Lemma 28.7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Morphisms, Definition 29.51.12. After shrinking V we may assume that $f^{-1}V \to V$ is flat (Morphisms, Proposition 29.27.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 10.164.4. □

0BGH **Lemma 54.13.2.** *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let $B \subset C$ be finite A-algebras. Assume that (a) B is a normal ring, and (b) the $\mathfrak{m}$-adic completion $C^\wedge$ is a normal ring. Then $B^\wedge$ is a normal ring.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B^\wedge & \longrightarrow & C^\wedge \end{array}$$

Recall that $\mathfrak{m}$-adic completion on the category of finite A-modules is exact because it is given by tensoring with the flat A-algebra $A^\wedge$ (Algebra, Lemma 10.97.2). We will use Serre's criterion (Algebra, Lemma 10.157.4) to prove that the Noetherian ring $B^\wedge$ is normal. Let $\mathfrak{q} \subset B^\wedge$ be a prime lying over $\mathfrak{p} \subset B$. If $\dim(B_\mathfrak{p}) \geq 2$, then $\mathrm{depth}(B_\mathfrak{p}) \geq 2$ and since $B_\mathfrak{p} \to B^\wedge_\mathfrak{q}$ is flat we find that $\mathrm{depth}(B^\wedge_\mathfrak{q}) \geq 2$ (Algebra, Lemma 10.163.2). If $\dim(B_\mathfrak{p}) \leq 1$, then $B_\mathfrak{p}$ is either a discrete valuation ring or a field. In that case $C_\mathfrak{p}$ is faithfully flat over $B_\mathfrak{p}$ (because it is finite and torsion free). Hence $B^\wedge_\mathfrak{p} \to C^\wedge_\mathfrak{p}$ is faithfully flat and the same holds after localizing at $\mathfrak{q}$. As $C^\wedge$ and hence any localization is (S_2) we conclude that $B^\wedge_\mathfrak{p}$ is (S_2) by Algebra, Lemma 10.164.5. All in all we find that (S_2) holds for $B^\wedge$. To prove that $B^\wedge$ is (R_1) we only have to consider primes $\mathfrak{q} \subset B^\wedge$ with $\dim(B^\wedge_\mathfrak{q}) \leq 1$. Since $\dim(B^\wedge_\mathfrak{q}) = \dim(B_\mathfrak{p}) + \dim(B^\wedge_\mathfrak{q}/\mathfrak{p}B^\wedge_\mathfrak{q})$ by Algebra, Lemma 10.112.6 we find that $\dim(B_\mathfrak{p}) \leq 1$ and we see that $B^\wedge_\mathfrak{q} \to C^\wedge_\mathfrak{q}$ is faithfully flat as before. We conclude using Algebra, Lemma 10.164.6. □

0BGI **Lemma 54.13.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \mathrm{Spec}(A)$ with X regular. Then*

1. *there exists a nonzero $f \in A$ such that A_f is regular,*
2. *the integral closure B of A in its fraction field is finite over A,*
3. *the $\mathfrak{m}$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and*
4. *the generic formal fibre of A is regular.*

Proof. Parts (1) and (2) follow from Lemma 54.13.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Schemes, Lemma 30.19.2 we see that C is a

finite A-module. As X is normal (Properties, Lemma 28.9.4) we see that C is normal domain (Properties, Lemma 28.7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Lemma 54.13.2 in order to prove (3) it suffices to show that the $\mathfrak{m}$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 10.97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak{m}$. There are finitely many of these and these are the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak{m}_i}$ and X by $X_i = X \times_{\mathrm{Spec}(C)} \mathrm{Spec}(C_{\mathfrak{m}_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal{O}) = C_{\mathfrak{m}_i}$ by Cohomology of Schemes, Lemma 30.5.2.)

Here A is a Noetherian local normal domain and $f : X \to \mathrm{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 54.11.2 $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Schemes, Lemma 30.5.2, we conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties, Lemma 28.9.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for Noetherian schemes). Thus $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Properties, Lemma 28.7.9. This proves (3).

Proof of (4). Let $\eta \in \mathrm{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular by Lemma 54.11.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \mathrm{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 10.164.4. □

54.14. Resolution

0BGJ Here is a definition.

0BGK **Definition 54.14.1.** Let Y be a Noetherian integral scheme. A *resolution of singularities* of Y is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

0BGL **Definition 54.14.2.** Let Y be a 2-dimensional Noetherian integral scheme. We say Y has a *resolution of singularities by normalized blowups* if there exists a sequence

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well.

0BGM **Lemma 54.14.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Assume A is normal and has dimension 2. If $\mathrm{Spec}(A)$ has a resolution of singularities, then $\mathrm{Spec}(A)$ has a resolution by normalized blowups.*

Proof. By Lemma 54.13.3 the completion $A^\wedge$ of A is normal. By Lemma 54.11.2 we see that $\mathrm{Spec}(A^\wedge)$ has a resolution. By Lemma 54.11.7 any sequence $Y_n \to Y_{n-1} \to \ldots \to \mathrm{Spec}(A^\wedge)$ of normalized blowups of comes from a sequence of normalized blowups $X_n \to \ldots \to \mathrm{Spec}(A)$. Moreover if Y_n is regular, then X_n is regular by Lemma 54.11.2. Thus it suffices to prove the lemma in case A is complete.

Assume in addition A is a complete. We will use that A is Nagata (Algebra, Proposition 10.162.16), excellent (More on Algebra, Proposition 15.52.3), and has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 15.42.2 applies). We will use this without further mention in the rest of the proof.

Let $X \to \mathrm{Spec}(A)$ be a resolution of singularities. Choose a sequence of normalized blowing ups

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \mathrm{Spec}(A)$$

dominating X (Lemma 54.5.3). The morphism $Y_n \to X$ is an isomorphism away from finitely many points of X. Hence we can apply Lemma 54.4.2 to find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X$$

in closed points such that X_m dominates Y_n. Diagram

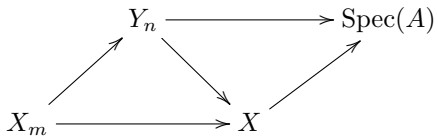

To prove the lemma it suffices to show that a finite number of normalized blowups of Y_n produce a regular scheme. By our diagram above we see that Y_n has a resolution (namely X_m). As Y_n is a normal surface this implies that Y_n has at most finitely many singularities $y_1, \ldots, y_t$ (because $X_m \to Y_n$ is an isomorphism away from the fibres of dimension 1, see Varieties, Lemma 33.17.3).

Let $x_a \in X$ be the image of y_a. Then $\mathcal{O}_{X,x_a}$ is regular and hence defines a rational singularity (Lemma 54.8.7). Apply Lemma 54.8.4 to $\mathcal{O}_{X,x_a} \to \mathcal{O}_{Y_n,y_a}$ to see that $\mathcal{O}_{Y_n,y_a}$ defines a rational singularity. By Lemma 54.9.8 there exists a finite sequence of blowups in singular closed points

$$Y_{a,n_a} \to Y_{a,n_a-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}_{Y_n,y_a})$$

such that Y_{a,n_a} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 54.6.4 with $n' = \sum n_a$ these sequences correspond to a sequence of blowups

$$Y_{n+n'} \to Y_{n+n'-1} \to \ldots \to Y_n$$

such that $Y_{n+n'}$ is normal and the local rings of $Y_{n+n'}$ are Gorenstein. Using the references given above we can dominate $Y_{n+n'}$ by a sequence of blowups $X_{m+m'} \to$

$\ldots \to X_m$ dominating $Y_{n+n'}$ as in the following

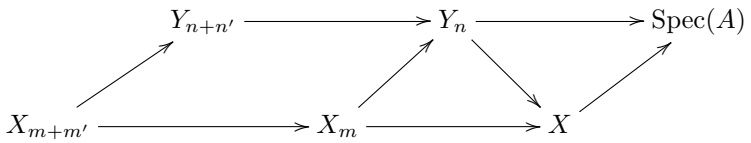

Thus again $Y_{n+n'}$ has a finite number of singular points $y'_1, \ldots, y'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{Y_{n+n'}, y'_b}$ are as in Lemma 54.12.3. Arguing exactly as above we conclude that the lemma is true. □

0BGN **Lemma 54.14.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Assume A is a normal domain of dimension 2. Then $\mathrm{Spec}(A)$ has a resolution of singularities.*

Proof. A Noetherian complete local ring is J-2 (More on Algebra, Proposition 15.48.7), Nagata (Algebra, Proposition 10.162.16), excellent (More on Algebra, Proposition 15.52.3), and has a dualizing complex (Dualizing Complexes, Lemma 47.22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 15.42.2 applies). In other words, the local rings which we encounter in the rest of the proof will have the required "excellency" properties required of them.

Choose $A_0 \subset A$ with A_0 a regular complete local ring and $A_0 \to A$ finite, see Algebra, Lemma 10.160.11. This induces a finite extension of fraction fields K/K_0. We will argue by induction on $[K : K_0]$. The base case is when the degree is 1 in which case $A_0 = A$ and the result is true.

Suppose there is an intermediate field $K_0 \subset L \subset K$, $K_0 \neq L \neq K$. Let $B \subset A$ be the integral closure of A_0 in L. By induction we choose a resolution of singularities $Y \to \mathrm{Spec}(B)$. Let X be the normalization of $Y \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. Picture:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(B) \end{array}$$

Since A is J-2 the regular locus of X is open. Since X is a normal surface we conclude that X has at worst finitely many singular points $x_1, \ldots, x_n$ which are closed points with $\dim(\mathcal{O}_{X,x_i}) = 2$. For each i let $y_i \in Y$ be the image. Since $\mathcal{O}^\wedge_{Y,y_i} \to \mathcal{O}^\wedge_{X,x_i}$ is finite of smaller degree than before we conclude by induction hypothesis that $\mathcal{O}^\wedge_{X,x_i}$ has resolution of singularities. By Lemma 54.14.3 there is a sequence

$$Z^\wedge_{i,n_i} \to \ldots \to Z^\wedge_{i,1} \to \mathrm{Spec}(\mathcal{O}^\wedge_{X,x_i})$$

of normalized blowups with $Z^\wedge_{i,n_i}$ regular. By Lemma 54.11.7 there is a corresponding sequence of normalized blowing ups

$$Z_{i,n_i} \to \ldots \to Z_{i,1} \to \mathrm{Spec}(\mathcal{O}_{X,x_i})$$

Then Z_{i,n_i} is a regular scheme by Lemma 54.11.2. By Lemma 54.6.5 we can fit these normalized blowing ups into a corresponding sequence

$$Z_n \to Z_{n-1} \to \ldots \to Z_1 \to X$$

and of course Z_n is regular too (look at the local rings). This proves the induction step.

Assume there is no intermediate field $K_0 \subset L \subset K$ with $K_0 \neq L \neq K$. Then either K/K_0 is separable or the characteristic to K is p and $[K : K_0] = p$. Then either Lemma 54.8.6 or 54.8.10 implies that reduction to rational singularities is possible. By Lemma 54.8.5 we conclude that there exists a normal modification $X \to \mathrm{Spec}(A)$ such that for every singular point x of X the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. Since A is J-2 we find that X has finitely many singular points $x_1, \ldots, x_n$. By Lemma 54.9.8 there exists a finite sequence of blowups in singular closed points

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}_{X,x_i})$$

such that X_{i,n_i} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 54.6.4 with $n = \sum n_a$ these sequences correspond to a sequence of blowups

$$X_n \to X_{n-1} \to \ldots \to X$$

such that X_n is normal and the local rings of X_n are Gorenstein. Again X_n has a finite number of singular points $x'_1, \ldots, x'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{X_n,x'_i}$ are as in Lemma 54.12.3. Arguing exactly as above we conclude that the lemma is true. □

We finally come to the main theorem of this chapter.

0BGP **Theorem 54.14.5** (Lipman)**.** *Let Y be a two dimensional integral Noetherian scheme. The following are equivalent* [**Lip78**, Theorem on page 151]

(1) *there exists an alteration $X \to Y$ with X regular,*
(2) *there exists a resolution of singularities of Y,*
(3) *Y has a resolution of singularities by normalized blowups,*
(4) *the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m$, and for each y_i the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal.*

Proof. The implications (3) ⇒ (2) ⇒ (1) are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 54.13.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms, Lemma 29.54.5. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Varieties, Lemma 33.17.2. Then f is flat over V by Algebra, Lemma 10.128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 10.157.4). Then V is regular by Algebra, Lemma 10.164.4. As Y^ν is Noetherian we conclude that $Y^\nu \setminus V = \{y_1, \ldots, y_m\}$ is finite. By Lemma 54.13.3 the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal. In this way we see that (1) ⇒ (4).

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in Y$ be the singular points. Applying Lemmas 54.14.4 and 54.14.3 we find there exists a finite sequence of normalized blowups

$$Y_{i,n_i} \to Y_{i,n_i-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}^\wedge_{Y,y_i})$$

such that Y_{i,n_i} is regular. By Lemma 54.11.7 there is a corresponding sequence of normalized blowing ups

$$X_{i,n_i} \to \ldots \to X_{i,1} \to \mathrm{Spec}(\mathcal{O}_{Y,y_i})$$

Then X_{i,n_i} is a regular scheme by Lemma 54.11.2. By Lemma 54.6.5 we can fit these normalized blowing ups into a corresponding sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. □

54.15. Embedded resolution

0BI3 Given a curve on a surface there is a blowing up which turns the curve into a strict normal crossings divisor. In this section we will use that a one dimensional locally Noetherian scheme is normal if and only if it is regular (Algebra, Lemma 10.119.7). We will also use that any point on a locally Noetherian scheme specializes to a closed point (Properties, Lemma 28.5.9).

0BI4 **Lemma 54.15.1.** *Let Y be a one dimensional integral Noetherian scheme. The following are equivalent*

(1) *there exists an alteration $X \to Y$ with X regular,*
(2) *there exists a resolution of singularities of Y,*
(3) *there exists a finite sequence $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ of blowups in closed points with Y_n regular, and*
(4) *the normalization $Y^\nu \to Y$ is finite.*

Proof. The implications (3) ⇒ (2) ⇒ (1) are immediate. The implication (1) ⇒ (4) follows from Lemma 54.13.1. Observe that a normal one dimensional scheme is regular hence the implication (4) ⇒ (2) is clear as well. Thus it remains to show that the equivalent conditions (1), (2), and (4) imply (3).

Let $f : X \to Y$ be a resolution of singularities. Since the dimension of Y is one we see that f is finite by Varieties, Lemma 33.17.2. We will construct factorizations

$$X \to \ldots \to Y_2 \to Y_1 \to Y$$

where $Y_i \to Y_{i-1}$ is a blowing up of a closed point and not an isomorphism as long as Y_{i-1} is not regular. Each of these morphisms will be finite (by the same reason as above) and we will get a corresponding system

$$f_*\mathcal{O}_X \supset \ldots \supset f_{2,*}\mathcal{O}_{Y_2} \supset f_{1,*}\mathcal{O}_{Y_1} \supset \mathcal{O}_Y$$

where $f_i : Y_i \to Y$ is the structure morphism. Since Y is Noetherian, this increasing sequence of coherent submodules must stabilize (Cohomology of Schemes, Lemma 30.10.1) which proves that for some n the scheme Y_n is regular as desired. To construct Y_i given Y_{i-1} we pick a singular closed point $y_{i-1} \in Y_{i-1}$ and we let $Y_i \to Y_{i-1}$ be the corresponding blowup. Since X is regular of dimension 1 (and hence the local rings at closed points are discrete valuation rings and in particular PIDs), the ideal sheaf $\mathfrak{m}_{y_{i-1}} \cdot \mathcal{O}_X$ is invertible. By the universal property of blowing up (Divisors, Lemma 31.32.5) this gives us a factorization $X \to Y_i$. Finally, $Y_i \to Y_{i-1}$ is not an isomorphism as $\mathfrak{m}_{y_{i-1}}$ is not an invertible ideal. □

0BI5 **Lemma 54.15.2.** *Let X be a Noetherian scheme. Let $Y \subset X$ be an integral closed subscheme of dimension 1 satisfying the equivalent conditions of Lemma 54.15.1. Then there exists a finite sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform of Y in X_n is a regular curve.

Proof. Let $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ be the sequence of blowups given to us by Lemma 54.15.1. Let $X_n \to X_{n-1} \to \ldots \to X_1 \to X$ be the corresponding sequence of blowups of X. This works because the strict transform is the blowup by Divisors, Lemma 31.33.2. □

Let X be a locally Noetherian scheme. Let $Y, Z \subset X$ be closed subschemes. Let $p \in Y \cap Z$ be a closed point. Assume that Y is integral of dimension 1 and that the generic point of Y is not contained in Z. In this situation we can consider the invariant

0BI6 (54.15.2.1) $$m_p(Y \cap Z) = \text{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{Y \cap Z,p})$$

This is an integer ≥ 1. Namely, if $I, J \subset \mathcal{O}_{X,p}$ are the ideals corresponding to Y, Z, then we see that $\mathcal{O}_{Y \cap Z,p} = \mathcal{O}_{X,p}/I + J$ has support equal to $\{\mathfrak{m}_p\}$ because we assumed that $Y \cap Z$ does not contain the unique point of Y specializing to p. Hence the length is finite by Algebra, Lemma 10.62.3.

0BI7 **Lemma 54.15.3.** *In the situation above let $X' \to X$ be the blowing up of X in p. Let $Y', Z' \subset X'$ be the strict transforms of Y, Z. If $\mathcal{O}_{Y,p}$ is regular, then*

(1) *$Y' \to Y$ is an isomorphism,*
(2) *Y' meets the exceptional fibre $E \subset X'$ in one point q and $m_q(Y \cap E) = 1$,*
(3) *if $q \in Z'$ too, then $m_q(Y \cap Z') < m_p(Y \cap Z)$.*

Proof. Since $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective and $\mathcal{O}_{Y,p}$ is a discrete valuation ring, we can pick an element $x_1 \in \mathfrak{m}_p$ mapping to a uniformizer in $\mathcal{O}_{Y,p}$. Choose an affine open $U = \text{Spec}(A)$ containing p such that $x_1 \in A$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to p. Let $I, J \subset A$ be the ideals defining Y, Z in $\text{Spec}(A)$. After shrinking U we may assume that $\mathfrak{m} = I + (x_1)$, in other words, that $V(x_1) \cap U \cap Y = \{p\}$ scheme theoretically. We conclude that p is an effective Cartier divisor on Y and since Y' is the blowing up of Y in p (Divisors, Lemma 31.33.2) we see that $Y' \to Y$ is an isomorphism by Divisors, Lemma 31.32.7. The relationship $\mathfrak{m} = I + (x_1)$ implies that $\mathfrak{m}^n \subset I + (x_1^n)$ hence we can define a map

$$\psi : A[\tfrac{\mathfrak{m}}{x_1}] \longrightarrow A/I$$

by sending $y/x_1^n \in A[\frac{\mathfrak{m}}{x_1}]$ to the class of a in A/I where a is chosen such that $y \equiv ax_1^n \bmod I$. Then ψ corresponds to the morphism of $Y \cap U$ into X' over U given by $Y' \cong Y$. Since the image of x_1 in $A[\frac{\mathfrak{m}}{x_1}]$ cuts out the exceptional divisor we conclude that $m_q(Y', E) = 1$. Finally, since $J \subset \mathfrak{m}$ implies that the ideal $J' \subset A[\frac{\mathfrak{m}}{x_1}]$ certainly contains the elements f/x_1 for $f \in J$. Thus if we choose $f \in J$ whose image $\overline{f}$ in A/I has minimal valuation equal to $m_p(Y \cap Z)$, then we see that $\psi(f/x_1) = \overline{f}/x_1$ in A/I has valuation one less proving the last part of the lemma. □

0BI8 **Lemma 54.15.4.** *Let X be a Noetherian scheme. Let $Y_i \subset X$, $i = 1, \ldots, n$ be an integral closed subschemes of dimension 1 each satisfying the equivalent conditions of Lemma 54.15.1. Then there exists a finite sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform $Y_i' \subset X_n$ of Y_i in X_n are pairwise disjoint regular curves.

Proof. It follows from Lemma 54.15.2 that we may assume Y_i is a regular curve for $i = 1, \ldots, n$. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (54.15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 54.15.3) by blowing up in all the points p where the maximum is attained. If the maximum is 1 then we can separate the curves using the same lemma by blowing up in all these points p. □

When our curve is contained on a regular surface we often want to turn it into a divisor with normal crossings.

0BIB **Lemma 54.15.5.** *Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme. There exists a sequence*

$$X_n \to \ldots \to X_1 \to X$$

of blowing ups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor.

Proof. Let $D \subset Z$ be the largest effective Cartier divisor contained in Z. Then $\mathcal{I}_Z \subset \mathcal{I}_D$ and the quotient is supported in closed points by Divisors, Lemma 31.15.8. Thus we can write $\mathcal{I}_Z = \mathcal{I}_{Z'}\mathcal{I}_D$ where $Z' \subset X$ is a closed subscheme which set theoretically consists of finitely many closed points. Applying Lemma 54.4.1 we find a sequence of blowups as in the statement of our lemma such that $\mathcal{I}_{Z'}\mathcal{O}_{X_n}$ is invertible. This proves the lemma. □

0BIC **Lemma 54.15.6.** *Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme such that every irreducible component $Y \subset Z$ of dimension 1 satisfies the equivalent conditions of Lemma 54.15.1. Then there exists a sequence*

$$X_n \to \ldots \to X_1 \to X$$

of blowups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor supported on a strict normal crossings divisor.

Proof. Let $X' \to X$ be a blowup in a closed point p. Then the inverse image $Z' \subset X'$ of Z is supported on the strict transform of Z and the exceptional divisor. The exceptional divisor is a regular curve (Lemma 54.3.1) and the strict transform Y' of each irreducible component Y is either equal to Y or the blowup of Y at p. Thus in this process we do not produce additional singular components of dimension 1. Thus it follows from Lemmas 54.15.5 and 54.15.4 that we may assume Z is an effective Cartier divisor and that all irreducible components Y of Z are regular. (Of course we cannot assume the irreducible components are pairwise disjoint because in each blowup of a point of Z we add a new irreducible component to Z, namely the exceptional divisor.)

Assume Z is an effective Cartier divisor whose irreducible components Y_i are regular. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (54.15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 54.15.3) by blowing up in all the points p where the maximum is attained (note that the "new" invariants $m_{q_i}(Y_i' \cap E)$ are always 1). If the maximum is 1 then, if $p \in Y_1 \cap \ldots \cap Y_r$ for some $r > 2$ and not any of the others (for example), then after blowing up p we see that $Y_1', \ldots, Y_r'$ do not meet in points above p and $m_{q_i}(Y_i', E) = 1$ where $Y_i' \cap E = \{q_i\}$. Thus continuing to blowup points where more than 3 of the components of Z meet, we reach the situation where for every closed point $p \in X$ there is

either (a) no curves Y_i passing through p, (b) exactly one curve Y_i passing through p and $\mathcal{O}_{Y_i,p}$ is regular, or (c) exactly two curves Y_i, Y_j passing through p, the local rings $\mathcal{O}_{Y_i,p}$, $\mathcal{O}_{Y_j,p}$ are regular and $m_p(Y_i \cap Y_j) = 1$. This means that $\sum Y_i$ is a strict normal crossings divisor on the regular surface X, see Étale Morphisms, Lemma 41.21.2. □

54.16. Contracting exceptional curves

0C2I Let X be a Noetherian scheme. Let $E \subset X$ be a closed subscheme with the following properties

(1) E is an effective Cartier divisor on X,
(2) there exists a field k and an isomorphism $\mathbf{P}^1_k \to E$ of schemes,
(3) the normal sheaf $\mathcal{N}_{E/X}$ pulls back to $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Such a closed subscheme is called an *exceptional curve of the first kind*.

Let X' be a Noetherian scheme and let $x \in X'$ be a closed point such that $\mathcal{O}_{X',x}$ is regular of dimension 2. Let $b : X \to X'$ be the blowing up of X' at x. In this case the exceptional fibre $E \subset X$ is an exceptional curve of the first kind. This follows from Lemma 54.3.1.

Question: Is every exceptional curve of the first kind obtained as the fibre of a blowing up as above? In other words, does there always exist a proper morphism of schemes $X \to X'$ such that E maps to a closed point $x \in X'$, such that $\mathcal{O}_{X',x}$ is regular of dimension 2, and such that X is the blowing up of X' at x. If true we say *there exists a contraction of* E.

0C5J **Lemma 54.16.1.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If a contraction $X \to X'$ of E exists, then it has the following universal property: for every morphism $\varphi : X \to Y$ such that $\varphi(E)$ is a point, there is a unique factorization $X \to X' \to Y$ of φ.*

Proof. Let $b : X \to X'$ be a contraction of E. As a topological space X' is the quotient of X by the relation identifying all points of E to one point. Namely, b is proper (Divisors, Lemma 31.32.13 and Morphisms, Lemma 29.43.5) and surjective, hence defines a submersive map of topological spaces (Topology, Lemma 5.6.5). On the other hand, the canonical map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. Namely, this is clear over the complement of the image point $x \in X'$ of E and on stalks at x the map is an isomorphism by part (4) of Lemma 54.3.4. Thus the pair $(X', \mathcal{O}_{X'})$ is constructed from X by taking the quotient as a topological space and endowing this with $b_*\mathcal{O}_X$ as structure sheaf.

Given φ we can let $\varphi' : X' \to Y$ be the unique map of topological spaces such that $\varphi = \varphi' \circ b$. Then the map

$$\varphi^\sharp : \varphi^{-1}\mathcal{O}_Y = b^{-1}((\varphi')^{-1}\mathcal{O}_Y) \to \mathcal{O}_X$$

is adjoint to a map

$$(\varphi')^\sharp : (\varphi')^{-1}\mathcal{O}_Y \to b_*\mathcal{O}_X = \mathcal{O}_{X'}$$

Then $(\varphi', (\varphi')^\sharp)$ is a morphism of ringed spaces from X' to Y such that we get the desired factorization. Since φ is a morphism of locally ringed spaces, it follows that φ' is too. Namely, the only thing to check is that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X',x}$ is local, where $y \in Y$ is the image of E under φ. This is true because an element $f \in \mathfrak{m}_y$ pulls back to a function on X which is zero in every point of E hence the

pull back of f to X' is a function defined on a neighbourhood of x in X' with the same property. Then it is clear that this function must vanish at x as desired. □

0C5K **Lemma 54.16.2.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a contraction of E, then it is unique up to unique isomorphism.*

Proof. This is immediate from the universal property of Lemma 54.16.1. □

0C2K **Lemma 54.16.3.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $E_n = nE$ and denote $\mathcal{O}_n$ its structure sheaf. Then*

$$A = \lim H^0(E_n, \mathcal{O}_n)$$

is a complete local Noetherian regular local ring of dimension 2 and $\mathrm{Ker}(A \to H^0(E_n, \mathcal{O}_n))$ is the nth power of its maximal ideal.

Proof. Recall that there exists an isomorphism $\mathbf{P}^1_k \to E$ such that the normal sheaf of E in X pulls back to $\mathcal{O}(-1)$. Then $H^0(E, \mathcal{O}_E) = k$. We will denote $\mathcal{O}_n(iE)$ the restriction of the invertible sheaf $\mathcal{O}_X(iE)$ to E_n for all $n \geq 1$ and $i \in \mathbf{Z}$. Recall that $\mathcal{O}_X(-nE)$ is the ideal sheaf of E_n. Hence for $d \geq 0$ we obtain a short exact sequence

$$0 \to \mathcal{O}_E(-(d+n)E) \to \mathcal{O}_{n+1}(-dE) \to \mathcal{O}_n(-dE) \to 0$$

Since $\mathcal{O}_E(-(d+n)E) = \mathcal{O}_{\mathbf{P}^1_k}(d+n)$ the first cohomology group vanishes for all $d \geq 0$ and $n \geq 1$. We conclude that the transition maps of the system $H^0(E_n, \mathcal{O}_n(-dE))$ are surjective. For $d = 0$ we get an inverse system of surjections of rings such that the kernel of each transition map is a nilpotent ideal. Hence $A = \lim H^0(E_n, \mathcal{O}_n)$ is a local ring with residue field k and maximal ideal

$$\lim \mathrm{Ker}(H^0(E_n, \mathcal{O}_n) \to H^0(E, \mathcal{O}_E)) = \lim H^0(E_n, \mathcal{O}_n(-E))$$

Pick x, y in this kernel mapping to a k-basis of $H^0(E, \mathcal{O}_E(-E)) = H^0(\mathbf{P}^1_k, \mathcal{O}(1))$. Then $x^d, x^{d-1}y, \ldots, y^d$ are elements of $\lim H^0(E_n, \mathcal{O}_n(-dE))$ which map to a basis of $H^0(E, \mathcal{O}_E(-dE)) = H^0(\mathbf{P}^1_k, \mathcal{O}(d))$. In this way we see that A is separated and complete with respect to the linear topology defined by the kernels

$$I_n = \mathrm{Ker}(A \longrightarrow H^0(E_n, \mathcal{O}_n))$$

We have $x, y \in I_1$, $I_d I_{d'} \subset I_{d+d'}$ and I_d/I_{d+1} is a free k-module on $x^d, x^{d-1}y, \ldots, y^d$. We will show that $I_d = (x, y)^d$. Namely, if $z_e \in I_e$ with $e \geq d$, then we can write

$$z_e = a_{e,0}x^d + a_{e,1}x^{d-1}y + \ldots + a_{e,d}y^d + z_{e+1}$$

where $a_{e,j} \in (x, y)^{e-d}$ and $z_{e+1} \in I_{e+1}$ by our description of I_d/I_{d+1}. Thus starting with some $z = z_d \in I_d$ we can do this inductively

$$z = \sum\nolimits_{e \geq d} \sum\nolimits_j a_{e,j} x^{d-j} y^j$$

with some $a_{e,j} \in (x, y)^{e-d}$. Then $a_j = \sum_{e \geq d} a_{e,j}$ exists (by completeness and the fact that $a_{e,j} \in I_{e-d}$) and we have $z = \sum a_{e,j} x^{d-j} y^j$. Hence $I_d = (x, y)^d$. Thus A is (x, y)-adically complete. Then A is Noetherian by Algebra, Lemma 10.97.5. It is clear that the dimension is 2 by the description of $(x, y)^d/(x, y)^{d+1}$ and Algebra, Proposition 10.60.9. Since the maximal ideal is generated by two elements it is regular. □

0C2L **Lemma 54.16.4.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a morphism $f : X \to Y$ such that*

(1) *Y is Noetherian,*
(2) *f is proper,*
(3) *f maps E to a point y of Y,*
(4) *f is quasi-finite at every point not in E,*

Then there exists a contraction of E and it is the Stein factorization of f.

Proof. We apply More on Morphisms, Theorem 37.53.4 to get a Stein factorization $X \to X' \to Y$. Then $X \to X'$ satisfies all the hypotheses of the lemma (some details omitted). Thus after replacing Y by X' we may in addition assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected.

Assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected. Note that $y \in Y$ is a closed point as f is closed and E is closed. The restriction $f^{-1}(Y \setminus \{y\}) \to Y \setminus \{y\}$ of f is a finite morphism (More on Morphisms, Lemma 37.44.1). Hence this restriction is an isomorphism since $f_*\mathcal{O}_X = \mathcal{O}_Y$ since finite morphisms are affine. To prove that $\mathcal{O}_{Y,y}$ is regular of dimension 2 we consider the isomorphism

$$\mathcal{O}_{Y,y}^\wedge \longrightarrow \lim H^0(X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n), \mathcal{O})$$

of Cohomology of Schemes, Lemma 30.20.7. Let $E_n = nE$ as in Lemma 54.16.3. Observe that

$$E_n \subset X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$$

because $E \subset X_y = X \times_Y \operatorname{Spec}(\kappa(y))$. On the other hand, since $E = f^{-1}(\{y\})$ set theoretically (because the fibres of f are geometrically connected), we see that the scheme theoretic fibre X_y is scheme theoretically contained in E_n for some $n > 0$. Namely, apply Cohomology of Schemes, Lemma 30.10.2 to the coherent $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_{X_y}$ and the ideal sheaf $\mathcal{I}$ of E and use that $\mathcal{I}^n$ is the ideal sheaf of E_n. This shows that

$$X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^m) \subset E_{nm}$$

Thus the inverse limit displayed above is equal to $\lim H^0(E_n, \mathcal{O}_n)$ which is a regular two dimensional local ring by Lemma 54.16.3. Hence $\mathcal{O}_{Y,y}$ is a two dimensional regular local ring because its completion is so (More on Algebra, Lemma 15.43.4 and 15.43.1).

We still have to prove that $f : X \to Y$ is the blowup $b : Y' \to Y$ of Y at y. We encourage the reader to find her own proof. First, we note that Lemma 54.16.3 also implies that $X_y = E$ scheme theoretically. Since the ideal sheaf of E is invertible, this shows that $f^{-1}\mathfrak{m}_y \cdot \mathcal{O}_X$ is invertible. Hence we obtain a factorization

$$X \to Y' \to Y$$

of the morphism f by the universal property of blowing up, see Divisors, Lemma 31.32.5. Recall that the exceptional fibre of $E' \subset Y'$ is an exceptional curve of the first kind by Lemma 54.3.1. Let $g : E \to E'$ be the induced morphism. Because for both E' and E the conormal sheaf is generated by (pullbacks of) a and b, we see that the canonical map $g^*\mathcal{C}_{E'/Y'} \to \mathcal{C}_{E/X}$ (Morphisms, Lemma 29.31.3) is surjective. Since both are invertible, this map is an isomorphism. Since $\mathcal{C}_{E/X}$ has positive degree, it follows that g cannot be a constant morphism. Hence g has finite fibres. Hence g is a finite morphism (same reference as above). However, since Y'

is regular (and hence normal) at all points of E' and since $X \to Y'$ is birational and an isomorphism away from E', we conclude that $X \to Y'$ is an isomorphism by Varieties, Lemma 33.17.3. □

0C5L **Lemma 54.16.5.** *Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. Then there is a short exact sequence*

$$0 \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to \mathbf{Z} \to 0$$

where the first map is pullback by b and the second map sends $\mathcal{L}$ to the degree of $\mathcal{L}$ on the exceptional curve E. The sequence is split by the map $n \mapsto \mathcal{O}_X(-nE)$.

Proof. Since $E = \mathbf{P}^1_k$ we see that the Picard group of E is $\mathbf{Z}$, see Divisors, Lemma 31.28.5. Hence we can think of the last map as $\mathcal{L} \mapsto \mathcal{L}|_E$. The degree of the restriction of $\mathcal{O}_X(E)$ to E is -1 by definition of exceptional curves of the first kind. Combining these remarks we see that it suffices to show that $\mathrm{Pic}(X') \to \mathrm{Pic}(X)$ is injective with image the invertible sheaves restricting to $\mathcal{O}_E$ on E.

Given an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ we claim the map $\mathcal{L}' \to b_*b^*\mathcal{L}'$ is an isomorphism. This is clear everywhere except possibly at the image point $x \in X'$ of E. To check it is an isomorphism on stalks at x we may replace X' by an open neighbourhood at x and assume $\mathcal{L}'$ is $\mathcal{O}_{X'}$. Then we have to show that the map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. This follows from Lemma 54.3.4 part (4).

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with $\mathcal{L}|_E = \mathcal{O}_E$. Then we claim (1) $b_*\mathcal{L}$ is invertible and (2) $b^*b_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Statements (1) and (2) are clear over $X' \setminus \{x\}$. Thus it suffices to prove (1) and (2) after base change to $\mathrm{Spec}(\mathcal{O}_{X',x})$. Computing b_* commutes with flat base change (Cohomology of Schemes, Lemma 30.5.2) and similarly for b^* and formation of the adjunction map. But if X' is the spectrum of a regular local ring then $\mathcal{L}$ is trivial by the description of the Picard group in Lemma 54.3.3. Thus the claim is proved.

Combining the claims proved in the previous two paragraphs we see that the map $\mathcal{L} \mapsto b_*\mathcal{L}$ is an inverse to the map

$$\mathrm{Pic}(X') \longrightarrow \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(E))$$

and the lemma is proved. □

0C5M **Remark 54.16.6.** Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. From Lemma 54.16.5 we obtain an identification

$$\mathrm{Pic}(X) = \mathrm{Pic}(X') \oplus \mathbf{Z}$$

where $\mathcal{L}$ corresponds to the pair $(\mathcal{L}', n)$ if and only if $\mathcal{L} = (b^*\mathcal{L}')(-nE)$, i.e., $\mathcal{L}(nE) = b^*\mathcal{L}'$. In fact the proof of Lemma 54.16.5 shows that $\mathcal{L}' = b_*\mathcal{L}(nE)$. Of course the assignment $\mathcal{L} \mapsto \mathcal{L}'$ is a group homomorphism.

0C2J **Lemma 54.16.7.** *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let n be the integer such that $\mathcal{L}|_E$ has degree n viewed as an invertible module on $\mathbf{P}^1$. Then*

1. *If $H^1(X, \mathcal{L}) = 0$ and $n \geq 0$, then $H^1(X, \mathcal{L}(iE)) = 0$ for $0 \leq i \leq n + 1$.*
2. *If $n \leq 0$, then $H^1(X, \mathcal{L}) \subset H^1(X, \mathcal{L}(E))$.*

Proof. Observe that $\mathcal{L}|_E = \mathcal{O}(n)$ by Divisors, Lemma 31.28.5. Use induction, the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(E) \to \mathcal{L}(E)|_E \to 0,$$

and use the fact that $H^1(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \geq -1$ and $H^0(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \leq -1$. Some details omitted. □

0C2M **Lemma 54.16.8.** *Let $S = \mathrm{Spec}(R)$ be an affine Noetherian scheme. Let $X \to S$ be a proper morphism. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $E \subset X$ be an exceptional curve of the first kind. Then*

(1) *there exists a contraction $b : X \to X'$ of E,*
(2) *X' is proper over S, and*
(3) *the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ is ample with $\mathcal{L}'$ as in Remark 54.16.6.*

Proof. Let n be the degree of $\mathcal{L}|_E$ as in Lemma 54.16.7. Observe that $n > 0$ as $\mathcal{L}$ is ample on E (Varieties, Lemma 33.44.14 and Properties, Lemma 28.26.3). After replacing $\mathcal{L}$ by a power we may assume $H^i(X, \mathcal{L}^{\otimes e}) = 0$ for all $i > 0$ and $e > 0$, see Cohomology of Schemes, Lemma 30.17.1. Finally, after replacing $\mathcal{L}$ by another power we may assume there exist global sections $t_0, \ldots, t_n$ of $\mathcal{L}$ which define a closed immersion $\psi : X \to \mathbf{P}^n_S$, see Morphisms, Lemma 29.39.4.

Set $\mathcal{M} = \mathcal{L}(nE)$. Then $\mathcal{M}|_E \cong \mathcal{O}_E$. Since we have the short exact sequence

$$0 \to \mathcal{M}(-E) \to \mathcal{M} \to \mathcal{O}_E \to 0$$

and since $H^1(X, \mathcal{M}(-E))$ is zero (by Lemma 54.16.7 and the fact that $n > 0$) we can pick a section s_{n+1} of $\mathcal{M}$ which generates $\mathcal{M}|_E$. Finally, denote $s_0, \ldots, s_n$ the sections of $\mathcal{M}$ we get from the sections $t_0, \ldots, t_n$ of $\mathcal{L}$ chosen above via $\mathcal{L} \subset \mathcal{L}(nE) = \mathcal{M}$. Combined the sections $s_0, \ldots, s_n, s_{n+1}$ generate $\mathcal{M}$ in every point of X and therefore define a morphism

$$\varphi : X \longrightarrow \mathbf{P}^{n+1}_S$$

over S, see Constructions, Lemma 27.13.1.

Below we will check the conditions of Lemma 54.16.4. Once this is done we see that the Stein factorization $X \to X' \to \mathbf{P}^{n+1}_S$ of φ is the desired contraction which proves (1). Moreover, the morphism $X' \to \mathbf{P}^{n+1}_S$ is finite hence X' is proper over S (Morphisms, Lemmas 29.44.11 and 29.41.4). This proves (2). Observe that X' has an ample invertible sheaf. Namely the pullback $\mathcal{M}'$ of $\mathcal{O}_{\mathbf{P}^{n+1}_S}(1)$ is ample by Morphisms, Lemma 29.37.7. Observe that $\mathcal{M}'$ pulls back to $\mathcal{M}$ on X (by Constructions, Lemma 27.13.1). Finally, $\mathcal{M} = \mathcal{L}(nE)$. Since in the arguments above we have replaced the original $\mathcal{L}$ by a positive power we conclude that the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ mentioned in (3) of the lemma is ample on X' by Properties, Lemma 28.26.2.

Easy observations: $\mathbf{P}^{n+1}_S$ is Noetherian and φ is proper. Details omitted.

Next, we observe that any point of $U = X \setminus E$ is mapped to the open subscheme W of $\mathbf{P}^{n+1}_S$ where one of the first $n+1$ homogeneous coordinates is nonzero. On the other hand, any point of E is mapped to a point where the first $n+1$ homogeneous coordinates are all zero, in particular into the complement of W. Moreover, it is clear that there is a factorization

$$U = \varphi^{-1}(W) \xrightarrow{\varphi|_U} W \xrightarrow{\mathrm{pr}} \mathbf{P}^n_S$$

of $\psi|_U$ where pr is the projection using the first $n+1$ coordinates and $\psi : X \to \mathbf{P}^n_S$ is the embedding chosen above. It follows that $\varphi|_U : U \to W$ is quasi-finite.

Finally, we consider the map $\varphi|_E : E \to \mathbf{P}^{n+1}_S$. Observe that for any point $x \in E$ the image $\varphi(x)$ has its first $n+1$ coordinates equal to zero, i.e., the morphism $\varphi|_E$ factors through the closed subscheme $\mathbf{P}^0_S \cong S$. The morphism $E \to S = \mathrm{Spec}(R)$ factors as $E \to \mathrm{Spec}(H^0(E, \mathcal{O}_E)) \to \mathrm{Spec}(R)$ by Schemes, Lemma 26.6.4. Since by assumption $H^0(E, \mathcal{O}_E)$ is a field we conclude that E maps to a point in $S \subset \mathbf{P}^{n+1}_S$ which finishes the proof. □

0C2N **Lemma 54.16.9.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. Let $E \subset X$ be an exceptional curve of the first kind which is in a fibre of f.*

(1) *If X is projective over S, then there exists a contraction $X \to X'$ of E and X' is projective over S.*
(2) *If X is quasi-projective over S, then there exists a contraction $X \to X'$ of E and X' is quasi-projective over S.*

Proof. Both cases follow from Lemma 54.16.8 using standard results on ample invertible modules and (quasi-)projective morphisms.

Proof of (1). Projectivity of f means that f is proper and there exists an f-ample invertible module $\mathcal{L}$, see Morphisms, Lemma 29.43.13 and Definition 29.40.1. Let $U \subset S$ be an affine open containing the image of E. By Lemma 54.16.8 there exists a contraction $c : f^{-1}(U) \to V'$ of E and an ample invertible module $\mathcal{N}'$ on V' whose pullback to $f^{-1}(U)$ is equal to $\mathcal{L}(nE)|_{f^{-1}(U)}$. Let $v \in V'$ be the closed point such that c is the blowing up of v. Then we can glue V' and $X \setminus E$ along $f^{-1}(U) \setminus E = V' \setminus \{v\}$ to get a scheme X' over S. The morphisms c and $\mathrm{id}_{X \setminus E}$ glue to a morphism $b : X \to X'$ which is the contraction of E. The inverse image of U in X' is proper over U. On the other hand, the restriction of $X' \to S$ to the complement of the image of v in S is isomorphic to the restriction of $X \to S$ to that open. Hence $X' \to S$ is proper (as being proper is local on the base by Morphisms, Lemma 29.41.3). Finally, $\mathcal{N}'$ and $\mathcal{L}|_{X \setminus E}$ restrict to isomorphic invertible modules over $f^{-1}(U) \setminus E = V' \setminus \{v\}$ and hence glue to an invertible module $\mathcal{L}'$ over X'. The restriction of $\mathcal{L}'$ to the inverse image of U in X' is ample because this is true for $\mathcal{N}'$. For affine opens of S avoiding the image of v, we see that the same is true because it holds for $\mathcal{L}$. Thus $\mathcal{L}'$ is $(X' \to S)$-relatively ample by Morphisms, Lemma 29.37.4 and (1) is proved.

Proof of (2). We can write X as an open subscheme of a scheme $\overline{X}$ projective over S by Morphisms, Lemma 29.43.12. By (1) there is a contraction $b : \overline{X} \to \overline{X}'$ and $\overline{X}'$ is projective over S. Then we let $X' \subset \overline{X}$ be the image of $X \to \overline{X}'$; this is an open as b is an isomorphism away from E. Then $X \to X'$ is the desired contraction. Note that X' is quasi-projective over S as it has an S-relatively ample invertible module by the construction in the proof of part (1). □

0C5N **Lemma 54.16.10.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type with X regular of dimension 2. Then X is quasi-projective over S.*

Proof. By Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) there exists a proper morphism $\pi : X' \to X$ which is an isomorphism over a dense open $U \subset X$

such that $X' \to S$ is H-quasi-projective. By Lemma 54.4.3 there exists a sequence of blowups in closed points

$$X_n \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $X_n \to X'$ extending the rational map $U \to X'$. Observe that $X_n \to X$ is projective by Divisors, Lemma 31.32.13 and Morphisms, Lemma 29.43.14. This implies that $X_n \to X'$ is projective by Morphisms, Lemma 29.43.15. Hence $X_n \to S$ is quasi-projective by Morphisms, Lemma 29.40.3 (and the fact that a projective morphism is quasi-projective, see Morphisms, Lemma 29.43.10). By Lemma 54.16.9 (and uniqueness of contractions Lemma 54.16.2) we conclude that $X_{n-1}, \ldots, X_0 = X$ are quasi-projective over S as desired. □

0C5P **Lemma 54.16.11.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a proper morphism with X regular of dimension 2. Then X is projective over S.*

Proof. This follows from Lemma 54.16.10 and Morphisms, Lemma 29.43.13. □

54.17. Factorization birational maps

0C5Q Proper birational morphisms between nonsingular surfaces are given by sequences of quadratic transforms.

0C5R **Lemma 54.17.1.** *Let $f : X \to Y$ be a proper birational morphism between integral Noetherian schemes regular of dimension 2. Then f is a sequence of blowups in closed points.*

Proof. Let $V \subset Y$ be the maximal open over which f is an isomorphism. Then V contains all codimension 1 points of V (Varieties, Lemma 33.17.3). Let $y \in Y$ be a closed point not contained in V. Then we want to show that f factors through the blowup $b : Y' \to Y$ of Y at y. Namely, if this is true, then at least one (and in fact exactly one) component of the fibre $f^{-1}(y)$ will map isomorphically onto the exceptional curve in Y' and the number of curves in fibres of $X \to Y'$ will be strictly less that the number of curves in fibres of $X \to Y$, so we conclude by induction. Some details omitted.

By Lemma 54.4.3 we know that there exists a sequence of blowing ups

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

in closed points lying over the fibre $f^{-1}(y)$ and a morphism $X' \to Y'$ such that

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is commutative. We want to show that the morphism $X' \to Y'$ factors through X and hence we can use induction on n to reduce to the case where $X' \to X$ is the blowup of X in a closed point $x \in X$ mapping to y.

Let $E \subset X'$ be the exceptional fibre of the blowing up $X' \to X$. If E maps to a point in Y', then we obtain the desired factorization by Lemma 54.16.1. We will

prove that if this is not the case we obtain a contradiction. Namely, if $f'(E)$ is not a point, then $E' = f'(E)$ must be the exceptional curve in Y'. Picture

$$\begin{array}{ccccc} E & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ E' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

Arguing as before f' is an isomorphism in an open neighbourhood of the generic point of E'. Hence $g : E \to E'$ is a finite birational morphism. Then the inverse of g (a rational map) is everywhere defined by Morphisms, Lemma 29.42.5 and g is an isomorphism. Consider the map

$$g^*\mathcal{C}_{E'/Y'} \longrightarrow \mathcal{C}_{E/X'}$$

of Morphisms, Lemma 29.31.3. Since the source and target are invertible modules of degree 1 on $E = E' = \mathbf{P}^1_\kappa$ and since the map is nonzero (as f' is an isomorphism in the generic point of E) we conclude it is an isomorphism. By Morphisms, Lemma 29.32.18 we conclude that $\Omega_{X'/Y'}|_E = 0$. This means that f' is unramified at every point of E (Morphisms, Lemma 29.35.14). Hence f' is quasi-finite at every point of E (Morphisms, Lemma 29.35.10). Hence the maximal open $V' \subset Y'$ over which f' is an isomorphism contains E' by Varieties, Lemma 33.17.3. This in turn implies that the inverse image of y in X' is E'. Hence the inverse image of y in X is x. Hence $x \in X$ is in the maximal open over which f is an isomorphism by Varieties, Lemma 33.17.3. This is a contradiction as we assumed that y is not in this open. □

0C5S **Lemma 54.17.2.** *Let S be a Noetherian scheme. Let X and Y be proper integral schemes over S which are regular of dimension 2. Then X and Y are S-birational if and only if there exists a diagram of S-morphisms*

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \to \ldots \to Y_1 \to Y_0 = Y$$

where each morphism is a blowup in a closed point.

Proof. Let $U \subset X$ be open and let $f : U \to Y$ be the given S-rational map (which is invertible as an S-rational map). By Lemma 54.4.3 we can factor f as $X_n \to \ldots \to X_1 \to X_0 = X$ and $f_n : X_n \to Y$. Since X_n is proper over S and Y separated over S the morphism f_n is proper. Clearly f_n is birational. Hence f_n is a composition of contractions by Lemma 54.17.1. We omit the proof of the converse. □

54.18. Other chapters

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CHAPTER 55

Semistable Reduction

0C2P

55.1. Introduction

0C2Q In this chapter we prove the semistable reduction theorem for curves. We will use the method of Artin and Winters from their paper [**AW71**].

It turns out that one can prove the semistable reduction theorem for curves without any results on desingularization. Namely, there is a way to establish the existence and projectivity of moduli of semistable curves using Geometric Invariant Theory (GIT) as developed by Mumford, see [**MFK94**]. This method was championed by Gieseker who proved the full result in his lecture notes [**Gie82**][1]. This is quite an amazing feat: it seems somewhat counter intuitive that one can prove such a result without ever truly studying families of curves over a positive dimensional base.

Historically the first proof of the semistable reduction theorem for curves can be found in the paper [**DM69**] by Deligne and Mumford. It proves the theorem by reducing the problem to the case of Abelian varieties which was already known at the time thanks to Grothendieck and others, see [**GRR72**] and [**DK73**]). The semistable reduction theorem for abelian varieties uses the theory of Néron models which in turn rests on a treatment of birational group laws over a base.

The method in the paper by Artin and Winters relies on desingularization of singularities of surfaces to obtain regular models. Given the existence of regular models, the proof consists in analyzing the possibilities for the special fibre and concluding using an inequality for torsion in the Picard group of a 1-dimensional scheme over a field. A similar argument can be found in a paper [**Sai87**] of Saito who uses étale cohomology directly and who obtains a stronger result in that he can characterize semistable reduction in terms of the action of the inertia on ℓ-adic étale cohomology.

A different approach one can use to prove the theorem is to use rigid analytic geometry techniques. Here we refer the reader to [**vdP84**] and [**AW12**].

The paper [**Tem10**] by Temkin uses valuation theoretic techniques (and proves a lot more besides); also Appendix A of this paper gives a nice overview of the different proofs and the relationship with desingularizations of 2 dimensional schemes.

Another overview paper that the reader may wish to consult is [**Abb00**] written by Ahmed Abbes.

55.2. Linear algebra

0C5T A couple of lemmas we will use later on.

[1] Gieseker's lecture notes are written over an algebraically closed field, but the same method works over $\mathbf{Z}$.

0C5U **Lemma 55.2.1.** *Let $A = (a_{ij})$ be a complex $n \times n$ matrix.* [**Tau49**, Theorem I]

(1) *If $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each i, then $\det(A)$ is nonzero.*
(2) *If there exists a real vector $m = (m_1, \ldots, m_n)$ with $m_i > 0$ such that $|a_{ii}m_i| > \sum_{j \neq i} |a_{ij}m_j|$ for each i, then $\det(A)$ is nonzero.*

Proof. If A is as in (1) and $\det(A) = 0$, then there is a nonzero vector z with $Az = 0$. Choose r with $|z_r|$ maximal. Then

$$|a_{rr}z_r| = |\sum\nolimits_{k \neq r} a_{rk}z_k| \leq \sum\nolimits_{k \neq r} |a_{rk}||z_k| \leq |z_r| \sum\nolimits_{k \neq r} |a_{rk}| < |a_{rr}||z_r|$$

which is a contradiction. To prove (2) apply (1) to the matrix $(a_{ij}m_j)$ whose determinant is $m_1 \ldots m_n \det(A)$. □

0C5V **Lemma 55.2.2.** *Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. For $I \subset \{1, \ldots, n\}$ let $x_I \in \mathbf{R}^n$ be the vector whose ith coordinate is m_i if $i \in I$ and 0 otherwise. If*

0C5W (55.2.2.1) $$-a_{ii}m_i \geq \sum\nolimits_{j \neq i} a_{ij}m_j$$

for each i, then $\mathrm{Ker}(A)$ is the vector space spanned by the vectors x_I such that

(1) *$a_{ij} = 0$ for $i \in I$, $j \notin I$, and*
(2) *equality holds in (55.2.2.1) for $i \in I$.*

Proof. After replacing a_{ij} by $a_{ij}m_j$ we may assume $m_i = 1$ for all i. If $I \subset \{1, \ldots, n\}$ such that (1) and (2) are true, then a simple computation shows that x_I is in the kernel of A. Conversely, let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ be a nonzero vector in the kernel of A. We will show by induction on the number of nonzero coordinates of x that x is in the span of the vectors x_I satisfying (1) and (2). Let $I \subset \{1, \ldots, n\}$ be the set of indices r with $|x_r|$ maximal. For $r \in I$ we have

$$|a_{rr}x_r| = |\sum\nolimits_{k \neq r} a_{rk}x_k| \leq \sum\nolimits_{k \neq r} a_{rk}|x_k| \leq |x_r| \sum\nolimits_{k \neq r} a_{rk} \leq |a_{rr}||x_r|$$

Thus equality holds everywhere. In particular, we see that $a_{rk} = 0$ if $r \in I$, $k \notin I$ and equality holds in (55.2.2.1) for $r \in I$. Then we see that we can subtract a suitable multiple of x_I from x to decrease the number of nonzero coordinates. □

0C5X **Lemma 55.2.3.** *Let $A = (a_{ij})$ be a symmetric real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. Assume*

(1) *$Am = 0$,*
(2) *there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.*

Then $x^tAx \leq 0$ with equality if and only if $x = qm$ for some $q \in \mathbf{R}$.

First proof. After replacing a_{ij} by $a_{ij}m_im_j$ we may assume $m_i = 1$ for all i. Condition (1) means $-a_{ii} = \sum_{j \neq i} a_{ij}$ for all i. Recall that $x^tAx = \sum_{i,j} x_ia_{ij}x_j$. Then

$$\begin{aligned} \sum\nolimits_{i \neq j} -a_{ij}(x_j - x_i)^2 &= \sum\nolimits_{i \neq j} -a_{ij}x_j^2 + 2a_{ij}x_ix_i - a_{ij}x_i^2 \\ &= \sum\nolimits_j a_{jj}x_j^2 + \sum\nolimits_{i \neq j} 2a_{ij}x_ix_i + \sum\nolimits_j a_{jj}x_i^2 \\ &= 2x^tAx \end{aligned}$$

This is clearly ≤ 0. If equality holds, then let I be the set of indices i with $x_i \neq x_1$. Then $a_{ij} = 0$ for $i \in I$ and $j \notin I$. Thus $I = \{1, \ldots, n\}$ by condition (2) and x is a multiple of $m = (1, \ldots, 1)$. □

Second proof. The matrix A has real eigenvalues by the spectral theorem. We claim all the eigenvalues are ≤ 0. Namely, since property (1) means $-a_{ii}m_i = \sum_{j \neq i} a_{ij}m_j$ for all i, we find that the matrix $A' = A - \lambda I$ for $\lambda > 0$ satisfies $|a'_{ii}m_i| > \sum a'_{ij}m_j = \sum |a'_{ij}m_j|$ for all i. Hence A' is invertible by Lemma 55.2.1. This implies that the symmetric bilinear form x^tAy is semi-negative definite, i.e., $x^tAx \leq 0$ for all x. It follows that the kernel of A is equal to the set of vectors x with $x^tAx = 0$. The description of the kernel in Lemma 55.2.2 gives the final statement of the lemma. □

0C6V **Lemma 55.2.4.** *Let L be a finite free $\mathbf{Z}$-module endowed with an integral symmetric bilinear positive definite form $\langle\ ,\ \rangle : L \times L \to \mathbf{Z}$. Let $A \subset L$ be a submodule with L/A torsion free. Set $B = \{b \in L \mid \langle a, b\rangle = 0,\ \forall a \in A\}$. Then we have injective maps*

$$A^\# /A \leftarrow L/(A \oplus B) \to B^\# /B$$

whose cokernels are quotients of $L^\#/L$. Here $A^\# = \{a' \in A \otimes \mathbf{Q} \mid \langle a, a'\rangle \in \mathbf{Z},\ \forall a \in A\}$ and similarly for B and L.

Proof. Observe that $L \otimes \mathbf{Q} = A \otimes \mathbf{Q} \oplus B \otimes \mathbf{Q}$ because the form is nondegenerate on A (by positivity). We denote $\pi_B : L \otimes \mathbf{Q} \to B \otimes \mathbf{Q}$ the projection. Observe that $\pi_B(x) \in B^\#$ for $x \in L$ because the form is integral. This gives an exact sequence

$$0 \to A \to L \xrightarrow{\pi_B} B^\# \to Q \to 0$$

where Q is the cokernel of $L \to B^\#$. Observe that Q is a quotient of $L^\#/L$ as the map $L^\# \to B^\#$ is surjective since it is the $\mathbf{Z}$-linear dual to $B \to L$ which is split as a map of $\mathbf{Z}$-modules. Dividing by $A \oplus B$ we get a short exact sequence

$$0 \to L/(A \oplus B) \to B^\# /B \to Q \to 0$$

This proves the lemma. □

0C6W **Lemma 55.2.5.** *Let L_0, L_1 be a finite free $\mathbf{Z}$-modules endowed with integral symmetric bilinear positive definite forms $\langle\ ,\ \rangle : L_i \times L_i \to \mathbf{Z}$. Let $d : L_0 \to L_1$ and $d^* : L_1 \to L_0$ be adjoint. If $\langle\ ,\ \rangle$ on L_0 is unimodular, then there is an isomorphism*

$$\Phi : \mathrm{Coker}(d^*d)_{torsion} \longrightarrow \mathrm{Im}(d)^\# / \mathrm{Im}(d)$$

with notation as in Lemma 55.2.4.

Proof. Let $x \in L_0$ be an element representing a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. Then for some $a > 0$ we can write $ax = \mathrm{d}^*\mathrm{d}(y)$. For any $z \in \mathrm{Im}(\mathrm{d})$, say $z = \mathrm{d}(y')$, we have

$$\langle (1/a)\mathrm{d}(y), z\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(y')\rangle = \langle x, y'\rangle \in \mathbf{Z}$$

Hence $(1/a)\mathrm{d}(y) \in \mathrm{Im}(\mathrm{d})^\#$. We define $\Phi(x) = (1/a)\mathrm{d}(y) \bmod \mathrm{Im}(\mathrm{d})$. We omit the proof that Φ is well defined, additive, and injective.

To prove Φ is surjective, let $z \in \mathrm{Im}(\mathrm{d})^\#$. Then z defines a linear map $L_0 \to \mathbf{Z}$ by the rule $x \mapsto \langle z, \mathrm{d}(x)\rangle$. Since the pairing on L_0 is unimodular by assumption we can find an $x' \in L_0$ with $\langle x', x\rangle = \langle z, \mathrm{d}(x)\rangle$ for all $x \in L_0$. In particular, we see that x' pairs to zero with $\mathrm{Ker}(\mathrm{d})$. Since $\mathrm{Im}(\mathrm{d}^*\mathrm{d}) \otimes \mathbf{Q}$ is the orthogonal complement of $\mathrm{Ker}(\mathrm{d}) \otimes \mathbf{Q}$ this means that x' defines a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. We claim that

$\Phi(x') = z$. Namely, write $ax' = \mathrm{d}^*\mathrm{d}(y)$ for some $y \in L_0$ and $a > 0$. For any $x \in L_0$ we get

$$\langle z, \mathrm{d}(x)\rangle = \langle x', x\rangle = \langle (1/a)\mathrm{d}^*\mathrm{d}(y), x\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(x)\rangle$$

Hence $z = \Phi(x')$ and the proof is complete. □

0C6X **Lemma 55.2.6.** *Let $A = (a_{ij})$ be a symmetric $n \times n$ integer matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be an integer vector with $m_i > 0$. Assume*

(1) *$Am = 0$,*
(2) *there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.*

Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. Then for ℓ a prime number coprime with all a_{ij} and m_i we have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Coker}(A)[\ell]) \leq 1 - n + e$$

Proof. By Lemma 55.2.3 the rank of A is $n - 1$. The composition

$$\mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n} \xrightarrow{(a_{ij})} \mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n}$$

has matrix $a_{ij}m_im_j$. Since the cokernel of the first and last maps are torsion of order prime to ℓ by our restriction on ℓ we see that it suffices to prove the lemma for the matrix with entries $a_{ij}m_im_j$. Thus we may assume $m = (1, \ldots, 1)$.

Assume $m = (1, \ldots, 1)$. Set $V = \{1, \ldots, n\}$ and $E = \{(i, j) \mid i < j \text{ and } a_{ij} > 0\}$. For $e = (i, j) \in E$ set $a_e = a_{ij}$. Define maps $s, t : E \to V$ by setting $s(i, j) = i$ and $t(i, j) = j$. Set $\mathbf{Z}(V) = \bigoplus_{i \in V} \mathbf{Z}i$ and $\mathbf{Z}(E) = \bigoplus_{e \in E} \mathbf{Z}e$. We define symmetric positive definite integer valued pairings on $\mathbf{Z}(V)$ and $\mathbf{Z}(E)$ by setting

$$\langle i, i\rangle = 1 \text{ for } i \in V, \quad \langle e, e\rangle = a_e \text{ for } e \in E$$

and all other pairings zero. Consider the maps

$$\mathrm{d} : \mathbf{Z}(V) \to \mathbf{Z}(E), \quad i \longmapsto \sum\nolimits_{e \in E,\ s(e)=i} e - \sum\nolimits_{e \in E,\ t(e)=i} e$$

and

$$\mathrm{d}^*(e) = a_e(s(e) - t(e))$$

A computation shows that

$$\langle d(x), y\rangle = \langle x, \mathrm{d}^*(y)\rangle$$

in other words, d and d^* are adjoint. Next we compute

$$\begin{aligned}\mathrm{d}^*\mathrm{d}(i) &= \mathrm{d}^*\Big(\sum\nolimits_{e \in E,\ s(e)=i} e - \sum\nolimits_{e \in E,\ t(e)=i} e\Big) \\ &= \sum\nolimits_{e \in E,\ s(e)=i} a_e(s(e) - t(e)) - \sum\nolimits_{e \in E,\ t(e)=i} a_e(s(e) - t(e))\end{aligned}$$

The coefficient of i in $\mathrm{d}^*\mathrm{d}(i)$ is

$$\sum\nolimits_{e \in E,\ s(e)=i} a_e + \sum\nolimits_{e \in E,\ t(e)=i} a_e = -a_{ii}$$

because $\sum_j a_{ij} = 0$ and the coefficient of $j \neq i$ in $\mathrm{d}^*\mathrm{d}(i)$ is $-a_{ij}$. Hence $\mathrm{Coker}(A) = \mathrm{Coker}(\mathrm{d}^*\mathrm{d})$.

Consider the inclusion

$$\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*) \subset \mathbf{Z}(E)$$

The left hand side is an orthogonal direct sum. Clearly $\mathbf{Z}(E)/\mathrm{Ker}(\mathrm{d}^*)$ is torsion free. We claim $\mathbf{Z}(E)/\mathrm{Im}(\mathrm{d})$ is torsion free as well. Namely, say $x = \sum x_e e \in \mathbf{Z}(E)$

and $a > 1$ are such that $ax = \mathrm{d}y$ for some $y = \sum y_i i \in \mathbf{Z}(V)$. Then $ax_e = y_{s(e)} - y_{t(e)}$. By property (2) we conclude that all y_i have the same congruence class modulo a. Hence we can write $y = ay' + (y_1, y_1, \ldots, y_1)$. Since $\mathrm{d}(y_1, y_1, \ldots, y_1) = 0$ we conclude that $x = \mathrm{d}(y')$ which is what we had to show.

Hence we may apply Lemma 55.2.4 to get injective maps

$$\mathrm{Im}(\mathrm{d})^\# / \mathrm{Im}(\mathrm{d}) \leftarrow \mathbf{Z}(E)/(\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*)) \to \mathrm{Ker}(\mathrm{d}^*)^\# / \mathrm{Ker}(\mathrm{d}^*)$$

whose cokernels are annihilated by the product of the a_e (which is prime to ℓ). Since $\mathrm{Ker}(\mathrm{d}^*)$ is a lattice of rank $1 - n + e$ we see that the proof is complete if we prove that there exists an isomorphism

$$\Phi : M_{torsion} \longrightarrow \mathrm{Im}(\mathrm{d})^\# / \mathrm{Im}(\mathrm{d})$$

This is proved in Lemma 55.2.5. □

55.3. Numerical types

0C6Y Part of the arguments will involve the combinatorics of the following data structures.

0C6Z **Definition 55.3.1.** A *numerical type* T is given by

$$n, m_i, a_{ij}, w_i, g_i$$

where $n \geq 1$ is an integer and m_i, a_{ij}, w_i, g_i are integers for $1 \leq i, j \leq n$ subject to the following conditions

(1) $m_i > 0$, $w_i > 0$, $g_i \geq 0$,
(2) the matrix $A = (a_{ij})$ is symmetric and $a_{ij} \geq 0$ for $i \neq j$,
(3) there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$, $j \notin I$,
(4) for each i we have $\sum_j a_{ij} m_j = 0$, and
(5) $w_i | a_{ij}$.

This is obviously a somewhat annoying type of structure to work with, but it is exactly what shows up in special fibres of proper regular models of smooth geometrically connected curves. Of course we only care about these types up to reordering the indices.

0C70 **Definition 55.3.2.** We say two numerical types n, m_i, a_{ij}, w_i, g_i and $n', m'_i, a'_{ij}, w'_i, g'_i$ are *equivalent types* if there exists a permutation σ of $\{1, \ldots, n\}$ such that $m_i = m'_{\sigma(i)}$, $a_{ij} = a'_{\sigma(i)\sigma(j)}$, $w_i = w'_{\sigma(i)}$, and $g_i = g'_{\sigma(i)}$.

A numerical type has a genus.

0C71 **Lemma 55.3.3.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type. Then the expression*

$$g = 1 + \sum m_i (w_i(g_i - 1) - \frac{1}{2} a_{ii})$$

is an integer.

Proof. To prove g is an integer we have to show that $\sum a_{ii}m_i$ is even. This we can see by computing modulo 2 as follows

$$\begin{aligned}\sum\nolimits_i a_{ii}m_i &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} a_{ii}m_i \\ &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j\neq i} a_{ij}m_j \\ &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j\neq i,\ m_j \text{ odd}} a_{ij}m_j \\ &\equiv \sum\nolimits_{i<j,\ m_i \text{ and } m_j \text{ odd}} a_{ij}(m_i + m_j) \\ &\equiv 0\end{aligned}$$

where we have used that $a_{ij} = a_{ji}$ and that $\sum_j a_{ij}m_j = 0$ for all i. □

0C72 **Definition 55.3.4.** We say n, m_i, a_{ij}, w_i, g_i is a *numerical type of genus g* if $g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ is the integer from Lemma 55.3.3.

We will prove below (Lemma 55.3.14) that the genus is almost always ≥ 0. But you can have numerical types with negative genus.

0C73 **Lemma 55.3.5.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n = 1$, then $a_{11} = 0$ and $g = 1 + m_1w_1(g_1 - 1)$. Moreover, we can classify all such numerical types as follows*

1. *If $g < 0$, then $g_1 = 0$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1 = 1 - g$.*
2. *If $g = 0$, then $m_1 = 1$, $w_1 = 1$, $g_1 = 0$ as in Lemma 55.6.1.*
3. *If $g = 1$, then we conclude $g_1 = 1$ but m_1, w_1 can be arbitrary positive integers; this is case (1) of Lemma 55.6.2.*
4. *If $g > 1$, then $g_1 > 1$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1(g_1-1) = g-1$.*

Proof. The lemma proves itself. □

0C74 **Lemma 55.3.6.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n > 1$, then $a_{ii} < 0$ for all i.*

Proof. Lemma 55.2.3 applies to the matrix A. □

0C75 **Lemma 55.3.7.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is < 0, then $g_i = 0$ and $a_{ii} = -w_i$.*

Proof. Follows immediately from Lemma 55.3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. □

0C76 **Definition 55.3.8.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type. We say i is a (-1)-*index* if $g_i = 0$ and $a_{ii} = -w_i$.

We can "contract" (-1)-indices.

0C77 **Lemma 55.3.9.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Then there is a numerical type T' given by $n', m'_i, a'_{ij}, w'_i, g'_i$ with*

1. *$n' = n - 1$,*
2. *$m'_i = m_i$,*
3. *$a'_{ij} = a_{ij} - a_{in}a_{jn}/a_{nn}$,*

(4) $w'_i = w_i/2$ *if* a_{in}/w_n *even and* a_{in}/w_i *odd and* $w'_i = w_i$ *else,*
(5) $g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w'_i w_n}$.

Moreover, we have $g = g'$.

Proof. Observe that $n > 1$ for example by Lemma 55.3.5 and hence $n' \geq 1$. We check conditions (1) – (5) of Definition 55.3.1 for $n', m'_i, a'_{ij}, w'_i, g'_i$.

Condition (1) is immediate.

Condition (2). Symmetry of $A' = (a'_{ij})$ is immediate and since $a_{nn} < 0$ by Lemma 55.3.6 we see that $a'_{ij} \geq a_{ij} \geq 0$ if $i \neq j$.

Condition (3). Suppose that $I \subset \{1, \ldots, n-1\}$ such that $a'_{ii'} = 0$ for $i \in I$ and $i' \in \{1, \ldots, n-1\} \backslash I$. Then we see that for each $i \in I$ and $i' \in I'$ we have $a_{in}a_{i'n} = 0$. Thus either $a_{in} = 0$ for all $i \in I$ and $I \subset \{1, \ldots, n\}$ is a contradiction for property (3) for T, or $a_{i'n} = 0$ for all $i' \in \{1, \ldots, n-1\} \setminus I$ and $I \cup \{n\} \subset \{1, \ldots, n\}$ is a contradiction for property (3) of T. Hence (3) holds for T'.

Condition (4). We compute

$$\sum\nolimits_{j=1}^{n-1} a'_{ij} m_j = \sum\nolimits_{j=1}^{n-1} (a_{ij} m_j - \frac{a_{in} a_{jn} m_j}{a_{nn}}) = -a_{in} m_n - \frac{a_{in}}{a_{nn}}(-a_{nn} m_n) = 0$$

as desired.

Condition (5). We have to show that w'_i divides $a_{in}a_{jn}/a_{nn}$. This is clear because $a_{nn} = -w_n$ and $w_n | a_{jn}$ and $w_i | a_{in}$.

To show that $g = g'$ we first write

$$\begin{aligned} g &= 1 + \sum\nolimits_{i=1}^{n} m_i (w_i(g_i - 1) - \frac{1}{2} a_{ii}) \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i (w_i(g_i - 1) - \frac{1}{2} a_{ii}) - \frac{1}{2} m_n w_n \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i (w_i(g_i - 1) - \frac{1}{2} a_{ii} - \frac{1}{2} a_{in}) \end{aligned}$$

Comparing with the expression for g' we see that it suffices if

$$w'_i(g'_i - 1) - \frac{1}{2} a'_{ii} = w_i(g_i - 1) - \frac{1}{2} a_{in} - \frac{1}{2} a_{ii}$$

for $i \leq n - 1$. In other words, we have

$$g'_i = \frac{2w_i(g_i - 1) - a_{in} - a_{ii} + a'_{ii} + 2w'_i}{2w'_i} = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w'_i w_n}$$

It is elementary to check that this is an integer ≥ 0 if we choose w'_i as in (4). □

0C78 **Lemma 55.3.10.** *Let* n, m_i, a_{ij}, w_i, g_i *be a numerical type. Let* e *be the number of pairs* (i, j) *with* $i < j$ *and* $a_{ij} > 0$. *Then the expression* $g_{top} = 1 - n + e$ *is* ≥ 0.

Proof. If not, then $e < n - 1$ which means there exists an i such that $a_{ij} = 0$ for all $j \neq i$. This contradicts assumption (3) of Definition 55.3.1. □

0C79 **Definition 55.3.11.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The *topological genus of* T is the nonnegative integer $g_{top} = 1 - n + e$ from Lemma 55.3.10.

We want to bound the genus by the topological genus. However, this will not always be the case, for example for numerical types with $n = 1$ as in Lemma 55.3.5. But it will be true for minimal numerical types which are defined as follows.

0C7A **Definition 55.3.12.** We say the numerical type n, m_i, a_{ij}, w_i, g_i of genus g is *minimal* if there does not exist an i with $g_i = 0$ and $a_{ii} = -w_i$, in other words, if there does not exist a (-1)-index.

We will prove that the genus g of a minimal type with $n > 1$ is greater than or equal to $\max(1, g_{top})$.

0C7B **Lemma 55.3.13.** *If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq 1$.*

Proof. This is true because $g = 1 + \sum \Phi_i$ with $\Phi_i = m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ non-negative by Lemma 55.3.7 and the definition of minimal types. □

0C7C **Lemma 55.3.14.** *If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq g_{top}$.*

Proof. The reader who is only interested in the case of numerical types associated to proper regular models can skip this proof as we will reprove this in the geometric situation later. We can write

$$g_{top} = 1 - n + \frac{1}{2}\sum\nolimits_{a_{ij}>0} 1 = 1 + \sum\nolimits_i (-1 + \frac{1}{2}\sum\nolimits_{j\neq i,\ a_{ij}>0} 1)$$

On the other hand, we have

$$\begin{aligned} g &= 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) \\ &= 1 + \sum m_i w_i g_i - \sum m_i w_i + \frac{1}{2}\sum\nolimits_{i\neq j} a_{ij}m_j \\ &= 1 + \sum\nolimits_i m_i w_i(-1 + g_i + \frac{1}{2}\sum\nolimits_{j\neq i} \frac{a_{ij}}{w_i}) \end{aligned}$$

The first equality is the definition, the second equality uses that $\sum a_{ij}m_j = 0$, and the last equality uses that uses $a_{ij} = a_{ji}$ and switching order of summation. Comparing with the formula for g_{top} we conclude that the lemma holds if

$$\Psi_i = m_i w_i(-1 + g_i + \frac{1}{2}\sum\nolimits_{j\neq i} \frac{a_{ij}}{w_i}) - (-1 + \frac{1}{2}\sum\nolimits_{j\neq i,\ a_{ij}>0} 1)$$

is ≥ 0 for each i. However, this may not be the case. Let us analyze for which indices we can have $\Psi_i < 0$. First, observe that

$$(-1 + g_i + \frac{1}{2}\sum\nolimits_{j\neq i} \frac{a_{ij}}{w_i}) \geq (-1 + \frac{1}{2}\sum\nolimits_{j\neq i,\ a_{ij}>0} 1)$$

because a_{ij}/w_i is a nonnegative integer. Since $m_i w_i$ is a positive integer we conclude that $\Psi_i \geq 0$ as soon as either $m_i w_i = 1$ or the left hand side of the inequality is ≥ 0 which happens if $g_i > 0$, or $a_{ij} > 0$ for at least two indices j, or if there is a j with $a_{ij} > w_i$. Thus

$$P = \{i : \Psi_i < 0\}$$

is the set of indices i such that $m_i w_i > 1$, $g_i = 0$, $a_{ij} > 0$ for a unique j, and $a_{ij} = w_i$ for this j. Moreover

$$i \in P \Rightarrow \Psi_i = \frac{1}{2}(-m_i w_i + 1)$$

The strategy of proof is to show that given $i \in P$ we can borrow a bit from Ψ_j where j is the neighbour of i, i.e., $a_{ij} > 0$. However, this won't quite work because j may be an index with $\Psi_j = 0$.

Consider the set

$$Z = \{j : g_j = 0 \text{ and } j \text{ has exactly two neighbours } i, k \text{ with } a_{ij} = w_j = a_{jk}\}$$

For $j \in Z$ we have $\Psi_j = 0$. We will consider sequences $M = (i, j_1, \ldots, j_s)$ where $s \geq 0$, $i \in P$, $j_1, \ldots, j_s \in Z$, and $a_{ij_1} > 0, a_{j_1 j_2} > 0, \ldots, a_{j_{s-1} j_s} > 0$. If our numerical type consists of two indices which are in P or more generally if our numerical type consists of two indices which are in P and all other indices in Z, then $g_{top} = 0$ and we win by Lemma 55.3.13. We may and do discard these cases.

Let $M = (i, j_1, \ldots, j_s)$ be a maximal sequence and let k be the second neighbour of j_s. (If $s = 0$, then k is the unique neighbour of i.) By maximality $k \notin Z$ and by what we just said $k \notin P$. Observe that $w_i = a_{ij_1} = w_{j_1} = a_{j_1 j_2} = \ldots = w_{j_s} = a_{j_s k}$. Looking at the definition of a numerical type we see that

$$\begin{aligned} m_i a_{ii} + m_{j_1} w_i &= 0, \\ m_i w_i + m_{j_1} a_{j_1 j_1} + m_{j_2} w_i &= 0, \\ &\ldots\ldots \\ m_{j_{s-1}} w_i + m_{j_s} a_{j_s j_s} + m_k w_i &= 0 \end{aligned}$$

The first equality implies $m_{j_1} \geq 2m_i$ because the numerical type is minimal. Then the second equality implies $m_{j_2} \geq 3m_i$, and so on. In any case, we conclude that $m_k \geq 2m_i$ (including when $s = 0$).

Let k be an index such that we have a $t > 0$ and pairwise distinct maximal sequences $M_1, \ldots, M_t$ as above, with $M_b = (i_b, j_{b,1}, \ldots, j_{b,s_b})$ such that k is a neighbour of j_{b,s_b} for $b = 1, \ldots, t$. We will show that $\Phi_j + \sum_{b=1,\ldots,t} \Phi_{i_b} \geq 0$. This will finish the proof of the lemma by what we said above. Let M be the union of the indices occurring in M_b, $b = 1, \ldots, t$. We write

$$\Psi_k = -\sum\nolimits_{b=1,\ldots,t} \Psi_{i_b} + \Psi'_k$$

where

$$\begin{aligned} \Psi'_k = m_k w_k \Bigg(-1 + g_k + \frac{1}{2} \sum\nolimits_{b=1,\ldots t} \Big(\frac{a_{k j_{b,s_b}}}{w_k} - \frac{m_{i_b} w_{i_b}}{m_k w_k} \Big) + \frac{1}{2} \sum\nolimits_{l \neq k,\ l \notin M} \frac{a_{kl}}{w_k} \Bigg) \\ - \Bigg(-1 + \frac{1}{2} \sum\nolimits_{l \neq k,\ l \notin M,\ a_{kl} > 0} 1 \Bigg) \end{aligned}$$

Assume $\Psi'_k < 0$ to get a contradiction. If the set $\{l : l \neq k,\ l \notin M,\ a_{kl} > 0\}$ is empty, then $\{1, \ldots, n\} = M \cup \{k\}$ and $g_{top} = 0$ because $e = n - 1$ in this case and the result holds by Lemma 55.3.13. Thus we may assume there is at least one such l which contributes $(1/2)a_{kl}/w_k \geq 1/2$ to the sum inside the first brackets. For each $b = 1, \ldots, t$ we have

$$\frac{a_{k j_{b,s_b}}}{w_k} - \frac{m_{i_b} w_{i_b}}{m_k w_k} = \frac{w_{i_b}}{w_k} \Big(1 - \frac{m_{i_b}}{m_k} \Big)$$

This expression is $\geq \frac{1}{2}$ because $m_k \geq 2m_{i_b}$ by the previous paragraph and is ≥ 1 if $w_k < w_{i_b}$. It follows that $\Psi'_k < 0$ implies $g_k = 0$. If $t \geq 2$ or $t = 1$ and $w_k < w_{i_1}$, then $\Psi'_k \geq 0$ (here we use the existence of an l as shown above) which is a contradiction too. Thus $t = 1$ and $w_k = w_{i_1}$. If there at least two nonzero terms in the sum over l or if there is one such k and $a_{kl} > w_k$, then $\Psi'_k \geq 0$ as well. The final possibility is that $t = 1$ and there is one l with $a_{kl} = w_k$. This is disallowed as this would mean $k \in Z$ contradicting the maximality of M_1. $\square$

0C7D **Lemma 55.3.15.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is 0, then $g_i = 0$ and $a_{ii} = -2w_i$.*

Proof. Follows immediately from Lemma 55.3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. □

It turns out that the indices satisfying this relation play an important role in the structure of minimal numerical types. Hence we give them a name.

0C7E **Definition 55.3.16.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. We say i is a (-2)-*index* if $g_i = 0$ and $a_{ii} = -2w_i$.

Given a minimal numerical type of genus g the (-2)-indices are exactly the indices which do not contribute a positive number to the genus in the formula

$$g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$$

Thus it will be somewhat tricky to bound the quantities associated with (-2)-indices as we will see later.

0C7F **Remark 55.3.17.** Let n, m_i, a_{ij}, w_i, g_i be a minimal numerical type with $n > 1$. Equality $g = g_{top}$ can hold in Lemma 55.3.14. For example, if $m_i = w_i = 1$ and $g_i = 0$ for all i and $a_{ij} \in \{0, 1\}$ for $i < j$.

55.4. The Picard group of a numerical type

0C7G Here is the definition.

0C7H **Definition 55.4.1.** Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The *Picard group of T* is the cokernel of the matrix (a_{ij}/w_i), more precisely

$$\mathrm{Pic}(T) = \mathrm{Coker}\left(\mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}, \quad e_i \mapsto \sum \frac{a_{ij}}{w_j} e_j\right)$$

where e_i denotes the ith standard basis vector for $\mathbf{Z}^{\oplus n}$.

0C7I **Lemma 55.4.2.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The Picard group of T is a finitely generated abelian group of rank 1.*

Proof. If $n = 1$, then $A = (a_{ij})$ is the zero matrix and the result is clear. For $n > 1$ the matrix A has rank $n - 1$ by either Lemma 55.2.2 or Lemma 55.2.3. Of course the rank is not affected by scaling the rows by $1/w_i$. This proves the lemma. □

0CE7 **Lemma 55.4.3.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Then $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$ where $A = (a_{ij})$.*

Proof. Since $\mathrm{Pic}(T)$ is the cokernel of (a_{ij}/w_i) we see that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}^{\oplus n} & \xrightarrow{A} & \mathbf{Z}^{\oplus n} & \longrightarrow & \mathrm{Coker}(A) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{diag}(w_1,\ldots,w_n)} & & \uparrow & & \\
0 & \longrightarrow & \mathbf{Z}^{\oplus n} & \xrightarrow{(a_{ij}/w_i)} & \mathbf{Z}^{\oplus n} & \longrightarrow & \mathrm{Pic}(T) & \longrightarrow & 0
\end{array}$$

with exact rows. By the snake lemma we conclude that $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$. □

0C7J **Lemma 55.4.4.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Let T' be the numerical type constructed in Lemma 55.3.9. There exists an injective map*

$$\mathrm{Pic}(T) \to \mathrm{Pic}(T')$$

whose cokernel is an elementary abelian 2-group.

Proof. Recall that $n' = n - 1$. Let e_i, resp., e'_i be the ith basis vector of $\mathbf{Z}^{\oplus n}$, resp. $\mathbf{Z}^{\oplus n-1}$. First we denote

$$q : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto 0 \text{ and } e_i \mapsto e'_i \text{ for } i \leq n-1$$

and we set

$$p : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto \sum\nolimits_{j=1}^{n-1} \frac{a_{nj}}{w'_j} e'_j \text{ and } e_i \mapsto \frac{w_i}{w'_i} e'_i \text{ for } i \leq n-1$$

A computation (which we omit) shows there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Z}^{\oplus n} & \xrightarrow{(a_{ij}/w_i)} & \mathbf{Z}^{\oplus n} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{Z}^{\oplus n'} & \xrightarrow{(a'_{ij}/w'_i)} & \mathbf{Z}^{\oplus n'} \end{array}$$

Since the cokernel of the top arrow is $\mathrm{Pic}(T)$ and the cokernel of the bottom arrow is $\mathrm{Pic}(T')$, we obtain the desired homomorphism of Picard groups. Since $\frac{w_i}{w'_i} \in \{1, 2\}$ we see that the cokernel of $\mathrm{Pic}(T) \to \mathrm{Pic}(T')$ is annihilated by 2 (because $2e'_i$ is in the image of p for all $i \leq n-1$). Finally, we show $\mathrm{Pic}(T) \to \mathrm{Pic}(T')$ is injective. Let $L = (l_1, \ldots, l_n)$ be a representative of an element of $\mathrm{Pic}(T)$ mapping to zero in $\mathrm{Pic}(T')$. Since q is surjective, a diagram chase shows that we can assume L is in the kernel of p. This means that $l_n a_{ni}/w'_i + l_i w_i/w'_i = 0$, i.e., $l_i = -a_{ni}/w_i l_n$. Thus L is the image of $-l_n e_n$ under the map (a_{ij}/w_j) and the lemma is proved. □

0C7K **Lemma 55.4.5.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. If the genus g of T is ≤ 0, then $\mathrm{Pic}(T) = \mathbf{Z}$.*

Proof. By induction on n. If $n = 1$, then the assertion is clear. If $n > 1$, then T is not minimal by Lemma 55.3.13. After replacing T by an equivalent type we may assume n is a (-1)-index. By Lemma 55.4.4 we find $\mathrm{Pic}(T) \subset \mathrm{Pic}(T')$. By Lemma 55.3.9 we see that the genus of T' is equal to the genus of T and we conclude by induction. □

55.5. Classification of proper subgraphs

0C7L In this section we assume given a numerical type n, m_i, a_{ij}, w_i, g_i of genus g. We will find a complete list of possible "subgraphs" consisting entirely of (-2)-indices (Definition 55.3.16) and at the same time we classify all possible minimal numerical types of genus 1. In other words, in this section we prove Proposition 55.5.17 and Lemma 55.6.2

Our strategy will be as follows. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Let $I \subset \{1, \ldots, n\}$ be a subset consisting of (-2)-indices such that there does not exist a nonempty proper subset $J \subset I$ with $a_{jj'} = 0$ for $j \in J$, $j' \in I \setminus J$. We work by induction on the cardinality $|I|$ of I. If $I = \{i\}$ consists of 1 index, then the only constraints on m_i, a_{ii}, and w_i are $w_i | a_{ii}$ from Definition 55.3.1 and $a_{ii} < 0$ from Lemma 55.3.6 and this will serve as our base case. In the induction step we first

apply the induction hypothesis to subsets $I' \subset I$ of size $|I'| < |I|$. This will put some constraints on the possible m_i, a_{ij}, w_i, $i, j \in I$. In particular, since $|I'| < |I| \leq n$ it will follow from $\sum a_{ij} m_j = 0$ and Lemma 55.2.3 that the sub matrices $(a_{ij})_{i,j \in I'}$ are negative definite and their determinant will have sign $(-1)^m$. For each possibility left over we compute the determinant of $(a_{ij})_{i,j \in I}$. If the determinant has sign $-(-1)^{|I|}$ then this case can be discarded because Sylvester's theorem tells us the matrix $(a_{ij})_{i,j \in I}$ is not negative semi-definite. If the determinant has sign $(-1)^{|I|}$, then $|I| < n$ and we (tentatively) conclude this case can occur as a possible proper subgraph and we list it in one of the lemmas in this section. If the determinant is 0, then we must have $|I| = n$ (by Lemma 55.2.3 again) and $g = 0$. In these cases we actually find all possible m_i, a_{ij}, w_i, $i, j \in I$ and list them in Lemma 55.6.2. After completing the argument we obtain all possible minimal numerical types of genus 1 with $n > 1$ because each of these necessarily consists entirely of (-2)-indices (and hence will show up in the induction process) by the formula for the genus and the remarks in the previous section. At the very end of the day the reader can go through the list of possibilities given in Lemma 55.6.2 to see that all configurations of proper subgraphs listed in this section as possible do in fact occur already for numerical types of genus 1.

Suppose that i and j are (-2)-indices with $a_{ij} > 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 55.2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} \\ a_{ij} & -2w_j \end{pmatrix}$$

is negative definite unless $n = 2$. The case $n = 2$ can happen: then the determinant $4w_1w_2 - a_{12}^2$ is zero. Using that $\text{lcm}(w_1, w_2)$ divides a_{12} the reader easily finds that the only possibilities are

$$(w_1, w_2, a_{12}) = (w, w, 2w), (w, 4w, 4w), \text{ or } (4w, w, 4w)$$

Observe that the case $(4w, w, 4w)$ is obtained from the case $(w, 4w, 4w)$ by switching the indices i, j. In these cases $g = 1$. This leads to cases (2) and (3) of Lemma 55.6.2. Assuming $n > 2$ we see that the determinant $4w_iw_j - a_{ij}^2$ of the displayed matrix is > 0 and we conclude that $a_{ij}^2/w_iw_j < 4$. On the other hand, we know that $\text{lcm}(w_i, w_j) | a_{ij}$ and hence a_{ij}^2/w_iw_j is an integer. Thus $a_{ij}^2/w_iw_j \in \{1, 2, 3\}$ and $w_i | w_j$ or vice versa. This leads to the following possibilities

$$(w_1, w_2, a_{12}) = (w, w, w), (w, 2w, 2w), (w, 3w, 3w), (2w, w, 2w), \text{ or } (3w, w, 3w)$$

Observe that the case $(2w, w, 2w)$ is obtained from the case $(w, 2w, 2w)$ by switching the indices i, j and similarly for the cases $(3w, w, 3w)$ and $(w, 3w, 3w)$. The first three solutions lead to cases (1), (2), and (3) of Lemma 55.5.1. In this lemma we wrote out the consequences for the integers m_i and m_j using that $\sum_l a_{kl} m_l = 0$ for each k in particular implies $a_{ii}m_i + a_{ij}m_j \leq 0$ for $k = i$ and $a_{ij}m_i + a_{jj}m_j \leq 0$ for $k = j$.

0C7M **Lemma 55.5.1.** *Classification of proper subgraphs of the form*

If $n > 2$, then given a pair i, j of (-2)-indices with $a_{ij} > 0$, then up to ordering we have the m's, a's, w's

0C7N (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & w \\ w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}$$

with w arbitrary and $2m_1 \geq m_2$ and $2m_2 \geq m_1$, or

0C7P (2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \end{pmatrix}$$

with w arbitrary and $m_1 \geq m_2$ and $2m_2 \geq m_1$, or

0C7Q (3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 3w \\ 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ 3w \end{pmatrix}$$

with w arbitrary and $2m_1 \geq 3m_2$ and $2m_2 \geq m_1$.

Proof. See discussion above. □

Suppose that i, j, and k are three (-2)-indices with $a_{ij} > 0$ and $a_{jk} > 0$. In other words, the index i "meets" j and j "meets" k. We will use without further mention that each pair (i, j), (i, k), and (j, k) is as listed in Lemma 55.5.1. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 55.2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} \\ a_{ij} & -2w_j & a_{jk} \\ a_{ik} & a_{jk} & -2w_k \end{pmatrix}$$

is negative definite unless $n = 3$. The case $n = 3$ can happen: then the determinant[2] of the matrix is zero and we obtain the equation

$$4 = \frac{a_{ij}^2}{w_i w_j} + \frac{a_{jk}^2}{w_j w_k} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k}$$

of integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k} \right)^2 = \frac{a_{ij}^2}{w_i w_j} \frac{a_{jk}^2}{w_j w_k} \frac{a_{ik}^2}{w_i w_k}$$

Since we have seen above that $\frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}$ are in $\{1, 2, 3\}$ and $\frac{a_{ik}^2}{w_i w_k}$ in $\{0, 1, 2, 3\}$, we conclude that the only possibilities are

$$(\frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{ik}^2}{w_i w_k}) = (1, 1, 1), (1, 3, 0), (2, 2, 0), \text{ or } (3, 1, 0)$$

Observe that the case $(3, 1, 0)$ is obtained from the case $(1, 3, 0)$ by reversing the order the indices i, j, k. In each of these cases $g = 1$; the reader can find these as cases (4), (5), (6), (7), (8), (9) of Lemma 55.6.2 with one case corresponding to $(1, 1, 1)$, two cases corresponding to $(1, 3, 0)$, and three cases corresponding to $(2, 2, 0)$. Assuming $n > 3$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{jk}^2}{w_j w_k} + \frac{a_{ij} a_{ik} a_{jk}}{w_i w_j w_k}$$

[2] It is $-8w_i w_j w_k + 2a_{ij}^2 w_k + 2a_{jk}^2 w_i + 2a_{ik}^2 w_j + 2a_{ij} a_{jk} a_{ik}$.

of integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{ik}^2}{w_iw_k}) = (1,1,0), (1,2,0), \text{ or } (2,1,0)$$

in particular $a_{ik} = 0$ (recall we are assuming $a_{ij} > 0$ and $a_{jk} > 0$). Observe that the case $(2,1,0)$ is obtained from the case $(1,2,0)$ by reversing the ordering of the indices i, j, k. The first two solutions lead to cases (1), (2), and (3) of Lemma 55.5.2 where we also wrote out the consequences for the integers m_i, m_j, and m_k.

0C7R **Lemma 55.5.2.** *Classification of proper subgraphs of the form*

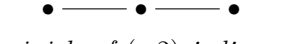

If $n > 3$, then given a triple i, j, k of (-2)-indices with at least two a_{ij}, a_{ik}, a_{jk} nonzero, then up to ordering we have the m's, a's, w's

0C7S (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & w \\ 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2$, or

0C7T (2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2$, or

0C7U (3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $m_3 \geq m_2$.

Proof. See discussion above. □

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index i "meets" j, j "meets" k, and k "meets" l. Then we see from Lemma 55.5.2 that $a_{ik} = a_{jl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & 0 & a_{il} \\ a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & a_{jk} & -2w_k & a_{kl} \\ a_{il} & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[3] of the matrix is zero and we obtain the equation

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} = 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

[3]It is $16w_iw_jw_kw_l - 4a_{ij}^2w_kw_l - 4a_{jk}^2w_iw_l - 4a_{kl}^2w_iw_j - 4a_{il}^2w_jw_k + a_{ij}^2a_{kl}^2 + a_{jk}^2a_{il}^2 - 2a_{ij}a_{il}a_{jk}a_{kl}$.

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}\right)^2 = \frac{a_{ij}^2}{w_iw_j}\frac{a_{jk}^2}{w_jw_k}\frac{a_{kl}^2}{w_kw_l}\frac{a_{il}^2}{w_iw_l}$$

Since we have seen above that $\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}$ are in $\{1,2\}$ and $\frac{a_{il}^2}{w_iw_l}$ in $\{0,1,2\}$, we conclude that the only possible solutions are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1,1,1,1) \text{ or } (2,1,2,0)$$

and case $g = 1$; the reader can find these as cases (10), (11), (12), and (13) of Lemma 55.6.2. Assuming $n > 4$ we obtain the inequality

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} > 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1,1,1,0), (1,1,2,0), (1,2,1,0), \text{ or } (2,1,1,0)$$

in particular $a_{il} = 0$ (recall that we assumed the other three to be nonzero). Observe that the case $(2,1,1,0)$ is obtained from the case $(1,1,2,0)$ by reversing the ordering of the indices i, j, k, l. The first three solutions lead to cases (1), (2), (3), and (4) of Lemma 55.5.3 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

0C7V **Lemma 55.5.3.** *Classification of proper subgraphs of the form*

If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{jk}, a_{kl} nonzero, then up to ordering we have the m's, a's, w's

0C7W (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, and $2m_4 \geq m_3$, or

0C7X (2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, and $2m_4 \geq m_3$, or

0C7Y (3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, *and* $m_4 \geq m_3$, *or*

0C7Z (4) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2 + m_4$, *and* $2m_4 \geq m_3$.

Proof. See discussion above. □

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{ij} > 0$, and $a_{il} > 0$. In other words, the index i "meets" the indices j, k, l. Then we see from Lemma 55.5.2 that $a_{jk} = a_{jl} = a_{kl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} & a_{il} \\ a_{ij} & -2w_j & 0 & 0 \\ a_{ik} & 0 & -2w_k & 0 \\ a_{il} & 0 & 0 & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[4] of the matrix is zero and we obtain the equation

$$4 = \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. Since we have seen above that $\frac{a_{ij}^2}{w_i w_j}, \frac{a_{ik}^2}{w_i w_k}, \frac{a_{il}^2}{w_i w_l}$ are in $\{1, 2\}$, we conclude that the only possibilities are up to reordering: $4 = 1 + 1 + 2$. In each of these cases $g = 1$; the reader can find these as cases (14) and (15) of Lemma 55.6.2. Assuming $n > 4$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. This implies that $\frac{a_{ij}^2}{w_i w_j} = \frac{a_{ik}^2}{w_i w_k} = \frac{a_{il}^2}{w_j w_l} = 1$ and that $w_i = w_j = w_k = w_l$. This leads to case (1) of Lemma 55.5.4 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

0C80 **Lemma 55.5.4.** *Classification of proper subgraphs of the form*

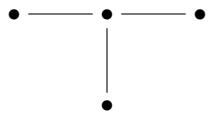

[4] It is $16w_iw_jw_kw_l - 4a_{ij}^2w_kw_l - 4a_{ik}^2w_jw_l - 4a_{il}^2w_jw_k$.

If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{ik}, a_{il} nonzero, then up to ordering we have the m's, a's, w's

0C81 (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2 + m_3 + m_4$, $2m_2 \geq m_1$, $2m_3 \geq m_1$, $2m_4 \geq m_1$. Observe that this implies $m_1 \geq \max(m_2, m_3, m_4)$.

Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index h "meets" i, i "meets" j, j "meets" k, and k "meets" l. Then we can apply Lemmas 55.5.2 and 55.5.3 to see that $a_{hj} = a_{hk} = a_{ik} = a_{il} = a_{jl} = 0$ and that the fractions $\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}$ are in $\{1, 2\}$ and the fraction $\frac{a_{hl}^2}{w_h w_l} \in \{0, 1, 2\}$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & a_{hl} \\ a_{hi} & -2w_i & a_{ij} & 0 & 0 \\ 0 & a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & 0 & a_{jk} & -2w_k & a_{kl} \\ a_{hl} & 0 & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 5$. The case $n = 5$ can happen: then the determinant[5] of the matrix is zero and we obtain the equation

$$\begin{aligned} 16 + \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k} \\ = 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l} \end{aligned}$$

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l}\right)^2 = \frac{a_{hi}^2}{w_h w_i}\frac{a_{ij}^2}{w_i w_j}\frac{a_{jk}^2}{w_j w_k}\frac{a_{kl}^2}{w_k w_l}\frac{a_{hl}^2}{w_h w_l}$$

We conclude the only possible solutions are

$$(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}) = (1,1,1,1,1), (1,1,2,1,0), (1,2,1,1,0), \text{ or } (2,1,1,2,0)$$

Observe that the case $(1,2,1,1,0)$ is obtained from the case $(1,1,2,1,0)$ by reversing the order of the indices h, i, j, k, l. In these cases $g = 1$; the reader can find these as cases (16), (17), (18), (19), (20), and (21) of Lemma 55.6.2 with one case

[5] It is $-32w_h w_i w_j w_k w_l + 8a_{hi}^2 w_j w_k w_l + 8a_{ij}^2 w_h w_k w_l + 8a_{jk}^2 w_h w_i w_l + 8a_{kl}^2 w_h w_i w_j + 8a_{hl}^2 w_i w_j w_k - 2a_{hi}^2 a_{jk}^2 w_l - 2a_{hi}^2 a_{kl}^2 w_j - 2a_{ij}^2 a_{kl}^2 w_h - 2a_{hl}^2 a_{ij}^2 w_k - 2a_{hl}^2 a_{jk}^2 w_i + 2a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}$.

corresponding to $(1,1,1,1,1)$, two cases corresponding to $(1,1,2,1,0)$, and three cases corresponding to $(2,1,1,2,0)$. Assuming $n > 5$ we obtain the inequality

$$16 + \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k}$$
$$> 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}) = (1,1,1,1,0), (1,1,1,2,0), \text{ or } (2,1,1,1,0)$$

in particular $a_{hl} = 0$ (recall that we assumed the other four to be nonzero). Observe that the case $(1,1,1,2,0)$ is obtained from the case $(2,1,1,1,0)$ by reversing the order of the indices h, i, j, k, l. The first two solutions lead to cases (1), (2), and (3) of Lemma 55.5.5 where we also wrote out the consequences for the integers m_h, m_i, m_j, m_k, and m_l.

0C82 **Lemma 55.5.5.** *Classification of proper subgraphs of the form*

If $n > 5$, then given five (-2)-indices h, i, j, k, l with $a_{hi}, a_{ij}, a_{jk}, a_{kl}$ nonzero, then up to ordering we have the m's, a's, w's

0C83 (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, and $2m_5 \geq m_4$, or

0C84 (2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, $2m_4 \geq m_3 + m_5$, and $2m_5 \geq m_4$, or

0C85 (3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, and $m_4 \geq m_3$.

Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, and $a_{hl} > 0$. In other words, the index h "meets" the indices i, j, k, l. Then we see from Lemma 55.5.2 that $a_{ij} = a_{ik} = a_{il} = a_{jk} = a_{jl} = a_{kl} = 0$ and by Lemma 55.5.4 that $w_h = w_i = w_j = w_k = w_l = w$ for some integer $w > 0$ and $a_{hi} = a_{hj} = a_{hk} = a_{hl} = -2w$. The corresponding matrix

$$\begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}$$

is singular. Hence this can only happen if $n = 5$ and $g = 1$. The reader can find this as case (22) Lemma 55.6.2.

0C86 **Lemma 55.5.6.** *Nonexistence of proper subgraphs of the form*

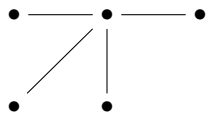

If $n > 5$, there do **not** *exist five (-2)-indices h, i, j, k with $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, and $a_{hl} > 0$.*

Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{jl} > 0$. In other words, the index h "meets" i and the index j "meets" the indices i, k, l. Then we see from Lemma 55.5.4 that $a_{ik} = a_{il} = a_{kl} = 0$, $w_i = w_j = w_k = w_l = w$, and $a_{ij} = a_{jk} = a_{jl} = w$ for some integer w. Applying Lemma 55.5.3 to the four tuples h, i, j, k and h, i, j, l we see that $a_{hj} = a_{hk} = a_{hl} = 0$, that $w_h = \frac{1}{2}w$, w, or $2w$, and that correspondingly $a_{hi} = w$, w, or $2w$. Since A is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & 0 \\ a_{hi} & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}$$

is negative definite unless $n = 5$. The reader computes that the determinant of the matrix is 0 when $w_h = \frac{1}{2}w$ or $2w$. This leads to cases (23) and (24) of Lemma 55.6.2. For $w_h = w$ we obtain case (1) of Lemma 55.5.7.

0C87 **Lemma 55.5.7.** *Classification of proper subgraphs of the form*

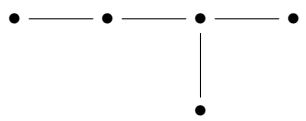

If $n > 5$, then given five (-2)-indices h, i, j, k, l with $a_{hi}, a_{ij}, a_{jk}, a_{jl}$ nonzero, then up to ordering we have the m's, a's, w's

0C88 (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_5$, $2m_4 \geq m_3$, *and* $2m_5 \geq m_3$.

Proof. See discussion above. □

Suppose that $t > 5$ and $i_1, \ldots, i_t$ are t distinct (-2)-indices such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$. We will prove by induction on t that if $n = t$ this leads to possibilities (25), (26), (27), (28) of Lemma 55.6.2 and if $n > t$ to cases (1), (2), and (3) of Lemma 55.5.8. First, if $a_{i_1 i_t}$ is nonzero, then it is clear from the result of Lemma 55.5.5 that $w_{i_1} = \ldots = w_{i_t} = w$ and that $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$ and $a_{i_1 i_t} = w$. Then the vector $(1, \ldots, 1)$ is in the kernel of the corresponding $t \times t$ matrix. Thus we must have $n = t$ and we see that the genus is 1 and that we are in case (25) of Lemma 55.6.2. Thus we may assume $a_{i_1 i_t} = 0$. By induction hypothesis (or Lemma 55.5.5 if $t = 6$) we see that $a_{i_j i_k} = 0$ if $k > j + 1$. Moreover, we have $w_{i_1} = \ldots = w_{i_{t-1}} = w$ for some integer w and $w_{i_1}, w_{i_t} \in \{\frac{1}{2}w, w, 2w\}$. Moreover, the value of w_{i_1}, resp. w_{i_t} being $\frac{1}{2}w$, w, or $2w$ implies that the value of $a_{i_1 i_2}$, resp. $a_{i_{t-1} i_t}$ is w, w, or $2w$. This gives 9 possibilities. In each case it is easy to decide what happens:

(1) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, \frac{1}{2}w)$, then we are in case (27) of Lemma 55.6.2,
(2) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, w)$ or $(w, \frac{1}{2}w)$ then we are in case (3) of Lemma 55.5.8,
(3) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, 2w)$ or $(2w, \frac{1}{2}w)$ then we are in case (26) of Lemma 55.6.2,
(4) if $(w_{i_1}, w_{i_t}) = (w, w)$ then we are in case (1) of Lemma 55.5.8,
(5) if $(w_{i_1}, w_{i_t}) = (w, 2w)$ or $(2w, w)$ then we are in case (2) of Lemma 55.5.8, and
(6) if $(w_{i_1}, w_{i_t}) = (2w, 2w)$ then we are in case (28) of Lemma 55.6.2.

0C89 **Lemma 55.5.8.** *Classification of proper subgraphs of the form*

Let $t > 5$ *and* $n > t$. *Then given* t *distinct* (-2)*-indices* $i_1, \ldots, i_t$ *such that* $a_{i_j i_{j+1}}$ *is nonzero for* $j = 1, \ldots, t-1$, *then up to reversing the order of these indices we have the* a*'s and* w*'s*

0C8A (1) *are given by* $w_{i_1} = w_{i_2} = \ldots = w_{i_t} = w$, $a_{i_j i_{j+1}} = w$, *and* $a_{i_j i_k} = 0$ *if* $k > j + 1$, *or*

0C8B (2) *are given by* $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = w$, $w_{j_t} = 2w$, $a_{i_j i_{j+1}} = w$ *for* $j < t - 1$, $a_{i_{t-1} i_t} = 2w$, *and* $a_{i_j i_k} = 0$ *if* $k > j + 1$, *or*

0C8C (3) *are given by* $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = 2w$, $w_{j_t} = w$, $a_{i_j i_{j+1}} = 2w$, *and* $a_{i_{t-1} i_t} = 2w$, *and* $a_{i_j i_k} = 0$ *if* $k > j + 1$.

Proof. See discussion above. □

Suppose that $t > 4$ and $i_1, \ldots, i_{t+1}$ are $t+1$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and such that $a_{j_{t-1} j_{t+1}} > 0$. See picture in Lemma 55.5.9. We

will prove by induction on t that if $n = t+1$ this leads to possibilities (29) and (30) of Lemma 55.6.2 and if $n > t+1$ to case (1) of Lemma 55.5.9. By induction hypothesis (or Lemma 55.5.7 in case $t = 5$) we see that $a_{i_j i_k}$ is zero outside of the required nonvanishing ones for $j, k \geq 2$. Moreover, we see that $w_2 = \ldots = w_{t+1} = w$ for some integer w and that the nonvanishing $a_{i_j i_k}$ for $j, k \geq 2$ are equal to w. Applying Lemma 55.5.8 (or Lemma 55.5.5 if $t = 5$) to the sequence $i_1, \ldots, i_t$ and to the sequence $i_1, \ldots, i_{t-1}, i_{t+1}$ we conclude that $a_{i_1 i_j} = 0$ for $j \geq 3$ and that w_1 is equal to $\frac{1}{2}w$, w, or $2w$ and that correspondingly $a_{i_1 i_2}$ is $w, w, 2w$. This gives 3 possibilities. In each case it is easy to decide what happens:

(1) If $w_1 = \frac{1}{2}w$, then we are in case (30) of Lemma 55.6.2.
(2) If $w_1 = w$, then we are in case (1) of Lemma 55.5.9.
(3) If $w_1 = 2w$, then we are in case (29) of Lemma 55.6.2.

0C8D **Lemma 55.5.9.** *Classification of proper subgraphs of the form*

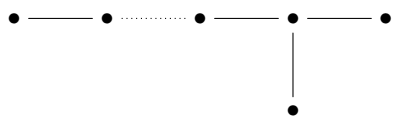

Let $t > 4$ and $n > t + 1$. Then given $t + 1$ distinct (-2)-indices $i_1, \ldots, i_{t+1}$ such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t - 1$ and $a_{i_{t-1} i_{t+1}}$ is nonzero, then we have the a's and w's

0C8E (1) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t+1}} = w$, $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$, $a_{i_{t-1} i_{t+1}} = w$ and $a_{i_j i_k} = 0$ for other pairs (j, k) with $j > k$.*

Proof. See discussion above. □

Suppose we are given 6 distinct (-2)-indices g, h, i, j, k, l such that $a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 55.5.10. Then we can apply Lemma 55.5.7 to see that we must be in the situation of Lemma 55.5.10. Since the determinant is $3w^6 > 0$ we conclude that in this case it never happens that $n = 6$!

0C8F **Lemma 55.5.10.** *Classification of proper subgraphs of the form*

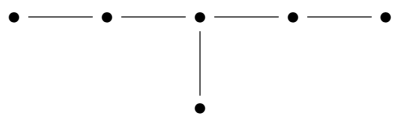

Let $n > 6$. Then given 6 distinct (-2)-indices $i_1, \ldots, i_6$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{36}$ are nonzero, then we have the m's, a's, and w's

0C8G (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & w \\ 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_6$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_3$, and $2m_6 \geq m_3$.

Proof. See discussion above. □

Suppose that $t \geq 4$ and $i_0, \ldots, i_{t+1}$ are $t+2$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and $a_{i_0 i_2} > 0$ and $a_{i_{t-1} i_{t+1}} > 0$. See picture in Lemma 55.5.11. Then we can apply Lemmas 55.5.7 and 55.5.9 to see that all other $a_{i_j i_k}$ for $j < k$ are zero and that $w_{i_0} = \ldots = w_{i_{t+1}} = w$ for some integer w and that the required nonzero off diagonal entries of A are equal to w. A computation shows that the determinant of the corresponding matrix is zero. Hence $n = t + 2$ and we are in case (31) of Lemma 55.6.2.

0C8H **Lemma 55.5.11.** *Nonexistence of proper subgraphs of the form*

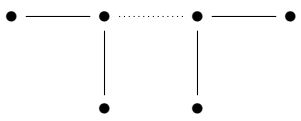

Assume $t \geq 4$ and $n > t + 2$. There do **not** *exist $t + 2$ distinct (-2)-indices $i_0, \ldots, i_{t+1}$ such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and $a_{i_0 i_2} > 0$ and $a_{i_{t-1} i_{t+1}} > 0$.*

Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero. See picture in Lemma 55.5.12. Then we can apply Lemma 55.5.7 to see that the corresponding matrix is

$$\begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}$$

Since the determinant is 0 we conclude that we must have $n = 7$ and $g = 1$ and we get case (32) of Lemma 55.6.2.

0C8I **Lemma 55.5.12.** *Nonexistence of proper subgraphs of the form*

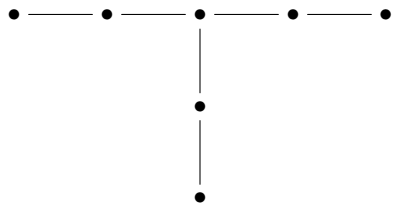

Assume $n > 7$. There do **not** *exist 7 distinct (-2)-indices f, g, h, i, j, k, l such that $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero.*

Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 55.5.13. Then we can apply Lemmas 55.5.7 and 55.5.9 to see that we must be in the situation of Lemma 55.5.13. Since the determinant is $-8w^7 > 0$ we conclude that in this case it never happens that $n = 7$!

0C8J **Lemma 55.5.13.** *Classification of proper subgraphs of the form*

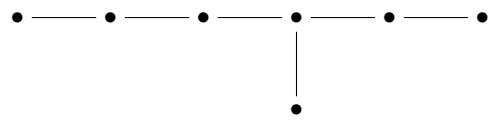

Let $n > 7$. Then given 7 distinct (-2)-indices $i_1, \ldots, i_7$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{47}$ are nonzero, then we have the m's, a's, and w's

0C8K (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5 + m_7$, $2m_5 \geq m_4 + m_6$, $2m_6 \geq m_5$, and $2m_7 \geq m_4$.

Proof. See discussion above. □

Suppose we are given 8 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like

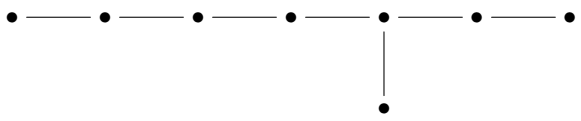

or like

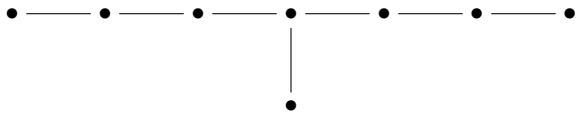

Arguing exactly as in the proof of Lemma 55.5.13 we see that the first pattern leads to case (1) in Lemma 55.5.14 and does not lead to a new case in Lemma 55.6.2. Arguing exactly as in the proof of Lemma 55.5.12 we see that the second pattern does not occur if $n > 8$, but leads to case (33) in Lemma 55.6.2 when $n = 8$.

0C8L **Lemma 55.5.14.** *Classification of proper subgraphs of the form*

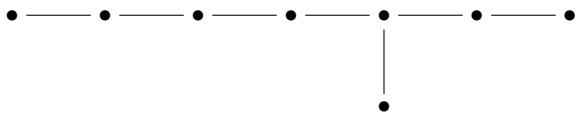

Let $n > 8$. Then given 8 distinct (-2)-indices $i_1, \ldots, i_8$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{65}, a_{57}$ are nonzero, then we have the m's, a's, and w's

0C8M (1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_4 + m_6 + m_8$, $2m_6 \geq m_5 + m_7$, $2m_7 \geq m_6$, *and* $2m_8 \geq m_5$.

Proof. See discussion above. □

0C8N **Lemma 55.5.15.** *Nonexistence of proper subgraphs of the form*

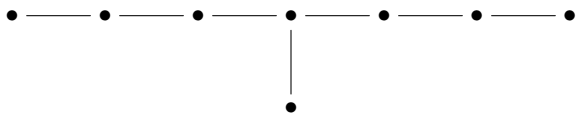

Assume $n > 8$. There do **not** *exist 8 distinct (-2)-indices e, f, g, h, i, j, k, l such that $a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ are nonzero.*

Proof. See discussion above. □

Suppose we are given 9 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like

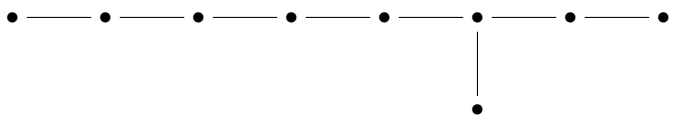

Arguing exactly as in the proof of Lemma 55.5.12 we see that this pattern does not occur if $n > 9$, but leads to case (34) in Lemma 55.6.2 when $n = 9$.

0C8P **Lemma 55.5.16.** *Nonexistence of proper subgraphs of the form*



Assume $n > 9$. There do **not** *exist 9 distinct (-2)-indices $d, e, f, g, h, i, j, k, l$ such that $a_{de}, a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ are nonzero.*

Proof. See discussion above. □

Collecting all the information together we find the following.

0C8Q **Proposition 55.5.17.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Let $I \subset \{1, \ldots, n\}$ be a proper subset of cardinality ≥ 2 consisting of (-2)-indices such that there does not exist a nonempty proper subset $I' \subset I$ with $a_{i'i} = 0$ for $i' \in I$, $i \in I \setminus I'$. Then up to reordering the m_i's, a_{ij}'s, w_i's for $i, j \in I$ are as listed in Lemmas 55.5.1, 55.5.2, 55.5.3, 55.5.4, 55.5.5, 55.5.7, 55.5.8, 55.5.9, 55.5.10, 55.5.13, or 55.5.14.*

Proof. This follows from the discussion above; see discussion at the start of Section 55.5. □

55.6. Classification of minimal type for genus zero and one

0C8R The title of the section explains it all.

0C8S **Lemma 55.6.1** (Genus zero)**.** *The only minimal numerical type of genus zero is* $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 1$, $g_1 = 0$.

Proof. Follows from Lemmas 55.3.13 and 55.3.5. □

0C8T **Lemma 55.6.2** (Genus one)**.** *The minimal numerical types of genus one are up to equivalence*

0C8U (1) $n = 1$, $a_{11} = 0$, $g_1 = 1$, $m_1, w_1 \geq 1$ *arbitrary,*

0C8V (2) $n = 2$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C8W (3) $n = 2$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 4w \\ 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C8X (4) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w \\ w & -2w & w \\ w & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C8Y (5) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 3w \\ 0 & 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 3w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C8Z (6) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \end{pmatrix}, \quad \begin{pmatrix} -6w & 3w & 0 \\ 3w & -6w & 3w \\ 0 & 3w & -2w \end{pmatrix}, \quad \begin{pmatrix} 3w \\ 3w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C90 (7) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 4w \\ 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C91 (8) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C92 (9) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C93 (10) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & w \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ w & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C94 (11) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 4w \\ 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C95 (12) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C96 (13) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C97 (14) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & 2w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ 2w & 0 & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C98 (15) $n = 4$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 2w & 2w \\ 2w & -4w & 0 & 0 \\ 2w & 0 & -4w & 0 \\ 2w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C99 (16) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & w \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9A (17) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9B (18) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9C (19) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 4w \\ 0 & 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9D (20) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9E (21) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9F (22) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9G (23) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9H (24) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 2w \\ 0 & 0 & 2w & -4w & 0 \\ 0 & 0 & 2w & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

0C9I (25) $n \geq 6$ *and we have an* n*-cycle generalizing (16):*
- (a) $m_1 = \ldots = m_n = m$,
- (b) $a_{12} = \ldots = a_{(n-1)n} = w$, $a_{1n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
- (c) $w_1 = \ldots = w_n = w$

with w *and* m *arbitrary,*

0C9J (26) $n \geq 6$ *and we have a chain generalizing (19):*
- (a) $m_1 = \ldots = m_{n-1} = 2m$, $m_n = m$,
- (b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-1)n} = 4w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
- (c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = 4w$

with w *and* m *arbitrary,*

0C9K (27) $n \geq 6$ *and we have a chain generalizing (20):*
- (a) $m_1 = \ldots = m_n = m$,
- (b) $a_{12} = \ldots = a_{(n-1)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
- (c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = w$

with w *and* m *arbitrary,*

0C9L (28) *$n \geq 6$ and we have a chain generalizing (21):*
(a) $m_1 = w$, $w_2 = \ldots = m_{n-1} = 2m$, $m_n = m$,
(b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-1)n} = 2w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = 2w$, $w_2 = \ldots = w_{n-1} = w$, $w_n = 2w$
with w and m arbitrary,

0C9M (29) *$n \geq 6$ and we have a type generalizing (23):*
(a) $m_1 = m$, $m_2 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = 2w$, $w_2 = \ldots = w_n = w$
with w and m arbitrary,

0C9N (30) *$n \geq 6$ and we have a type generalizing (24):*
(a) $m_1 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-2)n} = 2w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = w$, $w_2 = \ldots = w_n = 2w$
with w and m arbitrary,

0C9P (31) *$n \geq 6$ and we have a type generalizing (22):*
(a) $m_1 = m_2 = m$, $m_3 = \ldots = m_{n-2} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{13} = w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = \ldots = w_n = w$,
with w and m arbitrary,

0C9Q (32) *$n = 7$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9R (33) *$n = 8$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 3m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

0C9S (34) *$n = 9$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 5m \\ 6m \\ 4m \\ 2m \\ 3m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary.

Proof. This is proved in Section 55.5. See discussion at the start of Section 55.5. □

55.7. Bounding invariants of numerical types

0C9T In our proof of semistable reduction for curves we'll use a bound on Picard groups of numerical types of genus g which we will prove in this section.

0C9U **Lemma 55.7.1.** *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Given i, j with $a_{ij} > 0$ we have $m_i a_{ij} \leq m_j |a_{jj}|$ and $m_i w_i \leq m_j |a_{jj}|$.*

Proof. For every index j we have $m_j a_{jj} + \sum_{i \neq j} m_i a_{ij} = 0$. Thus if we have an upper bound on $|a_{jj}|$ and m_j, then we also get an upper bound on the nonzero (and hence positive) a_{ij} as well as m_i. Recalling that w_i divides a_{ij}, the reader easily sees the lemma is correct. □

0C9V **Lemma 55.7.2.** *Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g with $n > 1$ we have*

1. *the set $J \subset \{1, \ldots, n\}$ of non-(-2)-indices has at most $2g - 2$ elements,*
2. *for $j \in J$ we have $g_j < g$,*
3. *for $j \in J$ we have $m_j |a_{jj}| \leq 6g - 6$, and*
4. *for $j \in J$ and $i \in \{1, \ldots, n\}$ we have $m_i a_{ij} \leq 6g - 6$.*

Proof. Recall that $g = 1 + \sum m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$. For $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is > 0 and hence $\geq 1/2$. This uses Lemma 55.3.7, Definition 55.3.8, Definition 55.3.12, Lemma 55.3.15, and Definition 55.3.16; we will use these results without further mention in the following. Thus J has at most $2(g - 1)$ elements. This proves (1).

Recall that $-a_{ii} > 0$ for all i by Lemma 55.3.6. Hence for $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $> m_j w_j(g_j - 1)$. Thus

$$g - 1 > m_j w_j(g_j - 1) \Rightarrow g_j < (g - 1)/m_j w_j + 1$$

This indeed implies $g_j < g$ which proves (2).

For $j \in J$ if $g_j > 0$, then the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $\geq -\frac{1}{2}m_j a_{jj}$ and we immediately conclude that $m_j |a_{jj}| \leq 2(g - 1)$. Otherwise $a_{jj} = -k w_j$ for some integer $k \geq 3$ (because $j \in J$) and we get

$$m_j w_j(-1 + \frac{k}{2}) \leq g - 1 \Rightarrow m_j w_j \leq \frac{2(g - 1)}{k - 2}$$

Plugging this back into $a_{jj} = -km_jw_j$ we obtain

$$m_j|a_{jj}| \leq 2(g-1)\frac{k}{k-2} \leq 6(g-1)$$

This proves (3).

Part (4) follows from Lemma 55.7.1 and (3). □

0C9W **Lemma 55.7.3.** *Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g we have $m_i|a_{ij}| \leq 768g$.*

Proof. By Lemma 55.7.1 it suffices to show $m_i|a_{ii}| \leq 768g$ for all i. Let $J \subset \{1, \ldots, n\}$ be the set of non-(-2)-indices as in Lemma 55.7.2. Observe that J is nonempty as $g \geq 2$. Also $m_j|a_{jj}| \leq 6g$ for $j \in J$ by the lemma.

Suppose we have $j \in J$ and a sequence $i_1, \ldots, i_7$ of (-2)-indices such that a_{ji_1} and $a_{i_1i_2}$, $a_{i_2i_3}$, $a_{i_3i_4}$, $a_{i_4i_5}$, $a_{i_5i_6}$, and $a_{i_6i_7}$ are nonzero. Then we see from Lemma 55.7.1 that $m_{i_1}w_{i_1} \leq 6g$ and $m_{i_1}a_{ji_1} \leq 6g$. Because i_1 is a (-2)-index, we have $a_{i_1i_1} = -2w_{i_1}$ and we conclude that $m_{i_1}|a_{i_1i_1}| \leq 12g$. Repeating the argument we conclude that $m_{i_2}w_{i_2} \leq 12g$ and $m_{i_2}a_{i_1i_2} \leq 12g$. Then $m_{i_2}|a_{i_2i_2}| \leq 24g$ and so on. Eventually we conclude that $m_{i_k}|a_{i_ki_k}| \leq 2^k(6g) \leq 768g$ for $k = 1, \ldots, 7$.

Let $I \subset \{1, \ldots, n\} \setminus J$ be a maximal connected subset. In other words, there does not exist a nonempty proper subset $I' \subset I$ such that $a_{i'i} = 0$ for $i' \in I'$ and $i \in I \setminus I'$ and I is maximal with this property. In particular, since a numerical type is connected by definition, we see that there exists a $j \in J$ and $i \in I$ with $a_{ij} > 0$. Looking at the classification of such I in Proposition 55.5.17 and using the result of the previous paragraph, we see that $w_i|a_{ii}| \leq 768g$ for all $i \in I$ unless I is as described in Lemma 55.5.8 or Lemma 55.5.9. Thus we may assume the nonvanishing of $a_{ii'}$, $i, i' \in I$ has either the shape

(which has 3 subcases as detailed in Lemma 55.5.8) or the shape

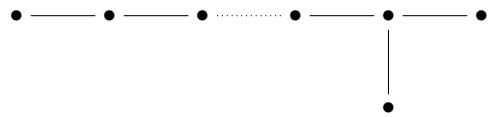

We will prove the bound holds for the first subcase of Lemma 55.5.8 and leave the other cases to reader (the argument is almost exactly the same in those cases).

After renumbering we may assume $I = \{1, \ldots, t\} \subset \{1, \ldots, n\}$ and there is an integer w such that

$$w = w_1 = \ldots = w_t = a_{12} = \ldots = a_{(t-1)t} = -\frac{1}{2}a_{i_1i_2} = \ldots = -\frac{1}{2}a_{(t-1)t}$$

The equalities $a_{ii}m_i + \sum_{j \neq i} a_{ij}m_j = 0$ imply that we have

$$2m_2 \geq m_1 + m_3, \ldots, 2m_{t-1} \geq m_{t-2} + m_t$$

Equality holds in $2m_i \geq m_{i-1} + m_{i+1}$ if and only if i does not "meet" any indices besides $i-1$ and $i+1$. And if i does meet another index, then this index is in J (by maximality of I). In particular, the map $\{1, \ldots, t\} \to \mathbf{Z}$, $i \mapsto m_i$ is concave.

Let $m = \max(m_i, i \in \{1, \ldots, t\})$. Then $m_i|a_{ii}| \leq 2mw$ for $i \leq t$ and our goal is to show that $2mw \leq 768g$. Let s, resp. s' in $\{1, \ldots, t\}$ be the smallest, resp. biggest

index with $m_s = m = m_{s'}$. By concavity we see that $m_i = m$ for $s \leq i \leq s'$. If $s > 1$, then we do not have equality in $2m_s \geq m_{s-1} + m_{s+1}$ and we see that s meets an index from J. In this case $2mw \leq 12g$ by the result of the second paragraph of the proof. Similarly, if $s' < t$, then s' meets an index from J and we get $2mw \leq 12g$ as well. But if $s = 1$ and $s' = t$, then we conclude that $a_{ij} = 0$ for all $j \in J$ and $i \in \{2, \ldots, t-1\}$. But as we've seen that there must be a pair $(i, j) \in I \times J$ with $a_{ij} > 0$, we conclude that this happens either with $i = 1$ or with $i = t$ and we conclude $2mw \leq 12g$ in the same manner as before (as $m_1 = m = m_t$ in this case). □

0C9X **Proposition 55.7.4.** *Let $g \geq 2$. For every numerical type T of genus g and prime number $\ell > 768g$ we have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(T)[\ell] \leq g$$

where $\mathrm{Pic}(T)$ *is as in Definition 55.4.1. If T is minimal, then we even have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(T)[\ell] \leq g_{top} \leq g$$

where g_{top} as in Definition 55.3.11.

Proof. Say T is given by n, m_i, a_{ij}, w_i, g_i. If T is not minimal, then there exists a (-1)-index. After replacing T by an equivalent type we may assume n is a (-1)-index. Applying Lemma 55.4.4 we find $\mathrm{Pic}(T) \subset \mathrm{Pic}(T')$ where T' is a numerical type of genus g (Lemma 55.3.9) with $n-1$ indices. Thus we conclude by induction on n provided we prove the lemma for minimal numerical types.

Assume that T is a minimal numerical type of genus ≥ 2. Observe that $g_{top} \leq g$ by Lemma 55.3.14. If $A = (a_{ij})$ then since $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$ by Lemma 55.4.3. Thus it suffices to prove the lemma for $\mathrm{Coker}(A)$. By Lemma 55.7.3 we see that $m_i|a_{ij}| \leq 768g$ for all i, j. Hence the result by Lemma 55.2.6. □

55.8. Models

0C2R In this chapter R will be a discrete valuation ring and K will be its fraction field. If needed we will denote $\pi \in R$ a uniformizer and $k = R/(\pi)$ its residue field.

Let V be an algebraic K-scheme (Varieties, Definition 33.20.1). A *model* for V will mean a flat finite type[6] morphism $X \to \mathrm{Spec}(R)$ endowed with an isomorphism $V \to X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$. We often will identify V and the generic fibre X_K of X and just write $V = X_K$. The special fibre is $X_k = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$. A *morphism of models $X \to X'$ for V* is a morphism $X \to X'$ of schemes over R which induces the identity on V.

We will say X *is a proper model of V* if X is a model of V and the structure morphism $X \to \mathrm{Spec}(R)$ is proper. Similarly for separated models, smooth models, and add more here. We will say X *is a regular model of V* if X is a model of V and X is a regular scheme. Similarly for normal models, reduced models, and add more here.

Let $R \subset R'$ be an extension of discrete valuation rings (More on Algebra, Definition 15.112.1). This induces an extension K'/K of fraction fields. Given an algebraic

[6] Occasionally it is useful to allow models to be locally of finite type over R, but we'll cross that bridge when we come to it.

scheme V over K, denote V' the base change $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$. Then there is a functor

$$\text{models for } V \text{ over } R \longrightarrow \text{models for } V' \text{ over } R'$$

sending X to $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$.

0C2S **Lemma 55.8.1.** *Let $V_1 \to V_2$ be a closed immersion of algebraic schemes over K. If X_2 is a model for V_2, then the scheme theoretic image of $V_1 \to X_2$ is a model for V_1.*

Proof. Using Morphisms, Lemma 29.6.3 and Example 29.6.4 this boils down to the following algebra statement. Let A_1 be a finite type R-algebra flat over R. Let $A_1 \otimes_R K \to B_2$ be a surjection. Then $A_2 = A_1/\operatorname{Ker}(A_1 \to B_2)$ is a finite type R-algebra flat over R such that $B_2 = A_2 \otimes_R K$. We omit the detailed proof; use More on Algebra, Lemma 15.22.11 to prove that A_2 is flat. □

0C2T **Lemma 55.8.2.** *Let X be a model of a geometrically normal variety V over K. Then the normalization $\nu : X^\nu \to X$ is finite and the base change of X^ν to the completion $R^\wedge$ is the normalization of the base change of X. Moreover, for each $x \in X^\nu$ the completion of $\mathcal{O}_{X^\nu,x}$ is normal.*

Proof. Observe that $R^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 15.43.5). Set $Y = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R^\wedge)$. Since $R^\wedge$ is a discrete valuation ring, we see that

$$Y \setminus Y_k = Y \times_{\mathrm{Spec}(R^\wedge)} \mathrm{Spec}(K^\wedge) = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^\wedge)$$

where $K^\wedge$ is the fraction field of $R^\wedge$. Since V is geometrically normal, we find that this is a normal scheme. Hence the first part of the lemma follows from Resolution of Surfaces, Lemma 54.11.6.

To prove the second part we may assume X and Y are normal (by the first part). If x is in the generic fibre, then $\mathcal{O}_{X,x} = \mathcal{O}_{V,x}$ is a normal local ring essentially of finite type over a field. Such a ring is excellent (More on Algebra, Proposition 15.52.3). If x is a point of the special fibre with image $y \in Y$, then $\mathcal{O}^\wedge_{X,x} = \mathcal{O}^\wedge_{Y,y}$ by Resolution of Surfaces, Lemma 54.11.1. In this case $\mathcal{O}_{Y,y}$ is a excellent normal local domain by the same reference as before as $R^\wedge$ is excellent. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 15.42.2 applies). This finishes the proof. □

0C2U **Lemma 55.8.3.** *Let X be a model of a smooth curve C over K. Then there exists a resolution of singularities of X and any resolution is a model of C.*

Proof. We check condition (4) of Lipman's theorem (Resolution of Surfaces, Theorem 54.14.5) hold. This is clear from Lemma 55.8.2 except for the statement that X^ν has finitely many singular points. To see this we can use that R is J-2 by More on Algebra, Proposition 15.48.7 and hence the nonsingular locus is open in X^ν. Since X^ν is normal of dimension ≤ 2, the singular points are closed, hence closedness of the singular locus means there are finitely many of them (as X is quasi-compact). Observe that any resolution of X is a modification of X (Resolution of Surfaces, Definition 54.14.1). This will be an isomorphism over the normal locus of X by Varieties, Lemma 33.17.3. Since the set of normal points includes $C = X_K$ we conclude any resolution is a model of C. □

0C2V **Definition 55.8.4.** Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A *minimal model* will be a regular, proper model X for C such that X does not contain an exceptional curve of the first kind (Resolution of Surfaces, Section 54.16).

Really such a thing should be called a minimal regular proper model or even a relatively minimal regular projective model. But as long as we stick to models over discrete valuation rings (as we will in this chapter), no confusion should arise.

Minimal models always exist (Proposition 55.8.6) and are unique when the genus is > 0 (Lemma 55.10.1).

0CD9 **Lemma 55.8.5.** *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If X is a regular proper model for C, then there exists a sequence of morphisms*

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind, and such that X_0 is a minimal model.

Proof. By Resolution of Surfaces, Lemma 54.16.11 we see that X is projective over R. Hence X has an ample invertible sheaf by More on Morphisms, Lemma 37.50.1 (we will use this below). Let $E \subset X$ be an exceptional curve of the first kind. See Resolution of Surfaces, Section 54.16. By Resolution of Surfaces, Lemma 54.16.8 we can contract E by a morphism $X \to X'$ such that X' is regular and is projective over R. Clearly, the number of irreducible components of X'_k is exactly one less than the number of irreducible components of X_k. Thus we can only perform a finite number of these contractions until we obtain a minimal model. □

0C2W **Proposition 55.8.6.** *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A minimal model exists.*

Proof. Choose a closed immersion $C \to \mathbf{P}^n_K$. Let X be the scheme theoretic image of $C \to \mathbf{P}^n_R$. Then $X \to \mathrm{Spec}(R)$ is a projective model of C by Lemma 55.8.1. By Lemma 55.8.3 there exists a resolution of singularities $X' \to X$ and X' is a model for C. Then $X' \to \mathrm{Spec}(R)$ is proper as a composition of proper morphisms. Then we may apply Lemma 55.8.5 to obtain a minimal model. □

55.9. The geometry of a regular model

0C5Y In this section we describe the geometry of a proper regular model X of a smooth projective curve C over K with $H^0(C, \mathcal{O}_C) = K$.

0C5Z **Lemma 55.9.1.** *Let X be a regular model of a smooth curve C over K.*

1. *the special fibre X_k is an effective Cartier divisor on X,*
2. *each irreducible component C_i of X_k is an effective Cartier divisor on X,*
3. *$X_k = \sum m_i C_i$ (sum of effective Cartier divisors) where m_i is the multiplicity of C_i in X_k,*
4. *$\mathcal{O}_X(X_k) \cong \mathcal{O}_X$.*

Proof. Recall that R is a discrete valuation ring with uniformizer π and residue field $k = R/(\pi)$. Because $X \to \mathrm{Spec}(R)$ is flat, the element π is a nonzerodivisor affine locally on X (see More on Algebra, Lemma 15.22.11). Thus if $U = \mathrm{Spec}(A) \subset X$ is an affine open, then

$$X_k \cap U = U_k = \mathrm{Spec}(A \otimes_R k) = \mathrm{Spec}(A/\pi A)$$

and π is a nonzerodivisor in A. Hence $X_k = V(\pi)$ is an effective Cartier divisor by Divisors, Lemma 31.13.2. Hence (1) is true.

The discussion above shows that the pair $(\mathcal{O}_X(X_k), 1)$ is isomorphic to the pair $(\mathcal{O}_X, \pi)$ which proves (4).

By Divisors, Lemma 31.15.11 there exist pairwise distinct integral effective Cartier divisors $D_i \subset X$ and integers $a_i \geq 0$ such that $X_k = \sum a_i D_i$. We can throw out those divisors D_i such that $a_i = 0$. Then it is clear (from the definition of addition of effective Cartier divisors) that $X_k = \bigcup D_i$ set theoretically. Thus $C_i = D_i$ are the irreducible components of X_k which proves (2). Let ξ_i be the generic point of C_i. Then $\mathcal{O}_{X,\xi_i}$ is a discrete valuation ring (Divisors, Lemma 31.15.4). The uniformizer $\pi_i \in \mathcal{O}_{X,\xi_i}$ is a local equation for C_i and the image of π is a local equation for X_k. Since $X_k = \sum a_i C_i$ we see that π and $\pi_i^{a_i}$ generate the same ideal in $\mathcal{O}_{X,\xi_i}$. On the other hand, the multiplicity of C_i in X_k is

$$m_i = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X_k,\xi_i} = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi) = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi_i^{a_i}) = a_i$$

See Chow Homology, Definition 42.9.2. Thus $a_i = m_i$ and (3) is proved. □

0C60 **Lemma 55.9.2.** *Let X be a regular model of a smooth curve C over K. Then*

(1) *$X \to \text{Spec}(R)$ is a Gorenstein morphism of relative dimension 1,*
(2) *each of the irreducible components C_i of X_k is Gorenstein.*

Proof. Since $X \to \text{Spec}(R)$ is flat, to prove (1) it suffices to show that the fibres are Gorenstein (Duality for Schemes, Lemma 48.25.3). The generic fibre is a smooth curve, which is regular and hence Gorenstein (Duality for Schemes, Lemma 48.24.3). For the special fibre X_k we use that it is an effective Cartier divisor on a regular (hence Gorenstein) scheme and hence Gorenstein for example by Dualizing Complexes, Lemma 47.21.6. The curves C_i are Gorenstein by the same argument. □

0C61 **Situation 55.9.3.** Let R be a discrete valuation ring with fraction field K, residue field k, and uniformizer π. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Let X be a regular proper model of C. Let $C_1, \ldots, C_n$ be the irreducible components of the special fibre X_k. Write $X_k = \sum m_i C_i$ as in Lemma 55.9.1.

0C62 **Lemma 55.9.4.** *In Situation 55.9.3 the special fibre X_k is connected.*

Proof. Consequence of More on Morphisms, Lemma 37.53.6. □

0C63 **Lemma 55.9.5.** *In Situation 55.9.3 there is an exact sequence*

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \text{Pic}(X) \to \text{Pic}(C) \to 0$$

where the first map sends 1 to $(m_1, \ldots, m_n)$ and the second maps sends the ith basis vector to $\mathcal{O}_X(C_i)$.

Proof. Observe that $C \subset X$ is an open subscheme. The restriction map $\text{Pic}(X) \to \text{Pic}(C)$ is surjective by Divisors, Lemma 31.28.3. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that there is an isomorphism $s : \mathcal{O}_C \to \mathcal{L}|_C$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_{\mathcal{L}}(s) = \sum a_i C_i$ for some $a_i \in \mathbf{Z}$ (Divisors, Definition 31.27.4). By Divisors, Lemma 31.27.6 (and the fact that X is normal) we conclude that $\mathcal{L} = \mathcal{O}_X(\sum a_i C_i)$. Finally, suppose that $\mathcal{O}_X(\sum a_i C_i) \cong \mathcal{O}_X$. Then there exists an element g of the function field of X with $\text{div}_X(g) = \sum a_i C_i$. In particular

the rational function g has no zeros or poles on the generic fibre C of X. Since C is a normal scheme this implies $g \in H^0(C, \mathcal{O}_C) = K$. Thus $g = \pi^a u$ for some $a \in \mathbf{Z}$ and $u \in R^*$. We conclude that $\mathrm{div}_X(g) = a \sum m_i C_i$ and the proof is complete. □

In Situation 55.9.3 for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ and every i we get an integer

$$\deg(\mathcal{L}|_{C_i}) = \chi(C_i, \mathcal{L}|_{C_i}) - \chi(C_i, \mathcal{O}_{C_i})$$

by taking the degree of the restriction of $\mathcal{L}$ to C_i relative to the ground field k[7] as in Varieties, Section 33.44.

0C64 **Lemma 55.9.6.** *In Situation 55.9.3 given $\mathcal{L}$ an invertible $\mathcal{O}_X$-module and $a = (a_1, \ldots, a_n) \in \mathbf{Z}^{\oplus n}$ we define*

$$\langle a, \mathcal{L} \rangle = \sum a_i \deg(\mathcal{L}|_{C_i})$$

Then $\langle , \rangle$ is bilinear and for $b = (b_1, \ldots, b_n) \in \mathbf{Z}^{\oplus n}$ we have

$$\Big\langle a, \mathcal{O}_X(\sum b_i C_i) \Big\rangle = \Big\langle b, \mathcal{O}_X(\sum a_i C_i) \Big\rangle$$

Proof. Bilinearity is immediate from the definition and Varieties, Lemma 33.44.7. To prove symmetry it suffices to assume a and b are standard basis vectors in $\mathbf{Z}^{\oplus n}$. Hence it suffices to prove that

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(\mathcal{O}_X(C_i)|_{C_j})$$

for all $1 \leq i, j \leq n$. If $i = j$ there is nothing to prove. If $i \neq j$, then the canonical section 1 of $\mathcal{O}_X(C_j)$ restricts to a nonzero (hence regular) section of $\mathcal{O}_X(C_j)|_{C_i}$ whose zero scheme is exactly $C_i \cap C_j$ (scheme theoretic intersection). In other words, $C_i \cap C_j$ is an effective Cartier divisor on C_i and

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(C_i \cap C_j)$$

by Varieties, Lemma 33.44.9. By symmetry we obtain the same (!) formula for the other side and the proof is complete. □

In Situation 55.9.3 it is often convenient to think of $\mathbf{Z}^{\oplus n}$ as the free abelian group on the set $\{C_1, \ldots, C_n\}$. We will indicate an element of this group as $\sum a_i C_i$; here we think of this as a formal sum although equivalently we may (and we sometimes do) think of such a sum as a Weil divisor on X supported on the special fibre X_k. Now Lemma 55.9.6 allows us to define a symmetric bilinear form $(\ \cdot\ )$ on this free abelian group by the rule

0C65 (55.9.6.1) $$\Big(\sum a_i C_i \cdot \sum b_j C_j\Big) = \Big\langle a, \mathcal{O}_X(\sum b_j C_j) \Big\rangle = \Big\langle b, \mathcal{O}_X(\sum a_i C_i) \Big\rangle$$

We will prove some properties of this bilinear form.

0C66 **Lemma 55.9.7.** *In Situation 55.9.3 the symmetric bilinear form (55.9.6.1) has the following properties*

(1) $(C_i \cdot C_j) \geq 0$ *if $i \neq j$ with equality if and only if $C_i \cap C_j = \emptyset$,*
(2) $(\sum m_i C_i \cdot C_j) = 0$,
(3) *there is no nonempty proper subset $I \subset \{1, \ldots, n\}$ such that $(C_i \cdot C_j) = 0$ for $i \in I$, $j \notin I$.*

[7] Observe that it may happen that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ is strictly bigger than k. In this case every invertible module on C_i has degree (as defined above) divisible by $[\kappa_i : k]$.

(4) $(\sum a_i C_i \cdot \sum a_i C_i) \leq 0$ *with equality if and only if there exists a* $q \in \mathbf{Q}$ *such that* $a_i = qm_i$ *for* $i = 1, \ldots, n$,

Proof. In the proof of Lemma 55.9.6 we saw that $(C_i \cdot C_j) = \deg(C_i \cap C_j)$ if $i \neq j$. This is ≥ 0 and > 0 if and only if $C_i \cap C_j \neq \emptyset$. This proves (1).

Proof of (2). This is true because by Lemma 55.9.1 the invertible sheaf associated to $\sum m_i C_i$ is trivial and the trivial sheaf has degree zero.

Proof of (3). This is expressing the fact that X_k is connected (Lemma 55.9.4) via the description of the intersection products given in the proof of (1).

Part (4) follows from (1), (2), and (3) by Lemma 55.2.3. □

0C67 **Lemma 55.9.8.** *In Situation 55.9.3 set* $d = \gcd(m_1, \ldots, m_n)$ *and let* $D = \sum (m_i/d) C_i$ *as an effective Cartier divisor. Then* $\mathcal{O}_X(D)$ *has order dividing* d *in* $\mathrm{Pic}(X)$ *and* $\mathcal{C}_{D/X}$ *an invertible* $\mathcal{O}_D$*-module of order dividing* d *in* $\mathrm{Pic}(D)$.

Proof. We have

$$\mathcal{O}_X(D)^{\otimes d} = \mathcal{O}_X(dD) = \mathcal{O}_X(X_k) = \mathcal{O}_X$$

by Lemma 55.9.1. We conclude as $\mathcal{C}_{D/X}$ is the pullback of $\mathcal{O}_X(-D)$. □

0C68 **Lemma 55.9.9.** *In Situation 55.9.3 let* $d = \gcd(m_1, \ldots, m_n)$. *Let* $D = \sum (m_i/d) C_i$ *as an effective Cartier divisor. Then there exists a sequence of effective Cartier divisors* [**AW71**, Lemma 2.6]

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = D$$

such that $Z_j = Z_{j-1} + C_{i_j}$ *for some* $i_j \in \{1, \ldots, n\}$ *for* $j = 1, \ldots, m$ *and such that* $H^0(Z_j, \mathcal{O}_{Z_j})$ *is a field finite over* k *for* $j = 0, \ldots m$.

Proof. The reduction $D_{red} = (X_k)_{red} = \sum C_i$ is connected (Lemma 55.9.4) and proper over k. Hence $H^0(D_{red}, \mathcal{O})$ is a field and a finite extension of k by Varieties, Lemma 33.9.3. Thus the result for $Z_0 = D_{red} = (X_k)_{red}$ is true. Suppose that we have already constructed

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_t \subset D$$

with $Z_j = Z_{j-1} + C_{i_j}$ for some $i_j \in \{1, \ldots, n\}$ for $j = 1, \ldots, t$ and such that $H^0(Z_j, \mathcal{O}_{Z_j})$ is a field finite over k for $j = 0, \ldots, t$. Write $Z_t = \sum a_i C_i$ with $1 \leq a_i \leq m_i/d$. If $a_i = m_i/d$ for all i, then $Z_t = D$ and the lemma is proved. If not, then $a_i < m_i/d$ for some i and it follows that $(Z_t \cdot Z_t) < 0$ by Lemma 55.9.7. This means that $(D - Z_t \cdot Z_t) > 0$ because $(D \cdot Z_t) = 0$ by the lemma. Thus we can find an i with $a_i < m_i/d$ such that $(C_i \cdot Z_t) > 0$. Set $Z_{t+1} = Z_t + C_i$ and $i_{t+1} = i$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-Z_t)|_{C_i} \to \mathcal{O}_{Z_{t+1}} \to \mathcal{O}_{Z_t} \to 0$$

of Divisors, Lemma 31.14.3. By our choice of i we see that $\mathcal{O}_X(-Z_t)|_{C_i}$ is an invertible sheaf of negative degree on the proper curve C_i, hence it has no nonzero global sections (Varieties, Lemma 33.44.12). We conclude that $H^0(\mathcal{O}_{Z_{t+1}}) \subset H^0(\mathcal{O}_{Z_t})$ is a field (this is clear but also follows from Algebra, Lemma 10.36.18) and a finite extension of k. Thus we have extended the sequence. Since the process must stop, for example because $t \leq \sum (m_i/d - 1)$, this finishes the proof. □

0C69 **Lemma 55.9.10.** *In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$. Let $D = \sum (m_i/d)C_i$ as an effective Cartier divisor on X. Then* [**AW71**, Lemma 2.6]

$$1 - g_C = d[\kappa : k](1 - g_D)$$

where g_C is the genus of C, g_D is the genus of D, and $\kappa = H^0(D, \mathcal{O}_D)$.

Proof. By Lemma 55.9.9 we see that κ is a field and a finite extension of k. Since also $H^0(C, \mathcal{O}_C) = K$ we see that the genus of C and D are defined (see Algebraic Curves, Definition 53.8.1) and we have $g_C = \dim_K H^1(C, \mathcal{O}_C)$ and $g_D = \dim_\kappa H^1(D, \mathcal{O}_D)$. By Derived Categories of Schemes, Lemma 36.32.2 we have

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k}) = \dim_k H^0(X_k, \mathcal{O}_{X_k}) - \dim_k H^1(X_k, \mathcal{O}_{X_k})$$

We claim that

$$\chi(X_k, \mathcal{O}_{X_k}) = d\chi(D, \mathcal{O}_D)$$

This will prove the lemma because

$$\chi(D, \mathcal{O}_D) = \dim_k H^0(D, \mathcal{O}_D) - \dim_k H^1(D, \mathcal{O}_D) = [\kappa : k](1 - g_D)$$

Observe that $X_k = dD$ as an effective Cartier divisor. To prove the claim we prove by induction on $1 \leq r \leq d$ that $\chi(rD, \mathcal{O}_{rD}) = r\chi(D, \mathcal{O}_D)$. The base case $r = 1$ is trivial. If $1 \leq r < d$, then we consider the short exact sequence

$$0 \to \mathcal{O}_X(rD)|_D \to \mathcal{O}_{(r+1)D} \to \mathcal{O}_{rD} \to 0$$

of Divisors, Lemma 31.14.3. By additivity of Euler characteristics (Varieties, Lemma 33.33.2) it suffices to prove that $\chi(D, \mathcal{O}_X(rD)|_D) = \chi(D, \mathcal{O}_D)$. This is true because $\mathcal{O}_X(rD)|_D$ is a torsion element of $\mathrm{Pic}(D)$ (Lemma 55.9.8) and because the degree of a line bundle is additive (Varieties, Lemma 33.44.7) hence zero for torsion invertible sheaves. □

0C6A **Lemma 55.9.11.** *In Situation 55.9.3 given a pair of indices i, j such that C_i and C_j are exceptional curves of the first kind and $C_i \cap C_j \neq \emptyset$, then $n = 2$, $m_1 = m_2 = 1$, $C_1 \cong \mathbf{P}^1_k$, $C_2 \cong \mathbf{P}^1_k$, C_1 and C_2 meet in a k-rational point, and C has genus 0.*

Proof. Choose isomorphisms $C_i = \mathbf{P}^1_{\kappa_i}$ and $C_j = \mathbf{P}^1_{\kappa_j}$. The scheme $C_i \cap C_j$ is a nonempty effective Cartier divisor in both C_i and C_j. Hence

$$(C_i \cdot C_j) = \deg(C_i \cap C_j) \geq \max([\kappa_i : k], [\kappa_j : k])$$

The first equality was shown in the proof of Lemma 55.9.6. On the other hand, the self intersection $(C_i \cdot C_i)$ is equal to the degree of $\mathcal{O}_X(C_i)$ on C_i which is $-[\kappa_i : k]$ as C_i is an exceptional curve of the first kind. Similarly for C_j. By Lemma 55.9.7

$$0 \geq (C_i + C_j)^2 = -[\kappa_i : k] + 2(C_i \cdot C_j) - [\kappa_j : k]$$

This implies that $[\kappa_i : k] = \deg(C_i \cap C_j) = [\kappa_j : k]$ and that we have $(C_i + C_j)^2 = 0$. Looking at the lemma again we conclude that $n = 2$, $\{1, 2\} = \{i, j\}$, and $m_1 = m_2$. Moreover, the scheme theoretic intersection $C_i \cap C_j$ consists of a single point p with residue field κ and $\kappa_i \to \kappa \leftarrow \kappa_j$ are isomorphisms. Let $D = C_1 + C_2$ as effective Cartier divisor on X. Observe that D is the scheme theoretic union of C_1 and C_2 (Divisors, Lemma 31.13.10) hence we have a short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_p \to 0$$

by Morphisms, Lemma 29.4.6. Since we know the cohomology of $C_i \cong \mathbf{P}^1_\kappa$ (Cohomology of Schemes, Lemma 30.8.1) we conclude from the long exact cohomology sequence that $H^0(D, \mathcal{O}_D) = \kappa$ and $H^1(D, \mathcal{O}_D) = 0$. By Lemma 55.9.10 we conclude

$$1 - g_C = d[\kappa : k](1 - 0)$$

where $d = m_1 = m_2$. It follows that $g_C = 0$ and $d = m_1 = m_2 = 1$ and $\kappa = k$. □

55.10. Uniqueness of the minimal model

0C9Y If the genus of the generic fibre is positive, then minimal models are unique (Lemma 55.10.1) and consequently have a suitable mapping property (Lemma 55.10.2).

0C6B **Lemma 55.10.1.** *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. There is a unique minimal model for C.*

Proof. We have already proven the hard part of the lemma which is the existence of a minimal model (whose proof relies on resolution of surface singularities), see Proposition 55.8.6. To prove uniqueness, suppose that X and Y are two minimal models. By Resolution of Surfaces, Lemma 54.17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \rightarrow \ldots \rightarrow Y_1 \rightarrow Y_0 = Y$$

where each morphism is a blowup in a closed point. The exceptional fibre of the morphism $X_n \to X_{n-1}$ is an exceptional curve of the first kind E. We claim that E is contracted to a point under the morphism $X_n = Y_m \to Y$. If this is true, then $X_n \to Y$ factors through X_{n-1} by Resolution of Surfaces, Lemma 54.16.1. In this case the morphism $X_{n-1} \to Y$ is still a sequence of contractions of exceptional curves by Resolution of Surfaces, Lemma 54.17.1. Hence by induction on n we conclude. (The base case $n = 0$ means that there is a sequence of contractions $X = Y_m \to \ldots \to Y_1 \to Y_0 = Y$ ending with Y. However as X is a minimal model it contains no exceptional curves of the first kind, hence $m = 0$ and $X = Y$.)

Proof of the claim. We will show by induction on m that any exceptional curve of the first kind $E \subset Y_m$ is mapped to a point by the morphism $Y_m \to Y$. If $m = 0$ this is clear because Y is a minimal model. If $m > 0$, then either $Y_m \to Y_{m-1}$ contracts E (and we're done) or the exceptional fibre $E' \subset Y_m$ of $Y_m \to Y_{m-1}$ is a second exceptional curve of the first kind. Since both E and E' are irreducible components of the special fibre and since $g_C > 0$ by assumption, we conclude that $E \cap E' = \emptyset$ by Lemma 55.9.11. Then the image of E in Y_{m-1} is an exceptional curve of the first kind (this is clear because the morphism $Y_m \to Y_{m-1}$ is an isomorphism in a neighbourhood of E). By induction we see that $Y_{m-1} \to Y$ contracts this curve and the proof is complete. □

0C9Z **Lemma 55.10.2.** *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. Let X be the minimal model for C (Lemma 55.10.1). Let Y be a regular proper model for C. Then there is a unique morphism of models $Y \to X$ which is a sequence of contractions of exceptional curves of the first kind.*

Proof. The existence and properties of the morphism $X \to Y$ follows immediately from Lemma 55.8.5 and the uniqueness of the minimal model. The morphism $Y \to X$ is unique because $C \subset Y$ is scheme theoretically dense and X is separated (see Morphisms, Lemma 29.7.10). □

0CA0 **Example 55.10.3.** If the genus of C is 0, then minimal models are indeed nonunique. Namely, consider the closed subscheme

$$X \subset \mathbf{P}^2_R$$

defined by $T_1T_2 - \pi T_0^2 = 0$. More precisely X is defined as $\text{Proj}(R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2))$. Then the special fibre X_k is a union of two exceptional curves C_1, C_2 both isomorphic to $\mathbf{P}^1_k$ (exactly as in Lemma 55.9.11). Projection from $(0 : 1 : 0)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_2 and inducing an isomorphism of C_1 with the special fiber of $\mathbf{P}^1_R$. Projection from $(0 : 0 : 1)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_1 and inducing an isomorphism of C_2 with the special fiber of $\mathbf{P}^1_R$. More precisely, these morphisms correspond to the graded R-algebra maps

$$R[T_0, T_1] \longrightarrow R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2) \longleftarrow R[T_0, T_2]$$

In Lemma 55.12.4 we will study this phenomenon.

55.11. A formula for the genus

0CA1 There is one more restriction on the combinatorial structure coming from a proper regular model.

0CA2 **Lemma 55.11.1.** *In Situation 55.9.3 suppose we have an effective Cartier divisors $D, D' \subset X$ such that $D' = D + C_i$ for some $i \in \{1, \ldots, n\}$ and $D' \subset X_k$. Then*

$$\chi(X_k, \mathcal{O}_{D'}) - \chi(X_k, \mathcal{O}_D) = \chi(X_k, \mathcal{O}_X(-D)|_{C_i}) = -(D \cdot C_i) + \chi(C_i, \mathcal{O}_{C_i})$$

Proof. The second equality follows from the definition of the bilinear form $(\ \cdot\ )$ in (55.9.6.1) and Lemma 55.9.6. To see the first equality we distinguish two cases. Namely, if $C_i \not\subset D$, then D' is the scheme theoretic union of D and C_i (by Divisors, Lemma 31.13.10) and we get a short exact sequence

$$0 \to \mathcal{O}_{D'} \to \mathcal{O}_D \times \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

by Morphisms, Lemma 29.4.6. Since we also have an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

(Divisors, Remark 31.14.11) we conclude that the claim holds by additivity of euler characteristics (Varieties, Lemma 33.33.2). On the other hand, if $C_i \subset D$ then we get an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

by Divisors, Lemma 31.14.3 and we immediately see the lemma holds. □

0CA3 **Lemma 55.11.2.** *In Situation 55.9.3 we have*

$$g_C = 1 + \sum\nolimits_{i=1,\ldots,n} m_i \left([\kappa_i : k](g_i - 1) - \frac{1}{2}(C_i \cdot C_i) \right)$$

where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and g_C is the genus of C.

Proof. Our basic tool will be Derived Categories of Schemes, Lemma 36.32.2 which shows that

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k})$$

Choose a sequence of effective Cartier divisors

$$X_k = D_m \supset D_{m-1} \supset \ldots \supset D_1 \supset D_0 = \emptyset$$

such that $D_{j+1} = D_j + C_{i_j}$ for each j. (It is clear that we can choose such a sequence by decreasing one nonzero multiplicity of D_{j+1} one step at a time.) Applying Lemma 55.11.1 starting with $\chi(\mathcal{O}_{D_0}) = 0$ we get

$$\begin{aligned} 1 - g_C &= \chi(X_k, \mathcal{O}_{X_k}) \\ &= \sum\nolimits_j \Big(-(D_j \cdot C_{i_j}) + \chi(C_{i_j}, \mathcal{O}_{C_{i_j}}) \Big) \\ &= -\sum\nolimits_j (C_{i_1} + C_{i_2} + \ldots + C_{i_{j-1}} \cdot C_{i_j}) + \sum\nolimits_j \chi(C_{i_j}, \mathcal{O}_{C_{i_j}}) \\ &= -\frac{1}{2} \sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i \chi(C_i, \mathcal{O}_{C_i}) \\ &= \frac{1}{2} \sum m_i (C_i \cdot C_i) + \sum m_i \chi(C_i, \mathcal{O}_{C_i}) \end{aligned}$$

Perhaps the last equality deserves some explanation. Namely, since $\sum_j C_{i_j} = \sum m_i C_i$ we have $(\sum_j C_{i_j} \cdot \sum_j C_{i_j}) = 0$ by Lemma 55.9.7. Thus we see that

$$0 = \sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i (C_i \cdot C_i)$$

by splitting this product into "nondiagonal" and "diagonal" terms. Note that κ_i is a field finite over k by Varieties, Lemma 33.26.2. Hence the genus of C_i is defined and we have $\chi(C_i, \mathcal{O}_{C_i}) = [\kappa_i : k](1 - g_i)$. Putting everything together and rearranging terms we get

$$g_C = -\frac{1}{2} \sum m_i (C_i \cdot C_i) + \sum m_i [\kappa_i : k](g_i - 1) + 1$$

which is what the lemma says too. □

0CA4 **Lemma 55.11.3.** *In Situation 55.9.3 with $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and g_i the genus of C_i the data*

$$n, m_i, (C_i \cdot C_j), [\kappa_i : k], g_i$$

is a numerical type of genus equal to the genus of C.

Proof. (In the proof of Lemma 55.11.2 we have seen that the quantities used in the statement of the lemma are well defined.) We have to verify the conditions (1) – (5) of Definition 55.3.1.

Condition (1) is immediate.

Condition (2). Symmetry of the matrix $(C_i \cdot C_j)$ follows from Equation (55.9.6.1) and Lemma 55.9.6. Nonnegativity of $(C_i \cdot C_j)$ for $i \neq j$ is part (1) of Lemma 55.9.7.

Condition (3) is part (3) of Lemma 55.9.7.

Condition (4) is part (2) of Lemma 55.9.7.

Condition (5) follows from the fact that $(C_i \cdot C_j)$ is the degree of an invertible module on C_i which is divisible by $[\kappa_i : k]$, see Varieties, Lemma 33.44.10.

The genus formula proved in Lemma 55.11.2 tells us that the numerical type has the genus as stated, see Definition 55.3.4. □

0CA5 **Definition 55.11.4.** In Situation 55.9.3 the *numerical type associated to X* is the numerical type described in Lemma 55.11.3.

Now we match minimality of the model with minimality of the type.

0CA6 **Lemma 55.11.5.** *In Situation 55.9.3. The following are equivalent*

(1) *X is a minimal model, and*
(2) *the numerical type associated to X is minimal.*

Proof. If the numerical type is minimal, then there is no i with $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$, see Definition 55.3.12. Certainly, this implies that none of the curves C_i are exceptional curves of the first kind.

Conversely, suppose that the numerical type is not minimal. Then there exists an i such that $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$. We claim this implies that C_i is an exceptional curve of the first kind. Namely, the invertible sheaf $\mathcal{O}_X(-C_i)|_{C_i}$ has degree $-(C_i \cdot C_i) = [\kappa_i : k]$ when C_i is viewed as a proper curve over k, hence has degree 1 when C_i is viewed as a proper curve over κ_i. Applying Algebraic Curves, Proposition 53.10.4 we conclude that $C_i \cong \mathbf{P}^1_{\kappa_i}$ as schemes over κ_i. Since the Picard group of $\mathbf{P}^1$ over a field is $\mathbf{Z}$, we see that the normal sheaf of C_i in X is isomorphic to $\mathcal{O}_{\mathbf{P}_{\kappa_i}}(-1)$ and the proof is complete. □

0CA7 **Remark 55.11.6.** Not every numerical type comes from a model for the silly reason that there exist numerical types whose genus is negative. There exist a minimal numerical types of positive genus which are not the numerical type associated to a model (over some dvr) of a smooth projective geometrically irreducible curve (over the fraction field of the dvr). A simple example is $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 6$, $g_1 = 1$. Namely, in this case the special fibre X_k would not be geometrically connected because it would live over an extension κ of k of degree 6. This is a contradiction with the fact that the generic fibre is geometrically connected (see More on Morphisms, Lemma 37.53.6). Similarly, $n = 2$, $m_1 = m_2 = 1$, $-a_{11} = -a_{22} = a_{12} = a_{21} = 6$, $w_1 = w_2 = 6$, $g_1 = g_2 = 1$ would be an example for the same reason (details omitted). But if the gcd of the w_i is 1 we do not have an example.

0CE8 **Lemma 55.11.7.** *In Situation 55.9.3 assume C has a K-rational point. Then*

(1) *X_k has a k-rational point x which is a smooth point of X_k over k,*
(2) *if $x \in C_i$, then $H^0(C_i, \mathcal{O}_{C_i}) = k$ and $m_i = 1$, and*
(3) *$H^0(X_k, \mathcal{O}_{X_k}) = k$ and X_k has genus equal to the genus of C.*

Proof. Since $X \to \mathrm{Spec}(R)$ is proper, the K-rational point extends to a morphism $a : \mathrm{Spec}(R) \to X$ by the valuative criterion of properness (Morphisms, Lemma 29.42.1). Let $x \in X$ be the image under a of the closed point of $\mathrm{Spec}(R)$. Then a corresponds to an R-algebra homomorphism $\psi : \mathcal{O}_{X,x} \to R$ (see Schemes, Section 26.13). It follows that $\pi \notin \mathfrak{m}_x^2$ (since the image of π in R is not in $\mathfrak{m}_R^2$). Hence $\mathcal{O}_{X_k,x} = \mathcal{O}_{X,x}/\pi\mathcal{O}_{X,x}$ is regular (Algebra, Lemma 10.106.3). Then $X_k \to \mathrm{Spec}(k)$ is smooth at x by Algebra, Lemma 10.140.5. It follows that x is contained in a unique irreducible component C_i of X_k, that $\mathcal{O}_{C_i,x} = \mathcal{O}_{X_k,x}$, and that $m_i = 1$. The fact that C_i has a k-rational point implies that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ (Varieties, Lemma 33.26.2) is equal to k. This proves (1). We have $H^0(X_k, \mathcal{O}_{X_k}) = k$ because $H^0(X_k, \mathcal{O}_{X_k})$ is a field extension of k (Lemma 55.9.9) which maps to $H^0(C_i, \mathcal{O}_{C_i}) = k$. The genus equality follows from Lemma 55.9.10. □

0CE9 **Lemma 55.11.8.** *In Situation 55.9.3 assume X is a minimal model,* $\gcd(m_1, \ldots, m_n) = 1$*, and $H^0((X_k)_{red}, \mathcal{O}) = k$. Then the map*

$$H^1(X_k, \mathcal{O}_{X_k}) \to H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}})$$

is surjective and has a nontrivial kernel as soon as $(X_k)_{red} \neq X_k$.

Proof. By vanishing of cohomology in degrees ≥ 2 over X_k (Cohomology, Proposition 20.20.7) any surjection of abelian sheaves on X_k induces a surjection on H^1. Consider the sequence

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = X_k$$

of Lemma 55.9.9. Since the field maps $H^0(Z_j, \mathcal{O}_{Z_j}) \to H^0((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) = k$ are injective we conclude that $H^0(Z_j, \mathcal{O}_{Z_j}) = k$ for $j = 0, \ldots, m$. It follows that $H^0(X_k, \mathcal{O}_{X_k}) \to H^0(Z_{m-1}, \mathcal{O}_{Z_{m-1}})$ is surjective. Let $C = C_{i_m}$. Then $X_k = Z_{m-1} + C$. Let $\mathcal{L} = \mathcal{O}_X(-Z_{m-1})|_C$. Then $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module. As in the proof of Lemma 55.9.9 there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_{X_k} \to \mathcal{O}_{Z_{m-1}} \to 0$$

of coherent sheaves on X_k. We conclude that we get a short exact sequence

$$0 \to H^1(C, \mathcal{L}) \to H^1(X_k, \mathcal{O}_{X_k}) \to H^1(Z_{m-1}, \mathcal{O}_{Z_{m-1}}) \to 0$$

The degree of $\mathcal{L}$ on C over k is

$$(C \cdot -Z_{m-1}) = (C \cdot C - X_k) = (C \cdot C)$$

Set $\kappa = H^0(C, \mathcal{O}_C)$ and $w = [\kappa : k]$. By definition of the degree of an invertible sheaf we see that

$$\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C) + (C \cdot C) = w(1 - g_C) + (C \cdot C)$$

where g_C is the genus of C. This expression is < 0 as X is minimal and hence C is not an exceptional curve of the first kind (see proof of Lemma 55.11.5). Thus $\dim_k H^1(C, \mathcal{L}) > 0$ which finishes the proof. □

0CEA **Lemma 55.11.9.** *In Situation 55.9.3 assume X_k has a k-rational point x which is a smooth point of $X_k \to \mathrm{Spec}(k)$. Then*

$$\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

where g_{geom} is as in Algebraic Curves, Section 53.18 and g_{top} is the topological genus (Definition 55.3.11) of the numerical type associated to X_k (Definition 55.11.4).

Proof. We are going to prove the inequality

$$\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$$

for all connected reduced effective Cartier divisors $D \subset (X_k)_{red}$ containing x by induction on the number of irreducible components of D. Here $g_{top}(D) = 1 - m + e$ where m is the number of irreducible components of D and e is the number of unordered pairs of components of D which meet.

Base case: D has one irreducible component. Then $D = C_i$ is the unique irreducible component containing x. In this case $\dim_k H^1(D, \mathcal{O}_D) = g_i$ and $g_{top}(D) = 0$. Since C_i has a k-rational smooth point it is geometrically integral (Varieties, Lemma 33.25.10). It follows that g_i is the genus of $C_{i,\overline{k}}$ (Algebraic Curves, Lemma 53.8.2). It also follows that $g_{geom}(D/k)$ is the genus of the normalization $C^\nu_{i,\overline{k}}$ of $C_{i,\overline{k}}$. Applying Algebraic Curves, Lemma 53.18.4 to the normalization morphism $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ we get

0CEB (55.11.9.1) $$\text{genus of } C_{i,\overline{k}} \geq \text{genus of } C^\nu_{i,\overline{k}}$$

Combining the above we conclude that $\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$ in this case.

Induction step. Suppose we have D with more than 1 irreducible component. Then we can write $D = C_i + D'$ where $x \in D'$ and D' is still connected. This is an exercise in graph theory we leave to the reader (hint: let C_i be the component of D which is farthest from x). We compute how the invariants change. As $x \in D'$ we have $H^0(D, \mathcal{O}_D) = H^0(D', \mathcal{O}_{D'}) = k$. Looking at the short exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{D'} \to \mathcal{O}_{C_i \cap D'} \to 0$$

(Morphisms, Lemma 29.4.6) and using additivity of euler characteristics we find

$$\begin{aligned} \dim_k H^1(D, \mathcal{O}_D) - \dim_k H^1(D', \mathcal{O}_{D'}) &= -\chi(\mathcal{O}_{C_i}) + \chi(\mathcal{O}_{C_i \cap D'}) \\ &= w_i(g_i - 1) + \sum\nolimits_{C_j \subset D'} a_{ij} \end{aligned}$$

Here as in Lemma 55.11.3 we set $w_i = [\kappa_i : k]$, $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and $a_{ij} = (C_i \cdot C_j)$. We have

$$g_{top}(D) - g_{top}(D') = -1 + \sum\nolimits_{C_j \subset D' \text{ meeting } C_i} 1$$

We have

$$g_{geom}(D/k) - g_{geom}(D'/k) = g_{geom}(C_i/k)$$

by Algebraic Curves, Lemma 53.18.1. Combining these with our induction hypothesis, we conclude that it suffices to show that

$$w_i g_i - g_{geom}(C_i/k) + \sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1)$$

is nonnegative. In fact, we have

0CEC (55.11.9.2) $$w_i g_i \geq [\kappa_i : k]_s g_i \geq g_{geom}(C_i/k)$$

The second inequality by Algebraic Curves, Lemma 53.18.5. On the other hand, since w_i divides a_{ij} (Varieties, Lemma 33.44.10) it is clear that

0CED (55.11.9.3) $$\sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1) \geq 0$$

because there is at least one $C_j \subset D'$ which meets C_i. □

0CEE **Lemma 55.11.10.** *If equality holds in Lemma 55.11.9 then*

(1) *the unique irreducible component of X_k containing x is a smooth projective geometrically irreducible curve over k,*
(2) *if $C \subset X_k$ is another irreducible component, then $\kappa = H^0(C, \mathcal{O}_C)$ is a finite separable extension of k, C has a κ-rational point, and C is smooth over κ*

Proof. Looking over the proof of Lemma 55.11.9 we see that in order to get equality, the inequalities (55.11.9.1), (55.11.9.2), and (55.11.9.3) have to be equalities.

Let C_i be the irreducible component containing x. Equality in (55.11.9.1) shows via Algebraic Curves, Lemma 53.18.4 that $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ is an isomorphism. Hence $C_{i,\overline{k}}$ is smooth and part (1) holds.

Next, let $C_i \subset X_k$ be another irreducible component. Then we may assume we have $D = D' + C_i$ as in the induction step in the proof of Lemma 55.11.9. Equality in (55.11.9.2) immediately implies that κ_i/k is finite separable. Equality in (55.11.9.3)

implies either $a_{ij} = 1$ for some j or that there is a unique $C_j \subset D'$ meeting C_i and $a_{ij} = w_i$. In both cases we find that C_i has a κ_i-rational point c and $c = C_i \cap C_j$ scheme theoretically. Since $\mathcal{O}_{X,c}$ is a regular local ring, this implies that the local equations of C_i and C_j form a regular system of parameters in the local ring $\mathcal{O}_{X,c}$. Then $\mathcal{O}_{C_i,c}$ is regular by (Algebra, Lemma 10.106.3). We conclude that $C_i \to \mathrm{Spec}(\kappa_i)$ is smooth at c (Algebra, Lemma 10.140.5). It follows that C_i is geometrically integral over κ_i (Varieties, Lemma 33.25.10). To finish we have to show that C_i is smooth over κ_i. Observe that

$$C_{i,\overline{k}} = C_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \coprod\nolimits_{\kappa_i \to \overline{k}} C_i \times_{\mathrm{Spec}(\kappa_i)} \mathrm{Spec}(\overline{k})$$

where there are $[\kappa_i : k]$-summands. Thus if C_i is not smooth over κ_i, then each of these curves is not smooth, then these curves are not normal and the normalization morphism drops the genus (Algebraic Curves, Lemma 53.18.4) which is disallowed because it would drop the geometric genus of C_i/k contradicting $[\kappa_i : k]g_i = g_{geom}(C_i/k)$. □

55.12. Blowing down exceptional curves

0CEF The following lemma tells us what happens with the intersection numbers when we contract an exceptional curve of the first kind in a regular proper model. We put this here mostly to compare with the numerical contractions introduced in Lemma 55.3.9. We will compare the geometric and numerical contractions in Remark 55.12.3.

0C6C **Lemma 55.12.1.** *In Situation 55.9.3 assume that C_n is an exceptional curve of the first kind. Let $f : X \to X'$ be the contraction of C_n. Let $C'_i = f(C_i)$. Write $X'_k = \sum m'_i C'_i$. Then X', C'_i, $i = 1, \ldots, n' = n-1$, and $m'_i = m_i$ is as in Situation 55.9.3 and we have*

1. *for $i, j < n$ we have $(C'_i \cdot C'_j) = (C_i \cdot C_j) - (C_i \cdot C_n)(C_j \cdot C_n)/(C_n \cdot C_n)$,*
2. *for $i < n$ if $C_i \cap C_n \neq \emptyset$, then there are maps $\kappa_i \leftarrow \kappa'_i \to \kappa_n$.*

Here $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and $\kappa'_i = H^0(C'_i, \mathcal{O}_{C'_i})$.

Proof. By Resolution of Surfaces, Lemma 54.16.8 we can contract C_n by a morphism $f : X \to X'$ such that X' is regular and is projective over R. Thus we see that X' is as in Situation 55.9.3. Let $x \in X'$ be the image of C_n. Since f defines an isomorphism $X \setminus C_n \to X' \setminus \{x\}$ it is clear that $m'_i = m_i$ for $i < n$.

Part (2) of the lemma is immediately clear from the existence of the morphisms $C_i \to C'_i$ and $C_n \to x \to C'_i$.

By Divisors, Lemma 31.32.11 the pullback $f^{-1}C'_i$ is defined. By Divisors, Lemma 31.15.11 we see that $f^{-1}C'_i = C_i + e_i C_n$ for some $e_i \geq 0$. Since $\mathcal{O}_X(C_i + e_i C_n) = \mathcal{O}_X(f^{-1}C'_i) = f^*\mathcal{O}_{X'}(C'_i)$ (Divisors, Lemma 31.14.5) and since the pullback of an invertible sheaf restricts to the trivial invertible sheaf on C_n we see that

$$0 = \deg_{C_n}(\mathcal{O}_X(C_i + e_i C_n)) = (C_i + e_i C_n \cdot C_n) = (C_i \cdot C_n) + e_i(C_n \cdot C_n)$$

As $f_j = f|_{C_j} : C_j \to C_j$ is a proper birational morphism of proper curves over k, we see that $\deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j})$ is the same as $\deg_{C_j}(f_j^*\mathcal{O}_{X'}(C'_i)|_{C'_j})$ (Varieties,

Lemma 33.44.4). Looking at the commutative diagram

$$\begin{array}{ccc} C_j & \longrightarrow & X \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f} \\ C'_j & \longrightarrow & X' \end{array}$$

and using Divisors, Lemma 31.14.5 we see that

$$(C'_i \cdot C'_j) = \deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j}) = \deg_{C_j}(\mathcal{O}_X(C_i + e_i C_n)) = (C_i + e_i C_n \cdot C_j)$$

Plugging in the formula for e_i found above we see that (1) holds. □

0CA8 **Remark 55.12.2.** In the situation of Lemma 55.12.1 we can also say exactly how the genus g_i of C_i and the genus g'_i of C'_i are related. The formula is

$$g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{(C_i \cdot C_n)^2 - w_n(C_i \cdot C_n)}{2w'_i w_n}$$

where $w_i = [\kappa_i : k]$, $w_n = [\kappa_n : k]$, and $w'_i = [\kappa'_i : k]$. To prove this we consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(-C'_i) \to \mathcal{O}_{X'} \to \mathcal{O}_{C'_i} \to 0$$

and its pullback to X which reads

$$0 \to \mathcal{O}_X(-C'_i - e_i C_n) \to \mathcal{O}_X \to \mathcal{O}_{C_i + e_i C_n} \to 0$$

with e_i as in the proof of Lemma 55.12.1. Since $Rf_* f^* \mathcal{L} = \mathcal{L}$ for any invertible module $\mathcal{L}$ on X' (details omitted), we conclude that

$$Rf_* \mathcal{O}_{C_i + e_i C_n} = \mathcal{O}_{C'_i}$$

as complexes of coherent sheaves on X'_k. Hence both sides have the same Euler characteristic and this agrees with the Euler characteristic of $\mathcal{O}_{C_i + e_i C_n}$ on X_k. Using the exact sequence

$$0 \to \mathcal{O}_{C_i + e_i C_n} \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{e_i C_n} \to \mathcal{O}_{C_i \cap e_i C_n} \to 0$$

and further filtering $\mathcal{O}_{e_i C_n}$ (details omitted) we find

$$\chi(\mathcal{O}_{C'_i}) = \chi(\mathcal{O}_{C_i}) - \binom{e_i + 1}{2}(C_n \cdot C_n) - e_i(C_i \cdot C_n)$$

Since $e_i = -(C_i \cdot C_n)/(C_n \cdot C_n)$ and $(C_n \cdot C_n) = -w_n$ this leads to the formula stated at the start of this remark. If we ever need this we will formulate this as a lemma and provide a detailed proof.

0CA9 **Remark 55.12.3.** Let $f : X \to X'$ be as in Lemma 55.12.1. Let n, m_i, a_{ij}, w_i, g_i be the numerical type associated to X and let $n', m'_i, a'_{ij}, w'_i, g'_i$ be the numerical type associated to X'. It is clear from Lemma 55.12.1 and Remark 55.12.2 that this agrees with the contraction of numerical types in Lemma 55.3.9 except for the value of w'_i. In the geometric situation w'_i is some positive integer dividing both w_i and w_n. In the numerical case we chose w'_i to be the largest possible integer dividing w_i such that g'_i (as given by the formula) is an integer. This works well in the numerical setting in that it helps compare the Picard groups of the numerical types, see Lemma 55.4.4 (although only injectivity is every used in the following and this injectivity works as well for smaller w'_i).

0CDA **Lemma 55.12.4.** *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus 0. If there is more than one minimal model for C, then the special fibre of every minimal model is isomorphic to $\mathbf{P}^1_k$.*

This lemma can be improved to say that the birational transformation between two nonisomorphic minimal models can be factored as a sequence of elementary transformations as in Example 55.10.3. If we ever need this, we will precisely formulate and prove this here.

Proof. Let X be some minimal model of C. The numerical type associated to X has genus 0 and is minimal (Definition 55.11.4 and Lemma 55.11.5). Hence by Lemma 55.6.1 we see that X_k is reduced, irreducible, has $H^0(X_k, \mathcal{O}_{X_k}) = k$, and has genus 0. Let Y be a second minimal model for C which is not isomorphic to X. By Resolution of Surfaces, Lemma 54.17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \rightarrow \ldots \rightarrow Y_1 \rightarrow Y_0 = Y$$

where each morphism is a blowup in a closed point. We will prove the lemma by induction on m. The base case is $m = 0$; it is true in this case because we assumed that Y is minimal hence this would mean $n = 0$, but X is not isomorphic to Y, so this does not happen, i.e., there is nothing to check.

Before we continue, note that $n + 1 = m + 1$ is equal to the number of irreducible components of the special fibre of $X_n = Y_m$ because both X_k and Y_k are irreducible. Another observation we will use below is that if $X' \to X''$ is a morphism of regular proper models for C, then $X' \to X''$ is an isomorphism over an open set of X'' whose complement is a finite set of closed points of the special fibre of X'', see Varieties, Lemma 33.17.3. In fact, any such $X' \to X''$ is a sequence of blowing ups in closed points (Resolution of Surfaces, Lemma 54.17.1) and the number of blowups is the difference in the number of irreducible components of the special fibres of X' and X''.

Let $E_i \subset Y_i$, $m \geq i \geq 1$ be the curve which is contracted by the morphism $Y_i \to Y_{i-1}$. Let i be the biggest index such that E_i has multiplicity > 1 in the special fibre of Y_i. Then the further blowups $Y_m \to \ldots \to Y_{i+1} \to Y_i$ are isomorphisms over E_i since otherwise E_j for some $j > i$ would have multiplicity > 1. Let $E \subset Y_m$ be the inverse image of E_i. By what we just said $E \subset Y_m$ is an exceptional curve of the first kind. Let $Y_m \to Y'$ be the contraction of E (which exists by Resolution of Surfaces, Lemma 54.16.9). The morphism $Y_m \to X$ has to contract E, because X_k is reduced. Hence there are morphisms $Y' \to Y$ and $Y' \to X$ (by Resolution of Surfaces, Lemma 54.16.1) which are compositions of at most $n - 1 = m - 1$ contractions of exceptional curves (see discussion above). We win by induction on m. Upshot: we may assume that the special fibres of all of the curves X_i and Y_i are reduced.

Since the fibres of X_i and Y_i are reduced, it has to be the case that the blowups $X_i \to X_{i-1}$ and $Y_i \to Y_{i-1}$ happen in closed points which are regular points of the special fibres. Namely, if X'' is a regular model for C and if $x \in X''$ is a closed point of the special fibre, and $\pi \in \mathfrak{m}_x^2$, then the exceptional fibre E of the blowup $X' \to X''$ at x has multiplicity at least 2 in the special fibre of X' (local computation omitted). Hence $\mathcal{O}_{X''_k,x} = \mathcal{O}_{X'',x}/\pi$ is regular (Algebra, Lemma 10.106.3) as claimed. In particular x is a Cartier divisor on the unique irreducible component Z' of X''_k it lies on (Varieties, Lemma 33.43.8). It follows that the strict

transform $Z \subset X'$ of Z' maps isomorphically to Z' (use Divisors, Lemmas 31.33.2 and 31.32.7). In other words, if an irreducible component Z of X_i is not contracted under the map $X_i \to X_j$ $(i > j)$ then it maps isomorphically to its image.

Now we are ready to prove the lemma. Let $E \subset Y_m$ be the exceptional curve of the first kind which is contracted by the morphism $Y_m \to Y_{m-1}$. If E is contracted by the morphism $Y_m = X_n \to X$, then there is a factorization $Y_{m-1} \to X$ (Resolution of Surfaces, Lemma 54.16.1) and moreover $Y_{m-1} \to X$ is a sequence of blowups in closed points (Resolution of Surfaces, Lemma 54.17.1). In this case we lower m and we win by induction. Finally, assume that E is not contracted by the morphism $Y_m \to X$. Then $E \to X_k$ is surjective as X_k is irreducible and by the above this means it is an isomorphism. Hence X_k is isomorphic to a projective line as desired. □

55.13. Picard groups of models

0CAA Assume $R, K, k, \pi, C, X, n, C_1, \ldots, C_n, m_1, \ldots, m_n$ are as in Situation 55.9.3. In Lemma 55.9.5 we found an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \mathrm{Pic}(X) \to \mathrm{Pic}(C) \to 0$$

We want to use this sequence to study the ℓ-torsion in the Picard groups for suitable primes ℓ.

0CAB **Lemma 55.13.1.** *In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$. If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which*

(1) *restricts to the trivial invertible module on C, and*
(2) *has degree 0 on each C_i,*

then $\mathcal{L}^{\otimes d} \cong \mathcal{O}_X$.

Proof. By Lemma 55.9.5 we have $\mathcal{L} \cong \mathcal{O}_X(\sum a_i C_i)$ for some $a_i \in \mathbf{Z}$. The degree of $\mathcal{L}|_{C_j}$ is $\sum_j a_j(C_i \cdot C_j)$. In particular $(\sum a_i C_i \cdot \sum a_i C_i) = 0$. Hence we see from Lemma 55.9.7 that $(a_1, \ldots, a_n) = q(m_1, \ldots, m_n)$ for some $q \in \mathbf{Q}$. Thus $\mathcal{L} = \mathcal{O}_X(lD)$ for some $l \in \mathbf{Z}$ where $D = \sum(m_i/d)C_i$ is as in Lemma 55.9.8 and we conclude. □

0CAC **Lemma 55.13.2.** *In Situation 55.9.3 let T be the numerical type associated to X. There exists a canonical map*

$$\mathrm{Pic}(C) \to \mathrm{Pic}(T)$$

whose kernel is exactly those invertible modules on C which are the restriction of invertible modules $\mathcal{L}$ on X with $\deg_{C_i}(\mathcal{L}|_{C_i}) = 0$ for $i = 1, \ldots, n$.

Proof. Recall that $w_i = [\kappa_i : k]$ where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and recall that the degree of any invertible module on C_i is divisible by w_i (Varieties, Lemma 33.44.10). Thus we can consider the map

$$\frac{\deg}{w} : \mathrm{Pic}(X) \to \mathbf{Z}^{\oplus n}, \quad \mathcal{L} \mapsto \left(\frac{\deg(\mathcal{L}|_{C_1})}{w_1}, \ldots, \frac{\deg(\mathcal{L}|_{C_n})}{w_n}\right)$$

The image of $\mathcal{O}_X(C_j)$ under this map is

$$((C_j \cdot C_1)/w_1, \ldots, (C_j \cdot C_n)/w_n) = (a_{1j}/w_1, \ldots, a_{nj}/w_n)$$

which is exactly the image of the jth basis vector under the map $(a_{ij}/w_i) : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}$ defining the Picard group of T, see Definition 55.4.1. Thus the canonical map of the lemma comes from the commutative diagram

$$\begin{array}{ccccccc} \mathbf{Z}^{\oplus n} & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \frac{\deg}{w}} & & \downarrow & & \\ \mathbf{Z}^{\oplus n} & \xrightarrow{(a_{ij}/w_i)} & \mathbf{Z}^{\oplus n} & \longrightarrow & \mathrm{Pic}(T) & \longrightarrow & 0 \end{array}$$

with exact rows (top row by Lemma 55.9.5). The description of the kernel is clear. □

0CAD **Lemma 55.13.3.** *In Situation 55.9.3 let $d = \gcd(m_1, \ldots, m_n)$ and let T be the numerical type associated to X. Let $h \geq 1$ be an integer prime to d. There exists an exact sequence*

$$0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$$

Proof. Taking h-torsion in the exact sequence of Lemma 55.9.5 we obtain the exactness of $0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h]$ because h is prime to d. Using the map of Lemma 55.13.2 we get a map $\mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$ which annihilates elements of $\mathrm{Pic}(X)[h]$. Conversely, if $\xi \in \mathrm{Pic}(C)[h]$ maps to zero in $\mathrm{Pic}(T)[h]$, then we can find an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}|_{C_i}) = 0$ for all i whose restriction to C is ξ. Then $\mathcal{L}^{\otimes h}$ is d-torsion by Lemma 55.13.1. Let d' be an integer such that $dd' \equiv 1 \bmod h$. Such an integer exists because h and d are coprime. Then $\mathcal{L}^{\otimes dd'}$ is an h-torsion invertible sheaf on X whose restriction to C is ξ. □

0CAE **Lemma 55.13.4.** *In Situation 55.9.3 let h be an integer prime to the characteristic of k. Then the map*

$$\mathrm{Pic}(X)[h] \longrightarrow \mathrm{Pic}((X_k)_{red})[h]$$

is injective.

Proof. Observe that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ is a finite order thickening of $(X_k)_{red}$ (this follows for example from Cohomology of Schemes, Lemma 30.10.2). Thus the canonical map $\mathrm{Pic}(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)) \to \mathrm{Pic}((X_k)_{red})$ identifies h torsion by More on Morphisms, Lemma 37.4.2 and our assumption on h. Thus if $\mathcal{L}$ is an h-torsion invertible sheaf on X which restricts to the trivial sheaf on $(X_k)_{red}$ then $\mathcal{L}$ restricts to the trivial sheaf on $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ for all n. We find

$$\begin{aligned} H^0(X, \mathcal{L})^\wedge &= \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{L}|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &\cong \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{O}_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &= R^\wedge \end{aligned}$$

using the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5) for the first and last equality and for example More on Algebra, Lemma 15.101.5 for the middle isomorphism. Since $H^0(X, \mathcal{L})$ is a finite R-module and R is a discrete valuation ring, this means that $H^0(X, \mathcal{L})$ is free of rank 1 as an R-module. Let $s \in H^0(X, \mathcal{L})$ be a basis element. Then tracing back through the isomorphisms above we see that $s|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}$ is a trivialization for all n. Since the vanishing locus of s is closed in X and $X \to \mathrm{Spec}(R)$ is proper we conclude that the vanishing locus of s is empty as desired. □

55.14. Semistable reduction

0CDB In this section we carefully define what we mean by semistable reduction.

0CDC **Example 55.14.1.** Let R be a discrete valuation ring with uniformizer π. Given $n \geq 0$, consider the ring map

$$R \longrightarrow A = R[x, y]/(xy - \pi^n)$$

Set $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. If $n = 0$, then $X \to S$ is smooth. For all n the morphism $X \to S$ is at-worst-nodal of relative dimension 1 as defined in Algebraic Curves, Section 53.20. If $n = 1$, then X is regular, but if $n > 1$, then X is not regular as (x, y) no longer generate the maximal ideal $\mathfrak{m} = (\pi, x, y)$. To ameliorate the situation in case $n > 1$ we consider the blowup $b : X' \to X$ of X in $\mathfrak{m}$. See Divisors, Section 31.32. By construction X' is covered by three affine pieces corresponding to the blowup algebras $A[\frac{\mathfrak{m}}{\pi}]$, $A[\frac{\mathfrak{m}}{x}]$, and $A[\frac{\mathfrak{m}}{y}]$.

The algebra $A[\frac{\mathfrak{m}}{\pi}]$ has generators $x' = x/\pi$ and $y' = y/\pi$ and $x'y' = \pi^{n-2}$. Thus this part of X' is the spectrum of $R[x', y'](x'y' - \pi^{n-2})$.

The algebra $A[\frac{\mathfrak{m}}{x}]$ has generators x, $u = \pi/x$ subject to the relation $xu - \pi$. Note that this ring contains $y/x = \pi^n/x^2 = u^2\pi^{n-2}$. Thus this part of X' is regular.

By symmetry the case of the algebra $A[\frac{\mathfrak{m}}{y}]$ is the same as the case of $A[\frac{\mathfrak{m}}{x}]$.

Thus we see that $X' \to S$ is at-worst-nodal of relative dimension 1 and that X' is regular, except for one point which has an affine open neighbourhood exactly as above but with n replaced by $n - 2$. Using induction on n we conclude that there is a sequence of blowing ups in closed points

$$X_{\lfloor n/2 \rfloor} \to \ldots \to X_1 \to X_0 = X$$

such that $X_{\lfloor n/2 \rfloor} \to S$ is at-worst-nodal of relative dimension 1 and $X_{\lfloor n/2 \rfloor}$ is regular.

0CDD **Lemma 55.14.2.** *Let R be a discrete valuation ring. Let X be a scheme which is at-worst-nodal of relative dimension 1 over R. Let $x \in X$ be a point of the special fibre of X over R. Then there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R') & & \end{array}$$

where $R \subset R'$ is an étale extension of discrete valuation rings, the morphism $U \to X$ is étale, the morphism $U \to \mathrm{Spec}(A)$ is étale, there is a point $x' \in U$ mapping to x, and

$$A = R'[u, v]/(uv) \quad \text{or} \quad A = R'[u, v]/(uv - \pi^n)$$

where $n \geq 0$ and $\pi \in R'$ is a uniformizer.

Proof. We have already proved this lemma in much greater generality, see Algebraic Curves, Lemma 53.21.1. All we have to do here is to translate the statement given there into the statement given above.

First, if the morphism $X \to \mathrm{Spec}(R)$ is smooth at x, then we can find an étale morphism $U \to \mathbf{A}^1_R = \mathrm{Spec}(R[u])$ for some affine open neighbourhood $U \subset X$ of x.

This is Morphisms, Lemma 29.36.20. After replacing the coordinate u by $u+1$ if necessary, we may assume that x maps to a point in the standard open $D(u) \subset \mathbf{A}^1_R$. Then $D(u) = \mathrm{Spec}(A)$ with $A = R[u,v]/(uv-1)$ and we see that the result is true in this case.

Next, assume that x is a singular point of the fibre. Then we may apply Algebraic Curves, Lemma 53.21.1 to get a diagram

$$\begin{array}{ccccccc} X & \longleftarrow & U & \longrightarrow & W & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longleftarrow & & & V & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[a]) \end{array}$$

with all the properties mentioned in the statement of the cited lemma. Let $x' \in U$ be the point mapping to x promised by the lemma. First we shrink V to an affine neighbourhood of the image of x'. Say $V = \mathrm{Spec}(R')$. Then $R \to R'$ is étale. Since R is a discrete valuation ring, we see that R' is a finite product of quasi-local Dedekind domains (use More on Algebra, Lemma 15.44.4). Hence (for example using prime avoidance) we find a standard open $D(f) \subset V = \mathrm{Spec}(R')$ containing the image of x' such that R'_f is a discrete valuation ring. Replacing R' by R'_f we reach the situation where $V = \mathrm{Spec}(R')$ with $R \subset R'$ an étale extension of discrete valuation rings (extensions of discrete valuation rings are defined in More on Algebra, Definition 15.112.1).

The morphism $V \to \mathrm{Spec}(\mathbf{Z}[a])$ is determined by the image h of a in R'. Then $W = \mathrm{Spec}(R'[u,v]/(uv-h))$. Thus the lemma holds with $A = R'[u,v]/(uv-h)$. If $h = 0$ then we clearly obtain the first case mentioned in the lemma. If $h \neq 0$ then we may write $h = \epsilon\pi^n$ for some $n \geq 0$ where ϵ is a unit of R'. Changing coordinates $u_{new} = \epsilon u$ and $v_{new} = v$ we obtain the second isomorphism type of A listed in the lemma. □

0CDE **Lemma 55.14.3.** *Let R be a discrete valuation ring. Let X be a quasi-compact scheme which is at-worst-nodal of relative dimension 1 with smooth generic fibre over R. Then there exists $m \geq 0$ and a sequence*

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that

(1) *$X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular,*
(2) *$X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1,*
(3) *X_m is regular.*

A slightly stronger statement (also true) would be that no matter how you blow up in singular points you eventually end up with a resolution and all the intermediate blowups are at-worst-nodal of relative dimension 1 over R.

Proof. Since X is quasi-compact we see that the special fibre X_k is quasi-compact. Since the singularities of X_k are at-worst-nodal, we see that X_k has a finite number of nodes and is otherwise smooth over k. As $X \to \mathrm{Spec}(R)$ is flat with smooth generic fibre it follows that X is smooth over R except at the finite number of nodes of X_k (use Morphisms, Lemma 29.34.14). It follows that X is regular at

every point except for possibly the nodes of its special fibre (see Algebra, Lemma 10.163.10). Let $x \in X$ be such a node. Choose a diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R') & & \end{array}$$

as in Lemma 55.14.2. Observe that the case $A = R'[u, v]/(uv)$ cannot occur, as this would mean that the generic fibre of X/R is singular (tiny detail omitted). Thus $A = R'[u, v]/(uv - \pi^n)$ for some $n \geq 0$. Since x is a singular point, we have $n \geq 2$, see discussion in Example 55.14.1.

After shrinking U we may assume there is a unique point $u \in U$ mapping to x. Let $w \in \mathrm{Spec}(A)$ be the image of u. We may also assume that u is the unique point of U mapping to w. Since the two horizontal arrows are étale we see that u, viewed as a closed subscheme of U, is the scheme theoretic inverse image of $x \in X$ and the scheme theoretic inverse image of $w \in \mathrm{Spec}(A)$. Since blowing up commutes with flat base change (Divisors, Lemma 31.32.3) we find a commutative diagram

$$\begin{array}{ccccc} X' & \longleftarrow & U' & \longrightarrow & W' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & U & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

with cartesian squares where the vertical arrows are the blowing up of x, u, w in $X, U, \mathrm{Spec}(A)$. The scheme W' was described in Example 55.14.1. We saw there that W' at-worst-nodal of relative dimension 1 over R'. Thus W' is at-worst-nodal of relative dimension 1 over R (Algebraic Curves, Lemma 53.20.7). Hence U' is at-worst-nodal of relative dimension 1 over R (see Algebraic Curves, Lemma 53.20.8). Since $X' \to X$ is an isomorphism over the complement of x, we conclude the same thing is true of X'/R (by Algebraic Curves, Lemma 53.20.8 again).

Finally, we need to argue that after doing a finite number of these blowups we arrive at a regular model X_m. This is rather clear because the "invariant" n decreases by 2 under the blowup described above, see computation in Example 55.14.1. However, as we want to avoid precisely defining this invariant and establishing its properties, we in stead argue as follows. If $n = 2$, then W' is regular and hence X' is regular at all points lying over x and we have decreased the number of singular points of X by 1. If $n > 2$, then the unique singular point w' of W' lying over w has $\kappa(w) = \kappa(w')$. Hence U' has a unique singular point u' lying over u with $\kappa(u) = \kappa(u')$. Clearly, this implies that X' has a unique singular point x' lying over x, namely the image of u'. Thus we can argue exactly as above that we get a commutative diagram

$$\begin{array}{ccccc} X'' & \longleftarrow & U'' & \longrightarrow & W'' \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & U' & \longrightarrow & W' \end{array}$$

with cartesian squares where the vertical arrows are the blowing up of x', u', w' in X', U', W'. Continuing like this we get a compatible sequence of blowups which stops after $\lfloor n/2 \rfloor$ steps. At the completion of this process the scheme $X^{(\lfloor n/2 \rfloor)}$ will

have one fewer singular point than X. Induction on the number of singular points completes the proof. □

0CDF **Lemma 55.14.4.** *Let R be a discrete valuation ring with fraction field K and residue field k. Assume $X \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1 over R. Let $X \to X'$ be the contraction of an exceptional curve $E \subset X$ of the first kind. Then X' is at-worst-nodal of relative dimension 1 over R.*

Proof. Namely, let $x' \in X'$ be the image of E. Then the only issue is to see that $X' \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1 in a neighbourhood of x'. The closed fibre of $X \to \mathrm{Spec}(R)$ is reduced, hence $\pi \in R$ vanishes to order 1 on E. This immediately implies that π viewed as an element of $\mathfrak{m}_{x'} \subset \mathcal{O}_{X',x'}$ but is not in $\mathfrak{m}_{x'}^2$. Since $\mathcal{O}_{X',x'}$ is regular of dimension 2 (by definition of contractions in Resolution of Surfaces, Section 54.16), this implies that $\mathcal{O}_{X'_k,x'}$ is regular of dimension 1 (Algebra, Lemma 10.106.3). On the other hand, the curve E has to meet at least one other component, say C of the closed fibre X_k. Say $x \in E \cap C$. Then x is a node of the special fibre X_k and hence $\kappa(x)/k$ is finite separable, see Algebraic Curves, Lemma 53.19.7. Since $x \mapsto x'$ we conclude that $\kappa(x')/k$ is finite separable. By Algebra, Lemma 10.140.5 we conclude that $X'_k \to \mathrm{Spec}(k)$ is smooth in an open neighbourhood of x'. Combined with flatness, this proves that $X' \to \mathrm{Spec}(R)$ is smooth in a neighbourhood of x' (Morphisms, Lemma 29.34.14). This finishes the proof as a smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1 (Algebraic Curves, Lemma 53.20.3). □

0CDG **Lemma 55.14.5.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent*

1. *there exists a proper model of C which is at-worst-nodal of relative dimension 1 over R,*
2. *there exists a minimal model of C which is at-worst-nodal of relative dimension 1 over R, and*
3. *any minimal model of C is at-worst-nodal of relative dimension 1 over R.*

Proof. To make sense out of this statement, recall that a minimal model is defined as a regular proper model without exceptional curves of the first kind (Definition 55.8.4), that minimal models exist (Proposition 55.8.6), and that minimal models are unique if the genus of C is > 0 (Lemma 55.10.1). Keeping this in mind the implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are clear.

Assume (1). Let X be a proper model of C which is at-worst-nodal of relative dimension 1 over R. Applying Lemma 55.14.3 we see that we may assume X is regular as well. Let

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

be as in Lemma 55.8.5. By Lemma 55.14.4 and induction this implies X_0 is at-worst-nodal of relative dimension 1 over R.

To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (see discussion above). On the other hand, if the minimal model is not unique, then the morphism $X \to \mathrm{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 55.12.4. □

0CDH **Definition 55.14.6.** Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has *semistable reduction* if the equivalent conditions of Lemma 55.14.5 are satisfied.

0CDI **Lemma 55.14.7.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent*

(1) *there exists a proper smooth model for C,*
(2) *there exists a minimal model for C which is smooth over R,*
(3) *any minimal model is smooth over R.*

Proof. If X is a smooth proper model, then the special fibre is connected (Lemma 55.9.4) and smooth, hence irreducible. This immediately implies that it is minimal. Thus (1) implies (2). To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (Lemma 55.10.1). On the other hand, if the minimal model is not unique, then the morphism $X \to \mathrm{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 55.12.4. □

0CDJ **Definition 55.14.8.** Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has *good reduction* if the equivalent conditions of Lemma 55.14.7 are satisfied.

55.15. Semistable reduction in genus zero

0CDK In this section we prove the semistable reduction theorem (Theorem 55.18.1) for genus zero curves.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If the genus of C is 0, then C is isomorphic to a conic, see Algebraic Curves, Lemma 53.10.3. Thus there exists a finite separable extension K'/K of degree at most 2 such that $C(K') \neq \emptyset$, see Algebraic Curves, Lemma 53.9.4. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.112.6. We will show that $C_{K'}$ has semistable reduction over $R'_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R' (of course in the current case there are at most two such ideals). After replacing R by $R'_{\mathfrak{m}}$ and C by $C_{K'}$ we reduce to the case discussed in the next paragraph.

In this paragraph R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, of genus 0, and C has a K-rational point. In this case $C \cong \mathbf{P}^1_K$ by Algebraic Curves, Proposition 53.10.4. Thus we can use $\mathbf{P}^1_R$ as a model and we see that C has both good and semistable reduction.

0CDL **Example 55.15.1.** Let $R = \mathbf{R}[[\pi]]$ and consider the scheme

$$X = V(T_1^2 + T_2^2 - \pi T_0^2) \subset \mathbf{P}^2_R$$

The base change of X to $\mathbf{C}[[\pi]]$ is isomorphic to the scheme defined in Example 55.10.3 because we have the factorization $T_1^2 + T_2^2 = (T_1 + iT_2)(T_1 - iT_2)$ over $\mathbf{C}$. Thus X is regular and its special fibre is irreducible yet singular, hence X is the unique minimal model of its generic fibre (use Lemma 55.12.4). It follows that an extension is needed even in genus 0.

55.16. Semistable reduction in genus one

0CEG In this section we prove the semistable reduction theorem (Theorem 55.18.1) for curves of genus one. We suggest the reader first read the proof in the case of genus ≥ 2 (Section 55.17). We are going to use as much as possible the classification of minimal numerical types of genus 1 given in Lemma 55.6.2.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is 1. Choose a prime $\ell \geq 7$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\mathrm{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$. See Algebraic Curves, Lemma 53.17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.112.6. We may replace R by $R'_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus 1, having a K-rational point, and with $\mathrm{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$ for some prime $\ell \geq 7$ different from the characteristic of k. We will prove that C has semistable reduction.

Let X be a minimal model for C, see Proposition 55.8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 55.11.4). Then T is a minimal numerical type (Lemma 55.11.5). As C has a rational point, there exists an i such that $m_i = w_i = 1$ by Lemma 55.11.7. Looking at the classification of minimal numerical types of genus 1 in Lemma 55.6.2 we see that $m = w = 1$ and that cases (3), (6), (7), (9), (11), (13), (15), (18), (19), (21), (24), (26), (28), (30) are disallowed (because there is no index where both w_i and m_i is equal to 1). Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. For the remaining cases we have

(A) $e = n - 1$ for cases (1), (2), (5), (8), (12), (14), (17), (20), (22), (23), (27), (29), (31), (32), (33), and (34), and
(B) $e = n$ for cases (4), (10), (16), and (25).

We will argue these cases separately.

Case (A). In this case $\mathrm{Pic}(T)[\ell]$ is trivial (the Picard group of a numerical type is defined in Section 55.4). The vanishing follows as $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$ (Lemma 55.4.3) and $\mathrm{Coker}(A)[\ell] = 0$ by Lemma 55.2.6 and the fact that ℓ was chosen relatively prime to a_{ij} and m_i. By Lemmas 55.13.3 and 55.13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2} \subset \mathrm{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 53.18.6 we obtain

$$2 \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}((X_k)_{red}/k)$$

By Algebraic Curves, Lemmas 53.18.1 and 53.18.5 we see that $g_{geom}((X_k)_{red}/k) \leq \sum w_i g_i$. The assumptions of Lemma 55.11.8 hold by Lemma 55.11.7 and we conclude that we have $\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \leq g = 1$. Combining these we see

$$2 \leq 1 + \sum w_i g_i$$

Looking at the list we conclude that the numerical type is given by $n = 1$, $w_1 = m_1 = g_1 = 1$. Because we have equality everywhere we see that $g_{geom}(C_1/k) = 1$.

On the other hand, we know that C_1 has a k-rational point x such that $C_1 \to \mathrm{Spec}(k)$ is smooth at x. It follows that C_1 is geometrically integral (Varieties, Lemma 33.25.10). Thus $g_{geom}(C_1/k) = 1$ is both equal to the genus of the normalization of $C_{1,\overline{k}}$ and the genus of $C_{1,\overline{k}}$. It follows that the normalization morphism $C^\nu_{1,\overline{k}} \to C_{1,\overline{k}}$ is an isomorphism (Algebraic Curves, Lemma 53.18.4). We conclude that C_1 is smooth over k as desired.

Case (B). Here we only conclude that there is an embedding

$$\mathbf{Z}/\ell\mathbf{Z} \subset \mathrm{Pic}(X_k)[\ell]$$

From the classification of types we see that $m_i = w_i = 1$ and $g_i = 0$ for each i. Thus each C_i is a genus zero curve over k. Moreover, for each i there is a j such that $C_i \cap C_j$ is a k-rational point. Then it follows that $C_i \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 53.10.4. In particular, since X_k is the scheme theoretic union of the C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = 1$. Also, by Varieties, Lemma 33.30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Pic}(X_{\overline{k}}) \geq 1$$

By Algebraic Curves, Proposition 53.17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 55.9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 48.25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 53.16.4. This finishes the proof in case (B).

0CEH **Example 55.16.1.** Let k be an algebraically closed field. Let Z be a smooth projective curve over k of positive genus g. Let $n \geq 1$ be an integer prime to the characteristic of k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Z$-module of order n, see Algebraic Curves, Lemma 53.17.1. Pick an isomorphism $\varphi : \mathcal{L}^{\otimes n} \to \mathcal{O}_Z$. Set $R = k[[\pi]]$ with fraction field $K = k((\pi))$. Denote Z_R the base change of Z to R. Let $\mathcal{L}_R$ be the pullback of $\mathcal{L}$ to Z_R. Consider the finite flat morphism

$$p : X \longrightarrow Z_R$$

such that

$$p_*\mathcal{O}_X = \mathrm{Sym}^*_{\mathcal{O}_{Z_R}}(\mathcal{L}_R)/(\varphi - \pi) = \mathcal{O}_{Z_R} \oplus \mathcal{L}_R \oplus \mathcal{L}_R^{\otimes 2} \oplus \ldots \oplus \mathcal{L}_R^{\otimes n-1}$$

More precisely, if $U = \mathrm{Spec}(A) \subset Z$ is an affine open such that $\mathcal{L}|_U$ is trivialized by a section s with $\varphi(s^{\otimes n}) = f$ (with f a unit), then

$$p^{-1}(U_R) = \mathrm{Spec}\left((A \otimes_R R[[\pi]])[x]/(x^n - \pi f)\right)$$

The reader verifies that the morphism $X_K \to Z_K$ of generic fibres is finite étale. Looking at the description of the structure sheaf we see that $H^0(X, \mathcal{O}_X) = R$ and $H^0(X_K, \mathcal{O}_{X_K}) = K$. By Riemann-Hurwitz (Algebraic Curves, Lemma 53.12.4) the genus of X_K is $n(g-1)+1$. In particular X_K has genus 1, if Z has genus 1. On the other hand, the scheme X is regular by the local equation above and the special fibre X_k is n times the reduced special fibre as an effective Cartier divisor. It follows that any finite extension K'/K over which X_K attains semistable reduction has to ramify with ramification index at least n (some details omitted). Thus there does not exist a universal bound for the degree of an extension over which a genus 1 curve attains semistable reduction.

55.17. Semistable reduction in genus at least two

0CEI In this section we prove the semistable reduction theorem (Theorem 55.18.1) for curves of genus ≥ 2. Fix $g \geq 2$.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is g. Choose a prime $\ell > 768g$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\mathrm{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$. See Algebraic Curves, Lemma 53.17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 15.112.6. We may replace R by $R'_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus g, having a K-rational point, and with $\mathrm{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$ for some prime $\ell \geq 768g$ different from the characteristic of k. We will prove that C has semistable reduction.

In the rest of this section we will use without further mention that the conclusions of Lemma 55.11.7 are true.

Let X be a minimal model for C, see Proposition 55.8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 55.11.4). Then T is a minimal numerical type of genus g (Lemma 55.11.5). By Proposition 55.7.4 we have

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(T)[\ell] \leq g_{top}$$

By Lemmas 55.13.3 and 55.13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g - g_{top}} \subset \mathrm{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 53.18.6 we obtain

$$2g - g_{top} \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}(X_k/k)$$

By Lemmas 55.11.8 and 55.11.9 we have

$$g \geq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

Elementary number theory tells us that the only way these 3 inequalities can hold is if they are all equalities. Looking at Lemma 55.11.8 we conclude that $m_i = 1$ for all i. Looking at Lemma 55.11.10 we conclude that every irreducible component of X_k is smooth over k.

In particular, since X_k is the scheme theoretic union of its irreducible components C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = g$. Also, by Varieties, Lemma 33.30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Pic}(X_{\overline{k}})[\ell]) \geq 2g - g_{top} = \dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) + g_{geom}(X_{\overline{k}})$$

By Algebraic Curves, Proposition 53.17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 55.9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 48.25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 53.16.4. This finishes the proof.

55.18. Semistable reduction for curves

0CDM In this section we finish the proof of the theorem. For $g \geq 2$ let $768g < \ell' < \ell$ be the first two primes $> 768g$ and set

0CEJ (55.18.0.1) $$B_g = (2g-2)(\ell^{2g})!$$

The precise form of B_g is unimportant; the point we are trying to make is that it depends only on g.

0CDN **Theorem 55.18.1.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Then there exists an extension of discrete valuation rings $R \subset R'$ which induces a finite separable extension of fraction fields K'/K such that $C_{K'}$ has semistable reduction. More precisely, we have the following* [**DM69**, Corollary 2.7]

1. *If the genus of C is zero, then there exists a degree 2 separable extension K'/K such that $C_{K'} \cong \mathbf{P}^1_{K'}$ and hence $C_{K'}$ is isomorphic to the generic fibre of the smooth projective scheme $\mathbf{P}^1_{R'}$ over the integral closure R' of R in K'.*
2. *If the genus of C is one, then there exists a finite separable extension K'/K such that $C_{K'}$ has semistable reduction over $R'_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'. Moreover, the special fibre of the (unique) minimal model of $C_{K'}$ over $R'_{\mathfrak{m}}$ is either a smooth genus one curve or a cycle of rational curves.*
3. *If the genus g of C is greater than one, then there exists a finite separable extension K'/K of degree at most B_g (55.18.0.1) such that $C_{K'}$ has semistable reduction over $R'_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'.*

Proof. For the case of genus zero, see Section 55.15. For the case of genus one, see Section 55.16. For the case of genus greater than one, see Section 55.17. To see that we have a bound on the degree $[K' : K]$ you can use the bound on the degree of the extension needed to make all ℓ or ℓ' torsion visible proved in Algebraic Curves, Lemma 53.17.2. (The reason for using ℓ and ℓ' is that we need to avoid the characteristic of the residue field k.) □

0CEK **Remark 55.18.2** (Improving the bound)**.** Results in the literature suggest that one can improve the bound given in the statement of Theorem 55.18.1. For example, in [**DM69**] it is shown that semistable reduction of C and its Jacobian are the same thing if the residue field is perfect and presumably this is true for general residue fields as well. For an abelian variety we have semistable reduction if the action of Galois on the ℓ-torsion is trivial for any $\ell \geq 3$ not equal to the residue characteristic. Thus we can presumably choose $\ell = 5$ in the formula (55.18.0.1) for B_g (but the proof would take a lot more work; if we ever need this we will make a precise statement and provide a proof here).

55.19. Other chapters

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CHAPTER 56

Functors and Morphisms

0GNG

56.1. Introduction

0GNH Let X and Y be schemes. This chapter circles around the relationship between functors $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ and morphisms of schemes $X \to Y$. More broadly speaking we study the relationship between $QCoh(\mathcal{O}_X)$ and X or, if X is Noetherian, the relationship between $Coh(\mathcal{O}_X)$ and X. This relationship was studied in [**Gab62**].

56.2. Functors on module categories

0GNI For a ring A let us denote Mod_A^{fp} the category of finitely presented A-modules.

0GNJ **Lemma 56.2.1.** *Let A be a ring. Let $\mathcal{B}$ be a category having filtered colimits. Let $F : Mod_A^{fp} \to \mathcal{B}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to \mathcal{B}$ which commutes with filtered colimits.*

Proof. This follows from Categories, Lemma 4.26.2. To see that the lemma applies observe that finitely presented A-modules are categorically compact objects of Mod_A by Algebra, Lemma 10.11.4. Also, every A-module is a filtered colimit of finitely presented A-modules by Algebra, Lemma 10.11.3. □

If a category $\mathcal{B}$ is additive and has filtered colimits, then $\mathcal{B}$ has arbitrary direct sums: any direct sum can be written as a filtered colimit of finite direct sums.

0GNK **Lemma 56.2.2.** *Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume $\mathcal{B}$ is additive and F is additive. Then F' is additive and commutes with arbitrary direct sums.*

Proof. To show that F' is additive it suffices to show that $F'(M) \oplus F'(M') \to F'(M \oplus M')$ is an isomorphism for any A-modules M, M', see Homology, Lemma 12.7.1. Write $M = \text{colim}_i M_i$ and $M' = \text{colim}_j M'_j$ as filtered colimits of finitely presented A-modules M_i. Then $F'(M) = \text{colim}_i F(M_i)$, $F'(M') = \text{colim}_j F(M'_j)$, and

$$\begin{aligned} F'(M \oplus M') &= F'(\text{colim}_{i,j} M_i \oplus M'_j) \\ &= \text{colim}_{i,j} F(M_i \oplus M'_j) \\ &= \text{colim}_{i,j} F(M_i) \oplus F(M'_j) \\ &= F'(M) \oplus F'(M') \end{aligned}$$

as desired. To show that F' commutes with direct sums, assume we have $M = \bigoplus_{i\in I} M_i$. Then $M = \operatorname{colim}_{I'\subset I \text{ finite}} \bigoplus_{i\in I'} M_i$ is a filtered colimit. We obtain

$$\begin{aligned} F'(M) &= \operatorname{colim}_{I'\subset I \text{ finite}} F'(\bigoplus\nolimits_{i\in I'} M_i) \\ &= \operatorname{colim}_{I'\subset I \text{ finite}} \bigoplus\nolimits_{i\in I'} F'(M_i) \\ &= \bigoplus\nolimits_{i\in I} F'(M_i) \end{aligned}$$

The second equality holds by the additivity of F' already shown. □

If a category $\mathcal{B}$ is additive, has filtered colimits, and has cokernels, then $\mathcal{B}$ has arbitrary colimits, see discussion above and Categories, Lemma 4.14.12.

0GNL **Lemma 56.2.3.** *Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume $\mathcal{B}$ is additive, has cokernels, and F is right exact. Then F' is additive, right exact, and commutes with arbitrary direct sums.*

Proof. Since F is right exact, F commutes with coproducts of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 12.7.1. Hence F' is additive and commutes with direct sums by Lemma 56.2.2. We urge the reader to prove that F' is right exact themselves instead of reading the proof below.

To show that F' is right exact, it suffices to show that F' commutes with coequalizers, see Categories, Lemma 4.23.3. Now, if $a, b : K \to L$ are maps of A-modules, then the coequalizer of a and b is the cokernel of $a - b : K \to L$. Thus let $K \to L \to M \to 0$ be an exact sequence of A-modules. We have to show that in

$$F'(K) \to F'(L) \to F'(M) \to 0$$

the second arrow is a cokernel for the first arrow in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \operatorname{colim}_{i\in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 10.11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $K \to L_i \to M_i \to 0$ over I. Since colimits commute with colimits by Categories, Lemma 4.14.10 and since cokernels are a type of coequalizer, it suffices to show that $F'(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F'(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \operatorname{colim}_{i\in I} L_i$ as a filtered colimit of finitely presented A-modules with the property that each L_i surjects onto M. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $K_i \to L_i \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. In this case the module $\operatorname{Ker}(L \to M)$ is finite (Algebra, Lemma 10.5.3). Thus we can write $K = \operatorname{colim}_{i\in I} K_i$ as a filtered colimit of finitely presented A-modules each surjecting onto $\operatorname{Ker}(L \to M)$. We obtain a system of short exact sequences $K_i \to L \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L) \to F(M)$ is a cokernel of $F(K_i) \to F(L)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is right exact. □

If a category $\mathcal{B}$ is additive and has kernels, then $\mathcal{B}$ has finite limits. Namely, finite products are direct sums which exist and the equalizer of $a, b : L \to M$ is the kernel

of $a - b : K \to L$ which exists. Thus all finite limits exist by Categories, Lemma 4.18.4.

0GNM **Lemma 56.2.4.** *Let A, $\mathcal{B}$, F be as in Lemma 56.2.1. Assume A is a coherent ring (Algebra, Definition 10.90.1), $\mathcal{B}$ is additive, has kernels, filtered colimits commute with taking kernels, and F is left exact. Then F' is additive, left exact, and commutes with arbitrary direct sums.*

Proof. Since A is coherent, the category Mod_A^{fp} is abelian with same kernels and cokernels as in Mod_A, see Algebra, Lemmas 10.90.4 and 10.90.3. Hence all finite limits exist in Mod_A^{fp} and Categories, Definition 4.23.1 applies. Since F is left exact, F commutes with products of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 12.7.1. Hence F' is additive and commutes with direct sums by Lemma 56.2.2. We urge the reader to prove that F' is left exact themselves instead of reading the proof below.

To show that F' is left exact, it suffices to show that F' commutes with equalizers, see Categories, Lemma 4.23.2. Now, if $a, b : L \to M$ are maps of A-modules, then the equalizer of a and b is the kernel of $a - b : L \to M$. Thus let $0 \to K \to L \to M$ be an exact sequence of A-modules. We have to show that in

$$0 \to F'(K) \to F'(L) \to F'(M)$$

the arrow $F'(K) \to F'(L)$ is a kernel for $F'(L) \to F'(M)$ in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \mathrm{colim}_{i \in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 10.11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $0 \to K \to L_i \to M_i$ over I. Since filtered colimits commute with taking kernels in $\mathcal{B}$ by assumption, it suffices to show that $F'(K) \to F'(L_i)$ is a kernel of $F'(L_i) \to F(M_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \mathrm{colim}_{i \in I} L_i$ as a filtered colimit of finitely presented A-modules. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $0 \to K_i \to L_i \to M$ over I. Repeating the argument already given, we reduce to showing $F'(K_i) \to F(L_i)$ is a kernel of $F(L_i) \to F(M)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. Since A is coherent, the A-module $K = \mathrm{Ker}(L \to M)$ is of finite presentation as the category of finitely presented A-modules is abelian (see references given above). In other words, all three modules K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is left exact. □

If a category $\mathcal{B}$ is additive and has cokernels, then $\mathcal{B}$ has finite colimits. Namely, finite coproducts are direct sums which exist and the coequalizer of $a, b : K \to L$ is the cokernel of $a - b : K \to L$ which exists. Thus all finite colimits exist by Categories, Lemma 4.18.7.

0GNN **Lemma 56.2.5.** *Let A be a ring. Let $\mathcal{B}$ be an additive category with cokernels. There is an equivalence of categories between*

(1) *the category of functors $F : Mod_A^{fp} \to \mathcal{B}$ which are right exact, and*
(2) *the category of pairs (K, κ) where $K \in \mathrm{Ob}(\mathcal{B})$ and $\kappa : A \to End_{\mathcal{B}}(K)$ is a ring homomorphism*

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Let (K, κ) be as in (2). We will construct a functor $F : \mathrm{Mod}_A^{fp} \to \mathcal{B}$ such that $F(A) = K$ endowed with the given A-action κ. Namely, given an integer $n \geq 0$ let us set

$$F(A^{\oplus n}) = K^{\oplus n}$$

Given an A-linear map $\varphi : A^{\oplus m} \to A^{\oplus n}$ with matrix $(a_{ij}) \in \mathrm{Mat}(n \times m, A)$ we define

$$F(\varphi) : F(A^{\oplus m}) = K^{\oplus m} \longrightarrow K^{\oplus n} = F(A^{\oplus n})$$

to be the map with matrix $(\kappa(a_{ij}))$. This defines an additive functor F from the full subcategory of Mod_A^{fp} with objects 0, A, $A^{\oplus 2}$, ... to $\mathcal{B}$; we omit the verification.

For each object M of Mod_A^{fp} choose a presentation

$$A^{\oplus m_M} \xrightarrow{\varphi_M} A^{\oplus n_M} \to M \to 0$$

of M as an A-module. Let us use the trivial presentation $0 \to A^{\oplus n} \xrightarrow{1} A^{\oplus n} \to 0$ if $M = A^{\oplus n}$ (this isn't necessary but simplifies the exposition). For each morphism $f : M \to N$ of Mod_A^{fp} we can choose a commutative diagram

0GNP (56.2.5.1)
$$\begin{array}{ccccccc} A^{\oplus m_M} & \xrightarrow{\varphi_M} & A^{\oplus n_M} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi_f} & & \downarrow{\scriptstyle \chi_f} & & \downarrow{\scriptstyle f} & & \\ A^{\oplus m_N} & \xrightarrow{\varphi_N} & A^{\oplus n_N} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Having made these choices we can define: for an object M of Mod_A^{fp} we set

$$F(M) = \mathrm{Coker}(F(\varphi_M) : F(A^{\oplus m_M}) \to F(A^{\oplus n_M}))$$

and for a morphism $f : M \to N$ of Mod_A^{fp} we set

$$F(f) = \text{the map } F(M) \to F(N) \text{ induced by } F(\psi_f) \text{ and } F(\chi_f) \text{ on cokernels}$$

Note that this rule extends the given functor F on the full subcategory consisting of the free modules $A^{\oplus n}$. We still have to show that F is a functor, that F is additive, and that F is right exact.

Let $f : M \to N$ be a morphism Mod_A^{fp}. We claim that the map $F(f)$ defined above is independent of the choices of ψ_f and χ_f in (56.2.5.1). Namely, say

$$\begin{array}{ccccccc} A^{\oplus m_M} & \xrightarrow{\varphi_M} & A^{\oplus n_M} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle f} & & \\ A^{\oplus m_N} & \xrightarrow{\varphi_N} & A^{\oplus n_N} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is also commutative. Denote $F(f)' : F(M) \to F(N)$ the map induced by $F(\psi)$ and $F(\chi)$. Looking at the commutative diagrams, by elementary commutative algebra there exists a map $\omega : A^{\oplus n_M} \to A^{\oplus m_N}$ such that $\chi = \chi_f + \varphi_N \circ \omega$. Applying F we find that $F(\chi) = F(\chi_f) + F(\varphi_N) \circ F(\omega)$. As $F(N)$ is the cokernel of $F(\varphi_N)$ we find that the map $F(A^{\oplus n_M}) \to F(M)$ equalizes $F(f)$ and $F(f)'$. Since a cokernel is an epimorphism, we conclude that $F(f) = F(f)'$.

Let us prove F is a functor. First, observe that $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ because we may pick the identities for ψ_f and χ_f in the diagram above in case $f = \mathrm{id}_M$. Second, suppose we have $f : M \to N$ and $g : L \to M$. Then we see that $\psi = \psi_f \circ \psi_g$ and $\chi = \chi_f \circ \chi_g$ fit into (56.2.5.1) for $f \circ g$. Hence these induce the correct map which exactly says that $F(f) \circ F(g) = F(f \circ g)$.

Let us prove that F is additive. Namely, suppose we have $f, g : M \to N$. Then we see that $\psi = \psi_f + \psi_g$ and $\chi = \chi_f + \chi_g$ fit into (56.2.5.1) for $f + g$. Hence these induce the correct map which exactly says that $F(f) + F(g) = F(f + g)$.

Finally, let us prove that F is right exact. It suffices to show that F commutes with coequalizers, see Categories, Lemma 4.23.3. For this, it suffices to prove that F commutes with cokernels. Let $K \to L \to M \to 0$ be an exact sequence of A-modules with K, L, M finitely presented. Since F is an additive functor, this certainly gives a complex

$$F(K) \to F(L) \to F(M) \to 0$$

and we have to show that the second arrow is the cokernel of the first in $\mathcal{B}$. In any case, we obtain a map $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. By elementary commutative algebra there exists a commutative diagram

$$\begin{array}{ccccccc} A^{\oplus m_M} & \xrightarrow{\varphi_M} & A^{\oplus n_M} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle 1} & & \\ K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Applying F to this diagram and using the construction of $F(M)$ as the cokernel of $F(\varphi_M)$ we find there exists a map $F(M) \to \mathrm{Coker}(F(K) \to F(L))$ which is a right inverse to the map $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. This first implies that $F(L) \to F(M)$ is an epimorphism always. Next, the above shows we have

$$\mathrm{Coker}(F(K) \to F(L)) = F(M) \oplus E$$

where the direct sum decomposition is compatible with both $F(M) \to \mathrm{Coker}(F(K) \to F(L))$ and $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. However, then the epimorphism $p : F(L) \to E$ becomes zero both after composition with $F(K) \to F(L)$ and after composition with $F(A^{n_M}) \to F(L)$. However, since $K \oplus A^{n_M} \to L$ is surjective (algebra argument omitted), we conclude that $F(K \oplus A^{n_M}) \to F(L)$ is an epimorphism (by the above) whence $E = 0$. This finishes the proof. □

0GNQ **Lemma 56.2.6.** *Let A be a ring. Let $\mathcal{B}$ be an additive category with arbitrary direct sums and cokernels. There is an equivalence of categories between*

1. *the category of functors $F : Mod_A \to \mathcal{B}$ which are right exact and commute with arbitrary direct sums, and*
2. *the category of pairs (K, κ) where $K \in \mathrm{Ob}(\mathcal{B})$ and $\kappa : A \to End_{\mathcal{B}}(K)$ is a ring homomorphism*

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Combine Lemmas 56.2.5 and 56.2.3. □

56.3. Functors between categories of modules

0GNR The following lemma is archetypical of the results in this chapter.

0GNS **Lemma 56.3.1.** *Let A and B be rings. Let $F : Mod_A \to Mod_B$ be a functor. The following are equivalent*

1. *F is isomorphic to the functor $M \mapsto M \otimes_A K$ for some $A \otimes_{\mathbf{Z}} B$-module K,*
2. *F is right exact and commutes with all direct sums,*
3. *F commutes with all colimits,*

(4) *F has a right adjoint G.*

Proof. If (1), then (4) as a right adjoint for $M \mapsto M \otimes_A K$ is $N \mapsto \mathrm{Hom}_B(K, N)$, see Differential Graded Algebra, Lemma 22.30.3. If (4), then (3) by Categories, Lemma 4.24.5. The implication (3) $\Rightarrow$ (2) is immediate from the definitions.

Assume (2). We will prove (1). By the discussion in Homology, Section 12.7 the functor F is additive. Hence F induces a ring map $A \to \mathrm{End}_B(F(M))$, $a \mapsto F(a \cdot \mathrm{id}_M)$ for every A-module M. We conclude that $F(M)$ is an $A \otimes_{\mathbf{Z}} B$-module functorially in M. Set $K = F(A)$. Define

$$M \otimes_A K = M \otimes_A F(A) \longrightarrow F(M), \quad m \otimes k \longmapsto F(\varphi_m)(k)$$

Here $\varphi_m : A \to M$ sends $a \to am$. The rule $(m, k) \mapsto F(\varphi_m)(k)$ is A-bilinear (and B-linear on the right) as required to obtain the displayed $A \otimes_{\mathbf{Z}} B$-linear map. This construction is functorial in M, hence defines a transformation of functors $- \otimes_A K \to F(-)$ which is an isomorphism when evaluated on A. For every A-module M we can choose an exact sequence

$$\bigoplus\nolimits_{j \in J} A \to \bigoplus\nolimits_{i \in I} A \to M \to 0$$

Using the maps constructed above we find a commutative diagram

$$\begin{array}{ccccccc}
(\bigoplus_{j \in J} A) \otimes_A K & \longrightarrow & (\bigoplus_{i \in I} A) \otimes_A K & \longrightarrow & M \otimes_A K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F(\bigoplus_{j \in J} A) & \longrightarrow & F(\bigoplus_{i \in I} A) & \longrightarrow & F(M) & \longrightarrow & 0
\end{array}$$

The lower row is exact as F is right exact. The upper row is exact as tensor product with K is right exact. Since F commutes with direct sums the left two vertical arrows are bijections. Hence we conclude. □

0GNT **Example 56.3.2.** Let R be a ring. Let A and B be R-algebras. Let K be a $A \otimes_R B$-module. Then we can consider the functor

0GNU (56.3.2.1) $$F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M \otimes_A K$$

This functor is R-linear, right exact, commutes with arbitrary direct sums, commutes with all colimits, has a right adjoint (Lemma 56.3.1).

0GNV **Lemma 56.3.3.** *Let R be a ring. Let A and B be R-algebras. There is an equivalence of categories between*

(1) *the category of R-linear functors $F : Mod_A \to Mod_B$ which are right exact and commute with arbitrary direct sums, and*
(2) *the category $Mod_{A \otimes_R B}$.*

given by sending K to the functor F in (56.3.2.1).

Proof. Let F be an object of the first category. By Lemma 56.3.1 we may assume $F(M) = M \otimes_A K$ functorially in M for some $A \otimes_{\mathbf{Z}} B$-module K. The R-linearity of F immediately implies that the $A \otimes_{\mathbf{Z}} B$-module structure on K comes from a (unique) $A \otimes_R B$-module structure on K. Thus we see that sending K to F as in (56.3.2.1) is essentially surjective.

To prove that our functor is fully faithful, we have to show that given $A \otimes_R B$-modules K and K' any transformation $t : F \to F'$ between the corresponding

functors, comes from a unique $\varphi : K \to K'$. Since $K = F(A)$ and $K' = F'(A)$ we can take φ to be the value $t_A : F(A) \to F'(A)$ of t at A. This maps is $A \otimes_R B$-linear by the definition of the $A \otimes B$-module structure on $F(A)$ and $F'(A)$ given in the proof of Lemma 56.3.1. □

0GNW **Remark 56.3.4.** Let R be a ring. Let A, B, C be R-algebras. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ and $F' : \mathrm{Mod}_B \to \mathrm{Mod}_C$ be R-linear, right exact functors which commute with arbitrary direct sums. If by the equivalence of Lemma 56.3.3 the object K in $\mathrm{Mod}_{A \otimes_R B}$ corresponds to F and the object K' in $\mathrm{Mod}_{B \otimes_R C}$ corresponds to F', then $K \otimes_B K'$ viewed as an object of $\mathrm{Mod}_{A \otimes_R C}$ corresponds to $F' \circ F$.

0GNX **Remark 56.3.5.** In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F is exact $\Leftrightarrow$ K is flat over A.

0GNY **Remark 56.3.6.** In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F sends finite A-modules to finite B-modules $\Leftrightarrow$ K is finite as a B-module.

0GNZ **Remark 56.3.7.** In the situation of Lemma 56.3.3 suppose that F corresponds to K. Then F sends finitely presented A-modules to finitely presented B-modules $\Leftrightarrow$ K is finitely presented as a B-module.

0GP0 **Lemma 56.3.8.** *Let A and B be rings. If*

$$F : Mod_A \longrightarrow Mod_B$$

is an equivalence of categories, then there exists an isomorphism $A \to B$ of rings and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. Since an equivalence commutes with all colimits, we see that Lemmas 56.3.1 applies. Let K be the $A \otimes_{\mathbf{Z}} B$-module such that F is isomorphic to the functor $M \mapsto M \otimes_A K$. Let K' be the $B \otimes_{\mathbf{Z}} A$-module such that a quasi-inverse of F is isomorphic to the functor $N \mapsto N \otimes_B K'$. By Remark 56.3.4 and Lemma 56.3.3 we have an isomorphism

$$\psi : K \otimes_B K' \longrightarrow A$$

of $A \otimes_{\mathbf{Z}} A$-modules. Similarly, we have an isomorphism

$$\psi' : K' \otimes_A K \longrightarrow B$$

of $B \otimes_{\mathbf{Z}} B$-modules. Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i \in K \otimes_B K'$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$K \xrightarrow{\psi^{-1} \otimes \mathrm{id}_K} K \otimes_B K' \otimes_A K \xrightarrow{\mathrm{id}_K \otimes \psi'} K$$

The composition is an isomorphism and given by

$$k \longmapsto \sum x_i \psi'(y_i \otimes k)$$

We conclude this automorphism factors as

$$K \to B^{\oplus n} \to K$$

as a map of B-modules. It follows that K is finite projective as a B-module.

We claim that K is invertible as a B-module. This is equivalent to asking the rank of K as a B-module to have the constant value 1, see More on Algebra, Lemma 15.118.2 and Algebra, Lemma 10.78.2. If not, then there exists a maximal ideal $\mathfrak{m} \subset B$ such that either (a) $K \otimes_B B/\mathfrak{m} = 0$ or (b) there is a surjection $K \to (B/\mathfrak{m})^{\oplus 2}$

of B-modules. Case (a) is absurd as $K' \otimes_A K \otimes_B N = N$ for all B-modules N. Case (b) would imply we get a surjection

$$A = K \otimes_B K' \longrightarrow (B/\mathfrak{m} \otimes_B K')^{\oplus 2}$$

of (right) A-modules. This is impossible as the target is an A-module which needs at least two generators: $B/\mathfrak{m} \otimes_B K'$ is nonzero as the image of the nonzero module $B/\mathfrak{m}$ under the quasi-inverse of F.

Since K is invertible as a B-module we see that $\mathrm{Hom}_B(K, K) = B$. Since $K = F(A)$ the action of A on K defines a ring isomorphism $A \to B$. The lemma follows. □

0GP1 **Lemma 56.3.9.** *Let R be a ring. Let A and B be R-algebras. If*

$$F : Mod_A \longrightarrow Mod_B$$

is an R-linear equivalence of categories, then there exists an isomorphism $A \to B$ of R-algebras and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. We get $A \to B$ and L from Lemma 56.3.8. To finish the proof, we need to show that the R-linearity of F forces $A \to B$ to be an R-algebra map. We omit the details. □

0GP2 **Remark 56.3.10.** Let A and B be rings. Let us endow Mod_A and Mod_B with the usual monoidal structure given by tensor products of modules. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be a functor of monoidal categories, see Categories, Definition 4.43.2. Here are some comments:

(1) Since $F(A)$ is a unit (by our definitions) we have $F(A) = B$.
(2) We obtain a multiplicative map $\varphi : A \to B$ by sending $a \in A$ to its action on $F(A) = B$.
(3) Take $A = B$ and $F(M) = M \otimes_A M$. In this case $\varphi(a) = a^2$.
(4) If F is additive, then φ is a ring map.
(5) Take $A = B = \mathbf{Z}$ and $F(M) = M/\text{torsion}$. Then $\varphi = \mathrm{id}_{\mathbf{Z}}$ but F is not the identity functor.
(6) If F is right exact and commutes with direct sums, then $F(M) = M \otimes_{A,\varphi} B$ by Lemma 56.3.1.

In other words, ring maps $A \to B$ are in bijection with isomorphism classes of functors of monoidal categories $\mathrm{Mod}_A \to \mathrm{Mod}_B$ which commute with all colimits.

56.4. Extending functors on categories of modules

0GP3 For a ring A let us denote Mod_A^{fp} the category of finitely presented A-modules.

0GP4 **Lemma 56.4.1.** *Let A and B be rings. Let $F : Mod_A^{fp} \to Mod_B^{fp}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to Mod_B$ which commutes with filtered colimits.*

Proof. Special case of Lemma 56.2.1. □

0GP5 **Remark 56.4.2.** With A, B, F, and F' as in Lemma 56.4.1. Observe that the tensor product of two finitely presented modules is finitely presented, see Algebra, Lemma 10.12.14. Thus we may endow Mod_A^{fp}, Mod_B^{fp}, Mod_A, and Mod_B with the usual monoidal structure given by tensor products of modules. In this case, if F is

a functor of monoidal categories, so is F'. This follows immediately from the fact that tensor products of modules commutes with filtered colimits.

0GP6 **Lemma 56.4.3.** *With A, B, F, and F' as in Lemma 56.4.1.*

(1) *If F is additive, then F' is additive and commutes with arbitrary direct sums, and*
(2) *if F is right exact, then F' is right exact.*

Proof. Follows from Lemmas 56.2.2 and 56.2.3. □

0GP7 **Remark 56.4.4.** Combining Remarks 56.3.10 and 56.4.2 and Lemma 56.4.3 we find the following. Given rings A and B the set of ring maps $A \to B$ is in bijection with the set of isomorphism classes of functors of monoidal categories $\mathrm{Mod}_A^{fp} \to \mathrm{Mod}_B^{fp}$ which are right exact.

0GP8 **Lemma 56.4.5.** *With A, B, F, and F' as in Lemma 56.4.1. Assume A is a coherent ring (Algebra, Definition 10.90.1). If F is left exact, then F' is left exact.*

Proof. Special case of Lemma 56.2.4. □

For a ring A let us denote Mod_A^{fg} the category of finitely generated A-modules (AKA finite A-modules).

0GP9 **Lemma 56.4.6.** *Let A and B be Noetherian rings. Let $F : Mod_A^{fg} \to Mod_B^{fg}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to Mod_B$ which commutes with filtered colimits. If F is additive, then F' is additive and commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is F'.*

Proof. See Lemmas 56.4.3 and 56.4.5. Also, use the finite A-modules are finitely presented A-modules, see Algebra, Lemma 10.31.4, and use that Noetherian rings are coherent, see Algebra, Lemma 10.90.5. □

56.5. Functors between categories of quasi-coherent modules

0FZA In this section we briefly study functors between categories of quasi-coherent modules.

0FZB **Example 56.5.1.** Let R be a ring. Let X and Y be schemes over R with X quasi-compact and quasi-separated. Let $\mathcal{K}$ be a quasi-coherent $\mathcal{O}_{X\times_R Y}$-module. Then we can consider the functor

0FZC (56.5.1.1) $$F : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_Y), \quad \mathcal{F} \longmapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K})$$

The morphism pr_2 is quasi-compact and quasi-separated (Schemes, Lemmas 26.19.3 and 26.21.12). Hence pushforward along this morphism preserves quasi-coherent modules, see Schemes, Lemma 26.24.1. Moreover, our functor is R-linear and commutes with arbitrary direct sums, see Cohomology of Schemes, Lemma 30.6.1.

The following lemma is a natural generalization of Lemma 56.3.3.

0FZD **Lemma 56.5.2.** *Let R be a ring. Let X and Y be schemes over R with X affine. There is an equivalence of categories between*

(1) *the category of R-linear functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which are right exact and commute with arbitrary direct sums, and*
(2) *the category $QCoh(\mathcal{O}_{X\times_R Y})$*

given by sending $\mathcal{K}$ to the functor F in (56.5.1.1).

Proof. Let $\mathcal{K}$ be an object of $QCoh(\mathcal{O}_{X\times_R Y})$ and $F_\mathcal{K}$ the functor (56.5.1.1). By the discussion in Example 56.5.1 we already know that F is R-linear and commutes with arbitrary direct sums. Since $\mathrm{pr}_2 : X \times_R Y \to Y$ is affine (Morphisms, Lemma 29.11.9) the functor $\mathrm{pr}_{2,*}$ is exact, see Cohomology of Schemes, Lemma 30.2.3. Hence F is right exact as well, in other words F is as in (1).

Let F be as in (1). Say $X = \mathrm{Spec}(A)$. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G} = F(\mathcal{O}_X)$. The functor F induces an R-linear map $A \to \mathrm{End}_{\mathcal{O}_Y}(\mathcal{G})$, $a \mapsto F(a\cdot\mathrm{id})$. Thus $\mathcal{G}$ is a sheaf of modules over

$$A \otimes_R \mathcal{O}_Y = \mathrm{pr}_{2,*}\mathcal{O}_{X\times_R Y}$$

By Morphisms, Lemma 29.11.7 we find that there is a unique quasi-coherent module $\mathcal{K}$ on $X \times_R Y$ such that $F(\mathcal{O}_X) = \mathcal{G} = \mathrm{pr}_{2,*}\mathcal{K}$ compatible with action of A and $\mathcal{O}_Y$. Denote $F_\mathcal{K}$ the functor given by (56.5.1.1). There is an equivalence $\mathrm{Mod}_A \to QCoh(\mathcal{O}_X)$ sending A to $\mathcal{O}_X$, see Schemes, Lemma 26.7.5. Hence we find an isomorphism $F \cong F_\mathcal{K}$ by Lemma 56.2.6 because we have an isomorphism $F(\mathcal{O}_X) \cong F_\mathcal{K}(\mathcal{O}_X)$ compatible with A-action by construction.

This shows that the functor sending $\mathcal{K}$ to $F_\mathcal{K}$ is essentially surjective. We omit the verification of fully faithfulness. □

0FZE **Remark 56.5.3.** Below we will use that for an affine morphism $h : T \to S$ we have $h_*\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{H} = h_*(\mathcal{G} \otimes_{\mathcal{O}_T} h^*\mathcal{H})$ for $\mathcal{G} \in QCoh(\mathcal{O}_T)$ and $\mathcal{H} \in QCoh(\mathcal{O}_S)$. This follows immediately on translating into algebra.

0FZF **Lemma 56.5.4.** *In Lemma 56.5.2 let F correspond to $\mathcal{K}$ in $QCoh(\mathcal{O}_{X\times_R Y})$. We have*

1. *If $f : X' \to X$ is an affine morphism, then $F \circ f_*$ corresponds to $(f \times id_Y)^*\mathcal{K}$.*
2. *If $g : Y' \to Y$ is a flat morphism, then $g^* \circ F$ corresponds to $(id_X \times g)^*\mathcal{K}$.*
3. *If $j : V \to Y$ is an open immersion, then $j^* \circ F$ corresponds to $\mathcal{K}|_{X\times_R V}$.*

Proof. Proof of (1). Consider the commutative diagram

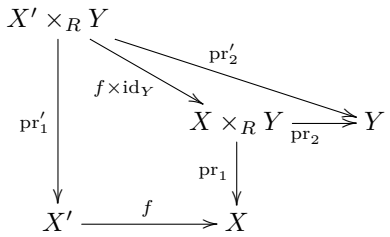

Let $\mathcal{F}'$ be a quasi-coherent module on X'. We have

$$\begin{aligned}\mathrm{pr}_{2,*}(\mathrm{pr}_1^* f_*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) &= \mathrm{pr}_{2,*}((f\times\mathrm{id}_Y)_*(\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) \\ &= \mathrm{pr}_{2,*}(f\times\mathrm{id}_Y)_*\left((\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times\mathrm{id}_Y)^*\mathcal{K})\right) \\ &= \mathrm{pr}_{2,*}'((\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times\mathrm{id}_Y)^*\mathcal{K})\end{aligned}$$

Here the first equality is affine base change for the left hand square in the diagram, see Cohomology of Schemes, Lemma 30.5.1. The second equality hold by Remark

56.5.3. The third equality is functoriality of pushforwards for modules. This proves (1).

Proof of (2). Consider the commutative diagram

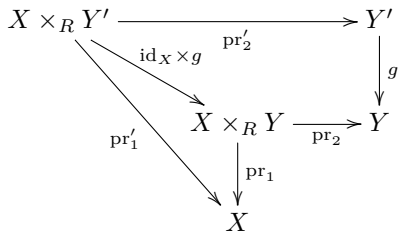

We have

$$\begin{aligned} g^*\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) &= \mathrm{pr}'_{2,*}((\mathrm{id}_X \times g)^*(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K})) \\ &= \mathrm{pr}'_{2,*}((\mathrm{pr}'_1)^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y'}} (\mathrm{id}_X \times g)^*\mathcal{K}) \end{aligned}$$

The first equality by flat base change for the square in the diagram, see Cohomology of Schemes, Lemma 30.5.2. The second equality by functoriality of pullback and the fact that a pullback of tensor products it the tensor product of the pullbacks.

Part (3) is a special case of (2). □

0GPA **Lemma 56.5.5.** *Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Let $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ be an R-linear, right exact functor which commutes with arbitrary direct sums. Then we can construct*

(1) *a quasi-coherent module $\mathcal{K}$ on $X \times_R Y$, and*
(2) *a natural transformation $t : F \to F_{\mathcal{K}}$ where $F_{\mathcal{K}}$ denotes the functor (56.5.1.1)*

such that $t : F \circ f_* \to F_{\mathcal{K}} \circ f_*$ is an isomorphism for every morphism $f : X' \to X$ whose source is an affine scheme.

Proof. Consider a morphism $f' : X' \to X$ with X' affine. Since the diagonal of X is affine, we see that f' is an affine morphism (Morphisms, Lemma 29.11.12). Thus $f'_* : QCoh(\mathcal{O}_{X'}) \to QCoh(\mathcal{O}_X)$ is an R-linear exact functor (Cohomology of Schemes, Lemma 30.2.3) which commutes with direct sums (Cohomology of Schemes, Lemma 30.6.1). Thus $F \circ f'_*$ is an R-linear, right exact functor which commutes with arbitrary direct sums. Whence $F \circ f'_* = F_{\mathcal{K}'}$ for some $\mathcal{K}'$ on $X' \times_R Y$ by Lemma 56.5.2. Moreover, given a morphism $f'' : X'' \to X'$ with X'' affine we obtain a canonical identification $(f'' \times \mathrm{id}_Y)^*\mathcal{K}' = \mathcal{K}''$ by the references already given combined with Lemma 56.5.4. These identifications satisfy a cocycle condition given another morphism $f''' : X''' \to X''$ which we leave it to the reader to spell out.

Choose an affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Since the diagonal of X is affine, we see that the intersections $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ are affine. As above the inclusion morphisms $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ are affine. Denote $\mathcal{K}_{i_0\ldots i_p}$ the quasi-coherent module on $U_{i_0\ldots i_p} \times_R Y$ corresponding to $F \circ j_{i_0\ldots i_p *}$ as above. By the above we obtain identifications

$$\mathcal{K}_{i_0\ldots i_p} = \mathcal{K}_{i_0\ldots \hat{i}_j \ldots i_p}|_{U_{i_0\ldots i_p} \times_R Y}$$

which satisfy the usual compatibilites for glueing. In other words, we obtain a unique quasi-coherent module $\mathcal{K}$ on $X \times_R Y$ whose restriction to $U_{i_0 \ldots i_p} \times_R Y$ is $\mathcal{K}_{i_0 \ldots i_p}$ compatible with the displayed identifications.

Next, we construct the transformation t. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$ and denote $(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p}$ the restriction of $\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K}$ to $U_{i_0 \ldots i_p} \times_R Y$. Observe that

$$\begin{aligned} F(j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}) &= \mathrm{pr}_{i_0 \ldots i_p, 2, *}(\mathrm{pr}_{i_0 \ldots i_p, 1}^* \mathcal{F}_{i_0 \ldots i_p} \otimes \mathcal{K}_{i_0 \ldots i_p}) \\ &= \mathrm{pr}_{i_0 \ldots i_p, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p} \end{aligned}$$

where $\mathrm{pr}_{i_0 \ldots i_p, 2} : U_{i_0 \ldots i_p} \times_R Y \to Y$ is the projection and similarly for the other projection. Moreover, these identifications are compatible with the displayed identifications in the previous paragraph. Recall, from Cohomology of Schemes, Lemma 30.7.1 that the relative Čech complex

$$\bigoplus \mathrm{pr}_{i_0, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \to \bigoplus \mathrm{pr}_{i_0 i_1, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \to \bigoplus \mathrm{pr}_{i_0 i_1 i_2, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1 i_2} \to \ldots$$

computes $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})$. Hence the cohomology sheaf in degree 0 is $F_{\mathcal{K}}(\mathcal{F})$. Thus we obtain the desired map $t : F(\mathcal{F}) \to F_{\mathcal{K}}(\mathcal{F})$ by contemplating the following commutative diagram

$$\begin{array}{ccccccc} & & F(\mathcal{F}) & \longrightarrow & \bigoplus F(j_{i_0 *}\mathcal{F}_{i_0}) & \longrightarrow & \bigoplus F(j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}) \\ & & \vdots & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{\mathcal{K}}(\mathcal{F}) & \longrightarrow & \bigoplus \mathrm{pr}_{i_0, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} & \longrightarrow & \bigoplus \mathrm{pr}_{i_0 i_1, 2, *}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \end{array}$$

We obtain the top row by applying F to the (exact) complex $0 \to \mathcal{F} \to \bigoplus j_{i_0 *}\mathcal{F}_{i_0} \to \bigoplus j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}$ (but since F is not exact, the top row is just a complex and not necessarily exact). The solid vertical arrows are the identifications above. This does indeed define the dotted arrow as desired. The arrow is functorial in $\mathcal{F}$; we omit the details.

We still have to prove the final assertion. Let $f : X' \to X$ be as in the statement of the lemma and let $\mathcal{K}'$ be the quasi-coherent module on $X' \times_R Y$ constructed in the first paragraph of the proof. If the morphism $f : X' \to X$ maps into one of the opens U_i, then the result follows from Lemma 56.5.4 because in this case we know that $\mathcal{K}_i = \mathcal{K}|_{U_i \times_R Y}$ pulls back to $\mathcal{K}$. In general, we obtain an affine open covering $X' = \bigcup U'_i$ with $U'_i = f^{-1}(U_i)$ and we obtain isomorphisms $\mathcal{K}'|_{U'_i} = f_i^*\mathcal{K}_i$ where $f_i : U'_i \to U_i$ is the induced morphism. These morphisms satisfy the compatibility conditions needed to glue to an isomorphism $\mathcal{K}' = f^*\mathcal{K}$ and we conclude. Some details omitted. □

0FZG **Lemma 56.5.6.** *In Lemma 56.5.2 or in Lemma 56.5.5 if F is an exact functor, then the corresponding object $\mathcal{K}$ of $QCoh(\mathcal{O}_{X \times_R Y})$ is flat over X.*

Proof. We may assume X is affine, so we are in the case of Lemma 56.5.2. By Lemma 56.5.4 we may assume Y is affine. In the affine case the statement translates into Remark 56.3.5. □

0FZH **Lemma 56.5.7.** *Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. There is an equivalence of categories between*

(1) *the category of R-linear exact functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commute with arbitrary direct sums, and*
(2) *the full subcategory of $QCoh(\mathcal{O}_{X \times_R Y})$ consisting of $\mathcal{K}$ such that*
   (a) *$\mathcal{K}$ is flat over X,*
   (b) *for $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have $R^q pr_{2,*}(pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) = 0$ for $q > 0$.*

given by sending $\mathcal{K}$ to the functor F in (56.5.1.1).

Proof. Let $\mathcal{K}$ be as in (2). The functor F in (56.5.1.1) commutes with direct sums. Since by (1) (a) the modules $\mathcal{K}$ is X-flat, we see that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we obtain a short exact sequence

$$0 \to \mathrm{pr}_1^*\mathcal{F}_1 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to \mathrm{pr}_1^*\mathcal{F}_2 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to \mathrm{pr}_1^*\mathcal{F}_3 \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K} \to 0$$

Since by (2)(b) the higher direct image $R^1\mathrm{pr}_{2,*}$ on the first term is zero, we conclude that $0 \to F(\mathcal{F}_1) \to F(\mathcal{F}_2) \to F(\mathcal{F}_3) \to 0$ is exact and we see that F is as in (1).

Let F be as in (1). Let $\mathcal{K}$ and $t : F \to F_{\mathcal{K}}$ be as in Lemma 56.5.5. By Lemma 56.5.6 we see that $\mathcal{K}$ is flat over X. To finish the proof we have to show that t is an isomorphism and the statement on higher direct images. Both of these follow from the fact that the relative Čech complex

$$\bigoplus \mathrm{pr}_{i_0,2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \to \bigoplus \mathrm{pr}_{i_0 i_1,2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \to \bigoplus \mathrm{pr}_{i_0 i_1 i_2,2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1 i_2} \to \ldots$$

computes $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})$. Please see proof of Lemma 56.5.5 for notation and for the reason why this is so. In the proof of Lemma 56.5.5 we also found that this complex is equal to F applied to the complex

$$\bigoplus j_{i_0 *}\mathcal{F}_{i_0} \to \bigoplus j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1} \to \bigoplus j_{i_0 i_1 i_2 *}\mathcal{F}_{i_0 i_1 i_2} \to \ldots$$

This complex is exact except in degree zero with cohomology sheaf equal to $\mathcal{F}$. Hence since F is an exact functor we conclude $F = F_{\mathcal{K}}$ and that (2)(b) holds.

We omit the proof that the construction that sends F to $\mathcal{K}$ is functorial and a quasi-inverse to the functor sending $\mathcal{K}$ to the functor $F_{\mathcal{K}}$ determined by (56.5.1.1). □

0GPB **Remark 56.5.8.** Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Lemma 56.5.7 may be generalized as follows: the functors (56.5.1.1) associated to quasi-coherent modules on $X \times_R Y$ are exactly those $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which have the following properties

(1) F is R-linear and commutes with arbitrary direct sums,
(2) $F \circ j_*$ is right exact when $j : U \to X$ is the inclusion of an affine open, and
(3) $0 \to F(\mathcal{F}) \to F(\mathcal{G}) \to F(\mathcal{H})$ is exact whenever $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence such that for all $x \in X$ the sequence on stalks $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is a split short exact sequence.

Namely, these assumptions are enough to get construct a transformation $t : F \to F_{\mathcal{K}}$ as in Lemma 56.5.5 and to show that it is an isomorphism. Moreover, properties (1), (2), and (3) do hold for functors (56.5.1.1). If we ever need this we will carefully state and prove this here.

0GPC **Lemma 56.5.9.** *Let R be a ring. Let X, Y, Z be schemes over R. Assume X and Y are quasi-compact and have affine diagonal. Let*

$$F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y) \quad \text{and} \quad G : QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_Z)$$

be R-linear exact functors which commute with arbitrary direct sums. Let $\mathcal{K}$ in $QCoh(\mathcal{O}_{X\times_R Y})$ and $\mathcal{L}$ in $QCoh(\mathcal{O}_{Y\times_R Z})$ be the corresponding "kernels", see Lemma 56.5.7. Then $G\circ F$ corresponds to $pr_{13,*}(pr_{12}^*\mathcal{K}\otimes_{\mathcal{O}_{X\times_R Y\times_R Z}} pr_{23}^*\mathcal{L})$ in $QCoh(\mathcal{O}_{X\times_R Z})$.

Proof. Since $G\circ F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ is R-linear, exact, and commutes with arbitrary direct sums, we find by Lemma 56.5.7 that there exists an $\mathcal{M}$ in $QCoh(\mathcal{O}_{X\times_R Z})$ corresponding to $G\circ F$. On the other hand, denote $\mathcal{E} = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{K}\otimes \mathrm{pr}_{23}^*\mathcal{L})$. Here and in the rest of the proof we omit the subscript from the tensor products. Let $U\subset X$ and $W\subset Z$ be affine open subschemes. To prove the lemma, we will construct an isomorphism

$$\Gamma(U\times_R W, \mathcal{E}) \cong \Gamma(U\times_R W, \mathcal{M})$$

compatible with restriction mappings for varying U and W.

First, we observe that

$$\Gamma(U\times_R W, \mathcal{E}) = \Gamma(U\times_R Y\times_R W, \mathrm{pr}_{12}^*\mathcal{K}\otimes \mathrm{pr}_{23}^*\mathcal{L})$$

by construction. Thus we have to show that the same thing is true for $\mathcal{M}$.

Write $U = \mathrm{Spec}(A)$ and denote $j : U\to X$ the inclusion morphism. Recall from the construction of $\mathcal{M}$ in the proof of Lemma 56.5.2 that

$$\Gamma(U\times_R W, \mathcal{M}) = \Gamma(W, G(F(j_*\mathcal{O}_U)))$$

where the A-module action on the right hand side is given by the action of A on $\mathcal{O}_U$. The correspondence between F and $\mathcal{K}$ tells us that $F(j_*\mathcal{O}_U) = b_*(a^*j_*\mathcal{O}_U\otimes\mathcal{K})$ where $a : X\times_R Y\to X$ and $b : X\times_R Y\to Y$ are the projection morphisms. Since j is an affine morphism, we have $a^*j_*\mathcal{O}_U = (j\times \mathrm{id}_Y)_*\mathcal{O}_{U\times_R Y}$ by Cohomology of Schemes, Lemma 30.5.1. Next, we have $(j\times\mathrm{id}_Y)_*\mathcal{O}_{U\times_R Y}\otimes\mathcal{K} = (j\times\mathrm{id}_Y)_*\mathcal{K}|_{U\times_R Y}$ by Remark 56.5.3 for example. Putting what we have found together we find

$$F(j_*\mathcal{O}_U) = (U\times_R Y\to Y)_*\mathcal{K}|_{U\times_R Y}$$

with obvious A-action. (This formula is implicit in the proof of Lemma 56.5.2.) Applying the functor G we obtain

$$G(F(j_*\mathcal{O}_U)) = t_*(s^*((U\times_R Y\to Y)_*\mathcal{K}|_{U\times_R Y})\otimes\mathcal{L})$$

where $s : Y\times_R Z\to Y$ and $t : Y\times_R Z\to Z$ are the projection morphisms. Again using affine base change (Cohomology of Schemes, Lemma 30.5.1) but this time for the square

$$\begin{array}{ccc} U\times_R Y\times_R Z & \longrightarrow & U\times_R Y \\ \downarrow & & \downarrow \\ Y\times_R Z & \longrightarrow & Y \end{array}$$

we obtain

$$s^*((U\times_R Y\to Y)_*\mathcal{K}|_{U\times_R Y}) = (U\times_R Y\times_R Z\to Y\times_R Z)_*\mathrm{pr}_{12}^*\mathcal{K}|_{U\times_R Y\times_R Z}$$

Using Remark 56.5.3 again we find

$$\begin{aligned} &(U\times_R Y\times_R Z\to Y\times_R Z)_*\mathrm{pr}_{12}^*\mathcal{K}|_{U\times_R Y\times_R Z}\otimes\mathcal{L} \\ &= (U\times_R Y\times_R Z\to Y\times_R Z)_*\left(\mathrm{pr}_{12}^*\mathcal{K}\otimes\mathrm{pr}_{23}^*\mathcal{L}\right)|_{U\times_R Y\times_R Z} \end{aligned}$$

Applying the functor $\Gamma(W, t_*(-)) = \Gamma(Y \times_R W, -)$ to this we obtain

$$\begin{aligned}\Gamma(U \times_R W, \mathcal{M}) &= \Gamma(W, G(F(j_*\mathcal{O}_U))) \\ &= \Gamma(Y \times_R W, (U \times_R Y \times_R Z \to Y \times_R Z)_*(\mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})|_{U \times_R Y \times_R Z}) \\ &= \Gamma(U \times_R Y \times_R W, \mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})\end{aligned}$$

as desired. We omit the verification that these isomorphisms are compatible with restriction mappings. □

0FZI **Lemma 56.5.10.** *Let R, X, Y, and $\mathcal{K}$ be as in Lemma 56.5.7 part (2). Then for any scheme T over R we have*

$$R^q pr_{13,*}(pr_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} pr_{23}^*\mathcal{K}) = 0$$

for $\mathcal{F}$ quasi-coherent on $T \times_R X$ and $q > 0$.

Proof. The question is local on T hence we may assume T is affine. In this case we can consider the diagram

$$\begin{array}{ccccc} T \times_R X & \longleftarrow & T \times_R X \times_R Y & \longrightarrow & T \times_R Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X \times_R Y & \longrightarrow & Y \end{array}$$

whose vertical arrows are affine. In particular the pushforward along $T \times_R Y \to Y$ is faithful and exact (Cohomology of Schemes, Lemma 30.2.3 and Morphisms, Lemma 29.11.7). Chasing around in the diagram using that higher direct images along affine morphisms vanish (see reference above) we see that it suffices to prove

$$R^q\mathrm{pr}_{2,*}(\mathrm{pr}_{23,*}(\mathrm{pr}_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \mathrm{pr}_{23}^*\mathcal{K})) = R^q\mathrm{pr}_{2,*}(\mathrm{pr}_{23,*}(\mathrm{pr}_{12}^*\mathcal{F}) \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}))$$

is zero which is true by assumption on $\mathcal{K}$. The equality holds by Remark 56.5.3. □

0FZJ **Lemma 56.5.11.** *In Lemma 56.5.7 let F and $\mathcal{K}$ correspond. If X is separated and flat over R, then there is a surjection $\mathcal{O}_X \boxtimes F(\mathcal{O}_X) \to \mathcal{K}$.*

Proof. Let $\Delta : X \to X \times_R X$ be the diagonal morphism and set $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$. Since Δ is a closed immersion have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X \times_R X} \to \mathcal{O}_\Delta \to 0$$

Since $\mathcal{K}$ is flat over X, the pullback $\mathrm{pr}_{23}^*\mathcal{K}$ to $X \times_R X \times_R Y$ is flat over $X \times_R X$. We obtain a short exact sequence

$$0 \to \mathrm{pr}_{12}^*\mathcal{I} \otimes \mathrm{pr}_{23}^*\mathcal{K} \to \mathrm{pr}_{23}^*\mathcal{K} \to \mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K} \to 0$$

on $X \times_R X \times_R Y$, see Modules, Lemma 17.20.4. Thus, by Lemma 56.5.10 we obtain a surjection

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{23}^*\mathcal{K}) \to \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K})$$

By flat base change (Cohomology of Schemes, Lemma 30.5.2) the source of this arrow is equal to $\mathrm{pr}_2^*\mathrm{pr}_{2,*}\mathcal{K} = \mathcal{O}_X \boxtimes F(\mathcal{O}_X)$. On the other hand the target is equal to

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K}) = \mathrm{pr}_{13,*}(\Delta \times \mathrm{id}_Y)_*\mathcal{K} = \mathcal{K}$$

which finishes the proof. The first equality holds for example by Cohomology, Lemma 20.54.4 and the fact that $\mathrm{pr}_{12}^*\mathcal{O}_\Delta = (\Delta \times \mathrm{id}_Y)_*\mathcal{O}_{X \times_R Y}$. □

56.6. Gabriel-Rosenberg reconstruction

0GPD The title of this section refers to results like Proposition 56.6.6. Besides Gabriel's original paper [**Gab62**], please consult [**Bra18**] which has a proof of the result for quasi-separated schemes and discusses the literature. In this section we will only prove Gabriel-Rosenberg reconstruction for quasi-compact and quasi-separated schemes.

0GPE **Lemma 56.6.1.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a categorically compact object of $QCoh(\mathcal{O}_X)$ if and only if $\mathcal{F}$ is of finite presentation.*

Proof. See Categories, Definition 4.26.1 for our notion of categorically compact objects in a category. If $\mathcal{F}$ is of finite presentation then it is categorically compact by Modules, Lemma 17.22.8. Conversely, any quasi-coherent module $\mathcal{F}$ can be written as a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ of finitely presented (hence quasi-coherent) $\mathcal{O}_X$-modules, see Properties, Lemma 28.22.7. If $\mathcal{F}$ is categorically compact, then we find some i and a morphism $\mathcal{F} \to \mathcal{F}_i$ which is a right inverse to the given map $\mathcal{F}_i \to \mathcal{F}$. We conclude that $\mathcal{F}$ is a direct summand of a finitely presented module, and hence finitely presented itself. □

0GPF **Lemma 56.6.2.** *Let X be an affine scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{E}$ be a nonzero quasi-coherent $\mathcal{O}_X$-module. If $Supp(\mathcal{E}) \subset Supp(\mathcal{F})$, then there exists a nonzero map $\mathcal{F} \to \mathcal{E}$.*

Proof. Let us translate the statement into algebra. Let A be a ring. Let M be a finitely presented A-module. Let N be a nonzero A-module. Assume $\operatorname{Supp}(N) \subset \operatorname{Supp}(M)$. To show: $\operatorname{Hom}_A(M, N)$ is nonzero. We may assume $N = A/I$ is cyclic (replace N by any nonzero cyclic submodule). Choose a presentation

$$A^{\oplus m} \xrightarrow{T} A^{\oplus n} \to M \to 0$$

Recall that $\operatorname{Supp}(M)$ is cut out by $\operatorname{Fit}_0(M)$ which is the ideal generated by the $n \times n$ minors of the matrix T. See More on Algebra, Lemma 15.8.4. The assumption $\operatorname{Supp}(N) \subset \operatorname{Supp}(M)$ now means that the elements of $\operatorname{Fit}_0(M)$ are nilpotent in A/I. Consider the exact sequence

$$0 \to \operatorname{Hom}_A(M, A/I) \to (A/I)^{\oplus n} \xrightarrow{T^t} (A/I)^{\oplus m}$$

We have to show that T^t cannot be injective; we urge the reader to find their own proof of this using the nilpotency of elements of $\operatorname{Fit}_0(M)$ in A/I. Here is our proof. Since $\operatorname{Fit}_0(M)$ is finitely generated, the nilpotency means that the annihilator $J \subset A/I$ of $\operatorname{Fit}_0(M)$ in A/I is nonzero. To show the non-injectivity of T^t we may localize at a prime. Choosing a suitable prime we may assume A is local and J is still nonzero. Then T^t has a nonzero kernel by More on Algebra, Lemma 15.15.6. □

0GPG **Lemma 56.6.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. The following two subcategories of $QCoh(\mathcal{O}_X)$ are equal*

1. *the full subcategory $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ whose objects are the quasi-coherent modules whose support is (set theoretically) contained in $Supp(\mathcal{F})$,*
2. *the smallest Serre subcategory $\mathcal{B} \subset QCoh(\mathcal{O}_X)$ containing $\mathcal{F}$ closed under extensions and arbitrary direct sums.*

Proof. Observe that the statement makes sense as finitely presented $\mathcal{O}_X$-modules are quasi-coherent. Since $\mathcal{A}$ is a Serre subcategory closed under extensions and direct sums and since $\mathcal{F}$ is an object of $\mathcal{A}$ we see that $\mathcal{B} \subset \mathcal{A}$. Thus it remains to show that $\mathcal{A}$ is contained in $\mathcal{B}$.

Let $\mathcal{E}$ be an object of $\mathcal{A}$. There exists a maximal submodule $\mathcal{E}' \subset \mathcal{E}$ which is in $\mathcal{B}$. Namely, suppose $\mathcal{E}_i \subset \mathcal{E}$, $i \in I$ is the set of subobjects which are objects of $\mathcal{B}$. Then $\bigoplus \mathcal{E}_i$ is in $\mathcal{B}$ and so is

$$\mathcal{E}' = \mathrm{Im}(\bigoplus \mathcal{E}_i \longrightarrow \mathcal{E})$$

This is clearly the maximal submodule we were looking for.

Now suppose that we have a nonzero map $\mathcal{G} \to \mathcal{E}/\mathcal{E}'$ with $\mathcal{G}$ in $\mathcal{B}$. Then $\mathcal{G}' = \mathcal{E} \times_{\mathcal{E}/\mathcal{E}'} \mathcal{G}$ is in $\mathcal{B}$ as an extension of $\mathcal{E}'$ and $\mathcal{G}$. Then the image $\mathcal{G}' \to \mathcal{E}$ would be strictly bigger than $\mathcal{E}'$, contradicting the maximality of $\mathcal{E}'$. Thus it suffices to show the claim in the following paragraph.

Let $\mathcal{E}$ be an nonzero object of $\mathcal{A}$. We claim that there is a nonzero map $\mathcal{G} \to \mathcal{E}$ with $\mathcal{G}$ in $\mathcal{B}$. We will prove this by induction on the minimal number n of affine opens U_i of X such that $\mathrm{Supp}(\mathcal{E}) \subset U_1 \cup \ldots \cup U_n$. Set $U = U_n$ and denote $j : U \to X$ the inclusion morphism. Denote $\mathcal{E}' = \mathrm{Im}(\mathcal{E} \to j_*\mathcal{E}|_U)$. Then the kernel $\mathcal{E}''$ of the surjection $\mathcal{E} \to \mathcal{E}'$ has support contained in $U_1 \cup \ldots \cup U_{n-1}$. Thus if $\mathcal{E}''$ is nonzero, then we win. In other words, we may assume that $\mathcal{E} \subset j_*\mathcal{E}|_U$. In particular, we see that $\mathcal{E}|_U$ is nonzero. By Lemma 56.6.2 there exists a nonzero map $\mathcal{F}|_U \to \mathcal{E}|_U$. This corresponds to a map

$$\varphi : \mathcal{F} \longrightarrow j_*(\mathcal{E}|_U)$$

whose restriction to U is nonzero. Setting $\mathcal{G} = \varphi^{-1}(\mathcal{E})$ we conclude. □

0GPH **Lemma 56.6.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subset such that $U = X \setminus Z$ is quasi-compact. Let $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ be the full subcategory whose objects are the quasi-coherent modules supported on Z. Then the restriction functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ induces an equivalence $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$.*

Proof. By the universal property of the quotient construction (Homology, Lemma 12.10.6) we certainly obtain an induced functor $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$. Denote $j : U \to X$ the inclusion morphism. Since j is quasi-compact and quasi-separated we obtain a functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$. The reader shows that this defines a quasi-inverse; details omitted. □

0GPI **Lemma 56.6.5.** *Let X be a quasi-compact and quasi-separated scheme. If $QCoh(\mathcal{O}_X)$ is equivalent to the category of modules over a ring, then X is affine.*

Proof. Say $F : \mathrm{Mod}_R \to QCoh(\mathcal{O}_X)$ is an equivalence. Then $\mathcal{F} = F(R)$ has the following properties:

(1) it is a finitely presented $\mathcal{O}_X$-module (Lemma 56.6.1),
(2) $\mathrm{Hom}_X(\mathcal{F}, -)$ is exact,
(3) $\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$ is a commutative ring,
(4) every object of $QCoh(\mathcal{O}_X)$ is a quotient of a direct sum of copies of $\mathcal{F}$.

Let $x \in X$ be a closed point. Consider the surjection

$$\mathcal{O}_X \to i_*\kappa(x)$$

where the target is the pushforward of $\kappa(x)$ by the inclusion morphism $i : x \to X$. We have

$$\mathrm{Hom}_X(\mathcal{F}, i_*\kappa(x)) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \kappa(x))$$

This first by (4) implies that $\mathcal{F}_x$ is nonzero. From (2) we deduce that every map $\mathcal{F}_x \to \kappa(x)$ lifts to a map $\mathcal{F}_x \to \mathcal{O}_{X,x}$ (as it even lifts to a global map $\mathcal{F} \to \mathcal{O}_X$). Since $\mathcal{F}_x$ is a finite $\mathcal{O}_{X,x}$-module, this implies that $\mathcal{F}_x$ is a (nonzero) finite free $\mathcal{O}_{X,x}$-module. Then since $\mathcal{F}$ is of finite presentation, this implies that $\mathcal{F}$ is finite free of positive rank in an open neighbourhood of x (Modules, Lemma 17.11.7). Since every closed subset of X contains a closed point (Topology, Lemma 5.12.8) this implies that $\mathcal{F}$ is finite locally free of positive rank. Similarly, the map

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_X(\mathcal{F}, i_*i^*\mathcal{F}) = \mathrm{Hom}_{\kappa(x)}(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x, \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x)$$

is surjective. By property (3) we conclude that the rank $\mathcal{F}_x$ must be 1. Hence $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module. But then we conclude that the functor

$$\mathcal{H} \longmapsto \Gamma(X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{F}, \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F})$$

on $QCoh(\mathcal{O}_X)$ is exact too. This implies that the first Ext group

$$\mathrm{Ext}^1_{QCoh(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

computed in the abelian category $QCoh(\mathcal{O}_X)$ vanishes for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. However, since $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ is closed under extensions (Schemes, Section 26.24) we see that Ext^1 between quasi-coherent modules computed in $QCoh(\mathcal{O}_X)$ is the same as computed in $Mod(\mathcal{O}_X)$. Hence we conclude that

$$H^1(X, \mathcal{H}) = \mathrm{Ext}^1_{Mod(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. This implies that X is affine for example by Cohomology of Schemes, Lemma 30.3.1. □

0GPJ **Proposition 56.6.6.** *Let X and Y be quasi-compact and quasi-separated schemes. If $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is an equivalence, then there exists an isomorphism $f : Y \to X$ of schemes and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.* Special case of [**Bra18**, Theorem 1.2]

Proof. Of course F is additive, exact, commutes with all limits, commutes with all colimits, commutes with direct sums, etc. Let $U \subset X$ be an affine open subscheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals such that $Z = V(\mathcal{I})$ is the complement of U in X, see Properties, Lemma 28.24.1. Then $\mathcal{O}_X/\mathcal{I}$ is a finitely presented $\mathcal{O}_X$-module. Hence $\mathcal{G} = F(\mathcal{O}_X/\mathcal{I})$ is a finitely presented $\mathcal{O}_Y$-module by Lemma 56.6.1. Denote $T \subset Y$ the support of $\mathcal{G}$ and set $V = Y \setminus T$. Since $\mathcal{G}$ is of finite presentation, the scheme V is a quasi-compact open of Y. By Lemma 56.6.3 we see that F induces an equivalence between

(1) the full subcategory of $QCoh(\mathcal{O}_X)$ consisting of modules supported on Z, and
(2) the full subcategory of $QCoh(\mathcal{O}_Y)$ consisting of modules supported on T.

By Lemma 56.6.4 we obtain a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_X) & \xrightarrow[F]{} & QCoh(\mathcal{O}_Y) \\ \downarrow & & \downarrow \\ QCoh(\mathcal{O}_U) & \xrightarrow{F_U} & QCoh(\mathcal{O}_V) \end{array}$$

where the vertical arrows are the restruction functors and the horizontal arrows are equivalences. By Lemma 56.6.5 we conclude that V is affine. For the affine case we have Lemma 56.3.8. Thus we find that there is an isomorphism $f_U : V \to U$ and an invertible $\mathcal{O}_V$-module $\mathcal{L}_U$ such that F_U is the functor $\mathcal{F} \mapsto f_U^*\mathcal{F} \otimes \mathcal{L}_U$.

The proof can be finished by noticing that the diagrams above satisfy an obvious compatibility with regards to inclusions of affine open subschemes of X. Thus the morphisms f_U and the invertible modules $\mathcal{L}_U$ glue. We omit the details. □

56.7. Functors between categories of coherent modules

0FZK The following lemma guarantees that we can use the material on functors between categories of quasi-coherent modules when we are given a functor between categories of coherent modules.

0FZL **Lemma 56.7.1.** *Let X and Y be Noetherian schemes. Let $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ be a functor. Then F extends uniquely to a functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commutes with filtered colimits. If F is additive, then its extension commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is its extension.*

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 4.26.2. To see that the lemma applies observe that coherent modules are of finite presentation (Modules, Lemma 17.12.2) and hence categorically compact objects of $Mod(\mathcal{O}_X)$ by Modules, Lemma 17.22.8. Finally, every quasi-coherent module is a filtered colimit of coherent ones for example by Properties, Lemma 28.22.3.

Assume F is additive. If $\mathcal{F} = \bigoplus_{j\in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \operatorname{colim}_{J'\subset J \text{ finite}} \bigoplus_{j\in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \operatorname{colim}_{J'\subset J \text{ finite}} F(\bigoplus\nolimits_{j\in J'} \mathcal{H}_j) \\ &= \operatorname{colim}_{J'\subset J \text{ finite}} \bigoplus\nolimits_{j\in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j\in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules, see Properties, Lemma 28.22.3. Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \operatorname{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \operatorname{colim}_{i\in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \operatorname{colim}_{i\in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \operatorname{colim}_{i\in I} F(\mathcal{F}''_i)$. Since filtered colimits are exact (Modules, Lemma 17.3.2) we conclude that exactness properties of F are inherited by its extension. □

0GPK **Lemma 56.7.2.** *Let X and Y be Noetherian schemes. Let $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ be an equivalence of categories. Then there is an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.*

Proof. By Lemma 56.7.1 we obtain a unique functor $F' : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ extending F. The same is true for the quasi-inverse of F and by the uniqueness we conclude that F' is an equivalence. By Proposition 56.6.6 we find an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F'(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Then f and $\mathcal{L}$ work for F as well. □

0GPL **Remark 56.7.3.** In Lemma 56.7.2 if X and Y are defined over a common base ring R and F is R-linear, then the isomorphism f will be a morphism of schemes over R.

0FZM **Lemma 56.7.4.** *Let $f : V \to X$ be a quasi-finite separated morphism of Noetherian schemes. If there exists a coherent $\mathcal{O}_V$-module $\mathcal{K}$ whose support is V such that $f_*\mathcal{K}$ is coherent and $R^q f_*\mathcal{K} = 0$, then f is finite.*

Proof. By Zariski's main theorem we can find an open immersion $j : V \to Y$ over X with $\pi : Y \to X$ finite, see More on Morphisms, Lemma 37.43.3. Since π is affine the functor π_* is exact and faithful on the category of coherent $\mathcal{O}_X$-modules. Hence we see that $j_*\mathcal{K}$ is coherent and that $R^q j_*\mathcal{K}$ is zero for $q > 0$. In other words, we reduce to the case discussed in the next paragraph.

Assume f is an open immersion. We may replace X by the scheme theoretic closure of V. Assume $X \setminus V$ is nonempty to get a contradiction. Choose a generic point $\xi \in X \setminus V$ of an irreducible component of $X \setminus V$. Looking at the situation after base change by $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to X$ using flat base change and using Local Cohomology, Lemma 51.8.2 we reduce to the algebra problem discussed in the next paragraph.

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite A-module whose support is $\mathrm{Spec}(A)$. Then $H^i_{\mathfrak{m}}(M) \neq 0$ for some i. This is true by Dualizing Complexes, Lemma 47.11.1 and the fact that M is not zero hence has finite depth. □

The next lemma can be generalized to the case where k is a Noetherian ring and X flat over k (all other assumptions stay the same).

0FZN **Lemma 56.7.5.** *Let k be a field. Let X, Y be finite type schemes over k with X separated. There is an equivalence of categories between*

1. *the category of k-linear exact functors $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$, and*
2. *the category of coherent $\mathcal{O}_{X\times Y}$-modules $\mathcal{K}$ which are flat over X and have support finite over Y*

given by sending $\mathcal{K}$ to the restriction of the functor (56.5.1.1) to $Coh(\mathcal{O}_X)$.

Proof. Let $\mathcal{K}$ be as in (2). By Lemma 56.5.7 the functor F given by (56.5.1.1) is exact and k-linear. Moreover, F sends $Coh(\mathcal{O}_X)$ into $Coh(\mathcal{O}_Y)$ for example by Cohomology of Schemes, Lemma 30.26.10.

Let us construct the quasi-inverse to the construction. Let F be as in (1). By Lemma 56.7.1 we can extend F to a k-linear exact functor on the categories of quasi-coherent modules which commutes with arbitrary direct sums. By Lemma 56.5.7 the extension corresponds to a unique quasi-coherent module $\mathcal{K}$, flat over X, such that $R^q\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{K}) = 0$ for $q > 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Since $F(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module, we conclude from Lemma 56.5.11 that $\mathcal{K}$ is coherent.

For a closed point $x \in X$ denote $\mathcal{O}_x$ the skyscraper sheaf at x with value the residue field of x. We have

$$F(\mathcal{O}_x) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{O}_x \otimes \mathcal{K}) = (x \times Y \to Y)_*(\mathcal{K}|_{x\times Y})$$

Since $x \times Y \to Y$ is finite, we see that the pushforward along this morphism is faithful. Hence if $y \in Y$ is in the image of the support of $\mathcal{K}|_{x\times Y}$, then y is in the support of $F(\mathcal{O}_x)$.

Let $Z \subset X \times Y$ be the scheme theoretic support Z of $\mathcal{K}$, see Morphisms, Definition 29.5.5. We first prove that $Z \to Y$ is quasi-finite, by proving that its fibres over closed points are finite. Namely, if the fibre of $Z \to Y$ over a closed point $y \in Y$ has dimension > 0, then we can find infinitely many pairwise distinct closed points $x_1, x_2, \ldots$ in the image of $Z_y \to X$. Since we have a surjection $\mathcal{O}_X \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_{x_i}$ we obtain a surjection

$$F(\mathcal{O}_X) \to \bigoplus\nolimits_{i=1,\ldots,n} F(\mathcal{O}_{x_i})$$

By what we said above, the point y is in the support of each of the coherent modules $F(\mathcal{O}_{x_i})$. Since $F(\mathcal{O}_X)$ is a coherent module, this will lead to a contradiction because the stalk of $F(\mathcal{O}_X)$ at y will be generated by $< n$ elements if n is large enough. Hence $Z \to Y$ is quasi-finite. Since $\mathrm{pr}_{2,*}\mathcal{K}$ is coherent and $R^q\mathrm{pr}_{2,*}\mathcal{K} = 0$ for $q > 0$ we conclude that $Z \to Y$ is finite by Lemma 56.7.4. □

0FZP **Lemma 56.7.6.** *Let $f : X \to Y$ be a finite type separated morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent module on X with support finite over Y and with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. Then there exists a section $s : Y \to X$ such that $\mathcal{F} \cong s_*\mathcal{L}$.*

Proof. Looking affine locally this translates into the following algebra problem. Let $A \to B$ be a ring map and let N be a B-module which is invertible as an A-module. Then the annihilator J of N in B has the property that $A \to B/J$ is an isomorphism. We omit the details. □

0FZQ **Lemma 56.7.7.** *Let $f : X \to Y$ be a finite type separated morphism of schemes with a section $s : Y \to X$. Let $\mathcal{F}$ be a finite type quasi-coherent module on X, set theoretically supported on $s(Y)$ with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. If Y is reduced, then $\mathcal{F} \cong s_*\mathcal{L}$.*

Proof. By Lemma 56.7.6 there exists a section $s' : Y \to X$ such that $\mathcal{F} = s'_*\mathcal{L}$. Since $s'(Y)$ and $s(Y)$ have the same underlying closed subset and since both are reduced closed subschemes of X, they have to be equal. Hence $s = s'$ and the lemma holds. □

0FZR **Lemma 56.7.8.** *Let k be a field. Let X, Y be finite type schemes over k with X separated and Y reduced. If there is a k-linear equivalence $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ of categories, then there is an isomorphism $f : Y \to X$ over k and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.*

Weak version of the result in [**Gab62**] stating that the category of quasi-coherent modules determines the isomorphism class of a scheme.

Proof using Gabriel-Rosenberg reconstruction. This lemma is a weak form of the results discussed in Lemma 56.7.2 and Remark 56.7.3. □

Proof not relying on Gabriel-Rosenberg reconstruction. By Lemma 56.7.5 we obtain a coherent $\mathcal{O}_{X\times Y}$-module $\mathcal{K}$ which is flat over X with support finite over Y such that F is given by the restriction of the functor (56.5.1.1) to $Coh(\mathcal{O}_X)$. If

we can show that $F(\mathcal{O}_X)$ is an invertible $\mathcal{O}_Y$-module, then by Lemma 56.7.6 we see that $\mathcal{K} = s_*\mathcal{L}$ for some section $s : Y \to X \times Y$ of pr_2 and some invertible $\mathcal{O}_Y$-module $\mathcal{L}$. This will show that F has the form indicated with $f = \mathrm{pr}_1 \circ s$. Some details omitted.

It remains to show that $F(\mathcal{O}_X)$ is invertible. We only sketch the proof and we omit some of the details. For a closed point $x \in X$ we denote $\mathcal{O}_x$ in $Coh(\mathcal{O}_X)$ the skyscraper sheaf at x with value $\kappa(x)$. First we observe that the only simple objects of the category $Coh(\mathcal{O}_X)$ are these skyscraper sheaves $\mathcal{O}_x$. The same is true for Y. Hence for every closed point $y \in Y$ there exists a closed point $x \in X$ such that $\mathcal{O}_y \cong F(\mathcal{O}_x)$. Moreover, looking at endomorphisms we find that $\kappa(x) \cong \kappa(y)$ as finite extensions of k. Then

$$\mathrm{Hom}_Y(F(\mathcal{O}_X), \mathcal{O}_y) \cong \mathrm{Hom}_Y(F(\mathcal{O}_X), F(\mathcal{O}_x)) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \kappa(x) \cong \kappa(y)$$

This implies that the stalk of the coherent $\mathcal{O}_Y$-module $F(\mathcal{O}_X)$ at $y \in Y$ can be generated by 1 generator (and no less) for each closed point $y \in Y$. It follows immediately that $F(\mathcal{O}_X)$ is locally generated by 1 element (and no less) and since Y is reduced this indeed tells us it is an invertible module. □

56.8. Other chapters

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CHAPTER 57

Derived Categories of Varieties

0FY0

57.1. Introduction

0FY1 In this chapter we continue the discussion started in Derived Categories of Schemes, Section 36.1. We will discuss Fourier-Mukai transforms, first studied by Mukai in [**Muk81**]. We will prove Orlov's theorem on derived equivalences ([**Orl97**]). We also discuss the countability of derived equivalence classes proved by Anel and Toën in [**AT09**].

A good introduction to this material is the book [**Huy06**] by Daniel Huybrechts. Some other papers which helped popularize this topic are

(1) the paper by Bondal and Kapranov, see [**BK89**]
(2) the paper by Bondal and Orlov, see [**BO01**]
(3) the paper by Bondal and Van den Bergh, see [**BV03**]
(4) the papers by Beilinson, see [**Bei78**] and [**Bei84**]
(5) the paper by Orlov, see [**Orl02**]
(6) the paper by Orlov, see [**Orl05**]
(7) the paper by Rouquier, see [**Rou08**]
(8) there are many more we could mention here.

57.2. Conventions and notation

0FY2 Let k be a field. A k-linear triangulated category $\mathcal{T}$ is a triangulated category (Derived Categories, Section 13.3) which is endowed with a k-linear structure (Differential Graded Algebra, Section 22.24) such that the translation functors $[n] : \mathcal{T} \to \mathcal{T}$ are k-linear for all $n \in \mathbf{Z}$.

Let k be a field. We denote Vect_k the category of k-vector spaces. For a k-vector space V we denote $V^\vee$ the k-linear dual of V, i.e., $V^\vee = \text{Hom}_k(V, k)$.

Let X be a scheme. We denote $D_{perf}(\mathcal{O}_X)$ the full subcategory of $D(\mathcal{O}_X)$ consisting of perfect complexes (Cohomology, Section 20.49). If X is Noetherian then $D_{perf}(\mathcal{O}_X) \subset D^b_{Coh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 36.11.6. If X is Noetherian and regular, then $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Lemma 36.11.8.

Let k be a field. Let X and Y be schemes over k. In this situation we will write $X \times Y$ instead of $X \times_{\text{Spec}(k)} Y$.

Let S be a scheme. Let X, Y be schemes over S. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module and let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. We set

$$\mathcal{F} \boxtimes \mathcal{G} = \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} \text{pr}_2^*\mathcal{G}$$

as $\mathcal{O}_{X\times_S Y}$-modules. If $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ then we set

$$K \boxtimes M = L\mathrm{pr}_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} L\mathrm{pr}_2^* M$$

as an object of $D(\mathcal{O}_{X\times_S Y})$. Thus our notation is potentially ambiguous, but context should make it clear which of the two is meant.

57.3. Serre functors

0FY3 The material in this section is taken from [**BK89**].

0FY4 **Lemma 57.3.1.** *Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$. The following are equivalent*

1. *there exists a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ and k-linear isomorphisms $c_{X,Y} : \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee$ functorial in $X, Y \in \mathrm{Ob}(\mathcal{T})$,*
2. *for every $X \in \mathrm{Ob}(\mathcal{T})$ the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$ is representable and the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$ is corepresentable.*

Proof. Condition (1) implies (2) since given (S, c) and $X \in \mathrm{Ob}(\mathcal{T})$ the object $S(X)$ represents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$ and the object $S^{-1}(X)$ corepresents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$.

Assume (2). We will repeatedly use the Yoneda lemma, see Categories, Lemma 4.3.5. For every X denote $S(X)$ the object representing the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$. Given $\varphi : X \to X'$, we obtain a unique arrow $S(\varphi) : S(X) \to S(X')$ determined by the corresponding transformation of functors $\mathrm{Hom}_{\mathcal{T}}(X, -)^\vee \to \mathrm{Hom}_{\mathcal{T}}(X', -)^\vee$. Thus S is a functor and we obtain the isomorphisms $c_{X,Y}$ by construction. It remains to show that S is an equivalence. For every X denote $S'(X)$ the object corepresenting the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(Y, X)^\vee$. Arguing as above we find that S' is a functor. We claim that S' is quasi-inverse to S. To see this observe that

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee = \mathrm{Hom}_{\mathcal{T}}(S'(S(X)), Y)$$

bifunctorially, i.e., we find $S' \circ S \cong \mathrm{id}_{\mathcal{T}}$. Similarly, we have

$$\mathrm{Hom}_{\mathcal{T}}(Y, X) = \mathrm{Hom}_{\mathcal{T}}(S'(X), Y)^\vee = \mathrm{Hom}_{\mathcal{T}}(Y, S(S'(X)))$$

and we find $S \circ S' \cong \mathrm{id}_{\mathcal{T}}$. □

0FY5 **Definition 57.3.2.** Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$. We say *a Serre functor exists* if the equivalent conditions of Lemma 57.3.1 are satisfied. In this case a *Serre functor* is a k-linear equivalence $S : \mathcal{T} \to \mathcal{T}$ endowed with k-linear isomorphisms $c_{X,Y} : \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(Y, S(X))^\vee$ functorial in $X, Y \in \mathrm{Ob}(\mathcal{T})$.

0H8B **Remark 57.3.3.** Let $X^0 \to X^1 \to X^2 \to X^0[1]$ and $Y^0 \to Y^1 \to Y^2 \to Y^0[1]$ be distinguished triangles in a triangulated category. For $p \in \mathbf{Z}$ write $p = 3n + i$ with $i \in \{0, 1, 2\}$ and set we set $X^p = X^i[n]$. Simlarly for Y^q. Consider the double complex with terms

$$K^{p,q} = \mathrm{Hom}(X^{-p}, Y^q)$$

The differential $d_1 : K^{p,q} \to K^{p+1,q}$ is given by the map $X^{-p-1} \to X^{-p}$ (equal to the corresponding map in the first distinguished triangle up the a shift) and the differential $d_2 : K^{p,q} \to K^{p,q+1}$ likewise by the map $Y^q \to Y^{q+1}$. From Derived Categories, Lemma 13.4.2 we see that the rows and columns of this double complex

are exact. Furthermore, we see that $K^{p,q} = K^{p+3,q-3}$ and these equalities are compatible with the differentials. Finally, axiom TR3 implies one additional property: given $\alpha \in K^{p,q}$ and $\beta \in K^{p-1,q+1}$ such that $d_2\alpha = d_1\beta$, there exists a $\gamma \in K^{p-2,q+2}$ such that $d_1\gamma = d_2\beta$ in $K^{p-1,q+2}$ and $d_2\gamma = d_1\alpha$ in $K^{p-2,q+3} = K^{p+1,q}$. (Hint: for $p = q = 0$ this is exactly the statement of TR3 and for other indices prove it by shifting.) A double complex with these properties is called a *matress* (see [**BK89**]).

0H8C **Remark 57.3.4.** Let k be a field and let $K^{\bullet,\bullet}$ be a double complex of finite dimensional k-vector spaces which is also a matress as in Remark 57.3.3. We claim that the dual double complex $L^{p,q} = \mathrm{Hom}_k(K^{-q,-p}, k)$ is also a matress. The exactness of rows and columns and the 3-periodicity are immediate. To see the additional condition holds, consider the linear map

$$\partial : K^{p,q} \oplus K^{p-1,q+1} \oplus K^{p-2,q+2} \longrightarrow K^{p,q+1} \oplus K^{p-1,q+2} \oplus K^{p-2,q+3}$$

sending (α, β, γ) to $(d_2\alpha - d_1\beta, d_2\gamma - d_1\alpha, d_1\gamma - d_2\beta)$ with identifications as in Remark 57.3.3. The condition of being a matress is that

$$\mathrm{Im}(\partial) \cap \big(0 \oplus K^{p-1,q+2} \oplus K^{p-2,q+3}\big) = \mathrm{Im}(\partial|_{0\oplus 0\oplus K^{p-2,q+2}})$$

Denoting $^\wedge$ the dual of a vector space or map, taking duals we get

$$\mathrm{Ker}(\partial^\wedge) + \big(L^{-p,-q-1} \oplus 0 \oplus 0\big) = \mathrm{Ker}(\mathrm{pr}_{L^{-p+2,-q-2}} \circ \partial^\wedge)$$

where the pr term indicates projection. This implies that $\mathrm{Im}(\partial^\wedge) \cap (L^{-p,-q} \oplus L^{-p+1,-q-1} \oplus 0)$ is equal to $\mathrm{Im}(\partial^\wedge|_{L^{-p,-q-1}\oplus 0\oplus 0})$ and this translates into the matress condition for $L^{\bullet,\bullet}$. Some details omitted.

0FY6 **Lemma 57.3.5.** *In the situation of Definition 57.3.2. If a Serre functor exists, then it is unique up to unique isomorphism and it is an exact functor of triangulated categories.*

Proof. Given a Serre functor S the object $S(X)$ represents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee$. Thus the object $S(X)$ together with the functorial identification $\mathrm{Hom}_{\mathcal{T}}(X, Y)^\vee = \mathrm{Hom}_{\mathcal{T}}(Y, S(X))$ is determined up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5). Moreover, for $\varphi : X \to X'$, the arrow $S(\varphi) : S(X) \to S(X')$ is uniquely determined by the corresponding transformation of functors $\mathrm{Hom}_{\mathcal{T}}(X, -)^\vee \to \mathrm{Hom}_{\mathcal{T}}(X', -)^\vee$.

For objects X, Y of $\mathcal{T}$ we have

$$\begin{aligned}
\mathrm{Hom}(Y, S(X)[1])^\vee &= \mathrm{Hom}(Y[-1], S(X))^\vee \\
&= \mathrm{Hom}(X, Y[-1]) \\
&= \mathrm{Hom}(X[1], Y) \\
&= \mathrm{Hom}(Y, S(X[1]))^\vee
\end{aligned}$$

By the Yoneda lemma we conclude that there is a unique isomorphism $S(X[1]) \to S(X)[1]$ inducing the isomorphism from top left to bottom right. Since each of the isomorphisms above is functorial in both X and Y we find that this defines an isomorphism of functors $S \circ [1] \to [1] \circ S$.

Let (A, B, C, f, g, h) be a distinguished triangle in $\mathcal{T}$. We have to show that the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ is distinguished. Here we use the canonical isomorphism $S(A[1]) \to S(A)[1]$ constructed above to identify the target

$S(A[1])$ of $S(h)$ with $S(A)[1]$. We first observe that for any X in $\mathcal{T}$ the triangle $(S(A), S(B), S(C), S(f), S(g), S(h))$ induces a long exact sequence

$$\ldots \to \mathrm{Hom}(X, S(A)) \to \mathrm{Hom}(X, S(B)) \to \mathrm{Hom}(X, S(C)) \to \mathrm{Hom}(X, S(A)[1]) \to \ldots$$

of finite dimensional k-vector spaces. Namely, this sequence is k-linear dual of the sequence

$$\ldots \leftarrow \mathrm{Hom}(A, X) \leftarrow \mathrm{Hom}(B, X) \leftarrow \mathrm{Hom}(C, X) \leftarrow \mathrm{Hom}(A[1], X) \leftarrow \ldots$$

which is exact by Derived Categories, Lemma 13.4.2. Next, we choose a distinguished triangle $(S(A), E, S(C), i, p, S(h))$ which is possible by axioms TR1 and TR2. We want to construct the dotted arrow making following diagram commute

$$\begin{array}{ccccccccc}
S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{S(f)} & S(B) & \xrightarrow{S(g)} & S(C) & \xrightarrow{S(h)} & S(A)[1] \\
\| & & \| & & \uparrow{\scriptstyle \varphi} & & \| & & \| \\
S(C)[-1] & \xrightarrow{S(h[-1])} & S(A) & \xrightarrow{i} & E & \xrightarrow{p} & S(C) & \xrightarrow{S(h)} & S(A)[1]
\end{array}$$

Namely, if we have φ, then we claim for any X the resulting map $\mathrm{Hom}(X, E) \to \mathrm{Hom}(X, S(B))$ will be an isomorphism of k-vector spaces. Namely, we will obtain a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}(X, S(C)[-1]) & \longrightarrow & \mathrm{Hom}(X, S(A)) & \longrightarrow & \mathrm{Hom}(X, S(B)) & \longrightarrow & \mathrm{Hom}(X, S(C)) & \longrightarrow & \mathrm{Hom}(X, S(A)[1]) \\
\| & & \| & & \uparrow{\scriptstyle \varphi} & & \| & & \| \\
\mathrm{Hom}(X, S(C)[-1]) & \longrightarrow & \mathrm{Hom}(X, S(A)) & \longrightarrow & \mathrm{Hom}(X, E) & \longrightarrow & \mathrm{Hom}(X, S(C)) & \longrightarrow & \mathrm{Hom}(X, S(A)[1])
\end{array}$$

with exact rows (see above) and we can apply the 5 lemma (Homology, Lemma 12.5.20) to see that the middle arrow is an isomorphism. By the Yoneda lemma we conclude that φ is an isomorphism.

To find φ, let us form the matress of Remark 57.3.3 from the distinguished triangles $A \to B \to C \to A[1]$ and $S(A) \to E \to S(C) \to S(A)[1]$. By Remark 57.3.4 the k-linear dual of this is also a matress. Using that S is a Serre functor, we see as above that this dual matress is the double complex made from the triangles (the second of which we do not yet know to be distinguished) $S(A) \to E \to S(C) \to S(A)[1]$ and $S(A) \to S(B) \to S(C) \to S(A)[1]$. But the matress condition exactly tells us that the maps $S(A) \to S(A)$ and $S(C) \to S(C)$ fit into a morphism of triangles, i.e., that we have a φ making the diagram above commute. □

57.4. Examples of Serre functors

0FY7 The lemma below is the standard example.

0FY8 **Lemma 57.4.1.** *Let k be a field. Let X be a proper scheme over k which is Gorenstein. Consider the complex $\omega_X^\bullet$ of Duality for Schemes, Lemmas 48.27.1. Then the functor*

$$S : D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X), \quad K \longmapsto S(K) = \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$$

is a Serre functor.

Proof. The statement make sense because $\dim \mathrm{Hom}_X(K, L) < \infty$ for $K, L \in D_{perf}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.7. Since X is Gorenstein the dualizing complex $\omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$, see Duality for

Schemes, Lemma 48.24.4. In particular, locally on X the complex $\omega_X^\bullet$ has one nonzero cohomology sheaf which is an invertible module, see Cohomology, Lemma 20.52.2. Thus $S(K)$ lies in $D_{perf}(\mathcal{O}_X)$. On the other hand, the invertibility of $\omega_X^\bullet$ clearly implies that S is a self-equivalence of $D_{perf}(\mathcal{O}_X)$. Finally, we have to find an isomorphism

$$c_{K,L} : \mathrm{Hom}_X(K, L) \longrightarrow \mathrm{Hom}_X(L, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} K)^\vee$$

bifunctorially in K, L. To do this we use the canonical isomorphisms

$$\mathrm{Hom}_X(K, L) = H^0(X, L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee)$$

and

$$\mathrm{Hom}_X(L, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} K) = H^0(X, \omega_X^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee)$$

given in Cohomology, Lemma 20.50.5. Since $(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee)^\vee = (K^\vee)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee$ and since there is a canonical isomorphism $K \to (K^\vee)^\vee$ we find these k-vector spaces are canonically dual by Duality for Schemes, Lemma 48.27.4. This produces the isomorphisms $c_{K,L}$. We omit the proof that these isomorphisms are functorial. □

57.5. Characterizing coherent modules

0FY9 This section is in some sense a continuation of the discussion in Derived Categories of Schemes, Section 36.34 and More on Morphisms, Section 37.69.

Before we can state the result we need some notation. Let k be a field. Let $n \geq 0$ be an integer. Let $S = k[X_0, \ldots, X_n]$. For an integer e denote $S_e \subset S$ the homogeneous polynomials of degree e. Consider the (noncommutative) k-algebra

$$R = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix}$$

(with $n + 1$ rows and columns) with obvious multiplication and addition.

0FYA **Lemma 57.5.1.** *With k, n, and R as above, for an object K of $D(R)$ the following are equivalent*

(1) $\sum_{i \in \mathbf{Z}} \dim_k H^i(K) < \infty$, *and*
(2) K *is a compact object.*

Proof. If K is a compact object, then K can be represented by a complex $M^\bullet$ which is finite projective as a graded R-module, see Differential Graded Algebra, Lemma 22.36.6. Since $\dim_k R < \infty$ we conclude $\sum \dim_k M^i < \infty$ and a fortiori $\sum \dim_k H^i(M^\bullet) < \infty$. (One can also easily deduce this implication from the easier Differential Graded Algebra, Proposition 22.36.4.)

Assume K satisfies (1). Consider the distinguished triangle of trunctions $\tau_{\leq m} K \to K \to \tau_{\geq m+1} K$, see Derived Categories, Remark 13.12.4. It is clear that both $\tau_{\leq m} K$ and $\tau_{\geq m+1} K$ satisfy (1). If we can show both are compact, then so is K, see Derived Categories, Lemma 13.37.2. Hence, arguing on the number of nonzero cohomology modules of K we may assume $H^i(K)$ is nonzero only for one i. Shifting, we may assume K is given by the complex consisting of a single finite dimensional R-module M sitting in degree 0.

Since $\dim_k(M) < \infty$ we see that M is Artinian as an R-module. Thus it suffices to show that every simple R-module represents a compact object of $D(R)$. Observe that

$$I = \begin{pmatrix} 0 & S_1 & S_2 & \ldots & \ldots \\ 0 & 0 & S_1 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

is a nilpotent two sided ideal of R and that R/I is a commutative k-algebra isomorphic to a product of $n+1$ copies of k (placed along the diagonal in the matrix, i.e., R/I can be lifted to a k-subalgebra of R). It follows that R has exactly $n+1$ isomorphism classes of simple modules $M_0, \ldots, M_n$ (sitting along the diagonal). Consider the right R-module P_i of row vectors

$$P_i = \begin{pmatrix} 0 & \ldots & 0 & S_0 & \ldots & S_{i-1} & S_i \end{pmatrix}$$

with obvious multiplication $P_i \times R \to P_i$. Then we see that $R \cong P_0 \oplus \ldots \oplus P_n$ as a right R-module. Since clearly R is a compact object of $D(R)$, we conclude each P_i is a compact object of $D(R)$. (We of course also conclude each P_i is projective as an R-module, but this isn't what we have to show in this proof.) Clearly, $P_0 = M_0$ is the first of our simple R-modules. For P_1 we have a short exact sequence

$$0 \to P_0^{\oplus n+1} \to P_1 \to M_1 \to 0$$

which proves that M_1 fits into a distinguished triangle whose other members are compact objects and hence M_1 is a compact object of $D(R)$. More generally, there exists a short exact sequence

$$0 \to C_i \to P_i \to M_i \to 0$$

where C_i is a finite dimensional R-module whose simple constituents are isomorphic to M_j for $j < i$. By induction, we first conclude that C_i determines a compact object of $D(R)$ whereupon we conclude that M_i does too as desired. □

0FYB **Lemma 57.5.2.** *Let k be a field. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$. The following are equivalent*

1. *K is in $D^b_{Coh}(\mathcal{O}_{\mathbf{P}^n_k})$,*
2. *$\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}^n_k, E \otimes^{\mathbf{L}} K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$,*
3. *$\sum_{i \in \mathbf{Z}} \dim_k \mathrm{Ext}^i_{\mathbf{P}^n_k}(E, K) < \infty$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$,*
4. *$\sum_{i \in \mathbf{Z}} \dim_k H^i(\mathbf{P}^n_k, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_k}(d)) < \infty$ for $d = 0, 1, \ldots, n$.*

Proof. Parts (2) and (3) are equivalent by Cohomology, Lemma 20.50.5. If (1) is true, then for E perfect the derived tensor product $E \otimes^{\mathbf{L}} K$ is in $D^b_{Coh}(\mathcal{O}_{\mathbf{P}^n_k})$ and we see that (2) holds by Derived Categories of Schemes, Lemma 36.11.3. It is clear that (2) implies (4) as $\mathcal{O}_{\mathbf{P}^n_k}(d)$ can be viewed as a perfect object of the derived category of $\mathbf{P}^n_k$. Thus it suffices to prove that (4) implies (1).

Assume (4). Let R be as in Lemma 57.5.1. Let $P = \bigoplus_{d=0,\ldots,n} \mathcal{O}_{\mathbf{P}^n_k}(-d)$. Recall that $R = \mathrm{End}_{\mathbf{P}^n_k}(P)$ whereas all other self-Exts of P are zero and that P determines an equivalence $- \otimes^{\mathbf{L}} P : D(R) \to D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$ by Derived Categories of Schemes,

Lemma 36.20.1. Say K corresponds to L in $D(R)$. Then

$$\begin{aligned} H^i(L) &= \mathrm{Ext}^i_{D(R)}(R, L) \\ &= \mathrm{Ext}^i_{\mathbf{P}^n_k}(P, K) \\ &= H^i(\mathbf{P}^n_k, K \otimes P^\vee) \\ &= \bigoplus\nolimits_{d=0,\ldots,n} H^i(\mathbf{P}^n_k, K \otimes \mathcal{O}(d)) \end{aligned}$$

by Differential Graded Algebra, Lemma 22.35.4 (and the fact that $- \otimes^{\mathbf{L}} P$ is an equivalence) and Cohomology, Lemma 20.50.5. Thus our assumption (4) implies that L satisfies condition (2) of Lemma 57.5.1 and hence is a compact object of $D(R)$. Therefore K is a compact object of $D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_k})$. Thus K is perfect by Derived Categories of Schemes, Proposition 36.17.1. Since $D_{perf}(\mathcal{O}_{\mathbf{P}^n_k}) = D^b_{Coh}(\mathcal{O}_{\mathbf{P}^n_k})$ by Derived Categories of Schemes, Lemma 36.11.8 we conclude (1) holds. □

0FYC **Lemma 57.5.3.** *Let X be a scheme proper over a field k. Let $K \in D^b_{Coh}(\mathcal{O}_X)$ and let E in $D(\mathcal{O}_X)$ be perfect. Then $\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Ext}^i_X(E, K) < \infty$.*

Proof. This follows for example by combining Derived Categories of Schemes, Lemmas 36.11.7 and 36.13.5. Alternative proof: combine Derived Categories of Schemes, Lemmas 36.11.6 and 36.11.3. □

0FYD **Lemma 57.5.4.** *Let X be a proper scheme over a field k. Let $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent*

(1) *$K \in D^b_{Coh}(\mathcal{O}_X)$, and*
(2) *$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Ext}^i_X(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.*

In the projective case this is [**Rou08**, Lemma 7.46] and implicit in [**BV03**, Theorem A.1]

Proof. The implication (1) ⇒ (2) follows from Lemma 57.5.3. The implication (2) ⇒ (1) follows from More on Morphisms, Lemma 37.69.6 (see Derived Categories of Schemes, Example 36.35.2 for the meaning of a relatively perfect object over a field); the easier proof in the projective case is in the next paragraph.

Assume (2) and X projective over k. Choose a closed immersion $i : X \to \mathbf{P}^n_k$. It suffices to show that Ri_*K is in $D^b_{Coh}(\mathbf{P}^n_k)$ since a quasi-coherent module $\mathcal{F}$ on X is coherent, resp. zero if and only if $i_*\mathcal{F}$ is coherent, resp. zero. For a perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_k})$, Li^*E is a perfect object of $D(\mathcal{O}_X)$ and

$$\mathrm{Ext}^q_{\mathbf{P}^n_k}(E, Ri_*K) = \mathrm{Ext}^q_X(Li^*E, K)$$

Hence by our assumption we see that $\sum_{q\in\mathbf{Z}} \dim_k \mathrm{Ext}^q_{\mathbf{P}^n_k}(E, Ri_*K) < \infty$. We conclude by Lemma 57.5.2. □

57.6. A representability theorem

0FYE The material in this section is taken from [**BV03**].

Let $\mathcal{T}$ be a k-linear triangulated category. In this section we consider k-linear cohomological functors H from $\mathcal{T}$ to the category of k-vector spaces. This will mean H is a functor

$$H : \mathcal{T}^{opp} \longrightarrow \mathrm{Vect}_k$$

which is k-linear such that for any distinguished triangle $X \to Y \to Z$ in $\mathcal{T}$ the sequence $H(Z) \to H(Y) \to H(X)$ is an exact sequence of k-vector spaces. See Derived Categories, Definition 13.3.5 and Differential Graded Algebra, Section 22.24.

0FYF **Lemma 57.6.1.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Let $X \in \mathrm{Ob}(\mathcal{D})$. The category of arrows $E \to X$ with $E \in \mathrm{Ob}(\mathcal{D}')$ is filtered.*

Proof. We check the conditions of Categories, Definition 4.19.1. The category is nonempty because it contains $0 \to X$. If $E_i \to X$, $i = 1, 2$ are objects, then $E_1 \oplus E_2 \to X$ is an object and there are morphisms $(E_i \to X) \to (E_1 \oplus E_2 \to X)$. Finally, suppose that $a, b : (E \to X) \to (E' \to X)$ are morphisms. Choose a distinguished triangle $E \xrightarrow{a-b} E' \to E''$ in $\mathcal{D}'$. By Axiom TR3 we obtain a morphism of triangles

$$\begin{array}{ccccc} E & \xrightarrow[a-b]{} & E' & \longrightarrow & E'' \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & X \end{array}$$

and we find that the resulting arrow $(E' \to X) \to (E'' \to X)$ equalizes a and b. □

0FYG **Lemma 57.6.2.** *Let k be a field. Let $\mathcal{D}$ be a k-linear triangulated category which has direct sums and is compactly generated. Denote $\mathcal{D}_c$ the full subcategory of compact objects. Let $H : \mathcal{D}_c^{opp} \to Vect_k$ be a k-linear cohomological functor such that $\dim_k H(X) < \infty$ for all $X \in \mathrm{Ob}(\mathcal{D}_c)$. Then H is isomorphic to the functor $X \mapsto \mathrm{Hom}(X, Y)$ for some $Y \in \mathrm{Ob}(\mathcal{D})$.* [**CKN01**, Lemma 2.14]

Proof. We will use Derived Categories, Lemma 13.37.2 without further mention. Denote $G : \mathcal{D}_c \to \mathrm{Vect}_k$ the k-linear homological functor which sends X to $H(X)^\vee$. For any object Y of $\mathcal{D}$ we set

$$G'(Y) = \mathrm{colim}_{X \to Y, X \in \mathrm{Ob}(\mathcal{D}_c)} G(X)$$

The colimit is filtered by Lemma 57.6.1. We claim that G' is a k-linear homological functor, the restriction of G' to $\mathcal{D}_c$ is G, and G' sends direct sums to direct sums.

Namely, suppose that $Y_1 \to Y_2 \to Y_3$ is a distinguished triangle. Let $\xi \in G'(Y_2)$ map to zero in $G'(Y_3)$. Since the colimit is filtered ξ is represented by some $X \to Y_2$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$. The fact that ξ maps to zero in $G'(Y_3)$ means the composition $X \to Y_2 \to Y_3$ factors as $X \to X' \to Y_3$ with $X' \in \mathcal{D}_c$ and g mapping to zero in $G(X')$. Choose a distinguished triangle $X'' \to X \to X'$. Then $X'' \in \mathrm{Ob}(\mathcal{D}_c)$. Since G is homological we find that g is the image of some $g'' \in G'(X'')$. By Axiom TR3 the maps $X \to Y_2$ and $X' \to Y_3$ fit into a morphism of distinguished triangles $(X'' \to X \to X') \to (Y_1 \to Y_2 \to Y_3)$ and we find that indeed ξ is the image of the element of $G'(Y_1)$ represented by $X'' \to Y_1$ and $g'' \in G(X'')$.

If $Y \in \mathrm{Ob}(\mathcal{D}_c)$, then $\mathrm{id} : Y \to Y$ is the final object in the category of arrows $X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$. Hence we see that $G'(Y) = G(Y)$ in this case and the statement on restriction holds. Let $Y = \bigoplus_{i \in I} Y_i$ be a direct sum. Let $a : X \to Y$ with $X \in \mathrm{Ob}(\mathcal{D}_c)$ and $g \in G(X)$ represent an element ξ of $G'(Y)$. The morphism $a : X \to Y$ can be uniquely written as a sum of morphisms $a_i : X \to Y_i$ almost all zero as X is a compact object of $\mathcal{D}$. Let $I' = \{i \in I \mid a_i \neq 0\}$. Then we can factor a as the composition

$$X \xrightarrow{(1,\ldots,1)} \bigoplus\nolimits_{i \in I'} X \xrightarrow{\bigoplus_{i \in I'} a_i} \bigoplus\nolimits_{i \in I} Y_i = Y$$

We conclude that $\xi = \sum_{i \in I'} \xi_i$ is the sum of the images of the elements $\xi_i \in G'(Y_i)$ corresponding to $a_i : X \to Y_i$ and $g \in G(X)$. Hence $\bigoplus G'(Y_i) \to G'(Y)$ is surjective. We omit the (trivial) verification that it is injective.

It follows that the functor $Y \mapsto G'(Y)^\vee$ is cohomological and sends direct sums to direct products. Hence by Brown representability, see Derived Categories, Proposition 13.38.2 we conclude that there exists a $Y \in \mathrm{Ob}(\mathcal{D})$ and an isomorphism $G'(Z)^\vee = \mathrm{Hom}(Z, Y)$ functorially in Z. For $X \in \mathrm{Ob}(\mathcal{D}_c)$ we have $G'(X)^\vee = G(X)^\vee = (H(X)^\vee)^\vee = H(X)$ because $\dim_k H(X) < \infty$ and the proof is complete. □

0FYH **Theorem 57.6.3.** *Let X be a proper scheme over a field k. Let $F : D_{perf}(\mathcal{O}_X)^{opp} \to Vect_k$ be a k-linear cohomological functor such that*

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(E, K)$ for some $K \in D^b_{Coh}(\mathcal{O}_X)$.

In the projective case this is [**BV03**, Theorem A.1]

Proof. The derived category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{perf}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 36.3.1, Theorem 36.15.3, and Proposition 36.17.1. By Lemma 57.6.2 we may assume $F(E) = \mathrm{Hom}_X(E, K)$ for some $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. Then it follows that K is in $D^b_{Coh}(\mathcal{O}_X)$ by Lemma 57.5.4. □

0H4A **Lemma 57.6.4.** *Let X be a proper scheme over a field k which is regular. Let $G : D_{perf}(\mathcal{O}_X) \to Vect_k$ be a k-linear homological functor such that*

$$\sum\nolimits_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(K, E)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

Proof. Consider the contravariant functor $E \mapsto E^\vee$ on $D_{perf}(\mathcal{O}_X)$, see Cohomology, Lemma 20.50.5. This functor is an exact anti-self-equivalence of $D_{perf}(\mathcal{O}_X)$. Hence we may apply Theorem 57.6.3 to the functor $F(E) = G(E^\vee)$ to find $K \in D_{perf}(\mathcal{O}_X)$ such that $G(E^\vee) = \mathrm{Hom}_X(E, K)$. It follows that $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$ and we conclude that taking $K^\vee$ works. □

57.7. Existence of adjoints

0FYM As a consequence of the results in the paper of Bondal and van den Bergh we get the following automatic existence of adjoints.

0FYN **Lemma 57.7.1.** *Let k be a field. Let X and Y be proper schemes over k. If X is regular, then any k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ has an exact right adjoint and an exact left adjoint.*

Proof. If an adjoint exists it is an exact functor by the very general Derived Categories, Lemma 13.7.1.

Let us prove the existence of a right adjoint. To see existence, it suffices to show that for $M \in D_{perf}(\mathcal{O}_Y)$ the contravariant functor $K \mapsto \mathrm{Hom}_Y(F(K), M)$ is representable. This functor is contravariant, k-linear, and cohomological. Hence by

Theorem 57.6.3 it suffices to show that

$$\sum\nolimits_{i\in\mathbf{Z}} \dim_k \mathrm{Ext}^i_Y(F(K), M) < \infty$$

This follows from Lemma 57.5.3.

For the existence of the left adjoint we argue in the same manner using Lemma 57.6.4 in stead of Theorem 57.6.3. □

57.8. Fourier-Mukai functors

0FYP These functors were first introduced in [**Muk81**].

0FYQ **Definition 57.8.1.** Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X\times_S Y})$. The exact functor

$$\Phi_K : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y), \quad M \longmapsto R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K)$$

of triangulated categories is called a *Fourier-Mukai functor* and K is called a *Fourier-Mukai kernel* for this functor. Moreover,

(1) if Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor,
(2) if Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor, and
(3) if X and Y are Noetherian and Φ_K sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ then the resulting exact functor $\Phi_K : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ is called a Fourier-Mukai functor. Similarly for D_{Coh}, D^+_{Coh}, D^-_{Coh}.

0FYR **Lemma 57.8.2.** *Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X\times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ if K is in $D_{QCoh}(\mathcal{O}_{X\times_S Y})$ and $X \to S$ is quasi-compact and quasi-separated.*

Proof. This follows from the fact that derived pullback preserves D_{QCoh} (Derived Categories of Schemes, Lemma 36.3.8), derived tensor products preserve D_{QCoh} (Derived Categories of Schemes, Lemma 36.3.9), the projection $\mathrm{pr}_2 : X \times_S Y \to Y$ is quasi-compact and quasi-separated (Schemes, Lemmas 26.19.3 and 26.21.12), and total direct image along a quasi-separated and quasi-compact morphism preserves D_{QCoh} (Derived Categories of Schemes, Lemma 36.4.1). □

0FYS **Lemma 57.8.3.** *Let S be a scheme. Let X, Y, Z be schemes over S. Assume $X \to S$, $Y \to S$, and $Z \to S$ are quasi-compact and quasi-separated. Let $K \in D_{QCoh}(\mathcal{O}_{X\times_S Y})$. Let $K' \in D_{QCoh}(\mathcal{O}_{Y\times_S Z})$. Consider the Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ and $\Phi_{K'} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Z)$. If X and Z are tor independent over S and $Y \to S$ is flat, then*

$$\Phi_{K'} \circ \Phi_K = \Phi_{K''} : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_Z)$$

where

$$K'' = Rpr_{13,*}(Lpr_{12}^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S Z}} Lpr_{23}^* K')$$

in $D_{QCoh}(\mathcal{O}_{X\times_S Z})$.

Proof. The statement makes sense by Lemma 57.8.2. We are going to use Derived Categories of Schemes, Lemmas 36.3.8, 36.3.9, and 36.4.1 and Schemes, Lemmas 26.19.3 and 26.21.12 without further mention. By Derived Categories of Schemes, Lemma 36.22.4 we see that $X \times_S Y$ and $Y \times_S Z$ are tor independent over Y. This means that we have base change for the cartesian diagram

$$\begin{array}{ccc} X \times_S Y \times_S Z & \longrightarrow & Y \times_S Z \\ \downarrow & & \downarrow p_Y^{YZ} \\ X \times_S Y & \xrightarrow{p_Y^{XY}} & Y \end{array}$$

for complexes with quasi-coherent cohomology sheaves, see Derived Categories of Schemes, Lemma 36.22.5. Abbreviating $p^* = Lp^*$, $p_* = Rp_*$ and $\otimes = \otimes^{\mathbf{L}}$ we have for $M \in D_{QCoh}(\mathcal{O}_X)$ the sequence of equalities

$$\begin{aligned} \Phi_{K'}(\Phi_K(M)) &= p_{Z,*}^{YZ}(p_Y^{YZ,*} p_{Y,*}^{XY}(p_X^{XY,*} M \otimes K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}\mathrm{pr}_{12}^*(p_X^{XY,*} M \otimes K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^* M \otimes \mathrm{pr}_{12}^* K) \otimes K') \\ &= p_{Z,*}^{YZ}(\mathrm{pr}_{23,*}(\mathrm{pr}_1^* M \otimes \mathrm{pr}_{12}^* K \otimes \mathrm{pr}_{23}^* K')) \\ &= \mathrm{pr}_{3,*}(\mathrm{pr}_1^* M \otimes \mathrm{pr}_{12}^* K \otimes \mathrm{pr}_{23}^* K') \\ &= p_{Z,*}^{XZ}\mathrm{pr}_{13,*}(\mathrm{pr}_1^* M \otimes \mathrm{pr}_{12}^* K \otimes \mathrm{pr}_{23}^* K') \\ &= p_{Z,*}^{XZ}(p_X^{XZ,*} M \otimes \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^* K \otimes \mathrm{pr}_{23}^* K')) \end{aligned}$$

as desired. Here we have used the remark on base change in the second equality and we have use Derived Categories of Schemes, Lemma 36.22.1 in the 4th and last equality. □

0FYT **Lemma 57.8.4.** *Let S be a scheme. Let X and Y be schemes over S. Let $K \in D(\mathcal{O}_{X \times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D_{perf}(\mathcal{O}_X)$ into $D_{perf}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:*

1. *S is Noetherian, $X \to S$ and $Y \to S$ are of finite type, $K \in D^b_{Coh}(\mathcal{O}_{X \times_S Y})$, the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(pr_2^{-1}\mathcal{O}_Y)$,*
2. *$X \to S$ is of finite presentation and K can be represented by a bounded complex $\mathcal{K}^\bullet$ of finitely presented $\mathcal{O}_{X \times_S Y}$-modules, flat over Y, with support proper over Y,*
3. *$X \to S$ is a proper flat morphism of finite presentation and K is perfect,*
4. *S is Noetherian, $X \to S$ is flat and proper, and K is perfect*
5. *$X \to S$ is a proper flat morphism of finite presentation and K is Y-perfect,*
6. *S is Noetherian, $X \to S$ is flat and proper, and K is Y-perfect.*

Proof. If M is perfect on X, then $L\mathrm{pr}_1^* M$ is perfect on $X \times_S Y$, see Cohomology, Lemma 20.49.6. We will use this without further mention below. We will also use that if $X \to S$ is of finite type, or proper, or flat, or of finite presentation, then the same thing is true for the base change $\mathrm{pr}_2 : X \times_S Y \to Y$, see Morphisms, Lemmas 29.15.4, 29.41.5, 29.25.8, and 29.21.4.

Part (1) follows from Derived Categories of Schemes, Lemma 36.27.1 combined with Derived Categories of Schemes, Lemma 36.11.6.

Part (2) follows from Derived Categories of Schemes, Lemma 36.30.1.

Part (3) follows from Derived Categories of Schemes, Lemma 36.30.4.

Part (4) follows from part (3) and the fact that a finite type morphism of Noetherian schemes is of finite presentation by Morphisms, Lemma 29.21.9.

Part (5) follows from Derived Categories of Schemes, Lemma 36.35.10 combined with Derived Categories of Schemes, Lemma 36.35.5.

Part (6) follows from part (5) in the same way that part (4) follows from part (3). □

0FYU **Lemma 57.8.5.** *Let S be a Noetherian scheme. Let X and Y be schemes of finite type over S. Let $K \in D^b_{Coh}(\mathcal{O}_{X\times_S Y})$. The corresponding Fourier-Mukai functor Φ_K sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ if at least one of the following conditions is satisfied:*

1. *the support of $H^i(K)$ is proper over Y for all i, and K has finite tor dimension as an object of $D(pr_1^{-1}\mathcal{O}_X)$,*
2. *K can be represented by a bounded complex $\mathcal{K}^\bullet$ of coherent $\mathcal{O}_{X\times_S Y}$-modules, flat over X, with support proper over Y,*
3. *the support of $H^i(K)$ is proper over Y for all i and X is a regular scheme,*
4. *K is perfect, the support of $H^i(K)$ is proper over Y for all i, and $Y \to S$ is flat.*

Furthermore in each case the support condition is automatic if $X \to S$ is proper.

Proof. Let M be an object of $D^b_{Coh}(\mathcal{O}_X)$. In each case we will use Derived Categories of Schemes, Lemma 36.11.3 to show that

$$\Phi_K(M) = R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K)$$

is in $D^b_{Coh}(\mathcal{O}_Y)$. The derived tensor product $L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K$ is a pseudo-coherent object of $D(\mathcal{O}_{X\times_S Y})$ (by Cohomology, Lemma 20.47.3, Derived Categories of Schemes, Lemma 36.10.3, and Cohomology, Lemma 20.47.5) whence has coherent cohomology sheaves (by Derived Categories of Schemes, Lemma 36.10.3 again). In each case the supports of the cohomology sheaves $H^i(L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K)$ is proper over Y as these supports are contained in the union of the supports of the $H^i(K)$. Hence in each case it suffices to prove that this tensor product is bounded below.

Case (1). By Cohomology, Lemma 20.27.4 we have

$$L\mathrm{pr}_1^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K \cong \mathrm{pr}_1^{-1} M \otimes^{\mathbf{L}}_{\mathrm{pr}_1^{-1}\mathcal{O}_X} K$$

with obvious notation. Hence the assumption on tor dimension and the fact that M has only a finite number of nonzero cohomology sheaves, implies the bound we want.

Case (2) follows because here the assumption implies that K has finite tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$ hence the argument in the previous paragraph applies.

In Case (3) it is also the case that K has finite tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$. Namely, choose affine opens $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ of X and Y mapping into the affine open $W = \mathrm{Spec}(R)$ of S. Then $K|_{U\times V}$ is given by a bounded complex of finite $A \otimes_R B$-modules $M^\bullet$. Since A is a regular ring of finite dimension we see that each M^i has finite projective dimension as an A-module

(Algebra, Lemma 10.110.8) and hence finite tor dimension as an A-module. Thus $M^\bullet$ has finite tor dimension as a complex of A-modules (More on Algebra, Lemma 15.67.8). Since $X \times Y$ is quasi-compact we conclude there exist $[a, b]$ such that for every point $z \in X \times Y$ the stalk K_z has tor amplitude in $[a, b]$ over $\mathcal{O}_{X,\mathrm{pr}_1(z)}$. This implies K has bounded tor dimension as an object of $D(\mathrm{pr}_1^{-1}\mathcal{O}_X)$, see Cohomology, Lemma 20.48.5. We conclude as in the previous to paragraphs.

Case (4). With notation as above, the ring map $R \to B$ is flat. Hence the ring map $A \to A \otimes_R B$ is flat. Hence any projective $A \otimes_R B$-module is A-flat. Thus any perfect complex of $A \otimes_R B$-modules has finite tor dimension as a complex of A-modules and we conclude as before. □

0FYV **Example 57.8.6.** Let $X \to S$ be a separated morphism of schemes. Then the diagonal $\Delta : X \to X \times_S X$ is a closed immersion and hence $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X = R\Delta_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{X\times_S X}$-module of finite type which is flat over X (under either projection). The Fourier-Mukai functor $\Phi_{\mathcal{O}_\Delta}$ is equal to the identity in this case. Namely, for any $M \in D(\mathcal{O}_X)$ we have

$$\begin{aligned} L\mathrm{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S X}} \mathcal{O}_\Delta &= L\mathrm{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S X}} R\Delta_*\mathcal{O}_X \\ &= R\Delta_*(L\Delta^*L\mathrm{pr}_1^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X) \\ &= R\Delta_*(M) \end{aligned}$$

The first equality we discussed above. The second equality is Cohomology, Lemma 20.54.4. The third because $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$ and we have Cohomology, Lemma 20.27.2. If we push this to X using $R\mathrm{pr}_{2,*}$ we obtain M by Cohomology, Lemma 20.28.2 and the fact that $\mathrm{pr}_2 \circ \Delta = \mathrm{id}_X$.

0FYW **Lemma 57.8.7.** *Let $X \to S$ and $Y \to S$ be morphisms of quasi-compact and quasi-separated schemes. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with pseudo-coherent kernel $K \in D_{QCoh}(\mathcal{O}_{X\times_S Y})$. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X\times_S Y})$ be the right adjoint to $Rpr_{2,*}$, see Duality for Schemes, Lemma 48.3.1. Denote*

Compare with discussion in [**Riz17**].

$$K' = (Y \times_S X \to X \times_S Y)^* R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}(K, a(\mathcal{O}_Y)) \in D_{QCoh}(\mathcal{O}_{Y\times_S X})$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical map

$$\mathrm{Hom}_X(M, \Phi'(N)) \longrightarrow \mathrm{Hom}_Y(\Phi(M), N)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism if

(1) *N is perfect, or*
(2) *K is perfect and $X \to S$ is proper flat and of finite presentation.*

Proof. By Lemma 57.8.2 we obtain a functor Φ as in the statement. Observe that $a(\mathcal{O}_Y)$ is in $D^+_{QCoh}(\mathcal{O}_{X\times_S Y})$ by Duality for Schemes, Lemma 48.3.5. Hence for K pseudo-coherent we have $K' \in D_{QCoh}(\mathcal{O}_{Y\times_S X})$ by Derived Categories of Schemes, Lemma 36.10.8 we we obtain Φ' as indicated.

We abbreviate $\otimes^{\mathbf{L}} = \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}}$ and $\mathcal{H}om = R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}$. Let M be in $D_{QCoh}(\mathcal{O}_X)$ and let N be in $D_{QCoh}(\mathcal{O}_Y)$. We have

$$\begin{aligned}\mathrm{Hom}_Y(\Phi(M), N) &= \mathrm{Hom}_Y(R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*M \otimes^{\mathbf{L}} K), N)\\ &= \mathrm{Hom}_{X\times_S Y}(L\mathrm{pr}_1^*M \otimes^{\mathbf{L}} K, a(N))\\ &= \mathrm{Hom}_{X\times_S Y}(L\mathrm{pr}_1^*M, R\mathcal{H}om(K, a(N)))\\ &= \mathrm{Hom}_X(M, R\mathrm{pr}_{1,*}R\mathcal{H}om(K, a(N)))\end{aligned}$$

where we have used Cohomology, Lemmas 20.42.2 and 20.28.1. There are canonical maps

$$L\mathrm{pr}_2^*N \otimes^{\mathbf{L}} R\mathcal{H}om(K, a(\mathcal{O}_Y)) \xrightarrow{\alpha} R\mathcal{H}om(K, L\mathrm{pr}_2^*N \otimes^{\mathbf{L}} a(\mathcal{O}_Y)) \xrightarrow{\beta} R\mathcal{H}om(K, a(N))$$

Here α is Cohomology, Lemma 20.42.6 and β is Duality for Schemes, Equation (48.8.0.1). Combining all of these arrows we obtain the functorial displayed arrow in the statement of the lemma.

The arrow α is an isomorphism by Derived Categories of Schemes, Lemma 36.10.9 as soon as either K or N is perfect. The arrow β is an isomorphism if N is perfect by Duality for Schemes, Lemma 48.8.1 or in general if $X \to S$ is flat proper of finite presentation by Duality for Schemes, Lemma 48.12.3. □

0FYX **Lemma 57.8.8.** *Let S be a Noetherian scheme. Let $Y \to S$ be a flat proper Gorenstein morphism and let $X \to S$ be a finite type morphism. Denote $\omega^\bullet_{Y/S}$ the relative dualizing complex of Y over S. Let $\Phi : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ be a Fourier-Mukai functor with perfect kernel $K \in D_{QCoh}(\mathcal{O}_{X\times_S Y})$. Denote*

$$K' = (Y \times_S X \to X \times_S Y)^*(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lpr_2^*\omega^\bullet_{Y/S}) \in D_{QCoh}(\mathcal{O}_{Y\times_S X})$$

and denote $\Phi' : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ the corresponding Fourier-Mukai transform. There is a canonical isomorphism

$$\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$.

Compare with discussion in [**Riz17**].

Proof. By Lemma 57.8.2 we obtain a functor Φ as in the statement.

Observe that formation of the relative dualizing complex commutes with base change in our setting, see Duality for Schemes, Remark 48.12.5. Thus $L\mathrm{pr}_2^*\omega^\bullet_{Y/S} = \omega^\bullet_{X\times_S Y/X}$. Moreover, we observe that $\omega^\bullet_{Y/S}$ is an invertible object of the derived category, see Duality for Schemes, Lemma 48.25.10, and a fortiori perfect.

To actually prove the lemma we're going to cheat. Namely, we will show that if we replace the roles of X and Y and K and K' then these are as in Lemma 57.8.7 and we get the result. It is clear that K' is perfect as a tensor product of perfect objects so that the discussion in Lemma 57.8.7 applies to it. To show that the procedure of Lemma 57.8.7 applied to K' on $Y \times_S X$ produces a complex isomorphic to K it suffices (details omitted) to show that

$$R\mathcal{H}om(R\mathcal{H}om(K, \omega^\bullet_{X\times_S Y/X}), \omega^\bullet_{X\times_S Y/X}) = K$$

This is clear because K is perfect and $\omega^\bullet_{X\times_S Y/X}$ is invertible; details omitted. Thus Lemma 57.8.7 produces a map

$$\mathrm{Hom}_Y(N, \Phi(M)) \longrightarrow \mathrm{Hom}_X(\Phi'(N), M)$$

functorial in M in $D_{QCoh}(\mathcal{O}_X)$ and N in $D_{QCoh}(\mathcal{O}_Y)$ which is an isomorphism because K' is perfect. This finishes the proof. □

0FYY **Lemma 57.8.9.** *Let S be a Noetherian scheme.*

(1) *For X, Y proper and flat over S and K in $D_{perf}(\mathcal{O}_{X\times_S Y})$ we obtain a Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.*
(2) *For X, Y, Z proper and flat over S, $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$, $K' \in D_{perf}(\mathcal{O}_{Y\times_S Z})$ the composition $\Phi_{K'} \circ \Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Z)$ is equal to $\Phi_{K''}$ with $K'' \in D_{perf}(\mathcal{O}_{X\times_S Z})$ computed as in Lemma 57.8.3,*
(3) *For X, Y, K, Φ_K as in (1) if $X \to S$ is Gorenstein, then $\Phi_{K'} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a right adjoint to Φ_K where $K' \in D_{perf}(\mathcal{O}_{Y\times_S X})$ is the pullback of $L\text{pr}_1^*\omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K^\vee$ by $Y \times_S X \to X \times_S Y$.*
(4) *For X, Y, K, Φ_K as in (1) if $Y \to S$ is Gorenstein, then $\Phi_{K''} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_X)$ is a left adjoint to Φ_K where $K'' \in D_{perf}(\mathcal{O}_{Y\times_S X})$ is the pullback of $L\text{pr}_2^*\omega^\bullet_{Y/S} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K^\vee$ by $Y \times_S X \to X \times_S Y$.*

Proof. Part (1) is immediate from Lemma 57.8.4 part (4).

Part (2) follows from Lemma 57.8.3 and the fact that $K'' = R\text{pr}_{13,*}(L\text{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S Z}} L\text{pr}_{23}^*K')$ is perfect for example by Derived Categories of Schemes, Lemma 36.27.4.

The adjointness in part (3) on all complexes with quasi-coherent cohomology sheaves follows from Lemma 57.8.7 with K' equal to the pullback of $R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}(K, a(\mathcal{O}_Y))$ by $Y \times_S X \to X \times_S Y$ where a is the right adjoint to $R\text{pr}_{2,*} : D_{QCoh}(\mathcal{O}_{X\times_S Y}) \to D_{QCoh}(\mathcal{O}_Y)$. Denote $f : X \to S$ the structure morphism of X. Since f is proper the functor $f^! : D^+_{QCoh}(\mathcal{O}_S) \to D^+_{QCoh}(\mathcal{O}_X)$ is the restriction to $D^+_{QCoh}(\mathcal{O}_S)$ of the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$, see Duality for Schemes, Section 48.16. Hence the relative dualizing complex $\omega^\bullet_{X/S}$ as defined in Duality for Schemes, Remark 48.12.5 is equal to $\omega^\bullet_{X/S} = f^!\mathcal{O}_S$. Since formation of the relative dualizing complex commutes with base change (see Duality for Schemes, Remark 48.12.5) we see that $a(\mathcal{O}_Y) = L\text{pr}_1^*\omega^\bullet_{X/S}$. Thus

$$R\mathcal{H}om_{\mathcal{O}_{X\times_S Y}}(K, a(\mathcal{O}_Y)) \cong L\text{pr}_1^*\omega^\bullet_{X/S} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} K^\vee$$

by Cohomology, Lemma 20.50.5. Finally, since $X \to S$ is assumed Gorenstein the relative dualizing complex is invertible: this follows from Duality for Schemes, Lemma 48.25.10. We conclude that $\omega^\bullet_{X/S}$ is perfect (Cohomology, Lemma 20.52.2) and hence K' is perfect. Therefore $\Phi_{K'}$ does indeed map $D_{perf}(\mathcal{O}_Y)$ into $D_{perf}(\mathcal{O}_X)$ which finishes the proof of (3).

The proof of (4) is the same as the proof of (3) except one uses Lemma 57.8.8 instead of Lemma 57.8.7. □

57.9. Resolutions and bounds

0FYZ The diagonal of a smooth proper scheme has a nice resolution.

0FZ0 **Lemma 57.9.1.** *Let R be a Noetherian ring. Let X, Y be finite type schemes over R having the resolution property. For any coherent $\mathcal{O}_{X\times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Let $U \subset X$ and $V \subset Y$ be affine open subschemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of the reduced induced closed subscheme structure on $X \setminus U$. Similarly, let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of the reduced induced closed subscheme structure on $Y \setminus V$. Then the ideal sheaf

$$\mathcal{J} = \mathrm{Im}(\mathrm{pr}_1^*\mathcal{I} \otimes_{\mathcal{O}_{X \times_R Y}} \mathrm{pr}_2^*\mathcal{I}' \to \mathcal{O}_{X \times_R Y})$$

satisfies $V(\mathcal{J}) = X \times_R Y \setminus U \times_R V$. For any section $s \in \mathcal{F}(U \times_R V)$ we can find an integer $n > 0$ and a map $\mathcal{J}^n \to \mathcal{F}$ whose restriction to $U \times_R V$ gives s, see Cohomology of Schemes, Lemma 30.10.5. By assumption we can choose surjections $\mathcal{E} \to \mathcal{I}$ and $\mathcal{G} \to \mathcal{I}'$. These produce corresponding surjections

$$\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{J} \quad \text{and} \quad \mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{J}^n$$

and hence a map $\mathcal{E}^{\otimes n} \boxtimes \mathcal{G}^{\otimes n} \to \mathcal{F}$ whose image contains the section s over $U \times_R V$. Since we can cover $X \times_R Y$ by a finite number of affine opens of the form $U \times_R V$ and since $\mathcal{F}|_{U \times_R V}$ is generated by finitely many sections (Properties, Lemma 28.16.1) we conclude that there exists a surjection

$$\bigoplus\nolimits_{j=1,\ldots,N} \mathcal{E}_j^{\otimes n_j} \boxtimes \mathcal{G}_j^{\otimes n_j} \to \mathcal{F}$$

where $\mathcal{E}_j$ is finite locally free on X and $\mathcal{G}_j$ is finite locally free on Y. Setting $\mathcal{E} = \bigoplus \mathcal{E}_j^{\otimes n_j}$ and $\mathcal{G} = \bigoplus \mathcal{G}_j^{\otimes n_j}$ we conclude that the lemma is true. □

0FZ1 **Lemma 57.9.2.** *Let R be a ring. Let X, Y be quasi-compact and quasi-separated schemes over R having the resolution property. For any finite type quasi-coherent $\mathcal{O}_{X \times_R Y}$-module $\mathcal{F}$ there exist a surjection $\mathcal{E} \boxtimes \mathcal{G} \to \mathcal{F}$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and $\mathcal{G}$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Follows from Lemma 57.9.1 by a limit argument. We urge the reader to skip the proof. Since $X \times_R Y$ is a closed subscheme of $X \times_{\mathbf{Z}} Y$ it is harmless if we replace R by $\mathbf{Z}$. We can write $\mathcal{F}$ as the quotient of a finitely presented $\mathcal{O}_{X \times_R Y}$-module by Properties, Lemma 28.22.8. Hence we may assume $\mathcal{F}$ is of finite presentation. Next we can write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$ and similarly $Y = \lim Y_j$, see Limits, Proposition 32.5.4. Then $\mathcal{F}$ will descend to $\mathcal{F}_{ij}$ on some $X_i \times_R Y_j$ (Limits, Lemma 32.10.2) and so does the property of having the resolution property (Derived Categories of Schemes, Lemma 36.36.9). Then we apply Lemma 57.9.1 to $\mathcal{F}_{ij}$ and we pullback. □

0FZ2 **Lemma 57.9.3.** *Let R be a Noetherian ring. Let X be a separated finite type scheme over R which has the resolution property. Set $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_X)$ where $\Delta : X \to X \times_R X$ is the diagonal of X/k. There exists a resolution*

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module.

Proof. Since X is separated, the diagonal morphism Δ is a closed immersion and hence $\mathcal{O}_\Delta$ is a coherent $\mathcal{O}_{X \times_R X}$-module (Cohomology of Schemes, Lemma 30.9.8). Thus the lemma follows immediately from Lemma 57.9.1. □

0FZ3 **Lemma 57.9.4.** *Let X be a regular Noetherian scheme of dimension $d < \infty$. Then*

(1) *for $\mathcal{F}$, $\mathcal{G}$ coherent $\mathcal{O}_X$-modules we have $\mathrm{Ext}^n_X(\mathcal{F}, \mathcal{G}) = 0$ for $n > d$, and*

(2) *for* $K, L \in D^b_{Coh}(\mathcal{O}_X)$ *and* $a \in \mathbf{Z}$ *if* $H^i(K) = 0$ *for* $i < a+d$ *and* $H^i(L) = 0$ *for* $i \geq a$ *then* $\mathrm{Hom}_X(K, L) = 0$.

Proof. To prove (1) we use the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G})$$

of Cohomology, Section 20.43. Let $x \in X$. We have

$$\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})_x = \mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

see Cohomology, Lemma 20.51.4 (this also uses that $\mathcal{F}$ is pseudo-coherent by Derived Categories of Schemes, Lemma 36.10.3). Set $d_x = \dim(\mathcal{O}_{X,x})$. Since $\mathcal{O}_{X,x}$ is regular the ring $\mathcal{O}_{X,x}$ has global dimension d_x, see Algebra, Proposition 10.110.1. Thus $\mathcal{E}xt^q_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is zero for $q > d_x$. It follows that the modules $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})$ have support of dimension at most $d - q$. Hence we have $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = 0$ for $p > d - q$ by Cohomology, Proposition 20.20.7. This proves (1).

Proof of (2). We may use induction on the number of nonzero cohomology sheaves of K and L. The case where these numbers are $0, 1$ follows from (1). If the number of nonzero cohomology sheaves of K is > 1, then we let $i \in \mathbf{Z}$ be minimal such that $H^i(K)$ is nonzero. We obtain a distinguished triangle

$$H^i(K)[-i] \to K \to \tau_{\geq i+1}K$$

(Derived Categories, Remark 13.12.4) and we get the vanishing of $\mathrm{Hom}(K, L)$ from the vanishing of $\mathrm{Hom}(H^i(K)[-i], L)$ and $\mathrm{Hom}(\tau_{\geq i+1}K, L)$ by Derived Categories, Lemma 13.4.2. Similarly if L has more than one nonzero cohomology sheaf. □

0FZ4 **Lemma 57.9.5.** *Let* X *be a regular Noetherian scheme of dimension* $d < \infty$. *Let* $K \in D^b_{Coh}(\mathcal{O}_X)$ *and* $a \in \mathbf{Z}$. *If* $H^i(K) = 0$ *for* $a < i < a + d$, *then* $K = \tau_{\leq a}K \oplus \tau_{\geq a+d}K$.

Proof. We have $\tau_{\leq a}K = \tau_{\leq a+d-1}K$ by the assumed vanishing of cohomology sheaves. By Derived Categories, Remark 13.12.4 we have a distinguished triangle

$$\tau_{\leq a}K \to K \to \tau_{\geq a+d}K \xrightarrow{\delta} (\tau_{\leq a}K)[1]$$

By Derived Categories, Lemma 13.4.11 it suffices to show that the morphism δ is zero. This follows from Lemma 57.9.4. □

0FZ5 **Lemma 57.9.6.** *Let* k *be a field. Let* X *be a quasi-compact separated smooth scheme over* k. *There exist finite locally free* $\mathcal{O}_X$*-modules* $\mathcal{E}$ *and* $\mathcal{G}$ *such that*

$$\mathcal{O}_\Delta \in \langle \mathcal{E} \boxtimes \mathcal{G} \rangle$$

in $D(\mathcal{O}_{X \times X})$ *where the notation is as in Derived Categories, Section 13.36.*

Proof. Recall that X is regular by Varieties, Lemma 33.25.3. Hence X has the resolution property by Derived Categories of Schemes, Lemma 36.36.8. Hence we may choose a resolution as in Lemma 57.9.3. Say $\dim(X) = d$. Since $X \times X$ is smooth over k it is regular. Hence $X \times X$ is a regular Noetherian scheme with $\dim(X \times X) = 2d$. The object

$$K = (\mathcal{E}_{2d} \boxtimes \mathcal{G}_{2d} \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)$$

of $D_{perf}(\mathcal{O}_{X\times X})$ has cohomology sheaves $\mathcal{O}_\Delta$ in degree 0 and $\mathrm{Ker}(\mathcal{E}_{2d}\boxtimes\mathcal{G}_{2d}\to\mathcal{E}_{2d-1}\boxtimes\mathcal{G}_{2d-1})$ in degree $-2d$ and zero in all other degrees. Hence by Lemma 57.9.5 we see that $\mathcal{O}_\Delta$ is a summand of K in $D_{perf}(\mathcal{O}_{X\times X})$. Clearly, the object K is in

$$\left\langle \bigoplus\nolimits_{i=0,\ldots,2d} \mathcal{E}_i\boxtimes\mathcal{G}_i \right\rangle \subset \left\langle \left(\bigoplus\nolimits_{i=0,\ldots,2d}\mathcal{E}_i\right)\boxtimes\left(\bigoplus\nolimits_{i=0,\ldots,2d}\mathcal{G}_i\right)\right\rangle$$

which finishes the proof. (The reader may consult Derived Categories, Lemmas 13.36.1 and 13.35.7 to see that our object is contained in this category.) □

0FZ6 **Lemma 57.9.7.** *Let k be a field. Let X be a scheme proper and smooth over k. Then $D_{perf}(\mathcal{O}_X)$ has a strong generator.*

Proof. Using Lemma 57.9.6 choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{G}$ such that $\mathcal{O}_\Delta \in \langle \mathcal{E}\boxtimes\mathcal{G}\rangle$ in $D(\mathcal{O}_{X\times X})$. We claim that $\mathcal{G}$ is a strong generator for $D_{perf}(\mathcal{O}_X)$. With notation as in Derived Categories, Section 13.35 choose $m, n \geq 1$ such that

$$\mathcal{O}_\Delta \in smd(add(\mathcal{E}\boxtimes\mathcal{G}[-m,m])^{\star n})$$

This is possible by Derived Categories, Lemma 13.36.2. Let K be an object of $D_{perf}(\mathcal{O}_X)$. Since $L\mathrm{pr}_1^*K\otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} -$ is an exact functor and since

$$L\mathrm{pr}_1^*K\otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}}(\mathcal{E}\boxtimes\mathcal{G}) = (K\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathcal{E})\boxtimes\mathcal{G}$$

we conclude from Derived Categories, Remark 13.35.5 that

$$L\mathrm{pr}_1^*K\otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta \in smd(add((K\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathcal{E})\boxtimes\mathcal{G}[-m,m])^{\star n})$$

Applying the exact functor $R\mathrm{pr}_{2,*}$ and observing that

$$R\mathrm{pr}_{2,*}\left((K\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathcal{E})\boxtimes\mathcal{G}\right) = R\Gamma(X, K\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathcal{E})\otimes_k\mathcal{G}$$

by Derived Categories of Schemes, Lemma 36.22.1 we conclude that

$$K = R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*K\otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta) \in smd(add(R\Gamma(X, K\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathcal{E})\otimes_k\mathcal{G}[-m,m])^{\star n})$$

The equality follows from the discussion in Example 57.8.6. Since K is perfect, there exist $a \leq b$ such that $H^i(X,K)$ is nonzero only for $i\in[a,b]$. Since X is proper, each $H^i(X,K)$ is finite dimensional. We conclude that the right hand side is contained in $smd(add(\mathcal{G}[-m+a, m+b])^{\star n})$ which is itself contained in $\langle\mathcal{G}\rangle_n$ by one of the references given above. This finishes the proof. □

0FZ7 **Lemma 57.9.8.** *Let k be a field. Let X be a proper smooth scheme over k. There exists integers $m, n \geq 1$ and a finite locally free $\mathcal{O}_X$-module $\mathcal{G}$ such that every coherent $\mathcal{O}_X$-module is contained in $smd(add(\mathcal{G}[-m,m])^{\star n})$ with notation as in Derived Categories, Section 13.35.*

Proof. In the proof of Lemma 57.9.7 we have shown that there exist $m', n \geq 1$ such that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathcal{F} \in smd(add(\mathcal{G}[-m'+a, m'+b])^{\star n})$$

for any $a \leq b$ such that $H^i(X,\mathcal{F})$ is nonzero only for $i\in[a,b]$. Thus we can take $a = 0$ and $b = \dim(X)$. Taking $m = \max(m', m'+b)$ finishes the proof. □

The following lemma is the boundedness result referred to in the title of this section.

0FZ8 **Lemma 57.9.9.** *Let k be a field. Let X be a smooth proper scheme over k. Let $\mathcal{A}$ be an abelian category. Let $H : D_{perf}(\mathcal{O}_X) \to \mathcal{A}$ be a homological functor (Derived Categories, Definition 13.3.5) such that for all K in $D_{perf}(\mathcal{O}_X)$ the object $H^i(K)$ is nonzero for only a finite number of $i \in \mathbf{Z}$. Then there exists an integer $m \geq 1$ such that $H^i(\mathcal{F}) = 0$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i \notin [-m, m]$. Similarly for cohomological functors.*

Proof. Combine Lemma 57.9.8 with Derived Categories, Lemma 13.35.8. □

0FZ9 **Lemma 57.9.10.** *Let k be a field. Let X, Y be finite type schemes over k. Let $K_0 \to K_1 \to K_2 \to \ldots$ be a system of objects of $D_{perf}(\mathcal{O}_{X\times Y})$ and $m \geq 0$ an integer such that*

(1) *$H^q(K_i)$ is nonzero only for $q \leq m$,*
(2) *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ the object*

$$Rpr_{2,*}(pr_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n)$$

has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m - n, m - n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$.

Then K_n has vanishing cohomology sheaves in degrees outside $[-m, m] \cup [-m - n, m - n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in degrees in $[-m, m]$. Moreover, if X and Y are smooth over k, then for n large enough we find $K_n = K \oplus C_n$ in $D_{perf}(\mathcal{O}_{X\times Y})$ where K has cohomology only indegrees $[-m, m]$ and C_n only in degrees $[-m - n, m - n]$ and the transition maps define isomorphisms between various copies of K.

Proof. Let Z be the scheme theoretic support of an $\mathcal{F}$ as in (2). Then $Z \to \mathrm{Spec}(k)$ is finite, hence $Z \times Y \to Y$ is finite. It follows that for an object M of $D_{QCoh}(\mathcal{O}_{X\times Y})$ with cohomology sheaves supported on $Z \times Y$ we have $H^i(R\mathrm{pr}_{2,*}(M)) = \mathrm{pr}_{2,*}H^i(M)$ and the functor $\mathrm{pr}_{2,*}$ is faithful on quasi-coherent modules supported on $Z \times Y$; details omitted. Hence we see that the objects

$$\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n$$

in $D_{perf}(\mathcal{O}_{X\times Y})$ have vanishing cohomology sheaves outside $[-m, m]\cup[-m-n, m-n]$ and for $n > 2m$ the transition maps induce isomorphisms on cohomology sheaves in $[-m, m]$. Let $z \in X \times Y$ be a closed point mapping to the closed point $x \in X$. Then we know that

$$K_{n,z} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y,z}} \mathcal{O}_{X\times Y,z}/\mathfrak{m}_x^t\mathcal{O}_{X\times Y,z}$$

has nonzero cohomology only in the intervals $[-m, m]\cup[-m-n, m-n]$. We conclude by More on Algebra, Lemma 15.101.2 that $K_{n,z}$ only has nonzero cohomology in degrees $[-m, m]\cup[-m-n, m-n]$. Since this holds for all closed points of $X \times Y$, we conclude K_n only has nonzero cohomology sheaves in degrees $[-m, m]\cup[-m-n, m-n]$. In exactly the same way we see that the maps $K_n \to K_{n+1}$ are isomorphisms on cohomology sheaves in degrees $[-m, m]$ for $n > 2m$.

If X and Y are smooth over k, then $X \times Y$ is smooth over k and hence regular by Varieties, Lemma 33.25.3. Thus we will obtain the direct sum decomposition of K_n as soon as $n > 2m + \dim(X \times Y)$ from Lemma 57.9.5. The final statement is clear from this. □

57.10. Sibling functors

0FZS In this section we prove some categorical result on the following notion.

0FZT **Definition 57.10.1.** Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. We say two exact functors of triangulated categories

$$F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$$

are *siblings*, or we say F' is a *sibling* of F, if the following two conditions are satisfied

(1) the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and
(2) $F(K) \cong F'(K)$ for any K in $D^b(\mathcal{A})$.

Sometimes the second condition is a consequence of the first.

0FZU **Lemma 57.10.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F, F' : D^b(\mathcal{A}) \longrightarrow \mathcal{D}$ be exact functors of triangulated categories. Assume*

(1) *the functors $F \circ i$ and $F' \circ i$ are isomorphic where $i : \mathcal{A} \to D^b(\mathcal{A})$ is the inclusion functor, and*
(2) *for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ we have $\mathrm{Ext}^q_{\mathcal{D}}(F(X), F(Y)) = 0$ for $q < 0$ (for example if F is fully faithful).*

Then F and F' are siblings.

Proof. Let $K \in D^b(\mathcal{A})$. We will show $F(K)$ is isomorphic to $F'(K)$. We can represent K by a bounded complex $A^\bullet$ of objects of $\mathcal{A}$. After replacing K by a translation we may assume $A^i = 0$ for $i > 0$. Choose $n \geq 0$ such that $A^{-i} = 0$ for $i > n$. The objects

$$M_i = (A^{-i} \to \ldots \to A^0)[-i], \quad i = 0, \ldots, n$$

form a Postnikov system in $D^b(\mathcal{A})$ for the complex $A^\bullet = A^{-n} \to \ldots \to A^0$ in $D^b(\mathcal{A})$. See Derived Categories, Example 13.41.2. Since both F and F' are exact functors of triangulated categories both

$$F(M_i) \quad \text{and} \quad F'(M_i)$$

form a Postnikov system in $\mathcal{D}$ for the complex

$$F(A^{-n}) \to \ldots \to F(A^0) = F'(A^{-n}) \to \ldots \to F'(A^0)$$

Since all negative Exts between these objects vanish by assumption we conclude by uniqueness of Postnikov systems (Derived Categories, Lemma 13.41.6) that $F(K) = F(M_n[n]) \cong F'(M_n[n]) = F'(K)$. □

0FZV **Lemma 57.10.3.** *Let F and F' be siblings as in Definition 57.10.1. Then*

(1) *if F is essentially surjective, then F' is essentially surjective,*
(2) *if F is fully faithful, then F' is fully faithful.*

Proof. Part (1) is immediate from property (2) for siblings.

Assume F is fully faithful. Denote $\mathcal{D}' \subset \mathcal{D}$ the essential image of F so that $F : D^b(\mathcal{A}) \to \mathcal{D}'$ is an equivalence. Since the functor F' factors through $\mathcal{D}'$ by property (2) for siblings, we can consider the functor $H = F^{-1} \circ F' : D^b(\mathcal{A}) \to D^b(\mathcal{A})$. Observe that H is a sibling of the identity functor. Since it suffices to prove that H is fully faithful, we reduce to the problem discussed in the next paragraph.

Set $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is fully faithful. Denote $a_X : X \to F(X)$ the functorial isomorphism for $X \in \mathrm{Ob}(\mathcal{A})$ given to us by Definition 57.10.1. For any K in $\mathcal{D}$ and distinguished triangle $K_1 \to K_2 \to K_3$ of $\mathcal{D}$ if the maps

$$F : \mathrm{Hom}(K, K_i[n]) \to \mathrm{Hom}(F(K), F(K_i[n]))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. This uses the 5-lemma Homology, Lemma 12.5.20 and Derived Categories, Lemma 13.4.2; details omitted. Similarly, if the maps

$$F : \mathrm{Hom}(K_i[n], K) \to \mathrm{Hom}(F(K_i[n]), F(K))$$

are isomorphisms for all $n \in \mathbf{Z}$ and $i = 1, 3$, then the same is true for $i = 2$ and all $n \in \mathbf{Z}$. Using the canonical truncations and induction on the number of nonzero cohomology objects, we see that it is enough to show

$$F : \mathrm{Ext}^q(X, Y) \to \mathrm{Ext}^q(F(X), F(Y))$$

is bijective for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ and all $q \in \mathbf{Z}$. Since F is a sibling of id we have $F(X) \cong X$ and $F(Y) \cong Y$ hence the right hand side is zero for $q < 0$. The case $q = 0$ is OK by our assumption that F is a sibling of the identity functor. It remains to prove the cases $q > 0$.

The case $q = 1$: Injectivity. An element ξ of $\mathrm{Ext}^1(X, Y)$ gives rise to a distinguished triangle

$$Y \to E \to X \xrightarrow{\xi} Y[1]$$

Observe that $E \in \mathrm{Ob}(\mathcal{A})$. Since F is a sibling of the identity functor we obtain a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ F(E) & \longrightarrow & F(X) \end{array}$$

whose vertical arrows are the isomorphisms a_E and a_X. By TR3 the distinguished triangle associated to ξ we started with is isomorphic to the distinguished triangle

$$F(Y) \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y[1]) = F(Y)[1]$$

Thus $\xi = 0$ if and only if $F(\xi)$ is zero, i.e., we see that $F : \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(F(X), F(Y))$ is injective.

The case $q = 1$: Surjectivity. Let θ be an element of $\mathrm{Ext}^1(F(X), F(Y))$. This defines an extension of $F(X)$ by $F(Y)$ in $\mathcal{A}$ which we may write as $F(E)$ as F is a sibling of the identity functor. We thus get a distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{\theta} F(Y[1]) = F(Y)[1]$$

for some morphisms $\alpha : Y \to E$ and $\beta : E \to X$. Since F is a sibling of the identity functor, the sequence $0 \to Y \to E \to X \to 0$ is a short exact sequence in $\mathcal{A}$! Hence we obtain a distinguished triangle

$$Y \xrightarrow{\alpha} E \xrightarrow{\beta} X \xrightarrow{\delta} Y[1]$$

for some morphism $\delta : X \to Y[1]$. Applying the exact functor F we obtain the distinguished triangle

$$F(Y) \xrightarrow{F(\alpha)} F(E) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\delta)} F(Y)[1]$$

Arguing as above, we see that these triangles are isomorphic. Hence there exists a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow[F(\delta)]{} & F(Y[1]) \\ \downarrow{\scriptstyle \gamma} & & {\scriptstyle \epsilon}\downarrow \\ F(X) & \xrightarrow{\theta} & F(Y[1]) \end{array}$$

for some isomorphisms γ, ϵ (we can say more but we won't need more information). We may write $\gamma = F(\gamma')$ and $\epsilon = F(\epsilon')$. Then we have $\theta = F(\epsilon' \circ \delta \circ (\gamma')^{-1})$ and we see the surjectivity holds.

The case $q > 1$: surjectivity. Using Yoneda extensions, see Derived Categories, Section 13.27, we find that for any element ξ in $\mathrm{Ext}^q(F(X), F(Y))$ we can find $F(X) = B_0, B_1, \ldots, B_{q-1}, B_q = F(Y) \in \mathrm{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \mathrm{Ext}^1(B_{i-1}, B_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. Write $B_i = F(A_i)$ (of course we have $A_i = B_i$ but we don't need to use this) so that

$$\xi_i = F(\eta_i) \in \mathrm{Ext}^1(F(A_{i-1}), F(A_i)) \quad \text{with} \quad \eta_i \in \mathrm{Ext}^1(A_{i-1}, A_i)$$

by surjectivity for $q = 1$. Then $\eta = \eta_q \circ \ldots \circ \eta_1$ is an element of $\mathrm{Ext}^q(X, Y)$ with $F(\eta) = \xi$.

The case $q > 1$: injectivity. An element ξ of $\mathrm{Ext}^q(X, Y)$ gives rise to a distinguished triangle

$$Y[q-1] \to E \to X \xrightarrow{\xi} Y[q]$$

Applying F we obtain a distinguished triangle

$$F(Y)[q-1] \to F(E) \to F(X) \xrightarrow{F(\xi)} F(Y)[q]$$

If $F(\xi) = 0$, then $F(E) \cong F(Y)[q-1] \oplus F(X)$ in $\mathcal{D}$, see Derived Categories, Lemma 13.4.11. Since F is a sibling of the identity functor we have $E \cong F(E)$ and hence

$$E \cong F(E) \cong F(Y)[q-1] \oplus F(X) \cong Y[q-1] \oplus X$$

In other words, E is isomorphic to the direct sum of its cohomology objects. This implies that the initial distinguished triangle is split, i.e., $\xi = 0$. □

Let us make a nonstandard definition. Let $\mathcal{A}$ be an abelian category. Let us say $\mathcal{A}$ *has enough negative objects* if given any $X \in \mathrm{Ob}(\mathcal{A})$ there exists an object N such that

(1) there is a surjection $N \to X$ and
(2) $\mathrm{Hom}(X, N) = 0$.

Let us prove a couple of lemmas about this notion in order to help with the proof of Proposition 57.10.6.

0GWF **Lemma 57.10.4.** *Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $X \in D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ with $H^i(X) = 0$ for $i > b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective and $\mathrm{Hom}(H^b(X), N) = 0$.*

Proof. Using the truncation functors we can represent X by a complex $A^a \to A^{a+1} \to \ldots \to A^b$ of objects of $\mathcal{A}$. Choose N in $\mathcal{A}$ such that there exists a surjection $t : N \to A^b$ and such that $\mathrm{Hom}(A^b, N) = 0$. Then the surjection t defines a map $N[-b] \to X$ as desired. □

0GWG **Lemma 57.10.5.** *Let $\mathcal{A}$ be an abelian category with enough negative objects. Let $f : X \to X'$ be a morphism of $D^b(\mathcal{A})$. Let $b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > b$ and $H^i(X') = 0$ for $i \geq b$. Then there exists a map $N[-b] \to X$ such that the induced map $N \to H^b(X)$ is surjective, such that $\mathrm{Hom}(H^b(X), N) = 0$, and such that the composition $N[-b] \to X \to X'$ is zero.*

Proof. We can represent f by a map $f^\bullet : A^\bullet \to B^\bullet$ of bounded complexes of objects of $\mathcal{A}$, see for example Derived Categories, Lemma 13.11.6. Consider the object

$$C = \mathrm{Ker}(A^b \to A^{b+1}) \times_{\mathrm{Ker}(B^b \to B^{b+1})} B^{b-1}$$

of $\mathcal{A}$. Since $H^b(B^\bullet) = 0$ we see that $C \to H^b(A^\bullet)$ is surjective. On the other hand, the map $C \to A^b \to B^b$ is the same as the map $C \to B^{b-1} \to B^b$ and hence the composition $C[-b] \to X \to X'$ is zero. Since $\mathcal{A}$ has enough negative objects, we can find an object N which has a surjection $N \to C \oplus H^b(X)$ such that $\mathrm{Hom}(C \oplus H^b(X), N) = 0$. Then N together with the map $N[-b] \to X$ is a solution to the problem posed by the lemma. □

We encourage the reader to read the original [**Orl97**, Proposition 2.16] for the marvellous ideas that go into the proof of the following proposition.

0FZW **Proposition 57.10.6.** *Let F and F' be siblings as in Definition 57.10.1. Assume that F is fully faithful and that $\mathcal{A}$ has enough negative objects (see above). Then F and F' are isomorphic functors.*

[**Orl97**, Proposition 2.16]; the fact that we do not need to assume vanishing of $\mathrm{Ext}^q(N, X)$ for $q > 0$ in the definition of negative objects above is due to [**CS14**].

Proof. By part (2) of Definition 57.10.1 the image of the functor F' is contained in the essential image of the functor F. Hence the functor $H = F^{-1} \circ F'$ is a sibling of the identity functor. This reduces us to the case described in the next paragraph.

Let $\mathcal{D} = D^b(\mathcal{A})$. We have to show a sibling $F : \mathcal{D} \to \mathcal{D}$ of the identity functor is isomorphic to the identity functor. Given an object X of $\mathcal{D}$ let us say X has *width* $w = w(X)$ if $w \geq 0$ is minimal such that there exists an integer $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a + w - 1]$. Since F is a sibling of the identity and since $F \circ [n] = [n] \circ F$ we are already given isomorphisms

$$c_X : X \to F(X)$$

for $w(X) \leq 1$ compatible with shifts. Moreover, if $X = A[-a]$ and $X' = A'[-a]$ for some $A, A' \in \mathrm{Ob}(\mathcal{A})$ then for any morphism $f : X \to X'$ the diagram

0FZX (57.10.6.1)
$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle c_X}\downarrow & & \downarrow{\scriptstyle c_{X'}} \\ F(X) & \xrightarrow{F(f)} & F(X') \end{array}$$

is commutative.

Next, let us show that for any morphism $f : X \to X'$ with $w(X), w(X') \leq 1$ the diagram (57.10.6.1) commutes. If X or X' is zero, this is clear. If not then we can write $X = A[-a]$ and $X' = A'[-a']$ for unique A, A' in $\mathcal{A}$ and $a, a' \in \mathbf{Z}$.

The case $a = a'$ was discussed above. If $a' > a$, then $f = 0$ (Derived Categories, Lemma 13.27.3) and the result is clear. If $a' < a$ then f corresponds to an element $\xi \in \mathrm{Ext}^q(A, A')$ with $q = a - a'$. Using Yoneda extensions, see Derived Categories, Section 13.27, we can find $A = A_0, A_1, \ldots, A_{q-1}, A_q = A' \in \mathrm{Ob}(\mathcal{A})$ and elements

$$\xi_i \in \mathrm{Ext}^1(A_{i-1}, A_i)$$

such that ξ is the composition $\xi_q \circ \ldots \circ \xi_1$. In other words, setting $X_i = A_i[-a+i]$ we obtain morphisms

$$X = X_0 \xrightarrow{f_1} X_1 \to \ldots \to X_{q-1} \xrightarrow{f_q} X_q = X'$$

whose compostion is f. Since the commutativity of (57.10.6.1) for $f_1, \ldots, f_q$ implies it for f, this reduces us to the case $q = 1$. In this case after shifting we may assume we have a distinguished triangle

$$A' \to E \to A \xrightarrow{f} A'[1]$$

Observe that E is an object of $\mathcal{A}$. Consider the following diagram

$$\begin{array}{ccccccc}
E & \longrightarrow & A & \xrightarrow{f} & A'[1] & \longrightarrow & E[1] \\
\downarrow{\scriptstyle c_E} & & \downarrow{\scriptstyle c_A} & \swarrow{\scriptstyle \epsilon} & \downarrow{\scriptstyle \gamma}\downarrow{\scriptstyle c_{A'}[1]} & & \downarrow{\scriptstyle c_E[1]} \\
F(E) & \longrightarrow & F(A) & \xrightarrow{F(f)} & F(A')[1] & \longrightarrow & F(E)[1]
\end{array}$$

whose rows are distinguished triangles. The square on the right commutes already but we don't yet know that the middle square does. By the axioms of a triangulated category we can find a morphism γ which does make the diagram commute. Then $\gamma - c_{A'}[1]$ composed with $F(A')[1] \to F(E)[1]$ is zero hence we can find $\epsilon : A'[1] \to F(A)$ such that $\gamma - c_{A'}[1] = F(f) \circ \epsilon$. However, any arrow $A'[1] \to F(A)$ is zero as it is a negative ext class between objects of $\mathcal{A}$. Hence $\gamma = c_{A'}[1]$ and we conclude the middle square commutes too which is what we wanted to show.

To finish the proof we are going to argue by induction on w that there exist isomorphisms $c_X : X \to F(X)$ for all X with $w(X) \leq w$ compatible with all morphisms between such objects. The base case $w = 1$ was shown above. Assume we know the result for some $w \geq 1$.

Let X be an object with $w(X) = w+1$. Pick $a \in \mathbf{Z}$ with $H^i(X) = 0$ for $i \notin [a, a+w]$. Set $b = a + w$ so that $H^b(X)$ is nonzero. Choose $N[-b] \to X$ as in Lemma 57.10.4. Choose a distinguished diagram

$$N[-b] \to X \to Y \to N[-b+1]$$

Computing the long exact cohomology sequence we find $w(Y) \leq w$. Hence by induction we find the solid arrows in the following diagram

$$\begin{array}{ccccccc}
N[-b] & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & N[-b+1] \\
\downarrow{\scriptstyle c_N[-b]} & & \vdots{\scriptstyle c_{N[-b]\to X}} & & \downarrow{\scriptstyle c_Y} & & \downarrow{\scriptstyle c_N[-b+1]} \\
F(N)[-b] & \longrightarrow & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(N)[-b+1]
\end{array}$$

We obtain the dotted arrow $c_{N[-b]\to X}$. By Derived Categories, Lemma 13.4.8 the dotted arrow is unique because $\mathrm{Hom}(X, F(N)[-b]) \cong \mathrm{Hom}(X, N[-b]) = 0$ by our choice of N. In fact, $c_{N[-b]\to X}$ is the unique dotted arrow making the square with vertices $X, Y, F(X), F(Y)$ commute.

Let $N'[-b] \to X$ be another map as in Lemma 57.10.4 and let us prove that $c_{N[-b]\to X} = c_{N'[-b]\to X}$. Observe that the map $(N \oplus N')[-b] \to X$ also satisfies the conditions of Lemma 57.10.4. Thus we may assume $N'[-b] \to X$ factors as $N'[-b] \to N[-b] \to X$ for some morphism $N' \to N$. Choose distinguished triangles $N[-b] \to X \to Y \to N[-b+1]$ and $N'[-b] \to X \to Y' \to N'[-b+1]$. By axiom TR3 we can find a morphism $g : Y' \to Y$ which joint with id_X and $N' \to N$ forms a morphism of triangles. Since we have (57.10.6.1) for g we conclude that

$$(F(X) \to F(Y)) \circ c_{N'[-b]\to X} = (F(X) \to F(Y)) \circ c_{N[-b]\to X}$$

The uniqueness of $c_{N[-b]\to X}$ pointed out in the construction above now shows that $c_{N'[-b]\to X} = c_{N[-b]\to X}$.

Thus we can now define for X of width $w+1$ the isomorphism $c_X : X \to F(X)$ as the common value of the maps $c_{N[-b]\to X}$ where $N[-b] \to X$ is as in Lemma 57.10.4. To finish the proof, we have to show that the diagrams (57.10.6.1) commute for all morphisms $f : X \to X'$ between objects with $w(X) \leq w+1$ and $w(X') \leq w+1$. Choose $a \leq b \leq a+w$ such that $H^i(X) = 0$ for $i \notin [a, b]$ and $a' \leq b' \leq a'+w$ such that $H^i(X') = 0$ for $i \notin [a', b']$. We will use induction on $(b'-a')+(b-a)$ to show the claim. (The base case is when this number is zero which is OK because $w \geq 1$.) We distinguish two cases.

Case I: $b' < b$. In this case, by Lemma 57.10.5 we may choose $N[-b] \to X$ as in Lemma 57.10.4 such that the composition $N[-b] \to X \to X'$ is zero. Choose a distuiguished triangle $N[-b] \to X \to Y \to N[-b+1]$. Since $N[-b] \to X'$ is zero, we find that f factors as $X \to Y \to X'$. Since $H^i(Y)$ is nonzero only for $i \in [a, b-1]$ we see by induction that (57.10.6.1) commutes for $Y \to X'$. The diagram (57.10.6.1) commutes for $X \to Y$ by construction if $w(X) = w+1$ and by our first induction hypothesis if $w(X) \leq w$. Hence (57.10.6.1) commutes for f.

Case II: $b' \geq b$. In this case we choose $N'[-b'] \to X'$ as in Lemma 57.10.4. We may also assume that $\mathrm{Hom}(H^{b'}(X), N') = 0$ (this is relevant only if $b' = b$), for example because we can replace N' by an object N'' which surjects onto $N' \oplus H^{b'}(X)$ and such that $\mathrm{Hom}(N' \oplus H^{b'}(X), N'') = 0$. We choose a distinguished triangle $N'[-b'] \to X' \to Y' \to N'[-b'+1]$. Since $\mathrm{Hom}(X, X') \to \mathrm{Hom}(X, Y')$ is injective by our choice of N' (details omitted) the same is true for $\mathrm{Hom}(X, F(X')) \to \mathrm{Hom}(X, F(Y'))$. Hence it suffices in this case to check that (57.10.6.1) commutes for the composition $X \to Y'$ of the morphisms $X \to X' \to Y'$. Since $H^i(Y')$ is nonzero only for $i \in [a', b'-1]$ we conclude by induction hypothesis. □

57.11. Deducing fully faithfulness

0G23 It will be useful for us to know when a functor is fully faithful we offer the following variant of [**Orl97**, Lemma 2.15].

0G24 **Lemma 57.11.1.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Let $S \subset \mathrm{Ob}(\mathcal{D})$ be a set of objects. Assume* Variant of [**Orl97**, Lemma 2.15]

(1) *F has both right and left adjoints,*
(2) *for $K \in \mathcal{D}$ if $\mathrm{Hom}(E, K[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,*
(3) *for $K \in \mathcal{D}$ if $\mathrm{Hom}(K, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$ then $K = 0$,*
(4) *the map $\mathrm{Hom}(E, E'[i]) \to \mathrm{Hom}(F(E), F(E')[i])$ induced by F is bijective for all $E, E' \in S$ and $i \in \mathbf{Z}$.*

Then F is fully faithful.

Proof. Denote F_r and F_l the right and left adjoints of F. For $E \in S$ choose a distinguished triangle

$$E \to F_r(F(E)) \to C \to E[1]$$

where the first arrow is the unit of the adjunction. For $E' \in S$ we have

$$\mathrm{Hom}(E', F_r(F(E))[i]) = \mathrm{Hom}(F(E'), F(E)[i]) = \mathrm{Hom}(E', E[i])$$

The last equality holds by assumption (4). Hence applying the homological functor $\mathrm{Hom}(E', -)$ (Derived Categories, Lemma 13.4.2) to the distinguished triangle above we conclude that $\mathrm{Hom}(E', C[i]) = 0$ for all $i \in \mathbf{Z}$ and $E' \in S$. By assumption (2) we conclude that $C = 0$ and $E = F_r(F(E))$.

For $K \in \mathrm{Ob}(\mathcal{D})$ choose a distinguished triangle

$$F_l(F(K)) \to K \to C \to F_l(F(K))[1]$$

where the first arrow is the counit of the adjunction. For $E \in S$ we have

$$\mathrm{Hom}(F_l(F(K)), E[i]) = \mathrm{Hom}(F(K), F(E)[i]) = \mathrm{Hom}(K, F_r(F(E))[i]) = \mathrm{Hom}(K, E[i])$$

where the last equality holds by the result of the first paragraph. Thus we conclude as before that $\mathrm{Hom}(C, E[i]) = 0$ for all $E \in S$ and $i \in \mathbf{Z}$. Hence $C = 0$ by assumption (3). Thus F is fully faithful by Categories, Lemma 4.24.4. □

0G02 **Lemma 57.11.2.** *Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ supported at x choose a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ supported at x such that $\mathcal{F}_x$ and $\mathcal{F}'_x$ are Matlis dual. Then there is an isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, M) = H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x])$$

where $d_x = \dim(\mathcal{O}_{X,x})$ functorial in M in $D_{perf}(\mathcal{O}_X)$.

Proof. Since $\mathcal{F}$ is supported at x we have

$$\mathrm{Hom}_X(\mathcal{F}, M) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M_x)$$

and similarly we have

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}'[-d_x]) = \mathrm{Tor}^{\mathcal{O}_{X,x}}_{d_x}(M_x, \mathcal{F}'_x)$$

Thus it suffices to show that given a Noetherian regular local ring A of dimension d and a finite length A-module N, if N' is the Matlis dual to N, then there exists a functorial isomorphism

$$\mathrm{Hom}_A(N, K) = \mathrm{Tor}^A_d(K, N')$$

for K in $D_{perf}(A)$. We can write the left hand side as $H^0(R\mathrm{Hom}_A(N, A) \otimes^{\mathbf{L}}_A K)$ by More on Algebra, Lemma 15.75.15 and the fact that N determines a perfect object of $D(A)$. Hence the formula holds because

$$R\mathrm{Hom}_A(N, A) = R\mathrm{Hom}_A(N, A[d])[-d] = N'[-d]$$

by Dualizing Complexes, Lemma 47.16.4 and the fact that $A[d]$ is a normalized dualizing complex over A (A is Gorenstein by Dualizing Complexes, Lemma 47.21.3). □

0G03 **Lemma 57.11.3.** *Let k be a field. Let X be a scheme of finite type over k which is regular. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D_{perf}(\mathcal{O}_X)$.*

(1) *If $\mathrm{Ext}^i_X(\mathcal{O}_x, K) = 0$ then there exists an open neighbourhood U of x such that $H^{i-d_x}(K)|_U = 0$ where $d_x = \dim(\mathcal{O}_{X,x})$.*
(2) *If $\mathrm{Hom}_X(\mathcal{O}_x, K[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.*
(3) *If $\mathrm{Ext}^i_X(K, \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K^\vee)|_U = 0$.*
(4) *If $\mathrm{Hom}_X(K, \mathcal{O}_x[i]) = 0$ for all $i \in \mathbf{Z}$, then K is zero in an open neighbourhood of x.*
(5) *If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ then there exists an open neighbourhood U of x such that $H^i(K)|_U = 0$.*
(6) *If $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x) = 0$ for $i \in \mathbf{Z}$ then K is zero in an open neighbourhood of x.*

Proof. Observe that $H^i(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ is equal to $K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \kappa(x)$. Hence part (5) follows from More on Algebra, Lemma 15.77.4. Part (6) follows from part (5). Part (1) follows from part (5), Lemma 57.11.2, and the fact that the Matlis dual of $\kappa(x)$ is $\kappa(x)$. Part (2) follows from part (1). Part (3) follows from part (5) and the fact that $\mathrm{Ext}^i(K, \mathcal{O}_x) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x)$ by Cohomology, Lemma 20.50.5. Part (4) follows from part (3) and the fact that $K \cong (K^\vee)^\vee$ by the lemma just cited. □

0GWZ **Lemma 57.11.4.** *Let X be a Noetherian scheme. Let $x \in X$ be a closed point and denote $\mathcal{O}_x$ the skyscraper sheaf at x with value $\kappa(x)$. Let K in $D^b_{Coh}(\mathcal{O}_X)$. Let $b \in \mathbf{Z}$. The following are equivalent*

(1) *$H^i(K)_x = 0$ for all $i > b$ and*
(2) *$\mathrm{Hom}_X(K, \mathcal{O}_x[-i]) = 0$ for all $i > b$.*

Proof. Consider the complex K_x in $D^b_{Coh}(\mathcal{O}_{X,x})$. There exist an integer $b_x \in \mathbf{Z}$ such that K_x can be represented by a bounded above complex

$$\ldots \to \mathcal{O}_{X,x}^{\oplus n_{b_x-2}} \to \mathcal{O}_{X,x}^{\oplus n_{b_x-1}} \to \mathcal{O}_{X,x}^{\oplus n_{b_x}} \to 0 \to \ldots$$

with $\mathcal{O}_{X,x}^{\oplus n_i}$ sitting in degree i where all the transition maps are given by matrices whose coefficients are in $\mathfrak{m}_x$. See More on Algebra, Lemma 15.76.6. The result follows easily from this (and the equivalent conditions hold if and only if $b \geq b_x$). □

0G25 **Lemma 57.11.5.** *Let k be a field. Let X and Y be proper schemes over k. Assume X is regular. Then a k-linear exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is fully faithful if and only if for any closed points $x, x' \in X$ the maps*

$$F : \mathrm{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \mathrm{Ext}^i_Y(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$. Here $\mathcal{O}_x$ is the skyscraper sheaf at x with value $\kappa(x)$.

Proof. By Lemma 57.7.1 the functor F has both a left and a right adjoint. Thus we may apply the criterion of Lemma 57.11.1 because assumptions (2) and (3) of that lemma follow from Lemma 57.11.3. □

0G26 **Lemma 57.11.6.** *Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume*

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for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then F is fully faithful.

Proof. By Lemma 57.11.5 it suffices to show that the maps

$$F : \mathrm{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_{x'}) \longrightarrow \mathrm{Ext}^i_X(F(\mathcal{O}_x), F(\mathcal{O}_{x'}))$$

are isomorphisms for all $i \in \mathbf{Z}$ and all closed points $x, x' \in X$. By assumption, the source and the target are isomorphic. If $x \neq x'$, then both sides are zero and the result is true. If $x = x'$, then it suffices to prove that the map is either injective or surjective. For $i < 0$ both sides are zero and the result is true. For $i = 0$ any nonzero map $\alpha : \mathcal{O}_x \to \mathcal{O}_x$ of $\mathcal{O}_X$-modules is an isomorphism. Hence $F(\alpha)$ is an isomorphism too and so $F(\alpha)$ is nonzero. Thus the result for $i = 0$. For $i = 1$ a nonzero element ξ in $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$ corresponds to a nonsplit short exact sequence

$$0 \to \mathcal{O}_x \to \mathcal{F} \to \mathcal{O}_x \to 0$$

Since $F(\mathcal{F}) \cong \mathcal{F}$ we see that $F(\mathcal{F})$ is a nonsplit extension of $\mathcal{O}_x$ by $\mathcal{O}_x$ as well. Since $\mathcal{O}_x \cong F(\mathcal{O}_x)$ is a simple $\mathcal{O}_X$-module and $\mathcal{F} \cong F(\mathcal{F})$ has length 2, we see that in the distinguished triangle

$$F(\mathcal{O}_x) \to F(\mathcal{F}) \to F(\mathcal{O}_x) \xrightarrow{F(\xi)} F(\mathcal{O}_x)[1]$$

the first two arrows must form a short exact sequence which must be isomorphic to the above short exact sequence and hence is nonsplit. It follows that $F(\xi)$ is nonzero and we conclude for $i = 1$. For $i > 1$ composition of ext classes defines a surjection

$$\mathrm{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \otimes \ldots \otimes \mathrm{Ext}^1(F(\mathcal{O}_x), F(\mathcal{O}_x)) \longrightarrow \mathrm{Ext}^i(F(\mathcal{O}_x), F(\mathcal{O}_x))$$

See Duality for Schemes, Lemma 48.15.4. Hence surjectivity in degree 1 implies surjectivity for $i > 0$. This finishes the proof. □

57.12. Special functors

0FZY In this section we prove some results on functors of a special type that we will use later in this chapter.

0FZZ **Definition 57.12.1.** Let k be a field. Let X, Y be finite type schemes over k. Recall that $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 36.11.2. We say two k-linear exact functors

$$F, F' : D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X)) \longrightarrow D^b_{Coh}(\mathcal{O}_Y)$$

are *siblings*, or we say F' is a *sibling* of F if F and F' are siblings in the sense of Definition 57.10.1 with abelian category being $Coh(\mathcal{O}_X)$. If X is regular then $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.6 and we use the same terminology for k-linear exact functors $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

0G00 **Lemma 57.12.2.** *Let k be a field. Let X, Y be finite type schemes over k with X separated. Let $F : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ be a k-linear exact functor sending $Coh(\mathcal{O}_X) \subset D^b_{Coh}(\mathcal{O}_X)$ into $Coh(\mathcal{O}_Y) \subset D^b_{Coh}(\mathcal{O}_Y)$. Then there exists a Fourier-Mukai functor $F' : D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_Y)$ whose kernel is a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{K}$ flat over X and with support finite over Y which is a sibling of F.*

Proof. Denote $H : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ the restriction of F. Since F is an exact functor of triangulated categories, we see that H is an exact functor of abelian categories. Of course H is k-linear as F is. By Functors and Morphisms, Lemma 56.7.5 we obtain a coherent $\mathcal{O}_{X\times Y}$-module $\mathcal{K}$ which is flat over X and has support finite over Y. Let F' be the Fourier-Mukai functor defined using $\mathcal{K}$ so that F' restricts to H on $Coh(\mathcal{O}_X)$. The functor F' sends $D^b_{Coh}(\mathcal{O}_X)$ into $D^b_{Coh}(\mathcal{O}_Y)$ by Lemma 57.8.5. Observe that F and F' satisfy the first and second condition of Lemma 57.10.2 and hence are siblings. □

0G01 **Remark 57.12.3.** If $F, F' : D^b_{Coh}(\mathcal{O}_X) \to \mathcal{D}$ are siblings, F is fully faithful, and X is reduced and projective over k then $F \cong F'$; this follows from Proposition 57.10.6 via the argument given in the proof of Theorem 57.13.3. However, in general we do not know whether siblings are isomorphic. Even in the situation of Lemma 57.12.2 it seems difficult to prove that the siblings F and F' are isomorphic functors. If X is smooth and proper over k and F is fully faithful, then $F \cong F'$ as is shown in [**Ola20**]. If you have a proof or a counter example in more general situations, please email stacks.project@gmail.com.

0GX0 **Lemma 57.12.4.** *Let k be a field. Let X, Y be proper schemes over k. Assume X is regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that*

(1) *$F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$,*
(2) *F is fully faithful.*

Then the essential image of G is contained in the essential image of F.

Proof. Recall that F and G have both adjoints, see Lemma 57.7.1. In particular the essential image $\mathcal{A} \subset D_{perf}(\mathcal{O}_Y)$ of F satisfies the equivalent conditions of Derived Categories, Lemma 13.40.7. We claim that G factors through $\mathcal{A}$. Since $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$ by Derived Categories, Lemma 13.40.7 it suffices to show that $\mathrm{Hom}_Y(G(M), N) = 0$ for all M in $D_{perf}(\mathcal{O}_X)$ and $N \in \mathcal{A}^\perp$. We have

$$\mathrm{Hom}_Y(G(M), N) = \mathrm{Hom}_X(M, G_r(N))$$

where G_r is the right adjoint to G. Thus it suffices to prove that $G_r(N) = 0$. Since $G(\mathcal{F}) \cong F(\mathcal{F})$ for $\mathcal{F}$ as in (1) we see that

$$\mathrm{Hom}_X(\mathcal{F}, G_r(N)) = \mathrm{Hom}_Y(G(\mathcal{F}), N) = \mathrm{Hom}_Y(F(\mathcal{F}), N) = 0$$

as N is in the right orthogonal to the essential image $\mathcal{A}$ of F. Of course, the same vanishing holds for $\mathrm{Hom}_X(\mathcal{F}, G_r(N)[i])$ for any $i \in \mathbf{Z}$. Thus $G_r(N) = 0$ by Lemma 57.11.3 and we win. □

0G27 **Lemma 57.12.5.** *Let k be a field. Let X be a proper scheme over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ there is an isomorphism $\mathcal{F} \cong F(\mathcal{F})$. Then there exists an automorphism $f : X \to X$ over k which induces the identity on the underlying topological space[1] and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that F and $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$ are siblings.*

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Proof. By Lemma 57.11.6 the functor F is fully faithful. By Lemma 57.12.4 the essential image of the identity functor is contained in the essential image of F, i.e.,

[1] This often forces f to be the identity, see Varieties, Lemma 33.32.1.

we see that F is essentially surjective. Thus F is an equivalence. Observe that the quasi-inverse F^{-1} satisfies the same assumptions as F.

Let $M \in D_{perf}(\mathcal{O}_X)$ and say $H^i(M) = 0$ for $i > b$. Since F is fully faithful, we see that

$$\mathrm{Hom}_X(M, \mathcal{O}_x[-i]) = \mathrm{Hom}_X(F(M), F(\mathcal{O}_x)[-i]) \cong \mathrm{Hom}_X(F(M), \mathcal{O}_x[-i])$$

for any $i \in \mathbf{Z}$ for any closed point x of X. Thus by Lemma 57.11.4 we see that $F(M)$ has vanishing cohomology sheaves in degrees $> b$.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By the above $F(\mathcal{F})$ has nonzero cohomology sheaves only in degrees ≤ 0. Set $\mathcal{G} = H^0(F(\mathcal{F}))$. Choose a distinguished triangle

$$K \to F(\mathcal{F}) \to \mathcal{G} \to K[1]$$

Then K has nonvanishing cohomology sheaves only in degrees ≤ -1. Applying F^{-1} we obtain a distinguished triangle

$$F^{-1}(K) \to \mathcal{F} \to F^{-1}(\mathcal{G}) \to F^{-1}(K')[1]$$

Since $F^{-1}(K)$ has nonvanishing cohomology sheaves only in degrees ≤ -1 (by the previous paragraph applied to F^{-1}) we see that the arrow $F^{-1}(K) \to \mathcal{F}$ is zero (Derived Categories, Lemma 13.27.3). Hence $K \to F(\mathcal{F})$ is zero, which implies that $F(\mathcal{F}) = \mathcal{G}$ by our choice of the first distinguished triangle.

From the preceding paragraph, we deduce that F preserves $Coh(\mathcal{O}_X)$ and indeed defines an equivalence $H : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_X)$. By Functors and Morphisms, Lemma 56.7.8 we get an automorphism $f : X \to X$ over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that $H(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Set $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$. Using Lemma 57.10.2 we see that F and F' are siblings. To see that f is the identity on the underlying topological space of X, we use that $F(\mathcal{O}_x) \cong \mathcal{O}_x$ and that the support of $\mathcal{O}_x$ is $\{x\}$. This finishes the proof. □

0G06 **Lemma 57.12.6.** *Let k be a field. Let X, Y be proper schemes over k. Assume X regular. Let $F, G : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ be k-linear exact functors such that*

(1) *$F(\mathcal{F}) \cong G(\mathcal{F})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$,*
(2) *F is fully faithful, and*
(3) *G is a Fourier-Mukai functor whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$.*

Then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ such that F and F' are siblings.

Proof. The essential image of G is contained in the essential image of F by Lemma 57.12.4. Consider the functor $H = F^{-1} \circ G$ which makes sense as F is fully faithful. By Lemma 57.12.5 we obtain an automorphism $f : X \to X$ and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that the functor $H' : K \mapsto f^*K \otimes \mathcal{L}$ is a sibling of H. In particular H is an auto-equivalence by Lemma 57.10.3 and H induces an auto-equivalence of $Coh(\mathcal{O}_X)$ (as this is true for its sibling functor H'). Thus the quasi-inverses H^{-1} and $(H')^{-1}$ exist, are siblings (small detail omitted), and $(H')^{-1}$ sends M to $(f^{-1})^*(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ which is a Fourier-Mukai functor (details omitted). Then of course $F = G \circ H^{-1}$ is a sibling of $G \circ (H')^{-1}$. Since compositions of Fourier-Mukai functors are Fourier-Mukai by Lemma 57.8.3 we conclude. □

57.13. Fully faithful functors

0G07 Our goal is to prove fully faithful functors between derived categories are siblings of Fourier-Mukai functors, following [**Orl97**] and [**Bal08**].

0G08 **Situation 57.13.1.** Here k is a field. We have proper smooth schemes X and Y over k. We have a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Before reading on, it makes sense to read at least some of Derived Categories, Section 13.41.

Recall that X is regular and hence has the resolution property (Varieties, Lemma 33.25.3 and Derived Categories of Schemes, Lemma 36.36.8). Thus on $X \times X$ we may choose a resolution

$$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_\Delta \to 0$$

where each $\mathcal{E}_i$ and $\mathcal{G}_i$ is a finite locally free $\mathcal{O}_X$-module, see Lemma 57.9.3. Using the complex

0G09 (57.13.1.1) $$\ldots \to \mathcal{E}_2 \boxtimes \mathcal{G}_2 \to \mathcal{E}_1 \boxtimes \mathcal{G}_1 \to \mathcal{E}_0 \boxtimes \mathcal{G}_0$$

in $D_{perf}(\mathcal{O}_{X\times X})$ as in Derived Categories, Example 13.41.2 if for each n we denote

$$M_n = (\mathcal{E}_n \boxtimes \mathcal{G}_n \to \ldots \to \mathcal{E}_0 \boxtimes \mathcal{G}_0)[-n]$$

we obtain an infinite Postnikov system for the complex (57.13.1.1). This means the morphisms $M_0 \to M_1[1] \to M_2[2] \to \ldots$ and $M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n$ and $\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1}$ satisfy certain conditions documented in Derived Categories, Definition 13.41.1. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{E}_n \boxtimes \mathcal{G}_n \to \mathcal{E}_{n-1} \boxtimes \mathcal{G}_{n-1})$$

Observe that since $\mathcal{O}_\Delta$ is flat over X via pr_1 the same is true for $\mathcal{F}_n$ for all n (this is a convenient though not essential observation). We have

$$H^q(M_n[n]) = \begin{cases} \mathcal{O}_\Delta & \text{if} \quad q = 0 \\ \mathcal{F}_n & \text{if} \quad q = -n \\ 0 & \text{if} \quad q \neq 0, -n \end{cases}$$

Thus for $n \geq \dim(X \times X)$ we have

$$M_n[n] \cong \mathcal{O}_\Delta \oplus \mathcal{F}_n[n]$$

in $D_{perf}(\mathcal{O}_{X\times X})$ by Lemma 57.9.5.

We are interested in the complex

0G0A (57.13.1.2) $$\ldots \to \mathcal{E}_2 \boxtimes F(\mathcal{G}_2) \to \mathcal{E}_1 \boxtimes F(\mathcal{G}_1) \to \mathcal{E}_0 \boxtimes F(\mathcal{G}_0)$$

in $D_{perf}(\mathcal{O}_{X\times Y})$ as the "totalization" of this complex should give us the kernel of the Fourier-Mukai functor we are trying to construct. For all $i, j \geq 0$ we have

$$\begin{aligned} \mathrm{Ext}^q_{X\times Y}(\mathcal{E}_i \boxtimes F(\mathcal{G}_i), \mathcal{E}_j \boxtimes F(\mathcal{G}_j)) &= \bigoplus\nolimits_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_Y(F(\mathcal{G}_i), F(\mathcal{G}_j)) \\ &= \bigoplus\nolimits_p \mathrm{Ext}^{q+p}_X(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \mathrm{Ext}^{-p}_X(\mathcal{G}_i, \mathcal{G}_j) \end{aligned}$$

The second equality holds because F is fully faithful and the first by Derived Categories of Schemes, Lemma 36.25.1. We find these Ext^q are zero for $q < 0$. Hence by Derived Categories, Lemma 13.41.6 we can build an infinite Postnikov system

$K_0, K_1, K_2, \ldots$ in $D_{perf}(\mathcal{O}_{X\times Y})$ for the complex (57.13.1.2). Parallel to what happens with $M_0, M_1, M_2, \ldots$ this means we obtain morphisms $K_0 \to K_1[1] \to K_2[2] \to \ldots$ and $K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$ and $\mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$ in $D_{perf}(\mathcal{O}_{X\times Y})$ satisfying certain conditions documented in Derived Categories, Definition 13.41.1.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module whose support has a finite number of points, i.e., with $\dim(\mathrm{Supp}(\mathcal{F})) = 0$. Consider the exact functor of triangulated categories

$$D_{perf}(\mathcal{O}_{X\times Y}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} N)$$

It follows that the objects $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_i)$ form a Postnikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$R\mathrm{pr}_{2,*}((\mathcal{F}\otimes\mathcal{E}_i)\boxtimes F(\mathcal{G}_i)) = \Gamma(X, \mathcal{F}\otimes\mathcal{E}_i)\otimes_k F(\mathcal{G}_i) = F(\Gamma(X, \mathcal{F}\otimes\mathcal{E}_i)\otimes_k \mathcal{G}_i)$$

Here we have used that $\mathcal{F}\otimes\mathcal{E}_i$ has vanishing higher cohomology as its support has dimension 0. On the other hand, applying the exact functor

$$D_{perf}(\mathcal{O}_{X\times X}) \longrightarrow D_{perf}(\mathcal{O}_Y), \quad N \longmapsto F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} N))$$

we find that the objects $F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} M_n))$ form a second infinite Postnikov system for the complex in $D_{perf}(\mathcal{O}_Y)$ with terms

$$F(R\mathrm{pr}_{2,*}((\mathcal{F}\otimes\mathcal{E}_i)\boxtimes\mathcal{G}_i)) = F(\Gamma(X, \mathcal{F}\otimes\mathcal{E}_i)\otimes_k \mathcal{G}_i)$$

This is the same as before! By uniqueness of Postnikov systems (Derived Categories, Lemma 13.41.6) which applies because

$$\mathrm{Ext}^q_Y(F(\Gamma(X, \mathcal{F}\otimes\mathcal{E}_i)\otimes_k \mathcal{G}_i), F(\Gamma(X, \mathcal{F}\otimes\mathcal{E}_j)\otimes_k \mathcal{G}_j)) = 0, \quad q < 0$$

as F is fully faithful, we find a system of isomorphisms

$$F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} M_n[n])) \cong R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n[n])$$

in $D_{perf}(\mathcal{O}_Y)$ compatible with the morphisms in $D_{perf}(\mathcal{O}_Y)$ induced by the morphisms

$$M_{n-1}[n-1] \to M_n[n] \quad \text{and} \quad K_{n-1}[n-1] \to K_n[n]$$
$$M_n \to \mathcal{E}_n \boxtimes \mathcal{G}_n \quad \text{and} \quad K_n \to \mathcal{E}_n \boxtimes F(\mathcal{G}_n)$$
$$\mathcal{E}_n \boxtimes \mathcal{G}_n \to M_{n-1} \quad \text{and} \quad \mathcal{E}_n \boxtimes F(\mathcal{G}_n) \to K_{n-1}$$

which are part of the structure of Postnikov systems. For n sufficiently large we obtain a direct sum decomposition

$$F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} M_n[n])) = F(\mathcal{F}) \oplus F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}_n))[n]$$

corresponding to the direct sum decomposition of M_n constructed above (we are using the flatness of $\mathcal{F}_n$ over X via pr_1 to write a usual tensor product in the formula above, but this isn't essential for the argument). By Lemma 57.9.9 we find there exists an integer $m \geq 0$ such that the first summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m, m]$ and the second summand in this direct sum decomposition has nonzero cohomology sheaves only in the interval $[-m-n, m+\dim(X)-n]$. We conclude the system $K_0 \to K_1[1] \to K_2[2] \to \ldots$ in $D_{perf}(\mathcal{O}_{X\times Y})$ satisfies the assumptions of Lemma 57.9.10 after possibly replacing m by a larger integer. We conclude we can write

$$K_n[n] = K \oplus C_n$$

for $n \gg 0$ compatible with transition maps and with C_n having nonzero cohomology sheaves only in the range $[-m-n, m-n]$. Denote G the Fourier-Mukai functor corresponding to K. Putting everything together we find

$$\begin{gathered} G(\mathcal{F}) \oplus R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} C_n) \cong \\ R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K_n[n]) \cong \\ F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} M_n[n])) \cong \\ F(\mathcal{F}) \oplus F(R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}_n))[n] \end{gathered}$$

Looking at the degrees that objects live in we conclude that for $n \gg m$ we obtain an isomorphism

$$F(\mathcal{F}) \cong G(\mathcal{F})$$

Moreover, recall that this holds for every coherent $\mathcal{F}$ on X whose support has dimension 0.

0G0B **Lemma 57.13.2.** *Let k be a field. Let X and Y be smooth proper schemes over k. Given a k-linear, exact, fully faithful functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ which is a sibling to F.*

Proof. Apply Lemma 57.12.6 to F and the functor G constructed above. □

The following theorem is also true without assuming X is projective, see [**Ola20**].

0G0C **Theorem 57.13.3** (Orlov)**.** *Let k be a field. Let X and Y be smooth proper schemes over k with X projective over k. Any k-linear fully faithful exact functor $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a Fourier-Mukai functor for some kernel in $D_{perf}(\mathcal{O}_{X\times Y})$.*

[**Orl97**, Theorem 2.2]; this is shown in [**Ola20**] without the assumption that X be projective

Proof. Let F' be the Fourier-Mukai functor which is a sibling of F as in Lemma 57.13.2. By Proposition 57.10.6 we have $F \cong F'$ provided we can show that $Coh(\mathcal{O}_X)$ has enough negative objects. However, if $X = \mathrm{Spec}(k)$ for example, then this isn't true. Thus we first decompose $X = \coprod X_i$ into its connected (and irreducible) components and we argue that it suffices to prove the result for each of the (fully faithful) composition functors

$$F_i : D_{perf}(\mathcal{O}_{X_i}) \to D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$$

Details omitted. Thus we may assume X is irreducible.

The case $\dim(X) = 0$. Here X is the spectrum of a finite (separable) extension k'/k and hence $D_{perf}(\mathcal{O}_X)$ is equivalent to the category of graded k'-vector spaces such that $\mathcal{O}_X$ corresponds to the trivial 1-dimensional vector space in degree 0. It is straightforward to see that any two siblings $F, F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ are isomorphic. Namely, we are given an isomorphism $F(\mathcal{O}_X) \cong F'(\mathcal{O}_X)$ compatible the action of the k-algebra $k' = \mathrm{End}_{D_{perf}(\mathcal{O}_X)}(\mathcal{O}_X)$ which extends canonically to an isomorphism on any graded k'-vector space.

The case $\dim(X) > 0$. Here X is a projective smooth variety of dimension > 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We have to show there exists a coherent module $\mathcal{N}$ such that

(1) there is a surjection $\mathcal{N} \to \mathcal{F}$ and
(2) $\mathrm{Hom}(\mathcal{F}, \mathcal{N}) = 0$.

Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. We claim that $\mathcal{N} = (\mathcal{L}^{\otimes n})^{\oplus r}$ will work for $n \ll 0$ and r large enough. Condition (1) follows from Properties, Proposition 28.26.13. Finally, we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{L}^{\otimes n}) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{L}^{\otimes n})) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{L}^{\otimes n})$$

Since the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is torsion free, this vanishes for $n \ll 0$ by Varieties, Lemma 33.48.1. This finishes the proof. □

0G0D **Proposition 57.13.4.** *Let k be a field. Let X and Y be smooth proper schemes over k. If $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ is a k-linear exact equivalence of triangulated categories then there exists a Fourier-Mukai functor $F' : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ whose kernel is in $D_{perf}(\mathcal{O}_{X\times Y})$ which is an equivalence and a sibling of F.*

Proof. The functor F' of Lemma 57.13.2 is an equivalence by Lemma 57.10.3. □

0G0E **Lemma 57.13.5.** *Let k be a field. Let X be a smooth proper scheme over k. Let $K \in D_{perf}(\mathcal{O}_{X\times X})$. If the Fourier-Mukai functor $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ is isomorphic to the identity functor, then $K \cong \Delta_*\mathcal{O}_X$ in ${}_{perf}(\mathcal{O}_{X\times X})$.*

Proof. Let i be the minimal integer such that the cohomology sheaf $H^i(K)$ is nonzero. Let $\mathcal{E}$ and $\mathcal{G}$ be finite locally free $\mathcal{O}_X$-modules. Then

$$\begin{aligned} H^i(X \times X, K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} (\mathcal{E} \boxtimes \mathcal{G})) &= H^i(X, R\mathrm{pr}_{2,*}(K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} (\mathcal{E} \boxtimes \mathcal{G}))) \\ &= H^i(X, \Phi_K(\mathcal{E}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \\ &\cong H^i(X, \mathcal{E} \otimes \mathcal{G}) \end{aligned}$$

which is zero if $i < 0$. On the other hand, we can choose $\mathcal{E}$ and $\mathcal{G}$ such that there is a surjection $\mathcal{E}^\vee \boxtimes \mathcal{G}^\vee \to H^i(K)$ by Lemma 57.9.1. In this case the left hand side of the equalities is nonzero. Hence we conclude that $H^i(K) = 0$ for $i < 0$.

Let i be the maximal integer such that $H^i(K)$ is nonzero. The same argument with $\mathcal{E}$ and $\mathcal{G}$ support of dimension 0 shows that $i \leq 0$. Hence we conclude that K is given by a single coherent $\mathcal{O}_{X\times X}$-module $\mathcal{K}$ sitting in degree 0.

Since $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{K})$ is $\mathcal{F}$, by taking $\mathcal{F}$ supported at closed points we see that the support of $\mathcal{K}$ is finite over X via pr_2. Since $R\mathrm{pr}_{2,*}(\mathcal{K}) \cong \mathcal{O}_X$ we conclude by Functors and Morphisms, Lemma 56.7.6 that $\mathcal{K} = s_*\mathcal{O}_X$ for some section $s : X \to X \times X$ of the second projection. Then $\Phi_K(M) = f^*M$ where $f = \mathrm{pr}_1 \circ s$ and this can happen only if s is the diagonal morphism as desired. □

57.14. A category of Fourier-Mukai kernels

0G0F Let S be a scheme. We claim there is a category with

(1) Objects are proper smooth schemes over S.
(2) Morphisms from X to Y are isomorphism classes of objects of $D_{perf}(\mathcal{O}_{X\times_S Y})$.
(3) Composition of the isomorphism class of $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$ and the isomorphism class of K' in $D_{perf}(\mathcal{O}_{Y\times_S Z})$ is the isomorphism class of

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S Z}} L\mathrm{pr}_{23}^*K')$$

which is in $D_{perf}(\mathcal{O}_{X\times_S Z})$ by Derived Categories of Schemes, Lemma 36.30.4.

(4) The identity morphism from X to X is the isomorphism class of $\Delta_{X/S,*}\mathcal{O}_X$ which is in $D_{perf}(\mathcal{O}_{X\times_S X})$ by More on Morphisms, Lemma 37.61.12 and the fact that $\Delta_{X/S}$ is a perfect morphism by Divisors, Lemma 31.22.11 and More on Morphisms, Lemma 37.61.7.

Let us check that associativity of composition of morphisms holds; we omit verifying that the identity morphisms are indeed identities. To see this suppose we have X, Y, Z, W and $c \in D_{perf}(\mathcal{O}_{X\times_S Y})$, $c' \in D_{perf}(\mathcal{O}_{Y\times_S Z})$, and $c'' \in D_{perf}(\mathcal{O}_{Z\times_S W})$. Then we have

$$\begin{aligned} c'' \circ (c' \circ c) &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{12}c \otimes \mathrm{pr}^{123,*}_{23}c') \otimes \mathrm{pr}^{134,*}_{34}c'') \\ &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}(\mathrm{pr}^{123,*}_{12}c \otimes \mathrm{pr}^{123,*}_{23}c') \otimes \mathrm{pr}^{134,*}_{34}c'') \\ &\cong \mathrm{pr}^{134}_{14,*}(\mathrm{pr}^{1234}_{134,*}(\mathrm{pr}^{1234,*}_{12}c \otimes \mathrm{pr}^{1234,*}_{23}c') \otimes \mathrm{pr}^{134,*}_{34}c'') \\ &\cong \mathrm{pr}^{134}_{14,*}\mathrm{pr}^{1234}_{134,*}((\mathrm{pr}^{1234,*}_{12}c \otimes \mathrm{pr}^{1234,*}_{23}c') \otimes \mathrm{pr}^{1234,*}_{34}c'') \\ &\cong \mathrm{pr}^{1234}_{14,*}((\mathrm{pr}^{1234,*}_{12}c \otimes \mathrm{pr}^{1234,*}_{23}c') \otimes \mathrm{pr}^{1234,*}_{34}c'') \end{aligned}$$

Here we use the notation

$$p^{1234}_{134} : X \times_S Y \times_S Z \times_S W \to X \times_S Z \times_S W \quad \text{and} \quad p^{134}_{14} : X \times_S Z \times_S W \to X \times_S W$$

the projections and similarly for other indices. We also write pr_* instead of $R\mathrm{pr}_*$ and pr^* instead of $L\mathrm{pr}^*$ and we drop all super and sub scripts on $\otimes$. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}^{134,*}_{13}\mathrm{pr}^{123}_{13,*} = \mathrm{pr}^{1234}_{134,*}\mathrm{pr}^{1234,*}_{123}$ by base change (Derived Categories of Schemes, Lemma 36.22.5). The third equality holds because pullbacks compose correctly and pass through tensor products, see Cohomology, Lemmas 20.27.2 and 20.27.3. The fourth equality follows from the "projection formula" for p^{1234}_{134}, see Derived Categories of Schemes, Lemma 36.22.1. The fifth equality is that proper pushforward is compatible with composition, see Cohomology, Lemma 20.28.2. Since tensor product is associative this concludes the proof of associativity of composition.

0G0G **Lemma 57.14.1.** *Let $S' \to S$ be a morphism of schemes. The rule which sends*

(1) *a smooth proper scheme X over S to $X' = S' \times_S X$, and*
(2) *the isomorphism class of an object K of $D_{perf}(\mathcal{O}_{X\times_S Y})$ to the isomorphism class of $L(X' \times_{S'} Y' \to X \times_S Y)^*K$ in $D_{perf}(\mathcal{O}_{X'\times_{S'}Y'})$*

is a functor from the category defined for S to the category defined for S'.

Proof. To see this suppose we have X, Y, Z and $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$ and $M \in D_{perf}(\mathcal{O}_{Y\times_S Z})$. Denote $K' \in D_{perf}(\mathcal{O}_{X'\times_{S'}Y'})$ and $M' \in D_{perf}(\mathcal{O}_{Y'\times_{S'}Z'})$ their pullbacks as in the statement of the lemma. The diagram

$$\begin{array}{ccc} X' \times_{S'} Y' \times_{S'} Z' & \longrightarrow & X \times_S Y \times_S Z \\ \downarrow{\scriptstyle \mathrm{pr}'_{13}} & & \downarrow{\scriptstyle \mathrm{pr}_{13}} \\ X' \times_{S'} Z' & \longrightarrow & X \times_S Z \end{array}$$

is cartesian and pr_{13} is proper and smooth. By Derived Categories of Schemes, Lemma 36.30.4 we see that the derived pullback by the lower horizontal arrow of the composition

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S Z}} L\mathrm{pr}_{23}^*M)$$

indeed is (canonically) isomorphic to

$$R\mathrm{pr}'_{13,*}(L(\mathrm{pr}'_{12})^*K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'\times_{S'}Y'\times_{S'}Z'}} L(\mathrm{pr}'_{23})^*M')$$

as desired. Some details omitted. □

57.15. Relative equivalences

0G0H In this section we prove some lemmas about the following concept.

0G0I **Definition 57.15.1.** Let S be a scheme. Let $X \to S$ and $Y \to S$ be smooth proper morphisms. An object $K \in D_{perf}(\mathcal{O}_{X\times_S Y})$ is said to be *the Fourier-Mukai kernel of a relative equivalence from X to Y over S* if there exist an object $K' \in D_{perf}(\mathcal{O}_{X\times_S Y})$ such that

$$\Delta_{X/S,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y\times_S X}} L\mathrm{pr}_{23}^*K')$$

in $D(\mathcal{O}_{X\times_S X})$ and

$$\Delta_{Y/S,*}\mathcal{O}_Y \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K' \otimes^{\mathbf{L}}_{\mathcal{O}_{Y\times_S X\times_S Y}} L\mathrm{pr}_{23}^*K)$$

in $D(\mathcal{O}_{Y\times_S Y})$. In other words, the isomorphism class of K defines an invertible arrow in the category defined in Section 57.14.

The language is intentionally cumbersome.

0G0J **Lemma 57.15.2.** *With notation as in Definition 57.15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Then the corresponding Fourier-Mukai functors $\Phi_K : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ (Lemma 57.8.2) and $\Phi_K : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ (Lemma 57.8.4) are equivalences.*

Proof. Immediate from Lemma 57.8.3 and Example 57.8.6. □

0G0K **Lemma 57.15.3.** *With notation as in Definition 57.15.1 let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Let $S_1 \to S$ be a morphism of schemes. Let $X_1 = S_1 \times_S X$ and $Y_1 = S_1 \times_S Y$. Then the pullback $K_1 = L(X_1 \times_{S_1} Y_1 \to X \times_S Y)^*K$ is the Fourier-Mukai kernel of a relative equivalence from X_1 to Y_1 over S_1.*

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y\times_S X})$ be the object assumed to exist in Definition 57.15.1. Denote K'_1 the pullback of K' by $Y_1 \times_{S_1} X_1 \to Y \times_S X$. Then it suffices to prove that we have

$$\Delta_{X_1/S_1,*}\mathcal{O}_X \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*K_1 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_1\times_{S_1}Y_1\times_{S_1}X_1}} L\mathrm{pr}_{23}^*K'_1)$$

in $D(\mathcal{O}_{X_1\times_{S_1}X_1})$ and similarly for the other condition. Since

$$\begin{array}{ccc} X_1 \times_{S_1} Y_1 \times_{S_1} X_1 & \longrightarrow & X \times_S Y \times_S X \\ \downarrow{\scriptstyle \mathrm{pr}_{13}} & & \downarrow{\scriptstyle \mathrm{pr}_{13}} \\ X_1 \times_{S_1} X_1 & \longrightarrow & X \times_S X \end{array}$$

is cartesian it suffices by Derived Categories of Schemes, Lemma 36.30.4 to prove that

$$\Delta_{X_1/S_1,*}\mathcal{O}_{X_1} \cong L(X_1 \times_{S_1} X_1 \to X \times_S X)^*\Delta_{X/S,*}\mathcal{O}_X$$

This in turn will be true if X and $X_1 \times_{S_1} X_1$ are tor independent over $X \times_S X$, see Derived Categories of Schemes, Lemma 36.22.5. This tor independence can be seen

directly but also follows from the more general More on Morphisms, Lemma 37.69.1 applied to the square with corners X, X, X, S and its base change by $S_1 \to S$. $\square$

0G0L **Lemma 57.15.4.** *Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. Let $0 \in I$. Let $X_0 \to S_0$ and $Y_0 \to S_0$ be smooth proper morphisms. We set $X_i = S_i \times_{S_0} X_0$ for $i \geq 0$ and $X = S \times_{S_0} X_0$ and similarly for Y_0. If K is the Fourier-Mukai kernel of a relative equivalence from X to Y over S then for some $i \geq 0$ there exists a Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i.*

Proof. Let $K' \in D_{perf}(\mathcal{O}_{Y \times_S X})$ be the object assumed to exist in Definition 57.15.1. Since $X \times_S Y = \lim X_i \times_{S_i} Y_i$ there exists an i and objects K_i and K'_i in $D_{perf}(\mathcal{O}_{Y_i \times_{S_i} X_i})$ whose pullbacks to $Y \times_S X$ give K and K'. See Derived Categories of Schemes, Lemma 36.29.3. By Derived Categories of Schemes, Lemma 36.30.4 the object

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\mathrm{pr}_{23}^* K'_i)$$

is perfect and its pullback to $X \times_S X$ is equal to

$$R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y \times_S X}} L\mathrm{pr}_{23}^* K') \cong \Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 57.15.3. On the other hand, since $X_i \to S$ is smooth and separated the object

$$\Delta_{i,*}\mathcal{O}_{X_i}$$

of $D(\mathcal{O}_{X_i \times_{S_i} X_i})$ is also perfect (by More on Morphisms, Lemmas 37.62.18 and 37.61.13) and its pullback to $X \times_S X$ is equal to

$$\Delta_{X/S,*}\mathcal{O}_X$$

See proof of Lemma 57.15.3. Thus by Derived Categories of Schemes, Lemma 36.29.3 after increasing i we may assume that

$$\Delta_{i,*}\mathcal{O}_{X_i} \cong R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* K_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i \times_{S_i} Y_i \times_{S_i} X_i}} L\mathrm{pr}_{23}^* K'_i)$$

as desired. The same works for the roles of K and K' reversed. $\square$

57.16. No deformations

0G0M The title of this section refers to Lemma 57.16.4

0G0N **Lemma 57.16.1.** *Let $(R, \mathfrak{m}, \kappa) \to (A, \mathfrak{n}, \lambda)$ be a flat local ring homorphism of local rings which is essentially of finite presentation. Let $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{n}/\mathfrak{m}A \subset A/\mathfrak{m}A$ be a regular sequence. Let $K \in D(A)$. Assume*

(1) *K is perfect,*
(2) *$K \otimes^{\mathbf{L}}_A A/\mathfrak{m}A$ is isomorphic in $D(A/\mathfrak{m}A)$ to the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$.*

Then K is isomorphic in $D(A)$ to a Koszul complex on a regular sequence $f_1, \ldots, f_r \in A$ lifting the given elements $\overline{f}_1, \ldots, \overline{f}_r$. Moreover, $A/(f_1, \ldots, f_r)$ is flat over R.

Proof. Let us use chain complexes in the proof of this lemma. The Koszul complex $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$ is defined in More on Algebra, Definition 15.28.2. By More on Algebra, Lemma 15.76.5 we can represent K by a complex

$$K_\bullet : A \to A^{\oplus r} \to \ldots \to A^{\oplus r} \to A$$

whose tensor product with $A/\mathfrak{m}A$ is equal (!) to $K_\bullet(\overline{f}_1, \ldots, \overline{f}_r)$. Denote $f_1, \ldots, f_r \in A$ the components of the arrow $A^{\oplus r} \to A$. These f_i are lifts of the $\overline{f}_i$. By Algebra, Lemma 10.128.6 $f_1, \ldots, f_r$ form a regular sequence in A and $A/(f_1, \ldots, f_r)$ is flat over R. Let $J = (f_1, \ldots, f_r) \subset A$. Consider the diagram

$$\begin{array}{ccc} K_\bullet & \xrightarrow{\varphi_\bullet} & K_\bullet(f_1, \ldots, f_r) \\ & \searrow \quad \swarrow & \\ & A/J & \end{array}$$

Since $f_1, \ldots, f_r$ is a regular sequence the south-west arrow is a quasi-isomorphism (see More on Algebra, Lemma 15.30.2). Hence we can find the dotted arrow making the diagram commute for example by Algebra, Lemma 10.71.4. Reducing modulo $\mathfrak{m}$ we obtain a commutative diagram

$$\begin{array}{ccc} K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) & \xrightarrow{\overline{\varphi}_\bullet} & K_\bullet(\overline{f}_1, \ldots, \overline{f}_r) \\ & \searrow \quad \swarrow & \\ & (A/\mathfrak{m}A)/(\overline{f}_1, \ldots, \overline{f}_r) & \end{array}$$

by our choice of $K_\bullet$. Thus $\overline{\varphi}$ is an isomorphism in the derived category $D(A/\mathfrak{m}A)$. It follows that $\overline{\varphi} \otimes^{\mathbf{L}}_{A/\mathfrak{m}A} \lambda$ is an isomorphism. Since $\overline{f}_i \in \mathfrak{n}/\mathfrak{m}A$ we see that

$$\mathrm{Tor}_i^{A/\mathfrak{m}A}(K_\bullet(\overline{f}_1, \ldots, \overline{f}_r), \lambda) = K_i(\overline{f}_1, \ldots, \overline{f}_r) \otimes_{A/\mathfrak{m}A} \lambda$$

Hence $\varphi_i \bmod \mathfrak{n}$ is invertible. Since A is local this means that φ_i is an isomorphism and the proof is complete. □

0G0P **Lemma 57.16.2.** *Let $R \to S$ be a finite type flat ring map of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(S)$ be perfect. Let $f_1, \ldots, f_r \in \mathfrak{q}S_\mathfrak{q}$ be a regular sequence such that $S_\mathfrak{q}/(f_1, \ldots, f_r)$ is flat over R and such that $K \otimes^{\mathbf{L}}_S S_\mathfrak{q}$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$. Then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that*

(1) *$f_1, \ldots, f_r$ are the images of $f'_1, \ldots, f'_r \in S_g$,*
(2) *$f'_1, \ldots, f'_r$ form a regular sequence in S_g,*
(3) *$S_g/(f'_1, \ldots, f'_r)$ is flat over R,*
(4) *$K \otimes^{\mathbf{L}}_S S_g$ is isomorphic to the Koszul complex on $f_1, \ldots, f_r$.*

Proof. We can find $g \in S$, $g \notin \mathfrak{q}$ with property (1) by the definition of localizations. After replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (2) holds, see Algebra, Lemma 10.68.6. By Algebra, Theorem 10.129.4 we find that $S_g/(f'_1, \ldots, f'_r)$ is flat over R in an open neighbourhood of $\mathfrak{q}$. Hence after once more replacing g by gg' for some $g' \in S$, $g' \notin \mathfrak{q}$ we may assume (3) holds as well. Finally, we get (4) for a further replacement by More on Algebra, Lemma 15.75.18. □

For a generalization of the following lemma, please see More on Morphisms of Spaces, Lemma 76.49.6.

0G0Q **Lemma 57.16.3.** *Let S be a Noetherian scheme. Let $s \in S$. Let $p : X \to Y$ be a morphism of schemes over S. Assume*

(1) *$Y \to S$ and $X \to S$ proper,*
(2) *X is flat over S,*

(3) $X_s \to Y_s$ *an isomorphism.*

Then there exists an open neighbourhood $U \subset S$ of s such that the base change $X_U \to Y_U$ is an isomorphism.

Proof. The morphism p is proper by Morphisms, Lemma 29.41.6. By Cohomology of Schemes, Lemma 30.21.2 there is an open $Y_s \subset V \subset Y$ such that $p|_{p^{-1}(V)} : p^{-1}(V) \to V$ is finite. By More on Morphisms, Theorem 37.16.1 there is an open $X_s \subset U \subset X$ such that $p|_U : U \to Y$ is flat. After removing the images of $X \setminus U$ and $Y \setminus V$ (which are closed subsets not containing s) we may assume p is flat and finite. Then p is open (Morphisms, Lemma 29.25.10) and $Y_s \subset p(X) \subset Y$ hence after shrinking S we may assume p is surjective. As $p_s : X_s \to Y_s$ is an isomorphism, the map

$$p^\sharp : \mathcal{O}_Y \longrightarrow p_*\mathcal{O}_X$$

of coherent $\mathcal{O}_Y$-modules (p is finite) becomes an isomorphism after pullback by $i : Y_s \to Y$ (by Cohomology of Schemes, Lemma 30.5.1 for example). By Nakayama's lemma, this implies that $\mathcal{O}_{Y,y} \to (p_*\mathcal{O}_X)_y$ is surjective for all $y \in Y_s$. Hence there is an open $Y_s \subset V \subset Y$ such that $p^\sharp|_V$ is surjective (Modules, Lemma 17.9.4). Hence after shrinking S once more we may assume $p^\sharp$ is surjective which means that p is a closed immersion (as p is already finite). Thus now p is a surjective flat closed immersion of Noetherian schemes and hence an isomorphism, see Morphisms, Section 29.26. □

0G0R **Lemma 57.16.4.** *Let k be a field. Let S be a finite type scheme over k with k-rational point s. Let $Y \to S$ be a smooth proper morphism. Let $X = Y_s \times S \to S$ be the constant family with fibre Y_s. Let K be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. Assume the restriction*

$$L(Y_s \times_S Y_s \to X \times_S Y)^* K \cong \Delta_{Y_s/k,*}\mathcal{O}_{Y_s}$$

in $D(\mathcal{O}_{Y_s \times Y_s})$. Then there is an open neighbourhood $s \in U \subset S$ such that $Y|_U$ is isomorphic to $Y_s \times U$ over U.

Proof. Denote $i : Y_s \times Y_s = X_s \times Y_s \to X \times_S Y$ the natural closed immersion. (We will write Y_s and not X_s for the fibre of X over s from now on.) Let $z \in Y_s \times Y_s = (X \times_S Y)_s \subset X \times_S Y$ be a closed point. As indicated we think of z both as a closed point of $Y_s \times Y_s$ as well as a closed point of $X \times_S Y$.

Case I: $z \notin \Delta_{Y_s/k}(Y_s)$. Denote $\mathcal{O}_z$ the coherent $\mathcal{O}_{Y_s \times Y_s}$-module supported at z whose value is $\kappa(z)$. Then $i_*\mathcal{O}_z$ is the coherent $\mathcal{O}_{X \times_S Y}$-module supported at z whose value is $\kappa(z)$. Our assumption means that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} i_*\mathcal{O}_z = Li^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{Y_s \times Y_s}} \mathcal{O}_z = 0$$

Hence by Lemma 57.11.3 we find an open neighbourhood $U(z) \subset X \times_S Y$ of z such that $K|_{U(z)} = 0$. In this case we set $Z(z) = \emptyset$ as closed subscheme of $U(z)$.

Case II: $z \in \Delta_{Y_s/k}(Y_s)$. Since Y_s is smooth over k we know that $\Delta_{Y_s/k} : Y_s \to Y_s \times Y_s$ is a regular immersion, see More on Morphisms, Lemma 37.62.18. Choose a regular sequence $\overline{f}_1, \ldots, \overline{f}_r \in \mathcal{O}_{Y_s \times Y_s, z}$ cutting out the ideal sheaf of $\Delta_{Y_s/k}(Y_s)$. Since a regular sequence is Koszul-regular (More on Algebra, Lemma 15.30.2) our assumption means that

$$K_z \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y, z}} \mathcal{O}_{Y_s \times Y_s, z} \in D(\mathcal{O}_{Y_s \times Y_s, z})$$

is represented by the Koszul complex on $\overline{f}_1, \ldots, \overline{f}_r$ over $\mathcal{O}_{Y_s \times Y_s, z}$. By Lemma 57.16.1 applied to $\mathcal{O}_{S,s} \to \mathcal{O}_{X \times_S Y, z}$ we conclude that $K_z \in D(\mathcal{O}_{X \times_S Y, z})$ is represented by the Koszul complex on a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_{X \times_S Y, z}$ lifting the regular sequence $\overline{f}_1, \ldots, \overline{f}_r$ such that moreover $\mathcal{O}_{X \times_S Y}/(f_1, \ldots, f_r)$ is flat over $\mathcal{O}_{S,s}$. By some limit arguments (Lemma 57.16.2) we conclude that there exists an affine open neighbourhood $U(z) \subset X \times_S Y$ of z and a closed subscheme $Z(z) \subset U(z)$ such that

(1) $Z(z) \to U(z)$ is a regular closed immersion,
(2) $K|_{U(z)}$ is quasi-isomorphic to $\mathcal{O}_{Z(z)}$,
(3) $Z(z) \to S$ is flat,
(4) $Z(z)_s = \Delta_{Y_s/k}(Y_s) \cap U(z)_s$ as closed subschemes of $U(z)_s$.

By property (2), for $z, z' \in Y_s \times Y_s$, we find that $Z(z) \cap U(z') = Z(z') \cap U(z)$ as closed subschemes. Hence we obtain an open neighbourhood

$$U = \bigcup\nolimits_{z \in Y_s \times Y_s \text{ closed}} U(z)$$

of $Y_s \times Y_s$ in $X \times_S Y$ and a closed subscheme $Z \subset U$ such that (1) $Z \to U$ is a regular closed immersion, (2) $Z \to S$ is flat, and (3) $Z_s = \Delta_{Y_s/k}(Y_s)$. Since $X \times_S Y \to S$ is proper, after replacing S by an open neighbourhood of s we may assume $U = X \times_S Y$. Since the projections $Z_s \to Y_s$ and $Z_s \to X_s$ are isomorphisms, we conclude that after shrinking S we may assume $Z \to Y$ and $Z \to X$ are isomorphisms, see Lemma 57.16.3. This finishes the proof. □

0G0S **Lemma 57.16.5.** *Let k be an algebraically closed field. Let X be a smooth proper scheme over k. Let $f : Y \to S$ be a smooth proper morphism with S of finite type over k. Let K be the Fourier-Mukai kernel of a relative equivalence from $X \times S$ to Y over S. Then S can be covered by open subschemes U such that there is a U-isomorphism $f^{-1}(U) \cong Y_0 \times U$ for some Y_0 proper and smooth over k.*

Proof. Choose a closed point $s \in S$. Since k is algebraically closed this is a k-rational point. Set $Y_0 = Y_s$. The restriction K_0 of K to $X \times Y_0$ is the Fourier-Mukai kernel of a relative equivalence from X to Y_0 over $\mathrm{Spec}(k)$ by Lemma 57.15.3. Let K_0' in $D_{perf}(\mathcal{O}_{Y_0 \times X})$ be the object assumed to exist in Definition 57.15.1. Then K_0' is the Fourier-Mukai kernel of a relative equivalence from Y_0 to X over $\mathrm{Spec}(k)$ by the symmetry inherent in Definition 57.15.1. Hence by Lemma 57.15.3 we see that the pullback

$$M = (Y_0 \times X \times S \to Y_0 \times X)^* K_0'$$

on $(Y_0 \times S) \times_S (X \times S) = Y_0 \times X \times S$ is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to $X \times S$ over S. Now consider the kernel

$$K_{new} = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{(Y_0 \times S) \times_S (X \times S) \times_S Y}} L\mathrm{pr}_{23}^* K)$$

on $(Y_0 \times S) \times_S Y$. This is the Fourier-Mukai kernel of a relative equivalence from $Y_0 \times S$ to Y over S since it is the composition of two invertible arrows in the category constructed in Section 57.14. Moreover, this composition passes through base change (Lemma 57.14.1). Hence we see that the pullback of K_{new} to $((Y_0 \times S) \times_S Y)_s = Y_0 \times Y_0$ is equal to the composition of K_0 and K_0' and hence equal to the identity in this category. In other words, we have

$$L(Y_0 \times Y_0 \to (Y_0 \times S) \times_S Y)^* K_{new} \cong \Delta_{Y_0/k,*} \mathcal{O}_{Y_0}$$

Thus by Lemma 57.16.4 we conclude that $Y \to S$ is isomorphic to $Y_0 \times S$ in an open neighbourhood of s. This finishes the proof. □

57.17. Countability

0G0T In this section we prove some elementary lemmas about countability of certain sets. Let $\mathcal{C}$ be a category. In this section we will say that $\mathcal{C}$ is *countable* if

(1) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is countable, and
(2) the set of isomorphism classes of objects of $\mathcal{C}$ is countable.

0G0U **Lemma 57.17.1.** *Let R be a countable Noetherian ring. Then the category of schemes of finite type over R is countable.*

Proof. Omitted. □

0G0V **Lemma 57.17.2.** *Let $\mathcal{A}$ be a countable abelian category. Then $D^b(\mathcal{A})$ is countable.*

Proof. It suffices to prove the statement for $D(\mathcal{A})$ as the others are full subcategories of this one. Since every object in $D(\mathcal{A})$ is a complex of objects of $\mathcal{A}$ it is immediate that the set of isomorphism classes of objects of $D^b(\mathcal{A})$ is countable. Moreover, for bounded complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{A}$ it is clear that $\mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, B^\bullet)$ is countable. We have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{colim}_{s:(A')^\bullet \to A^\bullet \text{ qis and } (A')^\bullet \text{ bounded}} \mathrm{Hom}_{K^b(\mathcal{A})}((A')^\bullet, B^\bullet)$$

by Derived Categories, Lemma 13.11.6. Thus this is a countable set as a countable colimit of □

0G0W **Lemma 57.17.3.** *Let X be a scheme of finite type over a countable Noetherian ring. Then the categories $D_{perf}(\mathcal{O}_X)$ and $D^b_{Coh}(\mathcal{O}_X)$ are countable.*

Proof. Observe that X is Noetherian by Morphisms, Lemma 29.15.6. Hence $D_{perf}(\mathcal{O}_X)$ is a full subcategory of $D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.6. Thus it suffices to prove the result for $D^b_{Coh}(\mathcal{O}_X)$. Recall that $D^b_{Coh}(\mathcal{O}_X) = D^b(Coh(\mathcal{O}_X))$ by Derived Categories of Schemes, Proposition 36.11.2. Hence by Lemma 57.17.2 it suffices to prove that $Coh(\mathcal{O}_X)$ is countable. This we omit. □

0G0X **Lemma 57.17.4.** *Let K be an algebraically closed field. Let S be a finite type scheme over K. Let $X \to S$ and $Y \to S$ be finite type morphisms. There exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties*

(1) *$S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$ and $Y_i = Y \times_S S_i$,*
(2) *$h_i : X_i \to Y_i$ is an isomorphism over S_i, and*
(3) *for any closed point $s \in S(K)$ if $X_s \cong Y_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.*

Proof. The field K is the filtered union of its countable subfields. Dually, $\mathrm{Spec}(K)$ is the cofiltered limit of the spectra of the countable subfields of K. Hence Limits, Lemma 32.10.1 guarantees that we can find a countable subfield k and morphisms $X_0 \to S_0$ and $Y_0 \to S_0$ of schemes of finite type over k such that $X \to S$ and $Y \to S$ are the base changes of these.

By Lemma 57.17.1 there is a countable set I and pairs $(S_{0,i} \to S_0, h_{0,i})$ such that

(1) $S_{0,i} \to S_0$ is a morphism of finite type, set $X_{0,i} = X_0 \times_{S_0} S_{0,i}$ and $Y_{0,i} = Y_0 \times_{S_0} S_{0,i}$,

(2) $h_{0,i} : X_{0,i} \to Y_{0,i}$ is an isomorphism over $S_{0,i}$.

such that every pair $(T \to S_0, h_T)$ with $T \to S_0$ of finite type and $h_T : X_0 \times_{S_0} T \to Y_0 \times_{S_0} T$ an isomorphism is isomorphic to one of these. Denote $(S_i \to S, h_i)$ the base change of $(S_{0,i} \to S_0, h_{0,i})$ by $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. We claim this works.

Let $s \in S(K)$ and let $h_s : X_s \to Y_s$ be an isomorphism over $K = \kappa(s)$. We can write K as the filtered union of its finitely generated k-subalgebras. Hence by Limits, Proposition 32.6.1 and Lemma 32.10.1 we can find such a finitely generated k-subalgebra $K \supset A \supset k$ such that

(1) there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(A) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ S & \longrightarrow & S_0 \end{array}$$

for some morphism $s' : \mathrm{Spec}(A) \to S_0$ over k,

(2) h_s is the base change of an isomorphism $h_{s'} : X_0 \times_{S_0,s'} \mathrm{Spec}(A) \to X_0 \times_{S_0,s'} \mathrm{Spec}(A)$ over A.

Of course, then $(s' : \mathrm{Spec}(A) \to S_0, h_{s'})$ is isomorphic to the pair $(S_{0,i} \to S_0, h_{0,i})$ for some $i \in I$. This concludes the proof because the commutative diagram in (1) shows that s is in the image of the base change of s' to $\mathrm{Spec}(K)$. □

0G0Y **Lemma 57.17.5.** *Let K be an algebraically closed field. There exists a countable set I and for $i \in I$ a pair $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ with the following properties*

(1) *S_i is a scheme of finite type over K,*
(2) *$X_i \to S_i$ and $Y_i \to S_i$ are proper smooth morphisms of schemes,*
(3) *$M_i \in D_{perf}(\mathcal{O}_{X_i \times_{S_i} Y_i})$ is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i, and*
(4) *for any smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$ there exists an $i \in I$ and a $s \in S_i(K)$ such that $X \cong (X_i)_s$ and $Y \cong (Y_i)_s$.*

Proof. Choose a countable subfield $k \subset K$ for example the prime field. By Lemmas 57.17.1 and 57.17.3 there exists a countable set of isomorphism classes of systems over k satisfying parts (1), (2), (3) of the lemma. Thus we can choose a countable set I and for each $i \in I$ such a system

$$(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$$

over k such that each isomorphism class occurs at least once. Denote $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ the base change of the displayed system to K. This system has properties (1), (2), (3), see Lemma 57.15.3. Let us prove property (4).

Consider smooth proper schemes X and Y over K such that there is a K-linear exact equivalence $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$. By Proposition 57.13.4 we may assume that there exists an object $M \in D_{perf}(\mathcal{O}_{X \times Y})$ such that $F = \Phi_M$ is the corresponding Fourier-Mukai functor. By Lemma 57.8.9 there is an M' in $D_{perf}(\mathcal{O}_{Y \times X})$ such that $\Phi_{M'}$ is the right adjoint to Φ_M. Since Φ_M is an equivalence, this means that $\Phi_{M'}$ is the quasi-inverse to Φ_M. By Lemma 57.8.9 we see that the Fourier-Mukai functors defined by the objects

$$A = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^* M \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times Y \times X}} L\mathrm{pr}_{23}^* M')$$

in $D_{perf}(\mathcal{O}_{X\times X})$ and

$$B = R\mathrm{pr}_{13,*}(L\mathrm{pr}_{12}^*M' \otimes^{\mathbf{L}}_{\mathcal{O}_{Y\times X\times Y}} L\mathrm{pr}_{23}^*M)$$

in $D_{perf}(\mathcal{O}_{Y\times Y})$ are isomorphic to $\mathrm{id} : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ and $\mathrm{id} : D_{perf}(\mathcal{O}_Y) \to D_{perf}(\mathcal{O}_Y)$ Hence $A \cong \Delta_{X/K,*}\mathcal{O}_X$ and $B \cong \Delta_{Y/K,*}\mathcal{O}_Y$ by Lemma 57.13.5. Hence we see that M is the Fourier-Mukai kernel of a relative equivalence from X to Y over K by definition.

We can write K as the filtered colimit of its finite type k-subalgebras $A \subset K$. By Limits, Lemma 32.10.1 we can find X_0, Y_0 of finite type over A whose base changes to K produces X and Y. By Limits, Lemmas 32.13.1 and 32.8.9 after enlarging A we may assume X_0 and Y_0 are smooth and proper over A. By Lemma 57.15.4 after enlarging A we may assume M is the pullback of some $M_0 \in D_{perf}(\mathcal{O}_{X_0\times_{\mathrm{Spec}(A)}Y_0})$ which is the Fourier-Mukai kernel of a relative equivalence from X_0 to Y_0 over $\mathrm{Spec}(A)$. Thus we see that $(S_0/k, X_0 \to S_0, Y_0 \to S_0, M_0)$ is isomorphic to $(S_{0,i}/k, X_{0,i} \to S_{0,i}, Y_{0,i} \to S_{0,i}, M_{0,i})$ for some $i \in I$. Since $S_i = S_{0,i} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ we conclude that (4) is true with $s : \mathrm{Spec}(K) \to S_i$ induced by the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \cong S_{0,i}$ we get from $A \subset K$. □

57.18. Countability of derived equivalent varieties

0G0Z In this section we prove a result of Anel and Toën, see [**AT09**].

0G10 **Definition 57.18.1.** Let k be a field. Let X and Y be smooth projective schemes over k. We say X and Y are *derived equivalent* if there exists a k-linear exact equivalence $D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_Y)$.

Here is the result

0G11 **Theorem 57.18.2.** *Let K be an algebraically closed field. Let $\mathbf{X}$ be a smooth proper scheme over K. There are at most countably many isomorphism classes of smooth proper schemes $\mathbf{Y}$ over K which are derived equivalent to $\mathbf{X}$.* Slight improvement of [**AT09**]

Proof. Choose a countable set I and for $i \in I$ systems $(S_i/K, X_i \to S_i, Y_i \to S_i, M_i)$ satisfying properties (1), (2), (3), and (4) of Lemma 57.17.5. Pick $i \in I$ and set $S = S_i$, $X = X_i$, $Y = Y_i$, and $M = M_i$. Clearly it suffice to show that the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. This we prove in the next paragraph.

Let S be a finite type scheme over K, let $X \to S$ and $Y \to S$ be proper smooth morphisms, and let $M \in D_{perf}(\mathcal{O}_{X\times_S Y})$ be the Fourier-Mukai kernel of a relative equivalence from X to Y over S. We will show the set of isomorphism classes of fibres Y_s for $s \in S(K)$ such that $X_s \cong \mathbf{X}$ is countable. By Lemma 57.17.4 applied to the families $\mathbf{X} \times S \to S$ and $X \to S$ there exists a countable set I and for $i \in I$ a pair $(S_i \to S, h_i)$ with the following properties

(1) $S_i \to S$ is a morphism of finite type, set $X_i = X \times_S S_i$,
(2) $h_i : \mathbf{X} \times S_i \to X_i$ is an isomorphism over S_i, and
(3) for any closed point $s \in S(K)$ if $\mathbf{X} \cong X_s$ over $K = \kappa(s)$ then s is in the image of $S_i \to S$ for some i.

Set $Y_i = Y \times_S S_i$. Denote $M_i \in D_{perf}(\mathcal{O}_{X_i\times_{S_i}Y_i})$ the pullback of M. By Lemma 57.15.3 M_i is the Fourier-Mukai kernel of a relative equivalence from X_i to Y_i over S_i. Since I is countable, by property (3) it suffices to prove that the set of

isomorphism classes of fibres $Y_{i,s}$ for $s \in S_i(K)$ is countable. In fact, this number is finite by Lemma 57.16.5 and the proof is complete. □

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CHAPTER 58

Fundamental Groups of Schemes

0BQ6

58.1. Introduction

0BQ7 In this chapter we discuss Grothendieck's fundamental group of a scheme and applications. A foundational reference is [**Gro71**]. A nice introduction is [**Len**]. Other references [**Mur67**] and [**GM71**].

58.2. Schemes étale over a point

04JI In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

04JJ **Definition 58.2.1.** Let G be a topological group. A *G-set*, sometimes called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 7.6.5) provided we endow G with the discrete topology.

Recall that if L/K is an infinite Galois extension then the Galois group $G = \mathrm{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 9.22.

03QR **Lemma 58.2.2.** *Let K be a field. Let K^{sep} be a separable closure of K. Consider the profinite group $G = Gal(K^{sep}/K)$. The functor*

$$\begin{array}{ccc} \textit{schemes étale over } K & \longrightarrow & \textit{G-Sets} \\ X/K & \longmapsto & \mathrm{Mor}_{\mathrm{Spec}(K)}(\mathrm{Spec}(K^{sep}), X) \end{array}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} \mathrm{Spec}(K_i)$ with each K_i a finite separable extension of K (Morphisms, Lemma 29.36.7). The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \mathrm{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 9.22.4) the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \mathrm{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. □

03QS **Remark 58.2.3.** Under the correspondence of Lemma 58.2.2, the coverings in the small étale site $\mathrm{Spec}(K)_{étale}$ of K correspond to surjective families of maps in *G-Sets*.

58.3. Galois categories

0BMQ In this section we discuss some of the material the reader can find in [**Gro71**, Exposé V, Sections 4, 5, and 6].

Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. There is a canonical injective map

0BS7 (58.3.0.1) $$\mathrm{Aut}(F) \longrightarrow \prod\nolimits_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$$

For a set E we endow $\mathrm{Aut}(E)$ with the compact open topology, see Topology, Example 5.30.2. Of course this is the discrete topology when E is finite, which is the case of interest in this section[1]. We endow $\mathrm{Aut}(F)$ with the topology induced from the product topology on the right hand side of (58.3.0.1). In particular, the action maps

$$\mathrm{Aut}(F) \times F(X) \longrightarrow F(X)$$

are continuous when $F(X)$ is given the discrete topology because this is true for the action maps $\mathrm{Aut}(E) \times E \to E$ for any set E. The universal property of our topology on $\mathrm{Aut}(F)$ is the following: suppose that G is a topological group and $G \to \mathrm{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \mathrm{Ob}(\mathcal{C})$ where $F(X)$ has the discrete topology. Then $G \to \mathrm{Aut}(F)$ is continuous.

The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite group.

0BMR **Lemma 58.3.1.** *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to Sets$ be a functor. The map (58.3.0.1) identifies $Aut(F)$ with a closed subgroup of $\prod_{X \in \mathrm{Ob}(\mathcal{C})} Aut(F(X))$. In particular, if $F(X)$ is finite for all X, then $Aut(F)$ is a profinite group.*

Proof. Let $\xi = (\gamma_X) \in \prod \mathrm{Aut}(F(X))$ be an element not in $\mathrm{Aut}(F)$. Then there exists a morphism $f : X \to X'$ of $\mathcal{C}$ and an element $x \in F(X)$ such that $F(f)(\gamma_X(x)) \neq \gamma_{X'}(F(f)(x))$. Consider the open neighbourhood $U = \{\gamma \in \mathrm{Aut}(F(X)) \mid \gamma(x) = \gamma_X(x)\}$ of γ_X and the open neighbourhood $U' = \{\gamma' \in \mathrm{Aut}(F(X')) \mid \gamma'(F(f)(x)) = \gamma_{X'}(F(f)(x))\}$. Then $U \times U' \times \prod_{X'' \neq X, X'} \mathrm{Aut}(F(X''))$ is an open neighbourhood of ξ not meeting $\mathrm{Aut}(F)$. The final statement follows from the fact that $\prod \mathrm{Aut}(F(X))$ is a profinite space if each $F(X)$ is finite. □

0BMS **Example 58.3.2.** Let G be a topological group. An important example will be the forgetful functor

0BMT (58.3.2.1) $$\mathit{Finite\text{-}G\text{-}Sets} \longrightarrow \mathit{Sets}$$

where *Finite-G-Sets* is the full subcategory of *G-Sets* whose objects are the finite G-sets. The category *G-Sets* of G-sets is defined in Definition 58.2.1.

[1] When we discuss the pro-étale fundamental group the general case will be of interest.

Let G be a topological group. The *profinite completion* of G will be the profinite group

$$G^\wedge = \lim_{U \subset G \text{ open, normal, finite index}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another. The universal property of the profinite completion is that any continuous map $G \to H$ to a profinite group H factors canonically as $G \to G^\wedge \to H$.

0BMU **Lemma 58.3.3.** *Let G be a topological group. The automorphism group of the functor (58.3.2.1) endowed with its profinite topology from Lemma 58.3.1 is the profinite completion of G.*

Proof. Denote F_G the functor (58.3.2.1). Any morphism $X \to Y$ in *Finite-G-Sets* commutes with the action of G. Thus any $g \in G$ defines an automorphism of F_G and we obtain a canonical homomorphism $G \to \mathrm{Aut}(F_G)$ of groups. Observe that any finite G-set X is a finite disjoint union of G-sets of the form G/H_i with canonical G-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_ig^{-1}$ is open, normal, and has finite index. Moreover U_i acts trivially on G/H_i hence $U = \bigcap U_i$ acts trivially on $F_G(X)$. Hence the action $G \times F_G(X) \to F_G(X)$ is continuous. By the universal property of the topology on $\mathrm{Aut}(F_G)$ the map $G \to \mathrm{Aut}(F_G)$ is continuous. By Lemma 58.3.1 and the universal property of profinite completion there is an induced continuous group homomorphism

$$G^\wedge \longrightarrow \mathrm{Aut}(F_G)$$

Moreover, since G/U acts faithfully on G/U this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 5.17.8.

Let $\gamma \in \mathrm{Aut}(F_G)$ and let $X \in \mathrm{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that γ and g induce the same action on $F_G(X)$. This will finish the proof. As before we see that X is a finite disjoint union of G/H_i. With U_i and U as above, the finite G-set $Y = G/U$ surjects onto G/H_i for all i and hence it suffices to find $g \in G$ such that γ and g induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of *Finite-G-Sets* commutes with the action of γ. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. □

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

0BMV **Lemma 58.3.4.** *Let G be a topological group. Let F : Finite-G-Sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the functor (58.3.2.1).*

Proof. Let X be a nonempty object of *Finite-G-Sets*. The diagram

$$\begin{array}{ccc} X & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \{*\} \end{array}$$

is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod\nolimits_{gU \in G/U} G/U$$

where the summand corresponding to gU corresponds to the orbit of (eU, gU) on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod\nolimits_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim_{U \subset G \text{ open, normal, finite idex}} F(G/U)$$

is nonempty (Categories, Lemma 4.21.7). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote F_G the functor (58.3.2.1). We can identify F_G with the functor

$$X \longmapsto \operatorname{colim}_U \operatorname{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$ (details omitted). Hence the element γ determines a well defined map

$$t : F_G \longrightarrow F$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. We will show that t induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any U. This implies in a straightforward manner that t is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(\mathrm{id}_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because F commutes with equalizers. It follows that if $g_1, \ldots, g_{|G/U|}$ is a system of representatives for G/U, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_i U) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. □

0BMW **Example 58.3.5.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathit{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$. By Lemma 58.3.1 we see that $G = \mathrm{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

0BMX (58.3.5.1) $$\mathcal{C} \longrightarrow \mathit{Finite\text{-}G\text{-}Sets}, \quad X \longmapsto F(X)$$

where $F(X)$ is endowed with the induced action of G. This action is continuous by our construction of the topology on $\mathrm{Aut}(F)$.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (58.3.5.1) is an equivalence. Our definition of a Galois category is as follows.

0BMY **Definition 58.3.6.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to Sets$ be a functor. The pair $(\mathcal{C}, F)$ is a *Galois category* if

(1) $\mathcal{C}$ has finite limits and finite colimits,

0BMZ (2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,

(3) $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$, and

(4) F reflects isomorphisms[2] and is exact[3].

Different from the definition in [**Gro71**, Exposé V, Definition 5.1]. Compare with [**BS13**, Definition 7.2.1].

Here we say $X \in \mathrm{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either Y is initial or $Y \to X$ is an isomorphism.

Warning: This definition is not the same (although eventually we'll see it is equivalent) as the definition given in most references. Namely, in [**Gro71**, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to *Finite-G-Sets* for some profinite group G. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor F. The price we'll pay for this later is that we'll have to work a bit harder to apply the results of this section.

0BN0 **Lemma 58.3.7.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in Arrows(\mathcal{C})$. Then*

(1) *F is faithful,*

(2) *$X \to Y$ is a monomorphism $\Leftrightarrow$ $F(X) \to F(Y)$ is injective,*

(3) *$X \to Y$ is an epimorphism $\Leftrightarrow$ $F(X) \to F(Y)$ is surjective,*

(4) *an object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,*

(5) *an object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,*

(6) *if X and Y are connected, then $X \to Y$ is an epimorphism,*

0BN1 (7) *if X is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,*

(8) *if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where X_i, Y_j are connected, then there is map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.*

Proof. Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let E be the equalizer of a and b. Then $F(E) = F(X)$ and we see that $E = X$ because F reflects isomorphisms.

Proof of (2). This is true because F turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and F reflects isomorphisms.

Proof of (3). This is true because F turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and F reflects isomorphisms.

Proof of (4). There exists an initial object A and certainly $F(A) = \emptyset$. On the other hand, if X is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

[2] Namely, given a morphism f of $\mathcal{C}$ if $F(f)$ is an isomorphism, then f is an isomorphism.

[3] This means that F commutes with finite limits and colimits, see Categories, Section 4.23.

Proof of (5). There exists a final object Z and certainly $F(Z)$ is a singleton. On the other hand, if X is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer E of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through E and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer E of a and b comes with a monomorphism $E \to X$ and $F(E) \subset F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either E is initial or $E = X$.

Proof of (8). For each i, j we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to X_i. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if s maps to an element of $F(Y_j) \subset F(Y)$, hence this happens for a unique $j = \alpha(i)$. □

By the lemma above we see that, given a connected object X of a Galois category $(\mathcal{C}, F)$, the automorphism group $\mathrm{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \mathrm{Aut}(X)$ we see that $F(g)(s) = s$ if and only if $g = \mathrm{id}_X$ by (7). We say X is *Galois* if equality holds. Equivalently, X is Galois if it is connected and $\mathrm{Aut}(X)$ acts transitively on $F(X)$.

0BN2 **Lemma 58.3.8.** *Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X of $\mathcal{C}$ there exists a Galois object Y and a morphism $Y \to X$.*

Proof. We will use the results of Lemma 58.3.7 without further mention. Let $n = |F(X)|$. Consider X^n endowed with its natural action of S_n. Let

$$X^n = \coprod\nolimits_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with s_i pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim s'_i are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then Z_t is the image of an connected component of X^{n-1} under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \longrightarrow X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying F it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of S_n on

$$F(X)^n = F(X^n) = \coprod\nolimits_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that s'_i are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of G on $F(Z_t)$ is transitive and the proof is complete. □

Here is a key lemma.

0BN3 **Lemma 58.3.9.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 58.3.5. For any connected X in $\mathcal{C}$ the action of G on $F(X)$ is transitive.*

Compare with [**BS13**, Definition 7.2.4].

Proof. We will use the results of Lemma 58.3.7 without further mention. Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let X_i be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on I by setting $i \geq i'$ if and only if there is a morphism $f_{ii'} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{ii'})(\gamma_i) = \gamma_{i'}$. With this normalization the morphism $f_{ii'}$ is unique. Observe that I is a directed partially ordered set: (Categories, Definition 4.21.1) if $i_1, i_2 \in I$ there exists a Galois object Y and a morphism $Y \to X_{i_1} \times X_{i_2}$ by Lemma 58.3.8 applied to a connected component of $X_{i_1} \times X_{i_2}$. Then $Y \cong X_i$ for some $i \in I$ and $i \geq i_1$, $i \geq I_2$.

We claim that the functor F is isomorphic to the functor F' which sends X to

$$F'(X) = \operatorname{colim}_I \operatorname{Mor}_{\mathcal{C}}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using (7) we find that t_X is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where X is connected by the definition of a Galois category. Then we may assume X is Galois by Lemma 58.3.8. In this case X is isomorphic to X_i for some i and we can choose the isomorphism $X_i \to X$ such that γ_i maps to γ (by definition of Galois objects). We conclude that t is an isomorphism.

Set $A_i = \operatorname{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{ii'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ii'}} & X_{i'} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h_{ii'}(a)} \\ X_i & \xrightarrow{f_{ii'}} & X_{i'} \end{array}$$

commutes. Namely, just let $h_{ii'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i'}) = F(f_{ii'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i'i''} \circ h_{ii'} = h_{ii''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_I A_i$$

This is a profinite group since the automorphism groups are finite. The map $A \to A_i$ is surjective for all i by Categories, Lemma 4.21.7.

Since elements of A act on the inverse system X_i we get an action of A (on the right) on F' by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the lemma is true. □

0BN4 **Proposition 58.3.10.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 58.3.5. The functor $F : \mathcal{C} \to$ Finite-G-Sets (58.3.5.1) an equivalence.*

This is a weak version of [**Gro71**, Exposé V]. The proof is borrowed from [**BS13**, Theorem 7.2.5].

Proof. We will use the results of Lemma 58.3.7 without further mention. In particular we know the functor is faithful. By Lemma 58.3.9 we know that for any connected X the action of G on $F(X)$ is transitive. Hence for general X the functor

F sends the connected components of X (which exist by an axiom of a Galois category) to the G-orbits of $F(X)$. Let X and Y be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of s is a union of connected components. Hence there exists a union of connected components Z of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence F is fully faithful.

To finish the proof we show that F is essentially surjective. It suffices to show that G/H is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on G there exists a finite collection of objects X_i such that

$$\mathrm{Ker}(G \longrightarrow \prod\nolimits_i \mathrm{Aut}(F(X_i)))$$

is contained in H. We may assume X_i is connected for all i. We can choose a Galois object Y mapping to a connected component of $\prod X_i$ using Lemma 58.3.8. Choose an isomorphism $F(Y) = G/U$ in G-*sets* for some open subgroup $U \subset G$. As Y is Galois, the group $\mathrm{Aut}(Y) = \mathrm{Aut}_{G\text{-}Sets}(G/U)$ acts transitively on $F(Y) = G/U$. This implies that U is normal. Since $F(Y)$ surjects onto $F(X_i)$ for each i we see that $U \subset H$. Let $M \subset \mathrm{Aut}(Y)$ be the finite subgroup corresponding to

$$(H/U)^{opp} \subset (G/U)^{opp} = \mathrm{Aut}_{G\text{-}Sets}(G/U) = \mathrm{Aut}(Y).$$

Set $X = Y/M$, i.e., X is the coequalizer of the arrows $m : Y \to Y$, $m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete. □

0BN5 **Lemma 58.3.11.** *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. There exists an isomorphism $t : F' \circ H \to F$. The choice of t determines a continuous homomorphism $h : G' = Aut(F') \to Aut(F) = G$ and a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \textit{Finite-G-Sets} & \xrightarrow{h} & \textit{Finite-G'-Sets} \end{array}$$

The map h is independent of t up to an inner automorphism of G. Conversely, given a continuous homomorphism $h : G' \to G$ there is an exact functor $H : \mathcal{C} \to \mathcal{C}'$ and an isomorphism t recovering h as above.

Proof. By Proposition 58.3.10 and Lemma 58.3.3 we may assume $\mathcal{C} = \textit{Finite-G-Sets}$ and F is the forgetful functor and similarly for $\mathcal{C}'$. Thus the existence of t follows from Lemma 58.3.4. The map h comes from transport of structure via t. The commutativity of the diagram is obvious. Uniqueness of h up to inner conjugation by an element of G comes from the fact that the choice of t is unique up to an element of G. The final statement is straightforward. □

58.4. Functors and homomorphisms

0BTQ Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$, $(\mathcal{C}'', F'')$ be Galois categories. Set $G = \mathrm{Aut}(F)$, $G' = \mathrm{Aut}(F')$, and $G'' = \mathrm{Aut}(F'')$. Let $H : \mathcal{C} \to \mathcal{C}'$ and $H' : \mathcal{C}' \to \mathcal{C}''$ be exact functors. Let $h : G' \to G$ and $h' : G'' \to G'$ be the corresponding continuous homomorphism as

in Lemma 58.3.11. In this section we consider the corresponding 2-commutative diagram

0BTR (58.4.0.1)
$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \\ \downarrow & & \downarrow & & \downarrow \\ \textit{Finite-}G\textit{-Sets} & \xrightarrow{h} & \textit{Finite-}G'\textit{-Sets} & \xrightarrow{h'} & \textit{Finite-}G''\textit{-Sets} \end{array}$$

and we relate exactness properties of the sequence $1 \to G'' \to G' \to G \to 1$ to properties of the functors H and H'.

0BN6 **Lemma 58.4.1.** *In diagram (58.4.0.1) the following are equivalent*

(1) $h : G' \to G$ *is surjective,*
(2) $H : \mathcal{C} \to \mathcal{C}'$ *is fully faithful,*
(3) *if* $X \in \mathrm{Ob}(\mathcal{C})$ *is connected, then* $H(X)$ *is connected,*
(4) *if* $X \in \mathrm{Ob}(\mathcal{C})$ *is connected and there is a morphism* $*' \to H(X)$ *in* $\mathcal{C}'$*, then there is a morphism* $* \to X$*, and*
(5) *for any object* X *of* $\mathcal{C}$ *the map* $\mathrm{Mor}_{\mathcal{C}}(*, X) \to \mathrm{Mor}_{\mathcal{C}'}(*', H(X))$ *is bijective.*

Here $*$ *and* $*'$ *are final objects of* $\mathcal{C}$ *and* $\mathcal{C}'$*.*

Proof. The implications (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (5) are clear.

Assume (3). Let X be a connected object of $\mathcal{C}$ and let $*' \to H(X)$ be a morphism. Since $H(X)$ is connected by (3) we see that $*' \to H(X)$ is an isomorphism. Hence the G'-set corresponding to $H(X)$ has exactly one element, which means the G-set corresponding to X has one element which means X is isomorphic to the final object of $\mathcal{C}$, in particular there is a map $* \to X$. In this way we see that (3) $\Rightarrow$ (4).

If (1) is true, then the functor *Finite-*G*-Sets* $\to$ *Finite-*G'*-Sets* is fully faithful: in this case a map of G-sets commutes with the action of G if and only if it commutes with the action of G'. Thus (1) $\Rightarrow$ (2).

If (1) is true, then for a G-set X the G-orbits and G'-orbits agree. Thus (1) $\Rightarrow$ (3).

To finish the proof it suffices to show that (4) implies (1). If (1) is false, i.e., if h is not surjective, then there is an open subgroup $U \subset G$ containing $h(G')$ which is not equal to G. Then the finite G-set $M = G/U$ has a transitive action but G' has a fixed point. The object X of $\mathcal{C}$ corresponding to M would contradict (3). In this way we see that (3) $\Rightarrow$ (1) and the proof is complete. □

0BS8 **Lemma 58.4.2.** *In diagram (58.4.0.1) the following are equivalent*

(1) $h \circ h'$ *is trivial, and*
(2) *the image of* $H' \circ H$ *consists of objects isomorphic to finite coproducts of final objects.*

Proof. We may replace H and H' by the canonical functors *Finite-*G*-Sets* $\to$ *Finite-*G'*-Sets* $\to$ *Finite-*G''*-Sets* determined by h and h'. Then we are saying that the action of G'' on every G-set is trivial if and only if the homomorphism $G'' \to G$ is trivial. This is clear. □

0BS9 **Lemma 58.4.3.** *In diagram (58.4.0.1) the following are equivalent*

(1) *the sequence* $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ *is exact in the following sense:* h *is surjective,* $h \circ h'$ *is trivial, and* $\mathrm{Ker}(h)$ *is the smallest closed normal subgroup containing* $\mathrm{Im}(h')$*,*

(2) *H is fully faithful and an object X' of $\mathcal{C}'$ is in the essential image of H if and only if $H'(X')$ is isomorphic to a finite coproduct of final objects, and*
(3) *H is fully faithful, $H \circ H'$ sends every object to a finite coproduct of final objects, and for an object X' of $\mathcal{C}'$ such that $H'(X')$ is a finite coproduct of final objects there exists an object X of $\mathcal{C}$ and an epimorphism $H(X) \to X'$.*

Proof. By Lemmas 58.4.1 and 58.4.2 we may assume that H is fully faithful, h is surjective, $H' \circ H$ maps objects to disjoint unions of the final object, and $h \circ h'$ is trivial. Let $N \subset G'$ be the smallest closed normal subgroup containing the image of h'. It is clear that $N \subset \mathrm{Ker}(h)$. We may assume the functors H and H' are the canonical functors *Finite-G-Sets* $\to$ *Finite-G'-Sets* $\to$ *Finite-G''-Sets* determined by h and h'.

Suppose that (2) holds. This means that for a finite G'-set X' such that G'' acts trivially, the action of G' factors through G. Apply this to $X' = G'/U'N$ where U' is a small open subgroup of G'. Then we see that $\mathrm{Ker}(h) \subset U'N$ for all U'. Since N is closed this implies $\mathrm{Ker}(h) \subset N$, i.e., (1) holds.

Suppose that (1) holds. This means that $N = \mathrm{Ker}(h)$. Let X' be a finite G'-set such that G'' acts trivially. This means that $\mathrm{Ker}(G' \to \mathrm{Aut}(X'))$ is a closed normal subgroup containing $\mathrm{Im}(h')$. Hence $N = \mathrm{Ker}(h)$ is contained in it and the G'-action on X' factors through G, i.e., (2) holds.

Suppose that (3) holds. This means that for a finite G'-set X' such that G'' acts trivially, there is a surjection of G'-sets $X \to X'$ where X is a G-set. Clearly this means the action of G' on X' factors through G, i.e., (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate. This finishes the proof. □

0BN7 **Lemma 58.4.4.** *In diagram (58.4.0.1) the following are equivalent*
(1) *h' is injective, and*
(2) *for every connected object X'' of $\mathcal{C}''$ there exists an object X' of $\mathcal{C}'$ and a diagram*
$$X'' \leftarrow Y'' \to H(X')$$
in $\mathcal{C}''$ where $Y'' \to X''$ is an epimorphism and $Y'' \to H(X')$ is a monomorphism.

Proof. We may replace H' by the corresponding functor between the categories of finite G'-sets and finite G''-sets.

Assume $h' : G'' \to G'$ is injective. Let $H'' \subset G''$ be an open subgroup. Since the topology on G'' is the induced topology from G' there exists an open subgroup $H' \subset G'$ such that $(h')^{-1}(H') \subset H''$. Then the desired diagram is
$$G''/H'' \leftarrow G''/(h')^{-1}(H') \to G'/H'$$
Conversely, assume (2) holds for the functor *Finite-G'-Sets* $\to$ *Finite-G''-Sets*. Let $g'' \in \mathrm{Ker}(h')$. Pick any open subgroup $H'' \subset G''$. By assumption there exists a finite G'-set X' and a diagram
$$G''/H'' \leftarrow Y'' \to X'$$
of G''-sets with the left arrow surjective and the right arrow injective. Since g'' is in the kernel of h' we see that g'' acts trivially on X'. Hence g'' acts trivially on Y''

and hence trivially on G''/H''. Thus $g'' \in H''$. As this holds for all open subgroups we conclude that g'' is the identity element as desired. □

0BTS **Lemma 58.4.5.** *In diagram (58.4.0.1) the following are equivalent*

(1) *the image of h' is normal, and*
(2) *for every connected object X' of $\mathcal{C}'$ such that there is a morphism from the final object of $\mathcal{C}''$ to $H'(X')$ we have that $H'(X')$ is isomorphic to a finite coproduct of final objects.*

Proof. This translates into the following statement for the continuous group homomorphism $h' : G'' \to G'$: the image of h' is normal if and only if every open subgroup $U' \subset G'$ which contains $h'(G'')$ also contains every conjugate of $h'(G'')$. The result follows easily from this; some details omitted. □

58.5. Finite étale morphisms

0BL6 In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let X be a scheme. We will use the notation $F\acute{E}t_X$ to denote the category of schemes finite and étale over X. Thus

(1) an object of $F\acute{E}t_X$ is a finite étale morphism $Y \to X$ with target X, and
(2) a morphism in $F\acute{E}t_X$ from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commute.

We will often call an object of $F\acute{E}t_X$ a *finite étale cover* of X (even if Y is empty). It turns out that there is a stack $p : F\acute{E}t \to Sch$ over the category of schemes whose fibre over X is the category $F\acute{E}t_X$ just defined. See Examples of Stacks, Section 95.6.

0BN8 **Example 58.5.1.** Let k be an algebraically closed field and $X = \mathrm{Spec}(k)$. In this case $F\acute{E}t_X$ is equivalent to the category of finite sets. This works more generally when k is separably algebraically closed. The reason is that a scheme étale over k is the disjoint union of spectra of fields finite separable over k, see Morphisms, Lemma 29.36.7.

0BN9 **Lemma 58.5.2.** *Let X be a scheme. The category $F\acute{E}t_X$ has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $F\acute{E}t_X \to F\acute{E}t_{X'}$ is exact.*

Proof. Finite limits and left exactness. By Categories, Lemma 4.18.4 it suffices to show that $F\acute{E}t_X$ has a final object and fibred products. This is clear because the category of all schemes over X has a final object (namely X) and fibred products. Also, fibred products of schemes finite étale over X are finite étale over X. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 4.23.2).

Finite colimits and right exactness. By Categories, Lemma 4.18.7 it suffices to show that $F\acute{E}t_X$ has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of $F\acute{E}t_X$. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \underline{\mathrm{Spec}}(\mathcal{C})$ and $Y = \underline{\mathrm{Spec}}(\mathcal{B})$ for some finite locally free $\mathcal{O}_X$-algebras $\mathcal{C}$ and $\mathcal{B}$. The morphisms a, b induce two maps $a^\sharp, b^\sharp : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{A} = \mathrm{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$\underline{\mathrm{Spec}}(\mathcal{A}) \longrightarrow X$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of $F\acute{E}t_X$ as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing X by the members of an étale covering (Descent, Lemmas 35.23.25 and 35.23.31). Thus by Étale Morphisms, Lemma 41.18.3 we may assume that $Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{j=1,\ldots,m} X$. Then

$$\mathcal{C} = \prod\nolimits_{1 \le j \le m} \mathcal{O}_X \quad \text{and} \quad \mathcal{B} = \prod\nolimits_{1 \le i \le n} \mathcal{O}_X$$

After a further replacement by the members of an open covering we may assume that a, b correspond to maps $a_s, b_s : \{1, \ldots, m\} \to \{1, \ldots, n\}$, i.e., the summand X of Z corresponding to the index j maps into the summand X of Y corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism a, resp. b. Let $\{1, \ldots, n\} \to T$ be the coequalizer of a_s, b_s. Then we see that

$$\mathcal{A} = \prod\nolimits_{t \in T} \mathcal{O}_X$$

whose spectrum is certainly finite étale over X. We omit the verification that this is compatible with base change. Thus base change is a right exact functor. □

0BNA **Remark 58.5.3.** Let X be a scheme. Consider the natural functors $F_1 : F\acute{E}t_X \to Sch$ and $F_2 : F\acute{E}t_X \to Sch/X$. Then

(1) The functors F_1 and F_2 commute with finite colimits.
(2) The functor F_2 commutes with finite limits,
(3) The functor F_1 commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 58.5.2 and Categories, Lemma 4.16.2. It is clear that F_1 and F_2 commute with finite coproducts. By the dual of Categories, Lemma 4.23.2 we need to show that F_1 and F_2 commute with coequalizers. In the proof of Lemma 58.5.2 we saw that coequalizers in $F\acute{E}t_X$ look étale locally like this

$$\coprod\nolimits_{j \in J} U \underset{b}{\overset{a}{\rightrightarrows}} \coprod\nolimits_{i \in I} U \longrightarrow \coprod\nolimits_{t \in \mathrm{Coeq}(a,b)} U$$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulated precisely in Descent, Lemma 35.13.8.

0BL7 **Lemma 58.5.4.** *Let X be a scheme. Given U, V finite étale over X there exists a scheme W finite étale over X such that*

$$\mathrm{Mor}_X(X, W) = \mathrm{Mor}_X(U, V)$$

and such that the same remains true after any base change.

Proof. By More on Morphisms, Lemma 37.68.4 there exists a scheme W representing $Mor_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 29.36.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing X by the members of an étale covering (Descent, Lemmas 35.23.25 and 35.23.6). Thus by Étale Morphisms, Lemma 41.18.3 we may assume that $U = \coprod_{i=1,\ldots,n} X$ and $V = \coprod_{j=1,\ldots,m} X$. In this case $W = \coprod_{\alpha:\{1,\ldots,n\}\to\{1,\ldots,m\}} X$ by inspection (details omitted) and the proof is complete. □

Let X be a scheme. A *geometric point* of X is a morphism $\mathrm{Spec}(k) \to X$ where k is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{x})$ to denote k. We say $\overline{x}$ *lies over* x to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 59.29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$|U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x}$$

Since $U_{\overline{x}}$ as a scheme over $\overline{x}$ is a disjoint union of copies of $\overline{x}$ (Morphisms, Lemma 29.36.7) we can also describe this set as

$$|U_{\overline{x}}| = \left\{ \begin{array}{c} \text{commutative} \\ \text{diagrams} \end{array} \quad \begin{array}{ccc} \overline{x} & \xrightarrow{\overline{u}} & U \\ & \searrow^{\overline{x}} & \downarrow \\ & & X \end{array} \right\}$$

The assignment $U \mapsto |U_{\overline{x}}|$ is a functor which is often denoted $F_{\overline{x}}$.

0BNB **Lemma 58.5.5.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point. The functor*

$$F_{\overline{x}} : F\acute{E}t_X \longrightarrow Sets, \quad Y \longmapsto |Y_{\overline{x}}|$$

defines a Galois category (Definition 58.3.6).

Proof. After identifying $F\acute{E}t_{\overline{x}}$ with the category of finite sets (Example 58.5.1) we see that our functor $F_{\overline{x}}$ is nothing but the base change functor for the morphism $\overline{x} \to X$. Thus we see that $F\acute{E}t_X$ has finite limits and finite colimits and that $F_{\overline{x}}$ is exact by Lemma 58.5.2. We will also use that finite limits in $F\acute{E}t_X$ agree with the corresponding finite limits in the category of schemes over X, see Remark 58.5.3.

If $Y' \to Y$ is a monomorphism in $F\acute{E}t_X$ then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Étale Morphisms, Theorem 41.14.1). Since Y' is finite over X and Y separated over X, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 29.44.14), hence closed (Morphisms, Lemma 29.44.11), hence it is the inclusion of an open and closed subscheme of Y. It follows that Y is a connected objects of the category $F\acute{E}t_X$ (as in Definition 58.3.6) if and only if Y is connected as a scheme. Then it follows from Topology, Lemma 5.7.7 that Y is a finite coproduct of its connected components both as a scheme and in the sense of Definition 58.3.6.

Let $Y \to Z$ be a morphism in $F\acute{E}t_X$ which induces a bijection $F_{\overline{x}}(Y) \to F_{\overline{x}}(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume Z is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to

show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of Z lying over $\overline{x}$ and since Z is connected and the degree is locally constant we conclude. □

58.6. Fundamental groups

0BQ8 In this section we define Grothendieck's algebraic fundamental group. The following definition makes sense thanks to Lemma 58.5.5.

0BNC **Definition 58.6.1.** Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X. The *fundamental group* of X with *base point* $\overline{x}$ is the group

$$\pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$$

of automorphisms of the fibre functor $F_{\overline{x}} : F\acute{E}t_X \to Sets$ endowed with its canonical profinite topology from Lemma 58.3.1.

Combining the above with the material from Section 58.3 we obtain the following theorem.

0BND **Theorem 58.6.2.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

(1) *The fibre functor $F_{\overline{x}}$ defines an equivalence of categories*

$$F\acute{E}t_X \longrightarrow Finite\text{-}\pi_1(X, \overline{x})\text{-}Sets$$

(2) *Given a second geometric point $\overline{x}'$ of X there exists an isomorphism $t : F_{\overline{x}} \to F_{\overline{x}'}$. This gives an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalences in (1). This isomorphism is independent of t up to inner conjugation.*

(3) *Given a morphism $f : X \to Y$ of connected schemes denote $\overline{y} = f \circ \overline{x}$. There is a canonical continuous homomorphism*

$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$$

such that the diagram

$$\begin{array}{ccc} F\acute{E}t_Y & \xrightarrow[base\ change]{} & F\acute{E}t_X \\ {\scriptstyle F_{\overline{y}}}\downarrow & & \downarrow{\scriptstyle F_{\overline{x}}} \\ Finite\text{-}\pi_1(Y, \overline{y})\text{-}Sets & \xrightarrow{f_*} & Finite\text{-}\pi_1(X, \overline{x})\text{-}Sets \end{array}$$

is commutative.

Proof. Part (1) follows from Lemma 58.5.5 and Proposition 58.3.10. Part (2) is a special case of Lemma 58.3.11. For part (3) observe that the diagram

$$\begin{array}{ccc} F\acute{E}t_Y & \longrightarrow & F\acute{E}t_X \\ {\scriptstyle F_{\overline{y}}}\downarrow & & \downarrow{\scriptstyle F_{\overline{x}}} \\ Sets & = & Sets \end{array}$$

is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \overline{x}$. Hence we can apply Lemma 58.3.11 with the implied transformation of functors to get (3). □

0BNE **Lemma 58.6.3.** *Let K be a field and set $X = \mathrm{Spec}(K)$. Let $\overline{K}$ be an algebraic closure and denote $\overline{x} : \mathrm{Spec}(\overline{K}) \to X$ the corresponding geometric point. Let $K^{sep} \subset \overline{K}$ be the separable algebraic closure.*

(1) *The functor of Lemma 58.2.2 induces an equivalence*

$$F\acute{E}t_X \longrightarrow \textit{Finite-Gal}(K^{sep}/K)\textit{-Sets}.$$

compatible with $F_{\overline{x}}$ and the functor Finite-Gal(K^{sep}/K)-Sets $\to$ Sets.

(2) *This induces a canonical isomorphism*

$$\textit{Gal}(K^{sep}/K) \longrightarrow \pi_1(X, \overline{x})$$

of profinite topological groups.

Proof. The functor of Lemma 58.2.2 is the same as the functor $F_{\overline{x}}$ because for any Y étale over X we have

$$\mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y) = \mathrm{Mor}_X(\mathrm{Spec}(K^{sep}), Y)$$

Namely, as seen in the proof of Lemma 58.2.2 we have $Y = \coprod_{i \in I} \mathrm{Spec}(L_i)$ with L_i/K finite separable over K. Hence any K-algebra homomorphism $L_i \to \overline{K}$ factors through K^{sep}. Also, note that $F_{\overline{x}}(Y)$ is finite if and only if I is finite if and only if $Y \to X$ is finite étale. This proves (1).

Part (2) is a formal consequence of (1), Lemma 58.3.11, and Lemma 58.3.3. (Please also see the remark below.) □

0BQ9 **Remark 58.6.4.** In the situation of Lemma 58.6.3 let us give a more explicit construction of the isomorphism $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$. Observe that $\mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}(\overline{K}/K)$ as $\overline{K}$ is the perfection of K^{sep}. Since $F_{\overline{x}}(Y) = \mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y)$ we may consider the map

$$\mathrm{Aut}(\overline{K}/K) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y), \quad (\sigma, \overline{y}) \mapsto \sigma \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma)$$

This is an action because

$$\sigma\tau \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma\tau) = \overline{y} \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma) = \sigma \cdot (\tau \cdot \overline{y})$$

The action is functorial in $Y \in F\acute{E}t_X$ and we obtain the desired map.

58.7. Galois covers of connected schemes

03SF Let X be a connected scheme with geometric point $\overline{x}$. Since $F_{\overline{x}} : F\acute{E}t_X \to \textit{Sets}$ is a Galois category (Lemma 58.5.5) the material in Section 58.3 applies. In this section we explicitly transfer some of the terminology and results to the setting of schemes and finite étale morphisms.

We will say a finite étale morphism $Y \to X$ is a *Galois cover* if Y defines a Galois object of $F\acute{E}t_X$. For a finite étale morphism $Y \to X$ with $G = \mathrm{Aut}_X(Y)$ the following are equivalent

(1) Y is a Galois cover of X,
(2) Y is connected and $|G|$ is equal to the degree of $Y \to X$,
(3) Y is connected and G acts transitively on $F_{\overline{x}}(Y)$, and
(4) Y is connected and G acts simply transitively on $F_{\overline{x}}(Y)$.

This follows immediately from the discussion in Section 58.3.

For any finite étale morphism $f : Y \to X$ with Y connected, there is a finite étale Galois cover $Y' \to X$ which dominates Y (Lemma 58.3.8).

The Galois objects of $F\acute{E}t_X$ correspond, via the equivalence

$$F_{\overline{x}} : F\acute{E}t_X \to \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$$

of Theorem 58.6.2, with the finite $\pi_1(X, \overline{x})$-*Sets* of the form $G = \pi_1(X, \overline{x})/H$ where H is a normal open subgroup. Equivalently, if G is a finite group and $\pi_1(X, \overline{x}) \to G$ is a continuous surjection, then G viewed as a $\pi_1(X, \overline{x})$-set corresponds to a Galois covering.

If $Y_i \to X$, $i = 1, 2$ are finite étale Galois covers with Galois groups G_i, then there exists a finite étale Galois cover $Y \to X$ whose Galois group is a subgroup of $G_1 \times G_2$. Namely, take the corresponding continuous homomorphisms $\pi_1(X, \overline{x}) \to G_i$ and let G be the image of the induced continuous homomorphism $\pi_1(X, \overline{x}) \to G_1 \times G_2$.

58.8. Topological invariance of the fundamental group

0BTT The main result of this section is that a universal homeomorphism of connected schemes induces an isomorphism on fundamental groups. See Proposition 58.8.4.

Instead of directly proving two schemes have the same fundamental group, we often prove that their categories of finite étale coverings are the same. This of course implies that their fundamental groups are equal provided they are connected.

0BQA **Lemma 58.8.1.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes such that the base change functor $F\acute{E}t_Y \to F\acute{E}t_X$ is an equivalence of categories. In this case*

(1) *f induces a homeomorphism $\pi_0(X) \to \pi_0(Y)$,*
(2) *if X or equivalently Y is connected, then $\pi_1(X, \overline{x}) = \pi_1(Y, \overline{y})$.*

Proof. Let $Y = Y_0 \amalg Y_1$ be a decomposition into nonempty open and closed subschemes. We claim that $f(X)$ meets both Y_i. Namely, if not, say $f(X) \subset Y_1$, then we can consider the finite étale morphism $V = Y_1 \to Y$. This is not an isomorphism but $V \times_Y X \to X$ is an isomorphism, which is a contradiction.

Suppose that $X = X_0 \amalg X_1$ is a decomposition into open and closed subschemes. Consider the finite étale morphism $U = X_1 \to X$. Then $U = X \times_Y V$ for some finite étale morphism $V \to Y$. The degree of the morphism $V \to Y$ is locally constant, hence we obtain a decomposition $Y = \coprod_{d \geq 0} Y_d$ into open and closed subschemes such that $V \to Y$ has degree d over Y_d. Since $f^{-1}(Y_d) = \emptyset$ for $d > 1$ we conclude that $Y_d = \emptyset$ for $d > 1$ by the above. And we conclude that $f^{-1}(Y_i) = X_i$ for $i = 0, 1$.

It follows that f^{-1} induces a bijection between the set of open and closed subsets of Y and the set of open and closed subsets of X. Note that X and Y are spectral spaces, see Properties, Lemma 28.2.4. By Topology, Lemma 5.12.10 the lattice of open and closed subsets of a spectral space determines the set of connected components. Hence $\pi_0(X) \to \pi_0(Y)$ is bijective. Since $\pi_0(X)$ and $\pi_0(Y)$ are profinite spaces (Topology, Lemma 5.22.5) we conclude that $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism by Topology, Lemma 5.17.8. This proves (1). Part (2) is immediate. □

The following lemma tells us that the fundamental group of a henselian pair is the fundamental group of the closed subset.

09ZS **Lemma 58.8.2.** *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. The functor*

$$F\acute{E}t_X \longrightarrow F\acute{E}t_Z, \quad U \longmapsto U \times_X Z$$

is an equivalence of categories.

Proof. This is a translation of More on Algebra, Lemma 15.13.2. □

The following lemma tells us that the fundamental group of a thickening is the same as the fundamental group of the original. We will use this in the proof of the strong proposition concerning universal homeomorphisms below.

0BQB **Lemma 58.8.3.** *Let $X \subset X'$ be a thickening of schemes. The functor*

$$F\acute{E}t_{X'} \longrightarrow F\acute{E}t_X, \quad U' \longmapsto U' \times_{X'} X$$

is an equivalence of categories.

Proof. For a discussion of thickenings see More on Morphisms, Section 37.2. Let $U' \to X'$ be an étale morphism such that $U = U' \times_{X'} X \to X$ is finite étale. Then $U' \to X'$ is finite étale as well. This follows for example from More on Morphisms, Lemma 37.3.4. Now, if $X \subset X'$ is a finite order thickening then this remark combined with Étale Morphisms, Theorem 41.15.2 proves the lemma. Below we will prove the lemma for general thickenings, but we suggest the reader skip the proof.

Let $X' = \bigcup X'_i$ be an affine open covering. Set $X_i = X \times_{X'} X'_i$, $X'_{ij} = X'_i \cap X'_j$, $X_{ij} = X \times_{X'} X'_{ij}$, $X'_{ijk} = X'_i \cap X'_j \cap X'_k$, $X_{ijk} = X \times_{X'} X'_{ijk}$. Suppose that we can prove the theorem for each of the thickenings $X_i \subset X'_i$, $X_{ij} \subset X'_{ij}$, and $X_{ijk} \subset X'_{ijk}$. Then the result follows for $X \subset X'$ by relative glueing of schemes, see Constructions, Section 27.2. Observe that the schemes X'_i, X'_{ij}, X'_{ijk} are each separated as open subschemes of affine schemes. Repeating the argument one more time we reduce to the case where the schemes X'_i, X'_{ij}, X'_{ijk} are affine.

In the affine case we have $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A'/I')$ where I' is a locally nilpotent ideal. Then (A', I') is a henselian pair (More on Algebra, Lemma 15.11.2) and the result follows from Lemma 58.8.2 (which is much easier in this case). □

The "correct" way to prove the following proposition would be to deduce it from the invariance of the étale site, see Étale Cohomology, Theorem 59.45.2.

0BQN **Proposition 58.8.4.** *Let $f : X \to Y$ be a universal homeomorphism of schemes. Then*

$$F\acute{E}t_Y \longrightarrow F\acute{E}t_X, \quad V \longmapsto V \times_Y X$$

is an equivalence. Thus if X and Y are connected, then f induces an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ of fundamental groups.

Proof. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 29.45.5. In particular, the diagonal $\Delta : X \to X \times_Y X$ is a thickening by Morphisms, Lemma

29.10.2. Thus by Lemma 58.8.3 we see that given a finite étale morphism $U \to X$ there is a unique isomorphism

$$\varphi : U \times_Y X \to X \times_Y U$$

of schemes finite étale over $X \times_Y X$ which pulls back under Δ to $\mathrm{id} : U \to U$ over X. Since $X \to X \times_Y X \times_Y X$ is a thickening as well (it is bijective and a closed immersion) we conclude that (U, φ) is a descent datum relative to X/Y. By Étale Morphisms, Proposition 41.20.6 we conclude that $U = X \times_Y V$ for some $V \to Y$ quasi-compact, separated, and étale. We omit the proof that $V \to Y$ is finite (hints: the morphism $U \to V$ is surjective and $U \to Y$ is integral). We conclude that $F\acute{E}t_Y \to F\acute{E}t_X$ is essentially surjective.

Arguing in the same manner as above we see that given $V_1 \to Y$ and $V_2 \to Y$ in $F\acute{E}t_Y$ any morphism $a : X \times_Y V_1 \to X \times_Y V_2$ over X is compatible with the canonical descent data. Thus a descends to a morphism $V_1 \to V_2$ over Y by Étale Morphisms, Lemma 41.20.3. □

58.9. Finite étale covers of proper schemes

0BQC In this section we show that the fundamental group of a connected proper scheme over a henselian local ring is the same as the fundamental group of its special fibre. We also prove a variant of this result for a henselian pair.

We also show that the fundamental group of a connected proper scheme over an algebraically closed field k does not change if we replace k by an algebraically closed extension.

Instead of stating and proving the results in the connected case we prove the results in general and we leave it to the reader to deduce the result for fundamental groups using Lemma 58.8.1.

0A48 **Lemma 58.9.1.** *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Then the functor*

$$F\acute{E}t_X \to F\acute{E}t_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof given here is an example of applying algebraization and approximation. We proceed in a number of stages.

Essential surjectivity when A is a complete local Noetherian ring. Let $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. By Étale Morphisms, Theorem 41.15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 37.3.3. In this case the morphism $U_0 \to \mathrm{Spec}(A/\mathfrak{m})$ is proper as X_0 is proper over $A/\mathfrak{m}$. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 30.28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 37.12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is a complete local Noetherian ring. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 41.14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when A is a henselian local Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to the maximal ideal. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \operatorname{colim} A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Theorem 16.13.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when A is a henselian local Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let $(A, \mathfrak{m})$ be a henselian local ring. Set $S = \operatorname{Spec}(A)$ and denote $s \in S$ the closed point. By Limits, Lemma 32.13.3 we can write $X \to \operatorname{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i of finite type over $\mathbf{Z}$. For each i let $s_i \in S_i$ be the image of s. Since $S = \lim S_i$ and $A = \mathcal{O}_{S,s}$ we have $A = \operatorname{colim} \mathcal{O}_{S_i, s_i}$. The ring $A_i = \mathcal{O}_{S_i, s_i}$ is a Noetherian local G-ring (More on Algebra, Proposition 15.50.12). By More on Algebra, Lemma 15.12.5 we see that $A = \operatorname{colim} A_i^h$. By More on Algebra, Lemma 15.50.8 the rings A_i^h are G-rings. Thus we see that $A = \operatorname{colim} A_i^h$ and

$$X = \lim(X_i \times_{S_i} \operatorname{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \operatorname{Spec}(A_i^h)$ (by Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} s_i)$. Thus we formally deduce the result for $X/\operatorname{Spec}(A)$ from the result for the $(X_i \times_{S_i} \operatorname{Spec}(A_i^h))/\operatorname{Spec}(A_i^h)$ which we dealt with above. $\square$

0GS2 **Lemma 58.9.2.** *Let (A, I) be a henselian pair. Let X be a proper scheme over A. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Then the functor*

$$F\acute{E}t_X \to F\acute{E}t_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 58.9.1.

Essential surjectivity when A is Noetherian and I-adically complete. Let $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I^{n+1})$. By Étale Morphisms, Theorem 41.15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 37.3.3. In this case the morphism $U_0 \to \mathrm{Spec}(A/I)$ is proper as X_0 is proper over A/I. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 30.28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 37.12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is Noetherian and I-adically complete. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 41.14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when (A, I) is a henselian pair and A is a Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to I. Observe that $A^\wedge$ is a Noetherian ring which is $IA^\wedge$-adically complete, see Algebra, Lemmas 10.97.4 and 10.97.6. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \mathrm{colim}\, A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Lemma 16.14.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of

$V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when $(A, I$ is a henselian pair and A is a Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let (A, I) be a henselian pair. Set $S = \mathrm{Spec}(A)$ and denote $S_0 = \mathrm{Spec}(A/I)$. By Limits, Lemma 32.13.3 we can write $X \to \mathrm{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i affine and of finite type over $\mathbf{Z}$. Write $S_i = \mathrm{Spec}(A_i)$ and denote $I_i \subset A_i$ the inverse image of I by the map $A_i \to A$. Set $S_{i,0} = \mathrm{Spec}(A_i/I_i)$. Since $S = \lim S_i$ we have $A = \mathrm{colim}\, A_i$. Thus we also have $I = \mathrm{colim}\, I_i$ and $A/I = \mathrm{colim}\, A_i/I_i$. The ring A_i is a Noetherian G-ring (More on Algebra, Proposition 15.50.12). Denote (A_i^h, I_i^h) the henselization of the pair (A_i, I_i). By More on Algebra, Lemma 15.12.5 we see that $A = \mathrm{colim}\, A_i^h$. By More on Algebra, Lemma 15.50.15 the rings A_i^h are G-rings. Thus we see that $A = \mathrm{colim}\, A_i^h$ and

$$X = \lim(X_i \times_{S_i} \mathrm{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \mathrm{Spec}(A_i^h)$ (by Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} S_{i,0})$. Thus we formally deduce the result for $X/\mathrm{Spec}(A)$ from the result for the $(X_i \times_{S_i} \mathrm{Spec}(A_i^h))/\mathrm{Spec}(A_i^h)$ which we dealt with above. □

0A49 **Lemma 58.9.3.** *Let k'/k be an extension of algebraically closed fields. Let X be a proper scheme over k. Then the functor*

$$U \longmapsto U_{k'}$$

is an equivalence of categories between schemes finite étale over X and schemes finite étale over $X_{k'}$.

Proof. Let us prove the functor is essentially surjective. Let $U' \to X_{k'}$ be a finite étale morphism. Write $k' = \mathrm{colim}\, A_i$ as a filtered colimit of finite type k-algebras. By Limits, Lemma 32.10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X_{k'}$ is U'. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 32.8.3 and 32.8.10). Since k is algebraically closed we can find a k-valued point t in $\mathrm{Spec}(A_i)$. Let $U = (U_i)_t$ be the fibre of U_i over t. Let A_i^h be the henselization of $(A_i)_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal corresponding to the point t. By Lemma 58.9.1 we see that $(U_i)_{A_i^h} = U \times \mathrm{Spec}(A_i^h)$ as schemes over $X_{A_i^h}$. Now since A_i^h is algebraic over A_i (see for example discussion in Smoothing Ring Maps, Example 16.13.3) and since k' is algebraically closed we can find a ring map $A_i^h \to k'$ extending the given inclusion $A_i \subset k'$. Hence we conclude that U' is isomorphic to the base change of U. The proof of fully faithfulness is exactly the same. □

58.10. Local connectedness

0BQD In this section we ask when $\pi_1(U) \to \pi_1(X)$ is surjective for U a dense open of a scheme X. We will see that this is the case (roughly) when $U \cap B$ is connected for any small "ball" B around a point $x \in X \setminus U$.

0BQE **Lemma 58.10.1.** *Let $f : X \to Y$ be a morphism of schemes. If $f(X)$ is dense in Y then the base change functor $F\acute{E}t_Y \to F\acute{E}t_X$ is faithful.*

Proof. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given W finite étale over Y and a morphism $s : X \to W$ over X there is at most one section $t : Y \to W$ such that $s = t \circ f$. Consider two sections $t_1, t_2 : Y \to W$ such that $s = t_1 \circ f = t_2 \circ f$. Since the equalizer of t_1 and t_2 is closed in Y (Schemes, Lemma 26.21.5) and since $f(X)$ is dense in Y we see that t_1 and t_2 agree on Y_{red}. Then it follows that t_1 and t_2 have the same image which is an open and closed subscheme of W mapping isomorphically to Y (Étale Morphisms, Proposition 41.6.1) hence they are equal. □

The condition in the following lemma that the punctured spectrum of the strict henselization is connected follows for example from the assumption that the local ring is geometrically unibranch, see More on Algebra, Lemma 15.107.5. There is a partial converse in Properties, Lemma 28.15.3.

0BLQ **Lemma 58.10.2.** *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. If the punctured spectrum of the strict henselization of A is connected, then*

$$F\acute{E}t_X \longrightarrow F\acute{E}t_U, \quad Y \longmapsto Y \times_X U$$

is a fully faithful functor.

Proof. Assume A is strictly henselian. In this case any finite étale cover Y of X is isomorphic to a finite disjoint union of copies of X. Thus it suffices to prove that any morphism $U \to U \amalg \ldots \amalg U$ over U, extends uniquely to a morphism $X \to X \amalg \ldots \amalg X$ over X. If U is connected (in particular nonempty), then this is true.

The general case. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given Y finite étale over X any morphism $s : U \to Y$ over X extends to a morphism $t : X \to Y$ over X. Let A^{sh} be the strict henselization of A and denote $X^{sh} = \mathrm{Spec}(A^{sh})$, $U^{sh} = U \times_X X^{sh}$, $Y^{sh} = Y \times_X X^{sh}$. By the first paragraph and our assumption on A, we can extend the base change $s^{sh} : U^{sh} \to Y^{sh}$ of s to $t^{sh} : X^{sh} \to Y^{sh}$. Set $A' = A^{sh} \otimes_A A^{sh}$. Then the two pullbacks t'_1, t'_2 of t^{sh} to $X' = \mathrm{Spec}(A')$ are extensions of the pullback s' of s to $U' = U \times_X X'$. As $A \to A'$ is flat we see that $U' \subset X'$ is (topologically) dense by going down for $A \to A'$ (Algebra, Lemma 10.39.19). Thus $t'_1 = t'_2$ by Lemma 58.10.1. Hence t^{sh} descends to a morphism $t : X \to Y$ for example by Descent, Lemma 35.13.7. □

In view of Lemma 58.10.2 it is interesting to know when the punctured spectrum of a ring (and of its strict henselization) is connected. There is a famous lemma due to Hartshorne which gives a sufficient condition, see Local Cohomology, Lemma 51.3.1.

0BQF **Lemma 58.10.3.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

1. *the underlying topological space of X is Noetherian, and*
2. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 32.20.3 we find an open $U \subset U'$ containing x and an extension $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ of φ. Since the underlying topological space of X is Noetherian this finishes the proof by Noetherian induction on the complement of the open over which φ is defined. □

0BSA **Lemma 58.10.4.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

(1) *$U \to X$ is quasi-compact,*
(2) *every point of $X \setminus U$ is closed, and*
(3) *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1.

Let $x \in X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. Since every point of $X \setminus U$ is closed V is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 32.20.3 (this uses that U is retrocompact in X) we find an open $U \subset U'_x$ containing x and an extension $\varphi'_x : (Y_1)_{U'_x} \to (Y_2)_{U'_x}$ of φ. Note that given two points $x, x' \in X \setminus U$ the morphisms φ'_x and $\varphi'_{x'}$ agree over $U'_x \cap U'_{x'}$ as U is dense in that open (Lemma 58.10.1). Thus we can extend φ to $\bigcup U'_x = X$ as desired. □

0BQG **Lemma 58.10.5.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

(1) *every quasi-compact open of X has finitely many irreducible components,*
(2) *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 58.10.1. We will prove existence by showing that we can enlarge U if $U \neq X$ and using Zorn's lemma to finish the proof.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 58.10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ (uniquely) to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. Choose an affine neighbourhood $W \subset X$ of x. Since $U \cap W$ is dense in W it contains the generic points $\eta_1, \ldots, \eta_n$ of W. Choose an affine open $W' \subset W \cap U$ containing $\eta_1, \ldots, \eta_n$. Set $V' = W' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By Limits, Lemma 32.20.3 applied to $x \in W \supset W'$ we find an open $W' \subset W'' \subset W$ with $x \in W''$ and a morphism $\varphi'' : (Y_1)_{W''} \to (Y_2)_{W''}$ agreeing

with φ over W'. Since W' is dense in $W'' \cap U$, we see by Lemma 58.10.1 that φ and φ'' agree over $U \cap W'$. Thus φ and φ'' glue to a morphism φ' over $U' = U \cup W''$ agreeing with φ over U. Observe that $x \in U'$ so that we've extended φ to a strictly larger open.

Consider the set $\mathcal{S}$ of pairs (U', φ') where $U \subset U'$ and φ' is an extension of φ. We endow $\mathcal{S}$ with a partial ordering in the obvious manner. If (U'_i, φ'_i) is a totally ordered subset, then it has a maximum (U', φ'). Just take $U' = \bigcup U'_i$ and let $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ be the morphism agreeing with φ'_i over U'_i. Thus Zorn's lemma applies and $\mathcal{S}$ has a maximal element. By the argument above we see that this maximal element is an extension of φ over all of X. □

0BSB **Lemma 58.10.6.** *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \operatorname{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Let U^{sh} be the punctured spectrum of the strict henselization A^{sh} of A. Assume U is quasi-compact and U^{sh} is connected. Then the sequence*

$$\pi_1(U^{sh}, \overline{u}) \to \pi_1(U, \overline{u}) \to \pi_1(X, \overline{u}) \to 1$$

is exact in the sense of Lemma 58.4.3 part (1).

Proof. The map $\pi_1(U) \to \pi_1(X)$ is surjective by Lemmas 58.10.2 and 58.4.1.

Write $X^{sh} = \operatorname{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X U^{sh}$. It follows that the composition $\pi_1(U^{sh}) \to \pi_1(U) \to \pi_1(X)$ is trivial, see Lemma 58.4.2.

To finish the proof, it suffices according to Lemma 58.4.3 to show the following: Given a finite étale morphism $V \to U$ such that $V \times_U U^{sh}$ is a disjoint union of copies of U^{sh}, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X U$ over U. The assumption implies that there exists a finite étale morphism $Y^{sh} \to X^{sh}$ and an isomorphism $V \times_U U^{sh} \cong Y^{sh} \times_{X^{sh}} U^{sh}$. Consider the following diagram

$$\begin{array}{ccccccc}
U & \longleftarrow & U^{sh} & \substack{\longleftarrow \\ \longleftarrow} & U^{sh} \times_U U^{sh} & \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} & U^{sh} \times_U U^{sh} \times_U U^{sh} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longleftarrow & X^{sh} & \substack{\longleftarrow \\ \longleftarrow} & X^{sh} \times_X X^{sh} & \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} & X^{sh} \times_X X^{sh} \times_X X^{sh}
\end{array}$$

Since $U \subset X$ is quasi-compact by assumption, all the downward arrows are quasi-compact open immersions. Let $\xi \in X^{sh} \times_X X^{sh}$ be a point not in $U^{sh} \times_U U^{sh}$. Then ξ lies over the closed point x^{sh} of X^{sh}. Consider the local ring homomorphism

$$A^{sh} = \mathcal{O}_{X^{sh}, x^{sh}} \to \mathcal{O}_{X^{sh} \times_X X^{sh}, \xi}$$

determined by the first projection $X^{sh} \times_X X^{sh}$. This is a filtered colimit of local homomorphisms which are localizations étale ring maps. Since A^{sh} is strictly henselian, we conclude that it is an isomorphism. Since this holds for every ξ in the complement it follows there are no specializations among these points and hence every such ξ is a closed point (you can also prove this directly). As the local ring at ξ is isomorphic to A^{sh}, it is strictly henselian and has connected punctured spectrum. Similarly for points ξ of $X^{sh} \times_X X^{sh} \times_X X^{sh}$ not in $U^{sh} \times_U U^{sh} \times_U U^{sh}$. It follows from Lemma 58.10.4 that pullback along the vertical arrows induce fully faithful functors on the categories of finite étale schemes. Thus the canonical descent datum

on $V \times_U U^{sh}$ relative to the fpqc covering $\{U^{sh} \to U\}$ translates into a descent datum for Y^{sh} relative to the fpqc covering $\{X^{sh} \to X\}$. Since $Y^{sh} \to X^{sh}$ is finite hence affine, this descent datum is effective (Descent, Lemma 35.37.1). Thus we get an affine morphism $Y \to X$ and an isomorphism $Y \times_X X^{sh} \to Y^{sh}$ compatible with descent data. By fully faithfulness of descent data (as in Descent, Lemma 35.35.11) we get an isomorphism $V \to U \times_X Y$. Finally, $Y \to X$ is finite étale as $Y^{sh} \to X^{sh}$ is, see Descent, Lemmas 35.23.31 and 35.23.25. □

Let X be an irreducible scheme. Let $\eta \in X$ be the generic point. The canonical morphism $\eta \to X$ induces a canonical map

0BQH (58.10.6.1) $$\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

The identification on the left hand side is Lemma 58.6.3.

0BQI **Lemma 58.10.7.** *Let X be an irreducible, geometrically unibranch scheme. For any nonempty open $U \subset X$ the canonical map*

$$\pi_1(U, \overline{u}) \longrightarrow \pi_1(X, \overline{u})$$

is surjective. The map (58.10.6.1) $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective as well.

Proof. By Lemma 58.8.3 we may replace X by its reduction. Thus we may assume that X is an integral scheme. By Lemma 58.4.1 the assertion of the lemma translates into the statement that the functors $F\acute{E}t_X \to F\acute{E}t_U$ and $F\acute{E}t_X \to F\acute{E}t_\eta$ are fully faithful.

The result for $F\acute{E}t_X \to F\acute{E}t_U$ follows from Lemma 58.10.5 and the fact that for a local ring A which is geometrically unibranch its strict henselization has an irreducible spectrum. See More on Algebra, Lemma 15.107.5.

Observe that the residue field $\kappa(\eta) = \mathcal{O}_{X,\eta}$ is the filtered colimit of $\mathcal{O}_X(U)$ over $U \subset X$ nonempty open affine. Hence $F\acute{E}t_\eta$ is the colimit of the categories $F\acute{E}t_U$ over such U, see Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. A formal argument then shows that fully faithfulness for $F\acute{E}t_X \to F\acute{E}t_\eta$ follows from the fully faithfulness of the functors $F\acute{E}t_X \to F\acute{E}t_U$. □

0BSC **Lemma 58.10.8.** *Let X be a scheme. Let $x_1, \ldots, x_n \in X$ be a finite number of closed points such that*

1. *$U = X \setminus \{x_1, \ldots, x_n\}$ is connected and is a retrocompact open of X, and*
2. *for each i the punctured spectrum U_i^{sh} of the strict henselization of $\mathcal{O}_{X,x_i}$ is connected.*

Then the map $\pi_1(U) \to \pi_1(X)$ is surjective and the kernel is the smallest closed normal subgroup of $\pi_1(U)$ containing the image of $\pi_1(U_i^{sh}) \to \pi_1(U)$ for $i = 1, \ldots, n$.

Proof. Surjectivity follows from Lemmas 58.10.4 and 58.4.1. We can consider the sequence of maps

$$\pi_1(U) \to \ldots \to \pi_1(X \setminus \{x_1, x_2\}) \to \pi_1(X \setminus \{x_1\}) \to \pi_1(X)$$

A group theory argument then shows it suffices to prove the statement on the kernel in the case $n = 1$ (details omitted). Write $x = x_1$, $U^{sh} = U_1^{sh}$, set $A = \mathcal{O}_{X,x}$, and

let A^{sh} be the strict henselization. Consider the diagram

$$\begin{array}{ccccc} U & \longleftarrow & \mathrm{Spec}(A) \setminus \{\mathfrak{m}\} & \longleftarrow & U^{sh} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A^{sh}) \end{array}$$

By Lemma 58.4.3 we have to show finite étale morphisms $V \to U$ which pull back to trivial coverings of U^{sh} extend to finite étale schemes over X. By Lemma 58.10.6 we know the corresponding statement for finite étale schemes over the punctured spectrum of A. However, by Limits, Lemma 32.20.1 schemes of finite presentation over X are the same thing as schemes of finite presentation over U and A glued over the punctured spectrum of A. This finishes the proof. □

58.11. Fundamental groups of normal schemes

0BQJ Let X be an integral, geometrically unibranch scheme. In the previous section we have seen that the fundamental group of X is a quotient of the Galois group of the function field K of X. Since the map is continuous the kernel is a normal closed subgroup of the Galois group. Hence this kernel corresponds to a Galois extension M/K by Galois theory (Fields, Theorem 9.22.4). In this section we will determine M when X is a normal integral scheme.

Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Consider the normalization $Y \to X$ of X in the morphism $\mathrm{Spec}(L) \to X$ as defined in Morphisms, Section 29.53. We will say (in this setting) that X *is unramified in* L if $Y \to X$ is an unramified morphism of schemes. In Lemma 58.13.4 we will elucidate this condition. Observe that the scheme theoretic fibre of $Y \to X$ over $\mathrm{Spec}(K)$ is $\mathrm{Spec}(L)$. Hence the field extension L/K is separable if X is unramified in L, see Morphisms, Lemmas 29.35.11.

0BQK **Lemma 58.11.1.** *In the situation above the following are equivalent*

(1) *X is unramified in L,*
(2) *$Y \to X$ is étale, and*
(3) *$Y \to X$ is finite étale.*

Proof. Observe that $Y \to X$ is an integral morphism. In each case the morphism $Y \to X$ is locally of finite type by definition. Hence we find that in each case $Y \to X$ is finite by Morphisms, Lemma 29.44.4. In particular we see that (2) is equivalent to (3). An étale morphism is unramified, hence (2) implies (1).

Conversely, assume $Y \to X$ is unramified. Since a normal scheme is geometrically unibranch (Properties, Lemma 28.15.2), we see that the morphism $Y \to X$ is étale by More on Morphisms, Lemma 37.37.2. We also give a direct proof in the next paragraph.

Let $x \in X$. We can choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$Y \times_X U = \coprod V_j \longrightarrow U$$

is a disjoint union of closed immersions, see Étale Morphisms, Lemma 41.17.3. Shrinking we may assume U is quasi-compact. Then U has finitely many irreducible components (Descent, Lemma 35.16.3). Since U is normal (Descent, Lemma

35.18.2) the irreducible components of U are open and closed (Properties, Lemma 28.7.5) and we may assume U is irreducible. Then U is an integral scheme whose generic point ξ maps to the generic point of X. On the other hand, we know that $Y \times_X U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$ by More on Morphisms, Lemma 37.19.2. Every point of $\mathrm{Spec}(L) \times_X U$ maps to ξ. Thus every V_j contains a point mapping to ξ by Morphisms, Lemma 29.53.9. Thus $V_j \to U$ is an isomorphism as $U = \overline{\{\xi\}}$. Thus $Y \times_X U \to U$ is étale. By Descent, Lemma 35.23.31 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x). □

0BQL **Lemma 58.11.2.** *Let X be a normal integral scheme with function field K. Let $Y \to X$ be a finite étale morphism. If Y is connected, then Y is an integral normal scheme and Y is the normalization of X in the function field of Y.*

Proof. The scheme Y is normal by Descent, Lemma 35.18.2. Since $Y \to X$ is flat every generic point of Y maps to the generic point of X by Morphisms, Lemma 29.25.9. Since $Y \to X$ is finite we see that Y has a finite number of irreducible components. Thus Y is the disjoint union of a finite number of integral normal schemes by Properties, Lemma 28.7.5. Thus if Y is connected, then Y is an integral normal scheme.

Let L be the function field of Y and let $Y' \to X$ be the normalization of X in L. By Morphisms, Lemma 29.53.4 we obtain a factorization $Y' \to Y \to X$ and $Y' \to Y$ is the normalization of Y in L. Since Y is normal it is clear that $Y' = Y$ (this can also be deduced from Morphisms, Lemma 29.54.8). □

0BQM **Proposition 58.11.3.** *Let X be a normal integral scheme with function field K. Then the canonical map (58.10.6.1)*

$$Gal(K^{sep}/K) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

is identified with the quotient map $Gal(K^{sep}/K) \to Gal(M/K)$ where $M \subset K^{sep}$ is the union of the finite subextensions L such that X is unramified in L.

Proof. The normal scheme X is geometrically unibranch (Properties, Lemma 28.15.2). Hence Lemma 58.10.7 applies to X. Thus $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective and top horizontal arrow of the commutative diagram

$$\begin{array}{ccc} F\acute{E}t_X & \longrightarrow & F\acute{E}t_\eta \\ \downarrow & \overset{c}{\searrow} & \downarrow \\ Finite\text{-}\pi_1(X, \overline{\eta})\text{-}sets & \longrightarrow & Finite\text{-}\mathrm{Gal}(K^{sep}/K)\text{-}sets \end{array}$$

is fully faithful. The left vertical arrow is the equivalence of Theorem 58.6.2 and the right vertical arrow is the equivalence of Lemma 58.6.3. The lower horizontal arrow is induced by the map of the proposition. By Lemmas 58.11.1 and 58.11.2 we see that the essential image of c consists of $\mathrm{Gal}(K^{sep}/K)$-*Sets* isomorphic to sets of the form

$$S = \mathrm{Hom}_K(\prod\nolimits_{i=1,\ldots,n} L_i, K^{sep}) = \coprod\nolimits_{i=1,\ldots,n} \mathrm{Hom}_K(L_i, K^{sep})$$

with L_i/K finite separable such that X is unramified in L_i. Thus if $M \subset K^{sep}$ is as in the statement of the lemma, then $\mathrm{Gal}(K^{sep}/M)$ is exactly the subgroup of $\mathrm{Gal}(K^{sep}/K)$ acting trivially on every object in the essential image of c. On the other hand, the essential image of c is exactly the category of S such that

the $\mathrm{Gal}(K^{sep}/K)$-action factors through the surjection $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{\eta})$. We conclude that $\mathrm{Gal}(K^{sep}/M)$ is the kernel. Hence $\mathrm{Gal}(K^{sep}/M)$ is a normal subgroup, M/K is Galois, and we have a short exact sequence

$$1 \to \mathrm{Gal}(K^{sep}/M) \to \mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(M/K) \to 1$$

by Galois theory (Fields, Theorem 9.22.4 and Lemma 9.22.5). The proof is done. □

0BSM **Lemma 58.11.4.** *Let $(A, \mathfrak{m})$ be a normal local ring. Set $X = \mathrm{Spec}(A)$. Let A^{sh} be the strict henselization of A. Let K and K^{sh} be the fraction fields of A and A^{sh}. Then the sequence*

$$\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(\mathrm{Spec}(K)) \to \pi_1(X) \to 1$$

is exact in the sense of Lemma 58.4.3 part (1).

Proof. Note that A^{sh} is a normal domain, see More on Algebra, Lemma 15.45.6. The map $\pi_1(\mathrm{Spec}(K)) \to \pi_1(X)$ is surjective by Proposition 58.11.3.

Write $X^{sh} = \mathrm{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X \mathrm{Spec}(K^{sh})$. It follows that the composition $\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(X)$ is trivial, see Lemma 58.4.2.

To finish the proof, it suffices according to Lemma 58.4.3 to show the following: Given a finite étale morphism $V \to \mathrm{Spec}(K)$ such that $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{sh})$ is a disjoint union of copies of $\mathrm{Spec}(K^{sh})$, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X \mathrm{Spec}(K)$ over $\mathrm{Spec}(K)$. Write $V = \mathrm{Spec}(L)$, so L is a finite product of finite separable extensions of K. Let $B \subset L$ be the integral closure of A in L. If $A \to B$ is étale, then we can take $Y = \mathrm{Spec}(B)$ and the proof is complete. By Algebra, Lemma 10.147.4 (and a limit argument we omit) we see that $B \otimes_A A^{sh}$ is the integral closure of A^{sh} in $L^{sh} = L \otimes_K K^{sh}$. Our assumption is that L^{sh} is a product of copies of K^{sh} and hence B^{sh} is a product of copies of A^{sh}. Thus $A^{sh} \to B^{sh}$ is étale. As $A \to A^{sh}$ is faithfully flat it follows that $A \to B$ is étale (Descent, Lemma 35.23.31) as desired. □

58.12. Group actions and integral closure

0BSN In this section we continue the discussion of More on Algebra, Section 15.111. Recall that a normal local ring is a domain by definition.

0BSP **Lemma 58.12.1.** *Let A be a normal domain whose fraction field K is separably algebraically closed. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal. Then the residue field $\kappa(\mathfrak{p})$ is algebraically closed.*

Proof. Assume the lemma is not true to get a contradiction. Then there exists a monic irreducible polynomial $P(T) \in \kappa(\mathfrak{p})[T]$ of degree $d > 1$. After replacing P by $a^d P(a^{-1}T)$ for suitable $a \in A$ (to clear denominators) we may assume that P is the image of a monic polynomial Q in $A[T]$. Observe that Q is irreducible in $K[T]$. Namely a factorization over K leads to a factorization over A by Algebra, Lemma 10.38.5 which we could reduce modulo $\mathfrak{p}$ to get a factorization of P. As K is separably closed, Q is not a separable polynomial (Fields, Definition 9.12.2). Then the characteristic of K is $p > 0$ and Q has vanishing linear term (Fields, Definition

9.12.2). However, then we can replace Q by $Q + aT$ where $a \in \mathfrak{p}$ is nonzero to get a contradiction. □

0BSQ **Lemma 58.12.2.** *A normal local ring with separably closed fraction field is strictly henselian.*

Proof. Let $(A, \mathfrak{m}, \kappa)$ be normal local with separably closed fraction field K. If $A = K$, then we are done. If not, then the residue field κ is algebraically closed by Lemma 58.12.1 and it suffices to check that A is henselian. Let $f \in A[T]$ be monic and let $a_0 \in \kappa$ be a root of multiplicity 1 of the reduction $\overline{f} \in \kappa[T]$. Let $f = \prod f_i$ be the factorization in $K[T]$. By Algebra, Lemma 10.38.5 we have $f_i \in A[T]$. Thus a_0 is a root of f_i for some i. After replacing f by f_i we may assume f is irreducible. Then, since the derivative f' cannot be zero in $A[T]$ as a_0 is a single root, we conclude that f is linear due to the fact that K is separably algebraically closed. Thus A is henselian, see Algebra, Definition 10.153.1. □

0BSS **Lemma 58.12.3.** *Let G be a finite group acting on a ring R. Let $R^G \to A$ be a ring map. Let $\mathfrak{q}' \subset A \otimes_{R^G} R$ be a prime lying over the prime $\mathfrak{q} \subset R$. Then*

$$I_{\mathfrak{q}} = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_{\kappa(\mathfrak{q})}\}$$

is equal to

$$I_{\mathfrak{q}'} = \{\sigma \in G \mid \sigma(\mathfrak{q}') = \mathfrak{q}' \text{ and } \sigma \bmod \mathfrak{q}' = id_{\kappa(\mathfrak{q}')}\}$$

Proof. Since $\mathfrak{q}$ is the inverse image of $\mathfrak{q}'$ and since $\kappa(\mathfrak{q}) \subset \kappa(\mathfrak{q}')$, we get $I_{\mathfrak{q}'} \subset I_{\mathfrak{q}}$. Conversely, if $\sigma \in I_{\mathfrak{q}}$, the σ acts trivially on the fibre ring $A \otimes_{R^G} \kappa(\mathfrak{q})$. Thus σ fixes all the primes lying over $\mathfrak{q}$ and induces the identity on their residue fields. □

0BST **Lemma 58.12.4.** *Let G be a finite group acting on a ring R. Let $\mathfrak{q} \subset R$ be a prime. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_{\mathfrak{q}}\}$$

Then $R^G \to R^I$ is étale at $R^I \cap \mathfrak{q}$.

Proof. The strategy of the proof is to use étale localization to reduce to the case where $R \to R^I$ is a local isomorphism at $R^I \cap \mathfrak{p}$. Let $R^G \to A$ be an étale ring map. We claim that if the result holds for the action of G on $A \otimes_{R^G} R$ and some prime $\mathfrak{q}'$ of $A \otimes_{R^G} R$ lying over $\mathfrak{q}$, then the result is true.

To check this, note that since $R^G \to A$ is flat we have $A = (A \otimes_{R^G} R)^G$, see More on Algebra, Lemma 15.111.7. By Lemma 58.12.3 the group I does not change. Then a second application of More on Algebra, Lemma 15.111.7 shows that $A \otimes_{R^G} R^I = (A \otimes_{R^G} R)^I$ (because $R^I \to A \otimes_{R^G} R^I$ is flat). Thus

$$\begin{array}{ccc} \mathrm{Spec}((A \otimes_{R^G} R)^I) & \longrightarrow & \mathrm{Spec}(R^I) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(R^G) \end{array}$$

is cartesian and the horizontal arrows are étale. Thus if the left vertical arrow is étale in some open neighbourhood W of $(A \otimes_{R^G} R)^I \cap \mathfrak{q}'$, then the right vertical arrow is étale at the points of the (open) image of W in $\mathrm{Spec}(R^I)$, see Descent, Lemma 35.14.5. In particular the morphism $\mathrm{Spec}(R^I) \to \mathrm{Spec}(R^G)$ is étale at $R^I \cap \mathfrak{q}$.

Let $\mathfrak{p} = R^G \cap \mathfrak{q}$. By More on Algebra, Lemma 15.111.8 the fibre of $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ over $\mathfrak{p}$ is finite. Moreover the residue field extensions at these points are algebraic, normal, with finite automorphism groups by More on Algebra, Lemma 15.111.9. Thus we may apply More on Morphisms, Lemma 37.42.1 to the integral ring map $R^G \to R$ and the prime $\mathfrak{p}$. Combined with the claim above we reduce to the case where $R = A_1 \times \ldots \times A_n$ with each A_i having a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ such that the residue field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Of course $\mathfrak{q}$ is one of these primes, say $\mathfrak{q} = \mathfrak{q}_1$.

It may not be the case that G permutes the factors A_i (this would be true if the spectrum of A_i were connected, for example if R^G was local). This we can fix as follows; we suggest the reader think this through for themselves, perhaps using idempotents instead of topology. Recall that the product decomposition gives a corresponding disjoint union decomposition of $\mathrm{Spec}(R)$ by open and closed subsets U_i. Since G is finite, we can refine this covering by a finite disjoint union decomposition $\mathrm{Spec}(R) = \coprod_{j \in J} W_j$ by open and closed subsets W_j, such that for all $j \in J$ there exists a $j' \in J$ with $\sigma(W_j) = W_{j'}$. The union of the W_j not meeting $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ is a closed subset not meeting the fibre over $\mathfrak{p}$ hence maps to a closed subset of $\mathrm{Spec}(R^G)$ not meeting $\mathfrak{p}$ as $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is closed. Hence after replacing R^G by a principal localization (permissible by the claim) we may assume each W_j meets one of the points $\mathfrak{q}_i$. Then we set $U_i = W_j$ if $\mathfrak{q}_i \in W_j$. The corresponding product decomposition $R = A_1 \times \ldots \times A_n$ is one where G permutes the factors A_i.

Thus we may assume we have a product decomposition $R = A_1 \times \ldots \times A_n$ compatible with G-action, where each A_i has a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ and the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Write $A' = A_2 \times \ldots \times A_n$ so that

$$R = A_1 \times A'$$

Since $\mathfrak{q} = \mathfrak{q}_1$ we find that every $\sigma \in I$ preserves the product decomposition above. Hence

$$R^I = (A_1)^I \times (A')^I$$

Observe that $I = D = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}$ because $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is purely inseparable. Since the action of G on primes over $\mathfrak{p}$ is transitive (More on Algebra, Lemma 15.111.8) we conclude that, the index of I in G is n and we can write $G = eI \amalg \sigma_2 I \amalg \ldots \amalg \sigma_n I$ so that $A_i = \sigma_i(A_1)$ for $i = 2, \ldots, n$. It follows that

$$R^G = (A_1)^I.$$

Thus the map $R^G \to R^I$ is étale at $R^I \cap \mathfrak{q}$ and the proof is complete. □

The following lemma generalizes More on Algebra, Lemma 15.113.8.

0BSU **Lemma 58.12.5.** *Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = Gal(L/K)$ and let B be the integral closure of A in L. Let $\mathfrak{q} \subset B$. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_{\kappa(\mathfrak{q})}\}$$

Then $(B^I)_{B^I \cap \mathfrak{q}}$ is a filtered colimit of étale A-algebras.

Proof. We can write L as the filtered colimit of finite Galois extensions of K. Hence it suffices to prove this lemma in case L/K is a finite Galois extension, see

Algebra, Lemma 10.154.3. Since $A = B^G$ as A is integrally closed in $K = L^G$ the result follows from Lemma 58.12.4. □

58.13. Ramification theory

0BSD In this section we continue the discussion of More on Algebra, Section 15.113 and we relate it to our discussion of the fundamental groups of schemes.

Let $(A, \mathfrak{m}, \kappa)$ be a normal local ring with fraction field K. Choose a separable algebraic closure K^{sep}. Let A^{sep} be the integral closure of A in K^{sep}. Choose maximal ideal $\mathfrak{m}^{sep} \subset A^{sep}$. Let $A \subset A^h \subset A^{sh}$ be the henselization and strict henselization. Observe that A^h and A^{sh} are normal rings as well (More on Algebra, Lemma 15.45.6). Denote K^h and K^{sh} their fraction fields. Since $(A^{sep})_{\mathfrak{m}^{sep}}$ is strictly henselian by Lemma 58.12.2 we can choose an A-algebra map $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$. Namely, first choose a κ-embedding[4] $\kappa(\mathfrak{m}^{sh}) \to \kappa(\mathfrak{m}^{sep})$ and then extend (uniquely) to an A-algebra homomorphism by Algebra, Lemma 10.155.10. We get the following diagram

$$\begin{array}{ccccccc} K^{sep} & \longleftarrow & K^{sh} & \longleftarrow & K^h & \longleftarrow & K \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (A^{sep})_{\mathfrak{m}^{sep}} & \longleftarrow & A^{sh} & \longleftarrow & A^h & \longleftarrow & A \end{array}$$

We can take the fundamental groups of the spectra of these rings. Of course, since K^{sep}, $(A^{sep})_{\mathfrak{m}^{sep}}$, and A^{sh} are strictly henselian, for them we obtain trivial groups. Thus the interesting part is the following

0BSV (58.13.0.1)
$$\begin{array}{ccccc} \pi_1(U^{sh}) & \longrightarrow & \pi_1(U^h) & \longrightarrow & \pi_1(U) \\ & \searrow^{1} & \downarrow & & \downarrow \\ & & \pi_1(X^h) & \longrightarrow & \pi_1(X) \end{array}$$

Here X^h and X are the spectra of A^h and A and U^{sh}, U^h, U are the spectra of K^{sh}, K^h, and K. The label 1 means that the map is trivial; this follows as it factors through the trivial group $\pi_1(X^{sh})$. On the other hand, the profinite group $G = \mathrm{Gal}(K^{sep}/K)$ acts on A^{sep} and we can make the following definitions

$$D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\} \supset I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = \mathrm{id}_{\kappa(\mathfrak{m}^{sep})}\}$$

These groups are sometimes called the *decomposition group* and the *inertia group* especially when A is a discrete valuation ring.

0BSW **Lemma 58.13.1.** *In the situation described above, via the isomorphism $\pi_1(U) = Gal(K^{sep}/K)$ the diagram (58.13.0.1) translates into the diagram*

$$\begin{array}{ccccc} I & \longrightarrow & D & \longrightarrow & Gal(K^{sep}/K) \\ & \searrow^{1} & \downarrow & & \downarrow \\ & & Gal(\kappa(\mathfrak{m}^{sh})/\kappa) & \longrightarrow & Gal(M/K) \end{array}$$

where $K^{sep}/M/K$ is the maximal subextension unramified with respect to A. Moreover, the vertical arrows are surjective, the kernel of the left vertical arrow is I

[4] This is possible because $\kappa(\mathfrak{m}^{sh})$ is a separable algebraic closure of κ and $\kappa(\mathfrak{m}^{sep})$ is an algebraic closure of κ by Lemma 58.12.1.

and the kernel of the right vertical arrow is the smallest closed normal subgroup of $Gal(K^{sep}/K)$ containing I.

Proof. By construction the group D acts on $(A^{sep})_{\mathfrak{m}^{sep}}$ over A. By the uniqueness of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ given the map on residue fields (Algebra, Lemma 10.155.10) we see that the image of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ is contained in $((A^{sep})_{\mathfrak{m}^{sep}})^I$. On the other hand, Lemma 58.12.5 shows that $((A^{sep})_{\mathfrak{m}^{sep}})^I$ is a filtered colimit of étale extensions of A. Since A^{sh} is the maximal such extension, we conclude that $A^{sh} = ((A^{sep})_{\mathfrak{m}^{sep}})^I$. Hence $K^{sh} = (K^{sep})^I$.

Recall that I is the kernel of a surjective map $D \to \mathrm{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa)$, see More on Algebra, Lemma 15.111.10. We have $\mathrm{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa) = \mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$ as we have seen above that these fields are the algebraic and separable algebraic closures of κ. On the other hand, any automorphism of A^{sh} over A is an automorphism of A^{sh} over A^h by the uniqueness in Algebra, Lemma 10.155.6. Furthermore, A^{sh} is the colimit of finite étale extensions $A^h \subset A'$ which correspond 1-to-1 with finite separable extension κ'/κ, see Algebra, Remark 10.155.4. Thus

$$\mathrm{Aut}(A^{sh}/A) = \mathrm{Aut}(A^{sh}/A^h) = \mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$$

Let κ'/κ be a finite Galois extension with Galois group G. Let $A^h \subset A'$ be the finite étale extension corresponding to $\kappa \subset \kappa'$ by Algebra, Lemma 10.153.7. Then it follows that $(A')^G = A^h$ by looking at fraction fields and degrees (small detail omitted). Taking the colimit we conclude that $(A^{sh})^{\mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)} = A^h$. Combining all of the above, we find $A^h = ((A^{sep})_{\mathfrak{m}^{sep}})^D$. Hence $K^h = (K^{sep})^D$.

Since U, U^h, U^{sh} are the spectra of the fields K, K^h, K^{sh} we see that the top lines of the diagrams correspond via Lemma 58.6.3. By Lemma 58.8.2 we have $\pi_1(X^h) = \mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$. The exactness of the sequence $1 \to I \to D \to \mathrm{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa) \to 1$ was pointed out above. By Proposition 58.11.3 we see that $\pi_1(X) = \mathrm{Gal}(M/K)$. Finally, the statement on the kernel of $\mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(M/K) = \pi_1(X)$ follows from Lemma 58.11.4. This finishes the proof. □

Let X be a normal integral scheme with function field K. Let K^{sep} be a separable algebraic closure of K. Let $X^{sep} \to X$ be the normalization of X in K^{sep}. Since $G = \mathrm{Gal}(K^{sep}/K)$ acts on K^{sep} we obtain a right action of G on X^{sep}. For $y \in X^{sep}$ define

$$D_y = \{\sigma \in G \mid \sigma(y) = y\} \supset I_y = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}_y = \mathrm{id}_{\kappa(y)}\}$$

similarly to the above. On the other hand, for $x \in X$ let $\mathcal{O}^{sh}_{X,x}$ be a strict henselization, let K^{sh}_x be the fraction field of $\mathcal{O}^{sh}_{X,x}$ and choose a K-embedding $K^{sh}_x \to K^{sep}$.

0BTD **Lemma 58.13.2.** *Let X be a normal integral scheme with function field K. With notation as above, the following three subgroups of $Gal(K^{sep}/K) = \pi_1(\mathrm{Spec}(K))$ are equal*

1. *the kernel of the surjection $Gal(K^{sep}/K) \longrightarrow \pi_1(X)$,*
2. *the smallest normal closed subgroup containing I_y for all $y \in X^{sep}$, and*
3. *the smallest normal closed subgroup containing $Gal(K^{sep}/K^{sh}_x)$ for all $x \in X$.*

Proof. The equivalence of (2) and (3) follows from Lemma 58.13.1 which tells us that I_y is conjugate to $\mathrm{Gal}(K^{sep}/K^{sh}_x)$ if y lies over x. By Lemma 58.11.4 we see

that $\mathrm{Gal}(K^{sep}/K^{sh}_x)$ maps trivially to $\pi_1(\mathrm{Spec}(\mathcal{O}_{X,x}))$ and therefore the subgroup $N \subset G = \mathrm{Gal}(K^{sep}/K)$ of (2) and (3) is contained in the kernel of $G \longrightarrow \pi_1(X)$.

To prove the other inclusion, since N is normal, it suffices to prove: given $N \subset U \subset G$ with U open normal, the quotient map $G \to G/U$ factors through $\pi_1(X)$. In other words, if L/K is the Galois extension corresponding to U, then we have to show that X is unramified in L (Section 58.11, especially Proposition 58.11.3). It suffices to do this when X is affine (we do this so we can refer to algebra results in the rest of the proof). Let $Y \to X$ be the normalization of X in L. The inclusion $L \subset K^{sep}$ induces a morphism $\pi : X^{sep} \to Y$. For $y \in X^{sep}$ the inertia group of $\pi(y)$ in $\mathrm{Gal}(L/K)$ is the image of I_y in $\mathrm{Gal}(L/K)$; this follows from More on Algebra, Lemma 15.111.11. Since $N \subset U$ all these inertia groups are trivial. We conclude that $Y \to X$ is étale by applying Lemma 58.12.4. (Alternative: you can use Lemma 58.11.4 to see that the pullback of Y to $\mathrm{Spec}(\mathcal{O}_{X,x})$ is étale for all $x \in X$ and then conclude from there with a bit more work.) □

0BTE **Example 58.13.3.** Let X be a normal integral Noetherian scheme with function field K. Purity of branch locus (see below) tells us that if X is regular, then it suffices in Lemma 58.13.2 to consider the inertia groups $I = \pi_1(\mathrm{Spec}(K^{sh}_x))$ for points x of codimension 1 in X. In general this is not enough however. Namely, let $Y = \mathbf{A}^n_k = \mathrm{Spec}(k[t_1, \ldots, t_n])$ where k is a field not of characteristic 2. Let $G = \{\pm 1\}$ be the group of order 2 acting on Y by multiplication on the coordinates. Set

$$X = \mathrm{Spec}(k[t_i t_j, i, j \in \{1, \ldots, n\}])$$

The embedding $k[t_i t_j] \subset k[t_1, \ldots, t_n]$ defines a degree 2 morphism $Y \to X$ which is unramified everywhere except over the maximal ideal $\mathfrak{m} = (t_i t_j)$ which is a point of codimension n in X.

0BTF **Lemma 58.13.4.** *Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Let $Y \to X$ be the normalization of X in L. The following are equivalent*

(1) *X is unramified in L as defined in Section 58.11,*
(2) *$Y \to X$ is an unramified morphism of schemes,*
(3) *$Y \to X$ is an étale morphism of schemes,*
(4) *$Y \to X$ is a finite étale morphism of schemes,*
(5) *for $x \in X$ the projection $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is unramified,*
(6) *same as in (5) but with $\mathcal{O}^h_{X,x}$,*
(7) *same as in (5) but with $\mathcal{O}^{sh}_{X,x}$,*
(8) *for $x \in X$ the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$.*

If L/K is Galois with Galois group G, then these are also equivalent to

(9) *for $y \in Y$ the group $I_y = \{g \in G \mid g(y) = y \text{ and } g \bmod \mathfrak{m}_y = id_{\kappa(y)}\}$ is trivial.*

Proof. The equivalence of (1) and (2) is the definition of (1). The equivalence of (2), (3), and (4) is Lemma 58.11.1. It is straightforward to prove that (4) ⇒ (5), (5) ⇒ (6), (6) ⇒ (7).

Assume (7). Observe that $\mathcal{O}^{sh}_{X,x}$ is a normal local domain (More on Algebra, Lemma 15.45.6). Let $L^{sh} = L \otimes_K K^{sh}_x$ where K^{sh}_x is the fraction field of $\mathcal{O}^{sh}_{X,x}$. Then $L^{sh} = \prod_{i=1,\ldots,n} L_i$ with L_i/K^{sh}_x finite separable. By Algebra, Lemma 10.147.4

(and a limit argument we omit) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ is the integral closure of $\mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ in L^{sh}. Hence by Lemma 58.11.1 (applied to the factors L_i of L^{sh}) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$ is finite étale. Looking at the generic point we see that the degree is equal to $[L : K]$ and hence we see that (8) is true.

Assume (8). Assume that $x \in X$ and that the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$. Observe that this means that Y has $\geq [L : K]$ geometric points lying over x. We will show that $Y \to X$ is finite étale over a neighbourhood of x. This will prove (1) holds. To prove this we may assume $X = \mathrm{Spec}(R)$, the point x corresponds to the prime $\mathfrak{p} \subset R$, and $Y = \mathrm{Spec}(S)$. We apply More on Morphisms, Lemma 37.42.1 and we find an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U = V_1 \amalg \ldots \amalg V_m$ such that V_i has a unique point v_i lying over u with $\kappa(v_i)/\kappa(u)$ purely inseparable. Shrinking U if necessary we may assume U is a normal integral scheme with generic point ξ (use Descent, Lemmas 35.16.3 and 35.18.2 and Properties, Lemma 28.7.5). By our remark on geometric points we see that $m \geq [L : K]$. On the other hand, by More on Morphisms, Lemma 37.19.2 we see that $\coprod V_i \to U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$. As $K \subset \kappa(\xi)$ is finite separable, we can write $\mathrm{Spec}(L) \times_X U = \mathrm{Spec}(\prod_{i=1,\ldots,n} L_i)$ with $L_i/\kappa(\xi)$ finite and $[L : K] = \sum [L_i : \kappa(\xi)]$. Since V_j is nonempty for each j and $m \geq [L : K]$ we conclude that $m = n$ and $[L_i : \kappa(\xi)] = 1$ for all i. Then $V_j \to U$ is an isomorphism in particular étale, hence $Y \times_X U \to U$ is étale. By Descent, Lemma 35.23.31 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x).

Assume L/K is Galois and (9) holds. Then $Y \to X$ is étale by Lemma 58.12.5. We omit the proof that (1) implies (9). □

In the case of infinite Galois extensions of discrete valuation rings we can say a tiny bit more. To do so we introduce the following notation. A subset $S \subset \mathbf{N}$ of integers is *multiplicativity directed* if $1 \in S$ and for $n, m \in S$ there exists $k \in S$ with $n|k$ and $m|k$. Define a partial ordering on S by the rule $n \geq_S m$ if and only if $m|n$. Given a field κ we obtain an inverse system of finite groups $\{\mu_n(\kappa)\}_{n \in S}$ with transition maps

$$\mu_n(\kappa) \longrightarrow \mu_m(\kappa), \quad \zeta \longmapsto \zeta^{n/m}$$

for $n \geq_S m$. Then we can form the profinite group

$$\lim_{n \in S} \mu_n(\kappa)$$

Observe that the limit is cofiltered (as S is directed). The construction is functorial in κ. In particular $\mathrm{Aut}(\kappa)$ acts on this profinite group. For example, if $S = \{1, n\}$, then this gives $\mu_n(\kappa)$. If $S = \{1, \ell, \ell^2, \ell^3, \ldots\}$ for some prime ℓ different from the characteristic of κ this produces $\lim_n \mu_{\ell^n}(\kappa)$ which is sometimes called the ℓ-adic Tate module of the multiplicative group of κ (compare with More on Algebra, Example 15.94.5).

0BUA **Lemma 58.13.5.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Let $G = Gal(L/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m} = id_{\kappa(\mathfrak{m})}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Aut(\kappa(\mathfrak{m})/\kappa_A) \to 1$$

The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

(1) *P is a normal subgroup of D,*
(2) *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
(3) *there is a multiplicatively directed $S \subset \mathbf{N}$ such that $\kappa(\mathfrak{m})$ contains a primitive nth root of unity for each $n \in S$ (elements of S are prime to p),*
(4) *there exists a canonical surjective map*

$$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$.

Proof. This is mostly a reformulation of the results on finite Galois extensions proved in More on Algebra, Section 15.113. The surjectivity of the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)$ is More on Algebra, Lemma 15.111.10. This gives the first exact sequence.

To construct the second short exact sequence let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$. Set $G_\lambda = \mathrm{Gal}(L_\lambda/K)$. Recall that G_λ is an inverse system of finite groups with surjective transition maps and that $G = \lim_{\lambda \in \Lambda} G_\lambda$, see Fields, Lemma 9.22.3. We let B_λ be the integral closure of A in L_λ. Then we set $\mathfrak{m}_\lambda = \mathfrak{m} \cap B_\lambda$ and we denote $P_\lambda, I_\lambda, D_\lambda$ the wild inertia, inertia, and decomposition group of $\mathfrak{m}_\lambda$, see More on Algebra, Lemma 15.113.5. For $\lambda \geq \lambda'$ the restriction defines a commutative diagram

$$\begin{array}{ccccccc} P_\lambda & \longrightarrow & I_\lambda & \longrightarrow & D_\lambda & \longrightarrow & G_\lambda \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_{\lambda'} & \longrightarrow & I_{\lambda'} & \longrightarrow & D_{\lambda'} & \longrightarrow & G_{\lambda'} \end{array}$$

with surjective vertical maps, see More on Algebra, Lemma 15.113.10.

From the definitions it follows immediately that $I = \lim I_\lambda$ and $D = \lim D_\lambda$ under the isomorphism $G = \lim G_\lambda$ above. Since $L = \mathrm{colim}\, L_\lambda$ we have $B = \mathrm{colim}\, B_\lambda$ and $\kappa(\mathfrak{m}) = \mathrm{colim}\, \kappa(\mathfrak{m}_\lambda)$. Since the transition maps of the system D_λ are compatible with the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak{m}_\lambda)/\kappa)$ (see More on Algebra, Lemma 15.113.10) we see that the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)$ is the limit of the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak{m}_\lambda)/\kappa)$.

There exist canonical maps

$$\theta_{\lambda,can} : I_\lambda \longrightarrow \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$$

where $n_\lambda = |I_\lambda|/|P_\lambda|$, where $\mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$ has order n_λ, such that $\theta_{\lambda,can}(\tau\sigma\tau^{-1}) = \tau(\theta_{\lambda,can}(\sigma))$ for $\tau \in D_\lambda$ and $\sigma \in I_\lambda$, and such that we get commutative diagrams

$$\begin{array}{ccc} I_\lambda & \xrightarrow[\theta_{\lambda,can}]{} & \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda)) \\ \downarrow & & \downarrow{\scriptstyle (-)^{n_\lambda/n_{\lambda'}}} \\ I_{\lambda'} & \xrightarrow{\theta_{\lambda',can}} & \mu_{n_{\lambda'}}(\kappa(\mathfrak{m}_{\lambda'})) \end{array}$$

see More on Algebra, Remark 15.113.11.

Let $S \subset \mathbf{N}$ be the collection of integers n_λ. Since Λ is directed, we see that S is multiplicatively directed. By the displayed commutative diagrams above we can take the limits of the maps $\theta_{\lambda,can}$ to obtain

$$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m})).$$

This map is continuous (small detail omitted). Since the transition maps of the system of I_λ are surjective and Λ is directed, the projections $I \to I_\lambda$ are surjective. For every λ the diagram

$$\begin{array}{ccc} I & \xrightarrow{\theta_{can}} & \lim_{n \in S} \mu_n(\kappa(\mathfrak{m})) \\ \downarrow & & \downarrow \\ I_\lambda & \xrightarrow{\theta_{\lambda,can}} & \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda)) \end{array}$$

commutes. Hence the image of θ_{can} surjects onto the finite group $\mu_{n_\lambda}(\kappa(\mathfrak{m})) = \mu_{n_\lambda}(\kappa(\mathfrak{m}_\lambda))$ of order n_λ (see above). It follows that the image of θ_{can} is dense. On the other hand θ_{can} is continuous and the source is a profinite group. Hence θ_{can} is surjective by a topological argument.

The property $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$ follows from the corresponding properties of the maps $\theta_{\lambda,can}$ and the compatibility of the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m}))$ with the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak{m}_\lambda))$. Setting $P = \mathrm{Ker}(\theta_{can})$ this implies that P is a normal subgroup of D. Setting $I_t = I/P$ we obtain the isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$ from the surjectivity of θ_{can}.

To finish the proof we show that $P = \lim P_\lambda$ which proves that P is a pro-p-group. Recall that the tame inertia group $I_{\lambda,t} = I_\lambda/P_\lambda$ has order n_λ. Since the transition maps $P_\lambda \to P_{\lambda'}$ are surjective and Λ is directed, we obtain a short exact sequence

$$1 \to \lim P_\lambda \to I \to \lim I_{\lambda,t} \to 1$$

(details omitted). Since for each λ the map $\theta_{\lambda,can}$ induces an isomorphism $I_{\lambda,t} \cong \mu_{n_\lambda}(\kappa(\mathfrak{m}))$ the desired result follows. □

0BUB **Lemma 58.13.6.** *Let A be a discrete valuation ring with fraction field K. Let K^{sep} be a separable closure of K. Let A^{sep} be the integral closure of A in K^{sep}. Let $\mathfrak{m}^{sep}$ be a maximal ideal of A^{sep}. Let $\mathfrak{m} = \mathfrak{m}^{sep} \cap A$, let $\kappa = A/\mathfrak{m}$, and let $\overline{\kappa} = A^{sep}/\mathfrak{m}^{sep}$. Then $\overline{\kappa}$ is an algebraic closure of κ. Let $G = Gal(K^{sep}/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = id_{\kappa(\mathfrak{m}^{sep})}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Gal(\kappa^{sep}/\kappa) \to 1$$

where $\kappa^{sep} \subset \overline{\kappa}$ is the separable closure of κ. The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

1. *P is a normal subgroup of D,*
2. *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*

(3) *there exists a canonical surjective map*

$$\theta_{can} : I \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$.

Proof. The field $\overline{\kappa}$ is the algebraic closure of κ by Lemma 58.12.1. Most of the statements immediately follow from the corresponding parts of Lemma 58.13.5. For example because $\mathrm{Aut}(\overline{\kappa}/\kappa) = \mathrm{Gal}(\kappa^{sep}/\kappa)$ we obtain the first sequence. Then the only other assertion that needs a proof is the fact that with S as in Lemma 58.13.5 the limit $\lim_{n \in S} \mu_n(\overline{\kappa})$ is equal to $\lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$. To see this it suffices to show that every integer n prime to p divides an element of S. Let $\pi \in A$ be a uniformizer and consider the splitting field L of the polynomial $X^n - \pi$. Since the polynomial is separable we see that L is a finite Galois extension of K. Choose an embedding $L \to K^{sep}$. Observe that if B is the integral closure of A in L, then the ramification index of $A \to B_{\mathfrak{m}^{sep} \cap B}$ is divisible by n (because π has an nth root in B; in fact the ramification index equals n but we do not need this). Then it follows from the construction of the S in the proof of Lemma 58.13.5 that n divides an element of S. □

58.14. Geometric and arithmetic fundamental groups

0BTU In this section we work out what happens when comparing the fundamental group of a scheme X over a field k with the fundamental group of $X_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of k.

0BTV **Lemma 58.14.1.** *Let I be a directed set. Let X_i be an inverse system of quasi-compact and quasi-separated schemes over I with affine transition morphisms. Let $X = \lim X_i$ as in Limits, Section 32.2. Then there is an equivalence of categories*

$$\mathrm{colim}\, F\acute{E}t_{X_i} = F\acute{E}t_X$$

If X_i is connected for all sufficiently large i and $\overline{x}$ is a geometric point of X, then

$$\pi_1(X, \overline{x}) = \lim \pi_1(X_i, \overline{x})$$

Proof. The equivalence of categories follows from Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. The second statement is formal given the statement on categories. □

0BTW **Lemma 58.14.2.** *Let k be a field with perfection k^{perf}. Let X be a connected scheme over k. Then $X_{k^{perf}}$ is connected and $\pi_1(X_{k^{perf}}) \to \pi_1(X)$ is an isomorphism.*

Proof. Special case of topological invariance of the fundamental group. See Proposition 58.8.4. To see that $\mathrm{Spec}(k^{perf}) \to \mathrm{Spec}(k)$ is a universal homeomorphism you can use Algebra, Lemma 10.46.10. □

0BTX **Lemma 58.14.3.** *Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-compact and quasi-separated scheme over k. If the base change $X_{\overline{k}}$ is connected, then there is a short exact sequence*

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k)) \to 1$$

of profinite topological groups.

Proof. Connected objects of $F\acute{E}t_{\mathrm{Spec}(k)}$ are of the form $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ with k'/k a finite separable extension. Then $X_{\mathrm{Spec}\, k'}$ is connected, as the morphism $X_{\overline{k}} \to X_{\mathrm{Spec}(k')}$ is surjective and $X_{\overline{k}}$ is connected by assumption. Thus $\pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is surjective by Lemma 58.4.1.

Before we go on, note that we may assume that k is a perfect field. Namely, we have $\pi_1(X_{k^{perf}}) = \pi_1(X)$ and $\pi_1(\mathrm{Spec}(k^{perf})) = \pi_1(\mathrm{Spec}(k))$ by Lemma 58.14.2.

It is clear that the composition of the functors $F\acute{E}t_{\mathrm{Spec}(k)} \to F\acute{E}t_X \to F\acute{E}t_{X_{\overline{k}}}$ sends objects to disjoint unions of copies of $X_{\mathrm{Spec}(\overline{k})}$. Therefore the composition $\pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is the trivial homomorphism by Lemma 58.4.2.

Let $U \to X$ be a finite étale morphism with U connected. Observe that $U \times_X X_{\overline{k}} = U_{\overline{k}}$. Suppose that $U_{\overline{k}} \to X_{\overline{k}}$ has a section $s : X_{\overline{k}} \to U_{\overline{k}}$. Then $s(X_{\overline{k}})$ is an open connected component of $U_{\overline{k}}$. For $\sigma \in \mathrm{Gal}(\overline{k}/k)$ denote s^σ the base change of s by $\mathrm{Spec}(\sigma)$. Since $U_{\overline{k}} \to X_{\overline{k}}$ is finite étale it has only a finite number of sections. Thus

$$\overline{T} = \bigcup s^\sigma(X_{\overline{k}})$$

is a finite union and we see that $\overline{T}$ is a $\mathrm{Gal}(\overline{k}/k)$-stable open and closed subset. By Varieties, Lemma 33.7.10 we see that $\overline{T}$ is the inverse image of a closed subset $T \subset U$. Since $U_{\overline{k}} \to U$ is open (Morphisms, Lemma 29.23.4) we conclude that T is open as well. As U is connected we see that $T = U$. Hence $U_{\overline{k}}$ is a (finite) disjoint union of copies of $X_{\overline{k}}$. By Lemma 58.4.5 we conclude that the image of $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is normal.

Let $V \to X_{\overline{k}}$ be a finite étale cover. Recall that $\overline{k}$ is the union of finite separable extensions of k. By Lemma 58.14.1 we find a finite separable extension k'/k and a finite étale morphism $U \to X_{k'}$ such that $V = X_{\overline{k}} \times_{X_{k'}} U = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$. Then the composition $U \to X_{k'} \to X$ is finite étale and $U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ contains $V = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$ as an open and closed subscheme. (Because $\mathrm{Spec}(\overline{k})$ is an open and closed subscheme of $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ via the multiplication map $k' \otimes_k \overline{k} \to \overline{k}$.) By Lemma 58.4.4 we conclude that $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is injective.

Finally, we have to show that for any finite étale morphism $U \to X$ such that $U_{\overline{k}}$ is a disjoint union of copies of $X_{\overline{k}}$ there is a finite étale morphism $V \to \mathrm{Spec}(k)$ and a surjection $V \times_{\mathrm{Spec}(k)} X \to U$. See Lemma 58.4.3. Arguing as above using Lemma 58.14.1 we find a finite separable extension k'/k such that there is an isomorphism $U_{k'} \cong \coprod_{i=1,\ldots,n} X_{k'}$. Thus setting $V = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k')$ we conclude. □

58.15. Homotopy exact sequence

0BUM In this section we discuss the following result. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups. See Proposition 58.15.2.

0BUN **Lemma 58.15.1.** *Let $f : X \to S$ be a proper morphism of schemes. Let $X \to S' \to S$ be the Stein factorization of f, see More on Morphisms, Theorem 37.53.5. If f is of finite presentation, flat, with geometrically reduced fibres, then $S' \to S$ is finite étale.* [**Gro71**, Expose X, Proposition 1.2, p. 262].

Proof. This follows from Derived Categories of Schemes, Lemma 36.32.8 and the information contained in More on Morphisms, Theorem 37.53.5. □

0C0J **Proposition 58.15.2.** *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Proof. Let $Y \to X$ be a finite étale morphism. Consider the Stein factorization

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

of $Y \to S$. By Lemma 58.15.1 the morphism $T \to S$ is finite étale. In this way we obtain a functor $F\acute{E}t_X \to F\acute{E}t_S$. For any finite étale morphism $U \to S$ a morphism $Y \to U \times_S X$ over X is the same thing as a morphism $Y \to U$ over S and such a morphism factors uniquely through the Stein factorization, i.e., corresponds to a unique morphism $T \to U$ (by the construction of the Stein factorization as a relative normalization in More on Morphisms, Lemma 37.53.1 and factorization by Morphisms, Lemma 29.53.4). Thus we see that the functors $F\acute{E}t_X \to F\acute{E}t_S$ and $F\acute{E}t_S \to F\acute{E}t_X$ are adjoints. Note that the Stein factorization of $U \times_S X \to S$ is U, because the fibres of $U \times_S X \to U$ are geometrically connected.

By the discussion above and Categories, Lemma 4.24.4 we conclude that $F\acute{E}t_S \to F\acute{E}t_X$ is fully faithful, i.e., $\pi_1(X) \to \pi_1(S)$ is surjective (Lemma 58.4.1).

It is immediate that the composition $F\acute{E}t_S \to F\acute{E}t_X \to F\acute{E}t_{X_{\overline{s}}}$ sends any U to a disjoint union of copies of $X_{\overline{s}}$. Hence $\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S)$ is trivial by Lemma 58.4.2.

Let $Y \to X$ be a finite étale morphism with Y connected such that $Y \times_X X_{\overline{s}}$ contains a connected component Z isomorphic to $X_{\overline{s}}$. Consider the Stein factorization T as above. Let $\overline{t} \in T_{\overline{s}}$ be the point corresponding to the fibre Z. Observe that T is connected (as the image of a connected scheme) and by the surjectivity above $T \times_S X$ is connected. Now consider the factorization

$$\pi : Y \longrightarrow T \times_S X$$

Let $\overline{x} \in X_{\overline{s}}$ be any closed point. Note that $\kappa(\overline{t}) = \kappa(\overline{s}) = \kappa(\overline{x})$ is an algebraically closed field. Then the fibre of π over $(\overline{t}, \overline{x})$ consists of a unique point, namely the unique point $\overline{z} \in Z$ corresponding to $\overline{x} \in X_{\overline{s}}$ via the isomorphism $Z \to X_{\overline{s}}$. We conclude that the finite étale morphism π has degree 1 in a neighbourhood of $(\overline{t}, \overline{x})$. Since $T \times_S X$ is connected it has degree 1 everywhere and we find that $Y \cong T \times_S X$. Thus $Y \times_X X_{\overline{s}}$ splits completely. Combining all of the above we see that Lemmas 58.4.3 and 58.4.5 both apply and the proof is complete. □

58.16. Specialization maps

0BUP In this section we construct specialization maps. Let $f : X \to S$ be a proper morphism of schemes with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points in S. Let $\overline{s}$ and $\overline{s}'$ be geometric points lying over s and s'. Then there is a specialization map

$$sp : \pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_{\overline{s}})$$

The construction of this map is as follows. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$, see Algebra, Definition 10.155.3. Since $s' \rightsquigarrow s$ the point s' corresponds to a point of $\mathrm{Spec}(\mathcal{O}_{S,s})$ and hence there is at least one point (and potentially many points) of $\mathrm{Spec}(A)$ over s' whose residue field is a separable algebraic extension of $\kappa(s')$. Since $\kappa(\overline{s}')$ is algebraically closed we can choose a morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ giving rise to a commutative diagram

$$\begin{array}{ccccc} \overline{s}' & \xrightarrow{\varphi} & \mathrm{Spec}(A) & \longleftarrow & \overline{s} \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

The specialization map is the composition

$$\pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_A) = \pi_1(X_{\kappa(s)^{sep}}) = \pi_1(X_{\overline{s}})$$

where the first equality is Lemma 58.9.1 and the second follows from Lemmas 58.14.2 and 58.9.3. By construction the specialization map fits into a commutative diagram

$$\begin{array}{ccc} \pi_1(X_{\overline{s}'}) & \xrightarrow{sp} & \pi_1(X_{\overline{s}}) \\ & \searrow \quad \swarrow & \\ & \pi_1(X) & \end{array}$$

provided that X is connected. The specialization map depends on the choice of $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ above and we will write sp_φ if we want to indicate this.

0C0K **Lemma 58.16.1.** *Consider a commutative diagram*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

of schemes where f and g are proper with geometrically connected fibres. Let $t' \rightsquigarrow t$ be a specialization of points in T and consider a specialization map $sp : \pi_1(Y_{\overline{t}'}) \to \pi_1(Y_{\overline{t}})$ as above. Then there is a commutative diagram

$$\begin{array}{ccc} \pi_1(Y_{\overline{t}'}) & \xrightarrow{sp} & \pi_1(Y_{\overline{t}}) \\ \downarrow & & \downarrow \\ \pi_1(X_{\overline{s}'}) & \xrightarrow{sp} & \pi_1(X_{\overline{s}}) \end{array}$$

of specialization maps where $\overline{s}$ and $\overline{s}'$ are the images of $\overline{t}$ and $\overline{t}'$.

Proof. Let B be the strict henselization of $\mathcal{O}_{T,t}$ with respect to $\kappa(t) \subset \kappa(t)^{sep} \subset \kappa(\overline{t})$. Pick $\psi : \overline{t}' \to \mathrm{Spec}(B)$ lifting $\overline{t}' \to T$ as in the construction of the specialization map. Let s and s' denote the images of t and t' in S. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$. Since $\kappa(\overline{s}) = \kappa(\overline{t})$, by the functoriality of strict henselization (Algebra, Lemma 10.155.10) we obtain a ring map $A \to B$ fitting into the commutative diagram

$$\begin{array}{ccccc} \overline{t}' & \xrightarrow{\psi} & \mathrm{Spec}(B) & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \overline{s}' & \xrightarrow{\varphi} & \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

Here the morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ is simply taken to be the composition $\overline{t}' \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Applying base change we obtain a commutative diagram

$$\begin{array}{ccc} Y_{\overline{t}'} & \longrightarrow & Y_B \\ \downarrow & & \downarrow \\ X_{\overline{s}'} & \longrightarrow & X_A \end{array}$$

and from the construction of the specialization map the commutativity of this diagram implies the commutativity of the diagram of the lemma. □

0C0L **Lemma 58.16.2.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s'' \rightsquigarrow s' \rightsquigarrow s$ be specializations of points of S. A composition of specialization maps $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is a specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct the first specialization map. Let $\mathcal{O}_{S,s'} \to A'$ be the strict henselization constructed using $\kappa(s') \subset \kappa(\overline{s}')$. By functoriality of strict henselization, there is a map $A \to A'$ such that the composition with $A' \to \kappa(\overline{s}')$ is the given map (Algebra, Lemma 10.154.6). Next, let $A' \to \kappa(\overline{s}'')$ be the map used to construct the second specialization map. Then it is clear that the composition of the first and second specialization maps is the specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$ constructed using $A \to A' \to \kappa(\overline{s}'')$. □

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian valuation ring with algebraically closed fraction field and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_\eta) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as both η and s have algebraically closed residue fields.

0C0M **Lemma 58.16.3.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. Then there exists a strictly henselian valuation ring R over S with algebraically closed fraction field such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a valuation ring with fraction field $\kappa(\overline{s}')$ dominating the image of A. See Algebra, Lemma 10.50.2. Observe that R is strictly henselian for example by Lemma 58.12.2 and Algebra, Lemma 10.50.3. Then the lemma is clear. □

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian discrete valuation ring and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_{\overline{\eta}}) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as s has algebraically closed residue field.

0C0N **Lemma 58.16.4.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. If S is Noetherian, then there exists a strictly henselian discrete valuation ring R over S such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a discrete valuation ring dominating the image of A, see Algebra, Lemma 10.119.13. Choose a diagram of fields

$$\begin{array}{ccc} \kappa(\overline{s}) & \longrightarrow & k \\ \uparrow & & \uparrow \\ A/\mathfrak{m}_A & \longrightarrow & R/\mathfrak{m}_R \end{array}$$

with k algebraically closed. Let R^{sh} be the strict henselization of R constructed using $R \to k$. We obtain a map $A \to R^{sh}$ from Algebra, Lemma 10.155.10. The ring R^{sh} is a discrete valuation ring by More on Algebra, Lemma 15.45.11. Denote η, o the generic and closed point of $\mathrm{Spec}(R^{sh})$. Since the diagram of schemes

$$\begin{array}{ccccc} \overline{\eta} & \longrightarrow & \mathrm{Spec}(R^{sh}) & \longleftarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{s}' & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & \overline{s} \end{array}$$

commutes, we obtain a commutative diagram

$$\begin{array}{ccc} \pi_1(X_{\overline{\eta}}) & \xrightarrow{sp_{R^{sh}}} & \pi_1(X_o) \\ \downarrow & & \downarrow \\ \pi_1(X_{\overline{s}'}) & \xrightarrow{sp} & X_{\overline{s}} \end{array}$$

of specialization maps by the construction of these maps. Since the vertical arrows are isomorphisms (Lemma 58.9.3), this proves the lemma. □

58.17. Restriction to a closed subscheme

0EJW In this section we prove some results about the restriction functor

$$F\acute{E}t_X \longrightarrow F\acute{E}t_Y, \quad U \longmapsto V = U \times_X Y$$

where X is a scheme and Y is a closed subscheme. Using the topological invariance of the fundamental group, we can relate the study of this functor to the completion functor on finite locally free modules.

In the following lemmas we use the concept of coherent formal modules defined in Cohomology of Schemes, Section 30.23. Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is *finite locally free* if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

0EL8 **Lemma 58.17.1.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor*

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is fully faithful.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given U finite étale over X and a morphism $t : Y \to U$ over X there exists a unique section $s : X \to U$ such that $t = s|_Y$. Picture

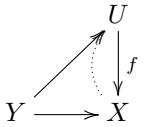

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_X$-algebra map

$$s^\sharp : f_*\mathcal{O}_U \longrightarrow \mathcal{O}_X$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 58.8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on X. Since $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_X$-module, we conclude that we get a unique $\mathcal{O}_X$-module map $\sigma : f_*\mathcal{O}_U \to \mathcal{O}_X$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{array}{ccc} f_*\mathcal{O}_U \otimes_{\mathcal{O}_X} f_*\mathcal{O}_U & \longrightarrow & f_*\mathcal{O}_U \\ \downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \end{array}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we conclude. □

0EL9 **Lemma 58.17.2.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor*

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is an equivalence on full subcategories of finite locally free objects (see above). Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is an equivalence.

Proof. The restriction functor is fully faithful by Lemma 58.17.1.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 58.8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$ as an $\mathcal{O}_X$-module on X which is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $Coh(X, \mathcal{I})$. By assumption there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_X$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $Coh(X, \mathcal{I})$. Hence by assumption we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. By faithfulness of the functor in the statement of the lemma, we conclude that μ defines a commutative $\mathcal{O}_X$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_X((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme $\pi : U \to X$ whose restriction to Y is isomorphic to U_1. The discriminant of π is the zero set of the section

$$\det(Q_\pi) : \mathcal{O}_X \longrightarrow \wedge^{top}(\pi_*\mathcal{O}_U)^{\otimes -2}$$

constructed in Discriminants, Section 49.3. Since the restriction of this to Y_n is an isomorphism for all n by Discriminants, Lemma 49.3.1 we conclude that it is an isomorphism. Thus π is étale by Discriminants, Lemma 49.3.1. □

0ELA **Lemma 58.17.3.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor*

$$\mathrm{colim}_{\mathcal{V}}\, Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $\mathrm{colim}_{\mathcal{V}}\, F\acute{E}t_V \to F\acute{E}t_Y$ is fully faithful.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 58.17.1; we urge the reader to skip it.

Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given U finite étale over $V \in \mathcal{V}$ and a morphism $t : Y \to U$ over V there exists a $V' \geq V$ and a morphism $s : V' \to U$ over V such that $t = s|_Y$. Picture

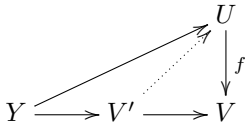

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_{V'}$-algebra map

$$s^\sharp : f_*\mathcal{O}_U|_{V'} \longrightarrow \mathcal{O}_{V'}$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 58.8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on V. Observe that $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_V$-module. Hence we get a $V' \geq V$ a map $\sigma : f_*\mathcal{O}_U|_{V'} \to \mathcal{O}_{V'}$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{array}{ccc} (f_*\mathcal{O}_U \otimes_{\mathcal{O}_V} f_*\mathcal{O}_U)|_{V'} & \longrightarrow & f_*\mathcal{O}_U|_{V'} \\ \downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{O}_{V'} \otimes_{\mathcal{O}_{V'}} \mathcal{O}_{V'} & \longrightarrow & \mathcal{O}_{V'} \end{array}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we deduce that there exists a $V'' \geq V'$ such that $\sigma|_{V''}$ is an algebra homomorphism as desired. □

0EK1 **Lemma 58.17.4.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor*

$$\text{colim}_{\mathcal{V}} \mathit{Coh}(\mathcal{O}_V) \longrightarrow \mathit{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines an equivalence of the full subcategories of finite locally free objects (see explanation above). Then the restriction functor

$$\text{colim}_{\mathcal{V}} F\acute{E}t_V \to F\acute{E}t_Y$$

is an equivalence.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 4.19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 58.17.2; we urge the reader to skip it.

The restriction functor is fully faithful by Lemma 58.17.3.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 58.8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$

as an $\mathcal{O}_X$-module on X which is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $Coh(X, \mathcal{I})$. By assumption there exists a $V \in \mathcal{V}$ and a finite locally free $\mathcal{O}_V$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_V$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_V} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $Coh(X, \mathcal{I})$. Hence by assumption after shrinking V we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. After possibly shrinking further we may assume μ defines a commutative $\mathcal{O}_V$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_V((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme over V whose restriction to Y is isomorphic to U_1. It follows that $U \to V$ is étale at all points lying over Y, see More on Morphisms, Lemma 37.12.3. Thus after shrinking V once more we may assume $U \to V$ is finite étale. This finishes the proof. □

0EJX **Lemma 58.17.5.** *Let X be a scheme and let $Y \subset X$ be a closed subscheme. If every connected component of X meets Y, then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is faithful.*

Proof. Let $a, b : U \to U'$ be two morphisms of schemes finite étale over X whose restriction to Y are the same. The image of a connected component of U is an connected component of X; this follows from Topology, Lemma 5.7.7 applied to the restriction of $U \to X$ to a connected component of X. Hence the image of every connected component of U meets Y by assumption. We conclude that $a = b$ after restriction to each connected component of U by Étale Morphisms, Proposition 41.6.3. Since the equalizer of a and b is an open subscheme of U (as the diagonal of U' over X is open) we conclude. □

0EJZ **Lemma 58.17.6.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Assume one of the following holds*

1. *X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism, or*
2. *X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$, or*
3. *for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism.*

Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is fully faithful.

Proof. This lemma follows formally from Lemma 58.17.1 and Algebraic and Formal Geometry, Lemma 52.15.1. □

0EK0 **Lemma 58.17.7.** *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$*

be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume one of the following holds

(1) *X is quasi-affine and*
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$
is an isomorphism, or

(2) *X has an ample invertible module $\mathcal{L}$ and*
$$\mathrm{colim}_{\mathcal{V}} \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$
is an isomorphism for all $m \gg 0$, or

(3) *for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map*
$$\mathrm{colim}_{V' \geq V} \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$
is an isomorphism.

Then the functor
$$\mathrm{colim}_{\mathcal{V}} F\acute{E}t_V \to F\acute{E}t_Y$$
is fully faithful.

Proof. This lemma follows formally from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.2. □

58.18. Pushouts and fundamental groups

0EK3 Here is the main result.

0EK4 **Lemma 58.18.1.** *In More on Morphisms, Situation 37.67.1, for example if $Z \to Y$ and $Z \to X$ are closed immersions of schemes, there is an equivalence of categories*
$$F\acute{E}t_{Y \amalg_Z X} \longrightarrow F\acute{E}t_Y \times_{F\acute{E}t_Z} F\acute{E}t_X$$

Proof. The pushout exists by More on Morphisms, Proposition 37.67.3. The functor is given by sending a scheme U finite étale over the pushout to the base changes $Y' = U \times_{Y \amalg_Z X} Y$ and $X' = U \times_{Y \amalg_Z X} X$ and the natural isomorphism $Y' \times_Y Z \to X' \times_X Z$ over Z. To prove this functor is an equivalence we use More on Morphisms, Lemma 37.67.7 to construct a quasi-inverse functor. The only thing left to prove is to show that given a morphism $U \to Y \amalg_Z X$ which is separated, quasi-finite and étale such that $X' \to X$ and $Y' \to Y$ are finite, then $U \to Y \amalg_Z X$ is finite. This can either be deduced from the corresponding algebra fact (More on Algebra, Lemma 15.6.7) or it can be seen because
$$X' \amalg Y' \to U$$
is surjective and X' and Y' are proper over $Y \amalg_Z X$ (this uses the description of the pushout in More on Morphisms, Proposition 37.67.3) and then we can apply Morphisms, Lemma 29.41.10 to conclude that U is proper over $Y \amalg_Z X$. Since a quasi-finite and proper morphism is finite (More on Morphisms, Lemma 37.44.1) we win. □

58.19. Finite étale covers of punctured spectra, I

0BLE We first prove some results á la Lefschetz.

0BLF **Situation 58.19.1.** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA.

Recall that for a scheme X the category of schemes finite étale over X is denoted $F\acute{E}t_X$, see Section 58.5. In Situation 58.19.1 we will study the base change functors

$$\begin{array}{ccc} F\acute{E}t_X & \longrightarrow & F\acute{E}t_U \\ \downarrow & & \downarrow \\ F\acute{E}t_{X_0} & \longrightarrow & F\acute{E}t_{U_0} \end{array}$$

In many case the right vertical arrow is faithful.

0BLG **Lemma 58.19.2.** *In Situation 58.19.1. Assume one of the following holds*

(1) $\dim(A/\mathfrak{p}) \geq 2$ *for every minimal prime* $\mathfrak{p} \subset A$ *with* $f \notin \mathfrak{p}$*, or*
(2) *every connected component of* U *meets* U_0*.*

Then

$$F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a faithful functor.

Proof. Case (2) is immediate from Lemma 58.17.5. Assumption (1) implies every irreducible component of U meets U_0, see Algebra, Lemma 10.60.13. Hence (1) follows from (2). □

Before we prove something more interesting, we need a couple of lemmas.

0BLH **Lemma 58.19.3.** *In Situation 58.19.1. Let* $V \to U$ *be a finite morphism. Let* $A^\wedge$ *be the* $\mathfrak{m}$*-adic completion of* A*, let* $X' = \mathrm{Spec}(A^\wedge)$ *and let* U' *and* V' *be the base changes of* U *and* V *to* X'*. If* $Y' \to X'$ *is a finite morphism such that* $V' = Y' \times_{X'} U'$*, then there exists a finite morphism* $Y \to X$ *such that* $V = Y \times_X U$ *and* $Y' = Y \times_X X'$*.*

Proof. This is a straightforward application of More on Algebra, Proposition 15.90.16. Namely, choose generators $f_1, \ldots, f_t$ of $\mathfrak{m}$. For each i write $V \times_U D(f_i) = \mathrm{Spec}(B_i)$. For $1 \leq i, j \leq n$ we obtain an isomorphism $\alpha_{ij} : (B_i)_{f_j} \to (B_j)_{f_i}$ of $A_{f_i f_j}$-algebras because the spectrum of both represent $V \times_U D(f_i f_j)$. Write $Y' = \mathrm{Spec}(B')$. Since $V \times_U U' = Y \times_{X'} U'$ we get isomorphisms $\alpha_i : B'_{f_i} \to B_i \otimes_A A^\wedge$. A straightforward argument shows that $(B', B_i, \alpha_i, \alpha_{ij})$ is an object of $\mathrm{Glue}(A \to A^\wedge, f_1, \ldots, f_t)$, see More on Algebra, Remark 15.90.10. Applying the proposition cited above (and using More on Algebra, Remark 15.90.20 to obtain the algebra structure) we find an A-algebra B such that $\mathrm{Can}(B)$ is isomorphic to $(B', B_i, \alpha_i, \alpha_{ij})$. Setting $Y = \mathrm{Spec}(B)$ we see that $Y \to X$ is a morphism which comes equipped with compatible isomorphisms $V \cong Y \times_X U$ and $Y' = Y \times_X X'$ as desired. □

0BLI **Lemma 58.19.4.** *In Situation 58.19.1 assume* A *is henselian or more generally that* $(A, (f))$ *is a henselian pair. Let* $A^\wedge$ *be the* $\mathfrak{m}$*-adic completion of* A*, let* $X' = \mathrm{Spec}(A^\wedge)$ *and let* U' *and* U'_0 *be the base changes of* U *and* U_0 *to* X'*. If* $F\acute{E}t_{U'} \to F\acute{E}t_{U'_0}$ *is fully faithful, then* $F\acute{E}t_U \to F\acute{E}t_{U_0}$ *is fully faithful.*

Proof. Assume $F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U'_0}$ is a fully faithful. Since $X' \to X$ is faithfully flat, it is immediate that the functor $V \to V_0 = V \times_U U_0$ is faithful. Since the category of finite étale coverings has an internal hom (Lemma 58.5.4) it suffices to prove the following: Given V finite étale over U we have

$$\mathrm{Mor}_U(U, V) = \mathrm{Mor}_{U_0}(U_0, V_0)$$

The we assume we have a morphism $s_0 : U_0 \to V_0$ over U_0 and we will produce a morphism $s : U \to V$ over U.

By our assumption there does exist a morphism $s' : U' \to V'$ whose restriction to V'_0 is the base change s'_0 of s_0. Since $V' \to U'$ is finite étale this means that $V' = s'(U') \amalg W'$ for some $W' \to U'$ finite and étale. Choose a finite morphism $Z' \to X'$ such that $W' = Z' \times_{X'} U'$. This is possible by Zariski's main theorem in the form stated in More on Morphisms, Lemma 37.43.3 (small detail omitted). Then

$$V' = s'(U') \amalg W' \longrightarrow X' \amalg Z' = Y'$$

is an open immersion such that $V' = Y' \times_{X'} U'$. By Lemma 58.19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. Write $Y = \mathrm{Spec}(B)$ so that $Y' = \mathrm{Spec}(B \otimes_A A^\wedge)$. Then $B \otimes_A A^\wedge$ has an idempotent e' corresponding to the open and closed subscheme X' of $Y' = X' \amalg Z'$.

The case A is henselian (slightly easier). The image $\overline{e}$ of e' in $B \otimes_A \kappa(\mathfrak{m}) = B/\mathfrak{m}B$ lifts to an idempotent e of B as A is henselian (because B is a product of local rings by Algebra, Lemma 10.153.3). Then we see that e maps to e' by uniqueness of lifts of idempotents (using that $B \otimes_A A^\wedge$ is a product of local rings). Let $Y_1 \subset Y$ be the open and closed subscheme corresponding to e. Then $Y_1 \times_X X' = s'(X')$ which implies that $Y_1 \to X$ is an isomorphism (by faithfully flat descent) and gives the desired section.

The case where $(A, (f))$ is a henselian pair. Here we use that s' is a lift of s'_0. Namely, let $Y_{0,1} \subset Y_0 = Y \times_X X_0$ be the closure of $s_0(U_0) \subset V_0 = Y_0 \times_{X_0} U_0$. As $X' \to X$ is flat, the base change $Y'_{0,1} \subset Y'_0$ is the closure of $s'_0(U'_0)$ which is equal to $X'_0 \subset Y'_0$ (see Morphisms, Lemma 29.25.16). Since $Y'_0 \to Y_0$ is submersive (Morphisms, Lemma 29.25.12) we conclude that $Y_{0,1}$ is open and closed in Y_0. Let $e_0 \in B/fB$ be the corresponding idempotent. By More on Algebra, Lemma 15.11.6 we can lift e_0 to an idempotent $e \in B$. Then we conclude as before. □

In Situation 58.19.1 fully faithfulness of the restriction functor $F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$ holds under fairly mild assumptions. In particular, the assumptions often do not imply U is a connected scheme, but the conclusion guarantees that U and U_0 have the same number of connected components.

0EK5 **Lemma 58.19.5.** *In Situation 58.19.1. Assume*

(a) *A has a dualizing complex,*
(b) *the pair $(A, (f))$ is henselian,*
(c) *one of the following is true*
  (i) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or*
  (ii) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ we have $\mathrm{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 2$.*

Then the restriction functor $F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$ is fully faithful.

Proof. Let A' be the $\mathfrak{m}$-adic completion of A. We will show that the hypotheses remain true for A'. This is clear for conditions (a) and (b). Condition (c)(ii) is preserved by Local Cohomology, Lemma 51.11.3. Next, assume (c)(i) holds. Since A is universally catenary (Dualizing Complexes, Lemma 47.17.4) we see that every irreducible component of $\mathrm{Spec}(A')$ not contained in $V(f)$ has dimension ≥ 3, see More on Algebra, Proposition 15.110.5. Since $A \to A'$ is flat with Gorenstein fibres, the condition that A_f is (S_2) implies that A'_f is (S_2). References used: Dualizing Complexes, Section 47.23, More on Algebra, Section 15.51, and Algebra, Lemma 10.163.4. Thus by Lemma 58.19.4 we may assume that A is a Noetherian complete local ring.

Assume A is a complete local ring in addition to the other assumptions. By Lemma 58.17.1 the result follows from Algebraic and Formal Geometry, Lemma 52.15.6. □

0BM6 **Lemma 58.19.6.** *In Situation 58.19.1. Assume* [**BdJ14**, Corollary 1.11]

(1) $H^1_{\mathfrak{m}}(A)$ *and* $H^2_{\mathfrak{m}}(A)$ *are annihilated by a power of* f*, and*
(2) A *is henselian or more generally* $(A, (f))$ *is a henselian pair.*

Then the restriction functor $F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$ *is fully faithful.*

Proof. By Lemma 58.19.4 we may assume that A is a Noetherian complete local ring. (The assumptions carry over; use Dualizing Complexes, Lemma 47.9.3.) By Lemma 58.17.1 the result follows from Algebraic and Formal Geometry, Lemma 52.15.5. □

0BLJ **Lemma 58.19.7.** *In Situation 58.19.1 assume* A *has depth* ≥ 3 *and* A *is henselian or more generally* $(A, (f))$ *is a henselian pair. Then the restriction functor* $F\acute{E}t_U \to F\acute{E}t_{U_0}$ *is fully faithful.*

Proof. The assumption of depth forces $H^1_{\mathfrak{m}}(A) = H^2_{\mathfrak{m}}(A) = 0$, see Dualizing Complexes, Lemma 47.11.1. Hence Lemma 58.19.6 applies. □

58.20. Purity in local case, I

0BM7 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$ be the punctured spectrum. We say *purity holds for* $(A, \mathfrak{m})$ if the restriction functor

$$F\acute{E}t_X \longrightarrow F\acute{E}t_U$$

is essentially surjective. In this section we try to understand how the question changes when one passes from X to a hypersurface X_0 in X, in other words, we study a kind of local Lefschetz property for the fundamental groups of punctured spectra. These results will be useful to proceed by induction on dimension in the proofs of our main results on local purity, namely, Lemma 58.21.3, Proposition 58.25.3, and Proposition 58.26.4.

0BM8 **Lemma 58.20.1.** *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring. Set* $X = \mathrm{Spec}(A)$ *and let* $U = X \setminus \{\mathfrak{m}\}$*. Let* $\pi : Y \to X$ *be a finite morphism such that* $\mathrm{depth}(\mathcal{O}_{Y,y}) \geq 2$ *for all closed points* $y \in Y$*. Then* Y *is the spectrum of* $B = \mathcal{O}_Y(\pi^{-1}(U))$*.*

Proof. Set $V = \pi^{-1}(U)$ and denote $\pi' : V \to U$ the restriction of π. Consider the $\mathcal{O}_X$-module map

$$\pi_*\mathcal{O}_Y \longrightarrow j_*\pi'_*\mathcal{O}_V$$

where $j : U \to X$ is the inclusion morphism. We claim Divisors, Lemma 31.5.11 applies to this map. If so, then $B = \Gamma(Y, \mathcal{O}_Y)$ and we see that the lemma holds.

Let $x \in X$ be the closed point. It suffices to show that $\text{depth}((\pi_*\mathcal{O}_Y)_x) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to x. By Algebra, Lemma 10.72.11 it suffices to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. Since this is the assumption of the lemma the proof is complete. □

0BLK **Lemma 58.20.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Assume A has depth ≥ 2. The following are equivalent*

(1) *$V = Y \times_X U$ for some $Y \to X$ finite étale,*
(2) *$B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.*

Proof. Denote $\pi : V \to U$ the given finite étale morphism. Assume Y as in (1) exists. Let $x \in X$ be the point corresponding to $\mathfrak{m}$. Let $y \in Y$ be a point mapping to x. We claim that $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. This is true because $Y \to X$ is étale and hence $A = \mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same depth (Algebra, Lemma 10.163.2). Hence Lemma 58.20.1 applies and $Y = \text{Spec}(B)$.

The implication (2) $\Rightarrow$ (1) is easier and the details are omitted. □

0BM9 **Lemma 58.20.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Assume A is normal of dimension ≥ 2. The functor*

$$F\text{Ét}_U \longrightarrow \left\{ \begin{matrix} \text{finite normal } A\text{-algebras } B \text{ such} \\ \text{that } \text{Spec}(B) \to X \text{ is étale over } U \end{matrix} \right\}, \quad V \longmapsto \Gamma(V, \mathcal{O}_V)$$

is an equivalence. Moreover, $V = Y \times_X U$ for some $Y \to X$ finite étale if and only if $B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.

Proof. Observe that $\text{depth}(A) \geq 2$ because A is normal (Serre's criterion for normality, Algebra, Lemma 10.157.4). Thus the final statement follows from Lemma 58.20.2. Given $\pi : V \to U$ finite étale, set $B = \Gamma(V, \mathcal{O}_V)$. If we can show that B is normal and finite over A, then we obtain the displayed functor. Since there is an obvious quasi-inverse functor, this is also all that we have to show.

Since A is normal, the scheme V is normal (Descent, Lemma 35.18.2). Hence V is a finite disjoint union of integral schemes (Properties, Lemma 28.7.6). Thus we may assume V is integral. In this case the function field L of V (Morphisms, Section 29.49) is a finite separable extension of the fraction field of A (because we get it by looking at the generic fibre of $V \to U$ and using Morphisms, Lemma 29.36.7). By Algebra, Lemma 10.161.8 the integral closure $B' \subset L$ of A in L is finite over A. By More on Algebra, Lemma 15.23.20 we see that B' is a reflexive A-module, which in turn implies that $\text{depth}_A(B') \geq 2$ by More on Algebra, Lemma 15.23.18.

Let $f \in \mathfrak{m}$. Then $B_f = \Gamma(V \times_U D(f), \mathcal{O}_V)$ (Properties, Lemma 28.17.1). Hence $B'_f = B_f$ because B_f is normal (see above), finite over A_f with fraction field L. It follows that $V = \text{Spec}(B') \times_X U$. Then we conclude that $B = B'$ from Lemma 58.20.1 applied to $\text{Spec}(B') \to X$. This lemma applies because the localizations $B'_{\mathfrak{m}'}$ of B' at maximal ideals $\mathfrak{m}' \subset B'$ lying over $\mathfrak{m}$ have depth ≥ 2 by Algebra, Lemma 10.72.11 and the remark on depth in the preceding paragraph. □

0BLL **Lemma 58.20.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \text{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. The following are equivalent*

(1) $V = Y \times_X U$ *for some* $Y \to X$ *finite étale, and*
(2) $V' = Y' \times_{X'} U'$ *for some* $Y' \to X'$ *finite étale.*

Proof. The implication (1) $\Rightarrow$ (2) follows from taking the base change of a solution $Y \to X$. Let $Y' \to X'$ be as in (2). By Lemma 58.19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. By descent we see that $Y \to X$ is finite étale (Algebra, Lemmas 10.83.2 and 10.143.3). This finishes the proof. □

The point of the following two lemmas is that the assumptions do not force A to have depth ≥ 3. For example if A is a complete normal local domain of dimension ≥ 3 and $f \in \mathfrak{m}$ is nonzero, then the assumptions are satisfied.

0EK6 **Lemma 58.20.5.** *In Situation 58.19.1. Let V be finite étale over U. Assume*

(a) *A has a dualizing complex,*
(b) *the pair $(A, (f))$ is henselian,*
(c) *one of the following is true*
  (i) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or*
  (ii) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ we have $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 2$.*
(d) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 58.20.4. (The assumptions carry over; see proof of Lemma 58.19.5.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 15.13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 15.11.4. Then we can use Lemma 58.19.5 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. □

0BLS **Lemma 58.20.6.** *In Situation 58.19.1. Let V be finite étale over U. Assume*

(1) *$H^1_\mathfrak{m}(A)$ and $H^2_\mathfrak{m}(A)$ are annihilated by a power of f,*
(2) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 58.20.4. (The assumptions carry over; use Dualizing Complexes, Lemma 47.9.3.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 15.13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 15.11.4. Then we can use Lemma 58.19.6 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. □

0BLM **Lemma 58.20.7.** *In Situation 58.19.1. Let V be finite étale over U. Assume*

(1) *A has depth ≥ 3,*
(2) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. The assumption of depth forces $H^1_\mathfrak{m}(A) = H^2_\mathfrak{m}(A) = 0$, see Dualizing Complexes, Lemma 47.11.1. Hence Lemma 58.20.6 applies. □

58.21. Purity of branch locus

0BJE We will use the discriminant of a finite locally free morphism. See Discriminants, Section 49.3.

0BJG **Lemma 58.21.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim(A) \geq 1$. Let $f \in \mathfrak{m}$. Then there exist a $\mathfrak{p} \in V(f)$ with $\dim(A_\mathfrak{p}) = 1$.*

Proof. By induction on $\dim(A)$. If $\dim(A) = 1$, then $\mathfrak{p} = \mathfrak{m}$ works. If $\dim(A) > 1$, then let $Z \subset \mathrm{Spec}(A)$ be an irreducible component of dimension > 1. Then $V(f) \cap Z$ has dimension > 0 (Algebra, Lemma 10.60.13). Pick a prime $\mathfrak{q} \in V(f) \cap Z$, $\mathfrak{q} \neq \mathfrak{m}$ corresponding to a closed point of the punctured spectrum of A; this is possible by Properties, Lemma 28.6.4. Then $\mathfrak{q}$ is not the generic point of Z. Hence $0 < \dim(A_\mathfrak{q}) < \dim(A)$ and $f \in \mathfrak{q}A_\mathfrak{q}$. By induction on the dimension we can find $f \in \mathfrak{p} \subset A_\mathfrak{q}$ with $\dim((A_\mathfrak{q})_\mathfrak{p}) = 1$. Then $\mathfrak{p} \cap A$ works. □

0BJH **Lemma 58.21.2.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$. Assume*

1. *f is flat,*
2. *f is quasi-finite at x,*
3. *x is not a generic point of an irreducible component of X,*
4. *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

Proof. Observe that the set of points where f is unramified is the same as the set of points where f is étale and that this set is open. See Morphisms, Definitions 29.35.1 and 29.36.1 and Lemma 29.36.16. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 35.23.31 and 35.31.1). Thus we can apply More on Morphisms, Lemma 37.41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over $y = f(x)$. Then it follows that f is finite locally free (Morphisms, Lemma 29.48.2).

Assume f is finite locally free and that x is the unique point of X lying over $y = f(x)$. By Discriminants, Lemma 49.3.1 we find a locally principal closed subscheme $D_\pi \subset Y$ such that $y' \in D_\pi$ if and only if there exists an $x' \in X$ with $f(x') = y'$ and f ramified at x'. Thus we have to prove that $y \notin D_\pi$. Assume $y \in D_\pi$ to get a contradiction.

By condition (3) we have $\dim(\mathcal{O}_{X,x}) \geq 1$. We have $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ by Algebra, Lemma 10.112.7. By Lemma 58.21.1 we can find $y' \in D_\pi$ specializing to y with $\dim(\mathcal{O}_{Y,y'}) = 1$. Choose $x' \in X$ with $f(x') = y'$ where f is ramified. Since f is finite it is closed, and hence $x' \rightsquigarrow x$. We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'}) = 1$ as before. This contradicts property (4). □

0BMA **Lemma 58.21.3.** *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Set $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Then*

1. *the functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective, i.e., purity holds for A,*
2. *any finite $A \to B$ with B normal which induces a finite étale morphism on punctured spectra is étale.*

Proof. Recall that a regular local ring is normal by Algebra, Lemma 10.157.5. Hence (1) and (2) are equivalent by Lemma 58.20.3. We prove the lemma by induction on d.

The case $d = 2$. In this case $A \to B$ is flat. Namely, we have going down for $A \to B$ by Algebra, Proposition 10.38.7. Then $\dim(B_{\mathfrak{m}'}) = 2$ for all maximal ideals $\mathfrak{m}' \subset B$ by Algebra, Lemma 10.112.7. Then $B_{\mathfrak{m}'}$ is Cohen-Macaulay by Algebra, Lemma 10.157.4. Hence and this is the important step Algebra, Lemma 10.128.1 applies to show $A \to B_{\mathfrak{m}'}$ is flat. Then Algebra, Lemma 10.39.18 shows $A \to B$ is flat. Thus we can apply Lemma 58.21.2 (or you can directly argue using the easier Discriminants, Lemma 49.3.1) to see that $A \to B$ is étale.

The case $d \geq 3$. Let $V \to U$ be finite étale. Let $f \in \mathfrak{m}_A$, $f \notin \mathfrak{m}_A^2$. Then A/fA is a regular local ring of dimension $d - 1 \geq 2$, see Algebra, Lemma 10.106.3. Let U_0 be the punctured spectrum of A/fA and let $V_0 = V \times_U U_0$. By Lemma 58.20.7 it suffices to show that V_0 is in the essential image of $F\acute{E}t_{\mathrm{Spec}(A/fA)} \to F\acute{E}t_{U_0}$. This follows from the induction hypothesis. □

0BMB **Lemma 58.21.4** (Purity of branch locus)**.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume*

(1) *$\mathcal{O}_{X,x}$ is normal,*
(2) *$\mathcal{O}_{Y,y}$ is regular,*
(3) *f is quasi-finite at x,*
(4) *$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) \geq 1$*
(5) *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

[**Nag59**] and [**Gro71**, Exp. X, Thm. 3.1]

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

An uninteresting case is when $d = 1$. In that case we are assuming that f is unramified at x and that $\mathcal{O}_{Y,y}$ is a discrete valuation ring (Algebra, Lemma 10.119.7). Then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ (otherwise the map would not be quasi-finite at x) and we see that f is flat at x. Since flat + unramified is étale we conclude (some details omitted).

The case $d \geq 2$. We will use induction on d to reduce to the case discussed in Lemma 58.21.3. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 35.23.31 and 35.31.1). Thus we can apply More on Morphisms, Lemma 37.41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over y. Here we use that étale extensions of local rings do not change dimension, normality, and regularity, see More on Algebra, Section 15.44 and Étale Morphisms, Section 41.19.

Next, we can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume that Y is the spectrum of a regular local ring. It follows that $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ as every point of X necessarily specializes to x.

The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite and necessarily injective (by equality of dimensions). We conclude we have going down for $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ by Algebra, Proposition 10.38.7 (and the fact that a regular ring is a normal ring by Algebra, Lemma 10.157.5). Pick $x' \in X$, $x' \neq x$ with image $y' = f(x')$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of a normal domain. Similarly, $\mathcal{O}_{Y,y'}$ is regular (see Algebra, Lemma 10.110.6). We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'})$ by Algebra, Lemma 10.112.7 (we

checked going down above). Of course these dimensions are strictly less than d as $x' \neq x$ and by induction on d we conclude that f is étale at x'.

Thus we arrive at the following situation: We have a finite local homomorphism $A \to B$ of Noetherian local rings of dimension $d \geq 2$, with A regular, B normal, which induces a finite étale morphism $V \to U$ on punctured spectra. Our goal is to show that $A \to B$ is étale. This follows from Lemma 58.21.3 and the proof is complete. □

The following lemma is sometimes useful to find the maximal open subset over which a finite étale morphism extends.

0EY6 **Lemma 58.21.5.** *Let $j : U \to X$ be an open immersion of locally Noetherian schemes such that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Let $\pi : V \to U$ be finite étale. Then*

(1) *$\mathcal{B} = j_*\pi_*\mathcal{O}_V$ is a reflexive coherent $\mathcal{O}_X$-algebra, set $Y = \underline{\text{Spec}}_X(\mathcal{B})$,*
(2) *$Y \to X$ is the unique finite morphism such that $V = Y \times_X U$ and $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$,*
(3) *$Y \to X$ is étale at y if and only if $Y \to X$ is flat at y, and*
(4) *$Y \to X$ is étale if and only if $\mathcal{B}$ is finite locally free as an $\mathcal{O}_X$-module.*

Moreover, (a) the construction of $\mathcal{B}$ and $Y \to X$ commutes with base change by flat morphisms $X' \to X$ of locally Noetherian schemes, and (b) if $V' \to U'$ is a finite étale morphism with $U \subset U' \subset X$ open which restricts to $V \to U$ over U, then there is a unique isomorphism $Y' \times_X U' = V'$ over U'.

Proof. Observe that $\pi_*\mathcal{O}_V$ is a finite locally free $\mathcal{O}_U$-module, in particular reflexive. By Divisors, Lemma 31.12.12 the module $j_*\pi_*\mathcal{O}_V$ is the unique reflexive coherent module on X restricting to $\pi_*\mathcal{O}_V$ over U. This proves (1).

By construction $Y \times_X U = V$. Since $\mathcal{B}$ is coherent, we see that $Y \to X$ is finite. We have $\text{depth}(\mathcal{B}_x) \geq 2$ for $x \in X \setminus U$ by Divisors, Lemma 31.12.11. Hence $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$ by Algebra, Lemma 10.72.11. Conversely, suppose that $\pi' : Y' \to X$ is a finite morphism such that $V = Y' \times_X U$ and $\text{depth}(\mathcal{O}_{Y',y'}) \geq 2$ for $y' \in Y' \setminus V$. Then $\pi'_*\mathcal{O}_{Y'}$ restricts to $\pi_*\mathcal{O}_V$ over U and satisfies $\text{depth}((\pi'_*\mathcal{O}_{Y'})_x) \geq 2$ for $x \in X \setminus U$ by Algebra, Lemma 10.72.11. Then $\pi'_*\mathcal{O}_{Y'}$ is canonically isomorphic to $j_*\pi_*\mathcal{O}_V$ for example by Divisors, Lemma 31.5.11. This proves (2).

If $Y \to X$ is étale at y, then $Y \to X$ is flat at y. Conversely, suppose that $Y \to X$ is flat at y. If $y \in V$, then $Y \to X$ is étale at y. If $y \notin V$, then we check (1), (2), (3), and (4) of Lemma 58.21.2 hold to see that $Y \to X$ is étale at y. Parts (1) and (2) are clear and so is (3) since $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. If $y' \rightsquigarrow y$ is a specialization and $\dim(\mathcal{O}_{Y,y'}) = 1$, then $y' \in V$ since otherwise the depth of this local ring would be 2 a contradiction by Algebra, Lemma 10.72.3. Hence $Y \to X$ is étale at y' and we conclude (4) of Lemma 58.21.2 holds too. This finishes the proof of (3).

Part (4) follows from (3) and the fact that $((Y \to X)_*\mathcal{O}_Y)_x$ is a flat $\mathcal{O}_{X,x}$-module if and only if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-module for all $y \in Y$ mapping to x, see Algebra, Lemma 10.39.18. Here we also use that a finite flat module over a Noetherian ring is finite locally free, see Algebra, Lemma 10.78.2 (and Algebra, Lemma 10.31.4).

As to the final assertions of the lemma, part (a) follows from flat base change, see Cohomology of Schemes, Lemma 30.5.2 and part (b) follows from the uniqueness in (2) applied to the restriction $Y \times_X U'$. □

0EY7 **Lemma 58.21.6.** *Let $j : U \to X$ be an open immersion of Noetherian schemes such that purity holds for $\mathcal{O}_{X,x}$ for all $x \notin U$. Then*

$$F\acute{E}t_X \longrightarrow F\acute{E}t_U$$

is essentially surjective.

Proof. Let $V \to U$ be a finite étale morphism. By Noetherian induction it suffices to extend $V \to U$ to a finite étale morphism to a strictly larger open subset of X. Let $x \in X \setminus U$ be the generic point of an irreducible component of $X \setminus U$. Then the inverse image U_x of U in $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the punctured spectrum of $\mathcal{O}_{X,x}$. By assumption $V_x = V \times_U U_x$ is the restriction of a finite étale morphism $Y_x \to \mathrm{Spec}(\mathcal{O}_{X,x})$ to U_x. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset X$ containing x and a morphism $V' \to U'$ of finite presentation whose restriction to U recovers $V \to U$ and whose restriction to $\mathrm{Spec}(\mathcal{O}_{X,x})$ recovering Y_x. Finally, the morphism $V' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 32.20.4. □

58.22. Finite étale covers of punctured spectra, II

0BLU In this section we prove some variants of the material discussed in Section 58.19. Suppose we have a Noetherian local ring $(A, \mathfrak{m})$ and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA. All of this is exactly as in Situation 58.19.1. The difference is that we will consider the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

In other words, we will not try to lift finite étale coverings of U_0 to all of U, but just to some open neighbourhood U' of U_0 in U.

0BLN **Lemma 58.22.1.** *In Situation 58.19.1. Let $U' \subset U$ be open and contain U_0. Assume for $\mathfrak{p} \subset A$ minimal with $\mathfrak{p} \in U'$, $\mathfrak{p} \notin U_0$ we have $\dim(A/\mathfrak{p}) \geq 2$. Then*

$$F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}, \quad V' \longmapsto V_0 = V' \times_{U'} U_0$$

is a faithful functor. Moreover, there exists a U' satisfying the assumption and any smaller open $U'' \subset U'$ containing U_0 also satisfies this assumption. In particular, the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is faithful.

Proof. By Algebra, Lemma 10.60.13 we see that $V(\mathfrak{p})$ meets U_0 for every prime $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p}) \geq 2$. Thus the displayed functor is faithful for a U as in the statement by Lemma 58.17.5. To see the existence of such a U' note that for $\mathfrak{p} \subset A$ with $\mathfrak{p} \in U$, $\mathfrak{p} \notin U_0$ with $\dim(A/\mathfrak{p}) = 1$ then $\mathfrak{p}$ corresponds to a closed point of U and hence $V(\mathfrak{p}) \cap U_0 = \emptyset$. Thus we can take U' to be the complement of the irreducible components of X which do not meet U_0 and have dimension 1. □

0DXX **Lemma 58.22.2.** *In Situation 58.19.1 assume*

(1) *A has a dualizing complex and is f-adically complete,*
(2) *every irreducible component of X not contained in X_0 has dimension ≥ 3.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is fully faithful.

Proof. To prove this we may replace A by its reduction by the topological invariance of the fundamental group, see Lemma 58.8.3. Then the result follows from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.7. □

0BLP **Lemma 58.22.3.** *In Situation 58.19.1 assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor.*
(3) *$H^1_{\mathfrak{m}}(A/fA)$ is a finite A-module.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is fully faithful.

Proof. Follows from Lemma 58.17.3 and Algebraic and Formal Geometry, Lemma 52.15.8. □

58.23. Finite étale covers of punctured spectra, III

0EK7 In this section we study when in Situation 58.19.1. the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is an equivalence of categories.

0DXY **Lemma 58.23.1.** *In Situation 58.19.1 assume*

(1) *A has a dualizing complex and is f-adically complete,*
(2) *one of the following is true*
   (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\mathrm{depth}(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) > 3$.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 58.17.4 and Algebraic and Formal Geometry, Lemma 52.24.1. □

0BLV **Lemma 58.23.2.** *In Situation 58.19.1 assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_{\mathfrak{m}}(A/fA)$ and $H^2_{\mathfrak{m}}(A/fA)$ are finite A-modules.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 58.17.4 and Algebraic and Formal Geometry, Lemma 52.24.2. □

0BLW **Remark 58.23.3.** Let $(A, \mathfrak{m})$ be a complete local Noetherian ring and $f \in \mathfrak{m}$ nonzero. Suppose that A_f is (S_2) and every irreducible component of $\mathrm{Spec}(A)$ has dimension ≥ 4. Then Lemma 58.23.1 tells us that the category

$$\mathrm{colim}_{U' \subset U \text{ open}, U_0 \subset U} \text{ category of schemes finite étale over } U'$$

is equivalent to the category of schemes finite étale over U_0. For example this holds if A is a normal domain of dimension ≥ 4!

58.24. Finite étale covers of punctured spectra, IV

0EK8 Let X, X_0, U, U_0 be as in Situation 58.19.1. In this section we ask when the restriction functor

$$F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$$

is essentially surjective. We will do this by taking results from Section 58.23 and then filling in the gaps using purity. Recall that we say *purity holds* for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective where $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

0EK9 **Lemma 58.24.1.** *In Situation 58.19.1 assume*

(1) *A has a dualizing complex and is f-adically complete,*
(2) *one of the following is true*
   (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\mathrm{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*
(3) *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.*

Then the restriction functor $F\acute{E}t_U \to F\acute{E}t_{U_0}$ is essentially surjective.

Proof. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 58.23.1 there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as in (3). By assumption we can find finite étale morphisms $V_i' \to \mathrm{Spec}(A_{\mathfrak{p}_i})$ agreeing with $V' \to U'$ over $U' \times_U \mathrm{Spec}(A_{\mathfrak{p}_i})$. By Limits, Lemma 32.20.1 applied n times we see that $V' \to U'$ extends to a finite étale morphism $V \to U$. □

0EKA **Lemma 58.24.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,*
(4) *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.*

Then the restriction functor $F\acute{E}t_U \to F\acute{E}t_{U_0}$ is essentially surjective.

Proof. The proof is identical to the proof of Lemma 58.24.1 using Lemma 58.23.2 in stead of Lemma 58.23.1. □

58.25. Purity in local case, II

0BPB This section is the continuation of Section 58.20. Recall that we say *purity holds* for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective where $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

0DXZ **Lemma 58.25.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

(1) *A has a dualizing complex and is f-adically complete,*
(2) *one of the following is true*
   (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
   (b) *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then $\text{depth}(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*
(3) *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$, and*
(4) *purity holds for A.*

Then purity holds for A/fA.

Proof. Denote $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$ the punctured spectrum. Similarly we have $X_0 = \text{Spec}(A/fA)$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 58.24.1 we find a finite étale morphism $V \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. By assumption (5) we find that $V \to U$ extends to a finite étale morphism $Y \to X$. Then the restriction of Y to X_0 is the desired extension of $V_0 \to U_0$. □

0BPC **Lemma 58.25.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,*
(4) *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$,*
(5) *purity holds for A.*

Then purity holds for A/fA.

Proof. The proof is identical to the proof of Lemma 58.25.1 using Lemma 58.24.2 in stead of Lemma 58.24.1. □

Now we can bootstrap the earlier results to prove that purity holds for complete intersections of dimension ≥ 3. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 23.8.

0BPD **Proposition 58.25.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 3, then purity holds for A in the sense that any finite étale cover of the punctured spectrum extends.*

Proof. By Lemma 58.20.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 58.21.3 which applies to regular rings of dimension ≥ 2.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 58.25.2 to $f_r \in A'$. This is permissible: condition (1) holds as $f_1, \ldots, f_r$ is a regular sequence, condition (2) holds as B and hence A' is complete, condition (3) holds as $A = A'/f_r A'$ is Cohen-Macaulay of dimension $\dim(A) \geq 3$, see Dualizing Complexes, Lemma 47.11.1, condition (4) holds by induction hypothesis as $\dim((A'_{f_r})_\mathfrak{p}) \geq 3$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_\mathfrak{p} = B_\mathfrak{q}/(f_1, \ldots, f_{r-1})$ for some $\mathfrak{q} \subset B$, condition (5) holds by induction hypothesis. □

58.26. Purity in local case, III

0EY8 In this section is a continuation of the discussion in Sections 58.20 and 58.25.

0EY9 **Lemma 58.26.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. Set $Y = \mathrm{Spec}(B)$ and $Y_0 = V(x_1, \ldots, x_d)$. For any open subscheme $V \subset Y$ with $V_0 = V \cap Y_0$ equal to $Y_0 \setminus \{\mathfrak{m}_B\}$ the restriction functor*

$$F\acute{E}t_V \longrightarrow F\acute{E}t_{V_0}$$

is fully faithful.

Proof. Set $I = (x_1, \ldots, x_d)$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Pushing forward modules by this affine morphism we get

$$\begin{aligned} \lim_n \Gamma(V_0, \mathcal{O}_V/I^n\mathcal{O}_V) &= \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) \\ &= \lim_n \Gamma(U, \mathcal{O}_U[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n) \\ &= \lim_n A[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n \\ &= B \end{aligned}$$

Namely, as the depth of A is ≥ 2 we have $\Gamma(U, \mathcal{O}_U) = A$, see Local Cohomology, Lemma 51.8.2. Thus for any $V \subset Y$ open as in the lemma we get

$$B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) = B$$

which implies both arrows are isomorphisms (small detail omitted). By Algebraic and Formal Geometry, Lemma 52.15.1 we conclude that $Coh(\mathcal{O}_V) \to Coh(V, I\mathcal{O}_V)$ is fully faithful on the full subcategory of finite locally free objects. Thus we conclude by Lemma 58.17.1. □

0EYA **Lemma 58.26.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. For any open $V \subset Y = \mathrm{Spec}(B)$ which contains*

1. *any prime $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A \neq \mathfrak{m}$,*
2. *the prime $\mathfrak{m}B$*

the functor $F\acute{E}t_Y \to F\acute{E}t_V$ is an equivalence. In particular purity holds for B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}$. In this case $A \to B_\mathfrak{q}$ is a flat local homomorphism and hence $\mathrm{depth}(B_\mathfrak{q}) \geq 2$ (Algebra, Lemma 10.163.2). Thus the functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1.

Let $W \to V$ be a finite étale morphism. Let $B \to C$ be the unique finite ring map such that $\mathrm{Spec}(C) \to Y$ is the finite morphism extending $W \to V$ constructed in Lemma 58.21.5. Observe that $C = \Gamma(W, \mathcal{O}_W)$.

Set $Y_0 = V(x_1, \ldots, x_d)$ and $V_0 = V \cap Y_0$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Thus we may view $W_0 = W \times_Y Y_0$ as a finite étale scheme over U. Then

$$W_0 \times_U (U \times_X Y) \quad \text{and} \quad W \times_V (U \times_X Y)$$

are schemes finite étale over $U \times_X Y$ which restrict to isomorphic finite étale schemes over V_0. By Lemma 58.26.1 applied to the open $U \times_X Y$ we obtain an isomorphism

$$W_0 \times_U (U \times_X Y) \longrightarrow W \times_V (U \times_X Y)$$

over $U \times_X Y$.

Observe that $C_0 = \Gamma(W_0, \mathcal{O}_{W_0})$ is a finite A-algebra by Lemma 58.21.5 applied to $W_0 \to U \subset X$ (exactly as we did for $B \to C$ above). Since the construction in Lemma 58.21.5 is compatible with flat base change and with change of opens, the isomorphism above induces an isomorphism

$$\Psi : C \longrightarrow C_0 \otimes_A B$$

of finite B-algebras. However, we know that $\mathrm{Spec}(C) \to Y$ is étale at all points above at least one point of Y lying over $\mathfrak{m} \in X$. Since Ψ is an isomorphism, we conclude that $\mathrm{Spec}(C_0) \to X$ is étale above $\mathfrak{m}$ (small detail omitted). Of course this means that $A \to C_0$ is finite étale and hence $B \to C$ is finite étale. □

0EYB **Lemma 58.26.3.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is smooth,*
(2) *S is Noetherian,*
(3) *for $s \in S$ with $\mathrm{depth}(\mathcal{O}_{S,s}) \leq 1$ we have $X_s = U_s$,*
(4) *$U_s \subset X_s$ is dense for all $s \in S$.*

Then $F\acute{E}t_X \to F\acute{E}t_U$ is an equivalence.

Proof. The functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1 (plus an application of Algebra, Lemma 10.163.2 to check the depth condition).

Let $\pi : V \to U$ be a finite étale morphism. Let $Y \to X$ be the finite morphism constructed in Lemma 58.21.5. We have to show that $Y \to X$ is finite étale. To show that this is true for all points $x \in X$ mapping to a given point $s \in S$ we may perform a base change by a flat morphism $S' \to S$ of Noetherian schemes such that s is in the image. This follows from the compatibility of the construction in Lemma 58.21.5 with flat base change.

After enlarging U we may assume $U \subset X$ is the maximal open over which $Y \to X$ is finite étale. Let $Z \subset X$ be the complement of U. To get a contradiction, assume $Z \neq \emptyset$. Let $s \in S$ be a point in the image of $Z \to S$ such that no strict generalization of s is in the image. Then after base change to $\mathrm{Spec}(\mathcal{O}_{S,s})$ we see that $S = \mathrm{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring of depth ≥ 2 and Z contained in the closed fibre X_s and nowhere dense in X_s. Choose a closed point $z \in Z$. Then $\kappa(z)/\kappa$ is finite (by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1). Choose a finite flat morphism $(S', s') \to (S, s)$ of local schemes realizing the residue field extension $\kappa(z)/\kappa$, see Algebra, Lemma 10.159.3. After doing a base change by $S' \to S$ we reduce to the case where $\kappa(z) = \kappa$.

By More on Morphisms, Lemma 37.38.5 there exists a locally closed subscheme $S' \subset X$ passing through z such that $S' \to S$ is étale at z. After performing the base change by $S' \to S$, we may assume there is a section $\sigma : S \to X$ such that $\sigma(s) = z$. Choose an affine neighbourhood $\mathrm{Spec}(B) \subset X$ of s. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \mathrm{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 10.139.4. Observe that $d > 0$ since otherwise we see that $X \to S$ is étale at z which would imply that z is a generic point of X_s and hence $z \in U$ by assumption (4). Similarly, if $d > 0$, then $\mathfrak{m}B^\wedge$ maps into U via the

morphism $\mathrm{Spec}(B^\wedge) \to X$. It suffices prove $Y \to X$ is finite étale after base change to $\mathrm{Spec}(B^\wedge)$. Since $B \to B^\wedge$ is flat (Algebra, Lemma 10.97.2) this follows from Lemma 58.26.2 and the uniqueness in the construction of $Y \to X$. □

0EYC **Proposition 58.26.4.** *Let $A \to B$ be a local homomorphism of local Noetherian rings. Assume A has depth ≥ 2, $A \to B$ is formally smooth for the $\mathfrak{m}_B$-adic topology, and $\dim(B) > \dim(A)$. For any open $V \subset Y = \mathrm{Spec}(B)$ which contains*

(1) *any prime $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A \neq \mathfrak{m}_A$,*
(2) *the prime $\mathfrak{m}_A B$*

the functor $F\acute{E}t_Y \to F\acute{E}t_V$ is an equivalence. In particular purity holds for B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}_A$. In this case $A \to B_\mathfrak{q}$ is a flat local homomorphism and hence $\mathrm{depth}(B_\mathfrak{q}) \geq 2$ (Algebra, Lemma 10.163.2). Thus the functor is fully faithful by Lemma 58.10.3 combined with Local Cohomology, Lemma 51.3.1.

Denote $A^\wedge$ and $B^\wedge$ the completions of A and B with respect to their maximal ideals. Observe that the assumptions of the proposition hold for $A^\wedge \to B^\wedge$, see More on Algebra, Lemmas 15.43.1, 15.43.2, and 15.37.4. By the uniqueness and compatibility with flat base change of the construction of Lemma 58.21.5 it suffices to prove the essential surjectivity for $A^\wedge \to B^\wedge$ and the inverse image of V (details omitted; compare with Lemma 58.20.4 for the case where V is the punctured spectrum). By More on Algebra, Proposition 15.49.2 this means we may assume $A \to B$ is regular.

Let $W \to V$ be a finite étale morphism. By Popescu's theorem (Smoothing Ring Maps, Theorem 16.12.1) we can write $B = \mathrm{colim}\, B_i$ as a filtered colimit of smooth A-algebras. We can pick an i and an open $V_i \subset \mathrm{Spec}(B_i)$ whose inverse image is V (Limits, Lemma 32.4.11). After increasing i we may assume there is a finite étale morphism $W_i \to V_i$ whose base change to V is $W \to V$, see Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. We may assume the complement of V_i is contained in the closed fibre of $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ as this is true for V (either choose V_i this way or use the lemma above to show this is true for i large enough). Let η be the generic point of the closed fibre of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Since $\eta \in V$, the image of η is in V_i. Hence after replacing V_i by an affine open neighbourhood of the image of the closed point of $\mathrm{Spec}(B)$, we may assume that the closed fibre of $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ is irreducible and that its generic point is contained in V_i (details omitted; use that a scheme smooth over a field is a disjoint union of irreducible schemes). At this point we may apply Lemma 58.26.3 to see that $W_i \to V_i$ extends to a finite étale morphism $\mathrm{Spec}(C_i) \to \mathrm{Spec}(B_i)$ and pulling back to $\mathrm{Spec}(B)$ we conclude that W is in the essential image of the functor $F\acute{E}t_Y \to F\acute{E}t_V$ as desired. □

58.27. Lefschetz for the fundamental group

0ELB Of course we have already proven a bunch of results of this type in the local case. In this section we discuss the projective case.

0ELC **Proposition 58.27.1.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have*

$$\mathit{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 1$$

Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is fully faithful. In fact, for any open subscheme $V \subset X$ containing Y the restriction functor $F\acute{E}t_V \to F\acute{E}t_Y$ is fully faithful.

Proof. The first statement is a formal consequence of Lemma 58.17.6 and Algebraic and Formal Geometry, Proposition 52.28.1. The second statement follows from Lemma 58.17.6 and Algebraic and Formal Geometry, Lemma 52.28.2. □

0ELD **Proposition 58.27.2.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Let $\mathcal{V}$ be the set of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have*

$$\mathit{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the restriction functor

$$\operatorname{colim}_{\mathcal{V}} F\acute{E}t_V \to F\acute{E}t_Y$$

is an equivalence.

Proof. This is a formal consequence of Lemma 58.17.4 and Algebraic and Formal Geometry, Proposition 52.28.7. □

0ELE **Proposition 58.27.3.** *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have*

$$\mathit{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

and that for $x \in X \setminus Y$ closed purity holds for $\mathcal{O}_{X,x}$. Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is an equivalence. If X or equivalently Y is connected, then

$$\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{y})$$

is an isomorphism for any geometric point $\overline{y}$ of Y.

Proof. Fully faithfulness holds by Proposition 58.27.1. By Proposition 58.27.2 any object of $F\acute{E}t_Y$ is isomorphic to the fibre product $U \times_V Y$ for some finite étale morphism $U \to V$ where $V \subset X$ is an open subscheme containing Y. The complement $T = X \setminus V$ is[5] a finite set of closed points of $X \setminus Y$. Say $T = \{x_1, \ldots, x_n\}$. By assumption we can find finite étale morphisms $V_i' \to \operatorname{Spec}(\mathcal{O}_{X,x_i})$ agreeing with $U \to V$ over $V \times_X \operatorname{Spec}(\mathcal{O}_{X,x_i})$. By Limits, Lemma 32.20.1 applied n times we see that $U \to V$ extends to a finite étale morphism $U' \to X$ as desired. See Lemma 58.8.1 for the final statement. □

58.28. Purity of ramification locus

0EA1 In this section we discuss the analogue of purity of branch locus for generically finite morphisms. Apparently, this result is due to Gabber. A special case is van der Waerden's purity theorem for the locus where a birational morphism from a normal variety to a smooth variety is not an isomorphism.

[5] Namely, T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 29.43.18) and hence finite over k (Morphisms, Lemma 29.44.11). Thus T is a finite set of closed points.

0EA2 **Lemma 58.28.1.** *Let A be a Noetherian normal local domain of dimension 2. Assume A is Nagata, has a dualizing module ω_A, and has a resolution of singularities $f : X \to \mathrm{Spec}(A)$. Let ω_X be as in Resolution of Surfaces, Remark 54.7.7. If $\omega_X \cong \mathcal{O}_X(E)$ for some effective Cartier divisor $E \subset X$ supported on the exceptional fibre, then A defines a rational singularity. If f is a minimal resolution, then $E = 0$.*

Proof. There is a trace map $Rf_*\omega_X \to \omega_A$, see Duality for Schemes, Section 48.7. By Grauert-Riemenschneider (Resolution of Surfaces, Proposition 54.7.8) we have $R^1 f_*\omega_X = 0$. Thus the trace map is a map $f_*\omega_X \to \omega_A$. Then we can consider

$$\mathcal{O}_{\mathrm{Spec}(A)} = f_*\mathcal{O}_X \to f_*\omega_X \to \omega_A$$

where the first map comes from the map $\mathcal{O}_X \to \mathcal{O}_X(E) = \omega_X$ which is assumed to exist in the statement of the lemma. The composition is an isomorphism by Divisors, Lemma 31.2.11 as it is an isomorphism over the punctured spectrum of A (by the assumption in the lemma and the fact that f is an isomorphism over the punctured spectrum) and A and ω_A are A-modules of depth 2 (by Algebra, Lemma 10.157.4 and Dualizing Complexes, Lemma 47.17.5). Hence $f_*\omega_X \to \omega_A$ is surjective whence an isomorphism. Thus $Rf_*\omega_X = \omega_A$ which by duality implies $Rf_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}(A)}$. Whence $H^1(X, \mathcal{O}_X) = 0$ which implies that A defines a rational singularity (see discussion in Resolution of Surfaces, Section 54.8 in particular Lemmas 54.8.7 and 54.8.1). If f is minimal, then $E = 0$ because the map $f^*\omega_A \to \omega_X$ is surjective by a repeated application of Resolution of Surfaces, Lemma 54.9.7 and $\omega_A \cong A$ as we've seen above. □

0EA3 **Lemma 58.28.2.** *Let $f : X \to \mathrm{Spec}(A)$ be a finite type morphism. Let $x \in X$ be a point. Assume*

(1) *A is an excellent regular local ring,*
(2) *$\mathcal{O}_{X,x}$ is normal of dimension 2,*
(3) *f is étale outside of $\overline{\{x\}}$.*

Then f is étale at x.

Proof. We first replace X by an affine open neighbourhood of x. Observe that $\mathcal{O}_{X,x}$ is an excellent local ring (More on Algebra, Lemma 15.52.2). Thus we can choose a minimal resolution of singularities $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$, see Resolution of Surfaces, Theorem 54.14.5. After possibly replacing X by an affine open neighbourhood of x we can find a proper morphism $b : X' \to X$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) = W$, see Limits, Lemma 32.20.1. After shrinking X further, we may assume X' is regular. Namely, we know W is regular and X' is excellent and the regular locus of the spectrum of an excellent ring is open. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is projective (as a sequence of normalized blowing ups), we may assume after shrinking X that b is projective (details omitted). Let $U = X \setminus \overline{\{x\}}$. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is an isomorphism over the punctured spectrum, we may assume $b : X' \to X$ is an isomorphism over U. Thus we may and will think of U as an open subscheme of X' as well. Set $f' = f \circ b : X' \to \mathrm{Spec}(A)$.

Since A is regular we see that $\mathcal{O}_Y$ is a dualizing complex for Y. Hence $f^!\mathcal{O}_Y$ is a dualzing complex on X (Duality for Schemes, Lemma 48.17.7). The Cohen-Macaulay locus of X is open by Duality for Schemes, Lemma 48.23.1 (this can also be proven using excellency). Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, after shrinking

X we may assume X is Cohen-Macaulay. Observe that an étale morphism is a local complete intersection. Thus Duality for Schemes, Lemma 48.29.3 applies with $r = 0$ and we get a map

$$\mathcal{O}_X \longrightarrow \omega_{X/Y} = H^0(f^!\mathcal{O}_Y)$$

which is an isomorphism over $X \setminus \overline{\{x\}}$. Since $\omega_{X/Y}$ is (S_2) by Duality for Schemes, Lemma 48.21.5 we find this map is an isomorphism by Divisors, Lemma 31.2.11. This already shows that X and in particular $\mathcal{O}_{X,x}$ is Gorenstein.

Set $\omega_{X'/Y} = H^0((f')^!\mathcal{O}_Y)$. Arguing in exactly the same manner as above we find that $(f')^!\mathcal{O}_Y = \omega_{X'/Y}[0]$ is a dualizing complex for X'. Since X' is regular the morphism $X' \to Y$ is a local complete intersection morphism, see More on Morphisms, Lemma 37.62.11. By Duality for Schemes, Lemma 48.29.2 there exists a map

$$\mathcal{O}_{X'} \longrightarrow \omega_{X'/Y}$$

which is an isomorphism over U. We conclude $\omega_{X'/Y} = \mathcal{O}_{X'}(E)$ for some effective Cartier divisor $E \subset X'$ disjoint from U.

Since $\omega_{X/Y} = \mathcal{O}_Y$ we see that $\omega_{X'/Y} = b^!f^!\mathcal{O}_Y = b^!\mathcal{O}_X$. Returning to $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ we see that $\omega_W = \mathcal{O}_W(E|_W)$. By Lemma 58.28.1 we find $E|_W = 0$. This means that $f' : X' \to Y$ is étale by (the already used) Duality for Schemes, Lemma 48.29.2. This immediately finishes the proof, as étaleness of f' forces b to be an isomorphism. □

0EA4 **Lemma 58.28.3** (Purity of ramification locus)**.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume*

(1) *$\mathcal{O}_{X,x}$ is normal of dimension ≥ 1,*
(2) *$\mathcal{O}_{Y,y}$ is regular,*
(3) *f is locally of finite type, and*
(4) *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is étale at x'.*

Then f is étale at x.

This result for complex spaces can be found on page 170 of [**Fis76**]. In general this is [**Zon14**, Theorem 2.4] attributed to Gabber.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x})$.

An uninteresting case is $d = 1$ since in that case the morphism f is étale at x by assumption. Assume $d \geq 2$.

We can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ without affecting the conclusion of the lemma, see Morphisms, Lemma 29.36.17. Thus we may assume $Y = \mathrm{Spec}(A)$ where A is a regular local ring and y corresponds to the maximal ideal $\mathfrak{m}$ of A.

Let $x' \rightsquigarrow x$ be a specialization with $x' \neq x$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of $\mathcal{O}_{X,x}$. If x' is not a generic point of X, then $1 \leq \dim(\mathcal{O}_{X,x'}) < d$ and we conclude that f is étale at x' by induction hypothesis. Thus we may assume that f is étale at all points specializing to x. Since the set of points where f is étale is open in X (by definition) we may after replacing X by an open neighbourhood of x assume that f is étale away from $\overline{\{x\}}$. In particular, we see that f is étale except at points lying over the closed point $y \in Y = \mathrm{Spec}(A)$.

Let $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $x' \in X'$ be the unique point lying over x. By the above we see that X' is étale over $\mathrm{Spec}(A^\wedge)$ away from the closed fibre and hence X' is normal away from the closed fibre. Since X is normal we conclude that X' is normal by Resolution of Surfaces, Lemma 54.11.6. Then if we can

show $X' \to \mathrm{Spec}(A^\wedge)$ is étale at x', then f is étale at x (by the aforementioned Morphisms, Lemma 29.36.17). Thus we may and do assume A is a regular complete local ring.

The case $d = 2$ now follows from Lemma 58.28.2.

Assume $d > 2$. Let $t \in \mathfrak{m}$, $t \notin \mathfrak{m}^2$. Set $Y_0 = \mathrm{Spec}(A/tA)$ and $X_0 = X \times_Y Y_0$. Then $X_0 \to Y_0$ is étale away from the fibre over the closed point. Since $d > 2$ we have $\dim(\mathcal{O}_{X_0,x}) = d - 1$ is ≥ 2. The normalization $X_0' \to X_0$ is surjective and finite (as we're working over a complete local ring and such rings are Nagata). Let $x' \in X_0'$ be a point mapping to x. By induction hypothesis the morphism $X_0' \to Y$ is étale at x'. From the inclusions $\kappa(y) \subset \kappa(x) \subset \kappa(x')$ we conclude that $\kappa(x)$ is finite over $\kappa(y)$. Hence x is a closed point of the fibre of $X \to Y$ over y. But since x is also a generic point of this fibre, we conclude that f is quasi-finite at x and we reduce to the case of purity of branch locus, see Lemma 58.21.4. □

58.29. Affineness of complement of ramification locus

0ECA Let $f : X \to Y$ be a finite type morphism of Noetherian schemes with X normal and Y regular. Let $V \subset X$ be the maximal open subscheme where f is étale. The discussion in [**DG67**, Chapter IV, Section 21.12] suggests that $V \to X$ might be an affine morphism. Observe that if $V \to X$ is affine, then we deduce purity of ramification locus (Lemma 58.28.3) by using Divisors, Lemma 31.16.4. Thus affineness of $V \to X$ is a "strong" form of purity for the ramification locus. In this section we prove $V \to X$ is affine when X and Y are equicharacteristic and excellent, see Theorem 58.29.3. It seems reasonable to guess the result remains true for X and Y of mixed characteristic (but still excellent).

0ECB **Lemma 58.29.1.** *Let $(A, \mathfrak{m})$ be a regular local ring which contains a field. Let $f : V \to \mathrm{Spec}(A)$ be étale and quasi-compact. Assume that $\mathfrak{m} \notin f(V)$ and assume that $g : V \to \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is affine. Then $H^i(V, \mathcal{O}_V)$, $i > 0$ is isomorphic to a direct sum of copies of the injective hull of the residue field of A.*

Proof. Denote $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ the punctured spectrum. Thus $g : V \to U$ is affine. We have $H^i(V, \mathcal{O}_V) = H^i(U, g_*\mathcal{O}_V)$ by Cohomology of Schemes, Lemma 30.2.4. The $\mathcal{O}_U$-module $g_*\mathcal{O}_V$ is quasi-coherent by Schemes, Lemma 26.24.1. For any quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$ the cohomology $H^i(U, \mathcal{F})$, $i > 0$ is $\mathfrak{m}$-power torsion, see for example Local Cohomology, Lemma 51.2.2. In particular, the A-modules $H^i(V, \mathcal{O}_V)$, $i > 0$ are $\mathfrak{m}$-power torsion. For any flat ring map $A \to A'$ we have $H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'})$ where $V' = V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ by flat base change Cohomology of Schemes, Lemma 30.5.2. If we take A' to be the completion of A (flat by More on Algebra, Section 15.43), then we see that

$$H^i(V, \mathcal{O}_V) = H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'}), \quad \text{for } i > 0$$

The first equality by the torsion property we just proved and More on Algebra, Lemma 15.90.3. Moreover, the injective hull of the residue field k is the same for A and A', see Dualizing Complexes, Lemma 47.7.4. In this way we reduce to the case $A = k[[x_1, \ldots, x_d]]$, see Algebra, Section 10.160.

Assume the characteristic of k is $p > 0$. Since $F : A \to A$, $a \mapsto a^p$ is flat (Local Cohomology, Lemma 51.17.6) and since $V \times_{\mathrm{Spec}(A),\mathrm{Spec}(F)} \mathrm{Spec}(A) \cong V$ as schemes

over $\mathrm{Spec}(A)$ by Étale Morphisms, Lemma 41.14.3 the above gives $H^i(V, \mathcal{O}_V) \otimes_{A,F} A \cong H^i(V, \mathcal{O}_V)$. Thus we get the result by Local Cohomology, Lemma 51.18.2.

Assume the characteristic of k is 0. By Local Cohomology, Lemma 51.19.3 there are additive operators D_j, $j = 1, \ldots, d$ on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to $\partial_j = \partial/\partial x_j$. Thus we get the result by Local Cohomology, Lemma 51.18.1. □

0ECC **Lemma 58.29.2.** *In the situation of Lemma 58.29.1 assume that $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$. Then V is affine.*

Proof. Let $k = A/\mathfrak{m}$. Since $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = \emptyset$, by cohomology and base change we have

$$R\Gamma(V, \mathcal{O}_V) \otimes^{\mathbf{L}}_A k = 0$$

See Derived Categories of Schemes, Lemma 36.22.5. Thus there is a spectral sequence (More on Algebra, Example 15.62.4)

$$E_2^{p,q} = \mathrm{Tor}_{-p}(k, H^q(V, \mathcal{O}_V)), \quad d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

and $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ converging to zero. By Lemma 58.29.1, Dualizing Complexes, Lemma 47.21.9, and our assumption $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$ we conclude that there is no nonzero differential entering or leaving the $(p, q) = (0, 0)$ spot. Thus $H^0(V, \mathcal{O}_V) \otimes_A k = 0$. This means that if $\mathfrak{m} = (x_1, \ldots, x_d)$ then we have an open covering $V = \bigcup V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ by affine open subschemes $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ (because V is affine over the punctured spectrum of A) such that $x_1, \ldots, x_d$ generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$. This implies V is affine by Properties, Lemma 28.27.3. □

0ECD **Theorem 58.29.3.** *Let Y be an excellent regular scheme over a field. Let $f : X \to Y$ be a finite type morphism of schemes with X normal. Let $V \subset X$ be the maximal open subscheme where f is étale. Then the inclusion morphism $V \to X$ is affine.*

Proof. Let $x \in X$ with image $y \in Y$. It suffices to prove that $V \cap W$ is affine for some affine open neighbourhood W of x. Since $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the limit of the schemes W, this holds if and only if

$$V_x = V \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$$

is affine (Limits, Lemma 32.4.13). Thus, if the theorem holds for the morphism $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$, then the theorem holds. In particular, we may assume Y is regular of finite dimension, which allows us to do induction on the dimension $d = \dim(Y)$. Combining this with the same argument again, we may assume that Y is local with closed point y and that $V \cap (X \setminus f^{-1}(\{y\}) \to X \setminus f^{-1}(\{y\})$ is affine.

Let $x \in X$ be a point lying over y. If $x \in V$, then there is nothing to prove. Observe that $f^{-1}(\{y\}) \cap V$ is a finite set of closed points (the fibres of an étale morphism are discrete). Thus after replacing X by an affine open neighbourhood of x we may assume $y \notin f(V)$. We have to prove that V is affine.

Let $e(V)$ be the maximum i with $H^i(V, \mathcal{O}_V) \neq 0$. As X is affine the integer $e(V)$ is the maximum of the numbers $e(V_x)$ where $x \in X \setminus V$, see Local Cohomology, Lemma 51.4.6 and the characterization of cohomological dimension in Local Cohomology, Lemma 51.4.1. We have $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 1$ by Local Cohomology, Lemma

51.4.7. If $\dim(\mathcal{O}_{X,x}) \geq 2$ then purity of ramification locus (Lemma 58.28.3) shows that V_x is strictly smaller than the punctured spectrum of $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is normal and excellent, this implies $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 2$ by Hartshorne-Lichtenbaum vanishing (Local Cohomology, Lemma 51.16.7). On the other hand, since $X \to Y$ is of finite type and $V \subset X$ is dense (after possibly replacing X by the closure of V), we see that $\dim(\mathcal{O}_{X,x}) \leq d$ by the dimension formula (Morphisms, Lemma 29.52.1). Whence $e(V) \leq \max(0, d-2)$. Thus V is affine by Lemma 58.29.2 if $d \geq 2$. If $d = 1$ or $d = 0$, then the punctured spectrum of $\mathcal{O}_{Y,y}$ is affine and hence V is affine. □

58.30. Specialization maps in the smooth proper case

0BUQ In this section we discuss the following result. Let $f : X \to S$ be a proper smooth morphism of schemes. Let $s \rightsquigarrow s'$ be a specialization of points in S. Then the specialization map

$$sp : \pi_1(X_{\overline{s}}) \longrightarrow \pi_1(X_{\overline{s}'})$$

of Section 58.16 is surjective and

(1) if the characteristic of $\kappa(s')$ is zero, then it is an isomorphism, or
(2) if the characteristic of $\kappa(s')$ is $p > 0$, then it induces an isomorphism on maximal prime-to-p quotients.

0C0P **Lemma 58.30.1.** *Let $f : X \to S$ be a flat proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If X_s is geometrically reduced, then the specialization map $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is surjective.*

Proof. Since X_s is geometrically reduced, we may assume all fibres are geometrically reduced after possibly shrinking S, see More on Morphisms, Lemma 37.26.7. Let $\mathcal{O}_{S,s} \to A \to \kappa(\overline{s}')$ be as in the construction of the specialization map, see Section 58.16. Thus it suffices to show that

$$\pi_1(X_{\overline{s}'}) \to \pi_1(X_A)$$

is surjective. This follows from Proposition 58.15.2 and $\pi_1(\mathrm{Spec}(A)) = \{1\}$. □

0C0Q **Proposition 58.30.2.** *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic to $\kappa(s)$ is zero, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is an isomorphism.

Proof. The map is surjective by Lemma 58.30.1. Thus we have to show it is injective.

We may assume S is affine. Then S is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to S$ is the base change of $X_0 \to S_0$ where S_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to S_0$ is smooth and proper. See Limits, Lemma 32.10.1, 32.8.9, and 32.13.1. By Lemma 58.16.1 we reduce to the case where the base is Noetherian.

Applying Lemma 58.16.4 we reduce to the case where the base S is the spectrum of a strictly henselian discrete valuation ring A and we are looking at the specialization map over A. Let K be the fraction field of A. Choose an algebraic closure $\overline{K}$ which corresponds to a geometric generic point $\overline{\eta}$ of $\mathrm{Spec}(A)$. For $\overline{K}/L/K$ finite separable,

let $B \subset L$ be the integral closure of A in L. This is a discrete valuation ring by More on Algebra, Remark 15.112.6.

Let $X \to \mathrm{Spec}(A)$ be as in the previous paragraph. To show injectivity of the specialization map it suffices to prove that every finite étale cover V of $X_{\overline{\eta}}$ is the base change of a finite étale cover $Y \to X$. Namely, then $\pi_1(X_{\overline{\eta}}) \to \pi_1(X) = \pi_1(X_s)$ is injective by Lemma 58.4.4.

Given V we can first descend V to $V' \to X_{K^{sep}}$ by Lemma 58.14.2 and then to $V'' \to X_L$ by Lemma 58.14.1. Let $Z \to X_B$ be the normalization of X_B in V''. Observe that Z is normal and that $Z_L = V''$ as schemes over X_L. Hence $Z \to X_B$ is finite étale over the generic fibre. The problem is that we do not know that $Z \to X_B$ is everywhere étale. Since $X \to \mathrm{Spec}(A)$ has geometrically connected smooth fibres, we see that the special fibre X_s is geometrically irreducible. Hence the special fibre of $X_B \to \mathrm{Spec}(B)$ is irreducible; let ξ_B be its generic point. Let $\xi_1, \ldots, \xi_r$ be the points of Z mapping to ξ_B. Our first (and it will turn out only) problem is now that the extensions

$$\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$$

of discrete valuation rings may be ramified. Let e_i be the ramification index of this extension. Note that since the characteristic of $\kappa(s)$ is zero, the ramification is tame!

To get rid of the ramification we are going to choose a further finite separable extension $K^{sep}/L'/L/K$ such that the ramification index e of the induced extensions B'/B is divisible by e_i. Consider the normalized base change Z' of Z with respect to $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$, see discussion in More on Morphisms, Section 37.65. Let $\xi_{i,j}$ be the points of Z' mapping to $\xi_{B'}$ and to ξ_i in Z. Then the local rings

$$\mathcal{O}_{Z',\xi_{i,j}}$$

are localizations of the integral closure of $\mathcal{O}_{Z,\xi_i}$ in $L' \otimes_L F_i$ where F_i is the fraction field of $\mathcal{O}_{Z,\xi_i}$; details omitted. Hence Abhyankar's lemma (More on Algebra, Lemma 15.115.4) tells us that

$$\mathcal{O}_{X_{B'},\xi_{B'}} \subset \mathcal{O}_{Z',\xi_{i,j}}$$

is unramified. We conclude that the morphism $Z' \to X_{B'}$ is étale away from codimension 1. Hence by purity of branch locus (Lemma 58.21.4) we see that $Z' \to X_{B'}$ is finite étale!

However, since the residue field extension induced by $A \to B'$ is trivial (as the residue field of A is algebraically closed being separably closed of characteristic zero) we conclude that Z' is the base change of a finite étale cover $Y \to X$ by applying Lemma 58.9.1 twice (first to get Y over A, then to prove that the pullback to B is isomorphic to Z'). This finishes the proof. □

Let G be a profinite group. Let p be a prime number. The *maximal prime-to-p quotient* is by definition

$$G' = \lim_{U \subset G \text{ open, normal, index prime to } p} G/U$$

If X is a connected scheme and p is given, then the maximal prime-to-p quotient of $\pi_1(X)$ is denoted $\pi_1'(X)$.

0C0R **Theorem 58.30.3.** *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic of $\kappa(s)$ is p, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is surjective and induces an isomorphism

$$\pi_1'(X_{\overline{s}'}) \cong \pi_1'(X_{\overline{s}})$$

of the maximal prime-to-p quotients

Proof. This is proved in exactly the same manner as Proposition 58.30.2 with the following differences

(1) Given X/A we no longer show that the functor $F\acute{E}t_X \to F\acute{E}t_{X_{\overline{\eta}}}$ is essentially surjective. We show only that Galois objects whose Galois group has order prime to p are in the essential image. This will be enough to conclude the injectivity of $\pi_1'(X_{\overline{s}'}) \to \pi_1'(X_{\overline{s}})$ by exactly the same argument.
(2) The extensions $\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$ are tamely ramified as the associated extension of fraction fields is Galois with group of order prime to p. See More on Algebra, Lemma 15.113.2.
(3) The extension κ_B/κ_A is no longer necessarily trivial, but it is purely inseparable. Hence the morphism $X_{\kappa_B} \to X_{\kappa_A}$ is a universal homeomorphism and induces an isomorphism of fundamental groups by Proposition 58.8.4.

□

58.31. Tame ramification

0BSE Let $X \to Y$ be a finite étale morphism of schemes of finite type over $\mathbf{Z}$. There are many ways to define what it means for f to be tamely ramified at ∞. The article [**KS10**] discusses to what extent these notions agree.

In this section we discuss a different more elementary question which precedes the notion of tameness at infinity. Please compare with the (slightly different) discussion in [**GM71**]. Assume we are given

(1) a locally Noetherian scheme X,
(2) a dense open $U \subset X$,
(3) a finite étale morphism $f : Y \to U$

such that for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring. Setting K_ξ equal to the fraction field of $\mathcal{O}_{X,\xi}$ we obtain a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(K_\xi) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{X,\xi}) & \longrightarrow & X \end{array}$$

of schemes. In particular, we see that $Y \times_U \mathrm{Spec}(K_\xi)$ is the spectrum of a finite separable algebra L_ξ/K_ξ. Then we say *Y is unramified over X in codimension* 1, resp. *Y is tamely ramified over X in codimension* 1 if L_ξ/K_ξ is unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$ for every (Z,ξ) as above, see More on Algebra, Definition 15.112.7. More precisely, we decompose L_ξ into a product of

finite separable field extensions of K_ξ and we require each of these to be unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$.

0EYD **Lemma 58.31.1.** *Let $X' \to X$ be a morphism of locally Noetherian schemes. Let $U \subset X$ be a dense open. Assume*

(1) *$U' = f^{-1}(U)$ is dense open in X',*
(2) *for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring,*
(3) *for every prime divisor $Z' \subset X'$ with $Z' \cap U' = \emptyset$ the local ring $\mathcal{O}_{X',\xi'}$ of X' at the generic point ξ' of Z' is a discrete valuation ring,*
(4) *if $\xi' \in X'$ is as in (3), then $\xi = f(\xi')$ is as in (2).*

Then if $f : Y \to U$ is finite étale and Y is unramified, resp. tamely ramified over X in codimension 1, then $Y' = Y \times_X X' \to U'$ is finite étale and Y' is unramified, resp. tamely ramified over X' in codimension 1.

Proof. The only interesting fact in this lemma is the commutative algebra result given in More on Algebra, Lemma 15.115.9. □

Using the terminology introduced above, we can reformulate our purity results obtained earlier in the following pleasing manner.

0H2W **Lemma 58.31.2.** *Let X be a locally Noetherian scheme. Let $U \subset X$ be open and dense. Let $Y \to U$ be a finite étale morphism. Assume*

(1) *Y is unramified over X in codimension 1, and*
(2) *$\mathcal{O}_{X,x}$ is regular for all $x \in X \setminus U$.*

Then there exists a finite étale morphism $Y' \to X$ whose restriction to U is Y.

Proof. Let $\xi \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$ of codimension 1. Then $\mathcal{O}_{X,\xi}$ is a discrete valuation ring. As in the discussion above, write $Y \times_U \mathrm{Spec}(K_\xi) = \mathrm{Spec}(L_\xi)$. Denote B_ξ the integral closure of $\mathcal{O}_{X,\xi}$ in L_ξ. Our assumption that Y is unramified over X in codimension 1 signifies that $\mathcal{O}_{X,\xi} \to B_\xi$ is finite étale. Thus we get $Y_\xi \to \mathrm{Spec}(\mathcal{O}_{X,\xi})$ finite étale and an isomorphism

$$Y \times_U \mathrm{Spec}(K_\xi) \cong Y_\xi \times_{\mathrm{Spec}(\mathcal{O}_{X,\xi})} \mathrm{Spec}(K_\xi)$$

over $\mathrm{Spec}(K_\xi)$. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset X$ containing ξ and a morphism $Y' \to U'$ of finite presentation whose restriction to U recovers Y and whose restriction to $\mathrm{Spec}(\mathcal{O}_{X,\xi})$ recovers Y_ξ. Finally, the morphism $Y' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 32.20.4. Repeating the argument with the other generic points of $X \setminus U$ of codimension 1 we may assume that we have a finite étale morphism $Y' \to U'$ extending $Y \to U$ to an open subscheme containing $U' \subset X$ containing U and all codimension 1 points of $X \setminus U$. We finish by applying Lemma 58.21.6 to $Y' \to U'$. Namely, all local rings $\mathcal{O}_{X,x}$ for $x \in X \setminus U'$ are regular and have $\dim(\mathcal{O}_{X,x}) \geq 2$. Hence we have purity for $\mathcal{O}_{X,x}$ by Lemma 58.21.3. □

0EYE **Lemma 58.31.3.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is unramified over X in codimension 1, then there exists a finite étale morphism $Y' \to X$ whose restriction to $X \setminus D$ is Y.*

Proof. This is a special case of Lemma 58.31.2. First, D is nowhere dense in X (see discussion in Divisors, Section 31.13) and hence $X \setminus D$ is dense in X. Second, the ring $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$ by Algebra, Lemma 10.106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. □

0EYF **Example 58.31.4** (Standard tamely ramified morphism)**.** Let A be a Noetherian ring. Let $f \in A$ be a nonzerodivisor such that A/fA is reduced. This implies that $A_{\mathfrak{p}}$ is a discrete valuation ring with uniformizer f for any minimal prime $\mathfrak{p}$ over f. Let $e \geq 1$ be an integer which is invertible in A. Set

$$C = A[x]/(x^e - f)$$

Then $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is a finite locally free morphism which is étale over the spectrum of A_f. The finite étale morphism

$$\mathrm{Spec}(C_f) \longrightarrow \mathrm{Spec}(A_f)$$

is tamely ramified over $\mathrm{Spec}(A)$ in codimension 1. The tameness follows immediately from the characterization of tamely ramified extensions in More on Algebra, Lemma 15.115.7.

Here is a version of Abhyankar's lemma for regular divisors.

0EYG **Lemma 58.31.5** (Abhyankar's lemma for regular divisor)**.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is tamely ramified over X in codimension 1, then étale locally on X the morphism $Y \to X$ is as given as a finite disjoint union of standard tamely ramified morphisms as described in Example 58.31.4.*

Proof. Before we start we note that $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$. This follows from Algebra, Lemma 10.106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. Below we will also use that regular rings are normal, see Algebra, Lemma 10.157.5.

To prove the lemma we may work locally on X. Thus we may assume $X = \mathrm{Spec}(A)$ and $D \subset X$ is given by a nonzerodivisor $f \in A$. Then $Y = \mathrm{Spec}(B)$ as a finite étale scheme over A_f. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of A over f. Then $A_i = A_{\mathfrak{p}_i}$ is a discrete valuation ring; denote its fraction field K_i. By assumption

$$K_i \otimes_{A_f} B = \prod L_{ij}$$

is a finite product of fields each tamely ramified with respect to A_i. Choose $e \geq 1$ sufficiently divisible (namely, divisible by all ramification indices for L_{ij} over A_i as in More on Algebra, Remark 15.112.6). Warning: at this point we do not know that e is invertible on A.

Consider the finite free A-algebra

$$A' = A[x]/(x^e - f)$$

Observe that $f' = x$ is a nonzerodivisor in A' and that $A'/f'A' \cong A/fA$ is a regular ring. Set $B' = B \otimes_A A' = B \otimes_{A_f} A'_{f'}$. By Abhyankar's lemma (More on Algebra, Lemma 15.115.4) we see that $\mathrm{Spec}(B')$ is unramified over $\mathrm{Spec}(A')$ in codimension 1. Namely, by Lemma 58.31.1 we see that $\mathrm{Spec}(B')$ is still at least tamely ramified over $\mathrm{Spec}(A')$ in codimension 1. But Abhyankar's lemma tells us that the ramification indices have all become equal to 1. By Lemma 58.31.3 we

conclude that $\mathrm{Spec}(B') \to \mathrm{Spec}(A'_{f'})$ extends to a finite étale morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A')$.

For a point $x \in D$ corresponding to $\mathfrak{p} \in V(f)$ denote A^{sh} a strict henselization of $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$. Observe that A^{sh} and $A^{sh}/fA^{sh} = (A/fA)^{sh}$ (Algebra, Lemma 10.156.4) are regular local rings, see More on Algebra, Lemma 15.45.10. Observe that A' has a unique prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ with identical residue field. Thus

$$(A')^{sh} = A^{sh} \otimes_A A' = A^{sh}[x]/(x^e - f)$$

is a strictly henselian local ring finite over A^{sh} (Algebra, Lemma 10.156.3). Since f' is a nonzerodivisor in $(A')^{sh}$ and since $(A')^{sh}/f'(A')^{sh} = A^{sh}/fA^{sh}$ is regular, we conclude that $(A')^{sh}$ is a regular local ring (see above). Observe that the induced extension

$$Q(A^{sh}) \subset Q((A')^{sh}) = Q(A^{sh})[x]/(x^e - f)$$

of fraction fields has degree e (and not less). Since $A' \to C$ is finite étale we see that $A^{sh} \otimes_A C$ is a finite product of copies of $(A')^{sh}$ (Algebra, Lemma 10.153.6). We have the inclusions

$$A^{sh}_f \subset A^{sh} \otimes_A B \subset A^{sh} \otimes_A B' = A^{sh} \otimes_A C_{f'}$$

and each of these rings is Noetherian and normal; this follows from Algebra, Lemma 10.163.9 for the ring in the middle. Taking total quotient rings, using the product decomposition of $A^{sh} \otimes_A C$ and using Fields, Lemma 9.24.3 we conclude that there is an isomorphism

$$Q(A^{sh}) \otimes_A B \cong \prod\nolimits_{i \in I} F_i, \quad F_i \cong Q(A^{sh})[x]/(x^{e_i} - f)$$

of $Q(A^{sh})$-algebras for some finite set I and integers $e_i | e$. Since $A^{sh} \otimes_A B$ is a normal ring, it must be the integral closure of A^{sh} in its total quotient ring. We conclude that we have an isomorphism

$$A^{sh} \otimes_A B \cong \prod A^{sh}_f[x]/(x^{e_i} - f)$$

over A^{sh}_f because the algebras $A^{sh}[x]/(x^{e_i} - f)$ are regular and hence normal. The discriminant of $A^{sh}[x]/(x^{e_i} - f)$ over A^{sh} is $e_i^{e_i} f^{e_i - 1}$ (up to sign; calculation omitted). Since $A_f \to B$ is finite étale we see that e_i must be invertible in A^{sh}_f. On the other hand, since $A_f \to B$ is tamely ramified over $\mathrm{Spec}(A)$ in codimension 1, by Lemma 58.31.1 the ring map $A^{sh}_f \to A^{sh} \otimes_A B$ is tamely ramified over $\mathrm{Spec}(A^{sh})$ in codimension 1. This implies e_i is nonzero in A^{sh}/fA^{sh} (as it must map to an invertible element of the fraction field of this domain by definition of tamely ramified extensions). We conclude that $V(e_i) \subset \mathrm{Spec}(A^{sh})$ has codimension ≥ 2 which is absurd unless it is empty. In other words, e_i is an invertible element of A^{sh}. We conclude that the pullback of Y to $\mathrm{Spec}(A^{sh})$ is indeed a finite disjoint union of standard tamely ramified morphisms.

To finish the proof, we write $A^{sh} = \mathrm{colim}\, A_\lambda$ as a filtered colimit of étale A-algebras A_λ. The isomorphism

$$A^{sh} \otimes_A B \cong \prod\nolimits_{i \in I} A^{sh}_f[x]/(x^{e_i} - f)$$

descends to an isomorphism

$$A_\lambda \otimes_A B \cong \prod\nolimits_{i \in I} (A_\lambda)_f[x]/(x^{e_i} - f)$$

for suitably large λ. After increasing λ a bit more we may assume e_i is invertible in A_λ. Then $\mathrm{Spec}(A_\lambda) \to \mathrm{Spec}(A)$ is the desired étale neighbourhood of x and the proof is complete. □

0EYH **Lemma 58.31.6.** *In the situation of Lemma 58.31.5 the normalization of X in Y is a finite locally free morphism $\pi : Y' \to X$ such that*

(1) *the restriction of Y' to $X \setminus D$ is isomorphic to Y,*
(2) *$D' = \pi^{-1}(D)_{red}$ is an effective Cartier divisor on Y', and*
(3) *D' is a regular scheme.*

Moreover, étale locally on X the morphism $Y' \to X$ is a finite disjoint union of morphisms

$$\mathrm{Spec}(A[x]/(x^e - f)) \to \mathrm{Spec}(A)$$

where A is a Noetherian ring, $f \in A$ is a nonzerodivisor with A/fA regular, and $e \geq 1$ is invertible in A.

Proof. This is just an addendum to Lemma 58.31.5 and in fact the truth of this lemma follows almost immediately if you've read the proof of that lemma. But we can also deduce the lemma from the result of Lemma 58.31.5. Namely, taking the normalization of X in Y commutes with étale base change, see More on Morphisms, Lemma 37.19.2. Hence we see that we may prove the statements on the local structure of $Y' \to X$ étale locally on X. Thus, by Lemma 58.31.5 we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring, that we have a nonzerodivisor $f \in A$ such that A/fA is regular, and that Y is a finite disjoint union of spectra of rings $A_f[x]/(x^e - f)$ where e is invertible in A. We omit the verification that the integral closure of A in $A_f[x]/(x^e - f)$ is equal to $A' = A[x]/(x^e - f)$. (To see this argue that the localizations of A' at primes lying over (f) are regular.) We omit the details. □

0EYI **Lemma 58.31.7.** *In the situation of Lemma 58.31.5 let $Y' \to X$ be as in Lemma 58.31.6. Let R be a discrete valuation ring with fraction field K. Let*

$$t : \mathrm{Spec}(R) \to X$$

be a morphism such that the scheme theoretic inverse image $t^{-1}D$ is the reduced closed point of $\mathrm{Spec}(R)$.

(1) *If $t|_{\mathrm{Spec}(K)}$ lifts to a point of Y, then we get a lift $t' : \mathrm{Spec}(R) \to Y'$ such that $Y' \to X$ is étale along $t'(\mathrm{Spec}(R))$.*
(2) *If $\mathrm{Spec}(K) \times_X Y$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}(K)$, then $Y' \to X$ is finite étale over an open neighbourhood of $t(\mathrm{Spec}(R))$.*

Proof. By the valuative criterion of properness applied to the finite morphism $Y' \to X$ we see that $\mathrm{Spec}(K)$-valued points of Y matching $t|_{\mathrm{Spec}(K)}$ as maps into X lift uniquely to morphisms $t' : \mathrm{Spec}(R) \to Y'$. Thus statement (1) make sense.

Choose an étale neighbourhood $(U, u) \to (X, t(\mathfrak{m}_R))$ such that $U = \mathrm{Spec}(A)$ and such that $Y' \times_X U \to U$ has a description as in Lemma 58.31.6 for some $f \in A$. Then $\mathrm{Spec}(R) \times_X U \to \mathrm{Spec}(R)$ is étale and surjective. If R' denotes the local ring of $\mathrm{Spec}(R) \times_X U$ lying over the closed point of $\mathrm{Spec}(R)$, then R' is a discrete valuation ring and $R \subset R'$ is an unramified extension of discrete valuation rings (More on Algebra, Lemma 15.44.4). The assumption on t signifies that the map $A \to R'$ corresponding to

$$\mathrm{Spec}(R') \to \mathrm{Spec}(R) \times_X U \to U$$

maps f to a uniformizer $\pi \in R'$. Now suppose that

$$Y' \times_X U = \coprod\nolimits_{i \in I} \mathrm{Spec}(A[x]/(x^{e_i} - f))$$

for some $e_i \geq 1$. Then we see that

$$\mathrm{Spec}(R') \times_U (Y' \times_X U) = \coprod\nolimits_{i \in I} \mathrm{Spec}(R'[x]/(x^{e_i} - \pi))$$

The rings $R'[x]/(x^{e_i} - f)$ are discrete valuation rings (More on Algebra, Lemma 15.115.2) and hence have no map into the fraction field of R' unless $e_i = 1$.

Proof of (1). In this case the map $t' : \mathrm{Spec}(R) \to Y'$ base changes to determine a corresponding map $t'' : \mathrm{Spec}(R') \to Y' \times_X U$ which must map into a summand corresponding to $i \in I$ with $e_i = 1$ by the discussion above. Thus clearly we see that $Y' \times_X U \to U$ is étale along the image of t''. Since being étale is a property one can check after étale base chamge, this proves (1).

Proof of (2). In this case the assumption implies that $e_i = 1$ for all $i \in I$. Thus $Y' \times_X U \to U$ is finite étale and we conclude as before. □

0EYJ **Lemma 58.31.8.** *Let S be an integral normal Noetherian scheme with generic point η. Let $f : X \to S$ be a smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 58.7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.*

Proof. First assume $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring R with closed point $s \in S$. Then X_s is an effective Cartier divisor in X and X_s is regular as a scheme smooth over a field. Moreover the generic fibre X_η is the open subscheme $X \setminus X_s$. It follows from More on Algebra, Lemma 15.113.2 and the assumption on G that Z is tamely ramified over X in codimension 1. Let $Z' \to X$ be as in Lemma 58.31.6. Observe that the action of G on Z extends to an action of G on Z'. By Lemma 58.31.7 we see that $Z' \to X$ is finite étale over an open neighbourhood of $\sigma(y)$. Since X_s is irreducible, this implies $Z \to X_\eta$ is unramified over X in codimension 1. Then we get a finite étale morphism $Y \to X$ whose restriction to X_η is Z by Lemma 58.31.3. Of course $Y \cong Z'$ (details omitted; hint: compute étale locally) and hence Y is a Galois cover with group G.

General case. Let $U \subset S$ be a maximal open subscheme such that there exists a finite étale Galois cover $Y \to X \times_S U$ with group G whose restriction to X_η is isomorphic to Z. Assume $U \neq S$ to get a contradiction. Let $s \in S \setminus U$ be a generic point of an irreducible component of $S \setminus U$. Then the inverse image U_s of U in $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the punctured spectrum of $\mathcal{O}_{S,s}$. We claim $Y \times_S U_s \to X \times_S U_s$ is the restriction of a finite étale Galois cover $Y'_s \to X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ with group G.

Let us first prove the claim produces the desired contradiction. By Limits, Lemma 32.20.3 we find an open subscheme $U \subset U' \subset S$ containing s and a morphism $Y'' \to U'$ of finite presentation whose restriction to U recovers $Y' \to U$ and whose restriction to $\mathrm{Spec}(\mathcal{O}_{S,s})$ recovers Y'_s. Moreover, by the equivalence of categories given in the lemma, we may assume after shrinking U' there is a morphism $Y'' \to U' \times_S X$ and there is an action of G on Y'' over $U' \times_S X$ compatible with the given morphisms and actions after base change to U and $\mathrm{Spec}(\mathcal{O}_{S,s})$. After shrinking U' further if necessary, we may assume $Y'' \to U \times_S X$ is finite étale, see Limits,

Lemma 32.20.4. This means we have found a strictly larger open of S over which Y extends to a finite étale Galois cover with group G which gives the contradiction we were looking for.

Proof of the claim. We may and do replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Then $S = \mathrm{Spec}(A)$ where $(A, \mathfrak{m})$ is a local normal domain. Also $U \subset S$ is the punctured spectrum and we have a finite étale Galois cover $Y \to X \times_S U$ with group G. If $\dim(A) = 1$, then we can construct the extension of Y to a Galois covering of X by the first paragraph of the proof. Thus we may assume $\dim(A) \geq 2$ and hence $\mathrm{depth}(A) \geq 2$ as S is normal, see Algebra, Lemma 10.157.4. Since $X \to S$ is flat, we conclude that $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$ for every point $x \in X$ mapping to s, see Algebra, Lemma 10.163.2. Let

$$Y' \longrightarrow X$$

be the finite morphism constructed in Lemma 58.21.5 using $Y \to X \times_S U$. Observe that we obtain a canonical G-action on Y. Thus all that remains is to show that Y' is étale over X. In fact, by Lemma 58.26.3 (for example) it even suffices to show that $Y' \to X$ is étale over the (unique) generic point of the fibre X_s. This we do by a local calculation in a (formal) neighbourhood of $\sigma(s)$.

Choose an affine open $\mathrm{Spec}(B) \subset X$ containing $\sigma(s)$. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \mathrm{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 10.139.4. Of course $B \to B^\wedge$ is flat (Algebra, Lemma 10.97.2) and the image of $\mathrm{Spec}(B^\wedge) \to X$ contains the generic point of X_s. Let $V \subset \mathrm{Spec}(B^\wedge)$ be the inverse image of U. Consider the finite étale morphism

$$W = Y \times_{(X \times_S U)} V \longrightarrow V$$

By the compatibility of the construction of Y' with flat base change in Lemma 58.21.5 we find that the base chang $Y' \times_X \mathrm{Spec}(B^\wedge) \to \mathrm{Spec}(B^\wedge)$ is constructed from $W \to V$ over $\mathrm{Spec}(B^\wedge)$ by the procedure in Lemma 58.21.5. Set $V_0 = V \cap V(x_1, \ldots, x_d) \subset V$ and $W_0 = W \times_V V_0$. This is a normal integral scheme which maps into $\sigma(S)$ by the morphism $\mathrm{Spec}(B^\wedge) \to X$ and in fact is identified with $\sigma(U)$. Hence we know that $W_0 \to V_0 = U$ completely decomposes as this is true for its generic fibre by our assumption on $Z \to X_\eta$ having a $\kappa(\eta)$-rational point lying over $\sigma(\eta)$ (and of course the G-action then implies the whole fibre $Z_{\sigma(\eta)}$ is a disjoint union of copies of the scheme $\eta = \mathrm{Spec}(\kappa(\eta))$). Finally, by Lemma 58.26.1 we have

$$W_0 \times_U V \cong W$$

This shows that W is a disjoint union of copies of V and hence $Y' \times_X \mathrm{Spec}(B^\wedge)$ is a disjoint union of copies of $\mathrm{Spec}(B^\wedge)$ and the proof is complete. □

0EZJ **Lemma 58.31.9.** *Let S be a quasi-compact and quasi-separated integral normal scheme with generic point η. Let $f : X \to S$ be a quasi-compact and quasi-separated smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 58.7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.*

Proof. If S is Noetherian, then this is the result of Lemma 58.31.8. The general case follows from this by a standard limit argument. We strongly urge the reader to skip the proof.

We can write $S = \lim S_i$ as a directed limit of a system of schemes with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 32.5.4. For each i let $S \to S'_i \to S_i$ be the normalization of S_i in S, see Morphisms, Section 29.53. Combining Algebra, Proposition 10.162.16 Morphisms, Lemmas 29.53.15 and 29.53.13 we conclude that S'_i is of finite type over $\mathbf{Z}$, finite over S_i, and that S'_i is an integral normal scheme such that $S \to S'_i$ is dominant. By Morphisms, Lemma 29.53.5 we obtain transition morphisms $S'_{i'} \to S'_i$ compatible with the transition morphisms $S_{i'} \to S_i$ and with the morphisms with source S. We claim that $S = \lim S'_i$. Proof of claim omitted (hint: look on affine opens over a chosen affine open in S_i for some i to translate this into a straightforward algebra problem). We conclude that we may write $S = \lim S_i$ as a directed limit of a system of normal integral schemes S_i with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$.

For some i we can find a smooth morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$. See Limits, Lemmas 32.10.1 and 32.8.9. After increasing i we may assume the section σ lifts to a section $\sigma_i : S_i \to X_i$ (by the equivalence of categories in Limits, Lemma 32.10.1). We may replace X_i by the open subscheme X_i^0 of it studied in More on Morphisms, Section 37.29 since the image of $X \to X_i$ clearly maps into it (openness by More on Morphisms, Lemma 37.29.6). Thus we may assume the fibres of $X_i \to S_i$ are geometrically connected. After increasing i we may assume $|G|$ is invertible on S_i. Let $\eta_i \in S_i$ be the generic point. Since X_η is the limit of the schemes X_{i,η_i} we can use the exact same arguments to descent $Z \to X_\eta$ to some finite étale Galois cover $Z_i \to X_{i,\eta_i}$ after possibly increasing i. See Lemma 58.14.1. After possibly increasing i once more we may assume Z_i has a $\kappa(\eta_i)$-rational point mapping to $\sigma_i(\eta_i)$. Then we apply the lemma in the Noetherian case and we pullback to X to conclude. □

58.32. Tricks in positive characteristic

0G1E In Piotr Achinger's paper [**Ach17**] it is shown that an affine scheme in positive characteristic is always a $K(\pi, 1)$. In this section we explain the more elementary parts of [**Ach17**]. Namely, we show that for a field k of positive characteristic an affine scheme étale over $\mathbf{A}^n_k$ is actually finite étale over $\mathbf{A}^n_k$ (by a different morphism). We also show that a closed immersion of connected affine schemes in positive characteristic induces an injective map on étale fundamental groups.

Let k be a field of characteristic $p > 0$. Let

$$k[x_1, \ldots, x_n] \longrightarrow A$$

be a surjection of finite type k-algebras whose source is the polynomial algebra on $x_1, \ldots, x_n$. Denote $I \subset k[x_1, \ldots, x_n]$ the kernel so that we have $A = k[x_1, \ldots, x_n]/I$. We do not assume A is nonzero (in other words, we allow the case where A is the zero ring and $I = k[x_1, \ldots, x_n]$). Finally, we assume given a finite étale ring map $\pi : A \to B$.

Suppose given $k, n, k[x_1, \ldots, x_n] \to A, I, \pi : A \to B$. Let C be a k-algebra. Consider commutative diagrams

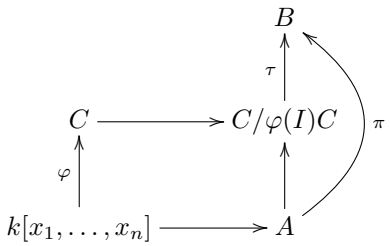

where φ is an étale k-algebra map and τ is a surjective k-algebra map. Let C, φ, τ be given. For any $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C as an algebra over $\mathrm{Im}(\varphi)$ let $s = s(r, y_1, \ldots, y_r) \in \{0, \ldots, r\}$ be the maximal element such that y_i is integral over $\mathrm{Im}(\varphi)$ for $1 \leq i \leq s$. We define $NF(C, \varphi, \tau)$ to be the minimum value of $r - s = r - s(r, y_1, \ldots, y_r)$ for all choices of r and $y_1, \ldots, y_r$ as above. Observe that $NF(C, \varphi, \tau)$ is 0 if and only if φ is finite.

0G1F **Lemma 58.32.1.** *In the situation above, if $NF(C, \varphi, \tau) > 0$, then there exist an étale k-algebra map φ' and a surjective k-algebra map τ' fitting into the commutative diagram*

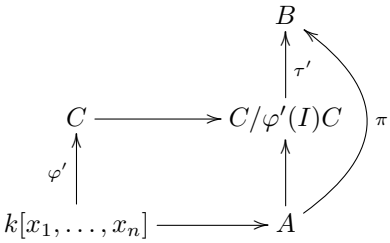

with $NF(C, \varphi', \tau') < NF(C, \varphi, \tau)$.

Proof. Choose $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C over $\mathrm{Im}(\varphi)$ and let $0 \leq s \leq r$ be such that $y_1, \ldots, y_s$ are integral over $\mathrm{Im}(\varphi)$ such that $r - s = NF(C, \varphi, \tau) > 0$. Since B is finite over A, the image of y_{s+1} in B satisfies a monic polynomial over A. Hence we can find $d \geq 1$ and $f_1, \ldots, f_d \in k[x_1, \ldots, x_n]$ such that

$$z = y_{s+1}^d + \varphi(f_1) y_{s+1}^{d-1} + \ldots + \varphi(f_d) \in J = \mathrm{Ker}(C \to C/\varphi(I)C \xrightarrow{\tau} B)$$

Since $\varphi : k[x_1, \ldots, x_n] \to C$ is étale, we can find a nonzero and nonconstant polynomial $g \in k[T_1, \ldots, T_{n+1}]$ such that

$$g(\varphi(x_1), \ldots, \varphi(x_n), z) = 0 \quad \text{in} \quad C$$

To see this you can use for example that $C \otimes_{\varphi, k[x_1, \ldots, x_n]} k(x_1, \ldots, x_n)$ is a finite product of finite separable field extensions of $k(x_1, \ldots, x_n)$ (see Algebra, Lemmas 10.143.4) and hence z satisfies a monic polynomial over $k(x_1, \ldots, x_n)$. Clearing denominators we obtain g.

The existence of g and Algebra, Lemma 10.115.2 produce integers $e_1, e_2, \ldots, e_n \geq 1$ such that z is integral over the subring C' of C generated by $t_1 = \varphi(x_1) + z^{pe_1}, \ldots, t_n = \varphi(x_n) + z^{pe_n}$. Of course, the elements $\varphi(x_1), \ldots, \varphi(x_n)$ are also

integral over C' as are the elements $y_1, \ldots, y_s$. Finally, by our choice of z the element y_{s+1} is integral over C' too.

Consider the ring map

$$\varphi' : k[x_1, \ldots, x_n] \longrightarrow C, \quad x_i \longmapsto t_i$$

with image C'. Since $\mathrm{d}(\varphi(x_i)) = \mathrm{d}(t_i) = \mathrm{d}(\varphi'(x_i))$ in $\Omega_{C/k}$ (and this is where we use the characteristic of k is $p > 0$) we conclude that φ' is étale because φ is étale, see Algebra, Lemma 10.151.9. Observe that $\varphi'(x_i) - \varphi(x_i) = t_i - \varphi(x_i) = z^{pe_i}$ is in the kernel J of the map $C \to C/\varphi(I)C \to B$ by our choice of z as an element of J. Hence for $f \in I$ the element

$$\varphi'(f) = f(t_1, \ldots, t_n) = f(\varphi(x_1) + z^{pe_1}, \ldots, \varphi(x_n) + z^{pe_n}) = \varphi(f) + \text{element of } (z)$$

is in J as well. In other words, $\varphi'(I)C \subset J$ and we obtain a surjection

$$\tau' : C/\varphi'(I)C \longrightarrow C/J \cong B$$

of algebras étale over A. Finally, the algebra C is generated by the elements $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_r$ over $C' = \mathrm{Im}(\varphi')$ with $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_{s+1}$ integral over $C' = \mathrm{Im}(\varphi')$. Hence $NF(C, \varphi', \tau') < r - s = NF(C, \varphi, \tau)$. This finishes the proof. □

0G1G **Lemma 58.32.2.** *Let k be a field of characteristic $p > 0$. Let $X \to \mathbf{A}^n_k$ be an étale morphism with X affine. Then there exists a finite étale morphism $X \to \mathbf{A}^n_k$.*

Proof. Write $X = \mathrm{Spec}(C)$. Set $A = 0$ and denote $I = k[x_1, \ldots, x_n]$. By assumption there exists some étale k-algebra map $\varphi : k[x_1, \ldots, x_n] \to C$. Denote $\tau : C/\varphi(I)C \to 0$ the unique surjection. We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 58.32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale. □

0G1H **Lemma 58.32.3.** *Let k be a field of characteristic $p > 0$. Let $Z \subset \mathbf{A}^n_k$ be a closed subscheme. Let $Y \to Z$ be finite étale. There exists a finite étale morphism $f : U \to \mathbf{A}^n_k$ such that there is an open and closed immersion $Y \to f^{-1}(Z)$ over Z.*

Proof. Let us turn the problem into algebra. Write $\mathbf{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$. Then $Z = \mathrm{Spec}(A)$ where $A = k[x_1, \ldots, x_n]/I$ for some ideal $I \subset k[x_1, \ldots, x_n]$. Write $Y = \mathrm{Spec}(B)$ so that $Y \to Z$ corresponds to the finite étale k-algebra map $A \to B$.

By Algebra, Lemma 10.143.10 there exists an étale ring map

$$\varphi : k[x_1, \ldots, x_n] \to C$$

and a surjective A-algebra map $\tau : C/\varphi(I)C \to B$. (We can even choose C, φ, τ such that τ is an isomorphism, but we won't use this). We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 58.32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale.

Let $f : U = \mathrm{Spec}(C) \to \mathbf{A}^n_k$ be the finite étale morphism corresponding to φ. The morphism $Y \to f^{-1}(Z) = \mathrm{Spec}(C/\varphi(I)C)$ induced by τ is a closed immersion as τ is surjective and open as it is an étale morphism by Morphisms, Lemma 29.36.18. This finishes the proof. □

Here is the main result.

0G1I **Proposition 58.32.4.** *Let p be a prime number. Let $i : Z \to X$ be a closed immersion of connected affine schemes over $\mathbf{F}_p$. For any geometric point $\overline{z}$ of Z the map*

$$\pi_1(Z, \overline{z}) \to \pi_1(X, \overline{z})$$

is injective.

Proof. Let $Y \to Z$ be a finite étale morphism. It suffices to construct a finite étale morphism $f : U \to X$ such that Y is isomorphic to an open and closed subscheme of $f^{-1}(Z)$, see Lemma 58.4.4. Write $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$ so the closed immersion $Y \to X$ is given by a surjection $R \to A$. We may write $A = \mathrm{colim}\, A_i$ as the filtered colimit of its $\mathbf{F}_p$-subalgebras of finite type. By Lemma 58.14.1 we can find an i and a finite étale morphism $Y_i \to Z_i = \mathrm{Spec}(A_i)$ such that $Y = Z \times_{Z_i} Y_i$.

Choose a surjection $\mathbf{F}_p[x_1, \ldots, x_n] \to A_i$. This determines a closed immersion

$$Z_i = \mathrm{Spec}(A_i) \longrightarrow X_i = \mathbf{A}^n_{\mathbf{F}_p} = \mathrm{Spec}(\mathbf{F}_p[x_1, \ldots, x_n])$$

By the universal property of polynomial algebras and since $R \to A$ is surjective, we can find a commutative diagram

$$\begin{array}{ccc} \mathbf{F}_p[x_1, \ldots, x_n] & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ R & \longrightarrow & A \end{array}$$

of $\mathbf{F}_p$-algebras. Thus we have a commutative diagram

$$\begin{array}{ccccc} Y_i & \longrightarrow & Z_i & \longrightarrow & X_i \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Z & \longrightarrow & X \end{array}$$

whose right square is cartesian. Clearly, if we can find $f_i : U_i \to X_i$ finite étale such that Y_i is isomorphic to an open and closed subscheme of $f_i^{-1}(Z_i)$, then the base change $f : U \to X$ of f_i by $X \to X_i$ is a solution to our problem. Thus we conclude by applying Lemma 58.32.3 to $Y_i \to Z_i \to X_i = \mathbf{A}^n_{\mathbf{F}_p}$. □

58.33. Other chapters

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(5) Topology
(6) Sheaves on Spaces
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Schemes

(26) Schemes

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Algebraic Spaces

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CHAPTER 59

Étale Cohomology

03N1

59.1. Introduction

03N2 This chapter is the first in a series of chapter on the étale cohomology of schemes. In this chapter we discuss the very basics of the étale topology and cohomology of abelian sheaves in this topology. Many of the topics discussed may be safely skipped on a first reading; please see the advice in the next section as to how to decide what to skip.

The initial version of this chapter was formed by the notes of the first part of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. The second part of the course can be found in the chapter on the trace formula, see The Trace Formula, Section 64.1.

59.2. Which sections to skip on a first reading?

04JG We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word "topos", or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 59.99.
(2) Recovering morphisms, Section 59.40.
(3) Push and pull, Section 59.41.
(4) Property (A), Section 59.42.
(5) Property (B), Section 59.43.
(6) Property (C), Section 59.44.
(7) Topological invariance of the small étale site, Section 59.45.
(8) Integral universally injective morphisms, Section 59.47.
(9) Big sites and pushforward, Section 59.48.
(10) Exactness of big lower shriek, Section 59.49.

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

59.3. Prologue

03N3 These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that

it fails to give the results that we would want is that if X is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is *flasque*. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are "not enough" open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if X is a smooth projective complex variety, then

$$H^{2\dim X}_{Betti}(X(\mathbf{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbf{Z},\ \mathbf{Z}/n\mathbf{Z},$$

where $X(\mathbf{C})$ means the set of complex points of X. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

59.4. The étale topology

03N4 It is very hard to simply "add" extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If X is a smooth projective variety over $\mathbf{C}$, then this means

(1) U is a disjoint union of smooth varieties, and
(2) φ is (analytically) locally an isomorphism.

The word "analytically" refers to the usual (transcendental) topology over $\mathbf{C}$. So the second condition means that the derivative of φ has full rank everywhere (and in particular all the components of U have the same dimension as X).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

$$\mathrm{Spec}(\mathbf{C}[t]) \longrightarrow \mathrm{Spec}(\mathbf{C}[t]), \quad t \longmapsto t^2$$

will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t = 0$. For a finite map between varieties over $\mathbf{C}$ to be étale all the fibers should have the same number of points. Removing the point $t = 0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely open subsets of X, even though their images always are.

03N5 **Definition 59.4.1.** A family of morphisms $\{\varphi_i : U_i \to X\}_{i \in I}$ is called an *étale covering* if each φ_i is an étale morphism and their images cover X, i.e., $X = \bigcup_{i \in I} \varphi_i(U_i)$.

This "defines" the étale topology. In other words, we can now say what the sheaves are. An *étale sheaf* $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on X is the data:

(1) for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,

(2) for each pair U, U' of étale schemes over X, and each morphism $U \to U'$ over X (which is automatically étale) a restriction map $\rho_U^{U'} : \mathcal{F}(U') \to \mathcal{F}(U)$

These data have to satisfy the condition that $\rho_U^U = \text{id}$ in case of the identity morphism $U \to U$ and that $\rho_U^{U'} \circ \rho_{U'}^{U''} = \rho_U^{U''}$ when we have morphisms $U \to U' \to U''$ of schemes étale over X as well as the following *sheaf axiom*:

(*) for every étale covering $\{\varphi_i : U_i \to U\}_{i \in I}$, the diagram

$$\mathcal{F}(U) \longrightarrow \Pi_{i \in I} \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram in the category of sets (resp. abelian groups, vector spaces, etc).

03N6 **Remark 59.4.2.** In the last statement, it is essential not to forget the case where $i = j$ which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

59.5. Feats of the étale topology

03N7 For a natural number $n \in \mathbf{N} = \{1, 2, 3, 4, \ldots\}$ it is true that

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}.$$

More generally, if X is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of $X(\mathbf{C})$, i.e.,

$$\dim_{\mathbf{F}_q} H^{2i}_{étale}(X, \mathbf{F}_q) = \dim_{\mathbf{F}_q} H^{2i}_{Betti}(X(\mathbf{C}), \mathbf{F}_q).$$

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}) = 0.$$

By contrast $H^2_{Betti}(\mathbf{P}^1(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}$ as the topological space $\mathbf{P}^1(\mathbf{C})$ is homeomorphic to a 2-sphere. There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

59.6. A computation

03N8 How do we compute the cohomology of $\mathbf{P}^1_{\mathbf{C}}$ with coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$? We use Čech cohomology. A covering of $\mathbf{P}^1_{\mathbf{C}}$ is given by the two standard opens U_0, U_1, which are both isomorphic to $\mathbf{A}^1_{\mathbf{C}}$, and whose intersection is isomorphic to $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\} = \mathbf{G}_{m,\mathbf{C}}$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_{étale}(U_0 \cap U_1, \Lambda) \to H^i_{étale}(\mathbf{P}^1_{\mathbf{C}}, \Lambda) \to H^i_{étale}(U_0, \Lambda) \oplus H^i_{étale}(U_1, \Lambda) \to H^i_{étale}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_{étale}(\mathbf{A}^1_{\mathbf{C}}, \Lambda) = 0 \quad \text{for } q \geq 1,$$

and

$$H^q_{étale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \text{ and} \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if X is an affine curve over $\mathbf{C}$, then

$$H^q_{étale}(X, \mathbf{Z}/n\mathbf{Z}) = 0 \quad \text{for } q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

Cohomology in degree 1. We use the following identifications:

$$H^1_{étale}(X, \mathbf{Z}/n\mathbf{Z}) = \left\{ \begin{array}{c} \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_{étale} \text{ endowed with an} \\ \text{action } \mathbf{Z}/n\mathbf{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbf{Z}/n\mathbf{Z}\text{-torsor.} \end{array} \right\} \Big/ \cong$$

$$= \left\{ \begin{array}{c} \text{morphisms } Y \to X \text{ which are finite étale together} \\ \text{with a free } \mathbf{Z}/n\mathbf{Z} \text{ action such that } X = Y/(\mathbf{Z}/n\mathbf{Z}). \end{array} \right\} \Big/ \cong .$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbf{A}^1_{\mathbf{C}}$ has no nontrivial finite étale covering and hence $H^1_{étale}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. It suffices to consider the case where Y is connected, which we assume. We are going to find out what Y can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the next section if you like). Say that this morphism is n to 1, and consider a projective compactification

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} \\ \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} & \hookrightarrow & \mathbf{P}^1_{\mathbf{C}} \end{array}$$

Even though φ is étale and does not ramify, $\bar{\varphi}$ may ramify at 0 and ∞. Say that the preimages of 0 are the points $y_1, \ldots, y_r$ with indices of ramification $e_1, \ldots e_r$, and that the preimages of ∞ are the points $y'_1, \ldots, y'_s$ with indices of ramification $d_1, \ldots d_s$. In particular, $\sum e_i = n = \sum d_j$. Applying the Riemann-Hurwitz formula, we get

$$2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)$$

and therefore $g_Y = 0$, $r = s = 1$ and $e_1 = d_1 = n$. Hence $Y \cong \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and it is easy to see that $\varphi(z) = \lambda z^n$ for some $\lambda \in \mathbf{C}^*$. After reparametrizing Y we may assume $\lambda = 1$. Thus our covering is given by taking the nth root of the coordinate on $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$.

Remember that we need to classify the coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ together with free $\mathbf{Z}/n\mathbf{Z}$-actions on them. In our case any such action corresponds to an automorphism of Y sending z to $\zeta_n z$, where ζ_n is a primitive nth root of unity. There are $\phi(n)$ such actions (here $\phi(n)$ means the Euler function). Thus there are exactly $\phi(n)$ connected finite étale coverings with a given free $\mathbf{Z}/n\mathbf{Z}$-action, each corresponding to a primitive nth root of unity. We leave it to the reader to see that the disconnected finite étale degree n coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ with a given free $\mathbf{Z}/n\mathbf{Z}$-action correspond one-to-one with nth roots of 1 which are not primitive. In other words, this computation shows that

$$H^1_{étale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}(\mu_n(\mathbf{C}), \mathbf{Z}/n\mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}.$$

The first identification is canonical, the second isn't, see Remark 59.69.5. Since the proof of Riemann-Hurwitz does not use the computation of cohomology, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

59.7. Nontorsion coefficients

03N9 To study nontorsion coefficients, one makes the following definition:

$$H^i_{étale}(X, \mathbf{Q}_\ell) := \left(\lim_n H^i_{étale}(X, \mathbf{Z}/\ell^n\mathbf{Z})\right) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The symbol $\lim_n$ denote the *limit* of the system of cohomology groups $H^i_{étale}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ indexed by n, see Categories, Section 4.21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

59.8. Sheaf theory

03NA At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding too much material here, just enough so the material later in this chapter makes sense.

59.9. Presheaves

03NB A reference for this section is Sites, Section 7.2.

03NC **Definition 59.9.1.** Let $\mathcal{C}$ be a category. A *presheaf of sets* (respectively, an *abelian presheaf*) on $\mathcal{C}$ is a functor $\mathcal{C}^{opp} \to Sets$ (resp. Ab).

Terminology. If $U \in \mathrm{Ob}(\mathcal{C})$, then elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U. For $\varphi : V \to U$ in $\mathcal{C}$, the map $\mathcal{F}(\varphi) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map* and is often denoted $s \mapsto s|_V$ or sometimes $s \mapsto \varphi^* s$. The notation $s|_V$ is ambiguous since the restriction map depends on φ, but it is a standard abuse of notation. We also use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Saying that $\mathcal{F}$ is a functor means that if $W \to V \to U$ are morphisms in $\mathcal{C}$ and $s \in \Gamma(U, \mathcal{F})$ then $(s|_V)|_W = s|_W$, with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms $\mathrm{id}_U : U \to U$ are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on $\mathcal{C}$ is denoted $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$). It is the category of functors from $\mathcal{C}^{opp}$ to $Sets$ (resp. Ab), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories $PSh(\mathcal{C})$ and $PAb(\mathcal{C})$ when the category

$\mathcal{C}$ is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn't small it should be listed in Categories, Remark 4.2.2.)

03ND **Example 59.9.2.** Given an object $X \in \mathrm{Ob}(\mathcal{C})$, we consider the rule

$$\begin{array}{ccc} U & \longmapsto & h_X(U) = \mathrm{Mor}_{\mathcal{C}}(U, X) \\ V \xrightarrow{\varphi} U & \longmapsto & (\psi \mapsto \psi \circ \varphi) : h_X(U) \to h_X(V). \end{array}$$

This defines a functor $h_X : \mathcal{C}^{opp} \to \mathit{Sets}$ and hence a presheaf. This is called the *representable presheaf associated to* X. It is not true that representable presheaves are sheaves in every topology on every site.

03NE **Lemma 59.9.3** (Yoneda)**.** *Let $\mathcal{C}$ be a category, and $X, Y \in \mathrm{Ob}(\mathcal{C})$. There is a natural bijection*

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y) \\ \psi & \longmapsto & h_\psi = \psi \circ - : h_X \to h_Y. \end{array}$$

Proof. See Categories, Lemma 4.3.5. □

59.10. Sites

03NF

03NG **Definition 59.10.1.** Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

(1) an object $U \in \mathcal{C}$,
(2) a set I (possibly empty), and
(3) for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

There is a notion of a *morphism of families of morphisms with fixed target*. A special case of that is the notion of a *refinement*. A reference for this material is Sites, Section 7.8.

03NH **Definition 59.10.2.** A *site*[1] consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called *coverings*, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I} \{i\} \times I_i}$$
is also a covering, and
(3) (base change) if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then
   (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and
   (b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

For us the category underlying a site is always "small", i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 7.6.3.

[1] What we call a site is a called a category endowed with a pretopology in [**AGV71**, Exposé II, Définition 1.3]. In [**Art62**] it is called a category with a Grothendieck topology.

03NI **Example 59.10.3.** If X is a topological space, then it has an associated site X_{Zar} defined as follows: the objects of X_{Zar} are the open subsets of X, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

59.11. Sheaves

03NJ

03NK **Definition 59.11.1.** A presheaf $\mathcal{F}$ of sets (resp. abelian presheaf) on a site $\mathcal{C}$ is said to be a *separated presheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ the map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i)$$

is injective. Here the map is $s \mapsto (s|_{U_i})_{i \in I}$. The presheaf $\mathcal{F}$ is a *sheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$, the diagram

03NL (59.11.1.1)
$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{i,j \in I} \mathcal{F}(U_i \times_U U_j),$$

where the first map is $s \mapsto (s|_{U_i})_{i \in I}$ and the two maps on the right are $(s_i)_{i \in I} \mapsto (s_i|_{U_i \times_U U_j})$ and $(s_i)_{i \in I} \mapsto (s_j|_{U_i \times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

03NM **Remark 59.11.2.** For the empty covering (where $I = \emptyset$), this implies that $\mathcal{F}(\emptyset)$ is an empty product, which is a final object in the corresponding category (a singleton, for both *Sets* and *Ab*).

03NN **Example 59.11.3.** Working this out for the site X_{Zar} associated to a topological space, see Example 59.10.3, gives the usual notion of sheaves.

03NO **Definition 59.11.4.** We denote $Sh(\mathcal{C})$ (resp. $Ab(\mathcal{C})$) the full subcategory of $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$) whose objects are sheaves. This is the *category of sheaves of sets* (resp. *abelian sheaves*) on $\mathcal{C}$.

59.12. The example of G-sets

03NP Let G be a group and define a site $\mathcal{T}_G$ as follows: the underlying category is the category of G-sets, i.e., its objects are sets endowed with a left G-action and the morphisms are equivariant maps; and the coverings of $\mathcal{T}_G$ are the families $\{\varphi_i : U_i \to U\}_{i \in I}$ satisfying $U = \bigcup_{i \in I} \varphi_i(U_i)$.

There is a special object in the site $\mathcal{T}_G$, namely the G-set G endowed with its natural action by left translations. We denote it ${}_GG$. Observe that there is a natural group isomorphism

$$\begin{array}{rccc} \rho: & G^{opp} & \longrightarrow & \mathrm{Aut}_{G\text{-}Sets}({}_GG) \\ & g & \longmapsto & (h \mapsto hg). \end{array}$$

In particular, for any presheaf $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ inherits a G-action via ρ. (Note that by contravariance of $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ is again a left G-set.) In fact, the functor

$$\begin{array}{ccc} Sh(\mathcal{T}_G) & \longrightarrow & G\text{-}Sets \\ \mathcal{F} & \longmapsto & \mathcal{F}({}_GG) \end{array}$$

is an equivalence of categories. Its quasi-inverse is the functor $X \mapsto h_X$. Without giving the complete proof (which can be found in Sites, Section 7.9) let us try to explain why this is true.

(1) If S is a G-set, we can decompose it into orbits $S = \coprod_{i \in I} O_i$. The sheaf axiom for the covering $\{O_i \to S\}_{i \in I}$ says that

$$\mathcal{F}(S) \longrightarrow \prod_{i \in I} \mathcal{F}(O_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j)$$

is an equalizer. Observing that fibered products in G-*Sets* are induced from fibered products in *Sets*, and using the fact that $\mathcal{F}(\emptyset)$ is a G-singleton, we get that

$$\prod_{i,j \in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i \in I} \mathcal{F}(O_i)$$

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that $\mathcal{F}(S) = \prod_{i \in I} \mathcal{F}(O_i)$.

(2) If S is the G-set $S = G/H$ and $\mathcal{F}$ is a sheaf on $\mathcal{T}_G$, then we claim that

$$\mathcal{F}(G/H) = \mathcal{F}({}_G G)^H$$

and in particular $\mathcal{F}(\{*\}) = \mathcal{F}({}_G G)^G$. To see this, let's use the sheaf axiom for the covering $\{{}_G G \to G/H\}$ of S. We have

$$\begin{array}{ccc} {}_G G \times_{G/H} {}_G G & \cong & G \times H \\ (g_1, g_2) & \longmapsto & (g_1, g_1^{-1} g_2) \end{array}$$

is a disjoint union of copies of ${}_G G$ (as a G-set). Hence the sheaf axiom reads

$$\mathcal{F}(G/H) \longrightarrow \mathcal{F}({}_G G) \rightrightarrows \prod_{h \in H} \mathcal{F}({}_G G)$$

where the two maps on the right are $s \mapsto (s)_{h \in H}$ and $s \mapsto (hs)_{h \in H}$. Therefore $\mathcal{F}(G/H) = \mathcal{F}({}_G G)^H$ as claimed.

This doesn't quite prove the claimed equivalence of categories, but it shows at least that a sheaf $\mathcal{F}$ is entirely determined by its sections over ${}_G G$. Details (and set theoretical remarks) can be found in Sites, Section 7.9.

59.13. Sheafification

03NQ

03NR **Definition 59.13.1.** Let $\mathcal{F}$ be a presheaf on the site $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$. We define the *zeroth Čech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

There is a canonical map $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$, $s \mapsto (s|_{U_i})_{i \in I}$. We say that a *morphism of coverings* from a covering $\mathcal{V} = \{V_j \to V\}_{j \in J}$ to $\mathcal{U}$ is a triple (χ, α, χ_j), where $\chi : V \to U$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism χ_j fits into a commutative diagram

$$\begin{array}{ccc} V_j & \xrightarrow{\chi_j} & U_{\alpha(j)} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\chi} & U. \end{array}$$

Given the data $\chi, \alpha, \{\chi_j\}_{j\in J}$ we define

$$\begin{array}{rcl} \check{H}^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{V}, \mathcal{F}) \\ (s_i)_{i\in I} & \longmapsto & \left(\chi_j^* \left(s_{\alpha(j)}\right)\right)_{j\in J}. \end{array}$$

We then claim that

(1) the map is well-defined, and
(2) depends only on χ and is independent of the choice of $\alpha, \{\chi_j\}_{j\in J}$.

We omit the proof of the first fact. To see part (2), consider another triple (ψ, β, ψ_j) with $\chi = \psi$. Then we have the commutative diagram

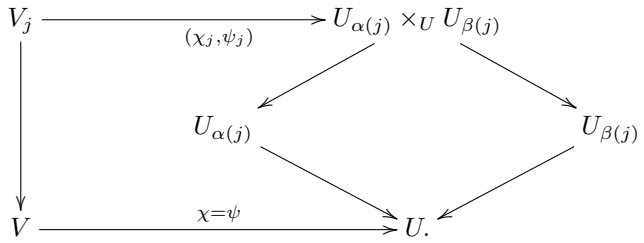

Given a section $s \in \mathcal{F}(\mathcal{U})$, its image in $\mathcal{F}(V_j)$ under the map given by $(\chi, \alpha, \{\chi_j\}_{j\in J})$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $(\psi, \beta, \{\psi_j\}_{j\in J})$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}^0(\mathcal{U}, \mathcal{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)}\times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)}\times_U U_{\beta(j)}}$$

pulled back by the map (χ_j, ψ_j) in the diagram.

03NS **Theorem 59.13.2.** *Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$.*

(1) *The rule*

$$U \mapsto \mathcal{F}^+(U) := \operatorname{colim}_{\mathcal{U} \text{ covering of } U} \check{H}^0(\mathcal{U}, \mathcal{F})$$

is a presheaf. And the colimit is a directed one.
(2) *There is a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*
(3) *If $\mathcal{F}$ is a separated presheaf then $\mathcal{F}^+$ is a sheaf and the map in (2) is injective.*
(4) *$\mathcal{F}^+$ is a separated presheaf.*
(5) *$\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf, and the canonical map induces a functorial isomorphism*

$$\operatorname{Hom}_{PSh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G})$$

for any $\mathcal{G} \in Sh(\mathcal{C})$.

Proof. See Sites, Theorem 7.10.10. □

In other words, this means that the natural map $\mathcal{F} \to \mathcal{F}^\#$ is a left adjoint to the forgetful functor $Sh(\mathcal{C}) \to PSh(\mathcal{C})$.

59.14. Cohomology

03NT The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

03NU **Theorem 59.14.1.** *The category of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. See Modules on Sites, Lemma 18.3.1 and Injectives, Theorem 19.7.4. □

So we can define cohomology as the right-derived functors of the sections functor: if $U \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{F} \in Ab(\mathcal{C})$,

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $Ab(\mathcal{C})$ is abelian, see Modules on Sites, Lemma 18.3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 21.2.

59.15. The fpqc topology

03NV Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

03NW **Definition 59.15.1.** Let T be a scheme. An *fpqc covering* of T is a family $\{\varphi_i : T_i \to T\}_{i \in I}$ such that

(1) each φ_i is a flat morphism and $\bigcup_{i \in I} \varphi_i(T_i) = T$, and
(2) for each affine open $U \subset T$ there exists a finite set K, a map $\mathbf{i} : K \to I$ and affine opens $U_{\mathbf{i}(k)} \subset T_{\mathbf{i}(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\mathbf{i}(k)}(U_{\mathbf{i}(k)})$.

03NX **Remark 59.15.2.** The first condition corresponds to fp, which stands for *fidèlement plat*, faithfully flat in french, and the second to qc, *quasi-compact*. The second part of the first condition is unnecessary when the second condition holds.

03NY **Example 59.15.3.** Examples of fpqc coverings.

(1) Any Zariski open covering of T is an fpqc covering.
(2) A family $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is a faithfully flat ring map.
(3) If $f : X \to Y$ is flat, surjective and quasi-compact, then $\{f : X \to Y\}$ is an fpqc covering.
(4) The morphism $\varphi : \coprod_{x \in \mathbf{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_k, x}) \to \mathbf{A}^1_k$, where k is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not an fpqc covering.
(5) Write $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$. Denote $i_x : D(x) \to \mathbf{A}^2_k$ and $i_y : D(y) \to \mathbf{A}^2_k$ the standard opens. Then the families $\{i_x, i_y, \mathrm{Spec}(k[[x, y]]) \to \mathbf{A}^2_k\}$ and $\{i_x, i_y, \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^2_k, 0}) \to \mathbf{A}^2_k\}$ are fpqc coverings.

03NZ **Lemma 59.15.4.** *The collection of fpqc coverings on the category of schemes satisfies the axioms of site.*

Proof. See Topologies, Lemma 34.9.8. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 34.9. It comes from our requirement that sites are "small", but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining "partial" fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

03X6 **Definition 59.15.5.** Let S be a scheme. The category of schemes over S is denoted Sch/S. Consider a functor $\mathcal{F} : (Sch/S)^{opp} \to Sets$, in other words a presheaf of sets. We say $\mathcal{F}$ *satisfies the sheaf property for the fpqc topology* if for every fpqc covering $\{U_i \to U\}_{i\in I}$ of schemes over S the diagram (59.11.1.1) is an equalizer diagram.

We similarly say that $\mathcal{F}$ *satisfies the sheaf property for the Zariski topology* if for every open covering $U = \bigcup_{i\in I} U_i$ the diagram (59.11.1.1) is an equalizer diagram. See Schemes, Definition 26.15.3. Clearly, this is equivalent to saying that for every scheme T over S the restriction of $\mathcal{F}$ to the opens of T is a (usual) sheaf.

03O1 **Lemma 59.15.6.** *Let $\mathcal{F}$ be a presheaf on Sch/S. Then $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf property with respect to the Zariski topology, and*
(2) *for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes over S, the sheaf axiom holds for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$. Namely, this means that*

$$\mathcal{F}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer diagram.

Proof. See Topologies, Lemma 34.9.14. □

An alternative way to think of a presheaf $\mathcal{F}$ on Sch/S which satisfies the sheaf condition for the fpqc topology is as the following data:

(1) for each T/S, a usual (i.e., Zariski) sheaf $\mathcal{F}_T$ on T_{Zar},
(2) for every map $f : T' \to T$ over S, a restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$

such that

(a) the restriction mappings are functorial,
(b) if $f : T' \to T$ is an open immersion then the restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over S, the diagram

$$\mathcal{F}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}_{\mathrm{Spec}(B\otimes_A B)}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on Sch/S satisfying the sheaf condition for the Zariski topology. By Lemma 59.15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

03O2 **Example 59.15.7.** Consider the presheaf

$$\begin{array}{rccc} \mathcal{F} : & (Sch/S)^{opp} & \longrightarrow & Ab \\ & T/S & \longmapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

The compatibility of differentials with localization implies that $\mathcal{F}$ is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case $S = \mathrm{Spec}(\mathbf{F}_p)$ and the morphism

$$\varphi : V = \mathrm{Spec}(\mathbf{F}_p[v]) \to U = \mathrm{Spec}(\mathbf{F}_p[u])$$

given by mapping u to v^p. The family $\{\varphi\}$ is an fpqc covering, yet the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ sends the generator $\mathrm{d}u$ to $\mathrm{d}(v^p) = 0$, so it is the zero map, and the diagram

$$\mathcal{F}(U) \xrightarrow{0} \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is not an equalizer. We will see later that $\mathcal{F}$ does in fact give rise to a sheaf on the étale and smooth sites.

03O3 **Lemma 59.15.8.** *Any representable presheaf on Sch/S satisfies the sheaf condition for the fpqc topology.*

Proof. See Descent, Lemma 35.13.7. □

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that *the fpqc topology is weaker than the canonical topology*, or that the fpqc topology is *subcanonical*. In the setting of sites this is discussed in Sites, Section 7.12.

03O4 **Remark 59.15.9.** The fpqc is finer than the Zariski, étale, smooth, syntomic, and fppf topologies. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf on the Zariski, étale, smooth, syntomic, and fppf sites. In particular representable presheaves will be sheaves on the étale site of a scheme for example.

03O5 **Example 59.15.10.** Let S be a scheme. Consider the additive group scheme $\mathbf{G}_{a,S} = \mathbf{A}^1_S$ over S, see Groupoids, Example 39.5.3. The associated representable presheaf is given by

$$h_{\mathbf{G}_{a,S}}(T) = \mathrm{Mor}_S(T, \mathbf{G}_{a,S}) = \Gamma(T, \mathcal{O}_T).$$

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

$$\begin{array}{rccc} \mathcal{O} : & (Sch/S)^{opp} & \longrightarrow & Rings \\ & T/S & \longmapsto & \Gamma(T, \mathcal{O}_T) \end{array}$$

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of $\mathcal{O}$-module, and so on and so forth.

59.16. Faithfully flat descent

03O6 In this section we discuss faithfully flat descent for quasi-coherent modules. More precisely, we will prove quasi-coherent modules satisfy effective descent with respect to fpqc coverings.

03O7 **Definition 59.16.1.** Let $\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. A *descent datum* for quasi-coherent sheaves with respect to $\mathcal{U}$ is a collection $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ where

(1) $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i, and

(2) $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ is an isomorphism of modules on $T_i \times_T T_j$,

such that the *cocycle condition* holds: the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{01}^*\varphi_{ij}} & \mathrm{pr}_1^*\mathcal{F}_j \\ & {\scriptstyle \mathrm{pr}_{02}^*\varphi_{ik}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & \mathrm{pr}_2^*\mathcal{F}_k & \end{array}$$

commute on $T_i \times_T T_j \times_T T_k$. This descent datum is called *effective* if there exist a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^*\mathcal{F} \cong \mathcal{F}_i$ compatible with the maps φ_{ij}, namely

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

03O8 **Theorem 59.16.2.** *If $\mathcal{V} = \{T_i \to T\}_{i \in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.*

Proof. See Descent, Proposition 35.5.2. □

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 6.33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_\bullet : B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

where B is in degree 0, $B \otimes_A B$ in degree 1, etc, and the maps are given by

$$\begin{aligned} b &\mapsto 1 \otimes b - b \otimes 1, \\ b_0 \otimes b_1 &\mapsto 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1, \\ &\text{etc.} \end{aligned}$$

03O9 **Lemma 59.16.3.** *If $A \to B$ is faithfully flat, then the complex $(B/A)_\bullet$ is exact in positive degrees, and $H^0((B/A)_\bullet) = A$.*

Proof. See Descent, Lemma 35.3.6. □

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where A is the only nonzero term, in degree 0. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing B with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A' = B$ and remark that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

03OA **Lemma 59.16.4.** *If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.*

Proof. See Descent, Lemma 35.3.6. □

03OB **Definition 59.16.5.** Let $A \to B$ be a ring map and N a B-module. A *descent datum* for N with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{array}{ccc} N \otimes_A B \otimes_A B & \xrightarrow{\varphi_{01}} & B \otimes_A N \otimes_A B \\ & {\scriptstyle \varphi_{02}} \searrow \quad \swarrow {\scriptstyle \varphi_{12}} & \\ & B \otimes_A B \otimes_A N & \end{array}$$

commutes where $\varphi_{01} = \varphi \otimes \mathrm{id}_B$ and similarly for φ_{12} and φ_{02}.

If $N' = B \otimes_A M$ for some A-module M, then it has a canonical descent datum given by the map

$$\begin{array}{rccc} \varphi_{\mathrm{can}} : & N' \otimes_A B & \to & B \otimes_A N' \\ & b_0 \otimes m \otimes b_1 & \mapsto & b_0 \otimes b_1 \otimes m. \end{array}$$

03OC **Definition 59.16.6.** A descent datum (N, φ) is called *effective* if there exists an A-module M such that $(N, \varphi) \cong (B \otimes_A M, \varphi_{\mathrm{can}})$, with the obvious notion of isomorphism of descent data.

Theorem 59.16.2 is a consequence the following result.

03OD **Theorem 59.16.7.** *If $A \to B$ is faithfully flat then descent data with respect to $A \to B$ are effective.*

Proof. See Descent, Proposition 35.3.9. See also Descent, Remark 35.3.11 for an alternative view of the proof. □

03OE **Remarks 59.16.8.** The results on descent of modules have several applications:

(1) The exactness of the Čech complex in positive degrees for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is faithfully flat. This will give some vanishing of cohomology.
(2) If (N, φ) is a descent datum with respect to a faithfully flat map $A \to B$, then the corresponding A-module is given by

$$M = \mathrm{Ker}\begin{pmatrix} N & \longrightarrow & B \otimes_A N \\ n & \longmapsto & 1 \otimes n - \varphi(n \otimes 1) \end{pmatrix}.$$

See Descent, Proposition 35.3.9.

59.17. Quasi-coherent sheaves

03OF We can apply the descent of modules to study quasi-coherent sheaves.

03OG **Proposition 59.17.1.** *For any quasi-coherent sheaf $\mathcal{F}$ on S the presheaf*

$$\begin{array}{rccc} \mathcal{F}^a : & Sch/S & \to & Ab \\ & (f : T \to S) & \mapsto & \Gamma(T, f^*\mathcal{F}) \end{array}$$

is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

Proof. This is proved in Descent, Lemma 35.8.1. We indicate the proof here. As established in Lemma 59.15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since $\Gamma(U, i^*\mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{\mathrm{Spec}(A)} = \widetilde{M}$ this corresponds to the fact that $M = H^0((B/A)_\bullet \otimes_A M)$, i.e., that

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

is exact by Lemma 59.16.4. □

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 18.23. Let $\mathcal{C}$ be a category, and let U be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over U, see Categories, Example 4.2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of V/U are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the *restriction* $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 7.25 for details.

03OH **Definition 59.17.2.** Let $\mathcal{C}$ be a *ringed site*, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called *quasi-coherent* if for all $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus\nolimits_{k \in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$

The direct sum over K is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k \in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

03OI **Remark 59.17.3.** In the case where $\mathcal{C}$ has a final object, e.g. S, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 18.23.3.

03OJ **Theorem 59.17.4** (Meta theorem on quasi-coherent sheaves)**.** *Let S be a scheme. Let $\mathcal{C}$ be a site. Assume that*

(1) *the underlying category $\mathcal{C}$ is a full subcategory of Sch/S,*
(2) *any Zariski covering of $T \in \mathrm{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,*
(3) *S/S is an object of $\mathcal{C}$,*
(4) *every covering of $\mathcal{C}$ is an fpqc covering of schemes.*

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. After some formal arguments this is exactly Theorem 59.16.2. Details omitted. In Descent, Proposition 35.8.9 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of S, as well as the small Zariski and étale sites of S. □

In other words, there is no difference between quasi-coherent modules on the scheme S and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(Sch/S)_{fppf}$, $S_{étale}$, etc can be found in Descent, Sections 35.8, 35.9, and 35.10. In this chapter we will sometimes refer to a "site as in Theorem 59.17.4" in order to conveniently state results which hold in any of those situations.

59.18. Čech cohomology

03OK Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{Zar}(S, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on S, and any site $\mathcal{C}$ as in Theorem 59.17.4. To this end, we introduce Čech cohomology on sites. See [**Art62**] and Cohomology on Sites, Sections 21.8, 21.9 and 21.10 for more details.

03OL **Definition 59.18.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Assume that all the fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exists in $\mathcal{C}$. Let $\mathcal{F} \in PAb(\mathcal{C})$ be an abelian presheaf. We define the *Čech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \ldots$$

where the first term is in degree 0 and the maps are the usual ones. The *Čech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

In the above definition, it is essential to allow the indices $(i_0, \ldots, i_p)$ to have repetitions.

03OM **Lemma 59.18.2.** *Notation and assumptions as in Definition 59.18.1. The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $PAb(\mathcal{C})$.*

In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in Ab, it is automatically an abelian category: a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in PAb if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact in Ab. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 21.9.1. □

This shows that $\check{H}^\bullet(\mathcal{U}, -)$ is a δ-functor. We now proceed to show that it is a universal δ-functor. We thus need to show that it is an *effaceable* functor. We start by recalling the Yoneda lemma.

03ON **Lemma 59.18.3** (Yoneda Lemma)**.** *For any presheaf $\mathcal{F}$ on a category $\mathcal{C}$ and $U \in \mathrm{Ob}(\mathcal{C})$ there is a functorial isomorphism*

$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

Proof. See Categories, Lemma 4.3.5. □

Given a set E we denote (in this section) $\mathbf{Z}[E]$ the free abelian group on E. In a formula $\mathbf{Z}[E] = \bigoplus_{e \in E} \mathbf{Z}$, i.e., $\mathbf{Z}[E]$ is a free $\mathbf{Z}$-module having a basis consisting of the elements of E. Using this notation we introduce the free abelian presheaf on a presheaf of sets.

03OO **Definition 59.18.4.** Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the *free abelian presheaf on* $\mathcal{G}$, denoted $\mathbf{Z}_\mathcal{G}$, by the rule

$$\mathbf{Z}_\mathcal{G}(U) = \mathbf{Z}[\mathcal{G}(U)]$$

for $U \in \mathrm{Ob}(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$ we write simply $\mathbf{Z}_U = \mathbf{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathbf{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{F}) = \mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

03OP **Lemma 59.18.5.** *Notation and assumptions as in Definition 59.18.1. The Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ can be described explicitly as follows*

$$\begin{aligned}\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) &= \left(\prod_{i_0 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}}, \mathcal{F}) \to \prod_{i_0, i_1 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U U_{i_1}}, \mathcal{F}) \to \ldots\right) \\ &= \mathrm{Hom}_{PAb(\mathcal{C})}\left(\left(\bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \ldots\right), \mathcal{F}\right)\end{aligned}$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 21.9.3. □

This reduces us to studying only the complex in the first argument of the last Hom.

03OQ **Lemma 59.18.6.** *Notation and assumptions as in Definition 59.18.1. The complex of abelian presheaves*

$$\mathbf{Z}^\bullet_\mathcal{U} \quad : \quad \bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 *in $PAb(\mathcal{C})$.*

Proof. For any $V \in \mathrm{Ob}(\mathcal{C})$ the complex of abelian groups $\mathbf{Z}^\bullet_\mathcal{U}(V)$ is

$$\mathbf{Z}\left[\textstyle\coprod_{i_0 \in I} \mathrm{Mor}_\mathcal{C}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\textstyle\coprod_{i_0, i_1 \in I} \mathrm{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})\right] \leftarrow \ldots =$$
$$\textstyle\bigoplus_{\varphi : V \to U} \left(\mathbf{Z}\left[\coprod_{i_0 \in I} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0, i_1 \in I} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \leftarrow \ldots\right)$$

where

$$\mathrm{Mor}_\varphi(V, U_i) = \{V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi\}.$$

Set $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$, so that

$$\mathbf{Z}^\bullet_\mathcal{U}(V) = \bigoplus_{\varphi : V \to U} (\mathbf{Z}[S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \ldots).$$

Thus it suffices to show that for each $S = S_\varphi$, the complex

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S \times S] \leftarrow \mathbf{Z}[S \times S \times S] \leftarrow \ldots$$

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix $s \in S$ and define $K : n_{(s_0, \ldots, s_p)} \mapsto n_{(s, s_0, \ldots, s_p)}$. One easily checks that K is a nullhomotopy for the operator

$$\delta : \eta_{(s_0, \ldots, s_p)} \mapsto \sum\nolimits_{i=0}^{p} (-1)^p \eta_{(s_0, \ldots, \hat{s}_i, \ldots, s_p)}.$$

See Cohomology on Sites, Lemma 21.9.4 for more details. □

03OR **Lemma 59.18.7.** *Notation and assumptions as in Definition 59.18.1. If $\mathcal{I}$ is an injective object of $PAb(\mathcal{C})$, then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$.*

Proof. The Čech complex is the result of applying the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ to the complex $\mathbf{Z}^\bullet_{\mathcal{U}}$, i.e.,

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = H^p(\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})).$$

But we have just seen that $\mathbf{Z}^\bullet_{\mathcal{U}}$ is exact in negative degrees, and the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact, hence $\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})$ is exact in positive degrees. □

03OS **Theorem 59.18.8.** *Notation and assumptions as in Definition 59.18.1. On $PAb(\mathcal{C})$ the functors $\check{H}^p(\mathcal{U}, -)$ are the right derived functors of $\check{H}^0(\mathcal{U}, -)$.*

Proof. By the Lemma 59.18.7, the functors $\check{H}^p(\mathcal{U}, -)$ are universal δ-functors since they are effaceable. So are the right derived functors of $\check{H}^0(\mathcal{U}, -)$. Since they agree in degree 0, they agree by the universal property of universal δ-functors. For more details see Cohomology on Sites, Lemma 21.9.6. □

03OT **Remark 59.18.9.** Observe that all of the preceding statements are about presheaves so we haven't made use of the topology yet.

59.19. The Čech-to-cohomology spectral sequence

03OU This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

03OV **Lemma 59.19.1.** *The forgetful functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ transforms injectives into injectives.*

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 21.10.1. □

03OW **Theorem 59.19.2.** *Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of $U \in \mathrm{Ob}(\mathcal{C})$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $V \mapsto H^q(V, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(\mathcal{C})$, and consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ and the maps

$$\begin{array}{ccc} \Gamma(U, I^\bullet) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet) \\ & & \uparrow \\ & & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \end{array}$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $Ab(\mathcal{C})$, hence by Lemma 59.19.1, it is a complex of injectives in $PAb(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees (Lemma 59.18.7), and the kernel of $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{I}^\bullet)$ is equal

to $\Gamma(U, \mathcal{I}^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the qth cohomology group of the pth column is

$$\prod_{i_0,\ldots,i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = \prod_{i_0,\ldots,i_p} \underline{H}^q(\mathcal{F})(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the E_2-page is as prescribed. For more details see Cohomology on Sites, Lemma 21.10.6. □

03OX **Remark 59.19.3.** This is a Grothendieck spectral sequence for the composition of functors

$$Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C}) \xrightarrow{\check{H}^0} Ab.$$

59.20. Big and small sites of schemes

03X7 Let S be a scheme. Let τ be one of the topologies we will be discussing. Thus $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = étale$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 59.25. In order to proceed with the discussion of cohomology of quasi-coherent sheaves it is convenient to introduce the big τ-site and in case $\tau \in \{étale, Zariski\}$, the small τ-site of S. In order to do this we first introduce the notion of a τ-covering.

03X8 **Definition 59.20.1.** (See Topologies, Definitions 34.7.1, 34.6.1, 34.5.1, 34.4.1, and 34.3.1.) Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i \in I}$ with fixed target is called a *τ-covering* if and only if each f_i is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

The class of all τ-coverings satisfies the axioms (1), (2) and (3) of Definition 59.10.2 (our definition of a site), see Topologies, Lemmas 34.7.3, 34.6.3, 34.5.3, 34.4.3, and 34.3.2.

Let us introduce the sites we will be working with. Contrary to what happens in [**AGV71**], we do not want to choose a universe. Instead we pick a "partial universe" (which is a suitably large set as in Sets, Section 3.5), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme S is contained in the partial universe. Having picked the underlying category we pick a suitably large set of τ-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of S (just as in [**AGV71**] the actual choice of universe doesn't matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

03XB **Definition 59.20.2.** Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$.

(1) A *big τ-site of S* is any of the sites $(Sch/S)_\tau$ constructed as explained above and in more detail in Topologies, Definitions 34.7.8, 34.6.8, 34.5.8, 34.4.8, and 34.3.7.

(2) If $\tau \in \{étale, Zariski\}$, then the *small τ-site of S* is the full subcategory S_τ of $(Sch/S)_\tau$ whose objects are schemes T over S whose structure morphism $T \to S$ is étale, resp. an open immersion. A covering in S_τ is a covering $\{U_i \to U\}$ in $(Sch/S)_\tau$ such that U is an object of S_τ.

The underlying category of the site $(Sch/S)_\tau$ has reasonable "closure" properties, i.e., given a scheme T in it any locally closed subscheme of T is isomorphic to an object of $(Sch/S)_\tau$. Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme $\mathrm{Spec}(R)$ one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in $(Sch/S)_\tau$ will take you outside of it. Also note that, given an object T of $(Sch/S)_\tau$ there will exist τ-coverings $\{T_i \to T\}_{i\in I}$ (as in Definition 59.20.1) which are not coverings in $(Sch/S)_\tau$ for example because the schemes T_i are not objects of the category $(Sch/S)_\tau$. But our choice of the sites $(Sch/S)_\tau$ is such that there always does exist a covering $\{U_j \to T\}_{j\in J}$ of $(Sch/S)_\tau$ which refines the covering $\{T_i \to T\}_{i\in I}$, see Topologies, Lemmas 34.7.7, 34.6.7, 34.5.7, 34.4.7, and 34.3.6. We will mostly ignore these issues in this chapter.

If $\mathcal{F}$ is a sheaf on $(Sch/S)_\tau$ or S_τ, then we denote

$$H^p_\tau(U, \mathcal{F}), \text{ in particular } H^p_\tau(S, \mathcal{F})$$

the cohomology groups of $\mathcal{F}$ over the object U of the site, see Section 59.14. Thus we have $H^p_{fppf}(S, \mathcal{F})$, $H^p_{syntomic}(S, \mathcal{F})$, $H^p_{smooth}(S, \mathcal{F})$, $H^p_{étale}(S, \mathcal{F})$, and $H^p_{Zar}(S, \mathcal{F})$. The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

03YX **Lemma 59.20.3.** *Let $\tau \in \{étale, Zariski\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(Sch/S)_\tau$, then the cohomology groups of $\mathcal{F}$ over S agree with the cohomology groups of $\mathcal{F}|_{S_\tau}$ over S.*

Proof. By Topologies, Lemmas 34.3.14 and 34.4.14 the functors $S_\tau \to (Sch/S)_\tau$ satisfy the hypotheses of Sites, Lemma 7.21.8. Hence our lemma follows from Cohomology on Sites, Lemma 21.7.2. □

The category of sheaves on the big or small étale site of S depends only on the full subcategory of $(Sch/S)_{étale}$ or $S_{étale}$ consisting of affines and one only needs to consider the standard étale coverings between them (as defined below). This gives rise to sites $(Aff/S)_{étale}$ and $S_{affine,étale}$, see Topologies, Definition 34.4.8. The comparison results are proven in Topologies, Lemmas 34.4.11 and 34.4.12. Here is our definition of standard coverings in some of the topologies we will consider in this chapter.

03X9 **Definition 59.20.4.** (See Topologies, Definitions 34.7.5, 34.6.5, 34.5.5, 34.4.5, and 34.3.4.) Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let T be an affine scheme. A *standard τ-covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, and each f_j flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in T and $T = \bigcup f_j(U_j)$.

03XA **Lemma 59.20.5.** *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Any τ-covering of an affine scheme can be refined by a standard τ-covering.*

Proof. See Topologies, Lemmas 34.7.4, 34.6.4, 34.5.4, 34.4.4, and 34.3.3. □

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma 59.21.2 below. For notation and terminology used in this lemma we refer to Topologies, Section 34.12.

03YY **Lemma 59.20.6.** *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $(Sch/S)_\tau$ and $(Sch'/S)_\tau$ be two big τ-sites of S, and assume that the first is contained in the second. In this case*

(1) *for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_\tau$ and any object U of $(Sch/S)_\tau$ we have*
$$H^p_\tau(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p_\tau(U, \mathcal{F}')$$
In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

(2) *for any abelian sheaf $\mathcal{F}$ on $(Sch/S)_\tau$ there is an abelian sheaf $\mathcal{F}'$ on $(Sch/S)'_\tau$ whose restriction to $(Sch/S)_\tau$ is isomorphic to $\mathcal{F}$.*

Proof. By Topologies, Lemma 34.12.2 the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 21.7.2. □

59.21. The étale topos

04HP A *topos* is the category of sheaves of sets on a site, see Sites, Definition 7.15.1. Hence it is customary to refer to the use the phrase "étale topos of a scheme" to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

04HQ **Definition 59.21.1.** Let S be a scheme.

(1) The *étale topos*, or the *small étale topos* of S is the category $Sh(S_{étale})$ of sheaves of sets on the small étale site of S.
(2) The *Zariski topos*, or the *small Zariski topos* of S is the category $Sh(S_{Zar})$ of sheaves of sets on the small Zariski site of S.
(3) For $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ a *big τ-topos* is the category of sheaves of set on a big τ-topos of S.

Note that the small Zariski topos of S is simply the category of sheaves of sets on the underlying topological space of S, see Topologies, Lemma 34.3.12. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

It turns out that the big or small étale topos only depends on the full subcategory of $(Sch/S)_{étale}$ or $S_{étale}$ consisting of affines, see Topologies, Lemmas 34.4.11 and 34.4.12. We will use this for example in the proof of the following lemma.

0958 **Lemma 59.21.2.** *Let S be a scheme. The étale topos of S is independent (up to canonical equivalence) of the construction of the small étale site in Definition 59.20.2.*

Proof. We have to show, given two big étale sites $Sch_{étale}$ and $Sch'_{étale}$ containing S, then $Sh(S_{étale}) \cong Sh(S'_{étale})$ with obvious notation. By Topologies, Lemma 34.12.1 we may assume $Sch_{étale} \subset Sch'_{étale}$. By Sets, Lemma 3.9.9 any affine scheme étale over S is isomorphic to an object of both $Sch_{étale}$ and $Sch'_{étale}$. Thus the induced functor $S_{affine,étale} \to S'_{affine,étale}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Topologies, Lemma 34.4.12. □

59.22. Cohomology of quasi-coherent sheaves

03OY We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ with $A \to B$ faithfully flat.

03OZ **Lemma 59.22.1.** *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(Sch/S)_\tau$, or on S_τ in case $\tau = étale$, and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a standard τ-covering of this site. Let $V = \coprod_{i\in I} U_i$. Then*

(1) *V is an affine scheme,*
(2) *$\mathcal{V} = \{V \to U\}$ is an fpqc covering and also a τ-covering unless $\tau = Zariski$,*
(3) *the Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ agree.*

Proof. The definition of a standard τ-covering is given in Topologies, Definition 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5. By definition each of the schemes U_i is affine and I is a finite set. Hence V is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 59.15.3. Excepting the Zariski case, the covering $\mathcal{V}$ is also a τ-covering, see Topologies, Definition 34.4.1, 34.5.1, 34.6.1, and 34.7.1.

Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation (21.8.2.1). Next, we observe that if $T = \coprod_{j\in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j\in J}$ is a Zariski covering, and so

03XC (59.22.1.1) $$\mathcal{F}(T) = \mathcal{F}(\coprod\nolimits_{j\in J} T_j) = \prod\nolimits_{j\in J} \mathcal{F}(T_j)$$

by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because

$$V \times_U \ldots \times_U V = \coprod\nolimits_{i_0,\ldots i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

as schemes. □

Note that Equality (59.22.1.1) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

03P0 **Remark 59.22.2.** In the statement of Lemma 59.22.1 the covering $\mathcal{U}$ is a refinement of $\mathcal{V}$ but not the other way around. Coverings of the form $\{V \to U\}$ do not form an initial subcategory of the category of all coverings of U. Yet it is still true that we can compute Čech cohomology $\check{H}^n(U, \mathcal{F})$ (which is defined as the colimit

over the opposite of the category of coverings $\mathcal{U}$ of U of the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$) in terms of the coverings $\{V \to U\}$. We will formulate a precise lemma (it only works for sheaves) and add it here if we ever need it.

03P1 **Lemma 59.22.3** (Locality of cohomology)**.** *Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, U an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of U in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then ξ is represented by a cocycle $\tilde{\xi} \in \mathcal{I}^p(U)$ with $d^p(\tilde{\xi}) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ in exact in $Ab(\mathcal{C})$, which means that there exists a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\tilde{\xi}|_{U_i} = d^{p-1}(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\tilde{\xi}|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 21.7.3. □

03P2 **Theorem 59.22.4.** *Let S be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(Sch/S)_\tau$ for $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ or $S_{étale}$. Then*

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

1. *the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme S, and*
2. *the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition 59.17.1) on the site $\mathcal{C}$.*

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that U is affine. Then we want to prove that $H^p_\tau(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on p.

p = 1 Pick $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 59.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. Applying the spectral sequence of Theorem 59.19.2, we see that ξ comes from a cohomology class $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \{\coprod_{i \in I} U_i \to U\}$. By Lemma 59.22.1, $\check{H}^\bullet(\mathcal{U}, \mathcal{F}^a) = \check{H}^\bullet(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ and $f^*\mathcal{F} = \widetilde{M}$ for some A-module M, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A)_\bullet \otimes_A M$. Now by Lemma 59.16.4, $H^p((B/A)_\bullet \otimes_A M) = 0$ for $p > 0$, hence $\check{\xi} = 0$ and so $\xi = 0$.

p > 1 Pick $\xi \in H^p_\tau(U, \mathcal{F}^a)$. By Lemma 59.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. We apply the spectral sequence of Theorem 59.19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that ξ must come from a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 59.16.4 again, we see that the Čech cohomology class $\check{\xi}$ must be zero, hence $\xi = 0$.

Next, assume that U is separated. Choose an affine open covering $U = \bigcup_{i \in I} U_i$ of U. The family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine since U is separated. So all rows of the spectral sequence of Theorem 59.19.2 are zero, except the zeroth row. Therefore

$$H^p_\tau(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma 30.2.6.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won't treat it. It follows from Descent, Proposition 35.9.3 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 21.7.1. □

03P3 **Remark 59.22.5.** Comment on Theorem 59.22.4. Since S is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

59.23. Examples of sheaves

03YZ Let S and τ be as in Section 59.20. We have already seen that any representable presheaf is a sheaf on $(Sch/S)_\tau$ or S_τ, see Lemma 59.15.8 and Remark 59.15.9. Here are some special cases.

03P4 **Definition 59.23.1.** On any of the sites $(Sch/S)_\tau$ or S_τ of Section 59.20.

(1) The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
(2) Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
(3) The *constant sheaf* $\underline{\mathbf{Z}/n\mathbf{Z}}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbf{Z}/n\mathbf{Z}$.

The first is a sheaf by Theorem 59.17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 39.4. The third by the finite étale group scheme $\mathbf{Z}/n\mathbf{Z}_S$ sometimes denoted $(\mathbf{Z}/n\mathbf{Z})_S$ which is just n copies of S endowed with the obvious group scheme structure over S, see Groupoids, Example 39.5.6 and the following remark.

03P5 **Remark 59.23.2.** Let G be an abstract group. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 59.20 the sheafification $\underline{G}$ of the constant presheaf associated to G in the *Zariski topology* of the site already gives

$$\Gamma(U, \underline{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme G_S according to Groupoids, Example 39.5.6. By Lemma 59.15.8 any representable presheaf satisfies the sheaf condition for the τ-topology as well, and hence we conclude that the Zariski sheafification $\underline{G}$ above is also the τ-sheafification.

04HS **Definition 59.23.3.** Let S be a scheme. The *structure sheaf* of S is the sheaf of rings $\mathcal{O}_S$ on any of the sites S_{Zar}, $S_{étale}$, or $(Sch/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{étale}}$ to stress the fact that we are working on the small étale site of S.

03P6 **Remark 59.23.4.** In the terminology introduced above a special case of Theorem 59.22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{étale}(X, \mathbf{G}_a) = H^p_{Zar}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of X.

59.24. Picard groups

03P7 The following theorem is sometimes called "Hilbert 90".

03P8 **Theorem 59.24.1.** *For any scheme X we have canonical identifications*

$$\begin{aligned} H^1_{fppf}(X, \mathbf{G}_m) &= H^1_{syntomic}(X, \mathbf{G}_m) \\ &= H^1_{smooth}(X, \mathbf{G}_m) \\ &= H^1_{étale}(X, \mathbf{G}_m) \\ &= H^1_{Zar}(X, \mathbf{G}_m) \\ &= \mathrm{Pic}(X) \\ &= H^1(X, \mathcal{O}_X^*) \end{aligned}$$

Proof. Let τ be one of the topologies considered in Section 59.20. By Cohomology on Sites, Lemma 21.6.1 we see that $H^1_\tau(X, \mathbf{G}_m) = H^1_\tau(X, \mathcal{O}_\tau^*) = \mathrm{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(Sch/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 59.17.4 or the more precise Descent, Proposition 35.8.9 we see that $\mathrm{Pic}(\mathcal{O}_\tau) = \mathrm{Pic}(X)$. The last equality is proved in the same way. □

59.25. The étale site

03P9 At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

59.26. Étale morphisms

03PA For more details, see Morphisms, Section 29.36 for the formal definition and Étale Morphisms, Sections 41.11, 41.12, 41.13, 41.14, 41.16, and 41.19 for a survey of interesting properties of étale morphisms.

Recall that an algebra A over an algebraically closed field k is *smooth* if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme X over k is *smooth* over k if it is locally of finite type and each affine open is the spectrum of a smooth k-algebra. If k is not algebraically closed then a k-algebra A is a smooth k-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is *smooth* if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 29.34. Using this we may define an étale morphism as follows.

03PB **Definition 59.26.1.** A morphism of schemes is *étale* if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

03PC **Proposition 59.26.2.** *Facts on étale morphisms.*

(1) *Let k be a field. A morphism of schemes $U \to \mathrm{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \mathrm{Spec}(k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*
(2) *Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:*
   (a) *φ is étale,*
   (b) *φ is locally finitely presented, flat, and all its fibres are étale,*
   (c) *φ is flat, unramified and locally of finite presentation.*
(3) *A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.*
(4) *The base change of an étale morphism is étale.*
(5) *Compositions of étale morphisms are étale.*
(6) *Fibre products and products of étale morphisms are étale.*
(7) *An étale morphism has relative dimension 0.*
(8) *Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.*
(9) *Étale morphisms are open.*
(10) *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 29.36.7. (2) Morphisms, Lemmas 29.36.8 and 29.36.16. (3) Algebra, Lemma 10.143.2. (4) Morphisms, Lemma 29.36.4. (5) Morphisms, Lemma 29.36.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 29.36.6 and 29.29.5. (8) See Algebra, Lemmas 10.163.7 and 10.163.5, see also more results of this kind in Étale Morphisms, Section 41.19. (9) See Morphisms, Lemma 29.25.10 and 29.36.12. (10) See Morphisms, Lemma 29.36.18. □

03PD **Definition 59.26.3.** A ring map $A \to B$ is called *standard étale* if $B \cong (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B. Then $A[t]/(f)$ is a finite free A-module of rank equal to the degree of the monic polynomial f. Hence B, as a localization of this free algebra is finitely presented and flat over A. To finish the proof that B is étale it suffices to show that the fibre rings

$$\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_A (A[t]/(f))_g \cong \kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(\mathfrak{p})[t]$ are the images of f and g. Let

$$\overline{f} = \overline{f}_1 \ldots \overline{f}_a \overline{f}_{a+1}^{e_1} \cdots \overline{f}_{a+b}^{e_b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \ldots, e_b > 0$. By assumption $\mathrm{d}\overline{f}/\mathrm{d}t$ is invertible in $\kappa(\mathfrak{p})[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f}) \cong \prod\nolimits_{i \in I} \kappa(\mathfrak{p})[t]/(\overline{f}_i)$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices i such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $\mathrm{d}\overline{f}/\mathrm{d}t$ in the factor $\kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is clearly equal to a unit times $\mathrm{d}\overline{f}_i/\mathrm{d}t$. Hence we conclude that $\kappa_i = \kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(\mathfrak{p})$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa_i/\kappa(\mathfrak{p})$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is *étale at a prime* $\mathfrak{q}$ of B if there exists $h \in B$, $h \notin \mathfrak{q}$ such that $A \to B_h$ is étale. Here is the result.

03PE **Theorem 59.26.4.** *A ring map $A \to B$ is étale at a prime $\mathfrak{q}$ if and only if there exists $g \in B$, $g \notin \mathfrak{q}$ such that B_g is standard étale over A.*

Proof. See Algebra, Proposition 10.144.4. □

59.27. Étale coverings

03PF We recall the definition.

03PG **Definition 59.27.1.** An *étale covering* of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

(1) each φ_i is an étale morphism,
(2) the U_i cover U, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

03PH **Lemma 59.27.2.** *Any étale covering is an fpqc covering.*

Proof. (See also Topologies, Lemma 34.9.7.) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since φ_i is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of V. Further, since V is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true *a fortiori* for étale coverings. For instance, the étale site is subcanonical.

03PI **Definition 59.27.3.** (For more details see Section 59.20, or Topologies, Section 34.4.) Let S be a scheme. The *big étale site over S* is the site $(Sch/S)_{étale}$, see Definition 59.20.2. The *small étale site over S* is the site $S_{étale}$, see Definition 59.20.2. We define similarly the *big* and *small Zariski sites* on S, denoted $(Sch/S)_{Zar}$ and S_{Zar}.

Loosely speaking the big étale site of S is made up out of schemes over S and coverings the étale coverings. The small étale site of S is made up out of schemes étale over S with coverings the étale coverings. Actually any morphism between objects of $S_{étale}$ is étale, in virtue of Proposition 59.26.2, hence to check that $\{U_i \to U\}_{i \in I}$ in $S_{étale}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the "opens" of the étale topology on S. It is a full subcategory of the big étale site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

03PJ **Proposition 59.27.4.** *Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(Sch/S)_{étale}$. Then $\mathcal{F}|_{S_{étale}}$ is a sheaf on $S_{étale}$ and*

$$H^p_{étale}(S, \mathcal{F}|_{S_{étale}}) = H^p_{étale}(S, \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a special case of Lemma 59.20.3. □

In accordance with the general notation introduced in Section 59.20 we write $H^p_{étale}(S, \mathcal{F})$ for the above cohomology group.

59.28. Kummer theory

03PK Let $n \in \mathbf{N}$ and consider the functor μ_n defined by

$$\begin{array}{ccc} Sch^{opp} & \longrightarrow & Ab \\ S & \longmapsto & \mu_n(S) = \{t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1\}. \end{array}$$

By Groupoids, Example 39.5.2 this is a representable functor, and the scheme representing it is denoted μ_n also. By Lemma 59.15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it also satisfies the sheaf condition for the étale, Zariski, etc topology).

03PL **Lemma 59.28.1.** *If $n \in \mathcal{O}_S^*$ then*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on both the small and big étale site of S.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 27.3 or 27.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, n and T^{n-1} are invertible in B, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of U we conclude that π is étale. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

03PM **Remark 59.28.2.** Lemma 59.28.1 is false when "étale" is replaced with "Zariski". Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 34.5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 59.24.1 and Lemma 59.28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0_{étale}(S, \mu_{n,S}) & \longrightarrow & \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{(\cdot)^n} & \Gamma(S, \mathcal{O}_S^*) \\ & & & \searrow & & & \\ & & H^1_{étale}(S, \mu_{n,S}) & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(S) \\ & & & \searrow & & & \\ & & H^2_{étale}(S, \mu_{n,S}) & \longrightarrow & \ldots & & \end{array}$$

at least if n is invertible on S. When n is not invertible on S we can apply the following lemma.

040N **Lemma 59.28.3.** *For any $n \in \mathbf{N}$ the sequence*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(Sch/S)_{fppf}$ and $(Sch/S)_{syntomic}$.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is weaker than the fppf topology, see Topologies, Lemma 34.7.2, it suffices to prove this for the syntomic topology. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find a syntomic cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 27.3 or 27.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, B is a global relative complete intersection over A, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of U we conclude that π is syntomic. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

040O **Remark 59.28.4.** Lemma 59.28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 59.24.1 and Lemma 59.28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0_{fppf}(S, \mu_{n,S}) & \longrightarrow & \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{(\cdot)^n} & \Gamma(S, \mathcal{O}_S^*) & \\ & & \searrow & & & & \\ & H^1_{fppf}(S, \mu_{n,S}) & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(S) & \\ & & \searrow & & & & \\ & H^2_{fppf}(S, \mu_{n,S}) & \longrightarrow & \ldots & & & \end{array}$$

for any scheme S and any integer n. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbf{N}$ and let S be any scheme. There is another more direct way to describe the first cohomology group with values in μ_n. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the nth tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves such that the diagram

$$\begin{array}{ccc} \mathcal{L}^{\otimes n} & \xrightarrow[\alpha]{} & \mathcal{O}_S \\ {\scriptstyle \varphi^{\otimes n}}\downarrow & & \downarrow{\scriptstyle 1} \\ (\mathcal{L}')^{\otimes n} & \xrightarrow{\alpha'} & \mathcal{O}_S \end{array}$$

commutes. Thus we have

(59.28.4.1)

040P
$$Isom_S((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \emptyset & \text{if} \quad \text{they are not isomorphic} \\ H^0(S, \mu_{n,S}) \cdot \varphi & \text{if} \quad \varphi \text{ isomorphism of pairs} \end{cases}$$

Moreover, given two pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ the tensor product

$$(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$$

is another pair. The pair $(\mathcal{O}_S, 1)$ is an identity for this tensor product operation, and an inverse is given by

$$(\mathcal{L}, \alpha)^{-1} = (\mathcal{L}^{\otimes -1}, \alpha^{\otimes -1}).$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$(\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)$$

is an isomorphism hence every element of this group has order dividing n. We warn the reader that this group is in general **not** the n-torsion in $\mathrm{Pic}(S)$.

040Q **Lemma 59.28.5.** *Let S be a scheme. There is a canonical identification*

$$H^1_{étale}(S, \mu_n) = \textit{group of pairs } (\mathcal{L}, \alpha) \textit{ up to isomorphism as above}$$

if n is invertible on S. In general we have

$$H^1_{fppf}(S, \mu_n) = \textit{group of pairs } (\mathcal{L}, \alpha) \textit{ up to isomorphism as above.}$$

The same result holds with fppf replaced by syntomic.

Proof. We first prove the second isomorphism. Let $(\mathcal{L}, \alpha)$ be a pair as above. Choose an affine open covering $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Say $s_i \in \mathcal{L}(U_i)$ is a generator. Then $\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}_S^*(U_i)$. Writing $U_i = \mathrm{Spec}(A_i)$ we see there exists a global relative complete intersection $A_i \to B_i = A_i[T]/(T^n - f_i)$ such that f_i maps to an nth power in B_i. In other words, setting $V_i = \mathrm{Spec}(B_i)$ we obtain a syntomic covering $\mathcal{V} = \{V_i \to S\}_{i \in I}$ and trivializations $\varphi_i : (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)$.

We will use this result (the existence of the covering $\mathcal{V}$) to associate to this pair a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$. We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps $V_i \times_S V_j$ we have the isomorphism

$$(\mathcal{O}_{V_i \times_S V_j}, 1) \xrightarrow{\mathrm{pr}_0^* \varphi_i^{-1}} (\mathcal{L}|_{V_i \times_S V_j}, \alpha|_{V_i \times_S V_j}) \xrightarrow{\mathrm{pr}_1^* \varphi_j} (\mathcal{O}_{V_i \times_S V_j}, 1)$$

of pairs. By (59.28.4.1) this is given by an element $\zeta_{ij} \in \mu_n(V_i \times_S V_j)$. We omit the verification that these ζ_{ij}'s give a 1-cocycle, i.e., give an element $(\zeta_{i_0 i_1}) \in \check{C}(\mathcal{V}, \mu_n)$ with $d(\zeta_{i_0 i_1}) = 0$. Thus its class is an element in $\check{H}^1(\mathcal{V}, \mu_n)$ and by Theorem 59.19.2 it maps to a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$.

Second construction: Using torsors. Consider the presheaf

$$\mu_n(\mathcal{L}, \alpha) : U \longmapsto Isom_U((\mathcal{O}_U, 1), (\mathcal{L}, \alpha)|_U)$$

on $(Sch/S)_{syntomic}$. We may view this as a subpresheaf of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, \mathcal{L})$ (internal hom sheaf, see Modules on Sites, Section 18.27). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (59.28.4.1) this sheaf has a free action of the sheaf $\mu_{n,S}$. Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover $\mathcal{V}$. Hence $\mu_n(\mathcal{L}, \alpha)$ is a $\mu_{n,S}$-torsor and by Cohomology on Sites, Lemma 21.4.3 we obtain a corresponding element of $H^1_{syntomic}(S, \mu_{n,S})$.

Ok, now we have to still show the following

(1) The two constructions give the same cohomology class.
(2) Isomorphic pairs give rise to the same cohomology class.
(3) The cohomology class of $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')$ is the sum of the cohomology classes of $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$.
(4) If the cohomology class is trivial, then the pair is trivial.
(5) Any element of $H^1_{syntomic}(S, \mu_{n,S})$ is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Čech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 21.4.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose $\xi \in H^1_{syntomic}(S, \mu_{n,S})$. Then ξ maps to an element $\overline{\xi} \in H^1_{syntomic}(S, \mathbf{G}_{m,S})$ with $n\overline{\xi} = 0$. By Theorem 59.24.1 we see that $\overline{\xi}$ corresponds to an invertible sheaf $\mathcal{L}$ whose nth tensor power is isomorphic to $\mathcal{O}_S$. Hence there exists a pair $(\mathcal{L}, \alpha')$ whose cohomology class ξ' has the same image $\overline{\xi'}$ in $H^1_{syntomic}(S, \mathbf{G}_{m,S})$. Thus it suffices to show that $\xi - \xi'$ is the class of a pair. By construction, and the long exact cohomology sequence above, we see that $\xi - \xi' = \partial(f)$ for some $f \in H^0(S, \mathcal{O}_S^*)$. Consider the pair $(\mathcal{O}_S, f)$. We omit the verification that the cohomology class

of this pair is $\partial(f)$, which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if n is invertible on S, then the covering $\mathcal{V}$ constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 59.28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 59.28.1. □

59.29. Neighborhoods, stalks and points

03PN We can associate to any geometric point of S a stalk functor which is exact. A map of sheaves on $S_{étale}$ is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme S has enough points. It also turns out that any point of the small étale topos of S (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of S can be understood in terms of geometric points and neighbourhoods.

03PO **Definition 59.29.1.** Let S be a scheme.

(1) A *geometric point* of S is a morphism $\mathrm{Spec}(k) \to S$ where k is algebraically closed. Such a point is usually denoted $\overline{s}$, i.e., by an overlined small case letter. We often use $\overline{s}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{s})$ to denote k.
(2) We say $\overline{s}$ *lies over* s to indicate that $s \in S$ is the image of $\overline{s}$.
(3) An *étale neighborhood* of a geometric point $\overline{s}$ of S is a commutative diagram

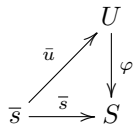

where φ is an étale morphism of schemes. We write $(U, \overline{u}) \to (S, \overline{s})$.
(4) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an S-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

03PP **Remark 59.29.2.** Since U and U' are étale over S, any S-morphism between them is also étale, see Proposition 59.26.2. In particular all morphisms of étale neighborhoods are étale.

04HT **Remark 59.29.3.** Let S be a scheme and $s \in S$ a point. In More on Morphisms, Definition 37.35.1 we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of (S, s). If $\overline{s}$ is a geometric point of S lying over s, then any étale neighbourhood $(U, \overline{u}) \to (S, \overline{s})$ gives rise to an étale neighbourhood (U, u) of (S, s) by taking $u \in U$ to be the unique point of U such that $\overline{u}$ lies over u. Conversely, given an étale neighbourhood (U, u) of (S, s) the residue field extension $\kappa(u)/\kappa(s)$ is finite separable (see Proposition 59.26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\overline{s})$ over $\kappa(s)$. In other words, we can find a geometric point $\overline{u}$ of U lying over u such that $(U, \overline{u})$ is an étale neighbourhood of $(S, \overline{s})$. We will use these observations to go between the two types of étale neighbourhoods.

03PQ **Lemma 59.29.4.** *Let S be a scheme, and let $\overline{s}$ be a geometric point of S. The category of étale neighborhoods is cofiltered. More precisely:*

(1) *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{s}$ in S. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Proposition 59.26.2. The map $\overline{s} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2. For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1,h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_S U'. \end{array}$$

Since $\overline{u}$ and $\overline{u}'$ agree over S with $\overline{s}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Proposition 59.26.2). Hence the vertical arrow (h_1, h_2) is étale by Remark 59.29.2 above. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem. □

03PR **Lemma 59.29.5.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $(U, \overline{u})$ be an étale neighborhood of $\overline{s}$. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is not empty for some i. Then look at the cartesian diagram

$$\begin{array}{ccc} \overline{s} \times_{\overline{u}, U, \varphi_i} U_i & \xrightarrow[\mathrm{pr}_2]{} & U_i \\ {\scriptstyle \sigma}\uparrow\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \varphi_i} \\ \mathrm{Spec}(k) = \overline{s} & \xrightarrow{\overline{u}} & U \end{array}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Proposition 59.26.2. Therefore, $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of k, by Proposition 59.26.2. Here $\overline{s} = \mathrm{Spec}(k)$. But k is algebraically closed, so all these extensions are trivial, and there exists a section σ of pr_1. The composition $\mathrm{pr}_2 \circ \sigma$ gives a map compatible with $\overline{u}$. □

040R **Definition 59.29.6.** Let S be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{étale}$. Let $\overline{s}$ be a geometric point of S. The *stalk* of $\mathcal{F}$ at $\overline{s}$ is

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in S.

By Lemma 59.29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

04FM **Lemma 59.29.7.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Consider the functor*

$$u : S_{étale} \longrightarrow Sets,$$
$$U \longmapsto |U_{\overline{s}}| = \{\overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s}\}.$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then u defines a point p of the site $S_{étale}$ (Sites, Definition 7.32.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

Proof. In the proof of Lemma 59.29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of U lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 59.29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 59.29.5. Hence Sites, Proposition 7.33.3 applies, so p is a point of the site $S_{étale}$. Finally, our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.32.1.1 which proves the final assertion. □

04FN **Remark 59.29.8.** Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ and $\overline{s}' : \mathrm{Spec}(k') \to S$ be two geometric points of S. A *morphism $a : \overline{s} \to \overline{s}'$ of geometric points* is simply a morphism $a : \mathrm{Spec}(k) \to \mathrm{Spec}(k')$ such that $\overline{s}' \circ a = \overline{s}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{s}'$ to the category of étale neighbourhoods of $\overline{s}$ by the rule $(U, \overline{u}') \mapsto (U, \overline{u}' \circ a)$. Hence we obtain a canonical map

$$\mathcal{F}_{\overline{s}'} = \mathrm{colim}_{(U, \overline{u}')} \mathcal{F}(U) \longrightarrow \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U) = \mathcal{F}_{\overline{s}}$$

from Categories, Lemma 4.14.8. Using the description of elements of stalks as triples this maps the element of $\mathcal{F}_{\overline{s}'}$ represented by the triple $(U, \overline{u}', \sigma)$ to the element of $\mathcal{F}_{\overline{s}}$ represented by the triple $(U, \overline{u}' \circ a, \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 7.37.2, that a defines a morphism $a : p \to p'$ between the points p, p' of the site $S_{étale}$ associated to $\overline{s}, \overline{s}'$ by Lemma 59.29.7. There are more general morphisms of points (corresponding to specializations of points of S) which we will describe later, and which will not be isomorphisms, see Section 59.75.

03PT **Lemma 59.29.9.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S.*

(1) *The stalk functor $PAb(S_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*

(2) *We have* $(\mathcal{F}^\#)_{\overline{s}} = \mathcal{F}_{\overline{s}}$ *for any presheaf of sets* $\mathcal{F}$ *on* $S_{étale}$.
(3) *The functor* $Ab(S_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ *is exact.*
(4) *Similarly the functors* $PSh(S_{étale}) \to Sets$ *and* $Sh(S_{étale}) \to Sets$ *given by the stalk functor* $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ *are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.*

Proof. Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 18.36. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ comes from a point of the small étale site of S, see Lemma 59.29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on $PAb(S_{étale})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_{\overline{s}} \longrightarrow (\mathcal{F}^\#)_{\overline{s}}$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 59.13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_{\overline{s}}$. There exists an étale neighborhood $(U, \overline{u}) \to (S, \overline{s})$ such that σ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_{\overline{s}}$ then there exists a morphism of étale neighborhoods $(U', \overline{u}') \to (U, \overline{u})$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U'\}_{i \in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all i. By Lemma 59.29.5 there exist $i \in I$ and a morphism $\overline{u}'_i : \overline{s} \to U'_i$ such that $(U'_i, \overline{u}'_i) \to (U', \overline{u}') \to (U, \overline{u})$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \overline{u}'_i) \to (U, \overline{u})$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves κ is injective.

To show that the functor $Ab(S_{étale}) \to Ab$ is exact, consider any short exact sequence in $Ab(S_{étale})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/^p\mathcal{G} \to 0,$$

where $/^p$ denotes the quotient in $PAb(S_{étale})$. Taking stalks at $\overline{s}$, we see that $(\mathcal{H}/^p\mathcal{G})_{\overline{s}} = (\mathcal{H}/\mathcal{G})_{\overline{s}} = 0$, since the sheafification of $\mathcal{H}/^p\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 = (\mathcal{H}/^p\mathcal{G})_{\overline{s}}$$

is exact, since taking stalks is exact as a functor from presheaves. □

03PU **Theorem 59.29.10.** *Let* S *be a scheme. A map* $a : \mathcal{F} \to \mathcal{G}$ *of sheaves of sets is injective (resp. surjective) if and only if the map on stalks* $a_{\overline{s}} : \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ *is injective (resp. surjective) for all geometric points of* S. *A sequence of abelian sheaves on* $S_{étale}$ *is exact if and only if it is exact on all stalks at geometric points of* S.

Proof. The necessity of exactness on stalks follows from Lemma 59.29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of sheaves such that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in \mathrm{Ob}(S_{étale})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\overline{u} \to U$ lying over u and an étale neighborhood $(V_u, \overline{v}_u) \to (U, \overline{u})$ such that $s|_{V_u} = \alpha(s_{V_u})$

for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u \in U}$ is an étale covering on which the restrictions of s are in the image of the map α. Thus, α is surjective, see Sites, Section 7.11. □

040S **Remarks 59.29.11.** On points of the geometric sites.

(1) Theorem 59.29.10 says that the family of points of $S_{étale}$ given by the geometric points of S (Lemma 59.29.7) is conservative, see Sites, Definition 7.38.1. In particular $S_{étale}$ has enough points.

04FP (2) Suppose $\mathcal{F}$ is a sheaf on the big étale site of S. Let $T \to S$ be an object of the big étale site of S, and let $\bar{t}$ be a geometric point of T. Then we define $\mathcal{F}_{\bar{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{étale}}$ of $\mathcal{F}$ to the small étale site of T. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in \mathrm{Ob}((Sch/S)_{étale})$.

(3) The big étale site of S also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

04HU **Lemma 59.29.12.** *Let S be a scheme.*

(1) *Let p be a point of the small étale site $S_{étale}$ of S given by a functor $u : S_{étale} \to Sets$. Then there exists a geometric point $\overline{s}$ of S such that p is isomorphic to the point of $S_{étale}$ associated to $\overline{s}$ in Lemma 59.29.7.*

(2) *Let $p : Sh(pt) \to Sh(S_{étale})$ be a point of the small étale topos of S. Then p comes from a geometric point of S, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 59.29.6.*

Proof. By Sites, Lemma 7.32.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 7.33.3 the functor u has the following properties: (a) $u(S) = \{*\}$, (b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\coprod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 7.32.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf h_U. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since p^{-1} is exact.

Consider the restriction of u to S_{Zar}. By Sites, Examples 7.33.5 and 7.33.6 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{*\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \notin S'$. Note that if $\varphi : U \to S$ is an object of $S_{étale}$ then $\varphi(U) \subset S$ is open (see Proposition 59.26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \Leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \to S$ lying over s, see Definition 59.29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{étale}$ with U affine. Note that φ is separated, and that the fibre U_s of φ over s is an affine scheme over $\mathrm{Spec}(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions k_i of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma 41.18.2 to get an étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and W having no point lying over $\overline{v}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking V a bit we can assume that V has exactly one point lying over s, and hence W has no point lying over s. By the above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod\nolimits_{i=1,\ldots,n} u(V)$$

Note that $u(V) \neq \emptyset$ as s is in the image of $V \to S$. In particular, we see that in this situation $u(U)$ is a finite set with n elements.

Consider the limit

$$\lim\nolimits_{(V,\overline{v})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{v})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{v})$ with V affine. By the previous paragraph (applied with the roles of V and U switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma 59.29.4. Hence by Categories, Section 4.20 the limit is nonempty. Pick an element x from this limit. This means we obtain a $x_{V,\overline{v}} \in u(V)$ for every étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{v}') \to (V, \overline{v})$ we have $u(\varphi)(x_{V',\overline{v}'}) = x_{V,\overline{v}}$.

We will use the choice of x to construct a functorial bijective map

$$c : |U_{\overline{s}}| \longrightarrow u(U)$$

for $U \in \mathrm{Ob}(S_{étale})$ which will conclude the proof. See Lemma 59.29.7 and its proof for a description of $|U_{\overline{s}}|$. First we claim that it suffices to construct the map for U affine. We omit the proof of this claim. Assume $U \to S$ in $S_{étale}$ with U affine, and let $\overline{u} : \overline{s} \to U$ be an element of $|U_{\overline{s}}|$. Choose a $(V, \overline{v})$ such that $U \times_S V$ decomposes as in the third paragraph of the proof. Then the pair $(\overline{u}, \overline{v})$ gives a geometric point of $U \times_S V$ lying over $\overline{v}$ and determines one of the components U_i of $U \times_S V$. More precisely, there exists a section $\sigma : V \to U \times_S V$ of the projection pr_U such that $(\overline{u}, \overline{v}) = \sigma \circ \overline{v}$. Set $c(\overline{u}) = u(\mathrm{pr}_U)(u(\sigma)(x_{V,\overline{v}})) \in u(U)$. We have to check this is independent of the choice of $(V, \overline{v})$. By Lemma 59.29.4 the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood $\varphi : (V', \overline{v}') \to (V, \overline{v})$ and a choice of a section $\sigma' : V' \to U \times_S V'$ of the projection such that $(\overline{u}, \overline{v'}) = \sigma' \circ \overline{v}'$ we have $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$. Consider the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\varphi} & V \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ U \times_S V' & \xrightarrow{1\times\varphi} & U \times_S V \end{array}$$

Now, it may not be the case that this diagram commutes. The reason is that the schemes V' and V may not be connected, and hence the decompositions used to construct σ' and σ above may not be unique. But we do know that $\sigma \circ \varphi \circ \overline{v}' = (1 \times \varphi) \circ \sigma' \circ \overline{v}'$ by construction. Hence, since $U \times_S V$ is étale over S, there exists an open neighbourhood $V'' \subset V'$ of $\overline{v'}$ such that the diagram does commute when restricted to V'', see Morphisms, Lemma 29.35.17. This means we may extend the

diagram above to

$$\begin{array}{ccccc} V'' & \longrightarrow & V' & \xrightarrow[\varphi]{} & V \\ \downarrow{\scriptstyle \sigma'|_{V''}} & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ U \times_S V'' & \longrightarrow & U \times_S V' & \xrightarrow{1\times\varphi} & U \times_S V \end{array}$$

such that the left square and the outer rectangle commute. Since u is a functor this implies that $x_{V'',\overline{v}'}$ maps to the same element in $u(U \times_S V)$ no matter which route we take through the diagram. On the other hand, it maps to the elements $x_{V',\overline{v}'}$ and $x_{V,\overline{v}}$ in $u(V')$ and $u(V)$. This implies the desired equality $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$.

In a similar manner one proves that the construction $c : |U_{\overline{s}}| \to u(U)$ is functorial in U; details omitted. And finally, by the results of the third paragraph it is clear that the map c is bijective which ends the proof of the lemma. □

59.30. Points in other topologies

06VW In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 7.38 and Topologies, Section 34.2 ff for the terminology used in this section. All of the geometric sites have enough points.

06VX **Lemma 59.30.1.** *Let S be a scheme. All of the following sites have enough points $S_{affine,Zar}$, S_{Zar}, $S_{affine,étale}$, $S_{étale}$, $(Sch/S)_{Zar}$, $(Aff/S)_{Zar}$, $(Sch/S)_{étale}$, $(Aff/S)_{étale}$, $(Sch/S)_{smooth}$, $(Aff/S)_{smooth}$, $(Sch/S)_{syntomic}$, $(Aff/S)_{syntomic}$, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$.*

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site $(Aff/S)_\tau$, see Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11. The result for the sites $(Aff/S)_\tau$ follows immediately from Deligne's result Sites, Lemma 7.39.4.

The result for S_{Zar} is clear. The result for $S_{affine,Zar}$ follows from Deligne's result. The result for $S_{étale}$ either follows from (the proof of) Theorem 59.29.10 or from Topologies, Lemma 34.4.12 and Deligne's result applied to $S_{affine,étale}$. □

The lemma above guarantees the existence of points, but it doesn't tell us what these points look like. We can explicitly construct *some* points as follows. Suppose $\overline{s} : \mathrm{Spec}(k) \to S$ is a geometric point with k algebraically closed. Consider the functor

$$u : (Sch/S)_{fppf} \longrightarrow Sets, \quad u(U) = U(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), U).$$

Note that $U \mapsto U(k)$ commutes with finite limits as $S(k) = \{\overline{s}\}$ and $(U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)$. Moreover, if $\{U_i \to U\}$ is an fppf covering, then $\coprod U_i(k) \to U(k)$ is surjective. By Sites, Proposition 7.33.3 we see that u defines a point p of $(Sch/S)_{fppf}$ with stalks

$$\mathcal{F}_p = \mathrm{colim}_{(U,x)} \mathcal{F}(U)$$

where the colimit is over pairs $U \to S$, $x \in U(k)$ as usual. But... this category has an initial object, namely $(\mathrm{Spec}(k), \mathrm{id})$, hence we see that

$$\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(k))$$

which isn't terribly interesting! In fact, in general these points won't form a conservative family of points. A more interesting type of point is described in the following remark.

06VY **Remark 59.30.2.** Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let (p, u) be a point of the site $(Aff/S)_{fppf}$, see Sites, Sections 7.32 and 7.33. Let $B = \mathcal{O}_p$ be the stalk of the structure sheaf at the point p. Recall that This is discussed in [**Sch14**].

$$B = \mathrm{colim}_{(U,x)} \mathcal{O}(U) = \mathrm{colim}_{(\mathrm{Spec}(C),x_C)} C$$

where $x_C \in u(\mathrm{Spec}(C))$. It can happen that $\mathrm{Spec}(B)$ is an object of $(Aff/S)_{fppf}$ and that there is an element $x_B \in u(\mathrm{Spec}(B))$ mapping to the compatible system x_C. In this case the system of neighbourhoods has an initial object and it follows that $\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(B))$ for any sheaf $\mathcal{F}$ on $(Aff/S)_{fppf}$. It is straightforward to see that if $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ defines a point of $Sh((Aff/S)_{fppf})$, then B has to be a local A-algebra such that for every faithfully flat, finitely presented ring map $B \to B'$ there is a section $B' \to B$. Conversely, for any such A-algebra B the functor $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(B))$ is the stalk functor of a point. Details omitted. It is not clear what a general point of the site $(Aff/S)_{fppf}$ looks like.

59.31. Supports of abelian sheaves

04FQ First we talk about supports of local sections.

04HV **Lemma 59.31.1.** *Let S be a scheme. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of S (see Sites, Example 7.10.2). Then there exists a unique open $W \subset S$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to S$ in $\mathrm{Ob}(S_{étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\{\varphi : U \to \varphi(U)\}$ is a covering. Hence we can take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$. □

04FR **Lemma 59.31.2.** *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subset U$ such that*

1. *$W \subset U$ is the largest Zariski open subset of U such that $\sigma|_W = 0$,*
2. *for every $\varphi : V \to U$ in $S_{étale}$ we have*
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
3. *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $S_{étale}$. Note that $\varphi(V) \subset U$ is an open subset and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. By Remark 59.29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over s is well defined.

04FS **Definition 59.31.3.** Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$.

(1) The *support of* $\mathcal{F}$ is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over s.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of* σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 59.31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \mathbf{A}^1_{\mathbf{C}}$. Let $i_t : \mathrm{Spec}(\mathbf{C}) \to S$ be the inclusion of the point $t \in \mathbf{C}$. We will see later that $\mathcal{F}_t = i_{t,*}(\mathbf{Z}/2\mathbf{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 59.46. Then

$$\bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{F}_n$$

is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 59.29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

04FT **Lemma 59.31.4.** *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $U \in \mathrm{Ob}(S_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in U.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 59.31.2 part (3). Parts (5) and (6) follow from the other parts. □

04FU **Lemma 59.31.5.** *The support of a sheaf of rings on $S_{étale}$ is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

59.32. Henselian rings

03QD We begin by stating a theorem which has already been used many times in the Stacks project. There are many versions of this result; here we just state the algebraic version.

03QE **Theorem 59.32.1.** *Let $A \to B$ be finite type ring map and $\mathfrak{p} \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that*

(1) $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,
(2) $B \otimes_A A' = B_1 \times \ldots \times B_r \times C$,
(3) $A' \to B_i$ *is finite and there exists a unique prime* $q_i \subset B_i$ *lying over* $\mathfrak{p}'$, *and*
(4) *all irreducible components of the fibre* $\mathrm{Spec}(C \otimes_{A'} \kappa(\mathfrak{p}'))$ *of* C *over* $\mathfrak{p}'$ *have dimension at least 1.*

Proof. See Algebra, Lemma 10.145.3, or see [**GD67**, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 37.41. □

Recall Hensel's lemma. There are many versions of this lemma. Here are two:

(f) if $f \in \mathbf{Z}_p[T]$ monic and $f \bmod p = g_0 h_0$ with $gcd(g_0, h_0) = 1$ then f factors as $f = gh$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,
(r) if $f \in \mathbf{Z}_p[T]$, monic $a_0 \in \mathbf{F}_p$, $\bar{f}(a_0) = 0$ but $\bar{f}'(a_0) \neq 0$ then there exists $a \in \mathbf{Z}_p$ with $f(a) = 0$ and $\bar{a} = a_0$.

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

03QF **Definition 59.32.2.** (See Algebra, Definition 10.153.1.) A local ring $(R, \mathfrak{m}, \kappa)$ is called *henselian* if for all $f \in R[T]$ monic, for all $a_0 \in \kappa$ such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and $a \bmod \mathfrak{m} = a_0$.

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 10.160.1) that a complete local ring is a local ring $(R, \mathfrak{m})$ such that $R \cong \lim_n R/\mathfrak{m}^n$, i.e., it is complete and separated for the $\mathfrak{m}$-adic topology.

03QG **Theorem 59.32.3.** *Complete local rings are henselian.*

Proof. Newton's method. See Algebra, Lemma 10.153.9. □

03QH **Theorem 59.32.4.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent:*

(1) *R is henselian,*
(2) *for any $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ in $\kappa[T]$ with $\gcd(g_0, h_0) = 1$, there exists a factorization $f = gh$ in $R[T]$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,*
(3) *any finite R-algebra S is isomorphic to a finite product of local rings finite over R,*
(4) *any finite type R-algebra A is isomorphic to a product $A \cong A' \times C$ where $A' \cong A_1 \times \ldots \times A_r$ is a product of finite local R-algebras and all the irreducible components of $C \otimes_R \kappa$ have dimension at least 1,*
(5) *if A is an étale R-algebra and $\mathfrak{n}$ is a maximal ideal of A lying over $\mathfrak{m}$ such that $\kappa \cong A/\mathfrak{n}$, then there exists an isomorphism $\varphi : A \cong R \times A'$ such that $\varphi(\mathfrak{n}) = \mathfrak{m} \times A' \subset R \times A'$.*

Proof. This is just a subset of the results from Algebra, Lemma 10.153.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 10.153.3 but is formulated slightly differently. □

03QJ **Lemma 59.32.5.** *If R is henselian and A is a finite R-algebra, then A is a finite product of henselian local rings.*

Proof. See Algebra, Lemma 10.153.4. □

03QK **Definition 59.32.6.** A local ring R is called *strictly henselian* if it is henselian and its residue field is separably closed.

03QI **Example 59.32.7.** In the case $R = \mathbf{C}[[t]]$, the étale R-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset \mathrm{Spec}(R)$, so any étale covering can be refined by the covering $\{\mathrm{id} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

03QL **Theorem 59.32.8.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\kappa \subset \kappa^{sep}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where*

(1) *R^h, R^{sh} are filtered colimits of étale R-algebras,*
(2) *R^h is henselian, R^{sh} is strictly henselian,*
(3) *$\mathfrak{m}R^h$ (resp. $\mathfrak{m}R^{sh}$) is the maximal ideal of R^h (resp. R^{sh}), and*
(4) *$\kappa = R^h/\mathfrak{m}R^h$, and $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$ as extensions of κ.*

Proof. The structure of R^h and R^{sh} is described in Algebra, Lemmas 10.155.1 and 10.155.2. □

The rings constructed in Theorem 59.32.8 are called respectively the *henselization* and the *strict henselization* of the local ring R, see Algebra, Definition 10.155.3. Many of the properties of R are reflected in its (strict) henselization, see More on Algebra, Section 15.45.

59.33. Stalks of the structure sheaf

04HW In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding "usual" point.

04HX **Lemma 59.33.1.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of κ in $\kappa(\overline{s})$. Then there is a canonical identification*

$$(\mathcal{O}_{S,s})^{sh} \cong (\mathcal{O}_S)_{\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 59.32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at the geometric point $\overline{s}$.

Proof. Let $\mathrm{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. Thus we have $\kappa(\mathfrak{p}) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$(\mathcal{O}_S)_{\overline{s}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that U is affine and $U \to S$ factors through $\mathrm{Spec}(A)$. In other words, we see that

$$(\mathcal{O}_S)_{\overline{s}} = \mathrm{colim}_{(B,\mathfrak{q},\phi)} B$$

where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and a $\kappa(\mathfrak{p})$-algebra map $\phi : \kappa(\mathfrak{q}) \to \kappa(\overline{s})$. Note that since $\kappa(\mathfrak{q})$ is finite separable over $\kappa(\mathfrak{p})$ the image of ϕ is contained in κ^{sep}. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 10.155.11. □

03PS **Definition 59.33.2.** Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over the point $s \in S$.

(1) The *étale local ring of S at $\overline{s}$* is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at $\overline{s}$. We sometimes call this the *strict henselization of $\mathcal{O}_{S,s}$* relative to the geometric point $\overline{s}$. Notation used: $\mathcal{O}^{sh}_{S,\overline{s}}$.
(2) The *henselization of $\mathcal{O}_{S,s}$* is the henselization of the local ring of S at s. See Algebra, Definition 10.155.3, and Theorem 59.32.8. Notation: $\mathcal{O}^h_{S,s}$.
(3) The *strict henselization of S at $\overline{s}$* is the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$.
(4) The *henselization of S at s* is the scheme $\mathrm{Spec}(\mathcal{O}^h_{S,s})$.

Let $f : T \to S$ be a morphism of schemes. Let $\overline{t}$ be a geometric point of T with image $\overline{s}$ in S. Let $t \in T$ and $s \in S$ be their images. Then we obtain a canonical commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\mathcal{O}^{sh}_{T,\overline{t}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}^h_{T,t}) & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}^h_{S,s}) & \longrightarrow & S \end{array}$$

of henselizations and strict henselizations of T and S. You can prove this by choosing affine neighbourhoods of t and s and using the functoriality of (strict) henselizations given by Algebra, Lemmas 10.155.8 and 10.155.12.

04HY **Lemma 59.33.3.** *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \mathrm{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category of étale neighbourhoods (U, u) of (S, s) such that $\kappa(s) = \kappa(u)$.

Proof. This lemma is a copy of More on Morphisms, Lemma 37.35.5. □

03QM **Remark 59.33.4.** Let S be a scheme. Let $s \in S$. If S is locally Noetherian then $\mathcal{O}^h_{S,s}$ is also Noetherian and it has the same completion:

$$\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}^h_{S,s}}.$$

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}^h_{S,s} \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}^h_{S,s}$, but this is not true for the completion in general. Insert future references here.

04HZ **Lemma 59.33.5.** *Let S be a scheme. The small étale site $S_{étale}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 18.40.4.*

Proof. This follows because the stalks $(\mathcal{O}_S)_{\overline{s}} = \mathcal{O}_{S,\overline{s}}^{sh}$ are local, and because $S_{étale}$ has enough points, see Lemma 59.33.1, Theorem 59.29.10, and Remarks 59.29.11. See Modules on Sites, Lemmas 18.40.2 and 18.40.3 for the fact that this implies the small étale site is locally ringed. □

59.34. Functoriality of small étale topos

04I0 So far we haven't yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 34.4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

04I1 (59.34.0.1) $$u : Y_{étale} \longrightarrow X_{étale}, \quad V/Y \longmapsto X \times_Y V/X.$$

This functor has the following important properties

(1) u(final object) = final object,
(2) u preserves fibre products,
(3) if $\{V_j \to V\}$ is a covering in $Y_{étale}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{étale}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a *morphism of sites*

$$f_{small} : X_{étale} \longrightarrow Y_{étale},$$

see Sites, Definition 7.14.1 and Sites, Proposition 7.14.7. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and f_{small}^{-1} (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

59.35. Direct images

03PV Let us define the pushforward of a presheaf.

03PW **Definition 59.35.1.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_*\mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where u is as in Equation (59.34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \longrightarrow PSh(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_*\mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \longrightarrow PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

03PX **Remark 59.35.2.** We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_*\mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$.

03PY **Definition 59.35.3.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 59.35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

called *direct image* again.

The functor f_* on abelian sheaves is left exact. (See Homology, Section 12.7 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact on $X_{étale}$, then for every $U/X \in \mathrm{Ob}(X_{étale})$ the sequence of abelian groups $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. Hence for every $V/Y \in \mathrm{Ob}(Y_{étale})$ the sequence of abelian groups $0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

04I2 **Definition 59.35.4.** Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^p f_*\}_{p \geq 1}$ of $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ are called *higher direct images*.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

59.36. Inverse image

03PZ In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 34.4.

03Q0 **Definition 59.36.1.** Let $f : X \to Y$ be a morphism of schemes. The *inverse image*, or *pullback*[2] functors are the functors

$$f^{-1} = f_{small}^{-1} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$

and

$$f^{-1} = f_{small}^{-1} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$

[2]We use the notation f^{-1} for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve f^* for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.

which are left adjoint to $f_* = f_{small,*}$. Thus f^{-1} is characterized by the fact that

$$\mathrm{Hom}_{Sh(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

functorially, for any $\mathcal{F} \in Sh(X_{étale})$ and $\mathcal{G} \in Sh(Y_{étale})$. We similarly have

$$\mathrm{Hom}_{Ab(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Ab(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

for $\mathcal{F} \in Ab(X_{étale})$ and $\mathcal{G} \in Ab(Y_{étale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 59.36.3 below. The general machinery shows that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

04I3 (59.36.1.1) $$U/X \longmapsto \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V/Y)$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 7.14.7 to the functor u of Equation (59.34.0.1) and use the description of $u_s = (u_p\ )^\#$ in Sites, Sections 7.13 and 7.5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

03Q1 **Lemma 59.36.2.** *Let $f : X \to Y$ be a morphism of schemes.*

(1) *The functor $f^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
(2) *The functor $f^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.*
(3) *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*
$$(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$
where $\overline{y} = f \circ \overline{x}$.
(4) *For any $V \to Y$ étale we have $f^{-1}h_V = h_{X \times_Y V}$.*

Proof. The exactness of f^{-1} on sheaves of sets is a consequence of Sites, Proposition 7.14.7 applied to our functor u of Equation (59.34.0.1). In fact the exactness of pullback is part of the definition of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $\mathcal{G}$ on $Y_{étale}$ the underlying sheaf of sets of $f^{-1}\mathcal{F}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{F}$, see Sites, Section 7.44. See also Modules on Sites, Lemma 18.31.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 59.29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 7.34.2. We will also prove (3) directly as follows. Note that by Lemma 59.29.9 taking stalks commutes with sheafification. Now recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U \longrightarrow \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V),$$

see Equation (59.36.1.1). Thus we have

$$\begin{aligned}(f^{-1}\mathcal{G})_{\overline{x}} &= \mathrm{colim}_{(U,\overline{u})} f^{-1}\mathcal{G}(U) \\ &= \mathrm{colim}_{(U,\overline{u})} \mathrm{colim}_{a:U \to X \times_Y V} \mathcal{G}(V) \\ &= \mathrm{colim}_{(V,\overline{v})} \mathcal{G}(V) \\ &= \mathcal{G}_{\overline{y}}\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$.

Part (4) can be proved in a similar manner by identifying the colimits which define $f^{-1}h_V$. Or you can use Yoneda's lemma (Categories, Lemma 4.3.5) and the functorial equalities

$$\mathrm{Mor}_{Sh(X_{étale})}(f^{-1}h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(Y_{étale})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$$

combined with the fact that representable presheaves are sheaves. See also Sites, Lemma 7.13.5 for a completely general result. □

The pair of functors (f_*, f^{-1}) define a morphism of small étale topoi

$$f_{small} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

03Q2 **Remark 59.36.3.** More generally, let $\mathcal{C}_1, \mathcal{C}_2$ be sites, and assume they have final objects and fibre products. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a functor satisfying:

(1) if $\{V_i \to V\}$ is a covering of $\mathcal{C}_2$, then $\{u(V_i) \to u(V)\}$ is a covering of $\mathcal{C}_1$ (we say that u is *continuous*), and
(2) u commutes with finite limits (i.e., u is left exact, i.e., u preserves fibre products and final objects).

Then one can define $f_* : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ by $f_*\mathcal{F}(V) = \mathcal{F}(u(V))$. Moreover, there exists an exact functor f^{-1} which is left adjoint to f_*, see Sites, Definition 7.14.1 and Proposition 7.14.7. Warning: It is not enough to require simply that u is continuous and commutes with fibre products in order to get a morphism of topoi.

59.37. Functoriality of big topoi

04DI Given a morphism of schemes $f : X \to Y$ there are a whole host of morphisms of topoi associated to f, see Topologies, Section 34.11 for a list. Perhaps the most used ones are the morphisms of topoi

$$f_{big} = f_{big,\tau} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

where $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. These each correspond to a continuous functor

$$(Sch/Y)_\tau \longrightarrow (Sch/X)_\tau, \quad V/Y \longmapsto X \times_Y V/X$$

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

$$f_{big} : (Sch/X)_\tau \longrightarrow (Sch/Y)_\tau.$$

See Topologies, Sections 34.3, 34.4, 34.5, 34.6, and 34.7. In particular, pushforward along f_{big} is given by the rule

$$(f_{big,*}\mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor f_{big}^{-1} which is very easy to describe. Namely, we have

$$(f_{big}^{-1}\mathcal{G})(U/X) = \mathcal{G}(U/Y)$$

where the structure morphism of U/Y is the composition of the structure morphism $U \to X$ with f, see Topologies, Lemmas 34.3.16, 34.4.16, 34.5.10, 34.6.10, and 34.7.12.

59.38. Functoriality and sheaves of modules

04I4 In this section we are going to reformulate some of the material explained in Descent, Sections 35.8, 35.9, and 35.10 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 59.34, 59.35, and 59.36 that this induces a morphism f_{small} of small étale sites. In Descent, Remark 35.8.4 we have seen that f also induces a natural map

$$f^\sharp_{small} : \mathcal{O}_{Y_{étale}} \longrightarrow f_{small,*}\mathcal{O}_{X_{étale}}$$

of sheaves of rings on $Y_{étale}$ such that $(f_{small}, f^\sharp_{small})$ is a morphism of ringed sites. See Modules on Sites, Definition 18.6.1 for the definition of a morphism of ringed sites. Let us just recall here that $f^\sharp_{small}$ is defined by the compatible system of maps

$$\mathrm{pr}_V^\sharp : \mathcal{O}(V) \longrightarrow \mathcal{O}(X \times_Y V)$$

for V varying over the objects of $Y_{étale}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g^\sharp_{small}) \circ (f_{small}, f^\sharp_{small}) = ((g \circ f)_{small}, (g \circ f)^\sharp_{small})$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 18.13.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f^*_{small} : Mod(\mathcal{O}_{Y_{étale}}) \longrightarrow Mod(\mathcal{O}_{X_{étale}}),$$
$$f_{small,*} : Mod(\mathcal{O}_{X_{étale}}) \longrightarrow Mod(\mathcal{O}_{Y_{étale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)^*_{small} = f^*_{small} \circ g^*_{small}$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on X and the category between all $\mathcal{O}_{X_{étale}}$-modules on $X_{étale}$. But the results of Descent, Sections 35.8, 35.9, and 35.10 tell us that there is not much difference between considering quasi-coherent modules on S and quasi-coherent modules on $S_{étale}$. (We have already seen this in Theorem 59.17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors f^*_{small} and f^* match for quasi-coherent sheaves, see Descent, Proposition 35.9.4. Moreover, the same is true for pushforward provided f is quasi-compact and quasi-separated, see Descent, Lemma 35.9.5.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then the morphism $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ becomes a morphism of ringed sites by a map

$$f^\sharp_{big} : \mathcal{O}_Y \longrightarrow f_{big,*}\mathcal{O}_X$$

see Descent, Remark 35.8.4. In fact it is given by the same construction as in the case of small sites explained above.

59.39. Comparing topologies

09XL In this section we start studying what happens when you compare sheaves with respect to different topologies.

09XM **Lemma 59.39.1.** *Let S be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $S_{étale}$. Let $s, t \in \mathcal{F}(S)$. Then there exists an open $W \subset S$ characterized by the following property: A morphism $f : T \to S$ factors through W if and only if $s|_T = t|_T$ (restriction is pullback by f_{small}).*

Proof. Consider the presheaf which assigns to $U \in \mathrm{Ob}(S_{étale})$ the empty set if $s|_U \neq t|_U$ and a singleton else. It is clear that this is a subsheaf of the final object of $Sh(S_{étale})$. By Lemma 59.31.1 we find an open $W \subset S$ representing this presheaf. For a geometric point $\overline{x}$ of S we see that $\overline{x} \in W$ if and only if the stalks of s and t at $\overline{x}$ agree. By the description of stalks of pullbacks in Lemma 59.36.2 we see that W has the desired property. □

09XN **Lemma 59.39.2.** *Let S be a scheme. Let $\tau \in \{Zariski, étale\}$. Consider the morphism*

$$\pi_S : (Sch/S)_\tau \longrightarrow S_\tau$$

of Topologies, Lemma 34.3.14 or 34.4.14. Let $\mathcal{F}$ be a sheaf on S_τ. Then $\pi_S^{-1}\mathcal{F}$ is given by the rule

$$(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T_\tau, f_{small}^{-1}\mathcal{F})$$

where $f : T \to S$. Moreover, $\pi_S^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.

Proof. Observe that we have a morphism $i_f : Sh(T_\tau) \to Sh(Sch/S)_\tau)$ such that $\pi_S \circ i_f = f_{small}$ as morphisms $T_\tau \to S_\tau$, see Topologies, Lemmas 34.3.13, 34.3.17, 34.4.13, and 34.4.17. Since pullback is transitive we see that $i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on T_τ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i.

Let $V \to T$ be an object of T_τ and let $t \in \mathcal{G}(V)$. For every i there is a largest open $W_i \subset T_i \times_T V$ such that the pullbacks of s_i and t agree as sections of the pullback of $\mathcal{G}$ to $W_i \subset T_i \times_T V$, see Lemma 59.39.1. Because s_i and s_j agree over $T_i \times_T T_j$ we find that W_i and W_j pullback to the same open over $T_i \times_T T_j \times_T V$. By Descent, Lemma 35.13.6 we find an open $W \subset V$ whose inverse image to $T_i \times_T V$ recovers W_i.

By construction of $g_{i,small}^{-1}\mathcal{G}$ there exists a τ-covering $\{T_{ij} \to T_i\}_{j \in J_i}$, for each j an open immersion or étale morphism $V_{ij} \to T$, a section $t_{ij} \in \mathcal{G}(V_{ij})$, and commutative diagrams

$$\begin{array}{ccc} T_{ij} & \longrightarrow & V_{ij} \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T \end{array}$$

such that $s_i|_{T_{ij}}$ is the pullback of t_{ij}. In other words, after replacing the covering $\{T_i \to T\}$ by $\{T_{ij} \to T\}$ we may assume there are factorizations $T_i \to V_i \to T$ with

$V_i \in \mathrm{Ob}(T_\tau)$ and sections $t_i \in \mathcal{G}(V_i)$ pulling back to s_i over T_i. By the result of the previous paragraph we find opens $W_i \subset V_i$ such that $t_i|_{W_i}$ "agrees with" every s_j over $T_j \times_T W_i$. Note that $T_i \to V_i$ factors through W_i. Hence $\{W_i \to T\}$ is a τ-covering and the lemma is proven. □

0A3H **Lemma 59.39.3.** *Let S be a scheme. Let $f : T \to S$ be a morphism such that*

(1) *f is flat and quasi-compact, and*
(2) *the geometric fibres of f are connected.*

Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(T, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\alpha \in \Gamma(T, f_{small}^{-1}\mathcal{F})$. Since $\{T \to S\}$ is an fpqc covering we can use Lemma 59.39.2 to see that suffices to prove that α pulls back to the same section over $T \times_S T$ by the two projections. Let $\overline{s} \to S$ be a geometric point. It suffices to show the agreement holds over $(T \times_S T)_{\overline{s}}$ as every geometric point of $T \times_S T$ is contained in one of these geometric fibres. In other words, we are trying to show that $\alpha|_{T_{\overline{s}}}$ pulls back to the same section over

$$(T \times_S T)_{\overline{s}} = T_{\overline{s}} \times_{\overline{s}} T_{\overline{s}}$$

by the two projections to $T_{\overline{s}}$. However, since $\mathcal{F}|_{T_{\overline{s}}}$ is the pullback of $\mathcal{F}|_{\overline{s}}$ it is a constant sheaf with value $\mathcal{F}_{\overline{s}}$. Since $T_{\overline{s}}$ is connected by assumption, any section of a constant sheaf is constant. Hence $\alpha|_{T_{\overline{s}}}$ corresponds to an element of $\mathcal{F}_{\overline{s}}$. Thus the two pullbacks to $(T \times_S T)_{\overline{s}}$ both correspond to this same element and we conclude. □

Here is a version of Lemma 59.39.3 where we do not assume that the morphism is flat.

0EZK **Lemma 59.39.4.** *Let S be a scheme. Let $f : X \to S$ be a morphism such that*

(1) *f is submersive, and*
(2) *the geometric fibres of f are connected.*

Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(X, f_{small}^{-1}\mathcal{F})$.

Proof. There is a canonical map $\Gamma(S, \mathcal{F}) \to \Gamma(X, f_{small}^{-1}\mathcal{F})$. Since f is surjective (because its fibres are connected) we see that this map is injective.

To show that the map is surjective, let $\tau \in \Gamma(X, f_{small}^{-1}\mathcal{F})$. It suffices to find an étale covering $\{U_i \to S\}$ and sections $\sigma_i \in \mathcal{F}(U_i)$ such that σ_i pulls back to $\tau|_{X \times_S U_i}$. Namely, the injectivity shown above guarantees that σ_i and σ_j restrict to the same section of $\mathcal{F}$ over $U_i \times_S U_j$. Thus we obtain a unique section $\sigma \in \mathcal{F}(S)$ which restricts to σ_i over U_i. Then the pullback of σ to X is τ because this is true locally.

Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Consider the image of τ in the stalk

$$(f_{small}^{-1}\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{s}}$$

See Lemma 59.36.2. We can find an étale neighbourhood $U \to S$ of $\overline{s}$ and a section $\sigma \in \mathcal{F}(U)$ mapping to this image in the stalk. Thus after replacing S by U and X by $X \times_S U$ we may assume there exits a section σ of $\mathcal{F}$ over S whose image in $(f_{small}^{-1}\mathcal{F})_{\overline{x}}$ is the same as τ.

By Lemma 59.39.1 there exists a maximal open $W \subset X$ such that $f_{small}^{-1}\sigma$ and τ agree over W and the formation of W commutes with further pullback. Observe that the pullback of $\mathcal{F}$ to the geometric fibre $X_{\overline{s}}$ is the pullback of $\mathcal{F}_{\overline{s}}$ viewed as a sheaf on $\overline{s}$ by $X_{\overline{s}} \to \overline{s}$. Hence we see that τ and σ give sections of the constant sheaf with value $\mathcal{F}_{\overline{s}}$ on $X_{\overline{s}}$ which agree in one point. Since $X_{\overline{s}}$ is connected by assumption, we conclude that W contains X_s. The same argument for different geometric fibres shows that W contains every fibre it meets. Since f is submersive, we conclude that W is the inverse image of an open neighbourhood of s in S. This finishes the proof. □

0A3I **Lemma 59.39.5.** *Let K/k be an extension of fields with k separably algebraically closed. Let S be a scheme over k. Denote $p : S_K = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to S$ the projection. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Then $\Gamma(S, \mathcal{F}) = \Gamma(S_K, p_{small}^{-1}\mathcal{F})$.*

Proof. Follows from Lemma 59.39.3. Namely, it is clear that p is flat and quasi-compact as the base change of $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. On the other hand, if $\overline{s} : \mathrm{Spec}(L) \to S$ is a geometric point, then the fibre of p over $\overline{s}$ is the spectrum of $K \otimes_k L$ which is irreducible hence connected by Algebra, Lemma 10.47.2. □

59.40. Recovering morphisms

04JH In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 59.40.5.

04I5 **Lemma 59.40.1.** *Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{small}, f_{small}^\sharp)$ associated to f is a morphism of locally ringed sites, see Modules on Sites, Definition 18.40.9.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 59.33.5. Moreover, we know that $X_{étale}$ has enough points, see Theorem 59.29.10 and Remarks 59.29.11. Hence it suffices to prove that $(f_{small}, f_{small}^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.40.8. To see this take a point p of $X_{étale}$. By Lemma 59.29.12 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 59.36.2 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of stalks

$$(\mathcal{O}_Y)_{\overline{y}} \longrightarrow (\mathcal{O}_X)_{\overline{x}}$$

is a local ring map. Suppose that $a \in (\mathcal{O}_Y)_{\overline{y}}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that a is the equivalence class of a triple $(V, \overline{v}, a)$ with $V \to Y$ étale, $\overline{v} : \overline{x} \to V$ over Y, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \overline{x} \times \overline{v}, \mathrm{pr}_V^\sharp(a))$ in the local ring $(\mathcal{O}_X)_{\overline{x}}$. But it is clear that being in the maximal ideal means that pulling back $\mathrm{pr}_V^\sharp(a)$ to an element of $\kappa(\overline{x})$ gives zero. Hence also pulling back a to $\kappa(\overline{x})$ is zero. Which means that a lies in the maximal ideal of $(\mathcal{O}_Y)_{\overline{y}}$. □

04IJ **Lemma 59.40.2.** *Let X, Y be schemes. Let $f : X \to Y$ be a morphism of schemes. Let t be a 2-morphism from $(f_{small}, f_{small}^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = id$.*

Proof. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{array}{ccc} f_{small}^{-1}\mathcal{O}_Y & \xleftarrow{\quad t \quad} & f_{small}^{-1}\mathcal{O}_Y \\ & {\scriptstyle f_{small}^\sharp} \searrow \quad \swarrow {\scriptstyle f_{small}^\sharp} & \\ & \mathcal{O}_X & \end{array}$$

is commutative. Suppose $V \to Y$ is étale with V affine. By Morphisms, Lemma 29.39.2 we may choose an immersion $i : V \to \mathbf{A}_Y^n$ over Y. In terms of sheaves this means that i induces an injection $h_i : h_V \to \prod_{j=1,\ldots,n} \mathcal{O}_Y$ of sheaves. The base change i' of i to X is an immersion (Schemes, Lemma 26.18.2). Hence $i' : X \times_Y V \to \mathbf{A}_X^n$ is an immersion, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j=1,\ldots,n} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X \times_Y V}$ of Lemma 59.36.2 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1} h_i} \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod_{j=1,\ldots,n} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1} h_V \to f_{small}^{-1} h_V$ fits into the commutative diagram

$$\begin{array}{ccccc} f_{small}^{-1} h_V & \xrightarrow{f^{-1} h_i} & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j=1,\ldots,n} \mathcal{O}_X \\ \downarrow {\scriptstyle t} & & \downarrow {\scriptstyle \prod t} & & \downarrow {\scriptstyle \mathrm{id}} \\ f_{small}^{-1} h_V & \xrightarrow{f^{-1} h_i} & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{array}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Topologies, Lemma 34.4.12 with Sites, Lemma 7.12.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. □

04LW **Lemma 59.40.3.** *Let X, Y be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a_{small}^\sharp) \cong (b_{small}, b_{small}^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let us argue this carefully since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that h_V is a subsheaf of the final sheaf $*$. Thus both $a_{small}^{-1} h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1} h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1} h_V = h_{a^{-1}(V)} \to b_{small}^{-1} h_V = h_{b^{-1}(V)}$$

has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that a and b are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \to \mathcal{O}_Y$. Pull this back and

apply t to get a commutative diagram

$$\begin{array}{ccccccc}
h_{b^{-1}(V)} & = & b_{small}^{-1}h_V & \xleftarrow{t} & a_{small}^{-1}h_V & = & h_{a^{-1}(V)} \\
 & & \downarrow{\scriptstyle b_{small}^{-1}(f)} & & \downarrow{\scriptstyle a_{small}^{-1}(f)} & & \\
 & & b_{small}^{-1}\mathcal{O}_Y & \xleftarrow{t} & a_{small}^{-1}\mathcal{O}_Y & & \\
 & & & {\scriptstyle b^\sharp}\searrow \quad \swarrow{\scriptstyle a^\sharp} & & & \\
 & & & \mathcal{O}_X & & &
\end{array}$$

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 18.8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a^\sharp_{small}(f) = b^\sharp_{small}(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open V and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes. □

04I6 **Lemma 59.40.4.** *Let X, Y be affine schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{small}, f^\sharp_{small})$, see Modules on Sites, Definition 18.8.1.

Proof. In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{étale}$, and similarly for $\mathcal{O}_Y$. Say $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Since $B = \Gamma(Y_{étale}, \mathcal{O}_Y)$, $A = \Gamma(X_{étale}, \mathcal{O}_X)$ we see that $g^\sharp$ induces a ring map $\varphi : B \to A$. Let $f = \mathrm{Spec}(\varphi) : X \to Y$ be the corresponding morphism of affine schemes. We will show this f does the job.

Let $V \to Y$ be an affine scheme étale over Y. Thus we may write $V = \mathrm{Spec}(C)$ with C an étale B-algebra. We can write

$$C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$$

with P_i polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in C, see for example Algebra, Lemma 10.143.2. If T is a scheme over Y, then a T-valued point of V is given by n sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf h_V on $Y_{étale}$ is the equalizer of the two maps

$$\prod\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \underset{b}{\overset{a}{\rightrightarrows}} \prod\nolimits_{j=1,\ldots,n} \mathcal{O}_Y$$

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$. Since g^{-1} is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

$$\begin{array}{ccccc}
g^{-1}h_V & \longrightarrow & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y & \underset{g^{-1}b}{\overset{g^{-1}a}{\rightrightarrows}} & \prod_{j=1,\ldots,n} g^{-1}\mathcal{O}_Y \\
\vdots & & \downarrow{\scriptstyle \prod g^\sharp} & & \downarrow{\scriptstyle \prod g^\sharp} \\
h_{X\times_Y V} & \longrightarrow & \prod_{i=1,\ldots,n} \mathcal{O}_X & \underset{b'}{\overset{a'}{\rightrightarrows}} & \prod_{j=1,\ldots,n} \mathcal{O}_X
\end{array}$$

Here b' is the zero map and a' is the map defined by the images $P_i' = \varphi(P_i) \in A[x_1, \ldots, x_n]$ via the same rule $a'(h_1, \ldots, h_n) = (P_1'(h_1, \ldots, h_n), \ldots, P_n'(h_1, \ldots, h_n))$. that a was defined by. The commutativity of the diagram follows from the fact that $\varphi = g^\sharp$ on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since $X \times_Y V$ is the affine scheme $\mathrm{Spec}(A \otimes_B C)$ and $A \otimes_B C = A[x_1, \ldots, x_n]/(P_1', \ldots, P_n')$. Thus we obtain a unique dotted arrow $g^{-1}h_V \to h_{X \times_Y V}$ fitting into the diagram

We claim that the map of sheaves $g^{-1}h_V \to h_{X \times_Y V}$ is an isomorphism. Since the small étale site of X has enough points (Theorem 59.29.10) it suffices to prove this on stalks. Hence let $\overline{x}$ be a geometric point of X, and denote p the associate point of the small étale topos of X. Set $q = g \circ p$. This is a point of the small étale topos of Y. By Lemma 59.29.12 we see that q corresponds to a geometric point $\overline{y}$ of Y. Consider the map of stalks

$$(g^\sharp)_p : (\mathcal{O}_Y)_{\overline{y}} = \mathcal{O}_{Y,q} = (g^{-1}\mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X,p} = (\mathcal{O}_X)_{\overline{x}}$$

Since $(g, g^\sharp)$ is a morphism of *locally* ringed topoi $(g^\sharp)_p$ is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 10.153.12 we conclude that $(g^{-1}h_V)_p \to (h_{X \times_Y V})_p$ is an isomorphism as desired.

We claim that the isomorphisms $g^{-1}h_V \to h_{X \times_Y V}$ are functorial. Namely, suppose that $V_1 \to V_2$ is a morphism of affine schemes étale over Y. Write $V_i = \mathrm{Spec}(C_i)$ with

$$C_i = B[x_{i,1}, \ldots, x_{i,n_i}]/(P_{i,1}, \ldots, P_{i,n_i})$$

The morphism $V_1 \to V_2$ is given by a B-algebra map $C_2 \to C_1$ which in turn is given by some polynomials $Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}]$ for $j = 1, \ldots, n_2$. Then it is an easy matter to show that the diagram of sheaves

$$\begin{array}{ccc} h_{V_1} & \longrightarrow & \prod_{i=1,\ldots,n_1} \mathcal{O}_Y \\ \downarrow & & \downarrow{\scriptstyle Q_1,\ldots,Q_{n_2}} \\ h_{V_2} & \longrightarrow & \prod_{i=1,\ldots,n_2} \mathcal{O}_Y \end{array}$$

is commutative, and pulling back to $X_{étale}$ we obtain the solid commutative diagram

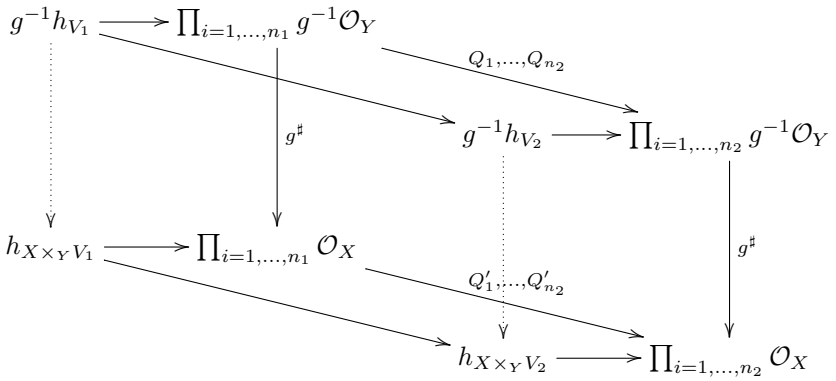

where $Q_j' \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of Q_j via φ. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we

see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to h_{X\times_Y V}$ is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that $f_{small,*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{étale}$. For every $V \in \mathrm{Ob}(X_{étale})$ affine we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= \mathrm{Mor}_{Sh(Y_{étale})}(h_V, g_*\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(X_{étale})}(g^{-1}h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(X_{étale})}(h_{X\times_Y V}, \mathcal{F}) \\ &= \mathcal{F}(X \times_Y V) \\ &= f_{small,*}\mathcal{F}(V)\end{aligned}$$

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X\times_Y V}$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in V as the isomorphisms $g^{-1}h_V \cong h_{X\times_Y V}$ are functorial. Now any sheaf on $Y_{étale}$ is determined by the restriction to the subcategory of affine schemes (Topologies, Lemma 34.4.12), and hence we obtain an isomorphism of functors $f_{small,*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{small,*} \cong g_*$ above, the maps $f^\sharp_{small}$ and $g^\sharp$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of B. We only need to check the result on sections over an affine V étale over Y (by Topologies, Lemma 34.4.12 again). Writing $V = \mathrm{Spec}(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions x_i are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it means that the diagram

$$\begin{array}{ccc} g^{-1}h_V & \longrightarrow & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \\ \downarrow & & \downarrow \prod g^\sharp \\ h_{X\times_Y V} & \longrightarrow & \prod_{i=1,\ldots,n} \mathcal{O}_X \end{array}$$

commutes. Thus the lemma is proved. □

Here is a version for general schemes.

04I7 **Theorem 59.40.5.** *Let X, Y be schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{small}, f^\sharp_{small})$. In other words, the construction

$$Sch \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. You can prove this theorem by carefully adjusting the arguments of the proof of Lemma 59.40.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 59.40.2. Unfortunately, this is a bit messy.

Let us prove existence when Y is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each i the inclusion morphism $j_i : U_i \to X$ induces a morphism of locally ringed topoi $(j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$ by Lemma 59.40.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(Y_{étale}), \mathcal{O}_Y)$$

see Modules on Sites, Lemma 18.40.10. By Lemma 59.40.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_{i,small}, f_{i,small}^\sharp) \longrightarrow (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp).$$

Set $U_{i,i'} = U_i \cap U_{i'}$, and denote $j_{i,i'} : U_{i,i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i,i'} = j_{i'} \circ j_{i',i}$ we see that

$$\begin{aligned}(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) \circ (j_{i,i',small}, j_{i,i',small}^\sharp) = \\ (g, g^\sharp) \circ (j_{i',small}, j_{i',small}^\sharp) \circ (j_{i',i,small}, j_{i',i,small}^\sharp)\end{aligned}$$

Hence by uniqueness (see Lemma 59.40.3) we conclude that $f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $\sharp$ to ease the notation)

$$\begin{array}{ccc} g \circ j_{i,small} \circ j_{i,i',small} & \xrightarrow{t_i \star \mathrm{id}_{j_{i,i',small}}} & f_{small} \circ j_{i,small} \circ j_{i,i',small} \\ \| & & \| \\ g \circ j_{i',small} \circ j_{i',i,small} & \xrightarrow{t_{i'} \star \mathrm{id}_{j_{i',i,small}}} & f_{small} \circ j_{i',small} \circ j_{i',i,small} \end{array}$$

The notation $\star$ indicates horizontal composition, see Categories, Definition 4.29.1 in general and Sites, Section 7.36 for our particular case. By the result of Lemma 59.40.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{étale}$ the isomorphisms $t_i : f_{small}^{-1}\mathcal{G}|_{U_i} \to g^{-1}\mathcal{G}|_{U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{small}^\sharp$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case Y is affine.

In the general case, let $V \subset Y$ be an affine open. Then h_V is a subsheaf of the final sheaf $*$ on $Y_{étale}$. As g is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{étale}$. Hence by Lemma 59.31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 18.40.12 there exists a commutative diagram of morphisms of locally ringed topoi

$$\begin{array}{ccc} (Sh(W_{étale}), \mathcal{O}_W) & \longrightarrow & (Sh(X_{étale}), \mathcal{O}_X) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ (Sh(V_{étale}), \mathcal{O}_V) & \longrightarrow & (Sh(Y_{étale}), \mathcal{O}_Y) \end{array}$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where g' is induced from g. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{small}, (f'_{small})^\sharp) \to (g', (g')^\sharp)$. Exactly as before these

morphisms f' (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms t are compatible on overlaps by Lemma 59.40.2 again and we obtain a global 2-isomorphism $(f_{small}, (f_{small})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted. □

59.41. Push and pull

04C6 Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over X with u in the image of h.
(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each j we have $X \times_Y V_j = \coprod W_{ij}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i.
(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over X.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{small,*}$. We will work this out in the next few sections.

59.42. Property (A)

04DJ Please see Section 59.41 for the definition of property (A).

04DK **Lemma 59.42.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (A).*

(1) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ reflects injections and surjections,*
(2) *$f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(3) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let U be an object of $X_{étale}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{étale}$ such that each W_i is an open and closed subscheme of $X \times_Y V_i$ for some object V_i of $Y_{étale}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{small,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{small,*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{small,*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with U as above. With W_i, V_i as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i)$ it suffices to show that for any $V \in \mathrm{Ob}(Y_{étale})$ any element $s \in \mathcal{F}(X \times_Y V)$ is étale locally on $X \times_Y V$ the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that s is the image of $s_j \in f_{small,*}\mathcal{G}(V_j) = \mathcal{G}(X \times_Y V_j)$. This proves $f_{small,*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma 18.15.1. □

04DL **Lemma 59.42.2.** *Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.*

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where U and Y are affine schemes. Denote $x \in X$ and $y \in Y$ the images of u. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma 29.36.6) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma 29.20.12). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on Morphisms, Lemma 37.41.5 applies to both morphisms. This means we may pick an étale neighbourhood $(V, v) \to (Y, y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

(1) W, R are open and closed in $X \times_Y V$,
(2) W', R' are open and closed in $U \times_Y V$,
(3) $W \to V$ and $W' \to V$ are finite,
(4) w, w' map to v,
(5) $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
(6) no other point of W or W' maps to v.

Here is a commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_Y V & \longleftarrow & W' \amalg R' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X \times_Y V & \longleftarrow & W \amalg R \\ \downarrow & & \downarrow & & \\ Y & \longleftarrow & V & & \end{array}$$

After shrinking V we may assume that W' maps into W: just remove the image the inverse image of R in W'; this is a closed set (as $W' \to V$ is finite) not containing v. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence $W' \to W$ is finite étale, and there is exactly one point in the fibre over w with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood W° of w, see Étale Morphisms, Lemma 41.14.2. Since $W \to V$ is finite the image of $W \setminus W^\circ$ is a closed subset T of V not containing v. Thus after replacing V by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. □

04DM **Lemma 59.42.3.** *Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.*

Proof. Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an X-morphism $W \to U$ with u in the image. We may shrink U and Y and assume U and Y are affine. In this case also X is affine, since an integral morphism is affine by definition. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $U = \mathrm{Spec}(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 10.143.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = \mathrm{Spec}(C') \to X' = \mathrm{Spec}(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 59.42.2. □

04C9 **Lemma 59.42.4.** *Let $f : X \to Y$ be a morphism of schemes. Denote $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ the associated morphism of small étale topoi. Assume at least one of the following*

(1) *f is integral, or*
(2) *f is separated and locally quasi-finite.*

Then the functor $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ has the following properties

(1) *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is always surjective,*
(2) *$f_{small,*}$ is faithful, and*
(3) *$f_{small,*}$ reflects injections and surjections.*

Proof. Combine Lemmas 59.42.2, 59.42.3, and 59.42.1. □

59.43. Property (B)

04DN Please see Section 59.41 for the definition of property (B).

04DO **Lemma 59.43.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (B) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ transforms surjections into surjections.*

Proof. This follows from Sites, Lemma 7.41.2. □

04DP **Lemma 59.43.2.** *Let $f : X \to Y$ be a morphism of schemes. Suppose*

(1) *$V \to Y$ is an étale morphism of schemes,*
(2) *$\{U_i \to X \times_Y V\}$ is an étale covering, and*
(3) *$v \in V$ is a point.*

Assume that for any such data there exists an étale neighbourhood $(V', v') \to (V, v)$, a disjoint union decomposition $X \times_Y V' = \coprod W_i'$, and morphisms $W_i' \to U_i$ over $X \times_Y V$. Then property (B) holds.

Proof. Omitted. □

04DQ **Lemma 59.43.3.** *Let $f : X \to Y$ be a finite morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 59.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is finite over Y, this also implies that X is affine. During the proof we may (finitely often) replace (V, v) by an étale neighbourhood (V', v') and correspondingly the covering $\{U_i \to X \times_Y V\}$ by $\{V' \times_V U_i \to X \times_Y V'\}$.

Since $X \times_Y V \to V$ is finite there exist finitely many (pairwise distinct) points $x_1, \ldots, x_n \in X \times_Y V$ mapping to v. We may apply More on Morphisms, Lemma 37.41.5 to $X \times_Y V \to V$ and the points $x_1, \ldots, x_n$ lying over v and find an étale neighbourhood $(V', v') \to (V, v)$ such that

$$X \times_Y V' = R \amalg \coprod T_a$$

with $T_a \to V'$ finite with exactly one point p_a lying over v' and moreover $\kappa(v') \subset \kappa(p_a)$ purely inseparable, and such that $R \to V'$ has empty fibre over v'. Because $X \to Y$ is finite, also $R \to V'$ is finite. Hence after shrinking V' we may assume that $R = \emptyset$. Thus we may assume that $X \times_Y V = X_1 \amalg \ldots \amalg X_n$ with exactly one

point $x_l \in X_l$ lying over v with moreover $\kappa(v) \subset \kappa(x_l)$ purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood (V, v).

For each l choose an i_l and a point $u_l \in U_{i_l}$ mapping to x_l. Now we apply property (A) for the finite morphism $X \times_Y V \to V$ and the étale morphisms $U_{i_l} \to X \times_Y V$ and the points u_l. This is permissible by Lemma 59.42.3 This gives produces an étale neighbourhood $(V', v') \to (V, v)$ and decompositions

$$X \times_Y V' = W_l \amalg R_l$$

and X-morphisms $a_l : W_l \to U_{i_l}$ whose image contains u_{i_l}. Here is a picture:

$$\begin{array}{ccccccccc} & & & & & & & & U_{i_l} \\ & & & & & & & & \downarrow \\ W_l & \longrightarrow & W_l \amalg R_l & = & X \times_Y V' & \longrightarrow & X \times_Y V & \longrightarrow & X \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & V' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

After replacing (V, v) by (V', v') we conclude that each x_l is contained in an open and closed neighbourhood W_l such that the inclusion morphism $W_l \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i. Replacing W_l by $W_l \cap X_l$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times_Y V \to V$ is finite we may shrink V and assume that $X \times_Y V = W_1 \amalg \ldots \amalg W_n$ as desired. □

04DR **Lemma 59.43.4.** *Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 59.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is integral over Y, this also implies that X and $X \times_Y V$ are affine. We may refine the covering $\{U_i \to X \times_Y V\}$, and hence we may assume that $\{U_i \to X \times_Y V\}_{i=1,\ldots,n}$ is a standard étale covering. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $V = \mathrm{Spec}(C)$, and $U_i = \mathrm{Spec}(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 10.143.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B'_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{\mathrm{Spec}(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n}$ with $X' = \mathrm{Spec}(B')$ finite over Y. In this case the result follows from Lemma 59.43.3. □

04C2 **Lemma 59.43.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral (for example finite). Then*

1. *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
2. *$f_{small}^{-1} f_{small,*} \mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
3. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful and reflects injections and surjections, and*
4. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact.*

Proof. Parts (2), (3) we have seen in Lemma 59.42.4. Part (1) follows from Lemmas 59.43.4 and 59.43.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 18.15.2. □

59.44. Property (C)

04DS Please see Section 59.41 for the definition of property (C).

04DT **Lemma 59.44.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections.*

Proof. Follows from Sites, Lemma 7.41.4. We omit the verification that property (C) implies that the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 7.41.4. □

04DU **Remark 59.44.2.** Property (C) holds if $f : X \to Y$ is an open immersion. Namely, if $U \in \mathrm{Ob}(X_{étale})$, then we can view U also as an object of $Y_{étale}$ and $U \times_Y X = U$. Hence property (C) does not imply that $f_{small,*}$ is exact as this is not the case for open immersions (in general).

04DV **Lemma 59.44.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that*

(1) *$X \times_Y V \to V$ has property (C), and*
(2) *$X \times_Y V \to V$ is closed.*

Then the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 7.42.3.

Proof. Let $V \to Y$ be an object of $Y_{étale}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{étale}$. By assumption (1) for each i we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \mathrm{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. □

04DW **Lemma 59.44.4.** *Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of X with a closed subset of Y. Then property (C) holds.*

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{étale}$ and a surjective morphism $X \times_Y V \to U$ over X. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{étale}$ and a morphism $h_u : X \times_Y V_u \to U$ over X with $u \in \mathrm{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair (V_u, h_u) as above. To do this we may shrink U and assume that U is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \rightsquigarrow u$ we may replace U by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in Y. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that g factors through $X \times_Y V$ and that the resulting $\{U \to X \times_Y V\}$ is an étale covering. Since f has property (B) , see Lemma 59.43.4, we see that there exists an étale covering $\{V_j \to V\}$ such

that $X \times_Y V_j \to X \times_Y V$ factor through U. This implies that $V' = \coprod V_j$ is étale over Y and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since u is the only point in its fibre it must be in the image of h and we win. □

We urge the reader to think of the following lemma as a way station[3] on the journey towards the ultimate truth regarding $f_{small,*}$ for integral universally injective morphisms.

04DX **Lemma 59.44.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume that f is universally injective and integral (for example a closed immersion). Then*

(1) *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections,*
(2) *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ commutes with pushouts and coequalizers (and more generally finite connected colimits),*
(3) *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
(4) *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any sheaf (of sets or of abelian groups) $\mathcal{F}$ on $X_{étale}$,*
(5) *the functor $f_{small,*}$ is faithful (on sheaves of sets and on abelian sheaves),*
(6) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact, and*
(7) *the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous.*

Proof. By Lemmas 59.42.3, 59.43.4 and 59.44.4 we know that the morphism f has properties (A), (B), and (C). Moreover, by Lemma 59.44.3 we know that the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. Now we have

(1) property (C) implies (1) by Lemma 59.44.1,
(2) almost continuous implies (2) by Sites, Lemma 7.42.6,
(3) property (B) implies (3) by Lemma 59.43.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 7.41.1 and Modules on Sites, Lemma 18.15.2. Property (7) we saw above. □

59.45. Topological invariance of the small étale site

04DY In the following theorem we show that the small étale site is a topological invariant in the following sense: If $f : X \to Y$ is a morphism of schemes which is a universal homeomorphism, then $X_{étale} \cong Y_{étale}$ as sites. This improves the result of Étale Morphisms, Theorem 41.15.2. We first prove the result for morphisms and then we state the result for categories.

0BTY **Theorem 59.45.1.** *Let X and Y be two schemes over a base scheme S. Let $S' \to S$ be a universal homeomorphism. Denote X' (resp. Y') the base change to S'. If X is étale over S, then the map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S'}(Y', X')$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. Thus we may and do assume both X and Y are étale over S. In other words, the theorem states that the base change functor is a fully faithful functor from the category of schemes étale over S to the category of schemes étale over S'.

[3] A way station is a place where people stop to eat and rest when they are on a long journey.

Consider the forgetful functor

$$\begin{array}{c}\text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S'\end{array} \longrightarrow \text{schemes } X' \text{ étale over } S' \tag{59.45.1.1}$$

0BTZ

We claim this functor is an equivalence. On the other hand, the functor

$$\text{schemes } X \text{ étale over } S \longrightarrow \begin{array}{c}\text{descent data } (X', \varphi') \text{ relative to } S'/S \\ \text{with } X' \text{ étale over } S'\end{array} \tag{59.45.1.2}$$

0BU0

is fully faithful by Étale Morphisms, Lemma 41.20.3. Thus the claim implies the theorem.

Proof of the claim. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 29.45.5. In particular, the diagonal $\Delta : S' \to S' \times_S S'$ is a thickening by Morphisms, Lemma 29.10.2. Thus by Étale Morphisms, Theorem 41.15.1 we see that given $X' \to S'$ étale there is a unique isomorphism

$$\varphi' : X' \times_S S' \to S' \times_S X'$$

of schemes étale over $S' \times_S S'$ which pulls back under Δ to $\mathrm{id} : X' \to X'$ over S'. Since $S' \to S' \times_S S' \times_S S'$ is a thickening as well (it is bijective and a closed immersion) we conclude that (X', φ') is a descent datum relative to S'/S. The canonical nature of the construction of φ' shows that it is compatible with morphisms between schemes étale over S'. In other words, we obtain a quasi-inverse $X' \mapsto (X', \varphi')$ of the functor (59.45.1.1). This proves the claim and finishes the proof of the theorem. □

04DZ **Theorem 59.45.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective (i.e., f is a universal homeomorphism, see Morphisms, Lemma 29.45.5). The functor* [**DG67**, IV Theorem 18.1.2]

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\textit{schemes } V \textit{ étale over } Y\} \leftrightarrow \{\textit{schemes } U \textit{ étale over } X\}$$

We give two proofs. The first uses effectivity of descent for quasi-compact, separated, étale morphisms relative to surjective integral morphisms. The second uses the material on properties (A), (B), and (C) discussed earlier in the chapter.

First proof. By Theorem 59.45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma

26.14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to show the lemma in case U and Y are affine. Recall that in the proof of Theorem 59.45.1 we showed that U comes with a unique descent datum (U, φ) relative to X/Y. By Étale Morphisms, Proposition 41.20.6 (which applies because $U \to X$ is quasi-compact and separated as well as étale by our reduction to the affine case) there exists an étale morphism $V \to Y$ such that $X \times_Y V \cong U$ and the proof is complete. □

Second proof. By Theorem 59.45.1 we see that the functor is fully faithful. It remains to show that the functor is essentially surjective. Let $U \to X$ be an étale morphism of schemes.

Suppose that the result holds if U and Y are affine. In that case, we choose an affine open covering $U = \bigcup U_i$ such that each U_i maps into an affine open of Y. By assumption (affine case) we can find étale morphisms $V_i \to Y$ such that $X \times_Y V_i \cong U_i$ as schemes over X. Let $V_{i,i'} \subset V_i$ be the open subscheme whose underlying topological space corresponds to $U_i \cap U_{i'}$. Because we have isomorphisms

$$X \times_Y V_{i,i'} \cong U_i \cap U_{i'} \cong X \times_Y V_{i',i}$$

as schemes over X we see by fully faithfulness that we obtain isomorphisms $\theta_{i,i'} : V_{i,i'} \to V_{i',i}$ of schemes over Y. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 26.14. Applying Schemes, Lemma 26.14.2 we obtain a scheme $V \to Y$ by glueing the schemes V_i along the identifications $\theta_{i,i'}$. It is clear that $V \to Y$ is étale and $X \times_Y V \cong U$ by construction.

Thus it suffices to prove that the functor

04E0 (59.45.2.1) $\{\text{affine schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{affine schemes } U \text{ étale over } X\}$

is essentially surjective when X and Y are affine.

Let $U \to X$ be an affine scheme étale over X. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to U over X. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 29.36.6 and 29.57.9. Hence we can do induction on the integer n bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 41.14.1), and the result is clear. Assume $n > 1$.

By Lemma 59.44.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over X. As U is quasi-compact we may replace W by a disjoint union of finitely many affine opens of W, hence we may assume that W is affine as well. Here is a diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_Y W & = & W_X \amalg R \\ \downarrow & & \downarrow & & \\ X & \longleftarrow & W_X & & \\ \downarrow & & \downarrow & & \\ Y & \longleftarrow & W & & \end{array}$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n-1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \amalg V'$ we find a scheme V'' étale over W whose base change to W_X is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find V over Y whose base change to X is isomorphic to U over X. Namely, by the fully faithfulness of the functor (59.45.2.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular φ satisfies the cocycle condition. Hence by Descent, Lemma 35.37.1 we see that φ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with φ. Note that $V \to Y$ is étale, by Descent, Lemma 35.23.31. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. □

05YX **Remark 59.45.3.** In the situation of Theorem 59.45.2 it is also true that $V \mapsto V_X$ induces an equivalence between those étale morphisms $V \to Y$ with V affine and those étale morphisms $U \to X$ with U affine. This follows for example from Limits, Proposition 32.11.2.

03SI **Proposition 59.45.4** (Topological invariance of étale cohomology)**.** *Let $X_0 \to X$ be a universal homeomorphism of schemes (for example the closed immersion defined by a nilpotent sheaf of ideals). Then*

(1) *the étale sites $X_{étale}$ and $(X_0)_{étale}$ are isomorphic,*
(2) *the étale topoi $Sh(X_{étale})$ and $Sh((X_0)_{étale})$ are equivalent, and*
(3) *$H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_0, \mathcal{F}|_{X_0})$ for all q and for any abelian sheaf $\mathcal{F}$ on $X_{étale}$.*

Proof. The equivalence of categories $X_{étale} \to (X_0)_{étale}$ is given by Theorem 59.45.2. We omit the proof that under this equivalence the étale coverings correspond. Hence (1) holds. Parts (2) and (3) follow formally from (1). □

59.46. Closed immersions and pushforward

04E1 Before stating and proving Proposition 59.46.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$Sch/X \longrightarrow Sch/Z, \quad U \longmapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme U_Z is a closed subscheme of U.

04FV **Lemma 59.46.1.** *Let $i : Z \to X$ be a closed immersion of schemes. Let U, U' be schemes étale over X. Let $h : U_Z \to U'_Z$ be a morphism over Z. Then there exists a diagram*

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ is an isomorphism and $h = b_Z \circ (a_Z)^{-1}$.

Proof. Consider the scheme $M = U \times_X U'$. The graph $\Gamma_h \subset M_Z$ of h is open. This is true for example as Γ_h is the image of a section of the étale morphism $\mathrm{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 41.6.1. Hence there exists an open subscheme $W \subset M$ whose intersection with the closed subset M_Z is Γ_h. Set $a = \mathrm{pr}_1|_W$ and $b = \mathrm{pr}_2|_W$. □

04FW **Lemma 59.46.2.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $V \to Z$ be an étale morphism of schemes. There exist étale morphisms $U_i \to X$ and morphisms $U_{i,Z} \to V$ such that $\{U_{i,Z} \to V\}$ is a Zariski covering of V.*

Proof. Since we only have to find a Zariski covering of V consisting of schemes of the form U_Z with U étale over X, we may Zariski localize on X and V. Hence we may assume X and V affine. In the affine case this is Algebra, Lemma 10.143.10. □

If $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point of X, then either $\overline{x}$ factors (uniquely) through the closed subscheme Z, or $Z_{\overline{x}} = \emptyset$. If $\overline{x}$ factors through Z we say that $\overline{x}$ is a geometric point of Z (because it is) and we use the notation "$\overline{x} \in Z$" to indicate this.

04FX **Lemma 59.46.3.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be a sheaf of sets on $Z_{étale}$. Let $\overline{x}$ be a geometric point of X. Then*

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \begin{cases} * & \text{if } \overline{x} \notin Z \\ \mathcal{G}_{\overline{x}} & \text{if } \overline{x} \in Z \end{cases}$$

where $*$ denotes a singleton set.

Proof. Note that $i_{small,*}\mathcal{G}|_{U_{étale}} = *$ is the final object in the category of étale sheaves on U, i.e., the sheaf which associates a singleton set to each scheme étale over U. This explains the value of $(i_{small,*}\mathcal{G})_{\overline{x}}$ if $\overline{x} \notin Z$.

Next, suppose that $\overline{x} \in Z$. Note that

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{G}(U_Z)$$

and on the other hand

$$\mathcal{G}_{\overline{x}} = \mathrm{colim}_{(V,\overline{v})} \mathcal{G}(V).$$

Let $\mathcal{C}_1 = \{(U, \overline{u})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in X, and let $\mathcal{C}_2 = \{(V, \overline{v})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in Z. The canonical map

$$\mathcal{G}_{\overline{x}} \longrightarrow (i_{small,*}\mathcal{G})_{\overline{x}}$$

corresponds to the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F(U, \overline{u}) = (U_Z, \overline{x})$. Now Lemmas 59.46.2 and 59.46.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 4.17.1. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 4.17.2. □

04CA **Proposition 59.46.4.** *Let $i : Z \to X$ be a closed immersion of schemes.*

(1) *The functor*

$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*

$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let's discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on Z the morphism $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 59.46.3 (and Theorem 59.29.10). This implies formally that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.41.1. It is clear that $i_{small,*}\mathcal{G}|_{U_{étale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on X such that $\mathcal{F}|_{U_{étale}} \cong *$. Consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1} \mathcal{F}$$

Combining Lemmas 59.46.3 and 59.36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

59.47. Integral universally injective morphisms

04FY Here is the general version of Proposition 59.46.4.

04FZ **Proposition 59.47.1.** *Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.*

(1) *The functor*

$$f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{étale}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) *the functor*

$$f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(X)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. We may factor f as

$$X \xrightarrow{h} Z \xrightarrow{i} Y$$

where h is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 59.46.4 to i and apply Theorem 59.45.2 to h. □

59.48. Big sites and pushforward

04E2 In this section we prove some technical results on $f_{big,*}$ for certain types of morphisms of schemes.

04C7 **Lemma 59.48.1.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{big}^{-1} f_{big,*} \mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(Sch/X)_\tau$.*

Proof. In this case the functor $(Sch/X)_\tau \to (Sch/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 7.21.7. □

04C8 **Remark 59.48.2.** In the situation of Lemma 59.48.1 it is true that the canonical map $\mathcal{F} \to f_{big}^{-1} f_{big!} \mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(Sch/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{big!}$ for sheaves of abelian groups is defined in Modules on Sites, Section 18.16 and is in general different from $f_{big!}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 18.16.4.

04E3 **Lemma 59.48.3.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of U.*

Proof. Let us prove the lemma when $\tau = syntomic$. The question is local on U. Thus we may assume that U is an affine scheme mapping into an affine of Y. Hence we reduce to proving the following case: $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A/I)$, and $U = \mathrm{Spec}(\overline{B})$, where $A/I \to \overline{B}$ be a syntomic ring map. By Algebra, Lemma 10.136.18 we can find elements $\overline{g}_i \in \overline{B}$ such that $\overline{B}_{\overline{g}_i} = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 10.137.19. In the étale case use Algebra, Lemma 10.143.10. □

04E4 **Lemma 59.48.4.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each j, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through U_i for some i.*

Proof. For each i we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over X, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 59.48.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps g_{ij} together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of Y. □

04C3 **Lemma 59.48.5.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$. The functor $V \mapsto X \times_Y V$ defines an almost cocontinuous functor (see Sites, Definition 7.42.3) $(Sch/Y)_\tau \to (Sch/X)_\tau$ between big τ sites.*

Proof. We have to show the following: given a morphism $V \to Y$ and any syntomic (resp. smooth, resp. étale) covering $\{U_i \to X \times_Y V\}$, there exists a smooth (resp. smooth, resp. étale) covering $\{V_j \to V\}$ such that for each j, either $X \times_Y V_j$ is

empty, or $X \times_Y V_j \to Z \times_Y V$ factors through one of the U_i. This follows on applying Lemma 59.48.4 above to the closed immersion $X \times_Y V \to V$. □

04C4 **Lemma 59.48.6.** *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$.*

(1) *The pushforward $f_{big,*} : Sh((Sch/X)_\tau) \to Sh((Sch/Y)_\tau)$ commutes with coequalizers and pushouts.*
(2) *The pushforward $f_{big,*} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$ is exact.*

Proof. This follows from Sites, Lemma 7.42.6, Modules on Sites, Lemma 18.15.3, and Lemma 59.48.5 above. □

04C5 **Remark 59.48.7.** In Lemma 59.48.6 the case $\tau = fppf$ is missing. The reason is that given a ring A, an ideal I and a faithfully flat, finitely presented ring map $A/I \to \overline{B}$, there is no reason to think that one can find *any* flat finitely presented ring map $A \to B$ with $B/IB \neq 0$ such that $A/I \to B/IB$ factors through $\overline{B}$. Hence the proof of Lemma 59.48.5 does not work for the fppf topology. In fact it is likely false that $f_{big,*} : Ab((Sch/X)_{fppf}) \to Ab((Sch/Y)_{fppf})$ is exact when f is a closed immersion. If you know an example, please email stacks.project@gmail.com.

59.49. Exactness of big lower shriek

04CB This is just the following technical result. Note that the functor $f_{big!}$ has nothing whatsoever to do with cohomology with compact support in general.

04CC **Lemma 59.49.1.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a morphism of schemes. Let*

$$f_{big} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

be the corresponding morphism of topoi as in Topologies, Lemma 34.3.16, 34.4.16, 34.5.10, 34.6.10, or 34.7.12.

(1) *The functor $f_{big}^{-1} : Ab((Sch/Y)_\tau) \to Ab((Sch/X)_\tau)$ has a left adjoint*
$$f_{big!} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$$
which is exact.
(2) *The functor $f_{big}^* : Mod((Sch/Y)_\tau, \mathcal{O}) \to Mod((Sch/X)_\tau, \mathcal{O})$ has a left adjoint*
$$f_{big!} : Mod((Sch/X)_\tau, \mathcal{O}) \to Mod((Sch/Y)_\tau, \mathcal{O})$$
which is exact.

Moreover, the two functors $f_{big!}$ agree on underlying sheaves of abelian groups.

Proof. Recall that f_{big} is the morphism of topoi associated to the continuous and cocontinuous functor $u : (Sch/X)_\tau \to (Sch/Y)_\tau$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big!}$ follows from Modules on Sites, Lemma 18.16.2, respectively Modules on Sites, Lemma 18.41.1. Note that if U is an object of $(Sch/X)_\tau$ then the functor u induces an equivalence of categories

$$u' : (Sch/X)_\tau/U \longrightarrow (Sch/Y)_\tau/U$$

because both sides of the arrow are equal to $(Sch/U)_\tau$. Hence the agreement of $f_{big!}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 18.41.2. The exactness of $f_{big!}$ follows from Modules on Sites, Lemma

18.16.3 as the functor u above which commutes with fibre products and equalizers. □

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

07AJ **Lemma 59.49.2.** *Let X be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (Sch/X)_\tau$ be full subcategories with the following properties:*

(1) *For an object U/X of $\mathcal{C}_t$,*
   (a) *if $\{U_i \to U\}$ is a covering of $(Sch/X)_\tau$, then U_i/X is an object of $\mathcal{C}_t$,*
   (b) *$U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_t$.*

(2) *X/X is an object of $\mathcal{C}_t$.*

We endow $\mathcal{C}_t$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(Sch/X)_\tau$ with $U \in \mathrm{Ob}(\mathcal{C}_t)$. Then

(a) *The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.*

Denote $g : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_t$ the restriction of $\mathcal{O}$ to $\mathcal{C}_t$. Denote $g_!$ the functor of Modules on Sites, Definition 18.16.1.

(b) *The canonical map $g_!\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.*

Proof. Assertion (a) is immediate from the definitions. In this proof all schemes are schemes over X and all morphisms of schemes are morphisms of schemes over X. Note that g^{-1} is given by restriction, so that for an object U of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}_1$ which associates to V in $\mathcal{C}_2$ the group

$$\mathrm{colim}_{V \to U} \mathcal{O}(U)$$

where U runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if

$$V \to U \to U'$$

are morphisms with $U, U' \in \mathrm{Ob}(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Surjectivity. Let V be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by h and the structure morphism $V \to X$. Writing $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to h. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let V be a scheme. Let $s = \sum_{i=1,\ldots,n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any i we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbf{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n$$

and we see that s is equivalent in the colimit to

$$\sum\nolimits_{i=1,\ldots,n}(f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on V, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbf{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that s is equivalent to zero in the colimit. □

59.50. Étale cohomology

03Q3 In the following sections we prove some basic results on étale cohomology. Here is an example of something we know for cohomology of topological spaces which also holds for étale cohomology.

0A50 **Lemma 59.50.1** (Mayer-Vietoris for étale cohomology)**.** *Let X be a scheme. Suppose that $X = U \cup V$ is a union of two opens. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ there exists a long exact cohomology sequence*

$$\begin{aligned} 0 \to H^0_{étale}(X, \mathcal{F}) \to H^0_{étale}(U, \mathcal{F}) \oplus H^0_{étale}(V, \mathcal{F}) \to H^0_{étale}(U \cap V, \mathcal{F}) \\ \to H^1_{étale}(X, \mathcal{F}) \to H^1_{étale}(U, \mathcal{F}) \oplus H^1_{étale}(V, \mathcal{F}) \to H^1_{étale}(U \cap V, \mathcal{F}) \to \ldots \end{aligned}$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Observe that if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V) \to 0$$

is exact. This is true in the first and middle spots as $\mathcal{I}$ is a sheaf. It is true on the right, because $\mathcal{I}(U) \to \mathcal{I}(U \cap V)$ is surjective by Cohomology on Sites, Lemma 21.12.6. Another way to prove it would be to show that the cokernel of the map $\mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is the first Čech cohomology group of $\mathcal{I}$ with respect to the covering $X = U \cup V$ which vanishes by Lemmas 59.18.7 and 59.19.1. Thus, if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution, then

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0$$

is a short exact sequence of complexes and the associated long exact cohomology sequence is the sequence of the statement of the lemma. □

0EYK **Lemma 59.50.2** (Relative Mayer-Vietoris)**.** *Let $f : X \to Y$ be a morphism of schemes. Suppose that $X = U \cup V$ is a union of two open subschemes. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every abelian sheaf $\mathcal{F}$ on $X_{étale}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

on $Y_{étale}$. This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ on $X_{étale}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any W in $Y_{étale}$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(W \times_Y X) \to \mathcal{I}^n(W \times_Y U) \oplus \mathcal{I}^n(W \times_Y V) \to \mathcal{I}^n(W \times_Y (U \cap V)) \to 0$$

was shown to be short exact in the proof of Lemma 59.50.1. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$. □

59.51. Colimits

03Q4 We recall that if $(\mathcal{F}_i, \varphi_{ii'})$ is a diagram of sheaves on a site $\mathcal{C}$ its colimit (in the category of sheaves) is the sheafification of the presheaf $U \mapsto \operatorname{colim}_i \mathcal{F}_i(U)$. See Sites, Lemma 7.10.14. If the system is directed, U is a quasi-compact object of $\mathcal{C}$ which has a cofinal system of coverings by quasi-compact objects, then $\mathcal{F}(U) = \operatorname{colim} \mathcal{F}_i(U)$, see Sites, Lemma 7.17.7. See Cohomology on Sites, Lemma 21.16.1 for a result dealing with higher cohomology groups of colimits of abelian sheaves.

In Cohomology on Sites, Lemma 21.16.6 we generalize this result to a system of sheaves on an inverse system of sites. Here is the corresponding notion in the case of a system of étale sheaves living on an inverse system of schemes.

0EZL **Definition 59.51.1.** Let I be a preordered set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I. A *system $(\mathcal{F}_i, \varphi_{i'i})$ of sheaves on $(X_i, f_{i'i})$* is given by

(1) a sheaf $\mathcal{F}_i$ on $(X_i)_{étale}$ for all $i \in I$,
(2) for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of sheaves on $(X_{i'})_{étale}$

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$.

In the situation of Definition 59.51.1, assume I is a directed set and the transition morphisms $f_{i'i}$ affine. Let $X = \lim X_i$ be the limit in the category of schemes, see Limits, Section 32.2. Denote $f_i : X \to X_i$ the projection morphisms and consider the maps

$$f_i^{-1}\mathcal{F}_i = f_{i'}^{-1}f_{i'i}^{-1}\mathcal{F}_i \xrightarrow{f_{i'}^{-1}\varphi_{i'i}} f_{i'}^{-1}\mathcal{F}_{i'}$$

This turns $f_i^{-1}\mathcal{F}_i$ into a system of sheaves on $X_{étale}$ over I (it is a good exercise to check this). We often want to know whether there is an isomorphism

$$H^q_{étale}(X, \operatorname{colim} f_i^{-1}\mathcal{F}_i) = \operatorname{colim} H^q_{étale}(X_i, \mathcal{F}_i)$$

It will turn out this is true if X_i is quasi-compact and quasi-separated for all i, see Theorem 59.51.3.

0EYL **Lemma 59.51.2.** *Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse system of schemes over I with affine transition morphisms. Let $X = \lim_{i \in I} X_i$. With notation as in Topologies, Lemma 34.4.12 we have*

$$X_{affine,étale} = \operatorname{colim}(X_i)_{affine,étale}$$

as sites in the sense of Sites, Lemma 7.18.2.

Proof. Let us first prove this when X and X_i are quasi-compact and quasi-separated for all i (as this is true in all cases of interest). In this case any object of $X_{affine,étale}$, resp. $(X_i)_{affine,étale}$ is of finite presentation over X. Moreover, the category of schemes of finite presentation over X is the colimit of the categories of schemes of finite presentation over X_i, see Limits, Lemma 32.10.1. The same holds for the subcategories of affine objects étale over X by Limits, Lemmas 32.4.13 and 32.8.10. Finally, if $\{U^j \to U\}$ is a covering of $X_{affine,étale}$ and if $U_i^j \to U_i$ is morphism of affine schemes étale over X_i whose base change to X is $U^j \to U$, then we see that the base change of $\{U_i^j \to U_i\}$ to some $X_{i'}$ is a covering for i' large enough, see Limits, Lemma 32.8.15.

In the general case, let U be an object of $X_{affine,étale}$. Then $U \to X$ is étale and separated (as U is separated) but in general not quasi-compact. Still, $U \to X$ is locally of finite presentation and hence by Limits, Lemma 32.10.5 there exists

an i, a quasi-compact and quasi-separated scheme U_i, and a morphism $U_i \to X_i$ which is locally of finite presentation whose base change to X is $U \to X$. Then $U = \lim_{i' \geq i} U_{i'}$ where $U_{i'} = U_i \times_{X_i} X_{i'}$. After increasing i we may assume U_i is affine, see Limits, Lemma 32.4.13. To check that $U_i \to X_i$ is étale for i sufficiently large, choose a finite affine open covering $U_i = U_{i,1} \cup \ldots \cup U_{i,m}$ such that $U_{i,j} \to U_i \to X_i$ maps into an affine open $W_{i,j} \subset X_i$. Then we can apply Limits, Lemma 32.8.10 to see that $U_{i,j} \to W_{i,j}$ is étale after possibly increasing i. In this way we see that the functor $\mathrm{colim}(X_i)_{affine,étale} \to X_{affine,étale}$ is essentially surjective. Fully faithfulness follows directly from the already used Limits, Lemma 32.10.5. The statement on coverings is proved in exactly the same manner as done in the first paragraph of the proof. □

Using the above we get the following general result on colimits and cohomology.

09YQ **Theorem 59.51.3.** *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Then*

$$\mathrm{colim}_{i \in I} H^p_{étale}(X_i, \mathcal{F}_i) = H^p_{étale}(X, \mathcal{F}).$$

for all $p \geq 0$.

Proof. By Topologies, Lemma 34.4.12 we can compute the cohomology of $\mathcal{F}$ on $X_{affine,étale}$. Thus the result by a combination of Lemma 59.51.2 and Cohomology on Sites, Lemma 21.16.6. □

The following two results are special cases of the theorem above.

03Q5 **Lemma 59.51.4.** *Let X be a quasi-compact and quasi-separated scheme. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ij})$ be a system of abelian sheaves on $X_{étale}$ over I. Then*

$$\mathrm{colim}_{i \in I} H^p_{étale}(X, \mathcal{F}_i) = H^p_{étale}(X, \mathrm{colim}_{i \in I} \mathcal{F}_i).$$

Proof. This is a special case of Theorem 59.51.3. We also sketch a direct proof. We prove it for all X at the same time, by induction on p.

(1) For any quasi-compact and quasi-separated scheme X and any étale covering $\mathcal{U}$ of X, show that there exists a refinement $\mathcal{V} = \{V_j \to X\}_{j \in J}$ with J finite and each V_j quasi-compact and quasi-separated such that all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ are also quasi-compact and quasi-separated.
(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$ and all X.
(3) Using the locality of cohomology (Lemma 59.22.3), the Čech-to-cohomology spectral sequence (Theorem 59.19.2) and the fact that the induction hypothesis applies to all $V_{j_0} \times_X \ldots \times_X V_{j_p}$ in the above situation, prove the induction step $p \to p + 1$.

□

03Q6 **Lemma 59.51.5.** *Let A be a ring, $(I, \leq)$ a directed set and (B_i, φ_{ij}) a system of A-algebras. Set $B = \mathrm{colim}_{i \in I} B_i$. Let $X \to \mathrm{Spec}(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ an abelian sheaf on $X_{étale}$. Denote*

$Y_i = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B_i)$, $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1}\mathcal{F}$ *and* $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. *Then*

$$H^p_{étale}(Y, \mathcal{G}) = \mathrm{colim}_{i \in I} H^p_{étale}(Y_i, \mathcal{G}_i).$$

Proof. This is a special case of Theorem 59.51.3. We also outline a direct proof as follows.

(1) Given $V \to Y$ étale with V quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_{Y_i} Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \mathrm{colim}\, B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 10.143.3.
(2) In the situation of (1) show that $\mathcal{G}(V) = \mathrm{colim}_{i' \geq i} \mathcal{G}_{i'}(V_{i'})$ where $V_{i'}$ is the base change of V_i to $Y_{i'}$.
(3) By (1), we see that for every étale covering $\mathcal{V} = \{V_j \to Y\}_{j \in J}$ with J finite and the V_js quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \{V_{ij} \to Y_i\}_{j \in J}$ such that $\mathcal{V} \cong \mathcal{V}_i \times_{Y_i} Y$.
(4) Show that (2) and (3) imply
$$\check{H}^*(\mathcal{V}, \mathcal{G}) = \mathrm{colim}_{i \in I} \check{H}^*(\mathcal{V}_i, \mathcal{G}_i).$$
(5) Cleverly use the Čech-to-cohomology spectral sequence (Theorem 59.19.2).

□

03Q8 **Lemma 59.51.6.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in Ab(X_{étale})$. Then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$(V \to Y) \longmapsto H^p_{étale}(X \times_Y V, \mathcal{F}|_{X \times_Y V}).$$

More generally, for $K \in D(X_{étale})$ we have that $R^p f_* K$ is the sheaf associated to the presheaf

$$(V \to Y) \longmapsto H^p_{étale}(X \times_Y V, K|_{X \times_Y V}).$$

Proof. This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 21.7.4 for the case of a sheaf and see Cohomology on Sites, Lemma 21.20.3 for the case of a complex of abelian sheaves. □

09Z1 **Lemma 59.51.7.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume the structure morphisms $g_i : X_i \to S$ and $g : X \to S$ are quasi-compact and quasi-separated. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of abelian sheaves on $(X_i, f_{i'i})$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Then*

$$\mathrm{colim}_{i \in I} R^p g_{i,*} \mathcal{F}_i = R^p g_* \mathcal{F}$$

for all $p \geq 0$.

Proof. Recall (Lemma 59.51.6) that $R^p g_{i,*} \mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_* \mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{étale}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if U is a quasi-compact and quasi-separated object, then we have

$$\mathrm{colim}\, H^p_{étale}(U \times_S X_i, \mathcal{F}_i) = H^p_{étale}(U \times_S X, \mathcal{F})$$

by Theorem 59.51.3. Thus the lemma follows. □

0EYM **Lemma 59.51.8.** *Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $h_{i'i} : S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms g_i, see Limits, Section 32.2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on $(X_i, f_{i'i})$. Set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$. Then*

$$R^p g_*\mathcal{F} = \operatorname{colim}_{i \in I} h_i^{-1} R^p g_{i,*}\mathcal{F}_i$$

for all $p \geq 0$.

Proof. How is the map of the lemma constructed? For $i' \geq i$ we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f_{i'}} & X_{i'} & \xrightarrow{f_{i'i}} & X_i \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{i'}} & & \downarrow{\scriptstyle g_i} \\ S & \xrightarrow{h_{i'}} & S_{i'} & \xrightarrow{h_{i'i}} & S_i \end{array}$$

If we combine the base change map $h_{i'i}^{-1} R g_{i,*}\mathcal{F}_i \to R g_{i',*} f_{i'i}^{-1}\mathcal{F}_i$ (Cohomology on Sites, Lemma 21.15.1 or Remark 21.19.3) with the map $R g_{i',*}\varphi_{i'i}$, then we obtain $\psi_{i'i} : h_{i'i}^{-1} R^p g_{i,*}\mathcal{F}_i \to R^p g_{i',*}\mathcal{F}_{i'}$. Similarly, using the left square in the diagram we obtain maps $\psi_i : h_i^{-1} R^p g_{i,*}\mathcal{F}_i \to R^p g_*\mathcal{F}$. The maps $h_{i'}^{-1}\psi_{i'i}$ and ψ_i are the maps used in the statement of the lemma. For this to make sense, we have to check that $\psi_{i''i} = \psi_{i''i'} \circ h_{i''i'}^{-1}\psi_{i'i}$ and $\psi_{i'} \circ h_{i'}^{-1}\psi_{i'i} = \psi_i$; this follows from Cohomology on Sites, Remark 21.19.5.

Proof of the equality. First proof using dimension shifting[4]. For any U affine and étale over X by Theorem 59.51.3 we have

$$g_*\mathcal{F}(U) = H^0(U \times_S X, \mathcal{F}) = \operatorname{colim} H^0(U_i \times_{S_i} X_i, \mathcal{F}_i) = \operatorname{colim} g_{i,*}\mathcal{F}_i(U_i)$$

where the colimit is over i large enough such that there exists an i and U_i affine étale over S_i whose base change is U over S (see Lemma 59.51.2). The right hand side is equal to $(\operatorname{colim} h_i^{-1} g_{i,*}\mathcal{F}_i)(U)$ by Sites, Lemma 7.18.4. This proves the lemma for $p = 0$. If $(\mathcal{G}_i, \varphi_{i'i})$ is a system with $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$ such that $\mathcal{G}_i$ is an injective abelian sheaf on X_i for all i, then for any U affine and étale over X by Theorem 59.51.3 we have

$$H^p(U \times_S X, \mathcal{G}) = \operatorname{colim} H^p(U_i \times_{S_i} X_i, \mathcal{G}_i) = 0$$

for $p > 0$ (same colimit as before). Hence $R^p g_*\mathcal{G} = 0$ and we get the result for $p > 0$ for such a system. In general we may choose a short exact sequence of systems

$$0 \to (\mathcal{F}_i, \varphi_{i'i}) \to (\mathcal{G}_i, \varphi_{i'i}) \to (\mathcal{Q}_i, \varphi_{i'i}) \to 0$$

where $(\mathcal{G}_i, \varphi_{i'i})$ is as above, see Cohomology on Sites, Lemma 21.16.5. By induction the lemma holds for $p - 1$ and by the above we have vanishing for p and $(\mathcal{G}_i, \varphi_{i'i})$. Hence the result for p and $(\mathcal{F}_i, \varphi_{i'i})$ by the long exact sequence of cohomology.

[4]You can also use this method to produce the maps in the lemma.

Second proof. Recall that $S_{affine,étale} = \text{colim}(S_i)_{affine,étale}$, see Lemma 59.51.2. Thus if U is an object of $S_{affine,étale}$, then we can write $U = U_i \times_{S_i} S$ for some i and some U_i in $(S_i)_{affine,étale}$ and

$$(\text{colim}_{i \in I} h_i^{-1} R^p g_{i,*} \mathcal{F}_i)(U) = \text{colim}_{i' \geq i} (R^p g_{i',*} \mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$$

by Sites, Lemma 7.18.4 and the construction of the transition maps in the system described above. Since $R^p g_{i',*} \mathcal{F}_{i'}$ is the sheaf associated to the presheaf $U_{i'} \mapsto H^p(U_{i'} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$ and since $R^p g_* \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p(U \times_S X, \mathcal{F})$ (Lemma 59.51.6) we obtain a canonical commutative diagram

$$\begin{array}{ccc} \text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'}) & \longrightarrow & \text{colim}_{i' \geq i} (R^p g_{i',*} \mathcal{F}_{i'})(U_i \times_{S_i} S_{i'}) \\ \downarrow & & \downarrow \\ H^p(U \times_S X, \mathcal{F}) & \longrightarrow & R^p g_* \mathcal{F}(U) \end{array}$$

Observe that the left hand vertical arrow is an isomorphism by Theorem 59.51.3. We're trying to show that the right hand vertical arrow is an isomorphism. However, we already know that the source and target of this arrow are sheaves on $S_{affine,étale}$. Hence it suffices to show: (1) an element in the target, locally comes from an element in the source and (2) an element in the source which maps to zero in the target locally vanishes. Part (1) follows immediately from the above and the fact that the lower horizontal arrow comes from a map of presheaves which becomes an isomorphism after sheafification. For part (2), say $\xi \in \text{colim}_{i' \geq i} (R^p g_{i',*} \mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ is in the kernel. Choose an $i' \geq i$ and $\xi_{i'} \in (R^p g_{i',*} \mathcal{F}_{i'})(U_i \times_{S_i} S_{i'})$ representing ξ. Choose a standard étale covering $\{U_{i',k} \to U_i \times_{S_i} S_{i'}\}_{k=1,\ldots,m}$ such that $\xi_{i'}|_{U_{i',k}}$ comes from $\xi_{i',k} \in H^p(U_{i',k} \times_{S_{i'}} X_{i'}, \mathcal{F}_{i'})$. Since it is enough to prove that ξ dies locally, we may replace U by the members of the étale covering $\{U_{i',k} \times_{S_{i'}} S \to U = U_i \times_{S_i} S\}$. After this replacement we see that ξ is the image of an element ξ' of the group $\text{colim}_{i' \geq i} H^p(U_i \times_{S_i} X_{i'}, \mathcal{F}_{i'})$ in the diagram above. Since ξ' maps to zero in $R^p g_* \mathcal{F}(U)$ we can do another replacement and assume that ξ' maps to zero in $H^p(U \times_S X, \mathcal{F})$. However, since the left vertical arrow is an isomorphism we then conclude $\xi' = 0$ hence $\xi = 0$ as desired. □

0EYN **Lemma 59.51.9.** *Let $X = \lim_{i \in I} X_i$ be a directed limit of schemes with affine transition morphisms $f_{i'i}$ and projection morphisms $f_i : X \to X_i$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then*

1. *there are canonical maps $\varphi_{i'i} : f_{i'i}^{-1} f_{i,*} \mathcal{F} \to f_{i',*} \mathcal{F}$ such that $(f_{i,*} \mathcal{F}, \varphi_{i'i})$ is a system of sheaves on $(X_i, f_{i'i})$ as in Definition 59.51.1, and*
2. *$\mathcal{F} = \text{colim}\, f_i^{-1} f_{i,*} \mathcal{F}$.*

Proof. Via Topologies, Lemma 34.4.12 and Lemma 59.51.2 this is a special case of Sites, Lemma 7.18.5. □

0DV2 **Lemma 59.51.10.** *Let I be a directed set. Let $g_i : X_i \to S_i$ be an inverse system of morphisms of schemes over I. Assume g_i is quasi-compact and quasi-separated and for $i' \geq i$ the transition morphisms $X_{i'} \to X_i$ and $S_{i'} \to S_i$ are affine. Let $g : X \to S$ be the limit of the morphisms g_i, see Limits, Section 32.2. Denote $f_i : X \to X_i$ and $h_i : S \to S_i$ the projections. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have*

$$R^p g_* \mathcal{F} = \text{colim}_{i \in I} h_i^{-1} R^p g_{i,*} (f_{i,*} \mathcal{F})$$

Proof. Formal combination of Lemmas 59.51.8 and 59.51.9. □

59.52. Colimits and complexes

0GIR In this section we discuss taking cohomology of systems of complexes in various settings, continuing the discussion for sheaves started in Section 59.51. We strongly urge the reader not to read this section unless absolutely necessary.

0EZM **Lemma 59.52.1.** *Let $X = \lim_{i\in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $\mathcal{F}_i^\bullet$ be a complex of abelian sheaves on $X_{i,étale}$. Let $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i^\bullet \to \mathcal{F}_{i'}^\bullet$ be a map of complexes on $X_{i,étale}$ such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Assume there is an integer a such that $\mathcal{F}_i^n = 0$ for $n < a$ and all $i \in I$. Then we have*

$$H^p_{étale}(X, \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet) = \operatorname{colim} H^p_{étale}(X_i, \mathcal{F}_i^\bullet)$$

where $f_i : X \to X_i$ is the projection.

Proof. This is a consequence of Theorem 59.51.3. Set $\mathcal{F}^\bullet = \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet$. The theorem tells us that

$$\operatorname{colim}_{i\in I} H^p_{étale}(X_i, \mathcal{F}_i^n) = H^p_{étale}(X, \mathcal{F}^n)$$

for all $n, p \in \mathbf{Z}$. Let us use the spectral sequences

$$E_{1,i}^{s,t} = H^t_{étale}(X_i, \mathcal{F}_i^s) \Rightarrow H^{s+t}_{étale}(X_i, \mathcal{F}_i^\bullet)$$

and

$$E_1^{s,t} = H^t_{étale}(X, \mathcal{F}^s) \Rightarrow H^{s+t}_{étale}(X, \mathcal{F}^\bullet)$$

of Derived Categories, Lemma 13.21.3. Since $\mathcal{F}_i^n = 0$ for $n < a$ (with a independent of i) we see that only a fixed finite number of terms $E_{1,i}^{s,t}$ (independent of i) and $E_1^{s,t}$ contribute to $H^q_{étale}(X_i, \mathcal{F}_i^\bullet)$ and $H^q_{étale}(X, \mathcal{F}^\bullet)$ and $E_1^{s,t} = \operatorname{colim} E_{i,i}^{s,t}$. This implies what we want. Some details omitted. (There is an alternative argument using "stupid" truncations of complexes which avoids using spectral sequences.) □

0GIS **Lemma 59.52.2.** *Let X be a quasi-compact and quasi-sepated scheme. Let $K_i \in D(X_{étale})$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $R\Gamma(X, \bigoplus_i K_i) = \bigoplus_i R\Gamma(X, K_i)$.*

Proof. We have to show that $H^p(X, \bigoplus_i K_i) = \bigoplus_i H^p(X, K_i)$ for all $p \in \mathbf{Z}$. Choose complexes $\mathcal{F}_i^\bullet$ representing K_i such that $\mathcal{F}_i^n = 0$ for $n < a$. The direct sum of the complexes $\mathcal{F}_i^\bullet$ represents the object $\bigoplus K_i$ by Injectives, Lemma 19.13.4. Since $\bigoplus \mathcal{F}^\bullet$ is the filtered colimit of the finite direct sums, the result follows from Lemma 59.52.1. □

0GIT **Lemma 59.52.3.** *Let S be a scheme. Let $X = \lim_{i\in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Let $K \in D^+(S_{étale})$. Then*

$$\operatorname{colim}_{i\in I} H^p_{étale}(X_i, K|_{X_i}) = H^p_{étale}(X, K|_X).$$

for all $p \in \mathbf{Z}$ where $K|_{X_i}$ and $K|_X$ are the pullbacks of K to X_i and X.

Proof. We may represent K by a bounded below complex $\mathcal{G}^\bullet$ of abelian sheaves on $S_{étale}$. Say $\mathcal{G}^n = 0$ for $n < a$. Denote $\mathcal{F}_i^\bullet$ and $\mathcal{F}^\bullet$ the pullbacks of this complex of X_i and X. These complexes represent the objects $K|_{X_i}$ and $K|_X$ and we have $\mathcal{F}^\bullet = \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet$ termwise. Hence the lemma follows from Lemma 59.52.1. □

0GIU **Lemma 59.52.4.** *Let I, $g_i : X_i \to S_i$, $g : X \to S$, f_i, g_i, h_i be as in Lemma 59.51.8. Let $0 \in I$ and $K_0 \in D^+(X_{0,étale})$. For $i \geq 0$ denote K_i the pullback of K_0 to X_i. Denote K the pullback of K_0 to X. Then*

$$R^p g_* K = \operatorname{colim}_{i \geq 0} h_i^{-1} R^p g_{i,*} K_i$$

for all $p \in \mathbf{Z}$.

Proof. Fix an integer $p_0 \in \mathbf{Z}$. Let a be an integer such that $H^j(K_0) = 0$ for $j < a$. We will prove the formula holds for all $p \leq p_0$ by descending induction on a. If $a > p_0$, then we see that the left and right hand side of the formula are zero for $p \leq p_0$ by trivial vanishing, see Derived Categories, Lemma 13.16.1. Assume $a \leq p_0$. Consider the distinguished triangle

$$H^a(K_0)[-a] \to K_0 \to \tau_{\geq a+1} K_0$$

Pulling back this distinguished triangle to X_i and X gives compatible distinguished triangles for K_i and K. For $p \leq p_0$ we consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_{i\geq 0} h_i^{-1} R^{p-1} g_{i,*}(\tau_{\geq a+1} K_i) & \xrightarrow{\alpha} & R^{p-1} g_*(\tau_{\geq a+1} K) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\geq 0} h_i^{-1} R^p g_{i,*}(H^a(K_i)[-a]) & \xrightarrow{\beta} & R^p g_*(H^a(K)[-a]) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\geq 0} h_i^{-1} R^p g_{i,*} K_i & \xrightarrow{\gamma} & R^p g_* K \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\geq 0} R^p g_{i,*} \tau_{\geq a+1} K_i & \xrightarrow{\delta} & R^p g_* \tau_{\geq a+1} K \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\geq 0} R^{p+1} g_{i,*}(H^a(K_i)[-a]) & \xrightarrow{\epsilon} & R^{p+1} g_*(H^a(K)[-a])
\end{array}$$

with exact columns. The arrows β and ϵ are isomorphisms by Lemma 59.51.8. The arrows α and δ are isomorphisms by induction hypothesis. Hence γ is an isomorphism as desired. □

0GIV **Lemma 59.52.5.** *Let I, $g_i : X_i \to S_i$, $g : X \to S$, $f_{ii'}$, f_i, g_i, h_i be as in Lemma 59.51.8. Let $\mathcal{F}_i^\bullet$ be a complex of abelian sheaves on $X_{i,étale}$. Let $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i^\bullet \to \mathcal{F}_{i'}^\bullet$ be a map of complexes on $X_{i,étale}$ such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Assume there is an integer a such that $\mathcal{F}_i^n = 0$ for $n < a$ and all $i \in I$. Then*

$$R^p g_*(\operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet) = \operatorname{colim}_{i \geq 0} h_i^{-1} R^p g_{i,*} \mathcal{F}_i^\bullet$$

for all $p \in \mathbf{Z}$.

Proof. This is a consequence of Lemma 59.51.8. Set $\mathcal{F}^\bullet = \operatorname{colim} f_i^{-1}\mathcal{F}_i^\bullet$. The lemma tells us that

$$\operatorname{colim}_{i\in I} h_i^{-1} R^p g_{i,*}\mathcal{F}_i^n = R^p g_*\mathcal{F}^n$$

for all $n, p \in \mathbf{Z}$. Let us use the spectral sequences

$$E_{1,i}^{s,t} = R^t g_{i,*}\mathcal{F}_i^s \Rightarrow R^{s+t} g_{i,*}\mathcal{F}_i^\bullet$$

and

$$E_1^{s,t} = R^t g_*\mathcal{F}^s \Rightarrow R^{s+t} g_*\mathcal{F}^\bullet$$

of Derived Categories, Lemma 13.21.3. Since $\mathcal{F}_i^n = 0$ for $n < a$ (with a independent of i) we see that only a fixed finite number of terms $E_{1,i}^{s,t}$ (independent of i) and $E_1^{s,t}$ contribute and $E_1^{s,t} = \operatorname{colim} E_{i,i}^{s,t}$. This implies what we want. Some details omitted. (There is an alternative argument using "stupid" truncations of complexes which avoids using spectral sequences.) □

0GIW **Lemma 59.52.6.** *Let $f : X \to Y$ be a quasi-compact and quasi-sepated morphism of schemes. Let $K_i \in D(X_{étale})$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $Rf_*(\bigoplus_i K_i) = \bigoplus_i Rf_*K_i$.*

Proof. We have to show that $R^p f_*(\bigoplus_i K_i) = \bigoplus_i R^p f_*K_i$ for all $p \in \mathbf{Z}$. Choose complexes $\mathcal{F}_i^\bullet$ representing K_i such that $\mathcal{F}_i^n = 0$ for $n < a$. The direct sum of the complexes $\mathcal{F}_i^\bullet$ represents the object $\bigoplus K_i$ by Injectives, Lemma 19.13.4. Since $\bigoplus \mathcal{F}^\bullet$ is the filtered colimit of the finite direct sums, the result follows from Lemma 59.52.5. □

59.53. Stalks of higher direct images

03Q7 The stalks of higher direct images can often be computed as follows.

03Q9 **Theorem 59.53.1.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{étale}$, and $\overline{s}$ a geometric point of S lying over $s \in S$. Then*

$$(R^n f_*\mathcal{F})_{\overline{s}} = H^n_{étale}(X \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F})$$

where $p : X \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. For $K \in D^+(X_{étale})$ and $n \in \mathbf{Z}$ we have

$$(R^n f_*K)_{\overline{s}} = H^n_{étale}(X \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$$

In fact, we have

$$(Rf_*K)_{\overline{s}} = R\Gamma_{étale}(X \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$$

in $D^+(Ab)$.

Proof. Let $\mathcal{I}$ be the category of étale neighborhoods of $\overline{s}$ on S. By Lemma 59.51.6 we have

$$(R^n f_*\mathcal{F})_{\overline{s}} = \operatorname{colim}_{(V,\overline{v})\in\mathcal{I}^{opp}} H^n_{étale}(X \times_S V, \mathcal{F}|_{X\times_S V}).$$

We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighbourhoods of $\overline{s}$. Observe that

$$\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V,\overline{v})\in\mathcal{I}} V$$

by Lemma 59.33.1 and Limits, Lemma 32.2.1. Since fibre products commute with limits we also obtain

$$X \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V,\overline{v})\in\mathcal{I}} X \times_S V$$

We conclude by Lemma 59.51.5. For the second variant, use the same argument using Lemma 59.52.3 instead of Lemma 59.51.5.

To see that the last statement is true, it suffices to produce a map $(Rf_*K)_{\overline{s}} \to R\Gamma_{étale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$ in $D^+(Ab)$ which realizes the ismorphisms on cohomology groups in degree n above for all n. To do this, choose a bounded below complex $\mathcal{J}^\bullet$ of injective abelian sheaves on $X_{étale}$ representing K. The complex $f_*\mathcal{J}^\bullet$ represents Rf_*K. Thus the complex

$$(f_*\mathcal{J}^\bullet)_{\overline{s}} = \mathrm{colim}_{(V,\overline{v})\in\mathcal{I}^{opp}}(f_*\mathcal{J}^\bullet)(V)$$

represents $(Rf_*K)_{\overline{s}}$. For each V we have maps

$$(f_*\mathcal{J}^\bullet)(V) = \Gamma(X \times_S V, \mathcal{J}^\bullet) \longrightarrow \Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{J}^\bullet)$$

and the target complex represents $R\Gamma_{étale}(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K)$ in $D^+(Ab)$. Taking the colimit of these maps we obtain the result. □

0GIX **Remark 59.53.2.** Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{étale})$. Let $\overline{s}$ be a geometric point of S. There are always canonical maps

$$(Rf_*K)_{\overline{s}} \longrightarrow R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}K) \longrightarrow R\Gamma(X_{\overline{s}}, K|_{X_{\overline{s}}})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection. Namely, consider the commutative diagram

$$\begin{array}{ccccc} X_{\overline{s}} & \longrightarrow & X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) & \xrightarrow{p} & X \\ \downarrow{\scriptstyle f_{\overline{s}}} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \overline{s} & \xrightarrow{i} & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) & \xrightarrow{j} & S \end{array}$$

We have the base change maps

$$i^{-1}Rf'_*(p^{-1}K) \to Rf_{\overline{s},*}(K|_{X_{\overline{s}}}) \quad \text{and} \quad j^{-1}Rf_*K \to Rf'_*(p^{-1}K)$$

(Cohomology on Sites, Remark 21.19.3) for the two squares in this diagram. Taking global sections we obtain the desired maps. By Cohomology on Sites, Remark 21.19.5 the composition of these two maps is the usual (base change) map $(Rf_*K)_{\overline{s}} \to R\Gamma(X_{\overline{s}}, K|_{X_{\overline{s}}})$.

59.54. The Leray spectral sequence

03QA

03QB **Lemma 59.54.1.** *Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $Ab(X_{étale})$. Let $V \in \mathrm{Ob}(Y_{étale})$. Then*

1. *for any covering $\mathcal{V} = \{V_j \to V\}_{j\in J}$ we have $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$,*
2. *$f_*\mathcal{I}$ is acyclic for the functor $\Gamma(V, -)$, and*
3. *if $g : Y \to Z$, then $f_*\mathcal{I}$ is acyclic for g_*.*

Proof. Observe that $\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{V} \times_Y X, \mathcal{I})$ which has vanishing higher cohomology groups by Lemma 59.18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 21.10.9 for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of R^pg_* in Lemma 59.51.6. □

Using the formalism of Grothendieck spectral sequences, this gives the following.

03QC **Proposition 59.54.2** (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p_{étale}(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{étale}(X, \mathcal{F}).$$

Proof. See Lemma 59.54.1 and see Derived Categories, Section 13.22. □

59.55. Vanishing of finite higher direct images

03QN The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

03QO **Lemma 59.55.1.** *Let R be a strictly henselian local ring. Set $S = \mathrm{Spec}(R)$ and let $\overline{s}$ be its closed point. Then the global sections functor $\Gamma(S, -) : Ab(S_{étale}) \to Ab$ is exact. In fact we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ for any sheaf of sets $\mathcal{F}$. In particular*

$$\forall p \geq 1, \quad H^p_{étale}(S, \mathcal{F}) = 0$$

for all $\mathcal{F} \in Ab(S_{étale})$.

Proof. If we show that $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ then $\Gamma(S, -)$ is exact as the stalk functor is exact. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{s}$. Pick an affine open neighborhood $\mathrm{Spec}(A)$ of $\overline{u}$ in U. Then $R \to A$ is étale and $\kappa(\overline{s}) = \kappa(\overline{u})$. By Theorem 59.32.4 we see that $A \cong R \times A'$ as an R-algebra compatible with maps to $\kappa(\overline{s}) = \kappa(\overline{u})$. Hence we get a section

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & U \\ & \nwarrow & \downarrow \\ & & S \end{array}$$

It follows that in the system of étale neighbourhoods of $\overline{s}$ the identity map $(S, \overline{s}) \to (S, \overline{s})$ is cofinal. Hence $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$. The final statement of the lemma follows as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 13.16.9. □

03QP **Proposition 59.55.2.** *Let $f : X \to Y$ be a finite morphism of schemes.*

(1) *For any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_* \mathcal{F})_{\overline{y}} = \prod\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Sh(X_{étale})$ and

$$(f_* \mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Ab(X_{étale})$.

(2) *For any $q \geq 1$ we have $R^q f_* \mathcal{F} = 0$ for $\mathcal{F}$ in $Ab(X_{étale})$.*

Proof. Let $X^{sh}_{\overline{y}}$ denote the fiber product $X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$. By Theorem 59.53.1 the stalk of $R^q f_* \mathcal{F}$ at $\overline{y}$ is computed by $H^q_{étale}(X^{sh}_{\overline{y}}, \mathcal{F})$. Since f is finite, $X^{sh}_{\overline{y}}$ is finite over $\mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$, thus $X^{sh}_{\overline{y}} = \mathrm{Spec}(A)$ for some ring A finite over $\mathcal{O}^{sh}_{Y,\overline{y}}$. Since the latter is strictly henselian, Lemma 59.32.5 implies that A is a finite product of henselian local rings $A = A_1 \times \ldots \times A_r$. Since the residue field of $\mathcal{O}^{sh}_{Y,\overline{y}}$ is separably closed the same is true for each A_i. Hence A_i is strictly henselian. This implies that $X^{sh}_{\overline{y}} = \coprod_{i=1}^r \mathrm{Spec}(A_i)$. The vanishing of Lemma 59.55.1 implies that

$(R^q f_*\mathcal{F})_{\overline{y}} = 0$ for $q > 0$ which implies (2) by Theorem 59.29.10. Part (1) follows from the corresponding statement of Lemma 59.55.1. □

0959 **Lemma 59.55.3.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f a finite morphism. For any sheaf of sets $\mathcal{F}$ on $X_{étale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 7.45. It suffices to check on stalks (Theorem 59.29.10). Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point. We have

$$\begin{aligned} (f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} &= \prod\nolimits_{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'} ((g')^{-1}\mathcal{F})_{\overline{x}'} \\ &= \prod\nolimits_{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'} \mathcal{F}_{g' \circ \overline{x}'} \\ &= \prod\nolimits_{\overline{x} : \mathrm{Spec}(k) \to X,\ f \circ \overline{x} = g \circ \overline{y}'} \mathcal{F}_{\overline{x}} \\ &= (f_*\mathcal{F})_{g \circ \overline{y}'} \\ &= (g^{-1}f_*\mathcal{F})_{\overline{y}'} \end{aligned}$$

The first equality by Proposition 59.55.2. The second equality by Lemma 59.36.2. The third equality holds because the diagram is a cartesian square and hence the map

$$\{\overline{x}' : \mathrm{Spec}(k) \to X',\ f' \circ \overline{x}' = \overline{y}'\} \longrightarrow \{\overline{x} : \mathrm{Spec}(k) \to X,\ f \circ \overline{x} = g \circ \overline{y}'\}$$

sending $\overline{x}'$ to $g' \circ \overline{x}'$ is a bijection. The fourth equality by Proposition 59.55.2. The fifth equality by Lemma 59.36.2. □

0EYP **Lemma 59.55.4.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f an integral morphism. For any sheaf of sets $\mathcal{F}$ on $X_{étale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. The question is local on Y and hence we may assume Y is affine. Then we can write $X = \lim X_i$ with $f_i : X_i \to Y$ finite (this is easy in the affine case, but see Limits, Lemma 32.7.3 for a reference). Denote $p_{i'i} : X_{i'} \to X_i$ the transition morphisms and $p_i : X \to X_i$ the projections. Setting $\mathcal{F}_i = p_{i,*}\mathcal{F}$ we obtain from Lemma 59.51.9 a system $(\mathcal{F}_i, \varphi_{i'i})$ with $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$. We get $f_*\mathcal{F} = \mathrm{colim}\, f_{i,*}\mathcal{F}_i$ from Lemma 59.51.7. Set $X'_i = Y' \times_Y X_i$ with projections f'_i and g'_i. Then $X' = \lim X'_i$

as limits commute with limits. Denote $p'_i : X' \to X'_i$ the projections. We have

$$\begin{aligned} g^{-1}f_*\mathcal{F} &= g^{-1}\operatorname{colim} f_{i,*}\mathcal{F}_i \\ &= \operatorname{colim} g^{-1}f_{i,*}\mathcal{F}_i \\ &= \operatorname{colim} f'_{i,*}(g'_i)^{-1}\mathcal{F}_i \\ &= f'_*(\operatorname{colim}(p'_i)^{-1}(g'_i)^{-1}\mathcal{F}_i) \\ &= f'_*(\operatorname{colim}(g')^{-1}p_i^{-1}\mathcal{F}_i) \\ &= f'_*(g')^{-1}\operatorname{colim} p_i^{-1}\mathcal{F}_i \\ &= f'_*(g')^{-1}\mathcal{F} \end{aligned}$$

as desired. For the first equality see above. For the second use that pullback commutes with colimits. For the third use the finite case, see Lemma 59.55.3. For the fourth use Lemma 59.51.7. For the fifth use that $g'_i \circ p'_i = p_i \circ g'$. For the sixth use that pullback commutes with colimits. For the seventh use $\mathcal{F} = \operatorname{colim} p_i^{-1}\mathcal{F}_i$. □

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

09Z2 **Lemma 59.55.5.** *Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of X over Y. For $\mathcal{F} \in Ab(Y_{étale})$ set $\mathcal{F}_n = f_{n,*}f_n^{-1}\mathcal{F}$. There is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

on $X_{étale}$. Moreover, there is a spectral sequence

$$E_1^{p,q} = H^q_{étale}(X_p, f_p^{-1}\mathcal{F})$$

converging to $H^{p+q}(Y_{étale}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{p,q} = H^q_{étale}(Y, \mathcal{F})$ converging to $H^{p+q}(Y_{étale}, \mathcal{F})$, see Derived Categories, Lemma 13.21.3. On the other hand, since $R^i f_{p,*}f_p^{-1}\mathcal{F} = 0$ for $i > 0$ (Proposition 59.55.2) we get

$$H^q_{étale}(X_p, f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, f_{p,*}f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, \mathcal{F}_p)$$

by Proposition 59.54.2 and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that X_n forms a simplicial scheme over Y, see Simplicial, Example 14.3.5. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_j^{-1}f_n^{-1}\mathcal{F} \to f_{n+1}^{-1}\mathcal{F}$. These maps induce maps

$$\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$$

for $j = 0, \ldots, n+1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_*f^{-1}\mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points (Theorem 59.29.10). Thus we let $\overline{y} : \operatorname{Spec}(k) \to Y$ be a geometric point. Let $E = \{\overline{x} : \operatorname{Spec}(k) \to X \mid f(\overline{x}) = \overline{y}\}$. Then E is a finite nonempty set and we see that

$$(\mathcal{F}_n)_{\overline{y}} = \bigoplus\nolimits_{e \in E^{n+1}} \mathcal{F}_{\overline{y}}$$

by Proposition 59.55.2 and Lemma 59.36.2. Thus we have to see that given an abelian group M the sequence

$$0 \to M \to \bigoplus\nolimits_{e \in E} M \to \bigoplus\nolimits_{e \in E^2} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n+2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 14.26.9 to the map $E \to \{*\}$. □

09Z3 **Remark 59.55.6.** In the situation of Lemma 59.55.5 if $\mathcal{G}$ is a sheaf of sets on $Y_{étale}$, then we have

$$\Gamma(Y, \mathcal{G}) = \text{Equalizer}(\ \Gamma(X_0, f_0^{-1}\mathcal{G}) \rightrightarrows \Gamma(X_1, f_1^{-1}\mathcal{G})\ )$$

This is proved in exactly the same way, by showing that the sheaf $\mathcal{G}$ is the equalizer of the two maps $f_{0,*}f_0^{-1}\mathcal{G} \to f_{1,*}f_1^{-1}\mathcal{G}$.

59.56. Galois action on stalks

03QW In this section we define an action of the absolute Galois group of a residue field of a point s of S on the stalk functor at any geometric point lying over s.

Galois action on stalks. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$. Define an action of σ on the stalk $\mathcal{F}_{\overline{s}}$ of a sheaf $\mathcal{F}$ as follows

04JK (59.56.0.1)
$$\begin{array}{ccc} \mathcal{F}_{\overline{s}} & \longrightarrow & \mathcal{F}_{\overline{s}} \\ (U, \overline{u}, t) & \longmapsto & (U, \overline{u} \circ \text{Spec}(\sigma), t). \end{array}$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 59.29.6. This is a left action, since if $\sigma_i \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$ then

$$\begin{aligned} \sigma_1 \cdot (\sigma_2 \cdot (U, \overline{u}, t)) &= \sigma_1 \cdot (U, \overline{u} \circ \text{Spec}(\sigma_2), t) \\ &= (U, \overline{u} \circ \text{Spec}(\sigma_2) \circ \text{Spec}(\sigma_1), t) \\ &= (U, \overline{u} \circ \text{Spec}(\sigma_1 \circ \sigma_2), t) \\ &= (\sigma_1 \circ \sigma_2) \cdot (U, \overline{u}, t) \end{aligned}$$

It is clear that this action is functorial in the sheaf $\mathcal{F}$. We note that we could have defined this action by referring directly to Remark 59.29.8.

03QX **Definition 59.56.1.** Let S be a scheme. Let $\overline{s}$ be a geometric point lying over the point s of S. Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\overline{s})$.

(1) In this situation the *absolute Galois group* of $\kappa(s)$ is $\text{Gal}(\kappa(s)^{sep}/\kappa(s))$. It is sometimes denoted $\text{Gal}_{\kappa(s)}$.
(2) The geometric point $\overline{s}$ is called *algebraic* if $\kappa(s) \subset \kappa(\overline{s})$ is an algebraic closure of $\kappa(s)$.

03QY **Example 59.56.2.** The geometric point $\text{Spec}(\mathbf{C}) \to \text{Spec}(\mathbf{Q})$ is not algebraic.

Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ be as in the definition. Note that as $\kappa(\overline{s})$ is algebraically closed the map

$$\text{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)^{sep}/\kappa(s)) = \text{Gal}_{\kappa(s)}$$

is surjective. Suppose $(U, \overline{u})$ is an étale neighbourhood of $\overline{s}$, and say $\overline{u}$ lies over the point u of U. Since $U \to S$ is étale, the residue field extension $\kappa(u)/\kappa(s)$ is finite separable. This implies the following

(1) If $\sigma \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s)^{sep})$ then σ acts trivially on $\mathcal{F}_{\overline{s}}$.
(2) More precisely, the action of $\mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$ determines and is determined by an action of the absolute Galois group $\mathrm{Gal}_{\kappa(s)}$ on $\mathcal{F}_{\overline{s}}$.
(3) Given $(U, \overline{u}, t)$ representing an element ξ of $\mathcal{F}_{\overline{s}}$ any element of $\mathrm{Gal}(\kappa(s)^{sep}/K)$ acts trivially, where $\kappa(s) \subset K \subset \kappa(s)^{sep}$ is the image of $\overline{u}^\sharp : \kappa(u) \to \kappa(\overline{s})$.

Altogether we see that $\mathcal{F}_{\overline{s}}$ becomes a $\mathrm{Gal}_{\kappa(s)}$-set (see Fundamental Groups, Definition 58.2.1). Hence we may think of the stalk functor as a functor

$$Sh(S_{étale}) \longrightarrow \mathrm{Gal}_{\kappa(s)}\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

and from now on we usually do think about the stalk functor in this way.

03QT **Theorem 59.56.3.** *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Taking stalks induces an equivalence of categories*

$$Sh(S_{étale}) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. Let us construct the inverse to this functor. In Fundamental Groups, Lemma 58.2.2 we have seen that given a G-set M there exists an étale morphism $X \to \mathrm{Spec}(K)$ such that $\mathrm{Mor}_K(\mathrm{Spec}(K^{sep}), X)$ is isomorphic to M as a G-set. Consider the sheaf $\mathcal{F}$ on $\mathrm{Spec}(K)_{étale}$ defined by the rule $U \mapsto \mathrm{Mor}_K(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $\mathcal{F}_{\overline{s}} = \mathrm{Mor}_K(\mathrm{Spec}(K^{sep}), X) = M$ as G-sets (details omitted). This gives the inverse of the functor and we win. □

04JL **Remark 59.56.4.** Another way to state the conclusion of Theorem 59.56.3 and Fundamental Groups, Lemma 58.2.2 is to say that every sheaf on $\mathrm{Spec}(K)_{étale}$ is representable by a scheme X étale over $\mathrm{Spec}(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 7.12.3. The reason is that in our construction of $\mathrm{Spec}(K)_{étale}$ we chose a sufficiently large set of schemes étale over $\mathrm{Spec}(K)$, whereas sheaves on $\mathrm{Spec}(K)_{étale}$ form a proper class.

04JM **Lemma 59.56.5.** *Assumptions and notations as in Theorem 59.56.3. There is a functorial bijection*

$$\Gamma(S, \mathcal{F}) = (\mathcal{F}_{\overline{s}})^G$$

Proof. We can prove this using formal arguments and the result of Theorem 59.56.3 as follows. Given a sheaf $\mathcal{F}$ corresponding to the G-set $M = \mathcal{F}_{\overline{s}}$ we have

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \mathrm{Mor}_{Sh(S_{étale})}(h_{\mathrm{Spec}(K)}, \mathcal{F}) \\ &= \mathrm{Mor}_{G\text{-}Sets}(\{*\}, M) \\ &= M^G \end{aligned}$$

Here the first identification is explained in Sites, Sections 7.2 and 7.12, the second results from Theorem 59.56.3 and the third is clear. We will also give a direct proof[5].

Suppose that $t \in \Gamma(S, \mathcal{F})$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is clearly invariant under the action of G. Conversely, suppose

[5]For the doubting Thomases out there.

that $(U, \overline{u}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is invariant. Then we may shrink U and assume $U = \mathrm{Spec}(L)$ for some finite separable field extension of K, see Proposition 59.26.2. In this case the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective since $U' \to U$ is a covering of $S_{étale}$. After enlarging L a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that

$$\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) = \coprod\nolimits_{\sigma \in \mathrm{Gal}(L/K)} \mathrm{Spec}(L)$$

where the maps $\mathrm{Spec}(L) \to \mathrm{Spec}(L \otimes_K L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of G implies that $t \in \mathcal{F}(\mathrm{Spec}(L))$ maps to the same element of $\mathcal{F}(\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $\mathcal{F}$ for the étale covering $\{\mathrm{Spec}(L) \to \mathrm{Spec}(K)\}$ kicks in and we conclude that t comes from a unique section of $\mathcal{F}$ over $\mathrm{Spec}(K)$. □

04JN **Remark 59.56.6.** Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S. By definition this means that k is algebraically closed. In particular the absolute Galois group of k is trivial. Hence by Theorem 59.56.3 the category of sheaves on $\mathrm{Spec}(k)_{étale}$ is equivalent to the category of sets. The equivalence is given by taking sections over $\mathrm{Spec}(k)$. This finally provides us with an alternative definition of the stalk functor. Namely, the functor

$$Sh(S_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

is isomorphic to the functor

$$Sh(S_{étale}) \longrightarrow Sh(\mathrm{Spec}(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{s}^*\mathcal{F}$$

To prove this rigorously one can use Lemma 59.36.2 part (3) with $f = \overline{s}$. Moreover, having said this the general case of Lemma 59.36.2 part (3) follows from functoriality of pullbacks.

59.57. Group cohomology

0A2H In the following, if we write $H^i(G, M)$ we will mean that G is a topological group and M a discrete G-module with continuous G-action and $H^i(G, -)$ is the ith right derived functor on the category Mod_G of such G-modules, see Definitions 59.57.1 and 59.57.2. This includes the case of an abstract group G, which simply means that G is viewed as a topological group with the discrete topology.

When the module has a nondiscrete topology, we will use the notation $H^i_{cont}(G, M)$ to indicate the continuous cohomology groups introduced in [**Tat76**], see Section 59.58.

04JP **Definition 59.57.1.** Let G be a topological group.

(1) A *G-module*, sometimes called a *discrete G-module*, is an abelian group M endowed with a left action $a : G \times M \to M$ by group homomorphisms such that a is continuous when M is given the discrete topology.
(2) A *morphism of G-modules* $f : M \to N$ is a G-equivariant homomorphism from M to N.
(3) The category of G-modules is denoted Mod_G.

Let R be a ring.

(1) An *R-G-module* is an R-module M endowed with a left action $a : G \times M \to M$ by R-linear maps such that a is continuous when M is given the discrete topology.
(2) A *morphism of R-G-modules* $f : M \to N$ is a G-equivariant R-module map from M to N.
(3) The category of R-G-modules is denoted $\mathrm{Mod}_{R,G}$.

The condition that $a : G \times M \to M$ is continuous is equivalent with the condition that the stabilizer of any $x \in M$ is open in G. If G is an abstract group then this corresponds to the notion of an abelian group endowed with a G-action provided we endow G with the discrete topology. Observe that $\mathrm{Mod}_{\mathbf{Z},G} = \mathrm{Mod}_G$.

The category Mod_G has enough injectives, see Injectives, Lemma 19.3.1. Consider the left exact functor

$$\mathrm{Mod}_G \longrightarrow Ab, \quad M \longmapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and ith right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

The same construction works for $H^0(G, -) : \mathrm{Mod}_{R,G} \to \mathrm{Mod}_R$. We will see in Lemma 59.57.3 that this agrees with the cohomology of the underlying G-module.

04JR **Definition 59.57.2.** Let G be a topological group. Let M be a discrete G-module with continuous G-action. In other words, M is an object of the category Mod_G introduced in Definition 59.57.1.

(1) The right derived functors $H^i(G, M)$ of $H^0(G, M)$ on the category Mod_G are called the *continuous group cohomology groups* of M.
(2) If G is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the *group cohomology groups* of M.
(3) If G is a Galois group, then the groups $H^i(G, M)$ are called the *Galois cohomology groups* of M.
(4) If G is the absolute Galois group of a field K, then the groups $H^i(G, M)$ are sometimes called the *Galois cohomology groups of K with coefficients in M*. In this case we sometimes write $H^i(K, M)$ instead of $H^i(G, M)$.

0DVD **Lemma 59.57.3.** *Let G be a topological group. Let R be a ring. For every $i \geq 0$ the diagram*

$$\begin{array}{ccc} Mod_{R,G} & \xrightarrow{H^i(G,-)} & Mod_R \\ \downarrow & & \downarrow \\ Mod_G & \xrightarrow{H^i(G,-)} & Ab \end{array}$$

whose vertical arrows are the forgetful functors is commutative.

Proof. Let us denote the forgetful functor $F : \mathrm{Mod}_{R,G} \to \mathrm{Mod}_G$. Then F has a left adjoint $H : \mathrm{Mod}_G \to \mathrm{Mod}_{R,G}$ given by $H(M) = M \otimes_{\mathbf{Z}} R$. Observe that every object of Mod_G is a quotient of a direct sum of modules of the form $\mathbf{Z}[G/U]$ where $U \subset G$ is an open subgroup. Here $\mathbf{Z}[G/U]$ denotes the G-modules of finite $\mathbf{Z}$-linear combinations of right U congruence classes in G endowed with left G-action. Thus every bounded above complex in Mod_G is quasi-isomorphic to a bounded above complex in Mod_G whose underlying terms are flat $\mathbf{Z}$-modules (Derived Categories,

Lemma 13.15.4). Thus it is clear that LH exists on $D^-(\mathrm{Mod}_G)$ and is computed by evaluating H on any complex whose terms are flat $\mathbf{Z}$-modules; this follows from Derived Categories, Lemma 13.15.7 and Proposition 13.16.8. We conclude from Derived Categories, Lemma 13.30.2 that

$$\mathrm{Ext}^i(\mathbf{Z}, F(M)) = \mathrm{Ext}^i(R, M)$$

for M in $\mathrm{Mod}_{R,G}$. Observe that $H^0(G, -) = \mathrm{Hom}(\mathbf{Z}, -)$ on Mod_G where $\mathbf{Z}$ denotes the G-module with trivial action. Hence $H^i(G, -) = \mathrm{Ext}^i(\mathbf{Z}, -)$ on Mod_G. Similarly we have $H^i(G, -) = \mathrm{Ext}^i(R, -)$ on $\mathrm{Mod}_{R,G}$. Combining everything we see that the lemma is true. □

0DVE **Lemma 59.57.4.** *Let G be a topological group. Let R be a ring. Let M, N be R-G-modules. If M is finite projective as an R-module, then $Ext^i(M, N) = H^i(G, M^\vee \otimes_R N)$ (for notation see proof).*

Proof. The module $M^\vee = \mathrm{Hom}_R(M, R)$ endowed with the contragredient action of G. Namely $(g \cdot \lambda)(m) = \lambda(g^{-1} \cdot m)$ for $g \in G$, $\lambda \in M^\vee$, $m \in M$. The action of G on $M^\vee \otimes_R N$ is the diagonal one, i.e., given by $g \cdot (\lambda \otimes n) = g \cdot \lambda \otimes g \cdot n$. Note that for a third R-G-module E we have $\mathrm{Hom}(E, M^\vee \otimes_R N) = \mathrm{Hom}(M \otimes_R E, N)$. Namely, this is true on the level of R-modules by Algebra, Lemmas 10.12.8 and 10.78.9 and the definitions of G-actions are chosen such that it remains true for R-G-modules. It follows that $M^\vee \otimes_R N$ is an injective R-G-module if N is an injective R-G-module. Hence if $N \to N^\bullet$ is an injective resolution, then $M^\vee \otimes_R N \to M^\vee \otimes_R N^\bullet$ is an injective resolution. Then

$$\mathrm{Hom}(M, N^\bullet) = \mathrm{Hom}(R, M^\vee \otimes_R N^\bullet) = (M^\vee \otimes_R N^\bullet)^G$$

Since the left hand side computes $\mathrm{Ext}^i(M, N)$ and the right hand side computes $H^i(G, M^\vee \otimes_R N)$ the proof is complete. □

0DVF **Lemma 59.57.5.** *Let G be a topological group. Let k be a field. Let V be a k-G-module. If G is topologically finitely generated and $\dim_k(V) < \infty$, then $\dim_k H^1(G, V) < \infty$.*

Proof. Let $g_1, \ldots, g_r \in G$ be elements which topologically generate G, i.e., this means that the subgroup generated by $g_1, \ldots, g_r$ is dense. By Lemma 59.57.4 we see that $H^1(G, V)$ is the k-vector space of extensions

$$0 \to V \to E \to k \to 0$$

of k-G-modules. Choose $e \in E$ mapping to $1 \in k$. Write

$$g_i \cdot e = v_i + e$$

for some $v_i \in V$. This is possible because $g_i \cdot 1 = 1$. We claim that the list of elements $v_1, \ldots, v_r \in V$ determine the isomorphism class of the extension E. Once we prove this the lemma follows as this means that our Ext vector space is isomorphic to a subquotient of the k-vector space $V^{\oplus r}$; some details omitted. Since E is an object of the category defined in Definition 59.57.1 we know there is an open subgroup U such that $u \cdot e = e$ for all $u \in U$. Now pick any $g \in G$. Then gU contains a word w in the elements $g_1, \ldots, g_r$. Say $gu = w$. Since the element $w \cdot e$ is determined by $v_1, \ldots, v_r$, we see that $g \cdot e = (gu) \cdot e = w \cdot e$ is too. □

0DV3 **Lemma 59.57.6.** *Let G be a profinite topological group. Then*

(1) *$H^i(G, M)$ is torsion for $i > 0$ and any G-module M, and*

(2) $H^i(G, M) = 0$ *if* M *is a* $\mathbf{Q}$*-vector space.*

Proof. Proof of (1). By dimension shifting we see that it suffices to show that $H^1(G, M)$ is torsion for every G-module M. Choose an exact sequence $0 \to M \to I \to N \to 0$ with I an injective object of the category of G-modules. Then any element of $H^1(G, M)$ is the image of an element $y \in N^G$. Choose $x \in I$ mapping to y. The stabilizer $U \subset G$ of x is open, hence has finite index r. Let $g_1, \ldots, g_r \in G$ be a system of representatives for G/U. Then $\sum g_i(x)$ is an invariant element of I which maps to ry. Thus r kills the element of $H^1(G, M)$ we started with. Part (2) follows as then $H^i(G, M)$ is both a $\mathbf{Q}$-vector space and torsion. □

59.58. Tate's continuous cohomology

0DVG Tate's continuous cohomology ([**Tat76**]) is defined by the complex of continuous inhomogeneous cochains. We can define this when M is an arbitrary topological abelian group endowed with a continuous G-action. Namely, we consider the complex

$$C^\bullet_{cont}(G, M) : M \to \mathrm{Maps}_{cont}(G, M) \to \mathrm{Maps}_{cont}(G \times G, M) \to \ldots$$

where the boundary map is defined for $n \geq 1$ by the rule

$$\begin{aligned} \mathrm{d}(f)(g_1, \ldots, g_{n+1}) = & g_1(f(g_2, \ldots, g_{n+1})) \\ & + \sum\nolimits_{j=1,\ldots,n} (-1)^j f(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

and for $n = 0$ sends $m \in M$ to the map $g \mapsto g(m) - m$. We define

$$H^i_{cont}(G, M) = H^i(C^\bullet_{cont}(G, M))$$

Since the terms of the complex involve continuous maps from G and self products of G into the topological module M, it is not clear that this turns a short exact sequence of topological modules into a long exact cohomology sequence. Another difficulty is that the category of topological abelian groups isn't an abelian category!

However, a short exact sequence of discrete G-modules does give rise to a short exact sequence of complexes of continuous cochains and hence a long exact cohomology sequence of continuous cohomology groups $H^i_{cont}(G, -)$. Therefore, on the category Mod_G of Definition 59.57.1 the functors $H^i_{cont}(G, M)$ form a cohomological δ-functor as defined in Homology, Section 12.12. Since the cohomology $H^i(G, M)$ of Definition 59.57.2 is a universal δ-functor (Derived Categories, Lemma 13.16.6) we obtain canonical maps

$$H^i(G, M) \longrightarrow H^i_{cont}(G, M)$$

for $M \in \mathrm{Mod}_G$. It is known that these maps are isomorphisms when G is an abstract group (i.e., G has the discrete topology) or when G is a profinite group (insert future reference here). If you know an example showing this map is not an isomorphism for a topological group G and $M \in \mathrm{Ob}(\mathrm{Mod}_G)$ please email stacks.project@gmail.com.

59.59. Cohomology of a point

03QQ As a consequence of the discussion in the preceding sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

04JQ **Lemma 59.59.1.** *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. The stalk functor induces an equivalence of categories*

$$Ab(S_{étale}) \longrightarrow Mod_G, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. In Theorem 59.56.3 we have seen the equivalence between sheaves of sets and G-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a G-module is just a G-set endowed with a commutative group law. □

03QU **Lemma 59.59.2.** *Notation and assumptions as in Lemma 59.59.1. Let $\mathcal{F}$ be an abelian sheaf on $\mathrm{Spec}(K)_{étale}$ which corresponds to the G-module M. Then*

(1) *in $D(Ab)$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,*
(2) *$H^0_{étale}(S, \mathcal{F}) = M^G$, and*
(3) *$H^q_{étale}(S, \mathcal{F}) = H^q(G, M)$.*

Proof. Combine Lemma 59.59.1 with Lemma 59.56.5. □

03QV **Example 59.59.3.** Sheaves on $\mathrm{Spec}(K)_{étale}$. Let $G = \mathrm{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

(1) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ corresponds to the module $\mathbf{Z}/n\mathbf{Z}$ with trivial G-action,
(2) the sheaf $\mathbf{G}_m|_{\mathrm{Spec}(K)_{étale}}$ corresponds to $(K^{sep})^*$ with its G-action,
(3) the sheaf $\mathbf{G}_a|_{\mathrm{Spec}(K^{sep})}$ corresponds to $(K^{sep}, +)$ with its G-action, and
(4) the sheaf $\mu_n|_{\mathrm{Spec}(K^{sep})}$ corresponds to $\mu_n(K^{sep})$ with its G-action.

By Remark 59.23.4 and Theorem 59.24.1 we have the following identifications for cohomology groups:

$$\begin{aligned} H^0_{étale}(S_{étale}, \mathbf{G}_m) &= \Gamma(S, \mathcal{O}_S^*) \\ H^1_{étale}(S_{étale}, \mathbf{G}_m) &= H^1_{Zar}(S, \mathcal{O}_S^*) = \mathrm{Pic}(S) \\ H^i_{étale}(S_{étale}, \mathbf{G}_a) &= H^i_{Zar}(S, \mathcal{O}_S) \end{aligned}$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{étale}$ we have

$$H^i(S_{étale}, \mathcal{F}) = H^i_{Zar}(S, \mathcal{F}),$$

see Theorem 59.22.4. In particular, this gives the following sequence of equalities

$$0 = \mathrm{Pic}(\mathrm{Spec}(K)) = H^1_{étale}(\mathrm{Spec}(K)_{étale}, \mathbf{G}_m) = H^1(G, (K^{sep})^*)$$

which is none other than Hilbert's 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(\mathrm{Spec}(K), \mathcal{O}) = H^i_{étale}(\mathrm{Spec}(K)_{étale}, \mathbf{G}_a) = H^i(G, K^{sep})$$

where the K^{sep} indicates K^{sep} as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

The following result is a curiosity and should be skipped on a first reading.

0D1W **Lemma 59.59.4.** *Let R be a local ring of dimension 0. Let $S = \mathrm{Spec}(R)$. Then every $\mathcal{O}_S$-module on $S_{étale}$ is quasi-coherent.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module on $S_{étale}$. We have to show that $\mathcal{F}$ is determined by the R-module $M = \Gamma(S, \mathcal{F})$. More precisely, if $\pi : X \to S$ is étale we have to show that $\Gamma(X, \mathcal{F}) = \Gamma(X, \pi^*\widetilde{M})$.

Let $\mathfrak{m} \subset R$ be the maximal ideal and let κ be the residue field. By Algebra, Lemma 10.153.10 the local ring R is henselian. If $X \to S$ is étale, then the underlying topological space of X is discrete by Morphisms, Lemma 29.36.7 and hence X is a disjoint union of affine schemes each having one point. Moreover, if $X = \mathrm{Spec}(A)$ is affine and has one point, then $R \to A$ is finite étale by Algebra, Lemma 10.153.5. We have to show that $\Gamma(X, \mathcal{F}) = M \otimes_R A$ in this case.

The functor $A \mapsto A/\mathfrak{m}A$ defines an equivalence of the category of finite étale R-algebras with the category of finite separable κ-algebras by Algebra, Lemma 10.153.7. Let us first consider the case where $A/\mathfrak{m}A$ is a Galois extension of κ with Galois group G. For each $\sigma \in G$ let $\sigma : A \to A$ denote the corresponding automorphism of A over R. Let $N = \Gamma(X, \mathcal{F})$. Then $\mathrm{Spec}(\sigma) : X \to X$ is an automorphism over S and hence pullback by this defines a map $\sigma : N \to N$ which is a σ-linear map: $\sigma(an) = \sigma(a)\sigma(n)$ for $a \in A$ and $n \in N$. We will apply Galois descent to the quasi-coherent module $\widetilde{N}$ on X endowed with the isomorphisms coming from the action on σ on N. See Descent, Lemma 35.6.2. This lemma tells us there is an isomorphism $N = N^G \otimes_R A$. On the other hand, it is clear that $N^G = M$ by the sheaf property for $\mathcal{F}$. Thus the required isomorphism holds.

The general case (with A local and finite étale over R) is deduced from the Galois case as follows. Choose $A \to B$ finite étale such that B is local with residue field Galois over κ. Let $G = \mathrm{Aut}(B/R) = \mathrm{Gal}(\kappa_B/\kappa)$. Let $H \subset G$ be the Galois group corresponding to the Galois extension κ_B/κ_A. Then as above one shows that $\Gamma(X, \mathcal{F}) = \Gamma(\mathrm{Spec}(B), \mathcal{F})^H$. By the result for Galois extensions (used twice) we get

$$\Gamma(X, \mathcal{F}) = (M \otimes_R B)^H = M \otimes_R A$$

as desired. □

59.60. Cohomology of curves

03R0 The next task at hand is to compute the étale cohomology of a smooth curve over an algebraically closed field with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology.

59.61. Brauer groups

03R1 Brauer groups of fields are defined using finite central simple algebras. In this section we review the relevant facts about Brauer groups, most of which are discussed in the chapter Brauer Groups, Section 11.1. For other references, see [**Ser62**], [**Ser97**] or [**Wei48**].

03R2 **Theorem 59.61.1.** *Let K be a field. For a unital, associative (not necessarily commutative) K-algebra A the following are equivalent*

(1) *A is finite central simple K-algebra,*

(2) *A is a finite dimensional K-vector space, K is the center of A, and A has no nontrivial two-sided ideal,*
(3) *there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong Mat(d \times d, \bar{K})$,*
(4) *there exists $d \geq 1$ such that $A \otimes_K K^{sep} \cong Mat(d \times d, K^{sep})$,*
(5) *there exist $d \geq 1$ and a finite Galois extension K'/K such that $A \otimes_K K' \cong Mat(d \times d, K')$,*
(6) *there exist $n \geq 1$ and a finite central skew field D over K such that $A \cong Mat(n \times n, D)$.*

The integer d is called the degree of A.

Proof. This is a copy of Brauer Groups, Lemma 11.8.6. □

03R4 **Lemma 59.61.2.** *Let A be a finite central simple algebra over K. Then*

$$\begin{array}{ccc} A \otimes_K A^{opp} & \longrightarrow & End_K(A) \\ a \otimes a' & \longmapsto & (x \mapsto axa') \end{array}$$

is an isomorphism of algebras over K.

Proof. See Brauer Groups, Lemma 11.4.10. □

03R3 **Definition 59.61.3.** Two finite central simple algebras A_1 and A_2 over K are called *similar*, or *equivalent* if there exist $m, n \geq 1$ such that $\mathrm{Mat}(n \times n, A_1) \cong \mathrm{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

By Brauer Groups, Lemma 11.5.1 this is an equivalence relation.

03R5 **Definition 59.61.4.** Let K be a field. The *Brauer group* of K is the set $\mathrm{Br}(K)$ of similarity classes of finite central simple algebras over K, endowed with the group law induced by tensor product (over K). The class of A in $\mathrm{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\mathrm{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma implies that inverses exist and that $-[A] = [A^{opp}]$. The Brauer group of a field is always torsion. In fact, we will see that $[A]$ has order dividing $\deg(A)$ for any finite central simple algebra A (see Lemma 59.62.2). In general the Brauer group is not finitely generated, for example the Brauer group of a non-Archimedean local field is $\mathbf{Q}/\mathbf{Z}$. The Brauer group of $\mathbf{C}(x, y)$ is uncountable.

03R6 **Lemma 59.61.5.** *Let K be a field and let K^{sep} be a separable algebraic closure. Then the set of isomorphism classes of central simple algebras of degree d over K is in bijection with the non-abelian cohomology $H^1(Gal(K^{sep}/K), PGL_d(K^{sep}))$.*

Sketch of proof. The Skolem-Noether theorem (see Brauer Groups, Theorem 11.6.1) implies that for any field L the group $\mathrm{Aut}_{L\text{-Algebras}}(\mathrm{Mat}_d(L))$ equals $\mathrm{PGL}_d(L)$. By Theorem 59.61.1, we see that central simple algebras of degree d correspond to forms of the K-algebra $\mathrm{Mat}_d(K)$. Combined we see that isomorphism classes of degree d central simple algebras correspond to elements of $H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep}))$. For more details on twisting, see for example [**Sil86**]. □

If A is a finite central simple algebra of degree d over a field K, we denote ξ_A the corresponding cohomology class in $H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep}))$. Consider the short exact sequence

$$1 \to (K^{sep})^* \to \mathrm{GL}_d(K^{sep}) \to \mathrm{PGL}_d(K^{sep}) \to 1,$$

which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map

$$\delta_d : H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{PGL}_d(K^{sep})) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*).$$

Explicitly, this is given as follows: if ξ is a cohomology class represented by the 1-cocycle (g_σ), then $\delta_d(\xi)$ is the class of the 2-cocycle

0A2I (59.61.5.1) $$(\sigma, \tau) \longmapsto \tilde{g}_\sigma^{-1} \tilde{g}_{\sigma\tau} \sigma(\tilde{g}_\tau^{-1}) \in (K^{sep})^*$$

where $\tilde{g}_\sigma \in \mathrm{GL}_d(K^{sep})$ is a lift of g_σ. Using this we can make explicit the map

$$\delta : \mathrm{Br}(K) \longrightarrow H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*), \quad [A] \longmapsto \delta_{\deg A}(\xi_A)$$

as follows. Assume A has degree d over K. Choose an isomorphism $\varphi : \mathrm{Mat}_d(K^{sep}) \to A \otimes_K K^{sep}$. For $\sigma \in \mathrm{Gal}(K^{sep}/K)$ choose an element $\tilde{g}_\sigma \in \mathrm{GL}_d(K^{sep})$ such that $\varphi^{-1} \circ \sigma(\varphi)$ is equal to the map $x \mapsto \tilde{g}_\sigma x \tilde{g}_\sigma^{-1}$. The class in H^2 is defined by the two cocycle (59.61.5.1).

03R7 **Theorem 59.61.6.** *Let K be a field with separable algebraic closure K^{sep}. The map $\delta : Br(K) \to H^2(Gal(K^{sep}/K), (K^{sep})^*)$ defined above is a group isomorphism.*

Sketch of proof. To prove that δ defines a group homomorphism, i.e., that $\delta(A \otimes_K B) = \delta(A) + \delta(B)$, one computes directly with cocycles.

Injectivity of δ. In the abelian case ($d = 1$), one has the identification

$$H^1(\mathrm{Gal}(K^{sep}/K), \mathrm{GL}_d(K^{sep})) = H^1_{\acute{e}tale}(\mathrm{Spec}(K), \mathrm{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of δ. (See [**Del77**].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a K-vector space V with a left A-module structure and such that $\dim_K V = \deg A$. In the case where V exists, one has $A \cong \mathrm{End}_K(V)$.

For surjectivity, pick a cohomology class $\xi \in H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$, then there exists a finite Galois extension $K^{sep}/K'/K$ such that ξ is the image of some $\xi' \in H^2(\mathrm{Gal}(K'|K), (K')^*)$. Then write down an explicit central simple algebra over K using the data K', ξ'. □

59.62. The Brauer group of a scheme

0A2J Let S be a scheme. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is called *Azumaya* if it is étale locally a matrix algebra, i.e., if there exists an étale covering $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ such that $\varphi_i^* \mathcal{A} \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{U_i})$ for some $d_i \geq 1$. Two such $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$$

as $\mathcal{O}_S$-algebras. The *Brauer group* of S is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

0A2K **Lemma 59.62.1.** *Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be finite locally free sheaves of $\mathcal{O}_S$-modules of positive rank. If there exists an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G}, \mathcal{G})$ of $\mathcal{O}_S$-algebras, then there exists an invertible sheaf $\mathcal{L}$ on S such that $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \cong \mathcal{G}$ and such that this isomorphism induces the given isomorphism of endomorphism algebras.*

Proof. Fix an isomorphism $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{G},\mathcal{G})$. Consider the sheaf $\mathcal{L} \subset \mathcal{H}om(\mathcal{F},\mathcal{G})$ generated as an $\mathcal{O}_S$-module by the local isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ such that conjugation by φ is the given isomorphism of endomorphism algebras. A local calculation (reducing to the case that $\mathcal{F}$ and $\mathcal{G}$ are finite free and S is affine) shows that $\mathcal{L}$ is invertible. Another local calculation shows that the evaluation map

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L} \longrightarrow \mathcal{G}$$

is an isomorphism. □

The argument given in the proof of the following lemma can be found in [**Sal81**].

0A2L **Lemma 59.62.2.** *Let S be a scheme. Let $\mathcal{A}$ be an Azumaya algebra which is locally free of rank d^2 over S. Then the class of $\mathcal{A}$ in the Brauer group of S is annihilated by d.* Argument taken from [**Sal81**].

Proof. Choose an étale covering $\{U_i \to S\}$ and choose isomorphisms $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i,\mathcal{F}_i)$ for some locally free $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ of rank d. (We may assume $\mathcal{F}_i$ is free.) Consider the composition

$$p_i : \mathcal{F}_i^{\otimes d} \to \wedge^d(\mathcal{F}_i) \to \mathcal{F}_i^{\otimes d}$$

The first arrow is the usual projection and the second arrow is the isomorphism of the top exterior power of $\mathcal{F}_i$ with the submodule of sections of $\mathcal{F}_i^{\otimes d}$ which transform according to the sign character under the action of the symmetric group on d letters. Then $p_i^2 = d!p_i$ and the rank of p_i is 1. Using the given isomorphism $\mathcal{A}|_{U_i} \to \mathcal{H}om(\mathcal{F}_i,\mathcal{F}_i)$ and the canonical isomorphism

$$\mathcal{H}om(\mathcal{F}_i,\mathcal{F}_i)^{\otimes d} = \mathcal{H}om(\mathcal{F}_i^{\otimes d},\mathcal{F}_i^{\otimes d})$$

we may think of p_i as a section of $\mathcal{A}^{\otimes d}$ over U_i. We claim that $p_i|_{U_i \times_S U_j} = p_j|_{U_i \times_S U_j}$ as sections of $\mathcal{A}^{\otimes d}$. Namely, applying Lemma 59.62.1 we obtain an invertible sheaf $\mathcal{L}_{ij}$ and a canonical isomorphism

$$\mathcal{F}_i|_{U_i \times_S U_j} \otimes \mathcal{L}_{ij} \longrightarrow \mathcal{F}_j|_{U_i \times_S U_j}.$$

Using this isomorphism we see that p_i maps to p_j. Since $\mathcal{A}^{\otimes d}$ is a sheaf on $S_{étale}$ (Proposition 59.17.1) we find a canonical global section $p \in \Gamma(S, \mathcal{A}^{\otimes d})$. A local calculation shows that

$$\mathcal{H} = \mathrm{Im}(\mathcal{A}^{\otimes d} \to \mathcal{A}^{\otimes d}, f \mapsto fp)$$

is a locally free module of rank d^d and that (left) multiplication by $\mathcal{A}^{\otimes d}$ induces an isomorphism $\mathcal{A}^{\otimes d} \to \mathcal{H}om(\mathcal{H},\mathcal{H})$. In other words, $\mathcal{A}^{\otimes d}$ is the trivial element of the Brauer group of S as desired. □

In this setting, the analogue of the isomorphism δ of Theorem 59.61.6 is a map

$$\delta_S : \mathrm{Br}(S) \to H^2_{étale}(S, \mathbf{G}_m).$$

It is true that δ_S is injective. If S is quasi-compact or connected, then $\mathrm{Br}(S)$ is a torsion group, so in this case the image of δ_S is contained in the *cohomological Brauer group* of S

$$\mathrm{Br}'(S) := H^2_{étale}(S, \mathbf{G}_m)_{\mathrm{torsion}}.$$

So if S is quasi-compact or connected, there is an inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. This is not always an equality: there exists a nonseparated singular surface S for which $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is a strict inclusion. If S is quasi-projective, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$.

However, it is not known whether this holds for a smooth proper variety over $\mathbf{C}$, say.

59.63. The Artin-Schreier sequence

0A3J Let p be a prime number. Let S be a scheme in characteristic p. The *Artin-Schreier* sequence is the short exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{F-1} \mathbf{G}_{a,S} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$.

0A3K **Lemma 59.63.1.** *Let p be a prime. Let S be a scheme of characteristic p.*

(1) *If S is affine, then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2$.*
(2) *If S is a quasi-compact and quasi-separated scheme of dimension d, then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2 + d$.*

Proof. Recall that the étale cohomology of the structure sheaf is equal to its cohomology on the underlying topological space (Theorem 59.22.4). The first statement follows from the Artin-Schreier exact sequence and the vanishing of cohomology of the structure sheaf on an affine scheme (Cohomology of Schemes, Lemma 30.2.2). The second statement follows by the same argument from the vanishing of Cohomology, Proposition 20.22.4 and the fact that S is a spectral space (Properties, Lemma 28.2.4). □

0A3L **Lemma 59.63.2.** *Let k be an algebraically closed field of characteristic $p > 0$. Let V be a finite dimensional k-vector space. Let $F : V \to V$ be a frobenius linear map, i.e., an additive map such that $F(\lambda v) = \lambda^p F(v)$ for all $\lambda \in k$ and $v \in V$. Then $F - 1 : V \to V$ is surjective with kernel a finite dimensional $\mathbf{F}_p$-vector space of dimension $\leq \dim_k(V)$.*

Proof. If $F = 0$, then the statement holds. If we have a filtration of V by F-stable subvector spaces such that the statement holds for each graded piece, then it holds for (V, F). Combining these two remarks we may assume the kernel of F is zero.

Choose a basis $v_1, \ldots, v_n$ of V and write $F(v_i) = \sum a_{ij} v_j$. Observe that $v = \sum \lambda_i v_i$ is in the kernel if and only if $\sum \lambda_i^p a_{ij} v_j = 0$. Since k is algebraically closed this implies the matrix (a_{ij}) is invertible. Let (b_{ij}) be its inverse. Then to see that $F - 1$ is surjective we pick $w = \sum \mu_i v_i \in V$ and we try to solve

$$(F-1)(\sum \lambda_i v_i) = \sum \lambda_i^p a_{ij} v_j - \sum \lambda_j v_j = \sum \mu_j v_j$$

This is equivalent to

$$\sum \lambda_j^p v_j - \sum b_{ij} \lambda_i v_j = \sum b_{ij} \mu_i v_j$$

in other words

$$\lambda_j^p - \sum b_{ij} \lambda_i = \sum b_{ij} \mu_i, \quad j = 1, \ldots, \dim(V).$$

The algebra

$$A = k[x_1, \ldots, x_n]/(x_j^p - \sum b_{ij} x_i - \sum b_{ij} \mu_i)$$

is standard smooth over k (Algebra, Definition 10.137.5) because the matrix (b_{ij}) is invertible and the partial derivatives of x_j^p are zero. A basis of A over k is the set of monomials $x_1^{e_1} \ldots x_n^{e_n}$ with $e_i < p$, hence $\dim_k(A) = p^n$. Since k is algebraically

closed we see that $\mathrm{Spec}(A)$ has exactly p^n points. It follows that $F-1$ is surjective and every fibre has p^n points, i.e., the kernel of $F-1$ is a group with p^n elements. $\square$

0A3M **Lemma 59.63.3.** *Let X be a separated scheme of finite type over a field k. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then $\dim_k H^d(X, \mathcal{F}) < \infty$ where $d = \dim(X)$.*

Proof. We will prove this by induction on d. The case $d = 0$ holds because in that case X is the spectrum of a finite dimensional k-algebra A (Varieties, Lemma 33.20.2) and every coherent sheaf $\mathcal{F}$ corresponds to a finite A-module $M = H^0(X, \mathcal{F})$ which has $\dim_k M < \infty$.

Assume $d > 0$ and the result has been shown for separated schemes of finite type of dimension $< d$. The scheme X is Noetherian. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow \dim_k H^d(X, \mathcal{F}) < \infty$$

We are going to use the result of Cohomology of Schemes, Lemma 30.12.4 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of cohomology

$$H^d(X, \mathcal{F}_1) \to H^d(X, \mathcal{F}) \to H^d(X, \mathcal{F}_2)$$

Thus if $\mathcal{P}$ holds for $\mathcal{F}_1$ and $\mathcal{F}_2$, then it holds for $\mathcal{F}$.

Let $Z \subset X$ be an integral closed subscheme. Let $\mathcal{I}$ be a coherent sheaf of ideals on Z. To finish the proof we have to show that $H^d(X, i_*\mathcal{I}) = H^d(Z, \mathcal{I})$ is finite dimensional. If $\dim(Z) < d$, then the result holds because the cohomology group will be zero (Cohomology, Proposition 20.20.7). In this way we reduce to the situation discussed in the following paragraph.

Assume X is a variety of dimension d and $\mathcal{F} = \mathcal{I}$ is a coherent ideal sheaf. In this case we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

where $i : Z \to X$ is the closed subscheme defined by $\mathcal{I}$. By induction hypothesis we see that $H^{d-1}(Z, \mathcal{O}_Z) = H^{d-1}(X, i_*\mathcal{O}_Z)$ is finite dimensional. Thus we see that it suffices to prove the result for the structure sheaf.

We can apply Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) to the morphism $X \to \mathrm{Spec}(k)$. Thus we get a diagram

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \xrightarrow{i} & \mathbf{P}^n_k \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle g'} & \swarrow & \\ & & \mathrm{Spec}(k) & & \end{array}$$

as in the statement of Chow's lemma. Also, let $U \subset X$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. We may assume X' is a variety as well, see Cohomology of Schemes, Remark 30.18.2. The morphism $i' = (i, \pi) : X' \to \mathbf{P}^n_X$ is a closed immersion (loc. cit.). Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_k}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_X}(1)$$

is π-relatively ample (for example by Morphisms, Lemma 29.39.7). Hence by Cohomology of Schemes, Lemma 30.16.2 there exists an $n \geq 0$ such that $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Choose any nonzero global section s of $\mathcal{L}^{\otimes n}$. Since $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$, the section s corresponds to section of $\mathcal{G}$, i.e., a map $\mathcal{O}_X \to \mathcal{G}$. Since $s|_U \neq 0$ as X' is a variety and $\mathcal{L}$ invertible, we see that $\mathcal{O}_X|_U \to \mathcal{G}|_U$ is nonzero. As $\mathcal{G}|_U = \mathcal{L}^{\otimes n}|_{\pi^{-1}(U)}$ is invertible we conclude that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is coherent and supported on a proper closed subscheme of X. Arguing as before using our induction hypothesis, we see that it suffices to prove $\dim H^d(X, \mathcal{G}) < \infty$.

By the Leray spectral sequence (Cohomology, Lemma 20.13.6) we see that $H^d(X, \mathcal{G}) = H^d(X', \mathcal{L}^{\otimes n})$. Let $\overline{X}' \subset \mathbf{P}^n_k$ be the closure of X'. Then $\overline{X}'$ is a projective variety of dimension d over k and $X' \subset \overline{X}'$ is a dense open. The invertible sheaf $\mathcal{L}$ is the restriction of $\mathcal{O}_{\overline{X}'}(n)$ to X. By Cohomology, Proposition 20.22.4 the map

$$H^d(\overline{X}', \mathcal{O}_{\overline{X}'}(n)) \longrightarrow H^d(X', \mathcal{L}^{\otimes n})$$

is surjective. Since the cohomology group on the left has finite dimension by Cohomology of Schemes, Lemma 30.14.1 the proof is complete. □

0A3N **Lemma 59.63.4.** *Let X be separated of finite type over an algebraically closed field k of characteristic $p > 0$. Then $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for $q \geq dim(X) + 1$.*

Proof. Let $d = \dim(X)$. By the vanishing established in Lemma 59.63.1 it suffices to show that $H^{d+1}_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$. By Lemma 59.63.3 we see that $H^d(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. Hence the long exact cohomology sequence associated to the Artin-Schreier sequence ends with

$$H^d(X, \mathcal{O}_X) \xrightarrow{F-1} H^d(X, \mathcal{O}_X) \to H^{d+1}_{étale}(X, \mathbf{Z}/p\mathbf{Z}) \to 0$$

By Lemma 59.63.2 the map $F - 1$ in this sequence is surjective. This proves the lemma. □

0A3P **Lemma 59.63.5.** *Let X be a proper scheme over an algebraically closed field k of characteristic $p > 0$. Then*

1. *$H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is a finite $\mathbf{Z}/p\mathbf{Z}$-module for all q, and*
2. *$H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q_{étale}(X_{k'}, \underline{\mathbf{Z}/p\mathbf{Z}})$ is an isomorphism if k'/k is an extension of algebraically closed fields.*

Proof. By Cohomology of Schemes, Lemma 30.19.2 and the comparison of cohomology of Theorem 59.22.4 the cohomology groups $H^q_{étale}(X, \mathbf{G}_a) = H^q(X, \mathcal{O}_X)$ are finite dimensional k-vector spaces. Hence by Lemma 59.63.2 the long exact cohomology sequence associated to the Artin-Schreier sequence, splits into short exact sequences

$$0 \to H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) \to H^q(X, \mathcal{O}_X) \xrightarrow{F-1} H^q(X, \mathcal{O}_X) \to 0$$

and moreover the $\mathbf{F}_p$-dimension of the cohomology groups $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is equal to the k-dimension of the vector space $H^q(X, \mathcal{O}_X)$. This proves the first statement. The second statement follows as $H^q(X, \mathcal{O}_X) \otimes_k k' \to H^q(X_{k'}, \mathcal{O}_{X_{k'}})$ is an isomorphism by flat base change (Cohomology of Schemes, Lemma 30.5.2). □

59.64. Locally constant sheaves

09Y8 This section is the analogue of Modules on Sites, Section 18.43 for the étale site.

03RU **Definition 59.64.1.** Let X be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$.

(1) Let E be a set. We say $\mathcal{F}$ is the *constant sheaf with value* E if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the constant values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{étale}$.

(1) Let A be an abelian group. We say $\mathcal{F}$ is the *constant sheaf with value* A if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto A$. Notation: $\underline{A}_X$ or $\underline{A}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as an abelian sheaf to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the constant values are finite abelian groups.

Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$.

(1) Let M be a Λ-module. We say $\mathcal{F}$ is the *constant sheaf with value* M if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$. Notation: $\underline{M}_X$ or $\underline{M}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as a sheaf of Λ-modules to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

095A **Lemma 59.64.2.** *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally constant sheaf of sets, abelian groups, or Λ-modules on $Y_{étale}$, the same is true for $f^{-1}\mathcal{G}$ on $X_{étale}$.*

Proof. Holds for any morphism of topoi, see Modules on Sites, Lemma 18.43.2. □

095B **Lemma 59.64.3.** *Let $f : X \to Y$ be a finite étale morphism of schemes. If $\mathcal{F}$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. The construction of f_* commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. □

03RV **Lemma 59.64.4.** *Let X be a scheme and $\mathcal{F}$ a sheaf of sets on $X_{étale}$. Then the following are equivalent*

(1) *$\mathcal{F}$ is finite locally constant, and*
(2) *$\mathcal{F} = h_U$ for some finite étale morphism $U \to X$.*

Proof. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus (2) implies (1). Conversely, if $\mathcal{F}$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma 35.39.1 we obtain a descent datum for schemes (U_i, φ_{ij}) relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma 35.37.1 and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas 35.23.25 and 35.23.31. □

095C **Lemma 59.64.5.** *Let X be a scheme.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $X_{étale}$. If $\mathcal{F}$ is of finite type, then there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.3. □

03RX **Lemma 59.64.6.** *Let X be a scheme.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.5. □

095D **Lemma 59.64.7.** *Let X be a scheme. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $X_{étale}$ is a locally constant sheaf of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.43.6. □

09BF **Lemma 59.64.8.** *Let X be a connected scheme. Let Λ be a ring and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules. Then there exists a Λ-module M and an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i} \cong \underline{M}|_{U_i}$.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As étale morphisms are open we see that $U_M = \bigcup_{i \in I_M} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. As X is connected only one U_M is nonempty and the lemma follows. □

59.65. Locally constant sheaves and the fundamental group

0DV4 We can relate locally constant sheaves to the fundamental group of a scheme in some cases.

0DV5 **Lemma 59.65.1.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

(1) *There is an equivalence of categories*

$$\left\{\begin{matrix}\text{finite locally constant} \\ \text{sheaves of sets on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1(X,\overline{x})\text{-sets}\right\}$$

(2) *There is an equivalence of categories*

$$\left\{\begin{matrix}\text{finite locally constant} \\ \text{sheaves of abelian groups on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1(X,\overline{x})\text{-modules}\right\}$$

(3) *Let Λ be a finite ring. There is an equivalence of categories*

$$\left\{\begin{matrix}\text{finite type, locally constant} \\ \text{sheaves of } \Lambda\text{-modules on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \pi_1(X,\overline{x})\text{-modules endowed} \\ \text{with commuting } \Lambda\text{-module structure}\end{matrix}\right\}$$

Proof. We observe that $\pi_1(X,\overline{x})$ is a profinite topological group, see Fundamental Groups, Definition 58.6.1. The left hand categories are defined in Section 59.64. The notation used in the right hand categories is taken from Fundamental Groups, Definition 58.2.1 for sets and Definition 59.57.1 for abelian groups. This explains the notation.

Assertion (1) follows from Lemma 59.64.4 and Fundamental Groups, Theorem 58.6.2. Parts (2) and (3) follow immediately from this by endowing the underlying (sheaves of) sets with additional structure. For example, a finite locally constant sheaf of abelian groups on $X_{étale}$ is the same thing as a finite locally constant sheaf of sets $\mathcal{F}$ together with a map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ satisfying the usual axioms. The equivalence in (1) sends products to products and hence sends $+$ to an addition on the corresponding finite $\pi_1(X,\overline{x})$-set. Since $\pi_1(X,\overline{x})$-modules are the same thing as $\pi_1(X,\overline{x})$-sets with a compatible abelian group structure we obtain (2). Part (3) is proved in exactly the same way. □

0GIY **Lemma 59.65.2.** *Let X be an irreducible, geometrically unibranch scheme. Let $\overline{x}$ be a geometric point of X. Let Λ be a ring. There is an equivalence of categories*

$$\left\{\begin{matrix}\text{finite type, locally constant} \\ \text{sheaves of } \Lambda\text{-modules on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \Lambda\text{-modules } M \text{ endowed} \\ \text{with a continuous } \pi_1(X,\overline{x})\text{-action}\end{matrix}\right\}$$

Proof. The proof given in Lemma 59.65.1 does not work as a finite Λ-module M may not have a finite underlying set.

Let $\nu : X^\nu \to X$ be the normalization morphism. By Morphisms, Lemma 29.54.12 this is a universal homeomorphism. By Fundamental Groups, Proposition 58.8.4 this induces an isomorphism $\pi_1(X^\nu,\overline{x}) \to \pi_1(X,\overline{x})$ and by Theorem 59.45.2 we get an equivalence of category between finite type, locally constant Λ-modules on $X_{étale}$ and on $X^\nu_{étale}$. This reduces us to the case where X is an integral normal scheme.

Assume X is an integral normal scheme. Let $\eta \in X$ be the generic point. Let $\overline{\eta}$ be a geometric point lying over η. By Fundamental Groups, Proposition 58.11.3 have a continuous surjection

$$\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta,\overline{\eta}) \longrightarrow \pi_1(X,\overline{\eta})$$

whose kernel is described in Fundamental Groups, Lemma 58.13.2. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $X_{étale}$. Let $M = \mathcal{F}_{\overline{\eta}}$ be the stalk of $\mathcal{F}$ at $\overline{\eta}$. We obtain a continuous action of $\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta))$ on M by Section 59.56. Our goal is to show that this action factors through the displayed surjection. Since $\mathcal{F}$ is of finite type, M is a finite Λ-module. Since $\mathcal{F}$ is locally constant, for every $x \in X$ the restriction of $\mathcal{F}$ to $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ is constant. Hence the action of $\mathrm{Gal}(K^{sep}/K^{sh}_x)$ (with notation as in Fundamental Groups, Lemma 58.13.2) on M is trivial. We conclude we have the factorization as desired.

On the other hand, suppose we have a finite Λ-module M with a continuous action of $\pi_1(X, \overline{\eta})$. We are going to construct an $\mathcal{F}$ such that $M \cong \mathcal{F}_{\overline{\eta}}$ as $\Lambda[\pi_1(X, \overline{\eta})]$-modules. Choose generators $m_1, \ldots, m_r \in M$. Since the action of $\pi_1(X, \overline{\eta})$ on M is continuous, for each i there exists an open subgroup N_i of the profinite group $\pi_1(X, \overline{\eta})$ such that every $\gamma \in H_i$ fixes m_i. We conclude that every element of the open subgroup $H = \bigcap_{i=1,\ldots,r} H_i$ fixes every element of M. After shrinking H we may assume H is an open normal subgroup of $\pi_1(X, \overline{\eta})$. Set $G = \pi_1(X, \overline{\eta})/H$. Let $f : Y \to X$ be the corresponding Galois finite étale G-cover. We can view $f_*\underline{\mathbf{Z}}$ as a sheaf of $\mathbf{Z}[G]$-modules on $X_{étale}$. Then we just take

$$\mathcal{F} = f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}[G]}} \underline{M}$$

We leave it to the reader to compute $\mathcal{F}_{\overline{\eta}}$. We also omit the verification that this construction is the inverse to the construction in the previous paragraph. □

0DV6 **Remark 59.65.3.** The equivalences of Lemmas 59.65.1 and 59.65.2 are compatible with pullbacks. For example, suppose $f : Y \to X$ is a morphism of connected schemes. Let $\overline{y}$ be geometric point of Y and set $\overline{x} = f(\overline{y})$. Then the diagram

$$\begin{array}{ccc} \text{finite locally constant sheaves of sets on } Y_{étale} & \longrightarrow & \text{finite } \pi_1(Y, \overline{y})\text{-sets} \\ \uparrow{\scriptstyle f^{-1}} & & \uparrow \\ \text{finite locally constant sheaves of sets on } X_{étale} & \longrightarrow & \text{finite } \pi_1(X, \overline{x})\text{-sets} \end{array}$$

is commutative, where the vertical arrow on the right comes from the continuous homomorphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$ induced by f. This follows immediately from the commutative diagram in Fundamental Groups, Theorem 58.6.2. A similar result holds for the other cases.

59.66. Méthode de la trace

03SH A reference for this section is [**AGV71**, Exposé IX, §5]. The material here will be used in the proof of Lemma 59.83.9 below.

Let $f : Y \to X$ be an étale morphism of schemes. There is a sequence

$$f_!, f^{-1}, f_*$$

of adjoint functors between $Ab(X_{étale})$ and $Ab(Y_{étale})$. The functor $f_!$ is discussed in Section 59.70. The adjunction map $\mathrm{id} \to f_*f^{-1}$ is called *restriction*. The adjunction map $f_!f^{-1} \to \mathrm{id}$ is often called the *trace map*. If f is finite étale, then $f_* = f_!$ (Lemma 59.70.7) and we can view this as a map $f_*f^{-1} \to \mathrm{id}$.

03SE **Definition 59.66.1.** Let $f : Y \to X$ be a finite étale morphism of schemes. The map $f_*f^{-1} \to \mathrm{id}$ described above and explicitly below is called the *trace*.

Let $f : Y \to X$ be a finite étale morphism of schemes. The trace map is characterized by the following two properties:

(1) it commutes with étale localization on X and
(2) if $Y = \coprod_{i=1}^{d} X$ then the trace map is the sum map $f_* f^{-1}\mathcal{F} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

By Étale Morphisms, Lemma 41.18.3 every finite étale morphism $f : Y \to X$ is étale locally on X of the form given in (2) for some integer $d \geq 0$. Hence we can define the trace map using the characterization given; in particular we do not need to know about the existence of $f_!$ and the agreement of $f_!$ with f_* in order to construct the trace map. This description shows that if f has constant degree d, then the composition

$$\mathcal{F} \xrightarrow{res} f_* f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$$

is multiplication by d. The "méthode de la trace" is the following observation: if $\mathcal{F}$ is an abelian sheaf on $X_{étale}$ such that multiplication by d on $\mathcal{F}$ is an isomorphism, then the map

$$H^n_{étale}(X, \mathcal{F}) \longrightarrow H^n_{étale}(Y, f^{-1}\mathcal{F})$$

is injective. Namely, we have

$$H^n_{étale}(Y, f^{-1}\mathcal{F}) = H^n_{étale}(X, f_* f^{-1}\mathcal{F})$$

by the vanishing of the higher direct images (Proposition 59.55.2) and the Leray spectral sequence (Proposition 59.54.2). Thus we can consider the maps

$$H^n_{étale}(X, \mathcal{F}) \to H^n_{étale}(Y, f^{-1}\mathcal{F}) = H^n_{étale}(X, f_* f^{-1}\mathcal{F}) \xrightarrow{trace} H^n_{étale}(X, \mathcal{F})$$

and the composition is an isomorphism (under our assumption on $\mathcal{F}$ and f). In particular, if $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$ then $H^q_{étale}(X, \mathcal{F}) = 0$ as well. Indeed, multiplication by d induces an isomorphism on $H^q_{étale}(X, \mathcal{F})$ which factors through $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$.

This is often combined with the following.

0A3R **Lemma 59.66.2.** *Let S be a connected scheme. Let ℓ be a prime number. Let $\mathcal{F}$ be a finite type, locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on $S_{étale}$. Then there exists a finite étale morphism $f : T \to S$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$.*

Proof. Choose a geometric point $\overline{s}$ of S. Via the equivalence of Lemma 59.65.1 the sheaf $\mathcal{F}$ corresponds to a finite dimensional $\mathbf{F}_\ell$-vector space V with a continuous $\pi_1(S, \overline{s})$-action. Let $G \subset \mathrm{Aut}(V)$ be the image of the homomorphism $\rho : \pi_1(S, \overline{s}) \to \mathrm{Aut}(V)$ giving the action. Observe that G is finite. The surjective continuous homomorphism $\overline{\rho} : \pi_1(S, \overline{s}) \to G$ corresponds to a Galois object $Y \to S$ of $F\acute{E}t_S$ with automorphism group $G = \mathrm{Aut}(Y/S)$, see Fundamental Groups, Section 58.7. Let $H \subset G$ be an ℓ-Sylow subgroup. We claim that $T = Y/H \to S$ works. Namely, let $\overline{t} \in T$ be a geometric point over $\overline{s}$. The image of $\pi_1(T, \overline{t}) \to \pi_1(S, \overline{s})$ is $(\overline{\rho})^{-1}(H)$ as follows from the functorial nature of fundamental groups. Hence the action of $\pi_1(T, \overline{t})$ on V corresponding to $f^{-1}\mathcal{F}$ is through the map $\pi_1(T, \overline{t}) \to H$, see Remark 59.65.3. As H is a finite ℓ-group, the irreducible constituents of the representation $\rho|_{\pi_1(T,\overline{t})}$ are each trivial of rank 1 (this is a simple lemma on representation theory of finite groups; insert future reference here). Via the equivalence of Lemma 59.65.1 this means $f^{-1}\mathcal{F}$ is a successive extension of constant sheaves with value $\underline{\mathbf{Z}/\ell\mathbf{Z}}_T$.

Moreover the degree of $T = Y/H \to S$ is prime to ℓ as it is equal to the index of H in G. □

0GIZ **Lemma 59.66.3.** *Let Λ be a Noetherian ring. Let ℓ be a prime number and $n \geq 1$. Let H be a finite ℓ-group. Let M be a finite $\Lambda[H]$-module annihilated by ℓ^n. Then there is a finite filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ by $\Lambda[H]$-submodules such that H acts trivially on M_{i+1}/M_i for all $i = 0, \ldots, t-1$.*

Proof. Omitted. Hint: Show that the augmentation ideal $\mathfrak{m}$ of the noncommutative ring $\mathbf{Z}/\ell^n\mathbf{Z}[H]$ is nilpotent. □

0GJ0 **Lemma 59.66.4.** *Let S be an irreducible, geometrically unibranch scheme. Let ℓ be a prime number and $n \geq 1$. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $S_{étale}$ which is annihilated by ℓ^n. Then there exists a finite étale morphism $f : T \to S$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a finite filtration whose successive quotients are of the form $\underline{M}_T$ for some finite Λ-modules M.*

Proof. Choose a geometric point $\overline{s}$ of S. Via the equivalence of Lemma 59.65.2 the sheaf $\mathcal{F}$ corresponds to a finite Λ-module M with a continuous $\pi_1(S, \overline{s})$-action. Let $G \subset \mathrm{Aut}(V)$ be the image of the homomorphism $\rho : \pi_1(S, \overline{s}) \to \mathrm{Aut}(M)$ giving the action. Observe that G is finite as M is a finite Λ-module (see proof of Lemma 59.65.2). The surjective continuous homomorphism $\overline{\rho} : \pi_1(S, \overline{s}) \to G$ corresponds to a Galois object $Y \to S$ of $F\acute{E}t_S$ with automorphism group $G = \mathrm{Aut}(Y/S)$, see Fundamental Groups, Section 58.7. Let $H \subset G$ be an ℓ-Sylow subgroup. We claim that $T = Y/H \to S$ works. Namely, let $\overline{t} \in T$ be a geometric point over $\overline{s}$. The image of $\pi_1(T, \overline{t}) \to \pi_1(S, \overline{s})$ is $(\overline{\rho})^{-1}(H)$ as follows from the functorial nature of fundamental groups. Hence the action of $\pi_1(T, \overline{t})$ on M corresponding to $f^{-1}\mathcal{F}$ is through the map $\pi_1(T, \overline{t}) \to H$, see Remark 59.65.3. Let $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ be as in Lemma 59.66.3. This induces a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t = f^{-1}\mathcal{F}$ such that the successive quotients are constant with value M_{i+1}/M_i. Finally, the degree of $T = Y/H \to S$ is prime to ℓ as it is equal to the index of H in G. □

59.67. Galois cohomology

0A2M In this section we prove a result on Galois cohomology (Proposition 59.67.4) using étale cohomology and the trick from Section 59.66. This will allow us to prove vanishing of higher étale cohomology groups over the spectrum of a field.

0DV7 **Lemma 59.67.1.** *Let ℓ be a prime number and n an integer > 0. Let S be a quasi-compact and quasi-separated scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed system of S-schemes each $X_i \to S$ being finite étale of constant degree relatively prime to ℓ. The following are equivalent:*

1. *there exists an ℓ-power torsion sheaf $\mathcal{G}$ on S such that $H^n_{étale}(S, \mathcal{G}) \neq 0$ and*
2. *there exists an ℓ-power torsion sheaf $\mathcal{F}$ on X such that $H^n_{étale}(X, \mathcal{F}) \neq 0$.*

In fact, given $\mathcal{G}$ we can take $\mathcal{F} = g^{-1}\mathcal{F}$ and given $\mathcal{F}$ we can take $\mathcal{G} = g_*\mathcal{F}$.

Proof. Let $g : X \to S$ and $g_i : X_i \to S$ denote the structure morphisms. Fix an ℓ-power torsion sheaf $\mathcal{G}$ on S with $H^n_{étale}(S, \mathcal{G}) \neq 0$. The system given by $\mathcal{G}_i = g_i^{-1}\mathcal{G}$

satisfy the conditions of Theorem 59.51.3 with colimit sheaf given by $g^{-1}\mathcal{G}$. This tells us that:

$$\text{colim}_{i\in I} H^n_{étale}(X_i, g_i^{-1}\mathcal{G}) = H^n_{étale}(X, \mathcal{G})$$

By virtue of the g_i being finite étale morphism of degree prime to ℓ we can apply "la méthode de la trace" and we find the maps

$$H^n_{étale}(S, \mathcal{G}) \to H^n_{étale}(X_i, g_i^{-1}\mathcal{G})$$

are all injective (and compatible with the transition maps). See Section 59.66. Thus, the colimit is non-zero, i.e., $H^n(X, g^{-1}\mathcal{G}) \neq 0$, giving us the desired result with $\mathcal{F} = g^{-1}\mathcal{G}$.

Conversely, suppose given an ℓ-power torsion sheaf $\mathcal{F}$ on X with $H^n_{étale}(X, \mathcal{F}) \neq 0$. We note that since the g_i are finite morphisms the higher direct images vanish (Proposition 59.55.2). Then, by applying Lemma 59.51.7 we may also conclude the same for g. The vanishing of the higher direct images tells us that $H^n_{étale}(X, \mathcal{F}) = H^n(S, g_*\mathcal{F}) \neq 0$ by Leray (Proposition 59.54.2) giving us what we want with $\mathcal{G} = g_*\mathcal{F}$. □

0DV8 **Lemma 59.67.2.** *Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^n_{étale}(\text{Spec}(K), \mathcal{F}) = 0$ for all ℓ-power torsion $\mathcal{F}$ if and only if $H^n_{étale}(\text{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. Write $L = \bigcup L_i$ as the union of its finite subextensions over K. Our choice of H implies that $[L_i : K]$ is prime to ℓ. Thus $\text{Spec}(L) = \lim_{i\in I} \text{Spec}(L_i)$ as in Lemma 59.67.1. Thus we may replace K by L and assume that the absolute Galois group G of K is a profinite pro-ℓ group.

Assume $H^n(\text{Spec}(K), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$. Let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\text{Spec}(K)_{étale}$. We will show that $H^n_{étale}(\text{Spec}(K), \mathcal{F}) = 0$. By the correspondence specified in Lemma 59.59.1 our sheaf $\mathcal{F}$ corresponds to an ℓ-power torsion G-module M. Any finite set of elements $x_1, \ldots, x_m \in M$ must be fixed by an open subgroup U by continuity. Let M' be the module spanned by the orbits of $x_1, \ldots, x_m$. This is a finite abelian ℓ-group as each x_i is killed by a power of ℓ and the orbits are finite. Since M is the filtered colimit of these submodules M', we see that $\mathcal{F}$ is the filtered colimit of the corresponding subsheaves $\mathcal{F}' \subset \mathcal{F}$. Applying Theorem 59.51.3 to this colimit, we reduce to the case where $\mathcal{F}$ is a finite locally constant sheaf.

Let M be a finite abelian ℓ-group with a continuous action of the profinite pro-ℓ group G. Then there is a G-invariant filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $M_{i+1}/M_i \cong \mathbf{Z}/\ell\mathbf{Z}$ with trivial G-action (this is a simple lemma on representation theory of finite groups; insert future reference here). Thus the corresponding sheaf $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

with successive quotients isomorphic to $\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Thus by induction and the long exact cohomology sequence we conclude. □

0DV9 **Lemma 59.67.3.** *Let ℓ be a prime number and n an integer > 0. Let K be a field with $G = Gal(K^{sep}/K)$ and let $H \subset G$ be a maximal pro-ℓ subgroup with L/K being the corresponding field extension. Then $H^q_{étale}(\mathrm{Spec}(K), \mathcal{F}) = 0$ for $q \geq n$ and all ℓ-torsion sheaves $\mathcal{F}$ if and only if $H^n_{étale}(\mathrm{Spec}(L), \underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. The forward direction is trivial, so we need only prove the reverse direction. We proceed by induction on q. The case of $q = n$ is Lemma 59.67.2. Now let $\mathcal{F}$ be an ℓ-power torsion sheaf on $\mathrm{Spec}(K)$. Let $f : \mathrm{Spec}(K^{sep}) \to \mathrm{Spec}(K)$ be the inclusion of a geometric point. Then consider the exact sequence:

$$0 \to \mathcal{F} \xrightarrow{res} f_* f^{-1}\mathcal{F} \to f_* f^{-1}\mathcal{F}/\mathcal{F} \to 0$$

Note that K^{sep} may be written as the filtered colimit of finite separable extensions. Thus f is the limit of a directed system of finite étale morphisms. We may, as was seen in the proof of Lemma 59.67.1, conclude that f has vanishing higher direct images. Thus, we may express the higher cohomology of $f_* f^{-1}\mathcal{F}$ as the higher cohomology on the geometric point which clearly vanishes. Hence, as everything here is still ℓ-torsion, we may use the inductive hypothesis in conjunction with the long-exact cohomology sequence to conclude the result for $q + 1$. □

03R8 **Proposition 59.67.4.** *Let K be a field with separable algebraic closure K^{sep}. Assume that for any finite extension K' of K we have $Br(K') = 0$. Then*

1. *$H^q(Gal(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$, and*
2. *$H^q(Gal(K^{sep}/K), M) = 0$ for any torsion $Gal(K^{sep}/K)$-module M and any $q \geq 2$,*

[**Ser97**, Chapter II, Section 3, Proposition 5]

Proof. Set $p = \mathrm{char}(K)$. By Lemma 59.59.2, Theorem 59.61.6, and Example 59.59.3 the proposition is equivalent to showing that if $H^2(\mathrm{Spec}(K'), \mathbf{G}_m|_{\mathrm{Spec}(K')_{étale}}) = 0$ for all finite extensions K'/K then:

- $H^q(\mathrm{Spec}(K), \mathbf{G}_m|_{\mathrm{Spec}(K)_{étale}}) = 0$ for all $q \geq 1$, and
- $H^q(\mathrm{Spec}(K), \mathcal{F}) = 0$ for any torsion sheaf $\mathcal{F}$ and any $q \geq 2$.

We prove the second part first. Since $\mathcal{F}$ is a torsion sheaf, we may use the ℓ-primary decomposition as well as the compatibility of cohomology with colimits (i.e, direct sums, see Theorem 59.51.3) to reduce to showing $H^q(\mathrm{Spec}(K), \mathcal{F}) = 0$, $q \geq 2$ for all ℓ-power torsion sheaves for every prime ℓ. This allows us to analyze each prime individually.

Suppose that $\ell \neq p$. For any extension K'/K consider the Kummer sequence (Lemma 59.28.1)

$$0 \to \mu_{\ell,\mathrm{Spec}\, K'} \to \mathbf{G}_{m,\mathrm{Spec}\, K'} \xrightarrow{(\cdot)^\ell} \mathbf{G}_{m,\mathrm{Spec}\, K'} \to 0$$

Since $H^q(\mathrm{Spec}\, K', \mathbf{G}_m|_{\mathrm{Spec}(K')_{étale}}) = 0$ for $q = 2$ by assumption and for $q = 1$ by Theorem 59.24.1 combined with $\mathrm{Pic}(K) = (0)$. Thus, by the long-exact cohomology sequence we may conclude that $H^2(\mathrm{Spec}\, K', \mu_\ell) = 0$ for any separable K'/K. Now let H be a maximal pro-ℓ subgroup of the absolute Galois group of K and let L be the corresponding extension. We can write L as the colimit of finite extensions, applying Theorem 59.51.3 to this colimit we see that $H^2(\mathrm{Spec}(L), \mu_\ell) = 0$. Now μ_ℓ must be the constant sheaf. If it weren't, that would imply there exists a Galois extension of degree relatively prime to ℓ of L which is not true by definition of L (namely, the extension one gets by adjoining the ℓth roots of unity to L). Hence, via Lemma 59.67.3, we conclude the result for $\ell \neq p$.

Now suppose that $\ell = p$. We consider the Artin-Schrier exact sequence (Section 59.63)

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_{\operatorname{Spec} K} \longrightarrow \mathbf{G}_{a,\operatorname{Spec} K} \xrightarrow{F-1} \mathbf{G}_{a,\operatorname{Spec} K} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$. Then note that the higher Cohomology of $\mathbf{G}_{a,\operatorname{Spec} K}$ vanishes, by Remark 59.23.4 and the vanishing of the higher cohomology of the structure sheaf of an affine scheme (Cohomology of Schemes, Lemma 30.2.2). Note this can be applied to any field of characteristic p. In particular, we can apply it to the field extension L defined by a maximal pro-p subgroup H. This allows us to conclude $H^n(\operatorname{Spec} L, \underline{\mathbf{Z}/p\mathbf{Z}}_{\operatorname{Spec} L}) = 0$ for $n \geq 2$, from which the result follows for $\ell = p$, by Lemma 59.67.3.

To finish the proof we still have to show that $H^q(\operatorname{Gal}(K^{sep}/K), (K^{sep})^*) = 0$ for all $q \geq 1$. Set $G = \operatorname{Gal}(K^{sep}/K)$ and set $M = (K^{sep})^*$ viewed as a G-module. We have already shown (above) that $H^1(G, M) = 0$ and $H^2(G, M) = 0$. Consider the exact sequence

$$0 \to A \to M \to M \otimes \mathbf{Q} \to B \to 0$$

of G-modules. By the above we have $H^i(G, A) = 0$ and $H^i(G, B) = 0$ for $i > 1$ since A and B are torsion G-modules. By Lemma 59.57.6 we have $H^i(G, M \otimes \mathbf{Q}) = 0$ for $i > 0$. It is a pleasant exercise to see that this implies that $H^i(G, M) = 0$ also for $i \geq 3$. □

03R9 **Definition 59.67.5.** A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an α is called a *nontrivial solution* of f.

03RA **Example 59.67.6.** An algebraically closed field is C_r.

In fact, we have the following simple lemma.

03RB **Lemma 59.67.7.** *Let k be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.*

Proof. This follows from dimension theory, for example in the form of Varieties, Lemma 33.34.2 applied $s - 1$ times. □

The following result computes the Brauer group of C_1 fields.

03RC **Theorem 59.67.8.** *Let K be a C_1 field. Then $Br(K) = 0$.*

Proof. Let D be a finite dimensional division algebra over K with center K. We have seen that

$$D \otimes_K K^{sep} \cong \operatorname{Mat}_d(K^{sep})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \operatorname{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree d map

$$\det = N_{\text{red}} : D \longrightarrow K$$

called the *reduced norm*. Since K is C_1, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\text{red}}(x) = 0$. This clearly implies that x is not invertible, which is a contradiction. Hence $\operatorname{Br}(K) = 0$. □

03RE **Definition 59.67.9.** Let k be a field. A *variety* is separated, integral scheme of finite type over k. A *curve* is a variety of dimension 1.

03RD **Theorem 59.67.10** (Tsen's theorem)**.** *The function field of a variety of dimension r over an algebraically closed field k is C_r.*

Proof. For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is C_r (exercise).

General case. Without loss of generality, we may assume X to be projective. Let $f \in k(X)[T_1, \ldots, T_n]_d$ with $0 < d^r < n$. Say the coefficients of f are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch (Varieties, Proposition 33.45.13) there exists a $c > 0$ such that

- the number of variables is $n \dim_k \Gamma(X, \mathcal{O}_X(eH)) \sim ne^r c$, and
- the number of equations is $\dim_k \Gamma(X, \mathcal{O}_X((de+1)H)) \sim (de+1)^r c$.

Since $n > d^r$, there are more variables than equations. The equations are homogeneous hence there is a solution by Lemma 59.67.7. □

03RF **Lemma 59.67.11.** *Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero: $Br(k(C)) = 0$.*

Proof. This is clear from Tsen's theorem, Theorem 59.67.10 and Theorem 59.67.8. □

03RG **Lemma 59.67.12.** *Let k be an algebraically closed field and K/k a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{étale}(\mathrm{Spec}(K), \mathbf{G}_m) = 0$.*

Proof. Recall that $H^q_{étale}(\mathrm{Spec}(K), \mathbf{G}_m) = H^q(\mathrm{Gal}(K^{sep}/K), (K^{sep})^*)$ by Lemma 59.59.2. Thus by Proposition 59.67.4 it suffices to show that if K'/K is a finite field extension, then $\mathrm{Br}(K') = 0$. Now observe that $K' = \mathrm{colim}\, K''$, where K'' runs over the finitely generated subextensions of k contained in K' of transcendence degree 1. Note that $\mathrm{Br}(K') = \mathrm{colim}\, \mathrm{Br}(K'')$ which reduces us to a finitely generated field extension K''/k of transcendence degree 1. Such a field is the function field of a curve over k, hence has trivial Brauer group by Lemma 59.67.11. □

59.68. Higher vanishing for the multiplicative group

03RH In this section, we fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \hookrightarrow X$ the inclusion of a closed point of X and $j : \eta \hookrightarrow X$ the inclusion of the generic point. We also denote X_0 the set of closed points of X.

03RI **Theorem 59.68.1** (The Fundamental Exact Sequence)**.** *There is a short exact sequence of étale sheaves on X*

$$0 \longrightarrow \mathbf{G}_{m,X} \longrightarrow j_*\mathbf{G}_{m,\eta} \longrightarrow \bigoplus\nolimits_{x \in X_0} i_{x*}\underline{\mathbf{Z}} \longrightarrow 0.$$

Proof. Let $\varphi : U \to X$ be an étale morphism. Then by properties of étale morphisms (Proposition 59.26.2), $U = \coprod_i U_i$ where each U_i is a smooth curve mapping to X. The above sequence for U is a product of the corresponding sequences for each U_i, so it suffices to treat the case where U is connected, hence irreducible. In this case, there is a well known exact sequence

$$1 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow k(U)^* \longrightarrow \bigoplus\nolimits_{y \in U_0} \mathbf{Z}_y.$$

This amounts to a sequence

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U^*) \longrightarrow \Gamma(\eta \times_X U, \mathcal{O}^*_{\eta \times_X U}) \longrightarrow \bigoplus\nolimits_{x \in X_0} \Gamma(x \times_X U, \underline{\mathbf{Z}})$$

which, unfolding definitions, is nothing but a sequence

$$0 \longrightarrow \mathbf{G}_m(U) \longrightarrow j_*\mathbf{G}_{m,\eta}(U) \longrightarrow \left(\bigoplus\nolimits_{x \in X_0} i_{x*}\underline{\mathbf{Z}}\right)(U).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall that if U is a nonsingular curve and D is a divisor on U, then there exists a Zariski open covering $\{U_j \to U\}$ of U such that $D|_{U_j} = \mathrm{div}(f_j)$ for some $f_j \in k(U)^*$. □

03RJ **Lemma 59.68.2.** *For any $q \geq 1$, $R^q j_*\mathbf{G}_{m,\eta} = 0$.*

Proof. We need to show that $(R^q j_*\mathbf{G}_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X.

Assume that $\bar{x}$ lies over a closed point x of X. Let $\mathrm{Spec}(A)$ be an affine open neighbourhood of x in X, and K the fraction field of A. Then

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \eta = \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K).$$

The ring $\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}^{sh}_{X,\bar{x}}$, so it is either $\mathcal{O}^{sh}_{X,\bar{x}}$ again, or its fraction field $K^{sh}_{\bar{x}}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$(R^q j_*\mathbf{G}_{m,\eta})_{(X,\bar{x})} = H^q_{\acute{e}tale}(\mathrm{Spec}\, K^{sh}_{\bar{x}}, \mathbf{G}_m).$$

Now recall that $\mathcal{O}^{sh}_{X,\bar{x}} = \mathrm{colim}_{(U,\bar{u}) \to \bar{x}} \mathcal{O}(U) = \mathrm{colim}_{A \subset B} B$ where $A \to B$ is étale, hence $K^{sh}_{\bar{x}}$ is an algebraic extension of $K = k(X)$, and we may apply Lemma 59.67.12 to get the vanishing.

Assume that $\bar{x} = \bar{\eta}$ lies over the generic point η of X (in fact, this case is superfluous). Then $\mathcal{O}^{sh}_{X,\bar{\eta}} = \kappa(\eta)^{sep}$ and thus

$$\begin{aligned}(R^q j_*\mathbf{G}_{m,\eta})_{\bar{\eta}} &= H^q_{\acute{e}tale}(\mathrm{Spec}(\kappa(\eta)^{sep}) \times_X \eta, \mathbf{G}_m) \\ &= H^q_{\acute{e}tale}(\mathrm{Spec}(\kappa(\eta)^{sep}), \mathbf{G}_m) \\ &= 0 \quad \text{for } q \geq 1\end{aligned}$$

since the corresponding Galois group is trivial. □

03RK **Lemma 59.68.3.** *For all $p \geq 1$, $H^p_{\acute{e}tale}(X, j_*\mathbf{G}_{m,\eta}) = 0$.*

Proof. The Leray spectral sequence reads

$$E_2^{p,q} = H^p_{\acute{e}tale}(X, R^q j_*\mathbf{G}_{m,\eta}) \Rightarrow H^{p+q}_{\acute{e}tale}(\eta, \mathbf{G}_{m,\eta}),$$

The right hand side vanishes for $p + q \geq 1$ by Lemma 59.67.12. The left hand side vanishes for $q \geq 1$ by Lemma 59.68.2. Thus we get the desired vanishing. □

03RL **Lemma 59.68.4.** *For all $q \geq 1$, $H^q_{\acute{e}tale}(X, \bigoplus_{x \in X_0} i_{x*}\underline{\mathbf{Z}}) = 0$.*

Proof. For X quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}})$. But then the inclusion i_x of a closed point is finite so $R^p i_{x*}\underline{\mathbf{Z}} = 0$ for all $p \geq 1$ by Proposition 59.55.2. Applying the Leray spectral sequence, we see that $H^q_{\acute{e}tale}(X, i_{x*}\underline{\mathbf{Z}}) = H^q_{\acute{e}tale}(x, \underline{\mathbf{Z}})$. Finally, since x is the spectrum of an algebraically closed field, all higher cohomology on x vanishes. □

Concluding this series of lemmata, we get the following result.

03RM **Theorem 59.68.5.** *Let X be a smooth curve over an algebraically closed field. Then*

$$H^q_{étale}(X, \mathbf{G}_m) = 0 \quad \text{for all } q \geq 2.$$

Proof. See discussion above. □

We also get the cohomology long exact sequence

$$0 \to H^0_{étale}(X, \mathbf{G}_m) \to H^0_{étale}(X, j_*\mathbf{G}_{m\eta}) \to H^0_{étale}(X, \bigoplus i_{x*}\underline{\mathbf{Z}}) \to H^1_{étale}(X, \mathbf{G}_m) \to 0$$

although this is the familiar

$$0 \to H^0_{Zar}(X, \mathcal{O}_X^*) \to k(X)^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

59.69. Picard groups of curves

03RN Our next step is to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

Let X be a smooth projective curve over an algebraically closed field k. Let $g = \dim_k H^1(X, \mathcal{O}_X)$ be the genus of X. There exists a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The abelian group $\mathrm{Pic}^0(X)$ can be identified with $\mathrm{Pic}^0(X) = \underline{\mathrm{Pic}}^0_{X/k}(k)$, i.e., the k-valued points of an abelian variety $\underline{\mathrm{Pic}}^0_{X/k}$ over k of dimension g. Consequently, if $n \in k^*$ then $\mathrm{Pic}^0(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ as abelian groups. See Picard Schemes of Curves, Section 44.6 and Groupoids, Section 39.9. This key fact, namely the description of the torsion in the Picard group of a smooth projective curve over an algebraically closed field does not appear to have an elementary proof.

03RQ **Lemma 59.69.1.** *Let X be a smooth projective curve of genus g over an algebraically closed field k and let $n \geq 1$ be invertible in k. Then there are canonical identifications*

$$H^q_{étale}(X, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0, \\ \mathrm{Pic}^0(X)[n] & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Since $\mu_n \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, this gives (noncanonical) identifications

$$H^q_{étale}(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } q = 0, \\ (\mathbf{Z}/n\mathbf{Z})^{2g} & \text{if } q = 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2, \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. Theorems 59.24.1 and 59.68.5 determine the étale cohomology of $\mathbf{G}_m$ on X in terms of the Picard group of X. The Kummer sequence $0 \to \mu_{n,X} \to \mathbf{G}_{m,X} \to$

$\mathbf{G}_{m,X} \to 0$ (Lemma 59.28.1) then gives us the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_n(k) & \longrightarrow & k^* & \xrightarrow{(\cdot)^n} & k^* \\ & & & & \searrow & & \\ & & H^1_{étale}(X, \mu_n) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(X) \\ & & & & \searrow & & \\ & & H^2_{étale}(X, \mu_n) & \longrightarrow & 0 & \longrightarrow & 0 \ldots \end{array}$$

The nth power map $k^* \to k^*$ is surjective since k is algebraically closed. So we need to compute the kernel and cokernel of the map $n : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\cdot)^n} & & \downarrow{\scriptstyle (\cdot)^n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

The group $\mathrm{Pic}^0(X)$ is the k-points of the group scheme $\underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 44.6.7. The same lemma tells us that $\underline{\mathrm{Pic}}^0_{X/k}$ is a g-dimensional abelian variety over k as defined in Groupoids, Definition 39.9.1. Hence the left vertical map is surjective by Groupoids, Proposition 39.9.11. Applying the snake lemma gives canonical identifications as stated in the lemma.

To get the noncanonical identifications of the lemma we need to show the kernel of $n : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$. This is also part of Groupoids, Proposition 39.9.11. □

0AMB **Lemma 59.69.2.** *Let $\pi : X \to Y$ be a nonconstant morphism of smooth projective curves over an algebraically closed field k and let $n \geq 1$ be invertible in k. The map*

$$\pi^* : H^2_{étale}(Y, \mu_n) \longrightarrow H^2_{étale}(X, \mu_n)$$

is given by multiplication by the degree of π.

Proof. Observe that the statement makes sense as we have identified both cohomology groups $H^2_{étale}(Y, \mu_n)$ and $H^2_{étale}(X, \mu_n)$ with $\mathbf{Z}/n\mathbf{Z}$ in Lemma 59.69.1. In fact, if $\mathcal{L}$ is a line bundle of degree 1 on Y with class $[\mathcal{L}] \in H^1_{étale}(Y, \mathbf{G}_m)$, then the coboundary of $[\mathcal{L}]$ is the generator of $H^2_{étale}(Y, \mu_n)$. Here the coboundary is the coboundary of the long exact sequence of cohomology associated to the Kummer sequence. Thus the result of the lemma follows from the fact that the degree of the line bundle $\pi^*\mathcal{L}$ on X is $\deg(\pi)$. Some details omitted. □

03RR **Lemma 59.69.3.** *Let X be an affine smooth curve over an algebraically closed field k and $n \in k^*$. Let $X \subset \overline{X}$ be a smooth projective compactification (Varieties, Remark 33.43.9). Let g be the genus of $\overline{X}$ and let r be the number of points of $\overline{X} \setminus X$. Then*

1. *$H^0_{étale}(X, \mu_n) = \mu_n(k)$;*
2. *$H^1_{étale}(X, \mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r-1}$, and*
3. *$H^q_{étale}(X, \mu_n) = 0$ for all $q \geq 2$.*

Proof. Write $X = \overline{X} - \{x_1, \ldots, x_r\}$. Then $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})/R$, where R is the subgroup generated by $\mathcal{O}_{\overline{X}}(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(X)$ is surjective, hence $\mathrm{Pic}(X)$ is divisible (see discussion in proof of Lemma 59.69.1). Applying the Kummer sequence, we get (1) and (3). For (2), recall that

$$\begin{aligned} H^1_{étale}(X, \mu_n) &= \{(\mathcal{L}, \alpha) | \mathcal{L} \in \mathrm{Pic}(X), \alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_X\}/\cong \\ &= \{(\bar{\mathcal{L}},\ D,\ \bar{\alpha})\}/\tilde{R} \end{aligned}$$

where $\bar{\mathcal{L}} \in \mathrm{Pic}^0(\overline{X})$, D is a divisor on $\overline{X}$ supported on $\{x_1, \ldots, x_r\}$ and $\bar{\alpha} : \bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}_{\bar{X}}(D)$ is an isomorphism. Note that D must have degree 0. Further $\tilde{R}$ is the subgroup of triples of the form $(\mathcal{O}_{\overline{X}}(D'), nD', 1^{\otimes n})$ where D' is supported on $\{x_1, \ldots, x_r\}$ and has degree 0. Thus, we get an exact sequence

$$0 \longrightarrow H^1_{étale}(\overline{X}, \mu_n) \longrightarrow H^1_{étale}(X, \mu_n) \longrightarrow \bigoplus_{i=1}^{r} \mathbf{Z}/n\mathbf{Z} \xrightarrow{\sum} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

where the middle map sends the class of a triple $(\bar{\mathcal{L}}, D, \bar{\alpha})$ with $D = \sum_{i=1}^r a_i(x_i)$ to the r-tuple $(a_i)_{i=1}^r$. It now suffices to use Lemma 59.69.1 to count ranks. □

03RS **Remark 59.69.4.** The "natural" way to prove the previous corollary is to excise X from $\bar{X}$. This is possible, we just haven't developed that theory.

0A44 **Remark 59.69.5.** Let k be an algebraically closed field. Let n be an integer prime to the characteristic of k. Recall that

$$\mathbf{G}_{m,k} = \mathbf{A}^1_k \setminus \{0\} = \mathbf{P}^1_k \setminus \{0, \infty\}$$

We claim there is a canonical isomorphism

$$H^1_{étale}(\mathbf{G}_{m,k}, \mu_n) = \mathbf{Z}/n\mathbf{Z}$$

What does this mean? This means there is an element 1_k in $H^1_{étale}(\mathbf{G}_{m,k}, \mu_n)$ such that for every morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ the pullback map on étale cohomology for the map $\mathbf{G}_{m,k'} \to \mathbf{G}_{m,k}$ maps 1_k to $1_{k'}$. (In particular this element is fixed under all automorphisms of k.) To see this, consider the $\mu_{n,\mathbf{Z}}$-torsor $\mathbf{G}_{m,\mathbf{Z}} \to \mathbf{G}_{m,\mathbf{Z}}$, $x \mapsto x^n$. By the identification of torsors with first cohomology, this pulls back to give our canonical elements 1_k. Twisting back we see that there are canonical identifications

$$H^1_{étale}(\mathbf{G}_{m,k}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{Hom}(\mu_n(k), \mathbf{Z}/n\mathbf{Z}),$$

i.e., these isomorphisms are compatible with respect to maps of algebraically closed fields, in particular with respect to automorphisms of k.

59.70. Extension by zero

03S2 The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

03S3 **Definition 59.70.1.** Let $j : U \to X$ be an étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{étale}) \to Sh(U_{étale})$ has a left adjoint $j^{Sh}_! : Sh(U_{étale}) \to Sh(X_{étale})$.
(2) The restriction functor $j^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$ has a left adjoint which is denoted $j_! : Ab(U_{étale}) \to Ab(X_{étale})$ and called *extension by zero*.

(3) Let Λ be a ring. The restriction functor $j^{-1} : Mod(X_{étale}, \Lambda) \to Mod(U_{étale}, \Lambda)$ has a left adjoint which is denoted $j_! : Mod(U_{étale}, \Lambda) \to Mod(X_{étale}, \Lambda)$ and called *extension by zero*.

If $\mathcal{F}$ is an abelian sheaf on $X_{étale}$, then $j_!\mathcal{F} \neq j_!^{Sh}\mathcal{F}$ in general. On the other hand $j_!$ for sheaves of Λ-modules agrees with $j_!$ on underlying abelian sheaves (Modules on Sites, Remark 18.19.6). The functor $j_!$ is characterized by the functorial isomorphism

$$\mathrm{Hom}_X(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_U(\mathcal{F}, j^{-1}\mathcal{G})$$

for all $\mathcal{F} \in Ab(U_{étale})$ and $\mathcal{G} \in Ab(X_{étale})$. Similarly for sheaves of Λ-modules.

To describe the functors in Definition 59.70.1 more explicitly, recall that j^{-1} is just the restriction via the functor $U_{étale} \to X_{étale}$. In other words, $j^{-1}\mathcal{G}(U') = \mathcal{G}(U')$ for U' étale over U. On the other hand, for $\mathcal{F} \in Ab(U_{étale})$ we consider the presheaf

0F4K (59.70.1.1) $$j_{p!}\mathcal{F} : X_{étale} \longrightarrow Ab, \quad V \longmapsto \bigoplus\nolimits_{V \to U} \mathcal{F}(V \to U)$$

Then $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$. This is proven in Modules on Sites, Lemma 18.19.2; more generally see the discussion in Modules on Sites, Sections 18.19 and 18.16.

03S4 **Exercise 59.70.2.** Prove directly that the functor $j_!$ defined as the sheafification of the functor $j_{p!}$ given in (59.70.1.1) is a left adjoint to j^{-1}.

03S5 **Proposition 59.70.3.** *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{F}$ in $Ab(U_{étale})$. If $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point of X, then*

$$(j_!\mathcal{F})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}:\mathrm{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.$$

In particular, $j_!$ is an exact functor.

Proof. Exactness of $j_!$ is very general, see Modules on Sites, Lemma 18.19.3. Of course it does also follow from the description of stalks. The formula for the stalk follows from Modules on Sites, Lemma 18.38.1 and the description of points of the small étale site in terms of geometric points, see Lemma 59.29.12.

For later use we note that the isomorphism

$$\begin{aligned}(j_!\mathcal{F})_{\overline{x}} &= (j_{p!}\mathcal{F})_{\overline{x}} \\ &= \mathrm{colim}_{(V,\overline{v})}\, j_{p!}\mathcal{F}(V) \\ &= \mathrm{colim}_{(V,\overline{v})} \bigoplus\nolimits_{\varphi:V\to U} \mathcal{F}(V \xrightarrow{\varphi} U) \\ &\to \bigoplus\nolimits_{\overline{u}:\mathrm{Spec}(k)\to U,\ j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.\end{aligned}$$

constructed in Modules on Sites, Lemma 18.38.1 sends $(V, \overline{v}, \varphi, s)$ to the class of s in the stalk of $\mathcal{F}$ at $\overline{u} = \varphi(\overline{v})$. □

0F70 **Lemma 59.70.4.** *Let $j : U \to X$ be an open immersion of schemes. For any abelian sheaf $\mathcal{F}$ on $U_{étale}$, the adjunction mappings $j^{-1}j_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to j^{-1}j_!\mathcal{F}$ are isomorphisms. In fact, $j_!\mathcal{F}$ is the unique abelian sheaf on $X_{étale}$ whose restriction to U is $\mathcal{F}$ and whose stalks at geometric points of $X \setminus U$ are zero.*

Proof. We encourage the reader to prove the first statement by working through the definitions, but here we just use that it is a special case of the very general Modules on Sites, Lemma 18.19.8. For the second statement, observe that if $\mathcal{G}$ is an abelian sheaf on $X_{étale}$ whose restriction to U is $\mathcal{F}$, then we obtain by adjointness a map $j_!\mathcal{F} \to \mathcal{G}$. This map is then an isomorphism at stalks of geometric points of U by Proposition 59.70.3. Thus if $\mathcal{G}$ has vanishing stalks at geometric points of $X \setminus U$, then $j_!\mathcal{F} \to \mathcal{G}$ is an isomorphism by Theorem 59.29.10. □

03S6 **Lemma 59.70.5** (Extension by zero commutes with base change)**.** *Let $f : Y \to X$ be a morphism of schemes. Let $j : V \to X$ be an étale morphism. Consider the fibre product*

$$\begin{array}{ccc} V' = Y \times_X V & \xrightarrow{j'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & X \end{array}$$

Then we have $j'_!f'^{-1} = f^{-1}j_!$ on abelian sheaves and on sheaves of modules.

Proof. This is true because $j'_!f'^{-1}$ is left adjoint to $f'_*(j')^{-1}$ and $f^{-1}j_!$ is left adjoint to $j^{-1}f_*$. Further $f'_*(j')^{-1} = j^{-1}f_*$ because f_* commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 18.20.1. □

0F4L **Lemma 59.70.6.** *Let $j : U \to X$ be separated and étale. Then there is a functorial injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ on abelian sheaves and sheaves of Λ-modules.*

Proof. We prove this in the case of abelian sheaves. Let us construct a canonical map

$$j_{p!}\mathcal{F} \to j_*\mathcal{F}$$

of abelian presheaves on $X_{étale}$ for any abelian sheaf $\mathcal{F}$ on $U_{étale}$ where $j_{p!}$ is as in (59.70.1.1). Sheafification of this map will be the desired map $j_!\mathcal{F} \to j_*\mathcal{F}$. Evaluating both sides on $V \to X$ étale we obtain

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U) \quad \text{and} \quad j_*\mathcal{F}(V) = \mathcal{F}(V \times_X U)$$

For each φ we have an open and closed immersion

$$\Gamma_\varphi = (1, \varphi) : V \longrightarrow V \times_X U$$

over U. It is open as it is a morphism between schemes étale over U and it is closed as it is a section of a scheme separated over V (Schemes, Lemma 26.21.11). Thus for a section $s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U)$ there exists a unique section s'_φ in $\mathcal{F}(V \times_X U)$ which pulls back to s_φ by Γ_φ and which restricts to zero on the complement of the image of Γ_φ.

To show that our map is injective suppose that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ is an element of $j_{p!}\mathcal{F}(V)$ in the formula above maps to zero in $j_*\mathcal{F}(V)$. Our task is to show that $\sum_{i=1,\ldots,n} s_{\varphi_i}$ restricts to zero on the members of an étale covering of V. Looking at all pairwise equalizers (which are open and closed in V) of the morphisms $\varphi_i : V \to U$ and working locally on V, we may assume the images of the morphisms $\Gamma_{\varphi_1}, \ldots, \Gamma_{\varphi_n}$ are pairwise disjoint. Since our assumption is that $\sum_{i=1,\ldots,n} s'_{\varphi_i} = 0$ we then immediately conclude that $s'_{\varphi_i} = 0$ for each i (by the disjointness of the supports of these sections), whence $s_{\varphi_i} = 0$ for all i as desired. □

03S7 **Lemma 59.70.7.** *Let $j : U \to X$ be finite and étale. Then the map $j_! \to j_*$ of Lemma 59.70.6 is an isomorphism on abelian sheaves and sheaves of Λ-modules.*

Proof. It suffices to check $j_!\mathcal{F} \to j_*\mathcal{F}$ is an isomorphism étale locally on X. Thus we may assume $U \to X$ is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. We omit the proof in this case. □

095L **Lemma 59.70.8.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. For every abelian sheaf $\mathcal{F}$ on $X_{étale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{étale}$.

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point $\overline{x}$ in X we have either $\overline{x} \in U$ in which case the map on the left hand side is an isomorphism on stalks and the stalk of $i_*i^{-1}\mathcal{F}$ is zero or $\overline{x} \in Z$ in which case the map on the right hand side is an isomorphism on stalks and the stalk of $j_!j^{-1}\mathcal{F}$ is zero. Here we have used the description of stalks of Lemma 59.46.3 and Proposition 59.70.3. □

0GJ1 **Lemma 59.70.9.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} U & \xrightarrow{j'} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & Y \end{array}$$

where f is finite and j is an open immersion. Then $f_* \circ j'_! = j_! \circ g_*$ as functors $Ab(U_{étale}) \to Ab(Y_{étale})$.

Proof. Let $\mathcal{F}$ be an object of $Ab(U_{étale})$. Let $\overline{y}$ be a geometric point of Y not contained in the open V. Then

$$(f_*j'_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x},\ f(\overline{x})=\overline{y}} (j'_!\mathcal{F})_{\overline{x}} = 0$$

by Proposition 59.55.2 and because the stalk of $j'_!\mathcal{F}$ at $\overline{x} \notin U$ are zero by Lemma 59.70.4. On the other hand, we have

$$j^{-1}f_*j'_!\mathcal{F} = g_*(j')^{-1}j'_!\mathcal{F} = g_*\mathcal{F}$$

by Lemmas 59.55.3 and 59.70.4. Hence by the characterization of $j_!$ in Lemma 59.70.4 we see that $f_*j'_!\mathcal{F} = j_!g_*\mathcal{F}$. We omit the verification that this identification is functorial in $\mathcal{F}$. □

59.71. Constructible sheaves

05BE Let X be a scheme. A *constructible locally closed subscheme* of X is a locally closed subscheme $T \subset X$ such that the underlying topological space of T is a constructible subset of X. If $T, T' \subset X$ are locally closed subschemes with the same underlying topological space, then $T_{étale} \cong T'_{étale}$ by the topological invariance of the étale site (Theorem 59.45.2). Thus in the following definition we may assume our locally closed subschemes are reduced.

03RW **Definition 59.71.1.** Let X be a scheme.

(1) A sheaf of sets on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(2) A sheaf of abelian groups on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(3) Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with h_U, $j_{U!}\mathbf{Z}/n\mathbf{Z}$, and $j_{U!}\underline{\Lambda}$ where U runs over all quasi-compact and quasi-separated objects of $X_{étale}$, and then take the smallest full subcategory of $Sh(X_{étale})$, $Ab(X_{étale})$, and $Mod(X_{étale}, \underline{\Lambda})$ containing these and closed under finite limits and colimits. It follows from Lemma 59.71.6 and Lemmas 59.73.5, 59.73.7, and 59.73.6 that this produces the same category if X is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \coprod U_i$ of a scheme by locally closed subschemes will be called a *partition* of U (compare with Topology, Section 5.28).

095E **Lemma 59.71.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. The following are equivalent*

(1) *$\mathcal{F}$ is constructible,*
(2) *there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and*
(3) *there exists a partition $X = \bigcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.*

A similar statement holds for abelian sheaves and sheaves of Λ-modules if Λ is Noetherian.

Proof. It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an i and an affine open neighbourhood $V_x \subset U_i$ of x. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each j there exists a finite decomposition $V_j = \coprod V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 5.15.5 each $V_{j,k}$ is constructible as a subset of X. By Topology, Lemma 5.28.7 we can find a finite stratification $X = \coprod X_l$ with constructible locally closed strata such that each $V_{j,k}$ is a union of X_l. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of U (for example by Properties, Lemma 28.2.1) and $U = \coprod U \cap X_i$ is a partition of U as in Definition 59.71.1. Thus (1) holds. □

09YR **Lemma 59.71.3.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) on $X_{étale}$. If there*

exist constructible locally closed subschemes $T_i \subset X$ such that (a) $X = \bigcup T_j$ and (b) $\mathcal{F}|_{T_j}$ is constructible, then $\mathcal{F}$ is constructible.

Proof. First, we can assume the covering is finite as X is quasi-compact in the spectral topology (Topology, Lemma 5.23.2 and Properties, Lemma 28.2.4). Observe that each T_i is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in X; details omitted). Thus we can find a finite partition $T_i = \coprod T_{i,j}$ into locally closed constructible parts of T_i such that $\mathcal{F}|_{T_{i,j}}$ is finite locally constant (Lemma 59.71.2). By Topology, Lemma 5.15.12 we see that $T_{i,j}$ is a constructible locally closed subscheme of X. Then we can apply Topology, Lemma 5.28.7 to $X = \bigcup T_{i,j}$ to find the desired partition of X. □

095F **Lemma 59.71.4.** *Let X be a scheme. Checking constructibility of a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) can be done Zariski locally on X.*

Proof. The statement means if $X = \bigcup U_i$ is an open covering such that $\mathcal{F}|_{U_i}$ is constructible, then $\mathcal{F}$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $\mathcal{F}|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma 59.71.2 that $\mathcal{F}|_U$ is constructible, i.e., a suitable partition of U exists. □

095G **Lemma 59.71.5.** *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules (with Λ Noetherian) on $Y_{étale}$, the same is true for $f^{-1}\mathcal{F}$ on $X_{étale}$.*

Proof. By Lemma 59.71.4 this reduces to the case where X and Y are affine. By Lemma 59.71.2 it suffices to find a finite partition of X by constructible locally closed subschemes such that $f^{-1}\mathcal{F}$ is finite locally constant on each of them. To find it we just pull back the partition of Y adapted to $\mathcal{F}$ and use Lemma 59.64.2. □

03RZ **Lemma 59.71.6.** *Let X be a scheme.*

(1) *The category of constructible sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of constructible abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. We prove (3). We will use the criterion of Homology, Lemma 12.10.3. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of constructible sheaves of Λ-modules. We have to show that $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is a short exact sequence of sheaves of Λ-modules with $\mathcal{F}$, $\mathcal{G}$ constructible. We have to show that $\mathcal{E}$ is constructible. In both cases we can replace X with the members of an affine open covering. Hence we may assume X is affine. Then we may further replace X by the members of a finite partition of X by constructible locally closed subschemes on which $\mathcal{F}$ and $\mathcal{G}$ are of finite type and locally constant. Thus we may apply Lemma 59.64.6 to conclude.

The proofs of (1) and (2) are very similar and are omitted. □

09YS **Lemma 59.71.7.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F} \to \mathcal{G}$ be a map of constructible sheaves of sets on $X_{étale}$. Then the set of points $x \in X$ where $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in X.*
(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*

Proof. Proof of (1). Let $X = \coprod X_i$ be a partition of X by locally closed constructible subschemes such that both $\mathcal{F}$ and $\mathcal{G}$ are finite locally constant over the parts (use Lemma 59.71.2 for both $\mathcal{F}$ and $\mathcal{G}$ and choose a common refinement). Then apply Lemma 59.64.5 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. □

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak "Noetherian" property.

095P **Lemma 59.71.8.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.*

(1) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of sets, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(2) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of abelian groups, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(3) *Let Λ be a Noetherian ring. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of Λ-modules, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 59.64.6). For each i let $T_i \subset X$ be the set of points $x \in X$ where $\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}}$ is not surjective. Because $\mathcal{F}_i$ and $\mathcal{F}$ are constructible T_i is a constructible subset of X (Lemma 59.71.7). Since the stalks of $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_i = \emptyset$ for i large enough. Thus $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $S_{i'} \subset X$ the set of points x such that the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}})$ is not equal to the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{i',\overline{x}})$. Because $\mathcal{F}_i$, $\mathcal{F}_{i'}$ and $\mathcal{F}$ are constructible $S_{i'}$ is a constructible subset of X (details omitted; hint: use Lemma 59.71.7). Since the stalks of $\mathcal{F}_i$ and $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).

Proof of (3). We will use without further mention that the category of constructible sheaves of Λ-modules is abelian (Lemma 59.64.6). For each i let $\mathcal{Q}_i$ be the cokernel of the map $\mathcal{F}_i \to \mathcal{F}$. The support T_i of $\mathcal{Q}_i$ is a constructible subset of X as $\mathcal{Q}_i$ is constructible (Lemma 59.71.7). Since the stalks of $\mathcal{F}$ are finite Λ-modules and since

$\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_i = \emptyset$ for i large enough. This proves the first assertion. For the second, assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $\mathcal{K}_{i'}$ the image of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ in $\mathcal{F}_{i'}$. The support $S_{i'}$ of $\mathcal{K}_{i'}$ is a constructible subset of X as $\mathcal{K}_{i'}$ is constructible. Since the stalks of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ are finite Λ-modules and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired. □

095I **Lemma 59.71.9.** *Let X be a scheme. Let Λ be a Noetherian ring. The tensor product of two constructible sheaves of Λ-modules on $X_{étale}$ is a constructible sheaf of Λ-modules.*

Proof. The question immediately reduces to the case where X is affine. Since any two partitions of X with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma 59.64.7. □

0GKB **Lemma 59.71.10.** *Let $\Lambda \to \Lambda'$ be a homomorphism of Noetherian rings. Let X be a scheme. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. Then $\mathcal{F} \otimes_{\underline{\Lambda}} \underline{\Lambda'}$ is a constructible sheaf of Λ'-modules.*

Proof. Omitted. Hint: affine locally you can use the same stratification. □

59.72. Auxiliary lemmas on morphisms

095J Some lemmas that are useful for proving functoriality properties of constructible sheaves.

03S0 **Lemma 59.72.1.** *Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \coprod_i X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all i.*

Proof. If $U \to X$ is separated, then this is More on Morphisms, Lemma 37.45.4. In general, we may assume X is affine. Choose a finite affine open covering $U = \bigcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_{j'} \to X$ and choose a common refinement $X = \coprod X_i$ of the resulting partitions. After refining the partition further we may assume X_i affine as well. Fix i and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_{jj'} = (U_j \cap U_{j'}) \times_X X_i \to X_i$ are finite étale. Hence V_j and $V_{jj'}$ are affine schemes and $V_{jj'} \subset V_j$ is closed as well as open (since $V_{jj'} \to X_i$ is proper, so Morphisms, Lemma 29.41.7 applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{jj'})$ is surjective, see Schemes, Lemma 26.21.7. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn't but we don't need this). □

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

03S1 **Lemma 59.72.2.** *Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If η is a generic point of an*

irreducible component of Y such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subset Y$ containing η such that $f^{-1}(V) \to V$ is finite.

Proof. This is Morphisms, Lemma 29.51.1. □

The statement of the following lemma can be strengthened a bit.

095K **Lemma 59.72.3.** *Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.*

(1) *There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subset Y' = X' \times_X Y$ such that*
   (a) *$Z' \subset Y'$ is a thickening, and*
   (b) *$Z' \to X'$ is a finite étale morphism.*

(2) *There exists a finite partition $X = \coprod X_i$ by locally closed, constructible, affine strata, and surjective finite locally free morphisms $X'_i \to X_i$ such that the reduction of $Y'_i = X'_i \times_X Y \to X'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (X'_i)_{red} \to (X'_i)_{red}$ for some n_i.*

Proof. Setting $X' = \coprod X'_i$ we see that (2) implies (1). Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Write A as a filtered colimit of finite type $\mathbf{Z}$-algebras A_i. Since B is an A-algebra of finite presentation, we see that there exists $0 \in I$ and a finite type ring map $A_0 \to B_0$ such that $B = \mathrm{colim}\, B_i$ with $B_i = A_i \otimes_{A_0} B_0$, see Algebra, Lemma 10.127.8. For i sufficiently large we see that $A_i \to B_i$ is quasi-finite, see Limits, Lemma 32.18.2. Thus we reduce to the case of finite type algebras over $\mathbf{Z}$, in particular we reduce to the Noetherian case. (Details omitted.)

Assume X and Y Noetherian. In this case any locally closed subset of X is constructible. By Lemma 59.72.2 and Noetherian induction we see that there is a finite partition $X = \coprod X_i$ of X by locally closed strata such that $Y \times_X X_i \to X_i$ is finite. We can refine this partition to get affine strata. Thus after replacing X by $X' = \coprod X_i$ we may assume $Y \to X$ is finite.

Assume X and Y Noetherian and $Y \to X$ finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism $U \to X$. Thus we have a partition $U = \coprod U_i$ and finite locally free morphisms $U'_i \to U_i$ such that $U'_i \times_X Y \to U'_i$ is isomorphic to $\coprod_{j=1}^{n_i} (U'_i)_{red} \to (U'_i)_{red}$ for some n_i. Then, by the argument in the previous paragraph, we can find a partition $X = \coprod X_j$ with locally closed affine strata such that $X_j \times_X U_i \to X_j$ is finite for all i, j. By Morphisms, Lemma 29.48.2 each $X_j \times_X U_i \to X_j$ is finite locally free. Hence $X_j \times_X U'_i \to X_j$ is finite locally free (Morphisms, Lemma 29.48.3). It follows that $X = \coprod X_j$ and $X'_j = \coprod_i X_j \times_X U'_i$ is a solution for $Y \to X$. Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma 29.48.6 we see we may assume that Y is a closed subscheme of an affine scheme Z which is (set theoretically) a finite union $Z = \bigcup_{i \in I} Z_i$ of closed subschemes mapping isomorphically to X. In this case we will find a finite partition of $X = \coprod X_j$ with affine locally closed strata that works (in other words $X'_j = X_j$). Set $T_i = Y \cap Z_i$. This is a closed subscheme of X. As X is Noetherian we can find a finite partition of $X = \coprod X_j$ by affine locally closed subschemes, such that each $X_j \times_X T_i$ is (set theoretically) a union of strata $X_j \times_X Z_i$. Replacing X by X_j we see that we may assume $I = I_1 \amalg I_2$ with $Z_i \subset Y$ for $i \in I_1$ and $Z_i \cap Y = \emptyset$ for $i \in I_2$. Replacing Z by $\bigcup_{i \in I_1} Z_i$ we see that we may

assume $Y = Z$. Finally, we can replace X again by the members of a partition as above such that for every $i, i' \subset I$ the intersection $Z_i \cap Z_{i'}$ is either empty or (set theoretically) equal to Z_i and $Z_{i'}$. This clearly means that Y is (set theoretically) equal to a disjoint union of the Z_i which is what we wanted to show. □

59.73. More on constructible sheaves

095M Let Λ be a Noetherian ring. Let X be a scheme. We often consider $X_{étale}$ as a ringed site with sheaf of rings $\underline{\Lambda}$. In case of abelian sheaves we often take $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for a suitable integer n.

03S8 **Lemma 59.73.1.** *Let $j : U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes.*

(1) *The sheaf h_U is a constructible sheaf of sets.*
(2) *The sheaf $j_!\underline{M}$ is a constructible abelian sheaf for a finite abelian group M.*
(3) *If Λ is a Noetherian ring and M is a finite Λ-module, then $j_!\underline{M}$ is a constructible sheaf of Λ-modules on $X_{étale}$.*

Proof. By Lemma 59.72.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of h_U to X_i is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 59.64.4. For cases (2) and (3) we note that

$$j_!(\underline{M})|_{X_i} = \pi_{i!}(\underline{M}) = \pi_{i*}(\underline{M})$$

by Lemmas 59.70.5 and 59.70.7. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 59.64.3. □

03SA **Lemma 59.73.2.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of sets.*
(2) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves.*
(3) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of Λ-modules.*

Proof. Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{étale}$. By Modules on Sites, Lemma 18.30.7 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer}\Big(\coprod\nolimits_{j=1,\ldots,m} h_{V_j} \rightrightarrows \coprod\nolimits_{i=1,\ldots,n} h_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. By Lemmas 59.73.1 and 59.71.6 these coequalizers are constructible. This proves (1).

Let Λ be a Noetherian ring. By Modules on Sites, Lemma 18.30.7 Λ-modules $\mathcal{F}$ is a filtered colimit of modules of the form

$$\text{Coker}\Big(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. By Lemmas 59.73.1 and 59.71.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n]$ is the n-torsion subsheaf. Then we can view $\mathcal{F}[n]$ as a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and apply (3). □

095Q **Lemma 59.73.3.** *Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be sheaf of Λ-modules on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*

Proof. One implication follows from Lemma 59.71.5. For the converse, assume $f^{-1}\mathcal{F}$ is constructible. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma 59.73.2. Since f^{-1} is a left adjoint it commutes with colimits (Categories, Lemma 4.24.5) and we see that $f^{-1}\mathcal{F} = \operatorname{colim} f^{-1}\mathcal{F}_i$. By Lemma 59.71.8 we see that $f^{-1}\mathcal{F}_i \to f^{-1}\mathcal{F}$ is surjective for all i large enough. Since f is surjective we conclude (by looking at stalks using Lemma 59.36.2 and Theorem 59.29.10) that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i large enough. Thus $\mathcal{F}$ is the quotient of a constructible sheaf $\mathcal{G}$. Applying the argument once more to $\mathcal{G} \times_{\mathcal{F}} \mathcal{G}$ or the kernel of $\mathcal{G} \to \mathcal{F}$ we conclude using that f^{-1} is exact and that the category of constructible sheaves (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside $Sh(Y_{étale})$, $Sh(X_{étale})$, $Ab(Y_{étale})$, $Ab(X_{étale})$, $Mod(Y_{étale}, \Lambda)$, and $Mod(X_{étale}, \Lambda)$, see Lemma 59.71.6. □

095H **Lemma 59.73.4.** *Let $f : X \to Y$ be a finite étale morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. By Lemma 59.71.4 it suffices to check this Zariski locally on Y and by Lemma 59.73.3 we may replace Y by an étale cover (the construction of f_* commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 41.18.3. Thus, in the case of sheaves of sets, the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are constructible sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. □

09Y9 **Lemma 59.73.5.** *Let X be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of $Sh(X_{étale})$ consisting of sheaves $\mathcal{F}$ which are coequalizers*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

such that $\mathcal{F}_i$, $i = 0, 1$ is a finite coproduct of sheaves of the form h_U with U a quasi-compact and quasi-separated object of $X_{étale}$.

Proof. In the proof of Lemma 59.73.2 we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets $\mathcal{F}$ we can find a surjection $\mathcal{F}_0 \to \mathcal{F}$ with $\mathcal{F}_0$ as in the lemma. Then we find our surjection $\mathcal{F}_1 \to \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ because the latter is constructible by Lemma 59.71.6.

By Topology, Lemma 5.28.7 we may choose a finite stratification $X = \coprod_{i \in I} X_i$ such that $\mathcal{F}$ is finite locally constant on each stratum. We will prove the result by induction on the cardinality of I. Let $i \in I$ be a minimal element in the partial

ordering of I. Then $X_i \subset X$ is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects U_α of $(X \setminus X_i)_{étale}$ and a surjective map $\coprod h_{U_\alpha} \to \mathcal{F}|_{X \setminus X_i}$. These determine a map

$$\coprod h_{U_\alpha} \to \mathcal{F}$$

which is surjective after restricting to $X \setminus X_i$. By Lemma 59.64.4 we see that $\mathcal{F}|_{X_i} = h_V$ for some scheme V finite étale over X_i. Let $\overline{v}$ be a geometric point of V lying over $\overline{x} \in X_i$. We may think of $\overline{v}$ as an element of the stalk $\mathcal{F}_{\overline{x}} = V_{\overline{x}}$. Thus we can find an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ and a section $s \in \mathcal{F}(U)$ whose stalk at $\overline{x}$ gives $\overline{v}$. Thinking of s as a map $s : h_U \to \mathcal{F}$, restricting to X_i we obtain a morphism $s|_{X_i} : U \times_X X_i \to V$ over X_i which maps $\overline{u}$ to $\overline{v}$. Since V is quasi-compact (finite over the closed subscheme X_i of the quasi-compact scheme X) a finite number $s^{(1)}, \ldots, s^{(m)}$ of these sections of $\mathcal{F}$ over $U^{(1)}, \ldots, U^{(m)}$ will determine a jointly surjective map

$$\coprod s^{(j)}|_{X_i} : \coprod U^{(j)} \times_X X_i \longrightarrow V$$

Then we obtain the surjection

$$\coprod h_{U_\alpha} \amalg \coprod h_{U^{(j)}} \to \mathcal{F}$$

as desired. □

095N **Lemma 59.73.6.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules is exactly the category of modules of the form*

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. In fact, we can even assume U_i and V_j affine.

Proof. In the proof of Lemma 59.73.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 59.71.6) it suffices to prove that given a constructible module $\mathcal{F}$ there is a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

for some affine objects U_i in $X_{étale}$. By Modules on Sites, Lemma 18.30.7 there is a surjection

$$\Psi : \bigoplus\nolimits_{i \in I} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

with U_i affine and the direct sum over a possibly infinite index set I. For every finite subset $I' \subset I$ set

$$T_{I'} = \mathrm{Supp}(\mathrm{Coker}(\bigoplus\nolimits_{i \in I'} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}))$$

By the very definition of constructible sheaves, the set $T_{I'}$ is a constructible subset of X. We want to show that $T_{I'} = \emptyset$ for some I'. Since every stalk $\mathcal{F}_{\overline{x}}$ is a finite type Λ-module and since Ψ is surjective, for every $x \in X$ there is an I' such that $x \notin T_{I'}$. In other words we have $\emptyset = \bigcap_{I' \subset I \text{ finite}} T_{I'}$. Since X is a spectral space by Properties, Lemma 28.2.4 the constructible topology on X is quasi-compact by Topology, Lemma 5.23.2. Thus $T_{I'} = \emptyset$ for some $I' \subset I$ finite as desired. □

09YT **Lemma 59.73.7.** *Let X be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form*

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\mathbf{Z}/m_j\mathbf{Z}}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\mathbf{Z}/n_i\mathbf{Z}}_{U_i}\right)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$ and m_j, n_i positive integers. In fact, we can even assume U_i and V_j affine.

Proof. This follows from Lemma 59.73.6 applied with $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and the fact that, since X is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer n (details omitted). □

09Z4 **Lemma 59.73.8.** *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$. Let $\mathcal{G} = \operatorname{colim} \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or Λ-modules. Then*

$$\mathrm{Mor}(\mathcal{F}, \mathcal{G}) = \operatorname{colim} \mathrm{Mor}(\mathcal{F}, \mathcal{G}_i)$$

in the category of sheaves of sets, abelian groups, or Λ-modules on $X_{étale}$.

Proof. The case of sheaves of sets. By Lemma 59.73.5 it suffices to prove the lemma for h_U where U is a quasi-compact and quasi-separated object of $X_{étale}$. Recall that $\mathrm{Mor}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 7.17.7.

In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 59.73.7 and 59.73.6. For the case of abelian sheaves, we add that $\mathrm{Mor}(j_{U!}\underline{\mathbf{Z}/n\mathbf{Z}}, \mathcal{G})$ is equal to the n-torsion elements of $\mathcal{G}(U)$. □

095R **Lemma 59.73.9.** *Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$, then $f_*\mathcal{F}$ is too.*

Proof. It suffices to prove this when X and Y are affine by Lemma 59.71.4. By Lemmas 59.55.3 and 59.73.3 we may base change to any affine scheme surjective over X. By Lemma 59.72.3 this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 59.45.2). The finite étale case is Lemma 59.73.4. □

09YU **Lemma 59.73.10.** *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.*

1. *The category of constructible sheaves of sets on $X_{étale}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{étale}$.*
2. *The category of constructible abelian sheaves on $X_{étale}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{étale}$.*
3. *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is the colimit of the categories of constructible sheaves of Λ-modules on $(X_i)_{étale}$.*

Proof. Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to "faithful", "fully faithful", and "essentially surjective".

Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_{\overline{x}} = b_{\overline{x}}$. By Lemma 59.71.7 the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into E. By Limits, Lemma 32.4.10 we find an $i \geq 0$ such that $X_i \to X_0$ maps into E. Hence $f_{i0}^{-1}a = f_{i0}^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a : f_0^{-1}\mathcal{F}_0 \to f_0^{-1}\mathcal{G}_0$ is a map. We claim there is an i and a map $a_i : f_{i0}^{-1}\mathcal{F}_0 \to f_{i0}^{-1}\mathcal{G}_0$ which pulls back to a on X. By Lemma 59.73.5 we can replace $\mathcal{F}_0$ by a finite coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{étale}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{étale}$, then

$$f_0^{-1}\mathcal{G}(U) = \operatorname{colim}_{i \geq 0} f_{i0}^{-1}\mathcal{G}(U_i)$$

where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. This is a special case of Theorem 59.51.3.

Essentially surjective. We have to show every constructible $\mathcal{F}$ on X is isomorphic to $f_i^{-1}\mathcal{F}$ for some constructible $\mathcal{F}_i$ on X_i. Applying Lemma 59.73.5 and using the results of the previous two paragraphs, we see that it suffices to prove this for h_U for some quasi-compact and quasi-separated object U of $X_{étale}$. In this case we have to show that U is the base change of a quasi-compact and quasi-separated scheme étale over X_i for some i. This follows from Limits, Lemmas 32.10.1 and 32.8.10.

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 59.73.6 instead of Lemma 59.73.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n. □

0GL2 **Lemma 59.73.11.** *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.*

(1) *The category of finite locally constant sheaves on $X_{étale}$ is the colimit of the categories of finite locally constant sheaves on $(X_i)_{étale}$.*
(2) *The category of finite locally constant abelian sheaves on $X_{étale}$ is the colimit of the categories of finite locally constant abelian sheaves on $(X_i)_{étale}$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{étale}$ is the colimit of the categories of finite type, locally constant sheaves of Λ-modules on $(X_i)_{étale}$.*

Proof. By Lemma 59.73.10 the functor in each case is fully faithful. By the same lemma, all we have to show to finish the proof in case (1) is the following: given a constructible sheaf $\mathcal{F}_i$ on X_i whose pullback $\mathcal{F}$ to X is finite locally constant, there exists an $i' \geq i$ such that the pullback $\mathcal{F}_{i'}$ of $\mathcal{F}_i$ to $X_{i'}$ is finite locally constant. By assumption there exists an étale covering $\mathcal{U} = \{U_j \to X\}_{j \in J}$ such that $\mathcal{F}|_{U_j} \cong \underline{S_j}$ for some finite set S_j. We may assume U_j is affine for all $j \in J$. Since X is quasi-compact, we may assume J finite. By Lemma 59.51.2 we can find an $i' \geq i$ and an étale covering $\mathcal{U}_{i'} = \{U_{i',j} \to X_{i'}\}_{j \in J}$ whose base change to X is $\mathcal{U}$. Then $\mathcal{F}_{i'}|_{U_{i',j}}$ and $\underline{S_j}$ are constructible sheaves on $(U_{i',j})_{étale}$ whose pullbacks to U_j are isomorphic. Hence after increasing i' we get that $\mathcal{F}_{i'}|_{U_{i',j}}$ and $\underline{S_j}$ are isomorphic.

Thus $\mathcal{F}_{i'}$ is finite locally constant. The proof in cases (2) and (3) is exactly the same. □

09BG **Lemma 59.73.12.** *Let X be an irreducible scheme with generic point η.*

(1) *Let $S' \subset S$ be an inclusion of sets. If we have $\underline{S'} \subset \mathcal{G} \subset \underline{S}$ in $Sh(X_{étale})$ and $S' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{S'}$.*
(2) *Let $A' \subset A$ be an inclusion of abelian groups. If we have $\underline{A'} \subset \mathcal{G} \subset \underline{A}$ in $Ab(X_{étale})$ and $A' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{A'}$.*
(3) *Let $M' \subset M$ be an inclusion of modules over a ring Λ. If we have $\underline{M'} \subset \mathcal{G} \subset \underline{M}$ in $Mod(X_{étale}, \underline{\Lambda})$ and $M' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{M'}$.*

Proof. This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point η is in the image. □

09Z5 **Lemma 59.73.13.** *Let X be an integral normal scheme with function field K. Let E be a set.*

(1) *Let $g : \mathrm{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_*\underline{E} = \underline{E}$.*
(2) *Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_*\underline{E} = \underline{E}$.*

Proof. Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}^{sh}_{X,\overline{x}}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 15.45.6 we see that $\mathcal{O}^{sh}_{X,\overline{x}}$ is a normal domain. Hence $\mathrm{Spec}(K) \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ is irreducible. It follows that the stalk $(g_*\underline{E})_{\overline{x}}$ is equal to E, see Theorem 59.53.1.

Proof of (2). Since g factors through j there is a map $j_*\underline{E} \to g_*\underline{E}$. This map is injective because for every scheme V étale over X the set $\mathrm{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $\underline{E} \to j_*\underline{E}$ and we conclude. □

0F0M **Lemma 59.73.14.** *Let X be a quasi-compact and quasi-separated scheme. Let $\eta \in X$ be a generic point of an irreducible component of X.*

(1) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.*
(2) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$ whose stalk $\mathcal{F}_{\overline{\eta}}$ is zero. Then $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subscheme not containing η.*

Proof. Proof of (1). We can write $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 59.73.2. Choose $i \in I$. Since $\mathcal{F}|_\eta$ is zero by assumption, we see that there exists an $i'(i) \geq i$ such that $\mathcal{F}_i|_\eta \to \mathcal{F}_{i'(i)}|_\eta$ is zero, see Lemma 59.71.8. Then $\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F}_{i'(i)})$ is a constructible abelian sheaf (Lemma 59.71.6) whose stalk at η is zero. Hence the support E_i of $\mathcal{G}_i$ is a constructible subset of X not containing η. Since η is a generic point of an irreducible component of X, we see that $\eta \notin Z_i = \overline{E_i}$ by Topology, Lemma 5.15.15. Define a new directed set I' by using the set I with ordering defined by the rule i_1 is bigger or equal to i_2 if and only if $i_1 \geq i'(i_2)$. Then the sheaves $\mathcal{G}_i$ form a system over I' with colimit $\mathcal{F}$ and the proof is complete.

The proof in case (2) is exactly the same and we omit it. □

59.74. Constructible sheaves on Noetherian schemes

03RY If X is a Noetherian scheme then any locally closed subset is a constructible locally closed subset (Topology, Lemma 5.16.1). Hence an abelian sheaf $\mathcal{F}$ on $X_{étale}$ is constructible if and only if there exists a finite partition $X = \coprod X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant. (By convention a partition of a topological space has locally closed parts, see Topology, Section 5.28.) In other words, we can omit the adjective "constructible" in Definition 59.71.1. Actually, the category of constructible sheaves on Noetherian schemes has some additional properties which we will catalogue in this section.

09BH **Proposition 59.74.1.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring.*

(1) *Any sub or quotient sheaf of a constructible sheaf of sets is constructible.*
(2) *The category of constructible abelian sheaves on $X_{étale}$ is a (strong) Serre subcategory of $Ab(X_{étale})$. In particular, every sub and quotient sheaf of a constructible abelian sheaf on $X_{étale}$ is constructible.*
(3) *The category of constructible sheaves of Λ-modules on $X_{étale}$ is a (strong) Serre subcategory of $Mod(X_{étale}, \Lambda)$. In particular, every submodule and quotient module of a constructible sheaf of Λ-modules on $X_{étale}$ is constructible.*

Proof. Proof of (1). Let $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{S}$ for some finite set S. Then $S' = \mathcal{G}_{\overline{\eta}} \subset S$ say $S' = \{s_1, \ldots, s_t\}$. Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset S$. Since σ_i maps to an element $s_i \in S' \subset S = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{S'} \subset \mathcal{G} \subset \underline{S}$. Then we see that $\underline{S'} = \mathcal{G}$ by Lemma 59.73.12.

Let $\mathcal{F} \to \mathcal{Q}$ be a surjection with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Then set $\mathcal{G} = \mathcal{F} \times_{\mathcal{Q}} \mathcal{F}$. By the first part of the proof we see that $\mathcal{G}$ is constructible as a subsheaf of $\mathcal{F} \times \mathcal{F}$. This in turn implies that $\mathcal{Q}$ is constructible, see Lemma 59.71.6.

Proof of (3). we already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 59.71.6. Thus it suffices to show the statement on submodules.

Let $\mathcal{G} \subset \mathcal{F}$ be a submodule of a constructible sheaf of Λ-modules on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{M}$ for some finite Λ-module M. Then $M' = \mathcal{G}_{\overline{\eta}} \subset M$. Pick finitely many elements $s_1, \ldots, s_t$ generating M' as a Λ-module. (This is possible as Λ is Noetherian and M is finite.) Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections

$\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset M$. Since σ_i maps to an element $s_i \in M' \subset M = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{M'} \subset \mathcal{G} \subset \underline{M}$. Then we see that $\underline{M'} = \mathcal{G}$ by Lemma 59.73.12.

Proof of (2). This follows in the usual manner from (3). Details omitted. □

The following lemma tells us that every object of the abelian category of constructible sheaves on X is "Noetherian", i.e., satisfies a.c.c. for subobjects.

09YV **Lemma 59.74.2.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring. Consider inclusions*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots \subset \mathcal{F}$$

in the category of sheaves of sets, abelian groups, or Λ-modules. If $\mathcal{F}$ is constructible, then for some n we have $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_{n+2} = \ldots$.

Proof. By Proposition 59.74.1 we see that $\mathcal{F}_i$ and $\operatorname{colim} \mathcal{F}_i$ are constructible. Then the lemma follows from Lemma 59.71.8. □

09Z6 **Lemma 59.74.3.** *Let X be a Noetherian scheme.*

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite abelian group.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite Λ-module.

Moreover, we may assume each Y_i is irreducible, reduced, maps onto an irreducible and reduced closed subscheme $Z_i \subset X$ such that $Y_i \to Z_i$ is finite étale over a nonempty open of Z_i.

Proof. Proof of (1). Because we have the ascending chain condition for subsheaves of $\mathcal{F}$ (Lemma 59.74.2), it suffices to show that for every point $x \in X$ we can find a map $\varphi : \mathcal{F} \to f_*\underline{E}$ where $f : Y \to X$ is finite and E is a finite set such that $\varphi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to (f_*S)_{\overline{x}}$ is injective. (This argument can be avoided by picking a partition of X as in Lemma 59.71.2 and constructing a $Y_i \to X$ for each irreducible component of each part.) Let $Z \subset X$ be the induced reduced scheme structure (Schemes, Definition 26.12.5) on $\overline{\{x\}}$. Since $\mathcal{F}$ is constructible, there is a finite separable extension $K/\kappa(x)$ such that $\mathcal{F}|_{\mathrm{Spec}(K)}$ is the constant sheaf with value E for some finite set E. Let $Y \to Z$ be the normalization of Z in $\mathrm{Spec}(K)$. By Morphisms, Lemma 29.53.13 we see that Y is a normal integral scheme. As $K/\kappa(x)$ is a finite extension, it is clear that K is the function field of Y. Denote

$g : \mathrm{Spec}(K) \to Y$ the inclusion. The map $\mathcal{F}|_{\mathrm{Spec}(K)} \to \underline{E}$ is adjoint to a map $\mathcal{F}|_Y \to g_*\underline{E} = \underline{E}$ (Lemma 59.73.13). This in turn is adjoint to a map $\varphi : \mathcal{F} \to f_*\underline{E}$. Observe that the stalk of φ at a geometric point $\overline{x}$ is injective: we may take a lift $\overline{y} \in Y$ of $\overline{x}$ and the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\overline{x}} & = & (\mathcal{F}|_Y)_{\overline{y}} \\ \downarrow & & \| \\ (f_*\underline{E})_{\overline{x}} & \longrightarrow & \underline{E}_{\overline{y}} \end{array}$$

proves the injectivity. We are not yet done, however, as the morphism $f : Y \to Z$ is integral but in general not finite[6].

To fix the problem stated in the last sentence of the previous paragraph, we write $Y = \lim_{i \in I} Y_i$ with Y_i irreducible, integral, and finite over Z. Namely, apply Properties, Lemma 28.22.13 to $f_*\mathcal{O}_Y$ viewed as a sheaf of $\mathcal{O}_Z$-algebras and apply the functor $\underline{\mathrm{Spec}}_Z$. Then $f_*\underline{E} = \mathrm{colim}\, f_{i,*}\underline{E}$ by Lemma 59.51.7. By Lemma 59.73.8 the map $\mathcal{F} \to f_*\underline{E}$ factors through $f_{i,*}\underline{E}$ for some i. Since $Y_i \to Z$ is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of Z (use Algebra, Lemma 10.140.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1). □

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

09Z7 **Lemma 59.74.4.** *Let X be a quasi-compact and quasi-separated scheme.* [**AGV71**, Exposee IX, Proposition 2.14]

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite abelian group.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite Λ-module.

Proof. We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 32.5.4 we can write $X = \lim_{t \in T} X_t$ with each X_t of finite type over $\mathrm{Spec}(\mathbf{Z})$ and with affine transition morphisms. By

[6] If X is a Nagata scheme, for example of finite type over a field, then $Y \to Z$ is finite.

Lemma 59.73.10 the category of constructible sheaves (of sets, abelian groups, or Λ-modules) on $X_{étale}$ is the colimit of the corresponding categories for X_t. Thus our constructible sheaf $\mathcal{F}$ is the pullback of a similar constructible sheaf $\mathcal{F}_t$ over X_t for some t. Then we apply the Noetherian case (Lemma 59.74.3) to find an injection

$$\mathcal{F}_t \longrightarrow \prod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i} \quad \text{or} \quad \mathcal{F}_t \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$

over X_t for some finite morphisms $f_i : Y_i \to X_t$. Since X_t is Noetherian the morphisms f_i are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 59.55.3), we conclude. □

0F0N **Lemma 59.74.5.** *Let X be a Noetherian scheme. Let $E \subset X$ be a subset closed under specialization.*

(1) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ is a filtered colimit of constructible abelian sheaves $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.*

(2) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$ whose support is contained in E. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ is a filtered colimit of constructible sheaves of Λ-modules $\mathcal{F}_i$ such that for each i the support of $\mathcal{F}_i$ is contained in a closed subset contained in E.*

Proof. Proof of (1). We can write $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ with $\mathcal{F}_i$ constructible abelian by Lemma 59.73.2. By Proposition 59.74.1 the image $\mathcal{F}'_i \subset \mathcal{F}$ of the map $\mathcal{F}_i \to \mathcal{F}$ is constructible. Thus $\mathcal{F} = \operatorname{colim} \mathcal{F}'_i$ and the support of $\mathcal{F}'_i$ is contained in E. Since the support of $\mathcal{F}'_i$ is constructible (by our definition of constructible sheaves), we see that its closure is also contained in E, see for example Topology, Lemma 5.23.6.

The proof in case (2) is exactly the same and we omit it. □

59.75. Specializations and étale sheaves

0GJ2 Topological picture: Let X be a topological space and let $x' \rightsquigarrow x$ be a specialization of points in X. Then every open neighbourhood of x contains x'. Hence for any sheaf $\mathcal{F}$ on X there is a *specialization map*

$$sp : \mathcal{F}_x \longrightarrow \mathcal{F}_{x'}$$

of stalks sending the equivalence class of the pair (U, s) in $\mathcal{F}_x$ to the equivalence class of the pair (U, s) in $\mathcal{F}_{x'}$; see Sheaves, Section 6.11 for the description of stalks in terms of equivalence classes of pairs. Of course this map is functorial in $\mathcal{F}$, i.e., sp is a transformation of functors.

For sheaves in the étale topology we can mimic this construction, see [**AGV71**, Exposee VIII, 7.7, page 397]. To do this suppose we have a scheme S, a geometric point $\overline{s}$ of S, and a geometric point $\overline{t}$ of $\operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. For any sheaf $\mathcal{F}$ on $S_{étale}$ we will construct the *specialization map*

$$sp : \mathcal{F}_{\overline{s}} \longrightarrow \mathcal{F}_{\overline{t}}$$

Here we have abused language: instead of writing $\mathcal{F}_{\overline{t}}$ we should write $\mathcal{F}_{p(\overline{t})}$ where $p : \operatorname{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to S$ is the canonical morphism. Recall that

$$\mathcal{F}_{\overline{s}} = \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit is over all étale neighbourhoods $(U, \overline{u})$ of $(S, \overline{s})$, see Section 59.29. Since $\mathcal{O}^{sh}_{S,\overline{s}}$ is the stalk of the structure sheaf, we find for every étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ a canonical map $\mathcal{O}_{U,u} \to \mathcal{O}^{sh}_{S,\overline{s}}$. Hence we get a unique factorization

$$\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to U \to S$$

If $\overline{v}$ denotes the image of $\overline{t}$ in U, then we see that $(U, \overline{v})$ is an étale neighbourhood of $(S, \overline{t})$. This construction defines a functor from the category of étale neighbourhoods of $(S, \overline{s})$ to the category of étale neighbourhoods of $(S, \overline{t})$. Thus we may define the map $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ by sending the equivalence class of $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$ to the equivalence class of $(U, \overline{v}, \sigma)$.

Let $K \in D(S_{étale})$. With $\overline{s}$ and $\overline{t}$ as above we have the *specialization map*

$$sp : K_{\overline{s}} \longrightarrow K_{\overline{t}} \quad \text{in} \quad D(Ab)$$

Namely, if K is represented by the complex $\mathcal{F}^\bullet$ of abelian sheaves, then we simply that the map

$$K_{\overline{s}} = \mathcal{F}^\bullet_{\overline{s}} \longrightarrow \mathcal{F}^\bullet_{\overline{t}} = K_{\overline{t}}$$

which is termwise given by the specialization maps for sheaves constructed above. This is independent of the choice of complex representing K by the exactness of the stalk functors (i.e., taking stalks of complexes is well defined on the derived category).

Clearly the construction is functorial in the sheaf $\mathcal{F}$ on $S_{étale}$. If we think of the stalk functors as morphisms of topoi $\overline{s}, \overline{t} : Sets \to Sh(S_{étale})$, then we may think of sp as a 2-morphism

$$Sets \;\underset{\overline{s}}{\overset{\overline{t}}{\Downarrow sp}}\; Sh(S_{étale})$$

of topoi.

0GJ3 **Remark 59.75.1** (Alternative description of sp)**.** Let S, $\overline{s}$, and $\overline{t}$ be as above. Another way to describe the specialization map is to use that

$$\mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F}) \quad \text{and} \quad \mathcal{F}_{\overline{t}} = \Gamma(\overline{t}, \overline{t}^{-1}p^{-1}\mathcal{F})$$

The first equality follows from Theorem 59.53.1 applied to $\mathrm{id}_S : S \to S$ and the second equality follows from Lemma 59.36.2. Then we can think of sp as the map

$$sp : \mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1}\mathcal{F}) \xrightarrow{\text{pullback by } \overline{t}} \Gamma(\overline{t}, \overline{t}^{-1}p^{-1}\mathcal{F}) = \mathcal{F}_{\overline{t}}$$

0GJ4 **Remark 59.75.2** (Yet another description of sp)**.** Let S, $\overline{s}$, and $\overline{t}$ be as above. Another alternative is to use the unique morphism

$$c : \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}) \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$$

over S which is compatible with the given morphism $\overline{t} \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ and the morphism $\overline{t} \to \mathrm{Spec}(\mathcal{O}^{sh}_{t,\overline{t}})$. The uniqueness and existence of the displayed arrow follows from Algebra, Lemma 10.154.6 applied to $\mathcal{O}_{S,s}$, $\mathcal{O}^{sh}_{S,\overline{t}}$, and $\mathcal{O}^{sh}_{S,\overline{s}} \to \kappa(\overline{t})$. We obtain

$$sp : \mathcal{F}_{\overline{s}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), \mathcal{F}) \xrightarrow{\text{pullback by } c} \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), \mathcal{F}) = \mathcal{F}_{\overline{t}}$$

(with obvious notational conventions). In fact this procedure also works for objects K in $D(S_{étale})$: the specialization map for K is equal to the map

$$sp : K_{\overline{s}} = R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) \xrightarrow{\text{pullback by } c} R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) = K_{\overline{t}}$$

The equality signs are valid as taking global sections over the strictly henselian schemes $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ and $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}})$ is exact (and the same as taking stalks at $\overline{s}$ and $\overline{t}$) and hence no subtleties related to the fact that K may be unbounded arise.

0GJ5 **Remark 59.75.3** (Lifting specializations)**.** Let S be a scheme and let $t \rightsquigarrow s$ be a specialization of point on S. Choose geometric points $\overline{t}$ and $\overline{s}$ lying over t and s. Since t corresponds to a point of $\mathrm{Spec}(\mathcal{O}_{S,s})$ by Schemes, Lemma 26.13.2 and since $\mathcal{O}_{S,s} \to \mathcal{O}^{sh}_{S,\overline{s}}$ is faithfully flat, we can find a point $t' \in \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ mapping to t. As $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is a limit of schemes étale over S we see that $\kappa(t')/\kappa(t)$ is a separable algebraic extension (usually not finite of course). Since $\kappa(\overline{t})$ is algebraically closed, we can choose an embedding $\kappa(t') \to \kappa(\overline{t})$ as extensions of $\kappa(t)$. This choice gives us a commutative diagram

$$\begin{array}{ccccc} \overline{t} & \longrightarrow & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) & \longleftarrow & \overline{s} \\ \downarrow & & \downarrow & & \downarrow \\ t & \longrightarrow & S & \longleftarrow & s \end{array}$$

of points and geometric points. Thus if $t \rightsquigarrow s$ we can always "lift" $\overline{t}$ to a geometric point of the strict henselization of S at $\overline{s}$ and get specialization maps as above.

0GJ6 **Lemma 59.75.4.** *Let $g : S' \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. Let $\overline{s}'$ be a geometric point of S', and let $\overline{t}'$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'})$. Denote $\overline{s} = g(\overline{s}')$ and $\overline{t} = h(\overline{t}')$ where $h : \mathrm{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the canonical morphism. For any sheaf $\mathcal{F}$ on $S_{étale}$ the specialization map*

$$sp : (f^{-1}\mathcal{F})_{\overline{s}'} \longrightarrow (f^{-1}\mathcal{F})_{\overline{t}'}$$

is equal to the specialization map $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ via the identifications $(f^{-1}\mathcal{F})_{\overline{s}'} = \mathcal{F}_{\overline{s}}$ and $(f^{-1}\mathcal{F})_{\overline{t}'} = \mathcal{F}_{\overline{t}}$ of Lemma 59.36.2.

Proof. Omitted. □

0GJ7 **Lemma 59.75.5.** *Let S be a scheme such that every quasi-compact open of S has finite number of irreducible components (for example if S has a Noetherian underlying topological space, or if S is locally Noetherian). Let $\mathcal{F}$ be a sheaf of sets on $S_{étale}$. The following are equivalent*

(1) *$\mathcal{F}$ is finite locally constant, and*
(2) *all stalks of $\mathcal{F}$ are finite sets and all specialization maps $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ are bijective.*

Proof. Assume (2). Let $\overline{s}$ be a geometric point of S lying over $s \in S$. In order to prove (1) we have to find an étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ such that $\mathcal{F}|_U$ is constant. We may and do assume S is affine.

Since $\mathcal{F}_{\overline{s}}$ is finite, we can choose $(U, \overline{u})$, $n \geq 0$, and pairwise distinct elements $\sigma_1, \ldots, \sigma_n \in \mathcal{F}(U)$ such that $\{\sigma_1, \ldots, \sigma_n\} \subset \mathcal{F}(U)$ maps bijectively to $\mathcal{F}_{\overline{s}}$ via the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$. Consider the map

$$\varphi : \underline{\{1, \ldots, n\}} \longrightarrow \mathcal{F}|_U$$

on $U_{étale}$ defined by $\sigma_1, \ldots, \sigma_n$. This map is a bijection on stalks at $\overline{u}$ by construction. Let us consider the subset

$$E = \{u' \in U \mid \varphi_{\overline{u}'} \text{ is bijective}\} \subset U$$

Here $\overline{u}'$ is any geometric point of U lying over u' (the condition is independent of the choice by Remark 59.29.8). The image $u \in U$ of $\overline{u}$ is in E. By our assumption on the specialization maps for $\mathcal{F}$, by Remark 59.75.3, and by Lemma 59.75.4 we see that E is closed under specializations and generalizations in the topological space U.

After shrinking U we may assume U is affine too. By Descent, Lemma 35.16.3 we see that U has a finite number of irreducible components. After removing the irreducible components which do not pass through u, we may assume every irreducible component of U passes through u. Since U is a sober topological space it follows that $E = U$ and we conclude that φ is an isomorphism by Theorem 59.29.10. Thus (1) follows.

We omit the proof that (1) implies (2). □

0GKC **Lemma 59.75.6.** *Let S be a scheme such that every quasi-compact open of S has finite number of irreducible components (for example if S has a Noetherian underlying topological space, or if S is locally Noetherian). Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $S_{étale}$. The following are equivalent*

(1) *$\mathcal{F}$ is a finite type, locally constant sheaf of Λ-modules, and*
(2) *all stalks of $\mathcal{F}$ are finite Λ-modules and all specialization maps $sp : \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{t}}$ are bijective.*

Proof. The proof of this lemma is the same as the proof of Lemma 59.75.5. Assume (2). Let $\overline{s}$ be a geometric point of S lying over $s \in S$. In order to prove (1) we have to find an étale neighbourhood $(U, \overline{u})$ of $(S, \overline{s})$ such that $\mathcal{F}|_U$ is constant. We may and do assume S is affine.

Since $M = \mathcal{F}_{\overline{s}}$ is a finite Λ-module and Λ is Noetherian, we can choose a presentation

$$\Lambda^{\oplus m} \xrightarrow{A} \Lambda^{\oplus n} \to M \to 0$$

for some matrix $A = (a_{ji})$ with coefficients in Λ. We can choose $(U, \overline{u})$ and elements $\sigma_1, \ldots, \sigma_n \in \mathcal{F}(U)$ such that $\sum a_{ji}\sigma_i = 0$ in $\mathcal{F}(U)$ and such that the images of σ_i in $\mathcal{F}_{\overline{s}} = M$ are the images of the standard basis element of Λ^n in the presentation of M given above. Consider the map

$$\varphi : \underline{M} \longrightarrow \mathcal{F}|_U$$

on $U_{étale}$ defined by $\sigma_1, \ldots, \sigma_n$. This map is a bijection on stalks at $\overline{u}$ by construction. Let us consider the subset

$$E = \{u' \in U \mid \varphi_{\overline{u}'} \text{ is bijective}\} \subset U$$

Here $\overline{u}'$ is any geometric point of U lying over u' (the condition is independent of the choice by Remark 59.29.8). The image $u \in U$ of $\overline{u}$ is in E. By our assumption on the specialization maps for $\mathcal{F}$, by Remark 59.75.3, and by Lemma 59.75.4 we see that E is closed under specializations and generalizations in the topological space U.

After shrinking U we may assume U is affine too. By Descent, Lemma 35.16.3 we see that U has a finite number of irreducible components. After removing

the irreducible components which do not pass through u, we may assume every irreducible component of U passes through u. Since U is a sober topological space it follows that $E = U$ and we conclude that φ is an isomorphism by Theorem 59.29.10. Thus (1) follows.

We omit the proof that (1) implies (2). □

0GJ8 **Lemma 59.75.7.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $K \in D^+(X_{étale})$. Let $\overline{s}$ be a geometric point of S and let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. We have a commutative diagram*

$$\begin{array}{ccc} (Rf_*K)_{\overline{s}} & \xrightarrow{\quad sp \quad} & (Rf_*K)_{\overline{t}} \\ \| & & \| \\ R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) & \longrightarrow & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) \end{array}$$

where the bottom horizontal arrow arises as pullback by the morphism $\mathrm{id}_X \times c$ where $c : \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the morphism introduced in Remark 59.75.2. The vertical arrows are given by Theorem 59.53.1.

Proof. This follows immediately from the description of *sp* in Remark 59.75.2. □

0GJ9 **Remark 59.75.8.** Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{étale})$. Let $\overline{s}$ be a geometric point of S and let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Let c be as in Remark 59.75.2. We can always make a commutative diagram

$$\begin{array}{ccccc} (Rf_*K)_{\overline{s}} & \longrightarrow & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), K) & \longrightarrow & R\Gamma(X_{\overline{s}}, K) \\ {\scriptstyle sp}\downarrow & & {\scriptstyle (\mathrm{id}_X \times c)^{-1}}\downarrow & & \\ (Rf_*K)_{\overline{t}} & \longrightarrow & R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{t}}), K) & \longrightarrow & R\Gamma(X_{\overline{t}}, K) \end{array}$$

where the horizontal arrows are those of Remark 59.53.2. In general there won't be a vertical map on the right between the cohomologies of K on the fibres fitting into this diagram, even in the case of Lemma 59.75.7.

59.76. Complexes with constructible cohomology

095V Let Λ be a ring. Denote $D(X_{étale}, \Lambda)$ the derived category of sheaves of Λ-modules on $X_{étale}$. We denote by $D^b(X_{étale}, \Lambda)$ (respectively D^+, D^-) the full subcategory of bounded (resp. above, below) complexes in $D(X_{étale}, \Lambda)$.

095W **Definition 59.76.1.** Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_c(X_{étale}, \Lambda)$ the full subcategory of $D(X_{étale}, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of Λ-modules.

This definition makes sense by Lemma 59.71.6 and Derived Categories, Section 13.17. Thus we see that $D_c(X_{étale}, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D(X_{étale}, \Lambda)$.

095X **Lemma 59.76.2.** *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

(1) *$K|_U \in D_c(U_{étale}, \Lambda)$ if $K \in D_c(X_{étale}, \Lambda)$, and*

(2) $j_!M \in D_c(X_{étale}, \Lambda)$ *if* $M \in D_c(U_{étale}, \Lambda)$ *and the morphism* j *is quasi-compact and quasi-separated.*

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 59.73.1. □

095Y **Lemma 59.76.3.** *Let* Λ *be a Noetherian ring. Let* $f : X \to Y$ *be a morphism of schemes. If* $K \in D_c(Y_{étale}, \Lambda)$ *then* $Lf^*K \in D_c(X_{étale}, \Lambda)$.

Proof. This follows as $f^{-1} = f^*$ is exact and Lemma 59.71.5. □

095Z **Lemma 59.76.4.** *Let* X *be a quasi-compact and quasi-separated scheme. Let* Λ *be a Noetherian ring. Let* $K \in D(X_{étale}, \Lambda)$ *and* $b \in \mathbf{Z}$ *such that* $H^b(K)$ *is constructible. Then there exist a sheaf* $\mathcal{F}$ *which is a finite direct sum of* $j_{U!}\underline{\Lambda}$ *with* $U \in \mathrm{Ob}(X_{étale})$ *affine and a map* $\mathcal{F}[-b] \to K$ *in* $D(X_{étale}, \Lambda)$ *inducing a surjection* $\mathcal{F} \to H^b(K)$.

Proof. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Modules on Sites, Lemma 18.30.6 we may choose a surjection $\bigoplus_{i\in I} j_{U_i!}\underline{\Lambda} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$ with U_i affine. For $I' \subset I$ finite, denote $\mathcal{H}_{I'} \subset H^b(K)$ the image of $\bigoplus_{i\in I'} j_{U_i!}\underline{\Lambda}$. By Lemma 59.71.8 we see that $\mathcal{H}_{I'} = H^b(K)$ for some $I' \subset I$ finite. The lemma follows taking $\mathcal{F} = \bigoplus_{i\in I'} j_{U_i!}\underline{\Lambda}$. □

0960 **Lemma 59.76.5.** *Let* X *be a quasi-compact and quasi-separated scheme. Let* Λ *be a Noetherian ring. Let* $K \in D^-(X_{étale}, \Lambda)$. *Then the following are equivalent*

(1) K *is in* $D_c(X_{étale}, \Lambda)$,
(2) K *can be represented by a bounded above complex whose terms are finite direct sums of* $j_{U!}\underline{\Lambda}$ *with* $U \in \mathrm{Ob}(X_{étale})$ *affine,*
(3) K *can be represented by a bounded above complex of flat constructible sheaves of* Λ*-modules.*

Proof. It is clear that (2) implies (3) and that (3) implies (1). Assume K is in $D_c^-(X_{étale}, \Lambda)$. Say $H^i(K) = 0$ for $i > b$. By induction on a we will construct a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ such that each $\mathcal{F}^i$ is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}^\bullet \to K$ which induces an isomorphism $H^i(\mathcal{F}^\bullet) \to H^i(K)$ for $i > a$ and a surjection $H^a(\mathcal{F}^\bullet) \to H^a(K)$. For $a = b$ this can be done by Lemma 59.76.4. Given such a datum choose a distinguished triangle

$$\mathcal{F}^\bullet \to K \to L \to \mathcal{F}^\bullet[1]$$

Then we see that $H^i(L) = 0$ for $i \geq a$. Choose $\mathcal{F}^{a-1}[-a+1] \to L$ as in Lemma 59.76.4. The composition $\mathcal{F}^{a-1}[-a+1] \to L \to \mathcal{F}^\bullet$ corresponds to a map $\mathcal{F}^{a-1} \to \mathcal{F}^a$ such that the composition with $\mathcal{F}^a \to \mathcal{F}^{a+1}$ is zero. By TR4 we obtain a map

$$(\mathcal{F}^{a-1} \to \ldots \to \mathcal{F}^b) \to K$$

in $D(X_{étale}, \Lambda)$. This finishes the induction step and the proof of the lemma. □

0961 **Lemma 59.76.6.** *Let* X *be a scheme. Let* Λ *be a Noetherian ring. Let* $K, L \in D_c^-(X_{étale}, \Lambda)$. *Then* $K \otimes^{\mathbf{L}}_{\Lambda} L$ *is in* $D_c^-(X_{étale}, \Lambda)$.

Proof. This follows from Lemmas 59.76.5 and 59.71.9. □

59.77. Tor finite with constructible cohomology

0F4M Let X be a scheme and let Λ be a Noetherian ring. An often used subcategory of the derived category $D_c(X_{étale}, \Lambda)$ defined in Section 59.76 is the full subcategory consisting of objects having (locally) finite tor dimension. Here is the formal definition.

03TQ **Definition 59.77.1.** Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_{ctf}(X_{étale}, \Lambda)$ the full subcategory of $D_c(X_{étale}, \Lambda)$ consisting of objects having locally finite tor dimension.

This is a strictly full, saturated triangulated subcategory of $D_c(X_{étale}, \Lambda)$ and $D(X_{étale}, \Lambda)$. By our conventions, see Cohomology on Sites, Definition 21.46.1, we see that

$$D_{ctf}(X_{étale}, \Lambda) \subset D^b_c(X_{étale}, \Lambda) \subset D^b(X_{étale}, \Lambda)$$

if X is quasi-compact. A good way to think about objects of $D_{ctf}(X_{étale}, \Lambda)$ is given in Lemma 59.77.3.

03TS **Remark 59.77.2.** Objects in the derived category $D_{ctf}(X_{étale}, \Lambda)$ in some sense have better global properties than the perfect objects in $D(\mathcal{O}_X)$. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. The following lemma shows this does not happen for D_{ctf} on a Noetherian scheme.

03TT **Lemma 59.77.3.** *Let Λ be a Noetherian ring. Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(X_{étale}, \Lambda)$. The following are equivalent*

1. *$K \in D_{ctf}(X_{étale}, \Lambda)$, and*
2. *K can be represented by a finite complex of constructible flat sheaves of Λ-modules.*

In fact, if K has tor amplitude in $[a, b]$ then we can represent K by a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ with $\mathcal{F}^p$ a constructible flat sheaf of Λ-modules.

Proof. It is clear that a finite complex of constructible flat sheaves of Λ-modules has finite tor dimension. It is also clear that it is an object of $D_c(X_{étale}, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbf{Z}$ such that $H^i(K \otimes^{\mathbf{L}}_{\Lambda} \mathcal{G}) = 0$ if $i \notin [a, b]$ for all sheaves of Λ-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Lemma 59.73.6 we can find finitely many affine schemes U_i étale over X and a surjection $\bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to H^b(K)$. After replacing U_i by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $\mathcal{F} = \bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$. This map determines a distinguished triangle

$$\mathcal{F}[-b] \to K \to L \to \mathcal{F}[-b+1]$$

in $D(X_{étale}, \Lambda)$. Since $D_{ctf}(X_{étale}, \Lambda)$ is a triangulated subcategory we see that L is in it too. In fact L has tor amplitude in $[a, b-1]$ as $\mathcal{F}$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}$

of flat constructible sheaves of Λ-modules representing L. The map $L \to \mathcal{F}[-b+1]$ corresponds to a map $\mathcal{F}^b \to \mathcal{F}$ annihilating the image of $\mathcal{F}^{b-1} \to \mathcal{F}^b$. Then it follows from axiom TR3 that K is represented by the complex

$$\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1} \to \mathcal{F}^b$$

which finishes the proof. □

03TR **Remark 59.77.4.** Let Λ be a Noetherian ring. Let X be a scheme. For a bounded complex $K^\bullet$ of constructible flat Λ-modules on $X_{étale}$ each stalk $K^p_{\overline{x}}$ is a finite projective Λ-module. Hence the stalks of the complex are perfect complexes of Λ-modules.

0962 **Lemma 59.77.5.** *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

(1) *$K|_U \in D_{ctf}(U_{étale}, \Lambda)$ if $K \in D_{ctf}(X_{étale}, \Lambda)$, and*
(2) *$j_!M \in D_{ctf}(X_{étale}, \Lambda)$ if $M \in D_{ctf}(U_{étale}, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.*

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where X is affine and then apply Lemma 59.77.3 to translate it into a statement about finite complexes of flat constructible sheaves of Λ-modules where the result follows from Lemma 59.73.1. □

0963 **Lemma 59.77.6.** *Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_{ctf}(Y_{étale}, \Lambda)$ then $Lf^*K \in D_{ctf}(X_{étale}, \Lambda)$.*

Proof. Apply Lemma 59.77.3 to reduce this to a question about finite complexes of flat constructible sheaves of Λ-modules. Then the statement follows as $f^{-1} = f^*$ is exact and Lemma 59.71.5. □

09BI **Lemma 59.77.7.** *Let X be a connected scheme. Let Λ be a Noetherian ring. Let $K \in D_{ctf}(X_{étale}, \Lambda)$ have locally constant cohomology sheaves. Then there exists a finite complex of finite projective Λ-modules $M^\bullet$ and an étale covering $\{U_i \to X\}$ such that $K|_{U_i} \cong \underline{M^\bullet}|_{U_i}$ in $D(U_{i,étale}, \Lambda)$.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is constant, say $K|_{U_i} \cong \underline{M_i^\bullet}_{U_i}$ for some finite complex of finite Λ-modules $M_i^\bullet$. See Cohomology on Sites, Lemma 21.53.1. Observe that $U_i \times_X U_j$ is empty if $M_i^\bullet$ is not isomorphic to $M_j^\bullet$ in $D(\Lambda)$. For each complex of Λ-modules $M^\bullet$ let $I_{M^\bullet} = \{i \in I \mid M_i^\bullet \cong M^\bullet \text{ in } D(\Lambda)\}$. As étale morphisms are open we see that $U_{M^\bullet} = \bigcup_{i \in I_{M^\bullet}} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_{M^\bullet}$ is a disjoint open covering of X. As X is connected only one $U_{M^\bullet}$ is nonempty. As K is in $D_{ctf}(X_{étale}, \Lambda)$ we see that $M^\bullet$ is a perfect complex of Λ-modules, see More on Algebra, Lemma 15.75.2. Hence we may assume $M^\bullet$ is a finite complex of finite projective Λ-modules. □

59.78. Torsion sheaves

0DDB A brief section on torsion abelian sheaves and their étale cohomology. Let $\mathcal{C}$ be a site. We have shown in Cohomology on Sites, Lemma 21.19.8 that any object in $D(\mathcal{C})$ whose cohomology sheaves are torsion sheaves, can be represented by a complex all of whose terms are torsion.

0DDC **Lemma 59.78.1.** *Let X be a quasi-compact and quasi-separated scheme.*

(1) *If $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then $H^n_{étale}(X, \mathcal{F})$ is a torsion abelian group for all n.*
(2) *If K in $D^+(X_{étale})$ has torsion cohomology sheaves, then $H^n_{étale}(X, K)$ is a torsion abelian group for all n.*

Proof. To prove (1) we write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[d]$ is the d-torsion subsheaf. By Lemma 59.51.4 we have $H^n_{étale}(X, \mathcal{F}) = \operatorname{colim} H^n_{étale}(X, \mathcal{F}[d])$. This proves (1) as $H^n_{étale}(X, \mathcal{F}[d])$ is annihilated by d.

To prove (2) we can use the spectral sequence $E_2^{p,q} = H^p_{étale}(X, H^q(K))$ converging to $H^n_{étale}(X, K)$ (Derived Categories, Lemma 13.21.3) and the result for sheaves. □

0DDD **Lemma 59.78.2.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes.*
(1) *If $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then $R^n f_* \mathcal{F}$ is a torsion abelian sheaf on $Y_{étale}$ for all n.*
(2) *If K in $D^+(X_{étale})$ has torsion cohomology sheaves, then Rf_*K is an object of $D^+(Y_{étale})$ whose cohomology sheaves are torsion abelian sheaves.*

Proof. Proof of (1). Recall that $R^n f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^n_{étale}(X \times_Y V, \mathcal{F})$ on $Y_{étale}$. See Cohomology on Sites, Lemma 21.7.4. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 59.78.1 to see that $H^n_{étale}(X \times_Y V, \mathcal{F})$ is torsion.

Proof of (2). Recall that $R^n f_* K$ is the sheaf associated to the presheaf $V \mapsto H^n_{étale}(X \times_Y V, K)$ on $Y_{étale}$. See Cohomology on Sites, Lemma 21.20.6. If we choose V affine, then $X \times_Y V$ is quasi-compact and quasi-separated because f is, hence we can apply Lemma 59.78.1 to see that $H^n_{étale}(X \times_Y V, K)$ is torsion. □

59.79. Cohomology with support in a closed subscheme

09XP Let X be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \operatorname{Supp}(s) \subset Z\}$$

be the sections with support in Z (Definition 59.31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{étale}) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 21.12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{étale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{étale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the *subsheaf of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \operatorname{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Definition 59.31.3. Using the equivalence of Proposition 59.46.4 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$Ab(X_{étale}) \longrightarrow Ab(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

09XQ **Lemma 59.79.1.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\mathrm{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Section 59.46) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

09XR **Lemma 59.79.2.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 21.14.2). □

0A45 **Lemma 59.79.3.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $j : U \to X$ be the inclusion of the complement of Z. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. There is a distinguished triangle*

$$i_*R\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to Rj_*(\mathcal{F}|_U) \to i_*R\mathcal{H}_Z(\mathcal{F})[1]$$

in $D(X_{étale})$. This produces an exact sequence

$$0 \to i_*\mathcal{H}_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to i_*\mathcal{H}^1_Z(\mathcal{F}) \to 0$$

and isomorphisms $R^pj_*(\mathcal{F}|_U) \cong i_*\mathcal{H}^{p+1}_Z(\mathcal{F})$ for $p \geq 1$.

Proof. To get the distinguished triangle, choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then we obtain a short exact sequence of complexes

$$0 \to i_*\mathcal{H}_Z(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet \to j_*(\mathcal{I}^\bullet|_U) \to 0$$

by the discussion above. Thus the distinguished triangle by Derived Categories, Section 13.12. □

Let X be a scheme and let $Z \subset X$ be a closed subscheme. We denote $D_Z(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of complexes whose cohomology sheaves are supported on Z. Note that $D_Z(X_{étale})$ only depends on the underlying closed subset of X.

0AEG **Lemma 59.79.4.** *Let $i : Z \to X$ be a closed immersion of schemes. The map $Ri_{small,*} = i_{small,*} : D(Z_{étale}) \to D(X_{étale})$ induces an equivalence $D(Z_{étale}) \to D_Z(X_{étale})$ with quasi-inverse*

$$i_{small}^{-1}|_{D_Z(X_{étale})} = R\mathcal{H}_Z|_{D_Z(X_{étale})}$$

Proof. Recall that i_{small}^{-1} and $i_{small,*}$ is an adjoint pair of exact functors such that $i_{small}^{-1} i_{small,*}$ is isomorphic to the identify functor on abelian sheaves. See Proposition 59.46.4 and Lemma 59.36.2. Thus $i_{small,*} : D(Z_{étale}) \to D_Z(X_{étale})$ is fully faithful and i_{small}^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{étale})$ and consider the adjunction map $K \to i_{small,*} i_{small}^{-1} K$. Using exactness of $i_{small,*}$ and i_{small}^{-1} this induces the adjunction maps $H^n(K) \to i_{small,*} i_{small}^{-1} H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{étale}) \to D_Z(X_{étale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i_{small}^{-1} K$ if K is an object of $D_Z(X_{étale})$. To do this we can use that $K = i_{small,*} i_{small}^{-1} K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i_{small}^{-1} K$. Since $i_{small,*}$ is the right adjoint to the exact functor i_{small}^{-1}, the complex $i_{small,*}\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_{small,*}\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. □

0A46 **Lemma 59.79.5.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and denote $\mathcal{F}^a$ the associated quasi-coherent sheaf on the small étale site of X (Proposition 59.17.1). Then*

1. *$H^q_Z(X, \mathcal{F})$ agrees with $H^q_Z(X_{étale}, \mathcal{F}^a)$,*
2. *if the complement of Z is retrocompact in X, then $i_*\mathcal{H}^q_Z(\mathcal{F}^a)$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules equal to $(i_*\mathcal{H}^q_Z(\mathcal{F}))^a$.*

Proof. Let $j : U \to X$ be the inclusion of the complement of Z. The statement (1) on cohomology groups follows from the long exact sequences for cohomology with supports and the agreements $H^q(X_{étale}, \mathcal{F}^a) = H^q(X, \mathcal{F})$ and $H^q(U_{étale}, \mathcal{F}^a) = H^q(U, \mathcal{F})$, see Theorem 59.22.4. If $j : U \to X$ is a quasi-compact morphism, i.e., if $U \subset X$ is retrocompact, then $R^q j_*$ transforms quasi-coherent sheaves into quasi-coherent sheaves (Cohomology of Schemes, Lemma 30.4.5) and commutes with taking associated sheaf on étale sites (Descent, Lemma 35.9.5). We conclude by applying Lemma 59.79.3. □

59.80. Schemes with strictly henselian local rings

0EZN In this section we collect some results about the étale cohomology of schemes whose local rings are strictly henselian. For example, here is a fun generalization of Lemma 59.55.1.

09AX **Lemma 59.80.1.** *Let S be a scheme all of whose local rings are strictly henselian. Then for any abelian sheaf $\mathcal{F}$ on $S_{étale}$ we have $H^i(S_{étale}, \mathcal{F}) = H^i(S_{Zar}, \mathcal{F})$.*

Proof. Let $\epsilon : S_{étale} \to S_{Zar}$ be the morphism of sites given by the inclusion functor. The Zariski sheaf $R^p\epsilon_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U, \mathcal{F})$. Thus the stalk at $x \in X$ is $\operatorname{colim} H^p_{étale}(U, \mathcal{F}) = H^p_{étale}(\operatorname{Spec}(\mathcal{O}_{X,x}), \mathcal{G}_x)$ where $\mathcal{G}_x$ denotes the pullback of $\mathcal{F}$ to $\operatorname{Spec}(\mathcal{O}_{X,x})$, see Lemma 59.51.5. Thus the

higher direct images of $R^p\epsilon_*\mathcal{F}$ are zero by Lemma 59.55.1 and we conclude by the Leray spectral sequence. □

0GY0 **Lemma 59.80.2.** *Let R be a ring all of whose local rings are strictly henselian. Let $\mathcal{F}$ be a sheaf on $\mathrm{Spec}(R)_{étale}$. Assume that for all $f, g \in R$ the kernel of*

$$H^1_{étale}(D(f+g), \mathcal{F}) \longrightarrow H^1_{étale}(D(f(f+g)), \mathcal{F}) \oplus H^1_{étale}(D(g(f+g)), \mathcal{F})$$

is zero. Then $H^q_{étale}(\mathrm{Spec}(R), \mathcal{F}) = 0$ for $q > 0$.

Proof. By Lemma 59.80.1 we see that étale cohomology of $\mathcal{F}$ agrees with Zariski cohomology on any open of $\mathrm{Spec}(R)$. We will prove by induction on i the statement: for $h \in R$ we have $H^q_{étale}(D(h), \mathcal{F}) = 0$ for $1 \leq q \leq i$. The base case $i = 0$ is trivial. Assume $i \geq 1$.

Let $\xi \in H^q_{étale}(D(h), \mathcal{F})$ for some $1 \leq q \leq i$ and $h \in R$. If $q < i$ then we are done by induction, so we assume $q = i$. After replacing R by R_h we may assume $\xi \in H^i_{étale}(\mathrm{Spec}(R), \mathcal{F})$; some details omitted. Let $I \subset R$ be the set of elements $f \in R$ such that $\xi|_{D(f)} = 0$. Since ξ is Zariski locally trivial, it follows that for every prime $\mathfrak{p}$ of R there exists an $f \in I$ with $f \notin \mathfrak{p}$. Thus if we can show that I is an ideal, then $1 \in I$ and we're done. It is clear that $f \in I$, $r \in R$ implies $rf \in I$. Thus we assume that $f, g \in I$ and we show that $f + g \in I$. If $q = i = 1$, then this is exactly the assumption of the lemma! Whence the result for $i = 1$. For $q = i > 1$, note that

$$D(f+g) = D(f(f+g)) \cup D(g(f+g))$$

By Mayer-Vietoris (Cohomology, Lemma 20.8.2 which applies as étale cohomology on open subschemes of $\mathrm{Spec}(R)$ equals Zariski cohomology) we have an exact sequence

$$\begin{array}{c} H^{i-1}_{étale}(D(fg(f+g)), \mathcal{F}) \\ \downarrow \\ H^i_{étale}(D(f+g), \mathcal{F}) \\ \downarrow \\ H^i_{étale}(D(f(f+g)), \mathcal{F}) \oplus H^i_{étale}(D(g(f+g)), \mathcal{F}) \end{array}$$

and the result follows as the first group is zero by induction. □

09AY **Lemma 59.80.3.** *Let S be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf $\mathcal{F}$ on $S_{étale}$ we have $H^i(S_{étale}, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Condition (1) implies that the underlying topological space of S is profinite, see Algebra, Lemma 10.26.5. Thus the higher cohomology groups of an abelian sheaf on the topological space S (i.e., Zariski cohomology) is trivial, see Cohomology, Lemma 20.22.3. The local rings are strictly henselian by Algebra, Lemma 10.153.10. Thus étale cohomology of S is computed by Zariski cohomology by Lemma 59.80.1 and the proof is done. □

The spectrum of an absolutely integrally closed ring is an example of a scheme all of whose local rings are strictly henselian, see More on Algebra, Lemma 15.14.7. It

turns out that normal domains with separably closed fraction fields have an even stronger property as explained in the following lemma.

09Z9 **Lemma 59.80.4.** *Let X be an integral normal scheme with separably closed function field.*

(1) *A separated étale morphism $U \to X$ is a disjoint union of open immersions.*
(2) *All local rings of X are strictly henselian.*

Proof. Let R be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a K-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of K. Let e_i, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since A is normal (Algebra, Lemma 10.163.9) the idempotents e_i are in A (Algebra, Lemma 10.37.12). Hence $A = \prod Ae_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that A is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma 29.10.2 and Étale Morphisms, Theorem 41.14.1.

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in X is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where U'_i is the set of point of U' which are specializations of ξ_i. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \overline{\{\xi_i\}}$ is an open and closed subscheme of U and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma 29.49.7 the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $\mathcal{O}_{X,x}$ as the local ring at $\overline{x}$ on the étale site of X (Lemma 59.33.1). It can also be proved directly, see Fundamental Groups, Lemma 58.12.2. □

0EZP **Lemma 59.80.5.** *Let $f : X \to Y$ be a morphism of schemes where X is an integral normal scheme with separably closed function field. Then $R^q f_* \underline{M} = 0$ for $q > 0$ and any abelian group M.*

Proof. Recall that $R^q f_* \underline{M}$ is the sheaf associated to the presheaf $V \mapsto H^q_{étale}(V \times_Y X, M)$ on $Y_{étale}$, see Lemma 59.51.6. If V is affine, then $V \times_Y X \to X$ is separated and étale. Hence $V \times_Y X = \coprod U_i$ is a disjoint union of open subschemes U_i of X, see Lemma 59.80.4. By Lemma 59.80.1 we see that $H^q_{étale}(U_i, M)$ is equal to $H^q_{Zar}(U_i, M)$. This vanishes by Cohomology, Lemma 20.20.2. □

09ZA **Lemma 59.80.6.** *Let X be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with V affine. Then V is a finite disjoint union of open subschemes of Z. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.*

Proof. By Algebra, Lemma 10.143.10 we can lift V to an affine scheme U étale over X. Apply Lemma 59.80.4 to $U \to X$ to get the first statement.

The final statement is a consequence of the first. Let $V = \coprod_{i=1,\ldots,n} V_i$ be a finite decomposition into open and closed subschemes with $V_i \to Z$ an open immersion. As $V \to Z$ is finite we see that $V_i \to Z$ is also closed. Let $U_i \subset Z$ be the image. Then we have a decomposition into open and closed subschemes

$$Z = \coprod_{(A,B)} \bigcap_{i \in A} U_i \cap \bigcap_{i \in B} U_i^c$$

where the disjoint union is over $\{1, \ldots, n\} = A \amalg B$ where A has at least one element. Each of the strata is contained in a single U_i and we find our section. □

09ZB **Lemma 59.80.7.** *Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^1_{étale}(Z, \underline{M}) = 0$.*

Proof. By Cohomology on Sites, Lemma 21.4.3 an element of $H^1_{étale}(Z, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $Z_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over Z, Lemma 59.64.4. Of course $V \to Z$ is surjective as a torsor is locally trivial. Since $V \to Z$ has a section by Lemma 59.80.6 we are done. □

09ZC **Lemma 59.80.8.** *Let X be a normal integral affine scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^q_{étale}(Z, \underline{M}) = 0$ for $q \geq 1$.*

Proof. Write $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(R')$ so that we have a surjection of rings $R \to R'$. All local rings of R' are strictly henselian by Lemma 59.80.4 and Algebra, Lemma 10.156.4. Furthermore, we see that for any $f' \in R'$ there is a surjection $R_f \to R'_{f'}$ where $f \in R$ is a lift of f'. Since R_f is a normal domain with separably closed fraction field we see that $H^1_{étale}(D(f'), \underline{M}) = 0$ by Lemma 59.80.7. Thus we may apply Lemma 59.80.2 to $Z = \mathrm{Spec}(R')$ to conclude. □

09ZD **Lemma 59.80.9.** *Let X be an affine scheme.*

(1) *There exists an integral surjective morphism $X' \to X$ such that for every closed subscheme $Z' \subset X'$, every finite abelian group M, and every $q \geq 1$ we have $H^q_{étale}(Z', \underline{M}) = 0$.*
(2) *For any closed subscheme $Z \subset X$, finite abelian group M, $q \geq 1$, and $\xi \in H^q_{étale}(Z, \underline{M})$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that ξ pulls back to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$.*

Proof. Write $X = \mathrm{Spec}(A)$. Write $A = \mathbf{Z}[x_i]/J$ for some ideal J. Let R be the integral closure of $\mathbf{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbf{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \mathrm{Spec}(A')$. This gives an example as in (1) by Lemma 59.80.8.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, ξ maps to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$. We may write X' as a limit $X' = \lim X'_i$ of schemes finite and of finite presentation over X; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 32.7.3. By Lemma 59.51.5 we see that ξ maps to zero in $H^q_{étale}(X'_i \times_X Z, \underline{M})$ for some i large enough. □

59.81. Absolutely integrally closed vanishing

0GY1 Recall that we say a ring R is absolutely integrally closed if every monic polynomial over R has a root in R (More on Algebra, Definition 15.14.1). In this section we prove that the étale cohomology of $\mathrm{Spec}(R)$ with coefficients in a finite torsion group vanishes in positive degrees (Proposition 59.81.5) thereby slightly improving the earlier Lemma 59.80.8. We suggest the reader skip this section.

0GY2 **Lemma 59.81.1.** *Let A be a ring. Let $a, b \in A$ such that $aA + bA = A$ and $a \bmod bA$ is a root of unity. Then there exists a monogenic extension $A \subset B$ and an element $y \in B$ such that $u = a - by$ is a unit.*

Proof. Say $a^n \equiv 1 \bmod bA$. In particular a^i is a unit modulo $b^m A$ for all $i, m \geq 1$. We claim there exist $a_1, \ldots, a_n \in A$ such that

$$1 = a^n + a_1 a^{n-1} b + a_2 a^{n-2} b^2 + \ldots + a_n b^n$$

Namely, since $1 - a^n \in bA$ we can find an element $a_1 \in A$ such that $1 - a^n - a_1 a^{n-1} b \in b^2 A$ using the unit property of a^{n-1} modulo bA. Next, we can find an element $a_2 \in A$ such that $1 - a^n - a_1 a^{n-1} b - a_2 a^{n-2} b^2 \in b^3 A$. And so on. Eventually we find $a_1, \ldots, a_{n-1} \in A$ such that $1 - (a^n + a_1 a^{n-1} b + a_2 a^{n-2} b^2 + \ldots + a_{n-1} a b^{n-1}) \in b^n A$. This allows us to find $a_n \in A$ such that the displayed equality holds.

With $a_1, \ldots, a_n$ as above we claim that setting

$$B = A[y]/(y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots + a_n)$$

works. Namely, suppose that $\mathfrak{q} \subset B$ is a prime ideal lying over $\mathfrak{p} \subset A$. To get a contradiction assume $u = a - by$ is in $\mathfrak{q}$. If $b \in \mathfrak{p}$ then $a \notin \mathfrak{p}$ as $aA + bA = A$ and hence u is not in $\mathfrak{q}$. Thus we may assume $b \notin \mathfrak{p}$, i.e., $b \notin \mathfrak{q}$. This implies that $y \bmod \mathfrak{q}$ is equal to $a/b \bmod \mathfrak{q}$. However, then we obtain

$$0 = y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots + a_n = b^{-n}(a^n + a_1 a^{n-1} b + a_2 a^{n-2} b^2 + \ldots + a_n b^n) = b^{-n}$$

a contradiction. This finishes the proof. □

In order to explain the proof we need to introduce some group schemes. Fix a prime number ℓ. Let

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{\ell-1} + x^{\ell-2} + \ldots + 1)$$

In other words A is the monogenic extension of $\mathbf{Z}$ generated by a primitive ℓth root of unity ζ. We set

$$\pi = \zeta - 1$$

A calculation (omitted) shows that ℓ is divisible by $\pi^{\ell-1}$ in A. Our first group scheme over A is

$$G = \mathrm{Spec}(A[s, \frac{1}{\pi s + 1}])$$

with group law given by the comultiplication

$$\mu : A[s, \frac{1}{\pi s + 1}] \longrightarrow A[s, \frac{1}{\pi s + 1}] \otimes_A A[s, \frac{1}{\pi s + 1}], \quad s \longmapsto \pi s \otimes s + s \otimes 1 + 1 \otimes s$$

With this choice we have

$$\mu(\pi s + 1) = (\pi s + 1) \otimes (\pi s + 1)$$

and hence we indeed have an A-algebra map as indicated. We omit the verification that this indeed defines a group law. Our second group scheme over A is

$$H = \mathrm{Spec}(A[t, \frac{1}{\pi^\ell t + 1}])$$

with group law given by the comultiplication

$$\mu : A[t, \frac{1}{\pi^\ell t + 1}] \longrightarrow A[t, \frac{1}{\pi^\ell t + 1}] \otimes_A A[t, \frac{1}{\pi^\ell t + 1}], \quad t \longmapsto \pi^\ell t \otimes t + t \otimes 1 + 1 \otimes t$$

The same verification as before shows that this defines a group law. Next, we observe that the polynomial

$$\Phi(s) = \frac{(\pi s + 1)^\ell - 1}{\pi^\ell}$$

is in $A[s]$ and of degree ℓ and monic in s. Namely, the coefficicient of s^i for $0 < i < \ell$ is equal to $\binom{\ell}{i}\pi^{i-\ell}$ and since $\pi^{\ell-1}$ divides ℓ in A this is an element of A. We obtain a ring map

$$A[t, \frac{1}{\pi^\ell t + 1}] \longrightarrow A[s, \frac{1}{\pi s + 1}], \quad t \longmapsto \Phi(s)$$

which the reader easily verifies is compatible with the comultiplications. Thus we get a morphism of group schemes

$$f : G \to H$$

The following lemma in particular shows that this morphism is faithfully flat (in fact we will see that it is finite étale surjective).

0GY3 **Lemma 59.81.2.** *We have*

$$A[s, \frac{1}{\pi s + 1}] = \left(A[t, \frac{1}{\pi^\ell t + 1}]\right)[s]/(\Phi(s) - t)$$

In particular, the Hopf algebra of G is a monogenic extension of the Hopf algebra of H.

Proof. Follows from the discussion above and the shape of $\Phi(s)$. In particular, note that using $\Phi(s) = t$ the element $\frac{1}{\pi^\ell t+1}$ becomes the element $\frac{1}{(\pi s+1)^\ell}$. □

Next, let us compute the kernel of f. Since the origin of H is given by $t = 0$ in H we see that the kernel of f is given by $\Phi(s) = 0$. Now observe that the A-valued points $\sigma_0, \ldots, \sigma_{\ell-1}$ of G given by

$$\sigma_i : s = \frac{\zeta^i - 1}{\pi} = \frac{\zeta^i - 1}{\zeta - 1} = \zeta^{i-1} + \zeta^{i-2} + \ldots + 1, \quad i = 0, 1, \ldots, \ell - 1$$

are certainly contained in $\mathrm{Ker}(f)$. Moreover, these are all pairwise distinct in **all** fibres of $G \to \mathrm{Spec}(A)$. Also, the reader computes that $\sigma_i +_G \sigma_j = \sigma_{i+j \bmod \ell}$. Hence we find a closed immersion of group schemes

$$\underline{\mathbf{Z}/\ell\mathbf{Z}}_A \longrightarrow \mathrm{Ker}(f)$$

sending i to σ_i. However, by construction $\mathrm{Ker}(f)$ is finite flat over $\mathrm{Spec}(A)$ of degree ℓ. Hence we conclude that this map is an isomorphism. All in all we conclude that we have a short exact sequence

0GY4 (59.81.2.1)
$$0 \to \underline{\mathbf{Z}/\ell\mathbf{Z}}_A \to G \to H \to 0$$

of group schemes over A.

0GY5 **Lemma 59.81.3.** *Let R be an A-algebra which is absolutely integrally closed. Then $G(R) \to H(R)$ is surjective.*

Proof. Let $h \in H(R)$ correspond to the A-algebra map $A[t, \frac{1}{\pi^\ell t+1}] \to R$ sending t to $a \in A$. Since $\Phi(s)$ is monic we can find $b \in A$ with $\Phi(b) = a$. By Lemma 59.81.2 sending s to b we obtain a unique A-algebra map $A[s, \frac{1}{\pi s+1}] \to R$ compatible with the map $A[t, \frac{1}{\pi^\ell t+1}] \to R$ above. This in turn corresponds to an element $g \in G(R)$ mapping to $h \in H(R)$. □

0GY6 **Lemma 59.81.4.** *Let R be an A-algebra which is absolutely integrally closed. Let $I, J \subset R$ be ideals with $I + J = R$. There exists a $g \in G(R)$ such that $g \bmod I = \sigma_0$ and $g \bmod J = \sigma_1$.*

Proof. Choose $x \in I$ such that $x \equiv 1 \bmod J$. We may and do replace I by xR and J by $(x-1)R$. Then we are looking for an $s \in R$ such that

(1) $1 + \pi s$ is a unit,
(2) $s \equiv 0 \bmod xR$, and
(3) $s \equiv 1 \bmod (x-1)R$.

The last two conditions say that $s = x + x(x-1)y$ for some $y \in R$. The first condition says that $1 + \pi s = 1 + \pi x + \pi x(x-1)y$ needs to be a unit of R. However, note that $1 + \pi x$ and $\pi x(x-1)$ generate the unit ideal of R and that $1 + \pi x$ is an ℓth root of 1 modulo $\pi x(x-1)$[7]. Thus we win by Lemma 59.81.1 and the fact that R is absolutely integrally closed. □

0GY7 **Proposition 59.81.5.** *Let R be an absolutely integrally closed ring. Let M be a finite abelian group. Then $H^i_{étale}(\mathrm{Spec}(R), \underline{M}) = 0$ for $i > 0$.*

Proof. Since any finite abelian group has a finite filtration whose subquotients are cyclic of prime order, we may assume $M = \mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number.

Observe that all local rings of R are strictly henselian, see More on Algebra, Lemma 15.14.7. Furthermore, any localization of R is also absolutely integrally closed by More on Algebra, Lemma 15.14.3. Thus Lemma 59.80.2 tells us it suffices to show that the kernel of

$$H^1_{étale}(D(f+g), \mathbf{Z}/\ell\mathbf{Z}) \longrightarrow H^1_{étale}(D(f(f+g)), \mathbf{Z}/\ell\mathbf{Z}) \oplus H^1_{étale}(D(g(f+g)), \mathbf{Z}/\ell\mathbf{Z})$$

is zero for any $f, g \in R$. After replacing R by R_{f+g} we reduce to the following claim: given $\xi \in H^1_{étale}(\mathrm{Spec}(R), \mathbf{Z}/\ell\mathbf{Z})$ and an affine open covering $\mathrm{Spec}(R) = U \cup V$ such that $\xi|_U$ and $\xi|_V$ are trivial, then $\xi = 0$.

Let $A = \mathbf{Z}[\zeta]$ as above. Since $\mathbf{Z} \subset A$ is monogenic, we can find a ring map $A \to R$. From now on we think of R as an A-algebra and we think of $\mathrm{Spec}(R)$ as a scheme over $\mathrm{Spec}(A)$. If we base change the short exact sequence (59.81.2.1) to $\mathrm{Spec}(R)$ and take étale cohomology we obtain

$$G(R) \to H(R) \to H^1_{étale}(\mathrm{Spec}(R), \mathbf{Z}/\ell\mathbf{Z}) \to H^1_{étale}(\mathrm{Spec}(R), G)$$

Please keep this in mind during the rest of the proof.

Let $\tau \in \Gamma(U \cap V, \mathbf{Z}/\ell\mathbf{Z})$ be a section whose boundary in the Mayer-Vietoris sequence (Lemma 59.50.1) gives ξ. For $i = 0, 1, \ldots, \ell - 1$ let $A_i \subset U \cap V$ be the open and

[7] Because $1 + \pi x$ is congruent to 1 modulo π, congruent to 1 modulo x, and congruent to $1 + \pi = \zeta$ modulo $x - 1$ and because we have $(\pi) \cap (x) \cap (x-1) = (\pi x(x-1))$ in $A[x]$.

closed subset where τ has the value $i \bmod \ell$. Thus we have a finite disjoint union decomposition

$$U \cap V = A_0 \amalg \ldots \amalg A_{\ell-1}$$

such that τ is constant on each A_i. For $i = 0, 1, \ldots, \ell - 1$ denote $\tau_i \in H^0(U \cap V, \mathbf{Z}/\ell\mathbf{Z})$ the element which is equal to 1 on A_i and equal to 0 on A_j for $j \neq i$. Then τ is a sum of multiples of the τ_i[8]. Hence it suffices to show that the cohomology class corresponding to τ_i is trivial. This reduces us to the case where τ takes only two distinct values, namely 1 and 0.

Assume τ takes only the values 1 and 0. Write

$$U \cap V = A \amalg B$$

where A is the locus where $\tau = 0$ and B is the locus where $\tau = 1$. Then A and B are disjoint closed subsets. Denote $\overline{A}$ and $\overline{B}$ the closures of A and B in $\mathrm{Spec}(R)$. Then we have a "banana": namely we have

$$\overline{A} \cap \overline{B} = Z_1 \amalg Z_2$$

with $Z_1 \subset U$ and $Z_2 \subset V$ disjoint closed subsets. Set $T_1 = \mathrm{Spec}(R) \setminus V$ and $T_2 = \mathrm{Spec}(R) \setminus U$. Observe that $Z_1 \subset T_1 \subset U$, $Z_2 \subset T_2 \subset V$, and $T_1 \cap T_2 = \emptyset$. Topologically we can write

$$\mathrm{Spec}(R) = \overline{A} \cup \overline{B} \cup T_1 \cup T_2$$

We suggest drawing a picture to visualize this. In order to prove that ξ is zero, we may and do replace R by its reduction (Proposition 59.45.4). Below, we think of A, $\overline{A}$, B, $\overline{B}$, T_1, T_2 as reduced closed subschemes of $\mathrm{Spec}(R)$. Next, as scheme structures on Z_1 and Z_2 we use

$$Z_1 = \overline{A} \cap (\overline{B} \cup T_1) \quad \text{and} \quad Z_2 = \overline{A} \cap (\overline{B} \cup T_2)$$

(scheme theoretic unions and intersections as in Morphisms, Definition 29.4.4).

Denote X the G-torsor over $\mathrm{Spec}(R)$ corresponding to the image of ξ in $H^1(\mathrm{Spec}(R), G)$. If X is trivial, then ξ comes from an element $h \in H(R)$ (see exact sequence of cohomology above). However, then by Lemma 59.81.3 the element h lifts to an element of $G(R)$ and we conclude $\xi = 0$ as desired. Thus our goal is to prove that X is trivial.

Recall that the embedding $\mathbf{Z}/\ell\mathbf{Z} \to G(R)$ sends $i \bmod \ell$ to $\sigma_i \in G(R)$. Observe that $\overline{A}$ is the spectrum of an absolutely integrally closed ring (namely a qotient of R). By Lemma 59.81.4 we can find $g \in G(\overline{A})$ with $g|_{\overline{A} \cap Z_1} = \sigma_0$ and $g|_{\overline{A} \cap Z_2} = \sigma_1$ (scheme theoretically). Then we can define

(1) $g_1 \in G(U)$ which is g on $\overline{A} \cap U$, which is σ_0 on $\overline{B} \cap U$, and σ_0 on T_1, and
(2) $g_2 \in G(V)$ which is g on $\overline{A} \cap V$, which is σ_1 on $\overline{B} \cap V$, and σ_1 on T_2.

Namely, to find g_1 as in (1) we glue the section σ_0 on $\Omega = (\overline{B} \cup T_1) \cap U$ to the restriction of the section g on $\Omega' = \overline{A} \cap U$. Note that $U = \Omega \cup \Omega'$ (scheme theoretically) because U is reduced and $\Omega \cap \Omega' = Z_1$ (scheme theoretically) by our choice of Z_1. Hence by Morphisms, Lemma 29.4.6 we have that U is the pushout of Ω and Ω' along Z_1. Thus we can find g_1. Similarly for the existence of g_2 in (2). Then we have

$$\tau = g_2|_{A \cup B} - g_1|_{A \cup B} \quad \text{(addition in group law)}$$

[8] Modulo calculation errors we have $\tau = \sum i\tau_i$.

and we see that X is trivial thereby finishing the proof. □

59.82. Affine analog of proper base change

09Z8 In this section we discuss a result by Ofer Gabber, see [**Gab94**]. This was also proved by Roland Huber, see [**Hub93b**]. We have already done some of the work needed for Gabber's proof in Section 59.80.

09ZE **Lemma 59.82.1.** *Let X be an affine scheme. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{étale}(Z, \mathcal{F}|_Z)$ for some $q > 0$. Then there exists an injective map $\mathcal{F} \to \mathcal{F}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ξ in $H^q_{étale}(Z, \mathcal{F}'|_Z)$ is zero.*

Proof. By Lemmas 59.73.2 and 59.51.4 we can find a map $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ a constructible abelian sheaf and ξ coming from an element ζ of $H^q_{étale}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ζ in $H^q_{étale}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $\mathcal{F}'$ to be the pushout

$$\mathcal{F}' = \mathcal{G}' \amalg_{\mathcal{G}} \mathcal{F}$$

and we conclude the result of the lemma holds. (Observe that restriction to Z is exact, so commutes with finite limits and colimits and moreover it commutes with arbitrary colimits as a left adjoint to pushforward.) Thus we may assume $\mathcal{F}$ is constructible.

Assume $\mathcal{F}$ is constructible. By Lemma 59.74.4 it suffices to prove the result when $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite abelian group and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 59.73.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 59.55.3) we see that the restriction of $f_*\underline{M}$ to Z is equal to the pushforward of $\underline{M}$ via $Y \times_X Z \to Z$. By the Leray spectral sequence (Proposition 59.54.2) and vanishing of higher direct images (Proposition 59.55.2), we find

$$H^q_{étale}(Z, f_*\underline{M}|_Z) = H^q_{étale}(Y \times_X Z, \underline{M}).$$

By Lemma 59.80.9 we can find a finite surjective morphism $Y' \to Y$ of finite presentation such that ξ maps to zero in $H^q(Y' \times_X Z, \underline{M})$. Denoting $f' : Y' \to X$ the composition $Y' \to Y \to X$ we claim the map

$$f_*\underline{M} \longrightarrow f'_*\underline{M}$$

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point, then

$$(f_*\underline{M})_{\overline{x}} = \bigoplus\nolimits_{f(\overline{y})=\overline{x}} M$$

by Proposition 59.55.2 and similarly for the other sheaf. Since $Y' \to Y$ is surjective and finite we see that the induced map on geometric points lifting $\overline{x}$ is surjective too and we conclude. □

The lemma above will take care of higher cohomology groups in Gabber's result. The following lemma will be used to deal with global sections.

09ZF **Lemma 59.82.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a closed immersion. Assume that*

(1) *for any sheaf $\mathcal{F}$ on X_{Zar} the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}\mathcal{F})$ is bijective, and*

(2) *for any finite morphism $X' \to X$ assumption (1) holds for $Z \times_X X' \to X'$.*

Then for any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.

Proof. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. There is a canonical (base change) map

$$i^{-1}(\mathcal{F}|_{X_{Zar}}) \longrightarrow (i^{-1}_{small}\mathcal{F})|_{Z_{Zar}}$$

of sheaves on Z_{Zar}. We will show this map is injective by looking at stalks. The stalk on the left hand side at $z \in Z$ is the stalk of $\mathcal{F}|_{X_{Zar}}$ at z. The stalk on the right hand side is the colimit over all elementary étale neighbourhoods $(U, u) \to (X, z)$ such that $U \times_X Z \to Z$ has a section over a neighbourhood of z. As étale morphisms are open, the image of $U \to X$ is an open neighbourhood U_0 of z in X. The map $\mathcal{F}(U_0) \to \mathcal{F}(U)$ is injective by the sheaf condition for $\mathcal{F}$ with respect to the étale covering $U \to U_0$. Taking the colimit over all U and U_0 we obtain injectivity on stalks.

It follows from this and assumption (1) that the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}_{small}\mathcal{F})$ is injective. By (2) the same thing is true on all X' finite over X.

Before we prove the surjectivity, let us introduce some notation. For any object U of $X_{étale}$ and $s \in \mathcal{F}(U)$ we denote $i^{-1}s \in (i^{-1}_{small}\mathcal{F})(U \times_X Z)$ the pullback (more precisely this is the image of s under the map of Sites, Lemma 7.5.3 combined with the map from the presheaf pullback to the sheaf pullback). This construction is compatible with pullback by morphisms in $X_{étale}$ and by (finite) morphisms $X' \to X$ in a manner which we leave to the reader to clarify.

Let $t \in \Gamma(Z, i^{-1}_{small}\mathcal{F})$. By construction of $i^{-1}_{small}\mathcal{F}$ there exists an étale covering $\{V_j \to Z\}_{j \in J}$, étale morphisms $U_j \to X$, sections $s_j \in \mathcal{F}(U_j)$ and morphisms $V_j \to U_j$ over X such that $t|_{V_j}$ is the pullback of $i^{-1}s_j$ by $V_j \to U_j \times_X Z$. Since $V_j \to U_j \times_X Z$ is étale, the image is an open subscheme. Since $U_j \times_X Z \subset U_j$ is closed, we may, after replacing U_j by an open subscheme, assume that $V_j \to U_j \times_X Z$ is surjective. Then $\{V_j \to U_j \times_X Z\}$ is an étale covering and we conclude that $t|_{U_j \times_X Z} = i^{-1}s_j$. Observe that every nonempty closed subscheme $T \subset X$ meets Z by assumption (1) applied to the sheaf $(T \to X)_*\underline{\mathbf{Z}}$ for example. Thus we see that $\coprod U_j \to X$ is surjective. By More on Morphisms, Lemma 37.45.7 we can find a finite surjective morphism $X' \to X$ such that $X' \to X$ Zariski locally factors through $\coprod U_j \to X$. Say $X' = \bigcup_{\alpha \in A} W_\alpha$ is an open covering and $j : A \to J$ is a map, such that there exist morphisms $g_\alpha : W_\alpha \to U_{j(\alpha)}$ over X. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X'_{étale}$. Denote $s'_\alpha \in \mathcal{F}'(W_\alpha)$ the pullback of $s_{j(\alpha)}$ by g_α. Set $Z' = X' \times_X Z$. Denote $i' : Z' \to X'$ the base change of i. Then $(i'_{small})^{-1}\mathcal{F}' = (Z' \to Z)^{-1}_{small} i^{-1}_{small}\mathcal{F}$. Via this identification we may denote $t' \in \Gamma(Z', (i')^{-1}_{small}\mathcal{F}')$ the pullback of t to Z'. Then $t'|_{W_\alpha \times_X Z}$ is equal to $(i')^{-1}s'_\alpha$ by construction. We conclude that t' is a section of the subsheaf

$$(i')^{-1}(\mathcal{F}'|_{X'_{Zar}}) \subset (i'_{small})^{-1}\mathcal{F}')|_{Z'_{Zar}}$$

By assumption (2) the section t' comes from a section $s' \in \Gamma(X', \mathcal{F}'|_{X'_{Zar}}) = \Gamma(X', \mathcal{F}')$. By the injectivity proved in the second paragraph, we conclude that the two pullbacks of s' to $X' \times_X X'$ are the same (after all this is true for the pullbacks of t' to $Z' \times_Z Z'$). Hence we conclude s' comes from a section of $\mathcal{F}$ over X by Remark 59.55.6. □

0CAM **Lemma 59.82.3.** *Let $Z \subset X$ be a closed subset of a topological space X. Assume*

(1) *X is a spectral space (Topology, Definition 5.23.1), and*
(2) *for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).*

If $Z = Z_1 \amalg Z_2$ with Z_i closed in Z, then there exists a decomposition $X = X_1 \amalg X_2$ with X_i closed in X and $Z_i = Z \cap X_i$.

Proof. Observe that Z_i is quasi-compact. Hence the set of points W_i specializing to Z_i is closed in the constructible topology by Topology, Lemma 5.24.7. Assumption (2) implies that $X = W_1 \amalg W_2$. Let $x \in \overline{W_1}$. By Topology, Lemma 5.23.6 part (1) there exists a specialization $x_1 \rightsquigarrow x$ with $x_1 \in W_1$. Thus $\overline{\{x\}} \subset \overline{\{x_1\}}$ and we see that $x \in W_1$. In other words, setting $X_i = W_i$ does the job. □

09ZG **Lemma 59.82.4.** *Let $Z \subset X$ be a closed subset of a topological space X. Assume*

(1) *X is a spectral space (Topology, Definition 5.23.1), and*
(2) *for $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty).*

Then for any sheaf $\mathcal{F}$ on X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

Proof. If $x \rightsquigarrow x'$ is a specialization of points, then there is a canonical map $\mathcal{F}_{x'} \to \mathcal{F}_x$ compatible with sections over opens and functorial in $\mathcal{F}$. Since every point of X specializes to a point of Z it follows that $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is injective. The difficult part is to show that it is surjective.

Denote $\mathcal{B}$ be the set of all quasi-compact opens of X. Write $\mathcal{F}$ as a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ where each $\mathcal{F}_i$ is as in Modules, Equation (17.19.2.1). See Modules, Lemma 17.19.2. Then $\mathcal{F}|_Z = \operatorname{colim} \mathcal{F}_i|_Z$ as restriction to Z is a left adjoint (Categories, Lemma 4.24.5 and Sheaves, Lemma 6.21.8). By Sheaves, Lemma 6.29.1 the functors $\Gamma(X, -)$ and $\Gamma(Z, -)$ commute with filtered colimits. Hence we may assume our sheaf $\mathcal{F}$ is as in Modules, Equation (17.19.2.1).

Suppose that we have an embedding $\mathcal{F} \subset \mathcal{G}$. Then we have

$$\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z) \cap \Gamma(X, \mathcal{G})$$

where the intersection takes place in $\Gamma(Z, \mathcal{G}|_Z)$. This follows from the first remark of the proof because we can check whether a global section of $\mathcal{G}$ is in $\mathcal{F}$ by looking at the stalks and because every point of X specializes to a point of Z.

By Modules, Lemma 17.19.4 there is an injection $\mathcal{F} \to \prod (Z_i \to X)_* \underline{S_i}$ where the product is finite, $Z_i \subset X$ is closed, and S_i is finite. Thus it suffices to prove surjectivity for the sheaves $(Z_i \to X)_* \underline{S_i}$. Observe that

$$\Gamma(X, (Z_i \to X)_* \underline{S_i}) = \Gamma(Z_i, \underline{S_i}) \quad \text{and} \quad \Gamma(X, (Z_i \to X)_* \underline{S_i}|_Z) = \Gamma(Z \cap Z_i, \underline{S_i})$$

Moreover, conditions (1) and (2) are inherited by Z_i; this is clear for (2) and follows from Topology, Lemma 5.23.5 for (1). Thus it suffices to prove the lemma in the case of a (finite) constant sheaf. This case is a restatement of Lemma 59.82.3 which finishes the proof. □

0CAF **Example 59.82.5.** Lemma 59.82.4 is false if X is not spectral. Here is an example: Let Y be a T_1 topological space, and $y \in Y$ a non-open point. Let $X = Y \amalg \{x\}$, endowed with the topology whose closed sets are $\emptyset$, $\{y\}$, and all $F \amalg \{x\}$, where F is a closed subset of Y. Then $Z = \{x, y\}$ is a closed subset of X, which satisfies assumption (2) of Lemma 59.82.4. But X is connected, while Z is not. The conclusion of the lemma thus fails for the constant sheaf with value $\{0, 1\}$ on X.

09ZH **Lemma 59.82.6.** *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. Recall that the spectrum of any ring is a spectral space, see Algebra, Lemma 10.26.2. By More on Algebra, Lemma 15.11.16 we see that $\overline{\{x\}} \cap Z$ is connected for every $x \in X$. By Lemma 59.82.4 we see that the statement is true for sheaves on X_{Zar}. For any finite morphism $X' \to X$ we have $X' = \mathrm{Spec}(A')$ and $Z \times_X X' = \mathrm{Spec}(A'/IA')$ with (A', IA') a henselian pair, see More on Algebra, Lemma 15.11.8 and we get the same statement for sheaves on $(X')_{Zar}$. Thus we can apply Lemma 59.82.2 to conclude. □

Finally, we can state and prove Gabber's theorem.

09ZI **Theorem 59.82.7** (Gabber)**.** *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. For any torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(Z, \mathcal{F}|_Z)$.*

Proof. The result holds for $q = 0$ by Lemma 59.82.6. Let $q \geq 1$. Suppose the result has been shown in all degrees $< q$. Let $\mathcal{F}$ be a torsion abelian sheaf. Let $\mathcal{F} \to \mathcal{F}'$ be an injective map of torsion abelian sheaves (to be chosen later) with cokernel $\mathcal{Q}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{Q} \to 0$$

of torsion abelian sheaves on $X_{étale}$. This gives a map of long exact cohomology sequences over X and Z part of which looks like

$$\begin{array}{ccccccc} H^{q-1}_{étale}(X, \mathcal{F}') & \longrightarrow & H^{q-1}_{étale}(X, \mathcal{Q}) & \longrightarrow & H^q_{étale}(X, \mathcal{F}) & \longrightarrow & H^q_{étale}(X, \mathcal{F}') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q-1}_{étale}(Z, \mathcal{F}'|_Z) & \longrightarrow & H^{q-1}_{étale}(Z, \mathcal{Q}|_Z) & \longrightarrow & H^q_{étale}(Z, \mathcal{F}|_Z) & \longrightarrow & H^q_{étale}(Z, \mathcal{F}'|_Z) \end{array}$$

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for $\mathcal{F}$. Let ξ be a nonzero element of $H^q_{étale}(X, \mathcal{F})$. By Lemma 59.82.1 applied with $Z = X$ (!) we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(X, \mathcal{Q})$ and bijectivity for $q - 1$ implies ξ does not map to zero in $H^q_{étale}(Z, \mathcal{F}|_Z)$.

Surjectivity for $\mathcal{F}$. Let ξ be an element of $H^q_{étale}(Z, \mathcal{F}|_Z)$. By Lemma 59.82.1 applied with $Z = Z$ we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(Z, \mathcal{Q}|_Z)$ and bijectivity for $q-1$ implies ξ is in the image of the vertical map. □

0A51 **Lemma 59.82.8.** *Let X be a scheme with affine diagonal which can be covered by $n + 1$ affine opens. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{A}$ be a torsion sheaf of rings on $X_{étale}$ and let $\mathcal{I}$ be an injective sheaf of $\mathcal{A}$-modules on $X_{étale}$. Then $H^q_{étale}(Z, \mathcal{I}|_Z) = 0$ for $q > n$.*

Proof. We will prove this by induction on n. If $n = 0$, then X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let A^h be the filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. Then (A^h, IA^h) is a henselian pair

and $A/I = A^h/IA^h$, see More on Algebra, Lemma 15.12.1 and its proof. Set $X^h = \mathrm{Spec}(A^h)$. By Theorem 59.82.7 we see that

$$H^q_{étale}(Z, \mathcal{I}|_Z) = H^q_{étale}(X^h, \mathcal{I}|_{X^h})$$

By Theorem 59.51.3 we have

$$H^q_{étale}(X^h, \mathcal{I}|_{X^h}) = \mathrm{colim}_{A \to B} H^q_{étale}(\mathrm{Spec}(B), \mathcal{I}|_{\mathrm{Spec}(B)})$$

where the colimit is over the A-algebras B as above. Since the morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are étale, the restriction $\mathcal{I}|_{\mathrm{Spec}(B)}$ is an injective sheaf of $\mathcal{A}|_{\mathrm{Spec}(B)}$-modules (Cohomology on Sites, Lemma 21.7.1). Thus the cohomology groups on the right are zero and we get the result in this case.

Induction step. We can use Mayer-Vietoris to do the induction step. Namely, suppose that $X = U \cup V$ where U is a union of n affine opens and V is affine. Then, using that the diagonal of X is affine, we see that $U \cap V$ is the union of n affine opens. Mayer-Vietoris gives an exact sequence

$$H^{q-1}_{étale}(U \cap V \cap Z, \mathcal{I}|_Z) \to H^q_{étale}(Z, \mathcal{I}|_Z) \to H^q_{étale}(U \cap Z, \mathcal{I}|_Z) \oplus H^q_{étale}(V \cap Z, \mathcal{I}|_Z)$$

and by our induction hypothesis we obtain vanishing for $q > n$ as desired. □

59.83. Cohomology of torsion sheaves on curves

03SB The goal of this section is to prove the basic finiteness and vanishing results for cohomology of torsion sheaves on curves, see Theorem 59.83.10. In Section 59.84 we will discuss constructible sheaves of torsion modules over a Noetherian ring.

0A52 **Situation 59.83.1.** Here k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$.

In Situation 59.83.1 we want to prove the following statements

0A53 (1) $H^q_{étale}(X, \mathcal{F}) = 0$ for $q > 2$,
0A54 (2) $H^q_{étale}(X, \mathcal{F}) = 0$ for $q > 1$ if X is affine,
0A55 (3) $H^q_{étale}(X, \mathcal{F}) = 0$ for $q > 1$ if $p = \mathrm{char}(k) > 0$ and $\mathcal{F}$ is p-power torsion,
0A56 (4) $H^q_{étale}(X, \mathcal{F})$ is finite if $\mathcal{F}$ is constructible and torsion prime to $\mathrm{char}(k)$,
0A57 (5) $H^q_{étale}(X, \mathcal{F})$ is finite if X is proper and $\mathcal{F}$ constructible,
0A58 (6) $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if $\mathcal{F}$ is torsion prime to $\mathrm{char}(k)$,
0A59 (7) $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for any extension k'/k of algebraically closed fields if X is proper,
0A5A (8) $H^2_{étale}(X, \mathcal{F}) \to H^2_{étale}(U, \mathcal{F})$ is surjective for all $U \subset X$ open.

Given any Situation 59.83.1 we will say that "statements (1) – (8) hold" if those statements that apply to the given situation are true. We start the proof with the following consequence of our computation of cohomology with constant coefficients.

0A5B **Lemma 59.83.2.** *In Situation 59.83.1 assume X is smooth and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$ for some prime number ℓ. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Since X is smooth, we see that X is a finite disjoint union of smooth curves. Hence we may assume X is a smooth curve.

Case I: ℓ different from the characteristic of k. This case follows from Lemma 59.69.1 (projective case) and Lemma 59.69.3 (affine case). Statement (6) on cohomology and extension of algebraically closed ground field follows from the fact that the

genus g and the number of "punctures" r do not change when passing from k to k'. Statement (8) follows as $H^2_{étale}(U, \mathcal{F})$ is zero as soon as $U \neq X$, because then U is affine (Varieties, Lemmas 33.43.2 and 33.43.10).

Case II: ℓ is equal to the characteristic of k. Vanishing by Lemma 59.63.4. Statements (5) and (7) follow from Lemma 59.63.5. □

Remark 59.83.3 (Invariance under extension of algebraically closed ground field).
0A47 Let k be an algebraically closed field of characteristic $p > 0$. In Section 59.63 we have seen that there is an exact sequence

$$k[x] \to k[x] \to H^1_{étale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to 0$$

where the first arrow maps $f(x)$ to $f^p - f$. A set of representatives for the cokernel is formed by the polynomials

$$\sum\nolimits_{p \nmid n} \lambda_n x^n$$

with $\lambda_n \in k$. (If k is not algebraically closed you have to add some constants to this as well.) In particular when k'/k is an algebraically closed extension, then the map

$$H^1_{étale}(\mathbf{A}^1_k, \mathbf{Z}/p\mathbf{Z}) \to H^1_{étale}(\mathbf{A}^1_{k'}, \mathbf{Z}/p\mathbf{Z})$$

is not an isomorphism in general. In particular, the map $\pi_1(\mathbf{A}^1_{k'}) \to \pi_1(\mathbf{A}^1_k)$ between étale fundamental groups (insert future reference here) is not an isomorphism either. Thus the étale homotopy type of the affine line depends on the algebraically closed ground field. From Lemma 59.83.2 above we see that this is a phenomenon which only happens in characteristic p with p-power torsion coefficients.

0A5C **Lemma 59.83.4.** *Let k be an algebraically closed field. Let X be a separated finite type scheme over k of dimension ≤ 1. Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of torsion abelian sheaves on X. If statements (1) – (8) hold for $\mathcal{F}_1$ and $\mathcal{F}_2$, then they hold for $\mathcal{F}$.*

Proof. This is mostly immediate from the definitions and the long exact sequence of cohomology. Also observe that $\mathcal{F}$ is constructible (resp. of torsion prime to the characteristic of k) if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible (resp. of torsion prime to the characteristic of k). See Proposition 59.74.1. Some details omitted. □

0A5D **Lemma 59.83.5.** *Let k be an algebraically closed field. Let $f : X \to Y$ be a finite morphism of separated finite type schemes over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. If statements (1) – (8) hold for $\mathcal{F}$, then they hold for $f_*\mathcal{F}$.*

Proof. Namely, we have $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(Y, f_*\mathcal{F})$ by the vanishing of $R^q f_*$ for $q > 0$ (Proposition 59.55.2) and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). For (8) use that formation of f_* commutes with arbitrary base change (Lemma 59.55.3). □

0GJA **Lemma 59.83.6.** *In Situation 59.83.1 assume $\mathcal{F}$ constructible. Let $j : X' \to X$ be the inclusion of a dense open subscheme. Then statements (1) – (8) hold for $\mathcal{F}$ if and only if they hold for $j_!j^{-1}\mathcal{F}$.*

Proof. Since X' is dense, we see that $Z = X \setminus X'$ has dimension 0 and hence is a finite set $Z = \{x_1, \ldots, x_n\}$ of k-rational points. Consider the short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of Lemma 59.70.8. Observe that $H^q_{étale}(X, i_*i^{-1}\mathcal{F}) = H^q_{étale}(Z, i^*\mathcal{F})$. Namely, $i : Z \to X$ is a closed immersion, hence finite, hence we have the vanishing of $R^q i_*$ for $q > 0$ by Proposition 59.55.2, and hence the equality follows from the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). Since Z is a disjoint union of spectra of algebraically closed fields, we conclude that $H^q_{étale}(Z, i^*\mathcal{F}) = 0$ for $q > 0$ and

$$H^0_{étale}(Z, i^{-1}\mathcal{F}) = \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{x_i}$$

which is finite as $\mathcal{F}_{x_i}$ is finite due to the assumption that $\mathcal{F}$ is constructible. The long exact cohomology sequence gives an exact sequence

$$0 \to H^0_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{étale}(X, \mathcal{F}) \to H^0_{étale}(Z, i^{-1}\mathcal{F}) \to H^1_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^1_{étale}(X, \mathcal{F}) \to 0$$

and isomorphisms $H^q_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^q_{étale}(X, \mathcal{F})$ for $q > 1$.

At this point it is easy to deduce each of (1) – (8) holds for $\mathcal{F}$ if and only if it holds for $j_!j^{-1}\mathcal{F}$. We make a few small remarks to help the reader: (a) if $\mathcal{F}$ is torsion prime to the characteristic of k, then so is $j_!j^{-1}\mathcal{F}$, (b) the sheaf $j_!j^{-1}\mathcal{F}$ is constructible, (c) we have $H^0_{étale}(Z, i^{-1}\mathcal{F}) = H^0_{étale}(Z_{k'}, i^{-1}\mathcal{F}|_{Z_{k'}})$, and (d) if $U \subset X$ is an open, then $U' = U \cap X'$ is dense in U. □

03SG **Lemma 59.83.7.** *In Situation 59.83.1 assume X is smooth. Let $j : U \to X$ an open immersion. Let ℓ be a prime number. Let $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Since X is smooth, it is a disjoint union of smooth curves and hence we may assume X is a curve (i.e., irreducible). Then either $U = \emptyset$ and there is nothing to prove or $U \subset X$ is dense. In this case the lemma follows from Lemmas 59.83.2 and 59.83.6. □

0A3Q **Lemma 59.83.8.** *In Situation 59.83.1 assume X reduced. Let $j : U \to X$ an open immersion. Let ℓ be a prime number and $\mathcal{F} = j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. The difference with Lemma 59.83.7 is that here we do not assume X is smooth. Let $\nu : X^\nu \to X$ be the normalization morphism. Then ν is finite (Varieties, Lemma 33.27.1) and X^ν is smooth (Varieties, Lemma 33.43.8). Let $j^\nu : U^\nu \to X^\nu$ be the inverse image of U. By Lemma 59.83.7 the result holds for $j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. By Lemma 59.83.5 the result holds for $\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$. In general it won't be true that $\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ is equal to $j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ but we can work around this as follows. As X is reduced the morphism $\nu : X^\nu \to X$ is an isomorphism over a dense open $j' : X' \to X$ (Varieties, Lemma 33.27.1). Over this open we have agreement

$$(j')^{-1}(\nu_*j^\nu_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = (j')^{-1}(j_!\underline{\mathbf{Z}/\ell\mathbf{Z}})$$

Using Lemma 59.83.6 twice for $j' : X' \to X$ and the sheaves above we conclude. □

03SD **Lemma 59.83.9.** *In Situation 59.83.1 assume X reduced. Let $j : U \to X$ an open immersion with U connected. Let ℓ be a prime number. Let $\mathcal{G}$ a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U. Let $\mathcal{F} = j_!\mathcal{G}$. Then statements (1) – (8) hold for $\mathcal{F}$.*

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to ℓ as in Lemma 59.66.2. The discussion in Section 59.66 gives maps

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \to \mathcal{G}$$

whose composition is an isomorphism. Hence it suffices to prove the lemma with $\mathcal{F} = j_!f_*f^{-1}\mathcal{G}$. By Zariski's Main theorem (More on Morphisms, Lemma 37.43.3) we can choose a diagram

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ U & \xrightarrow{j} & X \end{array}$$

with $\overline{f} : Y \to X$ finite and j' an open immersion with dense image. We may replace Y by its reduction (this does not change V as V is reduced being étale over U). Since f is finite and V dense in Y we have $V = U \times_X Y$. By Lemma 59.70.9 we have

$$j_!f_*f^{-1}\mathcal{G} = \overline{f}_*j'_!f^{-1}\mathcal{G}$$

By Lemma 59.83.5 it suffices to consider $j'_!f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 59.66.2, the fact that $j'_!$ is exact, and Lemma 59.83.4 reduces us to the case $\mathcal{F} = j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}$ which is Lemma 59.83.8. □

03SC **Theorem 59.83.10.** *If k is an algebraically closed field, X is a separated, finite type scheme of dimension ≤ 1 over k, and $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$, then*

(1) $H^q_{étale}(X, \mathcal{F}) = 0$ *for* $q > 2$,
(2) $H^q_{étale}(X, \mathcal{F}) = 0$ *for* $q > 1$ *if X is affine,*
(3) $H^q_{étale}(X, \mathcal{F}) = 0$ *for* $q > 1$ *if* $p = char(k) > 0$ *and $\mathcal{F}$ is p-power torsion,*
(4) $H^q_{étale}(X, \mathcal{F})$ *is finite if $\mathcal{F}$ is constructible and torsion prime to* $char(k)$,
(5) $H^q_{étale}(X, \mathcal{F})$ *is finite if X is proper and $\mathcal{F}$ constructible,*
(6) $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ *is an isomorphism for any extension k'/k of algebraically closed fields if $\mathcal{F}$ is torsion prime to* $char(k)$,
(7) $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ *is an isomorphism for any extension k'/k of algebraically closed fields if X is proper,*
(8) $H^2_{étale}(X, \mathcal{F}) \to H^2_{étale}(U, \mathcal{F})$ *is surjective for all $U \subset X$ open.*

Proof. The theorem says that in Situation 59.83.1 statements (1) – (8) hold. Our first step is to replace X by its reduction, which is permissible by Proposition 59.45.4. By Lemma 59.73.2 we can write $\mathcal{F}$ as a filtered colimit of constructible abelian sheaves. Taking cohomology commutes with colimits, see Lemma 59.51.4. Moreover, pullback via $X_{k'} \to X$ commutes with colimits as a left adjoint. Thus it suffices to prove the statements for a constructible sheaf.

In this paragraph we use Lemma 59.83.4 without further mention. Writing $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is ℓ_i-primary for some prime ℓ_i, we may assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.$$

Thus we see that it suffices to assume that $\mathcal{F}$ is ℓ-torsion. This means that $\mathcal{F}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces for some prime number ℓ.

By definition this means there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces. Since $\dim(X) \leq 1$ we may assume, after shrinking U, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of ≥ 2 components of U). By

Lemma 59.83.6 we reduce to the case $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{G}$ is a finite locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U.

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with U_i irreducible we have

$$j_!\mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{i!}(\mathcal{G}|_{U_i})$ is handled in Lemma 59.83.9. □

03RT **Theorem 59.83.11.** *Let X be a finite type, dimension 1 scheme over an algebraically closed field k. Let $\mathcal{F}$ be a torsion sheaf on $X_{étale}$. Then*

$$H^q_{étale}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$

If X affine then also $H^2_{étale}(X, \mathcal{F}) = 0$.

Proof. If X is separated, this follows immediately from the more precise Theorem 59.83.10. If X is nonseparated, choose an affine open covering $X = X_1 \cup \ldots \cup X_n$. By induction on n we may assume the vanishing holds over $U = X_1 \cup \ldots \cup X_{n-1}$. Then Mayer-Vietoris (Lemma 59.50.1) gives

$$H^2_{étale}(U, \mathcal{F}) \oplus H^2_{étale}(X_n, \mathcal{F}) \to H^2_{étale}(U \cap X_n, \mathcal{F}) \to H^3_{étale}(X, \mathcal{F}) \to 0$$

However, since $U \cap X_n$ is an open of an affine scheme and hence affine by our dimension assumption, the group $H^2_{étale}(U \cap X_n, \mathcal{F})$ vanishes by Theorem 59.83.10. □

0A5E **Lemma 59.83.12.** *Let k'/k be an extension of separably closed fields. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.*

Proof. We have seen this for algebraically closed fields in Theorem 59.83.10. Given $k \subset k'$ as in the statement of the lemma we can choose a diagram

$$\begin{array}{ccc} k' & \longrightarrow & \overline{k}' \\ \uparrow & & \uparrow \\ k & \longrightarrow & \overline{k} \end{array}$$

where $k \subset \overline{k}$ and $k' \subset \overline{k}'$ are the algebraic closures. Since k and k' are separably closed the field extensions $\overline{k}/k$ and $\overline{k}'/k'$ are algebraic and purely inseparable. In this case the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{k}'} \to X_{k'}$ are universal homeomorphisms. Thus the cohomology of $\mathcal{F}$ may be computed on $X_{\overline{k}}$ and the cohomology of $\mathcal{F}|_{X_{k'}}$ may be computed on $X_{\overline{k}'}$, see Proposition 59.45.4. Hence we deduce the general case from the case of algebraically closed fields. □

59.84. Cohomology of torsion modules on curves

0GJB In this section we repeat the arguments of Section 59.83 for constructible sheaves of modules over a Noetherian ring which are torsion. We start with the most interesting step.

0GJC **Lemma 59.84.1.** *Let Λ be a Noetherian ring, let M be a finite Λ-module which is annihilated by an integer $n > 0$, let k be an algebraically closed field, and let X be a separated, finite type scheme of dimension ≤ 1 over k. Then*

(1) $H^q_{étale}(X, \underline{M})$ *is a finite* Λ*-module if* n *is prime to* $char(k)$*,*
(2) $H^q_{étale}(X, \underline{M})$ *is a finite* Λ*-module if* X *is proper.*

Proof. If $n = \ell n'$ for some prime number ℓ, then we get a short exact sequence $0 \to M[\ell] \to M \to M' \to 0$ of finite Λ-modules and M' is annihilated by n'. This produces a corresponding short exact sequence of constant sheaves, which in turn gives rise to an exact sequence of cohomology modules

$$H^q_{étale}(X, \underline{M[n]}) \to H^q_{étale}(X, \underline{M}) \to H^q_{étale}(X, \underline{M'})$$

Thus, if we can show the result in case M is annihilated by a prime number, then by induction on n we win.

Let ℓ be a prime number such that ℓ annihilates M. Then we can replace Λ by the $\mathbf{F}_\ell$-algebra $\Lambda/\ell\Lambda$. Namely, the cohomology of $\mathcal{F}$ as a sheaf of Λ-modules is the same as the cohomology of $\mathcal{F}$ as a sheaf of $\Lambda/\ell\Lambda$-modules, for example by Cohomology on Sites, Lemma 21.12.4.

Assume ℓ be a prime number such that ℓ annihilates M and Λ. Let us reduce to the case where M is a finite free Λ-module. Namely, choose a short exact sequence

$$0 \to N \to \Lambda^{\oplus m} \to M \to 0$$

This determines an exact sequence

$$H^q_{étale}(X, \underline{\Lambda^{\oplus m}}) \to H^q_{étale}(X, \underline{M}) \to H^{q+1}_{étale}(X, \underline{N})$$

By descending induction on q we get the result for M if we know the result for $\Lambda^{\oplus m}$. Here we use that we know that our cohomology groups vanish in degrees > 2 by Theorem 59.83.10.

Let ℓ be a prime number and assume that ℓ annihilates Λ. It remains to show that the cohomology groups $H^q_{étale}(X, \underline{\Lambda})$ are finite Λ-modules. We will use a trick to show this; the "correct" argument uses a coefficient theorem which we will show later. Choose a basis $\Lambda = \bigoplus_{i \in I} \mathbf{F}_\ell e_i$ such that $e_0 = 1$ for some $0 \in I$. The choice of this basis determines an isomorphism

$$\underline{\Lambda} = \bigoplus \underline{\mathbf{F}_\ell} e_i$$

of sheaves on $X_{étale}$. Thus we see that

$$H^q_{étale}(X, \underline{\Lambda}) = H^q_{étale}(X, \bigoplus \underline{\mathbf{F}_\ell} e_i) = \bigoplus H^q_{étale}(X, \underline{\mathbf{F}_\ell}) e_i$$

since taking cohomology over X commutes with direct sums by Theorem 59.51.3 (or Lemma 59.51.4 or Lemma 59.52.2). Since we already know that $H^q_{étale}(X, \underline{\mathbf{F}_\ell})$ is a finite dimensional $\mathbf{F}_\ell$-vector space (by Theorem 59.83.10), we see that $H^q_{étale}(X, \underline{\Lambda})$ is free over Λ of the same rank. Namely, given a basis $\xi_1, \ldots, \xi_m$ of $H^q_{étale}(X, \underline{\mathbf{F}_\ell})$ we see that $\xi_1 e_0, \ldots, \xi_m e_0$ form a Λ-basis for $H^q_{étale}(X, \underline{\Lambda})$. □

0GJD **Lemma 59.84.2.** *Let* Λ *be a Noetherian ring, let* k *be an algebraically closed field, let* $f : X \to Y$ *be a finite morphism of separated finite type schemes over* k *of dimension* ≤ 1*, and let* $\mathcal{F}$ *be a sheaf of* Λ*-modules on* $X_{étale}$*. If* $H^q_{étale}(X, \mathcal{F})$ *is a finite* Λ*-module, then so is* $H^q_{étale}(Y, f_*\mathcal{F})$*.*

Proof. Namely, we have $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(Y, f_*\mathcal{F})$ by the vanishing of $R^q f_*$ for $q > 0$ (Proposition 59.55.2) and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). □

0GJE **Lemma 59.84.3.** *Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated finite type scheme over k of dimension ≤ 1, let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$, and let $j : X' \to X$ be the inclusion of a dense open subscheme. Then $H^q_{étale}(X, \mathcal{F})$ is a finite Λ-module if and only if $H^q_{étale}(X, j_!j^{-1}\mathcal{F})$ is a finite Λ-module.*

Proof. Since X' is dense, we see that $Z = X \setminus X'$ has dimension 0 and hence is a finite set $Z = \{x_1, \ldots, x_n\}$ of k-rational points. Consider the short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of Lemma 59.70.8. Observe that $H^q_{étale}(X, i_*i^{-1}\mathcal{F}) = H^q_{étale}(Z, i^*\mathcal{F})$. Namely, $i : Z \to X$ is a closed immersion, hence finite, hence we have the vanishing of R^qi_* for $q > 0$ by Proposition 59.55.2, and hence the equality follows from the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). Since Z is a disjoint union of spectra of algebraically closed fields, we conclude that $H^q_{étale}(Z, i^*\mathcal{F}) = 0$ for $q > 0$ and

$$H^0_{étale}(Z, i^{-1}\mathcal{F}) = \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{x_i}$$

which is a finite Λ-module $\mathcal{F}_{x_i}$ is finite due to the assumption that $\mathcal{F}$ is a constructible sheaf of Λ-modules. The long exact cohomology sequence gives an exact sequence

$$0 \to H^0_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{étale}(X, \mathcal{F}) \to H^0_{étale}(Z, i^{-1}\mathcal{F}) \to H^1_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^1_{étale}(X, \mathcal{F}) \to 0$$

and isomorphisms $H^0_{étale}(X, j_!j^{-1}\mathcal{F}) \to H^0_{étale}(X, \mathcal{F})$ for $q > 1$. The lemma follows easily from this. □

0GJF **Lemma 59.84.4.** *Let Λ be a Noetherian ring, let M be a finite Λ-module which is annihilated by an integer $n > 0$, let k be an algebraically closed field, let X be a separated, finite type scheme of dimension ≤ 1 over k, and let $j : U \to X$ be an open immersion. Then*

(1) *$H^q_{étale}(X, j_!\underline{M})$ is a finite Λ-module if n is prime to $char(k)$,*
(2) *$H^q_{étale}(X, j_!\underline{M})$ is a finite Λ-module if X is proper.*

Proof. Since $\dim(X) \leq 1$ there is an open $V \subset X$ which is disjoint from U such that $X' = U \cup V$ is dense open in X (details omitted). If $j' : X' \to X$ denotes the inclusion morphism, then we see that $j_!\underline{M}$ is a direct summand of $j'_!\underline{M}$. Hence it suffices to prove the lemma in case U is open and dense in X. This case follows from Lemmas 59.84.3 and 59.84.1. □

0GJG **Lemma 59.84.5.** *Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated finite type scheme over k of dimension ≤ 1, and let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be a short exact sequence of sheaves of Λ-modules on $X_{étale}$. If $H^q_{étale}(X, \mathcal{F}_i)$, $i = 1, 2$ are finite Λ-modules then $H^q_{étale}(X, \mathcal{F})$ is a finite Λ-module.*

Proof. Immediate from the long exact sequence of cohomology. □

0GJH **Lemma 59.84.6.** *Let Λ be a Noetherian ring, let k be an algebraically closed field, let X be a separated, finite type scheme of dimension ≤ 1 over k, let $j : U \to X$ be an open immersion with U connected, let ℓ be a prime number, let $n > 0$, and let $\mathcal{G}$ be a finite type, locally constant sheaf of Λ-modules on $U_{étale}$ annihilated by ℓ^n. Then*

(1) $H^q_{étale}(X, j_!\mathcal{G})$ *is a finite* Λ*-module if* ℓ *is prime to* $char(k)$,
(2) $H^q_{étale}(X, j_!\mathcal{G})$ *is a finite* Λ*-module if* X *is proper.*

Proof. Let $f : V \to U$ be a finite étale morphism of degree prime to ℓ as in Lemma 59.66.4. The discussion in Section 59.66 gives maps

$$\mathcal{G} \to f_* f^{-1}\mathcal{G} \to \mathcal{G}$$

whose composition is an isomorphism. Hence it suffices to prove the finiteness of $H^q_{étale}(X, j_! f_* f^{-1}\mathcal{G})$. By Zariski's Main theorem (More on Morphisms, Lemma 37.43.3) we can choose a diagram

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ U & \xrightarrow{j} & X \end{array}$$

with $\overline{f} : Y \to X$ finite and j' an open immersion with dense image. Since f is finite and V dense in Y we have $V = U \times_X Y$. By Lemma 59.70.9 we have

$$j_! f_* f^{-1}\mathcal{G} = \overline{f}_* j'_! f^{-1}\mathcal{G}$$

By Lemma 59.84.2 it suffices to consider $j'_! f^{-1}\mathcal{G}$. The existence of the filtration given by Lemma 59.66.4, the fact that $j'_!$ is exact, and Lemma 59.84.5 reduces us to the case $\mathcal{F} = j'_!\underline{M}$ for a finite Λ-module M which is Lemma 59.84.4. □

0GJI **Theorem 59.84.7.** *Let* Λ *be a Noetherian ring, let* k *be an algebraically closed field, let* X *be a separated, finite type scheme of dimension* ≤ 1 *over* k*, and let* $\mathcal{F}$ *be a constructible sheaf of* Λ*-modules on* $X_{étale}$ *which is torsion. Then*

0GJJ (1) $H^q_{étale}(X, \mathcal{F})$ *is a finite* Λ*-module if* $\mathcal{F}$ *is torsion prime to* $char(k)$,
0GJK (2) $H^q_{étale}(X, \mathcal{F})$ *is a finite* Λ*-module if* X *is proper.*

Proof. without further mention. Write $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is annihilated by $\ell_i^{n_i}$ for some prime ℓ_i and integer $n_i > 0$. By Lemma 59.84.5 it suffices to prove the theorem for $\mathcal{F}_i$. Thus we may and do assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ and integer $n > 0$.

Since $\mathcal{F}$ is constructible as a sheaf of Λ-modules, there is a dense open $U \subset X$ such that $\mathcal{F}|_U$ is a finite type, locally constant sheaf of Λ-modules. Since $\dim(X) \leq 1$ we may assume, after shrinking U, that $U = U_1 \amalg \ldots \amalg U_n$ is a disjoint union of irreducible schemes (just remove the closed points which lie in the intersections of ≥ 2 components of U). By Lemma 59.84.3 we reduce to the case $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{G}$ is a finite type, locally constant sheaf of Λ-modules on U (and annihilated by ℓ^n).

Since we chose $U = U_1 \amalg \ldots \amalg U_n$ with U_i irreducible we have

$$j_!\mathcal{G} = j_{1!}(\mathcal{G}|_{U_1}) \oplus \ldots \oplus j_{n!}(\mathcal{G}|_{U_n})$$

where $j_i : U_i \to X$ is the inclusion morphism. The case of $j_{i!}(\mathcal{G}|_{U_i})$ is handled in Lemma 59.84.6. □

59.85. First cohomology of proper schemes

0A5F In Fundamental Groups, Section 58.9 we have seen, in some sense, that taking $R^1 f_* \underline{G}$ commutes with base change if $f : X \to Y$ is a proper morphism and G is a finite group (not necessarily commutative). In this section we deduce a useful consequence of these results.

0A5G **Lemma 59.85.1.** *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Let M be a finite abelian group. Then $H^1_{étale}(X, \underline{M}) = H^1_{étale}(X_0, \underline{M})$.*

Proof. By Cohomology on Sites, Lemma 21.4.3 an element of $H^1_{étale}(X, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $X_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over X, Lemma 59.64.4. Conversely, a scheme V finite étale over X with an M-action which turns it into an M-torsor over X gives rise to a cohomology class. The same translation between cohomology classes over X_0 and torsors finite étale over X_0 holds. Thus the lemma is a consequence of the equivalence of categories of Fundamental Groups, Lemma 58.9.1. □

The following technical lemma is a key ingredient in the proof of the proper base change theorem. The argument works word for word for any proper scheme over A whose special fibre has dimension ≤ 1, but in fact the conclusion will be a consequence of the proper base change theorem and we only need this particular version in its proof.

0A5H **Lemma 59.85.2.** *Let A be a henselian local ring. Let $X = \mathbf{P}^1_A$. Let $X_0 \subset X$ be the closed fibre. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Then $H^q_{étale}(X_0, \mathcal{I}|_{X_0}) = 0$ for $q > 0$.*

Proof. Observe that X is a separated scheme which can be covered by 2 affine opens. Hence for $q > 1$ this follows from Gabber's affine variant of the proper base change theorem, see Lemma 59.82.8. Thus we may assume $q = 1$. Let $\xi \in H^1_{étale}(X_0, \mathcal{I}|_{X_0})$. Goal: show that ξ is 0. By Lemmas 59.73.2 and 59.51.4 we can find a map $\mathcal{F} \to \mathcal{I}$ with $\mathcal{F}$ a constructible sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules and ξ coming from an element ζ of $H^1_{étale}(X_0, \mathcal{F}|_{X_0})$. Suppose we have an injective map $\mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Since $\mathcal{I}$ is injective we can extend the given map $\mathcal{F} \to \mathcal{I}$ to a map $\mathcal{F}' \to \mathcal{I}$. In this situation we may replace $\mathcal{F}$ by $\mathcal{F}'$ and ζ by the image of ζ in $H^1_{étale}(X_0, \mathcal{F}'|_{X_0})$. Also, if $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is a direct sum, then we may replace $\mathcal{F}$ by $\mathcal{F}_i$ and ζ by the image of ζ in $H^1_{étale}(X_0, \mathcal{F}_i|_{X_0})$.

By Lemma 59.74.4 and the remarks above we may assume $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite $\mathbf{Z}/\ell\mathbf{Z}$-module and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 59.73.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 59.55.3) we see that the restriction of $f_*\underline{M}$ to X_0 is equal to the pushforward of $\underline{M}$ via the induced morphism $Y_0 \to X_0$ of special fibres. By the Leray spectral sequence (Proposition 59.54.2) and vanishing of higher direct images (Proposition 59.55.2), we find

$$H^1_{étale}(X_0, f_*\underline{M}|_{X_0}) = H^1_{étale}(Y_0, \underline{M}).$$

Since $Y \to \mathrm{Spec}(A)$ is proper we can use Lemma 59.85.1 to see that the $H^1_{étale}(Y_0, \underline{M})$ is equal to $H^1_{étale}(Y, \underline{M})$. Thus we see that our cohomology class ζ lifts to a cohomology class

$$\tilde{\zeta} \in H^1_{étale}(Y, \underline{M}) = H^1_{étale}(X, f_*\underline{M})$$

However, $\tilde{\zeta}$ maps to zero in $H^1_{étale}(X, \mathcal{I})$ as $\mathcal{I}$ is injective and by commutativity of

$$\begin{array}{ccc} H^1_{étale}(X, f_*\underline{M}) & \longrightarrow & H^1_{étale}(X, \mathcal{I}) \\ \downarrow & & \downarrow \\ H^1_{étale}(X_0, (f_*\underline{M})|_{X_0}) & \longrightarrow & H^1_{étale}(X_0, \mathcal{I}|_{X_0}) \end{array}$$

we conclude that the image ξ of ζ is zero as well. □

59.86. Preliminaries on base change

0EZQ If you are interested in either the smooth base change theorem or the proper base change theorem, you should skip directly to the corresponding sections. In this section and the next few sections we consider commutative diagrams

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

of schemes; we usually assume this diagram is cartesian, i.e., $Y = X \times_S T$. A commutative diagram as above gives rise to a commutative diagram

$$\begin{array}{ccc} X_{étale} & \xleftarrow{h_{small}} & Y_{étale} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle e_{small}} \\ S_{étale} & \xleftarrow{g_{small}} & T_{étale} \end{array}$$

of small étale sites. Let us use the notation

$$f^{-1} = f^{-1}_{small}, \quad g_* = g_{small,*}, \quad e^{-1} = e^{-1}_{small}, \text{ and } \quad h_* = h_{small,*}.$$

By Sites, Section 7.45 we get a base change or pullback map

$$f^{-1}g_*\mathcal{F} \longrightarrow h_*e^{-1}\mathcal{F}$$

for a sheaf $\mathcal{F}$ on $T_{étale}$. If $\mathcal{F}$ is an abelian sheaf on $T_{étale}$, then we get a derived base change map

$$f^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

see Cohomology on Sites, Lemma 21.15.1. Finally, if K is an arbitrary object of $D(T_{étale})$ there is a base change map

$$f^{-1}Rg_*K \longrightarrow Rh_*e^{-1}K$$

see Cohomology on Sites, Remark 21.19.3.

0EZR **Lemma 59.86.1.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Let $\{U_i \to X\}$ be an étale covering such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and consider the cartesian diagrams

$$\begin{array}{ccc} U_i & \xleftarrow[h_i]{} & U_i \times_X Y \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle e_i} \\ V_i & \xleftarrow{g_i} & V_i \times_S T \end{array}$$

Let $\mathcal{F}$ be a sheaf on $T_{étale}$. Let K in $D(T_{étale})$. Set $K_i = K|_{V_i \times_S T}$ and $\mathcal{F}_i = \mathcal{F}|_{V_i \times_S T}$.

1. *If $f_i^{-1} g_{i,*} \mathcal{F}_i = h_{i,*} e_i^{-1} \mathcal{F}_i$ for all i, then $f^{-1} g_* \mathcal{F} = h_* e^{-1} \mathcal{F}$.*
2. *If $f_i^{-1} R g_{i,*} K_i = R h_{i,*} e_i^{-1} K_i$ for all i, then $f^{-1} R g_* K = R h_* e^{-1} K$.*
3. *If $\mathcal{F}$ is an abelian sheaf and $f_i^{-1} R^q g_{i,*} \mathcal{F}_i = R^q h_{i,*} e_i^{-1} \mathcal{F}_i$ for all i, then $f^{-1} R^q g_* \mathcal{F} = R^q h_* e^{-1} \mathcal{F}$.*

Proof. Proof of (1). First we observe that

$$(f^{-1} g_* \mathcal{F})|_{U_i} = f_i^{-1}(g_* \mathcal{F}|_{V_i}) = f_i^{-1} g_{i,*} \mathcal{F}_i$$

The first equality because $U_i \to X \to S$ is equal to $U_i \to V_i \to S$ and the second equality because $g_* \mathcal{F}|_{V_i} = g_{i,*} \mathcal{F}_i$ by Sites, Lemma 7.28.2. Similarly we have

$$(h_* e^{-1} \mathcal{F})|_{U_i} = h_{i,*}(e^{-1} \mathcal{F}|_{U_i \times_X Y}) = h_{i,*} e_i^{-1} \mathcal{F}_i$$

Thus if the base change maps $f_i^{-1} g_{i,*} \mathcal{F}_i \to h_{i,*} e_i^{-1} \mathcal{F}_i$ are isomorphisms for all i, then the base change map $f^{-1} g_* \mathcal{F} \to h_* e^{-1} \mathcal{F}$ restricts to an isomorphism over U_i for all i and we conclude it is an isomorphism as $\{U_i \to X\}$ is an étale covering.

For the other two statements we replace the appeal to Sites, Lemma 7.28.2 by an appeal to Cohomology on Sites, Lemma 21.20.4. □

0EZS **Lemma 59.86.2.** *Consider a tower of cartesian diagrams of schemes*

$$\begin{array}{ccc} W & \xleftarrow[j]{} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xleftarrow[h]{} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Let K in $D(T_{étale})$. If

$$f^{-1} R g_* K \to R h_* e^{-1} K \quad \text{and} \quad i^{-1} R h_* e^{-1} K \to R j_* k^{-1} e^{-1} K$$

are isomorphisms, then $(f \circ i)^{-1} R g_* K \to R j_* (e \circ k)^{-1} K$ is an isomorphism. Similarly, if $\mathcal{F}$ is an abelian sheaf on $T_{étale}$ and if

$$f^{-1} R^q g_* \mathcal{F} \to R^q h_* e^{-1} \mathcal{F} \quad \text{and} \quad i^{-1} R^q h_* e^{-1} \mathcal{F} \to R^q j_* k^{-1} e^{-1} \mathcal{F}$$

are isomorphisms, then $(f \circ i)^{-1} R^q g_* \mathcal{F} \to R^q j_* (e \circ k)^{-1} \mathcal{F}$ is an isomorphism.

Proof. This is formal, provided one checks that the composition of these base change maps is the base change maps for the outer rectangle, see Cohomology on Sites, Remark 21.19.5. □

0EZT **Lemma 59.86.3.** *Let I be a directed set. Consider an inverse system of cartesian diagrams of schemes*

$$\begin{array}{ccc} X_i & \xleftarrow{h_i} & Y_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle e_i} \\ S_i & \xleftarrow{g_i} & T_i \end{array}$$

with affine transition morphisms and with g_i quasi-compact and quasi-separated. Set $X = \lim X_i$, $S = \lim S_i$, $T = \lim T_i$ and $Y = \lim Y_i$ to obtain the cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Let $(\mathcal{F}_i, \varphi_{i'i})$ be a system of sheaves on (T_i) as in Definition 59.51.1. Set $\mathcal{F} = \operatorname{colim} p_i^{-1}\mathcal{F}_i$ on T where $p_i : T \to T_i$ is the projection. Then we have the following

1. *If $f_i^{-1}g_{i,*}\mathcal{F}_i = h_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.*
2. *If $\mathcal{F}_i$ is an abelian sheaf for all i and $f_i^{-1}R^qg_{i,*}\mathcal{F}_i = R^qh_{i,*}e_i^{-1}\mathcal{F}_i$ for all i, then $f^{-1}R^qg_*\mathcal{F} = R^qh_*e^{-1}\mathcal{F}$.*

Proof. We prove (2) and we omit the proof of (1). We will use without further mention that pullback of sheaves commutes with colimits as it is a left adjoint. Observe that h_i is quasi-compact and quasi-separated as a base change of g_i. Denoting $q_i : Y \to Y_i$ the projections, observe that $e^{-1}\mathcal{F} = \operatorname{colim} e^{-1}p_i^{-1}\mathcal{F}_i = \operatorname{colim} q_i^{-1}e_i^{-1}\mathcal{F}_i$. By Lemma 59.51.8 this gives

$$R^qh_*e^{-1}\mathcal{F} = \operatorname{colim} r_i^{-1}R^qh_{i,*}e_i^{-1}\mathcal{F}_i$$

where $r_i : X \to X_i$ is the projection. Similarly, we have

$$f^{-1}Rg_*\mathcal{F} = f^{-1}\operatorname{colim} s_i^{-1}R^qg_{i,*}\mathcal{F}_i = \operatorname{colim} r_i^{-1}f_i^{-1}R^qg_{i,*}\mathcal{F}_i$$

where $s_i : S \to S_i$ is the projection. The lemma follows. □

0GJL **Lemma 59.86.4.** *Let I, X_i, Y_i, S_i, T_i, f_i, h_i, e_i, g_i, X, Y, S, T, f, h, e, g be as in the statement of Lemma 59.86.3. Let $0 \in I$ and let $K_0 \in D^+(T_{0,étale})$. For $i \in I$, $i \geq 0$ denote K_i the pullback of K_0 to T_i. Denote K the pullback of K_0 to T. If $f_i^{-1}Rg_{i,*}K_i = Rh_{i,*}e_i^{-1}K_i$ for all $i \geq 0$, then $f^{-1}Rg_*K = Rh_*e^{-1}K$.*

Proof. It suffices to show that the base change map $f^{-1}Rg_*K \to Rh_*e^{-1}K$ induces an isomorphism on cohomology sheaves. In other words, we have to show that $f^{-1}R^pg_*K \to R^ph_*e^{-1}K$ is an isomorphism for all $p \in \mathbf{Z}$ if we are given that $f_i^{-1}R^pg_{i,*}K_i \to R^ph_{i,*}e_i^{-1}K_i$ is an isomorphism for all $i \geq 0$ and $p \in \mathbf{Z}$. At this point we can argue exactly as in the proof of Lemma 59.86.3 replacing reference to Lemma 59.51.8 by a reference to Lemma 59.52.4. □

0EZU **Lemma 59.86.5.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

where $g : T \to S$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$. Let $q \geq 0$. The following are equivalent

(1) *For every geometric point $\overline{x}$ of X with image $\overline{s} = f(\overline{x})$ we have*
$$H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T, \mathcal{F}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S T, \mathcal{F})$$
(2) *$f^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Since $Y = X \times_S T$ we have $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y = \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S T$. Thus the map in (1) is the map of stalks at $\overline{x}$ for the map in (2) by Theorem 59.53.1 (and Lemma 59.36.2). Thus the result by Theorem 59.29.10. □

0EZV **Lemma 59.86.6.** *Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s}$ in S. Let $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ be a morphism with K a separably closed field. Let $\mathcal{F}$ be an abelian sheaf on $\mathrm{Spec}(K)_{étale}$. Let $q \geq 0$. The following are equivalent*

(1) $H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \mathrm{Spec}(K), \mathcal{F}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \mathrm{Spec}(K), \mathcal{F})$
(2) $H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \mathrm{Spec}(K), \mathcal{F}) = H^q(\mathrm{Spec}(K), \mathcal{F})$

Proof. Observe that $\mathrm{Spec}(K) \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is the spectrum of a filtered colimit of étale algebras over K. Since K is separably closed, each étale K-algebra is a finite product of copies of K. Thus we can write
$$\mathrm{Spec}(K) \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{i \in I} \coprod\nolimits_{a \in A_i} \mathrm{Spec}(K)$$
as a cofiltered limit where each term is a disjoint union of copies of $\mathrm{Spec}(K)$ over a finite set A_i. Note that A_i is nonempty as we are given $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. It follows that
$$\begin{aligned}\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \mathrm{Spec}(K) &= \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \big(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \mathrm{Spec}(K)\big) \\ &= \lim_{i \in I} \coprod\nolimits_{a \in A_i} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \mathrm{Spec}(K)\end{aligned}$$
Since taking cohomology in our setting commutes with limits of schemes (Theorem 59.51.3) we conclude. □

59.87. Base change for pushforward

0EZW This section is preliminary and should be skipped on a first reading. In this section we discuss for what morphisms $f : X \to S$ we have $f^{-1}g_* = h_* e^{-1}$ on all sheaves (of sets) for every cartesian diagram
$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$
with g quasi-compact and quasi-separated.

0EZX **Lemma 59.87.1.** *Consider the cartesian diagram of schemes*
$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Assume that f is flat and every object U of $X_{étale}$ has a covering $\{U_i \to U\}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ with $V_i \to S$ étale and $U_i \to V_i$ quasi-compact with geometrically connected fibres. Then for any sheaf $\mathcal{F}$ of sets on $T_{étale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ quasi-compact with geometrically connected fibres. Observe that $U \to V$ is flat (More on Flatness, Lemma 38.2.3). We claim that

$$\begin{aligned} f^{-1}g_*\mathcal{F}(U) &= g_*\mathcal{F}(V) \\ &= \mathcal{F}(V \times_S T) \\ &= e^{-1}\mathcal{F}(U \times_X Y) \\ &= h_*e^{-1}\mathcal{F}(U) \end{aligned}$$

Namely, thinking of U as an object of $X_{étale}$ and V as an object of $S_{étale}$ we see that the first equality follows from Lemma 59.39.3[9]. Thinking of $V \times_S T$ as an object of $T_{étale}$ the second equality follows from the definition of g_*. Observe that $U \times_X Y = U \times_S T$ (because $Y = X \times_S T$) and hence $U \times_X Y \to V \times_S T$ has geometrically connected fibres as a base change of $U \to V$. Thinking of $U \times_X Y$ as an object of $Y_{étale}$, we see that the third equality follows from Lemma 59.39.3 as before. Finally, the fourth equality follows from the definition of h_*.

Since by assumption every object of $X_{étale}$ has an étale covering to which the argument of the previous paragraph applies we see that the lemma is true. □

0EYS **Lemma 59.87.2.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

where f is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{étale}$.

Proof. Combine Lemma 59.87.1 with More on Morphisms, Lemma 37.46.3. □

0EZY **Lemma 59.87.3.** *Consider the cartesian diagrams of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Assume that S is the spectrum of a separably closed field. Then $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ on $T_{étale}$.

Proof. We may work locally on X. Hence we may assume X is affine. Then we can write X as a cofiltered limit of affine schemes of finite type over S. By Lemma 59.86.3 we may assume that X is of finite type over S. Then Lemma 59.87.1 applies because any scheme of finite type over a separably closed field is a finite disjoint

[9] Strictly speaking, we are also using that the restriction of $f^{-1}g_*\mathcal{F}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $g_*\mathcal{F}$ to $V_{étale}$. See Sites, Lemma 7.28.2.

union of connected and geometrically connected schemes (see Varieties, Lemma 33.7.6). □

0EZZ **Lemma 59.87.4.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

Assume that

(1) *f is flat and open,*
(2) *the residue fields of S are separably algebraically closed,*
(3) *given an étale morphism $U \to X$ with U affine we can write U as a finite disjoint union of open subschemes of X (for example if X is a normal integral scheme with separably closed function field),*
(4) *any nonempty open of a fibre X_s of f is connected (for example if X_s is irreducible or empty).*

Then for any sheaf $\mathcal{F}$ of sets on $T_{étale}$ we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$.

Proof. Omitted. Hint: the assumptions almost trivially imply the condition of Lemma 59.87.1. The for example in part (3) follows from Lemma 59.80.4. □

The following lemma doesn't really belong here but there does not seem to be a good place for it anywhere.

0EYR **Lemma 59.87.5.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation with geometrically reduced fibres. Then $f^{-1} : Sh(S_{étale}) \to Sh(X_{étale})$ commutes with products.*

Proof. Let I be a set and let $\mathcal{G}_i$ be a sheaf on $S_{étale}$ for $i \in I$. Let $U \to X$ be an étale morphism such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale and $U \to V$ flat of finite presentation with geometrically connected fibres. Then we have

$$\begin{aligned} f^{-1}(\prod \mathcal{G}_i)(U) &= (\prod \mathcal{G}_i)(V) \\ &= \prod \mathcal{G}_i(V) \\ &= \prod f^{-1}\mathcal{G}_i(U) \\ &= (\prod f^{-1}\mathcal{G}_i)(U) \end{aligned}$$

where we have used Lemma 59.39.3 in the first and third equality (we are also using that the restriction of $f^{-1}\mathcal{G}$ to $U_{étale}$ is the pullback via $U \to V$ of the restriction of $\mathcal{G}$ to $V_{étale}$, see Sites, Lemma 7.28.2). By More on Morphisms, Lemma 37.46.3 every object U of $X_{étale}$ has an étale covering $\{U_i \to U\}$ such that the discussion in the previous paragraph applies to U_i. The lemma follows. □

0F00 **Lemma 59.87.6.** *Let $f : X \to S$ be a flat morphism of schemes such that for every geometric point $\overline{x}$ of X the map*

$$\mathcal{O}^{sh}_{S,f(\overline{x})} \longrightarrow \mathcal{O}^{sh}_{X,\overline{x}}$$

has geometrically connected fibres. Then for every cartesian diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

with g quasi-compact and quasi-separated we have $f^{-1}g_*\mathcal{F} = h_*e^{-1}\mathcal{F}$ for any sheaf $\mathcal{F}$ of sets on $T_{étale}$.

Proof. It suffices to check equality on stalks, see Theorem 59.29.10. By Theorem 59.53.1 we have

$$(h_*e^{-1}\mathcal{F})_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y, e^{-1}\mathcal{F})$$

and we have similarly

$$(f^{-1}g_*^{-1}\mathcal{F})_{\overline{x}} = (g_*^{-1}\mathcal{F})_{f(\overline{x})} = \Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T, \mathcal{F})$$

These sets are equal by an application of Lemma 59.39.3 to the morphism

$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_X Y \longrightarrow \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})}) \times_S T$$

which is a base change of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$ because $Y = X \times_S T$. □

59.88. Base change for higher direct images

0F01 This section is the analogue of Section 59.87 for higher direct images. This section is preliminary and should be skipped on a first reading.

0F02 **Remark 59.88.1.** Let $f : X \to S$ be a morphism of schemes. Let n be an integer. We will say $BC(f, n, q_0)$ is true if for every commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} \\ S & \longleftarrow & S' & \xleftarrow{g} & T \end{array}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ annihilated by n the base change map

$$(f')^{-1}R^qg_*\mathcal{F} \longrightarrow R^qh_*e^{-1}\mathcal{F}$$

is an isomorphism for $q \leq q_0$.

0F03 **Lemma 59.88.2.** *With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Then for every commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} \\ S & \longleftarrow & S' & \xleftarrow{g} & T \end{array}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ annihilated by n

(1) *the base change map $(f')^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}$ is injective,*
(2) *if $\mathcal{F} \subset \mathcal{G}$ where $\mathcal{G}$ on $T_{étale}$ is annihilated by n, then*

$$\mathrm{Coker}\left((f')^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}\right) \subset \mathrm{Coker}\left((f')^{-1}R^qg_*\mathcal{G} \to R^qh_*e^{-1}\mathcal{G}\right)$$

(3) *if in (2) the sheaf $\mathcal{G}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules, then*

$$\mathrm{Coker}\left((f')^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}\right) \subset R^qh_*e^{-1}\mathcal{G}$$

Proof. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Consider the induced diagram

$$\begin{array}{ccccccc}
(f')^{-1}R^{q-1}g_*\mathcal{I} & \longrightarrow & (f')^{-1}R^{q-1}g_*\mathcal{Q} & \longrightarrow & (f')^{-1}R^qg_*\mathcal{F} & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow \\
R^{q-1}h_*e^{-1}\mathcal{I} & \longrightarrow & R^{q-1}h_*e^{-1}\mathcal{Q} & \longrightarrow & R^qh_*e^{-1}\mathcal{F} & \longrightarrow & R^qh_*e^{-1}\mathcal{I}
\end{array}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by $BC(f, n, q-1)$. We conclude that part (1) holds. The above also shows that

$$\mathrm{Coker}\left((f')^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}\right) \subset R^qh_*e^{-1}\mathcal{I}$$

hence part (3) holds. To prove (2) choose $\mathcal{F} \subset \mathcal{G} \subset \mathcal{I}$. □

0F04 **Lemma 59.88.3.** *With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams*

$$\begin{array}{ccccccc}
X & \leftarrow & X' & \xleftarrow[h]{} & Y & \xleftarrow[\pi']{} & Y' \\
{\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e'} \\
S & \leftarrow & S' & \xleftarrow{g} & T & \xleftarrow{\pi} & T'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X' & \xleftarrow[h'=h\circ\pi']{} & Y' \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e'} \\
S' & \xleftarrow{g'=g\circ\pi} & T'
\end{array}$$

where all squares are cartesian, g quasi-compact and quasi-separated, and π is integral surjective. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n and set $\mathcal{F}' = \pi^{-1}\mathcal{F}$. If the base change map

$$(f')^{-1}R^qg'_*\mathcal{F}' \longrightarrow R^qh'_*(e')^{-1}\mathcal{F}'$$

is an isomorphism, then the base change map $(f')^{-1}R^qg_*\mathcal{F} \to R^qh_*e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Observe that $\mathcal{F} \to \pi_*\pi^{-1}\mathcal{F}'$ is injective as π is surjective (check on stalks). Thus by Lemma 59.88.2 we see that it suffices to show that the base change map

$$(f')^{-1}R^qg_*\pi_*\mathcal{F}' \longrightarrow R^qh_*e^{-1}\pi_*\mathcal{F}'$$

is an isomorphism. This follows from the assumption because we have $R^qg_*\pi_*\mathcal{F}' = R^qg'_*\mathcal{F}'$, we have $e^{-1}\pi_*\mathcal{F}' = \pi'_*(e')^{-1}\mathcal{F}'$, and we have $R^qh_*\pi'_*(e')^{-1}\mathcal{F}' = R^qh'_*(e')^{-1}\mathcal{F}'$. This follows from Lemmas 59.55.4 and 59.43.5 and the relative leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). □

0F05 **Lemma 59.88.4.** *With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have $BC(f, n, q-1)$. Consider commutative diagrams*

$$\begin{array}{ccccccc}
X & \leftarrow & X' & \xleftarrow[\pi']{} & X'' & \xleftarrow[h']{} & Y \\
{\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle e} \\
S & \leftarrow & S' & \xleftarrow{\pi} & S'' & \xleftarrow{g'} & T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X' & \xleftarrow[h=h'\circ\pi']{} & Y \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} \\
S' & \xleftarrow{g=g'\circ\pi} & T
\end{array}$$

where all squares are cartesian, g' quasi-compact and quasi-separated, and π is integral. Let $\mathcal{F}$ be an abelian sheaf on $T_{étale}$ annihilated by n. If the base change map

$$(f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_* e^{-1}\mathcal{F}$$

is an isomorphism, then the base change map $(f'')^{-1}R^q g'_*\mathcal{F} \to R^q h'_* e^{-1}\mathcal{F}$ is an isomorphism.

Proof. Since π and π' are integral we have $R\pi_* = \pi_*$ and $R\pi'_* = \pi'_*$, see Lemma 59.43.5. We also have $(f')^{-1}\pi_* = \pi'_*(f'')^{-1}$. Thus we see that $\pi'_*(f'')^{-1}R^q g'_*\mathcal{F} = (f')^{-1}R^q g_*\mathcal{F}$ and $\pi'_* R^q h'_* e^{-1}\mathcal{F} = R^q h_* e^{-1}\mathcal{F}$. Thus the assumption means that our map becomes an isomorphism after applying the functor π'_*. Hence we see that it is an isomorphism by Lemma 59.43.5. □

0F06 **Lemma 59.88.5.** *Let T be a quasi-compact and quasi-separated scheme. Let P be a property for quasi-compact and quasi-separated schemes over T. Assume*

1. *If $T'' \to T'$ is a thickening of quasi-compact and quasi-separated schemes over T, then $P(T'')$ if and only if $P(T')$.*
2. *If $T' = \lim T_i$ is a limit of an inverse system of quasi-compact and quasi-separated schemes over T with affine transition morphisms and $P(T_i)$ holds for all i, then $P(T')$ holds.*
3. *If $Z \subset T'$ is a closed subscheme with quasi-compact complement $V \subset T'$ and $P(T')$ holds, then either $P(V)$ or $P(Z)$ holds.*

Then $P(T)$ implies $P(\mathrm{Spec}(K))$ for some morphism $\mathrm{Spec}(K) \to T$ where K is a field.

Proof. Consider the set $\mathfrak{T}$ of closed subschemes $T' \subset T$ such that $P(T')$. By assumption (2) this set has a minimal element, say T'. By assumption (1) we see that T' is reduced. Let $\eta \in T'$ be the generic point of an irreducible component of T'. Then $\eta = \mathrm{Spec}(K)$ for some field K and $\eta = \lim V$ where the limit is over the affine open subschemes $V \subset T'$ containing η. By assumption (3) and the minimality of T' we see that $P(V)$ holds for all these V. Hence $P(\eta)$ by (2) and the proof is complete. □

0F07 **Lemma 59.88.6.** *With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow[h]{} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} \\ S & \longleftarrow & S' & \xleftarrow{g} & \mathrm{Spec}(K) \end{array}$$

where $X' = X \times_S S'$, $Y = X' \times_{S'} \mathrm{Spec}(K)$, K is a field, and $\mathcal{F}$ is an abelian sheaf on $\mathrm{Spec}(K)$ annihilated by n such that $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism.

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow[h]{} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} \\ S & \longleftarrow & S' & \xleftarrow{g} & T \end{array}$$

with $X' = X \times_S S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and an abelian sheaf $\mathcal{F}$ on $T_{étale}$ annihilated by n such that the base change map $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is not an isomorphism. Of course we may and do replace S' by an affine open of S'; this implies that T is quasi-compact and quasi-separated. By Lemma 59.88.2 we see $(f')^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is injective. Pick a geometric point $\overline{x}$ of X' and an element ξ of $(R^q h_* q^{-1}\mathcal{F})_{\overline{x}}$ which is not in the image of the map $((f')^{-1}R^q g_*\mathcal{F})_{\overline{x}} \to (R^q h_* e^{-1}\mathcal{F})_{\overline{x}}$.

Consider a morphism $\pi : T' \to T$ with T' quasi-compact and quasi-separated and denote $\mathcal{F}' = \pi^{-1}\mathcal{F}$. Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{array}{ccccc} X' & \xleftarrow{h} & Y & \xleftarrow{\pi'} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e'} \\ S' & \xleftarrow{g} & T & \xleftarrow{\pi} & T' \end{array} \quad \text{and} \quad \begin{array}{ccc} X' & \xleftarrow{h' = h \circ \pi'} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e'} \\ S' & \xleftarrow{g' = g \circ \pi} & T' \end{array}$$

Using pullback maps we obtain a canonical commutative diagram

$$\begin{array}{ccc} (f')^{-1}R^q g_*\mathcal{F} & \longrightarrow & (f')^{-1}R^q g'_*\mathcal{F}' \\ \downarrow & & \downarrow \\ R^q h_* e^{-1}\mathcal{F} & \longrightarrow & R^q h'_*(e')^{-1}\mathcal{F}' \end{array}$$

of abelian sheaves on X'. Let $P(T')$ be the property

- The image ξ' of ξ in $(Rh'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$ is not in the image of the map $(f^{-1}R^q g'_*\mathcal{F}')_{\overline{x}} \to (R^q h'_*(e')^{-1}\mathcal{F}')_{\overline{x}}$.

We claim that hypotheses (1), (2), and (3) of Lemma 59.88.5 hold for P which proves our lemma.

Condition (1) of Lemma 59.88.5 holds for P because the étale topology of a scheme and a thickening of the scheme is the same. See Proposition 59.45.4.

Suppose that I is a directed set and that T_i is an inverse system over I of quasi-compact and quasi-separated schemes over T with affine transition morphisms. Set $T' = \lim T_i$. Denote $\mathcal{F}'$ and $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to T', resp. T_i. Consider the diagrams

$$\begin{array}{ccccc} X & \xleftarrow{h} & Y & \xleftarrow{\pi'_i} & Y_i \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_i} \\ S & \xleftarrow{g} & T & \xleftarrow{\pi_i} & T_i \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xleftarrow{h_i = h \circ \pi'_i} & Y_i \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e_i} \\ S & \xleftarrow{g_i = g \circ \pi_i} & T_i \end{array}$$

as in the previous paragraph. It is clear that $\mathcal{F}'$ on T' is the colimit of the pullbacks of $\mathcal{F}_i$ to T' and that $(e')^{-1}\mathcal{F}'$ is the colimit of the pullbacks of $e_i^{-1}\mathcal{F}_i$ to Y'. By Lemma 59.51.8 we have

$$R^q h'_*(e')^{-1}\mathcal{F}' = \operatorname{colim} R^q h_{i,*} e_i^{-1}\mathcal{F}_i \quad \text{and} \quad (f')^{-1}R^q g'_*\mathcal{F}' = \operatorname{colim}(f')^{-1}R^q g_{i,*}\mathcal{F}_i$$

It follows that if $P(T_i)$ is true for all i, then $P(T')$ holds. Thus condition (2) of Lemma 59.88.5 holds for P.

The most interesting is condition (3) of Lemma 59.88.5. Assume T' is a quasi-compact and quasi-separated scheme over T such that $P(T')$ is true. Let $Z \subset T'$

be a closed subscheme with complement $V \subset T'$ quasi-compact. Consider the diagram

$$\begin{array}{ccccc} Y' \times_{T'} Z & \xrightarrow{i'} & Y' & \xleftarrow{j'} & Y' \times_{T'} V \\ \downarrow{\scriptstyle e_Z} & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e_V} \\ Z & \xrightarrow{i} & T' & \xleftarrow{j} & V \end{array}$$

Choose an injective map $j^{-1}\mathcal{F}' \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on V. Looking at stalks we see that the map

$$\mathcal{F}' \to \mathcal{G} = j_*\mathcal{J} \oplus i_*i^{-1}\mathcal{F}'$$

is injective. Thus ξ' maps to a nonzero element of

$$\begin{aligned} &\text{Coker}\left(((f')^{-1}R^qg'_*\mathcal{G})_{\overline{x}} \to (R^qh'_*(e')^{-1}\mathcal{G})_{\overline{x}}\right) = \\ &\text{Coker}\left(((f')^{-1}R^qg'_*j_*\mathcal{J})_{\overline{x}} \to (R^qh'_*(e')^{-1}j_*\mathcal{J})_{\overline{x}}\right) \oplus \\ &\text{Coker}\left(((f')^{-1}R^qg'_*i_*i^{-1}\mathcal{F}')_{\overline{x}} \to (R^qh'_*(e')^{-1}i_*i^{-1}\mathcal{F}')_{\overline{x}}\right) \end{aligned}$$

by part (2) of Lemma 59.88.2. If ξ' does not map to zero in the second summand, then we use

$$(f')^{-1}R^qg'_*i_*i^{-1}\mathcal{F}' = (f')^{-1}R^q(g' \circ i)_*i^{-1}\mathcal{F}'$$

(because $Ri_* = i_*$ by Proposition 59.55.2) and

$$R^qh'_*(e')^{-1}i_*i^{-1}\mathcal{F} = R^qh'_*i'_*e_Z^{-1}i^{-1}\mathcal{F} = R^q(h' \circ i')_*e_Z^{-1}i^{-1}\mathcal{F}'$$

(first equality by Lemma 59.55.3 and the second because $Ri'_* = i'_*$ by Proposition 59.55.2) to we see that we have $P(Z)$. Finally, suppose ξ' does not map to zero in the first summand. We have

$$(e')^{-1}j_*\mathcal{J} = j'_*e_V^{-1}\mathcal{J} \quad \text{and} \quad R^aj'_*e_V^{-1}\mathcal{J} = 0, \quad a = 1, \ldots, q-1$$

by $BC(f, n, q-1)$ applied to the diagram

$$\begin{array}{ccccc} X & \longleftarrow & Y' & \xleftarrow{j'} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e_V} \\ S & \longleftarrow & T' & \xleftarrow{j} & V \end{array}$$

and the fact that $\mathcal{J}$ is injective. By the relative Leray spectral sequence for $h' \circ j'$ (Cohomology on Sites, Lemma 21.14.7) we deduce that

$$R^qh'_*(e')^{-1}j_*\mathcal{J} = R^qh'_*j'_*e_V^{-1}\mathcal{J} \longrightarrow R^q(h' \circ j')_*e_V^{-1}\mathcal{J}$$

is injective. Thus ξ maps to a nonzero element of $(R^q(h' \circ j')_*e_V^{-1}\mathcal{J})_{\overline{x}}$. Applying part (3) of Lemma 59.88.2 to the injection $j^{-1}\mathcal{F}' \to \mathcal{J}$ we conclude that $P(V)$ holds. □

0F08 **Lemma 59.88.7.** *With $f : X \to S$ and n as in Remark 59.88.1 assume for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow{h} & Y \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ S & \longleftarrow & S' & \longleftarrow & \text{Spec}(K) \end{array}$$

with both squares cartesian, where

(1) *S' is affine, integral, and normal with algebraically closed function field,*
(2) *K is algebraically closed and $\mathrm{Spec}(K) \to S'$ is dominant (in other words K is an extension of the function field of S')*

and there exists an integer $d|n$ such that $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Conversely, nonvanishing of $R^q h_*(\mathbf{Z}/d\mathbf{Z})$ in the lemma implies $BC(f, n, q)$ isn't true as Lemma 59.80.5 shows that $R^q(\mathrm{Spec}(K) \to S')_*\mathbf{Z}/d\mathbf{Z} = 0$.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 59.88.6. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). By Lemma 59.88.3 we may assume K is algebraically closed. Then $\mathcal{F}$ corresponds to a $\mathbf{Z}/n\mathbf{Z}$-module. Such a modules is a direct sum of copies of $\mathbf{Z}/d\mathbf{Z}$ for varying $d|n$ hence we may assume $\mathcal{F}$ is constant with value $\mathbf{Z}/d\mathbf{Z}$. By Lemma 59.88.4 we may replace S' by the normalization of S' in $\mathrm{Spec}(K)$ which finishes the proof. □

59.89. Smooth base change

0EYQ In this section we prove the smooth base change theorem.

0EYT **Lemma 59.89.1.** *Let K/k be an extension of fields. Let X be a smooth affine curve over k with a rational point $x \in X(k)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathrm{Spec}(K)$ annihilated by an integer n invertible in k. Let $q > 0$ and*

$$\xi \in H^q(X_K, (X_K \to \mathrm{Spec}(K))^{-1}\mathcal{F})$$

There exist

(1) *finite extensions K'/K and k'/k with $k' \subset K'$,*
(2) *a finite étale Galois cover $Z \to X_{k'}$ with group G*

such that the order of G divides a power of n, such that $Z \to X_{k'}$ is split over $x_{k'}$, and such that ξ dies in $H^q(Z_{K'}, (Z_{K'} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$.

Proof. For $q > 1$ we know that ξ dies in $H^q(X_{\overline{K}}, (X_{\overline{K}} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$ (Theorem 59.83.10). By Lemma 59.51.5 we see that this means there is a finite extension K'/K such that ξ dies in $H^q(X_{K'}, (X_{K'} \to \mathrm{Spec}(K))^{-1}\mathcal{F})$. Thus we can take $k' = k$ and $Z = X$ in this case.

Assume $q = 1$. Recall that $\mathcal{F}$ corresponds to a discrete module M with continuous Gal_K-action, see Lemma 59.59.1. Since M is n-torsion, it is the union of finite Gal_K-stable subgroups. Thus we reduce to the case where M is a finite abelian group annihilated by n, see Lemma 59.51.4. After replacing K by a finite extension we may assume that the action of Gal_K on M is trivial. Thus we may assume $\mathcal{F} = \underline{M}$ is the constant sheaf with value a finite abelian group M annihilated by n.

We can write M as a direct sum of cyclic groups. Any two finite étale Galois coverings whose Galois groups have order invertible in k, can be dominated by a third one whose Galois group has order invertible in k (Fundamental Groups, Section 58.7). Thus it suffices to prove the lemma when $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$.

Assume $M = \mathbf{Z}/d\mathbf{Z}$ where $d|n$. In this case $\overline{\xi} = \xi|_{X_{\overline{K}}}$ is an element of

$$H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) = H^1(X_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$$

See Theorem 59.83.10. This group classifies $\mathbf{Z}/d\mathbf{Z}$-torsors, see Cohomology on Sites, Lemma 21.4.3. The torsor corresponding to $\overline{\xi}$ (viewed as a sheaf on $X_{\overline{k},étale}$) in turn gives rise to a finite étale morphism $T \to X_{\overline{k}}$ endowed an action of $\mathbf{Z}/d\mathbf{Z}$ transitive

on the fibre of T over $x_{\overline{k}}$, see Lemma 59.64.4. Choose a connected component $T' \subset T$ (if $\overline{\xi}$ has order d, then T is already connected). Then $T' \to X_{\overline{k}}$ is a finite étale Galois cover whose Galois group is a subgroup $G \subset \mathbf{Z}/d\mathbf{Z}$ (small detail omitted). Moreover the element $\overline{\xi}$ maps to zero under the map $H^1(X_{\overline{k}}, \mathbf{Z}/d\mathbf{Z}) \to H^1(T', \mathbf{Z}/d\mathbf{Z})$ as this is one of the defining properties of T.

Next, we use a limit argument to choose a finite extension k'/k contained in $\overline{k}$ such that $T' \to X_{\overline{k}}$ descends to a finite étale Galois cover $Z \to X_{k'}$ with group G. See Limits, Lemmas 32.10.1, 32.8.3, and 32.8.10. After increasing k' we may assume that Z splits over $x_{k'}$. The image of ξ in $H^1(Z_{\overline{K}}, \mathbf{Z}/d\mathbf{Z})$ is zero by construction. Thus by Lemma 59.51.5 we can find a finite subextension $\overline{K}/K'/K$ containing k' such that ξ dies in $H^1(Z_{K'}, \mathbf{Z}/d\mathbf{Z})$ and this finishes the proof. □

0EYU **Theorem 59.89.2** (Smooth base change)**.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

where f is smooth and g quasi-compact and quasi-separated. Then

$$f^{-1}R^q g_* \mathcal{F} = R^q h_* e^{-1} \mathcal{F}$$

for any q and any abelian sheaf $\mathcal{F}$ on $T_{étale}$ all of whose stalks at geometric points are torsion of orders invertible on S.

First proof of smooth base change. This proof is very long but more direct (using less general theory) than the second proof given below.

The theorem is local on $X_{étale}$. More precisely, suppose we have $U \to X$ étale such that $U \to S$ factors as $U \to V \to S$ with $V \to S$ étale. Then we can consider the cartesian square

$$\begin{array}{ccc} U & \xleftarrow{h'} & U \times_X Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle e'} \\ V & \xleftarrow{g'} & V \times_S T \end{array}$$

and setting $\mathcal{F}' = \mathcal{F}|_{V \times_S T}$ we have $f^{-1}R^q g_* \mathcal{F}|_U = (f')^{-1} R^q g'_* \mathcal{F}'$ and $R^q h_* e^{-1}\mathcal{F}|_U = R^q h'_* (e')^{-1} \mathcal{F}'$ (as follows from the compatibility of localization with morphisms of sites, see Sites, Lemma 7.28.2 and and Cohomology on Sites, Lemma 21.20.4). Thus it suffices to produce an étale covering of X by $U \to X$ and factorizations $U \to V \to S$ as above such that the theorem holds for the diagram with f', h', g', e'.

By the local structure of smooth morphisms, see Morphisms, Lemma 29.36.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism

$X \to \mathbf{A}^d_S$. If we have a tower of cartesian diagrams

$$\begin{array}{ccc} W & \xleftarrow{j} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xleftarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & T \end{array}$$

and the theorem holds for the bottom and top squares, then the theorem holds for the outer rectangle; this is formal. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors $R^q h_*$ and $R^q g_*$ commute with filtered colimits by Lemma 59.51.7. Thus it suffices to prove the theorem for $\mathcal{F}[n]$. From now on we fix an integer n, we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules and we assume S is a scheme over $\operatorname{Spec}(\mathbf{Z}[1/n])$.

Next, we reduce to the case where T is affine. Since g is quasi-compact and quasi-separate and S is affine, the scheme T is quasi-compact and quasi-separated. Thus we can use the induction principle of Cohomology of Schemes, Lemma 30.4.1. Hence it suffices to show that if $T = W \cup W'$ is an open covering and the theorem holds for the squares

$$\begin{array}{ccc} X & \xleftarrow{i} & e^{-1}(W) \\ \downarrow & & \downarrow \\ S & \xleftarrow{a} & W \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{j} & e^{-1}(W') \\ \downarrow & & \downarrow \\ S & \xleftarrow{b} & W' \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{k} & e^{-1}(W \cap W') \\ \downarrow & & \downarrow \\ S & \xleftarrow{c} & W \cap W' \end{array}$$

then the theorem holds for the original diagram. To see this we consider the diagram

$$\begin{array}{ccccc} f^{-1}R^{q-1}c_*\mathcal{F}|_{W\cap W'} & \longrightarrow & f^{-1}R^q g_*\mathcal{F} & \longrightarrow & f^{-1}R^q a_*\mathcal{F}|_W \oplus f^{-1}R^q b_*\mathcal{F}|_{W'} \\ \downarrow\cong & & \downarrow & & \cong\downarrow \\ R^q k_* e^{-1}\mathcal{F}|_{e^{-1}(W\cap W')} & \longrightarrow & R^q h_* e^{-1}\mathcal{F} & \longrightarrow & R^q i_* e^{-1}\mathcal{F}|_{e^{-1}(W)} \oplus R^q j_* e^{-1}\mathcal{F}|_{e^{-1}(W')} \end{array}$$

whose rows are the long exact sequences of Lemma 59.50.2. Thus the 5-lemma gives the desired conclusion.

Summarizing, we may assume S, X, T, and Y affine, $\mathcal{F}$ is n torsion, $X \to S$ is smooth of relative dimension 1, and S is a scheme over $\mathbf{Z}[1/n]$. We will prove the theorem by induction on q. The base case $q = 0$ is handled by Lemma 59.87.2. Assume $q > 0$ and the theorem holds for all smaller degrees. Choose a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ where $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules.

Consider the induced diagram

$$\begin{array}{ccccccc} f^{-1}R^{q-1}g_*\mathcal{I} & \longrightarrow & f^{-1}R^{q-1}g_*\mathcal{Q} & \longrightarrow & f^{-1}R^qg_*\mathcal{F} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow \\ R^{q-1}h_*e^{-1}\mathcal{I} & \longrightarrow & R^{q-1}h_*e^{-1}\mathcal{Q} & \longrightarrow & R^qh_*e^{-1}\mathcal{F} & \longrightarrow & R^qh_*e^{-1}\mathcal{I} \end{array}$$

with exact rows. We have the zero in the right upper corner as $\mathcal{I}$ is injective. The left two vertical arrows are isomorphisms by induction hypothesis. Thus it suffices to prove that $R^qh_*e^{-1}\mathcal{I} = 0$.

Write $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ and say the morphism $T \to S$ is given by the ring map $A \to B$. We can write $A \to B = \mathrm{colim}_{i\in I}(A_i \to B_i)$ as a filtered colimit of maps of rings of finite type over $\mathbf{Z}[1/n]$ (see Algebra, Lemma 10.127.14). For $i \in I$ we set $S_i = \mathrm{Spec}(A_i)$ and $T_i = \mathrm{Spec}(B_i)$. For i large enough we can find a smooth morphism $X_i \to S_i$ of relative dimension 1 such that $X = X_i \times_{S_i} S$, see Limits, Lemmas 32.10.1, 32.8.9, and 32.18.4. Set $Y_i = X_i \times_{S_i} T_i$ to get squares

$$\begin{array}{ccc} X_i & \xleftarrow{h_i} & Y_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle e_i} \\ S_i & \xleftarrow{g_i} & T_i \end{array}$$

Observe that $\mathcal{I}_i = (T \to T_i)_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on T_i, see Cohomology on Sites, Lemma 21.14.2. We have $\mathcal{I} = \mathrm{colim}(T \to T_i)^{-1}\mathcal{I}_i$ by Lemma 59.51.9. Pulling back by e we get $e^{-1}\mathcal{I} = \mathrm{colim}(Y \to Y_i)^{-1}e_i^{-1}\mathcal{I}_i$. By Lemma 59.51.8 applied to the system of morphisms $Y_i \to X_i$ with limit $Y \to X$ we have

$$R^qh_*e^{-1}\mathcal{I} = \mathrm{colim}(X \to X_i)^{-1}R^qh_{i,*}e_i^{-1}\mathcal{I}_i$$

This reduces us to the case where T and S are affine of finite type over $\mathbf{Z}[1/n]$.

Summarizing, we have an integer $q \geq 1$ such that the theorem holds in degrees $< q$, the schemes S and T affine of finite type type over $\mathbf{Z}[1/n]$, we have $X \to S$ smooth of relative dimension 1 with X affine, and $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and we have to show that $R^qh_*e^{-1}\mathcal{I} = 0$. We will do this by induction on $\dim(T)$.

The base case is $T = \emptyset$, i.e., $\dim(T) < 0$. If you don't like this, you can take as your base case the case $\dim(T) = 0$. In this case $T \to S$ is finite (in fact even $T \to \mathrm{Spec}(\mathbf{Z}[1/n])$ is finite as the target is Jacobson; details omitted), so h is finite too and hence has vanishing higher direct images (see references below).

Assume $\dim(T) = d \geq 0$ and we know the result for all situations where T has lower dimension. Pick U affine and étale over X and a section ξ of $R^qh_*q^{-1}\mathcal{I}$ over U. We have to show that ξ is zero. Of course, we may replace X by U (and correspondingly Y by $U \times_X Y$) and assume $\xi \in H^0(X, R^qh_*e^{-1}\mathcal{I})$. Moreover, since $R^qh_*e^{-1}\mathcal{I}$ is a sheaf, it suffices to prove that ξ is zero locally on X. Hence we may replace X by the members of an étale covering. In particular, using Lemma 59.51.6 we may assume that ξ is the image of an element $\tilde{\xi} \in H^q(Y, e^{-1}\mathcal{I})$. In terms of $\tilde{\xi}$ our task is to show that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ for some étale covering $\{U_i \to X\}$.

By More on Morphisms, Lemma 37.38.8 we may assume that $X \to S$ factors as $X \to V \to S$ where $V \to S$ is étale and $X \to V$ is a smooth morphism of affine schemes of relative dimension 1, has a section, and has geometrically connected fibres. Observe that $\dim(V \times_S T) \leq \dim(T) = d$ for example by More on Algebra,

Lemma 15.44.2. Hence we may then replace S by V and T by $V \times_S T$ (exactly as in the discussion in the first paragraph of the proof). Thus we may assume $X \to S$ is smooth of relative dimension 1, geometrically connected fibres, and has a section $\sigma : S \to X$.

Let $\pi : T' \to T$ be a finite surjective morphism. We will use below that $\dim(T') \leq \dim(T) = d$, see Algebra, Lemma 10.112.3. Choose an injective map $\pi^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Then $\mathcal{I} \to \pi_*\mathcal{I}'$ is injective and hence has a splitting (as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules). Denote $\pi' : Y' = Y \times_T T' \to Y$ the base change of π and $e' : Y' \to T'$ the base change of e. Picture

$$\begin{array}{ccccc} X & \xleftarrow{h} & Y & \xleftarrow{\pi'} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e'} \\ S & \xleftarrow{g} & T & \xleftarrow{\pi} & T' \end{array}$$

By Proposition 59.55.2 and Lemma 59.55.3 we have $R\pi'_*(e')^{-1}\mathcal{I}' = e^{-1}\pi_*\mathcal{I}'$. Thus by the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.5) we have

$$H^q(Y', (e')^{-1}\mathcal{I}') = H^q(Y, e^{-1}\pi_*\mathcal{I}') \supset H^q(Y, e^{-1}\mathcal{I})$$

and this remains true after base change by any $U \to X$ étale. Thus we may replace T by T', $\mathcal{I}$ by $\mathcal{I}'$ and $\tilde{\xi}$ by its image in $H^q(Y', (e')^{-1}\mathcal{I}')$.

Suppose we have a factorization $T \to S' \to S$ where $\pi : S' \to S$ is finite. Setting $X' = S' \times_S X$ we can consider the induced diagram

$$\begin{array}{ccccc} X & \xleftarrow{\pi'} & X' & \xleftarrow{h'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{\pi} & S' & \xleftarrow{g} & T \end{array}$$

Since π' has vanishing higher direct images we see that $R^q h_* e^{-1}\mathcal{I} = \pi'_* R^q h'_* e^{-1}\mathcal{I}$ by the Leray spectral sequence. Hence $H^0(X, R^q h_* e^{-1}\mathcal{I}) = H^0(X', R^q h'_* e^{-1}\mathcal{I})$. Thus ξ is zero if and only if the corresponding section of $R^q h'_* e^{-1}\mathcal{I}$ is zero[10]. Thus we may replace S by S' and X by X'. Observe that $\sigma : S \to X$ base changes to $\sigma' : S' \to X'$ and hence after this replacement it is still true that $X \to S$ has a section σ and geometrically connected fibres.

We will use that S and T are Nagata schemes, see Algebra, Proposition 10.162.16 which will guarantee that various normalizations are finite, see Morphisms, Lemmas 29.53.15 and 29.54.11. In particular, we may first replace T by its normalization and then replace S by the normalization of S in T. Then $T \to S$ is a disjoint union of dominant morphisms of integral normal schemes, see Morphisms, Lemma 29.53.13. Clearly we may argue one connected component at a time, hence we may assume $T \to S$ is a dominant morphism of integral normal schemes.

[10]This step can also be seen another way. Namely, we have to show that there is an étale covering $\{U_i \to X\}$ such that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$. However, if we prove there is an étale covering $\{U'_j \to X'\}$ such that $\tilde{\xi}$ dies in $H^q(U'_i \times_{X'} Y, e^{-1}\mathcal{I})$, then by property (B) for $X' \to X$ (Lemma 59.43.3) there exists an étale covering $\{U_i \to X\}$ such that $U_i \times_X X'$ is a disjoint union of schemes over X' each of which factors through U'_j for some j. Thus we see that $\tilde{\xi}$ dies in $H^q(U_i \times_X Y, e^{-1}\mathcal{I})$ as desired.

Let $s \in S$ and $t \in T$ be the generic points. By Lemma 59.89.1 there exist finite field extensions $K/\kappa(t)$ and $k/\kappa(s)$ such that k is contained in K and a finite étale Galois covering $Z \to X_k$ with Galois group G of order dividing a power of n split over $\sigma(\mathrm{Spec}(k))$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_K, e^{-1}\mathcal{I}|_{Z_K})$. Let $T' \to T$ be the normalization of T in $\mathrm{Spec}(K)$ and let $S' \to S$ be the normalization of S in $\mathrm{Spec}(k)$. Then we obtain a commutative diagram

$$\begin{array}{ccc} S' & \longleftarrow & T' \\ \downarrow & & \downarrow \\ S & \longleftarrow & T \end{array}$$

whose vertical arrows are finite. By the arguments given above we may and do replace S and T by S' and T' (and correspondingly X by $X \times_S S'$ and Y by $Y \times_T T'$). After this replacement we conclude we have a finite étale Galois covering $Z \to X_s$ of the generic fibre of $X \to S$ with Galois group G of order dividing a power of n split over $\sigma(s)$ such that $\tilde{\xi}$ maps to zero in $H^q(Z_t, (Z_t \to Y)^{-1}e^{-1}\mathcal{I})$. Here $Z_t = Z \times_S t = Z \times_s t = Z \times_{X_s} Y_t$. Since n is invertible on S, by Fundamental Groups, Lemma 58.31.8 we can find a finite étale morphism $U \to X$ whose restriction to X_s is Z.

At this point we replace X by U and Y by $U \times_X Y$. After this replacement it may no longer be the case that the fibres of $X \to S$ are geometrically connected (there still is a section but we won't use this), but what we gain is that after this replacement $\tilde{\xi}$ maps to zero in $H^q(Y_t, e^{-1}\mathcal{I})$, i.e., $\tilde{\xi}$ restricts to zero on the generic fibre of $Y \to T$.

Recall that t is the spectrum of the function field of T, i.e., as a scheme t is the limit of the nonempty affine open subschemes of T. By Lemma 59.51.5 we conclude there exists a nonempty open subscheme $V \subset T$ such that $\tilde{\xi}$ maps to zero in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y \times_T V})$.

Denote $Z = T \setminus V$. Consider the diagram

$$\begin{array}{ccccc} Y \times_T Z & \xrightarrow[i']{} & Y & \xleftarrow[j']{} & Y \times_T V \\ \downarrow{\scriptstyle e_Z} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_V} \\ Z & \xrightarrow{i} & T & \xleftarrow{j} & V \end{array}$$

Choose an injection $i^{-1}\mathcal{I} \to \mathcal{I}'$ into an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on Z. Looking at stalks we see that the map

$$\mathcal{I} \to j_*\mathcal{I}|_V \oplus i_*\mathcal{I}'$$

is injective and hence splits as $\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules. Thus it suffices to show that $\tilde{\xi}$ maps to zero in

$$H^q(Y, e^{-1}j_*\mathcal{I}|_V) \oplus H^q(Y, e^{-1}i_*\mathcal{I}')$$

at least after replacing X by the members of an étale covering. Observe that

$$e^{-1}j_*\mathcal{I}|_V = j'_*e_V^{-1}\mathcal{I}|_V, \quad e^{-1}i_*\mathcal{I}' = i'_*e_Z^{-1}\mathcal{I}'$$

By induction hypothesis on q we see that

$$R^a j'_* e_V^{-1}\mathcal{I}|_V = 0, \quad a = 1, \ldots, q-1$$

By the Leray spectral sequence for j' and the vanishing above it follows that

$$H^q(Y, j'_*(e_V^{-1}\mathcal{I}|_V)) \longrightarrow H^q(Y \times_T V, e_V^{-1}\mathcal{I}_V) = H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y\times_T V})$$

is injective. Thus the vanishing of the image of $\tilde{\xi}$ in the first summand above because we know $\tilde{\xi}$ vanishes in $H^q(Y \times_T V, e^{-1}\mathcal{I}|_{Y\times_T V})$. Since $\dim(Z) < \dim(T) = d$ by induction the image of $\tilde{\xi}$ in the second summand

$$H^q(Y, e^{-1}i_*\mathcal{I}') = H^q(Y, i'_*e_Z^{-1}\mathcal{I}') = H^q(Y \times_T Z, e_Z^{-1}\mathcal{I}')$$

dies after replacing X by the members of a suitable étale covering. This finishes the proof of the smooth base change theorem. □

Second proof of smooth base change. This proof is the same as the longer first proof; it is shorter only in that we have split out the arguments used in a number of lemmas.

The case of $q = 0$ is Lemma 59.87.2. Thus we may assume $q > 0$ and the result is true for all smaller degrees.

For every $n \geq 1$ invertible on S, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$ by our assumption on the stalks of $\mathcal{F}$. The functors e^{-1} and f^{-1} commute with colimits as they are left adjoints. The functors R^qh_* and R^qg_* commute with filtered colimits by Lemma 59.51.7. Thus it suffices to prove the theorem for $\mathcal{F}[n]$. From now on we fix an integer n invertible on S and we work with sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules.

By Lemma 59.86.1 the question is étale local on X and S. By the local structure of smooth morphisms, see Morphisms, Lemma 29.36.20, we may assume X and S are affine and $X \to S$ factors through an étale morphism $X \to \mathbf{A}^d_S$. Writing $X \to S$ as the composition

$$X \to \mathbf{A}^{d-1}_S \to \mathbf{A}^{d-2}_S \to \ldots \to \mathbf{A}^1_S \to S$$

we conclude from Lemma 59.86.2 that it suffices to prove the theorem when X and S are affine and $X \to S$ has relative dimension 1.

By Lemma 59.88.7 it suffices to show that $R^qh_*\mathbf{Z}/d\mathbf{Z} = 0$ for $d|n$ whenever we have a cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & Y \\ \downarrow & & \downarrow \\ S & \longleftarrow & \operatorname{Spec}(K) \end{array}$$

where $X \to S$ is affine and smooth of relative dimension 1, S is the spectrum of a normal domain A with algebraically closed fraction field L, and K/L is an extension of algebraically closed fields.

Recall that $R^qh_*\mathbf{Z}/d\mathbf{Z}$ is the sheaf associated to the presheaf

$$U \longmapsto H^q(U \times_X Y, \mathbf{Z}/d\mathbf{Z}) = H^q(U \times_S \operatorname{Spec}(K), \mathbf{Z}/d\mathbf{Z})$$

on $X_{étale}$ (Lemma 59.51.6). Thus it suffices to show: given U and $\xi \in H^q(U \times_S \operatorname{Spec}(K), \mathbf{Z}/d\mathbf{Z})$ there exists an étale covering $\{U_i \to U\}$ such that ξ dies in $H^q(U_i \times_S \operatorname{Spec}(K), \mathbf{Z}/d\mathbf{Z})$.

Of course we may take U affine. Then $U \times_S \operatorname{Spec}(K)$ is a (smooth) affine curve over K and hence we have vanishing for $q > 1$ by Theorem 59.83.10.

Final case: $q = 1$. We may replace U by the members of an étale covering as in More on Morphisms, Lemma 37.38.8. Then $U \to S$ factors as $U \to V \to S$ where $U \to V$ has geometrically connected fibres, U, V are affine, $V \to S$ is étale, and there is a section $\sigma : V \to U$. By Lemma 59.80.4 we see that V is isomorphic to a (finite) disjoint union of (affine) open subschemes of S. Clearly we may replace S by one of these and X by the corresponding component of U. Thus we may assume $X \to S$ has geometrically connected fibres, has a section σ, and $\xi \in H^1(Y, \mathbf{Z}/d\mathbf{Z})$. Since K and L are algebraically closed we have

$$H^1(X_L, \mathbf{Z}/d\mathbf{Z}) = H^1(Y, \mathbf{Z}/d\mathbf{Z})$$

See Lemma 59.83.12. Thus there is a finite étale Galois covering $Z \to X_L$ with Galois group $G \subset \mathbf{Z}/d\mathbf{Z}$ which annihilates ξ. You can either see this by looking at the statement or proof of Lemma 59.89.1 or by using directly that ξ corresponds to a $\mathbf{Z}/d\mathbf{Z}$-torsor over X_L. Finally, by Fundamental Groups, Lemma 58.31.9 we find a (necessarily surjective) finite étale morphism $X' \to X$ whose restriction to X_L is $Z \to X_L$. Since ξ dies in X'_K this finishes the proof. □

The following immediate consequence of the smooth base change theorem is what is often used in practice.

0F09 **Lemma 59.89.3.** *Let S be a scheme. Let $S' = \lim S_i$ be a directed inverse limit of schemes S_i smooth over S with affine transition morphisms. Let $f : X \to S$ be quasi-compact and quasi-separated and form the fibre square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then

$$g^{-1}Rf_*E = R(f')_*(g')^{-1}E$$

for any $E \in D^+(X_{étale})$ whose cohomology sheaves $H^q(E)$ have stalks which are torsion of orders invertible on S.

Proof. Consider the spectral sequences

$$E_2^{p,q} = R^p f_* H^q(E) \quad \text{and} \quad {E'}_2^{p,q} = R^p f'_* H^q((g')^{-1}E) = R^p f'_*(g')^{-1}H^q(E)$$

converging to $R^n f_* E$ and $R^n f'_*(g')^{-1}E$. These spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Combining the smooth base change theorem (Theorem 59.89.2) with Lemma 59.86.3 we see that

$$g^{-1}R^p f_* H^q(E) = R^p(f')_*(g')^{-1}H^q(E)$$

Combining all of the above we get the lemma. □

59.90. Applications of smooth base change

0F0A In this section we discuss some more or less immediate consequences of the smooth base change theorem.

0F1C **Lemma 59.90.1.** *Let L/K be an extension of fields. Let $g : T \to S$ be a quasi-compact and quasi-separated morphism of schemes over K. Denote $g_L : T_L \to S_L$ the base change of g to* $\mathrm{Spec}(L)$. *Let $E \in D^+(T_{étale})$ have cohomology sheaves*

whose stalks are torsion of orders invertible in K. Let E_L be the pullback of E to $(T_L)_{étale}$. Then $Rg_{L,*}E_L$ is the pullback of Rg_*E to S_L.

Proof. If L/K is separable, then L is a filtered colimit of smooth K-algebras, see Algebra, Lemma 10.158.11. Thus the lemma in this case follows immediately from Lemma 59.89.3. In the general case, let K' and L' be the perfect closures (Algebra, Definition 10.45.5) of K and L. Then $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ and $\mathrm{Spec}(L') \to \mathrm{Spec}(L)$ are universal homeomorphisms as K'/K and L'/L are purely inseparable (see Algebra, Lemma 10.46.7). Thus we have $(T_{K'})_{étale} = T_{étale}$, $(S_{K'})_{étale} = S_{étale}$, $(T_{L'})_{étale} = (T_L)étale$, and $(S_{L'})_{étale} = (S_L)_{étale}$ by the topological invariance of étale cohomology, see Proposition 59.45.4. This reduces the lemma to the case of the field extension L'/K' which is separable (by definition of perfect fields, see Algebra, Definition 10.45.1). □

0F0B **Lemma 59.90.2.** *Let K/k be an extension of separably closed fields. Let X be a quasi-compact and quasi-separated scheme over k. Let $E \in D^+(X_{étale})$ have cohomology sheaves whose stalks are torsion of orders invertible in k. Then*

1. *the maps $H^q_{étale}(X, E) \to H^q_{étale}(X_K, E|_{X_K})$ are isomorphisms, and*
2. *$E \to R(X_K \to X)_*E|_{X_K}$ is an isomorphism.*

Proof. Proof of (1). First let $\overline{k}$ and $\overline{K}$ be the algebraic closures of k and K. The morphisms $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ and $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(K)$ are universal homeomorphisms as $\overline{k}/k$ and $\overline{K}/K$ are purely inseparable (see Algebra, Lemma 10.46.7). Thus $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_{\overline{k}}, \mathcal{F}_{X_{\overline{k}}})$ by the topological invariance of étale cohomology, see Proposition 59.45.4. Similarly for X_K and $X_{\overline{K}}$. Thus we may assume k and K are algebraically closed. In this case K is a limit of smooth k-algebras, see Algebra, Lemma 10.158.11. We conclude our lemma is a special case of Theorem 59.89.2 as reformulated in Lemma 59.89.3.

Proof of (2). For any quasi-compact and quasi-separated U in $X_{étale}$ the above shows that the restriction of the map $E \to R(X_K \to X)_*E|_{X_K}$ determines an isomorphism on cohomology. Since every object of $X_{étale}$ has an étale covering by such U this proves the desired statement. □

0F1D **Lemma 59.90.3.** *With $f : X \to S$ and n as in Remark 59.88.1 assume n is invertible on S and that for some $q \geq 1$ we have that $BC(f, n, q-1)$ is true, but $BC(f, n, q)$ is not. Then there exist a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow[h]{} & Y \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & S' & \longleftarrow & \mathrm{Spec}(K) \end{array}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K and there exists an integer $d|n$ such that $R^qh_*(\mathbf{Z}/d\mathbf{Z})$ is nonzero.

Proof. First choose a diagram and $\mathcal{F}$ as in Lemma 59.88.7. We may and do assume S' is affine (this is obvious, but see proof of the lemma in case of doubt). Let K'

be the function field of S' and let $Y' = X' \times_{S'} \mathrm{Spec}(K')$ to get the diagram

$$\begin{array}{ccccccc} X & \leftarrow & X' & \xleftarrow[h']{} & Y' & \leftarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & S' & \leftarrow & \mathrm{Spec}(K') & \leftarrow & \mathrm{Spec}(K) \end{array}$$

By Lemma 59.90.2 the total direct image $R(Y \to Y')_*\mathbf{Z}/d\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ in $D(Y'_{étale})$; here we use that n is invertible on S. Thus $Rh'_*\mathbf{Z}/d\mathbf{Z} = Rh_*\mathbf{Z}/d\mathbf{Z}$ by the relative Leray spectral sequence. This finishes the proof. □

59.91. The proper base change theorem

095S The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [**AGV71**, XII, Theorem 5.1] using Gabber's ideas (from the affine case) to slightly simplify the arguments.

0A0B **Lemma 59.91.1.** *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $Z = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on the topological space associated to X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. We will use Lemma 59.82.4 to prove this. First observe that the underlying topological space of X is spectral by Properties, Lemma 28.2.4. Let $Y \subset X$ be an irreducible closed subscheme. To finish the proof we show that $Y \cap Z = Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ is connected. Replacing X by Y we may assume that X is irreducible and we have to show that Z is connected. Let $X \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the Stein factorization of f (More on Morphisms, Theorem 37.53.5). Then $A \to B$ is integral and (B, IB) is a henselian pair (More on Algebra, Lemma 15.11.8). Thus we may assume the fibres of $X \to \mathrm{Spec}(A)$ are geometrically connected. On the other hand, the image $T \subset \mathrm{Spec}(A)$ of f is irreducible and closed as X is proper over A. Hence $T \cap V(I)$ is connected by More on Algebra, Lemma 15.11.16. Now $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to T \cap V(I)$ is a surjective closed map with connected fibres. The result now follows from Topology, Lemma 5.7.5. □

0A0C **Lemma 59.91.2.** *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $i : Z \to X$ be the closed immersion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ into X. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.*

Proof. This follows from Lemma 59.82.2 and 59.91.1 and the fact that any scheme finite over X is proper over $\mathrm{Spec}(A)$. □

0A3S **Lemma 59.91.3.** *Let A be a henselian local ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.*

Proof. This is a special case of Lemma 59.91.2. □

Let $f : X \to S$ be a morphism of schemes. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{s}$ is the scheme $X_{\overline{s}} = \mathrm{Spec}(k) \times_{\overline{s},S} X$ viewed as a scheme over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $X_{étale}$, then denote $\mathcal{F}_{\overline{s}} = p^{-1}_{small}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(X_{\overline{s}})_{étale}$. In the following we will consider the set

$$\Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

Let $s \in S$ be the image point of $\overline{s}$. Let $\kappa(s)^{sep}$ be the separable algebraic closure of $\kappa(s)$ in k as in Definition 59.56.1. By Lemma 59.39.5 pullback defines a bijection

$$\Gamma(X_{\kappa(s)^{sep}}, p_{sep}^{-1}\mathcal{F}) \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

where $p_{sep} : X_{\kappa(s)^{sep}} = \mathrm{Spec}(\kappa(s)^{sep}) \times_S X \to X$ is the projection.

0A3T **Lemma 59.91.4.** *Let $f : X \to S$ be a proper morphism of schemes. Let $\overline{s} \to S$ be a geometric point. For any sheaf $\mathcal{F}$ on $X_{étale}$ the canonical map*

$$(f_*\mathcal{F})_{\overline{s}} \longrightarrow \Gamma(X_{\overline{s}}, \mathcal{F}_{\overline{s}})$$

is bijective.

Proof. By Theorem 59.53.1 (for sheaves of sets) we have

$$(f_*\mathcal{F})_{\overline{s}} = \Gamma(X \times_S \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh}), p_{small}^{-1}\mathcal{F})$$

where $p : X \times_S \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{sh}) \to X$ is the projection. Since the residue field of the strictly henselian local ring $\mathcal{O}_{S,\overline{s}}^{sh}$ is $\kappa(s)^{sep}$ we conclude from the discussion above the lemma and Lemma 59.91.3. □

0A3U **Lemma 59.91.5.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{étale}$. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$.*

Proof. There is a canonical map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$. Namely, it is adjoint to the map

$$f_*\mathcal{F} \longrightarrow g_*f'_*(g')^{-1}\mathcal{F} = f_*g'_*(g')^{-1}\mathcal{F}$$

which is f_* applied to the canonical map $\mathcal{F} \to g'_*(g')^{-1}\mathcal{F}$. To check this map is an isomorphism we can compute what happens on stalks. Let $y' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image y in Y. By Lemma 59.91.4 the stalks are $\Gamma(X'_{y'}, \mathcal{F}_{y'})$ and $\Gamma(X_y, \mathcal{F}_y)$ respectively. Here the sheaves $\mathcal{F}_y$ and $\mathcal{F}_{y'}$ are the pullbacks of $\mathcal{F}$ by the projections $X_y \to X$ and $X'_{y'} \to X$. Since $X_y = X'_{y'}$ as schemes we see that the stalks agree. We omit the verification that this isomorphism is compatible with our map. □

At this point we start discussing the proper base change theorem. To do so we introduce some notation. consider a commutative diagram

0A29 (59.91.5.1)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of morphisms of schemes. Then we obtain a commutative diagram of sites

$$\begin{array}{ccc} X'_{étale} & \xrightarrow{g'_{small}} & X_{étale} \\ {\scriptstyle f'_{small}}\downarrow & & \downarrow{\scriptstyle f_{small}} \\ Y'_{étale} & \xrightarrow{g_{small}} & Y_{étale} \end{array}$$

For any object E of $D(X_{étale})$ we obtain a canonical base change map

0A2A (59.91.5.2)
$$g_{small}^{-1}Rf_{small,*}E \longrightarrow Rf'_{small,*}(g'_{small})^{-1}E$$

in $D(Y'_{étale})$. See Cohomology on Sites, Remark 21.19.3 where we use the constant sheaf $\mathbf{Z}$ as our sheaf of rings. We will usually omit the subscripts $_{small}$ in this formula. For example, if $E = \mathcal{F}[0]$ where $\mathcal{F}$ is an abelian sheaf on $X_{étale}$, the base change map is a map

0A4A (59.91.5.3) $$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

in $D(Y'_{étale})$.

The map (59.91.5.2) has no chance of being an isomorphism in the generality given above. The goal is to show it is an isomorphism if the diagram (59.91.5.1) is cartesian, $f : X \to Y$ proper, the cohomology sheaves of E are torsion, and E is bounded below. To study this question we introduce the following terminology. Let us say that *cohomology commutes with base change for $f : X \to Y$* if (59.91.5.3) is an isomorphism for every diagram (59.91.5.1) where $X' = Y' \times_Y X$ and every torsion abelian sheaf $\mathcal{F}$.

0A4B **Lemma 59.91.6.** *Let $f : X \to Y$ be a proper morphism of schemes. The following are equivalent*

1. *cohomology commutes with base change for f (see above),*
2. *for every prime number ℓ and every injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules $\mathcal{I}$ on $X_{étale}$ and every diagram (59.91.5.1) where $X' = Y' \times_Y X$ the sheaves $R^q f'_*(g')^{-1}\mathcal{I}$ are zero for $q > 0$.*

Proof. It is clear that (1) implies (2). Conversely, assume (2) and let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Y' \to Y$ be a morphism of schemes and let $X' = Y' \times_Y X$ with projections $g' : X' \to X$ and $f' : X' \to Y'$ as in diagram (59.91.5.1). We want to show the maps of sheaves

$$g^{-1}R^q f_*\mathcal{F} \longrightarrow R^q f'_*(g')^{-1}\mathcal{F}$$

are isomorphisms for all $q \geq 0$.

For every $n \geq 1$, let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$. The functors g^{-1} and $(g')^{-1}$ commute with arbitrary colimits (as left adjoints). Taking higher direct images along f or f' commutes with filtered colimits by Lemma 59.51.7. Hence we see that

$$g^{-1}R^q f_*\mathcal{F} = \operatorname{colim} g^{-1}R^q f_*\mathcal{F}[n] \quad \text{and} \quad R^q f'_*(g')^{-1}\mathcal{F} = \operatorname{colim} R^q f'_*(g')^{-1}\mathcal{F}[n]$$

Thus it suffices to prove the result in case $\mathcal{F}$ is annihilated by a positive integer n.

If $n = \ell n'$ for some prime number ℓ, then we obtain a short exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$$

Observe that $\mathcal{F}/\mathcal{F}[\ell]$ is annihilated by n'. Moreover, if the result holds for both $\mathcal{F}[\ell]$ and $\mathcal{F}/\mathcal{F}[\ell]$, then the result holds by the long exact sequence of higher direct images (and the 5 lemma). In this way we reduce to the case that $\mathcal{F}$ is annihilated by a prime number ℓ.

Assume $\mathcal{F}$ is annihilated by a prime number ℓ. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $D(X_{étale}, \mathbf{Z}/\ell\mathbf{Z})$. Applying assumption (2) and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we see that

$$f'_*(g')^{-1}\mathcal{I}^\bullet$$

computes $Rf'_*(g')^{-1}\mathcal{F}$. We conclude by applying Lemma 59.91.5. □

0A4C **Lemma 59.91.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume*

(1) *cohomology commutes with base change for f,*
(2) *cohomology commutes with base change for $g \circ f$, and*
(3) *f is surjective.*

Then cohomology commutes with base change for g.

Proof. We will use the equivalence of Lemma 59.91.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$. Choose an injective map of sheaves $f^{-1}\mathcal{I} \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Z_{étale}$. Since f is surjective the map $\mathcal{I} \to f_*\mathcal{J}$ is injective (look at stalks in geometric points). Since $\mathcal{I}$ is injective we see that $\mathcal{I}$ is a direct summand of $f_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $f_*\mathcal{J}$.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 59.91.5 we have $b^{-1}f_*\mathcal{J} = f'_*a^{-1}\mathcal{J}$. On the other hand, we know that $R^q f'_* a^{-1}\mathcal{J}$ and $R^q(g' \circ f')_* a^{-1}\mathcal{J}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 21.14.7)

$$R^p g'_* R^q f'_* a^{-1}\mathcal{J} \Rightarrow R^{p+q}(g' \circ f')_* a^{-1}\mathcal{J}$$

we conclude that $R^p g'_*(b^{-1}f_*\mathcal{J}) = R^p g'_*(f'_* a^{-1}\mathcal{J}) = 0$ for $p > 0$ as desired. □

0A4D **Lemma 59.91.8.** *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes. Assume*

(1) *cohomology commutes with base change for f, and*
(2) *cohomology commutes with base change for g.*

Then cohomology commutes with base change for $g \circ f$.

Proof. We will use the equivalence of Lemma 59.91.6 without further mention. Let ℓ be a prime number. Let $\mathcal{I}$ be an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Then $f_*\mathcal{I}$ is an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $Y_{étale}$ (Cohomology on Sites, Lemma 21.14.2). The result follows formally from this, but we will also spell it out.

Let $Z' \to Z$ be a morphism of schemes and set $Y' = Z' \times_Z Y$ and $X' = Z' \times_Z X = Y' \times_Y X$. Denote $a : X' \to X$, $b : Y' \to Y$, and $c : Z' \to Z$ the projections. Similarly for $f' : X' \to Y'$ and $g' : Y' \to Z'$. By Lemma 59.91.5 we have $b^{-1}f_*\mathcal{I} = f'_*a^{-1}\mathcal{I}$. On the other hand, we know that $R^q f'_* a^{-1}\mathcal{I}$ and $R^q(g')_* b^{-1}f_*\mathcal{I}$ are zero for $q > 0$. Using the spectral sequence (Cohomology on Sites, Lemma 21.14.7)

$$R^p g'_* R^q f'_* a^{-1}\mathcal{I} \Rightarrow R^{p+q}(g' \circ f')_* a^{-1}\mathcal{I}$$

we conclude that $R^p(g' \circ f')_* a^{-1}\mathcal{I} = 0$ for $p > 0$ as desired. □

0A4E **Lemma 59.91.9.** *Let $f : X \to Y$ be a finite morphism of schemes. Then cohomology commutes with base change for f.*

Proof. Observe that a finite morphism is proper, see Morphisms, Lemma 29.44.11. Moreover, the base change of a finite morphism is finite, see Morphisms, Lemma 29.44.6. Thus the result follows from Lemma 59.91.6 combined with Proposition 59.55.2. □

0A4F **Lemma 59.91.10.** *To prove that cohomology commutes with base change for every proper morphism of schemes it suffices to prove it holds for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.*

Proof. Let $f : X \to Y$ be a proper morphism of schemes. Let $Y = \bigcup Y_i$ be an affine open covering and set $X_i = f^{-1}(Y_i)$. If we can prove cohomology commutes with base change for $X_i \to Y_i$, then cohomology commutes with base change for f. Namely, the formation of the higher direct images commutes with Zariski (and even étale) localization on the base, see Lemma 59.51.6. Thus we may assume Y is affine.

Let Y be an affine scheme and let $X \to Y$ be a proper morphism. By Chow's lemma there exists a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_Y \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X' \to X$ is proper and surjective, see Limits, Lemma 32.12.1. Since $X \to Y$ is proper, we find that $X' \to Y$ is proper (Morphisms, Lemma 29.41.4). Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 29.41.7). It follows that $X' \to X \times_Y \mathbf{P}^n_Y = \mathbf{P}^n_X$ is a closed immersion (as an immersion with closed image).

By Lemma 59.91.7 it suffices to prove cohomology commutes with base change for π and $X' \to Y$. These morphisms both factor as a closed immersion followed by a projection $\mathbf{P}^n_S \to S$ (for some S). By Lemma 59.91.9 the result holds for closed immersions (as closed immersions are finite). By Lemma 59.91.8 it suffices to prove the result for projections $\mathbf{P}^n_S \to S$.

For every $n \geq 1$ there is a finite surjective morphism

$$\mathbf{P}^1_S \times_S \ldots \times_S \mathbf{P}^1_S \longrightarrow \mathbf{P}^n_S$$

given on coordinates by

$$((x_1 : y_1), (x_2 : y_2), \ldots, (x_n : y_n)) \longmapsto (F_0 : \ldots : F_n)$$

where $F_0, \ldots, F_n$ in $x_1, \ldots, y_n$ are the polynomials with integer coefficients such that

$$\prod(x_i t + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n$$

Applying Lemmas 59.91.7, 59.91.9, and 59.91.8 one more time we conclude that the lemma is true. □

095T **Theorem 59.91.11** (Proper Base Change)**.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then the base change map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. In the terminology introduced above, this means that cohomology commutes with base change for every proper morphism of schemes. By Lemma 59.91.10 it suffices to prove that cohomology commutes with base change for the morphism $\mathbf{P}^1_S \to S$ for every scheme S.

Let S be the spectrum of a strictly henselian local ring with closed point s. Set $X = \mathbf{P}^1_S$ and $X_0 = X_s = \mathbf{P}^1_s$. Let $\mathcal{F}$ be a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. The key to our proof is that

$$H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X_0, \mathcal{F}|_{X_0}).$$

Namely, choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by injective sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. Then $\mathcal{I}^\bullet|_{X_0}$ is a resolution of $\mathcal{F}|_{X_0}$ by right $H^0_{étale}(X_0, -)$-acyclic objects, see Lemma 59.85.2. Leray's acyclicity lemma tells us the right hand side is computed by the complex $H^0_{étale}(X_0, \mathcal{I}^\bullet|_{X_0})$ which is equal to $H^0_{étale}(X, \mathcal{I}^\bullet)$ by Lemma 59.91.3. This complex computes the left hand side.

Assume S is general and $\mathcal{F}$ is a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on $X_{étale}$. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S lying over $s \in S$. We have

$$(R^q f_*\mathcal{F})_{\overline{s}} = H^q_{étale}(\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}, \mathcal{F}|_{\mathbf{P}^1_{\mathcal{O}^{sh}_{S,\overline{s}}}}) = H^q_{étale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}})$$

where $\kappa(s)^{sep}$ is the residue field of $\mathcal{O}^{sh}_{S,\overline{s}}$, i.e., the separable algebraic closure of $\kappa(s)$ in k. The first equality by Theorem 59.53.1 and the second equality by the displayed formula in the previous paragraph.

Finally, consider any morphism of schemes $g : T \to S$ where S and $\mathcal{F}$ are as above. Set $f' : \mathbf{P}^1_T \to T$ the projection and let $g' : \mathbf{P}^1_T \to \mathbf{P}^1_S$ the morphism induced by g. Consider the base change map

$$g^{-1}R^q f_*\mathcal{F} \longrightarrow R^q f'_*(g')^{-1}\mathcal{F}$$

Let $\overline{t}$ be a geometric point of T with image $\overline{s} = g(\overline{t})$. By our discussion above the map on stalks at $\overline{t}$ is the map

$$H^q_{étale}(\mathbf{P}^1_{\kappa(s)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(s)^{sep}}}) \longrightarrow H^q_{étale}(\mathbf{P}^1_{\kappa(t)^{sep}}, \mathcal{F}|_{\mathbf{P}^1_{\kappa(t)^{sep}}})$$

Since $\kappa(s)^{sep} \subset \kappa(t)^{sep}$ this map is an isomorphism by Lemma 59.83.12.

This proves cohomology commutes with base change for $\mathbf{P}^1_S \to S$ and sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules. In particular, for an injective sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules the higher direct images of any base change are zero. In other words, condition (2) of Lemma 59.91.6 holds and the proof is complete. □

0DDE **Lemma 59.91.12.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Then the base change map (59.91.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.*

Proof. This is a simple consequence of the proper base change theorem (Theorem 59.91.11) using the spectral sequences

$$E_2^{p,q} = R^p f_* H^q(E) \quad \text{and} \quad E'^{p,q}_2 = R^p f'_*(g')^{-1} H^q(E)$$

converging to $R^n f_* E$ and $R^n f'_*(g')^{-1} E$. The spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Some details omitted. □

0DDF **Lemma 59.91.13.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $\overline{y} \to Y$ be a geometric point.*

1. *For a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $(R^n f_* \mathcal{F})_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.*
2. *For $E \in D^+(X_{étale})$ with torsion cohomology sheaves we have $(R^n f_* E)_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, E|_{X_{\overline{y}}})$.*

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to the scheme theoretic fibre $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 59.91.11. The second one is a special case of Lemma 59.91.12. □

59.92. Applications of proper base change

0A5I In this section we discuss some more or less immediate consequences of the proper base change theorem.

0DDG **Lemma 59.92.1.** *Let K/k be an extension of separably closed fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Then the map $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_K, \mathcal{F}|_{X_K})$ is an isomorphism for $q \geq 0$.*

Proof. Looking at stalks we see that this is a special case of Theorem 59.91.11. □

095U **Lemma 59.92.2.** *Let $f : X \to Y$ be a proper morphism of schemes all of whose fibres have dimension $\leq n$. Then for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $R^q f_* \mathcal{F} = 0$ for $q > 2n$.*

Proof. We will prove this by induction on n for all proper morphisms.

If $n = 0$, then f is a finite morphism (More on Morphisms, Lemma 37.44.1) and the result is true by Proposition 59.55.2.

If $n > 0$, then using Lemma 59.91.13 we see that it suffices to prove $H^i_{étale}(X, \mathcal{F}) = 0$ for $i > 2n$ and X a proper scheme, $\dim(X) \leq n$ over an algebraically closed field k and $\mathcal{F}$ is a torsion abelian sheaf on X.

If $n = 1$ this follows from Theorem 59.83.11. Assume $n > 1$. By Proposition 59.45.4 we may replace X by its reduction. Let $\nu : X^\nu \to X$ be the normalization. This is a surjective birational finite morphism (see Varieties, Lemma 33.27.1) and hence an isomorphism over a dense open $U \subset X$ (Morphisms, Lemma 29.50.5). Then we see that $c : \mathcal{F} \to \nu_* \nu^{-1} \mathcal{F}$ is injective (as ν is surjective) and an isomorphism over U. Denote $i : Z \to X$ the inclusion of the complement of U. Since U is dense in X we have $\dim(Z) < \dim(X) = n$. By Proposition 59.46.4 have $\mathrm{Coker}(c) = i_* \mathcal{G}$ for some abelian torsion sheaf $\mathcal{G}$ on $Z_{étale}$. Then $H^q_{étale}(X, \mathrm{Coker}(c)) = H^q_{étale}(Z, \mathcal{F})$ (by Proposition 59.55.2 and the Leray spectral sequence) and by induction hypothesis we conclude that the cokernel of c has cohomology in degrees $\leq 2(n-1)$. Thus it suffices to prove the result for $\nu_* \nu^{-1} \mathcal{F}$. As ν is finite this reduces us to showing that $H^i_{étale}(X^\nu, \nu^{-1} \mathcal{F})$ is zero for $i > 2n$. This case is treated in the next paragraph.

Assume X is integral normal proper scheme over k of dimension n. Choose a nonconstant rational function f on X. The graph $X' \subset X \times \mathbf{P}^1_k$ of f sits into a diagram

$$X \xleftarrow{b} X' \xrightarrow{f} \mathbf{P}^1_k$$

Observe that b is an isomorphism over an open subscheme $U \subset X$ whose complement is a closed subscheme $Z \subset X$ of codimension ≥ 2. Namely, U is the domain of definition of f which contains all codimension 1 points of X, see Morphisms, Lemmas 29.49.9 and 29.42.5 (combined with Serre's criterion for normality, see Properties, Lemma 28.12.5). Moreover the fibres of b have dimension ≤ 1 (as closed subschemes of $\mathbf{P}^1$). Hence $R^i b_* b^{-1}\mathcal{F}$ is nonzero only if $i \in \{0, 1, 2\}$ by induction. Choose a distinguished triangle

$$\mathcal{F} \to Rb_* b^{-1}\mathcal{F} \to Q \to \mathcal{F}[1]$$

Using that $\mathcal{F} \to b_* b^{-1}\mathcal{F}$ is injective as before and using what we just said, we see that Q has nonzero cohomology sheaves only in degrees $0, 1, 2$ sitting on Z. Moreover, these cohomology sheaves are torsion by Lemma 59.78.2. By induction we see that $H^i(X, Q)$ is zero for $i > 2 + 2\dim(Z) \leq 2 + 2(n-2) = 2n - 2$. Thus it suffices to prove that $H^i(X', b^{-1}\mathcal{F}) = 0$ for $i > 2n$. At this point we use the morphism

$$f : X' \to \mathbf{P}^1_k$$

whose fibres have dimension $< n$. Hence by induction we see that $R^i f_* b^{-1}\mathcal{F} = 0$ for $i > 2(n-1)$. We conclude by the Leray spectral sequence

$$H^i(\mathbf{P}^1_k, R^j f_* b^{-1}\mathcal{F}) \Rightarrow H^{i+j}(X', b^{-1}\mathcal{F})$$

and the fact that $\dim(\mathbf{P}^1_k) = 1$. □

When working with mod n coefficients we can do proper base change for unbounded complexes.

0F0C **Lemma 59.92.3.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Set $X' = Y' \times_Y X$ and denote $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Let $n \geq 1$ be an integer. Let $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. Then the base change map (59.91.5.2) $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.*

Proof. It is enough to prove this when Y and Y' are quasi-compact. By Morphisms, Lemma 29.28.5 we see that the dimension of the fibres of $f : X \to Y$ and $f' : X' \to Y'$ are bounded. Thus Lemma 59.92.2 implies that

$$f_* : Mod(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$

and

$$f'_* : Mod(X'_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y'_{étale}, \mathbf{Z}/n\mathbf{Z})$$

have finite cohomological dimension in the sense of Derived Categories, Lemma 13.32.2. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E. See Injectives, Theorem 19.12.6. By the usual proper base change theorem we find that $R^q f'_*(g')^{-1}\mathcal{I}^n = 0$ for $q > 0$, see Theorem 59.91.11. Hence we conclude by Derived Categories, Lemma 13.32.2 that we may compute $Rf'_*(g')^{-1}E$ by the complex $f'_*(g')^{-1}\mathcal{I}^\bullet$. Another application of the usual proper base change theorem shows that this is equal to $g^{-1}f_*\mathcal{I}^\bullet$ as desired. □

0F0E **Lemma 59.92.4.** *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^+(X_{étale})$ and $K \in D^+(\mathbf{Z})$. Then*

$$R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{K}) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

Proof. Say $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. We may represent K by a bounded below complex $K^\bullet$ of torsion free $\mathbf{Z}$-modules. (Choose a K-flat complex $L^\bullet$ representing K and then take $K^\bullet = \tau_{\geq b-1} L^\bullet$. This works because $\mathbf{Z}$ has global dimension 1. See More on Algebra, Lemma 15.67.2.) We may represent E by a bounded below complex $\mathcal{E}^\bullet$. Then $E \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{K}$ is represented by

$$\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \underline{K}^\bullet)$$

Using distinguished triangles

$$\sigma_{\geq -b+n+1} K^\bullet \to K^\bullet \to \sigma_{\leq -b+n} K^\bullet$$

and the trivial vanishing

$$H^n(X, \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathbf{Z}} \sigma_{\geq -a+n+1} \underline{K}^\bullet) = 0$$

and

$$H^n(R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}} \sigma_{\geq -a+n+1} K^\bullet) = 0$$

we reduce to the case where $K^\bullet$ is a bounded complex of flat $\mathbf{Z}$-modules. Repeating the argument we reduce to the case where $K^\bullet$ is equal to a single flat $\mathbf{Z}$-module sitting in some degree. Next, using the stupid trunctions for $\mathcal{E}^\bullet$ we reduce in exactly the same manner to the case where $\mathcal{E}^\bullet$ is a single abelian sheaf sitting in some degree. Thus it suffices to show that

$$H^n(X, \mathcal{E} \otimes_{\mathbf{Z}} \underline{M}) = H^n(X, \mathcal{E}) \otimes_{\mathbf{Z}} M$$

when M is a flat $\mathbf{Z}$-module and $\mathcal{E}$ is an abelian sheaf on X. In this case we write M is a filtered colimit of finite free $\mathbf{Z}$-modules (Lazard's theorem, see Algebra, Theorem 10.81.4). By Theorem 59.51.3 this reduces us to the case of finite free $\mathbf{Z}$-module M in which case the result is trivially true. □

0F0F **Lemma 59.92.5.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Let $K \in D^+(Y_{étale})$. Then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathbf{Z}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathbf{Z}} f^{-1}K)$$

in $D^+(Y_{étale})$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 21.50. We will check the equality on stalks. Recall that computing derived tensor products commutes with pullbacks. See Cohomology on Sites, Lemma 21.18.4. Thus we have

$$(E \otimes^{\mathbf{L}}_{\mathbf{Z}} f^{-1}K)_{\overline{x}} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathbf{Z}} K_{\overline{y}}$$

where $\overline{y}$ is the image of $\overline{x}$ in Y. Since $\mathbf{Z}$ has global dimension 1 we see that this complex has vanishing cohomology in degree $< -1 + a + b$ if $H^i(E) = 0$ for $i \geq a$ and $H^j(K) = 0$ for $j \geq b$. Moreover, since $H^i(E)$ is a torsion abelian sheaf for each i, the same is true for the cohomology sheaves of the complex $E \otimes^{\mathbf{L}}_{\mathbf{Z}} K$. Namely, we have

$$(E \otimes^{\mathbf{L}}_{\mathbf{Z}} f^{-1}K) \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q} = (E \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q}) \otimes^{\mathbf{L}}_{\mathbf{Q}} (f^{-1}K \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q})$$

which is zero in the derived category. In this way we see that Lemma 59.91.13 applies to both sides to see that it suffices to show

$$R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}} \otimes^{\mathbf{L}}_{\mathbf{Z}} (X_{\overline{y}} \to \overline{y})^{-1}K_{\overline{y}}) = R\Gamma(X_{\overline{y}}, E|_{X_{\overline{y}}}) \otimes^{\mathbf{L}}_{\mathbf{Z}} K_{\overline{y}}$$

This is shown in Lemma 59.92.4. □

59.93. Local acyclicity

0GJM In this section we deduce local acyclicity of smooth morphisms from the smooth base change theorem. In SGA 4 or SGA 4.5 the authors first prove a version of local acyclicity for smooth morphisms and then deduce the smooth base change theorem.

We will use the formulation of local acyclicity given by Deligne [**Del77**, Definition 2.12, page 242]. Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$ in S. Let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. We obtain a commutative diagram

$$\begin{array}{ccccc} F_{\overline{x},\overline{t}} = \overline{t} \times_{\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \overline{t} & \longrightarrow & \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) & \longrightarrow & S \end{array}$$

The scheme $F_{\overline{x},\overline{t}}$ is called a *variety of vanishing cycles of f at $\overline{x}$*. Let K be an object of $D(X_{étale})$. For any morphism of schemes $g : Y \to X$ we write $R\Gamma(Y, K)$ instead of $R\Gamma(Y_{étale}, g^{-1}_{small}K)$. Since $\mathcal{O}^{sh}_{X,\overline{x}}$ is strictly henselian we have $K_{\overline{x}} = R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}), K)$. Thus we obtain a canonical map

0GJN (59.93.0.1) $$\alpha_{K,\overline{x},\overline{t}} : K_{\overline{x}} \longrightarrow R\Gamma(F_{\overline{x},\overline{t}}, K)$$

by pulling back cohomology along $F_{\overline{x},\overline{t}} \to \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$.

0GJP **Definition 59.93.1.** Let $f : X \to S$ be a morphism of schemes. Let K be an object of $D(X_{étale})$. [**Del77**, Definition 2.12, page 242] and [**Del77**, Definition (1.3), page 54]

(1) Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$. We say f is *locally acyclic at $\overline{x}$ relative to K* if for every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ the map (59.93.0.1) is an isomorphism[11].
(2) We say f is *locally acyclic relative to K* if f is locally acyclic at $\overline{x}$ relative to K for every geometric point $\overline{x}$ of X.
(3) We say f is *universally locally acyclic relative to K* if for any morphism $S' \to S$ of schemes the base change $f' : X' \to S'$ is locally acyclic relative to the pullback of K to X'.
(4) We say f is *locally acyclic* if for all geometric points $\overline{x}$ of X and any integer n prime to the characteristic of $\kappa(\overline{x})$, the morphism f is locally acyclic at $\overline{x}$ relative to the constant sheaf with value $\mathbf{Z}/n\mathbf{Z}$.
(5) We say f is *universally locally acyclic* if for any morphism $S' \to S$ of schemes the base change $f' : X' \to S'$ is locally acyclic.

[11] We do not assume $\overline{t}$ is an algebraic geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Often using Lemma 59.90.2 one may reduce to this case.

Let M be an abelian group. Then local acyclicity of $f : X \to S$ with respect to the constant sheaf $\underline{M}$ boils down to the requirement that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0 \end{cases}$$

for any geometric point $\overline{x}$ of X and any geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. In this way we see that being locally acyclic corresponds to the vanishing of the higher cohomology groups of the geometric fibres $F_{\overline{x},\overline{t}}$ of the maps between the strict henselizations at $\overline{x}$ and $\overline{s}$.

0GJQ **Proposition 59.93.2.** *Let $f : X \to S$ be a smooth morphism of schemes. Then f is universally locally acyclic.*

Proof. Since the base change of a smooth morphism is smooth, it suffices to show that smooth morphisms are locally acyclic. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$. Let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. Since we are trying to prove a property of the ring map $\mathcal{O}^{sh}_{S,\overline{s}} \to \mathcal{O}^{sh}_{X,\overline{x}}$ (see discussion following Definition 59.93.1) we may and do replace $f : X \to S$ by the base change $X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Thus we may and do assume that S is the spectrum of a strictly henselian local ring and that $\overline{s}$ lies over the closed point of S.

We will apply Lemma 59.86.5 to the diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & X_{\overline{t}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\ S & \xleftarrow{g} & \overline{t} \end{array}$$

and the sheaf $\mathcal{F} = \underline{M}$ where $M = \mathbf{Z}/n\mathbf{Z}$ for some integer n prime to the characteristic of the residue field of $\overline{x}$. We know that the map $f^{-1}R^q g_*\mathcal{F} \to R^q h_* e^{-1}\mathcal{F}$ is an isomorphism by smooth base change, see Theorem 59.89.2 (the assumption on torsion holds by our choice of n). Thus Lemma 59.86.5 gives us the middle equality in

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) \times_S \overline{t}, \underline{M}) = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \times_S \overline{t}, \underline{M}) = H^q(\overline{t}, \underline{M})$$

For the outer two equalities we use that $S = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Since $\overline{t}$ is the spectrum of a separably closed field we conclude that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0 \end{cases}$$

which is what we had to show (see discussion following Definition 59.93.1). □

0GJR **Lemma 59.93.3.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a locally constant abelian sheaf on $X_{étale}$ such that for every geometric point $\overline{x}$ of X the abelian group $\mathcal{F}_{\overline{x}}$ is a torsion group all of whose elements have order prime to the characteristic of the residue field of $\overline{x}$. If f is locally acyclic, then f is locally acyclic relative to $\mathcal{F}$.*

Proof. Namely, let $\overline{x}$ be a geometric point of X. Since $\mathcal{F}$ is locally constant we see that the restriction of $\mathcal{F}$ to $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})$ is isomorphic to the constant sheaf $\underline{M}$ with $M = \mathcal{F}_{\overline{x}}$. By assumption we can write $M = \mathrm{colim}\, M_i$ as a filtered colimit of

finite abelian groups M_i of order prime to the characteristic of the residue field of $\overline{x}$. Consider a geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,f(\overline{x})})$. Since $F_{\overline{x},\overline{t}}$ is affine, we have

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \mathrm{colim}\, H^q(F_{\overline{x},\overline{t}}, \underline{M_i})$$

by Lemma 59.51.4. For each i we can write $M_i = \bigoplus \mathbf{Z}/n_{i,j}\mathbf{Z}$ as a finite direct sum for some integers $n_{i,j}$ prime to the characteristic of the residue field of $\overline{x}$. Since f is locally acyclic we see that

$$H^q(F_{\overline{x},\overline{t}}, \underline{\mathbf{Z}/n_{i,j}\mathbf{Z}}) = \begin{cases} \mathbf{Z}/n_{i,j}\mathbf{Z} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0 \end{cases}$$

See discussion following Definition 59.93.1. Taking the direct sums and the colimit we conclude that

$$H^q(F_{\overline{x},\overline{t}}, \underline{M}) = \begin{cases} M & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0 \end{cases}$$

and we win. □

0GJS **Lemma 59.93.4.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let K be an object of $D(X_{étale})$. Let $\overline{x}'$ be a geometric point of X' with image $\overline{x}$ in X. If

1. *f is locally acyclic at $\overline{x}$ relative to K and*
2. *g is locally quasi-finite, or $S' = \lim S_i$ is a directed inverse limit of schemes locally quasi-finite over S with affine transition morphisms, or $g : S' \to S$ is integral,*

then f' locally acyclic at $\overline{x}'$ relative to $(g')^{-1}K$.

Proof. Denote $\overline{s}'$ and $\overline{s}$ the images of $\overline{x}'$ and $\overline{x}$ in S' and S. Let $\overline{t}'$ be a geometric point of the spectrum of $\mathrm{Spec}(\mathcal{O}^{sh}_{S',\overline{s}'})$ and denote $\overline{t}$ the image in $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. By Algebra, Lemma 10.156.6 and our assumptions on g we have

$$\mathcal{O}^{sh}_{X,\overline{x}} \otimes_{\mathcal{O}^{sh}_{S,\overline{s}}} \mathcal{O}^{sh}_{S',\overline{s}'} \longrightarrow \mathcal{O}^{sh}_{X',\overline{x}'}$$

is an isomorphism. Since by our conventions $\kappa(\overline{t}) = \kappa(\overline{t}')$ we conclude that

$$F_{\overline{x}',\overline{t}'} = \mathrm{Spec}\left(\mathcal{O}^{sh}_{X',\overline{x}'} \otimes_{\mathcal{O}^{sh}_{S',\overline{s}'}} \kappa(\overline{t}')\right) = \mathrm{Spec}\left(\mathcal{O}^{sh}_{X,\overline{x}} \otimes_{\mathcal{O}^{sh}_{S,\overline{s}}} \kappa(\overline{t})\right) = F_{\overline{x},\overline{t}}$$

In other words, the varieties of vanishing cycles of f' at $\overline{x}'$ are examples of varieties of vanishing cycles of f at $\overline{x}$. The lemma follows immediately from this and the definitions. □

59.94. The cospecialization map

0GJT Let $f : X \to S$ be a morphism of schemes. Let $\overline{x}$ be a geometric point of X with image $\overline{s} = f(\overline{x})$ in S. Let $\overline{t}$ be a geometric point of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Let $K \in D(X_{étale})$. For any morphism $g : Y \to X$ of schemes we write $K|_Y$ instead of $g^{-1}_{small}K$ and $R\Gamma(Y, K)$ instead of $R\Gamma(Y_{étale}, g^{-1}_{small}K)$. We claim that if

1. *K is bounded below, i.e., $K \in D^+(X_{étale})$,*
2. *f is locally acyclic relative to K*

then there is a *cospecialization map*

$$cosp : R\Gamma(X_{\overline{t}}, K) \longrightarrow R\Gamma(X_{\overline{s}}, K)$$

which will be closely related to the specialization map considered in Section 59.75 and especially Remark 59.75.8.

To construct the map we consider the morphisms

$$X_{\overline{t}} \xrightarrow{h} X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \xleftarrow{i} X_{\overline{s}}$$

The unit of the adjunction between h^{-1} and Rh_* gives a map

$$\beta_{K,\overline{s},\overline{t}} : K|_{X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})} \longrightarrow Rh_*(K|_{X_{\overline{t}}})$$

in $D((X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}))_{\acute{e}tale})$. Lemma 59.94.1 below shows that the pullback $i^{-1}\beta_{K,\overline{s},\overline{t}}$ is an isomorphism under the assumptions above. Thus we can define the cospecialization map as the composition

$$\begin{aligned} R\Gamma(X_{\overline{t}}, K) &= R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), Rh_*(K|_{X_{\overline{t}}})) \\ &\xrightarrow{i^{-1}} R\Gamma(X_{\overline{s}}, i^{-1}Rh_*(K|_{X_{\overline{t}}})) \\ &\xrightarrow{(i^{-1}\beta_{K,\overline{s},\overline{t}})^{-1}} R\Gamma(X_{\overline{s}}, i^{-1}(K|_{X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})})) \\ &= R\Gamma(X_{\overline{s}}, K) \end{aligned}$$

0GJU **Lemma 59.94.1.** *The map $i^{-1}\beta_{K,\overline{s},\overline{t}}$ is an isomorphism.*

Proof. The construction of the maps h, i, $\beta_{K,\overline{s},\overline{t}}$ only depends on the base change of X and K to $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Thus we may and do assume that S is a strictly henselian scheme with closed point $\overline{s}$. Observe that the local acyclicity of f relative to K is preserved by this base change (for example by Lemma 59.93.4 or just directly by comparing strictly henselian rings in this very special case).

Let $\overline{x}$ be a geometric point of $X_{\overline{s}}$. Or equivalently, let $\overline{x}$ be a geometric point whose image by f is $\overline{s}$. Let us compute the stalk of $i^{-1}\beta_{K,\overline{s},\overline{t}}$ at $\overline{x}$. First, we have

$$(i^{-1}\beta_{K,\overline{s},\overline{t}})_{\overline{x}} = (\beta_{K,\overline{s},\overline{t}})_{\overline{x}}$$

since pullback preserves stalks, see Lemma 59.36.2. Since we are in the situation $S = \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ we see that $h : X_{\overline{t}} \to X$ has the property that $X_{\overline{t}} \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}}) = F_{\overline{x},\overline{t}}$. Thus we see that

$$(\beta_{K,\overline{s},\overline{t}})_{\overline{x}} : K_{\overline{x}} \longrightarrow Rh_*(K|_{X_{\overline{t}}})_{\overline{x}} = R\Gamma(F_{\overline{x},\overline{t}}, K)$$

where the equal sign is Theorem 59.53.1. It follows that the map $(\beta_{K,\overline{s},\overline{t}})_{\overline{x}}$ is none other than the map $\alpha_{K,\overline{x},\overline{t}}$ used in Definition 59.93.1. The result follows as we may check whether a map is an isomorphism in stalks by Theorem 59.29.10. □

The cospecialization map when it exists is trying to be the inverse of the specialization map.

0GJV **Lemma 59.94.2.** *In the situation above, if in addition f is quasi-compact and quasi-separated, then the diagram*

$$\begin{array}{ccc} (Rf_*K)_{\overline{s}} & \longrightarrow & R\Gamma(X_{\overline{s}}, K) \\ {\scriptstyle sp}\downarrow & & \uparrow{\scriptstyle cosp} \\ (Rf_*K)_{\overline{t}} & \longrightarrow & R\Gamma(X_{\overline{t}}, K) \end{array}$$

is commutative.

Proof. As in the proof of Lemma 59.94.1 we may replace S by $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$. Then our maps simplify to $h : X_{\overline{t}} \to X$, $i : X_{\overline{s}} \to X$, and $\beta_{K,\overline{s},\overline{t}} : K \to Rh_*(K|_{X_{\overline{t}}})$. Using that $(Rf_*K)_{\overline{s}} = R\Gamma(X, K)$ by Theorem 59.53.1 the composition of sp with the base change map $(Rf_*K)_{\overline{t}} \to R\Gamma(X_{\overline{t}}, K)$ is just pullback of cohomology along h. This is the same as the map

$$R\Gamma(X, K) \xrightarrow{\beta_{K,\overline{s},\overline{t}}} R\Gamma(X, Rh_*(K|_{X_{\overline{t}}})) = R\Gamma(X_{\overline{t}}, K)$$

Now the map $cosp$ first inverts the $=$ sign in this displayed formula, then pulls back along i, and finally applies the inverse of $i^{-1}\beta_{K,\overline{s},\overline{t}}$. Hence we get the desired commutativity. □

0GJW **Lemma 59.94.3.** *Let $f : X \to S$ be a morphism of schemes. Let $K \in D(X_{étale})$. Assume*

(1) *K is bounded below, i.e., $K \in D^+(X_{étale})$,*
(2) *f is locally acyclic relative to K,*
(3) *f is proper, and*
(4) *K has torsion cohomology sheaves.*

Then for every geometric point $\overline{s}$ of S and every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ both the specialization map $sp : (Rf_*K)_{\overline{s}} \to (Rf_*K)_{\overline{t}}$ and the cospecialization map $cosp : R\Gamma(X_{\overline{t}}, K) \to R\Gamma(X_{\overline{s}}, K)$ are isomorphisms.

Proof. By the proper base change theorem (in the form of Lemma 59.91.13) we have $(Rf_*K)_{\overline{s}} = R\Gamma(X_{\overline{s}}, K)$ and similarly for $\overline{t}$. The "correct" proof would be to show that the argument in Lemma 59.94.2 shows that sp and $cosp$ are inverse isomorphisms in this case. Instead we will show directly that $cosp$ is an isomorphism. From the discussion above we see that $cosp$ is an isomorphism if and only if pullback by i

$$R\Gamma(X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), Rh_*(K|_{X_{\overline{t}}})) \longrightarrow R\Gamma(X_{\overline{s}}, i^{-1}Rh_*(K|_{X_{\overline{t}}}))$$

is an isomorphism in $D^+(Ab)$. This is true by the proper base change theorem for the proper morphism $f' : X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ by the morphism $\overline{s} \to \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ and the complex $K' = Rh_*(K|_{X_{\overline{t}}})$. The complex K' is bounded below and has torsion cohomology sheaves by Lemma 59.78.2. Since $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is strictly henselian with $\overline{s}$ lying over the closed point, we see that the source of the displayed arrow equals $(Rf'_*K')_{\overline{s}}$ and the target equals $R\Gamma(X_{\overline{s}}, K')$ and the displayed map is an isomorphism by the already used Lemma 59.91.13. Thus we see that three out of the four arrows in the diagram of Lemma 59.94.2 are isomorphisms and we conclude. □

0GKD **Lemma 59.94.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Assume*

(1) *f is smooth and proper*
(2) *$\mathcal{F}$ is locally constant, and*
(3) *$\mathcal{F}_{\overline{x}}$ is a torsion group all of whose elements have order prime to the residue characteristic of $\overline{x}$ for every geometric point $\overline{x}$ of X.*

Then for every geometric point $\overline{s}$ of S and every geometric point $\overline{t}$ of $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ the specialization map $sp : (Rf_*\mathcal{F})_{\overline{s}} \to (Rf_*\mathcal{F})_{\overline{t}}$ is an isomorphism.

Proof. This follows from Lemmas 59.94.3 and 59.93.3 and Proposition 59.93.2. □

59.95. Cohomological dimension

0F0P We can deduce some bounds on the cohomological dimension of schemes and on the cohomological dimension of fields using the results in Section 59.83 and one, seemingly innocuous, application of the proper base change theorem (in the proof of Proposition 59.95.6).

0F0Q **Definition 59.95.1.** Let X be a quasi-compact and quasi-separated scheme. The *cohomological dimension of X* is the smallest element

$$\mathrm{cd}(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $H^i_{étale}(X, \mathcal{F}) = 0$ for $i > \mathrm{cd}(X)$. If $X = \mathrm{Spec}(A)$ we sometimes call this the cohomological dimension of A.

If the scheme is in characteristic p, then we often can obtain sharper bounds for the vanishing of cohomology of p-power torsion sheaves. We will address this elsewhere (insert future reference here).

0F0R **Lemma 59.95.2.** *Let $X = \lim X_i$ be a directed limit of a system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then $cd(X) \leq \max cd(X_i)$.*

Proof. Denote $f_i : X \to X_i$ the projections. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then we have $\mathcal{F} = \lim f_i^{-1} f_{i,*}\mathcal{F}$ by Lemma 59.51.9. Thus $H^q_{étale}(X, \mathcal{F}) = \mathrm{colim}\, H^q_{étale}(X_i, f_{i,*}\mathcal{F})$ by Theorem 59.51.3. The lemma follows. □

0F0S **Lemma 59.95.3.** *Let K be a field. Let X be a 1-dimensional affine scheme of finite type over K. Then $cd(X) \leq 1 + cd(K)$.*

Proof. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Consider the Leray spectral sequence for the morphism $f : X \to \mathrm{Spec}(K)$. We obtain

$$E_2^{p,q} = H^p(\mathrm{Spec}(K), R^q f_*\mathcal{F})$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$. The stalk of $R^q f_*\mathcal{F}$ at a geometric point $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(K)$ is the cohomology of the pullback of $\mathcal{F}$ to $X_{\overline{K}}$. Hence it vanishes in degrees ≥ 2 by Theorem 59.83.10. □

0F0T **Lemma 59.95.4.** *Let L/K be a field extension. Then we have $cd(L) \leq cd(K) + trdeg_K(L)$.*

Proof. If $\mathrm{trdeg}_K(L) = \infty$, then this is clear. If not then we can find a sequence of extensions $L = L_r/L_{r-1}/\ldots/L_1/L_0 = K$ such that $\mathrm{trdeg}_{L_i}(L_{i+1}) = 1$ and $r = \mathrm{trdeg}_K(L)$. Hence it suffices to prove the lemma in the case that $r = 1$. In this case we can write $L = \mathrm{colim}\, A_i$ as a filtered colimit of its finite type K-subalgebras.

By Lemma 59.95.2 it suffices to prove that $\mathrm{cd}(A_i) \leq 1 + \mathrm{cd}(K)$. This follows from Lemma 59.95.3. □

0F0U **Lemma 59.95.5.** *Let K be a field. Let X be a scheme of finite type over K. Let $x \in X$. Set $a = trdeg_K(\kappa(x))$ and $d = \dim_x(X)$. Then there is a map*

$$K(t_1, \ldots, t_a)^{sep} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

such that

1. *the residue field of $\mathcal{O}^{sh}_{X,x}$ is a purely inseparable extension of $K(t_1, \ldots, t_a)^{sep}$,*
2. *$\mathcal{O}^{sh}_{X,x}$ is a filtered colimit of finite type $K(t_1, \ldots, t_a)^{sep}$-algebras of dimension $\leq d - a$.*

Proof. We may assume X is affine. By Noether normalization, after possibly shrinking X again, we can choose a finite morphism $\pi : X \to \mathbf{A}^d_K$, see Algebra, Lemma 10.115.5. Since $\kappa(x)$ is a finite extension of the residue field of $\pi(x)$, this residue field has transcendence degree a over K as well. Thus we can find a finite morphism $\pi' : \mathbf{A}^d_K \to \mathbf{A}^d_K$ such that $\pi'(\pi(x))$ corresponds to the generic point of the linear subspace $\mathbf{A}^a_K \subset \mathbf{A}^d_K$ given by setting the last $d - a$ coordinates equal to zero. Hence the composition

$$X \xrightarrow{\pi' \circ \pi} \mathbf{A}^d_K \xrightarrow{p} \mathbf{A}^a_K$$

of $\pi' \circ \pi$ and the projection p onto the first a coordinates maps x to the generic point $\eta \in \mathbf{A}^a_K$. The induced map

$$K(t_1, \ldots, t_a)^{sep} = \mathcal{O}^{sh}_{\mathbf{A}^a_k, \eta} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

on étale local rings satisfies (1) since it is clear that the residue field of $\mathcal{O}^{sh}_{X,x}$ is an algebraic extension of the separably closed field $K(t_1, \ldots, t_a)^{sep}$. On the other hand, if $X = \mathrm{Spec}(B)$, then $\mathcal{O}^{sh}_{X,x} = \mathrm{colim}\, B_j$ is a filtered colimit of étale B-algebras B_j. Observe that B_j is quasi-finite over $K[t_1, \ldots, t_d]$ as B is finite over $K[t_1, \ldots, t_d]$. We may similarly write $K(t_1, \ldots, t_a)^{sep} = \mathrm{colim}\, A_i$ as a filtered colimit of étale $K[t_1, \ldots, t_a]$-algebras. For every i we can find an j such that $A_i \to K(t_1, \ldots, t_a)^{sep} \to \mathcal{O}^{sh}_{X,x}$ factors through a map $\psi_{i,j} : A_i \to B_j$. Then B_j is quasi-finite over $A_i[t_{a+1}, \ldots, t_d]$. Hence

$$B_{i,j} = B_j \otimes_{\psi_{i,j}, A_i} K(t_1, \ldots, t_a)^{sep}$$

has dimension $\leq d - a$ as it is quasi-finite over $K(t_1, \ldots, t_a)^{sep}[t_{a+1}, \ldots, t_d]$. The proof of (2) is now finished as $\mathcal{O}^{sh}_{X,x}$ is a filtered colimit[12] of the algebras $B_{i,j}$. Some details omitted. □

0F0V **Proposition 59.95.6.** *Let K be a field. Let X be an affine scheme of finite type over K. Then we have $cd(X) \leq \dim(X) + cd(K)$.*

Proof. We will prove this by induction on $\dim(X)$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$.

[12] Let R be a ring. Let $A = \mathrm{colim}_{i \in I} A_i$ be a filtered colimit of finitely presented R-algebras. Let $B = \mathrm{colim}_{j \in J} B_j$ be a filtered colimit of R-algebras. Let $A \to B$ be an R-algebra map. Assume that for all $i \in I$ there is a $j \in J$ and an R-algebra map $\psi_{i,j} : A_i \to B_j$. Say $(i', j', \psi_{i',j'}) \geq (i, j, \psi_{i,j})$ if $i' \geq i$, $j' \geq j$, and $\psi_{i,j}$ and $\psi_{i',j'}$ are compatible. Then the collection of triples forms a directed set and $B = \mathrm{colim}\, B_j \otimes_{\psi_{i,j} A_i} A$.

The case $\dim(X) = 0$. In this case the structure morphism $f : X \to \mathrm{Spec}(K)$ is finite. Hence we see that $R^i f_* \mathcal{F} = 0$ for $i > 0$, see Proposition 59.55.2. Thus $H^i_{étale}(X, \mathcal{F}) = H^i_{étale}(\mathrm{Spec}(K), f_* \mathcal{F})$ by the Leray spectral sequence for f (Cohomology on Sites, Lemma 21.14.5) and the result is clear.

The case $\dim(X) = 1$. This is Lemma 59.95.3.

Assume $d = \dim(X) > 1$ and the proposition holds for finite type affine schemes of dimension $< d$ over fields. By Noether normalization, see for example Varieties, Lemma 33.18.2, there exists a finite morphism $f : X \to \mathbf{A}^d_K$. Recall that $R^i f_* \mathcal{F} = 0$ for $i > 0$ by Proposition 59.55.2. By the Leray spectral sequence for f (Cohomology on Sites, Lemma 21.14.5) we conclude that it suffices to prove the result for $\pi_* \mathcal{F}$ on $\mathbf{A}^d_K$.

Interlude I. Let $j : X \to Y$ be an open immersion of smooth d-dimensional varieties over K (not necessarily affine) whose complement is the support of an effective Cartier divisor D. The sheaves $R^q j_* \mathcal{F}$ for $q > 0$ are supported on D. We claim that $(R^q j_* \mathcal{F})_{\overline{y}} = 0$ for $a = \mathrm{trdeg}_K(\kappa(y)) > d - q$. Namely, by Theorem 59.53.1 we have

$$(R^q j_* \mathcal{F})_{\overline{y}} = H^q(\mathrm{Spec}(\mathcal{O}^{sh}_{Y,y}) \times_Y X, \mathcal{F})$$

Choose a local equation $f \in \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ for D. Then we have

$$\mathrm{Spec}(\mathcal{O}^{sh}_{Y,y}) \times_Y X = \mathrm{Spec}(\mathcal{O}^{sh}_{Y,y}[1/f])$$

Using Lemma 59.95.5 we get an embedding

$$K(t_1, \ldots, t_a)^{sep}(x) = K(t_1, \ldots, t_a)^{sep}[x]_{(x)}[1/x] \longrightarrow \mathcal{O}^{sh}_{Y,y}[1/f]$$

Since the transcendence degree over K of the fraction field of $\mathcal{O}^{sh}_{Y,y}$ is d, we see that $\mathcal{O}^{sh}_{Y,y}[1/f]$ is a filtered colimit of $(d - a - 1)$-dimensional finite type algebras over the field $K(t_1, \ldots, t_a)^{sep}(x)$ which itself has cohomological dimension 1 by Lemma 59.95.4. Thus by induction hypothesis and Lemma 59.95.2 we obtain the desired vanishing.

Interlude II. Let Z be a smooth variety over K of dimension $d - 1$. Let $E_a \subset Z$ be the set of points $z \in Z$ with $\mathrm{trdeg}_K(\kappa(z)) \leq a$. Observe that E_a is closed under specialization, see Varieties, Lemma 33.20.3. Suppose that $\mathcal{G}$ is a torsion abelian sheaf on Z whose support is contained in E_a. Then we claim that $H^b_{étale}(Z, \mathcal{G}) = 0$ for $b > a + \mathrm{cd}(K)$. Namely, we can write $\mathcal{G} = \mathrm{colim}\, \mathcal{G}_i$ with $\mathcal{G}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 59.74.5. Then the induction hypothesis kicks in to imply the desired vanishing for $\mathcal{G}_i$[13]. Finally, we conclude by Theorem 59.51.3.

Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{A}^d_K & \xrightarrow{j} & \mathbf{P}^1_K \times_K \mathbf{A}^{d-1}_K \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & \\ & \mathbf{A}^{d-1}_K & \end{array}$$

[13]Here we first use Proposition 59.46.4 to write $\mathcal{G}_i$ as the pushforward of a sheaf on Z_i, the induction hypothesis gives the vanishing for this sheaf on Z_i, and the Leray spectral sequence for $Z_i \to Z$ gives the vanishing for $\mathcal{G}_i$.

Observe that j is an open immersion of smooth d-dimensional varieties whose complement is an effective Cartier divisor D. Thus we may use the results obtained in interlude I. We are going to study the relative Leray spectral sequence

$$E_2^{p,q} = R^p g_* R^q j_* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

Since $R^q j_* \mathcal{F}$ for $q > 0$ is supported on D and since $g|_D : D \to \mathbf{A}_K^{d-1}$ is an isomorphism, we find $R^p g_* R^q j_* \mathcal{F} = 0$ for $p > 0$ and $q > 0$. Moreover, we have $R^q j_* \mathcal{F} = 0$ for $q > d$. On the other hand, g is a proper morphism of relative dimension 1. Hence by Lemma 59.92.2 we see that $R^p g_* j_* \mathcal{F} = 0$ for $p > 2$. Thus the E_2-page of the spectral sequence looks like this

$$\begin{array}{ccc} g_* R^d j_* \mathcal{F} & 0 & 0 \\ \ldots & \ldots & \ldots \\ g_* R^2 j_* \mathcal{F} & 0 & 0 \\ g_* R^1 j_* \mathcal{F} & 0 & 0 \\ g_* j_* \mathcal{F} & R^1 g_* j_* \mathcal{F} & R^2 g_* j_* \mathcal{F} \end{array}$$

We conclude that $R^q f_* \mathcal{F} = g_* R^q j_* \mathcal{F}$ for $q > 2$. By interlude I we see that the support of $R^q f_* \mathcal{F}$ for $q > 2$ is contained in the set of points of $\mathbf{A}_K^{d-1}$ whose residue field has transcendence degree $\leq d - q$. By interlude II

$$H^p(\mathbf{A}_K^{d-1}, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - q + \mathrm{cd}(K) \text{ and } q > 2$$

On the other hand, by Theorem 59.53.1 we have $R^2 f_* \mathcal{F}_{\overline{\eta}} = H^2(\mathbf{A}^1_{\overline{\eta}}, \mathcal{F}) = 0$ (vanishing by the case of dimension 1) where η is the generic point of $\mathbf{A}_K^{d-1}$. Hence by interlude II again we see

$$H^p(\mathbf{A}_K^{d-1}, R^2 f_* \mathcal{F}) = 0 \text{ for } p > d - 2 + \mathrm{cd}(K)$$

Finally, we have

$$H^p(\mathbf{A}_K^{d-1}, R^q f_* \mathcal{F}) = 0 \text{ for } p > d - 1 + \mathrm{cd}(K) \text{ and } q = 0, 1$$

by induction hypothesis. Combining everything we just said with the Leray spectral sequence $H^p(\mathbf{A}_K^{d-1}, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(\mathbf{A}_K^d, \mathcal{F})$ we conclude. □

0F0W **Lemma 59.95.7.** *Let K be a field. Let X be an affine scheme of finite type over K. Let $E_a \subset X$ be the set of points $x \in X$ with $trdeg_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$ whose support is contained in E_a. Then $H^b_{étale}(X, \mathcal{F}) = 0$ for $b > a + cd(K)$.*

Proof. We can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ a torsion abelian sheaf supported on a closed subscheme Z_i contained in E_a, see Lemma 59.74.5. Then Proposition 59.95.6 gives the desired vanishing for $\mathcal{F}_i$. Details omitted; hints: first use Proposition 59.46.4 to write $\mathcal{F}_i$ as the pushforward of a sheaf on Z_i, use the vanishing for this sheaf on Z_i, and use the Leray spectral sequence for $Z_i \to Z$ to get the vanishing for $\mathcal{F}_i$. Finally, we conclude by Theorem 59.51.3. □

0F0X **Lemma 59.95.8.** *Let $f : X \to Y$ be an affine morphism of schemes of finite type over a field K. Let $E_a(X)$ be the set of points $x \in X$ with $trdeg_K(\kappa(x)) \leq a$. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$ whose support is contained in E_a. Then $R^q f_* \mathcal{F}$ has support contained in $E_{a-q}(Y)$.*

Proof. The question is local on Y hence we can assume Y is affine. Then X is affine too and we can choose a diagram

$$\begin{array}{ccc} X & \xrightarrow[i]{} & \mathbf{A}^{n+m}_K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ Y & \xrightarrow{j} & \mathbf{A}^n_K \end{array}$$

where the horizontal arrows are closed immersions and the vertical arrow on the right is the projection (details omitted). Then $j_*R^qf_*\mathcal{F} = R^q\mathrm{pr}_*i_*\mathcal{F}$ by the vanishing of the higher direct images of i and j, see Proposition 59.55.2. Moreover, the description of the stalks of j_* in the proposition shows that it suffices to prove the vanishing for $j_*R^qf_*\mathcal{F}$. Thus we may assume f is the projection morphism $\mathrm{pr} : \mathbf{A}^{n+m}_K \to \mathbf{A}^n_K$ and an abelian torsion sheaf $\mathcal{F}$ on $\mathbf{A}^{n+m}_K$ satisfying the assumption in the statement of the lemma.

Let y be a point in $\mathbf{A}^n_K$. By Theorem 59.53.1 we have

$$(R^q\mathrm{pr}_*\mathcal{F})_{\overline{y}} = H^q(\mathbf{A}^{n+m}_K \times_{A^n_K} \mathrm{Spec}(\mathcal{O}^{sh}_{Y,y}), \mathcal{F}) = H^q(\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}}, \mathcal{F})$$

Say $b = \mathrm{trdeg}_K(\kappa(y))$. From Lemma 59.95.5 we get an embedding

$$L = K(t_1, \ldots, t_b)^{sep} \longrightarrow \mathcal{O}^{sh}_{Y,y}$$

Write $\mathcal{O}^{sh}_{Y,y} = \mathrm{colim}\, B_i$ as the filtered colimit of finite type L-subalgebras $B_i \subset \mathcal{O}^{sh}_{Y,y}$ containing the ring $K[T_1, \ldots, T_n]$ of regular functions on $\mathbf{A}^n_K$. Then we get

$$\mathbf{A}^m_{\mathcal{O}^{sh}_{Y,y}} = \lim \mathbf{A}^m_{B_i}$$

If $z \in \mathbf{A}^m_{B_i}$ is a point in the support of $\mathcal{F}$, then the image x of z in $\mathbf{A}^{m+n}_K$ satisfies $\mathrm{trdeg}_K(\kappa(x)) \leq a$ by our assumption on $\mathcal{F}$ in the lemma. Since $\mathcal{O}^{sh}_{Y,y}$ is a filtered colimit of étale algebras over $K[T_1, \ldots, T_n]$ and since $B_i \subset \mathcal{O}^{sh}_{Y,y}$ we see that $\kappa(z)/\kappa(x)$ is algebraic (some details omitted). Then $\mathrm{trdeg}_K(\kappa(z)) \leq a$ and hence $\mathrm{trdeg}_L(\kappa(z)) \leq a - b$. By Lemma 59.95.7 we see that

$$H^q(\mathbf{A}^m_{B_i}, \mathcal{F}) = 0 \text{ for } q > a - b$$

Thus by Theorem 59.51.3 we get $(Rf_*\mathcal{F})_{\overline{y}} = 0$ for $q > a - b$ as desired. □

59.96. Finite cohomological dimension

0F0Y We continue the discussion started in Section 59.95.

0F0Z **Definition 59.96.1.** Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. The *cohomological dimension of f* is the smallest element

$$\mathrm{cd}(f) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $R^if_*\mathcal{F} = 0$ for $i > \mathrm{cd}(f)$.

0F10 **Lemma 59.96.2.** *Let K be a field.*

(1) *If $f : X \to Y$ is a morphism of finite type schemes over K, then $cd(f) < \infty$.*
(2) *If $cd(K) < \infty$, then $cd(X) < \infty$ for any finite type scheme X over K.*

Proof. Proof of (1). We may assume Y is affine. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove this. If X is affine too, then the result holds by Lemma 59.95.8. Thus it suffices to show that if $X = U \cup V$ and the result is true for $U \to Y$, $V \to Y$, and $U \cap V \to Y$, then it is true for f. This follows from the relative Mayer-Vietoris sequence, see Lemma 59.50.2.

Proof of (2). We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove this. If X is affine, then the result holds by Proposition 59.95.6. Thus it suffices to show that if $X = U \cup V$ and the result is true for U, V, and $U \cap V$, then it is true for X. This follows from the Mayer-Vietoris sequence, see Lemma 59.50.1. □

0F11 **Lemma 59.96.3.** *Cohomology and direct sums. Let $n \geq 1$ be an integer.*

(1) *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes with $cd(f) < \infty$. Then the functor*
$$Rf_* : D(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.

(2) *Let X be a quasi-compact and quasi-separated scheme with $cd(X) < \infty$. Then the functor*
$$R\Gamma(X, -) : D(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(\mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.

Proof. Proof of (1). Since $\mathrm{cd}(f) < \infty$ we see that
$$f_* : Mod(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow Mod(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$
has finite cohomological dimension in the sense of Derived Categories, Lemma 13.32.2. Let I be a set and for $i \in I$ let E_i be an object of $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. Choose a K-injective complex $\mathcal{I}_i^\bullet$ of $\mathbf{Z}/n\mathbf{Z}$-modules each of whose terms $\mathcal{I}_i^n$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules representing E_i. See Injectives, Theorem 19.12.6. Then $\bigoplus E_i$ is represented by the complex $\bigoplus \mathcal{I}_i^\bullet$ (termwise direct sum), see Injectives, Lemma 19.13.4. By Lemma 59.51.7 we have
$$R^q f_*(\bigoplus \mathcal{I}_i^n) = \bigoplus R^q f_*(\mathcal{I}_i^n) = 0$$
for $q > 0$ and any n. Hence we conclude by Derived Categories, Lemma 13.32.2 that we may compute $Rf_*(\bigoplus E_i)$ by the complex
$$f_*(\bigoplus \mathcal{I}_i^\bullet) = \bigoplus f_*(\mathcal{I}_i^\bullet)$$
(equality again by Lemma 59.51.7) which represents $\bigoplus Rf_* E_i$ by the already used Injectives, Lemma 19.13.4.

Proof of (2). This is identical to the proof of (1) and we omit it. □

0F0D **Lemma 59.96.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $n \geq 1$ be an integer. Then the functor*
$$Rf_* : D(X_{étale}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$$
commutes with direct sums.

Proof. It is enough to prove this when Y is quasi-compact. By Morphisms, Lemma 29.28.5 we see that the dimension of the fibres of $f : X \to Y$ is bounded. Thus Lemma 59.92.2 implies that $\mathrm{cd}(f) < \infty$. Hence the result by Lemma 59.96.3. □

0F12 **Lemma 59.96.5.** *Let X be a quasi-compact and quasi-separated scheme such that $cd(X) < \infty$. Let Λ be a torsion ring. Let $E \in D(X_{étale}, \Lambda)$ and $K \in D(\Lambda)$. Then*

$$R\Gamma(X, E \otimes^{\mathbf{L}}_{\Lambda} \underline{K}) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\Lambda} K$$

Proof. There is a canonical map from right to left by Cohomology on Sites, Section 21.50. Let $T(K)$ be the property that the statement of the lemma holds for $K \in D(\Lambda)$. We will check conditions (1), (2), and (3) of More on Algebra, Remark 15.59.11 hold for T to conclude. Property (1) holds because both sides of the equality commute with direct sums, see Lemma 59.96.3. Property (2) holds because we are comparing exact functors between triangulated categories and we can use Derived Categories, Lemma 13.4.3. Property (3) says the lemma holds when $K = \Lambda[k]$ for any shift $k \in \mathbf{Z}$ and this is obvious. □

0F0G **Lemma 59.96.6.** *Let $f : X \to Y$ be a proper morphism of schemes. Let Λ be a torsion ring. Let $E \in D(X_{étale}, \Lambda)$ and $K \in D(Y_{étale}, \Lambda)$. Then*

$$Rf_*E \otimes^{\mathbf{L}}_{\Lambda} K = Rf_*(E \otimes^{\mathbf{L}}_{\Lambda} f^{-1}K)$$

in $D(Y_{étale}, \Lambda)$.

Proof. There is a canonical map from left to right by Cohomology on Sites, Section 21.50. We will check the equality on stalks at $\overline{y}$. By the proper base change (in the form of Lemma 59.92.3 where $Y' = \overline{y}$) this reduces to the case where Y is the spectrum of an algebraically closed field. This is shown in Lemma 59.96.5 where we use that $\mathrm{cd}(X) < \infty$ by Lemma 59.92.2. □

59.97. Künneth in étale cohomology

0F13 We first prove a Künneth formula in case one of the factors is proper. Then we use this formula to prove a base change property for open immersions. This then gives a "base change by morphisms towards spectra of fields" (akin to smooth base change). Finally we use this to get a more general Künneth formula.

0F1E **Remark 59.97.1.** Consider a cartesian diagram in the category of schemes:

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & \swarrow^{c} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

Let Λ be a ring and let $E \in D(X_{étale}, \Lambda)$ and $K \in D(Y_{étale}, \Lambda)$. Then there is a canonical map

$$Rf_*E \otimes^{\mathbf{L}}_{\Lambda} Rg_*K \longrightarrow Rc_*(p^{-1}E \otimes^{\mathbf{L}}_{\Lambda} q^{-1}K)$$

For example we can define this using the canonical maps $Rf_*E \to Rc_*p^{-1}E$ and $Rg_*K \to Rc_*q^{-1}K$ and the relative cup product defined in Cohomology on Sites, Remark 21.19.7. Or you can use the adjoint to the map

$$c^{-1}(Rf_*E \otimes^{\mathbf{L}}_{\Lambda} Rg_*K) = p^{-1}f^{-1}Rf_*E \otimes^{\mathbf{L}}_{\Lambda} q^{-1}g^{-1}Rg_*K \to p^{-1}E \otimes^{\mathbf{L}}_{\Lambda} q^{-1}K$$

which uses the adjunction maps $f^{-1}Rf_*E \to E$ and $g^{-1}Rg_*K \to K$.

0F14 **Lemma 59.97.2.** *Let k be a separably closed field. Let X be a proper scheme over k. Let Y be a quasi-compact and quasi-separated scheme over k.*

(1) *If $E \in D^+(X_{étale})$ has torsion cohomology sheaves and $K \in D^+(Y_{étale})$, then*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}} pr_2^{-1}K) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}} R\Gamma(Y, K)$$

(2) *If $n \geq 1$ is an integer, Y is of finite type over k, $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$, and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$, then*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} pr_2^{-1}K) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, K)$$

Proof. Proof of (1). By Lemma 59.92.5 we have

$$R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathrm{pr}_2^{-1}K) = R\mathrm{pr}_{2,*}(\mathrm{pr}_1^{-1}E) \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

By proper base change (in the form of Lemma 59.91.12) this is equal to the object

$$\underline{R\Gamma(X, E)} \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

of $D(Y_{étale})$. Taking $R\Gamma(Y, -)$ on this object reproduces the left hand side of the equality in (1) by the Leray spectral sequence for pr_2. Thus we conclude by Lemma 59.92.4.

Proof of (2). This is exactly the same as the proof of (1) except that we use Lemmas 59.96.6, 59.92.3, and 59.96.5 as well as $\mathrm{cd}(Y) < \infty$ by Lemma 59.96.2. □

0F1F **Lemma 59.97.3.** *Let K be a separably closed field. Let X be a scheme of finite type over K. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$ whose support is contained in the set of closed points of X. Then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ and $\mathcal{F}$ is globally generated.*

Proof. (If $\mathcal{F}$ is torsion, then the vanishing follows immediately from Lemma 59.95.7.) By Lemma 59.74.5 we can write $\mathcal{F}$ as a filtered colimit of constructible sheaves $\mathcal{F}_i$ of $\mathbf{Z}$-modules whose supports $Z_i \subset X$ are finite sets of closed points. By Proposition 59.46.4 such a sheaf is of the form $(Z_i \to X)_*\mathcal{G}_i$ where $\mathcal{G}_i$ is a sheaf on Z_i. As K is separably closed, the scheme Z_i is a finite disjoint union of spectra of separably closed fields. Recall that $H^q(Z_i, \mathcal{G}_i) = H^q(X, \mathcal{F}_i)$ by the Leray spectral sequence for $Z_i \to X$ and vanising of higher direct images for this morphism (Proposition 59.55.2). By Lemmas 59.59.1 and 59.59.2 we see that $H^q(Z_i, \mathcal{G}_i)$ is zero for $q > 0$ and that $H^0(Z_i, \mathcal{G}_i)$ generates $\mathcal{G}_i$. We conclude the vanishing of $H^q(X, \mathcal{F}_i)$ for $q > 0$ and that $\mathcal{F}_i$ is generated by global sections. By Theorem 59.51.3 we see that $H^q(X, \mathcal{F}) = 0$ for $q > 0$. The proof is now done because a filtered colimit of globally generated sheaves of abelian groups is globally generated (details omitted). □

0F1G **Lemma 59.97.4.** *Let K be a separably closed field. Let X be a scheme of finite type over K. Let $Q \in D(X_{étale})$. Assume that $Q_{\overline{x}}$ is nonzero only if x is a closed point of X. Then*

$$Q = 0 \Leftrightarrow H^i(X, Q) = 0 \text{ for all } i$$

Proof. The implication from left to right is trivial. Thus we need to prove the reverse implication.

Assume Q is bounded below; this cases suffices for almost all applications. If Q is not zero, then we can look at the smallest i such that the cohomology sheaf $H^i(Q)$ is nonzero. By Lemma 59.97.3 we have $H^i(X, Q) = H^0(X, H^i(Q)) \neq 0$ and we conclude.

General case. Let $\mathcal{B} \subset \mathrm{Ob}(X_{étale})$ be the quasi-compact objects. By Lemma 59.97.3 the assumptions of Cohomology on Sites, Lemma 21.23.11 are satisfied. We

conclude that $H^q(U, Q) = H^0(U, H^q(Q))$ for all $U \in \mathcal{B}$. In particular, this holds for $U = X$. Thus the conclusion by Lemma 59.97.3 as Q is zero in $D(X_{étale})$ if and only if $H^q(Q)$ is zero for all q. □

0F1H **Lemma 59.97.5.** *Let K be a field. Let $j : U \to X$ be an open immersion of schemes of finite type over K. Let Y be a scheme of finite type over K. Consider the diagram*

$$\begin{array}{ccc} Y \times_{\mathrm{Spec}(K)} X & \xleftarrow{h} & Y \times_{\mathrm{Spec}(K)} U \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X & \xleftarrow{j} & U \end{array}$$

Then the base change map $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism for $\mathcal{F}$ an abelian sheaf on $U_{étale}$ whose stalks are torsion of orders invertible in K.

Proof. Write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. Since cohomology commutes with filtered colimits in our situation (for a precise reference see Lemma 59.86.3), it suffices to prove the lemma for $\mathcal{F}[n]$. Thus we may assume $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K (we will use this at the very end of the proof). In the proof we use the short hand $X \times_K Y$ for the fibre product over $\mathrm{Spec}(K)$. We will prove the lemma by induction on $\dim(X) + \dim(Y)$. The lemma is trivial if $\dim(X) \leq 0$, since in this case U is an open and closed subscheme of X. Choose a point $z \in X \times_K Y$. We will show the stalk at $\overline{z}$ is an isomorphism.

Suppose that $z \mapsto x \in X$ and assume $\mathrm{trdeg}_K(\kappa(x)) > 0$. Set $X' = \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ and denote $U' \subset X'$ the inverse image of U. Consider the base change

$$\begin{array}{ccc} Y \times_K X' & \xleftarrow{h'} & Y \times_K U' \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle p'} \\ X' & \xleftarrow{j'} & U' \end{array}$$

of our diagram by $X' \to X$. Observe that $X' \to X$ is a filtered colimit of étale morphisms. By smooth base change in the form of Lemma 59.89.3 the pullback of $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ to X' to $Y \times_K X'$ is the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to U'. (In this step it would suffice to use étale base change which is an essentially trivial result.) So it suffices to show that $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}'$ is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$. By Lemma 59.95.5 there is a separably closed field L/K such that $X' = \lim X_i$ with X_i affine of finite type over L and $\dim(X_i) < \dim(X)$. For i large enough there exists an open $U_i \subset X_i$ restricting to U' in X'. We may apply the induction hypothesis to the diagram

$$\begin{array}{ccc} Y \times_K X_i & \xleftarrow{h_i} & Y \times_K U_i \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle p_i} \\ X_i & \xleftarrow{j_i} & U_i \end{array} \quad \text{equal to} \quad \begin{array}{ccc} Y_L \times_L X_i & \xleftarrow{h_i} & Y_L \times_L U_i \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle p_i} \\ X_i & \xleftarrow{j_i} & U_i \end{array}$$

over the field L and the pullback of $\mathcal{F}$ to these diagrams. By Lemma 59.86.3 we conclude that the map $(q')^{-1}Rj'_*\mathcal{F}' \to Rj'_*(p')^{-1}\mathcal{F}$ is an isomorphism.

Suppose that $z \mapsto y \in Y$ and assume $\mathrm{trdeg}_K(\kappa(y)) > 0$. Let $Y' = \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$. By Lemma 59.95.5 there is a separably closed field L/K such that $Y' = \lim Y_i$ with Y_i affine of finite type over L and $\dim(Y_i) < \dim(Y)$. In particular Y' is a scheme over L. Denote with a subscript L the base change from schemes over K to schemes over L. Consider the commutative diagrams

$$\begin{array}{ccc} Y' \times_K X & \xleftarrow{h'} & Y' \times_K U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y \times_K X & \xleftarrow{h} & Y \times_K U \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X & \xleftarrow{j} & U \end{array} \quad \text{and} \quad \begin{array}{ccc} Y' \times_L X_L & \xleftarrow{h'} & Y' \times_L U_L \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle p'} \\ X_L & \xleftarrow{j_L} & U_L \\ \downarrow & & \downarrow \\ X & \xleftarrow{j} & U \end{array}$$

and observe the top and bottom rows are the same on the left and the right. By smooth base change we see that $f^{-1}Rh_*p^{-1}\mathcal{F} = Rh'_*(f')^{-1}p^{-1}\mathcal{F}$ (similarly to the previous paragraph). By smooth base change for $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ (Lemma 59.90.1) we see that $Rj_{L,*}\mathcal{F}_L$ is the pullback of $Rj_*\mathcal{F}$ to X_L. Combining these two observations, we conclude that it suffices to prove the base change map for the upper square in the diagram on the right is an isomorphism in order to prove that our original map is an isomorphism on stalks at $\overline{z}$[14]. Then using that $Y' = \lim Y_i$ and argueing exactly as in the previous paragraph we see that the induction hypothesis forces our map over $Y' \times_K X$ to be an isomorphism.

Thus any counter example with $\dim(X) + \dim(Y)$ minimal would only have non-isomorphisms $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ on stalks at closed points of $X \times_K Y$ (because a point z of $X \times_K Y$ is a closed point if and only if both the image of z in X and in Y are closed). Since it is enough to prove the isomorphism locally, we may assume X and Y are affine. However, then we can choose an open dense immersion $Y \to Y'$ with Y' projective. (Choose a closed immersion $Y \to \mathbf{A}^n_K$ and let Y' be the scheme theoretic closure of Y in $\mathbf{P}^n_K$.) Then $\dim(Y') = \dim(Y)$ and hence we get a "minimal" counter example with Y projective over K. In the next paragraph we show that this can't happen.

Consider a diagram as in the statement of the lemma such that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ is an isomorphism at all non-closed points of $X \times_K Y$ and such that Y is projective. The restriction of the map to $(X \times_K Y)_{K^{sep}}$ is the corresponding map for the diagram of the lemma base changed to K^{sep}. Thus we may and do assume K is separably algebraically closed. Choose a distinguished triangle

$$q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F} \to Q \to (q^{-1}Rj_*\mathcal{F})[1]$$

in $D((X \times_K Y)_{étale})$. Since Q is supported in closed points we see that it suffices to prove $H^i(X \times_K Y, Q) = 0$ for all i, see Lemma 59.97.4. Thus it suffices to prove that $q^{-1}Rj_*\mathcal{F} \to Rh_*p^{-1}\mathcal{F}$ induces an isomorphism on cohomology. Recall that $\mathcal{F}$ is annihilated by n invertible in K. By the Künneth formula of Lemma 59.97.2 we

[14]Here we use that a "vertical composition" of base change maps is a base change map as explained in Cohomology on Sites, Remark 21.19.4.

have

$$\begin{aligned} R\Gamma(X \times_K Y, q^{-1}Rj_*\mathcal{F}) &= R\Gamma(X, Rj_*\mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \\ &= R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z}) \end{aligned}$$

and

$$R\Gamma(X \times_K Y, Rh_*p^{-1}\mathcal{F}) = R\Gamma(U \times_K Y, p^{-1}\mathcal{F}) = R\Gamma(U, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, \mathbf{Z}/n\mathbf{Z})$$

This finishes the proof. □

0F1I **Lemma 59.97.6.** *Let K be a field. For any commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow[h]{} & Y \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle e} \\ \mathrm{Spec}(K) & \longleftarrow & S' & \xleftarrow{g} & T \end{array}$$

of schemes over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated, and every abelian sheaf $\mathcal{F}$ on $T_{étale}$ whose stalks are torsion of orders invertible in K the base change map

$$(f')^{-1}Rg_*\mathcal{F} \longrightarrow Rh_*e^{-1}\mathcal{F}$$

is an isomorphism.

Proof. The question is local on X, hence we may assume X is affine. By Limits, Lemma 32.7.2 we can write $X = \lim X_i$ as a cofiltered limit with affine transition morphisms of schemes X_i of finite type over K. Denote $X'_i = X_i \times_{\mathrm{Spec}(K)} S'$ and $Y_i = X'_i \times_{S'} T$. By Lemma 59.86.3 it suffices to prove the statement for the squares with corners X_i, Y_i, S_i, T_i. Thus we may assume X is of finite type over K. Similarly, we may write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}[n]$ where the colimit is over the multiplicative system of integers invertible in K. The same lemma used above reduces us to the case where $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n invertible in K.

We may replace K by its algebraic closure $\overline{K}$. Namely, formation of direct image commutes with base change to $\overline{K}$ according to Lemma 59.90.1 (works for both g and h). And it suffices to prove the agreement after restriction to $X'_{\overline{K}}$. Next, we may replace X by its reduction as we have the topological invariance of étale cohomology, see Proposition 59.45.4. After this replacement the morphism $X \to \mathrm{Spec}(K)$ is flat, finite presentation, with geometrically reduced fibres and the same is true for any base change, in particular for $X' \to S'$. Hence $(f')^{-1}g_*\mathcal{F} \to Rh_*e^{-1}\mathcal{F}$ is an isomorphism by Lemma 59.87.2.

At this point we may apply Lemma 59.90.3 to see that it suffices to prove: given a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X' & \xleftarrow[h]{} & Y \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longleftarrow & S' & \longleftarrow & \mathrm{Spec}(L) \end{array}$$

with both squares cartesian, where S' is affine, integral, and normal with algebraically closed function field K, then $R^qh_*(\mathbf{Z}/d\mathbf{Z})$ is zero for $q > 0$ and $d|n$.

Observe that this vanishing is equivalent to the statement that

$$(f')^{-1}R^q(\mathrm{Spec}(L) \to S')_*\mathbf{Z}/d\mathbf{Z} \longrightarrow R^q h_*\mathbf{Z}/d\mathbf{Z}$$

is an isomorphism, because the left hand side is zero for example by Lemma 59.80.5.

Write $S' = \mathrm{Spec}(B)$ so that L is the fraction field of B. Write $B = \bigcup_{i\in I} B_i$ as the union of its finite type K-subalgebras B_i. Let J be the set of pairs (i, g) where $i \in I$ and $g \in B_i$ nonzero with ordering $(i', g') \geq (i, g)$ if and only if $i' \geq i$ and g maps to an invertible element of $(B_{i'})_{g'}$. Then $L = \mathrm{colim}_{(i,g)\in J}(B_i)_g$. For $j = (i, g) \in J$ set $S_j = \mathrm{Spec}(B_i)$ and $U_j = \mathrm{Spec}((B_i)_g)$. Then

$$\begin{array}{ccc} X' & \xleftarrow[h]{} & Y \\ \downarrow & & \downarrow \\ S' & \longleftarrow & \mathrm{Spec}(L) \end{array} \quad \text{is the colimit of} \quad \begin{array}{ccc} X \times_K S_j & \xleftarrow[h_j]{} & X \times_K U_j \\ \downarrow & & \downarrow \\ S_j & \longleftarrow & U_j \end{array}$$

Thus we may apply Lemma 59.86.3 to see that it suffices to prove base change holds in the diagrams on the right which is what we proved in Lemma 59.97.5. □

0F1J **Lemma 59.97.7.** *Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow[p]{} & X' & \xleftarrow[h]{} & Y \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle e} \\ \mathrm{Spec}(K) & \longleftarrow & S' & \xleftarrow[g]{} & T \end{array}$$

of schemes with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$ and g quasi-compact and quasi-separated. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism if E in $D^+(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ has tor amplitude in $[a, \infty]$ for some $a \in \mathbf{Z}$ and F in $D^+(T_{étale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. This lemma is a generalization of Lemma 59.97.6 to objects of the derived category; the assertion of our lemma is true because in Lemma 59.97.6 the scheme X over K is arbitrary. We strongly urge the reader to skip the laborious proof (alternative: read only the last paragraph).

We may represent E by a bounded below K-flat complex $\mathcal{E}^\bullet$ consisting of flat $\mathbf{Z}/n\mathbf{Z}$-modules. See Cohomology on Sites, Lemma 21.46.4. Choose an integer b such that $H^i(F) = 0$ for $i < b$. Choose a large integer N and consider the short exact sequence

$$0 \to \sigma_{\geq N+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq N}\mathcal{E}^\bullet \to 0$$

of stupid truncations. This produces a distinguished triangle $E'' \to E \to E' \to E''[1]$ in $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$. For fixed F both sides of the arrow in the statement of the lemma are exact functors in E. Observe that

$$p^{-1}E'' \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \quad \text{and} \quad Rh_*(h^{-1}p^{-1}E'' \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

are sitting in degrees $\geq N + b$. Hence, if we can prove the lemma for the object E', then we see that the lemma holds in degrees $\leq N + b$ and we will conclude. Some details omitted. Thus we may assume E is represented by a bounded complex of

flat $\mathbf{Z}/n\mathbf{Z}$-modules. Doing another argument of the same nature, we may assume E is given by a single flat $\mathbf{Z}/n\mathbf{Z}$-module $\mathcal{E}$.

Next, we use the same arguments for the variable F to reduce to the case where F is given by a single sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules $\mathcal{F}$. Say $\mathcal{F}$ is annihilated by an integer $m|n$. If ℓ is a prime number dividing m and $m > \ell$, then we can look at the short exact sequence $0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$ and reduce to smaller m. This finally reduces us to the case where $\mathcal{F}$ is annihilated by a prime number ℓ dividing n. In this case observe that

$$p^{-1}\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*\mathcal{F} = p^{-1}(\mathcal{E}/\ell\mathcal{E}) \otimes^{\mathbf{L}}_{\mathbf{F}_\ell} (f')^{-1}Rg_*\mathcal{F}$$

by the flatness of $\mathcal{E}$. Similarly for the other term. This reduces us to the case where we are working with sheaves of $\mathbf{F}_\ell$-vector spaces which is discussed

Assume ℓ is a prime number invertible in K. Assume $\mathcal{E}$, $\mathcal{F}$ are sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$ and $T_{étale}$. We want to show that

$$p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} (f')^{-1}R^q g_*\mathcal{F} \longrightarrow R^q h_*(h^{-1}p^{-1}\mathcal{E} \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F})$$

is an isomorphism for every $q \geq 0$. This question is local on X hence we may assume X is affine. We can write $\mathcal{E}$ as a filtered colimit of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$, see Lemma 59.73.2. Since tensor products commute with filtered colimits and since higher direct images do too (Lemma 59.51.7) we may assume $\mathcal{E}$ is a constructible sheaf of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$. Then we can choose an integer m and finite and finitely presented morphisms $\pi_i : X_i \to X$, $i = 1, \ldots, m$ such that there is an injective map

$$\mathcal{E} \to \bigoplus\nolimits_{i=1,\ldots,m} \pi_{i,*}\mathbf{F}_\ell$$

See Lemma 59.74.4. Observe that the direct sum is a constructible sheaf as well (Lemma 59.73.9). Thus the cokernel is constructible too (Lemma 59.71.6). By dimension shifting, i.e., induction on q, on the category of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on $X_{étale}$, it suffices to prove the result for the sheaves $\pi_{i,*}\mathbf{F}_\ell$ (details omitted; hint: start with proving injectivity for $q = 0$ for all constructible $\mathcal{E}$). To prove this case we extend the diagram of the lemma to

$$\begin{array}{ccccc}
X_i & \xleftarrow{p_i} & X_i' & \xleftarrow{h_i} & Y_i \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i'} & & \downarrow{\scriptstyle \rho_i} \\
X & \xleftarrow{p} & X' & \xleftarrow{h} & Y \\
\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle e} \\
\mathrm{Spec}(K) & \longleftarrow & S' & \xleftarrow{g} & T
\end{array}$$

with all squares cartesian. In the equations below we are going to use that $R\pi_{i,*} = \pi_{i,*}$ and similarly for π_i', ρ_i, we are going to use the Leray spectral sequence, we are going to use Lemma 59.55.3, and we are going to use Lemma 59.96.6 (although this lemma is almost trivial for finite morphisms) for π_i, π_i', ρ_i. Doing so we see that

$$\begin{aligned}
p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^q g_*\mathcal{F} &= \pi'_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} (f')^{-1}R^q g_*\mathcal{F} \\
&= \pi'_{i,*}((\pi_i')^{-1}(f')^{-1}R^q g_*\mathcal{F})
\end{aligned}$$

Similarly, we have

$$\begin{aligned} R^qh_*(h^{-1}p^{-1}\pi_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) &= R^qh_*(\rho_{i,*}\mathbf{F}_\ell \otimes_{\mathbf{F}_\ell} e^{-1}\mathcal{F}) \\ &= R^qh_*(\rho_i^{-1}e^{-1}\mathcal{F}) \\ &= \pi'_{i,*}R^qh_{i,*}\rho_i^{-1}e^{-1}\mathcal{F}) \end{aligned}$$

Simce $R^qh_{i,*}\rho_i^{-1}e^{-1}\mathcal{F} = (\pi'_i)^{-1}(f')^{-1}R^qg_*\mathcal{F}$ by Lemma 59.97.6 we conclude. □

0F1N **Lemma 59.97.8.** *Let K be a field. Let $n \geq 1$ be invertible in K. Consider a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{p} & X' & \xleftarrow{h} & Y \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle e} \\ \mathrm{Spec}(K) & \longleftarrow & S' & \xleftarrow{g} & T \end{array}$$

of schemes of finite type over K with $X' = X \times_{\mathrm{Spec}(K)} S'$ and $Y = X' \times_{S'} T$. The canonical map

$$p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} (f')^{-1}Rg_*F \longrightarrow Rh_*(h^{-1}p^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} e^{-1}F)$$

is an isomorphism for E in $D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ and F in $D(T_{étale}, \mathbf{Z}/n\mathbf{Z})$.

Proof. We will reduce this to Lemma 59.97.7 using that our functors commute with direct sums. We suggest the reader skip the proof. Recall that derived tensor product commutes with direct sums. Recall that (derived) pullback commutes with direct sums. Recall that Rh_* and Rg_* commute with direct sums, see Lemmas 59.96.2 and 59.96.3 (this is where we use our schemes are of finite type over K).

To finish the proof we can argue as follows. First we write $E = \mathrm{hocolim}\tau_{\leq N}E$. Since our functors commute with direct sums, they commute with homotopy colimits. Hence it suffices to prove the lemma for E bounded above. Similarly for F we may assume F is bounded above. Then we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then

$$\mathcal{E}^\bullet = \mathrm{colim}\, \sigma_{\geq -N}\mathcal{E}^\bullet$$

(stupid truncations). Thus we may assume $\mathcal{E}^\bullet$ is a bounded complex of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. For F we choose a bounded above complex of flat(!) sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Then we reduce to the case where F is represented by a bounded complex of flat sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. At this point Lemma 59.97.7 kicks in and we conclude. □

0F1P **Lemma 59.97.9.** *Let k be a separably closed field. Let X and Y be finite type schemes over k. Let $n \geq 1$ be an integer invertible in k. Then for $E \in D(X_{étale}, \mathbf{Z}/n\mathbf{Z})$ and $K \in D(Y_{étale}, \mathbf{Z}/n\mathbf{Z})$ we have*

$$R\Gamma(X \times_{\mathrm{Spec}(k)} Y, pr_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} pr_2^{-1}K) = R\Gamma(X, E) \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R\Gamma(Y, K)$$

Proof. By Lemma 59.97.8 we have

$$R\mathrm{pr}_{1,*}(\mathrm{pr}_1^{-1}E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \mathrm{pr}_2^{-1}K) = E \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} \underline{R\Gamma(Y, K)}$$

We conclude by Lemma 59.96.5 which we may use because $\mathrm{cd}(X) < \infty$ by Lemma 59.96.2. □

59.98. Comparing chaotic and Zariski topologies

0F1K When constructing the structure sheaf of an affine scheme, we first construct the values on affine opens, and then we extend to all opens. A similar construction is often useful for constructing complexes of abelian groups on a scheme X. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Topologies, Definition 34.3.7. We remind the reader that the topos of $X_{affine,Zar}$ is the small Zariski topos of X, see Topologies, Lemma 34.3.11. In this section we denote X_{affine} the same underlying category with the chaotic topology, i.e., such that sheaves agree with presheaves. We obtain a morphisms of sites

$$\epsilon : X_{affine,Zar} \longrightarrow X_{affine}$$

as in Cohomology on Sites, Section 21.27.

0F1L **Lemma 59.98.1.** *In the situation above let K be an object of $D^+(X_{affine})$. Then K is in the essential image of the (fully faithful) functor $R\epsilon_*; D(X_{affine,Zar}) \to D(X_{affine})$ if and only if the following two conditions hold*

(1) *$R\Gamma(\emptyset, K)$ is zero in $D(Ab)$, and*
(2) *if $U = V \cup W$ with $U, V, W \subset X$ affine open and $V, W \subset U$ standard open (Algebra, Definition 10.17.3), then the map $c^K_{U,V,W,V\cap W}$ of Cohomology on Sites, Lemma 21.26.1 is a quasi-isomorphism.*

Proof. (The functor $R\epsilon_*$ is fully faithful by the discussion in Cohomology on Sites, Section 21.27.) Except for a snafu having to do with the empty set, this follows from the very general Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Schemes, Lemma 26.11.7 and Cohomology on Sites, Lemma 21.29.3.

To get around the snafu, denote $X_{affine,almost-chaotic}$ the site where the empty object $\emptyset$ has two coverings, namely, $\{\emptyset \to \emptyset\}$ and the empty covering (see Sites, Example 7.6.4 for a discussion). Then we have morphisms of sites

$$X_{affine,Zar} \to X_{affine,almost-chaotic} \to X_{affine}$$

The argument above works for the first arrow. Then we leave it to the reader to see that an object K of $D^+(X_{affine})$ is in the essential image of the (fully faithful) functor $D(X_{affine,almost-chaotic}) \to D(X_{affine})$ if and only if $R\Gamma(\emptyset, K)$ is zero in $D(Ab)$. □

59.99. Comparing big and small topoi

0757 Let S be a scheme. In Topologies, Lemma 34.4.14 we have introduced comparison morphisms $\pi_S : (Sch/S)_{étale} \to S_{étale}$ and $i_S : Sh(S_{étale}) \to Sh((Sch/S)_{étale})$ with $\pi_S \circ i_S = \mathrm{id}$ and $\pi_{S,*} = i_S^{-1}$. More generally, if $f : T \to S$ is an object of $(Sch/S)_{étale}$, then there is a morphism $i_f : Sh(T_{étale}) \to Sh((Sch/S)_{étale})$ such that $f_{small} = \pi_S \circ i_f$, see Topologies, Lemmas 34.4.13 and 34.4.17. In Descent, Remark 35.8.4 we have extended these to a morphism of ringed sites

$$\pi_S : ((Sch/S)_{étale}, \mathcal{O}) \to (S_{étale}, \mathcal{O}_S)$$

and morphisms of ringed topoi

$$i_S : (Sh(S_{étale}), \mathcal{O}_S) \to (Sh((Sch/S)_{étale}), \mathcal{O})$$

and

$$i_f : (Sh(T_{étale}), \mathcal{O}_T) \to (Sh((Sch/S)_{étale}, \mathcal{O}))$$

Note that the restriction $i_S^{-1} = \pi_{S,*}$ (see Topologies, Definition 34.4.15) transforms $\mathcal{O}$ into $\mathcal{O}_S$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_T$. See Descent, Remark 35.8.4. Hence $i_S^*\mathcal{F} = i_S^{-1}\mathcal{F}$ and $i_f^*\mathcal{F} = i_f^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_{étale}$. In particular i_S^* and i_f^* are exact functors. The functor i_S^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{S_{étale}}$ (and this does not conflict with the notation in Topologies, Definition 34.4.15).

0758 **Lemma 59.99.1.** *Let S be a scheme. Let T be an object of $(Sch/S)_{étale}$.*

(1) *If $\mathcal{I}$ is injective in $Ab((Sch/S)_{étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}$ is injective in $Ab(T_{étale})$,*
   (b) *$\mathcal{I}|_{S_{étale}}$ is injective in $Ab(S_{étale})$,*

(2) *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab((Sch/S)_{étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $Ab(T_{étale})$,*
   (b) *$\mathcal{I}^\bullet|_{S_{étale}}$ is a K-injective complex in $Ab(S_{étale})$,*

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{S,*} = i_S^{-1}$ is a right adjoint of the exact functor π_S^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies, Lemma 34.4.13 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Second proof. We can use that $i_f = i_T \circ f_{big}$ as is shown in Topologies, Lemma 34.4.17. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 21.7.1 and 21.20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_T^{-1} to conclude.

Let $S = \mathrm{Spec}(\mathbf{Z})$ and consider the map $2 : \mathcal{O}_S \to \mathcal{O}_S$. This is an injective map of $\mathcal{O}_S$-modules on $S_{étale}$. However, the pullback $\pi_S^*(2) : \mathcal{O} \to \mathcal{O}$ is not injective as we see by evaluating on $\mathrm{Spec}(\mathbf{F}_2)$. Now choose an injection $\alpha : \mathcal{O} \to \mathcal{I}$ into an injective $\mathcal{O}$-module $\mathcal{I}$ on $(Sch/S)_{étale}$. Then consider the diagram

$$\begin{array}{ccc} \mathcal{O}_S & \xrightarrow{\alpha|_{S_{étale}}} & \mathcal{I}|_{S_{étale}} \\ {\scriptstyle 2}\downarrow & \nearrow & \\ \mathcal{O}_S & & \end{array}$$

Then the dotted arrow cannot exist in the category of $\mathcal{O}_S$-modules because it would mean (by adjunction) that the injective map α factors through the noninjective map $\pi_S^*(2)$ which cannot be the case. Thus $\mathcal{I}|_{S_{étale}}$ is not an injective $\mathcal{O}_S$-module. □

Let $f : T \to S$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 34.4.17 (3) leads to a commutative diagram of ringed sites

$$\begin{array}{ccc} (T_{étale}, \mathcal{O}_T) & \xleftarrow{\pi_T} & ((Sch/T)_{étale}, \mathcal{O}) \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ (S_{étale}, \mathcal{O}_S) & \xleftarrow{\pi_S} & ((Sch/S)_{étale}, \mathcal{O}) \end{array}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_S^\sharp$, and $\pi_T^\sharp$. In particular this means that

0759 (59.99.1.1) $$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}})$$

for any sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (59.99.1.1) is an isomorphism of $\mathcal{O}_S$-modules on $S_{étale}$.

075A **Lemma 59.99.2.** *Let $f : T \to S$ be a morphism of schemes.*

(1) *For K in $D((Sch/T)_{étale})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(S_{étale})$.*
(2) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{S_{étale}} = Rf_{small,*}(K|_{T_{étale}})$ in $D(Mod(S_{étale}, \mathcal{O}_S))$.*

More generally, let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then

(3) *For K in $D((Sch/T)_{étale})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(S'_{étale})$.*
(4) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(Mod(S'_{étale}, \mathcal{O}_{S'}))$.*
(5) *For K in $D((Sch/T)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((Sch/S')_{étale})$.*
(6) *For K in $D((Sch/T)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(Mod(S'_{étale}, \mathcal{O}_{S'}))$.*

Proof. Part (1) follows from Lemma 59.99.1 and (59.99.1.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 59.99.1 and Topologies, Lemma 34.4.19 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.21.1.

Part (6) is Cohomology on Sites, Lemma 21.21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_S \circ f_{big} = f_{small} \circ \pi_T$ as morphisms of ringed sites. Hence we obtain $R\pi_{S,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{T,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{S,*}$ and $\pi_{T,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : T' \to S'$. □

Let S be a scheme and let $\mathcal{H}$ be an abelian sheaf on $(Sch/S)_{étale}$. Recall that $H^n_{étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Sch/S)_{étale}$.

0DDH **Lemma 59.99.3.** *Let $f : T \to S$ be a morphism of schemes. Then*

(1) *For K in $D(S_{étale})$ we have $H^n_{étale}(S, \pi_S^{-1}K) = H^n(S_{étale}, K)$.*
(2) *For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(S, L\pi_S^*K) = H^n(S_{étale}, K)$.*
(3) *For K in $D(S_{étale})$ we have $H^n_{étale}(T, \pi_S^{-1}K) = H^n(T_{étale}, f_{small}^{-1}K)$.*

(4) *For K in $D(S_{étale}, \mathcal{O}_S)$ we have $H^n_{étale}(T, L\pi_S^* K) = H^n(T_{étale}, Lf_{small}^* K)$.*
(5) *For M in $D((Sch/S)_{étale})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^{-1} M)$.*
(6) *For M in $D((Sch/S)_{étale}, \mathcal{O})$ we have $H^n_{étale}(T, M) = H^n(T_{étale}, i_f^* M)$.*

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 59.99.1 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_S^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 21.37.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_S^*$. Part (2) is a special case of (4). □

0DDI **Lemma 59.99.4.** *Let S be a scheme. For $K \in D(S_{étale})$ the map*

$$K \longrightarrow R\pi_{S,*}\pi_S^{-1} K$$

is an isomorphism.

Proof. This is true because both π_S^{-1} and $\pi_{S,*} = i_S^{-1}$ are exact functors and the composition $\pi_{S,*} \circ \pi_S^{-1}$ is the identity functor. □

0DDJ **Lemma 59.99.5.** *Let $f : T \to S$ be a proper morphism of schemes. Then we have*

(1) *$\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$ as functors $Sh(T_{étale}) \to Sh((Sch/S)_{étale})$,*
(2) *$\pi_S^{-1} Rf_{small,*} K = Rf_{big,*}\pi_T^{-1} K$ for K in $D^+(T_{étale})$ whose cohomology sheaves are torsion,*
(3) *$\pi_S^{-1} Rf_{small,*} K = Rf_{big,*}\pi_T^{-1} K$ for K in $D(T_{étale}, \mathbf{Z}/n\mathbf{Z})$, and*
(4) *$\pi_S^{-1} Rf_{small,*} K = Rf_{big,*}\pi_T^{-1} K$ for all K in $D(T_{étale})$ if f is finite.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $T_{étale}$. Let $g : S' \to S$ be an object of $(Sch/S)_{étale}$. Consider the fibre product

$$\begin{array}{ccc} T' & \xrightarrow{f'} & S' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{f} & S \end{array}$$

Then we have

$$(f_{big,*}\pi_T^{-1}\mathcal{F})(S') = (\pi_T^{-1}\mathcal{F})(T') = ((g'_{small})^{-1}\mathcal{F})(T') = (f'_{small,*}(g'_{small})^{-1}\mathcal{F})(S')$$

the second equality by Lemma 59.39.2. On the other hand

$$(\pi_S^{-1} f_{small,*}\mathcal{F})(S') = (g_{small}^{-1} f_{small,*}\mathcal{F})(S')$$

again by Lemma 59.39.2. Hence by proper base change for sheaves of sets (Lemma 59.91.5) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_S^{-1} f_{small,*}\mathcal{F} = f_{big,*}\pi_T^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_S^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_T^{-1}$.

Proof of (2). There is a canonical base change map $\pi_S^{-1} Rf_{small,*} K \to Rf_{big,*}\pi_T^{-1} K$ for any K in $D(T_{étale})$, see Cohomology on Sites, Remark 21.19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(S'_{étale}) \to Sh((Sch/S)_{étale})$ is an isomorphism for any object $g : S' \to S$ of

$(Sch/S)_{étale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$\begin{aligned} g_{small}^{-1}Rf_{small,*}K &= i_g^{-1}\pi_S^{-1}Rf_{small,*}K \\ &\to i_g^{-1}Rf_{big,*}\pi_T^{-1}K \\ &= Rf'_{small,*}(i_{g'}^{-1}\pi_T^{-1}K) \\ &= Rf'_{small,*}((g'_{small})^{-1}K) \end{aligned}$$

where we have used $\pi_S \circ i_g = g_{small}$, $\pi_T \circ i_{g'} = g'_{small}$, and Lemma 59.99.2. This map is an isomorphism by the proper base change theorem (Lemma 59.91.12) provided K is bounded below and the cohomology sheaves of K are torsion.

The proof of part (3) is the same as the proof of part (2), except we use Lemma 59.92.3 instead of Lemma 59.91.12.

Proof of (4). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Proposition 59.55.2 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 59.99.2 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). □

59.100. Comparing fppf and étale topologies

0DDK A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. We first review some material from Topologies, Sections 34.11 and 34.4.

Let S be a scheme and let $(Sch/S)_{fppf}$ be an fppf site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{fppf} \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the fppf sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0DDL **Lemma 59.100.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{fppf}$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the fppf topology. This is shown in Lemma 59.39.2 as well. □

In the situation of Lemma 59.100.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{fppf} \longrightarrow S_{étale}$$

0DDM **Lemma 59.100.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh((Sch/X)_{fppf}) & \xrightarrow{f_{big,fppf}} & Sh((Sch/Y)_{fppf}) \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh((Sch/X)_{étale}) & \xrightarrow{f_{big,étale}} & Sh((Sch/Y)_{étale}) \end{array}$$

and

$$\begin{array}{ccc} Sh((Sch/X)_{fppf}) & \xrightarrow{f_{big,fppf}} & Sh((Sch/Y)_{fppf}) \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 34.11. □

0DDN **Lemma 59.100.3.** *In Lemma 59.100.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.100.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,fppf,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,fppf,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. □

0DEU **Lemma 59.100.4.** *In Lemma 59.100.2 assume f is flat, locally of finite presentation, and surjective. Then the functor*

$$Sh(Y_{étale}) \longrightarrow \left\{ (\mathcal{G}, \mathcal{H}, \alpha) \middle| \begin{array}{l} \mathcal{G} \in Sh(X_{étale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}), \\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H} \text{ an isomorphism} \end{array} \right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \mathrm{id}$ by Lemma 59.100.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is

true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{étale}, X_{étale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 7.26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y.

Note that $\{X \to Y\}$ is an fppf covering. Working étale locally on Y, we may assume there exists a morphism $s : X' \to X$ such that the composition $f' = f \circ s : X' \to Y$ is surjective finite locally free, see More on Morphisms, Lemma 37.48.1. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{étale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case described in the next paragraph.

Assume f is surjective finite locally free. Let $(\mathcal{G}, \mathcal{H}, \alpha)$ be a triple. In this case consider the triple

$$(\mathcal{G}_1, \mathcal{H}_1, \alpha_1) = (f_{small}^{-1} f_{small,*}\mathcal{G}, f_{big,fppf,*} f_{big,fppf}^{-1}\mathcal{H}, \alpha_1)$$

where α_1 comes from the identifications

$$\begin{aligned} a_X^{-1} f_{small}^{-1} f_{small,*}\mathcal{G} &= f_{big,fppf}^{-1} a_Y^{-1} f_{small,*}\mathcal{G} \\ &= f_{big,fppf}^{-1} f_{big,fppf,*} a_X^{-1}\mathcal{G} \\ &\to f_{big,fppf}^{-1} f_{big,fppf,*} f_{big,fppf}^{-1}\mathcal{H} \end{aligned}$$

where the third equality is Lemma 59.100.3 and the arrow is given by α. This triple is in the image of our functor because $\mathcal{F}_1 = f_{small,*}\mathcal{F}$ is a solution (to see this use Lemma 59.100.3 again; details omitted). There is a canonical map of triples

$$(\mathcal{G}, \mathcal{H}, \alpha) \to (\mathcal{G}_1, \mathcal{H}_1, \alpha_1)$$

which uses the unit $\mathrm{id} \to f_{big,fppf,*} f_{big,fppf}^{-1}$ on the second entry (it is enough to prescribe morphisms on the second entry by the first paragraph of the proof). Since $\{f : X \to Y\}$ is an fppf covering the map $\mathcal{H} \to \mathcal{H}_1$ is injective (details omitted). Set

$$\mathcal{G}_2 = \mathcal{G}_1 \amalg_{\mathcal{G}} \mathcal{G}_1 \quad \mathcal{H}_2 = \mathcal{H}_1 \amalg_{\mathcal{H}} \mathcal{H}_1$$

and let α_2 be the induced isomorphism (pullback functors are exact, so this makes sense). Then $\mathcal{H}$ is the equalizer of the two maps $\mathcal{H}_1 \to \mathcal{H}_2$. Repeating the arguments above for the triple $(\mathcal{G}_2, \mathcal{H}_2, \alpha_2)$ we find an injective morphism of triples

$$(\mathcal{G}_2, \mathcal{H}_2, \alpha_2) \to (\mathcal{G}_3, \mathcal{H}_3, \alpha_3)$$

such that this last triple is in the image of our functor. Say it corresponds to $\mathcal{F}_3$ in $Sh(Y_{étale})$. By fully faithfulness we obtain two maps $\mathcal{F}_1 \to \mathcal{F}_3$ and we can let $\mathcal{F}$ be the equalizer of these two maps. By exactness of the pullback functors involved we find that $a_Y^{-1}\mathcal{F} = \mathcal{H}$ as desired. □

0F0H **Lemma 59.100.5.** *Consider the comparison morphism $\epsilon : (Sch/S)_{fppf} \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of finite morphisms of schemes. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form*

$\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $Ab(X_{étale})$. Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a finite morphism of schemes is a finite morphism of schemes, see Morphisms, Lemma 29.44.6.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.100.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (4).

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 37.48.1. □

0DDS **Lemma 59.100.6.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_{fppf})$ and an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a finite morphism $f : X \to Y$ in $(Sch/S)_{fppf}$ and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,fppf,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{étale})$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a finite morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.*
(5) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$.*

Proof. By Lemma 59.100.5 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the essential image of $a_X^{-1} : Ab(X_{étale}) \to Ab((Sch/X)_{fppf})$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason.

Part (5). We use that

$$\begin{aligned} R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K &= Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,étale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \end{aligned}$$

The first equality by the commutative diagram in Lemma 59.100.2 and Cohomology on Sites, Lemma 21.19.2. The second equality is (3). The third is Lemma 59.99.5 part (2). The fourth is (3) again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,fppf,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,fppf,*}a_X^{-1}K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ and M in $D^+((Sch/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by (1) because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

0DDT **Lemma 59.100.7.** *Let X be a scheme. For $K \in D^+(X_{étale})$ the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^qa_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 59.100.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^qa_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.100.6 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DDU **Lemma 59.100.8.** *For a scheme X and $a_X : Sh((Sch/X)_{fppf}) \to Sh(X_{étale})$ as above:*

(1) *$H^q(X_{étale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ for an abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(2) *$H^q(X_{étale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$.*

Example: if A is an abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 59.100.7 by Cohomology on Sites, Remark 21.14.4. □

59.101. Comparing fppf and étale topologies: modules

0DEV We continue the discussion in Section 59.100 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. The morphisms of sites ϵ_S, π_S, and their composition a_S introduced in Section 59.100 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow ((Sch/S)_{étale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_S : ((Sch/S)_{étale}, \mathcal{O}) \longrightarrow (S_{étale}, \mathcal{O}_S)$$

The third is the morphism

$$a_S : ((Sch/S)_{fppf}, \mathcal{O}) \longrightarrow (S_{étale}, \mathcal{O}_S)$$

We already know that the category of quasi-coherent modules on the scheme S is the same as the category of quasi-coherent modules on $(S_{étale}, \mathcal{O}_S)$, see Descent, Proposition 35.8.9. Since we are interested in stating a comparison between étale and fppf cohomology, we will in the rest of this section think of quasi-coherent sheaves in terms of the small étale site. Let us review what we already know about quasi-coherent modules on these sites.

0DEW **Lemma 59.101.1.** *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on $S_{étale}$.*

(1) *The rule*
$$\mathcal{F}^a : (Sch/S)_{étale} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{F})$$
satisfies the sheaf condition for fppf and a fortiori étale coverings,
(2) *$\mathcal{F}^a = \pi_S^*\mathcal{F}$ on $(Sch/S)_{étale}$,*
(3) *$\mathcal{F}^a = a_S^*\mathcal{F}$ on $(Sch/S)_{fppf}$,*
(4) *the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{étale}, \mathcal{O})$,*
(5) *the rule $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence between quasi-coherent $\mathcal{O}_S$-modules and quasi-coherent modules on $((Sch/S)_{fppf}, \mathcal{O})$,*
(6) *we have $\epsilon_{S,*}a_S^*\mathcal{F} = \pi_S^*\mathcal{F}$ and $a_{S,*}a_S^*\mathcal{F} = \mathcal{F}$,*
(7) *we have $R^i\epsilon_{S,*}(a_S^*\mathcal{F}) = 0$ and $R^i a_{S,*}(a_S^*\mathcal{F}) = 0$ for $i > 0$.*

Proof. We urge the reader to find their own proof of these results based on the material in Descent, Sections 35.8, 35.9, and 35.10.

We first explain why the notation in this lemma is consistent with our earlier use of the notation $\mathcal{F}^a$ in Sections 59.17 and 59.22 and in Descent, Section 35.8. Namely, we know by Descent, Proposition 35.8.9 that there exists a quasi-coherent module $\mathcal{F}_0$ on the scheme S (in other words on the small Zariski site) such that $\mathcal{F}$ is the restriction of the rule

$$\mathcal{F}_0^a : (Sch/S)_{étale} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$

to the subcategory $S_{étale} \subset (Sch/S)_{étale}$ where here f^* denotes usual pullback of sheaves of modules on schemes. Since $\mathcal{F}_0^a$ is pullback by the morphism of ringed sites

$$((Sch/S)_{étale}, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_{S_{Zar}})$$

by Descent, Remark 35.8.6 it follows immediately (from composition of pullbacks) that $\mathcal{F}^a = \mathcal{F}_0^a$. This proves the sheaf property even for fpqc coverings by Descent, Lemma 35.8.1 (see also Proposition 59.17.1). Then (2) and (3) follow again by Descent, Remark 35.8.6 and (4) and (5) follow from Descent, Proposition 35.8.9 (see also the meta result Theorem 59.17.4).

Part (6) is immediate from the description of the sheaf $\mathcal{F}^a = \pi_S^*\mathcal{F} = a_S^*\mathcal{F}$.

For any abelian $\mathcal{H}$ on $(Sch/S)_{fppf}$ the higher direct image $R^p\epsilon_{S,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(Sch/S)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Hence to prove $R^p\epsilon_{S,*}a_S^*\mathcal{F} = R^p\epsilon_{S,*}\mathcal{F}^a = 0$ for $p > 0$ it suffices to show that any scheme U over S has an étale covering $\{U_i \to U\}_{i\in I}$ such that $H^p_{fppf}(U_i, \mathcal{F}^a) = 0$ for $p > 0$. If we take an open covering by affines, then the required vanishing follows from comparison with usual cohomology (Descent, Proposition 35.9.3 or Theorem 59.22.4) and the vanishing of cohomology of quasi-coherent sheaves on affine schemes afforded by Cohomology of Schemes, Lemma 30.2.2.

To show that $R^p a_{S,*}a_S^{-1}\mathcal{F} = R^p a_{S,*}\mathcal{F}^a = 0$ for $p > 0$ we argue in exactly the same manner. This finishes the proof. □

0DEX **Lemma 59.101.2.** *Let S be a scheme. For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module on $S_{étale}$ the maps*

$$\pi_S^*\mathcal{F} \longrightarrow R\epsilon_{S,*}(a_S^*\mathcal{F}) \quad \text{and} \quad \mathcal{F} \longrightarrow Ra_{S,*}(a_S^*\mathcal{F})$$

are isomorphisms with $a_S : Sh((Sch/S)_{fppf}) \to Sh(S_{étale})$ as above.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 59.101.1. □

0H0U **Lemma 59.101.3.** *Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules. Consider the complex $\mathcal{F}^\bullet$ of presheaves of $\mathcal{O}$-modules on $(Aff/S)_{fppf}$ given by the rule*

$$(U/S) = (\mathrm{Spec}(B)/\mathrm{Spec}(A)) \longmapsto M^\bullet \otimes_A B$$

Then this is a complex of modules and the canonical map

$$M^\bullet \longrightarrow R\Gamma((Aff/S)_{fppf}, \mathcal{F}^\bullet)$$

is a quasi-isomorphism.

Proof. Each $\mathcal{F}^n$ is a sheaf of modules as it agrees with the restriction of the module $\mathcal{G}^n = (\widetilde{M^n})^a$ of Lemma 59.101.1 to $(Aff/S)_{fppf} \subset (Sch/S)_{fppf}$. Since this inclusion defines an equivalence of ringed topoi (Topologies, Lemma 34.7.11), we have

$$R\Gamma((Aff/S)_{fppf}, \mathcal{F}^\bullet) = R\Gamma((Sch/S)_{fppf}, \mathcal{G}^\bullet)$$

We observe that $M^\bullet = R\Gamma(S, \widetilde{M^\bullet})$ for example by Derived Categories of Schemes, Lemma 36.3.5. Hence we are trying to show the comparison map

$$R\Gamma(S, \widetilde{M^\bullet}) \longrightarrow R\Gamma((Sch/S)_{fppf}, (\widetilde{M^\bullet})^a)$$

is an isomorphism. If $M^\bullet$ is bounded below, then this holds by Descent, Proposition 35.9.3 and the first spectral sequence of Derived Categories, Lemma 13.21.3. For

the general case, let us write $M^\bullet = \lim M^\bullet_n$ with $M^\bullet_n = \tau_{\geq -n} M^\bullet$. Whence the system M^p_n is eventually constant with value M^p. We claim that

$$(\widetilde{M^\bullet})^a = R\lim (\widetilde{M^\bullet_n})^a$$

Namely, it suffices to show that the natural map from left to right induces an isomorphism on cohomology over any affine object $U = \mathrm{Spec}(B)$ of $(Sch/S)_{fppf}$. For $i \in \mathbf{Z}$ and $n > |i|$ we have

$$H^i(U, (\widetilde{M^\bullet_n})^a) = H^i(\tau_{\geq -n} M^\bullet \otimes_A B) = H^i(M^\bullet \otimes_A B)$$

The first equality holds by the bounded below case treated above. Thus we see from Cohomology on Sites, Lemma 21.23.2 that the claim holds. Then we finally get

$$\begin{aligned} R\Gamma((Sch/S)_{fppf}, (\widetilde{M^\bullet})^a) &= R\Gamma((Sch/S)_{fppf}, R\lim (\widetilde{M^\bullet_n})^a) \\ &= R\lim R\Gamma((Sch/S)_{fppf}, (\widetilde{M^\bullet_n})^a) \\ &= R\lim M^\bullet_n \\ &= M^\bullet \end{aligned}$$

as desired. The second equality holds because $R\lim$ commutes with $R\Gamma$, see Cohomology on Sites, Lemma 21.23.2. □

59.102. Comparing ph and étale topologies

0DDV A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. We first review some material from Topologies, Sections 34.11 and 34.4.

Let S be a scheme and let $(Sch/S)_{ph}$ be a ph site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_{ph}$ is continuous (by More on Morphisms, Lemma 37.48.7 and Topologies, Lemma 34.7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_{ph} \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the ph sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0DDW **Lemma 59.102.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. The rule*

$$(Sch/S)_{ph} \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f^{-1}_{small}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. In fact this sheaf is equal to $\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{étale}$ and $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ on $(Sch/S)_{ph}$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the ph topology. By Topologies, Lemma 34.8.15 it suffices to show that given a proper surjective morphism $V \to U$ of schemes over S we have an equalizer diagram

$$(\pi_S^{-1}\mathcal{F})(U) \longrightarrow (\pi_S^{-1}\mathcal{F})(V) \rightrightarrows (\pi_S^{-1}\mathcal{F})(V \times_U V)$$

Set $\mathcal{G} = \pi_S^{-1}\mathcal{F}|_{U_{étale}}$. Consider the commutative diagram

$$\begin{array}{ccc} V \times_U V & \longrightarrow & V \\ \downarrow & \overset{g}{\searrow} & \downarrow f \\ V & \xrightarrow[f]{} & U \end{array}$$

We have

$$(\pi_S^{-1}\mathcal{F})(V) = \Gamma(V, f^{-1}\mathcal{G}) = \Gamma(U, f_* f^{-1}\mathcal{G})$$

where we use f_* and f^{-1} to denote functorialities between small étale sites. Second, we have

$$(\pi_S^{-1}\mathcal{F})(V \times_U V) = \Gamma(V \times_U V, g^{-1}\mathcal{G}) = \Gamma(U, g_* g^{-1}\mathcal{G})$$

The two maps in the equalizer diagram come from the two maps

$$f_* f^{-1}\mathcal{G} \longrightarrow g_* g^{-1}\mathcal{G}$$

Thus it suffices to prove $\mathcal{G}$ is the equalizer of these two maps of sheaves. Let $\overline{u}$ be a geometric point of U. Set $\Omega = \mathcal{G}_{\overline{u}}$. Taking stalks at $\overline{u}$ by Lemma 59.91.4 we obtain the two maps

$$H^0(V_{\overline{u}}, \underline{\Omega}) \longrightarrow H^0((V \times_U V)_{\overline{u}}, \underline{\Omega}) = H^0(V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}, \underline{\Omega})$$

where $\underline{\Omega}$ indicates the constant sheaf with value Ω. Of course these maps are the pullback by the projection maps. Then it is clear that the sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection, and sections coming from pullback by projection onto the first factor are constant on the fibres of the first projection. The sections in the intersection of the images of these pullback maps are constant on all of $V_{\overline{u}} \times_{\overline{u}} V_{\overline{u}}$, i.e., these come from elements of Ω as desired. □

In the situation of Lemma 59.102.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_{ph} \longrightarrow S_{étale}$$

0DDX **Lemma 59.102.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_{ph}$. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh((Sch/X)_{ph}) & \xrightarrow[f_{big,ph}]{} & Sh((Sch/Y)_{ph}) \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh((Sch/X)_{étale}) & \xrightarrow{f_{big,étale}} & Sh((Sch/Y)_{étale}) \end{array}$$

and

$$\begin{array}{ccc} Sh((Sch/X)_{ph}) & \xrightarrow{f_{big,ph}} & Sh((Sch/Y)_{ph}) \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ *and* $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows from the discussion in Topologies, Section 34.11. □

0DDY **Lemma 59.102.3.** *In Lemma 59.102.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.102.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,ph,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,ph,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*} \pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. □

0F0I **Lemma 59.102.4.** *Consider the comparison morphism $\epsilon : (Sch/S)_{ph} \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms of schemes. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes and the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 29.41.5.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.102.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 59.78.2.

Cohomology on Sites, Property (5) follows from More on Morphisms, Lemma 37.48.1 combined with the fact that a finite morphism is proper and a surjective proper morphism defines a ph covering, see Topologies, Lemma 34.8.6. □

0DE4 **Lemma 59.102.5.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_{ph})$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a proper morphism $f : X \to Y$ in $(Sch/S)_{ph}$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,ph,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{étale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$.*

Proof. By Lemma 59.102.4 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the full subcategory of $Ab((Sch/X)_{ph})$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason. □

0DE5 **Lemma 59.102.6.** *Let X be a scheme. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 59.102.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.102.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DE6 **Lemma 59.102.7.** *For a scheme X and $a_X : Sh((Sch/X)_{ph}) \to Sh(X_{étale})$ as above:*

(1) $H^q(X_{étale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ *for a torsion abelian sheaf* $\mathcal{F}$ *on* $X_{étale}$,
(2) $H^q(X_{étale}, K) = H^q_{ph}(X, a_X^{-1}K)$ *for* $K \in D^+(X_{étale})$ *with torsion cohomology sheaves.*

Example: if A *is a torsion abelian group, then* $H^q_{étale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 59.102.6 by Cohomology on Sites, Remark 21.14.4. □

59.103. Comparing h and étale topologies

0EW7 A model for this section is the section on the comparison of the usual topology and the qc topology on locally compact topological spaces as discussed in Cohomology on Sites, Section 21.31. Moreover, this section is almost word for word the same as the section comparing the ph and étale topologies. We first review some material from Topologies, Sections 34.11 and 34.4 and More on Flatness, Section 38.34.

Let S be a scheme and let $(Sch/S)_h$ be an h site. On the same underlying category we have a second topology, namely the étale topology, and hence a second site $(Sch/S)_{étale}$. The identity functor $(Sch/S)_{étale} \to (Sch/S)_h$ is continuous (by More on Flatness, Lemma 38.34.6 and Topologies, Lemma 34.7.2) and defines a morphism of sites

$$\epsilon_S : (Sch/S)_h \longrightarrow (Sch/S)_{étale}$$

See Cohomology on Sites, Section 21.27. Please note that $\epsilon_{S,*}$ is the identity functor on underlying presheaves and that ϵ_S^{-1} associates to an étale sheaf the h sheafification. Let $S_{étale}$ be the small étale site. There is a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

given by the continuous functor $S_{étale} \to (Sch/S)_{étale}$, $U \mapsto U$. Namely, $S_{étale}$ has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies.

0EW8 **Lemma 59.103.1.** *With notation as above. Let* $\mathcal{F}$ *be a sheaf on* $S_{étale}$. *The rule*

$$(Sch/S)_h \longrightarrow Sets, \quad (f : X \to S) \longmapsto \Gamma(X, f_{small}^{-1}\mathcal{F})$$

is a sheaf and a fortiori a sheaf on $(Sch/S)_{étale}$. *In fact this sheaf is equal to* $\pi_S^{-1}\mathcal{F}$ *on* $(Sch/S)_{étale}$ *and* $\epsilon_S^{-1}\pi_S^{-1}\mathcal{F}$ *on* $(Sch/S)_h$.

Proof. The statement about the étale topology is the content of Lemma 59.39.2. To finish the proof it suffices to show that $\pi_S^{-1}\mathcal{F}$ is a sheaf for the h topology. However, in Lemma 59.102.1 we have shown that $\pi_S^{-1}\mathcal{F}$ is a sheaf even in the stronger ph topology. □

In the situation of Lemma 59.103.1 the composition of ϵ_S and π_S and the equality determine a morphism of sites

$$a_S : (Sch/S)_h \longrightarrow S_{étale}$$

0EW9 **Lemma 59.103.2.** *With notation as above. Let $f : X \to Y$ be a morphism of $(Sch/S)_h$. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh((Sch/X)_h) & \xrightarrow{f_{big,h}} & Sh((Sch/Y)_h) \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh((Sch/X)_{étale}) & \xrightarrow{f_{big,étale}} & Sh((Sch/Y)_{étale}) \end{array}$$

and

$$\begin{array}{ccc} Sh((Sch/X)_h) & \xrightarrow{f_{big,h}} & Sh((Sch/Y)_h) \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. The commutativity of the diagrams follows similarly to what was said in Topologies, Section 34.11. □

0EWA **Lemma 59.103.3.** *In Lemma 59.103.2 if f is proper, then we have $a_Y^{-1} \circ f_{small,*} = f_{big,h,*} \circ a_X^{-1}$.*

Proof. You can prove this by repeating the proof of Lemma 59.99.5 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 59.103.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,h,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,h,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 59.99.5. □

0F0J **Lemma 59.103.4.** *Consider the comparison morphism $\epsilon : (Sch/S)_h \to (Sch/S)_{étale}$. Let $\mathcal{P}$ denote the class of proper morphisms. For X in $(Sch/S)_{étale}$ denote $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf on $X_{étale}$ Then Cohomology on Sites, Properties (1), (2), (3), (4), and (5) of Cohomology on Sites, Situation 21.30.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab((Sch/X)_{étale})$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful for example by Lemma 59.99.4. If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab((Sch/X)_{étale})$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 59.99.4. In particular we see that $\pi_{X,*}\mathcal{G}$ is an abelian torsion sheaf on $X_{étale}$. Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Cohomology on Sites, Property (1) holds by the existence of fibre products of schemes, the fact that the base change of a proper morphism of schemes is a proper morphism of schemes, see Morphisms, Lemma 29.41.5, and the fact that the base

change of a morphism of finite presentation is a morphism of finite presentation, see Morphisms, Lemma 29.21.4.

Cohomology on Sites, Property (2) follows from the commutative diagram (3) in Topologies, Lemma 34.4.17.

Cohomology on Sites, Property (3) is Lemma 59.103.1.

Cohomology on Sites, Property (4) holds by Lemma 59.99.5 part (2) and the fact that $R^i f_{small}\mathcal{F}$ is torsion if $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$, see Lemma 59.78.2.

Cohomology on Sites, Property (5) is implied by More on Morphisms, Lemma 37.48.1 combined with the fact that a surjective finite locally free morphism is surjective, proper, and of finite presentation and hence defines a h-covering by More on Flatness, Lemma 38.34.7. □

0EWF **Lemma 59.103.5.** *With notation as above.*

(1) *For $X \in \mathrm{Ob}((Sch/S)_h)$ and an abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
(2) *For a proper morphism $f : X \to Y$ in $(Sch/S)_h$ and abelian torsion sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^i f_{small,*}\mathcal{F}) = R^i f_{big,h,*}(a_X^{-1}\mathcal{F})$ for all i.*
(3) *For a scheme X and K in $D^+(X_{étale})$ with torsion cohomology sheaves the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
(4) *For a proper morphism $f : X \to Y$ of schemes and K in $D^+(X_{étale})$ with torsion cohomology sheaves we have $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,h,*}(a_X^{-1}K)$.*

Proof. By Lemma 59.103.4 the lemmas in Cohomology on Sites, Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Cohomology on Sites, Lemma 21.30.2 is the full subcategory of $Ab((Sch/X)_h)$ consisting of sheaves of the form $a_X^{-1}\mathcal{F}$ where $\mathcal{F}$ is an abelian torsion sheaf on $X_{étale}$.

Part (1) is equivalent to (V_n) for all n which holds by Cohomology on Sites, Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Cohomology on Sites, Lemma 21.30.3.

Part (3) follows from Cohomology on Sites, Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}((Sch/X)_{étale})$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Cohomology on Sites, Lemma 21.30.8 part (2) for the same reason. □

0EWG **Lemma 59.103.6.** *Let X be a scheme. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh((Sch/X)_h) \to Sh(X_{étale})$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7)

applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 59.103.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 59.103.5 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 59.99.4 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0EWH **Lemma 59.103.7.** *For a scheme X and $a_X : Sh((Sch/X)_h) \to Sh(X_{étale})$ as above:*

(1) *$H^q(X_{étale}, \mathcal{F}) = H^q_h(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(2) *$H^q(X_{étale}, K) = H^q_h(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$ with torsion cohomology sheaves.*

Example: if A is a torsion abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_h(X, \underline{A})$.

Proof. This follows from Lemma 59.103.6 by Cohomology on Sites, Remark 21.14.4. □

59.104. Descending étale sheaves

0GEX We prove that étale sheaves "glue" in the fppf and h topology and related results. We have already shown the following related results

(1) Lemma 59.39.2 tells us that a sheaf on the small étale site of a scheme S determines a sheaf on the big étale site of S satisfying the sheaf condition for fpqc coverings (and a fortiori for Zariski, étale, smooth, syntomic, and fppf coverings),
(2) Lemma 59.100.1 is a restatement of the previous point for the fppf topology,
(3) Lemma 59.102.1 proves the same for the ph topology,
(4) Lemma 59.103.1 proves the same for the h topology,
(5) Lemma 59.100.4 is a version of fppf descent for étale sheaves, and
(6) Remark 59.55.6 tells us that we have descent of étale sheaves for finite surjective morphisms (we will clarify and strengthen this below).

In the chapter on simplicial spaces we will prove some additional results on this, see for example Simplicial Spaces, Sections 85.33 and 85.36.

In order to conveniently express our results we need some notation. Let $\mathcal{U} = \{f_i : X_i \to X\}$ be a family of morphisms of schemes with fixed target. A *descent datum* for étale sheaves with respect to $\mathcal{U}$ is a family $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ where

(1) $\mathcal{F}_i$ is in $Sh(X_{i,étale})$, and
(2) $\varphi_{ij} : \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j$ is an isomorphism in $Sh((X_i \times_X X_j)_{étale})$

such that the *cocycle condition* holds: the diagrams

$$\begin{array}{ccc} \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{01,small}^{-1}\varphi_{ij}} & \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j \\ & {\scriptstyle \mathrm{pr}_{02,small}^{-1}\varphi_{ik}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12,small}^{-1}\varphi_{jk}} & \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F}_k & \end{array}$$

commute in $Sh((X_i \times_X X_j \times_X X_k)_{étale})$. There is an obvious notion of *morphisms of descent data* and we obtain a category of descent data. A descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is called *effective* if there exist a $\mathcal{F}$ in $Sh(X_{étale})$ and isomorphisms $\varphi_i : f_{i,small}^{-1}\mathcal{F} \to \mathcal{F}_i$ in $Sh(X_{i,étale})$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \mathrm{pr}_{1,small}^{-1}(\varphi_j) \circ \mathrm{pr}_{0,small}^{-1}(\varphi_i^{-1})$$

Another way to say this is the following. Given an object $\mathcal{F}$ of $Sh(X_{étale})$ we obtain the *canonical descent datum* $(f_{i,small}^{-1}\mathcal{F}_i, c_{ij})$ where c_{ij} is the canonical isomorphism

$$c_{ij} : \mathrm{pr}_{0,small}^{-1} f_{i,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1} f_{j,small}^{-1}\mathcal{F}$$

The descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is effective if and only if it is isomorphic to the canonical descent datum associated to some $\mathcal{F}$ in $Sh(X_{étale})$.

If the family consists of a single morphism $\{X \to Y\}$, then we think of a descent datum as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is an object of $Sh(X_{étale})$ and φ is an isomorphism

$$\mathrm{pr}_{0,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}$$

in $Sh((X \times_Y X)_{étale})$ such that the cocycle condition holds:

$$\begin{array}{ccc} \mathrm{pr}_{0,small}^{-1}\mathcal{F} & \xrightarrow{\mathrm{pr}_{01,small}^{-1}\varphi} & \mathrm{pr}_{1,small}^{-1}\mathcal{F} \\ & {\scriptstyle \mathrm{pr}_{02,small}^{-1}\varphi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12,small}^{-1}\varphi} & \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F} & \end{array}$$

commutes in $Sh((X \times_Y X \times_Y X)_{étale})$. There is a notion of morphisms of descent data and effectivity exactly as before.

We first prove effective descent for surjective integral morphisms.

0GEY **Lemma 59.104.1.** *Let $f : X \to Y$ be a morphism of schemes which has a section. Then the functor*

$$Sh(Y_{étale}) \longrightarrow \textit{descent data for étale sheaves wrt } \{X \to Y\}$$

sending $\mathcal{G}$ in $Sh(Y_{étale})$ to the canonical descent datum is an equivalence of categories.

Proof. This is formal and depends only on functoriality of the pullback functors. We omit the details. Hint: If $s : Y \to X$ is a section, then a quasi-inverse is the functor sending $(\mathcal{F}, \varphi)$ to $s_{small}^{-1}\mathcal{F}$. □

0GEZ **Lemma 59.104.2.** *Let $f : X \to Y$ be a surjective integral morphism of schemes. The functor*

$$Sh(Y_{étale}) \longrightarrow \textit{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. In this proof we drop the subscript $_{small}$ from our pullback and pushforward functors. Denote $X_1 = X \times_Y X$ and denote $f_1 : X_1 \to Y$ the morphism $f \circ \mathrm{pr}_0 = f \circ \mathrm{pr}_1$. Let $(\mathcal{F}, \varphi)$ be a descent datum for $\{X \to Y\}$. Let us set $\mathcal{F}_1 = \mathrm{pr}_0^{-1}\mathcal{F}$. We

may think of φ as defining an isomorphism $\mathcal{F}_1 \to \mathrm{pr}_1^{-1}\mathcal{F}$. We claim that the rule which sends a descent datum $(\mathcal{F}, \varphi)$ to the sheaf

$$\mathcal{G} = \text{Equalizer}\Big(f_*\mathcal{F} \rightrightarrows f_{1,*}\mathcal{F}_1 \Big)$$

is a quasi-inverse to the functor in the statement of the lemma. The first of the two arrows comes from the map

$$f_*\mathcal{F} \to f_*\mathrm{pr}_{0,*}\mathrm{pr}_0^{-1}\mathcal{F} = f_{1,*}\mathcal{F}_1$$

and the second arrow comes from the map

$$f_*\mathcal{F} \to f_*\mathrm{pr}_{1,*}\mathrm{pr}_1^{-1}\mathcal{F} \xleftarrow{\varphi} f_*\mathrm{pr}_{0,*}\mathrm{pr}_0^{-1}\mathcal{F} = f_{1,*}\mathcal{F}_1$$

where the arrow pointing left is invertible. To prove this works we have to show that the canonical map $f^{-1}\mathcal{G} \to \mathcal{F}$ is an isomorphism; details omitted. In order to prove this it suffices to check after pulling back by any collection of morphisms $\mathrm{Spec}(k) \to Y$ where k is an algebraically closed field. Namely, the corresponding base changes $X_k \to X$ are jointly surjective and we can check whether a map of sheaves on $X_{étale}$ is an isomorphism by looking at stalks on geometric points, see Theorem 59.29.10. By Lemma 59.55.4 the construction of $\mathcal{G}$ from the descent datum $(\mathcal{F}, \varphi)$ commutes with any base change. Thus we may assume Y is the spectrum of an algebraically closed point (note that base change preserves the properties of the morphism f, see Morphisms, Lemma 29.9.4 and 29.44.6). In this case the morphism $X \to Y$ has a section, so we know that the functor is an equivalence by Lemma 59.104.1. However, the reader may show that the functor is an equivalence if and only if the construction above is a quasi-inverse; details omitted. This finishes the proof. □

0GF0 **Lemma 59.104.3.** *Let $f : X \to Y$ be a surjective proper morphism of schemes. The functor*

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. The exact same proof as given in Lemma 59.104.2 works, except the appeal to Lemma 59.55.4 should be replaced by an appeal to Lemma 59.91.5. □

0GF1 **Lemma 59.104.4.** *Let $f : X \to Y$ be a morphism of schemes. Let $Z \to Y$ be a surjective integral morphism of schemes or a surjective proper morphism of schemes. If the functors*

$$Sh(Z_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Z \to Z\}$$

and

$$Sh((Z \times_Y Z)_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y (Z \times_Y Z) \to Z \times_Y Z\}$$

are equivalences of categories, then

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence.

Proof. Formal consequence of the definitions and Lemmas 59.104.2 and 59.104.3. Details omitted. □

0GF2 **Lemma 59.104.5.** *Let $f : X \to Y$ be a morphism of schemes which is surjective, flat, locally of finite presentation. The functor*

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Exactly as in the proof of Lemma 59.104.2 we claim a quasi-inverse is given by the functor sending $(\mathcal{F}, \varphi)$ to

$$\mathcal{G} = \text{Equalizer}\left(f_*\mathcal{F} \rightrightarrows f_{1,*}\mathcal{F}_1 \right)$$

and in order to prove this it suffices to show that $f^{-1}\mathcal{G} \to \mathcal{F}$ is an isomorphism. This we may check locally, hence we may and do assume Y is affine. Then we can find a finite surjective morphism $Z \to Y$ such that there exists an open covering $Z = \bigcup W_i$ such that $W_i \to Y$ factors through X. See More on Morphisms, Lemma 37.48.6. Applying Lemma 59.104.4 we see that it suffices to prove the lemma after replacing Y by Z and $Z \times_Y Z$ and f by its base change. Thus we may assume f has sections Zariski locally. Of course, using that the problem is local on Y we reduce to the case where we have a section which is Lemma 59.104.1. □

0GF3 **Lemma 59.104.6.** *Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. The functor*

$$Sh(X_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. We have Lemma 59.104.5 for the morphism $f : \coprod X_i \to X$. Then a formal argument shows that descent data for f are the same thing as descent data for the covering, compare with Descent, Lemma 35.34.5. Details omitted. □

0GF4 **Lemma 59.104.7.** *Let $f : X' \to X$ be a proper morphism of schemes. Let $i : Z \to X$ be a closed immersion. Set $E = Z \times_X X'$. Picture*

$$\begin{array}{ccc} E & \xrightarrow[j]{} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

If f is an isomorphism over $X \setminus Z$, then the functor

$$Sh(X_{étale}) \longrightarrow Sh(X'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale})$$

is an equivalence of categories.

Proof. We will work with the 2-fibre product category as constructed in Categories, Example 4.31.3. The functor sends $\mathcal{F}$ to the triple $(f^{-1}\mathcal{F}, i^{-1}\mathcal{F}, c)$ where $c : j^{-1}f^{-1}\mathcal{F} \to g^{-1}i^{-1}\mathcal{F}$ is the canonical isomorphism. We will construct a quasi-inverse functor. Let $(\mathcal{F}', \mathcal{G}, \alpha)$ be an object of the right hand side of the arrow. We obtain an isomorphism

$$i^{-1}f_*\mathcal{F}' = g_*j^{-1}\mathcal{F}' \xrightarrow{g_*\alpha} g_*g^{-1}\mathcal{G}$$

The first equality is Lemma 59.91.5. Using this we obtain maps $i_*\mathcal{G} \to i_*g_*g^{-1}\mathcal{G}$ and $f'_*\mathcal{F}' \to i_*g_*g^{-1}\mathcal{G}$. We set

$$\mathcal{F} = f_*\mathcal{F}' \times_{i_*g_*g^{-1}\mathcal{G}} i_*\mathcal{G}$$

and we claim that $\mathcal{F}$ is an object of the left hand side of the arrow whose image in the right hand side is isomorphic to the triple we started out with. Let us compute the stalk of $\mathcal{F}$ at a geometric point $\overline{x}$ of X.

If $\overline{x}$ is not in Z, then on the one hand $\overline{x}$ comes from a unique geometric point $\overline{x}'$ of X' and $\mathcal{F}'_{\overline{x}'} = (f_*\mathcal{F}')_{\overline{x}}$ and on the other hand we have $(i_*\mathcal{G})_{\overline{x}}$ and $(i_*g_*g^{-1}\mathcal{G})_{\overline{x}}$ are singletons. Hence we see that $\mathcal{F}_{\overline{x}}$ equals $\mathcal{F}'_{\overline{x}'}$.

If $\overline{x}$ is in Z, i.e., $\overline{x}$ is the image of a geometric point $\overline{z}$ of Z, then we obtain $(i_*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{z}}$ and

$$(i_*g_*g^{-1}\mathcal{G})_{\overline{x}} = (g_*g^{-1}\mathcal{G})_{\overline{z}} = \Gamma(E_{\overline{z}}, g^{-1}\mathcal{G}|_{E_{\overline{z}}})$$

(by the proper base change for pushforward used above) and similarly

$$(f_*\mathcal{F}')_{\overline{x}} = \Gamma(X'_{\overline{x}}, \mathcal{F}'|_{X'_{\overline{x}}})$$

Since we have the identification $E_{\overline{z}} = X'_{\overline{x}}$ and since α defines an isomorphism between the sheaves $\mathcal{F}'|_{X'_{\overline{x}}}$ and $g^{-1}\mathcal{G}|_{E_{\overline{z}}}$ we conclude that we get

$$\mathcal{F}_{\overline{x}} = \mathcal{G}_{\overline{z}}$$

in this case.

To finish the proof, we observe that there are canonical maps $i^{-1}\mathcal{F} \to \mathcal{G}$ and $f^{-1}\mathcal{F} \to \mathcal{F}'$ compatible with α which on stalks produce the isomorphisms we saw above. We omit the careful construction of these maps. □

0GF5 **Lemma 59.104.8.** *Let S be a scheme. Then the category fibred in groupoids*

$$p : \mathcal{S} \longrightarrow (Sch/S)_h$$

whose fibre category over U is the category $Sh(U_{étale})$ of sheaves on the small étale site of U is a stack in groupoids.

Proof. To prove the lemma we will check conditions (1), (2), and (3) of More on Flatness, Lemma 38.37.13.

Condition (1) holds because we have glueing for sheaves (and Zariski coverings are étale coverings). See Sites, Lemma 7.26.4.

To see condition (2), suppose that $f : X \to Y$ is a surjective, flat, proper morphism of finite presentation over S with Y affine. Then we have descent for $\{X \to Y\}$ by either Lemma 59.104.5 or Lemma 59.104.3.

Condition (3) follows immediately from the more general Lemma 59.104.7. □

59.105. Blow up squares and étale cohomology

0EW4 Blow up squares are introduced in More on Flatness, Section 38.36. Using the proper base change theorem we can see that we have a Mayer-Vietoris type result for blow up squares.

0EW5 **Lemma 59.105.1.** *Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

For $K \in D^+(X_{étale})$ with torsion cohomology sheaves we have a distinguished triangle

$$K \to Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E) \to K[1]$$

in $D(X_{étale})$ where $c = i \circ \pi = b \circ j$.

Proof. The notation $K|_{X'}$ stands for $b_{small}^{-1}K$. Choose a bounded below complex $\mathcal{F}^\bullet$ of abelian sheaves representing K. Observe that $i_*(\mathcal{F}^\bullet|_Z)$ represents $Ri_*(K|_Z)$ because i_* is exact (Proposition 59.55.2). Choose a quasi-isomorphism $b_{small}^{-1}\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injective abelian sheaves on $X'_{étale}$. This map is adjoint to a map $\mathcal{F}^\bullet \to b_*(\mathcal{I}^\bullet)$ and $b_*(\mathcal{I}^\bullet)$ represents $Rb_*(K|_{X'})$. We have $\pi_*(\mathcal{I}^\bullet|_E) = (b_*\mathcal{I}^\bullet)|_Z$ by Lemma 59.91.5 and by Lemma 59.91.12 this complex represents $R\pi_*(K|_E)$. Hence the map

$$Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) \to Rc_*(K|_E)$$

is represented by the surjective map of bounded below complexes

$$i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet) \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)$$

To get our distinguished triangle it suffices to show that the canonical map $\mathcal{F}^\bullet \to i_*(\mathcal{F}^\bullet|_Z) \oplus b_*(\mathcal{I}^\bullet)$ maps quasi-isomorphically onto the kernel of the map of complexes displayed above (namely a short exact sequence of complexes determines a distinguished triangle in the derived category, see Derived Categories, Section 13.12). We may check this on stalks at a geometric point $\overline{x}$ of X. If $\overline{x}$ is not in Z, then $X' \to X$ is an isomorphism over an open neighbourhood of $\overline{x}$. Thus, if $\overline{x}'$ denotes the corresponding geometric point of X' in this case, then we have to show that

$$\mathcal{F}^\bullet_{\overline{x}} \to \mathcal{I}^\bullet_{\overline{x}'}$$

is a quasi-isomorphism. This is true by our choice of $\mathcal{I}^\bullet$. If $\overline{x}$ is in Z, then $b_(\mathcal{I}^\bullet)_{\overline{x}} \to i_*\left(b_*(\mathcal{I}^\bullet)|_Z\right)_{\overline{x}}$ is an isomorphism of complexes of abelian groups. Hence the kernel is equal to $i_*(\mathcal{F}^\bullet|_Z)_{\overline{x}} = \mathcal{F}^\bullet_{\overline{x}}$ as desired. □

0EW3 **Lemma 59.105.2.** *Let X be a scheme and let $K \in D^+(X_{étale})$ have torsion cohomology sheaves. Let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

Then there is a canonical long exact sequence

$$H^p_{étale}(X, K) \to H^p_{étale}(X', K|_{X'}) \oplus H^p_{étale}(Z, K|_Z) \to H^p_{étale}(E, K|_E) \to H^{p+1}_{étale}(X, K)$$

First proof. This follows immediately from Lemma 59.105.1 and the fact that

$$R\Gamma(X, Rb_*(K|_{X'})) = R\Gamma(X', K|_{X'})$$

(see Cohomology on Sites, Section 21.14) and similarly for the others. □

Second proof. By Lemma 59.102.7 these cohomology groups are the cohomology of X, X', E, Z with values in some complex of abelian sheaves on the site $(Sch/X)_{ph}$. (Namely, the object $a_X^{-1}K$ of the derived category, see Lemma 59.102.1 above and recall that $K|_{X'} = b_{small}^{-1}K$.) By More on Flatness, Lemma 38.36.1 the ph sheafification of the diagram of representable presheaves is cocartesian. Thus the lemma

follows from the very general Cohomology on Sites, Lemma 21.26.3 applied to the site $(Sch/X)_{ph}$ and the commutative diagram of the lemma. □

0EW6 **Lemma 59.105.3.** *Let X be a scheme and let $Z \subset X$ be a closed subscheme cut out by a quasi-coherent ideal of finite type. Consider the corresponding blow up square*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Suppose given

(1) *an object K' of $D^+(X'_{étale})$ with torsion cohomology sheaves,*
(2) *an object L of $D^+(Z_{étale})$ with torsion cohomology sheaves, and*
(3) *an isomorphism $\gamma : K'|_E \to L|_E$.*

Then there exists an object K of $D^+(X_{étale})$ and isomorphisms $f : K|_{X'} \to K'$, $g : K|_Z \to L$ such that $\gamma = g|_E \circ f^{-1}|_E$. Moreover, given

(1) *an object M of $D^+(X_{étale})$ with torsion cohomology sheaves,*
(2) *a morphism $\alpha : K' \to M|_{X'}$ of $D(X'_{étale})$,*
(3) *a morphism $\beta : L \to M|_Z$ of $D(Z_{étale})$,*

such that

$$\alpha|_E = \beta|_E \circ \gamma.$$

Then there exists a morphism $M \to K$ in $D(X_{étale})$ whose restriction to X' is $a \circ f$ and whose restriction to Z is $b \circ g$.

Proof. If K exists, then Lemma 59.105.1 tells us a distinguished triangle that it fits in. Thus we simply choose a distinguished triangle

$$K \to Ri_*(L) \oplus Rb_*(K') \to Rc_*(L|_E) \to K[1]$$

where $c = i \circ \pi = b \circ j$. Here the map $Ri_*(L) \to Rc_*(L|_E)$ is Ri_* applied to the adjunction mapping $E \to R\pi_*(L|_E)$. The map $Rb_*(K') \to Rc_*(L|_E)$ is the composition of the canonical map $Rb_*(K') \to Rc_*(K'|_E)) = R$ and $Rc_*(\gamma)$. The maps g and f of the statement of the lemma are the adjoints of these maps. If we restrict this distinguished triangle to Z then the map $Rb_*(K) \to Rc_*(L|_E)$ becomes an isomorphism by the base change theorem (Lemma 59.91.12) and hence the map $g : K|_Z \to L$ is an isomorphism. Looking at the distinguished triangle we see that $f : K|_{X'} \to K'$ is an isomorphism over $X' \setminus E = X \setminus Z$. Moreover, we have $\gamma \circ f|_E = g|_E$ by construction. Then since γ and g are isomorphisms we conclude that f induces isomorphisms on stalks at geometric points of E as well. Thus f is an isomorphism.

For the final statement, we may replace K' by $K|_{X'}$, L by $K|_Z$, and γ by the canonical identification. Observe that α and β induce a commutative square

$$\begin{array}{ccccccc} K & \longrightarrow & Ri_*(K|_Z) \oplus Rb_*(K|_{X'}) & \longrightarrow & Rc_*(K|_E) & \longrightarrow & K[1] \\ \downarrow & & {\scriptstyle\beta\oplus\alpha}\downarrow & & {\scriptstyle\alpha|_E}\downarrow & & \downarrow \\ M & \longrightarrow & Ri_*(M|_Z) \oplus Rb_*(M|_{X'}) & \longrightarrow & Rc_*(M|_E) & \longrightarrow & M[1] \end{array}$$

Thus by the axioms of a derived category we get a dotted arrow producing a morphism of distinguished triangles. □

59.106. Almost blow up squares and the h topology

0EWL In this section we continue the discussion in More on Flatness, Section 38.37. For the convenience of the reader we recall that an almost blow up square is a commutative diagram

0EWM (59.106.0.1)
$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow b \\ Z & \longrightarrow & X \end{array}$$

of schemes satisfying the following conditions:

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 28.24.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$.

We are going to give a criterion for "h sheafiness" for objects in the derived category of the big fppf site $(Sch/S)_{fppf}$ of a scheme S. On the same underlying category we have a second topology, namely the h topology (More on Flatness, Section 38.34). Recall that fppf coverings are h coverings (More on Flatness, Lemma 38.34.6). Hence we may consider the morphism

$$\epsilon : (Sch/S)_h \longrightarrow (Sch/S)_{fppf}$$

See Cohomology on Sites, Section 21.27. In particular, we have a fully faithful functor

$$R\epsilon_* : D((Sch/S)_h) \longrightarrow D((Sch/S)_{fppf})$$

and we can ask: what is the essential image of this functor?

0EWN **Lemma 59.106.1.** *With notation as above, if K is in the essential image of $R\epsilon_*$, then the maps $c^K_{X,Z,X',E}$ of Cohomology on Sites, Lemma 21.26.1 are quasi-isomorphisms.*

Proof. Denote $^\#$ sheafification in the h topology. We have seen in More on Flatness, Lemma 38.37.7 that $h^\#_X = h^\#_Z \amalg_{h^\#_E} h^\#_{X'}$. On the other hand, the map $h^\#_E \to h^\#_{X'}$ is injective as $E \to X'$ is a monomorphism. Thus this lemma is a special case of Cohomology on Sites, Lemma 21.29.3 (which itself is a formal consequence of Cohomology on Sites, Lemma 21.26.3). □

0EWQ **Proposition 59.106.2.** *Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$ is a quasi-isomorphism for every almost blow up square (59.106.0.1) in $(Sch/S)_h$ with X affine.*

Proof. We prove this by applying Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Lemma 59.106.1 and More on Flatness, Proposition 38.37.9. □

0EWR **Lemma 59.106.3.** *Let K be an object of $D^+((Sch/S)_{fppf})$. Then K is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$ if and only if $c^K_{X,X',Z,E}$*

is a quasi-isomorphism for every almost blow up square as in More on Flatness, Examples 38.37.10 and 38.37.11.

Proof. We prove this by applying Cohomology on Sites, Lemma 21.29.2 whose hypotheses hold by Lemma 59.106.1 and More on Flatness, Lemma 38.37.12 □

59.107. Cohomology of the structure sheaf in the h topology

0EWS Let p be a prime number. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with $p\mathcal{O} = 0$. Then we set $\mathrm{colim}_F \mathcal{O}$ equal to the colimit in the category of sheaves of rings of the system

$$\mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \mathcal{O} \xrightarrow{F} \ldots$$

where $F : \mathcal{O} \to \mathcal{O}$, $f \mapsto f^p$ is the Frobenius endomorphism.

0EWT **Lemma 59.107.1.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Consider the sheaf $\mathcal{O}^{perf} = \mathrm{colim}_F \mathcal{O}$ on $(Sch/S)_{fppf}$. Then $\mathcal{O}^{perf}$ is in the essential image of $R\epsilon_* : D((Sch/S)_h) \to D((Sch/S)_{fppf})$.*

Proof. We prove this using the criterion of Lemma 59.106.3. Before check the conditions, we note that for a quasi-compact and quasi-separated object X of $(Sch/S)_{fppf}$ we have

$$H^i_{fppf}(X, \mathcal{O}^{perf}) = \mathrm{colim}_F H^i_{fppf}(X, \mathcal{O})$$

See Cohomology on Sites, Lemma 21.16.1. We will also use that $H^i_{fppf}(X, \mathcal{O}) = H^i(X, \mathcal{O})$, see Descent, Proposition 35.9.3.

Let A, f, J be as in More on Flatness, Example 38.37.10 and consider the associated almost blow up square. Since X, X', Z, E are affine, we have no higher cohomology of $\mathcal{O}$. Hence we only have to check that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.38.2.

Let X, X', Z, E be as in More on Flatness, Example 38.37.11. Since X and Z are affine we have $H^p(X, \mathcal{O}_X) = H^p(Z, \mathcal{O}_X) = 0$ for $p > 0$. By More on Flatness, Lemma 38.38.1 we have $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$. Since $E = \mathbf{P}^1_Z$ and Z is affine we also have $H^p(E, \mathcal{O}_E) = 0$ for $p > 0$. As in the previous paragraph we reduce to checking that

$$0 \to \mathcal{O}^{perf}(X) \to \mathcal{O}^{perf}(X') \oplus \mathcal{O}^{perf}(Z) \to \mathcal{O}^{perf}(E) \to 0$$

is a short exact sequence. This was shown in (the proof of) More on Flatness, Lemma 38.38.2. □

0EWU **Proposition 59.107.2.** *Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. Then*

$$H^i((Sch/S)_h, \mathcal{O}^h) = \mathrm{colim}_F H^i(S, \mathcal{O})$$

Here on the left hand side by $\mathcal{O}^h$ we mean the h sheafification of the structure sheaf.

Proof. This is just a reformulation of Lemma 59.107.1. Recall that $\mathcal{O}^h = \mathcal{O}^{perf} = \operatorname{colim}_F \mathcal{O}$, see More on Flatness, Lemma 38.38.7. By Lemma 59.107.1 we see that $\mathcal{O}^{perf}$ viewed as an object of $D((Sch/S)_{fppf})$ is of the form $R\epsilon_* K$ for some $K \in D((Sch/S)_h)$. Then $K = \epsilon^{-1}\mathcal{O}^{perf}$ which is actually equal to $\mathcal{O}^{perf}$ because $\mathcal{O}^{perf}$ is an h sheaf. See Cohomology on Sites, Section 21.27. Hence $R\epsilon_*\mathcal{O}^{perf} = \mathcal{O}^{perf}$ (with apologies for the confusing notation). Thus the lemma now follows from Leray

$$R\Gamma_h(S, \mathcal{O}^{perf}) = R\Gamma_{fppf}(S, R\epsilon_*\mathcal{O}^{perf}) = R\Gamma_{fppf}(S, \mathcal{O}^{perf})$$

and the fact that

$$H^i_{fppf}(S, \mathcal{O}^{perf}) = H^i_{fppf}(S, \operatorname{colim}_F \mathcal{O}) = \operatorname{colim}_F H^i_{fppf}(S, \mathcal{O})$$

as S is quasi-compact and quasi-separated (see proof of Lemma 59.107.1). □

59.108. Other chapters

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CHAPTER 60

Crystalline Cohomology

07GI

60.1. Introduction

07GJ This chapter is based on a lecture series given by Johan de Jong held in 2012 at Columbia University. The goals of this chapter are to give a quick introduction to crystalline cohomology. A reference is the book [**Ber74**].

We have moved the more elementary purely algebraic discussion of divided power rings to a preliminary chapter as it is also useful in discussing Tate resolutions in commutative algebra. Please see Divided Power Algebra, Section 23.1.

60.2. Divided power envelope

07H7 The construction of the following lemma will be dubbed the divided power envelope. It will play an important role later.

07H8 **Lemma 60.2.1.** *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a homomorphism of divided power rings*

$$(A, I, \gamma) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

such that

$$\mathrm{Hom}_{(A,I,\gamma)}((D, \bar{J}, \bar{\gamma}), (C, K, \delta)) = \mathrm{Hom}_{(A,I)}((B, J), (C, K))$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ). Here the LHS is morphisms of divided power rings over (A, I, γ) and the RHS is morphisms of (ring, ideal) pairs over (A, I).

Proof. Denote $\mathcal{C}$ the category of divided power rings (C, K, δ). Consider the functor $F : \mathcal{C} \longrightarrow \mathit{Sets}$ defined by

$$F(C, K, \delta) = \left\{ (\varphi, \psi) \middle| \begin{array}{c} \varphi : (A, I, \gamma) \to (C, K, \delta) \text{ homomorphism of divided power rings} \\ \psi : (B, J) \to (C, K) \text{ an } A\text{-algebra homomorphism with } \psi(J) \subset K \end{array} \right\}$$

We will show that Divided Power Algebra, Lemma 23.3.3 applies to this functor which will prove the lemma. Suppose that $(\varphi, \psi) \in F(C, K, \delta)$. Let $C' \subset C$ be the subring generated by $\varphi(A)$, $\psi(B)$, and $\delta_n(\psi(f))$ for all $f \in J$. Let $K' \subset K \cap C'$ be the ideal of C' generated by $\varphi(I)$ and $\delta_n(\psi(f))$ for $f \in J$. Then $(C', K', \delta|_{K'})$ is a divided power ring and C' has cardinality bounded by the cardinal $\kappa = |A| \otimes |B|^{\aleph_0}$. Moreover, φ factors as $A \to C' \to C$ and ψ factors as $B \to C' \to C$. This proves assumption (1) of Divided Power Algebra, Lemma 23.3.3 holds. Assumption (2) is clear as limits in the category of divided power rings commute with the forgetful functor $(C, K, \delta) \mapsto (C, K)$, see Divided Power Algebra, Lemma 23.3.2 and its proof. □

07H9 **Definition 60.2.2.** Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \bar{J}, \bar{\gamma})$ constructed in Lemma 60.2.1 is called the *divided power envelope of J in B relative to (A, I, γ)* and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ is

$$\begin{array}{c}\text{ring maps } B \to C \\ \text{which map } J \text{ into } K\end{array} \longleftrightarrow \begin{array}{c}\text{divided power homomorphisms} \\ (D, \bar{J}, \bar{\gamma}) \to (C, K, \delta)\end{array}$$

and the correspondence is given by precomposing with the map $B \to D$ which corresponds to id_D. Here are some properties of $(D, \bar{J}, \bar{\gamma})$ which follow directly from the universal property. There are A-algebra maps

07HA (60.2.2.1) $$B \longrightarrow D \longrightarrow B/J$$

The first arrow maps J into $\bar{J}$ and $\bar{J}$ is the kernel of the second arrow. The elements $\bar{\gamma}_n(x)$ where $n > 0$ and x is an element in the image of $J \to D$ generate $\bar{J}$ as an ideal in D and generate D as a B-algebra.

07HB **Lemma 60.2.3.** *Let (A, I, γ) be a divided power ring. Let $\varphi : B' \to B$ be a surjection of A-algebras with kernel K. Let $IB \subset J \subset B$ be an ideal. Let $J' \subset B'$ be the inverse image of J. Write $D_{B',\gamma}(J') = (D', \bar{J}', \bar{\gamma})$. Then $D_{B,\gamma}(J) = (D'/K', \bar{J}'/K', \bar{\gamma})$ where K' is the ideal generated by the elements $\bar{\gamma}_n(k)$ for $n \geq 1$ and $k \in K$.*

Proof. Write $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. The universal property of D' gives us a homomorphism $D' \to D$ of divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that $D' \to D$ is surjective (see remarks above). It is clear that $\bar{\gamma}_n(k)$ is in the kernel for $n \geq 1$ and $k \in K$, i.e., we obtain a homomorphism $D'/K' \to D$. Conversely, there exists a divided power structure on $\bar{J}'/K' \subset D'/K'$, see Divided Power Algebra, Lemma 23.4.3. Hence the universal property of D gives an inverse $D \to D'/K'$ and we win. □

In the situation of Definition 60.2.2 we can choose a surjection $P \to B$ where P is a polynomial algebra over A and let $J' \subset P$ be the inverse image of J. The previous lemma describes $D_{B,\gamma}(J)$ in terms of $D_{P,\gamma}(J')$. Note that γ extends to a divided power structure γ' on IP by Divided Power Algebra, Lemma 23.4.2. Hence $D_{P,\gamma}(J') = D_{P,\gamma'}(J')$ is an example of a special case of divided power envelopes we describe in the following lemma.

07HC **Lemma 60.2.4.** *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J$, $t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : (B\langle x_t \rangle, IB\langle x_t \rangle + B\langle x_t \rangle_+, \delta) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all

$$\delta_n\left(\sum r_t x_t - r_0\right)$$

whenever $\sum r_t f_t = r_0$ in B for some $r_t \in B$, $r_0 \in I$.

Proof. In the statement of the lemma we think of $B\langle x_t\rangle$ as a divided power ring with ideal $J' = IB\langle x_t\rangle + B\langle x_t\rangle_+$, see Divided Power Algebra, Remark 23.5.2. The existence of Ψ follows from the universal property of divided power polynomial rings. Surjectivity of Ψ follows from the fact that its image is a divided power subring of D, hence equal to D by the universal property of D. It is clear that $x_t - f_t$ is in the kernel. Set

$$\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus\nolimits_{t\in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$$

If $(r_0, r_t) \in \mathcal{R}$ then it is clear that $\sum r_t x_t - r_0$ is in the kernel. As Ψ is a homomorphism of divided power rings and $\sum r_t x_t - r_0 \in J'$ it follows that $\delta_n(\sum r_t x_t - r_0)$ is in the kernel as well. Let $K \subset B\langle x_t\rangle$ be the ideal generated by $x_t - f_t$ and the elements $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. To show that $K = \mathrm{Ker}(\Psi)$ it suffices to show that δ extends to $B\langle x_t\rangle/K$. Namely, if so the universal property of D gives a map $D \to B\langle x_t\rangle/K$ inverse to Ψ. Hence we have to show that $K \cap J'$ is preserved by δ_n, see Divided Power Algebra, Lemma 23.4.3. Let $K' \subset B\langle x_t\rangle$ be the ideal generated by the elements

(1) $\delta_m(\sum r_t x_t - r_0)$ where $m > 0$ and $(r_0, r_t) \in \mathcal{R}$,
(2) $x_{t'}^{[m]}(x_t - f_t)$ where $m > 0$ and $t', t \in I$.

We claim that $K' = K \cap J'$. The claim proves that $K \cap J'$ is preserved by δ_n, $n > 0$ by the criterion of Divided Power Algebra, Lemma 23.4.3 (2)(c) and a computation of δ_n of the elements listed which we leave to the reader. To prove the claim note that $K' \subset K \cap J'$. Conversely, if $h \in K \cap J'$ then, modulo K' we can write

$$h = \sum r_t(x_t - f_t)$$

for some $r_t \in B$. As $h \in K \cap J' \subset J'$ we see that $r_0 = \sum r_t f_t \in I$. Hence $(r_0, r_t) \in \mathcal{R}$ and we see that

$$h = \sum r_t x_t - r_0$$

is in K' as desired. □

07KE **Lemma 60.2.5.** *Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then*

$$D_{B[x_i],\gamma}(JB[x_i] + (x_i)) = D_{B,\gamma}(J)\langle x_i\rangle$$

Proof. One possible proof is to deduce this from Lemma 60.2.4 as any relation between x_i in $B[x_i]$ is trivial. On the other hand, the lemma follows from the universal property of the divided power polynomial algebra and the universal property of divided power envelopes. □

Conditions (1) and (2) of the following lemma hold if $B \to B'$ is flat at all primes of $V(IB') \subset \mathrm{Spec}(B')$ and is very closely related to that condition, see Algebra, Lemma 10.99.8. It in particular says that taking the divided power envelope commutes with localization.

07HD **Lemma 60.2.6.** *Let (A, I, γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that*

(1) *$B/IB \to B'/IB'$ is flat, and*
(2) *$Tor_1^B(B', B/IB) = 0$.*

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_B(J) \otimes_B B' \longrightarrow D_{B'}(JB')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a B-algebra map $D \to D_{B'}(JB')$, whence a map as in the lemma. It suffices to show that the divided powers $\bar{\gamma}$ extend to $D \otimes_B B'$ since then the universal property of $D_{B'}(JB')$ will produce a map $D_{B'}(JB') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose a surjection $P \to B'$ where P is a polynomial algebra over B. In particular $B \to P$ is flat, hence $D \to D \otimes_B P$ is flat by Algebra, Lemma 10.39.7. Then $\bar{\gamma}$ extends to $D \otimes_B P$ by Divided Power Algebra, Lemma 23.4.2; we will denote this extension $\bar{\gamma}$ also. Set $\mathfrak{a} = \mathrm{Ker}(P \to B')$ so that we have the short exact sequence

$$0 \to \mathfrak{a} \to P \to B' \to 0$$

Thus $\mathrm{Tor}_1^B(B', B/IB) = 0$ implies that $\mathfrak{a} \cap IP = I\mathfrak{a}$. Now we have the following commutative diagram

$$\begin{array}{ccccc}
B/J \otimes_B \mathfrak{a} & \xrightarrow[\beta]{} & B/J \otimes_B P & \longrightarrow & B/J \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
D \otimes_B \mathfrak{a} & \xrightarrow{\alpha} & D \otimes_B P & \longrightarrow & D \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
\bar{J} \otimes_B \mathfrak{a} & \longrightarrow & \bar{J} \otimes_B P & \longrightarrow & \bar{J} \otimes_B B'
\end{array}$$

This diagram is exact even with 0's added at the top and the right. We have to show the divided powers on the ideal $\bar{J} \otimes_B P$ preserve the ideal $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$, see Divided Power Algebra, Lemma 23.4.3. Consider the exact sequence

$$0 \to \mathfrak{a}/I\mathfrak{a} \to P/IP \to B'/IB' \to 0$$

(which uses that $\mathfrak{a} \cap IP = I\mathfrak{a}$ as seen above). As B'/IB' is flat over B/IB this sequence remains exact after applying $B/J \otimes_{B/IB} -$, see Algebra, Lemma 10.39.12. Hence

$$\mathrm{Ker}(B/J \otimes_{B/IB} \mathfrak{a}/I\mathfrak{a} \to B/J \otimes_{B/IB} P/IP) = \mathrm{Ker}(\mathfrak{a}/J\mathfrak{a} \to P/JP)$$

is zero. Thus β is injective. It follows that $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$ is the image of $\bar{J} \otimes \mathfrak{a}$. Now if $f \in \bar{J}$ and $a \in \mathfrak{a}$, then $\bar{\gamma}_n(f \otimes a) = \bar{\gamma}_n(f) \otimes a^n$ hence the result is clear. □

The following lemma is a special case of [**dJ95**, Proposition 2.1.7] which in turn is a generalization of [**Ber74**, Proposition 2.8.2].

07HE **Lemma 60.2.7.** *Let $(B, I, \gamma) \to (B', I', \gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume*

(1) *$B/I \to B'/I'$ is flat, and*
(2) *$J' = JB' + I'$.*

Then the canonical map

$$D_{B,\gamma}(J) \otimes_B B' \longrightarrow D_{B',\gamma'}(J')$$

is an isomorphism.

Proof. Set $D = D_{B,\gamma}(J)$. Choose elements $f_t \in J$ which generate J/I. Set $\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t\in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$ as in the proof of Lemma 60.2.4. This lemma shows that

$$D = B\langle x_t\rangle/K$$

where K is generated by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. Thus we see that

07HF (60.2.7.1)
$$D \otimes_B B' = B'\langle x_t\rangle/K'$$

where K' is generated by the images in $B'\langle x_t\rangle$ of the generators of K listed above. Let $f'_t \in B'$ be the image of f_t. By assumption (1) we see that the elements $f'_t \in J'$ generate J'/I' and we see that $x_t - f'_t \in K'$. Set

$$\mathcal{R}' = \{(r'_0, r'_t) \in I' \oplus \bigoplus\nolimits_{t\in T} B' \mid \sum r'_t f'_t = r'_0 \text{ in } B'\}$$

To finish the proof we have to show that $\delta'_n(\sum r'_t x_t - r'_0) \in K'$ for $(r'_0, r'_t) \in \mathcal{R}'$, because then the presentation (60.2.7.1) of $D \otimes_B B'$ is identical to the presentation of $D_{B',\gamma'}(J')$ obtain in Lemma 60.2.4 from the generators f'_t. Suppose that $(r'_0, r'_t) \in \mathcal{R}'$. Then $\sum r'_t f'_t = 0$ in B'/I'. As $B/I \to B'/I'$ is flat by assumption (1) we can apply the equational criterion of flatness (Algebra, Lemma 10.39.11) to see that there exist an $m > 0$ and $r_{jt} \in B$ and $c_j \in B'$, $j = 1, \ldots, m$ such that

$$r_{j0} = \sum\nolimits_t r_{jt} f_t \in I \text{ for } j = 1, \ldots, m$$

and

$$i'_t = r'_t - \sum\nolimits_j c_j r_{jt} \in I' \text{ for all } t$$

Note that this also implies that $r'_0 = \sum_t i'_t f_t + \sum_j c_j r_{j0}$. Then we have

$$\begin{aligned}\delta'_n(\sum\nolimits_t r'_t x_t - r'_0) &= \delta'_n(\sum\nolimits_t i'_t x_t + \sum\nolimits_{t,j} c_j r_{jt} x_t - \sum\nolimits_t i'_t f_t - \sum\nolimits_j c_j r_{j0}) \\ &= \delta'_n(\sum\nolimits_t i'_t (x_t - f_t) + \sum\nolimits_j c_j (\sum\nolimits_t r_{jt} x_t - r_{j0}))\end{aligned}$$

Since $\delta_n(a+b) = \sum_{m=0,\ldots,n} \delta_m(a)\delta_{n-m}(b)$ and since $\delta_m(\sum i'_t(x_t - f_t))$ is in the ideal generated by $x_t - f_t \in K'$ for $m > 0$, it suffices to prove that $\delta_n(\sum c_j(\sum r_{jt}x_t - r_{j0}))$ is in K'. For this we use

$$\delta_n(\sum\nolimits_j c_j(\sum\nolimits_t r_{jt}x_t - r_{j0})) = \sum c_1^{n_1} \ldots c_m^{n_m} \delta_{n_1}(\sum r_{1t}x_t - r_{10}) \ldots \delta_{n_m}(\sum r_{mt}x_t - r_{m0})$$

where the sum is over $n_1 + \ldots + n_m = n$. This proves what we want. □

60.3. Some explicit divided power thickenings

07HG The constructions in this section will help us to define the connection on a crystal in modules on the crystalline site.

07HH **Lemma 60.3.1.** *Let (A, I, γ) be a divided power ring. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero and set $J = I \oplus M$. Set*

$$\delta_n(x + z) = \gamma_n(x) + \gamma_{n-1}(x)z$$

for $x \in I$ and $z \in M$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. We have to check conditions (1) – (5) of Divided Power Algebra, Definition 23.2.1. We will prove this directly for this case, but please see the proof of the next lemma for a method which avoids calculations. Conditions (1) and (3) are clear. Condition (2) follows from

$$\begin{aligned}\delta_n(x+z)\delta_m(x+z) &= (\gamma_n(x)+\gamma_{n-1}(x)z)(\gamma_m(x)+\gamma_{m-1}(x)z) \\ &= \gamma_n(x)\gamma_m(x)+\gamma_n(x)\gamma_{m-1}(x)z+\gamma_{n-1}(x)\gamma_m(x)z \\ &= \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)+\left(\frac{(n+m-1)!}{n!(m-1)!}+\frac{(n+m-1)!}{(n-1)!m!}\right)\gamma_{n+m-1}(x)z \\ &= \frac{(n+m)!}{n!m!}\delta_{n+m}(x+z)\end{aligned}$$

Condition (5) follows from

$$\begin{aligned}\delta_n(\delta_m(x+z)) &= \delta_n(\gamma_m(x)+\gamma_{m-1}(x)z) \\ &= \gamma_n(\gamma_m(x))+\gamma_{n-1}(\gamma_m(x))\gamma_{m-1}(x)z \\ &= \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)+\frac{((n-1)m)!}{(n-1)!(m!)^{n-1}}\gamma_{(n-1)m}(x)\gamma_{m-1}(x)z \\ &= \frac{(nm)!}{n!(m!)^n}(\gamma_{nm}(x)+\gamma_{nm-1}(x)z)\end{aligned}$$

by elementary number theory. To prove (4) we have to see that

$$\delta_n(x+x'+z+z') = \gamma_n(x+x')+\gamma_{n-1}(x+x')(z+z')$$

is equal to

$$\sum\nolimits_{i=0}^{n}(\gamma_i(x)+\gamma_{i-1}(x)z)(\gamma_{n-i}(x')+\gamma_{n-i-1}(x')z')$$

This follows easily on collecting the coefficients of 1, z, and z' and using condition (4) for γ. □

07HI **Lemma 60.3.2.** *Let (A, I, γ) be a divided power ring. Let M, N be A-modules. Let $q : M \times M \to N$ be an A-bilinear map. Let $B = A \oplus M \oplus N$ as an A-algebra with multiplication*

$$(x, z, w) \cdot (x', z', w') = (xx', xz' + x'z, xw' + x'w + q(z, z') + q(z', z))$$

and set $J = I \oplus M \oplus N$. Set

$$\delta_n(x, z, w) = (\gamma_n(x), \gamma_{n-1}(x)z, \gamma_{n-1}(x)w + \gamma_{n-2}(x)q(z, z))$$

for $(x, z, w) \in J$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. Suppose we want to prove that property (4) of Divided Power Algebra, Definition 23.2.1 is satisfied. Pick (x, z, w) and (x', z', w') in J. Pick a map

$$A_0 = \mathbf{Z}\langle s, s'\rangle \longrightarrow A, \quad s \longmapsto x, s' \longmapsto x'$$

which is possible by the universal property of divided power polynomial rings. Set $M_0 = A_0 \oplus A_0$ and $N_0 = A_0 \oplus A_0 \oplus M_0 \otimes_{A_0} M_0$. Let $q_0 : M_0 \times M_0 \to N_0$ be the obvious map. Define $M_0 \to M$ as the A_0-linear map which sends the basis vectors of M_0 to z and z'. Define $N_0 \to N$ as the A_0 linear map which sends the first two basis vectors of N_0 to w and w' and uses $M_0 \otimes_{A_0} M_0 \to M \otimes_A M \xrightarrow{q} N$ on the last summand. Then we see that it suffices to prove the identity (4) for the situation (A_0, M_0, N_0, q_0). Similarly for the other identities. This reduces us to the case of a

Z-torsion free ring A and A-torsion free modules. In this case all we have to do is show that

$$n!\delta_n(x, z, w) = (x, z, w)^n$$

in the ring A, see Divided Power Algebra, Lemma 23.2.2. To see this note that

$$(x, z, w)^2 = (x^2, 2xz, 2xw + 2q(z, z))$$

and by induction

$$(x, z, w)^n = (x^n, nx^{n-1}z, nx^{n-1}w + n(n-1)x^{n-2}q(z, z))$$

On the other hand,

$$n!\delta_n(x, z, w) = (n!\gamma_n(x), n!\gamma_{n-1}(x)z, n!\gamma_{n-1}(x)w + n!\gamma_{n-2}(x)q(z, z))$$

which matches. This finishes the proof. □

60.4. Compatibility

07HJ This section isn't required reading; it explains how our discussion fits with that of [**Ber74**]. Consider the following technical notion.

07HK **Definition 60.4.1.** Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say δ *is compatible with* γ if there exists a divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

Let p be a prime number. Let (A, I, γ) be a divided power ring. Let $A \to C$ be a ring map with p nilpotent in C. Assume that γ extends to IC (see Divided Power Algebra, Lemma 23.4.2). In this situation, the (big affine) crystalline site of $\mathrm{Spec}(C)$ over $\mathrm{Spec}(A)$ as defined in [**Ber74**] is the opposite of the category of systems

$$(B, J, \delta, A \to B, C \to B/J)$$

where

(1) (B, J, δ) is a divided power ring with p nilpotent in B,
(2) δ is compatible with γ, and
(3) the diagram

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

is commutative.

The conditions "γ extends to C and δ compatible with γ" are used in [**Ber74**] to ensure that the crystalline cohomology of $\mathrm{Spec}(C)$ is the same as the crystalline cohomology of $\mathrm{Spec}(C/IC)$. We will avoid this issue by working exclusively with C such that $IC = 0$[1]. In this case, for a system $(B, J, \delta, A \to B, C \to B/J)$ as above, the commutativity of the displayed diagram above implies $IB \subset J$ and compatibility is equivalent to the condition that $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings.

[1] Of course there will be a price to pay.

60.5. Affine crystalline site

07HL In this section we discuss the algebraic variant of the crystalline site. Our basic situation in which we discuss this material will be as follows.

07MD **Situation 60.5.1.** Here p is a prime number, (A, I, γ) is a divided power ring such that A is a $\mathbf{Z}_{(p)}$-algebra, and $A \to C$ is a ring map such that $IC = 0$ and such that p is nilpotent in C.

Usually the prime number p will be contained in the divided power ideal I.

07HM **Definition 60.5.2.** In Situation 60.5.1.

(1) A *divided power thickening* of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map $C \to B/J$ such that

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ & & C \\ & & \uparrow \\ A & \longrightarrow & A/I \end{array}$$

is commutative.

(2) A *homomorphism of divided power thickenings*

$$(B, J, \delta, C \to B/J) \longrightarrow (B', J', \delta', C \to B'/J')$$

is a homomorphism $\varphi : B \to B'$ of divided power A-algebras such that $C \to B/J \to B'/J'$ is the given map $C \to B'/J'$.

(3) We denote $\mathrm{CRIS}(C/A, I, \gamma)$ or simply $\mathrm{CRIS}(C/A)$ the category of divided power thickenings of C over (A, I, γ).

(4) We denote $\mathrm{Cris}(C/A, I, \gamma)$ or simply $\mathrm{Cris}(C/A)$ the full subcategory consisting of $(B, J, \delta, C \to B/J)$ such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = \mathrm{Ker}(B \to C)$ being understood.

Note that for a divided power thickening (B, J, δ) as above the ideal J is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. There is a canonical functor

07KF (60.5.2.1) $$\mathrm{CRIS}(C/A) \longrightarrow C\text{-algebras}, \quad (B, J, \delta) \longmapsto B/J$$

This category does not have equalizers or fibre products in general. It also doesn't have an initial object (= empty colimit) in general.

07HN **Lemma 60.5.3.** *In Situation 60.5.1.*

(1) *CRIS(C/A) has finite products (but not infinite ones),*
(2) *CRIS(C/A) has all finite nonempty colimits and (60.5.2.1) commutes with these, and*
(3) *Cris(C/A) has all finite nonempty colimits and Cris$(C/A) \to$ CRIS(C/A) commutes with them.*

Proof. The empty product, i.e., the final object in the category of divided power thickenings of C over (A, I, γ), is the zero ring viewed as an A-algebra endowed with the zero ideal and the unique divided powers on the zero ideal and finally endowed with the unique homomorphism of C to the zero ring. If $(B_t, J_t, \delta_t)_{t\in T}$ is a family of objects of $\mathrm{CRIS}(C/A)$ then we can form the product $(\prod_t B_t, \prod_t J_t, \prod_t \delta_t)$ as in Divided Power Algebra, Lemma 23.3.2. The map $C \to \prod B_t/\prod J_t = \prod B_t/J_t$ is clear. However, we are only guaranteed that p is nilpotent in $\prod_t B_t$ if T is finite.

Given two objects (B, J, γ) and (B', J', γ') of $\mathrm{CRIS}(C/A)$ we can form a cocartesian diagram

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

in the category of divided power rings. Then we see that we have

$$B''/J'' = B/J \otimes_{A/I} B'/J' \longleftarrow C \otimes_{A/I} C$$

see Divided Power Algebra, Remark 23.3.5. Denote $J'' \subset K \subset B''$ the ideal such that

$$\begin{array}{ccc} B''/J'' & \longrightarrow & B''/K \\ \uparrow & & \uparrow \\ C \otimes_{A/I} C & \longrightarrow & C \end{array}$$

is a pushout, i.e., $B''/K \cong B/J \otimes_C B'/J'$. Let $D_{B''}(K) = (D, \bar{K}, \bar{\delta})$ be the divided power envelope of K in B'' relative to (B'', J'', δ''). Then it is easily verified that $(D, \bar{K}, \bar{\delta})$ is a coproduct of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

Next, we come to coequalizers. Let $\alpha, \beta : (B, J, \delta) \to (B', J', \delta')$ be morphisms of $\mathrm{CRIS}(C/A)$. Consider $B'' = B'/(\alpha(b) - \beta(b))$. Let $J'' \subset B''$ be the image of J'. Let $D_{B''}(J'') = (D, \bar{J}, \bar{\delta})$ be the divided power envelope of J'' in B'' relative to (B', J', δ'). Then it is easily verified that $(D, \bar{J}, \bar{\delta})$ is the coequalizer of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

By Categories, Lemma 4.18.6 we have all finite nonempty colimits in $\mathrm{CRIS}(C/A)$. The constructions above shows that (60.5.2.1) commutes with them. This formally implies part (3) as $\mathrm{Cris}(C/A)$ is the fibre category of (60.5.2.1) over C. □

07KH **Remark 60.5.4.** In Situation 60.5.1 we denote $\mathrm{Cris}^\wedge(C/A)$ the category whose objects are pairs $(B \to C, \delta)$ such that

(1) B is a p-adically complete A-algebra,
(2) $B \to C$ is a surjection of A-algebras,
(3) δ is a divided power structure on $\mathrm{Ker}(B \to C)$,
(4) $A \to B$ is a homomorphism of divided power rings.

Morphisms are defined as in Definition 60.5.2. Then $\mathrm{Cris}(C/A) \subset \mathrm{Cris}^\wedge(C/A)$ is the full subcategory consisting of those B such that p is nilpotent in B. Conversely, any object $(B \to C, \delta)$ of $\mathrm{Cris}^\wedge(C/A)$ is equal to the limit

$$(B \to C, \delta) = \lim_e (B/p^e B \to C, \delta)$$

where for $e \gg 0$ the object $(B/p^eB \to C, \delta)$ lies in $\mathrm{Cris}(C/A)$, see Divided Power Algebra, Lemma 23.4.5. In particular, we see that $\mathrm{Cris}^\wedge(C/A)$ is a full subcategory of the category of pro-objects of $\mathrm{Cris}(C/A)$, see Categories, Remark 4.22.5.

07KG **Lemma 60.5.5.** *In Situation 60.5.1. Let $P \to C$ be a surjection of A-algebras with kernel J. Write $D_{P,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. Let $(D^\wedge, J^\wedge, \bar{\gamma}^\wedge)$ be the p-adic completion of D, see Divided Power Algebra, Lemma 23.4.5. For every $e \geq 1$ set $P_e = P/p^eP$ and $J_e \subset P_e$ the image of J and write $D_{P_e,\gamma}(J_e) = (D_e, \bar{J}_e, \bar{\gamma})$. Then for all e large enough we have*

(1) *$p^eD \subset \bar{J}$ and $p^eD^\wedge \subset \bar{J}^\wedge$ are preserved by divided powers,*
(2) *$D^\wedge/p^eD^\wedge = D/p^eD = D_e$ as divided power rings,*
(3) *$(D_e, \bar{J}_e, \bar{\gamma})$ is an object of Cris(C/A),*
(4) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is equal to $\lim_e (D_e, \bar{J}_e, \bar{\gamma})$, and*
(5) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is an object of Cris$^\wedge(C/A)$.*

Proof. Part (1) follows from Divided Power Algebra, Lemma 23.4.5. It is a general property of p-adic completion that $D/p^eD = D^\wedge/p^eD^\wedge$. Since D/p^eD is a divided power ring and since $P \to D/p^eD$ factors through P_e, the universal property of D_e produces a map $D_e \to D/p^eD$. Conversely, the universal property of D produces a map $D \to D_e$ which factors through D/p^eD. We omit the verification that these maps are mutually inverse. This proves (2). If e is large enough, then $p^eC = 0$, hence we see (3) holds. Part (4) follows from Divided Power Algebra, Lemma 23.4.5. Part (5) is clear from the definitions. □

07HP **Lemma 60.5.6.** *In Situation 60.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. With $(D_e, \bar{J}_e, \bar{\gamma})$ as in Lemma 60.5.5: for every object (B, J_B, δ) of CRIS(C/A) there exists an e and a morphism $D_e \to B$ of CRIS(C/A).*

Proof. We can find an A-algebra homomorphism $P \to B$ lifting the map $C \to B/J_B$. By our definition of $\mathrm{CRIS}(C/A)$ we see that $p^eB = 0$ for some e hence $P \to B$ factors as $P \to P_e \to B$. By the universal property of the divided power envelope we conclude that $P_e \to B$ factors through D_e. □

07KI **Lemma 60.5.7.** *In Situation 60.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. Let $(D, \bar{J}, \bar{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. For every object $(B \to C, \delta)$ of Cris$^\wedge(C/A)$ there exists a morphism $D \to B$ of Cris$^\wedge(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ compatible with maps to C. By our definition of $\mathrm{Cris}^\wedge(C/A)$ we see that $P \to B$ factors as $P \to D_{P,\gamma}(J) \to B$. As B is p-adically complete we can factor this map through D. □

60.6. Module of differentials

07HQ In this section we develop a theory of modules of differentials for divided power rings.

07HR **Definition 60.6.1.** Let A be a ring. Let (B, J, δ) be a divided power ring. Let $A \to B$ be a ring map. Let M be an B-module. A *divided power A-derivation* into

M is a map $\theta : B \to M$ which is additive, annihilates the elements of A, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and satisfies

$$\theta(\delta_n(x)) = \delta_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and all $x \in J$.

In the situation of the definition, just as in the case of usual derivations, there exists a *universal divided power A-derivation*

$$\mathrm{d}_{B/A,\delta} : B \to \Omega_{B/A,\delta}$$

such that any divided power A-derivation $\theta : B \to M$ is equal to $\theta = \xi \circ d_{B/A,\delta}$ for some unique B-linear map $\xi : \Omega_{B/A,\delta} \to M$. If $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then we can forget the divided powers on A and consider the divided power derivations of B over A. Here are some basic properties of the universal module of (divided power) differentials.

07HS **Lemma 60.6.2.** *Let A be a ring. Let (B, J, δ) be a divided power ring and $A \to B$ a ring map.*

(1) *Consider $B[x]$ with divided power ideal $(JB[x], \delta')$ where δ' is the extension of δ to $B[x]$. Then*
$$\Omega_{B[x]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[x] \oplus B[x]dx.$$
(2) *Consider $B\langle x\rangle$ with divided power ideal $(JB\langle x\rangle + B\langle x\rangle_+, \delta')$. Then*
$$\Omega_{B\langle x\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x\rangle \oplus B\langle x\rangle dx.$$
(3) *Let $K \subset J$ be an ideal preserved by δ_n for all $n > 0$. Set $B' = B/K$ and denote δ' the induced divided power on J/K. Then $\Omega_{B'/A,\delta'}$ is the quotient of $\Omega_{B/A,\delta} \otimes_B B'$ by the B'-submodule generated by dk for $k \in K$.*

Proof. These are proved directly from the construction of $\Omega_{B/A,\delta}$ as the free B-module on the elements $\mathrm{d}b$ modulo the relations

(1) $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$, $b, b' \in B$,
(2) $\mathrm{d}a = 0$, $a \in A$,
(3) $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$, $b, b' \in B$,
(4) $\mathrm{d}\delta_n(f) = \delta_{n-1}(f)\mathrm{d}f$, $f \in J$, $n > 1$.

Note that the last relation explains why we get "the same" answer for the divided power polynomial algebra and the usual polynomial algebra: in the first case x is an element of the divided power ideal and hence $\mathrm{d}x^{[n]} = x^{[n-1]}\mathrm{d}x$. □

Let (A, I, γ) be a divided power ring. In this setting the correct version of the powers of I is given by the divided powers

$$I^{[n]} = \text{ideal generated by } \gamma_{e_1}(x_1)\ldots\gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Of course we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

07HT **Lemma 60.6.3.** *Let $(A, I, \gamma) \to (B, J, \delta)$ be a homomorphism of divided power rings. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself over (A, I, γ), i.e., such that*

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B(1), J(1), \delta(1)) \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B, J, \delta) \end{array}$$

is cocartesian. Denote $K = \mathrm{Ker}(B(1) \to B)$. *Then* $K \cap J(1) \subset J(1)$ *is preserved by the divided power structure and*

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. The fact that $K \cap J(1) \subset J(1)$ is preserved by the divided power structure follows from the fact that $B(1) \to B$ is a homomorphism of divided power rings.

Recall that K/K^2 has a canonical B-module structure. Denote $s_0, s_1 : B \to B(1)$ the two coprojections and consider the map $\mathrm{d} : B \to K/(K^2 + (K \cap J(1))^{[2]})$ given by $b \mapsto s_1(b) - s_0(b)$. It is clear that d is additive, annihilates A, and satisfies the Leibniz rule. We claim that d is a divided power A-derivation. Let $x \in J$. Set $y = s_1(x)$ and $z = s_0(x)$. We will use δ instead of $\delta(1)$ for the divided power structure on $J(1)$. We have to show that $\delta_n(y) - \delta_n(z) = \delta_{n-1}(y)(y - z)$ modulo $K^2 + (K \cap J(1))^{[2]}$ for $n \geq 1$. The equality holds for $n = 1$. Assume $n > 1$. Note that $\delta_i(y - z)$ lies in $(K \cap J(1))^{[2]}$ for $i > 1$. Calculating modulo $K^2 + (K \cap J(1))^{[2]}$ we have

$$\delta_n(z) = \delta_n(z - y + y) = \sum\nolimits_{i=0}^{n} \delta_i(z - y)\delta_{n-i}(y) = \delta_{n-1}(y)\delta_1(z - y) + \delta_n(y)$$

This proves the desired equality.

Let M be a B-module. Let $\theta : B \to M$ be a divided power A-derivation. Set $D = B \oplus M$ where M is an ideal of square zero. Define a divided power structure on $J \oplus M \subset D$ by setting $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$ for $n > 1$, see Lemma 60.3.1. There are two divided power algebra homomorphisms $B \to D$: the first is given by the inclusion and the second by the map $b \mapsto b + \theta(b)$. Hence we get a canonical homomorphism $B(1) \to D$ of divided power algebras over (A, I, γ). This induces a map $K \to M$ which annihilates K^2 (as M is an ideal of square zero) and $(K \cap J(1))^{[2]}$ as $M^{[2]} = 0$. The composition $B \to K/K^2 + (K \cap J(1))^{[2]} \to M$ equals θ by construction. It follows that d is a universal divided power A-derivation and we win. □

07HU **Remark 60.6.4.** Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. The universal module $\Omega_{B/A,\delta}$ comes with a little bit of extra structure, namely the B-submodule N of $\Omega_{B/A,\delta}$ generated by $\mathrm{d}_{B/A,\delta}(J)$. In terms of the isomorphism given in Lemma 60.6.3 this corresponds to the image of $K \cap J(1)$ in $\Omega_{B/A,\delta}$. Consider the A-algebra $D = B \oplus \Omega^1_{B/A,\delta}$ with ideal $\bar{J} = J \oplus N$ and divided powers $\bar{\delta}$ as in the proof of the lemma. Then $(D, \bar{J}, \bar{\delta})$ is a divided power ring and the two maps $B \to D$ given by $b \mapsto b$ and $b \mapsto b + \mathrm{d}_{B/A,\delta}(b)$ are homomorphisms of divided power rings over A. Moreover, N is the smallest submodule of $\Omega_{B/A,\delta}$ such that this is true.

07HV **Lemma 60.6.5.** *In Situation 60.5.1. Let* (B, J, δ) *be an object of CRIS*(C/A). *Let* $(B(1), J(1), \delta(1))$ *be the coproduct of* (B, J, δ) *with itself in CRIS*(C/A). *Denote* $K = \mathrm{Ker}(B(1) \to B)$. *Then* $K \cap J(1) \subset J(1)$ *is preserved by the divided power structure and*

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. Word for word the same as the proof of Lemma 60.6.3. The only point that has to be checked is that the divided power ring $D = B \oplus M$ is an object of $\mathrm{CRIS}(C/A)$ and that the two maps $B \to D$ are morphisms of $\mathrm{CRIS}(C/A)$. Since $D/(J \oplus M) = B/J$ we can use $C \to B/J$ to view D as an object of $\mathrm{CRIS}(C/A)$ and the statement on morphisms is clear from the construction. □

07HW **Lemma 60.6.6.** *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map and let $IB \subset J \subset B$ be an ideal. Let $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ be the divided power envelope. Then we have*

$$\Omega_{D/A,\bar{\gamma}} = \Omega_{B/A} \otimes_B D$$

First proof. Let M be a D-module. We claim that an A-derivation $\vartheta : B \to M$ is the same thing as a divided power A-derivation $\theta : D \to M$. The claim implies the statement by the Yoneda lemma.

Consider the square zero thickening $D \oplus M$ of D. There is a divided power structure δ on $\bar{J} \oplus M$ if we set the higher divided power operations zero on M. In other words, we set $\delta_n(x+m) = \bar{\gamma}_n(x) + \bar{\gamma}_{n-1}(x)m$ for any $x \in \bar{J}$ and $m \in M$, see Lemma 60.3.1. Consider the A-algebra map $B \to D \oplus M$ whose first component is given by the map $B \to D$ and whose second component is ϑ. By the universal property we get a corresponding homomorphism $D \to D \oplus M$ of divided power algebras whose second component is the divided power A-derivation θ corresponding to ϑ. □

Second proof. We will prove this first when B is flat over A. In this case γ extends to a divided power structure γ' on IB, see Divided Power Algebra, Lemma 23.4.2. Hence $D = D_{B,\gamma'}(J)$ is equal to a quotient of the divided power ring (D', J', δ) where $D' = B\langle x_t \rangle$ and $J' = IB\langle x_t \rangle + B\langle x_t \rangle_+$ by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$, see Lemma 60.2.4 for notation and explanation. Write $\mathrm{d} : D' \to \Omega_{D'/A,\delta}$ for the universal derivation. Note that

$$\Omega_{D'/A,\delta} = \Omega_{B/A} \otimes_B D' \oplus \bigoplus D' \mathrm{d}x_t,$$

see Lemma 60.6.2. We conclude that $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{D'/A,\delta} \otimes_{D'} D$ by the submodule generated by d applied to the generators of the kernel of $D' \to D$ listed above, see Lemma 60.6.2. Since $\mathrm{d}(x_t - f_t) = -\mathrm{d}f_t + \mathrm{d}x_t$ we see that we have $\mathrm{d}x_t = \mathrm{d}f_t$ in the quotient. In particular we see that $\Omega_{B/A} \otimes_B D \to \Omega_{D/A,\gamma}$ is surjective with kernel given by the images of d applied to the elements $\delta_n(\sum r_t x_t - r_0)$. However, given a relation $\sum r_t f_t - r_0 = 0$ in B with $r_t \in B$ and $r_0 \in IB$ we see that

$$\begin{aligned} \mathrm{d}\delta_n(\sum r_t x_t - r_0) &= \delta_{n-1}(\sum r_t x_t - r_0)\mathrm{d}(\sum r_t x_t - r_0) \\ &= \delta_{n-1}(\sum r_t x_t - r_0)\left(\sum r_t \mathrm{d}(x_t - f_t) + \sum (x_t - f_t)\mathrm{d}r_t\right) \end{aligned}$$

because $\sum r_t f_t - r_0 = 0$ in B. Hence this is already zero in $\Omega_{B/A} \otimes_A D$ and we win in the case that B is flat over A.

In the general case we write B as a quotient of a polynomial ring $P \to B$ and let $J' \subset P$ be the inverse image of J. Then $D = D'/K'$ with notation as in Lemma 60.2.3. By the case handled in the first paragraph of the proof we have $\Omega_{D'/A,\bar{\gamma}'} = \Omega_{P/A} \otimes_P D'$. Then $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{P/A} \otimes_P D$ by the submodule generated by $\mathrm{d}\bar{\gamma}'_n(k)$ where k is an element of the kernel of $P \to B$, see Lemma 60.6.2 and the description of K' from Lemma 60.2.3. Since $\mathrm{d}\bar{\gamma}'_n(k) = \bar{\gamma}'_{n-1}(k)\mathrm{d}k$ we see again

that it suffices to divided by the submodule generated by $\mathrm{d}k$ with $k \in \mathrm{Ker}(P \to B)$ and since $\Omega_{B/A}$ is the quotient of $\Omega_{P/A} \otimes_A B$ by these elements (Algebra, Lemma 10.131.9) we win. □

07HZ **Remark 60.6.7.** Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Set $\Omega^i_{B/A,\delta} = \wedge^i_B \Omega_{B/A,\delta}$ where $\Omega_{B/A,\delta}$ is the target of the universal divided power A-derivation $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A,\delta}$. Note that $\Omega_{B/A,\delta}$ is the quotient of $\Omega_{B/A}$ by the B-submodule generated by the elements $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ for $x \in J$. We claim Algebra, Lemma 10.132.2 applies. To see this it suffices to verify the elements $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ of Ω_B are mapped to zero in $\Omega^2_{B/A,\delta}$. We observe that

$$\mathrm{d}(\delta_{n-1}(x)) \wedge \mathrm{d}x = \delta_{n-2}(x)\mathrm{d}x \wedge \mathrm{d}x = 0$$

in $\Omega^2_{B/A,\delta}$ as desired. Hence we obtain a *divided power de Rham complex*

$$\Omega^0_{B/A,\delta} \to \Omega^1_{B/A,\delta} \to \Omega^2_{B/A,\delta} \to \ldots$$

which will play an important role in the sequel.

07I0 **Remark 60.6.8.** Let $A \to B$ be a ring map. Let $\Omega_{B/A} \to \Omega$ be a quotient satisfying the assumptions of Algebra, Lemma 10.132.2. Let M be a B-module. A *connection* is an additive map

$$\nabla : M \longrightarrow M \otimes_B \Omega$$

such that $\nabla(bm) = b\nabla(m) + m \otimes \mathrm{d}b$ for $b \in B$ and $m \in M$. In this situation we can define maps

$$\nabla : M \otimes_B \Omega^i \longrightarrow M \otimes_B \Omega^{i+1}$$

by the rule $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes \mathrm{d}\omega$. This works because if $b \in B$, then

$$\begin{aligned}\nabla(bm \otimes \omega) - \nabla(m \otimes b\omega) &= \nabla(bm) \wedge \omega + bm \otimes \mathrm{d}\omega - \nabla(m) \wedge b\omega - m \otimes \mathrm{d}(b\omega) \\ &= b\nabla(m) \wedge \omega + m \otimes \mathrm{d}b \wedge \omega + bm \otimes \mathrm{d}\omega \\ &\quad - b\nabla(m) \wedge \omega - bm \otimes \mathrm{d}(\omega) - m \otimes \mathrm{d}b \wedge \omega = 0\end{aligned}$$

As is customary we say the connection is *integrable* if and only if the composition

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2$$

is zero. In this case we obtain a complex

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2 \xrightarrow{\nabla} M \otimes_B \Omega^3 \xrightarrow{\nabla} M \otimes_B \Omega^4 \to \ldots$$

which is called the de Rham complex of the connection.

07KJ **Remark 60.6.9.** Consider a commutative diagram of rings

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

Let $\Omega_{B/A} \to \Omega$ and $\Omega_{B'/A'} \to \Omega'$ be quotients satisfying the assumptions of Algebra, Lemma 10.132.2. Assume there is a map $\varphi : \Omega \to \Omega'$ which fits into a commutative

diagram

$$\begin{array}{ccc} \Omega_{B/A} & \longrightarrow & \Omega_{B'/A'} \\ \downarrow & & \downarrow \\ \Omega & \xrightarrow{\varphi} & \Omega' \end{array}$$

where the top horizontal arrow is the canonical map $\Omega_{B/A} \to \Omega_{B'/A'}$ induced by $\varphi : B \to B'$. In this situation, given any pair (M, ∇) where M is a B-module and $\nabla : M \to M \otimes_B \Omega$ is a connection we obtain a *base change* $(M \otimes_B B', \nabla')$ where

$$\nabla' : M \otimes_B B' \longrightarrow (M \otimes_B B') \otimes_{B'} \Omega' = M \otimes_B \Omega'$$

is defined by the rule

$$\nabla'(m \otimes b') = \sum m_i \otimes b' \mathrm{d}\varphi(b_i) + m \otimes \mathrm{d}b'$$

if $\nabla(m) = \sum m_i \otimes \mathrm{d}b_i$. If ∇ is integrable, then so is ∇', and in this case there is a canonical map of de Rham complexes (Remark 60.6.8)

07PY (60.6.9.1) $$M \otimes_B \Omega^\bullet \longrightarrow (M \otimes_B B') \otimes_{B'} (\Omega')^\bullet = M \otimes_B (\Omega')^\bullet$$

which maps $m \otimes \eta$ to $m \otimes \varphi(\eta)$.

07KK **Lemma 60.6.10.** *Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Let p be a prime number. Assume that A is a $\mathbf{Z}_{(p)}$-algebra and that p is nilpotent in B/J. Then we have*

$$\lim_e \Omega_{B_e/A,\bar{\delta}} = \lim_e \Omega_{B/A,\delta}/p^e\Omega_{B/A,\delta} = \lim_e \Omega_{B^\wedge/A,\delta^\wedge}/p^e\Omega_{B^\wedge/A,\delta^\wedge}$$

see proof for notation and explanation.

Proof. By Divided Power Algebra, Lemma 23.4.5 we see that δ extends to $B_e = B/p^eB$ for all sufficiently large e. Hence the first limit make sense. The lemma also produces a divided power structure $\delta^\wedge$ on the completion $B^\wedge = \lim_e B_e$, hence the last limit makes sense. By Lemma 60.6.2 and the fact that $\mathrm{d}p^e = 0$ (always) we see that the surjection $\Omega_{B/A,\delta} \to \Omega_{B_e/A,\bar{\delta}}$ has kernel $p^e\Omega_{B/A,\delta}$. Similarly for the kernel of $\Omega_{B^\wedge/A,\delta^\wedge} \to \Omega_{B_e/A,\bar{\delta}}$. Hence the lemma is clear. □

60.7. Divided power schemes

07I1 Some remarks on how to globalize the previous notions.

07I2 **Definition 60.7.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A *divided power structure* γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring.

To be sure this applies in particular to sheaves of rings on topological spaces. But it's good to be a little bit more general as the structure sheaf of the crystalline site lives on a... site! A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a *divided power topos* in this chapter. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given

a morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a *morphism of divided power topoi* if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{array}{ccc} f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \\ {\scriptstyle f^{-1}\gamma'_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\ f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \end{array}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all $U' \in \mathrm{Ob}(\mathcal{C}')$.

In the case of schemes we require the divided power ideal to be **quasi-coherent**. But apart from this the definition is exactly the same as in the case of topoi. Here it is.

07I3 **Definition 60.7.2.** A *divided power scheme* is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A *morphism of divided power schemes* $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $f^{-1}\mathcal{I}'\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_{S'}(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}_S(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open.

Recall that there is a 1-to-1 correspondence between quasi-coherent sheaves of ideals and closed immersions, see Morphisms, Section 29.2. Thus given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$. In many situations we only want to consider such triples (U, T, γ) when the morphism $U \to T$ is a thickening, see More on Morphisms, Definition 37.2.1.

07I4 **Definition 60.7.3.** A triple (U, T, γ) as above is called a *divided power thickening* if $U \to T$ is a thickening.

Fibre products of divided power schemes exist when one of the three is a divided power thickening. Here is a formal statement.

07ME **Lemma 60.7.4.** *Let $f : (T, \mathcal{J}, \delta) \to (S, \mathcal{I}, \gamma)$ and $f' : (T', \mathcal{J}', \delta') \to (S, \mathcal{I}, \gamma)$ be morphisms of divided power schemes. There exists a divided power scheme $(T'', \mathcal{J}'', \delta'')$ and a cartesian diagram*

$$\begin{array}{ccc} T & \longleftarrow & T'' \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \xleftarrow{f'} & T' \end{array}$$

in the category of divided power schemes. The morphsm $T'' \to T \times_S T'$ is a closed immersion and the morphism $T''_0 \to T_0 \times_{S_0} T'_0$ is an isomorphism.

Proof. Sketch. Note that the two final statements are compatible, via Spec(−), with what we have seen for pushouts of divided power algebras in Divided Power Algebra, Remark 23.3.5. Thus we construct T'' as a closed subscheme of $T \times_S T'$ as follows: for any affine opens $U \subset S$, $V \subset T$, $V' \subset T'$ with $f(V), f'(V') \subset U$ we consider the closed subscheme of $V \times_U V'$ determined by the construction in Divided Power Algebra, Remark 23.3.5. Since the schemes $V \times_U V'$ are the members of an open covering of $T \times_S T'$ we can proceed as follows: (1) we show that these closed subschemes glue, (2) the resulting divided power structures glue, and (3) the result of glueing is the fibre product in the category of divided power schemes. To see (1) is true amounts to showing that the construction of the pushout in the category of divided power rings commutes with localization (suitably formulated); this follows from the result of Divided Power Algebra, Lemma 23.4.2 which implies that divided powers extend to localizations (by flatness). □

We make the following observation. Suppose that (U, T, γ) is a divided power scheme. Assume that T is a scheme over $\mathbf{Z}_{(p)}$ and that p is locally nilpotent on U. Then

(1) p locally nilpotent on $T \Leftrightarrow U \to T$ is a thickening (see Divided Power Algebra, Lemma 23.2.6), and
(2) $p^e\mathcal{O}_T$ is locally on T preserved by γ for $e \gg 0$ (see Divided Power Algebra, Lemma 23.4.5).

This suggest that good results on divided power thickenings will be available under the following hypotheses.

07MF **Situation 60.7.5.** Here p is a prime number and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbf{Z}_{(p)}$. We set $S_0 = V(\mathcal{I}) \subset S$. Finally, $X \to S_0$ is a morphism of schemes such that p is locally nilpotent on X.

It is in this situation that we will define the big and small crystalline sites.

60.8. The big crystalline site

07I5 We first define the big site. Given a divided power scheme $(S, \mathcal{I}, \gamma)$ we say $(T, \mathcal{J}, \delta)$ is a divided power scheme over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes. Similarly, we say a divided power thickening (U, T, δ) is a divided power thickening over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes.

07I6 **Definition 60.8.1.** In Situation 60.7.5.

(1) A *divided power thickening of X relative to* $(S, \mathcal{I}, \gamma)$ is given by a divided power thickening (U, T, δ) over $(S, \mathcal{I}, \gamma)$ and an S-morphism $U \to X$.
(2) A *morphism of divided power thickenings of X relative to* $(S, \mathcal{I}, \gamma)$ is defined in the obvious manner.

The category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is denoted $\mathrm{CRIS}(X/S, \mathcal{I}, \gamma)$ or simply $\mathrm{CRIS}(X/S)$.

For any (U, T, δ) in $\mathrm{CRIS}(X/S)$ we have that p is locally nilpotent on T, see discussion preceding Situation 60.7.5. A good way to visualize all the data associated

to (U, T, δ) is the commutative diagram

$$\begin{array}{ccc} T & \longleftarrow & U \\ \downarrow & & \downarrow \\ & & X \\ \downarrow & & \downarrow \\ S & \longleftarrow & S_0 \end{array}$$

where $S_0 = V(\mathcal{I}) \subset S$. Morphisms of $\mathrm{CRIS}(X/S)$ can be similarly visualized as huge commutative diagrams. In particular, there is a canonical forgetful functor

07I7 (60.8.1.1) $$\mathrm{CRIS}(X/S) \longrightarrow Sch/X, \quad (U, T, \delta) \longmapsto U$$

as well as its one sided inverse (and left adjoint)

07I8 (60.8.1.2) $$Sch/X \longrightarrow \mathrm{CRIS}(X/S), \quad U \longmapsto (U, U, \emptyset)$$

which is sometimes useful.

07I9 **Lemma 60.8.2.** *In Situation 60.7.5. The category CRIS(X/S) has all finite nonempty limits, in particular products of pairs and fibre products. The functor (60.8.1.1) commutes with limits.*

Proof. Omitted. Hint: See Lemma 60.5.3 for the affine case. See also Divided Power Algebra, Remark 23.3.5. □

07IA **Lemma 60.8.3.** *In Situation 60.7.5. Let*

$$\begin{array}{ccc} (U_3, T_3, \delta_3) & \longrightarrow & (U_2, T_2, \delta_2) \\ \downarrow & & \downarrow \\ (U_1, T_1, \delta_1) & \longrightarrow & (U, T, \delta) \end{array}$$

be a fibre square in the category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$. If $T_2 \to T$ is flat and $U_2 = T_2 \times_T U$, then $T_3 = T_1 \times_T T_2$ (as schemes).

Proof. This is true because a divided power structure extends uniquely along a flat ring map. See Divided Power Algebra, Lemma 23.4.2. □

The lemma above means that the base change of a flat morphism of divided power thickenings is another flat morphism, and in fact is the "usual" base change of the morphism. This implies that the following definition makes sense.

07IB **Definition 60.8.4.** In Situation 60.7.5.

(1) A family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ of divided power thickenings of X/S is a *Zariski, étale, smooth, syntomic, or fppf covering* if and only if
   (a) $U_i = U \times_T T_i$ for all i and
   (b) $\{T_i \to T\}$ is a Zariski, étale, smooth, syntomic, or fppf covering.

(2) The *big crystalline site* of X over $(S, \mathcal{I}, \gamma)$, is the category $\mathrm{CRIS}(X/S)$ endowed with the Zariski topology.

(3) The topos of sheaves on $\mathrm{CRIS}(X/S)$ is denoted $(X/S)_{\mathrm{CRIS}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\mathrm{CRIS}}$[2].

[2]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

There are some obvious functorialities concerning these topoi.

07IC **Remark 60.8.5** (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Set $S_0 = V(\mathcal{I})$ and $S'_0 = V(\mathcal{I}')$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. Then we get a continuous and cocontinuous functor

$$\mathrm{CRIS}(X/S) \longrightarrow \mathrm{CRIS}(Y/S')$$

by letting (U, T, δ) correspond to (U, T, δ) with $U \to X \to Y$ as the S'-morphism from U to Y. Hence we get a morphism of topoi

$$f_{\mathrm{CRIS}} : (X/S)_{\mathrm{CRIS}} \longrightarrow (Y/S')_{\mathrm{CRIS}}$$

see Sites, Section 7.21.

07ID **Remark 60.8.6** (Comparison with Zariski site)**.** In Situation 60.7.5. The functor (60.8.1.1) is cocontinuous (details omitted) and commutes with products and fibred products (Lemma 60.8.2). Hence we obtain a morphism of topoi

$$U_{X/S} : (X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/X)_{Zar})$$

from the big crystalline topos of X/S to the big Zariski topos of X. See Sites, Section 7.21.

07IE **Remark 60.8.7** (Structure morphism)**.** In Situation 60.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 60.8.1 with S_0 instead of X. Hence we get a site $\mathrm{CRIS}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccccc} (X/S)_{\mathrm{CRIS}} & & \xrightarrow{f_{\mathrm{CRIS}}} & & (S_0/S)_{\mathrm{CRIS}} \\ \downarrow{\scriptstyle U_{X/S}} & & & & \downarrow{\scriptstyle U_{S_0/S}} \\ Sh((Sch/X)_{Zar}) & & \xrightarrow{f_{big}} & & Sh((Sch/S_0)_{Zar}) \\ & & & \searrow & \\ & & & & Sh((Sch/S)_{Zar}) \end{array}$$

by Remark 60.8.5. We think of the composition $(X/S)_{\mathrm{CRIS}} \to Sh((Sch/S)_{Zar})$ as the structure morphism of the big crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$(X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/S)_{Zar})$$

is defined as we can take the lower route through the diagram above. Thus it is the morphism of topoi corresponding to the cocontinuous functor $\mathrm{CRIS}(X/S) \to (Sch/S)_{Zar}$ given by the rule $(U, T, \delta)/S \mapsto U/S$, see Sites, Section 7.21.

07MG **Remark 60.8.8** (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 60.8.5 we obtain a commutative diagram of ringed topoi

$$\begin{array}{ccc} (X/S)_{\text{CRIS}} & \longrightarrow & (Y/S')_{\text{CRIS}} \\ \downarrow & & \downarrow \\ Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar}) \end{array}$$

where the vertical arrows are the structure morphisms.

60.9. The crystalline site

07IF Since (60.8.1.1) commutes with products and fibre products, we see that looking at those (U, T, δ) such that $U \to X$ is an open immersion defines a full subcategory preserved under fibre products (and more generally finite nonempty limits). Hence the following definition makes sense.

07IG **Definition 60.9.1.** In Situation 60.7.5.

(1) The (small) *crystalline site* of X over $(S, \mathcal{I}, \gamma)$, denoted $\text{Cris}(X/S, \mathcal{I}, \gamma)$ or simply $\text{Cris}(X/S)$ is the full subcategory of $\text{CRIS}(X/S)$ consisting of those (U, T, δ) in $\text{CRIS}(X/S)$ such that $U \to X$ is an open immersion. It comes endowed with the Zariski topology.
(2) The topos of sheaves on $\text{Cris}(X/S)$ is denoted $(X/S)_{\text{cris}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\text{cris}}$[3].

For any (U, T, δ) in $\text{Cris}(X/S)$ the morphism $U \to X$ defines an object of the small Zariski site X_{Zar} of X. Hence a canonical forgetful functor

07IH (60.9.1.1) $$\text{Cris}(X/S) \longrightarrow X_{Zar}, \quad (U, T, \delta) \longmapsto U$$

and a left adjoint

07II (60.9.1.2) $$X_{Zar} \longrightarrow \text{Cris}(X/S), \quad U \longmapsto (U, U, \emptyset)$$

which is sometimes useful.

We can compare the small and big crystalline sites, just like we can compare the small and big Zariski sites of a scheme, see Topologies, Lemma 34.3.14.

07IJ **Lemma 60.9.2.** *Assumptions as in Definition 60.8.1. The inclusion functor*

$$Cris(X/S) \to CRIS(X/S)$$

commutes with finite nonempty limits, is fully faithful, continuous, and cocontinuous. There are morphisms of topoi

$$(X/S)_{cris} \xrightarrow{i} (X/S)_{CRIS} \xrightarrow{\pi} (X/S)_{cris}$$

whose composition is the identity and of which the first is induced by the inclusion functor. Moreover, $\pi_* = i^{-1}$.

[3]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

Proof. For the first assertion see Lemma 60.8.2. This gives us a morphism of topoi $i : (X/S)_{\text{cris}} \to (X/S)_{\text{CRIS}}$ and a left adjoint $i_!$ such that $i^{-1}i_! = i^{-1}i_* = \text{id}$, see Sites, Lemmas 7.21.5, 7.21.6, and 7.21.7. We claim that $i_!$ is exact. If this is true, then we can define π by the rules $\pi^{-1} = i_!$ and $\pi_* = i^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_!$ is right exact and preserves fibre products (see references given). Hence it suffices to show that $i_!* = *$ where $*$ indicates the final object in the category of sheaves of sets. To see this it suffices to produce a set of objects (U_i, T_i, δ_i), $i \in I$ of $\text{Cris}(X/S)$ such that

$$\coprod\nolimits_{i \in I} h_{(U_i,T_i,\delta_i)} \to *$$

is surjective in $(X/S)_{\text{CRIS}}$ (details omitted; hint: use that $\text{Cris}(X/S)$ has products and that the functor $\text{Cris}(X/S) \to \text{CRIS}(X/S)$ commutes with them). In the affine case this follows from Lemma 60.5.6. We omit the proof in general. □

07IK **Remark 60.9.3** (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S_0' \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. By analogy with Topologies, Lemma 34.3.17 we define

$$f_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (Y/S')_{\text{cris}}$$

by the formula $f_{\text{cris}} = \pi_Y \circ f_{\text{CRIS}} \circ i_X$ where i_X and π_Y are as in Lemma 60.9.2 for X and Y and where f_{CRIS} is as in Remark 60.8.5.

07IL **Remark 60.9.4** (Comparison with Zariski site)**.** In Situation 60.7.5. The functor (60.9.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$u_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(X_{Zar})$$

relating the small crystalline topos of X/S with the small Zariski topos of X. See Sites, Section 7.21.

07KL **Lemma 60.9.5.** *In Situation 60.7.5. Let $X' \subset X$ and $S' \subset S$ be open subschemes such that X' maps into S'. Then there is a fully faithful functor $Cris(X'/S') \to Cris(X/S)$ which gives rise to a morphism of topoi fitting into the commutative diagram*

$$\begin{array}{ccc} (X'/S')_{cris} & \longrightarrow & (X/S)_{cris} \\ {\scriptstyle u_{X'/S'}}\downarrow & & \downarrow{\scriptstyle u_{X/S}} \\ Sh(X'_{Zar}) & \longrightarrow & Sh(X_{Zar}) \end{array}$$

Moreover, this diagram is an example of localization of morphisms of topoi as in Sites, Lemma 7.31.1.

Proof. The fully faithful functor comes from thinking of objects of $\text{Cris}(X'/S')$ as divided power thickenings (U, T, δ) of X where $U \to X$ factors through $X' \subset X$

(since then automatically $T \to S$ will factor through S'). This functor is clearly cocontinuous hence we obtain a morphism of topoi as indicated. Let $h_{X'} \in Sh(X_{Zar})$ be the representable sheaf associated to X' viewed as an object of X_{Zar}. It is clear that $Sh(X'_{Zar})$ is the localization $Sh(X_{Zar})/h_{X'}$. On the other hand, the category $\mathrm{Cris}(X/S)/u_{X/S}^{-1}h_{X'}$ (see Sites, Lemma 7.30.3) is canonically identified with $\mathrm{Cris}(X'/S')$ by the functor above. This finishes the proof. □

07IM **Remark 60.9.6** (Structure morphism)**.** In Situation 60.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 60.8.1 with S_0 instead of X. Hence we get a site $\mathrm{Cris}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of topoi

$$\begin{array}{ccccc} (X/S)_{\mathrm{cris}} & \xrightarrow[f_{\mathrm{cris}}]{} & (S_0/S)_{\mathrm{cris}} & & \\ \downarrow{\scriptstyle u_{X/S}} & & \downarrow{\scriptstyle u_{S_0/S}} & & \\ Sh(X_{Zar}) & \xrightarrow{f_{small}} & Sh(S_{0,Zar}) & \searrow & \\ & & & & Sh(S_{Zar}) \end{array}$$

see Remark 60.9.3. We think of the composition $(X/S)_{\mathrm{cris}} \to Sh(S_{Zar})$ as the structure morphism of the crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$\tau_{X/S} : (X/S)_{\mathrm{cris}} \longrightarrow Sh(S_{Zar})$$

is defined as we can take the lower route through the diagram above.

07MH **Remark 60.9.7** (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 60.9.3 we obtain a commutative diagram of ringed topoi

$$\begin{array}{ccc} (X/S)_{\mathrm{cris}} & \longrightarrow & (Y/S')_{\mathrm{cris}} \\ \downarrow & & \downarrow \\ Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar}) \end{array}$$

where the vertical arrows are the structure morphisms.

60.10. Sheaves on the crystalline site

07IN Notation and assumptions as in Situation 60.7.5. In order to discuss the small and big crystalline sites of X/S simultaneously in this section we let

$$\mathcal{C} = \mathrm{CRIS}(X/S) \quad \text{or} \quad \mathcal{C} = \mathrm{Cris}(X/S).$$

A sheaf $\mathcal{F}$ on $\mathcal{C}$ gives rise to a *restriction* $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$. Namely, $\mathcal{F}_T$ is the Zariski sheaf on the scheme T defined by the rule

$$\mathcal{F}_T(W) = \mathcal{F}(U \cap W, W, \delta|_W)$$

for $W \subset T$ is open. Moreover, if $f : T \to T'$ is a morphism between objects (U, T, δ) and (U', T', δ') of $\mathcal{C}$, then there is a canonical *comparison* map

07IP (60.10.0.1) $$c_f : f^{-1}\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T.$$

Namely, if $W' \subset T'$ is open then f induces a morphism

$$f|_{f^{-1}W'} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \longrightarrow (U' \cap W', W', \delta|_{W'})$$

of $\mathcal{C}$, hence we can use the restriction mapping $(f|_{f^{-1}W'})^*$ of $\mathcal{F}$ to define a map $\mathcal{F}_{T'}(W') \to \mathcal{F}_T(f^{-1}W')$. These maps are clearly compatible with further restriction, hence define an f-map from $\mathcal{F}_{T'}$ to $\mathcal{F}_T$ (see Sheaves, Section 6.21 and especially Sheaves, Definition 6.21.7). Thus a map c_f as in (60.10.0.1). Note that if f is an open immersion, then c_f is an isomorphism, because in that case $\mathcal{F}_T$ is just the restriction of $\mathcal{F}_{T'}$ to T.

Conversely, given Zariski sheaves $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$ and comparison maps c_f as above which (a) are isomorphisms for open immersions, and (b) satisfy a suitable cocycle condition, we obtain a sheaf on $\mathcal{C}$. This is proved exactly as in Topologies, Lemma 34.3.20.

The *structure sheaf* on $\mathcal{C}$ is the sheaf $\mathcal{O}_{X/S}$ defined by the rule

$$\mathcal{O}_{X/S} : (U, T, \delta) \longmapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf by the definition of coverings in $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules. In this case the comparison mappings (60.10.0.1) define a comparison map

07IQ (60.10.0.2) $$c_f : f^*\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T$$

of $\mathcal{O}_T$-modules.

Another type of example comes by starting with a sheaf $\mathcal{G}$ on $(Sch/X)_{Zar}$ or X_{Zar} (depending on whether $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$). Then $\underline{\mathcal{G}}$ defined by the rule

$$\underline{\mathcal{G}} : (U, T, \delta) \longmapsto \mathcal{G}(U)$$

is a sheaf on $\mathcal{C}$. In particular, if we take $\mathcal{G} = \mathbf{G}_a = \mathcal{O}_X$, then we obtain

$$\underline{\mathbf{G}_a} : (U, T, \delta) \longmapsto \Gamma(U, \mathcal{O}_U)$$

There is a surjective map of sheaves $\mathcal{O}_{X/S} \to \underline{\mathbf{G}_a}$ defined by the canonical maps $\Gamma(T, \mathcal{O}_T) \to \Gamma(U, \mathcal{O}_U)$ for objects (U, T, δ). The kernel of this map is denoted $\mathcal{J}_{X/S}$, hence a short exact sequence

$$0 \to \mathcal{J}_{X/S} \to \mathcal{O}_{X/S} \to \underline{\mathbf{G}_a} \to 0$$

Note that $\mathcal{J}_{X/S}$ comes equipped with a canonical divided power structure. After all, for each object (U, T, δ) the third component δ *is* a divided power structure on the kernel of $\mathcal{O}_T \to \mathcal{O}_U$. Hence the (big) crystalline topos is a divided power topos.

60.11. Crystals in modules

07IR It turns out that a crystal is a very general gadget. However, the definition may be a bit hard to parse, so we first give the definition in the case of modules on the crystalline sites.

07IS **Definition 60.11.1.** In Situation 60.7.5. Let $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathcal{C}$.

(1) We say $\mathcal{F}$ is *locally quasi-coherent* if for every object (U, T, δ) of $\mathcal{C}$ the restriction $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module.
(2) We say $\mathcal{F}$ is *quasi-coherent* if it is quasi-coherent in the sense of Modules on Sites, Definition 18.23.1.

(3) We say $\mathcal{F}$ is a *crystal in $\mathcal{O}_{X/S}$-modules* if all the comparison maps (60.10.0.2) are isomorphisms.

It turns out that we can relate these notions as follows.

07IT **Lemma 60.11.2.** *With notation $X/S, \mathcal{I}, \gamma, \mathcal{C}, \mathcal{F}$ as in Definition 60.11.1. The following are equivalent*

(1) *$\mathcal{F}$ is quasi-coherent, and*
(2) *$\mathcal{F}$ is locally quasi-coherent and a crystal in $\mathcal{O}_{X/S}$-modules.*

Proof. Assume (1). Let $f : (U', T', \delta') \to (U, T, \delta)$ be an object of $\mathcal{C}$. We have to prove (a) $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module and (b) $c_f : f^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism. The assumption means that we can find a covering $\{(T_i, U_i, \delta_i) \to (T, U, \delta)\}$ and for each i the restriction of $\mathcal{F}$ to $\mathcal{C}/(T_i, U_i, \delta_i)$ has a global presentation. Since it suffices to prove (a) and (b) Zariski locally, we may replace $f : (T', U', \delta') \to (T, U, \delta)$ by the base change to (T_i, U_i, δ_i) and assume that $\mathcal{F}$ restricted to $\mathcal{C}/(T, U, \delta)$ has a global presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \mathcal{F}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow 0$$

It is clear that this gives a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

and hence (a) holds. Moreover, the presentation restricts to T' to give a similar presentation of $\mathcal{F}_{T'}$, whence (b) holds.

Assume (2). Let (U, T, δ) be an object of $\mathcal{C}$. We have to find a covering of (U, T, δ) such that $\mathcal{F}$ has a global presentation when we restrict to the localization of $\mathcal{C}$ at the members of the covering. Thus we may assume that T is affine. In this case we can choose a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

as $\mathcal{F}_T$ is assumed to be a quasi-coherent $\mathcal{O}_T$-module. Then by the crystal property of $\mathcal{F}$ we see that this pulls back to a presentation of $\mathcal{F}_{T'}$ for any morphism $f : (U', T', \delta') \to (U, T, \delta)$ of $\mathcal{C}$. Thus the desired presentation of $\mathcal{F}|_{\mathcal{C}/(U,T,\delta)}$. □

07IU **Definition 60.11.3.** If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 60.11.2, then we say that $\mathcal{F}$ is a *crystal in quasi-coherent modules*. We say that $\mathcal{F}$ is a *crystal in finite locally free modules* if, in addition, $\mathcal{F}$ is finite locally free.

Of course, as Lemma 60.11.2 shows, this notation is somewhat heavy since a quasi-coherent module is always a crystal. But it is standard terminology in the literature.

07IV **Remark 60.11.4.** To formulate the general notion of a crystal we use the language of stacks and strongly cartesian morphisms, see Stacks, Definition 8.4.1 and Categories, Definition 4.33.1. In Situation 60.7.5 let $p : \mathcal{C} \to \mathrm{Cris}(X/S)$ be a stack. A *crystal in objects of $\mathcal{C}$ on X relative to S* is a *cartesian section* $\sigma : \mathrm{Cris}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \mathrm{id}$ and such that $\sigma(f)$ is strongly cartesian for all morphisms f of $\mathrm{Cris}(X/S)$. Similarly for the big crystalline site.

60.12. Sheaf of differentials

07IW In this section we will stick with the (small) crystalline site as it seems more natural. We globalize Definition 60.6.1 as follows.

07IX **Definition 60.12.1.** In Situation 60.7.5 let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. An *S-derivation* $D : \mathcal{O}_{X/S} \to \mathcal{F}$ is a map of sheaves such that for every object (U, T, δ) of $\mathrm{Cris}(X/S)$ the map

$$D : \Gamma(T, \mathcal{O}_T) \longrightarrow \Gamma(T, \mathcal{F})$$

is a divided power $\Gamma(V, \mathcal{O}_V)$-derivation where $V \subset S$ is any open such that $T \to S$ factors through V.

This means that D is additive, satisfies the Leibniz rule, annihilates functions coming from S, and satisfies $D(f^{[n]}) = f^{[n-1]}D(f)$ for a local section f of the divided power ideal $\mathcal{J}_{X/S}$. This is a special case of a very general notion which we now describe.

Please compare the following discussion with Modules on Sites, Section 18.33. Let $\mathcal{C}$ be a site, let $\mathcal{A} \to \mathcal{B}$ be a map of sheaves of rings on $\mathcal{C}$, let $\mathcal{J} \subset \mathcal{B}$ be a sheaf of ideals, let δ be a divided power structure on $\mathcal{J}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Then there is a notion of a *divided power $\mathcal{A}$-derivation* $D : \mathcal{B} \to \mathcal{F}$. This means that D is $\mathcal{A}$-linear, satisfies the Leibniz rule, and satisfies $D(\delta_n(x)) = \delta_{n-1}(x)D(x)$ for local sections x of $\mathcal{J}$. In this situation there exists a *universal divided power $\mathcal{A}$-derivation*

$$\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta} : \mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

Moreover, $\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta}$ is the composition

$$\mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

where the first map is the universal derivation constructed in the proof of Modules on Sites, Lemma 18.33.2 and the second arrow is the quotient by the submodule generated by the local sections $\mathrm{d}_{\mathcal{B}/\mathcal{A}}(\delta_n(x)) - \delta_{n-1}(x)\mathrm{d}_{\mathcal{B}/\mathcal{A}}(x)$.

We translate this into a relative notion as follows. Suppose $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a morphism of ringed topoi, $\mathcal{J} \subset \mathcal{O}$ a sheaf of ideals, δ a divided power structure on $\mathcal{J}$, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. In this situation we say $D : \mathcal{O} \to \mathcal{F}$ is a divided power $\mathcal{O}'$-derivation if D is a divided power $f^{-1}\mathcal{O}'$-derivation as defined above. Moreover, we write

$$\Omega_{\mathcal{O}/\mathcal{O}',\delta} = \Omega_{\mathcal{O}/f^{-1}\mathcal{O}',\delta}$$

which is the receptacle of the universal divided power $\mathcal{O}'$-derivation.

Applying this to the structure morphism

$$(X/S)_{\mathrm{Cris}} \longrightarrow Sh(S_{Zar})$$

(see Remark 60.9.6) we recover the notion of Definition 60.12.1 above. In particular, there is a universal divided power derivation

$$d_{X/S} : \mathcal{O}_{X/S} \to \Omega_{X/S}$$

Note that we omit from the notation the decoration indicating the module of differentials is compatible with divided powers (it seems unlikely anybody would ever consider the usual module of differentials of the structure sheaf on the crystalline site).

07IY **Lemma 60.12.2.** *Let $(T, \mathcal{J}, \delta)$ be a divided power scheme. Let $T \to S$ be a morphism of schemes. The quotient $\Omega_{T/S} \to \Omega_{T/S,\delta}$ described above is a quasi-coherent $\mathcal{O}_T$-module. For $W \subset T$ affine open mapping into $V \subset S$ affine open we have*

$$\Gamma(W, \Omega_{T/S,\delta}) = \Omega_{\Gamma(W,\mathcal{O}_W)/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 60.6.

Proof. Omitted. □

07IZ **Lemma 60.12.3.** *In Situation 60.7.5. For (U, T, δ) in $Cris(X/S)$ the restriction $(\Omega_{X/S})_T$ to T is $\Omega_{T/S,\delta}$ and the restriction $d_{X/S}|_T$ is equal to $d_{T/S,\delta}$.*

Proof. Omitted. □

07J0 **Lemma 60.12.4.** *In Situation 60.7.5. For any affine object (U, T, δ) of $Cris(X/S)$ mapping into an affine open $V \subset S$ we have*

$$\Gamma((U, T, \delta), \Omega_{X/S}) = \Omega_{\Gamma(T,\mathcal{O}_T)/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 60.6.

Proof. Combine Lemmas 60.12.2 and 60.12.3. □

07J1 **Lemma 60.12.5.** *In Situation 60.7.5. Let (U, T, δ) be an object of $Cris(X/S)$. Let*

$$(U(1), T(1), \delta(1)) = (U, T, \delta) \times (U, T, \delta)$$

in $Cris(X/S)$. Let $\mathcal{K} \subset \mathcal{O}_{T(1)}$ be the quasi-coherent sheaf of ideals corresponding to the closed immersion $\Delta : T \to T(1)$. Then $\mathcal{K} \subset \mathcal{J}_{T(1)}$ is preserved by the divided structure on $\mathcal{J}_{T(1)}$ and we have

$$(\Omega_{X/S})_T = \mathcal{K}/\mathcal{K}^{[2]}$$

Proof. Note that $U = U(1)$ as $U \to X$ is an open immersion and as (60.9.1.1) commutes with products. Hence we see that $\mathcal{K} \subset \mathcal{J}_{T(1)}$. Given this fact the lemma follows by working affine locally on T and using Lemmas 60.12.4 and 60.6.5. □

It turns out that $\Omega_{X/S}$ is not a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. But it does satisfy two closely related properties (compare with Lemma 60.11.2).

07KM **Lemma 60.12.6.** *In Situation 60.7.5. The sheaf of differentials $\Omega_{X/S}$ has the following two properties:*

(1) *$\Omega_{X/S}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $Cris(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*(\Omega_{X/S})_{T'} \to (\Omega_{X/S})_T$ is surjective.*

Proof. Part (1) follows from a combination of Lemmas 60.12.2 and 60.12.3. Part (2) follows from the fact that $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ is a quotient of $\Omega_{T/S}$ and that $f^*\Omega_{T'/S} \to \Omega_{T/S}$ is surjective. □

60.13. Two universal thickenings

07KN The constructions in this section will help us define a connection on a crystal in modules on the crystalline site. In some sense the constructions here are the "sheafified, universal" versions of the constructions in Section 60.3.

07J2 **Remark 60.13.1.** In Situation 60.7.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 60.12.3. We explicitly describe a first order thickening T' of T. Namely, set

$$\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$$

with algebra structure such that $\Omega_{T/S,\delta}$ is an ideal of square zero. Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Set $\mathcal{J}' = \mathcal{J} \oplus \Omega_{T/S,\delta}$. Define a divided power structure on $\mathcal{J}'$ by setting

$$\delta'_n(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega),$$

see Lemma 60.3.1. There are two ring maps

$$p_0, p_1 : \mathcal{O}_T \to \mathcal{O}_{T'}$$

The first is given by $f \mapsto (f, 0)$ and the second by $f \mapsto (f, \mathrm{d}_{T/S,\delta} f)$. Note that both are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}'$ and so is the quotient map $\mathcal{O}_{T'} \to \mathcal{O}_T$. Thus we get an object (U, T', δ') of $\mathrm{Cris}(X/S)$ and a commutative diagram

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle i} & \searrow^{\mathrm{id}} & \\ T & \xleftarrow{p_0} & T' & \xrightarrow{p_1} & T \end{array}$$

of $\mathrm{Cris}(X/S)$ such that i is a first order thickening whose ideal sheaf is identified with $\Omega_{T/S,\delta}$ and such that $p_1 - p_0 : \mathcal{O}_T \to \mathcal{O}_{T'}$ is identified with the universal derivation $\mathrm{d}_{T/S,\delta}$ composed with the inclusion $\Omega_{T/S,\delta} \to \mathcal{O}_{T'}$.

07J3 **Remark 60.13.2.** In Situation 60.7.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 60.12.3. We also write $\Omega^2_{T/S,\delta}$ for its second exterior power. We explicitly describe a second order thickening T'' of T. Namely, set

$$\mathcal{O}_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

with algebra structure defined in the following way

$$(f, \omega_1, \omega_2, \eta) \cdot (f', \omega'_1, \omega'_2, \eta') = (ff', f\omega'_1 + f'\omega_1, f\omega'_2 + f'\omega_2, f\eta' + f'\eta + \omega_1 \wedge \omega'_2 + \omega'_1 \wedge \omega_2).$$

Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Let $\mathcal{J}''$ be the inverse image of $\mathcal{J}$ under the projection $\mathcal{O}_{T''} \to \mathcal{O}_T$. Define a divided power structure on $\mathcal{J}''$ by setting

$$\delta''_n(f, \omega_1, \omega_2, \eta) = (\delta_n(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)\omega_1 \wedge \omega_2)$$

see Lemma 60.3.2. There are three ring maps $q_0, q_1, q_2 : \mathcal{O}_T \to \mathcal{O}_{T''}$ given by

$$\begin{aligned} q_0(f) &= (f, 0, 0, 0), \\ q_1(f) &= (f, \mathrm{d}f, 0, 0), \\ q_2(f) &= (f, \mathrm{d}f, \mathrm{d}f, 0) \end{aligned}$$

where $\mathrm{d} = \mathrm{d}_{T/S,\delta}$. Note that all three are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}''$. There are three ring maps $q_{01}, q_{12}, q_{02} : \mathcal{O}_{T'} \to \mathcal{O}_{T''}$ where $\mathcal{O}_{T'}$ is as in Remark 60.13.1. Namely, set

$$\begin{aligned} q_{01}(f,\omega) &= (f,\omega,0,0), \\ q_{12}(f,\omega) &= (f,\mathrm{d}f,\omega,\mathrm{d}\omega), \\ q_{02}(f,\omega) &= (f,\omega,\omega,0) \end{aligned}$$

These are also compatible with the given divided power structures. Let's do the verifications for q_{12}: Note that q_{12} is a ring homomorphism as

$$\begin{aligned} q_{12}(f,\omega)q_{12}(g,\eta) &= (f,\mathrm{d}f,\omega,\mathrm{d}\omega)(g,\mathrm{d}g,\eta,\mathrm{d}\eta) \\ &= (fg, f\mathrm{d}g + g\mathrm{d}f, f\eta + g\omega, f\mathrm{d}\eta + g\mathrm{d}\omega + \mathrm{d}f \wedge \eta + \mathrm{d}g \wedge \omega) \\ &= q_{12}(fg, f\eta + g\omega) = q_{12}((f,\omega)(g,\eta)) \end{aligned}$$

Note that q_{12} is compatible with divided powers because

$$\begin{aligned} \delta''_n(q_{12}(f,\omega)) &= \delta''_n((f,\mathrm{d}f,\omega,\mathrm{d}\omega)) \\ &= (\delta_n(f), \delta_{n-1}(f)\mathrm{d}f, \delta_{n-1}(f)\omega, \delta_{n-1}(f)\mathrm{d}\omega + \delta_{n-2}(f)\mathrm{d}(f) \wedge \omega) \\ &= q_{12}((\delta_n(f), \delta_{n-1}(f)\omega)) = q_{12}(\delta'_n(f,\omega)) \end{aligned}$$

The verifications for q_{01} and q_{02} are easier. Note that $q_0 = q_{01} \circ p_0$, $q_1 = q_{01} \circ p_1$, $q_1 = q_{12} \circ p_0$, $q_2 = q_{12} \circ p_1$, $q_0 = q_{02} \circ p_0$, and $q_2 = q_{02} \circ p_1$. Thus (U, T'', δ'') is an object of $\mathrm{Cris}(X/S)$ and we get morphisms

$$T'' \xrightarrow[\longrightarrow]{\longrightarrow} T' \xrightarrow[\longrightarrow]{\longrightarrow} T$$

of $\mathrm{Cris}(X/S)$ satisfying the relations described above. In applications we will use $q_i : T'' \to T$ and $q_{ij} : T'' \to T'$ to denote the morphisms associated to the ring maps described above.

60.14. The de Rham complex

07J4 In Situation 60.7.5. Working on the (small) crystalline site, we define $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_{X/S}} \Omega_{X/S}$ for $i \geq 0$. The universal S-derivation $\mathrm{d}_{X/S}$ gives rise to the *de Rham complex*

$$\mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

on $\mathrm{Cris}(X/S)$, see Lemma 60.12.4 and Remark 60.6.7.

60.15. Connections

07J5 In Situation 60.7.5. Given an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $\mathrm{Cris}(X/S)$ a *connection* is a map of abelian sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$ for local sections s, f of $\mathcal{F}$ and $\mathcal{O}_{X/S}$. Given a connection there are canonical maps $\nabla : \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{i+1}_{X/S}$ defined by the rule $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \mathrm{d}\omega$ as in Remark 60.6.8. We say the connection is *integrable* if $\nabla \circ \nabla = 0$. If ∇ is integrable we obtain the *de Rham complex*

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots$$

on $\mathrm{Cris}(X/S)$. It turns out that any crystal in $\mathcal{O}_{X/S}$-modules comes equipped with a canonical integrable connection.

07J6 **Lemma 60.15.1.** *In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in $\mathcal{O}_{X/S}$-modules on $Cris(X/S)$. Then $\mathcal{F}$ comes equipped with a canonical integrable connection.*

Proof. Say (U, T, δ) is an object of $\mathrm{Cris}(X/S)$. Let (U, T', δ') be the infinitesimal thickening of T by $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ constructed in Remark 60.13.1. It comes with projections $p_0, p_1 : T' \to T$ and a diagonal $i : T \to T'$. By assumption we get isomorphisms

$$p_0^*\mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'} \xleftarrow{c_1} p_1^*\mathcal{F}_T$$

of $\mathcal{O}_{T'}$-modules. Pulling $c = c_1^{-1} \circ c_0$ back to T by i we obtain the identity map of $\mathcal{F}_T$. Hence if $s \in \Gamma(T, \mathcal{F}_T)$ then $\nabla(s) = p_1^*s - c(p_0^*s)$ is a section of $p_1^*\mathcal{F}_T$ which vanishes on pulling back by i. Hence $\nabla(s)$ is a section of

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta}$$

because this is the kernel of $p_1^*\mathcal{F}_T \to \mathcal{F}_T$ as $\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$ by construction. It is easily verified that $\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}(f)$ using the description of d in Remark 60.13.1.

The collection of maps

$$\nabla : \Gamma(T, \mathcal{F}_T) \to \Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta})$$

so obtained is functorial in T because the construction of T' is functorial in T. Hence we obtain a connection.

To show that the connection is integrable we consider the object (U, T'', δ'') constructed in Remark 60.13.2. Because $\mathcal{F}$ is a sheaf we see that

$$\begin{array}{ccc} q_0^*\mathcal{F}_T & \xrightarrow{q_{01}^*c} & q_1^*\mathcal{F}_T \\ & {\scriptstyle q_{02}^*c} \searrow \quad \swarrow {\scriptstyle q_{12}^*c} & \\ & q_2^*\mathcal{F}_T & \end{array}$$

is a commutative diagram of $\mathcal{O}_{T''}$-modules. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^*s) = p_1^*s - \nabla(s)$. Write $\nabla(s) = \sum p_1^*s_i \cdot \omega_i$ where s_i is a local section of $\mathcal{F}_T$ and ω_i is a local section of $\Omega_{T/S,\delta}$. We think of ω_i as a local section of the structure sheaf of $\mathcal{O}_{T'}$ and hence we write product instead of tensor product. On the one hand

$$\begin{aligned} q_{12}^*c \circ q_{01}^*c(q_0^*s) &= q_{12}^*c(q_1^*s - \sum q_1^*s_i \cdot q_{01}^*\omega_i) \\ &= q_2^*s - \sum q_2^*s_i \cdot q_{12}^*\omega_i - \sum q_2^*s_i \cdot q_{01}^*\omega_i + \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i \end{aligned}$$

and on the other hand

$$q_{02}^*c(q_0^*s) = q_2^*s - \sum q_2^*s_i \cdot q_{02}^*\omega_i.$$

From the formulae of Remark 60.13.2 we see that $q_{01}^*\omega_i + q_{12}^*\omega_i - q_{02}^*\omega_i = \mathrm{d}\omega_i$. Hence the difference of the two expressions above is

$$\sum q_2^*s_i \cdot \mathrm{d}\omega_i - \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i$$

Note that $q_{12}^*\omega \cdot q_{01}^*\omega' = \omega' \wedge \omega = -\omega \wedge \omega'$ by the definition of the multiplication on $\mathcal{O}_{T''}$. Thus the expression above is $\nabla^2(s)$ viewed as a section of the subsheaf $\mathcal{F}_T \otimes \Omega^2_{T/S,\delta}$ of $q_2^*\mathcal{F}$. Hence we get the integrability condition. □

60.16. Cosimplicial algebra

07KP This section should be moved somewhere else. A *cosimplicial ring* is a cosimplicial object in the category of rings. Given a ring R, a *cosimplicial R-algebra* is a cosimplicial object in the category of R-algebras. A *cosimplicial ideal* in a cosimplicial ring A_* is given by an ideal $I_n \subset A_n$ for all n such that $A(f)(I_n) \subset I_m$ for all $f : [n] \to [m]$ in Δ.

Let A_* be a cosimplicial ring. Let $\mathcal{C}$ be the category of pairs (A, M) where A is a ring and M is a module over A. A morphism $(A, M) \to (A', M')$ consists of a ring map $A \to A'$ and an A-module map $M \to M'$ where M' is viewed as an A-module via $A \to A'$ and the A'-module structure on M'. Having said this we can define a *cosimplicial module M_* over A_** as a cosimplicial object (A_*, M_*) of $\mathcal{C}$ whose first entry is equal to A_*. A *homomorphism $\varphi_* : M_* \to N_*$ of cosimplicial modules over A_** is a morphism $(A_*, M_*) \to (A_*, N_*)$ of cosimplicial objects in $\mathcal{C}$ whose first component is 1_{A_*}.

A *homotopy* between homomorphisms $\varphi_*, \psi_* : M_* \to N_*$ of cosimplicial modules over A_* is a homotopy between the associated maps $(A_*, M_*) \to (A_*, N_*)$ whose first component is the trivial homotopy (dual to Simplicial, Example 14.26.3). We spell out what this means. Such a homotopy is a homotopy

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$$

between φ_* and ψ_* as homomorphisms of cosimplicial abelian groups such that for each n the map $h_n : M_n \to \prod_{\alpha \in \Delta[1]_n} N_n$ is A_n-linear. The following lemma is a version of Simplicial, Lemma 14.28.4 for cosimplicial modules.

07KQ **Lemma 60.16.1.** *Let A_* be a cosimplicial ring. Let $\varphi_*, \psi_* : K_* \to M_*$ be homomorphisms of cosimplicial A_*-modules.*

07KR (1) *If φ_* and ψ_* are homotopic, then*

$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} L_* \longrightarrow M_* \otimes_{A_*} L_*$$

are homotopic for any cosimplicial A_*-module L_*.

07KS (2) *If φ_* and ψ_* are homotopic, then*

$$\wedge^i(\varphi_*), \wedge^i(\psi_*) : \wedge^i(K_*) \longrightarrow \wedge^i(M_*)$$

are homotopic.

07KT (3) *If φ_* and ψ_* are homotopic, and $A_* \to B_*$ is a homomorphism of cosimplicial rings, then*

$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} B_* \longrightarrow M_* \otimes_{A_*} B_*$$

are homotopic as homomorphisms of cosimplicial B_*-modules.

07KU (4) *If $I_* \subset A_*$ is a cosimplicial ideal, then the induced maps*

$$\varphi_*^\wedge, \psi_*^\wedge : K_*^\wedge \longrightarrow M_*^\wedge$$

between completions are homotopic.

(5) *Add more here as needed, for example symmetric powers.*

Proof. Let $h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$ be the given homotopy. In degree n we have

$$h_n = (h_{n,\alpha}) : K_n \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} K_n$$

see Simplicial, Section 14.28. In order for a collection of $h_{n,\alpha}$ to form a homotopy, it is necessary and sufficient if for every $f : [n] \to [m]$ we have

$$h_{m,\alpha} \circ M_*(f) = N_*(f) \circ h_{n,\alpha \circ f}$$

see Simplicial, Equation (14.28.1.1). We also should have that $\psi_n = h_{n,0:[n]\to[1]}$ and $\varphi_n = h_{n,1:[n]\to[1]}$.

In each of the cases of the lemma we can produce the corresponding maps. Case (1). We can use the homotopy $h \otimes 1$ defined in degree n by setting

$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1_{L_n} : K_n \otimes_{A_n} L_n \longrightarrow M_n \otimes_{A_n} L_n.$$

Case (2). We can use the homotopy $\wedge^i h$ defined in degree n by setting

$$\wedge^i(h)_{n,\alpha} = \wedge^i(h_{n,\alpha}) : \wedge_{A_n}(K_n) \longrightarrow \wedge^i_{A_n}(M_n).$$

Case (3). We can use the homotopy $h \otimes 1$ defined in degree n by setting

$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1 : K_n \otimes_{A_n} B_n \longrightarrow M_n \otimes_{A_n} B_n.$$

Case (4). We can use the homotopy $h^\wedge$ defined in degree n by setting

$$(h^\wedge)_{n,\alpha} = h^\wedge_{n,\alpha} : K^\wedge_n \longrightarrow M^\wedge_n.$$

This works because each $h_{n,\alpha}$ is A_n-linear. □

60.17. Crystals in quasi-coherent modules

07J7 In Situation 60.5.1. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We are going to classify crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Before we do so we fix some notation.

Choose a polynomial ring $P = A[x_i]$ over A and a surjection $P \to C$ of A-algebras with kernel $J = \mathrm{Ker}(P \to C)$. Set

07J8 (60.17.0.1) $$D = \lim_e D_{P,\gamma}(J)/p^e D_{P,\gamma}(J)$$

for the p-adically completed divided power envelope. This ring comes with a divided power ideal $\bar{J}$ and divided power structure $\bar{\gamma}$, see Lemma 60.5.5. Set $D_e = D/p^e D$ and denote $\bar{J}_e$ the image of $\bar{J}$ in D_e. We will use the short hand

07J9 (60.17.0.2) $$\Omega_D = \lim_e \Omega_{D_e/A,\bar{\gamma}} = \lim_e \Omega_{D/A,\bar{\gamma}}/p^e \Omega_{D/A,\bar{\gamma}}$$

for the p-adic completion of the module of divided power differentials, see Lemma 60.6.10. It is also the p-adic completion of $\Omega_{D_{P,\gamma}(J)/A,\bar{\gamma}}$ which is free on $\mathrm{d}x_i$, see Lemma 60.6.6. Hence any element of Ω_D can be written uniquely as a sum $\sum f_i \mathrm{d}x_i$ with for all e only finitely many f_i not in $p^e D$. Moreover, the maps $\mathrm{d}_{D_e/A,\bar{\gamma}} : D_e \to \Omega_{D_e/A,\bar{\gamma}}$ fit together to define a divided power A-derivation

07JA (60.17.0.3) $$\mathrm{d} : D \longrightarrow \Omega_D$$

on p-adic completions.

We will also need the "products $\mathrm{Spec}(D(n))$ of $\mathrm{Spec}(D)$", see Proposition 60.21.1 and its proof for an explanation. Formally these are defined as follows. For $n \geq 0$ let $J(n) = \mathrm{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor product has $n+1$ factors. We set

07JF (60.17.0.4) $$D(n) = \lim_e D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n))/p^e D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n))$$

equal to the p-adic completion of the divided power envelope. We denote $\bar{J}(n)$ its divided power ideal and $\bar{\gamma}(n)$ its divided powers. We also introduce $D(n)_e = D(n)/p^eD(n)$ as well as the p-adically completed module of differentials

07L0 (60.17.0.5) $$\Omega_{D(n)} = \lim_e \Omega_{D(n)_e/A,\bar{\gamma}} = \lim_e \Omega_{D(n)/A,\bar{\gamma}}/p^e\Omega_{D(n)/A,\bar{\gamma}}$$

and derivation

07L1 (60.17.0.6) $$\mathrm{d} : D(n) \longrightarrow \Omega_{D(n)}$$

Of course we have $D = D(0)$. Note that the rings $D(0), D(1), D(2), \ldots$ form a cosimplicial object in the category of divided power rings.

07L2 **Lemma 60.17.1.** *Let D and $D(n)$ be as in (60.17.0.1) and (60.17.0.4). The coprojection $P \to P \otimes_A \ldots \otimes_A P$, $f \mapsto f \otimes 1 \otimes \ldots \otimes 1$ induces an isomorphism*

07L3 (60.17.1.1) $$D(n) = \lim_e D\langle \xi_i(j) \rangle / p^e D\langle \xi_i(j) \rangle$$

of algebras over D with

$$\xi_i(j) = x_i \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes x_i \otimes 1 \otimes \ldots \otimes 1$$

for $j = 1, \ldots, n$ where the second x_i is placed in the $j+1$st slot; recall that $D(n)$ is constructed starting with the $n+1$-fold tensor product of P over A.

Proof. We have

$$P \otimes_A \ldots \otimes_A P = P[\xi_i(j)]$$

and $J(n)$ is generated by J and the elements $\xi_i(j)$. Hence the lemma follows from Lemma 60.2.5. □

07L4 **Lemma 60.17.2.** *Let D and $D(n)$ be as in (60.17.0.1) and (60.17.0.4). Then $(D, \bar{J}, \bar{\gamma})$ and $(D(n), \bar{J}(n), \bar{\gamma}(n))$ are objects of $Cris^\wedge(C/A)$, see Remark 60.5.4, and*

$$D(n) = \coprod\nolimits_{j=0,\ldots,n} D$$

in $Cris^\wedge(C/A)$.

Proof. The first assertion is clear. For the second, if $(B \to C, \delta)$ is an object of $\mathrm{Cris}^\wedge(C/A)$, then we have

$$\mathrm{Mor}_{\mathrm{Cris}^\wedge(C/A)}(D, B) = \mathrm{Hom}_A((P, J), (B, \mathrm{Ker}(B \to C)))$$

and similarly for $D(n)$ replacing (P, J) by $(P \otimes_A \ldots \otimes_A P, J(n))$. The property on coproducts follows as $P \otimes_A \ldots \otimes_A P$ is a coproduct. □

In the lemma below we will consider pairs (M, ∇) satisfying the following conditions

07JB (1) M is a p-adically complete D-module,

07JC (2) $\nabla : M \to M \otimes^\wedge_D \Omega_D$ is a connection, i.e., $\nabla(fm) = m \otimes \mathrm{d}f + f\nabla(m)$,

07JD (3) ∇ is integrable (see Remark 60.6.8), and

07JE (4) ∇ is *topologically quasi-nilpotent*: If we write $\nabla(m) = \sum \theta_i(m)\mathrm{d}x_i$ for some operators $\theta_i : M \to M$, then for any $m \in M$ there are only finitely many pairs (i, k) such that $\theta_i^k(m) \notin pM$.

The operators θ_i are sometimes denoted $\nabla_{\partial/\partial x_i}$ in the literature. In the following lemma we construct a functor from crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$ to the category of such pairs. We will show this functor is an equivalence in Proposition 60.17.4.

07JG **Lemma 60.17.3.** *In the situation above there is a functor*

$$\begin{array}{c}\text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } Cris(X/S)\end{array} \longrightarrow \begin{array}{c}\text{pairs } (M,\nabla) \text{ satisfying} \\ \text{(1), (2), (3), and (4)}\end{array}$$

Proof. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on X/S. Set $T_e = \mathrm{Spec}(D_e)$ so that $(X, T_e, \bar{\gamma})$ is an object of $\mathrm{Cris}(X/S)$ for $e \gg 0$. We have morphisms

$$(X, T_e, \bar{\gamma}) \to (X, T_{e+1}, \bar{\gamma}) \to \ldots$$

which are closed immersions. We set

$$M = \lim_e \Gamma((X, T_e, \bar{\gamma}), \mathcal{F}) = \lim_e \Gamma(T_e, \mathcal{F}_{T_e}) = \lim_e M_e$$

Note that since $\mathcal{F}$ is locally quasi-coherent we have $\mathcal{F}_{T_e} = \widetilde{M_e}$. Since $\mathcal{F}$ is a crystal we have $M_e = M_{e+1}/p^e M_{e+1}$. Hence we see that $M_e = M/p^e M$ and that M is p-adically complete, see Algebra, Lemma 10.98.2.

By Lemma 60.15.1 we know that $\mathcal{F}$ comes endowed with a canonical integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$. If we evaluate this connection on the objects T_e constructed above we obtain a canonical integrable connection

$$\nabla : M \longrightarrow M \otimes^\wedge_D \Omega_D$$

To see that this is topologically nilpotent we work out what this means.

Now we can do the same procedure for the rings $D(n)$. This produces a p-adically complete $D(n)$-module $M(n)$. Again using the crystal property of $\mathcal{F}$ we obtain isomorphisms

$$M \otimes^\wedge_{D,p_0} D(1) \to M(1) \leftarrow M \otimes^\wedge_{D,p_1} D(1)$$

compare with the proof of Lemma 60.15.1. Denote c the composition from left to right. Pick $m \in M$. Write $\xi_i = x_i \otimes 1 - 1 \otimes x_i$. Using (60.17.1.1) we can write uniquely

$$c(m \otimes 1) = \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

for some $\theta_K(m) \in M$ where the sum is over multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. Set $\theta_i = \theta_K$ where K has a 1 in the ith spot and zeros elsewhere. We have

$$\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i.$$

as can be seen by comparing with the definition of ∇. Namely, the defining equation is $p_1^* m = \nabla(m) - c(p_0^* m)$ in Lemma 60.15.1 but the sign works out because in the Stacks project we consistently use $\mathrm{d}f = p_1(f) - p_0(f)$ modulo the ideal of the diagonal squared, and hence $\xi_i = x_i \otimes 1 - 1 \otimes x_i$ maps to $-\mathrm{d}x_i$ modulo the ideal of the diagonal squared.

Denote $q_i : D \to D(2)$ and $q_{ij} : D(1) \to D(2)$ the coprojections corresponding to the indices i, j. As in the last paragraph of the proof of Lemma 60.15.1 we see that

$$q_{02}^* c = q_{12}^* c \circ q_{01}^* c.$$

This means that

$$\sum\nolimits_{K''} \theta_{K''}(m) \otimes \prod \zeta_i^{\prime\prime[k_i'']} = \sum\nolimits_{K',K} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i^{\prime[k_i']} \prod \zeta_i^{[k_i]}$$

in $M \otimes^\wedge_{D,q_2} D(2)$ where

$$\zeta_i = x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1,$$
$$\zeta_i' = 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i,$$
$$\zeta_i'' = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i.$$

In particular $\zeta_i'' = \zeta_i + \zeta_i'$ and we have that $D(2)$ is the p-adic completion of the divided power polynomial ring in ζ_i, ζ_i' over $q_2(D)$, see Lemma 60.17.1. Comparing coefficients in the expression above it follows immediately that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ (this provides an alternative proof of the integrability of ∇) and that

$$\theta_K(m) = (\prod \theta_i^{k_i})(m).$$

In particular, as the sum expressing $c(m \otimes 1)$ above has to converge p-adically we conclude that for each i and each $m \in M$ only a finite number of $\theta_i^k(m)$ are allowed to be nonzero modulo p. □

07JH **Proposition 60.17.4.** *The functor*

$$\begin{matrix} \text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } Cris(X/S) \end{matrix} \longrightarrow \begin{matrix} \text{pairs } (M, \nabla) \text{ satisfying} \\ \text{(1), (2), (3), and (4)} \end{matrix}$$

of Lemma 60.17.3 is an equivalence of categories.

Proof. Let (M, ∇) be given. We are going to construct a crystal in quasi-coherent modules $\mathcal{F}$. Write $\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i$. Then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ and we can set $\theta_K(m) = (\prod \theta_i^{k_i})(m)$ for any multi-index $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$.

Let (U, T, δ) be any object of $\mathrm{Cris}(X/S)$ with T affine. Say $T = \mathrm{Spec}(B)$ and the ideal of $U \to T$ is $J_B \subset B$. By Lemma 60.5.6 there exists an integer e and a morphism

$$f : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$$

where $T_e = \mathrm{Spec}(D_e)$ as in the proof of Lemma 60.17.3. Choose such an e and f; denote $f : D \to B$ also the corresponding divided power A-algebra map. We will set $\mathcal{F}_T$ equal to the quasi-coherent sheaf of $\mathcal{O}_T$-modules associated to the B-module

$$M \otimes_{D,f} B.$$

However, we have to show that this is independent of the choice of f. Suppose that $g : D \to B$ is a second such morphism. Since f and g are morphisms in $\mathrm{Cris}(X/S)$ we see that the image of $f - g : D \to B$ is contained in the divided power ideal J_B. Write $\xi_i = f(x_i) - g(x_i) \in J_B$. By analogy with the proof of Lemma 60.17.3 we define an isomorphism

$$c_{f,g} : M \otimes_{D,f} B \longrightarrow M \otimes_{D,g} B$$

by the formula

$$m \otimes 1 \longmapsto \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

which makes sense by our remarks above and the fact that ∇ is topologically quasi-nilpotent (so the sum is finite!). A computation shows that

$$c_{g,h} \circ c_{f,g} = c_{f,h}$$

if given a third morphism $h : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$. It is also true that $c_{f,f} = 1$. Hence these maps are all isomorphisms and we see that the module $\mathcal{F}_T$ is independent of the choice of f.

If $a : (U', T', \delta') \to (U, T, \delta)$ is a morphism of affine objects of $\mathrm{Cris}(X/S)$, then choosing $f' = f \circ a$ it is clear that there exists a canonical isomorphism $a^*\mathcal{F}_T \to \mathcal{F}_{T'}$. We omit the verification that this map is independent of the choice of f. Using these maps as the restriction maps it is clear that we obtain a crystal in quasi-coherent modules on the full subcategory of $\mathrm{Cris}(X/S)$ consisting of affine objects. We omit the proof that this extends to a crystal on all of $\mathrm{Cris}(X/S)$. We also omit the proof that this procedure is a functor and that it is quasi-inverse to the functor constructed in Lemma 60.17.3. □

07L5 **Lemma 60.17.5.** *In Situation 60.5.1. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Then there are a choice of the data $A \to P \to C$ as above and homomorphisms of divided power A-algebras*

$$a : D \longrightarrow D', \quad b : D' \longrightarrow D$$

compatible with the maps $D \to C$ and $D' \to C$ such that $a \circ b = id_{D'}$. These maps induce an equivalence of categories of pairs (M, ∇) satisfying (1), (2), (3), and (4) over D and pairs (M', ∇') satisfying (1), (2), (3), and (4)[4] over D'. In particular, the equivalence of categories of Proposition 60.17.4 also holds for the corresponding functor towards pairs over D'.

Proof. Choose a polynomial algebra $P = A[y_1, \ldots, y_m]$ over A and a surjection $P \to P'$. Define $P \to C$ by the composition $P \to P' \to C$. We obtain a surjective map $a : D \to D'$ by functoriality of divided power envelopes and completion. Pick e large enough so that D_e is a divided power thickening of C over A. Then $D_e \to C$ is a surjection whose kernel is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. Setting $D'_e = D'/p^e D'$ we see that the kernel of $D_e \to D'_e$ is locally nilpotent. Hence by Algebra, Lemma 10.138.17 we can find a lift $\beta_e : P' \to D_e$ of the map $P' \to D'_e$. Note that $D_{e+i+1} \to D_{e+i} \times_{D'_{e+i}} D'_{e+i+1}$ is surjective with square zero kernel for any $i \geq 0$ because $p^{e+i}D \to p^{e+i}D'$ is surjective. Applying the usual lifting property (Algebra, Proposition 10.138.13) successively to the diagrams

$$\begin{array}{ccc} P' & \longrightarrow & D_{e+i} \times_{D'_{e+i}} D'_{e+i+1} \\ \uparrow & & \uparrow \\ A & \longrightarrow & D_{e+i+1} \end{array}$$

we see that we can find an A-algebra map $\beta : P' \to D$ whose composition with a is the given map $P' \to D'$. By the universal property of the divided power envelope we obtain a map $D_{P',\gamma}(J') \to D$. As D is p-adically complete we obtain $b : D' \to D$ such that $a \circ b = \mathrm{id}_{D'}$.

Consider the base change functors

$$F : (M, \nabla) \longmapsto (M \otimes^\wedge_{D,a} D', \nabla') \quad \text{and} \quad G : (M', \nabla') \longmapsto (M' \otimes^\wedge_{D',b} D, \nabla)$$

on modules with connections satisfying (1), (2), and (3). See Remark 60.6.9. Since $a \circ b = \mathrm{id}_{D'}$ we see that $F \circ G$ is the identity functor. Let us say that (M', ∇') has property (4) if this is true for $G(M', \nabla')$. A formal argument now shows that to finish the proof it suffices to show that $G(F(M, \nabla))$ is isomorphic to (M, ∇) in the

[4] This condition is tricky to formulate for (M', ∇') over D'. See proof.

case that (M, ∇) satisfies all four conditions (1), (2), (3), and (4). For this we use the functorial isomorphism

$$c_{\mathrm{id}_D, b \circ a} : M \otimes_{D, \mathrm{id}_D} D \longrightarrow M \otimes_{D, b \circ a} D$$

of the proof of Proposition 60.17.4 (which requires the topological quasi-nilpotency of ∇ which we have assumed). It remains to prove that this map is horizontal, i.e., compatible with connections, which we omit.

The last statement of the proof now follows. □

07L6 **Remark 60.17.6.** The equivalence of Proposition 60.17.4 holds if we start with a surjection $P \to C$ where P/A satisfies the strong lifting property of Algebra, Lemma 10.138.17. To prove this we can argue as in the proof of Lemma 60.17.5. (Details will be added here if we ever need this.) Presumably there is also a direct proof of this result, but the advantage of using polynomial rings is that the rings $D(n)$ are p-adic completions of divided power polynomial rings and the algebra is simplified.

60.18. General remarks on cohomology

07JI In this section we do a bit of work to translate the cohomology of modules on the cristalline site of an affine scheme into an algebraic question.

07JJ **Lemma 60.18.1.** *In Situation 60.7.5. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_{X/S}$-module on Cris(X/S). Then we have*

$$H^p((U, T, \delta), \mathcal{F}) = 0$$

for all $p > 0$ and all (U, T, δ) with T or U affine.

Proof. As $U \to T$ is a thickening we see that U is affine if and only if T is affine, see Limits, Lemma 32.11.1. Having said this, let us apply Cohomology on Sites, Lemma 21.10.9 to the collection $\mathcal{B}$ of affine objects (U, T, δ) and the collection Cov of affine open coverings $\mathcal{U} = \{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$. The Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F})$ for such a covering is simply the Čech complex of the quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}_T$ (here we are using the assumption that $\mathcal{F}$ is locally quasi-coherent) with respect to the affine open covering $\{T_i \to T\}$ of the affine scheme T. Hence the Čech cohomology is zero by Cohomology of Schemes, Lemma 30.2.6 and 30.2.2. Thus the hypothesis of Cohomology on Sites, Lemma 21.10.9 are satisfied and we win. □

07JK **Lemma 60.18.2.** *In Situation 60.7.5. Assume moreover X and S are affine schemes. Consider the full subcategory $\mathcal{C} \subset$ Cris(X/S) consisting of divided power thickenings (X, T, δ) endowed with the chaotic topology (see Sites, Example 7.6.6). For any locally quasi-coherent $\mathcal{O}_{X/S}$-module $\mathcal{F}$ we have*

$$R\Gamma(\mathcal{C}, \mathcal{F}|_{\mathcal{C}}) = R\Gamma(\mathit{Cris}(X/S), \mathcal{F})$$

Proof. Denote AffineCris(X/S) the fully subcategory of Cris(X/S) consisting of those objects (U, T, δ) with U and T affine. We turn this into a site by saying a family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}_{i \in I}$ of AffineCris(X/S) is a covering if and only if it is a covering of Cris(X/S). With this definition the inclusion functor

$$\mathrm{AffineCris}(X/S) \longrightarrow \mathrm{Cris}(X/S)$$

is a special cocontinuous functor as defined in Sites, Definition 7.29.2. The proof of this is exactly the same as the proof of Topologies, Lemma 34.3.10. Thus we

see that the topos of sheaves on $\mathrm{Cris}(X/S)$ is the same as the topos of sheaves on $\mathrm{AffineCris}(X/S)$ via restriction by the displayed inclusion functor. Therefore we have to prove the corresponding statement for the inclusion $\mathcal{C} \subset \mathrm{AffineCris}(X/S)$.

We will use without further mention that $\mathcal{C}$ and $\mathrm{AffineCris}(X/S)$ have products and fibre products (details omitted, see Lemma 60.8.2). The inclusion functor $u : \mathcal{C} \to \mathrm{AffineCris}(X/S)$ is fully faithful, continuous, and commutes with products and fibre products. We claim it defines a morphism of ringed sites

$$f : (\mathrm{AffineCris}(X/S), \mathcal{O}_{X/S}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{X/S}|_{\mathcal{C}})$$

To see this we will use Sites, Lemma 7.14.6. Note that $\mathcal{C}$ has fibre products and u commutes with them so the categories $\mathcal{I}^u_{(U,T,\delta)}$ are disjoint unions of directed categories (by Sites, Lemma 7.5.1 and Categories, Lemma 4.19.8). Hence it suffices to show that $\mathcal{I}^u_{(U,T,\delta)}$ is connected. Nonempty follows from Lemma 60.5.6: since U and T are affine that lemma says there is at least one object (X, T', δ') of $\mathcal{C}$ and a morphism $(U, T, \delta) \to (X, T', \delta')$ of divided power thickenings. Connectedness follows from the fact that $\mathcal{C}$ has products and that u commutes with them (compare with the proof of Sites, Lemma 7.5.2).

Note that $f_*\mathcal{F} = \mathcal{F}|_{\mathcal{C}}$. Hence the lemma follows if $R^p f_*\mathcal{F} = 0$ for $p > 0$, see Cohomology on Sites, Lemma 21.14.6. By Cohomology on Sites, Lemma 21.7.4 it suffices to show that $H^p(\mathrm{AffineCris}(X/S)/(X, T, \delta), \mathcal{F}) = 0$ for all (X, T, δ). This follows from Lemma 60.18.1 because the topos of the site $\mathrm{AffineCris}(X/S)/(X, T, \delta)$ is equivalent to the topos of the site $\mathrm{Cris}(X/S)/(X, T, \delta)$ used in the lemma. □

07JL **Lemma 60.18.3.** *In Situation 60.5.1. Set $\mathcal{C} = (Cris(C/A))^{opp}$ and $\mathcal{C}^\wedge = (Cris^\wedge(C/A))^{opp}$ endowed with the chaotic topology, see Remark 60.5.4 for notation. There is a morphism of topoi*

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}^\wedge)$$

such that if $\mathcal{F}$ is a sheaf of abelian groups on $\mathcal{C}$, then

$$R^p g_*\mathcal{F}(B \to C, \delta) = \begin{cases} \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 0 \\ R^1 \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 1 \\ 0 & \text{else} \end{cases}$$

where $B_e = B/p^e B$ for $e \gg 0$.

Proof. Any functor between categories defines a morphism between chaotic topoi in the same direction, for example because such a functor can be considered as a cocontinuous functor between sites, see Sites, Section 7.21. Proof of the description of $g_*\mathcal{F}$ is omitted. Note that in the statement we take $(B_e \to C, \delta)$ is an object of $\mathrm{Cris}(C/A)$ only for e large enough. Let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the transition maps

$$\mathcal{I}(B_e \to C, \delta) \leftarrow \mathcal{I}(B_{e+1} \to C, \delta)$$

are surjective as the morphisms

$$(B_e \to C, \delta) \longrightarrow (B_{e+1} \to C, \delta)$$

are monomorphisms in the category $\mathcal{C}$. Hence for an injective abelian sheaf both sides of the displayed formula of the lemma agree. Taking an injective resolution of $\mathcal{F}$ one easily obtains the result (sheaves are presheaves, so exactness is measured on the level of groups of sections over objects). □

07JM **Lemma 60.18.4.** *Let $\mathcal{C}$ be a category endowed with the chaotic topology. Let X be an object of $\mathcal{C}$ such that every object of $\mathcal{C}$ has a morphism towards X. Assume that $\mathcal{C}$ has products of pairs. Then for every abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the complex*

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \ldots$$

associated to the cosimplicial abelian group $[n] \mapsto \mathcal{F}(X^n)$.

Proof. Note that $H^q(X^p, \mathcal{F}) = 0$ for all $q > 0$ as any presheaf is a sheaf on $\mathcal{C}$. The assumption on X is that $h_X \to *$ is surjective. Using that $H^q(X, \mathcal{F}) = H^q(h_X, \mathcal{F})$ and $H^q(\mathcal{C}, \mathcal{F}) = H^q(*, \mathcal{F})$ we see that our statement is a special case of Cohomology on Sites, Lemma 21.13.2. □

60.19. Cosimplicial preparations

07JP In this section we compare crystalline cohomology with de Rham cohomology. We follow [**BdJ11**].

07L7 **Example 60.19.1.** Suppose that A_* is any cosimplicial ring. Consider the cosimplicial module M_* defined by the rule

$$M_n = \bigoplus\nolimits_{i=0,\ldots,n} A_n e_i$$

For a map $f : [n] \to [m]$ define $M_*(f) : M_n \to M_m$ to be the unique $A_*(f)$-linear map which maps e_i to $e_{f(i)}$. We claim the identity on M_* is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], M_*)$$

see Section 60.16. For $j \in \{0, \ldots, n+1\}$ we let $\alpha_j^n : [n] \to [1]$ be the map defined by $\alpha_j^n(i) = 0 \Leftrightarrow i < j$. Then $\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$ and correspondingly $\mathrm{Hom}(\Delta[1], M_*)_n = \prod_{j=0,\ldots,n+1} M_n$, see Simplicial, Sections 14.26 and 14.28. Instead of using this product representation, we think of an element in $\mathrm{Hom}(\Delta[1], M_*)_n$ as a function $\Delta[1]_n \to M_n$. Using this notation, we define h in degree n by the rule

$$h_n(e_i)(\alpha_j^n) = \begin{cases} e_i & \text{if} \quad i < j \\ 0 & \text{else} \end{cases}$$

We first check h is a morphism of cosimplicial modules. Namely, for $f : [n] \to [m]$ we will show that

07L8 (60.19.1.1) $$h_m \circ M_*(f) = \mathrm{Hom}(\Delta[1], M_*)(f) \circ h_n$$

The left hand side of (60.19.1.1) evaluated at e_i and then in turn evaluated at α_j^m is

$$h_m(e_{f(i)})(\alpha_j^m) = \begin{cases} e_{f(i)} & \text{if} \quad f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that $\alpha_j^m \circ f = \alpha_{j'}^n$ where $0 \leq j' \leq n+1$ is the unique index such that $f(i) < j$ if and only if $i < j'$. Thus the right hand side of (60.19.1.1) evaluated at e_i and then in turn evaluated at α_j^m is

$$M_*(f)(h_n(e_i)(\alpha_j^m \circ f) = M_*(f)(h_n(e_i)(\alpha_{j'}^n)) = \begin{cases} e_{f(i)} & \text{if} \quad i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of j' that the two answers are equal. Hence h is a map of cosimplicial modules. Let $0 : \Delta[0] \to \Delta[1]$ and $1 : \Delta[0] \to \Delta[1]$ be

the obvious maps, and denote $ev_0, ev_1 : \mathrm{Hom}(\Delta[1], M_*) \to M_*$ the corresponding evaluation maps. The reader verifies readily that the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence h is the desired homotopy between 0 and 1.

07L9 **Lemma 60.19.2.** *With notation as in (60.17.0.5) the complex*

$$\Omega_{D(0)} \to \Omega_{D(1)} \to \Omega_{D(2)} \to \ldots$$

is homotopic to zero as a $D(*)$-cosimplicial module.

Proof. We are going to use the principle of Simplicial, Lemma 14.28.4 and more specifically Lemma 60.16.1 which tells us that homotopic maps between (co)simplicial objects are transformed by any functor into homotopic maps. The complex of the lemma is equal to the p-adic completion of the base change of the cosimplicial module

$$M_* = \big(\Omega_{P/A} \to \Omega_{P \otimes_A P/A} \to \Omega_{P \otimes_A P \otimes_A P/A} \to \ldots\big)$$

via the cosimplicial ring map $P \otimes_A \ldots \otimes_A P \to D(n)$. This follows from Lemma 60.6.6, see comments following (60.17.0.2). Hence it suffices to show that the cosimplicial module M_* is homotopic to zero (uses base change and p-adic completion). We can even assume $A = \mathbf{Z}$ and $P = \mathbf{Z}[\{x_i\}_{i \in I}]$ as we can use base change with $\mathbf{Z} \to A$. In this case $P^{\otimes n+1}$ is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \ldots \otimes x_i \otimes \ldots \otimes 1$$

with x_i in the eth slot. The modules of the complex are free on the generators $\mathrm{d}x_i(e)$. Note that if $f : [n] \to [m]$ is a map then we see that

$$M_*(f)(\mathrm{d}x_i(e)) = \mathrm{d}x_i(f(e))$$

Hence we see that M_* is a direct sum over I of copies of the module studied in Example 60.19.1 and we win. □

07LA **Lemma 60.19.3.** *With notation as in (60.17.0.4) and (60.17.0.5), given any cosimplicial module M_* over $D(*)$ and $i > 0$ the cosimplicial module*

$$M_0 \otimes^\wedge_{D(0)} \Omega^i_{D(0)} \to M_1 \otimes^\wedge_{D(1)} \Omega^i_{D(1)} \to M_2 \otimes^\wedge_{D(2)} \Omega^i_{D(2)} \to \ldots$$

is homotopic to zero, where $\Omega^i_{D(n)}$ is the p-adic completion of the ith exterior power of $\Omega_{D(n)}$.

Proof. By Lemma 60.19.2 the endomorphisms 0 and 1 of $\Omega_{D(*)}$ are homotopic. If we apply the functor $\wedge^i$ we see that the same is true for the cosimplicial module $\wedge^i \Omega_{D(*)}$, see Lemma 60.16.1. Another application of the same lemma shows the p-adic completion $\Omega^i_{D(*)}$ is homotopy equivalent to zero. Tensoring with M_* we see that $M_* \otimes_{D(*)} \Omega^i_{D(*)}$ is homotopic to zero, see Lemma 60.16.1 again. A final application of the p-adic completion functor finishes the proof. □

60.20. Divided power Poincaré lemma

07LB Just the simplest possible version.

07LC **Lemma 60.20.1.** *Let A be a ring. Let $P = A\langle x_i \rangle$ be a divided power polynomial ring over A. For any A-module M the complex*

$$0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to M \otimes_A \Omega^2_{P/A,\delta} \to \ldots$$

is exact. Let D be the p-adic completion of P. Let Ω^i_D be the p-adic completion of the ith exterior power of $\Omega_{D/A,\delta}$. For any p-adically complete A-module M the complex

$$0 \to M \to M \otimes^\wedge_A D \to M \otimes^\wedge_A \Omega^1_D \to M \otimes^\wedge_A \Omega^2_D \to \ldots$$

is exact.

Proof. It suffices to show that the complex

$$E : (0 \to A \to P \to \Omega^1_{P/A,\delta} \to \Omega^2_{P/A,\delta} \to \ldots)$$

is homotopy equivalent to zero as a complex of A-modules. For every multi-index $K = (k_i)$ we can consider the subcomplex $E(K)$ which in degree j consists of

$$\bigoplus\nolimits_{I=\{i_1,\ldots,i_j\} \subset \mathrm{Supp}(K)} A \prod\nolimits_{i \notin I} x_i^{[k_i]} \prod\nolimits_{i \in I} x_i^{[k_i-1]} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_j}$$

Since $E = \bigoplus E(K)$ we see that it suffices to prove each of the complexes $E(K)$ is homotopic to zero. If $K = 0$, then $E(K) : (A \to A)$ is homotopic to zero. If K has nonempty (finite) support S, then the complex $E(K)$ is isomorphic to the complex

$$0 \to A \to \bigoplus\nolimits_{s \in S} A \to \wedge^2(\bigoplus\nolimits_{s \in S} A) \to \ldots \to \wedge^{\#S}(\bigoplus\nolimits_{s \in S} A) \to 0$$

which is homotopic to zero, for example by More on Algebra, Lemma 15.28.5. □

An alternative (more direct) approach to the following lemma is explained in Example 60.25.2.

07LD **Lemma 60.20.2.** *Let A be a ring. Let (B, I, δ) be a divided power ring where B is an A-algebra. Let $P = B\langle x_i \rangle$ be a divided power polynomial ring over B with divided power ideal $J = IP + B\langle x_i \rangle_+$ as usual. Let M be a B-module endowed with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_{B/A,\delta}$. Then the map of de Rham complexes*

$$M \otimes_B \Omega^*_{B/A,\delta} \longrightarrow M \otimes_P \Omega^*_{P/A,\delta}$$

is a quasi-isomorphism. Let D, resp. D' be the p-adic completion of B, resp. P and let Ω^i_D, resp. $\Omega^i_{D'}$ be the p-adic completion of $\Omega^i_{B/A,\delta}$, resp. $\Omega^i_{P/A,\delta}$. Let M be a p-adically complete D-module endowed with an integral connection $\nabla : M \to M \otimes^\wedge_D \Omega^1_D$. Then the map of de Rham complexes

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_{D'}$$

is a quasi-isomorphism.

Proof. Consider the decreasing filtration F^* on $\Omega^*_{B/A,\delta}$ given by the subcomplexes $F^i(\Omega^*_{B/A,\delta}) = \sigma_{\geq i}\Omega^*_{B/A,\delta}$. See Homology, Section 12.15. This induces a decreasing filtration F^* on $\Omega^*_{P/A,\delta}$ by setting

$$F^i(\Omega^*_{P/A,\delta}) = F^i(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.$$

We have a split short exact sequence

$$0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{P/B,\delta} \to 0$$

and the last module is free on $\mathrm{d}x_i$. It follows from this that $F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta}$ is a termwise split injection and that

$$\mathrm{gr}^i_F(\Omega^*_{P/A,\delta}) = \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. Thus we can define a filtration F^* on $M \otimes_B \Omega^*_{B/A,\delta}$ by setting

$$F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B F^i(\Omega^*_{P/A,\delta})$$

and we have

$$\mathrm{gr}^i_F(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. By Lemma 60.20.1 each of these complexes is quasi-isomorphic to $M \otimes_B \Omega^i_{B/A,\delta}$ placed in degree 0. Hence we see that the first displayed map of the lemma is a morphism of filtered complexes which induces a quasi-isomorphism on graded pieces. This implies that it is a quasi-isomorphism, for example by the spectral sequence associated to a filtered complex, see Homology, Section 12.24.

The proof of the second quasi-isomorphism is exactly the same. □

60.21. Cohomology in the affine case

07LE Let's go back to the situation studied in Section 60.17. We start with (A, I, γ) and $A/I \to C$ and set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. Then we choose a polynomial ring P over A and a surjection $P \to C$ with kernel J. We obtain D and $D(n)$ see (60.17.0.1) and (60.17.0.4). Set $T(n)_e = \mathrm{Spec}(D(n)/p^eD(n))$ so that $(X, T(n)_e, \delta(n))$ is an object of $\mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules and set

$$M(n) = \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$$

for $n = 0, 1, 2, 3, \ldots$. This forms a cosimplicial module over the cosimplicial ring $D(0), D(1), D(2), \ldots$.

07JN **Proposition 60.21.1.** *With notations as above assume that*

(1) *$\mathcal{F}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $Cris(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective.*

Then the complex

$$M(0) \to M(1) \to M(2) \to \ldots$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Using assumption (1) and Lemma 60.18.2 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is isomorphic to $R\Gamma(\mathcal{C}, \mathcal{F})$. Note that the categories $\mathcal{C}$ used in Lemmas 60.18.2 and 60.18.3 agree. Let $f : T \to T'$ be a closed immersion as in (2). Surjectivity of $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is equivalent to surjectivity of $\mathcal{F}_{T'} \to f_*\mathcal{F}_T$. Hence, if $\mathcal{F}$ satisfies (1) and (2), then we obtain a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_{T'} \to f_*\mathcal{F}_T \to 0$$

of quasi-coherent $\mathcal{O}_{T'}$-modules on T', see Schemes, Section 26.24 and in particular Lemma 26.24.1. Thus, if T' is affine, then we conclude that the restriction map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$ is surjective by the vanishing of $H^1(T', \mathcal{K})$, see Cohomology of Schemes, Lemma 30.2.2. Hence the transition maps of the inverse systems in Lemma 60.18.3 are surjective. We conclude that $R^pg_*(\mathcal{F}|_{\mathcal{C}}) = 0$ for all $p \geq 1$ where

g is as in Lemma 60.18.3. The object D of the category $\mathcal{C}^\wedge$ satisfies the assumption of Lemma 60.18.4 by Lemma 60.5.7 with

$$D \times \ldots \times D = D(n)$$

in $\mathcal{C}$ because $D(n)$ is the $n+1$-fold coproduct of D in $\mathrm{Cris}^\wedge(C/A)$, see Lemma 60.17.2. Thus we win. □

07LF **Lemma 60.21.2.** *Assumptions and notation as in Proposition 60.21.1. Then*

$$H^j(\mathit{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Using Lemma 60.12.6 it follows that $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}$ also satisfies assumptions (1) and (2) of Proposition 60.21.1. Write $M(n)_e = \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$ so that $M(n) = \lim_e M(n)_e$. Then

$$\begin{aligned} \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{H}) &= \lim_e M(n)_e \otimes_{D(n)_e} \Omega_{D(n)}/p^e\Omega_{D(n)} \\ &= \lim_e M(n)_e \otimes_{D(n)} \Omega_{D(n)} \end{aligned}$$

By Lemma 60.19.3 the cosimplicial modules

$$M(0)_e \otimes_{D(0)} \Omega^i_{D(0)} \to M(1)_e \otimes_{D(1)} \Omega^i_{D(1)} \to M(2)_e \otimes_{D(2)} \Omega^i_{D(2)} \to \ldots$$

are homotopic to zero. Because the transition maps $M(n)_{e+1} \to M(n)_e$ are surjective, we see that the inverse limit of the associated complexes are acyclic[5]. Hence the vanishing of cohomology of $\mathcal{H}$ by Proposition 60.21.1. □

07LG **Proposition 60.21.3.** *Assumptions as in Proposition 60.21.1 but now assume that $\mathcal{F}$ is a crystal in quasi-coherent modules. Let (M, ∇) be the corresponding module with connection over D, see Proposition 60.17.4. Then the complex*

$$M \otimes^\wedge_D \Omega^*_D$$

computes $R\Gamma(\mathit{Cris}(X/S), \mathcal{F})$.

Proof. We will prove this using the two spectral sequences associated to the double complex $K^{*,*}$ with terms

$$K^{a,b} = M \otimes^\wedge_D \Omega^a_{D(b)}$$

What do we know so far? Well, Lemma 60.19.3 tells us that each column $K^{a,*}$, $a > 0$ is acyclic. Proposition 60.21.1 tells us that the first column $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$. Hence the first spectral sequence associated to the double complex shows that there is a canonical quasi-isomorphism of $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ with $\mathrm{Tot}(K^{*,*})$.

Next, let's consider the rows $K^{*,b}$. By Lemma 60.17.1 each of the $b+1$ maps $D \to D(b)$ presents $D(b)$ as the p-adic completion of a divided power polynomial algebra over D. Hence Lemma 60.20.2 shows that the map

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_{D(b)} \Omega^*_{D(b)} = K^{*,b}$$

is a quasi-isomorphism. Note that each of these maps defines the *same* map on cohomology (and even the same map in the derived category) as the inverse is

[5] Actually, they are even homotopic to zero as the homotopies fit together, but we don't need this. The reason for this roundabout argument is that the limit $\lim_e M(n)_e \otimes_{D(n)} \Omega^i_{D(n)}$ isn't the p-adic completion of $M(n) \otimes_{D(n)} \Omega^i_{D(n)}$ as with the assumptions of the lemma we don't know that $M(n)_e = M(n)_{e+1}/p^e M(n)_{e+1}$. If $\mathcal{F}$ is a crystal then this does hold.

given by the co-diagonal map $D(b) \to D$ (corresponding to the multiplication map $P \otimes_A \ldots \otimes_A P \to P$). Hence if we look at the E_1 page of the second spectral sequence we obtain

$$E_1^{a,b} = H^a(M \otimes_D^\wedge \Omega_D^*)$$

with differentials

$$E_1^{a,0} \xrightarrow{0} E_1^{a,1} \xrightarrow{1} E_1^{a,2} \xrightarrow{0} E_1^{a,3} \xrightarrow{1} \ldots$$

as each of these is the alternation sum of the given identifications $H^a(M \otimes_D^\wedge \Omega_D^*) = E_1^{a,0} = E_1^{a,1} = \ldots$. Thus we see that the E_2 page is equal $H^a(M \otimes_D^\wedge \Omega_D^*)$ on the first row and zero elsewhere. It follows that the identification of $M \otimes_D^\wedge \Omega_D^*$ with the first row induces a quasi-isomorphism of $M \otimes_D^\wedge \Omega_D^*$ with $\mathrm{Tot}(K^{*,*})$. □

07LH **Lemma 60.21.4.** *Assumptions as in Proposition 60.21.3. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Let (M', ∇') be the pair over D' corresponding to $\mathcal{F}$, see Lemma 60.17.5. Then the complex*

$$M' \otimes_{D'}^\wedge \Omega_{D'}^*$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Choose $a : D \to D'$ and $b : D' \to D$ as in Lemma 60.17.5. Note that the base change $M = M' \otimes_{D',b} D$ with its connection ∇ corresponds to $\mathcal{F}$. Hence we know that $M \otimes_D^\wedge \Omega_D^*$ computes the crystalline cohomology of $\mathcal{F}$, see Proposition 60.21.3. Hence it suffices to show that the base change maps (induced by a and b)

$$M' \otimes_{D'}^\wedge \Omega_{D'}^* \longrightarrow M \otimes_D^\wedge \Omega_D^* \quad \text{and} \quad M \otimes_D^\wedge \Omega_D^* \longrightarrow M' \otimes_{D'}^\wedge \Omega_{D'}^*$$

are quasi-isomorphisms. Since $a \circ b = \mathrm{id}_{D'}$ we see that the composition one way around is the identity on the complex $M' \otimes_{D'}^\wedge \Omega_{D'}^*$. Hence it suffices to show that the map

$$M \otimes_D^\wedge \Omega_D^* \longrightarrow M \otimes_D^\wedge \Omega_D^*$$

induced by $b \circ a : D \to D$ is a quasi-isomorphism. (Note that we have the same complex on both sides as $M = M' \otimes_{D',b}^\wedge D$, hence $M \otimes_{D,b\circ a}^\wedge D = M' \otimes_{D',b\circ a\circ b}^\wedge D = M' \otimes_{D',b}^\wedge D = M$.) In fact, we claim that for any divided power A-algebra homomorphism $\rho : D \to D$ compatible with the augmentation to C the induced map $M \otimes_D^\wedge \Omega_D^* \to M \otimes_{D,\rho}^\wedge \Omega_D^*$ is a quasi-isomorphism.

Write $\rho(x_i) = x_i + z_i$. The elements z_i are in the divided power ideal of D because ρ is compatible with the augmentation to C. Hence we can factor the map ρ as a composition

$$D \xrightarrow{\sigma} D\langle \xi_i \rangle^\wedge \xrightarrow{\tau} D$$

where the first map is given by $x_i \mapsto x_i + \xi_i$ and the second map is the divided power D-algebra map which maps ξ_i to z_i. (This uses the universal properties of polynomial algebra, divided power polynomial algebras, divided power envelopes, and p-adic completion.) Note that there exists an *automorphism* α of $D\langle \xi_i \rangle^\wedge$ with $\alpha(x_i) = x_i - \xi_i$ and $\alpha(\xi_i) = \xi_i$. Applying Lemma 60.20.2 to $\alpha \circ \sigma$ (which maps x_i to x_i) and using that α is an isomorphism we conclude that σ induces a quasi-isomorphism of $M \otimes_D^\wedge \Omega_D^*$ with $M \otimes_{D,\sigma}^\wedge \Omega_{D\langle x_i \rangle^\wedge}^*$. On the other hand the map τ has as a left inverse the map $D \to D\langle x_i \rangle^\wedge$, $x_i \mapsto x_i$ and we conclude (using Lemma 60.20.2 once more) that τ induces a quasi-isomorphism of $M \otimes_{D,\sigma}^\wedge \Omega_{D\langle x_i \rangle^\wedge}^*$ with

$M \otimes^\wedge_{D,\tau\circ\sigma} \Omega^*_D$. Composing these two quasi-isomorphisms we obtain that ρ induces a quasi-isomorphism $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ as desired. □

60.22. Two counter examples

07LI Before we turn to some of the successes of crystalline cohomology, let us give two examples which explain why crystalline cohomology does not work very well if the schemes in question are either not proper over the base, or singular. The first example can be found in [**BO83**].

07LJ **Example 60.22.1.** Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x,y]/(x^2, xy, y^2)$. We choose the presentation

$$C = P/J = \mathbf{Z}_p[x,y]/(x^2, xy, y^2, p)$$

Let $D = D_{P,\gamma}(J)^\wedge$ with divided power ideal $(\bar{J}, \bar{\gamma})$ as in Section 60.17. We will denote x, y also the images of x and y in D. Consider the element

$$\tau = \bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) - \bar{\gamma}_p(xy)^2 \in D$$

We note that $p\tau = 0$ as

$$p!\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) = x^{2p}\bar{\gamma}_p(y^2) = \bar{\gamma}_p(x^2y^2) = x^py^p\bar{\gamma}_p(xy) = p!\bar{\gamma}_p(xy)^2$$

in D. We also note that $\mathrm{d}\tau = 0$ in Ω_D as

$$\begin{aligned}
\mathrm{d}(\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2)) &= \bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x^2 + \bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y^2 \\
&= 2x\bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x + 2y\bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y \\
&= 2/(p-1)!(x^{2p-1}\bar{\gamma}_p(y^2)\mathrm{d}x + y^{2p-1}\bar{\gamma}_p(x^2)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}\bar{\gamma}_p(xy^2)\mathrm{d}x + y^{p-1}\bar{\gamma}_p(x^2y)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}y^p\bar{\gamma}_p(xy)\mathrm{d}x + x^py^{p-1}\bar{\gamma}_p(xy)\mathrm{d}y) \\
&= 2\bar{\gamma}_{p-1}(xy)\bar{\gamma}_p(xy)(y\mathrm{d}x + x\mathrm{d}y) \\
&= \mathrm{d}(\bar{\gamma}_p(xy)^2)
\end{aligned}$$

Finally, we claim that $\tau \neq 0$ in D. To see this it suffices to produce an object $(B \to \mathbf{F}_p[x,y]/(x^2,xy,y^2), \delta)$ of $\mathrm{Cris}(C/S)$ such that τ does not map to zero in B. To do this take

$$B = \mathbf{F}_p[x,y,u,v]/(x^3, x^2y, xy^2, y^3, xu, yu, xv, yv, u^2, v^2)$$

with the obvious surjection to C. Let $K = \mathrm{Ker}(B \to C)$ and consider the map

$$\delta_p : K \longrightarrow K, \quad ax^2 + bxy + cy^2 + du + ev + fuv \longmapsto a^pu + c^pv$$

One checks this satisfies the assumptions (1), (2), (3) of Divided Power Algebra, Lemma 23.5.3 and hence defines a divided power structure. Moreover, we see that τ maps to uv which is not zero in B. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We draw the following conclusions

(1) $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ has p-torsion, and
(2) pulling back by Frobenius $F^* : H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \to H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not injective.

Namely, τ defines a nonzero torsion element of $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ by Proposition 60.21.3. Similarly, $F^*(\tau) = \sigma(\tau)$ where $\sigma : D \to D$ is the map induced by any lift of Frobenius on P. If we choose $\sigma(x) = x^p$ and $\sigma(y) = y^p$, then an easy computation shows that $F^*(\tau) = 0$.

The next example shows that even for affine n-space crystalline cohomology does not give the correct thing.

07LK **Example 60.22.2.** Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x_1, \ldots, x_r]$. We choose the presentation

$$C = P/J = P/pP \quad \text{with} \quad P = \mathbf{Z}_p[x_1, \ldots, x_r]$$

Note that pP has divided powers by Divided Power Algebra, Lemma 23.4.2. Hence setting $D = P^\wedge$ with divided power ideal (p) we obtain a situation as in Section 60.17. We conclude that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is represented by the complex

$$D \to \Omega^1_D \to \Omega^2_D \to \ldots \to \Omega^r_D$$

see Proposition 60.21.3. Assuming $r > 0$ we conclude the following

(1) The cristalline cohomology of the cristalline structure sheaf of $X = \mathbf{A}^r_{\mathbf{F}_p}$ over $S = \mathrm{Spec}(\mathbf{Z}_p)$ is zero except in degrees $0, \ldots, r$.
(2) We have $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) = \mathbf{Z}_p$.
(3) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is infinite and is not a torsion abelian group.
(4) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not separated for the p-adic topology.

While the first two statements are reasonable, parts (3) and (4) are disconcerting! The truth of these statements follows immediately from working out what the complex displayed above looks like. Let's just do this in case $r = 1$. Then we are just looking at the two term complex of p-adically complete modules

$$\mathrm{d} : D = \Big(\bigoplus\nolimits_{n \geq 0} \mathbf{Z}_p x^n\Big)^\wedge \longrightarrow \Omega^1_D = \Big(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p x^{n-1} \mathrm{d}x\Big)^\wedge$$

The map is given by $\mathrm{diag}(0, 1, 2, 3, 4, \ldots)$ except that the first summand is missing on the right hand side. Now it is clear that $\bigoplus_{n>0} \mathbf{Z}_p/n\mathbf{Z}_p$ is a subgroup of the cokernel, hence the cokernel is infinite. In fact, the element

$$\omega = \sum\nolimits_{e>0} p^e x^{p^{2e}-1} \mathrm{d}x$$

is clearly not a torsion element of the cokernel. But it gets worse. Namely, consider the element

$$\eta = \sum\nolimits_{e>0} p^e x^{p^e-1} \mathrm{d}x$$

For every $t > 0$ the element η is congruent to $\sum_{e>t} p^e x^{p^e-1} \mathrm{d}x$ modulo the image of d which is divisible by p^t. But η is not in the image of d because it would have to be the image of $a + \sum_{e>0} x^{p^e}$ for some $a \in \mathbf{Z}_p$ which is not an element of the left hand side. In fact, $p^N \eta$ is similarly not in the image of d for any integer N. This implies that η "generates" a copy of $\mathbf{Q}_p$ inside of $H^1_{\mathrm{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/\mathrm{Spec}(\mathbf{Z}_p))$.

60.23. Applications

07LL In this section we collect some applications of the material in the previous sections.

07LM **Proposition 60.23.1.** *In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on Cris(X/S). The truncation map of complexes*

$$(\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots) \longrightarrow \mathcal{F}[0],$$

while not a quasi-isomorphism, becomes a quasi-isomorphism after applying $Ru_{X/S,*}$. *In fact, for any* $i > 0$, *we have*

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0.$$

Proof. By Lemma 60.15.1 we get a de Rham complex as indicated in the lemma. We abbreviate $\mathcal{H} = \mathcal{F} \otimes \Omega^i_{X/S}$. Let $X' \subset X$ be an affine open subscheme which maps into an affine open subscheme $S' \subset S$. Then

$$(Ru_{X/S,*}\mathcal{H})|_{X'_{Zar}} = Ru_{X'/S',*}(\mathcal{H}|_{\mathrm{Cris}(X'/S')}),$$

see Lemma 60.9.5. Thus Lemma 60.21.2 shows that $Ru_{X/S,*}\mathcal{H}$ is a complex of sheaves on X_{Zar} whose cohomology on any affine open is trivial. As X has a basis for its topology consisting of affine opens this implies that $Ru_{X/S,*}\mathcal{H}$ is quasi-isomorphic to zero. □

07LN **Remark 60.23.2.** The proof of Proposition 60.23.1 shows that the conclusion

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for $i > 0$ is true for any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ which satisfies conditions (1) and (2) of Proposition 60.21.1. This applies to the following non-crystals: $\Omega^i_{X/S}$ for all i, and any sheaf of the form $\underline{\mathcal{F}}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, it applies to the sheaf $\underline{\mathcal{O}_X} = \underline{\mathbf{G}_a}$. But note that we need something like Lemma 60.15.1 to produce a de Rham complex which requires $\mathcal{F}$ to be a crystal. Hence (currently) the collection of sheaves of modules for which the full statement of Proposition 60.23.1 holds is exactly the category of crystals in quasi-coherent modules.

In Situation 60.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Proposition 60.23.1 allows us to construct a canonical map

07LP (60.23.2.1) $$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \longrightarrow R\Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T/S,\delta})$$

Namely, we have $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes \Omega^*_{X/S})$, we can restrict global cohomology classes to T, and $\Omega_{X/S}$ restricts to $\Omega_{T/S,\delta}$ by Lemma 60.12.3.

60.24. Some further results

07MI In this section we mention some results whose proof is missing. We will formulate these as a series of remarks and we will convert them into actual lemmas and propositions only when we add detailed proofs.

07MJ **Remark 60.24.1** (Higher direct images)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and X'. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\mathrm{Cris}(X/S)$. Then $Rf_{\mathrm{cris},*}\mathcal{F}$ can be computed as follows.

Given an object (U', T', δ') of $\mathrm{Cris}(X'/S')$ set $U = X \times_{X'} U' = f^{-1}(U')$ (an open subscheme of X). Denote (T_0, T, δ) the divided power scheme over S such that

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

is cartesian in the category of divided power schemes, see Lemma 60.7.4. There is an induced morphism $U \to T_0$ and we obtain a morphism $(U/T)_{\mathrm{cris}} \to (X/S)_{\mathrm{cris}}$, see Remark 60.9.3. Let $\mathcal{F}_U$ be the pullback of $\mathcal{F}$. Let $\tau_{U/T} : (U/T)_{\mathrm{cris}} \to T_{Zar}$ be the structure morphism. Then we have

07MK
$$(Rf_{\mathrm{cris},*}\mathcal{F})_{T'} = R(T \to T')_* \left(R\tau_{U/T,*}\mathcal{F}_U\right) \tag{60.24.1.1}$$

where the left hand side is the restriction (see Section 60.10).

Hints: First, show that $\mathrm{Cris}(U/T)$ is the localization (in the sense of Sites, Lemma 7.30.3) of $\mathrm{Cris}(X/S)$ at the sheaf of sets $f_{\mathrm{cris}}^{-1} h_{(U',T',\delta')}$. Next, reduce the statement to the case where $\mathcal{F}$ is an injective module and pushforward of modules using that the pullback of an injective $\mathcal{O}_{X/S}$-module is an injective $\mathcal{O}_{U/T}$-module on $\mathrm{Cris}(U/T)$. Finally, check the result holds for plain pushforward.

07ML **Remark 60.24.2** (Mayer-Vietoris)**.** In the situation of Remark 60.24.1 suppose we have an open covering $X = X' \cup X''$. Denote $X''' = X' \cap X''$. Let f', f'', and f'' be the restriction of f to X', X'', and X'''. Moreover, let $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ be the restriction of $\mathcal{F}$ to the crystalline sites of X', X'', and X'''. Then there exists a distinguished triangle

$$Rf_{\mathrm{cris},*}\mathcal{F} \longrightarrow Rf'_{\mathrm{cris},*}\mathcal{F}' \oplus Rf''_{\mathrm{cris},*}\mathcal{F}'' \longrightarrow Rf'''_{\mathrm{cris},*}\mathcal{F}''' \longrightarrow Rf_{\mathrm{cris},*}\mathcal{F}[1]$$

in $D(\mathcal{O}_{X'/S'})$.

Hints: This is a formal consequence of the fact that the subcategories $\mathrm{Cris}(X'/S)$, $\mathrm{Cris}(X''/S)$, $\mathrm{Cris}(X'''/S)$ correspond to open subobjects of the final sheaf on $\mathrm{Cris}(X/S)$ and that the last is the intersection of the first two.

07MM **Remark 60.24.3** (Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated[6] scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose an affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct a Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

Given $n \geq 0$ and $\lambda_0, \ldots, \lambda_n \in \Lambda$ write $U_{\lambda_0 \ldots \lambda_n} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_n}$. This is an affine scheme by assumption. Write $U_{\lambda_0 \ldots \lambda_n} = \mathrm{Spec}(C_{\lambda_0 \ldots \lambda_n})$. Set

$$P_{\lambda_0 \ldots \lambda_n} = P_{\lambda_0} \otimes_A \ldots \otimes_A P_{\lambda_n}$$

which comes with a canonical surjection onto $C_{\lambda_0 \ldots \lambda_n}$. Denote the kernel $J_{\lambda_0 \ldots \lambda_n}$ and set $D_{\lambda_0 \ldots \lambda_n}$ the p-adically completed divided power envelope of $J_{\lambda_0 \ldots \lambda_n}$ in $P_{\lambda_0 \ldots \lambda_n}$ relative to γ. Let $M_{\lambda_0 \ldots \lambda_n}$ be the $P_{\lambda_0 \ldots \lambda_n}$-module corresponding to the

[6] This assumption is not strictly necessary, as using hypercoverings the construction of the remark can be extended to the general case.

restriction of $\mathcal{F}$ to $\mathrm{Cris}(U_{\lambda_0\ldots\lambda_n}/S)$ via Proposition 60.17.4. By construction we obtain a cosimplicial divided power ring $D(*)$ having in degree n the ring

$$D(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} D_{\lambda_0\ldots\lambda_n}$$

(use that divided power envelopes are functorial and the trivial cosimplicial structure on the ring $P(*)$ defined similarly). Since $M_{\lambda_0\ldots\lambda_n}$ is the "value" of $\mathcal{F}$ on the objects $\mathrm{Spec}(D_{\lambda_0\ldots\lambda_n})$ we see that $M(*)$ defined by the rule

$$M(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} M_{\lambda_0\ldots\lambda_n}$$

forms a cosimplicial $D(*)$-module. Now we claim that we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = s(M(*))$$

Here $s(-)$ denotes the cochain complex associated to a cosimplicial module (see Simplicial, Section 14.25).

Hints: The proof of this is similar to the proof of Proposition 60.21.1 (in particular the result holds for any module satisfying the assumptions of that proposition).

07MN **Remark 60.24.4** (Alternating Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated quasi-compact scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose a finite affine open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ of X and a total ordering on Λ. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct an alternating Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

We are going to use the notation introduced in Remark 60.24.3. Denote $\Omega_{\lambda_0\ldots\lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0\ldots\lambda_n}$ over A compatible with the divided power structure. Let ∇ be the integrable connection on $M_{\lambda_0\ldots\lambda_n}$ coming from Proposition 60.17.4. Consider the double complex $M^{\bullet,\bullet}$ with terms

$$M^{n,m} = \bigoplus\nolimits_{\lambda_0<\ldots<\lambda_n} M_{\lambda_0\ldots\lambda_n} \otimes^\wedge_{D_{\lambda_0\ldots\lambda_n}} \Omega^m_{D_{\lambda_0\ldots\lambda_n}}.$$

For the differential d_1 (increasing n) we use the usual Čech differential and for the differential d_2 we use the connection, i.e., the differential of the de Rham complex. We claim that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = \mathrm{Tot}(M^{\bullet,\bullet})$$

Here $\mathrm{Tot}(-)$ denotes the total complex associated to a double complex, see Homology, Definition 12.18.3.

Hints: We have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S})$$

by Proposition 60.23.1. The right hand side of the formula is simply the alternating Čech complex for the covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ (which induces an open covering of the final sheaf of $\mathrm{Cris}(X/S)$) and the complex $\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S}$, see Proposition 60.21.3. Now the result follows from a general result in cohomology on sites, namely that the alternating Čech complex computes the cohomology provided it gives the correct answer on all the pieces (insert future reference here).

07MP **Remark 60.24.5** (Quasi-coherence)**.** In the situation of Remark 60.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is quasi-compact and quasi-separated. Then for a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ the sheaves $R^i f_{\text{cris},*}\mathcal{F}$ are locally quasi-coherent.

Hints: We have to show that the restrictions to T' are quasi-coherent $\mathcal{O}_{T'}$-modules, where (U', T', δ') is any object of $\text{Cris}(X'/S')$. It suffices to do this when T' is affine. We use the formula (60.24.1.1), the fact that $T \to T'$ is quasi-compact and quasi-separated (as T is affine over the base change of T' by $S \to S'$), and Cohomology of Schemes, Lemma 30.4.5 to see that it suffices to show that the sheaves $R^i\tau_{U/T,*}\mathcal{F}_U$ are quasi-coherent. Note that $U \to T_0$ is also quasi-compact and quasi-separated, see Schemes, Lemmas 26.21.14 and 26.21.14.

This reduces us to proving that $R^i\tau_{X/S,*}\mathcal{F}$ is quasi-coherent on S in the case that p locally nilpotent on S. Here $\tau_{X/S}$ is the structure morphism, see Remark 60.9.6. We may work locally on S, hence we may assume S affine (see Lemma 60.9.5). Induction on the number of affines covering X and Mayer-Vietoris (Remark 60.24.2) reduces the question to the case where X is also affine (as in the proof of Cohomology of Schemes, Lemma 30.4.5). Say $X = \text{Spec}(C)$ and $S = \text{Spec}(A)$ so that (A, I, γ) and $A \to C$ are as in Situation 60.5.1. Choose a polynomial algebra P over A and a surjection $P \to C$ as in Section 60.17. Let (M, ∇) be the module corresponding to $\mathcal{F}$, see Proposition 60.17.4. Applying Proposition 60.21.3 we see that $R\Gamma(\text{Cris}(X/S), \mathcal{F})$ is represented by $M \otimes_D \Omega^*_D$. Note that completion isn't necessary as p is nilpotent in A! We have to show that this is compatible with taking principal opens in $S = \text{Spec}(A)$. Suppose that $g \in A$. Then we conclude that similarly $R\Gamma(\text{Cris}(X_g/S_g), \mathcal{F})$ is computed by $M_g \otimes_{D_g} \Omega^*_{D_g}$ (again this uses that p-adic completion isn't necessary). Hence we conclude because localization is an exact functor on A-modules.

07MQ **Remark 60.24.6** (Boundedness)**.** In the situation of Remark 60.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is of finite type and quasi-separated. Then there exists an integer i_0 such that for any crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ we have $R^i f_{\text{cris},*}\mathcal{F} = 0$ for all $i > i_0$.

Hints: Arguing as in Remark 60.24.5 (using Cohomology of Schemes, Lemma 30.4.5) we reduce to proving that $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i \gg 0$ in the situation of Proposition 60.21.3 when C is a finite type algebra over A. This is clear as we can choose a finite polynomial algebra and we see that $\Omega^i_D = 0$ for $i \gg 0$.

07MR **Remark 60.24.7** (Specific boundedness)**.** In Situation 60.7.5 let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Assume that S_0 has a unique point and that $X \to S_0$ is of finite presentation.

(1) If $\dim X = d$ and X/S_0 has embedding dimension e, then $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i > d + e$.
(2) If X is separated and can be covered by q affines, and X/S_0 has embedding dimension e, then $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i > q + e$.

Hints: In case (1) we can use that

$$H^i(\text{Cris}(X/S), \mathcal{F}) = H^i(X_{Zar}, Ru_{X/S,*}\mathcal{F})$$

and that $Ru_{X/S,*}\mathcal{F}$ is locally calculated by a de Rham complex constructed using an embedding of X into a smooth scheme of dimension e over S (see Lemma 60.21.4).

These de Rham complexes are zero in all degrees $> e$. Hence (1) follows from Cohomology, Proposition 20.20.7. In case (2) we use the alternating Čech complex (see Remark 60.24.4) to reduce to the case X affine. In the affine case we prove the result using the de Rham complex associated to an embedding of X into a smooth scheme of dimension e over S (it takes some work to construct such a thing).

07MS **Remark 60.24.8** (Base change map)**.** In the situation of Remark 60.24.1 assume $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ are affine. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'/S'}$-module. Let $\mathcal{F}$ be the pullback of $\mathcal{F}'$. Then there is a canonical base change map

$$L(S' \to S)^* R\tau_{X'/S',*}\mathcal{F}' \longrightarrow R\tau_{X/S,*}\mathcal{F}$$

where $\tau_{X/S}$ and $\tau_{X'/S'}$ are the structure morphisms, see Remark 60.9.6. On global sections this gives a base change map

07MT (60.24.8.1) $$R\Gamma(\mathrm{Cris}(X'/S'), \mathcal{F}') \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$$

in $D(A)$.

Hint: Compose the very general base change map of Cohomology on Sites, Remark 21.19.3 with the canonical map $Lf^*_{\mathrm{cris}}\mathcal{F}' \to f^*_{\mathrm{cris}}\mathcal{F}' = \mathcal{F}$.

07MU **Remark 60.24.9** (Base change isomorphism)**.** The map (60.24.8.1) is an isomorphism provided all of the following conditions are satisfied:

(1) p is nilpotent in A',
(2) $\mathcal{F}'$ is a crystal in quasi-coherent $\mathcal{O}_{X'/S'}$-modules,
(3) $X' \to S'_0$ is a quasi-compact, quasi-separated morphism,
(4) $X = X' \times_{S'_0} S_0$,
(5) $\mathcal{F}'$ is a flat $\mathcal{O}_{X'/S'}$-module,
(6) $X' \to S'_0$ is a local complete intersection morphism (see More on Morphisms, Definition 37.62.2; this holds for example if $X' \to S'_0$ is syntomic or smooth),
(7) X' and S_0 are Tor independent over S'_0 (see More on Algebra, Definition 15.61.1; this holds for example if either $S_0 \to S'_0$ or $X' \to S'_0$ is flat).

Hints: Condition (1) means that in the arguments below p-adic completion does nothing and can be ignored. Using condition (3) and Mayer Vietoris (see Remark 60.24.2) this reduces to the case where X' is affine. In fact by condition (6), after shrinking further, we can assume that $X' = \mathrm{Spec}(C')$ and we are given a presentation $C' = A'/I'[x_1, \ldots, x_n]/(\bar{f}'_1, \ldots, \bar{f}'_c)$ where $\bar{f}'_1, \ldots, \bar{f}'_c$ is a Koszul-regular sequence in A'/I'. (This means that smooth locally $\bar{f}'_1, \ldots, \bar{f}'_c$ forms a regular sequence, see More on Algebra, Lemma 15.30.17.) We choose a lift of $\bar{f}'_i$ to an element $f'_i \in A'[x_1, \ldots, x_n]$. By (4) we see that $X = \mathrm{Spec}(C)$ with $C = A/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ where $f_i \in A[x_1, \ldots, x_n]$ is the image of f'_i. By property (7) we see that $\bar{f}_1, \ldots, \bar{f}_c$ is a Koszul-regular sequence in $A/I[x_1, \ldots, x_n]$. The divided power envelope of $I'A'[x_1, \ldots, x_n] + (f'_1, \ldots, f'_c)$ in $A'[x_1, \ldots, x_n]$ relative to γ' is

$$D' = A'[x_1, \ldots, x_n]\langle\xi_1, \ldots, \xi_c\rangle/(\xi_i - f'_i)$$

see Lemma 60.2.4. Then you check that $\xi_1 - f'_1, \ldots, \xi_n - f'_n$ is a Koszul-regular sequence in the ring $A'[x_1, \ldots, x_n]\langle\xi_1, \ldots, \xi_c\rangle$. Similarly the divided power envelope of $IA[x_1, \ldots, x_n] + (f_1, \ldots, f_c)$ in $A[x_1, \ldots, x_n]$ relative to γ is

$$D = A[x_1, \ldots, x_n]\langle\xi_1, \ldots, \xi_c\rangle/(\xi_i - f_i)$$

and $\xi_1 - f_1, \ldots, \xi_n - f_n$ is a Koszul-regular sequence in the ring $A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle$. It follows that $D' \otimes^{\mathbf{L}}_{A'} A = D$. Condition (2) implies $\mathcal{F}'$ corresponds to a pair (M', ∇) consisting of a D'-module with connection, see Proposition 60.17.4. Then $M = M' \otimes_{D'} D$ corresponds to the pullback $\mathcal{F}$. By assumption (5) we see that M' is a flat D'-module, hence

$$M = M' \otimes_{D'} D = M' \otimes_{D'} D' \otimes^{\mathbf{L}}_{A'} A = M' \otimes^{\mathbf{L}}_{A'} A$$

Since the modules of differentials $\Omega_{D'}$ and Ω_D (as defined in Section 60.17) are free D'-modules on the same generators we see that

$$M \otimes_D \Omega^\bullet_D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes_{D'} D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes^{\mathbf{L}}_{A'} A$$

which proves what we want by Proposition 60.21.3.

07MV **Remark 60.24.10** (Rlim)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A an algebra over $\mathbf{Z}_{(p)}$ with p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a scheme over S_0 with p locally nilpotent on X. Let $\mathcal{F}$ be any $\mathcal{O}_{X/S}$-module. For $e \gg 0$ we have $(p^e) \subset I$ is preserved by γ, see Divided Power Algebra, Lemma 23.4.5. Set $S_e = \mathrm{Spec}(A/p^e A)$ for $e \gg 0$. Then $\mathrm{Cris}(X/S_e)$ is a full subcategory of $\mathrm{Cris}(X/S)$ and we denote $\mathcal{F}_e$ the restriction of $\mathcal{F}$ to $\mathrm{Cris}(X/S_e)$. Then

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\lim_e R\Gamma(\mathrm{Cris}(X/S_e), \mathcal{F}_e)$$

Hints: Suffices to prove this for $\mathcal{F}$ injective. In this case the sheaves $\mathcal{F}_e$ are injective modules too, the transition maps $\Gamma(\mathcal{F}_{e+1}) \to \Gamma(\mathcal{F}_e)$ are surjective, and we have $\Gamma(\mathcal{F}) = \lim_e \Gamma(\mathcal{F}_e)$ because any object of $\mathrm{Cris}(X/S)$ is locally an object of one of the categories $\mathrm{Cris}(X/S_e)$ by definition of $\mathrm{Cris}(X/S)$.

07MW **Remark 60.24.11** (Comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) γ extends to a divided power structure on the ideal of X in Y so that (X, Y, γ) is an object of $\mathrm{Cris}(X/S)$,
(2) the restriction $\mathcal{F}_Y$ (see Section 60.10) comes endowed with a canonical integrable connection $\nabla : \mathcal{F}_Y \to \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and
(3) we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: See Divided Power Algebra, Lemma 23.4.2 for (1). See Lemma 60.15.1 for (2). For Part (3) note that there is a map, see (60.23.2.1). This map is an isomorphism when X is affine, see Lemma 60.21.4. This shows that $Ru_{X/S,*}\mathcal{F}$ and $\mathcal{F}_Y \otimes \Omega^\bullet_{Y/S}$ are quasi-isomorphic as complexes on $Y_{Zar} = X_{Zar}$. Since $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(X_{Zar}, Ru_{X/S,*}\mathcal{F})$ the result follows.

07MX **Remark 60.24.12** (Perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: By Remark 60.24.9 we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I \cong R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)})$$

By Remark 60.24.11 we have

$$R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)}) = R\Gamma(X, \mathcal{F}_X \otimes \Omega^\bullet_{X/S_0})$$

Using the stupid filtration on the de Rham complex we see that the last displayed complex is perfect in $D(A/I)$ as soon as the complexes

$$R\Gamma(X, \mathcal{F}_X \otimes \Omega^q_{X/S_0})$$

are perfect complexes in $D(A/I)$, see More on Algebra, Lemma 15.75.4. This is true by standard arguments in coherent cohomology using that $\mathcal{F}_X \otimes \Omega^q_{X/S_0}$ is a finite locally free sheaf and $X \to S_0$ is proper and flat (insert future reference here). Applying More on Algebra, Lemma 15.79.4 we see that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I^n$$

is a perfect object of $D(A/I^n)$ for all n. This isn't quite enough unless A is Noetherian. Namely, even though I is locally nilpotent by our assumption that p is nilpotent, see Divided Power Algebra, Lemma 23.2.6, we cannot conclude that $I^n = 0$ for some n. A counter example is $\mathbf{F}_p\langle x\rangle$. To prove it in general when $\mathcal{F} = \mathcal{O}_{X/S}$ the argument of `https://math.columbia.edu/~dejong/wordpress/?p=2227` works. When the coefficients $\mathcal{F}$ are non-trivial the argument of [**Fal99**] seems to be as follows. Reduce to the case $pA = 0$ by More on Algebra, Lemma 15.79.4. In this case the Frobenius map $A \to A$, $a \mapsto a^p$ factors as $A \to A/I \xrightarrow{\varphi} A$ (as $x^p = 0$ for $x \in I$). Set $X^{(1)} = X \otimes_{A/I,\varphi} A$. The absolute Frobenius morphism of X factors through a morphism $F_X : X \to X^{(1)}$ (a kind of relative Frobenius). Affine locally if $X = \mathrm{Spec}(C)$ then $X^{(1)} = \mathrm{Spec}(C \otimes_{A/I,\varphi} A)$ and F_X corresponds to $C \otimes_{A/I,\varphi} A \to C$, $c \otimes a \mapsto c^p a$. This defines morphisms of ringed topoi

$$(X/S)_{\mathrm{cris}} \xrightarrow{(F_X)_{\mathrm{cris}}} (X^{(1)}/S)_{\mathrm{cris}} \xrightarrow{u_{X^{(1)}/S}} Sh(X^{(1)}_{Zar})$$

whose composition is denoted Frob_X. One then shows that $R\mathrm{Frob}_{X,*}\mathcal{F}$ is representable by a perfect complex of $\mathcal{O}_{X^{(1)}}$-modules(!) by a local calculation.

07MY **Remark 60.24.13** (Complete perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: We know that $K = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is the derived limit $K = R\lim K_e$ of the cohomologies over A/p^eA, see Remark 60.24.10. Each K_e is a perfect complex of $D(A/p^eA)$ by Remark 60.24.12. Since A is p-adically complete the result follows from More on Algebra, Lemma 15.98.4.

07MZ **Remark 60.24.14** (Complete comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a proper smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a finite type crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) there exists a coherent $\mathcal{O}_Y$-module $\mathcal{F}_Y$ endowed with integrable connection
$$\nabla : \mathcal{F}_Y \longrightarrow \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$
such that $\mathcal{F}_Y/p^e\mathcal{F}_Y$ is the module with connection over A/p^eA found in Remark 60.24.11, and
(2) we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: The existence of $\mathcal{F}_Y$ is Grothendieck's existence theorem (insert future reference here). The isomorphism of cohomologies follows as both sides are computed as $R\lim$ of the versions modulo p^e (see Remark 60.24.10 for the left hand side; use the theorem on formal functions, see Cohomology of Schemes, Theorem 30.20.5 for the right hand side). Each of the versions modulo p^e are isomorphic by Remark 60.24.11.

60.25. Pulling back along purely inseparable maps

07PZ By an α_p-cover we mean a morphism of the form
$$X' = \mathrm{Spec}(C[z]/(z^p - c)) \longrightarrow \mathrm{Spec}(C) = X$$
where C is an $\mathbf{F}_p$-algebra and $c \in C$. Equivalently, X' is an α_p-torsor over X. An *iterated α_p-cover*[7] is a morphism of schemes in characteristic p which is locally on the target a composition of finitely many α_p-covers. In this section we prove that pullback along such a morphism induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. In fact, we prove a precise version of this result. We begin with a preliminary lemma whose formulation needs some notation.

Assume we have a ring map $B \to B'$ and quotients $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ satisfying the assumptions of Remark 60.6.9. Thus (60.6.9.1) provides a canonical map of complexes
$$c^\bullet_M : M \otimes_B \Omega^\bullet \longrightarrow M \otimes_B (\Omega')^\bullet$$
for all B-modules M endowed with integrable connection $\nabla : M \to M \otimes_B \Omega_B$.

Suppose we have $a \in B$, $z \in B'$, and a map $\theta : B' \to B'$ satisfying the following assumptions

07Q0 (1) $\mathrm{d}(a) = 0$,
07Q1 (2) $\Omega' = B' \otimes_B \Omega \oplus B'\mathrm{d}z$; we write $\mathrm{d}(f) = \mathrm{d}_1(f) + \partial_z(f)\mathrm{d}z$ with $\mathrm{d}_1(f) \in B' \otimes \Omega$ and $\partial_z(f) \in B'$ for all $f \in B'$,
07Q2 (3) $\theta : B' \to B'$ is B-linear,
07Q3 (4) $\partial_z \circ \theta = a$,
07Q4 (5) $B \to B'$ is universally injective (and hence $\Omega \to \Omega'$ is injective),
07Q5 (6) $af - \theta(\partial_z(f)) \in B$ for all $f \in B'$,
07Q6 (7) $(\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f)) \in \Omega$ for all $f \in B'$ where $\theta \otimes 1 : B' \otimes \Omega \to B' \otimes \Omega$

These conditions are not logically independent. For example, assumption (4) implies that $\partial_z(af - \theta(\partial_z(f))) = 0$. Hence if the image of $B \to B'$ is the collection of elements annihilated by ∂_z, then (6) follows. A similar argument can be made for condition (7).

[7] This is nonstandard notation.

07Q7 **Lemma 60.25.1.** *In the situation above there exists a map of complexes*

$$e_M^\bullet : M \otimes_B (\Omega')^\bullet \longrightarrow M \otimes_B \Omega^\bullet$$

such that $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by a.

Proof. In this proof all tensor products are over B. Assumption (2) implies that

$$M \otimes (\Omega')^i = (B' \otimes M \otimes \Omega^i) \oplus (B'\mathrm{d}z \otimes M \otimes \Omega^{i-1})$$

for all $i \geq 0$. A collection of additive generators for $M \otimes (\Omega')^i$ is formed by elements of the form $f\omega$ and elements of the form $f\mathrm{d}z \wedge \eta$ where $f \in B'$, $\omega \in M \otimes \Omega^i$, and $\eta \in M \otimes \Omega^{i-1}$.

For $f \in B'$ we write

$$\epsilon(f) = af - \theta(\partial_z(f)) \quad \text{and} \quad \epsilon'(f) = (\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f))$$

so that $\epsilon(f) \in B$ and $\epsilon'(f) \in \Omega$ by assumptions (6) and (7). We define $e_M^\bullet$ by the rules $e_M^i(f\omega) = \epsilon(f)\omega$ and $e_M^i(f\mathrm{d}z \wedge \eta) = \epsilon'(f) \wedge \eta$. We will see below that the collection of maps e_M^i is a map of complexes.

We define

$$h^i : M \otimes_B (\Omega')^i \longrightarrow M \otimes_B (\Omega')^{i-1}$$

by the rules $h^i(f\omega) = 0$ and $h^i(f\mathrm{d}z \wedge \eta) = \theta(f)\eta$ for elements as above. We claim that

$$\mathrm{d} \circ h + h \circ \mathrm{d} = a - c_M^\bullet \circ e_M^\bullet$$

Note that multiplication by a is a map of complexes by (1). Hence, since $c_M^\bullet$ is an injective map of complexes by assumption (5), we conclude that $e_M^\bullet$ is a map of complexes. To prove the claim we compute

$$\begin{aligned}
(\mathrm{d} \circ h + h \circ \mathrm{d})(f\omega) &= h\left(\mathrm{d}(f) \wedge \omega + f\nabla(\omega)\right) \\
&= \theta(\partial_z(f))\omega \\
&= af\omega - \epsilon(f)\omega \\
&= af\omega - c_M^i(e_M^i(f\omega))
\end{aligned}$$

The second equality because $\mathrm{d}z$ does not occur in $\nabla(\omega)$ and the third equality by assumption (6). Similarly, we have

$$\begin{aligned}
(\mathrm{d} \circ h + h \circ \mathrm{d})(f\mathrm{d}z \wedge \eta) &= \mathrm{d}(\theta(f)\eta) + h\left(\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta - f\mathrm{d}z \wedge \nabla(\eta)\right) \\
&= \mathrm{d}(\theta(f)) \wedge \eta + \theta(f)\nabla(\eta) - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta - \theta(f)\nabla(\eta) \\
&= \mathrm{d}_1(\theta(f)) \wedge \eta + \partial_z(\theta(f))\mathrm{d}z \wedge \eta - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta \\
&= af\mathrm{d}z \wedge \eta - \epsilon'(f) \wedge \eta \\
&= af\mathrm{d}z \wedge \eta - c_M^i(e_M^i(f\mathrm{d}z \wedge \eta))
\end{aligned}$$

The second equality because $\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta = -\mathrm{d}z \wedge \mathrm{d}_1(f) \wedge \eta$. The fourth equality by assumption (4). On the other hand it is immediate from the definitions that $e_M^i(c_M^i(\omega)) = \epsilon(1)\omega = a\omega$. This proves the lemma. □

07Q8 **Example 60.25.2.** A standard example of the situation above occurs when $B' = B\langle z \rangle$ is the divided power polynomial ring over a divided power ring (B, J, δ) with divided powers δ' on $J' = B'_+ + JB' \subset B'$. Namely, we take $\Omega = \Omega_{B,\delta}$ and $\Omega' = \Omega_{B',\delta'}$. In this case we can take $a = 1$ and

$$\theta(\sum b_m z^{[m]}) = \sum b_m z^{[m+1]}$$

Note that

$$f - \theta(\partial_z(f)) = f(0)$$

equals the constant term. It follows that in this case Lemma 60.25.1 recovers the crystalline Poincaré lemma (Lemma 60.20.2).

07N1 **Lemma 60.25.3.** *In Situation 60.5.1. Assume D and Ω_D are as in (60.17.0.1) and (60.17.0.2). Let $\lambda \in D$. Let D' be the p-adic completion of*

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$$

and let $\Omega_{D'}$ be the p-adic completion of the module of divided power differentials of D' over A. For any pair (M, ∇) over D satisfying (1), (2), (3), and (4) the canonical map of complexes (60.6.9.1)

$$c^\bullet_M : M \otimes^\wedge_D \Omega^\bullet_D \longrightarrow M \otimes^\wedge_D \Omega^\bullet_{D'}$$

has the following property: There exists a map $e^\bullet_M$ in the opposite direction such that both $c^\bullet_M \circ e^\bullet_M$ and $e^\bullet_M \circ c^\bullet_M$ are homotopic to multiplication by p.

Proof. We will prove this using Lemma 60.25.1 with $a = p$. Thus we have to find $\theta : D' \to D'$ and prove (1), (2), (3), (4), (5), (6), (7). We first collect some information about the rings D and D' and the modules Ω_D and $\Omega_{D'}$.

Writing

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda)) = D\langle\xi\rangle[z]/(z^p - \xi - \lambda)$$

we see that D' is the p-adic completion of the free D-module

$$\bigoplus\nolimits_{i=0,\ldots,p-1} \bigoplus\nolimits_{n\geq 0} z^i \xi^{[n]} D$$

where $\xi^{[0]} = 1$. It follows that $D \to D'$ has a continuous D-linear section, in particular $D \to D'$ is universally injective, i.e., (5) holds. We think of D' as a divided power algebra over A with divided power ideal $\overline{J}' = \overline{J}D' + (\xi)$. Then D' is also the p-adic completion of the divided power envelope of the ideal generated by $z^p - \lambda$ in D, see Lemma 60.2.4. Hence

$$\Omega_{D'} = \Omega_D \otimes^\wedge_D D' \oplus D'\mathrm{d}z$$

by Lemma 60.6.6. This proves (2). Note that (1) is obvious.

At this point we construct θ. (We wrote a PARI/gp script theta.gp verifying some of the formulas in this proof which can be found in the scripts subdirectory of the Stacks project.) Before we do so we compute the derivative of the elements $z^i\xi^{[n]}$. We have $\mathrm{d}z^i = iz^{i-1}\mathrm{d}z$. For $n \geq 1$ we have

$$\mathrm{d}\xi^{[n]} = \xi^{[n-1]}\mathrm{d}\xi = -\xi^{[n-1]}\mathrm{d}\lambda + pz^{p-1}\xi^{[n-1]}\mathrm{d}z$$

because $\xi = z^p - \lambda$. For $0 < i < p$ and $n \geq 1$ we have

$$\begin{aligned}
\mathrm{d}(z^i\xi^{[n]}) &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}\xi \\
&= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}(z^p - \lambda) \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i+p-1}\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i-1}(\xi + \lambda)\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + ((i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]})\mathrm{d}z
\end{aligned}$$

the last equality because $\xi\xi^{[n-1]} = n\xi^{[n]}$. Thus we see that

$$\begin{aligned}\partial_z(z^i) &= iz^{i-1}\\ \partial_z(\xi^{[n]}) &= pz^{p-1}\xi^{[n-1]}\\ \partial_z(z^i\xi^{[n]}) &= (i+pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]}\end{aligned}$$

Motivated by these formulas we define θ by the rules

$$\begin{array}{rcll}\theta(z^j) & = & p\frac{z^{j+1}}{j+1} & j = 0, \ldots p-1,\\ \theta(z^{p-1}\xi^{[m]}) & = & \xi^{[m+1]} & m \geq 1,\\ \theta(z^j\xi^{[m]}) & = & \frac{pz^{j+1}\xi^{[m]} - \theta(p\lambda z^j\xi^{[m-1]})}{(j+1+pm)} & 0 \leq j < p-1, m \geq 1\end{array}$$

where in the last line we use induction on m to define our choice of θ. Working this out we get (for $0 \leq j < p-1$ and $1 \leq m$)

$$\theta(z^j\xi^{[m]}) = \tfrac{pz^{j+1}\xi^{[m]}}{(j+1+pm)} - \tfrac{p^2\lambda z^{j+1}\xi^{[m-1]}}{(j+1+pm)(j+1+p(m-1))} + \ldots + \tfrac{(-1)^m p^{m+1}\lambda^m z^{j+1}}{(j+1+pm)\ldots(j+1)}$$

although we will not use this expression below. It is clear that θ extends uniquely to a p-adically continuous D-linear map on D'. By construction we have (3) and (4). It remains to prove (6) and (7).

Proof of (6) and (7). As θ is D-linear and continuous it suffices to prove that $p - \theta \circ \partial_z$, resp. $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ gives an element of D, resp. Ω_D when evaluated on the elements $z^i\xi^{[n]}$[8]. Set $D_0 = \mathbf{Z}_{(p)}[\lambda]$ and $D'_0 = \mathbf{Z}_{(p)}[z,\lambda]\langle\xi\rangle/(\xi - z^p + \lambda)$. Observe that each of the expressions above is an element of D'_0 or $\Omega_{D'_0}$. Hence it suffices to prove the result in the case of $D_0 \to D'_0$. Note that D_0 and D'_0 are torsion free rings and that $D_0 \otimes \mathbf{Q} = \mathbf{Q}[\lambda]$ and $D'_0 \otimes \mathbf{Q} = \mathbf{Q}[z,\lambda]$. Hence $D_0 \subset D'_0$ is the subring of elements annihilated by ∂_z and (6) follows from (4), see the discussion directly preceding Lemma 60.25.1. Similarly, we have $\mathrm{d}_1(f) = \partial_\lambda(f)\mathrm{d}\lambda$ hence

$$\left((\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta\right)(f) = \left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))\right)\mathrm{d}\lambda$$

Applying ∂_z to the coefficient we obtain

$$\begin{aligned}\partial_z\left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))\right) &= p\partial_\lambda(f) - \partial_z(\partial_\lambda(\theta(f)))\\ &= p\partial_\lambda(f) - \partial_\lambda(\partial_z(\theta(f)))\\ &= p\partial_\lambda(f) - \partial_\lambda(pf) = 0\end{aligned}$$

whence the coefficient does not depend on z as desired. This finishes the proof of the lemma. □

Note that an iterated α_p-cover $X' \to X$ (as defined in the introduction to this section) is finite locally free. Hence if X is connected the degree of $X' \to X$ is constant and is a power of p.

07Q9 **Lemma 60.25.4.** *Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme over $\mathbf{Z}_{(p)}$ with $p \in \mathcal{I}$. We set $S_0 = V(\mathcal{I}) \subset S$. Let $f : X' \to X$ be an iterated α_p-cover of schemes over S_0 with constant degree q. Let $\mathcal{F}$ be any crystal in quasi-coherent sheaves on X and set $\mathcal{F}' = f^*_{cris}\mathcal{F}$. In the distinguished triangle*

$$Ru_{X/S,*}\mathcal{F} \longrightarrow f_*Ru_{X'/S,*}\mathcal{F}' \longrightarrow E \longrightarrow Ru_{X/S,*}\mathcal{F}[1]$$

[8]This can be done by direct computation: It turns out that $p - \theta \circ \partial_z$ evaluated on $z^i\xi^{[n]}$ gives zero except for 1 which is mapped to p and ξ which is mapped to $-p\lambda$. It turns out that $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ evaluated on $z^i\xi^{[n]}$ gives zero except for $z^{p-1}\xi$ which is mapped to $-\lambda$.

the object E has cohomology sheaves annihilated by q.

Proof. Note that $X' \to X$ is a homeomorphism hence we can identify the underlying topological spaces of X and X'. The question is clearly local on X, hence we may assume X, X', and S affine and $X' \to X$ given as a composition

$$X' = X_n \to X_{n-1} \to X_{n-2} \to \ldots \to X_0 = X$$

where each morphism $X_{i+1} \to X_i$ is an α_p-cover. Denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. It suffices to prove that each of the maps

$$R\Gamma(\mathrm{Cris}(X_i/S), \mathcal{F}_i) \longrightarrow R\Gamma(\mathrm{Cris}(X_{i+1}/S), \mathcal{F}_{i+1})$$

fits into a triangle whose third member has cohomology groups annihilated by p. (This uses axiom TR4 for the triangulated category $D(X)$. Details omitted.)

Hence we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(C)$, $X' = \mathrm{Spec}(C')$ and $C' = C[z]/(z^p - c)$ for some $c \in C$. Choose a polynomial algebra P over A and a surjection $P \to C$. Let D be the p-adically completed divided power envelop of $\mathrm{Ker}(P \to C)$ in P as in (60.17.0.1). Set $P' = P[z]$ with surjection $P' \to C'$ mapping z to the class of z in C'. Choose a lift $\lambda \in D$ of $c \in C$. Then we see that the p-adically completed divided power envelope D' of $\mathrm{Ker}(P' \to C')$ in P' is isomorphic to the p-adic completion of $D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$, see Lemma 60.25.3 and its proof. Thus we see that the result follows from this lemma by the computation of cohomology of crystals in quasi-coherent modules in Proposition 60.21.3. □

The bound in the following lemma is probably not optimal.

07QA **Lemma 60.25.5.** *With notations and assumptions as in Lemma 60.25.4 the map*

$$f^* : H^i(\mathit{Cris}(X/S), \mathcal{F}) \longrightarrow H^i(\mathit{Cris}(X'/S), \mathcal{F}')$$

has kernel and cokernel annihilated by q^{i+1}.

Proof. This follows from the fact that E has nonzero cohomology sheaves in degrees -1 and up, so that the spectral sequence $H^a(\mathcal{H}^b(E)) \Rightarrow H^{a+b}(E)$ converges. This combined with the long exact cohomology sequence associated to a distinguished triangle gives the bound. □

In Situation 60.7.5 assume that $p \in \mathcal{I}$. Set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0.$$

Denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism.

07QB **Lemma 60.25.6.** *In the situation above, assume that $X \to S_0$ is smooth of relative dimension d. Then F_{X/S_0} is an iterated α_p-cover of degree p^d. Hence Lemmas 60.25.4 and 60.25.5 apply to this situation. In particular, for any crystal in quasi-coherent modules $\mathcal{G}$ on $\mathit{Cris}(X^{(1)}/S)$ the map*

$$F^*_{X/S_0} : H^i(\mathit{Cris}(X^{(1)}/S), \mathcal{G}) \longrightarrow H^i(\mathit{Cris}(X/S), F^*_{X/S_0, \mathit{cris}}\mathcal{G})$$

has kernel and cokernel annihilated by $p^{d(i+1)}$.

Proof. It suffices to prove the first statement. To see this we may assume that X is étale over $\mathbf{A}^d_{S_0}$, see Morphisms, Lemma 29.36.20. Denote $\varphi : X \to \mathbf{A}^d_{S_0}$ this étale morphism. In this case the relative Frobenius of X/S_0 fits into a diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{(1)} \\ \downarrow & & \downarrow \\ \mathbf{A}^d_{S_0} & \longrightarrow & \mathbf{A}^d_{S_0} \end{array}$$

where the lower horizontal arrow is the relative frobenius morphism of $\mathbf{A}^d_{S_0}$ over S_0. This is the morphism which raises all the coordinates to the pth power, hence it is an iterated α_p-cover. The proof is finished by observing that the diagram is a fibre square, see Étale Morphisms, Lemma 41.14.3. □

60.26. Frobenius action on crystalline cohomology

07N0 In this section we prove that Frobenius pullback induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. But in order to even formulate this we need to work in a special situation.

07N2 **Situation 60.26.1.** In Situation 60.7.5 assume the following

(1) $S = \mathrm{Spec}(A)$ for some divided power ring (A, I, γ) with $p \in I$,
(2) there is given a homomorphism of divided power rings $\sigma : A \to A$ such that $\sigma(x) = x^p \bmod pA$ for all $x \in A$.

In Situation 60.26.1 the morphism $\mathrm{Spec}(\sigma) : S \to S$ is a lift of the absolute Frobenius $F_{S_0} : S_0 \to S_0$ and since the diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow{F_{S_0}} & S_0 \end{array}$$

is commutative where $F_X : X \to X$ is the absolute Frobenius morphism of X. Thus we obtain a morphism of crystalline topoi

$$(F_X)_{\mathrm{cris}} : (X/S)_{\mathrm{cris}} \longrightarrow (X/S)_{\mathrm{cris}}$$

see Remark 60.9.3. Here is the terminology concerning F-crystals following the notation of Saavedra, see [**SR72**].

07N3 **Definition 60.26.2.** In Situation 60.26.1 an *F-crystal on X/S (relative to σ)* is a pair $(\mathcal{E}, F_\mathcal{E})$ given by a crystal in finite locally free $\mathcal{O}_{X/S}$-modules $\mathcal{E}$ together with a map

$$F_\mathcal{E} : (F_X)^*_{\mathrm{cris}}\mathcal{E} \longrightarrow \mathcal{E}$$

An F-crystal is called *nondegenerate* if there exists an integer $i \geq 0$ a map $V : \mathcal{E} \to (F_X)^*_{\mathrm{cris}}\mathcal{E}$ such that $V \circ F_\mathcal{E} = p^i \mathrm{id}$.

07N4 **Remark 60.26.3.** Let $(\mathcal{E}, F)$ be an F-crystal as in Definition 60.26.2. In the literature the nondegeneracy condition is often part of the definition of an F-crystal. Moreover, often it is also assumed that $F \circ V = p^n \mathrm{id}$. What is needed for the result below is that there exists an integer $j \geq 0$ such that $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are killed by p^j. If the rank of $\mathcal{E}$ is bounded (for example if X is quasi-compact), then both

of these conditions follow from the nondegeneracy condition as formulated in the definition. Namely, suppose R is a ring, $r \geq 1$ is an integer and $K, L \in \mathrm{Mat}(r \times r, R)$ are matrices with $KL = p^i 1_{r\times r}$. Then $\det(K)\det(L) = p^{ri}$. Let L' be the adjugate matrix of L, i.e., $L'L = LL' = \det(L)$. Set $K' = p^{ri}K$ and $j = ri + i$. Then we have $K'L = p^j 1_{r\times r}$ as $KL = p^i$ and

$$LK' = LK\det(L)\det(M) = LKLL'\det(M) = Lp^iL'\det(M) = p^j 1_{r\times r}$$

It follows that if V is as in Definition 60.26.2 then setting $V' = p^N V$ where $N > i \cdot \mathrm{rank}(\mathcal{E})$ we get $V' \circ F = p^{N+i}$ and $F \circ V' = p^{N+i}$.

07N5 **Theorem 60.26.4.** *In Situation 60.26.1 let $(\mathcal{E}, F_\mathcal{E})$ be a nondegenerate F-crystal. Assume A is a p-adically complete Noetherian ring and that $X \to S_0$ is proper smooth. Then the canonical map*

$$F_\mathcal{E} \circ (F_X)^*_{cris} : R\Gamma(Cris(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(Cris(X/S), \mathcal{E})$$

becomes an isomorphism after inverting p.

Proof. We first write the arrow as a composition of three arrows. Namely, set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0$$

and denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism. Denote $\mathcal{E}^{(1)}$ the base change of $\mathcal{E}$ by $\mathrm{Spec}(\sigma)$, in other words the pullback of $\mathcal{E}$ to $\mathrm{Cris}(X^{(1)}/S)$ by the morphism of crystalline topoi associated to the commutative diagram

$$\begin{array}{ccc} X^{(1)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathrm{Spec}(\sigma)} & S \end{array}$$

Then we have the base change map

07QC (60.26.4.1) $$R\Gamma(\mathrm{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\mathrm{Cris}(X^{(1)}/S), \mathcal{E}^{(1)})$$

see Remark 60.24.8. Note that the composition of $F_{X/S_0} : X \to X^{(1)}$ with the projection $X^{(1)} \to X$ is the absolute Frobenius morphism F_X. Hence we see that $F^*_{X/S_0}\mathcal{E}^{(1)} = (F_X)^*_{\mathrm{cris}}\mathcal{E}$. Thus pullback by F_{X/S_0} is a map

07N6 (60.26.4.2) $$F^*_{X/S_0} : R\Gamma(\mathrm{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \longrightarrow R\Gamma(\mathrm{Cris}(X/S), (F_X)^*_{\mathrm{cris}}\mathcal{E})$$

Finally we can use $F_\mathcal{E}$ to get a map

07QD (60.26.4.3) $$R\Gamma(\mathrm{Cris}(X/S), (F_X)^*_{\mathrm{cris}}\mathcal{E}) \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{E})$$

The map of the theorem is the composition of the three maps (60.26.4.1), (60.26.4.2), and (60.26.4.3) above. The first is a quasi-isomorphism modulo all powers of p by Remark 60.24.9. Hence it is a quasi-isomorphism since the complexes involved are perfect in $D(A)$ see Remark 60.24.13. The third map is a quasi-isomorphism after inverting p simply because $F_\mathcal{E}$ has an inverse up to a power of p, see Remark 60.26.3. Finally, the second is an isomorphism after inverting p by Lemma 60.25.6. □

60.27. Other chapters

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CHAPTER 61

Pro-étale Cohomology

0965 ## 61.1. Introduction

0966 The material in this chapter and more can be found in the preprint [**BS13**].

The goal of this chapter is to introduce the pro-étale topology and to develop the basic theory of cohomology of abelian sheaves in this topology. A secondary goal is to show how using the pro-étale topology simplifies the introduction of ℓ-adic cohomology in algebraic geometry.

Here is a brief overview of the history of ℓ-adic étale cohomology as we have understood it. In [**Gro77**, Exposés V and VI] Grothendieck et al developed a theory for dealing with ℓ-adic sheaves as inverse systems of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. In his second paper on the Weil conjectures ([**Del80**]) Deligne introduced a derived category of ℓ-adic sheaves as a certain 2-limit of categories of complexes of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules on the étale site of a scheme X. This approach is used in the paper by Beilinson, Bernstein, and Deligne ([**BBD82**]) as the basis for their beautiful theory of perverse sheaves. In a paper entitled "Continuous Étale Cohomology" ([**Jan88**]) Uwe Jannsen discusses an important variant of the cohomology of a ℓ-adic sheaf on a variety over a field. His paper is followed up by a paper of Torsten Ekedahl ([**Eke90**]) who discusses the adic formalism needed to work comfortably with derived categories defined as limits.

It turns out that, working with the pro-étale site of a scheme, one can avoid some of the technicalities these authors encountered. This comes at the expense of having to work with non-Noetherian schemes, even when one is only interested in working with ℓ-adic sheaves and cohomology of such on varieties over an algebraically closed field.

A very important and remarkable feature of the (small) pro-étale site of a scheme is that it has enough quasi-compact w-contractible objects. The existence of these objects implies a number of useful and (perhaps) unusual consequences for the derived category of abelian sheaves and for inverse systems of sheaves. This is exactly the feature that will allow us to handle the intricacies of working with ℓ-adic sheaves, but as we will see it has a number of other benefits as well.

61.2. Some topology

0967 Some preliminaries. We have defined *spectral spaces* and *spectral maps* of spectral spaces in Topology, Section 5.23. The spectrum of a ring is a spectral space, see Algebra, Lemma 10.26.2.

0968 **Lemma 61.2.1.** *Let X be a spectral space. Let $X_0 \subset X$ be the set of closed points. The following are equivalent*

(1) *Every open covering of X can be refined by a finite disjoint union decomposition $X = \coprod U_i$ with U_i open and closed in X.*
(2) *The composition $X_0 \to X \to \pi_0(X)$ is bijective.*

Moreover, if X_0 is closed in X and every point of X specializes to a unique point of X_0, then these conditions are satisfied.

Proof. We will use without further mention that X_0 is quasi-compact (Topology, Lemma 5.12.9) and $\pi_0(X)$ is profinite (Topology, Lemma 5.23.9). Picture

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & \pi_0(X) \end{array}$$

If (2) holds, the continuous bijective map $f : X_0 \to \pi_0(X)$ is a homeomorphism by Topology, Lemma 5.17.8. Given an open covering $X = \bigcup U_i$, we get an open covering $\pi_0(X) = \bigcup f(X_0 \cap U_i)$. By Topology, Lemma 5.22.4 we can find a finite open covering of the form $\pi_0(X) = \coprod V_j$ which refines this covering. Since $X_0 \to \pi_0(X)$ is bijective each connected component of X has a unique closed point, whence is equal to the set of points specializing to this closed point. Hence $\pi^{-1}(V_j)$ is the set of points specializing to the points of $f^{-1}(V_j)$. Now, if $f^{-1}(V_j) \subset X_0 \cap U_i \subset U_i$, then it follows that $\pi^{-1}(V_j) \subset U_i$ (because the open set U_i is closed under generalizations). In this way we see that the open covering $X = \coprod \pi^{-1}(V_j)$ refines the covering we started out with. In this way we see that (2) implies (1).

Assume (1). Let $x, y \in X$ be closed points. Then we have the open covering $X = (X \setminus \{x\}) \cup (X \setminus \{y\})$. It follows from (1) that there exists a disjoint union decomposition $X = U \amalg V$ with U and V open (and closed) and $x \in U$ and $y \in V$. In particular we see that every connected component of X has at most one closed point. By Topology, Lemma 5.12.8 every connected component (being closed) also does have a closed point. Thus $X_0 \to \pi_0(X)$ is bijective. In this way we see that (1) implies (2).

Assume X_0 is closed in X and every point specializes to a unique point of X_0. Then X_0 is a spectral space (Topology, Lemma 5.23.5) consisting of closed points, hence profinite (Topology, Lemma 5.23.8). Let $x, y \in X_0$ be distinct. By Topology, Lemma 5.22.4 we can find a disjoint union decomposition $X_0 = U_0 \amalg V_0$ with U_0 and V_0 open and closed and $x \in U_0$ and $y \in V_0$. Let $U \subset X$, resp. $V \subset X$ be the set of points specializing to U_0, resp. V_0. Observe that $X = U \amalg V$. By Topology, Lemma 5.24.7 we see that U is an intersection of quasi-compact open subsets. Hence U is closed in the constructible topology. Since U is closed under specialization, we see that U is closed by Topology, Lemma 5.23.6. By symmetry V is closed and hence U and V are both open and closed. This proves that x, y are not in the same connected component of X. In other words, $X_0 \to \pi_0(X)$ is injective. The map is also surjective by Topology, Lemma 5.12.8 and the fact that connected components are closed. In this way we see that the final condition implies (2). □

0969 **Example 61.2.2.** Let T be a profinite space. Let $t \in T$ be a point and assume that $T \setminus \{t\}$ is not quasi-compact. Let $X = T \times \{0, 1\}$. Consider the topology on X with a subbase given by the sets $U \times \{0, 1\}$ for $U \subset T$ open, $X \setminus \{(t, 0)\}$, and $U \times \{1\}$ for $U \subset T$ open with $t \notin U$. The set of closed points of X is $X_0 = T \times \{0\}$ and

$(t, 1)$ is in the closure of X_0. Moreover, $X_0 \to \pi_0(X)$ is a bijection. This example shows that conditions (1) and (2) of Lemma 61.2.1 do no imply the set of closed points is closed.

It turns out it is more convenient to work with spectral spaces which have the slightly stronger property mentioned in the final statement of Lemma 61.2.1. We give this property a name.

096A **Definition 61.2.3.** A spectral space X is *w-local* if the set of closed points X_0 is closed and every point of X specializes to a unique closed point. A continuous map $f : X \to Y$ of w-local spaces is *w-local* if it is spectral and maps any closed point of X to a closed point of Y.

We have seen in the proof of Lemma 61.2.1 that in this case $X_0 \to \pi_0(X)$ is a homeomorphism and that $X_0 \cong \pi_0(X)$ is a profinite space. Moreover, a connected component of X is exactly the set of points specializing to a given $x \in X_0$.

096B **Lemma 61.2.4.** *Let X be a w-local spectral space. If $Y \subset X$ is closed, then Y is w-local.*

Proof. The subset $Y_0 \subset Y$ of closed points is closed because $Y_0 = X_0 \cap Y$. Since X is w-local, every $y \in Y$ specializes to a unique point of X_0. This specialization is in Y, and hence also in Y_0, because $\overline{\{y\}} \subset Y$. In conclusion, Y is w-local. □

096C **Lemma 61.2.5.** *Let X be a spectral space. Let*

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

be a cartesian diagram in the category of topological spaces with T profinite. Then Y is spectral and $T = \pi_0(Y)$. If moreover X is w-local, then Y is w-local, $Y \to X$ is w-local, and the set of closed points of Y is the inverse image of the set of closed points of X.

Proof. Note that Y is a closed subspace of $X \times T$ as $\pi_0(X)$ is a profinite space hence Hausdorff (use Topology, Lemmas 5.23.9 and 5.3.4). Since $X \times T$ is spectral (Topology, Lemma 5.23.10) it follows that Y is spectral (Topology, Lemma 5.23.5). Let $Y \to \pi_0(Y) \to T$ be the canonical factorization (Topology, Lemma 5.7.9). It is clear that $\pi_0(Y) \to T$ is surjective. The fibres of $Y \to T$ are homeomorphic to the fibres of $X \to \pi_0(X)$. Hence these fibres are connected. It follows that $\pi_0(Y) \to T$ is injective. We conclude that $\pi_0(Y) \to T$ is a homeomorphism by Topology, Lemma 5.17.8.

Next, assume that X is w-local and let $X_0 \subset X$ be the set of closed points. The inverse image $Y_0 \subset Y$ of X_0 in Y maps bijectively onto T as $X_0 \to \pi_0(X)$ is a bijection by Lemma 61.2.1. Moreover, Y_0 is quasi-compact as a closed subset of the spectral space Y. Hence $Y_0 \to \pi_0(Y) = T$ is a homeomorphism by Topology, Lemma 5.17.8. It follows that all points of Y_0 are closed in Y. Conversely, if $y \in Y$ is a closed point, then it is closed in the fibre of $Y \to \pi_0(Y) = T$ and hence its image x in X is closed in the (homeomorphic) fibre of $X \to \pi_0(X)$. This implies $x \in X_0$ and hence $y \in Y_0$. Thus Y_0 is the collection of closed points of Y and for

each $y \in Y_0$ the set of generalizations of y is the fibre of $Y \to \pi_0(Y)$. The lemma follows. □

61.3. Local isomorphisms

096D We start with a definition.

096E **Definition 61.3.1.** Let $\varphi : A \to B$ be a ring map.

(1) We say $A \to B$ is a *local isomorphism* if for every prime $\mathfrak{q} \subset B$ there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $A \to B_g$ induces an open immersion $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$.
(2) We say $A \to B$ *identifies local rings* if for every prime $\mathfrak{q} \subset B$ the canonical map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is an isomorphism.

We list some elementary properties.

096F **Lemma 61.3.2.** *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

(1) *If $A \to B$ is a local isomorphism, then $A' \to B'$ is a local isomorphism.*
(2) *If $A \to B$ identifies local rings, then $A' \to B'$ identifies local rings.*

Proof. Omitted. □

096G **Lemma 61.3.3.** *Let $A \to B$ and $B \to C$ be ring maps.*

(1) *If $A \to B$ and $B \to C$ are local isomorphisms, then $A \to C$ is a local isomorphism.*
(2) *If $A \to B$ and $B \to C$ identify local rings, then $A \to C$ identifies local rings.*

Proof. Omitted. □

096H **Lemma 61.3.4.** *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism.*

(1) *If $A \to B$ and $A \to C$ are local isomorphisms, then $B \to C$ is a local isomorphism.*
(2) *If $A \to B$ and $A \to C$ identify local rings, then $B \to C$ identifies local rings.*

Proof. Omitted. □

096I **Lemma 61.3.5.** *Let $A \to B$ be a local isomorphism. Then*

(1) *$A \to B$ is étale,*
(2) *$A \to B$ identifies local rings,*
(3) *$A \to B$ is quasi-finite.*

Proof. Omitted. □

096J **Lemma 61.3.6.** *Let $A \to B$ be a local isomorphism. Then there exist $n \geq 0$, $g_1, \ldots, g_n \in B$, $f_1, \ldots, f_n \in A$ such that $(g_1, \ldots, g_n) = B$ and $A_{f_i} \cong B_{g_i}$.*

Proof. Omitted. □

096K **Lemma 61.3.7.** *Let $p : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $q : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be morphisms of locally ringed spaces. If $\mathcal{O}_Y = p^{-1}\mathcal{O}_X$, then*

$$\mathrm{Mor}_{LRS/(X,\mathcal{O}_X)}((Z, \mathcal{O}_Z), (Y, \mathcal{O}_Y)) \longrightarrow \mathrm{Mor}_{Top/X}(Z, Y), \quad (f, f^\sharp) \longmapsto f$$

is bijective. Here $LRS/(X, \mathcal{O}_X)$ is the category of locally ringed spaces over X and Top/X is the category of topological spaces over X.

Proof. This is immediate from the definitions. □

096L **Lemma 61.3.8.** *Let A be a ring. Set $X = \mathrm{Spec}(A)$. The functor*

$$B \longmapsto \mathrm{Spec}(B)$$

from the category of A-algebras B such that $A \to B$ identifies local rings to the category of topological spaces over X is fully faithful.

Proof. This follows from Lemma 61.3.7 and the fact that if $A \to B$ identifies local rings, then the pullback of the structure sheaf of $\mathrm{Spec}(A)$ via $p : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is equal to the structure sheaf of $\mathrm{Spec}(B)$. □

61.4. Ind-Zariski algebra

096M We start with a definition; please see Remark 61.6.9 for a comparison with the corresponding definition of the article [**BS13**].

096N **Definition 61.4.1.** A ring map $A \to B$ is said to be *ind-Zariski* if B can be written as a filtered colimit $B = \mathrm{colim}\, B_i$ with each $A \to B_i$ a local isomorphism.

An example of an Ind-Zariski map is a localization $A \to S^{-1}A$, see Algebra, Lemma 10.9.9. The category of ind-Zariski algebras is closed under several natural operations.

096P **Lemma 61.4.2.** *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-Zariski, then $A' \to B'$ is ind-Zariski.*

Proof. Omitted. □

096Q **Lemma 61.4.3.** *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-Zariski, then $A \to C$ is ind-Zariski.*

Proof. Omitted. □

096R **Lemma 61.4.4.** *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism. If $A \to B$ and $A \to C$ are ind-Zariski, then $B \to C$ is ind-Zariski.*

Proof. Omitted. □

096S **Lemma 61.4.5.** *A filtered colimit of ind-Zariski A-algebras is ind-Zariski over A.*

Proof. Omitted. □

096T **Lemma 61.4.6.** *Let $A \to B$ be ind-Zariski. Then $A \to B$ identifies local rings,*

Proof. Omitted. □

61.5. Constructing w-local affine schemes

096U An affine scheme X is called *w-local* if its underlying topological space is w-local (Definition 61.2.3). It turns out given any ring A there is a canonical faithfully flat ind-Zariski ring map $A \to A_w$ such that $\mathrm{Spec}(A_w)$ is w-local. The key to constructing A_w is the following simple lemma.

096V **Lemma 61.5.1.** *Let A be a ring. Set $X = \mathrm{Spec}(A)$. Let $Z \subset X$ be a locally closed subscheme which is of the form $D(f) \cap V(I)$ for some $f \in A$ and ideal $I \subset A$. Then*

(1) *there exists a multiplicative subset $S \subset A$ such that* $\mathrm{Spec}(S^{-1}A)$ *maps by a homeomorphism to the set of points of X specializing to Z,*
(2) *the A-algebra $A_Z^\sim = S^{-1}A$ depends only on the underlying locally closed subset $Z \subset X$,*
(3) *Z is a closed subscheme of* $\mathrm{Spec}(A_Z^\sim)$,

If $A \to A'$ is a ring map and $Z' \subset X' = \mathrm{Spec}(A')$ is a locally closed subscheme of the same form which maps into Z, then there is a unique A-algebra map $A_Z^\sim \to (A')_{Z'}^\sim$.

Proof. Let $S \subset A$ be the multiplicative set of elements which map to invertible elements of $\Gamma(Z, \mathcal{O}_Z) = (A/I)_f$. If $\mathfrak{p}$ is a prime of A which does not specialize to Z, then $\mathfrak{p}$ generates the unit ideal in $(A/I)_f$. Hence we can write $f^n = g + h$ for some $n \geq 0$, $g \in \mathfrak{p}$, $h \in I$. Then $g \in S$ and we see that $\mathfrak{p}$ is not in the spectrum of $S^{-1}A$. Conversely, if $\mathfrak{p}$ does specialize to Z, say $\mathfrak{p} \subset \mathfrak{q} \supset I$ with $f \notin \mathfrak{q}$, then we see that $S^{-1}A$ maps to $A_\mathfrak{q}$ and hence $\mathfrak{p}$ is in the spectrum of $S^{-1}A$. This proves (1).

The isomorphism class of the localization $S^{-1}A$ depends only on the corresponding subset $\mathrm{Spec}(S^{-1}A) \subset \mathrm{Spec}(A)$, whence (2) holds. By construction $S^{-1}A$ maps surjectively onto $(A/I)_f$, hence (3). The final statement follows as the multiplicative subset $S' \subset A'$ corresponding to Z' contains the image of the multiplicative subset S. □

Let A be a ring. Let $E \subset A$ be a finite subset. We get a stratification of $X = \mathrm{Spec}(A)$ into locally closed subschemes by looking at the vanishing behaviour of the elements of E. More precisely, given a disjoint union decomposition $E = E' \amalg E''$ we set

(61.5.1.1)

096W $$Z(E', E'') = \bigcap\nolimits_{f \in E'} D(f) \cap \bigcap\nolimits_{f \in E''} V(f) = D(\prod\nolimits_{f \in E'} f) \cap V(\sum\nolimits_{f \in E''} fA)$$

The points of $Z(E', E'')$ are exactly those $x \in X$ such that $f \in E'$ maps to a nonzero element in $\kappa(x)$ and $f \in E''$ maps to zero in $\kappa(x)$. Thus it is clear that

096X (61.5.1.2) $$X = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'')$$

set theoretically. Observe that each stratum is constructible.

096Y **Lemma 61.5.2.** *Let $X = \mathrm{Spec}(A)$ as above. Given any finite stratification $X = \coprod T_i$ by constructible subsets, there exists a finite subset $E \subset A$ such that the stratification (61.5.1.2) refines $X = \coprod T_i$.*

Proof. We may write $T_i = \bigcup_j U_{i,j} \cap V_{i,j}^c$ as a finite union for some $U_{i,j}$ and $V_{i,j}$ quasi-compact open in X. Then we may write $U_{i,j} = \bigcup D(f_{i,j,k})$ and $V_{i,j} = \bigcup D(g_{i,j,l})$. Then we set $E = \{f_{i,j,k}\} \cup \{g_{i,j,l}\}$. This does the job, because the stratification (61.5.1.2) is the one whose strata are labeled by the vanishing pattern of the elements of E which clearly refines the given stratification. □

We continue the discussion. Given a finite subset $E \subset A$ we set

096Z (61.5.2.1) $$A_E = \prod\nolimits_{E = E' \amalg E''} A_{Z(E', E'')}^\sim$$

with notation as in Lemma 61.5.1. This makes sense because (61.5.1.1) shows that each $Z(E', E'')$ has the correct shape. We take the spectrum of this ring and denote it

0970 (61.5.2.2) $$X_E = \mathrm{Spec}(A_E) = \coprod\nolimits_{E = E' \amalg E''} X_{E', E''}$$

with $X_{E',E''} = \mathrm{Spec}(A^\sim_{Z(E',E'')})$. Note that

0971 (61.5.2.3) $$Z_E = \coprod\nolimits_{E=E'\amalg E''} Z(E',E'') \longrightarrow X_E$$

is a closed subscheme. By construction the closed subscheme Z_E contains all the closed points of the affine scheme X_E as every point of $X_{E',E''}$ specializes to a point of $Z(E',E'')$.

Let $I(A)$ be the partially ordered set of all finite subsets of A. This is a directed partially ordered set. For $E_1 \subset E_2$ there is a canonical transition map $A_{E_1} \to A_{E_2}$ of A-algebras. Namely, given a decomposition $E_2 = E_2' \amalg E_2''$ we set $E_1' = E_1 \cap E_2'$ and $E_1'' = E_1 \cap E_2''$. Then observe that $Z(E_2', E_2'') \subset Z(E_1', E_1'')$ hence a unique A-algebra map $A^\sim_{Z(E_1',E_1'')} \to A^\sim_{Z(E_2',E_2'')}$ by Lemma 61.5.1. Using these maps collectively we obtain the desired ring map $A_{E_1} \to A_{E_2}$. Observe that the corresponding map of affine schemes

0972 (61.5.2.4) $$X_{E_2} \longrightarrow X_{E_1}$$

maps Z_{E_2} into Z_{E_1}. By uniqueness we obtain a system of A-algebras over $I(A)$ and we set

0973 (61.5.2.5) $$A_w = \mathrm{colim}_{E\in I(A)} A_E$$

This A-algebra is ind-Zariski and faithfully flat over A. Finally, we set $X_w = \mathrm{Spec}(A_w)$ and endow it with the closed subscheme $Z = \lim_{E\in I(A)} Z_E$. In a formula

0974 (61.5.2.6) $$X_w = \lim_{E\in I(A)} X_E \supset Z = \lim_{E\in I(A)} Z_E$$

0975 **Lemma 61.5.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. With $A \to A_w$, $X_w = \mathrm{Spec}(A_w)$, and $Z \subset X_w$ as above.*

(1) *$A \to A_w$ is ind-Zariski and faithfully flat,*
(2) *$X_w \to X$ induces a bijection $Z \to X$,*
(3) *Z is the set of closed points of X_w,*
(4) *Z is a reduced scheme, and*
(5) *every point of X_w specializes to a unique point of Z.*

In particular, X_w is w-local (Definition 61.2.3).

Proof. The map $A \to A_w$ is ind-Zariski by construction. For every E the morphism $Z_E \to X$ is a bijection, hence (2). As $Z \subset X_w$ we conclude $X_w \to X$ is surjective and $A \to A_w$ is faithfully flat by Algebra, Lemma 10.39.16. This proves (1).

Suppose that $y \in X_w$, $y \notin Z$. Then there exists an E such that the image of y in X_E is not contained in Z_E. Then for all $E \subset E'$ also y maps to an element of $X_{E'}$ not contained in $Z_{E'}$. Let $T_{E'} \subset X_{E'}$ be the reduced closed subscheme which is the closure of the image of y. It is clear that $T = \lim_{E\subset E'} T_{E'}$ is the closure of y in X_w. For every $E \subset E'$ the scheme $T_{E'} \cap Z_{E'}$ is nonempty by construction of $X_{E'}$. Hence $\lim T_{E'} \cap Z_{E'}$ is nonempty and we conclude that $T \cap Z$ is nonempty. Thus y is not a closed point. It follows that every closed point of X_w is in Z.

Suppose that $y \in X_w$ specializes to $z, z' \in Z$. We will show that $z = z'$ which will finish the proof of (3) and will imply (5). Let $x, x' \in X$ be the images of z and z'. Since $Z \to X$ is bijective it suffices to show that $x = x'$. If $x \neq x'$, then there exists an $f \in A$ such that $x \in D(f)$ and $x' \in V(f)$ (or vice versa). Set $E = \{f\}$ so that

$$X_E = \mathrm{Spec}(A_f) \amalg \mathrm{Spec}(A^\sim_{V(f)})$$

Then we see that z and z' map x_E and x'_E which are in different parts of the given decomposition of X_E above. But then it impossible for x_E and x'_E to be specializations of a common point. This is the desired contradiction.

Recall that given a finite subset $E \subset A$ we have Z_E is a disjoint union of the locally closed subschemes $Z(E', E'')$ each isomorphic to the spectrum of $(A/I)_f$ where I is the ideal generated by E'' and f the product of the elements of E'. Any nilpotent element b of $(A/I)_f$ is the class of g/f^n for some $g \in A$. Then setting $E' = E \cup \{g\}$ the reader verifies that b is pulls back to zero under the transition map $Z_{E'} \to Z_E$ of the system. This proves (4). □

0976 **Remark 61.5.4.** Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of A_w (Lemma 61.5.3) is at most κ. Namely, each A_E has cardinality at most κ and the set of finite subsets of A has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6.

0977 **Lemma 61.5.5** (Universal property of the construction)**.** *Let A be a ring. Let $A \to A_w$ be the ring map constructed in Lemma 61.5.3. For any ring map $A \to B$ such that* $\mathrm{Spec}(B)$ *is w-local, there is a unique factorization $A \to A_w \to B$ such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A_w)$ *is w-local.*

Proof. Denote $Y = \mathrm{Spec}(B)$ and $Y_0 \subset Y$ the set of closed points. Denote $f : Y \to X$ the given morphism. Recall that Y_0 is profinite, in particular every constructible subset of Y_0 is open and closed. Let $E \subset A$ be a finite subset. Recall that $A_w = \operatorname{colim} A_E$ and that the set of closed points of $\mathrm{Spec}(A_w)$ is the limit of the closed subsets $Z_E \subset X_E = \mathrm{Spec}(A_E)$. Thus it suffices to show there is a unique factorization $A \to A_E \to B$ such that $Y \to X_E$ maps Y_0 into Z_E. Since $Z_E \to X = \mathrm{Spec}(A)$ is bijective, and since the strata $Z(E', E'')$ are constructible we see that

$$Y_0 = \coprod f^{-1}(Z(E', E'')) \cap Y_0$$

is a disjoint union decomposition into open and closed subsets. As $Y_0 = \pi_0(Y)$ we obtain a corresponding decomposition of Y into open and closed pieces. Thus it suffices to construct the factorization in case $f(Y_0) \subset Z(E', E'')$ for some decomposition $E = E' \amalg E''$. In this case $f(Y)$ is contained in the set of points of X specializing to $Z(E', E'')$ which is homeomorphic to $X_{E',E''}$. Thus we obtain a unique continuous map $Y \to X_{E',E''}$ over X. By Lemma 61.3.7 this corresponds to a unique morphism of schemes $Y \to X_{E',E''}$ over X. This finishes the proof. □

Recall that the spectrum of a ring is profinite if and only if every point is closed. There are in fact a whole slew of equivalent conditions that imply this. See Algebra, Lemma 10.26.5 or Topology, Lemma 5.23.8.

0978 **Lemma 61.5.6.** *Let A be a ring such that* $\mathrm{Spec}(A)$ *is profinite. Let $A \to B$ be a ring map. Then* $\mathrm{Spec}(B)$ *is profinite in each of the following cases:*

(1) *if $\mathfrak{q}, \mathfrak{q}' \subset B$ lie over the same prime of A, then neither $\mathfrak{q} \subset \mathfrak{q}'$, nor $\mathfrak{q}' \subset \mathfrak{q}$,*
(2) *$A \to B$ induces algebraic extensions of residue fields,*
(3) *$A \to B$ is a local isomorphism,*
(4) *$A \to B$ identifies local rings,*
(5) *$A \to B$ is weakly étale,*
(6) *$A \to B$ is quasi-finite,*

(7) *$A \to B$ is unramified,*
(8) *$A \to B$ is étale,*
(9) *B is a filtered colimit of A-algebras as in (1) – (8),*
(10) *etc.*

Proof. By the references mentioned above (Algebra, Lemma 10.26.5 or Topology, Lemma 5.23.8) there are no specializations between distinct points of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ is profinite if and only if there are no specializations between distinct points of $\mathrm{Spec}(B)$. These specializations can only happen in the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. In this way we see that (1) is true.

The assumption in (2) implies all primes of B are maximal by Algebra, Lemma 10.35.9. Thus (2) holds. If $A \to B$ is a local isomorphism or identifies local rings, then the residue field extensions are trivial, so (3) and (4) follow from (2). If $A \to B$ is weakly étale, then More on Algebra, Lemma 15.105.17 tells us it induces separable algebraic residue field extensions, so (5) follows from (2). If $A \to B$ is quasi-finite, then the fibres are finite discrete topological spaces. Hence (6) follows from (1). Hence (3) follows from (1). Cases (7) and (8) follow from this as unramified and étale ring map are quasi-finite (Algebra, Lemmas 10.151.6 and 10.143.6). If $B = \mathrm{colim}\, B_i$ is a filtered colimit of A-algebras, then $\mathrm{Spec}(B) = \lim \mathrm{Spec}(B_i)$ in the category of topological spaces by Limits, Lemma 32.4.2. Hence if each $\mathrm{Spec}(B_i)$ is profinite, so is $\mathrm{Spec}(B)$ by Topology, Lemma 5.22.3. This proves (9). □

0979 **Lemma 61.5.7.** *Let A be a ring. Let $V(I) \subset \mathrm{Spec}(A)$ be a closed subset which is a profinite topological space. Then there exists an ind-Zariski ring map $A \to B$ such that $\mathrm{Spec}(B)$ is w-local, the set of closed points is $V(IB)$, and $A/I \cong B/IB$.*

Proof. Let $A \to A_w$ and $Z \subset Y = \mathrm{Spec}(A_w)$ as in Lemma 61.5.3. Let $T \subset Z$ be the inverse image of $V(I)$. Then $T \to V(I)$ is a homeomorphism by Topology, Lemma 5.17.8. Let $B = (A_w)^\sim_T$, see Lemma 61.5.1. It is clear that B is w-local with closed points $V(IB)$. The ring map $A/I \to B/IB$ is ind-Zariski and induces a homeomorphism on underlying topological spaces. Hence it is an isomorphism by Lemma 61.3.8. □

097A **Lemma 61.5.8.** *Let A be a ring such that $X = \mathrm{Spec}(A)$ is w-local. Let $I \subset A$ be the radical ideal cutting out the set X_0 of closed points in X. Let $A \to B$ be a ring map inducing algebraic extensions on residue fields at primes. Then*

(1) *every point of $Z = V(IB)$ is a closed point of $\mathrm{Spec}(B)$,*
(2) *there exists an ind-Zariski ring map $B \to C$ such that*
  (a) *$B/IB \to C/IC$ is an isomorphism,*
  (b) *the space $Y = \mathrm{Spec}(C)$ is w-local,*
  (c) *the induced map $p : Y \to X$ is w-local, and*
  (d) *$p^{-1}(X_0)$ is the set of closed points of Y.*

Proof. By Lemma 61.5.6 applied to $A/I \to B/IB$ all points of $Z = V(IB) = \mathrm{Spec}(B/IB)$ are closed, in fact $\mathrm{Spec}(B/IB)$ is a profinite space. To finish the proof we apply Lemma 61.5.7 to $IB \subset B$. □

61.6. Identifying local rings versus ind-Zariski

097B An ind-Zariski ring map $A \to B$ identifies local rings (Lemma 61.4.6). The converse does not hold (Examples, Section 110.46). However, it turns out that there is a kind

of structure theorem for ring maps which identify local rings in terms of ind-Zariski ring maps, see Proposition 61.6.6.

Let A be a ring. Let $X = \mathrm{Spec}(A)$. The space of connected components $\pi_0(X)$ is a profinite space by Topology, Lemma 5.23.9 (and Algebra, Lemma 10.26.2).

097C **Lemma 61.6.1.** *Let A be a ring. Let $X = \mathrm{Spec}(A)$. Let $T \subset \pi_0(X)$ be a closed subset. There exists a surjective ind-Zariski ring map $A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a homeomorphism of $\mathrm{Spec}(B)$ with the inverse image of T in X.*

Proof. Let $Z \subset X$ be the inverse image of T. Then Z is the intersection $Z = \bigcap Z_\alpha$ of the open and closed subsets of X containing Z, see Topology, Lemma 5.12.12. For each α we have $Z_\alpha = \mathrm{Spec}(A_\alpha)$ where $A \to A_\alpha$ is a local isomorphism (a localization at an idempotent). Setting $B = \mathrm{colim}\, A_\alpha$ proves the lemma. □

097D **Lemma 61.6.2.** *Let A be a ring and let $X = \mathrm{Spec}(A)$. Let T be a profinite space and let $T \to \pi_0(X)$ be a continuous map. There exists an ind-Zariski ring map $A \to B$ such that with $Y = \mathrm{Spec}(B)$ the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & \pi_0(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

is cartesian in the category of topological spaces and such that $\pi_0(Y) = T$ as spaces over $\pi_0(X)$.

Proof. Namely, write $T = \lim T_i$ as the limit of an inverse system finite discrete spaces over a directed set (see Topology, Lemma 5.22.2). For each i let $Z_i = \mathrm{Im}(T \to \pi_0(X) \times T_i)$. This is a closed subset. Observe that $X \times T_i$ is the spectrum of $A_i = \prod_{t \in T_i} A$ and that $A \to A_i$ is a local isomorphism. By Lemma 61.6.1 we see that $Z_i \subset \pi_0(X \times T_i) = \pi_0(X) \times T_i$ corresponds to a surjection $A_i \to B_i$ which is ind-Zariski such that $\mathrm{Spec}(B_i) = X \times_{\pi_0(X)} Z_i$ as subsets of $X \times T_i$. The transition maps $T_i \to T_{i'}$ induce maps $Z_i \to Z_{i'}$ and $X \times_{\pi_0(X)} Z_i \to X \times_{\pi_0(X)} Z_{i'}$. Hence ring maps $B_{i'} \to B_i$ (Lemmas 61.3.8 and 61.4.6). Set $B = \mathrm{colim}\, B_i$. Because $T = \lim Z_i$ we have $X \times_{\pi_0(X)} T = \lim X \times_{\pi_0(X)} Z_i$ and hence $Y = \mathrm{Spec}(B) = \lim \mathrm{Spec}(B_i)$ fits into the cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

of topological spaces. By Lemma 61.2.5 we conclude that $T = \pi_0(Y)$. □

09BJ **Example 61.6.3.** Let k be a field. Let T be a profinite topological space. There exists an ind-Zariski ring map $k \to A$ such that $\mathrm{Spec}(A)$ is homeomorphic to T. Namely, just apply Lemma 61.6.2 to $T \to \pi_0(\mathrm{Spec}(k)) = \{*\}$. In fact, in this case we have

$$A = \mathrm{colim}\, \mathrm{Map}(T_i, k)$$

whenever we write $T = \lim T_i$ as a filtered limit with each T_i finite.

097E **Lemma 61.6.4.** *Let $A \to B$ be ring map such that*

(1) $A \to B$ *identifies local rings,*
(2) *the topological spaces* $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ *are w-local,*
(3) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is w-local, and*
(4) $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ *is bijective.*

Then $A \to B$ is an isomorphism

Proof. Let $X_0 \subset X = \mathrm{Spec}(A)$ and $Y_0 \subset Y = \mathrm{Spec}(B)$ be the sets of closed points. By assumption Y_0 maps into X_0 and the induced map $Y_0 \to X_0$ is a bijection. As a space $\mathrm{Spec}(A)$ is the disjoint union of the spectra of the local rings of A at closed points. Similarly for B. Hence $X \to Y$ is a bijection. Since $A \to B$ is flat we have going down (Algebra, Lemma 10.39.19). Thus Algebra, Lemma 10.41.11 shows for any prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ we have $B_\mathfrak{q} = B_\mathfrak{p}$. Since $B_\mathfrak{q} = A_\mathfrak{p}$ by assumption, we see that $A_\mathfrak{p} = B_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A. Thus $A = B$ by Algebra, Lemma 10.23.1. □

097F **Lemma 61.6.5.** *Let $A \to B$ be ring map such that*

(1) $A \to B$ *identifies local rings,*
(2) *the topological spaces* $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ *are w-local, and*
(3) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is w-local.*

Then $A \to B$ is ind-Zariski.

Proof. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Let $X_0 \subset X$ and $Y_0 \subset Y$ be the set of closed points. Let $A \to A'$ be the ind-Zariski morphism of affine schemes such that with $X' = \mathrm{Spec}(A')$ the diagram

$$\begin{array}{ccc} X' & \longrightarrow & \pi_0(X') \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

is cartesian in the category of topological spaces and such that $\pi_0(X') = \pi_0(Y)$ as spaces over $\pi_0(X)$, see Lemma 61.6.2. By Lemma 61.2.5 we see that X' is w-local and the set of closed points $X'_0 \subset X'$ is the inverse image of X_0.

We obtain a continuous map $Y \to X'$ of underlying topological spaces over X identifying $\pi_0(Y)$ with $\pi_0(X')$. By Lemma 61.3.8 (and Lemma 61.4.6) this corresponds to a morphism of affine schemes $Y \to X'$ over X. Since $Y \to X$ maps Y_0 into X_0 we see that $Y \to X'$ maps Y_0 into X'_0, i.e., $Y \to X'$ is w-local. By Lemma 61.6.4 we see that $Y \cong X'$ and we win. □

The following proposition is a warm up for the type of result we will prove later.

097G **Proposition 61.6.6.** *Let $A \to B$ be a ring map which identifies local rings. Then there exists a faithfully flat, ind-Zariski ring map $B \to B'$ such that $A \to B'$ is ind-Zariski.*

Proof. Let $A \to A_w$, resp. $B \to B_w$ be the faithfully flat, ind-Zariski ring map constructed in Lemma 61.5.3 for A, resp. B. Since $\mathrm{Spec}(B_w)$ is w-local, there exists a unique factorization $A \to A_w \to B_w$ such that $\mathrm{Spec}(B_w) \to \mathrm{Spec}(A_w)$ is w-local by Lemma 61.5.5. Note that $A_w \to B_w$ identifies local rings, see Lemma 61.3.4. By Lemma 61.6.5 this means $A_w \to B_w$ is ind-Zariski. Since $B \to B_w$ is faithfully flat, ind-Zariski (Lemma 61.5.3) and the composition $A \to B \to B_w$ is ind-Zariski (Lemma 61.4.3) the proposition is proved. □

The proposition above allows us to characterize the affine, weakly contractible objects in the pro-Zariski site of an affine scheme.

09AZ **Lemma 61.6.7.** *Let A be a ring. The following are equivalent*

(1) *every faithfully flat ring map $A \to B$ identifying local rings has a retraction,*
(2) *every faithfully flat ind-Zariski ring map $A \to B$ has a retraction, and*
(3) *A satisfies*
   (a) *$\mathrm{Spec}(A)$ is w-local, and*
   (b) *$\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 61.6.6.

Assume (3)(a) and (3)(b). Let $A \to B$ be faithfully flat and ind-Zariski. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We will show that $A \to B$ has a retraction.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 61.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$.

Assume $\mathrm{Spec}(B)$ is w-local and the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. Choose a continuous section to the surjective continuous map $V(IB) \to V(I)$. This is possible as $V(I) \cong \pi_0(\mathrm{Spec}(A))$ is extremally disconnected, see Topology, Proposition 5.26.6. The image is a closed subspace $T \subset \pi_0(\mathrm{Spec}(B)) \cong V(IB)$ mapping homeomorphically onto $\pi_0(A)$. Replacing B by the ind-Zariski quotient ring constructed in Lemma 61.6.1 we see that we may assume $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ is bijective. At this point $A \to B$ is an isomorphism by Lemma 61.6.4.

Assume (1) or equivalently (2). Let $A \to A_w$ be the ring map constructed in Lemma 61.5.3. By (1) there is a retraction $A_w \to A$. Thus $\mathrm{Spec}(A)$ is homeomorphic to a closed subset of $\mathrm{Spec}(A_w)$. By Lemma 61.2.4 we see (3)(a) holds. Finally, let $T \to \pi_0(A)$ be a surjective map with T an extremally disconnected, quasi-compact, Hausdorff topological space (Topology, Lemma 5.26.9). Choose $A \to B$ as in Lemma 61.6.2 adapted to $T \to \pi_0(\mathrm{Spec}(A))$. By (1) there is a retraction $B \to A$. Thus we see that $T = \pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ has a section. A formal categorical argument, using Topology, Proposition 5.26.6, implies that $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected. □

09B0 **Lemma 61.6.8.** *Let A be a ring. There exists a faithfully flat, ind-Zariski ring map $A \to B$ such that B satisfies the equivalent conditions of Lemma 61.6.7.*

Proof. We first apply Lemma 61.5.3 to see that we may assume that $\mathrm{Spec}(A)$ is w-local. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(A))$, see Topology, Lemma 5.26.9. Note that T is profinite. Apply Lemma 61.6.2 to find an ind-Zariski ring map $A \to B$ such that $\pi_0(\mathrm{Spec}(B)) \to$

$\pi_0(\mathrm{Spec}(A))$ realizes $T \to \pi_0(\mathrm{Spec}(A))$ and such that

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \pi_0(\mathrm{Spec}(B)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \pi_0(\mathrm{Spec}(A)) \end{array}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(B)$ is w-local, that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is w-local, and that the set of closed points of $\mathrm{Spec}(B)$ is the inverse image of the set of closed points of $\mathrm{Spec}(A)$, see Lemma 61.2.5. Thus condition (3) of Lemma 61.6.7 holds for B. □

0A0D **Remark 61.6.9.** In each of Lemmas 61.6.1, 61.6.2, Proposition 61.6.6, and Lemma 61.6.8 we find an ind-Zariski ring map with some properties. In the paper [**BS13**] the authors use the notion of an ind-(Zariski localization) which is a filtered colimit of finite products of principal localizations. It is possible to replace ind-Zariski by ind-(Zariski localization) in each of the results listed above. However, we do not need this and the notion of an ind-Zariski homomorphism of rings as defined here has slightly better formal properties. Moreover, the notion of an ind-Zariski ring map is the natural analogue of the notion of an ind-étale ring map defined in the next section.

61.7. Ind-étale algebra

097H We start with a definition.

097I **Definition 61.7.1.** A ring map $A \to B$ is said to be *ind-étale* if B can be written as a filtered colimit of étale A-algebras.

The category of ind-étale algebras is closed under a number of natural operations.

097J **Lemma 61.7.2.** *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-étale, then $A' \to B'$ is ind-étale.*

Proof. This is Algebra, Lemma 10.154.1. □

097K **Lemma 61.7.3.** *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-étale, then $A \to C$ is ind-étale.*

Proof. This is Algebra, Lemma 10.154.2. □

097L **Lemma 61.7.4.** *A filtered colimit of ind-étale A-algebras is ind-étale over A.*

Proof. This is Algebra, Lemma 10.154.3. □

097M **Lemma 61.7.5.** *Let A be a ring. Let $B \to C$ be an A-algebra map of ind-étale A-algebras. Then C is an ind-étale B-algebra.*

Proof. This is Algebra, Lemma 10.154.5. □

097N **Lemma 61.7.6.** *Let $A \to B$ be ind-étale. Then $A \to B$ is weakly étale (More on Algebra, Definition 15.105.1).*

Proof. This follows from More on Algebra, Lemma 15.105.14. □

097P **Lemma 61.7.7.** *Let A be a ring and let $I \subset A$ be an ideal. The base change functor*

$$\textit{ind-étale } A\textit{-algebras} \longrightarrow \textit{ind-étale } A/I\textit{-algebras}, \quad C \longmapsto C/IC$$

has a fully faithful right adjoint v. In particular, given an ind-étale A/I-algebra $\overline{C}$ there exists an ind-étale A-algebra $C = v(\overline{C})$ such that $\overline{C} = C/IC$.

Proof. Let $\overline{C}$ be an ind-étale A/I-algebra. Consider the category $\mathcal{C}$ of factorizations $A \to B \to \overline{C}$ where $A \to B$ is étale. (We ignore some set theoretical issues in this proof.) We will show that this category is directed and that $C = \operatorname{colim}_{\mathcal{C}} B$ is an ind-étale A-algebra such that $\overline{C} = C/IC$.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 4.19.1). The category is nonempty as $A \to A \to \overline{C}$ is an object. Suppose that $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ are two objects of $\mathcal{C}$. Then $A \to B \otimes_A B' \to \overline{C}$ is another (use Algebra, Lemma 10.143.3). Suppose that $f, g : B \to B'$ are two maps between objects $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ of $\mathcal{C}$. Then a coequalizer is $A \to B' \otimes_{f,B,g} B' \to \overline{C}$. This is an object of $\mathcal{C}$ by Algebra, Lemmas 10.143.3 and 10.143.8. Thus the category $\mathcal{C}$ is directed.

Write $\overline{C} = \operatorname{colim} \overline{B_i}$ as a filtered colimit with $\overline{B_i}$ étale over A/I. For every i there exists $A \to B_i$ étale with $\overline{B_i} = B_i/IB_i$, see Algebra, Lemma 10.143.10. Thus $C \to \overline{C}$ is surjective. Since $C/IC \to \overline{C}$ is ind-étale (Lemma 61.7.5) we see that it is flat. Hence $\overline{C}$ is a localization of C/IC at some multiplicative subset $S \subset C/IC$ (Algebra, Lemma 10.108.2). Take an $f \in C$ mapping to an element of $S \subset C/IC$. Choose $A \to B \to \overline{C}$ in $\mathcal{C}$ and $g \in B$ mapping to f in the colimit. Then we see that $A \to B_g \to \overline{C}$ is an object of $\mathcal{C}$ as well. Thus f is an invertible element of C. It follows that $C/IC = \overline{C}$.

Next, we claim that for an ind-étale algebra D over A we have

$$\operatorname{Mor}_A(D, C) = \operatorname{Mor}_{A/I}(D/ID, \overline{C})$$

Namely, let $D/ID \to \overline{C}$ be an A/I-algebra map. Write $D = \operatorname{colim}_{i \in I} D_i$ as a colimit over a directed set I with D_i étale over A. By choice of $\mathcal{C}$ we obtain a transformation $I \to \mathcal{C}$ and hence a map $D \to C$ compatible with maps to $\overline{C}$. Whence the claim.

It follows that the functor v defined by the rule

$$\overline{C} \longmapsto v(\overline{C}) = \operatorname{colim}_{A \to B \to \overline{C}} B$$

is a right adjoint to the base change functor u as required by the lemma. The functor v is fully faithful because $u \circ v = \mathrm{id}$ by construction, see Categories, Lemma 4.24.4. □

61.8. Constructing ind-étale algebras

097Q Let A be a ring. Recall that any étale ring map $A \to B$ is isomorphic to a standard smooth ring map of relative dimension 0. Such a ring map is of the form

$$A \longrightarrow A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

where the determinant of the $n \times n$-matrix with entries $\partial f_i/\partial x_j$ is invertible in the quotient ring. See Algebra, Lemma 10.143.2.

Let $S(A)$ be the set of all *faithfully flat*[1] standard smooth A-algebras of relative dimension 0. Let $I(A)$ be the partially ordered (by inclusion) set of finite subsets E of $S(A)$. Note that $I(A)$ is a directed partially ordered set. For $E = \{A \to B_1, \ldots, A \to B_n\}$ set

$$B_E = B_1 \otimes_A \ldots \otimes_A B_n$$

Observe that B_E is a faithfully flat étale A-algebra. For $E \subset E'$, there is a canonical transition map $B_E \to B_{E'}$ of étale A-algebras. Namely, say $E = \{A \to B_1, \ldots, A \to B_n\}$ and $E' = \{A \to B_1, \ldots, A \to B_{n+m}\}$ then $B_E \to B_{E'}$ sends $b_1 \otimes \ldots \otimes b_n$ to the element $b_1 \otimes \ldots \otimes b_n \otimes 1 \otimes \ldots \otimes 1$ of $B_{E'}$. This construction defines a system of faithfully flat étale A-algebras over $I(A)$ and we set

$$T(A) = \operatorname{colim}_{E \in I(A)} B_E$$

Observe that $T(A)$ is a faithfully flat ind-étale A-algebra (Algebra, Lemma 10.39.20). By construction given any faithfully flat étale A-algebra B there is a (non-unique) A-algebra map $B \to T(A)$. Namely, pick some $(A \to B_0) \in S(A)$ and an isomorphism $B \cong B_0$. Then the canonical coprojection

$$B \to B_0 \to T(A) = \operatorname{colim}_{E \in I(A)} B_E$$

is the desired map.

097R **Lemma 61.8.1.** *Given a ring A there exists a faithfully flat ind-étale A-algebra C such that every faithfully flat étale ring map $C \to B$ has a retraction.*

Proof. Set $T^1(A) = T(A)$ and $T^{n+1}(A) = T(T^n(A))$. Let

$$C = \operatorname{colim} T^n(A)$$

This algebra is faithfully flat over each $T^n(A)$ and in particular over A, see Algebra, Lemma 10.39.20. Moreover, C is ind-étale over A by Lemma 61.7.4. If $C \to B$ is étale, then there exists an n and an étale ring map $T^n(A) \to B'$ such that $B = C \otimes_{T^n(A)} B'$, see Algebra, Lemma 10.143.3. If $C \to B$ is faithfully flat, then $\operatorname{Spec}(B) \to \operatorname{Spec}(C) \to \operatorname{Spec}(T^n(A))$ is surjective, hence $\operatorname{Spec}(B') \to \operatorname{Spec}(T^n(A))$ is surjective. In other words, $T^n(A) \to B'$ is faithfully flat. By our construction, there is a $T^n(A)$-algebra map $B' \to T^{n+1}(A)$. This induces a C-algebra map $B \to C$ which finishes the proof. □

097S **Remark 61.8.2.** Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of $T(A)$ is at most κ. Namely, each B_E has cardinality at most κ and the index set $I(A)$ has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. It follows that the ring constructed in the proof of Lemma 61.8.1 has cardinality at most κ as well.

097T **Remark 61.8.3.** The construction $A \mapsto T(A)$ is functorial in the following sense: If $A \to A'$ is a ring map, then we can construct a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & T(A) \\ \downarrow & & \downarrow \\ A' & \longrightarrow & T(A') \end{array}$$

[1] In the presence of flatness, e.g., for smooth or étale ring maps, this just means that the induced map on spectra is surjective. See Algebra, Lemma 10.39.16.

Namely, given $(A \to A[x_1, \ldots, x_n]/(f_1, \ldots, f_n))$ in $S(A)$ we can use the ring map $\varphi : A \to A'$ to obtain a corresponding element $(A' \to A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi))$ of $S(A')$ where f^φ means the polynomial obtained by applying φ to the coefficients of the polynomial f. Moreover, there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi) \end{array}$$

in the category of rings. For $E \subset S(A)$ finite, set $E' = \varphi(E)$ and define $B_E \to B_{E'}$ in the obvious manner. Taking the colimit gives the desired map $T(A) \to T(A')$, see Categories, Lemma 4.14.8.

097U **Lemma 61.8.4.** *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Then the same is true for every quotient ring A/I.*

Proof. Let $A/I \to \overline{B}$ be faithfully flat étale. By Algebra, Lemma 10.143.10 we can write $\overline{B} = B/IB$ for some étale ring map $A \to B$. The image U of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open and contains $V(I)$. Hence the complement $Z = \mathrm{Spec}(A) \setminus U$ is quasi-compact and disjoint from $V(I)$. Hence $Z \subset D(f_1) \cup \ldots \cup D(f_r)$ for some $r \geq 0$ and $f_i \in I$. Then $A \to B' = B \times \prod A_{f_i}$ is faithfully flat étale and $\overline{B} = B'/IB'$. Hence the retraction $B' \to A$ to $A \to B'$, induces a retraction to $A/I \to \overline{B}$. □

097V **Lemma 61.8.5.** *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Then every local ring of A at a maximal ideal is strictly henselian.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Let $A \to B$ be an étale ring map and let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. By the description of the strict henselization $A^{sh}_{\mathfrak{m}}$ in Algebra, Lemma 10.155.11 it suffices to show that $A_{\mathfrak{m}} = B_{\mathfrak{q}}$. Note that there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{m}$ and there are no specializations between them as an étale ring map is quasi-finite, see Algebra, Lemma 10.143.6. Thus $\mathfrak{q}_i$ is a maximal ideal and we can find $g \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, $g \notin \mathfrak{q}$ (Algebra, Lemma 10.15.2). After replacing B by B_g we see that $\mathfrak{q}$ is the only prime of B lying over $\mathfrak{m}$. The image $U \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open (Algebra, Proposition 10.41.8). Thus the complement $\mathrm{Spec}(A) \setminus U$ is closed and we can find $f \in A$, $f \notin \mathfrak{p}$ such that $\mathrm{Spec}(A) = U \cup D(f)$. The ring map $A \to B \times A_f$ is faithfully flat and étale, hence has a retraction $\sigma : B \times A_f \to A$ by assumption on A. Observe that σ is étale, hence flat as a map between étale A-algebras (Algebra, Lemma 10.143.8). Since $\mathfrak{q}$ is the only prime of $B \times A_f$ lying over A we find that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ has a retraction which is also flat. Thus $A_{\mathfrak{p}} \to B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ are flat local ring maps whose composition is the identity. Since a flat local homomorphism of local rings is injective we conclude these maps are isomorphisms as desired. □

097W **Lemma 61.8.6.** *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a retraction. Let $Z \subset \mathrm{Spec}(A)$ be a closed subscheme. Let $A \to A^\sim_Z$ be as constructed in Lemma 61.5.1. Then every faithfully flat étale ring map $A^\sim_Z \to C$ has a retraction.*

Proof. There exists an étale ring map $A \to B'$ such that $C = B' \otimes_A A^\sim_Z$ as $A^\sim_Z$-algebras. The image $U' \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is open and contains

$V(I)$, hence we can find $f \in I$ such that $\mathrm{Spec}(A) = U' \cup D(f)$. Then $A \to B' \times A_f$ is étale and faithfully flat. By assumption there is a retraction $B' \times A_f \to A$. Localizing we obtain the desired retraction $C \to A^\sim_Z$. □

097X **Lemma 61.8.7.** *Let $A \to B$ be a ring map inducing algebraic extensions on residue fields. There exists a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

with the following properties:

(1) *$A \to C$ is faithfully flat and ind-étale,*
(2) *$B \to D$ is faithfully flat and ind-étale,*
(3) *$\mathrm{Spec}(C)$ is w-local,*
(4) *$\mathrm{Spec}(D)$ is w-local,*
(5) *$\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local,*
(6) *the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$,*
(7) *the set of closed points of $\mathrm{Spec}(C)$ surjects onto $\mathrm{Spec}(A)$,*
(8) *the set of closed points of $\mathrm{Spec}(D)$ surjects onto $\mathrm{Spec}(B)$,*
(9) *for $\mathfrak{m} \subset C$ maximal the local ring $C_\mathfrak{m}$ is strictly henselian.*

Proof. There is a faithfully flat, ind-Zariski ring map $A \to A'$ such that $\mathrm{Spec}(A')$ is w-local and such that the set of closed points of $\mathrm{Spec}(A')$ maps onto $\mathrm{Spec}(A)$, see Lemma 61.5.3. Let $I \subset A'$ be the ideal such that $V(I)$ is the set of closed points of $\mathrm{Spec}(A')$. Choose $A' \to C'$ as in Lemma 61.8.1. Note that the local rings $C'_{\mathfrak{m}'}$ at maximal ideals $\mathfrak{m}' \subset C'$ are strictly henselian by Lemma 61.8.5. We apply Lemma 61.5.8 to $A' \to C'$ and $I \subset A'$ to get $C' \to C$ with $C'/IC' \cong C/IC$. Note that since $A' \to C'$ is faithfully flat, $\mathrm{Spec}(C'/IC')$ surjects onto the set of closed points of A' and in particular onto $\mathrm{Spec}(A)$. Moreover, as $V(IC) \subset \mathrm{Spec}(C)$ is the set of closed points of C and $C' \to C$ is ind-Zariski (and identifies local rings) we obtain properties (1), (3), (7), and (9).

Denote $J \subset C$ the ideal such that $V(J)$ is the set of closed points of $\mathrm{Spec}(C)$. Set $D' = B \otimes_A C$. The ring map $C \to D'$ induces algebraic residue field extensions. Keep in mind that since $V(J) \to \mathrm{Spec}(A)$ is surjective the map $T = V(JD') \to \mathrm{Spec}(B)$ is surjective too. Apply Lemma 61.5.8 to $C \to D'$ and $J \subset C$ to get $D' \to D$ with $D'/JD' \cong D/JD$. All of the remaining properties given in the lemma are immediate from the results of Lemma 61.5.8. □

61.9. Weakly étale versus pro-étale

097Y Recall that a ring homomorphism $A \to B$ is *weakly étale* if $A \to B$ is flat and $B \otimes_A B \to B$ is flat. We have proved some properties of such ring maps in More on Algebra, Section 15.105. In particular, if $A \to B$ is a local homomorphism, and A is a strictly henselian local rings, then $A = B$, see More on Algebra, Theorem 15.105.24. Using this theorem and the work we've done above we obtain the following structure theorem for weakly étale ring maps.

097Z **Proposition 61.9.1.** *Let $A \to B$ be a weakly étale ring map. Then there exists a faithfully flat, ind-étale ring map $B \to B'$ such that $A \to B'$ is ind-étale.*

Proof. The ring map $A \to B$ induces (separable) algebraic extensions of residue fields, see More on Algebra, Lemma 15.105.17. Thus we may apply Lemma 61.8.7 and choose a diagram

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

with the properties as listed in the lemma. Note that $C \to D$ is weakly étale by More on Algebra, Lemma 15.105.11. Pick a maximal ideal $\mathfrak{m} \subset D$. By construction this lies over a maximal ideal $\mathfrak{m}' \subset C$. By More on Algebra, Theorem 15.105.24 the ring map $C_{\mathfrak{m}'} \to D_{\mathfrak{m}}$ is an isomorphism. As every point of $\mathrm{Spec}(C)$ specializes to a closed point we conclude that $C \to D$ identifies local rings. Thus Proposition 61.6.6 applies to the ring map $C \to D$. Pick $D \to D'$ faithfully flat and ind-Zariski such that $C \to D'$ is ind-Zariski. Then $B \to D'$ is a solution to the problem posed in the proposition. □

61.10. The V topology and the pro-h topology

0EVM The V topology was introduced in Topologies, Section 34.10. The h topology was introduced in More on Flatness, Section 38.34. A kind of intermediate topology, namely the ph topology, was introduced in Topologies, Section 34.8.

Given a topology τ on a suitable category $\mathcal{C}$ of schemes, we can introduce a "pro-τ topology" on $\mathcal{C}$ as follows. Recall that for X in $\mathcal{C}$ we use h_X to denote the representable presheaf associated to X. Let us temporarily say a morphism $X \to Y$ of $\mathcal{C}$ is a τ-cover[2] if the τ-sheafification of $h_X \to h_Y$ is surjective. Then we can define the pro-τ topology as the coarsest topology such that

(1) the pro-τ topology is finer than the τ topology, and
(2) $X \to Y$ is a pro-τ-cover if Y is affine and $X = \lim X_\lambda$ is a directed limit of affine schemes X_λ over Y such that $h_{X_\lambda} \to h_Y$ is a τ-cover for all λ.

We use this pedantic formulation because we do not want to specify a choice of pro-τ coverings: for different τ different choices of collections of coverings are suitable. For example, in Section 61.12 we will see that in order to define the pro-étale topology looking at families of weakly étale morphisms with some finiteness property works well. More generally, the proposed construction given in this paragraph is meant mainly to motivate the results in this section and we will never implicitly define a pro-τ topology using this method.

The following lemma tells us that the pro-V topology is equal to the V topology.

0EVN **Lemma 61.10.1.** *Let Y be an affine scheme. Let $X = \lim X_i$ be a directed limit of affine schemes over Y. The following are equivalent*

(1) *$\{X \to Y\}$ is a standard V covering (Topologies, Definition 34.10.1), and*
(2) *$\{X_i \to Y\}$ is a standard V covering for all i.*

[2]This should not be confused with the notion of a covering. For example if $\tau = \textit{étale}$, any morphism $X \to Y$ which has a section is a τ-covering. But our definition of étale coverings $\{V_i \to Y\}_{i \in I}$ forces each $V_i \to Y$ to be étale.

Proof. A singleton $\{X \to Y\}$ is a standard V covering if and only if given a morphism $g : \mathrm{Spec}(V) \to Y$ there is an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \xrightarrow{g} & Y \end{array}$$

Thus (1) $\Rightarrow$ (2) is immediate from the definition. Conversely, assume (2) and let $g : \mathrm{Spec}(V) \to Y$ as above be given. Write $\mathrm{Spec}(V) \times_Y X_i = \mathrm{Spec}(A_i)$. Since $\{X_i \to Y\}$ is a standard V covering, we may choose a valuation ring W_i and a ring map $A_i \to W_i$ such that the composition $V \to A_i \to W_i$ is an extension of valuation rings. In particular, the quotient A_i' of A_i by its V-torsion is a faitfhully flat V-algebra. Flatness by More on Algebra, Lemma 15.22.10 and surjectivity on spectra because $A_i \to W_i$ factors through A_i'. Thus

$$A = \mathrm{colim}\, A_i'$$

is a faithfully flat V-algebra (Algebra, Lemma 10.39.20). Since $\{\mathrm{Spec}(A) \to \mathrm{Spec}(V)\}$ is a standard fpqc cover, it is a standard V cover (Topologies, Lemma 34.10.2) and hence we can choose $\mathrm{Spec}(W) \to \mathrm{Spec}(A)$ such that $V \to W$ is an extension of valuation rings. Since we can compose with the morphism $\mathrm{Spec}(A) \to X = \mathrm{Spec}(\mathrm{colim}\, A_i)$ the proof is complete. □

The following lemma tells us that the pro-h topology is equal to the pro-ph topology is equal to the V topology.

0EVP **Lemma 61.10.2.** *Let $X \to Y$ be a morphism of affine schemes. The following are equivalent*

(1) *$\{X \to Y\}$ is a standard V covering (Topologies, Definition 34.10.1),*
(2) *$X = \lim X_i$ is a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is a ph covering for each i, and*
(3) *$X = \lim X_i$ is a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is an h covering for each i.*

Proof. Proof of (2) $\Rightarrow$ (1). Recall that a V covering given by a single arrow between affines is a standard V covering, see Topologies, Definition 34.10.7 and Lemma 34.10.6. Recall that any ph covering is a V covering, see Topologies, Lemma 34.10.10. Hence if $X = \lim X_i$ as in (2), then $\{X_i \to Y\}$ is a standard V covering for each i. Thus by Lemma 61.10.1 we see that (1) is true.

Proof of (3) $\Rightarrow$ (2). This is clear because an h covering is always a ph covering, see More on Flatness, Definition 38.34.2.

Proof of (1) $\Rightarrow$ (3). This is the interesting direction, but the interesting content in this proof is hidden in More on Flatness, Lemma 38.34.1. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$. We can write $A = \mathrm{colim}\, A_i$ with A_i of finite presentation over R, see Algebra, Lemma 10.127.2. Set $X_i = \mathrm{Spec}(A_i)$. Then $\{X_i \to Y\}$ is a standard V covering for all i by (1) and Topologies, Lemma 34.10.6. Hence $\{X_i \to Y\}$ is an h covering by More on Flatness, Definition 38.34.2. This finishes the proof. □

The following lemma tells us, roughly speaking, that an h sheaf which is limit preserving satisfies the sheaf condition for V coverings. Please also compare with Remark 61.10.4.

0EVQ **Lemma 61.10.3.** *Let S be a scheme. Let F be a contravariant functor defined on the category of all schemes over S. If*

(1) *F satisfies the sheaf property for the h topology, and*
(2) *F is limit preserving (Limits, Remark 32.6.2),*

then F satisfies the sheaf property for the V topology.

Proof. We will prove this by verifying (1) and (2') of Topologies, Lemma 34.10.12. The sheaf property for Zariski coverings follows from the fact that F has the sheaf property for all h coverings. Finally, suppose that $X \to Y$ is a morphism of affine schemes over S such that $\{X \to Y\}$ is a V covering. By Lemma 61.10.2 we can write $X = \lim X_i$ as a directed limit of affine schemes over Y such that $\{X_i \to Y\}$ is an h covering for each i. We obtain

$$\begin{aligned}
&\text{Equalizer}(\ F(X) \rightrightarrows F(X \times_Y X)\ ) \\
&= \text{Equalizer}(\ \operatorname{colim} F(X_i) \rightrightarrows \operatorname{colim} F(X_i \times_Y X_i)\ ) \\
&= \operatorname{colim} \text{Equalizer}(\ F(X_i) \rightrightarrows F(X_i \times_Y X_i)\ ) \\
&= \operatorname{colim} F(Y) = F(Y)
\end{aligned}$$

which is what we wanted to show. The first equality because F is limit preserving and $X = \lim X_i$ and $X \times_Y X = \lim X_i \times_Y X_i$. The second equality because filtered colimits are exact. The third equality because F satisfies the sheaf property for h coverings. □

0EVR **Remark 61.10.4.** Let S be a scheme contained in a big site Sch_h. Let F be a sheaf of sets on $(Sch/S)_h$ such that $F(T) = \operatorname{colim} F(T_i)$ whenever $T = \lim T_i$ is a directed limit of affine schemes in $(Sch/S)_h$. In this situation F extends uniquely to a contravariant functor F' on the category of all schemes over S such that (a) F' satisfies the sheaf property for the h topology and (b) F' is limit preserving. See More on Flatness, Lemma 38.35.4. In this situation Lemma 61.10.3 tells us that F' satisfies the sheaf property for the V topology.

61.11. Constructing w-contractible covers

0980 In this section we construct w-contractible covers of affine schemes.

0981 **Definition 61.11.1.** Let A be a ring. We say A is *w-contractible* if every faithfully flat weakly étale ring map $A \to B$ has a retraction.

We remark that by Proposition 61.9.1 an equivalent definition would be to ask that every faithfully flat, ind-étale ring map $A \to B$ has a retraction. Here is a key observation that will allow us to construct w-contractible rings.

0982 **Lemma 61.11.2.** *Let A be a ring. The following are equivalent*

(1) *A is w-contractible,*
(2) *every faithfully flat, ind-étale ring map $A \to B$ has a retraction, and*
(3) *A satisfies*
   (a) *$\operatorname{Spec}(A)$ is w-local,*

(b) $\pi_0(\mathrm{Spec}(A))$ *is extremally disconnected, and*
(c) *for every maximal ideal* $\mathfrak{m} \subset A$ *the local ring* $A_\mathfrak{m}$ *is strictly henselian.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 61.9.1.

Assume (3)(a), (3)(b), and (3)(c). Let $A \to B$ be faithfully flat and ind-étale. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a retraction.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 61.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. In this case $A \to B$ identifies local rings by condition (3)(c) as it suffices to check this at maximal ideals of B which lie over maximal ideals of A. Thus $A \to B$ has a retraction by Lemma 61.6.7.

Assume (1) or equivalently (2). We have (3)(c) by Lemma 61.8.5. Properties (3)(a) and (3)(b) follow from Lemma 61.6.7. □

0983 **Proposition 61.11.3.** *For every ring* A *there exists a faithfully flat, ind-étale ring map* $A \to D$ *such that* D *is w-contractible.*

Proof. Applying Lemma 61.8.7 to $\mathrm{id}_A : A \to A$ we find a faithfully flat, ind-étale ring map $A \to C$ such that C is w-local and such that every local ring at a maximal ideal of C is strictly henselian. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(C))$, see Topology, Lemma 5.26.9. Note that T is profinite. Apply Lemma 61.6.2 to find an ind-Zariski ring map $C \to D$ such that $\pi_0(\mathrm{Spec}(D)) \to \pi_0(\mathrm{Spec}(C))$ realizes $T \to \pi_0(\mathrm{Spec}(C))$ and such that

$$\begin{array}{ccc} \mathrm{Spec}(D) & \longrightarrow & \pi_0(\mathrm{Spec}(D)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(C) & \longrightarrow & \pi_0(\mathrm{Spec}(C)) \end{array}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(D)$ is w-local, that $\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local, and that the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$, see Lemma 61.2.5. Thus it is still true that the local rings of D at its maximal ideals are strictly henselian (as they are isomorphic to the local rings at the corresponding maximal ideals of C). It follows from Lemma 61.11.2 that D is w-contractible. □

0984 **Remark 61.11.4.** Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of the ring D constructed in Proposition 61.11.3 is at most

$$\kappa^{2^{2^{2^\kappa}}}.$$

Namely, the ring map $A \to D$ is constructed as a composition

$$A \to A_w = A' \to C' \to C \to D.$$

Here the first three steps of the construction are carried out in the first paragraph of the proof of Lemma 61.8.7. For the first step we have $|A_w| \leq \kappa$ by Remark 61.5.4. We have $|C'| \leq \kappa$ by Remark 61.8.2. Then $|C| \leq \kappa$ because C is a localization of $(C')_w$ (it is constructed from C' by an application of Lemma 61.5.7 in the proof

of Lemma 61.5.8). Thus C has at most 2^κ maximal ideals. Finally, the ring map $C \to D$ identifies local rings and the cardinality of the set of maximal ideals of D is at most $2^{2^{2^\kappa}}$ by Topology, Remark 5.26.10. Since $D \subset \prod_{\mathfrak{m} \subset D} D_{\mathfrak{m}}$ we see that D has at most the size displayed above.

0985 **Lemma 61.11.5.** *Let $A \to B$ be a quasi-finite and finitely presented ring map. If the residue fields of A are separably algebraically closed and $\mathrm{Spec}(A)$ is Hausdorff and extremally disconnected, then $\mathrm{Spec}(B)$ is extremally disconnected.*

Proof. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Choose a finite partition $X = \coprod X_i$ and $X_i' \to X_i$ as in Étale Cohomology, Lemma 59.72.3. The map of topological spaces $\coprod X_i \to X$ (where the source is the disjoint union in the category of topological spaces) has a section by Topology, Proposition 5.26.6. Hence we see that X is topologically the disjoint union of the strata X_i. Thus we may replace X by the X_i and assume there exists a surjective finite locally free morphism $X' \to X$ such that $(X' \times_X Y)_{red}$ is isomorphic to a finite disjoint union of copies of X'_{red}. Picture

$$\begin{array}{ccc} \coprod_{i=1,\ldots,r} X' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

The assumption on the residue fields of A implies that this diagram is a fibre product diagram on underlying sets of points (details omitted). Since X is extremally disconnected and X' is Hausdorff (Lemma 61.5.6), the continuous map $X' \to X$ has a continuous section σ. Then $\coprod_{i=1,\ldots,r} \sigma(X) \to Y$ is a bijective continuous map. By Topology, Lemma 5.17.8 we see that it is a homeomorphism and the proof is done. □

0986 **Lemma 61.11.6.** *Let $A \to B$ be a finite and finitely presented ring map. If A is w-contractible, so is B.*

Proof. We will use the criterion of Lemma 61.11.2. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ and denote $f : Y \to X$ the induced morphism. As $f : Y \to X$ is a finite morphism, we see that the set of closed points Y_0 of Y is the inverse image of the set of closed points X_0 of X. Let $y \in Y$ with image $x \in X$. Then x specializes to a unique closed point $x_0 \in X$. Say $f^{-1}(\{x_0\}) = \{y_1, \ldots, y_n\}$ with y_i closed in Y. Since $R = \mathcal{O}_{X,x_0}$ is strictly henselian and since f is finite, we see that $Y \times_{f,X} \mathrm{Spec}(R)$ is equal to $\coprod_{i=1,\ldots,n} \mathrm{Spec}(R_i)$ where each R_i is a local ring finite over R whose maximal ideal corresponds to y_i, see Algebra, Lemma 10.153.3 part (10). Then y is a point of exactly one of these $\mathrm{Spec}(R_i)$ and we see that y specializes to exactly one of the y_i. In other words, every point of Y specializes to a unique point of Y_0. Thus Y is w-local. For every $y \in Y_0$ with image $x \in X_0$ we see that $\mathcal{O}_{Y,y}$ is strictly henselian by Algebra, Lemma 10.153.4 applied to $\mathcal{O}_{X,x} \to B \otimes_A \mathcal{O}_{X,x}$. It remains to show that Y_0 is extremally disconnected. To do this we look at $X_0 \times_X Y \to X_0$ where $X_0 \subset X$ is the reduced induced scheme structure. Note that the underlying topological space of $X_0 \times_X Y$ agrees with Y_0. Now the desired result follows from Lemma 61.11.5. □

0987 **Lemma 61.11.7.** *Let A be a ring. Let $Z \subset \mathrm{Spec}(A)$ be a closed subset of the form $Z = V(f_1, \ldots, f_r)$. Set $B = A^\sim_Z$, see Lemma 61.5.1. If A is w-contractible, so is B.*

Proof. Let $A_Z^\sim \to B$ be a weakly étale faithfully flat ring map. Consider the ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times B$$

this is faithful flat and weakly étale. If A is w-contractible, then there is a retraction σ. Consider the morphism

$$\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(A) \xrightarrow{\mathrm{Spec}(\sigma)} \coprod \mathrm{Spec}(A_{f_i}) \amalg \mathrm{Spec}(B)$$

Every point of $Z \subset \mathrm{Spec}(A_Z^\sim)$ maps into the component $\mathrm{Spec}(B)$. Since every point of $\mathrm{Spec}(A_Z^\sim)$ specializes to a point of Z we find a morphism $\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(B)$ as desired. □

61.12. The pro-étale site

0988 In this section we only discuss the actual definition and construction of the various pro-étale sites and the morphisms between them. The existence of weakly contractible objects will be done in Section 61.13.

The pro-étale topology is a bit like the fpqc topology (see Topologies, Section 34.9) in that the topos of sheaves on the small pro-étale site of a scheme depends on the choice of the underlying category of schemes. Thus we cannot speak of *the* pro-étale topos of a scheme. However, it will be true that the cohomology groups of a sheaf are unchanged if we enlarge our underlying category of schemes, see Section 61.31.

We will define pro-étale coverings using weakly étale morphisms of schemes, see More on Morphisms, Section 37.64. The reason is that, on the one hand, it is somewhat awkward to define the notion of a pro-étale morphism of schemes, and on the other, Proposition 61.9.1 assures us that we obtain the same sheaves[3] with the definition that follows.

0989 **Definition 61.12.1.** Let T be a scheme. A *pro-étale covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is weakly-étale and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. Here is a lemma that will allow us to recognize pro-étale coverings. It will also allow us to reduce many lemmas about pro-étale coverings to the corresponding results for fpqc coverings.

098A **Lemma 61.12.2.** *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i\in I}$ is a pro-étale covering,*
2. *each f_i is weakly étale and $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering,*
3. *each f_i is weakly étale and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
4. *each f_i is weakly étale and there exists an affine open covering $T = \bigcup_{\alpha\in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

[3] To be precise the pro-étale topology we obtain using our choice of coverings is the same as the one gotten from the general procedure explained in Section 61.10 starting with τ = *étale*.

(5) *each f_i is weakly étale, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Hence the lemma follows from Topologies, Lemma 34.9.2. □

098B **Lemma 61.12.3.** *Any étale covering and any Zariski covering is a pro-étale covering.*

Proof. This follows from the corresponding result for fpqc coverings (Topologies, Lemma 34.9.7), Lemma 61.12.2, and the fact that an étale morphism is a weakly étale morphism, see More on Morphisms, Lemma 37.64.9. □

098C **Lemma 61.12.4.** *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a pro-étale covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a pro-étale covering and for each i we have a pro-étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a pro-étale covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a pro-étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a pro-étale covering.*

Proof. This follows from the fact that composition and base changes of weakly étale morphisms are weakly étale (More on Morphisms, Lemmas 37.64.5 and 37.64.6), Lemma 61.12.2, and the corresponding results for fpqc coverings, see Topologies, Lemma 34.9.8. □

098D **Lemma 61.12.5.** *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a pro-étale covering of T. Then there exists a pro-étale covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. □

Thus we define the corresponding standard coverings of affines as follows.

098E **Definition 61.12.6.** Let T be an affine scheme. A *standard pro-étale covering* of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ where each T_j is affine, each f_i is weakly étale, and $T = \bigcup f_i(T_i)$.

We follow the general outline given in Topologies, Section 34.2 for constructing the big pro-étale site we will be working with. However, because we need a bit larger rings to accommodate for the size of certain constructions we modify the constructions slightly.

098G **Definition 61.12.7.** A *big pro-étale site* is any site $Sch_{pro\text{-}étale}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of pro-étale coverings Cov_0 among these schemes.
(2) Change the function *Bound* of Sets, Equation (3.9.1.1) into
$$Bound(\kappa) = \max\{\kappa^{2^{2^{2^\kappa}}}, \kappa^{\aleph_0}, \kappa^+\}.$$
(3) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0 and the function *Bound*.

(4) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of pro-étale coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

It will turn out, see Lemma 61.31.1, that the topology on a big pro-étale site $Sch_{pro\text{-}étale}$ is in some sense induced from the pro-étale topology on the category of all schemes.

098K **Definition 61.12.8.** Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S.

(1) The *big pro-étale site of* S, denoted $(Sch/S)_{pro\text{-}étale}$, is the site $Sch_{pro\text{-}étale}/S$ introduced in Sites, Section 7.25.
(2) The *small pro-étale site of* S, which we denote $S_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are those U/S such that $U \to S$ is weakly étale. A covering of $S_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$.
(3) The *big affine pro-étale site of* S, denoted $(Aff/S)_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ which is a standard pro-étale covering.

It is not completely clear that the small pro-étale site and the big affine pro-étale site are sites. We check this now.

098L **Lemma 61.12.9.** *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Both $S_{pro\text{-}étale}$ and $(Aff/S)_{pro\text{-}étale}$ are sites.*

Proof. Let us show that $S_{pro\text{-}étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{pro\text{-}étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$ we also have $U_i \in \mathrm{Ob}(S_{pro\text{-}étale})$. This follows from the definitions as the composition of weakly étale morphisms is weakly étale.

To show that $(Aff/S)_{pro\text{-}étale}$ is a site, reasoning as above, it suffices to show that the collection of standard pro-étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This follows from Lemma 61.12.2 and the corresponding result for standard fpqc coverings (Topologies, Lemma 34.9.11). □

098M **Lemma 61.12.10.** *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Let Sch be the category of all schemes.*

(1) *The categories $Sch_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have fibre products agreeing with fibre products in Sch.*
(2) *The categories $Sch_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{pro\text{-}étale}$ have equalizers agreeing with equalizers in Sch.*
(3) *The categories $(Sch/S)_{pro\text{-}étale}$, and $S_{pro\text{-}étale}$ both have a final object, namely S/S.*
(4) *The category $Sch_{pro\text{-}étale}$ has a final object agreeing with the final object of Sch, namely* $\mathrm{Spec}(\mathbf{Z})$.

Proof. The category $Sch_{pro\text{-}étale}$ contains $\mathrm{Spec}(\mathbf{Z})$ and is closed under products and fibre products by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{pro\text{-}étale})$. The fibre product $V \times_U W$ in $Sch_{pro\text{-}étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{pro\text{-}étale}$. This proves the result for $(Sch/S)_{pro\text{-}étale}$. If $U \to S$, $V \to U$ and $W \to U$ are weakly étale then so is $V \times_U W \to S$ (see More on Morphisms, Section 37.64) and hence we get fibre products for $S_{pro\text{-}étale}$. If U, V, W are affine, so is $V \times_U W$ and hence we get fibre products for $(Aff/S)_{pro\text{-}étale}$.

Let $a, b : U \to V$ be two morphisms in $Sch_{pro\text{-}étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/\mathrm{Spec}(\mathbf{Z})}, V \times_{\mathrm{Spec}(\mathbf{Z})} V, (a,b)} (U \times_{\mathrm{Spec}(\mathbf{Z})} U)$$

which is an object of $Sch_{pro\text{-}étale}$ by what we saw above. Thus $Sch_{pro\text{-}étale}$ has equalizers. If a and b are morphisms over S, then the equalizer (in the category of schemes) is also given by

$$V \times_{\Delta_{V/S}, V \times_S V, (a,b)} (U \times_S U)$$

hence we see that $(Sch/S)_{pro\text{-}étale}$ has equalizers. Moreover, if U and V are weakly-étale over S, then so is the equalizer above as a fibre product of schemes weakly étale over S. Thus $S_{pro\text{-}étale}$ has equalizers. The statements on final objects is clear. □

Next, we check that the big affine pro-étale site defines the same topos as the big pro-étale site.

098N **Lemma 61.12.11.** *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. The functor $(Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{pro\text{-}étale})$ to $Sh((Sch/S)_{pro\text{-}étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$. Being cocontinuous just means that any pro-étale covering of T/S, T affine, can be refined by a standard pro-étale covering of T. This is the content of Lemma 61.12.5. Hence (1) holds. We see u is continuous simply because a standard pro-étale covering is a pro-étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

098P **Lemma 61.12.12.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$. The functor $T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{pro\text{-}étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$. In other words, given a weakly étale morphism $j : U \to T$ corresponding to an object of $T_{pro\text{-}étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 61.12.10. Moreover, $T_{pro\text{-}étale}$ has equalizers and u commutes with them by Lemma 61.12.10. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

098Q **Lemma 61.12.13.** *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. The inclusion functor $S_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{pro\text{-}étale} \longrightarrow S_{pro\text{-}étale}$$

and a morphism of topoi

$$i_S : Sh(S_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 61.12.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$, in addition to the properties seen in the proof of Lemma 61.12.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

098R **Definition 61.12.14.** In the situation of Lemma 61.12.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small pro-étale site*, and for a sheaf $\mathcal{F}$ on the big pro-étale site we denote $\mathcal{F}|_{S_{pro\text{-}étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}\mathrm{Mor}_{Sh(S_{pro\text{-}étale})}(\mathcal{F}|_{S_{pro\text{-}étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{pro\text{-}étale})}(\mathcal{G}, \mathcal{F}|_{S_{pro\text{-}étale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{pro\text{-}étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{pro\text{-}étale}} = \mathcal{G}$.

098S **Lemma 61.12.15.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$. The functor*

$$u : (Sch/T)_{pro\text{-}étale} \longrightarrow (Sch/S)_{pro\text{-}étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{pro\text{-}étale} \longrightarrow (Sch/T)_{pro\text{-}étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 61.12.12). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given

U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

098T **Lemma 61.12.16.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 61.12.12 and i_T as in Lemma 61.12.13.*
(2) *The functor $S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*
$$f_{small} : Sh(T_{pro\text{-}étale}) \longrightarrow Sh(S_{pro\text{-}étale}).$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) *We have a commutative diagram of morphisms of sites*
$$\begin{array}{ccc} T_{pro\text{-}étale} & \xleftarrow[\pi_T]{} & (Sch/T)_{pro\text{-}étale} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{pro\text{-}étale} & \xleftarrow{\pi_S} & (Sch/S)_{pro\text{-}étale} \end{array}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemmas 61.12.4 and 61.12.10. Moreover, both $S_{pro\text{-}étale}$, $T_{pro\text{-}étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 7.14.7 the functor u corresponds to a morphism of sites $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 61.12.14 we have: for $\mathcal{F}$ a sheaf on the big pro-étale site of T

0F60 (61.12.16.1) $$(f_{big,*}\mathcal{F})|_{S_{pro\text{-}étale}} = f_{small,*}(\mathcal{F}|_{T_{pro\text{-}étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small pro-étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

098U **Lemma 61.12.17.** *Given schemes X, Y, Y in $Sch_{pro\text{-}étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 61.12.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 61.12.16. □

0F61 **Lemma 61.12.18.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Consider a cartesian diagram*

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in $Sch_{pro\text{-}étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{pro\text{-}étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{pro\text{-}étale}$ (use Lemmas 61.12.12 and 61.12.15). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

We can think about a sheaf on the big pro-étale site of S as a collection of sheaves on the small pro-étale site on schemes over S.

098V **Lemma 61.12.19.** *Let S be a scheme contained in a big pro-étale site $Sch_{pro\text{-}étale}$. A sheaf $\mathcal{F}$ on the big pro-étale site $(Sch/S)_{pro\text{-}étale}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{pro\text{-}étale})$ a sheaf $\mathcal{F}_T$ on $T_{pro\text{-}étale}$,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(a) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{pro\text{-}étale}$ the composition $c_g \circ g_{small}^{-1} c_f$ is equal to $c_{f \circ g}$, and*
(b) *if $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ is weakly étale then c_f is an isomorphism.*

Proof. Identical to the proof of Topologies, Lemma 34.4.20. □

098W **Lemma 61.12.20.** *Let S be a scheme. Let $S_{affine,pro\text{-}étale}$ denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects. A covering of $S_{affine,pro\text{-}étale}$ will be a standard pro-étale covering, see Definition 61.12.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{affine,pro\text{-}étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 7.29.1 by checking that the inclusion functor $S_{affine,pro\text{-}étale} \to S_{pro\text{-}étale}$ is a special cocontinuous functor (see Sites, Definition 7.29.2). □

098X **Lemma 61.12.21.** *Let S be an affine scheme. Let S_{app} denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects U such that $\mathcal{O}(S) \to \mathcal{O}(U)$ is ind-étale. A covering of S_{app} will be a standard pro-étale covering, see Definition 61.12.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{app}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{app})$.

Proof. By Lemma 61.12.20 we may replace $S_{pro\text{-}étale}$ by $S_{affine,pro\text{-}étale}$. The lemma follows from Sites, Lemma 7.29.1 by checking that the inclusion functor $S_{app} \to S_{affine,pro\text{-}étale}$ is a special cocontinuous functor, see Sites, Definition 7.29.2. The conditions of Sites, Lemma 7.29.1 follow immediately from the definition and the facts (a) any object U of $S_{affine,pro\text{-}étale}$ has a covering $\{V \to U\}$

with V ind-étale over X (Proposition 61.9.1) and (b) the functor u is fully faithful. □

098Z **Lemma 61.12.22.** *Let S be a scheme. The topology on each of the pro-étale sites $Sch_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ is subcanonical.*

Proof. Combine Lemma 61.12.2 and Descent, Lemma 35.13.7. □

61.13. Weakly contractible objects

0F4N In this section we prove the key fact that our pro-étale sites contain many weakly contractible objects. In fact, the proof of Lemma 61.13.3 is the reason for the shape of the function *Bound* in Definition 61.12.7 (although for readers who are ignoring set theoretical questions, this information is without content).
We first express the notion of w-contractible rings in terms of pro-étale coverings.

098F **Lemma 61.13.1.** *Let $T = \mathrm{Spec}(A)$ be an affine scheme. The following are equivalent*

(1) *A is w-contractible, and*
(2) *every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.*

Proof. Assume A is w-contractible. By Lemma 61.12.5 it suffices to prove we can refine every standard pro-étale covering $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ by a Zariski covering of T. The morphism $\coprod T_i \to T$ is a surjective weakly étale morphism of affine schemes. Hence by Definition 61.11.1 there exists a morphism $\sigma : T \to \coprod T_i$ over T. Then the Zariski covering $T = \coprod \sigma^{-1}(T_i)$ refines $\{f_i : T_i \to T\}$.

Conversely, assume (2). If $A \to B$ is faithfully flat and weakly étale, then $\{\mathrm{Spec}(B) \to T\}$ is a pro-étale covering. Hence there exists a Zariski covering $T = \coprod U_i$ and morphisms $U_i \to \mathrm{Spec}(B)$ over T. Since $T = \coprod U_i$ we obtain $T \to \mathrm{Spec}(B)$, i.e., an A-algebra map $B \to A$. This means A is w-contractible. □

098H **Lemma 61.13.2.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 61.12.7. Let $T = \mathrm{Spec}(A)$ be an affine object of $Sch_{pro\text{-}étale}$. The following are equivalent*

(1) *A is w-contractible,*
(2) *T is a weakly contractible (Sites, Definition 7.40.2) object of $Sch_{pro\text{-}étale}$, and*
(3) *every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.*

Proof. We have seen the equivalence of (1) and (3) in Lemma 61.13.1.

Assume (3) and let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves on $Sch_{pro\text{-}étale}$. Let $s \in \mathcal{G}(T)$. To prove (2) we will show that s is in the image of $\mathcal{F}(T) \to \mathcal{G}(T)$. We can find a covering $\{T_i \to T\}$ of $Sch_{pro\text{-}étale}$ such that s lifts to a section of $\mathcal{F}$ over T_i (Sites, Definition 7.11.1). By (3) we may assume we have a finite covering $T = \coprod_{j=1,\ldots,m} U_j$ by open and closed subsets and we have $t_j \in \mathcal{F}(U_j)$ mapping to $s|_{U_j}$. Since Zariski coverings are coverings in $Sch_{pro\text{-}étale}$ (Lemma 61.12.3) we conclude that $\mathcal{F}(T) = \prod \mathcal{F}(U_j)$. Thus $t = (t_1, \ldots, t_m) \in \mathcal{F}(T)$ is a section mapping to s.

Assume (2). Let $A \to D$ be as in Proposition 61.11.3. Then $\{V \to T\}$ is a covering of $Sch_{pro\text{-}étale}$. (Note that $V = \mathrm{Spec}(D)$ is an object of $Sch_{pro\text{-}étale}$ by Remark

61.11.4 combined with our choice of the function *Bound* in Definition 61.12.7 and the computation of the size of affine schemes in Sets, Lemma 3.9.5.) Since the topology on $Sch_{pro\text{-}étale}$ is subcanonical (Lemma 61.12.22) we see that $h_V \to h_T$ is a surjective map of sheaves (Sites, Lemma 7.12.4). Since T is assumed weakly contractible, we see that there is an element $f \in h_V(T) = \mathrm{Mor}(T, V)$ whose image in $h_T(T)$ is id_T. Thus $A \to D$ has a retraction $\sigma : D \to A$. Now if $A \to B$ is faithfully flat and weakly étale, then $D \to D \otimes_A B$ has the same properties, hence there is a retraction $D \otimes_A B \to D$ and combined with σ we get a retraction $B \to D \otimes_A B \to D \to A$ of $A \to B$. Thus A is w-contractible and (1) holds. □

098I **Lemma 61.13.3.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 61.12.7. For every object T of $Sch_{pro\text{-}étale}$ there exists a covering $\{T_i \to T\}$ in $Sch_{pro\text{-}étale}$ with each T_i affine and the spectrum of a w-contractible ring. In particular, T_i is weakly contractible in $Sch_{pro\text{-}étale}$.*

Proof. For those readers who do not care about set-theoretical issues this lemma is a trivial consequence of Lemma 61.13.2 and Proposition 61.11.3. Here are the details. Choose an affine open covering $T = \bigcup U_i$. Write $U_i = \mathrm{Spec}(A_i)$. Choose faithfully flat, ind-étale ring maps $A_i \to D_i$ such that D_i is w-contractible as in Proposition 61.11.3. The family of morphisms $\{\mathrm{Spec}(D_i) \to T\}$ is a pro-étale covering. If we can show that $\mathrm{Spec}(D_i)$ is isomorphic to an object, say T_i, of $Sch_{pro\text{-}étale}$, then $\{T_i \to T\}$ will be combinatorially equivalent to a covering of $Sch_{pro\text{-}étale}$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 61.12.7 and more precisely the application of Sets, Lemma 3.11.1 in the last step. To prove $\mathrm{Spec}(D_i)$ is isomorphic to an object of $Sch_{pro\text{-}étale}$, it suffices to prove that $|D_i| \leq Bound(\mathrm{size}(T))$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 61.12.7 and more precisely the application of Sets, Lemma 3.9.2 in step (3). Since $|A_i| \leq \mathrm{size}(U_i) \leq \mathrm{size}(T)$ by Sets, Lemmas 3.9.4 and 3.9.7 we get $|D_i| \leq \kappa^{2^{2^{2^\kappa}}}$ where $\kappa = \mathrm{size}(T)$ by Remark 61.11.4. Thus by our choice of the function *Bound* in Definition 61.12.7 we win. □

0990 **Lemma 61.13.4.** *Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ and if S is affine S_{app} have enough (affine) quasi-compact, weakly contractible objects, see Sites, Definition 7.40.2.*

Proof. Follows immediately from Lemma 61.13.3. □

0F4P **Lemma 61.13.5.** *Let S be a scheme. The pro-étale sites $Sch_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$ have the following property: for any object U there exists a covering $\{V \to U\}$ with V a weakly contractible object. If U is quasi-compact, then we may choose V affine and weakly contractible.*

Proof. Suppose that $V = \coprod_{j \in J} V_j$ is an object of $(Sch/S)_{pro\text{-}étale}$ which is the disjoint union of weakly contractible objects V_j. Since a disjoint union decomposition is a pro-étale covering we see that $\mathcal{F}(V) = \prod_{j \in J} \mathcal{F}(V_j)$ for any pro-étale sheaf $\mathcal{F}$. Let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. Since V_j is weakly contractible, the map $\mathcal{F}(V_j) \to \mathcal{G}(V_j)$ is surjective, see Sites, Definition 7.40.2. Thus $\mathcal{F}(V) \to \mathcal{G}(V)$ is surjective as a product of surjective maps of sets and we conclude that V is weakly contractible.

Choose a covering $\{U_i \to U\}_{i \in I}$ with U_i affine and weakly contractible as in Lemma 61.13.3. Take $V = \coprod_{i \in I} U_i$ (there is a set theoretic issue here which we will address below). Then $\{V \to U\}$ is the desired pro-étale covering by a weakly contractible

object (to check it is a covering use Lemma 61.12.2). If U is quasi-compact, then it follows immediately from Lemma 61.12.2 that we can choose a finite subset $I' \subset I$ such that $\{U_i \to U\}_{i \in I'}$ is still a covering and then $\{\coprod_{i \in I'} U_i \to U\}$ is the desired covering by an affine and weakly contractible object.

In this paragraph, which we urge the reader to skip, we address set theoretic problems. In order to know that the disjoint union lies in our partial universe, we need to bound the cardinality of the index set I. It is seen immediately from the construction of the covering $\{U_i \to U\}_{i \in I}$ in the proof of Lemma 61.13.3 that $|I| \leq \text{size}(U)$ where the size of a scheme is as defined in Sets, Section 3.9. Moreover, for each i we have $\text{size}(U_i) \leq \mathit{Bound}(\text{size}(U))$; this follows for the bound of the cardinality of $\Gamma(U_i, \mathcal{O}_{U_i})$ in the proof of Lemma 61.13.3 and Sets, Lemma 3.9.4. Thus $\text{size}(\coprod_{i \in I} U_i)) \leq \mathit{Bound}(\text{size}(U))$ by Sets, Lemma 3.9.5. Hence by construction of the big pro-étale site through Sets, Lemma 3.9.2 we see that $\coprod_{i \in I} U_i$ is isomorphic to an object of our site and the proof is complete. □

61.14. Weakly contractible hypercoverings

09A0 The results of Section 61.13 leads to the existence of hypercoverings made up out weakly contractible objects.

09A1 **Lemma 61.14.1.** *Let X be a scheme.*

(1) *For every object U of $X_{pro\text{-}étale}$ there exists a hypercovering K of U in $X_{pro\text{-}étale}$ such that each term K_n consists of a single weakly contractible object of $X_{pro\text{-}étale}$ covering U.*
(2) *For every quasi-compact and quasi-separated object U of $X_{pro\text{-}étale}$ there exists a hypercovering K of U in $X_{pro\text{-}étale}$ such that each term K_n consists of a single affine and weakly contractible object of $X_{pro\text{-}étale}$ covering U.*

Proof. Let $\mathcal{B} \subset \text{Ob}(X_{pro\text{-}étale})$ be the set of weakly contractible objects of $X_{pro\text{-}étale}$. Every object T of $X_{pro\text{-}étale}$ has a covering $\{T_i \to T\}_{i \in I}$ with I finite and $T_i \in \mathcal{B}$ by Lemma 61.13.5. By Hypercoverings, Lemma 25.12.6 we get a hypercovering K of U such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with I_n finite and $U_{n,i}$ weakly contractible. Then we can replace K by the hypercovering of U given by $\{U_n\}$ in degree n where $U_n = \coprod_{i \in I_n} U_{n,i}$ This is allowed by Hypercoverings, Remark 25.12.9.

Let $X_{qcqs,pro\text{-}étale} \subset X_{pro\text{-}étale}$ be the full subcategory consisting of quasi-compact and quasi-separated objects. A covering of $X_{qcqs,pro\text{-}étale}$ will be a finite pro-étale covering. Then $X_{qcqs,pro\text{-}étale}$ is a site, has fibre products, and the inclusion functor $X_{qcqs,pro\text{-}étale} \to X_{pro\text{-}étale}$ is continuous and commutes with fibre products. In particular, if K is a hypercovering of an object U in $X_{qcqs,pro\text{-}étale}$ then K is a hypercovering of U in $X_{pro\text{-}étale}$ by Hypercoverings, Lemma 25.12.5. Let $\mathcal{B} \subset \text{Ob}(X_{qcqs,pro\text{-}étale})$ be the set of affine and weakly contractible objects. By Lemma 61.13.3 and the fact that finite unions of affines are affine, for every object U of $X_{qcqs,pro\text{-}étale}$ there exists a covering $\{V \to U\}$ of $X_{qcqs,pro\text{-}étale}$ with $V \in \mathcal{B}$. By Hypercoverings, Lemma 25.12.6 we get a hypercovering K of U such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with I_n finite and $U_{n,i}$ affine and weakly contractible. Then we can replace K by the hypercovering of U given by $\{U_n\}$ in degree n where $U_n = \coprod_{i \in I_n} U_{n,i}$. This is allowed by Hypercoverings, Remark 25.12.9. □

In the following lemma we use the Čech complex $s(\mathcal{F}(K))$ associated to a hypercovering K in a site. See Hypercoverings, Section 25.5. If K is a hypercovering of

U and $K_n = \{U_n \to U\}$, then the Čech complex looks like this:

$$s(\mathcal{F}(K)) = (\mathcal{F}(U_0) \to \mathcal{F}(U_1) \to \mathcal{F}(U_2) \to \ldots)$$

where $s(\mathcal{F}(U_n))$ is placed in cohomological degree n.

09A2 **Lemma 61.14.2.** *Let X be a scheme. Let $E \in D^+(X_{pro\text{-}étale})$ be represented by a bounded below complex $\mathcal{E}^\bullet$ of abelian sheaves. Let K be a hypercovering of $U \in \mathrm{Ob}(X_{pro\text{-}étale})$ with $K_n = \{U_n \to U\}$ where U_n is a weakly contractible object of $X_{pro\text{-}étale}$. Then*

$$R\Gamma(U, E) = Tot(s(\mathcal{E}^\bullet(K)))$$

in $D(Ab)$.

Proof. If $\mathcal{E}$ is an abelian sheaf on $X_{pro\text{-}étale}$, then the spectral sequence of Hypercoverings, Lemma 25.5.3 implies that

$$R\Gamma(X_{pro\text{-}étale}, \mathcal{E}) = s(\mathcal{E}(K))$$

because the higher cohomology groups of any sheaf over U_n vanish, see Cohomology on Sites, Lemma 21.51.1.

If $\mathcal{E}^\bullet$ is bounded below, then we can choose an injective resolution $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ and consider the map of complexes

$$\mathrm{Tot}(s(\mathcal{E}^\bullet(K))) \longrightarrow \mathrm{Tot}(s(\mathcal{I}^\bullet(K)))$$

For every n the map $\mathcal{E}^\bullet(U_n) \to \mathcal{I}^\bullet(U_n)$ is a quasi-isomorphism because taking sections over U_n is exact. Hence the displayed map is a quasi-isomorphism by one of the spectral sequences of Homology, Lemma 12.25.3. Using the result of the first paragraph we see that for every p the complex $s(\mathcal{I}^p(K))$ is acyclic in degrees $n > 0$ and computes $\mathcal{I}^p(U)$ in degree 0. Thus the other spectral sequence of Homology, Lemma 12.25.3 shows $\mathrm{Tot}(s(\mathcal{I}^\bullet(K)))$ computes $R\Gamma(U, E) = \mathcal{I}^\bullet(U)$. □

09A3 **Lemma 61.14.3.** *Let X be a quasi-compact and quasi-separated scheme. The functor $R\Gamma(X, -) : D^+(X_{pro\text{-}étale}) \to D(Ab)$ commutes with direct sums and homotopy colimits.*

Proof. The statement means the following: Suppose we have a family of objects E_i of $D^+(X_{pro\text{-}étale})$ such that $\bigoplus E_i$ is an object of $D^+(X_{pro\text{-}étale})$. Then $R\Gamma(X, \bigoplus E_i) = \bigoplus R\Gamma(X, E_i)$. To see this choose a hypercovering K of X with $K_n = \{U_n \to X\}$ where U_n is an affine and weakly contractible scheme, see Lemma 61.14.1. Let N be an integer such that $H^p(E_i) = 0$ for $p < N$. Choose a complex of abelian sheaves $\mathcal{E}_i^\bullet$ representing E_i with $\mathcal{E}_i^p = 0$ for $p < N$. The termwise direct sum $\bigoplus \mathcal{E}_i^\bullet$ represents $\bigoplus E_i$ in $D(X_{pro\text{-}étale})$, see Injectives, Lemma 19.13.4. By Lemma 61.14.2 we have

$$R\Gamma(X, \bigoplus E_i) = \mathrm{Tot}(s((\bigoplus \mathcal{E}_i^\bullet)(K)))$$

and

$$R\Gamma(X, E_i) = \mathrm{Tot}(s(\mathcal{E}_i^\bullet(K)))$$

Since each U_n is quasi-compact we see that

$$\mathrm{Tot}(s((\bigoplus \mathcal{E}_i^\bullet)(K))) = \bigoplus \mathrm{Tot}(s(\mathcal{E}_i^\bullet(K)))$$

by Modules on Sites, Lemma 18.30.3. The statement on homotopy colimits is a formal consequence of the fact that $R\Gamma$ is an exact functor of triangulated categories and the fact (just proved) that it commutes with direct sums. □

09A4 **Remark 61.14.4.** Let X be a scheme. Because $X_{pro\text{-}étale}$ has enough weakly contractible objects for all K in $D(X_{pro\text{-}étale})$ we have $K = R\lim \tau_{\geq -n}K$ by Cohomology on Sites, Proposition 21.51.2. Since $R\Gamma$ commutes with $R\lim$ by Injectives, Lemma 19.13.6 we see that

$$R\Gamma(X, K) = R\lim R\Gamma(X, \tau_{\geq -n}K)$$

in $D(Ab)$. This will sometimes allow us to extend results from bounded below complexes to all complexes.

61.15. Compact generation

0994 In this section we prove that various derived categories associated to our pro-étale sites are compactly generated as defined in Derived Categories, Definition 13.37.5.

0F4Q **Lemma 61.15.1.** *Let S be a scheme. Let Λ be a ring.*

(1) *$D(S_{pro\text{-}étale})$ is compactly generated,*
(2) *$D(S_{pro\text{-}étale}, \Lambda)$ is compactly generated,*
(3) *$D(S_{pro\text{-}étale}, \mathcal{A})$ is compactly generated for any sheaf of rings $\mathcal{A}$ on $S_{pro\text{-}étale}$,*
(4) *$D((Sch/S)_{pro\text{-}étale})$ is compactly generated,*
(5) *$D((Sch/S)_{pro\text{-}étale}, \Lambda)$ is compactly generated, and*
(6) *$D((Sch/S)_{pro\text{-}étale}, \mathcal{A})$ is compactly generated for any sheaf of rings $\mathcal{A}$ on $(Sch/S)_{pro\text{-}étale}$,*

Proof. Proof of (3). Let U be an affine object of $S_{pro\text{-}étale}$ which is weakly contractible. Then $j_{U!}\mathcal{A}_U$ is a compact object of the derived category $D(S_{pro\text{-}étale}, \mathcal{A})$, see Cohomology on Sites, Lemma 21.52.6. Choose a set I and for each $i \in I$ an affine weakly contractible object U_i of $S_{pro\text{-}étale}$ such that every affine weakly contractible object of $S_{pro\text{-}étale}$ is isomorphic to one of the U_i. This is possible because $\mathrm{Ob}(S_{pro\text{-}étale})$ is a set. To finish the proof of (3) it suffices to show that $\bigoplus j_{U_i,!}\mathcal{A}_{U_i}$ is a generator of $D(S_{pro\text{-}étale}, \mathcal{A})$, see Derived Categories, Definition 13.36.3. To see this, let K be a nonzero object of $D(S_{pro\text{-}étale}, \mathcal{A})$. Then there exists an object T of our site $S_{pro\text{-}étale}$ and a nonzero element ξ of $H^n(K)(T)$. In other words, ξ is a nonzero section of the nth cohomology sheaf of K. We may assume K is represented by a complex $\mathcal{K}^\bullet$ of sheaves of $\mathcal{A}$-modules and ξ is the class of a section $s \in \mathcal{K}^n(T)$ with $\mathrm{d}(s) = 0$. Namely, ξ is locally represented as the class of a section (so you get the result after replacing T by a member of a covering of T). Next, we choose a covering $\{T_j \to T\}_{j\in J}$ as in Lemma 61.13.3. Since $H^n(K)$ is a sheaf, we see that for some j the restriction $\xi|_{T_j}$ remains nonzero. Thus $s|_{T_j}$ defines a nonzero map $j_{T_j,!}\mathcal{A}_{T_j} \to K$ in $D(S_{pro\text{-}étale}, \mathcal{A})$. Since $T_j \cong U_i$ for some $i \in I$ we conclude.

The exact same argument works for the big pro-étale site of S. □

61.16. Comparing topologies

0F62 This section is the analogue of Étale Cohomology, Section 59.39.

0F63 **Lemma 61.16.1.** *Let X be a scheme. Let $\mathcal{F}$ be a presheaf of sets on $X_{pro\text{-}étale}$ which sends finite disjoint unions to products. Then $\mathcal{F}^\#(W) = \mathcal{F}(W)$ if W is an affine weakly contractible object of $X_{pro\text{-}étale}$.*

Proof. Recall that $\mathcal{F}^\#$ is equal to $(\mathcal{F}^+)^+$, see Sites, Theorem 7.10.10, where $\mathcal{F}^+$ is the presheaf which sends an object U of $X_{pro\text{-}étale}$ to $\operatorname{colim} H^0(\mathcal{U}, \mathcal{F})$ where the colimit is over all pro-étale coverings $\mathcal{U}$ of U. Thus it suffices to prove that (a) $\mathcal{F}^+$ sends finite disjoint unions to products and (b) sends W to $\mathcal{F}(W)$. If $U = U_1 \amalg U_2$, then given a pro-étale covering $\mathcal{U} = \{f_j : V_j \to U\}$ of U we obtain pro-étale coverings $\mathcal{U}_i = \{f_j^{-1}(U_i) \to U_i\}$ and we clearly have

$$H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{U}_1, \mathcal{F}) \times H^0(\mathcal{U}_2, \mathcal{F})$$

because $\mathcal{F}$ sends finite disjoint unions to products (this includes the condition that $\mathcal{F}$ sends the empty scheme to the singleton). This proves (a). Finally, any pro-étale covering of W can be refined by a finite disjoint union decomposition $W = W_1 \amalg \ldots W_n$ by Lemma 61.13.2. Hence $\mathcal{F}^+(W) = \mathcal{F}(W)$ exactly because the value of $\mathcal{F}$ on W is the product of the values of $\mathcal{F}$ on the W_j. This proves (b). □

0F64 **Lemma 61.16.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf of sets on $Y_{pro\text{-}étale}$. If W is an affine weakly contractible object of $X_{pro\text{-}étale}$, then*

$$f_{small}^{-1}\mathcal{F}(W) = \operatorname{colim}_{W \to V} \mathcal{F}(V)$$

where the colimit is over morphisms $W \to V$ over Y with $V \in Y_{pro\text{-}étale}$.

Proof. Recall that $f_{small}^{-1}\mathcal{F}$ is the sheaf associated to the presheaf

$$u_p\mathcal{F} : U \mapsto \operatorname{colim}_{U \to V} \mathcal{F}(V)$$

on $X_{étale}$, see Sites, Sections 7.14 and 7.13; we've suppressed from the notation that the colimit is over the opposite of the category $\{U \to V, V \in Y_{pro\text{-}étale}\}$. By Lemma 61.16.1 it suffices to prove that $u_p\mathcal{F}$ sends finite disjoint unions to products. Suppose that $U = U_1 \amalg U_2$ is a disjoint union of open and closed subschemes. There is a functor

$$\{U_1 \to V_1\} \times \{U_2 \to V_2\} \longrightarrow \{U \to V\}, \quad (U_1 \to V_1, U_2 \to V_2) \longmapsto (U \to V_1 \amalg V_2)$$

which is initial (Categories, Definition 4.17.3). Hence the corresponding functor on opposite categories is cofinal and by Categories, Lemma 4.17.2 we see that $u_p\mathcal{F}$ on U is the colimit of the values $\mathcal{F}(V_1 \amalg V_2)$ over the product category. Since $\mathcal{F}$ is a sheaf it sends disjoint unions to products and we conclude $u_p\mathcal{F}$ does too. □

0F65 **Lemma 61.16.3.** *Let S be a scheme. Consider the morphism*

$$\pi_S : (Sch/S)_{pro\text{-}étale} \longrightarrow S_{pro\text{-}étale}$$

of Lemma 61.12.13. Let $\mathcal{F}$ be a sheaf on $S_{pro\text{-}étale}$. Then $\pi_S^{-1}\mathcal{F}$ is given by the rule

$$(\pi_S^{-1}\mathcal{F})(T) = \Gamma(T_{pro\text{-}étale}, f_{small}^{-1}\mathcal{F})$$

where $f : T \to S$. Moreover, $\pi_S^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to fpqc coverings.

Proof. Observe that we have a morphism $i_f : Sh(T_{pro\text{-}étale}) \to Sh((Sch/S)_{pro\text{-}étale})$ such that $\pi_S \circ i_f = f_{small}$ as morphisms $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$, see Lemma 61.12.12. Since pullback is transitive we see that $i_f^{-1}\pi_S^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on $T_{pro\text{-}étale}$ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i. We will prove this statement when T is affine and the covering is given by

a single surjective flat morphism $T' \to T$ of affines and omit the reduction of the general case to this case.

Let $g : T' \to T$ be a surjective flat morphism of affines and let $s' \in g_{small}^{-1}\mathcal{G}(T')$ be a section with $\mathrm{pr}_0^* s' = \mathrm{pr}_1^* s'$ on $T' \times_T T'$. Choose a surjective weakly étale morphism $W \to T'$ with W affine and weakly contractible, see Lemma 61.13.5. By Lemma 61.16.2 the restriction $s'|_W$ is an element of $\mathrm{colim}_{W \to U} \mathcal{G}(U)$. Choose $\phi : W \to U_0$ and $s_0 \in \mathcal{G}(U_0)$ corresponding to s'. Choose a surjective weakly étale morphism $V \to W \times_T W$ with V affine and weakly contractible. Denote $a, b : V \to W$ the induced morphisms. Since $a^*(s'|_W) = b^*(s'|_W)$ and since the category $\{V \to U, U \in T_{pro\text{-}étale}\}$ is cofiltered (this is clear but see Sites, Lemma 7.14.6 if in doubt), we see that the two morphisms $\phi \circ a, \phi \circ b : V \to U_0$ have to be equal. By the results in Descent, Section 35.13 (especially Descent, Lemma 35.13.7) it follows there is a unique morphism $T \to U_0$ such that ϕ is the composition of this morphism with the structure morphism $W \to T$ (small detail omitted). Then we can let s be the pullback of s_0 by this morphism. We omit the verification that s pulls back to s' on T'. □

61.17. Comparing big and small topoi

0F66 This section is the analogue of Étale Cohomology, Section 59.99. In the following we will often denote $\mathcal{F} \mapsto \mathcal{F}|_{S_{pro\text{-}étale}}$ the pullback functor i_S^{-1} corresponding to the morphism of topoi $i_S : Sh(S_{pro\text{-}étale}) \to Sh((Sch/S)_{pro\text{-}étale})$ of Lemma 61.12.13.

0F67 **Lemma 61.17.1.** *Let S be a scheme. Let T be an object of $(Sch/S)_{pro\text{-}étale}$.*

(1) *If $\mathcal{I}$ is injective in $Ab((Sch/S)_{pro\text{-}étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}$ is injective in $Ab(T_{pro\text{-}étale})$,*
   (b) *$\mathcal{I}|_{S_{pro\text{-}étale}}$ is injective in $Ab(S_{pro\text{-}étale})$,*

(2) *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab((Sch/S)_{pro\text{-}étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $Ab(T_{pro\text{-}étale})$,*
   (b) *$\mathcal{I}^\bullet|_{S_{pro\text{-}étale}}$ is a K-injective complex in $Ab(S_{pro\text{-}étale})$,*

Proof. Proof of (1)(a) and (2)(a): i_f^{-1} is a right adjoint of an exact functor $i_{f,!}$. Namely, recall that i_f corresponds to a cocontinuous functor $u : T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ which is continuous and commutes with fibre products and equalizers, see Lemma 61.12.12 and its proof. Hence we obtain $i_{f,!}$ by Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Then we conclude (1)(a) and (2)(a) hold by Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(b) and (2)(b) are special cases of (1)(a) and (2)(a) as $i_S = i_{\mathrm{id}_S}$. □

0F68 **Lemma 61.17.2.** *Let $f : T \to S$ be a morphism of schemes. For K in $D((Sch/T)_{pro\text{-}étale})$ we have*

$$(Rf_{big,*}K)|_{S_{pro\text{-}étale}} = Rf_{small,*}(K|_{T_{pro\text{-}étale}})$$

in $D(S_{pro\text{-}étale})$. More generally, let $S' \in \mathrm{Ob}((Sch/S)_{pro\text{-}étale})$ with structure morphism $g : S' \to S$. Consider the fibre product

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then for K in $D((Sch/T)_{pro\text{-}étale})$ we have

$$i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$$

in $D(S'_{pro\text{-}étale})$ and

$$g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$$

in $D((Sch/S')_{pro\text{-}étale})$.

Proof. The first equality follows from Lemma 61.17.1 and (61.12.16.1) on choosing a K-injective complex of abelian sheaves representing K. The second equality follows from Lemma 61.17.1 and Lemma 61.12.18 on choosing a K-injective complex of abelian sheaves representing K. The third equality follows similarly from Cohomology on Sites, Lemmas 21.7.1 and 21.20.1 and Lemma 61.12.18 on choosing a K-injective complex of abelian sheaves representing K. □

Let S be a scheme and let $\mathcal{H}$ be an abelian sheaf on $(Sch/S)_{pro\text{-}étale}$. Recall that $H^n_{pro\text{-}étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Sch/S)_{pro\text{-}étale}$.

0F69 **Lemma 61.17.3.** *Let $f : T \to S$ be a morphism of schemes. For K in $D(S_{pro\text{-}étale})$ we have*

$$H^n_{pro\text{-}étale}(S, \pi_S^{-1}K) = H^n(S_{pro\text{-}étale}, K)$$

and

$$H^n_{pro\text{-}étale}(T, \pi_S^{-1}K) = H^n(T_{pro\text{-}étale}, f_{small}^{-1}K).$$

For M in $D((Sch/S)_{pro\text{-}étale})$ we have

$$H^n_{pro\text{-}étale}(T, M) = H^n(T_{pro\text{-}étale}, i_f^{-1}M).$$

Proof. To prove the last equality represent M by a K-injective complex of abelian sheaves and apply Lemma 61.17.1 and work out the definitions. The second equality follows from this as $i_f^{-1} \circ \pi_S^{-1} = f_{small}^{-1}$. The first equality is a special case of the second one. □

0F6A **Lemma 61.17.4.** *Let S be a scheme. For $K \in D(S_{pro\text{-}étale})$ the map*

$$K \longrightarrow R\pi_{S,*}\pi_S^{-1}K$$

is an isomorphism.

Proof. This is true because both π_S^{-1} and $\pi_{S,*} = i_S^{-1}$ are exact functors and the composition $\pi_{S,*} \circ \pi_S^{-1}$ is the identity functor. □

61.18. Points of the pro-étale site

0991 We first apply Deligne's criterion to show that there are enough points.

0992 **Lemma 61.18.1.** *Let S be a scheme. The pro-étale sites $Sch_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have enough points.*

Proof. The big pro-étale topos of S is equivalent to the topos defined by $(Aff/S)_{pro\text{-}étale}$, see Lemma 61.12.11. The topos of sheaves on $S_{pro\text{-}étale}$ is equivalent to the topos associated to $S_{affine,pro\text{-}étale}$, see Lemma 61.12.20. The result for the sites $(Aff/S)_{pro\text{-}étale}$ and $S_{affine,pro\text{-}étale}$ follows immediately from Deligne's result Sites, Lemma 7.39.4. The case $Sch_{pro\text{-}étale}$ is handled because it is equal to $(Sch/\operatorname{Spec}(\mathbf{Z}))_{pro\text{-}étale}$. □

Let S be a scheme. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. We define a *pro-étale neighbourhood* of $\overline{s}$ to be a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{u}} & U \\ & \searrow^{\overline{s}} & \downarrow \\ & & S \end{array}$$

with $U \to S$ weakly étale.

0F6B **Lemma 61.18.2.** *Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. The category of pro-étale neighbourhoods of $\overline{s}$ is cofiltered.*

Proof. The proof is identitical to the proof of Étale Cohomology, Lemma 59.29.4 but using the corresponding facts about weakly étale morphisms proven in More on Morphisms, Lemmas 37.64.5, 37.64.6, and 37.64.13. □

0F6C **Lemma 61.18.3.** *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\mathcal{U} = \{\varphi_i : S_i \to S\}_{i \in I}$ be a pro-étale covering. Then there exist $i \in I$ and geometric point $\overline{s}_i$ of S_i mapping to $\overline{s}$.*

Proof. Immediate from the fact that $\coprod \varphi_i$ is surjective and that residue field extensions induced by weakly étale morphisms are separable algebraic (see for example More on Morphisms, Lemma 37.64.11. □

Let S be a scheme and let $\overline{s}$ be a geometric point of S. For $\mathcal{F}$ in $Sh(S_{pro\text{-}étale})$ define the *stalk of $\mathcal{F}$ at $\overline{s}$* by the formula

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit is over all pro-étale neighbourhoods $(U, \overline{u})$ of $\overline{s}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$. It follows from the two lemmas above that the functor

$$S_{pro\text{-}étale}Sets, \quad U \longmapsto \{\overline{u} \text{ geometric point of } U \text{ mapping to } \overline{s}\}$$

defines a point of the site $S_{pro\text{-}étale}$, see Sites, Definition 7.32.2 and Lemma 7.33.1. Hence the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defines a point of the topos $Sh(S_{pro\text{-}étale})$, see Sites, Definition 7.32.1 and Lemma 7.32.7. In particular this functor is exact and commutes with arbitrary colimits. In fact, this functor has another description.

0993 **Lemma 61.18.4.** *In the situation above the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an object of $X_{pro\text{-}étale}$ and there is a canonical isomorphism*

$$\mathcal{F}(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})) = \mathcal{F}_{\overline{s}}$$

functorial in $\mathcal{F}$.

Proof. The first statement is clear from the construction of the strict henselization as a filtered colimit of étale algebras over S, or by the characterization of weakly étale morphisms of More on Morphisms, Lemma 37.64.11. The second statement follows as by Olivier's theorem (More on Algebra, Theorem 15.105.24) the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an initial object of the category of pro-étale neighbourhoods of $\overline{s}$. □

Contrary to the situation with the étale topos of S it is not true that every point of $Sh(S_{pro\text{-}étale})$ is of this form, and it is not true that the collection of points associated to geometric points is conservative. Namely, suppose that $S = \mathrm{Spec}(k)$

where k is an algebraically closed field. Let A be a nonzero abelian group. Consider the sheaf $\mathcal{F}$ on $S_{pro\text{-}étale}$ defined by the

$$\mathcal{F}(U) = \frac{\{\text{functions } U \to A\}}{\{\text{locally constant functions}\}}$$

for U affine and by sheafification in general, see Example 61.19.12. Then $\mathcal{F}(U) = 0$ if $U = S = \mathrm{Spec}(k)$ but in general $\mathcal{F}$ is not zero. Namely, $S_{pro\text{-}étale}$ contains affine objects with infinitely many points. For example, let $E = \lim E_n$ be an inverse limit of finite sets with surjective transition maps, e.g., $E = \mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$. The scheme $U = \mathrm{Spec}(\mathrm{colim}\, \mathrm{Map}(E_n, k))$ is an object of $S_{pro\text{-}étale}$ because $\mathrm{colim}\, \mathrm{Map}(E_n, k)$ is weakly étale (even ind-Zariski) over k. Thus $\mathcal{F}(U)$ is nonzero as there exist maps $E \to A$ which aren't locally constant. Thus $\mathcal{F}$ is a nonzero abelian sheaf whose stalk at the unique geometric point of S is zero. Since we know that $S_{pro\text{-}étale}$ has enough points, we conclude there must be a point of the pro-étale site which does not come from the construction explained above.

The replacement for arguments using points, is to use affine weakly contractible objects. First, there are enough affine weakly contractible objects by Lemma 61.13.4. Second, if $W \in \mathrm{Ob}(S_{pro\text{-}étale})$ is affine weakly contractible, then the functor

$$Sh(S_{pro\text{-}étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is an exact functor $Sh(S_{pro\text{-}étale}) \to Sets$ which commutes with all limits. The functor

$$Ab(S_{pro\text{-}étale}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is exact and commutes with direct sums (as W is quasi-compact, see Sites, Lemma 7.17.7), hence commutes with all limits and colimits. Moreover, we can check exactness of a complex of abelian sheaves by evaluation at these affine weakly contractible objects of $S_{pro\text{-}étale}$, see Cohomology on Sites, Proposition 21.51.2.

A final remark is that the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ for W affine weakly contractible in general isn't a stalk functor of a point of $S_{pro\text{-}étale}$ because it doesn't preserve coproducts of sheaves of sets if W is disconnected. And in fact, W is disconnected as soon as W has more than 1 closed point, i.e., when W is not the spectrum of a strictly henselian local ring (which is the special case discussed above).

61.19. Comparison with the étale site

099R Let X be a scheme. With suitable choices of sites[4] the functor $u : X_{étale} \to X_{pro\text{-}étale}$ sending U/X to U/X defines a morphism of sites

$$\epsilon : X_{pro\text{-}étale} \longrightarrow X_{étale}$$

This follows from Sites, Proposition 7.14.7.

0GLZ **Lemma 61.19.1.** *With notation as above. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. The rule*

$$X_{pro\text{-}étale} \longrightarrow Sets, \quad (f : Y \to X) \longmapsto \Gamma(Y_{étale}, f_{étale}^{-1}\mathcal{F})$$

[4]Choose a big pro-étale site $Sch_{pro\text{-}étale}$ containing X as in Definition 61.12.7. Then let $Sch_{étale}$ be the site having the same underlying category as $Sch_{pro\text{-}étale}$ but whose coverings are exactly those pro-étale coverings which are also étale coverings. With these choices let $X_{étale}$ and $X_{pro\text{-}étale}$ be the subcategories defined in Definition 61.12.8 and Topologies, Definition 34.4.8. Compare with Topologies, Remark 34.11.1.

is a sheaf and is equal to $\epsilon^{-1}\mathcal{F}$. Here $f_{étale} : Y_{étale} \to X_{étale}$ is the morphism of small étale sites constructed in Étale Cohomology, Section 59.34.

Proof. By Lemma 61.12.2 any pro-étale covering is an fpqc covering. Hence the formula defines a sheaf on $X_{pro\text{-}étale}$ by Étale Cohomology, Lemma 59.39.2. Let $a : Sh(X_{étale}) \to Sh(X_{pro\text{-}étale})$ be the functor sending $\mathcal{F}$ to the sheaf given by the formula in the lemma. To show that $a = \epsilon^{-1}$ it suffices to show that a is a left adjoint to ϵ_*.

Let $\mathcal{G}$ be an object of $Sh(X_{pro\text{-}étale})$. Recall that $\epsilon_*\mathcal{G}$ is simply given by the restriction of $\mathcal{G}$ to the full subcategory $X_{étale}$. Let $f : Y \to X$ be an object of $X_{pro\text{-}étale}$. We view $Y_{étale}$ as a subcategory of $X_{pro\text{-}étale}$. The restriction maps of the sheaf $\mathcal{G}$ define a map

$$\epsilon_*\mathcal{G} = \mathcal{G}|_{X_{étale}} \longrightarrow f_{étale,*}(\mathcal{G}|_{Y_{étale}})$$

Namely, for U in $X_{étale}$ the value of $f_{étale,*}(\mathcal{G}|_{Y_{étale}})$ on U is $\mathcal{G}(Y \times_X U)$ and there is a restriction map $\mathcal{G}(U) \to \mathcal{G}(Y \times_X U)$. By adjunction this determines a map

$$f_{étale}^{-1}(\epsilon_*\mathcal{G}) \to \mathcal{G}|_{Y_{étale}}$$

Putting these together for all $f : Y \to X$ in $X_{pro\text{-}étale}$ we obtain a canonical map $a(\epsilon_*\mathcal{G}) \to \mathcal{G}$.

Let $\mathcal{F}$ be an object of $Sh(X_{étale})$. It is immediately clear that $\mathcal{F} = \epsilon_* a(\mathcal{F})$.

We claim the maps $\mathcal{F} \to \epsilon_* a(\mathcal{F})$ and $a(\epsilon_*\mathcal{G}) \to \mathcal{G}$ are the unit and counit of the adjunction (see Categories, Section 4.24). To see this it suffices to show that the corresponding maps

$$\mathrm{Mor}_{Sh(X_{pro\text{-}étale})}(a(\mathcal{F}), \mathcal{G}) \to \mathrm{Mor}_{Sh(X_{étale})}(\mathcal{F}, \epsilon^{-1}\mathcal{G})$$

and

$$\mathrm{Mor}_{Sh(X_{étale})}(\mathcal{F}, \epsilon^{-1}\mathcal{G}) \to \mathrm{Mor}_{Sh(X_{pro\text{-}étale})}(a(\mathcal{F}), \mathcal{G})$$

are mutually inverse. We omit the detailed verification. □

099T **Lemma 61.19.2.** *Let X be a scheme. For every sheaf $\mathcal{F}$ on $X_{étale}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^{-1}\mathcal{F}$ is an isomorphism, i.e., $\epsilon^{-1}\mathcal{F}(U) = \mathcal{F}(U)$ for U in $X_{étale}$.*

Proof. Follows immediately from the description of ϵ^{-1} in Lemma 61.19.1. □

099S **Lemma 61.19.3.** *Let X be a scheme. Let $Y = \lim Y_i$ be the limit of a directed inverse system of quasi-compact and quasi-separated objects of $X_{pro\text{-}étale}$ with affine transition morphisms. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have*

$$\epsilon^{-1}\mathcal{F}(Y) = \mathrm{colim}\, \epsilon^{-1}\mathcal{F}(Y_i)$$

Moreover, if Y_i is in $X_{étale}$ we have $\epsilon^{-1}\mathcal{F}(Y) = \mathrm{colim}\, \mathcal{F}(Y_i)$.

Proof. By the description of $\epsilon^{-1}\mathcal{F}$ in Lemma 61.19.1, the displayed formula is a special case of Étale Cohomology, Theorem 59.51.3. (When X, Y, and the Y_i are all affine, see the easier to parse Étale Cohomology, Lemma 59.51.5.) The final statement follows immediately from this and Lemma 61.19.2. □

099U **Lemma 61.19.4.** *Let X be an affine scheme. For injective abelian sheaf $\mathcal{I}$ on $X_{étale}$ we have $H^p(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{I}) = 0$ for $p > 0$.*

Proof. We are going to use Cohomology on Sites, Lemma 21.10.9 to prove this. Let $\mathcal{B} \subset \mathrm{Ob}(X_{pro\text{-}étale})$ be the set of affine objects U of $X_{pro\text{-}étale}$ such that $\mathcal{O}(X) \to \mathcal{O}(U)$ is ind-étale. Let Cov be the set of pro-étale coverings $\{U_i \to U\}_{i=1,\ldots,n}$ with $U \in \mathcal{B}$ such that $\mathcal{O}(U) \to \mathcal{O}(U_i)$ is ind-étale for $i = 1, \ldots, n$. Properties (1) and (2) of Cohomology on Sites, Lemma 21.10.9 hold for $\mathcal{B}$ and Cov by Lemmas 61.7.3, 61.7.2, and 61.12.5 and Proposition 61.9.1.

To check condition (3) suppose that $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ is an element of Cov. We have to show that the higher Cech cohomology groups of $\epsilon^{-1}\mathcal{I}$ with respect to $\mathcal{U}$ are zero. First we write $U_i = \lim_{a \in A_i} U_{i,a}$ as a directed inverse limit with $U_{i,a} \to U$ étale and $U_{i,a}$ affine. We think of $A_1 \times \ldots \times A_n$ as a direct set with ordering $(a_1, \ldots, a_n) \geq (a'_1, \ldots, a'_n)$ if and only if $a_i \geq a'_i$ for $i = 1, \ldots, n$. Observe that $\mathcal{U}_{(a_1,\ldots,a_n)} = \{U_{i,a_i} \to U\}_{i=1,\ldots,n}$ is an étale covering for all $a_1, \ldots, a_n \in A_1 \times \ldots \times A_n$. Observe that

$$U_{i_0} \times_U U_{i_1} \times_U \ldots \times_U U_{i_p} = \lim_{(a_1,\ldots,a_n) \in A_1 \times \ldots \times A_n} U_{i_0,a_{i_0}} \times_U U_{i_1,a_{i_1}} \times_U \ldots \times_U U_{i_p,a_{i_p}}$$

for all $i_0, \ldots, i_p \in \{1, \ldots, n\}$ because limits commute with fibred products. Hence by Lemma 61.19.3 and exactness of filtered colimits we have

$$\check{H}^p(\mathcal{U}, \epsilon^{-1}\mathcal{I}) = \mathrm{colim}\, \check{H}^p(\mathcal{U}_{(a_1,\ldots,a_n)}, \epsilon^{-1}\mathcal{I})$$

Thus it suffices to prove the vanishing for étale coverings of U!

Let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be an étale covering with U_i affine. Write $U = \lim_{b \in B} U_b$ as a directed inverse limit with U_b affine and $U_b \to X$ étale. By Limits, Lemmas 32.10.1, 32.4.13, and 32.8.10 we can choose a $b_0 \in B$ such that for $i = 1, \ldots, n$ there is an étale morphism $U_{i,b_0} \to U_{b_0}$ of affines such that $U_i = U \times_{U_{b_0}} U_{i,b_0}$. Set $U_{i,b} = U_b \times_{U_{b_0}} U_{i,b_0}$ for $b \geq b_0$. For b large enough the family $\mathcal{U}_b = \{U_{i,b} \to U_b\}_{i=1,\ldots,n}$ is an étale covering, see Limits, Lemma 32.8.15. Exactly as before we find that

$$\check{H}^p(\mathcal{U}, \epsilon^{-1}\mathcal{I}) = \mathrm{colim}\, \check{H}^p(\mathcal{U}_b, \epsilon^{-1}\mathcal{I}) = \mathrm{colim}\, \check{H}^p(\mathcal{U}_b, \mathcal{I})$$

the final equality by Lemma 61.19.2. Since each of the Čech complexes on the right hand side is acyclic in positive degrees (Cohomology on Sites, Lemma 21.10.2) it follows that the one on the left is too. This proves condition (3) of Cohomology on Sites, Lemma 21.10.9. Since $X \in \mathcal{B}$ the lemma follows. □

099V **Lemma 61.19.5.** *Let X be a scheme.*

(1) *For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $R\epsilon_*(\epsilon^{-1}\mathcal{F}) = \mathcal{F}$.*
(2) *For $K \in D^+(X_{étale})$ the map $K \to R\epsilon_*\epsilon^{-1}K$ is an isomorphism.*

Proof. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Recall that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I})$ is the sheaf associated to $U \mapsto H^q(U_{pro\text{-}étale}, \epsilon^{-1}\mathcal{I})$, see Cohomology on Sites, Lemma 21.7.4. By Lemma 61.19.4 we see that this is zero for $q > 0$ and U affine and étale over X. Since every object of $X_{étale}$ has a covering by affine objects, it follows that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I}) = 0$ for $q > 0$.

Let $K \in D^+(X_{étale})$. Choose a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves on $X_{étale}$ representing K. Then $\epsilon^{-1}K$ is represented by $\epsilon^{-1}\mathcal{I}^\bullet$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we see that $R\epsilon_*\epsilon^{-1}K$ is represented by $\epsilon_*\epsilon^{-1}\mathcal{I}^\bullet$. By Lemma 61.19.2 we conclude that $R\epsilon_*\epsilon^{-1}\mathcal{I}^\bullet = \mathcal{I}^\bullet$ and the proof of (2) is complete. Part (1) is a special case of (2). □

099W **Lemma 61.19.6.** *Let X be a scheme.*

(1) *For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have*
$$H^i(X_{étale}, \mathcal{F}) = H^i(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{F})$$
for all i.
(2) *For $K \in D^+(X_{étale})$ we have*
$$R\Gamma(X_{étale}, K) = R\Gamma(X_{pro\text{-}étale}, \epsilon^{-1}K)$$

Proof. Immediate consequence of Lemma 61.19.5 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). □

099X **Lemma 61.19.7.** *Let X be a scheme. Let $\mathcal{G}$ be a sheaf of (possibly noncommutative) groups on $X_{étale}$. We have*
$$H^1(X_{étale}, \mathcal{G}) = H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$$
where H^1 is defined as the set of isomorphism classes of torsors (see Cohomology on Sites, Section 21.4).

Proof. Since the functor ϵ^{-1} is fully faithful by Lemma 61.19.2 it is clear that the map $H^1(X_{étale}, \mathcal{G}) \to H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$ is injective. To show surjectivity it suffices to show that any $\epsilon^{-1}\mathcal{G}$-torsor $\mathcal{F}$ is étale locally trivial. To do this we may assume that X is affine. Thus we reduce to proving surjectivity for X affine.

Choose a covering $\{U \to X\}$ with (a) U affine, (b) $\mathcal{O}(X) \to \mathcal{O}(U)$ ind-étale, and (c) $\mathcal{F}(U)$ nonempty. We can do this by Proposition 61.9.1 and the fact that standard pro-étale coverings of X are cofinal among all pro-étale coverings of X (Lemma 61.12.5). Write $U = \lim U_i$ as a limit of affine schemes étale over X. Pick $s \in \mathcal{F}(U)$. Let $g \in \epsilon^{-1}\mathcal{G}(U \times_X U)$ be the unique section such that $g \cdot \mathrm{pr}_1^* s = \mathrm{pr}_2^* s$ in $\mathcal{F}(U \times_X U)$. Then g satisfies the cocycle condition
$$\mathrm{pr}_{12}^* g \cdot \mathrm{pr}_{23}^* g = \mathrm{pr}_{13}^* g$$
in $\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U)$. By Lemma 61.19.3 we have
$$\epsilon^{-1}\mathcal{G}(U \times_X U) = \operatorname{colim} \mathcal{G}(U_i \times_X U_i)$$
and
$$\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U) = \operatorname{colim} \mathcal{G}(U_i \times_X U_i \times_X U_i)$$
hence we can find an i and an element $g_i \in \mathcal{G}(U_i \times_X U_i)$ mapping to g satisfying the cocycle condition. The cocycle g_i then defines a torsor for $\mathcal{G}$ on $X_{étale}$ whose pullback is isomorphic to $\mathcal{F}$ by construction. Some details omitted (namely, the relationship between torsors and 1-cocycles which should be added to the chapter on cohomology on sites). □

09B1 **Lemma 61.19.8.** *Let X be a scheme. Let Λ be a ring.*
(1) *The essential image of the fully faithful functor $\epsilon^{-1} : Mod(X_{étale}, \Lambda) \to Mod(X_{pro\text{-}étale}, \Lambda)$ is a weak Serre subcategory $\mathcal{C}$.*
(2) *The functor ϵ^{-1} defines an equivalence of categories of $D^+(X_{étale}, \Lambda)$ with $D^+_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$ with quasi-inverse given by $R\epsilon_*$.*

Proof. To prove (1) we will prove conditions (1) – (4) of Homology, Lemma 12.10.3. Since ϵ^{-1} is fully faithful (Lemma 61.19.2) and exact, everything is clear except for condition (4). However, if
$$0 \to \epsilon^{-1}\mathcal{F}_1 \to \mathcal{G} \to \epsilon^{-1}\mathcal{F}_2 \to 0$$

is a short exact sequence of sheaves of Λ-modules on $X_{pro\text{-}étale}$, then we get

$$0 \to \epsilon_*\epsilon^{-1}\mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \epsilon_*\epsilon^{-1}\mathcal{F}_2 \to R^1\epsilon_*\epsilon^{-1}\mathcal{F}_1$$

which by Lemma 61.19.5 is the same as a short exact sequence

$$0 \to \mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \mathcal{F}_2 \to 0$$

Pulling pack we find that $\mathcal{G} = \epsilon^{-1}\epsilon_*\mathcal{G}$. This proves (1).

Part (2) follows from part (1) and Cohomology on Sites, Lemma 21.28.5. □

Let Λ be a ring. In Modules on Sites, Section 18.43 we have defined the notion of a locally constant sheaf of Λ-modules on a site. If M is a Λ-module, then $\underline{M}$ is of finite presentation as a sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module, see Modules on Sites, Lemma 18.42.5.

099Y **Lemma 61.19.9.** *Let X be a scheme. Let Λ be a ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{array}{c}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{étale}\\ \text{of finite presentation}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{pro\text{-}étale}\\ \text{of finite presentation}\end{array}\right\}$$

Proof. Let $\mathcal{F}$ be a locally constant sheaf of Λ-modules on $X_{pro\text{-}étale}$ of finite presentation. Choose a pro-étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As pro-étale coverings are fpqc coverings and by Descent, Lemma 35.13.6 we see that $U_M = \bigcup_{i \in I_M} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. We may replace X by U_M for some M and assume that $M_i = M$ for all i.

Consider the sheaf $\mathcal{I} = \mathit{Isom}(\underline{M}, \mathcal{F})$. This sheaf is a torsor for $\mathcal{G} = \mathit{Isom}(\underline{M}, \underline{M})$. By Modules on Sites, Lemma 18.43.4 we have $\mathcal{G} = \underline{G}$ where $G = \mathit{Isom}_\Lambda(M, M)$. Since torsors for the étale topology and the pro-étale topology agree by Lemma 61.19.7 it follows that $\mathcal{I}$ has sections étale locally on X. Thus $\mathcal{F}$ is étale locally a constant sheaf which is what we had to show. □

099Z **Lemma 61.19.10.** *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_{flc}(X_{étale}, \Lambda)$, resp. $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type. Then*

$$\epsilon^{-1} : D^+_{flc}(X_{étale}, \Lambda) \longrightarrow D^+_{flc}(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_{flc}(X_{étale}, \Lambda)$ and $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Modules on Sites, Lemma 18.43.5 and Derived Categories, Section 13.17. The statement of the lemma follows by combining Lemmas 61.19.8 and 61.19.9. □

09B2 **Lemma 61.19.11.** *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$. If K_1 is*

1. *in the essential image of $\epsilon^{-1} : D(X_{étale}, \Lambda/I) \to D(X_{pro\text{-}étale}, \Lambda/I)$, and*
2. *has tor amplitude in $[a, \infty)$ for some $a \in \mathbf{Z}$,*

then (1) and (2) hold for K_n as an object of $D(X_{pro\text{-}étale}, \Lambda/I^n)$.

Proof. Assertion (2) for K_n follows from the more general Cohomology on Sites, Lemma 21.46.9. Assertion (1) for K_n follows by induction on n from the distinguished triangles

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}}[1]$$

and the isomorphism

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} = K_1 \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

and the fact proven in Lemma 61.19.8 that the essential image of ϵ^{-1} is a triangulated subcategory of $D^+(X_{pro\text{-}étale}, \Lambda/I^n)$. □

0F6D **Example 61.19.12.** Let X be a scheme. Let A be an abelian group. Denote $fun(-, A)$ the sheaf on $X_{pro\text{-}étale}$ which maps U to the set of all maps $U \to A$ (of sets of points). Consider the sequence of sheaves

$$0 \to \underline{A} \to fun(-, A) \to \mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Since the constant sheaf is the pullback from the final topos we see that $\underline{A} = \epsilon^{-1}\underline{A}$. However, if A has more than one element, then neither $fun(-, A)$ nor $\mathcal{F}$ are pulled back from the étale site of X. To work out the values of $\mathcal{F}$ in some cases, assume that all points of X are closed with separably closed residue fields and U is affine. Then all points of U are closed with separably closed residue fields and we have

$$H^1_{pro\text{-}étale}(U, \underline{A}) = H^1_{étale}(U, \underline{A}) = 0$$

by Lemma 61.19.6 and Étale Cohomology, Lemma 59.80.3. Hence in this case we have

$$\mathcal{F}(U) = fun(U, A)/\underline{A}(U)$$

61.20. Derived completion in the constant Noetherian case

099L We continue the discussion started in Algebraic and Formal Geometry, Section 52.6; we assume the reader has read at least some of that section.

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Recall from Modules on Sites, Lemma 18.42.4 that

$$\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$$

is a flat $\underline{\Lambda}$-algebra and that the map $\underline{\Lambda} \to \underline{\Lambda}^\wedge$ identifies quotients by I. Hence Algebraic and Formal Geometry, Lemma 52.6.17 tells us that

$$D_{comp}(\mathcal{C}, \Lambda) = D_{comp}(\mathcal{C}, \underline{\Lambda}^\wedge)$$

In particular the cohomology sheaves $H^i(K)$ of an object K of $D_{comp}(\mathcal{C}, \Lambda)$ are sheaves of $\underline{\Lambda}^\wedge$-modules. For notational convenience we often work with $D_{comp}(\mathcal{C}, \Lambda)$.

099M **Lemma 61.20.1.** *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. The left adjoint to the inclusion functor $D_{comp}(\mathcal{C}, \Lambda) \to D(\mathcal{C}, \Lambda)$ of Algebraic and Formal Geometry, Proposition 52.6.12 sends K to*

$$K^\wedge = R\lim(K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n})$$

In particular, K is derived complete if and only if $K = R\lim(K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n})$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. By Algebraic and Formal Geometry, Lemma 52.6.9 we have

$$K^\wedge = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{K_n})$$

where $K_n = K(\Lambda, f_1^n, \ldots, f_r^n)$. In More on Algebra, Lemma 15.95.1 we have seen that the pro-systems $\{K_n\}$ and $\{\Lambda/I^n\}$ of $D(\Lambda)$ are isomorphic. Thus the lemma follows. □

099N **Lemma 61.20.2.** *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then*

(1) *Rf_* sends $D_{comp}(\mathcal{D}, \Lambda)$ into $D_{comp}(\mathcal{C}, \Lambda)$,*
(2) *the map $Rf_* : D_{comp}(\mathcal{D}, \Lambda) \to D_{comp}(\mathcal{C}, \Lambda)$ has a left adjoint $Lf^*_{comp} : D_{comp}(\mathcal{C}, \Lambda) \to D_{comp}(\mathcal{D}, \Lambda)$ which is Lf^* followed by derived completion,*
(3) *Rf_* commutes with derived completion,*
(4) *for K in $D_{comp}(\mathcal{D}, \Lambda)$ we have $Rf_*K = R\lim Rf_*(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$.*
(5) *for M in $D_{comp}(\mathcal{C}, \Lambda)$ we have $Lf^*_{comp}M = R\lim Lf^*(M \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$.*

Proof. We have seen (1) and (2) in Algebraic and Formal Geometry, Lemma 52.6.18. Part (3) follows from Algebraic and Formal Geometry, Lemma 52.6.19. For (4) let K be derived complete. Then

$$Rf_*K = Rf_*(R\lim K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}) = R\lim Rf_*(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$$

the first equality by Lemma 61.20.1 and the second because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3). This proves (4). To prove (5), by Lemma 61.20.1 we have

$$Lf^*_{comp}M = R\lim(Lf^*M \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$$

Since Lf^* commutes with derived tensor product by Cohomology on Sites, Lemma 21.18.4 and since $Lf^*\underline{\Lambda/I^n} = \underline{\Lambda/I^n}$ we get (5). □

61.21. Derived completion and weakly contractible objects

099P We continue the discussion in Section 61.20. In this section we will see how the existence of weakly contractible objects simplifies the study of derived complete modules.

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Although the general theory concerning $D_{comp}(\mathcal{C}, \Lambda)$ is quite satisfactory it is hard to explicitly give examples of derived complete complexes. We know that

(1) every object M of $D(\mathcal{C}, \Lambda/I^n)$ restricts to a derived complete object of $D(\mathcal{C}, \Lambda)$, and
(2) for every $K \in D(\mathcal{C}, \Lambda)$ the derived completion $K^\wedge = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$ is derived complete.

The first type of objects are trivially complete and perhaps not interesting. The problem with (2) is that derived completion in general is somewhat mysterious, even in case $K = \underline{\Lambda}$. Namely, by definition of homotopy limits there is a distinguished triangle

$$R\lim(\underline{\Lambda/I^n}) \to \prod \underline{\Lambda/I^n} \to \prod \underline{\Lambda/I^n} \to R\lim(\underline{\Lambda/I^n})[1]$$

in $D(\mathcal{C}, \Lambda)$ where the products are in $D(\mathcal{C}, \Lambda)$. These are computed by taking products of injective resolutions (Injectives, Lemma 19.13.4), so we see that the sheaf $H^p(\prod \underline{\Lambda/I^n})$ is the sheafification of the presheaf

$$U \longmapsto \prod H^p(U, \Lambda/I^n).$$

As an explicit example, if $X = \mathrm{Spec}(\mathbf{C}[t, t^{-1}])$, $\mathcal{C} = X_{étale}$, $\Lambda = \mathbf{Z}$, $I = (2)$, and $p = 1$, then we get the sheafification of the presheaf

$$U \mapsto \prod H^1(U_{étale}, \mathbf{Z}/2^n\mathbf{Z})$$

for U étale over X. Note that $H^1(X_{étale}, \mathbf{Z}/m\mathbf{Z})$ is cyclic of order m with generator α_m given by the finite étale $\mathbf{Z}/m\mathbf{Z}$-covering given by the equation $t = s^m$ (see Étale Cohomology, Section 59.6). Then the section

$$\alpha = (\alpha_{2^n}) \in \prod H^1(X_{étale}, \mathbf{Z}/2^n\mathbf{Z})$$

of the presheaf above does not restrict to zero on any nonempty étale scheme over X, whence the sheaf associated to the presheaf is not zero.

However, on the pro-étale site this phenomenon does not occur. The reason is that we have enough (quasi-compact) weakly contractible objects. In the following proposition we collect some results about derived completion in the Noetherian constant case for sites having enough weakly contractible objects (see Sites, Definition 7.40.2).

099Q **Proposition 61.21.1.** *Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has enough weakly contractible objects. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal.*

1. *The category of derived complete sheaves Λ-modules is a weak Serre subcategory of Mod$(\mathcal{C}, \Lambda)$.*
2. *A sheaf $\mathcal{F}$ of Λ-modules satisfies $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ if and only if $\mathcal{F}$ is derived complete and $\bigcap I^n\mathcal{F} = 0$.*
3. *The sheaf $\underline{\Lambda}^\wedge$ is derived complete.*
4. *If $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ is an inverse system of derived complete sheaves of Λ-modules, then $\lim \mathcal{F}_n$ is derived complete.*
5. *An object $K \in D(\mathcal{C}, \Lambda)$ is derived complete if and only if each cohomology sheaf $H^p(K)$ is derived complete.*
6. *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded above if and only if $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ is bounded above.*
7. *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded if $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has finite tor dimension.*

Proof. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. We will use the results of Cohomology on Sites, Lemma 21.51.1 and Proposition 21.51.2 without further mention.

Recall that $R\lim$ commutes with $R\Gamma(U, -)$, see Injectives, Lemma 19.13.6. Let $f \in I$. Recall that $T(K, f)$ is the homotopy limit of the system

$$\ldots \xrightarrow{f} K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{C}, \Lambda)$. Thus

$$R\Gamma(U, T(K, f)) = T(R\Gamma(U, K), f).$$

Since we can test isomorphisms of maps between objects of $D(\mathcal{C}, \Lambda)$ by evaluating at $U \in \mathcal{B}$ we conclude an object K of $D(\mathcal{C}, \Lambda)$ is derived complete if and only if for every $U \in \mathcal{B}$ the object $R\Gamma(U, K)$ is derived complete as an object of $D(\Lambda)$.

The remark above implies that items (1), (5) follow from the corresponding results for modules over rings, see More on Algebra, Lemmas 15.92.1 and 15.92.6. In the same way (2) can be deduced from More on Algebra, Proposition 15.92.5 as $(I^n\mathcal{F})(U) = I^n \cdot \mathcal{F}(U)$ for $U \in \mathcal{B}$ (by exactness of evaluating at U).

Proof of (4). The homotopy limit $R\lim \mathcal{F}_n$ is in $D_{comp}(X, \Lambda)$ (see discussion following Algebraic and Formal Geometry, Definition 52.6.4). By part (5) just proved we conclude that $\lim \mathcal{F}_n = H^0(R\lim \mathcal{F}_n)$ is derived complete. Part (3) is a special case of (4).

Proof of (6) and (7). Follows from Lemma 61.20.1 and Cohomology on Sites, Lemma 21.46.9 and the computation of homotopy limits in Cohomology on Sites, Proposition 21.51.2. □

61.22. Cohomology of a point

09B3 Let Λ be a Noetherian ring complete with respect to an ideal $I \subset \Lambda$. Let k be a field. In this section we "compute"

$$H^i(\mathrm{Spec}(k)_{pro\text{-}étale}, \underline{\Lambda}^\wedge)$$

where $\underline{\Lambda}^\wedge = \lim_m \underline{\Lambda/I^m}$ as before. Let k^{sep} be a separable algebraic closure of k. Then

$$\mathcal{U} = \{\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)\}$$

is a pro-étale covering of $\mathrm{Spec}(k)$. We will use the Čech to cohomology spectral sequence with respect to this covering. Set $U_0 = \mathrm{Spec}(k^{sep})$ and

$$\begin{aligned} U_n &= \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \\ &= \mathrm{Spec}(k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep}) \end{aligned}$$

($n+1$ factors). Note that the underlying topological space $|U_0|$ of U_0 is a singleton and for $n \geq 1$ we have

$$|U_n| = G \times \ldots \times G \quad (n \text{ factors})$$

as profinite spaces where $G = \mathrm{Gal}(k^{sep}/k)$. Namely, every point of U_n has residue field k^{sep} and we identify $(\sigma_1, \ldots, \sigma_n)$ with the point corresponding to the surjection

$$k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep} \longrightarrow k^{sep}, \quad \lambda_0 \otimes \lambda_1 \otimes \ldots \lambda_n \longmapsto \lambda_0\sigma_1(\lambda_1)\ldots\sigma_n(\lambda_n)$$

Then we compute

$$\begin{aligned} R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda}^\wedge) &= R\lim_m R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda/I^m}) \\ &= R\lim_m R\Gamma((U_n)_{étale}, \underline{\Lambda/I^m}) \\ &= \lim_m H^0(U_n, \underline{\Lambda/I^m}) \\ &= \mathrm{Maps}_{cont}(G \times \ldots \times G, \Lambda) \end{aligned}$$

The first equality because $R\Gamma$ commutes with derived limits and as $\Lambda^\wedge$ is the derived limit of the sheaves $\underline{\Lambda/I^m}$ by Proposition 61.21.1. The second equality by Lemma 61.19.6. The third equality by Étale Cohomology, Lemma 59.80.3. The fourth equality uses Étale Cohomology, Remark 59.23.2 to identify sections of the constant sheaf $\underline{\Lambda/I^m}$. Then it uses the fact that Λ is complete with respect to I and hence

equal to $\lim_m \Lambda/I^m$ as a topological space, to see that $\lim_m \mathrm{Map}_{cont}(G, \Lambda/I^m) = \mathrm{Map}_{cont}(G, \Lambda)$ and similarly for higher powers of G. At this point Cohomology on Sites, Lemmas 21.10.3 and 21.10.7 tell us that

$$\Lambda \to \mathrm{Maps}_{cont}(G, \Lambda) \to \mathrm{Maps}_{cont}(G \times G, \Lambda) \to \ldots$$

computes the pro-étale cohomology. In other words, we see that

$$H^i(\mathrm{Spec}(k)_{pro\text{-}étale}, \underline{\Lambda}^\wedge) = H^i_{cont}(G, \Lambda)$$

where the right hand side is Tate's continuous cohomology, see Étale Cohomology, Section 59.58. Of course, this is as it should be.

09B4 **Lemma 61.22.1.** *Let k be a field. Let $G = Gal(k^{sep}/k)$ be its absolute Galois group. Further,*

(1) *let M be a profinite abelian group with a continuous G-action, or*
(2) *let Λ be a Noetherian ring and $I \subset \Lambda$ an ideal an let M be an I-adically complete Λ-module with continuous G-action.*

Then there is a canonical sheaf $\underline{M}^\wedge$ on $\mathrm{Spec}(k)_{pro\text{-}étale}$ associated to M such that

$$H^i(\mathrm{Spec}(k), \underline{M}^\wedge) = H^i_{cont}(G, M)$$

as abelian groups or Λ-modules.

Proof. Proof in case (2). Set $M_n = M/I^nM$. Then $M = \lim M_n$ as M is assumed I-adically complete. Since the action of G is continuous we get continuous actions of G on M_n. By Étale Cohomology, Theorem 59.56.3 this action corresponds to a (locally constant) sheaf $\underline{M_n}$ of Λ/I^n-modules on $\mathrm{Spec}(k)_{étale}$. Pull back to $\mathrm{Spec}(k)_{pro\text{-}étale}$ by the comparison morphism ϵ and take the limit

$$\underline{M}^\wedge = \lim \epsilon^{-1}\underline{M_n}$$

to get the sheaf promised in the lemma. Exactly the same argument as given in the introduction of this section gives the comparison with Tate's continuous Galois cohomology. □

61.23. Functoriality of the pro-étale site

09A5 Let $f : X \to Y$ be a morphism of schemes. The functor $Y_{pro\text{-}étale} \to X_{pro\text{-}étale}$, $V \mapsto X \times_Y V$ induces a morphism of sites $f_{pro\text{-}étale} : X_{pro\text{-}étale} \to Y_{pro\text{-}étale}$, see Sites, Proposition 7.14.7. In fact, we obtain a commutative diagram of morphisms of sites

$$\begin{array}{ccc} X_{pro\text{-}étale} & \xrightarrow{\epsilon} & X_{étale} \\ \downarrow{\scriptstyle f_{pro\text{-}étale}} & & \downarrow{\scriptstyle f_{étale}} \\ Y_{pro\text{-}étale} & \xrightarrow{\epsilon} & Y_{étale} \end{array}$$

where ϵ is as in Section 61.19. In particular we have $\epsilon^{-1}f^{-1}_{étale} = f^{-1}_{pro\text{-}étale}\epsilon^{-1}$. Here is the corresponding result for pushforward.

09A6 **Lemma 61.23.1.** *Let $f : X \to Y$ be a morphism of schemes which is quasi-compact and quasi-separated.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. Then we have $f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}f_{étale,*}\mathcal{F}$.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then we have $Rf_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}Rf_{étale,*}\mathcal{F}$.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. There is a canonical map $\epsilon^{-1} f_{étale,*}\mathcal{F} \to f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F}$, see Sites, Section 7.45. To show it is an isomorphism we may work (Zariski) locally on Y, hence we may assume Y is affine. In this case every object of $Y_{pro\text{-}étale}$ has a covering by objects $V = \lim V_i$ which are limits of affine schemes V_i étale over Y (by Proposition 61.9.1 for example). Evaluating the map $\epsilon^{-1} f_{étale,*}\mathcal{F} \to f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F}$ on V we obtain a map

$$\operatorname{colim} \Gamma(X \times_Y V_i, \mathcal{F}) \longrightarrow \Gamma(X \times_Y V, \epsilon^*\mathcal{F})$$

See Lemma 61.19.3 for the left hand side. By Lemma 61.19.1 we have

$$\Gamma(X \times_Y V, \epsilon^*\mathcal{F}) = \Gamma(X \times_Y V, g_{étale}^{-1}\mathcal{F})$$

where $g : X \times_Y V \to X$ is the projection. Hence the result holds by Étale Cohomology, Lemma 59.51.5.

Proof of (2). Arguing in exactly the same manner as above we see that it suffices to show that

$$\operatorname{colim} H^i_{étale}(X \times_Y V_i, \mathcal{F}) \longrightarrow H^i_{étale}(X \times_Y V, \mathcal{F})$$

which follows once more from Étale Cohomology, Lemma 59.51.5. □

61.24. Finite morphisms and pro-étale sites

09A7 It is not clear that a finite morphism of schemes determines an exact pushforward on abelian pro-étale sheaves.

09A8 **Lemma 61.24.1.** *Let $f : Z \to X$ be a finite morphism of schemes which is locally of finite presentation. Then $f_{pro\text{-}étale,*} : Ab(Z_{pro\text{-}étale}) \to Ab(X_{pro\text{-}étale})$ is exact.*

Proof. The prove this we may work (Zariski) locally on X and assume that X is affine, say $X = \operatorname{Spec}(A)$. Then $Z = \operatorname{Spec}(B)$ for some finite A-algebra B of finite presentation. The construction in the proof of Proposition 61.11.3 produces a faithfully flat, ind-étale ring map $A \to D$ with D w-contractible. We may check exactness of a sequence of sheaves by evaluating on $U = \operatorname{Spec}(D)$ be such an object. Then $f_{pro\text{-}étale,*}\mathcal{F}$ evaluated at U is equal to $\mathcal{F}$ evaluated at $V = \operatorname{Spec}(D \otimes_A B)$. Since $D \otimes_A B$ is w-contractible by Lemma 61.11.6 evaluation at V is exact. □

61.25. Closed immersions and pro-étale sites

09A9 It is not clear (and likely false) that a closed immersion of schemes determines an exact pushforward on abelian pro-étale sheaves.

09BK **Lemma 61.25.1.** *Let $i : Z \to X$ be a closed immersion morphism of affine schemes. Denote X_{app} and Z_{app} the sites introduced in Lemma 61.12.21. The base change functor*

$$u : X_{app} \to Z_{app}, \quad U \longmapsto u(U) = U \times_X Z$$

is continuous and has a fully faithful left adjoint v. For V in Z_{app} the morphism $V \to v(V)$ is a closed immersion identifying V with $u(v(V)) = v(V) \times_X Z$ and every point of $v(V)$ specializes to a point of V. The functor v is cocontinuous and sends coverings to coverings.

Proof. The existence of the adjoint follows immediately from Lemma 61.7.7 and the definitions. It is clear that u is continuous from the definition of coverings in X_{app}.

Write $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let $V = \mathrm{Spec}(\overline{C})$ be an object of Z_{app} and let $v(V) = \mathrm{Spec}(C)$. We have seen in the statement of Lemma 61.7.7 that V equals $v(V) \times_X Z = \mathrm{Spec}(C/IC)$. Any $g \in C$ which maps to an invertible element of $C/IC = \overline{C}$ is invertible in C. Namely, we have the A-algebra maps $C \to C_g \to C/IC$ and by adjointness we obtain an C-algebra map $C_g \to C$. Thus every point of $v(V)$ specializes to a point of V.

Suppose that $\{V_i \to V\}$ is a covering in Z_{app}. Then $\{v(V_i) \to v(V)\}$ is a finite family of morphisms of Z_{app} such that every point of $V \subset v(V)$ is in the image of one of the maps $v(V_i) \to v(V)$. As the morphisms $v(V_i) \to v(V)$ are flat (since they are weakly étale) we conclude that $\{v(V_i) \to v(V)\}$ is jointly surjective. This proves that v sends coverings to coverings.

Let V be an object of Z_{app} and let $\{U_i \to v(V)\}$ be a covering in X_{app}. Then we see that $\{u(U_i) \to u(v(V)) = V\}$ is a covering of Z_{app}. By adjointness we obtain morphisms $v(u(U_i)) \to U_i$. Thus the family $\{v(u(U_i)) \to v(V)\}$ refines the given covering and we conclude that v is cocontinuous. □

09BL **Lemma 61.25.2.** *Let $Z \to X$ be a closed immersion morphism of affine schemes. The corresponding morphism of topoi $i = i_{pro\text{-}étale}$ is equal to the morphism of topoi associated to the fully faithful cocontinuous functor $v : Z_{app} \to X_{app}$ of Lemma 61.25.1. It follows that*

(1) *$i^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto \mathcal{F}(v(V))$,*
(2) *for a weakly contractible object V of Z_{app} we have $i^{-1}\mathcal{F}(V) = \mathcal{F}(v(V))$,*
(3) *$i^{-1} : Sh(X_{pro\text{-}étale}) \to Sh(Z_{pro\text{-}étale})$ has a left adjoint $i_!^{Sh}$,*
(4) *$i^{-1} : Ab(X_{pro\text{-}étale}) \to Ab(Z_{pro\text{-}étale})$ has a left adjoint $i_!$,*
(5) *$id \to i^{-1}i_!^{Sh}$, $id \to i^{-1}i_!$, and $i^{-1}i_* \to id$ are isomorphisms, and*
(6) *i_*, $i_!^{Sh}$ and $i_!$ are fully faithful.*

Proof. By Lemma 61.12.21 we may describe $i_{pro\text{-}étale}$ in terms of the morphism of sites $u : X_{app} \to Z_{app}$, $V \mapsto V \times_X Z$. The first statement of the lemma follows from Sites, Lemmas 7.22.1 and 7.22.2 (but with the roles of u and v reversed).

Proof of (1). By the description of i as the morphism of topoi associated to v this holds by the construction, see Sites, Lemma 7.21.1.

Proof of (2). Since the functor v sends coverings to coverings by Lemma 61.25.1 we see that the presheaf $\mathcal{G} : V \mapsto \mathcal{F}(v(V))$ is a separated presheaf (Sites, Definition 7.10.9). Hence the sheafification of $\mathcal{G}$ is $\mathcal{G}^+$, see Sites, Theorem 7.10.10. Next, let V be a weakly contractible object of Z_{app}. Let $\mathcal{V} = \{V_i \to V\}_{i=1,\ldots,n}$ be any covering in Z_{app}. Set $\mathcal{V}' = \{\coprod V_i \to V\}$. Since v commutes with finite disjoint unions (as a left adjoint or by the construction) and since $\mathcal{F}$ sends finite disjoint unions into products, we see that

$$H^0(\mathcal{V}, \mathcal{G}) = H^0(\mathcal{V}', \mathcal{G})$$

(notation as in Sites, Section 7.10; compare with Étale Cohomology, Lemma 59.22.1). Thus we may assume the covering is given by a single morphism, like so $\{V' \to V\}$. Since V is weakly contractible, this covering can be refined by the trivial covering

$\{V \to V\}$. It therefore follows that the value of $\mathcal{G}^+ = i^{-1}\mathcal{F}$ on V is simply $\mathcal{F}(v(V))$ and (2) is proved.

Proof of (3). Every object of Z_{app} has a covering by weakly contractible objects (Lemma 61.13.4). By the above we see that we would have $i_!^{Sh}h_V = h_{v(V)}$ for V weakly contractible if $i_!^{Sh}$ existed. The existence of $i_!^{Sh}$ then follows from Sites, Lemma 7.24.1.

Proof of (4). Existence of $i_!$ follows in the same way by setting $i_!\mathbf{Z}_V = \mathbf{Z}_{v(V)}$ for V weakly contractible in Z_{app}, using similar for direct sums, and applying Homology, Lemma 12.29.6. Details omitted.

Proof of (5). Let V be a contractible object of Z_{app}. Then $i^{-1}i_!^{Sh}h_V = i^{-1}h_{v(V)} = h_{u(v(V))} = h_V$. (It is a general fact that $i^{-1}h_U = h_{u(U)}$.) Since the sheaves h_V for V contractible generate $Sh(Z_{app})$ (Sites, Lemma 7.12.5) we conclude $\mathrm{id} \to i^{-1}i_!^{Sh}$ is an isomorphism. Similarly for the map $\mathrm{id} \to i^{-1}i_!$. Then $(i^{-1}i_*\mathcal{H})(V) = i_*\mathcal{H}(v(V)) = \mathcal{H}(u(v(V))) = \mathcal{H}(V)$ and we find that $i^{-1}i_* \to \mathrm{id}$ is an isomorphism.

The fully faithfulness statements of (6) now follow from Categories, Lemma 4.24.4. □

09AA **Lemma 61.25.3.** *Let $i : Z \to X$ be a closed immersion of schemes. Then*

(1) *$i^{-1}_{pro\text{-}étale}$ commutes with limits,*
(2) *$i_{pro\text{-}étale,*}$ is fully faithful, and*
(3) *$i^{-1}_{pro\text{-}étale}i_{pro\text{-}étale,*} \cong id_{Sh(Z_{pro\text{-}étale})}$.*

Proof. Assertions (2) and (3) are equivalent by Sites, Lemma 7.41.1. Parts (1) and (3) are (Zariski) local on X, hence we may assume that X is affine. In this case the result follows from Lemma 61.25.2. □

09AB **Lemma 61.25.4.** *Let $i : Z \to X$ be an integral universally injective and surjective morphism of schemes. Then $i_{pro\text{-}étale,*}$ and $i^{-1}_{pro\text{-}étale}$ are quasi-inverse equivalences of categories of pro-étale topoi.*

Proof. There is an immediate reduction to the case that X is affine. Then Z is affine too. Set $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Z)$. Then the categories of étale algebras over A and B are equivalent, see Étale Cohomology, Theorem 59.45.2 and Remark 59.45.3. Thus the categories of ind-étale algebras over A and B are equivalent. In other words the categories X_{app} and Z_{app} of Lemma 61.12.21 are equivalent. We omit the verification that this equivalence sends coverings to coverings and vice versa. Thus the result as Lemma 61.12.21 tells us the pro-étale topos is the topos of sheaves on X_{app}. □

09AC **Lemma 61.25.5.** *Let $i : Z \to X$ be a closed immersion of schemes. Let $U \to X$ be an object of $X_{pro\text{-}étale}$ such that*

(1) *U is affine and weakly contractible, and*
(2) *every point of U specializes to a point of $U \times_X Z$.*

Then $i^{-1}_{pro\text{-}étale}\mathcal{F}(U \times_X Z) = \mathcal{F}(U)$ for all abelian sheaves on $X_{pro\text{-}étale}$.

Proof. Since pullback commutes with restriction, we may replace X by U. Thus we may assume that X is affine and weakly contractible and that every point of X specializes to a point of Z. By Lemma 61.25.2 part (1) it suffices to show that $v(Z) = X$ in this case. Thus we have to show: If A is a w-contractible ring, $I \subset A$

an ideal contained in the Jacobson radical of A and $A \to B \to A/I$ is a factorization with $A \to B$ ind-étale, then there is a unique retraction $B \to A$ compatible with maps to A/I. Observe that $B/IB = A/I \times R$ as A/I-algebras. After replacing B by a localization we may assume $B/IB = A/I$. Note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective as the image contains $V(I)$ and hence all closed points and is closed under specialization. Since A is w-contractible there is a retraction $B \to A$. Since $B/IB = A/I$ this retraction is compatible with the map to A/I. We omit the proof of uniqueness (hint: use that A and B have isomorphic local rings at maximal ideals of A). □

09BM **Lemma 61.25.6.** *Let $i : Z \to X$ be a closed immersion of schemes. If $X \setminus i(Z)$ is a retrocompact open of X, then $i_{pro\text{-}étale,*}$ is exact.*

Proof. The question is local on X hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. There exist $f_1, \ldots, f_r \in I$ such that $Z = V(f_1, \ldots, f_r)$ set theoretically, see Algebra, Lemma 10.29.1. By Lemma 61.25.4 we may assume that $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$. In this case the functor $i_{pro\text{-}étale,*}$ is exact by Lemma 61.24.1. □

61.26. Extension by zero

09AD The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

09AE **Definition 61.26.1.** Let $j : U \to X$ be a weakly étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{pro\text{-}étale}) \to Sh(U_{pro\text{-}étale})$ has a left adjoint $j_!^{Sh} : Sh(X_{pro\text{-}étale}) \to Sh(U_{pro\text{-}étale})$.
(2) The restriction functor $j^{-1} : Ab(X_{pro\text{-}étale}) \to Ab(U_{pro\text{-}étale})$ has a left adjoint which is denoted $j_! : Ab(U_{pro\text{-}étale}) \to Ab(X_{pro\text{-}étale})$ and called *extension by zero*.
(3) Let Λ be a ring. The functor $j^{-1} : Mod(X_{pro\text{-}étale}, \Lambda) \to Mod(U_{pro\text{-}étale}, \Lambda)$ has a left adjoint $j_! : Mod(U_{pro\text{-}étale}, \Lambda) \to Mod(X_{pro\text{-}étale}, \Lambda)$ and called *extension by zero*.

As usual we compare this to what happens in the étale case.

09AF **Lemma 61.26.2.** *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{G}$ be an abelian sheaf on $U_{étale}$. Then $\epsilon^{-1} j_! \mathcal{G} = j_! \epsilon^{-1} \mathcal{G}$ as sheaves on $X_{pro\text{-}étale}$.*

Proof. This is true because both functors are left adjoint to $j^{-1}_{pro\text{-}étale} \epsilon_* = \epsilon_* j^{-1}_{étale}$. The equality holds by the discussion in Section 61.23. □

09AG **Lemma 61.26.3.** *Let $j : U \to X$ be a weakly étale morphism of schemes. Let $i : Z \to X$ be a closed immersion such that $U \times_X Z = \emptyset$. Let $V \to X$ be an affine object of $X_{pro\text{-}étale}$ such that every point of V specializes to a point of $V_Z = Z \times_X V$. Then $j_! \mathcal{F}(V) = 0$ for all abelian sheaves on $U_{pro\text{-}étale}$.*

Proof. Let $\{V_i \to V\}$ be a pro-étale covering. The lemma follows if we can refine this covering to a covering where the members have no morphisms into U over X (see construction of $j_!$ in Modules on Sites, Section 18.19). First refine the covering to get a finite covering with V_i affine. For each i let $V_i = \mathrm{Spec}(A_i)$ and let $Z_i \subset V_i$ be the inverse image of Z. Set $W_i = \mathrm{Spec}(A^\sim_{i,Z_i})$ with notation as in Lemma 61.5.1. Then $\coprod W_i \to V$ is weakly étale and the image contains all points of V_Z. Hence

the image contains all points of V by our assumption on specializations. Thus $\{W_i \to V\}$ is a pro-étale covering refining the given one. But each point in W_i specializes to a point lying over Z, hence there are no morphisms $W_i \to U$ over X. □

09BN **Lemma 61.26.4.** *Let $j : U \to X$ be an open immersion of schemes. Then $id \cong j^{-1}j_!$ and $j^{-1}j_* \cong id$ and the functors $j_!$ and j_* are fully faithful.*

Proof. See Modules on Sites, Lemma 18.19.8 (and Sites, Lemma 7.27.4 for the case of sheaves of sets) and Categories, Lemma 4.24.4. □

Here is the relationship between extension by zero and restriction to the complementary closed subscheme.

09AH **Lemma 61.26.5.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Assume that j is a quasi-compact morphism. For every abelian sheaf on $X_{pro\text{-}étale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$ where all the functors are for the pro-étale topology.

Proof. We obtain the maps by the adjointness properties of the functors involved. It suffices to show that $X_{pro\text{-}étale}$ has enough objects (Sites, Definition 7.40.2) on which the sequence evaluates to a short exact sequence. Let $V = \mathrm{Spec}(A)$ be an affine object of $X_{pro\text{-}étale}$ such that A is w-contractible (there are enough objects of this type). Then $V \times_X Z$ is cut out by an ideal $I \subset A$. The assumption that j is quasi-compact implies there exist $f_1, \ldots, f_r \in I$ such that $V(I) = V(f_1, \ldots, f_r)$. We obtain a faithfully flat, ind-Zariski ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)}$$

with $A^\sim_{V(I)}$ as in Lemma 61.5.1. Since $V_i = \mathrm{Spec}(A_{f_i}) \to X$ factors through U we have

$$j_!j^{-1}\mathcal{F}(V_i) = \mathcal{F}(V_i) \quad \text{and} \quad i_*i^{-1}\mathcal{F}(V_i) = 0$$

On the other hand, for the scheme $V^\sim = \mathrm{Spec}(A^\sim_{V(I)})$ we have

$$j_!j^{-1}\mathcal{F}(V^\sim) = 0 \quad \text{and} \quad \mathcal{F}(V^\sim) = i_*i^{-1}\mathcal{F}(V^\sim)$$

the first equality by Lemma 61.26.3 and the second by Lemmas 61.25.5 and 61.11.7. Thus the sequence evaluates to an exact sequence on $\mathrm{Spec}(A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)})$ and the lemma is proved. □

09BP **Lemma 61.26.6.** *Let $j : U \to X$ be a quasi-compact open immersion morphism of schemes. The functor $j_! : Ab(U_{pro\text{-}étale}) \to Ab(X_{pro\text{-}étale})$ commutes with limits.*

Proof. Since $j_!$ is exact it suffices to show that $j_!$ commutes with products. The question is local on X, hence we may assume X affine. Let $\mathcal{G}$ be an abelian sheaf on $U_{pro\text{-}étale}$. We have $j^{-1}j_*\mathcal{G} = \mathcal{G}$. Hence applying the exact sequence of Lemma 61.26.5 we get

$$0 \to j_!\mathcal{G} \to j_*\mathcal{G} \to i_*i^{-1}j_*\mathcal{G} \to 0$$

where $i : Z \to X$ is the inclusion of the reduced induced scheme structure on the complement $Z = X \setminus U$. The functors j_* and i_* commute with products as right adjoints. The functor i^{-1} commutes with products by Lemma 61.25.3. Hence

$j_!$ does because on the pro-étale site products are exact (Cohomology on Sites, Proposition 21.51.2). □

61.27. Constructible sheaves on the pro-étale site

09AI We stick to constructible sheaves of Λ-modules for a Noetherian ring. In the future we intend to discuss constructible sheaves of sets, groups, etc.

09AJ **Definition 61.27.1.** Let X be a scheme. Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{pro\text{-}étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

Again this does not give anything "new".

09AK **Lemma 61.27.2.** *Let X be a scheme. Let Λ be a Noetherian ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{array}{c}\text{constructible sheaves of} \\ \Lambda\text{-modules on } X_{étale}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{constructible sheaves of} \\ \Lambda\text{-modules on } X_{pro\text{-}étale}\end{array}\right\}$$

between constructible sheaves of Λ-modules on $X_{étale}$ and constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$.

Proof. By Lemma 61.19.2 the functor ϵ^{-1} is fully faithful and commutes with pullback (restriction) to the strata. Hence ϵ^{-1} of a constructible étale sheaf is a constructible pro-étale sheaf. To finish the proof let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{pro\text{-}étale}$ as in Definition 61.27.1. There is a canonical map

$$\epsilon^{-1}\epsilon_*\mathcal{F} \longrightarrow \mathcal{F}$$

We will show this map is an isomorphism. This will prove that $\mathcal{F}$ is in the essential image of ϵ^{-1} and finish the proof (details omitted).

Since it suffices to prove this locally on X we may assume X is quasi-compact and quasi-separated and that we have a finite partition $X = \coprod_{i=1,\ldots,n} X_i$ by constructible locally closed strata such that $\mathcal{F}|_{X_i}$ is locally constant of finite type. We will use induction on n. The base case $n = 1$ follows from Lemma 61.19.9. Take a point $x \in X$; then $x \in X_k$ for some k. It suffices to show the displayed map is an isomorphism in an open neighbourhood of x. Hence we may assume X_k is closed in X. Set $Z = X_k$ and denote $U \subset X$ the complement of Z. Observe that the induction hypothesis applies to the restriction of $\mathcal{F}$ to Z and to U. By Lemma 61.26.5 we have a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Functoriality gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} & \longrightarrow & \epsilon^{-1}\epsilon_*\mathcal{F} & \longrightarrow & \epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!j^{-1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_*i^{-1}\mathcal{F} & \longrightarrow & 0 \end{array}$$

By induction we know that on the one hand $\epsilon^{-1}\epsilon_*i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$ and $\epsilon^{-1}\epsilon_*j^{-1}\mathcal{F} \to j^{-1}\mathcal{F}$ are isomorphisms and on the other that $i^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{A}$ and $j^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{B}$ for

some constructible sheaves of Λ-modules $\mathcal{A}$ on $Z_{étale}$ and $\mathcal{B}$ on $U_{étale}$. Then

$$\epsilon^{-1}\epsilon_* j_! j^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_* j_! \epsilon^{-1}\mathcal{B} = \epsilon^{-1}\epsilon_*\epsilon^{-1} j_!\mathcal{B} = \epsilon^{-1} j_!\mathcal{B} = j_!\epsilon^{-1}\mathcal{B} = j_! j^{-1}\mathcal{F}$$

the second equality by Lemma 61.26.2, the third equality by Lemma 61.19.2, and the fourth equality by Lemma 61.26.2 again. Similarly, we have

$$\epsilon^{-1}\epsilon_* i_* i^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_* i_*\epsilon^{-1}\mathcal{A} = \epsilon^{-1}\epsilon_*\epsilon^{-1} i_*\mathcal{A} = \epsilon^{-1} i_*\mathcal{A} = i_*\epsilon^{-1}\mathcal{A} = i_* i^{-1}\mathcal{F}$$

this time using Lemma 61.23.1. By the five lemma we conclude the vertical map in the middle of the big diagram is an isomorphism. □

09B5 **Lemma 61.27.3.** *Let X be a scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$ is a weak Serre subcategory of $Mod(X_{pro\text{-}étale}, \Lambda)$.*

Proof. This is a formal consequence of Lemmas 61.27.2 and 61.19.8 and the result for the étale site (Étale Cohomology, Lemma 59.71.6). □

09AL **Lemma 61.27.4.** *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_c(X_{étale}, \Lambda)$, resp. $D_c(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are constructible sheaves of Λ-modules. Then*

$$\epsilon^{-1} : D_c^+(X_{étale}, \Lambda) \longrightarrow D_c^+(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_c(X_{étale}, \Lambda)$ and $D_c(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Étale Cohomology, Lemma 59.71.6 and Lemma 61.27.3 and Derived Categories, Section 13.17. The statement of the lemma follows by combining Lemmas 61.19.8 and 61.27.2. □

09BQ **Lemma 61.27.5.** *Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{pro\text{-}étale}, \Lambda)$. Then $K \otimes^{\mathbf{L}}_{\Lambda} L$ is in $D_c^-(X_{pro\text{-}étale}, \Lambda)$.*

Proof. Note that $H^i(K \otimes^{\mathbf{L}}_{\Lambda} L)$ is the same as $H^i(\tau_{\geq i-1}K \otimes^{\mathbf{L}}_{\Lambda} \tau_{\geq i-1}L)$. Thus we may assume K and L are bounded. In this case we can apply Lemma 61.27.4 to reduce to the case of the étale site, see Étale Cohomology, Lemma 59.76.6. □

09BR **Lemma 61.27.6.** *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. If K_1 is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I)$, then K_n is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I^n)$ for all n.*

Proof. Consider the distinguished triangles

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} = K_1 \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

By Lemma 61.27.5 we see that this tensor product has constructible cohomology sheaves (and vanishing when K_1 has vanishing cohomology). Hence by induction on n using Lemma 61.27.3 we see that each K_n has constructible cohomology sheaves. □

61.28. Constructible adic sheaves

09BS In this section we define the notion of a constructible Λ-sheaf as well as some variants.

09BT **Definition 61.28.1.** Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$.

(1) We say $\mathcal{F}$ is a *constructible Λ-sheaf* if $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ and each $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of Λ/I^n-modules.
(2) If $\mathcal{F}$ is a constructible Λ-sheaf, then we say $\mathcal{F}$ is *lisse* if each $\mathcal{F}/I^n\mathcal{F}$ is locally constant.
(3) We say $\mathcal{F}$ is *adic lisse*[5] if there exists a I-adically complete Λ-module M with M/IM finite such that $\mathcal{F}$ is locally isomorphic to
$$\underline{M}^\wedge = \lim \underline{M/I^nM}.$$
(4) We say $\mathcal{F}$ is *adic constructible*[6] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $\mathcal{F}|_{U_i}$ is adic lisse.

The definition of a constructible Λ-sheaf is equivalent to the one in [**Gro77**, Exposé VI, Definition 1.1.1] when $\Lambda = \mathbf{Z}_\ell$ and $I = (\ell)$. It is clear that we have the implications

$$\begin{array}{ccc} \text{lisse adic} & \Longrightarrow & \text{adic constructible} \\ \Downarrow & & \Downarrow \\ \text{lisse constructible } \Lambda\text{-sheaf} & \Longrightarrow & \text{constructible } \Lambda\text{-sheaf} \end{array}$$

The vertical arrows can be inverted in some cases (see Lemmas 61.28.2 and 61.28.5). In general neither the category of adic constructible sheaves nor the category of constructible Λ-sheaves is closed under kernels and cokernels.

Namely, let X be an affine scheme whose underlying topological space $|X|$ is homeomorphic to $\Lambda = \mathbf{Z}_\ell$, see Example 61.6.3. Denote $f : |X| \to \mathbf{Z}_\ell = \Lambda$ a homeomorphism. We can think of f as a section of $\underline{\Lambda}^\wedge$ over X and multiplication by f then defines a two term complex
$$\underline{\Lambda}^\wedge \xrightarrow{f} \underline{\Lambda}^\wedge$$
on $X_{pro\text{-}étale}$. The sheaf $\underline{\Lambda}^\wedge$ is adic lisse. However, the cokernel of the map above, is not adic constructible, as the isomorphism type of the stalks of this cokernel attains infinitely many values: $\mathbf{Z}/\ell^n\mathbf{Z}$ and $\mathbf{Z}_\ell$. The cokernel is a constructible $\mathbf{Z}_\ell$-sheaf. However, the kernel is not even a constructible $\mathbf{Z}_\ell$-sheaf as it is zero a non-quasi-compact open but not zero.

09BU **Lemma 61.28.2.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then there exists a finite partition $X = \coprod X_i$ by locally closed subschemes such that the restriction $\mathcal{F}|_{X_i}$ is lisse.*

[5] This may be nonstandard notation.
[6] This may be nonstandard notation.

Proof. Let $R = \bigoplus I^n/I^{n+1}$. Observe that R is a Noetherian ring. Since each of the sheaves $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of $\Lambda/I^n\Lambda$-modules also $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is a constructible sheaf of Λ/I-modules and hence the pullback of a constructible sheaf $\mathcal{G}_n$ on $X_{étale}$ by Lemma 61.27.2. Set $\mathcal{G} = \bigoplus \mathcal{G}_n$. This is a sheaf of R-modules on $X_{étale}$ and the map

$$\mathcal{G}_0 \otimes_{\Lambda/I} \underline{R} \longrightarrow \mathcal{G}$$

is surjective because the maps

$$\mathcal{F}/I\mathcal{F} \otimes \underline{I^n/I^{n+1}} \to I^n\mathcal{F}/I^{n+1}\mathcal{F}$$

are surjective. Hence $\mathcal{G}$ is a constructible sheaf of R-modules by Étale Cohomology, Proposition 59.74.1. Choose a partition $X = \coprod X_i$ such that $\mathcal{G}|_{X_i}$ is a locally constant sheaf of R-modules of finite type (Étale Cohomology, Lemma 59.71.2). We claim this is a partition as in the lemma. Namely, replacing X by X_i we may assume $\mathcal{G}$ is locally constant. It follows that each of the sheaves $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is locally constant. Using the short exact sequences

$$0 \to I^n\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

induction and Modules on Sites, Lemma 18.43.5 the lemma follows. □

09BV **Lemma 61.28.3.** *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$ such that*

(1) $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$,
(2) $\mathcal{F}/I^n\mathcal{F}$ *is a constant sheaf of Λ/I^n-modules,*
(3) $\mathcal{F}/I\mathcal{F}$ *is of finite type.*

Then $\mathcal{F} \cong \underline{M}^\wedge$ where M is a finite $\Lambda^\wedge$-module.

Proof. Pick a Λ/I^n-module M_n such that $\mathcal{F}/I^n\mathcal{F} \cong \underline{M_n}$. Since we have the surjections $\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F}$ we conclude that there exist surjections $M_{n+1} \to M_n$ inducing isomorphisms $M_{n+1}/I^nM_{n+1} \to M_n$. Fix a choice of such surjections and set $M = \lim M_n$. Then M is an I-adically complete Λ-module with $M/I^nM = M_n$, see Algebra, Lemma 10.98.2. Since M_1 is a finite type Λ-module (Modules on Sites, Lemma 18.42.5) we see that M is a finite $\Lambda^\wedge$-module. Consider the sheaves

$$\mathcal{I}_n = Isom(\underline{M_n}, \mathcal{F}/I^n\mathcal{F})$$

on $X_{pro\text{-}étale}$. Modding out by I^n defines a transition map

$$\mathcal{I}_{n+1} \longrightarrow \mathcal{I}_n$$

By our choice of M_n the sheaf $\mathcal{I}_n$ is a torsor under

$$Isom(\underline{M_n}, \underline{M_n}) = \underline{\mathrm{Isom}_\Lambda(M_n, M_n)}$$

(Modules on Sites, Lemma 18.43.4) since $\mathcal{F}/I^n\mathcal{F}$ is (étale) locally isomorphic to $\underline{M_n}$. It follows from More on Algebra, Lemma 15.101.4 that the system of sheaves $(\mathcal{I}_n)$ is Mittag-Leffler. For each n let $\mathcal{I}'_n \subset \mathcal{I}_n$ be the image of $\mathcal{I}_N \to \mathcal{I}_n$ for all $N \gg n$. Then

$$\ldots \to \mathcal{I}'_3 \to \mathcal{I}'_2 \to \mathcal{I}'_1 \to *$$

is a sequence of sheaves of sets on $X_{pro\text{-}étale}$ with surjective transition maps. Since $*(X)$ is a singleton (not empty) and since evaluating at X transforms surjective maps of sheaves of sets into surjections of sets, we can pick $s \in \lim \mathcal{I}'_n(X)$. The sections define isomorphisms $\underline{M}^\wedge \to \lim \mathcal{F}/I^n\mathcal{F} = \mathcal{F}$ and the proof is done. □

09BW **Lemma 61.28.4.** *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If $\mathcal{F}$ is a lisse constructible Λ-sheaf on $X_{pro\text{-}étale}$, then $\mathcal{F}$ is adic lisse.*

Proof. By Lemma 61.19.9 we have $\mathcal{F}/I^n\mathcal{F} = \epsilon^{-1}\mathcal{G}_n$ for some locally constant sheaf $\mathcal{G}_n$ of Λ/I^n-modules. By Étale Cohomology, Lemma 59.64.8 there exists a finite Λ/I^n-module M_n such that $\mathcal{G}_n$ is locally isomorphic to $\underline{M_n}$. Choose a covering $\{W_t \to X\}_{t\in T}$ with each W_t affine and weakly contractible. Then $\mathcal{F}|_{W_t}$ satisfies the assumptions of Lemma 61.28.3 and hence $\mathcal{F}|_{W_t} \cong \underline{N_t}^\wedge$ for some finite $\Lambda^\wedge$-module N_t. Note that $N_t/I^nN_t \cong M_n$ for all t and n. Hence $N_t \cong N_{t'}$ for all $t, t' \in T$, see More on Algebra, Lemma 15.101.5. This proves that $\mathcal{F}$ is adic lisse. □

09BX **Lemma 61.28.5.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then $\mathcal{F}$ is adic constructible.*

Proof. This is a consequence of Lemmas 61.28.2 and 61.28.4, the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.9.6), and the definitions. □

It will be useful to identify the constructible Λ-sheaves inside the category of derived complete sheaves of Λ-modules. It turns out that the naive analogue of More on Algebra, Lemma 15.95.5 is wrong in this setting. However, here is the analogue of More on Algebra, Lemma 15.92.7.

09BY **Lemma 61.28.6.** *Let X be a scheme. Let Λ be a ring and let $I \subset \Lambda$ be a finitely generated ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$. If $\mathcal{F}$ is derived complete and $\mathcal{F}/I\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. Assume that $\mathcal{F}/I\mathcal{F}$ is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $\mathcal{G} = \mathcal{F}/(f_1, \ldots, f_i)\mathcal{F}$ is nonzero. If i does not exist, then $\mathcal{F} = 0$ which is what we want to show. Then $\mathcal{G}$ is derived complete as a cokernel of a map between derived complete modules, see Proposition 61.21.1. By our choice of i we have that $f_{i+1} : \mathcal{G} \to \mathcal{G}$ is surjective. Hence

$$\lim(\ldots \to \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G})$$

is nonzero, contradicting the derived completeness of $\mathcal{G}$. □

09BZ **Lemma 61.28.7.** *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a derived complete sheaf of Λ-modules on $X_{pro\text{-}étale}$ with $\mathcal{F}/I\mathcal{F}$ a locally constant sheaf of Λ/I-modules of finite type. Then there exists an integer t and a surjective map*

$$(\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$$

Proof. Since X is weakly contractible, there exists a finite disjoint open covering $X = \coprod U_i$ such that $\mathcal{F}/I\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf associated to a finite Λ/I-module M_i. Choose finitely many generators m_{ij} of M_i. We can find sections $s_{ij} \in \mathcal{F}(X)$ restricting to m_{ij} viewed as a section of $\mathcal{F}/I\mathcal{F}$ over U_i. Let t be the total number of s_{ij}. Then we obtain a map

$$\alpha : \underline{\Lambda}^{\oplus t} \longrightarrow \mathcal{F}$$

which is surjective modulo I by construction. By Lemma 61.20.1 the derived completion of $\underline{\Lambda}^{\oplus t}$ is the sheaf $(\underline{\Lambda}^\wedge)^{\oplus t}$. Since $\mathcal{F}$ is derived complete we see that α factors through a map

$$\alpha^\wedge : (\underline{\Lambda}^\wedge)^{\oplus t} \longrightarrow \mathcal{F}$$

Then $\mathcal{Q} = \mathrm{Coker}(\alpha^\wedge)$ is a derived complete sheaf of Λ-modules by Proposition 61.21.1. By construction $\mathcal{Q}/I\mathcal{Q} = 0$. It follows from Lemma 61.28.6 that $\mathcal{Q} = 0$ which is what we wanted to show. □

61.29. A suitable derived category

09C0 Let X be a scheme. It will turn out that for many schemes X a suitable derived category of ℓ-adic sheaves can be gotten by considering the derived complete objects K of $D(X_{pro\text{-}étale}, \Lambda)$ with the property that $K \otimes^{\mathbf{L}}_{\Lambda} \mathbf{F}_\ell$ is bounded with constructible cohomology sheaves. Here is the general definition.

09C1 **Definition 61.29.1.** Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. An object K of $D(X_{pro\text{-}étale}, \Lambda)$ is called *constructible* if

(1) K is derived complete with respect to I,
(2) $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has constructible cohomology sheaves and locally has finite tor dimension.

We denote $D_{cons}(X, \Lambda)$ the full subcategory of constructible K in $D(X_{pro\text{-}étale}, \Lambda)$.

Recall that with our conventions a complex of finite tor dimension is bounded (Cohomology on Sites, Definition 21.46.1). In fact, let's collect everything proved so far in a lemma.

09C2 **Lemma 61.29.2.** *In the situation above suppose K is in $D_{cons}(X, \Lambda)$ and X is quasi-compact. Set $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. There exist a, b such that*

(1) *$K = R\lim K_n$ and $H^i(K) = 0$ for $i \notin [a, b]$,*
(2) *each K_n has tor amplitude in $[a, b]$,*
(3) *each K_n has constructible cohomology sheaves,*
(4) *each $K_n = \epsilon^{-1} L_n$ for some $L_n \in D_{ctf}(X_{étale}, \Lambda/I^n)$ (Étale Cohomology, Definition 59.77.1).*

Proof. By definition of local having finite tor dimension, we can find a, b such that K_1 has tor amplitude in $[a, b]$. Part (2) follows from Cohomology on Sites, Lemma 21.46.9. Then (1) follows as K is derived complete by the description of limits in Cohomology on Sites, Proposition 21.51.2 and the fact that $H^b(K_{n+1}) \to H^b(K_n)$ is surjective as $K_n = K_{n+1} \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. Part (3) follows from Lemma 61.27.6, Part (4) follows from Lemma 61.27.4 and the fact that L_n has finite tor dimension because K_n does (small argument omitted). □

09C3 **Lemma 61.29.3.** *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that the cohomology sheaves of $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ are locally constant. Then there exists a finite disjoint open covering $X = \coprod \overline{U_i}$ and for each i a finite collection of finite projective $\Lambda^\wedge$-modules $M^a, \ldots, M^b$ such that $K|_{U_i}$ is represented by a complex*

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^b})^\wedge$$

in $D(U_{i,pro\text{-}étale}, \Lambda)$ for some maps of sheaves of Λ-modules $(\underline{M^i})^\wedge \to (\underline{M^{i+1}})^\wedge$.

Proof. We freely use the results of Lemma 61.29.2. Choose a, b as in that lemma. We will prove the lemma by induction on $b - a$. Let $\mathcal{F} = H^b(K)$. Note that $\mathcal{F}$ is a derived complete sheaf of Λ-modules by Proposition 61.21.1. Moreover $\mathcal{F}/I\mathcal{F}$ is a locally constant sheaf of Λ/I-modules of finite type. Apply Lemma 61.28.7 to get a surjection $\rho : (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$.

If $a = b$, then $K = \mathcal{F}[-b]$. In this case we see that

$$\mathcal{F} \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I} = \mathcal{F}/I\mathcal{F}$$

As X is weakly contractible and $\mathcal{F}/I\mathcal{F}$ locally constant, we can find a finite disjoint union decomposition $X = \coprod U_i$ by affine opens U_i and Λ/I-modules $\overline{M}_i$ such that $\mathcal{F}/I\mathcal{F}$ restricts to $\underline{\overline{M}_i}$ on U_i. After refining the covering we may assume the map

$$\rho|_{U_i} \bmod I : \underline{\Lambda/I}^{\oplus t} \longrightarrow \underline{\overline{M}_i}$$

is equal to $\underline{\alpha_i}$ for some surjective module map $\alpha_i : \Lambda/I^{\oplus t} \to \overline{M}_i$, see Modules on Sites, Lemma 18.43.3. Note that each $\overline{M}_i$ is a finite Λ/I-module. Since $\mathcal{F}/I\mathcal{F}$ has tor amplitude in $[0, 0]$ we conclude that $\overline{M}_i$ is a flat Λ/I-module. Hence $\overline{M}_i$ is finite projective (Algebra, Lemma 10.78.2). Hence we can find a projector $\overline{p}_i : (\Lambda/I)^{\oplus t} \to (\Lambda/I)^{\oplus t}$ whose image maps isomorphically to $\overline{M}_i$ under the map α_i. We can lift $\overline{p}_i$ to a projector $p_i : (\Lambda^\wedge)^{\oplus t} \to (\Lambda^\wedge)^{\oplus t}$[7]. Then $M_i = \mathrm{Im}(p_i)$ is a finite I-adically complete $\Lambda^\wedge$-module with $M_i/IM_i = \overline{M}_i$. Over U_i consider the maps

$$\underline{M_i}^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}|_{U_i}$$

By construction the composition induces an isomorphism modulo I. The source and target are derived complete, hence so are the cokernel $\mathcal{Q}$ and the kernel $\mathcal{K}$. We have $\mathcal{Q}/I\mathcal{Q} = 0$ by construction hence $\mathcal{Q}$ is zero by Lemma 61.28.6. Then

$$0 \to \mathcal{K}/I\mathcal{K} \to \underline{\overline{M}_i} \to \mathcal{F}/I\mathcal{F} \to 0$$

is exact by the vanishing of Tor_1 see at the start of this paragraph; also use that $\underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge$ by Modules on Sites, Lemma 18.42.4 to see that $\underline{M_i}^\wedge/I\underline{M_i}^\wedge = \underline{\overline{M}_i}$. Hence $\mathcal{K}/I\mathcal{K} = 0$ by construction and we conclude that $\mathcal{K} = 0$ as before. This proves the result in case $a = b$.

If $b > a$, then we lift the map ρ to a map

$$\tilde{\rho} : (\underline{\Lambda}^\wedge)^{\oplus t}[-b] \longrightarrow K$$

in $D(X_{pro\text{-}étale}, \Lambda)$. This is possible as we can think of K as a complex of $\underline{\Lambda}^\wedge$-modules by discussion in the introduction to Section 61.20 and because $X_{pro\text{-}étale}$ is weakly contractible hence there is no obstruction to lifting the elements $\rho(e_s) \in H^0(X, \mathcal{F})$ to elements of $H^b(X, K)$. Fitting $\tilde{\rho}$ into a distinguished triangle

$$(\underline{\Lambda}^\wedge)^{\oplus t}[-b] \to K \to L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$$

we see that L is an object of $D_{cons}(X, \Lambda)$ such that $L \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has tor amplitude contained in $[a, b-1]$ (details omitted). By induction we can describe L locally as stated in the lemma, say L is isomorphic to

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^{b-1}})^\wedge$$

[7]Proof: by Algebra, Lemma 10.32.7 we can lift $\overline{p}_i$ to a compatible system of projectors $p_{i,n} : (\Lambda/I^n)^{\oplus t} \to (\Lambda/I^n)^{\oplus t}$ and then we set $p_i = \lim p_{i,n}$ which works because $\Lambda^\wedge = \lim \Lambda/I^n$.

The map $L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$ corresponds to a map $(\underline{M^{b-1}})^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t}$ which allows us to extend the complex by one. The corresponding complex is isomorphic to K in the derived category by the properties of triangulated categories. This finishes the proof. □

Motivated by what happens for constructible Λ-sheaves we introduce the following notion.

09C4 **Definition 61.29.4.** Let X be a scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $K \in D(X_{pro\text{-}étale}, \Lambda)$.

(1) We say K is *adic lisse*[8] if there exists a finite complex of finite projective $\Lambda^\wedge$-modules $M^\bullet$ such that K is locally isomorphic to
$$\underline{M^a}^\wedge \to \ldots \to \underline{M^b}^\wedge$$
(2) We say K is *adic constructible*[9] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $K|_{U_i}$ is adic lisse.

The difference between the local structure obtained in Lemma 61.29.3 and the structure of an adic lisse complex is that the maps $\underline{M^i}^\wedge \to \underline{M^{i+1}}^\wedge$ in Lemma 61.29.3 need not be constant, whereas in the definition above they are required to be constant.

09C5 **Lemma 61.29.5.** *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ is isomorphic in $D(X_{pro\text{-}étale}, \Lambda/I^n)$ to a complex of constant sheaves of Λ/I^n-modules. Then*
$$H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n)$$
has the Mittag-Leffler condition.

Proof. Say $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ is isomorphic to $\underline{E_n}$ for some object E_n of $D(\Lambda/I^n)$. Since $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I}$ has finite tor dimension and has finite type cohomology sheaves we see that E_1 is perfect (see More on Algebra, Lemma 15.75.2). The transition maps
$$K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^{n+1}} \to K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$$
locally come from (possibly many distinct) maps of complexes $E_{n+1} \to E_n$ in $D(\Lambda/I^{n+1})$ see Cohomology on Sites, Lemma 21.53.3. For each n choose one such map and observe that it induces an isomorphism $E_{n+1} \otimes^{\mathbf{L}}_{\Lambda/I^{n+1}} \Lambda/I^n \to E_n$ in $D(\Lambda/I^n)$. By More on Algebra, Lemma 15.98.4 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^n M^\bullet \to E_n$ in $D(\Lambda/I^n)$ compatible with the transition maps.

Now observe that for each finite collection of indices $n > m > k$ the triple of maps
$$H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n) \to H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^m) \to H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^k)$$
is isomorphic to
$$H^0(X, \underline{M^\bullet/I^n M^\bullet}) \to H^0(X, \underline{M^\bullet/I^m M^\bullet}) \to H^0(X, \underline{M^\bullet/I^k M^\bullet})$$

[8] This may be nonstandard notation

[9] This may be nonstandard notation.

Namely, choose any isomorphism

$$\underline{M^\bullet/I^nM^\bullet} \to K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n$$

induces similar isomorphisms module I^m and I^k and we see that the assertion is true. Thus to prove the lemma it suffices to show that the system $H^0(X, \underline{M^\bullet/I^nM^\bullet})$ has Mittag-Leffler. Since taking sections over X is exact, it suffices to prove that the system of Λ-modules

$$H^0(M^\bullet/I^nM^\bullet)$$

has Mittag-Leffler. Set $A = \Lambda^\wedge$ and consider the spectral sequence

$$\mathrm{Tor}^A_{-p}(H^q(M^\bullet), A/I^nA) \Rightarrow H^{p+q}(M^\bullet/I^nM^\bullet)$$

By More on Algebra, Lemma 15.27.3 the pro-systems $\{\mathrm{Tor}^A_{-p}(H^q(M^\bullet), A/I^nA)\}$ are zero for $p > 0$. Thus the pro-system $\{H^0(M^\bullet/I^nM^\bullet)\}$ is equal to the pro-system $\{H^0(M^\bullet)/I^nH^0(M^\bullet)\}$ and the lemma is proved. □

09C6 **Lemma 61.29.6.** *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If K is in $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda \underline{\Lambda/I}$ has locally constant cohomology sheaves, then K is adic lisse (Definition 61.29.4).*

Proof. Write $K_n = K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$. We will use the results of Lemma 61.29.2 without further mention. By Cohomology on Sites, Lemma 21.53.5 we see that K_n has locally constant cohomology sheaves for all n. We have $K_n = \epsilon^{-1}L_n$ some L_n in $D_{ctf}(X_{étale}, \Lambda/I^n)$ with locally constant cohomology sheaves. By Étale Cohomology, Lemma 59.77.7 there exist perfect $M_n \in D(\Lambda/I^n)$ such that L_n is étale locally isomorphic to $\underline{M_n}$. The maps $L_{n+1} \to L_n$ corresponding to $K_{n+1} \to K_n$ induces isomorphisms $L_{n+1} \otimes^{\mathbf{L}}_{\Lambda/I^{n+1}} \underline{\Lambda/I^n} \to L_n$. Looking locally on X we conclude that there exist maps $M_{n+1} \to M_n$ in $D(\Lambda/I^{n+1})$ inducing isomorphisms $M_{n+1} \otimes_{\Lambda/I^{n+1}} \Lambda/I^n \to M_n$, see Cohomology on Sites, Lemma 21.53.3. Fix a choice of such maps. By More on Algebra, Lemma 15.98.4 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to M_n$ in $D(\Lambda/I^n)$ compatible with the transition maps. To finish the proof we will show that K is locally isomorphic to

$$\underline{M^\bullet}^\wedge = \lim \underline{M^\bullet/I^nM^\bullet} = R\lim \underline{M^\bullet/I^nM^\bullet}$$

Let $E^\bullet$ be the dual complex to $M^\bullet$, see More on Algebra, Lemma 15.75.15 and its proof. Consider the objects

$$H_n = R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, K_n) = \underline{E^\bullet/I^nE^\bullet} \otimes^{\mathbf{L}}_{\Lambda/I^n} K_n$$

of $D(X_{pro\text{-}étale}, \Lambda/I^n)$. Modding out by I^n defines a transition map $H_{n+1} \to H_n$. Set $H = R\lim H_n$. Then H is an object of $D_{cons}(X, \Lambda)$ (details omitted) with $H \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n} = H_n$. Choose a covering $\{W_t \to X\}_{t \in T}$ with each W_t affine and weakly contractible. By our choice of $M^\bullet$ we see that

$$\begin{aligned} H_n|_{W_t} &\cong R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, \underline{M^\bullet/I^nM^\bullet}) \\ &= \underline{\mathrm{Tot}(E^\bullet/I^nE^\bullet \otimes_{\Lambda/I^n} M^\bullet/I^nM^\bullet)} \end{aligned}$$

Thus we may apply Lemma 61.29.5 to $H = R\lim H_n$. We conclude the system $H^0(W_t, H_n)$ satisfies Mittag-Leffler. Since for all $n \gg 1$ there is an element of $H^0(W_t, H_n)$ which maps to an isomorphism in

$$H^0(W_t, H_1) = \mathrm{Hom}(\underline{M^\bullet/IM^\bullet}, K_1)$$

we find an element $(\varphi_{t,n})$ in the inverse limit which produces an isomorphism mod I. Then

$$R\lim \varphi_{t,n} : \underline{M^\bullet}^\wedge|_{W_t} = R\lim \underline{M^\bullet/I^nM^\bullet}|_{W_t} \longrightarrow R\lim K_n|_{W_t} = K|_{W_t}$$

is an isomorphism. This finishes the proof. □

09C7 **Proposition 61.29.7.** *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$. Then K is adic constructible (Definition 61.29.4).*

Proof. This is a consequence of Lemma 61.29.6 and the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.9.6), and the definitions. □

61.30. Proper base change

09C8 In this section we explain how to prove the proper base change theorem for derived complete objects on the pro-étale site using the proper base change theorem for étale cohomology following the general theme that we use the pro-étale topology only to deal with "limit issues" and we use results proved for the étale topology to handle everything else.

09C9 **Theorem 61.30.1.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes giving rise to the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal such that Λ/I is torsion. Let K be an object of $D(X_{pro\text{-}étale})$ such that

1. *K is derived complete, and*
2. *$K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ is bounded below with cohomology sheaves coming from $X_{étale}$,*
3. *Λ/I^n is a perfect Λ-module*[10].

Then the base change map

$$Lg^*_{comp} Rf_* K \longrightarrow Rf'_* L(g')^*_{comp} K$$

is an isomorphism.

Proof. We omit the construction of the base change map (this uses only formal properties of derived pushforward and completed derived pullback, compare with Cohomology on Sites, Remark 21.19.3). Write $K_n = K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$. By Lemma 61.20.1 we have $K = R\lim K_n$ because K is derived complete. By Lemmas 61.20.2 and 61.20.1 we can unwind the left hand side

$$Lg^*_{comp} Rf_* K = R\lim Lg^*(Rf_* K) \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n} = R\lim Lg^* Rf_* K_n$$

the last equality because Λ/I^n is a perfect module and the projection formula (Cohomology on Sites, Lemma 21.50.1). Using Lemma 61.20.2 we can unwind the right hand side

$$Rf'_* L(g')^*_{comp} K = Rf'_* R\lim L(g')^* K_n = R\lim Rf'_* L(g')^* K_n$$

[10] This assumption can be removed if K is a constructible complex, see [**BS13**].

the last equality because Rf'_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.23.3). Thus it suffices to show the maps

$$Lg^*Rf_*K_n \longrightarrow Rf'_*L(g')^*K_n$$

are isomorphisms. By Lemma 61.19.8 and our second condition we can write $K_n = \epsilon^{-1}L_n$ for some $L_n \in D^+(X_{étale}, \Lambda/I^n)$. By Lemma 61.23.1 and the fact that ϵ^{-1} commutes with pullbacks we obtain

$$Lg^*Rf_*K_n = Lg^*Rf_*\epsilon^*L_n = Lg^*\epsilon^{-1}Rf_*L_n = \epsilon^{-1}Lg^*Rf_*L_n$$

and

$$Rf'_*L(g')^*K_n = Rf'_*L(g')^*\epsilon^{-1}L_n = Rf'_*\epsilon^{-1}L(g')^*L_n = \epsilon^{-1}Rf'_*L(g')^*L_n$$

(this also uses that L_n is bounded below). Finally, by the proper base change theorem for étale cohomology (Étale Cohomology, Theorem 59.91.11) we have

$$Lg^*Rf_*L_n = Rf'_*L(g')^*L_n$$

(again using that L_n is bounded below) and the theorem is proved. □

61.31. Change of partial universe

0F4R We advise the reader to skip this section: here we show that cohomology of sheaves in the pro-étale topology is independent of the choice of partial universe. Namely, the functor g_* of Lemma 61.31.2 below is an embedding of small pro-étale topoi which does not change cohomology. For big pro-étale sites we have Lemmas 61.31.3 and 61.31.4 saying essentially the same thing.

But first, as promised in Section 61.12 we prove that the topology on a big pro-étale site $Sch_{pro\text{-}étale}$ is in some sense induced from the pro-étale topology on the category of all schemes.

098J **Lemma 61.31.1.** *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 61.12.7. Let $T \in \mathrm{Ob}(Sch_{pro\text{-}étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary pro-étale covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$.*

Proof. Namely, we first let $\{V_k \to T\}$ be a covering as in Lemma 61.13.3. Then the pro-étale coverings $\{T_i \times_T V_k \to V_k\}$ can be refined by a finite disjoint open covering $V_k = V_{k,1} \amalg \ldots \amalg V_{k,n_k}$, see Lemma 61.13.1. Then $\{V_{k,i} \to T\}$ is a covering of $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$. □

We first state and prove the comparison for the small pro-étale sites. Note that we are not claiming that the small pro-étale topos of a scheme is independent of the choice of partial universe; this isn't true in contrast with the case of the small étale topos (Étale Cohomology, Lemma 59.21.2).

098Y **Lemma 61.31.2.** *Let S be a scheme. Let $S_{pro\text{-}étale} \subset S'_{pro\text{-}étale}$ be two small pro-étale sites of S as constructed in Definition 61.12.8. Then the inclusion functor satisfies the assumptions of Sites, Lemma 7.21.8. Hence there exist morphisms of topoi*

$$Sh(S_{pro\text{-}étale}) \xrightarrow{g} Sh(S'_{pro\text{-}étale}) \xrightarrow{f} Sh(S_{pro\text{-}étale})$$

whose composition is isomorphic to the identity and with $f_* = g^{-1}$. Moreover,

(1) *for $\mathcal{F}' \in Ab(S'_{pro\text{-}étale})$ we have $H^p(S'_{pro\text{-}étale}, \mathcal{F}') = H^p(S_{pro\text{-}étale}, g^{-1}\mathcal{F}')$,*

(2) *for* $\mathcal{F} \in Ab(S_{pro\text{-}étale})$ *we have*

$$H^p(S_{pro\text{-}étale}, \mathcal{F}) = H^p(S'_{pro\text{-}étale}, g_*\mathcal{F}) = H^p(S'_{pro\text{-}étale}, f^{-1}\mathcal{F}).$$

Proof. The inclusion functor is fully faithful and continuous. We have seen that $S_{pro\text{-}étale}$ and $S'_{pro\text{-}étale}$ have fibre products and final objects and that our functor commutes with these (Lemma 61.12.10). It follows from Lemma 61.31.1 that the inclusion functor is cocontinuous. Hence the existence of f and g follows from Sites, Lemma 7.21.8. The equality in (1) is Cohomology on Sites, Lemma 21.7.2. Part (2) follows from (1) as $\mathcal{F} = g^{-1}g_*\mathcal{F} = g^{-1}f^{-1}\mathcal{F}$. □

Next, we prove a corresponding result for the big pro-étale topoi.

0F4S **Lemma 61.31.3.** *Suppose given big sites* $Sch_{pro\text{-}étale}$ *and* $Sch'_{pro\text{-}étale}$ *as in Definition 61.12.7. Assume that* $Sch_{pro\text{-}étale}$ *is contained in* $Sch'_{pro\text{-}étale}$*. The inclusion functor* $Sch_{pro\text{-}étale} \to Sch'_{pro\text{-}étale}$ *satisfies the assumptions of Sites, Lemma 7.21.8. There are morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_{pro\text{-}étale}) &\longrightarrow Sh(Sch'_{pro\text{-}étale}) \\ f : Sh(Sch'_{pro\text{-}étale}) &\longrightarrow Sh(Sch_{pro\text{-}étale}) \end{aligned}$$

such that $f \circ g \cong id$*. For any object* S *of* $Sch_{pro\text{-}étale}$ *the inclusion functor* $(Sch/S)_{pro\text{-}étale} \to (Sch'/S)_{pro\text{-}étale}$ *satisfies the assumptions of Sites, Lemma 7.21.8 also. Hence similarly we obtain morphisms*

$$\begin{aligned} g : Sh((Sch/S)_{pro\text{-}étale}) &\longrightarrow Sh((Sch'/S)_{pro\text{-}étale}) \\ f : Sh((Sch'/S)_{pro\text{-}étale}) &\longrightarrow Sh((Sch/S)_{pro\text{-}étale}) \end{aligned}$$

with $f \circ g \cong id$*.*

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 7.21.8 are immediate for the functors $Sch_{pro\text{-}étale} \to Sch'_{pro\text{-}étale}$ and $(Sch/S)_{pro\text{-}étale} \to (Sch'/S)_{pro\text{-}étale}$. Property (a) holds by Lemma 61.31.1. Property (d) holds because fibre products in the categories $Sch_{pro\text{-}étale}$, $Sch'_{pro\text{-}étale}$ exist and are compatible with fibre products in the category of schemes. □

0F4T **Lemma 61.31.4.** *Let* S *be a scheme. Let* $(Sch/S)_{pro\text{-}étale}$ *and* $(Sch'/S)_{pro\text{-}étale}$ *be two big pro-étale sites of* S *as in Definition 61.12.8. Assume that the first is contained in the second. In this case*

(1) *for any abelian sheaf* $\mathcal{F}'$ *defined on* $(Sch'/S)_{pro\text{-}étale}$ *and any object* U *of* $(Sch/S)_{pro\text{-}étale}$ *we have*

$$H^p(U, \mathcal{F}'|_{(Sch/S)_{pro\text{-}étale}}) = H^p(U, \mathcal{F}')$$

In words: the cohomology of $\mathcal{F}'$ *over* U *computed in the bigger site agrees with the cohomology of* $\mathcal{F}'$ *restricted to the smaller site over* U*.*

(2) *for any abelian sheaf* $\mathcal{F}$ *on* $(Sch/S)_{pro\text{-}étale}$ *there is an abelian sheaf* $\mathcal{F}'$ *on* $(Sch/S)'_{pro\text{-}étale}$ *whose restriction to* $(Sch/S)_{pro\text{-}étale}$ *is isomorphic to* $\mathcal{F}$*.*

Proof. By Lemma 61.31.3 the inclusion functor $(Sch/S)_{pro\text{-}étale} \to (Sch'/S)_{pro\text{-}étale}$ satisfies the assumptions of Sites, Lemma 7.21.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 21.7.2. □

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CHAPTER 62

Relative Cycles

0H4B

62.1. Introduction

0H4C A foundational reference is [**SV00**].

In this chapter we only define what are called the universally integral relative cycles in [**SV00**]. This choice makes the theory somewhat simpler to develop than in the original, but of course we also lose something.

Fix a morphism $X \to S$ of finite type between Noetherian schemes. A family α of r-cycles on fibres of X/S is simply a collection $\alpha = (\alpha_s)_{s \in S}$ where $\alpha_s \in Z_r(X_s)$. It is immediately clear how to base change $g^*\alpha$ of α along any morphism $g : S' \to S$. Then we say α is a *relative r-cycle on X/S* if α is compatible with specializations, i.e., for any morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring, we require the generic fibre of $g^*\alpha$ to specialize to the closed fibre of $g^*\alpha$. See Section 62.6.

62.2. Conventions and notation

0H4D Please consult the chapter on Chow Homology and Chern Classes for our conventions and notation regarding cycles on schemes locally of finite type over a fixed Noetherian base, see Chow Homology, Section 42.7 ff.

In particular, if X is locally of finite type over a field k, then $Z_r(X)$ denotes the group of cycles of dimension r, see Chow Homology, Example 42.7.2 and Section 42.8. Given an integral closed subscheme $Z \subset X$ with $\dim(Z) = r$ we have $[Z] \in Z_r(X)$ and if X is quasi-compact, then $Z_r(X)$ is free abelian on these classes.

62.3. Cycles relative to fields

0H4E Let k be a field. Let X be a locally algebraic scheme over k. Let $r \geq 0$ be an integer. In this setting we have the group $Z_r(X)$ of r-cycles on X, see Section 62.2.

Base change. For any field extension k'/k there is a base change map $Z_r(X) \to Z_r(X_{k'})$, see Chow Homology, Section 42.67. Namely, given an integral closed subscheme $Z \subset X$ of dimension r we send $[Z] \in Z_r(X)$ to the r-cycle $[Z_{k'}]_r \in Z_r(X_{k'})$ associated to the closed subscheme $Z_{k'} \subset X_{k'}$ (of course in general $Z_{k'}$ is neither irreducible nor reduced). The base change map $Z_r(X) \to Z_r(X_{k'})$ is always injective.

0H4F **Lemma 62.3.1.** *Let K/k be a field extension. Let Z be an integral locally algebraic scheme over k. The multiplicity m_{Z',Z_K} of an irreducible component $Z' \subset Z_K$ is 1 or a power of the characteristic of k.*

Proof. If the characteristic of k is zero, then k is perfect and the multiplicity is always 1 since X_K is reduced by Varieties, Lemma 33.6.4. Assume the characteristic of k is $p > 0$. Let L be the function field of Z. Since Z is locally algebraic over k, the field extension L/k is finitely generated. The ring $K \otimes_k L$ is Noetherian (Algebra, Lemma 10.31.8). Translated into algebra, we have to show that the length of the artinian local ring $(K \otimes_k L)_{\mathfrak{q}}$ is a power of p for every minimal prime ideal $\mathfrak{q}$.

Let L'/L be a finite purely inseparable extension, say of degree p^n. Then $K \otimes_k L \subset K \otimes_k L'$ is a finite free ring map of degree p^n which induces a homeomorphism on spectra and purely inseparable residue field extensions. Hence for every minimal prime $\mathfrak{q}$ as above there is a unique minimal prime $\mathfrak{q}' \subset K \otimes_k L'$ lying over it and

$$p^n \text{length}((K \otimes_k L)_{\mathfrak{q}}) = [\kappa(\mathfrak{q}') : \kappa(\mathfrak{q})]\text{length}((K \otimes_k L')_{\mathfrak{q}'})$$

by Algebra, Lemma 10.52.12 applied to $M = (K \otimes_k L')_{\mathfrak{q}'} \cong (K \otimes_k L)_{\mathfrak{q}}^{\oplus p^n}$. Since $[\kappa(\mathfrak{q}') : \kappa(\mathfrak{q})]$ is a power of p we conclude that it suffices to prove the statement for L' and $\mathfrak{q}'$.

By the previous paragraph and Algebra, Lemma 10.45.3 we may assume that we have a subfield $L/k'/k$ such that L/k' is separable and k'/k is finite purely inseparable. Then $K \otimes_k k'$ is an Artinian local ring. The argument of the preceding paragraph (applied to $L = k$ and $L' = k'$) shows that $\text{length}(K \otimes_k k')$ is a power of p. Since L/k' is the localization of a smooth k'-algebra (Algebra, Lemma 10.158.10). Hence $S = (K \otimes_k L)_{\mathfrak{q}}$ is the localization of a smooth $R = K \otimes_k k'$-algebra at a minimal prime. Thus $R \to S$ is a flat local homomorphism of Artinian local rings and $\mathfrak{m}_R S = \mathfrak{m}_S$. It follows from Algebra, Lemma 10.52.13 that $\text{length}(K \otimes_k k') = \text{length}(R) = \text{length}(S) = \text{length}((K \otimes_k L)_{\mathfrak{q}})$ and the proof is finished. □

0H4G **Lemma 62.3.2.** *Let k be a field of characteristic $p > 0$ with perfect closure k^{perf}. Let X be an algebraic scheme over k. Let $r \geq 0$ be an integer. The cokernel of the injective map $Z_r(X) \to Z_r(X_{k^{perf}})$ is a p-power torsion module (More on Algebra, Definition 15.89.1).*

Proof. Since X is quasi-compact, the abelian group $Z_r(X)$ is free with basis given by the integral closed subschemes of dimension r. Similarly for $Z_r(X_{k^{perf}})$. Since $X_{k^{perf}} \to X$ is a homeomorphism, it follows that $Z_r(X) \to Z_r(X_{k^{perf}})$ is injective with torsion cokernel. Every element in the cokernel is p-power torsion by Lemma 62.3.1. □

62.4. Specialization of cycles

0H4H Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. There is a specialization map

$$sp_{X/R} : Z_r(X_K) \longrightarrow Z_r(X_\kappa)$$

defined as follows. For an integral closed subscheme $Z \subset X_K$ of dimension r we denote $\overline{Z}$ the scheme theoretic image of $Z \to X$. Then we let $sp_{X/R}$ be the unique $\mathbf{Z}$-linear map such that

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r$$

We briefly discuss why this is well defined. First, observe that the morphism $X_K \to X$ is quasi-compact and hence the morphism $Z \to X$ is quasi-compact. Thus taking the scheme theoretic image of $Z \to X$ commutes with flat base change by

Morphisms, Lemma 29.25.16. In particular, base changing back to X_K we see that $Z = \overline{Z}_K$. Since Z is integral, of course $\overline{Z}$ is integral too and in fact is equal to the unique integral closed subscheme whose generic point is the (image of the) generic point of Z. It follows from Varieties, Lemma 33.19.2 that Z_κ is equidimensional of dimension r.

0H4I **Lemma 62.4.1.** *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over R. Assume $\dim(Supp(\mathcal{F}_K)) \leq r$. Then $\dim(Supp(\mathcal{F}_\kappa)) \leq r$ and*

$$sp_{X/R}([\mathcal{F}_K]_r) = [\mathcal{F}_\kappa]_r$$

Proof. The statement on dimension follows from More on Morphisms, Lemma 37.18.4. Let x be a generic point of an integral closed subscheme $Z \subset X_\kappa$ of dimension r. To finish the proof we will show that the coefficient of $[Z]$ in the left (L) and right hand side (R) of equality are the same.

Let $A = \mathcal{O}_{X,x}$ and $M = \mathcal{F}_x$. Observe that M is a finite A-module flat over R. Let $\pi \in R$ be a uniformizer so that $A/\pi A = \mathcal{O}_{X_\kappa,x}$. By Chow Homology, Lemma 42.3.2 we have

$$\sum\nolimits_i \text{length}_A(A/(\pi, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) = \text{length}_A(M/\pi M)$$

where the sum is over the minimal primes $\mathfrak{q}_i$ in the support of M. Since π is a nonzerodivisor on M we see that $\pi \notin \mathfrak{q}_i$ and hence these primes correspond to those generic points $y_i \in X_K$ of the support of $\mathcal{F}_K$ which specialize to our chosen $x \in X_\kappa$. Thus the left hand side is the coefficient of $[Z]$ in (L). Of course $\text{length}_A(M/\pi M)$ is the coefficient of $[Z]$ in (R). This finishes the proof. □

0H4J **Lemma 62.4.2.** *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $W \subset X$ be a closed subscheme flat over R. Assume $\dim(W_K) \leq r$. Then $\dim(W_\kappa) \leq r$ and*

$$sp_{X/R}([W_K]_r) = [W_\kappa]_r$$

Proof. Taking $\mathcal{F} = \mathcal{O}_W$ this is a special case of Lemma 62.4.1. See Chow Homology, Lemma 42.10.3. □

0H4K **Lemma 62.4.3.** *Let R'/R be an extension of discrete valuation rings inducing fraction field extension K'/K and residue field extension κ'/κ (More on Algebra, Definition 15.112.1). Let X be locally of finite type over R. Denote $X' = X_{R'}$. Then the diagram*

$$\begin{array}{ccc} Z_r(X'_{K'}) & \xrightarrow{sp_{X'/R'}} & Z_r(X'_{\kappa'}) \\ \uparrow & & \uparrow \\ Z_r(X_K) & \xrightarrow{sp_{X/R}} & Z_r(X_\kappa) \end{array}$$

commutes where $r \geq 0$ and the vertical arrows are base change maps.

Proof. Observe that $X'_{K'} = X_{K'} = X_K \times_{\text{Spec}(K)} \text{Spec}(K')$ and similarly for closed fibres, so that the vertical arrows indeed make sense (see Section 62.3). Now if $Z \subset X_K$ is an integral closed subscheme with scheme theoretic image $\overline{Z} \subset X$,

then we see that $Z_{K'} \subset X_{K'}$ is a closed subscheme with scheme theoretic image $\overline{Z}_{R'} \subset X_{R'}$. The base change of $[Z]$ is $[Z_{K'}]_r = [\overline{Z}_{K'}]_r$ by definition. We have

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r \quad \text{and} \quad sp_{X'/R'}([\overline{Z}_{K'}]_r) = [(\overline{Z}_{R'})_{\kappa'}]_r$$

by Lemma 62.4.1. Since $(\overline{Z}_{R'})_{\kappa'} = (\overline{Z}_\kappa)_{\kappa'}$ we conclude. □

0H4L **Lemma 62.4.4.** *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $f : X' \to X$ be a morphism which is locally of finite type, flat, and of relative dimension e. Then the diagram*

$$\begin{array}{ccc} Z_{r+e}(X'_K) & \xrightarrow{sp_{X'/R}} & Z_{r+e}(X'_\kappa) \\ \uparrow & & \uparrow \\ Z_r(X_K) & \xrightarrow{sp_{X/R}} & Z_r(X_\kappa) \end{array}$$

commutes where $r \geq 0$ and the vertical arrows are given by flat pullback.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f^{-1}(Z) = X' \times_X Z$. Since R is a valuation ring, Z is flat over R. Hence Z' is flat over R and $sp_{X'/R}([Z'_K]_{r+e}) = [Z'_\kappa]_{r+e}$ by Lemma 62.4.2. Since by Chow Homology, Lemma 42.14.4 we have $f_K^*[Z_K] = [Z'_K]_{r+e}$ and $f_\kappa^*[Z_\kappa]_r = [Z'_\kappa]_{r+e}$ we win. □

0H4M **Lemma 62.4.5.** *Let R be a discrete valuation ring with fraction field K and residue field κ. Let $f : X \to Y$ be a proper morphism of schemes locally of finite type over R. Then the diagram*

$$\begin{array}{ccc} Z_r(X_K) & \xrightarrow{sp_{X/R}} & Z_r(X_\kappa) \\ \downarrow & & \downarrow \\ Z_r(Y_K) & \xrightarrow{sp_{Y/R}} & Z_r(Y_\kappa) \end{array}$$

commutes where $r \geq 0$ and the vertical arrows are given by proper pushforward.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f(Z) \subset Y$. Then Z' is an integral closed subscheme of Y dominating R. Thus $sp_{Y/R}([Z'_K]) = [Z'_\kappa]_r$.

We can think of $[Z]$ as an element of $Z_{r+1}(X)$. By definition we have $f_*[Z] = 0$ if $\dim(Z') < r + 1$ and $f_*[Z] = d[Z']$ if $Z \to Z'$ is generically finite of degree d. Since proper pushforward commutes with flat pullback by $Y_K \to Y$ (Chow Homology, Lemma 42.15.1) we see that correspondingly $f_{K,*}[Z_K] = 0$ or $f_{K,*}[Z_K] = d[Z'_K]$. Let us apply Chow Homology, Lemma 42.29.8 to the commutative diagram

$$\begin{array}{ccc} X_\kappa & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ Y_\kappa & \xrightarrow{j} & Y \end{array}$$

We obtain that $f_{\kappa,*}[Z_\kappa]_r = 0$ or $f_{\kappa,*}[Z_\kappa] = d[Z'_\kappa]_r$ because clearly $i^*[Z] = [Z_k]_r$ and $j^*[Z'] = [Z'_\kappa]_r$. Putting everything together we conclude. □

62.5. Families of cycles on fibres

0H4N Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A *family α of r-cycles on fibres of X/S* is a family

$$\alpha = (\alpha_s)_{s \in S}$$

indexed by the points s of the scheme S where $\alpha_s \in Z_r(X_s)$ is an r cycle on the scheme theoretic fibre X_s of f at s. There are various constructions we can perform on families of r-cycles on fibres.

Base change. Let

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

be a catesian square of morphisms of schemes with f locally of finite type. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the *base change* $g^*\alpha$ of α to be the family

$$g^*\alpha = (\alpha'_{s'})_{s' \in S'}$$

where $\alpha'_{s'} \in Z_r(X'_{s'})$ is the base change of the cycle α_s with $s' = g(s)$ as in Section 62.3 via the identitification $X'_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ of scheme theoretic fibres.

Restriction. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $U \subset X$ and $V \subset S$ be open subschemes with $f(U) \subset V$. Given a family α of r-cycles on fibres of X/S we can define the *restriction* $\alpha|_U$ of α to be the family of r-cycles on fibres of U/V

$$\alpha|_U = (\alpha_s|_{U_s})_{s \in V}$$

of restrictions to scheme theoretic fibres.

Flat pullback. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $r, e \geq 0$ be integers. Let $f : X' \to X$ be a flat morphism, locally of finite type, and of relative dimension e. Given a family α of r-cycles on fibres of X/S we define the *flat pullback* $f^*\alpha$ of α to be the family of $(r+e)$-cycles on fibres

$$f^*\alpha = (f_s^*\alpha_s)_{s \in S}$$

where $f_s^*\alpha_s \in Z_{r+e}(X'_s)$ is the flat pullback of the cycle α_s in $Z_r(X_s)$ by the flat morphism $f_s : X'_s \to X_s$ of relative dimension e of scheme theoretic fibres.

Proper pushforward. Let

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes with X and Y locally of finite type over S and f proper. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the *proper pushforward* $f_*\alpha$ of α to be the family of r-cycles on fibres of Y/S by

$$f_*\alpha = (f_{s,*}\alpha_s)_{s \in S}$$

where $f_{s,*}\alpha_s \in Z_r(Y_s)$ is the proper pushforward of the cycle α_s in $Z_r(X_s)$ by the proper morphism $f_s : X_s \to Y_s$ of scheme theoretic fibres.

0H4P **Lemma 62.5.1.** *We have the following compatibilities between the operations above: (1) base change is functorial, (2) restriction is a combination of base change and (a special case of) flat pullback, (3) flat pullback commutes with base change, (4) flat pullback is functorial, (5) proper pushforward commutes with base change, (6) proper pushforward is functorial, and (7) proper pushforward commutes with flat pullback.*

Proof. Each of these compatibilities follows directly from the corresponding results proved in the chapter on Chow homology applied to the fibres over S of the schemes in question. We omit the precise statements and the detailed proofs. Here are some references. Part (1): Chow Homology, Lemma 42.67.9. Part (2): Obvious. Part (3): Chow Homology, Lemma 42.67.5. Part (4): Chow Homology, Lemma 42.14.3. Part (5): Chow Homology, Lemma 42.67.6. Part (6): Chow Homology, Lemma 42.12.2. Part (7): Chow Homology, Lemma 42.15.1. □

0H4Q **Example 62.5.2.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For $s \in S$ denote $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to X_s. Assume $\dim(\mathrm{Supp}(\mathcal{F}_s)) \leq r$ for all $s \in S$. Then we can associate to $\mathcal{F}$ the family $[\mathcal{F}/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[\mathcal{F}/X/S]_r = ([\mathcal{F}_s]_r)_{s \in S}$$

where $[\mathcal{F}_s]_r$ is given by Chow Homology, Definition 42.10.2.

0H4R **Lemma 62.5.3.** *The construction in Example 62.5.2 is compatible with base change, restriction, and flat pullback.*

Proof. See Chow Homology, Lemmas 42.67.3 and 42.14.4. □

0H4S **Example 62.5.4.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $Z \subset X$ be a closed subscheme. For $s \in S$ denote Z_s the inverse image of Z in X_s or equivalently the scheme theoretic fibre of Z at s viewed as a closed subscheme of X_s. Assume $\dim(Z_s) \leq r$ for all $s \in S$. Then we can associate to Z the family $[Z/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[Z/X/S]_r = ([Z_s]_r)_{s \in S}$$

where $[Z_s]_r$ is given by Chow Homology, Definition 42.9.2.

0H4T **Lemma 62.5.5.** *The construction in Example 62.5.4 is compatible with base change, restriction, and flat pullback.*

Proof. Taking $\mathcal{F} = (Z \to X)_*\mathcal{O}_Z$ this is a special case of Lemma 62.5.3. See Chow Homology, Lemma 42.10.3. □

0H4U **Remark 62.5.6** (Support)**.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. We define the *support* of α to be

$$\mathrm{Supp}(\alpha) = \bigcup\nolimits_{s \in S} \mathrm{Supp}(\alpha_s) \subset X$$

Here $\mathrm{Supp}(\alpha_s) \subset X_s$ is the support of the cycle α_s, see Chow Homology, Definition 42.8.3. The support $\mathrm{Supp}(\alpha)$ is rarely a closed subset of X.

0H4V **Lemma 62.5.7.** *Taking the support as in Remark 62.5.6 is compatible with base change, restriction, and flat pullback.*

Proof. Omitted. □

0H4W **Lemma 62.5.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a surjective morphism of schemes. Set $S'' = S' \times_S S'$ and let $f' : X' \to S'$ and $f'' : X'' \to S''$ be the base changes of f. Let $x \in X$ with $\mathrm{trdeg}_{\kappa(f(x))}(\kappa(x)) = r$.*

1. *There exists an $x' \in X'$ mapping to x with $\mathrm{trdeg}_{\kappa(f'(x'))}(\kappa(x')) = r$.*
2. *If $x'_1, x'_2 \in X'$ are both as in (1), then there exists an $x'' \in X''$ with $\mathrm{trdeg}_{\kappa(f''(x''))}(\kappa(x'')) = r$ and $pr_i(x'') = x'_i$.*

Proof. Part (1) is Morphisms, Lemma 29.28.3. Let x'_1, x'_2 be as in (2). Then since $X'' = X' \times_X X'$ we see that there exists a $x'' \in X''$ mapping to both x'_1 and x'_2 (see for example Descent, Lemma 35.13.1). Denote $s'' \in S''$, $s'_i \in S'$, and $s \in S$ the images of x'', x'_i, and x. Denote $k = \kappa(s)$ and let $Z \subset X_k$ be the integral closed subscheme whose generic point is x. Then x'_i is a generic point of an irreducible component of $Z_{\kappa(s'_i)}$. Let $Z'' \subset Z_{\kappa(s'')}$ be an irreducible component containing x''. Denote $\xi'' \in Z''$ the generic point. Since $\xi'' \rightsquigarrow x''$ we see that ξ'' must also map to x'_i under the two projections. On the other hand, we see that $\mathrm{trdeg}_{\kappa(s'')}(\kappa(\xi'')) = r$ because it is a generic point of an irreducible component of the base change of Z. □

0H4X **Lemma 62.5.9.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a morphism of schemes and $X' = S' \times_S X$. Assume that for every $s \in S$ there exists a point $s' \in S'$ with $g(s') = s$ and such that $\kappa(s')/\kappa(s)$ is a separable extension of fields. Then*

1. *For families α_1 and α_2 of r-cycles on fibres of X/S if $g^*\alpha_1 = g^*\alpha_2$, then $\alpha_1 = \alpha_2$.*
2. *Given a family α' of r-cycles on fibres of X'/S' if $pr_1^*\alpha' = pr_2^*\alpha'$ as families of r-cycles on fibres of $(S' \times_S S') \times_S X/(S' \times_S S')$, then there is a unique family α of r-cycles on fibres of X/S such that $g^*\alpha = \alpha'$.*

Proof. Part (1) follows from the injectivity of the base change map discussed in Section 62.3. (This argument works as long as $S' \to S$ is surjective.)

Let α' be as in (2). Denote $\alpha'' = \mathrm{pr}_1^*\alpha' = \mathrm{pr}_2^*\alpha'$ the common value.

Let $(X/S)^{(r)}$ be the set of $x \in X$ with $\mathrm{trdeg}_{\kappa(f(x))}(\kappa(x)) = r$ and similarly define $(X'/S')^{(r)}$ and $(X''/S'')^{(r)}$ Taking coefficients, we may think of α' and α'' as functions $\alpha' : (X'/S')^{(r)} \to \mathbf{Z}$ and $\alpha'' : (X''/S'')^{(r)} \to \mathbf{Z}$. Given a function

$$\varphi : (X/S)^{(r)} \to \mathbf{Z}$$

we define $g^*\varphi : (X'/S')^{(r)} \to \mathbf{Z}$ by analogy with our base change operation. Namely, say $x' \in (X'/S')^{(r)}$ maps to $x \in X$, $s' \in S'$, and $s \in Z$. Denote $Z' \subset X'_{s'}$ and $Z \subset X_s$ the integral closed subschemes with generic points x' and x. Note that $\dim(Z') = r$. If $\dim(Z) < r$, then we set $(g^*\varphi)(x') = 0$. If $\dim(Z) = r$, then Z'

is an irreducible component of $Z_{s'}$ and hence has a multiplicity $m_{Z',Z_{s'}}$. Call this $m(x', g)$. Then we define

$$(g^*\varphi)(x') = m(x', g)\varphi(x)$$

Note that the coefficients $m(x', g)$ are always positive integers (see for example Lemma 62.3.1). We similarly have base change maps

$$\mathrm{pr}_1^*, \mathrm{pr}_2^* : \mathrm{Map}((X'/S')^{(r)}, \mathbf{Z}) \longrightarrow \mathrm{Map}((X''/S'')^{(r)}, \mathbf{Z})$$

It follows from the associativity of base change that we have $\mathrm{pr}_1^* \circ g^* = \mathrm{pr}_2^* \circ g^*$ (small detail omitted). To be explicitly, in terms of the maps of sets this equality just means that for $x'' \in (X''/S'')^{(r)}$ we have

$$m(x'', \mathrm{pr}_1)m(\mathrm{pr}_1(x''), g) = m(x'', \mathrm{pr}_2)m(\mathrm{pr}_2(x''), g)$$

provided that $\mathrm{pr}_1(x'')$ and $\mathrm{pr}_2(x'')$ are in $(X''/S'')^{(r)}$. By Lemma 62.5.8 and an elementary argument[1] using the previous displayed equation, it follows that there exists a unique map

$$\alpha : (X/S)^{(r)} \to \mathbf{Q}$$

such that $g^*\alpha = \alpha'$. To finish the proof it suffices to show that α has integer values (small detail omitted: one needs to see that α determines a locally finite sum on each fibre which follows from the corresponding fact for α'). Given any $x \in (X/S)^{(r)}$ with image $s \in S$ we can pick a point $s' \in S'$ such that $\kappa(s')/\kappa(s)$ is separable. Then we may choose $x' \in (X'/S')^{(r)}$ mapping to s and x and we see that $m(x', g) = 1$ because $Z_{s'}$ is reduced in this case. Whence $\alpha(x) = \alpha'(x')$ is an integer. □

0H4Y **Lemma 62.5.10.** *Let $g : S' \to S$ be a bijective morphism of schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'.*

Proof. Omitted. □

62.6. Relative cycles

0H4Z Here is the definition we will work with; see Section 62.15 for a comparison with the definitions in [**SV00**].

0H50 **Definition 62.6.1.** Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A *relative r-cycle on X/S* is a family α of r-cycles on fibres of X/S such that for every morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring we have

$$sp_{X'/S'}(\alpha_\eta) = \alpha_0$$

where $sp_{X'/S'}$ is as in Section 62.4 and α_η (resp. α_0) is the value of the base change $g^*\alpha$ of α at the generic (resp. closed) point of S'. The group of all relative r-cycles on X/S is denoted $z(X/S, r)$.

[1] Given $x \in (X/S)^{(r)}$ pick $x' \in (X'/S')^{(r)}$ mapping to x and set $\alpha(x) = \alpha'(x')/m(x', g)$. This is well defined by the formula and the lemma.

0H51 **Lemma 62.6.2.** *Let α be a relative r-cycle on X/S as in Definition 62.6.1. Then any restriction, base change, flat pullback, or proper pushforward of α is a relative r-cycle.*

Proof. For flat pullback use Lemma 62.4.4. Restriction is a special case of flat pullback. To see it holds for base change use that base change is transitive. For proper pushforward use Lemma 62.4.5. □

0H52 **Lemma 62.6.3.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. Let $\{g_i : S_i \to S\}$ be a h covering (More on Flatness, Definition 38.34.2). Then α is a relative r-cycle if and only if each base change $g_i^*\alpha$ is a relative r-cycle.*

Proof. If α is a relative r-cycle, then each base change $g_i^*\alpha$ is a relative r-cycle by Lemma 62.6.2. Assume each $g_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ and using that the base change of a h covering is a h covering (More on Flatness, Lemma 38.34.9) we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Since a h covering is a V covering (by definition), there is an i and a specialization $s' \rightsquigarrow s$ of points of S_i with $g_i(s') = \eta$ and $g_i(s) = 0$, see Topologies, Lemma 34.10.13. By Properties, Lemma 28.5.10 we can find a morphism $h : S' \to S_i$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to s' and maps the closed point $0'$ to s. Denote $\alpha' = h^*g_i^*\alpha$. By assumption we have $sp_{X'/S'}(\alpha'_{\eta'}) = \alpha'_{0'}$. Since $g = g_i \circ h : S' \to S$ is the morphism of schemes induced by an extension of discrete valuation rings we conclude that $sp_{X/S}$ and $sp_{X'/S'}$ are compatible with base change maps on the fibres, see Lemma 62.4.3. We conclude that $sp_{X/S}(\alpha_\eta) = \alpha_0$ because the base change map $Z_r(X_0) \to Z_r(X'_{0'})$ is injective as discussed in Section 62.3. □

0H53 **Lemma 62.6.4.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a family of r-cycles on fibres of X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is a relative r-cycle if and only if each flat pullback $f_i^*\alpha$ is a relative r-cycle.*

Proof. If α is a relative r-cycle, then each pull back $f_i^*\alpha$ is a relative r-cycle by Lemma 62.6.2. Assume each $f_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Denote $f_{i,0} : X_{i,0} \to X_0$ the base change of f_i to the closed point of S. Similarly for $f_{i,\eta}$. Observe that

$$f_{i,0}^* sp_{X/S}(\alpha_\eta) = sp_{X_i/S}(f_{i,\eta}^*\alpha_\eta) = f_{i,0}^*\alpha_0$$

Namely, the first equality holds by Lemma 62.4.4 and the second by assumption. Since the family of maps $f_{i,0}^* : Z_r(X_0) \to Z_r(X_{i,0})$ is jointly injective (due to the fact that $f_{i,0}$ is jointly surjective), we conclude what we want. □

0H54 **Lemma 62.6.5.** *Let S be a locally Noetherian scheme. Let $i : X \to Y$ be a closed immersion of schemes locally of finite type over S. Let $r \geq 0$. Let α be a family of r-cycles on fibres of X/S. Then α is a relative r-cycle on X/S if and only if $i_*\alpha$ is a relative r-cycle on Y/S.*

Proof. Since base change commutes with i_* (Lemma 62.5.1) it suffices to prove the following: if S is the spectrum of a discrete valuation ring with generic point η and closed point 0, then $sp_{X/S}(\alpha_\eta) = \alpha_0$ if and only if $sp_{Y/S}(i_{\eta,*}\alpha_\eta) = i_{0,*}\alpha_0$. This is true because $i_{0,*} : Z_r(X_0) \to Z_r(Y_0)$ is injective and because $i_{0,*}sp_{X/S}(\alpha_\eta) = sp_{Y/S}(i_{\eta,*}\alpha_\eta)$ by Lemma 62.4.5. □

The following lemma will be strengthened in Lemma 62.6.12.

0H55 **Lemma 62.6.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f locally of finite type. Let $r \geq 0$. Let α and β be relative r-cycles on X/S. The following are equivalent*

(1) *$\alpha = \beta$, and*
(2) *$\alpha_\eta = \beta_\eta$ for any generic point $\eta \in S$ of an irreducible component of S.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). For every $s \in S$ we can find an η as in (2) which specializes to s. By Properties, Lemma 28.5.10 we can find a morphism $g : S' \to S$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to η and maps the closed point 0 to s. Then α_s and β_s are elements of $Z_r(X_s)$ which base change to the same element of $Z_r(X_{0'})$, namely $sp_{X_{S'}/S'}(\alpha_{\eta'})$ where $\alpha_{\eta'}$ is the base change of α_η. Since the base change map $Z_r(X_s) \to Z_r(X_{0'})$ is injective as discussed in Section 62.3 we conclude $\alpha_s = \beta_s$. □

0H56 **Lemma 62.6.7.** *In the situation of Example 62.5.2 assume S is locally Noetherian and $\mathcal{F}$ is flat over S in dimensions $\geq r$ (More on Flatness, Definition 38.20.10). Then $[\mathcal{F}/X/S]_r$ is a relative r-cycle on X/S.*

Proof. By More on Flatness, Lemma 38.20.9 the hypothesis on $\mathcal{F}$ is preserved by any base change. Also, formation of $[\mathcal{F}/X/S]_r$ is compatible with any base change by Lemma 62.5.3. Since the condition of being compatible with specializations is checked after base change to the spectrum of a discrete valuation ring, this reduces us to the case where S is the spectrum of a valuation ring. In this case the set $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X by More on Flatness, Lemma 38.13.11. Since the complement of U in X has fibres of dimension $< r$ over S by assumption, we see that restriction along the inclusion $U \subset X$ induces an isomorphism on the groups of r-cycles on fibres after any base change, compatible with specialization maps and with formation of the relative cycle associated to $\mathcal{F}$. Thus it suffices to show compatibility with specializations for $[\mathcal{F}|_U/U/S]_r$. Since $\mathcal{F}|_U$ is flat over S, this follows from Lemma 62.4.1 and the definitions. □

0H57 **Lemma 62.6.8.** *In the situation of Example 62.5.4 assume S is locally Noetherian and Z is flat over S in dimensions $\geq r$. Then $[Z/X/S]_r$ is a relative r-cycle on X/S.*

Proof. The assumption means that $\mathcal{O}_Z$ is flat over S in dimensions $\geq r$. Thus applying Lemma 62.6.7 with $\mathcal{F} = (Z \to X)_*\mathcal{O}_Z$ we conclude. □

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism which is of finite type. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$

such that $Z \to S$ is flat and of relative dimension $\leq r$. By Lemma 62.6.8 for each $Z \in Hilb(X/S, r)$ we have an element $[Z/X/S]_r \in z(X/S, r)$. Thus we obtain a group homomorphism

0H58 (62.6.8.1) $$\text{free abelian group on } Hilb(X/S, r) \longrightarrow z(X/S, r)$$

sending $\sum n_i[Z_i]$ to $\sum n_i[Z_i/X/S]_r$. A key feature of relative r-cycles is that they are locally (on X and S in suitable topologies) in the image of this map.

0H59 **Lemma 62.6.9.** *Let $f : X \to S$ be a finite type morphism of schemes with S Noetherian. Let $r \geq 0$. Let α be a relative r-cycle on X/S. Then there is a proper, completely decomposed (More on Morphisms, Definition 37.78.1) morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (62.6.8.1).*

Proof. By Noetherian induction, we may assume the result holds for the pullback of α by any closed immersion $g : S' \to S$ which is not an isomorphism.

Let $S_1 \subset S$ be an irreducible component (viewed as an integral closed subscheme). Let $S_2 \subset S$ be the closure of the complement of S' (viewed as a reduced closed subscheme). If $S_2 \neq \emptyset$, then the result holds for the pullback of α by $S_1 \to S$ and $S_2 \to S$. If $g_1 : S_1' \to S_1$ and $g_2 : S_2' \to S_2$ are the corresponding completely decomposed proper morphisms, then $S' = S_1' \amalg S_2' \to S$ is a completely decomposed proper morphism and we see the result holds for S[2] . Thus we may assume $S' \to S$ is bijective and we reduce to the case described in the next paragraph.

Assume S is integral. Let $\eta \in S$ be the generic point and let $K = \kappa(\eta)$ be the function field of S. Then α_η is an r-cycle on X_K. Write $\alpha_\eta = \sum n_i[Y_i]$. Taking the closure of Y_i we obtain integral closed subschemes $Z_i \subset X$ whose base change to η is Y_i. By generic flatness (for example Morphisms, Proposition 29.27.1), we see that Z_i is flat over a nonempty open U of S for each i. Applying More on Flatness, Lemma 38.31.1 we can find a U-admissible blowing up $g : S' \to S$ such that the strict transform $Z_i' \subset X_{S'}$ of Z_i is flat over S'. Then $\beta = \sum n_i[Z_i'/X_{S'}/S']_r$ is in the image of (62.6.8.1) and $\beta = g^*\alpha$ by Lemma 62.6.6.

However, this does not finish the proof as $S' \to S$ may not be completely decomposed. This is easily fixed: denoting $T \subset S$ the complement of U (viewed as a closed subscheme), by Noetherian induction we can find a completely decomposed proper morphism $T' \to T$ such that $(T' \to S)^*\alpha$ is in the image of (62.6.8.1). Then $S' \amalg T' \to S$ does the job. □

0H5A **Lemma 62.6.10.** *Let $f : X \to S$ be a finite type morphism of schemes with S the spectrum of a discrete valuation ring. Let $r \geq 0$. Then (62.6.8.1) is surjective.*

Proof. This of course follows from Lemma 62.6.9 but we can also see it directly as follows. Say α is a relative r-cycle on X/S. Write $\alpha_\eta = \sum n_i[Z_i]$ (the sum is finite). Denote $\overline{Z}_i \subset X$ the closure of Z_i as in Section 62.4. Then $\alpha = \sum n_i[\overline{Z}_i/X/S]$. □

0H5B **Lemma 62.6.11.** *Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Assume S locally Noetherian and f smooth of relative dimension r. Let $\alpha \in z(X/S, r)$. Then the support of α is open and closed in X (see proof for a more precise result).*

[2]Namely, any closed subscheme of $S_1' \times_S X$ flat and of relative dimension $\leq r$ over S_1' may be viewed as a closed subscheme of $S' \times_S X$ flat and of relative dimension $\leq r$ over S'.

Proof. Let $x \in X$ with image $s \in S$. Since f is smooth, there is a unique irreducible component $Z(x)$ of X_s which contains x. Then $\dim(Z(x)) = r$. Let n_x be the coefficient of $Z(x)$ in the cycle α_s. We will show the function $x \mapsto n_x$ is locally constant on X.

Let $g : S' \to S$ be a morphism of locally Noetherian schemes. Let X' be the base change of X and let $\alpha' = g^*\alpha$ be the base change of α. Let $x' \in X'$ map to $s' \in S'$, $x \in X$, and $s \in S$. We claim $n_{x'} = n_x$. Namely, since $Z(x)$ is smooth over $\kappa(s)$ we see that $Z(x) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ is reduced. Since $Z(x')$ is an irreducible component of this scheme, we see that the coefficient $n_{x'}$ of $Z(x')$ in $\alpha'_{s'}$ is the same as the coefficient n_x of $Z(x)$ in α_s by the definition of base change in Section 62.3 thereby proving the claim.

Since X is locally Noetherian, to show that $x \mapsto n_x$ is locally constant, it suffices to show: if $x' \rightsquigarrow x$ is a specialization in X, then $n_{x'} = n_x$. Choose a morphism $S' \to X$ where S' is the spectrum of a discrete valuation ring mapping the generic point η to x' and the closed point 0 to x. See Properties, Lemma 28.5.10. Then the base change $X' \to S'$ of f by $S' \to S$ has a section $\sigma : S' \to X'$ such that $\sigma(\eta) \rightsquigarrow \sigma(0)$ is a specialization of points of X' mapping to $x' \rightsquigarrow x$ in X. Thus we reduce to the claim in the next paragraph.

Let S be the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have a section $\sigma : S \to X$. Claim: $n_{\sigma(\eta)} = n_{\sigma(0)}$. By the discussion in More on Morphisms, Section 37.29 and especially More on Morphisms, Lemma 37.29.6 after replacing X by an open subscheme, we may assume the fibres of $X \to S$ are connected. Since these fibres are smooth, they are irreducible. Then we see that $\alpha_\eta = n[X_\eta]$ with $n = n_{\sigma(\eta)}$ and the relation $sp_{X/S}(\alpha_\eta) = \alpha_0$ implies $\alpha_0 = n[X_0]$, i.e., $n_{\sigma(0)} = n$ as desired. □

0H5C **Lemma 62.6.12.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ and $\alpha, \beta \in z(X/S, r)$. The set $E = \{s \in S : \alpha_s = \beta_s\}$ is closed in S.*

Proof. The question is local on S, thus we may assume S is affine. Let $X = \bigcup U_i$ be an affine open covering. Let $E_i = \{s \in S : \alpha_s|_{U_{i,s}} = \beta_s|_{U_{i,s}}\}$. Then $E = \bigcap E_i$. Hence it suffices to prove the lemma for $U_i \to S$ and the restriction of α and β to U_i. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact. Set $\gamma = \alpha - \beta$. Then $E = \{s \in S : \gamma_s = 0\}$. By Lemma 62.6.8 there exists a jointly surjective finite family of proper morphisms $\{g_i : S_i \to S\}$ such that $g_i^*\gamma$ is in the image of (62.6.8.1). Observe that $E_i = g_i^{-1}(E)$ is the set of point $t \in S_i$ such that $(g_i^*\gamma)_t = 0$. If E_i is closed for all i, then $E = \bigcup g_i(E_i)$ is closed as well. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact and $\gamma = \sum n_i[Z_i/X/S]_r$ for a finite number of closed subschemes $Z_i \subset X$ flat and of relative dimension $\leq r$ over S. Set $X' = \bigcup Z_i$ (scheme theoretic union). Then $i : X' \to X$ is a closed immersion and X' has relative dimension $\leq r$ over S. Also $\gamma = i_*\gamma'$ where $\gamma' = \sum n_i[Z_i/X'/S]_r$. Since clearly $E = E' = \{s \in S : \gamma'_s = 0\}$ we reduce to the case discussed in the next paragraph.

Assume X has relative dimension $\leq r$ over S. Let $s \in S$, $s \notin E$. We will show that there exists an open neighbourhood $V \subset S$ of s such that $E \cap V$ is empty. The assumption $s \notin E$ means there exists an integral closed subscheme $Z \subset X_s$ of dimension r such that the coefficient n of $[Z]$ in γ_s is nonzero. Let $x \in Z$ be the generic point. Since $\dim(Z) = r$ we see that x is a generic point of an irreducible component (namely Z) of X_s. Thus after replacing X by an open neighbourhood of x, we may assume that Z is the only irreducible component of X_s. In particular, we have $\gamma_s = n[Z]$.

At this point we apply More on Morphisms, Lemma 37.47.1 and we obtain a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & & \downarrow{\scriptstyle \pi} \\ & & Y \\ & & \downarrow{\scriptstyle h} \\ S & = & S \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ & & \downarrow \\ & & y \\ & & \downarrow \\ s & = & s \end{array}$$

with all the properties listed there. Let $\gamma' = g^*\gamma$ be the flat pullback. Note that $E \subset E' = \{s \in S : \gamma'_s = 0\}$ and that $s \notin E'$ because the coefficient of Z' in γ'_s is nonzero, where $Z' \subset X'_s$ is the closure of x'. Similarly, set $\gamma'' = \pi_*\gamma'$. Then we have $E' \subset E'' = \{s \in S : \gamma''_s = 0\}$ and $s \notin E''$ because the coefficient of Z'' in γ''_s is nonzero, where $Z'' \subset Y_s$ is the closure of y. By Lemma 62.6.11 and openness of $Y \to S$ we see that an open neighbourhood of s is disjoint from E'' and the proof is complete. □

0H5D **Lemma 62.6.13.** *Let $S = \lim_{i \in I} S_i$ be the limit of a directed inverse system of Noetherian schemes with affine transition morphisms. Let $0 \in I$ and let $X_0 \to S_0$ be a finite type morphism of schemes. For $i \geq 0$ set $X_i = S_i \times_{S_0} X_0$ and set $X = S \times_{S_0} X_0$. If S is Noetherian too, then*

$$z(X/S, r) = \operatorname{colim}_{i \geq 0} z(X_i/S_i, r)$$

where the transition maps are given by base change of relative r-cycles.

Proof. Suppose that $i \geq 0$ and $\alpha_i, \beta_i \in z(X_i/S_i, r)$ map to the same element of $z(X/S, r)$. Then $S \to S_i$ maps into the closed subset $E \subset S_i$ of Lemma 62.6.12. Hence for some $j \geq i$ the morphism $S_j \to S_i$ maps into E, see Limits, Lemma 32.4.10. It follows that the base change of α_i and β_i to S_j agree. Thus the map is injective.

Let $\alpha \in z(X/S, r)$. Applying Lemma 62.6.9 a completely decomposed proper morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (62.6.8.1). Set $X' = S' \times_S X$. We write $g^*\alpha = \sum n_a[Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat and of relative dimension $\leq r$ over S'.

Now we bring the machinery of Limits, Section 32.10 ff to bear. We can find an $i \geq 0$ such that there exist

(1) a completely decomposed proper morphism $g_i : S'_i \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X'_i = S'_i \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X'_i$ flat and of relative dimension $\leq r$ over S'_i whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 32.10.1, 32.8.5, 32.8.7, 32.13.1, and 32.18.1 and More on Morphisms, Lemma 37.78.5. Consider $\alpha'_i = \sum n_a[Z_{ai}/X'_i/S'_i]_r \in z(X'_i/S'_i, r)$. The image of α'_i in $z(X'/S', r)$ agrees with the base change $g^*\alpha$ by construction.

Set $S''_i = S'_i \times_{S_i} S'_i$ and $X''_i = S''_i \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\mathrm{pr}_1, \mathrm{pr}_2 : S'' \to S'$ and $\mathrm{pr}_1, \mathrm{pr}_2 : S''_i \to S'_i$ the projections. The two base changes $\mathrm{pr}_1^*\alpha'_i$ and $\mathrm{pr}_1^*\alpha'_i$ map to the same element of $z(X''/S'', r)$ because $\mathrm{pr}_1^*g^*\alpha = \mathrm{pr}_1^*g^*\alpha$. Hence after increasing i we may assume that $\mathrm{pr}_1^*\alpha'_i = \mathrm{pr}_1^*\alpha'_i$ by the first paragraph of the proof. By Lemma 62.5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha'_i$ (this uses that $S'_i \to S_i$ is completely decomposed). By Lemma 62.6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 62.5.9 implies that the image of α_i in $z(X/S, r)$ is α and the proof is complete. □

0H5E **Lemma 62.6.14.** *Let S be a locally Noetherian scheme. Let $i : X \to X'$ be a thickening of schemes locally of finite type over S. Let $r \geq 0$. Then $i_* : z(X/S, r) \to z(X'/S, r)$ is a bijection.*

Proof. Since $i_s : X_s \to X'_s$ is a thickening it is clear that i_* induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of X'/S. Also, given a family α of r-cycles on the fibres of X/S $\alpha \in z(X/S, r) \Leftrightarrow i_*\alpha \in z(X'/S, r)$ by Lemma 62.6.5. The lemma follows. □

0H5F **Lemma 62.6.15.** *Let S be a locally Noetherian scheme. Let X be a scheme locally of finite type over S. Let $r \geq 0$. Let $U \subset X$ be an open such that $X \setminus U$ has relative dimension $< r$ over S, i.e., $\dim(X_s \setminus U_s) < r$ for all $s \in S$. Then restriction defines a bijection $z(X/S, r) \to z(U/S, r)$.*

Proof. Since $Z_r(X_s) \to Z_r(U_s)$ is a bijection by the dimension assumption, we see that restriction induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of U/S. These restriction maps $Z_r(X_s) \to Z_r(U_s)$ are compatible with base change and with specializations, see Lemma 62.5.1 and 62.4.4. The lemma follows easily from this; details omitted. □

0H5G **Lemma 62.6.16.** *Let $g : S' \to S$ be a universal homeomorphism of locally Noetherian schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection $z(X/S, r) \to z(X'/S', r)$.*

Proof. By Lemma 62.5.10 we have a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'. Say α is a families of r-cycles on fibres of X/S and $\alpha' = g^*\alpha$ is the base change. If R is a discrete valuation ring, then any morphism $h : \mathrm{Spec}(R) \to S$ factors as $g \circ h'$ for some unique morphism $h' : \mathrm{Spec}(R) \to S'$. Namely, the morphism $S' \times_S \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is a universal homomorphism inducing bijections on residue fields, and hence has a section (for example because R is a seminormal ring, see Morphisms, Section 29.47). Thus the condition that α is compatible with specializations (i.e., is a relative r-cycle) is equivalent to the condition that α' is compatible with specializations. □

62.7. Equidimensional relative cycles

0H5H Here is the definition.

0H5I **Definition 62.7.1.** Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S *equidimensional* if the support of α (Remark 62.5.6) is contained in a closed subset $W \subset X$ whose relative dimension over S is $\leq r$. The group of all equidimensional relative r-cycles on X/S is denoted $z_{equi}(X/S, r)$.

0H5J **Example 62.7.2.** There exist relative r-cycles which are not equidimensional. Namely, let k be a field and let $X = \mathrm{Spec}(k[x, y, t])$ over $S = \mathrm{Spec}(k[x, y])$. Let s be a point of S and denote $a, b \in \kappa(s)$ the images of x and y. Consider the family α of 0-cycles on X/S defined by [**SV00**, Example 3.1.9]

(1) $\alpha_s = 0$ if $b = 0$ and otherwise
(2) $\alpha_s = [p] - [q]$ where p, resp. q is the $\kappa(s)$-rational point of $\mathrm{Spec}(\kappa(s)[t])$ with $t = a/b$, resp. $t = (a + b^2)/b$.

We leave it to the reader to show that this is compatible with specializations; the idea is that a/b and $(a + b^2)/b = a/b + b$ limit to the same point in $\mathbf{P}^1$ over the residue field of any valuation v on $\kappa(s)$ with $v(b) > 0$. On the other hand, the closure of the support of α contains the whole fibre over $(0, 0)$.

0H5K **Lemma 62.7.3.** *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is equidimensional, then any restriction, base change, or flat pullback of α is equidimensional.*

Proof. Omitted. □

0H5L **Lemma 62.7.4.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is equidimensional we may work Zariski locally on X and S.*

Proof. Namely, the condition that α is equidimensional just means that the closure of the support of α has relative dimension $\leq r$ over S. Since taking closures commutes with restriction to opens, the lemma follows (small detail omitted). □

0H5M **Lemma 62.7.5.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be an fppf covering. Then α is equidimensional if and only if each base change $g_i^*\alpha$ is equidimensional.*

Proof. If α is equidimensional, then each $g_i^*\alpha$ is too by Lemma 62.7.3. Assume each $g_i^*\alpha$ is equidimensional. Denote W the closure of $\mathrm{Supp}(\alpha)$ in X. Since $g_i : S_i \to S$ is universally open (being flat and locally of finite presentation), so is the morphism $f_i : X_i = S_i \times_S X \to X$. Denote $\alpha_i = g_i^*\alpha$. We have $\mathrm{Supp}(\alpha_i) = f_i^{-1}(\mathrm{Supp}(\alpha))$ by Lemma 62.5.7. Since f_i is open, we see that $W_i = f_i^{-1}(W)$ is the closure of $\mathrm{Supp}(\alpha_i)$. Hence by assumption the morphism $W_i \to S_i$ has relative dimension $\leq r$. By Morphisms, Lemma 29.28.3 (and the fact that the morphisms $S_i \to S$ are jointly surjective) we conclude that $W \to S$ has relative dimension $\leq r$. □

0H5N **Lemma 62.7.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is equidimensional if and only if each flat pullback $f_i^*\alpha$ is equidimensional.*

Proof. Omitted. Hint: As in the proof of Lemma 62.7.5 one shows that the inverse image by f_i of the closure W of the support of α is the closure W_i of the support of $f_i^*\alpha$. Then $W \to S$ has relative dimension $\leq r$ holds if $W_i \to S$ has relative dimension $\leq r + e$ for all i. □

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Then we have $z(X/S, 0) = z_{equi}(X/S, 0)$ and $z(X/S, r) = 0$ for $r > 0$. Given $\alpha \in z(X/S, 0)$ let us define a map

$$w_\alpha : X \longrightarrow \mathbf{Z}, \quad x \mapsto \alpha(x)[\kappa(x) : \kappa(s)]_i \quad \text{where } s = f(x)$$

Here $\alpha(x)$ denotes the coefficient of x in the 0-cycle α_s on the fibre X_s and $[K : k]_i$ denotes the inseparable degree of a finite field extension. The following lemma shows that this map is a weighting of f (More on Morphisms, Definition 37.75.2) and that every weighting is of this form up to taking a multiple.

0H5P **Lemma 62.7.7.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Let $\alpha \in z(X/S, 0)$. The map $w_\alpha : X \to \mathbf{Z}$ constructed above is a weighting. Conversely, if X is quasi-compact, then given a weighting $w : X \to \mathbf{Z}$ there exists an integer $n > 0$ such that $nw = w_\alpha$ for some $\alpha \in z(X/S, 0)$. Finally, the integer n may be chosen to be a power of the prime p if S is a scheme over $\mathbf{F}_p$.*

Proof. First, let us show that the construction is compatible with base change: if $g : S' \to S$ is a morphism of locally Noetherian schemes, then $w_{g^*\alpha} = w_\alpha \circ g'$ where $g' : X' \to X$ is the projection $X' = S' \times_S X \to X$. Namely, let $x' \in X'$ with images s', s, x in S', S, X. Then the coefficient of $[x']$ in the base change of $[x]$ by $\kappa(s')/\kappa(s)$ is the length of the local ring $(\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q}$. Here $\mathfrak{q}$ is the prime ideal corresponding to x'. Thus compatibility with base change follows if

$$[\kappa(x) : \kappa(s)]_i = \text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})[\kappa(x') : \kappa(s')]_i$$

Let $k/\kappa(s')$ be an algebraically closure. Choose a prime $\mathfrak{p} \subset k \otimes_{\kappa(s)} \kappa(x)$ lying over $\mathfrak{q}$. Suppose we can show that

$$[\kappa(x) : \kappa(s)]_i = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p}) \quad \text{and} \quad [\kappa(x') : \kappa(s')]_i = \text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p})$$

Then we win because

$$\text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})\text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p}) = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p})$$

by Algebra, Lemma 10.52.13 and flatness of $\kappa(s') \otimes_{\kappa(s)} \kappa(x) \to k \otimes_{\kappa(s)} \kappa(x)$. To show the two equalities, it suffices to prove the first. Let $\kappa(x)/\kappa/\kappa(s)$ be the subfield constructed in Fields, Lemma 9.14.6. Then we see that

$$k \otimes_{\kappa(s)} \kappa(x) = \prod\nolimits_{\sigma:\kappa \to k} k \otimes_{\sigma,\kappa} \kappa(x)$$

and each of the factors is local of degree $[\kappa(x) : \kappa] = [\kappa(x) : \kappa(s)]_i$ as desired.

Let $\alpha \in z(X/S, 0)$ and choose a diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xleftarrow{g} & V \end{array}$$

as in More on Morphisms, Definition 37.75.2. Denote $\beta \in z(U/V, 0)$ the restriction of the base change $g^*\alpha$. By the compatibility with base change above we have $w_\beta = w_\alpha \circ h$ and it suffices to show that $\int_\pi w_\beta$ is locally constant on V. Next, note that

$$\begin{aligned} \left(\int_\pi w_\beta\right)(v) &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)]_i[\kappa(u) : \kappa(v)]_s \\ &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)] \end{aligned}$$

This last expression is the coefficient of v in $\pi_*\beta \in z(V/V, 0)$. By Lemma 62.6.11 this function is locally constand on V.

Conversely, let $w : X \to S$ be a weighting and X quasi-compact. Choose a sufficiently divisible integer n. Let α be the family of 0-cycles on fibres of X/S such that for $s \in S$ we have

$$\alpha_s = \sum\nolimits_{f(x) = s} \frac{nw(x)}{[\kappa(x) : \kappa(s)]_i}[x]$$

as a zero cycle on X_s. This makes sense since the fibres of f are universally bounded (Morphisms, Lemma 29.57.9) hence we can find n such that the right hand side is an integer for all $s \in S$. The final statement of the lemma also follows, provided we show α is a relative 0-cycle. To do this we have to show that α is compatible with specializations along discrete valuation rings. By the first paragraph of the proof our construction is compatible with base change (small detail omitted; it is the "inverse" construction we are discussing here). Also, the base change of a weighting is a weighting, see More on Morphisms, Lemma 37.75.3. Thus we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0. Let $w : X \to S$ be a weighting with X quasi-finite over S. Let α be the family of 0-cycles on fibres of X/S constructed in the previous paragraph (for a suitable n). We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Let $\beta \in z(X/S, 0)$ be the relative 0-cycle on X/S with $\beta_\eta = \alpha_\eta$ and $\beta_0 = sp_{X/S}(\alpha_\eta)$. Then $w' = w_\beta - nw : X \to \mathbf{Z}$ is a weighting (using the result above) and zero in the points of X which map to η. Now it is easy to see that a weighting which is zero on all points of X mapping to η has to be zero; details omitted. Hence $w' = 0$, i.e., $w_\beta = nw$, hence $\alpha = \beta$ as desired. □

62.8. Effective relative cycles

0H5Q Here is the definition.

0H5R **Definition 62.8.1.** Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S *effective* if α_s is an effective cycle (Chow Homology,

Definition 42.8.4) for all $s \in S$. The monoid of all effective relative r-cycles on X/S is denoted $z^{eff}(X/S, r)$.

Below we will show that an effective relative cycle is equidimensional, see Lemma 62.8.7.

0H5S **Lemma 62.8.2.** *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is effective, then any restriction, base change, flat pullback, or proper pushforward of α is effective.*

Proof. Omitted. □

0H5T **Lemma 62.8.3.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is effective we may work Zariski locally on X and S.*

Proof. Omitted. □

0H5U **Lemma 62.8.4.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $g : S' \to S$ be a surjective morphism. Then α is effective if and only if the base change $g^*\alpha$ is effective.*

Proof. Omitted. □

0H5V **Lemma 62.8.5.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is effective if and only if each flat pullback $f_i^*\alpha$ is effective.*

Proof. Omitted. □

0H5W **Lemma 62.8.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then $Supp(\alpha)$ is closed in X.*

Proof. Let $g : S' \to S$ be the inclusion of an irreducible component viewed as an integral closed subscheme. By Lemmas 62.8.2 and 62.5.7 it suffices to show that the support of the base change $g^*\alpha$ is closed in $S' \times_S S$. Thus we may assume S is an integral scheme with generic point η. We will show that $\mathrm{Supp}(\alpha)$ is the closure of $\mathrm{Supp}(\alpha_\eta)$. To do this, pick any $s \in S$. We can find a morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring mapping the generic point $\eta' \in S'$ to η and the closed point $0 \in S'$ to s, see Properties, Lemma 28.5.10. Then it suffices to prove that the support of $g^*\alpha$ is equal to the closure of $\mathrm{Supp}((g^\alpha)_{\eta'})$. This reduces us to the case discussed in the next paragraph.

Here S is the spectrum of a discrete valuation ring with generic point η and closed point 0. We have to show that $\mathrm{Supp}(\alpha)$ is the closure of $\mathrm{Supp}(\alpha_\eta)$. Since α is effective we may write $\alpha_\eta = \sum n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X_\eta$ integral closed of dimension r. Since $\alpha_0 = sp_{X/S}(\alpha_\eta)$ we know that $\alpha_0 = \sum n_i[\overline{Z}_{i,0}]_r$ where $\overline{Z}_i$ is the closure of Z_i. By Varieties, Lemma 33.19.2 we see that $\overline{Z}_{i,0}$ is equidimensional of dimension r. Since $n_i > 0$ we conclude that $\mathrm{Supp}(\alpha_0)$ is equal to the union of

the $\overline{Z}_{i,0}$ which is the fibre over 0 of $\bigcup \overline{Z}_i$ which in turn is the closure of $\bigcup Z_i$ as desired. □

0H5X **Lemma 62.8.7.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then α is equidimensional.*

Proof. Assume α is effective. By Lemma 62.8.6 the support $\mathrm{Supp}(\alpha)$ is closed in X. Thus α is equidimensional as the fibres of $\mathrm{Supp}(\alpha) \to S$ are the supports of the cycles α_s and hence have dimension r. □

0H5Y **Remark 62.8.8.** Let $f : X \to S$ be a morphism of schemes with S locally Noetherian and f locally of finite type. We can ask if the contravariant functor

$$\begin{matrix}\text{schemes } S' \text{ locally} \\ \text{of finite type over } S\end{matrix} \longrightarrow z^{eff}(X'/S', r) \text{ where } X' = S' \times_S X$$

is representable. Since $z(X'/S', r) = z(X'_{red}/S'_{red}, r)$ this cannot be true (we leave it to the reader to make an actual counter example). A better question would be if we can find a subcategory of the left hand side on which the functor is representable. Lemma 62.6.16 suggests we should restrict at least to the category of seminormal schemes over S.

If $S/\operatorname{Spec}(\mathbf{Q})$ is Nagata and f is a projective morphism, then it turns out that $S' \mapsto z^{eff}(X'/S', r)$ is representable on the category of seminormal S'. Roughly speaking this is the content of [**Kol96**, Theorem 3.21].

If S has points of positive characteristic, then this no longer works even if we replace seminormality with weak normality; a locally Noetherian scheme T is weakly normal if any birational universal homeomorphism $T' \to T$ has a section. An example is to consider 0-cycles of degree 2 on $X = \mathbf{A}^2_k$ over $S = \operatorname{Spec}(k)$ where k is a field of characteristic 2. Namely, over $W = X \times_S X$ we have a canonical relative 0-cycle $\alpha \in z^{eff}(X_W/W, 0)$: for $w = (x_1, x_2) \in W = X^2$ we have the cycle $\alpha_w = [x_1] + [x_2]$. This cycle is invariant under the involution $\sigma : W \to W$ switching the factors. Since W is smooth (hence normal, hence weakly normal), if $z(-/-, r)$ was representable by M on the category of weakly normal schemes of finite type over k we would get a σ-invariant morphism from W to M. This in turn would define a morphism from the quotient scheme $\mathrm{Sym}^2_S(X) = W/\langle\sigma\rangle$ to M. Since $\mathrm{Sym}^2_S(X)$ is normal, we would by the moduli property of M obtain a relative 0-cycle β on $X \times_S \mathrm{Sym}^2_S(X)/\mathrm{Sym}^2_S(X)$ whose pullback to W is α. However, there is no such cycle β. Namely, writing $X = \operatorname{Spec}(k[u, v])$ the scheme $\mathrm{Sym}^2_S(X)$ is the spectrum of

$$k[u_1 + u_2, u_1u_2, v_1 + v_2, v_1v_2, u_1v_1 + u_2v_2] \subset k[u_1, u_2, v_1, v_2]$$

The image of the diagonal $u_1 = u_2, v_1 = v_2$ in $\mathrm{Sym}^2_S(X)$ is the closed subscheme $V = \operatorname{Spec}(k[u_1^2, v_1^2])$; here we use that the characteristic of k is 2. Looking at the generic point η of V, the cycle β_η would be a zero cycle of degree 2 on $\mathbf{A}^2_{k(u_1^2, v_1^2)}$ whose pullback to $\mathbf{A}^2_{k(u_1,u_2)}$ would be 2[the point with coordinates(u_1, v_2)]. This is clearly impossible.

The discussion above does not contradict [**Kol96**, Theorem 4.13] as the Chow variety in that theorem only coarsely represents a functor (in fact 2 distinct functors, only one of which agrees with ours for projective X as one can see with some

work). Similarly, in [**SV00**, Section 4.4] it is shown that for projective X/S the h-sheafification of the presheaf $S' \mapsto z^{eff}(S' \times_S X/S', r)$ is equal to the h-sheafification of a representable functor.

0H5Z **Remark 62.8.9.** Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Let $Z \subset X$ be a closed subscheme. Assume

(1) S is Noetherian and geometrically unibranch,
(2) f is of finite type, and
(3) $Z \to S$ has relative dimension $\leq r$.

Then for all sufficiently divisible integers $n \geq 1$ there exists a unique effective relative r-cycle α on X/S such that $\alpha_\eta = n[Z_\eta]_r$ for every generic point η of S. This is a reformulation of [**SV00**, Theorem 3.4.2]. If we ever need this result, we will precisely state and prove it here.

62.9. Proper relative cycles

0H60 In our setting, the following is probably the correct definition.

0H61 **Definition 62.9.1.** Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a *proper relative cycle* if the support of α (Remark 62.5.6) is contained in a closed subset $W \subset X$ proper over S (Cohomology of Schemes, Definition 30.26.2). The group of all proper relative r-cycles on X/S is denoted $c(X/S, r)$.

By Cohomology of Schemes, Lemma 30.26.3 this just means that the closure of the support is proper over the base. To see that these form a group, use Cohomology of Schemes, Lemma 30.26.6.

0H62 **Lemma 62.9.2.** *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is proper, then any base change α is proper.*

Proof. Omitted. □

0H63 **Lemma 62.9.3.** *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be a h covering. Then α is proper if and only if each base change $g_i^*\alpha$ is proper.*

Proof. If α is proper, then each $g_i^*\alpha$ is too by Lemma 62.9.2. Assume each $g_i^*\alpha$ is proper. To prove that α is proper, it clearly suffices to work affine locally on S. Thus we may and do assume that S is affine. Then we can refine our covering $\{S_i \to S\}$ by a family $\{T_j \to S\}$ where $g : T \to S$ is a proper surjective morphism and $T = \bigcup T_j$ is an open covering. It follows that $\beta = g^*\alpha$ is proper on $Y = T \times_S X$ over T. By Lemma 62.5.7 we find that the support of β is the inverse image of the support of α by the morphism $f : Y \to X$. Hence the closure $W \subset Y$ of $f^{-1}\text{Supp}(\alpha)$ is proper over T. Since the morphism $T \to S$ is proper, it follows that W is proper over S. Then by Cohomology of Schemes, Lemma 30.26.5 the image $f(W) \subset X$ is a closed subset proper over S. Since $f(W)$ contains $\text{Supp}(\alpha)$ we conclude α is proper. □

62.10. Proper and equidimensional relative cycles

0H64 Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a *proper and equidimensional relative cycle* if α is both equidimensional (Definition 62.7.1) and proper (Definition 62.9.1). The group of all proper, equidimensional relative r-cycles on X/S is denoted $c_{equi}(X/S, r)$.

Similarly we say a relative r-cycle α on X/S is a *proper and effective relative cycle* if α is both effective (Definition 62.8.1) and proper (Definition 62.9.1). The monoid of all proper, effective relative r-cycles on X/S is denoted $c^{eff}(X/S, r)$. Observe that these are equidimensional by Lemma 62.8.7.

Thus we have the following diagram of inclusion maps

$$\begin{array}{ccccc} c^{eff}(X/S,r) & \longrightarrow & c_{equi}(X/S,r) & \longrightarrow & c(X/S,r) \\ \downarrow & & \downarrow & & \downarrow \\ z^{eff}(X/S,r) & \longrightarrow & z_{equi}(X/S,r) & \longrightarrow & z(X/S,r) \end{array}$$

62.11. Action on cycles

0H65 Let S be a locally Noetherian, universally catenary scheme endowed with a dimension function δ, see Chow Homology, Section 42.7. Let $X \to Y$ be a morphism of schemes over S, both locally of finite type over S. Let $r \geq 0$. Finally, let α be a family of r-cycles on fibres of X/Y. For $e \in \mathbf{Z}$ we will construct an operation

$$\alpha \cap - : Z_e(Y) \longrightarrow Z_{r+e}(X)$$

Namely, given $\beta \in Z_e(Y)$ write $\beta = \sum n_i[Z_i]$ where $Z_i \subset Y$ is an integral closed subscheme of δ-dimension e and the family Z_i is locally finite in the scheme Y. Let $y_i \in Z_i$ be the generic point. Write $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$. Thus $V_{ij} \subset X_{y_i}$ is an integral closed subscheme of dimension r and the family V_{ij} is locally finite in the scheme X_{y_i}. Then we set

$$\alpha \cap \beta = \sum n_i m_{ij} [\overline{V}_{ij}] \quad \in \quad Z_{r+e}(X)$$

Here $\overline{V}_{ij} \subset X$ is the scheme theoretic image of the morphism $V_{ij} \to X_{y_i} \to X$ or equivalently, $\overline{V}_{ij} \subset X$ is an integral closed subscheme mapping dominantly to $Z_i \subset Y$ whose generic fibre is V_{ij}. It follows readily that $\dim_\delta(\overline{V}_{ij}) = r + e$ and that the family of closed subschemes $\overline{V}_{ij} \subset X$ is locally finite (we omit the verifications). Hence $\alpha \cap \beta$ is indeed an element of $Z_{r+e}(X)$.

0H66 **Lemma 62.11.1.** *The construction above is bilinear, i.e., we have $(\alpha_1 + \alpha_2) \cap \beta = \alpha_1 \cap \beta + \alpha_2 \cap \beta$ and $\alpha \cap (\beta_1 + \beta_2) = \alpha \cap \beta_1 + \alpha \cap \beta_2$.*

Proof. Omitted. □

0H67 **Lemma 62.11.2.** *If $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, then $(\alpha \cap \beta)|_U$ is equal to $\alpha|_U \cap \beta|_V$.*

Proof. Immediate from the explicit description of $\alpha \cap \beta$ given above. □

0H68 **Lemma 62.11.3.** *Forming $\alpha \cap \beta$ is compatible with flat base change and flat pullback (see proof for elucidation).*

Proof. Let (S, δ), (S', δ'), $g : S' \to S$, and $c \in \mathbf{Z}$ be as in Chow Homology, Situation 42.67.1. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta \in Z_e(Y)$. Denote α' the base change of α by $Y' \to Y$. Denote $\beta' = g^*\beta \in Z_{e+c}(Y')$ the pullback of β by g, see Chow Homology, Section 42.67. Compatibility with base change means $\alpha' \cap \beta'$ is the base change of $\alpha \cap \beta$.

Proof of compatibility with base change. Since we are proving an equality of cycles on X', we may work locally on Y, see Lemma 62.11.2. Thus we may assume Y is affine. In particular β is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\beta = [Z]$ for some integral closed subscheme $Z \subset Y$ of δ-dimension e.

Let $y \in Z$ be the generic point. Write $\alpha_y = \sum m_j[V_j]$. Let $\overline{V}_j$ be the closure of V_j in X. Then we have

$$\alpha \cap \beta = \sum m_j[\overline{V}_j]$$

The base change of β is $\beta' = \sum [Z \times_S S']_{e+c}$ as a cycle on $Y' = Y \times_S S'$. Let $Z'_a \subset Z \times_S S'$ be the irreducible components, denote $y'_a \in Z'_a$ their generic points, and denote n_a the multiplicity of Z'_a in $Z \times_S S'$. We have

$$\beta' = \sum [Z \times_S S']_{e+c} = \sum n_a[Z'_a]$$

We have $\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r$ because α' is the base change of α by $Y' \to Y$. Let $V'_{jab} \subset V_{j,\kappa(y'_a)}$ be the irreducible components and denote m_{jab} the multiplicity of V'_{jab} in $V_{j,\kappa(y'_a)}$. We have

$$\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r = \sum m_j m_{jab}[V'_{jab}]$$

Thus we we have

$$\alpha' \cap \beta' = \sum n_a m_j m_{jab}[\overline{V}'_{jab}]$$

where $\overline{V}'_{jab}$ is the closure of V'_{jab} in X'. Thus to prove the desired equality it suffices to prove

(1) the irreducible components of $\overline{V}_j \times_S S'$ are the schemes $\overline{V}'_{jab}$ and
(2) the multiplicity of $\overline{V}'_{jab}$ in $\overline{V}_j \times_S S'$ is equal to $n_a m_{jab}$.

Note that $V_j \to \overline{V}_j$ is a birational morphism of integral schemes. The morphisms $V_j \times_S S' \to V_j$ and $\overline{V}_j \times_S S' \to \overline{V}_j$ are flat and hence map generic points of irreducible components to the (unique) generic points of V_j and $\overline{V}_j$. It follows that $V_j \times_S S' \to \overline{V}_j \times_S S'$ is a birational morphisms hence induces a bijection on irreducible components and identifies their multiplicities. This means that it suffices to prove that the irreducible components of $V_j \times_S S'$ are the schemes V'_{jab} and the multiplicity of V'_{jab} in $V_j \times_S S'$ is equal to $n_a m_{jab}$. However, then we are just saying that the diagram

$$\begin{array}{ccc} Z_r(V_j) & \longrightarrow & Z_{r+c}(V_j \times_S S') \\ \uparrow & & \uparrow \\ Z_0(\mathrm{Spec}(\kappa(y))) & \longrightarrow & Z_c(\mathrm{Spec}(\kappa(y)) \times_S S') \end{array}$$

is commutative where the horizontal arrows are base change by $\mathrm{Spec}(\kappa(y)) \times_S S' \to \mathrm{Spec}(\kappa(y))$ and the vertical arrows are flat pullback. This was shown in Chow Homology, Lemma 42.67.5.

The statement in the lemma on flat pullback means the following. Let (S, δ), $X \to Y$, α, and β be as in the construction of $\alpha \cap \beta$ above. Let $Y' \to Y$ be a flat morphism, locally of finite type, and of relative dimension c. Then we can let α' be the base change of α by $Y' \to Y$ and β' the flat pullback of β. Compatibility with flat pullback means $\alpha' \cap \beta'$ is the flat pullback of $\alpha \cap \beta$ by $X \times_Y Y' \to Y$. This is actually a special case of the discussion above if we set $S = Y$ and $S' = Y'$. □

0H69 **Lemma 62.11.4.** *Let (S, δ) and $f : X \to Y$ be as above. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with $\dim(\mathit{Supp}(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module with $\dim_\delta(\mathit{Supp}(\mathcal{G})) \leq e$. Set $\alpha = [\mathcal{F}/X/Y]_r$ (Example 62.5.2) and $\beta = [\mathcal{G}]_e$ (Chow Homology, Definition 42.10.2). If $\mathcal{F}$ is flat over Y, then $\alpha \cap \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$.*

Proof. Observe that

$$\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \mathrm{Supp}(\mathcal{F}) \cap f^{-1}\mathrm{Supp}(\mathcal{G}) = \bigcup\nolimits_{y \in \mathrm{Supp}(\mathcal{G})} \mathrm{Supp}(\mathcal{F}_y)$$

It follows that this is a closed subset of δ-dimension $\leq r + e$. Whence the expression $[\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$ makes sense.

We will use the notation $\beta = \sum n_i[Z_i]$, $y_i \in Z_i$, $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$, and $\overline{V}_{ij}$ introduced in the construction of $\alpha \cap \beta$. Since $\beta = [\mathcal{G}]_e$ we see that the Z_i are the irreducible components of $\mathrm{Supp}(\mathcal{G})$ which have δ-dimension e. Similarly, the V_{ij} are the irreducible components of $\mathrm{Supp}(\mathcal{F}_{y_i})$ having dimension r. It follows from this and the equation in the first paragraph that $\overline{V}_{ij}$ are the irreducible components of $\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ having δ-dimension $r + e$. Thus to prove the lemma it now suffices to show that

$$\mathrm{length}_{\mathcal{O}_{X,\xi_{ij}}}((\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_{\xi_{ij}}) = \mathrm{length}_{\mathcal{O}_{X_{y_i},\xi_{ij}}}((\mathcal{F}_{y_i})_{\xi_{ij}}) \cdot \mathrm{length}_{\mathcal{O}_{Y,y_i}}(\mathcal{G}_{y_i})$$

By the first paragraph of the proof the left hand side is equal to the length of the $B = \mathcal{O}_{X,\xi_{ij}}$-module

$$\mathcal{G}_{y_i} \otimes_{\mathcal{O}_{Y,y_i}} \mathcal{F}_{\xi_{ij}} = M \otimes_A N$$

Here $M = \mathcal{G}_{y_i}$ is a finite length $A = \mathcal{O}_{Y,y_i}$-module and $N = \mathcal{F}_{\xi_{ij}}$ is a finite B-module such that $N/\mathfrak{m}_A N$ has finite length. Since $\mathcal{F}$ is flat over Y the module N is A-flat. The right hand side of the formula is equal to

$$\mathrm{length}_B(N/\mathfrak{m}_A N) \cdot \mathrm{length}_A(M)$$

Thus the right and left hand side of the formula are additive in M (use flatness of N over A). Thus it suffices to prove the formula with $M = \kappa_A$ is the residue field in which case it is immediate. □

0H6A **Lemma 62.11.5.** *Let (S, δ) and $f : X \to Y$ be as above. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Set $\alpha = [Z/X/Y]_r$ (Example 62.5.4). Let $W \subset Y$ be a closed subscheme of δ-dimension $\leq e$. Set $\beta = [W]_e$ (Chow Homology, Definition 42.9.2). If Z is flat over Y, then $\alpha \cap \beta = [Z \times_Y W]_{r+e}$.*

Proof. This is a special case of Lemma 62.11.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. □

0H6B **Lemma 62.11.6.** *Let (S, δ) be as above. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta' \in Z_e(Y')$. Then we have $f_*(g^*\alpha \cap \beta') = \alpha \cap g_*\beta'$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Y, see Lemma 62.11.2. Thus we may assume Y is affine. Thus Y' is quasi-compact. In particular β' is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\beta' = [Z']$ for some integral closed subscheme $Z' \subset Y'$ of δ-dimension e. Set $Z = g(Z')$. This is an integral closed subscheme of Y of δ-dimension $\leq e$. For simplicity we are going to assume Z has δ-dimension equal to e and leave the other case (which is easier) to the reader. Let $y \in Z$ and $y' \in Z'$ be the generic points. Write $\alpha_y = \sum m_j[V_j]$ with $V_j \subset X_y$ integral closed subschemes of dimension r.

Assume first g is a closed immersion. Then $g_*\beta' = [Z]$ and $(g^*\alpha)_{y'} = \sum n_j[V_j]$; this makes sense because V_j is contained in the closed subscheme $X'_{y'}$ of X_y. Thus in this case the equality is obvious: in both cases we obtain $\sum m_j[\overline{V}_j]$ where $\overline{V}_j$ is the closure of V_j in the closed subscheme $X' \subset X$.

Back to the general case with $\beta' = [Z']$ as above. Set $W = Z \times_X Y$ and $W' = Z' \times_{X'} Y'$. Consider the cartesian squares

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} W' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Y' \end{array} \quad \begin{array}{ccc} W' & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

Since we know the result for the first two squares with by the previous paragraph, a formal argument shows that it suffices to prove the result for the last square and the element $\beta' = [Z'] \in Z_e(Z')$. This reduces us to the case discussed in the next paragraph.

Assume $Y' \to Y$ is a generically finite morphism of integral schemes of δ-dimension e and $\beta' = [Y']$. In this case both $f_*(g^*\alpha \cap \beta')$ and $\alpha \cap g_*\beta'$ are cycles which can be written as a sum of prime cycles dominant over Y. Thus we may replace Y by a nonempty open subscheme in order to check the equality. After such a replacement we may assume g is finite and flat, say of degree $d \geq 1$. Of course, this means that $g_*\beta' = g_*[Y'] = d[Y]$. Also $\beta' = [Y'] = g^*[Y]$. Hence

$$f_*(g^*\alpha \cap \beta') = f_*(g^*\alpha \cap g^*[Y]) = f_*f^*(\alpha \cap [Y]) = d(\alpha \cap [Y]) = \alpha \cap g_*\beta')$$

as desired. The second equality is Lemma 62.11.3 and the third equality is Chow Homology, Lemma 42.15.2. □

62.12. Action on chow groups

0H6C When α is a relative r-cycle, the operation $\alpha \cap -$ of Section 62.11 factors through rational equivalence and defines a bivariant class.

0H6D **Lemma 62.12.1.** *Let (S, δ) be as in Section 62.11. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Chow Homology, Definition 42.29.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow[i']{} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

as in Chow Homology, Remark 42.29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then $i^* f_* \alpha' = g_*(i')^* \alpha'$ in $Z_k(D)$ for any $\alpha' \in Z_{k+1}(X')$.

Proof. The statement makes sense as all operations are defined on the level of cycles, see Chow Homology, Remark 42.29.6 for the Gysin maps. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Chow Homology, Lemma 42.26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and both sides are zero by construction. □

0H6E **Lemma 62.12.2.** *Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. Let $(\mathcal{L}, s, i : D \to Y)$ be as in Chow Homology, Definition 42.29.1. Form the cartesian diagram*

$$\begin{array}{ccc} E & \xrightarrow[j]{} & X \\ \downarrow & & \downarrow \\ D & \xrightarrow{i} & Y \end{array}$$

See Chow Homology, Remark 42.29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then for $e \in \mathbf{Z}$ the diagram

$$\begin{array}{ccc} Z_e(D) & \xrightarrow[i^*\alpha\cap -]{} & Z_{e+r}(E) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ Z_{e+1}(Y) & \xrightarrow{\alpha\cap -} & Z_{r+e+1}(X) \end{array}$$

commutes where the vertical arrows i^* and j^* are the Gysin maps on cycles as in Chow Homology, Remark 42.29.6.

Proof. Preliminary remark. Suppose that $g : Y' \to Y$ is an envelope (Chow Homology, Definition 42.22.1). Denote $D', i', E', j', X', \alpha'$ the base changes of D, i, E, j, X, α by g and denote $f : X' \to X$ the projection. Assume the lemma holds for $D', i', E', j', X', Y', \alpha'$. Then, if $\beta' \in Z_{e+1}(Y')$, we have

$$\begin{aligned} i^*\alpha \cap i^* g_* \beta' &= i^*\alpha \cap f_*(i')^* \beta' \\ &= f_*(f^* i^* \alpha \cap (i')^* \beta') \\ &= f_*((i')^* \alpha' \cap (i')^* \beta') \\ &= f_*((j')^*(\alpha' \cap \beta')) \\ &= j^*(f_*(f^*\alpha \cap \beta')) \\ &= j^*(\alpha \cap g_* \beta') \end{aligned}$$

Here the first equality is Lemma 62.12.1, the second equality is Lemma 62.11.6, the third equality is the definition of α', the fourth equality is the assumption that our lemma holds for $D', i', E', j', X', \alpha'$, the fifth equality is Lemma 62.12.1, and the sixth equality is Lemma 62.11.6. Thus we see that our lemma holds for the image of $g_* : Z_{e+1}(Y') \to Z_e(Y)$. However, since g is completely decomposed this map is surjective and we conclude the lemma holds for D, i, E, j, X, Y, α.

Let $\beta \in Z_{e+1}(Y)$. We have to show that $(D \to Y)^*\alpha \cap i^*\beta = j^*(\alpha \cap \beta)$ as cycles on E. This question is local on E hence we can replace X and Y by open subschemes. (This uses that formation of the operators i^*, j^*, $\alpha \cap -$ and $(D \to Y)^*\alpha \cap -$ commute with localization. This is obvious for the Gysin maps and follows from Lemma 62.11.2 for the others.) Thus we may assume that X and Y are affine and we reduce to the case discussed in the next paragraph.

Assume X and Y are quasi-compact. By the first paragraph of the proof and Lemma 62.6.9 we may in addition assume that α is in the image of (62.6.8.1). By linearity of the operations in question, we may assume that $\alpha = [Z/X/Y]_r$ for some closed subscheme $Z \subset X$ which is flat and of relative dimension $\leq r$ over Y. Also, as Y is quasi-compact, the cycle β is a finite linear combination of prime cycles. Since the operations in question are linear, it suffices to prove the equality when $\beta = [W]$ for some integral closed subscheme $W \subset Y$ of δ-dimension $e + 1$.

If $W \subset D$, then on the one hand $i^*[W] = 0$ and on the other hand $\alpha \cap [W]$ is supported on E so also $j^*(\alpha \cap [W]) = 0$. Thus the equality holds in this case.

Say $W \not\subset D$. Then $i^*[W] = [D \cap W]_e$. Note that the pullback $i^*\alpha$ of $\alpha = [Z/X/Y]_r$ by i is $[(E \cap Z)/E/D]_r$ and that $(E \cap Z) = E \times_Y Z = D \times_Y Z$ is flat over D. Hence by Lemma 62.11.5 used twice we have

$$i^*\alpha \cap i^*[W] = [(E \cap Z) \times_D (D \cap W)]_{r+e} = [E \cap (Z \times_Y W)]_{r+e} = j^*(\alpha \cap [W])$$

as desired. □

0H6F **Proposition 62.12.3.** *Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. The rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates the operation*

$$g^*\alpha \cap - : Z_e(Y') \to Z_{r+e}(X')$$

where $X' = Y' \times_Y X$ factors through rational equivalence to define a bivariant class $c(\alpha) \in A^{-r}(X \to Y)$.

Proof. The operation factors through rational equivalence by Lemma 62.12.2 and Chow Homology, Lemma 42.35.1. The resulting operation on chow groups is a bivariant class by Chow Homology, Lemma 42.35.2 and Lemmas 62.11.6, 62.11.3, and 62.12.2. □

0H6G **Remark 62.12.4.** Let (S, δ) be as in Section 62.11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$. Let c be a rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates an operation

$$c \cap - : Z_e(Y') \to Z_{r+e}(X')$$

compatible with proper pushforward, flat pullback, and Gysin maps as in Lemma 62.12.2. Then we claim there is a relative r-cycle α on X/Y such that $c\cap = g^*\alpha\cap -$

for every g as above. If we ever need this, we will carefully state and prove this here.

62.13. Composition of families of cycles on fibres

0H6H Let $X \to Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let α be a family of r-cycles on fibres of X/Y and let β be a family of e-cycles on fibres of Y/S. Then we obtain a family of of $(r+e)$-cycles $\alpha \circ \beta$ on the fibres of X/S by setting

$$(\alpha \circ \beta)_s = (Y_s \to Y)^*\alpha \cap \beta_s$$

More precisely, the expression $(Y_s \to Y)^*\alpha$ denotes the base change of α by $Y_s \to Y$ to a family of r-cycles on the fibres of X_s/Y_s and the operation $- \cap -$ was defined and studied in Section 62.11[3].

0H6I **Lemma 62.13.1.** *The construction above is bilinear, i.e., we have* $(\alpha_1 + \alpha_2) \circ \beta \alpha_1 \circ \beta + \alpha_1 \circ \beta$ *and* $\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2$.

Proof. Omitted. Hint: on fibres the construction is bilinear by Lemma 62.11.1. □

0H6J **Lemma 62.13.2.** *If $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, then $(\alpha \circ \beta)|_U$ is equal to $\alpha|_U \circ \beta|_V$.*

Proof. Omitted. Hint: on fibres use Lemma 62.11.2. □

0H6K **Lemma 62.13.3.** *The formation of $\alpha \circ \beta$ is compatible with base change.*

Proof. Let $g : S' \to S$ be a morphism of schemes. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Denote α' the base change of α with respect to $Y' \to Y$. Denote β' the base change of β with respect to $S' \to S$. The assertion means that $\alpha' \circ \beta'$ is the base change of $\alpha \circ \beta$ by $g : S' \to S$.

Let $s' \in S'$ be a point with image $s \in S$. Then

$$(\alpha' \circ \beta')_{s'} = (Y'_{s'} \to Y')^*\alpha' \cap \beta'_{s'}$$

We observe that

$$(Y'_{s'} \to Y')^*\alpha' = (Y'_{s'} \to Y')^*(Y' \to Y)^*\alpha = (Y'_{s'} \to Y_s)^*(Y_s \to Y)^*\alpha$$

and that $\beta'_{s'}$ is the base change of β_s by $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$. Hence the result follows from Lemma 62.11.3 applied to $(Y_s \to Y)^*\alpha$, β_s, $X_s \to Y_s \to s$, and base change by $s' \to s$. □

0H6L **Lemma 62.13.4.** *Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type, with $\dim(\mathrm{Supp}(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type, with $\dim(\mathrm{Supp}(\mathcal{G}_s)) \leq e$ for all $s \in S$. If $\alpha = [\mathcal{F}/X/Y]_r$ and $\beta = [\mathcal{G}/Y/S]_e$ (Example 62.5.2) and $\mathcal{F}$ is flat over Y, then $\alpha \circ \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}/X/S]_{r+e}$.*

Proof. First we observe that $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Observe that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s = \mathcal{F}_s \otimes_{\mathcal{O}_{X_s}} f_s^*\mathcal{G}_s$$

by right exactness of tensor products. Moreover $\mathcal{F}_s$ is flat over Y_s as a base change of a flat module. Thus the equality $(\alpha \circ \beta)_s = [(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s]_{r+e}$ follows from Lemma 62.11.4. □

[3] To be sure, we use $s = \mathrm{Spec}(\kappa(s))$ as the base scheme with $\delta(s) = 0$.

0H6M **Lemma 62.13.5.** *Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Let $W \subset Y$ be a closed subscheme of relative dimension $\leq e$ over S. If $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]_e$ (Example 62.5.4) and Z is flat over Y, then $\alpha \circ \beta = [Z \times_Y W/X/S]_{r+e}$.*

Proof. This is a special case of Lemma 62.13.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. □

0H6N **Lemma 62.13.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let β' be a family of e-cycles on the fibres of Y'/S. Then we have $f_*(g^*(\alpha) \circ \beta') = \alpha \circ g_*\beta'$.

Proof. Unwinding the definitions, this follows from Lemma 62.11.6. □

0H6P **Lemma 62.13.7.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let $X \to Y \to Z$ be morphisms of schemes locally of finite type over S. Let $r, s, e \geq 0$. Then*

$$(\alpha \circ \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma) \quad \text{in} \quad Z_{r+s+e}(X)$$

where α is a family of r-cycles on fibres of X/Y, β is a family of s-cycles on fibres of Y/Z, and $\gamma \in Z_e(Z)$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Z, see Lemma 62.11.2. Thus we may assume Z is affine. In particular γ is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 62.11.1), it suffices to prove the equality when $\gamma = [W]$ for some integral closed subscheme $W \subset Z$ of δ-dimension e.

Let $z \in W$ be the generic point. Write $\beta_z = \sum m_j[V_j]$ in $Z_s(Y_z)$. Then $\beta \cap \gamma$ is equal to $\sum m_j[\overline{V}_j]$ where $\overline{V}_j \subset Y$ is an integral closed subscheme mapped by $Y \to Z$ into W with generic fibre V_j. Let $y_j \in V_j$ be the generic point. We may and do view also as the generic point of $\overline{V}_j$ (mapping to z in W). Write $\alpha_{y_j} = \sum n_{jk}[W_{jk}]$ in $Z_r(X_{y_j})$. Then $\alpha \cap (\beta \cap \gamma)$ is equal to

$$\sum m_j n_{jk} [\overline{W}_{jk}]$$

where $\overline{W}_{jk} \subset X$ is an integral closed subscheme mapped by $X \to Y$ into $\overline{V}_j$ with generic fibre W_{jk}.

On the other hand, let us consider

$$(\alpha \circ \beta)_z = (Y_z \to Y)^* \alpha \cap \beta_z = (Y_z \to Y)^* \alpha \cap (\sum m_j [V_j])$$

By the construction of $- \cap -$ this is equal to the cycle

$$\sum m_j n_{jk} [(\overline{W}_{jk})_z]$$

on X_z. Thus by definition we obtain

$$(\alpha \circ \beta) \cap [W] = \sum m_j n_{jk} [\widetilde{W}_{jk}]$$

where $\widetilde{W}_{jk} \subset X$ is an integral closed subscheme which is mapped by $X \to Z$ into W with generic fibre $(\overline{W}_{jk})_z$. Clearly, we must have $\widetilde{W}_{jk} = \overline{W}_{jk}$ and the proof is complete. □

62.14. Composition of relative cycles

0H6Q Let S be a locally Noetherian scheme. Let $X \to Y$ be a morphism of schemes locally of finite type over S. We are going to define a map

$$z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r + e), \quad \alpha \otimes \beta \longmapsto \alpha \circ \beta$$

using the construction in Section 62.13. We already know the construction is bilinear (Lemma 62.13.1) hence we obtain the displayed arrow once we show the following.

0H6R **Lemma 62.14.1.** *If α and β are relative cycles, then so is $\alpha \circ \beta$.*

Proof. The formation of $\alpha \circ \beta$ is compatible with base change by Lemma 62.13.3. Thus we may assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have to show that $sp_{X/S}((\alpha \circ \beta)_\eta) = (\alpha \circ \beta)_0$. Since we are trying to prove an equality of cycles, we may work locally on Y and X (this uses Lemmas 62.13.2 and 62.4.4 to see that the constructions commute with restriction). Thus we may assume X and Y are affine. By Lemma 62.6.9 we can find a completely decomposed proper morphism $g : Y' \to Y$ such that $g^*\alpha$ is in the image of (62.6.8.1).

Since the family of morphisms $g_\eta : Y'_\eta \to Y_\eta$ is completely decomposed, we can find $\beta'_\eta \in Z_e(Y'_\eta)$ such that $\beta_\eta = \sum g_{\eta,*}\beta'_\eta$, see Chow Homology, Lemma 42.22.4. Set $\beta'_0 = sp_{Y'/S}(\beta'_\eta)$ so that $\beta' = (\beta'_\eta, \beta'_0)$ is a relative e-cycle on Y'/S. Then $g_*\beta'$ and β are relative e-cycles on Y/S (Lemma 62.6.2) which have the same value at η and hence are equal (Lemma 62.6.6). By linearity (Lemma 62.13.1) it suffices to show that $\alpha \circ g_*\beta'$ is a relative $(r + e)$-cycle.

Set $X' = X \times_Y Y'$ and denote $f : X' \to X$ the projection. By Lemma 62.13.6 we see that $\alpha \circ g_*\beta' = f_*(g^*\alpha \circ \beta')$. By Lemma 62.6.2 it suffices to show that $g^*\alpha \circ \beta'$ is a relative $(r + e)$-cycle. Using Lemma 62.6.10 and bilinearity this reduces us to the case discussed in the next paragraph.

Assume $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]$ where $Z \subset X$ is a closed subscheme flat and of relative dimension $\leq r$ over Y and $W \subset Y$ is a closed subscheme flat and of relative dimension $\leq e$ over S. By Lemma 62.13.5 we see that

$$\alpha \circ \beta = [Z \times_X W/X/S]_{r+e}$$

and $Z \times_X W \subset X$ is a closed subscheme flat over S of relative dimension $\leq r + e$. This is a relative $(r + e)$-cycle by Lemma 62.6.8. □

0H6S **Lemma 62.14.2.** *Let $f : X \to Y$ and $g : Y \to S$ be a morphisms of schemes. Assume S locally Noetherian, g locally of finite type and flat of relative dimension $e \geq 0$, and f locally of finite type and flat of relative dimension $r \geq 0$. Then $[X/X/Y]_r \circ [Y/Y/S]_e = [X/X/S]_{r+e}$ in $z(X/S, r + e)$.*

Proof. Special case of Lemma 62.13.5. □

62.15. Comparison with Suslin and Voevodsky

0H6T We have tried to use the same notation as in [**SV00**], except that our notation for cycles is taken from Chow Homology, Section 42.8 ff. Here is a comparison:

(1) In [**SV00**, Section 3.1] there is a notion of a "relative cycle", of a "relative cycle of dimension r", and of a "equidimensional relative cycle of dimension r". There is no corresponding notion in this chapter. Consequently, the groups $Cycl(X/S, r)$, $Cycl_{equi}(X/S, r)$, $PropCycl(X/S, r)$, and $PropCycl_{equi}(X/S, r)$, have no counter parts in this chapter.
(2) On the bottom of [**SV00**, page 36] the groups $z(X/S, r)$, $c(X/S, r)$, $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ are defined. These agree with our notions when S is separated Noetherian and $X \to S$ is separated and of finite type.
(3) In [**SV00**] the symbol $z(X/S, r)$ is sometimes used for the presheaf $S' \mapsto z(S' \times_S X/S', r)$ on the category of schemes of finite type over S. Similarly for $c(X/S, r)$, $z_{equi}(X/S, r)$, and $c_{equi}(X/S, r)$.
(4) Base change, flat pullback, and proper pushforward as defined in [**SV00**] agrees with ours when both apply.
(5) For $\alpha \in z(X/S, r)$ the operation $\alpha \cap - : Z_e(S) \to Z_{e+r}(X)$ defined in Section 62.11 agrees with the operation $Cor(\alpha, -)$ in [**SV00**, Section 3.7] when both are defined.
(6) For $X \to Y \to S$ the composition law $z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r + e)$ defined in Section 62.14 agrees with the operation $Cor_{X/Y}(-, -)$ in [**SV00**, Corollary 3.7.5].

62.16. Relative cycles in the non-Noetherian case

0H6U We urge the reader to skip this section.

Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$ such that $Z \to S$ is flat, of finite presentation, and of relative dimension $\leq r$. We consider the group homomorphism

0H6V (62.16.0.1)
$$\begin{array}{c}\text{free abelian group}\\ \text{on } Hilb(X/S, r)\end{array} \longrightarrow \begin{array}{c}\text{families of } r\text{-cycles}\\ \text{on fibres of } X/S\end{array}$$

sending $\sum n_i[Z_i]$ to $\sum n_i[Z_i/X/S]_r$.

0H6W **Lemma 62.16.1.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles on fibres of X/S. The following are equivalent*

(1) *there exists a cartesian diagram*
$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_0 \end{array}$$
where $X_0 \to S_0$ is a finite type morphism of Noetherian schemes and $\alpha_0 \in z(X_0/S_0, r)$ such that α is the base change of α_0 by $S \to S_0$
(2) *there exists a completely decomposed proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (62.16.0.1).*

Proof. Let a diagram and $\alpha_0 \in z(X_0/S_0, r)$ as in (1) be given. By Lemma 62.6.9 there exists a proper surjective morphism $g_0 : S_0' \to S_0$ such that $g_0^*\alpha_0$ is in the image of (62.16.0.1). Namely, since S_0' is Noetherian, every closed subscheme of $S_0' \times_{S_0} X_0$ is of finite presentation over S_0'. Setting $S' = S \times_{S_0} S_0'$ and using base change by $S' \to S_0'$ we see that (2) holds.

Conversely, assume that (2) holds. Choose a surjective proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (62.16.0.1). Set $X' = S' \times_S X$. Write $g^*\alpha = \sum n_a[Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat, of finite presentation, and of relative dimension $\leq r$ over S'.

Write $S = \lim S_i$ as a directed limit with affine transition morphisms with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 32.5.4. We can find an i large enough such that there exist

(1) a completely decomposed proper morphism $g_i : S_i' \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X_i' = S_i' \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X_i'$ flat and of relative dimension $\leq r$ over S_i' whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 32.10.1, 32.8.5, 32.8.7, 32.8.15, 32.13.1, and 32.18.1 and and More on Morphisms, Lemma 37.78.5. Consider $\alpha_i' = \sum n_a[Z_{ai}/X_i'/S_i]_r \in z(X_i'/S_i', r)$. The base change of α_i' to a family of r-cycles on fibres of X'/S' agrees with the base change $g^*\alpha$ by construction.

Set $S_i'' = S_i' \times_{S_i} S_i'$ and $X_i'' = S_i'' \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\mathrm{pr}_1, \mathrm{pr}_2 : S'' \to S'$ and $\mathrm{pr}_1, \mathrm{pr}_2 : S_i'' \to S_i'$ the projections. The relative r-cycles $\mathrm{pr}_1^*\alpha_i'$ and $\mathrm{pr}_1^*\alpha_i'$ on X_i''/S_i'' base change to the same family of r-cycles on fibres of X''/S'' because $\mathrm{pr}_1^*g^*\alpha = \mathrm{pr}_1^*g^*\alpha$. Hence the morphism $S'' \to S_i''$ maps into $E = \{s \in S_i'' : (\mathrm{pr}_1^*\alpha_i')_s = (\mathrm{pr}_1^*\alpha_i')_s\}$. By Lemma 62.6.12 this is a closed subset. Since $S'' = \lim_{i' \geq i} S_{i'}''$ we see from Limits, Lemma 32.4.10 that for some $i' \geq i$ the morphism $S_{i'}'' \to S_i''$ maps into E. Therefore, after replacing i by i', we may assume that $\mathrm{pr}_1^*\alpha_i' = \mathrm{pr}_1^*\alpha_i'$. By Lemma 62.5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha_i'$ (this uses that $S_i' \to S_i$ is completely decomposed). By Lemma 62.6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 62.5.9 implies that the base change of α_i is α and we see (1) holds. □

Discussion. If $f : X \to S$, r, and α are as in Lemma 62.16.1, then it makes sense to say that α is a *relative r-cycle on X/S* if the equivalent conditions (1) and (2) of Lemma 62.16.1 hold. This definition has many good properties; for example it doesn't conflict with the earlier definition in case S is Noetherian and most of the results of Section 62.6 generalize to this setting.

We may still generalize further as follows. Assume S is arbitrary and $f : X \to S$ is locally of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles α on fibres of X/S. Then α is an *relative r-cycle on X/S* if for $U \subset X$ and $V \subset S$ affine open with $f(U) \subset V$ the restriction $\alpha|_U$ is a relative r-cycle on U/V as defined in the previous paragraph. Again many of the earlier results generalize to this setting.

If we ever need these generalizations we will carefully state and prove them here.

62.17. Other chapters

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CHAPTER 63

More Étale Cohomology

0F4U

63.1. Introduction

0F4V This chapter is the second in a series of chapters on the étale cohomology of schemes. To read the first chapter, please visit Étale Cohomology, Section 59.1.

The split with the previous chapter is roughly speaking that anything concerning "shriek functors" (cohomology with compact support and its right adjoint) and anything using this material goes into this chapter.

63.2. Growing sections

0F71 In this section we discuss results of the following type.

0F6F **Lemma 63.2.1.** *Let X be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\varphi : U' \to U$ be a morphism of $X_{étale}$. Let $Z' \subset U'$ be a closed subscheme such that $Z' \to U' \to U$ is a closed immersion with image $Z \subset U$. Then there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s' \in \mathcal{F}(U') \mid Supp(s') \subset Z'\}$$

which is given by restriction if $\varphi^{-1}(Z) = Z'$.

Proof. Consider the closed subscheme $Z'' = \varphi^{-1}(Z)$ of U'. Then $Z' \subset Z''$ is closed because Z' is closed in U'. On the other hand, $Z' \to Z''$ is an étale morphism (as a morphism between schemes étale over Z) and hence open. Thus $Z'' = Z' \amalg T$ for some closed subset T. The open covering $U' = (U' \setminus T) \cup (U' \setminus Z')$ shows that

$$\{s' \in \mathcal{F}(U') \mid \mathrm{Supp}(s') \subset Z'\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \mathrm{Supp}(s') \subset Z'\}$$

and the étale covering $\{U' \setminus T \to U, U \setminus Z \to U\}$ shows that

$$\{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \mathrm{Supp}(s') \subset Z'\}$$

This finishes the proof. □

0F6G **Lemma 63.2.2.** *Let X be a scheme. Let $Z \subset X$ be a locally closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Given $U, U' \subset X$ open containing Z as a closed subscheme, there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s \in \mathcal{F}(U') \mid Supp(s) \subset Z\}$$

which is given by restriction if $U' \subset U$.

Proof. Since Z is a closed subscheme of $U \cap U'$, it suffices to prove the lemma when $U' \subset U$. Then it is a special case of Lemma 63.2.1. □

Let us introduce a bit of nonstandard notation which will stand us in good stead later. Namely, in the situation of Lemma 63.2.2 above, let us denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\}$$

where $U \subset X$ is any choice of open subscheme containing Z as a closed subscheme. The reader who is troubled by the lack of precision this entails may choose $U = X \setminus \partial Z$ where $\partial Z = \overline{Z} \setminus Z$ is the "boundary" of Z in X. However, in many of the arguments below the flexibility of choosing different opens will play a role. Here are some properties of this construction:

0F6H (1) If $Z \subset Z'$ are locally closed subschemes of X and Z is closed in Z', then there is a natural injective map

$$H_Z(\mathcal{F}) \to H_{Z'}(\mathcal{F}).$$

0F6I (2) If $f : Y \to X$ is a morphism of schemes and $Z \subset X$ is a locally closed subscheme, then there is a natural pullback map $f^* : H_Z(\mathcal{F}) \to H_{f^{-1}Z}(f^{-1}\mathcal{F})$.

It will be convenient to extend our notation to the following situation: suppose that we have $W \in X_{étale}$ and a locally closed subscheme $Z \subset W$. Then we will denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset Z\} = H_Z(\mathcal{F}|_{W_{étale}})$$

where $U \subset W$ is any choice of open subscheme containing Z as a closed subscheme, exactly as above[1].

63.3. Sections with compact support

0F4W A reference for this section is [**AGV71**, Exposee XVII, Section 6]. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. In this section we define a functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ by taking $f_!\mathcal{F} \subset f_*\mathcal{F}$ to be the subsheaf of sections which have proper support relative to Y (suitably defined).

Warning: The functor $f_!$ is the zeroth cohomology sheaf of a functor $Rf_!$ on the derived category (insert future reference), but $Rf_!$ is not the derived functor of $f_!$.

0F4X **Lemma 63.3.1.** *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The rule*

$$Y_{étale} \longrightarrow Ab, \quad V \longmapsto \{s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V) \mid \textit{Supp}(s) \subset X_V \textit{ is proper over } V\}$$

is an abelian subsheaf of $f_*\mathcal{F}$.

Warning: This sheaf isn't the "correct one" if f is not separated.

Proof. Recall that the support of a section is closed (Étale Cohomology, Lemma 59.31.4) hence the material in Cohomology of Schemes, Section 30.26 applies. By the lemma above and Cohomology of Schemes, Lemma 30.26.6 we find that our subset of $f_*\mathcal{F}(V)$ is a subgroup. By Cohomology of Schemes, Lemma 30.26.4 we see that our rule defines a sub presheaf. Finally, suppose that we have $s \in f_*\mathcal{F}(V)$ and an étale covering $\{V_i \to V\}$ such that $s|_{V_i}$ has support proper over V_i. Observe that the support of $s|_{V_i}$ is the inverse image of the support of $s|_V$ (use the characterization of the support in terms of stalks and Étale Cohomology, Lemma 59.36.2). Whence the support of s is proper over V by Descent, Lemma 35.25.5. This proves that our rule satisfies the sheaf condition. □

[1] In fact, Lemma 63.2.1 shows, given Z over X which is isomorphic to a locally closed subscheme of some object W of $X_{étale}$, that the choice of W is irrelevant.

0F4Y **Lemma 63.3.2.** *Let $j : U \to X$ be a separated étale morphism. Let $\mathcal{F}$ be an abelian sheaf on $U_{étale}$. The image of the injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ of Étale Cohomology, Lemma 59.70.6 is the subsheaf of Lemma 63.3.1.*

An alternative would be to move this lemma later and prove this using the description of the stalks of both sheaves.

Proof. The construction of $j_!\mathcal{F} \to j_*\mathcal{F}$ in the proof of Étale Cohomology, Lemma 59.70.6 is via the construction of a map $j_{p!}\mathcal{F} \to j_*\mathcal{F}$ of presheaves whose image is clearly contained in the subsheaf of Lemma 63.3.1. Hence since $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$ we conclude the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ is contained in this subsheaf. Conversely, let $s \in j_*\mathcal{F}(V)$ have support Z proper over V. Then $Z \to V$ is finite with closed image $Z' \subset V$, see More on Morphisms, Lemma 37.44.1. The restriction of s to $V \setminus Z'$ is zero and the zero section is contained in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$. On the other hand, if $v \in Z'$, then we can find an étale neighbourhood $(V', v') \to (V, v)$ such that we have a decomposition $U_{V'} = W \amalg U'_1 \amalg \ldots \amalg U'_n$ into open and closed subschemes with $U'_i \to V'$ an isomorphism and with $T_{V'} \subset U'_1 \amalg \ldots \amalg U'_n$, see Étale Morphisms, Lemma 41.18.2. Inverting the isomorphisms $U'_i \to V'$ we obtain n morphisms $\varphi'_i : V' \to U$ and sections s'_i over V' by pulling back s. Then the section $\sum(\varphi'_i, s'_i)$ of $j_{p!}\mathcal{F}$ over V', see formula for $j_{p!}\mathcal{F}(V')$ in proof of Étale Cohomology, Lemma 59.70.6, maps to the restriction of s to V' by construction. We conclude that s is étale locally in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ and the proof is complete. □

0F4Z **Definition 63.3.3.** Let $f : X \to Y$ be a morphism of schemes which is separated (!) and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The subsheaf $f_!\mathcal{F} \subset f_*\mathcal{F}$ constructed in Lemma 63.3.1 is called the *direct image with compact support*.

By Lemma 63.3.2 this does not conflict with Étale Cohomology, Definition 59.70.1 as we have agreement when both definitions apply. Here is a sanity check.

0F51 **Lemma 63.3.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Then $f_! = f_*$.*

Proof. Immediate from the construction of $f_!$. □

A very useful observation is the following.

0F53 **Remark 63.3.5** (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $X' \subset X$ be an open subscheme. Denote $f' : X' \to Y$ the restriction of f. There is a canonical injective map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, let $V \in Y_{étale}$ and consider a section $s' \in f'_*(\mathcal{F}|_{X'})(V) = \mathcal{F}(X' \times_Y V)$ with support Z' proper over V. Then Z' is closed in $X \times_Y V$ as well, see Cohomology of Schemes, Lemma 30.26.5. Thus there is a unique section $s \in \mathcal{F}(X \times_Y V) = f_*\mathcal{F}(V)$ whose restriction to $X' \times_Y V$ is s' and whose restriction to $X \times_Y V \setminus Z'$ is zero, see Lemma 63.2.2. This construction is compatible with restriction maps and hence induces the desired map of sheaves $f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$ which is clearly injective. By

construction we obtain a commutative diagram

$$\begin{array}{ccc} f'_!(\mathcal{F}|_{X'}) & \longrightarrow & f_!\mathcal{F} \\ \downarrow & & \downarrow \\ f'_*(\mathcal{F}|_{X'}) & \longleftarrow & f_*\mathcal{F} \end{array}$$

functorial in $\mathcal{F}$. It is clear that for $X'' \subset X'$ open with $f'' = f|_{X''} : X'' \to Y$ the composition of the canonical maps $f''_!\mathcal{F}|_{X''} \to f'_!\mathcal{F}|_{X'} \to f_!\mathcal{F}$ just constructed is the canonical map $f''_!\mathcal{F}|_{X''} \to f_!\mathcal{F}$.

0F52 **Lemma 63.3.6.** *Let Y be a scheme. Let $j : X \to \overline{X}$ be an open immersion of schemes over Y with $\overline{X}$ proper over Y. Denote $f : X \to Y$ and $\overline{f} : \overline{X} \to Y$ the structure morphisms. For $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism (see proof)*

$$f_!\mathcal{F} \longrightarrow \overline{f}_!j_!\mathcal{F}$$

As we have $\overline{f}_! = \overline{f}_*$ by Lemma 63.3.4 we obtain $\overline{f}_* \circ j_! = f_!$ as functors $Ab(X_{étale}) \to Ab(Y_{étale})$.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 59.70.4. Thus the displayed arrow is the injective map $f_!(\mathcal{G}|_X) \to \overline{f}_!\mathcal{G}$ of Remark 63.3.5 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $V \in Y_{étale}$ and $s \in \overline{f}_!\mathcal{G}(V) = \overline{f}_*\mathcal{G}(V) = \mathcal{G}(\overline{X}_V)$ is a section, then the support of s is contained in the open $X_V \subset \overline{X}_V$. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. □

We want to relate the stalks of $f_!\mathcal{F}$ to sections with compact support on fibres. In order to state this, we need a definition.

0F72 **Definition 63.3.7.** Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let $H^0_c(X, \mathcal{F}) \subset H^0(X, \mathcal{F})$ be the set of sections whose support is proper over k. Elements of $H^0_c(X, \mathcal{F})$ are called *sections with compact support*.

Warning: This definition isn't the "correct one" if X isn't separated over k.

0F73 **Lemma 63.3.8.** *Let X be a proper scheme over a field k. Then $H^0_c(X, \mathcal{F}) = H^0(X, \mathcal{F})$.*

Proof. Immediate from the construction of H^0_c. □

0F74 **Remark 63.3.9** (Open embeddings and compactly supported sections)**.** Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Exactly as in Remark 63.3.5 for $X' \subset X$ open there is an injective map

$$H^0_c(X', \mathcal{F}|_{X'}) \longrightarrow H^0_c(X, \mathcal{F})$$

and these maps turn H^0_c into a "cosheaf" on the Zariski site of X.

0F75 **Lemma 63.3.10.** *Let k be a field. Let $j : X \to \overline{X}$ be an open immersion of schemes over k with $\overline{X}$ proper over k. For $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism (see proof)*

$$H^0_c(X, \mathcal{F}) \longrightarrow H^0_c(\overline{X}, j_!\mathcal{F}) = H^0(\overline{X}, j_!\mathcal{F})$$

where we have the equality on the right by Lemma 63.3.8.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 59.70.4. Thus the displayed arrow is the injective map $H^0_c(X, \mathcal{G}|_X) \to H^0_c(\overline{X}, \mathcal{G})$ of Remark 63.3.9 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $s \in H^0(\overline{X}, \mathcal{G})$ is a section, then the support of s is contained in the open X. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. □

0F76 **Lemma 63.3.11.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then there is a canonical isomorphism*

$$(f_!\mathcal{F})_{\overline{y}} \longrightarrow H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Recall that $(f_*\mathcal{F})_{\overline{y}} = \mathrm{colim}\, f_*\mathcal{F}(V)$ where the colimit is over the étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$. If $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$, then we can pullback s to a section of $\mathcal{F}$ over $(X_V)_{\overline{v}} = X_{\overline{y}}$. Thus we obtain a canonical map

$$c_{\overline{y}} : (f_*\mathcal{F})_{\overline{y}} \longrightarrow H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

We claim that this map induces a bijection between the subgroups $(f_!\mathcal{F})_{\overline{y}}$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. The claim implies the lemma, but is a little bit more precise in that it describes the identification of the lemma as given by pullbacks of sections of $\mathcal{F}$ to the geometric fibre of f.

Observe that any element $s \in (f_!\mathcal{F})_{\overline{y}} \subset (f_*\mathcal{F})_{\overline{y}}$ is mapped by $c_{\overline{y}}$ to an element of $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \subset H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. This is true because taking the support of a section commutes with pullback and because properness is preserved by base change. This at least produces the map in the statement of the lemma. To prove that it is an isomorphism we may work Zariski locally on Y and hence we may and do assume Y is affine.

An observation that we will use below is that given an open subscheme $X' \subset X$ and if $f' = f|_{X'}$, then we obtain a commutative diagram

$$\begin{array}{ccc} (f'_!(\mathcal{F}|_{X'}))_{\overline{y}} & \longrightarrow & H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \\ \downarrow & & \downarrow \\ (f_!\mathcal{F})_{\overline{y}} & \longrightarrow & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \end{array}$$

where the horizontal arrows are the maps constructed above and the vertical arrows are given in Remarks 63.3.5 and 63.3.9. The reason is that given an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$ and a section $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ whose support Z happens to be contained in X'_V and is proper over V, so that s gives rise to an element of both $(f'_!(\mathcal{F}|_{X'}))_{\overline{y}}$ and $(f_!\mathcal{F})_{\overline{y}}$ which correspond via the vertical arrow of the diagram, then these elements are mapped via the horizontal arrows to the pullback $s|_{X_{\overline{y}}}$ of s to $X_{\overline{y}}$ whose support $Z_{\overline{y}}$ is contained in $X'_{\overline{y}}$ and hence this restriction gives rise to a compatible pair of elements of $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$.

Suppose $s \in (f_!\mathcal{F})_{\overline{y}}$ maps to zero in $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Say s corresponds to $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ with support Z proper over V. We may assume that V is affine and hence Z is quasi-compact. Then we may choose a quasi-compact open $X' \subset X$ containing the image of Z. Then Z is contained in X'_V and hence s is the image of

an element $s' \in f'_!(\mathcal{F}|_{X'})(V)$ where $f' = f|_{X'}$ as in the previous paragraph. Then s' maps to zero in $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$. Hence in order to prove injectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

Suppose that $t \in H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Then the support of t is contained in a quasi-compact open subscheme $W \subset X_{\overline{y}}$. Hence we can find a quasi-compact open subscheme $X' \subset X$ such that $X'_{\overline{y}}$ contains W. Then it is clear that t is contained in the image of the injective map $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \to H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Hence in order to show surjectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

In this last paragraph of the proof we prove the lemma in case X is quasi-compact and Y is affine. By More on Flatness, Theorem 38.33.8 there exists a compactification $j : X \to \overline{X}$ over Y. Set $\mathcal{G} = j_!\mathcal{F}$ so that $\mathcal{F} = \mathcal{G}|_X$ by Étale Cohomology, Lemma 59.70.4. By the discussion above we get a commutative diagram

$$\begin{array}{ccc} (f_!\mathcal{F})_{\overline{y}} & \longrightarrow & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \\ \downarrow & & \downarrow \\ (\overline{f}_!\mathcal{G})_{\overline{y}} & \longrightarrow & H^0_c(\overline{X}_{\overline{y}}, \mathcal{G}|_{\overline{X}_{\overline{y}}}) \end{array}$$

By Lemmas 63.3.6 and 63.3.10 the vertical maps are isomorphisms. This reduces us to the case of the proper morphism $\overline{X} \to Y$. For a proper morphism our map is an isomorphism by Lemmas 63.3.4 and 63.3.8 and proper base change for pushforwards, see Étale Cohomology, Lemma 59.91.4. □

0F55 **Lemma 63.3.12.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f separated and locally of finite type. For any abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $f'_!(g')^{-1}\mathcal{F} = g^{-1}f_!\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 7.45. We claim that this map sends $g^{-1}f_!\mathcal{F}$ into the subsheaf $f'_!(g')^{-1}\mathcal{F}$ and induces the isomorphism in the lemma.

Choose a geometric point $\overline{y}' : \mathrm{Spec}(k) \to Y'$ and denote $\overline{y} = g \circ \overline{y}'$ the image in Y. There is a commutative diagram

$$\begin{array}{ccc} (f_*\mathcal{F})_{\overline{y}} & \longrightarrow & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \\ \downarrow & & \downarrow \\ (f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} & \longrightarrow & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}}) \end{array}$$

where the horizontal maps were used in the proof of Lemma 63.3.11 and the vertical maps are the pullback maps above. The diagram commutes because each of the four maps in question is given by pulling back local sections along a morphism of schemes and the underlying diagram of morphisms of schemes commutes. Since the

diagram in the statement of the lemma is cartesian we have $X'_{\overline{y}'} = X_{\overline{y}}$. Hence by Lemma 63.3.11 and its proof we obtain a commutative diagram

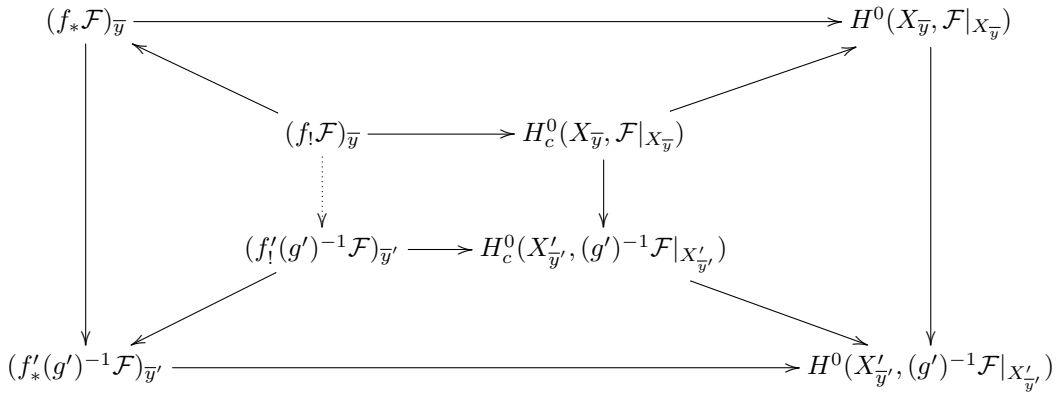

where the horizontal arrows of the inner square are isomorphisms and the two right vertical arrows are equalities. Also, the se, sw, ne, nw arrows are injective. It follows that there is a unique bijective dotted arrow fitting into the diagram. We conclude that $g^{-1}f_!\mathcal{F} \subset g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$ is mapped into the subsheaf $f'_!(g')^{-1}\mathcal{F} \subset f'_*(g')^{-1}\mathcal{F}$ because this is true on stalks, see Étale Cohomology, Theorem 59.29.10. The same theorem then implies that the induced map is an isomorphism and the proof is complete. □

0F50 **Lemma 63.3.13.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of schemes which are separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then $g_!f_!\mathcal{F} = (g \circ f)_!\mathcal{F}$ as subsheaves of $(g \circ f)_*\mathcal{F}$.*

Proof. We strongly urge the reader to prove this for themselves. Let $W \in Z_{étale}$ and $s \in (g \circ f)_*\mathcal{F}(W) = \mathcal{F}(X_W)$. Denote $T \subset X_W$ the support of s; this is a closed subset. Observe that s is a section of $(g \circ f)_!\mathcal{F}$ if and only if T is proper over W. We have $f_!\mathcal{F} \subset f_*\mathcal{F}$ and hence $g_!f_!\mathcal{F} \subset g_!f_*\mathcal{F} \subset g_*f_*\mathcal{F}$. On the other hand, s is a section of $g_!f_!\mathcal{F}$ if and only if (a) T is proper over Y_W and (b) the support T' of s viewed as section of $f_!\mathcal{F}$ is proper over W. If (a) holds, then the image of T in Y_W is closed and since $f_!\mathcal{F} \subset f_*\mathcal{F}$ we see that $T' \subset Y_W$ is the image of T (details omitted; look at stalks).

The conclusion is that we have to show a closed subset $T \subset X_W$ is proper over W if and only if T is proper over Y_W and the image of T in Y_W is proper over W. Let us endow T with the reduced induced closed subscheme structure. If T is proper over W, then $T \to Y_W$ is proper by Morphisms, Lemma 29.41.7 and the image of T in Y_W is proper over W by Cohomology of Schemes, Lemma 30.26.5. Conversely, if T is proper over Y_W and the image of T in Y_W is proper over W, then the morphism $T \to W$ is proper as a composition of proper morphisms (here we endow the closed image of T in Y_W with its reduced induced scheme structure to turn the question into one about morphisms of schemes), see Morphisms, Lemma 29.41.4. □

0F77 **Remark 63.3.14.** The isomorphisms between functors constructed above satisfy the following two properties:

(1) Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to T$ be composable morphisms of schemes which are separated and locally of finite type. Then the diagram

$$\begin{array}{ccc} (h \circ g \circ f)_! & \longrightarrow & (h \circ g)_! \circ f_! \\ \downarrow & & \downarrow \\ h_! \circ (g \circ f)_! & \longrightarrow & h_! \circ g_! \circ f_! \end{array}$$

commutes where the arrows are those of Lemma 63.3.13.

(2) Suppose that we have a diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{c} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{b} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{a} & Z \end{array}$$

with both squares cartesian and f and g separated and locally of finite type. Then the diagram

$$\begin{array}{ccccc} a^{-1} \circ (g \circ f)_! & & \longrightarrow & & (g' \circ f')_! \circ c^{-1} \\ \downarrow & & & & \downarrow \\ a^{-1} \circ g_! \circ f_! & \longrightarrow & g'_! \circ b^{-1} \circ f_! & \longrightarrow & g'_! \circ f'_! \circ c^{-1} \end{array}$$

commutes where the horizontal arrows are those of Lemma 63.3.12 the arrows are those of Lemma 63.3.13.

Part (1) holds true because we have a similar commutative diagram for pushforwards. Part (2) holds by the very general compatibility of base change maps for pushforwards (Sites, Remark 7.45.3) and the fact that the isomorphisms in Lemmas 63.3.12 and 63.3.13 are constructed using the corresponding maps of pushforwards.

0F54 **Lemma 63.3.15.** *Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $X = \bigcup_{i \in I} X_i$ be an open covering such that for all $i, j \in I$ there exists a k with $X_i \cup X_j \subset X_k$. Denote $f_i : X_i \to Y$ the restriction of f. Then*

$$f_! \mathcal{F} = \operatorname{colim}_{i \in I} f_{i,!}(\mathcal{F}|_{X_i})$$

functorially in $\mathcal{F} \in Ab(X_{étale})$ where the transition maps are the ones constructed in Remark 63.3.5.

Proof. It suffices to show that the canonical map from right to left is a bijection when evaluated on a quasi-compact object V of $Y_{étale}$. Observe that the colimit on the right hand side is directed and has injective transition maps. Thus we can use Sites, Lemma 7.17.7 to evaluate the colimit. Hence, the statement comes down to the observation that a closed subset $Z \subset X_V$ proper over V is quasi-compact and hence is contained in $X_{i,V}$ for some i. □

0F56 **Lemma 63.3.16.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Then functor $f_!$ commutes with direct sums.*

Proof. Let $\mathcal{F} = \bigoplus \mathcal{F}_i$. To show that the map $\bigoplus f_!\mathcal{F}_i \to f_!\mathcal{F}$ is an isomorphism, it suffices to show that these sheaves have the same sections over a quasi-compact object V of $Y_{étale}$. Replacing Y by V it suffices to show $H^0(Y, f_!\mathcal{F}) \subset H^0(X, \mathcal{F})$ is equal to $\bigoplus H^0(Y, f_!\mathcal{F}_i) \subset \bigoplus H^0(X, \mathcal{F}_i) \subset H^0(X, \bigoplus \mathcal{F}_i)$. In this case, by writing X as the union of its quasi-compact opens and using Lemma 63.3.15 we reduce to the case where X is quasi-compact as well. Then $H^0(X, \mathcal{F}) = \bigoplus H^0(X, \mathcal{F}_i)$ by Étale Cohomology, Theorem 59.51.3. Looking at supports of sections the reader easily concludes. □

0F57 **Lemma 63.3.17.** *Let $f : X \to Y$ be a morphism of schemes which is separated and locally quasi-finite. Then*

(1) *for $\mathcal{F}$ in $Ab(X_{étale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

functorially in $\mathcal{F}$, and

(2) *the functor $f_!$ is exact.*

Proof. The functor $f_!$ is left exact by construction. Right exactness may be checked on stalks (Étale Cohomology, Theorem 59.29.10). Thus it suffices to prove part (1).

Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. The scheme $X_{\overline{y}}$ has a discrete underlying topological space (Morphisms, Lemma 29.20.8) and all the residue fields at the points are equal to k (as finite extensions of k). Hence $\{\overline{x} : \mathrm{Spec}(k) \to X : f(\overline{x}) = \overline{y}\}$ is equal to the set of points of $X_{\overline{y}}$. Thus the computation of the stalk follows from the more general Lemma 63.3.11. □

63.4. Sections with finite support

0F6E In this section we extend the construction of Section 63.3 to not necessarily separated locally quasi-finite morphisms.

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Given V in $Y_{étale}$ denote $X_V = X \times_Y V$ the base change. We are going to consider the group of finite formal sums

0F6J (63.4.0.1) $$s = \sum\nolimits_{i=1,\ldots,n} (Z_i, s_i)$$

where $Z_i \subset X_V$ is a locally closed subscheme such that the morphism $Z_i \to V$ is finite[2] and where $s_i \in H_{Z_i}(\mathcal{F})$. Here, as in Section 63.2, we set

$$H_{Z_i}(\mathcal{F}) = \{s_i \in \mathcal{F}(U_i) \mid \mathrm{Supp}(s_i) \subset Z_i\}$$

where $U_i \subset X_V$ is an open subscheme containing Z_i as a closed subscheme. We are going to consider these formal sums modulo the following relations

0F6K (1) $(Z, s) + (Z, s') = (Z, s + s')$,

0F6L (2) $(Z, s) = (Z', s)$ if $Z \subset Z'$.

Observe that the second relation makes sense: since $Z \to V$ is finite and $Z' \to V$ is separated, the inclusion $Z \to Z'$ is closed and we can use the map discussed in (1).

Let us denote $f_{p!}\mathcal{F}(V)$ the quotient of the abelian group of formal sums (63.4.0.1) by these relations. The first relation tells us that $f_{p!}\mathcal{F}(V)$ is a quotient of the direct

[2]Since f is locally quasi-finite, the morphism $Z_i \to V$ is finite if and only if it is proper.

sum of the abelian groups $H_Z(\mathcal{F})$ over all locally closed subschemes $Z \subset X_V$ finite over V. The second relation tells us that we are really taking the colimit

0F6M (63.4.0.2) $$f_{p!}\mathcal{F}(V) = \operatorname{colim}_Z H_Z(\mathcal{F})$$

This formula will be a convenient abstract way to think about our construction.

Next, we observe that there is a natural way to turn this construction into a presheaf $f_{p!}\mathcal{F}$ of abelian groups on $Y_{étale}$. Namely, given $V' \to V$ in $Y_{étale}$ we obtain the base change morphism $X_{V'} \to X_V$. If $Z \subset X_V$ is a locally closed subscheme finite over V, then the scheme theoretic inverse image $Z' \subset X_{V'}$ is finite over V'. Moreover, if $U \subset X_V$ is an open such that Z is closed in U, then the inverse image $U' \subset X_{V'}$ is an open such that Z' is closed in U'. Hence the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ of $\mathcal{F}$ sends $H_Z(\mathcal{F})$ into $H_{Z'}(\mathcal{F})$; this is a special case of the functoriality discussed in (2) above. Clearly, these maps are compatible with inclusions $Z_1 \subset Z_2$ of such locally closed subschemes of X_V and we obtain a map

$$f_{p!}\mathcal{F}(V) = \operatorname{colim}_Z H_Z(\mathcal{F}) \longrightarrow \operatorname{colim}_{Z'} H_{Z'}(\mathcal{F}) = f_{p!}\mathcal{F}(V')$$

These maps indeed turn $f_{p!}\mathcal{F}$ into a presheaf of abelian groups on $Y_{étale}$. We omit the details.

A final observation is that the construction of $f_{p!}\mathcal{F}$ is functorial in $\mathcal{F}$ in $Ab(X_{étale})$. We conclude that given a locally quasi-finite morphism $f : X \to Y$ we have constructed a functor

$$f_{p!} : Ab(X_{étale}) \longrightarrow PAb(Y_{étale})$$

from the category of abelian sheaves on $X_{étale}$ to the category of abelian presheaves on $Y_{étale}$. Before we define $f_!$ as the sheafification of this functor, let us check that it agrees with the construction in Section 63.3 and with the construction in Étale Cohomology, Section 59.70 when both apply.

0F6N **Lemma 63.4.1.** *Let $f : X \to Y$ be a separated and locally quasi-finite morphism of schemes. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism(!)*

$$f_{p!}\mathcal{F} \longrightarrow f_!\mathcal{F}$$

of abelian presheaves which identifies the sheaf $f_!\mathcal{F}$ of Definition 63.3.3 with the presheaf $f_{p!}\mathcal{F}$ constructed above.

Proof. Let V be an object of $Y_{étale}$. If $Z \subset X_V$ is locally closed and finite over V, then, since f is separated, we see that the morphism $Z \to X_V$ is a closed immersion. Moreover, if Z_i, $i = 1, \ldots, n$ are closed subschemes of X_V finite over V, then $Z_1 \cup \ldots \cup Z_n$ (scheme theoretic union) is a closed subscheme finite over V. Hence in this case the colimit (63.4.0.2) defining $f_{p!}\mathcal{F}(V)$ is directed and we find that $f_{!p}\mathcal{F}(V)$ is simply equal to the set of sections of $\mathcal{F}(X_V)$ whose support is finite over V. Since any closed subset of X_V which is proper over V is actually finite over V (as f is locally quasi-finite) we conclude that this is equal to $f_!\mathcal{F}(V)$ by its very definition. □

0F6P **Lemma 63.4.2.** *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. Let $\overline{y} : \operatorname{Spec}(k) \to Y$ be a geometric point. Functorially in $\mathcal{F}$ in $Ab(X_{étale})$ we have*

$$(f_{p!}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Proof. Recall that the stalk at $\overline{y}$ of a presheaf is defined by the usual colimit over étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$, see Étale Cohomology, Definition 59.29.6. Accordingly suppose $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) is an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Then since

$$X_{\overline{y}} = (X_V)_{\overline{v}} \supset Z_{i,\overline{v}}$$

and since s_i is a section of $\mathcal{F}$ on an open neighbourhood of Z_i in X_V we can send s to

$$\sum\nolimits_{i=1,\ldots,n} \sum\nolimits_{\overline{x} \in Z_{i,\overline{v}}} (\text{class of } s_i \text{ in } \mathcal{F}_{\overline{x}}) \quad \in \quad \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

We omit the verification that this is compatible with restriction maps and that the relations (1) $(Z, s) + (Z, s') - (Z, s + s')$ and (2) $(Z, s) - (Z', s)$ if $Z \subset Z'$ are sent to zero. Thus we obtain a map

$$(f_{p!}\mathcal{F})_{\overline{y}} \longrightarrow \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Let us prove this arrow is surjective. For this it suffices to pick an $\overline{x}$ with $f(\overline{x}) = \overline{y}$ and prove that an element s in the summand $\mathcal{F}_{\overline{x}}$ is in the image. Let s correspond to the element $s \in \mathcal{F}(U)$ where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$. Since f is locally quasi-finite, the morphism $U \to Y$ is locally quasi-finite too. By More on Morphisms, Lemma 37.41.3 we can find an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$, an open subscheme

$$W \subset U \times_Y V,$$

and a geometric point $\overline{w}$ mapping to $\overline{u}$ and $\overline{v}$ such that $W \to V$ is finite and $\overline{w}$ is the only geometric point of W mapping to $\overline{v}$. (We omit the translation between the language of geometric points we are currently using and the language of points and residue field extensions used in the statement of the lemma.) Observe that $W \to X_V = X \times_Y V$ is étale. Choose an affine open neighbourhood $W' \subset X_V$ of the image $\overline{w}'$ of $\overline{w}$. Since $\overline{w}$ is the only point of W over $\overline{v}$ and since $W \to V$ is closed, after replacing V by an open neighbourhood of $\overline{v}$, we may assume $W \to X_V$ maps into W'. Then $W \to W'$ is finite and étale and there is a unique geometric point $\overline{w}$ of W lying over $\overline{w}'$. It follows that $W \to W'$ is an open immersion over an open neighbourhood of $\overline{w}'$ in W', see Étale Morphisms, Lemma 41.14.2. Shrinking V and W' we may assume $W \to W'$ is an isomorphism. Thus s may be viewed as a section s' of $\mathcal{F}$ over the open subscheme $W' \subset X_V$ which is finite over V. Hence by definition (W', s') defines an element of $j_{p!}\mathcal{F}(V)$ which maps to s as desired.

Let us prove the arrow is injective. To do this, let $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) be an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Assume s maps to zero under the map constructed above. First, after replacing $(V, \overline{v})$ by an étale neighbourhood of itself, we may assume there exist decompositions $Z_i = Z_{i,1} \amalg \ldots \amalg Z_{i,m_i}$ into open and closed subschemes such that each $Z_{i,j}$ has exactly one geometric point over $\overline{v}$. Say under the obvious direct sum decomposition

$$H_{Z_i}(\mathcal{F}) = \bigoplus H_{Z_{i,j}}(\mathcal{F})$$

the element s_i corresponds to $\sum s_{i,j}$. We may use relations (1) and (2) to replace s by $\sum_{i=1,\ldots,n} \sum_{j=1,\ldots,m_i}(Z_{i,j}, s_{i,j})$. In other words, we may assume Z_i has a unique geometric point lying over $\overline{v}$. Let $\overline{x}_1, \ldots, \overline{x}_m$ be the geometric points of X over $\overline{y}$ corresponding to the geometric points of our Z_i over $\overline{v}$; note that for one $j \in \{1, \ldots, m\}$ there may be multiple indices i for which $\overline{x}_j$ corresponds to a point

of Z_i. By More on Morphisms, Lemma 37.41.3 applied to both $X_V \to V$ after replacing $(V, \overline{v})$ by an étale neighbourhood of itself we may assume there exist open subschemes

$$W_j \subset X \times_Y V, \quad j = 1, \ldots, m$$

and a geometric point $\overline{w}_j$ of W_j mapping to $\overline{x}_j$ and $\overline{v}$ such that $W_j \to V$ is finite and $\overline{w}_j$ is the only geometric point of W_j mapping to $\overline{v}$. After shrinking V we may assume $Z_i \subset W_j$ for some j and we have the map $H_{Z_i}(\mathcal{F}) \to H_{W_j}(\mathcal{F})$. Thus by the relation (2) we see that our element is equivalent to an element of the form

$$\sum\nolimits_{j=1,\ldots,m} (W_j, t_j)$$

for some $t_j \in H_{W_j}(\mathcal{F})$. Clearly, this element is mapped simply to the class of t_j in the summand $\mathcal{F}_{\overline{x}_j}$. Since s maps to zero, we find that t_j maps to zero in $\mathcal{F}_{\overline{x}_j}$. This implies that t_j restricts to zero on an open neighbourhood of $\overline{w}_j$ in W_j, see Étale Cohomology, Lemma 59.31.2. Shrinking V once more we obtain $t_j = 0$ for all j as desired. □

0F6Q **Lemma 63.4.3.** *Let $f = j : U \to X$ be an étale morphism of schemes. Denote $j_{p!}$ the construction of Étale Cohomology, Equation (59.70.1.1) and denote $f_{p!}$ the construction above. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical map*

$$j_{p!}\mathcal{F} \longrightarrow f_{p!}\mathcal{F}$$

of abelian presheaves which identifies the sheaf $j_!\mathcal{F} = (j_{p!}\mathcal{F})^\#$ of Étale Cohomology, Definition 59.70.1 with $(f_{p!}\mathcal{F})^\#$.

Proof. Please read the proof of Étale Cohomology, Lemma 59.70.6 before reading the proof of this lemma. Let V be an object of $X_{étale}$. Recall that

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U)$$

Given φ we obtain an open subscheme $Z_\varphi \subset U_V = U \times_X V$, namely, the image of the graph of φ. Via φ we obtain an isomorphism $V \to Z_\varphi$ over U and we can think of an element

$$s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U) = \mathcal{F}(Z_\varphi) = H_{Z_\varphi}(\mathcal{F})$$

as a section of $\mathcal{F}$ over Z_φ. Since $Z_\varphi \subset U_V$ is open, we actually have $H_{Z_\varphi}(\mathcal{F}) = \mathcal{F}(Z_\varphi)$ and we can think of s_φ as an element of $H_{Z_\varphi}(\mathcal{F})$. Having said this, our map $j_{p!}\mathcal{F} \to f_{p!}\mathcal{F}$ is defined by the rule

$$\sum\nolimits_{i=1,\ldots,n} s_{\varphi_i} \longmapsto \sum\nolimits_{i=1,\ldots,n} (Z_{\varphi_i}, s_{\varphi_i})$$

with right hand side a sum as in (63.4.0.1). We omit the verification that this is compatible with restriction mappings and functorial in $\mathcal{F}$.

To finish the proof, we claim that given a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ there is a commutative diagram

$$\begin{array}{ccc} (j_{p!}\mathcal{F})_{\overline{y}} & \longrightarrow & \bigoplus_{j(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \\ \downarrow & & \| \\ (f_{p!}\mathcal{F})_{\overline{y}} & \longrightarrow & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \end{array}$$

where the top horizontal arrow is constructed in the proof of Étale Cohomology, Proposition 59.70.3, the bottom horizontal arrow is constructed in the proof of Lemma 63.4.2, the right vertical arrow is the obvious equality, and the left vertical arrow is the map defined in the previous paragraph on stalks. The claim follows in a straightforward manner from the explicit description of all of the arrows involved here and in the references given. Since the horizontal arrows are isomorphisms we conclude so is the left vertical arrow. Hence we find that our map induces an isomorphism on sheafifications by Étale Cohomology, Theorem 59.29.10. □

0F6R **Definition 63.4.4.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. We define the *direct image with compact support* to be the functor

$$f_! : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

defined by the formula $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$, i.e., $f_!\mathcal{F}$ is the sheafification of the presheaf $f_{p!}\mathcal{F}$ constructed above.

By Lemma 63.4.1 this does not conflict with Definition 63.3.3 (when both definitions apply) and by Lemma 63.4.3 this does not conflict with Étale Cohomology, Definition 59.70.1 (when both definitions apply).

0F5F **Lemma 63.4.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then*

(1) *for $\mathcal{F}$ in $Ab(X_{étale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

functorially in $\mathcal{F}$, and

(2) *the functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact and commutes with direct sums.*

Proof. The formula for the stalks is immediate (and in fact equivalent) to Lemma 63.4.2. The exactness of the functor follows immediately from this and the fact that exactness may be checked on stalks, see Étale Cohomology, Theorem 59.29.10. □

0F6S **Remark 63.4.6** (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \to Y$ the restriction of f. We claim there is a canonical map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, this map will be the sheafification of a canonical map

$$f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$$

constructed as follows. Let $V \in Y_{étale}$ and consider a section $s' = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (63.4.0.1) defining an element of $f'_{p!}(\mathcal{F}|_{X'})(V)$. Then $Z'_i \subset X'_V$ may also be viewed as a locally closed subscheme of X_V and we have $H_{Z'_i}(\mathcal{F}|_{X'}) = H_{Z'_i}(\mathcal{F})$. We will map s' to the exact same sum $s = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ but now viewed as an element of $f_{p!}\mathcal{F}(V)$. We omit the verification that this construction is compatible with restriction mappings and functorial in $\mathcal{F}$. This construction has the following properties:

(1) The maps $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ and $f'_!\mathcal{F}' \to f_!\mathcal{F}$ are compatible with the description of stalks given in Lemmas 63.4.2 and 63.4.5.

(2) If f is separated, then the map $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ is the same as the map constructed in Remark 63.3.5 via the isomorphism in Lemma 63.4.1.
(3) If $X'' \subset X'$ is another open, then the composition of $f''_{p!}(\mathcal{F}|_{X''}) \to f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$ is the map $f''_{p!}(\mathcal{F}|_{X''}) \to f_{p!}\mathcal{F}$ for the inclusion $X'' \subset X$. Sheafifying we conclude the same holds true for $f''_!(\mathcal{F}|_{X''}) \to f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$.
(4) The map $f'_!\mathcal{F}' \to f_!\mathcal{F}$ is injective because we can check this on stalks.

All of these statements are easily proven by representing elements as finite sums as above and considering what happens to these elements.

0F5H **Lemma 63.4.7.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $X = \bigcup_{i\in I} X_i$ be an open covering. Then there exists an exact complex*

$$\ldots \to \bigoplus\nolimits_{i_0,i_1,i_2} f_{i_0i_1i_2,!}\mathcal{F}|_{X_{i_0i_1i_2}} \to \bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}} \to f_!\mathcal{F} \to 0$$

functorial in $\mathcal{F} \in Ab(X_{étale})$, see proof for details.

Proof. Here as usual we set $X_{i_0\ldots i_p} = X_{i_0} \cap \ldots \cap X_{i_p}$ and we denote $f_{i_0\ldots i_p}$ the restriction of f to $X_{i_0\ldots i_p}$. The maps in the complex are the maps constructed in Remark 63.4.6 with sign rules as in the Čech complex. Exactness follows easily from the description of stalks in Lemma 63.4.5. Details omitted. □

0F5I **Remark 63.4.8** (Alternative construction)**.** Lemma 63.4.7 gives an alternative construction of the functor $f_!$ for locally quasi-finite morphisms f. Namely, given a locally quasi-finite morphism $f : X \to Y$ of schemes we can choose an open covering $X = \bigcup_{i\in I} X_i$ such that each $f_i : X_i \to Y$ is separated. For example choose an affine open covering of X. Then we can define $f_!\mathcal{F}$ as the cokernel of the penultimate map of the complex of the lemma, i.e.,

$$f_!\mathcal{F} = \text{Coker}\left(\bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}}\right)$$

where we can use the construction of $f_{i_0,!}$ and $f_{i_0i_1,!}$ in Section 63.3 because the morphisms f_{i_0} and $f_{i_0i_1}$ are separated. One can then compute the stalks of $f_!$ (using the separated case, namely Lemma 63.3.17) and obtain the result of Lemma 63.4.5. Having done so all the other results of this section can be deduced from this as well.

0F78 **Remark 63.4.9.** Let $g : Y' \to Y$ be a morphism of schemes. For an abelian presheaf $\mathcal{G}'$ on $Y'_{étale}$ let us denote $g_*\mathcal{G}'$ the presheaf $V \mapsto \mathcal{G}'(Y' \times_Y V)$. If $\alpha : \mathcal{G} \to g_*\mathcal{G}'$ is a map of abelian presheaves on $Y_{étale}$, then there is a unique map $\alpha^\# : \mathcal{G}^\# \to g_*((\mathcal{G}')^\#)$ of abelian sheaves on $Y_{étale}$ such that the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\alpha} & g_*\mathcal{G}' \\ \downarrow & & \downarrow \\ \mathcal{G}^\# & \xrightarrow{\alpha^\#} & g_*((\mathcal{G}')^\#) \end{array}$$

is commutative where the vertical maps come from the canonical maps $\mathcal{G} \to \mathcal{G}^\#$ and $\mathcal{G}' \to (\mathcal{G}')^\#$. If $\alpha' : g^{-1}\mathcal{G}^\# \to (\mathcal{G}')^\#$ is the map adjoint to $\alpha^\#$, then for a geometric point $\overline{y}' : \text{Spec}(k) \to Y'$ with image $\overline{y} = g \circ \overline{y}'$ in Y, the map

$$\alpha'_{\overline{y}'} : \mathcal{G}_{\overline{y}} = (\mathcal{G}^\#)_{\overline{y}} = (g^{-1}\mathcal{G}^\#)_{\overline{y}'} \longrightarrow (\mathcal{G}')^\#_{\overline{y}'} = \mathcal{G}'_{\overline{y}'}$$

is given by mapping the class in the stalk of a section s of $\mathcal{G}$ over an étale neighbourhood $(V, \overline{v})$ to the class of the section $\alpha(s)$ in $g_*\mathcal{G}'(V) = \mathcal{G}'(Y' \times_Y V)$ over the étale neighbourhood $(Y' \times_Y V, (\overline{y}', \overline{v}))$ in the stalk of $\mathcal{G}'$ at $\overline{y}'$.

0F5J **Lemma 63.4.10.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f locally quasi-finite. There is an isomorphism $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$ functorial for $\mathcal{F}$ in $Ab(X_{étale})$ which is compatible with the descriptions of stalks given in Lemma 63.4.5 (see proof for the precise statement).

Proof. With conventions as in Remark 63.4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_*f'_{p!}(g')^{-1}\mathcal{F}$$

of abelian presheaves on $Y_{étale}$. By the discussion in Remark 63.4.9 this will determine a canonical map $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$. Finally, we will show this map induces isomorphisms on stalks and conclude by Étale Cohomology, Theorem 59.29.10.

Construction of the map c. Let $V \in Y_{étale}$ and consider a section $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (63.4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. The value of $g_*f'_{p!}(g')^{-1}\mathcal{F}$ at V is $f'_{p!}(g')^{-1}\mathcal{F}(V')$ where $V' = V \times_Y Y'$. Denote $Z'_i \subset X'_{V'}$ the base change of Z_i to V'. By (2) there is a pullback map $H_{Z_i}(\mathcal{F}) \to H_{Z'_i}((g')^{-1}\mathcal{F})$. Denoting $s'_i \in H_{Z'_i}((g')^{-1}\mathcal{F})$ the image of s_i under pullback, we set $c(s) = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (63.4.0.1) defining an element of $f'_{p!}(g')^{-1}\mathcal{F}(V')$. We omit the verification that this construction is compatible the relations (1) and (2) and compatible with restriction mappings. The construction is clearly functorial in $\mathcal{F}$.

Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image $\overline{y} = g \circ \overline{y}'$ in Y. Observe that $X'_{\overline{y}'} = X_{\overline{y}}$ by transitivity of fibre products. Hence g' produces a bijection $\{f'(\overline{x}') = \overline{y}'\} \to \{f(\overline{x}) = \overline{y}\}$ and if $\overline{x}'$ maps to $\overline{x}$, then $((g')^{-1}\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}}$ by Étale Cohomology, Lemma 59.36.2. Now we claim that the diagram

$$\begin{array}{ccccc} (g^{-1}f_!\mathcal{F})_{\overline{y}'} & = & (f_!\mathcal{F})_{\overline{y}} & \longrightarrow & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \\ \downarrow & \swarrow & & & \downarrow \\ (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} & & \longrightarrow & & \bigoplus_{f'(\overline{x}')=\overline{y}'} (g')^{-1}\mathcal{F}_{\overline{x}'} \end{array}$$

commutes where the horizontal arrows are given in the proof of Lemma 63.4.2 and where the right vertical arrow is an equality by what we just said above. The southwest arrow is described in Remark 63.4.9 as the pullback map, i.e., simply given by our construction c above. Then the simple description of the image of a sum $\sum(Z_i, z_i)$ in the stalk at $\overline{x}$ given in the proof of Lemma 63.4.2 immediately shows the diagram commutes. This finishes the proof of the lemma. □

0F79 **Lemma 63.4.11.** *Let $f' : X \to Y'$ and $g : Y' \to Y$ be composable morphisms of schemes with f' and $f = g \circ f'$ locally quasi-finite and g separated and locally of finite type. Then there is a canonical isomorphism of functors $g_! \circ f'_! = f_!$. This isomorphism is compatible with*

(a) *covariance with respect to open embeddings as in Remarks 63.3.5 and 63.4.6,*
(b) *the base change isomorphisms of Lemmas 63.4.10 and 63.3.12, and*
(c) *equal to the isomorphism of Lemma 63.3.13 via the identifications of Lemma 63.4.1 in case f' is separated.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. With conventions as in Remark 63.4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_* f'_{p!}\mathcal{F}$$

of abelian presheaves on $Y_{étale}$. By the discussion in Remark 63.4.9 this will determine a canonical map $c^\# : f_!\mathcal{F} \to g_* f'_!\mathcal{F}$. We will show that $c^\#$ has image contained in the subsheaf $g_! f'_!\mathcal{F}$, thereby obtaining a map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$. Next, we will prove (a), (b), and (c) that. Finally, part (b) will allow us to show that c' is an isomorphism.

Construction of the map c. Let $V \in Y_{étale}$ and let $s = \sum(Z_i, s_i)$ be a sum as in (63.4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. Recall that $Z_i \subset X_V = X \times_Y V$ is a locally closed subscheme finite over V. Setting $V' = Y' \times_Y V$ we get $X_{V'} = X \times_{Y'} V' = X_V$. Hence $Z_i \subset X_{V'}$ is locally closed and Z_i is finite over V' because g is separated (Morphisms, Lemma 29.44.14). Hence we may set $c(s) = \sum(Z_i, s_i)$ but now viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_* f'_{p!}\mathcal{F})(V)$. The construction is clearly compatible with relations (1) and (2) and compatible with restriction mappings and hence we obtain the map c.

Observe that in the discussion above our section $c(s) = \sum(Z_i, s_i)$ of $f'_!\mathcal{F}$ over V' restricts to zero on $V' \setminus \mathrm{Im}(\coprod Z_i \to V')$. Since $\mathrm{Im}(\coprod Z_i \to V')$ is proper over V (for example by Morphisms, Lemma 29.41.10) we conclude that $c(s)$ defines a section of $g_! f'_!\mathcal{F} \subset g_* f'_!\mathcal{F}$ over V. Since every local section of $f_!\mathcal{F}$ locally comes from a local section of $f_{p!}\mathcal{F}$ we conclude that the image of $c^\#$ is contained in $g_! f'_!\mathcal{F}$. Thus we obtain an induced map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$ factoring $c^\#$ as predicted in the first paragraph of the proof.

Proof of (a). Let $Y'_1 \subset Y'$ be an open subscheme and set $X_1 = (f')^{-1}(W')$. We obtain a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{a} & X \\ {\scriptstyle f'_1}\downarrow & & \downarrow{\scriptstyle f'} \\ Y'_1 & \xrightarrow{b'} & Y' \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} \\ Y & = & Y \end{array}$$

with $f_1 = g_1 \circ f'_1$ and $f = g \circ f'$.

where the horizontal arrows are open immersions. Then our claim is that the diagram

$$\begin{array}{ccccc} f_{1,!}\mathcal{F}|_{X_1} & \xrightarrow[c'_1]{} & g_{1,!}f'_{1,!}\mathcal{F}|_{X_1} & & \\ \downarrow & & \| & & \\ & & g_{1,!}(f'_!\mathcal{F})|_{Y'_1} & & \\ & & \downarrow & & \\ f_!\mathcal{F} & \xrightarrow{c'} & g_!f'_!\mathcal{F} & \longrightarrow & g_*f'_!\mathcal{F} \end{array}$$

commutes where the left vertical arrow is Remark 63.4.6 and the right vertical arrow is Remark 63.3.5. The equality sign in the diagram comes about because f'_1 is the restriction of f' to Y'_1 and our construction of $f'_!$ is local on the base. Finally, to prove the commutativity we choose an object V of $Y_{\acute{e}tale}$ and a formal sum $s_1 = \sum(Z_{1,i}, s_{1,i})$ as in (63.4.0.1) defining an element of $f_{1,p!}\mathcal{F}|_{X_1}(V)$. Recall this means $Z_{1,i} \subset X_1 \times_Y V$ is locally closed finite over V and $s_{1,i} \in H_{Z_{1,i}}(\mathcal{F})$. Then we chase this section across the maps involved, but we only need to show we end up with the same element of $g_*f'_!\mathcal{F}(V) = f'_!\mathcal{F}(Y' \times_Y V)$. Going around both sides of the diagram the reader immediately sees we end up with the element $\sum(Z_{1,i}, s_{1,i})$ where now $Z_{1,i}$ is viewed as a locally closed subscheme of $X \times_{Y'} (Y' \times_Y V) = X \times_Y V$ finite over $Y' \times_Y V$.

Proof of (b). Let $b : Y_1 \to Y$ be a morphism of schemes. Let us form the commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow[a]{} & X \\ {\scriptstyle f'_1}\downarrow & & \downarrow{\scriptstyle f'} \\ Y'_1 & \xrightarrow[b']{} & Y' \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow{b} & Y \end{array} \qquad f_1 : X_1 \to Y_1, \quad f : X \to Y$$

with cartesian squares. We claim that our construction is compatible with the base change maps of Lemmas 63.4.10 and 63.3.12, i.e., that the top rectangle of the diagram

$$\begin{array}{ccccc} b^{-1}f_!\mathcal{F} & & \longrightarrow & & f_{1,!}a^{-1}\mathcal{F} \\ {\scriptstyle b^{-1}c'}\downarrow & & & & \downarrow{\scriptstyle c'_1} \\ b^{-1}g_!f'_!\mathcal{F} & \longrightarrow & g_{1,!}(b')^{-1}f'_!\mathcal{F} & \longrightarrow & g_{1,!}f'_{1,!}a^{-1}\mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ b^{-1}g_*f'_!\mathcal{F} & \longrightarrow & g_{1,*}(b')^{-1}f'_!\mathcal{F} & \longrightarrow & g_{1,*}f'_{1,!}a^{-1}\mathcal{F} \end{array}$$

commutes. The verification of this is completely routine and we urge the reader to skip it. Since the arrows going from the middle row down to the bottom row are injective, it suffices to show that the outer diagram commutes. To show this it suffices to take a local section of $b^{-1}f_!\mathcal{F}$ and show we end up with the same local section of $g_{1,*}f'_{1,!}a^{-1}\mathcal{F}$ going around either way. However, in fact it suffices to check

this for local sections which are of the pullback by b of a section $s = \sum(Z_i, s_i)$ of $f_{p!}\mathcal{F}(V)$ as above (since such pullbacks generate the abelian sheaf $b^{-1}f_!\mathcal{F}$). Denote V_1, V_1', and $Z_{1,i}$ the base change of V, $V' = Y' \times_Y V$, Z_i by $Y_1 \to Y$. Recall that Z_i is a locally closed subscheme of $X_V = X_{V'}$ and hence $Z_{1,i}$ is a locally closed subscheme of $(X_1)_{V_1} = (X_1)_{V_1'}$. Then $b^{-1}c'$ sends the pullback of s to the pullback of the local section $c(s)\sum(Z_i, s_i)$ viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_*f'_{p!}\mathcal{F})(V)$. The composition of the bottom two base change maps simply maps this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$. On the other hand, the base change map at the top of the diagram sends the pullback of s to $\sum(Z_{1,i}, s_{1,i})$ viewed as an element of $f_{1,!}a^{-1}\mathcal{F}(V_1)$. Then finally c_1' by its very construction does indeed map this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V_1') = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$ and the commutativity has been verified.

Proof of (c). This follows from comparing the definitions for both maps; we omit the details.

To finish the proof it suffices to show that the pullback of c' via any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ is an isomorphism. Namely, pulling back by $\overline{y}$ is the same thing as taking stalks and $\overline{y}$ (Étale Cohomology, Remark 59.56.6) and hence we can invoke Étale Cohomology, Theorem 59.29.10. By the compatibility (b) just shown, we conclude that we may assume Y is the spectrum of k and we have to show that c' is an isomorphism. To do this it suffices to show that the induced map

$$\bigoplus\nolimits_{x \in X} \mathcal{F}_x = H^0(Y, f_!\mathcal{F}) \longrightarrow H^0(Y, g_!f'_!\mathcal{F}) = H^0_c(Y', f'_!\mathcal{F})$$

is an isomorphism. The equalities hold by Lemmas 63.4.5 and 63.3.11. Recall that X is a disjoint union of spectra of Artinian local rings with residue field k, see Varieties, Lemma 33.20.2. Since the left and right hand side commute with direct sums (details omitted) we may assume that $\mathcal{F}$ is a skyscraper sheaf x_*A supported at some $x \in X$. Then $f'_!\mathcal{F}$ is the skyscraper sheaf at the image y' of x in Y by Lemma 63.4.5. In this case it is obvious that our construction produces the identity map $A \to H^0_c(Y', y'_*A) = A$ as desired. □

0F6T **Lemma 63.4.12.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then there is a canonical isomorphism of functors*

$$(g \circ f)_! \longrightarrow g_! \circ f_!$$

These isomorphisms satisfy the following properties:

1. *If f and g are separated, then the isomorphism agrees with Lemma 63.3.13.*
2. *If g is separated, then the isomorphism agrees with Lemma 63.4.11.*
3. *For a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$ the diagram*

$$\begin{array}{ccccc} ((g \circ f)_!\mathcal{F})_{\overline{z}} & & \longrightarrow & & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \\ \downarrow & & & & \| \\ (g_!f_!\mathcal{F})_{\overline{z}} & \longrightarrow & \bigoplus_{g(\overline{y})=\overline{z}} (f_!\mathcal{F})_{\overline{y}} & \longrightarrow & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \end{array}$$

is commutative where the horizontal arrows are given by Lemma 63.4.5.

(4) *Let $h : Z \to T$ be a third locally quasi-finite morphism of schemes. Then the diagram*

$$\begin{array}{ccc} (h \circ g \circ f)_! & \longrightarrow & (h \circ g)_! \circ f_! \\ \downarrow & & \downarrow \\ h_! \circ (g \circ f)_! & \longrightarrow & h_! \circ g_! \circ f_! \end{array}$$

commutes.

(5) *Suppose that we have a diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{c} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[b]{} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{a} & Z \end{array}$$

with both squares cartesian and f and g locally quasi-finite. Then the diagram

$$\begin{array}{ccccc} a^{-1} \circ (g \circ f)_! & & \longrightarrow & & (g' \circ f')_! \circ c^{-1} \\ \downarrow & & & & \downarrow \\ a^{-1} \circ g_! \circ f_! & \longrightarrow & g'_! \circ b^{-1} \circ f_! & \longrightarrow & g'_! \circ f'_! \circ c^{-1} \end{array}$$

commutes where the horizontal arrows are those of Lemma 63.4.10.

Proof. If f and g are separated, then this is a special case of Lemma 63.3.13. If g is separated, then this is a special case of Lemma 63.4.11 which moreover agrees with the case where f and g are separated.

Construction in the general case. Choose an open covering $Y = \bigcup Y_i$ such that the restriction $g_i : Y_i \to Z$ of g is separated. Set $X_i = f^{-1}(Y_i)$ and denote $f_i : X_i \to Y_i$ the restriction of f. Also denote $h = g \circ f$ and $h_i : X_i \to Z$ the restriction of h. Consider the following diagram

$$\begin{array}{ccccccc} \bigoplus_{i_0,i_1} h_{i_0 i_1,!}\mathcal{F}|_{X_{i_0 i_1}} & \longrightarrow & \bigoplus_{i_0} h_{i_0,!}\mathcal{F}|_{X_{i_0}} & \longrightarrow & h_!\mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ \bigoplus_{i_0,i_1} g_{i_0 i_1,!} f_{i_0 i_1,!}\mathcal{F}|_{X_{i_0 i_1}} & \longrightarrow & \bigoplus_{i_0} g_{i_0,!} f_{i_0,!}\mathcal{F}|_{X_{i_0}} & & & & \\ \downarrow & & \downarrow & & \vdots & & \\ \bigoplus_{i_0,i_1} g_{i_0 i_1,!}(f_!\mathcal{F})|_{Y_{i_0 i_1}} & \longrightarrow & \bigoplus_{i_0} g_{i_0,!}(f_!\mathcal{F})|_{Y_{i_0}} & \longrightarrow & g_! f_!\mathcal{F} & \longrightarrow & 0 \end{array}$$

By Lemma 63.4.7 the top and bottom row in the diagram are exact. By Lemma 63.4.11 the top left square commutes. The vertical arrows in the lower left square come about because $(f_!\mathcal{F})|_{Y_{i_0 i_1}} = f_{i_0 i_1,!}\mathcal{F}|_{X_{i_0 i_1}}$ and $(f_!\mathcal{F})|_{Y_{i_0}} = f_{i_0,!}\mathcal{F}|_{X_{i_0}}$ as the construction of $f_!$ is local on the base. Moreover, these equalities are (of course) compatible with the identifications $((f_!\mathcal{F})|_{Y_{i_0}})|_{Y_{i_0 i_1}} = (f_!\mathcal{F})|_{Y_{i_0 i_1}}$ and $(f_{i_0,!}\mathcal{F}|_{X_{i_0}})|_{Y_{i_0 i_1}} = f_{i_0 i_1,!}\mathcal{F}|_{X_{i_0 i_1}}$ which are used (together with the covariance for open embeddings for

$Y_{i_0 i_1} \subset Y_{i_0}$) to define the horizontal maps of the lower left square. Thus this square commutes as well. In this way we conclude there is a unique dotted arrow as indicated in the diagram and moreover this arrow is an isomorphism.

Proof of properties (1) – (5). Fix the open covering $Y = \bigcup Y_i$. Observe that if $Y \to Z$ happens to be separated, then we get a dotted arrow fitting into the huge diagram above by using the map of Lemma 63.4.11 (by the very properties of that lemma). This proves (2) and hence also (1) by the compatibility of the maps of Lemma 63.4.11 and Lemma 63.3.13. Next, for any scheme Z' over Z, we obtain the compatibility in (5) for the map $(g' \circ f')_! \to g'_! \circ f'_!$ constructed using the open covering $Y' = \bigcup b^{-1}(Y_i)$. This is clear from the corresponding compatibility of the maps constructed in Lemma 63.4.11. In particular, we can consider a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$. Since $X_{\overline{z}} \to Y_{\overline{z}} \to \mathrm{Spec}(k)$ are separated maps, we find that the base change of $(g \circ f)_! \mathcal{F} \to g_! f_! \mathcal{F}$ by $\overline{z}$ is equal to the map of Lemma 63.3.13. The reader then immediately sees that we obtain property (3). Of course, property (3) guarantees that our transformation of functors $(g \circ f)_! \to g_! \circ f_!$ constructed using the open covering $Y = \bigcup Y_i$ doesn't depend on the choice of this open covering. Finally, property (4) follows by looking at what happens on stalks using the already proven property (3). □

63.5. Weightings and trace maps for locally quasi-finite morphisms

0GKE A reference for this section is [**AGV71**, Exposee XVII, Proposition 6.2.5].

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f, see More on Morphisms, Definition 37.75.2. Let $\mathcal{F}$ be an abelian sheaf on $Y_{étale}$. In this section we will show that there exists map

$$\mathrm{Tr}_{f,w,\mathcal{F}} : f_! f^{-1} \mathcal{F} \longrightarrow \mathcal{F}$$

of abelian sheaves on $Y_{étale}$ characterized by the following property: on stalks at a geometric point $\overline{y}$ of Y we obtain the map

$$\bigoplus\nolimits_{f(\overline{x})=\overline{y}} w(\overline{x}) : (f_! f^{-1} \mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{y}} \longrightarrow \mathcal{F}_{\overline{y}}$$

Here as indicated the arrow is given by multiplication by the integer $w(\overline{x})$ on the summand corresponding to $\overline{x}$. The equality on the left of the arrow follows from Lemma 63.4.5 combined with Étale Cohomology, Lemma 59.36.2.

If the morphism $f : X \to Y$ is flat, locally quasi-finite, and locally of finite presentation, then there exists a canonical weighting and we obtain a canonical trace map whose formation is compatible with base change, see Example 63.5.5. If Y is a locally Noetherian unibranch scheme and $f : X \to Y$ is locally quasi-finite, then we can also define a (natural) weighting for f and we have trace maps in this case as well, see Example 63.5.7.

0GKF **Lemma 63.5.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of Λ-modues on $Y_{étale}$. There is a canonical isomorphism*

$$can : f_! \mathcal{F} \otimes_\Lambda \mathcal{G} \longrightarrow f_!(\mathcal{F} \otimes_\Lambda f^{-1} \mathcal{G})$$

of sheaves of Λ-modules on $Y_{étale}$.

Proof. Recall that $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$ by Definition 63.4.4 where $f_{p!}\mathcal{F}$ is the presheaf constructed in Section 63.4. Thus in order to construct the arrow it suffices to construct a map

$$f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G} \longrightarrow f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

of presheaves on $Y_{étale}$. Here the symbol $\otimes_{p,\Lambda}$ denotes the presheaf tensor product, see Modules on Sites, Section 18.26. Let V be an object of $Y_{étale}$. Recall that

$$f_{p!}\mathcal{F}(V) = \text{colim}_Z H_Z(\mathcal{F}) \quad \text{and} \quad f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})(V) = \text{colim}_Z H_Z(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

See Section 63.4. Our map will be defined on pure tensors by the rule

$$(Z, s) \otimes t \longmapsto (Z, s \otimes f^{-1}t)$$

(for notation see below) and extended by linearity to all of $(f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G})(V) = f_{p!}\mathcal{F}(V) \otimes_\Lambda \mathcal{G}(V)$. Here the notation used is as follows

(1) $Z \subset X_V$ is a locally closed subscheme finite over V,
(2) $s \in H_Z(\mathcal{F})$ which means that $s \in \mathcal{F}(U)$ with $\text{Supp}(s) \subset Z$ for some $U \subset X_V$ open such that $Z \subset U$ is closed, and
(3) $t \in \mathcal{G}(V)$ with image $f^{-1}t \in f^{-1}\mathcal{G}(U)$.

Since the support of $s \in \mathcal{F}(U)$ is contained in Z it is clear that the support of $s \otimes f^{-1}t$ is contained in Z as well. Thus considering the pair $(Z, s \otimes f^{-1}t)$ makes sense. It is immediate that the construction commutes with the transition maps in the colimit $\text{colim}_Z H_Z(\mathcal{F})$ and that it is compatible with restriction mappings. Finally, it is equally clear that the construction is compatible with the identifications of stalks of $f_!$ in Lemma 63.4.5. In other words, the map *can* we've produced on stalks at a geometric point $\overline{y}$ fits into a commutative diagram

$$\begin{array}{ccc} (f_!\mathcal{F} \otimes_\Lambda \mathcal{G})_{\overline{y}} & \xrightarrow{can_{\overline{y}}} & f_!(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})_{\overline{y}} \\ \downarrow & & \downarrow \\ (\bigoplus \mathcal{F}_{\overline{x}}) \otimes_\Lambda \mathcal{G}_{\overline{y}} & \longrightarrow & \bigoplus (\mathcal{F}_{\overline{x}} \otimes_\Lambda \mathcal{G}_{\overline{y}}) \end{array}$$

where the direct sums are over the geometric points $\overline{x}$ lying over $\overline{y}$, where the vertical arrows are the identifications of Lemma 63.4.5, and where the lower horizontal arrow is the obvious isomorphism. We conclude that *can* is an isomorphism as desired. □

0GKG **Lemma 63.5.2.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. For any abelian sheaf $\mathcal{F}$ on Y there exists a unique trace map $Tr_{f,w,\mathcal{F}} : f_!f^{-1}\mathcal{F} \to \mathcal{F}$ having the prescribed behaviour on stalks.*

Proof. By Lemma 63.5.1 we have an identification $f_!f^{-1}\mathcal{F} = f_!\underline{\mathbf{Z}} \otimes \mathcal{F}$ compatible with the description of stalks of these sheaves at geometric points. Hence it suffices to produce the map

$$\text{Tr}_{f,w,\underline{\mathbf{Z}}} : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

having the prescribed behaviour on stalks. By Definition 63.4.4 we have $f_!\underline{\mathbf{Z}} = (f_{p!}\underline{\mathbf{Z}})^\#$ where $f_{p!}\underline{\mathbf{Z}}$ is the presheaf constructed in Section 63.4. Thus it suffices to construct a map

$$f_{p!}\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

of presheaves on $Y_{étale}$. Let V be an object of $Y_{étale}$. Recall from Section 63.4 that

$$f_{p!}\underline{\mathbf{Z}}(V) = \text{colim}_Z H_Z(\underline{\mathbf{Z}})$$

Here the colimit is over the (partially ordered) collection of locally closed subschemes $Z \subset X_V$ which are finite over V. For each such Z we will define a map

$$H_Z(\underline{\mathbf{Z}}) \longrightarrow \underline{\mathbf{Z}}(V)$$

compatible with the maps defining the colimit.

Let $Z \subset X_V$ be locally closed and finite over V. Choose an open $U \subset X_V$ containing Z as a closed subset. An element s of $H_Z(\underline{\mathbf{Z}})$ is a section $s \in \underline{\mathbf{Z}}(U)$ whose support is contained in Z. Let $U_n \subset U$ be the open and closed subset where the value of s is $n \in \mathbf{Z}$. By the support condition we see that $Z \cap U_n = U_n$ for $n \neq 0$. Hence for $n \neq 0$, the open U_n is also closed in Z (as the complement of all the others) and we conclude that $U_n \to V$ is finite as Z is finite over V. By the very definition of a weighting this means the function $\int_{U_n \to V} w|_{U_n}$ is locally constant on V and we may view it as an element of $\underline{\mathbf{Z}}(V)$. Our construction sends (Z, s) to the element

$$\sum\nolimits_{n \in \mathbf{Z},\ n \neq 0} n \left(\int_{U_n \to V} w|_{U_n} \right) \quad \in \quad \underline{\mathbf{Z}}(V)$$

The sum is locally finite on V and hence makes sense; details omitted (in the whole discussion the reader may first choose affine opens and make sure all the schemes occurring in the argument are quasi-compact so the sum is finite). We omit the verification that this construction is compatible with the maps in the colimit and with the restriction mappings defining $f_{p!}\underline{\mathbf{Z}}$.

Let $\overline{y}$ be a geometric point of Y lying over the point $y \in Y$. Taking stalks at $\overline{y}$ the construction above determines a map

$$(f_!\underline{\mathbf{Z}})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{y}}$$

To finish the proof we will show this map is given by multiplication by $w(\overline{x})$ on the summand corresponding to $\overline{x}$. Namely, pick $\overline{x}$ lying over $\overline{y}$. We can find an étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ such that X_V contains an open U finite over V such that only the geometric point $\overline{x}$ is in U and not the other geometric points of X lifting $\overline{y}$. This follows from More on Morphisms, Lemma 37.41.3; some details omitted. Then $(U, 1)$ defines a section of $f_!\underline{\mathbf{Z}}$ over V which maps to 1 in the summand corresponding to $\overline{x}$ and zero in the other summands (see proof of Lemma 63.4.2) and our construction above sends $(U, 1)$ to $\int_{U \to V} w|_U$ which is constant with value $w(\overline{x})$ in a neighbourhood of $\overline{v}$ as desired. □

0GKH **Lemma 63.5.3.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. The trace maps constructed above have the following properties:*

1. *$Tr_{f,w,\mathcal{F}}$ is functorial in $\mathcal{F}$,*
2. *$Tr_{f,w,\mathcal{F}}$ is compatible with arbitrary base change,*
3. *given a ring Λ and K in $D(Y_{étale}, \Lambda)$ we obtain $Tr_{f,w,K} : f_!f^{-1}K \to K$ functorial in K and compatible with arbitrary base change.*

Proof. Part (1) either follows from the construction of the trace map in the proof of Lemma 63.5.2 or more simply because the characterization of the map forces it

to be true on all stalks. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of schemes. Then the function $w' = w \circ g' : X' \to \mathbf{Z}$ is a weighting of f' by More on Morphisms, Lemma 37.75.3. Statement (2) means that the diagram

$$\begin{array}{ccc} g^{-1}f_!f^{-1}\mathcal{F} & \xrightarrow{g^{-1}\mathrm{Tr}_{f,w,\mathcal{F}}} & g^{-1}\mathcal{F} \\ \| & & \| \\ f'_!(f')^{-1}g^{-1}\mathcal{F} & \xrightarrow{\mathrm{Tr}_{f',w',g^{-1}\mathcal{F}}} & g^{-1}\mathcal{F} \end{array}$$

is commutative where the left vertical equality is given by

$$g^{-1}f_!f^{-1}\mathcal{F} = f'_!(g')^{-1}f^{-1}\mathcal{F} = f'_!(f')^{-1}g^{-1}\mathcal{F}$$

with first equality sign given by Lemma 63.4.10 (base change for lower shriek). The commutativity of this diagram follows from the characterization of the action of our trace maps on stalks and the fact that the base change map of Lemma 63.4.10 respects the descriptions of stalks.

Given parts (1) and (2), part (3) follows as the functors $f^{-1} : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ and $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ are obtained by applying f^{-1} and $f_!$ to any complexes of modules representing the objects in question. □

0GL3 **Lemma 63.5.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. For $K \in D(Z_{étale}, \Lambda)$ the composition*

$$(g \circ f)_!(g \circ f)^{-1}K = g_!f_!f^{-1}g^{-1}K \xrightarrow{g_!\,Tr_{f,w_f,g^{-1}K}} g_!g^{-1}K \xrightarrow{Tr_{g,w_g,K}} K$$

is equal to $Tr_{g\circ f, w_{g\circ f}, K}$ where $w_{g\circ f}(x) = w_f(x)w_g(f(x))$.

Proof. We have $(g \circ f)_! = g_! \circ f_!$ by Lemma 63.4.12. In More on Morphisms, Lemma 37.75.5 we have seen that $w_{g\circ f}$ is a weighting for $g \circ f$ so the statement makes sense. To check equality compute on stalks. Details omitted. □

0GKI **Example 63.5.5** (Trace for flat quasi-finite)**.** Let $f : X \to Y$ be a morphism of schemes which is flat, locally quasi-finite, and locally of finite presentation. Then we obtain a canonical positive weighting $w : X \to \mathbf{Z}$ by setting

$$w(x) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x})[\kappa(x) : \kappa(f(x))]_i$$

See More on Morphisms, Lemma 37.75.7. Thus by Lemmas 63.5.2 and 63.5.3 for f we obtain trace maps

$$\mathrm{Tr}_{f,K} : f_!f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{étale}, \Lambda)$ and compatible with arbitrary base change. Note that any base change $f' : X' \to Y'$ of f satisfies the same properties and that w restricts to the canonical weighting for f'.

0GL4 **Remark 63.5.6.** Let $j : U \to X$ be an étale morphism of schemes. Then the trace map $\mathrm{Tr} : j_!j^{-1}K \to K$ of Example 63.5.5 is equal to the counit for the adjunction between $j_!$ and j^{-1}. We already used the terminology "trace" for this counit in Étale Cohomology, Section 59.66.

0GKJ **Example 63.5.7** (Trace for quasi-finite over normal)**.** Let Y be a geometrically unibranch and locally Noetherian scheme, for example Y could be a normal variety. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then there exists a positive weighting $w : X \to \mathbf{Z}$ for f which is roughly defined by sending x to the "generic separable degree" of $\mathcal{O}_{X,x}^{sh}$ over $\mathcal{O}_{Y,f(x)}^{sh}$. See More on Morphisms, Lemma 37.75.8. Thus by Lemmas 63.5.2 and 63.5.3 for f and w we obtain trace maps

$$\mathrm{Tr}_{f,w,K} : f_!f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{étale}, \Lambda)$ and compatible with arbitrary base change. However, in this case, given a base change $f' : X' \to Y'$ of f the restriction of w to X' in general does not have a "natural" interpretation in terms of the morphism f'.

63.6. Upper shriek for locally quasi-finite morphisms

0F58 For a locally quasi-finite morphism $f : X \to Y$ of schemes, the functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ commutes with direct sums and is exact, see Lemma 63.4.5. This suggests that it has a right adjoint which we will denote $f^!$.

Warning: This functor is the non-derived version!

0F59 **Lemma 63.6.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes.*

(1) *The functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ has a right adjoint $f^! : Ab(Y_{étale}) \to Ab(X_{étale})$.*
(2) *We have $f^!(\overline{y}_*A) = \prod_{f(\overline{x})=\overline{y}} \overline{x}_*A$.*
(3) *If Λ is a ring, then the functor $f_! : Mod(X_{étale}, \Lambda) \to Mod(Y_{étale}, \Lambda)$ has a right adjoint $f^! : Mod(Y_{étale}, \Lambda) \to Mod(X_{étale}, \Lambda)$ which agrees with $f^!$ on underlying abelian sheaves.*

Proof. Proof of (1). Let $E \subset \mathrm{Ob}(Ab(Y_{étale}))$ be the class consisting of products of skyscraper sheaves. We claim that

(a) every $\mathcal{G}$ in $Ab(Y_{étale})$ is a subsheaf of an element of E, and
(b) for every $\mathcal{G} \in E$ there exists an object $\mathcal{H}$ of $Ab(X_{étale})$ such that $\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{H})$ functorially in $\mathcal{F}$.

Once the claim has been verified, the dual of Homology, Lemma 12.29.6 produces the adjoint functor $f^!$.

Part (a) is true because we can map $\mathcal{G}$ to the sheaf $\prod \overline{y}_*\mathcal{G}_{\overline{y}}$ where the product is over all geometric points of Y. This is an injection by Étale Cohomology, Theorem 59.29.10. (This is the first step in the Godement resolution when done in the setting of abelian sheaves on topological spaces.)

Part (b) and part (2) of the lemma can be seen as follows. Suppose that $\mathcal{G} = \prod \overline{y}_*A_{\overline{y}}$ for some abelian groups $A_{\overline{y}}$. Then

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) = \prod \mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$$

Thus it suffices to find abelian sheaves $\mathcal{H}_{\overline{y}}$ on $X_{étale}$ representing the functors $\mathcal{F} \mapsto \mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$ and to take $\mathcal{H} = \prod \mathcal{H}_{\overline{y}}$. This reduces us to the case $\mathcal{H} = \overline{y}_*A$

for some fixed geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ and some fixed abelian group A. We claim that in this case $\mathcal{H} = \prod_{f(\overline{x})=\overline{y}} \overline{x}_* A$ works. This will finish the proof of parts (1) and (2) of the lemma. Namely, we have

$$\mathrm{Hom}(f_!\mathcal{F}, \overline{y}_* A) = \mathrm{Hom}_{Ab}((f_!\mathcal{F})_{\overline{y}}, A) = \mathrm{Hom}_{Ab}(\bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}, A)$$

by the description of stalks in Lemma 63.4.5 on the one hand and on the other hand we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}(\mathcal{F}, \overline{x}_* A) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}_{Ab}(\mathcal{F}_{\overline{x}}, A)$$

We leave it to the reader to identify these as functors of $\mathcal{F}$.

Proof of part (3). Observe that an object $Mod(X_{étale}, \Lambda)$ is the same thing as an object $\mathcal{F}$ of $Ab(X_{étale})$ together with a map $\Lambda \to \mathrm{End}(\mathcal{F})$. Hence the functors $f_!$ and $f^!$ in (1) define functors $f_!$ and $f^!$ as in (3). A straightforward computation shows that they are adjoints. □

0F5A **Lemma 63.6.2.** *Let $j : U \to X$ be an étale morphism. Then $j^! = j^{-1}$.*

Proof. This is true because $j_!$ as defined in Section 63.4 agrees with $j_!$ as defined in Étale Cohomology, Section 59.70, see Lemma 63.4.3. Finally, in Étale Cohomology, Section 59.70 the functor $j_!$ is defined as the left adjoint of j^{-1} and hence we conclude by uniqueness of adjoint functors. □

0F5B **Lemma 63.6.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be separated and locally quasi-finite morphisms. There is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$. Given a third locally quasi-finite morphism $h : Z \to T$ the diagram*

$$\begin{array}{ccc} (h \circ g \circ f)^! & \longrightarrow & f^! \circ (h \circ g)^! \\ \downarrow & & \downarrow \\ (g \circ f)^! \circ h^! & \longrightarrow & f^! \circ g^! \circ h^! \end{array}$$

commutes.

Proof. By uniqueness of adjoint functors, this immediately translates into the corresponding (dual) statement for the functors $f_!$. See Lemma 63.4.12. □

0F5C **Lemma 63.6.4.** *Let $j : U \to X$ and $j' : V \to U$ be étale morphisms. The isomorphism $(j \circ j')^{-1} = (j')^{-1} \circ j^{-1}$ and the isomorphism $(j \circ j')^! = (j')^! \circ j^!$ of Lemma 63.6.3 agree via the isomorphism of Lemma 63.6.2.*

Proof. Omitted. □

0F6U **Lemma 63.6.5.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f locally quasi-finite. For any abelian sheaf $\mathcal{F}$ on $Y'_{étale}$ we have $(g')_*(f')^!\mathcal{F} = f^! g_* \mathcal{F}$.

Proof. By uniqueness of adjoint functors, this follows from the corresponding (dual) statement for the functors $f_!$. See Lemma 63.4.10. □

0F5L **Remark 63.6.6.** The material in this section can be generalized to sheaves of pointed sets. Namely, for a site $\mathcal{C}$ denote $Sh^*(\mathcal{C})$ the category of sheaves of pointed sets. The constructions in this and the preceding section apply, mutatis mutandis, to sheaves of pointed sets. Thus given a locally quasi-finite morphism $f : X \to Y$ of schemes we obtain an adjoint pair of functors

$$f_! : Sh^*(X_{étale}) \longrightarrow Sh^*(Y_{étale}) \quad \text{and} \quad f^! : Sh^*(Y_{étale}) \longrightarrow Sh^*(X_{étale})$$

such that for every geometric point $\overline{y}$ of Y there are isomorphisms

$$(f_!\mathcal{F})_{\overline{y}} = \coprod\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

(coproduct taken in the category of pointed sets) functorial in $\mathcal{F} \in Sh^*(X_{étale})$ and isomorphisms

$$f^!(\overline{y}_* S) = \prod\nolimits_{f(\overline{x})=\overline{y}} \overline{x}_* S$$

functorial in the pointed set S. If $F : Ab(X_{étale}) \to Sh^*(X_{étale})$ and $F : Ab(Y_{étale}) \to Sh^*(Y_{étale})$ denote the forgetful functors, compatibility between the constructions will guarantee the existence of canonical maps

$$f_! F(\mathcal{F}) \longrightarrow F(f_!\mathcal{F})$$

functorial in $\mathcal{F} \in Ab(X_{étale})$ and

$$F(f^!\mathcal{G}) \longrightarrow f^! F(\mathcal{G})$$

functorial in $\mathcal{G} \in Ab(Y_{étale})$ which produce the obvious maps on stalks, resp. skyscraper sheaves. In fact, the transformation $F \circ f^! \to f^! \circ F$ is an isomorphism (because $f^!$ commutes with products).

63.7. Derived upper shriek for locally quasi-finite morphisms

0F5M We can take the derived versions of the functors in Section 63.6 and obtain the following.

0F5N **Lemma 63.7.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functors $f_!$ and $f^!$ of Definition 63.4.4 and Lemma 63.6.1 induce adjoint functors $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ and $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ on derived categories.*

In the separated case the functor $f_!$ is defined in Section 63.3.

Proof. This follows immediately from Derived Categories, Lemma 13.30.3, the fact that $f_!$ is exact (Lemma 63.4.5) and hence $Lf_! = f_!$ and the fact that we have enough K-injective complexes of Λ-modules on $Y_{étale}$ so that $Rf^!$ is defined. □

0GJX **Remark 63.7.2.** Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ of Lemma 63.7.1 sends complexes with torsion cohomology sheaves to complexes with torsion cohomology sheaves. This is immediate from the description of the stalks of $f_!$, see Lemma 63.4.5.

0GKK **Lemma 63.7.3.** *Let X be a scheme. Let $X = U \cup V$ with U and V open. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle*

$$j_{U\cap V!}K|_{U\cap V} \to j_{U!}K|_U \oplus j_{V!}K|_V \to K \to j_{U\cap V!}K|_{U\cap V}[1]$$

in $D(X_{étale}, \Lambda)$ with obvious notation.

Proof. Since the restriction functors and the lower shriek functors we use are exact, it suffices to show for any abelian sheaf $\mathcal{F}$ on $X_{étale}$ the sequence

$$0 \to j_{U\cap V!}\mathcal{F}|_{U\cap V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$

is exact. This can be seen by looking at stalks. □

0GKL **Lemma 63.7.4.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle*

$$j_!j^{-1}K \to K \to i_*i^{-1}K \to j_!j^{-1}K[1]$$

in $D(X_{étale}, \Lambda)$.

Proof. Immediate consequence of Étale Cohomology, Lemma 59.70.8 and the fact that the functors $j_!$, j^{-1}, i_*, i^{-1} are exact and hence their derived versions are computed by applying these functors to any complex of sheaves representing K. □

63.8. Preliminaries to derived lower shriek via compactifications

0F7A In this section we prove some lemmas on the existence of certain natural isomorphisms of functors which follow immediately from proper base change.

0F7B **Lemma 63.8.1.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f and f' proper and g and g' separated and locally quasi-finite. Let Λ be a ring. Functorially in $K \in D(X'_{étale}, \Lambda)$ there is a canonical map

$$g_!Rf'_*K \longrightarrow Rf_*(g'_!K)$$

in $D(Y_{étale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{étale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Then we can consider the map

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

where the first and third equality come from Lemma 63.3.4 and the second equality comes from Lemma 63.3.13 which tells us that both $g_! \circ f'_!$ and $f_! \circ g'_!$ are equal to $(g \circ f')_! = (f \circ g')_!$ as subsheaves of $(g \circ f')_* = (f \circ g')_*$.

Assume Λ is torsion, i.e., we are in case (b). With notation as above, it suffices to show that $f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$ is an isomorphism. The question is local on Y. Hence we may assume that the dimension of fibres of f is bounded, see Morphisms,

Lemma 29.28.5. Then we see that Rf_* has finite cohomological dimension, see Étale Cohomology, Lemma 59.92.2. Hence by Derived Categories, Lemma 13.32.2, if we show that $R^q f_*(g'_!\mathcal{J}) = 0$ for $q > 0$ and any injective sheaf of Λ-modules $\mathcal{J}$ on $X'_{étale}$, then the result follows.

The stalk of $R^q f_*(g'_!\mathcal{J})$ at a geometric point $\overline{y}$ is equal to $H^q(X_{\overline{y}}, (g'_!\mathcal{J})|_{X_{\overline{y}}})$ by Étale Cohomology, Lemma 59.91.13. Since formation of $g'_!$ commutes with base change (Lemma 63.3.12) this is equal to

$$H^q(X_{\overline{y}}, g'_{\overline{y},!}(\mathcal{J}|_{X'_{\overline{y}}}))$$

where $g'_{\overline{y}} : X'_{\overline{y}} \to X_{\overline{y}}$ is the induced morphism between geometric fibres. Since $Y' \to Y$ is locally quasi-finite, we see that $X'_{\overline{y}}$ is a disjoint union of the fibres $X'_{\overline{y}'}$ at geometric points $\overline{y}'$ of Y' lying over $\overline{y}$. Denote $g'_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ the restriction of $g'_{\overline{y}}$ to $X'_{\overline{y}'}$. Thus the previous cohomology group is equal to

$$H^q(X_{\overline{y}}, \bigoplus\nolimits_{\overline{y}'/\overline{y}} g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

for example by Lemma 63.3.15 (but it is also obvious from the definition of $g'_{\overline{y},!}$ in Section 63.3). Since taking étale cohomology over $X_{\overline{y}}$ commutes with direct sums (Étale Cohomology, Theorem 59.51.3) we conclude it suffices to show that

$$H^q(X_{\overline{y}}, g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

is zero. Observe that $g_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ is a morphism between proper scheme over $\overline{y}$ and hence is proper itself. As it is locally quasi-finite as well we conclude that $g_{\overline{y}'}$ is finite. Thus we see that $g'_{\overline{y}',!} = g'_{\overline{y}',*} = Rg'_{\overline{y}',*}$. By Leray we conlude that we have to show

$$H^q(X'_{\overline{y}'}, \mathcal{J}|_{X'_{\overline{y}'}})$$

is zero. As Λ is torsion, this follows from proper base change (Étale Cohomology, Lemma 59.91.13) as the higher direct images of $\mathcal{J}$ under f' are zero.

Proof in case (a). We will deduce this from case (b) by standard arguments. We will show that the induced map $g_!R^p f'_*K \to R^p f_*(g'_!K)$ is an isomorphism for all $p \in \mathbf{Z}$. Fix an integer $p_0 \in \mathbf{Z}$. Let a be an integer such that $H^j(K) = 0$ for $j < a$. We will prove $g_!R^p f'_*K \to R^p f_*(g'_!K)$ is an isomorphism for $p \leq p_0$ by descending induction on a. If $a > p_0$, then we see that the left and right hand side of the map are zero for $p \leq p_0$ by trivial vanishing, see Derived Categories, Lemma 13.16.1 (and use that $g_!$ and $g'_!$ are exact functors). Assume $a \leq p_0$. Consider the distinguished triangle

$$H^a(K)[-a] \to K \to \tau_{\geq a+1}K$$

By induction we have the result for $\tau_{\geq a+1}K$. In the next paragraph, we will prove the result for $H^a(K)[-a]$. Then five lemma applied to the map between long exact sequence of cohomology sheaves associated to the map of distinguished triangles

$$\begin{array}{ccccc} g_!Rf'_*(H^a(K)[-a]) & \longrightarrow & g_!Rf'_*K & \longrightarrow & g_!Rf'_*\tau_{\geq a+1}K \\ \downarrow & & \downarrow & & \downarrow \\ Rf_*(g'_!(H^a(K)[-a])) & \longrightarrow & Rf_*(g'_!K) & \longrightarrow & Rf_*(g'_!\tau_{|geqa+1}K) \end{array}$$

gives the result for K. Some details omitted.

Let $\mathcal{F}$ be a torsion abelian sheaf on $X'_{étale}$. To finish the proof we show that $g_!Rf'_*\mathcal{F} \to R^pf_*(g'_!\mathcal{F})$ is an isomorphism for all p. We can write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n] = \mathrm{Ker}(n : \mathcal{F} \to \mathcal{F})$. We have the isomorphism for $\mathcal{F}[n]$ by case (b). Since the functors $g_!$, $g'_!$, R^pf_*, $R^pf'_*$ commute with filtered colimits (follows from Lemma 63.3.17 and Étale Cohomology, Lemma 59.51.8) the proof is complete. □

0F7C **Lemma 63.8.2.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{l} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{m} & Z \end{array}$$

with f, f', g and g' proper and k, l, and m separated and locally quasi-finite. Then the isomorphisms of Lemma 63.8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $R(g\circ f)_* = Rg_*\circ Rf_*$ and $R(g'\circ f')_* = Rg'_*\circ Rf'_*$, then the isomorphism $m_!\circ Rg'_*\circ Rf'_* \to Rg_*\circ Rf_*\circ k_!$ of the outer rectangle is equal to the composition

$$m_! \circ Rg'_* \circ Rf'_* \to Rg_* \circ l_! \circ Rf'_* \to Rg_* \circ Rf_* \circ k_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{étale}$ and a quasi-isomorphism $k_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X_{étale}$. The proof of Lemma 63.8.1 shows that the canonical map

$$a : l_!f'_*\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism produces the second arrow on applying Rg_*. By Cohomology on Sites, Lemma 21.20.10 the complex $f_*\mathcal{I}^\bullet$, resp. $f'_*\mathcal{J}^\bullet$ is a K-injective complex of Λ-modules on $Y_{étale}$, resp. $Y'_{étale}$. (Using this is cheating and could be avoided.) In particular, the same reasoning gives that the canonical map

$$b : m_!g'_*f'_*\mathcal{J}^\bullet \to g_*f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism represents the first arrow. Finally, the proof of Lemma 63.8.1 show that $g_*l_!f'_!\mathcal{J}^\bullet$ represents $Rg_*(l_!f'_*\mathcal{J}^\bullet)$ because $f'_*\mathcal{J}^\bullet$ is K-injective. Hence $Rg_*(a) = g_*(a)$ and the composition $g_*(a)\circ b$ is the arrow of Lemma 63.8.1 for the rectangle. □

0F7D **Lemma 63.8.3.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccccc} X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h'} & Y' & \xrightarrow{h} & Y \end{array}$$

with f, f', and f'' proper and g, g', h, and h' separated and locally quasi-finite. Then the isomorphisms of Lemma 63.8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $(h \circ h')_! = h_! \circ h'_!$ and $(g \circ g')_! = g_! \circ g'_!$ using the equalities of Lemma 63.3.13, then the isomorphism $h_! \circ h'_! \circ Rf''_* \to Rf_* \circ g_! \circ g'_!$ of the outer rectangle is equal to the composition

$$h_! \circ h'_! \circ Rf''_* \to h_! \circ Rf'_* \circ g'_! \to Rf_* \circ g_! \circ g'_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X''_{étale}$ and a quasi-isomorphism $g'_!\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ to a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{étale}$. Next, choose a quasi-isomorphism $g_!\mathcal{J}^\bullet \to \mathcal{K}^\bullet$ to a K-injective complex $\mathcal{K}^\bullet$ of Λ-modules on $X_{étale}$. The proof of Lemma 63.8.1 shows that the canonical maps

$$h'_!f''_*\mathcal{I}^\bullet \to f'_*\mathcal{J}^\bullet \quad \text{and} \quad h_!f'_*\mathcal{J}^\bullet \to f_*\mathcal{K}^\bullet$$

are quasi-isomorphisms and these quasi-isomorphisms define the first and second arrow above. Since $g_!$ is an exact functor (Lemma 63.3.17) we find that $g_!g'_!\mathcal{I}^\bullet \to \mathcal{K}^\bullet$ is a quasi-ismorphism and hence the canonical map

$$h_!h'_!f''_*\mathcal{I}^\bullet \to f_*\mathcal{K}^\bullet$$

is a quasi-isomorphism and represents the map for the outer rectangle in the derived category. Clearly this map is the composition of the other two and the proof is complete. □

0F7E **Remark 63.8.4.** Consider a commutative diagram

$$\begin{array}{ccccc}
X'' & \xrightarrow[k']{} & X' & \xrightarrow[k]{} & X \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\
{\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z
\end{array}$$

of schemes whose vertical arrows are proper and whose horizontal arrows are separated and locally quasi-finite. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{array}{cc} A & B \\ C & D \end{array}$$

Then the maps of Lemma 63.8.1 for the squares are (where we use $Rf_* = f_*$, etc)

$$\begin{array}{ll} \gamma_A : l'_! \circ f''_* \to f'_* \circ k'_! & \gamma_B : l_! \circ f'_* \to f_* \circ k_! \\ \gamma_C : m'_! \circ g''_* \to g'_* \circ l'_! & \gamma_D : m_! \circ g'_* \to g_* \circ l_! \end{array}$$

For the 2×1 and 1×2 rectangles we have four further maps

$$\begin{array}{c} \gamma_{A+B} : (l \circ l')_! \circ f''_* \to f_* \circ (k \circ k')_* \\ \gamma_{C+D} : (m \circ m')_! \circ g''_* \to g_* \circ (l \circ l')_! \\ \gamma_{A+C} : m'_! \circ (g'' \circ f'')_* \to (g' \circ f')_* \circ k'_! \\ \gamma_{B+D} : m_! \circ (g' \circ f')_* \to (g \circ f)_* \circ k_! \end{array}$$

By Lemma 63.8.3 we have

$$\gamma_{A+B} = \gamma_B \circ \gamma_A, \quad \gamma_{C+D} = \gamma_D \circ \gamma_C$$

and by Lemma 63.8.2 we have

$$\gamma_{A+C} = \gamma_A \circ \gamma_C, \quad \gamma_{B+D} = \gamma_B \circ \gamma_D$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_B \star \mathrm{id}_{k'_!}) \circ (\mathrm{id}_{l_!} \star \gamma_A)$ with notation as in Categories, Section 4.28 and similarly for the others. Having said all of this we find (a priori) two transformations

$$m_! \circ m'_! \circ g''_* \circ f''_* \longrightarrow g_* \circ f_* \circ k_! \circ k'_!$$

namely

$$\gamma_B \circ \gamma_D \circ \gamma_A \circ \gamma_C = \gamma_{B+D} \circ \gamma_{A+C}$$

and

$$\gamma_B \circ \gamma_A \circ \gamma_D \circ \gamma_C = \gamma_{A+B} \circ \gamma_{C+D}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{array}{ccc} m_! \circ g'_* \circ l'_! \circ f''_* & \xrightarrow{\gamma_D} & g_* \circ l_! \circ l'_! \circ f''_* \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_A} \\ m_! \circ g'_* \circ f'_* \circ k'_! & \xrightarrow{\gamma_D} & g_* \circ l_! \circ f'_* \circ k'_! \end{array}$$

commutes. This is true because the squares A and D meet in only one point, more precisely by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

0F7F **Lemma 63.8.5.** *Let $b : Y_1 \to Y$ be a morphism of schemes. Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \quad \text{and let} \quad \begin{array}{ccc} X'_1 & \xrightarrow{g'_1} & X_1 \\ {\scriptstyle f'_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y'_1 & \xrightarrow{g_1} & Y_1 \end{array}$$

be the base change by b. Assume f and f' proper and g and g' separated and locally quasi-finite. For a ring Λ and K in $D(X'_{étale}, \Lambda)$ there is commutative diagram

$$\begin{array}{ccccc} b^{-1}g_!Rf'_*K & \longrightarrow & g_{1,!}(b')^{-1}Rf'_*K & \longrightarrow & g_{1,!}Rf'_{1,*}(a')^{-1}K \\ \downarrow & & & & \downarrow \\ b^{-1}Rf_*g'_!K & \longrightarrow & Rf_{1,*}a^{-1}g'_!K & \longrightarrow & Rf_{1,*}g'_{1,!}(a')^{-1}K \end{array}$$

in $D(Y_{1,étale}, \Lambda)$ where $a : X_1 \to X$, $a' : X'_1 \to X'$, $b' : Y'_1 \to Y'$ are the projections, the vertical maps are the arrows of Lemma 63.8.1 and the horizontal arrows are the base change map (from Étale Cohomology, Section 59.86) and the base change map of Lemma 63.3.12.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{étale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. The proof of Lemma 63.8.1 constructs $g_!Rf'_*K \to Rf_*g'_!K$ as

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

Choose a quasi-isomorphism $(a')^{-1}\mathcal{J}^\bullet \to \mathcal{J}_1^\bullet$ to a K-injective complex $\mathcal{J}_1^\bullet$ of sheaves of Λ-modules on $X'_{1,étale}$. Then we can pick a diagram of complexes

$$\begin{array}{ccccc} g'_{1,!}\mathcal{J}_1^\bullet & & \longrightarrow & & \mathcal{I}_1^\bullet \\ \uparrow & & & & \uparrow \\ g'_{1,!}(a')^{-1}\mathcal{J}^\bullet & = & a^{-1}g'_!\mathcal{J}^\bullet & \longrightarrow & a^{-1}\mathcal{I}^\bullet \end{array}$$

commuting up to homotopy where all arrows are quasi-isomorphisms, the equality comes from Lemma 63.3.4, and $\mathcal{I}_1^\bullet$ is a K-injective complex of sheaves of Λ-modules on $X_{1,étale}$. The map $g_{1,!}Rf'_{1,*}(a')^{-1}K \to Rf_{1,*}g'_{1,!}(a')^{-1}K$ is given by

$$g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet = g_{1,!}f'_{1,!}\mathcal{J}_1^\bullet = f_{1,!}g'_{1,!}\mathcal{J}_1^\bullet = f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \to f_{1,*}\mathcal{I}_1^\bullet$$

The identifications across the 3 equal signs in both arrows are compatible with pullback maps, i.e., the diagram

$$\begin{array}{ccccc} b^{-1}g_!f'_*\mathcal{J}^\bullet & \longrightarrow & g_{1,!}(b')^{-1}f'_*\mathcal{J}^\bullet & \longrightarrow & g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \\ \| & & & & \| \\ b^{-1}f_*g'_!\mathcal{J}^\bullet & \longrightarrow & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet & \longrightarrow & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \end{array}$$

of complexes of abelian sheaves commutes. To show this it is enough to show the diagram commutes with $g_!, g_{1,!}, g'_!, g'_{1,!}$ replaced by $g_*, g_{1,*}, g'_*, g'_{1,*}$ (because the shriek functors are defined as subfunctors of the $*$ functors and the base change maps are defined in a manner compatible with this, see proof of Lemma 63.3.12). For this new diagram the commutativity follows from the compatibility of pullback maps with horizontal and vertical stacking of diagrams, see Sites, Remarks 7.45.3 and 7.45.4 so that going around the diagram in either direction is the pullback map for the base change of $f \circ g' = g \circ f'$ by b. Since of course

$$\begin{array}{ccc} g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet & \longrightarrow & g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet \\ \| & & \| \\ f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet & \longrightarrow & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \end{array}$$

commutes, to finish the proof it suffices to show that

$$\begin{array}{ccccccc} b^{-1}f_*g'_!\mathcal{J}^\bullet & \longrightarrow & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet & \longrightarrow & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet & \longrightarrow & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \\ \downarrow & & \downarrow & & & & \downarrow \\ b^{-1}f_*\mathcal{I}^\bullet & \longrightarrow & f_{1,*}a^{-1}\mathcal{I}^\bullet & & \longrightarrow & & f_{1,*}\mathcal{I}_1^\bullet \end{array}$$

commutes in the derived category, which holds by our choice of maps earlier. □

0F7G **Lemma 63.8.6.** *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g} & \downarrow{\scriptstyle h} \\ & & Z \end{array}$$

with f and g locally quasi-finite and h proper. Let Λ be a ring. Funtorially in $K \in D(X_{étale}, \Lambda)$ there is a canonical map

$$g_!K \longrightarrow Rh_*(f_!K)$$

in $D(Z_{étale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. This is a special case of Lemma 63.8.1 if f and g are separated. We urge the reader to skip the proof in the general case as we'll mainly use the case where f and g are separated.

Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Choose a quasi-isomorphism $f_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $Y_{étale}$. Consider the map

$$g_!\mathcal{K}^\bullet = h_!f_!\mathcal{K}^\bullet = h_*f_!\mathcal{K}^\bullet \longrightarrow h_*\mathcal{I}^\bullet$$

where the equalities are Lemmas 63.4.11 and 63.3.4. This map of complexes determines the map $g_!K \to Rh_*(f_!K)$ of the statement of the lemma.

Assume Λ is torsion, i.e., we are in case (b). To check the map is an isomorphism we may work locally on Z. Hence we may assume that the dimension of fibres of h is bounded, see Morphisms, Lemma 29.28.5. Then we see that Rh_* has finite cohomological dimension, see Étale Cohomology, Lemma 59.92.2. Hence by Derived Categories, Lemma 13.32.2, if we show that $R^qh_*(f_!\mathcal{F}) = 0$ for $q > 0$ and any sheaf $\mathcal{F}$ of Λ-modules on $X_{étale}$, then $h_*f_!\mathcal{K}^\bullet \to h_*\mathcal{I}^\bullet$ is a quasi-isomorphism.

Observe that $\mathcal{G} = f_!\mathcal{F}$ is a sheaf of Λ-modules on Y whose stalks are nonzero only at points $y \in Y$ such that $\kappa(y)/\kappa(h(y))$ is a finite extension. This follows from the description of stalks of $f_!\mathcal{F}$ in Lemma 63.4.5 and the fact that both f and g are locally quasi-finite. Hence by the proper base change theorem (Étale Cohomology, Lemma 59.91.13) it suffices to show that $H^q(Y_{\overline{z}}, \mathcal{H}) = 0$ where $\mathcal{H}$ is a sheaf on the proper scheme $Y_{\overline{z}}$ over $\kappa(\overline{z})$ whose support is contained in the set of closed points. Thus the required vanishing by Étale Cohomology, Lemma 59.97.3.

Case (a) follows from case (b) by the exact same argument as used in the proof of Lemma 63.8.1 (using Lemma 63.4.5 instead of Lemma 63.3.17). □

63.9. Derived lower shriek via compactifications

0F7H Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Choose a compactification $j : X \to \overline{X}$ over Y, see More on Flatness, Theorem 38.33.8. Let Λ be a ring. Denote $D^+_{tors}(X_{étale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{étale}, \Lambda)$ consisting of objects K which are bounded below and whose cohomology sheaves are torsion. We will consider the functor

$$Rf_! = R\overline{f}_* \circ j_! : D^+_{tors}(X_{étale}, \Lambda) \longrightarrow D^+_{tors}(Y_{étale}, \Lambda)$$

where $\overline{f} : \overline{X} \to Y$ is the structure morphism. This makes sense: the functor $j_!$ sends $D^+_{tors}(X_{étale}, \Lambda)$ into $D^+_{tors}(\overline{X}_{étale}, \Lambda)$ by Remark 63.7.2 and $R\overline{f}_*$ sends $D^+_{tors}(\overline{X}_{étale}, \Lambda)$ into $D^+_{tors}(Y_{étale}, \Lambda)$ by Étale Cohomology, Lemma 59.78.2. If Λ is a torsion ring, then we define

$$Rf_! = R\overline{f}_* \circ j_! : D(X_{étale}, \Lambda) \longrightarrow D(Y_{étale}, \Lambda)$$

Here is the obligatory lemma.

0F7I **Lemma 63.9.1.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. The functors $Rf_!$ constructed above are, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. We will prove this for the functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ when Λ is a torsion ring; the case of the functor $Rf_! : D^+_{tors}(X_{étale}, \Lambda) \to D^+_{tors}(Y_{étale}, \Lambda)$ is proved in exactly the same way.

Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 38.33.8 and Lemmas 38.32.1 and 38.32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ j_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$. Let's be a little more explicit. Given a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{étale}$, we choose a quasi-isomorphism $j_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of sheaves of Λ-modules on $\overline{X}_{étale}$. Then our functor sends $\mathcal{K}^\bullet$ to $\overline{f}_*\mathcal{I}^\bullet$.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

using Lemma 63.8.6 in the first equality.

To finish the proof, since the category of compactifications of X over Y is cofiltered, it suffices to show compositions of morphisms of compactifications of X over Y are turned into compositions of isomorphisms of functors[3]. To do this, suppose that $j_3 : X \to \overline{X}_3$ is a third compactification and that $h : \overline{X}_2 \to \overline{X}_3$ is a morphism of compactifications. Then we have to show that the composition

$$R\overline{f}_{3,*} \circ j_{3,!} = R\overline{f}_{3,*} \circ Rh_* \circ j_{2,!} = R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

is equal to the isomorphism of functors constructed using simply j_3, $g \circ h$, and j_1. A calculation shows that it suffices to prove that the composition of the maps

$$j_{3,!} \to Rh_* \circ j_{2,!} \to Rh_* \circ Rg_* \circ j_{1,!}$$

of Lemma 63.8.6 agrees with the corresponding map $j_{3,!} \to R(h \circ g)_* \circ j_{1,!}$ via the identification $R(h \circ g)_* = Rh_* \circ Rg_*$. Since the map of Lemma 63.8.6 is a special case of the map of Lemma 63.8.1 (as j_1 and j_2 are separated) this follows immediately from Lemma 63.8.2. □

0F7J **Lemma 63.9.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then there is a canonical isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$.*

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and

[3]Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$.

Lemma 38.32.2 twice. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

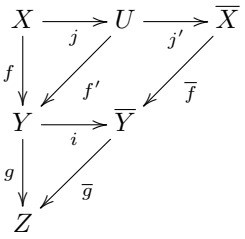

Then we have

$$\begin{aligned} R(g \circ f)_! &= R(\overline{g} \circ \overline{f})_* \circ (j' \circ j)_! \\ &= R\overline{g}_* \circ R\overline{f}_* \circ j'_! \circ j_! \\ &= R\overline{g}_* \circ i_! \circ Rf'_* \circ j_! \\ &= Rg_! \circ Rf_! \end{aligned}$$

The first equality is the definition of $R(g \circ f)_!$. The second equality uses the identifications $R(\overline{g} \circ \overline{f})_* = R\overline{g}_* \circ R\overline{f}_*$ and $(j' \circ j)_! = j'_! \circ j_!$ of Lemma 63.3.13. The identification $i_! \circ Rf'_* \to R\overline{f}_* \circ j_!$ used in the third equality is Lemma 63.8.1. The final fourth equality is the definition of $Rg_!$ and $Rf_!$. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

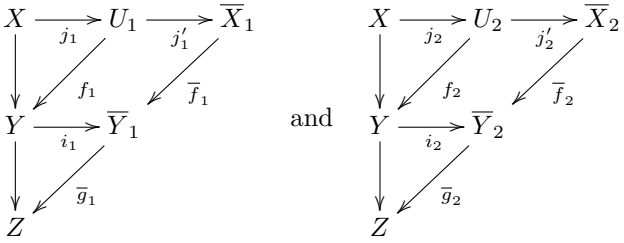

We can first choose a compactification $i : Y \to \overline{Y}$ of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 38.32.1. By More on Flatness, Lemma 38.32.3 and Categories, Lemmas 4.27.13 and 4.27.14 we can choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when

we have a commutative diagram as follows

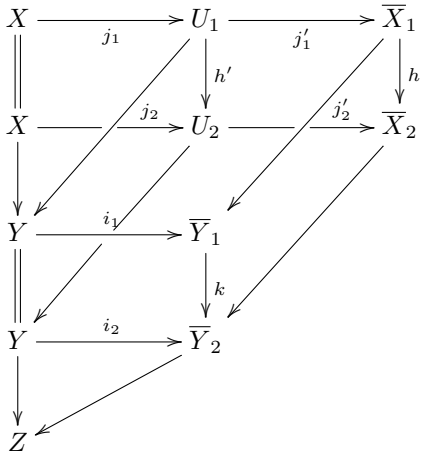

Each of the squares

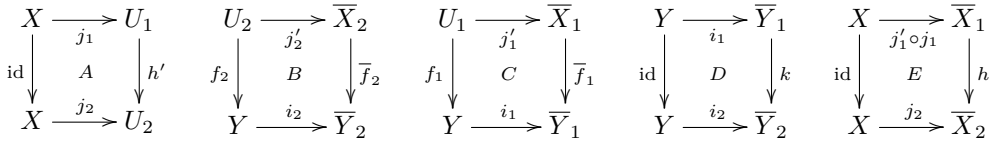

gives rise to an isomorphism as follows

$$\gamma_A : j_{2,!} \to Rh'_* \circ j_{1,!}$$
$$\gamma_B : i_{2,!} \circ Rf_{2,*} \to R\overline{f}_{2,*} \circ j'_{2,!}$$
$$\gamma_C : i_{1,!} \circ Rf_{1,*} \to R\overline{f}_{1,*} \circ j'_{1,!}$$
$$\gamma_D : i_{2,!} \to Rk_* \circ i_{1,!}$$
$$\gamma_E : j_{2,!} \to Rh_* \circ (j'_1 \circ j_1)_!$$

by applying the map from Lemma 63.8.1 (which is the same as the map in Lemma 63.8.6 in case the left vertical arrow is the identity). Let us write

$$F_1 = Rf_{1,*} \circ j_{1,!}$$
$$F_2 = Rf_{2,*} \circ j_{2,!}$$
$$G_1 = R\overline{g}_{1,*} \circ i_{1,!}$$
$$G_2 = R\overline{g}_{2,*} \circ i_{2,!}$$
$$C_1 = R(\overline{g}_1 \circ \overline{f}_1)_* \circ (j'_1 \circ j_1)_!$$
$$C_2 = R(\overline{g}_2 \circ \overline{f}_2)_* \circ (j'_2 \circ j_2)_!$$

The construction given in the first paragraph of the proof and in Lemma 63.9.1 uses

(1) γ_C for the map $G_1 \circ F_1 \to C_1$,
(2) γ_B for the map $G_2 \circ F_2 \to C_2$,
(3) γ_A for the map $F_2 \to F_1$,
(4) γ_D for the map $G_2 \to G_1$, and
(5) γ_E for the map $C_2 \to C_1$.

This implies that we have to show that the diagram

$$\begin{array}{ccc} C_2 & \xrightarrow{\gamma_E} & C_1 \\ {\scriptstyle \gamma_B}\uparrow & & \uparrow{\scriptstyle \gamma_C} \\ G_2 \circ F_2 & \xrightarrow{\gamma_D \circ \gamma_A} & G_1 \circ F_1 \end{array}$$

is commutative. We will use Lemmas 63.8.2 and 63.8.3 and with (abuse of) notation as in Remark 63.8.4 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_F \circ \gamma_A$ where

$$\begin{array}{ccc} U_1 & \xrightarrow{j_1'} & \overline{X}_1 \\ {\scriptstyle h'}\downarrow & F & \downarrow{\scriptstyle h} \\ U_2 & \xrightarrow{j_2'} & \overline{X}_2 \end{array}$$

Thus we see that

$$\gamma_E \circ \gamma_B = \gamma_F \circ \gamma_A \circ \gamma_B = \gamma_F \circ \gamma_B \circ \gamma_A$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 63.8.4). Thus it suffices to prove that $\gamma_C \circ \gamma_D = \gamma_F \circ \gamma_B$. Since both of these are equal to the map for the square

$$\begin{array}{ccc} U_1 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \overline{Y}_2 \end{array}$$

we conclude. □

0F7K **Lemma 63.9.3.** *Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then the diagram*

$$\begin{array}{ccc} Rh_! \circ Rg_! \circ Rf_! & \xrightarrow{\gamma_C} & R(h \circ g)_! \circ Rf_! \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_{A+B}} \\ Rh_! \circ R(g \circ f)_! & \xrightarrow{\gamma_{B+C}} & R(h \circ g \circ f)_! \end{array}$$

of isomorphisms of Lemma 63.9.2 commutes (for the meaning of the γ's see proof).

Proof. To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 38.33.8 and Lemma 38.32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$

under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \overline{X} \\
{\scriptstyle f}\downarrow & & \downarrow & A & \downarrow & B & \downarrow \\
Y & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \overline{Y} \\
{\scriptstyle g}\downarrow & & \downarrow & & \downarrow & C & \downarrow \\
Z & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & \overline{Z} \\
{\scriptstyle h}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & T & \longrightarrow & T & \longrightarrow & T
\end{array}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 63.9.2 we see that the maps in the first displayed diagram use the maps of Lemma 63.8.1 for the rectangles $A + B$, $B + C$, A, and C as indicated in the diagram in the statement of the lemma. Since by Lemmas 63.8.2 and 63.8.3 we have $\gamma_{A+B} = \gamma_B \circ \gamma_A$ and $\gamma_{B+C} = \gamma_B \circ \gamma_C$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 63.8.4). □

0F7L **Lemma 63.9.4.** *Consider a cartesian square*

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{g} & Y
\end{array}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then there is a canonical isomorphism

$$g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$$

Moreover, these isomorphisms are compatible with the isomorphisms of Lemma 63.9.2.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $j' : X' \to \overline{X}'$ and $\overline{f}' : \overline{X}' \to Y'$ denote the base changes of j and $\overline{f}$. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf'_! = R\overline{f}'_* \circ j'_!$ the isomorphism can be constructed via

$$g^{-1} \circ R\overline{f}_* \circ j_! \to R\overline{f}'_* \circ (\overline{g}')^{-1} \circ j_! \to R\overline{f}'_* \circ j'_! \circ (g')^{-1}$$

where the first arrow is the isomorphism given to us by the proper base change theorem (Étale Cohomology, Lemma 59.91.12 in the bounded below torsion case and Étale Cohomology, Lemma 59.92.3 in the case that Λ is torsion) and the second arrow is the isomorphism of Lemma 63.3.12.

To finish the proof we have to show two things: first we have to show that the isomorphism of functors so obtained does not depend on the choice of the compactification and second we have to show that if we vertically stack two base change diagrams as in the lemma, then these base change isomorphisms are compatible with the isomorphisms of Lemma 63.9.2. A straightforward argument which we omit shows that both follow if we can show that the isomorphisms

(1) $Rg_* \circ Rf_* = R(g \circ f)_*$ for $f : X \to Y$ and $g : Y \to Z$ proper,
(2) $g_! \circ f_! = (g \circ f)_!$ for $f : X \to Y$ and $g : Y \to Z$ separated and quasi-finite, and
(3) $g_! \circ Rf'_* = Rf_* \circ g'_!$ for $f : X \to Y$ and $f' : X' \to Y'$ proper and $g : Y' \to Y$ and $g' : X' \to X$ separated and quasi-finite with $f \circ g' = g \circ f'$

are compatible with base change. This holds for (1) by Cohomology on Sites, Remark 21.19.4, for (2) by Remark 63.3.14, and (3) by Lemma 63.8.5. □

0H6X **Remark 63.9.5.** Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Below we will construct a map

$$Rf_!K \longrightarrow Rf_*K$$

functorial for K in $D^+_{tors}(X_{étale}, \Lambda)$ or $D(X_{étale}, \Lambda)$ if Λ is torsion. This transformation of functors in both cases is compatible with

(1) the isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$ of Lemma 63.9.2 and the isomorphism $Rg_* \circ Rf_* \to R(g \circ f)_*$ of Cohomology on Sites, Lemma 21.19.2 and
(2) the isomorphism $g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$ of Lemma 63.9.4 and the base change map of Cohomology on Sites, Remark 21.19.3.

Namely, choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf_* = R\overline{f}_* \circ Rj_*$ it suffices to construct a transformation of functors $j_! \to Rj_*$. For this we use the canonical transformation $j_! \to j_*$ of Étale Cohomology, Lemma 59.70.6. We omit the proof that the resulting transformation is independent of the choice of compactification and we omit the proof of the compatibilities (1) and (2).

63.10. Properties of derived lower shriek

0G28 Here are some properties of derived lower shriek.

0G29 **Lemma 63.10.1.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring.*

(1) *Let $K_i \in D^+_{tors}(X_{étale}, \Lambda)$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $Rf_!(\bigoplus_i K_i) = \bigoplus_i Rf_!K_i$.*
(2) *If Λ is torsion, then the functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ commutes with direct sums.*

Proof. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{étale}) \to D(X_{étale})$ is a left adjoint to pullback $j^{-1} : D(X_{étale}) \to D(U_{étale})$ and hence commutes with direct sums, see Cohomology on Sites, Lemma 21.20.8. In the proper case we have $Rf_! = Rf_*$ and we get the result from Étale Cohomology, Lemma 59.52.6 in the bounded belo case and from Étale Cohomology, Lemma 59.96.4 in the case that our coefficient ring Λ is a torsion ring. □

0G2A **Lemma 63.10.2.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring. The functors $Rf_!$ constructed in Section 63.9 are bounded in the following sense: There exists an integer N such that for $E \in D^+_{tors}(X_{étale}, \Lambda)$ or $E \in D(X_{étale}, \Lambda)$ if Λ is torsion, we have*

(1) *$H^i(Rf_!(\tau_{\leq a}E) \to H^i(Rf_!(E))$ is an isomorphism for $i \leq a$,*

(2) $H^i(Rf_!(E)) \to H^i(Rf_!(\tau_{\geq b-N}E))$ *is an isomorphism for* $i \geq b$,
(3) *if* $H^i(E) = 0$ *for* $i \notin [a, b]$ *for some* $-\infty \leq a \leq b \leq \infty$, *then* $H^i(Rf_!(E)) = 0$ *for* $i \notin [a, b + N]$.

Proof. Assume Λ is torsion and consider the functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{étale}) \to D(X_{étale})$ is exact and hence the statement holds with $N = 0$ in this case. If f is proper then $Rf_! = Rf_*$, i.e., it is a right derived functor. Hence the bound on the left by Derived Categories, Lemma 13.16.1. Moreover in this case $f_* : Mod(X_{étale}, \Lambda) \to Mod(Y_{étale}, \Lambda)$ has bounded cohomological dimension by Morphisms, Lemma 29.28.5 and Étale Cohomology, Lemma 59.92.2. Thus we conclude by Derived Categories, Lemma 13.32.2.

Next, assume Λ is arbitrary and let us consider the functor $Rf_! : D^+_{tors}(X_{étale}, \Lambda) \to D^+_{tors}(Y_{étale}, \Lambda)$. Again we immediately reduce to the case where f is proper and $Rf_! = Rf_*$. Again part (1) is immediate. To show part (3) we can use induction on $b - a$, the distinguished triangles of trunctions, and Étale Cohomology, Lemma 59.92.2. Part (2) follows from (3). Details omitted. □

0GKM **Lemma 63.10.3.** *Let $f : X \to Y$ be a quasi-finite separated morphism of quasi-compact and quasi-separated schemes. Then the functors $Rf_!$ constructed in Section 63.9 agree with the restriction of the functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ constructed in Section 63.7 to their common domains of definition.*

Proof. By Zariski's main theorem (More on Morphisms, Lemma 37.43.3) we can find an open immersion $j : X \to \overline{X}$ and a finite morphism $\overline{f} : \overline{X} \to Y$ with $f = \overline{f} \circ j$. By construction we have $Rf_! = R\overline{f}_* \circ j_!$. Since $\overline{f}$ is finite, we have $R\overline{f}_* = \overline{f}_*$ by Étale Cohomology, Proposition 59.55.2. The lemma follows because $\overline{f}_* \circ j_! = f_!$ for example by Lemma 63.3.6. □

0GKN **Lemma 63.10.4.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let U and V be quasi-compact opens of X such that $X = U \cup V$. Denote $a : U \to Y$, $b : V \to Y$ and $c : U \cap V \to Y$ the restrictions of f. Let Λ be a ring. For K in $D^+_{tors}(X_{étale}, \Lambda)$ or $K \in D(X_{étale}, \Lambda)$ if Λ is torsion, we have a distinguished triangle*

$$Rc_!(K|_{U\cap V}) \to Ra_!(K|_U) \oplus Rb_!(K|_V) \to Rf_!K \to Rc_!(K|_{U\cap V})[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 63.7.3, the fact that $Rf_! \circ Rj_{U!} = Ra_!$ by Lemma 63.9.2, and the fact that $Rj_{U!} = j_{U!}$ by Lemma 63.10.3. □

0GKP **Lemma 63.10.5.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let U be a quasi-compact open of X with complement $Z \subset X$. Denote $g : U \to Y$ and $h : Z \to Y$ the restrictions of f. Let Λ be a ring. For K in $D^+_{tors}(X_{étale}, \Lambda)$ or $K \in D(X_{étale}, \Lambda)$ if Λ is torsion, we have a distinguished triangle*

$$Rg_!(K|_U) \to Rf_!K \to Rh_!(K|_Z) \to Rg_!(K|_U)[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 63.7.4, the fact that $Rf_! \circ Rj_! = Rg_!$ and $Rf_! \circ Ri_!$ by Lemma 63.9.2, and the fact that $Rj_! = j_!$ and $Ri_! = i_! = i_*$ by Lemma 63.10.3. □

0GKQ **Lemma 63.10.6.** *Let $f' : X' \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let $i : X \to X'$ be a thickening and denote $f = f' \circ i$. Let Λ be a ring. For K' in $D^+_{tors}(X'_{étale}, \Lambda)$ or $K' \in D(X'_{étale}, \Lambda)$ if Λ is torsion, we have $Rf_! i^{-1} K' = Rf'_! K'$.*

Proof. This is true because i^{-1} and $i_* = i_!$ inverse equivalences of categories by the topological invariance of the small étale topos (Étale Cohomology, Theorem 59.45.2) and we can apply Lemma 63.9.2. □

0GL5 **Lemma 63.10.7.** *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $E \in D(X_{étale}, \Lambda)$ and $K \in D(Y_{étale}, \Lambda)$. Then*

$$Rf_! E \otimes^{\mathbf{L}}_{\Lambda} K = Rf_!(E \otimes^{\mathbf{L}}_{\Lambda} f^{-1} K)$$

in $D(Y_{étale}, \Lambda)$.

Proof. Choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We have $j_! E \otimes^{\mathbf{L}}_{\Lambda} \overline{f}^{-1} K = j_!(E \otimes^{\mathbf{L}}_{\Lambda} f^{-1} K)$ by Cohomology on Sites, Lemma 21.20.9. Then we get the result by applying Étale Cohomology, Lemma 59.96.6 and using that $f^{-1} = j^{-1} \circ \overline{f}^{-1}$ and $Rf_! = R\overline{f}_* j_!$. □

0GL6 **Remark 63.10.8.** Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram

$$\begin{array}{ccc} D(X_{étale}, \Lambda_2) & \xrightarrow{res} & D(X_{étale}, \Lambda_1) \\ \downarrow{\scriptstyle Rf_!} & & \downarrow{\scriptstyle Rf_!} \\ D(Y_{étale}, \Lambda_2) & \xrightarrow{res} & D(Y_{étale}, \Lambda_1) \end{array}$$

commutes where res is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. Writing $Rf_! = R\overline{f}_* \circ j_!$ for a factorization $f = \overline{f} \circ j$ as in Section 63.9, we see that the result holds for $j_!$ by inspection and for $R\overline{f}_*$ by Cohomology on Sites, Lemma 21.20.7. On the other hand, also the diagram

$$\begin{array}{ccc} D(X_{étale}, \Lambda_1) & \xrightarrow[-\otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2]{} & D(X_{étale}, \Lambda_2) \\ \downarrow{\scriptstyle Rf_!} & & \downarrow{\scriptstyle Rf_!} \\ D(Y_{étale}, \Lambda_1) & \xrightarrow{-\otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2} & D(Y_{étale}, \Lambda_2) \end{array}$$

is commutative as follows from Lemma 63.10.7.

0GL7 **Remark 63.10.9.** Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring and let K and L be objects of $D(X_{étale}, \Lambda)$. We claim there is a canonical map

$$\alpha : Rf_* R\mathcal{H}om_{\Lambda}(K, L) \longrightarrow R\mathcal{H}om_{\Lambda}(Rf_! K, Rf_! L)$$

functorial in K and L. Namely, choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We first define a map

$$\beta : Rj_* R\mathcal{H}om_\Lambda(K, L) \longrightarrow R\mathcal{H}om_\Lambda(j_!K, j_!L)$$

By the construction of internal hom in the derived category, this is the same thing as defining a map

$$\beta' : Rj_* R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda j_!K \longrightarrow j_!L$$

See Cohomology on Sites, Section 21.35. The source of β' is equal to

$$j_! \left(R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K \right)$$

by Cohomology on Sites, Lemma 21.20.9. Hence we can set $\beta' = j_!\beta''$ where $\beta'' : R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K \to L$ corresponds to the identity on $R\mathcal{H}om_\Lambda(K, L)$ via the universal property of internal hom mentioned above. By Cohomology on Sites, Remark 21.35.10 we have a canonical map

$$\gamma : R\overline{f}_* R\mathcal{H}om_\Lambda(j_!K, j_!L) \longrightarrow R\mathcal{H}om_\Lambda(R\overline{f}_* j_!K, R\overline{f}_* j_!L)$$

Since $Rf_! = R\overline{f}_* j_!$ and $Rf_* = R\overline{f}_* Rj_*$ (by Leray) we obtain the desired map $\alpha = \gamma \circ R\overline{f}_*\beta$.

63.11. Derived upper shriek

0G2B We obtain $Rf^!$ by a Brown representability theorem.

0G2C **Lemma 63.11.1.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ has a right adjoint $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$.*

Proof. This follows from Injectives, Proposition 19.15.2 and Lemma 63.10.1 above. □

0GL8 **Lemma 63.11.2.** *Let $f : X \to Y$ be a separated quasi-finite morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ of Lemma 63.11.1 is the same as the functor $Rf^!$ of Lemma 63.7.1.*

Proof. Follows from uniqueness of adjoints as $Rf_! = f_!$ by Lemma 63.10.3. □

0GL9 **Lemma 63.11.3.** *Let $j : U \to X$ be a separated étale morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rj^! : D(X_{étale}, \Lambda) \to D(U_{étale}, \Lambda)$ is equal to j^{-1}.*

Proof. This is true because both $Rj^!$ and j^{-1} are right adjoints to $Rj_! = j_!$. See for example Lemmas 63.11.2 and 63.6.2. □

0GLA **Lemma 63.11.4.** *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. The functor $Rf^!$ sends $D^+(Y_{étale}, \Lambda)$ into $D^+(X_{étale}, \Lambda)$. More precisely, there exists an integer $N \geq 0$ such that if $K \in D(Y_{étale}, \Lambda)$ has $H^i(K) = 0$ for $i < a$ then $H^i(Rf^!K) = 0$ for $i < a - N$.*

Proof. Let N be the integer found in Lemma 63.10.2. By construction, for $K \in D(Y_{étale}, \Lambda)$ and $L \in\in D(X_{étale}, \Lambda)$ we have $\mathrm{Hom}_X(L, Rf^!K) = \mathrm{Hom}_Y(Rf_!L, K)$. Suppose $H^i(K) = 0$ for $i < a$. Then we take $L = \tau_{\leq a-N-1}Rf^!K$. By Lemma 63.10.2 the complex $Rf_!L$ has vanishing cohomology sheaves in degrees $\leq a - 1$. Hence $\mathrm{Hom}_Y(Rf_!L, K) = 0$ by Derived Categories, Lemma 13.27.3. Hence the canonical map $\tau_{\leq a-N-1}Rf^!K \to Rf^!K$ is zero which implies $H^i(Rf^!K) = 0$ for $i \leq a - N - 1$. □

Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion coefficient ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ we obtain a canonical map

0GLB (63.11.4.1)
$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \to R\mathcal{H}om_\Lambda(Rf_!L, Rf_!Rf^!K) \to R\mathcal{H}om_\Lambda(Rf_!L, K)$$

where the first arrow is Remark 63.10.9 and the second arrow is the counit $Rf_!Rf^!K \to K$ of the adjunction.

0GLC **Lemma 63.11.5.** *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (63.11.4.1)*

$$Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!L, K)$$

is an isomorphism.

Proof. To prove the lemma we have to show that for any $M \in D(Y_{étale}, \Lambda)$ the map (63.11.4.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K)) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_!L, K))$$

To see this we use the following string of equalities

$$\begin{aligned}
\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K)) &= \mathrm{Hom}_X(f^{-1}M, R\mathcal{H}om_\Lambda(L, Rf^!K)) \\
&= \mathrm{Hom}_X(f^{-1}M \otimes^{\mathbf{L}}_\Lambda L, Rf^!K) \\
&= \mathrm{Hom}_Y(Rf_!(f^{-1}M \otimes^{\mathbf{L}}_\Lambda L), K) \\
&= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_\Lambda Rf_!L, K) \\
&= \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_!L, K))
\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by Cohomology on Sites, Lemma 21.35.2. The third equality by construction of $Rf^!$. The fourth equality by Lemma 63.10.7 (this is the important step). The fifth by Cohomology on Sites, Lemma 21.35.2. □

0GLD **Lemma 63.11.6.** *Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (63.11.4.1) induces an isomorphism*

$$R\mathrm{Hom}_X(L, Rf^!K) \longrightarrow R\mathrm{Hom}_Y(Rf_!L, K)$$

of global derived homs.

Proof. By the construction in Cohomology on Sites, Section 21.36 we have
$R\mathrm{Hom}_X(L, Rf^!K) = R\Gamma(X, R\mathcal{H}om_\Lambda(L, Rf^!K)) = R\Gamma(Y, Rf_*R\mathcal{H}om_\Lambda(L, Rf^!K))$
(the second equality by Leray) and
$$R\mathrm{Hom}_Y(Rf_!L, K) = R\Gamma(Y, R\mathcal{H}om_\Lambda(Rf_!L, K))$$
Thus the lemma is a consequence of Lemma 63.11.5. □

0GLE **Lemma 63.11.7.** *Consider a cartesian square*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
of quasi-compact and quasi-separated schemes with f separated and of finite type. Then we have $Rf^! \circ Rg_* = Rg'_* \circ R(f')^!$.

Proof. By uniqueness of adjoint functors this follows from base change for derived lower shriek: we have $g^{-1} \circ Rf_! = Rf'_! \circ (g')^{-1}$ by Lemma 63.9.4. □

0GLF **Remark 63.11.8.** Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram
$$\begin{array}{ccc} D(X_{étale}, \Lambda_2) & \xrightarrow{res} & D(X_{étale}, \Lambda_1) \\ {\scriptstyle Rf^!}\uparrow & & \uparrow{\scriptstyle Rf^!} \\ D(Y_{étale}, \Lambda_2) & \xrightarrow{res} & D(Y_{étale}, \Lambda_1) \end{array}$$
commutes where res is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. This holds by uniquenss of adjoints, the second commutative diagram of Remark 63.10.8 and because we have
$$\mathrm{Hom}_{\Lambda_2}(K_1 \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2, K_2) = \mathrm{Hom}_{\Lambda_1}(K_1, res(K_2))$$
This equality either for objects living over $X_{étale}$ or on $Y_{étale}$ is a very special case of Cohomology on Sites, Lemma 21.19.1.

63.12. Compactly supported cohomology

0GJY Let k be a field. Let Λ be a ring. Let X be a separated scheme of finite type over k with structure morphism $f : X \to \mathrm{Spec}(k)$. In Section 63.9 we have defined the functor $Rf_! : D^+_{tors}(X_{étale}, \Lambda) \to D^+_{tors}(\mathrm{Spec}(k), \Lambda)$ and the functor $Rf_! : D(X_{étale}, \Lambda) \to D(\mathrm{Spec}(k), \Lambda)$ if Λ is a torsion ring. Composing with the global sections functor on $\mathrm{Spec}(k)$ we obtain what we will call the compactly supported cohomology.

0GJZ **Definition 63.12.1.** Let X be a separated scheme of finite type over a field k. Let Λ be a ring. Let K be an object of $D^+_{tors}(X_{étale}, \Lambda)$ or of $D(X_{étale}, \Lambda)$ in case Λ is torsion. The *cohomology of K with compact support* or the *compactly supported cohomology of K* is
$$R\Gamma_c(X, K) = R\Gamma(\mathrm{Spec}(k), Rf_!K)$$
where $f : X \to \mathrm{Spec}(k)$ is the structure morphism. We will write $H^i_c(X, K) = H^i(R\Gamma_c(X, K))$.

We will check that this definition doesn't conflict with Definition 63.3.7 by Lemma 63.12.3. The utility of this definition lies in the following result.

0GK0 **Lemma 63.12.2.** *Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Let K be an object of $D^+_{tors}(X_{étale}, \Lambda)$ or of $D(X_{étale}, \Lambda)$ in case Λ is torsion. Then there is a canonical isomorphism*

$$(Rf_!K)_{\overline{y}} \longrightarrow R\Gamma_c(X_{\overline{y}}, K|_{X_{\overline{y}}})$$

in $D(\Lambda)$ for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Immediate consequence of Lemma 63.9.4 and the definitions. □

0GK1 **Lemma 63.12.3.** *Let X be a separated scheme of finite type over a field k. If $\mathcal{F}$ is a torsion abelian sheaf, then the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 63.3.7 agrees with the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 63.12.1.*

Proof. Choose a compactification $j : X \to \overline{X}$ over k. In both cases the group is defined as $H^0(\overline{X}, j_!\mathcal{F})$. This is true for the first version by Lemma 63.3.10 and for the second version by construction. □

0GKR **Lemma 63.12.4.** *Let k be an algebraically closed field. Let X be a separated scheme of finite type type over k of dimension ≤ 1. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on X which is torsion. Then $H^q_c(X, \mathcal{F})$ is a finite Λ-module.*

Proof. This is a consequence of Étale Cohomology, Theorem 59.84.7. Namely, choose a compactification $j : X \to \overline{X}$. After replacing $\overline{X}$ by the scheme theoretic closure of X, we see that we may assume $\dim(\overline{X}) \leq 1$. Then $H^q_c(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F})$ and the theorem applies. □

0GKS **Remark 63.12.5** (Covariance of compactly supported cohomology)**.** Let k be a field. Let $f : X \to Y$ be a morphism of separated schemes of finite type over k. If X, Y, and f satisfies one of the following conditions

(1) f is étale, or
(2) f is flat and quasi-finite, or
(3) f is quasi-finite and Y is geometrically unibranch, or
(4) f is quasi-finite and there exists a weighting $w : X \to \mathbf{Z}$ of f

then compactly supported cohomology is covariant with respect to f. More precisely, let Λ be a ring. Let K be an object of $D^+_{tors}(Y_{étale}, \Lambda)$ or of $D(Y_{étale}, \Lambda)$ in case Λ is torsion. Under one of the assumptions (1) – (4) there is a canonical map

$$\mathrm{Tr}_{f,w,K} : f_!f^{-1}K \longrightarrow K$$

See Section 63.5 for the existence of the trace map and Examples 63.5.5 and 63.5.7 for cases (2) and (3). If $p : X \to \mathrm{Spec}(k)$ and $q : Y \to \mathrm{Spec}(k)$ denote the structure morphisms, then we have $Rq_! \circ f_! = Rp_!$ by Lemma 63.9.2 and the fact that $Rf_! = f_!$ for the quasi-finite separated morphism f by Lemma 63.10.3. Hence we can look

at the map

$$\begin{aligned} R\Gamma_c(X, f^{-1}K) &= R\Gamma(\mathrm{Spec}(k), Rp_! f^{-1}K) \\ &= R\Gamma(\mathrm{Spec}(k), Rq_! f_! f^{-1}K) \\ &\xrightarrow{Rq_!\mathrm{Tr}_{f,w,K}} R\Gamma(\mathrm{Spec}(k), Rq_! K) \\ &= R\Gamma_c(Y, K) \end{aligned}$$

In particular, if Λ is a torsion ring, then we obtain an arrow

$$\mathrm{Tr}_f : R\Gamma_c(X, \Lambda) \longrightarrow R\Gamma_c(Y, \Lambda)$$

This map has lots of additional properties, for example it is compatible with taking ground field extensions.

63.13. A constructibility result

0GKT We "compute" the cohomology of a smooth projective family of curves with constant coefficients.

0GKU **Lemma 63.13.1.** *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module viewed as an $\mathcal{O}_S$-module on $S_{étale}$. Let $F : \mathcal{E} \to \mathcal{E}$ be a homomorphism of abelian sheaves on $S_{étale}$ such that $F(ae) = a^p F(e)$ for local sections a, e of $\mathcal{O}_S$, $\mathcal{E}$ on $S_{étale}$. Then*

$$\mathrm{Coker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is zero and

$$\mathrm{Ker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is a constructible abelian sheaf on $S_{étale}$.

This lemma is a generalization of Étale Cohomology, Lemma 59.63.2.

Proof. We may assume $S = \mathrm{Spec}(A)$ where A is an $\mathbf{F}_p$-algebra and that $\mathcal{E}$ is the quasi-coherent module associated to the free A-module $Ae_1 \oplus \ldots \oplus Ae_n$. We write $F(e_i) = \sum a_{ij} e_j$.

Surjectivity of $F - 1$. It suffices to show that any element $\sum a_i e_i$, $a_i \in A$ is in the image of $F - 1$ after replacing A by a faithfully flat étale extension. Observe that

$$F(\sum x_i e_i) - \sum x_i e_i = \sum x_i^p a_{ij} e_j - \sum x_i e_i$$

Consider the A-algebra

$$A' = A[x_1, \ldots, x_n]/(a_i + x_i - \sum\nolimits_j a_{ji} x_j^p)$$

A computation shows that $\mathrm{d}x_i$ is zero in $\Omega_{A'/A}$ and hence $\Omega_{A'/A} = 0$. Since A' is of finite type over A, this implies that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is unramified and hence is quasi-finite. Since A' is generated by n elements and cut out by n equations, we conclude that A' is a global relative complete intersection over A. Thus A' is flat over A and we conclude that $A \to A'$ is étale (as a flat and unramified ring map). Finally, the reader can show that $A \to A'$ is faithfully flat by verifying directly that all geometric fibres of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ are nonempty, however this also follows from Étale Cohomology, Lemma 59.63.2. Finally, the element $\sum x_i e_i \in A'e_1 \oplus \ldots \oplus A'e_n$ maps to $\sum a_i e_i$ by $F - 1$.

Constructibility of the kernel. The calculations above show that $\mathrm{Ker}(F-1)$ is represented by the scheme

$$\mathrm{Spec}(A[x_1,\ldots,x_n]/(x_i - \sum\nolimits_j a_{ji}x_j^p))$$

over $S = \mathrm{Spec}(A)$. Since this is a scheme affine and étale over S we obtain the result from Étale Cohomology, Lemma 59.73.1. □

0GKV **Lemma 63.13.2.** *Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let ℓ be a prime number. Then $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ is a constructible.*

Proof. We may assume S is affine. Say $S = \mathrm{Spec}(A)$. Then, if we write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i and a morphism $f_i : X_i \to S_i = \mathrm{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth, proper, and of relative dimension 1, see Limits, Lemmas 32.13.1 32.8.9, and 32.18.4. By More on Morphisms, Lemma 37.53.8 we obtain an open subscheme $U_i \subset S_i$ such that the fibres of $f_i : X_i \to S_i$ over U_i are geometrically connected. Then $S \to S_i$ maps into U_i. We may replace $X \to S$ by $f_i : f_i^{-1}(U_i) \to U_i$ to reduce to the case discussed in the next paragraph.

Assume S is Noetherian. We may write $S = U \cup Z$ where U is the open subscheme defined by the nonvanishing of ℓ and $Z = V(\ell) \subset S$. Since the formation of $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ commutes with arbitrary base change (Étale Cohomology, Theorem 59.91.11), it suffices to prove the result over U and over Z. Thus we reduce to the following two cases: (a) ℓ is invertible on S and (b) ℓ is zero on S.

Case (a). We claim that in this case the sheaves $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite locally constant on S. First, by proper base change (in the form of Étale Cohomology, Lemma 59.91.13) and by finiteness (Étale Cohomology, Theorem 59.83.10) we see that the stalks of $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite. By Étale Cohomology, Lemma 59.94.4 all specialization maps are isomorphisms. We conclude the claim holds by Étale Cohomology, Lemma 59.75.6.

Case (b). Here $\ell = p$ is a prime and S is a scheme over $\mathrm{Spec}(\mathbf{F}_p)$. By the same references as above we already know that the stalks of $R^q f_* \underline{\mathbf{Z}/p\mathbf{Z}}$ are finite and zero for $q \geq 2$. It follows from Étale Cohomology, Lemma 59.39.3 that $f_* \underline{\mathbf{Z}/p\mathbf{Z}} = \underline{\mathbf{Z}/p\mathbf{Z}}$. It remains to prove that $R^1 f_* \underline{\mathbf{Z}/p\mathbf{Z}}$ is constructible. Consider the Artin-Schreyer sequence

$$0 \to \underline{\mathbf{Z}/p\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{F-1} \mathcal{O}_X \to 0$$

See Étale Cohomology, Section 59.63. Recall that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $R^1 f_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module of rank equal to the genera of the fibres of $X \to S$, see Algebraic Curves, Lemma 53.20.12. We conclude that we have a short exact sequence

$$0 \to \mathrm{Coker}(F-1 : \mathcal{O}_S \to \mathcal{O}_S) \to R^1 f_* \underline{\mathbf{Z}/p\mathbf{Z}} \to \mathrm{Ker}(F-1 : R^1 f_* \mathcal{O}_X \to R^1 f_* \mathcal{O}_X) \to 0$$

Applying Lemma 63.13.1 we win. □

0GKW **Lemma 63.13.3.** *Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let Λ be a Noetherian ring. Let M be*

a finite Λ-module annihilated by an integer $n > 0$. Then $R^q f_* \underline{M}$ is a constructible sheaf of Λ-modules on S.

Proof. If $n = \ell n'$ for some prime number ℓ, then we get a short exact sequence $0 \to M[\ell] \to M \to M' \to 0$ of finite Λ-modules and M' is annihilated by n'. This produces a corresponding short exact sequence of constant sheaves, which in turn gives rise to an exact sequence

$$R^{q-1} f_* \underline{M'} \to R^q f_* \underline{M[n]} \to R^q f_* \underline{M} \to R^q f_* \underline{M'} \to R^{q+1} f_* \underline{M[n]}$$

Thus, if we can show the result in case M is annihilated by a prime number, then by induction on n we win by Étale Cohomology, Lemma 59.71.6.

Let ℓ be a prime number such that ℓ annihilates M. Then we can replace Λ by the $\mathbf{F}_\ell$-algebra $\Lambda/\ell\Lambda$. Namely, the sheaf $R^q f_* \underline{M}$ where $\underline{M}$ is viewed as a sheaf of Λ-modules is the same as the sheaf $R^q f_* \underline{M}$ computed by viewing $\underline{M}$ as a sheaf of $\Lambda/\ell\Lambda$-modules, see Cohomology on Sites, Lemma 21.20.7.

Assume ℓ be a prime number such that ℓ annihilates M and Λ. Let us reduce to the case where M is a finite free Λ-module. Namely, choose a resolution

$$\ldots \to \Lambda^{\oplus m_2} \to \Lambda^{\oplus m_1} \to \Lambda^{\oplus m_0} \to M \to 0$$

Recall that f_* has finite cohomological dimension on sheaves of Λ-modules, see Étale Cohomology, Lemma 59.92.2 and Derived Categories, Lemma 13.32.2. Thus we see that $R^q f_* \underline{M}$ is the qth cohomology sheaf of the object

$$Rf_*(\underline{\Lambda^{\oplus m_a}} \to \ldots \to \underline{\Lambda^{\oplus m_0}})$$

in $D(S_{étale}, \Lambda)$ for some integer a large enough. Using the first spectral sequence of Derived Categories, Lemma 13.21.3 (or alternatively using an argument with truncations) we conclude that it suffices to prove that $R^q f_* \underline{\Lambda})$ is constructible.

At this point we can finally use that

$$(Rf_* \underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = Rf_* \underline{\Lambda}$$

by Étale Cohomology, Lemma 59.96.6. Since any module over the field $\mathbf{Z}/\ell\mathbf{Z}$ is flat we obtain

$$(R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = R^q f_* \underline{\Lambda}$$

Hence it suffices to prove the result for $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ by Étale Cohomology, Lemma 59.71.10. This case is Lemma 63.13.2. □

63.14. Complexes with constructible cohomology

0GK2 We continue the discussion started in Étale Cohomology, Section 59.76. In particular, for a scheme X and a Noetherian ring Λ we denote $D_c(X_{étale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{étale}, \Lambda)$ consisting of objects whose cohomology sheaves are constructible sheaves of Λ-modules.

0GK3 **Lemma 63.14.1.** *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite and of finite presentation. The functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ of Lemma 63.7.1 sends $D_c(X_{étale}, \Lambda)$ into $D_c(Y_{étale}, \Lambda)$.*

Proof. Since the functor $f_!$ is exact, it suffices to show that $f_!\mathcal{F}$ is constructible for any constructible sheaf $\mathcal{F}$ of Λ-modules on $X_{étale}$. The question is local on Y and hence we may and do assume Y is affine. Then X is quasi-compact and quasi-separated, see Morphisms, Definition 29.21.1. Say $X = \bigcup_{i=1,\ldots,n} X_i$ is a finite affine open covering. By Lemma 63.4.7 we see that it suffices to show that $f_{i,!}\mathcal{F}|_{X_i}$ and $f_{ii',!}\mathcal{F}|_{X_i \cap X_{i'}}$ are constructible where $f_i : X_i \to Y$ and $f_{ii'} : X_i \cap X_{i'} \to Y$ are the restrictions of f. Since X_i and $X_i \cap X_{i'}$ are quasi-compact and separated this means we may assume f is separated. By Zariski's main theorem (in the form of More on Morphisms, Lemma 37.43.4) we can choose a factorization $f = g \circ j$ where $j : X \to X'$ is an open immersion and $g : X' \to Y$ is finite and of finite presentation. Then $f_! = g_! \circ j_!$ by Lemma 63.3.13. By Étale Cohomology, Lemma 59.73.1 we see that $j_!\mathcal{F}$ is constructible on X'. The morphism g is finite hence $g_! = g_*$ by Lemma 63.3.4. Thus $f_!\mathcal{F} = g_!j_!\mathcal{F} = g_*j_!\mathcal{F}$ is constructible by Étale Cohomology, Lemma 59.73.9. □

0GKX **Lemma 63.14.2.** *Let S be a Noetherian affine scheme of finite dimension. Let $f : X \to S$ be a separated, affine, smooth morphism of relative dimension 1. Let Λ be a Noetherian ring which is torsion. Let M be a finite Λ-module. Then $Rf_!\underline{M}$ has constructible cohomology sheaves.*

Proof. We will prove the result by induction on $d = \dim(S)$.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^qf_!\underline{M}$ are finite Λ-modules. If $\overline{s}$ is a geometric point of S, then we have $(R^qf_!\underline{M})_{\overline{s}} = H^q_c(X_{\overline{s}}, \underline{M})$ by Lemma 63.12.2. This is a finite Λ-module by Lemma 63.12.4.

Induction step. It suffices to find a dense open $U \subset S$ such that $Rf_!\underline{M}|_U$ has constructible cohomology sheaves. Namely, the restriction of $Rf_!\underline{M}$ to the complement $S \setminus U$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_!\underline{M}$ commutes with all base change (Lemma 63.9.4). In fact, let $\eta \in S$ be a generic point of an irreducible component of S. Then it suffices to find an open neighbourhood U of η such that the restriction of $Rf_!\underline{M}$ to U is constructible. This is what we will do in the next paragraph.

Given a generic point $\eta \in S$ we choose a diagram

$$\begin{array}{ccccccccc}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n & \xleftarrow[j]{} & Y_1 \amalg \ldots \amalg Y_n & \xrightarrow[\nu]{} & X_V & \longrightarrow & X_U & \longrightarrow & X \\
 & \searrow & \downarrow & & \downarrow & & \downarrow & & \downarrow f \\
 & & T_1 \amalg \ldots \amalg T_n & \longrightarrow & V & \longrightarrow & U & \longrightarrow & S
\end{array}$$

as in More on Morphisms, Lemma 37.56.1. We will show that $Rf_!\underline{M}|_U$ is constructible. First, since $V \to U$ is finite and surjective, it suffices to show that the pullback to V is constructible, see Étale Cohomology, Lemma 59.73.3. Since formation of $Rf_!$ commutes with base change, we see that it suffices to show that $R(X_V \to V)_!\underline{M}$ is constructible. Let $W \subset X_V$ be the open subscheme given to us by More on Morphisms, Lemma 37.56.1 part (4). Let $Z \subset X_V$ be the reduced induced scheme structure on the complement of W in X_V. Then the fibres of $Z \to V$ have dimension 0 (as W is dense in the fibres) and hence $Z \to V$ is quasi-finite. From the distinguished triangle

$$R(W \to V)_!\underline{M} \to R(X_V \to V)_!\underline{M} \to R(Z \to V)_!\underline{M} \to \ldots$$

of Lemma 63.10.5 and from Lemma 63.14.1 we conclude that it suffices to show that $R(W \to V)_!\underline{M}$ has constructible cohomology sheaves. Next, we have

$$R(W \to V)_!\underline{M} = R(\nu^{-1}(W) \to V)_!\underline{M}$$

because the morphism $\nu : \nu^{-1}(W) \to W$ is a thickening and we may apply Lemma 63.10.6. Next, we let $Z' \subset \coprod \overline{Y}_i$ denote the complement of the open $j(\nu^{-1}(W))$. Again $Z' \to V$ is quasi-finite. Again use the distinguished triangle

$$R(\nu^{-1}(W) \to V)_!\underline{M} \to R(\coprod \overline{Y}_i \to V)_!\underline{M} \to R(Z' \to V)_!\underline{M} \to \ldots$$

to conclude that it suffices to prove

$$R(\coprod \overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(\overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(T_i \to V)_!R(\overline{Y}_i \to T_i)_!\underline{M}$$

has constructible cohomology sheaves (second equality by Lemma 63.9.2). The result for $R(\overline{Y}_i \to T_i)_!\underline{M}$ is Lemma 63.13.3 and we win because $T_i \to V$ is finite étale and we can apply Lemma 63.14.1. □

0GKY **Lemma 63.14.3.** *Let Y be a Noetherian affine scheme of finite dimension. Let Λ be a Noetherian ring which is torsion. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on an open subscheme $U \subset \mathbf{A}^1_Y$. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : U \to Y$ is the structure morphism.*

Proof. We may decompose Λ as a product $\Lambda = \Lambda_1 \times \ldots \times \Lambda_r$ where Λ_i is ℓ_i-primary for some prime ℓ_i. Thus we may assume there exists a prime ℓ and an integer $n > 0$ such that ℓ^n annihilates Λ (and hence $\mathcal{F}$).

Since U is Noetherian, we see that U has finitely many connected components. Thus we may assume U is connected. Let $g : U' \to U$ be the finite étale covering constructed in Étale Cohomology, Lemma 59.66.4. The discussion in Étale Cohomology, Section 59.66 gives maps

$$\mathcal{F} \to g_*g^{-1}\mathcal{F} \to \mathcal{F}$$

whose composition is an isomorphism. Hence it suffices to prove the result for $g_*g^{-1}\mathcal{F}$. On the other hand, we have $Rf_!g_*g^{-1}\mathcal{F} = R(f \circ g)_!g^{-1}\mathcal{F}$ by Lemma 63.9.2. Since $g^{-1}\mathcal{F}$ has a finite filtration by constant sheaves of Λ-modules of the form $\underline{M}$ for some finite Λ-module M (by our choice of g) this reduces us to the case proved in Lemma 63.14.2. □

0GKZ **Lemma 63.14.4.** *Let Y be an affine scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $\mathbf{A}^1_Y$ which is torsion. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : \mathbf{A}^1_Y \to Y$ is the structure morphism.*

Proof. Say $\mathcal{F}$ is annihilated by $n > 0$. Then we can replace Λ by $\Lambda/n\Lambda$ without changing $Rf_!\mathcal{F}$. Thus we may and do assume Λ is a torsion ring.

Say $Y = \mathrm{Spec}(R)$. Then, if we write $R = \bigcup R_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i such that $\mathcal{F}$ is the pullback of a constructible sheaf of Λ-modules on $\mathbf{A}^1_{R_i}$, see Étale Cohomology, Lemma 59.73.10. Hence we may assume Y is a Noetherian scheme of finite dimension.

Assume Y is a Noetherian scheme of finite dimension $d = \dim(Y)$ and Λ is torsion. We will prove the result by induction on d.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^q f_! \mathcal{F}$ are finite Λ-modules. If $\overline{y}$ is a geometric point of Y, then we have $(R^q f_! \mathcal{F})_{\overline{y}} = H^q_c(X_{\overline{y}}, \mathcal{F})$ by Lemma 63.12.2. This is a finite Λ-module by Lemma 63.12.4.

Induction step. It suffices to find a dense open $V \subset Y$ such that $Rf_! \mathcal{F}|_V$ has constructible cohomology sheaves. Namely, the restriction of $Rf_! \mathcal{F}$ to the complement $Y \setminus V$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_! \mathcal{F}$ commutes with all base change (Lemma 63.9.4). By definition of constructible sheaves of Λ-modules, there is a dense open subscheme $U \subset \mathbf{A}^1_Y$ such that $\mathcal{F}|_U$ is a finite type, locally constant sheaf of Λ-modules. Denote $Z \subset \mathbf{A}^1_Y$ the complement (viewed as a reduced closed subscheme). Note that U contains all the generic points of the fibres of $\mathbf{A}^1_Y \to Y$ over the generic points $\xi_1, \ldots, \xi_n$ of the irreducible components of Y. Hence $Z \to Y$ has finite fibres over $\xi_1, \ldots, \xi_n$. After replacing Y by a dense open (which is allowed), we may assume $Z \to Y$ is finite, see Morphisms, Lemma 29.51.1. By the distinguished triangle of Lemma 63.10.5 and the result for $Z \to Y$ (Lemma 63.14.1) we reduce to showing that $R(U \to Y)_! \mathcal{F}$ has constructible cohomology sheaves. This is Lemma 63.14.3. □

0GL0 **Theorem 63.14.5.** *Let $f : X \to Y$ be a separated morphism of finite presentation of quasi-compact and quasi-separated schemes. Let Λ be a Noetherian ring. Let K be an object of $D^+_{tors,c}(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $Rf_! K$ has constructible cohomology sheaves, i.e., $Rf_! K$ is in $D^+_{tors,c}(Y_{étale}, \Lambda)$ or in $D_c(Y_{étale}, \Lambda)$ in case Λ is torsion.*

Proof. The question is local on Y hence we may and do assume Y is affine. By the induction principle and Lemma 63.10.4 we reduce to the case where X is also affine.

Assume X and Y are affine. Since X is of finite presentation, we can choose a closed immersion $i : X \to \mathbf{A}^n_Y$ which is of finite presentation. If $p : \mathbf{A}^n_Y \to Y$ denotes the structure morphism, then we see that $Rf_! = Rp_! \circ Ri_!$ by Lemma 63.9.2. By Lemma 63.14.1 we have the result for $Ri_! = i_!$. Hence we may assume f is the projection morphism $\mathbf{A}^n_Y \to Y$. Since we can view f as the composition

$$X = \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \mathbf{A}^{n-2}_S \to \ldots \to \mathbf{A}^1_Y \to Y$$

we may assume $n = 1$.

Assume Y is affine and $X = \mathbf{A}^1_Y$. Since $Rf_!$ has finite cohomological dimension (Lemma 63.10.2) we may assume K is bounded below. Using the first spectral sequence of Derived Categories, Lemma 13.21.3 (or alternatively using an argument with truncations), we reduce to showing the result of Lemma 63.14.4. □

63.15. Applications

0GLG In this section we give some applications of Theorem 63.14.5.

0GLH **Lemma 63.15.1.** *Let k be an algebraically closed field. Let X be a finite type separated scheme over k. Let Λ be a Noetherian ring. Let K be an object of $D^+_{tors,c}(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $H^i_c(X, K)$ is a finite Λ-module for all $i \in \mathbf{Z}$.*

Proof. Immediate consequence of Theorem 63.14.5 and the definition of compactly supported cohomology in Section 63.12. □

0GLI **Proposition 63.15.2.** *Let $f : X \to S$ be a smooth proper morphism of schemes. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $X_{étale}$ such that for every geometric point $\overline{x}$ of X the stalk $\mathcal{F}_{\overline{x}}$ is annihilated by an integer $n > 0$ prime to the residue characteristic of $\overline{x}$. Then $R^i f_* \mathcal{F}$ is a finite type, locally constant sheaf of Λ-modules on $S_{étale}$ for all $i \in \mathbf{Z}$.*

Proof. The question is local on S and hence we may assume S is affine. For a point x of X denote $n_x \geq 1$ the smallest integer annihilating $\mathcal{F}_{\overline{x}}$ for some (equivalently any) geometric point $\overline{x}$ of X lying over x. Since X is quasi-compact (being proper over affine) there exists a finite étale covering $\{U_j \to X\}_{j=1,\ldots,m}$ such that $\mathcal{F}|_{U_j}$ is constant. Since $U_j \to X$ is open, we conclude that the function $x \mapsto n_x$ is locally constant and takes finitely many values. Accordingly we obtain a finite decomposition $X = X_1 \amalg \ldots \amalg X_N$ into open and closed subschemes such that $n_x = n$ if and only if $x \in X_n$. Then it suffices to prove the lemma for the induced morphisms $X_n \to S$ and the restriction of $\mathcal{F}$ to X_n. Thus we may and do assume there exists an integer $n > 0$ such that $\mathcal{F}$ is annihilated by n and such that n is prime to the residue characteristics of all residue fields of X.

Since f is smooth and proper the image $f(X) \subset S$ is open and closed. Hence we may replace S by $f(X)$ and assume $f(X) = S$. In particular, we see that we may assume n is invertible in the ring defining the affine scheme S.

In this paragraph we reduce to the case where S is Noetherian. Write $S = \mathrm{Spec}(A)$ for some $\mathbf{Z}[1/n]$-algebra A. Write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}[1/n]$-subalgebras. We can find an i and a morphism $f_i : X_i \to S_i = \mathrm{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth and proper, see Limits, Lemmas 32.13.1 32.8.9, and 32.18.4. By Étale Cohomology, Lemma 59.73.11 we see that there exists an i and a finite type, locally constant sheaf of Λ-modules $\mathcal{F}_i$ whose pullback to X is isomorphic to $\mathcal{F}$. As $\mathcal{F}$ is annihilated by n, we may replace $\mathcal{F}_i$ by $\mathrm{Ker}(n : \mathcal{F}_i \to \mathcal{F}_i)$ and assume the same thing is true for $\mathcal{F}_i$. This reduces us to the case discussed in the next paragraph.

Assume we have an integer $n \geq 1$, the base scheme S is Noetherian and lives over $\mathbf{Z}[1/n]$, and $\mathcal{F}$ is n-torsion. By Theorem 63.14.5 the sheaves $R^i f_* \mathcal{F}$ are constructible sheaves of Λ-modules. By Étale Cohomology, Lemma 59.94.3 the specialization maps of $R^i f_* \mathcal{F}$ are always isomorphisms. We conclude by Étale Cohomology, Lemma 59.75.6. □

63.16. More on derived upper shriek

0GLJ Let Λ be a torsion ring. Consider a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & U' \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g'} & \\ & Y & \end{array}$$

of quasi-compact and quasi-separated schemes with g and g' separated and of finite type and with j étale. This induces a canonical map

$$Rg_!\Lambda \longrightarrow Rg'_!\Lambda$$

in $D(Y_{étale}, \Lambda)$. Namely, by Lemmas 63.9.2 and 63.10.3 we have $Rg_! = Rg'_! \circ j_!$. On the other hand, since $j_!$ is left adjoint to j^{-1} we have the counit $\mathrm{Tr}_j : j_!\Lambda = j_!j^{-1}\Lambda \to \Lambda$; we also call this the trace map for j, see Remark 63.5.6. The map above is constructed as the composition

$$Rg_!\Lambda = Rg'_!j_!\Lambda \xrightarrow{Rg'_!\mathrm{Tr}_j} Rg'_!\Lambda$$

Given a second étale morphism $j' : U' \to U''$ for some $g'' : U'' \to Y$ separated and of finite type the composition

$$Rg_!\Lambda \longrightarrow Rg'_!\Lambda \longrightarrow Rg''_!\Lambda$$

of the maps for j and j' is equal to the map $Rg_!\Lambda \longrightarrow Rg''_!\Lambda$ constructed for $j' \circ j$. This follows from the corresponding statement on trace maps, see Lemma 63.5.4 for a more general case.

Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Then we obtain a functor

$$X_{affine,étale} \longrightarrow \left\{\begin{array}{c}\text{schemes separated of finite type over } Y \\ \text{with étale morphisms between them}\end{array}\right\}$$

Thus the construction above determines a functor $X^{opp}_{affine,étale} \to D(Y_{étale}, \Lambda)$ sending U to $R(U \to Y)_!\Lambda$.

0GLK **Lemma 63.16.1.** *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $K \in D(Y_{étale}, \Lambda)$. For $n \in \mathbf{Z}$ the cohomology sheaf $H^n(Rf^!K)$ restricted to $X_{affine,étale}$ is the sheaf associated to the presheaf*

$$U \longmapsto \mathrm{Hom}_Y(R(U \to Y)_!\Lambda, K[n])$$

See discussion above for the functorial nature of $R(U \to Y)_!\Lambda$.

Proof. Let $j : U \to X$ be an object of $X_{affine,étale}$ and set $g = f \circ j$. Recall that $\mathrm{Hom}_X(j_!\Lambda, M[n]) = H^n(U, M)$ for any M in $D(X_{étale}, \Lambda)$. Then $H^n(Rf^!K)$ is the sheaf associated to the presheaf

$$U \mapsto H^n(U, Rf^!K) = \mathrm{Hom}_X(j_!\Lambda, Rf^!K[n]) = \mathrm{Hom}_Y(Rf_!j_!\Lambda, K[n] = \mathrm{Hom}_Y(Rg_!\Lambda, K[n])$$

We omit the verification that the transition maps are given by the transition maps between the objects $Rg_!\Lambda = R(U \to Y)_!\Lambda$ we constructed above. □

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CHAPTER 64

The Trace Formula

0F5P

64.1. Introduction

0F5Q These are the notes of the second part of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the Stacks project and provide rigorous proofs of all the statements.

64.2. The trace formula

03SJ A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [**Del77**, Arcata]. Instead, we are going to study the trace formula for the frobenius, following the account of Deligne in [**Del77**, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript $_{étale}$. Let us now describe the formula we are after. Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

03SK (64.2.0.1)
$$\sum\nolimits_{x\in X(k)} \mathrm{Tr}(\mathrm{Frob}|\mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2}(-1)^i \mathrm{Tr}(\pi_X^*|H_c^i(X\otimes_k \bar{k},\mathcal{F}))$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

64.3. Frobenii

03SL In this section we will prove a "baffling" theorem. A topological analogue of the baffling theorem is the following.

03SO **Exercise 64.3.1.** Let X be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens U of X. Then g induces the identity on cohomology on X (for any coefficients).

We now turn to the statement for the étale site.

03SP **Lemma 64.3.2.** *Let X be a scheme and $g : X \to X$ a morphism. Assume that for all $\varphi : U \to X$ étale, there is an isomorphism*

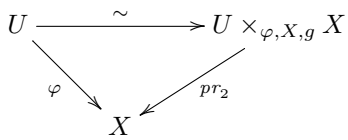

functorial in U. Then g induces the identity on cohomology (for any sheaf).

Proof. The proof is formal and without difficulty. □

Please see Varieties, Section 33.36 for a discussion of different variants of the Frobenius morphism.

03SN **Theorem 64.3.3** (The Baffling Theorem)**.** *Let X be a scheme in characteristic $p > 0$. Then the absolute frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers $j \geq 0$,*

$$F_X^* : H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \longrightarrow H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}})$$

is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with Čech cohomology, it suffices to pull back (using the fiber products on a site) the Čech cocycles. The general case is quite technical, see Hypercoverings, Theorem 25.10.1. To prove the theorem, we merely verify that the assumption of Lemma 64.3.2 holds for the frobenius.

Proof of Theorem 64.3.3. We need to verify the existence of a functorial isomorphism as above. For an étale morphism $\varphi : U \to X$, consider the diagram

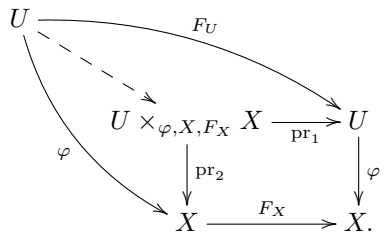

The dotted arrow is an étale morphism and a universal homeomorphism, so it is an isomorphism. See Étale Morphisms, Lemma 41.14.3. □

03SQ **Definition 64.3.4.** Let k be a finite field with $q = p^f$ elements. Let X be a scheme over k. The *geometric frobenius* of X is the morphism $\pi_X : X \to X$ over $\mathrm{Spec}(k)$ which equals F_X^f.

Since π_X is a morphism over k, we can base change it to any scheme over k. In particular we can base change it to the algebraic closure $\bar{k}$ and get a morphism $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$. Using π_X also for this base change should not be confusing as $X_{\bar{k}}$ does not have a geometric frobenius of its own.

03SR **Lemma 64.3.5.** *Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then there are canonical isomorphisms $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ and $\mathcal{F} \cong \pi_{X*}\mathcal{F}$.*

This is false for the fppf site.

Proof. Let $\varphi : U \to X$ be étale. Recall that $\pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi,X,\pi_X} X)$. Since $\pi_X = F_X^f$, it follows from the proof of Theorem 64.3.3 that there is a functorial

isomorphism

$$\begin{array}{ccc} U & \xrightarrow{\gamma_U} & U \times_{\varphi,X,\pi_X} X \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & X & \end{array}$$

where $\gamma_U = (\varphi, F_U^f)$. Now we define an isomorphism

$$\mathcal{F}(U) \longrightarrow \pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi,X,\pi_X} X)$$

by taking the restriction map of $\mathcal{F}$ along γ_U^{-1}. The other isomorphism is analogous. □

03SS **Remark 64.3.6.** It may or may not be the case that F_U^f equals π_U.

We continue discussion cohomology of sheaves on our scheme X over the finite field k with $q = p^f$ elements. Fix an algebraic closure $\bar{k}$ of k and write $G_k = \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of k. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We will define a left G_k-module structure cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as follows: if $\sigma \in G_k$, the diagram

$$\begin{array}{ccc} X_{\bar{k}} & \xrightarrow{\mathrm{Spec}(\sigma)\times \mathrm{id}_X} & X_{\bar{k}} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes. Thus we can set, for $\xi \in H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$

$$\sigma \cdot \xi := (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^* \xi \in H^j(X_{\bar{k}}, (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^{-1}\mathcal{F}|_{X_{\bar{k}}}) = H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}),$$

where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a G_k-module.

03ST **Lemma 64.3.7.** *In the situation above denote $\alpha : X \to \mathrm{Spec}(k)$ the structure morphism. Consider the stalk $(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ endowed with its natural Galois action as in Étale Cohomology, Section 59.56. Then the identification*

$$(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})} \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$$

from Étale Cohomology, Theorem 59.53.1 is an isomorphism of G_k-modules.

A similar result holds comparing $(R^j\alpha_!\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ with $H_c^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$.

Proof. Omitted. □

03SU **Definition 64.3.8.** The *arithmetic frobenius* is the map $\mathrm{frob}_k : \bar{k} \to \bar{k}$, $x \mapsto x^q$ of G_k.

03SV **Theorem 64.3.9.** *Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then for all $j \geq 0$, $frob_k$ acts on the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as the inverse of the map π_X^*.*

The map π_X^* is defined by the composition

$$H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) \xrightarrow{\pi_{X_{\bar{k}}}^*} H^j(X_{\bar{k}}, (\pi_X^{-1}\mathcal{F})|_{X_{\bar{k}}}) \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}).$$

where the last isomorphism comes from the canonical isomorphism $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ of Lemma 64.3.5.

Proof. The composition $X_{\bar{k}} \xrightarrow{\mathrm{Spec}(\mathrm{frob}_k)} X_{\bar{k}} \xrightarrow{\pi_X} X_{\bar{k}}$ is equal to $F^f_{X_{\bar{k}}}$, hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that $F^f_{X_{\bar{k}}} = \pi_X \circ \mathrm{Spec}(\mathrm{frob}_k) = \mathrm{Spec}(\mathrm{frob}_k) \circ \pi_X$. □

03SW **Definition 64.3.10.** If $x \in X(k)$ is a rational point and $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ the geometric point lying over x, we let $\pi_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ denote the action by frob_k^{-1} and call it the *geometric frobenius*[1]

We can now make a more precise statement (albeit a false one) of the trace formula (64.2.0.1). Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

03SX (64.3.10.1) $$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\pi_x | \mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2} (-1)^i \mathrm{Tr}(\pi_X^* | H^i_c(X_{\bar{k}}, \mathcal{F}))$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where $X = \mathbf{P}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. For any point, the Galois module $\mathcal{F}_{\bar{x}}$ is trivial, hence for any morphism φ acting on $\mathcal{F}_{\bar{x}}$, the left-hand side is

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\varphi | \mathcal{F}_{\bar{x}}) = \#\mathbf{P}^1_k(k) = q + 1.$$

Now $\mathbf{P}^1_k$ is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism $\pi : \mathbf{P}^1_k \to \mathbf{P}^1_k$, the right-hand side equals

$$\mathrm{Tr}(\pi^* | H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) + \mathrm{Tr}(\pi^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})).$$

The Galois module $H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}) = \mathbf{Z}/\ell\mathbf{Z}$ is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of π. For the second trace, we need to compute the pullback $\pi^* : H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$ for a map $\pi : \mathbf{P}^1_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$. This is a good exercise and the answer is multiplication by the degree of π (for a proof see Étale Cohomology, Lemma 59.69.2). In other words, this works as in the familiar situation of complex cohomology. In particular, if π is the geometric frobenius we get

$$\mathrm{Tr}(\pi_X^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q$$

and if π is the arithmetic frobenius then we get

$$\mathrm{Tr}(\mathrm{frob}_k^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q^{-1}.$$

The latter option is clearly wrong.

03SY **Remark 64.3.11.** The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

[1] This notation is not standard. This operator is denoted F_x in [**Del77**]. We will likely change this notation in the future.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of q. For example, let us consider the case where $X = \mathbf{A}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. The action on stalks is again trivial, so we only need look at the action on cohomology. But then π_X^* acts as the identity on $H^0(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$ and as multiplication by q on $H^2_c(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$.

64.4. Traces

03SZ We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring Λ with cardinality prime to p. Typically, Λ is the group ring $(\mathbf{Z}/\ell^n\mathbf{Z})[G]$ for some finite group G. By convention, all the Λ-modules considered will be left Λ-modules.

We introduce the following notation: We set $\Lambda^\natural$ to be the quotient of Λ by its additive subgroup generated by the commutators (i.e., the elements of the form $ab - ba$, $a, b \in \Lambda$). Note that $\Lambda^\natural$ is not a ring.

For instance, the module $(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural$ is the dual of the class functions, so

$$(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural = \bigoplus\nolimits_{\text{conjugacy classes of } G} \mathbf{Z}/\ell^n\mathbf{Z}.$$

For a free Λ-module, we have $\mathrm{End}_\Lambda(\Lambda^{\oplus m}) = \mathrm{Mat}_m(\Lambda)$. Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if $a \in \mathrm{End}(\Lambda^{\oplus m})$ has matrix A and $v \in \Lambda$, then $a(v) = vA$.

03T0 **Definition 64.4.1.** The *trace* of the endomorphism a is the sum of the diagonal entries of a matrix representing it. This defines an additive map $\mathrm{Tr} : \mathrm{End}_\Lambda(\Lambda^{\oplus m}) \to \Lambda^\natural$.

03T1 **Exercise 64.4.2.** Given maps

$$\Lambda^{\oplus m} \xrightarrow{a} \Lambda^{\oplus n} \quad \text{and} \quad \Lambda^{\oplus n} \xrightarrow{b} \Lambda^{\oplus m}$$

show that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

We extend the definition of the trace to a finite projective Λ-module P and an endomorphism φ of P as follows. Write P as the summand of a free Λ-module, i.e., consider maps $P \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} P$ with

(1) $\Lambda^{\oplus n} = \mathrm{Im}(a) \oplus \mathrm{Ker}(b)$; and
(2) $b \circ a = \mathrm{id}_P$.

Then we set $\mathrm{Tr}(\varphi) = \mathrm{Tr}(a\varphi b)$. It is easy to check that this is well-defined, using the previous exercise.

64.5. Why derived categories?

03T2 With this definition of the trace, let us now discuss another issue with the formula as stated. Let C be a smooth projective curve over k. Then there is a correspondence between finite locally constant sheaves $\mathcal{F}$ on $C_{étale}$ whose stalks are isomorphic to $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \mathrm{GL}_m(\mathbf{Z}/\ell^n\mathbf{Z}))$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to ρ. Then $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$, and there is a short exact sequence

$$0 \longrightarrow \pi_1(C_{\bar{k}}, \bar{c}) \longrightarrow \pi_1(C, \bar{c}) \longrightarrow G_k \longrightarrow 0.$$

For instance, let $\mathbf{Z} = \mathbf{Z}\sigma$ act on $\mathbf{Z}/\ell^2\mathbf{Z}$ via $\sigma(x) = (1+\ell)x$. The coinvariants are $(\mathbf{Z}/\ell^2\mathbf{Z})_\sigma = \mathbf{Z}/\ell\mathbf{Z}$, which is not a flat $\mathbf{Z}/\ell^2\mathbf{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for ℓ-adic coefficients. But $\mathbf{Q}_\ell = \mathbf{Z}_\ell[1/\ell]$ and $\mathbf{Z}_\ell = \lim \mathbf{Z}/\ell^n\mathbf{Z}$, and even for a flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $R\Gamma(C_{\bar{k}}, \mathcal{F}_\rho)$ in the derived category $D(\mathbf{Z}/\ell^n\mathbf{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

64.6. Derived categories

03T3 To set up notation, let $\mathcal{A}$ be an abelian category. Let $\mathrm{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and morphisms equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(A)$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\mathrm{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\mathrm{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\mathrm{Comp}^-(\mathcal{A}), K^-(\mathcal{A}), D^-(\mathcal{A})$ using bounded above complexes, and we can define $\mathrm{Comp}^b(\mathcal{A}), K^b(\mathcal{A}), D^b(\mathcal{A})$ using bounded complexes.

03T4 **Remark 64.6.1.** Notes on derived categories.

(1) There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.
(2) The categories $K(A)$ and $D(A)$ are endowed with the structure of a triangulated category.
(3) The categories $\mathrm{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

03T5 **Lemma 64.6.2.** *An object E of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i \ll 0$. Similar statements hold for D^- and D^b.*

Proof. Hint: use truncation functors. See Derived Categories, Lemma 13.11.5. □

03T6 **Lemma 64.6.3.** *Morphisms between objects in the derived category.*

(1) *Let $I^\bullet \in Comp^+(\mathcal{A})$ with I^n injective for all $n \in \mathbf{Z}$. Then*
$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet).$$
(2) *Let $P^\bullet \in Comp^-(\mathcal{A})$ with P^n is projective for all $n \in \mathbf{Z}$. Then*
$$\mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet).$$
(3) *If $\mathcal{A}$ has enough injectives and $\mathcal{I} \subset \mathcal{A}$ is the additive subcategory of injectives, then $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$ (as triangulated categories).*
(4) *If $\mathcal{A}$ has enough projectives and $\mathcal{P} \subset \mathcal{A}$ is the additive subcategory of projectives, then $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.*

Proof. Omitted. □

03T7 **Definition 64.6.4.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. We define the *total right derived functor of* F as the functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. We define the *total left derived functor of* G as the functor $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^-(\mathcal{A}) & \xrightarrow{LG} & D^-(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^-(\mathcal{P}) & \xrightarrow{G} & K^-(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma.

03T8 **Remark 64.6.5.** In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be ith homology of the complex $F(K^\bullet)$.

64.7. Filtered derived category

03T9 It turns out we have to do it all again and build the filtered derived category also.

03TA **Definition 64.7.1.** Let $\mathcal{A}$ be an abelian category.

(1) Let $\mathrm{Fil}(\mathcal{A})$ be the category of filtered objects (A, F) of $\mathcal{A}$, where F is a filtration of the form
$$A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0.$$
This is an additive category.
(2) We denote $\mathrm{Fil}^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects (A, F) have finite filtration. This is also an additive category.
(3) An object $I \in \mathrm{Fil}^f(\mathcal{A})$ is called *filtered injective* (respectively *projective*) provided that $\mathrm{gr}^p(I) = \mathrm{gr}^p_F(I) = F^pI/F^{p+1}I$ is injective (resp. projective) in $\mathcal{A}$ for all p.
(4) The category of complexes $\mathrm{Comp}(\mathrm{Fil}^f(\mathcal{A})) \supset \mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$ and its homotopy category $K(\mathrm{Fil}^f(\mathcal{A})) \supset K^+(\mathrm{Fil}^f(\mathcal{A}))$ are defined as before.
(5) A morphism $\alpha : K^\bullet \to L^\bullet$ of complexes in $\mathrm{Comp}(\mathrm{Fil}^f(\mathcal{A}))$ is called a *filtered quasi-isomorphism* provided that
$$\mathrm{gr}^p(\alpha) : \mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$$
is a quasi-isomorphism for all $p \in \mathbf{Z}$.
(6) We define $DF(\mathcal{A})$ (resp. $DF^+(\mathcal{A})$) by inverting the filtered quasi-isomorphisms in $K(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $K^+(\mathrm{Fil}^f(\mathcal{A}))$).

03TB **Lemma 64.7.2.** *If $\mathcal{A}$ has enough injectives, then $DF^+(\mathcal{A}) \cong K^+(\mathcal{I})$, where $\mathcal{I}$ is the full additive subcategory of $Fil^f(\mathcal{A})$ consisting of filtered injective objects.*

Similarly, if $\mathcal{A}$ has enough projectives, then $DF^-(\mathcal{A}) \cong K^-(\mathcal{P})$, where $\mathcal{P}$ is the full additive subcategory of $Fil^f(\mathcal{A})$ consisting of filtered projective objects.

Proof. Omitted. □

64.8. Filtered derived functors

03TC And then there are the filtered derived functors.

03TD **Definition 64.8.1.** Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. Define $RT : DF^+(\mathcal{A}) \to DF^+(\mathcal{B})$ to fit in the diagram

$$\begin{array}{ccc} DF^+(\mathcal{A}) & \xrightarrow{RT} & DF^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^+(\mathcal{I}) & \xrightarrow{T} & K^+(\mathrm{Fil}^f(\mathcal{B})). \end{array}$$

This is well-defined by the previous lemma. Let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. Define $LG : DF^-(\mathcal{A}) \to DF^-(\mathcal{B})$ to fit in the diagram

$$\begin{array}{ccc} DF^-(\mathcal{A}) & \xrightarrow{LG} & DF^-(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^-(\mathcal{P}) & \xrightarrow{G} & K^-(\mathrm{Fil}^f(\mathcal{B})). \end{array}$$

Again, this is well-defined by the previous lemma. The functors RT, resp. LG, are called the *filtered derived functor* of T, resp. G.

03TE **Proposition 64.8.2.** *In the situation above, we have*

$$gr^p \circ RT = RT \circ gr^p$$

where the RT on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

$$\begin{array}{ccc} DF^+(\mathcal{A}) & \xrightarrow{RT} & DF^+(\mathcal{B}) \\ {\scriptstyle gr^p}\downarrow & & \downarrow{\scriptstyle gr^p} \\ D^+(\mathcal{A}) & \xrightarrow{RT} & D^+(\mathcal{B}). \end{array}$$

Proof. Omitted. □

Given $K^\bullet \in DF^+(\mathcal{B})$, we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p K^\bullet) \Rightarrow H^{p+q}(\text{forget filt}(K^\bullet)).$$

64.9. Application of filtered complexes

03TF Let $\mathcal{A}$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $\mathcal{A}$. Consider $\widetilde{M} \in \mathrm{Fil}^f(\mathcal{A})$ to be M along with the filtration defined by

$$F^1M = L,\ F^nM = M \text{ for } n \leq 0, \text{ and } F^nM = 0 \text{ for } n \geq 2.$$

By definition, we have

$$\text{forget filt}(\widetilde{M}) = M, \quad \text{gr}^0(\widetilde{M}) = N, \quad \text{gr}^1(\widetilde{M}) = L$$

and $\text{gr}^n(\widetilde{M}) = 0$ for all other $n \neq 0, 1$. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then $RT(\widetilde{M}) \in DF^+(\mathcal{B})$ is a filtered complex with

$$\text{gr}^p(RT(\widetilde{M})) \overset{\text{qis}}{=} \begin{cases} 0 & \text{if} \quad p \neq 0, 1, \\ RT(N) & \text{if} \quad p = 0, \\ RT(L) & \text{if} \quad p = 1. \end{cases}$$

and $\text{forget filt}(RT(\widetilde{M})) \overset{\text{qis}}{=} RT(M)$. The spectral sequence applied to $RT(\widetilde{M})$ gives

$$E_1^{p,q} = R^{p+q}T(\text{gr}^p(\widetilde{M})) \Rightarrow R^{p+q}T(\text{forget filt}(\widetilde{M})).$$

Unwinding the spectral sequence gives us the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(L) & \longrightarrow & T(M) & \longrightarrow & T(N) \\ & & & & & & \searrow \\ & & R^1T(L) & \longrightarrow & R^1T(M) & \longrightarrow & \ldots \end{array}$$

This will be used as follows. Let X/k be a scheme of finite type. Let $\mathcal{F}$ be a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module. Then we want to show that the trace

$$\text{Tr}(\pi_X^* | R\Gamma_c(X_{\bar{k}}, \mathcal{F})) \in \mathbf{Z}/\ell^n\mathbf{Z}$$

is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\bar{k}}, -) \in D^+(\mathbf{Z}/\ell^n\mathbf{Z})$, but we will have to use the filtered derived category.

64.10. Perfectness

03TG Let Λ be a (possibly noncommutative) ring, Mod_Λ the category of left Λ-modules, $K(\Lambda) = K(\text{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\text{Mod}_\Lambda)$ the derived category.

03TH **Definition 64.10.1.** We denote by $K_{perf}(\Lambda)$ the category whose objects are bounded complexes of finite projective Λ-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{perf}(\Lambda) \to D(\Lambda)$ is fully faithful (Derived Categories, Lemma 13.19.8). Denote $D_{perf}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called *perfect* if it is in $D_{perf}(\Lambda)$.

03TI **Proposition 64.10.2.** *Let $K \in D_{perf}(\Lambda)$ and $f \in End_{D(\Lambda)}(K)$. Then the trace $Tr(f) \in \Lambda^\natural$ is well defined.*

Proof. We will use Derived Categories, Lemma 13.19.8 without further mention in this proof. Let $P^\bullet$ be a bounded complex of finite projective Λ-modules and let $\alpha : P^\bullet \to K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ corresponds to a morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ well defined up to homotopy. Set

$$\text{Tr}(f) = \sum_i (-1)^i \text{Tr}(f^i : P^i \to P^i) \in \Lambda^\natural.$$

Given $P^\bullet$ and α, this is independent of the choice of $f^\bullet$. Namely, any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1} (i \in \mathbf{Z})$. But

$$\begin{aligned} \mathrm{Tr}(dh) &= \sum_i (-1)^i \mathrm{Tr}(P^i \xrightarrow{dh} P^i) \\ &= \sum_i (-1)^i \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\sum_i (-1)^{i-1} \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\mathrm{Tr}(hd) \end{aligned}$$

and so $\sum_i (-1)^i \mathrm{Tr}((dh+hd)|_{P^i}) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. The compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet \quad \text{and} \quad P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\begin{aligned} \mathrm{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) &= \mathrm{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet}) \\ &= \mathrm{Tr}(f^\bullet|_{P^\bullet}) \end{aligned}$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence of the choice of $f^\bullet$ we saw above. □

64.11. Filtrations and perfect complexes

03TJ We now present a filtered version of the category of perfect complexes. An object (M, F) of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ is called *filtered finite projective* if for all p, $\mathrm{gr}_F^p(M)$ is finite and projective. We then consider the homotopy category $KF_{\mathrm{perf}}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$. We have a diagram of categories

$$\begin{array}{ccc} KF(\Lambda) & \supset & KF_{\mathrm{perf}}(\Lambda) \\ \downarrow & & \downarrow \\ DF(\Lambda) & \supset & DF_{\mathrm{perf}}(\Lambda) \end{array}$$

where the vertical functor on the right is fully faithful and the category $DF_{\mathrm{perf}}(\Lambda)$ is its essential image, as before.

03TK **Lemma 64.11.1** (Additivity)**.** *Let $K \in DF_{perf}(\Lambda)$ and $f \in End_{DF}(K)$. Then*

$$Tr(f|_K) = \sum\nolimits_{p \in \mathbf{Z}} Tr(f|_{gr^p K}).$$

Proof. By Proposition 64.10.2, we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\mathrm{Comp}(\mathrm{Fil}^f(\mathrm{Mod}_\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. □

03TL **Lemma 64.11.2.** *Let $P \in Fil^f(Mod_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $Fil^f(Mod_\Lambda)$. Then*

$$Tr(f|_P) = \sum\nolimits_p Tr(f|_{gr^p(P)}).$$

Proof. Omitted. □

64.12. Characterizing perfect objects

03TM For the commutative case see More on Algebra, Sections 15.65, 15.67, and 15.75.

03TN **Definition 64.12.1.** Let Λ be a (possibly noncommutative) ring. An object $K \in D(\Lambda)$ has *finite Tor-dimension* if there exist $a, b \in \mathbf{Z}$ such that for any right Λ-module N, we have $H^i(N \otimes^{\mathbf{L}}_{\Lambda} K) = 0$ for all $i \notin [a, b]$.

This in particular means that $K \in D^b(\Lambda)$ as we see by taking $N = \Lambda$.

03TO **Lemma 64.12.2.** *Let Λ be a left Noetherian ring and $K \in D(\Lambda)$. Then K is perfect if and only if the two following conditions hold:*

(1) *K has finite Tor-dimension, and*
(2) *for all $i \in \mathbf{Z}$, $H^i(K)$ is a finite Λ-module.*

Proof. See More on Algebra, Lemma 15.75.2 for the proof in the commutative case. □

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

03TP **Remark 64.12.3.** A variant of this lemma is to consider a Noetherian scheme X and the category $D_{perf}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules. Objects K of $D_{perf}(\mathcal{O}_X)$ can be characterized by having coherent cohomology sheaves and bounded tor dimension.

64.13. Cohomology of nice complexes

0964 The following is a special case of a more general result about compactly supported cohomology of objects of $D_{ctf}(X, \Lambda)$.

03TV **Proposition 64.13.1.** *Let X be a projective curve over a field k, Λ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then $R\Gamma(X_{\bar{k}}, K) \in D_{perf}(\Lambda)$.*

Sketch of proof. The first step is to show:

(1) *The cohomology of $R\Gamma(X_{\bar{k}}, K)$ is bounded.*

Consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, K)).$$

Since K is bounded and Λ is finite, the sheaves $\underline{H}^j(K)$ are torsion. Moreover, $X_{\bar{k}}$ has finite cohomological dimension, so the left-hand side is nonzero for finitely many i and j only. Therefore, so is the right-hand side.

(2) *The cohomology groups $H^{i+j}(R\Gamma(X_{\bar{k}}, K))$ are finite.*

Since the sheaves $\underline{H}^j(K)$ are constructible, the groups $H^i(X_{\bar{k}}, \underline{H}^j(K))$ are finite (Étale Cohomology, Section 59.83) so it follows by the spectral sequence again.

(3) *$R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension.*

Let N be a right Λ-module (in fact, since Λ is finite, it suffices to assume that N is finite). By the projection formula (change of module),

$$N \otimes^{\mathbf{L}}_{\Lambda} R\Gamma(X_{\bar{k}}, K) = R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K).$$

Therefore,

$$H^i(N \otimes^{\mathbf{L}}_{\Lambda} R\Gamma(X_{\bar{k}}, K)) = H^i(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K)).$$

Now consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(N \otimes^{\mathbf{L}}_{\Lambda} K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K)).$$

Since K has finite Tor-dimension, $\underline{H}^j(N \otimes^{\mathbf{L}}_{\Lambda} K)$ vanishes universally for j small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 64.12.2. □

64.14. Lefschetz numbers

03TW The fact that the total cohomology of a constructible complex of finite tor dimension is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

03TX **Definition 64.14.1.** Let Λ be a finite ring, X a projective curve over a finite field k and $K \in D_{ctf}(X, \Lambda)$ (for instance $K = \underline{\Lambda}$). There is a canonical map $c_K : \pi_X^{-1} K \to K$, and its base change $c_K|_{X_{\bar{k}}}$ induces an action denoted π_X^* on the perfect complex $R\Gamma(X_{\bar{k}}, K|_{X_{\bar{k}}})$. The *global Lefschetz number* of K is the trace $\mathrm{Tr}(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)})$ of that action. It is an element of $\Lambda^\natural$.

03TY **Definition 64.14.2.** With Λ, X, k, K as in Definition 64.14.1. Since $K \in D_{ctf}(X, \Lambda)$, for any geometric point $\bar{x}$ of X, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{perf}(\Lambda)$). As we have seen in Section 64.3, the Frobenius π_X acts on $K_{\bar{x}}$. The *local Lefschetz number* of K is the sum

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\pi_x|_{K_{\bar{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

03TZ **Theorem 64.14.3** (Lefschetz Trace Formula)**.** *Let X be a projective curve over a finite field k, Λ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then the global and local Lefschetz numbers of K are equal, i.e.,*

03U0 (64.14.3.1) $$Tr(\pi_X^*|_{R\Gamma(X_{\bar{k}}, K)}) = \sum\nolimits_{x \in X(k)} Tr(\pi_X|_{K_{\bar{x}}})$$

in $\Lambda^\natural$.

Proof. See discussion below. □

We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [**Del77**] (some of the things that are not adequately explained are listed in Section 64.21).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

03U1 **Theorem 64.14.4** (Weil)**.** *Let C be a nonsingular projective curve over an algebraically closed field k, and $\varphi : C \to C$ a k-endomorphism of C distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_\varphi$, where Δ_C is the diagonal, Γ_φ is the graph of φ, and the intersection number is taken on $C \times C$. Let $J = \underline{\mathrm{Pic}}^0_{C/k}$ be the jacobian of C and denote $\varphi^* : J \to J$ the action induced by φ by taking pullbacks. Then*

$$V(\varphi) = 1 - Tr_J(\varphi^*) + \deg \varphi.$$

Proof. The number $V(\varphi)$ is the number of fixed points of φ, it is equal to

$$V(\varphi) = \sum\nolimits_{c \in |C| : \varphi(c) = c} m_{\mathrm{Fix}(\varphi)}(c)$$

where $m_{\mathrm{Fix}(\varphi)}(c)$ is the multiplicity of c as a fixed point of φ, namely the order or vanishing of the image of a local uniformizer under $\varphi - \mathrm{id}_C$. Proofs of this theorem can be found in [**Lan02**] and [**Wei48**]. □

03U2 **Example 64.14.5.** Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by n. Then $\varphi^* = \varphi^t$ is multiplication by n on the jacobian, so it has trace $2n$ and degree n^2. On the other hand, the fixed points of φ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

Jacobians. We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil's proof. Let C be a nonsingular projective curve over an algebraically closed field k and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\mathrm{Pic}(C \times C)$, or $\mathrm{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \mathrm{pr}_1^*(A^1(C)) \oplus \mathrm{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim_{\mathrm{rat}} 0 \text{ and } Z|_{\{c_0\} \times C} \sim_{\mathrm{rat}} 0\}.$$

In other words, R is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \mathrm{End}(J)$ which maps a divisor Z in R to the endomorphism

$$\begin{array}{ccc} J & \to & J \\ [\mathcal{O}_C(D)] & \mapsto & (\mathrm{pr}_1|_Z)_*(\mathrm{pr}_2|_Z)^*(D). \end{array}$$

The aforementioned correspondence is the following. We denote by σ the automorphism of $C \times C$ that switches the factors.

$\mathrm{End}(J)$	R
composition of α, β	$\mathrm{pr}_{13*}(\mathrm{pr}_{12}{}^*(\alpha) \circ \mathrm{pr}_{23}{}^*(\beta))$
id_J	$\Delta_C - \{c_0\} \times C - C \times \{c_0\}$
φ^*	$\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C$
the trace form $\alpha, \beta \mapsto \mathrm{Tr}(\alpha\beta)$	$\alpha, \beta \mapsto -\int_{C\times C} \alpha.\sigma^*\beta$
the Rosati involution $\alpha \mapsto \alpha^\dagger$	$\alpha \mapsto \sigma^*\alpha$
positivity of Rosati $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$	Hodge index theorem on $C \times C$ $-\int_{C\times C} \alpha\sigma^*\alpha > 0.$

In fact, in light of the Kunneth formula, the subgroup R corresponds to the $1,1$ hodge classes in $H^1(C) \otimes H^1(C)$.

Weil's proof. Using this correspondence, we can prove the trace formula. We have

$$\begin{aligned} V(\varphi) &= \int_{C\times C} \Gamma_\varphi.\Delta \\ &= \int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) + \int_{C\times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right). \end{aligned}$$

Now, on the one hand

$$\int_{C\times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right) = 1 + \deg\varphi$$

and on the other hand, since R is the orthogonal of the ample divisor $\{c_0\} \times C + C \times \{c_0\}$,

$$\begin{aligned} &\int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = &\int_{C\times C} \left(\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C\right).\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = &-\mathrm{Tr}_J(\varphi^* \circ \mathrm{id}_J). \end{aligned}$$

Recapitulating, we have

$$V(\varphi) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi$$

which is the trace formula.

03U3 **Lemma 64.14.6.** *Consider the situation of Theorem 64.14.4 and let ℓ be a prime number invertible in k. Then*

$$\sum\nolimits_{i=0}^{2}(-1)^i Tr(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = V(\varphi) \mod \ell^n.$$

Sketch of proof. Observe first that the assumption makes sense because $H^i(C, \underline{\mathbf{Z}/\ell^n\mathbf{Z}})$ is a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module for all i. The trace of φ^* on the 0th degree cohomology is 1. The choice of a primitive ℓ^nth root of unity in k gives an isomorphism

$$H^i(C, \underline{\mathbf{Z}/\ell^n\mathbf{Z}}) \cong H^i(C, \mu_{\ell^n})$$

compatibly with the action of the geometric Frobenius. On the other hand, $H^1(C, \mu_{\ell^n}) = J[\ell^n]$. Therefore,

$$\begin{aligned} \mathrm{Tr}(\varphi^*|_{H^1(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) &= \mathrm{Tr}_J(\varphi^*) \mod \ell^n \\ &= \mathrm{Tr}_{\mathbf{Z}/\ell^n\mathbf{Z}}(\varphi^* : J[\ell^n] \to J[\ell^n]). \end{aligned}$$

Moreover, $H^2(C, \mu_{\ell^n}) = \mathrm{Pic}(C)/\ell^n \mathrm{Pic}(C) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ where φ^* is multiplication by $\deg\varphi$. Hence

$$\mathrm{Tr}(\varphi^*|_{H^2(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = \deg\varphi.$$

Thus we have

$$\sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi \mod \ell^n$$

and the corollary follows from Theorem 64.14.4. □

An alternative way to prove this corollary is to show that

$$X \mapsto H^*(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell \otimes \lim_n H^*(X, \mathbf{Z}/\ell^n\mathbf{Z})$$

defines a Weil cohomology theory on smooth projective varieties over k. Then the trace formula

$$V(\varphi) = \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\varphi^*|_{H^i(C,\mathbf{Q}_\ell)})$$

is a formal consequence of the axioms (it's an exercise in linear algebra, the proof is the same as in the topological case).

64.15. Preliminaries and sorites

03U4 Notation: We fix the notation for this section. We denote by A a commutative ring, Λ a (possibly noncommutative) ring with a ring map $A \to \Lambda$ which image lies in the center of Λ. We let G be a finite group, Γ a *monoid extension of G by* $\mathbf{N}$, meaning that there is an exact sequence

$$1 \to G \to \tilde{\Gamma} \to \mathbf{Z} \to 1$$

and Γ consists of those elements of $\tilde{\Gamma}$ which image is nonnegative. Finally, we let P be an $A[\Gamma]$-module which is finite and projective as an $A[G]$-module, and M a $\Lambda[\Gamma]$-module which is finite and projective as a Λ-module.

Our goal is to compute the trace of $1 \in \mathbf{N}$ acting over Λ on the coinvariants of G on $P \otimes_A M$, that is, the number

$$\mathrm{Tr}_\Lambda\left(1; (P \otimes_A M)_G\right) \in \Lambda^\natural.$$

The element $1 \in \mathbf{N}$ will correspond to the Frobenius.

03U5 **Lemma 64.15.1.** *Let $e \in G$ denote the neutral element. The map*

$$\begin{array}{ccc} \Lambda[G] & \longrightarrow & \Lambda^\natural \\ \sum \lambda_g \cdot g & \longmapsto & \lambda_e \end{array}$$

factors through $\Lambda[G]^\natural$. We denote $\varepsilon : \Lambda[G]^\natural \to \Lambda^\natural$ the induced map.

Proof. We have to show the map annihilates commutators. One has

$$\left(\sum \lambda_g g\right)\left(\sum \mu_g g\right) - \left(\sum \mu_g g\right)\left(\sum \lambda_g g\right) = \sum_g \left(\sum_{g_1 g_2 = g} \lambda_{g_1}\mu_{g_2} - \mu_{g_1}\lambda_{g_2}\right) g$$

The coefficient of e is

$$\sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_g \lambda_{g^{-1}}\right) = \sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_{g^{-1}} \lambda_g\right)$$

which is a sum of commutators, hence it zero in $\Lambda^\natural$. □

03U6 **Definition 64.15.2.** Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module P. We define

$$\mathrm{Tr}^G_\Lambda(f; P) := \varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f; P)\right)$$

to be the *G-trace of f on P*.

03U7 **Lemma 64.15.3.** *Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module P. Then*

$$Tr_\Lambda(f; P) = \#G \cdot Tr^G_\Lambda(f; P).$$

Proof. By additivity, reduce to the case $P = \Lambda[G]$. In that case, f is given by right multiplication by some element $\sum \lambda_g \cdot g$ of $\Lambda[G]$. In the basis $(g)_{g\in G}$, the matrix of f has coefficient $\lambda_{g_2^{-1}g_1}$ in the (g_1, g_2) position. In particular, all diagonal coefficients are λ_e, and there are $\#G$ such coefficients. □

03U8 **Lemma 64.15.4.** *The map $A \to \Lambda$ defines an A-module structure on $\Lambda^\natural$.*

Proof. This is clear. □

03U9 **Lemma 64.15.5.** *Let P be a finite projective $A[G]$-module and M a $\Lambda[G]$-module, finite projective as a Λ-module. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module, for the structure induced by the diagonal action of G.*

Note that $P \otimes_A M$ is naturally a Λ-module since M is. Explicitly, together with the diagonal action this reads

$$\left(\sum \lambda_g g\right)(p \otimes m) = \sum gp \otimes \lambda_g gm.$$

Proof. For any $\Lambda[G]$-module N one has

$$\mathrm{Hom}_{\Lambda[G]}\left(P \otimes_A M, N\right) = \mathrm{Hom}_{A[G]}\left(P, \mathrm{Hom}_\Lambda(M, N)\right)$$

where the G-action on $\mathrm{Hom}_\Lambda(M, N)$ is given by $(g \cdot \varphi)(m) = g\varphi(g^{-1}m)$. Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of P and M. □

03UA **Lemma 64.15.6.** *With assumptions as in Lemma 64.15.5, let $u \in End_{A[G]}(P)$ and $v \in End_{\Lambda[G]}(M)$. Then*

$$Tr^G_\Lambda\left(u \otimes v; P \otimes_A M\right) = Tr^G_A(u; P) \cdot Tr_\Lambda(v; M).$$

Sketch of proof. Reduce to the case $P = A[G]$. In that case, u is right multiplication by some element $a = \sum a_g g$ of $A[G]$, which we write $u = R_a$. There is an isomorphism of $\Lambda[G]$-modules

$$\begin{array}{rccc} \varphi: & A[G] \otimes_A M & \cong & (A[G] \otimes_A M)' \\ & g \otimes m & \longmapsto & g \otimes g^{-1} m \end{array}$$

where $(A[G] \otimes_A M)'$ has the module structure given by the left G-action, together with the Λ-linearity on M. This transport of structure changes $u \otimes v$ into $\sum_g a_g R_g \otimes g^{-1} v$. In other words,

$$\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum_g a_g R_g \otimes g^{-1} v.$$

Working out explicitly both sides of the equation, we have to show

$$\mathrm{Tr}_\Lambda^G \left(\sum_g a_g R_g \otimes g^{-1} v \right) = a_e \cdot \mathrm{Tr}_\Lambda(v; M).$$

This is done by showing that

$$\mathrm{Tr}_\Lambda^G \left(a_g R_g \otimes g^{-1} v \right) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \mathrm{Tr}_\Lambda(v; M) & \text{if } g = e \end{cases}$$

by reducing to $M = \Lambda$. □

Notation: Consider the monoid extension $1 \to G \to \Gamma \to \mathbf{N} \to 1$ and let $\gamma \in \Gamma$. Then we write $Z_\gamma = \{g \in G | g\gamma = \gamma g\}$.

03UB **Lemma 64.15.7.** *Let P be a $\Lambda[\Gamma]$-module, finite and projective as a $\Lambda[G]$-module, and $\gamma \in \Gamma$. Then*

$$Tr_\Lambda(\gamma, P) = \#Z_\gamma \cdot Tr_\Lambda^{Z_\gamma}(\gamma, P).$$

Proof. This follows readily from Lemma 64.15.3. □

03UC **Lemma 64.15.8.** *Let P be an $A[\Gamma]$-module, finite projective as $A[G]$-module. Let M be a $\Lambda[\Gamma]$-module, finite projective as a Λ-module. Then*

$$Tr_\Lambda^{Z_\gamma}(\gamma, P \otimes_A M) = Tr_A^{Z_\gamma}(\gamma, P) \cdot Tr_\Lambda(\gamma, M).$$

Proof. This follows directly from Lemma 64.15.6. □

03UD **Lemma 64.15.9.** *Let P be a $\Lambda[\Gamma]$-module, finite projective as $\Lambda[G]$-module. Then the coinvariants $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective Λ-module, endowed with an action of $\Gamma/G = \mathbf{N}$. Moreover, we have*

$$Tr_\Lambda(1; P_G) = \sum\nolimits'_{\gamma \mapsto 1} Tr_\Lambda^{Z_\gamma}(\gamma, P)$$

where $\sum'_{\gamma \mapsto 1}$ means taking the sum over the G-conjugacy classes in Γ.

Sketch of proof. We first prove this after multiplying by $\#G$.

$$\#G \cdot \mathrm{Tr}_\Lambda(1; P_G) = \mathrm{Tr}_\Lambda(\sum\nolimits_{\gamma \mapsto 1} \gamma, P_G) = \mathrm{Tr}_\Lambda(\sum\nolimits_{\gamma \mapsto 1} \gamma, P)$$

where the second equality follows by considering the commutative triangle

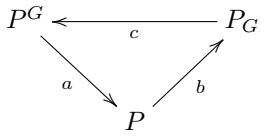

where a is the canonical inclusion, b the canonical surjection and $c = \sum_{\gamma \mapsto 1} \gamma$. Then we have

$$(\sum\nolimits_{\gamma \mapsto 1} \gamma)|_P = a \circ c \circ b \quad \text{and} \quad (\sum\nolimits_{\gamma \mapsto 1} \gamma)|_{P_G} = b \circ a \circ c$$

hence they have the same trace. We then have

$$\#G \cdot \mathrm{Tr}_\Lambda(1; P_G) = \sum_{\gamma \mapsto 1}{}' \frac{\#G}{\#Z_\gamma} \mathrm{Tr}_\Lambda(\gamma, P) = \#G \sum_{\gamma \mapsto 1}{}' \mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P).$$

To finish the proof, reduce to case Λ torsion-free by some universality argument. See [**Del77**] for details. □

03UE **Remark 64.15.10.** Let us try to illustrate the content of the formula of Lemma 64.15.8. Suppose that Λ, viewed as a trivial Γ-module, admits a finite resolution $0 \to P_r \to \ldots \to P_1 \to P_0 \to \Lambda \to 0$ by some $\Lambda[\Gamma]$-modules P_i which are finite and projective as $\Lambda[G]$-modules. In that case

$$H_* \left((P_\bullet)_G\right) = \mathrm{Tor}_*^{\Lambda[G]}(\Lambda, \Lambda) = H_*(G, \Lambda)$$

and

$$\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma} \mathrm{Tr}_\Lambda(\gamma, P_\bullet) = \frac{1}{\#Z_\gamma} \mathrm{Tr}(\gamma, \Lambda) = \frac{1}{\#Z_\gamma}.$$

Therefore, Lemma 64.15.8 says

$$\mathrm{Tr}_\Lambda(1, P_G) = \mathrm{Tr}\left(1|_{H_*(G,\Lambda)}\right) = \sum_{\gamma \mapsto 1}{}' \frac{1}{\#Z_\gamma}.$$

This can be interpreted as a point count on the stack BG. If $\Lambda = \mathbf{F}_\ell$ with ℓ prime to $\#G$, then $H_*(G, \Lambda)$ is $\mathbf{F}_\ell$ in degree 0 (and 0 in other degrees) and the formula reads

$$1 = \sum\nolimits_{\substack{\sigma\text{-conjugacy} \\ \text{classes} \langle\gamma\rangle}} \frac{1}{\#Z_\gamma} \mod \ell.$$

This is in some sense a "trivial" trace formula for G. Later we will see that (64.14.3.1) can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 64.30.

64.16. Proof of the trace formula

03UF

03UG **Theorem 64.16.1.** *Let k be a finite field and X a finite type, separated scheme of dimension at most 1 over k. Let Λ be a finite ring whose cardinality is prime to that of k, and $K \in D_{ctf}(X, \Lambda)$. Then*

03UH (64.16.1.1) $$Tr(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, K)}) = \sum\nolimits_{x \in X(k)} Tr(\pi_x|_{K_{\bar{x}}})$$

in $\Lambda^\natural$.

Please see Remark 64.16.2 for some remarks on the statement. Notation: For short, we write

$$T'(X,K) = \sum\nolimits_{x\in X(k)} \mathrm{Tr}(\pi_x|_{K_{\bar{x}}})$$

for the right-hand side of (64.16.1.1) and

$$T''(X,K) = \mathrm{Tr}(\pi_x^*|_{R\Gamma_c(X_{\bar{k}},K)})$$

for the left-hand side.

Proof of Theorem 64.16.1. The proof proceeds in a number of steps.

Step 1. *Let $j : \mathcal{U} \hookrightarrow X$ be an open immersion with complement $Y = X - \mathcal{U}$ and $i : Y \hookrightarrow X$. Then $T''(X,K) = T''(\mathcal{U}, j^{-1}K) + T''(Y, i^{-1}K)$ and $T'(X,K) = T'(\mathcal{U}, j^{-1}K) + T'(Y, i^{-1}K)$.*

This is clear for T'. For T'' use the exact sequence

$$0 \to j_!j^{-1}K \to K \to i_*i^{-1}K \to 0$$

to get a filtration on K. This gives rise to an object $\widetilde{K} \in DF(X,\Lambda)$ whose graded pieces are $j_!j^{-1}K$ and $i_*i^{-1}K$, both of which lie in $D_{ctf}(X,\Lambda)$. Then, by filtered derived abstract nonsense (INSERT REFERENCE), $R\Gamma_c(X_{\bar{k}},K) \in DF_{perf}(\Lambda)$, and it comes equipped with π_x^* in $DF_{perf}(\Lambda)$. By the discussion of traces on filtered complexes (INSERT REFERENCE) we get

$$\begin{aligned} \mathrm{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}},K)}) &= \mathrm{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}},j_!j^{-1}K)}) + \mathrm{Tr}(\pi_X^*|_{R\Gamma_c(X_{\bar{k}},i_*i^{-1}K)}) \\ &= T''(U, i^{-1}K) + T''(Y, i^{-1}K). \end{aligned}$$

Step 2. *The theorem holds if* $\dim X \leq 0$.

Indeed, in that case

$$R\Gamma_c(X_{\bar{k}},K) = R\Gamma(X_{\bar{k}},K) = \Gamma(X_{\bar{k}},K) = \bigoplus\nolimits_{\bar{x}\in X_{\bar{k}}} K_{\bar{x}}$$

which comes equipped with the endormophism π_X^*. Since the fixed points of $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$ are exactly the points $\bar{x} \in X_{\bar{k}}$ which lie over a k-rational point $x \in X(k)$ we get

$$\mathrm{Tr}\big(\pi_X^*|_{R\Gamma_c(X_{\bar{k}},K)}\big) = \sum\nolimits_{x\in X(k)} \mathrm{Tr}(\pi_x|_{K_{\bar{x}}}).$$

as desired.

Step 3. *It suffices to prove the equality $T'(\mathcal{U},\mathcal{F}) = T''(\mathcal{U},\mathcal{F})$ in the case where*

- *$\mathcal{U}$ is a smooth irreducible affine curve over k,*
- *$\mathcal{U}(k) = \emptyset$,*
- *$K = \mathcal{F}$ is a finite locally constant sheaf of Λ-modules on $\mathcal{U}$ whose stalk(s) are finite projective Λ-modules, and*
- *Λ is killed by a power of a prime ℓ and $\ell \in k^*$.*

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none[2]. We may assume that $\mathcal{U}$ is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of $\mathcal{F}$ come from unwinding the definition of $D_{ctf}(X,\Lambda)$ and those on Λ from considering its primary decomposition.

[2]At this point, there should be an evil laugh in the background.

For the remainder of the proof, we consider the situation

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathcal{U} & \longrightarrow & X \end{array}$$

where $\mathcal{U}$ is as above, f is a finite étale Galois covering, $\mathcal{V}$ is connected and the horizontal arrows are projective completions. Denoting $G = \mathrm{Aut}(\mathcal{V}|\mathcal{U})$, we also assume (as we may) that $f^{-1}\mathcal{F} = \underline{M}$ is constant, where the module $M = \Gamma(\mathcal{V}, f^{-1}\mathcal{F})$ is a $\Lambda[G]$-module which is finite and projective over Λ. This corresponds to the trivial monoid extension

$$1 \to G \to \Gamma = G \times \mathbf{N} \to \mathbf{N} \to 1.$$

In that context, using the reductions above, we need to show that $T''(\mathcal{U}, \mathcal{F}) = 0$.

Step 4. *There is a natural action of G on $f_*f^{-1}\mathcal{F}$ and the trace map $f_*f^{-1}\mathcal{F} \to \mathcal{F}$ defines an isomorphism*

$$(f_*f^{-1}\mathcal{F}) \otimes_{\Lambda[G]} \Lambda = (f_*f^{-1}\mathcal{F})_G \cong \mathcal{F}.$$

To prove this, simply unwind everything at a geometric point.

Step 5. *Let $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$. Then $f_*f^{-1}\mathcal{F} \cong (f_*\underline{A}) \otimes_{\underline{A}} \underline{M}$ with diagonal G-action.*

Step 6. *There is a canonical isomorphism $(f_*\underline{A} \otimes_{\underline{A}} \underline{M}) \otimes_{\Lambda[G]} \underline{\Lambda} \cong \mathcal{F}$.*

In fact, this is a derived tensor product, because of the projectivity assumption on $\mathcal{F}$.

Step 7. *There is a canonical isomorphism*

$$R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F}) = (R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A) \otimes^{\mathbf{L}}_A M) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda,$$

compatible with the action of $\pi^*_{\mathcal{U}}$.

This comes from the universal coefficient theorem, i.e., the fact that $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$, and the flatness of $\mathcal{F}$ as a Λ-module.

We have

$$\begin{aligned} \mathrm{Tr}(\pi^*_{\mathcal{U}}|_{R\Gamma_c(\mathcal{U}_{\bar{k}},\mathcal{F})}) &= {\sum_{g\in G}}' \mathrm{Tr}^{Z_g}_{\Lambda}\left((g,\pi^*_{\mathcal{U}})|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)\otimes^{\mathbf{L}}_A M}\right) \\ &= {\sum_{g\in G}}' \mathrm{Tr}^{Z_g}_{A}((g,\pi^*_{\mathcal{U}})|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}) \cdot \mathrm{Tr}_{\Lambda}(g|_M) \end{aligned}$$

where Γ acts on $R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F})$ by G and $(e, 1)$ acts via $\pi^*_{\mathcal{U}}$. So the monoidal extension is given by $\Gamma = G \times \mathbf{N} \to \mathbf{N}$, $\gamma \mapsto 1$. The first equality follows from Lemma 64.15.9 and the second from Lemma 64.15.8.

Step 8. *It suffices to show that $Tr^{Z_g}_A((g, \pi^*_{\mathcal{U}})|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}) \in A$ maps to zero in Λ.*

Recall that

$$\begin{aligned} \#Z_g \cdot \mathrm{Tr}^{Z_g}_A((g,\pi^*_{\mathcal{U}})|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}) &= \mathrm{Tr}_A((g,\pi^*_{\mathcal{U}})|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}) \\ &= \mathrm{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)}). \end{aligned}$$

The first equality is Lemma 64.15.7, the second is the Leray spectral sequence, using the finiteness of f and the fact that we are only taking traces over A. Now

since $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$ and $\#Z_g = \ell^a$ for some (fixed) a, it suffices to show the following result.

Step 9. *We have* $Tr_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V},A)}) = 0$ *in* A.

By additivity again, we have

$$\begin{aligned}\mathrm{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}}A)}) + \mathrm{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)}) \\ = \mathrm{Tr}_A((g^{-1}\pi_Y)^*|_{R\Gamma(Y_{\bar{k}},A)})\end{aligned}$$

The latter trace is the number of fixed points of $g^{-1}\pi_Y$ on Y, by Weil's trace formula Theorem 64.14.4. Moreover, by the 0-dimensional case already proven in step 2,

$$\mathrm{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)})$$

is the number of fixed points of $g^{-1}\pi_Y$ on $(Y-\mathcal{V})_{\bar{k}}$. Therefore,

$$\mathrm{Tr}_A((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)})$$

is the number of fixed points of $g^{-1}\pi_Y$ on $\mathcal{V}_{\bar{k}}$. But there are no such points: if $\bar{y} \in Y_{\bar{k}}$ is fixed under $g^{-1}\pi_Y$, then $\bar{f}(\bar{y}) \in X_{\bar{k}}$ is fixed under π_X. But $\mathcal{U}$ has no k-rational point, so we must have $\bar{f}(\bar{y}) \in (X-\mathcal{U})_{\bar{k}}$ and so $\bar{y} \notin \mathcal{V}_{\bar{k}}$, a contradiction. This finishes the proof. □

03UI **Remark 64.16.2.** Remarks on Theorem 64.16.1.

(1) This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement – in that generality.
(2) The complex $R\Gamma_c(X_{\bar{k}}, K)$ is defined by choosing an open immersion $j : X \hookrightarrow \bar{X}$ with $\bar{X}$ projective over k of dimension at most 1 and setting

$$R\Gamma_c(X_{\bar{k}}, K) := R\Gamma(\bar{X}_{\bar{k}}, j_!K).$$

This is independent of the choice of $\bar{X}$ follows from (insert reference here). We define $H^i_c(X_{\bar{k}}, K)$ to be the ith cohomology group of $R\Gamma_c(X_{\bar{k}}, K)$.

03UJ **Remark 64.16.3.** Even though all we did are reductions and mostly algebra, the trace formula Theorem 64.16.1 is much stronger than Weil's geometric trace formula (Theorem 64.14.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

64.17. Applications

03UK OK, having indicated the proof of the trace formula, let's try to use it for something.

64.18. On l-adic sheaves

03UL

03UM **Definition 64.18.1.** Let X be a Noetherian scheme. A *$\mathbf{Z}_\ell$-sheaf* on X, or simply an *ℓ-adic sheaf* $\mathcal{F}$ is an inverse system $\{\mathcal{F}_n\}_{n\geq 1}$ where

(1) $\mathcal{F}_n$ is a constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, and
(2) the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong \mathcal{F}_n$.

We say that $\mathcal{F}$ is *lisse* if each $\mathcal{F}_n$ is locally constant. A *morphism* of such is merely a morphism of inverse systems.

03UN **Lemma 64.18.2.** *Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an inverse system of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. Suppose that for all $k \geq 1$, the maps*

$$\mathcal{G}_{n+1}/\ell^k\mathcal{G}_{n+1} \to \mathcal{G}_n/\ell^k\mathcal{G}_n$$

are isomorphisms for all $n \gg 0$ (where the bound possibly depends on k). In other words, assume that the system $\{\mathcal{G}_n/\ell^k\mathcal{G}_n\}_{n\geq 1}$ is eventually constant, and call $\mathcal{F}_k$ the corresponding sheaf. Then the system $\{\mathcal{F}_k\}_{k\geq 1}$ forms a $\mathbf{Z}_\ell$-sheaf on X.

Proof. The proof is obvious. □

03UO **Lemma 64.18.3.** *The category of $\mathbf{Z}_\ell$-sheaves on X is abelian.*

Proof. Let $\Phi = \{\varphi_n\}_{n\geq 1} : \{\mathcal{F}_n\} \to \{\mathcal{G}_n\}$ be a morphism of $\mathbf{Z}_\ell$-sheaves. Set

$$\mathrm{Coker}(\Phi) = \left\{\mathrm{Coker}\left(\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n\right)\right\}_{n\geq 1}$$

and $\mathrm{Ker}(\Phi)$ is the result of Lemma 64.18.2 applied to the inverse system

$$\left\{\bigcap_{m\geq n} \mathrm{Im}\left(\mathrm{Ker}(\varphi_m) \to \mathrm{Ker}(\varphi_n)\right)\right\}_{n\geq 1}.$$

That this defines an abelian category is left to the reader. □

03UP **Example 64.18.4.** Let $X = \mathrm{Spec}(\mathbf{C})$ and $\Phi : \mathbf{Z}_\ell \to \mathbf{Z}_\ell$ be multiplication by ℓ. More precisely,

$$\Phi = \left\{\mathbf{Z}/\ell^n\mathbf{Z} \xrightarrow{\ell} \mathbf{Z}/\ell^n\mathbf{Z}\right\}_{n\geq 1}.$$

To compute the kernel, we consider the inverse system

$$\ldots \to \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z}.$$

Since the images are always zero, $\mathrm{Ker}(\Phi)$ is zero as a system.

03UQ **Remark 64.18.5.** If $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X and $\bar{x}$ is a geometric point then $M_n = \{\mathcal{F}_{n,\bar{x}}\}$ is an inverse system of finite $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that $M_{n+1} \to M_n$ is surjective and $M_n = M_{n+1}/\ell^n M_{n+1}$. It follows that

$$M = \lim_n M_n = \lim \mathcal{F}_{n,\bar{x}}$$

is a finite $\mathbf{Z}_\ell$-module. This follows from Algebra, Lemmas 10.98.2 and 10.96.12 and the fact that $M/\ell M = M_1$ is finite over $\mathbf{F}_\ell$. Therefore, $M \cong \mathbf{Z}_\ell^{\oplus r} \oplus \oplus_{i=1}^t \mathbf{Z}_\ell/\ell^{e_i}\mathbf{Z}_\ell$ for some $r, t \geq 0$, $e_i \geq 1$. The module $M = \mathcal{F}_{\bar{x}}$ is called the *stalk* of $\mathcal{F}$ at $\bar{x}$.

03UR **Definition 64.18.6.** A $\mathbf{Z}_\ell$-sheaf $\mathcal{F}$ is *torsion* if $\ell^n : \mathcal{F} \to \mathcal{F}$ is the zero map for some n. The abelian category of $\mathbf{Q}_\ell$-sheaves on X is the quotient of the abelian category of $\mathbf{Z}_\ell$-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are $\mathbf{Z}_\ell$-sheaves on X, and if $\mathcal{F}, \mathcal{G}$ are two such, then

$$\mathrm{Hom}_{\mathbf{Q}_\ell}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We denote by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbf{Q}_\ell$ the quotient functor (right adjoint to the inclusion). If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}'$ is a $\mathbf{Z}_\ell$-sheaf and $\bar{x}$ is a geometric point, then the *stalk* of $\mathcal{F}$ at $\bar{x}$ is $\mathcal{F}_{\bar{x}} = \mathcal{F}'_{\bar{x}} \otimes \mathbf{Q}_\ell$.

03US **Remark 64.18.7.** Since a $\mathbf{Z}_\ell$-sheaf is only defined on a Noetherian scheme, it is torsion if and only if its stalks are torsion.

03UT **Definition 64.18.8.** If X is a separated scheme of finite type over an algebraically closed field k and $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X, then we define

$$H^i(X, \mathcal{F}) := \lim_n H^i(X, \mathcal{F}_n) \quad \text{and} \quad H^i_c(X, \mathcal{F}) := \lim_n H^i_c(X, \mathcal{F}_n).$$

If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ for a $\mathbf{Z}_\ell$-sheaf $\mathcal{F}'$ then we set

$$H^i_c(X, \mathcal{F}) := H^i_c(X, \mathcal{F}') \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We call these the *ℓ-adic cohomology* of X with coefficients $\mathcal{F}$.

64.19. L-functions

03UU

03UV **Definition 64.19.1.** Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{étale}$. Then we set

$$L(X, \mathcal{F}) := \prod\nolimits_{x\in|X|} \det(1 - \pi_x^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} \in \Lambda[[T]]$$

where $|X|$ is the set of closed points of X, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over x. This definition clearly generalizes to the case where $\mathcal{F}$ is replaced by a $K \in D_{ctf}(X, \Lambda)$. We call this the *$L$-function of $\mathcal{F}$*.

03UW **Remark 64.19.2.** Intuitively, T should be thought of as $T = t^f$ where $p^f = \#k$. The definitions are then independent of the size of the ground field.

03UX **Definition 64.19.3.** Now assume that $\mathcal{F}$ is a $\mathbf{Q}_\ell$-sheaf on X. In this case we define

$$L(X, \mathcal{F}) := \prod\nolimits_{x\in|X|} \det(1 - \pi_x^* T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} \in \mathbf{Q}_\ell[[T]].$$

Note that this product converges since there are finitely many points of a given degree. We call this the *L-function of $\mathcal{F}$*.

64.20. Cohomological interpretation

03UY This is how Grothendieck interpreted the L-function.

03UZ **Theorem 64.20.1** (Finite Coefficients)**.** *Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{étale}$. Then*

$$L(X, \mathcal{F}) = \det(1 - \pi_X^* \ T|_{R\Gamma_c(X_{\bar{k}}, \mathcal{F})})^{-1} \in \Lambda[[T]].$$

Proof. Omitted. □

Thus far, we don't even know whether each cohomology group $H^i_c(X_{\bar{k}}, \mathcal{F})$ is free.

03V0 **Theorem 64.20.2** (Adic sheaves)**.** *Let X be a scheme of finite type over a finite field k, and $\mathcal{F}$ a $\mathbf{Q}_\ell$-sheaf on X. Then*

$$L(X, \mathcal{F}) = \prod\nolimits_i \det(1 - \pi_X^* T|_{H^i_c(X_{\bar{k}}, \mathcal{F})})^{(-1)^{i+1}} \in \mathbf{Q}_\ell[[T]].$$

Proof. This is sketched below. □

03V1 **Remark 64.20.3.** Since we have only developed some theory of traces and not of determinants, Theorem 64.20.1 is harder to prove than Theorem 64.20.2. We will only prove the latter, for the former see [**Del77**]. Observe also that there is no version of this theorem more general for $\mathbf{Z}_\ell$ coefficients since there is no ℓ-torsion.

We reduce the proof of Theorem 64.20.2 to a trace formula. Since $\mathbf{Q}_\ell$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More precisely, we apply $T\frac{d}{dT}\log$ to both sides. We have on the one hand

$$\begin{aligned} T\frac{d}{dT}\log L(X,\mathcal{F}) &= T\frac{d}{dT}\log\prod_{x\in|X|}\det(1-\pi_x^*T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} \\ &= \sum_{x\in|X|} T\frac{d}{dT}\log(\det(1-\pi_x^*T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1}) \\ &= \sum_{x\in|X|}\deg x\sum_{n\geq 1}\mathrm{Tr}((\pi_x^n)^*|_{\mathcal{F}_{\bar{x}}})T^{n\deg x} \end{aligned}$$

where the last equality results from the formula

$$T\frac{d}{dT}\log\left(\det(1-fT|_M)^{-1}\right) = \sum_{n\geq 1}\mathrm{Tr}(f^n|_M)T^n$$

which holds for any commutative ring Λ and any endomorphism f of a finite projective Λ-module M. On the other hand, we have

$$\begin{aligned} &T\frac{d}{dT}\log\left(\prod\nolimits_i \det(1-\pi_X^*T|_{H_c^i(X_{\bar{k}},\mathcal{F})})^{(-1)^{i+1}}\right) \\ &= \sum\nolimits_i (-1)^i \sum\nolimits_{n\geq 1}\mathrm{Tr}\left((\pi_X^n)^*|_{H_c^i(X_{\bar{k}},\mathcal{F})}\right)T^n \end{aligned}$$

by the same formula again. Now, comparing powers of T and using the Mobius inversion formula, we see that Theorem 64.20.2 is a consequence of the following equality

$$\sum_{d|n} d\sum_{\substack{x\in|X|\\ \deg x=d}}\mathrm{Tr}((\pi_X^{n/d})^*|_{\mathcal{F}_{\bar{x}}}) = \sum_i (-1)^i\mathrm{Tr}((\pi_X^n)^*|_{H_c^i(X_{\bar{k}},\mathcal{F})}).$$

Writing k_n for the degree n extension of k, $X_n = X\times_{\mathrm{Spec}\,k}\mathrm{Spec}(k_n)$ and ${}_n\mathcal{F} = \mathcal{F}|_{X_n}$, this boils down to

$$\sum_{x\in X_n(k_n)}\mathrm{Tr}(\pi_X^*|_{{}_n\mathcal{F}_{\bar{x}}}) = \sum_i (-1)^i\mathrm{Tr}((\pi_X^n)^*|_{H_c^i((X_n)_{\bar{k}},{}_n\mathcal{F})})$$

which is a consequence of Theorem 64.20.5.

03V3 **Theorem 64.20.4.** *Let X/k be as above, let Λ be a finite ring with $\#\Lambda\in k^*$ and $K\in D_{ctf}(X,\Lambda)$. Then $R\Gamma_c(X_{\bar{k}},K)\in D_{perf}(\Lambda)$ and*

$$\sum_{x\in X(k)} Tr(\pi_x|_{K_{\bar{x}}}) = Tr\left(\pi_X^*|_{R\Gamma_c(X_{\bar{k}},K)}\right).$$

Proof. Note that we have already proved this (REFERENCE) when $\dim X\leq 1$. The general case follows easily from that case together with the proper base change theorem. □

03V2 **Theorem 64.20.5.** *Let X be a separated scheme of finite type over a finite field k and $\mathcal{F}$ be a $\mathbf{Q}_\ell$-sheaf on X. Then $\dim_{\mathbf{Q}_\ell} H_c^i(X_{\bar{k}},\mathcal{F})$ is finite for all i, and is nonzero for $0\leq i\leq 2\dim X$ only. Furthermore, we have*

$$\sum_{x\in X(k)} Tr(\pi_x|_{\mathcal{F}_{\bar{x}}}) = \sum_i (-1)^i\, Tr\left(\pi_X^*|_{H_c^i(X_{\bar{k}},\mathcal{F})}\right).$$

Proof. We explain how to deduce this from Theorem 64.20.4. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let $\mathcal{F}$ be as in the theorem. We can write $\mathcal{F}$ as $\mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}' = \{\mathcal{F}'_n\}$ is a $\mathbf{Z}_\ell$-sheaf without torsion, i.e., $\ell : \mathcal{F}' \to \mathcal{F}'$ has trivial kernel in the category of $\mathbf{Z}_\ell$-sheaves. Then each $\mathcal{F}'_n$ is a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, so $\mathcal{F}'_n \in D_{ctf}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ and $\mathcal{F}'_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = \mathcal{F}'_n$. Note that the last equality holds also for standard (non-derived) tensor product, since $\mathcal{F}'_n$ is flat (it is the same equality). Therefore,

(1) the complex $K_n = R\Gamma_c(X_{\bar{k}}, \mathcal{F}'_n)$ is perfect, and it is endowed with an endomorphism $\pi_n : K_n \to K_n$ in $D(\mathbf{Z}/\ell^n\mathbf{Z})$,
(2) there are identifications
$$K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = K_n$$
in $D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, compatible with the endomorphisms π_{n+1} and π_n (see [**Del77**, Rapport 4.12]),
(3) the equality $\mathrm{Tr}(\pi_X^*|_{K_n}) = \sum_{x \in X(k)} \mathrm{Tr}\left(\pi_x|_{(\mathcal{F}'_n)_{\bar{x}}}\right)$ holds, and
(4) for each $x \in X(k)$, the elements $\mathrm{Tr}(\pi_x|_{\mathcal{F}'_{n,\bar{x}}}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element of $\mathbf{Z}_\ell$ which is equal to $\mathrm{Tr}(\pi_x|_{\mathcal{F}_{\bar{x}}}) \in \mathbf{Q}_\ell$.

It thus suffices to prove the following algebra lemma. □

03V4 **Lemma 64.20.6.** *Suppose we have $K_n \in D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, $\pi_n : K_n \to K_n$ and isomorphisms $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ compatible with π_{n+1} and π_n. Then*

(1) *the elements $t_n = Tr(\pi_n|_{K_n}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element $t_\infty = \{t_n\}$ of $\mathbf{Z}_\ell$,*
(2) *the $\mathbf{Z}_\ell$-module $H^i_\infty = \lim_n H^i(k_n)$ is finite and is nonzero for finitely many i only, and*
(3) *the operators $H^i(\pi_n) : H^i(K_n) \to H^i(K_n)$ are compatible and define $\pi^i_\infty : H^i_\infty \to H^i_\infty$ satisfying*
$$\sum (-1)^i Tr(\pi^i_\infty|_{H^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}) = t_\infty.$$

Proof. Since $\mathbf{Z}/\ell^n\mathbf{Z}$ is a local ring and K_n is perfect, each K_n can be represented by a finite complex $K_n^\bullet$ of finite free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that the map $K_n^p \to K_n^{p+1}$ has image contained in ℓK_n^{p+1}. It is a fact that such a complex is unique up to isomorphism. Moreover π_n can be represented by a morphism of complexes $\pi_n^\bullet : K_n^\bullet \to K_n^\bullet$ (which is unique up to homotopy). By the same token the isomorphism $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ is represented by a map of complexes
$$\varphi_n^\bullet : K_{n+1}^\bullet \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n^\bullet.$$
In fact, $\varphi_n^\bullet$ is an isomorphism of complexes, thus we see that

- there exist $a, b \in \mathbf{Z}$ independent of n such that $K_n^i = 0$ for all $i \notin [a, b]$, and
- the rank of K_n^i is independent of n.

Therefore, the module $K^i_\infty = \lim_n \{K_n^i, \varphi_n^i\}$ is a finite free $\mathbf{Z}_\ell$-module and $K^\bullet_\infty$ is a finite complex of finite free $\mathbf{Z}_\ell$-modules. By induction on the number of nonzero terms, one can prove that $H^i(K^\bullet_\infty) = \lim_n H^i(K_n^\bullet)$ (this is not true for unbounded complexes). We conclude that $H^i_\infty = H^i(K^\bullet_\infty)$ is a finite $\mathbf{Z}_\ell$-module. This proves

ii. To prove the remainder of the lemma, we need to overcome the possible non-commutativity of the diagrams

$$\begin{array}{ccc} K^\bullet_{n+1} & \xrightarrow{\varphi^\bullet_n} & K^\bullet_n \\ {\scriptstyle \pi^\bullet_{n+1}}\downarrow & & \downarrow{\scriptstyle \pi^\bullet_n} \\ K^\bullet_{n+1} & \xrightarrow[\varphi^\bullet_n]{} & K^\bullet_n. \end{array}$$

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace $\pi^\bullet_n$ for $n \geq 2$ by homotopic maps of complexes making these diagrams commute. Namely, if $h^i : K^i_{n+1} \to K^{i-1}_n$ is a homotopy, i.e.,

$$\pi^\bullet_n \circ \varphi^\bullet_n - \varphi^\bullet_n \circ \pi^\bullet_{n+1} = dh + hd,$$

then we choose $\tilde{h}^i : K^i_{n+1} \to K^{i-1}_{n+1}$ lifting h^i. This is possible because K^i_{n+1} free and $K^{i-1}_{n+1} \to K^{i-1}_n$ is surjective. Then replace $\pi^\bullet_n$ by $\tilde{\pi}^\bullet_n$ defined by

$$\tilde{\pi}^\bullet_{n+1} = \pi^\bullet_{n+1} + d\tilde{h} + \tilde{h}d.$$

With this choice of $\{\pi^\bullet_n\}$, the above diagrams commute, and the maps fit together to define an endomorphism $\pi^\bullet_\infty = \lim_n \pi^\bullet_n$ of $K^\bullet_\infty$. Then part *i* is clear: the elements $t_n = \sum (-1)^i \mathrm{Tr}\left(\pi^i_n|_{K^i_n}\right)$ fit into an element t_∞ of $\mathbf{Z}_\ell$. Moreover

$$\begin{aligned} t_\infty &= \sum (-1)^i \mathrm{Tr}_{\mathbf{Z}_\ell}(\pi^i_\infty|_{K^i_\infty}) \\ &= \sum (-1)^i \mathrm{Tr}_{\mathbf{Q}_\ell}(\pi^i_\infty|_{K^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}) \\ &= \sum (-1)^i \mathrm{Tr}(\pi_\infty|_{H^i(K^\bullet_\infty \otimes \mathbf{Q}_\ell)}) \end{aligned}$$

where the last equality follows from the fact that $\mathbf{Q}_\ell$ is a field, so the complex $K^\bullet_\infty \otimes \mathbf{Q}_\ell$ is quasi-isomorphic to its cohomology $H^i(K^\bullet_\infty \otimes \mathbf{Q}_\ell)$. The latter is also equal to $H^i(K^\bullet_\infty) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell = H^i_\infty \otimes \mathbf{Q}_\ell$, which finishes the proof of the lemma, and also that of Theorem 64.20.5. □

64.21. List of things which we should add above

03V5 What did we skip the proof of in the lectures so far:

(1) curves and their Jacobians,
(2) proper base change theorem,
(3) inadequate discussion of $R\Gamma_c$,
(4) more generally, given $f : X \to S$ finite type, separated S quasi-projective, discussion of $Rf_!$ on étale sheaves.
(5) discussion of $\otimes^{\mathbf{L}}$
(6) discussion of why $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$

64.22. Examples of L-functions

03V6 We use Theorem 64.20.2 for curves to give examples of L-functions

64.23. Constant sheaves

03V7 Let k be a finite field, X a smooth, geometrically irreducible curve over k and $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ the constant sheaf. If $\bar{x}$ is a geometric point of X, the Galois module $\mathcal{F}_{\bar{x}} = \mathbf{Q}_\ell$ is trivial, so

$$\det(1 - \pi_x^* \, T^{\deg x}|_{\mathcal{F}_{\bar{x}}})^{-1} = \frac{1}{1 - T^{\deg x}}.$$

Applying Theorem 64.20.2, we get

$$\begin{aligned} L(X, \mathcal{F}) &= \prod_{i=0}^{2} \det(1 - \pi_X^* T|_{H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell)})^{(-1)^{i+1}} \\ &= \frac{\det(1 - \pi_X^* T|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)})}{\det(1 - \pi_X^* T|_{H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell)}) \cdot \det(1 - \pi_X^* T|_{H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell)})}. \end{aligned}$$

To compute the latter, we distinguish two cases.

Projective case. Assume that X is projective, so $H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell) = H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$, and we have

$$H^i(X_{\bar{k}}, \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & \pi_X^* = 1 \quad \text{if } i = 0, \\ \mathbf{Q}_\ell^{2g} & \pi_X^* = ? \quad \text{if } i = 1, \\ \mathbf{Q}_\ell & \pi_X^* = q \quad \text{if } i = 2. \end{cases}$$

The identification of the action of π_X^* on H^2 comes from Étale Cohomology, Lemma 59.69.2 and the fact that the degree of π_X is $q = \#(k)$. We do not know much about the action of π_X^* on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\bar{\mathbf{Q}}_\ell$. Putting everything together, Theorem 64.20.2 yields the equality

$$\prod\nolimits_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_X^* T|_{H^1(X_{\bar{k}}, \mathbf{Q}_\ell)})}{(1 - T)(1 - qT)} = \frac{(1 - \alpha_1 T) \ldots (1 - \alpha_{2g} T)}{(1 - T)(1 - qT)}$$

from which we deduce the following result.

03V8 **Lemma 64.23.1.** *Let X be a smooth, projective, geometrically irreducible curve over a finite field k. Then*

(1) *the L-function $L(X, \mathbf{Q}_\ell)$ is a rational function,*
(2) *the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of π_X^* on $H^1(X_{\bar{k}}, \mathbf{Q}_\ell)$ are algebraic integers independent of ℓ,*
(3) *the number of rational points of X on k_n, where $[k_n : k] = n$, is*

$$\#X(k_n) = 1 - \sum\nolimits_{i=1}^{2g} \alpha_i^n + q^n,$$

(4) *for each i, $|\alpha_i| < q$.*

Proof. Part (3) is Theorem 64.20.5 applied to $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ on $X \otimes k_n$. For part (4), use the following result. □

03V9 **Exercise 64.23.2.** Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{z \in \mathbf{C} \mid |\arg z| < \varepsilon\}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \ldots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| \leq q$ for all i. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all i. We will come back to this later.

Affine case. Assume now that X is affine, say $X = \bar{X} - \{x_1, \ldots, x_n\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H^0_c(X_{\bar{k}}, \mathbf{Q}_\ell) = 0$ and $H^2_c(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$ so Theorem 64.20.2 reads

$$L(X, \mathbf{Q}_\ell) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det(1 - \pi_X^* T|_{H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell)})}{1 - qT}.$$

On the other hand, the previous case gives

$$\begin{aligned} L(X, \mathbf{Q}_\ell) &= L(\bar{X}, \mathbf{Q}_\ell) \prod_{i=1}^{n} \left(1 - T^{\deg x_i}\right) \\ &= \frac{\prod_{i=1}^{n}(1 - T^{\deg x_i}) \prod_{j=1}^{2g}(1 - \alpha_j T)}{(1 - T)(1 - qT)}. \end{aligned}$$

Therefore, we see that $\dim H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell) = 2g + \sum_{i=1}^{n} \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi^*_{\bar{X}}$ acting on the degree 1 cohomology are roots of unity. More precisely, each x_i gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j_! \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_{\ell, x_i} \to 0.$$

The long exact cohomology sequence reads

$$0 \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_\ell^{\oplus \deg x_i} \to H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell) \to H^1_c(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^{n} \mathbf{Q}_\ell^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H^2_c(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2_c(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$.

64.24. The Legendre family

03VA Let k be a finite field of odd characteristic, $X = \mathrm{Spec}(k[\lambda, \frac{1}{\lambda(\lambda-1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbf{P}^2_X$ whose affine equation is $y^2 = x(x-1)(x-\lambda)$. We set $\mathcal{F} = Rf^1_* \mathbf{Q}_\ell = \left\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\right\}_{n \geq 1} \otimes \mathbf{Q}_\ell$. In this situation, the following is true

- for each $n \geq 1$, the sheaf $R^1 f_*(\mathbf{Z}/\ell^n \mathbf{Z})$ is finite locally constant – in fact, it is free of rank 2 over $\mathbf{Z}/\ell^n \mathbf{Z}$,
- the system $\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\}_{n \geq 1}$ is a lisse ℓ-adic sheaf, and
- for all $x \in |X|$, $\det(1 - \pi_x \ T^{\deg x}|_{\mathcal{F}_{\bar{x}}}) = (1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})$ where α_x, β_x are the eigenvalues of the geometric frobenius of E_x acting on $H^1(E_{\bar{x}}, \mathbf{Q}_\ell)$.

Note that E_x is only defined over $\kappa(x)$ and not over k. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [**Del77**]. It follows that

$$L(E/X) := L(X, \mathcal{F}) = \prod_{x \in |X|} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$

Applying Theorem 64.20.2 we get

$$L(E/X) = \prod_{i=0}^{2} \det\left(1 - \pi_X^* T|_{H_c^i(X_{\bar{k}}, \mathcal{F})}\right)^{(-1)^{i+1}},$$

and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that $H_c^0(X_{\bar{k}}, \mathcal{F}) = H_c^2(X_{\bar{k}}, \mathcal{F}) = 0$, so we merely have

$$L(E/X) = \det(1 - \pi_X^* T|_{H_c^1(X, \mathcal{F})}).$$

To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism $f : E \to X$ over $\mathbf{Q}_\ell$, namely

$$H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell) \Rightarrow H_c^{i+j}(E_{\bar{k}}, \mathbf{Q}_\ell)$$

which degenerates. We have $f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell$ and $R^1 f_* \mathbf{Q}_\ell = \mathcal{F}$. The sheaf $R^2 f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell(-1)$ is the *Tate twist* of $\mathbf{Q}_\ell$, i.e., it is the sheaf $\mathbf{Q}_\ell$ where the Galois action is given by multiplication by $\#\kappa(x)$ on the stalk at $\bar{x}$. It follows that, for all $n \geq 1$,

$$\begin{aligned}
\#E(k_n) &= \sum\nolimits_i (-1)^i \mathrm{Tr}(\pi_E^{n\,*}|_{H_c^i(E_{\bar{k}}, \mathbf{Q}_\ell)}) \\
&= \sum\nolimits_{i,j} (-1)^{i+j} \mathrm{Tr}(\pi_X^{n\,*}|_{H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell)}) \\
&= (q^n - 2) + \mathrm{Tr}(\pi_X^{n\,*}|_{H_c^1(X_{\bar{k}}, \mathcal{F})}) + q^n(q^n - 2) \\
&= q^{2n} - q^n - 2 + \mathrm{Tr}(\pi_X^{n\,*}|_{H_c^1(X_{\bar{k}}, \mathcal{F})})
\end{aligned}$$

where the first equality follows from Theorem 64.20.5, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of $\mathbf{Q}_\ell$ under f. Alternatively, we could write

$$\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)$$

and use the trace formula for each curve. We can also find the number of k_n-rational points simply by counting. The zero section contributes $q^n - 2$ points (we omit the points where $\lambda = 0, 1$) hence

$$\#E(k_n) = q^n - 2 + \#\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\}.$$

Now we have

$$\#\{y^2 = x(x-1)(x-\lambda),\ \lambda \neq 0, 1\}$$

$$= \#\{y^2 = x(x-1)(x-\lambda) \text{ in } \mathbf{A}^3\} - \#\{y^2 = x^2(x-1)\} - \#\{y^2 = x(x-1)^2\}$$

$$= \#\{\lambda = \tfrac{-y^2}{x(x-1)} + x,\ x \neq 0, 1\} + \#\{y^2 = x(x-1)(x-\lambda), x = 0, 1\} - 2(q^n - \varepsilon_n)$$

$$= q^n(q^n - 2) + 2q^n - 2(q^n - \varepsilon_n)$$

$$= q^{2n} - 2q^n + 2\varepsilon_n$$

where $\varepsilon_n = 1$ if -1 is a square in k_n, 0 otherwise, i.e.,

$$\varepsilon_n = \frac{1}{2}\left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2}\left(1 + (-1)^{\frac{q^n - 1}{2}}\right).$$

Thus $\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n$. Comparing with the previous formula, we find

$$\mathrm{Tr}(\pi_X^{n\,*}|_{H_c^1(X_{\bar{k}}, \mathcal{F})}) = 2\varepsilon_n = 1 + (-1)^{\frac{q^n - 1}{2}},$$

which implies, by elementary algebra of complex numbers, that if -1 is a square in k_n^*, then $\dim H_c^1(X_{\bar{k}}, \mathcal{F}) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1-T)^2.$$

64.25. Exponential sums

03VB A standard problem in number theory is to evaluate sums of the form

$$S_{a,b}(p) = \sum_{x \in \mathbf{F}_p - \{0,1\}} e^{\frac{2\pi i x^a (x-1)^b}{p}}.$$

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = \mathrm{Spec}(\mathbf{F}_p[x, \frac{1}{x(x-1)}])$ and the affine curve $f : X \to \mathbf{P}^1 - \{0, 1, \infty\}$ over S given by the equation $y^{p-1} = x^a(x-1)^b$. This is a finite étale Galois cover with group $\mathbf{F}_p^*$ and there is a splitting

$$f_*(\bar{\mathbf{Q}}_\ell^*) = \bigoplus_{\chi : \mathbf{F}_p^* \to \bar{\mathbf{Q}}_\ell^*} \mathcal{F}_\chi$$

where χ varies over the characters of $\mathbf{F}_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $\mathbf{Q}_\ell$-sheaf on which $\mathbf{F}_p^*$ acts via χ on stalks. We get a corresponding decomposition

$$H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) = \bigoplus_\chi H^1(\mathbf{P}^1_{\bar{k}} - \{0, 1, \infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbf{P}^1 - \{0, 1, \infty\}$ for some suitable χ. It reads

$$S_{a,b}(p) = -\mathrm{Tr}(\pi_X^* | _{H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)}).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p-1) + 3(p-2) \approx p$$

so $g_X \approx p/2$, which also suggests that the χ-pieces are small.

64.26. Trace formula in terms of fundamental groups

03VC In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbf{P}^1$.

64.27. Fundamental groups

03VD This material is discussed in more detail in the chapter on fundamental groups. See Fundamental Groups, Section 58.1. Let X be a connected scheme and let $\overline{x} \to X$ be a geometric point. Consider the functor

$$\begin{array}{rccc} F_{\overline{x}} : & \begin{array}{c}\text{finite étale}\\ \text{schemes over } X\end{array} & \longrightarrow & \text{finite sets} \\ & Y/X & \longmapsto & F_{\overline{x}}(Y) = \left\{ \begin{array}{c}\text{geom points } \overline{y}\\ \text{of } Y \text{ lying over } \overline{x}\end{array} \right\} = Y_{\overline{x}} \end{array}$$

Set

$$\pi_1(X, \overline{x}) = Aut(F_{\overline{x}}) = \text{ set of automorphisms of the functor } F_{\overline{x}}$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \overline{x}) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y)$$

03VE **Definition 64.27.1.** A subgroup of the form $\mathrm{Stab}(\overline{y} \in F_{\overline{x}}(Y)) \subset \pi_1(X, \overline{x})$ is called *open*.

03VF **Theorem 64.27.2** (Grothendieck)**.** *Let X be a connected scheme.*

(1) *There is a topology on $\pi_1(X, \overline{x})$ such that the open subgroups form a fundamental system of open nbhds of $e \in \pi_1(X, \overline{x})$.*
(2) *With topology of (1) the group $\pi_1(X, \overline{x})$ is a profinite group.*
(3) *The functor*

$$\begin{array}{ccc} \text{schemes finite étale over } X & \to & \text{finite discrete continuous } \pi_1(X,\overline{x})\text{-sets} \\ Y/X & \mapsto & F_{\overline{x}}(Y) \text{ with its natural action} \end{array}$$

is an equivalence of categories.

Proof. See [**Gro71**]. □

03VG **Proposition 64.27.3.** *Let X be an integral normal Noetherian scheme. Let $\overline{y} \to X$ be an algebraic geometric point lying over the generic point $\eta \in X$. Then*

$$\pi_x(X, \overline{\eta}) = Gal(M/\kappa(\eta))$$

($\kappa(\eta)$, function field of X) where

$$\kappa(\overline{\eta}) \supset M \supset \kappa(\eta) = k(X)$$

is the max sub-extension such that for every finite sub extension $M \supset L \supset \kappa(\eta)$ the normalization of X in L is finite étale over X.

Proof. Omitted. □

Change of base point. For any $\overline{x}_1, \overline{x}_2$ geom. points of X there exists an isom. of fibre functions

$$\mathcal{F}_{\overline{x}_1} \cong \mathcal{F}_{\overline{x}_2}$$

(This is a path from $\overline{x}_1$ to $\overline{x}_2$.) Conjugation by this path gives isom

$$\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$$

well defined up to inner actions.

Functoriality. For any morphism $X_1 \to X_2$ of connected schemes any $\overline{x} \in X_1$ there is a canonical map

$$\pi_1(X_1, \overline{x}) \to \pi_1(X_2, \overline{x})$$

(Why? because the fibre functor ...)

Base field. Let X be a variety over a field k. Then we get

$$\pi_1(X, \overline{x}) \to \pi_1(Spec(k), \overline{x}) =^{\text{prop}} Gal(k^{\text{sep}}/k)$$

This map is surjective if and only if X is geometrically connected over k. So in the geometrically connected case we get s.e.s. of profinite groups

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to Gal(k^{\text{sep}}/k) \to 1$$

($\pi_1(X_{\overline{k}}, \overline{x})$: geometric fundamental group of X, $\pi_1(X, \overline{x})$: arithmetic fundamental group of X)

Comparison. If X is a variety over $\mathbf{C}$ then

$$\pi_1(X, \overline{x}) = \text{ profinite completion of } \pi_1(X(\mathbf{C})(\text{ usual topology}), x)$$

(have $x \in X(\mathbf{C})$)

Frobenii. X variety over k, $\#k < \infty$. For any $x \in X$ closed point, let

$$F_x \in \pi_1(x, \overline{x}) = \mathrm{Gal}(\kappa(x)^{\mathrm{sep}}/\kappa(x))$$

be the geometric frobenius. Let $\overline{\eta}$ be an alg. geom. gen. pt. Then

$$\pi_1(X, \overline{\eta}) \leftarrow^{\cong} \pi_1(X, \overline{x}) \underset{\leftarrow}{\overset{\text{functoriality}}{}} \pi_1(x, \overline{x})$$

Easy fact:

$$\begin{array}{ccc} \pi_1(X, \overline{\eta}) & \to^{\deg} \pi_1(\mathrm{Spec}(k), \overline{\eta})* & = Gal(k^{sep}/k) \\ & & || \\ & & \widehat{\mathbf{Z}} \cdot F_{\mathrm{Spec}(k)} \\ F_x & \mapsto & \deg(x) \cdot F_{\mathrm{Spec}(k)} \end{array}$$

Recall: $\deg(x) = [\kappa(x) : k]$

Fundamental groups and lisse sheaves. Let X be a connected scheme, $\overline{x}$ geom. pt. There are equivalences of categories

$$(\Lambda \text{ finite ring}) \quad \begin{array}{c}\text{fin. loc. const. sheaves of} \\ \Lambda\text{-modules of } X_{étale}\end{array} \quad \leftrightarrow \quad \begin{array}{c}\text{finite (discrete) } \Lambda\text{-modules} \\ \text{with continuous } \pi_1(X,\overline{x})\text{-action}\end{array}$$

$$(\ell \text{ a prime}) \quad \begin{array}{c}\text{lisse } \ell\text{-adic} \\ \text{sheaves}\end{array} \quad \leftrightarrow \quad \begin{array}{c}\text{finitely generated } \mathbf{Z}_\ell\text{-modules } M \text{ with continuous} \\ \pi_1(X,\overline{x})\text{-action where we use } \ell\text{-adic topology on } M\end{array}$$

In particular lisse $\mathbf{Q}_l$-sheaves correspond to continuous homomorphisms

$$\pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l), \quad r \geq 0$$

Notation: A module with action (M, ρ) corresponds to the sheaf $\mathcal{F}_\rho$.

Trace formulas. X variety over k, $\#k < \infty$.

(1) Λ finite ring $(\#\Lambda, \#k) = 1$

$$\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\Lambda)$$

continuous. For every $n \geq 1$ we have

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X|, \\ \deg(x) = d}} \mathrm{Tr}(\rho(F_x^{n/d})) \right) = \mathrm{Tr}\left((\pi_x^n)^* |_{R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

(2) $l \neq char(k)$ prime, $\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l)$. For any $n \geq 1$

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X| \\ \deg(x) = d}} \mathrm{Tr}\left(\rho(F_x^{n/d}) \right) \right) = \sum_{i=0}^{2 \dim X} (-1)^i \mathrm{Tr}\left(\pi_X^* |_{H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

Weil conjectures. (Deligne-Weil I, 1974) X smooth proj. over k, $\#k = q$, then the eigenvalues of π_X^* on $H^i(X_{\overline{k}}, \mathbf{Q}_l)$ are algebraic integers α with $|\alpha| = q^{1/2}$.

Deligne's conjectures. (almost completely proved by Lafforgue + ...) Let X be a normal variety over k finite

$$\rho : \pi_1(X, \overline{x}) \longrightarrow \mathrm{GL}_r(\mathbf{Q}_l)$$

continuous. Assume: ρ irreducible $\det(\rho)$ of finite order. Then

(1) there exists a number field E such that for all $x \in |X|$(closed points) the char. poly of $\rho(F_x)$ has coefficients in E.
(2) for any $x \in |X|$ the eigenvalues $\alpha_{x,i}$, $i = 1, \ldots, r$ of $\rho(F_x)$ have complex absolute value 1. (these are algebraic numbers not necessary integers)

(3) for every finite place λ(not dividing p), of E (maybe after enlarging E a bit) there exists

$$\rho\lambda : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(E_\lambda)$$

compatible with ρ. (some char. polys of F_x's)

03VH **Theorem 64.27.4** (Deligne, Weil II)**.** *For a sheaf $\mathcal{F}_\rho$ with ρ satisfying the conclusions of the conjecture above then the eigenvalues of π_X^* on $H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are algebraic numbers α with absolute values*

$$|\alpha| = q^{w/2}, \text{ for } w \in \mathbf{Z}, \ w \leq i$$

Moreover, if X smooth and proj. then $w = i$.

Proof. See [**Del80**]. □

64.28. Profinite groups, cohomology and homology

03VI Let G be a profinite group.

Cohomology. Consider the category of discrete modules with continuous G-action. This category has enough injectives and we can define

$$H^i(G, M) = R^iH^0(G, M) = R^i(M \mapsto M^G)$$

Also there is a derived version $RH^0(G, -)$.

Homology. Consider the category of compact abelian groups with continuous G-action. This category has enough projectives and we can define

$$H_i(G, M) = L_iH_0(G, M) = L_i(M \mapsto M_G)$$

and there is also a derived version.

Trivial duality. The functor $M \mapsto M^\wedge = \mathrm{Hom}_{cont}(M, S^1)$ exchanges the categories above and

$$H^i(G, M)^\wedge = H_i(G, M^\wedge)$$

Moreover, this functor maps torsion discrete G-modules to profinite continuous G-modules and vice versa, and if M is either a discrete or profinite continuous G-module, then $M^\wedge = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$.

Notes on Homology.

(1) If we look at Λ-modules for a finite ring Λ then we can identify

$$H_i(G, M) = Tor_i^{\Lambda[[G]]}(M, \Lambda)$$

where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of G.

(2) If G is a normal subgroup of Γ, and Γ is also profinite then
- $H^0(G, -)$: discrete Γ-module→ discrete Γ/G-modules
- $H_0(G, -)$: compact Γ-modules → compact Γ/G-modules

and hence the profinite group Γ/G acts on the cohomology groups of G with values in a Γ-module. In other words, there are derived functors

$$RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \longrightarrow D^+(\text{discrete } \Gamma/G\text{-modules})$$

and similarly for $LH_0(G, -)$.

64.29. Cohomology of curves, revisited

03VJ Let k be a field, X be geometrically connected, smooth curve over k. We have the fundamental short exact sequence

$$1 \to \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{Gal}(k^{sep}/k) \to 1$$

If Λ is a finite ring with $\#\Lambda \in k^*$ and M a finite Λ-module, and we are given

$$\rho : \pi_1(X, \overline{\eta}) \to \mathrm{Aut}_\Lambda(M)$$

continuous, then $\mathcal{F}_\rho$ denotes the associated sheaf on $X_{étale}$.

03VK **Lemma 64.29.1.** *There is a canonical isomorphism*

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$

as Gal(k^{sep}/k)-modules.

Here the subscript $_{\pi_1(X_{\overline{k}}, \overline{\eta})}$ indicates co-invariants, and (-1) indicates the Tate twist i.e., $\sigma \in \mathrm{Gal}(k^{sep}/k)$ acts via

$$\chi_{cycl}(\sigma)^{-1}.\sigma \text{ on RHS}$$

where

$$\chi_{cycl} : \mathrm{Gal}(k^{sep}/k) \to \prod\nolimits_{l \neq char(k)} \mathbf{Z}_l^*$$

is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $\mathcal{F}$ on X there is a maximal quotient $\mathcal{F} \to \mathcal{F}''$ with $\mathcal{F}''/X_{\overline{k}}$ a constant sheaf, hence

$$\mathcal{F}'' = (X \to \mathrm{Spec}(k))^{-1} F''$$

where F'' is a sheaf $\mathrm{Spec}(k)$, i.e., a $\mathrm{Gal}(k^{sep}/k)$-module. Then

$$H^2_c(X_{\overline{k}}, \mathcal{F}) \to H^2_c(X_{\overline{k}}, \mathcal{F}'') \to F''(-1)$$

is an isomorphism.

Proof of Lemma 64.29.1. Let $Y \to^{\varphi} X$ be the finite étale Galois covering corresponding to $\mathrm{Ker}(\rho) \subset \pi_1(X, \overline{\eta})$. So

$$\mathrm{Aut}(Y/X) = Ind(\rho)$$

is Galois group. Then $\varphi^*\mathcal{F}_\rho = \underline{M}_Y$ and

$$\varphi_*\varphi^*\mathcal{F}_\rho \to \mathcal{F}_\rho$$

which gives

$$\begin{aligned} & H^2_c(X_{\overline{k}}, \varphi_*\varphi^*\mathcal{F}_\rho) \to H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \varphi^*\mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \underline{M}) = \oplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \end{aligned}$$

$$\mathrm{Im}(\rho) \to H^2_c(Y_{\overline{k}}, \underline{M}) = \oplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \to_{\mathrm{Im}(\rho)\text{equivalent}} H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to^{\substack{\text{trivial Im}(\rho) \\ \text{action}}}$$

irreducible curve $C/\overline{k}$, $H^2_c(C, \underline{M}) = M$.

Since

$$\begin{matrix} \text{set of irreducible} \\ \text{components of } Y_k \end{matrix} = \frac{Im(\rho)}{Im(\rho|_{\pi_1(X_{\overline{k}}, \overline{\eta})})}$$

We conclude that $H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ is a quotient of $M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$. On the other hand, there is a surjection

$$\mathcal{F}_\rho \to \mathcal{F}'' = \begin{array}{c}\text{sheaf on } X \text{ associated to} \\ (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})} \leftarrow \pi_1(X, \overline{\eta})\end{array}$$

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

The twist in Galois action comes from the fact that $H^2_c(X_{\overline{k}}, \mu_n) =^{\text{can}} \mathbf{Z}/n\mathbf{Z}$. □

03VL **Remark 64.29.2.** Thus we conclude that if X is also projective then we have functorially in the representation ρ the identifications

$$H^0(X_{\overline{k}}, \mathcal{F}_\rho) = M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

and

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = M_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$

Of course if X is not projective, then $H^0_c(X_{\overline{k}}, \mathcal{F}_\rho) = 0$.

03VM **Proposition 64.29.3.** *Let X/k as before but $X_{\overline{k}} \neq \mathbf{P}^1_{\overline{k}}$ The functors $(M, \rho) \mapsto H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho)$ are the left derived functor of $(M, \rho) \mapsto H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ so*

$$H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho) = H_i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)(-1)$$

Moreover, there is a derived version, namely

$$R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho) = LH_0(\pi_1(X_{\overline{k}}, \overline{\eta}), M(-1)) = M(-1) \otimes^{\mathbf{L}}_{\Lambda[[\pi_1(X_{\overline{k}}, \overline{\eta})]]} \Lambda$$

in $D(\Lambda[[\widehat{\mathbf{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are the right derived functor of $(M, \rho) \mapsto M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$ so

$$H^i(X_{\overline{k}}, \mathcal{F}_\rho) = H^i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)$$

Moreover, in this case there is a derived version too.

Proof. (Idea) Show both sides are universal δ-functors. □

03VN **Remark 64.29.4.** By the proposition and Trivial duality then you get

$$H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho) \times H^i(X_{\overline{k}}, \mathcal{F}^\wedge_\rho(1)) \to \mathbf{Q}/\mathbf{Z}$$

a perfect pairing. If X is projective then this is Poincare duality.

64.30. Abstract trace formula

03VO Suppose given an extension of profinite groups,

$$1 \to G \to \Gamma \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

We say Γ *has an abstract trace formula* if and only if there exist

(1) an integer $q \geq 1$, and
(2) for every $d \geq 1$ a finite set S_d and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$ with $\deg(F_x) = d$

such that the following hold

(1) for all ℓ not dividing q have $\text{cd}_\ell(G) < \infty$, and

(2) for all finite rings Λ with $q \in \Lambda^*$, for all finite projective Λ-modules M with continuous Γ-action, for all $n > 0$ we have

$$\sum\nolimits_{d|n} d \left(\sum\nolimits_{x \in S_d} \mathrm{Tr}(F_x^{n/d}|_M) \right) = q^n \mathrm{Tr}(F^n|_{M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda})$$

in $\Lambda^\natural$.

Here $M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \widehat{\mathbf{Z}}$.

03VP **Remark 64.30.1.** Here are some observations concerning this notion.

(1) If modeling projective curves then we can use cohomology and we don't need factor q^n.
(2) The only examples I know are $\Gamma = \pi_1(X, \overline{\eta})$ where X is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\#k)^{\dim X}$. Modulo the proposition, we proved this for curves in this course.
(3) Given the integer q then the sets S_d are uniquely determined. (You can multiple q by an integer m and then replace S_d by m^d copies of S_d without changing the formula.)

03VQ **Example 64.30.2.** Fix an integer $q \geq 1$

$$\begin{array}{ccccccccc} 1 & \to & G = \widehat{\mathbf{Z}}^{(q)} & \to & \Gamma & \to & \widehat{\mathbf{Z}} & \to & 1 \\ & & = \prod_{l|q} \mathbf{Z}_l & & F & \mapsto & 1 & & \end{array}$$

with $FxF^{-1} = ux$, $u \in (\widehat{\mathbf{Z}}^{(q)})^*$. Just using the trivial modules $\mathbf{Z}/m\mathbf{Z}$ we see

$$q^n - (qu)^n \equiv \sum\nolimits_{d|n} d \# S_d$$

in $\mathbf{Z}/m\mathbf{Z}$ for all $(m, q) = 1$ (up to $u \to u^{-1}$) this implies $qu = a \in \mathbf{Z}$ and $|a| < q$. The special case $a = 1$ does occur with

$$\Gamma = \pi_1^t(\mathbf{G}_{m, \mathbf{F}_p}, \overline{\eta}), \quad \#S_1 = q - 1, \quad \text{and} \quad \#S_2 = \frac{(q^2 - 1) - (q - 1)}{2}$$

64.31. Automorphic forms and sheaves

03VR References: See especially the amazing papers [**Dri83**], [**Dri84**] and [**Dri80**] by Drinfeld.

Unramified cusp forms. Let k be a finite field of characteristic p. Let X geometrically irreducible projective smooth curve over k. Set $K = k(X)$ equal to the function field of X. Let v be a place of K which is the same thing as a closed point $x \in X$. Let K_v be the completion of K at v, which is the same thing as the fraction field of the completion of the local ring of X at x. Denote $O_v \subset K_v$ the ring of integers. We further set

$$O = \prod\nolimits_v O_v \subset \mathbf{A} = \prod\nolimits_v{}' K_v$$

and we let Λ be any ring with p invertible in Λ.

03VS **Definition 64.31.1.** An *unramified cusp form on $GL_2(\mathbf{A})$ with values in Λ*[3] is a function

$$f : \mathrm{GL}_2(\mathbf{A}) \to \Lambda$$

such that

[3] This is likely nonstandard notation.

(1) $f(x\gamma) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $\gamma \in \mathrm{GL}_2(K)$
(2) $f(ux) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $u \in \mathrm{GL}_2(O)$
(3) for all $x \in \mathrm{GL}_2(\mathbf{A})$,

$$\int_{\mathbf{A} \mod K} f\left(x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) dz = 0$$

see [**dJ01**, Section 4.1] for an explanation of how to make sense out of this for a general ring Λ in which p is invertible.

Hecke Operators. For v a place of K and f an unramified cusp form we set

$$T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x)dg,$$

and

$$U_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x\right)$$

Notations used: here $\pi_v \in O_v$ is a uniformizer

$$M_v = \{h \in Mat(2 \times 2, O_v) | \det h = \pi_v O_v^*\}$$

and $dg =$ is the Haar measure on $\mathrm{GL}_2(K_v)$ with $\int_{\mathrm{GL}_2(O_v)} dg = 1$. Explicitly we have

$$T_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x\right) + \sum_{i=1}^{q_v} f\left(\begin{pmatrix} 1 & 0 \\ -\pi_v^{-1}\lambda_i & \pi_v^{-1} \end{pmatrix} x\right)$$

with $\lambda_i \in O_v$ a set of representatives of $O_v/(\pi_v) = \kappa_v$, $q_v = \#\kappa_v$.

Eigenforms. An *eigenform* f is an unramified cusp form such that some value of f is a unit and $T_v f = t_v f$ and $U_v f = u_v f$ for some (uniquely determined) $t_v, u_v \in \Lambda$.

03VT **Theorem 64.31.2.** *Given an eigenform f with values in $\overline{\mathbf{Q}}_l$ and eigenvalues $u_v \in \overline{\mathbf{Z}}_l^*$ then there exists*

$$\rho : \pi_1(X) \to GL_2(E)$$

continuous, absolutely irreducible where E is a finite extension of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_l$ such that $t_v = Tr(\rho(F_v))$, and $u_v = q_v^{-1} \det(\rho(F_v))$ for all places v.

Proof. See [**Dri80**]. □

03VU **Theorem 64.31.3.** *Suppose $\mathbf{Q}_l \subset E$ finite, and*

$$\rho : \pi_1(X) \to GL_2(E)$$

absolutely irreducible, continuous. Then there exists an eigenform f with values in $\overline{\mathbf{Q}}_l$ whose eigenvalues t_v, u_v satisfy the equalities $t_v = Tr(\rho(F_v))$ and $u_v = q_v^{-1} \det(\rho(F_v))$.

Proof. See [**Dri83**]. □

03VV **Remark 64.31.4.** We now have, thanks to Lafforgue and many other mathematicians, complete theorems like this two above for GL_n and allowing ramification! In other words, the full global Langlands correspondence for GL_n is known for function fields of curves over finite fields. At the same time this does not mean there aren't a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.

Central character. If f is an eigenform then

$$\begin{array}{rccc} \chi_f : & O^*\backslash \mathbf{A}^*/K^* & \to & \Lambda^* \\ & (1, \ldots, \pi_v, 1, \ldots, 1) & \mapsto & u_v^{-1} \end{array}$$

is called the *central character*. If corresponds to the determinant of ρ via normalizations as above. Set

$$C(\Lambda) = \left\{ \begin{array}{c} \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \\ \text{such that } U_v f = \varphi_v^{-1} f \forall v \end{array} \right\}$$

03VW **Proposition 64.31.5.** *If Λ is Noetherian then $C(\Lambda)$ is a finitely generated Λ-module. Moreover, if Λ is a field with prime subfield $\mathbf{F} \subset \Lambda$ then*

$$C(\Lambda) = (C(\mathbf{F})) \otimes_{\mathbf{F}} \Lambda$$

compatibly with T_v acting.

Proof. See [**dJ01**, Proposition 4.7]. □

This proposition trivially implies the following lemma.

03VX **Lemma 64.31.6.** *Algebraicity of eigenvalues. If Λ is a field then the eigenvalues t_v for $f \in C(\Lambda)$ are algebraic over the prime subfield $\mathbf{F} \subset \Lambda$.*

Proof. Follows from Proposition 64.31.5. □

Combining all of the above we can do the following very useful trick.

03VY **Lemma 64.31.7.** *Switching l. Let E be a number field. Start with*

$$\rho : \pi_1(X) \to SL_2(E_\lambda)$$

absolutely irreducible continuous, where λ is a place of E not lying above p. Then for any second place λ' of E not lying above p there exists a finite extension $E'_{\lambda'}$ and a absolutely irreducible continuous representation

$$\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$$

which is compatible with ρ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne's conjecture!

Proof. To prove the switching lemma use Theorem 64.31.3 to obtain $f \in C(\overline{\mathbf{Q}}_l)$ eigenform ass. to ρ. Next, use Proposition 64.31.5 to see that we may choose $f \in C(E')$ with $E \subset E'$ finite. Next we may complete E' to see that we get $f \in C(E'_{\lambda'})$ eigenform with $E'_{\lambda'}$ a finite extension of $E_{\lambda'}$. And finally we use Theorem 64.31.2 to obtain $\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again. □

Speculation: If for a (topological) ring Λ we have

$$\left(\begin{array}{c} \rho : \pi_1(X) \to SL_2(\Lambda) \\ \text{abs irred} \end{array} \right) \leftrightarrow \text{ eigen forms in } C(\Lambda)$$

then all eigenvalues of $\rho(F_v)$ algebraic (won't work in an easy way if Λ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.

Conjecture. (See [**dJ01**]) For any continuous

$$\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbf{F}_l[[t]])$$

we have $\#\rho(\pi_1(X_{\overline{k}})) < \infty$.

A rephrasing in the language of sheaves: "For any lisse sheaf of $\overline{\mathbf{F}_l((t))}$-modules the geom monodromy is finite."

03VZ **Theorem 64.31.8.** *The Conjecture holds if $n \leq 2$.*

Proof. See [**dJ01**]. □

03W0 **Theorem 64.31.9.** *Conjecture holds if $l > 2n$ modulo some unproven things.*

Proof. See [**Gai07**]. □

It turns out the conjecture is useful for something. See work of Drinfeld on Kashiwara's conjectures. But there is also the much more down to earth application as follows.

03W1 **Theorem 64.31.10.** *(See* [**dJ01**, Theorem 3.5]*) Suppose*

$$\rho_0 : \pi_1(X) \to GL_n(\mathbf{F}_l)$$

is a continuous, $l \neq p$. Assume

(1) *Conj. holds for X,*
(2) *$\rho_0|_{\pi_1(X_{\overline{k}})}$ abs. irred., and*
(3) *l does not divide n.*

Then the universal deformation ring R_{univ} of ρ_0 is finite flat over $\mathbf{Z}_l$.

Explanation: There is a representation $\rho_{\mathrm{univ}} : \pi_1(X) \to \mathrm{GL}_n(R_{\mathrm{univ}})$ (Univ. Defo ring) R_{univ} loc. complete, residue field $\mathbf{F}_l$ and $(R_{\mathrm{univ}} \to \mathbf{F}_l) \circ \rho_{\mathrm{univ}} \cong \rho_0$. And given any $R \to \mathbf{F}_l$, R local complete and $\rho : \pi_1(X) \to \mathrm{GL}_n(R)$ then there exists $\psi : R_{\mathrm{univ}} \to R$ such that $\psi \circ \rho_{\mathrm{univ}} \cong \rho$. The theorem says that the morphism

$$\mathrm{Spec}(R_{\mathrm{univ}}) \longrightarrow \mathrm{Spec}(\mathbf{Z}_l)$$

is finite and flat. In particular, such a ρ_0 lifts to a $\rho : \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$.

Notes:

(1) The theorem on deformations is easy.
(2) Any result towards the conjecture seems hard.
(3) It would be interesting to have more conjectures on $\pi_1(X)$!

64.32. Counting points

03W2 Let X be a smooth, geometrically irreducible, projective curve over k and $q = \#k$. The trace formula gives: there exists algebraic integers $w_1, \ldots, w_{2g}$ such that

$$\#X(k_n) = q^n - \sum\nolimits_{i=1}^{2g_X} w_i^n + 1.$$

If $\sigma \in \mathrm{Aut}(X)$ then for all i, there exists j such that $\sigma(w_i) = w_j$.

Riemann-Hypothesis. For all i we have $|\omega_i| = \sqrt{q}$.

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs

- using intersection theory on $X \times X$, using the Hodge index theorem, and

- using the Jacobian of X.

There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbf{F}_q$-rational points of X, and that one can try to use this idea to give an upper bound for the number of points.

64.33. Precise form of Chebotarev

03W3 As a first application let us prove a precise form of Chebotarev for a finite étale Galois covering of curves. Let $\varphi : Y \to X$ be a finite étale Galois covering with group G. This corresponds to a homomorphism

$$\pi_1(X) \longrightarrow G = \mathrm{Aut}(Y/X)$$

Assume $Y_{\overline{k}} =$ irreducible. If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$|\#\{x \in X(k_n) \mid F_x \in C\} - \frac{\#C}{\#G} \cdot \#X(k_n)| \leq (\#C)(2g-2)\sqrt{q^n}$$

(Warning: Please check the coefficient $\#C$ on the right hand side carefully before using.)

Sketch. Write

$$\varphi_*(\overline{\mathbf{Q}}_l) = \oplus_{\pi \in \widehat{G}} \mathcal{F}_\pi$$

where $\widehat{G}$ is the set of isomorphism classes of irred representations of G over $\overline{\mathbf{Q}}_l$. For $\pi \in \widehat{G}$ let $\chi_\pi : G \to \overline{\mathbf{Q}}_l$ be the character of π. Then

$$H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l) = \oplus_{\pi \in \widehat{G}} H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l)_\pi =_{(\varphi \text{ finite })} \oplus_{\pi \in \widehat{G}} H^*(X_{\overline{k}}, \mathcal{F}_\pi)$$

If $\pi \neq 1$ then we have

$$H^0(X_{\overline{k}}, \mathcal{F}_\pi) = H^2(X_{\overline{k}}, \mathcal{F}_\pi) = 0, \quad \dim H^1(X_{\overline{k}}, \mathcal{F}_\pi) = (2g_X - 2)d_\pi^2$$

(can get this from trace formula for acting on ...) and we see that

$$|\sum_{x \in X(k_n)} \chi_\pi(\mathcal{F}_x)| \leq (2g_X - 2)d_\pi^2 \sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi \chi_\pi$, then $a_\pi = \langle 1_C, \chi_\pi \rangle$, and $a_1 = \langle 1_C, \chi_1 \rangle = \frac{\#C}{\#G}$ where

$$\langle f, h \rangle = \frac{1}{\#G} \sum_{g \in G} f(g)\overline{h(g)}$$

Thus we have the relation

$$\frac{\#C}{\#G} = ||1_C||^2 = \sum |a_\pi|^2$$

Final step:

$$\begin{aligned}\#\{x \in X(k_n) \mid F_x \in C\} &= \sum_{x \in X(k_n)} 1_C(x) \\ &= \sum_{x \in X(k_n)} \sum_{\pi} a_\pi \chi_\pi(F_x) \\ &= \underbrace{\frac{\#C}{\#G}\#X(k_n)}_{\text{term for } \pi=1} + \underbrace{\sum_{\pi \neq 1} a_\pi \sum_{x \in X(k_n)} \chi_\pi(F_x)}_{\text{error term (to be bounded by } E)}\end{aligned}$$

We can bound the error term by

$$\begin{aligned}|E| &\leq \sum_{\substack{\pi \in \widehat{G}, \\ \pi \neq 1}} |a_\pi|(2g-2)d_\pi^2\sqrt{q^n} \\ &\leq \sum_{\pi \neq 1} \frac{\#C}{\#G}(2g_X - 2)d_\pi^3\sqrt{q^n}\end{aligned}$$

By Weil's conjecture, $\#X(k_n) \sim q^n$. □

64.34. How many primes decompose completely?

03W4 This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled "How many primes decompose completely in an infinite unramified Galois extension of a global field?", see [**Iha83**]. Consider the fundamental exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

03W5 **Proposition 64.34.1.** *There exists a finite set $x_1, \ldots, x_n$ of closed points of X such that set of* **all** *frobenius elements corresponding to these points topologically generate $\pi_1(X)$.*

Another way to state this is: There exist $x_1, \ldots, x_n \in |X|$ such that the smallest normal closed subgroup Γ of $\pi_1(X)$ containing 1 frobenius element for each x_i is all of $\pi_1(X)$. i.e., $\Gamma = \pi_1(X)$.

Proof. Pick $N \gg 0$ and let

$$\{x_1, \ldots, x_n\} = \begin{array}{c}\text{set of all closed points of} \\ X \text{ of degree} \leq N \text{ over } k\end{array}$$

Let $\Gamma \subset \pi_1(X)$ be as in the variant statement for these points. Assume $\Gamma \neq \pi_1(X)$. Then we can pick a normal open subgroup U of $\pi_1(X)$ containing Γ with $U \neq \pi_1(X)$. By R.H. for X our set of points will have some x_{i_1} of degree N, some x_{i_2} of degree $N-1$. This shows $\deg : \Gamma \to \widehat{\mathbf{Z}}$ is surjective and so the same holds for U. This exactly means if $Y \to X$ is the finite étale Galois covering corresponding to U, then $Y_{\overline{k}}$ irreducible. Set $G = \mathrm{Aut}(Y/X)$. Picture

$$Y \to^G X, \quad G = \pi_1(X)/U$$

By construction all points of X of degree $\leq N$, split completely in Y. So, in particular

$$\#Y(k_N) \geq (\#G)\#X(k_N)$$

Use R.H. on both sides. So you get

$$q^N + 1 + 2g_Y q^{N/2} \geq \#G\#X(k_N) \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

Since $2g_Y - 2 = (\#G)(2g_X - 2)$, this means

$$q^N + 1 + (\#G)(2g_X - 1) + 1)q^{N/2} \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

Thus we see that G has to be the trivial group if N is large enough. □

Weird Question. Set $W_X = \deg^{-1}(\mathbf{Z}) \subset \pi_1(X)$. Is it true that for some finite set of closed points $x_1, \ldots, x_n$ of X the set of all frobenii corresponding to these points *algebraically* generate W_X?

By a Baire category argument this translates into the same question for all Frobenii.

64.35. How many points are there really?

03W6 If the genus of the curve is large relative to q, then the main term in the formula $\#X(k) = q - \sum \omega_i + 1$ is not q but the second term $\sum \omega_i$ which can (a priori) have size about $2g_X\sqrt{q}$. In the paper [**VD83**] the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about $g\sqrt{q}$.

Fix q and let k be a field with k elements. Set

$$A(q) = \limsup_{g_X \to \infty} \frac{\#X(k)}{g_X}$$

where X runs over geometrically irreducible smooth projective curves over k. With this definition we have the following results:

- RH $\Rightarrow A(q) \leq 2\sqrt{q}$
- Ihara $\Rightarrow A(q) \leq \sqrt{2q}$
- DV $\Rightarrow A(q) \leq \sqrt{q} - 1$ (actually this is sharp if q is a square)

Proof. Given X let $w_1, \ldots, w_{2g}$ and $g = g_X$ be as before. Set $\alpha_i = \frac{w_i}{\sqrt{q}}$, so $|\alpha_i| = 1$. If α_i occurs then $\overline{\alpha}_i = \alpha_i^{-1}$ also occurs. Then

$$N = \#X(k) \leq X(k_r) = q^r + 1 - (\sum_i \alpha_i^r)q^{r/2}$$

Rewriting we see that for every $r \geq 1$

$$-\sum_i \alpha_i^r \geq Nq^{-r/2} - q^{r/2} - q^{-r/2}$$

Observe that

$$0 \leq |\alpha_i^n + \alpha_i^{n-1} + \ldots + \alpha_i + 1|^2 = (n+1) + \sum_{j=1}^{n}(n+1-j)(\alpha_i^j + \alpha_i^{-j})$$

So

$$\begin{aligned} 2g(n+1) &\geq -\sum_i \left(\sum_{j=1}^{n}(n+1-j)(\alpha_i^j + \alpha_i^{-j}) \right) \\ &= -\sum_{j=1}^{n}(n+1-j)\left(\sum_i \alpha_i^j + \sum_i \alpha_i^{-j} \right) \end{aligned}$$

Take half of this to get

$$\begin{aligned} g(n+1) &\geq -\sum_{j=1}^{n}(n+1-j)(\sum_i \alpha_i^j) \\ &\geq N\sum_{j=1}^{n}(n+1-j)q^{-j/2} - \sum_{j=1}^{n}(n+1-j)(q^{j/2}+q^{-j/2}) \end{aligned}$$

This gives

$$\frac{N}{g} \leq \left(\sum_{j=1}^{n}\frac{n+1-j}{n+1}q^{-j/2}\right)^{-1} \cdot \left(1+\frac{1}{g}\sum_{j=1}^{n}\frac{n+1-j}{n+1}(q^{j/2}+q^{-j/2})\right)$$

Fix n let $g \to \infty$

$$A(q) \leq \left(\sum_{j=1}^{n}\frac{n+1-j}{n+1}q^{-j/2}\right)^{-1}$$

So

$$A(q) \leq \lim_{n\to\infty}(\ldots) = \left(\sum_{j=1}^{\infty}q^{-j/2}\right)^{-1} = \sqrt{q}-1$$

□

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Part 4

Algebraic Spaces

CHAPTER 65

Algebraic Spaces

025R

65.1. Introduction

025S Algebraic spaces were first introduced by Michael Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**]. Some of the foundational material was developed jointly with Knutson, who produced the book [**Knu71**]. Artin defined (see [**Art69c**, Definition 1.3]) an algebraic space as a sheaf for the étale topology which is locally in the étale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different from Artin's original definition. Namely, our algebraic spaces are sheaves for the fppf topology whose diagonal is representable and which have an étale "cover" by a scheme. Working with the fppf topology instead of the étale topology is just a technical point and scarcely makes any difference; we will show in Bootstrap, Section 80.12 that we would have gotten the same category of algebraic spaces if we had worked with the étale topology. In that same chapter we will prove that the condition on the diagonal can in some sense be removed, see Bootstrap, Section 80.6.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing as an étale equivalence relation, see the discussion in Section 65.9 and Theorem 65.10.5. The analogue of this theorem in Artin's setting is [**Art69c**, Theorem 1.5], or [**Knu71**, Proposition II.1.7]. In other words, the sheaf defined by an étale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an étale equivalence relation.

In Section 65.13 we introduce various separation axioms on algebraic spaces that we have found in the literature. Finally in Section 65.14 we give some weird and not so weird examples of algebraic spaces.

65.2. General remarks

025T We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. In Section 65.15 we discuss what changes if you change the big fppf site.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 35.13.7. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

65.3. Representable morphisms of presheaves

025U Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors, see Categories, Definition 4.8.2. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi, G} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By the Yoneda lemma, see Categories, Lemma 4.3.5, the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$. Suggestively we could represent this by the diagram

$$\begin{array}{ccccc} V_\xi & \rightsquigarrow & h_{V_\xi} & \longrightarrow & F \\ {\scriptstyle a_\xi}\downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\ U & \rightsquigarrow & h_U & \xrightarrow{\xi} & G \end{array}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

02W9 **Lemma 65.3.1.** *Let S be a scheme contained in Sch_{fppf} and let X, Y be objects of $(Sch/S)_{fppf}$. Let $f : X \to Y$ be a morphism of schemes. Then*

$$h_f : h_X \longrightarrow h_Y$$

is a representable transformation of functors.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

02WA **Lemma 65.3.2.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. Then*

$$b \circ a : F \longrightarrow H$$

is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

02WB **Lemma 65.3.3.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow[b']{} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

Then the base change a' is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

02WC **Lemma 65.3.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable transformation of functors.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 65.3.2 and 65.3.3. □

02WD **Lemma 65.3.5.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. If G is a sheaf, then so is F.*

Proof. Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a representable presheaf and hence (see remarks in Section 65.2) a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. □

05L9 **Lemma 65.3.6.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Then $\Delta_{F/G} : F \to F \times_G F$ is representable.*

Proof. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist a scheme V and a morphism $V \to U$ representing the fibre product $h_U \times_{\xi',G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $h_U \times_{\xi',G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $h_U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is a scheme. □

65.4. Lists of useful properties of morphisms of schemes

02WE For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

02WF **Remark 65.4.1.** Here is a list of properties/types of morphisms which are *stable under arbitrary base change*:

(1) closed, open, and locally closed immersions, see Schemes, Lemma 26.18.2,
(2) quasi-compact, see Schemes, Lemma 26.19.3,
(3) universally closed, see Schemes, Definition 26.20.1,
(4) (quasi-)separated, see Schemes, Lemma 26.21.12,
(5) monomorphism, see Schemes, Lemma 26.23.5
(6) surjective, see Morphisms, Lemma 29.9.4,
(7) universally injective, see Morphisms, Lemma 29.10.2,
(8) affine, see Morphisms, Lemma 29.11.9,

(9) quasi-affine, see Morphisms, Lemma 29.13.5,
(10) (locally) of finite type, see Morphisms, Lemma 29.15.4,
(11) (locally) quasi-finite, see Morphisms, Lemma 29.20.13,
(12) (locally) of finite presentation, see Morphisms, Lemma 29.21.4,
(13) locally of finite type of relative dimension d, see Morphisms, Lemma 29.29.2,
(14) universally open, see Morphisms, Definition 29.23.1,
(15) flat, see Morphisms, Lemma 29.25.8,
(16) syntomic, see Morphisms, Lemma 29.30.4,
(17) smooth, see Morphisms, Lemma 29.34.5,
(18) unramified (resp. G-unramified), see Morphisms, Lemma 29.35.5,
(19) étale, see Morphisms, Lemma 29.36.4,
(20) proper, see Morphisms, Lemma 29.41.5,
(21) H-projective, see Morphisms, Lemma 29.43.8,
(22) (locally) projective, see Morphisms, Lemma 29.43.9,
(23) finite or integral, see Morphisms, Lemma 29.44.6,
(24) finite locally free, see Morphisms, Lemma 29.48.4,
(25) universally submersive, see Morphisms, Lemma 29.24.2,
(26) universal homeomorphism, see Morphisms, Lemma 29.45.2.

Add more as needed.

02WG **Remark 65.4.2.** Of the properties of morphisms which are stable under base change (as listed in Remark 65.4.1) the following are also *stable under compositions*:

(1) closed, open and locally closed immersions, see Schemes, Lemma 26.24.3,
(2) quasi-compact, see Schemes, Lemma 26.19.4,
(3) universally closed, see Morphisms, Lemma 29.41.4,
(4) (quasi-)separated, see Schemes, Lemma 26.21.12,
(5) monomorphism, see Schemes, Lemma 26.23.4,
(6) surjective, see Morphisms, Lemma 29.9.2,
(7) universally injective, see Morphisms, Lemma 29.10.5,
(8) affine, see Morphisms, Lemma 29.11.8,
(9) quasi-affine, see Morphisms, Lemma 29.13.4,
(10) (locally) of finite type, see Morphisms, Lemma 29.15.3,
(11) (locally) quasi-finite, see Morphisms, Lemma 29.20.12,
(12) (locally) of finite presentation, see Morphisms, Lemma 29.21.3,
(13) universally open, see Morphisms, Lemma 29.23.3,
(14) flat, see Morphisms, Lemma 29.25.6,
(15) syntomic, see Morphisms, Lemma 29.30.3,
(16) smooth, see Morphisms, Lemma 29.34.4,
(17) unramified (resp. G-unramified), see Morphisms, Lemma 29.35.4,
(18) étale, see Morphisms, Lemma 29.36.3,
(19) proper, see Morphisms, Lemma 29.41.4,
(20) H-projective, see Morphisms, Lemma 29.43.7,
(21) finite or integral, see Morphisms, Lemma 29.44.5,
(22) finite locally free, see Morphisms, Lemma 29.48.3,
(23) universally submersive, see Morphisms, Lemma 29.24.3,
(24) universal homeomorphism, see Morphisms, Lemma 29.45.3.

Add more as needed.

02WH **Remark 65.4.3.** Of the properties mentioned which are stable under base change (as listed in Remark 65.4.1) the following are also *fpqc local on the base* (and a fortiori fppf local on the base):

(1) for immersions we have this for
   (a) closed immersions, see Descent, Lemma 35.23.21,
   (b) open immersions, see Descent, Lemma 35.23.18, and
   (c) quasi-compact immersions, see Descent, Lemma 35.23.23,
(2) quasi-compact, see Descent, Lemma 35.23.1,
(3) universally closed, see Descent, Lemma 35.23.3,
(4) (quasi-)separated, see Descent, Lemmas 35.23.2, and 35.23.6,
(5) monomorphism, see Descent, Lemma 35.23.33,
(6) surjective, see Descent, Lemma 35.23.7,
(7) universally injective, see Descent, Lemma 35.23.10,
(8) affine, see Descent, Lemma 35.23.20,
(9) quasi-affine, see Descent, Lemma 35.23.22,
(10) (locally) of finite type, see Descent, Lemmas 35.23.12, and 35.23.14,
(11) (locally) quasi-finite, see Descent, Lemma 35.23.26,
(12) (locally) of finite presentation, see Descent, Lemmas 35.23.13, and 35.23.15,
(13) locally of finite type of relative dimension d, see Descent, Lemma 35.23.27,
(14) universally open, see Descent, Lemma 35.23.4,
(15) flat, see Descent, Lemma 35.23.17,
(16) syntomic, see Descent, Lemma 35.23.28,
(17) smooth, see Descent, Lemma 35.23.29,
(18) unramified (resp. G-unramified), see Descent, Lemma 35.23.30,
(19) étale, see Descent, Lemma 35.23.31,
(20) proper, see Descent, Lemma 35.23.16,
(21) finite or integral, see Descent, Lemma 35.23.25,
(22) finite locally free, see Descent, Lemma 35.23.32,
(23) universally submersive, see Descent, Lemma 35.23.5,
(24) universal homeomorphism, see Descent, Lemma 35.23.11.

Note that the property of being an "immersion" may not be fpqc local on the base, but in Descent, Lemma 35.24.1 we proved that it is fppf local on the base.

65.5. Properties of representable morphisms of presheaves

02WI Here is the definition that makes this work.

025V **Definition 65.5.1.** With S, and $a : F \to G$ representable as above. Let $\mathcal{P}$ be a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 26.18.3, and
(2) is fppf local on the base, see Descent, Definition 35.22.1.

In this case we say that a has *property* $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the resulting morphism of schemes $V_\xi \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

02YN **Remark 65.5.2.** Consider the property $\mathcal{P}$ ="surjective". In this case there could be some ambiguity if we say "let $F \to G$ be a surjective map". Namely, we could mean the notion defined in Definition 65.5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 7.3.1, or, if both F and G are sheaves, we could mean a surjective map of sheaves, see Sites, Definition 7.11.1. If not mentioned otherwise when discussing morphisms of algebraic spaces we will always mean the first. See Lemma 65.5.9 for a case where surjectivity implies surjectivity as a map of sheaves.

Here is a sanity check.

02WJ **Lemma 65.5.3.** *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{P}$ be as in Definition 65.5.1. Then $h_X \longrightarrow h_Y$ has property $\mathcal{P}$ if and only if f has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 65.3.1. Proof omitted. □

02WK **Lemma 65.5.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1 which is stable under composition. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. If a and b have property $\mathcal{P}$ so does $b \circ a : F \longrightarrow H$.*

Proof. Note that the lemma makes sense by Lemma 65.3.2. Proof omitted. □

02WL **Lemma 65.5.5.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. Let $a : F \to G$ be a representable transformation of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{b'} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

If a has property $\mathcal{P}$ then also the base change a' has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 65.3.3. Proof omitted. □

03KD **Lemma 65.5.6.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. Let $a : F \to G$ be a representable transformation of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{b'} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

Assume that b induces a surjective map of fppf sheaves $H^\# \to G^\#$. In this case, if a' has property $\mathcal{P}$, then also a has property $\mathcal{P}$.

Proof. First we remark that by Lemma 65.3.3 the transformation a' is representable. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\xi \in G(U)$. By assumption there exists

an fppf covering $\{U_i \to U\}_{i\in I}$ and elements $\xi_i \in H(U_i)$ mapping to $\xi|_U$ via b. From general category theory it follows that for each i we have a fibre product diagram

$$\begin{array}{ccc} U_i \times_{\xi_i,H,a'} (H \times_{b,G,a} F) & \longrightarrow & U \times_{\xi,G,a} F \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array}$$

By assumption the left vertical arrow is a morphism of schemes which has property $\mathcal{P}$. Since $\mathcal{P}$ is local in the fppf topology this implies that also the right vertical arrow has property $\mathcal{P}$ as desired. □

02WM **Lemma 65.5.7.** *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Let $\mathcal{P}$ be a property as in Definition 65.5.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 65.3.4. Proof omitted. □

02YO **Lemma 65.5.8.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 65.5.1. Suppose that for any morphism of schemes $f : X \to Y$ we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$ then a has property $\mathcal{P}'$.*

Proof. Formal. □

05VM **Lemma 65.5.9.** *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be sheaves. Let $a : F \to G$ be representable, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is (representable by) a scheme and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective. Hence $\{T' \to T\}$ is an fppf covering such that $g|_{T'} \in G(T')$ comes from an element of $F(T')$, namely the map $T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.11.1. □

Here is a characterization of those functors for which the diagonal is representable.

025W **Lemma 65.5.10.** *Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. The following are equivalent:*

1. *the diagonal $F \to F \times F$ is representable,*
2. *for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $a \in F(U)$ the map $a : h_U \to F$ is representable,*
3. *for every pair $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $a \in F(U)$, $b \in F(V)$ the fibre product $h_U \times_{a,F,b} h_V$ is representable.*

Proof. This is completely formal, see Categories, Lemma 4.8.4. It depends only on the fact that the category $(Sch/S)_{fppf}$ has products of pairs of objects and fibre products, see Topologies, Lemma 34.7.10. □

In the situation of the lemma, for any morphism $\xi : h_U \to F$ as in the lemma, it makes sense to say that ξ has property $\mathcal{P}$, for any property as in Definition 65.5.1. In particular this holds for $\mathcal{P}$ = "surjective" and $\mathcal{P}$ = "étale", see Remark 65.4.3 above. We will use this remark in the definition of algebraic spaces below.

0CB7 **Lemma 65.5.11.** *Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 65.5.1. If for every $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and $a \in F(U)$, $b \in F(V)$ we have*

(1) *$h_U \times_{a,F,b} h_V$ is representable, say by the scheme W, and*
(2) *the morphism $W \to U \times_S V$ corresponding to $h_U \times_{a,F,b} h_V \to h_U \times h_V$ has property $\mathcal{P}$,*

then $\Delta : F \to F \times F$ is representable and has property $\mathcal{P}$.

Proof. Observe that Δ is representable by Lemma 65.5.10. We can formulate condition (2) as saying that the transformation $h_U \times_{a,F,b} h_V \to h_{U \times_S V}$ has property $\mathcal{P}$, see Lemma 65.5.3. Consider $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and $(a, b) \in (F \times F)(T)$. Observe that we have the commutative diagram

$$\begin{array}{ccc} F \times_{\Delta, F\times F, (a,b)} h_T & \longrightarrow & h_T \\ \downarrow & & \downarrow{\scriptstyle \Delta_{T/S}} \\ h_T \times_{a,F,b} h_T & \longrightarrow & h_{T \times_S T} \\ \downarrow & & \downarrow{\scriptstyle (a,b)} \\ F & \xrightarrow{\Delta} & F \times F \end{array}$$

both of whose squares are cartesian. In this way we see that the morphism $F \times_{F\times F} h_T \to h_T$ is the base change of a morphism having property $\mathcal{P}$ by $\Delta_{T/S}$. Since $\mathcal{P}$ is preserved under base change this finishes the proof. □

65.6. Algebraic spaces

025X Here is the definition.

025Y **Definition 65.6.1.** Let S be a scheme contained in Sch_{fppf}. An *algebraic space over S* is a presheaf

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Sets$$

with the following properties

(1) The presheaf F is a sheaf.
(2) The diagonal morphism $F \to F \times F$ is representable.
(3) There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a map $h_U \to F$ which is surjective, and étale.

There are two differences with the "usual" definition, for example the definition in Knutson's book [**Knu71**].

The first is that we require F to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Michael Artin's results on algebraic spaces. Built into his method is a condition which guarantees the result is locally of finite presentation over S. Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the category of algebraic spaces ends up being the same. See Bootstrap, Section 80.12.

The second is that we only require the diagonal map for F to be representable, whereas in [**Knu71**] it is required that it also be quasi-compact. If $F = h_U$ for

some scheme U over S this corresponds to the condition that U be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of F be representable, and simply add an additional hypothesis wherever this is necessary. In any case it has the pleasing consequence that the following lemma is true.

025Z **Lemma 65.6.2.** *A scheme is an algebraic space. More precisely, given a scheme $T \in \mathrm{Ob}((Sch/S)_{fppf})$ the representable functor h_T is an algebraic space.*

Proof. The functor h_T is a sheaf by our remarks in Section 65.2. The diagonal $h_T \to h_T \times h_T = h_{T \times T}$ is representable because $(Sch/S)_{fppf}$ has fibre products. The identity map $h_T \to h_T$ is surjective étale. □

0260 **Definition 65.6.3.** Let F, F' be algebraic spaces over S. A *morphism $f : F \to F'$ of algebraic spaces over S* is a transformation of functors from F to F'.

The category of algebraic spaces over S contains the category $(Sch/S)_{fppf}$ as a full subcategory via the Yoneda embedding $T/S \mapsto h_T$. From now on we no longer distinguish between a scheme T/S and the algebraic space it represents. Thus when we say "Let $f : T \to F$ be a morphism from the scheme T to the algebraic space F", we mean that $T \in \mathrm{Ob}((Sch/S)_{fppf})$, that F is an algebraic space over S, and that $f : h_T \to F$ is a morphism of algebraic spaces over S.

65.7. Fibre products of algebraic spaces

04T8 The category of algebraic spaces over S has both products and fibre products.

02X0 **Lemma 65.7.1.** *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Then $F \times G$ is an algebraic space, and is a product in the category of algebraic spaces over S.*

Proof. It is clear that $H = F \times G$ is a sheaf. The diagonal of H is simply the product of the diagonals of F and G. Hence it is representable by Lemma 65.3.4. Finally, if $U \to F$ and $V \to G$ are surjective étale morphisms, with $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, then $U \times V \to F \times G$ is surjective étale by Lemma 65.5.7. □

04T9 **Lemma 65.7.2.** *Let S be a scheme contained in Sch_{fppf}. Let H be a sheaf on $(Sch/S)_{fppf}$ whose diagonal is representable. Let F, G be algebraic spaces over S. Let $F \to H$, $G \to H$ be maps of sheaves. Then $F \times_H G$ is an algebraic space.*

Proof. We check the 3 conditions of Definition 65.6.1. A fibre product of sheaves is a sheaf, hence $F \times_H G$ is a sheaf. The diagonal of $F \times_H G$ is the left vertical arrow in

$$\begin{array}{ccc} F \times_H G & \longrightarrow & F \times G \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta_F \times \Delta_G} \\ (F \times F) \times_{(H \times H)} (G \times G) & \longrightarrow & (F \times F) \times (G \times G) \end{array}$$

which is cartesian. Hence Δ is representable as the base change of the morphism on the right which is representable, see Lemmas 65.3.4 and 65.3.3. Finally, let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and $a : U \to F$, $b : V \to G$ be surjective and étale. As Δ_H is representable, we see that $U \times_H V$ is a scheme. The morphism

$$U \times_H V \longrightarrow F \times_H G$$

is surjective and étale as a composition of the base changes $U \times_H V \to U \times_H G$ and $U \times_H G \to F \times_H G$ of the étale surjective morphisms $U \to F$ and $V \to G$, see Lemmas 65.3.2 and 65.3.3. This proves the last condition of Definition 65.6.1 holds and we conclude that $F \times_H G$ is an algebraic space. □

02X2 **Lemma 65.7.3.** *Let S be a scheme contained in Sch_{fppf}. Let $F \to H$, $G \to H$ be morphisms of algebraic spaces over S. Then $F \times_H G$ is an algebraic space, and is a fibre product in the category of algebraic spaces over S.*

Proof. It follows from the stronger Lemma 65.7.2 that $F \times_H G$ is an algebraic space. It is clear that $F \times_H G$ is a fibre product in the category of algebraic spaces over S since that is a full subcategory of the category of (pre)sheaves of sets on $(Sch/S)_{fppf}$. □

65.8. Glueing algebraic spaces

02WN In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

0F15 **Lemma 65.8.1.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F and G be sheaves on $(Sch/S)^{opp}_{fppf}$ and denote $F \amalg G$ the coproduct in the category of sheaves. The map $F \to F \amalg G$ is representable by open and closed immersions.*

Proof. Let U be a scheme and let $\xi \in (F \amalg G)(U)$. Recall the coproduct in the category of sheaves is the sheafification of the coproduct presheaf (Sites, Lemma 7.10.14). Thus there exists an fppf covering $\{g_i : U_i \to U\}_{i \in I}$ and a disjoint union decomposition $I = I' \amalg I''$ such that $U_i \to U \to F \amalg G$ factors through F, resp. G if and only if $i \in I'$, resp. $i \in I''$. Since F and G have empty intersection in $F \amalg G$ we conclude that $U_i \times_U U_j$ is empty if $i \in I'$ and $j \in I''$. Hence $U' = \bigcup_{i \in I'} g_i(U_i)$ and $U'' = \bigcup_{i \in I''} g_i(U_i)$ are disjoint open (Morphisms, Lemma 29.25.10) subschemes of U with $U = U' \amalg U''$. We omit the verification that $U' = U \times_{F \amalg G} F$. □

02WO **Lemma 65.8.2.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $U \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is representable by an open and closed subscheme U_i and $U \cong \coprod U_i$ as schemes.*

Proof. By Lemma 65.8.1 the map $F_i \to U$ is representable by open and closed immersions. Hence F_i is representable by an open and closed subscheme U_i of U. We have $U = \coprod U_i$ because we have $U \cong \coprod F_i$ as sheaves and we can test the equality on points. □

02WP **Lemma 65.8.3.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be an algebraic space over S. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $F \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is an algebraic space over S.*

Proof. The representability of $F \to F \times F$ implies that each diagonal morphism $F_i \to F_i \times F_i$ is representable (immediate from the definitions and the fact that $F \times_{(F \times F)} (F_i \times F_i) = F_i$). Choose a scheme U in $(Sch/S)_{fppf}$ and a surjective étale morphism $U \to F$ (this exist by hypothesis). The base change $U \times_F F_i \to F_i$ is surjective and étale by Lemma 65.5.5. On the other hand, $U \times_F F_i$ is a scheme by Lemma 65.8.1. Thus we have verified all the conditions in Definition 65.6.1 and F_i is an algebraic space. □

The condition on the size of I and the F_i in the following lemma may be ignored by those not worried about set theoretic questions.

02WQ **Lemma 65.8.4.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Suppose given a set I and algebraic spaces F_i, $i \in I$. Then $F = \coprod_{i \in I} F_i$ is an algebraic space provided I, and the F_i are not too "large": for example if we can choose surjective étale morphisms $U_i \to F_i$ such that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$, then F is an algebraic space.*

Proof. By construction F is a sheaf. We omit the verification that the diagonal morphism of F is representable. Finally, if U is an object of $(Sch/S)_{fppf}$ isomorphic to $\coprod_{i \in I} U_i$ then it is straightforward to verify that the resulting map $U \to \coprod F_i$ is surjective and étale. □

Here is the analogue of Schemes, Lemma 26.15.4.

02WR **Lemma 65.8.5.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

(1) *F is a sheaf,*
(2) *there exists an index set I and subfunctors $F_i \subset F$ such that*
   (a) *each F_i is an algebraic space,*
   (b) *each $F_i \to F$ is representable,*
   (c) *each $F_i \to F$ is an open immersion (see Definition 65.5.1),*
   (d) *the map $\coprod F_i \to F$ is surjective as a map of sheaves, and*
   (e) *$\coprod F_i$ is an algebraic space (set theoretic condition, see Lemma 65.8.4).*

Then F is an algebraic space.

Proof. Let T be an object of $(Sch/S)_{fppf}$. Let $T \to F$ be a morphism. By assumption (2)(b) and (2)(c) the fibre product $F_i \times_F T$ is representable by an open subscheme $V_i \subset T$. It follows that $(\coprod F_i) \times_F T$ is represented by the scheme $\coprod V_i$ over T. By assumption (2)(d) there exists an fppf covering $\{T_j \to T\}_{j \in J}$ such that $T_j \to T \to F$ factors through F_i, $i = i(j)$. Hence $T_j \to T$ factors through the open subscheme $V_{i(j)} \subset T$. Since $\{T_j \to T\}$ is jointly surjective, it follows that $T = \bigcup V_i$ is an open covering. In particular, the transformation of functors $\coprod F_i \to F$ is representable and surjective in the sense of Definition 65.5.1 (see Remark 65.5.2 for a discussion).

Next, let $T' \to F$ be a second morphism from an object in $(Sch/S)_{fppf}$. Write as above $T' = \bigcup V'_i$ with $V'_i = T' \times_F F_i$. To show that the diagonal $F \to F \times F$ is representable we have to show that $G = T \times_F T'$ is representable, see Lemma 65.5.10. Consider the subfunctors $G_i = G \times_F F_i$. Note that $G_i = V_i \times_{F_i} V'_i$, and hence is representable as F_i is an algebraic space. By the above the G_i form a Zariski covering of G. Hence by Schemes, Lemma 26.15.4 we see G is representable.

Choose a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to \coprod F_i$ (this exists by hypothesis). We may write $U = \coprod U_i$ with U_i the inverse image of F_i, see Lemma 65.8.2. We claim that $U \to F$ is surjective and étale. Surjectivity follows as $\coprod F_i \to F$ is surjective (see first paragraph of the proof) by applying Lemma 65.5.4. Consider the fibre product $U \times_F T$ where $T \to F$ is as above. We have to show that $U \times_F T \to T$ is étale. Since $U \times_F T = \coprod U_i \times_F T$ it suffices to show each $U_i \times_F T \to T$ is étale. Since $U_i \times_F T = U_i \times_{F_i} V_i$ this follows from the fact that $U_i \to F_i$ is étale and $V_i \to T$ is an open immersion (and Morphisms, Lemmas 29.36.9 and 29.36.3). □

65.9. Presentations of algebraic spaces

0261 Given an algebraic space we can find a "presentation" of it.

0262 **Lemma 65.9.1.** *Let F be an algebraic space over S. Let $f : U \to F$ be a surjective étale morphism from a scheme to F. Set $R = U \times_F U$. Then*

(1) *$j : R \to U \times_S U$ defines an equivalence relation on U over S (see Groupoids, Definition 39.3.1).*
(2) *the morphisms $s, t : R \to U$ are étale, and*
(3) *the diagram*
$$R \rightrightarrows U \longrightarrow F$$
is a coequalizer diagram in $Sh((Sch/S)_{fppf})$.

Proof. Let T/S be an object of $(Sch/S)_{fppf}$. Then $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$ which defines an equivalence relation on $U(T)$. The morphisms $s, t : R \to U$ are étale because the morphism $U \to F$ is étale.

To prove (3) we first show that $U \to F$ is a surjection of sheaves, see Sites, Definition 7.11.1. Let $\xi \in F(T)$ with T as above. Let $V = T \times_{\xi, F, f} U$. By assumption V is a scheme and $V \to T$ is surjective étale. Hence $\{V \to T\}$ is a covering for the fppf topology. Since $\xi|_V$ factors through U by construction we conclude $U \to F$ is surjective. Surjectivity implies that F is the coequalizer of the diagram by Sites, Lemma 7.11.3. □

This lemma suggests the following definitions.

02WS **Definition 65.9.2.** Let S be a scheme. Let U be a scheme over S. An *étale equivalence relation* on U over S is an equivalence relation $j : R \to U \times_S U$ such that $s, t : R \to U$ are étale morphisms of schemes.

0263 **Definition 65.9.3.** Let F be an algebraic space over S. A *presentation* of F is given by a scheme U over S and an étale equivalence relation R on U over S, and a surjective étale morphism $U \to F$ such that $R = U \times_F U$.

Equivalently we could ask for the existence of an isomorphism
$$U/R \cong F$$
where the quotient U/R is as defined in Groupoids, Section 39.20. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf U/R is an algebraic space. It will finally turn out this is always the case if R is an étale equivalence relation on U over S, see Theorem 65.10.5.

65.10. Algebraic spaces and equivalence relations

0264 Suppose given a scheme U over S and an étale equivalence relation R on U over S. We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf U/R (see Groupoids, Definition 39.20.1) has all the properties required of it in Definition 65.6.1.

02WT **Lemma 65.10.1.** *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Let $U' \to U$ be an étale morphism. Let R' be the restriction of R to U', see Groupoids, Definition 39.3.3. Then $j' : R' \to U' \times_S U'$ is an étale equivalence relation also.*

Proof. It is clear from the description of s', t' in Groupoids, Lemma 39.18.1 that $s', t' : R' \to U'$ are étale as compositions of base changes of étale morphisms (see Morphisms, Lemma 29.36.4 and 29.36.3). □

We will often use the following lemma to find open subspaces of algebraic spaces. A slight improvement (with more general hypotheses) of this lemma is Bootstrap, Lemma 80.7.1.

02WU **Lemma 65.10.2.** *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism. Assume*

(1) *j is an equivalence relation,*
(2) *$s, t : R \to U$ are surjective, flat and locally of finite presentation,*
(3) *g is flat and locally of finite presentation.*

Let $R' = R|_{U'}$ be the restriction of R to U'. Then $U'/R' \to U/R$ is representable, and is an open immersion.

Proof. By Groupoids, Lemma 39.3.2 the morphism $j' = (s', t') : R' \to U' \times_S U'$ defines an equivalence relation. Since g is flat and locally of finite presentation we see that g is universally open as well (Morphisms, Lemma 29.25.10). For the same reason s, t are universally open as well. Let $W^1 = g(U') \subset U$, and let $W = t(s^{-1}(W^1))$. Then W^1 and W are open in U. Moreover, as j is an equivalence relation we have $t(s^{-1}(W)) = W$ (see Groupoids, Lemma 39.19.2 for example).

By Groupoids, Lemma 39.20.5 the map of sheaves $F' = U'/R' \to F = U/R$ is injective. Let $a : T \to F$ be a morphism from a scheme into U/R. We have to show that $T \times_F F'$ is representable by an open subscheme of T.

The morphism a is given by the following data: an fppf covering $\{\varphi_j : T_j \to T\}_{j \in J}$ of T and morphisms $a_j : T_j \to U$ such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through $j : R \to U \times_S U$ via some (unique) maps $r_{jj'} : T_j \times_T T_{j'} \to R$. The system (a_j) corresponds to a in the sense that the diagrams

$$\begin{array}{ccc} T_j & \xrightarrow{a_j} & U \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & F \end{array}$$

commute.

Consider the open subsets $W_j = a_j^{-1}(W) \subset T_j$. Since $t(s^{-1}(W)) = W$ we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Descent, Lemma 35.13.6 this means there exists an open $W_T \subset T$ such that $\varphi_j^{-1}(W_T) = W_j$ for all $j \in J$. We claim that $W_T \to T$ represents $T \times_F F' \to T$.

First, let us show that $W_T \to T \to F$ is an element of $F'(W_T)$. Since $\{W_j \to W_T\}_{j \in J}$ is an fppf covering of W_T, it is enough to show that each $W_j \to U \to F$ is an element of $F'(W_j)$ (as F' is a sheaf for the fppf topology). Consider the

commutative diagram

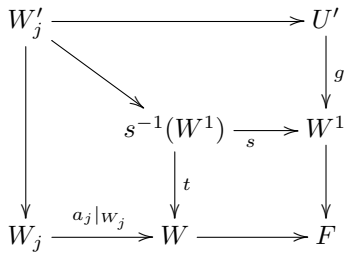

where $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$. Since t and g are surjective, flat and locally of finite presentation, so is $W'_j \to W_j$. Hence the restriction of the element $W_j \to U \to F$ to W'_j is an element of F' as desired.

Suppose that $f : T' \to T$ is a morphism of schemes such that $a|_{T'} \in F'(T')$. We have to show that f factors through the open W_T. Since $\{T' \times_T T_j \to T'\}$ is an fppf covering of T' it is enough to show each $T' \times_T T_j \to T$ factors through W_T. Hence we may assume f factors as $\varphi_j \circ f_j : T' \to T_j \to T$ for some j. In this case the condition $a|_{T'} \in F'(T')$ means that there exists some fppf covering $\{\psi_i : T'_i \to T'\}_{i \in I}$ and some morphisms $b_i : T'_i \to U'$ such that

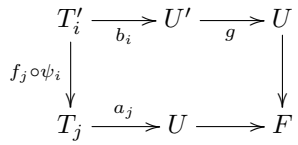

is commutative. This commutativity means that there exists a morphism $r'_i : T'_i \to R$ such that $t \circ r'_i = a_j \circ f_j \circ \psi_i$, and $s \circ r'_i = g \circ b_i$. This implies that $\mathrm{Im}(f_j \circ \psi_i) \subset W_j$ and we win. □

The following lemma is not completely trivial although it looks like it should be trivial.

02WV **Lemma 65.10.3.** *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. If the quotient U/R is an algebraic space, then $U \to U/R$ is étale and surjective. Hence $(U, R, U \to U/R)$ is a presentation of the algebraic space U/R.*

Proof. Denote $c : U \to U/R$ the morphism in question. Let T be a scheme and let $a : T \to U/R$ be a morphism. We have to show that the morphism (of schemes) $\pi : T \times_{a,U/R,c} U \to T$ is étale and surjective. The morphism a corresponds to an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = a \circ \varphi_i$. Hence

$$T_i \times_{\varphi_i,T} T \times_{a,U/R,c} U = T_i \times_{c \circ a_i,U/R,c} U = T_i \times_{a_i,U} U \times_{c,U/R,c} U = T_i \times_{a_i,U,t} R.$$

Since t is étale and surjective we conclude that the base change of π to T_i is surjective and étale. Since the property of being surjective and étale is local on the base in the fpqc topology (see Remark 65.4.3) we win. □

0265 **Lemma 65.10.4.** *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Assume that U is*

affine. Then the quotient $F = U/R$ is an algebraic space, and $U \to F$ is étale and surjective.

Proof. Since $j : R \to U \times_S U$ is a monomorphism we see that j is separated (see Schemes, Lemma 26.23.3). Since U is affine we see that $U \times_S U$ (which comes equipped with a monomorphism into the affine scheme $U \times U$) is separated. Hence we see that R is separated. In particular the morphisms s, t are separated as well as étale.

Since the composition $R \to U \times_S U \to U$ is locally of finite type we conclude that j is locally of finite type (see Morphisms, Lemma 29.15.8). As j is also a monomorphism it has finite fibres and we see that j is locally quasi-finite by Morphisms, Lemma 29.20.7. Altogether we see that j is separated and locally quasi-finite.

Our first step is to show that the quotient map $c : U \to F$ is representable. Consider a scheme T and a morphism $a : T \to F$. We have to show that the sheaf $G = T \times_{a,F,c} U$ is representable. As seen in the proofs of Lemmas 65.10.2 and 65.10.3 there exists an fppf covering $\{\varphi_i : T_i \to T\}_{i \in I}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = a \circ \varphi_i$. As in the proof of Lemma 65.10.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i, T} G &= T_i \times_{\varphi_i, T} T \times_{a, U/R, c} U \\ &= T_i \times_{c \circ a_i, U/R, c} U \\ &= T_i \times_{a_i, U} U \times_{c, U/R, c} U \\ &= T_i \times_{a_i, U, t} R \end{aligned}$$

Since t is separated and étale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 29.35.10 and 29.36.16) we see that the restriction of G to each T_i is representable by a morphism of schemes $X_i \to T_i$ which is separated and locally quasi-finite. By Descent, Lemma 35.39.1 we obtain a descent datum $(X_i, \varphi_{ii'})$ relative to the fppf-covering $\{T_i \to T\}$. Since each $X_i \to T_i$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 37.57.1 that this descent datum is effective. Hence by Descent, Lemma 35.39.1 (2) we conclude that G is representable as desired.

The second step of the proof is to show that $U \to F$ is surjective and étale. This is clear from the above since in the first step above we saw that $G = T \times_{a,F,c} U$ is a scheme over T which base changes to schemes $X_i \to T_i$ which are surjective and étale. Thus $G \to T$ is surjective and étale (see Remark 65.4.3). Alternatively one can reread the proof of Lemma 65.10.3 in the current situation.

The third and final step is to show that the diagonal map $F \to F \times F$ is representable. We first observe that the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is a fibre product square. By Lemma 65.3.4 the morphism $U \times_S U \to F \times F$ is representable (note that $h_U \times h_U = h_{U \times_S U}$). Moreover, by Lemma 65.5.7 the morphism $U \times_S U \to F \times F$ is surjective and étale (note also that étale and surjective occur in the lists of Remarks 65.4.3 and 65.4.2). It follows either from

Lemma 65.3.3 and the diagram above, or by writing $R \to F$ as $R \to U \to F$ and Lemmas 65.3.1 and 65.3.2 that $R \to F$ is representable as well. Let T be a scheme and let $a : T \to F \times F$ be a morphism. We have to show that $G = T \times_{a,F\times F,\Delta} F$ is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F\times F,a} T \longrightarrow T$$

is surjective and étale. Hence $\{T' \to T\}$ is an étale covering of T. Note also that

$$T' \times_T G = T' \times_{U\times_S U,j} R$$

as can be seen contemplating the following cube

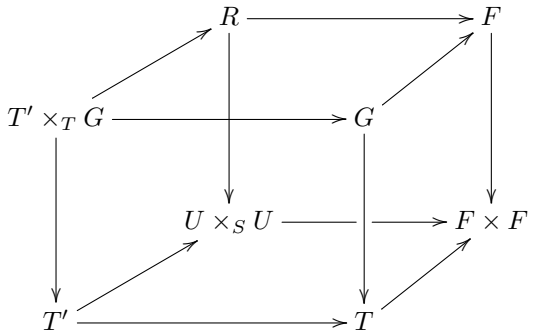

Hence we see that the restriction of G to T' is representable by a scheme X, and moreover that the morphism $X \to T'$ is a base change of the morphism j. Hence $X \to T'$ is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 35.39.1 we obtain a descent datum (X, φ) relative to the fppf-covering $\{T' \to T\}$. Since $X \to T'$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 37.57.1 that this descent datum is effective. Hence by Descent, Lemma 35.39.1 (2) we conclude that G is representable as desired. □

02WW **Theorem 65.10.5.** *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Then the quotient U/R is an algebraic space, and $U \to U/R$ is étale and surjective, in other words $(U, R, U \to U/R)$ is a presentation of U/R.*

Proof. By Lemma 65.10.3 it suffices to prove that U/R is an algebraic space. Let $U' \to U$ be a surjective, étale morphism. Then $\{U' \to U\}$ is in particular an fppf covering. Let R' be the restriction of R to U', see Groupoids, Definition 39.3.3. According to Groupoids, Lemma 39.20.6 we see that $U/R \cong U'/R'$. By Lemma 65.10.1 R' is an étale equivalence relation on U'. Thus we may replace U by U'.

We apply the previous remark to $U' = \coprod U_i$, where $U = \bigcup U_i$ is an affine open covering of U. Hence we may and do assume that $U = \coprod U_i$ where each U_i is an affine scheme.

Consider the restriction R_i of R to U_i. By Lemma 65.10.1 this is an étale equivalence relation. Set $F_i = U_i/R_i$ and $F = U/R$. It is clear that $\coprod F_i \to F$ is surjective. By Lemma 65.10.2 each $F_i \to F$ is representable, and an open immersion. By Lemma 65.10.4 applied to (U_i, R_i) we see that F_i is an algebraic space. Then by Lemma 65.10.3 we see that $U_i \to F_i$ is étale and surjective. From Lemma 65.8.4 it follows that $\coprod F_i$ is an algebraic space. Finally, we have verified all hypotheses of Lemma 65.8.5 and it follows that $F = U/R$ is an algebraic space. □

65.11. Algebraic spaces, retrofitted

02WX We start building our arsenal of lemmas dealing with algebraic spaces. The first result says that in Definition 65.6.1 we can weaken the condition on the diagonal as follows.

0BGQ **Lemma 65.11.1.** *Let S be a scheme contained in Sch_{fppf}. Let F be a sheaf on $(Sch/S)_{fppf}$ such that there exists $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a map $U \to F$ which is representable, surjective, and étale. Then F is an algebraic space.*

Proof. Set $R = U \times_F U$. This is a scheme as $U \to F$ is assumed representable. The projections $s, t : R \to U$ are étale as $U \to F$ is assumed étale. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 65.10.5 the quotient sheaf $F' = U/R$ is an algebraic space and $U \to F'$ is surjective and étale. Again since $R = U \times_F U$ we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of sheaves by Lemma 65.5.9. Thus $F' \to F$ is also surjective and we conclude $F' = F$ is an algebraic space. □

0BGR **Lemma 65.11.2.** *Let S be a scheme contained in Sch_{fppf}. Let G be an algebraic space over S, let F be a sheaf on $(Sch/S)_{fppf}$, and let $G \to F$ be a representable transformation of functors which is surjective and étale. Then F is an algebraic space.*

Proof. Pick a scheme U and a surjective étale morphism $U \to G$. Since G is an algebraic space $U \to G$ is representable. Hence the composition $U \to G \to F$ is representable, surjective, and étale. See Lemmas 65.3.2 and 65.5.4. Thus F is an algebraic space by Lemma 65.11.1. □

02WY **Lemma 65.11.3.** *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $G \to F$ be a representable transformation of functors. Then G is an algebraic space.*

Proof. By Lemma 65.3.5 we see that G is a sheaf. The diagram

$$\begin{array}{ccc} G \times_F G & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle \Delta_F} \\ G \times G & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence we see that $G \times_F G \to G \times G$ is representable by Lemma 65.3.3. By Lemma 65.3.6 we see that $G \to G \times_F G$ is representable. Hence $\Delta_G : G \to G \times G$ is representable as a composition of representable transformations, see Lemma 65.3.2. Finally, let U be an object of $(Sch/S)_{fppf}$ and let $U \to F$ be surjective and étale. By assumption $U \times_F G$ is representable by a scheme U'. By Lemma 65.5.5 the morphism $U' \to G$ is surjective and étale. This verifies the final condition of Definition 65.6.1 and we win. □

02WZ **Lemma 65.11.4.** *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $G \to F$ be a representable morphism. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and $q : U \to F$ surjective and étale. Set $V = G \times_F U$. Finally, let $\mathcal{P}$ be a property of morphisms of schemes as in Definition 65.5.1. Then $G \to F$ has property $\mathcal{P}$ if and only if $V \to U$ has property $\mathcal{P}$.*

Proof. (This lemma follows from Lemmas 65.5.5 and 65.5.6, but we give a direct proof here also.) It is clear from the definitions that if $G \to F$ has property $\mathcal{P}$, then $V \to U$ has property $\mathcal{P}$. Conversely, assume $V \to U$ has property $\mathcal{P}$. Let $T \to F$ be a morphism from a scheme to F. Let $T' = T \times_F G$ which is a scheme since $G \to F$ is representable. We have to show that $T' \to T$ has property $\mathcal{P}$. Consider the commutative diagram of schemes

$$\begin{array}{ccccc} V & \longleftarrow & T \times_F V & \longrightarrow & T \times_F G = T' \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & T \times_F U & \longrightarrow & T \end{array}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property $\mathcal{P}$ as a base change of $V \to U$. Finally, $\{T \times_F U \to T\}$ is a fppf covering as it is surjective étale, and hence we conclude that $T' \to T$ has property $\mathcal{P}$ as it is local on the base in the fppf topology. □

03I2 **Lemma 65.11.5.** *Let S be a scheme contained in Sch_{fppf}. Let $G \to F$ be a transformation of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

1. *$\mathcal{P}$ is preserved under any base change, fppf local on the base, and morphisms of type $\mathcal{P}$ satisfy descent for fppf coverings, see Descent, Definition 35.36.1,*
2. *G is a sheaf,*
3. *F is an algebraic space,*
4. *there exists a $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$ such that $V = G \times_F U$ is representable, and*
5. *$V \to U$ has $\mathcal{P}$.*

Then G is an algebraic space, $G \to F$ is representable and has property $\mathcal{P}$.

Proof. Let T be a scheme and let $T \to F$ be a morphism. Then $U \times_F T \to T$ is surjective étale, hence $\{U \times_F T \to T\}$ is a covering for the étale topology. Consider

$$W = G \times_F (U \times_F T) = V \times_F T = V \times_U (U \times_F T).$$

It is a scheme since F is an algebraic space. The morphism $W \to U \times_F T$ has property $\mathcal{P}$ since it is a base change of $V \to U$. There is an isomorphism

$$\begin{aligned} W \times_T (U \times_F T) &= (G \times_F (U \times_F T)) \times_T (U \times_F T) \\ &= (U \times_F T) \times_T (G \times_F (U \times_F T)) \\ &= (U \times_F T) \times_T W \end{aligned}$$

over $(U \times_F T) \times_T (U \times_F T)$. The middle equality maps $((g, (u_1, t)), (u_2, t))$ to $((u_1, t), (g, (u_2, t)))$. This defines a descent datum for $W/U \times_F T/T$, see Descent, Definition 35.34.1. This follows from Descent, Lemma 35.39.1. Namely we have a sheaf $G \times_F T$, whose base change to $U \times_F T$ is represented by W and the isomorphism above is the one from the proof of Descent, Lemma 35.39.1. By assumption on $\mathcal{P}$ the descent datum above is representable. Hence by the last statement of Descent, Lemma 35.39.1 we see that $G \times_F T$ is representable. This proves that $G \to F$ is a representable transformation of functors.

As $G \to F$ is representable, we see that G is an algebraic space by Lemma 65.11.3. The fact that $G \to F$ has property $\mathcal{P}$ now follows from Lemma 65.11.4. □

02X1 **Lemma 65.11.6.** *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $a : F \to G$ be a morphism. Given any $V \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $q : V \to G$ there exists a $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{a} & G \end{array}$$

with p surjective and étale.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ with surjective étale morphism $W \to F$. Next, put $U = W \times_G V$. Since G is an algebraic space we see that U is isomorphic to an object of $(Sch/S)_{fppf}$. As q is surjective étale, we see that $U \to W$ is surjective étale (see Lemma 65.5.5). Thus $U \to F$ is surjective étale as a composition of surjective étale morphisms (see Lemma 65.5.4). □

65.12. Immersions and Zariski coverings of algebraic spaces

02YT At this point an interesting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 65.5.1) but we have yet to define the notion of a *point*[1]. Thus the *Zariski topology* of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

02YU **Definition 65.12.1.** Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S.

(1) A morphism of algebraic spaces over S is called an *open immersion* if it is representable, and an open immersion in the sense of Definition 65.5.1.
(2) An *open subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an open immersion.
(3) A morphism of algebraic spaces over S is called a *closed immersion* if it is representable, and a closed immersion in the sense of Definition 65.5.1.
(4) A *closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is a closed immersion.
(5) A morphism of algebraic spaces over S is called an *immersion* if it is representable, and an immersion in the sense of Definition 65.5.1.
(6) A *locally closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 26.23.8 and Lemma 65.5.8), and hence the image of an immersion $G \to F$ of algebraic spaces is a subfunctor $F' \subset F$ which is (canonically) isomorphic to G. Thus some of the discussion of Schemes, Section 26.10 carries over to the setting of algebraic spaces.

02YV **Lemma 65.12.2.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

[1] We will associate a topological space to an algebraic space in Properties of Spaces, Section 66.4, and its opens will correspond exactly to the open subspaces defined below.

Proof. See Lemma 65.5.4 and Remarks 65.4.3 (see very last line of that remark) and 65.4.2. □

02YW **Lemma 65.12.3.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 65.5.5 and Remark 65.4.3 (see very last line of that remark). □

02YX **Lemma 65.12.4.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. Let F_1, F_2 be locally closed subspaces of F. If $F_1 \subset F_2$ as subfunctors of F, then F_1 is a locally closed subspace of F_2. Similarly for closed and open subspaces.*

Proof. Let $T \to F_2$ be a morphism with T a scheme. Since $F_2 \to F$ is a monomorphism, we see that $T \times_{F_2} F_1 = T \times_F F_1$. The lemma follows formally from this. □

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 65.8.5 we have already encountered such open coverings as a method for constructing algebraic spaces.

02YY **Definition 65.12.5.** Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *Zariski covering* $\{F_i \subset F\}_{i \in I}$ of F is given by a set I and a collection of open subspaces $F_i \subset F$ such that $\coprod F_i \to F$ is a surjective map of sheaves.

Note that if T is a schemes, and $a : T \to F$ is a morphism, then each of the fibre products $T \times_F F_i$ is identified with an open subscheme $T_i \subset T$. The final condition of the definition signifies exactly that $T = \bigcup_{i \in I} T_i$.

It is clear that the collection F_{Zar} of open subspaces of F is a set (as $(Sch/S)_{fppf}$ is a site, hence a set). Moreover, we can turn F_{Zar} into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions by Lemma 65.12.4). Finally, Definition 65.12.5 provides the notion of a Zariski covering $\{F_i \to F'\}_{i \in I}$ in the category F_{Zar}. Hence, just as in the case of a topological space (see Sites, Example 7.6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space F.

02YZ **Definition 65.12.6.** Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *small Zariski site* F_{Zar} of an algebraic space F is one of the sites described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the étale or fppf site of F to define quasi-coherent sheaves.

65.13. Separation conditions on algebraic spaces

02X3 A separation condition on an algebraic space F is a condition on the diagonal morphism $F \to F \times F$. Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 65.5.1).

02X4 **Lemma 65.13.1.** *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism. Then*

(1) *Δ is locally of finite type,*
(2) *Δ is a monomorphism,*
(3) *Δ is separated, and*
(4) *Δ is locally quasi-finite.*

Proof. Let $F = U/R$ be a presentation of F. As in the proof of Lemma 65.10.4 the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence according to Lemma 65.11.4 it suffices to show that j has the properties listed in the lemma. (Note that each of the properties (1) – (4) occur in the lists of Remarks 65.4.1 and 65.4.3.) Since j is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 26.23.3. As R is an étale equivalence relation we see that $s, t : R \to U$ are étale. Hence s, t are locally of finite type. Then it follows from Morphisms, Lemma 29.15.8 that j is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 29.20.7. □

Here are some common types of separation conditions, relative to the base scheme S. There is also an absolute notion of these conditions which we will discuss in Properties of Spaces, Section 66.3. Moreover, we will discuss separation conditions for a morphism of algebraic spaces in Morphisms of Spaces, Section 67.4.

02X5 **Definition 65.13.2.** Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism.

(1) We say F is *separated over S* if Δ is a closed immersion.
(2) We say F is *locally separated over S*[2] if Δ is an immersion.
(3) We say F is *quasi-separated over S* if Δ is quasi-compact.
(4) We say F is *Zariski locally quasi-separated over S*[3] if there exists a Zariski covering $F = \bigcup_{i \in I} F_i$ such that each F_i is quasi-separated.

Note that if the diagonal is quasi-compact (when F is separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 37.43.2).

65.14. Examples of algebraic spaces

02Z0 In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

02Z1 **Example 65.14.1.** Let k be a field of characteristic $\neq 2$. Let $U = \mathbf{A}^1_k$. Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

[2] In the literature this often refers to quasi-separated and locally separated algebraic spaces.
[3] This definition was suggested by B. Conrad.

where $\Delta = \{(x, x) \mid x \in \mathbf{A}^1_k\}$ and $\Gamma = \{(x, -x) \mid x \in \mathbf{A}^1_k, x \neq 0\}$. It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space by Theorem 65.10.5. Since R is quasi-compact we see that X is quasi-separated. On the other hand, X is not locally separated because the morphism j is not an immersion.

03FN **Example 65.14.2.** Let k be a field. Let k'/k be a degree 2 Galois extension with $\mathrm{Gal}(k'/k) = \{1, \sigma\}$. Let $S = \mathrm{Spec}(k[x])$ and $U = \mathrm{Spec}(k'[x])$. Note that

$$U \times_S U = \mathrm{Spec}((k' \otimes_k k')[x]) = \Delta(U) \amalg \Delta'(U)$$

where $\Delta' = (1, \sigma) : U \to U \times_S U$. Take

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ denotes the k'-rational point whose x-coordinate is zero. It is easy to see that R is an étale equivalence relation on U over S and hence $X = U/R$ is an algebraic space by Theorem 65.10.5. Here are some properties of X (some of which will not make sense until later):

(1) $X \to S$ is an isomorphism over $S \setminus \{0_S\}$,
(2) the morphism $X \to S$ is étale (see Properties of Spaces, Definition 66.16.2)
(3) the fibre 0_X of $X \to S$ over 0_S is isomorphic to $\mathrm{Spec}(k') = 0_U$,
(4) X is not a scheme because if it were, then $\mathcal{O}_{X,0_X}$ would be a local domain $(\mathcal{O}, \mathfrak{m}, \kappa)$ with fraction field $k(x)$, with $x \in \mathfrak{m}$ and residue field $\kappa = k'$ which is impossible,
(5) X is not separated, but it is locally separated and quasi-separated,
(6) there exists a surjective, finite, étale morphism $S' \to S$ such that the base change $X' = S' \times_S X$ is a scheme (namely, if we base change to $S' = \mathrm{Spec}(k'[x])$ then U splits into two copies of S' and X' becomes isomorphic to the affine line with 0 doubled, see Schemes, Example 26.14.3), and
(7) if we think of X as a finite type algebraic space over $\mathrm{Spec}(k)$, then similarly the base change $X_{k'}$ is a scheme but X is not a scheme.

In particular, this gives an example of a descent datum for schemes relative to the covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ which is not effective.

See also Examples, Lemma 110.66.1, which shows that descent data need not be effective even for a projective morphism of schemes. That example gives a smooth separated algebraic space of dimension 3 over $\mathbf{C}$ which is not a scheme.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

02Z2 **Lemma 65.14.3.** *Let $U \to S$ be a morphism of Sch_{fppf}. Let G be an abstract group. Let $G \to Aut_S(U)$ be a group homomorphism. Assume*

(*) *if $u \in U$ is a point, and $g(u) = u$ for some non-identity element $g \in G$, then g induces a nontrivial automorphism of $\kappa(u)$.*

Then

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is an étale equivalence relation and hence

$$F = U/R$$

is an algebraic space by Theorem 65.10.5.

Proof. In the statement of the lemma the symbol $\mathrm{Aut}_S(U)$ denotes the group of automorphisms of U over S. Assume $(*)$ holds. Let us show that

$$j : R = \coprod_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if T is a nonempty scheme, and $h : T \to U$ is a T-valued point such that $g \circ h = g' \circ h$ then $g = g'$. Suppose $T \neq \emptyset$, $h : T \to U$ and $g \circ h = g' \circ h$. Let $t \in T$. Consider the composition $\mathrm{Spec}(\kappa(t)) \to \mathrm{Spec}(\kappa(h(t))) \to U$. Then we conclude that $g^{-1} \circ g'$ fixes $u = h(t)$ and acts as the identity on its residue field. Hence $g = g'$ by $(*)$.

Thus if $(*)$ holds we see that j is a relation (see Groupoids, Definition 39.3.1). Moreover, it is an equivalence relation since on T-valued points for a connected scheme T we see that $R(T) = G \times U(T) \to U(T) \times U(T)$ (recall that we always work over S). Moreover, the morphisms $s, t : R \to U$ are étale since R is a disjoint product of copies of U. This proves that $j : R \to U \times_S U$ is an étale equivalence relation. □

Given a scheme U and an action of a group G on U we say the action of G on U is *free* if condition $(*)$ of Lemma 65.14.3 holds. This is equivalent to the notion of a free action of the constant group scheme G_S on U as defined in Groupoids, Definition 39.10.2. The lemma can be interpreted as saying that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

02Z3 **Definition 65.14.4.** Notation $U \to S$, G, R as in Lemma 65.14.3. If the action of G on U satisfies $(*)$ we say G *acts freely* on the scheme U. In this case the algebraic space U/R is denoted U/G and is called the *quotient of U by G*.

This notation is consistent with the notation U/G introduced in Groupoids, Definition 39.20.1. We will later make sense of the quotient as an algebraic stack without any assumptions on the action whatsoever; when we do this we will use the notation $[U/G]$. Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when G is finite.

02Z4 **Lemma 65.14.5.** *Notation and assumptions as in Lemma 65.14.3. Assume G is finite. Then*

(1) *if $U \to S$ is quasi-separated, then U/G is quasi-separated over S, and*
(2) *if $U \to S$ is separated, then U/G is separated over S.*

Proof. In the proof of Lemma 65.13.1 we saw that it suffices to prove the corresponding properties for the morphism $j : R \to U \times_S U$. If $U \to S$ is quasi-separated, then for every affine open $V \subset U$ which maps into an affine of S the opens $g(V) \cap V$ are quasi-compact. It follows that j is quasi-compact. If $U \to S$ is separated, the diagonal $\Delta_{U/S}$ is a closed immersion. Hence $j : R \to U \times_S U$ is a finite coproduct of closed immersions with disjoint images. Hence j is a closed immersion. □

02Z5 **Lemma 65.14.6.** *Notation and assumptions as in Lemma 65.14.3. If $\mathrm{Spec}(k) \to U/G$ is a morphism, then there exist*

(1) *a finite Galois extension k'/k,*
(2) *a finite subgroup $H \subset G$,*
(3) *an isomorphism $H \to Gal(k'/k)$, and*
(4) *an H-equivariant morphism $\mathrm{Spec}(k') \to U$.*

Conversely, such data determine a morphism $\mathrm{Spec}(k) \to U/G$.

Proof. Consider the fibre product $V = \mathrm{Spec}(k) \times_{U/G} U$. Here is a diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & U/G \end{array}$$

Then V is a nonempty scheme étale over $\mathrm{Spec}(k)$ and hence is a disjoint union $V = \coprod_{i \in I} \mathrm{Spec}(k_i)$ of spectra of fields k_i finite separable over k (Morphisms, Lemma 29.36.7). We have

$$\begin{aligned} V \times_{\mathrm{Spec}(k)} V &= (\mathrm{Spec}(k) \times_{U/G} U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_{U/G} U) \\ &= \mathrm{Spec}(k) \times_{U/G} U \times_{U/G} U \\ &= \mathrm{Spec}(k) \times_{U/G} U \times G \\ &= V \times G \end{aligned}$$

The action of G on U induces an action of $a : G \times V \to V$. The displayed equality means that $G \times V \to V \times_{\mathrm{Spec}(k)} V$, $(g, v) \mapsto (a(g, v), v)$ is an isomorphism. In particular we see that for every i we have an isomorphism $H_i \times \mathrm{Spec}(k_i) \to \mathrm{Spec}(k_i \otimes_k k_i)$ where $H_i \subset G$ is the subgroup of elements fixing $i \in I$. Thus H_i is finite and is the Galois group of k_i/k. We omit the converse construction. □

It follows from this lemma for example that if k'/k is a finite Galois extension, then $\mathrm{Spec}(k')/\mathrm{Gal}(k'/k) \cong \mathrm{Spec}(k)$. What happens if the extension is infinite? Here is an example.

02Z6 **Example 65.14.7.** Let $S = \mathrm{Spec}(\mathbf{Q})$. Let $U = \mathrm{Spec}(\overline{\mathbf{Q}})$. Let $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with obvious action on U. Then by construction property $(*)$ of Lemma 65.14.3 holds and we obtain an algebraic space

$$X = \mathrm{Spec}(\overline{\mathbf{Q}})/G \longrightarrow S = \mathrm{Spec}(\mathbf{Q}).$$

Of course this is totally ridiculous as an approximation of S! Namely, by the Artin-Schreier theorem, see [**Jac64**, Theorem 17, page 316], the only finite subgroups of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are $\{1\}$ and the conjugates of the order two group $\mathrm{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$. Hence, if $\mathrm{Spec}(k) \to X$ is a morphism with k algebraic over $\mathbf{Q}$, then it follows from Lemma 65.14.6 and the theorem just mentioned that either k is $\overline{\mathbf{Q}}$ or isomorphic to $\overline{\mathbf{Q}} \cap \mathbf{R}$.

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of $\mathrm{Spec}(\overline{\mathbf{Q}})$. The following example is much more reasonable in my opinion and may actually occur in "nature".

02Z7 **Example 65.14.8.** Let k be a field of characteristic zero. Let $U = \mathbf{A}^1_k$ and let $G = \mathbf{Z}$. As action we take $n(x) = x + n$, i.e., the action of $\mathbf{Z}$ on the affine line by translation. The only fixed point is the generic point and it is clearly the case that $\mathbf{Z}$ injects into the automorphism group of the field $k(x)$. (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : \mathrm{Spec}(k(x)) \longrightarrow X = \mathbf{A}^1_k/\mathbf{Z}$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism $\mathrm{Spec}(L) \to X$ of the spectrum of a field to X. (Contrary to what happens for schemes, see Schemes, Section 26.13.) In fact, since $\mathbf{Z}$ does not have any nontrivial finite subgroups we see from Lemma 65.14.6 that for any such factorization $k(x) = L$. Finally, γ is not a monomorphism since

$$\mathrm{Spec}(k(x)) \times_{\gamma,X,\gamma} \mathrm{Spec}(k(x)) \cong \mathrm{Spec}(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space X we should consider equivalence classes of morphisms from spectra of fields into X and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

02Z8 **Example 65.14.9.** Let k be a field. Let $A = \prod_{n \in \mathbf{N}} k$ be the infinite product. Set $U = \mathrm{Spec}(A)$ seen as a scheme over $S = \mathrm{Spec}(k)$. Note that the projection maps $\mathrm{pr}_n : A \to k$ define open and closed immersions $f_n : S \to U$. Set

$$R = U \amalg \coprod\nolimits_{(n,m) \in \mathbf{N}^2,\ n \neq m} S$$

with morphism j equal to $\Delta_{U/S}$ on the component U and $j = (f_n, f_m)$ on the component S corresponding to (n, m). It is clear from the remark above that s, t are étale. It is also clear that j is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field k is finite with q elements. Let us first discuss the topological space $|U|$ associated to the scheme U a little bit. All elements of A satisfy $x^q = x$. Hence every residue field of A is isomorphic to k, and all points of U are closed. But the topology on U isn't the discrete topology. Let $u_n \in |U|$ be the point corresponding to f_n. As mentioned above the points u_n are the open points (and hence isolated). This implies there have to be other points since we know U is quasi-compact, see Algebra, Lemma 10.17.8 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element x of A is of the form $x = ue$ where u is a unit and e is an idempotent. Hence a basis for the topology of A consists of open and closed subsets (see Algebra, Lemma 10.21.1.) So the topology on $|U|$ is totally disconnected, but nontrivial. Finally, note that $\{u_n\}$ is dense in $|U|$.

We will later define a topological space $|X|$ associated to X, see Properties of Spaces, Section 66.4. What can we say about $|X|$? It turns out that the map $|U| \to |X|$ is surjective and continuous. All the points u_n map to the same point x_0 of $|X|$, and none of the other points get identified. Since $\{u_n\}$ is dense in $|U|$ we conclude that the closure of x_0 in $|X|$ is $|X|$. In other words $|X|$ is irreducible and x_0 is a generic point of $|X|$. This seems bizarre since also x_0 is the image of a section $S \to X$ of the structure morphism $X \to S$ (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 29.20.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

65.15. Change of big site

03FO In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. Here is a precise statement of the result.

03FP **Lemma 65.15.1.** *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let S be an object of Sch_{fppf}. Let*

$$g : Sh((Sch/S)_{fppf}) \longrightarrow Sh((Sch'/S)_{fppf}),$$
$$f : Sh((Sch'/S)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

be the morphisms of topoi of Topologies, Lemma 34.12.2. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then

1. *if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$ over S, then $f^{-1}F$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$, and*
2. *if F is an algebraic space over S, then $f^{-1}F$ is an algebraic space over S also.*

Proof. Let $X \in \mathrm{Ob}((Sch/S)_{fppf})$. Let us write h_X for the representable sheaf on $(Sch/S)_{fppf}$ associated to X, and h'_X for the representable sheaf on $(Sch'/S)_{fppf}$ associated to X. By the description of f^{-1} in Topologies, Section 34.12 we see that $f^{-1}h_X = h'_X$. This proves (1).

Next, suppose that F is an algebraic space over S. By Lemma 65.9.1 this means that $F = h_U/h_R$ for some étale equivalence relation $R \to U \times_S U$ in $(Sch/S)_{fppf}$. Since f^{-1} is an exact functor we conclude that $f^{-1}F = h'_U/h'_R$. Hence $f^{-1}F$ is an algebraic space over S by Theorem 65.10.5. □

Note that this lemma is purely set theoretical and has virtually no content. Moreover, it is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

04W1 **Lemma 65.15.2.** *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote $Spaces/S$ the category of algebraic spaces over S defined using Sch_{fppf}. Similarly, denote $Spaces'/S$ the category of algebraic spaces over S defined using Sch'_{fppf}. The construction of Lemma 65.15.1 defines a fully faithful functor*

$$Spaces/S \longrightarrow Spaces'/S$$

whose essential image consists of those $X' \in \mathrm{Ob}(Spaces'/S)$ such that there exist $U, R \in \mathrm{Ob}((Sch/S)_{fppf})$[4] and morphisms

$$U \longrightarrow X' \quad \text{and} \quad R \longrightarrow U \times_{X'} U$$

[4]Requiring the existence of R is necessary because of our choice of the function *Bound* in Sets, Equation (3.9.1.1). The size of the fibre product $U \times_{X'} U$ can grow faster than *Bound* in

in $Sh((Sch'/S)_{fppf})$ which are surjective as maps of sheaves (for example if the displayed morphisms are surjective and étale).

Proof. In Sites, Lemma 7.21.8 we have seen that the functor $f^{-1} : Sh((Sch/S)_{fppf}) \to Sh((Sch'/S)_{fppf})$ is fully faithful (see discussion in Topologies, Section 34.12). Hence we see that the displayed functor of the lemma is fully faithful.

Suppose that $X' \in \mathrm{Ob}(Spaces'/S)$ such that there exists $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a map $U \to X'$ in $Sh((Sch'/S)_{fppf})$ which is surjective as a map of sheaves. Let $U' \to X'$ be a surjective étale morphism with $U' \in \mathrm{Ob}((Sch'/S)_{fppf})$. Let $\kappa = \mathrm{size}(U)$, see Sets, Section 3.9. Then U has an affine open covering $U = \bigcup_{i \in I} U_i$ with $|I| \leq \kappa$. Observe that $U' \times_{X'} U \to U$ is étale and surjective. For each i we can pick a quasi-compact open $U'_i \subset U'$ such that $U'_i \times_{X'} U_i \to U_i$ is surjective (because the scheme $U' \times_{X'} U_i$ is the union of the Zariski opens $W \times_{X'} U_i$ for $W \subset U'$ affine and because $U' \times_{X'} U_i \to U_i$ is étale hence open). Then $\coprod_{i \in I} U'_i \to X$ is surjective étale because of our assumption that $U \to X$ and hence $\coprod U_i \to X$ is a surjection of sheaves (details omitted). Because $U'_i \times_{X'} U \to U'_i$ is a surjection of sheaves and because U'_i is quasi-compact, we can find a quasi-compact open $W_i \subset U'_i \times_{X'} U$ such that $W_i \to U'_i$ is surjective as a map of sheaves (details omitted). Then $W_i \to U$ is étale and we conclude that $\mathrm{size}(W_i) \leq \mathrm{size}(U)$, see Sets, Lemma 3.9.7. By Sets, Lemma 3.9.11 we conclude that $\mathrm{size}(U'_i) \leq \mathrm{size}(U)$. Hence $\coprod_{i \in I} U'_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5.

Now let X', $U \to X'$ and $R \to U \times_{X'} U$ be as in the statement of the lemma. In the previous paragraph we have seen that we can find $U' \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U' \to X'$ in $Sh((Sch'/S)_{fppf})$. Then $U' \times_{X'} U \to U'$ is a surjection of sheaves, i.e., we can find an fppf covering $\{U'_i \to U'\}$ such that $U'_i \to U'$ factors through $U' \times_{X'} U \to U'$. By Sets, Lemma 3.9.12 we can find $\tilde{U} \to U'$ which is surjective, flat, and locally of finite presentation, with $\mathrm{size}(\tilde{U}) \leq \mathrm{size}(U')$, such that $\tilde{U} \to U'$ factors through $U' \times_{X'} U \to U'$. Then we consider

$$\begin{array}{ccccc} U' \times_{X'} U' & \longleftarrow & \tilde{U} \times_{X'} \tilde{U} & \longrightarrow & U \times_{X'} U \\ \downarrow & & \downarrow & & \downarrow \\ U' \times_S U' & \longleftarrow & \tilde{U} \times_S \tilde{U} & \longrightarrow & U \times_S U \end{array}$$

The squares are cartesian. We know the objects of the bottom row are represented by objects of $(Sch/S)_{fppf}$. By the result of the argument of the previous paragraph, the same is true for $U \times_{X'} U$ (as we have the surjection of sheaves $R \to U \times_{X'} U$ by assumption). Since $(Sch/S)_{fppf}$ is closed under fibre products (by construction), we see that $\tilde{U} \times_{X'} \tilde{U}$ is represented by an object of $(Sch/S)_{fppf}$. Finally, the map $\tilde{U} \times_{X'} \tilde{U} \to U' \times_{X'} U'$ is a surjection of fppf sheaves as $\tilde{U} \to U'$ is so. Thus we can once more apply the result of the previous paragraph to conclude that $R' = U' \times_{X'} U'$ is represented by an object of $(Sch/S)_{fppf}$. At this point Lemma 65.9.1 and Theorem 65.10.5 imply that $X = h_{U'}/h_{R'}$ is an object of $Spaces/S$ such that $f^{-1}X \cong X'$ as desired. □

terms of the size of U. We can illustrate this by setting $S = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(A[x_i, i \in I])$ and $R = \coprod_{(\lambda_i) \in A^I} \mathrm{Spec}(A[x_i, y_i]/(x_i - \lambda_i y_i))$. In this case the size of R grows like κ^κ where κ is the size of U.

65.16. Change of base scheme

03I3 In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic space over S can be viewed as an algebraic space over S'. And, given an algebraic space F' over S' there is a base change F'_S which is an algebraic space over S. We explain only what happens in case $S \to S'$ is a morphism of the big fppf site under consideration, if only S or S' is contained in the big site, then one first enlarges the big site as in Section 65.15.

03I4 **Lemma 65.16.1.** *Suppose given a big site Sch_{fppf}. Let $g : S \to S'$ be morphism of Sch_{fppf}. Let $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ be the corresponding localization functor. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then*

1. *for a scheme T' over S' we have $j_!F(T'/S') = \coprod_{\varphi : T' \to S} F(T' \xrightarrow{\varphi} S)$,*
2. *if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $j_!F$ is representable by $j(X)$ which is X viewed as a scheme over S', and*
3. *if F is an algebraic space over S, then $j_!F$ is an algebraic space over S', and if $F = U/R$ is a presentation, then $j_!F = j(U)/j(R)$ is a presentation.*

Let F' be a sheaf of sets on $(Sch/S')_{fppf}$. Then

4. *for a scheme T over S we have $j^{-1}F'(T/S) = F'(T/S')$,*
5. *if F' is representable by a scheme $X' \in \mathrm{Ob}((Sch/S')_{fppf})$, then $j^{-1}F'$ is representable, namely by $X'_S = S \times_{S'} X'$, and*
6. *if F' is an algebraic space, then $j^{-1}F'$ is an algebraic space, and if $F' = U'/R'$ is a presentation, then $j^{-1}F' = U'_S/R'_S$ is a presentation.*

Proof. The functors $j_!$, j_* and j^{-1} are defined in Sites, Lemma 7.25.8 where it is also shown that $j = j_{S/S'}$ is the localization of $(Sch/S')_{fppf}$ at the object S/S'. Hence all of the material on localization functors is available for j. The formula in (1) is Sites, Lemma 7.27.1. By definition $j_!$ is the left adjoint to restriction j^{-1}, hence $j_!$ is right exact. By Sites, Lemma 7.25.5 it also commutes with fibre products and equalizers. By Sites, Lemma 7.25.3 we see that $j_!h_X = h_{j(X)}$ hence (2) holds. If F is an algebraic space over S, then we can write $F = U/R$ (Lemma 65.9.1) and we get

$$j_!F = j(U)/j(R)$$

because $j_!$ being right exact commutes with coequalizers, and moreover $j(R) = j(U) \times_{j_!F} j(U)$ as $j_!$ commutes with fibre products. Since the morphisms $j(s), j(t) : j(R) \to j(U)$ are simply the morphisms $s, t : R \to U$ (but viewed as morphisms of schemes over S'), they are still étale. Thus $(j(U), j(R), s, t)$ is an étale equivalence relation. Hence by Theorem 65.10.5 we conclude that $j_!F$ is an algebraic space.

Proof of (4), (5), and (6). The description of j^{-1} is in Sites, Section 7.25. The restriction of the representable sheaf associated to X'/S' is the representable sheaf associated to $X'_S = S \times_{S'} Y'$ by Sites, Lemma 7.27.2. The restriction functor j^{-1} is exact, hence $j^{-1}F' = U'_S/R'_S$. Again by exactness the sheaf R'_S is still an equivalence relation on U'_S. Finally the two maps $R'_S \to U'_S$ are étale as base changes of the étale morphisms $R' \to U'$. Hence $j^{-1}F' = U'_S/R'_S$ is an algebraic space by Theorem 65.10.5 and we win. □

Note how the presentation $j_!F = j(U)/j(R)$ is just the presentation of F but viewed as a presentation by schemes over S'. Hence the following definition makes sense.

03I5 **Definition 65.16.2.** Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site.

(1) If F' is an algebraic space over S', then the *base change of F' to S* is the algebraic space $j^{-1}F'$ described in Lemma 65.16.1. We denote it F'_S.
(2) If F is an algebraic space over S, then F *viewed as an algebraic space over S'* is the algebraic space $j_!F$ over S' described in Lemma 65.16.1. We often simply denote this F; if not then we will write $j_!F$.

The algebraic space $j_!F$ comes equipped with a canonical morphism $j_!F \to S$ of algebraic spaces over S'. This is true simply because the sheaf $j_!F$ maps to h_S (see for example the explicit description in Lemma 65.16.1). In fact, in Sites, Lemma 7.25.4 we have seen that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of pairs $(\mathcal{F}', \mathcal{F}' \to h_S)$ consisting of a sheaf on $(Sch/S')_{fppf}$ and a map of sheaves $\mathcal{F}' \to h_S$. The equivalence assigns to the sheaf $\mathcal{F}$ the pair $(j_!\mathcal{F}, j_!\mathcal{F} \to h_S)$. This, combined with the above, leads to the following result for categories of algebraic spaces.

04SG **Lemma 65.16.3.** *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The construction above give an equivalence of categories*

$$\left\{\begin{array}{c}\text{category of algebraic}\\ \text{spaces over } S\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{category of pairs } (F', F' \to S) \text{ consisting}\\ \text{of an algebraic space } F' \text{ over } S' \text{ and a}\\ \text{morphism } F' \to S \text{ of algebraic spaces over } S'\end{array}\right\}$$

Proof. Let F be an algebraic space over S. The functor from left to right assigns the pair $(j_!F, j_!F \to S)$ to F which is an object of the right hand side by Lemma 65.16.1. Since this defines an equivalence of categories of sheaves by Sites, Lemma 7.25.4 to finish the proof it suffices to show: if F is a sheaf and $j_!F$ is an algebraic space, then F is an algebraic space. To do this, write $j_!F = U'/R'$ as in Lemma 65.9.1 with $U', R' \in \mathrm{Ob}((Sch/S')_{fppf})$. Then the compositions $U' \to j_!F \to S$ and $R' \to j_!F \to S$ are morphisms of schemes over S'. Denote U, R the corresponding objects of $(Sch/S)_{fppf}$. The two morphisms $R' \to U'$ are morphisms over S and hence correspond to morphisms $R \to U$. Since these are simply the same morphisms (but viewed over S) we see that we get an étale equivalence relation over S. As $j_!$ defines an equivalence of categories of sheaves (see reference above) we see that $F = U/R$ and by Theorem 65.10.5 we see that F is an algebraic space. □

The following lemma is a slight rephrasing of the above.

04SH **Lemma 65.16.4.** *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F' be a sheaf on $(Sch/S')_{fppf}$. The following are equivalent:*

(1) *The restriction $F'|_{(Sch/S)_{fppf}}$ is an algebraic space over S, and*
(2) *the sheaf $h_S \times F'$ is an algebraic space over S'.*

Proof. The restriction and the product match under the equivalence of categories of Sites, Lemma 7.25.4 so that Lemma 65.16.3 above gives the result. □

We finish this section with a lemma on a compatibility.

03I6 **Lemma 65.16.5.** *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F be an algebraic space over S. Let T be a scheme over S and let $f : T \to F$ be a morphism over S. Let $f' : T' \to F'$ be the morphism over S' we get from f by applying the equivalence of categories described in Lemma 65.16.3. For any property $\mathcal{P}$ as in Definition 65.5.1 we have $\mathcal{P}(f') \Leftrightarrow \mathcal{P}(f)$.*

Proof. Suppose that U is a scheme over S, and $U \to F$ is a surjective étale morphism. Denote U' the scheme U viewed as a scheme over S'. In Lemma 65.16.1 we have seen that $U' \to F'$ is surjective étale. Since

$$j(T \times_{f,F} U) = T' \times_{f',F'} U'$$

the morphism of schemes $T \times_{f,F} U \to U$ is identified with the morphism of schemes $T' \times_{f',F'} U' \to U'$. It is the same morphism, just viewed over different base schemes. Hence the lemma follows from Lemma 65.11.4. □

65.17. Other chapters

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Schemes

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CHAPTER 66

Properties of Algebraic Spaces

03BO

66.1. Introduction

03BP Please see Spaces, Section 65.1 for a brief introduction to algebraic spaces, and please read some of that chapter for our basic definitions and conventions concerning algebraic spaces. In this chapter we start introducing some basic notions and properties of algebraic spaces. A fundamental reference for the case of quasi-separated algebraic spaces is [**Knu71**].

The discussion is somewhat awkward at times since we made the design decision to first talk about properties of algebraic spaces by themselves, and only later about properties of morphisms of algebraic spaces. We make an exception for this rule regarding *étale morphisms* of algebraic spaces, which we introduce in Section 66.16. But until that section whenever we say a morphism has a certain property, it automatically means the source of the morphism is a scheme (or perhaps the morphism is representable).

Some of the material in the chapter (especially regarding points) will be improved upon in the chapter on decent algebraic spaces.

66.2. Conventions

03BQ The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$. The reason is that we want to avoid confusion when changing base schemes, as in Spaces, Section 65.16.

66.3. Separation axioms

03BR In this section we collect all the "absolute" separation conditions of algebraic spaces. Since in our language any algebraic space is an algebraic space over some definite base scheme, any absolute property of X over S corresponds to a conditions imposed on X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Here is the precise formulation.

03BS **Definition 66.3.1.** (Compare Spaces, Definition 65.13.2.) Consider a big fppf site $Sch_{fppf} = (Sch/\mathrm{Spec}(\mathbf{Z}))_{fppf}$. Let X be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let $\Delta : X \to X \times X$ be the diagonal morphism.

(1) We say X is *separated* if Δ is a closed immersion.
(2) We say X is *locally separated*[1] if Δ is an immersion.

[1] In the literature this often refers to quasi-separated and locally separated algebraic spaces.

(3) We say X is *quasi-separated* if Δ is quasi-compact.
(4) We say X is *Zariski locally quasi-separated*[2] if there exists a Zariski covering $X = \bigcup_{i \in I} X_i$ (see Spaces, Definition 65.12.5) such that each X_i is quasi-separated.

Let S is a scheme contained in Sch_{fppf}, and let X be an algebraic space over S. Then we say X is *separated*, *locally separated*, *quasi-separated*, or *Zariski locally quasi-separated* if X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2) has the corresponding property.

It is true that an algebraic space X over S which is separated (in the absolute sense above) is separated over S (and similarly for the other absolute separation properties above). This will be discussed in great detail in Morphisms of Spaces, Section 67.4. We will see in Lemma 66.6.6 that being Zariski locally separated is independent of the base scheme (hence equivalent to the absolute notion).

03DY **Lemma 66.3.2.** *Let S be a scheme. Let X be an algebraic space over S. We have the following implications among the separation axioms of Definition 66.3.1:*

(1) *separated implies all the others,*
(2) *quasi-separated implies Zariski locally quasi-separated.*

Proof. Omitted. □

0AHR **Lemma 66.3.3.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *X is a quasi-separated algebraic space,*
(2) *for $U \to X$, $V \to X$ with U, V quasi-compact schemes the fibre product $U \times_X V$ is quasi-compact,*
(3) *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is quasi-compact.*

Proof. Using Spaces, Lemma 65.16.3 we see that we may assume $S = \mathrm{Spec}(\mathbf{Z})$. Since $U \times_X V = X \times_{X \times X} (U \times V)$ and since $U \times V$ is quasi-compact if U and V are so, we see that (1) implies (2). It is clear that (2) implies (3). Assume (3). Choose a scheme W and a surjective étale morphism $W \to X$. Then $W \times W \to X \times X$ is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \to W \times W$$

is quasi-compact, see Spaces, Lemma 65.5.6. If $U \subset W$ and $V \subset W$ are affine opens, then $j^{-1}(U \times V) = U \times_X V$ is quasi-compact by assumption. Since the affine opens $U \times V$ form an affine open covering of $W \times W$ (Schemes, Lemma 26.17.4) we conclude by Schemes, Lemma 26.19.2. □

0AHS **Lemma 66.3.4.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *X is a separated algebraic space,*
(2) *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is affine and*
$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$
is surjective.

[2] This notion was suggested by B. Conrad.

Proof. Using Spaces, Lemma 65.16.3 we see that we may assume $S = \mathrm{Spec}(\mathbf{Z})$. Since $U \times_X V = X \times_{X \times X} (U \times V)$ and since $U \times V$ is affine if U and V are so, we see that (1) implies (2). Assume (2). Choose a scheme W and a surjective étale morphism $W \to X$. Then $W \times W \to X \times X$ is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \to W \times W$$

is a closed immersion, see Spaces, Lemma 65.5.6. If $U \subset W$ and $V \subset W$ are affine opens, then $j^{-1}(U \times V) = U \times_X V$ is affine by assumption and the map $U \times_X V \to U \times V$ is a closed immersion because the corresponding ring map is surjective. Since the affine opens $U \times V$ form an affine open covering of $W \times W$ (Schemes, Lemma 26.17.4) we conclude by Morphisms, Lemma 29.2.1. □

66.4. Points of algebraic spaces

03BT As is clear from Spaces, Example 65.14.8 a point of an algebraic space should not be defined as a monomorphism from the spectrum of a field. Instead we define them as equivalence classes of morphisms of spectra of fields exactly as explained in Schemes, Section 26.13.

Let S be a scheme. Let F be a presheaf on $(Sch/S)_{fppf}$. Let K be a field. Consider a morphism

$$\mathrm{Spec}(K) \longrightarrow F.$$

By the Yoneda Lemma this is given by an element $p \in F(\mathrm{Spec}(K))$. We say that two such pairs $(\mathrm{Spec}(K), p)$ and $(\mathrm{Spec}(L), q)$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(K) & \xrightarrow{p} & F. \end{array}$$

In other words, there are field extensions $K \to \Omega$ and $L \to \Omega$ such that p and q map to the same element of $F(\mathrm{Spec}(\Omega))$. We omit the verification that this defines an equivalence relation.

03BU **Definition 66.4.1.** Let S be a scheme. Let X be an algebraic space over S. A *point* of X is an equivalence class of morphisms from spectra of fields into X. The set of points of X is denoted $|X|$.

Note that if $f : X \to Y$ is a morphism of algebraic spaces over S, then there is an induced map $|f| : |X| \to |Y|$ which maps a representative $x : \mathrm{Spec}(K) \to X$ to the representative $f \circ x : \mathrm{Spec}(K) \to Y$.

03BV **Lemma 66.4.2.** *Let S be a scheme. Let X be a scheme over S. The points of X as a scheme are in canonical 1-1 correspondence with the points of X as an algebraic space.*

Proof. This is Schemes, Lemma 26.13.3. □

03H4 **Lemma 66.4.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Then the map of sets of points

$$|Z \times_Y X| \longrightarrow |Z| \times_{|Y|} |X|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to X$, $\mathrm{Spec}(L) \to Z$, then the assumption that they agree as elements of $|Y|$ means that there is a common extension M/K and M/L such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to X \to Y$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to Z \to Y$ agree. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to Z \times_Y X$. □

03H5 **Lemma 66.4.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $f : T \to X$ be a morphism from a scheme to X. The following are equivalent*

(1) *$f : T \to X$ is surjective (according to Spaces, Definition 65.5.1), and*
(2) *$|f| : |T| \to |X|$ is surjective.*

Proof. Assume (1). Let $x : \mathrm{Spec}(K) \to X$ be a morphism from the spectrum of a field into X. By assumption the morphism of schemes $\mathrm{Spec}(K) \times_X T \to \mathrm{Spec}(K)$ is surjective. Hence there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_X T$ such that the left square in the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \times_X T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & = & \mathrm{Spec}(K) & \xrightarrow{x} & X \end{array}$$

is commutative. This shows that $|f| : |T| \to |X|$ is surjective.

Assume (2). Let $Z \to X$ be a morphism where Z is a scheme. We have to show that the morphism of schemes $Z \times_X T \to T$ is surjective, i.e., that $|Z \times_X T| \to |Z|$ is surjective. This follows from (2) and Lemma 66.4.3. □

03BW **Lemma 66.4.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X, see Spaces, Definition 65.9.3. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|X|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that U is a scheme, $p : U \to X$ is a surjective, étale morphism, $R = U \times_X U$ is a scheme and defines an étale equivalence relation on U such that $X = U/R$ as sheaves. By Lemma 66.4.4 we see that $|U| \to |X|$ is surjective. By Lemma 66.4.3 the map

$$|R| \longrightarrow |U| \times_{|X|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|X|$. Combining these two statements we get the result of the lemma. □

03BX **Lemma 66.4.6.** *Let S be a scheme. There exists a unique topology on the sets of points of algebraic spaces over S with the following properties:*

(1) *if X is a scheme over S, then the topology on $|X|$ is the usual one (via the identification of Lemma 66.4.2),*
(2) *for every morphism of algebraic spaces $X \to Y$ over S the map $|X| \to |Y|$ is continuous, and*
(3) *for every étale morphism $U \to X$ with U a scheme the map of topological spaces $|U| \to |X|$ is continuous and open.*

Proof. Let X be an algebraic space over S. Let $p : U \to X$ be a surjective étale morphism where U is a scheme over S. We define $W \subset |X|$ is open if and only if $|p|^{-1}(W)$ is an open subset of $|U|$. This is a topology on $|X|$ (it is the quotient topology on $|X|$, see Topology, Lemma 5.6.2).

Let us prove that the topology is independent of the choice of the presentation. To do this it suffices to show that if U' is a scheme, and $U' \to X$ is an étale morphism, then the map $|U'| \to |X|$ (with topology on $|X|$ defined using $U \to X$ as above) is open and continuous; which in addition will prove that (3) holds. Set $U'' = U \times_X U'$, so that we have the commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

As $U \to X$ and $U' \to X$ are étale we see that both $U'' \to U$ and $U'' \to U'$ are étale morphisms of schemes. Moreover, $U'' \to U'$ is surjective. Hence we get a commutative diagram of maps of sets

$$\begin{array}{ccc} |U''| & \longrightarrow & |U'| \\ \downarrow & & \downarrow \\ |U| & \longrightarrow & |X| \end{array}$$

The lower horizontal arrow is surjective (see Lemma 66.4.4 or Lemma 66.4.5) and continuous by definition of the topology on $|X|$. The top horizontal arrow is surjective, continuous, and open by Morphisms, Lemma 29.36.13. The left vertical arrow is continuous and open (by Morphisms, Lemma 29.36.13 again.) Hence it follows formally that the right vertical arrow is continuous and open.

To finish the proof we prove (2). Let $a : X \to Y$ be a morphism of algebraic spaces. According to Spaces, Lemma 65.11.6 we can find a diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective and étale. This gives rise to the diagram

$$\begin{array}{ccc} |U| & \xrightarrow[\alpha]{} & |V| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ |X| & \xrightarrow{a} & |Y| \end{array}$$

where all but the lower horizontal arrows are known to be continuous and the two vertical arrows are surjective and open. It follows that the lower horizontal arrow is continuous as desired. □

03BY **Definition 66.4.7.** Let S be a scheme. Let X be an algebraic space over S. The underlying *topological space* of X is the set of points $|X|$ endowed with the topology constructed in Lemma 66.4.6.

It turns out that this topological space carries the same information as the small Zariski site X_{Zar} of Spaces, Definition 65.12.6.

03BZ **Lemma 66.4.8.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *The rule $X' \mapsto |X'|$ defines an inclusion preserving bijection between open subspaces X' (see Spaces, Definition 65.12.1) of X, and opens of the topological space $|X|$.*
(2) *A family $\{X_i \subset X\}_{i\in I}$ of open subspaces of X is a Zariski covering (see Spaces, Definition 65.12.5) if and only if $|X| = \bigcup |X_i|$.*

In other words, the small Zariski site X_{Zar} of X is canonically identified with a site associated to the topological space $|X|$ (see Sites, Example 7.6.4).

Proof. In order to prove (1) let us construct the inverse of the rule. Namely, suppose that $W \subset |X|$ is open. Choose a presentation $X = U/R$ corresponding to the surjective étale map $p : U \to X$ and étale maps $s, t : R \to U$. By construction we see that $|p|^{-1}(W)$ is an open of U. Denote $W' \subset U$ the corresponding open subscheme. It is clear that $R' = s^{-1}(W') = t^{-1}(W')$ is a Zariski open of R which defines an étale equivalence relation on W'. By Spaces, Lemma 65.10.2 the morphism $X' = W'/R' \to X$ is an open immersion. Hence X' is an algebraic space by Spaces, Lemma 65.11.3. By construction $|X'| = W$, i.e., X' is a subspace of X corresponding to W. Thus (1) is proved.

To prove (2), note that if $\{X_i \subset X\}_{i\in I}$ is a collection of open subspaces, then it is a Zariski covering if and only if the $U = \bigcup U \times_X X_i$ is an open covering. This follows from the definition of a Zariski covering and the fact that the morphism $U \to X$ is surjective as a map of presheaves on $(Sch/S)_{fppf}$. On the other hand, we see that $|X| = \bigcup |X_i|$ if and only if $U = \bigcup U \times_X X_i$ by Lemma 66.4.5 (and the fact that the projections $U \times_X X_i \to X_i$ are surjective and étale). Thus the equivalence of (2) follows. □

03IE **Lemma 66.4.9.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X' \subset X$ be an open subspace. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then f factors through X' if and only if $|f| : |Y| \to |X|$ factors through $|X'| \subset |X|$.*

Proof. By Spaces, Lemma 65.12.3 we see that $Y' = Y \times_X X' \to Y$ is an open immersion. If $|f|(|Y|) \subset |X'|$, then clearly $|Y'| = |Y|$. Hence $Y' = Y$ by Lemma 66.4.8. □

06NF **Lemma 66.4.10.** *Let S be a scheme. Let X be an algebraic spaces over S. Let U be a scheme and let $f : U \to X$ be an étale morphism. Let $X' \subset X$ be the open subspace corresponding to the open $|f|(|U|) \subset |X|$ via Lemma 66.4.8. Then f factors through a surjective étale morphism $f' : U \to X'$. Moreover, if $R = U \times_X U$, then $R = U \times_{X'} U$ and X' has the presentation $X' = U/R$.*

Proof. The existence of the factorization follows from Lemma 66.4.9. The morphism f' is surjective according to Lemma 66.4.4. To see f' is étale, suppose that $T \to X'$ is a morphism where T is a scheme. Then $T \times_X U = T \times_{X'} U$ as $X' \to X$ is a monomorphism of sheaves. Thus the projection $T \times_{X'} U \to T$ is étale as we assumed f étale. We have $U \times_X U = U \times_{X'} U$ as $X' \to X$ is a monomorphism. Then $X' = U/R$ follows from Spaces, Lemma 65.9.1. □

0H2X **Lemma 66.4.11.** *Let S be a scheme. Let X be an algebraic space over S. Let $p : \mathrm{Spec}(K) \to X$ and $q : \mathrm{Spec}(L) \to X$ be morphisms where K and L are fields. Assume p and q determine the same point of $|X|$ and p is a monomorphism. Then q factors uniquely through p.*

Proof. Since p and q define the same point of $|X|$, we see that the scheme

$$Y = \mathrm{Spec}(K) \times_{p,X,q} \mathrm{Spec}(L)$$

is nonempty. Since the base change of a monomorphism is a monomorphism this means that the projection morphism $Y \to \mathrm{Spec}(L)$ is a monomorphism. Hence $Y = \mathrm{Spec}(L)$, see Schemes, Lemma 26.23.11. We conclude that q factors through p. Uniqueness comes from the fact that p is a monomorphism. □

03E1 **Lemma 66.4.12.** *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

This map is injective.

Proof. This follows from Lemma 66.4.11. □

We will see in Decent Spaces, Lemma 68.11.1 that the map of Lemma 66.4.12 is a bijection when X is decent.

66.5. Quasi-compact spaces

03E2

03E3 **Definition 66.5.1.** Let S be a scheme. Let X be an algebraic space over S. We say X is *quasi-compact* if there exists a surjective étale morphism $U \to X$ with U quasi-compact.

03E4 **Lemma 66.5.2.** *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if $|X|$ is quasi-compact.*

Proof. Choose a scheme U and an étale surjective morphism $U \to X$. We will use Lemma 66.4.4. If U is quasi-compact, then since $|U| \to |X|$ is surjective we conclude that $|X|$ is quasi-compact. If $|X|$ is quasi-compact, then since $|U| \to |X|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still étale). Hence we win. □

040T **Lemma 66.5.3.** *A finite disjoint union of quasi-compact algebraic spaces is a quasi-compact algebraic space.*

Proof. This is clear from Lemma 66.5.2 and the corresponding topological fact. □

03IO **Example 66.5.4.** The space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ is a quasi-compact algebraic space.

04NN **Lemma 66.5.5.** *Let S be a scheme. Let X be an algebraic space over S. Every point of $|X|$ has a fundamental system of open quasi-compact neighbourhoods. In particular $|X|$ is locally quasi-compact in the sense of Topology, Definition 5.13.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |X|$ of topological spaces. To be a bit more precise, if $u \in U$ maps to $x \in |X|$, then the images of the affine neighbourhoods of u will give a fundamental system of quasi-compact open neighbourhoods of x. □

66.6. Special coverings

03FW In this section we collect some straightforward lemmas on the existence of étale surjective coverings of algebraic spaces.

03FX **Lemma 66.6.1.** *Let S be a scheme. Let X be an algebraic space over S. There exists a surjective étale morphism $U \to X$ where U is a disjoint union of affine schemes. We may in addition assume each of these affines maps into an affine open of S.*

Proof. Let $V \to X$ be a surjective étale morphism. Let $V = \bigcup_{i \in I} V_i$ be a Zariski open covering such that each V_i maps into an affine open of S. Then set $U = \coprod_{i \in I} V_i$ with induced morphism $U \to V \to X$. This is étale and surjective as a composition of étale and surjective representable transformations of functors (via the general principle Spaces, Lemma 65.5.4 and Morphisms, Lemmas 29.9.2 and 29.36.3). □

03FY **Lemma 66.6.2.** *Let S be a scheme. Let X be an algebraic space over S. There exists a Zariski covering $X = \bigcup X_i$ such that each algebraic space X_i has a surjective étale covering by an affine scheme. We may in addition assume each X_i maps into an affine open of S.*

Proof. By Lemma 66.6.1 we can find a surjective étale morphism $U = \coprod U_i \to X$, with U_i affine and mapping into an affine open of S. Let $X_i \subset X$ be the open subspace of X such that $U_i \to X$ factors through an étale surjective morphism $U_i \to X_i$, see Lemma 66.4.10. Since $U = \bigcup U_i$ we see that $X = \bigcup X_i$. As $U_i \to X_i$ is surjective it follows that $X_i \to S$ maps into an affine open of S. □

03H6 **Lemma 66.6.3.** *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if there exists an étale surjective morphism $U \to X$ with U an affine scheme.*

Proof. If there exists an étale surjective morphism $U \to X$ with U affine then X is quasi-compact by Definition 66.5.1. Conversely, if X is quasi-compact, then $|X|$ is quasi-compact. Let $U = \coprod_{i \in I} U_i$ be a disjoint union of affine schemes with an étale and surjective map $\varphi : U \to X$ (Lemma 66.6.1). Then $|X| = \bigcup \varphi(|U_i|)$ and by quasi-compactness there is a finite subset $i_1, \ldots, i_n$ such that $|X| = \bigcup \varphi(|U_{i_j}|)$. Hence $U_{i_1} \cup \ldots \cup U_{i_n}$ is an affine scheme with a finite surjective morphism towards X. □

The following lemma will be obsoleted by the discussion of separated morphisms in the chapter on morphisms of algebraic spaces.

03FZ **Lemma 66.6.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a separated scheme and $U \to X$ étale. Then $U \to X$ is separated, and $R = U \times_X U$ is a separated scheme.*

Proof. Let $X' \subset X$ be the open subscheme such that $U \to X$ factors through an étale surjection $U \to X'$, see Lemma 66.4.10. If $U \to X'$ is separated, then so is $U \to X$, see Spaces, Lemma 65.5.4 (as the open immersion $X' \to X$ is separated by Spaces, Lemma 65.5.8 and Schemes, Lemma 26.23.8). Moreover, since $U \times_{X'} U = U \times_X U$ it suffices to prove the result after replacing X by X', i.e., we may assume $U \to X$ surjective. Consider the commutative diagram

$$\begin{array}{ccc} R = U \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In the proof of Spaces, Lemma 65.13.1 we have seen that $j : R \to U \times_S U$ is separated. The morphism of schemes $U \to S$ is separated as U is a separated scheme, see Schemes, Lemma 26.21.13. Hence $U \times_S U \to U$ is separated as a base change, see Schemes, Lemma 26.21.12. Hence the scheme $U \times_S U$ is separated (by the same lemma). Since j is separated we see in the same way that R is separated. Hence $R \to U$ is a separated morphism (by Schemes, Lemma 26.21.13 again). Thus by Spaces, Lemma 65.11.4 and the diagram above we conclude that $U \to X$ is separated. □

07S4 **Lemma 66.6.5.** *Let S be a scheme. Let X be an algebraic space over S. If there exists a quasi-separated scheme U and a surjective étale morphism $U \to X$ such that either of the projections $U \times_X U \to U$ is quasi-compact, then X is quasi-separated.*

Proof. We may think of X as an algebraic space over $\mathbf{Z}$. Consider the cartesian diagram

$$\begin{array}{ccc} U \times_X U & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times U & \longrightarrow & X \times X \end{array}$$

Since U is quasi-separated the projection $U \times U \to U$ is quasi-separated (as a base change of a quasi-separated morphism of schemes, see Schemes, Lemma 26.21.12). Hence the assumption in the lemma implies j is quasi-compact by Schemes, Lemma 26.21.14. By Spaces, Lemma 65.11.4 we see that Δ is quasi-compact as desired. □

03W7 **Lemma 66.6.6.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

1. *X is Zariski locally quasi-separated over S,*
2. *X is Zariski locally quasi-separated,*
3. *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
4. *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i which maps into an affine open of S and a quasi-compact surjective étale morphism $U_i \to X_i$.*

Proof. Assume $U_i \to X_i \subset X$ are as in (3). To prove (4) choose for each i a finite affine open covering $U_i = U_{i1} \cup \ldots \cup U_{in_i}$ such that each U_{ij} maps into an affine open of S. The compositions $U_{ij} \to U_i \to X_i$ are étale and quasi-compact (see Spaces, Lemma 65.5.4). Let $X_{ij} \subset X_i$ be the open subspace corresponding to the image of $|U_{ij}| \to |X_i|$, see Lemma 66.4.10. Note that $U_{ij} \to X_{ij}$ is quasi-compact as $X_{ij} \subset X_i$ is a monomorphism and as $U_{ij} \to X$ is quasi-compact. Then $X = \bigcup X_{ij}$ is a covering as in (4). The implication (4) $\Rightarrow$ (3) is immediate.

Assume (4). To show that X is Zariski locally quasi-separated over S it suffices to show that X_i is quasi-separated over S. Hence we may assume there exists an affine scheme U mapping into an affine open of S and a quasi-compact surjective étale morphism $U \to X$. Consider the fibre product square

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

The right vertical arrow is surjective étale (see Spaces, Lemma 65.5.7) and $U \times_S U$ is affine (as U maps into an affine open of S, see Schemes, Section 26.17), and $U \times_X U$ is quasi-compact because the projection $U \times_X U \to U$ is quasi-compact as a base change of $U \to X$. It follows from Spaces, Lemma 65.11.4 that $\Delta_{X/S}$ is quasi-compact as desired.

Assume (1). To prove (3) there is an immediate reduction to the case where X is quasi-separated over S. By Lemma 66.6.2 we can find a Zariski open covering $X = \bigcup X_i$ such that each X_i maps into an affine open of S, and such that there exist affine schemes U_i and surjective étale morphisms $U_i \to X_i$. Since $U_i \to S$ maps into an affine open of S we see that $U_i \times_S U_i$ is affine, see Schemes, Section 26.17. As X is quasi-separated over S, the morphisms

$$R_i = U_i \times_{X_i} U_i = U_i \times_X U_i \longrightarrow U_i \times_S U_i$$

as base changes of $\Delta_{X/S}$ are quasi-compact. Hence we conclude that R_i is a quasi-compact scheme. This in turn implies that each projection $R_i \to U_i$ is quasi-compact. Hence, applying Spaces, Lemma 65.11.4 to the covering $U_i \to X_i$ and the morphism $U_i \to X_i$ we conclude that the morphisms $U_i \to X_i$ are quasi-compact as desired.

At this point we see that (1), (3), and (4) are equivalent. Since (3) does not refer to the base scheme we conclude that these are also equivalent with (2). □

The following lemma will turn out to be quite useful.

03IJ **Lemma 66.6.7.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a scheme. Let $\varphi : U \to X$ be an étale morphism such that the projections $R = U \times_X U \to U$ are quasi-compact; for example if φ is quasi-compact. Then the fibres of*

$$|U| \to |X| \quad \text{and} \quad |R| \to |X|$$

are finite.

Proof. Denote $R = U \times_X U$, and $s, t : R \to U$ the projections. Let $u \in U$ be a point, and let $x \in |X|$ be its image. The fibre of $|U| \to |X|$ over x is equal to $s(t^{-1}(\{u\}))$ by Lemma 66.4.3, and the fibre of $|R| \to |X|$ over x is $t^{-1}(s(t^{-1}(\{u\})))$.

Since $t : R \to U$ is étale and quasi-compact, it has finite fibres (as its fibres are disjoint unions of spectra of fields by Morphisms, Lemma 29.36.7 and quasi-compact). Hence we win. □

66.7. Properties of Spaces defined by properties of schemes

03E5 Any étale local property of schemes gives rise to a corresponding property of algebraic spaces via the following lemma.

03E8 **Lemma 66.7.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology, see Descent, Definition 35.15.1. The following are equivalent*

(1) *for some scheme U and surjective étale morphism $U \to X$ the scheme U has property $\mathcal{P}$, and*
(2) *for every scheme U and every étale morphism $U \to X$ the scheme U has property $\mathcal{P}$.*

If X is representable this is equivalent to $\mathcal{P}(X)$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. For the converse, choose a surjective étale morphism $U \to X$ with U a scheme that has $\mathcal{P}$ and let V be an étale X-scheme. Then $U \times_X V \to V$ is an étale surjection of schemes, so V inherits $\mathcal{P}$ from $U \times_X V$, which in turn inherits $\mathcal{P}$ from U (see discussion following Descent, Definition 35.15.1). The last claim is clear from (1) and Descent, Definition 35.15.1. □

03E6 **Definition 66.7.2.** Let $\mathcal{P}$ be a property of schemes which is local in the étale topology. Let S be a scheme. Let X be an algebraic space over S. We say X *has property* $\mathcal{P}$ if any of the equivalent conditions of Lemma 66.7.1 hold.

03E7 **Remark 66.7.3.** Here is a list of properties which are local for the étale topology (keep in mind that the fpqc, fppf, syntomic, and smooth topologies are stronger than the étale topology):

(1) locally Noetherian, see Descent, Lemma 35.16.1,
(2) Jacobson, see Descent, Lemma 35.16.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 35.17.1,
(4) Cohen-Macaulay, see Descent, Lemma 35.17.2,
(5) Gorenstein, see Duality for Schemes, Lemma 48.24.6,
(6) reduced, see Descent, Lemma 35.18.1,
(7) normal, see Descent, Lemma 35.18.2,
(8) locally Noetherian and (R_k), see Descent, Lemma 35.18.3,
(9) regular, see Descent, Lemma 35.18.4,
(10) Nagata, see Descent, Lemma 35.18.5.

Any étale local property of germs of schemes gives rise to a corresponding property of algebraic spaces. Here is the obligatory lemma.

04N2 **Lemma 66.7.4.** *Let $\mathcal{P}$ be a property of germs of schemes which is étale local, see Descent, Definition 35.21.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent*

(1) *for any $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$, and*
(2) *for some $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$.*

If X is representable, then this is equivalent to $\mathcal{P}(X, x)$.

Proof. Omitted. □

04RC **Definition 66.7.5.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{P}$ be a property of germs of schemes which is étale local. We say X *has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 66.7.4 hold.

0BBL **Remark 66.7.6.** Let P be a property of local rings. Assume that for any étale ring map $A \to B$ and $\mathfrak{q}$ is a prime of B lying over the prime $\mathfrak{p}$ of A, then $P(A_\mathfrak{p}) \Leftrightarrow P(B_\mathfrak{q})$. Then we obtain an étale local property of germs (U, u) of schemes by setting $\mathcal{P}(U, u) = P(\mathcal{O}_{U,u})$. In this situation we will use the terminology "the local ring of X at x has P" to mean X has property $\mathcal{P}$ at x. Here is a list of such properties P:

(1) Noetherian, see More on Algebra, Lemma 15.44.1,
(2) dimension d, see More on Algebra, Lemma 15.44.2,
(3) regular, see More on Algebra, Lemma 15.44.3,
(4) discrete valuation ring, follows from (2), (3), and Algebra, Lemma 10.119.7,
(5) reduced, see More on Algebra, Lemma 15.45.4,
(6) normal, see More on Algebra, Lemma 15.45.6,
(7) Noetherian and depth k, see More on Algebra, Lemma 15.45.8,
(8) Noetherian and Cohen-Macaulay, see More on Algebra, Lemma 15.45.9,
(9) Noetherian and Gorenstein, see Dualizing Complexes, Lemma 47.21.8.

There are more properties for which this holds, for example G-ring and Nagata. If we every need these we will add them here as well as references to detailed proofs of the corresponding algebra facts.

66.8. Constructible sets

0ECS

0ECT **Lemma 66.8.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $E \subset |X|$ be a subset. The following are equivalent*

(1) *for every étale morphism $U \to X$ where U is a scheme the inverse image of E in U is a locally constructible subset of U,*
(2) *for every étale morphism $U \to X$ where U is an affine scheme the inverse image of E in U is a constructible subset of U,*
(3) *for some surjective étale morphism $U \to X$ where U is a scheme the inverse image of E in U is a locally constructible subset of U.*

Proof. By Properties, Lemma 28.2.1 we see that (1) and (2) are equivalent. It is immediate that (1) implies (3). Thus we assume we have a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that $\varphi^{-1}(E)$ is locally constructible. Let $\varphi' : U' \to X$ be another étale morphism where U' is a scheme. Then we have

$$E'' = \mathrm{pr}_1^{-1}(\varphi^{-1}(E)) = \mathrm{pr}_2^{-1}((\varphi')^{-1}(E))$$

where $\mathrm{pr}_1 : U \times_X U' \to U$ and $\mathrm{pr}_2 : U \times_X U' \to U'$ are the projections. By Morphisms, Lemma 29.22.1 we see that E'' is locally constructible in $U \times_X U'$. Let $W' \subset U'$ be an affine open. Since pr_2 is étale and hence open, we can choose a quasi-compact open $W'' \subset U \times_X U'$ with $\mathrm{pr}_2(W'') = W'$. Then $\mathrm{pr}_2|_{W''} : W'' \to W'$ is quasi-compact. We have $W' \cap (\varphi')^{-1}(E) = \mathrm{pr}_2(E'' \cap W'')$ as φ is surjective, see Lemma 66.4.3. Thus $W' \cap (\varphi')^{-1}(E) = \mathrm{pr}_2(E'' \cap W'')$ is locally constructible by Morphisms, Theorem 29.22.3 as desired. □

0ECU **Definition 66.8.2.** Let S be a scheme. Let X be an algebraic space over S. Let $E \subset |X|$ be a subset. We say E is *étale locally constructible* if the equivalent conditions of Lemma 66.8.1 are satisfied.

Of course, if X is representable, i.e., X is a scheme, then this just means E is a locally constructible subset of the underlying topological space.

66.9. Dimension at a point

04N3 We can use Descent, Lemma 35.21.2 to define the dimension of an algebraic space X at a point x. This will give us a different notion than the topological one (i.e., the dimension of $|X|$ at x).

04N5 **Definition 66.9.1.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. We define the *dimension of X at x* to be the element $\dim_x(X) \in \{0, 1, 2, \ldots, \infty\}$ such that $\dim_x(X) = \dim_u(U)$ for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$. See Definition 66.7.5, Lemma 66.7.4, and Descent, Lemma 35.21.2.

Warning: It is **not** the case that $\dim_x(X) = \dim_x(|X|)$ in general. A counter example is the algebraic space X of Spaces, Example 65.14.9. Namely, let $x \in |X|$ be a point not equal to the generic point x_0 of $|X|$. Then we have $\dim_x(X) = 0$ but $\dim_x(|X|) = 1$. In particular, the dimension of X (as defined below) is different from the dimension of $|X|$.

04N6 **Definition 66.9.2.** Let S be a scheme. Let X be an algebraic space over S. The *dimension* $\dim(X)$ of X is defined by the rule

$$\dim(X) = \sup_{x \in |X|} \dim_x(X)$$

By Properties, Lemma 28.10.2 we see that this is the usual notion if X is a scheme. There is another integer that measures the dimension of a scheme at a point, namely the dimension of the local ring. This invariant is compatible with étale morphisms also, see Section 66.10.

66.10. Dimension of local rings

04N7 The dimension of the local ring of an algebraic space is a well defined concept.

0BAM **Lemma 66.10.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

1. *for some scheme U and étale morphism $a : U \to X$ and point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,*
2. *for any scheme U, any étale morphism $a : U \to X$, and any point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$.*

If X is a scheme, this is equivalent to $\dim(\mathcal{O}_{X,x}) = d$.

Proof. Combine Lemma 66.7.4 and Descent, Lemma 35.21.3. □

04NA **Definition 66.10.2.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The *dimension of the local ring of X at x* is the element $d \in \{0, 1, 2, \ldots, \infty\}$ satisfying the equivalent conditions of Lemma 66.10.1. In this case we will also say *x is a point of codimension d on X*.

Besides the lemma below we also point the reader to Lemmas 66.22.4 and 66.22.5.

0BAN **Lemma 66.10.3.** *Let S be a scheme. Let X be an algebraic space over S. The following quantities are equal:*

(1) *The dimension of X.*
(2) *The supremum of the dimensions of the local rings of X.*
(3) *The supremum of $\dim_x(X)$ for $x \in |X|$.*

Proof. The numbers in (1) and (3) are equal by Definition 66.9.2. Let $U \to X$ be a surjective étale morphism from a scheme U. The supremum of $\dim_x(X)$ for $x \in |X|$ is the same as the supremum of $\dim_u(U)$ for points u of U by definition. This is the same as the supremum of $\dim(\mathcal{O}_{U,u})$ by Properties, Lemma 28.10.2. This in turn is the same as (2) by definition. □

66.11. Generic points

0BAP Let T be a topological space. According to the second edition of EGA I, a *maximal point of T* is a generic point of an irreducible component of T. If $T = |X|$ is the topological space associated to an algebraic space X, there are at least two notions of maximal points: we can look at maximal points of T viewed as a topological space, or we can look at images of maximal points of U where $U \to X$ is an étale morphism and U is a scheme. The second notion corresponds to the set of points of codimension 0 (Lemma 66.11.1). The codimension 0 points are easier to work with for general algebraic spaces; the two notions agree for quasi-separated and more generally decent algebraic spaces (Decent Spaces, Lemma 68.20.1).

0BAQ **Lemma 66.11.1.** *Let S be a scheme and let X be an algebraic space over S. Let $x \in |X|$. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent*

(1) *x is a point of codimension 0 on X,*
(2) *for some $U \to X$ as above and $u \in U$ with $a(u) = x$, the point u is the generic point of an irreducible component of U, and*
(3) *for any $U \to X$ as above and any $u \in U$ mapping to x, the point u is the generic point of an irreducible component of U.*

If X is representable, this is equivalent to x being a generic point of an irreducible component of $|X|$.

Proof. Observe that a point u of a scheme U is a generic point of an irreducible component of U if and only if $\dim(\mathcal{O}_{U,u}) = 0$ (Properties, Lemma 28.10.4). Hence this follows from the definition of the codimension of a point on X (Definition 66.10.2). □

0BAR **Lemma 66.11.2.** *Let S be a scheme and let X be an algebraic space over S. The set of codimension 0 points of X is dense in $|X|$.*

Proof. If U is a scheme, then the set of generic points of irreducible components is dense in U (holds for any quasi-sober topological space). Thus if $U \to X$ is a surjective étale morphism, then the set of codimension 0 points of X is the image of a dense subset of $|U|$ (Lemma 66.11.1). Since $|X|$ has the quotient topology for $|U| \to |X|$ we conclude. □

66.12. Reduced spaces

03IP We have already defined reduced algebraic spaces in Section 66.7. Here we just prove some simple lemmas regarding reduced algebraic spaces.

0ABJ **Lemma 66.12.1.** *Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces. Then $|Z| \to |X|$ is a homeomorphism of $|Z|$ onto a locally closed subset of $|X|$.*

Proof. Let U be a scheme and $U \to X$ a surjective étale morphism. Then $Z \times_X U \to U$ is an immersion of schemes, hence gives a homeomorphism of $|Z \times_X U|$ with a locally closed subset T' of $|U|$. By Lemma 66.4.3 the subset T' is the inverse image of the image T of $|Z| \to |X|$. The map $|Z| \to |X|$ is injective because the transformation of functors $Z \to X$ is injective, see Spaces, Section 65.12. By Topology, Lemma 5.6.4 we see that T is locally closed in $|X|$. Moreover, the continuous map $|Z| \to T$ is a homeomorphism as the map $|Z \times_X U| \to T'$ is a homeomorphism and $|Z \times_Y U| \to |Z|$ is submersive. □

The following lemma will help us construct (locally) closed subspaces.

07TW **Lemma 66.12.2.** *Let S be a scheme. Let $j : R \to U \times_S U$ be an étale equivalence relation. Let $X = U/R$ be the associated algebraic space (Spaces, Theorem 65.10.5). There is a canonical bijection*

R-invariant locally closed subschemes Z' of U $\leftrightarrow$ locally closed subspaces Z of X

Moreover, if $Z \to X$ is closed (resp. open) if and only if $Z' \to U$ is closed (resp. open).

Proof. Denote $\varphi : U \to X$ the canonical map. The bijection sends $Z \to X$ to $Z' = Z \times_X U \to U$. It is immediate from the definition that $Z' \to U$ is an immersion, resp. closed immersion, resp. open immersion if $Z \to X$ is so. It is also clear that Z' is R-invariant (see Groupoids, Definition 39.19.1).

Conversely, assume that $Z' \to U$ is an immersion which is R-invariant. Let R' be the restriction of R to Z', see Groupoids, Definition 39.18.2. Since $R' = R \times_{s,U} Z' = Z' \times_{U,t} R$ in this case we see that R' is an étale equivalence relation on Z'. By Spaces, Theorem 65.10.5 we see $Z = Z'/R'$ is an algebraic space. By construction we have $U \times_X Z = Z'$, so $U \times_X Z \to Z$ is an immersion. Note that the property "immersion" is preserved under base change and fppf local on the base (see Spaces, Section 65.4). Moreover, immersions are separated and locally quasi-finite (see Schemes, Lemma 26.23.8 and Morphisms, Lemma 29.20.16). Hence by More on Morphisms, Lemma 37.57.1 immersions satisfy descent for fppf covering. This means all the hypotheses of Spaces, Lemma 65.11.5 are satisfied for $Z \to X$, $\mathcal{P}$ ="immersion", and the étale surjective morphism $U \to X$. We conclude that $Z \to X$ is representable and an immersion, which is the definition of a subspace (see Spaces, Definition 65.12.1).

It is clear that these constructions are inverse to each other and we win. □

03IQ **Lemma 66.12.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. There exists a unique closed subspace $Z \subset X$ with the following properties: (a) we have $|Z| = T$, and (b) Z is reduced.*

Proof. Let $U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$, so that $X = U/R$, see Spaces, Lemma 65.9.1. As usual we denote

$s, t : R \to U$ the two projection morphisms. By Lemma 66.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $s^{-1}(T') = t^{-1}(T')$. Let $Z' \subset U$ be the reduced induced scheme structure on T'. In this case the fibre products $Z' \times_{U,t} R$ and $Z' \times_{U,s} R$ are closed subschemes of R (Schemes, Lemma 26.18.2) which are étale over Z' (Morphisms, Lemma 29.36.4), and hence reduced (because being reduced is local in the étale topology, see Remark 66.7.3). Since they have the same underlying topological space (see above) we conclude that $Z' \times_{U,t} R = Z' \times_{U,s} R$. Thus we can apply Lemma 66.12.2 to obtain a closed subspace $Z \subset X$ whose pullback to U is Z'. By construction $|Z| = T$ and Z is reduced. This proves existence. We omit the proof of uniqueness. □

03JJ **Lemma 66.12.4.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume Y is reduced. A morphism $f : Y \to X$ factors through Z if and only if $f(|Y|) \subset |Z|$.*

Proof. Assume $f(|Y|) \subset |Z|$. Choose a diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{f} & X \end{array}$$

where U, V are schemes, and the vertical arrows are surjective and étale. The scheme V is reduced, see Lemma 66.7.1. Hence h factors through $a^{-1}(Z)$ by Schemes, Lemma 26.12.7. So $a \circ h$ factors through Z. As $Z \subset X$ is a subsheaf, and $V \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ we conclude that $X \to Y$ factors through Z. □

047X **Definition 66.12.5.** Let S be a scheme, and let X be an algebraic space over S. Let $Z \subset |X|$ be a closed subset. An *algebraic space structure on Z* is given by a closed subspace Z' of X with $|Z'|$ equal to Z. The *reduced induced algebraic space structure* on Z is the one constructed in Lemma 66.12.3. The *reduction X_{red} of X* is the reduced induced algebraic space structure on $|X|$.

66.13. The schematic locus

03JG Every algebraic space has a largest open subspace which is a scheme; this is more or less clear but we also write out the proof below. Of course this subspace may be empty, for example if $X = \mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ (the universal counter example). On the other hand, if X is for example quasi-separated, then this largest open subscheme is actually dense in X!

03JH **Lemma 66.13.1.** *Let S be a scheme. Let X be an algebraic space over S. There exists a largest open subspace $X' \subset X$ which is a scheme.*

Proof. Let $U \to X$ be an étale surjective morphism, where U is a scheme. Let $R = U \times_X U$. The open subspaces of X correspond $1-1$ with open subschemes of U which are R-invariant. Hence there is a set of them. Let X_i, $i \in I$ be the set of open subspaces of X which are schemes, i.e., are representable. Consider the open subspace $X' \subset X$ whose underlying set of points is the open $\bigcup |X_i|$ of $|X|$. By Lemma 66.4.4 we see that

$$\coprod X_i \longrightarrow X'$$

is a surjective map of sheaves on $(Sch/S)_{fppf}$. But since each $X_i \to X'$ is representable by open immersions we see that in fact the map is surjective in the Zariski topology. Namely, if $T \to X'$ is a morphism from a scheme into X', then $X_i \times_{X'} T$ is an open subscheme of T. Hence we can apply Schemes, Lemma 26.15.4 to see that X' is a scheme. □

In the rest of this section we say that an open subspace X' of an algebraic space X is *dense* if the corresponding open subset $|X'| \subset |X|$ is dense.

0BAS **Lemma 66.13.2.** *Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a quasi-separated scheme, then there exists a dense open subspace X' of X which is a scheme. More precisely, every point $x \in |X|$ of codimension 0 in X is contained in X'.*

Proof. Let $X' \subset X$ be the maximal open subspace which is a scheme (Lemma 66.13.1). Let $x \in |X|$ be a point of codimension 0 on X. By Lemma 66.11.2 it suffices to show $x \in X'$. Let $U \to X$ be as in the statement of the lemma. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. By Lemma 66.6.7 the fibre of $|U| \to |X|$ over x is finite, say $\{\eta_1, \ldots, \eta_n\}$. By Lemma 66.11.1 each η_i is the generic point of an irreducible component of U. By Properties, Lemma 28.29.1 we can find an affine open $W \subset U$ containing $\{\eta_1, \ldots, \eta_n\}$ (this is where we use that U is quasi-separated). By Groupoids, Lemma 39.24.1 we may assume that W is R-invariant. Since $W \subset U$ is an R-invariant affine open, the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 39.23.9. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 65.10.2 and it contains x by construction. □

We will improve the following proposition to the case of decent algebraic spaces in Decent Spaces, Theorem 68.10.2.

06NH **Proposition 66.13.3.** *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated (for example if X is quasi-separated), then there exists a dense open subspace X' of X which is a scheme. More precisely, every point $x \in |X|$ of codimension 0 on X is contained in X'.*

Proof. The question is local on X by Lemma 66.13.1. Thus by Lemma 66.6.6 we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Moreover $U \to X$ is separated (Lemma 66.6.4). Set $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Then s, t are surjective, quasi-compact, separated, and étale. Hence s, t are also quasi-finite and have finite fibres (Morphisms, Lemmas 29.36.6, 29.20.9, and 29.20.10). By Morphisms, Lemma 29.51.1 for every $\eta \in U$ which is the generic point of an irreducible component of U, there exists an open neighbourhood $V \subset U$ of η such that $s^{-1}(V) \to V$ is finite. By Descent, Lemma 35.23.25 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 40.6.4 which says that the largest open $W \subset U$ over which s is finite is R-invariant. By the above W contains every generic point of an irreducible component of U. The restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). By construction $s_W, t_W : R_W \to W$ are finite étale. Consider the

open subspace $X' = W/R_W \subset X$ (see Spaces, Lemma 65.10.2). By construction the inclusion map $X' \to X$ induces a bijection on points of codimension 0. This reduces us to Lemma 66.13.2. □

66.14. Obtaining a scheme

07S5 We have used in the previous section that the quotient U/R of an affine scheme U by an equivalence relation R is a scheme if the morphisms $s, t : R \to U$ are finite étale. This is a special case of the following result.

07S6 **Proposition 66.14.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$s, t : R \to U$ finite locally free,*
(2) *$j = (t, s)$ is an equivalence relation, and*
(3) *every nonempty closed subset Z of U contains a point u whose R-equivalence class $t(s^{-1}(\{u\}))$ is contained in an affine open of U[3].*

Then there exists a finite locally free morphism $U \to M$ of schemes over S such that $R = U \times_M U$ and such that M represents the quotient sheaf U/R in the fppf topology.

Proof. By assumption (3) and Groupoids, Lemma 39.24.1 we can find an open covering $U = \bigcup U_i$ such that each U_i is an R-invariant affine open of U. Set $R_i = R|_{U_i}$. Consider the fppf sheaves $F = U/R$ and $F_i = U_i/R_i$. By Spaces, Lemma 65.10.2 the morphisms $F_i \to F$ are representable and open immersions. By Groupoids, Proposition 39.23.9 the sheaves F_i are representable by affine schemes. If T is a scheme and $T \to F$ is a morphism, then $V_i = F_i \times_F T$ is open in T and we claim that $T = \bigcup V_i$. Namely, fppf locally on T we can lift $T \to F$ to a morphism $f : T \to U$ and in that case $f^{-1}(U_i) \subset V_i$. Hence we conclude that F is representable by a scheme, see Schemes, Lemma 26.15.4. □

For example, if U is isomorphic to a locally closed subscheme of an affine scheme or isomorphic to a locally closed subscheme of $\mathrm{Proj}(A)$ for some graded ring A, then the third assumption holds by Properties, Lemma 28.29.5. In particular we can apply this to free actions of finite groups and finite group schemes on quasi-affine or quasi-projective schemes. For example, the quotient X/G of a quasi-projective variety X by a free action of a finite group G is a scheme. Here is a detailed statement.

07S7 **Lemma 66.14.2.** *Let S be a scheme. Let $G \to S$ be a group scheme. Let $X \to S$ be a morphism of schemes. Let $a : G \times_S X \to X$ be an action. Assume that*

(1) *$G \to S$ is finite locally free,*
(2) *the action a is free,*
(3) *$X \to S$ is affine, or quasi-affine, or projective, or quasi-projective, or X is isomorphic to an open subscheme of an affine scheme, or X is isomorphic to an open subscheme of $\mathrm{Proj}(A)$ for some graded ring A, or $G \to S$ is radicial.*

Then the fppf quotient sheaf X/G is a scheme and $X \to X/G$ is an fppf G-torsor.

[3] Let $E \subset U$ be the set of points u such that $t(s^{-1}(\{u\}))$ is contained in an affine open of U. Condition (3) holds if $E = U$, or if every finite type point of U is in E, or if every $u \in U$ specializes to a point of E.

Proof. We first show that X/G is a scheme. Since the action is free the morphism $j = (a, \mathrm{pr}) : G \times_S X \to X \times_S X$ is a monomorphism and hence an equivalence relation, see Groupoids, Lemma 39.10.3. The maps $s, t : G \times_S X \to X$ are finite locally free as we've assumed that $G \to S$ is finite locally free. To conclude it now suffices to prove the last assumption of Proposition 66.14.1 holds. Since the action of G is over S it suffices to prove that any finite set of points in a fibre of $X \to S$ is contained in an affine open of X. If X is isomorphic to an open subscheme of an affine scheme or isomorphic to an open subscheme of $\mathrm{Proj}(A)$ for some graded ring A this follows from Properties, Lemma 28.29.5. If $X \to S$ is affine, or quasi-affine, or projective, or quasi-projective, we may replace S by an affine open and we get back to the case we just dealt with. If $G \to S$ is radicial, then the orbits of points on X under the action of G are singletons and the condition trivially holds. Some details omitted.

To see that $X \to X/G$ is an fppf G-torsor (Groupoids, Definition 39.11.3) we have to show that $G \times_S X \to X \times_{X/G} X$ is an isomorphism and that $X \to X/G$ fppf locally has sections. The second part is clear from the fact that $X \to X/G$ is surjective as a map of fppf sheaves (by construction). The first part follows from the isomorphism $R = U \times_M U$ in the conclusion of Proposition 66.14.1 (note that $R = G \times_S X$ in our case). □

0BBM **Lemma 66.14.3.** *Notation and assumptions as in Proposition 66.14.1. Then*

(1) *if U is quasi-separated over S, then U/R is quasi-separated over S,*
(2) *if U is quasi-separated, then U/R is quasi-separated,*
(3) *if U is separated over S, then U/R is separated over S,*
(4) *if U is separated, then U/R is separated, and*
(5) *add more here.*

Similar results hold in the setting of Lemma 66.14.2.

Proof. Since M represents the quotient sheaf we have a cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow[j]{} & U \times_S U \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times_S M \end{array}$$

of schemes. Since $U \times_S U \to M \times_S M$ is surjective finite locally free, to show that $M \to M \times_S M$ is quasi-compact, resp. a closed immersion, it suffices to show that $j : R \to U \times_S U$ is quasi-compact, resp. a closed immersion, see Descent, Lemmas 35.23.1 and 35.23.21. Since $j : R \to U \times_S U$ is a morphism over U and since R is finite over U, we see that j is quasi-compact as soon as the projection $U \times_S U \to U$ is quasi-separated (Schemes, Lemma 26.21.14). Since j is a monomorphism and locally of finite type, we see that j is a closed immersion as soon as it is proper (Étale Morphisms, Lemma 41.7.2) which will be the case as soon as the projection $U \times_S U \to U$ is separated (Morphisms, Lemma 29.41.7). This proves (1) and (3). To prove (2) and (4) we replace S by $\mathrm{Spec}(\mathbf{Z})$, see Definition 66.3.1. Since Lemma 66.14.2 is proved through an application of Proposition 66.14.1 the final statement is clear too. □

66.15. Points on quasi-separated spaces

06NI

Points can behave very badly on algebraic spaces in the generality introduced in the Stacks project. However, for quasi-separated spaces their behaviour is mostly like the behaviour of points on schemes. We prove a few results on this in this section; the chapter on decent spaces contains many more results on this, see for example Decent Spaces, Section 68.12.

06NJ **Lemma 66.15.1.** *Let S be a scheme. Let X be a Zariski locally quasi-separated algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.8.6).*

Proof. Combining Topology, Lemma 5.8.8 and Lemma 66.6.6 we see that we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$ with projection maps $s, t : R \to U$. Applying Lemma 66.6.7 we see that the fibres of s, t are finite. It follows all the assumptions of Topology, Lemma 5.19.8 are met, and we conclude that $|X|$ is Kolmogorov[4].

It remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Lemma 66.12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. Note that $U \times_X Z \to Z$ is a quasi-compact, surjective, étale morphism from an affine scheme to Z, hence Z is Zariski locally quasi-separated by Lemma 66.6.6. By Proposition 66.13.3 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 26.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in |Z'| \subset T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

0A4G **Lemma 66.15.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The topological space $|X|$ is a spectral space.*

Proof. By Topology, Definition 5.23.1 we have to check that $|X|$ is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. By Lemma 66.15.1 we see that $|X|$ is sober. By Lemma 66.5.2 we see that $|X|$ is quasi-compact. By Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $f : U \to X$. Since $|f| : |U| \to |X|$ is open and continuous and since $|U|$ has a basis of quasi-compact opens, we conclude that $|X|$ has a basis of quasi-compact opens. Finally, suppose that $A, B \subset |X|$ are quasi-compact open. Then $A = |X'|$ and $B = |X''|$ for some open subspaces $X', X'' \subset X$ (Lemma 66.4.8) and we can choose affine schemes V and W and surjective étale morphisms $V \to X'$ and $W \to X''$ (Lemma 66.6.3). Then $A \cap B$ is the image of $|V \times_X W| \to |X|$ (Lemma 66.4.3). Since $V \times_X W$ is quasi-compact as X is quasi-separated (Lemma 66.3.3) we conclude that $A \cap B$ is quasi-compact and the proof is finished. □

The following lemma can be used to prove that an algebraic space is isomorphic to the spectrum of a field.

03DZ **Lemma 66.15.3.** *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is quasi-separated, then $X \cong \mathrm{Spec}(k')$ where k/k' is a finite separable extension.*

[4]Actually we use here also Schemes, Lemma 26.11.1 (soberness schemes), Morphisms, Lemmas 29.36.12 and 29.25.9 (generalizations lift along étale morphisms), Lemma 66.4.5 (points on an algebraic space in terms of a presentation), and Lemma 66.4.6 (openness quotient map).

Proof. Set $R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k)$, so that we have a fibre product diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & \mathrm{Spec}(k) \\ {\scriptstyle t}\downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

By Spaces, Lemma 65.9.1 we know $X = \mathrm{Spec}(k)/R$ is the quotient sheaf. Because $\mathrm{Spec}(k) \to X$ is étale, the morphisms s and t are étale. Hence $R = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields, and both s and t induce finite separable field extensions $s, t : k \subset k_i$, see Morphisms, Lemma 29.36.7. Because

$$R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) = (\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)) \times_{X \times_S X, \Delta} X$$

and since Δ is quasi-compact by assumption we conclude that $R \to \mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is quasi-compact. Hence R is quasi-compact as $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is affine. We conclude that I is finite. This implies that s and t are finite locally free morphisms. Hence by Groupoids, Proposition 39.23.9 we conclude that $\mathrm{Spec}(k)/R$ is represented by $\mathrm{Spec}(k')$, with $k' \subset k$ finite locally free where

$$k' = \{x \in k \mid s_i(x) = t_i(x) \text{ for all } i \in I\}$$

It is easy to see that k' is a field. □

03E0 **Remark 66.15.4.** Lemma 66.15.3 holds for decent algebraic spaces, see Decent Spaces, Lemma 68.12.8. In fact a decent algebraic space with one point is a scheme, see Decent Spaces, Lemma 68.14.2. This also holds when X is locally separated, because a locally separated algebraic space is decent, see Decent Spaces, Lemma 68.15.2.

66.16. Étale morphisms of algebraic spaces

03FQ This section really belongs in the chapter on morphisms of algebraic spaces, but we need the notion of an algebraic space étale over another in order to define the small étale site of an algebraic space. Thus we need to do some preliminary work on étale morphisms from schemes to algebraic spaces, and étale morphisms between algebraic spaces. For more about étale morphisms of algebraic spaces, see Morphisms of Spaces, Section 67.39.

03EC **Lemma 66.16.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U, U' be schemes over S.*

1. *If $U \to U'$ is an étale morphism of schemes, and if $U' \to X$ is an étale morphism from U' to X, then the composition $U \to X$ is an étale morphism from U to X.*
2. *If $\varphi : U \to X$ and $\varphi' : U' \to X$ are étale morphisms towards X, and if $\chi : U \to U'$ is a morphism of schemes such that $\varphi = \varphi' \circ \chi$, then χ is an étale morphism of schemes.*
3. *If $\chi : U \to U'$ is a surjective étale morphism of schemes and $\varphi' : U' \to X$ is a morphism such that $\varphi = \varphi' \circ \chi$ is étale, then φ' is étale.*

Proof. Recall that our definition of an étale morphism from a scheme into an algebraic space comes from Spaces, Definition 65.5.1 via the fact that any morphism from a scheme into an algebraic space is representable.

Part (1) of the lemma follows from this, the fact that étale morphisms are preserved under composition (Morphisms, Lemma 29.36.3) and Spaces, Lemmas 65.5.4 and 65.5.3 (which are formal).

To prove part (2) choose a scheme W over S and a surjective étale morphism $W \to X$. Consider the base change $\chi_W : W \times_X U \to W \times_X U'$ of χ. As $W \times_X U$ and $W \times_X U'$ are étale over W, we conclude that χ_W is étale, by Morphisms, Lemma 29.36.18. On the other hand, in the commutative diagram

$$\begin{array}{ccc} W \times_X U & \longrightarrow & W \times_X U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

the two vertical arrows are étale and surjective. Hence by Descent, Lemma 35.14.4 we conclude that $U \to U'$ is étale.

To prove part (3) choose a scheme W over S and a morphism $W \to X$. As above we consider the diagram

$$\begin{array}{ccccc} W \times_X U & \longrightarrow & W \times_X U' & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U' & \longrightarrow & X \end{array}$$

Now we know that $W \times_X U \to W \times_X U'$ is surjective étale (as a base change of $U \to U'$) and that $W \times_X U \to W$ is étale. Thus $W \times_X U' \to W$ is étale by Descent, Lemma 35.14.4. By definition this means that φ' is étale. □

03FR **Definition 66.16.2.** Let S be a scheme. A morphism $f : X \to Y$ between algebraic spaces over S is called *étale* if and only if for every étale morphism $\varphi : U \to X$ where U is a scheme, the composition $f \circ \varphi$ is étale also.

If X and Y are schemes, then this agree with the usual notion of an étale morphism of schemes. In fact, whenever $X \to Y$ is a representable morphism of algebraic spaces, then this agrees with the notion defined via Spaces, Definition 65.5.1. This follows by combining Lemma 66.16.3 below and Spaces, Lemma 65.11.4.

03FS **Lemma 66.16.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *there exists a surjective étale morphism $\varphi : U \to X$, where U is a scheme, such that the composition $f \circ \varphi$ is étale (as a morphism of algebraic spaces),*
(3) *there exists a surjective étale morphism $\psi : V \to Y$, where V is a scheme, such that the base change $V \times_Y X \to V$ is étale (as a morphism of algebraic spaces),*
(4) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and the left vertical arrow is surjective such that the horizontal arrow is étale.

Proof. Let us prove that (4) implies (1). Assume a diagram as in (4) given. Let $W \to X$ be an étale morphism with W a scheme. Then we see that $W \times_X U \to U$ is étale. Hence $W \times_X U \to V$ is étale as the composition of the étale morphisms of schemes $W \times_X U \to U$ and $U \to V$. Therefore $W \times_X U \to Y$ is étale by Lemma 66.16.1 (1). Since also the projection $W \times_X U \to W$ is surjective and étale, we conclude from Lemma 66.16.1 (3) that $W \to Y$ is étale.

Let us prove that (1) implies (4). Assume (1). Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 65.11.6. By assumption the morphism $U \to Y$ is étale, and hence $U \to V$ is étale by Lemma 66.16.1 (2).

We omit the proof that (2) and (3) are also equivalent to (1). □

03FT **Lemma 66.16.4.** *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is immediate from the definition. □

03FU **Lemma 66.16.5.** *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \to Y$ be a morphism of algebraic spaces. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme W and a surjective étale morphism $W \to Z$. Then $U \to Y$ is étale, hence in the diagram

$$\begin{array}{ccc} W \times_Y U & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

the top horizontal arrow is étale. Moreover, the left vertical arrow is surjective and étale (verification omitted). Hence we conclude that the lower horizontal arrow is étale by Lemma 66.16.3. □

03FV **Lemma 66.16.6.** *Let S be a scheme. Let X, Y, Z be algebraic spaces. Let $g : X \to Z$, $h : Y \to Z$ be étale morphisms and let $f : X \to Y$ be a morphism such that $h \circ f = g$. Then f is étale.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\chi]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 65.11.6. By assumption the morphisms $\varphi : U \to X \to Z$ and $\psi : V \to Y \to Z$ are étale. Moreover, $\psi \circ \chi = \varphi$ by our assumption on f, g, h. Hence $U \to V$ is étale by Lemma 66.16.1 part (2). □

03IR **Lemma 66.16.7.** *Let S be a scheme. If $X \to Y$ is an étale morphism of algebraic spaces over S, then the associated map $|X| \to |Y|$ of topological spaces is open.*

Proof. This is clear from the diagram in Lemma 66.16.3 and Lemma 66.4.6. □

Finally, here is a fun lemma. It is not true that an algebraic space with an étale morphism towards a scheme is a scheme, see Spaces, Example 65.14.2. But it is true if the target is the spectrum of a field.

03KX **Lemma 66.16.8.** *Let S be a scheme. Let $X \to \mathrm{Spec}(k)$ be étale morphism over S, where k is a field. Then X is a scheme.*

Proof. Let U be an affine scheme, and let $U \to X$ be an étale morphism. By Definition 66.16.2 we see that $U \to \mathrm{Spec}(k)$ is an étale morphism. Hence $U = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k_i)$ is a finite disjoint union of spectra of finite separable extensions k_i of k, see Morphisms, Lemma 29.36.7. The $R = U \times_X U \to U \times_{\mathrm{Spec}(k)} U$ is a monomorphism and $U \times_{\mathrm{Spec}(k)} U$ is also a finite disjoint union of spectra of finite separable extensions of k. Hence by Schemes, Lemma 26.23.11 we see that R is similarly a finite disjoint union of spectra of finite separable extensions of k. This U and R are affine and both projections $R \to U$ are finite locally free. Hence U/R is a scheme by Groupoids, Proposition 39.23.9. By Spaces, Lemma 65.10.2 it is also an open subspace of X. By Lemma 66.13.1 we conclude that X is a scheme. □

66.17. Spaces and fpqc coverings

03W8 Let S be a scheme. An algebraic space over S is defined as a sheaf in the fppf topology with additional properties. Hence it is not immediately clear that it satisfies the sheaf property for the fpqc topology (see Topologies, Definition 34.9.13). In this section we give Gabber's argument showing this is true. However, when we say that the algebraic space X satisfies the sheaf property for the fpqc topology we really only consider fpqc coverings $\{f_i : T_i \to T\}_{i \in I}$ such that T, T_i are objects of the big site $(Sch/S)_{fppf}$ (as per our conventions, see Section 66.2).

0APL **Proposition 66.17.1** (Gabber)**.** *Let S be a scheme. Let X be an algebraic space over S. Then X satisfies the sheaf property for the fpqc topology.*

Proof. Since X is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines $f : T' \to T$ we have: $X(T)$ is the equalizer of the two maps $X(T') \to X(T' \times_T T')$. See Topologies, Lemma 34.9.14 (there is a little argument omitted here because the lemma cited is formulated for functors defined on the category of all schemes).

Let $a, b : T \to X$ be two morphisms such that $a \circ f = b \circ f$. We have to show $a = b$. Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Spaces, Lemma 65.13.1 the morphism $\Delta_{X/S}$ is a representable monomorphism. Hence $E \to T$ is a monomorphism of schemes. Our assumption that $a \circ f = b \circ f$ implies that $T' \to T$ factors (uniquely) through E. Consider the commutative diagram

$$\begin{array}{ccc} T' \times_T E & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ T' & \longrightarrow & T \end{array}$$

Since the projection $T' \times_T E \to T'$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $E \to T$ is an isomorphism by Descent, Lemma 35.23.19. This means $a = b$ as desired.

Next, let $c : T' \to X$ be a morphism such that the two compositions $T' \times_T T' \to T' \to X$ are the same. We have to find a morphism $a : T \to X$ whose composition with $T' \to T$ is c. Choose an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |X|$ contains the image of $|c| : |T'| \to |X|$. This is possible by Lemmas 66.4.6 and 66.6.1, the fact that a finite disjoint union of affines is affine, and the fact that $|T'|$ is quasi-compact (small argument omitted). Since $U \to X$ is separated (Lemma 66.6.4), we see that

$$V = U \times_{X,c} T' \longrightarrow T'$$

is a surjective, étale, separated morphism of schemes (to see that it is surjective use Lemma 66.4.3 and our choice of $U \to X$). The fact that $c \circ \mathrm{pr}_0 = c \circ \mathrm{pr}_1$ means that we obtain a descent datum on $V/T'/T$ (Descent, Definition 35.34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c\circ\mathrm{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c\circ\mathrm{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism $V \to T'$ is ind-quasi-affine by More on Morphisms, Lemma 37.66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 29.36.6). By More on Groupoids, Lemma 40.15.3 the descent datum is effective. Say $W \to T$ is a morphism such that there is an isomorphism $\alpha : T' \times_T W \to V$ compatible with the given descent datum on V and the canonical descent datum on $T' \times_T W$. Then $W \to T$ is surjective and étale (Descent, Lemmas 35.23.7 and 35.23.31). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions $b' \circ (\mathrm{pr}_0, 1), b' \circ (\mathrm{pr}_1, 1) : (T' \times_T T') \times_T W \to T' \times_T W \to U$ agree by our choice of α and the corresponding property of c (computation omitted). Hence b' descends to a morphism $b : W \to U$ by Descent, Lemma 35.13.7. The diagram

$$\begin{array}{ccccc} T' \times_T W & \longrightarrow & W & \xrightarrow[b]{} & U \\ \downarrow & & & & \downarrow \\ T' & & \xrightarrow{\quad c \quad} & & X \end{array}$$

is commutative. What this means is that we have proved the existence of a étale locally on T, i.e., we have an $a' : W \to X$. However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have $\mathrm{pr}_0^* a' = \mathrm{pr}_1^* a'$ as elements of $X(W \times_T W)$. Since X is an étale sheaf we find a unique $a \in X(T)$ restricting to a' on W. □

66.18. The étale site of an algebraic space

03EB In this section we define the small étale site of an algebraic space. This is the analogue of the small étale site $S_{étale}$ of a scheme. Lemma 66.16.1 implies that in the definition below any morphism between objects of the étale site of X is étale, and that any scheme étale over an object of $X_{étale}$ is also an object of $X_{étale}$.

03ED **Definition 66.18.1.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The *small étale site* $X_{étale}$ of X is defined as follows:

(1) An object of $X_{étale}$ is a morphism $\varphi : U \to X$ where $U \in \mathrm{Ob}((Sch/S)_{étale})$ is a scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{(U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{étale}$ is a covering if and only if $\{U_i \to U\}_{i \in I}$ is a covering of $(Sch/S)_{étale}$.

A consequence of our choice is that the étale site of an algebraic space in general does not have a final object! On the other hand, if X happens to be a scheme, then the definition above agrees with Topologies, Definition 34.4.8.

The above is our default site, but there are a couple of variants which we will also use. Namely, we can consider all *algebraic spaces* U which are étale over X and this produces the site $X_{spaces,étale}$ we define below or we can consider all *affine schemes* U which are étale over X and this produces the site $X_{affine,étale}$ we define below. The first of these two notions is used when discussing functoriality of the small étale site, see Lemma 66.18.8.

03G0 **Definition 66.18.2.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{spaces,étale}$ of X is defined as follows:

(1) An object of $X_{spaces,étale}$ is a morphism $\varphi : U \to X$ where U is an algebraic space over S and φ is an étale morphism of algebraic spaces over S,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{spaces,étale}$ is given by a morphism of algebraic spaces $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{spaces,étale}$ is a covering if and only if $|U| = \bigcup \varphi_i(|U_i|)$.

As usual we choose a set of coverings of this type, including at least the coverings in $X_{étale}$, as in Sets, Lemma 3.11.1 to turn $X_{spaces,étale}$ into a site.

Since the identity morphism of X is étale it is clear that $X_{spaces,étale}$ does have a final object. Let us show right away that the corresponding topos equals the small étale topos of X.

03G1 **Lemma 66.18.3.** *The functor*

$$X_{étale} \longrightarrow X_{spaces,étale}, \quad U/X \longmapsto U/X$$

is a special cocontinuous functor (Sites, Definition 7.29.2) and hence induces an equivalence of topoi $Sh(X_{étale}) \to Sh(X_{spaces,étale})$.

Proof. We have to show that the functor satisfies the assumptions (1) – (5) of Sites, Lemma 7.29.1. It is clear that the functor is continuous and cocontinuous, which proves assumptions (1) and (2). Assumptions (3) and (4) hold simply because the functor is fully faithful. Assumption (5) holds, because an algebraic space by definition has a covering by a scheme. □

03H7 **Remark 66.18.4.** Let us explain the meaning of Lemma 66.18.3. Let S be a scheme, and let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on the small étale site $X_{étale}$ of X. The lemma says that there exists a unique sheaf $\mathcal{F}'$ on $X_{spaces,étale}$ which restricts back to $\mathcal{F}$ on the subcategory $X_{étale}$. If $U \to X$ is an étale morphism of algebraic spaces, then how do we compute $\mathcal{F}'(U)$? Well, by definition of an algebraic space there exists a scheme U' and a surjective étale morphism $U' \to U$. Then $\{U' \to U\}$ is a covering in $X_{spaces,étale}$ and hence we get an equalizer diagram

$$\mathcal{F}'(U) \longrightarrow \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U').$$

Note that $U' \times_U U'$ is a scheme, and hence we may write $\mathcal{F}$ and not $\mathcal{F}'$. Thus we see how to compute $\mathcal{F}'$ when given the sheaf $\mathcal{F}$.

0H01 **Definition 66.18.5.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{affine,étale}$ of X is defined as follows:

(1) An object of $X_{affine,étale}$ is a morphism $\varphi : U \to X$ where $U \in \mathrm{Ob}((Sch/S)_{étale})$ is an affine scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{affine,étale}$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{affine,étale}$ is a covering if and only if $\{U_i \to U\}$ is a standard étale covering, see Topologies, Definition 34.4.5.

As usual we choose a set of coverings of this type, as in Sets, Lemma 3.11.1 to turn $X_{affine,étale}$ into a site.

04JS **Lemma 66.18.6.** *Let S be a scheme. Let X be an algebraic space over S. The functor $X_{affine,étale} \to X_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh(X_{affine,étale})$ to $Sh(X_{étale})$.*

Proof. Omitted. Hint: compare with the proof of Topologies, Lemma 34.4.11. □

04JT **Definition 66.18.7.** Let S be a scheme. Let X be an algebraic space over S. The *étale topos* of X, or more precisely the *small étale topos* of X is the category $Sh(X_{étale})$ of sheaves of sets on $X_{étale}$.

By Lemma 66.18.3 we have $Sh(X_{étale}) = Sh(X_{spaces,étale})$, so we can also think of this as the category of sheaves of sets on $X_{spaces,étale}$. Similarly, by Lemma 66.18.6 we see that $Sh(X_{étale}) = Sh(X_{affine,étale})$. It turns out that the topos is functorial with respect to morphisms of algebraic spaces. Here is a precise statement.

03G2 **Lemma 66.18.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *The continuous functor*

$$Y_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V \longmapsto X \times_Y V$$

induces a morphism of sites

$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$

(2) *The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 7.14.5).*

(3) *The morphism of topoi associated to $f_{spaces,étale}$ induces, via Lemma 66.18.3, a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.*
(4) *If f is a representable morphism of algebraic spaces, then f_{small} comes from a morphism of sites $X_{étale} \to Y_{étale}$, corresponding to the continuous functor $V \mapsto X \times_Y V$.*

Proof. Let us show that the functor described in (1) satisfies the assumptions of Sites, Proposition 7.14.7. Thus we have to show that $Y_{spaces,étale}$ has a final object (namely Y) and that the functor transforms this into a final object in $X_{spaces,étale}$ (namely X). This is clear as $X \times_Y Y = X$ in any category. Next, we have to show that $Y_{spaces,étale}$ has fibre products. This is true since the category of algebraic spaces has fibre products, and since $V \times_Y V'$ is étale over Y if V and V' are étale over Y (see Lemmas 66.16.4 and 66.16.5 above). OK, so the proposition applies and we see that we get a morphism of sites as described in (1).

Part (2) you get by unwinding the definitions. Part (3) is clear by using the equivalences for X and Y from Lemma 66.18.3 above. Part (4) follows, because if f is representable, then the functors above fit into a commutative diagram

$$\begin{array}{ccc} X_{étale} & \longrightarrow & X_{spaces,étale} \\ \uparrow & & \uparrow \\ Y_{étale} & \longrightarrow & Y_{spaces,étale} \end{array}$$

of categories. □

We can do a little bit better than the lemma above in describing the relationship between sheaves on X and sheaves on Y. Namely, we can formulate this in turns of f-maps, compare Sheaves, Definition 6.21.7, as follows.

03G3 **Definition 66.18.9.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. An *f-map* $\varphi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\varphi_{(U,V,g)} : \mathcal{G}(V) \to \mathcal{F}(U)$ indexed by commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ such that whenever given an extended diagram

$$\begin{array}{ccccc} U' & \longrightarrow & U & \longrightarrow & X \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

with $V' \to V$ and $U' \to U$ étale morphisms of schemes the diagram

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow[\varphi_{(U,V,g)}]{} & \mathcal{F}(U) \\ {\scriptstyle \text{restriction of } \mathcal{G}}\downarrow & & \downarrow{\scriptstyle \text{restriction of } \mathcal{F}} \\ \mathcal{G}(V') & \xrightarrow{\varphi_{(U',V',g')}} & \mathcal{F}(U') \end{array}$$

commutes.

03G4 **Lemma 66.18.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. There are canonical bijections between the following three sets:*

(1) *The set of maps $\mathcal{G} \to f_{small,*}\mathcal{F}$.*
(2) *The set of maps $f_{small}^{-1}\mathcal{G} \to \mathcal{F}$.*
(3) *The set of f-maps $\varphi : \mathcal{G} \to \mathcal{F}$.*

Proof. Note that (1) and (2) are the same because the functors $f_{small,*}$ and f_{small}^{-1} are a pair of adjoint functors. Suppose that $\alpha : f_{small}^{-1}\mathcal{G} \to \mathcal{F}$ is a map of sheaves on $Y_{étale}$. Let a diagram

$$\begin{array}{ccc} U & \xrightarrow{j_U} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j_V} & Y \end{array}$$

as in Definition 66.18.9 be given. By the commutativity of the diagram we also get a map $g_{small}^{-1}(j_V)^{-1}\mathcal{G} \to (j_U)^{-1}\mathcal{F}$ (compare Sites, Section 7.25 for the description of the localization functors). Hence we certainly get a map $\varphi_{(V,U,g)} : \mathcal{G}(V) = (j_V)^{-1}\mathcal{G}(V) \to (j_U)^{-1}\mathcal{F}(U) = \mathcal{F}(U)$. We omit the verification that this rule is compatible with further restrictions and defines an f-map from $\mathcal{G}$ to $\mathcal{F}$.

Conversely, suppose that we are given an f-map $\varphi = (\varphi_{(U,V,g)})$. Let $\mathcal{G}'$ (resp. $\mathcal{F}'$) denote the extension of $\mathcal{G}$ (resp. $\mathcal{F}$) to $Y_{spaces,étale}$ (resp. $X_{spaces,étale}$), see Lemma 66.18.3. Then we have to construct a map of sheaves

$$\mathcal{G}' \longrightarrow (f_{spaces,étale})_*\mathcal{F}'$$

To do this, let $V \to Y$ be an étale morphism of algebraic spaces. We have to construct a map of sets

$$\mathcal{G}'(V) \to \mathcal{F}'(X \times_Y V)$$

Choose an étale surjective morphism $V' \to V$ with V' a scheme, and after that choose an étale surjective morphism $U' \to X \times_U V'$ with U' a scheme. We get a morphism of schemes $g' : U' \to V'$ and also a morphism of schemes

$$g'' : U' \times_{X \times_Y V} U' \longrightarrow V' \times_V V'$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{F}'(X \times_Y V) & \longrightarrow & \mathcal{F}(U') & \rightrightarrows & \mathcal{F}(U' \times_{X \times_Y V} U') \\ \uparrow & & \uparrow{\scriptstyle \varphi_{(U',V',g')}} & & \uparrow{\scriptstyle \varphi_{(U'',V'',g'')}} \\ \mathcal{G}'(X \times_Y V) & \longrightarrow & \mathcal{G}(V') & \rightrightarrows & \mathcal{G}(V' \times_V V') \end{array}$$

The compatibility of the maps $\varphi_{\ldots}$ with restriction shows that the two right squares commute. The definition of coverings in $X_{spaces,étale}$ shows that the horizontal rows are equalizer diagrams. Hence we get the dotted arrow. We leave it to the reader to show that these arrows are compatible with the restriction mappings. □

If the morphism of algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

03LP **Lemma 66.18.11.** *Let S be a scheme, and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is étale. In this case there is a functor*

$$j : X_{étale} \to Y_{étale}, \quad (\varphi : U \to X) \mapsto (f \circ \varphi : U \to Y)$$

which is cocontinuous. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 7.21.1. Moreover, j is continuous as well, hence Sites, Lemma 7.21.5 applies. In particular $f_{small}^{-1}\mathcal{G}(U) = \mathcal{G}(jU)$ for all sheaves $\mathcal{G}$ on $Y_{étale}$.

Proof. Note that by our very definition of an étale morphism of algebraic spaces (Definition 66.16.2) it is indeed the case that the rule given defines a functor j as indicated. It is clear that j is cocontinuous and continuous, simply because a covering $\{U_i \to U\}$ of $j(\varphi : U \to X)$ in $Y_{étale}$ is the same thing as a covering of $(\varphi : U \to X)$ in $X_{étale}$. It remains to show that j induces the same morphism of topoi as f_{small}. To see this we consider the diagram

$$\begin{array}{ccc} X_{étale} & \longrightarrow & X_{spaces,étale} \\ \downarrow{\scriptstyle j} & & {\scriptstyle j_{spaces}}\swarrow \quad \uparrow{\scriptstyle v:V\mapsto X\times_Y V} \\ Y_{étale} & \longrightarrow & Y_{spaces,étale} \end{array}$$

of categories. Here the functor j_{spaces} is the obvious extension of j to the category $X_{spaces,étale}$. Thus the inner square is commutative. In fact j_{spaces} can be identified with the localization functor $j_X : Y_{spaces,étale}/X \to Y_{spaces,étale}$ discussed in Sites, Section 7.25. Hence, by Sites, Lemma 7.27.2 the cocontinuous functor j_{spaces} and the functor v of the diagram induce the same morphism of topoi. By Sites, Lemma 7.21.2 the commutativity of the inner square (consisting of cocontinuous functors between sites) gives a commutative diagram of associated morphisms of topoi. Hence, by the construction of f_{small} in Lemma 66.18.8 we win. □

The lemma above says that the pullback of $\mathcal{G}$ via an étale morphism $f : X \to Y$ of algebraic spaces is simply the restriction of $\mathcal{G}$ to the category $X_{étale}$. We will often use the short hand

03LQ (66.18.11.1) $$\mathcal{G}|_{X_{étale}} = f_{small}^{-1}\mathcal{G}$$

to indicate this. Note that the functor $j : X_{étale} \to Y_{étale}$ of the lemma in this situation is faithful, but not fully faithful in general. We will discuss this in a more technical fashion in Section 66.27.

03LR **Lemma 66.18.12.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. If g is étale, then

(1) *$f'_{small,*}(\mathcal{F}|_{X'}) = (f_{small,*}\mathcal{F})|_{Y'}$ in $Sh(Y'_{étale})$[5], and*
(2) *if $\mathcal{F}$ is an abelian sheaf, then $R^i f'_{small,*}(\mathcal{F}|_{X'}) = (R^i f_{small,*}\mathcal{F})|_{Y'}$.*

[5]Also $(f')^{-1}_{small}(\mathcal{G}|_{Y'}) = (f^{-1}_{small}\mathcal{G})|_{X'}$ because of commutativity of the diagram and (66.18.11.1)

Proof. Consider the following diagram of functors

$$\begin{array}{ccc} X'_{spaces,étale} & \xrightarrow[j]{} & X_{spaces,étale} \\ {\scriptstyle V'\mapsto V'\times_{Y'}X'}\uparrow & & \uparrow{\scriptstyle V\mapsto V\times_Y X} \\ Y'_{spaces,étale} & \xrightarrow{j} & Y_{spaces,étale} \end{array}$$

The horizontal arrows are localizations and the vertical arrows induce morphisms of sites. Hence the last statement of Sites, Lemma 7.28.1 gives (1). To see (2) apply (1) to an injective resolution of $\mathcal{F}$ and use that restriction is exact and preserves injectives (see Cohomology on Sites, Lemma 21.7.1). □

The following lemma says that you can think of a sheaf on the small étale site of an algebraic space as a compatible collection of sheaves on the small étale sites of schemes étale over the space. Please note that all the comparison mappings c_f in the lemma are isomorphisms, which is compatible with Topologies, Lemma 34.4.20 and the fact that all morphisms between objects of $X_{étale}$ are étale.

03LS **Lemma 66.18.13.** *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ on $X_{étale}$ is given by the following data:*

(1) *for every $U \in \mathrm{Ob}(X_{étale})$ a sheaf $\mathcal{F}_U$ on $U_{étale}$,*
(2) *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f_{small}^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $c_g \circ g_{small}^{-1}c_f$ is equal to $c_{f\circ g}$.

Proof. We may interpret g_{small}^{-1} as in Lemma 66.18.11. Then the lemma follows from a general fact about sites, see Sites, Lemma 7.26.6. □

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 65.9.3. In particular, we obtain a groupoid (U, R, s, t, c, e, i) such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 39.13.3.

05YY **Lemma 66.18.14.** *With S, $\varphi : U \to X$, and (U, R, s, t, c, e, i) as above. For any sheaf $\mathcal{F}$ on $X_{étale}$ the sheaf[6] $\mathcal{G} = \varphi^{-1}\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^{-1}\mathcal{G} \longrightarrow s^{-1}\mathcal{G}$$

such that the diagram

$$\begin{array}{ccccc} & & pr_1^{-1}t^{-1}\mathcal{G} & \xrightarrow[pr_1^{-1}\alpha]{} & pr_1^{-1}s^{-1}\mathcal{G} & & \\ & \swarrow & & & & \searrow & \\ pr_0^{-1}s^{-1}\mathcal{G} & & & & & & c^{-1}s^{-1}\mathcal{G} \\ & \nwarrow^{pr_0^{-1}\alpha} & & & & \nearrow_{c^{-1}\alpha} & \\ & & pr_0^{-1}t^{-1}\mathcal{G} & = & c^{-1}t^{-1}\mathcal{G} & & \end{array}$$

is a commutative. The functor $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ defines an equivalence of categories between sheaves on $X_{étale}$ and pairs $(\mathcal{G}, \alpha)$ as above.

[6] In this lemma and its proof we write simply φ^{-1} instead of φ_{small}^{-1} and similarly for all the other pullbacks.

First proof of Lemma 66.18.14. Let $\mathcal{C} = X_{spaces,étale}$. By Lemma 66.18.11 and its proof we have $U_{spaces,étale} = \mathcal{C}/U$ and the pullback functor φ^{-1} is just the restriction functor. Moreover, $\{U \to X\}$ is a covering of the site $\mathcal{C}$ and $R = U \times_X U$. The isomorphism α is just the canonical identification

$$\left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U\times_X U} = \left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U\times_X U}$$

and the commutativity of the diagram is the cocycle condition for glueing data. Hence this lemma is a special case of glueing of sheaves, see Sites, Section 7.26. □

Second proof of Lemma 66.18.14. The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see Lemma 66.18.8. In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α fits into the commutative diagram. The construction $\mathcal{F} \mapsto (\varphi^{-1}\mathcal{F}, \alpha)$ is clearly functorial in the sheaf $\mathcal{F}$. Hence we obtain the functor.

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a pair. Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Set $\mathcal{G}' = (V' \to V)^{-1}\mathcal{G}$. Since $R = U \times_X U$ with $t = \mathrm{pr}_0$ and $s = \mathrm{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $s', t' : V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : (t')^{-1}\mathcal{G}' \to (s')^{-1}\mathcal{G}'$. Having said this we simply define

$$\mathcal{F}(V) = \text{Equalizer}(\mathcal{G}(V') \rightrightarrows \mathcal{G}(V' \times_V V')).$$

We omit the verification that this defines a sheaf. To see that $\mathcal{G}(V) = \mathcal{F}(V)$ if there exists a morphism $V \to U$ note that in this case the equalizer is $H^0(\{V' \to V\}, \mathcal{G}) = \mathcal{G}(V)$. □

66.19. Points of the small étale site

04JU This section is the analogue of Étale Cohomology, Section 59.29.

0486 **Definition 66.19.1.** Let S be a scheme. Let X be an algebraic space over S.

(1) A *geometric point* of X is a morphism $\overline{x} : \mathrm{Spec}(k) \to X$, where k is an algebraically closed field. We often abuse notation and write $\overline{x} = \mathrm{Spec}(k)$.
(2) For every geometric point $\overline{x}$ we have the corresponding "image" point $x \in |X|$. We say that $\overline{x}$ is a *geometric point lying over* x.

It turns out that we can take stalks of sheaves on $X_{étale}$ at geometric points exactly in the same way as was done in the case of the small étale site of a scheme. In order to do this we define the notion of an étale neighbourhood as follows.

04JV **Definition 66.19.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.

(1) An *étale neighborhood* of $\overline{x}$ of X is a commutative diagram

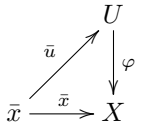

where φ is an étale morphism of algebraic spaces over S. We will use the notation $\varphi : (U, \overline{u}) \to (X, \overline{x})$ to indicate this situation.

(2) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an X-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Note that we allow U to be an algebraic space. When we take stalks of a sheaf on $X_{étale}$ we have to restrict to those U which are in $X_{étale}$, and so in this case we will only consider the case where U is a scheme. Alternately we can work with the site $X_{space,étale}$ and consider all étale neighbourhoods. And there won't be any difference because of the last assertion in the following lemma.

04JW **Lemma 66.19.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. The category of étale neighborhoods is cofiltered. More precisely:*

(1) *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{x}$ in X. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Moreover, given any étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ there exists a morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ where U' is a scheme.

Proof. For part (1), consider the fibre product $U = U_1 \times_X U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change and composition, see Lemmas 66.16.5 and 66.16.4. The map $\overline{u} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2.

For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1,h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_X U'. \end{array}$$

Since $\overline{u}$ and $\overline{u}'$ agree over X with $\overline{x}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over X, so is the fibre product $U' \times_X U'$ (as seen above in the case of $U_1 \times_X U_2$). Hence the vertical arrow (h_1, h_2) is étale by Lemma 66.16.6. Therefore U'' is étale over U' by base change, and hence also étale over X (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem posed by (2).

To see the final assertion, choose any surjective étale morphism $U' \to U$ where U' is a scheme. Then $U' \times_U \overline{u}$ is a scheme surjective and étale over $\overline{u} = \mathrm{Spec}(k)$ with k algebraically closed. It follows (see Morphisms, Lemma 29.36.7) that $U' \times_U \overline{u} \to \overline{u}$ has a section which gives us the desired $\overline{u}'$. □

05VN **Lemma 66.19.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point of X lying over $x \in |X|$. Let $\varphi : U \to X$ be an étale morphism of algebraic spaces and let $u \in |U|$ with $\varphi(u) = x$. Then there exists a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = \varphi \circ \overline{u}$.*

Proof. Choose an affine scheme U' with $u' \in U'$ and an étale morphism $U' \to U$ which maps u' to u. If we can prove the lemma for $(U', u') \to (X, x)$ then the lemma

follows. Hence we may assume that U is a scheme, in particular that $U \to X$ is representable. Then look at the cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U & \xrightarrow{\mathrm{pr}_2} & U \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \end{array}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Lemma 66.16.5. Therefore, the scheme $\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of finite separable extensions of k, see Morphisms, Lemma 29.36.7. But k is algebraically closed, so all these extensions are trivial, so $\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of copies of $\mathrm{Spec}(k)$ and each of these corresponds to a geometric point $\overline{u}$ with $\varphi \circ \overline{u} = \overline{x}$. By Lemma 66.4.3 the map

$$|\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U| \longrightarrow |\mathrm{Spec}(k)| \times_{|X|} |U|$$

is surjective, hence we can pick $\overline{u}$ to lie over u. □

04JX **Lemma 66.19.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ an étale neighborhood of $\overline{x}$. Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering in $X_{spaces,étale}$. Then there exist $i \in I$ and $\overline{u}_i : \overline{x} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. Let $u \in |U|$ be the image of $\overline{u}$. As $|U| = \bigcup_{i \in I} \varphi_i(|U_i|)$ there exists an i and a point $u_i \in U_i$ mapping to x. Apply Lemma 66.19.4 to $(U_i, u_i) \to (U, u)$ and $\overline{u}$ to get the desired geometric point. □

04JY **Definition 66.19.6.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $X_{étale}$. Let $\overline{x}$ be a geometric point of X. The *stalk* of $\mathcal{F}$ at $\overline{x}$ is

$$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{x}$ in X with $U \in \mathrm{Ob}(X_{étale})$.

By Lemma 66.19.3, this colimit is over a filtered index category, namely the opposite of the category of étale neighborhoods in $X_{étale}$. More precisely Lemma 66.19.3 says the opposite of the category of all étale neighbourhoods is filtered, and the full subcategory of those which are in $X_{étale}$ is a cofinal subcategory hence also filtered.

This means an element of $\mathcal{F}_{\overline{x}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$, $U'' \in \mathrm{Ob}(X_{étale})$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

This also implies that if $\mathcal{F}'$ is the sheaf on $X_{spaces,étale}$ corresponding to $\mathcal{F}$ on $X_{étale}$, then

04JZ (66.19.6.1) $$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{F}'(U)$$

where now the colimit is over all the étale neighbourhoods of $\overline{x}$. We will often jump between the point of view of using $X_{étale}$ and $X_{spaces,étale}$ without further mention.

In particular this means that if $\mathcal{F}$ is a presheaf of abelian groups, rings, etc then $\mathcal{F}_{\overline{x}}$ is an abelian group, ring, etc simply by the usual way of defining the group structure on a directed colimit of abelian groups, rings, etc.

04K0 **Lemma 66.19.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Consider the functor*

$$u : X_{étale} \longrightarrow Sets, \quad U \longmapsto |U_{\overline{x}}|$$

Then u defines a point p of the site $X_{étale}$ (Sites, Definition 7.32.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.32.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ defined above.

Proof. In the proof of Lemma 66.19.4 we have seen that the scheme $U_{\overline{x}} = \overline{x} \times_X U$ is a disjoint union of schemes isomorphic to $\overline{x}$. Thus we can also think of $|U_{\overline{x}}|$ as the set of geometric points of U lying over $\overline{x}$, i.e., as the collection of morphisms $\overline{u} : \overline{x} \to U$ fitting into the diagram of Definition 66.19.2. From this it follows that $u(X)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $X_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $X_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 66.19.5. Hence Sites, Proposition 7.33.3 applies, so p is a point of the site $X_{étale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.32.1.1 which proves the final assertion. □

04K1 **Lemma 66.19.8.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.*

(1) *The stalk functor $PAb(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*
(2) *We have $(\mathcal{F}^\#)_{\overline{x}} = \mathcal{F}_{\overline{x}}$ for any presheaf of sets $\mathcal{F}$ on $X_{étale}$.*
(3) *The functor $Ab(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*
(4) *Similarly the functors $PSh(X_{étale}) \to Sets$ and $Sh(X_{étale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.*

Proof. This result follows from the general material in Modules on Sites, Section 18.36. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ comes from a point of the small étale site of X, see Lemma 66.19.7. See the proof of Étale Cohomology, Lemma 59.29.9 for a direct proof of some of these statements in the setting of the small étale site of a scheme. □

We will see below that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is really the pullback along the morphism $\overline{x}$. In that sense the following lemma is a generalization of the lemma above.

04K2 **Lemma 66.19.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *The functor $f_{small}^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
(2) *The functor $f_{small}^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.*
(3) *For any étale morphism $V \to Y$ of algebraic spaces we have $f_{small}^{-1} h_V = h_{X \times_Y V}$.*
(4) *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*

$$(f_{small}^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$

where $\overline{y} = f \circ \overline{x}$.

Proof. Recall that f_{small} is defined via $f_{spaces,small}$ in Lemma 66.18.8. Parts (1), (2) and (3) are general consequences of the fact that $f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ is a morphism of sites, see Sites, Definition 7.14.1 for (2), Modules on Sites, Lemma 18.31.2 for (1), and Sites, Lemma 7.13.5 for (3).

Proof of (4). This statement is a special case of Sites, Lemma 7.34.2 via Lemma 66.19.7. We also provide a direct proof. Note that by Lemma 66.19.8. taking stalks commutes with sheafification. Let $\mathcal{G}'$ be the sheaf on $Y_{spaces,étale}$ whose restriction to $Y_{étale}$ is $\mathcal{G}$. Recall that $f^{-1}_{spaces,étale}\mathcal{G}'$ is the sheaf associated to the presheaf

$$U \longrightarrow \operatorname{colim}_{U \to X \times_Y V} \mathcal{G}'(V),$$

see Sites, Sections 7.13 and 7.5. Thus we have

$$\begin{aligned}(f^{-1}_{spaces,étale}\mathcal{G}')_{\overline{x}} &= \operatorname{colim}_{(U,\overline{u})} f^{-1}_{spaces,étale}\mathcal{G}'(U) \\ &= \operatorname{colim}_{(U,\overline{u})} \operatorname{colim}_{a:U \to X \times_Y V} \mathcal{G}'(V) \\ &= \operatorname{colim}_{(V,\overline{v})} \mathcal{G}'(V) \\ &= \mathcal{G}'_{\overline{y}}\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$. Since the stalk of $\mathcal{G}'$ (resp. $f^{-1}_{spaces,étale}\mathcal{G}'$) agrees with the stalk of $\mathcal{G}$ (resp. $f^{-1}_{small}\mathcal{G}$), see Equation (66.19.6.1) the result follows. □

04K3 **Remark 66.19.10.** This remark is the analogue of Étale Cohomology, Remark 59.56.6. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \operatorname{Spec}(k) \to X$ be a geometric point of X. By Étale Cohomology, Theorem 59.56.3 the category of sheaves on $\operatorname{Spec}(k)_{étale}$ is equivalent to the category of sets (by taking a sheaf to its global sections). Hence it follows from Lemma 66.19.9 part (4) applied to the morphism $\overline{x}$ that the functor

$$Sh(X_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{x}}$$

is isomorphic to the functor

$$Sh(X_{étale}) \longrightarrow Sh(\operatorname{Spec}(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{x}^*\mathcal{F}$$

Hence we may view the stalk functors as pullback functors along geometric morphisms (and not just some abstract morphisms of topoi as in the result of Lemma 66.19.7).

04K4 **Remark 66.19.11.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We claim that for any pair of geometric points $\overline{x}$ and $\overline{x}'$ lying over x the stalk functors are isomorphic. By definition of $|X|$ we can find a third geometric point $\overline{x}''$ so that there exists a commutative diagram

$$\begin{array}{ccc} \overline{x}'' & \longrightarrow & \overline{x}' \\ \downarrow & \overset{\overline{x}''}{\searrow} & \downarrow{\scriptstyle \overline{x}'} \\ \overline{x} & \xrightarrow{\overline{x}} & X. \end{array}$$

Since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is given by pullback along the morphism $\overline{x}$ (and similarly for the others) we conclude by functoriality of pullbacks.

The following theorem says that the small étale site of an algebraic space has enough points.

04K5 **Theorem 66.19.12.** *Let S be a scheme. Let X be an algebraic space over S. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective (resp. surjective) for all geometric points of X. A sequence of abelian sheaves on $X_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. We know the theorem is true if X is a scheme, see Étale Cohomology, Theorem 59.29.10. Choose a surjective étale morphism $f : U \to X$ where U is a scheme. Since $\{U \to X\}$ is a covering (in $X_{spaces,étale}$) we can check whether a map of sheaves is injective, or surjective by restricting to U. Now if $\overline{u} : \mathrm{Spec}(k) \to U$ is a geometric point of U, then $(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$ where $\overline{x} = f \circ \overline{u}$. (This is clear from the colimits defining the stalks at $\overline{u}$ and $\overline{x}$, but it also follows from Lemma 66.19.9.) Hence the result for U implies the result for X and we win. □

The following lemma should be skipped on a first reading.

04K6 **Lemma 66.19.13.** *Let S be a scheme. Let X be an algebraic space over S. Let $p : Sh(pt) \to Sh(X_{étale})$ be a point of the small étale topos of X. Then there exists a geometric point $\overline{x}$ of X such that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$.*

Proof. By Sites, Lemma 7.32.7 there is a one to one correspondence between points of the site and points of the associated topos. Hence we may assume that p is given by a functor $u : X_{étale} \to Sets$ which defines a point of the site $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ be an object whose structure morphism $j : U \to X$ is surjective. Note that h_U is a sheaf which surjects onto the final sheaf. Since taking stalks is exact we see that $(h_U)_p = u(U)$ is not empty (use Sites, Lemma 7.32.3). Pick $x \in u(U)$. By Sites, Lemma 7.35.1 we obtain a point $q : Sh(pt) \to Sh(U_{étale})$ such that $p = j_{small} \circ q$, so that $\mathcal{F}_p = (\mathcal{F}|_U)_q$ functorially. By Étale Cohomology, Lemma 59.29.12 there is a geometric point $\overline{u}$ of U and a functorial isomorphism $\mathcal{G}_q = \mathcal{G}_{\overline{u}}$ for $\mathcal{G} \in Sh(U_{étale})$. Set $\overline{x} = j \circ \overline{u}$. Then we see that $\mathcal{F}_{\overline{x}} \cong (\mathcal{F}|_U)_{\overline{u}}$ functorially in $\mathcal{F}$ on $X_{étale}$ by Lemma 66.19.9 and we win. □

66.20. Supports of abelian sheaves

04K7 First we talk about supports of local sections.

04K8 **Lemma 66.20.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of X (see Sites, Example 7.10.2). Then there exists a unique open $W \subset X$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to X$ in $\mathrm{Ob}(X_{spaces,étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\varphi(U) \subset X$ is an open subspace (Lemma 66.16.7) and $\{\varphi : U \to \varphi(U)\}$ is a covering in $X_{spaces,étale}$. Take $W = \bigcup_{\varphi:U\to S,\mathcal{F}(U)\neq\emptyset} \varphi(U)$ to conclude. □

04K9 **Lemma 66.20.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{spaces,étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subspace $W \subset U$ such that*

(1) *$W \subset U$ is the largest open subspace of U such that $\sigma|_W = 0$,*
(2) *for every $\varphi : V \to U$ in $X_{spaces,étale}$ we have*

$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$

(3) *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{x}} \Leftrightarrow \overline{u} \in W$$
where $\overline{x} = (U \to X) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $X_{spaces,étale}$. Note that $\varphi(V) \subset U$ is an open subspace (Lemma 66.16.7) and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{x}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. By Remark 66.19.11 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over x is well defined.

04KA **Definition 66.20.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$.

(1) The *support of $\mathcal{F}$* is the set of points $x \in |X|$ such that $\mathcal{F}_{\overline{x}} \neq 0$ for any (some) geometric point $\overline{x}$ lying over x.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of σ* is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 66.20.2).

04KB **Lemma 66.20.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in $|X|$.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(X)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $X_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 66.20.2 part (3). Parts (5) and (6) follow from the other parts. □

04KC **Lemma 66.20.5.** *The support of a sheaf of rings on the small étale site of an algebraic space is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

66.21. The structure sheaf of an algebraic space

04KD The structure sheaf of an algebraic space is the sheaf of rings of the following lemma.

03G6 **Lemma 66.21.1.** *Let S be a scheme. Let X be an algebraic space over S. The rule $U \mapsto \Gamma(U, \mathcal{O}_U)$ defines a sheaf of rings on $X_{étale}$.*

Proof. Immediate from the definition of a covering and Descent, Lemma 35.8.1. □

03G7 **Definition 66.21.2.** Let S be a scheme. Let X be an algebraic space over S. The *structure sheaf* of X is the sheaf of rings $\mathcal{O}_X$ on the small étale site $X_{étale}$ described in Lemma 66.21.1.

According to Lemma 66.18.13 the sheaf $\mathcal{O}_X$ corresponds to a system of étale sheaves $(\mathcal{O}_X)_U$ for U ranging through the objects of $X_{étale}$. It is clear from the proof of that lemma and our definition that we have simply $(\mathcal{O}_X)_U = \mathcal{O}_U$ where $\mathcal{O}_U$ is the structure sheaf of $U_{étale}$ as introduced in Descent, Definition 35.8.2. In particular, if X is a scheme we recover the sheaf $\mathcal{O}_X$ on the small étale site of X.

Via the equivalence $Sh(X_{étale}) = Sh(X_{spaces,étale})$ of Lemma 66.18.3 we may also think of $\mathcal{O}_X$ as a sheaf of rings on $X_{spaces,étale}$. It is explained in Remark 66.18.4 how to compute $\mathcal{O}_X(Y)$, and in particular $\mathcal{O}_X(X)$, when $Y \to X$ is an object of $X_{spaces,étale}$.

03G8 **Lemma 66.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there is a canonical map $f^\sharp : f_{small}^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that*

$$(f_{small}, f^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is a morphism of ringed topoi. Furthermore,

1. *The construction $f \mapsto (f_{small}, f^\sharp)$ is compatible with compositions.*
2. *If f is a morphism of schemes, then $f^\sharp$ is the map described in Descent, Remark 35.8.4.*

Proof. By Lemma 66.18.10 it suffices to give an f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. In other words, for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ we have to give a map of rings $(f^\sharp)_{(U,V,g)} : \Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$. Of course we just take $(f^\sharp)_{(U,V,g)} = g^\sharp$. It is clear that this is compatible with restriction mappings and hence indeed gives an f-map. We omit checking compatibility with compositions and agreement with the construction in Descent, Remark 35.8.4. □

0BGS **Lemma 66.21.4.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

1. *X is reduced,*
2. *for every $x \in |X|$ the local ring of X at x is reduced (Remark 66.7.6).*

In this case $\Gamma(X, \mathcal{O}_X)$ is a reduced ring and if $f \in \Gamma(X, \mathcal{O}_X)$ has $X = V(f)$, then $f = 0$.

Proof. The equivalence of (1) and (2) follows from Properties, Lemma 28.3.2 applied to affine schemes étale over X. The final statements follow the cited lemma and fact that $\Gamma(X, \mathcal{O}_X)$ is a subring of $\Gamma(U, \mathcal{O}_U)$ for some reduced scheme U étale over X. □

66.22. Stalks of the structure sheaf

04KE This section is the analogue of Étale Cohomology, Section 59.33.

04KF **Lemma 66.22.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{x}$ where U is a scheme. Then we have*

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,\overline{u}} = \mathcal{O}^{sh}_{U,u}$$

where the left hand side is the stalk of the structure sheaf of X, and the right hand side is the strict henselization of the local ring of U at the point u at which $\overline{u}$ is centered.

Proof. We know that the structure sheaf $\mathcal{O}_U$ on $U_{étale}$ is the restriction of the structure sheaf of X. Hence the first equality follows from Lemma 66.19.9 part (4). The second equality is explained in Étale Cohomology, Lemma 59.33.1. □

04KG **Definition 66.22.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$.

(1) The *étale local ring of X at $\overline{x}$* is the stalk of the structure sheaf $\mathcal{O}_X$ on $X_{étale}$ at $\overline{x}$. Notation: $\mathcal{O}_{X,\overline{x}}$.
(2) The *strict henselization of X at $\overline{x}$* is the scheme $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$.

The isomorphism type of the strict henselization of X at $\overline{x}$ (as a scheme over X) depends only on the point $x \in |X|$ and not on the choice of the geometric point lying over x, see Remark 66.19.11.

04KH **Lemma 66.22.3.** *Let S be a scheme. Let X be an algebraic space over S. The small étale site $X_{étale}$ endowed with its structure sheaf $\mathcal{O}_X$ is a locally ringed site, see Modules on Sites, Definition 18.40.4.*

Proof. This follows because the stalks $\mathcal{O}_{X,\overline{x}}$ are local, and because $S_{étale}$ has enough points, see Lemmas 66.22.1 and Theorem 66.19.12. See Modules on Sites, Lemma 18.40.2 and 18.40.3 for the fact that this implies the small étale site is locally ringed. □

04N9 **Lemma 66.22.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

(1) *the dimension of the local ring of X at x (Definition 66.10.2) is d,*
(2) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for some geometric point $\overline{x}$ lying over x, and*
(3) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (2) and (3) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 66.19.11. Using Lemma 66.22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: Given any local ring R we have $\dim(R) = \dim(R^{sh})$. This is More on Algebra, Lemma 15.45.7. □

0A4H **Lemma 66.22.5.** *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Let $x \in X$. Then (1) $\dim_x(X) = \dim_{f(x)}(Y)$ and (2) the dimension of the local ring of X at x equals the dimension of the local ring of Y at $f(x)$. If f is surjective, then (3) $\dim(X) = \dim(Y)$.*

Proof. Choose a scheme U and a point $u \in U$ and an étale morphism $U \to X$ which maps u to x. Then the composition $U \to Y$ is also étale and maps u to $f(x)$. Thus the statements (1) and (2) follow as the relevant integers are defined in terms of the behaviour of the scheme U at u. See Definition 66.9.1 for (1). Part (3) is an immediate consequence of (1), see Definition 66.9.2. □

0E01 **Lemma 66.22.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

(1) *the local ring of X at x is reduced (Remark 66.7.6),*
(2) *$\mathcal{O}_{X,\overline{x}}$ is reduced for some geometric point $\overline{x}$ lying over x, and*
(3) *$\mathcal{O}_{X,\overline{x}}$ is reduced for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (2) and (3) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 66.19.11. Using Lemma 66.22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: a local ring is reduced if and only if its strict henselization is reduced. This is More on Algebra, Lemma 15.45.4. □

66.23. Local irreducibility

06DJ A point on an algebraic space has a well defined étale local ring, which corresponds to the strict henselization of the local ring in the case of a scheme. In general we cannot see how many irreducible components of a scheme or an algebraic space pass through the given point from the étale local ring. We can only count the number of geometric branches.

06DK **Lemma 66.23.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

(1) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,*
(2) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ there is a unique irreducible component of U through u,*
(3) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ is unibranch,*
(4) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ is geometrically unibranch,*
(5) *$\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ by Lemma 66.22.1. In particular (4) and (5) are equivalent by More on Algebra, Lemma 15.107.5. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 37.36.2. □

06DL **Definition 66.23.2.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We say that X is *geometrically unibranch at x* if the equivalent

conditions of Lemma 66.23.1 hold. We say that X is *geometrically unibranch* if X is geometrically unibranch at every $x \in |X|$.

This is consistent with the definition for schemes (Properties, Definition 28.15.1).

0DQ3 **Lemma 66.23.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $n \in \{1, 2, \ldots\}$ be an integer. The following are equivalent*

(1) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of minimal primes of the local ring $\mathcal{O}_{U,u}$ is $\leq n$ and for at least one choice of U, a, u it is n,*
(2) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number irreducible components of U passing through u is $\leq n$ and for at least one choice of U, a, u it is n,*
(3) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of branches of U at u is $\leq n$ and for at least one choice of U, a, u it is n,*
(4) *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the number of geometric branches of U at u is n, and*
(5) *the number of minimal prime ideals of $\mathcal{O}_{X,\overline{x}}$ is n.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ by Lemma 66.22.1. Recall that the (geometric) number of branches of U at u is the number of minimal prime ideals of the (strict) henselization of $\mathcal{O}_{U,u}$. In particular (4) and (5) are equivalent. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 37.36.2. □

0DQ4 **Definition 66.23.4.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The *number of geometric branches of X at x* is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 66.23.3 hold, or else ∞.

66.24. Noetherian spaces

03E9 We have already defined locally Noetherian algebraic spaces in Section 66.7.

03EA **Definition 66.24.1.** Let S be a scheme. Let X be an algebraic space over S. We say X is *Noetherian* if X is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic space X is not just quasi-compact and locally Noetherian, but also quasi-separated. This does not conflict with the definition of a Noetherian scheme, as a locally Noetherian scheme is quasi-separated, see Properties, Lemma 28.5.4. This does not hold for algebraic spaces. Namely, $X = \mathbf{A}^1_k/\mathbf{Z}$, see Spaces, Example 65.14.8 is locally Noetherian and quasi-compact but not quasi-separated (hence not Noetherian according to our definitions).

A consequence of the choice made above is that an algebraic space of finite type over a Noetherian algebraic space is not automatically Noetherian, i.e., the analogue of Morphisms, Lemma 29.15.6 does not hold. The correct statement is that an algebraic space of finite presentation over a Noetherian algebraic space is Noetherian (see Morphisms of Spaces, Lemma 67.28.6).

A Noetherian algebraic space X is very close to being a scheme. In the rest of this section we collect some lemmas to illustrate this.

04ZF **Lemma 66.24.2.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is locally Noetherian then $|X|$ is a locally Noetherian topological space.*
(2) *If X is quasi-compact and locally Noetherian, then $|X|$ is a Noetherian topological space.*

Proof. Assume X is locally Noetherian. Choose a scheme U and a surjective étale morphism $U \to X$. As X is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 28.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |X|$ is open and surjective we conclude that $|X|$ is locally Noetherian by Topology, Lemma 5.9.3. This proves (1). If X is quasi-compact and locally Noetherian, then $|X|$ is quasi-compact and locally Noetherian. Hence $|X|$ is Noetherian by Topology, Lemma 5.12.14. □

04ZG **Lemma 66.24.3.** *Let S be a scheme. Let X be an algebraic space over S. If X is Noetherian, then $|X|$ is a sober Noetherian topological space.*

Proof. A quasi-separated algebraic space has an underlying sober topological space, see Lemma 66.15.1. It is Noetherian by Lemma 66.24.2. □

08AH **Lemma 66.24.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\overline{x}$ be a geometric point of X. Then $\mathcal{O}_{X,\overline{x}}$ is a Noetherian local ring.*

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ where U is a scheme. Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of the local ring of U at u, see Lemma 66.22.1. By our definition of Noetherian spaces the scheme U is locally Noetherian. Hence we conclude by More on Algebra, Lemma 15.45.3. □

66.25. Regular algebraic spaces

06LP We have already defined regular algebraic spaces in Section 66.7.

06LQ **Lemma 66.25.1.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The following are equivalent*

(1) *X is regular, and*
(2) *every étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption U is locally Noetherian. Moreover, every étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U and conversely, see Lemma 66.22.1. Thus by More on Algebra, Lemma 15.45.10 we see that (2) is equivalent to every local ring of U being regular, i.e., U being a regular scheme (see Properties, Lemma 28.9.2). This equivalent to (1) by Definition 66.7.2. □

We can use Descent, Lemma 35.21.4 to define what it means for an algebraic space X to be regular at a point x.

0AH9 **Definition 66.25.2.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. We say X *is regular at x* if $\mathcal{O}_{U,u}$ is a regular local ring for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$.

See Definition 66.7.5, Lemma 66.7.4, and Descent, Lemma 35.21.4.

0AHA **Lemma 66.25.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

(1) *X is regular at x, and*
(2) *the étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular for any (equivalently some) geometric point $\overline{x}$ lying over x.*

Proof. Let U be a scheme, $u \in U$ a point, and let $a : U \to X$ be an étale morphism mapping u to x. For any geometric point $\overline{x}$ of X lying over x, the étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U at u, see Lemma 66.22.1. Thus we conclude by More on Algebra, Lemma 15.45.10. □

0BGT **Lemma 66.25.4.** *A regular algebraic space is normal.*

Proof. This follows from the definitions and the case of schemes See Properties, Lemma 28.9.4. □

66.26. Sheaves of modules on algebraic spaces

03LT If X is an algebraic space, then a sheaf of modules on X is a sheaf of $\mathcal{O}_X$-modules on the small étale site of X where $\mathcal{O}_X$ is the structure sheaf of X. The category of sheaves of modules is denoted $Mod(\mathcal{O}_X)$.

Given a morphism $f : X \to Y$ of algebraic spaces, by Lemma 66.21.3 we get a morphism of ringed topoi and hence by Modules on Sites, Definition 18.13.1 we get well defined pullback and direct image functors

03LU (66.26.0.1) $$f^* : Mod(\mathcal{O}_Y) \longrightarrow Mod(\mathcal{O}_X), \quad f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of algebraic spaces over S, then we have $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$ simply because the morphisms of ringed topoi compose in the corresponding way (by the lemma).

03LV **Lemma 66.26.1.** *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, and $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$ for any sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$. In particular, $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. By the description of inverse image in Lemma 66.18.11 and the definition of the structure sheaves it is clear that $f_{small}^{-1}\mathcal{O}_Y = \mathcal{O}_X$. Since the pullback

$$f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

by definition we conclude that $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$. The exactness is clear because f_{small}^{-1} is exact, as f_{small} is a morphism of topoi. □

We continue our abuse of notation introduced in Equation (66.18.11.1) by writing

03LW (66.26.1.1) $$\mathcal{G}|_{X_{étale}} = f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$$

in the situation of the lemma above. We will discuss this in a more technical fashion in Section 66.27.

03LX **Lemma 66.26.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F} \in Mod(\mathcal{O}_X)$. If g is étale, then $f'_*(\mathcal{F}|_{X'}) = (f_*\mathcal{F})|_{Y'}$[7] and $R^i f'_*(\mathcal{F}|_{X'}) = (R^i f_*\mathcal{F})|_{Y'}$ in $Mod(\mathcal{O}_{Y'})$.

Proof. This is a reformulation of Lemma 66.18.12 in the case of modules. □

03LY **Lemma 66.26.3.** *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is given by the following data:*

(1) *for every $U \in \mathrm{Ob}(X_{étale})$ a sheaf $\mathcal{F}_U$ of $\mathcal{O}_U$-modules on $U_{étale}$,*
(2) *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f^*_{small}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $c_g \circ g^*_{small}c_f$ is equal to $c_{f\circ g}$.

Proof. Combine Lemmas 66.26.1 and 66.18.13, and use the fact that any morphism between objects of $X_{étale}$ is an étale morphism of schemes. □

66.27. Étale localization

04LX Reading this section should be avoided at all cost.

Let $X \to Y$ be an étale morphism of algebraic spaces. Then X is an object of $Y_{spaces,étale}$ and it is immediate from the definitions, see also the proof of Lemma 66.18.11, that

04LY (66.27.0.1) $$X_{spaces,étale} = Y_{spaces,étale}/X$$

where the right hand side is the localization of the site $Y_{spaces,étale}$ at the object X, see Sites, Definition 7.25.1. Moreover, this identification is compatible with the structure sheaves by Lemma 66.26.1. Hence the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$ is identified with the localization of the ringed site $(Y_{spaces,étale}, \mathcal{O}_Y)$ at the object X:

04LZ (66.27.0.2) $$(X_{spaces,étale}, \mathcal{O}_X) = (Y_{spaces,étale}/X, \mathcal{O}_Y|_{Y_{spaces,étale}/X})$$

The localization of a ringed site used on the right hand side is defined in Modules on Sites, Definition 18.19.1.

Assume now $X \to Y$ is an étale morphism of algebraic spaces and X is a scheme. Then X is an object of $Y_{étale}$ and it follows that

04M0 (66.27.0.3) $$X_{étale} = Y_{étale}/X$$

and

04M1 (66.27.0.4) $$(X_{étale}, \mathcal{O}_X) = (Y_{étale}/X, \mathcal{O}_Y|_{Y_{étale}/X})$$

as above.

Finally, if $X \to Y$ is an étale morphism of algebraic spaces and X is an affine scheme, then X is an object of $Y_{affine,étale}$ and

04M2 (66.27.0.5) $$X_{affine,étale} = Y_{affine,étale}/X$$

and

04M3 (66.27.0.6) $$(X_{affine,étale}, \mathcal{O}_X) = (Y_{affine,étale}/X, \mathcal{O}_Y|_{Y_{affine,étale}/X})$$

as above.

Next, we show that these localizations are compatible with morphisms.

[7] Also $(f')^*(\mathcal{G}|_{Y'}) = (f^*\mathcal{G})|_{X'}$ by commutativity of the diagram and (66.26.1.1)

04M4 **Lemma 66.27.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of algebraic spaces over S with p and q étale. Via the identifications (66.27.0.2) for $U \to X$ and $V \to Y$ the morphism of ringed topoi

$$(g_{spaces,étale}, g^\sharp) : (Sh(U_{spaces,étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{spaces,étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{spaces,étale,c}, f_c^\sharp)$ constructed in Modules on Sites, Lemma 18.20.2 starting with the morphism of ringed sites $(f_{spaces,étale}, f^\sharp)$ and the map $c : U \to V \times_Y X$ corresponding to g.

Proof. The morphism $(f_{spaces,étale,c}, f_c^\sharp)$ is defined as a composition $f' \circ j$ of a localization and a base change map. Similarly g is a composition $U \to V \times_Y X \to V$. Hence it suffices to prove the lemma in the following two cases: (1) $f = \mathrm{id}$, and (2) $U = X \times_Y V$. In case (1) the morphism $g : U \to V$ is étale, see Lemma 66.16.6. Hence $(g_{spaces,étale}, g^\sharp)$ is a localization morphism by the discussion surrounding Equations (66.27.0.1) and (66.27.0.2) which is exactly the content of the lemma in this case. In case (2) the morphism $g_{spaces,étale}$ comes from the morphism of ringed sites given by the functor $V_{spaces,étale} \to U_{spaces,étale}$, $V'/V \mapsto V' \times_V U/U$ which is also what the morphism f' is defined by, see Sites, Lemma 7.28.1. We omit the verification that $(f')^\sharp = g^\sharp$ in this case (both are the restriction of $f^\sharp$ to $U_{spaces,étale}$). □

04M5 **Lemma 66.27.2.** *Same notation and assumptions as in Lemma 66.27.1 except that we also assume U and V are schemes. Via the identifications (66.27.0.4) for $U \to X$ and $V \to Y$ the morphism of ringed topoi*

$$(g_{small}, g^\sharp) : (Sh(U_{étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{small,s}, f_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 starting with $(f_{small}, f^\sharp)$ and the map $s : h_U \to f_{small}^{-1} h_V$ corresponding to g.

Proof. Note that $(g_{small}, g^\sharp)$ is 2-isomorphic as a morphism of ringed topoi to the morphism of ringed topoi associated to the morphism of ringed sites $(g_{spaces,étale}, g^\sharp)$. Hence we conclude by Lemma 66.27.1 and Modules on Sites, Lemma 18.22.4. □

Finally, we discuss the relationship between sheaves of sets on the small étale site $Y_{étale}$ of an algebraic space Y and algebraic spaces étale over Y. Let S be a scheme and let Y be an algebraic space over S. Let $\mathcal{F}$ be an object of $Sh(Y_{étale})$. Consider the functor

$$X : (Sch/S)^{opp}_{fppf} \longrightarrow Sets$$

defined by the rule

$$X(T) = \{(y, s) \mid y : T \to Y \text{ is a morphism over } S \text{ and } s \in \Gamma(T, y_{small}^{-1}\mathcal{F})\}$$

Given a morphism $g : T' \to T$ the restriction map sends (y, s) to $(y \circ g, g_{small}^{-1} s)$. This makes sense as $y_{small} \circ g_{small} = (y \circ g)_{small}$ by Lemma 66.18.8. There is a canonical map $X \to Y$ sending the pair (y, s) to y.

0GF6 **Lemma 66.27.3.** *Let S be a scheme and let Y be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of sets on $Y_{étale}$. Provided a set theoretic condition is satisfied (see proof) we have*

(1) *the functor X associated to $\mathcal{F}$ above is an algebraic space,*
(2) *the map $X \to Y$ is an étale morphism of algebraic spaces,*
(3) *via the identification $Sh(Y_{étale}) = Sh(Y_{spaces,étale})$ we have $\mathcal{F} \cong h_X$,*
(4) *we have $\mathcal{F} \cong f_{small,!}*$. Here $*$ is the final object of the category $Sh(X_{étale})$ and $f_{small,!}$ exists by Lemma 66.18.11.*

Proof. Let us prove that X is a sheaf for the fppf topology. Namely, suppose that $\{g_i : T_i \to T\}$ is a covering of $(Sch/S)_{fppf}$ and $(y_i, s_i) \in X(T_i)$ satisfy the glueing condition, i.e., the restriction of (y_i, s_i) and (y_j, s_j) to $T_i \times_T T_j$ agree. Then since Y is a sheaf for the fppf topology, we see that the y_i give rise to a unique morphism $y : T \to Y$ such that $y_i = y \circ g_i$. Then we see that $y_{i,small}^{-1}\mathcal{F} = g_{i,small}^{-1}y_{small}^{-1}\mathcal{F}$. Hence the sections s_i glue uniquely to a section of $y_{small}^{-1}\mathcal{F}$ by Étale Cohomology, Lemma 59.39.2.

The construction that sends $\mathcal{F} \in \mathrm{Ob}(Sh(Y_{étale}))$ to $X \in \mathrm{Ob}((Sch/S)_{fppf})$ preserves finite limits and all colimits since each of the functors y_{small}^{-1} have this property. Of course, if $V \in \mathrm{Ob}(Y_{étale})$, then the construction sends the representable sheaf h_V on $Y_{étale}$ to the representable functor represented by V.

By Sites, Lemma 7.12.5 we can find a set I, for each $i \in I$ an object V_i of $Y_{étale}$ and a surjective map of sheaves

$$\coprod h_{V_i} \longrightarrow \mathcal{F}$$

on $Y_{étale}$. The set theoretic condition we need is that the index set I is not too large[8]. Then $V = \coprod V_i$ is an object of $(Sch/S)_{fppf}$ and therefore an object of $Y_{étale}$ and we have a surjective map $h_V \to \mathcal{F}$.

Observe that the product of h_V with itself in $Sh(Y_{étale})$ is $h_{V \times_Y V}$. Consider the fibre product

$$h_V \times_{\mathcal{F}} h_V \subset h_{V \times_Y V}$$

There is an open subscheme R of $V \times_Y V$ such that $h_V \times_{\mathcal{F}} h_V = h_R$, see Lemma 66.20.1 (small detail omitted). By the Yoneda lemma we obtain two morphisms $s, t : R \to V$ in $Y_{étale}$ and we find a coequalizer diagram

$$h_R \rightrightarrows h_V \longrightarrow \mathcal{F}$$

in $Sh(Y_{étale})$. Of course the morphisms s, t are étale and define an étale equivalence relation $(t, s) : R \to V \times_S V$.

By the discussion in the preceding two paragraphs we find a coequalizer diagram

$$R \rightrightarrows V \longrightarrow X$$

in $(Sch/S)_{fppf}$. Thus $X = V/R$ is an algebraic space by Spaces, Theorem 65.10.5. This proves (1). Part (2) follows because $V \to Y$ is étale. Part (3) is immediate from the definition of X and h_X. We omit the proof of part (4); it follows by matching the morphism associated to the cocontinuous functor j of Lemma 66.18.11 with the

[8]It suffices if the supremum of the cardinalities of the stalks of $\mathcal{F}$ at geometric points of Y is bounded by the size of some object of $(Sch/S)_{fppf}$.

description of $X_{spaces,étale}$ as the localization of $Y_{spaces,étale}$ at X discussed above and Sites, Lemma 7.25.3. □

66.28. Recovering morphisms

04KI In this section we prove that the rule which associates to an algebraic space its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 66.28.4.

04KJ **Lemma 66.28.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The morphism of ringed topoi $(f_{small}, f^\sharp)$ associated to f is a morphism of locally ringed topoi, see Modules on Sites, Definition 18.40.9.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 66.22.3. Moreover, we know that $X_{étale}$ has enough points, see Theorem 66.19.12. Hence it suffices to prove that $(f_{small}, f^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.40.8. To see this take a point p of $X_{étale}$. By Lemma 66.19.13 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 66.19.9 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of étale local rings

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. You can prove this directly, but instead we deduce it from the corresponding result for schemes. To do this choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\psi]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes, and the vertical arrows are surjective étale (see Spaces, Lemma 65.11.6). Choose a lift $\overline{u} : \overline{x} \to U$ (possible by Lemma 66.19.5). Set $\overline{v} = \psi \circ \overline{u}$. We obtain a commutative diagram of étale local rings

$$\begin{array}{ccc} \mathcal{O}_{U,\overline{u}} & \longleftarrow & \mathcal{O}_{V,\overline{v}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{X,\overline{x}} & \longleftarrow & \mathcal{O}_{Y,\overline{y}}. \end{array}$$

By Étale Cohomology, Lemma 59.40.1 the top horizontal arrow is a local ring map. Finally by Lemma 66.22.1 the vertical arrows are isomorphisms. Hence we win. □

04KK **Lemma 66.28.2.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let t be a 2-morphism from $(f_{small}, f^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = id$.*

Proof. Let X', resp. Y' be X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2. It is clear from the construction that $(X_{small}, \mathcal{O})$ is equal to $(X'_{small}, \mathcal{O})$ and similarly for Y. Hence we may work with X' and Y'. In other words we may assume that $S = \mathrm{Spec}(\mathbf{Z})$.

Assume $S = \mathrm{Spec}(\mathbf{Z})$, $f : X \to Y$ and t are as in the lemma. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{array}{ccc} f_{small}^{-1}\mathcal{O}_Y & \xleftarrow{\quad t \quad} & f_{small}^{-1}\mathcal{O}_Y \\ & {\scriptstyle f^\sharp} \searrow \quad \swarrow {\scriptstyle f^\sharp} & \\ & \mathcal{O}_X & \end{array}$$

is commutative. Suppose $V \to Y$ is étale with V affine. Write $V = \mathrm{Spec}(B)$. Choose generators $b_j \in B$, $j \in J$ for B as a $\mathbf{Z}$-algebra. Set $T = \mathrm{Spec}(\mathbf{Z}[\{x_j\}_{j\in J}])$. In the following we will use that $\mathrm{Mor}_{Sch}(U, T) = \prod_{j\in J} \Gamma(U, \mathcal{O}_U)$ for any scheme U without further mention. The surjective ring map $\mathbf{Z}[x_j] \to B$, $x_j \mapsto b_j$ corresponds to a closed immersion $V \to T$. We obtain a monomorphism

$$i : V \longrightarrow T_Y = T \times Y$$

of algebraic spaces over Y. In terms of sheaves on $Y_{étale}$ the morphism i induces an injection $h_i : h_V \to \prod_{j\in J} \mathcal{O}_Y$ of sheaves. The base change $i' : X \times_Y V \to T_X$ of i to X is a monomorphism too (Spaces, Lemma 65.5.5). Hence $i' : X \times_Y V \to T_X$ is a monomorphism, which in turn means that $h_{i'} : h_{X\times_Y V} \to \prod_{j\in J} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X\times_Y V}$ of Lemma 66.19.9 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1}h_i} \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod_{j\in J} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1} h_V \to f_{small}^{-1} h_V$ fits into the commutative diagram

$$\begin{array}{ccccc} f_{small}^{-1} h_V & \xrightarrow{f^{-1}h_i} & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j\in J} \mathcal{O}_X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \prod t} & & \downarrow{\scriptstyle \mathrm{id}} \\ f_{small}^{-1} h_V & \xrightarrow{f^{-1}h_i} & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j\in J} \mathcal{O}_X \end{array}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 66.18.6 with Sites, Lemma 7.12.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. □

04M6 **Lemma 66.28.3.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Any two morphisms $a, b : X \to Y$ of algebraic spaces over S for which there exists a 2-isomorphism $(a_{small}, a^\sharp) \cong (b_{small}, b^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. We may equivalently think of t as a transformation $t : a_{spaces,étale}^{-1} \to b_{spaces,étale}^{-1}$ since there is not difference between sheaves on $X_{étale}$ and sheaves on $X_{spaces,étale}$. Choose a commutative

diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective étale. Consider the diagram

$$\begin{array}{ccc} h_U & \xrightarrow{\alpha} & a^{-1}_{spaces,étale} h_V \\ \| & & \downarrow{\scriptstyle t} \\ h_U & \dashrightarrow & b^{-1}_{spaces,étale} h_V \end{array}$$

Since the sheaf $b^{-1}_{spaces,étale} h_V$ is isomorphic to $h_{V \times_{Y,b} X}$ we see that the dotted arrow comes from a morphism of schemes $\beta : U \to V$ fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\beta} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{b} & Y \end{array}$$

We claim that there exists a sequence of 2-isomorphisms

$$\begin{aligned} (\alpha_{small}, \alpha^\sharp) &\cong (\alpha_{spaces,étale}, \alpha^\sharp) \\ &\cong (a_{spaces,étale,c}, a_c^\sharp) \\ &\cong (b_{spaces,étale,d}, b_d^\sharp) \\ &\cong (\beta_{spaces,étale}, \beta^\sharp) \\ &\cong (\beta_{small}, \beta^\sharp) \end{aligned}$$

The first and the last 2-isomorphisms come from the identifications between sheaves on $U_{spaces,étale}$ and sheaves on $U_{étale}$ and similarly for V. The second and fourth 2-isomorphisms are those of Lemma 66.27.1 with $c : U \to X \times_{a,Y} V$ induced by α and $d : U \to X \times_{b,Y} V$ induced by β. The middle 2-isomorphism comes from the transformation t. Namely, the functor $a^{-1}_{spaces,étale,c}$ corresponds to the functor

$$(\mathcal{H} \to h_V) \longmapsto (a^{-1}_{spaces,étale} \mathcal{H} \times_{a^{-1}_{spaces,étale} h_V, \alpha} h_U \to h_U)$$

and similarly for $b^{-1}_{spaces,étale,d}$, see Sites, Lemma 7.28.3. This uses the identification of sheaves on $Y_{spaces,étale}/V$ as arrows $(\mathcal{H} \to h_V)$ in $Sh(Y_{spaces,étale})$ and similarly for U/X, see Sites, Lemma 7.25.4. Via this identification the structure sheaf $\mathcal{O}_V$ corresponds to the pair $(\mathcal{O}_Y \times h_V \to h_V)$ and similarly for $\mathcal{O}_U$, see Modules on Sites, Lemma 18.21.3. Since t switches α and β we see that t induces an isomorphism

$$t : a^{-1}_{spaces,étale} \mathcal{H} \times_{a^{-1}_{spaces,étale} h_V, \alpha} h_U \longrightarrow b^{-1}_{spaces,étale} \mathcal{H} \times_{b^{-1}_{spaces,étale} h_V, \beta} h_U$$

over h_U functorially in $(\mathcal{H} \to h_V)$. Also, t is compatible with $a_c^\sharp$ and $b_d^\sharp$ as t is compatible with $a^\sharp$ and $b^\sharp$ by our description of the structure sheaves $\mathcal{O}_U$ and $\mathcal{O}_V$ above. Hence, the morphisms of ringed topoi $(\alpha_{small}, \alpha^\sharp)$ and $(\beta_{small}, \beta^\sharp)$ are 2-isomorphic. By Étale Cohomology, Lemma 59.40.3 we conclude $\alpha = \beta$! Since $p : U \to X$ is a surjection of sheaves it follows that $a = b$. □

Here is the main result of this section.

04KL **Theorem 66.28.4.** *Let X, Y be algebraic spaces over* $\mathrm{Spec}(\mathbf{Z})$. *Let*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of algebraic spaces $f : X \to Y$ such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$. In other words, the construction

$$\textit{Spaces}/\mathrm{Spec}(\mathbf{Z}) \longrightarrow \textit{Locally ringed topoi}, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. The uniqueness we have seen in Lemma 66.28.3. Thus it suffices to prove existence. In this proof we will freely use the identifications of Equation (66.27.0.4) as well as the result of Lemma 66.27.2.

Let $U \in \mathrm{Ob}(X_{étale})$, let $V \in \mathrm{Ob}(Y_{étale})$ and let $s \in g^{-1}h_V(U)$ be a section. We may think of s as a map of sheaves $s : h_U \to g^{-1}h_V$. By Modules on Sites, Lemma 18.22.3 we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(X_{étale}/U), \mathcal{O}_U) & \xrightarrow[(j,j^\sharp)]{} & (Sh(X_{étale}), \mathcal{O}_X) \\ \downarrow{\scriptstyle (g_s, g_s^\sharp)} & & \downarrow{\scriptstyle (g,g^\sharp)} \\ (Sh(V_{étale}), \mathcal{O}_V) & \longrightarrow & (Sh(Y_{étale}), \mathcal{O}_Y). \end{array}$$

By Étale Cohomology, Theorem 59.40.5 we obtain a unique morphism of schemes $f_s : U \to V$ such that $(g_s, g_s^\sharp)$ is 2-isomorphic to $(f_{s,small}, f_s^\sharp)$. The construction $(U, V, s) \rightsquigarrow f_s$ just explained satisfies the following functoriality property: Suppose given morphisms $a : U' \to U$ in $X_{étale}$ and $b : V' \to V$ in $Y_{étale}$ and a map $s' : h_{U'} \to g^{-1}h_{V'}$ such that the diagram

$$\begin{array}{ccc} h_{U'} & \xrightarrow[s']{} & g^{-1}h_{V'} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g^{-1}b} \\ h_U & \xrightarrow{s} & g^{-1}h_V \end{array}$$

commutes. Then the diagram

$$\begin{array}{ccc} U' & \xrightarrow[f_{s'}]{} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{f_s} & u(V) \end{array}$$

of schemes commutes. The reason this is true is that the same condition holds for the morphisms $(g_s, g_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 and the uniqueness in Étale Cohomology, Theorem 59.40.5.

The problem is to glue the morphisms f_s to a morphism of algebraic spaces. To do this first choose a scheme V and a surjective étale morphism $V \to Y$. This means that $h_V \to *$ is surjective and hence $g^{-1}h_V \to *$ is surjective too. This means there exists a scheme U and a surjective étale morphism $U \to X$ and a morphism $s : h_U \to g^{-1}h_V$. Next, set $R = V \times_Y V$ and $R' = U \times_X U$. Then we get $g^{-1}h_R = g^{-1}h_V \times g^{-1}h_V$ as g^{-1} is exact. Thus s induces a morphism

$s \times s : h_{R'} \to g^{-1}h_R$. Applying the constructions above we see that we get a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} R' & \xrightarrow{f_{s\times s}} & R \\ \downarrow\downarrow & & \downarrow\downarrow \\ U & \xrightarrow{f_s} & V \end{array}$$

Since we have $X = U/R'$ and $Y = V/R$ (see Spaces, Lemma 65.9.1) we conclude that this diagram defines a morphism of algebraic spaces $f : X \to Y$ fitting into an obvious commutative diagram. Now we still have to show that $(f_{small}, f^\sharp)$ is 2-isomorphic to $(g, g^\sharp)$. Let $t_V : f_{s,small}^{-1} \to g_s^{-1}$ and $t_R : f_{s\times s,small}^{-1} \to g_{s\times s}^{-1}$ be the 2-isomorphisms which are given to us by the construction above. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then we see that t_V defines an isomorphism

$$f_{small}^{-1}\mathcal{G}|_{U_{étale}} = f_{s,small}^{-1}\mathcal{G}|_{V_{étale}} \xrightarrow{t_V} g_s^{-1}\mathcal{G}|_{V_{étale}} = g^{-1}\mathcal{G}|_{U_{étale}}.$$

Moreover, this isomorphism pulled back to R' via either projection $R' \to U$ is the isomorphism

$$f_{small}^{-1}\mathcal{G}|_{R'_{étale}} = f_{s\times s,small}^{-1}\mathcal{G}|_{R_{étale}} \xrightarrow{t_R} g_{s\times s}^{-1}\mathcal{G}|_{R_{étale}} = g^{-1}\mathcal{G}|_{R'_{étale}}.$$

Since $\{U \to X\}$ is a covering in the site $X_{spaces,étale}$ this means the first displayed isomorphism descends to an isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$ of sheaves (small detail omitted). The isomorphism is functorial in $\mathcal{G}$ since t_V and t_R are transformations of functors. Finally, t is compatible with $f^\sharp$ and $g^\sharp$ as t_V and t_R are (some details omitted). This finishes the proof of the theorem. □

05YZ **Lemma 66.28.5.** *Let X, Y be algebraic spaces over $\mathbf{Z}$. If*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is an isomorphism of ringed topoi, then there exists a unique morphism $f : X \to Y$ of algebraic spaces such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$ and moreover f is an isomorphism of algebraic spaces.

Proof. By Theorem 66.28.4 it suffices to show that $(g, g^\sharp)$ is a morphism of locally ringed topoi. By Modules on Sites, Lemma 18.40.8 (and since the site $X_{étale}$ has enough points) it suffices to check that the map $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ induced by $g^\sharp$ is a local ring map where $q = f \circ p$ and p is any point of $X_{étale}$. As it is an isomorphism this is clear. □

66.29. Quasi-coherent sheaves on algebraic spaces

03G5 In Descent, Sections 35.8, 35.9, and 35.10 we have seen that for a scheme U, there is no difference between a quasi-coherent $\mathcal{O}_U$-module on U, or a quasi-coherent $\mathcal{O}$-module on the small étale site of U. Hence the following definition is compatible with our original notion of a quasi-coherent sheaf on a scheme (Schemes, Section 26.24), when applied to a representable algebraic space.

03G9 **Definition 66.29.1.** Let S be a scheme. Let X be an algebraic space over S. A *quasi-coherent $\mathcal{O}_X$-module* is a quasi-coherent module on the ringed site $(X_{étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on X is denoted $QCoh(\mathcal{O}_X)$.

Note that as being quasi-coherent is an intrinsic notion (see Modules on Sites, Lemma 18.23.2) this is equivalent to saying that the corresponding $\mathcal{O}_X$-module on $X_{spaces,étale}$ is quasi-coherent.

As usual, quasi-coherent sheaves behave well with respect to pullback.

03GA **Lemma 66.29.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. □

Note that this pullback functor agrees with the usual pullback functor between quasi-coherent sheaves of modules if X and Y happen to be schemes, see Descent, Proposition 35.9.4. Here is the obligatory lemma comparing this with quasi-coherent sheaves on the objects of the small étale site of X.

03LZ **Lemma 66.29.3.** *Let S be a scheme. Let X be an algebraic space over S. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is given by the following data:*

(1) *for every $U \in \mathrm{Ob}(X_{étale})$ a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_U$ on $U_{étale}$,*
(2) *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f^*_{small}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $c_g \circ g^*_{small} c_f$ is equal to $c_{f \circ g}$.

Proof. Combine Lemmas 66.29.2 and 66.26.3. □

05VP **Lemma 66.29.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and let $\overline{x}$ be a geometric point lying over x. Finally, let $\varphi : (U, \overline{u}) \to (X, \overline{x})$ be an étale neighbourhood where U is a scheme. Then*

$$(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$$

where $u \in U$ is the image of $\overline{u}$.

Proof. Note that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ by Lemma 66.22.1 hence the tensor product makes sense. Moreover, from Definition 66.19.6 it is clear that

$$\mathcal{F}_{\overline{u}} = \mathrm{colim}(\varphi^*\mathcal{F})_u$$

where the colimit is over $\varphi : (U, \overline{u}) \to (X, \overline{x})$ as in the lemma. Hence there is a canonical map from left to right in the statement of the lemma. We have a similar colimit description for $\mathcal{O}_{X,\overline{x}}$ and by Lemma 66.29.3 we have

$$((\varphi')^*\mathcal{F})_{u'} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U',u'}$$

whenever $(U', \overline{u}') \to (U, \overline{u})$ is a morphism of étale neighbourhoods. To complete the proof we use that $\otimes$ commutes with colimits. □

05VQ **Lemma 66.29.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\overline{x}$ be a geometric point of X and let $\overline{y} = f \circ \overline{x}$ be the image in Y. Then there is a canonical isomorphism*

$$(f^*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}}$$

of the stalk of the pullback with the tensor product of the stalk with the local ring of X at $\overline{x}$.

Proof. Since $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$ this follows from the description of stalks of pullbacks in Lemma 66.19.9 and the fact that taking stalks commutes with tensor products. A more direct way to see this is as follows. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective étale. By Lemma 66.19.4 we can choose a geometric point $\overline{u}$ of U such that $\overline{x} = p \circ \overline{u}$. Set $\overline{v} = \alpha \circ \overline{u}$. Then we see that

$$\begin{aligned} (f^*\mathcal{G})_{\overline{x}} &= (p^*f^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (\alpha^*q^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{Y,\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \\ &= \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \end{aligned}$$

Here we have used Lemma 66.29.4 (twice) and the corresponding result for pullbacks of quasi-coherent sheaves on schemes, see Sheaves, Lemma 6.26.4. □

03M0 **Lemma 66.29.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

1. *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,*
2. *there exists an étale morphism $f : Y \to X$ of algebraic spaces over S with $|f| : |Y| \to |X|$ surjective such that $f^*\mathcal{F}$ is quasi-coherent on Y,*
3. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and*
4. *for every affine scheme U and étale morphism $\varphi : U \to X$ the restriction $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module.*

Proof. It is clear that (1) implies (2) by considering id_X. Assume $f : Y \to X$ is as in (2), and let $V \to Y$ be a surjective étale morphism from a scheme towards Y. Then the composition $V \to X$ is surjective étale as well and by Lemma 66.29.2 the pullback of $\mathcal{F}$ to V is quasi-coherent as well. Hence we see that (2) implies (3).

Let $U \to X$ be as in (3). Let us use the abuse of notation introduced in Equation (66.26.1.1). As $\mathcal{F}|_{U_{étale}}$ is quasi-coherent there exists an étale covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_{i,étale}}$ has a global presentation, see Modules on Sites, Definition 18.17.1 and Lemma 18.23.3. Let $V \to X$ be an object of $X_{étale}$. Since $U \to X$ is surjective and étale, the family of maps $\{U_i \times_X V \to V\}$ is an étale covering of V. Via the morphisms $U_i \times_X V \to U_i$ we can restrict the global presentations of $\mathcal{F}|_{U_{i,étale}}$ to get a global presentation of $\mathcal{F}|_{(U_i \times_X V)_{étale}}$ Hence the sheaf $\mathcal{F}$ on $X_{étale}$ satisfies the condition of Modules on Sites, Definition 18.23.1 and hence is quasi-coherent.

The equivalence of (3) and (4) comes from the fact that any scheme has an affine open covering. □

03M1 **Lemma 66.29.7.** *Let S be a scheme. Let X be an algebraic space over S. The category $QCoh(\mathcal{O}_X)$ of quasi-coherent sheaves on X has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.*
(5) *Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation (see Section 66.30), then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. If X is a scheme, then this is Descent, Lemma 35.10.3. We will reduce the lemma to this case by étale localization.

Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Our notation will be that $Mod(\mathcal{O}_U) = Mod(U_{étale}, \mathcal{O}_U)$ and $QCoh(\mathcal{O}_U) = QCoh(U_{étale}, \mathcal{O}_U)$; in other words, even though U is a scheme we think of quasi-coherent modules on U as modules on the small étale site of U. By Lemma 66.29.2 we have a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_X) & \xrightarrow{\varphi^*} & QCoh(\mathcal{O}_U) \\ \downarrow & & \downarrow \\ Mod(\mathcal{O}_X) & \xrightarrow{\varphi^*} & Mod(\mathcal{O}_U) \end{array}$$

The bottom horizontal arrow is the restriction functor (66.26.1.1) $\mathcal{G} \mapsto \mathcal{G}|_{U_{étale}}$. This functor has both a left adjoint and a right adjoint, see Modules on Sites, Section 18.19, hence commutes with all limits and colimits. Moreover, we know that an object of $Mod(\mathcal{O}_X)$ is in $QCoh(\mathcal{O}_X)$ if and only if its restriction to U is in $QCoh(\mathcal{O}_U)$, see Lemma 66.29.6. With these preliminaries out of the way we can start the proof.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of quasi-coherent $\mathcal{O}_X$-modules. By the discussion above we have

$$\Big(\bigoplus \mathcal{F}_i\Big)|_{U_{étale}} = \bigoplus \mathcal{F}_i|_{U_{étale}}$$

Each of the modules $\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent. Hence the direct sum is quasi-coherent by the case of schemes. Hence $\bigoplus \mathcal{F}_i$ is quasi-coherent as a module restricting to a quasi-coherent module on U.

Proof of (2). Let $\mathcal{I} \to QCoh(\mathcal{O}_X)$, $i \mapsto \mathcal{F}_i$ be a diagram. Then

$$(\operatorname{colim} \mathcal{F}_i)|_{U_{étale}} = \operatorname{colim} \mathcal{F}_i|_{U_{étale}}$$

by the discussion above and we conclude in the same manner.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh(\mathcal{O}_X)$. Then we have $\operatorname{Ker}(a)|_{U_{étale}} = \operatorname{Ker}(a|_{U_{étale}})$ and $\operatorname{Coker}(a)|_{U_{étale}} = \operatorname{Coker}(a|_{U_{étale}})$ and we conclude in the same manner.

Proof of (4). The restriction $0 \to \mathcal{F}_1|_{U_{étale}} \to \mathcal{F}_2|_{U_{étale}} \to \mathcal{F}_3|_{U_{étale}} \to 0$ is short exact. Hence we have the 2-out-of-3 property for this sequence and we conclude as before.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_X)$. Then we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_{U_{étale}} = \mathcal{F}|_{U_{étale}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{étale}}$$

and we conclude as before.

Proof of (6). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_X)$ with $\mathcal{F}$ of finite presentation. We have

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_{U_{étale}} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$$

Namely, restriction is a localization, see Section 66.27, especially formula (66.27.0.4)) and formation of internal hom commutes with localization, see Modules on Sites, Lemma 18.27.2. Thus we conclude as before. □

It is in general not the case that the pushforward of a quasi-coherent sheaf along a morphism of algebraic spaces is quasi-coherent. We will return to this issue in Morphisms of Spaces, Section 67.11.

66.30. Properties of modules

05VR In Modules on Sites, Sections 18.17, 18.23, and Definition 18.28.1 we have defined a number of intrinsic properties of modules of $\mathcal{O}$-module on any ringed topos. If X is an algebraic space, we will apply these notions freely to modules on the ringed site $(X_{étale}, \mathcal{O}_X)$, or equivalently on the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$.

Global properties $\mathcal{P}$:

(a) *free*,
(b) *finite free*,
(c) *generated by global sections*,
(d) *generated by finitely many global sections*,
(e) having a *global presentation*, and
(f) having a *global finite presentation*.

Local properties $\mathcal{P}$:

(g) *locally free*,
(f) *finite locally free*,
(h) *locally generated by sections*,
(i) *locally generated by r sections*,
(j) *finite type*,
(k) *quasi-coherent* (see Section 66.29),
(l) *of finite presentation*,
(m) *coherent*, and
(n) *flat*.

Here are some results which follow immediately from the definitions:

(1) In each case, except for $\mathcal{P}$ ="coherent", the property is preserved under pullback, see Modules on Sites, Lemmas 18.17.2, 18.23.4, and 18.39.1.
(2) Each of the properties above (including coherent) are preserved under pullbacks by étale morphisms of algebraic spaces (because in this case pullback is given by restriction, see Lemma 66.18.11).
(3) Assume $f : Y \to X$ is a surjective étale morphism of algebraic spaces. For each of the local properties (g) – (m), the fact that $f^*\mathcal{F}$ has $\mathcal{P}$ implies that $\mathcal{F}$ has $\mathcal{P}$. This follows as $\{Y \to X\}$ is a covering in $X_{spaces,étale}$ and Modules on Sites, Lemma 18.23.3.

(4) If X is a scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, and $\mathcal{P}$ any property except "coherent" or "locally free", then $\mathcal{P}$ for $\mathcal{F}$ on $X_{étale}$ is equivalent to the corresponding property for $\mathcal{F}|_{X_{Zar}}$, i.e., it corresponds to $\mathcal{P}$ for $\mathcal{F}$ when we think of it as a quasi-coherent sheaf on the scheme X. See Descent, Lemma 35.8.10.
(5) If X is a locally Noetherian scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, then $\mathcal{F}$ is coherent on $X_{étale}$ if and only if $\mathcal{F}|_{X_{Zar}}$ is coherent, i.e., it corresponds to the usual notion of a coherent sheaf on the scheme X being coherent. See Descent, Lemma 35.8.10.

66.31. Locally projective modules

060P Recall that in Properties, Section 28.21 we defined the notion of a locally projective quasi-coherent module.

060Q **Lemma 66.31.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *for some scheme U and surjective étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U, and*
(2) *for any scheme U and any étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U.*

Proof. Let $U \to X$ be as in (1) and let $V \to X$ be étale where V is a scheme. Then $\{U \times_X V \to V\}$ is an fppf covering of schemes. Hence if $\mathcal{F}|_U$ is locally projective, then $\mathcal{F}|_{U \times_X V}$ is locally projective (see Properties, Lemma 28.21.3) and hence $\mathcal{F}|_V$ is locally projective, see Descent, Lemma 35.7.7. □

060R **Definition 66.31.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if the equivalent conditions of Lemma 66.31.1 are satisfied.

060S **Lemma 66.31.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Denote $\psi : U \to V$ the induced morphism. Then

$$f^*\mathcal{G}|_U = \psi^*(\mathcal{G}|_V)$$

Hence the lemma follows from the definition and the result in the case of schemes, see Properties, Lemma 28.21.3. □

66.32. Quasi-coherent sheaves and presentations

03M2 Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 65.9.3. In particular, we obtain a groupoid (U, R, s, t, c), such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 39.13.3. In Groupoids, Definition 39.14.1 we have the defined the notion of a quasi-coherent sheaf on an arbitrary groupoid. With these notions in place we have the following observation.

03M3 **Proposition 66.32.1.** *With S, $\varphi : U \to X$, and (U, R, s, t, c) as above. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $\varphi^*\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^*\varphi^*\mathcal{F} \longrightarrow s^*\varphi^*\mathcal{F}$$

which satisfies the conditions of Groupoids, Definition 39.14.1 and therefore defines a quasi-coherent sheaf on (U, R, s, t, c). The functor $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ defines an equivalence of categories

$$\begin{array}{c} \textit{Quasi-coherent} \\ \mathcal{O}_X\textit{-modules} \end{array} \longleftrightarrow \begin{array}{c} \textit{Quasi-coherent modules} \\ \textit{on } (U, R, s, t, c) \end{array}$$

Proof. In the statement of the proposition, and in this proof we think of a quasi-coherent sheaf on a scheme as a quasi-coherent sheaf on the small étale site of that scheme. This is permissible by the results of Descent, Sections 35.8, 35.9, and 35.10.

The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see discussion surrounding Equation (66.26.0.1). In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α satisfies condition (1) of Groupoids, Definition 39.14.1. To see condition (2) of the definition it suffices to see that α is an isomorphism which is clear. The construction $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ is clearly functorial in the quasi-coherent sheaf $\mathcal{F}$. Hence we obtain the functor from left to right in the displayed formula of the lemma.

Conversely, suppose that $(\mathcal{F}, \alpha)$ is a quasi-coherent sheaf on (U, R, s, t, c). Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Moreover, the quasi-coherent sheaf $\mathcal{F}$ pulls back to a quasi-coherent sheaf $\mathcal{F}'$ on V'. Since $R = U \times_X U$ with $t = \mathrm{pr}_0$ and $s = \mathrm{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : \mathrm{pr}_0^*\mathcal{F}' \to \mathrm{pr}_1^*\mathcal{F}'$, and the pair $(\mathcal{F}', \alpha')$ is a descend datum for quasi-coherent sheaves with respect to $\{V' \to V\}$. By Descent, Proposition 35.5.2 this descent datum is effective, and we obtain a quasi-coherent $\mathcal{O}_V$-module $\mathcal{F}_V$ on $V_{étale}$. To see that this gives a quasi-coherent sheaf on $X_{étale}$ we have to show (by Lemma 66.29.3) that for any morphism $f : V_1 \to V_2$ in $X_{étale}$ there is a canonical isomorphism $c_f : \mathcal{F}_{V_1} \to \mathcal{F}_{V_2}$ compatible with compositions of morphisms. We omit the verification. We also omit the verification that this defines a functor from the category on the right to the category on the left which is inverse to the functor described above. □

077V **Proposition 66.32.2.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.*
(2) *The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*[9]

$$Q : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

[9]This functor is sometimes called the *coherator*.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 28.23.4. We advise the reader to read that proof first.

Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 66.29.7 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

To construct a generator, choose a presentation $X = U/R$ so that (U, R, s, t, c) is an étale groupoid scheme and in particular s and t are flat morphisms of schemes. Pick a cardinal κ as in Groupoids, Lemma 39.15.7. Pick a collection $(\mathcal{E}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) as in Groupoids, Lemma 39.15.6. Let $\mathcal{F}_t$ be the quasi-coherent module on X which corresponds to the quasi-coherent module $(\mathcal{E}_t, \alpha_t)$ via the equivalence of categories of Proposition 66.32.1. Then we see that every quasi-coherent module $\mathcal{H}$ is the directed colimit of its quasi-coherent submodules which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful. □

66.33. Morphisms towards schemes

05Z0 Here is the analogue of Schemes, Lemma 26.6.4.

05Z1 **Lemma 66.33.1.** *Let X be an algebraic space over $\mathbf{Z}$. Let T be an affine scheme. The map*

$$\mathrm{Mor}(X, T) \longrightarrow \mathrm{Hom}(\Gamma(T, \mathcal{O}_T), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. We construct the inverse of the map. Let $\varphi : \Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. Choose a presentation $X = U/R$, see Spaces, Definition 65.9.3. By Schemes, Lemma 26.6.4 the composition

$$\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$$

corresponds to a unique morphism of schemes $g : U \to T$. By the same lemma the two compositions $R \to U \to T$ are equal. Hence we obtain a morphism $f : X =$

$U/R \to T$ such that $U \to X \to T$ equals g. By construction the diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U) & \longleftarrow & \Gamma(X, \mathcal{O}_X) \\ & \nwarrow^{g^\sharp} & \uparrow{\scriptstyle \varphi} \quad \uparrow{\scriptstyle f^\sharp} \\ & & \Gamma(T, \mathcal{O}_T) \end{array}$$

commutes. Hence $f^\sharp$ equals φ because $U \to X$ is an étale covering and $\mathcal{O}_X$ is a sheaf on $X_{étale}$. The uniqueness of f follows from the uniqueness of g. □

66.34. Quotients by free actions

071R Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group. Let $a : G \to \mathrm{Aut}(X)$ be a homomorphism, i.e., a is an *action* of G on X. We will say the action is *free* if for every scheme T over S the map

$$G \times X(T) \longrightarrow X(T)$$

is free. (We cannot use a criterion as in Spaces, Lemma 65.14.3 because points may not have well defined residue fields.) In case the action is free we're going to construct the quotient X/G as an algebraic space. This is a special case of the general Bootstrap, Lemma 80.11.7 that we will prove later.

071S **Lemma 66.34.1.** *Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group with a free action on X. Then the quotient sheaf X/G is an algebraic space.*

Proof. The statement means that the sheaf F associated to the presheaf

$$T \longmapsto X(T)/G$$

is an algebraic space. To see this we will construct a presentation. Namely, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Set $V = \coprod_{g \in G} U$ and set $\psi : V \to X$ equal to $a(g) \circ \varphi$ on the component corresponding to $g \in G$. Let G act on V by permuting the components, i.e., $g_0 \in G$ maps the component corresponding to g to the component corresponding to $g_0 g$ via the identity morphism of U. Then ψ is a G-equivariant morphism, i.e., we reduce to the case dealt with in the next paragraph.

Assume that there exists a G-action on U and that $U \to X$ is surjective, étale and G-equivariant. In this case there is an induced action of G on $R = U \times_X U$ compatible with the projection mappings $t, s : R \to U$. Now we claim that

$$X/G = U/\coprod\nolimits_{g \in G} R$$

where the map

$$j : \coprod\nolimits_{g \in G} R \longrightarrow U \times_S U$$

is given by $(r, g) \mapsto (t(r), g(s(r)))$. Note that j is a monomorphism: If $(t(r), g(s(r))) = (t(r'), g'(s(r')))$, then $t(r) = t(r')$, hence r and r' have the same image in X under both s and t, hence $g = g'$ (as G acts freely on X), hence $s(r) = s(r')$, hence $r = r'$ (as R is an equivalence relation on U). Moreover j is an equivalence relation (details omitted). Both projections $\coprod_{g \in G} R \to U$ are étale, as s and t are étale. Thus

j is an étale equivalence relation and $U/\coprod_{g\in G} R$ is an algebraic space by Spaces, Theorem 65.10.5. There is a map

$$U/\coprod\nolimits_{g\in G} R \longrightarrow X/G$$

induced by the map $U \to X$. We omit the proof that it is an isomorphism of sheaves. □

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CHAPTER 67

Morphisms of Algebraic Spaces

03H8

67.1. Introduction

03H9 In this chapter we introduce some types of morphisms of algebraic spaces. A reference is [**Knu71**].

The goal is to extend the definition of each of the types of morphisms of schemes defined in the chapters on schemes, and on morphisms of schemes to the category of algebraic spaces. Each case is slightly different and it seems best to treat them all separately.

67.2. Conventions

040V The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

67.3. Properties of representable morphisms

03HA Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces. In Spaces, Section 65.5 we defined what it means for f to have property $\mathcal{P}$ in case $\mathcal{P}$ is a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 26.18.3, and
(2) is fppf local on the base, see Descent, Definition 35.22.1.

Namely, in this case we say f has property $\mathcal{P}$ if and only if for every scheme U and any morphism $U \to Y$ the morphism of schemes $X \times_Y U \to U$ has property $\mathcal{P}$.

According to the lists in Spaces, Section 65.4 this applies to the following properties: (1)(a) closed immersions, (1)(b) open immersions, (1)(c) quasi-compact immersions, (2) quasi-compact, (3) universally-closed, (4) (quasi-)separated, (5) monomorphism, (6) surjective, (7) universally injective, (8) affine, (9) quasi-affine, (10) (locally) of finite type, (11) (locally) quasi-finite, (12) (locally) of finite presentation, (13) locally of finite type of relative dimension d, (14) universally open, (15) flat, (16) syntomic, (17) smooth, (18) unramified (resp. G-unramified), (19) étale, (20) proper, (21) finite or integral, (22) finite locally free, (23) universally submersive, (24) universal homeomorphism, and (25) immersion.

In this chapter we will redefine these notions for not necessarily representable morphisms of algebraic spaces. Whenever we do this we will make sure that the new definition agrees with the old one, in order to avoid ambiguity.

Note that the definition above applies whenever X is a scheme, since a morphism from a scheme to an algebraic space is representable. And in particular it applies when both X and Y are schemes. In Spaces, Lemma 65.5.3 we have seen that in this case the definitions match, and no ambiguity arise.

Furthermore, in Spaces, Lemma 65.5.5 we have seen that the property of representable morphisms of algebraic spaces so defined is stable under arbitrary base change by a morphism of algebraic spaces. And finally, in Spaces, Lemmas 65.5.4 and 65.5.7 we have seen that if $\mathcal{P}$ is stable under compositions, which holds for the properties (1)(a), (1)(b), (1)(c), (2) – (25), except (13) above, then taking products of representable morphisms preserves property $\mathcal{P}$ and compositions of representable morphisms preserves property $\mathcal{P}$.

We will use these facts below, and whenever we do we will simply refer to this section as a reference.

67.4. Separation axioms

03HJ It makes sense to list some a priori properties of the diagonal of a morphism of algebraic spaces.

03HK **Lemma 67.4.1.** *Let S be a scheme contained in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism. Then*

(1) *$\Delta_{X/Y}$ is representable,*
(2) *$\Delta_{X/Y}$ is locally of finite type,*
(3) *$\Delta_{X/Y}$ is a monomorphism,*
(4) *$\Delta_{X/Y}$ is separated, and*
(5) *$\Delta_{X/Y}$ is locally quasi-finite.*

Proof. We are going to use the fact that $\Delta_{X/S}$ is representable (by definition of an algebraic space) and that it satisfies properties (2) – (5), see Spaces, Lemma 65.13.1. Note that we have a factorization

$$X \longrightarrow X \times_Y X \longrightarrow X \times_S X$$

of the diagonal $\Delta_{X/S} : X \to X \times_S X$. Since $X \times_Y X \to X \times_S X$ is a monomorphism, and since $\Delta_{X/S}$ is representable, it follows formally that $\Delta_{X/Y}$ is representable. In particular, the rest of the statements now make sense, see Section 67.3.

Choose a surjective étale morphism $U \to X$, with U a scheme. Consider the diagram

$$\begin{array}{ccccc} R = U \times_X U & \longrightarrow & U \times_Y U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \times_Y X & \longrightarrow & X \times_S X \end{array}$$

Both squares are cartesian, hence so is the outer rectangle. The top row consists of schemes, and the vertical arrows are surjective étale morphisms. By Spaces, Lemma 65.11.4 the properties (2) – (5) for $\Delta_{X/Y}$ are equivalent to those of $R \to U \times_Y U$. In the proof of Spaces, Lemma 65.13.1 we have seen that $R \to U \times_S U$ has properties (2) – (5). The morphism $U \times_Y U \to U \times_S U$ is a monomorphism of schemes. These facts imply that $R \to U \times_Y U$ have properties (2) – (5).

Namely: For (3), note that $R \to U \times_Y U$ is a monomorphism as the composition $R \to U \times_S U$ is a monomorphism. For (2), note that $R \to U \times_Y U$ is locally of finite type, as the composition $R \to U \times_S U$ is locally of finite type (Morphisms, Lemma 29.15.8). A monomorphism which is locally of finite type is locally quasi-finite because it has finite fibres (Morphisms, Lemma 29.20.7), hence (5). A monomorphism is separated (Schemes, Lemma 26.23.3), hence (4). □

03HL **Definition 67.4.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is *locally separated*[1] if $\Delta_{X/Y}$ is an immersion.
(3) We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

This definition makes sense since $\Delta_{X/Y}$ is representable, and hence we know what it means for it to have one of the properties described in the definition. We will see below (Lemma 67.4.13) that this definition matches the ones we already have for morphisms of schemes and representable morphisms.

03KK **Lemma 67.4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f is locally separated and f is quasi-separated.*

Proof. This is true, via the general principle Spaces, Lemma 65.5.8, because a closed immersion of schemes is an immersion and is quasi-compact. □

03KL **Lemma 67.4.4.** *All of the separation axioms listed in Definition 67.4.2 are stable under base change.*

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. By the results of Section 67.3 each of the properties of the diagonal used in Definition 67.4.2 is stable under base change. Hence the lemma is true. □

03KN **Lemma 67.4.5.** *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Z \to T$ is locally separated, then i is an immersion,*
(3) *if $Z \to T$ is separated, then i is a closed immersion, and*
(4) *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow[i]{} & X \times_T Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Delta_{Z/T}} & Z \times_T Z \end{array}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 67.4.1, and the material in Section 67.3. □

[1] In the literature this term often refers to quasi-separated and locally separated morphisms.

03KO **Lemma 67.4.6.** *Let S be a scheme. Let T be an algebraic space over S. Let $g : X \to Y$ be a morphism of algebraic spaces over T. Consider the graph $i : X \to X \times_T Y$ of g. Then*

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Y \to T$ is locally separated, then i is an immersion,*
(3) *if $Y \to T$ is separated, then i is a closed immersion, and*
(4) *if $Y \to T$ is quasi-separated, then i is quasi-compact.*

Proof. This is a special case of Lemma 67.4.5 applied to the morphism $X = X \times_Y Y \to X \times_T Y$. □

03KP **Lemma 67.4.7.** *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Let $s : T \to X$ be a section of f (in a formula $f \circ s = id_T$). Then*

(1) *s is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if f is locally separated, then s is an immersion,*
(3) *if f is separated, then s is a closed immersion, and*
(4) *if f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 67.4.6 applied to $g = s$ so the morphism $i = s : T \to T \times_T X$. □

03KQ **Lemma 67.4.8.** *All of the separation axioms listed in Definition 67.4.2 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 67.4.5 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see Section 67.3. □

04ZH **Lemma 67.4.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is separated and f is separated, then X is separated.*
(2) *If Y is quasi-separated and f is quasi-separated, then X is quasi-separated.*
(3) *If Y is locally separated and f is locally separated, then X is locally separated.*
(4) *If Y is separated over S and f is separated, then X is separated over S.*
(5) *If Y is quasi-separated over S and f is quasi-separated, then X is quasi-separated over S.*
(6) *If Y is locally separated over S and f is locally separated, then X is locally separated over S.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 67.4.8 and Spaces, Definition 65.13.2. Parts (1), (2), and (3) reduce to parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. □

03KR **Lemma 67.4.10.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is separated then so is f.*
(2) *If $g \circ f$ is locally separated then so is f.*
(3) *If $g \circ f$ is quasi-separated then so is f.*

Proof. Consider the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal morphism of $g \circ f$. In any case the last morphism is a monomorphism. Hence for any scheme T and morphism $T \to X \times_Y X$ we have the equality

$$X \times_{(X \times_Y X)} T = X \times_{(X \times_Z X)} T.$$

Hence the result is clear. □

04ZI **Lemma 67.4.11.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is separated then X is separated over S.*
(2) *If X is locally separated then X is locally separated over S.*
(3) *If X is quasi-separated then X is quasi-separated over S.*

Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(4) *If X is separated over S then f is separated.*
(5) *If X is locally separated over S then f is locally separated.*
(6) *If X is quasi-separated over S then f is quasi-separated.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 67.4.10 and Spaces, Definition 65.13.2. Parts (1), (2), and (3) follow from parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. □

03KM **Lemma 67.4.12.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 67.4.2. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Properties of Spaces, Definition 66.3.1),*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

Proof. We will repeatedly use Lemma 67.4.4 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Let us prove that (3) and (4) are equivalent. Note that if Z is an affine scheme, then the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is a separated morphism as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$. If $Z \times_Y X \to Z$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$ as a composition (see Lemma 67.4.8). Hence the algebraic space $Z \times_Y X$ is $\mathcal{P}$.

Conversely, if the algebraic space $Z \times_Y X$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$, and hence by Lemma 67.4.10 we see that $Z \times_Y X \to Z$ is $\mathcal{P}$.

Let us prove that (3) implies (5). Assume (3). Let V be a scheme and let $V \to Y$ be étale surjective. We have to show that $V \times_Y X \to V$ has property $\mathcal{P}$. In other words, we have to show that the morphism

$$V \times_Y X \longrightarrow (V \times_Y X) \times_V (V \times_Y X) = V \times_Y X \times_Y X$$

has the corresponding property (i.e., is a closed immersion, immersion, or quasi-compact). Let $V = \bigcup V_j$ be an affine open covering of V. By assumption we know that each of the morphisms

$$V_j \times_Y X \longrightarrow V_j \times_Y X \times_Y X$$

does have the corresponding property. Since being a closed immersion, immersion, quasi-compact immersion, or quasi-compact is Zariski local on the target, and since the V_j cover V we get the desired conclusion.

Let us prove that (5) implies (1). Let $V \to Y$ be as in (5). Then we have the fibre product diagram

$$\begin{array}{ccc} V \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ V \times_Y X \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

By assumption the left vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact. It follows from Spaces, Lemma 65.5.6 that also the right vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact.

It is clear that (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Choose schemes V_i and surjective étale morphisms $V_i \to Y_i$. Note that the morphisms $V_i \times_Y X \to V_i$ have $\mathcal{P}$ as they are base changes of the morphisms $f^{-1}(Y_i) \to Y_i$. Set $V = \coprod V_i$. Then $V \to Y$ is a morphism as in (5) (details omitted). Hence (6) implies (5) and we are done. □

03KY **Lemma 67.4.13.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S.*

(1) *The morphism f is locally separated.*
(2) *The morphism f is (quasi-)separated in the sense of Definition 67.4.2 above if and only if f is (quasi-)separated in the sense of Section 67.3.*

In particular, if $f : X \to Y$ is a morphism of schemes over S, then f is (quasi-)separated in the sense of Definition 67.4.2 if and only if f is (quasi-)separated as a morphism of schemes.

Proof. This is the equivalence of (1) and (2) of Lemma 67.4.12 combined with the fact that any morphism of schemes is locally separated, see Schemes, Lemma 26.21.2. □

67.5. Surjective morphisms

03MC We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be surjective.

03MD **Lemma 67.5.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is surjective (in the sense of Section 67.3) if and only if $|f| : |X| \to |Y|$ is surjective.*

Proof. Namely, if $f : X \to Y$ is representable, then it is surjective if and only if for every scheme T and every morphism $T \to Y$ the base change $f_T : T \times_Y X \to T$ of f is a surjective morphism of schemes, in other words, if and only if $|f_T|$ is surjective. By Properties of Spaces, Lemma 66.4.3 the map $|T \times_Y X| \to |T| \times_{|Y|} |X|$ is always surjective. Hence $|f_T| : |T \times_Y X| \to |T|$ is surjective if $|f| : |X| \to |Y|$ is surjective. Conversely, if $|f_T|$ is surjective for every $T \to Y$ as above, then by taking T to be the spectrum of a field we conclude that $|X| \to |Y|$ is surjective. □

This clears the way for the following definition.

03ME **Definition 67.5.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *surjective* if the map $|f| : |X| \to |Y|$ of associated topological spaces is surjective.

03MF **Lemma 67.5.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is surjective,*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a surjective morphism,*
5. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is surjective,*
6. *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

   where U, V are schemes and the vertical arrows are surjective étale such that the top horizontal arrow is surjective, and
7. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is surjective.*

Proof. Omitted. □

03MG **Lemma 67.5.4.** *The composition of surjective morphisms is surjective.*

Proof. This is immediate from the definition. □

03MH **Lemma 67.5.5.** *The base change of a surjective morphism is surjective.*

Proof. Follows immediately from Properties of Spaces, Lemma 66.4.3. □

67.6. Open morphisms

03Z0 For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally open. Hence before we give the natural definition we check that it agrees with this in the representable case.

03Z1 **Lemma 67.6.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open (in the sense of Section 67.3), and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is open.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally open. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

03Z2 **Definition 67.6.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *open* if the map of topological spaces $|f| : |X| \to |Y|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is open, i.e., the base change $Z \times_Y X \to Z$ is open.

Note that an étale morphism of algebraic spaces is universally open, see Properties of Spaces, Definition 66.16.2 and Lemmas 66.16.7 and 66.16.5.

03Z3 **Lemma 67.6.3.** *The base change of a universally open morphism of algebraic spaces by any morphism of algebraic spaces is universally open.*

Proof. This is immediate from the definition. □

03Z4 **Lemma 67.6.4.** *The composition of a pair of (universally) open morphisms of algebraic spaces is (universally) open.*

Proof. Omitted. □

03Z5 **Lemma 67.6.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open, and*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally open.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\
\downarrow & & \downarrow & & \downarrow & \swarrow & \\
\coprod |W_i| & = & |W| & \longrightarrow & |Z| & &
\end{array}$$

We have to show the south-east arrow is open. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 66.16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are open. Hence it follows that the right vertical arrow is open.

Assume $V \to Y$ is as in (4). We will show that f is universally open. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc}
|(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\
& \searrow & \downarrow & & \downarrow \\
& & |V \times_Y Z| & \longrightarrow & |Z|
\end{array}$$

The south-west arrow is open by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 66.16.7). It follows that the right vertical arrow is open.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is open. By a simple topological argument this implies that $Z \times_Y X \to Z$ is open. Hence (1) holds. □

06DN **Lemma 67.6.6.** *Let S be a scheme. Let $p : X \to \mathrm{Spec}(k)$ be a morphism of algebraic spaces over S where k is a field. Then $p : X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The composition $U \to \mathrm{Spec}(k)$ is universally open (as a morphism of schemes) by Morphisms, Lemma 29.23.4. Let $Z \to \mathrm{Spec}(k)$ be a morphism of schemes. Then

$U \times_{\mathrm{Spec}(k)} Z \to X \times_{\mathrm{Spec}(k)} Z$ is surjective, see Lemma 67.5.5. Hence the first of the maps

$$|U \times_{\mathrm{Spec}(k)} Z| \to |X \times_{\mathrm{Spec}(k)} Z| \to |Z|$$

is surjective. Since the composition is open by the above we conclude that the second map is open as well. Whence p is universally open by Lemma 67.6.5. □

67.7. Submersive morphisms

0411 For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally submersive. Hence before we give the natural definition we check that it agrees with this in the representable case.

0CFQ **Lemma 67.7.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally submersive (in the sense of Section 67.3), and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is submersive.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally submersive. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

0412 **Definition 67.7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *submersive*[2] if the continuous map $|X| \to |Y|$ is submersive, see Topology, Definition 5.6.3.
(2) We say f is *universally submersive* if for every morphism of algebraic spaces $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CFR **Lemma 67.7.3.** *The base change of a universally submersive morphism of algebraic spaces by any morphism of algebraic spaces is universally submersive.*

Proof. This is immediate from the definition. □

0CFS **Lemma 67.7.4.** *The composition of a pair of (universally) submersive morphisms of algebraic spaces is (universally) submersive.*

Proof. Omitted. □

[2]This is very different from the notion of a submersion of differential manifolds.

67.8. Quasi-compact morphisms

03HC By Section 67.3 we know what it means for a representable morphism of algebraic spaces to be quasi-compact. In order to formulate the definition for a general morphism of algebraic spaces we make the following observation.

03HD **Lemma 67.8.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact (in the sense of Section 67.3), and*
(2) *for every quasi-compact algebraic space Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact.*

Proof. Assume (1), and let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. By Properties of Spaces, Definition 66.5.1 there exists a quasi-compact scheme U and a surjective étale morphism $U \to Z$. Since f is representable and quasi-compact we see by definition that $U \times_Y X$ is a scheme, and that $U \times_Y X \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact scheme. The morphism $U \times_Y X \to Z \times_Y X$ is étale and surjective (as the base change of the representable étale and surjective morphism $U \to Z$, see Section 67.3). Hence by definition $Z \times_Y X$ is quasi-compact.

Assume (2). Let $Z \to Y$ be a morphism, where Z is a scheme. We have to show that $p : Z \times_Y X \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_Y Z$ and the scheme $U \times_Y Z$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Schemes, Section 26.19. □

This motivates the following definition.

03HE **Definition 67.8.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-compact* if for every quasi-compact algebraic space Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is quasi-compact.

By Lemma 67.8.1 above this agrees with the already existing notion for representable morphisms of algebraic spaces.

0EMK **Lemma 67.8.3.** *Let S be a scheme. If $f : X \to Y$ is a quasi-compact morphism of algebraic spaces over S, then the underlying map $|f| : |X| \to |Y|$ of topological space is quasi-compact.*

Proof. Let $V \subset |Y|$ be quasi-compact open. By Properties of Spaces, Lemma 66.4.8 there is an open subspace $Y' \subset Y$ with $V = |Y'|$. Then Y' is a quasi-compact algebraic space by Properties of Spaces, Lemma 66.5.2 and hence $X' = Y' \times_Y X$ is a quasi-compact algebraic space by Definition 67.8.2. On the other hand, $X' \subset X$ is an open subspace (Spaces, Lemma 65.12.3) and $|X'| = |f|^{-1}(|X'|) = |f|^{-1}(V)$ by Properties of Spaces, Lemma 66.4.3. We conclude using Properties of Spaces, Lemma 66.5.2 again that $|X'|$ is a quasi-compact open of $|X|$ as desired. □

03HF **Lemma 67.8.4.** *The base change of a quasi-compact morphism of algebraic spaces by any morphism of algebraic spaces is quasi-compact.*

Proof. Omitted. Hint: Transitivity of fibre products. □

03HG **Lemma 67.8.5.** *The composition of a pair of quasi-compact morphisms of algebraic spaces is quasi-compact.*

Proof. Omitted. Hint: Transitivity of fibre products. □

040W **Lemma 67.8.6.** *Let S be a scheme.*

(1) *If $X \to Y$ is a surjective morphism of algebraic spaces over S, and X is quasi-compact then Y is quasi-compact.*
(2) *If*

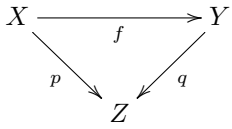

is a commutative diagram of morphisms of algebraic spaces over S and f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Assume X is quasi-compact and $X \to Y$ is surjective. By Definition 67.5.2 the map $|X| \to |Y|$ is surjective, hence we see Y is quasi-compact by Properties of Spaces, Lemma 66.5.2 and the topological fact that the image of a quasi-compact space under a continuous map is quasi-compact, see Topology, Lemma 5.12.7. Let f, p, q be as in (2). Let $T \to Z$ be a morphism whose source is a quasi-compact algebraic space. By assumption $T \times_Z X$ is quasi-compact. By Lemma 67.5.5 the morphism $T \times_Z X \to T \times_Z Y$ is surjective. Hence by part (1) we see $T \times_Z Y$ is quasi-compact too. Thus q is quasi-compact. □

04ZJ **Lemma 67.8.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a universally open and surjective morphism of algebraic spaces such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. As g is universally open and surjective, we see that $Y' \times_Y Z \to Z$ is open and surjective. As every point of $|Y' \times_Y Z|$ has a fundamental system of quasi-compact open neighbourhoods (see Properties of Spaces, Lemma 66.5.5) we can find a quasi-compact open $W \subset |Y' \times_Y Z|$ which surjects onto Z. Denote $f'' : W \times_Y X \to W$ the base change of f' by $W \to Y'$. By assumption $W \times_Y X$ is quasi-compact. As $W \to Z$ is surjective we see that $W \times_Y X \to Z \times_Y X$ is surjective. Hence $Z \times_Y X$ is quasi-compact by Lemma 67.8.6. Thus f is quasi-compact. □

03KG **Lemma 67.8.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism of algebraic spaces $Z \times_Y X \to Z$ is quasi-compact,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a quasi-compact morphism of algebraic spaces, and*
(5) *there exists a surjective étale morphism $Y' \to Y$ of algebraic spaces such that $Y' \times_Y X \to Y'$ is a quasi-compact morphism of algebraic spaces, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-compact.*

Proof. We will use Lemma 67.8.4 without further mention. It is clear that (1) implies (2) and that (2) implies (3). Assume (3). Let Z be a quasi-compact algebraic space over S, and let $Z \to Y$ be a morphism. By Properties of Spaces, Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $U \to Z$. Then $U \times_Y X \to Z \times_Y X$ is a surjective morphism of algebraic spaces, see Lemma 67.5.5. By assumption $|U \times_Y X|$ is quasi-compact. It surjects onto $|Z \times_Y X|$, hence we conclude that $|Z \times_Y X|$ is quasi-compact, see Topology, Lemma 5.12.7. This proves that (3) implies (1).

The implications (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) are clear. The implication (5) $\Rightarrow$ (1) follows from Lemma 67.8.7 and the fact that an étale morphism of algebraic spaces is universally open (see discussion following Definition 67.6.2).

Of course (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Let Z be affine and let $Z \to Y$ be a morphism. Then there exists a finite standard affine covering $Z = Z_1 \cup \ldots \cup Z_n$ such that each $Z_j \to Y$ factors through Y_{i_j} for some i_j. Hence the algebraic space

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} f^{-1}(Y_{i_j})$$

is quasi-compact. Since $Z \times_Y X = \bigcup_{j=1,\ldots,n} Z_j \times_Y X$ is a Zariski covering we see that $|Z \times_Y X| = \bigcup_{j=1,\ldots,n} |Z_j \times_Y X|$ (see Properties of Spaces, Lemma 66.4.8) is a finite union of quasi-compact spaces, hence quasi-compact. Thus we see that (6) implies (3). □

The following (and the next) lemma guarantees in particular that a morphism $X \to \mathrm{Spec}(A)$ is quasi-compact as soon as X is a quasi-compact algebraic space

03KS **Lemma 67.8.9.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 67.4.7 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 67.4.4). The second map is quasi-compact as it is the base change of $g \circ f$, see Lemma 67.8.4. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 67.8.5. □

073B **Lemma 67.8.10.** *Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.*

(1) *If X is quasi-compact and Y is quasi-separated, then f is quasi-compact.*
(2) *If X is quasi-compact and quasi-separated and Y is quasi-separated, then f is quasi-compact and quasi-separated.*
(3) *A fibre product of quasi-compact and quasi-separated algebraic spaces is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 67.8.9 with $Z = S = \mathrm{Spec}(\mathbf{Z})$. Part (2) follows from (1) and Lemma 67.4.10. For (3) let $X \to Y$ and $Z \to Y$ be morphisms of quasi-compact and quasi-separated algebraic spaces. Then $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of $X \to Y$ using (2) and Lemmas 67.8.4 and 67.4.4. Hence $X \times_Y Z$ is quasi-compact and quasi-separated as

an algebraic space quasi-compact and quasi-separated over Z, see Lemmas 67.4.9 and 67.8.5. □

67.9. Universally closed morphisms

03HH For a representable morphism of algebraic spaces we have already defined (in Section 67.3) what it means to be universally closed. Hence before we give the natural definition we check that it agrees with this in the representable case.

03XD **Lemma 67.9.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed (in the sense of Section 67.3), and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is closed.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally closed. By Properties of Spaces, Section 66.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

03HI **Definition 67.9.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *closed* if the map of topological spaces $|X| \to |Y|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is closed, i.e., the base change $Z \times_Y X \to Z$ is closed.

03IS **Lemma 67.9.3.** *The base change of a universally closed morphism of algebraic spaces by any morphism of algebraic spaces is universally closed.*

Proof. This is immediate from the definition. □

03IU **Lemma 67.9.4.** *The composition of a pair of (universally) closed morphisms of algebraic spaces is (universally) closed.*

Proof. Omitted. □

03IT **Lemma 67.9.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally closed.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\
\downarrow & & \downarrow & & \downarrow & \swarrow & \\
\coprod |W_i| & = & |W| & \longrightarrow & |Z| & &
\end{array}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 66.16.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc}
|(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\
& \searrow & \downarrow & & \downarrow \\
& & |V \times_Y Z| & \longrightarrow & |Z|
\end{array}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 66.16.7). It follows that the right vertical arrow is closed.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is closed. By a simple topological argument this implies that $Z \times_Y X \to Z$ is closed. Hence (1) holds. □

03IV **Example 67.9.6.** Strange example of a universally closed morphism. Let $\mathbf{Q} \subset k$ be a field of characteristic zero. Let $X = \mathbf{A}^1_k/\mathbf{Z}$ as in Spaces, Example 65.14.8. We claim the structure morphism $p : X \to \mathrm{Spec}(k)$ is universally closed. Namely, if Z/k is a scheme, and $T \subset |X \times_k Z|$ is closed, then T corresponds to a $\mathbf{Z}$-invariant closed subset of $T' \subset |\mathbf{A}^1 \times Z|$. It is easy to see that this implies that T' is the inverse image of a subset T'' of Z. By Morphisms, Lemma 29.25.12 we have that

$T'' \subset Z$ is closed. Of course T'' is the image of T. Hence p is universally closed by Lemma 67.9.5.

04XW **Lemma 67.9.7.** *Let S be a scheme. A universally closed morphism of algebraic spaces over S is quasi-compact.*

Proof. This proof is a repeat of the proof in the case of schemes, see Morphisms, Lemma 29.41.8. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Lemma 67.8.8 there exists an affine scheme Z and a morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is not quasi-compact. To achieve our goal it suffices to show that $Z \times_Y X \to Z$ is not universally closed, hence we may assume that $Y = \mathrm{Spec}(B)$ for some ring B.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are quasi-compact open subspaces of X. For example, choose a surjective étale morphism $U \to X$ where U is a scheme, choose an affine open covering $U = \bigcup U_i$ and let $X_i \subset X$ be the image of U_i. We will use later that the morphisms $X_i \to Y$ are quasi-compact, see Lemma 67.8.9. Let $T = \mathrm{Spec}(B[a_i; i \in I])$. Let $T_i = D(a_i) \subset T$. Let $Z \subset T \times_Y X$ be the reduced closed subspace whose underlying closed set of points is $|T \times_Y Z| \setminus \bigcup_{i \in I} |T_i \times_Y X_i|$, see Properties of Spaces, Lemma 66.12.3. (Note that $T_i \times_Y X_i$ is an open subspace of $T \times_Y X$ as $T_i \to T$ and $X_i \to X$ are open immersions, see Spaces, Lemmas 65.12.3 and 65.12.2.) Here is a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & T \times_Y X & \xrightarrow[q]{} & X \\ & \searrow & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f} \\ & & T & \xrightarrow{p} & Y \end{array}$$

It suffices to prove that the image $f_T(|Z|)$ is not closed in $|T|$.

We claim there exists a point $y \in Y$ such that there is no affine open neighborhood V of y in Y such that X_V is quasi-compact. If not then we can cover Y with finitely many such V and for each V the morphism $Y_V \to V$ is quasi-compact by Lemma 67.8.9 and then Lemma 67.8.8 implies f quasi-compact, a contradiction. Fix a $y \in Y$ as in the claim.

Let $t \in T$ be the point lying over y with $\kappa(t) = \kappa(y)$ such that $a_i = 1$ in $\kappa(t)$ for all i. Suppose $z \in |Z|$ with $f_T(z) = t$. Then $q(t) \in X_i$ for some i. Hence $f_T(z) \notin T_i$ by construction of Z, which contradicts the fact that $t \in T_i$ by construction. Hence we see that $t \in |T| \setminus f_T(|Z|)$.

Assume $f_T(|Z|)$ is closed in $|T|$. Then there exists an element $g \in B[a_i; i \in I]$ with $f_T(|Z|) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(y)$. Hence this coefficient is invertible on some affine open neighborhood V of y. Let J be the finite set of $j \in I$ such that the variable a_j appears in g. Since X_V is not quasi-compact and each $X_{i,V}$ is quasi-compact, we may choose a point $x \in |X_V| \setminus \bigcup_{j \in J} |X_{j,V}|$. In other words, $x \in |X| \setminus \bigcup_{j \in J} |X_j|$ and x lies above some $v \in V$. Since g has a coefficient that is invertible on V, we can find a point $t' \in T$ lying above v such that $t' \notin V(g)$ and $t' \in V(a_i)$ for all $i \notin J$. This is true because $V(a_i; i \in I \setminus J) = \mathrm{Spec}(B[a_j; j \in J])$ and the set of points of this scheme lying over v is bijective with $\mathrm{Spec}(\kappa(v)[a_j; j \in J])$ and g restricts to a nonzero element of this polynomial ring by construction. In

other words $t' \notin T_i$ for each $i \notin J$. By Properties of Spaces, Lemma 66.4.3 we can find a point z of $X \times_Y T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin |X_j|$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in |Z|$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(|Z|)$ closed" is wrong and we conclude indeed that f_T is not closed as desired. □

The target of a separated algebraic space under a surjective universally closed morphism is separated.

05Z2 **Lemma 67.9.8.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a surjective universally closed morphism of algebraic spaces over B.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over B, then Y is quasi-separated over B.*
(4) *If X is separated over B, then Y is separated over B.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = B = \mathrm{Spec}(\mathbf{Z})$ (see Properties of Spaces, Definition 66.3.1). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/B}} & X \times_B X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/B}} & Y \times_B Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_B X \to X \times_B Y \to Y \times_B Y$. Hence it is also quasi-compact, see Lemma 67.9.7.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is quasi-compact. Then if Z is quasi-compact and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/B}$ is quasi-compact, i.e., Y is quasi-separated over B.

Assume X is separated over B, i.e., $\Delta_{X/B}$ is a closed immersion. Then if Z is affine, and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X \to Z$ is universally closed by our remarks above. We conclude that $\Delta_{Y/B}$ is universally closed. It follows that $\Delta_{Y/B}$ is representable, locally of finite type, a monomorphism (see Lemma 67.4.1) and universally closed, hence a closed immersion, see Étale Morphisms, Lemma 41.7.2 (and also the abstract principle Spaces, Lemma 65.5.8). Thus Y is separated over B. □

67.10. Monomorphisms

042K A representable morphism $X \to Y$ of algebraic spaces is a monomorphism according to Section 67.3 if for every scheme Z and morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is representable by a monomorphism of schemes. This means exactly that $Z \times_Y X \to Z$ is an injective map of sheaves on $(Sch/S)_{fppf}$. Since this is supposed to hold for all Z and all maps $Z \to Y$ this is in turn equivalent to the map $X \to Y$ being an injective map of sheaves on $(Sch/S)_{fppf}$. Thus we may define a monomorphism of a (possibly nonrepresentable[3]) morphism of algebraic spaces as follows.

[3]We do not know whether any monomorphism of algebraic spaces is representable. For a discussion see More on Morphisms of Spaces, Section 76.4.

042L **Definition 67.10.1.** Let S be a scheme. A morphism of algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves, i.e., a monomorphism in the category of sheaves on $(Sch/S)_{fppf}$.

The following lemma shows that this also means that it is a monomorphism in the category of algebraic spaces over S.

042M **Lemma 67.10.2.** *Let S be a scheme. Let $j : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *j is a monomorphism (as in Definition 67.10.1),*
(2) *j is a monomorphism in the category of algebraic spaces over S, and*
(3) *the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. Note that $X \times_Y X$ is both the fibre product in the category of sheaves on $(Sch/S)_{fppf}$ and the fibre product in the category of algebraic spaces over S, see Spaces, Lemma 65.7.3. The equivalence of (1) and (3) is a general characterization of injective maps of sheaves on any site. The equivalence of (2) and (3) is a characterization of monomorphisms in any category with fibre products. □

042N **Lemma 67.10.3.** *A monomorphism of algebraic spaces is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 67.10.2 above. □

042O **Lemma 67.10.4.** *A composition of monomorphisms is a monomorphism.*

Proof. True because a composition of injective sheaf maps is injective. □

042P **Lemma 67.10.5.** *The base change of a monomorphism is a monomorphism.*

Proof. This is a general fact about fibre products in a category of sheaves. □

042Q **Lemma 67.10.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is a monomorphism,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ is a monomorphism, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a monomorphism.*

Proof. We will use without further mention that a base change of a monomorphism is a monomorphism, see Lemma 67.10.5. In particular it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) (by taking V to be a disjoint union of affine schemes étale over Y, see Properties of Spaces, Lemma 66.6.1). Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{G} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{H} & \xrightarrow{d} & \mathcal{I} \end{array} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. Thus (4) implies (1). Proof of the equivalence of (5) and (1) is omitted. □

042R **Lemma 67.10.7.** *An immersion of algebraic spaces is a monomorphism. In particular, any immersion is separated.*

Proof. Let $f : X \to Y$ be an immersion of algebraic spaces. For any morphism $Z \to Y$ with Z representable the base change $Z \times_Y X \to Z$ is an immersion of schemes, hence a monomorphism, see Schemes, Lemma 26.23.8. Hence f is representable, and a monomorphism. □

We will improve on the following lemma in Decent Spaces, Lemma 68.19.1.

06MG **Lemma 67.10.8.** *Let S be a scheme. Let k be a field and let $Z \to \mathrm{Spec}(k)$ be a monomorphism of algebraic spaces over S. Then either $Z = \emptyset$ or $Z = \mathrm{Spec}(k)$.*

Proof. By Lemmas 67.10.3 and 67.4.9 we see that Z is a separated algebraic space. Hence there exists an open dense subspace $Z' \subset Z$ which is a scheme, see Properties of Spaces, Proposition 66.13.3. By Schemes, Lemma 26.23.11 we see that either $Z' = \emptyset$ or $Z' \cong \mathrm{Spec}(k)$. In the first case we conclude that $Z = \emptyset$ and in the second case we conclude that $Z' = Z = \mathrm{Spec}(k)$ as $Z \to \mathrm{Spec}(k)$ is a monomorphism which is an isomorphism over Z'. □

06RV **Lemma 67.10.9.** *Let S be a scheme. If $X \to Y$ is a monomorphism of algebraic spaces over S, then $|X| \to |Y|$ is injective.*

Proof. Immediate from the definitions. □

67.11. Pushforward of quasi-coherent sheaves

03M7 We first prove a simple lemma that relates pushforward of sheaves of modules for a morphism of algebraic spaces to pushforward of sheaves of modules for a morphism of schemes.

03M8 **Lemma 67.11.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \to X$ be a surjective étale morphism from a scheme to X. Set $R = U \times_X U$ and denote $t, s : R \to U$ the projection morphisms as usual. Denote $a : U \to Y$ and $b : R \to Y$ the induced morphisms. For any object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ there exists an exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R)$$

where the second arrow is the difference $t^* - s^*$.

Proof. We denote $\mathcal{F}$ also its extension to a sheaf of modules on $X_{spaces,étale}$, see Properties of Spaces, Remark 66.18.4. Let $V \to Y$ be an object of $Y_{étale}$. Then $V \times_Y X$ is an object of $X_{spaces,étale}$, and by definition $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. Since $U \to X$ is surjective étale, we see that $\{V \times_Y U \to V \times_Y X\}$ is a covering. Also, we have $(V \times_Y U) \times_X (V \times_Y U) = V \times_Y R$. Hence, by the sheaf condition of $\mathcal{F}$ on $X_{spaces,étale}$ we have a short exact sequence

$$0 \to \mathcal{F}(V \times_Y X) \to \mathcal{F}(V \times_Y U) \to \mathcal{F}(V \times_Y R)$$

where the second arrow is the difference of restricting via t or s. This exact sequence is functorial in V and hence we obtain the lemma. □

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. By Descent, Proposition 35.9.4 the functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ agrees with the usual functor if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

04CF (67.11.1.1)
$$f'_*(\mathcal{F}|_U) = (f_*\mathcal{F})|_V,$$

see Properties of Spaces, Lemma 66.26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

The next level of generality is to consider an arbitrary quasi-compact and quasi-separated morphism of algebraic spaces.

03M9 **Lemma 67.11.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. For this it suffices to show that for any affine scheme V and étale morphism $V \to Y$ the restriction of $f_*\mathcal{F}$ to V is quasi-coherent, see Properties of Spaces, Lemma 66.29.6. Let $f' : V \times_Y X \to V$ be the base change of f by $V \to Y$. Note that f' is also quasi-compact and quasi-separated, see Lemmas 67.8.4 and 67.4.4. By (67.11.1.1) we know that the restriction of $f_*\mathcal{F}$ to V is f'_* of the restriction of $\mathcal{F}$ to $V \times_Y X$. Hence we may replace f by f', and assume that Y is an affine scheme.

Assume Y is an affine scheme. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. By Lemma 67.11.1 we get an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R).$$

where $R = U \times_X U$. As $X \to Y$ is quasi-separated we see that $R \to U \times_Y U$ is a quasi-compact monomorphism. This implies that R is a quasi-compact separated scheme (as U and Y are affine at this point). Hence $a : U \to Y$ and $b : R \to Y$ are quasi-compact and quasi-separated morphisms of schemes. Thus by Descent, Proposition 35.9.4 the sheaves $a_*(\mathcal{F}|_U)$ and $b_*(\mathcal{F}|_R)$ are quasi-coherent (see also the discussion preceding this lemma). This implies that $f_*\mathcal{F}$ is a kernel of quasi-coherent modules, and hence itself quasi-coherent, see Properties of Spaces, Lemma 66.29.7. □

Higher direct images are discussed in Cohomology of Spaces, Section 69.3.

67.12. Immersions

03HB Open, closed and locally closed immersions of algebraic spaces were defined in Spaces, Section 65.12. Namely, a morphism of algebraic spaces is a *closed immersion* (resp. *open immersion*, resp. *immersion*) if it is representable and a closed immersion (resp. open immersion, resp. immersion) in the sense of Section 67.3.

In particular these types of morphisms are stable under base change and compositions of morphisms in the category of algebraic spaces over S, see Spaces, Lemmas 65.12.2 and 65.12.3.

03M4 **Lemma 67.12.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is a closed immersion (resp. open immersion, resp. immersion),*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a closed immersion (resp. open immersion, resp. immersion), and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a closed immersion (resp. open immersion, resp. immersion).*

Proof. Using that a base change of a closed immersion (resp. open immersion, resp. immersion) is another one it is clear that (1) implies (2) and (2) implies (3). Also (3) implies (4) since we can take V to be a disjoint union of affines, see Properties of Spaces, Lemma 66.6.1.

Assume $V \to Y$ is as in (4). Let $\mathcal{P}$ be the property closed immersion (resp. open immersion, resp. immersion) of morphisms of schemes. Note that property $\mathcal{P}$ is preserved under any base change and fppf local on the base (see Section 67.3). Moreover, morphisms of type $\mathcal{P}$ are separated and locally quasi-finite (in each of the three cases, see Schemes, Lemma 26.23.8, and Morphisms, Lemma 29.20.16). Hence by More on Morphisms, Lemma 37.57.1 the morphisms of type $\mathcal{P}$ satisfy descent for fppf covering. Thus Spaces, Lemma 65.11.5 applies and we see that $X \to Y$ is representable and has property $\mathcal{P}$, in other words (1) holds.

The equivalence of (1) and (5) follows from the fact that $\mathcal{P}$ is Zariski local on the target (since we saw above that $\mathcal{P}$ is in fact fppf local on the target). □

0AGC **Lemma 67.12.2.** *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S.*

(1) *If $Z \to X$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism, then $Z \to Y$ is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism.*
(2) *If $Z \to X$ is an immersion and $Y \to X$ is locally separated, then $Z \to Y$ is an immersion.*
(3) *If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Y \times_X Z & \longrightarrow & Z \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & X \end{array}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence representable, locally of finite type, locally quasi-finite, separated, and a monomorphism by Lemma 67.4.7. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence is representable, locally of finite type, locally quasi-finite, separated, and a monomorphism (as each of these properties of morphisms of schemes is stable under base change, see Spaces, Remark 65.4.1). Hence the same is true for the composition (as each of these properties of morphisms of schemes is stable under composition, see Spaces, Remark 65.4.2). This proves (1). The other results are proved in exactly the same manner. □

04CD **Lemma 67.12.3.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then $|i| : |Z| \to |X|$ is a homeomorphism onto a locally closed subset, and i is a closed immersion if and only if the image $|i|(|Z|) \subset |X|$ is a closed subset.*

Proof. The first statement is Properties of Spaces, Lemma 66.12.1. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption $T = U \times_X Z$ is a scheme and the morphism $j : T \to U$ is an immersion of schemes. By Lemma 67.12.1 the morphism i is a closed immersion if and only if j is a closed immersion. By Schemes, Lemma 26.10.4 this is true if and only if $j(T)$ is closed in U. However, the subset $j(T) \subset U$ is the inverse image of $|i|(|Z|) \subset |X|$, see Properties of Spaces, Lemma 66.4.3. This finishes the proof. □

04CE **Remark 67.12.4.** Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Since i is a monomorphism we may think of $|Z|$ as a subset of $|X|$; in the rest of this remark we do so. Let $\partial|Z|$ be the boundary of $|Z|$ in the topological space $|X|$. In a formula

$$\partial|Z| = \overline{|Z|} \setminus |Z|.$$

Let ∂Z be the reduced closed subspace of X with $|\partial Z| = \partial|Z|$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 66.12.5. By construction we see that $|Z|$ is closed in $|X| \setminus |\partial Z| = |X \setminus \partial Z|$. Hence it is true that any immersion of algebraic spaces can be factored as a closed immersion followed by an open immersion (but not the other way in general, see Morphisms, Example 29.3.4).

06EC **Remark 67.12.5.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|X|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $U \subset X$ be the open subspace of X with $|U| = |X| \setminus \partial T$, see Properties of Spaces, Lemma 66.4.8. Let Z be the reduced closed subspace of U with $|Z| = T$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 66.12.5. By construction $Z \to U$ is a closed immersion of

algebraic spaces and $U \to X$ is an open immersion, hence $Z \to X$ is an immersion of algebraic spaces over S (see Spaces, Lemma 65.12.2). Note that Z is a reduced algebraic space and that $|Z| = T$ as subsets of $|X|$. We sometimes say Z is the *reduced induced subspace structure* on T.

081U **Lemma 67.12.6.** *Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces over S. Assume $Z \to X$ is quasi-compact. There exists a factorization $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is a closed immersion.*

Proof. Let U be a scheme and let $U \to X$ be surjective étale. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Set $T = Z \times_U X$. Let $\overline{T} \subset U$ be the scheme theoretic image of $T \to U$. Note that $s^{-1}\overline{T} = t^{-1}\overline{T}$ as taking scheme theoretic images of quasi-compact morphisms commute with flat base change, see Morphisms, Lemma 29.25.16. Hence we obtain a closed subspace $\overline{Z} \subset X$ whose pullback to U is $\overline{T}$, see Properties of Spaces, Lemma 66.12.2. By Morphisms, Lemma 29.7.7 the morphism $T \to \overline{T}$ is an open immersion. It follows that $Z \to \overline{Z}$ is an open immersion and we win. □

67.13. Closed immersions

03MA In this section we elucidate some of the results obtained previously on immersions of algebraic spaces. See Spaces, Section 65.12 and Section 67.12 in this chapter. This section is the analogue of Morphisms, Section 29.2 for algebraic spaces.

03MB **Lemma 67.13.1.** *Let S be a scheme. Let X be an algebraic space over S. For every closed immersion $i : Z \to X$ the sheaf $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective and its kernel is a quasi-coherent sheaf of ideals. The rule $Z \mapsto \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ defines an inclusion reversing bijection*

$$\begin{array}{c} \text{closed subspaces} \\ Z \subset X \end{array} \longrightarrow \begin{array}{c} \text{quasi-coherent sheaves} \\ \text{of ideals } \mathcal{I} \subset \mathcal{O}_X \end{array}$$

Moreover, given a closed subscheme Z corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ a morphism of algebraic spaces $h : Y \to X$ factors through Z if and only if the map $h^*\mathcal{I} \to h^*\mathcal{O}_X = \mathcal{O}_Y$ is zero.

Proof. Let $U \to X$ be a surjective étale morphism whose source is a scheme. Consider the diagram

$$\begin{array}{ccc} U \times_X Z & \longrightarrow & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ U & \longrightarrow & X \end{array}$$

By Lemma 67.12.1 we see that i is a closed immersion if and only if i' is a closed immersion. By Properties of Spaces, Lemma 66.26.2 we see that $i'_*\mathcal{O}_{U \times_X Z}$ is the restriction of $i_*\mathcal{O}_Z$ to U. Hence the assertions on $\mathcal{O}_X \to i_*\mathcal{O}_Z$ are equivalent to the corresponding assertions on $\mathcal{O}_U \to i'_*\mathcal{O}_{U \times_X Z}$. And since i' is a closed immersion of schemes, these results follow from Morphisms, Lemma 29.2.1.

Let us prove that given a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ the formula

$$Z(T) = \{h : T \to X \mid h^*\mathcal{I} \to \mathcal{O}_T \text{ is zero}\}$$

defines a closed subspace of X. It is clearly a subfunctor of X. To show that $Z \to X$ is representable by closed immersions, let $\varphi : U \to X$ be a morphism from a scheme

towards X. Then $Z \times_X U$ is represented by the analogous subfunctor of U corresponding to the sheaf of ideals $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. By Properties of Spaces, Lemma 66.29.2 the $\mathcal{O}_U$-module $\varphi^*\mathcal{I}$ is quasi-coherent on U, and hence $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$ is a quasi-coherent sheaf of ideals on U. By Schemes, Lemma 26.4.6 we conclude that $Z \times_X U$ is represented by the closed subscheme of U associated to $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. Thus Z is a closed subspace of X.

In the formula for Z above the inputs T are schemes since algebraic spaces are sheaves on $(Sch/S)_{fppf}$. We omit the verification that the same formula remains true if T is an algebraic space. □

083Q **Definition 67.13.2.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. The *inverse image* $f^{-1}(Z)$ *of the closed subspace* Z is the closed subspace $Z \times_X Y$ of Y.

This definition makes sense by Lemma 67.12.1. If $\mathcal{I} \subset \mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to Z via Lemma 67.13.1 then $f^{-1}\mathcal{I}\mathcal{O}_Y = \mathrm{Im}(f^*\mathcal{I} \to \mathcal{O}_Y)$ is the sheaf of ideals corresponding to $f^{-1}(Z)$.

04CG **Lemma 67.13.3.** *A closed immersion of algebraic spaces is quasi-compact.*

Proof. This follows from Schemes, Lemma 26.19.5 by general principles, see Spaces, Lemma 65.5.8. □

04CH **Lemma 67.13.4.** *A closed immersion of algebraic spaces is separated.*

Proof. This follows from Schemes, Lemma 26.23.8 by general principles, see Spaces, Lemma 65.5.8. □

04E5 **Lemma 67.13.5.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S.*

(1) *The functor*
$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*
$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in $|Z|$.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. Set $V = Z \times_X U$. Then V is a scheme and $i' : V \to U$ is a closed immersion of schemes. By Properties of Spaces, Lemma 66.18.12 for any sheaf $\mathcal{G}$ on Z we have
$$(i_{small}^{-1} i_{small,*}\mathcal{G})|_V = (i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V)$$
By Étale Cohomology, Proposition 59.46.4 the map $(i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V) \to \mathcal{G}|_V$ is an isomorphism. Since $V \to Z$ is surjective and étale this implies that $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism. This clearly implies that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.41.1. To prove the statement on the essential image, consider a sheaf of

sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$. As in the proof of Étale Cohomology, Proposition 59.46.4 we consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1} \mathcal{F}.$$

As in the first part we see that the restriction of this map to U is an isomorphism by the corresponding result for the case of schemes. Since U is an étale covering of X we conclude it is an isomorphism. □

0DK1 **Lemma 67.13.6.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\overline{z}$ be a geometric point of Z with image $\overline{x}$ in X. Then $(i_{small,*}\mathcal{F})_{\overline{z}} = \mathcal{F}_{\overline{x}}$ for any sheaf $\mathcal{F}$ on $Z_{étale}$.*

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$. Then the stalk $(i_{small,*}\mathcal{F})_{\overline{z}}$ is the stalk of $i_{small,*}\mathcal{F}|_U$ at $\overline{u}$. By Properties of Spaces, Lemma 66.18.12 we may replace X by U and Z by $Z \times_X U$. Then $Z \to X$ is a closed immersion of schemes and the result is Étale Cohomology, Lemma 59.46.3. □

The following lemma holds more generally in the setting of a closed immersion of topoi (insert future reference here).

04G0 **Lemma 67.13.7.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Z_{étale}$. Let $\varphi : \mathcal{A} \to i_{small,*}\mathcal{B}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi*

$$f : (Sh(Z_{étale}), \mathcal{B}) \longrightarrow (Sh(X_{étale}), \mathcal{A}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G} \longrightarrow f_*(f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{F} \otimes_{\mathcal{B}} f^*f_*\mathcal{G} = f^*(\mathcal{F} \otimes_{\mathcal{A}} f_*\mathcal{G}) \longrightarrow f^*\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$$

coming from $\mathrm{id} : f^*\mathcal{F} \to f^*\mathcal{F}$ and the adjunction map $f^*f_*\mathcal{G} \to \mathcal{G}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 66.19.12). Let $\overline{z} : \mathrm{Spec}(k) \to Z$ be a geometric point with image $\overline{x} = i \circ \overline{z} : \mathrm{Spec}(k) \to X$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{G}_{\overline{z}} = (\mathcal{F}_{\overline{x}} \otimes_{\mathcal{A}_{\overline{x}}} \mathcal{B}_{\overline{z}}) \otimes_{\mathcal{B}_{\overline{z}}} \mathcal{G}_{\overline{z}}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 66.19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 67.13.6. □

67.14. Closed immersions and quasi-coherent sheaves

04CI This section is the analogue of Morphisms, Section 29.4.

04CJ **Lemma 67.14.1.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z.*

(1) *For any $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$.*
(2) *The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.*
(3) *The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{I}\mathcal{F} = 0$.

Proof. During this proof we work exclusively with sheaves on the small étale sites, and we use $i_*, i^{-1}, \ldots$ to denote pushforward and pullback of sheaves of abelian groups instead of $i_{small,*}, i_{small}^{-1}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By Lemma 67.13.7 applied with $\mathcal{A} = \mathcal{O}_X$ and $\mathcal{G} = \mathcal{B} = \mathcal{O}_Z$ we see that $i_*i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. By Lemma 67.13.1 we see that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

It follows from properties of the tensor product that $\mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z = \mathcal{F}/\mathcal{I}\mathcal{F}$. This proves (1) (except that we omit the verification that the map is induced by the adjunction mapping).

Let $\mathcal{G}$ be any $\mathcal{O}_Z$-module. By Lemma 67.13.5 we see that $i^{-1}i_*\mathcal{G} = \mathcal{G}$. Hence to prove (2) we have to show that the canonical map $\mathcal{G} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{G}$ is an isomorphism. This follows from general properties of tensor products if we can show that $i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. By Lemma 67.13.5 it suffices to prove that $i_*i^{-1}\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Since the surjective map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ factors through this map we see that (2) holds.

Finally we prove the most interesting part of the lemma, namely part (3). A closed immersion is quasi-compact and separated, see Lemmas 67.13.3 and 67.13.4. Hence Lemma 67.11.2 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X. Thus we obtain our functor $i_*^{QCoh} : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$. It is clear from part (2) that i_*^{QCoh} is fully faithful since it has a left inverse, namely i^*.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{G}) = 0$ for any $\mathcal{O}_Z$-module, since $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ which is the map we use to put an $\mathcal{O}_X$-module structure on $i_*\mathcal{G}$. Next, suppose that $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{F} = 0$. Then we see that $\mathcal{F}$ is an $i_*\mathcal{O}_Z$-module because $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Hence in particular its support is contained in $|Z|$. We apply Lemma 67.13.5 to see that $\mathcal{F} \cong i_*\mathcal{G}$ for some $\mathcal{O}_Z$-module $\mathcal{G}$. The only small detail left over is to see why $\mathcal{G}$ is quasi-coherent. This is true because $\mathcal{G} \cong i^*\mathcal{F}$ by part (2) and Properties of Spaces, Lemma 66.29.2. □

Let $i : Z \to X$ be a closed immersion of algebraic spaces. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

04CK **Lemma 67.14.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a*

unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

04CL **Lemma 67.14.3.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. There is a functor[4] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 67.14.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subspaces (see Lemma 67.13.1) we can define scheme theoretic intersections and unions of closed subschemes.

0CYZ **Definition 67.14.4.** Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The *scheme theoretic intersection* of Z and Y is the closed subspace of X cut out by $\mathcal{I} + \mathcal{J}$. Then *scheme theoretic union* of Z and Y is the closed subspace of X cut out by $\mathcal{I} \cap \mathcal{J}$.

It is clear that formation of scheme theoretic intersection commutes with étale localization and the same is true for scheme theoretic union.

0CZ0 **Lemma 67.14.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z, Y \subset X$ be closed subspaces. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and*

$$\begin{array}{ccc} Z \cap Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

[4] This is likely nonstandard notation.

is a cartesian diagram of algebraic spaces over S, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 67.13.1. Since formation of the scheme theoretic intersection commutes with étale localization we conclude the diagram is cartesian by the case of schemes. See Morphisms, Lemma 29.4.5. □

0CZ1 **Lemma 67.14.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $Y, Z \subset X$ be closed subspaces. Let $Y \cup Z$ be the scheme theoretic union of Y and Z. Let $Y \cap Z$ be the scheme theoretic intersection of Y and Z. Then $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions, there is a short exact sequence*

$$0 \to \mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y \times \mathcal{O}_Z \to \mathcal{O}_{Y \cap Z} \to 0$$

of $\mathcal{O}_Z$-modules, and the diagram

$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup Z \end{array}$$

is cocartesian in the category of algebraic spaces over S, i.e., $Y \cup Z = Y \amalg_{Y \cap Z} Z$.

Proof. The morphisms $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions by Lemma 67.13.1. In the short exact sequence we use the equivalence of Lemma 67.14.1 to think of quasi-coherent modules on closed subspaces of X as quasi-coherent modules on X. For the first map in the sequence we use the canonical maps $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y$ and $\mathcal{O}_{Y \cup Z} \to \mathcal{O}_Z$ and for the second map we use the canonical map $\mathcal{O}_Y \to \mathcal{O}_{Y \cap Z}$ and the negative of the canonical map $\mathcal{O}_Z \to \mathcal{O}_{Y \cap Z}$. Then to check exactness we may work étale locally and deduce exactness from the case of schemes (Morphisms, Lemma 29.4.6).

To show the diagram is cocartesian, suppose we are given an algebraic space T over S and morphisms $f : Y \to T$, $g : Z \to T$ agreeing as morphisms $Y \cap Z \to T$. Goal: Show there exists a unique morphism $h : Y \cup Z \to T$ agreeing with f and g. To construct h we may work étale locally on $Y \cup Z$ (as $Y \cup Z$ is an étale sheaf being an algebraic space). Hence we may assume that X is a scheme. In this case we know that $Y \cup Z$ is the pushout of Y and Z along $Y \cap Z$ in the category of schemes by Morphisms, Lemma 29.4.6. Choose a scheme T' and a surjective étale morphism $T' \to T$. Set $Y' = T' \times_{T,f} Y$ and $Z' = T' \times_{T,g} Z$. Then Y' and Z' are schemes and we have a canonical isomorphism $\varphi : Y' \times_Y (Y \cap Z) \to Z' \times_Z (Y \cap Z)$ of schemes. By More on Morphisms, Lemma 37.67.8 the pushout $W' = Y' \amalg_{Y' \times_Y (Y \cap Z), \varphi} Z'$ exists in the category of schemes. The morphism $W' \to Y \cup Z$ is étale by More on Morphisms, Lemma 37.67.9. It is surjective as $Y' \to Y$ and $Z' \to Z$ are surjective. The morphisms $f' : Y' \to T'$ and $g' : Z' \to T'$ glue to a unique morphism of schemes $h' : W' \to T'$. By uniqueness the composition $W' \to T' \to T$ descends to the desired morphism $h : Y \cup Z \to T$. Some details omitted. □

67.15. Supports of modules

07TX In this section we collect some elementary results on supports of quasi-coherent modules on algebraic spaces. Let X be an algebraic space. The support of an abelian sheaf on $X_{étale}$ has been defined in Properties of Spaces, Section 66.20. We

use the same definition for supports of modules. The following lemma tells us this agrees with the notion as defined for quasi-coherent modules on schemes.

07TY **Lemma 67.15.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. Then*

$$Supp(\varphi^*\mathcal{F}) = |\varphi|^{-1}(Supp(\mathcal{F}))$$

where the left hand side is the support of $\varphi^*\mathcal{F}$ as a quasi-coherent module on the scheme U.

Proof. Let $u \in U$ be a (usual) point and let $\overline{x}$ be a geometric point lying over u. By Properties of Spaces, Lemma 66.29.4 we have $(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is the strict henselization by Properties of Spaces, Lemma 66.22.1 we see that it is faithfully flat (see More on Algebra, Lemma 15.45.1). Thus we see that $(\varphi^*\mathcal{F})_u = 0$ if and only if $\mathcal{F}_{\overline{x}} = 0$. This proves the lemma. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

07TZ **Lemma 67.15.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then*

1. *The support of $\mathcal{F}$ is closed.*
2. *For a geometric point $\overline{x}$ lying over $x \in |X|$ we have*
$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_{\overline{x}} \neq 0 \Leftrightarrow \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \kappa(\overline{x}) \neq 0.$$
3. *For any morphism of algebraic spaces $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. By Lemma 67.15.1 the inverse image of the support of $\mathcal{F}$ is the support of $\varphi^*\mathcal{F}$ which is closed by Morphisms, Lemma 29.5.3. Thus (1) follows from the definition of the topology on $|X|$.

The first equivalence in (2) is the definition of support. The second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.20.1.

Let $f : Y \to X$ be as in (3). Note that $f^*\mathcal{F}$ is of finite type by Properties of Spaces, Section 66.30. For the final assertion, let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ on X. Recall that

$$(f^*\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}},$$

see Properties of Spaces, Lemma 66.29.5. Hence $(f^*\mathcal{F})_{\overline{y}} \otimes \kappa(\overline{y})$ is nonzero if and only if $\mathcal{F}_{\overline{x}} \otimes \kappa(\overline{x})$ is nonzero. By (2) this implies $x \in \mathrm{Supp}(\mathcal{F})$ if and only if $y \in \mathrm{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Our next task is to show that the scheme theoretic support of a finite type quasi-coherent module (see Morphisms, Definition 29.5.5) also makes sense for finite type quasi-coherent modules on algebraic spaces.

07U0 **Lemma 67.15.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. There exists a smallest closed subspace $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

(1) *If U is a scheme and $\varphi : U \to X$ is an étale morphism then $Z \times_X U$ is the scheme theoretic support of $\varphi^*\mathcal{F}$.*
(2) *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
(3) *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
(4) *The support of $\mathcal{G}$ and of $\mathcal{F}$ is $|Z|$.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Let $R = U \times_X U$ with projections $s, t : R \to U$. Let $i' : Z' \to U$ be the scheme theoretic support of $\varphi^*\mathcal{F}$ and let $\mathcal{G}'$ be the (unique up to unique isomorphism) finite type quasi-coherent $\mathcal{O}_{Z'}$-module with $i'_*\mathcal{G}' = \varphi^*\mathcal{F}$, see Morphisms, Lemma 29.5.4. As $s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F}$ we see that $R' = s^{-1}Z' = t^{-1}Z'$ as closed subschemes of R by Morphisms, Lemma 29.25.14. Thus we may apply Properties of Spaces, Lemma 66.12.2 to find a closed subspace $i : Z \to X$ whose pullback to U is Z'. Writing $s', t' : R' \to Z'$ the projections and $j' : R' \to R$ the given closed immersion, we see that

$$j'_*(s')^*\mathcal{G}' = s^*i'_*\mathcal{G}' = s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F} = t^*i'_*\mathcal{G}' = j'_*(t')^*\mathcal{G}'$$

(the first and the last equality by Cohomology of Schemes, Lemma 30.5.2). Hence the uniqueness of Morphisms, Lemma 29.25.14 applied to $R' \to R$ gives an isomorphism $\alpha : (t')^*\mathcal{G}' \to (s')^*\mathcal{G}'$ compatible with the canonical isomorphism $t^*\varphi^*\mathcal{F} = s^*\varphi^*\mathcal{F}$ via j'_*. Clearly α satisfies the cocycle condition, hence we may apply Properties of Spaces, Proposition 66.32.1 to obtain a quasi-coherent module $\mathcal{G}$ on Z whose restriction to Z' is $\mathcal{G}'$ compatible with α. Again using the equivalence of the proposition mentioned above (this time for X) we conclude that $i_*\mathcal{G} \cong \mathcal{F}$.

This proves existence. The other properties of the lemma follow by comparing with the result for schemes using Lemma 67.15.1. Detailed proofs omitted. □

07U1 **Definition 67.15.4.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. The *scheme theoretic support of $\mathcal{F}$* is the closed subspace $Z \subset X$ constructed in Lemma 67.15.3.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 67.14.1).

67.16. Scheme theoretic image

082W Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

082X **Lemma 67.16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There exists a closed subspace $Z \subset Y$ such that f factors through Z and such that for any other closed subspace $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 67.13.1. In general the lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 67.14.2. □

Suppose that in the situation of Lemma 67.16.1 above X and Y are representable. Then the closed subspace $Z \subset Y$ found in the lemma agrees with the closed subscheme $Z \subset Y$ found in Morphisms, Lemma 29.6.1. The reason is that closed

subspaces (or subschemes) are in an inclusion reversing correspondence with quasi-coherent ideal sheaves on $X_{étale}$ and X. As the category of quasi-coherent modules on $X_{étale}$ and X are the same (Properties of Spaces, Section 66.29) we conclude. Thus the following definition agrees with the earlier definition for morphisms of schemes.

082Y **Definition 67.16.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *scheme theoretic image* of f is the smallest closed subspace $Z \subset Y$ through which f factors, see Lemma 67.16.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subspaces of Y.

082Z **Lemma 67.16.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

(1) *the sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
(2) *the scheme theoretic image Z is the closed subspace corresponding to $\mathcal{I}$,*
(3) *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$, and*
(4) *the image $|f|(|X|) \subset |Z|$ is a dense subset of $|Z|$.*

Proof. To prove (3) it suffices to prove (1) and (2) since the formation of $\mathcal{I}$ commutes with étale localization. If (1) holds then in the proof of Lemma 67.16.1 we showed (2). Let us prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is étale local we may assume Y is an affine scheme. As f is quasi-compact, we can find an affine scheme U and a surjective étale morphism $U \to X$. Denote f' the composition $U \to X \to Y$. Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_U$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Lemma 67.11.2 the sheaf $f'_*\mathcal{O}_U$ is quasi-coherent on Y. Hence $\mathcal{I}$ is quasi-coherent as a kernel of a map between coherent modules. Finally, part (4) follows from parts (1), (2), and (3) as the ideal $\mathcal{I}$ will be the unit ideal in any point of $|Y|$ which is not contained in the closure of $|f|(|X|)$. □

0830 **Lemma 67.16.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X is reduced. Then*

(1) *the scheme theoretic image Z of f is the reduced induced algebraic space structure on $\overline{|f|(|X|)}$, and*
(2) *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$.*

Proof. Part (1) is true because the reduced induced algebraic space structure on $\overline{|f|(|X|)}$ is the smallest closed subspace of Y through which f factors, see Properties of Spaces, Lemma 66.12.4. Part (2) follows from (1), the fact that $|V| \to |Y|$ is open, and the fact that being reduced is preserved under étale localization. □

089B **Lemma 67.16.5.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. Let Z be the scheme theoretic image of f. Let $z \in |Z|$.*

There exists a valuation ring A with fraction field K and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ *maps to* z.

Proof. Choose an affine scheme V with a point $z' \in V$ and an étale morphism $V \to Y$ mapping z' to z. Let $Z' \subset V$ be the scheme theoretic image of $X \times_Y V \to V$. By Lemma 67.16.3 we have $Z' = Z \times_Y V$. Thus $z' \in Z'$. Since f is quasi-compact and V is affine we see that $X \times_Y V$ is quasi-compact. Hence there exists an affine scheme W and a surjective étale morphism $W \to X \times_Y V$. Then $Z' \subset V$ is also the scheme theoretic image of $W \to V$. By Morphisms, Lemma 29.6.5 we can choose a diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K) & \longrightarrow & W & \longrightarrow & X \times_Y V & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z'. Composing with $Z' \to Z$ and $W \to X \times_Y V \to X$ we obtain a solution. □

0CP2 **Lemma 67.16.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a commutative diagram of algebraic spaces over S. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z_2 & \longrightarrow & Y_2 \end{array}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subspace of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. □

0CP3 **Lemma 67.16.7.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace such that $V \to Y$ is quasi-compact. Let $s : V \to X$ be a morphism such that $f \circ s = id_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.*

Proof. By Lemma 67.8.9 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 67.16.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Lemma 67.4.7), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. □

67.17. Scheme theoretic closure and density

0831 This section is the analogue of Morphisms, Section 29.7.

0832 **Lemma 67.17.1.** *Let S be a scheme. Let $W \subset S$ be a scheme theoretically dense open subscheme (Morphisms, Definition 29.7.1). Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Then $f^{-1}(W)$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Morphisms, Lemma 29.7.5. Assume $V \subset X$ is an open and $g \in \Gamma(V, \mathcal{O}_V)$ is a function which restricts to zero on $f^{-1}(W) \cap V$. We have to show that $g = 0$. Assume $g \neq 0$ to get a contradiction. By More on Morphisms, Lemma 37.45.6 we may shrink V, find an open $U \subset S$ fitting into a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S, \end{array}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$. Say $(g_1, \ldots, g_r) \in \Gamma(U, \mathcal{F}^{\oplus r})$ maps to g. Then we see that $g_i|_{W \cap U} = 0$ because $g|_{f^{-1}W \cap V} = 0$. Hence $g_i = 0$ because $\mathcal{F} \subset \mathcal{O}_U$ and W is scheme theoretically dense in S. This implies $g = 0$ which is the desired contradiction. □

0833 **Lemma 67.17.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. The following are equivalent*

(1) *for every étale morphism $\varphi : V \to X$ (of algebraic spaces) the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*
(2) *there exists a scheme V and a surjective étale morphism $\varphi : V \to X$ such that the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*

Proof. Observe that if $V \to V'$ is a morphism of algebraic spaces étale over X, and $Z \subset V$, resp. $Z' \subset V'$ is the scheme theoretic closure of $U \times_X V$, resp. $U \times_X V'$ in V, resp. V', then Z maps into Z'. Thus if $V \to V'$ is surjective and étale then $Z = V$ implies $Z' = V'$. Next, note that an étale morphism is flat, locally of finite presentation, and locally quasi-finite (see Morphisms, Section 29.36). Thus Lemma 67.17.1 implies that if V and V' are schemes, then $Z' = V'$ implies $Z = V$. A formal argument using that every algebraic space has an étale covering by a scheme shows that (1) and (2) are equivalent. □

It follows from Lemma 67.17.2 that the following definition is compatible with the definition in the case of schemes.

0834 **Definition 67.17.3.** Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.
(2) We say U is *scheme theoretically dense in X* if the equivalent conditions of Lemma 67.17.2 are satisfied.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X. This is somewhat inelegant. But with suitable finiteness conditions we will see that it does hold.

0835 **Lemma 67.17.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. If $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 67.16.3 part (3). □

0836 **Lemma 67.17.5.** *Let S be a scheme. Let $j : U \to X$ be an open immersion of algebraic spaces over S. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any algebraic space V étale over X. Hence the scheme theoretic closure of $U \times_X V$ in V is equal to V, see proof of Lemma 67.16.1. Conversely, assume the scheme theoretic closure of $U \times_X V$ is equal to V for all V étale over X. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine, say $V = \mathrm{Spec}(A)$, étale over X and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \times_X U, \mathcal{O})$. In this case the scheme theoretic closure of $V \times_X U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

0837 **Lemma 67.17.6.** *Let S be a scheme. Let X be an algebraic space over S. If U, V are scheme theoretically dense open subspaces of X, then so is $U \cap V$.*

Proof. Let $W \to X$ be any étale morphism. Consider the map $\mathcal{O}(W) \to \mathcal{O}(W \times_X V) \to \mathcal{O}(W \times_X (V \cap U))$. By Lemma 67.17.5 both maps are injective. Hence the composite is injective. Hence by Lemma 67.17.5 $U \cap V$ is scheme theoretically dense in X. □

088G **Lemma 67.17.7.** *Let S be a scheme. Let $h : Z \to X$ be an immersion of algebraic spaces over S. Assume either $Z \to X$ is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subspace of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. In both cases the formation of $\overline{Z}$ commutes with étale localization, see Lemmas 67.16.3 and 67.16.4. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 29.7.7. □

084N **Lemma 67.17.8.** *Let S be a scheme. Let B be an algebraic space over S. Let $f, g : X \to Y$ be morphisms of algebraic spaces over B. Let $U \subset X$ be an open subspace such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to B$ is separated, then $f = g$.*

Proof. As $Y \to B$ is separated the fibre product $Y \times_{\Delta, Y \times_B Y, (f,g)} X$ is a closed subspace $Z \subset X$. As $f|_U = g|_U$ we see that $U \subset Z$. Hence $Z = X$ as U is assumed scheme theoretically dense in X. □

67.18. Dominant morphisms

0ABK We copy the definition of a dominant morphism of schemes to get the notion of a dominant morphism of algebraic spaces. We caution the reader that this definition

is not well behaved unless the morphism is quasi-compact and the algebraic spaces satisfy some separation axioms.

0ABL **Definition 67.18.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called *dominant* if the image of $|f| : |X| \to |Y|$ is dense in $|Y|$.

67.19. Universally injective morphisms

03MT We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be universally injective. For a field K over S (recall this means that we are given a structure morphism $\mathrm{Spec}(K) \to S$) and an algebraic space X over S we write $X(K) = \mathrm{Mor}_S(\mathrm{Spec}(K), X)$. We first translate the condition for representable morphisms into a condition on the functor of points.

03MU **Lemma 67.19.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is universally injective (in the sense of Section 67.3) if and only if for all fields K the map $X(K) \to Y(K)$ is injective.*

Proof. We are going to use Morphisms, Lemma 29.10.2 without further mention. Suppose that f is universally injective. Then for any field K and any morphism $\mathrm{Spec}(K) \to Y$ the morphism of schemes $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$ is universally injective. Hence there exists at most one section of the morphism $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$. Hence the map $X(K) \to Y(K)$ is injective. Conversely, suppose that for every field K the map $X(K) \to Y(K)$ is injective. Let $T \to Y$ be a morphism from a scheme into Y, and consider the base change $f_T : T \times_Y X \to T$. For any field K we have

$$(T \times_Y X)(K) = T(K) \times_{Y(K)} X(K)$$

by definition of the fibre product, and hence the injectivity of $X(K) \to Y(K)$ guarantees the injectivity of $(T \times_Y X)(K) \to T(K)$ which means that f_T is universally injective as desired. □

Next, we translate the property that the transformation between field valued points is injective into something more geometric.

040X **Lemma 67.19.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *the map $X(K) \to Y(K)$ is injective for every field K over S*
2. *for every morphism $Y' \to Y$ of algebraic spaces over S the induced map $|Y' \times_Y X| \to |Y'|$ is injective, and*
3. *the diagonal morphism $X \to X \times_Y X$ is surjective.*

Proof. Assume (1). Let $g : Y' \to Y$ be a morphism of algebraic spaces, and denote $f' : Y' \times_Y X \to Y'$ the base change of f. Let K_i, $i = 1, 2$ be fields and let $\varphi_i : \mathrm{Spec}(K_i) \to Y' \times_Y X$ be morphisms such that $f' \circ \varphi_1$ and $f' \circ \varphi_2$ define the same element of $|Y'|$. By definition this means there exists a field Ω and embeddings $\alpha_i : K_i \subset \Omega$ such that the two morphisms $f' \circ \varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y'$ are equal.

Here is the corresponding commutative diagram

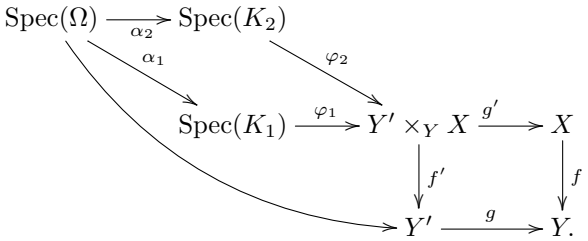

In particular the compositions $g \circ f' \circ \varphi_i \circ \alpha_i$ are equal. By assumption (1) this implies that the morphism $g' \circ \varphi_i \circ \alpha_i$ are equal, where $g' : Y' \times_Y X \to X$ is the projection. By the universal property of the fibre product we conclude that the morphisms $\varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y' \times_Y X$ are equal. In other words φ_1 and φ_2 define the same point of $Y' \times_Y X$. We conclude that (2) holds.

Assume (2). Let K be a field over S, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. Denote $c : \mathrm{Spec}(K) \to Y$ the common value. By assumption $|\mathrm{Spec}(K) \times_{c,Y} X| \to |\mathrm{Spec}(K)|$ is injective. This means there exists a field Ω and embeddings $\alpha_i : K \to \Omega$ such that

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow[\alpha_1]{} & \mathrm{Spec}(K) \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle a} \\ \mathrm{Spec}(K) & \xrightarrow{b} & \mathrm{Spec}(K) \times_{c,Y} X \end{array}$$

is commutative. Composing with the projection to $\mathrm{Spec}(K)$ we see that $\alpha_1 = \alpha_2$. Denote the common value α. Then we see that $\{\alpha : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)\}$ is a fpqc covering of $\mathrm{Spec}(K)$ such that the two morphisms a, b become equal on the members of the covering. By Properties of Spaces, Proposition 66.17.1 we conclude that $a = b$. We conclude that (1) holds.

Assume (3). Let $x, x' \in |X|$ be a pair of points such that $f(x) = f(x')$ in $|Y|$. By Properties of Spaces, Lemma 66.4.3 we see there exists a $x'' \in |X \times_Y X|$ whose projections are x and x'. By assumption and Properties of Spaces, Lemma 66.4.4 there exists a $x''' \in |X|$ with $\Delta_{X/Y}(x''') = x''$. Thus $x = x'$. In other words f is injective. Since condition (3) is stable under base change we see that f satisfies (2).

Assume (2). Then in particular $|X \times_Y X| \to |X|$ is injective which implies immediately that $|\Delta_{X/Y}| : |X| \to |X \times_Y X|$ is surjective, which implies that $\Delta_{X/Y}$ is surjective by Properties of Spaces, Lemma 66.4.4. □

By the two lemmas above the following definition does not conflict with the already defined notion of a universally injective representable morphism of algebraic spaces.

03MV **Definition 67.19.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *universally injective* if for every morphism $Y' \to Y$ the induced map $|Y' \times_Y X| \to |Y'|$ is injective.

To be sure this means that any or all of the equivalent conditions of Lemma 67.19.2 hold.

05VS **Remark 67.19.4.** A universally injective morphism of schemes is separated, see Morphisms, Lemma 29.10.3. This is not the case for morphisms of algebraic spaces. Namely, the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$ constructed in Spaces, Example 65.14.1 comes equipped with a morphism $X \to \mathbf{A}^1_k$ which maps the point with coordinate x to the point with coordinate x^2. This is an isomorphism away from 0, and there is a unique point of X lying above 0. As X isn't separated this is a universally injective morphism of algebraic spaces which is not separated.

03MW **Lemma 67.19.5.** *The base change of a universally injective morphism is universally injective.*

Proof. Omitted. Hint: This is formal. □

03MX **Lemma 67.19.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is universally injective,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
(4) *there exists a scheme Z and a surjective morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is universally injective, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally injective.*

Proof. We will use that being universally injective is preserved under base change (Lemma 67.19.5) without further mention in this proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume $g : Z \to Y$ as in (4). Let $y : \mathrm{Spec}(K) \to Y$ be a morphism from the spectrum of a field into Y. By assumption we can find an extension field $\alpha : K \subset K'$ and a morphism $z : \mathrm{Spec}(K') \to Z$ such that $y \circ \alpha = g \circ z$ (with obvious abuse of notation). By assumption the morphism $Z \times_Y X \to Z$ is universally injective, hence there is at most one lift of $g \circ z : \mathrm{Spec}(K') \to Y$ to a morphism into X. Since $\{\alpha : \mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is a fpqc covering this implies there is at most one lift of $y : \mathrm{Spec}(K) \to Y$ to a morphism into X, see Properties of Spaces, Proposition 66.17.1. Thus we see that (1) holds.

We omit the verification that (5) is equivalent to (1). □

03MY **Lemma 67.19.7.** *A composition of universally injective morphisms is universally injective.*

Proof. Omitted. □

67.20. Affine morphisms

03WD We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be affine.

03WE **Lemma 67.20.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is affine (in the sense of Section 67.3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine.*

Proof. This follows directly from the definition of an affine morphism of schemes (Morphisms, Definition 29.11.1). □

This clears the way for the following definition.

03WF **Definition 67.20.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme.

03WG **Lemma 67.20.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and affine,*
(2) *f is affine,*
(3) *for every affine scheme V and étale morphism $V \to Y$ the scheme $X \times_Y V$ is affine,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is affine, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is affine.*

Proof. It is clear that (1) implies (2), that (2) implies (3), and that (3) implies (4) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (4). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of affine morphisms satisfies all the required properties (see Morphisms, Lemmas 29.11.9 and Descent, Lemmas 35.23.20 and 35.37.1). The conclusion of applying this lemma is that f is representable and affine, i.e., (1) holds.

The equivalence of (1) and (5) follows from the fact that being affine is Zariski local on the target (the reference above shows that being affine is in fact fpqc local on the target). □

03WH **Lemma 67.20.4.** *The composition of affine morphisms is affine.*

Proof. Omitted. Hint: Transitivity of fibre products. □

03WI **Lemma 67.20.5.** *The base change of an affine morphism is affine.*

Proof. Omitted. Hint: Transitivity of fibre products. □

07U2 **Lemma 67.20.6.** *A closed immersion is affine.*

Proof. Follows immediately from the corresponding statement for morphisms of schemes, see Morphisms, Lemma 29.11.10. □

081V **Lemma 67.20.7.** *Let S be a scheme. Let X be an algebraic space over S. There is an anti-equivalence of categories*

$$\begin{matrix} \text{algebraic spaces} \\ \text{affine over } X \end{matrix} \longleftrightarrow \begin{matrix} \text{quasi-coherent sheaves} \\ \text{of } \mathcal{O}_X\text{-algebras} \end{matrix}$$

which associates to $f : Y \to X$ the sheaf $f_*\mathcal{O}_Y$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. This lemma is the analogue of Morphisms, Lemma 29.11.5. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We will construct an affine morphism of algebraic spaces $\pi : Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ with $\pi_*\mathcal{O}_Y \cong \mathcal{A}$. To do this, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Denote $\psi : R \to X$ the composition $\psi = \varphi \circ s = \varphi \circ t$. By the aforementioned lemma there exists an affine morphisms of schemes $\pi_0 : V \to U$ and $\pi_1 : W \to R$ with $\pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$ and $\pi_{1,*}\mathcal{O}_W \cong \psi^*\mathcal{A}$. Since the construction is compatible with base change there exist morphisms $s', t' : W \to V$ such that the diagrams

$$\begin{array}{ccc} W & \xrightarrow[s']{} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{s} & U \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow[t']{} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{t} & U \end{array}$$

are cartesian. It follows that s', t' are étale. It is a formal consequence of the above that $(t', s') : W \to V \times_S V$ is a monomorphism. We omit the verification that $W \to V \times_S V$ is an equivalence relation (hint: think about the pullback of $\mathcal{A}$ to $U \times_X U \times_X U = R \times_{s,U,t} R$). The quotient sheaf $Y = V/W$ is an algebraic space, see Spaces, Theorem 65.10.5. By Groupoids, Lemma 39.20.7 we see that $Y \times_X U \cong V$. Hence $Y \to X$ is affine by Lemma 67.20.3. Finally, the isomorphism of

$$(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U} = \pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$$

is compatible with glueing isomorphisms, whence $(Y \to X)_*\mathcal{O}_Y \cong \mathcal{A}$ by Properties of Spaces, Proposition 66.32.1. We omit the verification that this construction is compatible with base change. □

081W **Definition 67.20.8.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *relative spectrum of $\mathcal{A}$ over X*, or simply the *spectrum of $\mathcal{A}$ over X* is the affine morphism $\underline{\mathrm{Spec}}(\mathcal{A}) \to X$ corresponding to $\mathcal{A}$ under the equivalence of categories of Lemma 67.20.7.

Forming the relative spectrum commutes with arbitrary base change.

081X **Remark 67.20.9.** Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Then f has a canonical factorization

$$Y \longrightarrow \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y) \longrightarrow X$$

This makes sense because $f_*\mathcal{O}_Y$ is quasi-coherent by Lemma 67.11.2. The morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ comes from the canonical $\mathcal{O}_Y$-algebra map $f^*f_*\mathcal{O}_Y \to \mathcal{O}_Y$ which corresponds to a canonical morphism $Y \to Y \times_X \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ over Y (see Lemma 67.20.7) whence a factorization of f as above.

08AI **Lemma 67.20.10.** *Let S be a scheme. Let $f : Y \to X$ be an affine morphism of algebraic spaces over S. Let $\mathcal{A} = f_*\mathcal{O}_Y$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{array}{c}\text{category of quasi-coherent} \\ \mathcal{O}_Y\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of quasi-coherent} \\ \mathcal{A}\text{-modules}\end{array}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_X$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. □

08GB **Lemma 67.20.11.** *Let S be a scheme. Let B be an algebraic space over S. Suppose $g : X \to Y$ is a morphism of algebraic spaces over B.*

(1) *If X is affine over B and $\Delta : Y \to Y \times_B Y$ is affine, then g is affine.*
(2) *If X is affine over B and Y is separated over B, then g is affine.*
(3) *A morphism from an affine scheme to an algebraic space with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1) is affine.*
(4) *A morphism from an affine scheme to a separated algebraic space is affine.*

Proof. Proof of (1). The base change $X \times_B Y \to Y$ is affine by Lemma 67.20.5. The morphism $(1, g) : X \to X \times_B Y$ is the base change of $Y \to Y \times_B Y$ by the morphism $X \times_B Y \to Y \times_B Y$. Hence it is affine by Lemma 67.20.5. The composition of affine morphisms is affine (see Lemma 67.20.4) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 67.20.6) and Y/B separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

09TF **Lemma 67.20.12.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to X$ is affine.*

Proof. Let $U \to X$ be an étale morphism with U affine. To prove the lemma we have to show that $\mathrm{Spec}(A) \times_X U$ is affine, see Lemma 67.20.3. Since X is quasi-separated the scheme $\mathrm{Spec}(A) \times_X U$ is quasi-compact. Moreover, the projection morphism $\mathrm{Spec}(A) \times_X U \to \mathrm{Spec}(A)$ is étale. Hence this morphism has finite discrete fibers and moreover the topology on $\mathrm{Spec}(A)$ is discrete. Thus $\mathrm{Spec}(A) \times_X U$ is a scheme whose underlying topological space is a finite discrete set. We are done by Schemes, Lemma 26.11.8. □

67.21. Quasi-affine morphisms

03WJ We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be quasi-affine.

03WK **Lemma 67.21.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is quasi-affine (in the sense of Section 67.3) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is quasi-affine.*

Proof. This follows directly from the definition of a quasi-affine morphism of schemes (Morphisms, Definition 29.13.1). □

This clears the way for the following definition.

03WL **Definition 67.21.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by a quasi-affine scheme.

03WM **Lemma 67.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and quasi-affine,*
(2) *f is quasi-affine,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is quasi-affine, and*

(4) *there exists a Zariski covering* $Y = \bigcup Y_i$ *such that each of the morphisms* $f^{-1}(Y_i) \to Y_i$ *is quasi-affine.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is a quasi-affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is quasi-affine. This means we can apply Spaces, Lemma 65.11.5 because the class of quasi-affine morphisms satisfies all the required properties (see Morphisms, Lemmas 29.13.5 and Descent, Lemmas 35.23.22 and 35.38.1). The conclusion of applying this lemma is that f is representable and quasi-affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being quasi-affine is Zariski local on the target (the reference above shows that being quasi-affine is in fact fpqc local on the target). □

03WN **Lemma 67.21.4.** *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Omitted. □

03WO **Lemma 67.21.5.** *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Omitted. □

086S **Lemma 67.21.6.** *Let S be a scheme. A quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ is quasi-affine if and only if the canonical factorization $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (Remark 67.20.9) is an open immersion.*

Proof. Let $U \to X$ be a surjective morphism where U is a scheme. Since we may check whether f is quasi-affine after base change to U (Lemma 67.21.3), since $f_*\mathcal{O}_Y|_V$ is equal to $(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U}$ (Properties of Spaces, Lemma 66.26.2), and since formation of relative spectrum commutes with base change (Lemma 67.20.7), we see that the assertion reduces to the case that X is a scheme. If X is a scheme and either f is quasi-affine or $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ is an open immersion, then Y is a scheme as well. Thus we reduce to Morphisms, Lemma 29.13.3. □

67.22. Types of morphisms étale local on source-and-target

03MI Given a property of morphisms of schemes which is *étale local on the source-and-target*, see Descent, Definition 35.32.3 we may use it to define a corresponding property of morphisms of algebraic spaces, namely by imposing either of the equivalent conditions of the lemma below.

03MJ **Lemma 67.22.1.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale. The following are equivalent

(1) *for any diagram as above the morphism h has property $\mathcal{P}$, and*
(2) *for some diagram as above with $a : U \to X$ surjective the morphism h has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for representable morphisms f this is also equivalent to f having property $\mathcal{P}$ in the sense of Section 67.3.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate (taking into account Spaces, Lemma 65.11.6). Assume

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are two diagrams as in the lemma. Assume $U \to X$ is surjective and h has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_X U' & \longrightarrow & U' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (h,h')} & & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_Y V' & \longrightarrow & V' \end{array}$$

By Descent, Lemma 35.32.5 we see that h has $\mathcal{P}$ implies (h, h') has $\mathcal{P}$ and since $U \times_X U' \to U'$ is surjective this implies (by the same lemma) that h' has $\mathcal{P}$.

If X and Y are representable, then Descent, Lemma 35.32.5 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to X$ the base change $Z \times_Y X \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_Y U & \longrightarrow & Z \times_Y V \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is étale and surjective and the right vertical arrow is étale. Thus Descent, Lemma 35.32.5 once again kicks in and shows that $Z \times_Y X \to Z$ has property $\mathcal{P}$. □

04RD **Definition 67.22.2.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. We say a morphism $f : X \to Y$ of algebraic spaces over S *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 67.22.1 hold.

Here are a couple of obvious remarks.

0AML **Remark 67.22.3.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under compositions. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under composition.

0AMM **Remark 67.22.4.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Suppose that moreover $\mathcal{P}$ is stable under base change. Then the class of morphisms of algebraic spaces having property $\mathcal{P}$ is stable under base change.

Given a property of morphisms of germs of schemes which is *étale local on the source-and-target*, see Descent, Definition 35.33.1 we may use it to define a corresponding property of morphisms of algebraic spaces at a point, namely by imposing either of the equivalent conditions of the lemma below.

04NC **Lemma 67.22.5.** *Let $\mathcal{Q}$ be a property of morphisms of germs which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point of X. Consider the diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

where U and V are schemes, a, b are étale, and u, v, x, y are points of the corresponding spaces. The following are equivalent

(1) *for any diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$, and*
(2) *for some diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$.*

If X and Y are representable, then this is also equivalent to $\mathcal{Q}((X, x) \to (Y, y))$.

Proof. Omitted. Hint: Very similar to the proof of Lemma 67.22.1. □

04RE **Definition 67.22.6.** Let $\mathcal{Q}$ be a property of morphisms of germs of schemes which is étale local on the source-and-target. Let S be a scheme. Given a morphism $f : X \to Y$ of algebraic spaces over S and a point $x \in |X|$ we say that f *has property $\mathcal{Q}$ at x* if the equivalent conditions of Lemma 67.22.5 hold.

The following lemma should not be used blindly to go from a property of morphisms to a property of morphisms at a point. For example if $\mathcal{P}$ is the property of being flat, then the property Q in the following lemma means "f is flat in an open neighbourhood of x" which is not the same as "f is flat at x".

04RF **Lemma 67.22.7.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs associated to $\mathcal{P}$ in Descent, Lemma 35.33.2. Then*

(1) *$\mathcal{Q}$ is étale local on the source-and-target.*
(2) *given a morphism of algebraic spaces $f : X \to Y$ and $x \in |X|$ the following are equivalent*
   (a) *f has $\mathcal{Q}$ at x, and*
   (b) *there is an open neighbourhood $X' \subset X$ of x such that $X' \to Y$ has $\mathcal{P}$.*
(3) *given a morphism of algebraic spaces $f : X \to Y$ the following are equivalent:*

(a) *f has $\mathcal{P}$,*
(b) *for every $x \in |X|$ the morphism f has $\mathcal{Q}$ at x.*

Proof. See Descent, Lemma 35.33.2 for (1). The implication (1)(a) $\Rightarrow$ (2)(b) follows on letting $X' = a(U) \subset X$ given a diagram as in Lemma 67.22.5. The implication (2)(b) $\Rightarrow$ (1)(a) is clear. The equivalence of (3)(a) and (3)(b) follows from the corresponding result for morphisms of schemes, see Descent, Lemma 35.33.3. □

04RG **Remark 67.22.8.** We will apply Lemma 67.22.7 above to all cases listed in Descent, Remark 35.32.7 except "flat". In each case we will do this by defining f to have property $\mathcal{P}$ at x if f has $\mathcal{P}$ in a neighbourhood of x.

67.23. Morphisms of finite type

03XE The property "locally of finite type" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.15.4, and Descent, Lemmas 35.23.12. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XF **Definition 67.23.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ locally of finite type.
(2) Let $x \in |X|$. We say f is of *finite type at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite type.
(3) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Consider the algebraic space $\mathbf{A}^1_k/\mathbf{Z}$ of Spaces, Example 65.14.8. The morphism $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is of finite type.

03XG **Lemma 67.23.2.** *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.15.3. □

03XH **Lemma 67.23.3.** *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.15.4. □

040Y **Lemma 67.23.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite type,*
(2) *for every $x \in |X|$ the morphism f is of finite type at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite type,*

(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite type,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite type,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, $U \to X$ is surjective, and the top horizontal arrow is locally of finite type, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite type.*

Proof. Each of the conditions (2), (3), (4), (5), (6), (7), and (9) imply condition (8) in a straightforward manner. For example, if (5) holds, then we can choose a scheme V which is a disjoint union of affines and a surjective morphism $V \to Y$ (see Properties of Spaces, Lemma 66.6.1). Then $V \times_Y X \to V$ is locally of finite type by (5). Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is locally of finite type by Lemma 67.23.2. Hence (8) is true.

The conditions (1), (7), and (8) are equivalent by definition.

To finish the proof, we show that (1) implies all of the conditions (2), (3), (4), (5), (6), and (9). For (2) this is immediate. For (3), (4), (5), and (9) this follows from the fact that being locally of finite type is preserved under base change, see Lemma 67.23.3. For (6) we can take $U = X$ and we're done. □

04ZK **Lemma 67.23.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and Y is locally Noetherian, then X is locally Noetherian.*

Proof. Let

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

be a commutative diagram where U, V are schemes and the vertical arrows are surjective étale. If f is locally of finite type, then $U \to V$ is locally of finite type. If Y is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 29.15.6 we see that U is locally Noetherian, which means that X is locally Noetherian. □

0462 **Lemma 67.23.6.** *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where U, V, W are schemes, the vertical arrows are étale and surjective, see Spaces, Lemma 65.11.6. At this point we can use Lemma 67.23.4 and Morphisms, Lemma 29.15.8 to conclude. □

06ED **Lemma 67.23.7.** *An immersion is locally of finite type.*

Proof. Follows from the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemmas 29.15.5. □

67.24. Points and geometric points

0485 In this section we make some remarks on points and geometric points (see Properties of Spaces, Definition 66.19.1). One way to think about a geometric point of X is to consider a geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ of S and a lift of $\overline{s}$ to a morphism $\overline{x}$ into X. Here is a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \\ & \searrow^{\overline{s}} & \downarrow \\ & & S. \end{array}$$

We often say "let k be an algebraically closed field over S" to indicate that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to S$. In this situation we write

$$X(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), X) = \{\overline{x} \in X \text{ lying over } \overline{s}\}$$

for the set of k-valued points of X. In this case the map $X(k) \to |X|$ maps into the subset $|X_s| \subset |X|$. Here $X_s = \mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. As $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism, also the base change $X_s \to X$ is a monomorphism, and $|X_s|$ is indeed a subset of $|X|$.

0487 **Lemma 67.24.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. The following are equivalent:*

1. *f is surjective, and*
2. *for every algebraically closed field k over S the induced map $X(k) \to Y(k)$ is surjective.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 65.11.6. Since f is locally of finite type we see that $U \to V$ is locally of finite type.

Assume (1) and let $\overline{y} \in Y(k)$. Then $U \to Y$ is surjective and locally of finite type by Lemmas 67.5.4 and 67.23.2. Let $Z = U \times_{Y,\overline{y}} \mathrm{Spec}(k)$. This is a scheme. The projection $Z \to \mathrm{Spec}(k)$ is surjective and locally of finite type by Lemmas 67.5.5 and 67.23.3. It follows from Varieties, Lemma 33.14.1 that Z has a k valued point $\overline{z}$. The image $\overline{x} \in X(k)$ of $\overline{z}$ maps to $\overline{y}$ as desired.

Assume (2). By Properties of Spaces, Lemma 66.4.4 it suffices to show that $|X| \to |Y|$ is surjective. Let $y \in |Y|$. Choose a $u \in U$ mapping to y. Let $k \supset \kappa(u)$ be an algebraic closure. Denote $\overline{u} \in U(k)$ the corresponding point and $\overline{y} \in Y(k)$ its image. By assumption there exists a $\overline{x} \in X(k)$ mapping to $\overline{y}$. Then it is clear that the image $x \in |X|$ of $\overline{x}$ maps to y. □

In order to state the next lemma we introduce the following notation. Given a scheme T we denote

$$\lambda(T) = \sup\{\aleph_0, |\kappa(t)|; t \in T\}.$$

In words $\lambda(T)$ is the smallest infinite cardinal bounding all the cardinalities of residue fields of T. Note that if R is a ring then the cardinality of any residue field $\kappa(\mathfrak{p})$ of R is bounded by the cardinality of R (details omitted). This implies that $\lambda(T) \leq \mathrm{size}(T)$ where $\mathrm{size}(T)$ is the size of the scheme T as introduced in Sets, Section 3.9. If L/K is a finitely generated field extension then $|K| \leq |L| \leq \max\{\aleph_0, |K|\}$. It follows that if $T' \to T$ is a morphism of schemes which is locally of finite type then $\lambda(T') \leq \lambda(T)$, and if $T' \to T$ is also surjective then equality holds. Next, suppose that S is a scheme and that X is an algebraic space over S. In this case we define

$$\lambda(X) := \lambda(U)$$

where U is any scheme over S which has a surjective étale morphism towards X. The reason that this is independent of the choice of U is that given a pair of such schemes U and U' the fibre product $U \times_X U'$ is a scheme which admits a surjective étale morphism to both U and U', whence $\lambda(U) = \lambda(U \times_X U') = \lambda(U')$ by the discussion above.

0488 **Lemma 67.24.2.** *Let S be a scheme. Let X, Y be algebraic spaces over S.*

(1) *As k ranges over all algebraically closed fields over S the collection of geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(2) *As k ranges over all algebraically closed fields over S with $|k| \geq \lambda(Y)$ and $|k| > \lambda(X)$ the geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(3) *For any geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X_s|$$
is surjective.
(4) *Let $X \to Y$ be a morphism of algebraic spaces over S. For any geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X| \times_{|Y|} Y(k)$$
is surjective.
(5) *Let $X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*
   (a) *the map $X \to Y$ is surjective,*
   (b) *for all algebraically closed fields k over S with $|k| > \lambda(X)$, and $|k| \geq \lambda(Y)$ the map $X(k) \to Y(k)$ is surjective.*

Proof. To prove part (1) choose a surjective étale morphism $V \to Y$ where V is a scheme. For each $v \in V$ choose an algebraic closure $\kappa(v) \subset k_v$. Consider the morphisms $\overline{x} : \mathrm{Spec}(k_v) \to V \to Y$. By construction of $|Y|$ these cover $|Y|$.

To prove part (2) we will use the following two facts whose proofs we omit: (i) If K is a field and $\overline{K}$ is algebraic closure then $|\overline{K}| \leq \max\{\aleph_0, |K|\}$. (ii) For any algebraically closed field k and any cardinal $\aleph$, $\aleph \geq |k|$ there exists an extension of algebraically closed fields k'/k with $|k'| = \aleph$. Now we set $\aleph = \max\{\lambda(X), \lambda(Y)\}^+$. Here $\lambda^+ > \lambda$ indicates the next bigger cardinal, see Sets, Section 3.6. Now (i) implies that the fields k_u constructed in the first paragraph of the proof all have cardinality bounded by $\lambda(X)$. Hence by (ii) we can find extensions $k_u \subset k'_u$ such that $|k'_u| = \aleph$. The morphisms $\overline{x}' : \mathrm{Spec}(k'_u) \to X$ cover $|X|$ as desired. To really finish the proof of (2) we need to show that the schemes $\mathrm{Spec}(k'_u)$ are (isomorphic to) objects of Sch_{fppf} because our conventions are that all schemes are objects of Sch_{fppf}; the rest of this paragraph should be skipped by anyone who is not interested in set theoretical considerations. By construction there exists an object T of Sch_{fppf} such that $\lambda(X)$ and $\lambda(Y)$ are bounded by $\mathrm{size}(T)$. By our construction of the category Sch_{fppf} in Topologies, Definitions 34.7.6 as the category Sch_α constructed in Sets, Lemma 3.9.2 we see that any scheme whose size is $\leq \mathrm{size}(T)^+$ is isomorphic to an object of Sch_{fppf}. See the expression for the function *Bound* in Sets, Equation (3.9.1.1). Since $\aleph \leq \mathrm{size}(T)^+$ we conclude.

The notation X_s in part (3) means the fibre product $\mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. Hence part (2) follows from (4) with $Y = \mathrm{Spec}(\kappa(s))$.

Let us prove (4). Let $X \to Y$ be a morphism of algebraic spaces over S. Let k be an algebraically closed field over S of cardinality $> \lambda(X)$. Let $\overline{y} \in Y(k)$ and $x \in |X|$ which map to the same element y of $|Y|$. We have to find $\overline{x} \in X(k)$ mapping to x and $\overline{y}$. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 65.11.6. Choose a $u \in |U|$ which maps to x, and denote $v \in |V|$ the image. We will think of $u = \mathrm{Spec}(\kappa(u))$ and $v = \mathrm{Spec}(\kappa(v))$ as schemes. Note that $V \times_Y \mathrm{Spec}(k)$ is a scheme étale over k. Hence it is a disjoint union of spectra of finite separable extensions of k, see Morphisms, Lemma 29.36.7. As v maps to y we see that $v \times_Y \mathrm{Spec}(k)$ is a nonempty scheme. As $v \to V$ is a monomorphism, we see that $v \times_Y \mathrm{Spec}(k) \to V \times_Y \mathrm{Spec}(k)$ is a monomorphism. Hence $v \times_Y \mathrm{Spec}(k)$ is a disjoint union of spectra of finite separable extensions of k, by Schemes, Lemma 26.23.11. We conclude that the morphism $v \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ has a section, i.e., we can find a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ lying over v and over $\overline{y}$. Finally we consider the scheme

$$u \times_{V,\overline{v}} \mathrm{Spec}(k) = \mathrm{Spec}(\kappa(u) \otimes_{\kappa(v)} k)$$

where $\kappa(v) \to k$ is the field map defining the morphism $\overline{v}$. Since the cardinality of k is larger than the cardinality of $\kappa(u)$ by assumption we may apply Algebra, Lemma 10.35.12 to see that any maximal ideal $\mathfrak{m} \subset \kappa(u) \otimes_{\kappa(v)} k$ has a residue field which is algebraic over k and hence equal to k. Such a maximal ideal will hence produce

a morphism $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u and mapping to $\overline{v}$. The composition $\mathrm{Spec}(k) \to U \to X$ will be the desired geometric point $\overline{x} \in X(k)$. This concludes the proof of part (4).

Part (5) is a formal consequence of parts (2) and (4) and Properties of Spaces, Lemma 66.4.4. □

67.25. Points of finite type

06EE Let S be a scheme. Let X be an algebraic space over S. A finite type point $x \in |X|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

06EF **Lemma 67.25.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *There exists a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type and represents x.*
(2) *There exists a scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to X$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to X$ is representable and locally of finite type (by the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemmas 29.36.11 and 29.21.8). Hence we see (1) holds by Lemma 67.23.2.

Conversely, assume $\mathrm{Spec}(k) \to X$ is locally of finite type and represents x. Let $U \to X$ be a surjective étale morphism where U is a scheme. By assumption $U \times_X \mathrm{Spec}(k) \to U$ is locally of finite type. Pick a finite type point v of $U \times_X \mathrm{Spec}(k)$ (there exists at least one, see Morphisms, Lemma 29.16.4). By Morphisms, Lemma 29.16.5 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 29.16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

06EG **Definition 67.25.2.** Let S be a scheme. Let X be an algebraic space over S. We say a point $x \in |X|$ is a *finite type point*[5] if the equivalent conditions of Lemma 67.25.1 are satisfied. We denote $X_{\text{ft-pts}}$ the set of finite type points of X.

We can describe the set of finite type points as follows.

06EH **Lemma 67.25.3.** *Let S be a scheme. Let X be an algebraic space over S. We have*

$$X_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to X \text{ étale}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes étale over X or over all affine schemes étale over X.

Proof. Immediate from Lemma 67.25.1. □

06EI **Lemma 67.25.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $f(X_{ft\text{-}pts}) \subset Y_{ft\text{-}pts}$.*

[5] This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Proof. Take $x \in X_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \text{Spec}(k) \to X$. Then $f \circ x$ is locally of finite type by Lemma 67.23.2. Hence $f(x) \in Y_{\text{ft-pts}}$. □

06EJ **Lemma 67.25.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and surjective, then $f(X_{ft\text{-}pts}) = Y_{ft\text{-}pts}$.*

Proof. We have $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$ by Lemma 67.25.4. Let $y \in |Y|$ be a finite type point. Represent y by a morphism $\text{Spec}(k) \to Y$ which is locally of finite type. As f is surjective the algebraic space $X_k = \text{Spec}(k) \times_Y X$ is nonempty, therefore has a finite type point $x \in |X_k|$ by Lemma 67.25.3. Now $X_k \to X$ is a morphism which is locally of finite type as a base change of $\text{Spec}(k) \to Y$ (Lemma 67.23.3). Hence the image of x in X is a finite type point by Lemma 67.25.4 which maps to y by construction. □

06EK **Lemma 67.25.6.** *Let S be a scheme. Let X be an algebraic space over S. For any locally closed subset $T \subset |X|$ we have*

$$T \neq \emptyset \Rightarrow T \cap X_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |X|$ we see that $T \cap X_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : Z \to X$ be the reduced induce subspace structure on T, see Remark 67.12.5. Any immersion is locally of finite type, see Lemma 67.23.7. Hence by Lemma 67.25.4 we see $Z_{\text{ft-pts}} \subset X_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with an étale morphism towards Z has at least one closed point. Hence Z has at least one finite type point by Lemma 67.25.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic space.

06EL **Lemma 67.25.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

1. *x is a finite type point,*
2. *there exists an algebraic space Z whose underlying topological space $|Z|$ is a singleton, and a morphism $f : Z \to X$ which is locally of finite type such that $\{x\} = |f|(|Z|)$, and*
3. *there exists an algebraic space Z and a morphism $f : Z \to X$ with the following properties:*
   (a) *there is a surjective étale morphism $z : \text{Spec}(k) \to Z$ where k is a field,*
   (b) *f is locally of finite type,*
   (c) *f is a monomorphism, and*
   (d) *$x = f(z)$.*

Proof. Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ with $\varphi(u) = x$, see Lemma 67.25.3. Set $u = \text{Spec}(\kappa(u))$ as usual. The projection morphisms $u \times_X u \to u$ are the compositions

$$u \times_X u \to u \times_X U \to u \times_X X = u$$

where the first arrow is a closed immersion (a base change of $u \to U$) and the second arrow is étale (a base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of finite separable extensions of k (see Morphisms, Lemma

29.36.7) and therefore the closed subscheme $u \times_X u$ is a disjoint union of finite separable extension of k, i.e., $u \times_X u \to u$ is étale. By Spaces, Theorem 65.10.5 we see that $Z = u/u \times_X u$ is an algebraic space. By construction the diagram

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is commutative with étale vertical arrows. Hence $Z \to X$ is locally of finite type (see Lemma 67.23.4). By construction the morphism $Z \to X$ is a monomorphism and the image of z is x. Thus (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of X by Lemma 67.25.4 (and Lemma 67.25.6 to see that $Z_{\text{ft-pts}}$ is nonempty, i.e., the unique point of Z is a finite type point of Z). □

67.26. Nagata spaces

0BAT See Properties of Spaces, Section 66.7 for the definition of a Nagata algebraic space.

0BAU **Lemma 67.26.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is Nagata and f locally of finite type then X is Nagata.*

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let U be a scheme and let $U \to X \times_Y V$ be a surjective étale morphism. If Y is Nagata, then V is a Nagata scheme. If $X \to Y$ is locally of finite type, then $U \to V$ is locally of finite type. Hence V is a Nagata scheme by Morphisms, Lemma 29.18.1. Then X is Nagata by definition. □

0BAV **Lemma 67.26.2.** *The following types of algebraic spaces are Nagata.*

(1) *Any algebraic space locally of finite type over a Nagata scheme.*
(2) *Any algebraic space locally of finite type over a field.*
(3) *Any algebraic space locally of finite type over a Noetherian complete local ring.*
(4) *Any algebraic space locally of finite type over $\mathbf{Z}$.*
(5) *Any algebraic space locally of finite type over a Dedekind ring of characteristic zero.*
(6) *And so on.*

Proof. The first property holds by Lemma 67.26.1. Thus the others hold as well, see Morphisms, Lemma 29.18.2. □

67.27. Quasi-finite morphisms

03XI The property "locally quasi-finite" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.20.13, and Descent, Lemma 35.23.26. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally quasi-finite as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XJ **Definition 67.27.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally quasi-finite* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ locally quasi-finite.
(2) Let $x \in |X|$. We say f is *quasi-finite at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally quasi-finite.
(3) A morphism of algebraic spaces $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and quasi-compact.

The last part is compatible with the notion of quasi-finiteness for morphisms of schemes by Morphisms, Lemma 29.20.9.

0ABM **Lemma 67.27.2.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of algebraic spaces over S. Denote $f' : X' \to Y'$ the base change of f by g. Denote $g' : X' \to X$ the projection. Assume f is locally of finite type. Let $W \subset |X|$, resp. $W' \subset |X'|$ be the set of points where f, resp. f' is quasi-finite.*

(1) *$W \subset |X|$ and $W' \subset |X'|$ are open,*
(2) *$W' = (g')^{-1}(W)$, i.e., formation of the locus where f is quasi-finite commutes with base change,*
(3) *the base change of a locally quasi-finite morphism is locally quasi-finite, and*
(4) *the base change of a quasi-finite morphism is quasi-finite.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to Y' \times_Y V$. Set $U' = V' \times_V U$ so that $U' \to X'$ is a surjective étale morphism as well. Picture

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array} \quad \text{lying over} \quad \begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

Choose $u \in |U|$ with image $x \in |X|$. The property of being "locally quasi-finite" is étale local on the source-and-target, see Descent, Remark 35.32.7. Hence Lemmas 67.22.5 and 67.22.7 apply and we see that $f : X \to Y$ is quasi-finite at x if and only if $U \to V$ is quasi-finite at u. Similarly for $f' : X' \to Y'$ and the morphism $U' \to V'$. Hence parts (1), (2), and (3) reduce to Morphisms, Lemmas 29.20.13 and 29.56.2. Part (4) follows from (3) and Lemma 67.8.4. □

03XK **Lemma 67.27.3.** *The composition of quasi-finite morphisms is quasi-finite. The same holds for locally quasi-finite.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.20.12. □

03XL **Lemma 67.27.4.** *A base change of a quasi-finite morphism is quasi-finite. The same holds for locally quasi-finite.*

Proof. Immediate consequence of Lemma 67.27.2. □

The following lemma characterizes locally quasi-finite morphisms as those morphisms which are locally of finite type and have "discrete fibres". However, this is not the same thing as asking $|X| \to |Y|$ to have discrete fibres as the discussion in Examples, Section 110.51 shows.

06RW **Lemma 67.27.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Assume f is locally of finite type. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field the space $|X_k|$ is discrete. Here $X_k = \mathrm{Spec}(k) \times_Y X$.*

Proof. Assume f is locally quasi-finite. Let $\mathrm{Spec}(k) \to Y$ be as in (2). Choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_k = \mathrm{Spec}(k) \times_Y U \to X_k$ is an étale morphism of algebraic spaces by Properties of Spaces, Lemma 66.16.5. By Lemma 67.27.4 we see that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. By definition this means that $U_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Hence $|U_k|$ is discrete by Morphisms, Lemma 29.20.8. Since $|U_k| \to |X_k|$ is surjective and open we conclude that $|X_k|$ is discrete.

Conversely, assume (2). Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Note that $U \to V$ is locally of finite type as f is locally of finite type. Picture

$$\begin{array}{ccccc} U & \longrightarrow & X \times_Y V & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

If f is not locally quasi-finite then $U \to V$ is not locally quasi-finite. Hence there exists a specialization $u \rightsquigarrow u'$ for some $u, u' \in U$ lying over the same point $v \in V$, see Morphisms, Lemma 29.20.6. We claim that u, u' do not have the same image in $X_v = \mathrm{Spec}(\kappa(v)) \times_Y X$ which will contradict the assumption that $|X_v|$ is discrete as desired. Let $d = \mathrm{trdeg}_{\kappa(v)}(\kappa(u))$ and $d' = \mathrm{trdeg}_{\kappa(v)}(\kappa(u'))$. Then we see that $d > d'$ by Morphisms, Lemma 29.28.7. Note that U_v (the fibre of $U \to V$ over v) is the fibre product of U and X_v over $X \times_Y V$, hence $U_v \to X_v$ is étale (as a base change of the étale morphism $U \to X \times_Y V$). If $u, u' \in U_v$ map to the same element of $|X_v|$ then there exists a point $r \in R_v = U_v \times_{X_v} U_v$ with $t(r) = u$ and $s(r) = u'$, see Properties of Spaces, Lemma 66.4.3. Note that $s, t : R_v \to U_v$ are étale morphisms of schemes over $\kappa(v)$, hence $\kappa(u) \subset \kappa(r) \supset \kappa(u')$ are finite separable extensions of fields over $\kappa(v)$ (see Morphisms, Lemma 29.36.7). We conclude that the transcendence degrees are equal. This contradiction finishes the proof. □

040Z **Lemma 67.27.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally quasi-finite,*
(2) *for every $x \in |X|$ the morphism f is quasi-finite at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally quasi-finite,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally quasi-finite,*

(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally quasi-finite,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally quasi-finite, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally quasi-finite.*

Proof. Omitted. □

03XM **Lemma 67.27.7.** *An immersion is locally quasi-finite.*

Proof. Omitted. □

03XN **Lemma 67.27.8.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is locally quasi-finite, then $X \to Y$ is locally quasi-finite.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Apply Morphisms, Lemma 29.20.17 to the top row. □

0ABN **Lemma 67.27.9.** *Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let $y \in |Y|$. There are at most finitely many points of $|X|$ lying over y at which f is quasi-finite.*

Proof. Choose a field k and a morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class determined by y. The fibre $X_k = \mathrm{Spec}(k) \times_Y X$ is an algebraic space of finite type over a field, in particular quasi-compact. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 66.4.3). Moreover, the set of points where $X_k \to \mathrm{Spec}(k)$ is quasi-finite maps onto the set of points lying over y where f is quasi-finite by Lemma 67.27.2. Choose an affine scheme U and a surjective étale morphism $U \to X_k$ (Properties of Spaces, Lemma 66.6.3). Then $U \to \mathrm{Spec}(k)$ is a morphism of finite type and there are at most a finite number of points where this morphism is quasi-finite, see Morphisms, Lemma 29.20.14. Since $X_k \to \mathrm{Spec}(k)$ is quasi-finite at a point x' if and only if it is the image of a point of U where $U \to \mathrm{Spec}(k)$ is quasi-finite, we conclude. □

0463 **Lemma 67.27.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.*

Proof. A monomorphism is separated, see Lemma 67.10.3. By Lemma 67.27.6 it suffices to prove the lemma after performing a base change by $Z \to Y$ with Z affine. Hence we may assume that Y is an affine scheme. Choose an affine scheme U and an étale morphism $U \to X$. Since $X \to Y$ is locally of finite type the morphism of affine schemes $U \to Y$ is of finite type. Since $X \to Y$ is a monomorphism we have $U \times_X U = U \times_Y U$. In particular the maps $U \times_Y U \to U$ are étale. Let $y \in Y$. Then either U_y is empty, or $\mathrm{Spec}(\kappa(u)) \times_{\mathrm{Spec}(\kappa(y))} U_y$ is isomorphic to the fibre of $U \times_Y U \to U$ over u for some $u \in U$ lying over y. This implies that the fibres of $U \to Y$ are finite discrete sets (as $U \times_Y U \to U$ is an étale morphism of affine schemes, see Morphisms, Lemma 29.36.7). Hence $U \to Y$ is quasi-finite, see Morphisms, Lemma 29.20.6. As $U \to X$ was an arbitrary étale morphism with U affine this implies that $X \to Y$ is locally quasi-finite. □

67.28. Morphisms of finite presentation

03XO The property "locally of finite presentation" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.21.4, and Descent, Lemma 35.23.13. Hence, by Lemma 67.22.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03XP **Definition 67.28.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally of finite presentation* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P}$ ="locally of finite presentation".
(2) Let $x \in |X|$. We say f is of *finite presentation at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite presentation[6].
(3) A morphism of algebraic spaces $f : X \to Y$ is *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

03XQ **Lemma 67.28.2.** *The composition of morphisms of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.21.3. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 67.8.5 and 67.4.8). □

03XR **Lemma 67.28.3.** *A base change of a morphism of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

[6]It seems awkward to use "locally of finite presentation at x", but the current terminology may be misleading in the sense that "of finite presentation at x" does **not** mean that there is an open neighbourhood $X' \subset X$ such that $f|_{X'}$ is of finite presentation.

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.21.4. Also use the result for quasi-compact and for quasi-separated morphisms (Lemmas 67.8.4 and 67.4.4). □

0410 **Lemma 67.28.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite presentation,*
(2) *for every $x \in |X|$ the morphism f is of finite presentation at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite presentation,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite presentation,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite presentation,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally of finite presentation, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite presentation.*

Proof. Omitted. □

0464 **Lemma 67.28.5.** *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite presentation. This means there exists a diagram as in Lemma 67.22.1 with h locally of finite presentation and surjective vertical arrow a. By Morphisms, Lemma 29.21.8 h is locally of finite type. Hence $X \to Y$ is locally of finite type by definition. If f is of finite presentation then it is quasi-compact and it follows that f is of finite type. □

04ZL **Lemma 67.28.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is of finite presentation and Y is Noetherian, then X is Noetherian.*

Proof. Assume f is of finite presentation and Y Noetherian. By Lemmas 67.28.5 and 67.23.5 we see that X is locally Noetherian. As f is quasi-compact and Y is quasi-compact we see that X is quasi-compact. As f is of finite presentation it is quasi-separated (see Definition 67.28.1) and as Y is Noetherian it is quasi-separated (see Properties of Spaces, Definition 66.24.1). Hence X is quasi-separated by Lemma 67.4.9. Hence we have checked all three conditions of Properties of Spaces, Definition 66.24.1 and we win. □

06G4 **Lemma 67.28.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If Y is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : X \to Y$ locally of finite type and Y locally Noetherian. This means there exists a diagram as in Lemma 67.22.1 with h locally of finite type and surjective vertical arrow a. By Morphisms, Lemma 29.21.9 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

06G5 **Lemma 67.28.8.** *Let S be a scheme. Let Y be an algebraic space over S which is quasi-compact and quasi-separated. If X is of finite presentation over Y, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

05WT **Lemma 67.28.9.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. If X is locally of finite presentation over Z, and Y is locally of finite type over Z, then f is locally of finite presentation.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Then choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Finally choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By definition U is locally of finite presentation over W and V is locally of finite type over W. By Morphisms, Lemma 29.21.11 the morphism $U \to V$ is locally of finite presentation. Hence f is locally of finite presentation. □

084P **Lemma 67.28.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is of finite type over X (by Lemma 67.23.3). Thus the first statement holds by Lemma 67.28.9. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is separated (Lemma 67.4.1). □

06CN **Lemma 67.28.11.** *An open immersion of algebraic spaces is locally of finite presentation.*

Proof. An open immersion is by definition representable, hence we can use the general principle Spaces, Lemma 65.5.8 and Morphisms, Lemma 29.21.5. □

084Q **Lemma 67.28.12.** *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By Lemma 67.28.4 we see that $i' : Z \times_X U \to U$ is of finite presentation if and only if i is. By Properties of Spaces, Section 66.30 we see that $\mathcal{I}$ is of finite type if and only if $\mathcal{I}|_U = \mathrm{Ker}(\mathcal{O}_U \to i'_*\mathcal{O}_{Z\times_X U})$ is. Hence the result follows from the case of schemes, see Morphisms, Lemma 29.21.7. □

67.29. Constructible sets

0ECV This section is the continuation of Properties of Spaces, Section 66.8.

0ECW **Lemma 67.29.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E \subset |Y|$ be a subset. If E is étale locally constructible in Y, then $f^{-1}(E)$ is étale locally constructible in X.*

Proof. Choose a scheme V and a surjective étale morphism $\varphi : V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to X$ is surjective étale and the inverse image of $f^{-1}(E)$ in U is the inverse image of $\varphi^{-1}(E)$ by $U \to V$. Thus the lemma follows from the case of schemes for $U \to V$ (Morphisms, Lemma 29.22.1) and the definition (Properties of Spaces, Definition 66.8.2). □

0ECX **Theorem 67.29.2** (Chevalley's Theorem)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and locally of finite presentation. Then the image of every étale locally constructible subset of $|X|$ is an étale locally constructible subset of $|Y|$.*

Proof. Let $E \subset |X|$ be étale locally constructible. Let $V \to Y$ be an étale morphism with V affine. It suffices to show that the inverse image of $f(E)$ in V is constructible, see Properties of Spaces, Definition 66.8.2. Since f is quasi-compact $V \times_Y X$ is a quasi-compact algebraic space. Choose an affine scheme U and a surjective étale morphism $U \to V \times_Y X$ (Properties of Spaces, Lemma 66.6.3). By Properties of Spaces, Lemma 66.4.3 the inverse image of $f(E)$ in V is the image under $U \to V$ of the inverse image of E in U. Thus the result follows from the case of schemes, see Morphisms, Lemma 29.22.2. □

67.30. Flat morphisms

03MK The property "flat" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.25.8 and Descent, Lemma 35.23.17. Hence, by Lemma 67.22.1 above, we may define the notion of a flat morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ML **Definition 67.30.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *flat* if the equivalent conditions of Lemma 67.22.1 with $\mathcal{P}$ ="flat".

(2) Let $x \in |X|$. We say f is *flat at* x if the equivalent conditions of Lemma 67.22.5 hold with $\mathcal{Q}$ ="induced map local rings is flat".

Note that the second part makes sense by Descent, Lemma 35.33.4.

We do a quick sanity check.

08EW **Lemma 67.30.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if f is flat at all points of $|X|$.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. By definition f is flat if and only if h is flat (Definition 67.22.2). By definition f is flat at $x \in |X|$ if and only if h is flat at some (equivalently any) $u \in U$ which maps to x (Definition 67.22.6). Thus the lemma follows from the fact that a morphism of schemes is flat if and only if it is flat at all points of the source (Morphisms, Definition 29.25.1). □

03MN **Lemma 67.30.3.** *The composition of flat morphisms is flat.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.25.6. □

03MO **Lemma 67.30.4.** *The base change of a flat morphism is flat.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.25.8. □

03MM **Lemma 67.30.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is flat,*
(2) *for every $x \in |X|$ the morphism f is flat at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is flat,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is flat,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is flat,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is flat, and

(9) *there exists a Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is flat.*

Proof. Omitted. □

042S **Lemma 67.30.6.** *A flat morphism locally of finite presentation is universally open.*

Proof. Let $f : X \to Y$ be a flat morphism locally of finite presentation of algebraic spaces over S. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. By Lemmas 67.30.5 and 67.28.4 the morphism α is flat and locally of finite presentation. Hence by Morphisms, Lemma 29.25.10 we see that α is universally open. Hence $X \to Y$ is universally open according to Lemma 67.6.5. □

0413 **Lemma 67.30.7.** *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, surjective morphism of algebraic spaces over S. A subset $T \subset |Y|$ is open (resp. closed) if and only $f^{-1}(|T|)$ is open (resp. closed) in $|X|$. In other words f is submersive, and in fact universally submersive.*

Proof. Choose affine schemes V_i and étale morphisms $V_i \to Y$ such that $V = \coprod V_i \to Y$ is surjective, see Properties of Spaces, Lemma 66.6.1. For each i the algebraic space $V_i \times_Y X$ is quasi-compact. Hence we can find an affine scheme U_i and a surjective étale morphism $U_i \to V_i \times_Y X$, see Properties of Spaces, Lemma 66.6.3. Then the composition $U_i \to V_i \times_Y X \to V_i$ is a surjective, flat morphism of affines. Of course then $U = \coprod U_i \to X$ is surjective and étale and $U \to V \times_Y X$ is surjective. Moreover, the morphism $U \to V$ is the disjoint union of the morphisms $U_i \to V_i$. Hence $U \to V$ is surjective, quasi-compact and flat. Consider the diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

By definition of the topology on $|Y|$ the set T is closed (resp. open) if and only if $g^{-1}(T) \subset |V|$ is closed (resp. open). The same holds for $f^{-1}(T)$ and its inverse image in $|U|$. Since $U \to V$ is quasi-compact, surjective, and flat we win by Morphisms, Lemma 29.25.12. □

04NG **Lemma 67.30.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$. Let $\overline{y} = f \circ \overline{x}$. The following are equivalent*

(1) *f is flat at x, and*
(2) *the map on étale local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x. We can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 66.19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 66.22.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings with flat horizontal arrows. We have to show that the left vertical arrow is flat if and only if the right vertical arrow is. Algebra, Lemma 10.39.9 tells us $\mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{O}_{X,\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 38.2.5. □

073C **Lemma 67.30.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if the morphism of sites $(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$ associated to f is flat.*

Proof. Flatness of $(f_{small}, f^\sharp)$ is defined in terms of flatness of $\mathcal{O}_X$ as a $f^{-1}_{small}\mathcal{O}_Y$-module. This can be checked at stalks, see Modules on Sites, Lemma 18.39.3 and Properties of Spaces, Theorem 66.19.12. But we've already seen that flatness of f can be checked on stalks, see Lemma 67.30.8. □

089C **Lemma 67.30.10.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of the scheme theoretic support as given in Lemma 67.15.3 and using the characterization of flat morphisms in terms of étale coverings in Lemma 67.30.5 we reduce to the case of schemes which is Morphisms, Lemma 29.25.14. □

089D **Lemma 67.30.11.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $V \to Y$ be a quasi-compact open immersion. If V is scheme theoretically dense in Y, then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. Using the characterization of scheme theoretically dense opens in Lemma 67.17.2 and using the characterization of flat morphisms in terms of étale coverings in Lemma 67.30.5 we reduce to the case of schemes which is Morphisms, Lemma 29.25.15. □

089E **Lemma 67.30.12.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $g : V \to Y$ be a quasi-compact morphism of algebraic spaces. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Let $Y' \to Y$ be a surjective étale morphism such that Y' is a disjoint union of affine schemes (Properties of Spaces, Lemma 66.6.1). Let $X' \to X \times_Y Y'$ be a surjective étale morphism such that X' is a disjoint union of affine schemes. By Lemma 67.30.5 the morphism $X' \to Y'$ is flat. Set $V' = V \times_Y Y'$. By Lemma 67.16.3 the inverse image of Z in Y' is the scheme theoretic image of $V' \to Y'$ and the inverse image of Z' in X' is the scheme theoretic image of $V' \times_{Y'} X' \to X'$. Since $X' \to X$ is surjective étale, it suffices to prove the result in the case of the morphisms $X' \to Y'$ and $V' \to Y'$. Thus we may assume X and Y are affine schemes. In this case V is a quasi-compact algebraic space. Choose an affine scheme W and a surjective étale morphism $W \to V$ (Properties of Spaces, Lemma 66.6.3). It is clear that the scheme theoretic image of $V \to Y$ agrees with the scheme theoretic image of $W \to Y$ and similarly for $V \times_Y X \to Y$ and $W \times_Y X \to X$. Thus we reduce to the case of schemes which is Morphisms, Lemma 29.25.16. □

67.31. Flat modules

05VT In this section we define what it means for a module to be flat at a point. To do this we will use the notion of the stalk of a sheaf on the small étale site $X_{étale}$ of an algebraic space, see Properties of Spaces, Definition 66.19.6.

05VU **Lemma 67.31.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$. The following are equivalent*

(1) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x the module $a^*\mathcal{F}$ is flat at u over V,

(2) *the stalk $\mathcal{F}_{\overline{x}}$ is flat over the étale local ring $\mathcal{O}_{Y,\overline{y}}$ where $\overline{x}$ is any geometric point lying over x and $\overline{y} = f \circ \overline{x}$.*

Proof. During this proof we fix a geometric proof $\overline{x} : \mathrm{Spec}(k) \to X$ over x and we denote $\overline{y} = f \circ \overline{x}$ its image in Y. Given a diagram as in (1) we can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 66.19.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 66.22.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings. Finally, we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$

by Properties of Spaces, Lemma 66.29.4. Thus Algebra, Lemma 10.39.9 tells us $(a^*\mathcal{F})_u$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 38.2.5. □

05VV **Definition 67.31.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) Let $x \in |X|$. We say $\mathcal{F}$ is *flat at x over Y* if the equivalent conditions of Lemma 67.31.1 hold.
(2) We say $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at all $x \in |X|$.

Having defined this we have the obligatory base change lemma. This lemma implies that formation of the flat locus of a quasi-coherent sheaf commutes with flat base change.

05VW **Lemma 67.31.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $x' \in |X'|$ with image $x \in |X|$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and denote $\mathcal{F}' = (g')^*\mathcal{F}$.

(1) *If $\mathcal{F}$ is flat at x over Y then $\mathcal{F}'$ is flat at x' over Y'.*
(2) *If g is flat at $f'(x')$ and $\mathcal{F}'$ is flat at x' over Y', then $\mathcal{F}$ is flat at x over Y.*

In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y'.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme endowed with a surjective étale morphism $U' = V' \times_V U \to Y' \times_Y X = X'$. Pick $u' \in U'$ mapping to $x' \in |X'|$. Then we can check flatness of $\mathcal{F}'$ at x' over Y' in terms of flatness of $\mathcal{F}'|_{U'}$ at u' over V'. Hence the lemma follows from More on Morphisms, Lemma 37.15.2. □

The following lemma discusses "composition" of flat morphisms in terms of modules. It also shows that flatness satisfies a kind of top down descent.

05VX **Lemma 67.31.4.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$.*

(1) *If $\mathcal{F}$ is flat at x over Y and Y is flat at y over Z, then $\mathcal{F}$ is flat at x over Z.*
(2) *Let $x : \mathrm{Spec}(K) \to X$ be a representative of x. If*
   (a) *$\mathcal{F}$ is flat at x over Y,*
   (b) *$x^*\mathcal{F} \neq 0$, and*
   (c) *$\mathcal{F}$ is flat at x over Z,*
   then Y is flat at y over Z.
(3) *Let $\overline{x}$ be a geometric point of X lying over x with image $\overline{y}$ in Y. If $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module and $\mathcal{F}$ is flat at x over Z, then Y is flat at y over Z.*

Proof. Pick $\overline{x}$ and $\overline{y}$ as in part (3) and denote $\overline{z}$ the induced geometric point of Z. Via the characterization of flatness in Lemmas 67.31.1 and 67.30.8 the lemma reduces to a purely algebraic question on the local ring map $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}}$ and the module $\mathcal{F}_{\overline{x}}$. Part (1) follows from Algebra, Lemma 10.39.4. We remark that condition (2)(b) guarantees that $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{F}_{\overline{x}}$ is nonzero. Hence (2)(a) + (2)(b) imply that $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module, see Algebra, Lemma 10.39.15. Thus (2) is a special case of (3). Finally, (3) follows from Algebra, Lemma 10.39.10. □

Sometimes the base change happens "up on top". Here is a precise statement.

05VY **Lemma 67.31.5.** *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in |X|$ with image $y \in |Y|$. If f is flat at x, then*

$$\mathcal{G} \text{ flat over } Z \text{ at } y \Leftrightarrow f^*\mathcal{G} \text{ flat over } Z \text{ at } x.$$

In particular: If f is surjective and flat, then $\mathcal{G}$ is flat over Z, if and only if $f^*\mathcal{G}$ is flat over Z.

Proof. Pick a geometric point $\overline{x}$ of X and denote $\overline{y}$ the image in Y and $\overline{z}$ the image in Z. Via the characterization of flatness in Lemmas 67.31.1 and 67.30.8 and the description of the stalk of $f^*\mathcal{G}$ at $\overline{x}$ of Properties of Spaces, Lemma 66.29.5 the lemma reduces to a purely algebraic question on the local ring maps $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ and the module $\mathcal{G}_{\overline{y}}$. This algebraic statement is Algebra, Lemma 10.39.9. □

0CVU **Lemma 67.31.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = Supp(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.*

Proof. Choose a surjective étale morphism $\varphi : V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Then it suffices to prove the lemma for $U \to V$ and the quasi-coherent $\mathcal{O}_V$-module $\varphi^*\mathcal{F}$. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 29.25.11. □

67.32. Generic flatness

06QR This section is the analogue of Morphisms, Section 29.27.

06QS **Proposition 67.32.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*

(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let $X_V = V \times_Y X$ and let $\mathcal{F}_V$ be the restriction of $\mathcal{F}$ to X_V. Suppose that the result holds for the morphism $X_V \to V$ and the sheaf $\mathcal{F}_V$. Then there exists an open subscheme $V' \subset V$ such that $X_{V'} \to V'$ is flat and of finite presentation and $\mathcal{F}_{V'}$ is an $\mathcal{O}_{X_{V'}}$-module of finite presentation flat over V'. Let $W \subset Y$ be the image of the étale morphism $V' \to Y$, see Properties of Spaces, Lemma 66.4.10. Then $V' \to W$ is a surjective étale morphism, hence we see that $X_W \to W$ is flat, locally of finite presentation, and quasi-compact by Lemmas 67.28.4, 67.30.5, and 67.8.8. By the discussion in Properties of Spaces, Section 66.30 we see that $\mathcal{F}_W$ is of finite presentation as a $\mathcal{O}_{X_W}$-module and by Lemma 67.31.3 we see that $\mathcal{F}_W$ is flat over W. This argument reduces the proposition to the case where Y is a scheme.

Suppose we can prove the proposition when Y is an affine scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S with Y a scheme, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Choose an affine open covering $Y = \bigcup V_j$. By assumption we can find dense open $W_j \subset V_j$ such that $X_{W_j} \to W_j$ is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_{W_j}}$ is flat over W_j and of finite presentation as an $\mathcal{O}_{X_{W_j}}$-module. In this situation we simply take $W = \bigcup W_j$ and we win. Hence we reduce the proposition to the case where Y is an affine scheme.

Let Y be an affine scheme over S, let $f : X \to Y$ be a finite type morphism of algebraic spaces over S, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Since f is of finite type it is quasi-compact, hence X is quasi-compact. Thus we can find an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. Note that $U \to Y$ is of finite type (this is what it means for f to be of finite type in this case). Hence we can apply Morphisms, Proposition 29.27.2 to see that there exists a dense open $W \subset Y$ such that $U_W \to W$ is flat and of finite presentation and such that $\mathcal{F}|_{U_W}$ is flat over W and of finite presentation as an $\mathcal{O}_{U_W}$-module. According to our definitions this means that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$. □

We cannot improve the result of the lemma above to requiring $X_W \to W$ to be of finite presentation as $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z} \to \mathrm{Spec}(\mathbf{Q})$ gives a counter example. The problem is that the diagonal morphism $\Delta_{X/Y}$ may not be quasi-compact, i.e., f may not be quasi-separated. Clearly, this is also the only problem.

06QT **Proposition 67.32.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*
(2) *f is quasi-separated,*
(3) *f is of finite type, and*
(4) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat and of finite presentation and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. This follows immediately from Proposition 67.32.1 and the fact that "of finite presentation" = "locally of finite presentation" + "quasi-compact" + "quasi-separated". □

67.33. Relative dimension

04NH In this section we define the relative dimension of a morphism of algebraic spaces at a point, and some closely related properties.

04NM **Definition 67.33.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Let $d, r \in \{0, 1, 2, \ldots, \infty\}$.

(1) We say the *dimension of the local ring of the fibre of f at x* is d if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 35.33.6.
(2) We say the *transcendence degree of $x/f(x)$* is r if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_r$ described in Descent, Lemma 35.33.7.
(3) We say *f has relative dimension d at x* if the equivalent conditions of Lemma 67.22.5 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 35.33.8.

Let us spell out what this means. Namely, choose some diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

as in Lemma 67.22.5. Then we have

$$\begin{array}{rcc} \text{relative dimension of } f \text{ at } x & = & \dim_u(U_v) \\ \text{dimension of local ring of the fibre of } f \text{ at } x & = & \dim(\mathcal{O}_{U_v,u}) \\ \text{transcendence degree of } x/f(x) & = & \operatorname{trdeg}_{\kappa(v)}(\kappa(u)) \end{array}$$

Note that if $Y = \operatorname{Spec}(k)$ is the spectrum of a field, then the relative dimension of X/Y at x is the same as $\dim_x(X)$, the transcendence degree of $x/f(x)$ is the transcendence degree over k, and the dimension of the local ring of the fibre of f at x is just the dimension of the local ring at x, i.e., the relative notions become absolute notions in that case.

06LR **Definition 67.33.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $d \in \{0, 1, 2, \ldots\}$.

(1) We say f has *relative dimension $\leq d$* if f has relative dimension $\leq d$ at all $x \in |X|$.
(2) We say f has *relative dimension d* if f has relative dimension d at all $x \in |X|$.

Having relative dimension *equal* to d means roughly speaking that all nonempty fibres are equidimensional of dimension d.

06RX **Lemma 67.33.3.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and let $y \in |Y|$, $z \in |Z|$ be the images. Assume $X \to Y$ is locally quasi-finite and $Y \to Z$ locally of finite type. Then the transcendence degree of x/z is equal to the transcendence degree of y/z.*

Proof. We can choose commutative diagrams

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array} \qquad \begin{array}{ccccc} u & \longrightarrow & v & \longrightarrow & w \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & y & \longrightarrow & z \end{array}$$

where U, V, W are schemes and the vertical arrows are étale. By definition the morphism $U \to V$ is locally quasi-finite which implies that $\kappa(v) \subset \kappa(u)$ is finite, see Morphisms, Lemma 29.20.5. Hence the result is clear. □

0ECY **Lemma 67.33.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, Y is Jacobson (Properties of Spaces, Remark 66.7.3), and $x \in |X|$ is a finite type point of X, then the transcendence degree of $x/f(x)$ is 0.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. By Lemma 67.25.5 we can find a finite type point $u \in U$ mapping to x. After shrinking U we may assume $u \in U$ is closed (Morphisms, Lemma 29.16.4). Let $v \in V$ be the image of u. By Morphisms, Lemma 29.16.8 the extension $\kappa(u)/\kappa(v)$ is finite. This finishes the proof. □

0AFH **Lemma 67.33.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian algebraic spaces over S which is flat, locally of finite type and of relative dimension d. For every point x in $|X|$ with image y in $|Y|$ we have $\dim_x(X) = \dim_y(Y) + d$.*

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1) and by definition of having relative dimension d, this reduces to the corresponding statement for schemes (Morphisms, Lemma 29.29.6). □

67.34. Morphisms and dimensions of fibres

04NP This section is the analogue of Morphisms, Section 29.28. The formulations in this section are a bit awkward since we do not have local rings of algebraic spaces at points.

04NQ **Lemma 67.34.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume f is locally of finite type. Then we have*

$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{dimension of local ring of the fibre of } f \textit{ at } x \\ + \\ \textit{transcendence degree of } x/f(x) \end{array}$$

where the notation is as in Definition 67.33.1.

Proof. This follows immediately from Morphisms, Lemma 29.28.1 applied to $h : U \to V$ and $u \in U$ as in Lemma 67.22.5. □

04NR **Lemma 67.34.2.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and set $y = f(x)$. Assume f and g locally of finite type. Then*

(1)
$$\begin{array}{c} \textit{relative dimension of } g \circ f \textit{ at } x \\ \leq \\ \textit{relative dimension of } f \textit{ at } x \\ + \\ \textit{relative dimension of } g \textit{ at } y \end{array}$$

(2) *equality holds in (1) if for some morphism $\mathrm{Spec}(k) \to Z$ from the spectrum of a field in the class of $g(f(x)) = g(y)$ the morphism $X_k \to Y_k$ is flat at x, for example if f is flat at x,*

(3)
$$\begin{array}{c} \textit{transcendence degree of } x/g(f(x)) \\ = \\ \textit{transcendence degree of } x/f(x) \\ + \\ \textit{transcendence degree of } f(x)/g(f(x)) \end{array}$$

Proof. Choose a diagram
$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$
with U, V, W schemes and vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Choose $u \in U$ mapping to x. Set v, w equal to the images of u in V, W. Apply Morphisms, Lemma 29.28.2 to the top row and the points u, v, w. Details omitted. □

04NS **Lemma 67.34.3.** *Let S be a scheme. Let*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
be a fibre product diagram of algebraic spaces over S. Let $x' \in |X'|$. Set $x = g'(x')$. Assume f locally of finite type. Then

(1)
$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{relative dimension of } f' \textit{ at } x' \end{array}$$

(2) *we have*

$$\begin{gathered} \textit{dimension of local ring of the fibre of } f' \textit{ at } x' \\ - \\ \textit{dimension of local ring of the fibre of } f \textit{ at } x \\ = \\ \textit{transcendence degree of } x/f(x) \\ - \\ \textit{transcendence degree of } x'/f'(x') \end{gathered}$$

and the common value is ≥ 0*,*

(3) *given* x *and* $y' \in |Y'|$ *mapping to the same* $y \in |Y|$ *there exists a choice of* x' *such that the integer in (2) is* 0*.*

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Choose a surjective étale morphism $V' \to V \times_Y Y'$ with V' a scheme. Set $U' = V' \times_V U$. Then the induced morphism $U' \to X'$ is also surjective and étale (argument omitted). Choose $u' \in U'$ mapping to x'. At this point parts (1) and (2) follow by applying Morphisms, Lemma 29.28.3 to the diagram of schemes involving U', U, V', V and the point u'. To prove (3) first choose $v \in V$ mapping to y. Then using Properties of Spaces, Lemma 66.4.3 we can choose $v' \in V'$ mapping to y' and v and $u \in U$ mapping to x and v. Finally, according to Morphisms, Lemma 29.28.3 we can choose $u' \in U'$ mapping to v' and u such that the integer is zero. Then taking $x' \in |X'|$ the image of u' works. □

04NT **Lemma 67.34.4.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S*. Let* $n \geq 0$*. Assume* f *is locally of finite type. The set*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

is open in $|X|$*.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. By Morphisms, Lemma 29.28.4 the set U_n of points where h has relative dimension $\leq n$ is open in U. By our definition of relative dimension for morphisms of algebraic spaces at points we see that $U_n = a^{-1}(W_n)$. The lemma follows by definition of the topology on $|X|$. □

04NU **Lemma 67.34.5.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S *Let* $n \geq 0$*. Assume* f *is locally of finite presentation. The open*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

of Lemma 67.34.4 is retrocompact in $|X|$*. (See Topology, Definition 5.12.1.)*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. In the proof of Lemma 67.34.4 we have seen that $a^{-1}(W_n) = U_n$ is the corresponding set for the morphism h. By Morphisms, Lemma 29.28.6 we see that U_n is retrocompact in U. The lemma follows by definition of the topology on $|X|$, compare with Properties of Spaces, Lemma 66.5.5 and its proof. □

04NV **Lemma 67.34.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then f is locally quasi-finite if and only if f has relative dimension 0 at each $x \in |X|$.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 65.11.6. The definitions imply that h is locally quasi-finite if and only if f is locally quasi-finite, and that f has relative dimension 0 at all $x \in |X|$ if and only if h has relative dimension 0 at all $u \in U$. Hence the result follows from the result for h which is Morphisms, Lemma 29.29.5. □

04NW **Lemma 67.34.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then there exists a canonical open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is locally quasi-finite, and such that the relative dimension of f at any $x \in |X|$, $x \notin |X'|$ is ≥ 1. Formation of X' commutes with arbitrary base change.*

Proof. Combine Lemmas 67.34.4, 67.34.6, and 67.34.3. □

06LS **Lemma 67.34.8.** *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow[p]{} & F \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is locally of finite type and where k is a field over S. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$X \longrightarrow Y$$

is locally quasi-finite.

Proof. Let $X' \subset X$ be the open subspace over which f is locally quasi-finite found in Lemma 67.34.7. Since the formation of X' commutes with arbitrary base change we see that $z \in X' \times_Y \mathrm{Spec}(k)$. Hence the lemma is clear. □

67.35. The dimension formula

0BAW The analog of the dimension formula (Morphisms, Lemma 29.52.1) is a bit tricky to formulate, because we would have to define integral algebraic spaces (we do this later) as well as universally catenary algebraic spaces. However, the following version is straightforward.

0BAX **Lemma 67.35.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Then we have*

$$\begin{aligned} &\text{the dimension of the local ring of } X \text{ at } x \leq \\ &\text{the dimension of the local ring of } Y \text{ at } y + E - \\ &\text{the transcendence degree of } x/y \end{aligned}$$

Here E is the maximum of the transcendence degrees of $\xi/f(\xi)$ where $\xi \in |X|$ runs over the points specializing to x at which the local ring of X has dimension 0.

Proof. Choose an affine scheme V, an étale morphism $V \to Y$, and a point $v \in V$ mapping to y. Choose an affine scheme U , an étale morphism $U \to X \times_Y V$ and a point $u \in U$ mapping to v in V and x in X. Unwinding Definition 67.33.1 and Properties of Spaces, Definition 66.10.2 we have to show that

$$\dim(\mathcal{O}_{U,u}) \leq \dim(\mathcal{O}_{V,v}) + E - \text{trdeg}_{\kappa(v)}(\kappa(u))$$

Let $\xi_U \in U$ be a generic point of an irreducible component of U which contains u. Then ξ_U maps to a point $\xi \in |X|$ which is in the list used to define the quantity E and in fact every ξ used in the definition of E occurs in this manner (small detail omitted). In particular, there are only a finite number of these ξ and we can take the maximum (i.e., it really is a maximum and not a supremum). The transcendence degree of ξ over $f(\xi)$ is $\text{trdeg}_{\kappa(\xi_V)}(\kappa(\xi_U))$ where $\xi_V \in V$ is the image of ξ_U. Thus the lemma follows from Morphisms, Lemma 29.52.2. □

0BAY **Lemma 67.35.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f is locally of finite type. Then*

$$\dim(X) \leq \dim(Y) + E$$

where E is the supremum of the transcendence degrees of $\xi/f(\xi)$ where ξ runs through the points at which the local ring of X has dimension 0.

Proof. Immediate consequence of Lemma 67.35.1 and Properties of Spaces, Lemma 66.10.3. □

67.36. Syntomic morphisms

03Z6 The property "syntomic" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.30.4 and Descent, Lemma 35.23.28. Hence, by Lemma 67.22.1 above, we may define the notion of a syntomic morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03Z7 **Definition 67.36.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *syntomic* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P}$ ="syntomic".
(2) Let $x \in |X|$. We say f is *syntomic at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is syntomic.

03Z8 **Lemma 67.36.2.** *The composition of syntomic morphisms is syntomic.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.30.3. □

03Z9 **Lemma 67.36.3.** *The base change of a syntomic morphism is syntomic.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.30.4. □

03ZA **Lemma 67.36.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is syntomic,*
(2) *for every $x \in |X|$ the morphism f is syntomic at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a syntomic morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is syntomic,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is syntomic,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is syntomic, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is syntomic.*

Proof. Omitted. □

0DEY **Lemma 67.36.5.** *A syntomic morphism is locally of finite presentation.*

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 29.30.6). □

0DEZ **Lemma 67.36.6.** *A syntomic morphism is flat.*

Proof. Follows immediately from the case of schemes (Morphisms, Lemma 29.30.7). □

0DF0 **Lemma 67.36.7.** *A syntomic morphism is universally open.*

Proof. Combine Lemmas 67.36.5, 67.36.6, and 67.30.6. □

67.37. Smooth morphisms

03ZB The property "smooth" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.34.5 and Descent, Lemma 35.23.29. Hence, by Lemma 67.22.1 above, we may define the notion of a smooth morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ZC **Definition 67.37.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *smooth* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P}$ ="smooth".
(2) Let $x \in |X|$. We say f is *smooth at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is smooth.

03ZD **Lemma 67.37.2.** *The composition of smooth morphisms is smooth.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.34.4. □

03ZE **Lemma 67.37.3.** *The base change of a smooth morphism is smooth.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.34.5. □

03ZF **Lemma 67.37.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is smooth,*
(2) *for every $x \in |X|$ the morphism f is smooth at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a smooth morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is smooth,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is smooth,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is smooth, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is smooth.*

Proof. Omitted. □

04AJ **Lemma 67.37.5.** *A smooth morphism of algebraic spaces is locally of finite presentation.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.8 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. □

06MH **Lemma 67.37.6.** *A smooth morphism of algebraic spaces is locally of finite type.*

Proof. Combine Lemmas 67.37.5 and 67.28.5. □

04TA **Lemma 67.37.7.** *A smooth morphism of algebraic spaces is flat.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.8 h is flat. Hence $X \to Y$ is flat by definition. □

06CP **Lemma 67.37.8.** *A smooth morphism of algebraic spaces is syntomic.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 67.22.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 29.34.7 h is syntomic. Hence $X \to Y$ is syntomic by definition. □

0DZI **Lemma 67.37.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There is a maximal open subspace $U \subset X$ such that $f|_U : U \to Y$ is smooth. Moreover, formation of this open commutes with base change by*

(1) *morphisms which are flat and locally of finite presentation,*
(2) *flat morphisms provided f is locally of finite presentation.*

Proof. The existence of U follows from the fact that the property of being smooth is Zariski (and even étale) local on the source, see Lemma 67.37.4. Moreover, this lemma allows us to translate properties (1) and (2) into the case of morphisms of schemes. The case of schemes is Morphisms, Lemma 29.34.15. Some details omitted. □

0AFI **Lemma 67.37.10.** *Let X and Y be locally Noetherian algebraic spaces over a scheme S, and let $f : X \to Y$ be a smooth morphism. For every point $x \in |X|$ with image $y \in |Y|$,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y)$$

where $\dim_x(X_y)$ *is the relative dimension of* f *at* x *as in Definition 67.33.1.*

Proof. By definition of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1), this reduces to the corresponding statement for schemes (Morphisms, Lemma 29.34.21). □

67.38. Unramified morphisms

03ZG The property "unramified" (resp. "G-unramified") of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 35.32.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 29.35.5 and Descent, Lemma 35.23.30. Hence, by Lemma 67.22.1 above, we may define the notion of an unramified morphism (resp. G-unramified morphism) of algebraic spaces as follows and it agrees with the already existing notion defined in Section 67.3 when the morphism is representable.

03ZH **Definition 67.38.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *unramified* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ unramified.
(2) Let $x \in |X|$. We say f is *unramified at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is unramified.
(3) We say f is *G-unramified* if the equivalent conditions of Lemma 67.22.1 hold with $\mathcal{P} =$ G-unramified.
(4) Let $x \in |X|$. We say f is *G-unramified at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is G-unramified.

Because of the following lemma, from here on we will only develop theory for unramified morphisms, and whenever we want to use a G-unramified morphism we will simply say "an unramified morphism locally of finite presentation".

04G1 **Lemma 67.38.2.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S*. Then* f *is G-unramified if and only if* f *is unramified and locally of finite presentation.*

Proof. Consider any diagram as in Lemma 67.22.1. Then all we are saying is that the morphism h is G-unramified if and only if it is unramified and locally of finite presentation. This is clear from Morphisms, Definition 29.35.1. □

03ZI **Lemma 67.38.3.** *The composition of unramified morphisms is unramified.*

Proof. See Remark 67.22.3 and Morphisms, Lemma 29.35.4. □

03ZJ **Lemma 67.38.4.** *The base change of an unramified morphism is unramified.*

Proof. See Remark 67.22.4 and Morphisms, Lemma 29.35.5. □

03ZK **Lemma 67.38.5.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S*. The following are equivalent:*

(1) f *is unramified,*
(2) *for every* $x \in |X|$ *the morphism* f *is unramified at* x*,*
(3) *for every scheme* Z *and any morphism* $Z \to Y$ *the morphism* $Z \times_Y X \to Z$ *is unramified,*

(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an unramified morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is unramified,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is unramified,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is unramified, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is unramified.*

Proof. Omitted. □

05VZ **Lemma 67.38.6.** *An unramified morphism of algebraic spaces is locally of finite type.*

Proof. Via a diagram as in Lemma 67.22.1 this translates into Morphisms, Lemma 29.35.9. □

05W0 **Lemma 67.38.7.** *If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. Via a diagram as in Lemma 67.22.1 this translates into Morphisms, Lemma 29.35.10. □

06CQ **Lemma 67.38.8.** *An immersion of algebraic spaces is unramified.*

Proof. Let $i : X \to Y$ be an immersion of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is an immersion of schemes, hence unramified (see Morphisms, Lemmas 29.35.7 and 29.35.8). Thus by definition i is unramified. □

05W1 **Lemma 67.38.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is unramified, then the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion.*
(2) *If f is locally of finite type and $\Delta_{X/Y}$ is an open immersion, then f is unramified.*

Proof. We know in any case that $\Delta_{X/Y}$ is a representable monomorphism, see Lemma 67.4.1. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\Delta_{U/V}} & U \times_V U & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_{V/Y}} \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \longrightarrow & V \times_Y V \end{array}$$

with cartesian right square. The left vertical arrow is surjective étale. The right vertical arrow is étale as a morphism between schemes étale over Y, see Properties of Spaces, Lemma 66.16.6. Hence the middle vertical arrow is étale too (but it need not be surjective).

Assume f is unramified. Then $U \to V$ is unramified, hence $\Delta_{U/V}$ is an open immersion by Morphisms, Lemma 29.35.13. Looking at the left square of the diagram above we conclude that $\Delta_{X/Y}$ is an étale morphism, see Properties of Spaces, Lemma 66.16.3. Hence $\Delta_{X/Y}$ is a representable étale monomorphism, which implies that it is an open immersion by Étale Morphisms, Theorem 41.14.1. (See also Spaces, Lemma 65.5.8 for the translation from schemes language into the language of functors.)

Assume that f is locally of finite type and that $\Delta_{X/Y}$ is an open immersion. This implies that $U \to V$ is locally of finite type too (by definition of a morphism of algebraic spaces which is locally of finite type). Looking at the displayed diagram above we conclude that $\Delta_{U/V}$ is étale as a morphism between schemes étale over $X \times_Y X$, see Properties of Spaces, Lemma 66.16.6. But since $\Delta_{U/V}$ is the diagonal of a morphism between schemes we see that it is in any case an immersion, see Schemes, Lemma 26.21.2. Hence it is an open immersion, and we conclude that $U \to V$ is unramified by Morphisms, Lemma 29.35.13. This in turn means that f is unramified by definition. □

05W2 **Lemma 67.38.10.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces over S. Assume that $X \to Z$ is locally of finite type. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is unramified. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of Morphisms, Lemma 29.35.15 and in fact we will deduce the lemma from it. By Definition 67.38.1 the set $\{x \in |X| : f \text{ is unramified at } x\}$ is open in X. Hence we only need to prove the final statement. By Lemma 67.23.6 the morphism $X \to Y$ is locally of finite type. By Lemma 67.23.3 the morphism $X' \to Y'$ is locally of finite type.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 66.16.7). This combined with Lemma 67.38.5 implies that $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism $U \to V$ is unramified. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism $U' \to V'$ is unramified. Now, by construction the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is cartesian (in the category of schemes). Hence the aforementioned Morphisms, Lemma 29.35.15 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

06G6 **Lemma 67.38.11.** *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is unramified, then $X \to Y$ is unramified.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 65.11.6.) Apply Morphisms, Lemma 29.35.16 to the top row. □

67.39. Étale morphisms

03XS The notion of an étale morphism of algebraic spaces was defined in Properties of Spaces, Definition 66.16.2. Here is what it means for a morphism to be étale at a point.

04RH **Definition 67.39.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. We say f is *étale at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is étale.

03XT **Lemma 67.39.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *for every $x \in |X|$ the morphism f is étale at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an étale morphism,*

(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is étale,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is étale,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ surjective such that the top horizontal arrow is étale, and
(9) *there exist Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is étale.*

Proof. Combine Properties of Spaces, Lemmas 66.16.3, 66.16.5 and 66.16.4. Some details omitted. □

0465 **Lemma 67.39.3.** *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 66.16.4. □

0466 **Lemma 67.39.4.** *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 66.16.5. □

03XU **Lemma 67.39.5.** *An étale morphism of algebraic spaces is locally quasi-finite.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces, see Properties of Spaces, Definition 66.16.2. By Properties of Spaces, Lemma 66.16.3 we see this means there exists a diagram as in Lemma 67.22.1 with h étale and surjective vertical arrow a. By Morphisms, Lemma 29.36.6 h is locally quasi-finite. Hence $X \to Y$ is locally quasi-finite by definition. □

04XX **Lemma 67.39.6.** *An étale morphism of algebraic spaces is smooth.*

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses the fact that an étale morphism of schemes is smooth (by definition of an étale morphism of schemes). □

0467 **Lemma 67.39.7.** *An étale morphism of algebraic spaces is flat.*

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.12. □

0468 **Lemma 67.39.8.** *An étale morphism of algebraic spaces is locally of finite presentation.*

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.11. □

06LT **Lemma 67.39.9.** *An étale morphism of algebraic spaces is locally of finite type.*

Proof. An étale morphism is locally of finite presentation and a morphism locally of finite presentation is locally of finite type, see Lemmas 67.39.8 and 67.28.5. □

06CR **Lemma 67.39.10.** *An étale morphism of algebraic spaces is unramified.*

Proof. The proof is identical to the proof of Lemma 67.39.5. It uses Morphisms, Lemma 29.36.5. □

05W3 **Lemma 67.39.11.** *Let S be a scheme. Let X, Y be algebraic spaces étale over an algebraic space Z. Any morphism $X \to Y$ over Z is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 66.16.6. □

06LU **Lemma 67.39.12.** *A locally finitely presented, flat, unramified morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be a locally finitely presented, flat, unramified morphism of algebraic spaces. By Properties of Spaces, Lemma 66.16.3 we see this means there exists a diagram as in Lemma 67.22.1 with h locally finitely presented, flat, unramified and surjective vertical arrow a. By Morphisms, Lemma 29.36.16 h is étale. Hence $X \to Y$ is étale by definition. □

67.40. Proper morphisms

03ZL The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic spaces.

03ZM **Definition 67.40.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *proper* if f is separated, finite type, and universally closed.

083R **Lemma 67.40.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is proper,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is proper, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is proper.*

Proof. Combine Lemmas 67.4.12, 67.23.4, 67.8.8, and 67.9.5. □

04WP **Lemma 67.40.3.** *A base change of a proper morphism is proper.*

Proof. See Lemmas 67.4.4, 67.23.3, and 67.9.3. □

04XY **Lemma 67.40.4.** *A composition of proper morphisms is proper.*

Proof. See Lemmas 67.4.8, 67.23.2, and 67.9.4. □

04XZ **Lemma 67.40.5.** *A closed immersion of algebraic spaces is a proper morphism of algebraic spaces.*

Proof. As a closed immersion is by definition representable this follows from Spaces, Lemma 65.5.8 and the corresponding result for morphisms of schemes, see Morphisms, Lemma 29.41.6. □

04NX **Lemma 67.40.6.** *Let S be a scheme. Consider a commutative diagram of algebraic spaces*

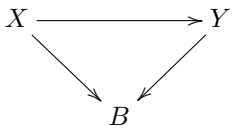

over S.

(1) *If $X \to B$ is universally closed and $Y \to B$ is separated, then the morphism $X \to Y$ is universally closed. In particular, the image of $|X|$ in $|Y|$ is closed.*
(2) *If $X \to B$ is proper and $Y \to B$ is separated, then the morphism $X \to Y$ is proper.*

Proof. Assume $X \to B$ is universally closed and $Y \to B$ is separated. We factor the morphism as $X \to X \times_B Y \to Y$. The first morphism is a closed immersion, see Lemma 67.4.6 hence universally closed. The projection $X \times_B Y \to Y$ is the base change of a universally closed morphism and hence universally closed, see Lemma 67.9.3. Thus $X \to Y$ is universally closed as the composition of universally closed morphisms, see Lemma 67.9.4. This proves (1). To deduce (2) combine (1) with Lemmas 67.4.10, 67.8.9, and 67.23.6. □

08AJ **Lemma 67.40.7.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces over B. If X is universally closed over B and f is surjective then Y is universally closed over B. In particular, if also Y is separated and of finite type over B, then Y is proper over B.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to B$, $q : Y \to B$ the structure morphisms. Let $B' \to B$ be a morphism of algebraic spaces over S. The base change $f' : X_{B'} \to Y_{B'}$ is surjective (Lemma 67.5.5), and the base change $p' : X_{B'} \to B'$ is closed. If $T \subset Y_{B'}$ is closed, then $(f')^{-1}(T) \subset X_{B'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

0AGD **Lemma 67.40.8.** *Let S be a scheme. Let*

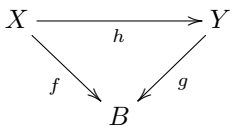

be a commutative diagram of morphism of algebraic spaces over S. Assume

(1) *$X \to B$ is a proper morphism,*
(2) *$Y \to B$ is separated and locally of finite type,*

Then the scheme theoretic image $Z \subset Y$ of h is proper over B and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 67.16. Observe that h is quasi-compact (Lemma 67.8.10) hence $|h|(|X|) \subset |Z|$ is dense (Lemma 67.16.3). On the other hand $|h|(|X|)$ is closed in $|Y|$ (Lemma 67.40.6) hence $X \to Z$ is surjective. Thus $Z \to B$ is a proper (Lemma 67.40.7). □

04Y0 **Lemma 67.40.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is separated,*
(2) *$\Delta_{X/Y} : X \to X \times_Y X$ is universally closed, and*
(3) *$\Delta_{X/Y} : X \to X \times_Y X$ is proper.*

Proof. The implication (1) ⇒ (3) follows from Lemma 67.40.5. We will use Spaces, Lemma 65.5.8 without further mention in the rest of the proof. Recall that $\Delta_{X/Y}$ is a representable monomorphism which is locally of finite type, see Lemma 67.4.1. Since proper ⇒ universally closed for morphisms of schemes we conclude that (3) implies (2). If $\Delta_{X/Y}$ is universally closed then Étale Morphisms, Lemma 41.7.2 implies that it is a closed immersion. Thus (2) ⇒ (1) and we win. □

67.41. Valuative criteria

03IW The section introduces the basics on valuative criteria for morphisms of algebraic spaces. Here is a list of references to further results

(1) the valuative criterion for universal closedness can be found in Section 67.42,
(2) the valuative criterion of separatedness can be found in Section 67.43,
(3) the valuative criterion for properness can be found in Section 67.44,
(4) additional converse statements can be found in Decent Spaces, Section 68.16 and Decent Spaces, Lemma 68.17.11, and
(5) in the Noetherian case it is enough to check the criterion for discrete valuation rings as is shown in Cohomology of Spaces, Section 69.19 and Limits of Spaces, Section 70.21, and
(6) refined versions of the valuative criteria in the Noetherian case can be found in Limits of Spaces, Section 70.22.

We first formally state the definition and then we discuss how this differs from the case of morphisms of schemes.

03IX **Definition 67.41.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say *f satisfies the uniqueness part of the valuative criterion* if given any commutative solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists at most one dotted arrow (without requiring existence). We say *f satisfies the existence part of the valuative criterion* if given any solid diagram as above there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to$

X such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

We say f *satisfies the valuative criterion* if f satisfies both the existence and uniqueness part.

The formulation of the existence part of the valuative criterion is slightly different for morphisms of algebraic spaces, since it may be necessary to extend the fraction field of the valuation ring. In practice this difference almost never plays a role.

(1) Checking the uniqueness part of the valuative criterion never involves any fraction field extensions, hence this is exactly the same as in the case of schemes.
(2) It is necessary to allow for field extensions in general, see Example 67.41.6.
(3) For morphisms of algebraic spaces it always suffices to take a finite separable extensions K'/K in the existence part of the valuative criterion, see Lemma 67.41.3.
(4) If $f : X \to Y$ is a separated morphism of algebraic spaces, then we can always take $K = K'$ when we check the existence part of the valuative criterion, see Lemma 67.41.5.
(5) For a quasi-compact and quasi-separated morphism $f : X \to Y$, we get an equivalence between "f is separated and universally closed" and "f satisfies the usual valuative criterion", see Lemma 67.43.3. The valuative criterion for properness is the usual one, see Lemma 67.44.1.

As a first step in the theory, we show that the criterion is identical to the criterion as formulated for morphisms of schemes in case the morphism of algebraic spaces is representable.

03K8 **Lemma 67.41.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable. The following are equivalent*

(1) *f satisfies the existence part of the valuative criterion as in Definition 67.41.1,*
(2) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. It suffices to show that given a commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \overset{\varphi}{\nearrow} & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 67.41.1, then we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram too. Set $X_A = \mathrm{Spec}(A) \times_Y X$. As f is representable we see that X_A is a scheme. The morphism φ gives a morphism $\varphi' : \mathrm{Spec}(A') \to X_A$. Let $x \in X_A$ be the image of the closed point of $\varphi' : \mathrm{Spec}(A') \to X_A$. Then we have the following commutative diagram of rings

$$\begin{array}{ccccc} K' & \longleftarrow & K & \longleftarrow & \mathcal{O}_{X_A,x} \\ \uparrow & & & \swarrow & \uparrow \\ A' & \longleftarrow & A & \longleftarrow & A \end{array}$$

Since A is a valuation ring, and since A' dominates A, we see that $K \cap A' = A$. Hence the ring map $\mathcal{O}_{X_A,x} \to K$ has image contained in A. Whence a morphism $\mathrm{Spec}(A) \to X_A$ (see Schemes, Section 26.13) as desired. □

03KH **Lemma 67.41.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

1. *f satisfies the existence part of the valuative criterion as in Definition 67.41.1,*
2. *f satisfies the existence part of the valuative criterion as in Definition 67.41.1 modified by requiring the extension K'/K to be finite separable.*

Proof. We have to show that (1) implies (2). Suppose given a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. Choose a scheme U and a surjective étale morphism $U \to X$. Then

$$\mathrm{Spec}(A') \times_X U \longrightarrow \mathrm{Spec}(A')$$

is surjective étale. Let p be a point of $\mathrm{Spec}(A') \times_X U$ mapping to the closed point of $\mathrm{Spec}(A')$. Let $p' \rightsquigarrow p$ be a generalization of p mapping to the generic point of $\mathrm{Spec}(A')$. Such a generalization exists because generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 29.25.9. Then p' corresponds to a point of the scheme $\mathrm{Spec}(K') \times_X U$. Note that

$$\mathrm{Spec}(K') \times_X U = \mathrm{Spec}(K') \times_{\mathrm{Spec}(K)} (\mathrm{Spec}(K) \times_X U)$$

Hence p' maps to a point $q' \in \mathrm{Spec}(K) \times_X U$ whose residue field is a finite separable extension of K. Finally, $p' \rightsquigarrow p$ maps to a specialization $u' \rightsquigarrow u$ on the scheme U.

With all this notation we get the following diagram of rings

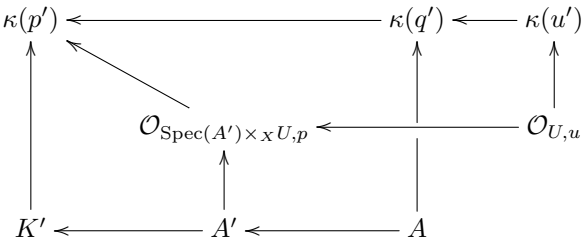

This means that the ring $B \subset \kappa(q')$ generated by the images of A and $\mathcal{O}_{U,u}$ maps to a subring of $\kappa(p')$ contained in the image B' of $\mathcal{O}_{\mathrm{Spec}(A')\times_X U,p} \to \kappa(p')$. Note that B' is a local ring. Let $\mathfrak{m} \subset B$ be the maximal ideal. By construction $A \cap \mathfrak{m}$, (resp. $\mathcal{O}_{U,u} \cap \mathfrak{m}$, resp. $A' \cap \mathfrak{m}$) is the maximal ideal of A (resp. $\mathcal{O}_{U,u}$, resp. A'). Set $\mathfrak{q} = B \cap \mathfrak{m}$. This is a prime ideal such that $A \cap \mathfrak{q}$ is the maximal ideal of A. Hence $B_\mathfrak{q} \subset \kappa(q')$ is a local ring dominating A. By Algebra, Lemma 10.50.2 we can find a valuation ring $A_1 \subset \kappa(q')$ with field of fractions $\kappa(q')$ dominating $B_\mathfrak{q}$. The (local) ring map $\mathcal{O}_{U,u} \to A_1$ gives a morphism $\mathrm{Spec}(A_1) \to U \to X$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(q')) & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

is commutative. Since the fraction field of A_1 is $\kappa(q')$ and since $\kappa(q')/K$ is finite separable by construction the lemma is proved. □

0ARH **Lemma 67.41.4.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Suppose given a diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. Then the dotted arrow exists making the diagram commute.

Proof. We have to show that we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram.

Consider the base change $X_A = \mathrm{Spec}(A) \times_Y X$ of X. Then $X_A \to \mathrm{Spec}(A)$ is a separated morphism of algebraic spaces (Lemma 67.4.4). Base changing all the morphisms of the diagram above we obtain

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X_A \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & = & \mathrm{Spec}(A) \end{array}$$

Thus we may replace X by X_A, assume that $Y = \mathrm{Spec}(A)$ and that we have a diagram as above. We may and do replace X by a quasi-compact open subspace containing the image of $|\mathrm{Spec}(A')| \to |X|$.

The morphism $\mathrm{Spec}(A') \to X$ is quasi-compact by Lemma 67.8.9. Let $Z \subset X$ be the scheme theoretic image of $\mathrm{Spec}(A') \to X$. Then Z is a reduced (Lemma 67.16.4), quasi-compact (as a closed subspace of X), separated (as a closed subspace of X) algebraic space over A. Consider the base change

$$\mathrm{Spec}(K') = \mathrm{Spec}(A') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \to X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = X_K$$

of the morphism $\mathrm{Spec}(A') \to X$ by the flat morphism of schemes $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$. By Lemma 67.30.12 we see that the scheme theoretic image of this morphism is the base change Z_K of Z. On the other hand, by assumption (i.e., the commutative diagram above) this morphism factors through a morphism $\mathrm{Spec}(K) \to Z_K$ which is a section to the structure morphism $Z_K \to \mathrm{Spec}(K)$. As Z_K is separated, this section is a closed immersion (Lemma 67.4.7). We conclude that $Z_K = \mathrm{Spec}(K)$.

Let $V \to Z$ be a surjective étale morphism with V an affine scheme (Properties of Spaces, Lemma 66.6.3). Say $V = \mathrm{Spec}(B)$. Then $V \times_Z \mathrm{Spec}(A') = \mathrm{Spec}(C)$ is affine as Z is separated. Note that $B \to C$ is injective as V is the scheme theoretic image of $V \times_Z \mathrm{Spec}(A') \to V$ by Lemma 67.16.3. On the other hand, $A' \to C$ is étale as corresponds to the base change of $V \to Z$. Since A' is a torsion free A-module, the flatness of $A' \to C$ implies C is a torsion free A-module, hence B is a torsion free A-module. Note that being torsion free as an A-module is equivalent to being flat (More on Algebra, Lemma 15.22.10). Next, we write

$$V \times_Z V = \mathrm{Spec}(B')$$

Note that the two ring maps $B \to B'$ are étale as $V \to Z$ is étale. The canonical surjective map $B \otimes_A B \to B'$ becomes an isomorphism after tensoring with K over A because $Z_K = \mathrm{Spec}(K)$. However, $B \otimes_A B$ is torsion free as an A-module by our remarks above. Thus $B' = B \otimes_A B$. It follows that the base change of the ring map $A \to B$ by the faithfully flat ring map $A \to B$ is étale (note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective as $X \to \mathrm{Spec}(A)$ is surjective). Hence $A \to B$ is étale (Descent, Lemma 35.23.31), in other words, $V \to X$ is étale. Since we have $V \times_Z V = V \times_{\mathrm{Spec}(A)} V$ we conclude that $Z = \mathrm{Spec}(A)$ as algebraic spaces (for example by Spaces, Lemma 65.9.1) and the proof is complete. □

0A3W **Lemma 67.41.5.** *Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. The following are equivalent*

(1) *f satisfies the existence part of the valuative criterion as in Definition 67.41.1,*
(2) *given any commutative solid diagram*

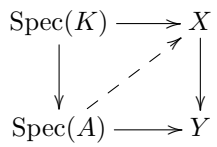

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. We have to show that (1) implies (2). Suppose given a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in part (2). By (1) there exists a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 67.41.1 with $K \subset K'$ arbitrary. By Lemma 67.41.4 we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram, i.e., (2) holds. □

03KI **Example 67.41.6.** Consider the algebraic space X constructed in Spaces, Example 65.14.2. Recall that it is Galois twist of the affine line with zero doubled. The Galois twist is with respect to a degree two Galois extension k'/k of fields. As such it comes with a morphism

$$\pi : X \longrightarrow S = \mathbf{A}^1_k$$

which is quasi-compact. We claim that π is universally closed. Namely, after base change by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ the morphism π is identified with the morphism

$$\text{affine line with zero doubled} \longrightarrow \text{affine line}$$

which is universally closed (some details omitted). Since the morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is universally closed and surjective, a diagram chase shows that π is universally closed. On the other hand, consider the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k((x))) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \pi \\ \mathrm{Spec}(k[[x]]) & \longrightarrow & \mathbf{A}^1_k \end{array}$$

Since the unique point of X above $0 \in \mathbf{A}^1_k$ corresponds to a monomorphism $\mathrm{Spec}(k') \to X$ it is clear there cannot exist a dotted arrow! This shows that a finite separable field extension is needed in general.

03IY **Lemma 67.41.7.** *The base change of a morphism of algebraic spaces which satisfies the existence part of (resp. uniqueness part of) the valuative criterion by any morphism of algebraic spaces satisfies the existence part of (resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over the scheme S. Let $Z \to Y$ be any morphism of algebraic spaces over S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

Then the set of north-west dotted arrows making the diagram commute is in 1-1 correspondence with the set of west-north-west dotted arrows making the diagram commute. This proves the lemma in the case of "uniqueness". For the existence part, assume f satisfies the existence part of the valuative criterion. If we are given a solid commutative diagram as above, then by assumption there exists an extension K'/K of fields and a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ fitting into the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

And by the remarks above the skew arrow corresponds to an arrow $\mathrm{Spec}(A') \to Z \times_Y X$ as desired. □

03IZ **Lemma 67.41.8.** *The composition of two morphisms of algebraic spaces which satisfy the (existence part of, resp. uniqueness part of) the valuative criterion satisfies the (existence part of, resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ & & Y \\ & \nearrow & \downarrow g \\ \mathrm{Spec}(A) & \longrightarrow & Z \end{array}$$

If we have the uniqueness part for g, then there exists at most one north-west dotted arrow making the diagram commute. If we also have the uniqueness part for f, then we have at most one north-north-west dotted arrow making the diagram commute. The proof in the existence case comes from contemplating the following diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K'') & \longrightarrow & \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & & \nearrow & & & \downarrow f \\ & & & & & & Y \\ & & & \nearrow & & & \downarrow g \\ \mathrm{Spec}(A'') & \longrightarrow & \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Z \end{array}$$

Namely, the existence part for g gives us the extension K', the valuation ring A' and the arrow $\mathrm{Spec}(A') \to Y$, whereupon the existence part for f gives us the extension K'', the valuation ring A'' and the arrow $\mathrm{Spec}(A'') \to X$. □

67.42. Valuative criterion for universal closedness

03K9 The existence part of the valuative criterion implies universal closedness for quasi-compact morphisms, see Lemma 67.42.1. In the case of schemes, this is an "if and

only if" statement, but for morphisms of algebraic spaces this is wrong. Example 67.9.6 shows that $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is universally closed, but it is easy to see that the existence part of the valuative criterion fails. We revisit this topic in Decent Spaces, Section 68.16 and show the converse holds if the source of the morphism is a decent space (see also Decent Spaces, Lemma 68.17.11 for a relative version).

03KA **Lemma 67.42.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 67.8.4 and 67.41.7 properties (1) and (2) are preserved under any base change. By Lemma 67.9.5 we only have to show that $|T \times_Y X| \to |T|$ is closed, whenever T is an affine scheme over S mapping into Y. Hence it suffices to prove: If Y is an affine scheme, $f : X \to Y$ is quasi-compact and satisfies the existence part of the valuative criterion, then $f : |X| \to |Y|$ is closed. In this situation X is a quasi-compact algebraic space. By Properties of Spaces, Lemma 66.6.3 there exists an affine scheme U and a surjective étale morphism $\varphi : U \to X$. Let $T \subset |X|$ closed. The inverse image $\varphi^{-1}(T) \subset U$ is closed, and hence is the set of points of an affine closed subscheme $Z \subset U$. Thus, by Algebra, Lemma 10.41.5 we see that $f(T) = f(\varphi(|Z|)) \subset |Y|$ is closed if it is closed under specialization.

Let $y' \rightsquigarrow y$ be a specialization in Y with $y' \in f(T)$. Choose a point $x' \in T \subset |X|$ mapping to y' under f. We may represent x' by a morphism $\mathrm{Spec}(K) \to X$ for some field K. Thus we have the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow[x']{} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathcal{O}_{Y,y}) & \longrightarrow & Y, \end{array}$$

see Schemes, Section 26.13 for the existence of the left vertical map. Choose a valuation ring $A \subset K$ dominating the image of the ring map $\mathcal{O}_{Y,y} \to K$ (this is possible since the image is a local ring and not a field as $y' \neq y$, see Algebra, Lemma 10.50.2). By assumption there exists a field extension K'/K and a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ fitting into the commutative diagram. Since A' dominates A, and A dominates $\mathcal{O}_{Y,y}$ we see that the closed point of $\mathrm{Spec}(A')$ maps to a point $x \in X$ with $f(x) = y$ which is a specialization of x'. Hence $x \in T$ as T is closed, and hence $y \in f(T)$ as desired. □

The following lemma will be generalized in Decent Spaces, Lemma 68.17.11.

0A3X **Lemma 67.42.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is quasi-separated and universally closed, then f satisfies the existence part of the valuative criterion.*
(2) *If f is quasi-compact and quasi-separated, then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. If (1) is true then combined with Lemma 67.42.1 we obtain (2). Assume f is quasi-separated and universally closed. Assume given a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 67.41.1. A formal argument shows that the existence of the desired diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

follows from existence in the case of the morphism $X_A \to \mathrm{Spec}(A)$. Since being quasi-separated and universally closed are preserved by base change, the lemma follows from the result in the next paragraph.

Consider a solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & = & \mathrm{Spec}(A) \end{array}$$

where A is a valuation ring with field of fractions K. By Lemma 67.8.9 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$ is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |X|$ in the closure of x mapping to the closed point of $\mathrm{Spec}(A)$. By Lemma 67.16.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & X \end{array}$$

such that the closed point of $\mathrm{Spec}(A')$ maps to $x' \in |X|$. It follows that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. □

0A3Y **Lemma 67.42.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and separated. Then the following are equivalent*

(1) *f is universally closed,*
(2) *the existence part of the valuative criterion holds as in Definition 67.41.1, and*

(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Since f is separated parts (2) and (3) are equivalent by Lemma 67.41.5. The equivalence of (3) and (1) follows from Lemma 67.42.2. □

089F **Lemma 67.42.4.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $\mathrm{Spec}(A) \to Y$ be a morphism where A is a valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|Y|$ in the image of $|X| \to |Y|$, then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $A \to A'$ is an extension of valuation rings (More on Algebra, Definition 15.124.1).

Proof. The base change $X_A \to \mathrm{Spec}(A)$ is flat (Lemma 67.30.4) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|X_A| \to |\mathrm{Spec}(A)|$ (Properties of Spaces, Lemma 66.4.3). Thus we may assume $Y = \mathrm{Spec}(A)$. Let $U \to X$ be a surjective étale morphism where U is a scheme. Let $u \in U$ map to the closed point of $\mathrm{Spec}(A)$. Consider the flat local ring map $A \to B = \mathcal{O}_{U,u}$. By Algebra, Lemma 10.39.16 there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{q}$ lies over $(0) \subset A$. By Algebra, Lemma 10.50.2 we can find a valuation ring $A' \subset \kappa(\mathfrak{q})$ dominating $B/\mathfrak{q}$. The induced morphism $\mathrm{Spec}(A') \to U \to X$ is a solution to the problem posed by the lemma. □

089G **Lemma 67.42.5.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. If*

(1) *f and h are quasi-compact,*
(2) *$|h|(|U|)$ is dense in $|X|$, and*

given any commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a valuation ring with field of fractions K

(3) *there exists at most one dotted arrow making the diagram commute, and*

(4) *there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes*

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

then f is universally closed. If moreover

(5) *f is quasi-separated*

then f is separated and universally closed.

Proof. Assume (1), (2), (3), and (4). We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Lemma 67.42.1. To do this, consider a commutative diagram

089H (67.42.5.1)
$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Properties of Spaces, Lemma 66.12.4. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 67.16.4.

Reduction to the case Y affine. Choose an étale morphism $\mathrm{Spec}(R) \to Y$ such that the closed point of $\mathrm{Spec}(A)$ maps to an element of $\mathrm{Im}(|\mathrm{Spec}(R)| \to |Y|)$. By Lemma 67.42.4 we can find a local ring map $A \to A'$ of valuation rings and a morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(R)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

Since in Definition 67.41.1 we allow for extensions of valuation rings it is clear that we may replace A by A', Y by $\mathrm{Spec}(R)$, X by $X \times_Y \mathrm{Spec}(R)$ and U by $U \times_Y \mathrm{Spec}(R)$.

From now on we assume that $Y = \mathrm{Spec}(R)$ is an affine scheme. Let $\mathrm{Spec}(B) \to X$ be an étale morphism from an affine scheme such that the morphism $\mathrm{Spec}(K) \to X$ is in the image of $|\mathrm{Spec}(B)| \to |X|$. Since we may replace K by an extension $K' \supset K$ and A by a valuation ring $A' \subset K'$ dominating A (which exists by Algebra, Lemma 10.50.2), we may assume the morphism $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(B)$ (by definition of $|X|$). In other words, we may think of K as a B-algebra. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat as a composition $\mathrm{Spec}(P) \to \mathrm{Spec}(B) \to X$. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma

67.30.12. By Lemma 67.16.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.50.10. As C is an R-algebra with fraction field K', we obtain a solid commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K_1') & \dashrightarrow & \mathrm{Spec}(K') & \longrightarrow & U & \longrightarrow & X \\ \vdots & & \downarrow & & & & \downarrow \\ \mathrm{Spec}(C_1) & \dashrightarrow & \mathrm{Spec}(C) & & \longrightarrow & & Y \end{array}$$

as in the statement of the lemma. Thus assumption (4) produces $C \to C_1$ and the dotted arrows making the diagram commute. Let $A_1' = (C_1)_{\mathfrak{p}}$ be the localization of C_1 at a prime $\mathfrak{p} \subset C_1$ lying over $\mathfrak{m}_{A'} \subset C$. Since $C \to C_1$ is flat by More on Algebra, Lemma 15.22.10 such a prime $\mathfrak{p}$ exists by Algebra, Lemmas 10.39.17 and 10.39.16. Note that A' is the localization of C at $\mathfrak{m}_{A'}$ and that A_1' is a valuation ring (Algebra, Lemma 10.50.9). In other words, $A' \to A_1'$ is a local ring map of valuation rings. Assumption (3) implies

$$\begin{array}{ccccc} \mathrm{Spec}(A_1') & \longrightarrow & \mathrm{Spec}(C_1) & \longrightarrow & X \\ \downarrow & & & & \uparrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) & \longrightarrow & \mathrm{Spec}(B) \end{array}$$

commutes. Hence the restriction of the morphism $\mathrm{Spec}(C_1) \to X$ to $\mathrm{Spec}(C_1/\mathfrak{p})$ restricts to the composition

$$\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(A'/\mathfrak{m}_{A'}) = \mathrm{Spec}(K'') \to \mathrm{Spec}(K) \to X$$

on the generic point of $\mathrm{Spec}(C_1/\mathfrak{p})$. Moreover, $C_1/\mathfrak{p}$ is a valuation ring (Algebra, Lemma 10.50.9) dominating A'' which dominates A. Thus the morphism $\mathrm{Spec}(C_1/\mathfrak{p}) \to X$ witnesses the existence part of the valuative criterion for the diagram (67.42.5.1) as desired.

Next, suppose that (5) is satisfied as well, i.e., the morphism $\Delta : X \to X \times_S X$ is quasi-compact. In this case assumptions (1) – (4) hold for h and Δ. Hence the first part of the proof shows that Δ is universally closed. By Lemma 67.40.9 we conclude that f is separated. □

67.43. Valuative criterion of separatedness

03KT First we prove a converse and then we state the criterion.

03KU **Lemma 67.43.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 67.41.1 be given. Suppose there are two distinct morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. Then $Z = \mathrm{Spec}(A) \times_{(a,b),X\times_Y X,\Delta} X$. If f is separated, then Δ is a closed immersion, and this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. □

03KV **Lemma 67.43.2** (Valuative criterion separatedness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. Assumption (1) means $\Delta_{X/Y}$ is quasi-compact. We claim the morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \times_Y X \end{array}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over Y. By assumption (2) we see that $a = b$. Hence using a as the dotted arrow works. Hence Lemma 67.42.1 applies, and we see that $\Delta_{X/Y}$ is universally closed. Since always $\Delta_{X/Y}$ is locally of finite type and separated, we conclude from More on Morphisms, Lemma 37.44.1 that $\Delta_{X/Y}$ is a finite morphism (also, use the general principle of Spaces, Lemma 65.5.8). At this point $\Delta_{X/Y}$ is a representable, finite monomorphism, hence a closed immersion by Morphisms, Lemma 29.44.15. □

0A3Z **Lemma 67.43.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and quasi-separated. Then the following are equivalent*

(1) *f is separated and universally closed,*
(2) *the valuative criterion holds as in Definition 67.41.1,*
(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Since f is quasi-separated, the uniqueness part of the valutative criterion implies f is separated (Lemma 67.43.2). Conversely, if f is separated, then it satisfies the uniqueness part of the valuative criterion (Lemma 67.43.1). Having said this, we see that in each of the three cases the morphism f is separated and satisfies the uniqueness part of the valuative criterion. In this case the lemma is a formal consequence of Lemma 67.42.3. □

67.44. Valuative criterion of properness

0CKZ Here is a statement.

0A40 **Lemma 67.44.1** (Valuative criterion for properness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is of finite type and quasi-separated. Then the following are equivalent*

(1) *f is proper,*
(2) *the valuative criterion holds as in Definition 67.41.1,*
(3) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 26.20.3.

Proof. Formal consequence of Lemma 67.43.3 and the definitions. □

67.45. Integral and finite morphisms

03ZN We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be integral (resp. finite).

03ZO **Lemma 67.45.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is integral, resp. finite (in the sense of Section 67.3), if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine and integral, resp. finite, over Z.*

Proof. This follows directly from the definition of an integral (resp. finite) morphism of schemes (Morphisms, Definition 29.44.1). □

This clears the way for the following definition.

03ZP **Definition 67.45.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say that f is *integral* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme integral over Z.
(2) We say that f is *finite* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme finite over Z.

03ZQ **Lemma 67.45.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and integral (resp. finite),*
(2) *f is integral (resp. finite),*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is integral (resp. finite), and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is integral (resp. finite).*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of integral (resp. finite) morphisms satisfies all the required properties (see Morphisms, Lemmas 29.44.6 and Descent, Lemmas 35.23.24, 35.23.25, and 35.37.1). The conclusion of applying this lemma is that f is representable and integral (resp. finite), i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being integral (resp. finite) is Zariski local on the target (the reference above shows that being integral or finite is in fact fpqc local on the target). □

03ZR **Lemma 67.45.4.** *The composition of integral (resp. finite) morphisms is integral (resp. finite).*

Proof. Omitted. □

03ZS **Lemma 67.45.5.** *The base change of an integral (resp. finite) morphism is integral (resp. finite).*

Proof. Omitted. □

0414 **Lemma 67.45.6.** *A finite morphism of algebraic spaces is integral. An integral morphism of algebraic spaces which is locally of finite type is finite.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 67.45.3. Hence this lemma follows from the same lemma for the case of schemes, see Morphisms, Lemma 29.44.4. □

0415 **Lemma 67.45.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 67.45.3, 67.20.3, and 67.9.5. Hence the result follows from Morphisms, Lemma 29.44.7. □

04NY **Lemma 67.45.8.** *A finite morphism of algebraic spaces is quasi-finite.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. By Definition 67.45.2 and Lemmas 67.8.8 and 67.27.6 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 29.44.10. □

04NZ **Lemma 67.45.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is finite, and*
(2) *f is affine and proper.*

Proof. In both cases the morphism is representable, and you can check the condition after base change to an affine scheme mapping into Y, see Lemmas 67.45.3, 67.20.3, and 67.40.2. Hence the result follows from Morphisms, Lemma 29.44.11. □

081Y **Lemma 67.45.10.** *A closed immersion is finite (and a fortiori integral).*

Proof. Omitted. □

0CZ2 **Lemma 67.45.11.** *Let S be a scheme. Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of algebraic spaces over S. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.*

Proof. Follows from the case of schemes (Morphisms, Lemma 29.44.13) by étale localization. □

081Z **Lemma 67.45.12.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 67.45.5. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Lemma 67.4.7. A closed immersion is finite (resp. integral), see Lemma 67.45.10. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 67.45.4. Thus we win. □

67.46. Finite locally free morphisms

03ZT We have already defined in Section 67.3 what it means for a representable morphism of algebraic spaces to be finite locally free.

03ZU **Lemma 67.46.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is finite locally free (in the sense of Section 67.3) if and only if f is affine and the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Assume f is finite locally free (as defined in Section 67.3). This means that for every morphism $V \to Y$ whose source is a scheme the base change $f' : V \times_Y X \to V$ is a finite locally free morphism of schemes. This in turn means (by the definition of a finite locally free morphism of schemes) that $f'_*\mathcal{O}_{V \times_Y X}$ is a finite locally free $\mathcal{O}_V$-module. We may choose $V \to Y$ to be surjective and étale. By Properties of Spaces, Lemma 66.26.2 we conclude the restriction of $f_*\mathcal{O}_X$ to V is

finite locally free. Hence by Modules on Sites, Lemma 18.23.3 applied to the sheaf $f_*\mathcal{O}_X$ on $Y_{spaces,étale}$ we conclude that $f_*\mathcal{O}_X$ is finite locally free.

Conversely, assume f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. Again by Properties of Spaces, Lemma 66.26.2 we see that $f'_*\mathcal{O}_{V\times_Y X}$ is finite locally free. Hence $f' : V \times_Y X \to V$ is finite locally free (as it is also affine). By Spaces, Lemma 65.11.5 we conclude that f is finite locally free (use Morphisms, Lemma 29.48.4 Descent, Lemmas 35.23.32 and 35.37.1). Thus we win. □

This clears the way for the following definition.

03ZV **Definition 67.46.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of rank d.

03ZW **Lemma 67.46.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and finite locally free,*
(2) *f is finite locally free,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is finite locally free, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ is finite locally free.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 66.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 66.13.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 65.11.5 because the class of finite locally free morphisms satisfies all the required properties (see Morphisms, Lemma 29.48.4 Descent, Lemmas 35.23.32 and 35.37.1). The conclusion of applying this lemma is that f is representable and finite locally free, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being finite locally free is Zariski local on the target (the reference above shows that being finite locally free is in fact fpqc local on the target). □

03ZX **Lemma 67.46.4.** *The composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

03ZY **Lemma 67.46.5.** *The base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

0416 **Lemma 67.46.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If Y is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. In each of the three cases the morphism is representable and you can check the property after base change by a surjective étale morphism $V \to Y$, see Lemmas 67.45.3, 67.46.3, 67.30.5, and 67.28.4. If Y is locally Noetherian, then V is locally Noetherian. Hence the result follows from the corresponding result in the schemes case, see Morphisms, Lemma 29.48.2. □

67.47. Rational maps

0EML This section is the analogue of Morphisms, Section 29.49. We will use without further mention that the intersection of dense opens of a topological space is a dense open.

0EMM **Definition 67.47.1.** Let S be a scheme. Let X, Y be algebraic spaces over S.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of algebraic spaces over S defined on dense open subspaces U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some dense open subspace $W \subset U \cap V$.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) Given morphisms $X \to B$ and $Y \to B$ of algebraic spaces over S we say that a rational map from X to Y is a *B-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is a morphism over B.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In many cases we will consider in the future, the algebraic spaces X and Y will contain a dense open subspaces X' and Y' which are schemes. In that case a rational map from X to Y is the same as an S-rational map from X' to Y' in the sense of Morphisms, Definition 67.47.1. Then all of the theory developed for schemes can be brought to bear.

0EMN **Definition 67.47.2.** Let S be a scheme. Let X be an algebraic space over S. A *rational function on X* is a rational map from X to $\mathbf{A}^1_S$.

Looking at the discussion following Morphisms, Definition 29.49.3 we find that this is the same as the notion defined there in case X happens to be a scheme.

Recall that we have the canonical identification

$$\mathrm{Mor}_S(T, \mathbf{A}^1_S) = \mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$$

for any scheme T over S, see Schemes, Example 26.15.2. Hence $\mathbf{A}^1_S$ is a ring-object in the category of schemes over S. In other words, addition and multiplication define morphisms

$$+ : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S \quad \text{and} \quad * : \mathbf{A}^1_S \times_S \mathbf{A}^1_S \to \mathbf{A}^1_S$$

satisfying the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_S$ has a natural ring structure.

0EMP **Definition 67.47.3.** Let S be a scheme. Let X be an algebraic space over S. The *ring of rational functions on X* is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

We will define function fields for integral algebraic spaces later, see Spaces over Fields, Section 72.4.

0EMQ **Definition 67.47.4.** Let S be a scheme. Let φ be a rational map between two algebraic spaces X and Y over S. We say φ is *defined in a point* $x \in |X|$ if there exists a representative (U, f) of φ with $x \in |U|$. The *domain of definition* of φ is the set of all points where φ is defined.

The domain of definition is viewed as an open subspace of X via Properties of Spaces, Lemma 66.4.8. With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

0EMR **Lemma 67.47.5.** *Let S be a scheme. Let X and Y be algebraic spaces over S. Assume X is reduced and Y is separated over S. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ of algebraic spaces representing φ.*

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subspace $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V \cap V'}$ and $g'|_{V \cap V'}$ is a closed subspace of $V \cap V'$ because it is the base change of $\Delta : Y \to Y \times_S Y$ by the morphism $V \cap V' \to Y \times_S Y$ given by $g|_{V \cap V'}$ and $g'|_{V \cap V'}$. Now $W \subset E$ implies that $|E| = |V \cap V'|$. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V \cap V'} = g'|_{V \cap V'}$, see Properties of Spaces, Lemma 66.12.4. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$ because $\coprod V \to U$ is a surjection of fppf sheaves and $\coprod_{V,V'} V \cap V' = (\coprod V) \times_U (\coprod V)$. □

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our spaces are irreducible and we look at dominant rational maps, then we can compose rational maps.

0EMS **Definition 67.47.6.** Let S be a scheme. Let X and Y be algebraic spaces over S. Assume $|X|$ and $|Y|$ are irreducible. A rational map from X to Y is called *dominant* if any representative $f : U \to Y$ is a dominant morphism in the sense of Definition 67.18.1.

We can compose a dominant rational map φ between irreducible algebraic spaces X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $|U| \subset |X|$ open dense and $g : V \to Z$ with $|V| \subset |Y|$ open dense. Then $W = |f|^{-1}(V) \subset |X|$ is open nonempty (because the image of $|f|$ is dense and hence must meet the nonempty open V) and hence dense as $|X|$ is irreducible. We define $\psi \circ \varphi$ as the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible algebraic spaces over S and whose morphisms are dominant rational maps.

0EMT **Definition 67.47.7.** Let S be a scheme. Let X and Y be algebraic spaces over S with $|X|$ and $|Y|$ irreducible. We say X and Y are *birational* if X and Y are isomorphic in the category of irreducible algebraic spaces over S and dominant rational maps.

If X and Y are birational irreducible algebraic spaces, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all algebraic spaces Z (functorially in Z). For "general" irreducible algebraic spaces this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions; sometimes these two notions are equivalent (insert future reference here).

0EMU **Lemma 67.47.8.** *Let S be a scheme. Let X and Y be algebraic space over S with $|X|$ and $|Y|$ irreducible. Then X and Y are birational if and only if there are nonempty open subspaces $U \subset X$ and $V \subset Y$ which are isomorphic as algebraic spaces over S.*

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. After shrinking U we may assume $f : U \to Y$ factors through V. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller open we may assume that $U \to V \to X$ is the inclusion of U into X. By symmetry we find there exists an open subspace $V' \subset V$ such that $g|_{V'} : V' \to X$ factors through $U \subset X$ and such that $V' \to U \to Y$ is the identity. The inverse image of $|V'|$ by $|U| \to |V|$ is an open of $|U|$ and hence equal to $|U'|$ for some open subspace $U' \subset U$, see Properties of Spaces, Lemma 66.4.8. Then $U' \subset U \to V$ factors as $U' \to V'$. Similarly $V' \to U$ factors as $V' \to U'$. The reader finds that $U' \to V'$ and $V' \to U'$ are mutually inverse morphisms of algebraic spaces over S and the proof is complete. □

67.48. Relative normalization of algebraic spaces

0BAZ This section is the analogue of Morphisms, Section 29.53.

0820 **Lemma 67.48.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. There exists a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}' \subset \mathcal{A}$ such that for any affine object U of $X_{étale}$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.*

Proof. Let U be an object of $X_{étale}$. Then U is a scheme. Denote $\mathcal{A}|_U$ the restriction to the Zariski site. Then $\mathcal{A}|_U$ is a quasi-coherent sheaf of $\mathcal{O}_U$-algebras hence we can apply Morphisms, Lemma 29.53.1 to find a quasi-coherent subalgebra $\mathcal{A}'_U \subset \mathcal{A}|_U$ such that the value of $\mathcal{A}'_U$ on any affine open $W \subset U$ is as given in the statement of the lemma. If $f : U' \to U$ is a morphism in $X_{étale}$, then $\mathcal{A}|_{U'} = f^*(\mathcal{A}|_U)$ where f^* means pullback by the morphism f in the Zariski topology; this holds because $\mathcal{A}$ is quasi-coherent (see introduction to Properties of Spaces, Section 66.29 and the references to the discussion in the chapter on descent on schemes). Since f is étale we find that More on Morphisms, Lemma 37.19.1 says that we get a canonical isomorphism $f^*(\mathcal{A}'_U) = \mathcal{A}'_{U'}$. This immediately tells us that we obtain a sub presheaf $\mathcal{A}' \subset \mathcal{A}$ of $\mathcal{O}_X$-algebras over $X_{étale}$ which is a sheaf for the Zariski topology and has the right values on affine objects. But the fact that each $\mathcal{A}'_U$ is quasi-coherent on the scheme U and that for $f : U' \to U$ étale we have $\mathcal{A}'_{U'} = f^*(\mathcal{A}'_U)$ implies that $\mathcal{A}'$ is quasi-coherent on $X_{étale}$ as well (as this is a local property and we have the references above describing quasi-coherent modules on $U_{étale}$ in exactly this manner). □

0821 **Definition 67.48.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 67.48.1 above.

We will apply this in particular when $\mathcal{A} = f_*\mathcal{O}_Y$ for a quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ (see Lemma 67.11.2). We can then take the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra (Lemma 67.20.7) to obtain the normalization of X in Y.

0822 **Definition 67.48.3.** Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the morphism of algebraic spaces

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over S. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

To get the factorization, use Remark 67.20.9 and functoriality of the $\underline{\mathrm{Spec}}$ construction.

0ABP **Lemma 67.48.4.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $Y \to X' \to X$ be the normalization of X in Y.*

(1) *If $W \to X$ is an étale morphism of algebraic spaces over S, then $W \times_X X'$ is the normalization of W in $W \times_X Y$.*
(2) *If Y and X are representable, then X' is representable and is canonically isomorphic to the normalization of the scheme X in the scheme Y as constructed in Morphisms, Section 29.54.*

Proof. It is immediate from the construction that the formation of the normalization of X in Y commutes with étale base change, i.e., part (1) holds. On the other hand, if X and Y are schemes, then for $U \subset X$ affine open, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U))$ and hence $\nu^{-1}(U)$ is the spectrum of exactly the same ring as we get in the corresponding construction for schemes. □

Here is a characterization of this construction.

0823 **Lemma 67.48.5.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*
(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f'}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\nu} & X \end{array}$$

for a unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 67.48.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 67.45.2 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine U étale over X (by Definition 67.45.2) we see from Lemma 67.48.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 67.20.7 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 67.45.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

0AYF **Lemma 67.48.6.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.*

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. □

0AYG **Lemma 67.48.7.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If $x' \in |X'|$ is a point of codimension 0 (Properties of Spaces, Definition 66.10.2), then x' is the image of some $y \in |Y|$ of codimension 0.*

Proof. By Lemma 67.48.4 and the definitions, we may assume that $X = \mathrm{Spec}(A)$ is affine. Then $X' = \mathrm{Spec}(A')$ where A' is the integral closure of A in $\Gamma(Y, \mathcal{O}_Y)$ and x' corresponds to a minimal prime of A'. Choose a surjective étale morphism $V \to Y$ where $V = \mathrm{Spec}(B)$ is affine. Then $A' \to B$ is injective, hence every minimal prime of A' is the image of a minimal prime of B, see Algebra, Lemma 10.30.5. The lemma follows. □

0824 **Lemma 67.48.8.** *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two algebraic spaces. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.*

Proof. Omitted. □

0A0Q **Lemma 67.48.9.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact, quasi-separated and universally closed morphisms of algebraic spaces over S. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$. In other words, the normalization of Y in X is equal to the factorization*

$$X \longrightarrow \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X) \longrightarrow Y$$

of Remark 67.20.9.

Proof. The question is étale local on Y, hence we may reduce to the case where $Y = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & \mathbf{A}^1_Y \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & Y & \end{array}$$

Let $Z \subset \mathbf{A}^1_Y$ be the scheme theoretic image of h, see Definition 67.16.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_Y \to Y$ is separated, see Lemma 67.8.9. By Lemma 67.16.3 the morphism $X \to Z$ has dense image on underlying topological spaces. By Lemma 67.40.6 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 67.40.7 to conclude that $Z \to Y$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_Y$, we see that $Z \to Y$ is affine and proper, hence integral by Lemma 67.45.7. Writing $\mathbf{A}^1_Y = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

0825 **Lemma 67.48.10.** *Let S be a scheme. Let $f : Y \to X$ be an integral morphism of algebraic spaces over S. Then the integral closure of X in Y is equal to Y.*

Proof. By Lemma 67.45.7 this is a special case of Lemma 67.48.9. □

0BB0 **Lemma 67.48.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *Y is Nagata,*
(2) *f is quasi-separated of finite type,*
(3) *X is reduced.*

Then the normalization $\nu : Y' \to Y$ of Y in X is finite.

Proof. The question is étale local on Y, see Lemma 67.48.4. Thus we may assume $Y = \mathrm{Spec}(R)$ is affine. Then R is a Noetherian Nagata ring and we have to show that the integral closure of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact we see that X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Then $\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_X)$. Since R is Noetherian it suffices to show that the integral closure of R in $\Gamma(U, \mathcal{O}_U)$ is finite over R. As $U \to Y$ is of finite type this follows from Morphisms, Lemma 29.53.15. □

67.49. Normalization

07U3 This section is the analogue of Morphisms, Section 29.54.

0BB1 **Lemma 67.49.1.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *there is a surjective étale morphism $U \to X$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,*
(2) *for every scheme U and every étale morphism $U \to X$ every quasi-compact open of U has finitely many irreducible components,*

(3) *for every quasi-compact algebraic space Y étale over X the set of codimension 0 points of Y (Properties of Spaces, Definition 66.10.2) is finite, and*
(4) *for every quasi-compact algebraic space Y étale over X the space $|Y|$ has finitely many irreducible components.*

If X is representable this means that every quasi-compact open of X has finitely many irreducible components.

Proof. The equivalence of (1) and (2) and the final statement follow from Descent, Lemma 35.16.3 and Properties of Spaces, Lemma 66.7.1. It is clear that (4) implies (1) and (2) by considering only those Y which are schemes. Similarly, (3) implies (1) and (2) since for a scheme the codimension 0 points are the generic points of its irreducible components, see for example Properties of Spaces, Lemma 66.11.1.

Conversely, assume (2) and let $Y \to X$ be an étale morphism of algebraic spaces with Y quasi-compact. Then we can choose an affine scheme V and a surjective étale morphism $V \to Y$ (Properties of Spaces, Lemma 66.6.3). Since V has finitely many irreducible components by (2) and since $|V| \to |Y|$ is surjective and continuous, we conclude that $|Y|$ has finitely many irreducible components by Topology, Lemma 5.8.5. Thus (4) holds. Similarly, by Properties of Spaces, Lemma 66.11.1 the images of the generic points of the irreducible components of V are the codimension 0 points of Y and we conclude that there are finitely many, i.e., (3) holds. □

0GMB **Lemma 67.49.2.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Then X satisfies the equivalent conditions of Lemma 67.49.1.*

Proof. If $U \to X$ is étale and U is a scheme, then U is a locally Noetherian scheme, see Properties of Spaces, Section 66.7. A locally Noetherian scheme has a locally finite set of irreducible components (Divisors, Lemma 31.26.1). Thus we conclude that X passes condition (2) of the lemma. □

0GMC **Lemma 67.49.3.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Then for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.*

Proof. Via Properties of Spaces, Lemma 66.11.1 and étale localization this translates into the case of a morphism of schemes and generic points of irreducible components. Here the result follows as generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 29.25.9. □

0GMD **Lemma 67.49.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is flat and locally of finite type and assume Y satisfies the equivalent conditions of Lemma 67.49.1. Then X satisfies the equivalent conditions of Lemma 67.49.1 and for $x \in |X|$ we have: x has codimension 0 in $X \Rightarrow f(x)$ has codimension 0 in Y.*

Proof. The last statement follows from Lemma 67.49.3. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. It suffices to show that every quasi-compact open of U has finitely many irreducible components. We will use the results of Properties of Spaces, Lemma 66.11.1 without further mention. By what we've already shown, the codimension 0 points of U lie above codimension 0 points in U and these are locally finite by assumption. Hence it suffices to show that for $v \in V$

of codimension 0 the codimension 0 points of the scheme theoretic fibre $U_v = U \times_V v$ are locally finite. This is true because U_v is a scheme locally of finite type over $\kappa(v)$, hence locally Noetherian and we can apply Lemma 67.49.2 for example. □

07U4 **Lemma 67.49.5.** *Let S be a scheme. For every algebraic space X over S satisfying the equivalent conditions of Lemma 67.49.1 there exists a morphism of algebraic spaces*

$$\nu_X : X^\nu \longrightarrow X$$

with the following properties

(1) *if X satisfies the equivalent conditions of Lemma 67.49.1 then X^ν is normal and ν_X is integral,*
(2) *if X is a scheme such that every quasi-compact open has finitely many irreducible components, then $\nu_X : X^\nu \to X$ is the normalization of X constructed in Morphisms, Section 29.54,*
(3) *if $f : X \to Y$ is a morphism of algebraic spaces over S which both satisfy the equivalent conditions of Lemma 67.49.1 and every codimension 0 point of X is mapped by f to a codimension 0 point of Y, then there is a unique morphism $f^\nu : X^\nu \to Y^\nu$ of algebraic spaces over S such that $\nu_Y \circ f^\nu = f \circ \nu_X$, and*
(4) *if $f : X \to Y$ is an étale or smooth morphism of algebraic spaces and Y satisfies the equivalent conditions of Lemma 67.49.1, then the hypotheses of (3) hold and the morphism f^ν induces an isomorphism $X^\nu \to X \times_Y Y^\nu$.*

Proof. Consider the category $\mathcal{C}$ whose objects are the schemes U over S such that every quasi-compact open of U has finitely many irreducible components and whose morphisms are those morphisms $g : U \to V$ of schemes over S such that every generic point of an irreducible component of U is mapped to the generic point of an irreducible component of V. We have already shown that

(a) for $U \in \mathrm{Ob}(\mathcal{C})$ we have a normalization morphism $\nu_U : U^\nu \to U$ as in Morphisms, Definition 29.54.1,
(b) for $U \in \mathrm{Ob}(\mathcal{C})$ the morphism ν_U is integral and U^ν is a normal scheme, see Morphisms, Lemma 29.54.5,
(c) for every $g : U \to V \in \mathrm{Arrows}(\mathcal{C})$ there is a unique morphism $g^\nu : U^\nu \to V^\nu$ such that $\nu_V \circ g^\nu = g \circ \nu_U$, see Morphisms, Lemma 29.54.5 part (4) applied to the composition $X^\nu \to X \to Y$,
(d) if $V \in \mathrm{Ob}(\mathcal{C})$ and $g : U \to V$ is étale or smooth, then $U \in \mathrm{Ob}(\mathcal{C})$ and $g \in \mathrm{Arrows}(\mathcal{C})$ and the morphism g^ν induces an isomorphism $U^\nu \to U \times_V V^\nu$, see Lemma 67.49.4 and More on Morphisms, Lemma 37.19.3.

Our task is to extend this construction to the corresponding category of algebraic spaces X over S.

Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. Let $U \to X$ be a surjective étale morphism where U is a scheme. Set $R = U \times_X U$ with projections $s, t : R \to U$ and $j = (t, s) : R \to U \times_S U$ so that $X = U/R$, see Spaces, Lemma 65.9.1. Observe that U and R are objects of $\mathcal{C}$ by our assumptions on X and that the morphisms s and t are étale morphisms of schemes over S. By (a) we have the normalization morphisms $\nu_U : U^\nu \to U$ and $\nu_R : R^\nu \to R$, by (d) we have morphisms $s^\nu : R^\nu \to U^\nu$, $t^\nu : R^\nu \to U^\nu$ which define isomorphisms $R^\nu \to R \times_{s,U} U^\nu$ and $R^\nu \to U^\nu \times_{U,t} R$. It follows that s^ν and t^ν are

étale (as they are isomorphic to base changes of étale morphisms). The induced morphism $j^\nu = (t^\nu, s^\nu) : R^\nu \to U^\nu \times_S U^\nu$ is a monomorphism as it is equal to the composition

$$\begin{aligned} R^\nu &\to (U^\nu \times_{U,t} R) \times_R (R \times_{s,U} U^\nu) \\ &= U^\nu \times_{U,t} R \times_{s,U} U^\nu \\ &\xrightarrow{j} U^\nu \times_U (U \times_S U) \times_U U^\nu \\ &= U^\nu \times_S U^\nu \end{aligned}$$

The first arrow is the diagonal morphism of ν_R. (This tells us that R^ν is a subscheme of the restriction of R to U^ν.) A formal computation with fibre products using property (d) shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the étale morphism $c : R \times_{s,U,t} R \to R$ extends uniquely to c^ν by (d). The morphism c^ν is compatible with the projection $\mathrm{pr}_{13} : U^\nu \times_S U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$. Similarly, there are morphisms $i^\nu : R^\nu \to R^\nu$ compatible with the morphism $U^\nu \times_S U^\nu \to U^\nu \times_S U^\nu$ which switches factors and there is a morphism $e^\nu : U^\nu \to R^\nu$ compatible with the diagonal morphism $U^\nu \to U^\nu \times_S U^\nu$. All in all it follows that $j^\nu : R^\nu \to U^\nu \times_S U^\nu$ is an étale equivalence relation. At this point we may and do set $X^\nu = U^\nu / R^\nu$ (Spaces, Theorem 65.10.5). Then we see that we have $U^\nu = X^\nu \times_X U$ by Groupoids, Lemma 39.20.7.

What have we shown in the previous paragraph is this: for every algebraic space X over S satisfying the equivalent conditions of Lemma 67.49.1 if we choose a surjective étale morphism $g : U \to X$ where U is a scheme, then we obtain a cartesian diagram

$$\begin{array}{ccc} X^\nu & \xleftarrow[g^\nu]{} & U^\nu \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_U} \\ X & \xleftarrow{g} & U \end{array}$$

of algebraic spaces. This immediately implies that X^ν is a normal algebraic space and that ν_X is an integral morphism. This gives part (1) of the lemma.

We will show below that the morphism $\nu_X : X^\nu \to X$ up to unique isomorphism is independent of the choice of g, but for now, if X is a scheme, we choose $\mathrm{id} : X \to X$ so that it is clear that we have part (2) of the lemma.

We still have to prove parts (3) and (4). Let $g : U \to X$ and $\nu_X : X^\nu \to X$ and $g^\nu : U^\nu \to X^\nu$ be as above. Let Z be a normal scheme and let $h : Z \to U$ and $a : Z \to X^\nu$ be morphisms over S such that $g \circ h = \nu_X \circ a$ and such that every irreducible component of Z dominates an irreducible component of U (via h). By Morphisms, Lemma 29.54.5 part (4) we obtain a unique morphism $h^\nu : Z \to U^\nu$ such that $h = \nu_U \circ h^\nu$. Picture:

$$\begin{array}{ccccc} & & \overset{a}{\frown} & & \\ X^\nu & \xleftarrow[g^\nu]{} & U^\nu & \xleftarrow{h^\nu} & Z \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_U} & \swarrow{\scriptstyle h} & \\ X & \xleftarrow{g} & U & & \end{array}$$

Observe that $a = g^\nu \circ h^\nu$. Namely, since the square with corners X^ν, X, U^ν, U is cartesian, this follows immediately from the fact that h^ν is unique (given h). In

other words, given $h : Z \to U$ as above (and not a) there is a unique morphism $a : Z \to X^\nu$ with $\nu_X \circ a = g \circ h$.

Let $f : X \to Y$ be as in part (3) of the statement of the lemma. Suppose we have chosen surjective étale morphisms $U \to X$ and $V \to Y$ where U and V are schemes such that f lifts to a morphism $g : U \to V$. Then $g \in \text{Arrows}(\mathcal{C})$ and we obtain a unique morphism $g^\nu : U^\nu \to V^\nu$ compatible with ν_U and ν_V. However, then the two morphisms

$$R^\nu = U^\nu \times_{X^\nu} U^\nu \to U^\nu \to V^\nu \to Y^\nu$$

must be the same by our comments in the previous paragraph (applied with Y in stead of X). Since X^ν is constructed by taking the quotient of U^ν by R^ν it follows that we obtain a (unique) morphism $f^\nu : X^\nu \to Y^\nu$ as stated in (3).

To see that the construction of X^ν is independent of the choice of $g : U \to X$ surjective étale, apply the construction in the previous paragraph to $\text{id} : X \to X$ and a morphism $U' \to U$ between étale coverings of X. This is enough because given any two étale coverings of X there is a third one which dominates both. The reader shows that the morphism between the two normalizations constructed using either $U' \to X$ or $U \to X$ becomes an isomorphism after base change to U' and hence was an isomorphism. We omit the details.

We omit the proof of (4) which is similar; hint use part (d) above. □

This leads us to the following definition.

0BB2 **Definition 67.49.6.** Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. We define the *normalization* of X as the morphism

$$\nu_X : X^\nu \longrightarrow X$$

constructed in Lemma 67.49.5.

The definition applies to locally Noetherian algebraic spaces, see Lemma 67.49.2. Usually the normalization is defined only for reduced algebraic spaces. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

0BB3 **Lemma 67.49.7.** *Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. We may check this étale locally on X and hence reduce to the case of schemes which is Morphisms, Lemma 29.54.2. Some details omitted. □

0BB4 **Lemma 67.49.8.** *Let S be a scheme. Let X be an algebraic space over S satisfying the equivalent conditions of Lemma 67.49.1.*

(1) *The normalization X^ν is normal.*
(2) *The morphism $\nu : X^\nu \to X$ is integral and surjective.*
(3) *The map $|\nu| : |X^\nu| \to |X|$ induces a bijection between the sets of points of codimension 0 (Properties of Spaces, Definition 66.10.2).*
(4) *Let $Z \to X$ be a morphism. Assume Z is a normal algebraic space and that for $z \in |Z|$ we have: z has codimension 0 in $Z \Rightarrow f(z)$ has codimension 0 in X. Then there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Properties (1), (2), and (3) follow from the corresponding results for schemes (Morphisms, Lemma 29.54.5) combined with the fact that a point of a scheme is a generic point of an irreducible component if and only if the dimension of the local ring is zero (Properties, Lemma 28.10.4).

Let $Z \to X$ be a morphism as in (4). Let U be a scheme and let $U \to X$ be a surjective étale morphism. Choose a scheme V and a surjective étale morphism $V \to U \times_X Z$. The condition on codimension 0 points assures us that $V \to U$ maps generic points of irreducible components of V to generic points of irreducible components of U. Thus we obtain a unique factorization $V \to U^\nu \to U$ by Morphisms, Lemma 29.54.5. The uniqueness guarantees us that the two maps

$$V \times_{U \times_X Z} V \to V \to U^\nu$$

agree because these maps are the unique factorization of the map $V \times_{U \times_X Z} V \to V \to U$. Since the algebraic space $U \times_X Z$ is equal to the quotient $V/V \times_{U \times_X Z} V$ (see Spaces, Section 65.9) we find a canonical morphism $U \times_X Z \to U^\nu$. Picture

$$\begin{array}{ccccc} U \times_X Z & \longrightarrow & U^\nu & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ Z & \dashrightarrow & X^\nu & \longrightarrow & X \end{array}$$

To obtain the dotted arrow we note that the construction of the arrow $U \times_X Z \to U^\nu$ is functorial in the étale morphism $U \to X$ (precise formulation and proof omitted). Hence if we set $R = U \times_X U$ with projections $s, t : R \to U$, then we obtain a morphism $R \times_X Z \to R^\nu$ commuting with $s, t : R \to U$ and $s^\nu, t^\nu : R^\nu \to U^\nu$. Recall that $X^\nu = U^\nu/R^\nu$, see proof of Lemma 67.49.5. Since $X = U/R$ a simple sheaf theoretic argument shows that $Z = (U \times_X Z)/(R \times_X Z)$. Thus the morphisms $U \times_X Z \to U^\nu$ and $R \times_X Z \to R^\nu$ define a morphism $Z \to X^\nu$ as desired. □

0BB5 **Lemma 67.49.9.** *Let S be a scheme. Let X be a Nagata algebraic space over S. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Since X being Nagata is locally Noetherian, Definition 67.49.6 applies. By construction of X^ν in Lemma 67.49.5 we immediately reduce to the case of schemes which is Morphisms, Lemma 29.54.11. □

67.50. Separated, locally quasi-finite morphisms

0417 In this section we prove that an algebraic space which is locally quasi-finite and separated over a scheme, is representable. This implies that a separated and locally quasi-finite morphism is representable (see Lemma 67.51.1). But first... a lemma (which will be obsoleted by Proposition 67.50.2).

03XW **Lemma 67.50.1.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccccc} V' & \longrightarrow & T' \times_T X & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & T' & \longrightarrow & T \end{array}$$

of algebraic spaces over S. Assume

(1) *$T' \to T$ is an étale morphism of affine schemes,*
(2) *$X \to T$ is a separated, locally quasi-finite morphism,*

(3) V' *is an open subspace of* $T' \times_T X$, *and*
(4) $V' \to T'$ *is quasi-affine.*

In this situation the image U *of* V' *in* X *is a quasi-compact open subspace of* X *which is representable.*

Proof. We first make some trivial observations. Note that V' is representable by Lemma 67.21.3. It is also quasi-compact (as a quasi-affine scheme over an affine scheme, see Morphisms, Lemma 29.13.2). Since $T' \times_T X \to X$ is étale (Properties of Spaces, Lemma 66.16.5) the map $|T' \times_T X| \to |X|$ is open, see Properties of Spaces, Lemma 66.16.7. Let $U \subset X$ be the open subspace corresponding to the image of $|V'|$, see Properties of Spaces, Lemma 66.4.8. As $|V'|$ is quasi-compact we see that $|U|$ is quasi-compact, hence U is a quasi-compact algebraic space, by Properties of Spaces, Lemma 66.5.2.

By Morphisms, Lemma 29.57.9 the morphism $T' \to T$ is universally bounded. Hence we can do induction on the integer n bounding the degree of the fibres of $T' \to T$, see Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $T' \to T$ is an open immersion (see Étale Morphisms, Theorem 41.14.1), and the result is clear. Assume $n > 1$.

Consider the affine scheme $T'' = T' \times_T T'$. As $T' \to T$ is étale we have a decomposition (into open and closed affine subschemes) $T'' = \Delta(T') \amalg T^*$. Namely $\Delta = \Delta_{T'/T}$ is open by Morphisms, Lemma 29.35.13 and closed because $T' \to T$ is separated as a morphism of affines. As a base change the degrees of the fibres of the second projection $\mathrm{pr}_1 : T' \times_T T' \to T'$ are bounded by n, see Morphisms, Lemma 29.57.5. On the other hand, $\mathrm{pr}_1|_{\Delta(T')} : \Delta(T') \to T'$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 29.57.8 we conclude the degrees of the fibres of the restriction $\mathrm{pr}_1|_{T^*} : T^* \to T'$ are bounded by $n - 1$. Hence the induction hypothesis applied to the diagram

$$\begin{array}{ccccc} p_0^{-1}(V') \cap X^* & \longrightarrow & X^* & \xrightarrow[p_1|_{X^*}]{} & X' \\ & \searrow & \downarrow & & \downarrow \\ & & T^* & \xrightarrow{\mathrm{pr}_1|_{T^*}} & T' \end{array}$$

gives that $p_1(p_0^{-1}(V') \cap X^*)$ is a quasi-compact scheme. Here we set $X'' = T'' \times_T X$, $X^* = T^* \times_T X$, and $X' = T' \times_T X$, and $p_0, p_1 : X'' \to X'$ are the base changes of $\mathrm{pr}_0, \mathrm{pr}_1$. Most of the hypotheses of the lemma imply by base change the corresponding hypothesis for the diagram above. For example $p_0^{-1}(V') = T'' \times_{T'} V'$ is a scheme quasi-affine over T'' as a base change. Some verifications omitted.

By Properties of Spaces, Lemma 66.13.1 we conclude that

$$p_1(p_0^{-1}(V')) = V' \cup p_1(p_0^{-1}(V') \cap X^*)$$

is a quasi-compact scheme. Moreover, it is clear that $p_1(p_0^{-1}(V'))$ is the inverse image of the quasi-compact open subspace $U \subset X$ discussed in the first paragraph of the proof. In other words, $T' \times_T U$ is a scheme! Note that $T' \times_T U$ is quasi-compact and separated and locally quasi-finite over T', as $T' \times_T X \to T'$ is locally quasi-finite and separated being a base change of the original morphism $X \to T$ (see Lemmas 67.4.4 and 67.27.4). This implies by More on Morphisms, Lemma 37.43.2 that $T' \times_T U \to T'$ is quasi-affine.

By Descent, Lemma 35.39.1 this gives a descent datum on $T' \times_T U/T'$ relative to the étale covering $\{T' \to W\}$, where $W \subset T$ is the image of the morphism $T' \to T$. Because U' is quasi-affine over T' we see from Descent, Lemma 35.38.1 that this datum is effective, and by the last part of Descent, Lemma 35.39.1 this implies that U is a scheme as desired. Some minor details omitted. □

03XX **Proposition 67.50.2.** *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces over S. Assume*

(1) *T is representable,*
(2) *f is locally quasi-finite, and*
(3) *f is separated.*

Then X is representable.

Proof. Let $T = \bigcup T_i$ be an affine open covering of the scheme T. If we can show that the open subspaces $X_i = f^{-1}(T_i)$ are representable, then X is representable, see Properties of Spaces, Lemma 66.13.1. Note that $X_i = T_i \times_T X$ and that locally quasi-finite and separated are both stable under base change, see Lemmas 67.4.4 and 67.27.4. Hence we may assume T is an affine scheme.

By Properties of Spaces, Lemma 66.6.2 there exists a Zariski covering $X = \bigcup X_i$ such that each X_i has a surjective étale covering by an affine scheme. By Properties of Spaces, Lemma 66.13.1 again it suffices to prove the proposition for each X_i. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$. This reduces us to the situation in the next paragraph.

Assume we have

$$U \longrightarrow X \longrightarrow T$$

where U and T are affine schemes, $U \to X$ is étale surjective, and $X \to T$ is separated and locally quasi-finite. By Lemmas 67.39.5 and 67.27.3 the morphism $U \to T$ is locally quasi-finite. Since U and T are affine it is quasi-finite. Set $R = U \times_X U$. Then $X = U/R$, see Spaces, Lemma 65.9.1. As $X \to T$ is separated the morphism $R \to U \times_T U$ is a closed immersion, see Lemma 67.4.5. In particular R is an affine scheme also. As $U \to X$ is étale the projection morphisms $t, s : R \to U$ are étale as well. In particular s and t are quasi-finite, flat and of finite presentation (see Morphisms, Lemmas 29.36.6, 29.36.12 and 29.36.11).

Let (U, R, s, t, c) be the groupoid associated to the étale equivalence relation R on U. Let $u \in U$ be a point, and denote $p \in T$ its image. We are going to use More on Groupoids, Lemma 40.13.2 for the groupoid (U, R, s, t, c) over the scheme T with points p and u as above. By the discussion in the previous paragraph all the assumptions (1) – (7) of that lemma are satisfied. Hence we get an étale neighbourhood $(T', p') \to (T, p)$ and disjoint union decompositions

$$U_{T'} = U' \amalg W, \quad R_{T'} = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), (g), and (h) of the aforementioned More on Groupoids, Lemma 40.13.2. We may and do assume that T' is affine (after possibly shrinking T'). Conclusion (h) implies that $R' = U' \times_{X_{T'}} U'$ with projection mappings identified with the restrictions of s' and t'. Thus $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'})$ of conclusion (g) is an étale equivalence relation. By Spaces, Lemma 65.10.2 we conclude that U'/R' is an open subspace of $X_{T'}$. By conclusion (d) the schemes U', R' are affine and the morphisms $s'|_{R'}, t'|_{R'}$ are finite

étale. Hence Groupoids, Proposition 39.23.9 kicks in and we see that U'/R' is an affine scheme.

We conclude that for every pair of points (u, p) as above we can find an étale neighbourhood $(T', p') \to (T, p)$ with $\kappa(p) = \kappa(p')$ and a point $u' \in U_{T'}$ mapping to u such that the image x' of u' in $|X_{T'}|$ has an open neighbourhood V' in $X_{T'}$ which is an affine scheme. We apply Lemma 67.50.1 to obtain an open subspace $W \subset X$ which is a scheme, and which contains x (the image of u in $|X|$). Since this works for every x we see that X is a scheme by Properties of Spaces, Lemma 66.13.1. This ends the proof. □

67.51. Applications

05W4 An alternative proof of the following lemma is to see it as a consequence of Zariski's main theorem for (nonrepresentable) morphisms of algebraic spaces as discussed in More on Morphisms of Spaces, Section 76.34. Namely, More on Morphisms of Spaces, Lemma 76.34.2 implies that a quasi-finite and separated morphism of algebraic spaces is quasi-affine and therefore representable.

0418 **Lemma 67.51.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally quasi-finite and separated, then f is representable.*

Proof. This is immediate from Proposition 67.50.2 and the fact that being locally quasi-finite and separated is preserved under any base change, see Lemmas 67.27.4 and 67.4.4. □

05W5 **Lemma 67.51.2.** *Let S be a scheme. Let $f : X \to Y$ be an étale and universally injective morphism of algebraic spaces over S. Then f is an open immersion.*

Proof. Let $T \to Y$ be a morphism from a scheme into Y. If we can show that $X \times_Y T \to T$ is an open immersion, then we are done. Since being étale and being universally injective are properties of morphisms stable under base change (see Lemmas 67.39.4 and 67.19.5) we may assume that Y is a scheme. Note that the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is étale, a monomorphism, and surjective by Lemma 67.19.2. Hence we see that $\Delta_{X/Y}$ is an isomorphism (see Spaces, Lemma 65.5.9), in particular we see that X is separated over Y. It follows that X is a scheme too, by Proposition 67.50.2. Finally, $X \to Y$ is an open immersion by the fundamental theorem for étale morphisms of schemes, see Étale Morphisms, Theorem 41.14.1. □

67.52. Zariski's Main Theorem (representable case)

0ABQ This is the version you can prove using that normalization commutes with étale localization. Before we can prove more powerful versions (for non-representable morphisms) we need to develop more tools. See More on Morphisms of Spaces, Section 76.34.

0ABR **Lemma 67.52.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is representable, of finite type, and separated. Let Y' be the*

normalization of Y in X. Picture:

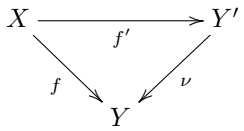

Then there exists an open subspace $U' \subset Y'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. Let $W \to Y$ be a surjective étale morphism where W is a scheme. Then $W \times_Y X$ is a scheme as well. By Lemma 67.48.4 the algebraic space $W \times_Y Y'$ is representable and is the normalization of the scheme W in the scheme $W \times_Y X$. Picture

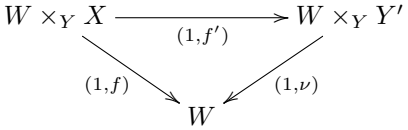

By More on Morphisms, Lemma 37.43.1 the result of the lemma holds over W. Let $V' \subset W \times_Y Y'$ be the open subscheme such that

(1) $(1, f')^{-1}(V') \to V'$ is an isomorphism, and
(2) $(1, f')^{-1}(V') \subset W \times_Y X$ is the set of points at which $(1, f)$ is quasi-finite.

By Lemma 67.34.7 there is a maximal open set of points $U \subset X$ where f is quasi-finite and $W \times_Y U = (1, f')^{-1}(V')$. The morphism $f'|_U : U \to Y'$ is an open immersion by Lemma 67.12.1 as its base change to W is the isomorphism $(1, f')^{-1}(V') \to V'$ followed by the open immersion $V' \to W \times_Y Y'$. Setting $U' = \mathrm{Im}(U \to Y')$ finishes the proof (omitted: the verification that $(f')^{-1}(U') = U$). □

In the following lemma we can drop the assumption of being representable as we've shown that a locally quasi-finite separated morphism is representable.

0ABS **Lemma 67.52.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

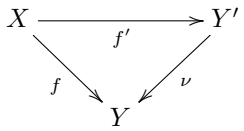

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. By Lemma 67.51.1 the morphism f is representable. Hence we may apply Lemma 67.52.1. Thus there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Lemma 67.8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. □

67.53. Universal homeomorphisms

05Z3 The class of universal homeomorphisms of schemes is closed under composition and arbitrary base change and is fppf local on the base. See Morphisms, Lemmas 29.45.3 and 29.45.2 and Descent, Lemma 35.23.11. Thus, if we apply the discussion in Section 67.3 to this notion we see that we know what it means for a representable morphism of algebraic spaces to be a universal homeomorphism.

05Z4 **Lemma 67.53.1.** *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is a universal homeomorphism (in the sense of Section 67.3) if and only if for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.*

Proof. If for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$, then the same is true whenever Z is a scheme, which formally implies that f is a universal homeomorphism in the sense of Section 67.3. Conversely, if f is a universal homeomorphism in the sense of Section 67.3 then $X \to Y$ is integral, universally injective and surjective (by Spaces, Lemma 65.5.8 and Morphisms, Lemma 29.45.5). Hence f is universally closed, see Lemma 67.45.7 and universally injective and (universally) surjective, i.e., f is a universal homeomorphism. □

05Z5 **Definition 67.53.2.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is called a *universal homeomorphism* if and only if for every morphism of algebraic spaces $Z \to Y$ the base change $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

This definition does not clash with the pre-existing definition for representable morphisms of algebraic spaces by our Lemma 67.53.1. For morphisms of algebraic spaces it is not the case that universal homeomorphisms are always integral.

05Z6 **Example 67.53.3.** This is a continuation of Remark 67.19.4. Consider the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$. There are morphisms

$$\mathbf{A}^1_k \longrightarrow X \longrightarrow \mathbf{A}^1_k$$

such that the first arrow is étale surjective, the second arrow is universally injective, and the composition is the map $x \mapsto x^2$. Hence the composition is universally closed. Thus it follows that the map $X \to \mathbf{A}^1_k$ is a universal homeomorphism, but $X \to \mathbf{A}^1_k$ is not separated.

Let S be a scheme. Let $f : X \to Y$ be a universal homeomorphism of algebraic spaces over S. Then f is universally closed, hence is quasi-compact, see Lemma 67.9.7. But f need not be separated (see example above), and not even quasi-separated: an example is to take infinite dimensional affine space $\mathbf{A}^\infty = \mathrm{Spec}(k[x_1, x_2, \ldots])$ modulo the equivalence relation given by flipping finitely many signs of nonzero coordinates (details omitted).

First we state the obligatory lemmas.

0CFT **Lemma 67.53.4.** *The base change of a universal homeomorphism of algebraic spaces by any morphism of algebraic spaces is a universal homeomorphism.*

Proof. This is immediate from the definition. □

0CFU **Lemma 67.53.5.** *The composition of a pair of universal homeomorphisms of algebraic spaces is a universal homeomorphism.*

Proof. Omitted. □

08AK **Lemma 67.53.6.** *Let S be a scheme. Let X be an algebraic space over S. The canonical closed immersion $X_{red} \to X$ (see Properties of Spaces, Definition 66.12.5) is a universal homeomorphism.*

Proof. Omitted. □

We put the following result here as we do not currently have a better place to put it.

0AEH **Lemma 67.53.7.** *Let S be a scheme. Let $f : Y \to X$ be a universally injective, integral morphism of algebraic spaces over S.*

(1) *The functor*
$$f_{small,*} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $|X| \setminus f(|Y|)$ is isomorphic to $*$, and

(2) *the functor*
$$f_{small,*} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(|Y|)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. Since f is integral it is universally closed (Lemma 67.45.7). In particular, $f(|Y|)$ is a closed subset of $|X|$ and the statements make sense. The rest of the proof is identical to the proof of Lemma 67.13.5 except that we use Étale Cohomology, Proposition 59.47.1 instead of Étale Cohomology, Proposition 59.46.4. □

67.54. Other chapters

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CHAPTER 68

Decent Algebraic Spaces

06NK

68.1. Introduction

06NL In this chapter we study "local" properties of general algebraic spaces, i.e., those algebraic spaces which aren't quasi-separated. Quasi-separated algebraic spaces are studied in [**Knu71**]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn't worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

68.2. Conventions

06NM The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

68.3. Universally bounded fibres

03JK We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 29.57 for similar definitions and results on morphisms of schemes.

03JL **Definition 68.3.1.** Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $f : U \to X$ be a morphism over S. We say the *fibres of f are universally bounded*[1] if there exists an integer n such that for all fields k and all morphisms $\mathrm{Spec}(k) \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $\mathrm{Spec}(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 29.57.1 by virtue of Morphisms, Lemma 29.57.2.

03JM **Lemma 68.3.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $V \to U$ be a morphism of schemes over S, and let $U \to X$ be a morphism from U to X. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.*

[1] This is probably nonstandard notation.

Proof. Let n be an integer which works for $V \to U$, and let m be an integer which works for $U \to X$ in Definition 68.3.1. Let $\mathrm{Spec}(k) \to X$ be a morphism, where k is a field. Consider the morphisms

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U \longrightarrow \mathrm{Spec}(k).$$

By assumption the scheme $\mathrm{Spec}(k) \times_X U$ is finite of degree at most m over k, and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 29.57.4 we conclude that $\mathrm{Spec}(k) \times_X V$ is finite over k of degree at most nm. □

03JN **Lemma 68.3.3.** *Let S be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over S. Let $U \to X$ be a morphism from a scheme to X. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.*

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) □

03JO **Lemma 68.3.4.** *Let S be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over S. Let $f : U \to X$ be a morphism from a scheme towards X. Let $f' : U \times_X Y \to Y$ be the base change of f. If*

$$\mathrm{Im}(|f| : |U| \to |X|) \subset \mathrm{Im}(|g| : |Y| \to |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\mathrm{Spec}(k) \times_Y (U \times_X Y)$ as in Definition 68.3.1 for the morphism f'. We claim that n works for f also. Namely, suppose that $x : \mathrm{Spec}(k) \to X$ is a morphism from the spectrum of a field. Then either $\mathrm{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 66.4.3 and the assumption of the lemma we see that this means there exists a field extension k'/k and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

Hence we see that

$$\mathrm{Spec}(k') \times_Y (U \times_X Y) = \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

Since the scheme $\mathrm{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) □

03JP **Lemma 68.3.5.** *Let S be a scheme. Let X be an algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & X & \end{array}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\mathrm{Spec}(k) \times_X U$ as in Definition 68.3.1. We claim n also works for h. Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field to X. Consider the morphism of schemes

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\mathrm{Spec}(k)$. It follows from Morphisms, Lemma 29.57.10 that the degree of the scheme on the right is also bounded by n as desired. □

03JQ **Lemma 68.3.6.** *Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $\varphi : U \to X$ be a morphism over S. If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \to |X|$ has at most n elements.*

Proof. The integer n of Definition 68.3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : \mathrm{Spec}(k) \to X$. Then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x} & X \end{array}$$

which shows (via Properties of Spaces, Lemma 66.4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. □

68.4. Finiteness conditions and points

03JR In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

03II **Remark 68.4.1.** Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 29.36.12 and 29.25.9).
(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 29.36.16, 29.35.10, and 29.20.6).
(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 29.20.9 and 29.20.10).
(4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 29.36.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 41.11.

03JS **Lemma 68.4.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each i the fibre of $|U_i| \to |X|$ over x is finite, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over x is finite.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards X. Consider the fibre product diagrams

$$\begin{array}{ccc} U \times_X U_i & \xrightarrow{p_i} & U_i \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ U & \xrightarrow{\varphi} & X \end{array} \qquad \begin{array}{ccc} \coprod U \times_X U_i & \xrightarrow{\coprod p_i} & \coprod U_i \\ {\scriptstyle \coprod q_i}\downarrow & & \downarrow{\scriptstyle \coprod \varphi_i} \\ U & \xrightarrow{\varphi} & X \end{array}$$

Since q_i is étale it is open (see Remark 68.4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U. The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 29.57.9 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \to U_i$ are finite. Hence by Properties of Spaces, Lemma 66.4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. □

03JU **Lemma 68.4.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a scheme U, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(2) *for every scheme U, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(3) *there exists a morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x such that the projections $\mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact, and*
(4) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x.*

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards X, and let u, u' be points of U lying over x such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = \mathrm{Spec}(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma 26.23.7), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards X. Let $w, w' \in W$ be points of W mapping to

x. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{array}{ccc} W \times_X U & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \varphi} \\ W & \xrightarrow{\psi} & X \end{array}$$

As x is the image of u and u' we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 66.4.3. As p, q are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 68.4.1. Then we get a commutative diagram

$$\begin{array}{ccccc} w \times_X w' & \longleftarrow & \tilde{w} \times_X \tilde{w}' & \longrightarrow & u \times_X u' \\ \downarrow & & \downarrow & & \downarrow \\ w \times_X w' & \longleftarrow & \tilde{w} \times_S \tilde{w}' & \longrightarrow & u \times_S u' \end{array}$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme U such that x is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to x. Then we have seen in the previous paragraph that $u = \mathrm{Spec}(\kappa(u)) \to X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 68.4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 26.23.11 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field into X, in the equivalence class of x such that the two projections $t, s : R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $\mathrm{Spec}(k)$. By Spaces, Theorem 65.10.5 we know that the quotient sheaf $X' = \mathrm{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 39.20.6 the map $X' \to X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 66.15.3 applies to X', and we see that $X' = \mathrm{Spec}(k')$ for some field k'. Hence we get a factorization

$$\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k') \longrightarrow X$$

which shows that $\mathrm{Spec}(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\mathrm{Spec}(k) \to X$ be a monomorphism with k a field in the equivalence class of x. Let $U \to X$ be a surjective étale morphism from a scheme U to X. Let $u \in U$ be a point over x. Since $\mathrm{Spec}(k) \times_X u$ is nonempty, and since $\mathrm{Spec}(k) \times_X u \to u$ is a monomorphism we conclude that $\mathrm{Spec}(k) \times_X u = u$ (see Schemes, Lemma 26.23.11). Hence $u \to U \to X$ factors through $\mathrm{Spec}(k) \to X$, here is a picture

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

Since the right vertical arrow is étale this implies that $\kappa(u)/k$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\mathrm{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. □

040U **Lemma 68.4.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:*

(1) *x is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both*
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite nonempty scheme over k.*

Proof. Assume (1). This clearly implies the first condition of Lemma 68.4.3 and hence we obtain a monomorphism $\mathrm{Spec}(k) \to X$ in the class of x. Taking the fibre product we see that $\mathrm{Spec}(k) \times_X U \to \mathrm{Spec}(k)$ is a scheme étale over $\mathrm{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over k, i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over x is clear since this fibre is equal to $|\mathrm{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 66.4.3. The finiteness of the fibre of $|R| \to |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(\mathrm{Spec}(k) \times_X U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

which is finite over k. Thus (1) holds. □

03JV **Lemma 68.4.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *for every affine scheme U, any étale morphism $\varphi : U \to X$ setting $R = U \times_X U$ the fibres of both*
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and for each i, setting $R_i = U_i \times_X U_i$ the fibres of both*
$$|U_i| \longrightarrow |X| \quad \text{and} \quad |R_i| \longrightarrow |X|$$
over x are finite,

(3) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and for any affine scheme U and étale morphism $U \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k,*

(4) *there exists a quasi-compact monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x,*

(5) *there exists a quasi-compact morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and*

(6) *every morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x is quasi-compact.*

Proof. The equivalence of (1) and (3) follows on applying Lemma 68.4.4 to every étale morphism $U \to X$ with U affine. It is clear that (3) implies (2). Assume $U_i \to X$ and R_i are as in (2). We conclude from Lemma 68.4.2 that for any affine scheme U and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over x is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 68.4.3 (1) applies to $U_i \to X$ for some i such that x is in the image of $|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times_{|X|} |U|$ is finite over (u_a, u_b), $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \to X$ the scheme $\mathrm{Spec}(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 68.4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $\mathrm{Spec}(k) \to X$ is a monomorphism as in (3), then
$$\coprod \mathrm{Spec}(k) \times_X U_i \longrightarrow \coprod U_i$$
is quasi-compact (because over each U_i we see that $\mathrm{Spec}(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\mathrm{Spec}(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.8.

It is immediate that (4) implies (5). Conversely, let $\mathrm{Spec}(k) \to X$ be a quasi-compact morphism in the equivalence class of x. Let $U \to X$ be an étale morphism with U affine. Consider the fibre product
$$\begin{array}{ccc} F & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$
Then $F \to U$ is quasi-compact, hence F is quasi-compact. On the other hand, $F \to \mathrm{Spec}(k)$ is étale, hence F is a finite disjoint union of spectra of finite separable extensions of k (Remark 68.4.1). Since the image of $|F| \to |U|$ is the fibre of $|U| \to |X|$ over x (Properties of Spaces, Lemma 66.4.3), we conclude that the fibre of $|U| \to |X|$ over x is finite. The scheme $F \times_{\mathrm{Spec}(k)} F$ is also a finite union of spectra of fields because it is also quasi-compact and étale over $\mathrm{Spec}(k)$. There is

a monomorphism $F \times_X F \to F \times_{\mathrm{Spec}(k)} F$, hence $F \times_X F$ is a finite disjoint union of spectra of fields (Schemes, Lemma 26.23.11). Thus the image of $F \times_X F \to U \times_X U = R$ is finite. Since this image is the fibre of $|R| \to |X|$ over x by Properties of Spaces, Lemma 66.4.3 we conclude that (1) holds. At this point we know that (1) – (5) are equivalent.

It is clear that (6) implies (5). Conversely, assume $\mathrm{Spec}(k) \to X$ is as in (4) and let $\mathrm{Spec}(k') \to X$ be another morphism with k' a field in the equivalence class of x. By Properties of Spaces, Lemma 66.4.11 we have a factorization $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to X$ of the given morphism. This is a composition of quasi-compact morphisms and hence quasi-compact (Morphisms of Spaces, Lemma 67.8.5) as desired. □

03JT **Lemma 68.4.6.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards X, covering X, such that each φ_i has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards X. For each i consider the fibre product diagram

$$\begin{array}{ccc} U \times_X U_i & \xrightarrow{p_i} & U_i \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ U & \xrightarrow{\psi} & X \end{array}$$

Since q_i is étale it is open (see Remark 68.4.1). Moreover, we have $U = \bigcup \mathrm{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can finite finitely many $i_1, \ldots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$ such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 29.57.9 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 68.3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 68.3.5. In other words (2) holds. □

03IH **Lemma 68.4.7.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.*

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 29.36.3 and Spaces, Lemmas 65.5.4 and 65.5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 66.4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \to U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 65.11.4 the morphisms $U_i \to X_i$ are quasi-compact. □

68.5. Conditions on algebraic spaces

03JW In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 68.6 to get a feeling for the meaning of these conditions.

03JX **Lemma 68.5.1.** *Let S be a scheme. Let X be an algebraic space over S. Consider the following conditions on X:*

- (α) *For every $x \in |X|$, the equivalent conditions of Lemma 68.4.2 hold.*
- (β) *For every $x \in |X|$, the equivalent conditions of Lemma 68.4.3 hold.*
- (γ) *For every $x \in |X|$, the equivalent conditions of Lemma 68.4.5 hold.*
- (δ) *The equivalent conditions of Lemma 68.4.6 hold.*
- (ϵ) *The equivalent conditions of Lemma 68.4.7 hold.*
- (ζ) *The space X is Zariski locally quasi-separated.*
- (η) *The space X is quasi-separated*
- (θ) *The space X is representable, i.e., X is a scheme.*
- (ι) *The space X is a quasi-separated scheme.*

We have

$$\begin{array}{ccccccccccc} & & & (\theta) & & & & & & & \\ & & \nearrow & & \searrow & & & & & & \\ (\iota) & & & & & (\zeta) & \Longrightarrow & (\epsilon) & \Longrightarrow & (\delta) & \Longrightarrow & (\gamma) & \Longleftrightarrow & (\alpha)+(\beta) \\ & & \searrow & & \nearrow & & & & & & \\ & & & (\eta) & & & & & & & \end{array}$$

Proof. The implication $(\gamma) \Leftrightarrow (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ), i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 66.6.6.

Assume (ϵ). By Lemma 68.4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \to X$ has universally bounded fibres, since then the family of all these morphisms $W \to X$ covers X. To do this we consider the diagram

$$\begin{array}{ccc} W \times_X U_i & \xrightarrow{p} & U_i \\ {\scriptstyle q}\downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

Since $W \to X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 29.57.9 and we conclude that p has universally bounded fibres. From Lemma 68.3.4 we conclude that $W \to X$ has universally bounded fibres as well.

Assume (δ). Let U be an affine scheme, and let $U \to X$ be an étale morphism. By assumption the fibres of the morphism $U \to X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \to U$ are universally bounded, see Lemma 68.3.3. And by Lemma 68.3.2 also the fibres of $R \to X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite, see Lemma 68.3.6. In other words, the equivalent conditions of Lemma 68.4.5 hold. This proves that $(\delta) \Rightarrow (\gamma)$. □

03KE **Lemma 68.5.2.** *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (α), (β), (γ), (δ), (ϵ), (ζ), or (θ) of algebraic spaces listed in Lemma 68.5.1. Then if X is an algebraic space over S, and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$, then X has $\mathcal{P}$.*

Proof. Let X be an algebraic space over S, and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$.

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U, and étale morphism $\varphi : U \to X_i$ the fibre of $\varphi : |U| \to |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \to X$. Since $X = \bigcup X_i$ is a Zariski open covering there exits a finite affine open covering $U = U_1 \cup \ldots \cup U_n$ such that each $U_j \to X$ factors through some X_{i_j}. By assumption the fibres of $|U_j| \to |X_{i_j}|$ over x are finite for $j = 1, \ldots, n$. Clearly this means that the fibre of $|U| \to |X|$ over x is finite. This proves the result for (α).

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i. Hence the same follows for X as $X_i \to X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 68.5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \to X_i$ with universally bounded fibres which cover X_i. These schemes also give an étale surjective morphism $\coprod U_{ij} \to X$ and $U_{ij} \to X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \to X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \to U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \to X_i$ is surjective. In this case the compositions $U_{ij} \to X_i \to X$ are étale (combine Morphisms, Lemmas 29.36.3 and 29.36.9 and Spaces, Lemmas 65.5.4 and 65.5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \to X$ is surjective. Hence X satisfies (ϵ).

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 66.13.1. □

03KF **Lemma 68.5.3.** *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (β), (γ), (δ), (ϵ), or (θ) of algebraic spaces listed in Lemma 68.5.1. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y has property $\mathcal{P}$, so does X.*

Proof. Assume $f : X \to Y$ is a representable morphism of algebraic spaces, and assume that Y has $\mathcal{P}$. Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y, i.e., to a monomorphism $\mathrm{Spec}(k') \to X_y$. As $X_y \to X$ is a monomorphism also we see that x is represented by the monomorphism $\mathrm{Spec}(k') \to X_y \to X$. In other words (β) holds for X.

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

Also, by definition of property (γ) via Lemma 68.4.5 (2) there exist schemes V_i and étale morphisms $V_i \to Y$ such that $\coprod V_i \to Y$ is surjective and for each i, setting $R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \mathrm{Spec}(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = ((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X, again via the second condition of Lemma 68.4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 68.3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 68.4.6 applies and we see that Y also has property (δ).

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 65.5.5. Let $V_i \to Y$ be as in Lemma 68.4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence X satisfies Lemma 68.4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 4.8.3. □

68.6. Reasonable and decent algebraic spaces

03I7 In Lemma 68.5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

(α)	fibres of étale morphisms from affines are finite
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fibres of étale morphisms from affines are universally bounded
(ϵ)	cover by étale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X, but they are in some sense separation conditions on X.

03I8 **Definition 68.6.1.** Let S be a scheme. Let X be an algebraic space over S.

(1) We say X is *decent* if for every point $x \in X$ the equivalent conditions of Lemma 68.4.5 hold, in other words property (γ) of Lemma 68.5.1 holds.
(2) We say X is *reasonable* if the equivalent conditions of Lemma 68.4.6 hold, in other words property (δ) of Lemma 68.5.1 holds.
(3) We say X is *very reasonable* if the equivalent conditions of Lemma 68.4.7 hold, i.e., property (ϵ) of Lemma 68.5.1 holds.

We have the following implications among these conditions on algebraic spaces:

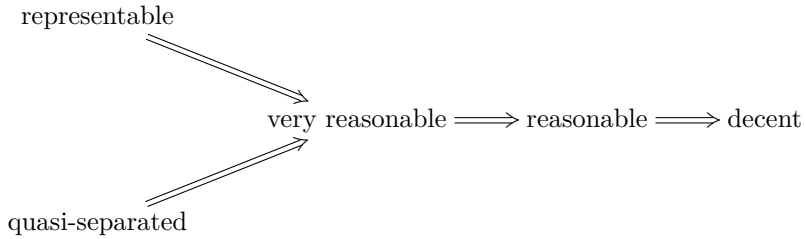

The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces X where one has a good relationship between the topology of $|X|$ and properties of X itself.

03ID **Example 68.6.2.** The algebraic space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ constructed in Spaces, Example 65.14.8 is not decent as its "generic point" cannot be represented by a monomorphism from the spectrum of a field.

03JY **Remark 68.6.3.** Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \to Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y, see Lemma 68.17.8 but we don't know if this is true for the property "very reasonable". Below we give another technical property enjoyed by reasonable algebraic spaces.

03K0 **Lemma 68.6.4.** *Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated*

algebraic spaces X_i such that $X = \operatorname{colim}_i X_i$ (colimit in the category of sheaves). Moreover we can arrange it such that

(1) *for every quasi-compact scheme T over S we have $\operatorname{colim} X_i(T) = X(T)$,*
(2) *the transition morphisms $X_i \to X_{i'}$ of the system and the coprojections $X_i \to X$ are surjective and étale, and*
(3) *if X is a scheme, then the algebraic spaces X_i are schemes and the transition morphisms $X_i \to X_{i'}$ and the coprojections $X_i \to X$ are local isomorphisms.*

Proof. We sketch the proof. By Properties of Spaces, Lemma 66.6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \to X$ is universally bounded. Hence there exists an integer N such that the "fibres" of $U \to X$ have degree at most N, see Definition 68.3.1. Denote $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

(1) $A \subset R'$,
(2) R' is quasi-compact, and
(3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$ is a groupoid scheme.

Note that $e : U \to R$ is open as it is a section of the étale morphism $s : R \to U$, see Étale Morphisms, Proposition 41.6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s,U,t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S, and we consider k-points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{array}{c} \text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n-1. \end{array} \right\}$$

But as the fibres of $U(k) \to X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$\operatorname{colim}_{R' \subset R} U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see Spaces, Theorem 65.10.5). Since R' is quasi-compact, and U affine the morphism $R' \to U \times_{\operatorname{Spec}(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$\operatorname{colim}_{R' \subset R} U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R. In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R'.

Properties (1) and (2) follow from the discussion above. If X is a scheme, then if we choose U to be a finite disjoint union of affine opens of X we will obtain (3). Details omitted. □

0ABT **Lemma 68.6.5.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y is decent (resp. reasonable), then so is X.*

Proof. Translation of Lemma 68.5.3. □

0ABU **Lemma 68.6.6.** *Let S be a scheme. Let $X \to Y$ be an étale morphism of algebraic spaces over S. If Y is decent, resp. reasonable, then so is X.*

Proof. Let U be an affine scheme and $U \to X$ an étale morphism. Set $R = U \times_X U$ and $R' = U \times_Y U$. Note that $R \to R'$ is a monomorphism.

Let $x \in |X|$. To show that X is decent, we have to show that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite. But if Y is decent, then the fibres of $|U| \to |Y|$ and $|R'| \to |Y|$ are finite. Hence the result for "decent".

To show that X is reasonable, we have to show that the fibres of $U \to X$ are universally bounded. However, if Y is reasonable, then the fibres of $U \to Y$ are universally bounded, which immediately implies the same thing for the fibres of $U \to X$. Hence the result for "reasonable". □

68.7. Points and specializations

03K1 There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 110.51.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y.

03IM **Lemma 68.7.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 68.4.2 then $u = u'$.*

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 68.4.2. Let U be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \rightsquigarrow u$. We may and do replace U by an affine neighbourhood of u. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 66.4.5. Because generalizations lift along the étale morphism t (Remark 68.4.1) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set

$u''' = s(r'')$, and so on. Here is a picture:

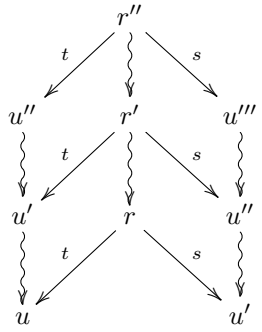

In Remark 68.4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s. Hence if $u^{(n+1)} = u^{(n)}$ for some n, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 68.4.2, as desired. □

0H1Q **Lemma 68.7.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If X is locally Noetherian, then $u = u'$.*

Proof. The discussion in Schemes, Section 26.13 shows that $\mathcal{O}_{U,u'}$ is a localization of the Noetherian local ring $\mathcal{O}_{U,u}$. By Properties of Spaces, Lemma 66.10.1 we have $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{U,u'})$. By dimension theory for Noetherian local rings we conclude $u = u'$. □

03K2 **Lemma 68.7.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 68.4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. We may replace U by an affine open neighbourhood of u. Hence we may assume that U is affine. As x is in the image of the open map $|U| \to |X|$, so is x'. Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 66.4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 68.4.5 we see that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x' are finite. Say $\{u'_1, \ldots, u'_n\} \subset U$ and $\{r'_1, \ldots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \ldots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \ldots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i. Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \to U$, see Remark 68.4.1, we can find $r' \rightsquigarrow r$

with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$-invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 66.4.5 and 66.4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 66.4.3). As generalizations lift along t, and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. □

0B7W **Lemma 68.7.4.** *Let S be a scheme. Let $f : Y \to X$ be a flat morphism of algebraic spaces over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 68.4.5 (for example if X is decent, X is quasi-separated, or X is representable). Then for every $y \in |Y|$ with $f(y) = x$, there exists a point $y' \in |Y|$, $y' \rightsquigarrow y$ with $f(y') = x'$.*

Proof. (The parenthetical statement holds by the definition of decent spaces and the implications between the different separation conditions mentioned in Section 68.6.) Choose a scheme V and a surjective étale morphism $V \to Y$. Choose $v \in V$ mapping to y. Then we see that it suffices to prove the lemma for $V \to X$. Thus we may assume Y is a scheme. Choose a scheme U and a surjective étale morphism $U \to X$. Choose $u \in U$ mapping to x. By Lemma 68.7.3 we may choose $u' \rightsquigarrow u$ mapping to x'. By Properties of Spaces, Lemma 66.4.3 we may choose $z \in U \times_X Y$ mapping to y and u. Thus we reduce to the case of the flat morphism of schemes $U \times_X Y \to U$ which is Morphisms, Lemma 29.25.9. □

68.8. Stratifying algebraic spaces by schemes

0A4I In this section we prove that a quasi-compact and quasi-separated algebraic space has a finite stratification by locally closed subspaces each of which is a scheme and such that the glueing of the parts is by elementary distinguished squares. We first prove a slightly weaker result for reasonable algebraic spaces.

07S8 **Lemma 68.8.1.** *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = X$$

such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \coprod_{r=0,\ldots,n} X_r$ is characterized by the following universal property: Given $g : T \to X$ the projection $W \times_X T \to T$ is finite locally free of degree r if and only if $g(|T|) \subset |X_r|$.

Proof. Let n be an integer bounding the degrees of the fibres of $W \to X$. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 37.45.3 to $W \times_X U \to U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting

$R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of R. In other words the closed subsets $Y_r \subset U$ are R-invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 66.12.5.

Let $g : T \to X$ be a morphism of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \to X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 67.46). Similarly, the morphism of schemes $W \times_X V \to V$ is finite locally free of degree r if and only if the morphism of schemes

$$W \times_X (U \times_X V) \longrightarrow U \times_X V$$

is finite locally free of degree r (see Descent, Lemma 35.23.32). By construction this happens if and only if $|U \times_X V| \to |U|$ maps into $|Y_r|$, which is true if and only if $|V| \to |X|$ maps into $|Z_r|$. □

086T **Lemma 68.8.2.** *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces*

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to X$ where k is a field factors through X_d if and only if $W \times_X \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 37.45.5 to $W \times_X U \to U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of R. In other words the open subschemes $U_d \subset U$ are R-invariant. This means that U_d is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 66.12.2). □

0BBN **Lemma 68.8.3.** *Let S be a scheme. Let X be a quasi-compact algebraic space over S. There exist open subspaces*

$$\ldots \subset U_4 \subset U_3 \subset U_2 \subset U_1 = X$$

with the following properties:

1. *setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism,*
2. *if $x \in |X|$ can be represented by a quasi-compact morphism $\mathrm{Spec}(k) \to X$ from a field, then $x \in T_p$ for some p.*

Proof. By Properties of Spaces, Lemma 66.6.3 we can choose an affine scheme U and a surjective étale morphism $U \to X$. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

where the fibre product has p factors. Since U is separated, the morphism $U \to X$ is separated and all fibre products $U \times_X \ldots \times_X U$ are separated schemes. Since $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Similarly, all the diagonal morphisms are open and closed immersions and W_p is an open and closed subscheme of $U \times_X \ldots \times_X U$. Moreover, the morphism

$$U \times_X \ldots \times_X U \longrightarrow U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$$

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 67.4.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \ldots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 37.45.1. Therefore, the same is true for W_p. There is a free action of the symmetric group S_p on W_p over X (because we threw out the fix point locus from $U \times_X \ldots \times_X U$). By the above and Properties of Spaces, Proposition 66.14.1 the quotient $V_p = W_p/S_p$ is a scheme. Since the action of S_p on W_p was over X, there is a morphism $V_p \to X$. Since $W_p \to X$ is étale and since $W_p \to V_p$ is surjective étale, it follows that also $V_p \to X$ is étale, see Properties of Spaces, Lemma 66.16.3. Observe that V_p is a separated scheme by Properties of Spaces, Lemma 66.14.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \to X$. By construction a morphism $\mathrm{Spec}(k) \to X$ with k algebraically closed, factors through U_p if and only if $U \times_X \mathrm{Spec}(k)$ has $\geq p$ points; as usual observe that $U \times_X \mathrm{Spec}(k)$ is scheme theoretically a disjoint union of (possibly infinitely many) copies of $\mathrm{Spec}(k)$, see Remark 68.4.1. It follows that the U_p give a filtration of X as stated in the lemma. Moreover, our morphism $\mathrm{Spec}(k) \to X$ factors through T_p if and only if $U \times_X \mathrm{Spec}(k)$ has exactly p points. In this case we see that $V_p \times_X \mathrm{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of V_p. Then $Z_p \to T_p$ is an étale morphism between algebraic spaces which induces a bijection on k-valued points for any algebraically closed field k. To be sure this implies that $Z_p \to T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 67.51.2 hence an isomorphism and (1) has been proved.

Let $x : \mathrm{Spec}(k) \to X$ be a quasi-compact morphism where k is a field. Then the composition $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k) \to X$ is quasi-compact as well (Morphisms of Spaces, Lemma 67.8.5). In this case the scheme $U \times_X \mathrm{Spec}(\overline{k})$ is quasi-compact. In view of the fact (seen above) that it is a disjoint union of copies of $\mathrm{Spec}(\overline{k})$ we find that it has finitely many points. If the number of points is p, then we see that indeed $x \in T_p$ and the proof is finished. □

07S9 **Lemma 68.8.4.** *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 68.8.3. Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 68.6.1. Then we see that $U_{n+1} = \emptyset$ and the proof is complete. □

07SA **Lemma 68.8.5.** *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

such that each $T_p = U_p \backslash U_{p+1}$ (with reduced induced subspace structure) is a scheme.

Proof. Immediate consequence of Lemma 68.8.4. □

The following result is almost identical to [**GR71**, Proposition 5.7.8].

07ST **Lemma 68.8.6.** *Let X be a quasi-compact and quasi-separated algebraic space over* Spec($\mathbf{Z}$). *There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

This result is almost identical to [**GR71**, Proposition 5.7.8].

Proof. The proof of this lemma is identical to the proof of Lemma 68.8.3. Observe that a quasi-separated space is reasonable, see Lemma 68.5.1 and Definition 68.6.1. Hence we find that $U_{n+1} = \emptyset$ as in Lemma 68.8.4. At the end of the argument we add that since X is quasi-separated the schemes $U \times_X \ldots \times_X U$ are all quasi-compact. Hence the schemes W_p are quasi-compact. Hence the quotients $V_p = W_p/S_p$ by the symmetric group S_p are quasi-compact schemes. □

The following lemma probably belongs somewhere else.

0ECZ **Lemma 68.8.7.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let $E \subset |X|$ be a subset. Then E is étale locally constructible (Properties of Spaces, Definition 66.8.2) if and only if E is a locally constructible subset of the topological space $|X|$ (Topology, Definition 5.15.1).*

Proof. Assume $E \subset |X|$ is a locally constructible subset of the topological space $|X|$. Let $f : U \to X$ be an étale morphism where U is a scheme. We have to show that $f^{-1}(E)$ is locally constructible in U. The question is local on U and X, hence we may assume that X is quasi-compact, $E \subset |X|$ is constructible, and U is affine. In this case $U \to X$ is quasi-compact, hence $f : |U| \to |X|$ is quasi-compact. Observe that retrocompact opens of $|X|$, resp. U are the same thing as quasi-compact opens of $|X|$, resp. U, see Topology, Lemma 5.27.1. Thus $f^{-1}(E)$ is constructible by Topology, Lemma 5.15.3.

Conversely, assume E is étale locally constructible. We want to show that E is locally constructible in the topological space $|X|$. The question is local on X, hence we may assume that X is quasi-compact as well as quasi-separated. We will show that in this case E is constructible in $|X|$. Choose open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and surjective étale morphisms $f_p : V_p \to U_p$ inducing isomorphisms $f_p^{-1}(T_p) \to T_p = U_p \setminus U_{p+1}$ where V_p is a quasi-compact separated scheme as in Lemma 68.8.6.

By definition the inverse image $E_p \subset V_p$ of E is locally constructible in V_p. Then E_p is constructible in V_p by Properties, Lemma 28.2.5. Thus $E_p \cap |f_p^{-1}(T_p)| = E \cap |T_p|$ is constructible in $|T_p|$ by Topology, Lemma 5.15.7 (observe that $V_p \setminus f_p^{-1}(T_p)$ is quasi-compact as it is the inverse image of the quasi-compact space U_{p+1} by the quasi-compact morphism f_p). Thus

$$E = (|T_n| \cap E) \cup (|T_{n-1}| \cap E) \cup \ldots \cup (|T_1| \cap E)$$

is constructible by Topology, Lemma 5.15.14. Here we use that $|T_p|$ is constructible in $|X|$ which is clear from what was said above. □

68.9. Integral cover by a scheme

0D2T Here we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, integral morphism $Y \to X$. After we develop some theory about limits of algebraic spaces, we will prove that one can do this with a finite morphism, see Limits of Spaces, Section 70.16.

0G2D **Lemma 68.9.1.** *Let S be a scheme. Let $j : V \to Y$ be a quasi-compact open immersion of algebraic spaces over S. Let $\pi : Z \to V$ be an integral morphism. Then there exists an integral morphism $\nu : Y' \to Y$ such that Z is V-isomorphic to the inverse image of V in Y'.*

Proof. Since both j and π are quasi-compact and separated, so is $j \circ \pi$. Let $\nu : Y' \to Y$ be the normalization of Y in Z, see Morphisms of Spaces, Section 67.48. Of course ν is integral, see Morphisms of Spaces, Lemma 67.48.5. The final statement follows formally from Morphisms of Spaces, Lemmas 67.48.4 and 67.48.10. □

09YB **Lemma 68.9.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.*

(1) *There exists a surjective integral morphism $Y \to X$ where Y is a scheme,*
(2) *given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Part (1) is the special case of part (2) where $U = X$. Choose a surjective étale morphism $U' \to U$ where U' is a scheme. It is clear that we may replace U by U' and hence we may assume U is a scheme. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U' again, we see that we may assume U is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Lemma 68.5.1). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Morphisms of Spaces, Lemma 67.52.2 and we obtain a factorization

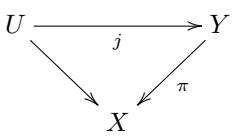

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Then $U \times_X Y$ is a quasi-compact, separated scheme (being finite over U) and we have

$$U \times_X Y = U \amalg W$$

Here the first summand is the image of $U \to U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 67.4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subspace containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of Y by Lemma 68.8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 68.9.1). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have an open neighbourhood of z in Z (which is also an open neighbourhood of z in Z') which factors through $W \subset U \times_X Y$ and hence through U. □

0GUL **Lemma 68.9.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.*

(1) *There exists a surjective integral morphism $Y \to X$ where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,*
(2) *given a surjective étale morphism $V \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \to X$ factors through V.*

Proof. The proof is the (roughly) same as the proof of Lemma 68.9.2 with additional technical comments to obtain the dense quasi-compact open U (and unfortunately changes in notation to keep track of U).

Part (1) is the special case of part (2) where $V = X$.

Proof of (2). Choose a surjective étale morphism $V' \to V$ where V' is a scheme. It is clear that we may replace V by V' and hence we may assume V is a scheme. Since X is quasi-compact, there exist finitely many affine opens $V_i \subset V$ such that $V' = \coprod V_i \to X$ is surjective. After replacing V by V' again, we see that we may assume V is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $V \to X$ (see Lemma 68.5.1).

By induction on $d \geq 1$ we will prove the following induction hypothesis (H_d):

- for any quasi-compact and quasi-separated algebraic space X with finitely many irreducible components, for any $m \geq 0$, for any quasi-compact and separated schemes V_j, $j = 1, \ldots, m$, for any étale morphisms $\varphi_j : V_j \to X$, $j = 1, \ldots, m$ such that d bounds the degree of the geometric fibres of $\varphi_j : V_j \to X$ and $\varphi = \coprod \varphi_j : V = \coprod V_j \to X$ is surjective, the statement of the lemma holds for $\varphi : V \to X$.

If $d = 1$, then each φ_j is an open immersion. Hence X is a scheme and the result holds with $Y = V$. Assume $d > 1$, assume (H_{d-1}) and let m, $\varphi : V_j \to X$, $j = 1, \ldots, m$ be as in (H_d).

Let $\eta_1, \ldots, \eta_n \in |X|$ be the generic points of the irreducible components of $|X|$. By Properties of Spaces, Proposition 66.13.3 there is an open subscheme $U \subset X$ with $\eta_1, \ldots, \eta_n \in U$. By shrinking U we may assume U affine and by Morphisms, Lemma 29.51.1 we may assume each $\varphi_j : V_j \to X$ is finite étale over U. Of course, we see that U is quasi-compact and dense in X and that $\varphi_j^{-1}(U)$ is dense in V_j. In particular each V_j has finitely many irreducible components.

Fix $j \in \{1, \ldots, m\}$. As in Morphisms of Spaces, Lemma 67.52.2 we let Y_j be the normalization of X in V_j. We obtain a factorization

$$\begin{array}{ccc} V_j & \longrightarrow & Y_j \\ & {\scriptstyle \varphi_j} \searrow \quad \swarrow {\scriptstyle \pi_j} & \\ & X & \end{array}$$

with π_j integral and $V_j \to Y_j$ a quasi-compact open immersion. Since Y_j is the normalization of X in V_j, we see from Morphisms of Spaces, Lemmas 67.48.4 and 67.48.10 that $\varphi_j^{-1}(U) \to \pi_j^{-1}(U)$ is an isomorphism. Thus π_j is finite étale over U. Observe that V_j is scheme theoretically dense in Y_j because Y_j is the normalization of X in V_j (follows from the characterization of relative normalization in Morphisms of Spaces, Lemma 67.48.5). Since V_j is quasi-compact we see that $|V_j| \subset |Y_j|$ is dense, see Morphisms of Spaces, Section 67.17 (and especially Morphisms of Spaces, Lemma 67.17.7). It follows that $|Y_j|$ has finitely many irreducible components. Then $V_j \times_X Y_j$ is a quasi-compact, separated scheme (being finite over V_j) and

$$V_j \times_X Y_j = V_j \amalg W_j$$

Here the first summand is the image of $V_j \to V_j \times_X Y_j$ (which is closed by Morphisms of Spaces, Lemma 67.4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement.

The étale morphism $W_j \to Y_j$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $V_j \subset Y_j$ by inspection. Since $|V_j| \subset |Y_j|$ is dense, it holds for all geometric points of Y_j by Lemma 68.8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). By (H_{d-1}) applied to $V_j \amalg W_j \to Y_j$ we find a surjective integral morphism $Y_j' \to Y_j$ with Y_j' a scheme, which Zariski locally factors through $V_j \amalg W_j$, and which is finite étale over a quasi-compact dense open $U_j \subset Y_j$. After shrinking U we may and do assume that $\pi_j^{-1}(U) \subset U_j$ (we may and do choose the same U for all j; some details omitted).

We claim that
$$Y = \coprod\nolimits_{j=1,\ldots,m} Y'_j \longrightarrow X$$
is the solution to our problem. First, this morphism is integral as on each summand we have the composition $Y'_j \to Y \to X$ of integral morphisms (Morphisms of Spaces, Lemma 67.45.4). Second, this morphism Zariski locally factors through $V = \coprod V_j$ because we saw above that each $Y'_j \to Y_j$ factors Zariski locally through $V_j \amalg W_j = V_j \times_X Y_j$. Finally, since both $Y'_j \to Y_j$ and $Y_j \to X$ are finite étale over U, so is the composition. This finishes the proof. □

68.10. Schematic locus

06NN In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We first prove this for reasonable algebraic spaces.

03JI **Proposition 68.10.1.** *Let S be a scheme. Let X be an algebraic space over S. If X is reasonable, then there exists a dense open subspace of X which is a scheme.*

Proof. By Properties of Spaces, Lemma 66.13.1 the question is local on X. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.1). Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 68.6.1. We will argue by induction on n that whenever

(1) $U \to X$ is a surjective étale morphism whose fibres have degree $\leq n$, and
(2) U is isomorphic to a locally closed subscheme of an affine scheme

then the schematic locus is dense in X.

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 68.8.1 using the morphism $U \to X$. Let $U_n \subset U$ be the inverse image of X_n. Then $U_n \to X_n$ is finite locally free of degree n. Hence X_n is a scheme by Properties of Spaces, Proposition 66.14.1 (and the fact that any finite set of points of U_n is contained in an affine open of U_n, see Properties, Lemma 28.29.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 5.21.1). Let $U' \subset U$ be the inverse image. Then $U' \to X'$ is surjective étale and has degrees of fibres bounded by $n-1$. By induction we see that the schematic locus of X' is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. □

086U **Theorem 68.10.2** (David Rydh)**.** *Let S be a scheme. Let X be an algebraic space over S. If X is decent, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 66.13.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since X' is not dense in X, there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing X by X'' we get a nonempty decent algebraic space X which does not contain *any* open subspace which is a scheme.

Choose a nonempty affine scheme U and an étale morphism $U \to X$. We may and do replace X by the open subscheme corresponding to the image of $|U| \to |X|$.

Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \ldots$$

constructed in Lemma 68.8.2 for the morphism $U \to X$. Note that $X_0 = X_1$ as $U \to X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \ldots$ be the induced sequence of open subschemes of U.

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \ldots$ with $V_n \subset U_n$. Namely, having constructed $V_1, \ldots, V_{n-1}$ we can always choose V_n unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \mathrm{Im}(|V_{n-1}| \to |X|)$ is contained in $|X| \setminus |X_n|$. Hence $V_{n-1} \to X'$ is an étale morphism whose fibres have degree bounded by $n - 1$. In other words, X' is reasonable (by definition), hence X' contains a nonempty open subscheme by Proposition 68.10.1. This is a contradiction which shows that we can pick V_n.

By Limits, Lemma 32.4.3 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\mathrm{Spec}(k) \to V_\infty$. The composition $\mathrm{Spec}(k) \to V_\infty \to U \to X$ has image contained in all X_d by construction. In other words, the fibred $U \times_X \mathrm{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. □

0BA1 **Lemma 68.10.3.** *Let S be a scheme. Let $X \to Y$ be a surjective finite locally free morphism of algebraic spaces over S. For $y \in |Y|$ the following are equivalent*

(1) *y is in the schematic locus of Y, and*
(2) *there exists an affine open $U \subset X$ containing the preimage of y.*

Proof. If $y \in Y$ is in the schematic locus, then it has an affine open neighbourhood $V \subset Y$ and the inverse image U of V in X is an open finite over V, hence affine. Thus (1) implies (2).

Conversely, assume that $U \subset X$ as in (2) is given. Set $R = X \times_Y X$ and denote the projections $s, t : R \to X$. Consider $Z = R \setminus s^{-1}(U) \cap t^{-1}(U)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of X which can loosely be described as the set of points of X which are R-equivalent to a point of $X \setminus U$. Hence $U' = X \setminus t(Z)$ is an R-invariant, open subspace of X contained in U which contains the fibre of $X \to Y$ over y. Since $X \to Y$ is open (Morphisms of Spaces, Lemma 67.30.6) the image of U' is an open subspace $V' \subset Y$. Since U' is R-invariant and $R = X \times_Y X$, we see that U' is the inverse image of V' (use Properties of Spaces, Lemma 66.4.3). After replacing Y by V' and X by U' we see that we may assume X is a scheme isomorphic to an open subscheme of an affine scheme.

Assume X is a scheme isomorphic to an open subscheme of an affine scheme. In this case the fppf quotient sheaf X/R is a scheme, see Properties of Spaces, Proposition 66.14.1. Since Y is a sheaf in the fppf topology, obtain a canonical map $X/R \to Y$ factoring $X \to Y$. Since $X \to Y$ is surjective finite locally free, it is surjective as a map of sheaves (Spaces, Lemma 65.5.9). We conclude that $X/R \to Y$ is surjective as a map of sheaves. On the other hand, since $R = X \times_Y X$ as sheaves we conclude that $X/R \to Y$ is injective as a map of sheaves. Hence $X/R \to Y$ is an isomorphism and we see that Y is representable. □

At this point we have several different ways for proving the following lemma.

06NG **Lemma 68.10.4.** *Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a scheme, then there exists a dense open subspace of X which is a scheme.*

First proof. The morphism $U \to X$ is finite locally free. Hence there is a decomposition of X into open and closed subspaces $X_d \subset X$ such that $U \times_X X_d \to X_d$ is finite locally free of degree d. Thus we may assume $U \to X$ is finite locally free of degree d. In this case, let $U_i \subset U$, $i \in I$ be the set of affine opens. For each i the morphism $U_i \to X$ is étale and has universally bounded fibres (namely, bounded by d). In other words, X is reasonable and the result follows from Proposition 68.10.1. □

Second proof. The question is local on X (Properties of Spaces, Lemma 66.13.1), hence may assume X is quasi-compact. Then U is quasi-compact. Then there exists a dense open subscheme $W \subset U$ which is separated (Properties, Lemma 28.29.3). Set $Z = U \setminus W$. Let $R = U \times_X U$ and $s, t : R \to U$ the projections. Then $t^{-1}(Z)$ is nowhere dense in R (Topology, Lemma 5.21.6) and hence $\Delta = s(t^{-1}(Z))$ is an R-invariant closed nowhere dense subset of U (Morphisms, Lemma 29.48.7). Let $u \in U \setminus \Delta$ be a generic point of an irreducible component. Since these points are dense in $U \setminus \Delta$ and since Δ is nowhere dense, it suffices to show that the image $x \in X$ of u is in the schematic locus of X. Observe that $t(s^{-1}(\{u\})) \subset W$ is a finite set of generic points of irreducible components of W (compare with Properties of Spaces, Lemma 66.11.1). By Properties, Lemma 28.29.1 we can find an affine open $V \subset W$ such that $t(s^{-1}(\{u\})) \subset V$. Since $t(s^{-1}(\{u\}))$ is the fibre of $|U| \to |X|$ over x, we conclude by Lemma 68.10.3. □

Third proof. (This proof is essentially the same as the second proof, but uses fewer references.) Assume X is an algebraic space, U a scheme, and $U \to X$ is a finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. Claim: The union of the R-invariant affine opens of U is topologically dense in U.

Proof of the claim. Let $W \subset U$ be an affine open. Set $W' = t(s^{-1}(W)) \subset U$. Since $s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subset U$ is a quasi-compact open. By Properties, Lemma 28.29.3 there exists a dense open $W'' \subset W'$ which is a separated scheme. Set $\Delta' = W' \setminus W''$. This is a nowhere dense closed subset of W''. Since $t|_{s^{-1}(W)} : s^{-1}(W) \to W'$ is open (because it is étale) we see that the inverse image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subset s^{-1}(W)$ is a nowhere dense closed subset (see Topology, Lemma 5.21.6). Hence, by Morphisms, Lemma 29.48.7 we see that

$$\Delta = s\left((t|_{s^{-1}(W)})^{-1}(\Delta')\right)$$

is a nowhere dense closed subset of W. Pick any point $\eta \in W$, $\eta \notin \Delta$ which is a generic point of an irreducible component of W (and hence of U). By our choices above the finite set $t(s^{-1}(\{\eta\})) = \{\eta_1, \ldots, \eta_n\}$ is contained in the separated scheme W''. Note that the fibres of s is are finite discrete spaces, and that generalizations lift along the étale morphism t, see Morphisms, Lemmas 29.36.12 and 29.25.9. In this way we see that each η_i is a generic point of an irreducible component of W''. Thus, by Properties, Lemma 28.29.1 we can find an affine open $V \subset W''$ such that $\{\eta_1, \ldots, \eta_n\} \subset V$. By Groupoids, Lemma 39.24.1 this implies that η is contained in an R-invariant affine open subscheme of U. The claim follows as W was chosen

as an arbitrary affine open of U and because the set of generic points of irreducible components of $W \setminus \Delta$ is dense in W.

Using the claim we can finish the proof. Namely, if $W \subset U$ is an R-invariant affine open, then the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 39.19.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows in particular that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 39.23.9. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 65.10.2. Hence having a dense collection of points contained in R-invariant affine open of U certainly implies that the schematic locus of X (see Properties of Spaces, Lemma 66.13.1) is open dense in X. □

68.11. Residue fields and henselian local rings

0EMV For a decent algebraic space we can define the residue field and the henselian local ring at a point. For example, the following lemma tells us the residue field of a point on a decent space is defined.

03K4 **Lemma 68.11.1.** *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

This map is always injective. If X is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 66.4.12 that the map is an injection in general. By Lemma 68.5.1 it is surjective when X is decent (actually one can say this is part of the definition of being decent). □

Let S be a scheme. Let X be an algebraic space over S. If a point $x \in |X|$ can be represented by a monomorphism $\mathrm{Spec}(k) \to X$, then the field k is unique up to unique isomorphism. For a decent algebraic space such a monomorphism exists for every point by Lemma 68.11.1 and hence the following definition makes sense.

0EMW **Definition 68.11.2.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The *residue field of X at x* is the unique field $\kappa(x)$ which comes equipped with a monomorphism $\mathrm{Spec}(\kappa(x)) \to X$ representing x.

Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ be a point. Set $y = f(x) \in |Y|$. Then the composition $\mathrm{Spec}(\kappa(x)) \to Y$ is in the equivalence class defining y and hence factors through $\mathrm{Spec}(\kappa(y)) \to Y$. In other words we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\kappa(x)) & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\kappa(y)) & \xrightarrow{y} & Y \end{array}$$

The left vertical morphism corresponds to a homomorphism $\kappa(y) \to \kappa(x)$ of fields. We will often simply call this the homomorphism induced by f.

0EMX **Lemma 68.11.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ be a point with image $y = f(x) \in |Y|$. The following are equivalent*

(1) *f induces an isomorphism $\kappa(y) \to \kappa(x)$, and*
(2) *the induced morphism $\mathrm{Spec}(\kappa(x)) \to Y$ is a monomorphism.*

Proof. Immediate from the discussion above. □

The following lemma tells us that the henselian local ring of a point on a decent algebraic space is defined.

0BBP **Lemma 68.11.4.** *Let S be a scheme. Let X be a decent algebraic space over S. For every point $x \in |X|$ there exists an étale morphism*

$$(U, u) \longrightarrow (X, x)$$

where U is an affine scheme, u is the only point of U lying over x, and the induced homomorphism $\kappa(x) \to \kappa(u)$ is an isomorphism.

Proof. We may assume that X is quasi-compact by replacing X with a quasi-compact open containing x. Recall that x can be represented by a quasi-compact (mono)morphism from the spectrum a field (by definition of decent spaces). Thus the lemma follows from Lemma 68.8.3. □

0BGU **Definition 68.11.5.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in X$ be a point. An *elementary étale neighbourhood* is an étale morphism $(U, u) \to (X, x)$ where U is a scheme, $u \in U$ is a point mapping to x, and the morphism $u = \mathrm{Spec}(\kappa(u)) \to X$ is a monomorphism. A *morphism of elementary étale neighbourhoods* $(U, u) \to (U', u')$ is defined as a morphism $U \to U'$ over X mapping u to u'.

If X is not decent then the category of elementary étale neighbourhoods may be empty.

0BGV **Lemma 68.11.6.** *Let S be a scheme. Let X be a decent algebraic space over S. Let x be a point of X. The category of elementary étale neighborhoods of (X, x) is cofiltered (see Categories, Definition 4.20.1).*

Proof. The category is nonempty by Lemma 68.11.4. Suppose that we have two elementary étale neighbourhoods $(U_i, u_i) \to (X, x)$. Then consider $U = U_1 \times_X U_2$. Since $\mathrm{Spec}(\kappa(u_i)) \to X$, $i = 1, 2$ are both monomorphisms in the class of x (Lemma 68.11.3) , we see that

$$u = \mathrm{Spec}(\kappa(u_1)) \times_X \mathrm{Spec}(\kappa(u_2))$$

is the spectrum of a field $\kappa(u)$ such that the induced maps $\kappa(u_i) \to \kappa(u)$ are isomorphisms. Then $u \to U$ is a point of U and we see that $(U, u) \to (X, x)$ is an elementary étale neighbourhood dominating (U_i, u_i). If $a, b : (U_1, u_1) \to (U_2, u_2)$ are two morphisms between our elementary étale neighbourhoods, then we consider the scheme

$$U = U_1 \times_{(a,b),(U_2 \times_X U_2),\Delta} U_2$$

Using Properties of Spaces, Lemma 66.16.6 we see that $U \to X$ is étale. Moreover, in exactly the same manner as before we see that U has a point u such that $(U, u) \to (X, x)$ is an elementary étale neighbourhood. Finally, $U \to U_1$ equalizes a and b and the proof is finished. □

0BGW **Definition 68.11.7.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The *henselian local ring of X at x*, is

$$\mathcal{O}^h_{X,x} = \operatorname{colim} \Gamma(U, \mathcal{O}_U)$$

where the colimit is over the elementary étale neighbourhoods $(U, u) \to (X, x)$.

Here is the analogue of Properties of Spaces, Lemma 66.22.1.

0EMY **Lemma 68.11.8.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. Let $(U, u) \to (X, x)$ be an elementary étale neighbourhood. Then*

$$\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$$

In words: the henselian local ring of X at x is equal to the henselization $\mathcal{O}^h_{U,u}$ of the local ring $\mathcal{O}_{U,u}$ of U at u.

Proof. Since the category of elementary étale neighbourhood of (X, x) is cofiltered (Lemma 68.11.6) we see that the category of elementary étale neighbourhoods of (U, u) is initial in the category of elementary étale neighbourhood of (X, x). Then the equality follows from More on Morphisms, Lemma 37.35.5 and Categories, Lemma 4.17.2 (initial is turned into cofinal because the colimit defining henselian local rings is over the opposite of the category of elementary étale neighbourhoods). □

0EMZ **Lemma 68.11.9.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over $x \in |X|$. The étale local ring $\mathcal{O}_{X,\overline{x}}$ of X at $\overline{x}$ (Properties of Spaces, Definition 66.22.2) is the strict henselization of the henselian local ring $\mathcal{O}^h_{X,x}$ of X at x.*

Proof. Follows from Lemma 68.11.8, Properties of Spaces, Lemma 66.22.1 and the fact that $(R^h)^{sh} = R^{sh}$ for a local ring $(R, \mathfrak{m}, \kappa)$ and a given separable algebraic closure κ^{sep} of κ. This equality follows from Algebra, Lemma 10.154.7. □

0EN0 **Lemma 68.11.10.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The residue field of the henselian local ring of X at x (Definition 68.11.7) is the residue field of X at x (Definition 68.11.2).*

Proof. Choose an elementary étale neighbourhood $(U, u) \to (X, x)$. Then $\kappa(u) = \kappa(x)$ and $\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$ (Lemma 68.11.8). The residue field of $\mathcal{O}^h_{U,u}$ is $\kappa(u)$ by Algebra, Lemma 10.155.1 (the output of this lemma is the construction/definition of the henselization of a local ring, see Algebra, Definition 10.155.3). □

0EPL **Remark 68.11.11.** Let S be a scheme. Let $f : X \to Y$ be a morphism of decent algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Choose an elementary étale neighbourhood $(V, v) \to (Y, y)$ (possible by Lemma 68.11.4). Then $V \times_Y X$ is an algebraic space étale over X which has a unique point x' mapping to x in X and to v in V. (Details omitted; use that all points can be represented by monomorphisms from spectra of fields.) Choose an elementary étale neighbourhood $(U, u) \to (V \times_Y X, x')$. Then we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\operatorname{Spec}(\mathcal{O}_{X,\overline{x}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}^h_{X,x}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{U,u}) & \longrightarrow & U & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}(\mathcal{O}_{Y,\overline{y}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}^h_{Y,y}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{V,v}) & \longrightarrow & V & \longrightarrow & Y
\end{array}$$

This comes from the identifications $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$, $\mathcal{O}^h_{X,x} = \mathcal{O}^h_{U,u}$, $\mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$, $\mathcal{O}^h_{Y,y} = \mathcal{O}^h_{V,v}$ see in Lemma 68.11.8 and Properties of Spaces, Lemma 66.22.1 and the functoriality of the (strict) henselization discussed in Algebra, Sections 10.154 and 10.155.

68.12. Points on decent spaces

03IG In this section we prove some properties of points on decent algebraic spaces. The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 65.14.9 shows that this is **not** true in general.

03K5 **Lemma 68.12.1.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \rightsquigarrow u$, then $u = u'$.*

Proof. Combine Lemmas 68.5.1 and 68.7.1. □

03IL **Lemma 68.12.2.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. Combine Lemmas 68.5.1 and 68.7.3. □

03K3 **Lemma 68.12.3.** *Let S be a scheme. Let X be a decent algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.8.6).*

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \to X$ such that x_1, x_2 are both in the image of $|U| \to |X|$. By Lemma 68.12.2 we can find a specialization $u_1 \rightsquigarrow u_2$ in U mapping to $x_1 \rightsquigarrow x_2$. By Lemma 68.12.2 we can find $u_2' \rightsquigarrow u_1$ mapping to $x_2 \rightsquigarrow x_1$. This means that $u_2' \rightsquigarrow u_2$ is a specialization between points of U mapping to the same point of X, namely x_2. This is not possible, unless $u_2' = u_2$, see Lemma 68.12.1. Hence also $u_1 = u_2$ as desired. □

03K6 **Proposition 68.12.4.** *Let S be a scheme. Let X be a decent algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.8.6).*

Proof. We have seen in Lemma 68.12.3 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 66.12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence Z is a decent algebraic space by Lemma 68.5.3. By Theorem 68.10.2 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 26.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

For decent algebraic spaces dimension works as expected.

0A4J **Lemma 68.12.5.** *Let S be a scheme. Dimension as defined in Properties of Spaces, Section 66.9 behaves well on decent algebraic spaces X over S.*

(1) *If $x \in |X|$, then $\dim_x(|X|) = \dim_x(X)$, and*

(2) $\dim(|X|) = \dim(X)$.

Proof. Proof of (1). Choose a scheme U with a point $u \in U$ and an étale morphism $h : U \to X$ mapping u to x. By definition the dimension of X at x is $\dim_u(|U|)$. Thus we may pick U such that $\dim_x(X) = \dim(|U|)$. Let d be an integer. If $\dim(U) \geq d$, then there exists a sequence of nontrivial specializations $u_d \rightsquigarrow \ldots \rightsquigarrow u_0$ in U. Taking the image we find a corresponding sequence $h(u_d) \rightsquigarrow \ldots \rightsquigarrow h(u_0)$ each of which is nontrivial by Lemma 68.12.1. Hence we see that the image of $|U|$ in $|X|$ has dimension at least d. Conversely, suppose that $x_d \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations in $|X|$ with x_0 in the image of $|U| \to |X|$. Then we can lift this to a sequence of specializations in U by Lemma 68.12.2.

Part (2) is an immediate consequence of part (1), Topology, Lemma 5.10.2, and Properties of Spaces, Section 66.9. □

0ABW **Lemma 68.12.6.** *Let S be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Then the dimension of the local ring of Y at y is $\geq$ to the dimension of the local ring of X at x.*

Proof. The definition of the dimension of the local ring of a point on an algebraic space is given in Properties of Spaces, Definition 66.10.2. Choose an étale morphism $(V, v) \to (Y, y)$ where V is a scheme. Choose an étale morphism $U \to V \times_Y X$ and a point $u \in U$ mapping to $x \in |X|$ and $v \in V$. Then $U \to V$ is locally quasi-finite and we have to prove that

$$\dim(\mathcal{O}_{V,v}) \geq \dim(\mathcal{O}_{U,u})$$

This is Algebra, Lemma 10.125.4. □

0ED0 **Lemma 68.12.7.** *Let S be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over S. Then* $\dim(X) \leq \dim(Y)$.

Proof. This follows from Lemma 68.12.6 and Properties of Spaces, Lemma 66.10.3. □

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 66.15.3. We will improve this lemma in Lemma 68.14.2.

03IK **Lemma 68.12.8.** *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism* $\mathrm{Spec}(k) \to X$. *If X is decent, then $X \cong \mathrm{Spec}(k')$ where k/k' is a finite separable extension.*

Proof. The assumption implies that $|X| = \{x\}$ is a singleton. Since X is decent we can find a quasi-compact monomorphism $\mathrm{Spec}(k') \to X$ whose image is x. Then the projection $U = \mathrm{Spec}(k') \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism, whence $U = \mathrm{Spec}(k)$, see Schemes, Lemma 26.23.11. Hence the projection $\mathrm{Spec}(k) = U \to \mathrm{Spec}(k')$ is étale and we win. □

68.13. Reduced singleton spaces

06QU A *singleton* space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 65.14.7. We develop a tiny bit of machinery to be able to talk about these.

06QV **Lemma 68.13.1.** *Let S be a scheme. Let Z be an algebraic space over S. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to Z$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & Z \end{array}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 67.31.5 that $\mathrm{Spec}(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. □

06QW **Lemma 68.13.2.** *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

1. *Z is reduced and $|Z|$ is a singleton,*
2. *there exists a surjective flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field, and*
3. *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 67.30.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k. Hence $W \times_Z \mathrm{Spec}(k)$ is a disjoint union of spectra of fields (see Remark 68.4.1), in particular reduced. Since $W \times_Z \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 35.19.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 67.23.2 and 67.39.9. It is also flat and surjective by Lemma 68.13.1. Hence (3) holds. □

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

06QX **Lemma 68.13.3.** *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

1. *Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and*

(2) *there exists a locally finitely presented, surjective, flat morphism* $\mathrm{Spec}(k) \to Z$ *where* k *is a field.*

Proof. Assume (2) holds. By Lemma 68.13.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k, hence a disjoint union of spectra of fields (see Remark 68.4.1), hence locally Noetherian. Since $W \times_Z \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 35.16.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 67.28.2 and 67.39.8. It is also flat and surjective by Lemma 68.13.1. Hence (2) holds. □

06QY **Lemma 68.13.4.** *Let* S *be a scheme. Let* $Z' \to Z$ *be a monomorphism of algebraic spaces over* S*. Assume there exists a field* k *and a locally finitely presented, surjective, flat morphism* $\mathrm{Spec}(k) \to Z$*. Then either* Z' *is empty or* $Z' = Z$*.*

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z \mathrm{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 66.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 67.10.8. We conclude that $\mathrm{Spec}(k) \to Z$ factors through Z'. But as $\mathrm{Spec}(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $\mathrm{Spec}(k) \to Z$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$ (see Spaces, Remark 65.5.2) and we conclude that $Z' = Z$. □

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

06QZ **Lemma 68.13.5.** *Let* S *be a scheme. Let* X *be an algebraic space over* S*. Let* $x \in |X|$*. Then there exists a unique monomorphism* $Z \to X$ *of algebraic spaces over* S *such that* Z *is an algebraic space which satisfies the equivalent conditions of Lemma 68.13.3 and such that the image of* $|Z| \to |X|$ *is* $\{x\}$*.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 65.9). Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint union of spectra of fields, using Remark 68.4.1, and using Schemes, Lemma 26.23.11 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces,

Theorem 65.10.5. As R' is the restriction of R by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 39.20.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 67.10.9. By Properties of Spaces, Lemma 66.4.3 we see that

$$|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 68.13.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 66.4.3. Hence the two projections are isomorphisms by Lemma 68.13.4 and we win. □

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 100.11.8.

06R0 **Definition 68.13.6.** Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The *residual space of X at x*[2] is the monomorphism $Z_x \to X$ constructed in Lemma 68.13.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow Z_x.$$

The residual space is often given by a monomorphism from the spectrum of a field.

0H1R **Lemma 68.13.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The residual space Z_x of X at x is isomorphic to the spectrum of a field if and only if x can be represented by a monomorphism $\mathrm{Spec}(k) \to X$ where k is a field. If X is decent, this holds for all $x \in |X|$.*

Proof. Since $Z_x \to X$ is a monomorphism, if $Z_x = \mathrm{Spec}(k)$ for some field k, then x is represented by the monomorphism $\mathrm{Spec}(k) = Z_x \to X$. Conversely, if $\mathrm{Spec}(k) \to X$ is a monomorphism which represents x, then $Z_x \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism whose source is nonempty by Properties of Spaces, Lemma 66.4.3. Hence $Z_x \times_X \mathrm{Spec}(k) = \mathrm{Spec}(k)$ by Morphisms of Spaces, Lemma 67.10.8. Hence we get a monomorphism $\mathrm{Spec}(k) \to Z_x$. This is an isomorphism by Lemma 68.13.4. The final statement follows from Lemma 68.11.1. □

The residual space is a regular algebraic space by the following lemma.

06R1 **Lemma 68.13.8.** *A reduced, locally Noetherian singleton algebraic space Z is regular.*

[2] This is nonstandard notation.

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S. Let $W \to Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma 68.13.3). The scheme $T = W \times_Z \mathrm{Spec}(k)$ is étale over k in particular regular, see Remark 68.4.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 35.19.2. By definition this means that Z is regular. □

0H1S **Lemma 68.13.9.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point. Assume*

(1) *$|f|(|Y|)$ is contained in $\{x\} \subset |X|$,*
(2) *Y is reduced, and*
(3) *X is locally Noetherian.*

Then f factors through the residual space Z_x of X at x.

Proof. Preliminary remark: since $Z_x \to X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \to Y$ such that $Y' \to X$ factors through Z_x. A remark here is that Y' is reduced as well.

Let U be an affine scheme and let $U \to X$ be an étale morphism such that x is in the image of $|U| \to |X|$. Since X is locally Noetherian, U is a Noetherian affine scheme. By assumption (1) we see that $Y' = U \times_X Y \to Y$ is surjective as well as étale. Denote $E \subset |U|$ the set of points mapping to x. There are no nontrivial specializations between the elements of E, see Lemma 68.7.2. The morphism $Y' \to U$ maps $|Y'|$ into E. By our construction of Z_x in the proof of Lemma 68.13.5 we know that $\coprod_{u \in E} u \to X$ factors through Z_x. Hence it suffices to prove that $Y' \to U$ factors through $\coprod_{u \in E} u \to X$. After replacing Y' by an étale covering by a scheme (which we are allowed by our preliminary remark), this follows from Morphisms, Lemma 29.58.2. □

0H1T **Lemma 68.13.10.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point. Assume*

(1) *$|f|(|Y|)$ is contained in $\{x\} \subset |X|$,*
(2) *Y is reduced, and*
(3) *x can be represented by a quasi-compact monomorphism $x : \mathrm{Spec}(k) \to X$ where k is a field (for example if X is decent).*

Then f factors through the residual space $Z_x = \mathrm{Spec}(k)$ of X at x.

Proof. By Lemma 68.13.7 we have $Z_x = \mathrm{Spec}(k)$.

Preliminary remark: since $\mathrm{Spec}(k) \to X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \to Y$ such that $Y' \to X$ factors through Z_x. A remark here is that Y' is reduced as well.

After replacing X by a quasi-compact open neighbourhood of x, we may assume X quasi-compact. By Lemma 68.8.3, x is a point of $T \subset U \subset X$ where $T \to U$ (resp. $U \to X$) is a closed (resp. open) immersion, and T is a scheme. By Properties of Spaces, Lemma 66.4.9, f factors through U, so we may assume $U = X$. Then f factors through T because Y is reduced, see Properties of Spaces, Lemma 66.12.4. So we may assume that $X = T$ is a scheme. By our preliminary remark we may assume Y is a scheme too. This reduces us to Morphisms, Lemma 29.58.1. □

0H2Y **Example 68.13.11.** Here is a counter example to Lemmas 68.13.9 and 68.13.10 in case X is neither locally Noetherian nor decent. Let k be a field. Let G be an infinite profinite group. Let Y be G viewed as a zero-dimensional affine k-group scheme, i.e., $Y = \mathrm{Spec}(\text{locally constant maps } G \to k)$. Let Γ be G viewed as a discrete k-group scheme, acting on Y by translations. Put $X = Y/\Gamma$. This is a one-point algebraic space, with projection $q : Y \to X$. Let $e \in G$ be the origin (any element would do), and view it as a k-point of Y. We get a k-point $x : \mathrm{Spec}(k) \to X$ which is a monomorphism since it is a section of $X \to \mathrm{Spec}(k)$. We claim that (although Y is affine and reduced and $|X| = \{x\}$), the morphism q does not factor through any morphism $\mathrm{Spec}(K) \to X$, where K is a field. Otherwise it would factor through x by Properties of Spaces, Lemma 66.4.11. Now the pullback of q by x is $\Gamma \to \mathrm{Spec}(k)$, with the projection $\Gamma \to Y$ being the orbit map $g \mapsto g \cdot e$. The latter has no section, whence the claim.

0H1U **Lemma 68.13.12.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ with residual space $Z_x \subset X$. Assume X is locally Noetherian. Then x is a closed point of $|X|$ if and only if the morphism $Z_x \to X$ is a closed immersion.*

Proof. If $Z_x \to X$ is a closed immersion, then x is a closed point of $|X|$, see Morphisms of Spaces, Lemma 67.12.3. Conversely, assume x is a closed point of $|X|$. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 66.12.3). Then Z is locally Noetherian by Morphisms of Spaces, Lemmas 67.23.7 and 67.23.5. Since also Z is reduced and $|Z| = \{x\}$ it $Z = Z_x$ is the residual space by definition. □

68.14. Decent spaces

047Y In this section we collect some useful facts on decent spaces.

0BB6 **Lemma 68.14.1.** *Any locally Noetherian decent algebraic space is quasi-separated.*

Proof. Namely, let X be an algebraic space (over some base scheme, for example over $\mathbf{Z}$) which is decent and locally Noetherian. Let $U \to X$ and $V \to X$ be étale morphisms with U and V affine schemes. We have to show that $W = U \times_X V$ is quasi-compact (Properties of Spaces, Lemma 66.3.3). Since X is locally Noetherian, the schemes U, V are Noetherian and W is locally Noetherian. Since X is decent, the fibres of the morphism $W \to U$ are finite. Namely, we can represent any $x \in |X|$ by a quasi-compact monomorphism $\mathrm{Spec}(k) \to X$. Then U_k and V_k are finite disjoint unions of spectra of finite separable extensions of k (Remark 68.4.1) and we see that $W_k = U_k \times_{\mathrm{Spec}(k)} V_k$ is finite. Let n be the maximum degree of a fibre of $W \to U$ at a generic point of an irreducible component of U. Consider the stratification

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

associated to $W \to U$ in More on Morphisms, Lemma 37.45.5. By our choice of n above we conclude that U_{n+1} is empty. Hence we see that the fibres of $W \to U$ are universally bounded. Then we can apply More on Morphisms, Lemma 37.45.3 to find a stratification

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = U$$

by closed subsets such that with $S_r = Z_r \setminus Z_{r-1}$ the morphism $W \times_U S_r \to S_r$ is finite locally free. Since U is Noetherian, the schemes S_r are Noetherian, whence

the schemes $W \times_U S_r$ are Noetherian, whence $W = \coprod W \times_U S_r$ is quasi-compact as desired. □

047Z **Lemma 68.14.2.** *Let S be a scheme. Let X be a decent algebraic space over S.*

(1) *If $|X|$ is a singleton then X is a scheme.*
(2) *If $|X|$ is a singleton and X is reduced, then $X \cong \mathrm{Spec}(k)$ for some field k.*

Proof. Assume $|X|$ is a singleton. It follows immediately from Theorem 68.10.2 that X is a scheme, but we can also argue directly as follows. Choose an affine scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Then U and R have finitely many points by Lemma 68.4.5 (and the definition of a decent space). All of these points are closed in U and R by Lemma 68.12.1. It follows that U and R are affine schemes. We may shrink U to a singleton space. Then U is the spectrum of a henselian local ring, see Algebra, Lemma 10.153.10. The projections $R \to U$ are étale, hence finite étale because U is the spectrum of a 0-dimensional henselian local ring, see Algebra, Lemma 10.153.3. It follows that X is a scheme by Groupoids, Proposition 39.23.9.

Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field. □

049D **Remark 68.14.3.** We will see in Limits of Spaces, Lemma 70.15.3 that an algebraic space whose reduction is a scheme is a scheme.

07U5 **Lemma 68.14.4.** *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that $\kappa(s) \subset k$ is algebraic. Then the image x of $\mathrm{Spec}(k) \to X$ is a closed point of $|X|$.

Proof. Suppose that $x \rightsquigarrow x'$ for some $x' \in |X|$. Choose an étale morphism $U \to X$ where U is a scheme and a point $u' \in U'$ mapping to x'. Choose a specialization $u \rightsquigarrow u'$ in U with u mapping to x in X, see Lemma 68.12.2. Then u is the image of a point w of the scheme $W = \mathrm{Spec}(k) \times_X U$. Since the projection $W \to \mathrm{Spec}(k)$ is étale we see that $\kappa(w) \supset k$ is finite. Hence $\kappa(w) \supset \kappa(s)$ is algebraic. Hence $\kappa(u) \supset \kappa(s)$ is algebraic. Thus u is a closed point of U by Morphisms, Lemma 29.20.2. Thus $u = u'$, whence $x = x'$. □

08AL **Lemma 68.14.5.** *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that the field extension $k/\kappa(s)$ is finite. Then $\mathrm{Spec}(k) \to X$ is a finite morphism. If $\kappa(s) = k$ then $\mathrm{Spec}(k) \to X$ is a closed immersion.

Proof. By Lemma 68.14.4 the image point $x \in |X|$ is closed. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 66.12.3). Note that Z is a decent algebraic space by Lemma 68.6.5. By Lemma 68.14.2 we see that $Z = \mathrm{Spec}(k')$ for some field k'. Of course $k \supset k' \supset \kappa(s)$. Then $\mathrm{Spec}(k) \to Z$ is a finite morphism of schemes and $Z \to X$ is a finite morphism as it is a closed immersion. Hence $\mathrm{Spec}(k) \to X$ is finite (Morphisms of Spaces, Lemma 67.45.4). If $k = \kappa(s)$, then $\mathrm{Spec}(k) = Z$ and $\mathrm{Spec}(k) \to X$ is a closed immersion. □

0AHB **Lemma 68.14.6.** *Let S be a scheme. Suppose X is a decent algebraic space over S. Let $x \in |X|$ be a closed point. Then x can be represented by a closed immersion $i : \mathrm{Spec}(k) \to X$ from the spectrum of a field.*

Proof. We know that x can be represented by a quasi-compact monomorphism $i : \mathrm{Spec}(k) \to X$ where k is a field (Definition 68.6.1). Let $U \to X$ be an étale morphism where U is an affine scheme. As x is closed and X decent, the fibre F of $|U| \to |X|$ over x consists of closed points (Lemma 68.12.1). As i is a monomorphism, so is $U_k = U \times_X \mathrm{Spec}(k) \to U$. In particular, the map $|U_k| \to F$ is injective. Since U_k is quasi-compact and étale over a field, we see that U_k is a finite disjoint union of spectra of fields (Remark 68.4.1). Say $U_k = \mathrm{Spec}(k_1) \amalg \ldots \amalg \mathrm{Spec}(k_r)$. Since $\mathrm{Spec}(k_i) \to U$ is a monomorphism, we see that its image u_i has residue field $\kappa(u_i) = k_i$. Since $u_i \in F$ is a closed point we conclude the morphism $\mathrm{Spec}(k_i) \to U$ is a closed immersion. As the u_i are pairwise distinct, $U_k \to U$ is a closed immersion. Hence i is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). This finishes the proof. □

68.15. Locally separated spaces

088H It turns out that a locally separated algebraic space is decent.

088I **Lemma 68.15.1.** *Let A be a ring. Let k be a field. Let $\mathfrak{p}_n$, $n \geq 1$ be a sequence of pairwise distinct primes of A. Moreover, for each n let $k \to \kappa(\mathfrak{p}_n)$ be an embedding. Then the closure of the image of*

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(\kappa(\mathfrak{p}_n) \otimes_k \kappa(\mathfrak{p}_m)) \longrightarrow \mathrm{Spec}(A \otimes A)$$

meets the diagonal.

Proof. Set $k_n = \kappa(\mathfrak{p}_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \mathrm{Spec}(k_n)$ the open and closed point corresponding to $A \to k_n$. Then $\mathrm{Spec}(A) = Z \amalg \{x_n\}$ where Z is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where e_n is the idempotent of A corresponding to the factor k_n and Z is nonempty as the ideal generated by the e_n is not equal to A. We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

$$(\prod k_n) \otimes_k (\prod k_m) \longrightarrow \prod\nolimits_{n \neq m} k_n \otimes_k k_m$$

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\mathrm{Spec}(A))$ is $\Delta(Z)$. Thus

it suffices to show that

$$\coprod\nolimits_{n\neq m} \operatorname{Spec}(k_n \otimes_k k_m) \longrightarrow \operatorname{Spec}(\prod\nolimits_{n\neq m} k_n \otimes_k k_m)$$

has dense image. This follows as the family of ring maps $\prod_{n\neq m} k_n \otimes_k k_m \to k_n \otimes_k k_m$ is jointly injective. □

088J **Lemma 68.15.2** (David Rydh)**.** *A locally separated algebraic space is decent.*

Proof. Let S be a scheme and let X be a locally separated algebraic space over S. We may assume $S = \operatorname{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 66.3.1. Unadorned fibre products will be over $\mathbf{Z}$. Let $x \in |X|$. Choose a scheme U, an étale morphism $U \to X$, and a point $u \in U$ mapping to x in $|X|$. As usual we identify $u = \operatorname{Spec}(\kappa(u))$. As X is locally separated the morphism

$$u \times_X u \to u \times u$$

is an immersion (Morphisms of Spaces, Lemma 67.4.5). Hence More on Groupoids, Lemma 40.11.5 tells us that it is a closed immersion (use Schemes, Lemma 26.10.4). As $u \times_X u \to u \times_X U$ is a monomorphism (base change of $u \to U$) and as $u \times_X U \to u$ is étale we conclude that $u \times_X u$ is a disjoint union of spectra of fields (see Remark 68.4.1 and Schemes, Lemma 26.23.11). Since it is also closed in the affine scheme $u \times u$ we conclude $u \times_X u$ is a finite disjoint union of spectra of fields. Thus x can be represented by a monomorphism $\operatorname{Spec}(k) \to X$ where k is a field, see Lemma 68.4.3.

Next, let $U = \operatorname{Spec}(A)$ be an affine scheme and let $U \to X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \operatorname{Spec}(k)$ is finite. Write $F = \coprod_{i\in I} \operatorname{Spec}(k_i)$ as the disjoint union of finite separable extensions of k. We have to show that I is finite. Set $R = U \times_X U$. As X is locally separated, the morphism $j : R \to U \times U$ is an immersion. Let $U' \subset U \times U$ be an open such that j factors through a closed immersion $j' : R \to U'$. Let $e : U \to R$ be the diagonal map. Using that e is a morphism between schemes étale over U such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \amalg W$ for some open and closed subscheme $W \subset R$. Since j' is a closed immersion we conclude that $j'(W) \subset U'$ is closed and disjoint from $j'(e(U))$. Therefore $\overline{j(W)} \cap \Delta(U) = \emptyset$ in $U \times U$. Note that W contains $\operatorname{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i'$, $i, i' \in I$. By Lemma 68.15.1 we conclude that I is finite as desired. □

68.16. Valuative criterion

06NP For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

03KJ **Proposition 68.16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact, and X is decent. Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 67.42.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & Y \end{array}$$

be a diagram as in Morphisms of Spaces, Definition 67.41.1. Let $X_A = \mathrm{Spec}(A) \times_Y X$, so that we have

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X_A \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

By Morphisms of Spaces, Lemma 67.8.4 we see that $X_A \to \mathrm{Spec}(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that X_A is decent also, see Lemma 68.5.3. Moreover, as f is universally closed, we see that $X_A \to \mathrm{Spec}(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $\mathrm{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\mathrm{Spec}(K) \to X$. Let $y \in |Y| = |\mathrm{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\mathrm{Spec}(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y. Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 68.7.3 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & X \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{array}{ccc} K' & \longleftarrow & \mathcal{O}_{U,u} \\ \uparrow & & \uparrow \\ K & & \\ & \nwarrow & \\ & & A \end{array}$$

By Algebra, Lemma 10.50.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K'. Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 67.41.1 and the proposition is proved. □

68.17. Relative conditions

03KW This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

03KZ **Definition 68.17.1.** Let S be a scheme. We say an algebraic space X over S *has property* (β) if X has the corresponding property of Lemma 68.5.1. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *has property* (β) if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ has property (β).
(2) We say f is *decent* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
(3) We say f is *reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.
(4) We say f is *very reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 68.17.10 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

03M5 **Lemma 68.17.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We have the following implications among the conditions on f:*

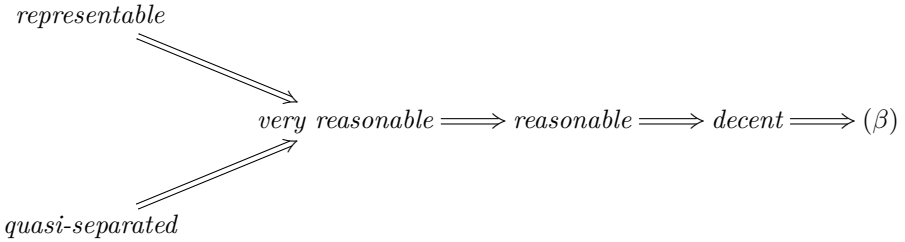

Proof. This is clear from the definitions, Lemma 68.5.1 and Morphisms of Spaces, Lemma 67.4.12. □

Here is another sanity check.

0ABX **Lemma 68.17.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If X is decent (resp. is reasonable, resp. has property (β) of Lemma 68.5.1), then f is decent (resp. reasonable, resp. has property (β)).*

Proof. Let T be a scheme and let $T \to Y$ be a morphism. Then $T \to Y$ is representable, hence the base change $T \times_Y X \to X$ is representable. Hence if X is decent (or reasonable), then so is $T \times_Y X$, see Lemma 68.6.5. Similarly, for property (β), see Lemma 68.5.3. □

03L0 **Lemma 68.17.4.** *Having property (β), being decent, or being reasonable is preserved under arbitrary base change.*

Proof. This is immediate from the definition. □

0ABY **Lemma 68.17.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\omega \in \{\beta, decent, reasonable\}$. Suppose that Y has property (ω) and $f : X \to Y$ has (ω). Then X has (ω).*

Proof. Let us prove the lemma in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X. Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. Then x corresponds to a point x' of $\mathrm{Spec}(k) \times_Y X$. By assumption x' is represented by a monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$. Clearly the composition $\mathrm{Spec}(k') \to X$ is a monomorphism representing x.

Let us prove the lemma in case $\omega = decent$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $\mathrm{Spec}(k) \times_Y X$ is decent. Hence the monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \mathrm{Spec}(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 67.8.4 and 67.8.5). As the point x was arbitrary this implies X is decent.

Let us prove the lemma in case $\omega = reasonable$. Choose $V \to Y$ étale with V an affine scheme. Choose $U \to V \times_Y X$ étale with U an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 68.3.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on f we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 68.3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. □

03L1 **Lemma 68.17.6.** *Having property (β), being decent, or being reasonable is preserved under compositions.*

Proof. Let $\omega \in \{\beta, decent, reasonable\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Assume f and g both have property (ω). Then we have to show that for any scheme T and morphism $T \to Z$ the space $T \times_Z X$ has (ω). By Lemma 68.17.4 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω), and that $f : X \to Y$ is a morphism with (ω). Then X has (ω). This is the content of Lemma 68.17.5. □

0ABZ **Lemma 68.17.7.** *Let S be a scheme. Let $f : X \to Y$, $g : Z \to Y$ be morphisms of algebraic spaces over S. If X and Z are decent (resp. reasonable, resp. have property (β) of Lemma 68.5.1), then so does $X \times_Y Z$.*

Proof. Namely, by Lemma 68.17.3 the morphism $X \to Y$ has the property. Then the base change $X \times_Y Z \to Z$ has the property by Lemma 68.17.4. And finally this implies $X \times_Y Z$ has the property by Lemma 68.17.5. □

03L2 **Lemma 68.17.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable\}$. Assume*

(1) *f is quasi-compact,*
(2) *f is étale,*
(3) *$|f| : |X| \to |Y|$ is surjective, and*
(4) *the algebraic space X has property $\mathcal{P}$.*

Then Y has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point

$x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $\mathrm{Spec}(k) \to X$, with k a field. By Lemma 68.4.3 it suffices to show that the projections $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

$$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_Y X \longrightarrow \mathrm{Spec}(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 66.16.8 we see that $\mathrm{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k. By Schemes, Lemma 26.23.11 we see that the first arrow identifies $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k)$ with a finite disjoint union of spectra of finite separable field extensions of k. Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = \textit{decent}$. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : \mathrm{Spec}(k) \to Y$. Pick such a y. Pick an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |Y|$ contains y. By Lemma 68.4.5 it suffices to show that U_y is a finite scheme over k. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a quasi-compact étale algebraic space over k. Hence by Properties of Spaces, Lemma 66.16.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k. Say $X_y = \{x_1, \ldots, x_n\}$ so x_i is given by $x_i : \mathrm{Spec}(k_i) \to X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = \mathrm{Spec}(k_i) \times_X U$ are finite over k_i. Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = \textit{reasonable}$. Pick an affine scheme V and an étale morphism $V \to Y$. We have the show the fibres of $V \to Y$ are universally bounded. The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \to V \times_Y X$, see Properties of Spaces, Lemma 66.6.3. Here is a picture (solid diagram)

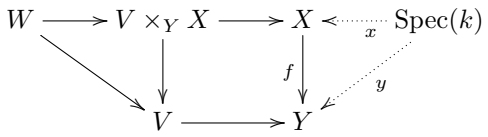

The morphism $W \to X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \to X$. We claim that the same integer works for bounding the fibres of $V \to Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$\mathrm{Spec}(k) \times_{x,X} W \to \mathrm{Spec}(k) \times_{x,X} (V \times_Y X) = \mathrm{Spec}(k) \times_{y,Y} V$$

which shows that the degree of $\mathrm{Spec}(k) \times_{y,Y} V$ over k is less than or equal to the degree of $\mathrm{Spec}(k) \times_{x,X} W$ over k, i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 68.3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) $\square$

03L3 **Lemma 68.17.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable, very\ reasonable\}$. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

If $\mathcal{P} \in \{(\beta), decent, reasonable\}$, then this is also equivalent to

(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. The implication (3) $\Rightarrow$ (1) can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over S. Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 68.5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) $\Rightarrow$ (5) is trivial. Let $V \to Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram

$$\begin{array}{ccc} Z \times_Y V & \xrightarrow{q} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{array}{ccccc} V \times_Y X & \longleftarrow & (\coprod W_i) \times_Y X & \longrightarrow & Z \times_Y X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & \coprod W_i & \longrightarrow & Z \end{array}$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 68.5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 68.17.8 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. □

03L4 **Remark 68.17.10.** An informal description of the properties (β), decent, reasonable, very reasonable was given in Section 68.6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

03M6 **Lemma 68.17.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 68.17.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 67.42.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 68.16.1 we have seen that it suffices to show, for any valuation ring A, and any morphism $\mathrm{Spec}(A) \to Y$, that the base change $f_A : X_A \to \mathrm{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 68.16.1 applies to the morphism f_A and we win. □

68.18. Points of fibres

0AC0 Let S be a scheme. Consider a cartesian diagram

0AC1 (68.18.0.1)
$$\begin{array}{ccc} W & \xrightarrow{q} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces over S. Let $x \in |X|$ and $z \in |Z|$ be points mapping to the same point $y \in |Y|$. We may ask: When is the set

0AC2 (68.18.0.2)
$$F_{x,z} = \{w \in |W| \text{ such that } p(w) = x \text{ and } q(w) = z\}$$

finite?

0AC3 **Example 68.18.1.** If X, Y, Z are schemes, then the set $F_{x,z}$ is equal to the spectrum of $\kappa(x) \otimes_{\kappa(y)} \kappa(z)$ (Schemes, Lemma 26.17.5). Thus we obtain a finite set if either $\kappa(y) \subset \kappa(x)$ is finite or if $\kappa(y) \subset \kappa(z)$ is finite. In particular, this is always the case if g is quasi-finite at z (Morphisms, Lemma 29.20.5).

0AC4 **Example 68.18.2.** Let K be a characteristic 0 field endowed with an automorphism σ of infinite order. Set $Y = \mathrm{Spec}(K)/\mathbf{Z}$ and $X = \mathbf{A}^1_K/\mathbf{Z}$ where $\mathbf{Z}$ acts on K via σ and on $\mathbf{A}^1_K = \mathrm{Spec}(K[t])$ via $t \mapsto t + 1$. Let $Z = \mathrm{Spec}(K)$. Then $W = \mathbf{A}^1_K$. Picture
$$\begin{array}{ccc} \mathbf{A}^1_K & \xrightarrow{q} & \mathrm{Spec}(K) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{A}^1_K/\mathbf{Z} & \xrightarrow{f} & \mathrm{Spec}(K)/\mathbf{Z} \end{array}$$
Take x corresponding to $t = 0$ and z the unique point of $\mathrm{Spec}(K)$. Then we see that $F_{x,z} = \mathbf{Z}$ as a set.

0AC5 **Lemma 68.18.3.** *In the situation of (68.18.0.1) if $Z' \to Z$ is a morphism and $z' \in |Z'|$ maps to z, then the induced map $F_{x,z'} \to F_{x,z}$ is surjective.*

Proof. Set $W' = X \times_Y Z' = W \times_Z Z'$. Then $|W'| \to |W| \times_{|Z|} |Z'|$ is surjective by Properties of Spaces, Lemma 66.4.3. Hence the surjectivity of $F_{x,z'} \to F_{x,z}$. □

0AC6 **Lemma 68.18.4.** *In diagram (68.18.0.1) the set (68.18.0.2) is finite if f is of finite type and f is quasi-finite at x.*

Proof. The morphism q is quasi-finite at every $w \in F_{x,z}$, see Morphisms of Spaces, Lemma 67.27.2. Hence the lemma follows from Morphisms of Spaces, Lemma 67.27.9. □

0AC7 **Lemma 68.18.5.** *In diagram (68.18.0.1) the set (68.18.0.2) is finite if y can be represented by a monomorphism $\mathrm{Spec}(k) \to Y$ where k is a field and g is quasi-finite at z. (Special case: Y is decent and g is étale.)*

Proof. By Lemma 68.18.3 applied twice we may replace Z by $Z_k = \mathrm{Spec}(k) \times_Y Z$ and X by $X_k = \mathrm{Spec}(k) \times_Y X$. We may and do replace Y by $\mathrm{Spec}(k)$ as well. Note that $Z_k \to \mathrm{Spec}(k)$ is quasi-finite at z by Morphisms of Spaces, Lemma 67.27.2. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Z_k$ mapping v to z. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to X_k$ mapping u to x. Again by Lemma 68.18.3 it suffices to show $F_{u,v}$ is finite for the diagram

$$\begin{array}{ccc} U \times_{\mathrm{Spec}(k)} V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

The morphism $V \to \mathrm{Spec}(k)$ is quasi-finite at v (follows from the general discussion in Morphisms of Spaces, Section 67.22 and the definition of being quasi-finite at a point). At this point the finiteness follows from Example 68.18.1. The parenthetical remark of the statement of the lemma follows from the fact that on decent spaces points are represented by monomorphisms from fields and from the fact that an étale morphism of algebraic spaces is locally quasi-finite. □

0AC8 **Lemma 68.18.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$ and assume that y is represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \to Y$. Then $|X_k| \to |X|$ is a homeomorphism onto $f^{-1}(\{y\}) \subset |X|$ with induced topology.*

Proof. We will use Properties of Spaces, Lemma 66.16.7 and Morphisms of Spaces, Lemma 67.10.9 without further mention. Let $V \to Y$ be an étale morphism with V affine such that there exists a $v \in V$ mapping to y. Since $\mathrm{Spec}(k) \to Y$ is quasi-compact there are a finite number of points of V mapping to y (Lemma 68.4.5). After shrinking V we may assume v is the only one. Choose a scheme U and a surjective étale morphism $U \to X$. Consider the commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U_V & \longleftarrow & U_v \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X_V & \longleftarrow & X_v \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & \longleftarrow & v \end{array}$$

Since $U_v \to U_V$ identifies U_v with a subset of U_V with the induced topology (Schemes, Lemma 26.18.5), and since $|U_V| \to |X_V|$ and $|U_v| \to |X_v|$ are surjective and open, we see that $|X_v| \to |X_V|$ is a homeomorphism onto its image (with induced topology). On the other hand, the inverse image of $f^{-1}(\{y\})$ under the open map $|X_V| \to |X|$ is equal to $|X_v|$. We conclude that $|X_v| \to f^{-1}(\{y\})$ is open.

The morphism $X_v \to X$ factors through X_k and $|X_k| \to |X|$ is injective with image $f^{-1}(\{y\})$ by Properties of Spaces, Lemma 66.4.3. Using $|X_v| \to |X_k| \to f^{-1}(\{y\})$ the lemma follows because $X_v \to X_k$ is surjective. □

0AC9 **Lemma 68.18.7.** *Let X be an algebraic space locally of finite type over a field k. Let $x \in |X|$. Consider the conditions*

(1) $\dim_x(|X|) = 0$,
(2) *x is closed in $|X|$ and if $x' \rightsquigarrow x$ in $|X|$ then $x' = x$,*
(3) *x is an isolated point of $|X|$,*
(4) $\dim_x(X) = 0$,
(5) *$X \to \mathrm{Spec}(k)$ is quasi-finite at x.*

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) is equivalent to the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemmas 67.34.7 and 67.34.8.

Let $U \to X$ be an étale morphism where U is an affine scheme and let $u \in U$ be a point mapping to x. Moreover, if x is a closed point, e.g., in case (2) or (3), then we may and do assume that u is a closed point. Observe that $\dim_u(U) = \dim_x(X)$ by definition and that this is equal to $\dim(\mathcal{O}_{U,u})$ if u is a closed point, see Algebra, Lemma 10.114.6.

If $\dim_x(X) > 0$ and u is closed, by the arguments above we can choose a nontrivial specialization $u' \rightsquigarrow u$ in U. Then the transcendence degree of $\kappa(u')$ over k exceeds the transcendence degree of $\kappa(u)$ over k. It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k are well defined, see Morphisms of Spaces, Definition 67.33.1. This applies in particular in cases (2) and (3) and we conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite at x, then $U \to \mathrm{Spec}(k)$ is locally quasi-finite at u, hence u is an isolated point of U (Morphisms, Lemma 29.20.6). It follows that (5) implies (2) and (3) as $|U| \to |X|$ is continuous and open.

Assume X is decent and (1) holds. Then $\dim_x(X) = \dim_x(|X|)$ by Lemma 68.12.5 and the proof is complete. □

0ACA **Lemma 68.18.8.** *Let X be an algebraic space locally of finite type over a field k. Consider the conditions*

(1) *$|X|$ is a finite set,*
(2) *$|X|$ is a discrete space,*
(3) $\dim(|X|) = 0$,
(4) $\dim(X) = 0$,
(5) *$X \to \mathrm{Spec}(k)$ is locally quasi-finite,*

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) implies the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemma 67.34.7.

Let $U \to X$ be a surjective étale morphism where U is a scheme.

If $\dim(U) > 0$, then choose a nontrivial specialization $u \rightsquigarrow u'$ in U and the transcendence degree of $\kappa(u)$ over k exceeds the transcendence degree of $\kappa(u')$ over k.

It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k is well defined, see Morphisms of Spaces, Definition 67.33.1. We conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite, then U is locally Noetherian (Morphisms, Lemma 29.15.6) of dimension 0 (Morphisms, Lemma 29.29.5) and hence is a disjoint union of spectra of Artinian local rings (Properties, Lemma 28.10.5). Hence U is a discrete topological space, and since $|U| \to |X|$ is continuous and open, the same is true for $|X|$. In other words, (4) implies (2) and (3).

Assume X is decent and (1) holds. Then we may choose U above to be affine. The fibres of $|U| \to |X|$ are finite (this is a part of the defining property of decent spaces). Hence U is a finite type scheme over k with finitely many points. Hence U is quasi-finite over k (Morphisms, Lemma 29.20.7) which by definition means that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. □

0ACB **Lemma 68.18.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $F = f^{-1}(\{y\})$ with induced topology from $|X|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$. Let $\tilde{x} \in |X_k|$ map to $x \in |X|$. Consider the following conditions*

0ACC (1) $\dim_x(F) = 0$,
0ACD (2) *x is isolated in F,*
0ACE (3) *x is closed in F and if $x' \rightsquigarrow x$ in F, then $x = x'$,*
0ACF (4) $\dim_{\tilde{x}}(|X_k|) = 0$,
0ACG (5) *$\tilde{x}$ is isolated in $|X_k|$,*
0ACH (6) *$\tilde{x}$ is closed in $|X_k|$ and if $\tilde{x}' \rightsquigarrow \tilde{x}$ in $|X_k|$, then $\tilde{x} = \tilde{x}'$,*
0ACI (7) $\dim_{\tilde{x}}(X_k) = 0$,
0ACJ (8) *f is quasi-finite at x.*

Then we have

$$(4) \underset{f \text{ decent}}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then all conditions are equivalent.

Proof. By Lemma 68.18.7 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is quasi-finite at $\tilde{x}$. Thus by Morphisms of Spaces, Lemma 67.27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 68.18.7 shows that (4) implies (5).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 68.18.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. □

0ACK **Lemma 68.18.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $y \in |Y|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$ and let $F = f^{-1}(\{y\})$ with the induced topology from $|X|$. Consider the following conditions*

0ACL (1) *F is finite,*
0ACM (2) *F is a discrete topological space,*
0ACN (3) $\dim(F) = 0$,
0ACP (4) *$|X_k|$ is a finite set,*
0ACQ (5) *$|X_k|$ is a discrete space,*
0ACR (6) $\dim(|X_k|) = 0$,
0ACS (7) $\dim(X_k) = 0$,
0ACT (8) *f is quasi-finite at all points of $|X|$ lying over y.*

Then we have

$$(1) \Longleftarrow (4) \underset{f \text{ decent}}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then (1) implies all the other conditions.

Proof. By Lemma 68.18.8 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Thus by Morphisms of Spaces, Lemma 67.27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 68.18.8 shows that (4) implies (5).

The map $|X_k| \to F$ is surjective by Properties of Spaces, Lemma 66.4.3 and we see (4) $\Rightarrow$ (1).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 68.18.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. □

68.19. Monomorphisms

06RY Here is another case where monomorphisms are representable. Please see More on Morphisms of Spaces, Section 76.4 for more information.

06RZ **Lemma 68.19.1.** *Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S. Let $f : X \to Y$ be a monomorphism of algebraic spaces over S. Then f is representable, i.e., X is a scheme.*

Proof. This immediately reduces to the case $Y = \mathrm{Spec}(A)$ where A is a zero dimensional local ring, i.e., $\mathrm{Spec}(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 67.10.9) we see that $U \to X$ is surjective. Note that $U \times_X U = U \times_Y U = \mathrm{Spec}(B \otimes_A B)$. Thus we see that the ring maps $B \to B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a $A/\mathfrak{m}_A$-vector space. Since $B \to B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 29.57.8 and 29.57.9). Every prime of B lies over $\mathfrak{m}_A$ (the unique prime of A). Hence $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}_A)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 10.53.2

and 10.53.6. Hence all prime ideals of B are maximal and $B = B_1 \times \ldots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 10.53.5. Thus $B \to B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 10.153.10. Thus X is an affine scheme by Groupoids, Proposition 39.23.9. □

68.20. Generic points

0BB7 This section is a continuation of Properties of Spaces, Section 66.11.

0ABV **Lemma 68.20.1.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The following are equivalent*

(1) *x is a generic point of an irreducible component of $|X|$,*
(2) *for any étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,*
(3) *for some étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,*
(4) *the dimension of the local ring of X at x is zero, and*
(5) *x is a point of codimension 0 on X*

Proof. Conditions (4) and (5) are equivalent for any algebraic space by definition, see Properties of Spaces, Definition 66.10.2. Observe that any Y as in (2) and (3) is decent by Lemma 68.6.6. Thus it suffices to prove the equivalence of (1) and (4) as then the equivalence with (2) and (3) follows since the dimension of the local ring of Y at y is equal to the dimension of the local ring of X at x. Let $f : U \to X$ be an étale morphism from an affine scheme and let $u \in U$ be a point mapping to x.

Assume (1). Let $u' \rightsquigarrow u$ be a specialization in U. Then $f(u') = f(u) = x$. By Lemma 68.12.1 we see that $u' = u$. Hence u is a generic point of an irreducible component of U. Thus $\dim(\mathcal{O}_{U,u}) = 0$ and we see that (4) holds.

Assume (4). The point x is contained in an irreducible component $T \subset |X|$. Since $|X|$ is sober (Proposition 68.12.4) we T has a generic point x'. Of course $x' \rightsquigarrow x$. Then we can lift this specialization to $u' \rightsquigarrow u$ in U (Lemma 68.12.2). This contradicts the assumption that $\dim(\mathcal{O}_{U,u}) = 0$ unless $u' = u$, i.e., $x' = x$. □

0ED1 **Lemma 68.20.2.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $T \subset |X|$ be an irreducible closed subset. Let $\xi \in T$ be the generic point (Proposition 68.12.4). Then $\mathrm{codim}(T, |X|)$ (Topology, Definition 5.11.1) is the dimension of the local ring of X at ξ (Properties of Spaces, Definition 66.10.2).*

Proof. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to X$ sending u to ξ. Then any sequence of nontrivial specializations $\xi_e \rightsquigarrow \ldots \rightsquigarrow \xi_0 = \xi$ can be lifted to a sequence $u_e \rightsquigarrow \ldots \rightsquigarrow u_0 = u$ in U by Lemma 68.12.2. Conversely, any sequence of nontrivial specializations $u_e \rightsquigarrow \ldots \rightsquigarrow u_0 = u$ in U maps to a sequence of nontrivial specializations $\xi_e \rightsquigarrow \ldots \rightsquigarrow \xi_0 = \xi$ by Lemma 68.12.1. Because $|X|$ and U are sober topological spaces we conclude that the codimension of T in $|X|$ and of $\overline{\{u\}}$ in U are the same. In this way the lemma reduces to the schemes case which is Properties, Lemma 28.10.3. □

0BB8 **Lemma 68.20.3.** *Let S be a scheme. Let X be an algebraic space over S. Assume*

(1) *every quasi-compact scheme étale over X has finitely many irreducible components, and*
(2) *every $x \in |X|$ of codimension 0 on X can be represented by a monomorphism $\mathrm{Spec}(k) \to X$.*

Then X is a reasonable algebraic space.

Proof. Let U be an affine scheme and let $a : U \to X$ be an étale morphism. We have to show that the fibres of a are universally bounded. By assumption (1) the scheme U has finitely many irreducible components. Let $u_1, \ldots, u_n \in U$ be the generic points of these irreducible components. Let $\{x_1, \ldots, x_m\} \subset |X|$ be the image of $\{u_1, \ldots, u_n\}$. Each x_j is a point of codimension 0. By assumption (2) we may choose a monomorphism $\mathrm{Spec}(k_j) \to X$ representing x_j. By Properties of Spaces, Lemma 66.11.1 we have

$$U \times_X \mathrm{Spec}(k_j) = \coprod\nolimits_{a(u_i)=x_j} \mathrm{Spec}(\kappa(u_i))$$

This is a scheme finite over $\mathrm{Spec}(k_j)$ of degree $d_j = \sum_{a(u_i)=x_j} [\kappa(u_i) : k_j]$. Set $n = \max d_j$.

Observe that a is separated (Properties of Spaces, Lemma 66.6.4). Consider the stratification

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

associated to $U \to X$ in Lemma 68.8.2. By our choice of n above we conclude that X_{n+1} is empty. Namely, if not, then $a^{-1}(X_{n+1})$ is a nonempty open of U and hence would contain one of the x_i. This would mean that X_{n+1} contains $x_j = a(u_i)$ which is impossible. Hence we see that the fibres of $U \to X$ are universally bounded (in fact by the integer n). □

0BB9 **Lemma 68.20.4.** *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *X is decent and $|X|$ has finitely many irreducible components,*
(2) *every quasi-compact scheme étale over X has finitely many irreducible components, there are finitely many $x \in |X|$ of codimension 0 on X, and each of these can be represented by a monomorphism $\mathrm{Spec}(k) \to X$,*
(3) *there exists a dense open $X' \subset X$ which is a scheme, X' has finitely many irreducible components with generic points $\{x'_1, \ldots, x'_m\}$, and the morphism $x'_j \to X$ is quasi-compact for $j = 1, \ldots, m$.*

Moreover, if these conditions hold, then X is reasonable and the points $x'_j \in |X|$ are the generic points of the irreducible components of $|X|$.

Proof. In the proof we use Properties of Spaces, Lemma 66.11.1 without further mention. Assume (1). Then X has a dense open subscheme X' by Theorem 68.10.2. Since the closure of an irreducible component of $|X'|$ is an irreducible component of $|X|$, we see that $|X'|$ has finitely many irreducible components. Thus (3) holds.

Assume $X' \subset X$ is as in (3). Let $\{x'_1, \ldots, x'_m\}$ be the generic points of the irreducible components of X'. Let $a : U \to X$ be an étale morphism with U a quasi-compact scheme. To prove (2) it suffices to show that U has finitely many irreducible components whose generic points lie over $\{x'_1, \ldots, x'_m\}$. It suffices to prove this for the members of a finite affine open cover of U, hence we may and do assume U is affine. Note that $U' = a^{-1}(X') \subset U$ is a dense open. Since $U' \to X'$ is

an étale morphism of schemes, we see the generic points of irreducible components of U' are the points lying over $\{x'_1, \ldots, x'_m\}$. Since $x'_j \to X$ is quasi-compact there are finitely many points of U lying over x'_j (Lemma 68.4.5). Hence U' has finitely many irreducible components, which implies that the closures of these irreducible components are the irreducible components of U. Thus (2) holds.

Assume (2). This implies (1) and the final statement by Lemma 68.20.3. (We also use that a reasonable algebraic space is decent, see discussion following Definition 68.6.1.) □

68.21. Generically finite morphisms

0BBA This section discusses for morphisms of algebraic spaces the material discussed in Morphisms, Section 29.51 and Varieties, Section 33.17 for morphisms of schemes.

0ACZ **Lemma 68.21.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is quasi-separated of finite type. Let $y \in |Y|$ be a point of codimension 0 on Y. The following are equivalent:*

(1) *the space $|X_k|$ is finite where $\mathrm{Spec}(k) \to Y$ represents y,*
(2) *$X \to Y$ is quasi-finite at all points of $|X|$ over y,*
(3) *there exists an open subspace $Y' \subset Y$ with $y \in |Y'|$ such that $Y' \times_Y X \to Y'$ is finite.*

If Y is decent these are also equivalent to

(4) *the set $f^{-1}(\{y\})$ is finite.*

Proof. The equivalence of (1) and (2) follows from Lemma 68.18.10 (and the fact that a quasi-separated morphism is decent by Lemma 68.17.2).

Assume the equivalent conditions of (1) and (2). Choose an affine scheme V and an étale morphism $V \to Y$ mapping a point $v \in V$ to y. Then v is a generic point of an irreducible component of V by Properties of Spaces, Lemma 66.11.1. Choose an affine scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is of finite type. The morphism $U \to V$ is quasi-finite at every point lying over v by (2). It follows that the fibre of $U \to V$ over v is finite (Morphisms, Lemma 29.20.14). By Morphisms, Lemma 29.51.1 after shrinking V we may assume that $U \to V$ is finite. Let

$$R = U \times_{V \times_Y X} U$$

Since f is quasi-separated, we see that $V \times_Y X$ is quasi-separated and hence R is a quasi-compact scheme. Moreover the morphisms $R \to V$ is quasi-finite as the composition of an étale morphism $R \to U$ and a finite morphism $U \to V$. Hence we may apply Morphisms, Lemma 29.51.1 once more and after shrinking V we may assume that $R \to V$ is finite as well. This of course implies that the two projections $R \to V$ are finite étale. It follows that $V/R = V \times_Y X$ is an affine scheme, see Groupoids, Proposition 39.23.9. By Morphisms, Lemma 29.41.9 we conclude that $V \times_Y X \to V$ is proper and by Morphisms, Lemma 29.44.11 we conclude that $V \times_Y X \to V$ is finite. Finally, we let $Y' \subset Y$ be the open subspace of Y corresponding to the image of $|V| \to |Y|$. By Morphisms of Spaces, Lemma 67.45.3 we conclude that $Y' \times_Y X \to Y'$ is finite as the base change to V is finite and as $V \to Y'$ is a surjective étale morphism.

If Y is decent and f is quasi-separated, then we see that X is decent too; use Lemmas 68.17.2 and 68.17.5. Hence Lemma 68.18.10 applies to show that (4)

implies (1) and (2). On the other hand, we see that (2) implies (4) by Morphisms of Spaces, Lemma 67.27.9. □

0AD0 **Lemma 68.21.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is quasi-separated and locally of finite type and Y quasi-separated. Let $y \in |Y|$ be a point of codimension 0 on Y. The following are equivalent:*

(1) *the set $f^{-1}(\{y\})$ is finite,*
(2) *the space $|X_k|$ is finite where $\mathrm{Spec}(k) \to Y$ represents y,*
(3) *there exist open subspaces $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, $y \in |Y'|$, and $f^{-1}(\{y\}) \subset |X'|$ such that $f|_{X'} : X' \to Y'$ is finite.*

Proof. Since quasi-separated algebraic spaces are decent, the equivalence of (1) and (2) follows from Lemma 68.18.10. To prove that (1) and (2) imply (3) we may and do replace Y by a quasi-compact open containing y. Since $f^{-1}(\{y\})$ is finite, we can find a quasi-compact open subspace of $X' \subset X$ containing the fibre. The restriction $f|_{X'} : X' \to Y$ is quasi-compact and quasi-separated by Morphisms of Spaces, Lemma 67.8.10 (this is where we use that Y is quasi-separated). Applying Lemma 68.21.1 to $f|_{X'} : X' \to Y$ we see that (3) holds. We omit the proof that (3) implies (2). □

0BBB **Lemma 68.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X, resp. Y. Let $y \in Y^0$. The following are equivalent*

(1) *$f^{-1}(\{y\}) \subset X^0$,*
(2) *f is quasi-finite at all points lying over y,*
(3) *f is quasi-finite at all $x \in X^0$ lying over y.*

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let U be a scheme and let $U \to V \times_Y X$ be a surjective étale morphism. Then f is quasi-finite at the image x of a point $u \in U$ if and only if $U \to V$ is quasi-finite at u. Moreover, $x \in X^0$ if and only if u is the generic point of an irreducible component of U (Properties of Spaces, Lemma 66.11.1). Thus the lemma reduces to the case of the morphism $U \to V$, i.e., to Morphisms, Lemma 29.51.4. □

0BBC **Lemma 68.21.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X, resp. Y. Assume*

(1) *Y is decent,*
(2) *X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,*
(3) *either f is quasi-compact or f is separated.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemmas 68.20.4 and 68.20.1 we may assume Y is a scheme with finitely many irreducible components. Shrinking further we may assume Y is an irreducible affine scheme with generic point y. Then the fibre of f over y is finite.

Assume f is quasi-compact and Y affine irreducible. Then X is quasi-compact and we may choose an affine scheme U and a surjective étale morphism $U \to X$. Then $U \to Y$ is of finite type and the fibre of $U \to Y$ over y is the set U^0 of generic points

of irreducible components of U (Properties of Spaces, Lemma 66.11.1). Hence U^0 is finite (Morphisms, Lemma 29.20.14) and after shrinking Y we may assume that $U \to Y$ is finite (Morphisms, Lemma 29.51.1). Next, consider $R = U \times_X U$. Since the projection $s : R \to U$ is étale we see that $R^0 = s^{-1}(U^0)$ lies over y. Since $R \to U \times_Y U$ is a monomorphism, we conclude that R^0 is finite as $U \times_Y U \to Y$ is finite. And R is separated (Properties of Spaces, Lemma 66.6.4). Thus we may shrink Y once more to reach the situation where R is finite over Y (Morphisms, Lemma 29.51.5). In this case it follows that $X = U/R$ is finite over Y by exactly the same arguments as given in the proof of Lemma 68.21.1 (or we can simply apply that lemma because it follows immediately that X is quasi-separated as well).

Assume f is separated and Y affine irreducible. Choose $V \subset Y$ and $U \subset X$ as in Lemma 68.21.2. Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Morphisms of Spaces, Lemmas 67.40.6 and 67.45.9). Thus $f^{-1}(V) = U \amalg W$ for some open subspace W of X. However, since U contains all the codimension 0 points of X we conclude that $W = \emptyset$ (Properties of Spaces, Lemma 66.11.2) as desired. □

68.22. Birational morphisms

0ACU The following definition of a birational morphism of algebraic spaces seems to be the closest to our definition (Morphisms, Definition 29.50.1) of a birational morphism of schemes.

0ACV **Definition 68.22.1.** Let S be a scheme. Let X and Y algebraic spaces over S. Assume X and Y are decent and that $|X|$ and $|Y|$ have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

(1) $|f|$ induces a bijection between the set of generic points of irreducible components of $|X|$ and the set of generic points of the irreducible components of $|Y|$, and
(2) for every generic point $x \in |X|$ of an irreducible component the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism (see clarification below).

Clarification: Since X and Y are decent the topological spaces $|X|$ and $|Y|$ are sober (Proposition 68.12.4). Hence condition (1) makes sense. Moreover, because we have assumed that $|X|$ and $|Y|$ have finitely many irreducible components, we see that the generic points $x_1, \ldots, x_n \in |X|$, resp. $y_1, \ldots, y_n \in |Y|$ are contained in any dense open of $|X|$, resp. $|Y|$. In particular, they are contained in the schematic locus of X, resp. Y by Theorem 68.10.2. Thus we can define $\mathcal{O}_{X,x_i}$, resp. $\mathcal{O}_{Y,y_i}$ to be the local ring of this scheme at x_i, resp. y_i.

We conclude that if the morphism $f : X \to Y$ is birational, then there exist dense open subspaces $X' \subset X$ and $Y' \subset Y$ such that

(1) $f(X') \subset Y'$,
(2) X' and Y' are representable, and
(3) $f|_{X'} : X' \to Y'$ is birational in the sense of Morphisms, Definition 29.50.1.

However, we do insist that X and Y are decent with finitely many irreducible components. Other ways to characterize decent algebraic spaces with finitely many irreducible components are given in Lemma 68.20.4. In most cases birational morphisms are isomorphisms over dense opens.

0ACW **Lemma 68.22.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows immediately from the definitions. See Morphisms of Spaces, Definition 67.18.1. □

0BBD **Lemma 68.22.3.** *Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.*

Proof. Let $X' \subset X$ and $Y' \subset Y$ be the maximal open subspaces which are representable, see Lemma 68.20.4. By Lemma 68.21.3 the fibre of f over y is consists of points of codimension 0 of X and is therefore contained in X'. Hence $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) = X' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{Y',y})$ and the result follows from Morphisms, Lemma 29.50.3. □

0BBE **Lemma 68.22.4.** *Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume one of the following conditions is satisfied*

(1) *f is locally of finite type and Y reduced (i.e., integral),*
(2) *f is locally of finite presentation.*

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism.

Proof. By Lemma 68.20.4 we may assume that X and Y are schemes. In this case the result is Morphisms, Lemma 29.50.5. □

0BBF **Lemma 68.22.5.** *Let S be a scheme. Let $f : X \to Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume*

(1) *either f is quasi-compact or f is separated, and*
(2) *either f is locally of finite type and Y is reduced or f is locally of finite presentation.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Proof. By Lemma 68.20.4 we may assume Y is a scheme. By Lemma 68.21.4 we may assume that f is finite. Then X is a scheme too and the result follows from Morphisms, Lemma 29.51.6. □

0B4D **Lemma 68.22.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational and $V \to Y$ is an étale morphism with V affine, then $X \times_Y V$ is decent with finitely many irreducible components and $X \times_Y V \to V$ is birational.*

Proof. The algebraic space $U = X \times_Y V$ is decent (Lemma 68.6.6). The generic points of V and U are the elements of $|V|$ and $|U|$ which lie over generic points of $|Y|$ and $|X|$ (Lemma 68.20.1). Since Y is decent we conclude there are finitely many generic points on V. Let $\xi \in |X|$ be a generic point of an irreducible component.

By the discussion following Definition 68.22.1 we have a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,\xi}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{Y,f(\xi)}) & \longrightarrow & Y \end{array}$$

whose horizontal morphisms are monomorphisms identifying local rings and where the left vertical arrow is an isomorphism. It follows that in the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,\xi}) \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{Y,f(\xi)}) \times_Y V & \longrightarrow & V \end{array}$$

the vertical arrow on the left is an isomorphism. The horizontal arrows have image contained in the schematic locus of U and V and identify local rings (some details omitted). Since the image of the horizontal arrows are the points of $|U|$, resp. $|V|$ lying over ξ, resp. $f(\xi)$ we conclude. □

0BBG **Lemma 68.22.7.** *Let S be a scheme. Let $f : X \to Y$ be a birational morphism between algebraic spaces over S which are decent and have finitely many irreducible components. Then the normalizations $X^\nu \to X$ and $Y^\nu \to Y$ exist and there is a commutative diagram*

$$\begin{array}{ccc} X^\nu & \longrightarrow & Y^\nu \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

of algebraic spaces over S. The morphism $X^\nu \to Y^\nu$ is birational.

Proof. By Lemma 68.20.4 we see that X and Y satisfy the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 and the normalizations are defined. By Morphisms of Spaces, Lemma 67.49.8 the algebraic space X^ν is normal and maps codimension 0 points to codimension 0 points. Since f maps codimension 0 points to codimension 0 points (this is the same as generic points on decent spaces by Lemma 68.20.1) we obtain from Morphisms of Spaces, Lemma 67.49.8 a factorization of the composition $X^\nu \to X \to Y$ through Y^ν.

Observe that X^ν and Y^ν are decent for example by Lemma 68.6.5. Moreover the maps $X^\nu \to X$ and $Y^\nu \to Y$ induce bijections on irreducible components (see references above) hence X^ν and Y^ν both have a finite number of irreducible components and the map $X^\nu \to Y^\nu$ induces a bijection between their generic points. To prove that $X^\nu \to Y^\nu$ is birational, it therefore suffices to show it induces an isomorphism on local rings at these points. To do this we may replace X and Y by open neighbourhoods of their generic points, hence we may assume X and Y are affine irreducible schemes with generic points x and y. Since f is birational the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism. Let $x^\nu \in X^\nu$ and $y^\nu \in Y^\nu$ be the points lying over x and y. By construction of the normalization we see that $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and similarly on Y. Thus the map $\mathcal{O}_{X^\nu,x^\nu} \to \mathcal{O}_{Y^\nu,y^\nu}$ is an isomorphism as well. □

0B4E **Lemma 68.22.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *X and Y are decent and have finitely many irreducible components,*
(2) *f is integral and birational,*
(3) *Y is normal, and*
(4) *X is reduced.*

Then f is an isomorphism.

Proof. Let $V \to Y$ be an étale morphism with V affine. It suffices to show that $U = X \times_Y V \to V$ is an isomorphism. By Lemma 68.22.6 and its proof we see that U and V are decent and have finitely many irreducible components and that $U \to V$ is birational. By Properties, Lemma 28.7.5 V is a finite disjoint union of integral schemes. Thus we may assume V is integral. As f is birational, we see that U is irreducible and reduced, i.e., integral (note that U is a scheme as f is integral, hence representable). Thus we may assume that X and Y are integral schemes and the result follows from the case of schemes, see Morphisms, Lemma 29.54.8. □

0BBH **Lemma 68.22.9.** *Let S be a scheme. Let $f : X \to Y$ be an integral birational morphism of decent algebraic spaces over S which have finitely many irreducible components. Then there exists a factorization $Y^\nu \to X \to Y$ and $Y^\nu \to X$ is the normalization of X.*

Proof. Consider the map $X^\nu \to Y^\nu$ of Lemma 68.22.7. This map is integral by Morphisms of Spaces, Lemma 67.45.12. Hence it is an isomorphism by Lemma 68.22.8. □

68.23. Jacobson spaces

0BA2 We have defined the Jacobson property for algebraic spaces in Properties of Spaces, Remark 66.7.3. For representable algebraic spaces it agrees with the property discussed in Properties, Section 28.6. The relationship between the Jacobson property and the behaviour of the topological space $|X|$ is not evident for general algebraic spaces $|X|$. However, a decent (for example quasi-separated or locally separated) algebraic space X is Jacobson if and only if $|X|$ is Jacobson (see Lemma 68.23.4).

0BA3 **Lemma 68.23.1.** *Let S be a scheme. Let X be a Jacobson algebraic space over S. Any algebraic space locally of finite type over X is Jacobson.*

Proof. Let $U \to X$ be a surjective étale morphism where U is a scheme. Then U is Jacobson (by definition) and for a morphism of schemes $V \to U$ which is locally of finite type we see that V is Jacobson by the corresponding result for schemes (Morphisms, Lemma 29.16.9). Thus if $Y \to X$ is a morphism of algebraic spaces which is locally of finite type, then setting $V = U \times_X Y$ we see that Y is Jacobson by definition. □

0BA4 **Lemma 68.23.2.** *Let S be a scheme. Let X be a Jacobson algebraic space over S. For $x \in X_{ft\text{-}pts}$ and $g : W \to X$ locally of finite type with W a scheme, if $x \in \mathrm{Im}(|g|)$, then there exists a closed point of W mapping to x.*

Proof. Let $U \to X$ be an étale morphism with U a scheme and with $u \in U$ closed mapping to x, see Morphisms of Spaces, Lemma 67.25.3. Observe that W, $W \times_X U$, and U are Jacobson schemes by Lemma 68.23.1. Hence finite type points on these schemes are the same thing as closed points by Morphisms, Lemma 29.16.8. The inverse image $T \subset W \times_X U$ of u is a nonempty (as x in the image of $W \to X$) closed subset. By Morphisms, Lemma 29.16.7 there is a closed point t of $W \times_X U$

which maps to u. As $W \times_X U \to W$ is locally of finite type the image of t in W is closed by Morphisms, Lemma 29.16.8. □

0BA5 **Lemma 68.23.3.** *Let S be a scheme. Let X be a decent Jacobson algebraic space over S. Then $X_{ft\text{-}pts} \subset |X|$ is the set of closed points.*

Proof. If $x \in |X|$ is closed, then we can represent x by a closed immersion $\mathrm{Spec}(k) \to X$, see Lemma 68.14.6. Hence x is certainly a finite type point.

Conversely, let $x \in |X|$ be a finite type point. We know that x can be represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \to X$ where k is a field (Definition 68.6.1). On the other hand, by definition, there exists a morphism $\mathrm{Spec}(k') \to X$ which is locally of finite type and represents x (Morphisms, Definition 29.16.3). We obtain a factorization $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to X$. Let $U \to X$ be any étale morphism with U affine and consider the morphisms

$$\mathrm{Spec}(k') \times_X U \to \mathrm{Spec}(k) \times_X U \to U$$

The quasi-compact scheme $\mathrm{Spec}(k) \times_X U$ is étale over $\mathrm{Spec}(k)$ hence is a finite disjoint union of spectra of fields (Remark 68.4.1). Moreover, the first morphism is surjective and locally of finite type (Morphisms, Lemma 29.15.8) hence surjective on finite type points (Morphisms, Lemma 29.16.6) and the composition (which is locally of finite type) sends finite type points to closed points as U is Jacobson (Morphisms, Lemma 29.16.8). Thus the image of $\mathrm{Spec}(k) \times_X U \to U$ is a finite set of closed points hence closed. Since this is true for every affine U and étale morphism $U \to X$, we conclude that $x \in |X|$ is closed. □

0BA6 **Lemma 68.23.4.** *Let S be a scheme. Let X be a decent algebraic space over S. Then X is Jacobson if and only if $|X|$ is Jacobson.*

Proof. Assume X is Jacobson and that $T \subset |X|$ is a closed subset. By Morphisms of Spaces, Lemma 67.25.6 we see that $T \cap X_{\mathrm{ft\text{-}pts}}$ is dense in T. By Lemma 68.23.3 we see that $X_{\mathrm{ft\text{-}pts}}$ are the closed points of $|X|$. Thus $|X|$ is indeed Jacobson.

Assume $|X|$ is Jacobson. Let $f : U \to X$ be an étale morphism with U an affine scheme. We have to show that U is Jacobson. If $x \in |X|$ is closed, then the fibre $F = f^{-1}(\{x\})$ is a finite (by definition of decent) closed (by construction of the topology on $|X|$) subset of U. Since there are no specializations between points of F (Lemma 68.12.1) we conclude that every point of F is closed in U. If U is not Jacobson, then there exists a non-closed point $u \in U$ such that $\{u\}$ is locally closed (Topology, Lemma 5.18.3). We will show that $f(u) \in |X|$ is closed; by the above u is closed in U which is a contradiction and finishes the proof. To prove this we may replace U by an affine open neighbourhood of u. Thus we may assume that $\{u\}$ is closed in U. Let $R = U \times_X U$ with projections $s, t : R \to U$. Then $s^{-1}(\{u\}) = \{r_1, \ldots, r_m\}$ is finite (by definition of decent spaces). After replacing U by a smaller affine open neighbourhood of u we may assume that $t(r_j) = u$ for $j = 1, \ldots, m$. It follows that $\{u\}$ is an R-invariant closed subset of U. Hence $\{f(u)\}$ is a locally closed subset of X as it is closed in the open $|f|(|U|)$ of $|X|$. Since $|X|$ is Jacobson we conclude that $f(u)$ is closed in $|X|$ as desired. □

0ED2 **Lemma 68.23.5.** *Let S be a scheme. Let X be a decent locally Noetherian algebraic space over S. Let $x \in |X|$. Then*

$$W = \{x' \in |X| : x' \rightsquigarrow x,\ x' \neq x\}$$

is a Noetherian, spectral, sober, Jacobson topological space.

Proof. We may replace by any open subspace containing x. Thus we may assume that X is quasi-compact. Then $|X|$ is a Noetherian topological space (Properties of Spaces, Lemma 66.24.2). Thus W is a Noetherian topological space (Topology, Lemma 5.9.2).

Combining Lemma 68.14.1 with Properties of Spaces, Lemma 66.15.2 we see that $|X|$ is a spectral topological space. By Topology, Lemma 5.24.7 we see that $W \cup \{x\}$ is a spectral topological space. Now W is a quasi-compact open of $W \cup \{x\}$ and hence W is spectral by Topology, Lemma 5.23.5.

Let $E \subset W$ be an irreducible closed subset. Then if $Z \subset |X|$ is the closure of E we see that $x \in Z$. There is a unique generic point $\eta \in Z$ by Proposition 68.12.4. Of course $\eta \in W$ and hence $\eta \in E$. We conclude that E has a unique generic point, i.e., W is sober.

Let $x' \in W$ be a point such that $\{x'\}$ is locally closed in W. To finish the proof we have to show that x' is a closed point of W. If not, then there exists a nontrivial specialization $x' \rightsquigarrow x'_1$ in W. Let U be an affine scheme, $u \in U$ a point, and let $U \to X$ be an étale morphism mapping u to x. By Lemma 68.12.2 we can choose specializations $u' \rightsquigarrow u'_1 \rightsquigarrow u$ mapping to $x' \rightsquigarrow x'_1 \rightsquigarrow x$. Let $\mathfrak{p}' \subset \mathcal{O}_{U,u}$ be the prime ideal corresponding to u'. The existence of the specializations implies that $\dim(\mathcal{O}_{U,u}/\mathfrak{p}') \geq 2$. Hence every nonempty open of $\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{p}')$ is infinite by Algebra, Lemma 10.61.1. By Lemma 68.12.1 we obtain a continuous map

$$\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{p}') \setminus \{\mathfrak{m}_u/\mathfrak{p}'\} \longrightarrow W$$

Since the generic point of the LHS maps to x' the image is contained in $\overline{\{x'\}}$. We conclude the inverse image of $\{x'\}$ under the displayed arrow is nonempty open hence infinite. However, the fibres of $U \to X$ are finite as X is decent and we conclude that $\{x'\}$ is infinite. This contradiction finishes the proof. □

68.24. Local irreducibility

0DQ5 We have already defined the geometric number of branches of an algebraic space at a point in Properties of Spaces, Section 66.23. The number of branches of an algebraic space at a point can only be defined for decent algebraic spaces.

0DQ6 **Lemma 68.24.1.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

1. *for any elementary étale neighbourhood $(U, u) \to (X, x)$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,*
2. *for any elementary étale neighbourhood $(U, u) \to (X, x)$ there is a unique irreducible component of U through u,*
3. *for any elementary étale neighbourhood $(U, u) \to (X, x)$ the local ring $\mathcal{O}_{U,u}$ is unibranch,*
4. *the henselian local ring $\mathcal{O}^h_{X,x}$ has a unique minimal prime.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. The ring $\mathcal{O}^h_{X,x}$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 68.11.7. In particular (3) and (4) are equivalent by More on

Algebra, Lemma 15.107.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 37.36.2. □

0DQ7 **Definition 68.24.2.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. We say that X is *unibranch at* x if the equivalent conditions of Lemma 68.24.1 hold. We say that X is *unibranch* if X is unibranch at every $x \in |X|$.

This is consistent with the definition for schemes (Properties, Definition 28.15.1).

0DQ8 **Lemma 68.24.3.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a point. Let $n \in \{1, 2, \ldots\}$ be an integer. The following are equivalent*

(1) *for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number of minimal primes of the local ring $\mathcal{O}_{U,u}$ is $\leq n$ and for at least one choice of (U, u) it is n,*
(2) *for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number irreducible components of U passing through u is $\leq n$ and for at least one choice of (U, u) it is n,*
(3) *for any elementary étale neighbourhood $(U, u) \to (X, x)$ the number of branches of U at u is $\leq n$ and for at least one choice of (U, u) it is n,*
(4) *the number of minimal prime ideals of $\mathcal{O}^h_{X,x}$ is n.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u. The ring $\mathcal{O}_{X,x}$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 68.11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 15.107.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 37.36.2. □

0DQ9 **Definition 68.24.4.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The *number of branches of X at x* is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 68.24.3 hold, or else ∞.

68.25. Catenary algebraic spaces

0ED3 This section extends the material in Properties, Section 28.11 and Morphisms, Section 29.17 to algebraic spaces.

0ED4 **Definition 68.25.1.** Let S be a scheme. Let X be a decent algebraic space over S. We say X is *catenary* if $|X|$ is catenary (Topology, Definition 5.11.4).

If X is representable, then this is equivalent to the corresponding notion for the scheme representing X.

0ED5 **Lemma 68.25.2.** *Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let X be a decent algebraic space over S such that the structure morphism $X \to S$ is locally of finite type. Let $\delta_X : |X| \to \mathbf{Z}$ be the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$. Then δ_X is a dimension function on $|X|$.*

Proof. Let $\varphi : U \to X$ be a surjective étale morphism where U is a scheme. Then the similarly defined function δ_U is a dimension function on U by Morphisms,

Lemma 29.52.3. On the other hand, by the definition of relative transcendence degree in (Morphisms of Spaces, Definition 67.33.1) we see that $\delta_U(u) = \delta_X(\varphi(u))$.

Let $x \rightsquigarrow x'$ be a specialization of points in $|X|$. by Lemma 68.12.2 we can find a specialization $u \rightsquigarrow u'$ of points of U with $\varphi(u) = x$ and $\varphi(u') = x'$. Moreover, we see that $x = x'$ if and only if $u = u'$, see Lemma 68.12.1. Thus the fact that δ_U is a dimension function implies that δ_X is a dimension function, see Topology, Definition 5.20.1. □

0ED6 **Lemma 68.25.3.** *Let S be a locally Noetherian and universally catenary scheme. Let X be an algebraic space over S such that X is decent and such that the structure morphism $X \to S$ is locally of finite type. Then X is catenary.*

Proof. The question is local on S (use Topology, Lemma 5.11.5). Thus we may assume that S has a dimension function, see Topology, Lemma 5.20.4. Then we conclude that $|X|$ has a dimension function by Lemma 68.25.2. Since $|X|$ is sober (Proposition 68.12.4) we conclude that $|X|$ is catenary by Topology, Lemma 5.20.2. □

By Lemma 68.25.3 the following definition is compatible with the already existing notion for representable algebraic spaces.

0ED7 **Definition 68.25.4.** Let S be a scheme. Let X be a decent and locally Noetherian algebraic space over S. We say X is *universally catenary* if for every morphism $Y \to X$ of algebraic spaces which is locally of finite type and with Y decent, the algebraic space Y is catenary.

If X is an algebraic space, then the condition "X is decent and locally Noetherian" is equivalent to "X is quasi-separated and locally Noetherian". This is Lemma 68.14.1. Thus another way to understand the definition above is that X is universally catenary if and only if Y is catenary for all morphisms $Y \to X$ which are quasi-separated and locally of finite type.

0ED8 **Lemma 68.25.5.** *Let S be a scheme. Let X be a decent, locally Noetherian, and universally catenary algebraic space over S. Then any decent algebraic space locally of finite type over X is universally catenary.*

Proof. This is formal from the definitions and the fact that compositions of morphisms locally of finite type are locally of finite type (Morphisms of Spaces, Lemma 67.23.2). □

0ED9 **Lemma 68.25.6.** *Let S be a scheme. Let $f : Y \to X$ be a surjective finite morphism of decent and locally Noetherian algebraic spaces. Let $\delta : |X| \to \mathbf{Z}$ be a function. If $\delta \circ |f|$ is a dimension function, then δ is a dimension function.*

Proof. Let $x \mapsto x'$, $x \neq x'$ be a specialization in $|X|$. Choose $y \in |Y|$ with $|f|(y) = x$. Since $|f|$ is closed (Morphisms of Spaces, Lemma 67.45.9) we find a specialization $y \rightsquigarrow y'$ with $|f|(y') = x'$. Thus we conclude that $\delta(x) = \delta(|f|(y)) > \delta(|f|(y')) = \delta(x')$ (see Topology, Definition 5.20.1). If $x \rightsquigarrow x'$ is an immediate specialization, then $y \rightsquigarrow y'$ is an immediate specialization too: namely if $y \rightsquigarrow y'' \rightsquigarrow y'$, then $|f|(y'')$ must be either x or x' and there are no nontrivial specializations between points of fibres of $|f|$ by Lemma 68.18.10. □

The discussion will be continued in More on Morphisms of Spaces, Section 76.32.

68.26. Other chapters

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CHAPTER 69

Cohomology of Algebraic Spaces

071T

69.1. Introduction

071U In this chapter we write about cohomology of algebraic spaces. Although we prove some results on cohomology of abelian sheaves, we focus mainly on cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter "Cohomology of Schemes". Some of the results in this chapter can be found in [**Knu71**].

An important missing ingredient in this chapter is the *induction principle*, i.e., the analogue for quasi-compact and quasi-separated algebraic spaces of Cohomology of Schemes, Lemma 30.4.1. This is formulated precisely and proved in detail in Derived Categories of Spaces, Section 75.9. Instead of the induction principle, in this chapter we use the alternating Čech complex, see Section 69.6. It is designed to prove vanishing statements such as Proposition 69.7.2, but in some cases the induction principle is a more powerful and perhaps more "standard" tool. We encourage the reader to take a look at the induction principle after reading some of the material in this section.

69.2. Conventions

071V The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

69.3. Higher direct images

071Y Let S be a scheme. Let X be a representable algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent module on X (see Properties of Spaces, Section 66.29). By Descent, Proposition 35.9.3 the cohomology groups $H^i(X, \mathcal{F})$ agree with the usual cohomology group computed in the Zariski topology of the corresponding quasi-coherent module on the scheme representing X.

More generally, let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y. Let $\mathcal{F}$ be a quasi-coherent module on X. By Descent, Lemma 35.9.5 the sheaf $R^i f_* \mathcal{F}$ agrees with the usual higher direct image computed for the Zariski topology of the quasi-coherent module on the scheme representing X mapping to the scheme representing Y.

More generally still, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let

$V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

071Z (69.3.0.1) $$R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V,$$

see Properties of Spaces, Lemma 66.26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $R^i f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

Next, we prove that higher direct images of quasi-coherent sheaves are quasi-coherent for any quasi-compact and quasi-separated morphism of algebraic spaces. In the proof we use a trick; a "better" proof would use a relative Čech complex, as discussed in Sheaves on Stacks, Sections 96.18 and 96.19 ff.

0720 **Lemma 69.3.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then $R^i f_*$ transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $V \to Y$ be an étale morphism where V is an affine scheme. Set $U = V \times_Y X$ and denote $f' : U \to V$ the induced morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. By Properties of Spaces, Lemma 66.26.2 we have $R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V$. Since the property of being a quasi-coherent module is local in the étale topology on Y (see Properties of Spaces, Lemma 66.29.6) we may replace Y by V, i.e., we may assume Y is an affine scheme.

Assume Y is affine. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $g : U \to X$, see Properties of Spaces, Lemma 66.6.3. Picture

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ & \searrow^{f \circ g} & \downarrow f \\ & & Y \end{array}$$

The morphism $g : U \to X$ is representable, separated and quasi-compact because X is quasi-separated. Hence the lemma holds for g (by the discussion above the lemma). It also holds for $f \circ g : U \to Y$ (as this is a morphism of affine schemes).

In the situation described in the previous paragraph we will show by induction on n that IH_n: for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaves $R^i f \mathcal{F}$ are quasi-coherent for $i \leq n$. The case $n = 0$ follows from Morphisms of Spaces, Lemma 67.11.2. Assume IH_n. In the rest of the proof we show that IH_{n+1} holds.

Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_U$-module. Consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_* \mathcal{H} \Rightarrow R^{p+q}(f \circ g)_* \mathcal{H}$$

Cohomology on Sites, Lemma 21.14.7. As $R^q g_* \mathcal{H}$ is quasi-coherent by IH_n all the sheaves $R^p f_* R^q g_* \mathcal{H}$ are quasi-coherent for $p \leq n$. The sheaves $R^{p+q}(f \circ g)_* \mathcal{H}$ are all quasi-coherent (in fact zero for $p + q > 0$ but we do not need this). Looking in degrees $\leq n + 1$ the only module which we do not yet know is quasi-coherent is $E_2^{n+1,0} = R^{n+1} f_* g_* \mathcal{H}$. Moreover, the differentials $d_r^{n+1,0} : E_r^{n+1,0} \to E_r^{n+1+r,1-r}$ are zero as the target is zero. Using that $QCoh(\mathcal{O}_X)$ is a weak Serre subcategory

of $Mod(\mathcal{O}_X)$ (Properties of Spaces, Lemma 66.29.7) it follows that $R^{n+1}f_*g_*\mathcal{H}$ is quasi-coherent (details omitted).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $\mathcal{H} = g^*\mathcal{F}$. The adjunction mapping $\mathcal{F} \to g_*g^*\mathcal{F} = g_*\mathcal{H}$ is injective as $U \to X$ is surjective étale. Consider the exact sequence

$$0 \to \mathcal{F} \to g_*\mathcal{H} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of the first map and in particular quasi-coherent. Applying the long exact cohomology sequence we obtain

$$R^nf_*g_*\mathcal{H} \to R^nf_*\mathcal{G} \to R^{n+1}f_*\mathcal{F} \to R^{n+1}f_*g_*\mathcal{H} \to R^{n+1}f_*\mathcal{G}$$

The cokernel of the first arrow is quasi-coherent and we have seen above that $R^{n+1}f_*g_*\mathcal{H}$ is quasi-coherent. Thus $R^{n+1}f_*\mathcal{F}$ has a 2-step filtration where the first step is quasi-coherent and the second a submodule of a quasi-coherent sheaf. Since $\mathcal{F}$ is an arbitrary quasi-coherent $\mathcal{O}_X$-module, this result also holds for $\mathcal{G}$. Thus we can choose an exact sequence $0 \to \mathcal{A} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ with $\mathcal{A}$, $\mathcal{B}$ quasi-coherent $\mathcal{O}_Y$-modules. Then the kernel $\mathcal{K}$ of $R^{n+1}f_*g_*\mathcal{H} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ is quasi-coherent, whereupon we obtain a map $\mathcal{K} \to \mathcal{A}$ whose kernel $\mathcal{K}'$ is quasi-coherent too. Hence $R^{n+1}f_*\mathcal{F}$ sits in an exact sequence

$$R^nf_*g_*\mathcal{H} \to R^nf_*\mathcal{G} \to R^{n+1}f_*\mathcal{F} \to \mathcal{K}' \to 0$$

with all modules quasi-coherent except for possibly $R^{n+1}f_*\mathcal{F}$. We conclude that $R^{n+1}f_*\mathcal{F}$ is quasi-coherent, i.e., IH_{n+1} holds as desired. □

08EX **Lemma 69.3.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any affine object V of $Y_{étale}$ we have*

$$H^q(V \times_Y X, \mathcal{F}) = H^0(V, R^qf_*\mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Since formation of Rf_* commutes with étale localization (Properties of Spaces, Lemma 66.26.2) we may replace Y by V and assume $Y = V$ is affine. Consider the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology on Sites, Lemma 21.14.5. By Lemma 69.3.1 we see that the sheaves $R^qf_*\mathcal{F}$ are quasi-coherent. By Cohomology of Schemes, Lemma 30.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. □

69.4. Finite morphisms

0DK2 Here are some results which hold for all abelian sheaves (in particular also quasi-coherent modules). We **warn** the reader that these lemmas do not hold for finite morphisms of schemes and the Zariski topology.

0A4K **Lemma 69.4.1.** *Let S be a scheme. Let $f : X \to Y$ be an integral (for example finite) morphism of algebraic spaces. Then $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ is an exact functor and $R^pf_* = 0$ for $p > 0$.*

Proof. By Properties of Spaces, Lemma 66.18.12 we may compute the higher direct images on an étale cover of Y. Hence we may assume Y is a scheme. This implies that X is a scheme (Morphisms of Spaces, Lemma 67.45.3). In this case we may

apply Étale Cohomology, Lemma 59.43.5. For the finite case the reader may wish to consult the less technical Étale Cohomology, Proposition 59.55.2. □

0DK3 **Lemma 69.4.2.** *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y with lifts $\overline{x}_1, \ldots, \overline{x}_n$ in X. Then*

$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i}$$

for any sheaf $\mathcal{F}$ on $X_{étale}$.

Proof. Choose an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$. Then the stalk $(f_*\mathcal{F})_{\overline{y}}$ is the stalk of $f_*\mathcal{F}|_V$ at $\overline{v}$. By Properties of Spaces, Lemma 66.18.12 we may replace Y by V and X by $X \times_Y V$. Then $Z \to X$ is a finite morphism of schemes and the result is Étale Cohomology, Proposition 59.55.2. □

0DK4 **Lemma 69.4.3.** *Let S be a scheme. Let $\pi : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Y_{étale}$. Let $\varphi : \mathcal{B} \to \pi_*\mathcal{A}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi*

$$f = (\pi, \varphi) : (Sh(X_{étale}), \mathcal{A}) \longrightarrow (Sh(Y_{étale}), \mathcal{B}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F} \longrightarrow f_*(f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{G} \otimes_{\mathcal{A}} f^*f_*\mathcal{F} = f^*(\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F}) \longrightarrow f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}$$

coming from $\mathrm{id} : f^*\mathcal{G} \to f^*\mathcal{G}$ and the adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 66.19.12). Let $\overline{y}$ be a geometric point of Y and let $\overline{x}_1, \ldots, \overline{x}_n$ be the geometric points of X lying over $\overline{y}$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{y}}} \left(\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i} \right) = \bigoplus\nolimits_{i=1,\ldots,n} (\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{x}}} \mathcal{A}_{\overline{x}_i}) \otimes_{\mathcal{A}_{\overline{x}_i}} \mathcal{F}_{\overline{x}_i}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 66.19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 69.4.2. □

We end this section with an insanely general projection formula for finite morphisms.

0DK5 **Lemma 69.4.4.** *With S, X, Y, π, $\mathcal{A}$, $\mathcal{B}$, φ, and f as in Lemma 69.4.3 we have*

$$K \otimes^{\mathbf{L}}_{\mathcal{B}} Rf_*M = Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{A}} M)$$

in $D(\mathcal{B})$ for any $K \in D(\mathcal{B})$ and $M \in D(\mathcal{A})$.

Proof. Since f_* is exact (Lemma 69.4.1) the functor Rf_* is computed by applying f_* to any representative complex. Choose a complex $\mathcal{K}^\bullet$ of $\mathcal{B}$-modules representing K which is K-flat with flat terms, see Cohomology on Sites, Lemma 21.17.11. Then

$f^*\mathcal{K}^\bullet$ is K-flat with flat terms, see Cohomology on Sites, Lemma 21.18.1. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{A}$-modules representing M. Then we have to show

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{B}} f_*\mathcal{M}^\bullet) = f_*\mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{A}} \mathcal{M}^\bullet)$$

because by our choices these complexes represent the right and left hand side of the formula in the lemma. Since f_* commutes with direct sums (for example by the description of the stalks in Lemma 69.4.2), this reduces to the equalities

$$\mathcal{K}^n \otimes_{\mathcal{B}} f_*\mathcal{M}^m = f_*(f^*\mathcal{K}^n \otimes_{\mathcal{A}} \mathcal{M}^m)$$

which are true by Lemma 69.4.3. □

69.5. Colimits and cohomology

073D The following lemma in particular applies to diagrams of quasi-coherent sheaves.

073E **Lemma 69.5.1.** *Let S be a scheme. Let X be an algebraic space over S. If X is quasi-compact and quasi-separated, then*

$$\mathrm{colim}_i\, H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \mathrm{colim}_i\, \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $X_{étale}$.

Proof. This follows from Cohomology on Sites, Lemma 21.16.1. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ be the set of quasi-compact and quasi-separated spaces étale over X. Note that if $U \in \mathcal{B}$ then, because U is quasi-compact, the collection of finite coverings $\{U_i \to U\}$ with $U_i \in \mathcal{B}$ is cofinal in the set of coverings of U in $X_{spaces,étale}$. By Morphisms of Spaces, Lemma 67.8.10 the set $\mathcal{B}$ satisfies all the assumptions of Cohomology on Sites, Lemma 21.16.1. Since $X \in \mathcal{B}$ we win. □

07U6 **Lemma 69.5.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $X_{étale}$. Then for any $p \geq 0$ we have*

$$R^p f_*\mathcal{F} = \mathrm{colim}\, R^p f_*\mathcal{F}_i.$$

Proof. We will use that the morphism of topoi $f_{small} : X_{small} \to Y_{small}$ comes from the morphism of sites $f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ corresponding to the continuous functor $V \longmapsto X \times_Y V$, see Properties of Spaces, Lemma 66.18.8. We will apply Cohomology on Sites, Lemma 21.16.4 to this morphism of sites. Since every object of $Y_{spaces,étale}$ has a covering by affine objects, it suffices to show that for V affine and étale over Y we have $H^p(X \times_Y V, \mathcal{F}) = \mathrm{colim}\, H^p(X \times_Y V, \mathcal{F}_i)$. Since V is affine, the algebraic space $X \times_Y V$ is quasi-compact and quasi-separated. Hence we can apply Lemma 69.5.1 to conclude. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic spaces.

07U7 **Lemma 69.5.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation. Then we have*

$$\mathrm{colim}_i\, \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i\, \mathcal{F}_i).$$

In particular, $\mathrm{Hom}_X(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_X)$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 18.27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 7.17.8 part (4) for the site $X_{étale}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 7.17.9. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{étale})$ be the set of affine objects. Then

(1) Since X is quasi-compact, there exists a $U \in \mathcal{B}$ such that $U \to X$ is surjective (Properties of Spaces, Lemma 66.6.3), hence $h_U^\# \to *$ is surjective.
(2) For $U \in \mathcal{B}$ every étale covering $\{U_i \to U\}_{i \in I}$ of U can be refined by a finite étale covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$ (Topologies, Lemma 34.4.4).
(3) For $U, U' \in \mathrm{Ob}(X_{étale})$ we have $h_U^\# \times h_{U'}^\# = h_{U \times_X U'}^\#$. If $U, U' \in \mathcal{B}$, then $U \times_X U'$ is quasi-compact because X is quasi-separated, see Morphisms of Spaces, Lemma 67.8.10 for example. Hence we can find a surjective étale morphism $U'' \to U \times_X U'$ with $U'' \in \mathcal{B}$ (Properties of Spaces, Lemma 66.6.3). In other words, we have morphisms $U'' \to U$ and $U'' \to U'$ such that the map $h_{U''}^\# \to h_U^\# \times h_{u'}^\#$ is surjective.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Properties of Spaces, Lemma 66.29.7. □

69.6. The alternating Čech complex

0721 Let S be a scheme. Let $f : U \to X$ be an étale morphism of algebraic spaces over S. The functor

$$j : U_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V/U \longmapsto V/X$$

induces an equivalence of $U_{spaces,étale}$ with the localization $X_{spaces,étale}/U$, see Properties of Spaces, Section 66.27. Hence there exist functors

$$f_! : Ab(U_{étale}) \longrightarrow Ab(X_{étale}), \quad f_! : Mod(\mathcal{O}_U) \longrightarrow Mod(\mathcal{O}_X),$$

which are left adjoint to

$$f^{-1} : Ab(X_{étale}) \longrightarrow Ab(U_{étale}), \quad f^* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_U)$$

see Modules on Sites, Section 18.19. Warning: This functor, a priori, has nothing to do with cohomology with compact supports! We dubbed this functor "extension by zero" in the reference above. Note that the two versions of $f_!$ agree as $f^* = f^{-1}$ for sheaves of $\mathcal{O}_X$-modules.

As we are going to use this construction below let us recall some of its properties. Given an abelian sheaf $\mathcal{G}$ on $U_{étale}$ the sheaf $f_!$ is the sheafification of the presheaf

$$V/X \longmapsto f_!\mathcal{G}(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

see Modules on Sites, Lemma 18.19.2. Moreover, if $\mathcal{G}$ is an $\mathcal{O}_U$-module, then $f_!\mathcal{G}$ is the sheafification of the exact same presheaf of abelian groups which is endowed with an $\mathcal{O}_X$-module structure in an obvious way (see loc. cit.). Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point. Then there is a canonical identification

$$(f_!\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

where the sum is over all $\overline{u} : \mathrm{Spec}(k) \to U$ such that $f \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.38.1 and Properties of Spaces, Lemma 66.19.13. In the following

we are going to study the sheaf $f_!\underline{\mathbf{Z}}$. Here $\underline{\mathbf{Z}}$ denotes the constant sheaf on $X_{étale}$ or $U_{étale}$.

0722 **Lemma 69.6.1.** *Let S be a scheme. Let $f_i : U_i \to X$ be étale morphisms of algebraic spaces over S. Then there are isomorphisms*

$$f_{1,!}\underline{\mathbf{Z}} \otimes_{\mathbf{Z}} f_{2,!}\underline{\mathbf{Z}} \longrightarrow f_{12,!}\underline{\mathbf{Z}}$$

where $f_{12} : U_1 \times_X U_2 \to X$ is the structure morphism and

$$(f_1 \amalg f_2)_!\underline{\mathbf{Z}} \longrightarrow f_{1,!}\underline{\mathbf{Z}} \oplus f_{2,!}\underline{\mathbf{Z}}$$

Proof. Once we have defined the map it will be an isomorphism by our description of stalks above. To define the map it suffices to work on the level of presheaves. Thus we have to define a map

$$\left(\bigoplus\nolimits_{\varphi_1 \in \mathrm{Mor}_X(V,U_1)} \mathbf{Z}\right) \otimes_{\mathbf{Z}} \left(\bigoplus\nolimits_{\varphi_2 \in \mathrm{Mor}_X(V,U_2)} \mathbf{Z}\right) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V,U_1 \times_X U_2)} \mathbf{Z}$$

We map the element $1_{\varphi_1} \otimes 1_{\varphi_2}$ to the element $1_{\varphi_1 \times \varphi_2}$ with obvious notation. We omit the proof of the second equality. □

Another important feature is the trace map

$$\mathrm{Tr}_f : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}.$$

The trace map is adjoint to the map $\mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$ (which is an isomorphism). If $\overline{x}$ is above, then Tr_f on stalks at $\overline{x}$ is the map

$$(\mathrm{Tr}_f)_{\overline{x}} : (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{x}}$$

which sums the given integers. This is true because it is adjoint to the map $1 : \mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$. In particular, if f is surjective as well as étale then Tr_f is surjective.

Assume that $f : U \to X$ is a surjective étale morphism of algebraic spaces. Consider the *Koszul complex* associated to the trace map we discussed above

$$\ldots \to \wedge^3 f_!\underline{\mathbf{Z}} \to \wedge^2 f_!\underline{\mathbf{Z}} \to f_!\underline{\mathbf{Z}} \to \underline{\mathbf{Z}} \to 0$$

Here the exterior powers are over the sheaf of rings $\underline{\mathbf{Z}}$. The maps are defined by the rule

$$e_1 \wedge \ldots \wedge e_n \longmapsto \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \mathrm{Tr}_f(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

where $e_1, \ldots, e_n$ are local sections of $f_!\underline{\mathbf{Z}}$. Let $\overline{x}$ be a geometric point of X and set $M_{\overline{x}} = (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus_{\overline{u}} \mathbf{Z}$. Then the stalk of the complex above at $\overline{x}$ is the complex

$$\ldots \to \wedge^3 M_{\overline{x}} \to \wedge^2 M_{\overline{x}} \to M_{\overline{x}} \to \mathbf{Z} \to 0$$

which is exact because $M_{\overline{x}} \to \mathbf{Z}$ is surjective, see More on Algebra, Lemma 15.28.5. Hence if we let $K^\bullet = K^\bullet(f)$ be the complex with $K^i = \wedge^{i+1} f_!\underline{\mathbf{Z}}$, then we obtain a quasi-isomorphism

0723 (69.6.1.1) $$K^\bullet \longrightarrow \underline{\mathbf{Z}}[0]$$

We use the complex $K^\bullet$ to define what we call the alternating Čech complex associated to $f : U \to X$.

0724 **Definition 69.6.2.** Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. The *alternating Čech complex*[1] $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ associated to $\mathcal{F}$ and f is the complex

$$\mathrm{Hom}(K^0, \mathcal{F}) \to \mathrm{Hom}(K^1, \mathcal{F}) \to \mathrm{Hom}(K^2, \mathcal{F}) \to \ldots$$

with Hom groups computed in $Ab(X_{étale})$.

The reader may verify that if $U = \coprod U_i$ and $f|_{U_i} : U_i \to X$ is the open immersion of a subspace, then $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ agrees with the complex introduced in Cohomology, Section 20.23 for the Zariski covering $X = \bigcup U_i$ and the restriction of $\mathcal{F}$ to the Zariski site of X. What is more important however, is to relate the cohomology of the alternating Čech complex to the cohomology.

0725 **Lemma 69.6.3.** *Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. There exists a canonical map*

$$\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D(Ab)$. Moreover, there is a spectral sequence with E_1-page

$$E_1^{p,q} = \mathrm{Ext}^q_{Ab(X_{étale})}(K^p, \mathcal{F})$$

converging to $H^{p+q}(X, \mathcal{F})$ where $K^p = \wedge^{p+1} f_! \underline{\mathbf{Z}}$.

Proof. Recall that we have the quasi-isomorphism $K^\bullet \to \underline{\mathbf{Z}}[0]$, see (69.6.1.1). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$. Consider the double complex $\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet)$ with terms $\mathrm{Hom}(K^p, \mathcal{I}^q)$. The differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $K^{p+1} \to K^p$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$ the associated total complex, see Homology, Section 12.18. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

Because $K^\bullet$ is a resolution of $\underline{\mathbf{Z}}$ we see that the complexes

$$\mathrm{Hom}(K^\bullet, \mathcal{I}^q) : \mathrm{Hom}(K^0, \mathcal{I}^q) \to \mathrm{Hom}(K^1, \mathcal{I}^q) \to \mathrm{Hom}(K^2, \mathcal{I}^q) \to \ldots$$

are acyclic in positive degrees and have H^0 equal to $\Gamma(X, \mathcal{I}^q)$. Hence by Homology, Lemma 12.25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))) = H^n(X, \mathcal{F})$.

The map $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ of the lemma is the composition of $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to \mathrm{Tot}(\mathrm{Hom}(K^\bullet, \mathcal{I}^\bullet))$ with the inverse of the displayed quasi-isomorphism.

Finally, consider the spectral sequence $({}'E_r, {}'d_r)$. We have

$$E_1^{p,q} = q\text{th cohomology of } \mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

This proves the lemma. □

It follows from the lemma that it is important to understand the ext groups $\mathrm{Ext}_{Ab(X_{étale})}(K^p, \mathcal{F})$, i.e., the right derived functors of $\mathcal{F} \mapsto \mathrm{Hom}(K^p, \mathcal{F})$.

[1] This may be nonstandard notation

0726 **Lemma 69.6.4.** *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

where the fibre product has $p+1$ factors. There is a free action of S_{p+1} on W_p over X and

$$\mathrm{Hom}(K^p, \mathcal{F}) = S_{p+1}\textit{-anti-invariant elements of } \mathcal{F}(W_p)$$

functorially in $\mathcal{F}$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Because $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Hence W_p is an open and closed subspace of $U^{p+1} = U \times_X \ldots \times_X U$. The action of S_{p+1} on W_p is free as we've thrown out the fixed points of the action. By Lemma 69.6.1 we see that

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} = f_!^{p+1}\underline{\mathbf{Z}} = (W_p \to X)_!\underline{\mathbf{Z}} \oplus \textit{Rest}$$

where $f^{p+1} : U^{p+1} \to X$ is the structure morphism. Looking at stalks over a geometric point $\overline{x}$ of X we see that

$$\Big(\bigoplus\nolimits_{\overline{u} \mapsto \overline{x}} \mathbf{Z}\Big)^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}}_{\overline{x}}$$

is the quotient whose kernel is generated by all tensors $1_{\overline{u}_0} \otimes \ldots \otimes 1_{\overline{u}_p}$ where $\overline{u}_i = \overline{u}_j$ for some $i \neq j$. Thus the quotient map

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

factors through $(W_p \to X)_!\underline{\mathbf{Z}}$, i.e., we get

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

This already proves that $\mathrm{Hom}(K^p, \mathcal{F})$ is (functorially) a subgroup of

$$\mathrm{Hom}((W_p \to X)_!\underline{\mathbf{Z}}, \mathcal{F}) = \mathcal{F}(W_p)$$

To identify it with the S_{p+1}-anti-invariants we have to prove that the surjection $(W_p \to X)_!\underline{\mathbf{Z}} \to \wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the maximal S_{p+1}-anti-invariant quotient. In other words, we have to show that $\wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the quotient of $(W_p \to X)_!\underline{\mathbf{Z}}$ by the subsheaf generated by the local sections $s - \mathrm{sign}(\sigma)\sigma(s)$ where s is a local section of $(W_p \to X)_!\underline{\mathbf{Z}}$. This can be checked on the stalks, where it is clear. □

0727 **Lemma 69.6.5.** *Let S be a scheme. Let W be an algebraic space over S. Let G be a finite group acting freely on W. Let $U = W/G$, see Properties of Spaces, Lemma 66.34.1. Let $\chi : G \to \{+1, -1\}$ be a character. Then there exists a rank 1 locally free sheaf of $\mathbf{Z}$-modules $\underline{\mathbf{Z}}(\chi)$ on $U_{étale}$ such that for every abelian sheaf $\mathcal{F}$ on $U_{étale}$ we have*

$$H^0(W, \mathcal{F}|_W)^\chi = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi))$$

Proof. The quotient morphism $q : W \to U$ is a G-torsor, i.e., there exists a surjective étale morphism $U' \to U$ such that $W \times_U U' = \coprod_{g \in G} U'$ as spaces with G-action over U'. (Namely, $U' = W$ works.) Hence $q_*\underline{\mathbf{Z}}$ is a finite locally free $\mathbf{Z}$-module with an action of G. For any geometric point $\overline{u}$ of U, then we get G-equivariant isomorphisms

$$(q_*\underline{\mathbf{Z}})_{\overline{u}} = \bigoplus\nolimits_{\overline{w} \mapsto \overline{u}} \mathbf{Z} = \bigoplus\nolimits_{g \in G} \mathbf{Z} = \mathbf{Z}[G]$$

where the second $=$ uses a geometric point $\overline{w}_0$ lying over $\overline{u}$ and maps the summand corresponding to $g \in G$ to the summand corresponding to $g(\overline{w}_0)$. We have

$$H^0(W, \mathcal{F}|_W) = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})$$

because $q_*\mathcal{F}|_W = \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}$ as one can check by restricting to U'. Let

$$\underline{\mathbf{Z}}(\chi) = (q_*\underline{\mathbf{Z}})^\chi \subset q_*\underline{\mathbf{Z}}$$

be the subsheaf of sections that transform according to χ. For any geometric point $\overline{u}$ of U we have

$$\underline{\mathbf{Z}}(\chi)_{\overline{u}} = \mathbf{Z} \cdot \sum\nolimits_g \chi(g) g \subset \mathbf{Z}[G] = (q_*\underline{\mathbf{Z}})_{\overline{u}}$$

It follows that $\underline{\mathbf{Z}}(\chi)$ is locally free of rank 1 (more precisely, this should be checked after restricting to U'). Note that for any $\mathbf{Z}$-module M the χ-semi-invariants of $M[G]$ are the elements of the form $m \cdot \sum_g \chi(g) g$. Thus we see that for any abelian sheaf $\mathcal{F}$ on U we have

$$(\mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})^\chi = \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$$

because we have equality at all stalks. The result of the lemma follows by taking global sections. □

Now we can put everything together and obtain the following pleasing result.

0728 **Lemma 69.6.6.** *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

(with $p+1$ factors) as in Lemma 69.6.4. Let $\chi_p : S_{p+1} \to \{+1, -1\}$ be the sign character. Let $U_p = W_p/S_{p+1}$ and $\underline{\mathbf{Z}}(\chi_p)$ be as in Lemma 69.6.5. Then the spectral sequence of Lemma 69.6.3 has E_1-page

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$$

and converges to $H^{p+q}(X, \mathcal{F})$.

Proof. Note that since the action of S_{p+1} on W_p is over X we do obtain a morphism $U_p \to X$. Since $W_p \to X$ is étale and since $W_p \to U_p$ is surjective étale, it follows that also $U_p \to X$ is étale, see Morphisms of Spaces, Lemma 67.39.2. Therefore an injective object of $Ab(X_{étale})$ restricts to an injective object of $Ab(U_{p,étale})$, see Cohomology on Sites, Lemma 21.7.1. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is an auto-equivalence of $Ab(U_p)$, whence transforms injective objects into injective objects and is exact (because $\underline{\mathbf{Z}}(\chi_p)$ is an invertible $\underline{\mathbf{Z}}$-module). Thus given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$ the complex

$$\Gamma(U_p, \mathcal{I}^0|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^1|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^2|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \ldots$$

computes $H^*(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$. On the other hand, by Lemma 69.6.5 it is equal to the complex of S_{p+1}-anti-invariants in

$$\Gamma(W_p, \mathcal{I}^0) \to \Gamma(W_p, \mathcal{I}^1) \to \Gamma(W_p, \mathcal{I}^2) \to \ldots$$

which by Lemma 69.6.4 is equal to the complex

$$\mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

which computes $\mathrm{Ext}^*_{Ab(X_{étale})}(K^p, \mathcal{F})$. Putting everything together we win. □

69.7. Higher vanishing for quasi-coherent sheaves

0729 In this section we show that given a quasi-compact and quasi-separated algebraic space X there exists an integer $n = n(X)$ such that the cohomology of any quasi-coherent sheaf on X vanishes beyond degree n.

072A **Lemma 69.7.1.** *With S, W, G, U, χ as in Lemma 69.6.5. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, then so is $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$.*

Proof. The $\mathcal{O}_U$-module structure is clear. To check that $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$ is quasi-coherent it suffices to check étale locally. Hence the lemma follows as $\underline{\mathbf{Z}}(\chi)$ is finite locally free as a $\underline{\mathbf{Z}}$-module. □

The following proposition is interesting even if X is a scheme. It is the natural generalization of Cohomology of Schemes, Lemma 30.4.2. Before we state it, observe that given an étale morphism $f : U \to X$ from an affine scheme towards a quasi-separated algebraic space X the fibres of f are universally bounded, in particular there exists an integer d such that the fibres of $|U| \to |X|$ all have size at most d; this is the implication $(\eta) \Rightarrow (\delta)$ of Decent Spaces, Lemma 68.5.1.

072B **Proposition 69.7.2.** *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and separated. Let U be an affine scheme, and let $f : U \to X$ be a surjective étale morphism. Let d be an upper bound for the size of the fibres of $|U| \to |X|$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq d$.*

Proof. We will use the spectral sequence of Lemma 69.6.6. The lemma applies since f is separated as U is separated, see Morphisms of Spaces, Lemma 67.4.10. Since X is separated the scheme $U \times_X \ldots \times_X U$ is a closed subscheme of $U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$ hence is affine. Thus W_p is affine. Hence $U_p = W_p/S_{p+1}$ is an affine scheme by Groupoids, Proposition 39.23.9. The discussion in Section 69.3 shows that cohomology of quasi-coherent sheaves on W_p (as an algebraic space) agrees with the cohomology of the corresponding quasi-coherent sheaf on the underlying affine scheme, hence vanishes in positive degrees by Cohomology of Schemes, Lemma 30.2.2. By Lemma 69.7.1 the sheaves $\mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)$ are quasi-coherent. Hence $H^q(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $q > 0$. By our definition of the integer d we see that $W_p = \emptyset$ for $p \geq d$. Hence also $H^0(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $p \geq d$. This proves the proposition. □

In the following lemma we establish that a quasi-compact and quasi-separated algebraic space has finite cohomological dimension for quasi-coherent modules. We are explicit about the bound only because we will use it later to prove a similar result for higher direct images.

072C **Lemma 69.7.3.** *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Then we can choose*

1. *an affine scheme U,*
2. *a surjective étale morphism $f : U \to X$,*
3. *an integer d bounding the degrees of the fibres of $U \to X$,*
4. *for every $p = 0, 1, \ldots, d$ a surjective étale morphism $V_p \to U_p$ from an affine scheme V_p where U_p is as in Lemma 69.6.6, and*
5. *an integer d_p bounding the degree of the fibres of $V_p \to U_p$.*

Moreover, whenever we have (1) – (5), then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq \max(d_p + p)$.

Proof. Since X is quasi-compact we can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 66.6.3. By Decent Spaces, Lemma 68.5.1 the fibres of f are universally bounded, hence we can find d. We have $U_p = W_p/S_{p+1}$ and $W_p \subset U \times_X \ldots \times_X U$ is open and closed. Since X is quasi-separated the schemes W_p are quasi-compact, hence U_p is quasi-compact. Since U is separated, the schemes W_p are separated, hence U_p is separated by (the absolute version of) Spaces, Lemma 65.14.5. By Properties of Spaces, Lemma 66.6.3 we can find the morphisms $V_p \to W_p$. By Decent Spaces, Lemma 68.5.1 we can find the integers d_p.

At this point the proof uses the spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

see Lemma 69.6.6. By definition of the integer d we see that $U_p = 0$ for $p \geq d$. By Proposition 69.7.2 and Lemma 69.7.1 we see that $H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero for $q \geq d_p$ for $p = 0, \ldots, d$. Whence the lemma. □

69.8. Vanishing for higher direct images

073F We apply the results of Section 69.7 to obtain vanishing of higher direct images of quasi-coherent sheaves for quasi-compact and quasi-separated morphisms. This is useful because it allows one to argue by descending induction on the cohomological degree in certain situations.

073G **Lemma 69.8.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *f is quasi-compact and quasi-separated, and*
(2) *Y is quasi-compact.*

Then there exists an integer $n(X \to Y)$ such that for any algebraic space Y', any morphism $Y' \to Y$ and any quasi-coherent sheaf $\mathcal{F}'$ on $X' = Y' \times_Y X$ the higher direct images $R^i f'_* \mathcal{F}'$ are zero for $i \geq n(X \to Y)$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is an affine scheme, see Properties of Spaces, Lemma 66.6.3. Suppose we prove the result for the base change $f_V : V \times_Y X \to V$. Then the result holds for f with $n(X \to Y) = n(X_V \to V)$. Namely, if $Y' \to Y$ and $\mathcal{F}'$ are as in the lemma, then $R^i f'_* \mathcal{F}'|_{V \times_Y Y'}$ is equal to $R^i f'_{V,*} \mathcal{F}'|_{X'_V}$ where $f'_V : X'_V = V \times_Y Y' \times_Y X \to V \times_Y Y' = Y'_V$, see Properties of Spaces, Lemma 66.26.2. Thus we may assume that Y is an affine scheme.

Moreover, to prove the vanishing for all $Y' \to Y$ and $\mathcal{F}'$ it suffices to do so when Y' is an affine scheme. In this case, $R^i f'_* \mathcal{F}'$ is quasi-coherent by Lemma 69.3.1. Hence it suffices to prove that $H^i(X', \mathcal{F}') = 0$, because $H^i(X', \mathcal{F}') = H^0(Y', R^i f'_* \mathcal{F}')$ by Cohomology on Sites, Lemma 21.14.6 and the vanishing of higher cohomology of quasi-coherent sheaves on affine algebraic spaces (Proposition 69.7.2).

Choose $U \to X$, d, $V_p \to U_p$ and d_p as in Lemma 69.7.3. For any affine scheme Y' and morphism $Y' \to Y$ denote $X' = Y' \times_Y X$, $U' = Y' \times_Y U$, $V'_p = Y' \times_Y V_p$. Then $U' \to X'$, $d' = d$, $V'_p \to U'_p$ and $d'_p = d$ is a collection of choices as in Lemma 69.7.3 for the algebraic space X' (details omitted). Hence we see that $H^i(X', \mathcal{F}') = 0$ for $i \geq \max(p + d_p)$ and we win. □

073H **Lemma 69.8.2.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $R^i f_* \mathcal{F} = 0$ for $i > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. Recall that an affine morphism of algebraic spaces is representable. Hence this follows from (69.3.0.1) and Cohomology of Schemes, Lemma 30.2.3. □

0D2U **Lemma 69.8.3.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 69.8.2 and the Leray spectral sequence. See Cohomology on Sites, Lemma 21.14.6. □

69.9. Cohomology with support in a closed subspace

0A4L This section is the analogue of Cohomology, Sections 20.21 and 20.34 and Étale Cohomology, Section 59.79 for abelian sheaves on algebraic spaces.

Let S be a scheme. Let X be an algebraic space over S and let $Z \subset X$ be a closed subspace. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the sections with support in Z (Properties of Spaces, Definition 66.20.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{étale}) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Let $U \subset X$ be the open subspace which is the complement of Z. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 21.12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{étale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{étale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the *subsheaf of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Properties of Spaces, Definition 66.20.3. Using the equivalence of Morphisms of Spaces, Lemma 67.13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$Ab(X_{étale}) \longrightarrow Ab(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

0A4M **Lemma 69.9.1.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\mathrm{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Lemma 69.4.1) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

0A4N **Lemma 69.9.2.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact (Lemma 69.4.1) and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 21.14.2). □

0A4P **Lemma 69.9.3.** *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \subset Y$ be a closed subspace such that $f^{-1}(Z) \to Z$ is an isomorphism of algebraic spaces. Let $\mathcal{F}$ be an abelian sheaf on X. Then*

$$\mathcal{H}^q_Z(\mathcal{F}) = \mathcal{H}^q_{f^{-1}(Z)}(f^{-1}\mathcal{F})$$

as abelian sheaves on $Z = f^{-1}(Z)$ and we have $H^q_Z(Y, \mathcal{F}) = H^q_{f^{-1}(Z)}(X, f^{-1}\mathcal{F})$.

Proof. Because f is étale an injective resolution of $\mathcal{F}$ pulls back to an injective resolution of $f^{-1}\mathcal{F}$. Hence it suffices to check the equality for $\mathcal{H}_Z(-)$ which follows from the definitions. The proof for cohomology with supports is the same. Some details omitted. □

Let S be a scheme and let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. We denote $D_T(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of objects whose cohomology sheaves are supported on T.

0AEI **Lemma 69.9.4.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. The map $Ri_* = i_* : D(Z_{étale}) \to D(X_{étale})$ induces an equivalence $D(Z_{étale}) \to D_{|Z|}(X_{étale})$ with quasi-inverse*

$$i^{-1}|_{D_Z(X_{étale})} = R\mathcal{H}_Z|_{D_{|Z|}(X_{étale})}$$

Proof. Recall that i^{-1} and i_* is an adjoint pair of exact functors such that $i^{-1}i_*$ is isomorphic to the identify functor on abelian sheaves. See Properties of Spaces, Lemma 66.19.9 and Morphisms of Spaces, Lemma 67.13.5. Thus $i_* : D(Z_{étale}) \to D_Z(X_{étale})$ is fully faithful and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{étale})$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{étale}) \to D_Z(X_{étale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(X_{étale})$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i^{-1}K$. Since i_* is the right adjoint to the exact functor i^{-1}, the complex $i_*\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. □

69.10. Vanishing above the dimension

0A4Q Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. In this case $|X|$ is a spectral space, see Properties of Spaces, Lemma 66.15.2. Moreover, the dimension of X (as defined in Properties of Spaces, Definition 66.9.2) is equal to the Krull dimension of $|X|$, see Decent Spaces, Lemma 68.12.5. We will show that for quasi-coherent sheaves on X we have vanishing of cohomology above the dimension. This result is already interesting for quasi-separated algebraic spaces of finite type over a field.

0A4R **Lemma 69.10.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume $\dim(X) \leq d$ for some integer d. Let $\mathcal{F}$ be a quasi-coherent sheaf $\mathcal{F}$ on X.*

(1) *$H^q(X, \mathcal{F}) = 0$ for $q > d$,*
(2) *$H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for any quasi-compact open $U \subset X$,*
(3) *$H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ for any closed subspace $Z \subset X$ whose complement is quasi-compact.*

Proof. By Properties of Spaces, Lemma 66.22.5 every algebraic space Y étale over X has dimension $\leq d$. If Y is quasi-separated, the dimension of Y is equal to the Krull dimension of $|Y|$ by Decent Spaces, Lemma 68.12.5. Also, if Y is a scheme, then étale cohomology of $\mathcal{F}$ over Y, resp. étale cohomology of $\mathcal{F}$ with support in a closed subscheme, agrees with usual cohomology of $\mathcal{F}$, resp. usual cohomology with support in the closed subscheme. See Descent, Proposition 35.9.3 and Étale Cohomology, Lemma 59.79.5. We will use these facts without further mention.

By Decent Spaces, Lemma 68.8.6 there exist an integer n and open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

As $U_n = V_n$ is a scheme, our initial remarks imply the cohomology of $\mathcal{F}$ over U_n vanishes in degrees $> d$ by Cohomology, Proposition 20.22.4. Suppose we have shown, by induction, that $H^q(U_{p+1}, \mathcal{F}|_{U_{p+1}}) = 0$ for $q > d$. It suffices to show $H^q_{T_p}(U_p, \mathcal{F})$ for $q > d$ is zero in order to conclude the vanishing of cohomology of $\mathcal{F}$ over U_p in degrees $> d$. However, we have

$$H^q_{T_p}(U_p, \mathcal{F}) = H^q_{f_p^{-1}(T_p)}(V_p, \mathcal{F})$$

by Lemma 69.9.3 and as V_p is a scheme we obtain the desired vanishing from Cohomology, Proposition 20.22.4. In this way we conclude that (1) is true.

To prove (2) let $U \subset X$ be a quasi-compact open subspace. Consider the open subspace $U' = U \cup U_n$. Let $Z = U' \setminus U$. Then $g : U_n \to U'$ is an étale morphism such

that $g^{-1}(Z) \to Z$ is an isomorphism. Hence by Lemma 69.9.3 we have $H^q_Z(U', \mathcal{F}) = H^q_Z(U_n, \mathcal{F})$ which vanishes in degree $> d$ because U_n is a scheme and we can apply Cohomology, Proposition 20.22.4. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Assume, by induction, that we have reduced our problem to the case where U contains U_{p+1}. Then we set $U' = U \cup U_p$, set $Z = U' \setminus U$, and we argue using the morphism $f_p : V_p \to U'$ which is étale and has the property that $f_p^{-1}(Z) \to Z$ is an isomorphism. In other words, we again see that

$$H^q_Z(U', \mathcal{F}) = H^q_{f_p^{-1}(Z)}(V_p, \mathcal{F})$$

and we again see this vanishes in degrees $> d$. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Eventually we reach the stage where $U_1 = X \subset U$ which finishes the proof.

A formal argument shows that (2) implies (3). □

69.11. Cohomology and base change, I

073I Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : Y' \to Y$ is a morphism of algebraic spaces over S. Denote $X' = X_{Y'} = Y' \times_Y X$ the base change of X and denote $f' : X' \to Y'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

073J (69.11.0.1)
$$\begin{array}{ccccc} \mathcal{F}' = (g')^*\mathcal{F} & & X' & \xrightarrow{g'} & X & & \mathcal{F} \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ Rf'_*\mathcal{F}' & & Y' & \xrightarrow{g} & Y & & Rf_*\mathcal{F} \end{array}$$

Here is the simplest case of the base change property we have in mind.

07U8 **Lemma 69.11.1.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every diagram (69.11.0.1) we have*

$$g^* f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. By the discussion surrounding (69.3.0.1) this reduces to the case of an affine morphism of schemes which is treated in Cohomology of Schemes, Lemma 30.5.1. □

073K **Lemma 69.11.2** (Flat base change)**.** *Let S be a scheme. Consider a cartesian diagram of algebraic spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

(1) *the base change map of Cohomology on Sites, Lemma 21.15.1 is an isomorphism*
$$g^* R^i f_*\mathcal{F} \longrightarrow R^i f'_*\mathcal{F}',$$
(2) *if $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.*

Proof. The morphism g' is flat by Morphisms of Spaces, Lemma 67.30.4. Note that flatness of g and g' is equivalent to flatness of the morphisms of small étale ringed sites, see Morphisms of Spaces, Lemma 67.30.9. Hence we can apply Cohomology on Sites, Lemma 21.15.1 to obtain a base change map

$$g^* R^p f_* \mathcal{F} \longrightarrow R^p f'_* \mathcal{F}'$$

To prove this map is an isomorphism we can work locally in the étale topology on Y'. Thus we may assume that Y and Y' are affine schemes. Say $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$. In this case we are really trying to show that the map

$$H^p(X, \mathcal{F}) \otimes_A B \longrightarrow H^p(X_B, \mathcal{F}_B)$$

is an isomorphism where $X_B = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B. In other words, it suffices to prove (2).

Fix $A \to B$ a flat ring map and let X be a quasi-compact and quasi-separated algebraic space over A. Note that $g' : X_B \to X$ is affine as a base change of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Hence the higher direct images $R^i(g')_* \mathcal{F}_B$ are zero by Lemma 69.8.2. Thus $H^p(X_B, \mathcal{F}_B) = H^p(X, g'_* \mathcal{F}_B)$, see Cohomology on Sites, Lemma 21.14.6. Moreover, we have

$$g'_* \mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$$

where $\underline{A}$, $\underline{B}$ denotes the constant sheaf of rings with value A, B. Namely, it is clear that there is a map from right to left. For any affine scheme U étale over X we have

$$\begin{aligned} g'_* \mathcal{F}_B(U) &= \mathcal{F}_B(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U) \\ &= \Gamma(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U, (\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U \to U)^* \mathcal{F}|_U) \\ &= B \otimes_A \mathcal{F}(U) \end{aligned}$$

hence the map is an isomorphism. Write $B = \mathrm{colim}\, M_i$ as a filtered colimit of finite free A-modules M_i using Lazard's theorem, see Algebra, Theorem 10.81.4. We deduce that

$$\begin{aligned} H^p(X, g'_* \mathcal{F}_B) &= H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{B}) \\ &= H^p(X, \mathrm{colim}_i\, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F}) \otimes_A M_i \\ &= H^p(X, \mathcal{F}) \otimes_A \mathrm{colim}_i\, M_i \\ &= H^p(X, \mathcal{F}) \otimes_A B \end{aligned}$$

The first equality because $g'_* \mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$ as seen above. The second because $\otimes$ commutes with colimits. The third equality because cohomology on X commutes with colimits (see Lemma 69.5.1). The fourth equality because M_i is finite free (i.e., because cohomology commutes with finite direct sums). The fifth because $\otimes$ commutes with colimits. The sixth by choice of our system. □

69.12. Coherent modules on locally Noetherian algebraic spaces

07U9 This section is the analogue of Cohomology of Schemes, Section 30.9. In Modules on Sites, Definition 18.23.1 we have defined coherent modules on any ringed topos. We use this notion to define coherent modules on locally Noetherian algebraic spaces.

Although it is possible to work with coherent modules more generally we resist the urge to do so.

07UA **Definition 69.12.1.** Let S be a scheme. Let X be a locally Noetherian algebraic space over S. A quasi-coherent module $\mathcal{F}$ on X is called *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on the site $X_{étale}$ in the sense of Modules on Sites, Definition 18.23.1.

This definition is compatible with the already existing notion of a coherent module on a locally Noetherian scheme; see assertion (5) of Properties of Spaces, Section 66.30 (or more directly Descent, Lemma 35.8.10). Thus from now on, if X is a locally Noetherian scheme over S, we will not distinguish between a coherent module on X viewed as a scheme or a coherent module on X viewed as an algebraic space; this is compatible with the corresponding identifications of categories of quasi-coherent modules discussed in Properties of Spaces, Section 66.29.

Having said the above, the following lemma gives an understandable characterization of coherent modules on locally Noetherian algebraic spaces.

07UB **Lemma 69.12.2.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is coherent,*
(2) *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
(3) *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
(4) *for any étale morphism $\varphi : U \to X$ where U is a scheme the pullback $\varphi^*\mathcal{F}$ is a coherent module on U, and*
(5) *there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that the pullback $\varphi^*\mathcal{F}$ is a coherent module on U.*

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. To be sure, if X is a locally Noetherian algebraic space and $U \to X$ is an étale morphism, then U is locally Noetherian, see Properties of Spaces, Section 66.7. The lemma then follows from the points (1) – (5) made in Properties of Spaces, Section 66.30 and the corresponding result for coherent modules on locally Noetherian schemes, see Cohomology of Schemes, Lemma 30.9.1. □

07UC **Lemma 69.12.3.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (66.26.1.1). By Lemma 69.12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent by checking whether $f^*\mathcal{F}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 30.9.2. □

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules. This does not hold for modules on a general ringed topos.

07UD **Lemma 69.12.4.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (66.26.1.1). By Lemma 69.12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{G}$ is coherent by checking whether $f^*\mathcal{H}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 30.9.3. □

07UE **Lemma 69.12.5.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S,. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. Via Lemma 69.12.2 this follows from the result for schemes, see Cohomology of Schemes, Lemma 30.9.4. □

07UF **Lemma 69.12.6.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $\overline{x}$ be a geometric point of X lying over $x \in |X|$.*

1. *If $\mathcal{F}_{\overline{x}} = 0$ then there exists an open neighbourhood $X' \subset X$ of x such that $\mathcal{F}|_{X'} = 0$.*
2. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is injective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is injective.*
3. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is surjective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is surjective.*
4. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is bijective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is an isomorphism.*

Proof. Let $\varphi : U \to X$ be an étale morphism where U is a scheme and let $u \in U$ be a point mapping to x. By Properties of Spaces, Lemmas 66.29.4 and 66.22.1 as well as More on Algebra, Lemma 15.45.1 we see that $\varphi_{\overline{x}}$ is injective, surjective, or bijective if and only if $\varphi_u : \varphi^*\mathcal{F}_u \to \varphi^*\mathcal{G}_u$ has the corresponding property. Thus we can apply the schemes version of this lemma to see that (after possibly shrinking U) the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$ is injective, surjective, or an isomorphism. Let $X' \subset X$ be the open subspace corresponding to $|\varphi|(|U|) \subset |X|$, see Properties of Spaces, Lemma 66.4.8. Since $\{U \to X'\}$ is a covering for the étale topology, we conclude that $\varphi|_{X'}$ is injective, surjective, or an isomorphism as desired. Finally, observe that (1) follows from (2) by looking at the map $\mathcal{F} \to 0$. □

07UG **Lemma 69.12.7.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and $\mathcal{G}$ the quasi-coherent $\mathcal{O}_Z$-module such that $i_*\mathcal{G} = \mathcal{F}$, see Morphisms of Spaces, Definition 67.15.4. Then $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module.*

Proof. The statement of the lemma makes sense as a coherent module is in particular of finite type. Moreover, as $Z \to X$ is a closed immersion it is locally of finite type and hence Z is locally Noetherian, see Morphisms of Spaces, Lemmas 67.23.7 and 67.23.5. Finally, as $\mathcal{G}$ is of finite type it is a coherent $\mathcal{O}_Z$-module by Lemma 69.12.2 □

08AM **Lemma 69.12.8.** *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of locally Noetherian algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms of Spaces, Lemma 67.14.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms of Spaces, Lemma 67.14.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. To check that $\mathcal{G}$ is coherent we can work étale locally (Lemma 69.12.2). Choosing an étale covering by a scheme we conclude that $\mathcal{G}$ is coherent by the case of schemes (Cohomology of Schemes, Lemma 30.9.8). Hence the functor is fully faithful and the proof is done. □

07UH **Lemma 69.12.9.** *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^p f_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is a finite morphism of locally Noetherian schemes. By (69.3.0.1) we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.9.9. □

69.13. Coherent sheaves on Noetherian spaces

07UI In this section we mention some properties of coherent sheaves on Noetherian algebraic spaces.

07UJ **Lemma 69.13.1.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). If $\mathcal{F}_n|_U = \mathcal{F}_{n+1}|_U = \ldots$ then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.1. □

07UK **Lemma 69.13.2.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subspace $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $Supp(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). Note that $\mathcal{I}^n\mathcal{F}|_U = 0$ if and only if $\mathcal{I}^n\mathcal{F} = 0$ and similarly for the condition on the support. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.2. □

07UL **Lemma 69.13.3** (Artin-Rees)**.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 67.23.5). The equality of the lemma holds if and only if it holds after restricting to U. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 30.10.3. □

07UM **Lemma 69.13.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subspace and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. Let W be an affine scheme and let $W \to X$ be a surjective étale morphism (see Properties of Spaces, Lemma 66.6.3). Set $R = W \times_X W$. Then W and R are Noetherian schemes, see Morphisms of Spaces, Lemma 67.23.5. Hence the result hold for the restrictions of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{I}$, U, Z to W and R by Cohomology of Schemes, Lemma 30.10.5. It follows formally that the result holds over X. □

69.14. Devissage of coherent sheaves

07UN This section is the analogue of Cohomology of Schemes, Section 30.12.

07UP **Lemma 69.14.1.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathrm{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\mathrm{Supp}(\mathcal{G}') \subset Z'$ and $\mathrm{Supp}(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subspace structure on Z, see Properties of Spaces, Lemma 66.12.3. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every geometric point $\overline{x}$ of $Z' \setminus Z$ we have $\mathcal{I}_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$ and hence $\mathcal{G}_{n,\overline{x}} = 0$. Thus we see that $\mathrm{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian algebraic space. Hence by Lemma 69.13.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\mathrm{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

In the following we will freely use the scheme theoretic support of finite type modules as defined in Morphisms of Spaces, Definition 67.15.4.

07UQ **Lemma 69.14.2.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that the scheme theoretic support of $\mathcal{F}$ is a reduced $Z \subset X$ with $|Z|$ irreducible. Then there exist an integer $r > 0$, a nonzero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$, and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

whose cokernel is supported on a proper closed subspace of Z.

Proof. By assumption there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with support Z and $\mathcal{F} \cong i_*\mathcal{G}$, see Lemma 69.12.7. Hence it suffices to prove the lemma for the case $Z = X$ and $i = \mathrm{id}$.

By Properties of Spaces, Proposition 66.13.3 there exists a dense open subspace $U \subset X$ which is a scheme. Note that U is a Noetherian integral scheme. After shrinking U we may assume that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ (for example by Cohomology of Schemes, Lemma 30.12.2 or by a direct algebra argument). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals whose associated closed subspace is the complement of U in X (see for example Properties of Spaces, Section 66.12). By Lemma 69.13.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our isomorphism over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective: namely, if σ is a nonzero section of $\mathcal{I}^{\oplus r}$ over a scheme W étale over X, then because X hence W is reduced the support of σ contains a nonempty open of W. But the kernel of $(\mathcal{I}^n)^{\oplus r} \to \mathcal{F}$ is zero over a dense open, hence σ cannot be a section of the kernel. □

07UR **Lemma 69.14.3.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists a reduced closed subspace $Z_j \subset X$ with $|Z_j|$ irreducible and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} T \subset |X| \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 66.24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold. Clearly $T \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 69.14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that T is irreducible.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 66.12.3. By Lemma 69.13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Then the scheme theoretic support of $\mathcal{F}$ is T, see Morphisms of Spaces, Lemma 67.14.1. Hence we can apply Lemma 69.14.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is a proper closed subset of T. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of T. But then clearly $\mathcal{F}$ does too, which is our final contradiction. □

07US **Lemma 69.14.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *For every reduced closed subspace $Z \subset X$ with $|Z|$ irreducible and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 69.14.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. □

Here is a more useful variant of the lemma above.

07UT **Lemma 69.14.5.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*

(3) *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on Z such that*

   (a) *$Supp(\mathcal{G}) = Z$,*

   (b) *for every nonzero quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ such that $Supp(\mathcal{G}/\mathcal{G}')$ is proper closed in $|Z|$ and such that $\mathcal{P}$ holds for $i_*\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} T \subset |X| \text{ nonempty closed such that there exists a coherent sheaf} \\ \mathcal{F} \text{ with } \mathrm{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 66.24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold.

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 69.14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\text{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has $\mathcal{P}$. Hence $\mathcal{F}$ has property $\mathcal{P}$ by (1), a contradiction.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 66.12.3. By Lemma 69.13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does by (1). In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Denote $i : Z \to X$ the closed subspace corresponding to $\mathcal{J}$. Then $\mathcal{F} = i_*\mathcal{H}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}$, see Morphisms of Spaces, Lemma 67.14.1 and Lemma 69.12.7. Let $\mathcal{G}$ be the coherent sheaf on Z satisfying (3)(a) and (3)(b). We apply Lemma 69.14.2 to get injective maps

$$\mathcal{I}_1^{\oplus r_1} \to \mathcal{H} \quad \text{and} \quad \mathcal{I}_2^{\oplus r_2} \to \mathcal{G}$$

where the support of the cokernels are proper closed in Z. Hence we find an nonempty open $V \subset Z$ such that

$$\mathcal{H}_V^{\oplus r_2} \cong \mathcal{G}_V^{\oplus r_1}$$

Let $\mathcal{I} \subset \mathcal{O}_Z$ be a quasi-coherent ideal sheaf cutting out $Z \setminus V$ we obtain (Lemma 69.13.4) a map

$$\mathcal{I}^n\mathcal{G}^{\oplus r_1} \longrightarrow \mathcal{H}^{\oplus r_2}$$

which is an isomorphism over V. The kernel is supported on $Z \setminus V$ hence annihilated by some power of $\mathcal{I}$, see Lemma 69.13.2. Thus after increasing n we may assume the displayed map is injective, see Lemma 69.13.3. Applying (3)(b) we find $\mathcal{G}' \subset \mathcal{I}^n\mathcal{G}$ such that

$$(i_*\mathcal{G}')^{\oplus r_1} \longrightarrow i_*\mathcal{H}^{\oplus r_2} = \mathcal{F}^{\oplus r_2}$$

is injective with cokernel supported in a proper closed subset of Z and such that property $\mathcal{P}$ holds for $i_*\mathcal{G}'$. By (1) property $\mathcal{P}$ holds for $(i_*\mathcal{G}')^{\oplus r_1}$. By (1) and minimality of $T = |Z|$ property $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r_2}$. And finally by (2) property $\mathcal{P}$ holds for $\mathcal{F}$ which is the desired contradiction. □

08AN **Lemma 69.14.6.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

1. *For any short exact sequence of coherent sheaves on X if two out of three have property $\mathcal{P}$ so does the third.*
2. *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*
3. *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on X whose scheme theoretic support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. We will show that conditions (1) and (2) of Lemma 69.14.4 hold. This is clear for condition (1). To show that (2) holds, let

$$\mathcal{T} = \left\{ \begin{array}{l} i : Z \to X \text{ reduced closed subspace with } |Z| \text{ irreducible such} \\ \text{that } i_*\mathcal{I} \text{ does not have } \mathcal{P} \text{ for some quasi-coherent } \mathcal{I} \subset \mathcal{O}_Z \end{array} \right\}$$

If $\mathcal{T}$ is nonempty, then since X is Noetherian, we can find an $i : Z \to X$ which is minimal in $\mathcal{T}$. We will show that this leads to a contradiction.

Let $\mathcal{G}$ be the sheaf whose scheme theoretic support is Z whose existence is assumed in assumption (3). Let $\varphi : i_*\mathcal{I}^{\oplus r} \to \mathcal{G}$ be as in Lemma 69.14.2. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \text{Coker}(\varphi)$$

be a filtration as in Lemma 69.14.3. By minimality of Z and assumption (1) we see that $\text{Coker}(\varphi)$ has property $\mathcal{P}$. As φ is injective we conclude using assumption (1) once more that $i_*\mathcal{I}^{\oplus r}$ has property $\mathcal{P}$. Using assumption (2) we conclude that $i_*\mathcal{I}$ has property $\mathcal{P}$.

Finally, if $\mathcal{J} \subset \mathcal{O}_Z$ is a second quasi-coherent sheaf of ideals, set $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ and consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{I} \to \mathcal{I}/\mathcal{K} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{J} \to \mathcal{J}/\mathcal{K} \to 0$$

Arguing as above, using the minimality of Z, we see that $i_*\mathcal{I}/\mathcal{K}$ and $i_*\mathcal{J}/\mathcal{K}$ satisfy $\mathcal{P}$. Hence by assumption (1) we conclude that $i_*\mathcal{K}$ and then $i_*\mathcal{J}$ satisfy $\mathcal{P}$. In other words, Z is not an element of $\mathcal{T}$ which is the desired contradiction. □

69.15. Limits of coherent modules

07UU A colimit of coherent modules (on a locally Noetherian algebraic space) is typically not coherent. But it is quasi-coherent as any colimit of quasi-coherent modules on an algebraic space is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. Conversely, if the algebraic space is Noetherian, then every quasi-coherent module is a filtered colimit of coherent modules.

07UV **Lemma 69.15.1.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_X$-submodule which contains both of them (see Lemmas 69.12.3 and 69.12.4). In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to X$ a surjective étale morphism. Set $R = U \times_X U$ so that $X = U/R$ as usual. By Properties of Spaces, Proposition 66.32.1 we see that $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$. Hence we reduce to showing the corresponding thing for $QCoh(U, R, s, t, c)$. Thus the result follows from the more general Groupoids, Lemma 39.15.4. □

07UW **Lemma 69.15.2.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S with Y Noetherian. Then every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Write $f_*\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ with $\mathcal{H}_i$ a coherent $\mathcal{O}_Y$-module, see Lemma 69.15.1. By Lemma 69.12.2 the modules $\mathcal{H}_i$ are $\mathcal{O}_Y$-modules of finite presentation. Hence $f^*\mathcal{H}_i$ is an $\mathcal{O}_X$-module of finite presentation, see Properties of Spaces, Section 66.30. We claim the map

$$\operatorname{colim} f^*\mathcal{H}_i = f^*f_*\mathcal{F} \to \mathcal{F}$$

is surjective as f is assumed affine, Namely, choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then U is a scheme, $f' : U \to V$ is affine, and $U \to X$ is surjective étale. By Properties of Spaces, Lemma 66.26.2 we see that $f'_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ and similarly for pullbacks. Thus the restriction of $f^*f_*\mathcal{F} \to \mathcal{F}$ to U is the map

$$f^*f_*\mathcal{F}|_U = (f')^*(f_*\mathcal{F})|_V) = (f')^*f'_*(\mathcal{F}|_U) \to \mathcal{F}|_U$$

which is surjective as f' is an affine morphism of schemes. Hence the claim holds.

We conclude that every quasi-coherent module on X is a quotient of a filtered colimit of finitely presented modules. In particular, we see that $\mathcal{F}$ is a cokernel of a map

$$\operatorname{colim}_{j\in J} \mathcal{G}_j \longrightarrow \operatorname{colim}_{i\in I} \mathcal{H}_i$$

with $\mathcal{G}_j$ and $\mathcal{H}_i$ finitely presented. Note that for every $j \in I$ there exist $i \in I$ and a morphism $\alpha : \mathcal{G}_j \to \mathcal{H}_i$ such that

$$\begin{array}{ccc} \mathcal{G}_j & \xrightarrow{\alpha} & \mathcal{H}_i \\ \downarrow & & \downarrow \\ \operatorname{colim}_{j\in J} \mathcal{G}_j & \longrightarrow & \operatorname{colim}_{i\in I} \mathcal{H}_i \end{array}$$

commutes, see Lemma 69.5.3. In this situation $\operatorname{Coker}(\alpha)$ is a finitely presented $\mathcal{O}_X$-module which comes endowed with a map $\operatorname{Coker}(\alpha) \to \mathcal{F}$. Consider the set K of triples (i, j, α) as above. We say that $(i, j, \alpha) \leq (i', j', \alpha')$ if and only if $i \leq i'$, $j \leq j'$, and the diagram

$$\begin{array}{ccc} \mathcal{G}_j & \xrightarrow{\alpha} & \mathcal{H}_i \\ \downarrow & & \downarrow \\ \mathcal{G}_{j'} & \xrightarrow{\alpha'} & \mathcal{H}_{i'} \end{array}$$

commutes. It follows from the above that K is a directed partially ordered set,

$$\mathcal{F} = \operatorname{colim}_{(i,j,\alpha)\in K} \operatorname{Coker}(\alpha),$$

and we win. □

69.16. Vanishing of cohomology

07UX In this section we show that a quasi-compact and quasi-separated algebraic space is affine if it has vanishing higher cohomology for all quasi-coherent sheaves. We do this in a sequence of lemmas all of which will become obsolete once we prove Proposition 69.16.7.

07UY **Situation 69.16.1.** Here S is a scheme and X is a quasi-compact and quasi-separated algebraic space over S with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. We set $A = \Gamma(X, \mathcal{O}_X)$.

We would like to show that the canonical morphism

$$p : X \longrightarrow \mathrm{Spec}(A)$$

(see Properties of Spaces, Lemma 66.33.1) is an isomorphism. If M is an A-module we denote $M \otimes_A \mathcal{O}_X$ the quasi-coherent module $p^*\widetilde{M}$.

07UZ **Lemma 69.16.2.** *In Situation 69.16.1 for an A-module M we have $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ and $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$.*

Proof. The equality $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ follows from the equality $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$ as $p_*(M \otimes_A \mathcal{O}_X)$ is a quasi-coherent module on $\mathrm{Spec}(A)$ by Morphisms of Spaces, Lemma 67.11.2. Observe that $\Gamma(X, \bigoplus_{i\in I} \mathcal{O}_X) = \bigoplus_{i\in I} A$ by Lemma 69.5.1. Hence the lemma holds for free modules. Choose a short exact sequence $F_1 \to F_0 \to M$ where F_0, F_1 are free A-modules. Since $H^1(X, -)$ is zero the global sections functor is right exact. Moreover the pullback p^* is right exact as well. Hence we see that

$$\Gamma(X, F_1 \otimes_A \mathcal{O}_X) \to \Gamma(X, F_0 \otimes_A \mathcal{O}_X) \to \Gamma(X, M \otimes_A \mathcal{O}_X) \to 0$$

is exact. The result follows. □

The following lemma shows that Situation 69.16.1 is preserved by base change of $X \to \mathrm{Spec}(A)$ by $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$.

07V0 **Lemma 69.16.3.** *In Situation 69.16.1.*

(1) *Given an affine morphism $X' \to X$ of algebraic spaces, we have $H^1(X', \mathcal{F}') = 0$ for every quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$.*
(2) *Given an A-algebra A' setting $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ the morphism $X' \to X$ is affine and $\Gamma(X', \mathcal{O}_{X'}) = A'$.*

Proof. Part (1) follows from Lemma 69.8.2 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.5). Let $A \to A'$ be as in (2). Then $X' \to X$ is affine because affine morphisms are preserved under base change (Morphisms of Spaces, Lemma 67.20.5) and the fact that a morphism of affine schemes is affine. The equality $\Gamma(X', \mathcal{O}_{X'}) = A'$ follows as $(X' \to X)_*\mathcal{O}_{X'} = A' \otimes_A \mathcal{O}_X$ by Lemma 69.11.1 and thus

$$\Gamma(X', \mathcal{O}_{X'}) = \Gamma(X, (X' \to X)_*\mathcal{O}_{X'}) = \Gamma(X, A' \otimes_A \mathcal{O}_X) = A'$$

by Lemma 69.16.2. □

07V1 **Lemma 69.16.4.** *In Situation 69.16.1. Let $Z_0, Z_1 \subset |X|$ be disjoint closed subsets. Then there exists an $a \in A$ such that $Z_0 \subset V(a)$ and $Z_1 \subset V(a-1)$.*

Proof. We may and do endow Z_0, Z_1 with the reduced induced subspace structure (Properties of Spaces, Definition 66.12.5) and we denote $i_0 : Z_0 \to X$ and $i_1 : Z_1 \to X$ the corresponding closed immersions. Since $Z_0 \cap Z_1 = \emptyset$ we see that the canonical map of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow i_{0,*}\mathcal{O}_{Z_0} \oplus i_{1,*}\mathcal{O}_{Z_1}$$

is surjective (look at stalks at geometric points). Since $H^1(X, -)$ is zero on the kernel of this map the induced map of global sections is surjective. Thus we can find $a \in A$ which maps to the global section $(0, 1)$ of the right hand side. □

07V4 **Lemma 69.16.5.** *In Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally injective.*

Proof. Let $A \to k$ be a ring homomorphism where k is a field. It suffices to show that $\mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$ has at most one point (see Morphisms of Spaces, Lemma 67.19.6). Using Lemma 69.16.3 we may assume that A is a field and we have to show that $|X|$ has at most one point.

Let's think of X as an algebraic space over $\mathrm{Spec}(k)$ and let's use the notation $X(K)$ to denote K-valued points of X for any extension K/k, see Morphisms of Spaces, Section 67.24. If K/k is an algebraically closed field extension of large transcendence degree, then we see that $X(K) \to |X|$ is surjective, see Morphisms of Spaces, Lemma 67.24.2. Hence, after replacing k by K, we see that it suffices to prove that $X(k)$ is a singleton (in the case $A = k$).

Let $x, x' \in X(k)$. By Decent Spaces, Lemma 68.14.4 we see that x and x' are closed points of $|X|$. Hence x and x' map to distinct points of $\mathrm{Spec}(k)$ if $x \neq x'$ by Lemma 69.16.4. We conclude that $x = x'$ as desired. □

07V5 **Lemma 69.16.6.** *In Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is separated.*

Proof. By Decent Spaces, Lemma 68.9.2 we can find a scheme Y and a surjective integral morphism $Y \to X$. Since an integral morphism is affine, we can apply Lemma 69.16.3 to see that $H^1(Y, \mathcal{G}) = 0$ for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$. Since $Y \to X$ is quasi-compact and X is quasi-compact, we see that Y is quasi-compact. Since Y is a scheme, we may apply Cohomology of Schemes, Lemma 30.3.1 to see that Y is affine. Hence Y is separated. Note that an integral morphism is affine and universally closed, see Morphisms of Spaces, Lemma 67.45.7. By Morphisms of Spaces, Lemma 67.9.8 we see that X is a separated algebraic space. □

07V6 **Proposition 69.16.7.** *A quasi-compact and quasi-separated algebraic space is affine if and only if all higher cohomology groups of quasi-coherent sheaves vanish. More precisely, any algebraic space as in Situation 69.16.1 is an affine scheme.*

Proof. Choose an affine scheme $U = \mathrm{Spec}(B)$ and a surjective étale morphism $\varphi : U \to X$. Set $R = U \times_X U$. As p is separated (Lemma 69.16.6) we see that R is a closed subscheme of $U \times_{\mathrm{Spec}(A)} U = \mathrm{Spec}(B \otimes_A B)$. Hence $R = \mathrm{Spec}(C)$ is affine too and the ring map

$$B \otimes_A B \longrightarrow C$$

is surjective. Let us denote the two maps $s, t : B \to C$ as usual. Pick $g_1, \ldots, g_m \in B$ such that $s(g_1), \ldots, s(g_m)$ generate C over $t : B \to C$ (which is possible as $t : B \to C$ is of finite presentation and the displayed map is surjective). Then $g_1, \ldots, g_m$ give global sections of $\varphi_*\mathcal{O}_U$ and the map

$$\mathcal{O}_X[z_1, \ldots, z_n] \longrightarrow \varphi_*\mathcal{O}_U, \quad z_j \longmapsto g_j$$

is surjective: you can check this by restricting to U. Namely, $\varphi^*\varphi_*\mathcal{O}_U = t_*\mathcal{O}_R$ (by Lemma 69.11.2) hence you get exactly the condition that $s(g_i)$ generate C over $t : B \to C$. By the vanishing of H^1 of the kernel we see that

$$\Gamma(X, \mathcal{O}_X[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n] \longrightarrow \Gamma(X, \varphi_*\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = B$$

is surjective. Thus we conclude that B is a finite type A-algebra. Hence $X \to \mathrm{Spec}(A)$ is of finite type and separated. By Lemma 69.16.5 and Morphisms of Spaces, Lemma 67.27.5 it is also locally quasi-finite. Hence $X \to \mathrm{Spec}(A)$ is representable by Morphisms of Spaces, Lemma 67.51.1 and X is a scheme. Finally X is affine, hence equal to $\mathrm{Spec}(A)$, by an application of Cohomology of Schemes, Lemma 30.3.1. □

0D2V **Lemma 69.16.8.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. Then X is an affine scheme.*

Proof. The assumption implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then X is affine by Proposition 69.16.7. □

0D2W **Lemma 69.16.9.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Then X is a scheme and $\mathcal{L}$ is ample on X.*

Proof. Let $s \in H^0(X, \mathcal{L}^{\otimes d})$ be a global section. Let $U \subset X$ be the open subspace over which s is a generator of $\mathcal{L}^{\otimes d}$. In particular we have $\mathcal{L}^{\otimes d}|_U \cong \mathcal{O}_U$. We claim that U is affine.

Proof of the claim. We will show that $H^1(U, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$. This will prove the claim by Proposition 69.16.7. Denote $j : U \to X$ the inclusion morphism. Since étale locally the morphism j is affine (by Morphisms, Lemma 29.11.11) we see that j is affine (Morphisms of Spaces, Lemma 67.20.3). Hence we have

$$H^1(U, \mathcal{F}) = H^1(X, j_*\mathcal{F})$$

by Lemma 69.8.2 (and Cohomology on Sites, Lemma 21.14.6). Write $j_*\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Lemma 69.15.1. Then

$$H^1(X, j_*\mathcal{F}) = \mathrm{colim}\, H^1(X, \mathcal{F}_i)$$

by Lemma 69.5.1. Thus it suffices to show that $H^1(X, \mathcal{F}_i)$ maps to zero in $H^1(U, j^*\mathcal{F}_i)$. By assumption there exists an $n \geq 1$ such that

$$H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} (\mathcal{O}_X \oplus \mathcal{L} \oplus \ldots \oplus \mathcal{L}^{\otimes d-1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

Hence there exists an $a \geq 0$ such that $H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) = 0$. On the other hand, the map

$$s^a : \mathcal{F}_i \longrightarrow \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}$$

is an isomorphism after restriction to U. Contemplating the commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{F}_i) & \longrightarrow & H^1(U, j^*\mathcal{F}_i) \\ \downarrow{\scriptstyle s^a} & & \downarrow{\scriptstyle \cong} \\ H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) & \longrightarrow & H^1(U, j^*(\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad})) \end{array}$$

we conclude that the map $H^1(X, \mathcal{F}_i) \to H^1(U, j^*\mathcal{F}_i)$ is zero and the claim holds.

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$ (this also uses that a quasi-separated

algebraic space is decent, see Decent Spaces, Section 68.6). Thus $\mathcal{O}_X \to i_*\mathcal{O}_{\mathrm{Spec}(k)}$ is surjective. Let $\mathcal{I} \subset \mathcal{O}_X$ be the kernel and choose $d \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$. Then

$$H^0(X, \mathcal{L}^{\otimes d}) \to H^0(X, i_*\mathcal{O}_{\mathrm{Spec}(k)} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = H^0(\mathrm{Spec}(k), i^*\mathcal{L}^{\otimes d}) \cong k$$

is surjective by the long exact cohomology sequence. Hence there exists an $s \in H^0(X, \mathcal{L}^{\otimes d})$ such that $x \in U$ where U is the open subspace corresponding to s as above. Thus x is in the schematic locus (see Properties of Spaces, Lemma 66.13.1) of X by our claim.

To conclude that X is a scheme, it suffices to show that any open subset of $|X|$ which contains all the closed points is equal to $|X|$. This follows from the fact that $|X|$ is a Noetherian topological space, see Properties of Spaces, Lemma 66.24.3. Finally, if X is a scheme, then we can apply Cohomology of Schemes, Lemma 30.3.3 to conclude that $\mathcal{L}$ is ample. □

69.17. Finite morphisms and affines

07VN This section is the analogue of Cohomology of Schemes, Section 30.13.

0GF7 **Lemma 69.17.1.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume f is finite, surjective and X locally Noetherian. Let $i : Z \to X$ be a closed immersion. Denote $i' : Z' \to Y$ the inverse image of Z (Morphisms of Spaces, Section 67.13) and $f' : Z' \to Z$ the induced morphism. Then $\mathcal{G} = f'_*\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_Z$-module whose support is Z.*

Proof. Observe that f' is the base change of f and hence is finite and surjective by Morphisms of Spaces, Lemmas 67.5.5 and 67.45.5. Note that Y, Z, and Z' are locally Noetherian by Morphisms of Spaces, Lemma 67.23.5 (and the fact that closed immersions and finite morphisms are of finite type). By Lemma 69.12.9 we see that $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. The support of $\mathcal{G}$ is closed in $|Z|$, see Morphisms of Spaces, Lemma 67.15.2. Hence if the support of $\mathcal{G}$ is not equal to $|Z|$, then after replacing X by an open subspace we may assume $\mathcal{G} = 0$ but $Z \neq \emptyset$. This would mean that $f'_*\mathcal{O}_{Z'} = 0$. In particular the section $1 \in \Gamma(Z', \mathcal{O}_{Z'}) = \Gamma(Z, f'_*\mathcal{O}_{Z'})$ would be zero which would imply $Z' = \emptyset$ is the empty algebraic space. This is impossible as $Z' \to Z$ is surjective. □

0GF8 **Lemma 69.17.2.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$ (with notation as explained in the proof).*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ on $Y_{étale}$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is étale local on X and hence we may assume X is an affine scheme. In this case the result is a consequence of the corresponding result for schemes, see Cohomology of Schemes, Lemma 30.13.2. □

07VP **Lemma 69.17.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume*

(1) *f finite,*
(2) *f surjective,*
(3) *Y affine, and*
(4) *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then it follows that X is affine from Proposition 69.16.7.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 69.14.5. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 69.17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 69.14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 69.17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 69.17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. The morphism f' is affine, hence $R^1 f'_*\mathcal{I}' = 0$ by Lemma 69.8.2. As Z' is affine (as a closed subscheme of an affine scheme) we have $H^1(Z', \mathcal{I}') = 0$. Hence the Leray spectral sequence (in the form Cohomology on Sites, Lemma 21.14.6) implies that $H^1(Z, f'_*\mathcal{I}') = 0$. Since $i : Z \to X$ is affine we conclude that $R^1 i_* f'_*\mathcal{I}' = 0$ hence $H^1(X, i_* f'_*\mathcal{I}') = 0$ by Leray again. In other words, we have $H^1(X, i_*\mathcal{G}') = 0$ as desired. □

69.18. A weak version of Chow's lemma

089I In this section we quickly prove the following lemma in order to help us prove the basic results on cohomology of coherent modules on proper algebraic spaces.

089J **Lemma 69.18.1.** *Let A be a ring. Let X be an algebraic space over* $\mathrm{Spec}(A)$ *whose structure morphism $X \to \mathrm{Spec}(A)$ is separated of finite type. Then there exists a proper surjective morphism $X' \to X$ where X' is a scheme which is H-quasi-projective over* $\mathrm{Spec}(A)$.

Proof. Let W be an affine scheme and let $f : W \to X$ be a surjective étale morphism. There exists an integer d such that all geometric fibres of f have $\leq d$ points (because X is a separated algebraic hence reasonable, see Decent Spaces, Lemma 68.5.1). Picking d minimal we get a nonempty open $U \subset X$ such that

$f^{-1}(U) \to U$ is finite étale of degree d, see Decent Spaces, Lemma 68.8.1. Let

$$V \subset W \times_X W \times_X \ldots \times_X W$$

(d factors in the fibre product) be the complement of all the diagonals. Because $W \to X$ is separated the diagonal $W \to W \times_X W$ is a closed immersion. Since $W \to X$ is étale the diagonal $W \to W \times_X W$ is an open immersion, see Morphisms of Spaces, Lemmas 67.39.10 and 67.38.9. Hence the diagonals are open and closed subschemes of the quasi-compact scheme $W \times_X \ldots \times_X W$. In particular we conclude V is a quasi-compact scheme. Choose an open immersion $W \subset Y$ with Y H-projective over A (this is possible as W is affine and of finite type over A; for example we can use Morphisms, Lemmas 29.39.2 and 29.43.11). Let

$$Z \subset Y \times_A Y \times_A \ldots \times_A Y$$

be the scheme theoretic image of the composition $V \to W \times_X \ldots \times_X W \to Y \times_A \ldots \times_A Y$. Observe that this morphism is quasi-compact since V is quasi-compact and $Y \times_A \ldots \times_A Y$ is separated. Note that $V \to Z$ is an open immersion as $V \to Y \times_A \ldots \times_A Y$ is an immersion, see Morphisms, Lemma 29.7.7. The projection morphisms give d morphisms $g_i : Z \to Y$. These morphisms g_i are projective as Y is projective over A, see material in Morphisms, Section 29.43. We set

$$X' = \bigcup g_i^{-1}(W) \subset Z$$

There is a morphism $X' \to X$ whose restriction to $g_i^{-1}(W)$ is the composition $g_i^{-1}(W) \to W \to X$. Namely, these morphisms agree over V hence agree over $g_i^{-1}(W) \cap g_j^{-1}(W)$ by Morphisms of Spaces, Lemma 67.17.8. Claim: the morphism $X' \to X$ is proper.

If the claim holds, then the lemma follows by induction on d. Namely, by construction X' is H-quasi-projective over $\mathrm{Spec}(A)$. The image of $X' \to X$ contains the open U as V surjects onto U. Denote T the reduced induced algebraic space structure on $X \setminus U$. Then $T \times_X W$ is a closed subscheme of W, hence affine. Moreover, the morphism $T \times_X W \to T$ is étale and every geometric fibre has $< d$ points. By induction hypothesis there exists a proper surjective morphism $T' \to T$ where T' is a scheme H-quasi-projective over $\mathrm{Spec}(A)$. Since T is a closed subspace of X we see that $T' \to X$ is a proper morphism. Thus the lemma follows by taking the proper surjective morphism $X' \amalg T' \to X$.

Proof of the claim. By construction the morphism $X' \to X$ is separated and of finite type. We will check conditions (1) – (4) of Morphisms of Spaces, Lemma 67.42.5 for the morphisms $V \to X'$ and $X' \to X$. Conditions (1) and (2) we have seen above. Condition (3) holds as $X' \to X$ is separated (as a morphism whose source is a separated algebraic space). Thus it suffices to check liftability to X' for diagrams

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

where R is a valuation ring with fraction field K. Note that the top horizontal map is given by d pairwise distinct K-valued points $w_1, \ldots, w_d$ of W. In fact, this is a complete set of inverse images of the point $x \in X(K)$ coming from the diagram.

Since $W \to X$ is surjective, we can, after possibly replacing R by an extension of valuation rings, lift the morphism $\mathrm{Spec}(R) \to X$ to a morphism $w : \mathrm{Spec}(R) \to W$, see Morphisms of Spaces, Lemma 67.42.4. Since $w_1, \ldots, w_d$ is a complete collection of inverse images of x we see that $w|_{\mathrm{Spec}(K)}$ is equal to one of them, say w_i. Thus we see that we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle g_i} \\ \mathrm{Spec}(R) & \xrightarrow{w} & Y \end{array}$$

By the valuative criterion of properness for the projective morphism g_i we can lift w to $z : \mathrm{Spec}(R) \to Z$, see Morphisms, Lemma 29.43.5 and Schemes, Proposition 26.20.6. The image of z is in $g_i^{-1}(W) \subset X'$ and the proof is complete. □

69.19. Noetherian valuative criterion

0ARI We prove a version of the valuative criterion for properness using discrete valuation rings. More precise (and therefore more technical) versions can be found in Limits of Spaces, Section 70.21.

0ARJ **Lemma 69.19.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *Y is locally Noetherian,*
(2) *f is locally of finite type and quasi-separated,*
(3) *for every commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K its fraction field, there is at most one dotted arrow making the diagram commute.

Then f is separated.

Proof. We have to show that the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 67.4.1. Choose an affine scheme U and an étale morphism $U \to X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \to U$ is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 67.23.2, 67.23.3, and 67.23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \to U$ is of finite type. Consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & U \end{array}$$

of morphisms of schemes where A is a discrete valuation ring with fraction field K. We can interpret the composition $\mathrm{Spec}(A) \to U \to X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$. Hence our assumption (3) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 32.15.3 we conclude that $V \to U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 41.7.2) as desired. □

0ARK **Lemma 69.19.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *Y is locally Noetherian,*
(2) *f is of finite type and quasi-separated,*
(3) *for every commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dotted arrow making the diagram commute.

Then f is proper.

Proof. It suffices to prove f is universally closed because f is separated by Lemma 69.19.1. To do this we may work étale locally on Y (Morphisms of Spaces, Lemma 67.9.5). Hence we may assume $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme. Choose $X' \to X$ as in the weak form of Chow's lemma (Lemma 69.18.1). We claim that $X' \to \mathrm{Spec}(A)$ is universally closed. The claim implies the lemma by Morphisms of Spaces, Lemma 67.40.7. To prove this, according to Limits, Lemma 32.15.4 it suffices to prove that in every solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & \overset{a}{\nearrow} & & \overset{b}{\nearrow} & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a dvr with fraction field K we can find the dotted arrow a. By assumption we can find the dotted arrow b. Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a. □

0ARL **Remark 69.19.3** (Variant for complete discrete valuation rings)**.** In Lemmas 69.19.1 and 69.19.2 it suffices to consider complete discrete valuation rings. To be precise in Lemma 69.19.1 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a complete discrete valuation ring with fraction field K there exists at most one dotted arrow making the diagram commute. Namely, given any diagram as in Lemma 69.19.1 (3) the completion $A^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 15.43.5) and the uniqueness of the arrow $\mathrm{Spec}(A^\wedge) \to X$ implies the uniqueness of the arrow $\mathrm{Spec}(A) \to X$ for example by Properties of Spaces, Proposition 66.17.1. Similarly in Lemma 69.19.2 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a complete discrete valuation ring with fraction field K there exists an extension $A \subset A'$ of complete discrete valuation rings inducing a fraction field extension $K \subset K'$ such that there exists a unique arrow $\mathrm{Spec}(A') \to X$ making the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

commute. Namely, given any diagram as in Lemma 69.19.2 part (3) the existence of any commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

for *any* extension $A \subset B$ of discrete valuation rings will imply there exists an arrow $\mathrm{Spec}(A) \to X$ fitting into the diagram. This was shown in Morphisms of Spaces, Lemma 67.41.4. In fact, it follows from these considerations that it suffices to look for dotted arrows in diagrams for any class of discrete valuation rings such that, given any discrete valuation ring, there is an extension of it that is in the class. For example, we could take complete discrete valuation rings with algebraically closed residue field.

69.20. Higher direct images of coherent sheaves

08AP In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent. First we prove a helper lemma.

08AQ **Lemma 69.20.1.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow[i]{} & \mathbf{P}^n_Y \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

of algebraic spaces over S. Assume i is a closed immersion and Y Noetherian. Set $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_Y}(1)$. Let $\mathcal{F}$ be a coherent module on X. Then there exists an integer d_0 such that for all $d \geq d_0$ we have $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$.

Proof. Checking whether $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes d})$ is zero can be done étale locally on Y, see Equation (69.3.0.1). Hence we may assume Y is the spectrum of a Noetherian ring. In this case X is a scheme and the result follows from Cohomology of Schemes, Lemma 30.16.2. □

08AR **Lemma 69.20.2.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $i \geq 0$.*

Proof. We first remark that X is a locally Noetherian algebraic space by Morphisms of Spaces, Lemma 67.23.5. Hence the statement of the lemma makes sense. Moreover, computing $R^i f_*\mathcal{F}$ commutes with étale localization on Y (Properties of Spaces, Lemma 66.26.2) and checking whether $R^i f_*\mathcal{F}$ coherent can be done étale locally on Y (Lemma 69.12.2). Hence we may assume that $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme.

Assume $Y = \mathrm{Spec}(A)$ is an affine scheme. Note that f is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.7). Thus it is of finite presentation, hence X is Noetherian (Morphisms of Spaces, Lemma 67.28.6). Thus Lemma 69.14.6 applies to the category of coherent modules of X. For a coherent sheaf $\mathcal{F}$ on X we say $\mathcal{P}$ holds if and only if $R^i f_*\mathcal{F}$ is a coherent module on $\mathrm{Spec}(A)$. We will show that conditions (1), (2), and (3) of Lemma 69.14.6 hold for this property thereby finishing the proof of the lemma.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1}f_*\mathcal{F}_3 \to R^p f_*\mathcal{F}_1 \to R^p f_*\mathcal{F}_2 \to R^p f_*\mathcal{F}_3 \to R^{p+1}f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemmas 69.12.3 and 69.12.4. Hence property $\mathcal{P}$ holds for the third as well.

Verification of condition (2). This follows immediately from the fact that $R^i f_*(\mathcal{F}_1 \oplus \mathcal{F}_2) = R^i f_*\mathcal{F}_1 \oplus R^i f_*\mathcal{F}_2$ and that a summand of a coherent module is coherent (see lemmas cited above).

Verification of condition (3). Let $i : Z \to X$ be a closed immersion with Z reduced and $|Z|$ irreducible. Set $g = f \circ i : Z \to \mathrm{Spec}(A)$. Let $\mathcal{G}$ be a coherent module on Z whose scheme theoretic support is equal to Z such that $R^p g_*\mathcal{G}$ is coherent for all p. Then $\mathcal{F} = i_*\mathcal{G}$ is a coherent module on X whose scheme theoretic support is Z such that $R^p f_*\mathcal{F} = R^p g_*\mathcal{G}$. To see this use the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) and the fact that $R^q i_*\mathcal{G} = 0$ for $q > 0$ by Lemma 69.8.2 and the fact that a closed immersion is affine. (Morphisms of Spaces, Lemma 67.20.6). Thus we reduce to finding a coherent sheaf $\mathcal{G}$ on Z with support equal to Z such that $R^p g_*\mathcal{G}$ is coherent for all p.

We apply Lemma 69.18.1 to the morphism $Z \to \mathrm{Spec}(A)$. Thus we get a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\pi} & Z' & \xrightarrow{i} & \mathbf{P}^n_A \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle g'} & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

with $\pi : Z' \to Z$ proper surjective and i an immersion. Since $Z \to \mathrm{Spec}(A)$ is proper we conclude that g' is proper (Morphisms of Spaces, Lemma 67.40.4). Hence i is a closed immersion (Morphisms of Spaces, Lemmas 67.40.6 and 67.12.3). It follows that the morphism $i' = (i, \pi) : \mathbf{P}^n_A \times_{\mathrm{Spec}(A)} Z' = \mathbf{P}^n_Z$ is a closed immersion (Morphisms of Spaces, Lemma 67.4.6). Set

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_A}(1) = (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

We may apply Lemma 69.20.1 to $\mathcal{L}$ and π as well as $\mathcal{L}$ and g'. Hence for all $d \gg 0$ we have $R^p\pi_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes d}$. By the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes d} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes d}$$

and by choice of d the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes d}$ are those with $p = q = 0$. This implies that $R^p g_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$. Applying Cohomology of Schemes, Lemma 30.16.3 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$ is coherent.

We still have to check that the support of $\mathcal{G}$ is Z. This follows from the fact that $\mathcal{L}^{\otimes d}$ has lots of global sections. We spell it out here. Note that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq 0$ because the same is true for $\mathcal{O}_{\mathbf{P}^n}(d)$. Pick a point $z \in Z'$ mapping to the generic point ξ of Z which we can do as π is surjective. (Observe that Z does indeed have a generic point as $|Z|$ is irreducible and Z is Noetherian, hence quasi-separated, hence $|Z|$ is a sober topological space by Properties of Spaces, Lemma 66.15.1.) Pick $s \in \Gamma(Z', \mathcal{L}^{\otimes d})$ which does not vanish at z. Since $\Gamma(Z, \mathcal{G}) = \Gamma(Z', \mathcal{L}^{\otimes d})$ we may think of s as a global section of $\mathcal{G}$. Choose a geometric point $\overline{z}$ of Z' lying over z and denote $\overline{\xi} = g' \circ \overline{z}$ the corresponding geometric point of Z. The adjunction map

$$(g')^*\mathcal{G} = (g')^* g'_* \mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d}$$

induces a map of stalks $\mathcal{G}_{\overline{\xi}} \to \mathcal{L}_{\overline{z}}$, see Properties of Spaces, Lemma 66.29.5. Moreover the adjunction map sends the pullback of s (viewed as a section of $\mathcal{G}$) to s (viewed as a section of $\mathcal{L}^{\otimes d}$). Thus the image of s in the vector space which is the source of the arrow

$$\mathcal{G}_{\overline{\xi}} \otimes \kappa(\overline{\xi}) \longrightarrow \mathcal{L}_{\overline{z}}^{\otimes d} \otimes \kappa(\overline{z})$$

isn't zero since by choice of s the image in the target of the arrow is nonzero. Hence ξ is in the support of $\mathcal{G}$ (Morphisms of Spaces, Lemma 67.15.2). Since $|Z|$ is irreducible and Z is reduced we conclude that the scheme theoretic support of $\mathcal{G}$ is all of Z as desired. □

08AS **Lemma 69.20.3.** *Let A be a Noetherian ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Lemma 69.20.2. Namely, by Lemma 69.3.1 we know that $R^if_*\mathcal{F}$ is a quasi-coherent sheaf. Hence it is the quasi-coherent sheaf associated to the A-module $\Gamma(\mathrm{Spec}(A), R^if_*\mathcal{F}) = H^i(X, \mathcal{F})$. The equality holds by Cohomology on Sites, Lemma 21.14.6 and vanishing of higher cohomology groups of quasi-coherent modules on affine schemes (Cohomology of Schemes, Lemma 30.2.2). By Lemma 69.12.2 we see $R^if_*\mathcal{F}$ is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. Hence Lemma 69.20.2 gives us the conclusion. □

08AT **Lemma 69.20.4.** *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type. For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms of Spaces, Lemma 67.40.3. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.31.1). This implies that X' is a Noetherian algebraic space (Morphisms of Spaces, Lemma 67.28.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Morphisms of Spaces, Lemma 67.20.7. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms of Spaces, Lemma 67.20.10. Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 69.12.2). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 69.8.2 and Cohomology on Sites, Lemma 21.14.6. Thus the lemma follows from Lemma 69.20.3. □

69.21. Ample invertible sheaves and cohomology

0GF9 Here is a criterion for ampleness on proper algebraic spaces over affine bases in terms of vanishing of cohomology after twisting.

0GFA **Lemma 69.21.1.** *Let R be a Noetherian ring. Let X be a proper algebraic space over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent*

1. *X is a scheme and $\mathcal{L}$ is ample on X,*
2. *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$, and*
3. *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$.*

Proof. The implication (1) ⇒ (2) follows from Cohomology of Schemes, Lemma 30.17.1. The implication (2) ⇒ (3) is trivial. The implication (3) ⇒ (1) is Lemma 69.16.9. □

0GFB **Lemma 69.21.2.** *Let R be a Noetherian ring. Let $f : Y \to X$ be a morphism of algebraic spaces proper over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume f is finite and surjective. The following are equivalent*

(1) *X is a scheme and $\mathcal{L}$ is ample, and*
(2) *Y is a scheme and $f^*\mathcal{L}$ is ample.*

Proof. Assume (1). Then Y is a scheme as a finite morphism is representable (by schemes), see Morphisms of Spaces, Lemma 67.45.3. Hence (2) follows from Cohomology of Schemes, Lemma 30.17.2.

Assume (2). Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which implies $\mathcal{L}$ is ample by Lemma 69.21.1. We are going to apply Lemma 69.14.5. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor.

To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 69.17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 69.14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 69.17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 69.17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. Recall that f'_*, i_*, and i'_* transform coherent modules into coherent modules, see Lemmas 69.12.9 and 69.12.8. As Y is a scheme and $\mathcal{L}$ is ample we see from Lemma 69.21.1 that there exists an n_0 such that

$$H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0$$

for $n \geq n_0$ and $p > 0$. Now we get

$$\begin{aligned}
H^p(X, i_*\mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(Z, \mathcal{G}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}) \\
&= H^p(Z, f'_*\mathcal{I}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})) \\
&= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (f')^*i^*\mathcal{L}^{\otimes n})) \\
&= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\
&= H^p(Z', \mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\
&= H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0
\end{aligned}$$

Here we have used the projection formula and the Leray spectral sequence (see Cohomology on Sites, Sections 21.50 and 21.14) and Lemma 69.4.1. This verifies property (3)(b) of Lemma 69.14.5 as desired. □

69.22. The theorem on formal functions

08AU This section is the analogue of Cohomology of Schemes, Section 30.20. We encourage the reader to read that section first.

08AV **Situation 69.22.1.** Here A is a Noetherian ring and $I \subset A$ is an ideal. Also, $f : X \to \mathrm{Spec}(A)$ is a proper morphism of algebraic spaces and $\mathcal{F}$ is a coherent sheaf on X.

In this situation we denote $I^n\mathcal{F}$ the quasi-coherent submodule of $\mathcal{F}$ generated as an $\mathcal{O}_X$-module by products of local sections of $\mathcal{F}$ and elements of I^n. In other words, it is the image of the map $f^*\widetilde{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$.

08AW **Lemma 69.22.2.** *In Situation 69.22.1. Set $B = \bigoplus_{n\geq 0} I^n$. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.*

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 69.20.4. □

08AX **Lemma 69.22.3.** *In Situation 69.22.1. For every $p \geq 0$ there exists an integer $c \geq 0$ such that*

(1) *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
(2) *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 69.22.2 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $B = \bigoplus_{n\geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n - c)$. Consider the commutative diagram of A-modules

$$\begin{array}{ccccc} I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & H^p(X, I^{n+m}\mathcal{F}) \\ \downarrow & & & & \downarrow \\ I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) & & \longrightarrow & & H^p(X, I^n\mathcal{F}) \end{array}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

08AY **Lemma 69.22.4.** *In Situation 69.22.1. Fix $p \geq 0$.*

(1) *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$
(2) *The inverse system*
$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n\in\mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 12.31.2).
(3) *In fact for any p and n there exists a $c_2(n) \geq n$ such that*
$$\mathrm{Im}(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 69.22.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\mathrm{Ker}(H^p(X,\mathcal{F}) \to H^p(X,\mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X,I^n\mathcal{F}) \to H^p(X,\mathcal{F}))$$

Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X,\mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^n\mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I^{n+m}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^{n+m}\mathcal{F} & \longrightarrow & 0 \end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^p(X,I^n\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{F}/I^n\mathcal{F}) & \xrightarrow{\delta} & H^{p+1}(X,I^n\mathcal{F}) \\ \uparrow & & \uparrow{\scriptstyle 1} & & \uparrow & & \uparrow{\scriptstyle a} \\ H^p(X,I^{n+m}\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{F}/I^{n+m}\mathcal{F}) & \longrightarrow & H^{p+1}(X,I^{n+m}\mathcal{F}) \end{array}$$

If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X,I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 10.51.3) there exists an integer $c_3(n)$ such that

$$I^N H^{p+1}(X,I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X,\mathcal{F}/I^n\mathcal{F})\right)$$

for all $N \geq c_3(n)$. As $H^p(X,\mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then

$$\mathrm{Im}(H^p(X,\mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X,\mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X,\mathcal{F}) \to H^p(X,\mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

08AZ **Theorem 69.22.5** (Theorem on formal functions)**.** *In Situation 69.22.1. Fix $p \geq 0$. The system of maps*

$$H^p(X,\mathcal{F})/I^nH^p(X,\mathcal{F}) \longrightarrow H^p(X,\mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X,\mathcal{F})^\wedge \longrightarrow \lim_n H^p(X,\mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X,\mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 69.22.4. We spell out the details. Set $M = H^p(X,\mathcal{F})$ and $M_n = H^p(X,\mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 12.31 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2,3,\ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 69.22.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 69.22.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

08B0 **Lemma 69.22.6.** *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 69.22.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 69.20.3) hence I-adically complete (Algebra, Lemma 10.97.1) and we see that completion on the left hand side is not necessary. □

08B1 **Lemma 69.22.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume*

(1) *Y locally Noetherian,*
(2) *f proper, and*
(3) *$\mathcal{F}$ coherent.*

Let $\overline{y}$ be a geometric point of Y. Consider the "infinitesimal neighbourhoods"

$$\begin{array}{ccc} X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X & \xrightarrow[i_n]{} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) & \xrightarrow{c_n} & Y \end{array}$$

of the fibre $X_1 = X_{\overline{y}}$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})^\wedge_{\overline{y}} \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}^\wedge_{Y,\overline{y}}$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 69.22.5. Let us spell it out. Note that $\mathcal{O}_{Y,\overline{y}}$ is a Noetherian local ring, see Properties of Spaces, Lemma 66.24.4. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism as it identifies local rings. Denote $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 69.11.2. Moreover, we have canonical identifications $X_n = X'_n$ for all $n \geq 1$.

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a strictly henselian Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $\overline{y} \to Y$ is equal to $\mathrm{Spec}(A/\mathfrak{m}) \to Y$. It follows that

$$(R^p f_* \mathcal{F})_{\overline{y}} = \Gamma(Y, R^p f_* \mathcal{F}) = H^p(X, \mathcal{F})$$

because $(Y, \overline{y})$ is an initial object in the category of étale neighbourhoods of $\overline{y}$. The morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 69.12.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

0A4S **Lemma 69.22.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$X_{\overline{y}}$ has discrete underlying topological space.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_* \mathcal{F})_{\overline{y}} = 0$ for all $p > 0$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 69.22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. By Morphisms of Spaces, Lemma 67.34.8 the morphism $f : X \to Y$ is quasi-finite at each of the points of the fibre of $X \to Y$ over $\overline{y}$. It follows that $X_{\overline{y}} \to \overline{y}$ is separated and quasi-finite. Hence $X_{\overline{y}}$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. Since it is quasi-compact its underlying topological space is a finite discrete space. Then it is an affine scheme by Schemes, Lemma 26.11.8. By Lemma 69.17.3 it follows that the algebraic spaces X_n are affine schemes as well. Moreover, the underlying topological of each X_n is the same as that of X_1. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^p f_* \mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 69.22.7. Note that $R^p f_* \mathcal{F}$ is coherent by Lemma 69.20.2 and hence $R^p f_* \mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 10.97.1 this implies that $(R^p f_* \mathcal{F})_{\overline{y}} = 0$. □

0A4T **Lemma 69.22.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$\dim(X_{\overline{y}}) = d$.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_* \mathcal{F})_{\overline{y}} = 0$ for all $p > d$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 69.22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of $X_{\overline{y}}$. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Lemma 69.10.1. Hence we see that $(R^p f_* \mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 69.22.7 for $p > d$. Note that $R^p f_* \mathcal{F}$ is coherent by Lemma 69.20.2 and hence $R^p f_* \mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 10.97.1 this implies that $(R^p f_* \mathcal{F})_{\overline{y}} = 0$. □

69.23. Applications of the theorem on formal functions

0A4U We will add more here as needed.

0A4V **Lemma 69.23.1.** *(For a more general version see More on Morphisms of Spaces, Lemma 76.35.1). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian. The following are equivalent*

(1) *f is finite, and*
(2) *f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.*

Proof. A finite morphism is proper according to Morphisms of Spaces, Lemma 67.45.9. A finite morphism is quasi-finite according to Morphisms of Spaces, Lemma 67.45.8. A quasi-finite morphism has discrete fibres X_k, see Morphisms of Spaces, Lemma 67.27.5. Hence a finite morphism is proper and has discrete fibres X_k.

Assume f is proper with discrete fibres X_k. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms of Spaces, Lemma 67.45.7 whereupon it follows from Morphisms of Spaces, Lemma 67.45.6 that f is finite.

To show that f is affine we may assume that Y is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. We will show that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 69.15.1 and 69.5.1. Then it follows that X is affine from Proposition 69.16.7. By Lemma 69.22.8 we conclude that the stalks of $R^1 f_* \mathcal{F}$ are zero for all geometric points of Y. In other words, $R^1 f_* \mathcal{F} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{F}) = H^1(Y, f_* \mathcal{F})$. Since Y is affine, and $f_* \mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) we conclude $H^1(Y, f_* \mathcal{F}) = 0$ from Cohomology of Schemes, Lemma 30.2.2. Hence $H^1(X, \mathcal{F}) = 0$ as desired. □

As a consequence we have the following useful result.

0A4W **Lemma 69.23.2.** *(For a more general version see More on Morphisms of Spaces, Lemma 76.35.2). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

(1) *Y is locally Noetherian,*
(2) *f is proper, and*
(3) *$|X_{\overline{y}}|$ is finite.*

Then there exists an open neighbourhood $V \subset Y$ of $\overline{y}$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the geometric points of X lying over $\overline{y}$ by Morphisms of Spaces, Lemma 67.34.8. By Morphisms of Spaces, Lemma 67.34.7 the set of points at which f is quasi-finite is an open subspace $U \subset X$. Let $Z = X \setminus U$. Then $\overline{y} \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of $\overline{y}$ with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ has discrete fibres X_k (Morphisms of Spaces, Lemma 67.27.5) which are quasi-compact hence finite. Thus $f^{-1}(V) \to V$ is finite by Lemma 69.23.1. □

69.24. Other chapters

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CHAPTER 70

Limits of Algebraic Spaces

07SB

70.1. Introduction

07SC In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed sets (Categories, Definition 4.21.1) with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [**CLO12**]. Another approach is due to David Rydh (see [**Ryd08**]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

70.2. Conventions

07SD The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

70.3. Morphisms of finite presentation

049I In this section we generalize Limits, Proposition 32.6.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

049J **Definition 70.3.1.** Let S be a scheme.

(1) A functor $F : (Sch/S)^{opp}_{fppf} \to Sets$ is said to be *limit preserving* or *locally of finite presentation* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = \mathrm{colim}\, F(T_i).$$
We sometimes say that F is *locally of finite presentation over* S.

(2) Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. A transformation of functors $a : F \to G$ is *limit preserving* or *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)^{opp}_{fppf} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is *relatively limit preserving* over G.

[1]The characterization (2) in Lemma 70.3.2 may be easier to parse.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

049K (70.3.1.1) $$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} *$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 7.10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 7.25, for information on localization, and especially Sites, Remark 7.25.10 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 67.28.1 and one using that $X \to Y$ is a transformation of functors so that Definition 70.3.1 applies (we will use the terminology "limit preserving" for this notion as much as possible). We will show in Proposition 70.3.10 that these two definitions agree.

06BC **Lemma 70.3.2.** *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)^{opp}_{fppf} \to Sets$. The following are equivalent*

(1) *$a : F \to G$ is limit preserving, and*
(2) *for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets*

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \end{array}$$

is a fibre product diagram.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\operatorname{colim}_i G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 70.3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $\operatorname{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\operatorname{colim} F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\operatorname{colim}_i G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\operatorname{colim}_i F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\operatorname{colim}_i F_y(T'_i)$ mapping to $x_{T'}$ and we win. □

049L **Lemma 70.3.3.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are limit preserving, then*

$$b \circ a : F \longrightarrow H$$

is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Consider the diagram of sets

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \operatorname{colim}_i H(T_i) & \longrightarrow & H(T) \end{array}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

0GDY **Lemma 70.3.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If $b \circ a$ and b are limit preserving, then a is limit preserving.*

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Consider the diagram of sets

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \operatorname{colim}_i H(T_i) & \longrightarrow & H(T) \end{array}$$

By assumption the lower square and the outer rectangle are fibre products of sets. Hence the upper square is a fibre product square too which proves the lemma. □

049M **Lemma 70.3.5.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow[b']{} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

If a is limit preserving, then the base change a' is limit preserving.

Proof. Omitted. Hint: This is formal. □

0GDZ **Lemma 70.3.6.** *Let S be a scheme contained in Sch_{fppf}. Let $E, F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$, and $c : G \to E$ be transformations of functors. If c, $c \circ a$, and $c \circ b$ are limit preserving, then $F \times_G H \to E$ is too.*

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 70.3.2. Then we have

$$\operatorname{colim}(F \times_G H)(T_i) = \operatorname{colim} F(T_i) \times_{\operatorname{colim} G(T_i)} \operatorname{colim} H(T_i)$$

as filtered colimits commute with finite products. Our goal is thus to show that

$$\begin{array}{ccc} \operatorname{colim} F(T_i) \times_{\operatorname{colim} G(T_i)} \operatorname{colim} H(T_i) & \longrightarrow & F(T) \times_{G(T)} H(T) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i E(T_i) & \longrightarrow & E(T) \end{array}$$

is a fibre product diagram. This follows from the observation that given maps of sets $E' \to E$, $F \to G$, $H \to G$, and $G \to E$ we have

$$E' \times_E (F \times_G H) = (E' \times_E F) \times_{(E' \times_E G)} (E' \times_E H)$$

Some details omitted. □

049O **Lemma 70.3.7.** *Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. If F is limit preserving then its sheafification $F^\#$ is limit preserving.*

Proof. Assume F is limit preserving. It suffices to show that F^+ is limit preserving, since $F^\# = (F^+)^+$, see Sites, Theorem 7.10.10. Let T be an affine scheme over S, and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 34.7.4. Hence we can write

$$F^+(T) = \operatorname{colim}_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F).$$

Any $\mathcal{V} = \{T_k \to T\}_{k=1,\ldots,n}$ in the colimit may be written as $\mathcal{V}_i \times_{T_i} T$ for some i and some standard fppf covering $\mathcal{V}_i = \{T_{i,k} \to T_i\}_{k=1,\ldots,n}$ of T_i. Denote $\mathcal{V}_{i'} = \{T_{i',k} \to T_{i'}\}_{k=1,\ldots,n}$ the base change for $i' \geq i$. Then we see that

$$\begin{aligned} \operatorname{colim}_{i' \geq i} \check{H}^0(\mathcal{V}_i, F) &= \operatorname{colim}_{i' \geq i} \text{Equalizer}\Big(\prod F(T_{i',k}) \rightrightarrows \prod F(T_{i',k} \times_{T_{i'}} T_{i',l}) \Big) \\ &= \text{Equalizer}\Big(\operatorname{colim}_{i' \geq i} \prod F(T_{i',k}) \rightrightarrows \operatorname{colim}_{k' \geq k} \prod F(T_{i',k} \times_{T_{i'}} T_{i',l}) \Big) \\ &= \text{Equalizer}\Big(\prod F(T_k) \rightrightarrows \prod F(T_k \times_T T_l) \Big) \\ &= \check{H}^0(\mathcal{V}, F) \end{aligned}$$

Here the second equality holds because filtered colimits are exact. The third equality holds because F is limit preserving and because $\lim_{i' \geq i} T_{i',k} = T_k$ and $\lim_{i' \geq i} T_{i',k} \times_{T_{i'}} T_{i',l} = T_k \times_T T_l$ by Limits, Lemma 32.2.3. If we use this for all coverings at the same time we obtain

$$\begin{aligned} F^+(T) &= \operatorname{colim}_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F) \\ &= \operatorname{colim}_{i \in I} \operatorname{colim}_{\mathcal{V}_i \text{ standard covering } T_i} \check{H}^0(T \times_{T_i} \mathcal{V}_i, F) \\ &= \operatorname{colim}_{i \in I} F^+(T_i) \end{aligned}$$

The switch of the order of the colimits is allowed by Categories, Lemma 4.14.10. □

049P **Lemma 70.3.8.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

1. *F is a sheaf, and*
2. *there exists an fppf covering $\{U_j \to S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is limit preserving.*

Then F is limit preserving.

Proof. Let T be an affine scheme over S. Let I be a directed set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\operatorname{colim} F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 32.2.3.

Suppose that $x, x' \in \operatorname{colim} F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x_i' \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x_i' map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x_i'|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x_i'|_{T_{i'}}$ as desired. This proves that the map $\operatorname{colim} F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. □

049Q **Lemma 70.3.9.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be functors. If $a : F \to G$ is a transformation which is limit preserving, then the induced transformation of sheaves $F^\# \to G^\#$ is limit preserving.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F^\#_y : (Sch/T)^{opp}_{fppf} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 70.3.1 is limit preserving. By Equation (70.3.1.1) we see that $F^\#_y$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F^\#_{y_j}$. If we can show that $F^\#_{y_j}$ is limit preserving then Lemma 70.3.8 guarantees that $F^\#_y$ is limit preserving and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F^\#_y = (F_y)^\#$. This follows from Equation (70.3.1.1). Thus this case follows from Lemma 70.3.7. □

04AK **Proposition 70.3.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 67.28.1.*
2. *The morphism $f : X \to Y$ is limit preserving as a transformation of functors, see Definition 70.3.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_Y X$ is limit preserving over T in the sense of Definition 70.3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is limit preserving as a functor $X : (Sch/S)^{opp}_{fppf} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 65.9.3. It follows from Morphisms of Spaces, Definition 67.28.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 32.6.1 we have

$$U(T) = \operatorname{colim} U(T_i), \quad R(T) = \operatorname{colim} R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is limit preserving. Hence by Lemma 70.3.7 its sheafification $X = U/R$ is limit preserving too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 70.3.5 the transformation of functors $V \times_Y X \to V$ is limit preserving. By Morphisms of Spaces, Lemma 67.39.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence limit preserving as a transformation of functors by the first part of the proof. By Lemma 70.3.3 the composition $U \to V \times_Y X \to V$ is limit preserving as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 32.6.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 32.6.1 we characterize $U \to V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \operatorname{colim} V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) $\Rightarrow$ (1) of Limits, Proposition 32.6.1 we see that the question reduces to the case that U and V are affine. Say $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 10.127.3 only A-algebras of cardinality $\leq |B|$ are used. □

05N0 **Remark 70.3.11.** Here is an important special case of Proposition 70.3.10. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 32.6.2. In fact, we will see in Lemma 70.3.12 below that it suffices if the map

$$\operatorname{colim} X(T_i) \longrightarrow X(T)$$

is surjective whenever $T = \lim T_i$ is a directed limit of affine schemes over S.

0CM6 **Lemma 70.3.12.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map*

$$\operatorname{colim} X(T_i) \longrightarrow X(T) \times_{Y(T)} \operatorname{colim} Y(T_i)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 70.3.10 part (2) it suffices to check surjectivity in the criterion of Lemma 70.3.2.

Proof. Choose a scheme V and a surjective étale morphism $g : V \to Y$. Next, choose a scheme U and a surjective étale morphism $h : U \to V \times_Y X$. It suffices to show for $T = \lim T_i$ as in the lemma that the map

$$\operatorname{colim} U(T_i) \longrightarrow U(T) \times_{V(T)} \operatorname{colim} V(T_i)$$

is surjective, because then $U \to V$ will be locally of finite presentation by Limits, Lemma 32.6.3 (modulo a set theoretic remark exactly as in the proof of Proposition 70.3.10). Thus we take $a : T \to U$ and $b_i : T_i \to V$ which determine the same morphism $T \to V$. Picture

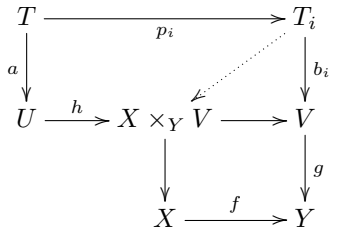

By the assumption of the lemma after increasing i we can find a morphism $c_i : T_i \to X$ such that $h \circ a = (b_i, c_i) \circ p_i : T_i \to V \times_Y X$ and such that $f \circ c_i = g \circ b_i$. Since h is an étale morphism of algebraic spaces (and hence locally of finite presentation), we have the surjectivity of

$$\operatorname{colim} U(T_i) \longrightarrow U(T) \times_{(X \times_Y V)(T)} \operatorname{colim}(X \times_Y V)(T_i)$$

by Proposition 70.3.10. Hence after increasing i again we can find the desired morphism $a_i : T_i \to U$ with $a = a_i \circ p_i$ and $b_i = (U \to V) \circ a_i$. □

70.4. Limits of algebraic spaces

07SE The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 67.20.5).

07SF **Lemma 70.4.1.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S. If the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,*

(1) *each of the morphisms $f_i : X \to X_i$ is affine,*
(2) *for any $i \in I$ and any morphism of algebraic spaces $T \to X_i$ we have*

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S.

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S. Choose an element $0 \in I$ (this is possible as a directed set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 32.2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \to U$ are the base change of the two projections $R_0 \to U_0$ by the morphism $U \to U_0$, in particular étale. The morphism $R \to U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \to U \times_S U$ is an étale equivalence relation. We claim that the natural map

07SG (70.4.1.1) $$U/R \longrightarrow \lim X_i$$

is an isomorphism of fppf sheaves on the category of schemes over S. The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 65.10.5.

Let Z be a scheme and let $a : Z \to \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \to X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \to X_i$ above. Hence we obtain a morphism $W_0 \to \lim_{i \geq 0} U_i = U$. Since $W_0 \to Z$ is surjective and étale, we conclude that (70.4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \to U/R$ are two morphisms which are equalized by (70.4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \to U$ which give rise to a and b. The condition that a, b are equalized by (70.4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \to U \to U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \to U_i \times_S U_i$ factors through R_i, say by some morphism $c_i : Z \to R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that $c = \lim c_i : Z \to R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R.

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \to U_i$ is affine by construction. □

07SH **Lemma 70.4.2.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let $X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \to X_0$ is a morphism of algebraic spaces. Then*

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S.

Proof. The limit X is an algebraic space by Lemma 70.4.1. The equality is formal, see Categories, Lemma 4.14.10. □

0CUH **Lemma 70.4.3.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i}) \to (Y_i, g_{i'i})$ be a morphism of inverse systems over I of algebraic spaces over S. Assume*

1. *the morphisms $f_{i'i} : X_{i'} \to X_i$ are affine,*
2. *the morphisms $g_{i'i} : Y_{i'} \to Y_i$ are affine,*
3. *the morphisms $X_i \to Y_i$ are closed immersions.*

Then $\lim X_i \to \lim Y_i$ is a closed immersion.

Proof. Observe that $\lim X_i$ and $\lim Y_i$ exist by Lemma 70.4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $V_0 \to Y_0$. Then the morphisms $V_i = Y_i \times_{Y_0} V_0 \to U_i = X_i \times_{Y_0} V_0$ are closed immersions of affine schemes. Hence the morphism $V = Y \times_{Y_0} V_0 \to U = X \times_{Y_0} V_0$ is a closed immersion because $V = \lim V_i$, $U = \lim U_i$ and because a limit of closed immersions of affine schemes is a closed immersion: a filtered colimit of surjective ring maps is surjective. Since the étale morphisms $V \to Y$ form an étale covering of Y as we vary our choice of $V_0 \to Y_0$ we see that the lemma is true. □

0CUI **Lemma 70.4.4.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse systems over I of algebraic spaces over S. If X_i is reduced for all i, then X is reduced.*

Proof. Observe that $\lim X_i$ exists by Lemma 70.4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $U_0 \to X_0$. Then the affine schemes $U_i = X_i \times_{X_0} U_0$ are reduced. Hence $U = X \times_{X_0} U_0$ is a reduced affine scheme as a limit of reduced affine schemes: a filtered colimit of reduced rings is reduced. Since the étale morphisms $U \to X$ form an étale covering of X as we vary our choice of $U_0 \to X_0$ we see that the lemma is true. □

0CP4 **Lemma 70.4.5.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. The equivalent conditions (1) and (2) of Proposition 70.3.10 are also equivalent to*

(3) *for every directed limit $T = \lim T_i$ of quasi-compact and quasi-separated algebraic spaces T_i over S with affine transition morphisms the diagram of sets*

$$\begin{array}{ccc} \operatorname{colim}_i \operatorname{Mor}(T_i, X) & \longrightarrow & \operatorname{Mor}(T, X) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i \operatorname{Mor}(T_i, Y) & \longrightarrow & \operatorname{Mor}(T, Y) \end{array}$$

is a fibre product diagram.

Proof. It is clear that (3) implies (2). We will assume (2) and prove (3). The proof is rather formal and we encourage the reader to find their own proof.

Let us first prove that (3) holds when T_i is in addition assumed separated for all i. Choose $i \in I$ and choose a surjective étale morphism $U_i \to T_i$ where U_i is affine. Using Lemma 70.4.2 we see that with $U = U_i \times_{T_i} T$ and $U_{i'} = U_i \times_{T_i} T_{i'}$ we have $U = \lim_{i' \geq i} U_{i'}$. Of course U and $U_{i'}$ are affine (see Lemma 70.4.1). Since T_i is separated, the fibre product $V_i = U_i \times_{T_i} U_i$ is an affine scheme as well and we obtain affine schemes $V = V_i \times_{T_i} T$ and $V_{i'} = V_i \times_{T_i} T_{i'}$ with $V = \lim_{i' \geq i} V_{i'}$. Observe that $U \to T$ and $U_i \to T_i$ are surjective étale and that $V = U \times_T U$ and $V_{i'} = U_{i'} \times_{T_{i'}} U_{i'}$. Note that $\operatorname{Mor}(T, X)$ is the equalizer of the two maps $\operatorname{Mor}(U, X) \to \operatorname{Mor}(V, X)$; this is true for example because X as a sheaf on $(Sch/S)_{fppf}$ is the coequalizer of the two maps $h_V \to h_u$. Similarly $\operatorname{Mor}(T_{i'}, X)$ is the equalizer of the two maps $\operatorname{Mor}(U_{i'}, X) \to \operatorname{Mor}(V_{i'}, X)$. And of course the same thing is true with X replaced with Y. Condition (2) says that the diagrams of in (3) are fibre products in the case of $U = \lim U_i$ and $V = \lim V_i$. It follows formally that the same thing is true for $T = \lim T_i$.

In the general case, choose an affine scheme U, an $i \in I$, and a surjective étale morphism $U \to T_i$. Repeating the argument of the previous paragraph we still achieve the proof: the schemes $V_{i'}$, V are no longer affine, but they are still quasi-compact and separated and the result of the preceding paragraph applies. □

70.5. Descending properties

0826 This section is the analogue of Limits, Section 32.4.

0CUJ **Lemma 70.5.1.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). If each X_i is decent (for example quasi-separated or locally separated) then $|X| = \lim_i |X_i|$ as sets.*

Proof. There is a canonical map $|X| \to \lim |X_i|$. Choose $0 \in I$. If $W_0 \subset X_0$ is an open subspace, then we have $f_0^{-1}W_0 = \lim_{i \geq 0} f_{i0}^{-1}W_0$, see Lemma 70.4.1. Hence, if we can prove the lemma for inverse systems where X_0 is quasi-compact, then the lemma follows in general. Thus we may and do assume X_0 is quasi-compact.

Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 70.4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 32.4.6 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \to \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i. This is a finite nonempty set by the definition of decent spaces (Decent Spaces, Definition 68.6.1). Hence $\lim S_i$ is nonempty, see Categories, Lemma 4.21.7. Let $(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \to \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u_i' \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u_i') \in |U_i| \times |U_i|$. This is a finite set by the definition of decent spaces (Decent Spaces, Definition 68.6.1). Moreover T_i is nonempty as we've assumed that x and x' map to the same point of X_i. Hence $\lim T_i$ is nonempty, see Categories, Lemma 4.21.7. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired.

Parenthetical statement: A quasi-separated algebraic space is decent, see Decent Spaces, Section 68.6 (the key observation to this is Properties of Spaces, Lemma 66.6.7). A locally separated algebraic space is decent by Decent Spaces, Lemma 68.15.2. □

086V **Lemma 70.5.2.** *With same notation and assumptions as in Lemma 70.5.1 we have $|X| = \lim_i |X_i|$ as topological spaces.*

Proof. We will use the criterion of Topology, Lemma 5.14.3. We have seen that $|X| = \lim_i |X_i|$ as sets in Lemma 70.5.1. The maps $f_i : X \to X_i$ are morphisms of algebraic spaces hence determine continuous maps $|X| \to |X_i|$. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset |X_i|$. Finally, let $x \in |X|$ and let $x \in V \subset |X|$ be an open neighbourhood. We have to find an i and an open neighbourhood $W_i \subset |X_i|$ of the image x with $f_i^{-1}(W_i) \subset V$. Choose $0 \in I$. Choose a scheme U_0 and a surjective

étale morphism $U_0 \to X_0$. Set $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$ for $i \geq 0$. Then $U = \lim_{i \geq 0} U_i$ in the category of schemes by Lemma 70.4.1. Choose $u \in U$ mapping to x. By the result for schemes (Limits, Lemma 32.4.2) we can find an $i \geq 0$ and an open neighbourhood $E_i \subset U_i$ of the image of u whose inverse image in U is contained in the inverse image of V in U. Then we can set $W_i \subset |X_i|$ equal to the image of E_i. This works because $|U_i| \to |X_i|$ is open. □

086W **Lemma 70.5.3.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). If each X_i is quasi-compact and nonempty, then $|X|$ is nonempty.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 32.4.3) and thus X is nonempty. □

0CUK **Lemma 70.5.4.** *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). Let $x \in |X|$ with images $x_i \in |X_i|$. If each X_i is decent, then $\overline{\{x\}} = \lim_i \overline{\{x_i\}}$ as sets and as algebraic spaces if endowed with reduced induced scheme structure.*

Proof. Set $Z = \overline{\{x\}} \subset |X|$ and $Z_i = \overline{\{x_i\}} \subset |X_i|$. Since $|X| \to |X_i|$ is continuous we see that Z maps into Z_i for each i. Hence we obtain an injective map $Z \to \lim Z_i$ because $|X| = \lim |X_i|$ as sets (Lemma 70.5.1). Suppose that $x' \in |X|$ is not in Z. Then there is an open subset $U \subset |X|$ with $x' \in U$ and $x \notin U$. Since $|X| = \lim |X_i|$ as topological spaces (Lemma 70.5.2) we can write $U = \bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset |X_j|$, see Topology, Lemma 5.14.2. Then we see that for some $j \in J$ we have $f_j(x') \in U_j$ and $f_j(x) \notin U_j$. In other words, we see that $f_j(x') \notin Z_j$. Thus $Z = \lim Z_i$ as sets.

Next, endow Z and Z_i with their reduced induced scheme structures, see Properties of Spaces, Definition 66.12.5. The transition morphisms $X_{i'} \to X_i$ induce affine morphisms $Z_{i'} \to Z_i$ and the projections $X \to X_i$ induce compatible morphisms $Z \to Z_i$. Hence we obtain morphisms $Z \to \lim Z_i \to X$ of algebraic spaces. By Lemma 70.4.3 we see that $\lim Z_i \to X$ is a closed immersion. By Lemma 70.4.4 the algebraic space $\lim Z_i$ is reduced. By the above $Z \to \lim Z_i$ is bijective on points. By uniqueness of the reduced induced closed subscheme structure we find that this morphism is an isomorphism of algebraic spaces. □

084R **Situation 70.5.5.** Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

07SI **Lemma 70.5.6.** *Notation and assumptions as in Situation 70.5.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on X_0. Set $\mathcal{F}_i = f_{0i}^* \mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^* \mathcal{F}_0$. Then*

$$\Gamma(X, \mathcal{F}) = \operatorname{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 66.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$

and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 32.4.7 implies that

$$\mathcal{F}(U) = \operatorname{colim} \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \operatorname{colim} \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \operatorname{Ker}(\mathcal{F}(U) \to \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \operatorname{Ker}(\mathcal{F}_i(U_i) \to \mathcal{F}_i(R_i))$ □

0827 **Lemma 70.5.7.** *Notation and assumptions as in Situation 70.5.5. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U.*

Proof. Follows formally from the construction of limits in Lemma 70.4.1 and the corresponding result for schemes: Limits, Lemma 32.4.11. □

The following lemma will be superseded by the stronger Lemma 70.6.10.

084S **Lemma 70.5.8.** *Notation and assumptions as in Situation 70.5.5. Let $f_0 : Y_0 \to Z_0$ be a morphism of algebraic spaces over X_0. Assume (a) $Y_0 \to X_0$ and $Z_0 \to X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \to Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 32.8.11 to see that $Y_0 \times_{X_0} U_i \to Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \to X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). □

084T **Lemma 70.5.9.** *Notation and assumptions as in Situation 70.5.5. If X is separated, then X_i is separated for some $i \in I$.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \to X_i$ and $U \to X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \to U_i$ and $s : R \to U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \to U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \to U \times U$ is a closed immersion. Since $R \to U$ is of finite type, there exists an i such that the morphism $R \to U_i \times U$ is a closed immersion (Limits, Lemma 32.4.16). Fix such an $i \in I$. Apply Limits, Lemma 32.8.5 to the system of morphisms $R_{i'} \to U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \to U_i \times U_{i'}$ is a closed immersion for i' sufficiently large. This implies immediately that $R_{i'} \to U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. □

07SQ **Lemma 70.5.10.** *Notation and assumptions as in Situation 70.5.5. If X is affine, then there exists an i such that X_i is affine.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \to X$ is an étale morphism of affine schemes. Hence we can write $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$ and

$$B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$$

such that $\Delta = \det(\partial g_\lambda/\partial x_\mu)$ is invertible in B, see Algebra, Lemma 10.143.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \operatorname{colim} A_i$ by Lemma 70.5.6. After increasing 0 we may assume we have $g_{1,i}, \ldots, g_{n,i} \in A_i[x_1, \ldots, x_n]$ mapping to $g_1, \ldots, g_n$. Set

$$B_i = A_i[x_1, \ldots, x_n]/(g_{1,i}, \ldots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i}/\partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \to B_i$ is an étale ring map. After increasing 0 we may assume also that $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)$ is surjective, see Limits, Lemma 32.8.15. Increasing 0 yet again we may choose elements $h_{1,i}, \ldots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$ which map to the classes of $x_1, \ldots, x_n$ in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

084U (70.5.10.1)
$$\begin{array}{ccc} X_i & \longleftarrow & U_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_i) & \longleftarrow & \mathrm{Spec}(B_i) \end{array}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0. The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 70.5.8 guarantees that there exists an i such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (70.5.10.1) is cartesian. Thus Descent, Lemma 35.39.1 applied to the fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)\}$ combined with Descent, Lemma 35.37.1 give that $X_i \to \mathrm{Spec}(A_i)$ is representable by a scheme affine over $\mathrm{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \mathrm{Spec}(A_i)$ but we don't need this.) □

07SR **Lemma 70.5.11.** *Notation and assumptions as in Situation 70.5.5. If X is a scheme, then there exists an i such that X_i is a scheme.*

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 70.5.7 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \to X$. By Lemma 70.5.10 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 66.13). □

0828 **Lemma 70.5.12.** *Let S be a scheme. Let B be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \to X$ be a morphism of algebraic spaces over B.*

(1) *If $Y \to X$ is a closed immersion, X_i quasi-compact, and $Y \to B$ locally of finite type, then $Y \to X_i$ is a closed immersion for i large enough.*
(2) *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*

(3) *If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to B$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to B$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.*
(4) *If $Y \to X$ is a monomorphism, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is a monomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \to B$ such that the image of $|X_0| \to |B|$ is contained in $|W| \to |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \to X_0 \times_B W$ such that $U_0 \to X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \to Y$, resp. $U \to X$, resp. $U_i \to X_i$ be the base change of $U_0 \to X_0$ (for $i \geq 0$). It suffices to prove that $V \to U_i$ is a closed immersion for i sufficiently large. Thus we reduce to proving the result for $V \to U = \lim U_i$ over W. This follows from the case of schemes, which is Limits, Lemma 32.4.16.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 70.5.7 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \to U_0$ is of finite presentation (because $U \to B$ is locally of finite presentation and $U_0 \to B$ is locally of finite type and Morphisms of Spaces, Lemma 67.28.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \to X_i$ are affine hence separated, the transition morphisms $Z_{i'} \to Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 67.4.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \to X$ is a monomorphism. Choose $0 \in I$. Since $Y \to X_0$ is locally of finite type (Morphisms of Spaces, Lemma 67.23.6) the morphism $Y \to Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10). The morphisms $Z_i \to Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i. This proves (4) and the lemma. □

086X **Lemma 70.5.13.** *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y is quasi-separated,*

(2) X_i *is quasi-compact and quasi-separated,*
(3) *the morphism* $X \to Y$ *is separated.*

Then $X_i \to Y$ *is separated for all* i *large enough.*

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \to Y$ such that the image of $|W| \to |Y|$ contains the image of $|X_0| \to |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \to W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \to X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \to X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 70.5.9. □

0A0R **Lemma 70.5.14.** *Let* S *be a scheme. Let* Y *be an algebraic space over* S*. Let* $X = \lim X_i$ *be a directed limit of algebraic spaces over* Y *with affine transition morphisms. Assume*

(1) Y *quasi-compact and quasi-separated,*
(2) X_i *quasi-compact and quasi-separated,*
(3) $X \to Y$ *affine.*

Then $X_i \to Y$ *is affine for* i *large enough.*

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is affine and it suffices to check that $X_i \times_Y W$ is affine for some i (Morphisms of Spaces, Lemma 67.20.3). This follows from Lemma 70.5.10. □

0A0S **Lemma 70.5.15.** *Let* S *be a scheme. Let* Y *be an algebraic space over* S*. Let* $X = \lim X_i$ *be a directed limit of algebraic spaces over* Y *with affine transition morphisms. Assume*

(1) Y *quasi-compact and quasi-separated,*
(2) X_i *quasi-compact and quasi-separated,*
(3) *the transition morphisms* $X_{i'} \to X_i$ *are finite,*
(4) $X_i \to Y$ *locally of finite type*
(5) $X \to Y$ *integral.*

Then $X_i \to Y$ *is finite for* i *large enough.*

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is finite over W and it suffices to check that $X_i \times_Y W$ is finite over W for some i (Morphisms of Spaces, Lemma 67.45.3). By Lemma 70.5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 32.4.19. □

0A0T **Lemma 70.5.16.** *Let* S *be a scheme. Let* Y *be an algebraic space over* S*. Let* $X = \lim X_i$ *be a directed limit of algebraic spaces over* Y *with affine transition morphisms. Assume*

(1) Y *quasi-compact and quasi-separated,*
(2) X_i *quasi-compact and quasi-separated,*
(3) *the transition morphisms* $X_{i'} \to X_i$ *are closed immersions,*
(4) $X_i \to Y$ *locally of finite type*

(5) $X \to Y$ *is a closed immersion.*

Then $X_i \to Y$ is a closed immersion for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is a closed subspace of W and it suffices to check that $X_i \times_Y W$ is a closed subspace W for some i (Morphisms of Spaces, Lemma 67.12.1). By Lemma 70.5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 32.4.20. □

70.6. Descending properties of morphisms

084V This section is the analogue of Section 70.5 for properties of morphisms. We will work in the following situation.

084W **Situation 70.6.1.** Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 70.4.1). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of algebraic spaces over B_0. Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to B_i and let $f : X \to Y$ be the base change of f_0 to B.

07SL **Lemma 70.6.2.** *With notation and assumptions as in Situation 70.6.1. If*

(1) *f is étale,*
(2) *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is étale if and only if $U_i \to V_i$ is étale and similarly $X \to Y$ is étale if and only if $U \to V$ is étale (Morphisms of Spaces, Lemma 67.39.2). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.8.10. □

0CN2 **Lemma 70.6.3.** *With notation and assumptions as in Situation 70.6.1. If*

(1) *f is smooth,*
(2) *f_0 is locally of finite presentation,*

then f_i is smooth for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is smooth if and only if $U_i \to V_i$ is smooth and similarly $X \to Y$ is smooth if and only if $U \to V$ is smooth (Morphisms of Spaces, Definition 67.37.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.8.9. □

07SN **Lemma 70.6.4.** *With notation and assumptions as in Situation 70.6.1. If*

(1) *f is surjective,*
(2) *f_0 is locally of finite presentation,*

then f_i is surjective for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \to V_i$ is $U \to V$, and the morphisms $U_i \to X_i \times_{Y_i} V_i$ and $U \to X \times_Y V$ are surjective (as base changes of $U_0 \to X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \to Y_i$ is surjective if and only if $U_i \to V_i$ is surjective and similarly $X \to Y$ is surjective if and only if $U \to V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 32.8.15). □

084X **Lemma 70.6.5.** *Notation and assumptions as in Situation 70.6.1. If*

(1) *f is universally injective,*
(2) *f_0 is locally of finite type,*

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \to Y$ is universally injective if and only if the diagonal $X \to X \times_Y X$ is surjective (Morphisms of Spaces, Definition 67.19.3 and Lemma 67.19.2). Observe that $X_0 \to X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10). Hence the lemma follows from Lemma 70.6.4 by considering the morphism $X_0 \to X_0 \times_{Y_0} X_0$. □

084Y **Lemma 70.6.6.** *Notation and assumptions as in Situation 70.6.1. If f is affine, then f_i is affine for some $i \geq 0$.*

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 67.20.3). □

084Z **Lemma 70.6.7.** *Notation and assumptions as in Situation 70.6.1. If*

(1) *f is finite,*
(2) *f_0 is locally of finite type,*

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is finite by Limits, Lemma 32.8.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 67.45.3). □

0850 **Lemma 70.6.8.** *Notation and assumptions as in Situation 70.6.1. If*

(1) *f is a closed immersion,*
(2) *f_0 is locally of finite type,*

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V. By Lemma 70.5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is a closed immersion by Limits, Lemma 32.8.5. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 67.45.3). □

0851 **Lemma 70.6.9.** *Notation and assumptions as in Situation 70.6.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 70.6.8 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \to X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) □

0852 **Lemma 70.6.10.** *Notation and assumptions as in Situation 70.6.1. If*

(1) *f is an isomorphism,*
(2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 67.51.2. Thus the lemma follows from Lemmas 70.6.2, 70.6.4, and 70.6.5. □

07SM **Lemma 70.6.11.** *Notation and assumptions as in Situation 70.6.1. If*

(1) *f is a monomorphism,*
(2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 70.6.10 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

08K0 **Lemma 70.6.12.** *Notation and assumptions as in Situation 70.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module and denote $\mathcal{F}_i$ the pullback to X_i and $\mathcal{F}$ the pullback to X. If*

(1) *$\mathcal{F}$ is flat over Y,*
(2) *$\mathcal{F}_0$ is of finite presentation, and*
(3) *f_0 is locally of finite presentation,*

then $\mathcal{F}_i$ is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $\mathcal{F}_i$ is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly $\mathcal{F}$ is flat over Y if and only if $\mathcal{F}|_U$ is flat over V (Morphisms of Spaces, Definition 67.30.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 32.10.4. □

08K1 **Lemma 70.6.13.** *Assumptions and notation as in Situation 70.6.1. If*

(1) *f is proper, and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume Y_0 is affine.

By Lemma 70.6.9 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 69.18.1 we can choose a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{\pi} & X_0' & \longrightarrow & \mathbf{P}^n_{Y_0} \\ & \searrow & \downarrow & \swarrow & \\ & & Y_0 & & \end{array}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 67.40.4 and 67.40.3 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemma 67.40.6). By Morphisms of Spaces, Lemma 67.40.7 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 70.6.8 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. □

0D4K **Lemma 70.6.14.** *Assumptions and notation as in Situation 70.6.1. Let $d \geq 0$. If*

(1) *f has relative dimension $\leq d$ (Morphisms of Spaces, Definition 67.33.2), and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. In this situation $X_i \to Y_i$ has relative dimension $\leq d$ if and only if $U_i \to V_i$ has relative dimension $\leq d$ (as defined in Morphisms, Definition 29.29.1). To see the equivalence, use that the definition for morphisms of algebraic spaces involves Morphisms of Spaces, Definition 67.33.1 which uses étale localization. The same is true for $X \to Y$ and $U \to V$. Since f_0 is

locally of finite type, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the more general Limits, Lemma 32.18.1. □

70.7. Descending relative objects

07SJ The following lemma is typical of the type of results in this section.

07SK **Lemma 70.7.1.** *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S. Assume*

(1) *the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
(2) *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i.

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 66.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \to X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U. Choose an affine scheme W and a surjective étale morphism $W \to V$. Then $W \to Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 65.9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 32.10.1 we can find an index i and a morphism of schemes $W_i \to U_i$ of finite presentation whose base change to U gives $W \to U$. Similarly we can find, after possibly increasing i, a scheme R'_i of finite presentation over R_i whose base change to R is R'. The projection morphisms $s', t' : R' \to W$ are morphisms over the projection morphisms $s, t : R \to U$. Hence we can view s', resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \to U$ given by $R' \to R$ followed by s, resp. t). Hence we can apply Limits, Lemma 32.10.1 again to see that, after possibly increasing i, there exist morphisms $s'_i, t'_i : R'_i \to W_i$, whose base change to U is S', t'. By Limits, Lemmas 32.8.10 and 32.8.14 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \to W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 65.10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y.

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i, i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0. Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \to Z$ be a morphism of algebraic spaces over X. Choose a surjective étale morphism $V_0 \to Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y. The composition $V \to Y \to Z \to Z_0$ comes from a (essentially unique) morphism

$V_i \to Z_0$ for some $i \geq 0$ by Proposition 70.3.10 (applied to $Z_0 \to X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \to V_i \to Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \to Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. □

07V7 **Lemma 70.7.2.** *With notation and assumptions as in Lemma 70.7.1. The category of $\mathcal{O}_X$-modules of finite presentation is the colimit over I of the categories $\mathcal{O}_{X_i}$-modules of finite presentation.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 70.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s,U,t} R = \operatorname{colim} R_i \times_{s_i,U_i,t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \operatorname{colim} QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \operatorname{colim} QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s,U,t} R}) = \operatorname{colim} QCoh(\mathcal{O}_{R_i \times_{s_i,U_i,t_i} R_i})$ by Limits, Lemma 32.10.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 66.32.1. Thus the result follows formally. □

0D2X **Lemma 70.7.3.** *With notation and assumptions as in Lemma 70.7.1. Then*

(1) *any finite locally free $\mathcal{O}_X$-module is the pullback of a finite locally free $\mathcal{O}_{X_i}$-module for some i,*
(2) *any invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_{X_i}$-module for some i.*

Proof. Proof of (2). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Since invertible modules are of finite presentation we can find an i and modules $\mathcal{L}_i$ and $\mathcal{N}_i$ of finite presentation over X_i such that $f_i^*\mathcal{L}_i \cong \mathcal{L}$ and $f_i^*\mathcal{N}_i \cong \mathcal{L}^{\otimes -1}$, see Lemma 70.7.2. Since pullback commutes with tensor product we see that $f_i^*(\mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i)$ is isomorphic to $\mathcal{O}_X$. Since the tensor product of finitely presented modules is finitely presented, the same lemma implies that $f_{i'i}^*\mathcal{L}_i \otimes_{\mathcal{O}_{X_{i'}}} f_{i'i}^*\mathcal{N}_i$ is isomorphic to $\mathcal{O}_{X_{i'}}$ for some $i' \geq i$. It follows that $f_{i'i}^*\mathcal{L}_i$ is invertible (Modules on Sites, Lemma 18.32.2) and the proof is complete.

Proof of (1). Omitted. Hint: argue as in the proof of (2) using that a module (on a locally ringed site) is finite locally free if and only if it has a dual, see Modules on Sites, Section 18.29. Alternatively, argue as in the proof for schemes, see Limits, Lemma 32.10.3. □

70.8. Absolute Noetherian approximation

07SS The following result is [**CLO12**, Theorem 1.2.2]. A key ingredient in the proof is Decent Spaces, Lemma 68.8.6.

07SU **Proposition 70.8.1.** *Let X be a quasi-compact and quasi-separated algebraic space over* $\operatorname{Spec}(\mathbf{Z})$. *There exist a directed set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that*

Our proof follows closely the proof given in [**CLO12**, Theorem 1.2.2].

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each X_i is quasi-separated and of finite type over $\mathbf{Z}$, and*
(3) $X = \lim X_i$.

Proof. We apply Decent Spaces, Lemma 68.8.6 to get open subspaces $U_p \subset X$, schemes V_p, and morphisms $f_p : V_p \to U_p$ with properties as stated. Note that $f_n : V_n \to U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 67.51.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms, see Limits, Proposition 32.5.4. Thus, applying descending induction on p, we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \to X$ with the following properties

(1) X is a quasi-compact and quasi-separated algebraic space,
(2) V is a quasi-compact and separated scheme,
(3) $U \subset X$ is a quasi-compact open subspace,
(4) $(U_i, g_{ii'})$ is a directed inverse system of quasi-separated algebraic spaces of finite type over $\mathbf{Z}$ with affine transition morphisms whose limit is U,
(5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
(6) $f : V \to X$ is a surjective étale morphism such that $f^{-1}(Z) \to Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X.

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U. Hence we may apply Lemmas 70.7.1 and 70.6.2 to find an index $0 \in I$ and an étale morphism $W_0 \to U_0$ of finite presentation whose base change to U produces W. Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may assume the W_i are schemes, see Lemma 70.5.11. Moreover, W_i is of finite type over $\mathbf{Z}$.

Apply Limits, Lemma 32.5.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over $\mathbf{Z}$ with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{array}{ccc} W_i & \longrightarrow & V_i \\ {\scriptstyle\Delta}\downarrow & & \downarrow \\ W_i \times_{U_i} W_i & \longrightarrow & R_i \end{array}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 26.14). Note that the étale projection maps $W_i \times_{U_i} W_i \to W_i$ extend to étale morphisms $s_i, t_i : R_i \to V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \to V_i \times V_i$ is an étale equivalence relation on V_i. Note that $W_i \times_{U_i} W_i$ is quasi-compact (as

U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$ the diagram

07SV (70.8.1.1)
$$\begin{array}{ccc} R_i & \longrightarrow & R_{i'} \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_{i'}} \\ V_i & \longrightarrow & V_{i'} \end{array}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 65.10.5). As V_i is of finite type over $\mathbf{Z}$ and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over $\mathbf{Z}$ (see Properties of Spaces, Lemma 66.6.5 and Morphisms of Spaces, Lemmas 67.8.6 and 67.23.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \to X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{array}{ccc} V_i & \longrightarrow & V_{i'} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_{i'} \end{array}$$

are cartesian as (70.8.1.1) is cartesian, see Groupoids, Lemma 39.20.7. Since $V_i \to V_{i'}$ is affine, this implies that $X_i \to X_{i'}$ is affine, see Morphisms of Spaces, Lemma 67.20.3. Thus we can form the limit $X' = \lim X_i$ by Lemma 70.4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \to X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 70.4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W. Property (6) implies the projections $V \times_X V \to V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V. Since $V \to X$ and $V \to X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 65.9.1 and we win. □

70.9. Applications

07V8 The following lemma can also be deduced directly from Decent Spaces, Lemma 68.8.6 without passing through absolute Noetherian approximation.

07V9 **Lemma 70.9.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. We may view X as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Proposition 70.8.1 and write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$. Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 67.28.6. The morphism $X \to X_i$ is affine, see Lemma 70.4.1. Conclusion by Cohomology of Spaces, Lemma 69.15.2. □

The rest of this section consists of straightforward applications of Lemma 70.9.1.

0829 **Lemma 70.9.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 70.9.1. Then the images $\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

086Y **Lemma 70.9.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each $\mathcal{F}_i$ is an $\mathcal{O}_X$-module of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 70.9.1). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose an étale surjection $U \to X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \mathrm{colim}\, \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 70.9.2). Then $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

Let X be an algebraic space. In the following lemma we use the notion of a *finitely presented quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

082A **Lemma 70.9.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

Proof. First we write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 70.9.1. For each i let $\mathcal{B}_i = \mathrm{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \mathrm{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \mathrm{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 70.9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \mathrm{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be an algebraic space. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

082B **Lemma 70.9.5.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 70.9.2. □

Let X be an algebraic space. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

086Z **Lemma 70.9.6.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras with surjective transition maps.*

Proof. By Lemma 70.9.3 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \mathrm{colim}\, \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 70.9.2). Set

$$\mathcal{A}_i = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjective, and $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an étale surjective map $U \to X$ where U is an affine scheme. Take generators $f_1, \ldots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$-algebra we see that for each j there exists a monic polynomial $P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

082C **Lemma 70.9.7.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

(1) *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
(2) *$\mathcal{A}$ is a directed colimit of finite and finitely presented $\mathcal{O}_X$-algebras.*

Proof. By Lemma 70.9.5 we have $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-sub algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (use Algebra, Lemma 10.36.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \mathrm{colim}\, \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 70.9.1. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 69.5.3. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 70.9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 70.9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

0853 **Lemma 70.9.8.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 67.11.2). Let $\mathcal{H} = \operatorname{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \to j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \operatorname{Im}(\mathcal{H}' \to \mathcal{F})$ to conclude. □

70.10. Relative approximation

09NR We discuss variants of Proposition 70.8.1 over a base.

0GS3 **Lemma 70.10.1.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over $\mathbf{Z}$. Then there exists a direct set I and an inverse system $(f_i : X_i \to Y_i)$ of morphisms algebraic spaces over I, such that the transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$ are affine, such that X_i and Y_i are quasi-separated and of finite type over $\mathbf{Z}$, and such that $(X \to Y) = \lim(X_i \to Y_i)$.*

Proof. Write $X = \lim_{a \in A} X_a$ and $Y = \lim_{b \in B} Y_b$ as in Proposition 70.8.1, i.e., with X_a and Y_b quasi-separated and of finite type over $\mathbf{Z}$ and with affine transition morphisms.

Fix $b \in B$. By Lemma 70.4.5 applied to Y_b and $X = \lim X_a$ over $\mathbf{Z}$ we find there exists an $a \in A$ and a morphism $f_{a,b} : X_a \to Y_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & Y_b \end{array}$$

commute. Let I be the set of triples $(a, b, f_{a,b})$ we obtain in this manner.

Let $(a, b, f_{a,b})$ and $(a', b', f_{a',b'})$ be in I. Let $b'' \leq \min(b, b')$. By Lemma 70.4.5 again, there exists an $a'' \geq \max(a, a')$ such that the compositions $X_{a''} \to X_a \to Y_b \to Y_{b''}$ and $X_{a''} \to X_{a'} \to Y_{b'} \to Y_{b''}$ are equal. We endow I with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b \geq b',\ \text{and } g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b,b'} : Y_b \to Y_{b'}$ are the transition morphisms. The remarks above show that I is directed and that the maps $I \to A$, $(a, b, f_{a,b}) \mapsto a$ and $I \to B$, $(a, b, f_{a,b})$ are cofinal. If for $i = (a, b, f_{a,b})$ we set $X_i = X_a$, $Y_i = Y_b$, and $f_i = f_{a,b}$, then we get an inverse system of morphisms over I and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} S_i = \lim_{b \in B} Y_b = Y$$

by Categories, Lemma 4.17.4 (recall that limits over I are really limits over the opposite category associated to I and hence cofinal turns into initial). This finishes the proof. □

09NS **Lemma 70.10.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *Y is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed inverse system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y.

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. By Lemma 70.10.1 we can write $(X \to Y) = \lim(X_i \to Y_i)$ for some directed inverse system of morphisms of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. Since limits commute with limits (Categories, Lemma 4.14.10) we have $X = \lim X_i \times_{Y_i} Y$. For $i \geq i'$ the transition morphism $X_i \times_{Y_i} Y \to X_{i'} \times_{Y_{i'}} Y$ is affine as the composition

$$X_i \times_{Y_i} Y \to X_i \times_{Y_{i'}} Y \to X_{i'} \times_{Y_{i'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 67.4.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 67.20.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 67.20.4). The morphisms f_i are of finite presentation (Morphisms of Spaces, Lemmas 67.28.7 and 67.28.9) and hence the base changes $X_i \times_{f_i, Y_i} Y \to Y$ are of finite presentation (Morphisms of Spaces, Lemma 67.28.3). □

70.11. Finite type closed in finite presentation

07SP This section is the analogue of Limits, Section 32.9.

0870 **Lemma 70.11.1.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y.*

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.4 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is affine and of finite presentation and $X = \lim X_i$. □

09YA **Lemma 70.11.2.** *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y.*

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.7 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation and $X = \lim X_i$. □

07VR **Lemma 70.11.3.** *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 70.9.6 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ with surjective transition maps. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. □

0A0U **Lemma 70.11.4.** *Let S be a scheme. Let $f : X \to Y$ be a closed immersion of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the morphisms $X_i \to Y$ are closed immersions of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subspace of Y. By Lemma 70.9.2 we can write $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ as the filtered colimit of its finite type quasi-coherent submodules. Let X_i be the closed subspace of X cut out by $\mathcal{I}_i$. Then $X_i \to Y$ is a closed immersion of finite presentation, and $X = \lim X_i$. Some details omitted. □

0871 **Lemma 70.11.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is locally of finite type and quasi-affine, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Morphisms of Spaces, Lemma 67.21.6 we can find a factorization $X \to Z \to Y$ where $X \to Z$ is a quasi-compact open immersion and $Z \to Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 70.11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 70.5.7). Let U_i be the inverse image of U_0 in Z_i, so $U = \lim U_i$. By Lemma 70.5.12 we see that $X \to U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. □

0872 **Lemma 70.11.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume:*

(1) *f is of locally of finite type.*
(2) *X is quasi-compact and quasi-separated, and*
(3) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ of algebraic spaces over Y.

Proof. By Proposition 70.8.1 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,Y} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 67.28.7. Hence the base change $X_{i,Y} \to Y$ is of finite presentation by Morphisms of Spaces, Lemma 67.28.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \to X \times Y$ is a monomorphism. By Lemma 70.5.12 we see that $X \to X_{i,Y}$ is a monomorphism for i large enough. Fix such an i. Note that $X \to X_{i,Y}$ is locally of finite type (Morphisms of Spaces, Lemma 67.23.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 67.27.10). Hence $X \to X_{i,Y}$ is representable. Hence $X \to X_{i,Y}$ is quasi-affine because we can use the principle Spaces, Lemma 65.5.8 and the result for morphisms of schemes More on Morphisms, Lemma 37.43.2. Thus Lemma 70.11.5 gives a factorization $X \to X' \to X_{i,Y}$ with $X \to X'$ a closed immersion and $X' \to X_{i,Y}$ of finite presentation. Finally, $X' \to Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 67.28.2). □

0873 **Proposition 70.11.7.** *Let S be a scheme. $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is of finite type and separated, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Lemma 70.11.6 there is a closed immersion $X \to Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Lemma 70.9.2 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \mathrm{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by $\mathcal{I}_a$. These form an inverse system indexed by A.

The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \operatorname{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Z$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the morphisms $X_a \to Y$ are of finite presentation. Thus it suffices to show that $X_a \to Y$ is separated for some $a \in A$. This follows from Lemma 70.5.13 as we have assumed that $X \to Y$ is separated. □

70.12. Approximating proper morphisms

0A0V

0A0W **Lemma 70.12.1.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of algebraic spaces X_i proper and of finite presentation over Y and with transition morphisms and morphisms $X \to X_i$ closed immersions.*

Proof. By Proposition 70.11.7 we can find a closed immersion $X \to X'$ with X' separated and of finite presentation over Y. By Lemma 70.11.4 we can write $X = \lim X_i$ with $X_i \to X'$ a closed immersion of finite presentation. We claim that for all i large enough the morphism $X_i \to Y$ is proper which finishes the proof.

To prove this we may assume that Y is an affine scheme, see Morphisms of Spaces, Lemma 67.40.2. Next, we use the weak version of Chow's lemma, see Cohomology of Spaces, Lemma 69.18.1, to find a diagram

$$\begin{array}{ccccc} X' & \xleftarrow{\pi} & X'' & \longrightarrow & \mathbf{P}^n_Y \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X'' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X'' \to X'$ is proper and surjective. Denote $X''_i \subset X''$, resp. $\pi^{-1}(X)$ the scheme theoretic inverse image of $X_i \subset X'$, resp. $X \subset X'$. Then $\lim X''_i = \pi^{-1}(X)$. Since $\pi^{-1}(X) \to Y$ is proper (Morphisms of Spaces, Lemmas 67.40.4), we see that $\pi^{-1}(X) \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemmas 67.40.6 and 67.12.3). Hence for i large enough we find that $X''_i \to \mathbf{P}^n_Y$ is a closed immersion by Lemma 70.5.16. Thus X''_i is proper over Y. For such i the morphism $X_i \to Y$ is proper by Morphisms of Spaces, Lemma 67.40.7. □

0A0X **Lemma 70.12.2.** *Let $f : X \to Y$ be a proper morphism of algebraic spaces over $\mathbf{Z}$ with Y quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to Y_i)$ of morphisms of algebraic spaces over I, such that the transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$ are affine, such that f_i is proper and of finite presentation, such that Y_i is of finite presentation over $\mathbf{Z}$, and such that $(X \to Y) = \lim(X_i \to Y_i)$.*

Proof. By Lemma 70.12.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \to Y$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 70.8.1) we can write $Y = \lim_{j \in J} Y_j$ with Y_j of finite presentation over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to Y_j$ of finite presentation with $X_k \cong Y \times_{Y_j} X_{k,j}$ as algebraic spaces over Y, see Lemma 70.7.1. After increasing

j we may assume $X_{k,j} \to Y_j$ is proper, see Lemma 70.6.13. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to Y_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $Y_{j'} \to Y_j$ whose base change to Y gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 70.7.1). These morphisms form the transition morphisms of the system. Some details omitted. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 67.15.4.

08K2 **Lemma 70.12.3.** *Assumptions and notation as in Situation 70.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

(1) *f_0 is locally of finite type,*
(2) *$\mathcal{F}_0$ is of finite type,*
(3) *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms of Spaces, Lemma 67.15.2 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 29.41.9. By Lemma 70.6.13 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4. □

70.13. Embedding into affine space

088K Some technical lemmas to be used in the proof of Chow's lemma later.

088L **Lemma 70.13.1.** *Let S be a scheme. Let $f : U \to X$ be a morphism of algebraic spaces over S. Assume U is an affine scheme, f is locally of finite type, and X quasi-separated and locally separated. Then there exists an immersion $U \to \mathbf{A}^n_X$ over X.*

Proof. Say $U = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}\, A_i$ as a filtered colimit of finite type $\mathbf{Z}$-subalgebras. For each i the morphism $U \to U_i = \mathrm{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X. In the limit the morphism $U \to X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 67.4.6. By Lemma 70.5.12 we see that $U \to X \times U_i$ is an immersion for some i. Since U_i is isomorphic to a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ the lemma follows. □

088M **Remark 70.13.2.** We have seen in Examples, Section 110.29 that Lemma 70.13.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 70.13.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

088N **Lemma 70.13.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \to \mathbf{A}^n_X$.*

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 67.28.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 66.13.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 5.9.2 and Properties, Lemma 28.5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}^n_X \amalg \mathbf{A}^m_X \longrightarrow \mathbf{A}^{\max(n,m)+1}_X$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \to X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 66.15.1) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y. Let $\mathcal{I} \subset X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 66.12.5). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by $\mathcal{I}^n$. By construction $V \to Y \to X$ factors through Z. Because $\mathbf{A}^n_Z \to \mathbf{A}^n_X$ is an immersion, we may replace X by Z and Y by V. Hence we reach the situation where Y and X are irreducible and $Y \to X$ maps the generic point of Y onto the generic point of X.

Assume Y and X are irreducible, Y is a scheme, and $Y \to X$ maps the generic point of Y onto the generic point of X. By Properties of Spaces, Proposition 66.13.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U. By Morphisms, Lemma 29.39.2 we may factor $V \to U$ as $V \to \mathbf{A}^n_U \to U$. Composing with $\mathbf{A}^n_U \to \mathbf{A}^n_X$ we obtain the desired immersion. □

70.14. Sections with support in a closed subset

0854 This section is the analogue of Properties, Section 28.24.

0855 **Lemma 70.14.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:*

(1) *$U \to X$ is quasi-compact,*
(2) *U is quasi-compact, and*
(3) *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.*

Proof. Let W be an affine scheme and let $\varphi : W \to X$ be a surjective étale morphism, see Properties of Spaces, Lemma 66.6.3. If (1) holds, then $\varphi^{-1}(U) \to W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as $|\varphi^{-1}(U)| \to |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 67.8.10). Hence $\varphi^{-1}(U) \to W$ is a quasi-compact morphism of schemes by Properties, Lemma 28.24.1. It follows that $U \to X$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.8. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 66.12.3 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subspace structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. As U is quasi-compact it follows from Lemma 70.9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if $\mathcal{I}$ is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 28.24.1) applied to $\varphi^{-1}\mathcal{I}$ on W. Thus (2) holds. □

0856 **Lemma 70.14.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object U of $X_{étale}$ the module*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

(1) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
(2) *for affine U in $X_{étale}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
(3) *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 28.24.2. □

0857 **Definition 70.14.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 70.14.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

0858 **Lemma 70.14.4.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) so that Lemma 70.14.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$. □

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

0859 **Lemma 70.14.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object $\varphi : W \to X$ of $X_{étale}$ the module*

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \text{the support of } s \text{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \to X$ is quasi-compact, then

(1) *for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,*
(2) *for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,*

(3) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 28.24.5. □

085A **Definition 70.14.6.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 70.14.5 above is called the *subsheaf of sections supported on T*.

085B **Lemma 70.14.7.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \to Y$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T.*

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \to Y$ is quasi-compact, so is $U \to X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 70.14.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and $\mathcal{F}$. The equality of the given quasi-coherent modules is immediate from the definitions. □

70.15. Characterizing affine spaces

07VQ This section is the analogue of Limits, Section 32.11.

07VS **Lemma 70.15.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 67.45.7. By Morphisms of Spaces, Lemma 67.9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 70.11.3 we can write $X = \lim X_a$ with each $X_a \to Y$ finite and of finite presentation. By Lemma 70.5.10 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \to Y$ is finite, surjective, and of finite presentation.

By Proposition 70.8.1 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over $\mathbf{Z}$. By Lemma 70.7.1 we can find $0 \in I$ and a morphism $X_0 \to Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 70.6.7 we see that $X_i \to Y_i$ is finite for i large enough. By Lemma 70.6.4 we see that $X_i \to Y_i$ is surjective for i large enough. By Lemma 70.5.10 we see that X_i is affine for i large enough. Hence for i large enough we can apply Cohomology of Spaces, Lemma 69.17.3 to conclude that Y_i is affine. This implies that Y is affine and we conclude. □

07VT **Proposition 70.15.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that X is affine and f is surjective and universally closed*[2]*. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 65.16.2). By Morphisms of Spaces, Lemma 67.9.8 we see that Y is a separated algebraic space. Then by Morphisms of Spaces, Lemma 67.20.11 we find that f is affine. Whereupon by Morphisms of Spaces, Lemma 67.45.7 we see that f is integral.

By the preceding paragraph, we may assume $f : X \to Y$ is surjective and integral, X is affine, and Y is separated. Since f is surjective and X is quasi-compact we also deduce that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, see Morphisms of Spaces, Lemma 67.11.2. By Lemma 70.9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_Y$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_Y$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_Y \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_Y$-algebra. Hence

$$X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \longrightarrow Y$$

is a finite morphism of algebraic spaces. Here $\underline{\mathrm{Spec}}$ is the construction of Morphisms of Spaces, Lemma 67.20.7. It is clear that $X = \lim_i X_i$. Hence by Lemma 70.5.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to Y$ factors through each X_i we see that $X_i \to Y$ is surjective. Hence we conclude that Y is affine by Lemma 70.15.1. □

The following corollary of the result above can be found in [**CLO12**].

07VU **Lemma 70.15.3.** *Let S be a scheme. Let X be an algebraic space over S. If X_{red} is a scheme, then X is a scheme.* [**CLO12**, 3.1.12]

Proof. Let $U' \subset X_{red}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{red}| = |X|$. Then $U' \to U$ is surjective and integral. Hence U is affine by Proposition 70.15.2. Thus every point is contained in an open subscheme of X, i.e., X is a scheme. □

07VV **Lemma 70.15.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral and induces a bijection $|X| \to |Y|$. Then X is a scheme if and only if Y is a scheme.*

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X. Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 66.4.8. Then $U \to V$ is integral and surjective, hence V is an affine scheme by Proposition 70.15.2. This concludes the proof. □

08B2 **Lemma 70.15.5.** *Let S be a scheme. Let $f : X \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Assume*

(1) *$B' \to B$ is a closed immersion,*
(2) *$|B'| \to |B|$ is bijective,*

[2] An integral morphism is universally closed, see Morphisms of Spaces, Lemma 67.45.7.

(3) $X \times_B B' \to B'$ *is a closed immersion, and*
(4) $X \to B$ *is of finite type or* $B' \to B$ *is of finite presentation.*

Then $f : X \to B$ *is a closed immersion.*

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \to X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \to B$ be an étale morphism with U affine. Then $X' \times_B U \to X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \to B$ and $X' \to B'$ are representable) so is $X \times_B U$ by Lemma 70.15.3. Hence $X \to B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 29.45.7. □

70.16. Finite cover by a scheme

0ACX As an application of the limit results of this chapter, we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, finite morphism $Y \to X$. We will rely on the already proven result that we can find a finite integral cover by a scheme, which was proved in Decent Spaces, Section 68.9.

09YC **Proposition 70.16.1.** *Let* S *be a scheme. Let* X *be a quasi-compact and quasi-separated algebraic space over* S.

(1) *There exists a surjective finite morphism* $Y \to X$ *of finite presentation where* Y *is a scheme,*
(2) *given a surjective étale morphism* $U \to X$ *we may choose* $Y \to X$ *such that for every* $y \in Y$ *there is an open neighbourhood* $V \subset Y$ *such that* $V \to X$ *factors through* U.

Proof. Part (1) is the special case of (2) with $U = X$. Let $Y \to X$ be as in Decent Spaces, Lemma 68.9.2. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Lemma 70.11.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 70.5.11. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Lemma 70.5.7. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Proposition 70.3.10. This finishes the proof. □

0GUM **Lemma 70.16.2.** *Let* S *be a scheme. Let* $f : X \to Y$ *be an integral morphism of algebraic spaces over* S. *Assume* Y *quasi-compact and quasi-separated. Let* $V \subset Y$ *be a quasi-compact open subspace such that* $f^{-1}(V) \to V$ *is finite and of finite presentation. Then* X *can be written as a directed limit* $X = \lim X_i$ *where* $f_i : X_i \to Y$ *are finite and of finite presentation such that* $f^{-1}(V) \to f_i^{-1}(V)$ *is an isomorphism for all* i.

Proof. This lemma is a slight refinement of Proposition 70.16.1. Consider the integral quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A} = f_*\mathcal{O}_X$. In the next paragraph, we will write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ such that $\mathcal{A}_i|_V = \mathcal{A}|_V$. Having done this we set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 67.20.8. By construction $X_i \to Y$ is finite and of finite presentation, $X = \lim X_i$, and $f_i^{-1}(V) = f^{-1}(V)$.

The proof of the assertion on algebras is similar to the proof of part (2) of Lemma 70.9.7. First, write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_Y$-modules using Lemma 70.9.1. Since $\mathcal{A}|_V$ is a finite type $\mathcal{O}_V$-module we may and do assume that $\mathcal{F}_i|_V \to \mathcal{A}|_V$ is surjective for all i. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$. We have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (as finite type quasi-coherent subalgebras of the integral quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A}$ over $\mathcal{O}_X$). The restriction of $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ to V is $\mathcal{A}|_V$ by the surjectivity above. Hence $\mathcal{J}_i|_V$ is finitely generated as an ideal sheaf of $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)|_V$ due to the fact that $\mathcal{A}|_V$ is finitely presented as an $\mathcal{O}_Y$-algebra. Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. We may and do assume that $\mathcal{E}_{ik}|_V$ generates $\mathcal{J}_i|_V$ as a sheaf of ideal of $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)|_V$ by the statement on finite generation above. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 69.5.3. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 70.9.6). Moreover we have $\mathcal{A}_{ik}|_V = \mathcal{A}|_V$ by construction. Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 70.9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

0GUN **Lemma 70.16.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.*

(1) *There exists a surjective finite morphism $f : Y \to X$ of finite presentation where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,*
(2) *given a surjective étale morphism $V \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \to X$ factors through V.*

Proof. Part (1) is the special case of (2) with $V = X$.

Proof of (2). Let $\pi : Y \to X$ be as in Decent Spaces, Lemma 68.9.3 and let $U \subset X$ be a quasi-compact dense open such that $\pi^{-1}(U) \to U$ is finite étale. Choose a finite affine open covering $Y = \bigcup W_j$ such that $W_j \to X$ factors through V. We can write $Y = \lim Y_i$ with $\pi_i : Y_i \to X$ finite and of finite presentation such that $\pi^{-1}(U) \to \pi_i^{-1}(U)$ is an isomorphism, see Lemma 70.16.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 70.5.11. For large enough i we can find affine opens $W_{i,j} \subset Y_i$ whose inverse image in Y recovers W_j, see Lemma 70.5.7.

For even larger i the morphisms $W_j \to V$ over X come from morphisms $W_{i,j} \to U$ over X, see Proposition 70.3.10. This finishes the proof. □

0GUP **Lemma 70.16.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. There exists a $t \geq 0$ and closed subspaces*

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \ldots t-1$ there exists a scheme Y_i, a surjective, finite, and finitely presented morphism $Y_i \to Z_i$ which is finite étale over $Z_i \setminus Z_{i+1}$.

Proof. We may view X as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Proposition 70.8.1. It follows that we can find an affine morphism $X \to X_0$ with X_0 of finite presentation over $\mathbf{Z}$. If we can prove the lemma for X_0, then we can pull back the stratification and the morphisms to X and get the result for X; some details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over $\mathbf{Z}$. Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. Any finite morphism towards X is of finite presentation, so we can ignore that requirement in the rest of the proof. By Lemma 70.16.3 there exists a surjective finite morphism $Y \to X$ which is finite étale over a dense open $U \subset X$. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \ldots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \to Z_1$ is a thickening and there exist finite surjective morphisms $Y_{1,i} \to Z_{1,i}$ which are finite étale over $Z_{1,i} \setminus Z_{1,i+1}$. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Y_i = Y_{1,i-1}$ for $i \geq 1$ and the lemma is proved. □

70.17. Obtaining schemes

0B7X A few more techniques to show an algebraic space is a scheme. The first is that we can show there is a minimal closed subspace which is not a scheme.

0B7Y **Lemma 70.17.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. If X is not a scheme, then there exists a closed subspace $Z \subset X$ such that Z is not a scheme, but every proper closed subspace $Z' \subset Z$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let $\mathcal{Z}$ be the set of closed subspaces Z which are not schemes ordered by inclusion. By assumption $\mathcal{Z}$ contains X, hence is nonempty. If Z_α is a totally ordered subset of $\mathcal{Z}$, then $Z = \bigcap Z_\alpha$ is in $\mathcal{Z}$. Namely,

$$Z = \lim Z_\alpha$$

and the transition morphisms are affine. Thus we may apply Lemma 70.5.11 to see that if Z were a scheme, then so would one of the Z_α. (This works even if $Z = \emptyset$, but note that by Lemma 70.5.3 this cannot happen.) Thus $\mathcal{Z}$ has minimal elements by Zorn's lemma. □

Now we can prove a little bit about these minimal non-schemes.

0B7Z **Lemma 70.17.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that every proper closed subspace $Z \subset X$ is a scheme, but X is not a scheme. Then X is reduced and irreducible.*

Proof. We see that X is reduced by Lemma 70.15.3. Choose closed subsets $T_1 \subset |X|$ and $T_2 \subset |X|$ such that $|X| = T_1 \cup T_2$. If T_1 and T_2 are proper closed subsets, then the corresponding reduced induced closed subspaces $Z_1, Z_2 \subset X$ (Properties of Spaces, Definition 66.12.5) are schemes and so is $Z = Z_1 \times_X Z_2 = Z_1 \cap Z_2$ as a closed subscheme of either Z_1 or Z_2. Observe that the coproduct $Z_1 \amalg_Z Z_2$ exists in the category of schemes, see More on Morphisms, Lemma 37.67.8. One way to proceed, is to show that $Z_1 \amalg_Z Z_2$ is isomorphic to X, but we cannot use this here as the material on pushouts of algebraic spaces comes later in the theory. Instead we will use Lemma 70.15.1 to find an affine neighbourhood of every point. Namely, let $x \in |X|$. If $x \notin Z_1$, then x has a neighbourhood which is a scheme, namely, $X \setminus Z_1$. Similarly if $x \notin Z_2$. If $x \in Z = Z_1 \cap Z_2$, then we choose an affine open $U \subset Z_1 \amalg_Z Z_2$ containing z. Then $U_1 = Z_1 \cap U$ and $U_2 = Z_2 \cap U$ are affine opens whose intersections with Z agree. Since $|Z_1| = T_1$ and $|Z_2| = T_2$ are closed subsets of $|X|$ which intersect in $|Z|$, we find an open $W \subset |X|$ with $W \cap T_1 = |U_1|$ and $W \cap T_2 = |U_2|$. Let W denote the corresponding open subspace of X. Then $x \in |W|$ and the morphism $U_1 \amalg U_2 \to W$ is a surjective finite morphism whose source is an affine scheme. Thus W is an affine scheme by Lemma 70.15.1. □

A key point in the following lemma is that we only need to check the condition in the images of points of X.

0B80 **Lemma 70.17.3.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism from an algebraic space to a scheme S. If for every $x \in |X|$ with image $s = f(x) \in S$ the algebraic space $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then X is a scheme.*

Proof. Let $x \in |X|$. It suffices to find an open neighbourhood U of $s = f(x)$ such that $X \times_S U$ is a scheme. As $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is a scheme, then, since $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_S(U)$ where the colimit is over affine open neighbourhoods of s in S we see that

$$X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}) = \lim X \times_S U$$

By Lemma 70.5.11 we see that $X \times_S U$ is a scheme for some U. □

Instead of restricting to local rings as in Lemma 70.17.3, we can restrict to closed subschemes of the base.

0B81 **Lemma 70.17.4.** *Let $\varphi : X \to \mathrm{Spec}(A)$ be a quasi-compact and quasi-separated morphism from an algebraic space to an affine scheme. If X is not a scheme, then there exists an ideal $I \subset A$ such that the base change $X_{A/I}$ is not a scheme, but for every $I \subset I'$, $I \neq I'$ the base change $X_{A/I'}$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let $\mathcal{I}$ be the set of ideals I such that $X_{A/I}$ is not a scheme. By assumption $\mathcal{I}$ contains (0). If I_α is a chain of ideals in $\mathcal{I}$, then $I = \bigcup I_\alpha$ is in $\mathcal{I}$. Namely, $A/I = \mathrm{colim}\, A/I_\alpha$, hence

$$X_{A/I} = \lim X_{A/I_\alpha}$$

Thus we may apply Lemma 70.5.11 to see that if $X_{A/I}$ were a scheme, then so would be one of the X_{A/I_α}. Thus $\mathcal{I}$ has maximal elements by Zorn's lemma. □

70.18. Glueing in closed fibres

0E8Y Applying our theory above to the spectrum of a local ring we obtain a few pleasing glueing results for relative algebraic spaces. We first prove a helper lemma (which will be vastly generalized in Bootstrap, Section 80.11).

0E8Z **Lemma 70.18.1.** *Let $S = U \cup W$ be an open covering of a scheme. Then the functor*

$$FP_S \longrightarrow FP_U \times_{FP_{U\cap W}} FP_W$$

given by base change is an equivalence where FP_T is the category of algebraic spaces of finite presentation over the scheme T.

Proof. First, since $S = U \cup W$ is a Zariski covering, we see that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of triples $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ where $\mathcal{F}_U$ is a sheaf on $(Sch/U)_{fppf}$, $\mathcal{F}_W$ is a sheaf on $(Sch/W)_{fppf}$, and

$$\varphi : \mathcal{F}_U|_{(Sch/U\cap W)_{fppf}} \longrightarrow \mathcal{F}_W|_{(Sch/U\cap W)_{fppf}}$$

is an isomorphism. See Sites, Lemma 7.26.5 (note that no other gluing data are necessary because $U \times_S U = U$, $W \times_S W = W$ and that the cocycle condition is automatic for the same reason). Now, if the sheaf $\mathcal{F}$ on $(Sch/S)_{fppf}$ maps to $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ via this equivalence, then $\mathcal{F}$ is an algebraic space if and only if $\mathcal{F}_U$ and $\mathcal{F}_W$ are algebraic spaces. This follows immediately from Algebraic Spaces, Lemma 65.8.5 as $\mathcal{F}_U \to \mathcal{F}$ and $\mathcal{F}_W \to \mathcal{F}$ are representable by open immersions and cover $\mathcal{F}$. Finally, in this case the algebraic space $\mathcal{F}$ is of finite presentation over S if and only if $\mathcal{F}_U$ is of finite presentation over U and $\mathcal{F}_W$ is of finite presentation over W by Morphisms of Spaces, Lemmas 67.8.8, 67.4.12, and 67.28.4. □

0E90 **Lemma 70.18.2.** *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{FP_V} FP_{\mathrm{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. Let $W \subset S$ be an open neighbourhood of s. The functor

$$FP_S \to FP_U \times_{FP_{W\setminus\{s\}}} FP_W$$

is an equivalence of categories by Lemma 70.18.1. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of algebraic spaces of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 70.7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 32.2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0E91 **Lemma 70.18.3.** *Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories*

$$\mathrm{colim}_{s \in U' \supset U \text{ open}} FP_{U'} \longrightarrow FP_U \times_{FP_V} FP_{\mathrm{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. Let $W \subset S$ be an open neighbourhood of s. By Lemma 70.18.1 the functor

$$FP_{U \cup W} \longrightarrow FP_U \times_{FP_{U \cap W}} FP_W$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of algebraic spaces of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 70.7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 32.2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0E92 **Lemma 70.18.4.** *Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where FP_T is the category of algebraic spaces of finite presentation over T.

Proof. For $n = 1$ this is Lemma 70.18.2. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of FP_{S_i} and $f : X \to U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 70.18.2 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i, which is isomorphic to X over U, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_n} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n-1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. □

70.19. Application to modifications

0BGX Using limits we can describe the category of modifications of a decent algebraic space over a closed point in terms of the henselian local ring.

0BGY **Lemma 70.19.1.** *Let S be a scheme. Consider a separated étale morphism $f : V \to W$ of algebraic spaces over S. Assume there exists a closed subspace $T \subset W$ such that $f^{-1}T \to T$ is an isomorphism. Then, with $W^0 = W \setminus T$ and $V^0 = f^{-1}W^0$ the base change functor*

$$\left\{ \begin{array}{c} g : X \to W \text{ morphism of algebraic spaces} \\ g^{-1}(W^0) \to W^0 \text{ is an isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} h : Y \to V \text{ morphism of algebraic spaces} \\ h^{-1}(V^0) \to V^0 \text{ is an isomorphism} \end{array} \right\}$$

is an equivalence of categories.

Proof. Since $V \to W$ is separated we see that $V \times_W V = \Delta(V) \amalg U$ for some open and closed subspace U of $V \times_W V$. By the assumption that $f^{-1}T \to T$ is an isomorphism we see that $U \times_W T = \emptyset$, i.e., the two projections $U \to V$ maps into V^0.

Given $h : Y \to V$ in the right hand category, consider the contravariant functor X on $(Sch/S)_{fppf}$ defined by the rule

$$X(T) = \{(w, y) \mid w : T \to W,\ y : T \times_{w,W} V \to Y \text{ morphism over } V\}$$

Denote $g : X \to W$ the map sending $(w, y) \in X(T)$ to $w \in W(T)$. Since $h^{-1}V^0 \to V^0$ is an isomorphism, we see that if $w : T \to W$ maps into W^0, then there is a unique choice for h. In other words $X \times_{g,W} W^0 = W^0$. On the other hand, consider a T-valued point (w, y, v) of $X \times_{g,W,f} V$. Then $w = f \circ v$ and

$$y : T \times_{f\circ v,W} V \longrightarrow V$$

is a morphism over V. Consider the morphism

$$T \times_{f\circ v,W} V \xrightarrow{(v,\mathrm{id}_V)} V \times_W V = V \amalg U$$

The inverse image of V is T embedded via $(\mathrm{id}_T, v) : T \to T \times_{f\circ v,W} V$. The composition $y' = y \circ (\mathrm{id}_T, v) : T \to Y$ is a morphism with $v = h \circ y'$ which determines y because the restriction of y to the other part is uniquely determined as U maps into V^0 by the second projection. It follows that $X \times_{g,W,f} V \to Y$, $(w, y, v) \mapsto y'$ is an isomorphism.

Thus if we can show that X is an algebraic space, then we are done. Since $V \to W$ is separated and étale it is representable by Morphisms of Spaces, Lemma 67.51.1 (and Morphisms of Spaces, Lemma 67.39.5). Of course $W^0 \to W$ is representable and étale as it is an open immersion. Thus

$$W^0 \amalg Y = X \times_{g,W} W^0 \amalg X \times_{g,W,f} V = X \times_{g,W} (W^0 \amalg V) \longrightarrow X$$

is representable, surjective, and étale by Spaces, Lemmas 65.3.3 and 65.5.5. Thus X is an algebraic space by Spaces, Lemma 65.11.2. □

0BGZ **Lemma 70.19.2.** *Notation and assumptions as in Lemma 70.19.1. Let $g : X \to W$ correspond to $h : Y \to V$ via the equivalence. Then g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, and add more here if and only if h is so.*

Proof. If g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, finite, so is h as a base change of g by Morphisms of Spaces, Lemmas 67.8.4, 67.4.4, 67.28.3, 67.23.3, 67.40.3, 67.45.5. Conversely, let P be a property of morphisms of algebraic spaces which is étale local on the base and which holds for the identity morphism of any algebraic space. Since $\{W^0 \to W, V \to W\}$ is an étale covering, to prove that g has P it suffices to show that h has P. Thus we conclude using Morphisms of Spaces, Lemmas 67.8.8, 67.4.12, 67.28.4, 67.23.4, 67.40.2, 67.45.3. □

0BH0 **Lemma 70.19.3.** *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point such that $U = X \setminus \{x\} \to X$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}^h_{X,x}) \setminus \{\mathfrak{m}^h_x\}$ the base change functor*

$$\left\{\begin{array}{l} f : Y \to X \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array}\right\} \longrightarrow \left\{\begin{array}{c} g : Y \to \mathrm{Spec}(\mathcal{O}^h_{X,x}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{array}\right\}$$

is an equivalence of categories.

Proof. Let $a : (W, w) \to (X, x)$ be an elementary étale neighbourhood of x with W affine as in Decent Spaces, Lemma 68.11.4. Since x is a closed point of X and w is the unique point of W lying over x, we see that w is a closed point of W. Since a is étale and identifies residue fields at x and w, it follows that a induces an isomorphism $a^{-1}x \to x$ (as closed subspaces of X and W). Thus we may apply Lemma 70.19.1 and 70.19.2 to reduce the problem to the case where X is an affine scheme.

Assume X is an affine scheme. Recall that $\mathcal{O}^h_{X,x}$ is the colimit of $\Gamma(U, \mathcal{O}_U)$ over affine elementary étale neighbourhoods $(U, u) \to (X, x)$. Recall that the category of these neighbourhoods is cofiltered, see Decent Spaces, Lemma 68.11.6 or More on Morphisms, Lemma 37.35.4. Then $\mathrm{Spec}(\mathcal{O}^h_{X,x}) = \lim U$ and $V = \lim U \setminus \{u\}$ (Lemma 70.4.1) where the limits are taken over the same category. Thus by Lemma 70.7.1 The category on the right is the colimit of the categories for the pairs (U, u). And by the material in the first paragraph, each of these categories is equivalent to the category for the pair (X, x). This finishes the proof. □

70.20. Universally closed morphisms

0CM7 In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

0CM8 **Lemma 70.20.1.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces over S. Let $z \in |Z|$ and let $T \subset |X \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If f is quasi-compact, then there exists an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

1. *the morphism $b : Z' \to Y$ is locally of finite presentation,*
2. *with $z' = a(v)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and*
3. *the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.*

Moreover, we may assume V and Z' are affine schemes and if Z is a scheme we may assume V is an affine open neighbourhood of z.

Proof. We will deduce this from the corresponding result for morphisms of schemes. Let $y \in |Y|$ be the image of z. First we choose an affine étale neighbourhood $(U, u) \to (Y, y)$ and then we choose an affine étale neighbourhood $(V, v) \to (Z, z)$ such that the morphism $V \to Y$ factors through U. Then we may replace

(1) $X \to Y$ by $X \times_Y U \to U$,
(2) $Z \to Y$ by $V \to U$,
(3) z by v, and
(4) T by its inverse image in $|(X \times_Y U) \times_U V| = |X \times_Y V|$.

In fact, below we will show that after replacing V by an affine open neighbourhood of v there will be a morphism $a : V \to Z'$ for some $Z' \to U$ of finite presentation and a closed subset T' of $|(X \times_Y U) \times_U Z'| = |X \times_Y Z'|$ such that T maps into T' and $a(v) \notin \mathrm{Im}(T' \to |Z'|)$. Thus we may and do assume that Z and Y are affine schemes with the proviso that we need to find a solution where V is an open neighbourhood of z.

Since f is quasi-compact and Y is affine, the algebraic space X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $W \to X$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 32.14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

(1) the morphism $b : Z' \to Y$ is locally of finite presentation,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to Z')$, and
(3) $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{array}{ccccc} W \times_Y Z & \longleftarrow & W \times_Y V & \xrightarrow{a_1} & W \times_Y Z' \\ \downarrow & & \downarrow c & & \downarrow q \\ X \times_Y Z & \longleftarrow & X \times_Y V & \xrightarrow{a_2} & X \times_Y Z' \end{array}$$

has cartesian squares and the vertical maps are, surjective, étale and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |X \times_Y V|$ by c. By Properties of Spaces, Lemma 66.4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 5.6.4 we get

$$q^{-1}\left(\overline{a_2(T_2)}\right) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \to |Z'|$ does not contain z' since the same is true for T'. Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |X \times_Y Z'|$ as a solution to the problem posed by the lemma. □

0CM9 **Lemma 70.20.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $Z \to Y$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. It is clear that (1) implies (2). Suppose that $|X \times_Y Z| \to |Z|$ is not closed for some morphism of algebraic spaces $Z \to Y$ over S. This means that there exists some closed subset $T \subset |X \times_Y Z|$ such that $\mathrm{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 70.20.1. We find an étale neighbourhood $(V, v) \to (Z, z)$, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

(1) the morphism $b : Z' \to Y$ is locally of finite presentation,
(2) with $z' = a(v)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \to |X \times_Y Z'|$.

We claim that z' is in the closure of $\mathrm{Im}(T' \to |Z'|)$ which implies that $|X \times_Y Z'| \to |Z'|$ is not closed. The claim shows that (2) implies (1). To see the claim is true we contemplate following commutative diagram

$$\begin{array}{ccccc} X \times_Y Z & \longleftarrow & X \times_Y V & \longrightarrow & X \times_Y Z' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & V & \xrightarrow{a} & Z' \end{array}$$

Let $T_V \subset |X \times_Y V|$ be the inverse image of T. By Properties of Spaces, Lemma 66.4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that v is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|X \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(v)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to Y$ be as in (3). If we can show that $X \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Spaces, Lemma 67.9.5. Thus it suffices to show that $f : X \to Y$ satisfies (2) if f is a quasi-compact morphism of algebraic spaces, Y is a scheme, and $|\mathbf{A}^n \times X| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation. We have to show the map $|X \times_Y Z| \to |Z|$ is closed. This question is étale local on Z hence we may assume Z is affine (some details omitted). Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 29.39.2. Consider the cartesian diagram

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & \mathbf{A}^n \times X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbf{A}^n \times Y \end{array} \qquad \text{inducing the cartesian square} \qquad \begin{array}{ccc} |X \times_Y Z| & \longrightarrow & |\mathbf{A}^n \times X| \\ \downarrow & & \downarrow \\ |Z| & \longrightarrow & |\mathbf{A}^n \times Y| \end{array}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Spaces, Lemma 66.12.1). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. □

0CMA **Lemma 70.20.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated and of finite type. The following are equivalent*

(1) *The morphism f is proper.*
(2) *For any morphism $Y \to Z$ which is locally of finite presentation the map $|X \times_Y Z| \to |Z|$ is closed, and*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 70.20.2. □

70.21. Noetherian valuative criterion

0CMB We have already proved some results in Cohomology of Spaces, Section 69.19. The corresponding section for schemes is Limits, Section 32.15.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CMC **Lemma 70.21.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f finite type and Y locally Noetherian. Let $y \in |Y|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. Moreover, we can assume that the point $x \in |X|$ corresponding to $\mathrm{Spec}(K) \to X$ is a codimension 0 point[3] and that K is the residue field of a point on a scheme étale over X.

Proof. Choose an affine scheme V, a point $v \in V$ and an étale morphism $V \to Y$ mapping v to y. The map $|V| \to |Y|$ is open and by Properties of Spaces, Lemma 66.4.3 the image of $|X \times_Y V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|X \times_Y V| \to |V|$. If we prove the lemma for $X \times_Y V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : X \to Y$ and $y \in |Y|$ as in the lemma where Y is an affine scheme. Since f is quasi-compact, we conclude that X is quasi-compact. Hence we can choose an affine scheme W and a surjective étale morphism $W \to X$. Then the image of $|f|$ is the same as the image of $W \to Y$. In this way we reduce to the case of schemes which is Limits, Lemma 32.15.1. □

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of algebraic spaces over a base scheme S having the following

[3] See discussion in Properties of Spaces, Section 66.11.

shape

0H1V (70.21.1.1)

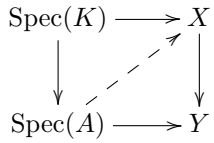

with A a valuation ring and K its field of fractions.

0H1W **Lemma 70.21.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-separated and locally of finite type and Y is locally Noetherian. The following are equivalent:*

1. *The morphism f is separated.*
2. *For any diagram (70.21.1.1) there is at most one dotted arrow.*
3. *For all diagrams (70.21.1.1) with A a discrete valuation ring there is at most one dotted arrow.*
4. *For all diagrams (70.21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 on X there is at most one dotted arrow.*

Proof. We have (1) $\Rightarrow$ (2) by Morphisms of Spaces, Lemma 67.43.1. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate. It remains to show (4) implies (1).

Assume (4). We have to show that the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 67.4.1. Choose an affine scheme U and an étale morphism $U \to X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \to U$ is a closed immersion (Morphisms of Spaces, Lemma 67.12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 67.23.2, 67.23.3, and 67.23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \to U$ is separated and of finite type. Consider a commutative diagram

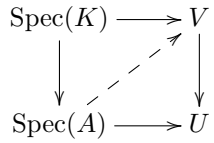

of morphisms of schemes where A is a discrete valuation ring with fraction field K and where K is the residue field of a generic point of the Noetherian scheme V. Since $V \to X$ is étale (as a base change of the étale morphism $U \to X \times_Y X$) we see that the image of $\mathrm{Spec}(K) \to V \to X$ is a point of codimension 0, see Properties of Spaces, Section 66.10. We can interpret the composition $\mathrm{Spec}(A) \to U \to X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$ and that this restriction maps to a point of codimension 0. Hence our assumption (4) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 32.15.3 we conclude that $V \to U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 41.7.2) as desired. □

0H1X **Lemma 70.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-separated and of finite type and Y is locally Noetherian. The following are equivalent:*

(1) *f is proper,*
(2) *f satisfies the valuative criterion, see Morphisms of Spaces, Definition 67.41.1,*
(3) *for any diagram (70.21.1.1) there exists exactly one dotted arrow,*
(4) *for all diagrams (70.21.1.1) with A a discrete valuation ring there exists exactly one dotted arrow, and*
(5) *for all diagrams (70.21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 on X there exists exactly one dotted arrow*[4].

Proof. We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Morphisms of Spaces, Lemma 67.44.1. It is clear that (3) $\Rightarrow$ (4) $\Rightarrow$ (5). To finish the proof we will now show (5) implies (1).

Assume (5). By Lemma 70.21.2 we see that f is separated. To finish the proof it suffices to show that f is universally closed. Let $V \to Y$ be an étale morphism where V is an affine scheme. It suffices to show that the base change $V \times_Y X \to V$ is universally closed, see Morphisms of Spaces, Lemma 67.9.5. Let

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & V \times_Y X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & V & \longrightarrow & Y \end{array}$$

of algebraic spaces over S be a commutative diagram where A is a discrete valuation ring with fraction field K and where $\mathrm{Spec}(K) \to V \times_Y X$ maps to a point of codimension 0 of the algebraic space $V \times_Y X$. Since $V \times_Y X \to X$ is étale it follows that the image of $\mathrm{Spec}(K) \to X$ is a point of codimension 0 of X. Thus by (5) we obtain the longer of the two dotted arrows fitting into the diagram. Then of course we obtain the shorter one as well. It follows that our assumptions hold for the morphism $V \times_Y X \to V$ and we reduce to the case discussed in the next paragraph.

Aassume Y is a Noetherian affine scheme. In this case X is a separated Noetherian algebraic space (we already know f is separated) of finite type over Y. (In particular, the algebraic space X has a dense open subspace which is a scheme by Properties of Spaces, Proposition 66.13.3 although strictly speaking we will not need this.) Choose a quasi-projective scheme X' over Y and a proper surjective morphism $X' \to X$ as in the weak form of Chow's lemma (Cohomology of Spaces, Lemma 69.18.1). We may replace X' by the disjoint union of the irreducible components which dominate an irreducible component of X; details omitted. In particular, we may assume that generic points of the scheme X' map to points of codimension 0 of X (in this case these are exactly the generic points of X). We claim that $X' \to Y$ is proper. The claim implies X is proper over Y by Morphisms of Spaces, Lemma 67.40.7. To prove this, according to Limits, Lemma 32.15.3 it

[4]There is a sharper formulation where in the existence part one only requires the dotted arrow exists after an extension of discrete valuation rings.

suffices to prove that in every solid commutative diagram

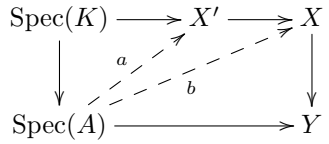

where A is a dvr with fraction field K and where K is the residue field of a generic point of X' we can find the dotted arrow a (we already know uniqueness as X' is separated). By assumption (5) we can find the dotted arrow b. Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a. □

0H1Y **Lemma 70.21.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f is of finite type. Then the following are equivalent*

(1) *f is universally closed,*
(2) *f satisfies the existence part of the valuative criterion,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $|\mathbf{A}^n \times X \times_Y V| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$,*
(4) *for all diagrams (70.21.1.1) with A a discrete valuation ring there there exists a finite separable extension K'/K of fields, a discrete valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes*

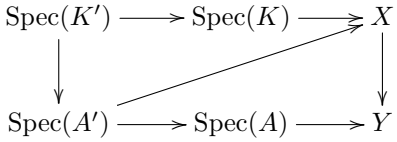

(5) *for all diagrams (70.21.1.1) with A a discrete valuation ring there there exists a field extension K'/K, a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes*

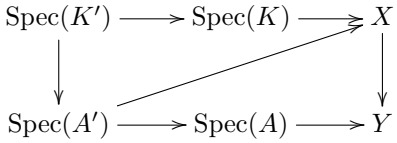

Proof. Parts (1), (2), and (3) are equivalent by Lemma 70.20.2 and Morphisms of Spaces, Lemma 67.42.1. These equivalent conditions imply part (4) as Morphisms of Spaces, Lemma 67.41.3 tells us that we may always choose K'/K finite separable in the existence part of the valuative criterion and this automatically forces A' to be a discrete valuation ring by Krull-Akizuki (Algebra, Lemma 10.119.12). The implication (4) ⇒ (5) is immediate. In the rest of the proof we show that (5) implies (1).

Assume (5). Chose an affine scheme V and an étale morphism $V \to Y$. It suffices to show that the base change of f to V is universally closed, see Morphisms of Spaces, Lemma 67.9.5. Exactly as in the proof of Lemma 70.21.3 we see that assumption

(5) is inherited by this base change; details omitted. This reduces us to the case discussed in the next paragraph.

Assume Y is a Noetherian affine scheme and we have (5). To prove that f is universally closed it suffices to show that $|X \times \mathbf{A}^n| \to |Y \times \mathbf{A}^n|$ is closed for all n (by the discussion above). Since assumption (5) is inherited by the product morphism $X \times \mathbf{A}^n \to Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|X| \to |Y|$ is closed.

Assume Y is a Noetherian affine scheme and we have (5). Let $T \subset |X|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace X by the reduced induced closed subspace structure on T; we omit the verification that property (5) is preserved by this replacement. Thus we reduce to proving that the image of $|X| \to |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|X| \to |Y|$. By Lemma 70.21.1 we may choose a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. It follows immediately from property (5) that y is in the image of $|X| \to |Y|$ and the proof is complete. □

70.22. Refined Noetherian valuative criteria

0H1Z This section is the analogue of Limits, Section 32.16. One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria.

0CMD **Lemma 70.22.1.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to Y$ be an étale morphism where V is an affine scheme. By Morphisms of Spaces, Lemma 67.9.5 it suffices to prove that the base change $X \times_Y V \to V$ is universally closed. By Properties of Spaces, Lemma 66.4.3 the image I of $|U \times_Y V| \to |X \times_Y V|$ is the inverse image of the image of $|h|$. Since $|X \times_Y V| \to |X|$ is open (Properties of Spaces, Lemma 66.16.7) we conclude that I is dense in $|X \times_Y V|$. Therefore the assumptions of the lemma are satisfied for the morphisms $U \times_Y V \to X \times_Y V \to V$. Hence we may assume Y is an affine scheme.

Assume Y is an affine scheme. Then U is quasi-compact. Choose an affine scheme and a surjective étale morphism $W \to U$. Then we may and do replace U by W and assume that U is affine. By the weak version of Chow's lemma (Cohomology of Spaces, Lemma 69.18.1) we can choose a surjective proper morphism $X' \to X$ where X' is a scheme. Then $U' = X' \times_X U$ is a scheme and $U' \to X'$ is of finite type. We may replace X' by the scheme theoretic image of $h' : U' \to X'$ and hence $h'(U')$ is dense in X'. We claim that for every diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U' & \xrightarrow{h} & X' \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle f'} \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. Namely, we first get an arrow $\mathrm{Spec}(A) \to X$ by the assumption of the lemma and then we lift this to an arrow $\mathrm{Spec}(A) \to X'$ using the valuative criterion for properness (Morphisms of Spaces, Lemma 67.44.1). The morphism $X' \to Y$ is separated as a composition of a proper and a separated morphism. Thus by the case of schemes the morphism $X' \to Y$ is proper (Limits, Lemma 32.16.1). By Morphisms of Spaces, Lemma 67.40.7 we conclude that $X \to Y$ is proper. □

0CME **Lemma 70.22.2.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |U| \to |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 70.22.1 to the morphisms $U \to X$ and $\Delta : X \to X \times_Y X$. We check the conditions. Observe that Δ is quasi-compact because f is quasi-separated. Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle \Delta} \\ \mathrm{Spec}(A) & & \xrightarrow{(a,b)} & & X \times_Y X \end{array}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. □

0CMF **Lemma 70.22.3.** *Let S be a scheme. Let $f : X \to Y$ and $h : U \to X$ be morphisms of algebraic spaces over S. Assume that Y is locally Noetherian, that f*

and h are of finite type, that f is quasi-separated, and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 70.22.2 and 70.22.1. □

70.23. Descending finite type spaces

0CP5 This section continues the theme of Section 70.11 in the spirit of the results discussed in Section 70.7. It is also the analogue of Limits, Section 32.22 for algebraic spaces.

0CP6 **Situation 70.23.1.** Let S be a scheme, for example $\mathrm{Spec}(\mathbf{Z})$. Let $B = \lim_{i \in I} B_i$ be the limit of a directed inverse system of Noetherian spaces over S with affine transition morphisms $B_{i'} \to B_i$ for $i' \geq i$.

0CP7 **Lemma 70.23.2.** *In Situation 70.23.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Then there exists an $i \in I$ and a diagram*

0CP8 (70.23.2.1)
$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_i \end{array}$$

such that $W \to B_i$ is of finite type and such that the induced morphism $X \to B \times_{B_i} W$ is a closed immersion.

Proof. By Lemma 70.11.6 we can find a closed immersion $X \to X'$ over B where X' is an algebraic space of finite presentation over B. By Lemma 70.7.1 we can find an i and a morphism of finite presentation $X'_i \to B_i$ whose pull back is X'. Set $W = X'_i$. □

0CP9 **Lemma 70.23.3.** *In Situation 70.23.1. Let $X \to B$ be a quasi-separated and finite type morphism of algebraic spaces. Given $i \in I$ and a diagram*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_i \end{array}$$

as in (70.23.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to B_{i'} \times_{B_i} W$ commutes with étale localization (Morphisms of Spaces, Lemma 67.16.3). Hence we may and do assume W is affine and maps into an affine U_i étale over B_i. Then

$$B_{i'} \times_{B_i} W = B_{i'} \times_{B_i} U_i \times_{U_i} W = U_{i'} \times_{U_i} W$$

where $U_{i'} = B_{i'} \times_{B_i} U_i$ is affine as the transition morphisms are affine. Thus the lemma follows from the case of schemes which is Limits, Lemma 32.22.3. □

0CPA **Lemma 70.23.4.** *In Situation 70.23.1. Let $f : X \to Y$ be a morphism of algebraic spaces quasi-separated and of finite type over B. Let*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_2} \end{array}$$

be diagrams as in (70.23.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 70.23.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(B_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to B_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 70.3.10 as improved by Lemma 70.4.5 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over B_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_i & \longrightarrow & X_{i_0} \\ \downarrow & \swarrow & \vdots & & \downarrow h \\ Y & \longrightarrow & Y_i & \longrightarrow & V \\ \downarrow & \searrow & \downarrow & & \downarrow \\ B & \longrightarrow & B_i & \longrightarrow & B_{i_0} \end{array}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to B_i \times_{B_{i_2}} V$ we find that the morphism $X_i \to B_i \times_{B_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms of Spaces, Lemma 67.16.6). This proves existence.

Uniqueness. Let $E_i \to X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. We have $E_i = Y_i \times_{\Delta, Y_i \times_{B_i} Y_i, (f_i, g_i)} X_i$. Hence $E_i \to X_i$ is a monomorphism of finite presentation as a base change of the diagonal of Y_i over B_i, see Morphisms of Spaces, Lemmas 67.4.1 and 67.28.10. Since X_i is a closed subspace of $B_i \times_{B_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(B_i \times_{B_{i_0}} X_{i_0})} (B_i \times_{B_{i_0}} E_{i_0}) = X_i \times_{X_{i_0}} E_{i_0}$$

Similarly, we have $X = X \times_{X_{i_0}} E_{i_0}$. Hence we conclude that $E_i = X_i$ for i large enough by Lemma 70.6.10. □

0CPB **Remark 70.23.5.** In Situation 70.23.1 Lemmas 70.23.2, 70.23.3, and 70.23.4 tell us that the category of algebraic spaces quasi-separated and of finite type over B is equivalent to certain types of inverse systems of algebraic spaces over $(B_i)_{i \in I}$, namely the ones produced by applying Lemma 70.23.3 to a diagram of the form

(70.23.2.1). For example, given $X \to B$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to B_{i_1}$ and $X \to V_2 \to B_{i_2}$ as in (70.23.2.1), then applying Lemma 70.23.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CPC **Lemma 70.23.6.** *Notation and assumptions as in Lemma 70.23.4. If f is flat and of finite presentation, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.*

Proof. By Lemma 70.7.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 70.6.12. As discussed in Remark 70.23.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \to U \to B_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms of Spaces, Lemma 67.30.12). Observe that $Y_{i'} \to B_{i'} \times_{B_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to B_{i'} \times_{B_i} U$ is equal to the closed subspace $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. □

0CPD **Lemma 70.23.7.** *Notation and assumptions as in Lemma 70.23.4. If f is smooth, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is smooth.*

Proof. Combine Lemmas 70.23.6 and 70.6.3. □

0CPE **Lemma 70.23.8.** *Notation and assumptions as in Lemma 70.23.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.*

Proof. By the discussion in Remark 70.23.5 the choice of i_1 and W fitting into a diagram as in (70.23.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to Y$ proper and of finite presentation, see Lemma 70.12.1. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 70.10.2 and 70.6.13. With this choice of W it is immediate from the construction that for $i \geq i_3$ the algebraic space X_i is a closed subspace of $Y_i \times_{Y_{i_3}} W \subset B_i \times_{B_{i_3}} W$ and hence proper over Y_i. □

0CPF **Lemma 70.23.9.** *In Situation 70.23.1 suppose that we have a cartesian diagram*

$$\begin{array}{ccc} X^1 & \xrightarrow{p} & X^3 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle a} \\ X^2 & \xrightarrow{b} & X^4 \end{array}$$

of algebraic spaces quasi-separated and of finite type over B. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{array}{ccc} X^j & \longrightarrow & W^j \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_j} \end{array}$$

as in (70.23.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 70.23.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding morphisms of inverse systems constructed in Lemma 70.23.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 70.23.5 the choice of W^1 fitting into a diagram as in (70.23.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 70.23.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \to X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \to X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 70.6.10. □

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CHAPTER 71

Divisors on Algebraic Spaces

0838 **71.1. Introduction**

0839 In this chapter we study divisors on algebraic spaces and related topics. A basic reference for algebraic spaces is [**Knu71**].

71.2. Associated and weakly associated points

0CTV In the case of schemes we have introduced two competing notions of associated points. Namely, the usual associated points (Divisors, Section 31.2) and the weakly associated points (Divisors, Section 31.5). For a general algebraic space the notion of an associated point is basically useless and we don't even bother to introduce it. If the algebraic space is locally Noetherian, then we allow ourselves to use the phrase "associated point" instead of "weakly associated point" as the notions are the same for Noetherian schemes (Divisors, Lemma 31.5.8). Before we make our definition, we need a lemma.

0CTW **Lemma 71.2.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. The following are equivalent*

1. *for some étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is weakly associated to $f^*\mathcal{F}$,*
2. *for every étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is weakly associated to $f^*\mathcal{F}$,*
3. *the maximal ideal of $\mathcal{O}_{X,\overline{x}}$ is a weakly associated prime of the stalk $\mathcal{F}_{\overline{x}}$.*

If X is locally Noetherian, then these are also equivalent to

4. *for some étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is associated to $f^*\mathcal{F}$,*
5. *for every étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x, the point u is associated to $f^*\mathcal{F}$,*
6. *the maximal ideal of $\mathcal{O}_{X,\overline{x}}$ is an associated prime of the stalk $\mathcal{F}_{\overline{x}}$.*

Proof. Choose a scheme U with a point u and an étale morphism $f : U \to X$ mapping u to x. Lift $\overline{x}$ to a geometric point of U over u. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ where the strict henselization is with respect to our chosen lift of $\overline{x}$, see Properties of Spaces, Lemma 66.22.1. Finally, we have

$$\mathcal{F}_{\overline{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}^{sh}_{U,u}$$

by Properties of Spaces, Lemma 66.29.4. Hence the equivalence of (1), (2), and (3) follows from More on Flatness, Lemma 38.2.9. If X is locally Noetherian, then any U as above is locally Noetherian, hence we see that (1), resp. (2) are equivalent to (4), resp. (5) by Divisors, Lemma 31.5.8. On the other hand, in the locally Noetherian case the local ring $\mathcal{O}_{X,\overline{x}}$ is Noetherian too (Properties of Spaces,

Lemma 66.24.4). Hence the equivalence of (3) and (6) by the same lemma (or by Algebra, Lemma 10.66.9). □

0CTX **Definition 71.2.2.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$.

(1) We say x is *weakly associated* to $\mathcal{F}$ if the equivalent conditions (1), (2), and (3) of Lemma 71.2.1 are satisfied.
(2) We denote $\text{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
(3) The *weakly associated points of* X are the weakly associated points of $\mathcal{O}_X$.

If X is locally Noetherian we will say x *is associated to* $\mathcal{F}$ if and only if x is weakly associated to $\mathcal{F}$ and we set $\text{Ass}(\mathcal{F}) = \text{WeakAss}(\mathcal{F})$. Finally (still assuming X is locally Noetherian), we will say x *is an associated point of* X if and only if x is a weakly associated point of X.

At this point we can prove the obligatory lemmas.

0CTY **Lemma 71.2.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}(\mathcal{F}) \subset \text{Supp}(\mathcal{F})$.*

Proof. This is immediate from the definitions. The support of an abelian sheaf on X is defined in Properties of Spaces, Definition 66.20.3. □

0CTZ **Lemma 71.2.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\text{WeakAss}(\mathcal{F}_2) \subset \text{WeakAss}(\mathcal{F}_1) \cup \text{WeakAss}(\mathcal{F}_3)$ and $\text{WeakAss}(\mathcal{F}_1) \subset \text{WeakAss}(\mathcal{F}_2)$.*

Proof. For every geometric point $\overline{x} \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,\overline{x}} \to \mathcal{F}_{2,\overline{x}} \to \mathcal{F}_{3,\overline{x}} \to 0$ is a short exact sequence of $\mathcal{O}_{X,\overline{x}}$-modules. Hence the lemma follows from Algebra, Lemma 10.66.4. □

0CU0 **Lemma 71.2.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow \text{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Then $\mathcal{F}$ is zero if and only if $f^*\mathcal{F}$ is zero. Hence the lemma follows from the definition and the lemma in the case of schemes, see Divisors, Lemma 31.5.5. □

0CUL **Lemma 71.2.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. If*

(1) *$x \in \text{Supp}(\mathcal{F})$*
(2) *x is a codimension 0 point of X (Properties of Spaces, Definition 66.10.2).*

Then $x \in \text{WeakAss}(\mathcal{F})$. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module with scheme theoretic support Z (Morphisms of Spaces, Definition 67.15.4) and x is a codimension 0 point of Z, then $x \in \text{WeakAss}(\mathcal{F})$.

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the stalk $\mathcal{F}_{\overline{x}}$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_{\overline{x}})$ is nonempty by Algebra, Lemma 10.66.5. On the other hand, the spectrum of $\mathcal{O}_{X,\overline{x}}$ is a singleton. Hence x is a weakly associated point of $\mathcal{F}$ by definition. The final statement follows as $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Z,\overline{z}}$ is a surjection, the spectrum of $\mathcal{O}_{Z,\overline{z}}$ is a singleton, and $\mathcal{F}_{\overline{x}}$ is a nonzero module over $\mathcal{O}_{Z,\overline{z}}$. □

0CUM **Lemma 71.2.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. If*

(1) *X is decent (for example quasi-separated or locally separated),*
(2) *$x \in Supp(\mathcal{F})$*
(3) *x is not a specialization of another point in $Supp(\mathcal{F})$.*

Then $x \in WeakAss(\mathcal{F})$.

Proof. (A quasi-separated algebraic space is decent, see Decent Spaces, Section 68.6. A locally separated algebraic space is decent, see Decent Spaces, Lemma 68.15.2.) Choose a scheme U, a point $u \in U$, and an étale morphism $f : U \to X$ mapping u to x. By Decent Spaces, Lemma 68.12.1 if $u' \rightsquigarrow u$ is a nontrivial specialization, then $f(u') \neq x$. Hence we see that $u \in \mathrm{Supp}(f^*\mathcal{F})$ is not a specialization of another point of $\mathrm{Supp}(f^*\mathcal{F})$. Hence $u \in \mathrm{WeakAss}(f^*\mathcal{F})$ by Divisors, Lemma 71.2.6. □

0CUN **Lemma 71.2.8.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap W$ is finite for every quasi-compact open $W \subset |X|$.*

Proof. Choose a quasi-compact scheme U and an étale morphism $U \to X$ such that W is the image of $|U| \to |X|$. Then U is a Noetherian scheme and we may apply Divisors, Lemma 31.2.5 to conclude. □

0CUP **Lemma 71.2.9.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \to X$ is an étale morphism such that $WeakAss(\mathcal{F}) \subset \mathrm{Im}(|U| \to |X|)$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\mathcal{F}'|_U = 0$. This implies that $\mathrm{WeakAss}(\mathcal{F}') \cap \mathrm{Im}(|U| \to |X|) = \emptyset$ (by the definition of weakly associated points). On the other hand, $\mathrm{WeakAss}(\mathcal{F}') \subset \mathrm{WeakAss}(\mathcal{F})$ by Lemma 71.2.4. We conclude $\mathrm{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 71.2.5. □

0CUQ **Lemma 71.2.10.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$ be a point which is not in the image of $|f|$. Then y is not weakly associated to $f_*\mathcal{F}$.*

Proof. By Morphisms of Spaces, Lemma 67.11.2 the $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is quasi-coherent hence the lemma makes sense. Choose an affine scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. We may replace $f : X \to Y$, $\mathcal{F}$, y by $X \times_Y V \to V$, $\mathcal{F}|_{X \times_Y V}$, v. Thus we may assume Y is an affine scheme. In this case X is quasi-compact, hence we can choose an affine scheme U and a surjective étale morphism $U \to X$. Denote $g : U \to Y$ the composition. Then $f_*\mathcal{F} \subset g_*(\mathcal{F}|_U)$. By Lemma 71.2.4 we reduce to the case of schemes which is Divisors, Lemma 31.5.9. □

0CUR **Lemma 71.2.11.** *Let S be a scheme. Let X be an algebraic space over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in |X|$ at least one of the following happens*

(1) *$\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective, or*
(2) *$x \notin WeakAss(\mathcal{F})$.*

Then φ is injective.

Proof. The assumptions imply that $\text{WeakAss}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 71.2.5. □

0EN1 **Lemma 71.2.12.** *Let S be a scheme. Let X be a reduced algebraic space over S. Then the weakly associated point of X are exactly the codimension 0 points of X.*

Proof. Working étale locally this follows from Divisors, Lemma 31.5.12 and Properties of Spaces, Lemma 66.11.1. □

71.3. Morphisms and weakly associated points

0CU1

0CU2 **Lemma 71.3.1.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$\textit{WeakAss}_S(f_*\mathcal{F}) \subset f(\textit{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is an affine morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.1. □

0CU8 **Lemma 71.3.2.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$\textit{WeakAss}_Y(f_*\mathcal{F}) = f(\textit{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is an affine morphism of schemes and U is locally Noetherian. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.2. □

0CU9 **Lemma 71.3.3.** *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\textit{WeakAss}(f_*\mathcal{F}) = f(\textit{WeakAss}(\mathcal{F}))$.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then $U \to V$ is a finite morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.3. □

0CUA **Lemma 71.3.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ and $y = f(x) \in |Y|$. If*

(1) *$y \in \textit{WeakAss}_S(\mathcal{G})$,*
(2) *f is flat at x, and*
(3) *the dimension of the local ring of the fibre of f at x is zero (Morphisms of Spaces, Definition 67.33.1),*

then $x \in \textit{WeakAss}(f^*\mathcal{G})$.

Proof. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to V \times_Y X$ mapping v to a point lying over v and x. This is possible because there is a $t \in |V \times_Y X|$ mapping to (v, y) by Properties of Spaces, Lemma 66.4.3. By definition we see that the dimension of $\mathcal{O}_{U_v,u}$ is zero. Hence u is a generic point of the fiber U_v. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \to V$. This case is treated in Divisors, Lemma 31.6.4. □

0CUS **Lemma 71.3.5.** *Let K/k be a field extension. Let X be an algebraic space over k. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback $\mathcal{F}_K$, then x is a weakly associated point of $\mathcal{F}$.*

Proof. This is the translation of Divisors, Lemma 31.6.5 into the language of algebraic spaces. We omit the details of the translation. □

0CUT **Lemma 71.3.6.** *Let S be a scheme. Let $f : X \to Y$ be a finite flat morphism of algebraic spaces. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then*

$$x \in \mathit{WeakAss}(g^*\mathcal{G}) \Leftrightarrow y \in \mathit{WeakAss}(\mathcal{G})$$

Proof. Follows immediately from the case of schemes (More on Flatness, Lemma 38.2.7) by étale localization. □

0CUU **Lemma 71.3.7.** *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then*

$$x \in \mathit{WeakAss}(f^*\mathcal{G}) \Leftrightarrow y \in \mathit{WeakAss}(\mathcal{G})$$

Proof. This is immediate from the definition of weakly associated points and in fact the corresponding lemma for the case of schemes (More on Flatness, Lemma 38.2.8) is the basis for our definition. □

71.4. Relative weak assassin

0CUV We need a couple of lemmas to define this gadget.

0CUW **Lemma 71.4.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. The following are equivalent*

1. *for some scheme V, point $v \in V$, and étale morphism $V \to Y$ mapping v to y, the algebraic space X_v is locally Noetherian,*
2. *for every scheme V, point $v \in V$, and étale morphism $V \to Y$ mapping v to y, the algebraic space X_v is locally Noetherian, and*
3. *there exists a field k and a morphism $\mathrm{Spec}(k) \to Y$ representing y such that X_k is locally Noetherian.*

If there exists a field k_0 and a monomorphism $\mathrm{Spec}(k_0) \to Y$ representing y, then these are also equivalent to

4. *the algebraic space X_{k_0} is locally Noetherian.*

Proof. Observe that $X_v = v \times_Y X = \mathrm{Spec}(\kappa(v)) \times_Y X$. Hence the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are clear. Assume that $\mathrm{Spec}(k) \to Y$ is a morphism from the spectrum of a field such that X_k is locally Noetherian. Let $V \to Y$ be an étale morphism from

a scheme V and let $v \in V$ a point mapping to y. Then the scheme $v \times_Y \mathrm{Spec}(k)$ is nonempty. Choose a point $w \in v \times_Y \mathrm{Spec}(k)$. Consider the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \to Y$ is étale and since w may be viewed as a point of $V \times_Y \mathrm{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 29.36.7). Thus $X_w \to X_k$ is a finite étale morphism as a base change of $w \to \mathrm{Spec}(k)$. Hence X_w is locally Noetherian (Morphisms of Spaces, Lemma 67.23.5). The morphism $X_w \to X_v$ is a surjective, affine, flat morphism as a base change of the surjective, affine, flat morphism $w \to v$. Then the fact that X_w is locally Noetherian implies that X_v is locally Noetherian. This can be seen by picking a surjective étale morphism $U \to X$ and then using that $U_w \to U_v$ is surjective, affine, and flat. Working affine locally on the scheme U_v we conclude that U_w is locally Noetherian by Algebra, Lemma 10.164.1.

Finally, it suffices to prove that (3) implies (4) in case we have a monomorphism $\mathrm{Spec}(k_0) \to Y$ in the class of y. Then $\mathrm{Spec}(k) \to Y$ factors as $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0) \to Y$. The argument given above then shows that X_k being locally Noetherian impies that X_{k_0} is locally Noetherian. □

0CUX **Definition 71.4.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. We say *the fibre of f over y is locally Noetherian* if the equivalent conditions (1), (2), and (3) of Lemma 71.4.1 are satisfied. We say *the fibres of f are locally Noetherian* if this holds for every $y \in |Y|$.

Of course, the usual way to guarantee locally Noetherian fibres is to assume the morphism is locally of finite type.

0CUY **Lemma 71.4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then the fibres of f are locally Noetherian.*

Proof. This follows from Morphisms of Spaces, Lemma 67.23.5 and the fact that the spectrum of a field is Noetherian. □

0CUZ **Lemma 71.4.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ and $y = f(x) \in |Y|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider commutative diagrams*

$$\begin{array}{ccccc} X & \longleftarrow & X \times_Y V & \longleftarrow & X_v \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & \longleftarrow & v \end{array} \qquad \begin{array}{ccccc} X & \longleftarrow & U & \longleftarrow & U_v \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & \longleftarrow & v \end{array} \qquad \begin{array}{ccccc} x & \longleftarrow & x' & \longleftarrow & u \\ \downarrow & & \downarrow & \swarrow & \\ y & \longleftarrow & v & & \end{array}$$

where V and U are schemes, $V \to Y$ and $U \to X \times_Y V$ are étale, $v \in V$, $x' \in |X_v|$, $u \in U$ are points related as in the last diagram. Denote $\mathcal{F}|_{X_v}$ and $\mathcal{F}|_{U_v}$ the pullbacks of $\mathcal{F}$. The following are equivalent

1. *for some V, v, x' as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,*
2. *for every $V \to Y, v, x'$ as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,*
3. *for some U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,*
4. *for every U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,*
5. *for some field k and morphism $\mathrm{Spec}(k) \to Y$ representing y and some $t \in |X_k|$ mapping to x, the point t is a weakly associated point of $\mathcal{F}|_{X_k}$.*

If there exists a field k_0 and a monomorphism $\mathrm{Spec}(k_0) \to Y$ representing y, then these are also equivalent to

(6) *x_0 is a weakly associated point of $\mathcal{F}|_{X_{k_0}}$ where $x_0 \in |X_{k_0}|$ is the unique point mapping to x.*

If the fibre of f over y is locally Noetherian, then in conditions (1), (2), (3), (4), and (6) we may replace "weakly associated" with "associated".

Proof. Observe that given V, v, x' as in the lemma we can find $U \to X \times_Y V$ and $u \in U$ mapping to x' and then the morphism $U_v \to X_v$ is étale. Thus it is clear that (1) and (3) are equivalent as well as (2) and (4). Each of these implies (5). We will show that (5) implies (2). Suppose given V, v, x' as well as $\mathrm{Spec}(k) \to X$ and $t \in |X_k|$ such that the point t is a weakly associated point of $\mathcal{F}|_{X_k}$. We can choose a point $w \in v \times_Y \mathrm{Spec}(k)$. Then we obtain the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \to Y$ is étale and since w may be viewed as a point of $V \times_Y \mathrm{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 29.36.7). Thus $X_w \to X_k$ is a finite étale morphism as a base change of $w \to \mathrm{Spec}(k)$. Thus any point x'' of X_w lying over t is a weakly associated point of $\mathcal{F}|_{X_w}$ by Lemma 71.3.7. We may pick x'' mapping to x' (Properties of Spaces, Lemma 66.4.3). Then Lemma 71.3.5 implies that x' is a weakly associated point of $\mathcal{F}|_{X_v}$.

To finish the proof it suffices to show that the equivalent conditions (1) – (5) imply (6) if we are given $\mathrm{Spec}(k_0) \to Y$ as in (6). In this case the morphism $\mathrm{Spec}(k) \to Y$ of (5) factors uniquely as $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0) \to Y$. Then x_0 is the image of t under the morphism $X_k \to X_{k_0}$. Hence the same lemma as above shows that (6) is true. □

0CV0 **Definition 71.4.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of $\mathcal{F}$ in X over Y* is the set $\mathrm{WeakAss}_{X/Y}(\mathcal{F}) \subset |X|$ consisting of those $x \in |X|$ such that the equivalent conditions of Lemma 71.4.4 are satisfied. If the fibres of f are locally Noetherian (Definition 71.4.2) then we use the notation $\mathrm{Ass}_{X/Y}(\mathcal{F})$.

With this notation we can formulate some of the results already proven for schemes.

0CV1 **Lemma 71.4.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Assume*

(1) *$\mathcal{F}$ is flat over Y,*
(2) *X and Y are locally Noetherian, and*
(3) *the fibres of f are locally Noetherian.*

Then

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \{x \in Ass_{X/Y}(\mathcal{F}) \text{ such that } f(x) \in Ass_Y(\mathcal{G})\}$$

Proof. Via étale localization, this is an immediate consequence of the result for schemes, see Divisors, Lemma 31.3.1. The result for schemes is more general only because we haven't defined associated points for non-Noetherian algebraic spaces (hence we need to assume X and the fibres of $X \to Y$ are locally Noetherian to even be able to formulate this result). □

0CV2 **Lemma 71.4.7.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and set $\mathcal{F}' = (g')^*\mathcal{F}$. If f is locally of finite type, then

(1) *$x' \in Ass_{X'/Y'}(\mathcal{F}') \Rightarrow g'(x') \in Ass_{X/Y}(\mathcal{F})$*
(2) *if $x \in Ass_{X/Y}(\mathcal{F})$, then given $y' \in |Y'|$ with $f(x) = g(y')$, there exists an $x' \in Ass_{X'/Y'}(\mathcal{F}')$ with $g'(x') = x$ and $f'(x') = y'$.*

Proof. This follows from the case of schemes by étale localization. We write out the details completely. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \to X'$ is surjective and étale.

Proof of (1). Choose $u' \in U'$ mapping to x'. Denote $v' \in V'$ the image of u'. Then $x' \in \mathrm{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \mathrm{Ass}(\mathcal{F}|_{U'_{v'}})$ by definition (writing Ass instead of WeakAss makes sense as $U'_{v'}$ is locally Noetherian). Applying Divisors, Lemma 31.7.3 we see that the image $u \in U$ of u' is in $\mathrm{Ass}(\mathcal{F}|_{U_v})$ where $v \in V$ is the image of u. This in turn means $g'(x') \in \mathrm{Ass}_{X/Y}(\mathcal{F})$.

Proof of (2). Choose $u \in U$ mapping to x. Denote $v \in V$ the image of u. Then $x \in \mathrm{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \mathrm{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $v' \in V'$ mapping to $y' \in |Y'|$ and to $v \in V$ (possible by Properties of Spaces, Lemma 66.4.3). Let $t \in \mathrm{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$ be a generic point of an irreducible component. Let $u' \in U'$ be the image of t. Applying Divisors, Lemma 31.7.3 we see that $u' \in \mathrm{Ass}(\mathcal{F}'|_{U'_{v'}})$. This in turn means $x' \in \mathrm{Ass}_{X'/Y'}(\mathcal{F}')$ where $x' \in |X'|$ is the image of u'. □

0CV3 **Lemma 71.4.8.** *With notation and assumptions as in Lemma 71.4.7. Assume g is locally quasi-finite, or more generally that for every $y' \in |Y'|$ the transcendence degree of $y'/g(y')$ is 0. Then $Ass_{X'/Y'}(\mathcal{F}')$ is the inverse image of $Ass_{X/Y}(\mathcal{F})$.*

Proof. The transcendence degree of a point over its image is defined in Morphisms of Spaces, Definition 67.33.1. Let $x' \in |X'|$ with image $x \in |X|$. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \to X'$ is surjective and étale. Choose $u \in U$ mapping to x. Denote $v \in V$ the image of u. Then $x \in \mathrm{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \mathrm{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $u' \in U'$ mapping to $x' \in |X'|$ and to $u \in U$ (possible by Properties of Spaces, Lemma 66.4.3). Let $v' \in V'$ be the image of u'. Then $x' \in \mathrm{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \mathrm{Ass}(\mathcal{F}'|_{U'_{v'}})$ by definition. Now the lemma follows from the discussion in Divisors, Remark 31.7.4 applied to $u' \in \mathrm{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$. □

0CV4 **Lemma 71.4.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $i : Z \to X$ be a finite morphism. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $WeakAss_{X/Y}(i_*\mathcal{G}) = i(WeakAss_{Z/Y}(\mathcal{G}))$.*

Proof. Follows from the case of schemes (Divisors, Lemma 31.8.3) by étale localization. Details omitted. □

0CVV **Lemma 71.4.10.** *Let Y be a scheme. Let X be an algebraic space of finite presentation over Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subspace such that $U \to Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid Ass_{X_y}(\mathcal{F}_y) \subset |U_y|\}$$

is locally constructible in Y.

Proof. Note that since Y is a scheme, it makes sense to take the fibres $X_y = \mathrm{Spec}(\kappa(y)) \times_Y X$. (Also, by our definitions, the set $\mathrm{Ass}_{X_y}(\mathcal{F}_y)$ is exactly the fibre of $\mathrm{Ass}_{X/Y}(\mathcal{F}) \to Y$ over y, but we won't need this.) The question is local on Y, indeed, we have to show that E is constructible if Y is affine. In this case X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to X$. Then $\mathrm{Ass}_{X_y}(\mathcal{F}_y)$ is the image of $\mathrm{Ass}_{W_y}(\varphi^*\mathcal{F}_y)$ for all $y \in Y$. Hence the lemma follows from the case of schemes for the open $\varphi^{-1}(U) \subset W$ and the morphism $W \to Y$. The case of schemes is More on Morphisms, Lemma 37.25.5. □

71.5. Fitting ideals

0CZ3 This section is the continuation of the discussion in Divisors, Section 31.9. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. In this situation we can construct the Fitting ideals

$$0 = \mathrm{Fit}_{-1}(\mathcal{F}) \subset \mathrm{Fit}_0(\mathcal{F}) \subset \mathrm{Fit}_1(\mathcal{F}) \subset \ldots \subset \mathcal{O}_X$$

as the sequence of quasi-coherent sheaves ideals characterized by the following property: for every affine $U = \mathrm{Spec}(A)$ étale over X if $\mathcal{F}|_U$ corresponds to the A-module M, then $\mathrm{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\mathrm{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $A \to B$ is an étale ring map, then the ith Fitting ideal of $M \otimes_A B$ over B is equal to $\mathrm{Fit}_i(M)B$ by More on Algebra, Lemma 15.8.4 part (3). More precisely (perhaps), the existence of the quasi-coherent sheaves of ideals $\mathrm{Fit}_0(\mathcal{O}_X)$ follows (for example) from the description of quasi-coherent sheaves in Properties of Spaces, Lemma 66.29.3 and the pullback property given in Divisors, Lemma 31.9.1.

The advantage of constructing the Fitting ideals in this way is that we see immediately that formation of Fitting ideals commutes with étale localization hence many properties of the Fitting ideals immediately reduce to the corresponding properties in the case of schemes. Often we will use the discussion in Properties of Spaces, Section 66.30 to do the translation between properties of quasi-coherent sheaves on schemes and on algebraic spaces.

0CZ4 **Lemma 71.5.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_Y$-module. Then $f^{-1}Fit_i(\mathcal{F}) \cdot \mathcal{O}_X = Fit_i(f^*\mathcal{F})$.*

Proof. Reduces to Divisors, Lemma 31.9.1 by étale localization. □

0CZ5 **Lemma 71.5.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Then $Fit_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.*

Proof. Reduces to Divisors, Lemma 31.9.2 by étale localization. □

0CZ6 **Lemma 71.5.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $Z_0 \subset X$ be the closed subspace cut out by $Fit_0(\mathcal{F})$. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then*

1. *$Z \subset Z_0 \subset X$ as closed subspaces,*
2. *$|Z| = |Z_0| = Supp(\mathcal{F})$ as closed subsets of $|X|$,*
3. *there exists a finite type, quasi-coherent $\mathcal{O}_{Z_0}$-module $\mathcal{G}_0$ with*

$$(Z_0 \to X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that formation of Z commutes with étale localization, see Morphisms of Spaces, Definition 67.15.4 (which uses Morphisms of Spaces, Lemma 67.15.3 to define Z). Hence (1) and (2) follow from the case of schemes, see Divisors, Lemma 31.9.3. To get $\mathcal{G}_0$ as in part (3) we can use that we have $\mathcal{G}$ on Z as in Morphisms of Spaces, Lemma 67.15.3 and set $\mathcal{G}_0 = (Z \to Z_0)_*\mathcal{G}$. □

0CZ7 **Lemma 71.5.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Then $\mathcal{F}$ can be generated by r elements in an étale neighbourhood of x if and only if $Fit_r(\mathcal{F})_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$.*

Proof. Reduces to Divisors, Lemma 31.9.4 by étale localization (as well as the description of the local ring in Properties of Spaces, Section 66.22 and the fact that the strict henselization of a local ring is faithfully flat to see that the equality over the strict henselization is equivalent to the equality over the local ring). □

0CZ8 **Lemma 71.5.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $r \geq 0$. The following are equivalent*

1. *$\mathcal{F}$ is finite locally free of rank r*
2. *$Fit_{r-1}(\mathcal{F}) = 0$ and $Fit_r(\mathcal{F}) = \mathcal{O}_X$, and*
3. *$Fit_k(\mathcal{F}) = 0$ for $k < r$ and $Fit_k(\mathcal{F}) = \mathcal{O}_X$ for $k \geq r$.*

Proof. Reduces to Divisors, Lemma 31.9.5 by étale localization. □

0CZ9 **Lemma 71.5.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The closed subspaces*

$$X = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$

defined by the Fitting ideals of $\mathcal{F}$ have the following properties

1. *The intersection $\bigcap Z_r$ is empty.*
2. *The functor $(Sch/X)^{opp} \to Sets$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

   is representable by the open subspace $X \setminus Z_r$.
3. *The functor $F_r : (Sch/X)^{opp} \to Sets$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

   is representable by the locally closed subspace $Z_{r-1} \setminus Z_r$ of X.

If $\mathcal{F}$ is of finite presentation, then $Z_r \to X$, $X \setminus Z_r \to X$, and $Z_{r-1} \setminus Z_r \to X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 31.9.6 by étale localization. □

0CZA **Lemma 71.5.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $X = Z_{-1} \subset Z_0 \subset Z_1 \subset \ldots$ be as in Lemma 71.5.6. Set $X_r = Z_{r-1} \setminus Z_r$. Then $X' = \coprod_{r \geq 0} X_r$ represents the functor*

$$F_{flat} : Sch/X \longrightarrow Sets, \qquad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{X_r}$ is locally free of rank r and the morphisms $X_r \to X$ and $X' \to X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 31.9.7 by étale localization. □

71.6. Effective Cartier divisors

083A For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else. Note that in Morphisms of Spaces, Section 67.13 we discussed the correspondence between closed subspaces and quasi-coherent sheaves of ideals. Moreover, in Properties of Spaces, Section 66.30, we discussed properties of quasi-coherent modules, in particular "locally generated by 1 element". These references show that the following definition is compatible with the definition for schemes.

083B **Definition 71.6.1.** Let S be a scheme. Let X be an algebraic space over S.

(1) A *locally principal closed subspace* of X is a closed subspace whose sheaf of ideals is locally generated by 1 element.
(2) An *effective Cartier divisor* on X is a closed subspace $D \subset X$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subspace, but the converse is not always true. Effective Cartier divisors are closed subspaces of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

083C **Lemma 71.6.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be a closed subspace. The following are equivalent:*

(1) *The subspace D is an effective Cartier divisor on X.*
(2) *For some scheme U and surjective étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(3) *For every scheme U and every étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(4) *For every $x \in |D|$ there exists an étale morphism $(U, u) \to (X, x)$ of pointed algebraic spaces such that $U = \mathrm{Spec}(A)$ and $D \times_X U = \mathrm{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.*

Proof. The equivalence of (1) – (3) follows from Definition 71.6.1 and the references preceding it. Assume (1) and let $x \in |D|$. Choose a scheme W and a surjective étale morphism $W \to X$. Choose $w \in D \times_X W$ mapping to x. By (3) $D \times_X W$ is an effective Cartier divisor on W. Hence we can find affine étale neighbourhood U by choosing an affine open neighbourhood of w in W as in Divisors, Lemma 31.13.2.

Assume (4). Then we see that $\mathcal{I}_D|_U$ is invertible by Divisors, Lemma 31.13.2. Since we can find an étale covering of X by the collection of all such U and $X \setminus D$, we conclude that $\mathcal{I}_D$ is an invertible $\mathcal{O}_X$-module. □

083D **Lemma 71.6.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a locally principal closed subspace. Let $U = X \setminus Z$. Then $U \to X$ is an affine morphism.*

Proof. The question is étale local on X, see Morphisms of Spaces, Lemmas 67.20.3 and Lemma 71.6.2. Thus this follows from the case of schemes which is Divisors, Lemma 31.13.3. □

083S **Lemma 71.6.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$. Then $U \to X$ is an affine morphism and U is scheme theoretically dense in X.*

Proof. Affineness is Lemma 71.6.3. The density question is étale local on X by Morphisms of Spaces, Definition 67.17.3. Thus this follows from the case of schemes which is Divisors, Lemma 31.13.4. □

083T **Lemma 71.6.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $x \in |D|$. If $\dim_x(X) < \infty$, then $\dim_x(D) < \dim_x(X)$.*

Proof. Both the definition of an effective Cartier divisor and of the dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1) are étale local. Hence this lemma follows from the case of schemes which is Divisors, Lemma 31.13.5. □

083U **Definition 71.6.6.** Let S be a scheme. Let X be an algebraic space over S. Given effective Cartier divisors D_1, D_2 on X we set $D = D_1 + D_2$ equal to the closed subspace of X corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

083V **Lemma 71.6.7.** *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Étale locally this reduces to the following simple algebra fact: if $f_1, f_2 \in A$ are nonzerodivisors of a ring A, then $f_1 f_2 \in A$ is a nonzerodivisor. □

083W **Lemma 71.6.8.** *Let S be a scheme. Let X be an algebraic space over S. Let Z, Y be two closed subspaces of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. By Lemma 71.6.2 this reduces to the case of schemes which is Divisors, Lemma 31.13.9. □

Recall that we have defined the inverse image of a closed subspace under any morphism of algebraic spaces in Morphisms of Spaces, Definition 67.13.2.

083X **Lemma 71.6.9.** *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a locally principal closed subspace. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subspace of X'.*

Proof. Omitted. □

083Y **Definition 71.6.10.** Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $D \subset X$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subspace $f^{-1}(D) \subset X'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice.

083Z **Lemma 71.6.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

(1) *$f(x) \notin |D|$ for any weakly associated point x of X,*
(2) *f is flat, and*
(3) *add more here as needed.*

Proof. Working étale locally this lemma reduces to the case of schemes, see Divisors, Lemma 31.13.13. □

0840 **Lemma 71.6.12.** *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let D_1, D_2 be effective Cartier divisors on X. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. □

71.7. Effective Cartier divisors and invertible sheaves

0CPG Since an effective Cartier divisor has an invertible ideal sheaf (Definition 71.6.1) the following definition makes sense.

0841 **Definition 71.7.1.** Let S be a scheme. Let X be an algebraic space over S and let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.

(1) The *invertible sheaf $\mathcal{O}_X(D)$ associated to D* is defined by
$$\mathcal{O}_X(D) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{\otimes -1}.$$
(2) The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_X(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_X$.
(3) We write $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\otimes -1} = \mathcal{I}_D$.
(4) Given a second effective Cartier divisor $D' \subset X$ we define $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_X(D)$ turns addition of effective Cartier divisors (Definition 71.6.6) into addition in the Picard group of X (Lemma 71.7.3). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on X as a commutative monoid $\mathrm{EffCart}(X)$ whose zero element is the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_X(D - D')$ defines a group homomorphism
$$\mathrm{EffCart}(X)^{gp} \longrightarrow \mathrm{Pic}(X)$$
where the left hand side is the group completion of $\mathrm{EffCart}(X)$. In other words, when we write $\mathcal{O}_X(D - D')$ we may think of $D - D'$ as an element of $\mathrm{EffCart}(X)^{gp}$.

0B4F **Lemma 71.7.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Then for the conormal sheaf we have $\mathcal{C}_{D/X} = \mathcal{I}_D|D = \mathcal{O}_X(D)^{\otimes -1}|_D$.*

Proof. Omitted. □

0842 **Lemma 71.7.3.** *Let S be a scheme. Let X be an algebraic space over S. Let D_1, D_2 be effective Cartier divisors on X. Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

0843 **Definition 71.7.4.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

0844 **Lemma 71.7.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,\overline{x}}$ is not a zerodivisor.*
(3) *for any affine $U = \mathrm{Spec}(A)$ étale over X the restriction $f|_U$ is a nonzerodivisor of A, and*
(4) *there exists a scheme U and a surjective étale morphism $U \to X$ such that $f|_U$ is a regular section of $\mathcal{O}_U$.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules on Sites, Lemma 18.32.4 for the dual invertible sheaf.)

0845 **Definition 71.7.6.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subspace $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

0846 **Lemma 71.7.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
(2) *For any morphism of algebraic spaces $f : Y \to X$ over S we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subspace of X.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor on X if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

0847 **Lemma 71.7.8.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*

(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\text{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{matrix}\text{pairs } (\mathcal{L}, s) \text{ consisting of an invertible} \\ \mathcal{O}_X\text{-module and a regular global section}\end{matrix}\right\}$$

Proof. Omitted. □

71.8. Effective Cartier divisors on Noetherian spaces

0CPH In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

0B4G **Lemma 71.8.1.** *Let S be a scheme and let X be a locally Noetherian algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. If X is (S_k), then D is (S_{k-1}).*

Proof. By our definition of the property (S_k) for algebraic spaces (Properties of Spaces, Section 66.7) and Lemma 71.6.2 this follows from the case of schemes (Divisors, Lemma 31.15.5). □

0B4H **Lemma 71.8.2.** *Let S be a scheme and let X be a locally Noetherian normal algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1).*

Proof. By our definition of normality for algebraic spaces (Properties of Spaces, Section 66.7) and Lemma 71.6.2 this follows from the case of schemes (Divisors, Lemma 31.15.6). □

The following lemma can sometimes be used to produce effective Cartier divisors.

0DML **Lemma 71.8.3.** *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S. Let $U \subset X$ be a dense affine open. Then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. We claim that the reduced induced algebraic space structure D on $X \setminus U$ (Properties of Spaces, Definition 66.12.5) is the desired effective Cartier divisor. The construction of D commutes with étale localization, see proof of Properties of Spaces, Lemma 66.12.3. Let $X' \to X$ be a surjective étale morphism with X' affine. Since X is separated, we see that $U' = X' \times_X U$ is affine. Since $|X'| \to |X|$ is open, we see that U' is dense in X'. Since $D' = X' \times_X D$ is the reduced induced scheme structure on $X' \setminus U'$, we conclude that D' is an effective Cartier divisor by Divisors, Lemma 31.16.6 and its proof. This is what we had to show. □

0DMM **Lemma 71.8.4.** *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S. Then every invertible $\mathcal{O}_X$-module is isomorphic to*

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Choose a dense affine open $U \subset X$ such that $\mathcal{L}|_U$ is trivial. This is possible because X has a dense open subspace which is a scheme, see Properties of Spaces, Proposition 66.13.3. Denote $s : \mathcal{O}_U \to \mathcal{L}|_U$ the

trivialization. The complement of U is an effective Cartier divisor D. We claim that for some $n > 0$ the map s extends uniquely to a map

$$s : \mathcal{O}_X(-nD) \longrightarrow \mathcal{L}$$

The claim implies the lemma because it shows that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nD)$ has a regular global section hence is isomorphic to $\mathcal{O}_X(D')$ for some effective Cartier divisor D' by Lemma 71.7.8. To prove the claim we may work étale locally. Thus we may assume X is an affine Noetherian scheme. Since $\mathcal{O}_X(-nD) = \mathcal{I}^n$ where $\mathcal{I} = \mathcal{O}_X(-D)$ is the ideal sheaf of D in X, this case follows from Cohomology of Schemes, Lemma 30.10.5. □

The following lemma really belongs to a different section.

0DMC **Lemma 71.8.5.** *Let R be a valuation ring with fraction field K. Let X be an algebraic space over R such that $X \to \mathrm{Spec}(R)$ is smooth. For every effective Cartier divisor $D \subset X_K$ there exists an effective Cartier divisor $D' \subset X$ with $D'_K = D$.*

Proof. Let $D' \subset X$ be the scheme theoretic image of $D \to X_K \to X$. Since this morphism is quasi-compact, formation of D' commutes with flat base change, see Morphisms of Spaces, Lemma 67.30.12. In particular we find that $D'_K = D$. Hence, we may assume X is affine. Say $X = \mathrm{Spec}(A)$. Then $X_K = \mathrm{Spec}(A \otimes_R K)$ and D corresponds to an ideal $I \subset A \otimes_R K$. We have to show that $J = I \cap A$ cuts out an effective Cartier divisor in X. First, observe that A/J is flat over R (as a torsion free R-module, see More on Algebra, Lemma 15.22.10), hence J is finitely generated by More on Algebra, Lemma 15.25.6 and Algebra, Lemma 10.5.3. Thus it suffices to show that $J_\mathfrak{q} \subset A_\mathfrak{q}$ is generated by a single element for each prime $\mathfrak{q} \subset A$. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R_\mathfrak{p}$ is a valuation ring (Algebra, Lemma 10.50.9). Observe further that $A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}$ is a regular ring by Algebra, Lemma 10.140.3. Thus we may apply More on Algebra, Lemma 15.122.3 to see that $I(A_\mathfrak{q} \otimes_R K)$ is generated by a single element $f \in A_\mathfrak{p} \otimes_R K$. After clearing denominators we may assume $f \in A_\mathfrak{q}$. Let $\mathfrak{c} \subset R_\mathfrak{p}$ be the content ideal of f (see More on Algebra, Definition 15.24.1 and More on Flatness, Lemma 38.19.6). Since $R_\mathfrak{p}$ is a valuation ring and since $\mathfrak{c}$ is finitely generated (More on Algebra, Lemma 15.24.2) we see $\mathfrak{c} = (\pi)$ for some $\pi \in R_\mathfrak{p}$ (Algebra, Lemma 10.50.15). After relacing f by $\pi^{-1}f$ we see that $f \in A_\mathfrak{q}$ and $f \notin \mathfrak{p}A_\mathfrak{q}$. Claim: $I_\mathfrak{q} = (f)$ which finishes the proof. To see the claim, observe that $f \in I_\mathfrak{q}$. Hence we have a surjection $A_\mathfrak{q}/(f) \to A_\mathfrak{q}/I_\mathfrak{q}$ which is an isomorphism after tensoring over R with K. Thus we are done if $A_\mathfrak{q}/(f)$ is $R_\mathfrak{p}$-flat. This follows from Algebra, Lemma 10.128.5 and our choice of f. □

71.9. Relative effective Cartier divisors

0EPM The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

0EPN **Lemma 71.9.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset X$ be a closed subspace. Assume*

(1) *D is an effective Cartier divisor, and*
(2) *$D \to Y$ is a flat morphism.*

Then for every morphism of schemes $g : Y' \to Y$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = Y' \times_Y X$ where $g' : X' \to X$ is the projection.

Proof. Using Lemma 71.6.2 the property of being an effective Cartier divisor is étale local. Thus this lemmma immediately reduces to the case of schemes which is Divisors, Lemma 31.18.1. □

This lemma is the motivation for the following definition.

0EPP **Definition 71.9.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. A *relative effective Cartier divisor* on X/Y is an effective Cartier divisor $D \subset X$ such that $D \to Y$ is a flat morphism of algebraic spaces.

71.10. Meromorphic functions and sections

0EN2 This section is the analogue of Divisors, Section 31.23. Beware: it is even easier to make mistakes with this material in the case of algebraic space, than it is in the case of schemes!

Let S be a scheme. Let X be an algebraic space over S. For any scheme U étale over X we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 71.7.4. The restriction of a regular section to V/U étale is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

on $X_{étale}$ and its sheafification, see Modules on Sites, Section 18.44.

0EN3 **Definition 71.10.1.** Let S be a scheme. Let X be an algebraic space over S. The *sheaf of meromorphic functions on* X is the sheaf $\mathcal{K}_X$ on $X_{étale}$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective. Moreover, by the compatibility of sheafification and taking stalks we see that

$$\mathcal{K}_{X,\overline{x}} = \mathcal{S}_{\overline{x}}^{-1}\mathcal{O}_{X,\overline{x}}$$

for any geometric point $\overline{x}$ of X. The set $\mathcal{S}_{\overline{x}}$ is a subset of the set of nonzerodivisors of $\mathcal{O}_{X,\overline{x}}$, but in general not equal to this.

0EN4 **Lemma 71.10.2.** *Let S be a scheme. Let X be an algebraic space over S. For U affine and étale over X the set $\mathcal{S}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$.*

Proof. Follows from Lemma 71.7.5. □

Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X_{étale}$. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules on Sites, Lemma 18.44.2.

0EN5 **Definition 71.10.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X_{étale}$.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).

(2) A *meromorphic section of* $\mathcal{F}$ is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{K}_{X,\overline{x}} = \mathcal{S}_{\overline{x}}^{-1}\mathcal{F}_{\overline{x}}$$

for any geometric point $\overline{x}$ of X. However, one has to be careful since it may not be the case that $\mathcal{S}_{\overline{x}}$ is the set of nonzerodivisors in the étale local ring $\mathcal{O}_{X,\overline{x}}$ as we pointed out above. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

0EN6 **Lemma 71.10.4.** *Let S be a scheme. Let X be an algebraic space over S. Assume*

(a) *every weakly associated point of X is a point of codimension 0, and*
(b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1.*

Then

(1) *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
(2) *for $U \in X_{étale}$ affine $\mathcal{K}_X(U)$ is the total ring of fractions of $\mathcal{O}_X(U)$,*
(3) *for a geometric point $\overline{x}$ the set $\mathcal{S}_{\overline{x}}$ the set of nonzerodivisors of $\mathcal{O}_{X,\overline{x}}$, and*
(4) *for a geometric point $\overline{x}$ the ring $\mathcal{K}_{X,\overline{x}}$ is the total ring of fractions of $\mathcal{O}_{X,\overline{x}}$.*

Proof. By Lemma 71.7.5 we see that $U \in X_{étale}$ affine $\mathcal{S}_X(U) \subset \mathcal{O}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$. Thus the presheaf $\mathcal{S}^{-1}\mathcal{O}_X$ is equal to

$$U \longmapsto Q(\mathcal{O}_X(U))$$

on $X_{affine,étale}$, with notation as in Algebra, Example 10.9.8. Observe that the codimension 0 points of X correspond to the generic points of U, see Properties of Spaces, Lemma 66.11.1. Hence if $U = \mathrm{Spec}(A)$, then A is a ring with finitely many minimal primes such that any weakly associated prime of A is minimal. The same is true for any étale extension of A (because the spectrum of such is an affine scheme étale over X hence can play the role of A in the previous sentence). In order to show that our presheaf is a sheaf and quasi-coherent it suffices to show that

$$Q(A) \otimes_A B \longrightarrow Q(B)$$

is an isomorphism when $A \to B$ is an étale ring map, see Properties of Spaces, Lemma 66.29.3. (To define the displayed arrow, observe that since $A \to B$ is flat it maps nonzerodivisors to nonzerodivisors.) By Algebra, Lemmas 10.25.4 and 10.66.7. we have

$$Q(A) = \prod\nolimits_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}} \quad \text{and} \quad Q(B) = \prod\nolimits_{\mathfrak{q} \subset B \text{ minimal}} B_{\mathfrak{q}}$$

Since $A \to B$ is étale, the minimal primes of B are exactly the primes of B lying over the minimal primes of A (for example by More on Algebra, Lemma 15.44.2). By Algebra, Lemmas 10.153.10, 10.153.3 (13), and 10.153.5 we see that $A_{\mathfrak{p}} \otimes_A B$ is a finite product of local rings finite étale over $A_{\mathfrak{p}}$. This clearly implies that $A_{\mathfrak{p}} \otimes_A B = \prod_{\mathfrak{q} \text{ lies over } \mathfrak{p}} B_{\mathfrak{q}}$ as desired.

At this point we know that (1) and (2) hold. Proof of (3). Let $s \in \mathcal{O}_{X,\overline{x}}$ be a nonzerodivisor. Then we can find an étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ and $f \in \mathcal{O}_X(U)$ mapping to s. Let $u \in U$ be the point determined by $\overline{u}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is faithfully flat (as a strict henselization), we see that f maps to a nonzerodivisor in $\mathcal{O}_{U,u}$. By Divisors, Lemma 31.23.6 after shrinking U we find that

f is a nonzerodivisor and hence a section of $\mathcal{S}_X(U)$. Part (4) follows from (3) by computing stalks. □

0EN7 **Lemma 71.10.5.** *Let S be a scheme. Let X be an algebraic space over S. Assume*

(a) *every weakly associated point of X is a point of codimension 0, and*
(b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1.*
(c) *X is representable by a scheme X_0 (awkward but temporary notation).*

Then the sheaf of meromorphic functions $\mathcal{K}_X$ is the quasi-coherent sheaf of $\mathcal{O}_X$-algebras associated to the quasi-coherent sheaf of meromorphic functions $\mathcal{K}_{X_0}$.

Proof. For the equivalence between $QCoh(\mathcal{O}_X)$ and $QCoh(\mathcal{O}_{X_0})$, please see Properties of Spaces, Section 66.29. The lemma is true because $\mathcal{K}_X$ and $\mathcal{K}_{X_0}$ are quasi-coherent and have the same value on corresponding affine opens of X and X_0 by Lemma 71.10.4 and Divisors, Lemma 31.23.6. □

0EN8 **Definition 71.10.6.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that *pullbacks of meromorphic functions are defined for f* if for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

with $U \in X_{étale}$ and $V \in Y_{étale}$ and any section $s \in \mathcal{S}_Y(V)$ the pullback $f^\sharp(s) \in \mathcal{O}_X(U)$ is an element of $\mathcal{S}_X(U)$.

In this case there is an induced map $f^\sharp : f_{small}^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(X_{étale}), \mathcal{K}_X) & \longrightarrow & (Sh(X_{étale}), \mathcal{O}_X) \\ \downarrow{\scriptstyle f_{small}} & & \downarrow{\scriptstyle f_{small}} \\ (Sh(Y_{étale}), \mathcal{K}_Y) & \longrightarrow & (Sh(Y_{étale}), \mathcal{O}_Y) \end{array}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

0EN9 **Lemma 71.10.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Pullbacks of meromorphic sections are defined in each of the following cases*

(1) *weakly associated points of X are mapped to points of codimension 0 on Y,*
(2) *f is flat,*
(3) *add more here as needed.*

Proof. Working étale locally, this translates into the case of schemes, see Divisors, Lemma 31.23.5. To do the translation use Lemma 71.7.5 (description of regular sections), Definition 71.2.2 (definition of weakly associated points), and Properties of Spaces, Lemma 66.11.1 (description of codimension 0 points). □

0ENA **Lemma 71.10.8.** *Let S be a scheme. Let X be an algebraic space over S. Assume*

(a) *every weakly associated point of X is a point of codimension 0, and*

(b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1,*
(c) *every codimension 0 point of X can be represented by a monomorphism $\mathrm{Spec}(k) \to X$.*

Let $X^0 \subset |X|$ be the set of codimension 0 points of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13; this makes sense because X^0 is contained in the schematic locus of X. Similarly, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we obtain the formula

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

for the sheaf of meromorphic sections of $\mathcal{F}$. Finally, the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. By Decent Spaces, Lemma 68.20.3 and Section 68.6 we see that X is decent[1]. Thus $X^0 \subset |X|$ is the set of generic points of irreducible components (Decent Spaces, Lemma 68.20.1) and X^0 is locally finite in $|X|$ by (b). It follows that X^0 is contained in every dense open subset of $|X|$. In particular, X^0 is contained in the schematic locus (Decent Spaces, Theorem 68.10.2). Thus the local rings $\mathcal{O}_{X,\eta}$ and the morphisms j_η are defined.

Observe that a locally finite direct sum of sheaves of modules is equal to the product. This and the fact that X^0 is locally finite in $|X|$ explains the equalities between direct sums and products in the statement. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that the second equality follows from the first.

Let $j : Y = \coprod_{\eta \in X^0} \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ be the product of the morphisms j_η. We have to show that $\mathcal{K}_X = j_*\mathcal{O}_Y$. Observe that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 71.10.7. This gives a map

$$\mathcal{K}_X \longrightarrow j_*\mathcal{O}_Y.$$

Let $U \in X_{étale}$ be affine. By Lemma 71.10.4 the left hand side evaluates to total ring of fractions of $\mathcal{O}_X(U)$. On the other hand, the right hand side is equal to the product of the local rings of U at the codimension 0 points, i.e., the generic points of U. These two rings are equal (as we already saw in the proof of Lemma 71.10.4) by Algebra, Lemmas 10.25.4 and 10.66.7. Thus our map is an isomorphism.

Finally, we have to show that $R(X) = \Gamma(X, \mathcal{K}_X)$. This follows from the case of schemes (Divisors, Lemma 31.23.6) applied to the schematic locus $X' \subset X$. Namely, the ring of rational functions of X is by definition the same as the ring of rational functions on X' as it is a dense open subspace of X (see above). Certainly, $R(X')$ agrees with the ring of rational functions when X' is viewed as a scheme. On the other hand, by our description of $\mathcal{K}_X$ above, and the fact, seen above, that $X^0 \subset |X'|$ is contained in any dense open, we see that $\Gamma(X, \mathcal{K}_X) = \Gamma(X', \mathcal{K}_{X'})$. Finally, use the compatibility recorded in Lemma 71.10.5. □

[1]Conversely, if X is decent, then condition (c) holds automatically.

0ENB **Definition 71.10.9.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective.

Let us spell out when (regular) meromorphic sections can be pulled back.

0ENC **Lemma 71.10.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that pullbacks of meromorphic functions are defined for f (see Definition 71.10.6).*

(1) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
(2) *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. □

0EPQ **Lemma 71.10.11.** *Let S be a scheme. Let X be an algebraic space over S satisfying (a), (b), and (c) of Lemma 71.10.8. Then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ has a regular meromorphic section.*

Proof. With notation as in Lemma 71.10.8 the stalk $\mathcal{L}_\eta$ of $\mathcal{L}$ at is defined for all $\eta \in X^0$ and it is a rank 1 free $\mathcal{O}_{X,\eta}$-module. Pick a generator $s_\eta \in \mathcal{L}_\eta$ for all $\eta \in X^0$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 71.10.8 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. □

71.11. Relative Proj

0848 This section revisits the construction of the relative proj in the setting of algebraic spaces. The material in this section corresponds to the material in Constructions, Section 27.16 and Divisors, Section 31.30 in the case of schemes.

0849 **Situation 71.11.1.** Here S is a scheme, X is an algebraic space over S, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra.

In Situation 71.11.1 we are going to define a functor $F : (Sch/S)^{opp}_{fppf} \to Sets$ which will turn out to be an algebraic space. We will follow (mutatis mutandis) the procedure of Constructions, Section 27.16. First, given a scheme T over S we define a *quadruple over T* to be a system $(d, f : T \to X, \mathcal{L}, \psi)$

(1) $d \geq 1$ is an integer,
(2) $f : T \to X$ is a morphism over S,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

We say two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are *equivalent*[2] if and only if we have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$. Given a quadruple $(d, f, \mathcal{L}, \psi)$ and a morphism $h : T' \to T$ we have the pullback $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$. Pullback

[2]This definition is motivated by Constructions, Lemma 27.16.4. The advantage of choosing this one is that it clearly defines an equivalence relation.

preserves the equivalence relation. Finally, for a *quasi-compact* scheme T over S we set

$$F(T) = \text{the set of equivalence classes of quadruples over } T$$

and for an arbitrary scheme T over S we set

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. Thus we have defined our functor

084A (71.11.1.1) $$F : Sch^{opp} \longrightarrow Sets$$

There is a morphism $F \to X$ of functors sending the quadruple $(d, f, \mathcal{L}, \psi)$ to f.

084B **Lemma 71.11.2.** *In Situation 71.11.1. The functor F above is an algebraic space. For any morphism $g : Z \to X$ where Z is a scheme there is a canonical isomorphism $\underline{Proj}_Z(g^*\mathcal{A}) = Z \times_X F$ compatible with further base change.*

Proof. It suffices to prove the second assertion, see Spaces, Lemma 65.11.3. Let $g : Z \to X$ be a morphism where Z is a scheme. Let F' be the functor of quadruples associated to the graded quasi-coherent $\mathcal{O}_Z$-algebra $g^*\mathcal{A}$. Then there is a canonical isomorphism $F' = Z \times_X F$, sending a quadruple $(d, f : T \to Z, \mathcal{L}, \psi)$ for F' to $(d, g \circ f, \mathcal{L}, \psi)$ (details omitted, see proof of Constructions, Lemma 27.16.1). By Constructions, Lemmas 27.16.4, 27.16.5, and 27.16.6 and Definition 27.16.7 we see that F' is representable by $\underline{\mathrm{Proj}}_Z(g^*\mathcal{A})$. □

The lemma above tells us the following definition makes sense.

084C **Definition 71.11.3.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over X*, or the *homogeneous spectrum of $\mathcal{A}$ over X*, or the *relative Proj of $\mathcal{A}$ over X* is the algebraic space F over X of Lemma 71.11.2. We denote it $\pi : \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$.

In particular the structure morphism of the relative Proj is representable by construction. We can also think about the relative Proj via glueing. Let $\varphi : U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$ with projection morphisms $s, t : R \to U$. By Lemma 71.11.2 there exists a canonical isomorphism

$$\gamma : \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \longrightarrow \underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X U$$

over U. Let $\alpha : t^*\varphi^*\mathcal{A} \to s^*\varphi^*\mathcal{A}$ be the canonical isomorphism of Properties of Spaces, Proposition 66.32.1. Then the diagram

$$\begin{array}{ccc} & \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,s} R = \underline{\mathrm{Proj}}_R(s^*\varphi^*\mathcal{A}) & \\ {\scriptstyle s^*\gamma}\nearrow & & \\ \underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X R & & \downarrow {\scriptstyle \text{induced by } \alpha} \\ {\scriptstyle t^*\gamma}\searrow & & \\ & \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,t} R = \underline{\mathrm{Proj}}_R(t^*\varphi^*\mathcal{A}) & \end{array}$$

is commutative (the equal signs come from Constructions, Lemma 27.16.10). Thus, if we denote $\mathcal{A}_U$, $\mathcal{A}_R$ the pullback of $\mathcal{A}$ to U, R, then $P = \underline{\mathrm{Proj}}_X(\mathcal{A})$ has an

étale covering by the scheme $P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $P_U \times_P P_U$ is equal to $P_R = \underline{\mathrm{Proj}}_R(\mathcal{A}_R)$. Using these remarks we can argue in the usual fashion using étale localization to transfer results on the relative proj from the case of schemes to the case of algebraic spaces.

084D **Lemma 71.11.4.** *In Situation 71.11.1. The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$ and a canonical homomorphism of graded algebras*

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$$

whose base change to any scheme over X agrees with Constructions, Lemma 27.15.5.

Proof. As in the discussion following Definition 71.11.3 choose a scheme U and a surjective étale morphism $U \to X$, set $R = U \times_X U$ with projections $s, t : R \to U$, $\mathcal{A}_U = \mathcal{A}|_U$, $\mathcal{A}_R = \mathcal{A}|_R$, and $\pi : P = \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$, $\pi_U : P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $\pi_R : P_R = \underline{\mathrm{Proj}}_U(\mathcal{A}_R)$. By the Constructions, Lemma 27.15.5 we have a quasi-coherent sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{P_U}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{P_U}(n)$ and a canonical map $\psi_U : \pi_U^*\mathcal{A}_U \to \bigoplus_{n\geq 0} \mathcal{O}_{P_U}(n)$ and similarly for P_R. By Constructions, Lemma 27.16.10 the pullback of $\mathcal{O}_{P_U}(n)$ and ψ_U by either projection $P_R \to P_U$ is equal to $\mathcal{O}_{P_R}(n)$ and ψ_R. By Properties of Spaces, Proposition 66.32.1 we obtain $\mathcal{O}_P(n)$ and ψ. We omit the verification of compatibility with pullback to arbitrary schemes over X. □

Having constructed the relative Proj we turn to some basic properties.

085C **Lemma 71.11.5.** *Let S be a scheme. Let $g : X' \to X$ be a morphism of algebraic spaces over S and let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then there is a canonical isomorphism*

$$r : \underline{Proj}_{X'}(g^*\mathcal{A}) \longrightarrow X' \times_X \underline{Proj}_X(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*pr_2^*\left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}$-algebras.

Proof. Let F be the functor (71.11.1.1) and let F' be the corresponding functor defined using $g^*\mathcal{A}$ on X'. We claim there is a canonical isomorphism $r : F' \to X' \times_X F$ of functors (and of course r is the isomorphism of the lemma). It suffices to construct the bijection $r : F'(T) \to X'(T) \times_{X(T)} F(T)$ for quasi-compact schemes T over S. First, if $\xi = (d', f', \mathcal{L}', \psi')$ is a quadruple over T for F', then we can set $r(\xi) = (f', (d', g \circ f', \mathcal{L}', \psi'))$. This makes sense as $(g \circ f')^*\mathcal{A}^{(d)} = (f')^*(g^*\mathcal{A})^{(d)}$. The inverse map sends the pair $(f', (d, f, \mathcal{L}, \psi))$ to the quadruple $(d, f', \mathcal{L}, \psi)$. We omit the proof of the final assertion (hint: reduce to the case of schemes by étale localization and apply Constructions, Lemma 27.16.10). □

084E **Lemma 71.11.6.** *In Situation 71.11.1 the morphism $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is separated.*

Proof. By Morphisms of Spaces, Lemma 67.4.12 and the construction of the relative Proj this follows from the case of schemes which is Constructions, Lemma 27.16.9. □

084F **Lemma 71.11.7.** *In Situation 71.11.1. If one of the following holds*

(1) *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
(2) *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
(3) *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is quasi-compact.

Proof. By Morphisms of Spaces, Lemma 67.8.8 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.1. □

084G **Lemma 71.11.8.** *In Situation 71.11.1. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_X$-algebras, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is of finite type.*

Proof. By Morphisms of Spaces, Lemma 67.23.4 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.2. □

084H **Lemma 71.11.9.** *In Situation 71.11.1. If $\mathcal{O}_X \to \mathcal{A}_0$ is an integral algebra map[3] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is universally closed.*

Proof. By Morphisms of Spaces, Lemma 67.9.5 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.3. □

084I **Lemma 71.11.10.** *In Situation 71.11.1. The following conditions are equivalent*

(1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
(2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_X$-algebra.*

If these conditions hold, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is proper.

Proof. By Morphisms of Spaces, Lemma 67.40.2 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 31.30.3. □

085D **Lemma 71.11.11.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. With $P = \underline{Proj}_X(\mathcal{A})$ we have*

(1) *P represents the functor F_1 which associates to T over S the set of isomorphism classes of triples $(f, \mathcal{L}, \psi)$, where $f : T \to X$ is a morphism over S, $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a map of graded $\mathcal{O}_T$-algebras inducing a surjection $f^*\mathcal{A}_1 \to \mathcal{L}$,*
(2) *the canonical map $\pi^*\mathcal{A}_1 \to \mathcal{O}_P(1)$ is surjective, and*
(3) *each $\mathcal{O}_P(n)$ is invertible and the multiplication maps induce isomorphisms $\mathcal{O}_P(n) \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) = \mathcal{O}_P(n+m)$.*

Proof. Omitted. See Constructions, Lemma 27.16.11 for the case of schemes. □

71.12. Functoriality of relative proj

085E This section is the analogue of Constructions, Section 27.18.

[3] In other words, the integral closure of $\mathcal{O}_X$ in $\mathcal{A}_0$, see Morphisms of Spaces, Definition 67.48.2, equals $\mathcal{A}_0$.

085F **Lemma 71.12.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a map of quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A}) \to X$ and $Q = \underline{Proj}_X(\mathcal{B}) \to X$. There is a canonical open subspace $U(\psi) \subset Q$ and a canonical morphism of algebraic spaces*

$$r_\psi : U(\psi) \longrightarrow P$$

over X and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_P(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any scheme W étale over X the triple

$$(U(\psi) \times_X W, \quad r_\psi|_{U(\psi) \times_X W} : U(\psi) \times_X W \to P \times_X W, \quad \theta|_{U(\psi) \times_X W})$$

is equal to the triple associated to $\psi : \mathcal{A}|_W \to \mathcal{B}|_W$ of Constructions, Lemma 27.18.1.

Proof. This lemma follows from étale localization and the case of schemes, see discussion following Definition 71.11.3. Details omitted. □

085G **Lemma 71.12.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A})$, $Q = \underline{Proj}_X(\mathcal{B})$ and $R = \underline{Proj}_X(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_X$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi \circ \varphi} = r_\varphi \circ r_\psi|_{U(\psi \circ \varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi \circ \varphi}$$

with obvious notation.

Proof. Omitted. □

085H **Lemma 71.12.3.** *With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

1. *$U(\psi) = Q$,*
2. *$r_\psi : Q \to R$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).*

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.3) by étale localization. □

085I **Lemma 71.12.4.** *With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Q$,*
2. *$r_\psi : Q \to P$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.*

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.4) by étale localization. □

085J **Lemma 71.12.5.** *With hypotheses and notation as in Lemma 71.12.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to P$ *is a closed immersion, and*
(3) *the maps* $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ *are isomorphisms.*

Proof. Follows from the case of schemes (Constructions, Lemma 27.18.5) by étale localization. □

71.13. Invertible sheaves and morphisms into relative Proj

0D2Y It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme and Y an algebraic space over S.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_Y$-algebra.
(3) Denote $\pi : \underline{\mathrm{Proj}}_Y(\mathcal{A}) \to Y$ the relative Proj of $\mathcal{A}$ over Y.
(4) Let $f : X \to Y$ be a morphism of algebraic spaces over S.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data let $U(\psi) \subset X$ be the open subspace with

$$|U(\psi)| = \bigcup\nolimits_{d \geq 1} \{\text{locus where } f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d} \text{ is surjective}\}$$

Formation of $U(\psi) \subset X$ commutes with pullback by any morphism $X' \to X$.

0D2Z **Lemma 71.13.1.** *With assumptions and notation as above. The morphism* ψ *induces a canonical morphism of algebraic spaces over* Y

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{Proj}_Y(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$*-algebras*

$$\theta : r_{\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{Proj}_Y(\mathcal{A})}(d) \right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) *For* $V \to Y$ *étale and* $d \geq 0$ *the diagram*

$$\begin{array}{ccc} \mathcal{A}_d(V) & \xrightarrow{\quad\psi\quad} & \Gamma(V \times_Y X, \mathcal{L}^{\otimes d}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle restrict} \\ \Gamma(V \times_Y \underline{Proj}_Y(\mathcal{A}), \mathcal{O}_{\underline{Proj}_Y(\mathcal{A})}(d)) & \xrightarrow{\theta} & \Gamma(V \times_Y U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

(2) *For any* $d \geq 1$ *and any morphism* $W \to X$ *where* W *is a scheme such that* $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ *is surjective we have (a)* $W \to X$ *factors through* $U(\psi)$ *and (b) composition of* $W \to U(\psi)$ *with* $r_{\mathcal{L},\psi}$ *agrees with the morphism* $W \to \underline{Proj}_Y(\mathcal{A})$ *which exists by the construction of* $\underline{Proj}_Y(\mathcal{A})$*, see Definition 71.11.3.*

(3) *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where X' and Y' are schemes, set $\mathcal{A}' = g^*\mathcal{A}$ and $\mathcal{L}' = (g')^*\mathcal{L}$ and denote $\psi' : (f')^*\mathcal{A} \to \bigoplus_{d\geq 0}(\mathcal{L}')^{\otimes d}$ the pullback of ψ. Let $U(\psi')$, $r_{\psi',\mathcal{L}'}$, and θ' be the open, morphism, and homomorphism constructed in Constructions, Lemma 71.13.1. Then $U(\psi') = (g')^{-1}(U(\psi))$ and $r_{\psi',\mathcal{L}'}$ agrees with the base change of $r_{\psi,\mathcal{L}}$ via the isomorphism $\underline{Proj}_{Y'}(\mathcal{A}') = Y' \times_Y \underline{Proj}_Y(\mathcal{A})$ of Lemma 71.11.5. Moreover, θ' is the pullback of θ.

Proof. Omitted. Hints: First we observe that for a quasi-compact scheme W over X the following are equivalent

(1) $W \to X$ factors through $U(\psi)$, and
(2) there exists a d such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective.

This gives a description of $U(\psi)$ as a subfunctor of X on our base category $(Sch/S)_{fppf}$. For such a W and d we consider the quadruple $(d, W \to Y, \mathcal{L}|_W, \psi^{(d)}|_W)$. By definition of $\underline{\mathrm{Proj}}_Y(\mathcal{A})$ we obtain a morphism $W \to \underline{\mathrm{Proj}}_Y(\mathcal{A})$. By our notion of equivalence of quadruples one sees that this morphism is independent of the choice of d. This clearly defines a transformation of functors $r_{\psi,\mathcal{L}} : U(\psi) \to \underline{\mathrm{Proj}}_Y(\mathcal{A})$, i.e., a morphism of algebraic spaces. By construction this morphism satisfies (2). Since the morphism constructed in Constructions, Lemma 27.19.1 satisfies the same property, we see that (3) is true.

To construct θ and check the compatibility (1) of the lemma, work étale locally on Y and X, arguing as in the discussion following Definition 71.11.3. □

71.14. Relatively ample sheaves

0D30 This section is the analogue of Morphisms, Section 29.37 for algebraic spaces. Our definition of a relatively ample invertible sheaf is as follows.

0D31 **Definition 71.14.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/Y*, or *f-ample* if $f : X \to Y$ is representable and for every morphism $Z \to Y$ where Z is a scheme, the pullback $\mathcal{L}_Z$ of $\mathcal{L}$ to $X_Z = Z \times_Y X$ is ample on X_Z/Z as in Morphisms, Definition 29.37.1.

We will almost always reduce questions about relatively ample invertible sheaves to the case of schemes. Thus in this section we have mainly sanity checks.

0D32 **Lemma 71.14.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume Y is a scheme. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/Y in the sense of Definition 71.14.1, and*
(2) *X is a scheme and $\mathcal{L}$ is ample on X/Y in the sense of Morphisms, Definition 29.37.1.*

Proof. This follows from the definitions and Morphisms, Lemma 29.37.9 (which says that being relatively ample for schemes is preserved under base change). □

0D33 **Lemma 71.14.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y' \to Y$ be a morphism of algebraic spaces over S. Let $f' : X' \to Y'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.*

Proof. This follows immediately from the definition! (Hint: transitivity of base change.) □

0D34 **Lemma 71.14.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If there exists an f-ample invertible sheaf, then f is representable, quasi-compact, and separated.*

Proof. This is clear from the definitions and Morphisms, Lemma 29.37.3. (If in doubt, take a look at the principle of Algebraic Spaces, Lemma 65.5.8.) □

0D35 **Lemma 71.14.5.** *Let $V \to U$ be a surjective étale morphism of affine schemes. Let X be an algebraic space over U. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y = V \times_U X$ and let $\mathcal{N}$ be the pullback of $\mathcal{L}$ to Y. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/U, and*
(2) *$\mathcal{N}$ is ample on Y/V.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 71.14.3. Assume (2). This implies that $Y \to V$ is quasi-compact and separated (Lemma 71.14.4) and Y is a scheme. It follows that the morphism $f : X \to U$ is quasi-compact and separated (Morphisms of Spaces, Lemmas 67.8.8 and 67.4.12). Set $\mathcal{A} = \bigoplus_{d \geq 0} f_* \mathcal{L}^{\otimes d}$. This is a quasi-coherent sheaf of graded $\mathcal{O}_U$-algebras (Morphisms of Spaces, Lemma 67.11.2). By adjunction we have a map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$. Applying Lemma 71.13.1 we obtain an open subspace $U(\psi) \subset X$ and a morphism

$$r_{\mathcal{L},\psi} : U(\psi) \to \underline{\mathrm{Proj}}_U(\mathcal{A})$$

Since $h : V \to U$ is étale we have $\mathcal{A}|_V = (Y \to V)_*(\bigoplus_{d \geq 0} \mathcal{N}^{\otimes d})$, see Properties of Spaces, Lemma 66.26.2. It follows that the pullback ψ' of ψ to Y is the adjunction map for the situation $(Y \to V, \mathcal{N})$ as in Morphisms, Lemma 29.37.4 part (5). Since $\mathcal{N}$ is ample on Y/V we conclude from the lemma just cited that $U(\psi') = Y$ and that $r_{\mathcal{N},\psi'}$ is an open immersion. Since Lemma 71.13.1 tells us that the formation of $r_{\mathcal{L},\psi}$ commutes with base change, we conclude that $U(\psi) = X$ and that we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{r'} & \underline{\mathrm{Proj}}_V(\mathcal{A}|_V) & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{r} & \underline{\mathrm{Proj}}_U(\mathcal{A}) & \longrightarrow & U \end{array}$$

whose squares are fibre products. We conclude that r is an open immersion by Morphisms of Spaces, Lemma 67.12.1. Thus X is a scheme. Then we can apply Morphisms, Lemma 29.37.4 part (5) to conclude that $\mathcal{L}$ is ample on X/U. □

0D36 **Lemma 71.14.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/Y,*
(2) *for every scheme Z and every morphism $Z \to Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback $\mathcal{L}_Z$ is ample on X_Z/Z,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback $\mathcal{L}_Z$ is ample on X_Z/Z,*

(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the algebraic space $X_V = V \times_Y X$ is a scheme and the pullback $\mathcal{L}_V$ is ample on X_V/V.*

Proof. Parts (1) and (2) are equivalent by definition. The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds and $Z \to Y$ is as in (2), then we see that $X_Z \to Z$ is affine locally on Z representable. Hence X_Z is a scheme for example by Properties of Spaces, Lemma 66.13.1. Then it follows that $\mathcal{L}_Z$ is ample on X_Z/Z because it holds locally on Z and we can use Morphisms, Lemma 29.37.4. Thus (1), (2), and (3) are equivalent. Clearly these conditions imply (4).

Assume (4). Let $Z \to Y$ be a morphism with Z affine. Then $U = V \times_Y Z \to Z$ is a surjective étale morphism such that the pullback of $\mathcal{L}_Z$ by $X_U \to X_Z$ is relatively ample on X_U/U. Of course we may replace U by an affine open. It follows that $\mathcal{L}_Z$ is ample on X_Z/Z by Lemma 71.14.5. Thus (4) $\Rightarrow$ (3) and the proof is complete. □

0GUQ **Lemma 71.14.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample.*

Proof. Follows from the case of schemes, see Morphisms, Lemma 29.37.6. □

71.15. Relative ampleness and cohomology

0D37 This section contains some results related to the results in Cohomology of Schemes, Sections 30.21 and 30.17.

The following lemma is just an example of what we can do.

0D38 **Lemma 71.15.1.** *Let R be a Noetherian ring. Let X be an algebraic space over R such that the structure morphism $f : X \to \mathrm{Spec}(R)$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/R (Definition 71.14.1),*
(2) *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Cohomology of Schemes, Lemma 30.16.1 because assumption (1) implies that X is a scheme. The implication (2) $\Rightarrow$ (1) is Cohomology of Spaces, Lemma 69.16.9. □

0D39 **Lemma 71.15.2.** *Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that X_y is a scheme and $\mathcal{L}_y$ is ample on X_y. Then there exists a d_0 such that for all $d \geq d_0$ we have*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^* R^p f_* \mathcal{F} = R^p f'_* \mathcal{F}'$ by Cohomology of Spaces, Lemma 69.11.2. Moreover, the fibres X_y and X'_y are identified. Hence we may

assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to $\mathfrak{m}$. In this case $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \to \mathrm{Spec}(\kappa)$ the projection.

Let $B = \mathrm{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^* \widetilde{B}$ of quasi-coherent graded $\mathcal{O}_{X_y}$-algebras. We will use notation as in Cohomology of Spaces, Section 69.22 with I replaced by $\mathfrak{m}$. Since X_y is the closed subspace of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ as a coherent $\mathcal{O}_{X_y}$-module, see Cohomology of Spaces, Lemma 69.12.8. Then $\bigoplus_{n \geq 0} \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is a quasi-coherent graded $\mathcal{B}$-module of finite type because it is generated in degree zero over $\mathcal{B}$ abd because the degree zero part is $\mathcal{F}_y = \mathcal{F}/\mathfrak{m}\mathcal{F}$ which is a coherent $\mathcal{O}_{X_y}$-module. Hence by Cohomology of Schemes, Lemma 30.19.3 part (2) there exists a d_0 such that

$$H^p(X_y, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Cohomology of Spaces, Lemma 69.8.3 this is the same as the statement that $H^p(X, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

(1) $H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
(2) $H^0(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Cohomology of Spaces, Theorem 69.22.5) we find that the $\mathfrak{m}$-adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A-module by Cohomology of Spaces, Lemma 69.20.3 it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Cohomology of Spaces, Lemma 69.22.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. □

0D3A **Lemma 71.15.3.** *(For a more general version see Descent on Spaces, Lemma 74.13.2). Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is proper. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that X_y is a scheme and $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$ (as in Definition 71.14.1).*

Proof. Pick d_0 as in Lemma 71.15.2 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \ldots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \ldots, s_{y,r})} : X_y \to \mathbf{P}^r_{\kappa(y)}.$$

This is possible by Morphisms, Lemma 29.39.4 but we also use Morphisms, Lemma 29.41.7 to see that φ_y is a closed immersion and Constructions, Section 27.13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0, after replacing Y by an open neighbourhood of y, we can choose $s_0, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \ldots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subspace where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $|X_y| \subset U = \bigcup |X_{s_i}|$. Since $X \to Y$ is closed, we see that there is an open neighbourhood $y \in V \subset Y$ such that $|f|^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \ldots, s_r)} : X \longrightarrow \mathbf{P}^r_Y$$

with $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}^r_Y}(1)$ whose base change to y gives φ_y (strictly speaking we need to write out a proof that the construction of morphisms into projective space given in Constructions, Section 27.13 also works to describe morphisms of algebraic spaces into projective space; we omit the details).

We will finish the proof by a sleight of hand; the "correct" proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y. Namely, by Cohomology of Spaces, Lemma 69.23.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}^r_{\kappa(y)}$ of $\mathbf{P}^r_Y \to Y$ above y. Using that $\mathbf{P}^r_Y \to Y$ is closed, after shrinking Y we may assume that φ is finite. In particular X is a scheme. Then $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}^r_Y}(1)$ is ample by the very general Morphisms, Lemma 29.37.7. □

71.16. Closed subspaces of relative proj

085K Some auxiliary lemmas about closed subspaces of relative proj. This section is the analogue of Divisors, Section 31.31.

085L **Lemma 71.16.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subspace. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} \pi_* \left((i_* \mathcal{O}_Z)(d) \right)$$

If π is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{I})$.

Proof. The morphism π is separated by Lemma 71.11.6. As π is quasi-compact, π_* transforms quasi-coherent modules into quasi-coherent modules, see Morphisms of Spaces, Lemma 67.11.2. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra. The functoriality morphism $Z' = \underline{\mathrm{Proj}}_X(\mathcal{B}) \to \underline{\mathrm{Proj}}_X(\mathcal{A})$ is everywhere defined and a closed immersion, see Lemma 71.12.3. Hence it suffices to prove $Z = Z'$ as closed subspaces of P.

Having said this, the question is étale local on the base and we reduce to the case of schemes (Divisors, Lemma 31.31.1) by étale localization. □

In case the closed subspace is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

085M **Lemma 71.16.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a*

closed subscheme. If π is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$.

Proof. The reader can redo the arguments used in the case of schemes. However, we will show the lemma follows from the case of schemes by a trick. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 71.16.1. Choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 66.6.3. By the case of schemes (Divisors, Lemma 31.31.4) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_U$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_U \subset \mathcal{A}_d|_U$ such that $Z \times_X U$ is equal to $\underline{\mathrm{Proj}}_U(\mathcal{A}|_U/\mathcal{F}'\mathcal{A}|_U)$. By Limits of Spaces, Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_U$. Let $Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 71.12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X U \subset Z \times_X U$ by our choice of $\mathcal{F}$. Thus $Z = Z'$ as desired. □

085N **Lemma 71.16.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subspace. Let $U \subset X$ be an open. Assume that*

(1) *π is quasi-compact,*
(2) *i of finite presentation,*
(3) *$|U| \cap |\pi|(|i|(|Z|)) = \emptyset$,*
(4) *U is quasi-compact,*
(5) *$\mathcal{A}_n$ is a finite type $\mathcal{O}_X$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. We use the same trick as in the proof of Lemma 71.16.2 to reduce to the case of schemes. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 71.16.1. Choose an affine scheme W and a surjective étale morphism $W \to X$, see Properties of Spaces, Lemma 66.6.3. By the case of schemes (Divisors, Lemma 31.31.5) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_W$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_W \subset \mathcal{A}_d|_W$ such that (a) $Z \times_X W$ is equal to $\underline{\mathrm{Proj}}_W(\mathcal{A}|_W/\mathcal{F}'\mathcal{A}|_W)$ and (b) the support of $\mathcal{A}_d|_W/\mathcal{F}'$ is disjoint from $U \times_X W$. By Limits of Spaces, Lemma 70.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_W$. Let $Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 71.12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X W \subset Z \times_X W$ by our choice of $\mathcal{F}$. Thus $Z = Z'$. Finally, we see that $\mathcal{A}_d/\mathcal{F}$ is supported on $X \setminus U$ as $\mathcal{A}_d|_W/\mathcal{F}|_W$ is a quotient of $\mathcal{A}_d|_W/\mathcal{F}'$ which is supported on $W \setminus U \times_X W$. Thus the lemma follows. □

0B4I **Lemma 71.16.4.** *Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There is a bijection*

$$\left\{\begin{array}{c}\text{sections } \sigma \text{ of the} \\ \text{morphism } \mathbf{P}(\mathcal{E}) \to X\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{surjections } \mathcal{E} \to \mathcal{L} \text{ where} \\ \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module}\end{array}\right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\mathrm{Ker}(\mathcal{E} \to \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Because the constructions are compatible with base change, it suffices to check the statement étale locally on X. Thus we may assume X is a scheme and the result is Divisors, Lemma 31.31.6. □

71.17. Blowing up

085P Blowing up is an important tool in algebraic geometry.

085Q **Definition 71.17.1.** Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subspace corresponding to $\mathcal{I}$ (Morphisms of Spaces, Lemma 67.13.1). The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\mathrm{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n \right) \longrightarrow X$$

The *exceptional divisor* of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" algebraic space over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blowup of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Lemma 71.11.11.

085R **Lemma 71.17.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine scheme étale over X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $X' \to X$ is the blowup of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$U \times_X X' = \mathit{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of schemes over U, where the right hand side is the homogeneous spectrum of the Rees algebra of I in A. Moreover, $U \times_X X'$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. Note that the restriction $\mathcal{I}|_U$ is equal to the pullback of $\mathcal{I}$ via the morphism $U \to X$, see Properties of Spaces, Section 66.26. Thus the lemma follows on combining Lemma 71.11.2 with Divisors, Lemma 31.32.2. □

085S **Lemma 71.17.3.** *Let S be a scheme. Let $X_1 \to X_2$ be a flat morphism of algebraic spaces over S. Let $Z_2 \subset X_2$ be a closed subspace. Let Z_1 be the inverse image of Z_2 in X_1. Let X_i' be the blowup of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_1' & \longrightarrow & X_2' \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over S.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (see Morphisms of Spaces, Definition 67.13.2 and discussion following the definition). By Lemma 71.11.5 we see that $X_1 \times_{X_2} X_2'$ is the relative Proj of $\bigoplus_{n\geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X_2' = X_1'$. □

085T **Lemma 71.17.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The blowing up $b : X' \to X$ of Z in X has the following properties:*

(1) *$b|_{b^{-1}(X\setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
(2) *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
(3) *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 31.32.4. □

085U **Lemma 71.17.5** (Universal property blowing up)**.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $\mathcal{C}$ be the full subcategory of $(Spaces/X)$ consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 71.17.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By Lemma 71.11.11. the triple $(f : Y \to X, \mathcal{I}_D, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 71.6.4. Thus the morphism $Y \to X'$ is unique by Morphisms of Spaces, Lemma 67.17.8 (also b is separated by Lemma 71.11.6). □

085V **Lemma 71.17.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 71.17.5). □

085W **Lemma 71.17.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in $\mathcal{I}$ is reduced.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 31.32.8. □

0BH1 **Lemma 71.17.8.** *Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be the blowup of X in a closed subspace. If X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 then so does X'.*

Proof. Follows immediately from the lemma cited in the statement, the étale local description of blowing ups in Lemma 71.17.2, and Divisors, Lemma 31.32.10. □

085X **Lemma 71.17.9.** *Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be a blowup of X in a closed subspace. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 71.6.10).*

Proof. By Lemmas 71.17.2 and 71.6.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

085Y **Lemma 71.17.10.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 71.17.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 71.17.9. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 71.17.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 71.6.8. Thus a morphism $c' : Y \to X'$ over X by Lemma 71.17.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

085Z **Lemma 71.17.11.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then $b : X' \to X$ is a proper morphism.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 71.17.3) we can prove each of these statements after base change to U (see Morphisms of Spaces, Lemma 67.40.2). This reduces us to the case of schemes. In this case the morphism b is projective by Divisors, Lemma 31.32.13 hence proper by Morphisms, Lemma 29.43.5. □

0860 **Lemma 71.17.12.** *Let S be a scheme and let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Let $Z \subset X$ be a closed subspace of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subspace of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subspace $Y \subset X$ of finite presentation, such that*

(1) *$|Y| = |Z| \cup |b|(|Z'|)$, and*
(2) *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms of Spaces, Lemma 67.28.12. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open subspace of X by Limits of Spaces, Lemma 70.14.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 71.17.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open subspace in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open subspace in X. By Lemma 71.16.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $|X| \setminus |U|$ we see that $|V(\mathcal{J})|$ is contained in $|X| \setminus |U|$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $|Z| \subset |V(\mathcal{J})|$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $|V(\mathcal{J})|$ equals $|Z| \cup |b|(|Z'|)$. It follows that also $|V(\mathcal{I}\mathcal{J})| = |Z| \cup |b|(|Z'|)$. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 71.17.10. Hence it suffices to show that the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 71.17.6 and 71.17.10.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 71.17.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \mathrm{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. □

71.18. Strict transform

0861 This section is the analogue of Divisors, Section 31.33. Let S be a scheme, let B be an algebraic space over S, and let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of B in Z and denote $E \subset B'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider an algebraic space X over B and form the cartesian diagram

$$\begin{array}{ccccc} \mathrm{pr}_{B'}^{-1}E & \longrightarrow & X \times_B B' & \xrightarrow{\mathrm{pr}_X} & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{B'}} & & \downarrow{\scriptstyle f} \\ E & \longrightarrow & B' & \longrightarrow & B \end{array}$$

Since E is an effective Cartier divisor (Lemma 71.17.4) we see that $\mathrm{pr}_{B'}^{-1}E \subset X \times_B B'$ is locally principal (Lemma 71.6.9). Thus the inclusion morphism of the complement of $\mathrm{pr}_{B'}^{-1}E$ in $X \times_B B'$ is affine and in particular quasi-compact (Lemma 71.6.3).

Consequently, for a quasi-coherent $\mathcal{O}_{X \times_B B'}$-module $\mathcal{G}$ the subsheaf of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$ is a quasi-coherent submodule, see Limits of Spaces, Definition 70.14.6. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_B B'}$, then this subsheaf is an ideal of $\mathcal{G}$.

0862 **Definition 71.18.1.** With $Z \subset B$ and $f : X \to B$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of B in Z is the quotient $\mathcal{F}'$ of $\mathrm{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$.
(2) The *strict transform* of X is the closed subspace $X' \subset X \times_B B'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_B B'}$ supported on $|\mathrm{pr}_{B'}^{-1}E|$.

Note that taking the strict transform along a blowup depends on the closed subspace used for the blowup (and not just on the morphism $B' \to B$).

0863 **Lemma 71.18.2** (Étale localization and strict transform)**.** *In the situation of Definition 71.18.1. Let*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & B \end{array}$$

be a commutative diagram of morphisms with U and V schemes and étale horizontal arrows. Let $V' \to V$ be the blowup of V in $Z \times_B V$. Then

(1) *$V' = V \times_B B'$ and the maps $V' \to B'$ and $U \times_V V' \to X \times_B B'$ are étale,*
(2) *the strict transform U' of U relative to $V' \to V$ is equal to $X' \times_X U$ where X' is the strict transform of X relative to $B' \to B$, and*
(3) *for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the restriction of the strict transform $\mathcal{F}'$ to $U \times_V V'$ is the strict transform of $\mathcal{F}|_U$ relative to $V' \to V$.*

Proof. Part (1) follows from the fact that blowup commutes with flat base change (Lemma 71.17.3), the fact that étale morphisms are flat, and that the base change of an étale morphism is étale. Part (3) then follows from the fact that taking the sheaf of sections supported on a closed commutes with pullback by étale morphisms, see Limits of Spaces, Lemma 70.14.5. Part (2) follows from (3) applied to $\mathcal{F} = \mathcal{O}_X$. □

0864 **Lemma 71.18.3.** *In the situation of Definition 71.18.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subspace $f^{-1}Z$ of X.*
(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_B B'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 71.17.5) there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

whence a morphism $i : X'' \to X \times_B B'$. The first assertion of the lemma is that i is a closed immersion with image X'. The second assertion of the lemma is that

$\mathcal{F}' = i_*\mathcal{F}''$ where $\mathcal{F}''$ is the strict transform of $\mathcal{F}$ with respect to the blowing up $X'' \to X$. We can check these assertions étale locally on X, hence we reduce to the case of schemes (Divisors, Lemma 31.33.2). Some details omitted. □

0865 **Lemma 71.18.4.** *In the situation of Definition 71.18.1.*

(1) *If X is flat over B at all points lying over Z, then the strict transform of X is equal to the base change $X \times_B B'$.*
(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over B at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.3) by étale localization (Lemma 71.18.2). □

0866 **Lemma 71.18.5.** *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of Z in B. Let $g : X \to Y$ be an affine morphism of spaces over B. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_B B' \to Y \times_B B'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.4) by étale localization (Lemma 71.18.2). □

0867 **Lemma 71.18.6.** *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $D \subset B$ be an effective Cartier divisor. Let $Z' \subset B$ be the closed subspace cut out by the product of the ideal sheaves of Z and D. Let $B' \to B$ be the blowup of B in Z.*

(1) *The blowup of B in Z' is isomorphic to $B' \to B$.*
(2) *Let $f : X \to B$ be a morphism of algebraic spaces and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If the subsheaf of $\mathcal{F}$ of sections supported on $|f^{-1}D|$ is zero, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of B in Z'.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.5) by étale localization (Lemma 71.18.2). □

0868 **Lemma 71.18.7.** *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up with center Z. Let $Z' \subset B'$ be a closed subspace. Let $B'' \to B'$ be the blowing up with center Z'. Let $Y \subset B$ be a closed subscheme such that $|Y| = |Z| \cup |b|(|Z'|)$ and the composition $B'' \to B$ is isomorphic to the blowing up of B in Y. In this situation, given any scheme X over B and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

(1) *the strict transform of $\mathcal{F}$ with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $B' \to B$ of B in Z, and*
(2) *the strict transform of X with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of X with respect to the blowup $B' \to B$ of B in Z.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.6) by étale localization (Lemma 71.18.2). □

0869 **Lemma 71.18.8.** *In the situation of Definition 71.18.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to B$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $B' \to B$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 31.33.7) by étale localization (Lemma 71.18.2). □

0D0P **Lemma 71.18.9.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_B$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 71.5. Let $B' \to B$ be the blowup of B in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ can locally be generated by $\leq k$ sections.*

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 31.35.1) by étale localization (Lemma 71.18.2). □

0D0Q **Lemma 71.18.10.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_B$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 71.5. Assume that $\mathcal{F}$ is locally free of rank k on $B \setminus Z_k$. Let $B' \to B$ be the blowup of B in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ is locally free of rank k.*

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 31.35.2) by étale localization (Lemma 71.18.2). □

71.19. Admissible blowups

086A To have a bit more control over our blowups we introduce the following standard terminology.

086B **Definition 71.19.1.** Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms of Spaces, Lemma 67.28.12. In particular, a U-admissible blowup is a proper morphism, see Lemma 71.17.11. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 71.17.4) we will often abuse notation and think of U as an open subspace of X' as well.

086C **Lemma 71.19.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 71.17.12. □

086D **Lemma 71.19.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U, V \subset X$ be quasi-compact open subspaces. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$. By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

086E **Lemma 71.19.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 71.17.10. □

086F **Lemma 71.19.5.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let U, V be quasi-compact disjoint open subspaces of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Limits of Spaces, Lemma 70.14.1. Then $|V(\mathcal{I}\mathcal{J})| = |X|$. Hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I}+\mathcal{J}$. This is $U \cup V$-admissible as $|V(\mathcal{I}+\mathcal{J})| = |X| \setminus |U| \cup |V|$. We will show that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ as in the statement of the lemma.

Since $|V(\mathcal{I} + \mathcal{J})|$ is the complement of $|U \cup V|$ we conclude that $V \cup U$ is scheme theoretically dense in X', see Lemmas 71.17.4 and 71.6.4. Thus if such a decomposition $X' = X'_1 \amalg X'_2$ into open and closed subspaces exists, then X'_1 is the scheme theoretic closure of U in X' and similarly X'_2 is the scheme theoretic closure of V in X'. Since $U \to X'$ and $V \to X'$ are quasi-compact taking scheme theoretic closures commutes with étale localization (Morphisms of Spaces, Lemma 67.16.3). Hence to verify the existence of X'_1 and X'_2 we may work étale locally on X. This reduces us to the case of schemes which is treated in the proof of Divisors, Lemma 31.34.5. □

71.20. Other chapters

Preliminaries

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CHAPTER 72

Algebraic Spaces over Fields

06DR

72.1. Introduction

06DS This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [**Knu71**].

72.2. Conventions

06LX The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

72.3. Generically finite morphisms

0ACY This section continues the discussion in Decent Spaces, Section 68.21 and the analogue for morphisms of algebraic spaces of Varieties, Section 33.17.

0AD1 **Lemma 72.3.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is locally Noetherian. Let $y \in |Y|$ be a point of codimension ≤ 1 on Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied*

1. *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
2. *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
3. *f is quasi-finite at every $x \in X^0$,*
4. *f is quasi-finite at a dense set of points of $|X|$,*
5. *add more here.*

Then f is quasi-finite at every point of X lying over y.

Proof. We want to reduce the proof to the case of schemes. To do this we choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where U, V are schemes and where the horizontal arrows are étale and surjective. Pick $v \in V$ mapping to y. Observe that V is locally Noetherian and that $\dim(\mathcal{O}_{V,v}) \leq 1$ (see Properties of Spaces, Definitions 66.10.2 and Remark 66.7.3). The fibre U_v of $U \to V$ over v surjects onto $f^{-1}(\{y\}) \subset |X|$. The inverse image of

X^0 in U is exactly the set of generic points of irreducible components of U (Properties of Spaces, Lemma 66.11.1). If $\eta \in U$ is such a point with image $x \in X^0$, then the transcendence degree of $x/f(x)$ is the transcendence degree of $\kappa(\eta)$ over $\kappa(g(\eta))$ (Morphisms of Spaces, Definition 67.33.1). Observe that $U \to V$ is quasi-finite at $u \in U$ if and only if f is quasi-finite at the image of u in X.

Case (1). Here case (1) of Varieties, Lemma 33.17.1 applies and we conclude that $U \to V$ is quasi-finite at all points of U_v. Hence f is quasi-finite at every point lying over y.

Case (2). Let $u \in U$ be a generic point of an irreducible component whose image in V specializes to v. Then the image $x \in X^0$ of u has the property that $f(x) \rightsquigarrow y$. Hence we see that case (2) of Varieties, Lemma 33.17.1 applies and we conclude as before.

Case (3) follows from case (3) of Varieties, Lemma 33.17.1.

In case (4), since $|U| \to |X|$ is open, we see that the set of points where $U \to V$ is quasi-finite is dense as well. Hence case (4) of Varieties, Lemma 33.17.1 applies. □

0AD2 **Lemma 72.3.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper and Y is locally Noetherian. Let $y \in Y$ be a point of codimension ≤ 1 in Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied*

1. *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
2. *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
3. *f is quasi-finite at every $x \in X^0$,*
4. *f is quasi-finite at a dense set of points of $|X|$,*
5. *add more here.*

Then there exists an open subspace $Y' \subset Y$ containing y such that $Y' \times_Y X \to Y'$ is finite.

Proof. By Lemma 72.3.1 the morphism f is quasi-finite at every point lying over y. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point lying over y. Then $|X_{\overline{y}}|$ is a discrete space (Decent Spaces, Lemma 68.18.10). Since $X_{\overline{y}}$ is quasi-compact as f is proper we conclude that $|X_{\overline{y}}|$ is finite. Thus we can apply Cohomology of Spaces, Lemma 69.23.2 to conclude. □

0BBQ **Lemma 72.3.3.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $f : Y \to X$ be a birational proper morphism of algebraic spaces with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains*

1. *every point of codimension 0 in X,*
2. *every $x \in |X|$ of codimension 1 on X such that the local ring of X at x is normal (Properties of Spaces, Remark 66.7.6), and*
3. *every $x \in |X|$ such that the fibre of $|Y| \to |X|$ over x is finite and such that the local ring of X at x is normal.*

Proof. Part (1) follows from Decent Spaces, Lemma 68.22.5 (and the fact that the Noetherian algebraic spaces X and Y are quasi-separated and hence decent). Part (2) follows from part (3) and Lemma 72.3.2 (and the fact that finite morphisms have finite fibres). Let $x \in |X|$ be as in (3). By Cohomology of Spaces, Lemma 69.23.2 (which applies by Decent Spaces, Lemma 68.18.10) we may assume f is finite.

Choose an affine scheme X' and an étale morphism $X' \to X$ and a point $x' \in X$ mapping to x. It suffices to show there exists an open neighbourhood U' of $x' \in X'$ such that $Y \times_X X' \to X'$ is an isomorphism over U' (namely, then U contains the image of U' in X, see Spaces, Lemma 65.5.6). Then $Y \times_X X' \to X$ is a finite birational (Decent Spaces, Lemma 68.22.6) morphism. Since a finite morphism is affine we reduce to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. This is treated in Varieties, Lemma 33.17.3. □

72.4. Integral algebraic spaces

0AD3 We have not yet defined the notion of an integral algebraic space. The problem is that being integral is not an étale local property of schemes. We could use the property, that X is reduced and $|X|$ is irreducible, given in Properties, Lemma 28.3.4 to define integral algebraic spaces. In this case the algebraic space described in Spaces, Example 65.14.9 would be integral which does not seem right. To avoid this type of pathology we will in addition assume that X is a decent algebraic space, although perhaps a weaker alternative exists.

0AD4 **Definition 72.4.1.** Let S be a scheme. We say an algebraic space X over S is *integral* if it is reduced, decent, and $|X|$ is irreducible.

In this case the irreducible topological space $|X|$ is sober (Decent Spaces, Proposition 68.12.4). Hence it has a unique generic point x. In fact, in Decent Spaces, Lemma 68.20.4 we characterized decent algebraic spaces with finitely many irreducible components. Applying that lemma we see that an algebraic space X is integral if it is reduced, has an irreducible dense open subscheme X' with generic point x' and the morphism $x' \to X$ is quasi-compact.

0END **Lemma 72.4.2.** *Let S be a scheme. Let X be an integral algebraic space over S. Let $\eta \in |X|$ be the generic point of X. There are canonical identifications*

$$R(X) = \mathcal{O}^h_{X,\eta} = \kappa(\eta)$$

where $R(X)$ is the ring of rational functions defined in Morphisms of Spaces, Definition 67.47.3, $\kappa(\eta)$ is the residue field defined in Decent Spaces, Definition 68.11.2, and $\mathcal{O}^h_{X,\eta}$ is the henselian local ring defined in Decent Spaces, Definition 68.11.5. In particular, these rings are fields.

Proof. Since X is a scheme in an open neighbourhood of η (see discussion above), this follows immediately from the corresponding result for schemes, see Morphisms, Lemma 29.49.5. We also use: the henselianization of a field is itself and that our definitions of these objects for algebraic spaces are compatible with those for schemes. Details omitted. □

This leads to the following definition.

0ENE **Definition 72.4.3.** Let S be a scheme. Let X be an integral algebraic space over S. The *function field*, or the *field of rational functions* of X is the field $R(X)$ of Lemma 72.4.2.

We may occasionally indicate this field $k(X)$ instead of $R(X)$.

0BH2 **Lemma 72.4.4.** *Let S be a scheme. Let X be an integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. If $f, g \in R$ are nonzero and $fg = 0$ then $X = V(f) \cup V(g)$ where $V(f)$ denotes the closed subspace of X cut out by f. Since X is irreducible, we see that either $V(f) = X$ or $V(g) = X$. Then either $f = 0$ or $g = 0$ by Properties of Spaces, Lemma 66.21.4. □

Here is a lemma about normal integral algebraic spaces.

0AYH **Lemma 72.4.5.** *Let S be a scheme. Let X be a normal integral algebraic space over S. For every $x \in |X|$ there exists a normal integral affine scheme U and an étale morphism $U \to X$ such that x is in the image.*

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that x is in the image. Let u_i, $i \in I$ be the generic points of irreducible components of U. Then each u_i maps to the generic point of X (Decent Spaces, Lemma 68.20.1). By our definition of a decent space (Decent Spaces, Definition 68.6.1), we see that I is finite. Hence $U = \mathrm{Spec}(A)$ where A is a normal ring with finitely many minimal primes. Thus $A = \prod_{i \in I} A_i$ is a product of normal domains by Algebra, Lemma 10.37.16. Then $U = \coprod U_i$ with $U_i = \mathrm{Spec}(A_i)$ and x is in the image of $U_i \to X$ for some i. This proves the lemma. □

0BH3 **Lemma 72.4.6.** *Let S be a scheme. Let X be a normal integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Then R is a domain by Lemma 72.4.4. Let $f = a/b$ be an element of the fraction field of R which is integral over R. For any $U \to X$ étale with U a scheme there is at most one $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$. Namely, U is reduced and the generic points of U map to the generic point of X which implies that $b|_U$ is a nonzerodivisor. For every $x \in |X|$ we choose $U \to X$ as in Lemma 72.4.5. Then there is a unique $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$ because $\Gamma(U, \mathcal{O}_U)$ is a normal domain by Properties, Lemma 28.7.9. By the uniqueness mentioned above these f_U glue and define a global section f of the structure sheaf, i.e., of R. □

0ENF **Lemma 72.4.7.** *Let S be a scheme. Let X be a decent algebraic space over S. There are canonical bijections between the following sets:*

(1) *the set of points of X, i.e., $|X|$,*
(2) *the set of irreducible closed subsets of $|X|$,*
(3) *the set of integral closed subspaces of X.*

The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends Z to $|Z|$.

Proof. Our map defines a bijection between (1) and (2) as $|X|$ is sober by Decent Spaces, Proposition 68.12.4. Given $T \subset |X|$ closed and irreducible, there is a unique reduced closed subspace $Z \subset X$ such that $|Z| = T$, namely, Z is the reduced induced subspace structure on T, see Properties of Spaces, Definition 66.12.5. This is an integral algebraic space because it is decent, reduced, and irreducible. □

72.5. Morphisms between integral algebraic spaces

0ENG The following lemma characterizes dominant morphisms of finite degree between integral algebraic spaces.

0AD5 **Lemma 72.5.1.** *Let S be a scheme. Let X, Y be integral algebraic spaces over S Let $x \in |X|$ and $y \in |Y|$ be the generic points. Let $f : X \to Y$ be locally of finite type. Assume f is dominant (Morphisms of Spaces, Definition 67.18.1). The following are equivalent:*

(1) *the transcendence degree of x/y is 0,*
(2) *the extension $\kappa(x)/\kappa(y)$ (see proof) is finite,*
(3) *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite,*
(4) *f is quasi-finite at x, and*
(5) *x is the only point of $|X|$ mapping to y.*

If f is separated or if f is quasi-compact, then these are also equivalent to

(6) *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. By elementary topology, we see that $f(x) = y$ as f is dominant. Let $Y' \subset Y$ be the schematic locus of Y and let $X' \subset f^{-1}(Y')$ be the schematic locus of $f^{-1}(Y')$. By the discussion above, using Decent Spaces, Proposition 68.12.4 and Theorem 68.10.2, we see that $x \in |X'|$ and $y \in |Y'|$. Then $f|_{X'} : X' \to Y'$ is a morphism of integral schemes which is locally of finite type. Thus we see that (1), (2), (3) are equivalent by Morphisms, Lemma 29.51.7.

Condition (4) implies condition (1) by Morphisms of Spaces, Lemma 67.33.3 applied to $X \to Y \to Y$. On the other hand, condition (3) implies condition (4) as a finite morphism is quasi-finite and as $x \in U$ because x is the generic point. Thus (1) – (4) are equivalent.

Assume the equivalent conditions (1) – (4). Suppose that $x' \mapsto y$. Then $x \rightsquigarrow x'$ is a specialization in the fibre of $|X| \to |Y|$ over y. If $x' \neq x$, then f is not quasi-finite at x by Decent Spaces, Lemma 68.18.9. Hence $x = x'$ and (5) holds. Conversely, if (5) holds, then (5) holds for the morphism of schemes $X' \to Y'$ (see above) and we can use Morphisms, Lemma 29.51.7 to see that (1) holds.

Observe that (6) implies the equivalent conditions (1) – (5) without any further assumptions on f. To finish the proof we have to show the equivalent conditions (1) – (5) imply (6). This follows from Decent Spaces, Lemma 68.21.4. □

0AD6 **Definition 72.5.2.** Let S be a scheme. Let X and Y be integral algebraic spaces over S. Let $f : X \to Y$ be locally of finite type and dominant. Assume any of the equivalent conditions (1) – (5) of Lemma 72.5.1. Let $x \in |X|$ and $y \in |Y|$ be the generic points. Then the positive integer

$$\deg(X/Y) = [\kappa(x) : \kappa(y)]$$

is called the *degree of X over Y*.

0ENH **Lemma 72.5.3.** *Let S be a scheme. Let X, Y, Z be integral algebraic spaces over S. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume any of the equivalent conditions (1) – (5) of Lemma 72.5.1 hold for f and g. Then*

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 9.7.7. □

72.6. Weil divisors

0ENI This section is the analogue of Divisors, Section 31.26.

We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral algebraic spaces. Since we are not assuming our algebraic spaces are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. In the quasi-compact case our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subspaces are locally finite.

0EE5 **Lemma 72.6.1.** *Let S be a scheme and let X be a locally Noetherian algebraic space over S. If $T \subset |X|$ is a closed subset, then the collection of irreducible components of T is locally finite.*

Proof. The topological space $|X|$ is locally Noetherian (Properties of Spaces, Lemma 66.24.2). A Noetherian topological space has a finite number of irreducible components and a subspace of a Noetherian space is Noetherian (Topology, Lemma 5.9.2). Thus the lemma follows from the definition of locally finite (Topology, Definition 5.28.4). □

Let S be a scheme. Let X be a decent algebraic space over S. Let Z be an integral closed subspace of X and let $\xi \in |Z|$ be the generic point. Then the codimension of $|Z|$ in $|X|$ is equal to the dimension of the local ring of X at ξ by Decent Spaces, Lemma 68.20.2. Recall that we also indicate this by saying that ξ *is a point of codimension* 1 *on* X, see Properties of Spaces, Definition 66.10.2.

0ENJ **Definition 72.6.2.** Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S.

(1) A *prime divisor* is an integral closed subspace $Z \subset X$ of codimension 1, i.e., the generic point of $|Z|$ is a point of codimension 1 on X.
(2) A *Weil divisor* is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{|Z| : n_Z \neq 0\}$ is locally finite in $|X|$ (Topology, Definition 5.28.4).

The group of all Weil divisors on X is denoted $\mathrm{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we need to define the order of vanishing of a rational function on a locally Noetherian integral algebraic space X along a prime divisor Z. Let $\xi \in |Z|$ be the generic point. Here we run into the problem that the local ring $\mathcal{O}_{X,\xi}$ doesn't exist and the henselian local ring $\mathcal{O}^h_{X,\xi}$ may not be a domain, see Example 72.6.11. To get around this we use the following lemma.

0ENK **Lemma 72.6.3.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. Then the henselian local ring $\mathcal{O}^h_{X,\xi}$ is a reduced 1-dimensional Noetherian local ring and there is a canonical injective map*

$$R(X) \longrightarrow Q(\mathcal{O}^h_{X,\xi})$$

from the function field $R(X)$ of X into the total ring of fractions.

Proof. We will use the results of Decent Spaces, Section 68.11. Let $(U, u) \to (X, \xi)$ be an elementary étale neighbourhood. Observe that U is locally Noetherian and reduced. Thus $\mathcal{O}_{U,u}$ is a 1-dimensional (by our definition of prime divisors) reduced Noetherian ring. After replacing U by an affine open neighbourhood of u we may assume U is Noetherian and affine. After replacing U by a smaller open, we may assume every irreducible component of U passes through u. Since $U \to X$ is open and X irreducible, $U \to X$ is dominant. Hence we obtain a ring map $R(X) \to R(U)$ by composing rational maps, see Morphisms of Spaces, Section 67.47. Since $R(X)$ is a field, this map is injective. By our choice of U we see that $R(U)$ is the total quotient ring $Q(\mathcal{O}_{U,u})$, see Morphisms, Lemma 29.49.5 and Algebra, Lemma 10.25.4.

At this point we have proved all the statements in the lemma with $\mathcal{O}_{U,u}$ in stead of $\mathcal{O}^h_{X,\xi}$. However, $\mathcal{O}^h_{X,\xi}$ is the henselization of $\mathcal{O}_{U,u}$. Thus $\mathcal{O}^h_{X,\xi}$ is a 1-dimensional reduced Noetherian ring, see More on Algebra, Lemmas 15.45.4, 15.45.7, and 15.45.3. Since $\mathcal{O}_{U,u} \to \mathcal{O}^h_{X,\xi}$ is faithfully flat by More on Algebra, Lemma 15.45.1 it sends nonzerodivisors to nonzerodivisors. Therefore we obtain a canonical map $Q(\mathcal{O}_{U,u}) \to Q(\mathcal{O}^h_{X,\xi})$ and we obtain our map. We omit the verification that the map is independent of the choice of $(U, u) \to (X, x)$; a slightly better approach would be to first observe that $\operatorname{colim} Q(\mathcal{O}_{U,u}) = Q(\mathcal{O}^h_{X,\xi})$. □

0ENL **Definition 72.6.4.** Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the *order of vanishing of f along Z* as the integer

$$\operatorname{ord}_Z(f) = \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) - \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/b\mathcal{O}^h_{X,\xi})$$

where $a, b \in \mathcal{O}^h_{X,\xi}$ are nonzerodivisors such that the image of f in $Q(\mathcal{O}^h_{X,\xi})$ (Lemma 72.6.3) is equal to a/b. This is well defined by Algebra, Lemma 10.121.1.

If $\mathcal{O}^h_{X,\xi}$ happens to be a domain, then we obtain

$$\operatorname{ord}_Z(f) = \operatorname{ord}_{\mathcal{O}^h_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 10.121.2. Note that for $f, g \in R(X)^*$ we have

$$\operatorname{ord}_Z(fg) = \operatorname{ord}_Z(f) + \operatorname{ord}_Z(g).$$

Of course it can happen that $\operatorname{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\operatorname{ord}_Z(f) > 0$ is the *order of pole of f along Z*. It is important to note that the condition $\operatorname{ord}_Z(f) \geq 0$ is **not** equivalent to the condition $f \in \mathcal{O}^h_{X,\xi}$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

0ENM **Lemma 72.6.5.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. If the prime divisor $Z \subset X$ meets the schematic locus of X, then the order of vanishing $\operatorname{ord}_Z(f)$ of Definition 72.6.4 agrees with the order of vanishing of Divisors, Definition 31.26.3.*

Proof. After shrinking X we may assume X is an integral Noetherian scheme. If $\xi \in Z$ denotes the generic point, then we find that $\mathcal{O}^h_{X,\xi}$ is the henselization of $\mathcal{O}_{X,\xi}$ (Decent Spaces, Lemma 68.11.8). To prove the lemma it suffices and is necessary to show that

$$\operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/a\mathcal{O}_{X,\xi}) = \operatorname{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi})$$

This follows immediately from Algebra, Lemma 10.52.13 (and the fact that $\mathcal{O}_{X,\xi} \to \mathcal{O}^h_{X,\xi}$ is a flat local ring homomorphism of local Noetherian rings). □

0ENN **Lemma 72.6.6.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. Then the collections*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } f \text{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \text{ a prime divisor and } ord_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 72.6.1 gives the desired result. □

This lemma allows us to make the following definition.

0ENP **Definition 72.6.7.** Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. The *principal Weil divisor associated to f* is the Weil divisor

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\mathrm{ord}_Z(f)$ is as in Definition 72.6.4. This makes sense by Lemma 72.6.6.

0ENQ **Lemma 72.6.8.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f, g \in R(X)^*$. Then*

$$div_X(fg) = div_X(f) + div_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. □

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

0ENR **Definition 72.6.9.** Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. The *Weil divisor class group* of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\mathrm{Cl}(X)$.

By construction we obtain an exact complex

0ENS (72.6.9.1) $$R(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0$$

which we can think of as a presentation of $\mathrm{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

0ENT **Example 72.6.10.** This is a continuation of Morphisms of Spaces, Example 67.53.3. Consider the algebraic space $X = \mathbf{A}^1_k/\{t \sim -t \mid t \neq 0\}$. This is a smooth algebraic space over the field k. There is a universal homeomorphism

$$X \longrightarrow \mathbf{A}^1_k = \mathrm{Spec}(k[t])$$

which is an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. We conclude that X is Noetherian and integral. Since $\dim(X) = 1$, we see that the prime divisors of X are the closed points of X. Consider the unique closed point $x \in |X|$ lying over $0 \in \mathbf{A}^1_k$. Since

$X \setminus \{x\}$ maps isomorphically to $\mathbf{A}^1 \setminus \{0\}$ we see that the classes in $\mathrm{Cl}(X)$ of closed points different from x are zero. However, the divisor of t on X is $2[x]$. We conclude that $\mathrm{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$.

0ENU **Example 72.6.11.** Let k be a field. Let

$$U = \mathrm{Spec}(k[x, y]/(xy))$$

be the union of the coordinate axes in $\mathbf{A}^2_k$. Denote $\Delta : U \to U \times_k U$ the diagonal and $\Delta' : U \to U \times_k U$ the map $u \mapsto (u, \sigma(u))$ where $\sigma : U \to U$, $(x, y) \mapsto (y, x)$ is the automorphism flipping the coordinate axes. Set

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ is the origin. It is easy to see that R is an étale equivalence relation on U. The quotient $X = U/R$ is an algebraic space. The morphism $U \to \mathbf{A}^1_k$, $(x, y) \mapsto x + y$ is R-invariant and hence defines a morphism

$$X \longrightarrow \mathbf{A}^1_k$$

This morphism is a universal homeomorphism and an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. It follows that X is integral and Noetherian. Exactly as in Example 72.6.10 the reader shows that $\mathrm{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$ with generator corresponding to the unique closed point $x \in |X|$ mapping to $0 \in \mathbf{A}^1_k$. However, in this case the henselian local ring of X at x isn't a domain, as it is the henselization of $\mathcal{O}_{U,0_U}$.

72.7. The Weil divisor class associated to an invertible module

0ENV In this section we go through exactly the same progression as in Section 72.6 to define a canonical map $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ on a locally Noetherian integral algebraic space.

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. By Divisors on Spaces, Lemma 71.10.11 there exists a regular meromorphic section $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$. In fact, by Divisors on Spaces, Lemma 71.10.8 this is the same thing as a nonzero element in $\mathcal{L}_\eta$ where $\eta \in |X|$ is the generic point. The same lemma tells us that if $\mathcal{L} = \mathcal{O}_X$, then s is the same thing as a nonzero rational function on X (so what we will do below matches the construction in Section 72.6).

Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. We are going to define the order of vanishing of s along Z. Consider the canonical morphism

$$c_\xi : \mathrm{Spec}(\mathcal{O}^h_{X,\xi}) \longrightarrow X$$

whose source is the spectrum of the henselian local ring of X as ξ (Decent Spaces, Definition 68.11.7). The pullback $\mathcal{L}_\xi = c_\xi^*\mathcal{L}$ is an invertible module and hence trivial; choose a generator s_ξ of $\mathcal{L}_\xi$. Since c_ξ is flat, pullbacks of meromorphic functions and (regular) sections are defined for c_ξ, see Divisors on Spaces, Definition 71.10.6 and Lemmas 71.10.7 and 71.10.10. Thus we get

$$c_\xi^*(s) = f s_\xi$$

for some nonzerodivisor $f \in Q(\mathcal{O}^h_{X,\xi})$. Here we are using Divisors, Lemma 31.24.2 to identify the space of meromorphic sections of $\mathcal{L}_\xi \cong \mathcal{O}_{\mathrm{Spec}(\mathcal{O}^h_{X,\xi})}$ in terms of the total ring of fractions of $\mathcal{O}^h_{X,\xi}$. Let us agree to denote this element

$$s/s_\xi = f \in Q(\mathcal{O}^h_{X,\xi})$$

Observe that $f = s/s_\xi$ is replaced by uf where $u \in \mathcal{O}^h_{X,\xi}$ is a unit if we change our choice of s_ξ.

0EPR **Definition 72.7.1.** Let S be a scheme. Let X be a locally Noetherian integral algebraic algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ with generic point $\xi \in |Z|$ we define the *order of vanishing of s along Z* as the integer

$$\mathrm{ord}_{Z,\mathcal{L}}(s) = \mathrm{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) - \mathrm{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/b\mathcal{O}^h_{X,\xi})$$

where $a, b \in \mathcal{O}^h_{X,\xi}$ are nonzerodivisors such that the element s/s_ξ of $Q(\mathcal{O}^h_{X,\xi})$ constructed above is equal to a/b. This is well defined by the above and Algebra, Lemma 10.121.1.

As explained above, a regular meromorphic section s of $\mathcal{O}_X$ can be written $s = f \cdot 1$ where f is a nonzero rational function on X and we have $\mathrm{ord}_Z(f) = \mathrm{ord}_{Z,\mathcal{O}_X}(s)$. As in the case of principal divisors we have the following lemma.

0EPS **Lemma 72.7.2.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } ord_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 72.6.1. gives the desired result. □

0EPT **Lemma 72.7.3.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 72.7.2 guarantees that the sums are indeed Weil divisors. □

0EPU **Definition 72.7.4.** Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z] \in \mathrm{Div}(X)$$

where the sum is over prime divisors. This is well defined by Lemma 72.7.2.

(2) We define *Weil divisor class associated to $\mathcal{L}$* as the image of $\mathrm{div}_{\mathcal{L}}(s)$ in $\mathrm{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 72.7.3.

As expected this construction is additive in the invertible module.

0EPV **Lemma 72.7.5.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ and*

$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$

in $Div(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in |Z|$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$ with notation as described earlier in this section. Then $s_\xi \otimes t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$ in $Q(\mathcal{O}^h_{X,\xi})$. Applying the additivity of Algebra, Lemma 10.121.1 we conclude that

$$\mathrm{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \mathrm{div}_{\mathcal{L},Z}(s) + \mathrm{div}_{\mathcal{N},Z}(t)$$

Some details omitted. □

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. By the constructions and lemmas above we obtain a homomorphism of abelian groups

0EPW (72.7.5.1) $$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$$

which assigns to an invertible module its Weil divisor class.

0EPX **Lemma 72.7.6.** *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. If X is normal, then the map (72.7.5.1) $\mathrm{Pic}(X) \to Cl(X)$ is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\mathrm{div}_{\mathcal{L}}(s) = \mathrm{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\mathrm{div}_{\mathcal{L}}(t) = 0$, see Lemma 72.7.3. We claim that t defines a trivialization of $\mathcal{L}$. The claim finishes the proof of the lemma. Our proof of the claim is a bit awkward as we don't yet have a lot of theory at our dispposal; we suggest the reader skip the proof.

We may check our claim étale locally. Let $U \in X_{étale}$ be affine such that $\mathcal{L}|_U$ is trivial. Say $s_U \in \Gamma(U, \mathcal{L}|_U)$ is a trivialization. By Properties, Lemma 28.7.5 we may also assume U is integral. Write $U = \mathrm{Spec}(A)$ as the spectrum of a normal Noetherian domain A with fraction field K. We may write $t|_U = fs_U$ for some element f of K, see Divisors on Spaces, Lemma 71.10.4 for example. Let $\mathfrak{p} \subset A$ be a height one prime corresponding to a codimension 1 point $u \in U$ which maps to a codimension 1 point $\xi \in |X|$. Choose a trivialization s_ξ of $\mathcal{L}_\xi$ as in the beginning of this section. Choose a geometric point $\overline{u}$ of U lying over u. Then

$$(\mathcal{O}^h_{X,\xi})^{sh} = \mathcal{O}_{X,\overline{u}} = \mathcal{O}^{sh}_{U,u} = (A_{\mathfrak{p}})^{sh}$$

see Decent Spaces, Lemmas 68.11.9 and Properties of Spaces, Lemma 66.22.1. The normality of X shows that all of these are discrete valuation rings. The trivializations s_U and s_ξ differ by a unit as sections of $\mathcal{L}$ pulled back to $\mathrm{Spec}(\mathcal{O}_{X,\overline{u}})$.

Write $t = f_\xi s_\xi$ with $f_\xi \in Q(\mathcal{O}^h_{X,\xi})$. We conclude that f_ξ and f differ by a unit in $Q(\mathcal{O}_{X,\overline{u}})$. If $Z \subset X$ denotes the prime divisor corresponding to ξ (Lemma 72.4.7), then $0 = \mathrm{ord}_{Z,\mathcal{L}}(t) = \mathrm{ord}_{\mathcal{O}^h_{X,\xi}}(f_\xi)$ and since $\mathcal{O}^h_{X,\xi}$ is a discrete valuation ring we see that f_ξ is a unit. Thus f is a unit in $\mathcal{O}_{X,\overline{u}}$ and hence in particular $f \in A_{\mathfrak{p}}$. This implies $f \in A$ by Algebra, Lemma 10.157.6. We conclude that $t \in \Gamma(X, \mathcal{L})$. Repeating the argument with t^{-1} viewed as a meromorphic section of $\mathcal{L}^{\otimes -1}$ finishes the proof. □

72.8. Modifications and alterations

0AD7 Using our notion of an integral algebraic space we can define a modification as follows.

0AD8 **Definition 72.8.1.** Let S be a scheme. Let X be an integral algebraic space over S. A *modification of X* is a birational proper morphism $f : X' \to X$ of algebraic spaces over S with X' integral.

For birational morphisms of algebraic spaces, see Decent Spaces, Definition 68.22.1.

0AD9 **Lemma 72.8.2.** *Let $f : X' \to X$ be a modification as in Definition 72.8.1. There exists a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is an isomorphism.*

Proof. By Lemma 72.5.1 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Morphisms of Spaces, Proposition 67.32.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Morphisms of Spaces, Lemma 67.46.6). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. □

0ADA **Definition 72.8.3.** Let S be a scheme. Let X be an integral algebraic space over S. An *alteration of X* is a proper dominant morphism $f : Y \to X$ of algebraic spaces over S with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$.

If $f : Y \to X$ is a dominant and proper morphism between integral algebraic spaces, then it is an alteration as soon as the induced extension of residue fields in generic points is finite. Here is the precise statement.

0ADB **Lemma 72.8.4.** *Let S be a scheme. Let $f : X \to Y$ be a proper dominant morphism of integral algebraic spaces over S. Then f is an alteration if and only if any of the equivalent conditions (1) – (6) of Lemma 72.5.1 hold.*

Proof. Immediate consequence of the lemma referenced in the statement. □

0DMN **Lemma 72.8.5.** *Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Assume Y is integral. Then there exists an integral closed subspace $X' \subset X$ such that $f' = f|_{X'} : X' \to Y$ is an alteration.*

Proof. Let $V \subset Y$ be a nonempty open affine (Decent Spaces, Theorem 68.10.2). Let $\eta \in V$ be the generic point. Then X_η is a nonempty proper algebraic space over η. Choose a closed point $x \in |X_\eta|$ (exists because $|X_\eta|$ is a quasi-compact, sober topological space, see Decent Spaces, Proposition 68.12.4 and Topology, Lemma 5.12.8.) Let X' be the reduced induced closed subspace structure on $\overline{\{x\}} \subset |X|$ (Properties of Spaces, Definition 66.12.5. Then $f' : X' \to Y$ is surjective as the

image contains η. Also f' is proper as a composition of a closed immersion and a proper morphism. Finally, the fibre X'_η has a single point; to see this use Decent Spaces, Lemma 68.18.6 for both $X \to Y$ and $X' \to Y$ and the point η. Since Y is decent and $X' \to Y$ is separated we see that X' is decent (Decent Spaces, Lemmas 68.17.2 and 68.17.5). Thus f' is an alteration by Lemma 72.8.4. □

72.9. Schematic locus

06LY We have already proven a number of results on the schematic locus of an algebraic space. Here is a list of references:

(1) Properties of Spaces, Sections 66.13 and 66.14,
(2) Decent Spaces, Section 68.10,
(3) Properties of Spaces, Lemma 66.15.3 ⇐ Decent Spaces, Lemma 68.12.8 ⇐ Decent Spaces, Lemma 68.14.2,
(4) Limits of Spaces, Section 70.15, and
(5) Limits of Spaces, Section 70.17.

There are some cases where certain types of morphisms of algebraic spaces are automatically representable, for example separated, locally quasi-finite morphisms (Morphisms of Spaces, Lemma 67.51.1), and flat monomorphisms (More on Morphisms of Spaces, Lemma 76.4.1). In Section 72.10 we will study what happens with the schematic locus under extension of base field.

06LZ **Lemma 72.9.1.** *Let S be a scheme. Let X be an algebraic space over S. Assume X satisfies at least one of the following conditions*

(1) *X is quasi-separated and $\dim(X) = 0$,*
(2) *X is locally of finite type over a field k and $\dim(X) = 0$,*
(3) *X is Noetherian and $\dim(X) = 0$, or*
(4) *add more here.*

Then X is a separated scheme and any quasi-compact open of X is affine.

Proof. If we prove that any quasi-compact open of X is affine, then X is a separated scheme. Thus we may assume X is quasi-compact and we aim to show that X is affine. Cases (2) and (3) follow immediately from case (1) but we will give a separate proofs of (2) and (3) as these proofs use significantly less theory.

Proof of (3). Let U be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 66.9.2 we see that $\dim(U) = 0$ and $\dim(R) = 0$. Since R is a locally Noetherian scheme of dimension 0, we see that R is a disjoint union of spectra of Artinian local rings (Properties, Lemma 28.10.5). Since we assumed that X is Noetherian (so quasi-separated) we conclude that R is quasi-compact. Hence R is an affine scheme (use Schemes, Lemma 26.6.8). The étale morphisms $s, t : R \to U$ induce finite residue field extensions. Hence s and t are finite by Algebra, Lemma 10.54.4 (small detail omitted). Thus Groupoids, Proposition 39.23.9 shows that $X = U/R$ is an affine scheme.

Proof of (2) – almost identical to the proof of (3). Let U be an affine scheme and let $U \to X$ be a surjective étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 66.9.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to \mathrm{Spec}(k)$ is locally of finite type as the

composition of the étale morphism $U \to X$ and $X \to \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemmas 67.23.2 and 67.39.9. Similarly, $R \to \mathrm{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 33.20.2 we see that U and R are disjoint unions of spectra of local Artinian k-algebras finite over k. The same thing is therefore true of $U \times_{\mathrm{Spec}(k)} U$. As

$$R = U \times_X U \longrightarrow U \times_{\mathrm{Spec}(k)} U$$

is a monomorphism, we see that R is a finite(!) union of spectra of finite k-algebras. It follows that R is affine, see Schemes, Lemma 26.6.8. Applying Varieties, Lemma 33.20.2 once more we see that R is finite over k. Hence s, t are finite, see Morphisms, Lemma 29.44.14. Thus Groupoids, Proposition 39.23.9 shows that $X = U/R$ is an affine scheme.

Cohomological proof of (1). By Cohomology of Spaces, Lemma 69.10.1 we have vanishing of higher cohomology groups for all quasi-coherent sheaves $\mathcal{F}$ on X. Hence X is affine (in particular a scheme) by Cohomology of Spaces, Proposition 69.16.7.

Geometric proof of (1). Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and étale morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 68.8.6 (we will use all their properties below). Then $\dim(V_p) = 0$ by our definition of dimension of algebraic spaces. Thus Properties, Lemma 28.10.6 applies to each V_p. Then $f_p^{-1}(U_{p+1}) \subset V_p$ is quasi-compact open and hence is affine as well as closed. It follows that $|T_p| \subset |U_p|$ (see locus citatus) is open as well as closed. Hence X is a disjoint union of open and closed subspaces whose reduced structures are schemes. It follows that X is a scheme (Limits of Spaces, Lemma 70.15.3). Then the proof is finished by the case of schemes that we already referenced above. □

The following lemma tells us that a quasi-separated algebraic space is a scheme away from codimension 1.

0ADC **Lemma 72.9.2.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let $x \in |X|$. The following are equivalent*

(1) *x is a point of codimension 0 on X,*
(2) *the local ring of X at x has dimension 0, and*
(3) *x is a generic point of an irreducible component of $|X|$.*

If true, then there exists an open subspace of X containing x which is a scheme.

Proof. The equivalence of (1), (2), and (3) follows from Decent Spaces, Lemma 68.20.1 and the fact that a quasi-separated algebraic space is decent (Decent Spaces, Section 68.6). However in the next paragraph we will give a more elementary proof of the equivalence.

Note that (1) and (2) are equivalent by definition (Properties of Spaces, Definition 66.10.2). To prove the equivalence of (1) and (3) we may assume X is quasi-compact. Choose

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and $f_i : V_i \to U_i$ as in Decent Spaces, Lemma 68.8.6. Say $x \in U_i$, $x \notin U_{i+1}$. Then $x = f_i(y)$ for a unique $y \in V_i$. If (1) holds, then y is a generic point of an irreducible

component of V_i (Properties of Spaces, Lemma 66.11.1). Since $f_i^{-1}(U_{i+1})$ is a quasi-compact open of V_i not containing y, there is an open neighbourhood $W \subset V_i$ of y disjoint from $f_i^{-1}(V_i)$ (see Properties, Lemma 28.2.2 or more simply Algebra, Lemma 10.26.4). Then $f_i|_W : W \to X$ is an isomorphism onto its image and hence $x = f_i(y)$ is a generic point of $|X|$. Conversely, assume (3) holds. Then f_i maps $\overline{\{y\}}$ onto the irreducible component $\overline{\{x\}}$ of $|U_i|$. Since $|f_i|$ is bijective over $\overline{\{x\}}$, it follows that $\overline{\{y\}}$ is an irreducible component of U_i. Thus x is a point of codimension 0.

The final statement of the lemma is Properties of Spaces, Proposition 66.13.3. □

The following lemma says that a separated locally Noetherian algebraic space is a scheme in codimension 1, i.e., away from codimension 2.

0ADD **Lemma 72.9.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. If X is separated, locally Noetherian, and the dimension of the local ring of X at x is ≤ 1 (Properties of Spaces, Definition 66.10.2), then there exists an open subspace of X containing x which is a scheme.*

Proof. (Please see the remark below for a different approach avoiding the material on finite groupoids.) We can replace X by an quasi-compact neighbourhood of x, hence we may assume X is quasi-compact, separated, and Noetherian. There exists a scheme U and a finite surjective morphism $U \to X$, see Limits of Spaces, Proposition 70.16.1. Let $R = U \times_X U$. Then $j : R \to U \times_S U$ is an equivalence relation and we obtain a groupoid scheme (U, R, s, t, c) over S with s, t finite and U Noetherian and separated. Let $\{u_1, \ldots, u_n\} \subset U$ be the set of points mapping to x. Then $\dim(\mathcal{O}_{U,u_i}) \leq 1$ by Decent Spaces, Lemma 68.12.6.

By More on Groupoids, Lemma 40.14.10 there exists an R-invariant affine open $W \subset U$ containing the orbit $\{u_1, \ldots, u_n\}$. Since $U \to X$ is finite surjective the continuous map $|U| \to |X|$ is closed surjective, hence submersive by Topology, Lemma 5.6.5. Thus $f(W)$ is open and there is an open subspace $X' \subset X$ with $f : W \to X'$ a surjective finite morphism. Then X' is an affine scheme by Cohomology of Spaces, Lemma 69.17.3 and the proof is finished. □

0ADE **Remark 72.9.4.** Here is a sketch of a proof of Lemma 72.9.3 which avoids using More on Groupoids, Lemma 40.14.10.

Step 1. We may assume X is a reduced Noetherian separated algebraic space (for example by Cohomology of Spaces, Lemma 69.17.3 or by Limits of Spaces, Lemma 70.15.3) and we may choose a finite surjective morphism $Y \to X$ where Y is a Noetherian scheme (by Limits of Spaces, Proposition 70.16.1).

Step 2. After replacing X by an open neighbourhood of x, there exists a birational finite morphism $X' \to X$ and a closed subscheme $Y' \subset X' \times_X Y$ such that $Y' \to X'$ is surjective finite locally free. Namely, because X is reduced there is a dense open subspace $U \subset X$ over which Y is flat (Morphisms of Spaces, Proposition 67.32.1). Then we can choose a U-admissible blowup $b : \tilde{X} \to X$ such that the strict transform $\tilde{Y}$ of Y is flat over $\tilde{X}$, see More on Morphisms of Spaces, Lemma 76.39.1. (An alternative is to use Hilbert schemes if one wants to avoid using the result on blowups). Then we let $X' \subset \tilde{X}$ be the scheme theoretic closure of $b^{-1}(U)$ and $Y' = X' \times_{\tilde{X}} \tilde{Y}$. Since x is a codimension 1 point, we see that $X' \to X$ is finite over a neighbourhood of x (Lemma 72.3.2).

Step 3. After shrinking X to a smaller neighbourhood of x we get that X' is a scheme. This holds because Y' is a scheme and $Y' \to X'$ being finite locally free and because every finite set of codimension 1 points of Y' is contained in an affine open. Use Properties of Spaces, Proposition 66.14.1 and Varieties, Proposition 33.42.7.

Step 4. There exists an affine open $W' \subset X'$ containing all points lying over x which is the inverse image of an open subspace of X. To prove this let $Z \subset X$ be the closure of the set of points where $X' \to X$ is not an isomorphism. We may assume $x \in Z$ otherwise we are already done. Then x is a generic point of an irreducible component of Z and after shrinking X we may assume Z is an affine scheme (Lemma 72.9.2). Then the inverse image $Z' \subset X'$ is an affine scheme as well. Say $x_1, \ldots, x_n \in Z'$ are the points mapping to x. Then we can find an affine open W' in X' whose intersection with Z' is the inverse image of a principal open of Z containing x. Namely, we first pick an affine open $W' \subset X'$ containing $x_1, \ldots, x_n$ using Varieties, Proposition 33.42.7. Then we pick a principal open $D(f) \subset Z$ containing x whose inverse image $D(f|_{Z'})$ is contained in $W' \cap Z'$. Then we pick $f' \in \Gamma(W', \mathcal{O}_{W'})$ restricting to $f|_{Z'}$ and we replace W' by $D(f') \subset W'$. Since $X' \to X$ is an isomorphism away from $Z' \to Z$ the choice of W' guarantees that the image $W \subset X$ of W' is open with inverse image W' in X'.

Step 5. Then $W' \to W$ is a finite surjective morphism and W is a scheme by Cohomology of Spaces, Lemma 69.17.3 and the proof is complete.

72.10. Schematic locus and field extension

0B82 It can happen that a nonrepresentable algebraic space over a field k becomes representable (i.e., a scheme) after base change to an extension of k. See Spaces, Example 65.14.2. In this section we address this issue.

0B83 **Lemma 72.10.1.** *Let k be a field. Let X be an algebraic space over k. If there exists a purely inseparable field extension k'/k such that $X_{k'}$ is a scheme, then X is a scheme.*

Proof. The morphism $X_{k'} \to X$ is integral, surjective, and universally injective. Hence this lemma follows from Limits of Spaces, Lemma 70.15.4. □

0B84 **Lemma 72.10.2.** *Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-separated algebraic space over k.*

(1) *If there exists a field extension K/k such that X_K is a scheme, then $X_{\overline{k}}$ is a scheme.*
(2) *If X is quasi-compact and there exists a field extension K/k such that X_K is a scheme, then $X_{k'}$ is a scheme for some finite separable extension k' of k.*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we see that the first part of the lemma follows from the second part (some details omitted). Thus we assume X is quasi-compact and we assume given an extension K/k with X_K representable. Write $K = \bigcup A$ as the colimit of finitely generated k-subalgebras A. By Limits of Spaces, Lemma 70.5.11 we see that X_A is a scheme for some A. Choose a maximal ideal $\mathfrak{m} \subset A$. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A/\mathfrak{m}$ is a finite extension

of k. Thus we see that $X_{k'}$ is a scheme. If $k' \supset k$ is not separable, let $k'/k''/k$ be the subextension found in Fields, Lemma 9.14.6. Since k'/k'' is purely inseparable, by Lemma 72.10.1 the algebraic space $X_{k''}$ is a scheme. Since $k''|k$ is separable the proof is complete. □

0B86 **Lemma 72.10.3.** *Let k'/k be a finite Galois extension with Galois group G. Let X be an algebraic space over k. Then G acts freely on the algebraic space $X_{k'}$ and $X = X_{k'}/G$ in the sense of Properties of Spaces, Lemma 66.34.1.*

Proof. Omitted. Hints: First show that $\mathrm{Spec}(k) = \mathrm{Spec}(k')/G$. Then use compatibility of taking quotients with base change. □

0B87 **Lemma 72.10.4.** *Let S be a scheme. Let X be an algebraic space over S and let G be a finite group acting freely on X. Set $Y = X/G$ as in Properties of Spaces, Lemma 66.34.1. For $y \in |Y|$ the following are equivalent*

(1) *y is in the schematic locus of Y, and*
(2) *there exists an affine open $U \subset X$ containing the preimage of y.*

Proof. It follows from the construction of $Y = X/G$ in Properties of Spaces, Lemma 66.34.1 that the morphism $X \to Y$ is surjective and étale. Of course we have $X \times_Y X = X \times G$ hence the morphism $X \to Y$ is even finite étale. It is also surjective. Thus the lemma follows from Decent Spaces, Lemma 68.10.3. □

0B85 **Lemma 72.10.5.** *Let k be a field. Let X be a quasi-separated algebraic space over k. If there exists a purely transcendental field extension K/k such that X_K is a scheme, then X is a scheme.*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we may assume X is quasi-compact (some details omitted). Recall (Fields, Definition 9.26.1) that the assumption on the extension K/k signifies that K is the fraction field of a polynomial ring (in possibly infinitely many variables) over k. Thus $K = \bigcup A$ is the union of subalgebras each of which is a localization of a finite polynomial algebra over k. By Limits of Spaces, Lemma 70.5.11 we see that X_A is a scheme for some A. Write

$$A = k[x_1, \ldots, x_n][1/f]$$

for some nonzero $f \in k[x_1, \ldots, x_n]$.

If k is infinite then we can finish the proof as follows: choose $a_1, \ldots, a_n \in k$ with $f(a_1, \ldots, a_n) \neq 0$. Then $(a_1, \ldots, a_n)$ define an k-algebra map $A \to k$ mapping x_i to a_i and $1/f$ to $1/f(a_1, \ldots, a_n)$. Thus the base change $X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X$ is a scheme as desired.

In this paragraph we finish the proof in case k is finite. In this case we write $X = \lim X_i$ with X_i of finite presentation over k and with affine transition morphisms (Limits of Spaces, Lemma 70.10.2). Using Limits of Spaces, Lemma 70.5.11 we see that $X_{i,A}$ is a scheme for some i. Thus we may assume $X \to \mathrm{Spec}(k)$ is of finite presentation. Let $x \in |X|$ be a closed point. We may represent x by a closed immersion $\mathrm{Spec}(\kappa) \to X$ (Decent Spaces, Lemma 68.14.6). Then $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(k)$ is of finite type, hence κ is a finite extension of k (by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1; some details omitted). Say $[\kappa : k] = d$. Choose an integer $n \gg 0$ prime to d and let k'/k be the extension of degree n. Then k'/k is Galois with $G = \mathrm{Aut}(k'/k)$ cyclic of order n. If n is large enough there will be

k-algebra homomorphism $A \to k'$ by the same reason as above. Then $X_{k'}$ is a scheme and $X = X_{k'}/G$ (Lemma 72.10.3). On the other hand, since n and d are relatively prime we see that

$$\mathrm{Spec}(\kappa) \times_X X_{k'} = \mathrm{Spec}(\kappa) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') = \mathrm{Spec}(\kappa \otimes_k k')$$

is the spectrum of a field. In other words, the fibre of $X_{k'} \to X$ over x consists of a single point. Thus by Lemma 72.10.4 we see that x is in the schematic locus of X as desired. □

0BA7 **Remark 72.10.6.** Let k be a finite field. Let K/k be a geometrically irreducible field extension. Then K is the limit of geometrically irreducible finite type k-algebras A. Given A the estimates of Lang and Weil [**LW54**], show that for $n \gg 0$ there exists an k-algebra homomorphism $A \to k'$ with k'/k of degree n. Analyzing the argument given in the proof of Lemma 72.10.5 we see that if X is a quasi-separated algebraic space over k and X_K is a scheme, then X is a scheme. If we ever need this result we will precisely formulate it and prove it here.

0B88 **Lemma 72.10.7.** *Let k be a field with algebraic closure $\overline{k}$. Let X be an algebraic space over k such that*

(1) *X is decent and locally of finite type over k,*
(2) *$X_{\overline{k}}$ is a scheme, and*
(3) *any finite set of $\overline{k}$-rational points of $X_{\overline{k}}$ is contained in an affine.*

Then X is a scheme.

Proof. If K/k is an extension, then the base change X_K is decent (Decent Spaces, Lemma 68.6.5) and locally of finite type over K (Morphisms of Spaces, Lemma 67.23.3). By Lemma 72.10.1 it suffices to prove that X becomes a scheme after base change to the perfection of k, hence we may assume k is a perfect field (this step isn't strictly necessary, but makes the other arguments easier to think about). By covering X by quasi-compact opens we see that it suffices to prove the lemma in case X is quasi-compact (small detail omitted). In this case $|X|$ is a sober topological space (Decent Spaces, Proposition 68.12.4). Hence it suffices to show that every closed point in $|X|$ is contained in the schematic locus of X (use Properties of Spaces, Lemma 66.13.1 and Topology, Lemma 5.12.8).

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can find a closed immersion $\mathrm{Spec}(l) \to X$ representing x. Then $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ is of finite type (Morphisms of Spaces, Lemma 67.23.2) and we conclude that l is a finite extension of k by the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1). It is separable because k is perfect. Thus the scheme

$$\mathrm{Spec}(l) \times_X X_{\overline{k}} = \mathrm{Spec}(l) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(l \otimes_k \overline{k})$$

is the disjoint union of a finite number of $\overline{k}$-rational points. By assumption (3) we can find an affine open $W \subset X_{\overline{k}}$ containing these points.

By Lemma 72.10.2 we see that $X_{k'}$ is a scheme for some finite extension k'/k. After enlarging k' we may assume that there exists an affine open $U' \subset X_{k'}$ whose base change to $\overline{k}$ recovers W (use that $X_{\overline{k}}$ is the limit of the schemes $X_{k''}$ for $k' \subset k'' \subset \overline{k}$ finite and use Limits, Lemmas 32.4.11 and 32.4.13). We may assume that k'/k is a Galois extension (take the normal closure Fields, Lemma 9.16.3 and use that k is perfect). Set $G = \mathrm{Gal}(k'/k)$. By construction the G-invariant closed subscheme

$\mathrm{Spec}(l) \times_X X_{k'}$ is contained in U'. Thus x is in the schematic locus by Lemmas 72.10.3 and 72.10.4. □

The following two lemmas should go somewhere else. Please compare the next lemma to Decent Spaces, Lemma 68.18.8.

06S0 **Lemma 72.10.8.** *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is locally quasi-finite over k,*
(2) *X is locally of finite type over k and has dimension 0,*
(3) *X is a scheme and is locally quasi-finite over k,*
(4) *X is a scheme and is locally of finite type over k and has dimension 0, and*
(5) *X is a disjoint union of spectra of Artinian local k-algebras A over k with $\dim_k(A) < \infty$.*

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X. Hence by Morphisms of Spaces, Lemma 67.34.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 72.9.1 (and trivial implications) that (1) – (4) are equivalent. Finally, Varieties, Lemma 33.20.2 shows that (1) – (4) are equivalent with (5). □

06S1 **Lemma 72.10.9.** *Let k be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over k. If Y is locally quasi-finite over k so is X.*

Proof. Assume Y is locally quasi-finite over k. By Lemma 72.10.8 we see that $Y = \coprod \mathrm{Spec}(A_i)$ where each A_i is an Artinian local ring finite over k. By Decent Spaces, Lemma 68.19.1 we see that X is a scheme. Consider $X_i = f^{-1}(\mathrm{Spec}(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to prove. If X_i has one point, then $X_i = \mathrm{Spec}(B_i)$ with B_i a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 10.20.1 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus B_i is a finite local k-algebra, and we conclude by Lemma 72.10.8 that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. □

72.11. Geometrically reduced algebraic spaces

0DMP If X is a reduced algebraic space over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced algebraic spaces.

0DMQ **Definition 72.11.1.** Let k be a field. Let X be an algebraic space over k.

(1) Let $x \in |X|$ be a point. We say X is *geometrically reduced at x* if $\mathcal{O}_{X,\overline{x}}$ is geometrically reduced over k.
(2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

Observe that if X is geometrically reduced at x, then the local ring of X at x is reduced (Properties of Spaces, Lemma 66.22.6). Similarly, if X is geometrically reduced over k, then X is reduced (by Properties of Spaces, Lemma 66.21.4). The following lemma in particular implies this definition does not clash with the corresponding property for schemes over a field.

0DMR **Lemma 72.11.2.** *Let k be a field. Let X be an algebraic space over k. Let $x \in |X|$. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *for some étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u,*
(3) *for any étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u.*

Proof. Recall that the local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$, see Properties of Spaces, Lemma 66.22.1. By Varieties, Lemma 33.6.2 we find that U is geometrically reduced at u if and only if $\mathcal{O}_{U,u}$ is geometrically reduced over k. Thus we have to show: if A is a local k-algebra, then A is geometrically reduced over k if and only if A^{sh} is geometrically reduced over k. We check this using the definition of geometrically reduced algebras (Algebra, Definition 10.43.1). Let K/k be a field extension. Since $A \to A^{sh}$ is faithfully flat (More on Algebra, Lemma 15.45.1) we see that $A \otimes_k K \to A^{sh} \otimes_k K$ is faithfully flat (Algebra, Lemma 10.39.7). Hence if $A^{sh} \otimes_k K$ is reduced, so is $A \otimes_k K$ by Algebra, Lemma 10.164.2. Conversely, recall that A^{sh} is a colimit of étale A-algebra, see Algebra, Lemma 10.155.2. Thus $A^{sh} \otimes_k K$ is a filtered colimit of étale $A \otimes_k K$-algebras. We conclude by Algebra, Lemma 10.163.7. □

0DMS **Lemma 72.11.3.** *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically reduced,*
(2) *for some surjective étale morphism $U \to X$ where U is a scheme, U is geometrically reduced,*
(3) *for any étale morphism $U \to X$ where U is a scheme, U is geometrically reduced.*

Proof. Immediate from the definitions and Lemma 72.11.2. □

The notion isn't interesting in characteristic zero.

0E02 **Lemma 72.11.4.** *Let X be an algebraic space over a perfect field k (for example k has characteristic zero).*

(1) *For $x \in |X|$, if $\mathcal{O}_{X,\overline{x}}$ is reduced, then X is geometrically reduced at x.*
(2) *If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Algebra, Lemma 10.43.6 and the definition of a perfect field (Algebra, Definition 10.45.1). The second statement follows from the first. □

0E03 **Lemma 72.11.5.** *Let k be a field of characteristic $p > 0$. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically reduced over k,*
(2) *$X_{k'}$ is reduced for every field extension k'/k,*
(3) *$X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,*
(4) *$X_{k^{1/p}}$ is reduced,*
(5) *$X_{k^{perf}}$ is reduced, and*
(6) *$X_{\bar{k}}$ is reduced.*

Proof. Choose a surjective étale morphism $U \to X$ where U is a scheme. Via Lemma 72.11.3 the lemma follows from the result for U over k. See Varieties, Lemma 33.6.4. □

0E04 **Lemma 72.11.6.** *Let k be a field. Let X be an algebraic space over k. Let k'/k be a field extension. Let $x \in |X|$ be a point and let $x' \in |X_{k'}|$ be a point lying over x. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *$X_{k'}$ is geometrically reduced at x'.*

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. Choose an étale morphism $U \to X$ where U is a scheme and a point $u \in U$ mapping to $x \in |X|$. By Properties of Spaces, Lemma 66.4.3 we may choose a point $u' \in U_{k'} = U \times_X X_{k'}$ mapping to both u and x'. By Lemma 72.11.2 the lemma follows from the lemma for U, u, u' which is Varieties, Lemma 33.6.6. □

0E05 **Lemma 72.11.7.** *Let k be a field. Let $f : X \to Y$ be a morphism of algebraic spaces over k. Let $x \in |X|$ be a point with image $y \in |Y|$.*

(1) *if f is étale at x, then X is geometrically reduced at $x \Leftrightarrow Y$ is geometrically reduced at y,*
(2) *if f is surjective étale, then X is geometrically reduced $\Leftrightarrow Y$ is geometrically reduced.*

Proof. Part (1) is clear because $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{Y,\overline{y}}$ if f is étale at x. Part (2) follows immediately from part (1). □

72.12. Geometrically connected algebraic spaces

0A0Y If X is a connected algebraic space over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected algebraic spaces.

0A0Z **Definition 72.12.1.** Let X be an algebraic space over the field k. We say X is *geometrically connected* over k if the base change $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected algebraic spaces are nonempty.

0A10 **Lemma 72.12.2.** *Let X be an algebraic space over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, for any field extension k''/k there exists a common field extension k'''/k of k'/k and k''/k, see Fields, Lemma 9.6.9. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. □

0A11 **Lemma 72.12.3.** *Let k be a field. Let X, Y be algebraic spaces over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. Let $y \in |Y|$ be represented by a morphism $\mathrm{Spec}(K) \to Y$ where K is a field. The fibre of $|X \times_k Y| \to |Y|$ over y is the image of $|X_K| \to |X \times_k Y|$ by Properties of Spaces, Lemma 66.4.3. Thus these fibres are connected by our assumption that X is geometrically connected. By Morphisms of Spaces, Lemma 67.6.6 the map $|p|$ is open. Thus we may apply Topology, Lemma 5.7.6 to conclude. □

0A12 **Lemma 72.12.4.** *Let k'/k be an extension of fields. Let X be an algebraic space over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 10.48.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Varieties, Lemma 33.7.5. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 72.12.3. □

0A13 **Lemma 72.12.5.** *Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 72.12.4 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. □

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be an algebraic space over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

0A14 (72.12.5.1) $$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

0A15 **Lemma 72.12.6.** *Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

1. *there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
2. *there exists an open subgroup $H \subset Gal(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a quasi-compact open $W \subset U_{\overline{k}}$ whose image in $X_{\overline{k}}$ is V. This is possible because $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is continuous and because $|U_{\overline{k}}|$ has a basis of quasi-compact opens. We can apply Varieties, Lemma 33.7.9 to $W \subset U_{\overline{k}}$ to obtain the lemma. □

0A16 **Lemma 72.12.7.** *Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let X be an algebraic space over k. Let $\overline{T} \subset |X_{\overline{k}}|$ have the following properties*

(1) *$\overline{T}$ is a closed subset of $|X_{\overline{k}}|$,*
(2) *for every $\sigma \in Gal(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset |X|$ whose inverse image in $|X_{k'}|$ is $\overline{T}$.

Proof. Let $T \subset |X|$ be the image of $\overline{T}$. Since $|X_{\overline{k}}| \to |X|$ is surjective, the statement means that T is closed and that its inverse image is $\overline{T}$. Choose a scheme U and a surjective étale morphism $U \to X$. By the case of schemes (see Varieties, Lemma 33.7.10) there exists a closed subset $T' \subset |U|$ whose inverse image in $|U_{\overline{k}}|$ is the inverse image of $\overline{T}$. Since $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is surjective, we see that T' is the inverse image of T via $|U| \to |X|$. By our construction of the topology on $|X|$ this means that T is closed. In the same manner one sees that $\overline{T}$ is the inverse image of T. □

0A17 **Lemma 72.12.8.** *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically connected,*
(2) *for every finite separable field extension k'/k the algebraic space $X_{k'}$ is connected.*

Proof. This proof is identical to the proof of Varieties, Lemma 33.7.11 except that we replace Varieties, Lemma 33.7.7 by Lemma 72.12.5, we replace Varieties, Lemma 33.7.9 by Lemma 72.12.6, and we replace Varieties, Lemma 33.7.10 by Lemma 72.12.7. We urge the reader to read that proof in stead of this one.

It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 72.12.5 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty algebraic subspaces of $X_{\overline{k}}$.

Suppose that $W \subset X$ is any quasi-compact open subspace. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed subspaces of $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 72.12.6 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open subspace such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\overline{k}/k'/k$ and a decomposition $W_{0,k'} = U_0' \amalg V_0'$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U_0')_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V_0')_{\overline{k}}$. Let $H = \mathrm{Gal}(\overline{k}/k') \subset \mathrm{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open subspace $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed subspaces. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open subspaces, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 72.12.7 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. □

72.13. Geometrically irreducible algebraic spaces

0DMT Spaces, Example 65.14.9 shows that it is best not to think about irreducible algebraic spaces in complete generality[1]. For decent (for example quasi-separated) algebraic spaces this kind of disaster doesn't happen. Thus we make the following definition only under the assumption that our algebraic space is decent.

0DMU **Definition 72.13.1.** Let k be a field. Let X be a decent algebraic space over k. We say X is *geometrically irreducible* if the topological space $|X_{k'}|$ is irreducible[2] for any field extension k' of k.

Observe that $X_{k'}$ is a decent algebraic space (Decent Spaces, Lemma 68.6.5). Hence the topological space $|X_{k'}|$ is sober. Decent Spaces, Proposition 68.12.4.

72.14. Geometrically integral algebraic spaces

0DMV Recall that integral algebraic spaces are by definition decent, see Section 72.4.

0DMW **Definition 72.14.1.** Let X be an algebraic space over the field k. We say X is *geometrically integral* over k if the algebraic space $X_{k'}$ is integral (Definition 72.4.1) for every field extension k' of k.

In particular X is a decent algebraic space. We can relate this to being geometrically reduced and geometrically irreducible as follows.

0DMX **Lemma 72.14.2.** *Let k be a field. Let X be a decent algebraic space over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. This is an immediate consequence of the definitions because our notion of integral (in the presence of decency) is equivalent to reduced and irreducible. □

0DMY **Lemma 72.14.3.** *Let k be a field. Let X be a proper algebraic space over k.*

(1) *$A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,*
(2) *$A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of $|X|$,*

[1] To be sure, if we say "the algebraic space X is irreducible", we probably mean to say "the topological space $|X|$ is irreducible".

[2] An irreducible space is nonempty.

(3) *if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,*
(4) *if X is geometrically reduced, then k_i is finite separable over k,*
(5) *if X is geometrically connected, then A is geometrically irreducible over k,*
(6) *if X is geometrically irreducible, then A is geometrically irreducible over k,*
(7) *if X is geometrically reduced and connected, then $A = k$, and*
(8) *if X is geometrically integral, then $A = k$.*

Proof. By Cohomology of Spaces, Lemma 69.20.3 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local rings by Algebra, Lemma 10.53.2 and Algebra, Proposition 10.60.7. If $X = Y \amalg Z$ with Y and Z open subspaces of X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of $|X|$. Finally, as $|X|$ is a Noetherian topological space (by Morphisms of Spaces, Lemma 67.28.6 and Properties of Spaces, Lemma 66.24.2) its connected components are open. Hence the connected components of $|X|$ correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced (Properties of Spaces, Lemma 66.21.4). Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 10.25.1). This proves (3).

If X is geometrically reduced, then same thing is true for $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (see Cohomology of Spaces, Lemma 69.11.2 for equality). This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 10.44.2 and 10.44.4. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). □

0DMZ **Lemma 72.14.4.** *Let k be a field. Let X be a proper integral algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are both nonzero, then $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Let $s \in H^0(X, \mathcal{L})$ and $t \in H^0(X, \mathcal{L}^{\otimes -1})$ be nonzero sections. Let $x \in |X|$ be a point in the support of s. Choose an affine étale neighbourhood $(U, u) \to (X, x)$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $s|_U$ corresponds to a nonzero regular function on the reduced (because X is reduced) scheme U and hence is nonvanishing in a generic point of an irreducible component of U. By Decent Spaces, Lemma 68.20.1 we conclude that the generic point η of $|X|$ is in the support of s. The same is true for t. Then of course st must be nonzero because the local ring of X at η is a field (by aforementioned lemma the local ring has dimension zero, as X is reduced the

local ring is reduced, and Algebra, Lemma 10.25.1). However, we have seen that $K = H^0(X, \mathcal{O}_X)$ is a field in Lemma 72.14.3. Thus st is everywhere nonzero and we see that $s : \mathcal{O}_X \to \mathcal{L}$ is an isomorphism. □

72.15. Dimension

0EDA In this section we continue the discussion about dimension. Here is a list of previous material:

(1) dimension is defined in Properties of Spaces, Section 66.9,
(2) dimension of local ring is defined in Properties of Spaces, Section 66.10,
(3) a couple of results in Properties of Spaces, Lemmas 66.22.4 and 66.22.5,
(4) relative dimension is defined in Morphisms of Spaces, Section 67.33,
(5) results on dimension of fibres in Morphisms of Spaces, Section 67.34,
(6) a weak form of the dimension formula Morphisms of Spaces, Section 67.35,
(7) a result on smoothness and dimension Morphisms of Spaces, Lemma 67.37.10,
(8) dimension is $\dim(|X|)$ for decent spaces Decent Spaces, Lemma 68.12.5,
(9) quasi-finite maps and dimension Decent Spaces, Lemmas 68.12.6 and 68.12.7.

In More on Morphisms of Spaces, Section 76.31 we will discuss jumping of dimension in fibres of a finite type morphism.

0EDB **Lemma 72.15.1.** *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces. Then $\dim(X) \leq \dim(Y)$. If f is surjective then $\dim(X) = \dim(Y)$.*

Proof. Choose $V \to Y$ surjective étale with V a scheme. Then $U = X \times_Y V$ is a scheme and $U \to V$ is integral (and surjective if f is surjective). By Properties of Spaces, Lemma 66.22.5 we have $\dim(X) = \dim(U)$ and $\dim(Y) = \dim(V)$. Thus the result follows from the case of schemes which is Morphisms, Lemma 29.44.9. □

0EDC **Lemma 72.15.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *Y is locally Noetherian,*
(2) *X and Y are integral algebraic spaces,*
(3) *f is dominant, and*
(4) *f is locally of finite type.*

If $x \in |X|$ and $y \in |Y|$ are the generic points, then

$$\dim(X) \leq \dim(Y) + \textit{transcendence degree of } x/y.$$

If f is proper, then equality holds.

Proof. Recall that $|X|$ and $|Y|$ are irreducible sober topological spaces, see discussion following Definition 72.4.1. Thus the fact that f is dominant means that $|f|$ maps x to y. Moreover, $x \in |X|$ is the unique point at which the local ring of X has dimension 0, see Decent Spaces, Lemma 68.20.1. By Morphisms of Spaces, Lemma 67.35.1 we see that the dimension of the local ring of X at any point $x' \in |X|$ is at most the dimension of the local ring of Y at $y' = f(x')$ plus the transcendence degree of x/y. Since the dimension of X, resp. dimension of Y is the supremum of the dimensions of the local rings at x', resp. y' (Properties of Spaces, Lemma 66.10.3) we conclude the inequality holds.

Assume f is proper. Let $V \subset Y$ be a nonempty quasi-compact open subspace. If we can prove the equality for the morphism $f^{-1}(V) \to V$, then we get the equality for $X \to Y$. Thus we may assume that X and Y are quasi-compact. Observe that X is quasi-separated as a locally Noetherian decent algebraic space, see Decent Spaces, Lemma 68.14.1. Thus we may choose $Y' \to Y$ finite surjective where Y' is a scheme, see Limits of Spaces, Proposition 70.16.1. After replacing Y' by a suitable closed subscheme, we may assume Y' is integral, see for example the more general Lemma 72.8.5. By the same lemma, we may choose a closed subspace $X' \subset X \times_Y Y'$ such that X' is integral and $X' \to X$ is finite surjective. Now X' is also locally Noetherian (Morphisms of Spaces, Lemma 67.23.5) and we can use Limits of Spaces, Proposition 70.16.1 once more to choose a finite surjective morphism $X'' \to X'$ with X'' a scheme. As before we may assume that X'' is integral. Picture

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

By Lemma 72.15.1 we have $\dim(X'') = \dim(X)$ and $\dim(Y') = \dim(Y)$. Since X and Y have open neighbourhoods of x, resp. y which are schemes, we readily see that the generic points $x'' \in X''$, resp. $y' \in Y'$ are the unique points mapping to x, resp. y and that the residue field extensions $\kappa(x'')/\kappa(x)$ and $\kappa(y')/\kappa(y)$ are finite. This implies that the transcendence degree of x''/y' is the same as the transcendence degree of x/y. Thus the equality follows from the case of schemes whicn is Morphisms, Lemma 29.52.4. □

72.16. Spaces smooth over fields

06M0 This section is the analogue of Varieties, Section 33.25.

06M1 **Lemma 72.16.1.** *Let k be a field. Let X be an algebraic space smooth over k. Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2). Hence U is regular by Varieties, Lemma 33.25.3. By Properties of Spaces, Definition 66.7.2 this means that X is regular. □

07W4 **Lemma 72.16.2.** *Let k be a field. Let X be an algebraic space smooth over $\mathrm{Spec}(k)$. The set of $x \in |X|$ which are image of morphisms $\mathrm{Spec}(k') \to X$ with $k' \supset k$ finite separable is dense in $|X|$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2). Hence we can apply Varieties, Lemma 33.25.6 to see that the closed points of U whose residue fields are finite separable over k are dense. This implies the lemma by our definition of the topology on $|X|$. □

72.17. Euler characteristics

0DN0 In this section we prove some elementary properties of Euler characteristics of coherent sheaves on algebraic spaces proper over fields.

0DN1 **Definition 72.17.1.** Let k be a field. Let X be a proper algebraic over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of* $\mathcal{F}$ is the integer

$$\chi(X, \mathcal{F}) = \sum\nolimits_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Spaces, Lemma 69.7.3) and each of these spaces is finite dimensional (Cohomology of Spaces, Lemma 69.20.3). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

0DN2 **Lemma 72.17.2.** *Let k be a field. Let X be a proper algebraic space over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. □

0EDD **Lemma 72.17.3.** *Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Spaces, Lemmas 69.20.2 and 69.8.1. By Cohomology on Sites, Lemma 21.14.5 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + 1 = n$ (this is the meaning of convergence of the spectral sequence), we conclude. □

72.18. Numerical intersections

0DN3 In this section we play around with the Euler characteristic of coherent sheaves on proper algebraic spaces to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

0DN4 **Lemma 72.18.1.** *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map*

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the scheme theoretic support of $\mathcal{F}$.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 69.12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 21.50.1) and Cohomology of Spaces, Lemma 69.8.3. Since $|Z| = \mathrm{Supp}(\mathcal{F})$ we see that it suffices to show

$$P_{\mathcal{F}}(n_1, \ldots, n_r) : (n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(X)$. Let us say property $\mathcal{P}$ holds for the coherent $\mathcal{O}_X$-module $\mathcal{F}$ if the above is true.

We will prove this statement by devissage, more precisely we will check conditions (1), (2), and (3) of Cohomology of Spaces, Lemma 69.14.6 are satisfied.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. By Lemma 72.17.2 we have

$$P_{\mathcal{F}_2}(n_1, \ldots, n_r) = P_{\mathcal{F}_1}(n_1, \ldots, n_r) + P_{\mathcal{F}_3}(n_1, \ldots, n_r)$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then so does the third.

Condition (2) follows because $P_{\mathcal{F}^{\oplus m}}(n_1, \ldots, n_r) = mP_{\mathcal{F}}(n_1, \ldots, n_r)$.

Proof of (3). Let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. We have to find a coherent module $\mathcal{G}$ on X whose support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$. We will give two constructions: one using Chow's lemma and one using a finite cover by a scheme.

Proof existence $\mathcal{G}$ using a finite cover by a scheme. Choose $\pi : Z' \to Z$ finite surjective where Z' is a scheme, see Limits of Spaces, Proposition 70.16.1. Set $\mathcal{G} = i_*\pi_*\mathcal{O}_{Z'} = (i \circ \pi)_*\mathcal{O}_{Z'}$. Note that Z' is proper over k and that the support of $\mathcal{G}$ is Y (details omitted). We have

$$R(\pi \circ i)_*(\mathcal{O}_{Z'}) = \mathcal{G} \quad \text{and} \quad R(\pi \circ i)_*(\pi^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$$

The first equality holds because $i \circ \pi$ is affine (Cohomology of Spaces, Lemma 69.8.2) and the second equality follows from the first and the projection formula (Cohomology on Sites, Lemma 21.50.1). Using Leray (Cohomology on Sites, Lemma 21.14.6) we obtain

$$P_{\mathcal{G}}(n_1, \ldots, n_r) = \chi(Z', \pi^*i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

By the case of schemes (Varieties, Lemma 33.45.1) this is a numerical polynomial in $n_1, \ldots, n_r$ of degree at most $\dim(Z')$. We conclude because $\dim(Z') \leq \dim(Z) \leq \dim(X)$. The first inequality follows from Decent Spaces, Lemma 68.12.7.

Proof existence $\mathcal{G}$ using Chow's lemma. We apply Cohomology of Spaces, Lemma 69.18.1 to the morphism $Z \to \mathrm{Spec}(k)$. Thus we get a surjective proper morphism $f : Y \to Z$ over $\mathrm{Spec}(k)$ where Y is a closed subscheme of $\mathbf{P}^m_k$ for some m. After replacing Y by a closed subscheme we may assume that Y is integral and $f : Y \to Z$ is an alteration, see Lemma 72.8.5. Denote $\mathcal{O}_Y(n)$ the pullback of $\mathcal{O}_{\mathbf{P}^m_k}(n)$. Pick $n > 0$ such that $R^p f_* \mathcal{O}_Y(n) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 69.20.1. We claim that $\mathcal{G} = i_* f_* \mathcal{O}_Y(n)$ satisfies $\mathcal{P}$. Namely, by the case of schemes (Varieties, Lemma 33.45.1) we know that

$$(n_1, \ldots, n_r) \longmapsto \chi(Y, \mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(Y)$. On the other hand, by the projection formula (Cohomology on Sites, Lemma 21.50.1)

$$\begin{aligned} i_* R f_* \left(\mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) \right) &= i_* R f_* \mathcal{O}_Y(n) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r} \\ &= \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r} \end{aligned}$$

the last equality by our choice of n. By Leray (Cohomology on Sites, Lemma 21.14.6) we get

$$\chi(Y, \mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = P_{\mathcal{G}}(n_1, \ldots, n_r)$$

and we conclude because $\dim(Y) \leq \dim(Z) \leq \dim(X)$. The first inequality holds by Morphisms of Spaces, Lemma 67.35.2 and the fact that $Y \to Z$ is an alteration (and hence the induced extension of residue fields in generic points is finite). □

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

0EDE **Lemma 72.18.2.** *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(\mathit{Supp}(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $\mathit{Supp}(\mathcal{F})$ of dimension d. Let $\overline{x}_i$ be a geometric generic point of Z_i and set $m_i = \mathit{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$. Then*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. We first prove a slightly weaker statement. Namely, say $\dim(X) = N$ and let $X_i \subset X$ be the irreducible components of dimension N. Let $\overline{x}_i$ be a geometric generic point of X_i. The étale local ring $\mathcal{O}_{X,\overline{x}_i}$ is Noetherian of dimension 0, hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ the length

$$m_i(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$$

is an integer ≥ 0. We claim that

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i(\mathcal{F}) \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< N$. We will prove this using Cohomology of Spaces, Lemma 69.14.6. For any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have $E(\mathcal{F}) = E(\mathcal{F}') + E(\mathcal{F}'')$. This follows from additivity of Euler characteristics (Lemma 72.17.2) and additivity of lengths (Algebra, Lemma

10.52.3). This immediately implies properties (1) and (2) of Cohomology of Spaces, Lemma 69.14.6. Finally, property (3) holds because for $\mathcal{G} = \mathcal{O}_Z$ for any $Z \subset X$ irreducible reduced closed subspace. Namely, if $Z = Z_{i_0}$ for some i_0, then $m_i(\mathcal{G}) = \delta_{i_0 i}$ and we conclude $E(\mathcal{G}) = 0$. If $Z \neq Z_i$ for any i, then $m_i(\mathcal{G}) = 0$ for all i, $\dim(Z) < N$ and we get the result from Lemma 72.18.1.

Proof of the statement as in the lemma. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 69.12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 21.50.1) and Cohomology of Spaces, Lemma 69.8.3. Since $|Z| = \mathrm{Supp}(\mathcal{F})$ we see that $Z_i \subset Z$ for all i and we see that these are the irreducible components of Z of dimension d. We may and do think of $\overline{x}_i$ as a geometric point of Z. The map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ determines a surjection

$$\mathcal{O}_{X,\overline{x}_i} \to \mathcal{O}_{Z,\overline{x}_i}$$

Via this map we have an isomorphism of modules $\mathcal{G}_{\overline{x}_i} = \mathcal{F}_{\overline{x}_i}$ as $\mathcal{F} = i_*\mathcal{G}$. This implies that

$$m_i = \mathrm{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i}) = \mathrm{length}_{\mathcal{O}_{Z,\overline{x}_i}}(\mathcal{G}_{\overline{x}_i})$$

Thus we see that the expression in the lemma is equal to

$$\chi(Z, \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i \, \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

and the result follows from the discussion in the first paragraph (applied with Z in stead of X). □

0EDF **Definition 72.18.3.** Let k be a field. Let X be a proper algebraic space over k. Let $i : Z \to X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4. We prove a few lemmas for these intersection numbers.

0EDG **Lemma 72.18.4.** *In the situation of Definition 72.18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.*

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a $\mathbf{Z}$-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2}\ldots\binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. □

0EDH **Lemma 72.18.5.** *In the situation of Definition 72.18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have*

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 72.18.1 the function

$$(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}_i')^{\otimes n_i'} \otimes (\mathcal{L}_i'')^{\otimes n_i''} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d+1$ variables. □

0EDI **Lemma 72.18.6.** *In the situation of Definition 72.18.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = length_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{O}_{Z,\overline{x}_i})$ where $\overline{x}_i$ is a geometric generic point of Z_i. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 72.18.2 and the definitions. □

0EDJ **Lemma 72.18.7.** *Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $Z \subset Y$ be an integral closed subspace of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is as in Definition 72.5.2 or 0 if $\dim(f(Z)) < d$.

Proof. In the statement $f(Z) \subset X$ is the scheme theoretic image of f and it is also the reduced induced algebraic space structure on the closed subset $f(|Z|) \subset X$, see Morphisms of Spaces, Lemma 67.16.4. Then Z and $f(Z)$ are reduced, proper (hence decent) algebraic spaces over k, whence integral (Definition 72.4.1). The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 72.17.3 and the projection formula (Cohomology, Lemma 20.54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 72.18.1 and the result is true. Assume $\dim(f(Z)) = d$. Then by dimension theory (Lemma 72.15.2) we find that the equivalent conditions (1) – (5) of Lemma 72.5.1 hold. Thus $\deg(Z \to f(Z))$ is well defined. By the already used Lemma 72.5.1 we find $f : Z \to f(Z)$ is finite over a nonempty open V of $f(Z)$; after possibly shrinking V we may assume V is a scheme. Let $\xi \in V$ be the generic point. Thus $\deg(f : Z \to f(Z))$ the length of the stalk of $f_*\mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$ and the stalk of $R^i f_* \mathcal{O}_X$ at ξ is zero for $i > 0$ (for example by Cohomology of Spaces, Lemma 69.4.1). Thus the terms $\chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z))\chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 72.18.2. The desired result follows. □

0EDK **Lemma 72.18.8.** *Let k be a field. Let X be a proper algebraic space over k. Let $Z \subset X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity (Lemma 72.17.2) that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. □

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CHAPTER 73

Topologies on Algebraic Spaces

03Y4

73.1. Introduction

03Y5 In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [**Gro71**], [**BLR90**], [**LMB00**] and [**Knu71**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

73.2. The general procedure

03Y6 In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 34.2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S. (The above is entirely as prescribed in Topologies, Section 34.7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \to U \times_S U$ with $U, R \in \mathrm{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 65.9.1. Hence we can find a full subcategory $Spaces/S$ of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of $Spaces/S$. We fix a choice of such a category.

In the sections below, given a topology τ, the big site $(Spaces/S)_\tau$ (resp. the big site $(Spaces/X)_\tau$ of an algebraic space X over S) has as underlying category the category $Spaces/S$ (resp. the subcategory $Spaces/X$ of $Spaces/S$, see Categories, Example 4.2.13). The procedure for turning this into a site is as usual by defining a class of τ-coverings and using Sets, Lemma 3.11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site* $X_{étale}$ *of an algebraic space* X has already been defined in Properties of Spaces, Definition 66.18.1. Its objects are schemes étale over X, of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 34.4.8) is the site $X_{spaces,étale}$ of Properties of Spaces, Definition 66.18.2.

These two sites define the same topos, see Properties of Spaces, Lemma 66.18.3. We will not redefine these in this chapter; instead we will simply use them.

73.3. Zariski topology

03YD In Spaces, Section 65.12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaced by open immersions.

041G **Definition 73.3.1.** Let S be a scheme, and let X be an algebraic space over S. A *Zariski covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces over S such that each f_i is an open immersion and such that

$$|X| = \bigcup_{i\in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

041H **Lemma 73.3.2.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a Zariski covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a Zariski covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is a Zariski covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a Zariski covering.*

Proof. Omitted. □

73.4. Étale topology

03YC In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 34.4.

041E **Definition 73.4.1.** Let S be a scheme, and let X be an algebraic space over S. An *étale covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces over S such that each f_i is étale and such that

$$|X| = \bigcup_{i\in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

0DF1 **Lemma 73.4.2.** *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism (this follows from Morphisms, Lemma 29.36.9 via Spaces, Lemma 65.5.8 as immersions are representable). □

041F **Lemma 73.4.3.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a étale covering of X.*

(2) *If $\{X_i \to X\}_{i\in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a étale covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is a étale covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a étale covering.*

Proof. Omitted. □

The following lemma tells us that the sites $(Spaces/X)_{étale}$ and $(Spaces/X)_{smooth}$ have the same categories of sheaves.

0CFV **Lemma 73.4.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $\{X_i \to X\}_{i\in I}$ be a smooth covering of X. Then there exists an étale covering $\{U_j \to X\}_{j\in J}$ of X which refines $\{X_i \to X\}_{i\in I}$.*

Proof. First choose a scheme U and a surjective étale morphism $U \to X$. For each i choose a scheme W_i and a surjective étale morphism $W_i \to X_i$. Then $\{W_i \to X\}_{i\in I}$ is a smooth covering which refines $\{X_i \to X\}_{i\in I}$. Hence $\{W_i \times_X U \to U\}_{i\in I}$ is a smooth covering of schemes. By More on Morphisms, Lemma 37.38.7 we can choose an étale covering $\{U_j \to U\}$ which refines $\{W_i \times_X U \to U\}$. Then $\{U_j \to X\}_{j\in J}$ is an étale covering refining $\{X_i \to X\}_{i\in I}$. □

0DBX **Definition 73.4.5.** Let S be a scheme. A big étale site $(Spaces/S)_{étale}$ is any site constructed as follows:

(1) Choose a big étale site $(Sch/S)_{étale}$ as in Topologies, Section 34.4.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category $Spaces/S$ and the class of étale coverings of Definition 73.4.1.

Having defined this, we can localize to get the étale site of an algebraic space.

0DBY **Definition 73.4.6.** Let S be a scheme. Let $(Spaces/S)_{étale}$ be as in Definition 73.4.5. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{étale}$. Then the big étale site $(Spaces/X)_{étale}$ of X is the localization of the site $(Spaces/S)_{étale}$ at X introduced in Sites, Section 7.25.

Recall that given an algebraic space X over S as in the definition, we already have defined the small étale sites $X_{spaces,étale}$ and $X_{étale}$, see Properties of Spaces, Section 66.18. We will silently identify the corresponding topoi using the inclusion functor $X_{étale} \subset X_{spaces,étale}$ (Properties of Spaces, Lemma 66.18.3) and we will call it the small étale topos of X. Next, we establish some relationships between the topoi associated to these sites.

0DF2 **Lemma 73.4.7.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of $(Spaces/S)_{étale}$. The inclusion functor $Y_{spaces,étale} \to (Spaces/X)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(Y_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Spaces/X)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : Y_{spaces,étale} \to (Spaces/X)_{étale}$. In other words, given an étale morphism $j : U \to Y$ corresponding to an object of $Y_{spaces,étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. The category $Y_{spaces,étale}$ has fibre products and equalizers and u commutes with them. It is immediate that u cocontinuous. The functor u is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

0DF3 **Lemma 73.4.8.** *Let S be a scheme. Let X be an object of $(Spaces/S)_{étale}$. The inclusion functor $X_{spaces,étale} \to (Spaces/X)_{étale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites*

$$\pi_X : (Spaces/X)_{étale} \longrightarrow X_{spaces,étale}$$

and a morphism of topoi

$$i_X : Sh(X_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

such that $\pi_X \circ i_X = id$. Moreover, $i_X = i_{id_X}$ with i_{id_X} as in Lemma 73.4.7. In particular the functor $i_X^{-1} = \pi_{X,*}$ is described by the rule $i_X^{-1}(\mathcal{G})(U/X) = \mathcal{G}(U/X)$.

Proof. In this case the functor $u : X_{spaces,étale} \to (Spaces/X)_{étale}$, in addition to the properties seen in the proof of Lemma 73.4.7 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

0DF4 **Definition 73.4.9.** In the situation of Lemma 73.4.8 the functor $i_X^{-1} = \pi_{X,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we often denote $\mathcal{F}|_{X_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\begin{aligned}\mathrm{Mor}_{Sh(X_{étale})}(\mathcal{F}|_{X_{étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Spaces/X)_{étale})}(\mathcal{F}, i_{X,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(X_{étale})}(\mathcal{G}, \mathcal{F}|_{X_{étale}}) &= \mathrm{Mor}_{Sh((Spaces/X)_{étale})}(\pi_X^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{X,*}\mathcal{G})|_{X_{étale}} = \mathcal{G}$ and we have $(\pi_X^{-1}\mathcal{G})|_{X_{étale}} = \mathcal{G}$.

0DF5 **Lemma 73.4.10.** *Let S be a scheme. Let $f : Y \to X$ be a morphism in $(Spaces/S)_{étale}$. The functor*

$$u : (Spaces/Y)_{étale} \longrightarrow (Spaces/X)_{étale}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{étale} \longrightarrow (Spaces/Y)_{étale}, \quad (U \to X) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 73.4.7). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given

U/Y and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DF6 **Lemma 73.4.11.** *Let S be a scheme. Let $f : Y \to X$ be a morphism in $(Spaces/S)_{étale}$.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 73.4.7 and i_T as in Lemma 73.4.8.*
(2) *The functor $X_{spaces,étale} \to T_{spaces,étale}$, $(U \to X) \mapsto (U \times_X Y \to Y)$ is continuous and induces a morphism of sites*
$$f_{spaces,étale} : Y_{spaces,étale} \longrightarrow X_{spaces,étale}$$
The corresponding morphism of small étale topoi is denoted
$$f_{small} : Sh(Y_{étale}) \to Sh(X_{étale})$$
We have $f_{small,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$.
(3) *We have a commutative diagram of morphisms of sites*
$$\begin{array}{ccc} Y_{spaces,étale} & \xleftarrow{\pi_Y} & (Spaces/Y)_{étale} \\ {\scriptstyle f_{spaces,étale}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ X_{spaces,étale} & \xleftarrow{\pi_X} & (Spaces/X)_{étale} \end{array}$$
so that $f_{small} \circ \pi_Y = \pi_X \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_X \circ f_{big} \circ i_Y = \pi_X \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_Y$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : X_{spaces,étale} \to Y_{spaces,étale}$, $u(U \to X) = (U \times_X Y \to Y)$ was shown to give rise to a morphism of sites and correspong morphism of small étale topoi in Properties of Spaces, Lemma 66.18.8. The description of the pushforward is clear.

Part (3) follows because π_X and π_Y are given by the inclusion functors and $f_{spaces,étale}$ and f_{big} by the base change functors $U \mapsto U \times_X Y$.

Statement (4) follows from (3) by precomposing with i_Y. □

In the situation of the lemma, using the terminology of Definition 73.4.9 we have: for $\mathcal{F}$ a sheaf on the big étale site of Y
$$(f_{big,*}\mathcal{F})|_{X_{étale}} = f_{small,*}(\mathcal{F}|_{Y_{étale}}),$$
This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of Y, resp. X is given by $\pi_{Y,*}$, resp. $\pi_{X,*}$. A similar formula involving pullbacks and restrictions is false.

0DF7 **Lemma 73.4.12.** *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ in $(Spaces/S)_{étale}$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.4.10. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 73.4.11. □

0DF8 **Lemma 73.4.13.** *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

in $(Spaces/S)_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/X' that $U' \times_{X'} Y' = U' \times_X Y$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Spaces/Y)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{X'} Y')$ on $X'_{étale}$ (use Lemmas 73.4.7 and 73.4.10). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

0DF9 **Remark 73.4.14.** The sites $(Spaces/X)_{étale}$ and $X_{spaces,étale}$ come with structure sheaves. For the small étale site we have seen this in Properties of Spaces, Section 66.21. The structure sheaf $\mathcal{O}$ on the big étale site $(Spaces/X)_{étale}$ is defined by assigning to an object U the global sections of the structure sheaf of U. This makes sense because after all U is an algebraic space itself hence has a structure sheaf. Since $\mathcal{O}_U$ is a sheaf on the étale site of U, the presheaf $\mathcal{O}$ so defined satisfies the sheaf condition for coverings of U, i.e., $\mathcal{O}$ is a sheaf. We can upgrade the morphisms i_f, π_X, i_X, f_{small}, and f_{big} defined above to morphisms of ringed sites, respectively topoi. Let us deal with these one by one.

(1) In Lemma 73.4.7 denote $\mathcal{O}$ the structure sheaf on $(Spaces/X)_{étale}$. We have $(i_f^{-1}\mathcal{O})(U/Y) = \mathcal{O}_U(U) = \mathcal{O}_Y(U)$ by construction. Hence an isomorphism $i_f^\sharp : i_f^{-1}\mathcal{O} \to \mathcal{O}_Y$.
(2) In Lemma 73.4.8 it was noted that i_X is a special case of i_f with $f = \mathrm{id}_X$ hence we are back in case (1).
(3) In Lemma 73.4.8 the morphism π_X satisfies $(\pi_{X,*}\mathcal{O})(U) = \mathcal{O}(U) = \mathcal{O}_X(U)$. Hence we can use this to define $\pi_X^\sharp : \mathcal{O}_X \to \pi_{X,*}\mathcal{O}$.
(4) In Lemma 73.4.11 the extension of f_{small} to a morphism of ringed topoi was discussed in Properties of Spaces, Lemma 66.21.3.
(5) In Lemma 73.4.11 the functor f_{big}^{-1} is simply the restriction via the inclusion functor $(Spaces/Y)_{étale} \to (Spaces/X)_{étale}$. Let $\mathcal{O}_1$ be the structure sheaf on $(Spaces/X)_{étale}$ and let $\mathcal{O}_2$ be the structure sheaf on $(Spaces/Y)_{étale}$. We obtain a canonical isomorphism $f_{big}^\sharp : f_{big}^{-1}\mathcal{O}_1 \to \mathcal{O}_2$.

Moreover, with these definitions compositions work out correctly too. We omit giving a detailed statement and proof.

73.5. Smooth topology

03YB In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 34.5.

041C **Definition 73.5.1.** Let S be a scheme, and let X be an algebraic space over S. A *smooth covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic

spaces over S such that each f_i is smooth and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

0DFA **Lemma 73.5.2.** *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth (Morphisms of Spaces, Lemma 67.39.6), and Lemma 73.4.2. □

041D **Lemma 73.5.3.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a smooth covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a smooth covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a smooth covering.*

Proof. Omitted. □

To be continued...

73.6. Syntomic topology

03YA In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 34.6.

041A **Definition 73.6.1.** Let S be a scheme, and let X be an algebraic space over S. A *syntomic covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is syntomic and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

0DFB **Lemma 73.6.2.** *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic (Morphisms of Spaces, Lemma 67.37.8), and Lemma 73.5.2. □

041B **Lemma 73.6.3.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a syntomic covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a syntomic covering.*

(3) *If $\{X_i \to X\}_{i\in I}$ is a syntomic covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

To be continued...

73.7. Fppf topology

03Y7 In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 34.7.

03Y8 **Definition 73.7.1.** Let S be a scheme, and let X be an algebraic space over S. An *fppf covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces over S such that each f_i is flat and locally of finite presentation and such that

$$|X| = \bigcup\nolimits_{i\in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 34.7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

0DFC **Lemma 73.7.2.** *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation (Morphisms of Spaces, Lemmas 67.36.5 and 67.36.6) and Lemma 73.6.2. □

03Y9 **Lemma 73.7.3.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fppf covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fppf covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is an fppf covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fppf covering.*

Proof. Omitted. □

042T **Lemma 73.7.4.** *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fppf covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fppf covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fppf covering. □

0469 **Lemma 73.7.5.** *Let S be a scheme. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fppf covering of algebraic spaces over S. Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. This follows from Spaces, Lemma 65.5.9. See also Spaces, Remark 65.5.2 in case you are confused about the meaning of this lemma. □

0DBV **Definition 73.7.6.** Let S be a scheme. A big fppf site $(Spaces/S)_{fppf}$ is any site constructed as follows:

(1) Choose a big fppf site $(Sch/S)_{fppf}$ as in Topologies, Section 34.7.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category $Spaces/S$ and the class of fppf coverings of Definition 73.7.1.

Having defined this, we can localize to get the fppf site of an algebraic space.

0DBW **Definition 73.7.7.** Let S be a scheme. Let $(Spaces/S)_{fppf}$ be as in Definition 73.7.6. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{fppf}$. Then the big fppf site $(Spaces/X)_{fppf}$ of X is the localization of the site $(Spaces/S)_{fppf}$ at X introduced in Sites, Section 7.25.

Next, we establish some relationships between the topoi associated to these sites.

0DFD **Lemma 73.7.8.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor*

$$u : (Spaces/Y)_{fppf} \longrightarrow (Spaces/X)_{fppf}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{fppf} \longrightarrow (Spaces/Y)_{fppf}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{fppf}) \longrightarrow Sh((Spaces/X)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DFE **Lemma 73.7.9.** *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.7.8. □

73.8. The ph topology

0DFF In this section we define the ph topology. This is the topology generated by étale coverings and proper surjective morphisms, see Lemma 73.8.7.

0DFG **Definition 73.8.1.** Let S be a scheme and let X be an algebraic space over S. A *ph covering of X* is a family of morphisms $\{X_i \to X\}_{i \in I}$ of algebraic spaces over S such that f_i is locally of finite type and such that for every $U \to X$ with U affine there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{X_i \times_X U \to U\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a ph covering of schemes by Topologies, Definition 34.8.4.

0DFH **Lemma 73.8.2.** *Any fppf covering is a ph covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is a ph covering.*

Proof. We will show that an fppf covering is a ph covering, and then the rest follows from Lemma 73.7.2. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over a base scheme S. Let U be an affine scheme and let $U \to X$ be a morphism. We can refine the fppf covering $\{X_i \times_U U \to U\}_{i \in I}$ by an fppf covering $\{T_i \to U\}_{i \in I}$ where T_i is a scheme (Lemma 73.7.4). Then we can find a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining $\{T_i \to U\}_{i \in I}$ by More on Morphisms, Lemma 37.48.7 (and the definition of ph coverings for schemes). Thus $\{X_i \to X\}_{i \in I}$ is a ph covering by definition. □

0DFI **Lemma 73.8.3.** *Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Then $\{Y \to X\}$ is a ph covering.*

Proof. Let $U \to X$ be a morphism with U affine. By Chow's lemma (in the weak form given as Cohomology of Spaces, Lemma 69.18.1) we see that there is a surjective proper morphism of schemes $V \to U$ which factors through $Y \times_X U \to U$. Taking any finite affine open cover of V we obtain a standard ph covering of U refining $\{X \times_Y U \to U\}$ as desired. □

0DFJ **Lemma 73.8.4.** *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a ph covering of X.*
2. *If $\{X_i \to X\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a ph covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i \in I}$ a ph covering as in (3). By Morphisms of Spaces, Lemma 67.23.3 the morphisms $X' \times_X X_i \to X'$ are locally of finite type. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i \in I}$ means there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering. This proves (3).

Let $\{X_i \to X\}_{i \in I}$ and $\{X_{ij} \to X_i\}_{j \in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard ph coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 34.8.3 the family $\{Z_{j,l} \to Z\}$ can be refined by a standard ph covering. Hence we conclude that $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering. □

0DFK **Definition 73.8.5.** Let S be a scheme. A big ph site $(Spaces/S)_{ph}$ is any site constructed as follows:

(1) Choose a big ph site $(Sch/S)_{ph}$ as in Topologies, Section 34.8.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 73.2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category $Spaces/S$ and the class of ph coverings of Definition 73.8.1.

Having defined this, we can localize to get the ph site of an algebraic space.

0DFL **Definition 73.8.6.** Let S be a scheme. Let $(Spaces/S)_{ph}$ be as in Definition 73.8.5. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{ph}$. Then the big ph site $(Spaces/X)_{ph}$ of X is the localization of the site $(Spaces/S)_{ph}$ at X introduced in Sites, Section 7.25.

Here is the promised characterization of ph sheaves.

0DFM **Lemma 73.8.7.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $(Spaces/X)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for étale coverings, and*
(2) *if $f : V \to U$ is a proper surjective morphism of $(Spaces/X)_{ph}$, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(Spaces/X)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}$ restricting to s_i over T_i. Let $\{U_j \to T\}$ be an étale covering with U_j affine. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties. □

Next, we establish some relationships between the topoi associated to these sites.

0DFN **Lemma 73.8.8.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor*

$$u : (Spaces/Y)_{ph} \longrightarrow (Spaces/X)_{ph}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{ph} \longrightarrow (Spaces/Y)_{ph}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{ph}) \longrightarrow Sh((Spaces/X)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DFP **Lemma 73.8.9.** *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 73.8.8. □

0DK6 **Lemma 73.8.10.** *Let S be a scheme. Let X be an algebraic space over S. Let P be a property of objects in $(Spaces/X)_{fppf}$ such that whenever $\{U_i \to U\}$ is a covering in $(Spaces/X)_{fppf}$, then*

$$P(U_{i_0} \times_U \ldots \times_U U_{i_p}) \text{ for all } p \geq 0,\ i_0, \ldots, i_p \in I \Rightarrow P(U)$$

If $P(U)$ for all U affine and flat, locally of finite presentation over X, then $P(X)$.

Proof. Let U be a separated algebraic space locally of finite presentation over X. Then we can choose an étale covering $\{U_i \to U\}_{i \in I}$ with V_i affine. Since U is separated, we conclude that $U_{i_0} \times_U \ldots \times_U U_{i_p}$ is always affine. Whence $P(U_{i_0} \times_U \ldots \times_U U_{i_p})$ always. Hence $P(U)$ holds. Choose a scheme U which is a disjoint union of affines and a surjective étale morphism $U \to X$. Then $U \times_X \ldots \times_X U$ (with $p+1$ factors) is a separated algebraic space étale over X. Hence $P(U \times_X \ldots \times_X U)$ by the above. We conclude that $P(X)$ is true. □

73.9. Fpqc topology

03MP We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 34.9. We will show in Descent on Spaces, Proposition 74.4.1 that quasi-coherent sheaves descent along these.

03MQ **Definition 73.9.1.** Let S be a scheme, and let X be an algebraic space over S. An *fpqc covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \to X$ there exists a standard fpqc covering $\{g_j : Z_j \to Z\}_{j=1,\ldots,m}$ which refines the family $\{X_i \times_X Z \to Z\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : Z_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 34.9.12.

0DFQ **Lemma 73.9.2.** *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 73.7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering of algebraic spaces over S. By definition this means that the f_i are flat which checks the first condition of Definition 73.9.1. To check the second, let $V \to U$ be a morphism with V affine. We may choose an étale covering $\{V_{ij} \to V \times_U U_i\}$ with V_{ij} affine. Then the compositions $f_{ij} : V_{ij} \to V \times_U U_i \to V$ are flat and locally of finite presentation as compositions of such (Morphisms of Spaces, Lemmas 67.28.2, 67.30.3, 67.39.7,

and 67.39.8). Hence these morphisms are open (Morphisms of Spaces, Lemma 67.30.6) and we see that $|V| = \bigcup_{i\in I}\bigcup_{j\in J_i} f_{ij}(|V_{ij}|)$ is an open covering of $|V|$. Since $|V|$ is quasi-compact, this covering has a finite refinement. Say $V_{i_1j_1}, \ldots, V_{i_Nj_N}$ do the job. Then $\{V_{i_kj_k} \to V\}_{k=1,\ldots,N}$ is a standard fpqc covering of V refinining the family $\{U_i \times_U V \to V\}$. This finishes the proof. □

03MR **Lemma 73.9.3.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fpqc covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i\in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 67.30.4 the morphisms $X' \times_X X_i \to X'$ are flat. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i\in I}$ means there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \to X\}_{i\in I}$ and $\{X_{ij} \to X_i\}_{j\in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 34.9.11 the family $\{Z_{j,l} \to Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering. □

03MS **Lemma 73.9.4.** *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is a family of morphisms of algebraic spaces with target X. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering of U.*

Proof. If $\{X_i \to X\}_{i\in I}$ is an fpqc covering, then so is $\{U \times_X X_i \to U\}_{i\in I}$ by Lemma 73.9.3. Assume that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering. Let $h : Z \to X$ be a morphism from an affine scheme towards X. Then we see that $U \times_X Z \to Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens $W_1, \ldots, W_t$ of $U \times_X Z$ whose images cover Z. For each j we may apply the condition that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering to the morphism $W_j \to U$, and obtain a standard fpqc covering $\{W_{jl} \to W_j\}$ which refines $\{W_j \times_X X_i \to W_j\}_{i\in I}$. Hence $\{W_{jl} \to Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 34.9.11) which refines $\{Z \times_X X_i \to X\}$ and we win. □

0419 **Lemma 73.9.5.** *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fpqc covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fpqc covering. □

To be continued...

73.10. Other chapters

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CHAPTER 74

Descent and Algebraic Spaces

03YE

74.1. Introduction

03YF In the chapter on topologies on algebraic spaces (see Topologies on Spaces, Section 73.1) we introduced étale, fppf, smooth, syntomic and fpqc coverings of algebraic spaces. In this chapter we discuss what kind of structures over algebraic spaces can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**].

74.2. Conventions

041I The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

74.3. Descent data for quasi-coherent sheaves

04W2 This section is the analogue of Descent, Section 35.2 for algebraic spaces. It makes sense to read that section first.

04W3 **Definition 74.3.1.** Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on X_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{X_i \times_X X_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{02}^*\varphi_{ik}} & \mathrm{pr}_2^*\mathcal{F}_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & \mathrm{pr}_1^*\mathcal{F}_j & \end{array}$$

of $\mathcal{O}_{X_i \times_X X_j \times_X X_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{X_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the

diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^*\mathcal{F}_j \\ {\scriptstyle \mathrm{pr}_0^*\psi_i}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\psi_j} \\ \mathrm{pr}_0^*\mathcal{F}'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^*\mathcal{F}'_j \end{array}$$

commute.

04W4 **Lemma 74.3.2.** *Let S be a scheme. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of algebraic spaces over S with fixed targets. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.
(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
(3) *Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Let $g : U \to V$ be a morphism of algebraic spaces. The lemma above tells us that there is a well defined pullback functor between the categories of descent data relative to families of maps with target V and U provided there is a morphism between those families of maps which "lives over g".

04W5 **Definition 74.3.3.** Let S be a scheme. Let $\{U_i \to U\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{U \to U\}$ the *trivial descent datum*.
(2) The pullback of the trivial descent datum to $\{U_i \to U\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{U_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on U such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{U_i}, can)$.

04W6 **Lemma 74.3.4.** *Let S be a scheme. Let U be an algebraic space over S. Let $\{U_i \to U\}$ be a Zariski covering of U, see Topologies on Spaces, Definition 73.3.1. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to U\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_U$-modules to the category of descent data with respect to $\{U_i \to U\}$ is fully faithful.*

Proof. Omitted. □

74.4. Fpqc descent of quasi-coherent sheaves

04W7 The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

04W8 **Proposition 74.4.1.** *Let S be a scheme. Let $\{X_i \to X\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 73.9.1. Any descent datum on quasi-coherent sheaves for $\{X_i \to X\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data with respect to $\{X_i \to X\}$ is fully faithful.*

Proof. This is more or less a formal consequence of the corresponding result for schemes, see Descent, Proposition 35.5.2. Here is a strategy for a proof:

(1) The fact that $\{X_i \to X\}$ is a refinement of the trivial covering $\{X \to X\}$ gives, via Lemma 74.3.2, a functor $QCoh(\mathcal{O}_X) \to DD(\{X_i \to X\})$ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data for the given family.
(2) In order to prove the proposition we will construct a quasi-inverse functor $back : DD(\{X_i \to X\}) \to QCoh(\mathcal{O}_X)$.
(3) Applying again Lemma 74.3.2 we see that there is a functor $DD(\{X_i \to X\}) \to DD(\{T_j \to X\})$ if $\{T_j \to X\}$ is a refinement of the given family. Hence in order to construct the functor *back* we may assume that each X_i is a scheme, see Topologies on Spaces, Lemma 73.9.5. This reduces us to the case where all the X_i are schemes.
(4) A quasi-coherent sheaf on X is by definition a quasi-coherent $\mathcal{O}_X$-module on $X_{étale}$. Now for any $U \in \mathrm{Ob}(X_{étale})$ we get an fppf covering $\{U_i \times_X X_i \to U\}$ by schemes and a morphism $g : \{U_i \times_X X_i \to U\} \to \{X_i \to X\}$ of coverings lying over $U \to X$. Given a descent datum $\xi = (\mathcal{F}_i, \varphi_{ij})$ we obtain a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_{\xi,U}$ corresponding to the pullback $g^*\xi$ of Lemma 74.3.2 to the covering of U and using effectivity for fppf covering of schemes, see Descent, Proposition 35.5.2.
(5) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is functorial in ξ. Omitted.
(6) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is compatible with morphisms $U \to U'$ of the site $X_{étale}$, so that the system of sheaves $\mathcal{F}_{\xi,U}$ corresponds to a quasi-coherent $\mathcal{F}_\xi$ on $X_{étale}$, see Properties of Spaces, Lemma 66.29.3. Details omitted.
(7) Check that $back : \xi \mapsto \mathcal{F}_\xi$ is quasi-inverse to the functor constructed in (1). Omitted.

This finishes the proof. □

74.5. Quasi-coherent modules and affines

0H02 Let S be a scheme. Let X be an algebraic space over S. Recall that $X_{affine,étale}$ is the full subcategory of $X_{étale}$ whose objects are affine turned into a site by declaring the coverings to be the standard étale coverings. See Properties of Spaces, Definition 66.18.5. By Properties of Spaces, Lemma 66.18.6 we have an equivalence of topoi $g : Sh(X_{affine,étale}) \to Sh(X_{étale})$ whose pullback functor is given by restriction. Recall that $\mathcal{O}_X$ denotes the structure sheaf on $X_{étale}$. Then we obtain an equivalence

0H03 (74.5.0.1) $$(Sh(X_{affine,étale}), \mathcal{O}_X|_{X_{affine,étale}}) \longrightarrow (Sh(X_{étale}), \mathcal{O}_X)$$

of ringed topoi. We will often write $\mathcal{O}_X$ in stead of $\mathcal{O}_X|_{X_{affine,étale}}$. Having said this we can compare quasi-coherent modules as well.

0H04 **Lemma 74.5.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules on $X_{affine,étale}$. The following are equivalent*

1. *for every morphism $U \to U'$ of $X_{affine,étale}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}_X(U')} \mathcal{O}_X(U) \to \mathcal{F}(U)$ is an isomorphism,*
2. *$\mathcal{F}$ is a quasi-coherent module on the ringed site $(X_{affine,étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 18.23.1,*
3. *$\mathcal{F}$ corresponds to a quasi-coherent module on X via the equivalence (74.5.0.1),*

Proof. Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $X_{affine,étale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$, reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}_X(U) \to \prod \mathcal{O}_X(U_i)$ is faithfully flat (by Descent, Lemma 35.9.1 and the fact that coverings in $X_{affine,étale}$ are standard étale coverings) and we may apply Descent, Lemma 35.3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $X_{affine,étale}$. Namely, for U in $X_{affine,étale}$, set $R = \mathcal{O}_X(U)$ and choose a presentation

$$\bigoplus\nolimits_{k\in K} R \longrightarrow \bigoplus\nolimits_{l\in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $X_{affine,étale}$ we obtain a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_X(U') \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_X(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $X_{affine,etale}/U$ has a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_X|_{X_{affine,étale}/U} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_X|_{X_{affine,étale}/U} \longrightarrow \mathcal{F}|_{X_{affine,étale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (74.5.0.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

Let us assume (3) and prove (1). Namely, let $\mathcal{G}$ be a quasi-coherent module on X corresponding to $\mathcal{F}$. Let $h : U \to U' \to X$ be a morphism of $X_{affine,étale}$. Denote $f : U \to X$ and $f' : U' \to X$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3. □

74.6. Descent of finiteness properties of modules

060T This section is the analogue for the case of algebraic spaces of Descent, Section 35.7. The goal is to show that one can check a quasi-coherent module has a certain finiteness conditions by checking on the members of a covering. We will repeatedly use the following proof scheme. Suppose that X is an algebraic space, and that $\{X_i \to X\}$ is a fppf (resp. fpqc) covering. Let $U \to X$ be a surjective étale morphism

such that U is a scheme. Then there exists an fppf (resp. fpqc) covering $\{Y_j \to X\}$ such that

(1) $\{Y_j \to X\}$ is a refinement of $\{X_i \to X\}$,
(2) each Y_j is a scheme, and
(3) each morphism $Y_j \to X$ factors though U, and
(4) $\{Y_j \to U\}$ is an fppf (resp. fpqc) covering of U.

Namely, first refine $\{X_i \to X\}$ by an fppf (resp. fpqc) covering such that each X_i is a scheme, see Topologies on Spaces, Lemma 73.7.4, resp. Lemma 73.9.5. Then set $Y_i = U \times_X X_i$. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type, of finite presentation, etc if and only if the quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}|_U$ is of finite type, of finite presentation, etc. Hence we can use the existence of the refinement $\{Y_j \to X\}$ to reduce the proof of the following lemmas to the case of schemes. We will indicate this by saying that "*the result follows from the case of schemes by étale localization*".

060U **Lemma 74.6.1.** *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.1, by étale localization. □

060V **Lemma 74.6.2.** *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.3, by étale localization. □

060W **Lemma 74.6.3.** *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.5, by étale localization. □

060X **Lemma 74.6.4.** *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.6, by étale localization. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties of Spaces, Section 66.31. It is also proved there that this notion is preserved under pullback.

060Y **Lemma 74.6.5.** *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 35.7.7, by étale localization. □

We also add here two results which are related to the results above, but are of a slightly different nature.

060Z **Lemma 74.6.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 35.7.9. □

0610 **Lemma 74.6.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 35.7.10. □

74.7. Fpqc coverings

04P0 This section is the analogue of Descent, Section 35.13. At the moment we do not know if all of the material for fpqc coverings of schemes holds also for algebraic spaces.

04P1 **Lemma 74.7.1.** *Let S be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Suppose that for each i we have an open subspace $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subspaces of $T_i \times_T T_j$. Then there exists a unique open subspace $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. By Topologies on Spaces, Lemma 73.9.5 we may assume each T_i is a scheme. Choose a scheme U and a surjective étale morphism $U \to T$. Then $\{T_i \times_T U \to U\}$ is an fpqc covering of U and $T_i \times_T U$ is a scheme for each i. Hence we see that the collection of opens $W_i \times_T U$ comes from a unique open subscheme $W' \subset U$ by Descent, Lemma 35.13.6. As $U \to X$ is open we can define $W \subset X$ the Zariski open which is the image of W', see Properties of Spaces, Section 66.4. We omit the verification that this works, i.e., that W_i is the inverse image of W for each i. □

04P2 **Lemma 74.7.2.** *Let S be a scheme. Let $\{T_i \to T\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 73.9.1. Then given an algebraic space B over S the sequence*

$$\mathrm{Mor}_S(T, B) \longrightarrow \prod_i \mathrm{Mor}_S(T_i, B) \rightrightarrows \prod_{i,j} \mathrm{Mor}_S(T_i \times_T T_j, B)$$

is an equalizer diagram. In other words, every representable functor on the category of algebraic spaces over S satisfies the sheaf condition for fpqc coverings.

Proof. We know this is true if $\{T_i \to T\}$ is an fpqc covering of schemes, see Properties of Spaces, Proposition 66.17.1. This is the key fact and we encourage the reader to skip the rest of the proof which is formal. Choose a scheme U and a surjective étale morphism $U \to T$. Let U_i be a scheme and let $U_i \to T_i \times_T U$ be a surjective étale morphism. Then $\{U_i \to U\}$ is an fpqc covering. This follows from Topologies on Spaces, Lemmas 73.9.3 and 73.9.4. By the above we have the result for $\{U_i \to U\}$.

What this means is the following: Suppose that $b_i : T_i \to B$ is a family of morphisms with $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_j \to B$. Then we let $a_i : U_i \to B$ be the composition of $U_i \to T_i$ with b_i. By what was said above we find a unique morphism $a : U \to B$ such that a_i is the composition of a with $U_i \to U$. The uniqueness guarantees that $a \circ \mathrm{pr}_0 = a \circ \mathrm{pr}_1$ as morphisms $U \times_T U \to B$. Then since $T = U/(U \times_T U)$ as a sheaf, we find that a comes from a unique morphism $b : T \to B$. Chasing diagrams we find that b is the morphism we are looking for. □

74.8. Descent of finiteness and smoothness properties of morphisms

06NQ The following type of lemma is occasionally useful.

06NR **Lemma 74.8.1.** *Let S be a scheme. Let $X \to Y \to Z$ be morphism of algebraic spaces. Let P be one of the following properties of morphisms of algebraic spaces over S: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $X \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$. Then $Y \to Z$ is P.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By assumption we can find an fppf covering $\{V_i \to V\}$ and lifts $V_i \to X$ of the morphism $V_i \to Y$. Since $U \to X$ is surjective étale we see that over the members of the fppf covering $\{V_i \times_X U \to V\}$ we have lifts into U. Hence $U \to V$ induces a surjection of sheaves on $(Sch/S)_{fppf}$. By our definition of what it means to have property P for a morphism of algebraic spaces (see Morphisms of Spaces, Definition 67.30.1, Definition 67.23.1, and Definition 67.28.1) we see that $U \to W$ has P and we have to show $V \to W$ has P. Thus we reduce the question to the case of morphisms of schemes which is treated in Descent, Lemma 35.14.8. □

A more standard case of the above lemma is the following. (The version with "flat" follows from Morphisms of Spaces, Lemma 67.31.5.)

0AHC **Lemma 74.8.2.** *Let S be a scheme. Let*

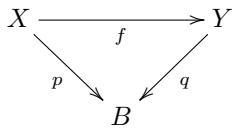

be a commutative diagram of morphisms of algebraic spaces over S. Assume that f is surjective, flat, and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. Since $\{X \to Y\}$ is an fppf covering, it induces a surjection of fppf sheaves (Topologies on Spaces, Lemma 73.7.5) and the lemma is a special case of Lemma 74.8.1. On the other hand, an easier argument is to deduce it from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.23.4 and 67.28.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.3. □

0AHD **Lemma 74.8.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) *f is surjective, and syntomic (resp. smooth, resp. étale),*
(2) *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.36.4, 67.37.4, and 67.39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.4. □

Actually we can strengthen this result as follows.

0AHE **Lemma 74.8.4.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 67.37.4 and 67.39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.5. □

0AHF **Lemma 74.8.5.** *Let S be a scheme. Let*

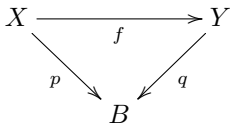

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is syntomic.*

Then both q and f are syntomic.

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemma 67.36.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 67.30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 35.14.7. □

74.9. Descending properties of spaces

06DP In this section we put some results of the following kind.

06DQ **Lemma 74.9.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. If f is flat at x and X is geometrically unibranch at x, then Y is geometrically unibranch at $f(x)$.*

Proof. Consider the map of étale local rings $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$. By Morphisms of Spaces, Lemma 67.30.8 this is flat. Hence if $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime, so does $\mathcal{O}_{Y,f(\overline{x})}$ (by going down, see Algebra, Lemma 10.39.19). □

06MI **Lemma 74.9.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat and surjective and X is reduced, then Y is reduced.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat, the morphism of schemes $U \to V$ is surjective and flat. In this way we reduce the problem to the case of schemes (as reducedness of X and Y is defined in terms of reducedness of U and V, see Properties of Spaces, Section 66.7). The case of schemes is Descent, Lemma 35.19.1. □

06MJ **Lemma 74.9.3.** *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is locally Noetherian, then Y is locally Noetherian.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective, flat, and locally of finite presentation the morphism of schemes $U \to V$ is surjective, flat, and locally of finite presentation. In this way we reduce the problem to the case of schemes (as being locally Noetherian for X and Y is defined in terms of being locally Noetherian of U and V, see Properties of Spaces, Section 66.7). In the case of schemes the result follows from Descent, Lemma 35.16.1. □

06MK **Lemma 74.9.4.** *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. By Lemma 74.9.3 we know that Y is locally Noetherian. Choose a scheme V and a surjective étale morphism $V \to Y$. It suffices to prove that the local rings of V are all regular local rings, see Properties, Lemma 28.9.2. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat the morphism of schemes $U \to V$ is surjective and flat. By assumption U is a regular scheme in particular all of its local rings are regular (by the lemma above). Hence the lemma follows from Algebra, Lemma 10.110.9. □

0GB3 **Lemma 74.9.5.** *Let $f : X \to Y$ be a smooth morphism of algebraic spaces. If Y is reduced, then X is reduced. If f is surjective and X is reduced, then Y is reduced.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes, the vertical arrows are surjective and étale, and $U \to X \times_Y V$ is surjective étale. Observe that X is a reduced algebraic space if and only if U is a reduced scheme by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7. Similarly for Y and V. The morphism $U \to V$ is a smooth morphism of schemes, see Morphisms of Spaces, Lemma 67.37.4. Since being reduced is local for the smooth topology for schemes (Descent, Lemma 35.18.1) we see that U is reduced if V is reduced. On the other hand, if $X \to Y$ is surjective, then $U \to V$ is surjective and in this case if U is reduced, then V is reduced. □

74.10. Descending properties of morphisms

03YG In this section we introduce the notion of when a property of morphisms of algebraic spaces is local on the target in a topology. Please compare with Descent, Section 35.22.

03YH **Definition 74.10.1.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ of algebraic spaces and any morphism of algebraic spaces $f : X \to Y$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

06EM **Lemma 74.10.2.** *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subspace $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

06R2 **Lemma 74.10.3.** *Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. For any morphism of algebraic spaces $f : X \to Y$ over S there exists a largest open subspace $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

(1) *if $g : Y' \to Y$ is a morphism of algebraic spaces which is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then g factors through $W(f)$,*
(2) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
(3) *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union $W_{set} \subset |Y|$ of the images $g(|Y'|) \subset |Y|$ of morphisms $g : Y' \to Y$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms of Spaces, Lemma 67.30.6) we see that W_{set} is an open subset of $|Y|$. Denote $W \subset Y$ the open subspace whose underlying set of points is W_{set}, see Properties of Spaces, Lemma 66.4.8. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 74.10.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. □

041J **Lemma 74.10.4.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *if $X_i \to Y_i$, $i = 1, 2$ have property $\mathcal{P}$ so does $X_1 \amalg X_2 \to Y_1 \amalg Y_2$,*
(2) *a morphism of algebraic spaces $f : X \to Y$ has property $\mathcal{P}$ if and only if for every affine scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f has property $\mathcal{P}$, and*
(3) *for any surjective flat morphism of affine schemes $Z' \to Z$ over S and a morphism $f : X \to Z$ from an algebraic space to Z we have*

$$f' : Z' \times_Z X \to Z' \text{ has } \mathcal{P} \Rightarrow f \text{ has } \mathcal{P}.$$

Then $\mathcal{P}$ is fpqc local on the base.

Proof. If $\mathcal{P}$ has property (2), then it is automatically stable under any base change. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $f_i : Y_i \times_Y X \to Y_i$ has property $\mathcal{P}$. Our goal is to show that f has $\mathcal{P}$. Let Z be an affine scheme, and let $Z \to Y$ be a morphism. By (2) it suffices to show that the morphism of algebraic spaces $Z \times_Y X \to Z$ has $\mathcal{P}$. Since $\{Y_i \to Y\}_{i\in I}$ is an fpqc covering we know there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to Y_{i_j}$ over Y for suitable indices $i_j \in I$. Since f_{i_j} has $\mathcal{P}$ we see that

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} (Y_{i_j} \times_Y X) \longrightarrow Z_j$$

has $\mathcal{P}$ as a base change of f_{i_j} (see first remark of the proof). Set $Z' = \coprod_{j=1,\ldots,n} Z_j$, so that $Z' \to Z$ is a flat and surjective morphism of affine schemes over S. By (1) we conclude that $Z' \times_Y X \to Z'$ has property $\mathcal{P}$. Since this is the base change of the morphism $Z \times_Y X \to Z$ by the morphism $Z' \to Z$ we conclude that $Z \times_Y X \to Z$ has property $\mathcal{P}$ as desired. □

74.11. Descending properties of morphisms in the fpqc topology

041K In this section we find a large number of properties of morphisms of algebraic spaces which are local on the base in the fpqc topology. Please compare with Descent, Section 35.23 for the case of morphisms of schemes.

041L **Lemma 74.11.1.** *Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-compact" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.8.8. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-compact. We have to show that f is quasi-compact. To see this, using Morphisms of Spaces, Lemma 67.8.8 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the fibre product $Y \times_Z X$ is quasi-compact. Here is a picture:

041M (74.11.1.1)

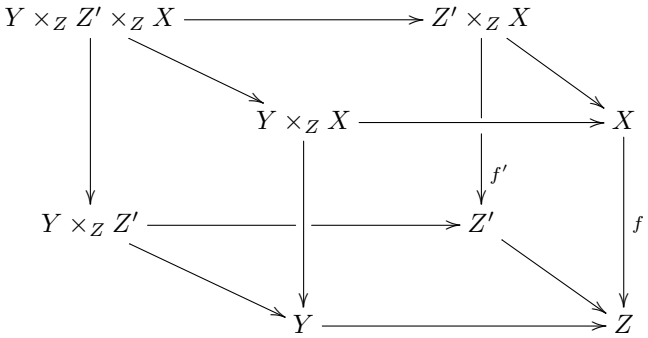

Note that all squares are cartesian and the bottom square consists of affine schemes. The assumption that f' is quasi-compact combined with the fact that $Y \times_Z Z'$ is affine implies that $Y \times_Z Z' \times_Z X$ is quasi-compact. Since

$$Y \times_Z Z' \times_Z X \longrightarrow Y \times_Z X$$

is surjective as a base change of $Z' \to Z$ we conclude that $Y \times_Z X$ is quasi-compact, see Morphisms of Spaces, Lemma 67.8.6. This finishes the proof. □

041N **Lemma 74.11.2.** *Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-separated" is fpqc local on the base on algebraic spaces over S.*

Proof. A base change of a quasi-separated morphism is quasi-separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is quasi-separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is quasi-compact. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.1 that Δ is quasi-compact, i.e., f is quasi-separated. □

041O **Lemma 74.11.3.** *Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally closed" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.9.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is universally closed. We have to show that f is universally closed. To see this, using Morphisms of Spaces, Lemma 67.9.5 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the map $|Y \times_Z X| \to |Y|$ is closed. Consider the cube (74.11.1.1). The assumption that f' is universally closed implies that $|Y \times_Z Z' \times_Z X| \to |Y \times_Z Z'|$ is closed. As $Y \times_Z Z' \to Y$ is quasi-compact, surjective, and flat as a base change of $Z' \to Z$ we see the map $|Y \times_Z Z'| \to |Y|$ is submersive, see Morphisms, Lemma 29.25.12. Moreover the map

$$|Y \times_Z Z' \times_Z X| \longrightarrow |Y \times_Z Z'| \times_{|Y|} |Y \times_Z X|$$

is surjective, see Properties of Spaces, Lemma 66.4.3. It follows by elementary topology that $|Y \times_Z X| \to |Y|$ is closed. □

041P **Lemma 74.11.4.** *Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally open" is fpqc local on the base on algebraic spaces over S.*

Proof. The proof is the same as the proof of Lemma 74.11.3. □

0CFW **Lemma 74.11.5.** *The property $\mathcal{P}(f) =$"f is universally submersive" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 74.11.3. □

041Q **Lemma 74.11.6.** *The property $\mathcal{P}(f) =$"f is surjective" is fpqc local on the base.*

Proof. Omitted. (Hint: Use Properties of Spaces, Lemma 66.4.3.) □

041R **Lemma 74.11.7.** *The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.9.5. Let $Z' \to Z$ be a flat surjective morphism of affine schemes over S and let $f : X \to Z$ be a morphism from an algebraic space to Z. Assume that the base change $f' : X' \to Z'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $Z' \to Z$ is surjective there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to Z'$ such that the following solid diagram commutes

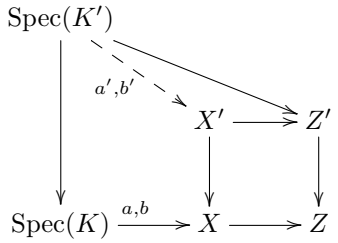

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to Z'$ is universally injective we get $a' = b'$. This forces $a = b$ as $\{\mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is an fpqc covering, see Properties of Spaces, Proposition 66.17.1. Hence f is universally injective as desired. □

0CFX **Lemma 74.11.8.** *The property $\mathcal{P}(f) =$"f is a universal homeomorphism" is fpqc local on the base.*

Proof. This can be proved in exactly the same manner as Lemma 74.11.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. See Morphisms of Spaces, Lemma 67.5.5 and Morphisms of Spaces, Definitions 67.19.3, 67.5.2, 67.6.2, 67.53.2. Thus the lemma follows from Lemmas 74.11.6, 74.11.7, and 74.11.4. □

041S **Lemma 74.11.9.** *The property $\mathcal{P}(f) =$"f is locally of finite type" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.23.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite type. We have to show that f is locally of finite type. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.23.4 again, it is enough to show that $U \to Z$ is locally of finite type. Since f' is locally of finite type, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite type. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite type by Descent, Lemma 35.23.12 as desired. □

041T **Lemma 74.11.10.** *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.28.4. Let $Z' \to Z$ be a

surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite presentation. We have to show that f is locally of finite presentation. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.28.4 again, it is enough to show that $U \to Z$ is locally of finite presentation. Since f' is locally of finite presentation, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite presentation. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite presentation by Descent, Lemma 35.23.13 as desired. □

041U **Lemma 74.11.11.** *The property $\mathcal{P}(f) =$"f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 74.11.1 and 74.11.9. □

041V **Lemma 74.11.12.** *The property $\mathcal{P}(f) =$"f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 74.11.1, 74.11.2 and 74.11.10. □

041W **Lemma 74.11.13.** *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.30.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is flat. We have to show that f is flat. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.30.5 again, it is enough to show that $U \to Z$ is flat. Since f' is flat, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is flat. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is flat by Descent, Lemma 35.23.17 as desired. □

041X **Lemma 74.11.14.** *The property $\mathcal{P}(f) =$"f is an open immersion" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.12.1. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is an open immersion. We have to show that $X \to Z$ is an open immersion. Note that $|X'| \subset |Z'|$ corresponds to an open subscheme $U' \subset Z'$ (isomorphic to X') with the property that $\mathrm{pr}_0^{-1}(U') = \mathrm{pr}_1^{-1}(U')$ as open subschemes of $Z' \times_Z Z'$. Hence there exists an open subscheme $U \subset Z$ such that $X' = (Z' \to Z)^{-1}(U)$, see Descent, Lemma 35.13.6. By Properties of Spaces, Proposition 66.17.1 we see that X satisfies the sheaf condition for the fpqc topology. Now we have the

fpqc covering $\mathcal{U} = \{U' \to U\}$ and the element $U' \to X' \to X \in \check{H}^0(\mathcal{U}, X)$. By the sheaf condition we obtain a morphism $U \to X$ such that

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow \cong & & \downarrow \\ X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

is commutative. On the other hand, we know that for any scheme T over S and T-valued point $T \to X$ the composition $T \to X \to Z$ is a morphism such that $Z' \times_Z T \to Z'$ factors through U'. Clearly this means that $T \to Z$ factors through U. In other words the map of sheaves $U \to X$ is bijective and we win. □

041Y **Lemma 74.11.15.** *The property $\mathcal{P}(f) =$"f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 74.11.6 and 74.11.14. □

041Z **Lemma 74.11.16.** *The property $\mathcal{P}(f) =$"f is affine" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.20.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 35.34.1 (verification omitted, compare with Descent, Lemma 35.39.1). Since $X' \to Z'$ is affine this descent datum is effective, see Descent, Lemma 35.37.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is affine (by construction or by Descent, Lemma 35.23.20). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X with respect to this covering (see Properties of Spaces, Proposition 66.17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 74.11.15. This proves that X is representable by an affine scheme and we win. □

0420 **Lemma 74.11.17.** *The property $\mathcal{P}(f) =$"f is a closed immersion" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.12.1. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a closed immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is affine. Hence by Lemma 74.11.16 we see that X is a scheme and $X \to Z$ is affine. It follows from Descent, Lemma 35.23.21 that $X \to Z$ is a closed immersion as desired. □

0421 **Lemma 74.11.18.** *The property $\mathcal{P}(f) =$"f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is a closed immersion. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.17 that Δ is a closed immersion, i.e., f is separated. □

0422 **Lemma 74.11.19.** *The property $\mathcal{P}(f) =$"f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 74.11.3, 74.11.18 and 74.11.11. □

0423 **Lemma 74.11.20.** *The property $\mathcal{P}(f) =$"f is quasi-affine" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.21.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 35.34.1 (verification omitted, compare with Descent, Lemma 35.39.1). Since $X' \to Z'$ is quasi-affine this descent datum is effective, see Descent, Lemma 35.38.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is quasi-affine (by construction or by Descent, Lemma 35.23.22). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see Properties of Spaces, Proposition 66.17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 74.11.15. This proves that X is representable by a quasi-affine scheme and we win. □

0424 **Lemma 74.11.21.** *The property $\mathcal{P}(f) =$"f is a quasi-compact immersion" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemmas 67.12.1 and 67.8.8. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a quasi-compact immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is quasi-affine. Hence by Lemma 74.11.20 we see that X is a scheme and $X \to Z$ is quasi-affine. It follows from Descent, Lemma 35.23.23 that $X \to Z$ is a quasi-compact immersion as desired. □

0425 **Lemma 74.11.22.** *The property $\mathcal{P}(f) =$"f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms of Spaces, Lemma 67.45.7. Hence the lemma follows on combining Lemmas 74.11.3 and 74.11.16. □

0426 **Lemma 74.11.23.** *The property $\mathcal{P}(f) =$"f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms of Spaces, Lemma 67.45.6. Hence the lemma follows on combining Lemmas 74.11.9 and 74.11.22. □

0427 **Lemma 74.11.24.** *The properties $\mathcal{P}(f) =$"f is locally quasi-finite" and $\mathcal{P}(f) =$"f is quasi-finite" are fpqc local on the base.*

Proof. We have already seen that "quasi-compact" is fpqc local on the base, see Lemma 74.11.1. Hence it is enough to prove the lemma for "locally quasi-finite". We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.27.6. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally quasi-finite. We have to show that f is locally quasi-finite. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.27.6 again, it is enough to show that $U \to Z$ is locally quasi-finite. Since f' is locally quasi-finite, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally quasi-finite. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally quasi-finite by Descent, Lemma 35.23.26 as desired. □

0428 **Lemma 74.11.25.** *The property $\mathcal{P}(f) =$"f is syntomic" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.36.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is syntomic. We have to show that f is syntomic. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.36.4 again, it is enough to show that $U \to Z$ is syntomic. Since f' is syntomic, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is

syntomic. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is syntomic by Descent, Lemma 35.23.28 as desired. □

0429 **Lemma 74.11.26.** *The property $\mathcal{P}(f) =$"f is smooth" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.37.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is smooth. We have to show that f is smooth. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.37.4 again, it is enough to show that $U \to Z$ is smooth. Since f' is smooth, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is smooth. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is smooth by Descent, Lemma 35.23.29 as desired. □

042A **Lemma 74.11.27.** *The property $\mathcal{P}(f) =$"f is unramified" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.38.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is unramified. We have to show that f is unramified. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.38.5 again, it is enough to show that $U \to Z$ is unramified. Since f' is unramified, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is unramified. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is unramified by Descent, Lemma 35.23.30 as desired. □

042B **Lemma 74.11.28.** *The property $\mathcal{P}(f) =$"f is étale" is fpqc local on the base.*

Proof. We will use Lemma 74.10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 67.39.2. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is étale. We have to show that f is étale. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 67.39.2 again, it is enough to show that $U \to Z$ is étale. Since f' is étale, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is étale. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is étale by Descent, Lemma 35.23.31 as desired. □

042C **Lemma 74.11.29.** *The property $\mathcal{P}(f) =$"f is finite locally free" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms of Spaces, Lemma 67.46.6). Hence this follows from Lemmas 74.11.23, 74.11.13, and 74.11.10. □

042D **Lemma 74.11.30.** *The property $\mathcal{P}(f) =$"f is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to Y_i$ of f is a monomorphism. We have to show that f is a monomorphism.

First proof. Note that f is a monomorphism if and only if $\Delta : X \to X \times_Y X$ is an isomorphism. By applying this to f_i we see that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an isomorphism. The base change of an fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 73.9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.11.15 that Δ is an isomorphism, i.e., f is a monomorphism.

Second proof. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If we can show that $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{G} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{H} & \xrightarrow{d} & \mathcal{I} \end{array} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. This reduces the problem to the case where Y is a scheme. Moreover, in this case we may assume that the algebraic spaces Y_i are schemes also, since we can always refine the covering to place ourselves in this situation, see Topologies on Spaces, Lemma 73.9.5.

Assume $\{Y_i \to Y\}$ is an fpqc covering of schemes. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : Y_i \times_Y T \to X_i$ are the base changes. In particular the compositions $Y_i \times_Y T \to T \to X$ are equal. Since $\{Y_i \times_Y T \to T\}$ is an fpqc covering we deduce that $a = b$ from Properties of Spaces, Proposition 66.17.1. □

74.12. Descending properties of morphisms in the fppf topology

042E In this section we find some properties of morphisms of algebraic spaces for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

042U **Lemma 74.12.1.** *The property $\mathcal{P}(f)$ = "f is an immersion" is fppf local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Let $f_i : X_i \to Y_i$ be the base change of f.

If f is an immersion, then each f_i is an immersion by Spaces, Lemma 65.12.3. This proves the direct implication in Definition 74.10.1.

Conversely, assume each f_i is an immersion. By Morphisms of Spaces, Lemma 67.10.7 this implies each f_i is separated. By Morphisms of Spaces, Lemma 67.27.7 this implies each f_i is locally quasi-finite. Hence we see that f is locally quasi-finite and separated, by applying Lemmas 74.11.18 and 74.11.24. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable!

By Morphisms of Spaces, Lemma 67.12.1 it suffices to show that for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ is an immersion. By Topologies on Spaces, Lemma 73.7.4 we can find an fppf covering $\{Z_i \to Z\}$ by schemes which refines the pullback of the covering $\{Y_i \to Y\}$ to Z. Hence we see that $Z \times_Y X \to Z$ (which is a morphism of schemes according to the result of the preceding paragraph) becomes an immersion after pulling back to the members of an fppf (by schemes) of Z. Hence $Z \times_Y X \to Z$ is an immersion by the result for schemes, see Descent, Lemma 35.24.1. □

042F **Lemma 74.12.2.** *The property $\mathcal{P}(f) =$"f is locally separated" is fppf local on the base.*

Proof. A base change of a locally separated morphism is locally separated, see Morphisms of Spaces, Lemma 67.4.4. Hence the direct implication in Definition 74.10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is locally separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an immersion. The base change of a fppf covering is an fppf covering, see Topologies on Spaces, Lemma 73.7.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fppf covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 74.12.1 that Δ is an immersion, i.e., f is locally separated. □

74.13. Application of descent of properties of morphisms

0D3B This section is the analogue of Descent, Section 35.25.

0D3C **Lemma 74.13.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering. Let $f_i : X_i \to Y_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X/Y, and*
(2) *$\mathcal{L}_i$ is ample on X_i/Y_i for every $i \in I$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Divisors on Spaces, Lemma 71.14.3. Assume (2). To check $\mathcal{L}$ is ample on X/Y we may work étale locally on Y, see Divisors on Spaces, Lemma 71.14.6. Thus we may assume that Y is a scheme and then we may in turn assume each Y_i is a scheme too, see Topologies on Spaces, Lemma 73.9.5. In other words, we may assume that $\{Y_i \to Y\}$ is an fpqc covering of schemes.

By Divisors on Spaces, Lemma 71.14.4 we see that $X_i \to Y_i$ is representable (i.e., X_i is a scheme), quasi-compact, and separated. Hence f is quasi-compact and separated by Lemmas 74.11.1 and 74.11.18. This means that $\mathcal{A} = \bigoplus_{d \geq 0} f_* \mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra (Morphisms of Spaces, Lemma 67.11.2). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Spaces, Lemma 69.11.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d \geq 0} f_{i,*} \mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^* \mathcal{A}$. It follows that the natural maps $\psi_d : f^* \mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give

the natural maps $\psi_{i,d} : f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/Y_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 29.37.4. If $x \in |X|$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x}_i}$ is flat hence faithfully flat, we conclude that for every $x \in |X|$ there exists a $d \geq 1$ such that $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Divisors on Spaces, Lemma 71.13.1 corresponding to the map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\mathrm{Proj}}_Y(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to Y_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 29.37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 74.11.14. Hence X is a scheme and we conclude $\mathcal{L}$ is ample on X/Y by Morphisms, Lemma 29.37.4. □

0D3D **Lemma 74.13.2.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. There exists an open subspace $V \subset Y$ characterized by the following property: A morphism $Y' \to Y$ of algebraic spaces factors through V if and only if the pullback $\mathcal{L}'$ of $\mathcal{L}$ to $X' = Y' \times_Y X$ is ample on X'/Y' (as in Divisors on Spaces, Definition 71.14.1).*

Proof. Suppose that the lemma holds whenever Y is a scheme. Let U be a scheme and let $U \to Y$ be a surjective étale morphism. Let $R = U \times_Y U$ with projections $t, s : R \to U$. Denote $X_U = U \times_Y X$ and $\mathcal{L}_U$ the pullback. Then we get an open subscheme $V' \subset U$ as in the lemma for $(X_U \to U, \mathcal{L}_U)$. By the functorial characterization we see that $s^{-1}(V') = t^{-1}(V')$. Thus there is an open subspace $V \subset Y$ such that V' is the inverse image of V in U. In particular $V' \to V$ is surjective étale and we conclude that $\mathcal{L}_V$ is ample on X_V/V (Divisors on Spaces, Lemma 71.14.6). Now, if $Y' \to Y$ is a morphism such that $\mathcal{L}'$ is ample on X'/Y', then $U \times_Y Y' \to Y'$ must factor through V' and we conclude that $Y' \to Y$ factors through V. Hence $V \subset Y$ is as in the statement of the lemma. In this way we reduce to the case dealt with in the next paragraph.

Assume Y is a scheme. Since the question is local on Y we may assume Y is an affine scheme. We will show the following:

(A) If $\mathrm{Spec}(k) \to Y$ is a morphism such that $\mathcal{L}_k$ is ample on X_k/k, then there is an open neighbourhood $V \subset Y$ of the image of $\mathrm{Spec}(k) \to Y$ such that $\mathcal{L}_V$ is ample on X_V/V.

It is clear that (A) implies the truth of the lemma.

Let $X \to Y$, $\mathcal{L}$, $\mathrm{Spec}(k) \to Y$ be as in (A). By Lemma 74.13.1 we may assume that $k = \kappa(y)$ is the residue field of a point y of Y.

As Y is affine we can find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of algebraic spaces with Y_i of finite presentation over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper and of finite presentation, and such that $X \to Y = \lim(X_i \to Y_i)$. See Limits of Spaces, Lemma 70.12.2. After shrinking I we may assume Y_i is an (affine) scheme for all i, see Limits of Spaces, Lemma 70.5.10. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits of Spaces, Lemma 70.7.3.

Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \operatorname{colim} \kappa(y_i)$. Hence $X_y = \lim X_{i,y_i}$ and after shrinking I we may assume X_{i,y_i} is a scheme for all i, see Limits of Spaces, Lemma 70.5.11. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 32.4.15. By Divisors on Spaces, Lemma 71.15.3 we find an open neighbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 29.37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 29.37.7). □

74.14. Properties of morphisms local on the source

06EN In this section we define what it means for a property of morphisms of algebraic spaces to be local on the source. Please compare with Descent, Section 35.26.

06EP **Definition 74.14.1.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism $f : X \to Y$ of algebraic spaces over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ of algebraic spaces we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

06EQ **Lemma 74.14.2.** *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. smooth, resp. étale, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

06ER **Lemma 74.14.3.** *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Suppose that $\mathcal{P}$ is a property of morphisms of schemes over S which is étale local on the source-and-target. Denote $\mathcal{P}_{spaces}$ the corresponding property of morphisms of algebraic spaces over S, see Morphisms of Spaces, Definition 67.22.2. If $\mathcal{P}$ is local on the source for the τ-topology, then $\mathcal{P}_{spaces}$ is local on the source for the τ-topology.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a τ-covering of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. For each i choose a scheme U_i and a surjective étale morphism $U_i \to X_i \times_X U$.

Note that $\{X_i \times_X U \to U\}_{i \in I}$ is a τ-covering. Note that each $\{U_i \to X_i \times_X U\}$ is an étale covering, hence a τ-covering. Hence $\{U_i \to U\}_{i \in I}$ is a τ-covering of algebraic spaces over S. But since U and each U_i is a scheme we see that $\{U_i \to U\}_{i \in I}$ is a τ-covering of schemes over S.

Now we have

$$\begin{aligned} f \text{ has } \mathcal{P}_{spaces} &\Leftrightarrow U \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } U_i \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}_{spaces}. \end{aligned}$$

the first and last equivalence by the definition of $\mathcal{P}_{spaces}$ the middle equivalence because we assumed $\mathcal{P}$ is local on the source in the τ-topology. □

74.15. Properties of morphisms local in the fpqc topology on the source

06ES Here are some properties of morphisms that are fpqc local on the source.

06ET **Lemma 74.15.1.** *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.30.1 and Descent, Lemma 35.27.1. □

74.16. Properties of morphisms local in the fppf topology on the source

06EU Here are some properties of morphisms that are fppf local on the source.

06EV **Lemma 74.16.1.** *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fppf local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.28.1 and Descent, Lemma 35.28.1. □

06EW **Lemma 74.16.2.** *The property $\mathcal{P}(f) =$"f is locally of finite type" is fppf local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.23.1 and Descent, Lemma 35.28.2. □

06EX **Lemma 74.16.3.** *The property $\mathcal{P}(f) =$"f is open" is fppf local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.6.2 and Descent, Lemma 35.28.3. □

06EY **Lemma 74.16.4.** *The property $\mathcal{P}(f) =$"f is universally open" is fppf local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.6.2 and Descent, Lemma 35.28.4. □

74.17. Properties of morphisms local in the syntomic topology on the source

06EZ Here are some properties of morphisms that are syntomic local on the source.

06F0 **Lemma 74.17.1.** *The property $\mathcal{P}(f) =$"f is syntomic" is syntomic local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.36.1 and Descent, Lemma 35.29.1. □

74.18. Properties of morphisms local in the smooth topology on the source

06F1 Here are some properties of morphisms that are smooth local on the source.

06F2 **Lemma 74.18.1.** *The property $\mathcal{P}(f) =$"f is smooth" is smooth local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.37.1 and Descent, Lemma 35.30.1. □

74.19. Properties of morphisms local in the étale topology on the source

06F3 Here are some properties of morphisms that are étale local on the source.

06F4 **Lemma 74.19.1.** *The property $\mathcal{P}(f) =$"f is étale" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.39.1 and Descent, Lemma 35.31.1. □

06F5 **Lemma 74.19.2.** *The property $\mathcal{P}(f) =$"f is locally quasi-finite" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.27.1 and Descent, Lemma 35.31.2. □

06F6 **Lemma 74.19.3.** *The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source.*

Proof. Follows from Lemma 74.14.3 using Morphisms of Spaces, Definition 67.38.1 and Descent, Lemma 35.31.3. □

74.20. Properties of morphisms smooth local on source-and-target

06F7 Let $\mathcal{P}$ be a property of morphisms of algebraic spaces. There is an intuitive meaning to the phrase "$\mathcal{P}$ is smooth local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both smooth local on the source and smooth local on the target. We have discussed a similar phenomenon (for the étale topology and the category of schemes) in great detail in Descent, Section 35.32 (for a quick overview take a look at Descent, Remark 35.32.8). However, there is an important difference between the case of the smooth and the étale topology. To see this difference we encourage the reader to ponder the difference between Descent, Lemma 35.32.4 and Lemma 74.20.2 as well as the difference between Descent, Lemma 35.32.5 and Lemma 74.20.3. Namely, in the étale setting the choice of the étale "covering" of the target is immaterial, whereas in the smooth setting it is not.

06F8 **Definition 74.20.1.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *smooth local on source-and-target* if

(1) (stable under precomposing with smooth maps) if $f : X \to Y$ is smooth and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with smooth vertical arrows and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being smooth local on the target, smooth local on the source, and stable under post-composing by smooth morphisms.

06F9 **Lemma 74.20.2.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth, then $g \circ f$ has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 74.20.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 74.20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth. For every $x \in |X|$ we can think of $(X, x) \to (X, x)$ as a smooth morphism of pairs, $Y \to Z$ is a smooth morphism, and $h = f$ is as in part (3) of Definition 74.20.1. Thus we see that $g \circ f$ has $\mathcal{P}$. □

The following lemma is the analogue of Morphisms, Lemma 29.14.4.

06FA **Lemma 74.20.3.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism of pairs $a : (U, u) \to (X, x)$, a smooth morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*

(c) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b smooth and a surjective the morphism h has $\mathcal{P}$,

(d) *for any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth and $U \to X \times_Y V$ smooth the morphism h has $\mathcal{P}$,

(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*

(f) *there exists a smooth covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*

(g) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ a smooth covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 74.20.1. The equivalence of (a) and (e) is Lemma 74.20.2 part (2). The equivalence of (a) and (f) is Lemma 74.20.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism of pairs $a_x : (U_x, u_x) \to (X, x)$, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with smooth morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

06FB **Lemma 74.20.4.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $Y \to Z$ is a smooth morphism then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 74.14.2 we see that $\mathcal{P}$ is stable under precomposing with smooth morphisms. By Lemma 74.10.2 we see that $\mathcal{P}$ is

stable under smooth base change. Hence it suffices to prove part (3) of Definition 74.20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 74.20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $a_x : U_x \to X$, a point $u_x \in |U_x|$ mapping to x, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b smooth, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. Because a is surjective and $\mathcal{P}$ is smooth local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This follows as we assumed that $\mathcal{P}$ is stable under postcomposing with a smooth morphism and as b is smooth. □

06FC **Remark 74.20.5.** Using Lemma 74.20.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is smooth local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 74.20.4 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 74.15.1 and 74.11.13,
(2) locally of finite presentation, see Lemmas 74.16.1 and 74.11.10,
(3) locally finite type, see Lemmas 74.16.2 and 74.11.9,
(4) universally open, see Lemmas 74.16.4 and 74.11.4,
(5) syntomic, see Lemmas 74.17.1 and 74.11.25,
(6) smooth, see Lemmas 74.18.1 and 74.11.26,
(7) add more here as needed.

74.21. Properties of morphisms étale-smooth local on source-and-target

0CFY This section is the analogue of Section 74.20 for properties of morphisms which are étale local on the source and smooth local on the target. We give this property a ridiculously long name in order to avoid using it too much.

0CFZ **Definition 74.21.1.** Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *étale-smooth local on source-and-target* if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth and $U \to X \times_Y V$ étale and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being étale local on the target, smooth local on the source, and stable under post-composing by étale morphisms.

0CG0 **Lemma 74.21.2.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is étale-smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) *$\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 74.21.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 74.21.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $X_i \to X \times_Y Y_i$ is an étale morphism (because it is an isomorphism), $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 74.20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 74.20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. The morphism $X \to Y \times_Z X$ is étale as a morphism between algebraic spaces étale over X (Properties of Spaces, Lemma 66.16.6). Also $Y \to Z$ is étale hence a smooth morphism. Thus the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{g \circ f} & Z \end{array}$$

works for every $x \in |X|$ in part (3) of Definition 74.20.1 and we conclude that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 74.21.1 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$

has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms of Spaces, Lemma 67.39.11. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 74.21.1 part (1). □

0CG1 **Lemma 74.21.3.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is etale-smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism $b : V \to Y$, an étale morphism $a : U \to V \times_Y X$, and a point $u \in |U|$ mapping to x such that $U \to V$ has $\mathcal{P}$,*
(c) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth, $U \to V \times_Y X$ étale, and a surjective the morphism h has $\mathcal{P}$,
(d) *for any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth and $U \to X \times_Y V$ étale, the morphism h has $\mathcal{P}$,
(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 74.21.1. The equivalence of (a) and (e) is Lemma 74.21.2 part (2). The equivalence of (a) and (f) is Lemma 74.21.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and $u_x \in |U_x|$ mapping to x such that $U_x \to V_x$ has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with étale morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

0CG2 **Lemma 74.21.4.** *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open embedding then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale-smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 74.14.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 74.10.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 74.20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 74.20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and a point $u_x \in |U_x|$ mapping to x such that $h_x : U_x \to V_x$ has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$.

Let $a_x : U_x \to X$ be the composition $U_x \to V_x \times_Y X \to X$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth, $U \to V \times_Y X$ étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. In the next paragraph we prove that we may assume U, V, X, Y are schemes; we encourage the reader to skip it.

Let X, Y, U, V, a, b, f, h be as in the previous paragraph. We have to show f has $\mathcal{P}$. Let $X' \to X$ be a surjective étale morphism with X_i a scheme. Set $U' = X' \times_X U$. Then $U' \to X'$ is surjective and $U' \to X' \times_Y V$ is étale. Since $\mathcal{P}$ is étale local on the source, we see that $U' \to V$ has $\mathcal{P}$ and that it suffices to show that $X' \to Y$ has $\mathcal{P}$. In other words, we may assume that X is a scheme. Next, choose a surjective étale morphism $Y' \to Y$ with Y' a scheme. Set $V' = V \times_Y Y'$, $X' = X \times_Y Y'$, and $U' = U \times_Y Y'$. Then $U' \to X'$ is surjective and $U' \to X' \times_{Y'} V'$ is étale. Since $\mathcal{P}$ is smooth local on the target, we see that $U' \to V'$ has $\mathcal{P}$ and that it suffices to prove $X' \to Y'$ has $\mathcal{P}$. Thus we may assume both X and Y are schemes. Choose a surjective étale morphism $V' \to V$ with V' a scheme. Set $U' = U \times_V V'$. Then $U' \to X$ is surjective and $U' \to X \times_Y V'$ is étale. Since $\mathcal{P}$ is smooth local on the source, we see that $U' \to V'$ has $\mathcal{P}$. Thus we may replace U, V by U', V' and assume X, Y, V are schemes. Finally, we replace U by a scheme surjective étale over U and we see that we may assume U, V, X, Y are all schemes.

If U, V, X, Y are schemes, then f has $\mathcal{P}$ by Descent, Lemma 35.32.11. □

0CG3 **Remark 74.21.5.** Using Lemma 74.21.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is etale local on the source and the lemma which implies the property is

smooth local on the target. In each case the third assumption of Lemma 74.21.4 is trivial to check, and we omit it. Here is the list:

(1) étale, see Lemmas 74.19.1 and 74.11.28,
(2) locally quasi-finite, see Lemmas 74.19.2 and 74.11.24,
(3) unramified, see Lemmas 74.19.3 and 74.11.27, and
(4) add more here as needed.

Of course any property listed in Remark 74.20.5 is a fortiori an example that could be listed here.

74.22. Descent data for spaces over spaces

0ADF This section is the analogue of Descent, Section 35.34 for algebraic spaces. Most of the arguments in this section are formal relying only on the definition of a descent datum.

0ADG **Definition 74.22.1.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Let $V \to Y$ be a morphism of algebraic spaces. A *descent datum for* $V/Y/X$ is an isomorphism $\varphi : V \times_X Y \to Y \times_X V$ of algebraic spaces over $Y \times_X Y$ satisfying the *cocycle condition* that the diagram

$$\begin{array}{ccc} V \times_X Y \times_X Y & \xrightarrow{\varphi_{02}} & Y \times_X Y \times_X V \\ & {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\ & Y \times_X V \times_X Y & \end{array}$$

commutes (with obvious notation).
(2) We also say that the pair $(V/Y, \varphi)$ is a *descent datum relative to* $Y \to X$.
(3) A *morphism* $f : (V/Y, \varphi) \to (V'/Y, \varphi')$ *of descent data relative to* $Y \to X$ is a morphism $f : V \to V'$ of algebraic spaces over Y such that the diagram

$$\begin{array}{ccc} V \times_X Y & \xrightarrow{\varphi} & Y \times_X V \\ {\scriptstyle f \times \mathrm{id}_Y} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_Y \times f} \\ V' \times_X Y & \xrightarrow{\varphi'} & Y \times_X V' \end{array}$$

commutes.

0ADH **Remark 74.22.2.** Let S be a scheme. Let $Y \to X$ be a morphism of algebraic spaces over S. Let $(V/Y, \varphi)$ be a descent datum relative to $Y \to X$. We may think of the isomorphism φ as an isomorphism

$$(Y \times_X Y) \times_{\mathrm{pr}_0, Y} V \longrightarrow (Y \times_X Y) \times_{\mathrm{pr}_1, Y} V$$

of algebraic spaces over $Y \times_X Y$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^* V \to \mathrm{pr}_1^* V$[1]. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi = \mathrm{pr}_{12}^* \varphi \circ \mathrm{pr}_{01}^* \varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

[1] Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 74.22.1 and 74.22.3 we should have the opposite direction to what was done in Definition 74.3.1 by the general principle that "functions" and "spaces" are dual.

0ADI **Definition 74.22.3.** Let S be a scheme. Let $\{X_i \to X\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A *descent datum* (V_i, φ_{ij}) *relative to the family* $\{X_i \to X\}$ is given by an algebraic space V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_X X_j \to X_i \times_X V_j$ of algebraic spaces over $X_i \times_X X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} V_i \times_X X_j \times_X X_k & \xrightarrow{\quad\quad\quad\quad} & X_i \times_X X_j \times_X V_k \\ & \searrow^{\mathrm{pr}_{01}^*\varphi_{ij}} \quad \mathrm{pr}_{02}^*\varphi_{ik} \quad \nearrow_{\mathrm{pr}_{12}^*\varphi_{jk}} & \\ & X_i \times_X V_j \times_X X_k & \end{array}$$

of algebraic spaces over $X_i \times_X X_j \times_X X_k$ commutes (with obvious notation).

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms $\psi_i : V_i \to V_i'$ of algebraic spaces over X_i such that all the diagrams

$$\begin{array}{ccc} V_i \times_X X_j & \xrightarrow{\varphi_{ij}} & X_i \times_X V_j \\ \downarrow{\scriptstyle \psi_i \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \times \psi_j} \\ V_i' \times_X X_j & \xrightarrow{\varphi_{ij}'} & X_i \times_X V_j' \end{array}$$

commute.

0ADJ **Remark 74.22.4.** Let S be a scheme. Let $\{X_i \to X\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to X\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_X X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_X X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of algebraic spaces over $X_i \times_X X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_X X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi_{ik} = \mathrm{pr}_{12}^*\varphi_{jk} \circ \mathrm{pr}_{01}^*\varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

0ADK **Lemma 74.22.5.** *Let S be a scheme. Let $\{X_i \to X\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Set $Y = \coprod_{i\in I} X_i$. There is a canonical equivalence of categories*

$$\begin{array}{c} \textit{category of descent data} \\ \textit{relative to the family } \{X_i \to X\}_{i\in I} \end{array} \longrightarrow \begin{array}{c} \textit{category of descent data} \\ \textit{relative to } Y/X \end{array}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i\in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $Y \times_X Y = \coprod_{ij} X_i \times_X X_j$ and similarly for higher fibre products. Giving a morphism $V \to Y$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. □

0ADL **Lemma 74.22.6.** *Pullback of descent data. Let S be a scheme.*

(1) *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ X' & \xrightarrow{h} & X \end{array}$$

be a commutative diagram of algebraic spaces over S. The construction

$$(V \to Y, \varphi) \longmapsto f^*(V \to Y, \varphi) = (V' \to Y', \varphi')$$

where $V' = Y' \times_Y V$ and where φ' is defined as the composition

$$\begin{array}{ccccc} V' \times_{X'} Y' & = & (Y' \times_Y V) \times_{X'} Y' & = & (Y' \times_{X'} Y') \times_{Y \times_X Y} (V \times_X Y) \\ & & & & \downarrow{\scriptstyle id \times \varphi} \\ Y' \times_{X'} V' & = & Y' \times_{X'} (Y' \times_Y V) & = & (Y' \times_X Y') \times_{Y \times_X Y} (Y \times_X V) \end{array}$$

defines a functor from the category of descent data relative to $Y \to X$ to the category of descent data relative to $Y' \to X'$.

(2) *Given two morphisms $f_i : Y' \to Y$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 74.22.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : Y' \to Y \times_X Y$ with $f_i = \mathrm{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= Y' \times_{f_0, Y} V \\ &= Y' \times_{\mathrm{pr}_0 \circ r, Y} V \\ &= Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\mathrm{pr}_0, Y} V \\ &\xrightarrow{\varphi} Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\mathrm{pr}_1, Y} V \\ &= Y' \times_{\mathrm{pr}_1 \circ r, Y} V \\ &= Y' \times_{f_1, Y} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. □

0ADM **Definition 74.22.7.** With $S, X, X', Y, Y', f, a, a', h$ as in Lemma 74.22.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

0ADN **Lemma 74.22.8.** *Let S be a scheme. Let $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$ and $\mathcal{U} = \{X_j \to X\}_{i \in I}$ be families of morphisms with fixed target. Let $\alpha : I' \to I$, $g : X' \to X$ and $g_i : X'_i \to X_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.*

(1) *Let (V_i, φ_{ij}) be a descent datum relative to the family $\mathcal{U}$. The system*

$$\left(g_i^*V_{\alpha(i)}, (g_i \times g_j)^*\varphi_{\alpha(i)\alpha(j)}\right)$$

(with notation as in Remark 74.22.4) is a descent datum relative to $\mathcal{U}'$.

(2) *This construction defines a functor between the category of descent data relative to $\mathcal{U}$ and the category of descent data relative to $\mathcal{U}'$.*
(3) *Given a second $\beta : I' \to I$, $h : X' \to X$ and $h'_i : X'_i \to X_{\beta(i)}$ morphism of families of maps with fixed target, then if $g = h$ the two resulting functors between descent data are canonically isomorphic.*
(4) *These functors agree, via Lemma 74.22.5, with the pullback functors constructed in Lemma 74.22.6.*

Proof. This follows from Lemma 74.22.6 via the correspondence of Lemma 74.22.5. □

0ADP **Definition 74.22.9.** With $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$, $\mathcal{U} = \{X_i \to X\}_{i \in I}$, $\alpha : I' \to I$, $g : X' \to X$, and $g_i : X'_i \to X_{\alpha(i)}$ as in Lemma 74.22.8 the functor

$$(V_i, \varphi_{ij}) \longmapsto (g_i^* V_{\alpha(i)}, (g_i \times g_j)^* \varphi_{\alpha(i)\alpha(j)})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{U}'$ have the same target X, and if $\mathcal{U}'$ refines $\mathcal{U}$ (see Sites, Definition 7.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 74.22.8 that the choice of the pair does not matter (up to a canonical isomorphism).

0ADQ **Definition 74.22.10.** Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Given an algebraic space U over X we have the *trivial descent datum* of U relative to $\mathrm{id} : X \to X$, namely the identity morphism on U.
(2) By Lemma 74.22.6 we get a *canonical descent datum* on $Y \times_X U$ relative to $Y \to X$ by pulling back the trivial descent datum via f. We often denote $(Y \times_X U, can)$ this descent datum.
(3) A descent datum (V, φ) relative to Y/X is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(Y \times_X U, can)$ for some algebraic space U over X.

Thus being effective means there exists an algebraic space U over X and an isomorphism $\psi : V \to Y \times_X U$ over Y such that φ is equal to the composition

$$V \times_X Y \xrightarrow{\psi \times \mathrm{id}_Y} Y \times_X U \times_S Y = Y \times_X Y \times_X U \xrightarrow{\mathrm{id}_Y \times \psi^{-1}} Y \times_X V$$

There is a slight problem here which is that this definition (in spirit) conflicts with the definition given in Descent, Definition 35.34.10 in case Y and X are schemes. However, it will always be clear from context which version we mean.

0ADR **Definition 74.22.11.** Let S be a scheme. Let $\{X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) Given an algebraic space U over X we have a *canonical descent datum* on the family of algebraic spaces $X_i \times_X U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_X U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists an algebraic space U over X such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_X U, can)$.

74.23. Descent data in terms of sheaves

0ADS This section is the analogue of Descent, Section 35.39. It is slightly different as algebraic spaces are already sheaves.

0ADT **Lemma 74.23.1.** *Let S be a scheme. Let $\{X_i \to X\}_{i\in I}$ be an fppf covering of algebraic spaces over S (Topologies on Spaces, Definition 73.7.1). There is an equivalence of categories*

$$\left\{\begin{matrix}\text{descent data } (V_i, \varphi_{ij}) \\ \text{relative to } \{X_i \to X\}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{sheaves } F \text{ on } (Sch/S)_{fppf} \text{ endowed} \\ \text{with a map } F \to X \text{ such that each} \\ X_i \times_X F \text{ is an algebraic space}\end{matrix}\right\}.$$

Moreover,

1. *the algebraic space $X_i \times_X F$ on the right hand side corresponds to V_i on the left hand side, and*
2. *the sheaf F is an algebraic space[2] if and only if the corresponding descent datum (X_i, φ_{ij}) is effective.*

Proof. Let us construct the functor from right to left. Let $F \to X$ be a map of sheaves on $(Sch/S)_{fppf}$ such that each $V_i = X_i \times_X F$ is an algebraic space. We have the projection $V_i \to X_i$. Then both $V_i \times_X X_j$ and $X_i \times_X V_j$ represent the sheaf $X_i \times_X F \times_X X_j$ and hence we obtain an isomorphism

$$\varphi_{ii'} : V_i \times_X X_j \to X_i \times_X V_j$$

It is straightforward to see that the maps φ_{ij} are morphisms over $X_i \times_X X_j$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (V_i, \varphi_{ij})$.

Let us construct a functor from left to right. The isomorphisms φ_{ij} give isomorphisms

$$\varphi_{ij} : V_i \times_X X_j \longrightarrow X_i \times_X V_j$$

over $X_i \times X_j$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} V_i \times_X X_j \underset{\mathrm{pr}_1 \circ \varphi_{ij}}{\overset{\mathrm{pr}_0}{\rightrightarrows}} \coprod_i V_i \longrightarrow F$$

The cocycle condition guarantees that F comes with a map $F \to X$ and that $X_i \times_X F$ is isomorphic to V_i. The functor from left to right is given by this construction $(V_i, \varphi_{ij}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. □

74.24. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields

[2] We will see later that this is always the case if I is not too large, see Bootstrap, Lemma 80.11.3.

(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
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CHAPTER 75

Derived Categories of Spaces

08EY

75.1. Introduction

08EZ In this chapter we discuss derived categories of modules on algebraic spaces. There do not seem to be good introductory references addressing this topic; it is covered in the literature by referring to papers dealing with derived categories of modules on algebraic stacks, for example see [**Ols07b**].

75.2. Conventions

08F0 If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

75.3. Generalities

08GD In this section we put some general results on cohomology of unbounded complexes of modules on algebraic spaces.

08GE **Lemma 75.3.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $U = V \times_Y X$ and denote $g : U \to V$ the projection morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Cohomology on Sites, Lemma 21.20.1. Hence the result follows from Properties of Spaces, Lemma 66.26.2. □

08GF **Definition 75.3.2.** Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset |X|$ be a closed subset. We say E is *supported on T* if the cohomology sheaves $H^i(E)$ are supported on T.

75.4. Derived category of quasi-coherent modules on the small étale site

071P Let X be a scheme. In this section we show that $D_{QCoh}(\mathcal{O}_X)$ can be defined in terms of the small étale site $X_{étale}$ of X. Denote $\mathcal{O}_{étale}$ the structure sheaf on $X_{étale}$. Consider the morphism of ringed sites

08H7 (75.4.0.1) $$\epsilon : (X_{étale}, \mathcal{O}_{étale}) \longrightarrow (X_{Zar}, \mathcal{O}_X).$$

denoted $\mathrm{id}_{small,étale,Zar}$ in Descent, Lemma 35.8.5.

08H8 **Lemma 75.4.1.** *The morphism ϵ of (75.4.0.1) is a flat morphism of ringed sites. In particular the functor $\epsilon^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_{étale})$ is exact. Moreover, if $\epsilon^*\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. The flatness of the morphism ϵ is Descent, Lemma 35.10.1. Here is another proof. We have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_X$-module. To do this it suffices to check $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is flat for any geometric point $\overline{x}$ of X, see Modules on Sites, Lemma 18.39.3, Sites, Lemma 7.34.2, and Étale Cohomology, Remarks 59.29.11. By Étale Cohomology, Lemma 59.33.1 we see that $\mathcal{O}_{étale,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is faithfully flat by More on Algebra, Lemma 15.45.1.

The exactness of ϵ^* follows from the flatness of ϵ by Modules on Sites, Lemma 18.31.2.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\epsilon^*\mathcal{F} = 0$, then with notation as above

$$0 = \epsilon^*\mathcal{F}_{\overline{x}} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{étale,\overline{x}}$$

(Modules on Sites, Lemma 18.36.4) for all geometric points $\overline{x}$. By faithful flatness of $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ we conclude $\mathcal{F}_x = 0$ for all $x \in X$. □

Let X be a scheme. Notation as in (75.4.0.1). Recall that $\epsilon^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{étale})$ is an equivalence by Descent, Proposition 35.8.9 and Remark 35.8.6. Moreover, $QCoh(\mathcal{O}_{étale})$ forms a Serre subcategory of $Mod(\mathcal{O}_{étale})$ by Descent, Lemma 35.10.2. Hence we can let $D_{QCoh}(\mathcal{O}_{étale})$ be the triangulated subcategory of $D(\mathcal{O}_{étale})$ whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 13.17. The functor ϵ^* is exact (Lemma 75.4.1) hence induces $\epsilon^* : D(\mathcal{O}_X) \to D(\mathcal{O}_{étale})$ and since pullbacks of quasi-coherent modules are quasi-coherent also $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$.

071Q **Lemma 75.4.2.** *Let X be a scheme. The functor $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$ defined above is an equivalence.*

Proof. We will prove this by showing the functor $R\epsilon_* : D(\mathcal{O}_{étale}) \to D(\mathcal{O}_X)$ induces a quasi-inverse. We will use freely that ϵ_* is given by restriction to $X_{Zar} \subset X_{étale}$ and the description of $\epsilon^* = \mathrm{id}^*_{small,étale,Zar}$ in Descent, Lemma 35.8.5.

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^*\mathcal{F}$ is an isomorphism by the fact that $\mathcal{F}^a$ (Descent, Definition 35.8.2) is a sheaf as proved in Descent, Lemma 35.8.1. Conversely, every quasi-coherent $\mathcal{O}_{étale}$-module $\mathcal{H}$ is of the form $\epsilon^*\mathcal{F}$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, see Descent, Proposition 35.8.9. Then $\mathcal{F} = \epsilon_*\mathcal{H}$ by what we just said and we conclude that the adjunction map $\epsilon^*\epsilon_*\mathcal{H} \to \mathcal{H}$ is an isomorphism for all quasi-coherent $\mathcal{O}_{étale}$-modules $\mathcal{H}$.

Let E be an object of $D_{QCoh}(\mathcal{O}_{étale})$ and denote $\mathcal{H}^q = H^q(E)$ its qth cohomology sheaf. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^q) = 0$ for all $p > 0$, all $q \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. By Cohomology on Sites, Lemma 21.23.11 this means that

$$H^q(U, E) = H^0(U, \mathcal{H}^q)$$

for all $U \in \mathcal{B}$. In particular, we find that this holds for affine opens $U \subset X$. It follows that the qth cohomology of $R\epsilon_*E$ over U is the value of the sheaf $\epsilon_*\mathcal{H}^q$ over U. Applying sheafification we obtain

$$H^q(R\epsilon_*E) = \epsilon_*\mathcal{H}^q$$

which in particular shows that $R\epsilon_*$ induces a functor $D_{QCoh}(\mathcal{O}_{étale}) \to D_{QCoh}(\mathcal{O}_X)$. Since ϵ^* is exact we then obtain $H^q(\epsilon^* R\epsilon_* E) = \epsilon^*\epsilon_*\mathcal{H}^q = \mathcal{H}^q$ (by discussion above). Thus the adjunction map $\epsilon^* R\epsilon_* E \to E$ is an isomorphism.

Conversely, for $F \in D_{QCoh}(\mathcal{O}_X)$ the adjunction map $F \to R\epsilon_*\epsilon^* F$ is an isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^* F$ are isomorphic to $\epsilon_* H^m(\epsilon^* F) = \epsilon_*\epsilon^* H^m(F) = H^m(F)$. □

75.5. Derived category of quasi-coherent modules

071W Let S be a scheme. Lemma 75.4.2 shows that the category $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of complexes of $\mathcal{O}_S$-modules on the scheme S or by complexes of $\mathcal{O}$-modules on the small étale site of S. Hence the following definition is compatible with the definition in the case of schemes.

071X **Definition 75.5.1.** Let S be a scheme. Let X be an algebraic space over S. The *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves* is denoted $D_{QCoh}(\mathcal{O}_X)$.

This makes sense by Properties of Spaces, Lemma 66.29.7 and Derived Categories, Section 13.17. Thus we obtain a canonical functor

08F1 (75.5.1.1) $$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

Observe that a flat morphism $f : Y \to X$ of algebraic spaces induces an exact functor $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$, see Morphisms of Spaces, Lemma 67.30.9 and Modules on Sites, Lemma 18.31.2. In particular $Lf^* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$ is computed on any representative complex (Derived Categories, Lemma 13.16.9). We will write $Lf^* = f^*$ when f is flat and we have $H^i(f^*E) = f^*H^i(E)$ for E in $D(\mathcal{O}_X)$ in this case. We will use this often when f is étale. Of course in the étale case the pullback functor is just the restriction to $Y_{étale}$, see Properties of Spaces, Equation (66.26.1.1).

08F2 **Lemma 75.5.2.** *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is in $D_{QCoh}(\mathcal{O}_X)$,*
2. *for every étale morphism $\varphi : U \to X$ where U is an affine scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
3. *for every étale morphism $\varphi : U \to X$ where U is a scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
4. *there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$, and*
5. *there exists a surjective étale morphism of algebraic spaces $f : Y \to X$ such that Lf^*E is an object of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. This follows immediately from the discussion preceding the lemma and Properties of Spaces, Lemma 66.29.6. □

08F3 **Lemma 75.5.3.** *Let S be a scheme. Let X be an algebraic space over S. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 66.29.7. □

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh}.

0D3E **Lemma 75.5.4.** *Let S be a scheme. Let X be an algebraic space over S. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \to H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then*

(1) *$H^q(K) = \lim H^q(K_n)$,*
(2) *$R\lim H^q(K_n) = \lim H^q(K_n)$, and*
(3) *for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.*

Proof. Let $\mathcal{B} \subset \mathrm{Ob}(X_{étale})$ be the set of affine objects. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by the discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by similar reasoning. Part (1) follows from Cohomology on Sites, Lemma 21.23.12 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology on Sites, Lemma 21.23.5. □

08F4 **Lemma 75.5.5.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. Since $a^* \circ Lf^* = Lh^* \circ b^*$ the result follows from Lemma 75.5.2 and the case of schemes which is Derived Categories of Schemes, Lemma 36.3.8. □

08F5 **Lemma 75.5.6.** *Let S be a scheme. Let X be an algebraic space over S. For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Let $\varphi : U \to X$ be a surjective étale morphism from a scheme U. Since $\varphi^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = \varphi^*K \otimes^{\mathbf{L}}_{\mathcal{O}_U} \varphi^*L$ we see from Lemma 75.5.2 that this follows from the case of schemes which is Derived Categories of Schemes, Lemma 36.3.9. □

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

08F6 **Lemma 75.5.7.** *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the map $E \to R\lim \tau_{\geq -n}E$ of Derived Categories, Remark 13.34.4 is an isomorphism*[1].

[1] In particular, E has a K-injective representative, see Derived Categories, Lemma 13.34.5.

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$ as U is an affine scheme. See discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2. Thus the lemma follows from Cohomology on Sites, Lemma 21.23.10 with $d = 0$. □

08F7 **Lemma 75.5.8.** *Let S be a scheme. Let X be an algebraic space over S. Let $F : Mod(\mathcal{O}_X) \to Ab$ be a functor and $N \geq 0$ an integer. Assume that*

(1) *F is left exact,*
(2) *F commutes with countable direct products,*
(3) *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$

(1) *$H^i(RF(\tau_{\leq a}E) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,*
(2) *$H^i(RF(E)) \to H^i(RF(\tau_{\geq b-N+1}E))$ is an isomorphism for $i \geq b$,*
(3) *if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + N - 1]$.*

Proof. Statement (1) is Derived Categories, Lemma 13.16.1.

Proof of statement (2). Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 75.5.7. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 19.13.6. Thus for every $i \in \mathbf{Z}$ we have a short exact sequence

$$0 \to R^1 \lim H^{i-1}(RF(E_n)) \to H^i(RF(E)) \to \lim H^i(RF(E_n)) \to 0$$

see More on Algebra, Remark 15.87.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals $H^i(RF(E_{-b+N-1})$ for any b with $i \geq b$.

For every n we have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 13.12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for $i + n \geq N$ and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

for $i + n + 1 \geq N$. We conclude that

$$H^i(RF(E_n)) \to H^i(RF(E_{n-1}))$$

is an isomorphism for $n \geq N - i$. Thus the systems $H^i(RF(E_n))$ all satisfy the ML condition and the $R^1 \lim$ term in our short exact sequence is zero (see discussion in More on Algebra, Section 15.87). Moreover, the system $H^i(RF(E_n))$ is constant starting with $n = N - i - 1$ as desired.

Proof of (3). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from (1). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + n$ from part (2). □

75.6. Total direct image

08F9 The following lemma is the analogue of Cohomology of Spaces, Lemma 69.8.1.

08FA **Lemma 75.6.1.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S.*

(1) *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
(2) *If Y is quasi-compact, there exists an integer $N = N(X, Y, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.*
(3) *In fact, if Y is quasi-compact we can find $N = N(X, Y, f)$ such that for every morphism of algebraic spaces $Y' \to Y$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to Y'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. This question is local on Y, hence we may assume Y is quasi-compact. Pick $N = N(X, Y, f)$ as in Cohomology of Spaces, Lemma 69.8.1. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$. Moreover $R^pf_*\mathcal{F}$ is quasi-coherent for all p by Cohomology of Spaces, Lemma 69.3.1. These statements remain true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N. In this case we can for example use the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

(Derived Categories, Lemma 13.21.3), the quasi-coherence of $R^pf_*H^q(E)$, and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let V be an affine object of $Y_{étale}$. We have $H^p(V \times_Y X, \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Spaces, Lemma 69.3.2. Hence we may apply Lemma 75.5.8 to the functor $\Gamma(V \times_Y X, -)$ to see that

$$R\Gamma(V, Rf_*E) = R\Gamma(V \times_Y X, E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all V affine in $Y_{étale}$ we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \to E \to \tau_{\geq -n}E \to (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 13.12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that R^qf_*E is equal to $R^qf_*\tau_{\geq -n}E$ for some n and the result above applies. □

08FB **Lemma 75.6.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 75.6.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited.

Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 13.21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Spaces, Lemma 69.5.2. □

08GH **Remark 75.6.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of representable algebraic spaces X and Y over S. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then the diagram

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal[\text{Lemma 75.4.2}]{} & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle Lf_0^*}\uparrow & & \uparrow{\scriptstyle Lf^*} \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal[\text{Lemma 75.4.2}]{} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 75.5.5 and Derived Categories of Schemes, Lemma 36.3.8) is commutative. This follows as the equivalences $D_{QCoh}(\mathcal{O}_{X_0}) \to D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 75.4.2 come from pulling back by the (flat) morphisms of ringed sites $\epsilon : X_{étale} \to X_{0,Zar}$ and $\epsilon : Y_{étale} \to Y_{0,Zar}$ and the diagram of ringed sites

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow{\epsilon} & X_{étale} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{étale} \end{array}$$

is commutative (details omitted). If f is quasi-compact and quasi-separated, equivalently if f_0 is quasi-compact and quasi-separated, then we claim

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal[\text{Lemma 75.4.2}]{} & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle Rf_{0,*}}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal[\text{Lemma 75.4.2}]{} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 75.6.1 and Derived Categories of Schemes, Lemma 36.4.1) is commutative as well. This also follows from the commutative diagram of sites displayed above as the proof of Lemma 75.4.2 shows that the functor $R\epsilon_*$ gives the equivalences $D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{X_0})$ and $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{Y_0})$.

08II **Lemma 75.6.4.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is étale local on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 36.5.2) via Lemma 75.4.2 and Remark 75.6.3. □

08IJ **Lemma 75.6.5.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_Y)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Spaces, Lemma 69.8.2). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 36.5.3) via Lemma 75.4.2 and Remark 75.6.3. □

75.7. Being proper over a base

0CZB This section is the analogue of Cohomology of Schemes, Section 30.26. As usual with material having to do with topology on the sets of points, we have to be careful translating the material to algebraic spaces.

0CZC **Lemma 75.7.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T \subset |X|$ be a closed subset. The following are equivalent*

1. *the morphism $Z \to Y$ is proper if Z is the reduced induced algebraic space structure on T (Properties of Spaces, Definition 66.12.5),*
2. *for some closed subspace $Z \subset X$ with $|Z| = T$ the morphism $Z \to Y$ is proper, and*
3. *for any closed subspace $Z \subset X$ with $|Z| = T$ the morphism $Z \to Y$ is proper.*

Proof. The implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let Z' and Z'' be closed subspaces with $T = |Z'| = |Z''|$ such that $Z' \to Y$ is a proper morphism of algebraic spaces. We have to show that $Z'' \to Y$ is proper too. Let $Z''' = Z' \cup Z''$ be the scheme theoretic union, see Morphisms of Spaces, Definition 67.14.4. Then Z''' is another closed subspace with $|Z'''| = T$. This follows for example from the description of scheme theoretic unions in Morphisms of Spaces, Lemma 67.14.6. Since $Z'' \to Z'''$ is a closed immersion it suffices to prove that $Z''' \to Y$ is proper (see Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4). The morphism $Z' \to Z'''$ is a bijective closed immersion and in particular surjective and universally closed. Then the fact that $Z' \to Y$ is separated implies that $Z''' \to Y$ is separated, see Morphisms of Spaces, Lemma 67.9.8. Moreover $Z''' \to Y$ is locally of finite type as $X \to Y$ is locally of finite type (Morphisms of Spaces, Lemmas 67.23.7 and 67.23.2). Since $Z' \to Y$ is quasi-compact and $Z' \to Z'''$ is a universal homeomorphism we see that $Z''' \to Y$ is quasi-compact. Finally, since $Z' \to Y$ is universally closed, we see that the same thing is true for $Z''' \to Y$ by Morphisms of Spaces, Lemma 67.40.7. This finishes the proof. □

0CZD **Definition 75.7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T \subset |X|$ be a closed subset. We say T *is proper over* Y if the equivalent conditions of Lemma 75.7.1 are satisfied.

The lemma used in the definition above is false if the morphism $f : X \to Y$ is not locally of finite type. Therefore we urge the reader not to use this terminology if f is not locally of finite type.

0CZE **Lemma 75.7.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T' \subset T \subset |X|$ be closed subsets. If T is proper over Y, then the same is true for T'.*

Proof. Omitted. □

0CZF **Lemma 75.7.4.** *Let S be a scheme. Consider a cartesian diagram of algebraic spaces over S*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f locally of finite type. If T is a closed subset of $|X|$ proper over Y, then $|g'|^{-1}(T)$ is a closed subset of $|X'|$ proper over Y'.

Proof. Observe that the statement makes sense as f' is locally of finite type by Morphisms of Spaces, Lemma 67.23.3. Let $Z \subset X$ be the reduced induced closed subspace structure on T. Denote $Z' = (g')^{-1}(Z)$ the scheme theoretic inverse image. Then $Z' = X' \times_X Z = (Y' \times_Y X) \times_X Z = Y' \times_Y Z$ is proper over Y' as a base change of Z over Y (Morphisms of Spaces, Lemma 67.40.3). On the other hand, we have $T' = |Z'|$. Hence the lemma holds. □

0CZG **Lemma 75.7.5.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which are locally of finite type over B.*

(1) *If Y is separated over B and $T \subset |X|$ is a closed subset proper over B, then $|f|(T)$ is a closed subset of $|Y|$ proper over B.*
(2) *If f is universally closed and $T \subset |X|$ is a closed subset proper over B, then $|f|(T)$ is a closed subset of Y proper over B.*
(3) *If f is proper and $T \subset |Y|$ is a closed subset proper over B, then $|f|^{-1}(T)$ is a closed subset of $|X|$ proper over B.*

Proof. Proof of (1). Assume Y is separated over B and $T \subset |X|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T and apply Morphisms of Spaces, Lemma 67.40.8 to $Z \to Y$ over B to conclude.

Proof of (2). Assume f is universally closed and $T \subset |X|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T and let Z' be the reduced induced closed subspace structure on $|f|(T)$. We obtain an induced morphism $Z \to Z'$. Denote $Z'' = f^{-1}(Z')$ the scheme theoretic inverse image. Then $Z'' \to Z'$ is universally closed as a base change of f (Morphisms of Spaces, Lemma 67.40.3). Hence $Z \to Z'$ is universally closed as a composition of the closed immersion $Z \to Z''$ and $Z'' \to Z'$ (Morphisms of Spaces, Lemmas 67.40.5 and 67.40.4). We conclude that $Z' \to B$ is separated by Morphisms of Spaces, Lemma 67.9.8. Since $Z \to B$ is quasi-compact and $Z \to Z'$ is surjective we see that $Z' \to B$ is quasi-compact. Since $Z' \to B$ is the composition of $Z' \to Y$ and $Y \to B$ we see that $Z' \to B$ is locally of finite type (Morphisms of Spaces, Lemmas 67.23.7 and

67.23.2). Finally, since $Z \to B$ is universally closed, we see that the same thing is true for $Z' \to B$ by Morphisms of Spaces, Lemma 67.40.7. This finishes the proof.

Proof of (3). Assume f is proper and $T \subset |Y|$ is a closed subset proper over B. Let Z be the reduced induced closed subspace structure on T. Denote $Z' = f^{-1}(Z)$ the scheme theoretic inverse image. Then $Z' \to Z$ is proper as a base change of f (Morphisms of Spaces, Lemma 67.40.3). Whence $Z' \to B$ is proper as the composition of $Z' \to Z$ and $Z \to B$ (Morphisms of Spaces, Lemma 67.40.4). This finishes the proof. □

0CZH **Lemma 75.7.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $T_i \subset |X|$, $i = 1, \ldots, n$ be closed subsets. If T_i, $i = 1, \ldots, n$ are proper over Y, then the same is true for $T_1 \cup \ldots \cup T_n$.*

Proof. Let Z_i be the reduced induced closed subscheme structure on T_i. The morphism

$$Z_1 \amalg \ldots \amalg Z_n \longrightarrow X$$

is finite by Morphisms of Spaces, Lemmas 67.45.10 and 67.45.11. As finite morphisms are universally closed (Morphisms of Spaces, Lemma 67.45.9) and since $Z_1 \amalg \ldots \amalg Z_n$ is proper over S we conclude by Lemma 75.7.5 part (2) that the image $Z_1 \cup \ldots \cup Z_n$ is proper over S. □

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Then the support $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of $|X|$, see Morphisms of Spaces, Lemma 67.15.2. Hence it makes sense to say "the support of $\mathcal{F}$ is proper over Y".

0CZI **Lemma 75.7.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *the support of $\mathcal{F}$ is proper over Y,*
2. *the scheme theoretic support of $\mathcal{F}$ (Morphisms of Spaces, Definition 67.15.4) is proper over Y, and*
3. *there exists a closed subspace $Z \subset X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $Z \to Y$ is proper, and (b) $(Z \to X)_*\mathcal{G} = \mathcal{F}$.*

Proof. The support $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of $|X|$, see Morphisms of Spaces, Lemma 67.15.2. Hence we can apply Definition 75.7.2. Since the scheme theoretic support of $\mathcal{F}$ is a closed subspace whose underlying closed subset is $\mathrm{Supp}(\mathcal{F})$ we see that (1) and (2) are equivalent by Definition 75.7.2. It is clear that (2) implies (3). Conversely, if (3) is true, then $\mathrm{Supp}(\mathcal{F}) \subset |Z|$ and hence $\mathrm{Supp}(\mathcal{F})$ is proper over Y for example by Lemma 75.7.3. □

0CZJ **Lemma 75.7.8.** *Let S be a scheme. Consider a cartesian diagram of algebraic spaces over S*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. If the support of $\mathcal{F}$ is proper over Y, then the support of $(g')^*\mathcal{F}$ is proper over Y'.

Proof. Observe that the statement makes sense because $(g') * \mathcal{F}$ is of finite type by Modules on Sites, Lemma 18.23.4. We have $\mathrm{Supp}((g')^*\mathcal{F}) = |g'|^{-1}(\mathrm{Supp}(\mathcal{F}))$ by Morphisms of Spaces, Lemma 67.15.2. Thus the lemma follows from Lemma 75.7.4. □

0CZK **Lemma 75.7.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$, $\mathcal{G}$ be finite type, quasi-coherent $\mathcal{O}_X$-module.*

(1) *If the supports of $\mathcal{F}$, $\mathcal{G}$ are proper over Y, then the same is true for $\mathcal{F} \oplus \mathcal{G}$, for any extension of $\mathcal{G}$ by $\mathcal{F}$, for $\mathrm{Im}(u)$ and $\mathrm{Coker}(u)$ given any $\mathcal{O}_X$-module map $u : \mathcal{F} \to \mathcal{G}$, and for any quasi-coherent quotient of $\mathcal{F}$ or $\mathcal{G}$.*
(2) *If Y is locally Noetherian, then the category of coherent $\mathcal{O}_X$-modules with support proper over Y is a Serre subcategory (Homology, Definition 12.10.1) of the abelian category of coherent $\mathcal{O}_X$-modules.*

Proof. Proof of (1). Let T, T' be the support of $\mathcal{F}$ and $\mathcal{G}$. Then all the sheaves mentioned in (1) have support contained in $T \cup T'$. Thus the assertion itself is clear from Lemmas 75.7.3 and 75.7.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Properties of Spaces, Section 66.29. For "finite type" we refer the reader to Properties of Spaces, Section 66.30.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as X is locally Noetherian by Morphisms of Spaces, Lemma 67.23.5 and by the description of the category of coherent modules in Cohomology of Spaces, Section 69.12. □

08GC **Lemma 75.7.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with support proper over Y. Then $R^p f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $p \geq 0$.*

Proof. By Lemma 75.7.7 there exists a closed immersion $i : Z \to X$ with $g = f \circ i : Z \to Y$ proper and $\mathcal{F} = i_*\mathcal{G}$ for some coherent module $\mathcal{G}$ on Z. We see that $R^p g_*\mathcal{G}$ is coherent on S by Cohomology of Spaces, Lemma 69.20.2. On the other hand, $R^q i_*\mathcal{G} = 0$ for $q > 0$ (Cohomology of Spaces, Lemma 69.12.9). By Cohomology on Sites, Lemma 21.14.7 we get $R^p f_*\mathcal{F} = R^p g_*\mathcal{G}$ and the lemma follows. □

75.8. Derived category of coherent modules

08GI Let S be a scheme. Let X be a locally Noetherian algebraic space over S. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.10 and Cohomology of Spaces, Lemma 69.12.3. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

08GJ (75.8.0.1) $$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

08GK **Lemma 75.8.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over Y for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$.*

Proof. Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 13.21.3. By assumption and Lemma 75.7.10 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D_{Coh}(\mathcal{O}_Y)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Spaces, Lemma 69.8.1 or from Lemma 75.6.1. □

0D0R **Lemma 75.8.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^+_{Coh}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^+_{Coh}(\mathcal{O}_Y)$.*

Proof. The proof is the same as the proof of Lemma 75.8.1. You can also deduce it from Lemma 75.8.1 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a}E \to E \to \tau_{\geq a+1}E \to \tau_{\leq a}E[1]$. □

0D0S **Lemma 75.8.3.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. If L is in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D^+_{Coh}(\mathcal{O}_X)$.*

Proof. We can check whether an object of $D(\mathcal{O}_X)$ is in $D_{Coh}(\mathcal{O}_X)$ étale locally on X, see Cohomology of Spaces, Lemma 69.12.2. Hence this lemma follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.11.5. □

0D0T **Lemma 75.8.4.** *Let A be a Noetherian ring. Let X be a proper algebraic space over A. For L in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, the A-modules $\mathrm{Ext}^n_{\mathcal{O}_X}(K, L)$ are finite.*

Proof. Recall that

$$\mathrm{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\mathrm{Spec}(A), Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology on Sites, Lemma 21.35.1 and Cohomology on Sites, Section 21.14. Thus the result follows from Lemmas 75.8.3 and 75.8.2. □

75.9. Induction principle

08GL In this section we discuss an induction principle for algebraic spaces analogous to what is Cohomology of Schemes, Lemma 30.4.1 for schemes. To formulate it we introduce the notion of an *elementary distinguished square*; this terminology is borrowed from [**MV99**]. The principle as formulated here is implicit in the paper [**GR71**] by Raynaud and Gruson. A related principle for algebraic stacks is [**Ryd10**, Theorem D] by David Rydh.

08GM **Definition 75.9.1.** Let S be a scheme. A commutative diagram

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & W \end{array}$$

of algebraic spaces over S is called an *elementary distinguished square* if

(1) U is an open subspace of W and j is the inclusion morphism,
(2) f is étale, and
(3) setting $T = W \setminus U$ (with reduced induced subspace structure) the morphism $f^{-1}(T) \to T$ is an isomorphism.

We will indicate this by saying: "Let $(U \subset W, f : V \to W)$ be an elementary distinguished square."

Note that if $(U \subset W, f : V \to W)$ is an elementary distinguished square, then we have $W = U \cup f(V)$. Thus $\{U \to W, V \to W\}$ is an étale covering of W. It turns out that these étale coverings have nice properties and that in some sense there are "enough" of them.

08GN **Lemma 75.9.2.** *Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *If $V' \subset V$ and $U \subset U' \subset W$ are open subspaces and $W' = U' \cup f(V')$ then $(U' \subset W', f|_{V'} : V' \to W')$ is an elementary distinguished square.*
(2) *If $p : W' \to W$ is a morphism of algebraic spaces, then $(p^{-1}(U) \subset W', V \times_W W' \to W')$ is an elementary distinguished square.*
(3) *If $S' \to S$ is a morphism of schemes, then $(S' \times_S U \subset S' \times_S W, S' \times_S V \to S' \times_S W)$ is an elementary distinguished square.*

Proof. Omitted. □

08GP **Lemma 75.9.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let P be a property of the quasi-compact and quasi-separated objects of $X_{spaces,étale}$. Assume that*

(1) *P holds for every affine object of $X_{spaces,étale}$,*
(2) *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
   (a) *W is a quasi-compact and quasi-separated object of $X_{spaces,étale}$,*
   (b) *U is quasi-compact,*
   (c) *V is affine, and*
   (d) *P holds for U, V, and $U \times_W V$,*

   then P holds for W.

Then P holds for every quasi-compact and quasi-separated object of $X_{spaces,étale}$ and in particular for X.

Proof. We first claim that P holds for every representable quasi-compact and quasi-separated object of $X_{spaces,étale}$. Namely, suppose that $U \to X$ is étale and U is a quasi-compact and quasi-separated scheme. By assumption (1) property P holds for every affine open of U. Moreover, if $W, V \subset U$ are quasi-compact open with V affine and P holds for W, V, and $W \cap V$, then P holds for $W \cup V$ by (2) (as the pair $(W \subset W \cup V, V \to W \cup V)$ is an elementary distinguished square). Thus P holds for U by the induction principle for schemes, see Cohomology of Schemes, Lemma 30.4.1.

To finish the proof it suffices to prove P holds for X (because we can simply replace X by any quasi-compact and quasi-separated object of $X_{spaces,étale}$ we want to prove the result for). We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 68.8.6. We will prove that P holds for U_p by descending induction on p. Note that P holds for U_{n+1} by (1) as an empty algebraic space is affine. Assume P holds for U_{p+1}. Note that $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, but (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = U_{p+1}$ and

$$W_{p,i} = U_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X. Then we have

$$U_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = U_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 75.9.2. Note that P holds for each $V_{p,1}$ (as affine schemes) and for $W_{p,i} \times_{W_{p,i+1}} V_{p,i+1}$ as this is a quasi-compact open of $V_{p,i+1}$ and hence P holds for it by the first paragraph of this proof. Thus (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = U_p$. □

08GQ **Lemma 75.9.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$. Let P be a property of the elements of $\mathcal{B}$. Assume that*

(1) *every $W \in \mathcal{B}$ is quasi-compact and quasi-separated,*
(2) *if $W \in \mathcal{B}$ and $U \subset W$ is quasi-compact open, then $U \in \mathcal{B}$,*
(3) *if $V \in \mathrm{Ob}(X_{spaces,étale})$ is affine, then (a) $V \in \mathcal{B}$ and (b) P holds for V,*
(4) *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
   (a) *$W \in \mathcal{B}$,*
   (b) *U is quasi-compact,*
   (c) *V is affine, and*
   (d) *P holds for U, V, and $U \times_W V$,*
   then P holds for W.

Then P holds for every $W \in \mathcal{B}$.

Proof. This is proved in exactly the same manner as the proof of Lemma 75.9.3. (We remark that (4)(d) makes sense as $U \times_W V$ is a quasi-compact open of V hence an element of $\mathcal{B}$ by conditions (2) and (3).) □

08GR **Remark 75.9.5.** How to choose the collection $\mathcal{B}$ in Lemma 75.9.4? Here are some examples:

(1) If X is quasi-compact and separated, then we can choose $\mathcal{B}$ to be the set of quasi-compact and separated objects of $X_{spaces,étale}$. Then $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a). With this choice of $\mathcal{B}$ Lemma 75.9.4 reproduces Lemma 75.9.3.
(2) If X is quasi-compact with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1), then we can choose $\mathcal{B}$ to be the set of objects of $X_{spaces,étale}$ which are quasi-compact and have affine diagonal over $\mathbf{Z}$. Again $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a).
(3) If X is quasi-compact and quasi-separated, then the smallest subset $\mathcal{B}$ which contains X and satisfies (1), (2), and (3)(a) is given by the rule

$W \in \mathcal{B}$ if and only if either W is a quasi-compact open subspace of X, or W is a quasi-compact open of an affine object of $X_{spaces,étale}$.

Here is a variant where we extend the truth from an open to larger opens.

09IT **Lemma 75.9.6.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open subspace. Let P be a property of quasi-compact open subspaces of X. Assume that*

(1) *P holds for W, and*
(2) *for every elementary distinguished square $(W_1 \subset W_2, f : V \to W_2)$ where such that*
   (a) *W_1, W_2 are quasi-compact open subspaces of X,*
   (b) *$W \subset W_1$,*
   (c) *V is affine, and*
   (d) *P holds for W_1,*
   then P holds for W_2.

Then P holds for X.

Proof. We can deduce this from Lemma 75.9.4, but instead we will give a direct argument by explicitly redoing the proof of Lemma 75.9.3. We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 68.8.6. We will prove that P holds for $W_p = W \cup U_p$ by descending induction on p. This will finish the proof as $W_1 = X$. Note that P holds for $W_{n+1} = W \cap U_{n+1} = W$ by (1). Assume P holds for W_{p+1}. Observe that $W_p \setminus W_{p+1}$ (with reduced induced subspace structure) is a closed subspace of $U_p \setminus U_{p+1}$. Since $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, the same is true for $(W_{p+1} \subset W_p, f_p : V_p \to W_p)$. However (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = W_{p+1}$ and

$$W_{p,i} = W_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X containing W. Then we have

$$W_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = W_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 75.9.2. Now (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = W_p$. □

75.10. Mayer-Vietoris

08GS In this section we prove that an elementary distinguished triangle gives rise to various Mayer-Vietoris sequences.

Let S be a scheme. Let $U \to X$ be an étale morphism of algebraic spaces over S. In Properties of Spaces, Section 66.27 it was shown that $U_{spaces,étale} = X_{spaces,étale}/U$ compatible with structure sheaves. Hence in this situation we often think of the morphism $j_U : U \to X$ as a localization morphism (see Modules on Sites, Definition 18.19.1). In particular we think of pullback j_U^* as restriction to U and we often

denote it by $|_U$; this is compatible with Properties of Spaces, Equation (66.26.1.1). In particular we see that

08GT (75.10.0.1) $$(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$$

if $\overline{u}$ is a geometric point of U and $\overline{x}$ the image of $\overline{u}$ in X. Moreover, restriction has an exact left adjoint $j_{U!}$, see Modules on Sites, Lemmas 18.19.2 and 18.19.3. Finally, recall that if $\mathcal{G}$ is an $\mathcal{O}_X$-module, then

08GU (75.10.0.2) $$(j_{U!}\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

for any geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ where the direct sum is over those morphisms $\overline{u} : \mathrm{Spec}(k) \to U$ such that $j_U \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.38.1 and Properties of Spaces, Lemma 66.19.13.

08GV **Lemma 75.10.1.** *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*
$$0 \to j_{U\times_X V!}\mathcal{F}|_{U\times_X V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$
(2) *For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*
$$j_{U\times_X V!}E|_{U\times_X V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U\times_X V!}E|_{U\times_X V}[1]$$
in $D(\mathcal{O}_X)$.

Proof. To show the sequence of (1) is exact we may check on stalks at geometric points by Properties of Spaces, Theorem 66.19.12. Let $\overline{x}$ be a geometric point of X. By Equations (75.10.0.1) and (75.10.0.2) taking stalks at $\overline{x}$ we obtain the sequence
$$0 \to \bigoplus\nolimits_{(\overline{u},\overline{v})} \mathcal{F}_{\overline{x}} \to \bigoplus\nolimits_{\overline{u}} \mathcal{F}_{\overline{x}} \oplus \bigoplus\nolimits_{\overline{v}} \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to 0$$
This sequence is exact because for every $\overline{x}$ there either is exactly one $\overline{u}$ mapping to $\overline{x}$, or there is no $\overline{u}$ and exactly one $\overline{v}$ mapping to $\overline{x}$.

Proof of (2). We have seen in Cohomology on Sites, Section 21.20 that the restriction functors and the extension by zero functors on derived categories are computed by just applying the functor to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules
$$0 \to j_{U\times_X V!}\mathcal{E}^\bullet|_{U\times_X V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$
which is short exact by (1). □

08GW **Lemma 75.10.2.** *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*
$$0 \to \mathcal{F} \to j_{U,*}\mathcal{F}|_U \oplus j_{V,*}\mathcal{F}|_V \to j_{U\times_X V,*}\mathcal{F}|_{U\times_X V} \to 0$$
(2) *For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*
$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\times_X V,*}E|_{U\times_X V} \to E[1]$$
in $D(\mathcal{O}_X)$.

Proof. Let W be an object of $X_{étale}$. We claim the sequence

$$0 \to \mathcal{F}(W) \to \mathcal{F}(W \times_X U) \oplus \mathcal{F}(W \times_X V) \to \mathcal{F}(W \times_X U \times_X V)$$

is exact and that an element of the last group can locally on W be lifted to the middle one. By Lemma 75.9.2 the pair $(W \times_X U \subset W, V \times_X W \to W)$ is an elementary distinguished square. Thus we may assume $W = X$ and it suffices to prove the same thing for

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V)$$

We have seen that

$$0 \to j_{U \times_X V!}\mathcal{O}_{U \times_X V} \to j_{U!}\mathcal{O}_U \oplus j_{V!}\mathcal{O}_V \to \mathcal{O}_X \to 0$$

is a exact sequence of $\mathcal{O}_X$-modules in Lemma 75.10.1 and applying the right exact functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$ gives the sequence above. This also means that the obstruction to lifting $s \in \mathcal{F}(U \times_X V)$ to an element of $\mathcal{F}(U) \oplus \mathcal{F}(V)$ lies in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$. By locality of cohomology (Cohomology on Sites, Lemma 21.7.3) this obstruction vanishes étale locally on X and the proof of (1) is complete.

Proof of (2). Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then $\mathcal{I}^\bullet|U$ is a K-injective complex (Cohomology on Sites, Lemma 21.20.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \times_X V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

This sequence is exact by (1). □

08JK **Lemma 75.10.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $(U \subset X, V \to X)$ be an elementary distinguished square. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \times_X V} : U \times_X V \to Y$ the restrictions. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \times_X V}) \to Rf_*E[1]$$

in $D(\mathcal{O}_Y)$. This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Cohomology on Sites, Lemma 21.20.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

To see this is a short exact sequence of complexes we argue as follows. Pick an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. Apply f_* to the short exact sequence

$$0 \to \mathcal{I} \to j_{U,*}\mathcal{I}|_U \oplus j_{V,*}\mathcal{I}|_V \to j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

of Lemma 75.10.2 and use that $R^1 f_*\mathcal{I} = 0$ to get a short exact sequence

$$0 \to f_*\mathcal{I} \to f_*j_{U,*}\mathcal{I}|_U \oplus f_*j_{V,*}\mathcal{I}|_V \to f_*j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

The proof is finished by observing that $a_* = f_* j_{U,*}$ and similarly for b_* and c_*. □

08H9 **Lemma 75.10.4.** *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\ldots \longrightarrow \mathrm{Ext}^{-1}(E_{U\times_X V}, F_{U\times_X V}) \longrightarrow \mathrm{Hom}(E,F) \longrightarrow \mathrm{Hom}(E_U,F_U)\oplus \mathrm{Hom}(E_V,F_V) \longrightarrow \mathrm{Hom}(E_{U\times_X V}, F_{U\times_X V})$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 75.10.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Cohomology on Sites, Lemma 21.20.8. □

0CRS **Lemma 75.10.5.** *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$R\Gamma(X,E) \to R\Gamma(U,E) \oplus R\Gamma(V,E) \to R\Gamma(U \times_X V, E) \to R\Gamma(X,E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X,E) \to H^n(U,E) \oplus H^n(V,E) \to H^n(U \times_X V, E) \to H^{n+1}(X,E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. In the proof of Lemma 75.10.2 we found a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\times_X V,*}\mathcal{I}^\bullet|_{U\times_X V} \to 0$$

Since $H^1(X, \mathcal{I}^n) = 0$, we see that taking global sections gives an exact sequence of complexes

$$0 \to \Gamma(X,\mathcal{I}^\bullet) \to \Gamma(U,\mathcal{I}^\bullet) \oplus \Gamma(V,\mathcal{I}^\bullet) \to \Gamma(U \times_X V, \mathcal{I}^\bullet) \to 0$$

Since these complexes represent $R\Gamma(X,E)$, $R\Gamma(U,E)$, $R\Gamma(V,E)$, and $R\Gamma(U\times_X V,E)$ we get a distinguished triangle by Derived Categories, Section 13.12 and especially Lemma 13.12.1. □

08HA **Lemma 75.10.6.** *Let S be a scheme. Let $j : U \to X$ be a étale morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $W = V \times_X U$ and denote $j_W : W \to V$ the projection morphism. Then $(j_!E)|_V = j_{W!}(E|_W)$ for E in $D(\mathcal{O}_U)$.*

Proof. This is true because $(j_!\mathcal{F})|_V = j_{W!}(\mathcal{F}|_W)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ as follows immediately from the construction of the functors $j_!$ and $j_{W!}$, see Modules on Sites, Lemma 18.19.2. □

08GG **Lemma 75.10.7.** *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Set $T = |X| \setminus |U|$.*

(1) *If E is an object of $D(\mathcal{O}_X)$ supported on T, then (a) $E \to Rj_*(E|_V)$ and (b) $j_!(E|_V) \to E$ are isomorphisms.*
(2) *If F is an object of $D(\mathcal{O}_V)$ supported on $j^{-1}T$, then (a) $F \to (j_!F)|_V$, (b) $(Rj_*F)|_V \to F$, and (c) $j_!F \to Rj_*F$ are isomorphisms.*

Proof. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. Then we see that $E|_U = 0$ and $E|_{U \times_X V} = 0$ as T doesn't meet U and $j^{-1}T$ doesn't meet $U \times_X V$. Thus (1)(a) follows from Lemma 75.10.2. In exactly the same way (1)(b) follows from Lemma 75.10.1.

Let F be an object of $D(\mathcal{O}_V)$ whose cohomology sheaves are supported on $j^{-1}T$. By Lemma 75.3.1 we have $(Rj_*F)|_U = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. Similarly $(j_!F)|_U = j_{W!}(F|_W) = 0$ by Lemma 75.10.6. Thus $j_!F$ and Rj_*F are supported on T and $(j_!F)|_V$ and $(Rj_*F)|_V$ are supported on $j^{-1}(T)$. To check that the maps (2)(a), (b), (c) are isomorphisms in the derived category, it suffices to check that these map induce isomorphisms on stalks of cohomology sheaves at geometric points of T and $j^{-1}(T)$ by Properties of Spaces, Theorem 66.19.12. This we may do after replacing X by V, U by $U \times_X V$, V by $V \times_X V$ and F by $F|_{V \times_X V}$ (restriction via first projection), see Lemmas 75.3.1, 75.10.6, and 75.9.2. Since $V \times_X V \to V$ has a section this reduces (2) to the case that $j : V \to X$ has a section.

Assume j has a section $\sigma : X \to V$. Set $V' = \sigma(X)$. This is an open subspace of V. Set $U' = j^{-1}(U)$. This is another open subspace of V. Then $(U' \subset V, V' \to V)$ is an elementary distinguished square. Observe that $F|_{U'} = 0$ and $F|_{V' \cap U'} = 0$ because F is supported on $j^{-1}(T)$. Denote $j' : V' \to V$ the open immersion and $j_{V'} : V' \to X$ the composition $V' \to V \to X$ which is the inverse of σ. Set $F' = \sigma^* F$. The distinguished triangles of Lemmas 75.10.1 and 75.10.2 show that $F = j'_!(F|_{V'})$ and $F = Rj'_*(F|_{V'})$. It follows that $j_!F = j_!j'_!(F|_{V'}) = j_{V'!}F = F'$ because $j_{V'} : V' \to X$ is an isomorphism and the inverse of σ. Similarly, $Rj_*F = Rj_*Rj'_*F = Rj_{V',*}F = F'$. This proves (2)(c). To prove (2)(a) and (2)(b) it suffices to show that $F = F'|_V$. This is clear because both F and $F'|_V$ restrict to zero on U' and $U' \cap V'$ and the same object on V'. □

We can glue complexes!

08HB **Lemma 75.10.8.** *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. Suppose given*

(1) *an object A of $D(\mathcal{O}_U)$,*
(2) *an object B of $D(\mathcal{O}_V)$, and*
(3) *an isomorphism $c : A|_{U \times_X V} \to B|_{U \times_X V}$.*

Then there exists an object F of $D(\mathcal{O}_X)$ and isomorphisms $f : F|_U \to A$, $g : F|_V \to B$ such that $c = g|_{U \times_X V} \circ f^{-1}|_{U \times_X V}$. Moreover, given

(1) *an object E of $D(\mathcal{O}_X)$,*
(2) *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
(3) *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*

such that

$$a|_{U \times_X V} = b|_{U \times_X V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is $a \circ f$ and whose restriction to V is $b \circ g$.

Proof. Denote j_U, j_V, $j_{U\times_X V}$ the corresponding morphisms towards X. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U\times_X V,*}(B|_{U\times_X V}) \to F[1]$$

Here the map $Rj_{V,*}B \to Rj_{U\times_X V,*}(B|_{U\times_X V})$ is the obvious one. The map $Rj_{U,*}A \to Rj_{U\times_X V,*}(B|_{U\times_X V})$ is the composition of $Rj_{U,*}A \to Rj_{U\times_X V,*}(A|_{U\times_X V})$ with $Rj_{U\times_X V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U\times_X V,*}(B|_{U\times_X V}))|_U \to F|_U[1]$$

Denote $j : U \times_X V \to U$. Compatibility of restriction and total direct image (Lemma 75.3.1) shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U\times_X V,*}(B|_{U\times_X V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U\times_X V})$. Hence the second arrow of the last displayed equation has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism.

To see that the morphism $F|_V \to B$ is an isomorphism we will use a trick. Namely, choose a distinguished triangle

$$F|_V \to B \to B' \to F[1]|_V$$

in $D(\mathcal{O}_V)$. Since $F|_U \to A$ is an isomorphism, and since we have the isomorphism $c : A|_{U\times_X V} \to B|_{U\times_X V}$ the restriction of $F|_V \to B$ is an isomorphism over $U \times_X V$. Thus B' is supported on $j_V^{-1}(T)$ where $T = |X| \setminus |U|$. On the other hand, there is a morphism of distinguished triangles

$$\begin{array}{ccccccc}
F & \longrightarrow & Rj_{U,*}F|_U \oplus Rj_{V,*}F|_V & \longrightarrow & Rj_{U\times_X V,*}F|_{U\times_X V} & \longrightarrow & F[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & Rj_{U,*}A \oplus Rj_{V,*}B & \longrightarrow & Rj_{U\times_X V,*}(B|_{U\times_X V}) & \longrightarrow & F[1]
\end{array}$$

The all of the vertical maps in this diagram are isomorphisms, except for the map $Rj_{V,*}F|_V \to Rj_{V,*}B$, hence that is an isomorphism too (Derived Categories, Lemma 13.4.3). This implies that $Rj_{V,*}B' = 0$. Hence $B' = 0$ by Lemma 75.10.7.

The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 75.10.4. □

75.11. The coherator

08GX Let S be a scheme. Let X be an algebraic space over S. The *coherator* is a functor

$$Q_X : \mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$. It exists for any algebraic space X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties of Spaces, Proposition 66.32.2. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\mathit{QCoh}(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$ by Derived Categories, Lemma 13.30.3.

In this section we will study the functor RQ_X. In Section 75.19 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ (when it exists).

08GY **Lemma 75.11.1.** *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Spaces, Lemma 69.8.2. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.9. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. The statement is étale local on Y hence we may assume Y affine. As an affine morphism is representable we reduce to the case of schemes by the compatibility of Remark 75.6.3. The case of schemes is Derived Categories of Schemes, Lemma 36.7.1. □

08GZ **Lemma 75.11.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \to D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Morphisms of Spaces, Lemma 67.11.2. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 66.32.2). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology on Sites, Lemma 21.19.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.30.3.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. □

08H0 **Lemma 75.11.3.** *Let S be a scheme. Let X be an affine algebraic space over S. Set $A = \Gamma(X, \mathcal{O}_X)$. Then*

(1) *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,*
(2) *$RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
(3) *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (75.5.1.1).*

Proof. Let $X_0 = \mathrm{Spec}(A)$ be the affine scheme representing X. Recall that there is a morphism of ringed sites $\epsilon : X_{étale} \to X_{0,Zar}$ which induces equivalences

$$QCoh(\mathcal{O}_X) \underset{\epsilon^*}{\overset{\epsilon_*}{\rightleftarrows}} QCoh(\mathcal{O}_{X_0})$$

see Lemma 75.4.2. Hence we see that $Q_X = \epsilon^* \circ Q_{X_0} \circ \epsilon_*$ by uniqueness of adjoint functors. Hence (1) follows from the description of Q_{X_0} in Derived Categories of Schemes, Lemma 36.7.3 and the fact that $\Gamma(X_0, \epsilon_*\mathcal{F}) = \Gamma(X, \mathcal{F})$. Part (2) follows from (1) and the fact that the functor from A-modules to quasi-coherent $\mathcal{O}_X$-modules is exact. The third assertion now follows from the result for schemes (Derived Categories of Schemes, Lemma 36.7.3) and Lemma 75.4.2. □

Next, we prove a criterion for when the functor $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence.

09TG **Lemma 75.11.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Suppose that for every étale morphism $j : V \to W$ with $W \subset X$ quasi-compact open and V affine the right derived functor*

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_W))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_V) \to QCoh(\mathcal{O}_W)$ fits into a commutative diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_V)) & \xrightarrow[i_V]{} & D_{QCoh}(\mathcal{O}_V) \\ \Phi \downarrow & & \downarrow Rj_* \\ D(QCoh(\mathcal{O}_W)) & \xrightarrow{i_W} & D_{QCoh}(\mathcal{O}_W) \end{array}$$

Then the functor (75.5.1.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. We first use the induction principle to prove i_X is fully faithful. More precisely, we will use Lemma 75.9.6. Let $(U \subset W, V \to W)$ be an elementary distinguished square with V affine and U, W quasi-compact open in X. Assume that i_U is fully faithful. We have to show that i_W is fully faithful. We may replace X by W, i.e., we may assume $W = X$ (we do this just to simplify the notation – observe that the condition in the statement of the lemma is preserved under this operation).

Suppose that A, B are objects of $D(QCoh(\mathcal{O}_X))$. We want to show that

$$\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))$$

is bijective. Let $T = |X| \setminus |U|$.

Assume first $i_X(B)$ is supported on T. In this case the map

$$i_X(B) \to Rj_{V,*}(i_X(B)|_V) = Rj_{V,*}(i_V(B|_V))$$

is a quasi-isomorphism (Lemma 75.10.7). By assumption we have an isomorphism $i_X(\Phi(B|_V)) \to Rj_{V,*}(i_V(B|_V))$ in $D(\mathcal{O}_X)$. Moreover, Φ and $-|_V$ are adjoint functors on the derived categories of quasi-coherent modules (by Derived Categories, Lemma 13.30.3). The adjunction map $B \to \Phi(B|_V)$ becomes an isomorphism after applying i_X, whence is an isomorphism in $D(QCoh(\mathcal{O}_X))$. Hence

$$\begin{aligned}
\mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, B) &= \mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, \Phi(B|_V)) \\
&= \mathrm{Mor}_{D(QCoh(\mathcal{O}_V))}(A|_V, B|_V) \\
&= \mathrm{Mor}_{D(\mathcal{O}_V)}(i_V(A|_V), i_V(B|_V)) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), Rj_{V,*}(i_V(B|_V))) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))
\end{aligned}$$

as desired. Here we have used that i_V is fully faithful (Lemma 75.11.3).

In general, choose any complex $\mathcal{B}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules representing B. Next, choose any quasi-isomorphism $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ of complexes of quasi-coherent modules on U. As $j_U : U \to X$ is quasi-compact and quasi-separated the functor $j_{U,*}$ transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus there is a canonical map $\mathcal{B}^\bullet \to j_{U,*}(\mathcal{B}^\bullet|_U) \to j_{U,*}\mathcal{C}^\bullet$ of complexes of quasi-coherent modules on X. Set $B'' = j_{U,*}\mathcal{C}^\bullet$ in $D(QCoh(\mathcal{O}_X))$ and choose a distinguished triangle

$$B \to B'' \to B' \to B[1]$$

in $D(QCoh(\mathcal{O}_X))$. Since the first arrow of the triangle restricts to an isomorphism over U we see that B' is supported on T. Hence in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1]) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'')) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'))
\end{array}$$

we have exact columns and the top and bottom horizontal arrows are bijective. Finally, choose a complex $\mathcal{A}^\bullet$ of quasi-coherent modules representing A.

Let $\alpha : i_X(A) \to i_X(B)$ be a morphism between in $D(\mathcal{O}_X)$. The restriction $\alpha|_U$ comes from a morphism in $D(QCoh(\mathcal{O}_U))$ as i_U is fully faithful. Hence there exists a choice of $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ as above such that $\alpha|_U$ is represented by an

actual map of complexes $\mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$. This corresponds to a map of complexes $\mathcal{A} \to j_{U,*}\mathcal{C}^\bullet$. In other words, the image of α in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B''))$ comes from an element of $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. A diagram chase then shows that α comes from a morphism $A \to B$ in $D(QCoh(\mathcal{O}_X))$. Finally, suppose that $a : A \to B$ is a morphism of $D(QCoh(\mathcal{O}_X))$ which becomes zero in $D(\mathcal{O}_X)$. After choosing $\mathcal{B}^\bullet$ suitably, we may assume a is represented by a morphism of complexes $a^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$. Since i_U is fully faithul the restriction $a^\bullet|_U$ is zero in $D(QCoh(\mathcal{O}_U))$. Thus we can choose s such that $s \circ a^\bullet|_U : \mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$ is homotopic to zero. Applying the functor $j_{U,*}$ we conclude that $\mathcal{A}^\bullet \to j_{U,*}\mathcal{C}^\bullet$ is homotopic to zero. Thus a maps to zero in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. Thus we may assume that a is the image of an element of $b \in \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1])$. The image of b in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1])$ comes from a $\gamma \in \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B''[-1])$ (as a maps to zero in the group on the right). Since we've seen above the horizontal arrows are surjective, we see that γ comes from a c in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B''[-1])$ which implies $a = 0$ as desired.

At this point we know that i_X is fully faithful for our original X. Since RQ_X is its right adjoint, we see that $RQ_X \circ i_X = \mathrm{id}$ (Categories, Lemma 4.24.4). To finish the proof we show that for any E in $D_{QCoh}(\mathcal{O}_X)$ the map $i_X(RQ_X(E)) \to E$ is an isomorphism. Choose a distinguished triangle

$$i_X(RQ_X(E)) \to E \to E' \to i_X(RQ_X(E))[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. A formal argument using the above shows that $i_X(RQ_X(E')) = 0$. Thus it suffices to prove that for $E \in D_{QCoh}(\mathcal{O}_X)$ the condition $i_X(RQ_X(E)) = 0$ implies that $E = 0$. Consider an étale morphism $j : V \to X$ with V affine. By Lemmas 75.11.3 and 75.11.2 and our assumption we have

$$Rj_*(E|_V) = Rj_*(i_V(RQ_V(E|_V))) = i_X(\Phi(RQ_V(E|_V))) = i_X(RQ_X(Rj_*(E|_V)))$$

Choose a distinguished triangle

$$E \to Rj_*(E|_V) \to E' \to E[1]$$

Apply RQ_X to get a distinguished triangle

$$0 \to RQ_X(Rj_*(E|_V)) \to RQ_X(E') \to 0[1]$$

in other words the map in the middle is an isomorphism. Combined with the string of equalities above we find that our first distinguished triangle becomes a distinguished triangle

$$E \to i_X(RQ_X(E')) \to E' \to E[1]$$

where the middle morphism is the adjunction map. However, the composition $E \to E'$ is zero, hence $E \to i_X(RQ_X(E'))$ is zero by adjunction! Since this morphism is isomorphic to the morphism $E \to Rj_*(E|_V)$ adjoint to $\mathrm{id} : E|_V \to E|_V$ we conclude that $E|_V$ is zero. Since this holds for all affine V étale over X we conclude E is zero as desired. □

08H1 **Proposition 75.11.5.** *Let S be a scheme. Let X be a quasi-compact algebraic space over S with affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1). Then the functor (75.5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let $V \to W$ be an étale morphism with V affine and W a quasi-compact open subspace of X. Then the morphism $V \to W$ is affine as W has affine diagonal over $\mathbf{Z}$ and V is affine (Morphisms of Spaces, Lemma 67.20.11). Lemma 75.11.1 then guarantees that the assumption of Lemma 75.11.4 holds. Hence we conclude. □

0CSR **Lemma 75.11.6.** *Let S be a scheme and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y are quasi-compact and have affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1). Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 75.11.5. Hence we can identify these categories (and similarly for other quasi-compact algebraic spaces with affine diagonal) and then the statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ be the set of objects which are quasi-compact and have affine diagonal. For $U \in \mathcal{B}$ and any morphism $g : U \to Z$ where Z is a quasi-compact algebraic space over S with affine diagonal, denote

$$\Phi_g : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_Z))$$

the derived extension of g_*. Let $P(U) =$ "for any K in $D(QCoh(\mathcal{O}_U))$ and any $g : U \to Z$ as above the map $\Phi_g(K) \to Rg_*K$ is an isomorphism". By Remark 75.9.5 conditions (1), (2), and (3)(a) of Lemma 75.9.4 hold and we are left with proving (3)(b) and (4).

Checking condition (3)(b). Let U be an affine scheme étale over X. Let $g : U \to Z$ be as above. Since the diagonal of Z is affine the morphism $g : U \to Z$ is affine (Morphisms of Spaces, Lemma 67.20.11). Hence $P(U)$ holds by Lemma 75.11.1.

Checking condition (4). Let $(U \subset W, V \to W)$ be an elementary distinguished square in $X_{spaces,étale}$ with U, W, V in $\mathcal{B}$ and V affine. Assume that P holds for U, V, and $U \times_W V$. We have to show that P holds for W. Let $g : W \to Z$ be a morphism to a quasi-compact algebraic space with affine diagonal. Let K be an object of $D(QCoh(\mathcal{O}_W))$. Consider the distinguished triangle

$$K \to Rj_{U,*}K|_U \oplus Rj_{V,*}K|_V \to Rj_{U\times_W V,*}K|_{U\times_W V} \to K[1]$$

in $D(\mathcal{O}_W)$. By the two-out-of-three property mentioned above, it suffices to show that $\Phi_g(Rj_{U,*}K|_U) \to Rg_*(Rj_{U,*}K|_U)$ is an isomorphism and similarly for V and $U \times_W V$. This is discussed in the next paragraph.

Let $j : U \to W$ be a morphism $X_{spaces,étale}$ with U, W in $\mathcal{B}$ and P holds for U. Let $g : W \to Z$ be a morphism to a quasi-compact algebraic space with affine

diagonal. To finish the proof we have to show that $\Phi_g(Rj_*K) \to Rg_*(Rj_*K)$ is an isomorphism for any K in $D(QCoh(\mathcal{O}_U))$. Let $\mathcal{I}^\bullet$ be a K-injective complex in $QCoh(\mathcal{O}_U)$ representing K. From $P(U)$ applied to j we see that $j_*\mathcal{I}^\bullet$ represents Rj_*K. Since $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_W)$, see Derived Categories, Lemma 13.31.9. Hence $\Phi_g(Rj_*K)$ is represented by $g_*j_*\mathcal{I}^\bullet = (g \circ j)_*\mathcal{I}^\bullet$. By $P(U)$ applied to $g \circ j$ we see that this represents $R_{g\circ j,*}(K) = Rg_*(Rj_*K)$. This finishes the proof. □

75.12. The coherator for Noetherian spaces

09TH We need a little bit more about injective modules to treat the case of a Noetherian algebraic space.

09TI **Lemma 75.12.1.** *Let S be a Noetherian affine scheme. Every injective object of $QCoh(\mathcal{O}_S)$ is a filtered colimit $\operatorname{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to S)_*\mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Let $S = \operatorname{Spec}(A)$. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_S)$. Since $QCoh(\mathcal{O}_S)$ is equivalent to the category of A-modules we see that $\mathcal{J}$ is equal to $\widetilde{J}$ for some injective A-module J. By Dualizing Complexes, Proposition 47.5.9 we can write $J = \bigoplus E_\alpha$ with E_α indecomposable and therefore isomorphic to the injective hull of a reside field at a point. Thus (because finite disjoint unions of Artinian schemes are Artinian) we may assume that J is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Then $J = \bigcup J[\mathfrak{p}^n]$ where $J[\mathfrak{p}^n]$ is the injective hull of $\kappa(\mathfrak{p})$ over $A/\mathfrak{p}^nA_\mathfrak{p}$, see Dualizing Complexes, Lemma 47.7.3. Thus $\widetilde{J}$ is the colimit of the sheaves $(Z_n \to X)_*\mathcal{G}_n$ where $Z_n = \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{p}^nA_\mathfrak{p})$ and $\mathfrak{G}_n$ the coherent sheaf associated to the finite $A/\mathfrak{p}^nA_\mathfrak{p}$-module $J[\mathfrak{p}^n]$. Finiteness follows from Dualizing Complexes, Lemma 47.6.1. □

09TJ **Lemma 75.12.2.** *Let S be an affine scheme. Let X be a Noetherian algebraic space over S. Every injective object of $QCoh(\mathcal{O}_X)$ is a direct summand of a filtered colimit $\operatorname{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Choose an affine scheme U and a surjective étale morphism $j : U \to X$ (Properties of Spaces, Lemma 66.6.3). Then U is a Noetherian affine scheme. Choose an injective object $\mathcal{J}'$ of $QCoh(\mathcal{O}_U)$ such that there exists an injection $\mathcal{J}|_U \to \mathcal{J}'$. Then

$$\mathcal{J} \to j_*\mathcal{J}'$$

is an injective morphism in $QCoh(\mathcal{O}_X)$, hence identifies $\mathcal{J}$ as a direct summand of $j_*\mathcal{J}'$. Thus the result follows from the corresponding result for $\mathcal{J}'$ proved in Lemma 75.12.1. □

09TK **Lemma 75.12.3.** *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, and quasi-separated morphism of algebraic spaces over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then $f_*\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_Y)$.*

Proof. Since f is quasi-compact and quasi-separated, the functor f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 67.11.2). The functor f^* is a left adjoint to f_* which transforms injections into injections. Hence the result follows from Homology, Lemma 12.29.1 □

09TL **Lemma 75.12.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then*

(1) *$H^p(U, \mathcal{J}|_U) = 0$ for $p > 0$ and for every quasi-compact and quasi-separated algebraic space U étale over X,*
(2) *for any morphism $f : X \to Y$ of algebraic spaces over S with Y quasi-separated we have $R^p f_* \mathcal{J} = 0$ for $p > 0$.*

Proof. Proof of (1). Write $\mathcal{J}$ as a direct summand of $\operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i = (Z_i \to X)_* \mathcal{G}_i$ as in Lemma 75.12.2. It is clear that it suffices to prove the vanishing for $\operatorname{colim} \mathcal{F}_i$. Since pullback commutes with colimits and since U is quasi-compact and quasi-separated, it suffices to prove $H^p(U, \mathcal{F}_i|_U) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 69.5.1. Observe that $Z_i \to X$ is an affine morphism, see Morphisms of Spaces, Lemma 67.20.12. Thus

$$\mathcal{F}_i|_U = (Z_i \times_X U \to U)_* \mathcal{G}_i' = R(Z_i \times_X U \to U)_* \mathcal{G}_i'$$

where $\mathcal{G}_i'$ is the pullback of $\mathcal{G}_i$ to $Z_i \times_X U$, see Cohomology of Spaces, Lemma 69.11.1. Since $Z_i \times_X U$ is affine we conclude that $\mathcal{G}_i'$ has no higher cohomology on $Z_i \times_X U$. By the Leray spectral sequence we conclude the same thing is true for $\mathcal{F}_i|_U$ (Cohomology on Sites, Lemma 21.14.6).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \to Y$ be an étale morphism with V affine. Then $V \times_Y X \to X$ is an étale morphism and $V \times_Y X$ is a quasi-compact and quasi-separated algebraic space étale over X (details omitted). Hence $H^p(V \times_Y X, \mathcal{J})$ is zero by part (1). Since $R^p f_* \mathcal{J}$ is the sheaf associated to the presheaf $V \mapsto H^p(V \times_Y X, \mathcal{J})$ the result is proved. □

09TM **Lemma 75.12.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over S. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Since X and Y are Noetherian the morphism is quasi-compact and quasi-separated (see Morphisms of Spaces, Lemma 67.8.10). Thus f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 67.11.2. Next, let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 66.32.2), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_* \mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 75.6.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1}\mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2}\mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on X. By Lemma 75.6.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^m f_*\mathcal{I}^\bullet$ is isomorphic to $R^m f_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This case follows from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) with required vanishing because of Lemma 75.12.4. □

09TN **Proposition 75.12.6.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Then the functor (75.5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Follows immediately from Lemmas 75.12.5 and 75.11.4. □

75.13. Pseudo-coherent and perfect complexes

08HC In this section we study the general notions defined in Cohomology on Sites, Sections 21.44, 21.45, 21.46, and 21.47 for the étale site of an algebraic space. In particular we match this with what happens for schemes.

First we compare the notion of a pseudo-coherent complex on a scheme and on its associated small étale site.

08HD **Lemma 75.13.1.** *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite type as an $\mathcal{O}_X$-module, and*
(2) *$\epsilon^*\mathcal{F}$ is of finite type as an $\mathcal{O}_{étale}$-module on the small étale site of X.*

Here ϵ is as in (75.4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Modules on Sites, Lemma 18.23.4. Assume (2). By assumption there exists an étale covering $\{f_i : X_i \to X\}$ such that $\epsilon^*\mathcal{F}|_{(X_i)_{étale}}$ is generated by finitely many sections. Let $x \in X$. We will show that $\mathcal{F}$ is generated by finitely many sections in a neighbourhood of x. Say x is in the image of $X_i \to X$ and denote $X' = X_i$. Let $s_1, \ldots, s_n \in \Gamma(X', \epsilon^*\mathcal{F}|_{X'_{étale}})$ be generating sections. As $\epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{étale}$ we can find an étale morphism $X'' \to X'$ such that x is in the image of $X' \to X$ and such that $s_i|_{X''} = \sum s_{ij} \otimes a_{ij}$ for some sections $s_{ij} \in \epsilon^{-1}\mathcal{F}(X'')$ and $a_{ij} \in \mathcal{O}_{étale}(X'')$. Denote $U \subset X$ the image of $X'' \to X$. This is an open subscheme as $f'' : X'' \to X$ is étale (Morphisms, Lemma 29.36.13). After possibly shrinking X'' more we may assume s_{ij} come from elements $t_{ij} \in \mathcal{F}(U)$ as follows from the construction of the inverse image functor ϵ^{-1}. Now we claim that t_{ij} generate $\mathcal{F}|_U$ which finishes the proof of the lemma. Namely, the corresponding map $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ has the property that its pullback

by f'' to X'' is surjective. Since $f'' : X'' \to U$ is a surjective flat morphism of schemes, this implies that $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ is surjective by looking at stalks and using that $\mathcal{O}_{U,f''(z)} \to \mathcal{O}_{X'',z}$ is faithfully flat for all $z \in X''$. □

In the situation above the morphism of sites ϵ is flat hence defines a pullback on complexes of modules.

08HE **Lemma 75.13.2.** *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *E is m-pseudo-coherent, and*
(2) *ϵ^*E is m-pseudo-coherent on the small étale site of X.*

Here ϵ is as in (75.4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Cohomology on Sites, Lemma 21.45.3. Assume ϵ^*E is m-pseudo-coherent. We will use without further mention that ϵ^* is an exact functor and that therefore

$$\epsilon^* H^i(E) = H^i(\epsilon^* E).$$

To show that E is m-pseudo-coherent we may work locally on X, hence we may assume that X is quasi-compact (for example affine). Since X is quasi-compact every étale covering $\{U_i \to X\}$ has a finite refinement. Thus we see that ϵ^*E is an object of $D^-(\mathcal{O}_{étale})$, see comments following Cohomology on Sites, Definition 21.45.1. By Lemma 75.4.1 it follows that E is an object of $D^-(\mathcal{O}_X)$.

Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(E)$ is nonzero; then n is also the largest integer such that $H^n(\epsilon^*E)$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is clearly true. If $n \geq m$, then $H^n(\epsilon^*E)$ is a finite $\mathcal{O}_{étale}$-module, see Cohomology on Sites, Lemma 21.45.7. Hence $H^n(E)$ is a finite $\mathcal{O}_X$-module, see Lemma 75.13.1. After replacing X by the members of an open covering, we may assume there exists a surjection $\mathcal{O}_X^{\oplus t} \to H^n(E)$. We may locally on X lift this to a map of complexes $\alpha : \mathcal{O}_X^{\oplus t}[-n] \to E$ (details omitted). Choose a distinguished triangle

$$\mathcal{O}_X^{\oplus t}[-n] \to E \to C \to \mathcal{O}_X^{\oplus t}[-n+1]$$

Then C has vanishing cohomology in degrees $\geq n$. On the other hand, the complex ϵ^*C is m-pseudo-coherent, see Cohomology on Sites, Lemma 21.45.4. Hence by induction we see that C is m-pseudo-coherent. Applying Cohomology on Sites, Lemma 21.45.4 once more we conclude. □

08HF **Lemma 75.13.3.** *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then*

(1) *E has tor amplitude in $[a, b]$ if and only if ϵ^*E has tor amplitude in $[a, b]$.*
(2) *E has finite tor dimension if and only if ϵ^*E has finite tor dimension.*

Here ϵ is as in (75.4.0.1).

Proof. The easy implication follows from Cohomology on Sites, Lemma 21.46.5. For the converse, assume that ϵ^*E has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As ϵ is a flat morphism of ringed sites (Lemma 75.4.1) we have

$$\epsilon^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = \epsilon^* E \otimes^{\mathbf{L}}_{\mathcal{O}_{étale}} \epsilon^* \mathcal{F}$$

Thus the (assumed) vanishing of cohomology sheaves on the right hand side implies the desired vanishing of the cohomology sheaves of $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ via Lemma 75.4.1. □

0DK7 **Lemma 75.13.4.** *Let $f : X \to Y$ be a morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Then*

(1) *E as an object of $D(f^{-1}\mathcal{O}_Y)$ has tor amplitude in $[a, b]$ if and only if ϵ^*E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}_{small}\mathcal{O}_{Y_{étale}})$.*
(2) *E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$ if and only if ϵ^*E locally has finite tor dimension as an object of $D(f^{-1}_{small}\mathcal{O}_{Y_{étale}})$.*

Here ϵ is as in (75.4.0.1).

Proof. The easy direction in (1) follows from Cohomology on Sites, Lemma 21.46.5. Let $x \in X$ be a point and let $\overline{x}$ be a geometric point lying over x. Let $y = f(x)$ and denote $\overline{y}$ the geometric point of Y coming from $\overline{x}$. Then $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,y}$ (Sheaves, Lemma 6.21.5) and

$$(f^{-1}_{small}\mathcal{O}_{Y_{étale}})_{\overline{x}} = \mathcal{O}_{Y_{étale},\overline{y}} = \mathcal{O}^{sh}_{Y,y}$$

is the strict henselization (by Étale Cohomology, Lemmas 59.36.2 and 59.33.1). Since the stalk of $\mathcal{O}_{X_{étale}}$ at X is $\mathcal{O}^{sh}_{X,x}$ we obtain

$$(\epsilon^*E)_{\overline{x}} = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}^{sh}_{X,x}$$

by transitivity of pullbacks. If ϵ^*E has tor amplitude in $[a, b]$ as a complex of $f^{-1}_{small}\mathcal{O}_{Y_{étale}}$-modules, then $(\epsilon^*E)_{\overline{x}}$ has tor amplitude in $[a, b]$ as a complex of $\mathcal{O}^{sh}_{Y,y}$-modules (because taking stalks is a pullback and lemma cited above). By More on Flatness, Lemma 38.2.6 we find the tor amplitude of $(\epsilon^*E)_{\overline{x}}$ as a complex of $\mathcal{O}_{Y,y}$-modules is in $[a, b]$. Since $\mathcal{O}_{X,x} \to \mathcal{O}^{sh}_{X,x}$ is faithfully flat (More on Algebra, Lemma 15.45.1) and since $(\epsilon^*E)_{\overline{x}} = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}^{sh}_{X,x}$ we may apply More on Algebra, Lemma 15.67.18 to conclude the tor amplitude of E_x as a complex of $\mathcal{O}_{Y,y}$-modules is in $[a, b]$. By Cohomology, Lemma 20.48.5 we conclude that E as an object of $D(f^{-1}\mathcal{O}_Y)$ has tor amplitude in $[a, b]$. This gives the reverse implication in (1). Part (2) follows formally from (1). □

08HG **Lemma 75.13.5.** *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if ϵ^*E is a perfect object of $D(\mathcal{O}_{étale})$. Here ϵ is as in (75.4.0.1).*

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 21.47.5. For the converse, we can use the equivalence of Cohomology on Sites, Lemma 21.47.4 and the corresponding results for pseudo-coherent and complexes of finite tor dimension, namely Lemmas 75.13.2 and 75.13.3. Some details omitted. □

08JL **Lemma 75.13.6.** *Let S be a scheme. Let X be an algebraic space over S. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

08IK **Lemma 75.13.7.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

(1) *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
(2) *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we can find an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Observe that U is Noetherian. Note that E is m-pseudo-coherent if and only if $E|_U$ is m-pseudo-coherent (follows from the definition or from Cohomology on Sites, Lemma 21.45.2). Similarly, $H^i(E)$ is coherent if and only if $H^i(E)|_U = H^i(E|_U)$ is coherent (see Cohomology of Spaces, Lemma 69.12.2). Thus we may assume that X is representable.

If X is representable by a scheme X_0 then (Lemma 75.4.2) we can write $E = \epsilon^* E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (75.4.0.1). In this case E is m-pseudo-coherent if and only if E_0 is by Lemma 75.13.2. Similarly, $H^i(E_0)$ is of finite type (i.e., coherent) if and only if $H^i(E)$ is by Lemma 75.13.1. Finally, $H^i(E_0) = 0$ if and only if $H^i(E) = 0$ by Lemma 75.4.1. Thus we reduce to the case of schemes which is Derived Categories of Schemes, Lemma 36.10.3. □

08IL **Lemma 75.13.8.** *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

1. *E has tor amplitude in $[a, b]$, and*
2. *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $j : U \to X$ be an étale morphism with U affine. As X is quasi-separated $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Since it suffices to prove that $E|_U$ has tor amplitude in $[a, b]$ we reduce to the case where X is representable.

If X is representable by a scheme X_0 then (Lemma 75.4.2) we can write $E = \epsilon^* E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (75.4.0.1). For every quasi-coherent module $\mathcal{F}_0$ on X_0 the module $\epsilon^*\mathcal{F}_0$ is quasi-coherent on X and

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \epsilon^*\mathcal{F}_0) = \epsilon^* H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$$

as ϵ is flat (Lemma 75.4.1). Moreover, the vanishing of these sheaves for $i \notin [a, b]$ implies the same thing for $H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$ by the same lemma. Thus we've reduced the problem to the case of schemes which is treated in Derived Categories of Schemes, Lemma 36.10.6. □

08JP **Lemma 75.13.9.** *Let X be a scheme. Let E, F be objects of $D(\mathcal{O}_X)$. Assume either*

1. *E is pseudo-coherent and F lies in $D^+(\mathcal{O}_X)$, or*
2. *E is perfect and F arbitrary,*

then there is a canonical isomorphism

$$\epsilon^* R\mathcal{H}om(E, F) \longrightarrow R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$$

Here ϵ is as in (75.4.0.1).

Proof. Recall that ϵ is flat (Lemma 75.4.1) and hence $\epsilon^* = L\epsilon^*$. There is a canonical map from left to right by Cohomology on Sites, Remark 21.35.11. To

see this is an isomorphism we can work locally, i.e., we may assume X is an affine scheme.

In case (1) we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules, see Derived Categories of Schemes, Lemma 36.13.3. We may also represent F by a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules. Applying Cohomology, Lemma 20.46.11 we see that $R\mathcal{H}om(E, F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

Applying Cohomology on Sites, Lemma 21.44.10 we see that $R\mathcal{H}om(\epsilon^*E, \epsilon^*F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}^p, \epsilon^*\mathcal{F}^q)$$

Thus the statement of the lemma boils down to the true fact that the canonical map

$$\epsilon^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}, \epsilon^*\mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$ and finite free $\mathcal{O}_X$-module $\mathcal{E}$.

In case (2) we can represent E by a strictly perfect complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules, use Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7 and the fact that a perfect complex of modules is represented by a finite complex of finite projective modules. Thus we can do the exact same proof as above, replacing the reference to Cohomology, Lemma 20.46.11 by a reference to Cohomology, Lemma 20.46.9. □

0A8A **Lemma 75.13.10.** *Let S be a scheme. Let X be an algebraic space over S. Let L, K be objects of $D(\mathcal{O}_X)$. If either*

(1) *L in $D^+_{QCoh}(\mathcal{O}_X)$ and K is pseudo-coherent,*
(2) *L in $D_{QCoh}(\mathcal{O}_X)$ and K is perfect,*

then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.

Proof. This follows from the analogue for schemes (Derived Categories of Schemes, Lemma 36.10.8) via the criterion of Lemma 75.5.2, the criterion of Lemmas 75.13.2 and 75.13.5, and the result of Lemma 75.13.9. □

0E4Q **Lemma 75.13.11.** *Let S be a scheme. Let X be an algebraic space over S. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

of Cohomology on Sites, Lemma 21.35.7 is an isomorphism in the following cases

(1) *M perfect, or*
(2) *K is perfect, or*
(3) *M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.*

Proof. Checking whether or not the map is an isomorphism can be done étale locally hence we may assume X is an affine scheme. Then we can write K, L, M as $\epsilon^*K_0, \epsilon^*L_0, \epsilon^*M_0$ for some K_0, L_0, M_0 in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.4.2. Then we see that Lemma 75.13.9 reduces cases (1) and (3) to the case of schemes which is Derived Categories of Schemes, Lemma 36.10.9. If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because K will be represented (after localizing further) by a finite complex of finite free modules in which case it is clear. □

75.14. Approximation by perfect complexes

08HH In this section we continue the discussion started in Derived Categories of Schemes, Section 36.14.

08HI **Definition 75.14.1.** Let S be a scheme. Let X be an algebraic space over S. Consider triples (T, E, m) where

(1) $T \subset |X|$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Please read the remarks following Derived Categories of Schemes, Definition 36.14.1 to see why.

08HJ **Definition 75.14.2.** Let S be a scheme. Let X be an algebraic space over S. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset |X|$ such that the morphism $X \setminus T \to X$ is quasi-compact there exists an integer r such that for every triple (T, E, m) as in Definition 75.14.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

08HK **Lemma 75.14.3.** *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic space over S. Let E be a perfect object of $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ where $T = |X| \setminus |U|$. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being perfect is local on $X_{étale}$. Thus it suffices to check that Rj_*E is perfect when restricted to U and V. We have $Rj_*E|_V = E$ by Lemma 75.10.7 which is perfect. We have $Rj_*E|_U = 0$ because $E|_{V \setminus j^{-1}(T)} = 0$ (use Lemma 75.3.1). □

08HL **Lemma 75.14.4.** *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Let T be a closed subset of $|X| \setminus |U|$ and let (T, E, m) be a triple as in Definition 75.14.1. If*

(1) *approximation holds for $(j^{-1}T, E|_V, m)$, and*
(2) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $P \to E|_V$ be an approximation of the triple $(j^{-1}T, E|_V, m)$ over V. Then Rj_*P is a perfect object of $D(\mathcal{O}_X)$ by Lemma 75.14.3. On the other hand, $Rj_*P = j_!P$ by Lemma 75.10.7. We see that $j_!P$ is supported on T for example by (75.10.0.2). Hence we obtain an approximation $Rj_*P = j_!P \to j_!(E|_V) \to E$. □

08HM **Lemma 75.14.5.** *Let S be a scheme. Let X be an algebraic space over S which is representable by an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 75.14.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Let X_0 be an affine scheme representing X. Let $T_0 \subset X_0$ by the closed subset corresponding to T. Let $\epsilon : X_{étale} \to X_{0,Zar}$ be the morphism (75.4.0.1). We may write $E = \epsilon^* E_0$ for some object E_0 of $D_{QCoh}(\mathcal{O}_{X_0})$, see Lemma 75.4.2. Then E_0 is m-pseudo-coherent, see Lemma 75.13.2. Comparing stalks of cohomology sheaves (see proof of Lemma 75.4.1) we see that $H^i(E_0)$ is supported on T_0 for $i \geq m - r + 1$. By Derived Categories of Schemes, Lemma 36.14.4 there exists an approximation $P_0 \to E_0$ of (T_0, E_0, m). By Lemma 75.13.5 we see that $P = \epsilon^* P_0$ is a perfect object of $D(\mathcal{O}_X)$. Pulling back we obtain an approximation $P = \epsilon^* P_0 \to \epsilon^* E_0 = E$ as desired. □

08HN **Lemma 75.14.6.** *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume U quasi-compact, V affine, and $U \times_X V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation by perfect complexes holds on X.*

Proof. Let $T \subset |X|$ be a closed subset with $X \setminus T \to X$ quasi-compact. Let r_U be the integer of Definition 75.14.2 adapted to the pair $(U, T \cap |U|)$. Set $T' = T \setminus |U|$. Endow T' with the induced reduced subspace structure. Since $|T'|$ is contained in $|X| \setminus |U|$ we see that $j^{-1}(T') \to T'$ is an isomorphism. Moreover, $V \setminus j^{-1}(T')$ is quasi-compact as it is the fibre product of $U \times_X V$ with $X \setminus T$ over X and we've assumed $U \times_X V$ quasi-compact and $X \setminus T \to X$ quasi-compact. Let r' be the number of affines needed to cover $V \setminus j^{-1}(T')$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m - r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 75.14.5 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 75.14.4 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap |U|, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. In the rest of the proof we will use the equivalence of Lemma 75.4.2 (and the compatibilities of Remark 75.6.3) for the representable algebraic spaces V and $U \times_X V$. We will also use the fact that $(m-r)$-pseudo-coherence, resp. perfectness on the Zariski site and étale site agree, see Lemmas 75.13.2 and 75.13.5. Thus we can use the results of Derived Categories of Schemes, Section 36.13 for the open immersion $U \times_X V \subset V$. In this way Derived Categories of Schemes, Lemma 36.13.10 implies there exists a perfect object Q in $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ and an isomorphism $Q|_{U \times_X V} \to (P \oplus P[1])|_{U \times_X V}$. By Derived Categories of Schemes, Lemma 36.13.7 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \times_X V} \longrightarrow (P \oplus P[1])|_{U \times_X V} \longrightarrow P|_{U \times_X V} \longrightarrow E|_{U \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 75.10.8 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective

on restriction to U. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m-r)$-pseudo-coherent (Cohomology on Sites, Lemma 21.45.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then R'' is a perfect complex (Cohomology on Sites, Lemma 21.47.6) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

08HP **Theorem 75.14.7.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Lemma 75.9.3 and Lemmas 75.14.6 and 75.14.5. □

75.15. Generating derived categories

09IU This section is the analogue of Derived Categories of Schemes, Section 36.15. However, we first prove the following lemma which is the analogue of Derived Categories of Schemes, Lemma 36.13.11.

09IV **Lemma 75.15.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is a quasi-compact morphism. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_W$ be a map where P is a perfect object of $D(\mathcal{O}_W)$ supported on $T \cap W$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_W$ in $D(\mathcal{O}_W)$ compatible α and $\beta|_W$.*

Proof. We will use the induction principle of Lemma 75.9.6 to prove this. Thus we immediately reduce to the case where we have an elementary distinguished square $(W \subset X, f : V \to X)$ with V affine and $P \to E|_W$ as in the statement of the lemma. In the rest of the proof we will use Lemma 75.4.2 (and the compatibilities of Remark 75.6.3) for the representable algebraic spaces V and $W \times_X V$. We will also use the fact that perfectness on the Zariski site and étale site agree, see Lemma 75.13.5.

By Derived Categories of Schemes, Lemma 36.13.10 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $f^{-1}T$ and an isomorphism $Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times_X V}$. By Derived Categories of Schemes, Lemma 36.13.7 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $f^{-1}T$) and assume that the map

$$Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times V} \longrightarrow P|_{W \times_X V} \longrightarrow E|_{W \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 75.10.8 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_W$ is isomorphic to $P \oplus P[1] \to E|_W$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

09IW **Remark 75.15.2.** The proof of Lemma 75.15.1 shows that

$$R|_W = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_W$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_W$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_W$ and P above any given degree.

09IX **Lemma 75.15.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let W be a quasi-compact open subspace of X. Let P be a perfect object of $D(\mathcal{O}_W)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 75.15.1. □

09IY **Theorem 75.15.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

(1) *$E = 0$, and*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.*

Proof. We will prove this using the induction principle of Lemma 75.9.3.

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact such that the theorem holds for U and V is an affine scheme. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 75.15.3 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rf_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 75.10.7 we have $K' = f_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 36.15.2 (using also Lemma 75.4.2) the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \times_X V \to V)_* E|_{U \times_X V}$. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

0E4R **Remark 75.15.5.** Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 75.15.4). Then the following are equivalent

(1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
(2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
(3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_*. These conditions hold if f is affine (Lemma 75.6.4) or if f is an open immersion, or if f is a composition of such.

The following result is an strengthening of Theorem 75.15.4 proved using exactly the same methods. Let $T \subset |X|$ be a closed subset where X is an algebraic space. Let's denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T.

0AEC **Lemma 75.15.6.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.*

Proof. We will prove this using the induction principle of Lemma 75.9.3. The property is true for representable quasi-compact and quasi-separated objects of the site $X_{spaces,étale}$ by Derived Categories of Schemes, Lemma 36.15.4.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square such that the lemma holds for U and V is affine. To finish the proof we have to show that the result holds for X. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T\cap U}(\mathcal{O}_U)$. Using Lemma 75.15.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then $f^{-1}Z$ is a closed subset of V such that $V \setminus f^{-1}Z$ is quasi-compact. As X is quasi-separated, it follows that $f^{-1}Z \cap f^{-1}T = f^{-1}(Z \cap T)$ is a closed subset of V such that $W = V \setminus f^{-1}(Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $f^{-1}(Z \cap T) = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $f^{-1}(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 75.10.7 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 36.15.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{array}{ccccc} W & \xrightarrow{j} & V & \xleftarrow{} & Z \cap T \\ {\scriptstyle j'}\uparrow & \nearrow_{j''} & & \nwarrow & \downarrow \\ V \setminus f^{-1}Z & & & & Z \end{array}$$

Since E is supported on T we see that $E|_W$ is supported on $f^{-1}T \cap W = f^{-1}T \cap (V \setminus f^{-1}Z)$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

Here the second equality is part (1) of Cohomology, Lemma 20.33.6 which applies because V is a scheme and E has quasi-coherent cohomology sheaves hence pushforward along the quasi-compact open immersion j' agrees with pushforward on the underlying schemes, see Remark 75.6.3. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

75.16. Compact and perfect objects

09M7 This section is the analogue of Derived Categories of Schemes, Section 36.17.

09M8 **Proposition 75.16.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. If K is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology on Sites, Lemma 21.48.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums and since $H^0(X, -)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.6.2 we conclude that K is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine scheme U étale over X, see Cohomology on Sites, Lemma 21.47.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 75.6.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a perfect object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine, in particular a quasi-compact and quasi-separated scheme. Via Lemma 75.4.2 and 75.13.5 we reduce to the case of schemes, i.e., to Derived Categories of Schemes, Proposition 36.17.1. □

0GFC **Remark 75.16.2.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let G be a perfect object of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$. By Theorem 75.15.4 there is at least one of these. Combining Lemma 75.5.3 with Proposition 75.16.1 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf}(\mathcal{O}_X)$.

The following result is a strengthening of Proposition 75.16.1. Let $T \subset |X|$ be a closed subset where X is an algebraic space. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T. Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0AED **Lemma 75.16.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 75.16.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 75.15.6. Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 13.37.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. □

0GFD **Remark 75.16.4.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. Let G be a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$ which is a generator for $D_{QCoh,T}(\mathcal{O}_X)$. By Lemma 75.15.6 there is at least one of these. Combining the fact that $D_{QCoh,T}(\mathcal{O}_X)$ has direct sums with Lemma 75.16.3 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf,T}(\mathcal{O}_X)$.

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0AEE **Lemma 75.16.5.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that the complement $U \subset X$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

(1) *P is perfect and E supported on T, or*
(2) *P pseudo-coherent, E supported on T, and E bounded below.*

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 75.13.10 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 75.15.6. Write

$$K = \mathrm{hocolim} K_n$$

as in Derived Categories, Lemma 13.37.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α, see Derived Categories, Lemma 13.33.9. By Derived Categories, Lemma 13.37.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. □

75.17. Derived categories as module categories

09M9 The section is the analogue of Derived Categories of Schemes, Section 36.18.

09MA **Lemma 75.17.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \text{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property P. Let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.20. Then we have

$$P \otimes_E K^\bullet = \text{hocolim } F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. □

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

09MB **Lemma 75.17.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K be a perfect object of $D(\mathcal{O}_X)$. Then*

1. *there exist integers $a \leq b$ such that $\text{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$ for $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and*
2. *if L is bounded, then $\text{Ext}^n_{D(\mathcal{O}_X)}(K, L)$ is zero for all but finitely many n.*

Proof. Part (2) follows from (1) as $\text{Ext}^n_{D(\mathcal{O}_X)}(K, L) = \text{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\text{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology on Sites, Lemma 21.48.4. Note that $P = K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 75.5.6 and 75.13.6 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

shows that $H^j(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ is zero if $H^q(L) = 0$ for $q \in [j-b, j-a]$. Let N be the integer $\max(d_p + p)$ of Cohomology of Spaces, Lemma 69.7.3. Then $H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ H^{-N+1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ \ldots,\ H^0(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

are zero. Namely, by the lemma cited and Lemma 75.5.8, we have

$$H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$ which is zero by Derived Categories, Lemma 13.16.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b-N, -a]$. □

The following is the analogue of Derived Categories of Schemes, Theorem 36.18.2.

09MC **Theorem 75.17.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 75.15.4 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.35. Since $K^\bullet$ is K-injective we have

09MD (75.17.3.1) $$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 75.17.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 75.16.1. Combined with (75.17.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.35.6. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 22.35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 75.17.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.7.2. □

0DK8 **Remark 75.17.4** (Variant with support)**.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. The analogue of Theorem 75.17.3 holds for $D_{QCoh,T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the

theorem, using Lemmas 75.15.6 and 75.16.3 and a variant of Lemma 75.17.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

0DK9 **Remark 75.17.5** (Uniqueness of dga)**.** Let X be a quasi-compact and quasi-separated algebraic space over a ring R. By the construction of the proof of Theorem 75.17.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \mathrm{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \mathrm{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \mathrm{d})$$

is given by the functor $- \otimes^{\mathbf{L}}_A \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 22.37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

75.18. Characterizing pseudo-coherent complexes, I

0DKA This material will be continued in More on Morphisms of Spaces, Section 76.51. We can characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DKB **Lemma 75.18.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $K \in D(\mathcal{O}_X)$. The following are equivalent*

1. *K is pseudo-coherent, and*
2. *$K = hocolim K_n$ where K_n is perfect and $\tau_{\geq -n} K_n \to \tau_{\geq -n} K$ is an isomorphism for all n.*

Proof. The implication (2) $\Rightarrow$ (1) is true on any ringed site. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology on Sites, Lemma 21.47.4. Then it follows from the definitions that $\tau_{\geq -n} K_n$ is $(-n+1)$-pseudo-coherent and hence $\tau_{\geq -n} K$ is $(-n+1)$-pseudo-coherent, hence K is $(-n+1)$-pseudo-coherent. This is true for all n, hence K is pseudo-coherent, see Cohomology on Sites, Definition 21.45.1.

Assume (1). We start by choosing an approximation $K_1 \to K$ of $(X, K, -2)$ by a perfect complex K_1, see Definitions 75.14.1 and 75.14.2 and Theorem 75.14.7. Suppose by induction we have

$$K_1 \to K_2 \to \ldots \to K_n \to K$$

with K_i perfect such that such that $\tau_{\geq -i}K_i \to \tau_{\geq -i}K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 75.17.2 for the perfect object K_n. Choose an approximation $K_{n+1} \to K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \to K \to C \to K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \to K \to C$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can factor $K_n \to K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \mathrm{hocolim} K_n$. This follows by an application of Derived Categories, Lemma 13.33.8 to the functors $H^i(-) : D(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ and our choice of K_n. □

0DKC **Lemma 75.18.2.** *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T. The following are equivalent*

(1) *K is pseudo-coherent, and*
(2) *$K = \mathrm{hocolim} K_n$ where K_n is perfect, supported on T, and $\tau_{\geq -n}K_n \to \tau_{\geq -n}K$ is an isomorphism for all n.*

Proof. The proof of this lemma is exactly the same as the proof of Lemma 75.18.1 except that in the choice of the approximations we use the triples (T, K, m). □

75.19. The coherator revisited

0CR3 In Section 75.11 we constructed and studied the right adjoint RQ_X to the canonical functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : \mathit{Mod}(\mathcal{O}_X) \to \mathit{QCoh}(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote[2] it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated algebraic spaces have such a right adjoint.

0CR4 **Lemma 75.19.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint.*

First proof. We will use the induction principle in Lemma 75.9.3 to prove this. If $D(\mathit{QCoh}(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 75.11 is a right adjoint to the functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine algebraic spaces, see Lemma 75.11.3. Thus we see that it suffices to show: if $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact and V affine and the lemma holds for U, V, and $U \times_X V$, then the lemma holds for X.

[2]This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 13.40.7. Consider the distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\times_X V,*}E|_{U\times_X V} \to E[1]$$

in $D(\mathcal{O}_X)$ of Lemma 75.10.2. By Derived Categories, Lemma 13.40.5 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U\times_X V,*}E|_{U\times_X V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an étale morphism corresponding with U quasi-compact and quasi-separated and the lemma is true for U. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_*L \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\mathrm{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 13.40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \to Rj_*L \to Rj_*Q \to Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.6.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\mathrm{Hom}(M, Rj_*Q) = \mathrm{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 75.5.5. This finishes the proof. □

Second proof. The adjoint exists by Derived Categories, Proposition 13.38.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 75.5.3). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 75.15.4 and because perfect objects are compact by Proposition 75.16.1. □

0CR5 **Lemma 75.19.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \to D$ exist for X and Y, then*

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 75.6.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 75.5.5) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. □

0CR6 **Remark 75.19.3.** Let S be a scheme. Let $(U \subset X, f : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume X, U, V are quasi-compact

and quasi-separated. By Lemma 75.19.1 the functors DQ_X, DQ_U, DQ_V, $DQ_{U\times_X V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \to Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \to Rj_{U\times_X V,*}DQ_{U\times_X V}(K|_{U\times_X V}) \to$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Lemma 75.10.2 and using Lemma 75.19.2 three times.

0CSS **Lemma 75.19.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The functor DQ_X of Lemma 75.19.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine étale over X and $i \notin [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \notin [a, b + N]$.*

Proof. We will prove this using the induction principle of Lemma 75.9.3.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemma 75.11.3.

Let $(U \subset W, f : V \to W)$ be an elementary distinguished square with W quasi-compact and quasi-separated, $U \subset W$ quasi-compact open, V affine such that the lemma holds for U, V, and $U \times_W V$. Say with integers $N(U)$, $N(V)$, and $N(U \times_W V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine W étale over X and all $i \notin [a, b]$. Then $K|_U$, $K|_V$, $K|_{U\times_W V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U\cap V}(K|_{U\times_W V})$ have vanishing cohomology sheaves outside the interval $[a, b + \max(N(U), N(V), N(U \times_W V))$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U\times_W V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 75.6.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U\cap V,*}DQ_{U\times_W V}(K|_{U\times_W V})$ have vanishing cohomology sheaves outside the interval $[a, b+N]$. Then finally we conclude by the distinguished triangle of Remark 75.19.3. □

0CST **Example 75.19.5.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves on X. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R\lim \mathcal{F}_n$ for this. For any affine U étale over X we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D^b_{QCoh}(\mathcal{O}_X)$ and has vanishing cohomology sheaves in negative degrees by Lemma 75.19.4.

75.20. Cohomology and base change, IV

08IM This section is the analogue of Derived Categories of Schemes, Section 36.22.

08IN **Lemma 75.20.1.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ we have*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology on Sites, Lemmas 21.18.4 and 21.19.1. To check it is an isomorphism we may work étale locally on Y. Hence we reduce to the case that Y is an affine scheme.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 75.6.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $^\sim : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence by Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.59.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

08IP **Definition 75.20.2.** Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. We say X and Y are *Tor independent over B* if and only if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \\ {\scriptstyle \overline{y}}\downarrow & \swarrow{\scriptstyle \overline{b}} & \downarrow \\ Y & \longrightarrow & B \end{array}$$

of geometric points the rings $\mathcal{O}_{X,\overline{x}}$ and $\mathcal{O}_{Y,\overline{y}}$ are Tor independent over $\mathcal{O}_{B,\overline{b}}$ (see More on Algebra, Definition 15.61.1).

The following lemma shows in particular that this definition agrees with our definition in the case of representable algebraic spaces.

08IQ **Lemma 75.20.3.** *Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. The following are equivalent*

(1) *X and Y are Tor independent over B,*

(2) *for every commutative diagram*

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & B & \longleftarrow & Y \end{array}$$

with étale vertical arrows U and V are Tor independent over W,

(3) *for some commutative diagram as in (2) with (a) $W \to B$ étale surjective, (b) $U \to X \times_B W$ étale surjective, (c) $V \to Y \times_B W$ étale surjective, the spaces U and V are Tor independent over W, and*

(4) *for some commutative diagram as in (3) with U, V, W schemes, the schemes U and V are Tor independent over W in the sense of Derived Categories of Schemes, Definition 36.22.2.*

Proof. For an étale morphism $\varphi : U \to X$ of algebraic spaces and geometric point $\overline{u}$ the map of local rings $\mathcal{O}_{X,\varphi(\overline{u})} \to \mathcal{O}_{U,\overline{u}}$ is an isomorphism. Hence the equivalence of (1) and (2) follows. So does the implication (1) $\Rightarrow$ (3). Assume (3) and pick a diagram of geometric points as in Definition 75.20.2. The assumptions imply that we can first lift $\overline{b}$ to a geometric point $\overline{w}$ of W, then lift the geometric point $(\overline{x}, \overline{b})$ to a geometric point $\overline{u}$ of U, and finally lift the geometric point $(\overline{y}, \overline{b})$ to a geometric point $\overline{v}$ of V. Use Properties of Spaces, Lemma 66.19.4 to find the lifts. Using the remark on local rings above we conclude that the condition of the definition is satisfied for the given diagram.

Having made these initial points, it is clear that (4) comes down to the statement that Definition 75.20.2 agrees with Derived Categories of Schemes, Definition 36.22.2 when X, Y, and B are schemes.

Let $\overline{x}, \overline{b}, \overline{y}$ be as in Definition 75.20.2 lying over the points x, y, b. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{X,x}^{sh}$ (Properties of Spaces, Lemma 66.22.1) and similarly for the other two. By Algebra, Lemma 10.155.12 we see that $\mathcal{O}_{X,\overline{x}}$ is a strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}}$. In particular, the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is flat (More on Algebra, Lemma 15.45.1). By More on Algebra, Lemma 15.61.3 we see that

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}) = \mathrm{Tor}_i^{\mathcal{O}_{B,\overline{b}}}(\mathcal{O}_{X,\overline{x}}, \mathcal{O}_{Y,\overline{y}})$$

Hence it follows that if X and Y are Tor independent over B as schemes, then X and Y are Tor independent as algebraic spaces over B.

For the converse, we may assume X, Y, and B are affine. Observe that the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}$$

is flat by the observations given above. Moreover, the image of the map on spectra includes all primes $\mathfrak{s} \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}$ lying over $\mathfrak{m}_x$ and $\mathfrak{m}_y$. Hence from this and the displayed formula of Tor's above we see that if X and Y are Tor independent over B as algebraic spaces, then

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})_{\mathfrak{s}} = 0$$

for all $i > 0$ and all $\mathfrak{s}$ as above. By More on Algebra, Lemma 15.61.6 applied to the ring maps $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ and $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ this implies that X and Y are Tor independent over B. □

08IR **Lemma 75.20.4.** *Let S be a scheme. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If X and Y' are Tor independent over Y, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*L(g')^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology on Sites, Remark 21.19.3 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*L(g')^*E$. To check this is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 75.6.4 (and use Lemmas 75.5.5, 75.5.6, and 75.6.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that g' is affine as well (Morphisms of Spaces, Lemma 67.20.5). By Lemma 75.6.5 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{Y'}$. Thus by Lemma 75.20.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. This follows from our assumption that X and Y' are Tor independent over Y. Namely, to check it we may work étale locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and $Y' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (see Lemma 75.20.3 and More on Algebra, Lemma 15.61.6), i.e., $\mathrm{Tor}^R_i(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. □

The following lemma will be used in the chapter on dualizing complexes.

0E4S **Lemma 75.20.5.** *Let $g : S' \to S$ be a morphism of affine schemes. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of quasi-compact and quasi-separated algebraic spaces. Assume g and f Tor independent. Write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\mathrm{Hom}_X(M, K) \otimes^{\mathbf{L}}_R R' \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

(1) *$M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or*

(2) *$M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and R' has finite tor dimension over R.*

Proof. There is a canonical map $R\mathrm{Hom}_X(M, K) \to R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology on Sites, Section 21.36. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $- \otimes^{\mathbf{L}}_R R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\mathrm{Hom}_X(M, R(g')_*L(g')^*K) = R\mathrm{Hom}_X(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

by Lemma 75.6.5. In both cases the complex $R\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 75.13.10. There is a natural map

$$R\mathcal{H}om(M, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases Lemma 75.13.11. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_X$ the second equality by Lemma 75.20.4. Using Derived Categories of Schemes, Lemma 36.10.4, Lemma 75.13.3, and Cohomology on Sites, Lemma 21.46.5 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 75.6.5. Hence the result follows from Lemma 75.20.4. □

0E4T **Remark 75.20.6.** With notation as in Lemma 75.20.5. The diagram

$$\begin{array}{ccc} R\mathrm{Hom}_X(M, Rg'_*L) \otimes^{\mathbf{L}}_R R' & \longrightarrow & R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle a} \\ R\mathrm{Hom}_X(M, R(g')_*L) & = & R\mathrm{Hom}_{X'}(L(g')^*M, L) \end{array}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \to L$. The multiplication map is the adjunction map $K' \otimes^{\mathbf{L}}_R R' \to K'$ for any $K' \in D(R')$.

0DKD **Lemma 75.20.7.** *Let S be a scheme. Consider a cartesian square of algebraic spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

over S. Assume g and f Tor independent.

(1) *If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_{Y'}$-modules.*
(2) *If $\mathcal{G}$ is an $\mathcal{O}_X$-module flat over Y, then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.*

Proof. We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12. If $\overline{x}'$ is a geometric point of X' with image $\overline{x}$ in X, then

$$(L(g')^*E)_{\overline{x}'} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$$

Let $\overline{y}'$ in Y' and $\overline{y}$ in Y be the image of $\overline{x}'$ and $\overline{x}$. Since X and Y' are tor independent over Y, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$ in $D(\mathcal{O}_{Y',\overline{y}'})$. Thus (1) follows from More on Algebra, Lemma 15.67.13. To see (2) observe that flatness of $\mathcal{G}$ is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0,0]$. Applying (1) we conclude. □

75.21. Cohomology and base change, V

0DKE This section is the analogue of Derived Categories of Schemes, Section 36.26. In Section 75.20 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 15.61. In this section we analyze when one can get a base change result "one complex at a time".

To make this work, let S be a base scheme and suppose we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of algebraic spaces over S (usually we will assume it is cartesian). Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. For a geometric point $\overline{x}'$ of X' consider the geometric points $\overline{x} = g'(\overline{x}')$, $\overline{y}' = f'(\overline{x}')$, $\overline{y} = f(\overline{x}) = g(\overline{y}')$ of X, Y', Y. Then we can consider the maps

$$K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'} \to K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'} \to K'_{\overline{x}'}$$

where the first arrow is More on Algebra, Equation (15.61.0.1) and the second comes from $(L(g')^*K)_{\overline{x}'} = K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$ and the given map $L(g')^*K \to K'$. For each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$-module structure on $H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})$. Putting everything together we obtain canonical maps

0DKF (75.21.0.1) $$H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}) \otimes_{(\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})} \mathcal{O}_{X',\overline{x}'} \longrightarrow H^i(K'_{\overline{x}'})$$

of $\mathcal{O}_{X',\overline{x}'}$-modules.

0DKG **Lemma 75.21.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

(1) *for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (75.21.0.1) is an isomorphism,*

(2) *for any commutative diagram*

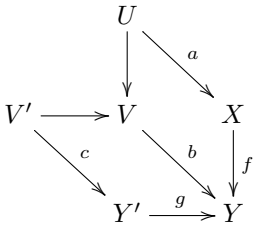

with a, b, c étale, U, V, V' schemes, and with $U' = V' \times_V U$ the equivalent conditions of Derived Categories of Schemes, Lemma 36.26.1 hold for $(U \to X)^*K$ and $(U' \to X')^*K'$, and

(3) *there is some diagram as in (2) with $U' \to X'$ surjective.*

Proof. Observe that (1) is étale local on X'. Working through formal implications of what is known, we see that it suffices to prove condition (1) of this lemma is equivalent to condition (1) of Derived Categories of Schemes, Lemma 36.26.1 if X, Y, Y', X' are representable by schemes X_0, Y_0, Y_0', X_0'. Denote f_0, g_0, g_0', f_0' the morphisms between these schemes corresponding to f, g, g', f'. We may assume $K = \epsilon^* K_0$ and $K' = \epsilon^* K_0'$ for some objects $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ and $K_0' \in D_{QCoh}(\mathcal{O}_{X_0'})$, see Lemma 75.4.2. Moreover, the map $Lg^*K \to K'$ is the pullback of a map $L(g_0)^*K_0 \to K_0'$ with notation as in Remark 75.6.3. Recall that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$ (Properties of Spaces, Lemma 66.22.1) and that we have

$$K_{\overline{x}} = K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}} \quad \text{and} \quad K'_{\overline{x}'} = K'_{0,x'} \otimes^{\mathbf{L}}_{\mathcal{O}_{X',x'}} \mathcal{O}_{X',\overline{x}'}$$

(akin to Properties of Spaces, Lemma 66.29.4). Consider the commutative diagram

$$\begin{array}{ccc} H^i(K_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}) \otimes_{(\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'})} \mathcal{O}_{X',\overline{x}'} & \longrightarrow & H^i(K'_{\overline{x}'}) \\ \uparrow & & \uparrow \\ H^i(K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'})} \mathcal{O}_{X',x'} & \longrightarrow & H^i(K'_{0,x'}) \end{array}$$

We have to show that the lower horizontal arrow is an isomorphism if and only if the upper horizontal arrow is an isomorphism. Since $\mathcal{O}_{X',x'} \to \mathcal{O}_{X',\overline{x}'}$ is faithfully flat (More on Algebra, Lemma 15.45.1) it suffices to show that the top arrow is the base change of the bottom arrow by this map. This follows immediately from the relationships between stalks given above for the objects on the right. For the objects on the left it suffices to show that

$$\begin{aligned} & H^i\left((K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}}) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}\right) \\ & = H^i(K_{0,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'})} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}) \end{aligned}$$

This follows from More on Algebra, Lemma 15.61.5. The flatness assumptions of this lemma hold by what was said above as well as Algebra, Lemma 10.155.12 implying that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\overline{y}}$ and that $\mathcal{O}_{Y',\overline{y}'}$ is the strict henselization of $\mathcal{O}_{Y',y'} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\overline{y}}$. □

0DKH **Lemma 75.21.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

(1) *the equivalent conditions of Lemma 75.21.1 hold, and*
(2) *f is quasi-compact and quasi-separated,*

then the composition $Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K'$ is an isomorphism.

Proof. To check the map is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In this case, we will use the induction principle of Lemma 75.9.3 to prove that for a quasi-compact and quasi-separated algebraic space U étale over X the similarly constructed map $Lg^*R(U \to Y)_*K|_U \to R(U' \to Y')_*K'|_{U'}$ is an isomorphism. Here $U' = X' \times_{g',X} U = Y' \times_{g,Y} U$.

If U is a scheme (for example affine), then the result holds. Namely, then Y, Y', U, U' are schemes, K and K' come from objects of the derived category of the underlying schemes by Lemma 75.4.2 and the condition of Derived Categories of Schemes, Lemma 36.26.1 holds for these complexes by Lemma 75.21.1. Thus (by the compatibilities explained in Remark 75.6.3) we can apply the result in the case of schemes which is Derived Categories of Schemes, Lemma 36.26.2.

The induction step. Let $(U \subset W, V \to W)$ be an elementary distinguished square with W a quasi-compact and quasi-separated algebraic space étale over X, with U quasi-compact, V affine and the result holds for U, V, and $U \times_W V$. To easy notation we replace W by X (this is permissible at this point). Denote $a : U \to Y$, $b : V \to Y$, and $c : U \times_X V \to Y$ the obvious morphisms. Let $a' : U' \to Y'$, $b' : V' \to Y'$ and $c' : U' \times_{X'} V' \to Y'$ be the base changes of a, b, and c. Using the distinguished triangles from relative Mayer-Vietoris (Lemma 75.10.3) we obtain a commutative diagram

$$\begin{array}{ccc} Lg^*Rf_*K & \longrightarrow & Rf'_*K' \\ \downarrow & & \downarrow \\ Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V & \longrightarrow & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \\ \downarrow & & \downarrow \\ Lg^*Rc_*K|_{U \times_X V} & \longrightarrow & Rc'_*K'|_{U' \times_{X'} V'} \\ \downarrow & & \downarrow \\ Lg^*Rf_*K[1] & \longrightarrow & Rf'_*K'[1] \end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. □

0DKI **Lemma 75.21.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 75.21.1 hold, then

(1) *for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \to L(g')^*E \otimes^{\mathbf{L}} K'$,*
(2) *if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$, and*
(3) *if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 75.21.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$.*

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 75.5.5, 75.5.6, and 75.13.10 and Cohomology on Sites, Lemmas 21.45.3 and 21.47.5. Having said this, we can check the maps (75.21.0.1) are isomorphisms in case (1) by computing the source and target of (75.21.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 15.59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, L(g')^*K) \to R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology on Sites, Remark 21.35.11 and the second arrow comes from the given map $L(g')^*K \to K'$. To prove the maps (75.21.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X,x}$-modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. □

0A1K **Lemma 75.21.4.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a bounded above complex of quasi-coherent $\mathcal{O}_X$-modules flat over Y. Then formation of*

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in other words $X' = Y' \times_Y X$. The lemma asserts that

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \longrightarrow Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 75.21.2 and 75.21.3 to see that it

suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 75.21.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}^\bullet_{\overline{x}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y',\overline{y}'}$ computes the derived tensor product by our assumptions on the complex $\mathcal{G}^\bullet$. □

08JQ **Lemma 75.21.5.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. If*

(1) *E is perfect, $\mathcal{G}^\bullet$ is a bounded above, and $\mathcal{G}^n$ is flat over Y, or*
(2) *E is pseudo-coherent, $\mathcal{G}^\bullet$ is bounded, and $\mathcal{G}^n$ is flat over Y,*

then formation of

$$Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in other words $X' = Y' \times_Y X$. The lemma asserts that

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_*R\mathcal{H}om(L(g')^*E, (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 75.21.2 and 75.21.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 75.21.1. This was shown in the proof of Lemma 75.21.4. □

75.22. Producing perfect complexes

0A1L The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation.

08IS **Lemma 75.22.1.** *Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ such that*

(1) *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
(2) *the support of $H^i(E)$ is proper over Y for all i,*
(3) *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.

Proof. By Lemma 75.8.1 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$. Hence Rf_*E is pseudo-coherent (Lemma 75.13.7). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology on Sites, Lemma 21.47.4. By Lemma 75.13.8 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}$ has universally bounded cohomology for all

quasi-coherent sheaves $\mathcal{F}$ on Y. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset |X|$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over Y by assumptions (1) and (2) and Lemma 75.7.6. In particular there exists a quasi-compact open subspace $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. We have assumed f is quasi-separated. Thus

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F} = Rf_* \left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} \right) = Rf_* \left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F} \right)$$

by Lemma 75.20.1 and Cohomology on Sites, Lemma 21.18.5. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. □

0DKJ **Lemma 75.22.2.** *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over B with support proper over B. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_B)$.*

Proof. The object K is perfect by Lemma 75.22.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus\nolimits_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers $a, b, r_a, \ldots, r_b$. This immediately implies the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are coherent. The hypothesis on the tor dimension also follows as $\mathcal{G}^i$ is flat over $f^{-1}\mathcal{O}_Y$. □

0DKK **Lemma 75.22.3.** *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over B with support proper over B. Then $K = Rf_*R\mathcal{H}om(E, \mathcal{G})$ is a perfect object of $D(\mathcal{O}_B)$.*

Proof. Since E is a perfect complex there exists a dual perfect complex $E^\vee$, see Cohomology on Sites, Lemma 21.48.4. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 75.22.2. □

75.23. A projection formula for Ext

08JM Lemma 75.23.3 (or similar results in the literature) is sometimes useful to verify properties of an obstruction theory needed to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems. Suppose that $f : X \to Y$ is a proper, flat, finitely presented morphism of algebraic spaces and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}^i_X(E, f^*\mathcal{F}) = \mathrm{Ext}^i_Y((Rf_*E^\vee)^\vee, \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on Y. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 75.20.1.

0A1M **Lemma 75.23.1.** *Assumptions and notation as in Lemma 75.22.2. Then there are functorial isomorphisms*

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology on Sites, Lemma 21.18.5, the second as $\mathcal{G}^n$ is a flat $f^{-1}\mathcal{O}_B$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \\ &= H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology on Sites, Remark 21.14.4), and the third equality by Lemma 75.20.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) & \longrightarrow & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) & \longrightarrow & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}^n$ is flat over B. We omit the verification of the commutativity of the displayed diagram. □

08JN **Lemma 75.23.2.** *Assumption and notation as in Lemma 75.22.3. Then there are functorial isomorphisms*

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 75.22.3 let $E^\vee$ be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$. Since we also have

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E^\vee$, the existence of the isomorphisms follows from Lemma 75.23.1 applied to $E^\vee$ and $\mathcal{G}^\bullet$. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms

fit into commutative diagrams

$$\begin{array}{ccc} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}^n$ is flat over B. We omit the verification of the commutativity of the displayed diagram. □

08JR **Lemma 75.23.3.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S, $E \in D(\mathcal{O}_X)$, and $\mathcal{F}^\bullet$ a complex of $\mathcal{O}_X$-modules. Assume*

(1) *B is Noetherian,*
(2) *f is locally of finite type and quasi-separated,*
(3) *$E \in D^-_{Coh}(\mathcal{O}_X)$,*
(4) *$\mathcal{G}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-module flat over B with support proper over B.*

Then the following two statements are true

(A) *for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_B)$ and functorial maps*

$$\alpha^i_{\mathcal{F}} : \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$, and

(B) *there exists a pseudo-coherent $L \in D(\mathcal{O}_B)$ and functorial isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{O}_B}(L, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps.

Proof. Proof of (A). Suppose $\mathcal{G}^i$ is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of $\mathcal{G}^i$. Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m - 1 + a)$ (possible by Theorem 75.14.7). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.27.3. This reduces us to the case that E is a perfect complex which is Lemma 75.23.2. The statement on boundaries is explained in the proof of Lemma 75.23.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 75.13.7. By Lemma 75.18.1 we can write $E = \text{hocolim} E_n$ with E_n perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n^\vee$ be the dual perfect complex (Cohomology on Sites, Lemma 21.48.4). We obtain an inverse system $\ldots \to E_3^\vee \to E_2^\vee \to E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system

$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

perfect on Y, see Lemma 75.22.2. By Lemma 75.23.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent $\mathcal{F}$ on Y we have functorial canonical maps

$$\begin{array}{ccc} & \text{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) & \\ \swarrow & & \searrow \\ H^i(Y, K_{n+1} \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}) & \longrightarrow & H^i(Y, K_n \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}) \end{array}$$

which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(\mathcal{O}_Y)$ such that for any quasi-coherent $\mathcal{F}$ on Y the maps

$$\text{Ext}^i_{\mathcal{O}_Y}(L_{n+1}, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_Y}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 75.18.1. The mapping property of homotopy colimits gives that $\text{Ext}^i_{\mathcal{O}_Y}(L, \mathcal{F}) = \text{Ext}^i_{\mathcal{O}_Y}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. □

0DKL **Remark 75.23.4.** The pseudo-coherent complex L of part (B) of Lemma 75.23.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes we have canonical functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_{Y'}}(Lg^*L, \mathcal{F}') \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on Y'. Obsere that we do **not** use derived pullback on $\mathcal{G}^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

75.24. Limits and derived categories

09RG In this section we collect some results about the derived category of an algebraic space which is the limit of an inverse system of algebraic spaces. More precisely, we will work in the following setting.

09RH **Situation 75.24.1.** Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We denote $f_i : X \to X_i$ the projection. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

09RI **Lemma 75.24.2.** *In Situation 75.24.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{X_0})$. Set $E_i = Lf_{i0}^* E_0$ and $K_i = Lf_{i0}^* K_0$ for $i \geq 0$ and set $E = Lf_0^* E_0$ and $K = Lf_0^* K_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(E_i, K_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(E, K)$$

is an isomorphism if either

(1) *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$, or*
(2) *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has finite tor dimension.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the canonical map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0 by the induction principle of Lemma 75.9.3. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Lemma 75.10.4. Thus it suffices to prove the lemma when X_0 is affine.

If X_0 is affine, then the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.29.2. To see this use the equivalence of Lemma 75.4.2 and use the translation of properties explained in Lemmas 75.13.2, 75.13.3, and 75.13.5. □

09RJ **Lemma 75.24.3.** *In Situation 75.24.1 the category of perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for every U_0 by the induction principle of Lemma 75.9.3. First, we observe that we already know the functor is fully faithful by Lemma 75.24.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have an elementary distinguished square $(U_0 \subset X_0, V_0 \to X_0)$ and that P holds for U_0, V_0, and $U_0 \times_{X_0} V_0$. Let E be a perfect object of $D(\mathcal{O}_X)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (R(X \to X_i)_* E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (R(X \to X_i)_* E)|_{V_i}$$

the maps adjoint to the isomorphisms $L(U \to U_i)^* E_{U,i} \to E|_U$ and $L(V \to V_i)^* E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{X_i} V_i} \to E_{V,i}|_{U_i \times_{X_i} V_i}$ which pulls back to the identifications

$$L(U \to U_i)^* E_{U,i}|_{U \times_X V} \to E|_{U \times_X V} \to L(V \to V_i)^* E_{V,i}|_{U \times_X V}.$$

Apply Lemma 75.10.8 to get an object E_i on X_i and a map $d : E_i \to R(X \to X_i)_* E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $L(X \to X_i)^* E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when X_0 is affine. This follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.29.3. To see this use the equivalence of Lemma 75.4.2 and use the translation of Lemma 75.13.5. □

75.25. Cohomology and base change, VI

0A1N A final section on cohomology and base change continuing the discussion of Sections 75.20, 75.21, and 75.22. An easy to grok special case is given in Remark 75.25.2.

0A1P **Lemma 75.25.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.4. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 75.22.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 70.7.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits of Spaces, Lemma 70.6.12. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits of Spaces, Lemma 70.12.3. Finally, by Lemma 75.24.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. By Lemma 75.22.2 we have that $K_i = Rf_{i,*}(E_i \otimes^{\mathbf{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i^\bullet)$ is perfect on $\mathrm{Spec}(R_i)$ where $f_i : X_i \to \mathrm{Spec}(R_i)$ is the structure morphism. By the base change result (Lemma 75.21.4) the pullback of K_i to $Y = \mathrm{Spec}(R)$ is K and we conclude. □

0A1Q **Remark 75.25.2.** Let R be a ring. Let X be an algebraic space of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with support proper over R. By Lemma 75.25.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

0CTL **Lemma 75.25.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent*

object. Let $\mathcal{G}^\bullet$ be a bounded above complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 75.25.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is étale local on Y, hence we may assume Y is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ has cohomological dimension N as explained in Lemma 75.6.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m-N-1-b)$. This is possible by Theorem 75.14.7. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m-N-1-b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m-N-1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G})$ are zero in degrees $\geq m-1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}) \to Rf_*(E \otimes^{\mathbf{L}} \mathcal{G})$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet \longrightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m-N-1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet) \to Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 75.25.1 the source is a perfect complex. We conclude that K is m-pseudo-coherent as desired. □

0CTM **Lemma 75.25.4.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of finite presentation of algebraic spaces over S.*

(1) *Let $E \in D(\mathcal{O}_X)$ be perfect and f flat. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.*
(2) *Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.*

Proof. Special cases of Lemma 75.25.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. □

0CTN **Lemma 75.25.5.** *Let S be a scheme. Let $f : X \to Y$ be a flat proper morphism of finite presentation of algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.*

More generally, if $f : X \to Y$ is proper and E on X is pseudo-coherent relative to Y (More on Morphisms of Spaces, Definition 76.45.3), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). The case of this for schemes is proved in [**Kie72**].

Proof. Special case of Lemma 75.25.3 applied with $\mathcal{G} = \mathcal{O}_X$. □

0D3F **Lemma 75.25.6.** *Let R be a ring. Let X be an algebraic space and let $f : X \to \mathrm{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R-modules with surjective transition maps. Then the canonical map*

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\mathrm{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \mathrm{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 75.15.4) it suffices to check this for perfect E. By Lemma 75.5.4 we have $\lim \mathcal{O}_X \otimes_R M_n = R\lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$ of Cohomology on Sites, Section 21.36 commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R\lim R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology on Sites, Lemma 21.48.4. We have

$$R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*M_n) = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n$$

by Lemma 75.20.1. From Lemma 75.25.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 15.103.3 we obtain

$$R\lim R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R \lim M_n$$

as desired. □

0CWH **Lemma 75.25.7.** *Let A be a ring. Let X be an algebraic space over A which is quasi-compact and quasi-separated. Let $K \in D^-_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.*

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ has cohomological dimension N as explained in Lemma 75.6.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X. It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 75.14.7. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \to R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent as desired. □

0A1R **Lemma 75.25.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over Y, with support proper over Y. Then*

$$K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 75.21.5. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 75.22.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 70.7.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits of Spaces, Lemma 70.6.12. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits of Spaces, Lemma 70.12.3. Finally, by Lemma 75.13.5 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 75.23.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

75.26. Perfect complexes

0D1X We first talk about jumping loci for betti numbers of perfect complexes. First we have to define betti numbers.

Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $x \in |X|$. We want to define $\beta_i(x) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$. To do this, choose a morphism $f : \mathrm{Spec}(k) \to X$ in the equivalence class of x. Then Lf^*E is an object of $D(\mathrm{Spec}(k)_{\acute{e}tale}, \mathcal{O})$. By Étale Cohomology, Lemma 59.59.4 and Theorem 59.17.4 we find that $D(\mathrm{Spec}(k)_{\acute{e}tale}, \mathcal{O}) = D(k)$ is the derived category of k-vector spaces. Hence Lf^*E is a complex of k-vector spaces and we can take $\beta_i(x) = \dim_k H^i(Lf^*E)$. It is easy to see that this does not depend on the choice of the representative in x. Moreover, if X is a scheme, this is the same as the notion used in Derived Categories of Schemes, Section 36.31.

0D1Y **Lemma 75.26.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function*

$$\beta_i : |X| \longrightarrow \{0, 1, 2, \ldots\}$$

defined above. Then we have

1. *formation of β_i commutes with arbitrary base change,*
2. *the functions β_i are upper semi-continuous, and*
3. *the level sets of β_i are étale locally constructible.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Then $L\varphi^*E$ is a pseudo-coherent complex on the scheme U (use Lemma 75.13.2) and we can apply the result for schemes, see Derived Categories of Schemes, Lemma 36.31.1. The meaning of part (3) is that the inverse image of the level sets to U are locally constructible, see Properties of Spaces, Definition 66.8.2. □

0E0R **Lemma 75.26.2.** *Let Y be a scheme and let X be an algebraic space over Y such that the structure morphism $f : X \to Y$ is flat, proper, and of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over Y. For fixed $i \in \mathbf{Z}$ consider the function*

$$\beta_i : |Y| \to \{0, 1, 2, \ldots\}, \quad y \longmapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y)$$

Then we have

1. *formation of β_i commutes with arbitrary base change,*
2. *the functions β_i are upper semi-continuous, and*
3. *the level sets of β_i are locally constructible in Y.*

Proof. By cohomology and base change (more precisely by Lemma 75.25.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of Y whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_y, \mathcal{F}_y) = H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y))$$

Thus the lemma follows from Lemma 75.26.1. □

0D1Z **Lemma 75.26.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect. The function*

$$\chi_E : |X| \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \beta_i(x)$$

is locally constant on X.

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.2. □

0D20 **Lemma 75.26.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subspace $U \subset X$ characterized by the following*

1. *$E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank r,*
2. *a morphism $f : Y \to X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i.*

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.3. □

0E6A **Lemma 75.26.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is proper, flat, and of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over Y. Fix $i, r \in \mathbf{Z}$. Then there exists an open subspace $V \subset Y$ with the following property: A morphism $T \to Y$ factors through V if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i.*

Proof. By cohomology and base change (Lemma 75.25.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of Y whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 75.26.4. □

0D21 **Lemma 75.26.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $r \geq 0$. Then there exists a locally closed subspace $j : Z \to X$ characterized by the following*

(1) *$H^a(Lj^*E)$ is a locally free $\mathcal{O}_Z$-module of rank r, and*
(2) *a morphism $f : Y \to X$ factors through Z if and only if for all morphisms $g : Y' \to Y$ the $\mathcal{O}_{Y'}$-module $H^a(L(f \circ g)^*E)$ is locally free of rank r.*

Moreover, $j : Z \to X$ is of finite presentation and we have

(3) *if $f : Y \to X$ factors as $Y \xrightarrow{g} Z \to X$, then $H^a(Lf^*E) = g^*H^a(Lj^*E)$,*
(4) *if $\beta_a(x) \leq r$ for all $x \in |X|$, then j is a closed immersion and given $f : Y \to X$ the following are equivalent*
   (a) *$f : Y \to X$ factors through Z,*
   (b) *$H^0(Lf^*E)$ is a locally free $\mathcal{O}_Y$-module of rank r,*

   and if $r = 1$ these are also equivalent to
   (c) *$\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$ is injective.*

Proof. Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 36.31.4. □

0E6B **Lemma 75.26.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *for a morphism $\mathrm{Spec}(k) \to Y$ where k is a field, we have $k = H^0(X_k, \mathcal{O}_{X_k})$.*

Then we have

(a) *$f_*\mathcal{O}_X = \mathcal{O}_Y$ and this holds after any base change,*
(b) *étale locally on Y we have*

$$Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$$

in $D(\mathcal{O}_Y)$ where P is perfect of tor amplitude in $[1, \infty)$.

Proof. It suffices to prove (a) and (b) étale locally on Y, thus we may and do assume Y is an affine scheme. By cohomology and base change (Lemma 75.25.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. In particular, for $y \in Y$ we see that $H^0(E \otimes^{\mathbf{L}} \kappa(y)) = H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$. Thus $\beta_0(y) \leq 1$ for all $y \in Y$ with notation as in Lemma 75.26.1. Apply Lemma 75.26.6 with $a = 0$ and $r = 1$. We obtain a universal closed subscheme $j : Z \to Y$ with $H^0(Lj^*E)$ invertible characterized by the equivalence of (4)(a), (b), and (c) of the lemma. Since formation of E commutes with base change, we have

$$Lf^*E = R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X}$$

The morphism $\mathrm{pr}_1 : X \times_Y X$ has a section namely the diagonal morphism Δ for X over Y. We obtain maps

$$\mathcal{O}_X \longrightarrow R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} \longrightarrow \mathcal{O}_X$$

in $D(\mathcal{O}_X)$ whose composition is the identity. Thus $R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} = \mathcal{O}_X \oplus E'$ in $D(\mathcal{O}_X)$. Thus $\mathcal{O}_X$ is a direct summand of $H^0(Lf^*E)$ and we conclude that $X \to Y$

factors through Z by the equivalence of (4)(c) and (4)(a) of the lemma cited above. Since $\{X \to Y\}$ is an fppf covering, we have $Z = Y$. Thus $f_*\mathcal{O}_X$ is an invertible $\mathcal{O}_Y$-module. We conclude $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism because a ring map $A \to B$ such that B is invertible as an A-module is an isomorphism. Since the assumptions are preserved under base change, we see that (a) is true.

Proof of (b). Above we have seen that for every $y \in Y$ the map $\mathcal{O}_Y \to H^0(E \otimes^{\mathbf{L}} \kappa(y))$ is surjective. Thus we may apply More on Algebra, Lemma 15.77.2 to see that in an open neighbourhood of y we have a decomposition $Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$ □

0E0S **Lemma 75.26.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the geometric fibres of f are reduced and connected.*

Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ and this holds after any base change.

Proof. By Lemma 75.26.7 it suffices to show that $k = H^0(X_k, \mathcal{O}_{X_k})$ for all morphisms $\mathrm{Spec}(k) \to Y$ where k is a field. This follows from Spaces over Fields, Lemma 72.14.3 and the fact that X_k is geometrically connected and geometrically reduced. □

75.27. Other applications

0CRT In this section we state and prove some results that can be deduced from the theory worked out above.

0CRU **Lemma 75.27.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine schemes étale over X. Then for any quasi-compact and quasi-separated étale morphism $U \to X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets*

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology on Sites, Lemma 21.48.4 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Lemma 75.9.3.

When $U = \mathrm{Spec}(A)$ is affine the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 36.33.2.

To finish the proof it suffices to show: if $(U \subset W, V \to W)$ is an elementary distinguished triangle and the result holds for U, V, and $U \times_W V$, then the result holds for W. This is an immediate consequence of the Mayer-Vietoris sequence, see Lemma 75.10.5. □

0CRV **Lemma 75.27.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume the sets of sections of $\mathcal{O}_X$ over affines étale over X are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

(1) *$K = hocolim E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and*
(2) *the cohomology sheaves $H^i(K)$ have countable sets of sections over affines étale over X.*

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 13.33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on X.

Assume (2). Choose a K-injective complex $\mathcal{K}^\bullet$ representing K. Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex $\mathcal{I}^\bullet$. According to Theorem 75.17.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A, \mathrm{d})$ where (A, d) is the differential graded algebra

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \mathrm{Ext}^i(E, E)$ and $H^i(M) = \mathrm{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, \mathrm{d})$. By Differential Graded Algebra, Lemma 22.38.3 it suffices show that $\mathrm{Ext}^i(E, E)$ and $\mathrm{Ext}^i(E, K)$ are countable for each i. This follows from Lemma 75.27.1. □

0CRW **Lemma 75.27.3.** *Let A be a ring. Let $f : U \to X$ be a flat morphism of algebraic spaces of finite presentation over A. Then*

(1) *there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that*

$$R\Gamma(U, Lf^*K) = \mathit{hocolim}\ R\,\mathrm{Hom}_X(L_n, K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and

(2) *there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that*

$$R\Gamma(U, Lf^*K) = \mathit{hocolim}\ R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$

in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 75.20.1 we have

$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$

functorially in K. Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 75.6.2. Similarly, $- \otimes^{\mathbf{L}} K$ commutes with derived colimits because $- \otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove (2) it suffices to write $Rf_*\mathcal{O}_U = \mathrm{hocolim} E_n$ for a system of perfect objects E_n of $D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology on Sites, Lemma 21.48.4.

Write $A = \mathrm{colim}\, A_i$ with A_i of finite type over $\mathbf{Z}$. By Limits of Spaces, Lemma 70.7.1 we can find an i and morphisms $U_i \to X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to $\mathrm{Spec}(A)$ recovers $U \to X \to \mathrm{Spec}(A)$. After increasing i we may assume that $f_i : U_i \to X_i$ is flat, see Limits of Spaces, Lemma 70.6.12. By Lemma 75.20.4 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \to X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i. Hence we may assume that U and X are of finite type over $\mathbf{Z}$.

Assume $f : U \to X$ is a morphism of algebraic spaces of finite type over $\mathbf{Z}$. To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes.

To see this we apply Lemma 75.27.2. Thus it suffices to show that $R^if_*\mathcal{O}_U$ has countable sets of sections over affines étale over X. This follows from Lemma 75.27.1 applied to the structure sheaf. □

75.28. The resolution property

0GUR This section is the analogue of Derived Categories of Schemes, Section 36.36 for algebraic spaces; please read that section first. It is currently not known if a smooth proper algebraic space over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com.

We can make the following definition although it scarcely makes sense to consider it for general algebraic spaces.

0GUS **Definition 75.28.1.** Let S be a scheme. Let X be an algebraic space over S. We say X has the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

If X is a quasi-compact and quasi-separated algebraic space, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Limits of Spaces, Lemma 70.9.3. If X is a Noetherian algebraic space, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Spaces, Lemma 69.12.2.

0GUT **Lemma 75.28.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *Y is quasi-compact and quasi-separated and has the resolution property,*
(2) *there exists an f-ample invertible module on X (Divisors on Spaces, Definition 71.14.1).*

Then X has the resolution property.

Proof. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an f-ample invertible module. Choose an affine scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. Then $\mathcal{L}|_U$ is ample on U. By Properties, Proposition 28.26.13 we know there exists finitely many maps $s_i : \mathcal{L}^{\otimes n_i}|_U \to \mathcal{F}|_U$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of s_i as a section over V of the sheaf $\mathcal{H}_{-n_i}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_i$ and maps $\mathcal{E}_i \to \mathcal{H}_{-n_i}$ such that s_i is in the image. Then the corresponding maps

$$f^*\mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_i} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Limits of Spaces, Lemma 70.9.2 for each i we can find a finite type quasi-coherent submodule $\mathcal{H}'_i \subset \mathcal{H}_{-n_i}$ which contains the section s_i over V. Thus the resolution property of Y produces surjections $\mathcal{E}_i \to \mathcal{H}'_i$ and we conclude. □

0GUU **Lemma 75.28.3.** *Let S be a scheme. Let $f : X \to Y$ be an affine or quasi-affine morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. If Y has the resolution property, so does X.*

Proof. By Divisors on Spaces, Lemma 71.14.7 this is a special case of Lemma 75.28.2. □

Here is a case where one can prove the resolution property goes down.

0GUV **Lemma 75.28.4.** *Let S be a scheme. Let $f : X \to Y$ be a surjective finite locally free morphism of algebraic spaces over S. If X has the resolution property, so does Y.*

Proof. The condition means that f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of positive rank. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. By assumption there exists a surjection $\mathcal{E} \to f^*\mathcal{G}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Since f_* is exact (Cohomology of Spaces, Section 69.4) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since $f_*\mathcal{E}$ is finite locally free, we conclude. □

For more on the resolution property of algebraic spaces, please see More on Morphisms of Spaces, Section 76.56.

75.29. Detecting Boundedness

0GFE In this section, we show that compact generators of D_{QCoh} of a quasi-compact, quasi-separated scheme, as constructed in Section 75.15, have a special property. We recommend reading that section first as it is very similar to this one.

0GFF **Lemma 75.29.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

(1) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ *for $i \gg 0$, and*
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a}E) = 0$ *for $i \gg 0$.*

Proof. Using the triangle $\tau_{<a}E \to E \to \tau_{\geq a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a}E)[i]) = 0$$

for $i \gg 0$. As P is perfect this is true by Lemma 75.17.2. □

0GFG **Lemma 75.29.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

(1) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ *for $i \ll 0$, and*
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a}E) = 0$ *for $i \ll 0$.*

Proof. Using the triangle $\tau_{\leq a}E \to E \to \tau_{>a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a}E)[i]) = 0$$

for $i \ll 0$. As P is perfect this is true by Lemma 75.17.2. □

0GFH **Proposition 75.29.3.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

(1) $E \in D^-_{QCoh}(\mathcal{O}_X)$,
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ *for* $i \gg 0$,
(3) $\mathrm{Ext}^i_X(G, E) = 0$ *for* $i \gg 0$,
(4) $R\mathrm{Hom}_X(G, E)$ *is in* $D^-(\mathbf{Z})$,
(5) $H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ *for* $i \gg 0$,
(6) $R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ *is in* $D^-(\mathbf{Z})$,
(7) *for every perfect object* P *of* $D(\mathcal{O}_X)$
   (a) *the assertions (2), (3), (4) hold with* G *replaced by* P, *and*
   (b) $H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ *for* $i \gg 0$,
   (c) $R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ *is in* $D^-(\mathbf{Z})$.

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \gg 0$ by Lemma 75.17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 21.36. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 21.48.4 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 75.16.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\mathrm{Hom}_X(P, E)$ is in $D^-(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 75.9.3.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 36.40.5.

Now assume $(U \subset X, j : V \to X)$ is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over S and assume the implication (2) – (7) $\Rightarrow$ (1) is known for U, V, and $U \times_X V$. To finish the proof we have to show the implication (2) – (7) $\Rightarrow$ (1) for X. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 75.15.3 and Theorem 75.15.4 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$.

Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retro-compact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rj_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). By assumption, we know $R\mathrm{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\mathrm{Hom}_{\mathcal{O}_X}(K', E)$ are bounded above.

By Lemma 75.10.7 we have $K' = j_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for $i \gg 0$. Therefore, we may apply Derived Categories of Schemes, Lemma 36.40.1 to $E|_V$ to obtain an integer a such that $\tau_{\geq a}(E|_V) = \tau_{\geq a}R(U \times_X V \to V)_*(E|_{U \times_X V})$. Then $\tau_{\geq a}E = \tau_{\geq a}R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 75.29.1 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \gg 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^-_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^-_{QCoh} (by Lemma 75.6.1), we get $\tau_{\geq a}E \in D^-_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^-_{QCoh}(\mathcal{O}_X)$. □

0GFI **Proposition 75.29.4.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$E \in D^+_{QCoh}(\mathcal{O}_X)$,*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \ll 0$,*
3. *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \ll 0$,*
4. *$R\mathrm{Hom}_X(G, E)$ is in $D^+(\mathbf{Z})$,*
5. *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
6. *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$,*
7. *for every perfect object P of $D(\mathcal{O}_X)$*
   (a) *the assertions (2), (3), (4) hold with G replaced by P, and*
   (b) *$H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
   (c) *$R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \ll 0$ by Lemma 75.17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 21.36. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 21.48.4 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 75.16.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\mathrm{Hom}_X(P, E)$ is in $D^+(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 75.9.3.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 36.40.6.

Now assume $(U \subset X, j : V \to X)$ is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over S and assume the implication (2) – (7) $\Rightarrow$ (1) is known for U, V, and $U \times_X V$. To finish the proof we have to

show the implication (2) – (7) $\Rightarrow$ (1) for X. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 75.15.3 and Theorem 75.15.4 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$.

Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 75.9.1). This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rj_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 75.14.3 and 75.10.7). By assumption, we know $R\mathrm{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\mathrm{Hom}_{\mathcal{O}_X}(K', E)$ are bounded below.

By Lemma 75.10.7 we have $K' = j_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for $i \ll 0$. Therefore, we may apply Derived Categories of Schemes, Lemma 36.40.2 to $E|_V$ to obtain an integer a such that $\tau_{\leq a}(E|_V) = \tau_{\leq a}R(U \times_X V \to V)_*(E|_{U \times_X V})$. Then $\tau_{\leq a}E = \tau_{\leq a}R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 75.29.2 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \ll 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^+_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^+_{QCoh} (by Lemma 75.6.1), we get $\tau_{\leq a}E \in D^+_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^+_{QCoh}(\mathcal{O}_X)$. □

75.30. Quasi-coherent objects in the derived category

0H05 Let S be a scheme. Let X be an algebraic space over S. Recall that $X_{affine,étale}$ denotes the category of affine objects of $X_{étale}$ with topology given by standard étale coverings, see Properties of Spaces, Definition 66.18.5. We remind the reader that the topos of $X_{affine,étale}$ is the small étale topos of X, see Properties of Spaces, Lemma 66.18.6. The site $X_{étale}$ comes with a structure sheaf $\mathcal{O}_X$ whose restriction to $X_{affine,étale}$ we also denote $\mathcal{O}_X$. Then there is an equivalence of ringed topoi

$$(Sh(X_{affine,étale}), \mathcal{O}_X) \longrightarrow (Sh(X_{étale}), \mathcal{O}_X)$$

See Descent on Spaces, Equation (74.5.0.1) and the discussion in Descent on Spaces, Section 74.5.

In this section we denote X_{affine} the underlying category of $X_{affine,étale}$ endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf $\mathcal{O}_X$ becomes a sheaf on X_{affine} as well. We obtain a morphisms of ringed sites

$$\epsilon : (X_{affine,étale}, \mathcal{O}_X) \longrightarrow (X_{affine}, \mathcal{O}_X)$$

as in Cohomology on Sites, Section 21.27. In this section we will identify $D_{QCoh}(\mathcal{O}_X)$ with the category $QC(X_{affine}, \mathcal{O}_X)$ introduced in Cohomology on Sites, Section 21.43.

0H06 **Lemma 75.30.1.** *In the situation above there are canonical exact equivalences between the following triangulated categories*

(1) $D_{QCoh}(\mathcal{O}_X)$,
(2) $D_{QCoh}(X_{affine,étale}, \mathcal{O}_X)$,
(3) $D_{QCoh}(X_{affine}, \mathcal{O}_X)$, *and*
(4) $QC(X_{affine}, \mathcal{O}_X)$.

Proof. If $U \to V \to X$ are étale morphisms with U and V affine, then the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is flat. Hence the equivalence between (3) and (4) is a special case of Cohomology on Sites, Lemma 21.43.11 (the proof also clarifies the statement).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi $(Sh(X_{affine,étale}), \mathcal{O}_X) \to (Sh(X_{étale}), \mathcal{O}_X)$ and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 18.23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (1) and (2).

To get an exact equivalence between the triangulated categories in (2) and (3) we will apply Cohomology on Sites, Lemma 21.29.1 to the morphism $\epsilon : (X_{affine,étale}, \mathcal{O}_X) \to (X_{affine}, \mathcal{O}_X)$ above. We take $\mathcal{B} = \mathrm{Ob}(X_{affine})$ and we take $\mathcal{A} \subset PMod(X_{affine}, \mathcal{O})$ to be the full subcategory of those presheaves $\mathcal{F}$ such that $\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \to \mathcal{F}(U)$ is an isomorphism. Observe that by Descent on Spaces, Lemma 74.5.1 objects of $\mathcal{A}$ are exactly those sheaves in the étale topology which are quasi-coherent modules on $(X_{affine,étale}, \mathcal{O}_X)$. On the other hand, by Modules on Sites, Lemma 18.24.3, the objects of $\mathcal{A}$ are exactly the quasi-coherent modules on $(X_{affine}, \mathcal{O}_X)$, i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 21.29.1 applies, then we do indeed get the canonical equivalence between the categories of (2) and (3) using ϵ^* and $R\epsilon_*$.

We have to verify 4 conditions:

(1) Every object of $\mathcal{A}$ is a sheaf for the étale topology. This we have seen above.
(2) $\mathcal{A}$ is a weak Serre subcategory of $Mod(X_{affine,étale}, \mathcal{O}_X)$. Above we have seen that $\mathcal{A} = QCoh(X_{affine,étale}, \mathcal{O}_X)$ and we have seen above that these, via the equivalence $Mod(X_{affine,étale}, \mathcal{O}) = Mod(X_{étale}, \mathcal{O}_X)$, correspond to the quasi-coherent modules on X. Thus the result by Properties of Spaces, Lemma 66.29.7 and Homology, Lemma 12.10.3.
(3) Every object of X_{affine} has a covering in the chaotic topology whose members are elements of $\mathcal{B}$. This holds because $\mathcal{B}$ contains all objects.
(4) For every object U of X_{affine} and $\mathcal{F}$ in $\mathcal{A}$ we have $H^p_{Zar}(U, \mathcal{F}) = 0$ for $p > 0$. This holds by the vanishing of cohomology of quasi-coherent modules on affines, see discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2.

This finishes the proof. □

0H07 **Remark 75.30.2.** Let S be a scheme. Let X be an algebraic space over S. We will later show that also $QC((Aff/X), \mathcal{O})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_X)$. See Sheaves on Stacks, Proposition 96.26.4.

75.31. Other chapters

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CHAPTER 76

More on Morphisms of Spaces

049F

76.1. Introduction

049G In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [**Knu71**].

76.2. Conventions

049H The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

76.3. Radicial morphisms

0480 It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

0481 **Definition 76.3.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $\mathrm{Spec}(K) \to Y$ where K is a field the reduction $(\mathrm{Spec}(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

0482 **Lemma 76.3.2.** *A radicial morphism of algebraic spaces is universally injective.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $\mathrm{Spec}(K) \to Y$ there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 67.19.2. □

0483 **Example 76.3.3.** It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [\mathrm{Spec}(\overline{\mathbf{Q}})/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \mathrm{Spec}(\mathbf{Q})$$

of Spaces, Example 65.14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

0484 **Lemma 76.3.4.** *Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.*

(1) *If f is decent then f is radicial.*

(2) *If f is quasi-separated then f is radicial.*
(3) *If f is locally separated then f is radicial.*

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 76.3.1 the morphism $(\mathrm{Spec}(K) \times_Y X)_{red} \to \mathrm{Spec}(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to \mathrm{Spec}(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to \mathrm{Spec}(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 29.10.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 68.17.4, 68.17.6, and 68.17.2 (to see that an immersion is decent) we see that the discussion in the first paragraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 68.14.2 we conclude that $X \cong \mathrm{Spec}(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 68.17.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 67.4.4, 67.4.8, and 67.10.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 66.5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 66.6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{\mathrm{Spec}(K)} U$$

As $X \to \mathrm{Spec}(K)$ is universally injective j is surjective, and as $X \to \mathrm{Spec}(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 26.10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). □

049E **Remark 76.3.5.** Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $\mathrm{Spec}(K) \to Y$ where K is a field

(1) the space $|\mathrm{Spec}(K) \times_Y X|$ is a singleton,
(2) there exists a monomorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \times_Y X$, and
(3) $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism *weakly radicial*. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to \mathrm{Spec}(\overline{\mathbf{Q}})$ of the morphism in Example 76.3.3 is weakly radicial, but not radicial. The analogue

of Lemma 76.3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

0AGE **Lemma 76.3.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is locally of finite type,*
(2) *for every étale morphism $V \to Y$ the map $|X \times_Y V| \to |V|$ is injective.*

Then f is universally injective.

Proof. The question is étale local on Y by Morphisms of Spaces, Lemma 67.19.6. Hence we may assume that Y is a scheme. Then Y is in particular decent and by Decent Spaces, Lemma 68.18.9 we see that f is locally quasi-finite. Let $y \in Y$ be a point and let X_y be the scheme theoretic fibre. Assume X_y is not empty. By Spaces over Fields, Lemma 72.10.8 we see that X_y is a scheme which is locally quasi-finite over $\kappa(y)$. Since $|X_y| \subset |X|$ is the fibre of $|X| \to |Y|$ over y we see that X_y has a unique point x. The same is true for $X_y \times_{\mathrm{Spec}(\kappa(y))} \mathrm{Spec}(k)$ for any finite separable extension $k/\kappa(y)$ because we can realize k as the residue field at a point lying over y in an étale scheme over Y, see More on Morphisms, Lemma 37.35.2. Thus X_y is geometrically connected, see Varieties, Lemma 33.7.11. This implies that the finite extension $\kappa(x)/\kappa(y)$ is purely inseparable.

We conclude (in the case that Y is a scheme) that for every $y \in Y$ either the fibre X_y is empty, or $(X_y)_{red} = \mathrm{Spec}(\kappa(x))$ with $\kappa(y) \subset \kappa(x)$ purely inseparable. Hence f is radicial (some details omitted), whence universally injective by Lemma 76.3.2. □

76.4. Monomorphisms

0B89 This section is the continuation of Morphisms of Spaces, Section 67.10. We would like to know whether or not every monomorphism of algebraic spaces is representable. If you can prove this is true or have a counterexample, please email stacks.project@gmail.com. For the moment this is known in the following cases

(1) for monomorphisms which are locally of finite type (more generally any separated, locally quasi-finite morphism is representable by Morphisms of Spaces, Lemma 67.51.1 and a monomorphism which is locally of finite type is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.10),
(2) if the target is a disjoint union of spectra of zero dimensional local rings (Decent Spaces, Lemma 68.19.1), and
(3) for flat monomorphisms (see below).

0B8A **Lemma 76.4.1** (David Rydh)**.** *A flat monomorphism of algebraic spaces is representable by schemes.*

Proof. Let $f : X \to Y$ be a flat monomorphism of algebraic spaces. To prove f is representable, we have to show $X \times_Y V$ is a scheme for every scheme V mapping to Y. Since being a scheme is local (Properties of Spaces, Lemma 66.13.1), we may assume V is affine. Thus we may assume $Y = \mathrm{Spec}(B)$ is an affine scheme. Next, we can assume that X is quasi-compact by replacing X by a quasi-compact open. The space X is separated as $X \to X \times_{\mathrm{Spec}(B)} X$ is an isomorphism. Applying Limits of Spaces, Lemma 70.17.3 we reduce to the case where B is local, $X \to \mathrm{Spec}(B)$ is a flat monomorphism, and there exists a point $x \in X$ mapping to the closed

point of $\mathrm{Spec}(B)$. Then $X \to \mathrm{Spec}(B)$ is surjective as generalizations lift along flat morphisms of separated algebraic spaces, see Decent Spaces, Lemma 68.7.4. Hence we see that $\{X \to \mathrm{Spec}(B)\}$ is an fpqc cover. Then $X \to \mathrm{Spec}(B)$ is a morphism which becomes an isomorphism after base change by $X \to \mathrm{Spec}(B)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 74.11.15. □

The following is (in some sense) a variant of the lemma above.

0B8B **Lemma 76.4.2.** *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact monomorphism of algebraic spaces such that for every $T \to Y$ the map*

$$\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X \times_Y T}$$

is injective. Then f is an isomorphism (and hence representable by schemes).

Proof. The question is étale local on Y, hence we may assume $Y = \mathrm{Spec}(A)$ is affine. Then X is quasi-compact and we may choose an affine scheme $U = \mathrm{Spec}(B)$ and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 66.6.3). Note that $U \times_X U = \mathrm{Spec}(B \otimes_A B)$. Hence the category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category $DD_{B/A}$ of descent data on modules for $A \to B$. See Properties of Spaces, Proposition 66.32.1, Descent, Definition 35.3.1, and Descent, Subsection 35.4.14. On the other hand,

$$A \to B$$

is a universally injective ring map. Namely, given an A-module M we see that $A \oplus M \to B \otimes_A (A \oplus M)$ is injective by the assumption of the lemma. Hence $DD_{B/A}$ is equivalent to the category of A-modules by Descent, Theorem 35.4.22. Thus pullback along $f : X \to \mathrm{Spec}(A)$ determines an equivalence of categories of quasi-coherent modules. In particular f^* is exact on quasi-coherent modules and we see that f is flat (small detail omitted). Moreover, it is clear that f is surjective (for example because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective). Hence we see that $\{X \to \mathrm{Spec}(A)\}$ is an fpqc cover. Then $X \to \mathrm{Spec}(A)$ is a morphism which becomes an isomorphism after base change by $X \to \mathrm{Spec}(A)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 74.11.15. □

0B8C **Lemma 76.4.3.** *A quasi-compact flat surjective monomorphism of algebraic spaces is an isomorphism.*

Proof. Such a morphism satisfies the assumptions of Lemma 76.4.2. □

76.5. Conormal sheaf of an immersion

04CM Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 67.13.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 67.14.1. This quasi-coherent $\mathcal{O}_Z$-module is the *conormal sheaf of Z in X* and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 67.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of

Spaces, Remark 67.12.4. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

04CN **Definition 76.5.1.** Let $i : Z \to X$ be an immersion. The *conormal sheaf* $\mathcal{C}_{Z/X}$ *of* Z *in* X or the *conormal sheaf of* i is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

04CO **Lemma 76.5.2.** *Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 67.13.1. □

04CP **Lemma 76.5.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 67.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 67.13.1). Thus we obtain a commutative

diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & g^{-1}\mathcal{I}' & \longrightarrow & g^{-1}\mathcal{O}_{X'} & \longrightarrow & g^{-1}i'_*\mathcal{O}_{Z'} & \longrightarrow & 0 \end{array}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. □

04G2 **Lemma 76.5.4.** *Let S be a scheme. The conormal sheaf of Definition 76.5.1, and its functoriality of Lemma 76.5.3 satisfy the following properties:*

(1) *If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 29.31.1.*
(2) *If in Lemma 76.5.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 29.31.3.*
(3) *Given a commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow[i]{} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ Z'' & \xrightarrow{i''} & X'' \end{array}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 76.5.2. □

04CQ **Lemma 76.5.5.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow[i]{} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 76.5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ U' & \longrightarrow & X' \end{array}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 65.11.6. Then using Lemmas 76.5.2 and 76.5.4 we see that the

question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 29.31.4. □

06BD **Lemma 76.5.6.** *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 76.5.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 76.5.2 and 76.5.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 29.31.5. □

76.6. The normal cone of an immersion

09RM Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 67.13.1. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms of Spaces, Lemma 67.14.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms of Spaces, Section 67.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal algebra of i as the conormal algebra of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 67.12.4. It is often denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

09RN **Definition 76.6.1.** Let $i : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of i* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

09RP (76.6.1.1) $$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

09RQ (76.6.1.2) $$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

09RR **Lemma 76.6.2.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X,*}|_{Z_U} = \mathcal{C}_{Z_U/U,*}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma follows from the fact that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 67.13.1. □

09RS **Lemma 76.6.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 67.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 67.13.1). Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & g^{-1}\mathcal{I}' & \longrightarrow & g^{-1}\mathcal{O}_{X'} & \longrightarrow & g^{-1}i'_*\mathcal{O}_{Z'} & \longrightarrow & 0 \end{array}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^n = g^{-1}((\mathcal{I}')^n)$ for all $n \geq 1$. Hence the diagram induces a map $g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{C}_{Z/X,n}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives maps $f^{-1}\mathcal{C}_{Z'/X',n} \to \mathcal{C}_{Z/X,n}$, which induce the desired map. □

09RT **Lemma 76.6.4.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a cartesian square of algebraic spaces over S with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 76.6.3 is surjective. If g is flat, then it is an isomorphism.

Proof. We may check the statement after étale localizing X'. In this case we may assume $X' \to X$ is a morphism of schemes, hence Z and Z' are schemes and the result follows from the case of schemes, see Divisors, Lemma 31.19.4. □

We use the same conventions for cones and vector bundles over algebraic spaces as we do for schemes (where we use the conventions of EGA), see Constructions, Sections 27.7 and 27.6. In particular, a vector bundle is a very general gadget (and not locally isomorphic to an affine space bundle).

09RU **Definition 76.6.5.** Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. The *normal cone* $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Morphisms of Spaces, Definition 67.20.8. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z. Moreover, the canonical surjection (76.6.1.2) of graded algebras defines a canonical closed immersion

09RV (76.6.5.1) $$C_Z X \longrightarrow N_Z X$$

of cones over Z.

76.7. Sheaf of differentials of a morphism

04CR We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 29.32) as well as Modules on Sites, Section 18.33. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{étale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 35.8.2.

04CS **Lemma 76.7.1.** *Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{étale} \to Y_{étale}$ be the associated morphism of small étale sites, see Descent, Remark 35.8.4. Then there is a canonical isomorphism*

$$(\Omega_{X/Y})^a = \Omega_{X_{étale}/Y_{étale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{étale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 29.32.1, and the second module is the one from Modules on Sites, Definition 18.33.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 37.9.9. This means that there is a natural $\mathcal{O}_{Y_{étale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{étale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X.

Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 35.8.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a "usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \mathrm{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Consider the morphism of ringed sites $\mathrm{id}_{small,étale,Zar} : X_{étale} \to X_{Zar}$ discussed in Descent, Remark 35.8.4 and Lemma 35.8.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to $\mathrm{id}_{small,étale,Zar,*}$, since $\mathrm{id}^*_{small,étale,Zar}$ is left adjoint to $\mathrm{id}_{small,étale,Zar,*}$ and since $(\Omega_{X/Y})^a = \mathrm{id}^*_{small,étale,Zar}\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{étale}/Y_{étale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\mathrm{d}(f)) = \mathrm{d}_{X/Y}(f)$ and $c(\mathrm{d}_{X/Y}(f)) = \mathrm{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. □

This clears the way for the following definition. For an alternative, see Remark 76.7.5.

04CT **Definition 76.7.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *sheaf of differentials* $\Omega_{X/Y}$ *of* X *over* Y is sheaf of differentials (Modules on Sites, Definition 18.33.10) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 66.21.3. The *universal* Y*-derivation* will be denoted $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 76.7.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 29.32.1. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

04CU **Lemma 76.7.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 66.18.11 and Equation (66.18.11.1) we may think of the restriction of a sheaf on $X_{étale}$ to $U_{étale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 18.33.6 we have

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{\mathcal{O}_{U_{étale}}/a^{-1}_{small}f^{-1}_{small}\mathcal{O}_{Y_{étale}}}$$

Since $a^{-1}_{small}f^{-1}_{small}\mathcal{O}_{Y_{étale}} = \psi^{-1}_{small}b^{-1}_{small}\mathcal{O}_{Y_{étale}} = \psi^{-1}_{small}\mathcal{O}_{V_{étale}}$ we see that the lemma holds. □

04CV **Lemma 76.7.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Choose a diagram as in Lemma 76.7.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent (for example by Morphisms, Lemma 29.32.7). Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 66.29.6. □

04CW **Remark 76.7.5.** Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 66.32 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 29.34.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 76.7.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

04CX **Lemma 76.7.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ maps to $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 18.33.11. □

05Z7 **Lemma 76.7.7.** *Let S be a scheme. Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{g} & X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f\circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f\circ g}$ in terms of the effect these maps have on local sections. □

05Z8 **Lemma 76.7.8.** *Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 76.7.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.9, of this result via étale localization, see Lemma 76.7.3. □

05Z9 **Lemma 76.7.9.** *Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.*

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.14, of this result via étale localization, see Lemma 76.7.3. □

05ZA **Lemma 76.7.10.** *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $d_{X/B}$ and the second arrow comes from Lemma 76.7.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 29.32.15 and will be a consequence of that lemma by étale localization, see Lemmas 76.7.3 and 76.5.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/B}$" here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 67.14.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

05ZB **Lemma 76.7.11.** *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B, and assume i (étale locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 76.7.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i over B, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 76.7.3 and 76.5.2. □

05ZC **Lemma 76.7.12.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 76.7.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.10 and étale localization, see Lemma 76.7.3. □

05ZD **Lemma 76.7.13.** *Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 76.7.6*

$$p^*\Omega_{X/B} \oplus q^*\Omega_{Y/B} \longrightarrow \Omega_{X \times_B Y/B}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.11 and étale localization, see Lemma 76.7.3. □

05ZE **Lemma 76.7.14.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.12 and étale localization, see Lemma 76.7.3. □

05ZF **Lemma 76.7.15.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Follows from the schemes version, see Morphisms, Lemma 29.32.13 and étale localization, see Lemma 76.7.3. □

0CK5 **Lemma 76.7.16.** *Let S be a scheme. Let $f : X \to Y$ be a smooth morphism of algebraic spaces over S. Then the module of differentials $\Omega_{X/Y}$ is finite locally free.*

Proof. The statement is étale local on X and Y by Lemma 76.7.3. Hence this follows from the case of schemes, see Morphisms, Lemma 29.34.12. □

76.8. Topological invariance of the étale site

05ZG We show that the site $X_{spaces,étale}$ is a "topological invariant". It then follows that $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too, see Lemma 76.8.2.

05ZH **Theorem 76.8.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 67.53.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b),V \times_Y V,\Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is

a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\mathrm{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 67.51.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \mathrm{pr}_0$ and $s_X = \mathrm{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\mathrm{pr}_0, \mathrm{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 65.10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 74.7.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatible isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme W_X. Hence by Étale Cohomology, Theorem 59.45.2 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. □

07VW **Lemma 76.8.2.** *With assumption and notation as in Theorem 76.8.1 the equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$ restricts to equivalences of categories $Y_{étale} \to X_{étale}$ and $Y_{affine,étale} \to X_{affine,étale}$.*

Proof. This is just the statement that given an object $V \in Y_{spaces,étale}$ we have V is a(n affine) scheme if and only if $V \times_Y X$ is a(n affine) scheme. Since $V \times_Y X \to V$

is integral, universally injective, and surjective (as a base change of $X \to Y$) this follows from Limits of Spaces, Lemma 70.15.4 and Proposition 70.15.2. □

05ZI **Remark 76.8.3.** A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 67.53.3. In fact Theorem 76.8.1 does not hold for universal homeomorphisms. To see this, let k be an algebraically closed field of characteristic 0 and let Email by Lenny Taelman dated May 1, 2016.

$$\mathbf{A}^1 \to X \to \mathbf{A}^1$$

be as in Morphisms of Spaces, Example 67.53.3. Recall that the first morphism is étale and identifies t with $-t$ for $t \in \mathbf{A}^1_k \setminus \{0\}$ and that the second morphism is our universal homeomorphism. Since $\mathbf{A}^1_k$ has no nontrivial connected finite étale coverings (because k is algebraically closed of characteristic zero; details omitted), it suffices to construct a nontrivial connected finite étale covering $Y \to X$. To do this, let Y be the affine line with zero doubled (Schemes, Example 26.14.3). Then $Y = Y_1 \cup Y_2$ with $Y_i = \mathbf{A}^1_k$ glued along $\mathbf{A}^1_k \setminus \{0\}$. To define the morphism $Y \to X$ we use the morphisms

$$Y_1 \xrightarrow{1} \mathbf{A}^1_k \to X \quad \text{and} \quad Y_2 \xrightarrow{-1} \mathbf{A}^1_k \to X.$$

These glue over $Y_1 \cap Y_2$ by the construction of X and hence define a morphism $Y \to X$. In fact, we claim that

$$\begin{array}{ccc} Y & \longleftarrow & Y_1 \amalg Y_2 \\ \downarrow & & \downarrow \\ X & \longleftarrow & \mathbf{A}^1_k \end{array}$$

is a cartesian square. We omit the details; you can use for example Groupoids, Lemma 39.20.7. Since $\mathbf{A}^1_k \to X$ is étale and surjective, this proves that $Y \to X$ is finite étale of degree 2 which gives the desired example.

More simply, you can argue as follows. The scheme Y has a free action of the group $G = \{+1, -1\}$ where -1 acts by swapping Y_1 and Y_2 and changing the sign of the coordinate. Then $X = Y/G$ (see Spaces, Definition 65.14.4) and hence $Y \to X$ is finite étale. You can also show directly that there exists a universal homeomorphism $X \to \mathbf{A}^1_k$ by using $t \mapsto t^2$ on affine spaces. In fact, this X is the same as the X above.

76.9. Thickenings

05ZJ The following terminology may not be completely standard, but it is convenient.

05ZK **Definition 76.9.1.** Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a *thickening* of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a *first order thickening* of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) We say X' is a *finite order thickening* of X if X is a closed subspace of X and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X is nilpotent, i.e., there exists an integer $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$.

(4) We say X' is an *nth order thickening* of X if X is a closed subspace of X and $\mathcal{I}^{n+1} = 0$ where $\mathcal{I} \subset \mathcal{O}_{X'}$ is the quasi-coherent sheaf of ideals defining X.
(5) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(6) Let B be an algebraic space. We similarly define *thickenings over B*, and *morphisms of thickenings over B*. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

05ZL (76.9.1.1) $$X_{spaces,étale} = X'_{spaces,étale}$$

via the pullback functor, see Theorem 76.8.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

05ZM (76.9.1.2) $$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'})$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 66.28.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$\begin{array}{ccc} X_{spaces,étale} & \longleftarrow & Y_{spaces,étale} \\ \uparrow & & \uparrow \\ X'_{spaces,étale} & \longleftarrow & Y'_{spaces,étale} \end{array}$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$\begin{array}{ccc} f^{-1}_{spaces,étale}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \\ {\scriptstyle i_Y^\sharp}\uparrow & & \uparrow{\scriptstyle i_X^\sharp} \\ f^{-1}_{spaces,étale}\mathcal{O}_{Y'} & \xrightarrow{(f')^\sharp} & \mathcal{O}_{X'} \end{array}$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

05ZN **Lemma 76.9.2.** *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras*

$$\alpha : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$\begin{array}{ccc} f^{-1}_{spaces,étale}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \\ {\scriptstyle i_Y^\sharp}\uparrow & & \uparrow{\scriptstyle i_X^\sharp} \\ f^{-1}_{spaces,étale}\mathcal{O}_{Y'} & \xrightarrow{\alpha} & \mathcal{O}_{X'} \end{array}$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 66.28.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,étale}, \alpha) : (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,étale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 18.40.9), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 66.28.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 66.28.4. We omit the verification that f' is a morphism over B. □

05ZP **Lemma 76.9.3.** *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.*

Proof. Let $U' \to X'$ be the object of $X'_{spaces,étale}$ corresponding to the object $U \to X$ of $X_{spaces,étale}$ via (76.9.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 67.51.2. □

Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. This gives some control over the structure sheaf of $\mathcal{O}_{X'}$ but technically the class of finite order thickenings $X \subset X'$ is much easier to handle. Namely, if X' is an nth order thickening we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

05ZQ **Lemma 76.9.4.** *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,étale}$. Then*

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$ via (76.9.1.2).

Proof. Let $U' \to X'$ be the étale morphism of algebraic spaces such that $U = X \times_{X'} U'$, see Theorem 76.8.1. By Limits of Spaces, Lemma 70.15.1 we see that U' is an affine scheme. Hence $\Gamma(U, \mathcal{O}_{X'}) = \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ is surjective as

$U \to U'$ is a closed immersion of affine schemes. Below we give a direct proof for finite order thickenings which is the case most used in practice. □

Proof for finite order thickenings. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 67.14.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{étale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. □

05ZR **Lemma 76.9.5.** *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.*

Proof. Note that $X'_{red} = X_{red}$. Hence if X is a scheme, then X'_{red} is a scheme. Thus the result follows from Limits of Spaces, Lemma 70.15.3. Below we give a direct proof for finite order thickenings which is the case most often used in practice. □

Proof for finite order thickenings. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 66.13.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 76.9.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 76.9.4 the map $A' \to A$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 66.33.1 we obtain a canonical morphism $f : X' \to \mathrm{Spec}(A')$ which fits into the following commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \| & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \end{array}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 67.51.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to \mathrm{Spec}(A')$ is étale, see Properties of Spaces, Definition 66.16.2. Write $U' = \mathrm{Spec}(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = \mathrm{Spec}(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where

$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 76.9.4. The morphism $U' \to X' \to \mathrm{Spec}(A')$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\widetilde{I})^a$, see Descent, Definition 35.8.2 for notation and Descent, Proposition 35.8.9 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 10.143.11. □

05ZS **Lemma 76.9.6.** *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. The functor*

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{étale} \to X_{étale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,étale} \to X_{spaces,étale}$, see Theorem 76.8.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 76.9.5. □

First order thickening are described as follows.

05ZT **Lemma 76.9.7.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{étale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

1. *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
2. *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 76.9.4 and 76.9.5. We first handle the case $B = S = \mathrm{Spec}(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{étale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,étale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 35.8.9. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 10.143.11. Hence we see that

$$U \longmapsto U' = \mathrm{Spec}(\mathcal{A}(U))$$

is a functor from $X_{affine,étale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{étale}$. To see this, if U is an object of $X_{étale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 35.9.4. Hence by More on Morphisms, Lemma 37.2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 65.9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 65.10.5). The morphism $X = U/R \to U'/R' = X'$ is a first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{étale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{étale}} \longrightarrow \mathcal{A}|_{U_{étale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{étale}$. Hence by Properties of Spaces, Lemma 66.18.14 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 76.9.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. □

09ZX **Lemma 76.9.8.** *Let S be a scheme. Let $Y \subset Y'$ be a thickening of algebraic spaces over S. Let $X' \to Y'$ be a morphism and set $X = Y \times_{Y'} X'$. Then $(X \subset X') \to (Y \subset Y')$ is a morphism of thickenings. If $Y \subset Y'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

0BPH **Lemma 76.9.9.** *Let S be a scheme. If $X \subset X'$ and $X' \subset X''$ are thickenings of algebraic spaces over S, then so is $X \subset X''$.*

Proof. Omitted. □

0BPI **Lemma 76.9.10.** *The property of being a thickening is fpqc local. Similarly for first order thickenings.*

Proof. The statement means the following: Let S be a scheme and let $X \to X'$ be a morphism of algebraic spaces over S. Let $\{g_i : X'_i \to X'\}$ be an fpqc covering of algebraic spaces such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent on Spaces, Lemma 74.11.17 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

76.10. Morphisms of thickenings

0CG4 If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of algebraic spaces, then often properties of the morphism f are inherited by f'. There are several variants.

09ZY **Lemma 76.10.1.** *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*
(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is universally injective if and only if f' is universally injective,*
(8) *f is universally open if and only if f' is universally open,*
(9) *f is representable if and only if f' is representable, and*
(10) *add more here.*

Proof. Observe that $Y \to Y'$ and $X \to X'$ are integral and universal homeomorphisms. This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Limits of Spaces, Proposition 70.15.2 which tells us that there is a 1-to-1 correspondence between affine schemes étale over X and X' and between affine schemes étale over Y and Y'. Part (5) follows from (1) and (4) by Morphisms of Spaces, Lemma 67.45.7. Finally, note that

$$X \times_Y X = X \times_{Y'} X \to X \times_{Y'} X' \to X' \times_{Y'} X'$$

is a thickening (the two arrows are thickenings by Lemma 76.9.8). Hence applying (3) and (4) to the morphism $(X \subset X') \to (X \times_Y X \to X' \times_{Y'} X')$ we obtain (6). Finally, part (9) follows from the fact that an algebraic space thickening of a scheme is again a scheme, see Lemma 76.9.5. □

09ZZ **Lemma 76.10.2.** *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S such that $X = Y \times_{Y'} X'$. If $X \subset X'$ is a finite order thickening, then*

(1) *f is a closed immersion if and only if f' is a closed immersion,*
(2) *f is locally of finite type if and only if f' is locally of finite type,*
(3) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(4) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(5) *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
(6) *f is unramified if and only if f' is unramified,*
(7) *f is proper if and only if f' is proper,*
(8) *f is a finite morphism if and only if f' is an finite morphism,*
(9) *f is a monomorphism if and only if f' is a monomorphism,*
(10) *f is an immersion if and only if f' is an immersion, and*
(11) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 37.3.3. This proves (2), (3), (4), (5), and (6).

The properties of morphisms in (1), (7), (8), (9), (10) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemmas 67.40.3, 67.45.5, and 67.10.5.

The interesting direction in (1), (7), (8), (9), (10) is to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $Y \subset Y'$ is a first order thickening, see discussion on finite order thickenings above.

Proof of (1). Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$ and $U = X \times_Y V$. Then $U \to V$ is a closed immersion, which implies that U is a scheme, which in turn implies that U' is a scheme (Lemma 76.9.5). Thus we can apply the lemma in the case of schemes (More on Morphisms, Lemma 37.3.3) to $(U \subset U') \to (V \subset V')$ to conclude.

Proof of (7). Follows by combining (2) with results of Lemma 76.10.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 76.10.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 29.44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $X' \to X' \times_{Y'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{Y'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{Y'}$ cutting out Y in Y'. Indeed, $X = X \times_Y X = (X' \times_{Y'} X') \times_{Y'} Y$ is contained in X'. The conormal sheaf of the closed immersion $\Delta : X' \to X' \times_{Y'} X'$ is equal to $\Omega_{X'/Y'}$ (this is the analogue of Morphisms, Lemma 29.32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 76.7.11 and 76.7.13; some details omitted). Thus it suffices to show that $\Omega_{X'/Y'} = 0$ which follows from (5) and the corresponding statement for X/Y.

Proof of (10). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 67.12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (1). This finishes the proof. □

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

0BPJ **Lemma 76.10.3.** *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings of algebraic spaces over S. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then*

(1) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(2) *f is finite if and only if f' is finite,*
(3) *f is a closed immersion if and only if f' is a closed immersion,*
(4) *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
(5) *f is unramified if and only if f' is unramified,*
(6) *f is a monomorphism if and only if f' is a monomorphism,*
(7) *f is an immersion if and only if f' is an immersion,*

(8) *f is proper if and only if f' is proper, and*
(9) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 37.3.4. This proves (1), (4), and (5).

The properties in (2), (3), (6), (7), and (8) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemmas 67.40.3, 67.45.5, and 67.10.5. Hence in each case we need only to prove that if f has the desired property, so does f'.

Case (2) follows from case (5) of Lemma 76.10.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms of Spaces, Lemma 67.45.6).

Case (3). This follows immediately from Limits of Spaces, Lemma 70.15.5.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ induces a bijection $|X'| \to |X' \times_{Y'} X'|$, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.10. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero. (The conormal sheaf of the closed immersion $\Delta_{X'/Y'}$ is equal to $\Omega_{X'/Y'}$; this is the analogue of Morphisms, Lemma 29.32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 76.7.11 and 76.7.13; some details omitted.) In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 10.21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 67.12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 76.10.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. □

76.11. Picard groups of thickenings

0DNL Some material on Picard groups of thickenings.

0DNM **Lemma 76.11.1.** *Let S be a scheme. Let $X \subset X'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X', \mathcal{O}_{X'}^*) \longrightarrow H^0(X, \mathcal{O}_X^*) \\ \longrightarrow H^1(X, \mathcal{I}) \longrightarrow \mathrm{Pic}(X') \longrightarrow \mathrm{Pic}(X) \\ \longrightarrow H^2(X, \mathcal{I}) \longrightarrow \ldots \longrightarrow \ldots \end{aligned}$$

of abelian groups.

Proof. Recall that $X_{étale} = X'_{étale}$, see Lemma 76.9.6 and more generally the discussion in Section 76.9. The sequence of the lemma is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

on $X_{étale}$ where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed site with the first cohomology group of the sheaf of invertible functions, see Cohomology on Sites, Lemma 21.6.1. □

76.12. Infinitesimal neighbourhoods

05ZU A natural construction of finite order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 67.12.4). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 67.13.1. For $n \geq 1$ we can consider the closed subspace $Z_n \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^{n+1}$.

05ZV **Definition 76.12.1.** Let $i : Z \to X$ be an immersion of algebraic spaces.

(1) The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z_1$ over X described above.
(2) The *nth order infinitesimal neighbourhood* of Z in X is the nth order thickening $Z \subset Z_n$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

05ZW **Lemma 76.12.2.** *Let $i : Z \to X$ be an immersion of algebraic spaces.*

(1) *The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

(2) *For $n \geq 1$ the nth order infinitesimal neighbourhood Z_n of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is an nth order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z_n)$ of thickenings over X.

Proof. We will only prove (1). Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of b as a morphism into U, see Properties of Spaces, Lemma 66.4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 67.13.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 67.13.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. □

05ZX **Lemma 76.12.3.** *Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 76.5.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. □

76.13. Formally smooth, étale, unramified transformations

04G3 Recall that a ring map $R \to A$ is called *formally smooth*, resp. *formally étale*, resp. *formally unramified* (see Algebra, Definition 10.138.1, resp. Definition 10.150.1, resp. Definition 10.148.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

049S **Definition 76.13.1.** Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{array}{ccc} F & \longleftarrow & T \\ {\scriptstyle a}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ G & \longleftarrow & T' \end{array}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is *formally smooth* if given any solid diagram as above there exists a dotted arrow making the diagram commute[1].
(2) We say a is *formally étale* if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is *formally unramified* if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

04G4 **Lemma 76.13.2.** *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.*

Proof. Formal from the definition. □

049T **Lemma 76.13.3.** *Composition.*

(1) *A composition of formally smooth transformations of functors is formally smooth.*
(2) *A composition of formally étale transformations of functors is formally étale.*
(3) *A composition of formally unramified transformations of functors is formally unramified.*

Proof. This is formal. □

049U **Lemma 76.13.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow[b']{} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

(1) *If a is formally smooth, then the base change a' is formally smooth.*
(2) *If a is formally étale, then the base change a' is formally étale.*
(3) *If a is formally unramified, then the base change a' is formally unramified.*

Proof. This is formal. □

04AL **Lemma 76.13.5.** *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors.*

(1) *If a is smooth then a is formally smooth.*

[1] This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

(2) *If a is étale, then a is formally étale.*
(3) *If a is unramified, then a is formally unramified.*

Proof. Consider a solid commutative diagram

$$\begin{array}{ccc} F & \longleftarrow & T \\ {\scriptstyle a}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ G & \longleftarrow & T' \end{array}$$

as in Definition 76.13.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 37.11.7 (resp. Lemma 37.8.9, resp. Lemma 37.6.8) we can fill in (resp. uniquely fill in, resp. fill in at most one way) the dotted arrow in the diagram

$$\begin{array}{ccc} F \times_G T' & \longleftarrow & T \\ \downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ T' & \longleftarrow & T' \end{array}$$

an hence we also obtain the corresponding assertion in the first diagram. □

04CY **Lemma 76.13.6.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.*

(1) *If b is formally smooth, then $b \circ a$ is formally smooth.*
(2) *If b is formally étale, then $b \circ a$ is formally étale.*
(3) *If b is formally unramified, then $b \circ a$ is formally unramified.*

Conversely, consider a solid commutative diagram

$$\begin{array}{ccc} G & \longleftarrow & T \\ {\scriptstyle b}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ H & \longleftarrow & T' \end{array}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

(4) *If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that*

$$\begin{array}{ccccc} G & \longleftarrow & T & \longleftarrow & T \times_{T'} U' \\ {\scriptstyle b}\downarrow & \swarrow & & & \downarrow \\ H & \longleftarrow & T' & \longleftarrow & U' \end{array}$$

commutes and t is in the image of $U' \to T'$.
(5) *If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.*
(6) *If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.*

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is

representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 76.13.5 and Lemma 76.13.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 41.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$\begin{array}{ccccc} F & \longleftarrow & W & \longleftarrow & T \times_{T'} U' \\ {\scriptstyle b\circ a}\downarrow & \nwarrow & & & \downarrow \\ H & \longleftarrow & T' & \longleftarrow & U' \end{array}$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 41.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. □

04CZ **Remark 76.13.7.** It is tempting to think that in the situation of Lemma 76.13.6 we have "b formally smooth" $\Leftrightarrow$ "$b \circ a$ formally smooth". However, this is likely not true in general.

04G5 **Lemma 76.13.8.** *Let S be a scheme. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.*

(1) *If $b \circ a$ is formally unramified then a is formally unramified.*
(2) *If $b \circ a$ is formally étale then a is formally étale.*
(3) *If $b \circ a$ is formally smooth then a is formally smooth.*

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. □

76.14. Formally unramified morphisms

04G6 In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

04G7 **Definition 76.14.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally unramified* if it is formally unramified as a transformation of functors as in Definition 76.13.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 76.13. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 76.14.6.

04G8 **Lemma 76.14.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified,*
(2) *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 37.6.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.*

Proof. Assume f is formally unramified. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 76.13.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 76.13.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 76.13.6 that $X \to Y$ is formally unramified, i.e., (1) holds. □

05ZY **Lemma 76.14.3.** *Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 76.14.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 66.6.1) and a morphism $T' \to X$ is determined by its restriction to U'. □

05ZZ **Lemma 76.14.4.** *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

0600 **Lemma 76.14.5.** *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

04G9 **Lemma 76.14.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified, and*
(2) *$\Omega_{X/Y} = 0$.*

Proof. This is a combination of Lemma 76.14.2, More on Morphisms, Lemma 37.6.7, and Lemma 76.7.3. □

04GA **Lemma 76.14.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is unramified,*
(2) *the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and*
(3) *the morphism f is locally of finite type and formally unramified.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 29.35.2 and Lemma 76.14.6. □

05W6 **Lemma 76.14.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified.*

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 76.14.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 67.38.9) and surjective (Morphisms of Spaces, Lemma 67.19.2) hence an isomorphism, i.e., f is a monomorphism. In this

way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 67.27.10 and 67.51.1. □

05W8 **Lemma 76.14.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is universally closed, unramified, and a monomorphism,*
(3) *f is universally closed, unramified, and universally injective,*
(4) *f is universally closed, locally of finite type, and a monomorphism,*
(5) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (2) – (5) follows immediately from Lemma 76.14.8. Moreover, if (2) – (5) are satisfied then f is representable. Similarly, if (1) is satisfied then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 41.7.2. □

76.15. Universal first order thickenings

0601 Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

0602 (76.15.0.1)

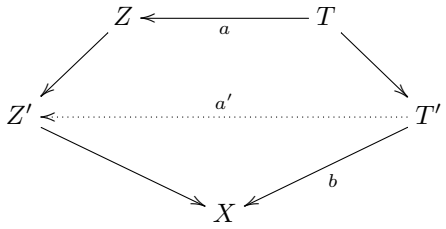

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

0603 **Lemma 76.15.1.** *Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent*

(1) *$Z \subset Z'$ is a universal first order thickening,*
(2) *for any diagram (76.15.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and*
(3) *for any diagram (76.15.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are formal. Assume (3) a assume given an arbitrary diagram (76.15.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 65.9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines,

so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i,j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence $a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. □

0604 **Lemma 76.15.2.** *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.*

Proof. This is formal. Namely, by Lemma 76.15.1 it suffices to consider solid commutative diagrams (76.15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. □

0605 **Lemma 76.15.3.** *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.*

(1) *If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 76.8.1) is a universal first order thickening of Z over X.*
(2) *If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.*

Proof. Proof of (1). By Lemma 76.15.1 it suffices to consider solid commutative diagrams (76.15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 76.13.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 76.8.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. □

0606 **Lemma 76.15.4.** *Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. Choose any commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 76.14.2. Combining Lemma 76.15.1 and More on Morphisms, Lemma 37.7.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 76.15.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 65.11.6). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 76.15.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 76.15.2 part (2) that Z' is a universal first order thickening of Z over X. □

0607 **Definition 76.15.5.** Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 76.15.4.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{étale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

0608 **Lemma 76.15.6.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 76.12.1,*
(3) *the conormal sheaf of i in the sense of Definition 76.5.1 agrees with the conormal sheaf of i in the sense of Definition 76.15.5.*

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 29.35.7 and 29.35.8 an immersion is unramified (via

the abstract principle of Spaces, Lemma 65.5.8). Hence it is formally unramified by Lemma 76.14.7. The other assertions follow by combining Lemmas 76.12.2 and 76.12.3 and the definitions. □

0609 **Lemma 76.15.7.** *Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.*

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

$$\begin{array}{ccc} Z' & \xleftarrow{c} & T \\ \downarrow & \nwarrow^{a,b} & \downarrow \\ X & \longleftarrow & T' \end{array}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

060A **Lemma 76.15.8.** *Let S be a scheme Consider a commutative diagram of algebraic spaces over S*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

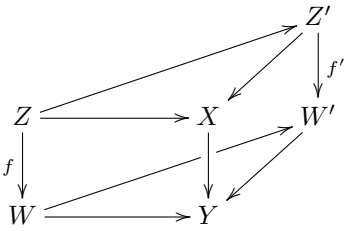

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 76.5.3 applied to $(Z \subset Z') \to (W \subset W')$. □

060B **Lemma 76.15.9.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 76.15.8 is surjective.

Proof. The morphism h is formally unramified by Lemma 76.14.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 76.5.5 for why this implies that the map of conormal sheaves is surjective. □

060C **Lemma 76.15.10.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 67.30.4. Hence the first statement follows from the description of W' in Lemma 76.15.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 76.5.5 for why this implies that the map of conormal sheaves is an isomorphism. □

060D **Lemma 76.15.11.** *Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over a base scheme S. Let*

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 76.14.2. The compatibility of universal first order thickenings is a consequence of Lemmas 76.15.2 and 76.15.3. □

060E **Lemma 76.15.12.** *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$dh' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 76.7.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma 76.15.11 and Lemma 76.7.3. In this case the result is More on Morphisms, Lemma 37.7.9. □

060F **Lemma 76.15.13.** *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $d_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 76.7.10. Hence the result follows on applying Lemma 76.15.12. □

06BE **Lemma 76.15.14.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 76.15.8 and the second from Lemma 76.15.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 76.15.11 and Lemma 76.7.3. In this case the result is More on Morphisms, Lemma 37.7.11. □

06BF **Lemma 76.15.15.** *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.*

(1) *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*

(2) *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 76.15.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 76.15.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 76.5.6 we have an exact sequence

$$(i')^*\mathcal{C}_{Y\times_{Y'}Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y\times_{Y'}Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Lemma 76.5.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y\times_{Y'}Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y\times_{Y'}Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

76.16. Formally étale morphisms

04GB In this section we work out what it means that a morphism of algebraic spaces is formally étale.

04GC **Definition 76.16.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally étale* if it is formally étale as a transformation of functors as in Definition 76.13.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 76.13.

04GD **Lemma 76.16.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally étale,*
(2) *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 37.8.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally étale.*

Proof. Assume f is formally étale. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 76.13.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 76.13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 76.13.6 that $X \to Y$ is formally étale, i.e., (1) holds. □

0611 **Lemma 76.16.3.** *Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{} & T' \end{array}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 76.16.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 76.14.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. □

0612 **Lemma 76.16.4.** *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

0613 **Lemma 76.16.5.** *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

0614 **Lemma 76.16.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:*

(1) *f is formally étale,*
(2) *f is formally unramified and the universal first order thickening of X over Y is equal to X,*
(3) *f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and*
(4) *$\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 76.14.6 and Definition 76.15.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. □

0615 **Lemma 76.16.7.** *An unramified flat morphism is formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.8.7 and étale localization, see Lemmas 76.14.2 and 76.16.2 and Morphisms of Spaces, Lemma 67.30.5. □

0616 **Lemma 76.16.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is étale, and*
(2) *the morphism f is locally of finite presentation and formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.8.9 and étale localization, see Lemma 76.16.2 and Morphisms of Spaces, Lemmas 67.28.4 and 67.39.2. □

76.17. Infinitesimal deformations of maps

0617 In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (76.9.1.1) and (76.9.1.2).

0618 **Lemma 76.17.1.** *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 76.5.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}_{X/X'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle (a')^\sharp} \uparrow{\scriptstyle (b')^\sharp} & & \uparrow & & \\ 0 & \longrightarrow & a^{-1}\mathcal{C}_{Y/Y'} & \longrightarrow & a^{-1}\mathcal{O}_{Y'} & \longrightarrow & a^{-1}\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

0619 (76.17.1.1) $$D = \mathrm{d}_{Y/B} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

04D0 **Lemma 76.17.2.** *Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let*

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 76.17.1 hold and the derivation D and θ are related by Equation (76.17.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (76.17.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 76.9.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. □

0CK6 **Remark 76.17.3.** Assumptions and notation as in Lemma 76.17.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (76.17.1.1).

061A **Lemma 76.17.4.** *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that*

(1) *a' is a morphism over B,*
(2) *$a'|_U = a|_U$, and*
(3) *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Then the rule

061B (76.17.4.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 74.7.2. □

061C **Lemma 76.17.5.** *Same notation and assumptions as in Lemma 76.17.4. We identify sheaves on X and X' via (76.9.1.1). There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (76.17.4.1). Moreover, the action is simply transitive for any object U' of $(X')_{spaces,étale}$ over which the sheaf (76.17.4.1) has a section.

Proof. This is a combination of Lemmas 76.17.1, 76.17.2, and 76.17.4. □

061D **Remark 76.17.6.** A special case of Lemmas 76.17.1, 76.17.2, 76.17.4, and 76.17.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 76.17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } B \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$.

0CK7 **Remark 76.17.7.** Another special case of Lemmas 76.17.1, 76.17.2, 76.17.4, and 76.17.5 is where B itself is a thickening $Z \subset Z' = B$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{array}{ccc} (X \subset X') & \overset{(a,?)}{\dashrightarrow} & (Y \subset Y') \\ & {\scriptstyle (g,g')} \searrow \quad \swarrow {\scriptstyle (h,h')} & \\ & (Z \subset Z') & \end{array}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 76.17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

0CK8 **Lemma 76.17.8.** *Let S be a scheme. Consider a commutative diagram of first order thickenings*

$$\begin{array}{ccc} (T_2 \subset T_2') & \xrightarrow[(a_2,a_2')]{} & (X_2 \subset X_2') \\ {\scriptstyle (h,h')}\downarrow & & \downarrow{\scriptstyle (f,f')} \\ (T_1 \subset T_1') & \xrightarrow{(a_1,a_1')} & (X_1 \subset X_1') \end{array} \quad \text{and a commutative diagram} \quad \begin{array}{ccc} X_2' & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ X_1' & \longrightarrow & B_1 \end{array}$$

of algebraic spaces over S with $X_2 \to X_1$ and $B_2 \to B_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/B_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/B_2} = h^*a_1^*\Omega_{X_1/B_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{array}{ccc} T_2' & \xrightarrow[\theta_2 \cdot a_2']{} & X_2' \\ \downarrow & & \downarrow \\ T_1' & \xrightarrow{\theta_1 \cdot a_1'} & X_1' \end{array}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 76.17.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ we get as the construction of the sheaf of differentials is compatible with étale localization (both on source and target), see Lemma 76.7.3. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on étale locally. Thus we may assume T_i', X_i', B_2, and B_1 are schemes and in this case the lemma follows from More on Morphisms, Lemma 37.9.10. Alternative proof: using Lemma 76.9.2 it suffices to show a certain diagram of sheaves of rings on X_1' is commutative; then argue exactly as in the proof of the aforementioned More on Morphisms, Lemma 37.9.10 to see that this is indeed the case. □

76.18. Infinitesimal deformations of algebraic spaces

06BG The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

06BH **Lemma 76.18.1.** *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings of algebraic spaces over S. Assume that f is flat. Then the following are equivalent*

(1) *f' is flat and $X = Y \times_{Y'} X'$, and*
(2) *the canonical map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 37.10.1. □

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 76.23.

0CG5 **Lemma 76.18.2.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (B \subset B') & \end{array}$$

of thickenings of algebraic spaces over S. Assume

(1) *X' is flat over B',*
(2) *f is flat,*
(3) *$B \subset B'$ is a finite order thickening, and*
(4) *$X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.*

Then f' is flat and Y' is flat over B' at all points in the image of f'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of schemes. See More on Morphisms, Lemma 37.10.2. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. □

Many properties of morphisms of schemes are preserved under flat deformations.

0CG6 **Lemma 76.18.3.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (B \subset B') & \end{array}$$

of thickenings of algebraic spaces over S. Assume $B \subset B'$ is a finite order thickening, X' flat over B', $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

0CG7 (1) *f is representable if and only if f' is representable,*
0CG8 (2) *f is flat if and only if f' is flat,*
0CG9 (3) *f is an isomorphism if and only if f' is an isomorphism,*
0CGA (4) *f is an open immersion if and only if f' is an open immersion,*
0CGB (5) *f is quasi-compact if and only if f' is quasi-compact,*
0CGC (6) *f is universally closed if and only if f' is universally closed,*
0CGD (7) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
0CGE (8) *f is a monomorphism if and only if f' is a monomorphism,*
0CGF (9) *f is surjective if and only if f' is surjective,*
0CGG (10) *f is universally injective if and only if f' is universally injective,*
0CGH (11) *f is affine if and only if f' is affine,*
0CGI (12) *f is locally of finite type if and only if f' is locally of finite type,*
0CGJ (13) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
0CGK (14) *f is locally of finite presentation if and only if f' is locally of finite presentation,*
0CGL (15) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
0CGM (16) *f is universally open if and only if f' is universally open,*
0CGN (17) *f is syntomic if and only if f' is syntomic,*
0CGP (18) *f is smooth if and only if f' is smooth,*
0CGQ (19) *f is unramified if and only if f' is unramified,*
0CGR (20) *f is étale if and only if f' is étale,*
0CGS (21) *f is proper if and only if f' is proper,*
0CGT (22) *f is integral if and only if f' is integral,*
0CGU (23) *f is finite if and only if f' is finite,*
0CGV (24) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(25) *add more here.*

Proof. Case (1) follows from Lemma 76.10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (14), (15), (17), (18), (19), (20) follow from the corresponding cases of More on Morphisms, Lemma 37.10.3.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (16) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.4.4, 67.10.5, 67.20.5, 67.40.3, 67.45.5, and 67.46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (20) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 76.10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 76.10.1.

The case (21). See Lemma 76.10.2.

The case (22). See Lemma 76.10.1.

The case (23). See Lemma 76.10.2.

The case (24). Assume f finite locally free. By (23) we see that f' is finite. By (2) we see that f' is flat. By (14) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 76.23.

0CGW **Lemma 76.18.4.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (B \subset B') & \end{array}$$

of thickenings of algebraic spaces over S. Assume

(1) *$Y' \to B'$ is locally of finite type,*
(2) *$X' \to B'$ is flat and locally of finite presentation,*
(3) *f is flat, and*
(4) *$X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.*

Then f' is flat and for all $y' \in |Y'|$ in the image of $|f'|$ the morphism $Y' \to B'$ is flat at y'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$

and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of schemes. See More on Morphisms, Lemma 37.10.4. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. □

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

0CGX **Lemma 76.18.5.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (B \subset B') & \end{array}$$

of thickenings of algebraic spaces over S. Assume $Y' \to B'$ locally of finite type, $X' \to B'$ flat and locally of finite presentation, $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

0CGY (1) *f is representable if and only if f' is representable,*
0CGZ (2) *f is flat if and only if f' is flat,*
0CH0 (3) *f is an isomorphism if and only if f' is an isomorphism,*
0CH1 (4) *f is an open immersion if and only if f' is an open immersion,*
0CH2 (5) *f is quasi-compact if and only if f' is quasi-compact,*
0CH3 (6) *f is universally closed if and only if f' is universally closed,*
0CH4 (7) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
0CH5 (8) *f is a monomorphism if and only if f' is a monomorphism,*
0CH6 (9) *f is surjective if and only if f' is surjective,*
0CH7 (10) *f is universally injective if and only if f' is universally injective,*
0CH8 (11) *f is affine if and only if f' is affine,*
0CH9 (12) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
0CHA (13) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
0CHB (14) *f is universally open if and only if f' is universally open,*
0CHC (15) *f is syntomic if and only if f' is syntomic,*
0CHD (16) *f is smooth if and only if f' is smooth,*
0CHE (17) *f is unramified if and only if f' is unramified,*
0CHF (18) *f is étale if and only if f' is étale,*
0CHG (19) *f is proper if and only if f' is proper,*
0CHH (20) *f is finite if and only if f' is finite,*
0CHI (21) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(22) *add more here.*

Proof. Case (1) follows from Lemma 76.10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (15), (16), (17), (18) follow from the corresponding cases of More on Morphisms, Lemma 37.10.5.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (14) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.4.4, 67.10.5, 67.20.5, 67.40.3, 67.45.5, and 67.46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (18) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 76.10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 76.10.1.

The case (19). See Lemma 76.10.3.

The case (20). See Lemma 76.10.3.

The case (21). Assume f finite locally free. By (20) we see that f' is finite. By (2) we see that f' is flat. Also f' is locally finite presentation by Morphisms of Spaces, Lemma 67.28.9. Hence f' is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

76.19. Formally smooth morphisms

049R In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 76.19.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

060G **Definition 76.19.1.** Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 76.13.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 76.14.3 and 76.16.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 76.19.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 76.13.

061E **Lemma 76.19.2.** *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

061F **Lemma 76.19.3.** *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 10.138.2 for the algebraic version. □

061G **Lemma 76.19.4.** *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

Here is a helper lemma which will be superseded by Lemma 76.19.10.

061H **Lemma 76.19.5.** *Let S be a scheme. Let*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 76.13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 76.13.3 the composition $U \to Y$ is formally smooth. By Lemma 76.13.8 we see $\psi : U \to V$ is formally smooth. □

The following lemma is the main result of this section. It implies, combined with Limits of Spaces, Proposition 70.3.10, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of "simple" properties of the transformation of functors $X \to Y$.

04AM **Lemma 76.19.6** (Infinitesimal lifting criterion)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation, and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 76.19.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 67.28.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 37.11.7. Hence f is smooth, see Morphisms of Spaces, Lemma 67.37.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{} & T' \end{array}$$

as in Definition 76.19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 76.17.4 as in the special case discussed in Remark 76.17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 76.17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.4.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 76.13.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by

Lemma 76.13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 76.13.6 part (4).

Finally we prove (II). By Lemma 76.7.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 66.30). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of Spaces, Lemma 66.29.7. Thus by Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2 we have

$$H^1(T_{spaces,étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. □

Smooth morphisms satisfy strong local lifting property, see Lemma 76.19.7. If in the lemma we assume T' is affine, then we do not know if it is necessary to take an étale covering. More precisely, if we have a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ \downarrow & \nearrow & \downarrow \\ Y & \longleftarrow & T' \end{array}$$

of algebraic spaces where $X \to Y$ is smooth and $T \to T'$ is a thickening of affine schemes, the does a dotted arrow making the diagram commute always exist? If you know the answer, or if you have a reference, please email stacks.project@gmail.com.

0CHJ **Lemma 76.19.7.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ \downarrow & & \downarrow \\ Y & \longleftarrow & T' \end{array}$$

of algebraic spaces over S where $X \to Y$ is smooth and $T \to T'$ is a thickening. Then there exists an étale covering $\{T'_i \to T'\}$ such that we can find the dotted arrow in

$$\begin{array}{ccccc} X & \longleftarrow & T & \longleftarrow & T \times_{T'} T'_i \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ Y & \longleftarrow & T' & \longleftarrow & T'_i \end{array}$$

making the diagram commute (for all i).

Proof. Choose an étale covering $\{Y_i \to Y\}$ with each Y_i affine. After replacing T' by the induced étale covering we may assume Y is affine.

Assume Y is affine. Choose an étale covering $\{X_i \to X\}$. This gives rise to an étale covering of T. This étale covering of T comes from an étale covering of T' (by Theorem 76.8.1, see discussion in Section 76.9). Hence we may assume X is affine.

Assume X and Y are affine. We can do one more étale covering of T' and assume T' is affine. In this case the lemma follows from Algebra, Lemma 10.138.17. □

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of

differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 66.31.

061I **Lemma 76.19.8.** *Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 76.19.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 37.11.6. Hence by Algebra, Lemma 10.138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 28.21. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above – details omitted.) □

061J **Lemma 76.19.9.** *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{étale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{étale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{étale}, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 66.31.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 35.8.9) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{étale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i\in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i\in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 21.12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 35.9.3 and Cohomology of Schemes, Lemma 30.2.2. □

061K **Lemma 76.19.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally smooth,*
(2) *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 37.6.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.*

Proof. We have seen that (1) implies (2) and (3) in Lemma 76.19.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 76.19.6.

Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \longleftarrow & T' \end{array}$$

as in Definition 76.19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 76.17.4 as in the special case discussed in Remark 76.17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 76.17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.4.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 76.13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 76.13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 76.13.6 part (4).

Finally we prove (II). By Lemma 76.19.8 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 74.6.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 66.31.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 76.19.9 as desired. □

06CS **Lemma 76.19.11.** *The property $\mathcal{P}(f) =$"f is formally smooth" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{array}{ccc} U_i & \xrightarrow{\psi_i} & V_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 66.6.1. By Lemma 76.19.10 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 10.138.16. Some details omitted. □

06BI **Lemma 76.19.12.** *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

Lemma 76.7.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 37.11.11, by étale localization, see Lemmas 76.19.10 and 76.7.3. □

06BJ **Lemma 76.19.13.** *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 76.15.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 76.15.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 76.7.11. □

06BK **Lemma 76.19.14.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow f \\ & & Y \end{array}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 76.15.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 76.15.13 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & Z' & \xrightarrow{a} & X \\ & \searrow^{j'} & \downarrow k & & \downarrow f \\ & & Z'' & \xrightarrow{b} & Y \end{array}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concerning conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique

morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{array}{ccc} U & \longrightarrow & Z' \\ \downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle k} \\ U'' & \longrightarrow & Z'' \end{array}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

76.20. Smoothness over a Noetherian base

0APM This section is the analogue of More on Morphisms, Section 37.12.

0APN **Lemma 76.20.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume that Y is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth at x,*
(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\alpha} & \operatorname{Spec}(B) \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{\beta} & \operatorname{Spec}(B') \end{array}$$

where $B' \to B$ is a surjection of local rings with $\operatorname{Ker}(B' \to B)$ of square zero, and α mapping the closed point of $\operatorname{Spec}(B)$ to x there exists a dotted arrow making the diagram commute, and
(3) *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 10.141.1).*

Proof. Condition (1) means there is an open subspace $X' \subset X$ such that $X' \to Y$ is smooth. Hence (1) implies conditions (2) and (3) by Lemma 76.19.6. Condition (2) implies condition (3) trivially. Assume (3). Choose a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ Y & \longleftarrow & V \end{array}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Next, consider a diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \xleftarrow{\alpha} & \operatorname{Spec}(B) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ Y & \longleftarrow & V & \xleftarrow{\beta} & \operatorname{Spec}(B') \end{array}$$

as in (3) but for $u \in U \to V$. Let $\gamma : \operatorname{Spec}(B') \to X$ be the arrow we get from our assumption that (3) holds for X. Because $U \to X$ is étale and hence formally étale (Lemma 76.16.8) the morphism γ has a unique lift to U compatible with α. Then because $V \to Y$ is étale hence formally étale this lift is compatible with β. Hence (3) holds for $u \in U \to V$ and we conclude that $U \to V$ is smooth at u by More

on Morphisms, Lemma 37.12.1. This proves that $X \to Y$ is smooth at x, thereby finishing the proof. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms of Spaces, Section 67.25).

0APP **Lemma 76.20.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*
(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{} & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\nwarrow} & \downarrow{\scriptstyle i} \\ Y & \xleftarrow[\beta]{} & \mathrm{Spec}(B') \end{array}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 76.19.6) says f is formally smooth and (2) holds.

Assume f is not smooth. The set of points $x \in X$ where f is not smooth forms a closed subset T of $|X|$. By Morphisms of Spaces, Lemma 67.25.6, there exists a point $x \in T \subset X$ with $x \in X_{\text{ft-pts}}$. Choose a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{} & U \\ \downarrow & & \downarrow \\ Y & \xleftarrow{} & V \end{array} \qquad \begin{array}{c} u \\ \top \\ \downarrow \\ v \end{array}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Then u is a finite type point of U. Since $U \to V$ is not smooth at the point u, by More on Morphisms, Lemma 37.12.1 there is a diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & U & \xleftarrow{} & \mathrm{Spec}(B) \\ \downarrow & & \downarrow & \overset{\alpha}{\nwarrow} & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{} & V & \xleftarrow[\beta]{} & \mathrm{Spec}(B') \end{array}$$

with $B' \to B$ a small extension of (Artinian) local rings such that the residue field of B is equal to $\kappa(v)$ and such that the dotted arrow does not exist. Since $U \to V$ is of finite type, we see that v is a finite type point of V. By Morphisms, Lemma 29.16.2 the morphism β is of finite type, hence the composition $\mathrm{Spec}(B) \to Y$ is of finite type also. Arguing exactly as in the proof of Lemma 76.20.1 (using that $U \to X$ and $V \to Y$ are étale hence formally étale) we see that there cannot be an arrow $\mathrm{Spec}(B) \to X$ fitting into the outer rectangle of the last displayed diagram. In other words, (2) doesn't hold and the proof is complete. □

Here is a useful application.

0APQ **Lemma 76.20.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $Z \subset Y$ be a closed subspace with nth infinitesimal neighbourhood $Z_n \subset Y$. Set $X_n = Z_n \times_Y X$.*

(1) *If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.*
(2) *If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.*

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 76.20.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. □

76.21. The naive cotangent complex

0D0U This section is the continuation of Modules on Sites, Section 18.35 which in turn continues the discussion in Algebra, Section 10.134.

0D0V **Definition 76.21.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *naive cotangent complex of f* is the complex defined in Modules on Sites, Definition 18.35.4 for the morphism of ringed topoi f_{small} between the small étale sites of X and Y, see Properties of Spaces, Lemma 66.21.3. Notation: NL_f or $NL_{X/Y}$.

The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $NL_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

0D0W **Lemma 76.21.2.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the naive cotangent complex commutes with pullback (Modules on Sites, Lemma 18.35.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ by Properties of Spaces, Lemma 66.26.1. Tracing through the definitions we conclude that $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$. □

0D0X **Lemma 76.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $NL_{X/Y} = \epsilon^* NL_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 75.4 and NL_{X_0/Y_0} is as in More on Morphisms, Definition 37.13.1.*

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1} f_0^{-1} \mathcal{O}_{Y_0} \to f_{small}^{-1} \mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1} \mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow{\epsilon} & X_{étale} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{étale} \end{array}$$

see Derived Categories of Spaces, Remark 75.6.3. Thus we obtain a canonical map

$$\epsilon^{-1} NL_{X_0/Y_0} = \epsilon^{-1} NL_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = NL_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \to NL_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = NL_{X/Y}$$

by functoriality of the naive cotangent complex. To see that the induced map $\epsilon^* NL_{X_0/Y_0} \to NL_{X/Y}$ is an isomorphism in $D(\mathcal{O}_X)$ we may check on stalks at geometric points (Properties of Spaces, Theorem 66.19.12). Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$NL_{X/Y,\overline{x}} = NL_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

This is true because taking stalks at $\overline{x}$ is the same as taking inverse image via $\overline{x} : \mathrm{Spec}(k) \to X$ and we may apply Modules on Sites, Lemma 18.35.3. On the other hand we have

$$(\epsilon^* NL_{X_0/Y_0})_{\overline{x}} = NL_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = NL_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 7.34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 18.36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 66.22.1). Thus the result follows from More on Algebra, Lemma 15.33.8. □

0D0Y **Lemma 76.21.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.*

Proof. By construction of the naive cotangent complex in Modules on Sites, Section 18.35 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$ (by our construction of $\Omega_{X/Y}$ in Section 76.7). The sheaf of differentials is quasi-coherent (by Lemma 76.7.4). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking étale locally (allowed by Lemma 76.21.2 and Properties of Spaces, Lemma 66.29.6) reducing to the case of schemes (Lemma 76.21.3) and finally using the result in the case of schemes (More on Morphisms, Lemma 37.13.3). □

0D0Z **Lemma 76.21.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $NL_{X/Y}$ is étale locally on X quasi-isomorphic to a complex*

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. Formation of the naive cotangent complex commutes with étale localization by Lemma 76.21.2. This reduces us to the case of schemes by Lemma 76.21.3. The result in the case of schemes is More on Morphisms, Lemma 37.13.4. □

0D10 **Lemma 76.21.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is formally smooth,*
(2) *$H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.*

Proof. This follows from Lemma 76.19.10, Lemma 76.21.2, Lemma 76.21.3 and the case of schemes which is More on Morphisms, Lemma 37.13.5. □

0D11 **Lemma 76.21.7.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is formally étale,*
(2) *$H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. Assume (1). A formally étale morphism is a formally smooth morphism. Thus $H^{-1}(NL_{X/Y}) = 0$ by Lemma 76.21.6. On the other hand, a formally étale morphism if formally unramified hence we have $\Omega_{X/Y} = 0$ by Lemma 76.14.6. Conversely, if (2) holds, then f is formally smooth by Lemma 76.21.6 and formally unramified by Lemma 76.14.6 and hence formally étale by Lemmas 76.19.4. □

0D12 **Lemma 76.21.8.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is smooth, and*
(2) *f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.*

Proof. This follows from Lemma 76.19.10, Lemma 76.21.2, Lemma 76.21.3 and the case of schemes which is More on Morphisms, Lemma 37.13.7. □

76.22. Openness of the flat locus

05WU This section is analogue of More on Morphisms, Section 37.15. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 67.31.

05WV **Theorem 76.22.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 65.11.6. By More on Morphisms, Theorem 37.15.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 67.31.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 66.4.6. □

05WW **Lemma 76.22.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 76.22.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 67.31.3. □

76.23. Critère de platitude par fibres

05WX Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

05WY **Lemma 76.23.1.** *In the situation above the following are equivalent*

(1) *Pick a geometric point $\overline{x}$ of X lying over x. Set $\overline{y} = f \circ \overline{x}$ and $\overline{z} = g \circ \overline{x}$. Then the module $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.*
(2) *Pick a morphism $x : \mathrm{Spec}(K) \to X$ in the equivalence class of x. Set $z = g \circ x$, $X_z = \mathrm{Spec}(K) \times_{z,Z} X$, $Y_z = \mathrm{Spec}(K) \times_{z,Z} Y$, and $\mathcal{F}_z$ the pullback of $\mathcal{F}$ to X_z. Then $\mathcal{F}_z$ is flat at x over Y_z (as defined in Morphisms of Spaces, Definition 67.31.2).*
(3) *Pick a commutative diagram*

$$\begin{array}{ccccc} & & U & & V \\ & {\scriptstyle a}\swarrow & & \searrow\swarrow{\scriptstyle b} & \\ X & & \xrightarrow{f} & Y & & W \\ & {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} & \swarrow{\scriptstyle c} & \\ & & Z & & & \end{array}$$

where U, V, W are schemes, and a, b, c are étale, and a point $u \in U$ mapping to x. Let $w \in W$ be the image of u. Let $\mathcal{F}_w$ be the pullback of $\mathcal{F}$ to the fibre U_w of $U \to W$ at w. Then $\mathcal{F}_w$ is flat over V_w at u.

Proof. Note that in (2) the morphism $x : \mathrm{Spec}(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, the condition in (2) is independent of the choice of $\mathrm{Spec}(K) \to X$ in the equivalence class of x (details omitted; this will also follow from the arguments below because the other conditions do not depend on this choice). Also note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ as in (1). By Properties of Spaces, Lemma 66.19.4 we can choose a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : \mathrm{Spec}(k) \to V$ and $\overline{w} : \mathrm{Spec}(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ and similarly for Y and Z, see Properties of Spaces, Lemma 66.22.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}^{sh}_{U,u}$$

see Properties of Spaces, Lemma 66.29.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u/\mathfrak{m}_w(a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w\mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w\mathcal{O}^{sh}_{W,w}$ we see that

$$\begin{aligned}
\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} &= (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U,u}/\mathfrak{m}_w\mathcal{O}^{sh}_{U,u} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U_w,\overline{u}} \\
&= (\mathcal{F}_w)_{\overline{u}}
\end{aligned}$$

the penultimate equality by Algebra, Lemma 10.156.4 and the last equality by Properties of Spaces, Lemma 66.29.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}^{sh}_{V_w,\overline{v}} = \mathcal{O}^{sh}_{V,v}/\mathfrak{m}_w\mathcal{O}^{sh}_{V,v} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 67.31.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and a morphism $\tilde{x} : \mathrm{Spec}(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} \mathrm{Spec}(K)$ is a nonempty scheme). Set $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to U \to W$ be

the composition. We obtain a commutative diagram

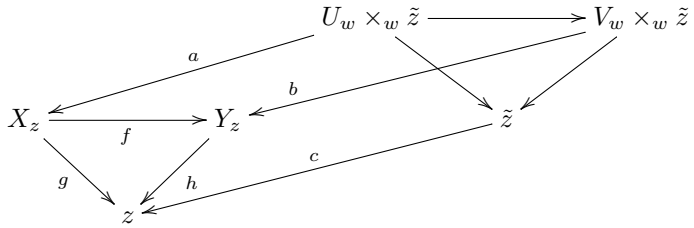

where $z = \mathrm{Spec}(K)$ and $w = \mathrm{Spec}(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{array}{ccccc} X_z & \longleftarrow & U_w \times_w \tilde{z} & \longrightarrow & U_w \\ \downarrow & & \downarrow & & \downarrow \\ Y_z & \longleftarrow & V_w \times_w \tilde{z} & \longrightarrow & V_w \end{array}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 67.31.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. □

05WZ **Definition 76.23.2.** Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

(1) We say *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 76.23.1 are satisfied.
(2) We say *the fibre of X over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 76.23.1 hold with $\mathcal{F} = \mathcal{O}_X$.
(3) We say *the fibre of X over z is flat over the fibre of Y over z* if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state a version of the criterion as follows. The Noetherian version can be found in Section 76.24.

05X0 **Theorem 76.23.3.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over Z.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $\mathrm{Supp}(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 76.23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 37.16.2. □

05X1 **Lemma 76.23.4.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and*
(4) *Y is locally of finite type over Z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 76.23.3. □

05X2 **Lemma 76.23.5.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over Z, and*
(4) *Y is locally of finite type over Z.*

Then the set

$$A = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 37.16.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes. Namely, choose a diagram as in Lemma 76.23.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 37.16.4. □

05X3 **Lemma 76.23.6.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z, and*
(3) *Y is locally of finite type over Z.*

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change $Z' \to Z$.

Proof. This is a special case of Lemma 76.23.5. □

0CZS **Lemma 76.23.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in |X|$ with image $y \in |Y|$. If $\mathcal{F}$ is flat at x over Y, then the following are equivalent*

(1) *$(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,*
(2) *$(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a free $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,*
(3) *$\mathcal{F}_{\overline{y}}$ is finite free in an étale neighbourhood of $\overline{x}$ in $X_{\overline{y}}$, and*
(4) *$\mathcal{F}$ is finite free in an étale neighbourhood of x in X.*

Here $\overline{x}$ is a geometric point of X lying over x and $\overline{y} = f \circ \overline{x}$.

Proof. Pick a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale such that there is a point $u \in U$ mapping to x. Let $v \in V$ be the image of u. Applying Lemma 76.23.1 to $\mathrm{id} : X \to X$ over Y we see that (1) translates into the condition "$\mathcal{F}|_{U_v}$ is flat over U_v at u". In other words, (1) is equivalent to $(\mathcal{F}|_{U_v})_u$ being a flat $\mathcal{O}_{U_v,u}$-module. By the case of schemes (More on Morphisms, Lemma 37.16.7), we find that this implies that $\mathcal{F}|_U$ is finite free in an open neighbourhood of u. In this way we see that (1) implies (4). The implications (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are immediate. For the implication (3) $\Rightarrow$ (2) use the description of local rings and stalks in Properties of Spaces, Lemmas 66.22.1 and 66.29.4. □

0CZT **Lemma 76.23.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over Y. Then the set*

$$\{x \in |X| : \mathcal{F} \text{ free in an étale neighbourhood of } x\}$$

is open in $|X|$ and its formation commutes with arbitrary base change $Y' \to Y$.

Proof. Openness holds trivially. Let $Y' \to Y$ be a morphism of algebraic spaces, set $X' = Y' \times_Y X$, and let $x' \in |X'|$ be a point lying over $x \in |X|$. By Lemma 76.23.7 we see that x is in our set if and only if $(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module. Similarly, x' is in the analogue of our set for the pullback $\mathcal{F}'$ of $\mathcal{F}$ to X' if and only if $(\mathcal{F}'_{\overline{y}'})_{\overline{x}'}$ is a flat $\mathcal{O}_{X'_{\overline{y}'},\overline{x}'}$-module (with obvious notation). These two assertions are equivalent by Lemma 76.23.1 applied to the morphism $\mathrm{id} : X \to X$ over Y. Thus the statement on base change holds. □

76.24. Flatness over a Noetherian base

08VN Here is the "Critère de platitude par fibres" in the Noetherian case.

0APR **Theorem 76.24.1.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X, Y, Z locally Noetherian, and*
(2) *$\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Choose a diagram as in Lemma 76.23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 37.16.1. □

0APS **Lemma 76.24.2.** *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X, Y, Z locally Noetherian,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 76.24.1. □

Just like for checking smoothness, if the base is Noetherian it suffices to check flatness over Artinian rings. Here is a sample statement.

08VP **Lemma 76.24.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let X be an algebraic space locally of finite presentation over $S = \mathrm{Spec}(A)$. For $n \geq 1$ set $S_n = \mathrm{Spec}(A/I^n)$ and $X_n = S_n \times_S X$. Let $\mathcal{F}$ be coherent $\mathcal{O}_X$-module. If for every $n \geq 1$ the pullback $\mathcal{F}_n$ of $\mathcal{F}$ to X is flat over S_n, then the (open) locus where $\mathcal{F}$ is flat over X contains the inverse image of $V(I)$ under $X \to S$.*

Proof. The locus where $\mathcal{F}$ is flat over S is open in $|X|$ by Theorem 76.22.1. The statement is insensitive to replacing X by the members of an étale covering, hence we may assume X is an affine scheme. In this case the result follows immediately from Algebra, Lemma 10.99.11. Some details omitted. □

76.25. Normalization revisited

082D Normalization commutes with smooth base change.

082E **Lemma 76.25.1.** *Let S be a scheme. Let $f : Y \to X$ be a smooth morphism of algebraic spaces over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.*

Proof. By our construction of the integral closure, see Morphisms of Spaces, Definition 67.48.2, this reduces immediately to the case where X and Y are affine. In this case the result is Algebra, Lemma 10.147.4. □

082F **Lemma 76.25.2** (Normalization commutes with smooth base change)**.** *Let S be a scheme. Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow f \\ X_2 & \xrightarrow{\varphi} & X_1 \end{array}$$

be a fibre square of algebraic spaces over S. Assume f is quasi-compact and quasi-separated and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms of Spaces, Lemma 67.45.5). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms of Spaces, Lemma 67.48.5. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$ as the construction of the relative spectrum commutes with arbitrary base change. By Cohomology of Spaces, Lemma 69.11.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 76.25.1. □

76.26. Cohen-Macaulay morphisms

0E0T This is the analogue of More on Morphisms, Section 37.22.

0E0U **Lemma 76.26.1.** *The property of morphisms of germs of schemes*

$$\mathcal{P}((X, x) \to (S, s)) =$$

the local ring $\mathcal{O}_{X_s,x}$ of the fibre is Noetherian and Cohen-Macaulay

is étale local on the source-and-target (Descent, Definition 35.33.1).

Proof. Given a diagram as in Descent, Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 35.33.5. Thus the strict henselizations of the local rings $\mathcal{O}_{U'_{v'},u'}$ and $\mathcal{O}_{U_v,u}$ are the same. We conclude by More on Algebra, Lemma 15.45.9. □

0E0V **Definition 76.26.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 71.4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 67.22.5 hold for the property $\mathcal{P}$ described in Lemma 76.26.1.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

0E0W **Lemma 76.26.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay,*
(2) *f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces, and*
(3) *f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces.*

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) *f is Cohen-Macaulay at x, and*
(b) *$\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Cohen-Macaulay.*

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Cohen-Macaulay if and only if the algebraic space X_v is Cohen-Macaulay. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see More on Morphisms, Lemma 37.22.2 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 67.30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}^{sh}_{V,v} \to \mathcal{O}^{sh}_{U,u}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 66.22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 10.156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Cohen-Macaulay if and only if its strict henselization is. This is More on Algebra, Lemma 15.45.9. □

0E0X **Lemma 76.26.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.*

(1) *If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.*
(2) *If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.*
(3) *If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.*
(4) *If $f \circ g$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.*

Proof. Working étale locally this follows from the corresponding result for schemes, see More on Morphisms, Lemma 37.22.4. Alternatively, we can use the equivalence of (a) and (b) in Lemma 76.26.3. Thus we consider the local homomorphism of

Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Algebra, Lemma 10.163.3. □

0E0Y **Lemma 76.26.5.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(\overline{x})}$ is Cohen-Macaulay for all $x \in |X|$.*

Proof. After translating into algebra using Lemma 76.26.3 (compare with the proof of Lemma 76.26.4) this follows from Algebra, Lemma 10.163.3. □

0E0Z **Lemma 76.26.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.*

1. *If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
2. *If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*
3. *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote $u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see More on Morphisms, Lemma 37.22.6. □

0E10 **Lemma 76.26.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let*

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale vertical arrows and U and V schemes. Let $u \in U$ with image $x \in |X|$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by

definition). Thus we reduce to the case of the morphism $U \to V$. See More on Morphisms, Lemma 37.22.7. □

0E11 **Lemma 76.26.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is locally of finite presentation and Cohen-Macaulay. Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to Y$ has relative dimension d.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale vertical arrows and U and V schemes. Then $U \to V$ is locally of finite presentation and Cohen-Macaulay (immediate from our definitions). Thus we have a decomposition $U = \coprod_{d \geq 0} U_d$ into open and closed subschemes with $f|_{U_d} : U_d \to V$ of relative dimension d, see Morphisms, Lemma 29.29.4. Let $u \in U$ with image $x \in |X|$. Then f has relative dimension d at x if and only if $U \to V$ has relative dimension d at u (this follows from our definitions). In this way we see that U_d is the inverse image of a subset $X_d \subset |X|$ which is necessarily open and closed. Denoting X_d the corresponding open and closed algebraic subspace of X we see that the lemma is true. □

76.27. Gorenstein morphisms

0E12 This is the analogue of Duality for Schemes, Section 48.25.

0E13 **Lemma 76.27.1.** *The property of morphisms of germs of schemes*

$$\mathcal{P}((X, x) \to (S, s)) =$$
the local ring $\mathcal{O}_{X_s, x}$ of the fibre is Noetherian and Gorenstein

is étale local on the source-and-target (Descent, Definition 35.33.1).

Proof. Given a diagram as in Descent, Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 35.33.5. Thus $\mathcal{O}_{U_v, u} \to \mathcal{O}_{U'_{v'}, u'}$ is the localization of an étale ring map. Hence the first is Noetherian if and only if the second is Noetherian, see More on Algebra, Lemma 15.44.1. Then, since $\mathcal{O}_{U'_{v'}, u'}/\mathfrak{m}_u \mathcal{O}_{U'_{v'}, u'} = \kappa(u')$ (Algebra, Lemma 10.143.5) is a Gorenstein ring, we see that $\mathcal{O}_{U_v, u}$ is Gorenstein if and only if $\mathcal{O}_{U'_{v'}, u'}$ is Gorenstein by Dualizing Complexes, Lemma 47.21.8. □

0E14 **Definition 76.27.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 71.4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 67.22.5 hold for the property $\mathcal{P}$ described in Lemma 76.27.1.
(2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X.

Here is a translation.

0E15 **Lemma 76.27.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Gorenstein,*
(2) *f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces, and*
(3) *f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces.*

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) *f is Gorenstein at x, and*
(b) *$\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Gorenstein.*

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Gorenstein if and only if the algebraic space X_v is Gorenstein. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see Duality for Schemes, Lemma 48.24.4 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 67.30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}^{sh}_{V,v} \to \mathcal{O}^{sh}_{U,u}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 66.22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 10.156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Gorenstein if and only if its strict henselization is. This follows immediately from Dualizing Complexes, Lemma 47.22.3 and the definition of a Gorenstein local ring as a Noetherian local ring which is a dualizing complex over itself. □

0E16 **Lemma 76.27.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.*

(1) *If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.*
(2) *If f and g are Gorenstein, then $g \circ f$ is Gorenstein.*
(3) *If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.*
(4) *If $f \circ g$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.*

Proof. Working étale locally this follows from the corresponding result for schemes, see Duality for Schemes, Lemma 48.25.6. Alternatively, we can use the equivalence of (a) and (b) in Lemma 76.27.3. Thus we consider the local homomorphism of Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Dualizing Complexes, Lemma 47.21.8. □

0E17 **Lemma 76.27.5.** *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(\overline{x})}$ is Gorenstein for all $x \in |X|$.*

Proof. After translating into algebra using Lemma 76.27.3 (compare with the proof of Lemma 76.27.4) this follows from Dualizing Complexes, Lemma 47.21.8. □

0E18 **Lemma 76.27.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.*

(1) *If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.*
(2) *If f is flat at x and f' is Gorenstein at x', then f is Gorenstein at x.*
(3) *If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.*

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote $u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see Duality for Schemes, Lemma 48.25.8. □

0E19 **Lemma 76.27.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let*

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Let $u \in U$ with image $x \in |X|$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Thus we reduce to the case of the morphism $U \to V$ of schemes. Openness is proven in Duality for Schemes, Lemma 48.25.11 and compatibility with base change in Duality for Schemes, Lemma 48.25.9. □

76.28. Slicing Cohen-Macaulay morphisms

06LV Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the *closed subspace of X cut out by $f_1, \ldots, f_r$*. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 67.13.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 39.19.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

06LW **Lemma 76.28.1.** *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let k'/k be an extension field. Then $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 67.30.8. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\overline{z}'}$ is a regular sequence too, see Algebra, Lemma 10.68.5. Thus, during the proof we may replace k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : \mathrm{Spec}(k) \to F$ of the structure morphism $F \to \mathrm{Spec}(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $\mathrm{Spec}(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : \mathrm{Spec}(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k) = U \times_V \mathrm{Spec}(k) \to U \times_Y \mathrm{Spec}(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : \mathrm{Spec}(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements

$f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k), \overline{u}} = \mathcal{O}_{U \times_V \mathrm{Spec}(k), \overline{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 38.2.5 and Morphisms of Spaces, Lemma 67.30.8) we see from Algebra, Lemma 10.68.5 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 37.23.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 66.4.6 and 66.4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 67.30.1) as we have the commutative diagram

$$\begin{array}{ccc} V(f_1, \ldots, f_r) \times_X U' & \longrightarrow & V \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ V(f_1, \ldots, f_r) \cap X' & \longrightarrow & Y \end{array}$$

with a, b étale and a surjective. □

76.29. Reduced fibres

0E06 This section is the analogue of More on Morphisms, Section 37.26.

0E07 **Lemma 76.29.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. The following are equivalent*

1. *for some morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is geometrically reduced over k,*
2. *for every morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is geometrically reduced over k,*
3. *for every morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is reduced.*

Proof. This follows immediately from Spaces over Fields, Lemma 72.11.6 and the definition of the equivalence relation defining $|X|$ given in Properties of Spaces, Section 66.4. □

0E08 **Definition 76.29.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. We say *the fibre of $f : X \to Y$ at y is geometrically reduced* if the equivalent conditions of Lemma 76.29.1 hold.

Here are the obligatory lemmas.

0E09 **Lemma 76.29.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of algebraic spaces over S. Denote $f' : X' \to Y'$ the base change of f by g. Then*

$$\{y' \in |Y'| : \text{the fibre of } f' : X' \to Y' \text{ at } y' \text{ is geometrically reduced}\}$$
$$= g^{-1}(\{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}).$$

Proof. For $y' \in |Y'|$ choose a morphism $\mathrm{Spec}(k) \to Y'$ in the equivalence class of y'. Then $g(y')$ is represented by the composition $\mathrm{Spec}(k) \to Y' \to Y$. Hence $X' \times_{Y'} \mathrm{Spec}(k) = X \times_Y \mathrm{Spec}(k)$ and the result follows from the definition. □

0E0A **Lemma 76.29.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is quasi-compact and locally of finite presentation. Then the set*

$$E = \{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}$$

is étale locally constructible.

Proof. Choose an affine scheme V and an étale morphism $V \to Y$. The meaning of the statement is that the inverse image of E in $|V|$ is constructible. By Lemma 76.29.3 we may replace Y by V, i.e., we may assume that Y is an affine scheme. Then X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$. For a morphism $\mathrm{Spec}(k) \to Y$ the morphism between fibres $U_k \to X_k$ is surjective étale. Hence U_k is geometrically reduced over k if and only if X_k is geometrically reduced over k, see Spaces over Fields, Lemma 72.11.7. Thus the set E for $X \to Y$ is the same as the set E for $U \to Y$. In this way we see that the lemma follows from the case of schemes, see More on Morphisms, Lemma 37.26.5. □

0E0B **Lemma 76.29.5.** *Let X be an algebraic space over a discrete valuation ring R whose structure morphism $X \to \mathrm{Spec}(R)$ is proper and flat. If the special fibre is reduced, then both X and the generic fibre X_η are reduced.*

Proof. Choose an étale morphism $U \to X$ where U is an affine scheme. Then U is of finite type over R. Let $u \in U$ be in the special fibre. The local ring $A = \mathcal{O}_{U,u}$ is essentially of finite type over R, hence Noetherian. Let $\pi \in R$ be a uniformizer. Since X is flat over R, we see that $\pi \in \mathfrak{m}_A$ is a nonzerodivisor on A and since the special fibre of X is reduced, we have that $A/\pi A$ is reduced. If $a \in A$, $a \neq 0$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \notin \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 10.51.4). If a is nilpotent, so is a', because π is a nonzerodivisor. But a' maps to a nonzero element of the reduced ring $A/\pi A$ so this is impossible. Hence A is reduced. It follows that there exists an open neighbourhood of u in U which is reduced (small detail omitted; use that U is Noetherian). Thus we can find an étale morphism $U \to X$ with U a reduced scheme, such that every point of the special fibre of X is in the image. Since X is proper over R it follows that $U \to X$ is surjective. Hence X is reduced. Since the generic fibre of $U \to \mathrm{Spec}(R)$ is reduced as well (on affine pieces it is computed by taking localizations), we conclude the same thing is true for the generic fibre. □

0E0C **Lemma 76.29.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat, proper, and of finite presentation, then the set*

$$E = \{y \in |Y| : \text{the fibre of } f : X \to Y \text{ at } y \text{ is geometrically reduced}\}$$

is open in $|Y|$.

Proof. By Lemma 76.29.3 formation of E commutes with base change. To check a subset of $|Y|$ is open, we may replace Y by the members of an étale covering. Thus we may assume Y is affine. Then Y is a cofiltered limit of affine schemes of

finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits of Spaces, Lemma 70.7.1, 70.6.12, and 70.6.13. Since the formation of E commutes with base change (see above), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 76.29.4. Hence it suffices to show the set is stable under generalization (Topology, Lemma 5.19.10). By Properties, Lemma 28.5.10 we reduce to the case where $Y = \mathrm{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Spaces over Fields, Lemmas 72.11.4 (characteristic zero) and 72.11.5 (positive characteristic). There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 10.120.18. After replacing R by R' we reduce to Lemma 76.29.5. □

76.30. Connected components of fibres

0E1A This section is the analogue of More on Morphisms, Section 37.28.

0E1B **Lemma 76.30.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let*

$$n_{X/Y} : |Y| \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of X_k where $\mathrm{Spec}(k) \to Y$ is in the equivalence class of y with k algebraically closed. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Let $\mathrm{Spec}(k') \to Y'$ be in the equivalence class of y' with k' algebraically closed. Then we can choose a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(k') & \longrightarrow & Y' \\ & \searrow & & & \downarrow \\ & & \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

where K is an algebraically closed field. The result follows as the morphisms of schemes

$$X'_{k'} \longleftarrow (X'_{k'})_K = (X_k)_K \longrightarrow X_k$$

induce bijections between connected components, see Spaces over Fields, Lemma 72.12.4. To use this to prove the function is well defined take $Y' = Y$. □

76.31. Dimension of fibres

0D4L This section is the analogue of More on Morphisms, Section 37.30.

0D4M **Lemma 76.31.1.** *Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let $y \in |Y|$. The following quantities are the same*

(1) *$d = -\infty$ if y is not in the image of $|f|$ and otherwise the minimal integer d such that f has relative dimension $\leq d$ at every $x \in |X|$ mapping to y,*

(2) *the dimension of the algebraic space* $X_k = \mathrm{Spec}(k) \times_Y X$ *for any morphism* $\mathrm{Spec}(k) \to Y$ *in the equivalence class defining* y.

Proof. To parse this one has to consult Morphisms of Spaces, Definition 67.33.1, Properties of Spaces, Definition 66.9.2, Properties of Spaces, Definition 66.9.1. We will show that the numbers in (1) and (2) are equal for a fixed morphism $\mathrm{Spec}(k) \to Y$. Choose an étale morphism $V \to Y$ where V is an affine scheme and a point $v \in V$ mapping to y. Since $V \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale (by Properties of Spaces, Lemma 66.4.3) we can find a finite separable extension k'/k (by Morphisms, Lemma 29.36.7) and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

We may replace $X \to Y$ by $V \times_Y X \to V$ and X_k by $X_{k'} = \mathrm{Spec}(k') \times_V (V \times_Y X)$ because this does not change the dimensions in question by Properties of Spaces, Lemma 66.22.5 and Morphisms of Spaces, Lemma 67.34.3. Thus we may assume that Y is an affine scheme. In this case we may assume that $k = \kappa(y)$ because the dimension of $X_{\kappa(y)}$ and X_k are the same by the aforementioned Morphisms of Spaces, Lemma 67.34.3 and the fact that for an algebraic space Z over a field K the relative dimension of Z at a point $z \in |Z|$ is the same as $\dim_z(Z)$ by definition. Assume Y is affine and $k = \kappa(y)$. Then X is quasi-compact we can choose an affine scheme U and an surjective étale morphism $U \to X$. Then $\dim(X_k) = \dim(U_k) = \max \dim_u(U_k)$ is equal to the number given in (1) by definition. □

0D4N **Lemma 76.31.2.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a finite type morphism of algebraic spaces over* S. *Let*

$$n_{X/Y} : |Y| \to \{-\infty, 0, 1, 2, 3, \ldots\}$$

be the function which associates to $y \in |Y|$ *the integer discussed in Lemma 76.31.1. If* $g : Y' \to Y$ *is a morphism then*

$$n_{X'/Y'} = n_{X/Y} \circ |g|$$

where $X' \to Y'$ *is the base change of* f.

Proof. This follows immediately from Lemma 76.31.1. □

0D4P **Lemma 76.31.3.** *Let* S *be a scheme. Let* $f : X \to Y$ *be a flat morphism of finite presentation of algebraic spaces over* S. *Let* $n_{X/Y}$ *be the function on* Y *giving the dimension of fibres of* f *introduced in Lemma 76.31.2. Then* $n_{X/Y}$ *is lower semi-continuous.*

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. If we can show that the composition $n_{X/Y} \circ |g|$ is lower semi-continuous, then the lemma follows as $|g|$ is open. Hence we may assume Y is a scheme. Working locally we may assume V is an affine scheme. Then we can choose an affine scheme U and a surjective étale morphism $U \to X$. Then $n_{X/Y} = n_{U/Y}$. Hence we may assume X and Y are both schemes. In this case the lemma follows from More on Morphisms, Lemma 37.30.4. □

0D4Q **Lemma 76.31.4.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 76.31.2. Then $n_{X/Y}$ is upper semi-continuous.*

Proof. Let $Z_d = \{x \in |X| : \text{the fibre of } f \text{ at } x \text{ has dimension } > d\}$. Then Z_d is a closed subset of $|X|$ by Morphisms of Spaces, Lemma 67.34.4. Since f is proper $f(Z_d)$ is closed in $|Y|$. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. □

0D4R **Lemma 76.31.5.** *Let S be a scheme. Let $f : X \to Y$ be a proper, flat, finitely presented morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 76.31.2. Then $n_{X/Y}$ is locally constant.*

Proof. Immediate consequence of Lemmas 76.31.3 and 76.31.4. □

76.32. Catenary algebraic spaces

0EDL This section continues the discussion started in Decent Spaces, Section 68.25. The following lemma will be used in the proof of the next one.

0EDM **Lemma 76.32.1.** *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Let $y \in |Y|$ be a point which can be represented by a closed immersion $y : \mathrm{Spec}(k) \to Y$. Then there exists a factorization $X \to X' \to Y$ of f such that*

(1) *$X' \to Y$ is integral,*
(2) *$X \to X'$ is an isomorphism over $X' \setminus X'_y$,*
(3) *X'_y has a unique point x' with $\kappa(x') = k$.*

Moreover, if f is finite and Y is locally Noetherian, then $X' \to Y$ is finite.

Proof. By Morphisms of Spaces, Lemma 67.11.2 the sheaves $f_*\mathcal{O}_X$, $(X_y \to Y)_*\mathcal{O}_{X_y}$, and $y_*\mathcal{O}_{\mathrm{Spec}(k)}$ are quasi-coherent sheaves of $\mathcal{O}_Y$-algebras. Consider the maps

$$f_*\mathcal{O}_Y \longrightarrow (X_y \to Y)_*\mathcal{O}_{X_y} \longleftarrow y_*\mathcal{O}_{\mathrm{Spec}(k)}$$

The fibre product is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A}'$ and we can define $X' \to Y$ as the relative spectrum of $\mathcal{A}'$ over Y, see Morphisms, Lemma 29.11.5. This construction commutes with arbitrary change of base. In particular, it is clear that over the open subspace $|Y| \setminus \{y\}$ the morphism $X \to X'$ is an isomorphism and over $|Y| \setminus \{y\}$ the morphism $X' \to Y$ is integral. It remains to prove the statements in a small neighbourhood of y. Choose an affine scheme $V = \mathrm{Spec}(R)$ and an étale morphism $\varphi : V \to Y$ such that y is in the image of φ. Then V_y is a closed subscheme of V étale over k, whence consists of finitely many points each with residue field separable over k (see Decent Spaces, Remark 68.4.1). After shrinking V we may assume there is a unique closed point $v = \mathrm{Spec}(l) \to V$ mapping to y with l/k finite separable. We may write $V \times_Y X = \mathrm{Spec}(C)$ with $R \to C$ an integral ring map. The stated compatibility with base change gives us that $U \times_X Y' = \mathrm{Spec}(C')$ where

$$C' = C \times_{C \otimes_R l} l$$

Since $R \to l$ is surjective, also $C \to C \otimes_R l$ is surjective and we see that this is a fibre product of the kind studied in More on Algebra, Situation 15.6.1 (with A, A', B, B' corresponding to $C \otimes_R l, C, l, C'$). Observe that C' is an R-subalgebra of C and hence is integral over R; this proves (1). Finally, More on Algebra, Lemma 15.6.2

shows that $V \times_X Y' = \mathrm{Spec}(C')$ has a unique point y'' lying over v with residue l (this corresponds with the obvious surjective map $C' \to l$). Thus $X_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$ has a unique point with residue field l. Since l/k is finite separable, this implies X'_y has a unique point with residue field k, i.e., (3) holds.

To prove the final statement, observe that if Y is locally Noetherian, then R is a Noetherian ring and if f is finite, then $R \to C$ is finite. Then C' is a finite type R-algebra by More on Algebra, Lemma 15.5.1. This proves that $X' \to Y$ is finite. □

0EDN **Lemma 76.32.2.** *Let S be a scheme. Let B be an algebraic space over S. Let $\delta : |B| \to \mathbf{Z}$ be a function. Assume B is decent, locally Noetherian, and universally catenary and δ is a dimension function. If X is a decent algebraic space over B whose structure morphism $f : X \to B$ is locally of finite type we define $\delta_X : |X| \to \mathbf{Z}$ by the rule*

$$\delta_X(x) = \delta(f(x)) + \textit{transcendence degreeof } x/f(x)$$

(Morphisms of Spaces, Definition 67.33.1). Then δ_X is a dimension function.

Proof. The problem is local on B. Thus we may assume B is quasi-compact. By Decent Spaces, Lemma 68.14.1 we see B is quasi-separated. By Limits of Spaces, Proposition 70.16.1 we can choose a finite surjective morphism $\pi : Y \to X$ where Y is a scheme. Claim: δ_Y is a dimension function.

The claim implies the lemma. With $X \to B$ as in the lemma set $Z = Y \times_B X$ with projections $p : Z \to Y$ and $q : Z \to X$. Then we have

$$\delta_Z(z) = \delta_Y(p(z)) + \text{transcendence degreeof } z/p(z)$$

and $\delta_Z(z) = \delta_X(q(z))$. This follows from Morphisms of Spaces, Lemma 67.34.2 and the fact that these transcendence degrees are zero for finite morphisms. By Decent Spaces, Lemma 68.25.2 and the claim we find that δ_Z is a dimension function. Then we find that δ_X is a dimension function by Decent Spaces, Lemma 68.25.6.

Proof of the claim. Consider a specialization $y \rightsquigarrow y'$, $y \neq y'$ of points of the Noetherian scheme Y. Then $\delta_Y(y) > \delta_Y(y')$ because there are no specializations between points in fibres of Y (see Decent Spaces, Lemma 68.18.10). Using this for a chain of specializations we find

$$\delta_Y(y) - \delta_Y(y') \geq \mathrm{codim}(\overline{\{y'\}}, \overline{\{y\}})$$

Our task is to show equality. By Properties, Lemma 28.5.9 we can choose a specialization $y' \rightsquigarrow y_0$. It suffices to show $\delta_Y(y) - \delta_Y(y_0) = \mathrm{codim}(\overline{\{y_0\}}, \overline{\{y\}})$ because this will imply the equality for both $y \rightsquigarrow y'$ and $y' \rightsquigarrow y_0$.

Choose a maximal chain $y = y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ of specializations in Y. Set $b = \pi(y)$ and $b_0 = \pi(y_0)$. Choose a maximal chain $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$ of specializations in $|B|$. We have to show $e = c$. Since π is closed (Morphisms of Spaces, Lemma 67.45.9) we can find a sequence of specializations $y = y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ mapping to $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$. Observe that $y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ is a maximal chain as well. If $y_0 = y'_0$, then because Y is catenary, we conclude that $e = c$ as desired. In the next paragraph we reduce to this case by sleight of hand and we conclude in the same manner.

Since π is closed we see that b_0 is a closed point of $|B|$. By Decent Spaces, Lemma 68.14.6 we can represent b_0 by a closed immersion $b_0 : \mathrm{Spec}(k) \to B$. By Lemma 76.32.1 we can find a factorization

$$Y \to Y' \to X$$

with $\pi' : Y' \to X$ finite and $Y \to Y'$ a morphism which map y_0 and y_0' to the same point and is an isomorphism away from the inverse image of b_0. (Of course Y' won't be a scheme but this doesn't matter for the argument that follows.) Clearly the maximal chains of specializations $y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ and $y_e' \rightsquigarrow y_{e-1}' \rightsquigarrow \ldots \rightsquigarrow y_0'$ map to maximal chains of specializations in Y' having the same start and end. Since B is universally catenary, we see that $|Y'|$ is catenary and we conclude as before. □

76.33. Étale localization of morphisms

082G The section is the analogue of More on Morphisms, Section 37.41.

082H **Lemma 76.33.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Let $x_1, \ldots, x_n \in |X|$ mapping to y. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated,*
(3) *f is quasi-finite at $x_1, \ldots, x_n$, and*
(4) *f is quasi-compact or Y is decent.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite, every point of the fibre of $|V| \to |U|$ over u maps to an x_i, and the fibre of $|W| \to |U|$ over u contains no point mapping to an x_i.

Proof. Let $(U, u) \to (Y, y)$ be an étale morphism of algebraic spaces and consider the set of $w \in |U \times_Y X|$ mapping to $u \in |U|$ and one of the $x_i \in |X|$. By Decent Spaces, Lemma 68.18.4 (if f is of finite type) or Decent Spaces, Lemma 68.18.5 (if Y is decent) this set is finite. It follows that we may replace f by the base change $U \times_Y X \to U$ and $x_1, \ldots, x_n$ by the set of these w. In particular we may and do assume that Y is an affine scheme, whence X is a separated algebraic space.

Choose an affine scheme Z and an étale morphism $Z \to X$ such that $x_1, \ldots, x_n$ are in the image of $|Z| \to |X|$. The fibres of $|Z| \to |X|$ are finite, see Properties of Spaces, Lemma 66.6.7 (or the more general discussion in Decent Spaces, Section 68.6). Let $\{z_1, \ldots, z_m\} \subset |Z|$ be the preimage of $\{x_1, \ldots, x_n\}$. By More on Morphisms, Lemma 37.41.4 there exists an étale morphism $(U, u) \to (Y, y)$ such that $U \times_Y Z = Z_1 \amalg Z_2$ with $Z_1 \to U$ finite and $(Z_1)_y = \{z_1, \ldots, z_m\}$. We may assume that U is affine and hence Z_1 is affine too.

Since f is separated, the image V of $Z_1 \to X$ is both open and closed (Morphisms of Spaces, Lemma 67.40.6). Set $W = X \setminus V$ to get a decomposition as in the lemma. To finish the proof we have to show that $V \to U$ is finite. As $Z_1 \to V$ is surjective and étale, V is the quotient of Z_1 by the étale equivalence relation $R = Z_1 \times_V Z_1$, see Spaces, Lemma 65.9.1. Since f is separated, $V \to U$ is separated and R is closed in $Z_1 \times_U Z_1$. Since $Z_1 \to U$ is finite, the projections $s, t : R \to Z_1$ are finite. Thus V is an affine scheme by Groupoids, Proposition 39.23.9. By Morphisms, Lemma

29.41.9 we conclude that $V \to U$ is proper and by Morphisms, Lemma 29.44.11 we conclude that $V \to U$ is finite, thereby finishing the proof. □

0ADU **Lemma 76.33.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *f is quasi-finite at x.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite and there exists a point $v \in |V|$ which maps to x in $|X|$ and u in $|U|$.

Proof. Pick a scheme U, a point $u \in U$, and an étale morphism $U \to Y$ mapping u to y. There exists a point $x' \in |U \times_Y X|$ mapping to x in $|X|$ and u in $|U|$ (Properties of Spaces, Lemma 66.4.3). To finish, apply Lemma 76.33.1 to the morphism $U \times_Y X \to U$ and the point x'. It applies because U is a scheme and hence u comes from the monomorphism $\mathrm{Spec}(\kappa(u)) \to U$. □

76.34. Zariski's Main Theorem

05W7 In this section we apply the results of the previous section to prove Zariski's main theorem for morphisms of algebraic spaces. This section is the analogue of More on Morphisms, Section 37.43.

082I **Lemma 76.34.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is of finite type and separated. Let Y' be the normalization of Y in X. Picture:*

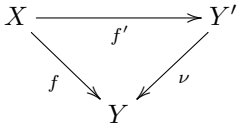

Then there exists an open subspace $U' \subset Y'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms of Spaces, Lemma 67.34.7 there is an open subspace $U \subset X$ corresponding to the points of $|X|$ where f is quasi-finite. We have to prove

(a) the image of $|U| \to |Y'|$ is $|U'|$ for some open subspace U' of Y',
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Since formation of U commutes with arbitrary base change (Morphisms of Spaces, Lemma 67.34.7), since formation of the normalization Y' commutes with smooth base change (Lemma 76.25.2), since étale morphisms are open, and since "being an isomorphism" is fpqc local on the base (Descent on Spaces, Lemma 74.11.15), it suffices to prove (a), (b), (c) étale locally on Y (some details omitted). Thus we may assume Y is an affine scheme. This implies that Y' is an (affine) scheme as well.

Let $x \in |U|$. Claim: there exists an open neighbourhood $f'(x) \in V \subset Y'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is a scheme (as an open of Y'), locally of finite type over Y (as an open subspace of X), and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset Y'$ and Y' is integral over Y). Hence the fibres of $V \to Y$ are discrete (Morphisms, Lemma 29.20.2) and $(f')^{-1}V \to Y$ is locally quasi-finite (Morphisms, Lemma 29.20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $y = f(x) \in |Y|$. Let $(T, t) \to (Y, y)$ be an étale morphism of pointed schemes. Denote by a subscript $_T$ the base change to T. Let $z \in X_T$ be a point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms of Spaces, Lemma 67.34.7. Note that

$$X_T \xrightarrow{f'_T} Y'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 76.25.2. Suppose that the claim holds for $z \in U_T \subset X_T \to Y'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(z) \in V' \subset Y'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $Y'_T \to Y'$ is étale hence the image $V \subset Y'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(z) \in V'$. Observe that

$$\begin{array}{ccc} (f'_T)^{-1}V' & \longrightarrow & (f')^{-1}(V) \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is a fibre square (as $Y'_T \times_{Y'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent on Spaces, Lemma 74.11.15. In other words, the claim holds for $x \in U \subset X \to Y' \to Y$.

By the result of the previous paragraph to prove the claim for $x \in |U|$, we may replace Y by an étale neighbourhood T of $y = f(x)$ and x by any point lying over x in $T \times_Y X$. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subspaces where $V \to Y$ is finite and $x \in V$, see Lemma 76.33.1. Since X is a disjoint union of V and W over Y and since $V \to Y$ is finite we see that the normalization of Y in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of Y in W, see Morphisms of Spaces, Lemmas 67.48.8, 67.45.6, and 67.48.10. The claim follows and we win. □

The following lemma is a duplicate of Morphisms of Spaces, Lemma 67.52.2. The reason for having two copies of the same lemma is that the proofs are somewhat different. The proof given below rests on Zariski's Main Theorem for nonrepresentable morphisms of algebraic spaces as presented above, whereas the proof of Morphisms of Spaces, Lemma 67.52.2 rests on Morphisms of Spaces, Proposition 67.50.2 to reduce to the case of morphisms of schemes.

082J **Lemma 76.34.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"'] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 76.34.1. Namely, by that lemma there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Morphisms of Spaces, Lemma 67.8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. □

082K **Lemma 76.34.3** (Zariski's Main Theorem)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated and assume that Y is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "j"'] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to Y' \to Y$ be as in the conclusion of Lemma 76.34.2. By Limits of Spaces, Lemma 70.9.7 we can write $\nu_*\mathcal{O}_{Y'} = \operatorname{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{Y'}$. Then $\pi_i : T_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i) \to Y$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $Y' = \lim T_i$.

By Limits of Spaces, Lemma 70.5.7 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in Y' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits of Spaces, Lemma 70.4.1. By Limits of Spaces, Lemma 70.5.12 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms of Spaces, Lemma 67.12.6 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

0874 **Lemma 76.34.4.** *With notation and hypotheses as in Lemma 76.34.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over Y.*

Proof. By Limits of Spaces, Lemma 70.11.3 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits of Spaces, Lemma 70.5.7 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits of Spaces,

Lemma 70.5.12 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

76.35. Applications of Zariski's Main Theorem, I

0F43 A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

0A4X **Lemma 76.35.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is finite,*
(2) *f is proper and locally quasi-finite,*
(3) *f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field,*
(4) *f is universally closed, separated, locally of finite type and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.*

Proof. We have (1) $\Rightarrow$ (2) by Morphisms of Spaces, Lemmas 67.45.9, 67.45.8. We have (2) $\Rightarrow$ (3) by Morphisms of Spaces, Lemma 67.27.5. By definition (3) implies (4).

Assume (4). Since f is universally closed it is quasi-compact (Morphisms of Spaces, Lemma 67.9.7). Pick a point y of $|Y|$. We represent y by a morphism $\mathrm{Spec}(k) \to Y$. Note that $|X_k|$ is finite discrete as a quasi-compact discrete space. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 66.4.3). By Morphisms of Spaces, Lemma 67.34.8 we see that $X \to Y$ is quasi-finite at all the points of the fibre of $|X| \to |Y|$ over y. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$ and decomposition $X_U = V \amalg W$ as in Lemma 76.33.1 adapted to all the points of $|X|$ lying over y. Note that $W_u = \emptyset$ because we used all the points in the fibre of $|X| \to |Y|$ over y. Since f is universally closed we see that the image of $|W|$ in $|U|$ is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $y \in |Y|$ was arbitrary this means there exists a family $\{U_i \to Y\}$ of étale morphisms whose images cover Y such that the base changes $X_{U_i} \to U_i$ are finite. We conclude that f is finite by Morphisms of Spaces, Lemma 67.45.3. □

As a consequence we have the following useful result.

0A4Y **Lemma 76.35.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Assume*

(1) *f is proper, and*
(2) *f is quasi-finite at all $x \in |X|$ lying over y (Decent Spaces, Lemma 68.18.10).*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Morphisms of Spaces, Lemma 67.34.7 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $y \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of y with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ is finite by Lemma 76.35.1. □

0AEJ **Lemma 76.35.3.** *Let S be a scheme. Let*

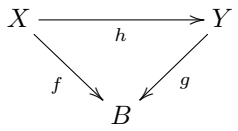

be a commutative diagram of morphism of algebraic spaces over S. Let $b \in B$ and let $\mathrm{Spec}(k) \to B$ be a morphism in the equivalence class of b. Assume

(1) *$X \to B$ is a proper morphism,*
(2) *$Y \to B$ is separated and locally of finite type,*
(3) *one of the following is true*
   (a) *the image of $|X_k| \to |Y_k|$ is finite,*
   (b) *the image of $|f|^{-1}(\{b\})$ in $|Y|$ is finite and B is decent.*

Then there is an open subspace $B' \subset B$ containing b such that $X_{B'} \to Y_{B'}$ factors through a closed subspace $Z \subset Y_{B'}$ finite over B'.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms of Spaces, Section 67.16. By Morphisms of Spaces, Lemma 67.40.8 the morphism $X \to Z$ is surjective and $Z \to B$ is proper. Thus

$$\{x \in |X| \text{ lying over } b\} \to \{z \in |Z| \text{ lying over } b\}$$

and $|X_k| \to |Z_k|$ are surjective. We see that either (3)(a) or (3)(b) imply that $Z \to B$ is quasi-finite all points of $|Z|$ lying over b by Decent Spaces, Lemma 68.18.10. Hence $Z \to B$ is finite in an open neighbourhood of b by Lemma 76.35.2. □

76.36. Stein factorization

0A18 Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

0A19 **Lemma 76.36.1.** *Let S be a scheme. Let $f : X \to Y$ be a universally closed and quasi-separated morphism of algebraic spaces over S. There exists a factorization*

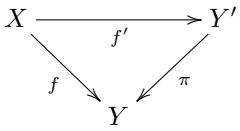

with the following properties:

(1) *the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,*
(2) *the morphism $\pi : Y' \to Y$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
(4) *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
(5) *Y' is the normalization of Y in X as defined in Morphisms of Spaces, Definition 67.48.3.*

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms of Spaces, Lemma 67.9.7 the morphism f is quasi-compact. We just define Y' as the normalization of Y in X, so (5) and (2) hold automatically. By Morphisms of Spaces, Lemma 67.48.9 we see that (4) holds. The morphism f' is universally closed by Morphisms of Spaces, Lemma 67.40.6. It is quasi-compact by

Morphisms of Spaces, Lemma 67.8.9 and quasi-separated by Morphisms of Spaces, Lemma 67.4.10.

To show the remaining statements we may assume the base Y is affine (as taking normalization commutes with étale localization). Say $Y = \mathrm{Spec}(R)$. Then $Y' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{Y'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Morphisms of Spaces, Lemma 67.11.2, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset Y' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the Jacobson radical of A and we conclude that $V(I) = Y'$ as desired. □

0E1C **Lemma 76.36.2.** *In Lemma 76.36.1 assume in addition that f is locally of finite type and Y affine. Then for $y \in Y$ the fibre $\pi^{-1}(\{y\}) = \{y_1, \ldots, y_n\}$ is finite and the field extensions $\kappa(y_i)/\kappa(y)$ are finite.*

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{y\})$, see Algebra, Lemma 10.36.20. As f' is surjective, we find that $|X_y| \to \pi^{-1}(\{y\})$ is surjective. Observe that X_y is a quasi-separated algebraic space of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus $|X_y|$ is a Noetherian topological space (Morphisms of Spaces, Lemma 67.28.6). A topological argument (omitted) now shows that $\pi^{-1}(\{y\})$ is finite. For each i we can pick a finite type point $x_i \in |X_y|$ mapping to y_i (Morphisms of Spaces, Lemma 67.25.6). We conclude that $\kappa(y_i)/\kappa(y)$ is finite: x_i can be represented by a morphism $\mathrm{Spec}(k_i) \to X_y$ of finite type (by our definition of finite type points) and hence $\mathrm{Spec}(k_i) \to y = \mathrm{Spec}(\kappa(y))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(y)$ is finite (Morphisms, Lemma 29.16.1). □

Let $f : X \to Y$ be a morphism of algebraic spaces and let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Then the *fibre of f over $\overline{y}$* is the algebraic space $X_{\overline{y}} = X \times_{Y,\overline{y}} \mathrm{Spec}(k)$ over k. If Y is a scheme and $y \in Y$ is a point, then we denote $X_y = X \times_Y \mathrm{Spec}(\kappa(y))$ the fibre as usual.

0A1A **Lemma 76.36.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Then $X_{\overline{y}}$ is connected, if and only if for every étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ where V is a scheme the base change $X_V \to V$ has connected fibre X_v.*

Proof. Since the category of étale neighbourhoods of $\overline{y}$ is cofiltered and contains a cofinal collection of schemes (Properties of Spaces, Lemma 66.19.3) we may replace Y by one of these neighbourhoods and assume that Y is a scheme. Let $y \in Y$ be the point corresponding to $\overline{y}$. Then X_y is geometrically connected over $\kappa(y)$ if and only if $X_{\overline{y}}$ is connected and if and only if $(X_y)_{k'}$ is connected for every finite separable extension k' of $\kappa(y)$. See Spaces over Fields, Section 72.12 and especially Lemma 72.12.8. By More on Morphisms, Lemma 37.35.2 there exists an affine étale neighbourhood $(V, v) \to (Y, y)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k'$ any given finite separable extension. The lemma follows. □

0A1B **Theorem 76.36.4** (Stein factorization; Noetherian case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

with the following properties:

(1) *the morphism f' is proper with connected geometric fibres,*
(2) *the morphism $\pi : Y' \to Y$ is finite,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
(4) *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
(5) *Y' is the normalization of Y in X, see Morphisms, Definition 29.53.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 76.36.1. Note that besides the conclusions of Lemma 76.36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 67.4.10) and finite type (Morphisms of Spaces, Lemma 67.23.6). Hence f' is proper. By Cohomology of Spaces, Lemma 69.20.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that the geometric fibres of f' are connected. It is clear from the discussion above that we may replace Y by Y'. Then Y is locally Noetherian, $f : X \to Y$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $\overline{y}$ be a geometric point of Y. At this point we apply the theorem on formal functions, more precisely Cohomology of Spaces, Lemma 69.22.7. It tells us that

$$\mathcal{O}^\wedge_{Y,\overline{y}} = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where $X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}^n_{\overline{y}}) \times_Y X$. Note that $X_1 = X_{\overline{y}} \to X_n$ is a (finite order) thickening and hence the underlying topological space of X_n is equal to that of $X_{\overline{y}}$. Thus, if $X_{\overline{y}} = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subspaces, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{Y,\overline{y}}$ is a local ring. Thus the assumption was wrong, i.e., $X_{\overline{y}}$ is connected as desired. □

0A1C **Theorem 76.36.5** (Stein factorization; general case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

with the following properties:

(1) *the morphism f' is proper with connected geometric fibres,*
(2) *the morphism $\pi : Y' \to Y$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
(4) *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*

(5) *Y' is the normalization of Y in X (Morphisms of Spaces, Definition 67.48.3).*

Proof. We may apply Lemma 76.36.1 to get the morphism $f' : X \to Y'$. Note that besides the conclusions of Lemma 76.36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 67.4.10) and finite type (Morphisms of Spaces, Lemma 67.23.6). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has connected geometric fibres.

It is clear from the discussion that we may replace Y by Y'. Then $f : X \to Y$ is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that these conditions are preserved under flat base change (Morphisms of Spaces, Lemma 67.40.3 and Cohomology of Spaces, Lemma 69.11.2). Let $\overline{y}$ be a geometric point of Y. By Lemma 76.36.3 and the remark just made we reduce to the case where Y is a scheme, $y \in Y$ is a point, $f : X \to Y$ is a proper algebraic space over Y with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and we have to show the fibre X_y is connected. Replacing Y by an affine neighbourhood of y we may assume that $Y = \mathrm{Spec}(R)$ is affine. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ signifies that the ring map $R \to \Gamma(X, \mathcal{O}_X)$ is bijective.

By Limits of Spaces, Lemma 70.12.2 we can write $(X \to Y) = \lim(X_i \to Y_i)$ with $X_i \to Y_i$ proper and of finite presentation and Y_i Noetherian. For i large enough Y_i is affine (Limits of Spaces, Lemma 70.5.10). Say $Y_i = \mathrm{Spec}(R_i)$. Let $R_i' = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that we have ring maps $R_i \to R_i' \to R$. Namely, we have the first because X_i is an algebraic space over R_i and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathrm{colim}\, R_i'$ by Limits of Spaces, Lemma 70.5.6. Then

$$\begin{array}{ccccc} X & \longrightarrow & X_i & & \\ \downarrow & & \downarrow & & \\ Y & \longrightarrow & Y_i' & \longrightarrow & Y_i \end{array}$$

is commutative with $Y_i' = \mathrm{Spec}(R_i')$. Let $y_i' \in Y_i'$ be the image of y. We have $X_y = \lim X_{i,y_i'}$ because $X = \lim X_i$, $Y = \lim Y_i'$, and $\kappa(y) = \mathrm{colim}\, \kappa(y_i')$. Now let $X_y = U \amalg V$ with U and V open and closed. Then U, V are the inverse images of opens U_i, V_i in $X_{i,y_i'}$ (Limits of Spaces, Lemma 70.5.7). By Theorem 76.36.4 the fibres of $X_i \to Y_i'$ are connected, hence either U or V is empty. This finishes the proof. □

Here is an application.

0AYI **Lemma 76.36.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is proper,*
(2) *Y is integral (Spaces over Fields, Definition 72.4.1) with generic point ξ,*
(3) *Y is normal,*
(4) *X is reduced,*
(5) *every generic point of an irreducible component of $|X|$ maps to ξ,*
(6) *we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.*

Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f has geometrically connected fibres.

Proof. Apply Theorem 76.36.5 to get a factorization $X \to Y' \to Y$. It is enough to show that $Y' = Y$. It suffices to show that $Y' \times_Y V \to V$ is an isomorphism,

where $V \to Y$ is an étale morphism and V an affine integral scheme, see Spaces over Fields, Lemma 72.4.5. The formation of Y' commutes with étale base change, see Morphisms of Spaces, Lemma 67.48.4. The generic points of $X \times_Y V$ lie over the generic points of X (Decent Spaces, Lemma 68.20.1) hence map to the generic point of V by assumption (5). Moreover, condition (6) is preserved under the base change by $V \to Y$, for example by flat base change (Cohomology of Spaces, Lemma 69.11.2). Thus it suffices to prove the lemma in case Y is a normal integral affine scheme.

Assume Y is a normal integral affine scheme. We will show $Y' \to Y$ is an isomorphism by an application of Morphisms, Lemma 29.54.8. Namely, Y' is reduced because X is reduced (Morphisms of Spaces, Lemma 67.48.6). The morphism $Y' \to Y$ is integral by the theorem cited above. Since Y is decent and $X \to Y$ is separated, we see that X is decent too; to see this use Decent Spaces, Lemmas 68.17.2 and 68.17.5. By assumption (5), Morphisms of Spaces, Lemma 67.48.7, and Decent Spaces, Lemma 68.20.1 we see that every generic point of an irreducible component of $|Y'|$ maps to ξ. On the other hand, since Y' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre Y'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence Y' is an integral scheme with function field equal to the function field of Y. This finishes the proof. □

Here is another application.

0E1D **Lemma 76.36.7.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If f is proper, flat, and of finite presentation, then the function $n_{X/Y} : |Y| \to \mathbf{Z}$ counting the number of geometric connected components of fibres of f (Lemma 76.30.1) is lower semi-continuous.*

Proof. The question is étale local on Y, hence we may and do assume Y is an affine scheme. Let $y \in Y$. Set $n = n_{X/S}(y)$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of y such that $n_{X/S}|_V \geq n$. Let $X \to Y' \to Y$ be the Stein factorization as in Theorem 76.36.5. By Lemma 76.36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and the extensions $\kappa(y'_i)/\kappa(y)$ are finite. More on Morphisms, Lemma 37.42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. Then the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over Y. Hence the image of $X_i \to Y$ is open (Morphisms of Spaces, Lemma 67.30.6). Thus in a neighbourhood of y we see that $n_{X/Y}$ is at least n. □

0E1E **Lemma 76.36.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the geometric fibres of f are reduced.*

Then the function $n_{X/S} : |Y| \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f (Lemma 76.30.1) is locally constant.

Proof. By Lemma 76.36.7 the function $n_{X/Y}$ is lower semincontinuous. Thus it suffices to show it is upper semi-continuous. To do this we may work étale locally on

Y, hence we may assume Y is an affine scheme. For $y \in Y$ consider the $\kappa(y)$-algebra

$$A = H^0(X_y, \mathcal{O}_{X_y})$$

By Spaces over Fields, Lemma 72.14.3 and the fact that X_y is geometrically reduced A is finite product of finite separable extensions of $\kappa(y)$. Hence $A \otimes_{\kappa(y)} \kappa(\overline{y})$ is a product of $\beta_0(y) = \dim_{\kappa(y)} A$ copies of $\kappa(\overline{y})$. Thus $X_{\overline{y}}$ has $\beta_0(y)$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on Y. Thus $n_{X/Y}$ is upper semi-continuous by Derived Categories of Spaces, Lemma 75.26.2. This finishes the proof. □

0E0D **Lemma 76.36.9.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $X \to Y' \to Y$ be the Stein factorization of f (Theorem 76.36.5). If f is of finite presentation, flat, with geometrically reduced fibres (Definition 76.29.2), then $Y' \to Y$ is finite étale.*

Proof. Formation of the Stein factorization commutes with flat base change, see Lemma 76.36.1. Thus we may work étale locally on Y and we may assume Y is an affine scheme. Then Y' is an affine scheme and $Y' \to Y$ is integral.

Let $y \in Y$. Set n be the number of connected components of the geometric fibre $X_{\overline{y}}$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. By Lemma 76.36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and for each i we can pick a finite type point $x_i \in |X_y|$ mapping to y'_i the extension $\kappa(y'_i)/\kappa(y)$ is finite. Thus More on Morphisms, Lemma 37.42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. In this case the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are proper, flat, of finite presentation, with geometrically reduced fibres over Y. It suffices to prove the lemma for each of the morphisms $X_i \to Y$. This reduces us to the case where $X_{\overline{y}}$ is connected.

Assume that $X_{\overline{y}}$ is connected. By Lemma 76.36.8 we see that $X \to Y$ has geometrically connected fibres in a neighbourhood of y. Thus we may assume the fibres of $X \to Y$ are geometrically connected. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ by Derived Categories of Spaces, Lemma 75.26.8 which finishes the proof. □

The proof of the following lemma uses Stein factorization for schemes which is why it ended up in this section.

0CWI **Lemma 76.36.10.** *Let (A, I) be a henselian pair. Let X be an algebraic space separated and of finite type over A. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subspace such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then there exists an open and closed subspace $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.*

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By a weak version of Chow's lemma (in the form of Cohomology of Spaces, Lemma 69.18.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. The lemma holds for (X', Y') by More on Morphisms, Lemma 37.53.9. Let $W' \subset X'$ be the open and closed subscheme proper over A such

that $Y' = W'_0$. By Morphisms of Spaces, Lemma 67.40.6 $Q_1 = \varphi(|W'|) \subset |X|$ and $Q_2 = \varphi(|X' \setminus W'|) \subset |X|$ are closed subsets and by Morphisms of Spaces, Lemma 67.40.7 any closed subspace structure on Q_1 is proper over A. The image of $Q_1 \cap Q_2$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $Q_1 \cap Q_2$ is nonempty, then we find that $Q_1 \cap Q_2$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that Q_1 is open and closed in $|X|$. Let $W \subset X$ be the corresponding open and closed subspace. Then W is proper over A with $W_0 = Y$. □

76.37. Extending properties from an open

0875 In this section we collect a number of results of the form: If $f : X \to Y$ is a flat morphism of algebraic spaces and f satisfies some property over a dense open of Y, then f satisfies the same property over all of Y.

0876 **Lemma 76.37.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type and flat over Y,*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(4) *$\mathcal{F}|_{f^{-1}V}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. It suffices to prove the pullback of $\mathcal{F}$ to a scheme surjective and étale over X is of finite presentation. Hence we may assume X is a scheme. Similarly, we can replace Y by a scheme surjective and étale and over Y (the inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 67.17). Thus we reduce to the case of schemes which is More on Flatness, Lemma 38.11.1. □

0877 **Lemma 76.37.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite type and flat,*
(2) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(3) *$f|_{f^{-1}V} : f^{-1}V \to V$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 76.37.1 except one uses More on Flatness, Lemma 38.11.2. □

0878 **Lemma 76.37.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite type. Let $V \subset Y$ be an open subspace such that $|V| \subset |Y|$ is dense and such that $X_V \to V$ has relative dimension $\leq d$. If also either*

(1) *f is locally of finite presentation, or*
(2) *$V \to Y$ is quasi-compact,*

then $f : X \to Y$ has relative dimension $\leq d$.

Proof. We may replace Y by its reduction, hence we may assume Y is reduced. Then V is scheme theoretically dense in Y, see Morphisms of Spaces, Lemma

67.17.7. By definition the property of having relative dimension $\leq d$ can be checked on an étale covering, see Morphisms of Spaces, Sections 67.33. Thus it suffices to prove f has relative dimension $\leq d$ after replacing X by a scheme surjective and étale over X. Similarly, we can replace Y by a scheme surjective and étale and over Y. The inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 67.17. Since a scheme theoretically dense open of a scheme is in particular dense, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.3. □

0B4J **Lemma 76.37.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and proper. Let $V \to Y$ be an open subspace with $|V| \subset |Y|$ dense such that $X_V \to V$ is finite. If also either f is locally of finite presentation or $V \to Y$ is quasi-compact, then f is finite.*

Proof. By Lemma 76.37.3 the fibres of f have dimension zero. By Morphisms of Spaces, Lemma 67.34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable. We can check whether f is finite étale locally on Y, hence we may assume Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.4. □

0879 **Lemma 76.37.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(V) \to V$ is an isomorphism, and*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*

then f is an open immersion.

Proof. Applying Lemma 76.37.2 we see that f is locally of finite presentation. Applying Lemma 76.37.3 we see that f has relative dimension ≤ 0. By Morphisms of Spaces, Lemma 67.34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 67.51.1 this implies that f is representable. By Descent on Spaces, Lemma 74.11.14 we can check whether f is an open immersion étale locally on Y. Hence we may assume that Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 38.11.5. □

76.38. Blowing up and flatness

087A Instead of redoing the work in More on Flatness, Section 38.30 we prove an analogue of More on Flatness, Lemma 38.30.5 which tells us that the problem of finding a suitable blowup is often étale local on the base.

087B **Lemma 76.38.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\varphi : W \to X$ be a quasi-compact separated étale morphism. Let $U \subset X$ be a quasi-compact open subspace. Let $\mathcal{I} \subset \mathcal{O}_W$ be a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}) \cap \varphi^{-1}(U) = \emptyset$. Then there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that*

(1) *$V(\mathcal{J}) \cap U = \emptyset$, and*
(2) *$\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathcal{I}\mathcal{I}'$ for some finite type quasi-coherent ideal $\mathcal{I}' \subset \mathcal{O}_W$.*

Proof. Choose a factorization $W \to Y \to X$ where $j : W \to Y$ is a quasi-compact open immersion and $\pi : Y \to X$ is a finite morphism of finite presentation (Lemma

76.34.4). Let $V = j(W) \cup \pi^{-1}(U) \subset Y$. Note that $\mathcal{I}$ on $W \cong j(W)$ and $\mathcal{O}_{\pi^{-1}(U)}$ glue to a finite type quasi-coherent sheaf of ideals $\mathcal{I}_1 \subset \mathcal{O}_V$. By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_2 \subset \mathcal{O}_Y$ such that $\mathcal{I}_2|_V = \mathcal{I}_1$. In other words, $\mathcal{I}_2 \subset \mathcal{O}_Y$ is a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_2)$ is disjoint from $\pi^{-1}(U)$ and $j^{-1}\mathcal{I}_2 = \mathcal{I}$. Denote $i : Z \to Y$ the corresponding closed immersion which is of finite presentation (Morphisms of Spaces, Lemma 67.28.12). In particular the composition $\tau = \pi \circ i : Z \to X$ is finite and of finite presentation (Morphisms of Spaces, Lemmas 67.28.2 and 67.45.4).

Let $\mathcal{F} = \tau_*\mathcal{O}_Z$ which we think of as a quasi-coherent $\mathcal{O}_X$-module. By Descent on Spaces, Lemma 74.6.7 we see that $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{J} = \mathrm{Fit}_0(\mathcal{F})$. (Insert reference to fitting modules on ringed topoi here.) This is a finite type quasi-coherent sheaf of ideals on X (as $\mathcal{F}$ is of finite presentation, see More on Algebra, Lemma 15.8.4). Part (1) of the lemma holds because $|\tau|(|Z|) \cap |U| = \emptyset$ by our choice of $\mathcal{I}_2$ and because the 0th Fitting ideal of the trivial module equals the structure sheaf. To prove (2) note that $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathrm{Fit}_0(\varphi^*\mathcal{F})$ because taking Fitting ideals commutes with base change. On the other hand, as $\varphi : W \to X$ is separated and étale we see that $(1, j) : W \to W \times_X Y$ is an open and closed immersion. Hence $W \times_Y Z = V(\mathcal{I}) \amalg Z'$ for some finite and finitely presented morphism of algebraic spaces $\tau' : Z' \to W$. Thus we see that

$$\begin{aligned} \mathrm{Fit}_0(\varphi^*\mathcal{F}) &= \mathrm{Fit}_0((W \times_Y Z \to W)_*\mathcal{O}_{W \times_Y Z}) \\ &= \mathrm{Fit}_0(\mathcal{O}_W/\mathcal{I}) \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \\ &= \mathcal{I} \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \end{aligned}$$

the second equality by More on Algebra, Lemma 15.8.4 translated in sheaves on ringed topoi. Setting $\mathcal{I}' = \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'})$ finishes the proof of the lemma. □

087C **Theorem 76.38.2.** *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset B$ be a quasi-compact open subspace. Assume*

(1) *X is quasi-compact,*
(2) *X is locally of finite presentation over B,*
(3) *$\mathcal{F}$ is a module of finite type,*
(4) *$\mathcal{F}_U$ is of finite presentation, and*
(5) *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_B B'}$-module of finite presentation and flat over B'.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to X$. Because strict transform commutes with étale localization (Divisors on Spaces, Lemma 71.18.2) it suffices to prove the result with X replaced by V. Hence we may assume that $X \to B$ is representable (in addition to the hypotheses of the lemma).

Assume that $X \to B$ is representable. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to B$. Note that $X \times_B W$ is a scheme. By the case of schemes (More on Flatness, Theorem 38.30.7) we can find a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ such that (a) $|V(\mathcal{I})| \cap |\varphi^{-1}(U)| = \emptyset$ and (b) the strict transform of $\mathcal{F}|_{X \times_B W}$ with respect to the blowing up $W' \to W$ in $\mathcal{I}$ becomes flat over W' and is a module of finite presentation. Choose a finite type sheaf of ideals $\mathcal{J} \subset \mathcal{O}_B$ as in Lemma 76.38.1. Let $B' \to B$ be the blowing up of $\mathcal{J}$. We claim that this blowup

works. Namely, it is clear that $B' \to B$ is U-admissible by our choice of ideal $\mathcal{J}$. Moreover, the base change $B' \times_B W \to W$ is the blowup of W in $\varphi^{-1}\mathcal{J} = \mathcal{I}\mathcal{I}'$ (compatibility of blowup with flat base change, see Divisors on Spaces, Lemma 71.17.3). Hence there is a factorization

$$W \times_B B' \to W' \to W$$

where the first morphism is a blowup as well, see Divisors on Spaces, Lemma 71.17.10). The restriction of $\mathcal{F}'$ (which lives on $B' \times_B X$) to $W \times_B B' \times_B X$ is the strict transform of $\mathcal{F}|_{X \times_B W}$ (Divisors on Spaces, Lemma 71.18.2) and hence is the twice repeated strict transform of $\mathcal{F}|_{X \times_B W}$ by the two blowups displayed above (Divisors on Spaces, Lemma 71.18.7). After the first blowup our sheaf is already flat over the base and of finite presentation (by construction). Whence this holds after the second strict transform as well (since this is a pullback by Divisors on Spaces, Lemma 71.18.4). Thus we see that the restriction of $\mathcal{F}'$ to an étale cover of $B' \times_B X$ has the desired properties and the theorem is proved. □

76.39. Applications

087D In this section we apply the result on flattening by blowing up.

087E **Lemma 76.39.1.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is of finite type and quasi-separated, and*
(4) *$f^{-1}(U) \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is flat and of finite presentation over B'.

Proof. Let $B' \to B$ be a U-admissible blowup. Note that the strict transform of X is quasi-compact and quasi-separated over B' as X is quasi-compact and quasi-separated over B. Hence we only need to worry about finding a U-admissible blowup such that the strict transform becomes flat and locally of finite presentation. We cannot directly apply Theorem 76.38.2 because X is not locally of finite presentation over B.

Choose an affine scheme V and a surjective étale morphism $V \to X$. (This is possible as X is quasi-compact as a finite type space over the quasi-compact space B.) Then it suffices to show the result for the morphism $V \to B$ (as strict transform commutes with étale localization, see Divisors on Spaces, Lemma 71.18.2). Hence we may assume that $X \to B$ is separated as well as finite type. In this case we can find a closed immersion $i : X \to Y$ with $Y \to B$ separated and of finite presentation, see Limits of Spaces, Proposition 70.11.7.

Apply Theorem 76.38.2 to $\mathcal{F} = i_*\mathcal{O}_X$ on Y/B. We find a U-admissible blowup $B' \to B$ such that strict transform of $\mathcal{F}$ is flat over B' and of finite presentation. Let X' be the strict transform of X under the blowup $B' \to B$. Let $i' : X' \to Y \times_B B'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors on Spaces, Lemma 71.18.5), we see that $i'_*\mathcal{O}_{X'}$ is flat over B' and of finite presentation as a $\mathcal{O}_{Y \times_B B'}$-module. Thus $X' \to B'$ is flat and locally of finite presentation. This implies the lemma by our earlier remarks. □

0B4K **Lemma 76.39.2.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is proper, and*
(4) *$f^{-1}(U) \to U$ is finite locally free.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is finite locally free over B'.

Proof. By Lemma 76.39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 76.37.4. Hence f is finite locally free by Morphisms of Spaces, Lemma 67.46.6. □

0GUW **Lemma 76.39.3.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is proper, and*
(4) *$f^{-1}(U) \to U$ is an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X maps isomorphically to B'.

Proof. By Lemma 76.39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 76.37.4 and an open immersion by Lemma 76.37.5. Hence f is an open immersion whose image is closed and contains the dense open U, whence f is an isomorphism. □

087F **Lemma 76.39.4.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *f is of finite type*
(4) *$f^{-1}(U) \to U$ is an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ such that U is scheme theoretically dense in B' and such that the strict transform X' of X maps isomorphically to an open subspace of B'.

Proof. This lemma is a generalization of Lemma 76.39.3. As the composition of U-admissible blowups is U-admissible (Divisors on Spaces, Lemma 71.19.2) we can proceed in stages. Pick a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_B$ with $|B| \setminus |U| = |V(\mathcal{I})|$. Replace B by the blowup of B in $\mathcal{I}$ and X by the strict transform of X. After this replacement $B \setminus U$ is the support of an effective Cartier divisor D (Divisors on Spaces, Lemma 71.17.4). In particular U is scheme theoretically dense in B (Divisors on Spaces, Lemma 71.6.4). Next, we do another U-admissible blowup to get to the situation where $X \to B$ is flat and of finite presentation, see Lemma 76.39.1. Note that U is still scheme theoretically dense in B. Hence $X \to B$ is an open immersion by Lemma 76.37.5. □

The following lemma says that a modification can be dominated by a blowup.

087G **Lemma 76.39.5.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is proper,*
(4) *$f^{-1}(U) \to U$ us an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ which dominates X, i.e., such that there exists a factorization $B' \to X \to B$ of the blowup morphism.

Proof. By Lemma 76.39.3 we may find a U-admissible blowup $B' \to B$ such that the strict transform X' maps isomorphically to B'. Then we can use $B' = X' \to X$ as the factorization. □

0CPI **Lemma 76.39.6.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $U \subset W \subset Y$ be open subspaces. Let $f : X \to W$ and let $s : U \to X$ be morphisms such that $f \circ s = id_U$. Assume*

(1) *f is proper,*
(2) *Y is quasi-compact and quasi-separated, and*
(3) *U and W are quasi-compact.*

Then there exists a U-admissible blowup $b : Y' \to Y$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms of Spaces, Lemma 67.16.7. By Lemma 76.39.5 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors on Spaces, Lemma 71.19.3 to extend $W' \to W$ to a U-admissible blowup of Y. □

76.40. Chow's lemma

088P In this section we prove Chow's lemma (Lemma 76.40.5). We encourage the reader to take a look at Cohomology of Spaces, Section 69.18 for a weak version of Chow's lemma that is easy to prove and sufficient for many applications.

Since we have yet to define projective morphisms of algebraic spaces, the statements of lemmas (see for example Lemma 76.40.2) will involve representable proper morphisms, rather than projective ones.

088Q **Lemma 76.40.1.** *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $U \to X_1$ and $U \to X_2$ be open immersions of algebraic spaces over Y and assume U, X_1, X_2 of finite type and separated over Y. Then there exists a commutative diagram*

$$\begin{array}{ccccc} X_1' & \longrightarrow & X & \longleftarrow & X_2' \\ \downarrow & \swarrow & \uparrow & \nearrow & \downarrow \\ X_1 & \longleftarrow & U & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over Y where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over Y.

Proof. Throughout the proof all the algebraic spaces will be separated of finite type over Y. This in particular implies these algebraic spaces are quasi-compact and quasi-separated and that the morphisms between them will be quasi-compact and separated. See Morphisms of Spaces, Sections 67.4 and 67.8. We will use that if $U \to W$ is an immersion of such spaces over Y, then the scheme theoretic image Z of U in W is a closed subspace of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $|U| \subset |Z|$ is dense. See Morphisms of Spaces, Lemma 67.17.7.

Let $X_{12} \subset X_1 \times_Y X_2$ be the scheme theoretic image of $U \to X_1 \times_Y X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms of Spaces, Lemma 67.16.7. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subspace of X_i^i, see Lemma 76.39.4. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 71.18.3. Let X'_{12} be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 71.17.10 for what this entails. We obtain a commutative diagram

$$\begin{array}{ccc} X'_{12} & \longrightarrow & X_{12}^2 \\ \downarrow & & \downarrow \\ X_{12}^1 & \longrightarrow & X_{12} \end{array}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X'_i \to X_i^i$ restricting to $X'_{12} \to X_{12}^i$, see Divisors on Spaces, Lemma 71.19.3. Then $X'_{12} \subset X'_i$ is an open subspace and the diagram

$$\begin{array}{ccc} X'_{12} & \longrightarrow & X'_i \\ \downarrow & & \downarrow \\ X_{12}^i & \longrightarrow & X_i^i \end{array}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X'_{12} \to X'_1 \times_Y X'_2$ is an immersion and proper (use that $X'_{12} \to X_{12}$ is proper and $X_{12} \to X_1 \times_Y X_2$ is closed and $X'_1 \times_Y X'_2 \to X_1 \times_Y X_2$ is separated and apply Morphisms of Spaces, Lemma 67.40.6). Thus $X'_{12} \to X'_1 \times_Y X'_2$ is a closed immersion. If we define X by glueing X'_1 and X'_2 along the common open subspace X'_{12}, then $X \to Y$ is of finite type and separated[2]. As compositions of U-admissible blowups are U-admissible blowups (Divisors on Spaces, Lemma 71.19.2) the lemma is proved. □

088R **Lemma 76.40.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \subset X$ be an open subspace. Assume*

(1) *U is quasi-compact,*
(2) *Y is quasi-compact and quasi-separated,*
(3) *there exists an immersion $U \to \mathbf{P}^n_Y$ over Y,*
(4) *f is of finite type and separated.*

[2]Because we may check closedness of the diagonal $X \to X \times_Y X$ over the four open parts $X'_i \times_Y X'_j$ of $X \times_Y X$ where it is clear.

Then there exists a commutative diagram

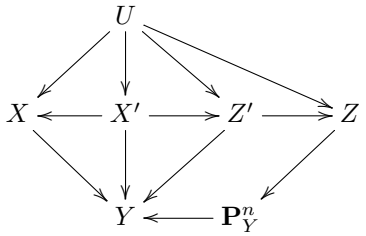

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. Let $Z \subset \mathbf{P}^n_Y$ be the scheme theoretic image of the immersion $U \to \mathbf{P}^n_Y$. Since $U \to \mathbf{P}^n_Y$ is quasi-compact we see that $U \subset Z$ is a (scheme theoretically) dense open subspace (Morphisms of Spaces, Lemma 67.17.7). Apply Lemma 76.40.1 to find a diagram

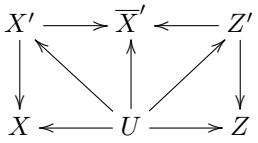

with properties as listed in the statement of that lemma. As $X' \to X$ and $Z' \to Z$ are U-admissible blowups we find that U is a scheme theoretically dense open of both X' and Z' (see Divisors on Spaces, Lemmas 71.17.4 and 71.6.4). Since $Z' \to Z \to Y$ is proper we see that $Z' \subset \overline{X}'$ is a closed subspace (see Morphisms of Spaces, Lemma 67.40.6). It follows that $X' \subset Z'$ (scheme theoretically), hence X' is an open subspace of Z' (small detail omitted) and the lemma is proved. □

088S **Lemma 76.40.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated, of finite type, and Y Noetherian. Then there exists a dense open subspace $U \subset X$ and a commutative diagram*

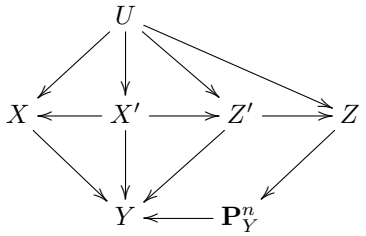

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. By Limits of Spaces, Lemma 70.13.3 there exists a dense open subspace $U \subset X$ and an immersion $U \to \mathbf{A}^n_Y$ over Y. Composing with the open immersion $\mathbf{A}^n_Y \to \mathbf{P}^n_Y$ we obtain a situation as in Lemma 76.40.2 and the result follows. □

088T **Remark 76.40.4.** In Lemmas 76.40.2 and 76.40.3 the morphism $g : Z' \to Y$ is a composition of projective morphisms. Presumably (by the analogue for algebraic spaces of Morphisms, Lemma 29.37.8) there exists a g-ample invertible sheaf on Z'. If we ever need this, then we will state and prove this here.

The following result is [**Knu71**, IV Theorem 3.1]. Note that the immersion $X' \to \mathbf{P}^n_Y$ is quasi-compact, hence can be factored as $X' \to Z' \to \mathbf{P}^n_Y$ where the first morphism is an open immersion and the second morphism a closed immersion (Morphisms of Spaces, Lemma 67.17.7).

088U **Lemma 76.40.5** (Chow's lemma)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and Noetherian. Then there exists a commutative diagram* [**Knu71**, IV Theorem 3.1]

$$\begin{array}{ccccc} X & \longleftarrow & X' & \longrightarrow & \mathbf{P}^n_Y \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. In this first paragraph of the proof we reduce the lemma to the case where Y is of finite type over $\mathrm{Spec}(\mathbf{Z})$. We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. We can write $Y = \lim Y_i$ as a directed limit of separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1 and Lemma 70.5.9. For all i sufficiently large we can find a separated finite type morphism $X_i \to Y_i$ such that $X = Y \times_{Y_i} X_i$, see Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of $|X|$ (X is Noetherian as a finite type separated algebraic space over the Noetherian algebraic space Y and therefore $|X|$ is a Noetherian topological space). By Limits of Spaces, Lemma 70.5.2 we find that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are distinct for i large enough. We may replace X_i by the scheme theoretic image of the (quasi-compact, in fact affine) morphism $X \to X_i$. After this replacement we see that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are the generic points of the irreducible components of $|X_i|$, see Morphisms of Spaces, Lemma 67.16.3. Having said this, suppose we can find a diagram

$$\begin{array}{ccccc} X_i & \longleftarrow & X'_i & \longrightarrow & \mathbf{P}^n_{Y_i} \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X'_i \to X_i$ is a U_i-admissible blowup for some dense open $U_i \subset X_i$ and the morphism $X'_i \to \mathbf{P}^n_{Y_i}$ is an immersion. Then the strict transform $X' \to X$ of X relative to $X'_i \to X_i$ is a U-admissible blowing up where $U \subset X$ is the inverse image of U_i in X. Because of our carefully chosen index i it follows that $\eta_1, \ldots, \eta_n \in |U|$ and $U \subset X$ is dense. Moreover, $X' \to \mathbf{P}^n_Y$ is an immersion as X' is closed in $X'_i \times_{X_i} X = X'_i \times_{Y_i} Y$ which comes with an immersion into $\mathbf{P}^n_Y$. Thus we have reduced to the situation of the following paragraph.

Assume that Y is separated of finite type over $\mathrm{Spec}(\mathbf{Z})$. Then $X \to \mathrm{Spec}(\mathbf{Z})$ is separated of finite type as well. We apply Lemma 76.40.3 to $X \to \mathrm{Spec}(\mathbf{Z})$ to find

a dense open subspace $U \subset X$ and a commutative diagram

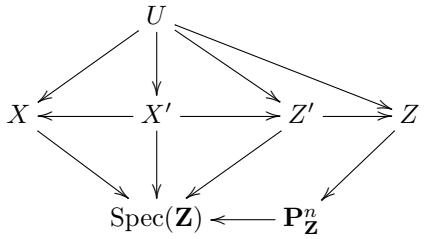

with all the properties listed in the lemma. Note that Z has an ample invertible sheaf, namely $\mathcal{O}_{\mathbf{P}^n}(1)|_Z$. Hence $Z' \to Z$ is a H-projective morphism by Morphisms, Lemma 29.43.16. It follows that $Z' \to \mathrm{Spec}(\mathbf{Z})$ is H-projective by Morphisms, Lemma 29.43.7. Thus there exists a closed immersion $Z' \to \mathbf{P}^m_{\mathrm{Spec}(\mathbf{Z})}$ for some $m \geq 0$. It follows that the diagonal morphism

$$X' \to Y \times \mathbf{P}^m_{\mathbf{Z}} = \mathbf{P}^m_Y$$

is an immersion (because the composition with the projection to $\mathbf{P}^m_{\mathbf{Z}}$ is an immersion) and we win. □

76.41. Variants of Chow's Lemma

089K In this section we prove a number of variants of Chow's lemma dealing with morphisms between non-Noetherian algebraic spaces. The Noetherian versions are Lemma 76.40.3 and Lemma 76.40.5.

089L **Lemma 76.41.1.** *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a separated morphism of finite type. Then there exists a commutative diagram*

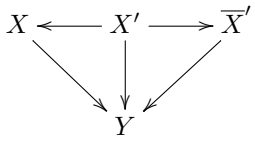

where $X' \to X$ is proper surjective, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Proposition 70.11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1. By Limits of Spaces, Lemma 70.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 70.6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 76.40.3. □

089M **Lemma 76.41.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and quasi-compact. Then there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \longrightarrow & \mathbf{P}^n_Y \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X' \to X$ is proper surjective morphism and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. By Limits of Spaces, Proposition 70.11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 70.8.1. By Limits of Spaces, Lemma 70.5.9 we may assume that Y_i is separated for all i. By Limits of Spaces, Lemma 70.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 70.6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 76.40.5. □

76.42. Grothendieck's existence theorem

089N In this section we discuss Grothendieck's existence theorem for algebraic spaces. Instead of developing a theory of "formal algebraic spaces" we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal space".

Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ of such inverse systems is simply a compatible system of morphisms $\alpha_n : \mathcal{F}_n \to \mathcal{G}_n$. Let us denote the category of these inverse systems with $Coh(X, \mathcal{I})$. We will develop some theory regarding these systems that will parallel to the corresponding results in the case of schemes, see Cohomology of Schemes, Sections 30.24, 30.25, 30.27, and 30.28.

Functoriality. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over a scheme S, and let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. In this situation there is a functor

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. Compare with Cohomology of Schemes, Lemma 30.23.9. If f is étale, then we may think of this as simply the restriction of the system to X, see Properties of Spaces, Equation 66.26.1.1.

Étale descent. Let S be a scheme. Let $U_0 \to X$ be a surjective étale morphism of Noetherian algebraic spaces. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I}_i = \mathcal{I}|_{U_i}$. In this situation we obtain a diagram of categories

$$Coh(X, \mathcal{I}) \longrightarrow Coh(U_0, \mathcal{I}_0) \rightrightarrows Coh(U_1, \mathcal{I}_1) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} Coh(U_2, \mathcal{I}_2)$$

an the first arrow presents $Coh(X, \mathcal{I})$ as the homotopy limit of the right part of the diagram. More precisely, given a *descent datum*, i.e., a pair $((\mathcal{G}_n), \varphi)$ where $(\mathcal{G}_n)$ is an object of $Coh(U_0, \mathcal{I}_0)$ and $\varphi : \mathrm{pr}_0^*(\mathcal{G}_n) \to \mathrm{pr}_1^*(\mathcal{G}_n)$ is an isomorphism in $Coh(U_1, \mathcal{I}_1)$ satisfying the cocycle condition in $Coh(U_2, \mathcal{I}_2)$, then there exists a unique object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ whose associated canonical descent datum is isomorphic to $((\mathcal{G}_n), \varphi)$. Compare with Descent on Spaces, Definition 74.3.3. The proof of this statement follows immediately by applying Descent on Spaces, Proposition 74.4.1 to the descent data $(\mathcal{G}_n, \varphi_n)$ for varying n.

089P **Lemma 76.42.1.** *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $Coh(X, \mathcal{I})$ is abelian.*
(2) *Exactness in $Coh(X, \mathcal{I})$ can be checked étale locally.*
(3) *For any flat morphism $f : X' \to X$ of Noetherian algebraic spaces the functor $f^* : Coh(X, \mathcal{I}) \to Coh(X', f^{-1}\mathcal{I}\mathcal{O}_{X'})$ is exact.*

Proof. Proof of (1). Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X$. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$ as in our discussion of étale descent above. The categories $Coh(U_i, \mathcal{I}_i)$ are abelian (Cohomology of Schemes, Lemma 30.23.2) and the pullback functors are exact functors $Coh(U_0, \mathcal{I}_0) \to Coh(U_1, \mathcal{I}_1)$ and $Coh(U_1, \mathcal{I}_1) \to Coh(U_2, \mathcal{I}_2)$ (Cohomology of Schemes, Lemma 30.23.9). The lemma then follows formally from the description of $Coh(X, \mathcal{I})$ as a category of descent data. Some details omitted; compare with the proof of Groupoids, Lemma 39.14.6.

Part (2) follows immediately from the discussion in the previous paragraph. In the situation of (3) choose a commutative diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

where U' and U are affine schemes and the vertical morphisms are surjective étale. Then $U' \to U$ is a flat morphism of Noetherian schemes (Morphisms of Spaces, Lemma 67.30.5) whence the pullback functor $Coh(U, \mathcal{I}\mathcal{O}_U) \to Coh(U', \mathcal{I}\mathcal{O}_{U'})$ is exact by Cohomology of Schemes, Lemma 30.23.9. Since we can check exactness in $Coh(X, \mathcal{O}_X)$ on U and similarly for X', U' the assertion follows. □

08B3 **Lemma 76.42.2.** *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. We can check on an affine étale cover of X by Lemma 76.42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.3. □

Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

08B4 (76.42.2.1) $$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

08B5 **Lemma 76.42.3.** *The functor (76.42.2.1) is exact.*

Proof. It suffices to check this étale locally on X, see Lemma 76.42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.4. □

08B6 **Lemma 76.42.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge).$$

Proof. Since $\mathcal{H}$ is a sheaf on $X_{étale}$ and since we have étale descent for objects of $Coh(X, \mathcal{I})$ it suffices to prove this étale locally. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.5. □

We introduce the setting that we will focus on throughout the rest of this section.

08B7 **Situation 76.42.5.** Here A is a Noetherian ring complete with respect to an ideal I. Also $f : X \to \mathrm{Spec}(A)$ is a finite type separated morphism of algebraic spaces and $\mathcal{I} = I\mathcal{O}_X$.

In this situation we denote

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose support is proper over $\mathrm{Spec}(A)$, or equivalently whose scheme theoretic support is proper over $\mathrm{Spec}(A)$, see Derived Categories of Spaces, Lemma 75.7.7. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (76.42.2.1) induces a functor

08B8 (76.42.5.1) $$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I})$$

Our first result is that this functor is fully faithful.

08B9 **Lemma 76.42.6.** *In Situation 76.42.5. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$ is proper over* $\mathrm{Spec}(A)$*. Then the map*

$$\mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$$

coming from (76.42.2.1) is a bijection. In particular, (76.42.5.1) is fully faithful.

Proof. Let $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_X$-module because its restriction of schemes étale over X is coherent by Modules, Lemma 17.22.6. By Lemma 76.42.4 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that Z is a closed subspace such that $|Z|$ is contained in the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$. Hence $Z \to \mathrm{Spec}(A)$ is proper by assumption (see Derived Categories of Spaces, Section 75.7). Write $\mathcal{H} = i_*\mathcal{H}'$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}'$. We have $i_*(\mathcal{H}'/I^n\mathcal{H}') = \mathcal{H}/I^n\mathcal{H}$. Hence we obtain

$$\begin{aligned}\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) &= \lim_n H^0(Z, \mathcal{H}'/\mathcal{I}^n\mathcal{H}') \\ &= H^0(Z, \mathcal{H}') \\ &= H^0(X, \mathcal{H}) \\ &= \mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G})\end{aligned}$$

the second equality by the theorem on formal functions (Cohomology of Spaces, Lemma 69.22.6). This proves the lemma. □

08BA **Remark 76.42.7.** Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine scheme $U = \mathrm{Spec}(A)$ and a surjective étale morphism $U \to X$ denote $I, K \subset A$ the ideals corresponding to the restrictions $\mathcal{I}|_U, \mathcal{K}|_U$. Denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Cohomology of Schemes, Lemma 30.23.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any (or some) affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we refer to Cohomology of Schemes, Remark 30.25.1.

08BB **Lemma 76.42.8.** *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
(2) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
(3) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
(4) *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness and étale descent for objects of $Coh(X, \mathcal{I})$ and $Coh(\mathcal{O}_X)$ implies it suffices to construct $(\mathcal{F}, a, \beta)$ étale locally on X. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 30.23.6. □

08BC **Lemma 76.42.9.** *In Situation 76.42.5. Let $\mathcal{K} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $X_e \subset X$ be the closed subspace cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. Assume*

(1) *the functor $Coh_{\text{support proper over } A}(\mathcal{O}_{X_e}) \to Coh_{\text{support proper over } A}(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
(2) *there exists an object $\mathcal{H}$ of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$ and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (76.42.5.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Cohomology of Spaces, Lemma 69.12.8. In particular we think of

$$Coh_{\text{support proper over } A}(\mathcal{O}_{X_e}) \subset Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

as the full subcategory of consisting of modules annihilated by $\mathcal{K}^e$ and

$$Coh_{\text{support proper over } A}(X_e, \mathcal{I}_e) \subset Coh_{\text{support proper over } A}(X, \mathcal{I})$$

as the full subcategory of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (76.42.5.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. We obtain an object $(\mathcal{G}_e)$ of the category $Coh_{\text{support proper over } A}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$ (details omitted; see Cohomology of Schemes, Remark 30.25.1). Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

(details omitted; see Cohomology of Schemes, Remark 30.25.1). We apply Lemma 76.42.8 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 76.42.8 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight.

To finish the proof of the lemma we still have to show that the support of $\mathcal{F}$ is proper over A. By construction the kernel of $a : \mathcal{F} \to \mathcal{H}$ is annihilated by a power of $\mathcal{K}$. Hence the support of this kernel is contained in the support of $\mathcal{G}_1$. Since $\mathcal{G}_1$ is an object of $Coh_{\text{support proper over } A}(\mathcal{O}_{X_1})$ we see this is proper over A. Combined with the fact that the support of $\mathcal{H}$ is proper over A we conclude that the support of $\mathcal{F}$ is proper over A by Derived Categories of Spaces, Lemma 75.7.6. □

08BD **Lemma 76.42.10.** *Let S be a scheme. Let $f : X \to Y$ be a representable proper morphism of Noetherian algebraic spaces over S. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Assume f is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Cohomology of Spaces, Lemma 69.20.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness and étale descent for $Coh(Y, \mathcal{J})$ it suffices to prove this étale locally on Y. Thus we may assume Y is the spectrum of a Noetherian ring. As f is representable we see that X is a scheme as well. Thus we reduce to the case of schemes, see proof of Cohomology of Schemes, Lemma 30.25.3. □

08BE **Theorem 76.42.11** (Grothendieck's existence theorem)**.** *In Situation 76.42.5 the functor (76.42.5.1) is an equivalence.*

Proof. We will use the equivalence of categories of Cohomology of Spaces, Lemma 69.12.8 without further mention in the proof of the theorem. By Lemma 76.42.6 the functor is fully faithful. Thus we need to prove the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Cohomology of Spaces, Lemma 69.13.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is in the essential image, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 76.40.5) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Note that Y is a scheme and f representable. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 29.43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $|T_n| = |T_1|$, hence T_n is proper over A for all n (Morphisms of Spaces, Lemma 67.40.7). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms of Spaces, Lemma 67.40.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 29.41.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Cohomology of Schemes, Lemma 30.24.3) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \text{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\text{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the Jacobson radical of A (Algebra, Lemma 10.96.6). But we've seen above that

$Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Cohomology of Spaces, Lemma 69.20.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 76.42.10 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ in the category $Coh_{\text{support proper over } A}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $|Z_0| \subset f(|Z'|)$. Hence $Z_0 \to \mathrm{Spec}(A)$ is proper (Morphisms of Spaces, Lemma 67.40.7). Thus $\mathcal{H}$ is an object of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$. Since each of the sheaves of ideals $\mathcal{K}^e$ is an element of Ξ we see that the assumptions of Lemma 76.42.9 are satisfied and we conclude. □

08BF **Remark 76.42.12** (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \ldots & & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & & S \end{array}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\mathrm{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 76.42.11 says that the completion functor

$$\begin{array}{c} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{array} \longrightarrow \begin{array}{c} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

76.43. Grothendieck's algebraization theorem

0A00 This section is the analogue of Cohomology of Schemes, Section 30.28. However, this section is missing the result on algebraization of deformations of proper algebraic spaces endowed with ample invertible sheaves, as a proper algebraic space which comes with an ample invertible sheaf is already a scheme. We do have an algebraization result on proper algebraic spaces of relative dimension 1. Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

08BG **Lemma 76.43.1.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose*

given a commutative diagram

$$\begin{array}{ccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of algebraic spaces with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of algebraic spaces $Z \to X$ such that $Z_n = Z \times_S S_n$ for all $n \geq 1$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Limits of Spaces, Lemma 70.15.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 76.42.12, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 76.42.3) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. □

0A01 **Lemma 76.43.2.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of algebraic spaces with cartesian squares. Assume that

(1) *$Y_1 \to X_1$ is a finite morphism, and*
(2) *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of algebraic spaces $Y \to X$ such that $Y_n = Y \times_S S_n$ for all $n \geq 1$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of f_n to X_1 is f_1. Thus Lemma 76.10.2 shows that f_n is a finite morphism. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 76.42.12, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fully faithful (Theorem 76.42.11) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. □

0A4Z **Lemma 76.43.3.** *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be algebraic spaces over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

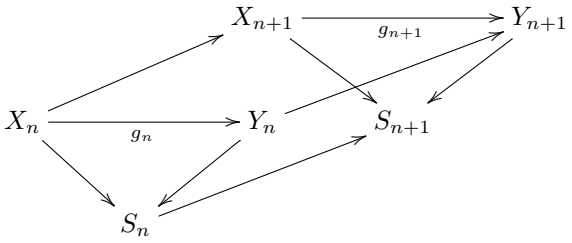

Assume that

(1) *$X \to S$ is proper, and*
(2) *$Y \to S$ is separated of finite type.*

Then there exists a unique morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Morphisms of Spaces, Lemma 67.4.7). Thus by Lemma 76.43.1 there exists a closed subspace $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms of Spaces, Lemma 67.40.6) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is quasi-finite on an open subspace of Z containing every point of Z_0 for example by Morphisms of Spaces, Lemma 67.34.7. As Z is proper over S this open neighbourhood is all of Z. We conclude that $p : Z \to X$ is finite by Zariski's main theorem (for example apply Lemma 76.34.3 and use properness of Z over X to see that the immersion is a closed immersion). Applying the equivalence of Theorem 76.42.11 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. □

0GHK **Remark 76.43.4.** We can ask if in Grothendieck's algebraization theorem (in the form of Lemma 76.43.3), we can get by with weaker separation axioms on the target. Let us be more precise. Let A, I, S, S_n, X, Y, X_n, Y_n, and g_n be as in the statement of Lemma 76.43.3 and assume that

(1) $X \to S$ is proper, and
(2) $Y \to S$ is locally of finite type.

Does there exist a morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n? We don't know the answer in general; if you do please email stacks.project@gmail.com. If $Y \to S$ is separated, then the result holds by the lemma (there is an immediate reduction to the case where X is finite type over S, by choosing a quasi-compact open containing the image of g_1). If we only assume $Y \to S$ is quasi-separated, then the result is true as well. First, as before we may assume Y is quasi-compact as well as quasi-separated. Then we can use either [**Bha16**] or from [**HR19**] to algebraize (g_n). Namely, to apply the first reference,

we use

$$D_{perf}(X) \to \lim D_{perf}(X_n) \xrightarrow{\lim Lg_n^*} \lim D_{perf}(Y_n) = D_{perf}(Y)$$

where the last step uses a Grothendieck existence result for the derived category of the proper algebraic space Y over R (compare with Flatness on Spaces, Remark 77.13.7). The paper cited shows that this arrow determines a morphism $Y \to X$ as desired. To apply the second reference we use the same argument with coherent modules:

$$Coh(\mathcal{O}_X) \to \lim Coh(\mathcal{O}_{X_n}) \xrightarrow{\lim g_n^*} \lim Coh(\mathcal{O}_{Y_n}) = Coh(\mathcal{O}_Y)$$

where the final equality is a consequence of Grothendieck's existence theorem (Theorem 76.42.11). The second reference tells us that this functor corresponds to a morphism $Y \to X$ over R. If we ever need this generalization we will precisely state and carefully prove the result here.

0E7R **Lemma 76.43.5.** *Let $(A, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/\mathfrak{m}^n)$. Consider a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots \end{array}$$

of algebraic spaces with cartesian squares. If $\dim(X_1) \leq 1$, then there exists a projective morphism of schemes $X \to S$ and isomorphisms $X_n \cong X \times_S S_n$ compatible with i_n.

Proof. By Spaces over Fields, Lemma 72.9.3 the algebraic space X_1 is a scheme. Hence X_1 is a proper scheme of dimension ≤ 1 over κ. By Varieties, Lemma 33.43.4 we see that X_1 is H-projective over κ. Let $\mathcal{L}_1$ be an ample invertible sheaf on X_1.

We are going to show that $\mathcal{L}_1$ lifts to a compatible system $\{\mathcal{L}_n\}$ of invertible sheaves on $\{X_n\}$. Observe that X_n is a scheme too by Lemma 76.9.5. Recall that $X_1 \to X_n$ induces homeomorphisms of underlying topological spaces. In the rest of the proof we do not distinguish between sheaves on X_n and sheaves on X_1. Suppose, given a lift $\mathcal{L}_n$ to X_n. We consider the exact sequence

$$1 \to (1 + \mathfrak{m}^n \mathcal{O}_{X_{n+1}})^* \to \mathcal{O}_{X_{n+1}}^* \to \mathcal{O}_{X_n}^* \to 1$$

of sheaves on X_{n+1}. The class of $\mathcal{L}_n$ in $H^1(X_n, \mathcal{O}_{X_n}^*)$ (see Cohomology, Lemma 20.6.1) can be lifted to an element of $H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^*)$ if and only if the obstruction in $H^2(X_{n+1}, (1 + \mathfrak{m}^n \mathcal{O}_{X_{n+1}})^*)$ is zero. As X_1 is a Noetherian scheme of dimension ≤ 1 this cohomology group vanishes (Cohomology, Proposition 20.20.7).

By Grothendieck's algebraization theorem (Cohomology of Schemes, Theorem 30.28.4) we find a projective morphism of schemes $X \to S = \mathrm{Spec}(A)$ and a compatible system of isomorphisms $X_n = S_n \times_S X$. □

0AE7 **Lemma 76.43.6.** *Let $(A, \mathfrak{m}, \kappa)$ be a complete Noetherian local ring. Let X be an algebraic space over $\mathrm{Spec}(A)$. If $X \to \mathrm{Spec}(A)$ is proper and $\dim(X_\kappa) \leq 1$, then X is a scheme projective over A.*

Proof. Set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 76.43.5 there exists a projective morphism $Y \to \mathrm{Spec}(A)$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 76.43.3 we see that $X \cong Y$ and the proof is complete. □

76.44. Regular immersions

06BL This section is the analogue of Divisors, Section 31.21 for morphisms of algebraic spaces. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 31.21 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [**BGI71**, Exposee VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 67.3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 67.22.1 to define them either. We replace this lemma instead by the following.

06BM **Lemma 76.44.1.** *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{array}{ccc} X \times_Y V & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

(1) *for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and*
(2) *for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{array}{ccc} X \times_Y V & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X \times_Y V' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$

has $\mathcal{P}$. To do this consider the diagram

$$\begin{array}{ccccc} X \times_Y V & \longleftarrow & (X \times_Y V) \times_X (X \times_Y V') & \longrightarrow & X \times_Y V' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V \times_Y V' & \longrightarrow & V' \end{array}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 35.22.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\mathrm{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. □

Note that "being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 35.23.34. Hence the following definition now makes sense.

06BN **Definition 76.44.2.** Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a *Koszul-regular immersion* if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is a Koszul-regular immersion".
(2) We say i is an *H_1-regular immersion* if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is an H_1-regular immersion".
(3) We say i is a *quasi-regular immersion* if i is representable and the equivalent conditions of Lemma 76.44.1 hold with $\mathcal{P}(f) =$"f is a quasi-regular immersion".

06BP **Lemma 76.44.3.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 31.21.2. □

09RW **Lemma 76.44.4.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Assume X is locally Noetherian. Then i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Via Definition 76.44.2 (and the definition of a locally Noetherian algebraic space in Properties of Spaces, Section 66.7) this immediately translates to the case of schemes which is Divisors, Lemma 31.21.3. □

09RX **Lemma 76.44.5.** *Let S be a scheme. Let $i : Z \to X$ be a Koszul-regular, H_1-regular, or quasi-regular immersion of algebraic spaces over S. Let $X' \to X$ be a flat morphism of algebraic spaces over S. Then the base change $i' : Z \times_X X' \to X'$ is a Koszul-regular, H_1-regular, or quasi-regular immersion.*

Proof. Via Definition 76.44.2 (and the definition of a flat morphism of algebraic spaces in Morphisms of Spaces, Section 67.30) this lemma reduces to the case of schemes, see Divisors, Lemma 31.21.4. □

09RY **Lemma 76.44.6.** *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (76.6.1.2) is an isomorphism.*

Proof. Follows from the case of schemes (Divisors, Lemma 31.21.5) via étale localization (use Definition 76.44.2 and Lemma 76.6.2). □

09RZ **Lemma 76.44.7.** *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces over S. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Lemma 76.5.6*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and (étale) locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 76.44.6 and Lemma 76.44.3) it suffices to prove that the sequence is exact. It suffices to show that the first map is injective as the sequence is already right exact in general. After étale localization on X this reduces to the case of schemes, see Divisors, Lemma 31.21.6. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.69.8. The other types of regular immersions are preserved under composition.

09S0 **Lemma 76.44.8.** *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S.*

(1) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(2) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(3) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.7. □

09S1 **Lemma 76.44.9.** *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 76.5.6 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.8. □

09S2 **Lemma 76.44.10.** *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume X is locally Noetherian. The following are equivalent*

(1) *i and j are Koszul regular immersions,*

(2) *i and $j \circ i$ are Koszul regular immersions,*
(3) *$j \circ i$ is a Koszul regular immersion and the conormal sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Immediate from the case of schemes, see Divisors, Lemma 31.21.9. □

76.45. Relative pseudo-coherence

0CSV This section is the analogue of More on Morphisms, Section 37.59. However, in the treatment of this material for algebraic spaces we have decided to work exclusively with objects in the derived category whose cohomology sheaves are quasi-coherent. There are two reasons for this: (1) it greatly simplifies the exposition and (2) we currently have no use for the more general notion.

0CSW **Remark 76.45.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of representable algebraic spaces over S which is locally of finite type. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then f_0 is locally of finite type. If E is an object of $D_{QCoh}(\mathcal{O}_X)$, then E is the pullback of a unique object E_0 in $D_{QCoh}(\mathcal{O}_{X_0})$, see Derived Categories of Spaces, Lemma 75.4.2. In this situation the phrase "E is m-pseudo-coherent relative to Y" will be taken to mean "E_0 is m-pseudo-coherent relative to Y_0" as defined in More on Morphisms, Section 37.59.

0CSX **Lemma 76.45.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $m \in \mathbf{Z}$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. With notation as explained in Remark 76.45.1 the following are equivalent:*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is m-pseudo-coherent relative to V,
(2) *for some commutative diagram as in (1) with $U \to X$ surjective, the complex $E|_U$ is m-pseudo-coherent relative to V,*
(3) *for every commutative diagram as in (1) with U and V affine the complex $R\Gamma(U, E)$ of $\mathcal{O}_X(U)$-modules is m-pseudo-coherent relative to $\mathcal{O}_Y(V)$.*

Proof. Part (1) implies (3) by More on Morphisms, Lemma 37.59.7.

Assume (3). Pick any commutative diagram as in (1) with $U \to X$ surjective. Choose an affine open covering $V = \bigcup V_j$ and affine open coverings $(U \to V)^{-1}(V_j) = \bigcup U_{ij}$. By (3) and More on Morphisms, Lemma 37.59.7 we see that $E|_U$ is m-pseudo-coherent relative to V. Thus (3) implies (2).

Assume (2). Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, the morphism $U \to X$ is surjective, and $E|_U$ is m-pseudo-coherent relative to V. Next, suppose given a second commutative diagram

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale vertical arrows and U', V' schemes. We want to show that $E|_{U'}$ is m-pseudo-coherent relative to V'. The morphism $U'' = U \times_X U' \to U'$ is surjective étale and $U'' \to V'$ factors through $V'' = V' \times_Y V$ which is étale over V'. Hence it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V'', see More on Morphisms, Lemmas 37.70.1 and 37.70.2. Using the second lemma once more it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V. This is true by More on Morphisms, Lemma 37.59.16 and the fact that an étale morphism of schemes is pseudo-coherent by More on Morphisms, Lemma 37.60.6. □

0CSY **Definition 76.45.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is *m-pseudo-coherent relative to Y* if the equivalent conditions of Lemma 76.45.2 are satisfied.
(2) We say E is *pseudo-coherent relative to Y* if E is m-pseudo-coherent relative to Y for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is *m-pseudo-coherent relative to Y* if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is m-pseudo-coherent relative to Y.
(4) We say $\mathcal{F}$ is *pseudo-coherent relative to Y* if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is pseudo-coherent relative to Y.

Most of the properties of pseudo-coherent complexes relative to a base will follow immediately from the corresponding properties in the case of schemes. We will add the relevant lemmas here as needed.

0DII **Lemma 76.45.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let E in $D_{QCoh}(\mathcal{O}_X)$. If f is flat and locally of finite presentation, then the following are equivalent*

(1) *E is pseudo-coherent relative to Y, and*
(2) *E is pseudo-coherent on X.*

Proof. By étale localization and the definitions we may assume X and Y are schemes. For the case of schemes this follows from More on Morphisms, Lemma 37.59.18. □

76.46. Pseudo-coherent morphisms

06BQ This section is the analogue of More on Morphisms, Section 37.60 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property "pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.60.10 and 37.60.13 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1

we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 37.60 when the algebraic spaces in question are representable.

06BR **Definition 76.46.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *pseudo-coherent* if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="pseudo-coherent".
(2) Let $x \in |X|$. We say f is *pseudo-coherent at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

06BS **Lemma 76.46.2.** *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.3. □

06BT **Lemma 76.46.3.** *A composition of pseudo-coherent morphisms is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.4. □

06BU **Lemma 76.46.4.** *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

06BV **Lemma 76.46.5.** *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.6. □

06BW **Lemma 76.46.6.** *Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.7. □

06BX **Lemma 76.46.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.60.9. □

76.47. Perfect morphisms

06BY This section is the analogue of More on Morphisms, Section 37.61 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property "perfect" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.61.10 and 37.61.14 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1 we may define

the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 37.61 when the algebraic spaces in question are representable.

06BZ **Definition 76.47.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *perfect* if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="perfect".
(2) Let $x \in |X|$. We say f is *perfect at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

06C0 **Lemma 76.47.2.** *A flat base change of a perfect morphism is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.3. □

06C1 **Lemma 76.47.3.** *A composition of perfect morphisms is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.4. □

06C2 **Lemma 76.47.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.61.5. □

0E4U **Lemma 76.47.5.** *Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a perfect proper morphism of algebraic spaces. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.*

Proof. We claim that Derived Categories of Spaces, Lemma 75.22.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and Y affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_Y$-modules on the étale site. By Derived Categories of Spaces, Lemma 75.13.4 it suffices to check this on the Zariski site. This is equivalent to being perfect for finite type morphisms of schemes by More on Morphisms, Lemma 37.61.11. □

76.48. Local complete intersection morphisms

06C3 This section is the analogue of More on Morphisms, Section 37.62 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

The property "being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 37.62.19 and 37.62.20 and Descent, Lemma 35.32.6. By Morphisms of Spaces, Lemma 67.22.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already

existing notion defined in More on Morphisms, Section 37.62 when the algebraic spaces in question are representable.

06C4 **Definition 76.48.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}(f) =$"f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is *Koszul at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

06C5 **Lemma 76.48.2.** *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y. Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[i]{} & P \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 76.44.2. By Definition 76.48.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 37.62.3. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

06C6 **Lemma 76.48.3.** *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.4. □

Beware that a base change of a Koszul morphism is not Koszul in general.

06C7 **Lemma 76.48.4.** *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.6. □

06C8 **Lemma 76.48.5.** *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.7. □

06C9 **Lemma 76.48.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 37.62.8. □

0CHK **Lemma 76.48.7.** *Let S be a scheme. A Koszul-regular immersion of algebraic spaces over S is a local complete intersection morphism.*

Proof. Let $i : X \to Y$ be a Koszul-regular immersion of algebraic spaces over S. By definition there exists a surjective étale morphism $V \to Y$ where V is a scheme such that $X \times_Y V$ is a scheme and the base change $X \times_Y V \to V$ is a Koszul-regular immersion of schemes. By More on Morphisms, Lemma 37.62.9 we see that $X \times_Y V \to V$ is a local complete intersection morphism. From Definition 76.48.1 we conclude that i is a local complete intersection morphism of algebraic spaces. □

0CHL **Lemma 76.48.8.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume $Y \to Z$ is smooth and $X \to Z$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 37.62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. □

0CHM **Lemma 76.48.9.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is fpqc local on the base.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}$ be an fpqc covering (Topologies on Spaces, Definition 73.9.1). Let $f_i : X_i \to Y_i$ be the base change of f by $Y_i \to Y$. If f is a local complete intersection morphism, then each f_i is a local complete intersection morphism by Lemma 76.48.4.

Conversely, assume each f_i is a local complete intersection morphism. We may replace the covering by a refinement (again because flat base change preserves the property of being a local complete intersection morphism). Hence we may assume Y_i is a scheme for each i, see Topologies on Spaces, Lemma 73.9.5. Choose a scheme

V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. We have to show that $U \to V$ is a local complete intersection morphism of schemes. By Topologies on Spaces, Lemma 73.9.4 we have that $\{Y_i \times_Y V \to V\}$ is an fpqc covering of schemes. By the case of schemes (More on Morphisms, Lemma 37.62.19) it suffices to prove the base change

$$U \times_Y Y_i = U \times_V (V \times_Y Y_i) \longrightarrow V$$

of $U \to V$ by $V \times_Y Y_i \to V$ is a local complete intersection morphism. We can write this as the composition

$$U \times_Y Y_i \longrightarrow (V \times_Y X) \times_Y Y_i = V \times_Y X_i \longrightarrow V \times_Y Y_i$$

The first arrow is an étale morphism of schemes (as a base change of $U \to V \times_Y X$) and the second arrow is a local complete intersection morphism of schemes as a flat base change of f_i. The result follows as being a local complete intersection morphism is syntomic local on the source and since étale morphisms are syntomic (More on Morphisms, Lemma 37.62.20 and Morphisms, Lemma 29.36.10). □

0CHN **Lemma 76.48.10.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.*

Proof. This follows from Descent on Spaces, Lemma 74.14.3 and More on Morphisms, Lemma 37.62.20. □

06CA **Lemma 76.48.11.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 37.62.21 and in fact we will deduce the lemma from it. By Definition 76.48.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 66.4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 66.16.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of

the set T' of points in U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 37.62.21 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

06CB **Lemma 76.48.12.** *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 37.62.22, by étale localization, see Lemma 76.15.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) □

06CC **Lemma 76.48.13.** *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 76.5.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 76.15.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 37.62.23) as by definition the morphism $W \to V$ is a local complete intersection morphism. □

76.49. When is a morphism an isomorphism?

05X7 More generally we can ask: "When does a morphism have property $\mathcal{P}$?" A more precise question is the following. Suppose given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f) =$"f is flat" which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter "More on Flatness", starting with More on Flatness, Section 38.20.

05X8 **Lemma 76.49.1.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 67.38.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p|\left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 66.4.9) that W has the desired universal property. □

05X9 **Lemma 76.49.2.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is closed, and*
(3) *$p_2 : X \times_Y X \to Z$ is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 76.49.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 67.38.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 67.19.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2|\left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|)\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 66.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 66.4.3 in the commutative diagram

$$\begin{array}{ccc} |X_{Z'}| & \longrightarrow & |(X \times_Y X)_{Z'}| \\ \downarrow & & \downarrow \\ |p|^{-1}(\{z\}) & \longrightarrow & |p_2|^{-1}(\{z\}) \end{array}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

05XA **Lemma 76.49.3.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is universally closed, and*
(3) *$q : Y \to Z$ is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see Lemma 76.14.9. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 67.40.6. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 67.9.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 76.49.2 are satisfied. The projection $\mathrm{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 67.9.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 67.9.4). This finishes the proof of the lemma. □

05XB **Lemma 76.49.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) p *is locally of finite presentation,*
(2) p *is flat,*
(3) p *is closed, and*
(4) q *is locally of finite type.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By Lemma 76.23.6 the set

$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| (|X| \setminus A)$$

i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. □

05XC **Lemma 76.49.5.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) p *is locally of finite presentation,*
(2) p *is flat,*
(3) p *is closed,*
(4) q *is locally of finite type, and*
(5) q *is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 76.49.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 67.28.9. Hence f is open, see Morphisms of Spaces, Lemma 67.30.6. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |q| (|Y| \setminus |f|(|X|)) .$$

in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 66.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma

66.4.3 in the commutative diagram

$$\begin{array}{ccc} |X_{Z'}| & \longrightarrow & |Y_{Z'}| \\ \downarrow & & \downarrow \\ |p|^{-1}(\{z\}) & \longrightarrow & |q|^{-1}(\{z\}) \end{array}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

05XD **Lemma 76.49.6.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is universally closed,*
(4) *q is locally of finite type,*
(5) *q is closed, and*
(6) *q is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 76.49.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 76.49.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 29.26.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) □

06CE **Lemma 76.49.7.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is flat and locally of finite presentation,*
(2) *p is closed, and*
(3) *q is flat and locally of finite presentation,*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By Lemma 76.48.11 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 66.4.8) with underlying set of points

$$|W| = |Z| \setminus |p|\left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 66.4.9) that W has the desired universal property. □

76.50. Exact sequences of differentials and conormal sheaves

06CD In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 76.7.6 or Lemma 76.15.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Lemma 76.7.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.12.

(2) If g is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 76.15.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.13.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$

see Lemma 76.15.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 76.19.14.

(4) if g and f are formally unramified, then there is a canonical exact sequence

$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$

see Lemma 76.15.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 76.48.13.

76.51. Characterizing pseudo-coherent complexes, II

0CTP In this section we discuss a characterization of pseudo-coherent complexes in terms of cohomology. Earlier material on pseudo-coherent complexes on algebraic spaces may be found in Derived Categories of Spaces, Section 75.13 and in Derived Categories of Spaces, Section 75.18. The analogue of this section for schemes is More on Morphisms, Section 37.69. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 76.51.2.

0CTQ **Lemma 76.51.1.** *Let S be a scheme. Consider a commutative diagram of algebraic spaces*

$$\begin{array}{ccc} Z' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X' & \longrightarrow & B' \end{array}$$

over S. Let $B \to B'$ be a morphism. Denote by X and Y the base changes of X' and Y' to B. Assume $Y' \to B'$ and $Z' \to X'$ are flat. Then $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$.

Proof. By Derived Categories of Spaces, Lemma 75.20.3 we may check tor independence étale locally on $X \times_B Y$ and Z'. This[3] reduces the lemma to the case of schemes which is More on Morphisms, Lemma 37.69.1. □

0CTR **Lemma 76.51.2** (Derived Chow's lemma)**.** *Let A be a ring. Let X be a separated algebraic space of finite presentation over A. Let $x \in |X|$. Then there exist an $n \geq 0$, a closed subspace $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in |Z|$, an open $V \subset \mathbf{P}^n_A$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that*

(1) *$Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,*
(2) *$c : Z \to \mathbf{P}^n_A$ is a closed immersion over V, set $W = c^{-1}(V)$,*
(3) *the restriction of $b : Z \to X$ to W is étale, $z \in W$, and $b(z) = x$,*
(4) *$E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,*
(5) *E is pseudo-coherent and supported on Z.*

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and an algebraic space X' separated and of finite presentation over A' whose base change to A is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Let $x' \in |X'|$ be the image of x. If we can prove the lemma for $(X'/A', x')$, then the lemma follows for $(X/A, x)$. Namely, if n', Z', z', V', E' provide the solution for $(X'/A', x')$, then we can let $n = n'$, let $Z \subset X \times \mathbf{P}^n$ be the inverse image of Z', let $z \in Z$ be the unique point mapping to x, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology on Sites, Lemma 21.45.3. It only remains to check (5). To see this set $W = c^{-1}(V)$ and $W' = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{array}{ccc} W & \longrightarrow & W' \\ {\scriptstyle (b,c)}\downarrow & & \downarrow{\scriptstyle (b',c')} \\ X \times_A V & \longrightarrow & X' \times_{A'} V' \end{array}$$

[3]Here is the argument in more detail. Choose a surjective étale morphism $W' \to B'$ with W' a scheme. Choose a surjective étale morphism $W \to B \times_{B'} W'$ with W a scheme. Choose a surjective étale morphism $U' \to X' \times_{B'} W'$ with U' a scheme. Choose a surjective étale morphism $V' \to Y' \times_{B'} W'$ with V' a scheme. Observe that $U' \times_{W'} V' \to X' \times_{B'} Y'$ is surjective étale. Choose a surjective étale morphism $T' \to Z' \times_{X' \times_{B'} Y'} U' \times_{W'} V'$ with T' a scheme. Denote U and V the base changes of U' and V' to W. Then the lemma says that $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$ as algebraic spaces if and only if $U \times_W V$ and T' are Tor independent over $U' \times_{W'} V'$ as schemes. Thus it suffices to prove the lemma for the square with corners T', U', V', W' and base change by $W \to W'$. The flatness of $Y' \to B'$ and $Z' \to X'$ implies flatness of $V' \to W'$ and $T' \to U'$.

By Lemma 76.51.1 $X \times_A V$ and W' are tor-independent over $X' \times_{A'} V'$. Thus the derived pullback of $(b', c')_*\mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_*\mathcal{O}_W$ by Derived Categories of Spaces, Lemma 75.20.4. This also uses that $R(b', c')_*\mathcal{O}_{Z'} = (b', c')_*\mathcal{O}_{Z'}$ because (b', c') is a closed immersion and similarly for $(b, c)_*\mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_*\mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_*\mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an étale morphism $U \to X$ where U is an affine scheme and a point $u \in U$ mapping to x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Let $z \in Z$ be the image of u. Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by Morphisms of Spaces, Lemma 67.16.7. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (2) holds, that $W = (\mathrm{id}_U, j)(U)$, and hence that (3) holds. Part (1) holds because A is Noetherian.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian algebraic spaces. Hence we can take $E = (b, c)_*\mathcal{O}_Z$ in this case: (4) is clear and for (5) see Derived Categories of Spaces, Lemma 75.13.7. This finishes the proof. □

0CTS **Lemma 76.51.3.** *Let X/A, $x \in |X|$, and n, Z, z, V, E be as in Lemma 76.51.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have*

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(W \to V)_*K|_W$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $W \to V$ is the finitely presented closed immersion $c|_W : W \to V$.

Proof. Since $W = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c|_W$ is a closed immersion. It is of finite presentation because W and V are of finite presentation over A, see Morphisms of Spaces, Lemma 67.28.9. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Denote $i = (b, c)|_W : W \to X \times_A V$ the given closed immersion. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_*\mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_*\mathcal{O}_W = Ri_*\mathcal{O}_W$. Using Derived Categories of Spaces, Lemma 75.20.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(W \to V)_*K|_W$$

which is what we want. (Note that restricting to W and derived pulling back via $W \to X$ is the same thing as W is étale over X.) □

0CTT **Lemma 76.51.4.** *Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A (Definition 76.45.3).*

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in |X|$. We will show that K is pseudo-coherent relative to A in an étale neighbourhood of x. This will prove the lemma by our definition of relative pseudo-coherence.

Choose n, Z, z, V, E as in Lemma 76.51.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} & R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))) \end{aligned}$$

by Derived Categories of Spaces, Lemma 75.20.1. By Derived Categories of Spaces, Lemma 75.25.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 36.34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 76.51.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(W \to V)_* K|_W$$

Since $W \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_W$ is pseudo-coherent relative to A for example by More on Morphisms, Lemma 37.59.18. □

0GFJ **Lemma 76.51.5.** *Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. In view of Lemma 76.51.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 75.29.3 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Spaces, Lemma 75.25.7. □

76.52. Relatively perfect objects

0DKM In this section we introduce a notion from [**Lie06a**]. This notion has been discussed for morphisms of schemes in Derived Categories of Schemes, Section 36.35.

0DKN **Definition 76.52.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is *perfect relative to* Y or *Y-perfect* if E is pseudo-coherent (Cohomology on Sites, Definition 21.45.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$ (Cohomology on Sites, Definition 21.46.1).

Please see Derived Categories of Schemes, Remark 36.35.14 for a discussion; here we just mention that E being pseudo-coherent is the same thing as E being pseudo-coherent relative to Y by Lemma 76.45.4. Moreover, pseudo-coherence of E implies $E \in D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 75.13.6.

0DKP **Example 76.52.2.** Let k be a field. Let X be an algebraic space of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{Coh}(X)$ (by Derived Categories of Spaces, Lemma 75.13.7). Thus being relatively perfect does **not** mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 15.84. We can link the notion for algebraic spaces with the algebraic notion as follows.

0DKQ **Lemma 76.52.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent:*

(1) *E is Y-perfect,*
(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 36.35.1,
(3) *for some commutative diagram as in (2) with $U \to X$ surjective, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 36.35.1,*
(4) *for every commutative diagram as in (2) with U and V affine the complex $R\Gamma(U, E)$ is $\mathcal{O}_Y(V)$-perfect.*

Proof. To make sense of parts (2), (3), (4) of the lemma, observe that the object $E|_U$ of $D_{QCoh}(\mathcal{O}_U)$ corresponds to an object E_0 of $D_{QCoh}(\mathcal{O}_{U_0})$ where U_0 denotes the scheme underlying U, see Derived Categories of Spaces, Lemma 75.4.2. Moreover, in this case E_0 is pseudo-coherent if and only if $E|_U$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 75.13.2. Also, $E|_U$ locally has finite tor dimension over $f^{-1}\mathcal{O}_Y|_U = g^{-1}\mathcal{O}_V$ if and only if E_0 locally has finite tor dimension over $g_0^{-1}\mathcal{O}_{V_0}$ by Derived Categories of Spaces, Lemma 75.13.4. Here $g_0 : U_0 \to V_0$ is the morphism of schemes representing $g : U \to V$ (notation as in Derived Categories of Spaces, Remark 75.6.3). Finally, observe that "being pseudo-coherent" is étale local and of course "having locally finite tor dimension" is étale local. Thus we see that it suffices to check Y-perfectness étale locally and by the above discussion we see that (1) implies (2) and (3) implies (1). Since part (4) is equivalent to (2) and (3) by Derived Categories of Schemes, Lemma 36.35.3 the proof is complete. □

0DKR **Lemma 76.52.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of Y-perfect objects is a saturated[4] triangulated subcategory.*

Proof. This follows from Cohomology on Sites, Lemmas 21.45.4, 21.45.6, 21.46.6, and 21.46.8. □

[4]Derived Categories, Definition 13.6.1.

0DKS **Lemma 76.52.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is Y-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is Y-perfect if K is perfect and M is Y-perfect.*

Proof. Reduce to the case of schemes using Lemma 76.52.3 and then apply Derived Categories of Schemes, Lemma 36.35.5. □

0DKT **Lemma 76.52.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Set $X' = Y' \times_Y X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is Y-perfect, then $L(g')^*K$ is Y'-perfect.*

Proof. Reduce to the case of schemes using Lemma 76.52.3 and then apply Derived Categories of Schemes, Lemma 36.35.6. □

0DKU **Situation 76.52.7.** Let S be a scheme. Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $g_{i'i} : Y_{i'} \to Y_i$. We assume that Y_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : Y \to Y_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to Y_0$. We set $X_i = Y_i \times_{Y_0} X_0$ and $X = Y \times_{Y_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

0DKV **Lemma 76.52.8.** *In Situation 76.52.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to Y_0)^{-1}\mathcal{O}_{Y_0})$

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. Then P holds for W_0 by Mayer-Vietoris for hom in the derived category, see Derived Categories of Spaces, Lemma 75.10.4.

We first consider $U_0 = W_0 \times_{Y_0} X_0$ with W_0 a quasi-compact and quasi-separated object of $(Y_0)_{spaces,étale}$. By the induction principle of Derived Categories of Spaces, Lemma 75.9.3 applied to these W_0 and the previous paragraph, we find that it is enough to prove P for $U_0 = W_0 \times_{Y_0} X_0$ with W_0 affine. In other words, we have reduced to the case where Y_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and Y_0 are affine, then we are back in the case of schemes which is proved in Derived Categories of Schemes, Lemma 36.35.8. The reader may use Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, 75.13.2, and 75.13.4 to accomplish

the translation of the statement into a statement involving only schemes and derived categories of modules on schemes. □

0DKW **Lemma 76.52.9.** *In Situation 76.52.7 the category of Y-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of Y_i-perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\operatorname{colim}_{i\geq 0} D_{Y_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{Y\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We observe that we already know this functor is fully faithful by Lemma 76.52.8. Thus it suffices to prove essential surjectivity.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. We claim that P holds for W_0. We will use the notation $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0$, and similarly for V_0 and W_0. We will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \times_W V \to U_i \times_{W_i} V_i$, and $W \to W_i$. Suppose E is an Y-perfect object of $D(\mathcal{O}_W)$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an Y_i-perfect object $E_{U,i}$ on U_i, an Y_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^* E_{U,i} \to E|_U$ and $Lf_i^* E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^* E_{U,i} \to E|_U$ and $Lf_i^* E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{W_i} V_i} \to E_{V,i}|_{U_i \times_{W_i} V_i}$ which pulls back to the identifications

$$Lf_i^* E_{U,i}|_{U\times_W V} \to E|_{U\times_W V} \to Lf_i^* E_{V,i}|_{U\times_W V}.$$

Apply Derived Categories of Spaces, Lemma 75.10.8 to get an object E_i on W_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is Y_i-perfect (because being relatively perfect is an étale local property) and that d is adjoint to an isomorphism $Lf_i^* E_i \to E$.

By exactly the same argument as used in the proof of Lemma 76.52.8 using the induction principle (Derived Categories of Spaces, Lemma 75.9.3) we reduce to the case where both X_0 and Y_0 are affine: first work with quasi-compact and quasi-separated objects in $(Y_0)_{spaces,étale}$ to reduce to Y_0 affine, then work with quasi-compact and quasi-separated object in $(X_0)_{spaces,étale}$ to reduce to X_0 affine. In the affine case the result follows from the case of schemes which is Derived Categories of Schemes, Lemma 36.35.9. The translation into the case for schemes is done by Lemma 76.52.3. □

0DKX **Lemma 76.52.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be Y-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.*

Proof. The statement on base change is Derived Categories of Spaces, Lemma 75.21.4 (with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0). Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where Y is Noetherian affine by a limit argument.

The question is étale local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 70.7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemmas 70.6.13 and 70.6.12 we may assume X_i is proper and flat over R_i. By Lemma 76.52.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Derived Categories of Spaces, Lemma 75.22.1 to $X_i \to \mathrm{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. □

0DKY **Lemma 76.52.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E, K \in D(\mathcal{O}_X)$. Assume*

(1) *Y is quasi-compact and quasi-separated,*
(2) *f is proper, flat, and of finite presentation,*
(3) *E is Y-perfect,*
(4) *K is pseudo-coherent.*

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_Y)$ such that

$$Rf_*R\,\mathcal{H}om(K, E) = R\,\mathcal{H}om(L, \mathcal{O}_Y)$$

and the same is true after arbitrary base change: given

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

cartesian, then we have

$$Rf'_*R\,\mathcal{H}om(L(g')^*K, L(g')^*E) = R\,\mathcal{H}om(Lg^*L, \mathcal{O}_{Y'})$$

Proof. Since Y is quasi-compact and quasi-separated, the same is true for X. By Derived Categories of Spaces, Lemma 75.18.1 we can write $K = \mathrm{hocolim} K_n$ with K_n perfect and $K_n \to K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology on Sites, Lemma 21.48.4). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 76.52.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is Y-perfect. Thus we may apply Lemma 76.52.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on Y with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_*R\,\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^*M_n = Rf'_*((L(g')^*K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*E) = Rf'_*R\,\mathcal{H}om(L(g')^*K_n, L(g')^*E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\,\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Derived Categories of Spaces, Lemma 75.20.7. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Pick $y \in |Y|$. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$; this is possible by Decent Spaces, Lemma 68.11.4. Take Y' equal to the spectrum of the residue field at u. Pull back to see that $C_n|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \kappa(u)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 15.76.7 we see that the perfect complex $C_n|_U$ has tor amplitude in $[n + a, m_n]$ for some integer m_n and after possibly shrinking U. Thus C_n has tor amplitude in $[n + a, m_n]$ for some integer m_n (because Y is quasi-compact). In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n - a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 75.18.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology on Sites, Lemma 21.48.8) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_Y) = R\mathcal{H}om(L, \mathcal{O}_Y)$$

This proves the formula over Y. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. □

0DKZ **Remark 76.52.12.** The reader may have noticed the similarity between Lemma 76.52.11 and Derived Categories of Spaces, Lemma 75.23.3. Indeed, the pseudo-coherent complex L of Lemma 76.52.11 may be characterized as the unique pseudo-coherent complex on Y such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_Y}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_Y)$. If we ever need this we will formulate a precise result here and give a detailed proof.

0GFK **Lemma 76.52.13.** *Let S be a scheme. Let X be an algebraic space over S such that the structure morphism $f : X \to S$ is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is S-perfect, and*
2. *E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.*

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 76.52.3. This case is handled by Derived Categories of Schemes, Lemma 36.35.13. □

0GFL **Lemma 76.52.14.** *Let A be a ring. Let X be an algebraic space separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is* $\text{Spec}(A)$*-perfect.*

Proof. By Lemma 76.51.5, K is pseudo-coherent relative to A. By Lemma 76.45.4, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 75.29.4 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Lemma 76.52.13, we will be done if we can show $Li^*(K)$ is bounded

below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Theorem 75.15.4. We have

$$\begin{aligned} R\operatorname{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) &= R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ &= R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology on Sites, Lemma 21.48.4; we omit some details. The second equality follows from Derived Categories of Spaces, Lemma 75.20.4 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Spaces, Remark 75.15.5. Hence Derived Categories of Spaces, Proposition 75.29.4 now implies that $Li^*(K)$ is bounded below and we win. □

76.53. Theorem of the cube

0D22 This section is the analogue of More on Morphisms, Section 37.33. The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

0D23 **Lemma 76.53.1.** *Let S be a scheme. Let $f : X \to Y$ be a flat, proper morphism of finite presentation of algebraic spaces over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : Y' \to Y$ consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Assume $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is an isomorphism for all $g : Y' \to Y$. Then there exists an immersion $j : Z \to Y$ of finite presentation such that a morphism $g : Y' \to Y$ factors through Z if and only if there exists a finite locally free $\mathcal{O}_{Y'}$-module $\mathcal{N}$ with $(f')^*\mathcal{N} \cong (g')^*\mathcal{L}$.

Proof. Let $y : \operatorname{Spec}(k) \to Y$ be a field valued point. Then the fibre X_y of f at y is connected by our assumption that $H^0(X_y, \mathcal{O}_{X_y}) = k$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms of Spaces, Lemma 67.30.6) and closed we conclude that there is a decomposition $Y = \coprod Y_r$ of Y into open and closed subspaces such that $\mathcal{E}$ has constant rank r on the inverse image of Y_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Spaces, Lemma 75.25.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of Y and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : Y' \to Y$ we have $f'_*((g')^*\mathcal{E}) = H^0(Lg^*E)$ and $f'_*((g')^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to Y$ and $j' : Z' \to Y$ be the locally closed immersions constructed in Derived Categories of Spaces, Lemma 75.26.6 for E and E' with $a = 0$ and $r = r$; these are characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are locally free modules of rank r compatible with pullback.

Let $g : Y' \to Y$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology on Sites, Lemma 21.50.1, we see that the modules

$$f'_*((g')^*\mathcal{E}) \cong f'_*((f')^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_{Y'}} f'_*\mathcal{O}_{X'} \cong \mathcal{N} \quad \text{and similarly} \quad f'_*((g')^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are locally free of rank r and remain locally free of rank r after any further base change $Y'' \to Y'$. Hence in this case $g : Y' \to Y$ factors through j and through j'. Thus we may replace Y by $Z \times_Y Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are locally free $\mathcal{O}_Y$-modules of rank r whose formation commutes with arbitrary change of base.

In this situation if $g : Y' \to Y$ is a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$f'_*((g')^*\mathcal{E}) \otimes_{\mathcal{O}_{Y'}} f'_*((g')^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_{Y'}$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on Y' we have a basis of sections $\sigma_1, \ldots, \sigma_r$ in $f'_*((g')^*\mathcal{L})$ and $\tau_1, \ldots, \tau_r$ in $f'_*((g')^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i\tau_j = \delta_{ij}$. Thinking of σ_i as a section of $(g')^*\mathcal{L}$ on X' and τ_j as a section of $(g')^*\mathcal{L}^\vee$ on X', we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X'}^{\oplus r} \longrightarrow (g')^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : (g')^*\mathcal{E} \longrightarrow \mathcal{O}_{X'}^{\oplus r}$$

In other words, we see that $(f')^*f'_*(g')^*\mathcal{E} \cong (g')^*\mathcal{E}$. But the condition that the cup product is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. □

76.54. Descent of finiteness properties of complexes

0DL0 This section is the analogue of More on Morphisms, Section 37.70 and Derived Categories of Schemes, Section 36.12.

0DL1 **Lemma 76.54.1.** *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fpqc covering of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology on Sites, Lemma 21.45.3. Thus it suffices to assume Lf_i^*E is m-pseudo-coherent and to prove that E is m-pseudo-coherent. Then first we may assume X_i is a scheme for all i, see Topologies on Spaces, Lemma 73.9.5. Next, choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_i = U \times_X X_i$ is a scheme and we obtain an fpqc covering $\{U_i \to U\}$ of schemes, see Topologies on Spaces, Lemma 73.9.4. We know the result is true for $\{U_i \to U\}_{i \in I}$ by the case for schemes, see Derived Categories of Schemes, Lemma 36.12.2. On the other hand, the restriction $E|_U$ comes from an object of $D_{QCoh}(\mathcal{O}_U)$ (defined using the Zariski topology and the "usual" structure sheaf of U), see Derived Categories of Spaces, Lemma 75.4.2. The lemma follows as the two notions of pseudo-coherent (étale and Zariski) agree by Derived Categories of Spaces, Lemma 75.13.2. □

0DL2 **Lemma 76.54.2.** *Let S be a scheme. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces and set $X_i = Y_i \times_Y X$ with projections $f_i : X_i \to Y_i$ and $g'_i : X_i \to X$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$. Then the following are equivalent*

(1) *E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_Y)$, and*
(2) *$L(g_i')^*E$ has tor amplitude in $[a, b]$ as a object of $D(f_i^{-1}\mathcal{O}_{Y_i})$ for all i.*

Also true if "tor amplitude in $[a, b]$" is replaced by "locally finite tor dimension".

Proof. Pullback preserves "tor amplitude in $[a, b]$" by Derived Categories of Spaces, Lemma 75.20.7 Observe that Y_i and X are tor independent over Y as $Y_i \to Y$ is flat. Let us assume (2) and prove (1). We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12. Let $\overline{x}$ be a geometric point of X. Choose an i and a geometric point $\overline{x}_i$ in X_i with image $\overline{x}$ in X. Then

$$(L(g_i')^*E)_{\overline{x}_i} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X,\overline{x}_i}$$

Let $\overline{y}_i$ in Y_i and $\overline{y}$ in Y be the image of $\overline{x}_i$ and $\overline{x}$. Since X and Y_i are tor independent over Y, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y_i,\overline{y}_i}$ in $D(\mathcal{O}_{Y_i,\overline{y}_i})$. Since we have assume the tor amplitude of this is in $[a, b]$, we conclude that the tor amplitude of $E_{\overline{x}}$ in $D(\mathcal{O}_{Y,\overline{y}})$ is in $[a, b]$ by More on Algebra, Lemma 15.67.17. Thus (1) follows.

Using some elementary topology the case "locally finite tor dimension" follows too. □

The following lemmas do not really belong in this section.

0DL3 **Lemma 76.54.3.** *Let S be a scheme. Let $i : X \to X'$ be a finite order thickening of algebraic spaces. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.*

Proof. We first prove K' has quasi-coherent cohomology sheaves; we urge the reader to skip this part. To do this, we may reduce to the case of a first order thickening, see Section 76.9. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology of Spaces, Lemma 69.4.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Spaces, Lemmas 75.13.6, 75.5.6, and 75.6.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is étale local on X' hence we may assume X' is affine. In this case the result follows from the case of schemes (More on Morphisms, Lemma 37.71.1). The translation into the language of schemes uses Derived Categories of Spaces, Lemmas 75.4.2 and 75.13.2 and Remark 75.6.3. □

0DL4 **Lemma 76.54.4.** *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y' \end{array}$$

of algebraic spaces over S. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Definition 76.52.1). By Lemma 76.54.3 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Spaces, Lemma 75.13.6. By Lemma 76.52.3 this reduces us to the case of schemes. The case of schemes is More on Morphisms, Lemma 37.71.2. □

0DL5 **Lemma 76.54.5.** *Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$. Let X be an algebraic space over R whose structure morphism $f : X \to S$ is proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.*

Proof. Choose a surjective étale morphism $U \to X$ with U affine. Choose a closed immersion $U \to \mathbf{A}^d_S$. Set $U_0 = S_0 \times_S U$. The complex $E_0|_{U_0}$ has tor amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $\overline{x}$ be a geometric point of X. We will show that the tor amplitude of $E_{\overline{x}}$ over R is in $[a - d, b]$. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology on Sites, Lemma 21.46.10 and Properties of Spaces, Theorem 66.19.12.

Let $x \in |X|$ be the point determined by $\overline{x}$. Recall that $|X| \to |S|$ is closed (by definition of proper morphisms). Since I is contained in the Jacobson radical, any nonempty closed subset of S contains a point of the closed subscheme S_0. Hence we can find a specialization $x \rightsquigarrow x_0$ in $|X|$ with $x_0 \in |X_0|$. Choose $u_0 \in U_0$ mapping to x_0. By Decent Spaces, Lemma 68.7.4 (or by Decent Spaces, Lemma 68.7.3 which applies directly to étale morphisms) we find a specialization $u \rightsquigarrow u_0$ in U such that u maps to x. We may lift $\overline{x}$ to a geometric point $\overline{u}$ of U lying over u. Then we have $E_{\overline{x}} = (E|_U)_{\overline{u}}$.

Write $U = \mathrm{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d-variables. The restriction $E|_U$ corresponds (by Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 and Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes^{\mathbf{L}}_A A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $u \rightsquigarrow u_0$. Then

$$E_{\overline{x}} = (E|_U)_{\overline{u}} = E_u \otimes^{\mathbf{L}}_{\mathcal{O}_{U,u}} \mathcal{O}_{U,\overline{u}} = K_{\mathfrak{q}} \otimes^{\mathbf{L}}_{A_{\mathfrak{q}}} A^{sh}_{\mathfrak{q}}$$

(some details omitted). Since $A_{\mathfrak{q}} \to A^{sh}_{\mathfrak{q}}$ is flat, the tor amplitude of this as an R-module is the same as the tor amplitude of $K_{\mathfrak{q}}$ as an R-module (More on Algebra, Lemma 15.67.18). Also, $K_{\mathfrak{q}}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a - d, b]$ as a complex of R-modules.

Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}_0) &= (K \otimes^{\mathbf{L}}_R R/I) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \\ &= (K \otimes^{\mathbf{L}}_A A/IA) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a, b this complex has cohomology only in degrees in the interval $[a, b]$. Thus we may finally apply More on Algebra, Lemma 15.84.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. □

76.55. Families of nodal curves

0DSD This section is the continuation of Algebraic Curves, Section 53.20. Please also see that section for our choice of terminology.

The property "at-worst-nodal of relative dimension 1" of morphisms of schemes is étale local on the source-and-target, see Descent, Lemma 35.32.6 and Algebraic Curves, Lemmas 53.20.8, 53.20.9, and 53.20.7. It is also stable under base change and fpqc local on the target, see Algebraic Curves, Lemmas 53.20.4 and 53.20.9. Hence, by Morphisms of Spaces, Lemma 67.22.1 we may define the notion of an at-worst-nodal morphism of relative dimension 1 for algebraic spaces as follows and it agrees with the already existing notion defined in Morphisms of Spaces, Section 67.3 when the morphism is representable.

0DSE **Definition 76.55.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *at-worst-nodal of relative dimension* 1 if the equivalent conditions of Morphisms of Spaces, Lemma 67.22.1 hold with $\mathcal{P}$ ="at-worst-nodal of relative dimension 1".

0DSF **Lemma 76.55.2.** *The property of being at-worst-nodal of relative dimension* 1 *is preserved under base change.*

Proof. See Morphisms of Spaces, Remark 67.22.4 and Algebraic Curves, Lemma 53.20.4. □

0DSG **Lemma 76.55.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is at-worst-nodal of relative dimension* 1,
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension* 1,
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension* 1,
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is at-worst-nodal of relative dimension* 1,
5. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is at-worst-nodal of relative dimension* 1,
6. *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is at-worst-nodal of relative dimension 1,

(7) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is at-worst-nodal of relative dimension 1, and

(8) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is at-worst-nodal of relative dimension 1.*

Proof. Omitted. □

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

0DSH **Lemma 76.55.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Then there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with arbitrary base change.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective. By the lemma for the case of schemes (Algebraic Curves, Lemma 53.20.5) we find a maximal open subscheme $U' \subset U$ such that $h|_{U'} : U' \to V$ is at-worst-nodal of relative dimension 1 and such that formation of U' commutes with base change. Let $X' \subset X$ be the open subspace whose points correspond to the open subset $\mathrm{Im}(|U'| \to |X|)$. By Lemma 76.55.3 we see that $X' \to Y$ is at-worst-nodal of relative dimension 1 and that X' is the largest open subspace with this property (this also implies that U' is the inverse image of X' in U, but we do not need this). Since the same is true after base change the proof is complete. □

76.56. The resolution property

0GUX We continue the discussion in Derived Categories of Spaces, Section 75.28.

0GUY **Situation 76.56.1.** Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $V \to X$ be a surjective étale morphism where V is an affine scheme (such a thing exists by Properties of Spaces, Lemma 66.6.3). Choose a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

where j is an open immersion and π is a finite morphism of algebraic spaces (such a diagram exists by Lemma 76.34.3). Let $\mathcal{I} \subset \mathcal{O}_Y$ be a finite type quasi-coherent

sheaf of ideals on Y with $V(\mathcal{I}) = Y \setminus j(V)$ (such a sheaf of ideals exists by Limits of Spaces, Lemma 70.14.1).

0GUZ **Lemma 76.56.2.** *In Situation 76.56.1, assume X is Noetherian. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exist $r \geq 0$, integers $n_1, \ldots, n_r \geq 0$, and a surjection*

$$\bigoplus\nolimits_{i=1,\ldots,r} \pi_*(\mathcal{I}^{n_i}) \longrightarrow \mathcal{F}$$

of $\mathcal{O}_X$-modules.

Proof. Denote $\omega_{Y/X}$ the coherent $\mathcal{O}_Y$-module such that there is an isomorphism

$$\pi_*\omega_{Y/X} \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X)$$

of $\pi_*\mathcal{O}_Y$-modules, see Morphisms of Spaces, Lemma 67.20.10 and Descent on Spaces, Lemma 74.6.6. The canonical map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ produces a canonical map

$$\mathrm{Tr}_\pi : \pi_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

Since V is Noetherian affine we may choose sections

$$s_1, \ldots, s_r \in \Gamma(V, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X})$$

generating the coherent module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_{Y/X}$ over V. By Cohomology of Spaces, Lemma 69.13.4 we can choose integers $n_i \geq 0$ such that s_i extends to a map $s_i' : \mathcal{I}^{n_i} \to \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$. Pushing to X we obtain maps

$$\sigma_i : \pi_*\mathcal{I}^{n_i} \xrightarrow{\pi_* s_i'} \pi_*(\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{F} \otimes_{\mathcal{O}_X} \pi_*\omega_{Y/X} \xrightarrow{\mathrm{Tr}_\pi} \mathcal{F}$$

where the equality sign is Cohomology of Spaces, Lemma 69.4.3. To finish the proof we will show that the sum of these maps is surjective.

Let $x \in |X|$ be a point of X. Let $v \in |V|$ be a point mapping to x. We may choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$U \times_X Y = W \coprod W'$$

(disjoint union of algebraic spaces) such that $W \to U$ is an isomorphism and such that the unique point $w \in W$ lying over u maps to v in $V \subset Y$. To see this is true use Lemma 76.33.2 and Étale Morphisms, Lemma 41.18.1. After shrinking U further if necessary we may assume W maps into $V \subset Y$ by the projection. Since the formation of $\omega_{Y/X}$ commutes with étale localization we see that

$$\pi_*\omega_{Y/X}|_U = (\pi|_W)_*\omega_{W/U} \oplus (\pi|_{W'})_*\omega_{W'/U}$$

We have $(\pi|_W)_*\omega_{W/U} = \mathcal{O}_U$ and this isomorphism is given by the trace map $\mathrm{Tr}_\pi|_U$ restricted to the first summand in the decomposition above. Since W maps into V we see that $\mathcal{I}^{n_i}|_W = \mathcal{O}_W$. Hence

$$\pi_*(\mathcal{I}^{n_i})|_U = \mathcal{O}_U \oplus (W' \to U)_*(\mathcal{I}^{n_i}|_{W'})$$

Chasing diagrams the reader sees (details omitted) that $\sigma_i|_U$ on the summand $\mathcal{O}_U$ is the map $\mathcal{O}_U \to \mathcal{F}$ corresponding to the section

$$s_i|_W \in \Gamma(W, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \Gamma(W, \mathcal{F}|_W \otimes_{\mathcal{O}_W} \omega_{W/U}) = \Gamma(U, \mathcal{F})$$

Since the sections s_i generate the module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$ over V and since W maps into V we conclude that the restriction of $\bigoplus \sigma_i$ to U is surjective. Since x was an arbitrary point the proof is complete. □

0GV0 **Lemma 76.56.3.** *In Situation 76.56.1, assume X is Noetherian. Then X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.*

Proof. The module $\pi_*\mathcal{I}$ is coherent by Cohomology of Spaces, Lemma 69.12.9. Hence if X has the resolution property then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module. Conversely, assume given a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Observe that for all $n \geq 1$ there is a surjection

$$\pi_*\mathcal{I} \otimes_{\mathcal{O}_X} \pi_*\mathcal{I}^n \longrightarrow \pi_*\mathcal{I}^{n+1}$$

Hence $\mathcal{E}^{\otimes n}$ surjects onto $\pi_*\mathcal{I}^n$ for all $n \geq 1$. We conclude that X has the resolution property if we combine this with the result of Lemma 76.56.2. □

0GV1 **Lemma 76.56.4.** *In Situation 76.56.1, the algebraic space X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.*

Proof. The pushforward $\pi_*\mathcal{G}$ of a finite type quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_X$-module by Descent on Spaces, Lemma 74.6.6. In particular, if X has the resolution property, then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module by Derived Categories of Spaces, Definition 75.28.1.

Assume that we have a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. In the rest of the proof we show that X has the resolution property by reducing to the Noetherian case handled in Lemma 76.56.3. We suggest the reader skip the rest of the proof.

A first reduction is that we may view X as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2. (This doesn't affect the conditions nor the conclusion of the lemma.)

By Limits of Spaces, Lemma 70.11.3 we can write $Y = \lim Y_i$ with Y_i finite and of finite presentation over X and where the transition maps are closed immersions. Consider the closed subspace $Z = V(\mathcal{I})$ of Y. Since $\mathcal{I}$ is of finite type, the morphism $Z \to Y$ is of finite presentation. Hence we can find an i and a morphism $Z_i \to Y_i$ of finite presentation whose base change to Y is $Z \to Y$, see Limits of Spaces, Lemma 70.7.1. For $i' \geq i$ denote $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. After increasing i we may assume $Z_i \to Y_i$ is a closed immersion (of finite presentation), see Limits of Spaces, Lemma 70.6.8. Denote $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ the ideal sheaf of Z_i and denote $\pi_i : Y_i \to X$ the structure morphism. Similarly for $i' \geq i$. Since $Z = \lim_{i' \geq i} Z_{i'}$ we have

$$\pi_*\mathcal{I} = \mathrm{colim}\, \pi_{i',*}\mathcal{I}_{i'}$$

The transition maps in the system are all surjective as follows from the surjectivity of the maps $\pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i',*}\mathcal{O}_{Y_{i'}}$ and the fact that $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. By Cohomology of Spaces, Lemma 69.5.3 for some $i' \geq i$ the map $\mathcal{E} \to \pi_*\mathcal{I}$ lifts to a map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$. After increasing i' this map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$ becomes surjective (since if not the colimit of the cokernels, having surjective transition maps, is nonzero). This reduces us to the case discussed in the next paragraph.

Assume X is an algebraic space over $\mathbf{Z}$ and that $Y \to X$ is of finite presentation. By absolute Noetherian approximation we can write $X = \lim X_i$ as a directed limit, where each X_i is a quasi-separated algebraic space of finite type over $\mathbf{Z}$ and the transition morphisms are affine, see Limits of Spaces, Proposition 70.8.1. Since $\pi : Y \to X$ is of finite presentation we can find an i and a morphism $\pi_i : Y_i \to X_i$

of finite presentation whose base change to X is π, see Limits of Spaces, Lemma 70.7.1. After increasing i we may assume π_i is finite, see Limits of Spaces, Lemma 70.6.7. Next, we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is $\mathcal{E}$, see Limits of Spaces, Lemma 70.7.3. We may also assume there is a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose pullback to X is the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$, see Limits of Spaces, Lemma 70.7.2. The cokernel

$$\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i \to 0$$

is a coherent $\mathcal{O}_{Y_i}$-module whose pullback to X is the (finitely presented) cokernel $\mathcal{Q}$ of the map $\mathcal{E} \to \pi_*\mathcal{O}_Y$. In other words, we have $\mathcal{Q} = \pi_*(\mathcal{O}_Y/\mathcal{I})$. Consider the map

$$\mathcal{E}_i \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \longrightarrow \pi_{i,*}\mathcal{O}_{Y_i} \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i$$

where the second arrow is given by the algebra structure on $\pi_{i,*}\mathcal{O}_{Y_i}$. The pullback of this map to Y is zero because the image of $\mathcal{E} \to \pi_*\mathcal{O}_Y$ is the ideal $\pi_*\mathcal{I}$. Hence by Limits of Spaces, Lemma 70.7.2 after increasing i we may assume the displayed composition is zero. This exactly means that the imag of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ is of the form $\pi_{i,*}\mathcal{I}_i$ for some coherent ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$. Since $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ pulls back to $\mathcal{E} \to \pi_*\mathcal{O}_Y$ we see that the pullback of $\mathcal{I}_i$ to Y generates $\mathcal{I}$. Denote $V_i \subset Y_i$ the open subspace whose complement is $V(\mathcal{I}_i) \subset Y_i$. Then V is the inverse image of V_i by the comments above. After increasing i we may assume that V_i is affine and that $\pi_i|_{V_i} : V_i \to X_i$ is étale, see Limits of Spaces, Lemmas 70.5.10 and 70.6.2. Having said all of this, we may apply Lemma 76.56.3 to conclude that X_i has the resolution property. Since $X \to X_i$ is affine we conclude that X has the resolution property too by Derived Categories of Spaces, Lemma 75.28.3. □

0GV2 **Lemma 76.56.5.** *Let S be a scheme. Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated algebraic spaces over S with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.*

Proof. If X_i has the resolution property, then X does by Derived Categories of Spaces, Lemma 75.28.3. Assume X has the resolution property. Choose $i \in I$. We may choose an affine scheme V_i and a surjective étale morphism $V_i \to X_i$ (Properties of Spaces, Lemma 66.6.3). We may choose an embedding $j : V_i \to Y_i$ with Y_i finite and finitely presented over X_i (Lemma 76.34.4). We may choose a finite type quasi-coherent ideal $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ such that $V_i = Y_i \setminus V(\mathcal{I}_i)$ (Limits of Spaces, Lemma 70.14.1). Denote $V \to Y \to X$ the base changes of $V_i \to Y_i \to X_i$ to X. Denote $\mathcal{I} \subset \mathcal{O}_Y$ the pullback of the ideal $\mathcal{I}_i$. By the easy direction of Lemma 76.56.4 there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \pi_*\mathcal{I}$. Note that since $\pi_i : Y_i \to X_i$ is finite and of finite presentation we also have that $\pi : Y \to X$ is finite and of finite presentation and that the $\mathcal{O}_{X_i}$-modules $\pi_{i,*}\mathcal{O}_{Y_i}$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ are of finite presentation and pullback to X to give $\pi_*\mathcal{O}_Y$ and $\pi_*(\mathcal{O}_Y/\mathcal{I})$. Thus by Limits of Spaces, Lemma 70.7.2 after increasing i we can find a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ and a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose base change to X recovers the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\mathrm{Coker}(\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i})$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ to X agree as quotients of $\pi_*\mathcal{O}_Y$. Hence by Limits of Spaces, Lemma 70.7.2 we may assume that these agree, in other words that the image of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{X_i}$ is equal to $\pi_{i,*}\mathcal{I}_i$. Then we conclude that X_i has the resolution property by Lemma 76.56.4. □

0GV3 **Lemma 76.56.6.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space with the resolution property. Then X has affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 66.3.1).*

Proof. We could prove this as in the case of schemes, but instead we will deduce the lemma from the case of schemes. First, we may and do assume $S = \mathrm{Spec}(\mathbf{Z})$. Next, we choose a scheme Y and a surjective integral morphism $f : Y \to X$, see Decent Spaces, Lemma 68.9.2. Then f is affine, hence Y has the resolution property by Derived Categories of Spaces, Lemma 75.28.3. Hence by the case of schemes, the scheme Y has affine diagonal, see Derived Categories of Schemes, Lemma 36.36.10. Next, we consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_Y} & Y \times_{\mathbf{Z}} Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_X} & X \times_{\mathbf{Z}} X \end{array}$$

Observe that the right vertical arrow is integral, in particular affine. Let $W \to X \times_{\mathbf{Z}} X$ be a morphism with W affine. Then we see that

$$Y \times_{X \times_{\mathbf{Z}} X} W = Y \times_{\Delta_Y, Y \times_{\mathbf{Z}} Y} (Y \times_{\mathbf{Z}} Y) \times_{X \times_{\mathbf{Z}} X} W$$

is affine. On the other hand, $Y \to X$ is integral and surjective hence

$$Y \times_{X \times_{\mathbf{Z}} X} W \longrightarrow X \times_{X \times_{\mathbf{Z}} X} W$$

is integral surjective as the base change of $Y \to X$ to W. We conclude that the target of this arrow is affine by Limits of Spaces, Proposition 70.15.2. It follows that Δ_X is affine as desired. □

76.57. Blowing up and the resolution property

0GV4 We prove that the resolution property is satisfied after a blowing up.

0GV5 **Lemma 76.57.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that $|X|$ has finitely many irreducible components. There exists a dense quasi-compact open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the algebraic space X' has the resolution property.*

Proof. By Limits of Spaces, Lemma 70.16.3 there exists a surjective, finite, and finitely presented morphism $f : Y \to X$ where Y is a scheme and a quasi-compact dense open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale. By More on Morphisms, Lemma 37.80.2 there is a quasi-compact dense open $V \subset Y$ and a V-admissible blowing up $Y' \to Y$ such that Y' has an ample family of invertible modules. After shrinking U we may assume that $f^{-1}(U) \subset V$ (details omitted). Hence $f' : Y' \to X$ is finite étale over U and in particular, the morphism $(f')^{-1}(U) \to U$ is finite locally free. By Lemma 76.39.2 there is a U-admissible blowing up $X' \to X$ such that the strict transform Y'' of Y' is finite locally free over X'. Picture

$$\begin{array}{ccccc} Y'' & \xrightarrow{g} & Y' & \longrightarrow & Y \\ \downarrow & & & & \downarrow \\ X' & & \longrightarrow & & X \end{array}$$

Since $g : Y'' \to Y'$ is a blowing up (Divisors on Spaces, Lemma 71.18.3) in the inverse image of the center of $X' \to X$, we see that $g : Y'' \to Y'$ is projective

and that there exists some g-ample invertible module on Y''. Hence by More on Morphisms, Lemma 37.79.1 we see that Y'' has an ample family of invertible modules. Hence Y'' has the resolution property, see Derived Categories of Schemes, Lemma 36.36.7. We conclude that X' has the resolution property by Derived Categories of Spaces, Lemma 75.28.4. □

0GV6 **Lemma 76.57.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. There exists a $t \geq 0$ and closed subspaces*

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \ldots t-1$ there exists a $(Z_i \setminus Z_{i-1})$-admissible blowing up $Z_i' \to Z_i$ such that Z_i' has the resolution property.

Proof. In this paragraph we use absolute Noetherian approximation to reduce to the case of algebraic spaces of finite presentation over $\mathrm{Spec}(\mathbf{Z})$. We may view X as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 65.16.2 and Properties of Spaces, Definition 66.3.1. Thus we may apply Limits of Spaces, Proposition 70.8.1. It follows that we can find an affine morphism $X \to X_0$ with X_0 of finite presentation over $\mathbf{Z}$. If we can prove the lemma for X_0, then we can pull back the stratification and the centers of the blowing ups to X and get the result for X; this uses that the resolution property goes up along affine morphisms (Derived Categories of Spaces, Lemma 75.28.3) and that the strict transform of an affine morphism is affine – details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over $\mathbf{Z}$. Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. By Lemma 76.57.1 there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has the resolution property. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \ldots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \to Z_1$ is a thickening and there exist $(Z_{1,i} \setminus Z_{1,i+1})$-admissible blowing ups $Z_{1,i}' \to Z_{1,i}$ such that $Z_{1,i}'$ has the resolution property. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Z_i' = Z_{1,i-1}'$ for $i \geq 1$ and the lemma is proved. □

76.58. Other chapters

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CHAPTER 77

Flatness on Algebraic Spaces

0CU3

77.1. Introduction

0CU4 In this chapter, we discuss some advanced results on flat modules and flat morphisms in the setting of algebraic spaces. We strongly encourage the reader to take a look at the corresponding chapter in the setting of schemes first, see More on Flatness, Section 38.1. A reference is the paper [**GR71**] by Raynaud and Gruson.

77.2. Impurities

0CV5 The section is the analogue of More on Flatness, Section 38.15.

0CV6 **Situation 77.2.1.** Let S be a scheme. Let $f : X \to Y$ be a finite type, decent[1] morphism of algebraic spaces over S. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally $y \in |Y|$ is a point of Y.

In this situation consider a scheme T, a morphism $g : T \to Y$, a point $t \in T$ with $g(t) = y$, a specialization $t' \rightsquigarrow t$ in T, and a point $\xi \in |X_T|$ lying over t'. Here $X_T = T \times_Y X$. Picture

0CV7 (77.2.1.1)
$$\begin{array}{ccc} \xi & & X_T \longrightarrow X \\ \downarrow & & {\scriptstyle f_T}\downarrow \quad \downarrow{\scriptstyle f} \\ t' \rightsquigarrow t \longmapsto y & & T \xrightarrow{g} Y \end{array}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

0CV8 **Definition 77.2.2.** In Situation 77.2.1 we say a diagram (77.2.1.1) defines an *impurity of $\mathcal{F}$ above y* if $\xi \in \mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$ and $t \notin f_T(\overline{\{\xi\}})$. We will indicate this by saying "let $(g : T \to Y, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y".

Another way to say this is: $(g : T \to Y, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above y if there exists no specialization $\xi \rightsquigarrow \theta$ in the topological space $|X_T|$ with $f_T(\theta) = t$. Specializations in non-decent algebraic spaces do not behave well. If the morphism f is decent, then X_T is a decent algebraic space for all morphisms $g : T \to Y$ as above, see Decent Spaces, Definition 68.17.1.

0CV9 **Lemma 77.2.3.** *In Situation 77.2.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over Y. Then for some i the triple $(T_i \to Y, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above y.*

[1] Quasi-separated morphisms are decent, see Decent Spaces, Lemma 68.17.2. For any morphism $\mathrm{Spec}(k) \to Y$ where k is a field, the algebraic space X_k is of finite presentation over k because it is of finite type over k and quasi-separated by Decent Spaces, Lemma 68.14.1.

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t_i' = p_i(t')$. Finally $\xi_i \in |X_{T_i}|$ is the image of ξ. By Divisors on Spaces, Lemma 71.4.7 we have $\xi_i \in \mathrm{Ass}_{X_{T_i}/T_i}(\mathcal{F}_{T_i})$. Thus the only point is to show that $t_i \notin f_{T_i}(\overline{\{\xi_i\}})$ for some i.

Let $Z_i \subset X_{T_i}$ be the reduced induced scheme structure on $\overline{\{\xi_i\}} \subset |X_{T_i}|$ and let $Z \subset X_T$ be the reduced induced scheme structure on $\overline{\{\xi\}} \subset |X_T|$. Then $Z = \lim Z_i$ by Limits of Spaces, Lemma 70.5.4 (the lemma applies because each X_{T_i} is decent). Choose a field k and a morphism $\mathrm{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \mathrm{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \mathrm{Spec}(k) = \lim Z_i \times_{T_i} \mathrm{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \mathrm{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence $Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \mathrm{Spec}(k)$ is empty for some i by Limits of Spaces, Lemma 70.5.3. Since the image of the composition $\mathrm{Spec}(k) \to T \to T_i$ is t_i we obtain what we want. □

Impurities go up along flat base change.

0CVA **Lemma 77.2.4.** *In Situation 77.2.1. Let $(Y_1, y_1) \to (Y, y)$ be a morphism of pointed algebraic spaces over S. Assume $Y_1 \to Y$ is flat at y_1. If $(T \to Y, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above y, then there exists an impurity $(T_1 \to Y_1, t_1' \rightsquigarrow t_1, \xi_1)$ of the pullback $\mathcal{F}_1$ of $\mathcal{F}$ to $X_1 = Y_1 \times_Y X$ over y_1 such that T_1 is étale over $Y_1 \times_Y T$.*

Proof. Choose an étale morphism $T_1 \to Y_1 \times_Y T$ where T_1 is a scheme and let $t_1 \in T_1$ be a point mapping to y_1 and t. It is possible to find a pair (T_1, t_1) like this by Properties of Spaces, Lemma 66.4.3. The morphism of schemes $T_1 \to T$ is flat at t_1 (use Morphisms of Spaces, Lemma 67.30.4 and the definition of flat morphisms of algebraic spaces) there exists a specialization $t_1' \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Morphisms, Lemma 29.25.9. Choose a point $\xi_1 \in |X_{T_1}|$ mapping to t_1' and ξ with $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. point of $\mathrm{Spec}(\kappa(t_1') \otimes_{\kappa(t')} \kappa(\xi))$. This is possible by Divisors on Spaces, Lemma 71.4.7. As the closure Z_1 of $\{\xi_1\}$ in $|X_{T_1}|$ maps into the closure of $\{\xi\}$ in $|X_T|$ we conclude that the image of Z_1 in $|T_1|$ cannot contain t_1. Hence $(T_1 \to Y_1, t_1' \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_1$ above Y_1. □

0CVB **Lemma 77.2.5.** *In Situation 77.2.1. Let $\overline{y}$ be a geometric point lying over y. Let $\mathcal{O} = \mathcal{O}_{Y,\overline{y}}$ be the étale local ring of Y at $\overline{y}$. Denote $Y^{sh} = \mathrm{Spec}(\mathcal{O})$, $X^{sh} = X \times_Y Y^{sh}$, and $\mathcal{F}^{sh}$ the pullback of $\mathcal{F}$ to X^{sh}. The following are equivalent*

1. *there exists an impurity $(Y^{sh} \to Y, y' \rightsquigarrow \overline{y}, \xi)$ of $\mathcal{F}$ above y,*
2. *every point of $Ass_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$ specializes to a point of the closed fibre $X_{\overline{y}}$,*
3. *there exists an impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y such that $(T, t) \to (Y, y)$ is an étale neighbourhood, and*
4. *there exists an impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y such that $T \to Y$ is quasi-finite at t.*

Proof. That parts (1) and (2) are equivalent is immediate from the definition.

Recall that $\mathcal{O} = \mathcal{O}_{Y,\overline{y}}$ is the filtered colimit of $\mathcal{O}(V)$ over the category of étale neighbourhoods $(V, \overline{v}) \to (Y, \overline{y})$ (Properties of Spaces, Lemma 66.19.3). Moreover, it suffices to consider affine étale neighbourhoods V. Hence $Y^{sh} = \mathrm{Spec}(\mathcal{O}) = \lim \mathrm{Spec}(\mathcal{O}(V)) = \lim V$. Thus we see that (1) implies (3) by Lemma 77.2.3.

Since an étale morphism is locally quasi-finite (Morphisms of Spaces, Lemma 67.39.5) we see that (3) implies (4).

Finally, assume (4). After replacing T by an open neighbourhood of t we may assume $T \to Y$ is locally quasi-finite. By Lemma 77.2.4 we find an impurity $(T_1 \to Y^{sh}, t_1' \rightsquigarrow t_1, \xi_1)$ with $T_1 \to T \times_Y Y^{sh}$ étale. Since an étale morphism is locally quasi-finite and using Morphisms of Spaces, Lemma 67.27.4 and Morphisms, Lemma 29.20.12 we see that $T_1 \to Y^{sh}$ is locally quasi-finite. As $\mathcal{O}$ is strictly henselian, we can apply More on Morphisms, Lemma 37.41.1 to see that after replacing T_1 by an open and closed neighbourhood of t_1 we may assume that $T_1 \to Y^{sh} = \mathrm{Spec}(\mathcal{O})$ is finite. Let $\theta \in |X^{sh}|$ be the image of ξ_1 and let $y' \in \mathrm{Spec}(\mathcal{O})$ be the image of t_1'. By Divisors on Spaces, Lemma 71.4.7 we see that $\theta \in \mathrm{Ass}_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$. Since $\pi : X_{T_1} \to X^{sh}$ is finite, it induces a closed map $|X_{T_1}| \to |X^{sh}|$. Hence the image of $\overline{\{\xi_1\}}$ is $\overline{\{\theta\}}$. It follows that $(Y^{sh} \to Y, y' \rightsquigarrow \overline{y}, \theta)$ is an impurity of $\mathcal{F}$ above y and the proof is complete. □

77.3. Relatively pure modules

0CVC This section is the analogue of More on Flatness, Section 38.16.

0CVD **Definition 77.3.1.** In Situation 77.2.1.

(1) We say $\mathcal{F}$ is *pure above* y if **none** of the equivalent conditions of Lemma 77.2.5 hold.
(2) We say $\mathcal{F}$ is *universally pure above* y if there does not exist any impurity of $\mathcal{F}$ above y.
(3) We say that X is *pure above* y if $\mathcal{O}_X$ is pure above y.
(4) We say $\mathcal{F}$ is *universally Y-pure*, or *universally pure relative to* Y if $\mathcal{F}$ is universally pure above y for every $y \in |Y|$.
(5) We say $\mathcal{F}$ is *Y-pure*, or *pure relative to* Y if $\mathcal{F}$ is pure above y for every $y \in |Y|$.
(6) We say that X is *Y-pure* or *pure relative to* Y if $\mathcal{O}_X$ is pure relative to Y.

The obligatory lemmas follow.

0CVE **Lemma 77.3.2.** *In Situation 77.2.1.*

(1) *$\mathcal{F}$ is universally pure above y, and*
(2) *for every morphism $(Y', y') \to (Y, y)$ of pointed algebraic spaces the pullback $\mathcal{F}_{Y'}$ is pure above y'.*

In particular, $\mathcal{F}$ is universally pure relative to Y if and only if every base change $\mathcal{F}_{Y'}$ of $\mathcal{F}$ is pure relative to Y'.

Proof. This is formal. □

0CVF **Lemma 77.3.3.** *In Situation 77.2.1. Let $(Y', y') \to (Y, y)$ be a morphism of pointed algebraic spaces. If $Y' \to Y$ is quasi-finite at y' and $\mathcal{F}$ is pure above y, then $\mathcal{F}_{Y'}$ is pure above y'.*

Proof. It $(T \to Y', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{Y'}$ above y' with $T \to Y'$ quasi-finite at t, then $(T \to Y, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above y with $T \to Y$ quasi-finite at t, see Morphisms of Spaces, Lemma 67.27.3. Hence the lemma follows immediately from the definition of purity. □

Purity satisfies flat descent.

0CVG **Lemma 77.3.4.** *In Situation 77.2.1. Let $(Y_1, y_1) \to (Y, y)$ be a morphism of pointed algebraic spaces. Assume $Y_1 \to Y$ is flat at y_1.*

(1) *If $\mathcal{F}_{Y_1}$ is pure above y_1, then $\mathcal{F}$ is pure above y.*
(2) *If $\mathcal{F}_{Y_1}$ is universally pure above y_1, then $\mathcal{F}$ is universally pure above y.*

Proof. This is true because impurities go up along a flat base change, see Lemma 77.2.4. For example part (1) follows because by any impurity $(T \to Y, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above y with $T \to Y$ quasi-finite at t by the lemma leads to an impurity $(T_1 \to Y_1, t_1' \rightsquigarrow t_1, \xi_1)$ of the pullback $\mathcal{F}_1$ of $\mathcal{F}$ to $X_1 = Y_1 \times_Y X$ over y_1 such that T_1 is étale over $Y_1 \times_Y T$. Hence $T_1 \to Y_1$ is quasi-finite at t_1 because étale morphisms are locally quasi-finite and compositions of locally quasi-finite morphisms are locally quasi-finite (Morphisms of Spaces, Lemmas 67.39.5 and 67.27.3). Similarly for part (2). □

0CVH **Lemma 77.3.5.** *In Situation 77.2.1. Let $i : Z \to X$ be a closed immersion and assume that $\mathcal{F} = i_*\mathcal{G}$ for some finite type, quasi-coherent sheaf $\mathcal{G}$ on Z. Then $\mathcal{G}$ is (universally) pure above y if and only if $\mathcal{F}$ is (universally) pure above y.*

Proof. This follows from Divisors on Spaces, Lemma 71.4.9. □

0CVI **Lemma 77.3.6.** *In Situation 77.2.1.*

(1) *If the support of $\mathcal{F}$ is proper over Y, then $\mathcal{F}$ is universally pure relative to Y.*
(2) *If f is proper, then $\mathcal{F}$ is universally pure relative to Y.*
(3) *If f is proper, then X is universally pure relative to Y.*

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$ (Morphisms of Spaces, Definition 67.15.4). Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$. In case (1) $Z \to Y$ is proper by assumption. Thus by Lemma 77.3.5 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to Y, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above y. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

77.4. Flat finite type modules

0CVJ Please compare with More on Flatness, Sections 38.10, 38.13, and 38.26. Most of these results have immediate consequences of algebraic spaces by étale localization.

0CWJ **Lemma 77.4.1.** *Let S be a scheme. Let $X \to Y$ be a finite type morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$ be a point. There exists an étale morphism $(Y', y') \to (Y, y)$ with Y' an affine scheme and étale morphisms $h_i : W_i \to X_{Y'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/W_i/Y'$ over y', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to W_i and such that $|(X_{Y'})_{y'}| \subset \bigcup h_i(W_i)$.*

Proof. The question is étale local on Y hence we may assume Y is an affine scheme. Then X is quasi-compact, hence we can choose an affine scheme X' and a surjective étale morphism $X' \to X$. Then we may apply More on Flatness, Lemma 38.5.8 to $X' \to Y$, $(X' \to Y)^*\mathcal{F}$, and y to get what we want. □

0CWK **Lemma 77.4.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $y \in |Y|$ and $F = f^{-1}(\{y\}) \subset |X|$. Then the set*

$$\{x \in F \mid \mathcal{F} \text{ flat over } Y \text{ at } x\}$$

is open in F.

Proof. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Y$ mapping v to y. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $|U_v| \to F$ is an open continuous map of topological spaces as $|U| \to |X|$ is continuous and open. Hence the result follows from the case of schemes which is More on Flatness, Lemma 38.10.4. □

0CVK **Lemma 77.4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. If $\mathcal{F}$ is flat at x over Y, then*

$$x \in \mathit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow y \in \mathit{WeakAss}_Y(\mathcal{G}) \text{ and } x \in \mathit{Ass}_{X/Y}(\mathcal{F}).$$

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective étale. Choose $u \in U$ mapping to x. Let $\mathcal{E} = \mathcal{F}|_U$ and $\mathcal{H} = \mathcal{G}|_V$. Let $v \in V$ be the image of u. Then $x \in \mathrm{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $u \in \mathrm{WeakAss}_X(\mathcal{E} \otimes_{\mathcal{O}_X} g^*\mathcal{H})$ by Divisors on Spaces, Definition 71.2.2. Similarly, $y \in \mathrm{WeakAss}_Y(\mathcal{G})$ if and only if $v \in \mathrm{WeakAss}_V(\mathcal{H})$. Finally, we have $x \in \mathrm{Ass}_{X/Y}(\mathcal{F})$ if and only if $u \in \mathrm{Ass}_{U_v}(\mathcal{E}|_{U_v})$ by Divisors on Spaces, Definition 71.4.5. Observe that flatness of $\mathcal{F}$ at x is equivalent to flatness of $\mathcal{E}$ at u, see Morphisms of Spaces, Definition 67.31.2. The equivalence for $g : U \to V$, $\mathcal{E}$, $\mathcal{H}$, u, and v is More on Flatness, Lemma 38.13.3. □

0CVL **Lemma 77.4.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over Y. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then we have*

$$\mathit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \mathit{Ass}_{X/Y}(\mathcal{F}) \cap |f|^{-1}(\mathit{WeakAss}_Y(\mathcal{G}))$$

Proof. Immediate consequence of Lemma 77.4.3. □

0DLR **Theorem 77.4.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *$X \to Y$ is locally of finite presentation,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
(3) *the set of weakly associated points of Y is locally finite in Y.*

Then $U = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over Y.

Proof. Condition (3) means that if $V \to Y$ is a surjective étale morphism where V is a scheme, then the weakly associated points of V are locally finite on the scheme V. (Recall that the weakly associated points of V are exactly the inverse image of the weakly associated points of Y by Divisors on Spaces, Definition 71.2.2.) Having said this the question is étale local on X and Y, hence we may assume X and Y are schemes. Thus the result follows from More on Flatness, Theorem 38.13.6. □

0CVW **Lemma 77.4.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over Y at all $x \in F$.*

Then there exists an étale morphism $(Y', y') \to (Y, y)$ where Y' is a scheme and a commutative diagram of algebraic spaces

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{Y',y'}) \end{array}$$

such that $X' \to X \times_Y \mathrm{Spec}(\mathcal{O}_{Y',y'})$ is étale, $|X'_{y'}| \to F$ is surjective, X' is affine, and $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{Y',y'}$-module.

Proof. Choose an étale morphism $(Y', y') \to (Y, y)$ where Y' is an affine scheme. Then $X \times_Y Y'$ is quasi-compact. Choose an affine scheme X' and a surjective étale morphism $X' \to X \times_Y Y'$. Picture

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \longleftarrow & Y' \end{array}$$

Then $\mathcal{F}' = g^*\mathcal{F}$ is flat over Y' at all points of $X'_{y'}$, see Morphisms of Spaces, Lemma 67.31.3. Hence we can apply the lemma in the case of schemes (More on Flatness, Lemma 38.12.11) to the morphism $X' \to Y'$, the quasi-coherent sheaf $g^*\mathcal{F}$, and the point y'. This gives an étale morphism $(Y'', y'') \to (Y', y')$ and a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{g'} & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y' & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{Y'',y''}) \end{array}$$

To get what we want we take $(Y'', y'') \to (Y, y)$ and $g \circ g' : X'' \to X$. □

0CWL **Theorem 77.4.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in |X|$ with image $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Consider the conditions*

(1) *$\mathcal{F}$ is flat at x over Y, and*

(2) *for every $x' \in F \cap Ass_{X/Y}(\mathcal{F})$ which specializes to x we have that $\mathcal{F}$ is flat at x' over Y.*

Then we always have (2) $\Rightarrow$ (1). If X and Y are decent, then (1) $\Rightarrow$ (2).

Proof. Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a point $u \in U$ mapping to x. Let $v \in V$ be the image of u. We will deduce the result from the corresponding result for $\mathcal{F}|_U = (U \to X)^*\mathcal{F}$ and the point u. U_v. This works because $\mathrm{Ass}_{U/V}(\mathcal{F}|_U) \cap |U_v|$ is equal to $\mathrm{Ass}_{U_v}(\mathcal{F}|_{U_v})$ and equal to the inverse image of $F \cap \mathrm{Ass}_{X/Y}(\mathcal{F})$. Since the map $|U_v| \to F$ is continuous we see that specializations in $|U_v|$ map to specializations in F, hence condition (2) is inherited by $U \to V$, $\mathcal{F}|_U$, and the point u. Thus More on Flatness, Theorem 38.26.1 applies and we conclude that (1) holds.

If Y is decent, then we can represent y by a quasi-compact monomorphism $\mathrm{Spec}(k) \to Y$ (by definition of decent spaces, see Decent Spaces, Definition 68.6.1). Then $F = |X_k|$, see Decent Spaces, Lemma 68.18.6. If in addition X is decent (or more generally if f is decent, see Decent Spaces, Definition 68.17.1 and Decent Spaces, Lemma 68.17.3), then X_y is a decent space too. Furthermore, specializations in F can be lifted to specializations in $U_v \to X_y$, see Decent Spaces, Lemma 68.12.2. Having said this it is clear that the reverse implication holds, because it holds in the case of schemes. □

0CWM **Lemma 77.4.8.** *Let S be a local scheme with closed point s. Let $f : X \to S$ be a morphism from an algebraic space X to S which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that*

(1) *every point of $Ass_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s*[2]*,*
(2) *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 77.4.7. □

77.5. Flat finitely presented modules

0CVX This is the analogue of More on Flatness, Section 38.12.

0CVY **Proposition 77.5.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat at x over Y.*

Then there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (Y, y) & \longleftarrow & (Y', y') \end{array}$$

[2]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

whose horizontal arrows are étale such that X', Y' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(Y', \mathcal{O}_{Y'})$-module.

Proof. As formulated this proposition immediately reduces to the case of schemes, which is More on Flatness, Proposition 38.12.4. □

0CVZ **Lemma 77.5.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat over Y at all $x \in F$.*

Then there exists a commutative diagram of algebraic spaces

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xleftarrow{h} & Y' \end{array}$$

such that h and g are étale, there is a point $y' \in |Y'|$ mapping to y, we have $F \subset g(|X'|)$, the algebraic spaces X', Y' are affine, and $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(Y', \mathcal{O}_{Y'})$-module.

Proof. As formulated this lemma immediately reduces to the case of schemes, which is More on Flatness, Lemma 38.12.5. □

77.6. A criterion for purity

0CW0 This section is the analogue of More on Flatness, Section 38.18.

0CW1 **Lemma 77.6.1.** *Let S be a scheme. Let X be a decent algebraic space locally of finite type over S. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ such that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in Ass_{X/S}(\mathcal{F})$. If the closure of $\{x'\}$ in $|X|$ meets $|X_s|$, then the closure meets $Ass_{X/S}(\mathcal{F}) \cap |X_s|$.*

Proof. Observe that $|X_s| \subset |X|$ is the set of points of $|X|$ lying over $s \in S$, see Decent Spaces, Lemma 68.18.6. Let $t \in |X_s|$ be a specialization of x' in $|X|$. Choose an affine scheme U and a point $u \in U$ and an étale morphism $\varphi : U \to X$ mapping u to t. By Decent Spaces, Lemma 68.12.2 we can choose a specialization $u' \rightsquigarrow u$ with u' mapping to x'. Set $g = f \circ \varphi$. Observe that $s' = g(u') = f(x')$ specializes to s. By our definition of $\mathrm{Ass}_{X/S}(\mathcal{F})$ we have $u' \in \mathrm{Ass}_{U/S}(\varphi^*\mathcal{F})$. By the schemes version of this lemma (More on Flatness, Lemma 38.18.1) we see that there is a specialization $u' \rightsquigarrow u$ with $u \in \mathrm{Ass}_{U_s}(\varphi^*\mathcal{F}_s) = \mathrm{Ass}_{U/S}(\varphi^*\mathcal{F}) \cap U_s$. Hence $x = \varphi(u) \in \mathrm{Ass}_{X/S}(\mathcal{F})$ lies over s and the lemma is proved. □

0CW2 **Lemma 77.6.2.** *Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be a morphism of algebraic spaces over Y with X locally of finite type over Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Ass_{X/Y}(\mathcal{F}) \subset g(|X'|)$, then for any morphism $Z \to Y$ we have $Ass_{X_Z/Z}(\mathcal{F}_Z) \subset g_Z(|X'_Z|)$.*

Proof. By Properties of Spaces, Lemma 66.4.3 the map $|X'_Z| \to |X_Z| \times_{|X|} |X'|$ is surjective as X'_Z is equal to $X_Z \times_X X'$. By Divisors on Spaces, Lemma 71.4.7 the map $|X_Z| \to |X|$ sends $\mathrm{Ass}_{X_Z/Z}(\mathcal{F}_Z)$ into $\mathrm{Ass}_{X/Y}(\mathcal{F})$. The lemma follows. □

0CW3 **Lemma 77.6.3.** *Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be an étale morphism of algebraic spaces over Y. Assume the structure morphisms $X' \to Y$ and $X \to Y$ are decent and of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$. Set $F = f^{-1}(\{y\}) \subset |X|$.*

(1) *If $Ass_{X/Y}(\mathcal{F}) \subset g(|X'|)$ and $g^*\mathcal{F}$ is (universally) pure above y, then $\mathcal{F}$ is (universally) pure above y.*
(2) *If $\mathcal{F}$ is pure above y, $g(|X'|)$ contains F, and Y is affine local with closed point y, then $Ass_{X/Y}(\mathcal{F}) \subset g(|X'|)$.*
(3) *If $\mathcal{F}$ is pure above y, $\mathcal{F}$ is flat at all points of F, $g(|X'|)$ contains $Ass_{X/Y}(\mathcal{F}) \cap F$, and Y is affine local with closed point y, then $Ass_{X/Y}(\mathcal{F}) \subset g(|X'|)$.*
(4) *Add more here.*

Proof. The assumptions on $X \to Y$ and $X' \to Y$ guarantee that we may apply the material in Sections 77.2 and 77.3 to these morphisms and the sheaves $\mathcal{F}$ and $g^*\mathcal{F}$. Since g is étale we see that $\mathrm{Ass}_{X'/Y}(g^*\mathcal{F})$ is the inverse image of $\mathrm{Ass}_{X/Y}(\mathcal{F})$ and the same remains true after base change.

Proof of (1). Assume $\mathrm{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$. Suppose that $(T \to Y, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above y. Since $\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset g_T(|X'_T|)$ by Lemma 77.6.2 we can choose a point $\xi' \in |X'_T|$ mapping to ξ. By the above we see that $(T \to Y, t' \rightsquigarrow t, \xi')$ is an impurity of $g^*\mathcal{F}$ above y'. This implies (1) is true.

Proof of (2). This follows from the fact that $g(|X'|)$ is open in $|X|$ and the fact that by purity every point of $\mathrm{Ass}_{X/Y}(\mathcal{F})$ specializes to a point of F.

Proof of (3). This follows from the fact that $g(|X'|)$ is open in $|X|$ and the fact that by purity combined with Lemma 77.6.1 every point of $\mathrm{Ass}_{X/Y}(\mathcal{F})$ specializes to a point of $\mathrm{Ass}_{X/Y}(\mathcal{F}) \cap F$. □

0CW4 **Lemma 77.6.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in |Y|$. Assume*

(1) *f is decent and of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *$\mathcal{F}$ is flat over Y at all points lying over y, and*
(4) *$\mathcal{F}$ is pure above y.*

Then $\mathcal{F}$ is universally pure above y.

Proof. Consider the morphism $\mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism from the spectrum of a strictly henselian local ring which maps the closed point to y. By Lemma 77.3.4 we reduce to the case described in the next paragraph.

Assume Y is the spectrum of a strictly henselian local ring R with closed point y. By Lemma 77.4.6 there exists an étale morphism $g : X' \to X$ with $g(|X'|) \supset |X_y|$, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free R-module. Then $g^*\mathcal{F}$ is universally pure relative to Y, see More on Flatness, Lemma 38.17.4. Hence it suffices to prove that $g(|X'|)$ contains $\mathrm{Ass}_{X/Y}(\mathcal{F})$ by Lemma 77.6.3 part (1). This in turn follows from Lemma 77.6.3 part (2). □

0CW5 **Lemma 77.6.5.** *Let S be a scheme. Let $f : X \to Y$ be a decent, finite type morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over Y. In this case $\mathcal{F}$ is pure relative to Y if and only if $\mathcal{F}$ is universally pure relative to Y.*

Proof. Immediate consequence of Lemma 77.6.4 and the definitions. □

0CW6 **Lemma 77.6.6.** *Let Y be an algebraic space over a scheme S. Let $g : X' \to X$ be a flat morphism of algebraic spaces over Y with X locally of finite type over Y. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over Y. If $Ass_{X/Y}(\mathcal{F}) \subset g(|X'|)$ then the canonical map*

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_Z \to (g_Z)_* g_Z^* \mathcal{F}_Z$ is injective for any morphism $Z \to Y$. Since the assumption on the relative assassin is preserved by base change (Lemma 77.6.2) it suffices to prove the injectivity of the displayed arrow.

Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors on Spaces, Lemma 71.2.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors on Spaces, Lemma 71.2.4. As $\mathcal{F}$ is flat we see from Lemma 77.4.4 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/Y}(\mathcal{F})$. By assumption any point x of $\mathrm{Ass}_{X/Y}(\mathcal{F})$ is the image of some $x' \in |X'|$. Since g is flat the local ring map $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X',\overline{x}'}$ is faithfully flat, hence the map

$$\mathcal{F}_{\overline{x}} \longrightarrow (g^*\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X',\overline{x}'}$$

is injective (see Algebra, Lemma 10.82.11). Since the displayed arrow factors through $\mathcal{F}_{\overline{x}} \to (g_* g^* \mathcal{F})_{\overline{x}}$, we conclude that $\mathcal{K}_{\overline{x}} = 0$. Hence x cannot be a weakly associated point of $\mathcal{K}$ and we win. □

77.7. Flattening functors

083E This section is the analogue of More on Flatness, Section 38.20. We urge the reader to skip this section on a first reading.

083F **Situation 77.7.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_B T \to X$. In this situation we can consider the functor

083G (77.7.1.1) $$F_{iso} : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

In Situation 77.7.1 we sometimes think of the functors F_{iso}, F_{inj}, F_{surj}, and F_{zero} as functors $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_{iso} \to B$, $F_{inj} \to B$, $F_{surj} \to B$, and $F_{zero} \to B$. Namely, if T is a scheme over S, then an element $h \in F_{iso}(T)$ is a morphism $h : T \to B$ such that the base change of u via h is an isomorphism. In particular, when we say that F_{iso} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

083H **Lemma 77.7.2.** *In Situation 77.7.1. Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3. In particular, for every $x \in |X_T|$ there exists an $i \in I$ and an $x_i \in |X_i|$ mapping to x. Since $\mathcal{O}_{X_T,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is flat, hence faithfully flat (see Morphisms of Spaces, Section 67.30). we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. The lemma follows. □

083I **Lemma 77.7.3.** *In Situation 77.7.1 let $X' \to X$ be a flat morphism of algebraic spaces. Denote $u' : \mathcal{F}' \to \mathcal{G}'$ the pullback of u to X'. Denote F'_{iso}, F'_{inj}, F'_{surj}, F'_{zero} the functors on Sch/B associated to u'.*

(1) *If $\mathcal{G}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{G}$, then $F_{surj} = F'_{surj}$ and $F_{zero} = F'_{zero}$.*
(2) *If $\mathcal{F}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{F}$, then $F_{inj} = F'_{inj}$ and $F_{zero} = F'_{zero}$.*
(3) *If $\mathcal{F}$ and $\mathcal{G}$ are of finite type and the image of $|X'| \to |X|$ contains the supports of $\mathcal{F}$ and $\mathcal{G}$, then $F_{iso} = F'_{iso}$.*

Proof. let $v : \mathcal{H} \to \mathcal{E}$ be a map of quasi-coherent modules on an algebraic space Y and let $\varphi : Y' \to Y$ be a surjective flat morphism of algebraic spaces, then v is an isomorphism, injective, surjective, or zero if and only if $\varphi^* v$ is an isomorphism, injective, surjective, or zero. Namely, for every $y \in |Y|$ there exists a $y' \in |Y'|$ and the map of local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{Y',\overline{y'}}$ is faithfully flat (see Morphisms of Spaces, Section 67.30). Of course, to check for injectivity or being zero it suffices to look at the points in the support of $\mathcal{H}$, and to check for surjectivity it suffices to look at points in the support of $\mathcal{E}$. Moreover, under the finite type assumptions as in the statement of the lemma, taking the supports commutes with base change, see Morphisms of Spaces, Lemma 67.15.2. Thus the lemma is clear. □

Recall that we've defined the scheme theoretic support of a finite type quasi-coherent module in Morphisms of Spaces, Definition 67.15.4.

083J **Lemma 77.7.4.** *In Situation 77.7.1.*

(1) *If $\mathcal{G}$ is of finite type and the scheme theoretic support of $\mathcal{G}$ is quasi-compact over B, then F_{surj} is limit preserving.*
(2) *If $\mathcal{F}$ of finite type and the scheme theoretic support of $\mathcal{F}$ is quasi-compact over B, then F_{zero} is limit preserving.*
(3) *If $\mathcal{F}$ is of finite type, $\mathcal{G}$ is of finite presentation, and the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ are quasi-compact over B, then F_{iso} is limit preserving.*

Proof. Proof of (1). Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{G}$ and think of $\mathcal{G}$ as a finite type quasi-coherent module on Z. We may replace X by Z and u by the map $i^*\mathcal{F} \to \mathcal{G}$ (details omitted). Hence we may assume f is quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove (1) we have to show that u_i is surjective for some i. Pick $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Since f is quasi-compact we see X_0 is quasi-compact. Hence we may choose a surjective étale morphism $\varphi_0 : W_0 \to X_0$ where W_0 is an affine scheme. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes

endowed with a surjective étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$. Hence $\varphi^* u_T$ is surjective and it suffices to prove that $\varphi_i^* u_i$ is surjective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.127.5 part (2).

Proof of (2). Assume $\mathcal{F}$ is of finite type with scheme theoretic support $Z \subset B$ quasi-compact over B. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is zero. Set $X_i = T_i \times_B X$ and denote $u_i : \mathcal{F}_i \to \mathcal{G}_i$ the pullback. Choose $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Set $Z_0 = Z \times_X X_0$. By Morphisms of Spaces, Lemma 67.15.2 the support of $\mathcal{F}_i$ is $|Z_0|$. Since $|Z_0|$ is quasi-compact we can find an affine scheme W_0 and an étale morphism $W_0 \to X_0$ such that $|Z_0| \subset \mathrm{Im}(|W_0| \to |X_0|)$. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$ and that the support of $\mathcal{F}_T$ and $\mathcal{F}_i$ is contained in the image of $|W| \to |X_T|$ and $|W_i| \to |X_i|$. Now $\varphi^* u_T$ is injective and it suffices to prove that $\varphi_i^* u_i$ is injective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.127.5 part (1).

Proof of (3). This can be proven in exactly the same manner as in the previous two paragraphs using Algebra, Lemma 10.127.5 part (3). We can also deduce it from (1) and (2) as follows. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is an isomorphism. By part (1) there exists an $0 \in I$ such that u_{T_0} is surjective. Set $\mathcal{K} = \mathrm{Ker}(u_{T_0})$ and consider the map of quasi-coherent modules $v : \mathcal{K} \to \mathcal{F}_{T_0}$. For $i \geq 0$ the base change v_{T_i} is zero if and only if u_i is an isomorphism. Moreover, v_T is zero. Since $\mathcal{G}_{T_0}$ is of finite presentation, $\mathcal{F}_{T_0}$ is of finite type, and u_{T_0} is surjective we conclude that $\mathcal{K}$ is of finite type (Modules on Sites, Lemma 18.24.2). It is clear that the support of $\mathcal{K}$ is contained in the support of $\mathcal{F}_{T_0}$ which is quasi-compact over T_0. Hence we can apply part (2) to see that v_{T_i} is zero for some i. □

0CVM **Lemma 77.7.5.** *In Situation 77.7.1 suppose given an exact sequence*

$$\mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{H} \to 0$$

Then we have $F_{v,iso} = F_{u,zero}$ with obvious notation.

Proof. Since pullback is right exact we see that $\mathcal{F}_T \to \mathcal{G}_T \to \mathcal{H}_T \to 0$ is exact for every scheme T over B. Hence u_T is surjective if and only if v_T is an isomorphism. □

0CW7 **Lemma 77.7.6.** *In Situation 77.7.1 suppose given an affine morphism $i : Z \to X$ and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{H}$ such that $\mathcal{G} = i_* \mathcal{H}$. Let $v : i^* \mathcal{F} \to \mathcal{H}$ be the map adjoint to u. Then*

1. *$F_{v,zero} = F_{u,zero}$, and*
2. *if i is a closed immersion, then $F_{v,surj} = F_{u,surj}$.*

Proof. Let T be a scheme over B. Denote $i_T : Z_T \to X_T$ the base change of i and $\mathcal{H}_T$ the pullback of $\mathcal{H}$ to Z_T. Observe that $(i^* \mathcal{F})_T = i_T^* \mathcal{F}_T$ and $i_{T,*} \mathcal{H}_T = (i_* \mathcal{H})_T$. The first statement follows from commutativity of pullbacks and the second from Cohomology of Spaces, Lemma 69.11.1. Hence we see that u_T and v_T are adjoint maps as well. Thus $u_T = 0$ if and only if $v_T = 0$. This proves (1). In case (2) we

see that u_T is surjective if and only if v_T is surjective because u_T factors as

$$\mathcal{F}_T \to i_{T,*}i_T^*\mathcal{F}_T \xrightarrow{i_{T,*}v_T} i_{T,*}\mathcal{H}_T$$

and the fact that $i_{T,*}$ is an exact functor fully faithfully embedding the category of quasi-coherent modules on Z_T into the category of quasi-coherent $\mathcal{O}_{X_T}$-modules. See Morphisms of Spaces, Lemma 67.14.1. □

0CW8 **Lemma 77.7.7.** *In Situation 77.7.1 suppose given an affine morphism $g : X \to X'$. Set $u' = f_*u : f_*\mathcal{F} \to f_*\mathcal{G}$. Then $F_{u,iso} = F_{u',iso}$, $F_{u,inj} = F_{u',inj}$, $F_{u,surj} = F_{u',surj}$, and $F_{u,zero} = F_{u',zero}$.*

Proof. By Cohomology of Spaces, Lemma 69.11.1 we have $g_{T,*}u_T = u'_T$. Moreover, $g_{T,*} : QCoh(\mathcal{O}_{X_T}) \to QCoh(\mathcal{O}_X)$ is a faithful, exact functor reflecting isomorphisms, injective maps, and surjective maps. □

0CWX **Situation 77.7.8.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over Y we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_Y T \to X$. Since the base change of a flat module is flat we obtain a functor

0CWY (77.7.8.1) $$F_{flat} : (Sch/Y)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 77.7.8 we sometimes think of F_{flat} as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_{flat} \to Y$. Namely, if T is a scheme over S, then an element $h \in F_{flat}(T)$ is a morphism $h : T \to Y$ such that the base change of $\mathcal{F}$ via h is flat over T. In particular, when we say that F_{flat} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

0CWZ **Lemma 77.7.9.** *In Situation 77.7.8.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of algebraic spaces over Y, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see Morphisms of Spaces, Lemma 67.31.3. Part (2) follows from Limits of Spaces, Lemma 70.6.12 if f is also quasi-separated (i.e., f is of finite presentation). For the general case, first reduce to the case where the base is affine and then cover X by finitely many affines to reduce to the quasi-separated case. Details omitted. □

77.8. Making a map zero

0CW9 This section has no analogue in the corresponding chapter on schemes.

0CWA **Situation 77.8.1.** Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{G}$ flat over S.

083K **Lemma 77.8.2.** *In Situation 77.8.1. Let $T \to S$ be a quasi-compact morphism of schemes such that the base change u_T is zero. Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) the base change u_Z is*

zero. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module and the scheme theoretic support of $\mathcal{F}$ is quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Let $U \to X$ be a surjective étale morphism of algebraic spaces where $U = \coprod U_i$ is a disjoint union of affine schemes (see Properties of Spaces, Lemma 66.6.1). By Lemma 77.7.3 we see that we may replace X by U. In other words, we may assume that $X = \coprod X_i$ is a disjoint union of affine schemes X_i. Suppose that we can prove the lemma for $u_i = u|_{X_i}$. Then we find a closed subscheme $Z_i \subset S$ such that $T \to S$ factors through Z_i and u_{i,Z_i} is zero. If $Z_i = \operatorname{Spec}(R/I_i) \subset \operatorname{Spec}(R) = S$, then taking $Z = \operatorname{Spec}(R/\sum I_i)$ works. Thus we may assume that $X = \operatorname{Spec}(A)$ is affine.

Choose a finite affine open covering $T = T_1 \cup \ldots \cup T_m$. It is clear that we may replace T by $\coprod_{j=1,\ldots,m} T_j$. Hence we may assume T is affine. Say $T = \operatorname{Spec}(R')$. Let $u : M \to N$ be the homomorphisms of A-modules corresponding to $u : \mathcal{F} \to \mathcal{G}$. Then N is a flat R-module as $\mathcal{G}$ is flat over S. The assumption of the lemma means that the composition

$$M \otimes_R R' \to N \otimes_R R'$$

is zero. Let $z \in M$. By Lazard's theorem (Algebra, Theorem 10.81.4) and the fact that $\otimes$ commutes with colimits we can find free R-module F_z, an element $\tilde{z} \in F_z$, and a map $F_z \to N$ such that $u(z)$ is the image of $\tilde{z}$ and $\tilde{z}$ maps to zero in $F_z \otimes_R R'$. Choose a basis $\{e_{z,\alpha}\}$ of F_z and write $\tilde{z} = \sum f_{z,\alpha} e_{z,\alpha}$ with $f_{z,\alpha} \in R$. Let $I \subset R$ be the ideal generated by the elements $f_{z,\alpha}$ with z ranging over all elements of M. By construction I maps to zero in R' and the elements $\tilde{z}$ map to zero in F_z/IF_z whence in N/IN. Thus $Z = \operatorname{Spec}(R/I)$ is a solution to the problem in this case.

Assume $\mathcal{F}$ is of finite type with quasi-compact scheme theoretic support. Write $Z = \operatorname{Spec}(R/I)$. Write $I = \bigcup I_\lambda$ as a filtered union of finitely generated ideals. Set $Z_\lambda = \operatorname{Spec}(R/I_\lambda)$, so $Z = \operatorname{colim} Z_\lambda$. Since u_Z is zero, we see that u_{Z_λ} is zero for some λ by Lemma 77.7.4. This finishes the proof of the lemma. □

083L **Lemma 77.8.3.** *Let A be a ring. Let $u : M \to N$ be a map of A-modules. If N is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

(1) *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is zero, and*
(2) *$\varphi(I) = 0$.*

Proof. As N is projective we can find a projective A-module C such that $F = N \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume N is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\operatorname{Im}(u)$ with respect to some (fixed) basis of N. □

0CWB **Lemma 77.8.4.** *In Situation 77.8.1. Let $T \subset S$ be a subset. Let $s \in S$ be in the closure of T. For $t \in T$, let u_t be the pullback of u to X_t and let u_s be the pullback of u to X_s. If X is locally of finite presentation over S, $\mathcal{G}$ is of finite presentation[3], and $u_t = 0$ for all $t \in T$, then $u_s = 0$.*

[3]It would suffice if X is locally of finite type over S and $\mathcal{G}$ is finitely presented relative to S, but this notion hasn't yet been defined in the setting of algebraic spaces. The definition for schemes is given in More on Morphisms, Section 37.58.

Proof. To check whether u_s is zero, is étale local on the fibre X_s. Hence we may pick a point $x \in |X_s| \subset |X|$ and check in an étale neighbourhood. Choose

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

as in Proposition 77.5.1. Let $T' \subset S'$ be the inverse image of T. Observe that s' is in the closure of T' because $S' \to S$ is open. Hence we reduce to the algebra problem described in the next paragraph.

We have an R-module map $u : M \to N$ such that N is projective as an R-module and such that $u_t : M \otimes_R \kappa(t) \to N \otimes_R \kappa(t)$ is zero for each $t \in T$. Problem: show that $u_s = 0$. Let $I \subset R$ be the ideal defined in Lemma 77.8.3. Then I maps to zero in $\kappa(t)$ for all $t \in T$. Hence $T \subset V(I)$. Since s is in the closure of T we see that $s \in V(I)$. Hence $u_s = 0$. □

It would be interesting to find a "simple" direct proof of either Lemma 77.8.5 or Lemma 77.8.6 using arguments like those used in Lemmas 77.8.2 and 77.8.4. A "classical" proof of this lemma when $f : X \to B$ is a projective morphism and B a Noetherian scheme would be: (a) choose a relatively ample invertible sheaf $\mathcal{O}_X(1)$, (b) set $u_n : f_*\mathcal{F}(n) \to f_*\mathcal{G}(n)$, (c) observe that $f_*\mathcal{G}(n)$ is a finite locally free sheaf for all $n \gg 0$, and (d) F_{zero} is represented by the vanishing locus of u_n for some $n \gg 0$.

0CWC **Lemma 77.8.5.** *In Situation 77.7.1. Assume*

(1) *f is of finite presentation, and*
(2) *$\mathcal{G}$ is of finite presentation, flat over B, and pure relative to B.*

Then F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. By Lemma 77.6.5 the module $\mathcal{G}$ is universally pure relative to B. In order to prove that F_{zero} is an algebraic space, it suffices to show that $F_{zero} \to B$ is representable, see Spaces, Lemma 65.11.3. Let $B' \to B$ be a morphism where B' is a scheme and let $u' : \mathcal{F}' \to \mathcal{G}'$ be the pullback of u to $X' = X_{B'}$. Then the associated functor F'_{zero} equals $F_{zero} \times_B B'$. This reduces us to the case that B is a scheme.

Assume B is a scheme. We will show that F_{zero} is representable by a closed subscheme of B. By Lemma 77.7.2 and Descent, Lemmas 35.37.2 and 35.39.1 the question is local for the étale topology on B. Let $b \in B$. We first replace B by an affine neighbourhood of b. Choose a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ B & \longleftarrow & B' \end{array}$$

and $b' \in B'$ mapping to $b \in B$ as in Lemma 77.5.2. As we are working étale locally, we may replace B by B' and assume that we have a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

with B and X' affine such that $\Gamma(X', g^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module and $g(|X'|) \supset |X_b|$. Let $U \subset X$ be the open subspace with $|U| = g(|X'|)$. By Divisors on Spaces, Lemma 71.4.10 the set

$$E = \{t \in B : \mathrm{Ass}_{X_t}(\mathcal{G}_t) \subset |U_t|\} = \{t \in B : \mathrm{Ass}_{X/B}(\mathcal{G}) \cap |X_t| \subset |U_t|\}$$

is constructible in B. By Lemma 77.6.3 part (2) we see that E contains $\mathrm{Spec}(\mathcal{O}_{B,b})$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of b. Hence after replacing B by a smaller affine neighbourhood of b we may assume that $\mathrm{Ass}_{X/B}(\mathcal{G}) \subset g(|X'|)$.

From Lemma 77.6.6 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to B$ is a morphism of affine schemes and $\Gamma(X, \mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module. This case follows immediately from Lemma 77.8.3.

We still have to show that $F_{zero} \to B$ is of finite presentation if $\mathcal{F}$ is of finite type. This follows from Lemma 77.7.4 combined with Limits of Spaces, Proposition 70.3.10. □

083M **Lemma 77.8.6.** *In Situation 77.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation flat over B,*
(3) *the support of $\mathcal{G}$ is proper over B.*

Then the functor F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. If f is of finite presentation, then this follows immediately from Lemmas 77.8.5 and 77.3.6. This is the only case of interest and we urge the reader to skip the rest of the proof, which deals with the possibility (allowed by the assumptions in this lemma) that f is not quasi-separated or quasi-compact.

Let $i : Z \to X$ be the closed subspace cut out by the zeroth fitting ideal of $\mathcal{G}$ (Divisors on Spaces, Section 71.5). Then $Z \to B$ is proper by assumption (see Derived Categories of Spaces, Section 75.7). On the other hand i is of finite presentation (Divisors on Spaces, Lemma 71.5.2 and Morphisms of Spaces, Lemma 67.28.12). There exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{H}$ of finite type with $i_*\mathcal{H} = \mathcal{G}$ (Divisors on Spaces, Lemma 71.5.3). In fact $\mathcal{H}$ is of finite presentation as an $\mathcal{O}_Z$-module by Algebra, Lemma 10.6.4 (details omitted). Then F_{zero} is the same as the functor F_{zero} for the map $i^*\mathcal{F} \to \mathcal{H}$ adjoint to u, see Lemma 77.7.6. The sheaf $\mathcal{H}$ is flat relative to B because the same is true for $\mathcal{G}$ (check on stalks; details omitted). Moreover, note that if $\mathcal{F}$ is of finite type, then $i^*\mathcal{F}$ is of finite type. Hence we have reduced the lemma to the case discussed in the first paragraph of the proof. □

77.9. Flattening a map

0CVN This section is the analogue of More on Flatness, Section 38.23. In particular the following result is a variant of More on Flatness, Theorem 38.23.3.

0CWD **Theorem 77.9.1.** *In Situation 77.7.1 assume*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, flat over B, and pure relative to B, and*
(3) *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to B$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and denote $v : \mathcal{K} \to \mathcal{F}$ the inclusion. By Lemma 77.7.5 we see that $F_{u,iso} = F_{v,zero}$. By Lemma 77.8.5 applied to v we see that $F_{u,iso} = F_{v,zero}$ is representable by a closed subspace of B. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 18.24.2. Hence we also obtain the final statement of the lemma. □

083N **Lemma 77.9.2.** *In Situation 77.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is locally of finite presentation and flat over B,*
(3) *the support of $\mathcal{F}$ is proper over B, and*
(4) *u is surjective.*

Then the functor F_{iso} is an algebraic space and $F_{iso} \to B$ is a closed immersion. If $\mathcal{G}$ is of finite presentation, then $F_{iso} \to B$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and denote $v : \mathcal{K} \to \mathcal{F}$ the inclusion. By Lemma 77.7.5 we see that $F_{u,iso} = F_{v,zero}$. By Lemma 77.8.6 applied to v we see that $F_{u,iso} = F_{v,zero}$ is representable by a closed subspace of B. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 18.24.2. Hence we also obtain the final statement of the lemma. □

We will use the following (easy) result when discussing the Quot functor.

09TP **Lemma 77.9.3.** *In Situation 77.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{G}$ is of finite type,*
(3) *the support of $\mathcal{G}$ is proper over B.*

Then F_{surj} is an algebraic space and $F_{surj} \to B$ is an open immersion.

Proof. Consider $\mathrm{Coker}(u)$. Observe that $\mathrm{Coker}(u_T) = \mathrm{Coker}(u)_T$ for any T/B. Note that formation of the support of a finite type quasi-coherent module commutes with pullback (Morphisms of Spaces, Lemma 67.15.1). Hence F_{surj} is representable by the open subspace of B corresponding to the open set

$$|B| \setminus |f|(\mathrm{Supp}(\mathrm{Coker}(u)))$$

see Properties of Spaces, Lemma 66.4.8. This is an open because $|f|$ is closed on $\mathrm{Supp}(\mathcal{G})$ and $\mathrm{Supp}(\mathrm{Coker}(u))$ is a closed subset of $\mathrm{Supp}(\mathcal{G})$. □

77.10. Flattening in the local case

0CWN This section is the analogue of More on Flatness, Section 38.24.

0CWP **Lemma 77.10.1.** *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of algebraic spaces which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset |X_s|$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

(1) *$\mathcal{F}_T$ is flat over T at all points of $|X_t|$ which map to a point of $E \subset |X_s|$, and*
(2) *$\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset |X_s|$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. Choose a scheme U and an étale morphism $\varphi : U \to X$. Let $E' \subset |U_s|$ be the inverse image of E. If $E' \to E$ is surjective, then condition (1) is equivalent to: $(\varphi^*\mathcal{F})_T$ is flat over T at all points of $|U_t|$ which map to a point of $E' \subset |U_t|$. Choosing φ to be surjective, we reduced to the case of schemes which is More on Flatness, Lemma 38.24.3. If E is closed and quasi-compact, then we may choose U to be affine such that $E' \to E$ is surjective. Then E' is closed and quasi-compact and the final statement follows from the final statement of More on Flatness, Lemma 38.24.3. □

77.11. Universal flattening

0CWQ This section is the analogue of More on Flatness, Section 38.27. Our main aim is to prove Lemma 77.11.8. However, we do not see a way to deduce this result from the corresponding result for schemes directly. Hence we have to redevelop some of the material here. But first a definition.

0CWR **Definition 77.11.1.** Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 77.7.8 is an algebraic space. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

This is a bit unsatisfactory, because here the definition of universal flattening does not agree with the one used in the case of schemes, as we don't know whether every monomorphism of algebraic spaces is representable (More on Morphisms of Spaces, Section 76.4). Hopefully no confusion will ever result from this.

0CWS **Lemma 77.11.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent*

(1) *for some commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with surjective, étale vertical arrows where U and V are schemes, the sheaf $\varphi^*\mathcal{F}$ is flat over V in dimensions $\geq n$ (More on Flatness, Definition 38.20.10),

(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale vertical arrows where U and V are schemes, the sheaf $\varphi^*\mathcal{F}$ is flat over V in dimensions $\geq n$, and

(3) *for $x \in |X|$ such that $\mathcal{F}$ is not flat at x over Y the transcendence degree of $x/f(x)$ is $< n$ (Morphisms of Spaces, Definition 67.33.1).*

If this is true, then it remains true after any base change $Y' \to Y$.

Proof. Suppose that we have a diagram as in (1). Then the equivalence of the conditions in More on Flatness, Lemma 38.20.9 shows that (1) and (3) are equivalent. But condition (3) is inherited by $\varphi^*\mathcal{F}$ for any $U \to V$ as in (2). Whence we see that (3) implies (2) by the result for schemes again. The result for schemes also implies the statement on base change. □

0CWT **Definition 77.11.3.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say *$\mathcal{F}$ is flat over Y in dimensions $\geq n$* if the equivalent conditions of Lemma 77.11.2 are satisfied.

0CWU **Situation 77.11.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over Y we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_Y T \to X$. Note that $f_T : X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms of Spaces, Lemma 67.23.3 and Modules on Sites, Lemma 18.23.4). Let $n \geq 0$. By Definition 77.11.3 and Lemma 77.11.2 we obtain a functor

(77.11.4.1)

0CWV $$F_n : (Sch/Y)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 77.11.4 we sometimes think of F_n as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_n \to Y$. Namely, if T is a scheme over S, then an element $h \in F_n(T)$ is a morphism $h : T \to Y$ such that the base change of $\mathcal{F}$ via h is flat over T in $\dim \geq n$. In particular, when we say that F_n is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

0CWW **Lemma 77.11.5.** *In Situation 77.11.4.*

(1) *The functor F_n satisfies the sheaf property for the fpqc topology.*

(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Proof of (1). Suppose that $\{T_i \to T\}$ is an fpqc covering of a scheme T over Y. We have to show that if $F_n(T_i)$ is nonempty for all i, then $F_n(T)$ is

nonempty. Choose a diagram as in part (1) of Lemma 77.11.2. Denote F'_n the corresponding functor for $\varphi^*\mathcal{F}$ and the morphism $U \to V$. By More on Flatness, Lemma 38.20.12 we have the sheaf property for F'_n. Thus we get the sheaf property for F_n because for $T \to Y$ we have $F_n(T) = F'_n(V \times_Y T)$ by Lemma 77.11.2 and because $\{V \times_Y T_i \to V \times_Y T\}$ is an fpqc covering.

Proof of (2). Suppose that $T = \lim_{i \in I} T_i$ is a filtered limit of affine schemes T_i over Y and assume that $F_n(T)$ is nonempty. We have to show that $F_n(T_i)$ is nonempty for some i. Choose a diagram as in part (1) of Lemma 77.11.2. Fix $i \in I$ and choose an affine open $W_i \subset V \times_Y T_i$ mapping surjectively onto T_i. For $i' \geq i$ let $W_{i'}$ be the inverse image of W_i in $V \times_Y T_{i'}$ and let $W \subset V \times_Y T$ be the inverse image of W_i. Then $W = \lim_{i' \geq i} W_i$ is a filtered limit of affine schemes over V. By Lemma 77.11.2 again it suffices to show that $F'_n(W_{i'})$ is nonempty for some $i' \geq i$. But we know that $F'_n(W)$ is nonempty because of our assumption that $F_n(T) = F'_n(V \times_Y T)$ is nonempty. Thus we can apply More on Flatness, Lemma 38.20.12 to conclude. □

0CX0 **Lemma 77.11.6.** *In Situation 77.11.4. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (77.11.4.1) associated to $(f' : X' \to Y, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/Y}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Choose $U \to X$, $V \to Y$, $U \to V$ as in part (1) of Lemma 77.11.2. Choose a surjective étale morphism $U' \to U \times_X X'$ where U' is a scheme. Then we have the lemma for the two functors $F_{U,n}$ and $F_{U',n}$ determined by $U' \to U$ and $\mathcal{F}|_U$ over V, see More on Flatness, Lemma 38.27.2. On the other hand, Lemma 77.11.2 tells us that given $T \to Y$ we have $F_n(T) = F_{U,n}(V \times_Y T)$ and $F'_n(T) = F_{U',n}(V \times_Y T)$. This proves the lemma. □

0CX1 **Theorem 77.11.7.** *In Situation 77.11.4. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to Y. Then F_n is an algebraic space and $F_n \to Y$ is a monomorphism of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 77.11.5. Since $F_n \to Y$ is a monomorphism of sheaves on $(Sch/S)_{fppf}$ we see that $\Delta : F_n \to F_n \times F_n$ is the pullback of the diagonal $\Delta_Y : Y \to Y \times_S Y$. Hence the representability of Δ_Y implies the same thing for F_n. Therefore it suffices to prove that there exists a scheme W over S and a surjective étale morphism $W \to F_n$.

To construct $W \to F_n$ choose an étale covering $\{Y_i \to Y\}$ with Y_i a scheme. Let $X_i = X \times_Y Y_i$ and let $\mathcal{F}_i$ be the pullback of $\mathcal{F}$ to X_i. Then $\mathcal{F}_i$ is pure relative to Y_i either by definition or by Lemma 77.3.3. The other assumptions of the theorem are preserved as well. Finally, the restriction of F_n to Y_i is the functor F_n corresponding to $X_i \to Y_i$ and $\mathcal{F}_i$. Hence it suffices to show: Given $\mathcal{F}$ and $f : X \to Y$ as in the statement of the theorem where Y is a scheme, the functor F_n is representable by a scheme Z_n and $Z_n \to Y$ is a monomorphism of finite presentation.

Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 29.20.15). Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemmas 35.23.33 and 35.23.15 we see that the question is local for the étale topology on Y.

In particular the situation is local for the Zariski topology on Y and we may assume that Y is affine. In this case the dimension of the fibres of f is bounded above, hence

we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to Y$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_Y X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to Y$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 77.3.3 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $Y_{n+1} = Y$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over Y.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over the affine scheme Y. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on Y it suffices to show that for every $y \in Y$ there exists an étale neighbourhood $(Y', y') \to (Y, y)$ such that the result holds after base change to Y'. Thus by Lemma 77.4.1 we may assume there exist étale morphisms $h_j : W_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/W_j/Y$ over y, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to W_j and such that $|X_y| \subset \bigcup h_j(W_j)$. Since h_j is étale, by Lemma 77.11.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over Y. Set $W = \bigcup h_j(W_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_y we see that

$$E = \{t \in |Y| : \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of y. By Divisors on Spaces, Lemma 71.4.10 E is a constructible subset of Y. We have seen that $\mathrm{Spec}(\mathcal{O}_{Y,y}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of y. Hence after shrinking Y we may assume that $E = Y$. It follows from Lemma 77.11.6 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod W_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $W_j \to Y$ and the sheaf $\mathcal{F}_j$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to Y$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_Y \ldots \times_Y Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in More on Flatness, Lemma 38.27.4. □

Thus we finally obtain the desired result.

0CX2 **Lemma 77.11.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to Y, then there exists a universal flattening $Y' \to Y$ of $\mathcal{F}$. Moreover $Y' \to Y$ is a monomorphism of finite presentation.*
2. *If f is of finite presentation and X is pure relative to Y, then there exists a universal flattening $Y' \to Y$ of X. Moreover $Y' \to Y$ is a monomorphism of finite presentation.*
3. *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $Y' \to Y$ of $\mathcal{F}$. Moreover $Y' \to Y$ is a monomorphism of finite presentation.*
4. *If f is proper and of finite presentation then there exists a universal flattening $Y' \to Y$ of X.*

Proof. These statements follow immediately from Theorem 77.11.7 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 77.3.6. □

77.12. Grothendieck's Existence Theorem

0CX3 This section is the analogue of More on Flatness, Section 38.28 and continues the discussion in More on Morphisms of Spaces, Section 76.42. We will work in the following situation.

0CX4 **Situation 77.12.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have an algebraic space X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subspace of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms of Spaces, Lemma 67.14.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CX5 **Lemma 77.12.2.** *In Situation 77.12.1 consider*

$$K = R\lim_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Spaces, Example 75.19.5. □

0CX6 **Lemma 77.12.3.** *In Situation 77.12.1 let K be as in Lemma 77.12.2. For any perfect object E of $D(\mathcal{O}_X)$ we have*

1. *$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$,*
2. *$N = R\mathrm{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\mathrm{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes^{\mathbf{L}}_A A_n$ in $D(A_n)$.*

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\mathrm{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\mathrm{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology on Sites, Section 21.36. Hence by Derived Categories of Spaces, Lemma 75.25.8 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.98.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, K) = R\lim R\mathrm{Hom}_X(E, \mathcal{F}_n) = R\lim R\mathrm{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology on Sites, Lemma 21.48.4. □

0CX7 **Lemma 77.12.4.** *In Situation 77.12.1 let K be as in Lemma 77.12.2. Then K is pseudo-coherent relative to A.*

Proof. Combinging Lemma 77.12.3 and Derived Categories of Spaces, Lemma 75.25.7 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms of Spaces, Lemma 76.51.4. □

0CX8 **Lemma 77.12.5.** *In Situation 77.12.1 let K be as in Lemma 77.12.2. For any étale morphism $U \to X$ with U quasi-compact and quasi-separated we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \times_X X_n$.

Proof. Fix n. By Derived Categories of Spaces, Lemma 75.27.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 77.12.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0CX9 **Lemma 77.12.6.** *In Situation 77.12.1 let K be as in Lemma 77.12.2. Denote $X_0 \subset |X|$ the closed subset consisting of points lying over the closed subset $\text{Spec}(A_1) = \text{Spec}(A_2) = \ldots$ of $\text{Spec}(A)$. There exists an open subspace $W \subset X$ containing X_0 such that*

(1) *$H^i(K)|_W$ is zero unless $i = 0$,*
(2) *$\mathcal{F} = H^0(K)|_W$ is of finite presentation, and*
(3) *$\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.*

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \to X$ be an étale morphism with U affine and $u \in U$ a point mapping to x. Since $|U| \to |X|$ is open it suffices to find an open neighbourhood of u in U where (1), (2), and (3) are true. Say $U = \text{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 77.12.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \text{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 77.12.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point u is maximal. Hence in an open neighbourhood of u inside $\mathrm{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. □

0CXA **Lemma 77.12.7.** *In Situation 77.12.1 let K be as in Lemma 77.12.2. Let $W \subset X$ be as in Lemma 77.12.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.*

Proof. Fix $n \geq 1$. Let $I_n = \mathrm{Ker}(A \to A_n)$. By More on Algebra, Lemma 15.11.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the scheme theoretic support of $\mathcal{F}$. This is a closed subspace as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 77.12.6 we see that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\mathrm{Spec}(A/I)$. By More on Morphisms of Spaces, Lemma 76.36.10 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, $|Z_2|$ does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. □

0CXB **Theorem 77.12.8** (Grothendieck Existence Theorem)**.** *In Situation 77.12.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 77.12.2, 77.12.3, 77.12.4, 77.12.5, 77.12.6, and 77.12.7 to get an open subspace $W \subset X$ containing all points lying over $\mathrm{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$ for all $n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 77.3.6 we see that $\mathcal{F}$ is universally pure relative to $\mathrm{Spec}(A)$. By Theorem 77.11.7 (for explanation, see Lemma 77.11.8) there exists a universal flattening $S' \to \mathrm{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \mathrm{Spec}(A)$ is a monomorphism of finite presentation. In particular S' is a scheme (this follows from the proof of the theorem but it also follows a postoriori by Morphisms of Spaces, Proposition 67.50.2). Since the base change of $\mathcal{F}$ to $\mathrm{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)$ factors (uniquely) through S' for each n. By More on Flatness, Lemma 38.28.8 we see that $S' = \mathrm{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\mathrm{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. □

77.13. Grothendieck's Existence Theorem, bis

0DIJ In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 77.12 for quasi-coherent modules. This section is the analogue of More on Flatness, Section 38.29 for algebraic spaces. The classical case (for algebraic spaces) is discussed in More on Morphisms of Spaces, Section 76.42. We will work in the following situation.

0DIK **Situation 77.13.1.** Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have an algebraic space X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subspace of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li^*_{n-1}K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIL **Lemma 77.13.2.** *In Situation 77.13.1 consider*

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Spaces, Lemma 75.19.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Spaces, Lemma 75.19.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \to X$ be étale with U affine. Then there is a canonical exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology on Sites, Lemma 21.23.2. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Spaces, Lemma 75.13.6) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 36.3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology on Sites, Lemma 21.45.7). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too (argue by checking on stalks for example; small detail omitted). Thus $a_{n-1} = a_n$ for all n and we conclude. □

0DIM **Lemma 77.13.3.** *In Situation 77.13.1 let K be as in Lemma 77.13.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology*

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes^{\mathbf{L}}_A A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Spaces, Lemma 75.25.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes^{\mathbf{L}}_{A_n} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.98.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. □

0DIN **Lemma 77.13.4.** *In Situation 77.13.1 let K be as in Lemma 77.13.2. Then K is pseudo-coherent on X.*

Proof. Combinging Lemma 77.13.3 and Derived Categories of Spaces, Lemma 75.25.7 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms of Spaces, Lemma 76.51.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms of Spaces, Lemma 76.45.4. □

0DIP **Lemma 77.13.5.** *In Situation 77.13.1 let K be as in Lemma 77.13.2. For any étale morphism $U \to X$ with U quasi-compact and quasi-separated we have*

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \times_X X_n$.

Proof. Fix n. By Derived Categories of Spaces, Lemma 75.27.3 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 77.13.3 and the fact that $- \otimes^{\mathbf{L}}_A A_n$ commutes with homotopy colimits we obtain the result. □

0DIQ **Theorem 77.13.6** (Derived Grothendieck Existence Theorem)**.** *In Situation 77.13.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 77.13.2, 77.13.3, 77.13.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine U in Lemma 77.13.5 it follows immediately that K restricts to K_n over X_n. □

0DIR **Remark 77.13.7.** The result in this section can be generalized. It is probably correct if we only assume $X \to \text{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported

on a closed subset proper over A. If we ever need this, we will formulate a precise statement and prove it here.

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CHAPTER 78

Groupoids in Algebraic Spaces

0437

78.1. Introduction

0438 This chapter is devoted to generalities concerning groupoids in algebraic spaces. We recommend reading the beautiful paper [**KM97**] by Keel and Mori.

A lot of what we say here is a repeat of what we said in the chapter on groupoid schemes, see Groupoids, Section 39.1. The discussion of quotient stacks is new here.

78.2. Conventions

0439 The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

78.3. Notation

043A Let S be a scheme; this will be our base scheme and all algebraic spaces will be over S. Let B be an algebraic space over S; this will be our base algebraic space, and often other algebraic spaces, and schemes will be over B. If we say that X is an algebraic space over B, then we mean that X is an algebraic space over S which comes equipped with structure morphism $X \to B$. Moreover, we try to reserve the letter T to denote a "test" scheme over B. In other words T is a scheme which comes equipped with a structure morphism $T \to B$. In this situation we denote $X(T)$ for the set of T-valued points of X *over* B. In a formula:

$$X(T) = \mathrm{Mor}_B(T, X).$$

Similarly, given a second algebraic space Y over B we set

$$X(Y) = \mathrm{Mor}_B(Y, X).$$

Suppose we are given algebraic spaces X, Y over B as above and a morphism $f : X \to Y$ over B. For any scheme T over B we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which is functorial in the scheme T over B. As f is a map of sheaves on $(Sch/S)_{fppf}$ over the sheaf B it is clear that f determines and is determined by this rule. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are algebraic spaces over B, and if $m : X \times_B Y \to Z \times_B Z$ is a morphism

of algebraic spaces over B, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

Finally, given two maps $f, g : X \to Y$ of algebraic spaces over B, if the induced maps $f, g : X(T) \to Y(T)$ are equal for every scheme T over B, then $f = g$, and hence also $f, g : X(Z) \to Y(Z)$ are equal for every third algebraic space Z over B. Hence, for example, to check the axioms for an group algebraic space G over B, it suffices to check commutativity of diagram on T-valued points where T is a scheme over B as we do in Definition 78.5.1 below.

78.4. Equivalence relations

043B Please refer to Groupoids, Section 39.3 for notation.

043C **Definition 78.4.1.** Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B.

(1) A *pre-relation* on U over B is any morphism $j : R \to U \times_B U$ of algebraic spaces over B. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A *relation* on U over B is a monomorphism $j : R \to U \times_B U$ of algebraic spaces over B.
(3) A *pre-equivalence relation* is a pre-relation $j : R \to U \times_B U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all schemes T over B.
(4) We say a morphism $R \to U \times_B U$ of algebraic spaces over B is an *equivalence relation on U over B* if and only if for every T over B the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

043D **Lemma 78.4.2.** *Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Finally, set*

$$R' = (U' \times_B U') \times_{U \times_B U} R \xrightarrow{j'} U' \times_B U'$$

Then j' is a pre-relation on U' over B. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

043E **Definition 78.4.3.** Let $B \to S$ as in Section 78.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The pre-relation $j' : R' \to U' \times_B U'$ of Lemma 78.4.2 is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

043F **Lemma 78.4.4.** *Let $B \to S$ as in Section 78.3. Let $j : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Consider the relation on $|U|$ defined by the rule*

$$x \sim y \Leftrightarrow \exists\ r \in |R| : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in |R|$ with $t(r) = x$, $s(r) = y$ and pick $r' \in |R|$ with $t(r') = y$, $s(r') = z$. We may pick a field K such that r and r' can be represented by morphisms $r, r' : \mathrm{Spec}(K) \to R$ with $s \circ r = t \circ r'$. Denote $x = t \circ r$, $y = s \circ r = t \circ r'$, and $z = s \circ r'$, so $x, y, z : \mathrm{Spec}(K) \to U$. By construction $(x, y) \in j(R(K))$ and $(y, z) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x, z) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

78.5. Group algebraic spaces

043G Please refer to Groupoids, Section 39.4 for notation.

043H **Definition 78.5.1.** Let $B \to S$ as in Section 78.3.

(1) A *group algebraic space over B* is a pair (G, m), where G is an algebraic space over B and $m : G \times_B G \to G$ is a morphism of algebraic spaces over B with the following property: For every scheme T over B the pair $(G(T), m)$ is a group.
(2) A *morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B* is a morphism $\psi : G \to G'$ of algebraic spaces over B such that for every T/B the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group algebraic space over the algebraic space B. By the discussion in Groupoids, Section 39.4 we obtain morphisms of algebraic spaces over B (identity) $e : B \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group.

Let (G, m), (G', m') be group algebraic spaces over B. Let $f : G \to G'$ be a morphism of algebraic spaces over B. It follows from the definition that f is a morphism of group algebraic spaces over B if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_B G & \xrightarrow{f \times f} & G' \times_B G' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{f} & G' \end{array}$$

043I **Lemma 78.5.2.** *Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let $B' \to B$ be a morphism of algebraic spaces. The pullback $(G_{B'}, m_{B'})$ is a group algebraic space over B'.*

Proof. Omitted. □

78.6. Properties of group algebraic spaces

06P5 In this section we collect some simple properties of group algebraic spaces which hold over any base.

06P6 **Lemma 78.6.1.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then $G \to B$ is separated (resp. quasi-separated, resp. locally separated) if and only if the identity morphism $e : B \to G$ is a closed immersion (resp. quasi-compact, resp. an immersion).*

Proof. We recall that by Morphisms of Spaces, Lemma 67.4.7 we have that e is a closed immersion (resp. quasi-compact, resp. an immersion) if $G \to B$ is separated (resp. quasi-separated, resp. locally separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow[\Delta_{G/B}]{} & G \times_B G \\ \downarrow & & \downarrow{\scriptstyle (g,g') \mapsto m(i(g),g')} \\ B & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/B}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact, resp. an immersion) so is $\Delta_{G/B}$, see Spaces, Lemma 65.12.3 (resp. Morphisms of Spaces, Lemma 67.8.4, resp. Spaces, Lemma 65.12.3). □

0DSI **Lemma 78.6.2.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Assume $G \to B$ is locally of finite type. Then $G \to B$ is unramified (resp. locally quasi-finite) if and only if $G \to B$ is unramified (resp. quasi-finite) at $e(b)$ for all $b \in |B|$.*

Proof. By Morphisms of Spaces, Lemma 67.38.10 (resp. Morphisms of Spaces, Lemma 67.27.2) there is a maximal open subspace $U \subset G$ such that $U \to B$ is unramified (resp. locally quasi-finite) and formation of U commutes with base change. Thus we reduce to the case where $B = \mathrm{Spec}(k)$ is the spectrum of a field. Let $g \in G(K)$ be a point with values in an extension K/k. Then to check whether or not g is in U, we may base change to K. Hence it suffices to show

$$G \to \mathrm{Spec}(k) \text{ is unramified at } e \Leftrightarrow G \to \mathrm{Spec}(k) \text{ is unramified at } g$$

for a k-rational point g (resp. similarly for quasi-finite at g and e). Since translation by g is an automorphism of G over k this is clear. □

0DSJ **Lemma 78.6.3.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Assume $G \to B$ is locally of finite type.*

1. *There exists a maximal open subspace $U \subset B$ such that $G_U \to U$ is unramified and formation of U commutes with base change.*
2. *There exists a maximal open subspace $U \subset B$ such that $G_U \to U$ is locally quasi-finite and formation of U commutes with base change.*

Proof. By Morphisms of Spaces, Lemma 67.38.10 (resp. Morphisms of Spaces, Lemma 67.27.2) there is a maximal open subspace $W \subset G$ such that $W \to B$ is unramified (resp. locally quasi-finite). Moreover formation of W commutes with base change. By Lemma 78.6.2 we see that $U = e^{-1}(W)$ in either case. □

78.7. Examples of group algebraic spaces

06P7 If $G \to S$ is a group scheme over the base scheme S, then the base change G_B to any algebraic space B over S is an group algebraic space over B by Lemma 78.5.2. We will frequently use this in the examples below.

043J **Example 78.7.1** (Multiplicative group algebraic space)**.** Let $B \to S$ as in Section 78.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{m,B} = B \times_S \mathbf{G}_{m,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the multiplicative group scheme over S, see Groupoids, Example 39.5.1.

043K **Example 78.7.2** (Roots of unity as a group algebraic space)**.** Let $B \to S$ as in Section 78.3. Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T over B the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the group algebraic space

$$\mu_{n,B} = B \times_S \mu_{n,S}$$

over B. Here $\mu_{n,S}$ is the group scheme of nth roots of unity over S, see Groupoids, Example 39.5.2.

043L **Example 78.7.3** (Additive group algebraic space)**.** Let $B \to S$ as in Section 78.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{a,B} = B \times_S \mathbf{G}_{a,S}$$

over B. Here $\mathbf{G}_{a,S}$ is the additive group scheme over S, see Groupoids, Example 39.5.3.

043M **Example 78.7.4** (General linear group algebraic space)**.** Let $B \to S$ as in Section 78.3. Let $n \geq 1$. Consider the functor which associates to any scheme T over B the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathrm{GL}_{n,B} = B \times_S \mathrm{GL}_{n,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the general linear group scheme over S, see Groupoids, Example 39.5.4.

043N **Example 78.7.5.** Let $B \to S$ as in Section 78.3. Let $n \geq 1$. The determinant defines a morphism of group algebraic spaces

$$\det : \mathrm{GL}_{n,B} \longrightarrow \mathbf{G}_{m,B}$$

over B. It is the base change of the determinant morphism over S from Groupoids, Example 39.5.5.

043O **Example 78.7.6** (Constant group algebraic space)**.** Let $B \to S$ as in Section 78.3. Let G be an abstract group. Consider the functor which associates to any scheme T over B the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the group algebraic space

$$G_B = B \times_S G_S$$

over B. Here G_S is the constant group scheme introduced in Groupoids, Example 39.5.6.

78.8. Actions of group algebraic spaces

043P Please refer to Groupoids, Section 39.10 for notation.

043Q **Definition 78.8.1.** Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B.

(1) An *action of G on the algebraic space X/B* is a morphism $a : G \times_B X \to X$ over B such that for every scheme T over B the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are algebraic spaces over B each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

043R (78.8.1.1)
$$\begin{array}{ccc} G \times_B G \times_B X & \xrightarrow{1_G \times a} & G \times_B X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ G \times_B X & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} G \times_B X & \xrightarrow{a} & X \\ {\scriptstyle e \times 1_X}\uparrow & \nearrow_{1_X} & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_B X & \xrightarrow{\mathrm{id} \times f} & G \times_B Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

06P8 **Definition 78.8.2.** Let $B \to S$, $G \to B$, and $X \to B$ as in Definition 78.8.1. Let $a : G \times_B X \to X$ be an action of G on X/B. We say the action is *free* if for every scheme T over B the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

06P9 **Lemma 78.8.3.** *Situation as in Definition 78.8.2, The action a is free if and only if*

$$G \times_B X \to X \times_B X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism of algebraic spaces.

Proof. Immediate from the definitions. □

78.9. Principal homogeneous spaces

04TV This section is the analogue of Groupoids, Section 39.11. We suggest reading that section first.

04TW **Definition 78.9.1.** Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism $G \times_B X \to X \times_B X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over B where G acts on G by left multiplication.

It is clear that if $B' \to B$ is a morphism of algebraic spaces then the pullback $X_{B'}$ of a pseudo G-torsor over B is a pseudo $G_{B'}$-torsor over B'.

04TX **Lemma 78.9.2.** *In the situation of Definition 78.9.1.*

(1) *The algebraic space X is a pseudo G-torsor if and only if for every scheme T over B the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to B$ has a section.*

Proof. Omitted. □

04TY **Definition 78.9.3.** Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be a pseudo G-torsor over B.

(1) We say X is a *principal homogeneous space*, or more precisely a *principal homogeneous G-space over B* if there exists a fpqc covering[1] $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section (i.e., is a trivial pseudo G_{B_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section.
(3) If X is a principal homogeneous G-space over B, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a principal homogeneous G-space over B, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-principal homogeneous space over B" to indicate that X is an algebraic space over B equipped with an action of G which turns it into a principal homogeneous space over B. Next we show that this agrees with the notation introduced earlier when both apply.

04TZ **Lemma 78.9.4.** *Let S be a scheme. Let (G, m) be a group algebraic space over S. Let X be an algebraic space over S, and let $a : G \times_S X \to X$ be an action of G on X. Then X is a G-torsor in the $fppf$-topology in the sense of Definition 78.9.3 if and only if X is a G-torsor on $(Sch/S)_{fppf}$ in the sense of Cohomology on Sites, Definition 21.4.1.*

Proof. Omitted. □

0DSK **Lemma 78.9.5.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be a pseudo G-torsor over B. Assume G and X locally of finite type over B.*

(1) *If $G \to B$ is unramified, then $X \to B$ is unramified.*
(2) *If $G \to B$ is locally quasi-finite, then $X \to B$ is locally quasi-finite.*

Proof. Proof of (1). By Morphisms of Spaces, Lemma 67.38.10 we reduce to the case where B is the spectrum of a field. If X is empty, then the result holds. If X is nonempty, then after increasing the field, we may assume X has a point. Then $G \cong X$ and the result holds.

[1] The default type of torsor in Groupoids, Definition 39.11.3 is a pseudo torsor which is trivial on an fpqc covering. Since G, as an algebraic space, can be seen a sheaf of groups there already is a notion of a G-torsor which corresponds to fppf-torsor, see Lemma 78.9.4. Hence we use "principal homogeneous space" for a pseudo torsor which is fpqc locally trivial, and we try to avoid using the word torsor in this situation.

The proof of (2) works in exactly the same way using Morphisms of Spaces, Lemma 67.27.2. □

78.10. Equivariant quasi-coherent sheaves

043S Please compare with Groupoids, Section 39.12.

043T **Definition 78.10.1.** Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on the algebraic space X over B. An *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply a *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G \times_B X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_B X \to X$ is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} & \xrightarrow{\mathrm{pr}_{12}^*\alpha} & \mathrm{pr}_2^*\mathcal{F} \\ {\scriptstyle (1_G\times a)^*\alpha}\uparrow & & \uparrow{\scriptstyle (m\times 1_X)^*\alpha} \\ (1_G \times a)^*a^*\mathcal{F} & = & (m \times 1_X)^*a^*\mathcal{F} \end{array}$$

is a commutative in the category of $\mathcal{O}_{G \times_B G \times_B X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (78.8.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

043U **Lemma 78.10.2.** *Let $B \to S$ as in Section 78.3. Let G be a group algebraic space over B. Let $f : X \to Y$ be a G-equivariant morphism between algebraic spaces over B endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of quasi-coherent G-equivariant sheaves on Y to the category of quasi-coherent G-equivariant sheaves on X.*

Proof. Omitted. □

78.11. Groupoids in algebraic spaces

043V Please refer to Groupoids, Section 39.13 for notation.

043W **Definition 78.11.1.** Let $B \to S$ as in Section 78.3.

(1) A *groupoid in algebraic spaces over B* is a quintuple (U, R, s, t, c) where U and R are algebraic spaces over B, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of algebraic spaces over B with the following property: For any scheme T over B the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) A *morphism* $f : (U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in algebraic spaces over* B is given by morphisms of algebraic spaces $f : U \to U'$ and $f : R \to R'$ over B with the following property: For any scheme T over B the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Note that there are unique morphisms of algebraic spaces $e : U \to R$ and $i : R \to R$ over B such that for every scheme T over B the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) of Groupoids, Section 39.13. Conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over B. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid in algebraic spaces over B we denote

$$j = (t, s) : R \longrightarrow U \times_B U$$

which is compatible with our conventions in Section 78.4 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B" to stress the existence of identity and inverse.

043X **Lemma 78.11.2.** *Let $B \to S$ as in Section 78.3. Given a groupoid in algebraic spaces (U, R, s, t, c) over B the morphism $j : R \to U \times_B U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. □

043Y **Lemma 78.11.3.** *Let $B \to S$ as in Section 78.3. Given an equivalence relation $j : R \to U \times_B U$ over B there is a unique way to extend it to a groupoid in algebraic spaces (U, R, s, t, c) over B.*

Proof. Omitted. This is a nice exercise in the definitions. □

043Z **Lemma 78.11.4.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram*

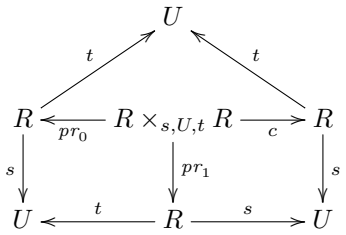

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

0450 **Lemma 78.11.5.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. The diagram*

04P3 (78.11.5.1)
$$\begin{array}{ccccc}
R \times_{t,U,t} R & \xrightarrow[pr_0]{pr_1} & R & \xrightarrow{t} & U \\
\downarrow{\scriptstyle pr_0 \times c \circ (i,1)} & & \downarrow{\scriptstyle id_R} & & \downarrow{\scriptstyle id_U} \\
R \times_{s,U,t} R & \xrightarrow[pr_0]{c} & R & \xrightarrow{t} & U \\
\downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle s} & & \\
R & \xrightarrow[t]{s} & U & &
\end{array}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 78.11.4. □

0DTA **Lemma 78.11.6.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $B' \to B$ be a morphism of algebraic spaces. Then the base changes $U' = B' \times_B U$, $R' = B' \times_B R$ endowed with the base changes s', t', c' of the morphisms s, t, c form a groupoid in algebraic spaces (U', R', s', t', c') over B' and the projections determine a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B.*

Proof. Omitted. Hint: $R' \times_{s',U',t'} R' = B' \times_B (R \times_{s,U,t} R)$. □

78.12. Quasi-coherent sheaves on groupoids

0440 Please compare with Groupoids, Section 39.14.

0441 **Definition 78.12.1.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{array}{ccccc}
 & & \mathrm{pr}_1^* t^*\mathcal{F} & \xrightarrow[\mathrm{pr}_1^*\alpha]{} & \mathrm{pr}_1^* s^*\mathcal{F} & & \\
 & \swarrow & \| & & \| & \searrow & \\
\mathrm{pr}_0^* s^*\mathcal{F} & & & & & & c^* s^*\mathcal{F} \\
 & \nwarrow{\scriptstyle \mathrm{pr}_0^*\alpha} & & & & \nearrow{\scriptstyle c^*\alpha} & \\
 & & \mathrm{pr}_0^* t^*\mathcal{F} & = & c^* t^*\mathcal{F} & &
\end{array}$$

is a commutative in the category of $\mathcal{O}_{R \times_{s,U,t} R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 78.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

077W **Lemma 78.12.2.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 78.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. □

0442 **Lemma 78.12.3.** *Let $B \to S$ as in Section 78.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid in algebraic spaces over B. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. □

0GPM **Lemma 78.12.4.** *Let $B \to S$ as in Section 78.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B. Assume that*

(1) *$f : U \to U'$ is quasi-compact and quasi-separated,*
(2) *the square*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian, and
(3) *s' and t' are flat.*

Then pushforward f_* given by

$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U, R, s, t, c) to the category of quasi-coherent sheaves on (U', R', s', t', c') which is right adjoint to pullback as defined in Lemma 78.12.3.

Proof. Since $U \to U'$ is quasi-compact and quasi-separated we see that f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces,

Lemma 67.11.2). Moreover, since the squares

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

are cartesian we find that $(t')^*f_*\mathcal{F} = f_*t^*\mathcal{F}$ and $(s')^*f_*\mathcal{F} = f_*s^*\mathcal{F}$, see Cohomology of Spaces, Lemma 69.11.2. Thus it makes sense to think of $f_*\alpha$ as a map $(t')^*f_*\mathcal{F} \to (s')^*f_*\mathcal{F}$. A similar argument shows that $f_*\alpha$ satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and push-forward on modules on ringed spaces are adjoint. Some details omitted. □

077X **Lemma 78.12.5.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Properties of Spaces, Lemma 66.29.7. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

06VZ **Lemma 78.12.6.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^* \operatorname{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^* \operatorname{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^* \operatorname{Ker}(\varphi) \to s^* \operatorname{Ker}(\varphi)$ and $\lambda : t^* \operatorname{Coker}(\varphi) \to s^* \operatorname{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\operatorname{Ker}(\varphi), \kappa)$ and $(\operatorname{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\operatorname{Coim}(\varphi) = \operatorname{Im}(\varphi)$ follows because it holds over U. □

78.13. Colimits of quasi-coherent modules

0GPN This section is the analogue of Groupoids, Section 39.15.

0GPP **Lemma 78.13.1.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume s, t are flat, quasi-compact, and quasi-separated. For any quasi-coherent module $\mathcal{G}$ on U, there exists a canonical isomorphism $\alpha : t^*s_*t^*\mathcal{G} \to s^*s_*t^*\mathcal{G}$ which turns $(s_*t^*\mathcal{G}, \alpha)$ into a quasi-coherent module on (U, R, s, t, c). This construction defines a functor*

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$.

Proof. The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Morphisms of Spaces, Lemma 67.11.2. Hence $s_*t^*\mathcal{G}$ is quasi-coherent. With notation as in Lemma 78.11.4 we have

$$t^*s_*t^*\mathcal{G} = \mathrm{pr}_{1,*}\mathrm{pr}_0^*t^*\mathcal{G} = \mathrm{pr}_{1,*}c^*t^*\mathcal{G} = s^*s_*t^*\mathcal{G}$$

The middle equality because $t \circ c = t \circ \mathrm{pr}_0$ as morphisms $R \times_{s,U,t} R \to U$, and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Spaces, Lemma 69.11.2.

To verify the cocycle condition of Definition 78.12.1 for α and the adjointness property we describe the construction $\mathcal{G} \mapsto (s_*t^*\mathcal{G}, \alpha)$ in another way. Consider the groupoid scheme $(R, R \times_{t,U,t} R, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ associated to the equivalence relation $R \times_{t,U,t} R$ on R, see Lemma 78.11.3. There is a morphism

$$f : (R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by $s : R \to U$ and $R \times_{t,U,t} R \to R$ given by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$; we omit the verification of the commutativity of the required diagrams. Since $t, s : R \to U$ are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{array}{ccc} R \times_{t,U,t} R & \xrightarrow[(r_0,r_1)\mapsto r_0^{-1}\circ r_1]{} & R \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow{\quad s \quad} & U \end{array}$$

by Lemma 78.11.5 it follows that Lemma 78.12.4 applies to f. Thus pushforward and pullback of quasi-coherent modules along f are adjoint functors. To finish the proof we will identify these functors with the functors described above. To do this, note that

$$t^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(R, R \times_{t,U,t} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is an equivalence by the theory of descent of quasi-coherent sheaves as $\{t : R \to U\}$ is an fpqc covering, see Descent on Spaces, Proposition 74.4.1.

Pushforward along f precomposed with the equivalence t^* sends $\mathcal{G}$ to $(s_*t^*\mathcal{G}, \alpha)$; we omit the verification that the isomorphism α obtained in this fashion is the same as the one constructed above.

Pullback along f postcomposed with the inverse of the equivalence t^* sends $(\mathcal{F}, \beta)$ to the descent relative to $\{t : R \to U\}$ of the module $s^*\mathcal{F}$ endowed with the descent datum γ on $R \times_{t,U,t} R$ which is the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. Consider the isomorphism $\beta : t^*\mathcal{F} \to s^*\mathcal{F}$. The canonical descent datum (Descent on Spaces, Definition 74.3.3) on $t^*\mathcal{F}$ relative to $\{t : R \to U\}$ translates via β into the map

$$\mathrm{pr}_0^*s^*\mathcal{F} \xrightarrow{\mathrm{pr}_0^*\beta^{-1}} \mathrm{pr}_0^*t^*\mathcal{F} \xrightarrow{can} \mathrm{pr}_1^*t^*\mathcal{F} \xrightarrow{\mathrm{pr}_1^*\beta} \mathrm{pr}_1^*s^*\mathcal{F}$$

Since β satisfies the cocycle condition, this is equal to the pullback of β by $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$. To see this take the actual cocycle relation in Definition 78.12.1 and pull it back by the morphism $(\mathrm{pr}_0, c \circ (i, 1)) : R \times_{t,U,t} R \to R \times_{s,U,t} R$ which also plays a role in the commutative diagram of Lemma 78.11.5. It follows that $(s^*\mathcal{F}, \gamma)$ is isomorphic to $(t^*\mathcal{F}, can)$. All in all, we conclude that pullback by f postcomposed with the inverse of the equivalence t^* is isomorphic to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$. $\square$

0GPQ **Remark 78.13.2.** In the situation of Lemma 78.13.1 denote

$$F : QCoh(U, R, s, t, c) \to QCoh(\mathcal{O}_U), \quad (\mathcal{F}, \beta) \mapsto \mathcal{F}$$

the forgetful functor and denote

$$G : QCoh(\mathcal{O}_U) \to QCoh(U, R, s, t, c), \quad \mathcal{G} \mapsto (s_* t^* \mathcal{G}, \alpha)$$

the right adjoint constructed in the lemma. Then the unit $\eta : \mathrm{id} \to G \circ F$ of the adjunction evaluated on $(\mathcal{F}, \beta)$ is given by the map

$$\mathcal{F} \to s_* s^* \mathcal{F} \xrightarrow{\beta^{-1}} s_* t^* \mathcal{F}$$

We omit the verification.

0GPR **Lemma 78.13.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module, and let $\varphi : \mathcal{G} \to f^*\mathcal{F}$ be a module map. Assume*

(1) *φ is injective,*
(2) *f is quasi-compact, quasi-separated, flat, and surjective,*
(3) *X, Y are locally Noetherian, and*
(4) *$\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.*

Then $\mathcal{F} \cap f_*\mathcal{G}$ defined as the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f_* f^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F} \cap f_* \mathcal{G} & \longrightarrow & f_* \mathcal{G} \end{array}$$

is a coherent $\mathcal{O}_X$-module.

Proof. We will freely use the characterization of coherent modules of Cohomology of Spaces, Lemma 69.12.2 as well as the fact that coherent modules form a Serre subcategory of $QCoh(\mathcal{O}_X)$, see Cohomology of Spaces, Lemma 69.12.4. If f has a section σ, then we see that $\mathcal{F} \cap f_*\mathcal{G}$ is contained in the image of $\sigma^*\mathcal{G} \to \sigma^* f^* \mathcal{F} = \mathcal{F}$, hence coherent. In general, to show that $\mathcal{F} \cap f_*\mathcal{G}$ is coherent, it suffices the show that $f^*(\mathcal{F} \cap f_*\mathcal{G})$ is coherent (see Descent on Spaces, Lemma 74.6.1). Since f is flat this is equal to $f^*\mathcal{F} \cap f^* f_* \mathcal{G}$. Since f is flat, quasi-compact, and quasi-separated we see $f^* f_* \mathcal{G} = p_* q^* \mathcal{G}$ where $p, q : Y \times_X Y \to Y$ are the projections, see Cohomology of Spaces, Lemma 69.11.2. Since p has a section we win. □

Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume that U is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module.

0GPS **Lemma 78.13.4.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Assume that*

(1) *U, R are Noetherian,*
(2) *s, t are flat, quasi-compact, and quasi-separated.*

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is a filtered colimit of coherent modules.

Proof. We will use the characterization of Cohomology of Spaces, Lemma 69.12.2 of coherent modules on locally Noetherian algebraic spaces without further mention. We can write $\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ as the filtered colimit of coherent submodules $\mathcal{H}_i \subset \mathcal{F}$, see Cohomology of Spaces, Lemma 69.15.1. Given a quasi-coherent sheaf $\mathcal{H}$ on U we denote $(s_*t^*\mathcal{H}, \alpha)$ the quasi-coherent sheaf on (U, R, s, t, c) of Lemma 78.13.1. Consider the adjunction map $(\mathcal{F}, \beta) \to (s_*t^*\mathcal{F}, \alpha)$ in $QCoh(U, R, s, t, c)$, see Remark 78.13.2. Set

$$(\mathcal{F}_i, \beta_i) = (\mathcal{F}, \beta) \times_{(s_*t^*\mathcal{F}, \alpha)} (s_*t^*\mathcal{H}_i, \alpha)$$

in $QCoh(U, R, s, t, c)$. Since restriction to U is an exact functor on $QCoh(U, R, s, t, c)$ by the proof of Lemma 78.12.6 we obtain a pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_*t^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_*t^*\mathcal{H}_i \end{array}$$

in other words $\mathcal{F}_i = \mathcal{F} \cap s_*t^*\mathcal{H}_i$. By the description of the adjunction map in Remark 78.13.2 this diagram is isomorphic to the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_*s^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_*t^*\mathcal{H}_i \end{array}$$

where the right vertical arrow is the result of applying s_* to the map

$$t^*\mathcal{H}_i \to t^*\mathcal{F} \xrightarrow{\beta} s^*\mathcal{F}$$

This arrow is injective as t is a flat morphism. It follows that $\mathcal{F}_i$ is coherent by Lemma 78.13.3. Finally, because s is quasi-compact and quasi-separated we see that s_* commutes with colimits (see Cohomology of Schemes, Lemma 30.6.1). Hence $s_*t^*\mathcal{F} = \operatorname{colim} s_*t^*\mathcal{H}_i$ and hence $(\mathcal{F}, \beta) = \operatorname{colim}(\mathcal{F}_i, \beta_i)$ as desired. □

78.14. Crystals in quasi-coherent sheaves

077Y Let (I, Φ, j) be a pair consisting of a set I and a pre-relation $j : \Phi \to I \times I$. Assume given for every $i \in I$ a scheme X_i and for every $\phi \in \Phi$ a morphism of schemes $f_\phi : X_{i'} \to X_i$ where $j(\phi) = (i, i')$. Set $X = (\{X_i\}_{i \in I}, \{f_\phi\}_{\phi \in \Phi})$. Define a *crystal in quasi-coherent modules on* X as a rule which associates to every $i \in \mathrm{Ob}(\mathcal{I})$ a quasi-coherent sheaf $\mathcal{F}_i$ on X_i and for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ an isomorphism

$$\alpha_\phi : f_\phi^*\mathcal{F}_i \longrightarrow \mathcal{F}_{i'}$$

of quasi-coherent sheaves on $X_{i'}$. These crystals in quasi-coherent modules form an additive category $CQC(X)$[2]. This category has colimits (proof is the same as the proof of Lemma 78.12.5). If all the morphisms f_ϕ are flat, then $CQC(X)$ is abelian (proof is the same as the proof of Lemma 78.12.6). Let κ be a cardinal.

[2]We could single out a set of triples $\phi, \phi', \phi'' \in \Phi$ with $j(\phi) = (i, i')$, $j(\phi') = (i', i'')$, and $j(\phi'') = (i, i'')$ such that $f_{\phi''} = f_\phi \circ f_{\phi'}$ and require that $\alpha_{\phi'} \circ f_{\phi'}^*\alpha_\phi = \alpha_{\phi''}$ for these triples. This would define an additive subcategory. For example the data (I, Φ) could be the set of objects and arrows of an index category and X could be a diagram of schemes over this index category. The result of Lemma 78.14.1 immediately gives the corresponding result in the subcategory.

We say that a crystal in quasi-coherent modules $\mathcal{F}$ on X is *κ-generated* if each $\mathcal{F}_i$ is κ-generated (see Properties, Definition 28.23.1).

077Z **Lemma 78.14.1.** *In the situation above, if all the morphisms f_ϕ are flat, then there exists a cardinal κ such that every object $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ of $CQC(X)$ is the directed colimit of its κ-generated submodules.*

Proof. In the lemma and in this proof a *submodule* of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ means the data of a quasi-coherent submodule $\mathcal{G}_i \subset \mathcal{F}_i$ for all i such that $\alpha_\phi(f_\phi^*\mathcal{G}_i) = \mathcal{G}_{i'}$ as subsheaves of $\mathcal{F}_{i'}$ for all $\phi \in \Phi$. This makes sense because since f_ϕ is flat the pullback f_ϕ^* is exact, i.e., preserves subsheaves. The proof will be a variant to the proof of Properties, Lemma 28.23.3. We urge the reader to read that proof first.

We claim that it suffices to prove the lemma in case all the schemes X_i are affine. To see this let

$$J = \coprod\nolimits_{i\in I} \{U \subset X_i \text{ affine open}\}$$

and let

$$\begin{aligned}\Psi = &\coprod\nolimits_{\phi\in\Phi} \{(U, V) \mid U \subset X_i, V \subset X_{i'} \text{ affine open with } f_\phi(U) \subset V\} \\ &\amalg \coprod\nolimits_{i\in I} \{(U, U') \mid U, U' \subset X_i \text{ affine open with } U \subset U'\}\end{aligned}$$

endowed with the obvious map $\Psi \to J \times J$. Then our $(\mathcal{F}, \alpha)$ induces a crystal in quasi-coherent sheaves $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ on $Y = (J, \Psi)$ by setting $\mathcal{H}_{(i,U)} = \mathcal{F}_i|_U$ for $(i, U) \in J$ and setting β_ψ for $\psi \in \Psi$ equal to the restriction of α_ϕ to U if $\psi = (\phi, U, V)$ and equal to $\mathrm{id} : (\mathcal{F}_i|_{U'})|_U \to \mathcal{F}_i|_U$ when $\psi = (i, U, U')$. Moreover, submodules of $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ correspond 1-to-1 with submodules of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. We omit the proof (hint: use Sheaves, Section 6.30). Moreover, it is clear that if κ works for Y, then the same κ works for X (by the definition of κ-generated modules). Hence it suffices to proof the lemma for crystals in quasi-coherent sheaves on Y.

Assume that all the schemes X_i are affine. Let κ be an infinite cardinal larger than the cardinality of I or Φ. Let $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ be an object of $CQC(X)$. For each i write $X_i = \mathrm{Spec}(A_i)$ and $M_i = \Gamma(X_i, \mathcal{F}_i)$. For every $\phi \in \Phi$ with $j(\phi) = (i, i')$ the map α_ϕ translates into an $A_{i'}$-module isomorphism

$$\alpha_\phi : M_i \otimes_{A_i} A_{i'} \longrightarrow M_{i'}$$

Using the axiom of choice choose a rule

$$(\phi, m) \longmapsto S(\phi, m')$$

where the source is the collection of pairs (ϕ, m') such that $\phi \in \Phi$ with $j(\phi) = (i, i')$ and $m' \in M_{i'}$ and where the output is a finite subset $S(\phi, m') \subset M_i$ so that

$$m' = \alpha_\phi\left(\sum\nolimits_{m\in S(\phi,m')} m \otimes a'_m\right)$$

for some $a'_m \in A_{i'}$.

Having made these choices we claim that any section of any $\mathcal{F}_i$ over any X_i is in a κ-generated submodule. To see this suppose that we are given a collection

$\mathcal{S} = \{S_i\}_{i\in I}$ of subsets $S_i \subset M_i$ each with cardinality at most κ. Then we define a new collection $\mathcal{S}' = \{S_i'\}_{i\in I}$ with

$$S_i' = S_i \cup \bigcup\nolimits_{(\phi,m'),\ j(\phi)=(i,i'),\ m'\in S_{i'}} S(\phi, m')$$

Note that each S_i' still has cardinality at most κ. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $S_i^{(\infty)} = \bigcup_{n\geq 0} S_i^{(n)}$ and $\mathcal{S}^{(\infty)} = \{S_i^{(\infty)}\}_{i\in I}$. By construction, for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ and every $m' \in S_{i'}^{(\infty)}$ we can write m' as a finite linear combination of images $\alpha_\phi(m \otimes 1)$ with $m \in S_i^{(\infty)}$. Thus we see that setting N_i equal to the A_i-submodule of M_i generated by $S_i^{(\infty)}$ the corresponding quasi-coherent submodules $\widetilde{N_i} \subset \mathcal{F}_i$ form a κ-generated submodule. This finishes the proof. □

0780 **Lemma 78.14.2.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then there exists a set T and a family of objects $(\mathcal{F}_t, \alpha_t)_{t\in T}$ of $QCoh(U, R, s, t, c)$ such that every object $(\mathcal{F}, \alpha)$ is the directed colimit of its submodules isomorphic to one of the objects $(\mathcal{F}_t, \alpha_t)$.*

Proof. This lemma is a generalization of Groupoids, Lemma 39.15.7 which deals with the case of a groupoid in schemes. We can't quite use the same argument, so we use the material on "crystals of quasi-coherent sheaves" we developed above.

Choose a scheme W and a surjective étale morphism $W \to U$. Choose a scheme V and a surjective étale morphism $V \to W \times_{U,s} R$. Choose a scheme V' and a surjective étale morphism $V' \to R \times_{t,U} W$. Consider the collection of schemes

$$I = \{W, W \times_U W, V, V', V \times_R V'\}$$

and the set of morphisms of schemes

$$\Phi = \{\mathrm{pr}_i : W \times_U W \to W, V \to W, V' \to W, V \times_R V' \to V, V \times_R V' \to V'\}$$

Set $X = (I, \Phi)$. Recall that we have defined a category $CQC(X)$ of crystals of quasi-coherent sheaves on X. There is a functor

$$QCoh(U, R, s, t, c) \longrightarrow CQC(X)$$

which assigns to $(\mathcal{F}, \alpha)$ the sheaf $\mathcal{F}|_W$ on W, the sheaf $\mathcal{F}|_{W\times_U W}$ on $W \times_U W$, the pullback of $\mathcal{F}$ via $V \to W \times_{U,s} R \to W \to U$ on V, the pullback of $\mathcal{F}$ via $V' \to R \times_{t,U} W \to W \to U$ on V', and finally the pullback of $\mathcal{F}$ via $V \times_R V' \to V \to W \times_{U,s} R \to W \to U$ on $V \times_R V'$. As comparison maps $\{\alpha_\phi\}_{\phi\in\Phi}$ we use the obvious ones (coming from associativity of pullbacks) except for the map $\phi = \mathrm{pr}_{V'} : V \times_R V' \to V'$ we use the pullback of $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ to $V \times_R V'$. This

makes sense because of the following commutative diagram

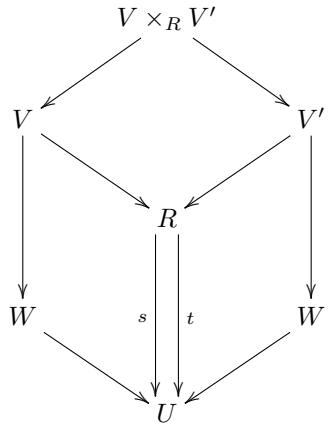

The functor displayed above isn't an equivalence of categories. However, since $W \to U$ is surjective étale it is faithful[3]. Since all the morphisms in the diagram above are flat we see that it is an exact functor of abelian categories. Moreover, we claim that given $(\mathcal{F}, \alpha)$ with image $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ there is a 1-to-1 correspondence between quasi-coherent submodules of $(\mathcal{F}, \alpha)$ and $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. Namely, given a submodule of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ compatibility of the submodule over W with the projection maps $W \times_U W \to W$ will guarantee the submodule comes from a quasi-coherent submodule of $\mathcal{F}$ (by Properties of Spaces, Proposition 66.32.1) and compatibility with $\alpha_{\mathrm{pr}_{V'}}$ will insure this subsheaf is compatible with α (details omitted).

Choose a cardinal κ as in Lemma 78.14.1 for the system $X = (I, \Phi)$. It is clear from Properties, Lemma 28.23.2 that there is a set of isomorphism classes of κ-generated crystals in quasi-coherent sheaves on X. Hence the result is clear. □

78.15. Groupoids and group spaces

0443 Please compare with Groupoids, Section 39.16.

0444 **Lemma 78.15.1.** *Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B with identity e_G and inverse i_G. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Then we get a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner:*

1. *We set $U = X$, and $R = G \times_B X$.*
2. *We set $s : R \to U$ equal to $(g, x) \mapsto x$.*
3. *We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.*
4. *We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.*
5. *We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.*
6. *We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.*

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

[3] In fact the functor is fully faithful, but we won't need this.

0445 **Lemma 78.15.2.** *Let $B \to S$ as in Section 78.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Let (U, R, s, t, c) be the groupoid in algebraic spaces constructed in Lemma 78.15.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_B X \to X$, see Definitions 78.10.1 and 78.12.1. Using the translation in Lemma 78.15.1 the commutativity requirements of the two definitions match up exactly. □

78.16. The stabilizer group algebraic space

0446 Please compare with Groupoids, Section 39.17. Given a groupoid in algebraic spaces we get a group algebraic space as follows.

0447 **Lemma 78.16.1.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The algebraic space G defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle j=(t,s)} \\ U & \xrightarrow{\Delta} & U \times_B U \end{array}$$

is a group algebraic space over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is a monomorphism we see that $G = j^{-1}(\Delta_{U/B})$ is a subsheaf of R. Thinking of it in this way, the structure morphism $G = j^{-1}(\Delta_{U/B}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

0448 **Definition 78.16.2.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The group algebraic space $j^{-1}(\Delta_{U/B}) \to U$ is called the *stabilizer of the groupoid in algebraic spaces* (U, R, s, t, c).

In the literature the stabilizer group algebraic space is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

0449 **Lemma 78.16.3.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B, and let G/U be its stabilizer. Denote R_t/U the algebraic space R seen as an algebraic space over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/B we define $a(g, r) = c(g, r)$. □

78.17. Restricting groupoids

044A Please refer to Groupoids, Section 39.18 for notation.

044B **Lemma 78.17.1.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces. Consider the following diagram*

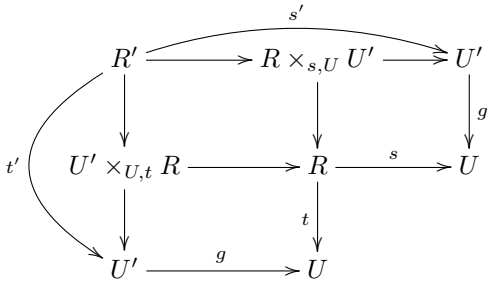

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid in algebraic spaces over B and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, for any scheme T over B the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see Groupoids, Section 39.18) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. □

044C **Definition 78.17.2.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The morphism of groupoids in algebraic spaces $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 78.17.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

044D **Lemma 78.17.3.** *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 78.17.2 and 78.4.3 agree via the constructions of Lemmas 78.11.2 and 78.11.3.*

Proof. What we are saying here is that R' of Lemma 78.17.1 is also equal to

$$R' = (U' \times_B U') \times_{U \times_B U} R \longrightarrow U' \times_B U'$$

In fact this might have been a clearer way to state that lemma. □

78.18. Invariant subspaces

044E In this section we discuss briefly the notion of an invariant subspace.

044F **Definition 78.18.1.** Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over the base B.

(1) We say an open subspace $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) A locally closed subspace $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R.

(3) A monomorphism of algebraic spaces $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over R.

For an open subspace $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-invariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant locally closed subspace, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

044G **Lemma 78.18.2.** *Let $B \to S$ as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B.*

(1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.*
(2) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subspaces.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open we see that $W' = s(t^{-1}(W))$ is an open subspace of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)},U,t} R \xrightarrow{\mathrm{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s,U,s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\mathrm{pr}_0} R.$$

The algebraic spaces A, B on the left of the arrows above are open subspaces of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 78.11.4 the diagram

$$\begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow[(\mathrm{pr}_1, c)]{} & R \times_{s,U,s} R \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_0} & \\ & R & \end{array}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\mathrm{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over images of affines étale over U we see (2). □

78.19. Quotient sheaves

044H Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. For each scheme S' over S we can take the equivalence relation $\sim_{S'}$ generated by the image of $j(S') : R(S') \to U(S') \times U(S')$. Hence we get a presheaf

044I (78.19.0.1)
$$\begin{array}{ccc} (Sch/S)^{opp}_{fppf} & \longrightarrow & Sets, \\ S' & \longmapsto & U(S')/\sim_{S'} \end{array}$$

Note that since j is a morphism of algebraic spaces over B and into $U \times_B U$ there is a canonical transformation of presheaves from the presheaf (78.19.0.1) to B.

044J **Definition 78.19.1.** Let $B \to S$ and the pre-relation $j : R \to U \times_B U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (78.19.0.1) on $(Sch/S)_{fppf}$. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G over B on U as in Lemma 78.15.1 then we denote the quotient sheaf U/G.

This means exactly that the diagram

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_{fppf}$. Again there is a canonical map of sheaves $U/R \to B$ as j is a morphism of algebraic spaces over B into $U \times_B U$.

044K **Remark 78.19.2.** A variant of the construction above would have been to sheafify the functor

$$\begin{array}{ccc} (Spaces/B)^{opp}_{fppf} & \longrightarrow & Sets, \\ X & \longmapsto & U(X)/\sim_X \end{array}$$

where now $\sim_X \subset U(X) \times U(X)$ is the equivalence relation generated by the image of $j : R(X) \to U(X) \times U(X)$. Here of course $U(X) = \mathrm{Mor}_B(X, U)$ and $R(X) = \mathrm{Mor}_B(X, R)$. In fact, the result would have been the same, via the identifications of (insert future reference in Topologies of Spaces here).

044L **Definition 78.19.3.** In the situation of Definition 78.19.1. We say that the pre-relation j has a *quotient representable by an algebraic space* if the sheaf U/R is an algebraic space. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable by a scheme. We will say a groupoid in algebraic spaces (U, R, s, t, c) over B has a *representable quotient* (resp. *quotient representable by an algebraic space*) if the quotient U/R with $j = (t, s)$ is representable (resp. an algebraic space).

If the quotient U/R is representable by M (either a scheme or an algebraic space over S), then it comes equipped with a canonical structure morphism $M \to B$ as we've seen above.

The following lemma characterizes M representing the quotient. It applies for example if $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

044M **Lemma 78.19.4.** *In the situation of Definition 78.19.1. Assume there is an algebraic space M over S, and a morphism $U \to M$ such that*

1. *the morphism $U \to M$ equalizes s, t,*
2. *the map $U \to M$ is a surjection of sheaves, and*
3. *the induced map $(t, s) : R \to U \times_M U$ is a surjection of sheaves.*

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $U \to M$ factors through U/R. Condition (2) says that $U/R \to M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

046O **Lemma 78.19.5.** *Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. For a scheme S' over S and $a, b \in U(S')$ the following are equivalent:*

(1) *a and b map to the same element of $(U/R)(S')$, and*
(2) *there exists an fppf covering $\{f_i : S_i \to S'\}$ of S' and morphisms $r_i : S_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.*

In other words, in this case the map of sheaves

$$R \longrightarrow U \times_{U/R} U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (78.19.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 78.4.1. □

046P **Lemma 78.19.6.** *Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B and $g : U' \to U$ a morphism of algebraic spaces over B. Let $j' : R' \to U' \times_B U'$ be the restriction of j to U'. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If $U' \to U$ is surjective as a map of sheaves, for example if $\{g : U' \to U\}$ is an fppf covering (see Topologies on Spaces, Definition 73.7.1), then $U'/R' \to U/R$ is an isomorphism of sheaves.

Proof. Suppose $\xi, \xi' \in (U'/R')(S')$ are sections which map to the same section of U/R. Then we can find an fppf covering $\mathcal{S} = \{S_i \to S'\}$ of S' such that $\xi|_{S_i}, \xi'|_{S_i}$ are given by $a_i, a_i' \in U'(S_i)$. By Lemma 78.19.5 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : S_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(S_i)$ and this shows that a_i and a_i' define the same section of U'/R' over S_i. By the sheaf condition this implies $\xi = \xi'$.

If $U' \to U$ is a surjective map of sheaves, then $U'/R' \to U/R$ is surjective also. Finally, if $\{g : U' \to U\}$ is a fppf covering, then the map of sheaves $U' \to U$ is surjective, see Topologies on Spaces, Lemma 73.7.5. □

044N **Lemma 78.19.7.** *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U, \qquad U' \times_{g,U,t} R \xrightarrow{h} U$$

is a surjection of fppf sheaves then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is an fppf-covering, or if $U' \to U$ is a surjection of sheaves, or if $\{g : U' \to U\}$ is a covering in the fppf topology.

Proof. Injectivity follows on combining Lemmas 78.11.2 and 78.19.6. To see surjectivity (see Sites, Section 7.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence

we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f_i' = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f_i' \in U'(T_i)$ maps to $g \circ f_i' \in U(T_i)$ and that $g \circ f_i' \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (78.19.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). □

78.20. Quotient stacks

044O In this section and the next few sections we describe a kind of generalization of Section 78.19 above and Groupoids, Section 39.20. It is different in the following way: We are going to take quotient stacks instead of quotient sheaves.

Let us assume we have a scheme S, and algebraic space B over S and a groupoid in algebraic spaces (U, R, s, t, c) over B. Given these data we consider the functor

044P (78.20.0.1)
$$\begin{array}{ccc} (Sch/S)_{fppf}^{opp} & \longrightarrow & Groupoids \\ S' & \longmapsto & (U(S'), R(S'), s, t, c) \end{array}$$

By Categories, Example 4.37.1 this "presheaf in groupoids" corresponds to a category fibred in groupoids over $(Sch/S)_{fppf}$. In this chapter we will denote this

$$[U/_pR] \to (Sch/S)_{fppf}$$

where the subscript $_p$ is there to distinguish from the quotient stack.

044Q **Definition 78.20.1.** Quotient stacks. Let $B \to S$ be as above.

(1) Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The *quotient stack*
$$p : [U/R] \longrightarrow (Sch/S)_{fppf}$$
of (U, R, s, t, c) is the stackification (see Stacks, Lemma 8.9.1) of the category fibred in groupoids $[U/_pR]$ over $(Sch/S)_{fppf}$ associated to (78.20.0.1).

(2) Let (G, m) be a group algebraic space over B. Let $a : G \times_B X \to X$ be an action of G on an algebraic space over B. The *quotient stack*
$$p : [X/G] \longrightarrow (Sch/S)_{fppf}$$
is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B of Lemma 78.15.1.

Thus $[U/R]$ and $[X/G]$ are stacks in groupoids over $(Sch/S)_{fppf}$. These stacks will be very important later on and hence it makes sense to give a detailed description. Recall that given an algebraic space X over S we use the notation $\mathcal{S}_X \to (Sch/S)_{fppf}$ to denote the stack in sets associated to the sheaf X, see Categories, Lemma 4.38.6 and Stacks, Lemma 8.6.2.

044R **Lemma 78.20.2.** *Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). There are canonical 1-morphisms $\pi : \mathcal{S}_U \to [U/R]$, and $[U/R] \to \mathcal{S}_B$ of stacks in groupoids over $(Sch/S)_{fppf}$. The composition $\mathcal{S}_U \to \mathcal{S}_B$ is the 1-morphism associated to the structure morphism $U \to B$.*

Proof. During this proof let us denote $[U/_pR]$ the category fibred in groupoids associated to the presheaf in groupoids (78.20.0.1). By construction of the stackification there is a 1-morphism $[U/_pR] \to [U/R]$. The 1-morphism $\mathcal{S}_U \to [U/R]$ is simply the composition $\mathcal{S}_U \to [U/_pR] \to [U/R]$, where the first arrow associates to the scheme S'/S and morphism $x : S' \to U$ over S the object $x \in U(S')$ of the fibre category of $[U/_pR]$ over S'.

To construct the 1-morphism $[U/R] \to \mathcal{S}_B$ it is enough to construct the 1-morphism $[U/_pR] \to \mathcal{S}_B$, see Stacks, Lemma 8.9.2. On objects over S'/S we just use the map

$$U(S') \longrightarrow B(S')$$

coming from the structure morphism $U \to B$. And clearly, if $a \in R(S')$ is an "arrow" with source $s(a) \in U(S')$ and target $t(a) \in U(S')$, then since s and t are morphisms *over* B these both map to the same element $\overline{a}$ of $B(S')$. Hence we can map an arrow $a \in R(S')$ to the identity morphism of $\overline{a}$. (This is good because the fibre category $(\mathcal{S}_B)_{S'}$ only contains identities.) We omit the verification that this rule is compatible with pullback on these split fibred categories, and hence defines a 1-morphism $[U/_pR] \to \mathcal{S}_B$ as desired.

We omit the verification of the last statement. □

044S **Lemma 78.20.3.** *Assumptions and notation as in Lemma 78.20.2. There exists a canonical* 2*-morphism* $\alpha : \pi \circ s \to \pi \circ t$ *making the diagram*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow[s]{} & \mathcal{S}_U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

2*-commutative.*

Proof. Let S' be a scheme over S. Let $r : S' \to R$ be a morphism over S. Then $r \in R(S')$ is an isomorphism between the objects $s \circ r, t \circ r \in U(S')$. Moreover, this construction is compatible with pullbacks. This gives a canonical 2-morphism $\alpha_p : \pi_p \circ s \to \pi_p \circ t$ where $\pi_p : \mathcal{S}_U \to [U/_pR]$ is as in the proof of Lemma 78.20.2. Thus even the diagram

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow[s]{} & \mathcal{S}_U \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \pi_p} \\ \mathcal{S}_U & \xrightarrow{\pi_p} & [U/_pR] \end{array}$$

is 2-commutative. Thus a fortiori the diagram of the lemma is 2-commutative. □

04M7 **Remark 78.20.4.** In future chapters we will use the ambiguous notation where instead of writing $\mathcal{S}_X$ for the stack in sets associated to X we simply write X. Using this notation the diagram of Lemma 78.20.3 becomes the familiar diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\pi} & [U/R] \end{array}$$

In the following sections we will show that this diagram has many good properties. In particular we will show that it is a 2-fibre product (Section 78.22) and that it is close to being a 2-coequalizer of s and t (Section 78.23).

78.21. Functoriality of quotient stacks

04Y3 A morphism of groupoids in algebraic spaces gives an associated morphism of quotient stacks.

046Q **Lemma 78.21.1.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be a morphism of groupoids in algebraic spaces over B. Then f induces a canonical 1-morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R'].$$

Proof. Denote $[U/_pR]$ and $[U'/_pR']$ the categories fibred in groupoids over the base site $(Sch/S)_{fppf}$ associated to the functors (78.20.0.1). It is clear that f defines a 1-morphism $[U/_pR] \to [U'/_pR']$ which we can compose with the stackyfication functor for $[U'/R']$ to get $[U/_pR] \to [U'/R']$. Then, by the universal property of the stackyfication functor $[U/_pR] \to [U/R]$, see Stacks, Lemma 8.9.2 we get $[U/R] \to [U'/R']$. □

Let $B \to S$ and $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be as in Lemma 78.21.1. In this situation, we define a third groupoid in algebraic spaces over B as follows, using the language of T-valued points where T is a (varying) scheme over B:

(1) $U'' = U \times_{f,U',t'} R'$ so that a T-valued point is a pair (u, r') with $f(u) = t'(r')$,
(2) $R'' = R \times_{f \circ s,U',t'} R'$ so that a T-valued point is a pair (r, r') with $f(s(r)) = t'(r')$,
(3) $s'' : R'' \to U''$ is given by $s''(r, r') = (s(r), r')$,
(4) $t'' : R'' \to U''$ is given by $t''(r, r') = (t(r), c'(f(r), r'))$,
(5) $c'' : R'' \times_{s'',U'',t''} R'' \to R''$ is given by $c''((r_1, r_1'), (r_2, r_2')) = (c(r_1, r_2), r_2')$.

The formula for c'' makes sense as $s''(r_1, r_1') = t''(r_2, r_2')$. It is clear that c'' is associative. The identity e'' is given by $e''(u, r) = (e(u), r)$. The inverse of (r, r') is given by $(i(r), c'(f(r), r'))$. Thus we do indeed get a groupoid in algebraic spaces over B.

Clearly the maps $U'' \to U$ and $R'' \to R$ define a morphism $g : (U'', R'', s'', t'', c'') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, the maps $U'' \to U'$, $(u, r') \mapsto s'(r')$ and $R'' \to U'$, $(r, r') \mapsto s'(r')$ show that in fact $(U'', R'', s'', t'', c'')$ is a groupoid in algebraic spaces over U'.

04Y4 **Lemma 78.21.2.** *Notation and assumption as in Lemma 78.21.1. Let $(U'', R'', s'', t'', c'')$ be the groupoid in algebraic spaces over B constructed above. There is a 2-commutative square*

$$\begin{array}{ccc} [U''/R''] & \xrightarrow[[g]]{} & [U/R] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

which identifies $[U''/R'']$ with the 2-fibre product.

Proof. The maps $[f]$ and $[g]$ come from an application of Lemma 78.21.1 and the other two maps come from Lemma 78.20.2 (and the fact that $(U'', R'', s'', t'', c'')$ lives over U'). To show the 2-fibre product property, it suffices to prove the lemma for the diagram

$$\begin{array}{ccc} [U''/_pR''] & \xrightarrow[[g]]{} & [U/_pR] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/_pR'] \end{array}$$

of categories fibred in groupoids, see Stacks, Lemma 8.9.3. In other words, it suffices to show that an object of the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T corresponds to a T-valued point of U'' and similarly for morphisms. And of course this is exactly how we constructed U'' and R'' in the first place.

In detail, an object of $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T is a triple (u', u, r') where u' is a T-valued point of U', u is a T-valued point of U, and r' is a morphism from u' to $f(u)$ in $[U'/R']_T$, i.e., r' is a T-valued point of R with $s'(r') = u'$ and $t'(r') = f(u)$. Clearly we can forget about u' without losing information and we see that these objects are in one-to-one correspondence with T-valued points of R''.

Similarly for morphisms: Let (u'_1, u_1, r'_1) and (u'_2, u_2, r'_2) be two objects of the fibre product over T. Then a morphism from (u'_2, u_2, r'_2) to (u'_1, u_1, r'_1) is given by $(1, r)$ where $1 : u'_1 \to u'_2$ means simply $u'_1 = u'_2$ (this is so because $\mathcal{S}_U$ is fibred in sets), and r is a T-valued point of R with $s(r) = u_2$, $t(r) = u_1$ and moreover $c'(f(r), r'_2) = r'_1$. Hence the arrow

$$(1, r) : (u'_2, u_2, r'_2) \to (u'_1, u_1, r'_1)$$

is completely determined by knowing the pair (r, r'_2). Thus the functor of arrows is represented by R'', and moreover the morphisms s'', t'', and c'' clearly correspond to source, target and composition in the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$. □

78.22. The 2-cartesian square of a quotient stack

04M8 In this section we compute the *Isom*-sheaves for a quotient stack and we deduce that the defining diagram of a quotient stack is a 2-fibre product.

044V **Lemma 78.22.1.** *Assume $B \to S$, (U, R, s, t, c) and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 78.20.2. Let S' be a scheme over S. Let $x, y \in \mathrm{Ob}([U/R]_{S'})$ be objects of the quotient stack over S'. If $x = \pi(x')$ and $y = \pi(y')$ for some morphisms $x', y' : S' \to U$, then*

$$Isom(x, y) = S' \times_{(y', x'), U \times_S U} R$$

as sheaves over S'.

Proof. Let $[U/_pR]$ be the category fibred in groupoids associated to the presheaf in groupoids (78.20.0.1) as in the proof of Lemma 78.20.2. By construction the sheaf $Isom(x, y)$ is the sheaf associated to the presheaf $Isom(x', y')$. On the other hand, by definition of morphisms in $[U/_pR]$ we have

$$Isom(x', y') = S' \times_{(y', x'), U \times_S U} R$$

and the right hand side is an algebraic space, therefore a sheaf. □

04M9 **Lemma 78.22.2.** *Assume $B \to S$, (U, R, s, t, c), and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 78.20.2. The 2-commutative square*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

of Lemma 78.20.3 is a 2-fibre product of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. According to Stacks, Lemma 8.5.6 the lemma makes sense. It also tells us that we have to show that the functor

$$\mathcal{S}_R \longrightarrow \mathcal{S}_U \times_{[U/R]} \mathcal{S}_U$$

which maps $r : T \to R$ to $(T, t(r), s(r), \alpha(r))$ is an equivalence, where the right hand side is the 2-fibre product as described in Categories, Lemma 4.32.3. This is, after spelling out the definitions, exactly the content of Lemma 78.22.1. (Alternative proof: Work out the meaning of Lemma 78.21.2 in this situation will give you the result also.) □

044W **Lemma 78.22.3.** *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ has the following property: There exists a fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is representable by an algebraic space.*

Proof. Follows immediately from Lemma 78.22.1 and the fact that both x and y locally in the fppf topology come from objects of $\mathcal{S}_U$ by construction of the quotient stack. □

78.23. The 2-coequalizer property of a quotient stack

04MA On a groupoid we have the composition, which leads to a cocycle condition for the canonical 2-morphism of the lemma above. To give the precise formulation we will use the notation introduced in Categories, Sections 4.28 and 4.29.

044T **Lemma 78.23.1.** *Assumptions and notation as in Lemmas 78.20.2 and 78.20.3. The vertical composition of*

$$\mathcal{S}_{R \times_{s,U,t} R} \;\underset{\pi \circ t \circ pr_1 = \pi \circ s \circ pr_0}{\overset{\pi \circ s \circ pr_1 = \pi \circ s \circ c}{\Rightarrow}}\; [U/R]$$

(with $\Downarrow \alpha \star id_{pr_1}$ and $\Downarrow \alpha \star id_{pr_0}$, and $\pi \circ t \circ pr_0 = \pi \circ t \circ c$)

is the 2-morphism $\alpha \star id_c$. In a formula $\alpha \star id_c = (\alpha \star id_{pr_0}) \circ (\alpha \star id_{pr_1})$.

Proof. We make two remarks:

(1) The formula $\alpha \star \mathrm{id}_c = (\alpha \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\alpha \star \mathrm{id}_{\mathrm{pr}_1})$ only makes sense if you realize the *equalities* $\pi \circ s \circ \mathrm{pr}_1 = \pi \circ s \circ c$, $\pi \circ t \circ \mathrm{pr}_1 = \pi \circ s \circ \mathrm{pr}_0$, and $\pi \circ t \circ \mathrm{pr}_0 = \pi \circ t \circ c$. Namely, the second one implies the vertical composition $\circ$ makes sense, and the other two guarantee the two sides of the formula are 2-morphisms with the same source and target.

(2) The reason the lemma holds is that composition in the category fibred in groupoids $[U/_pR]$ associated to the presheaf in groupoids (78.20.0.1) comes from the composition law $c : R \times_{s,U,t} R \to R$.

We omit the proof of the lemma. □

Note that, in the situation of the lemma, we actually have the equalities $s \circ \mathrm{pr}_1 = s \circ c$, $t \circ \mathrm{pr}_1 = s \circ \mathrm{pr}_0$, and $t \circ \mathrm{pr}_0 = t \circ c$ before composing with π. Hence the formula in the lemma below makes sense in exactly the same way that the formula in the lemma above makes sense.

044U **Lemma 78.23.2.** *Assumptions and notation as in Lemmas 78.20.2 and 78.20.3. The 2-commutative diagram of Lemma 78.20.3 is a 2-coequalizer in the following sense: Given*

(1) *a stack in groupoids $\mathcal{X}$ over $(Sch/S)_{fppf}$,*
(2) *a 1-morphism $f : \mathcal{S}_U \to \mathcal{X}$, and*
(3) *a 2-arrow $\beta : f \circ s \to f \circ t$*

such that

$$\beta \star id_c = (\beta \star id_{pr_0}) \circ (\beta \star id_{pr_1})$$

then there exists a 1-morphism $[U/R] \to \mathcal{X}$ which makes the diagram

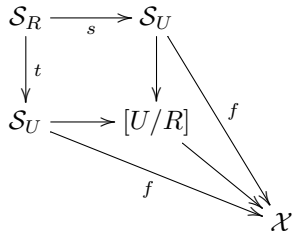

2-*commute.*

Proof. Suppose given $\mathcal{X}$, f and β as in the lemma. By Stacks, Lemma 8.9.2 it suffices to construct a 1-morphism $g : [U/_pR] \to \mathcal{X}$. First we note that the 1-morphism $\mathcal{S}_U \to [U/_pR]$ is bijective on objects. Hence on objects we can set $g(x) = f(x)$ for $x \in \mathrm{Ob}(\mathcal{S}_U) = \mathrm{Ob}([U/_pR])$. A morphism $\varphi : x \to y$ of $[U/_pR]$ arises from a commutative diagram

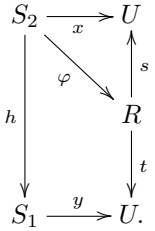

Thus we can set $g(\varphi)$ equal to the composition

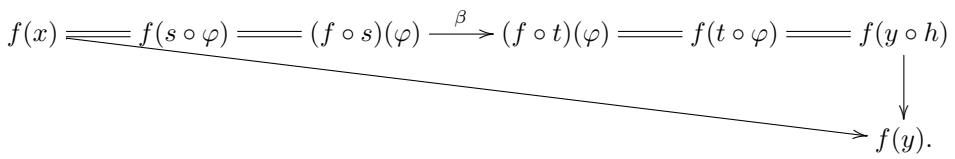

The vertical arrow is the result of applying the functor f to the canonical morphism $y \circ h \to y$ in $\mathcal{S}_U$ (namely, the strongly cartesian morphism lifting h with target y). Let us verify that f so defined is compatible with composition, at least on fibre categories. So let S' be a scheme over S, and let $a : S' \to R \times_{s,U,t} R$ be a morphism. In this situation we set $x = s \circ \mathrm{pr}_1 \circ a = s \circ c \circ a$, $y = t \circ \mathrm{pr}_1 \circ a = s \circ \mathrm{pr}_0 \circ a$, and $z = t \circ \mathrm{pr}_0 \circ a = t \circ \mathrm{pr}_0 \circ c$ to get a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow[c\circ a]{} & z \\ & {\scriptstyle \mathrm{pr}_1\circ a} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_0\circ a} & \\ & y & \end{array}$$

in the fibre category $[U/_pR]_{S'}$. Moreover, any commutative triangle in this fibre category has this form. Then we see by our definitions above that f maps this to a commutative diagram if and only if the diagram

$$\begin{array}{ccccc} & & (f\circ s)(c\circ a) & \xrightarrow[\beta]{} & (f\circ t)(c\circ a) & & \\ & \swarrow\!\!\swarrow & & & & \searrow\!\!\searrow & \\ (f\circ s)(\mathrm{pr}_1\circ a) & & & & & & (f\circ t)(\mathrm{pr}_0\circ a) \\ & {\scriptstyle\beta}\searrow & & & & \nearrow{\scriptstyle\beta} & \\ & & (f\circ t)(\mathrm{pr}_1\circ a) & = & (f\circ s)(\mathrm{pr}_0\circ a) & & \end{array}$$

is commutative which is exactly the condition expressed by the formula in the lemma. We omit the verification that f maps identities to identities and is compatible with composition for arbitrary morphisms. □

78.24. Explicit description of quotient stacks

04MB In order to formulate the result we need to introduce some notation. Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). Let T be a scheme over S. Let $\mathcal{T} = \{T_i \to T\}_{i\in I}$ be an fppf covering. A $[U/R]$-*descent datum* relative to $\mathcal{T}$ is given by a system (u_i, r_{ij}) where

(1) for each i a morphism $u_i : T_i \to U$, and
(2) for each i, j a morphism $r_{ij} : T_i \times_T T_j \to R$

such that

(a) as morphisms $T_i \times_T T_j \to U$ we have

$$s \circ r_{ij} = u_i \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{ij} = u_j \circ \mathrm{pr}_1,$$

(b) as morphisms $T_i \times_T T_j \times_T T_k \to R$ we have

$$c \circ (r_{jk} \circ \mathrm{pr}_{12}, r_{ij} \circ \mathrm{pr}_{01}) = r_{ik} \circ \mathrm{pr}_{02}.$$

A *morphism* $(u_i, r_{ij}) \to (u'_i, r'_{ij})$ *between two* $[U/R]$-*descent data* over the same covering $\mathcal{T}$ is a collection $(r_i : T_i \to R)$ such that

(α) as morphisms $T_i \to U$ we have

$$u_i = s \circ r_i \quad \text{and} \quad u'_i = t \circ r_i$$

(β) as morphisms $T_i \times_T T_j \to R$ we have

$$c \circ (r'_{ij}, r_i \circ \mathrm{pr}_0) = c \circ (r_j \circ \mathrm{pr}_1, r_{ij}).$$

There is a natural composition law on morphisms of descent data relative to a fixed covering and we obtain a category of descent data. This category is a groupoid. Finally, if $\mathcal{T}' = \{T'_j \to T\}_{j\in J}$ is a second fppf covering which refines $\mathcal{T}$ then there is a notion of pullback of descent data. This is particularly easy to describe explicitly in this case. Namely, if $\alpha : J \to I$ and $\varphi_j : T'_j \to T_{\alpha(i)}$ is the morphism of coverings, then the pullback of the descent datum $(u_i, r_{ii'})$ is simply

$$(u_{\alpha(i)} \circ \varphi_j, r_{\alpha(j)\alpha(j')} \circ \varphi_j \times \varphi_{j'}).$$

Pullback defined in this manner defines a functor from the category of descent data over $\mathcal{T}$ to the category of descend data over $\mathcal{T}'$.

044X **Lemma 78.24.1.** *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 78.20.1 (1). Let $\pi : \mathcal{S}_U \to [U/R]$ be as in Lemma 78.20.2. Let T be a scheme over S.*

(1) *for every object x of the fibre category $[U/R]_T$ there exists an fppf covering $\{f_i : T_i \to T\}_{i\in I}$ such that $f_i^* x \cong \pi(u_i)$ for some $u_i \in U(T_i)$,*
(2) *the composition of the isomorphisms*

$$\pi(u_i \circ pr_0) = pr_0^*\pi(u_i) \cong pr_0^* f_i^* x \cong pr_1^* f_j^* x \cong pr_1^*\pi(u_j) = \pi(u_j \circ pr_1)$$

are of the form $\pi(r_{ij})$ for certain morphisms $r_{ij} : T_i \times_T T_j \to R$,
(3) *the system (u_i, r_{ij}) forms a $[U/R]$-descent datum as defined above,*
(4) *any $[U/R]$-descent datum (u_i, r_{ij}) arises in this manner,*
(5) *if x corresponds to (u_i, r_{ij}) as above, and $y \in \mathrm{Ob}([U/R]_T)$ corresponds to (u'_i, r'_{ij}) then there is a canonical bijection*

$$\mathrm{Mor}_{[U/R]_T}(x, y) \longleftrightarrow \left\{ \begin{array}{c} \text{morphisms } (u_i, r_{ij}) \to (u'_i, r'_{ij}) \\ \text{of } [U/R]\text{-descent data} \end{array} \right\}$$

(6) *this correspondence is compatible with refinements of fppf coverings.*

Proof. Statement (1) is part of the construction of the stackyfication. Part (2) follows from Lemma 78.22.1. We omit the verification of (3). Part (4) is a translation of the fact that in a stack all descent data are effective. We omit the verifications of (5) and (6). □

78.25. Restriction and quotient stacks

046R In this section we study what happens to the quotient stack when taking a restriction.

046S **Lemma 78.25.1.** *Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is fully faithful if and only if R is the restriction of R' via the morphism $f : U \to U'$.

Proof. Let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/R']_T$. The functor $[f]$ is fully faithful if and only if the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

is an isomorphism for every T, x, y. We may test this locally on T (in the fppf topology). Hence, by Lemma 78.24.1 we may assume that x, y come from $a, b \in$

$U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 78.22.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R'.$$

This is an isomorphism if R is the restriction, because in that case $R = (U \times_B U) \times_{U' \times_B U'} R'$, see Lemma 78.17.3 and its proof. Conversely, if the last displayed map is an isomorphism for all T, a, b, then it follows that $R = (U \times_B U) \times_{U' \times_B U'} R'$, i.e., R is the restriction of R'. □

046T **Lemma 78.25.2.** *Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is an equivalence if and only if

(1) *(U, R, s, t, c) is the restriction of (U', R', s', t', c') via $f : U \to U'$, and*
(2) *the map*

$$U \times_{f,U',t'} R' \xrightarrow[pr_1]{} R' \xrightarrow[s']{} U', \quad \text{with } h : U \times_{f,U',t'} R' \to U'$$

is a surjection of sheaves.

Part (2) holds for example if $\{h : U \times_{f,U',t'} R' \to U'\}$ is an fppf covering, or if $f : U \to U'$ is a surjection of sheaves, or if $\{f : U \to U'\}$ is an fppf covering.

Proof. We already know that part (1) is equivalent to fully faithfulness by Lemma 78.25.1. Hence we may assume that (1) holds and that $[f]$ is fully faithful. Our goal is to show, under these assumptions, that $[f]$ is an equivalence if and only if (2) holds. We may use Stacks, Lemma 8.4.8 which characterizes equivalences.

Assume (2). We will use Stacks, Lemma 8.4.8 to prove $[f]$ is an equivalence. Suppose that T is a scheme and $x' \in \mathrm{Ob}([U'/R']_T)$. There exists a covering $\{g_i : T_i \to T\}$ such that $g_i^* x'$ is the image of some element $a_i' \in U'(T_i)$, see Lemma 78.24.1. Hence we may assume that x' is the image of $a' \in U'(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $b_i : T_i \to U \times_{g,U',t'} R'$ such that $a' \circ \varphi_i = h \circ b_i$. Denote $a_i = \mathrm{pr}_0 \circ b_i : T_i \to U$. Then we see that $a_i \in U(T_i)$ maps to $f \circ a_i \in U'(T_i)$ and that $f \circ a_i \cong_{T_i} h \circ b_i = a' \circ \varphi_i$, where $\cong_{T_i}$ denotes isomorphism in the fibre category $[U'/R']_{T_i}$. Namely, the element of $R'(T_i)$ giving the isomorphism is $\mathrm{pr}_1 \circ b_i$. This means that the restriction of x to T_i is in the essential image of the functor $[U/R]_{T_i} \to [U'/R']_{T_i}$ as desired.

Assume $[f]$ is an equivalence. Let $\xi' \in [U'/R']_{U'}$ denote the object corresponding to the identity morphism of U'. Applying Stacks, Lemma 8.4.8 we see there exists an fppf covering $\mathcal{U}' = \{g_i' : U_i' \to U'\}$ such that $(g_i')^* \xi' \cong [f](\xi_i)$ for some ξ_i in $[U/R]_{U_i'}$. After refining the covering $\mathcal{U}'$ (using Lemma 78.24.1) we may assume ξ_i comes from a morphism $a_i : U_i' \to U$. The fact that $[f](\xi_i) \cong (g_i')^* \xi'$ means that, after possibly refining the covering $\mathcal{U}'$ once more, there exist morphisms $r_i' : U_i' \to R'$ with $t' \circ r_i' = f \circ a_i$ and $s' \circ r_i' = \mathrm{id}_{U'} \circ g_i'$. Picture

$$\begin{array}{ccccc} U & & \xleftarrow{\quad a_i \quad} & & U_i' \\ \downarrow{\scriptstyle f} & & & \swarrow{\scriptstyle r_i'} & \downarrow{\scriptstyle g_i'} \\ U' & \xleftarrow{t'} & R' & \xrightarrow{s'} & U' \end{array}$$

Thus $(a_i, r'_i) : U'_i \to U \times_{g,U',t'} R'$ are morphisms such that $h \circ (a_i, r'_i) = g'_i$ and we conclude that $\{h : U \times_{g,U',t'} R' \to U'\}$ can be refined by the fppf covering $\mathcal{U}'$ which means that h induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 73.7.5. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid in algebraic spaces). □

04ZN **Lemma 78.25.3.** *Notation and assumption as in Lemma 78.21.1. Assume that*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian. Then

$$\begin{array}{ccc} \mathcal{S}_U & \longrightarrow & [U/R] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

is a 2-fibre product square.

Proof. Applying the inverse isomorphisms $i : R \to R$ and $i' : R' \to R'$ to the (first) cartesian diagram of the statement of the lemma we see that

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian as well. By Lemma 78.21.2 we have a 2-fibre square

$$\begin{array}{ccc} [U''/R''] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

where $U'' = U \times_{f,U',t'} R'$ and $R'' = R \times_{f \circ s,U',t'} R'$. By the above we see that $(t, f) : R \to U''$ is an isomorphism, and that

$$R'' = R \times_{f \circ s,U',t'} R' = R \times_{s,U} U \times_{f,U',t'} R' = R \times_{s,U,t} \times R.$$

Explicitly the isomorphism $R \times_{s,U,t} R \to R''$ is given by the rule $(r_0, r_1) \mapsto (r_0, f(r_1))$. Moreover, s'', t'', c'' translate into the maps

$$R \times_{s,U,t} R \to R, \quad s''(r_0, r_1) = r_1, \quad t''(r_0, r_1) = c(r_0, r_1)$$

and

$$\begin{array}{rccc} c'' : & (R \times_{s,U,t} R) \times_{s'',R,t''} (R \times_{s,U,t} R) & \longrightarrow & R \times_{s,U,t} R, \\ & ((r_0, r_1), (r_2, r_3)) & \longmapsto & (c(r_0, r_2), r_3). \end{array}$$

Precomposing with the isomorphism

$$R \times_{s,U,s} R \longrightarrow R \times_{s,U,t} R, \quad (r_0, r_1) \longmapsto (c(r_0, i(r_1)), r_1)$$

we see that t'' and s'' turn into pr_0 and pr_1 and that c'' turns into $\mathrm{pr}_{02} : R \times_{s,U,s} R \times_{s,U,s} R \to R \times_{s,U,s} R$. Hence we see that there is an isomorphism $[U''/R''] \cong [R/R \times_{s,U,s} R]$ where as a groupoid in algebraic spaces $(R, R \times_{s,U,s} R, s'', t'', c'')$ is the restriction of the trivial groupoid $(U, U, \mathrm{id}, \mathrm{id}, \mathrm{id})$ via $s : R \to U$. Since $s : R \to U$ is a surjection of fppf sheaves (as it has a right inverse) the morphism

$$[U''/R''] \cong [R/R \times_{s,U,s} R] \longrightarrow [U/U] = \mathcal{S}_U$$

is an equivalence by Lemma 78.25.2. This proves the lemma. □

78.26. Inertia and quotient stacks

06PA The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.34.

06PB **Lemma 78.26.1.** *Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 78.16.2. Set $R' = R \times_{s,U} G$ and set*

(1) $s' : R' \to G$, $(r, g) \mapsto g$,
(2) $t' : R' \to G$, $(r, g) \mapsto c(r, c(g, i(r)))$,
(3) $c' : R' \times_{s',G,t'} R' \to R'$, $((r_1, g_1), (r_2, g_2) \mapsto (c(r_1, r_2), g_1)$.

Then (G, R', s', t', c') is a groupoid in algebraic spaces over B and

$$\mathcal{I}_{[U/R]} = [G/R'].$$

i.e., the associated quotient stack is the inertia stack of $[U/R]$.

Proof. By Stacks, Lemma 8.8.5 it suffices to prove that $\mathcal{I}_{[U/_pR]} = [G/_pR']$. Let T be a scheme over S. Recall that an object of the inertia fibred category of $[U/_pR]$ over T is given by a pair (x, g) where x is an object of $[U/_pR]$ over T and g is an automorphism of x in its fibre category over T. In other words, $x : T \to U$ and $g : T \to R$ such that $x = s \circ g = t \circ g$. This means exactly that $g : T \to G$. A morphism in the inertia fibred category from $(x, g) \to (y, h)$ over T is given by $r : T \to R$ such that $s(r) = x$, $t(r) = y$ and $c(r, g) = c(h, r)$, see the commutative diagram in Categories, Lemma 4.34.1. In a formula

$$h = c(r, c(g, i(r))) = c(c(r, g), i(r)).$$

The notation $s(r)$, etc is a short hand for $s \circ r$, etc. The composition of $r_1 : (x_2, g_2) \to (x_1, g_1)$ and $r_2 : (x_1, g_1) \to (x_2, g_2)$ is $c(r_1, r_2) : (x_1, g_1) \to (x_3, g_3)$.

Note that in the above we could have written g in stead of (x, g) for an object of $\mathcal{I}_{[U/_pR]}$ over T as x is the image of g under the structure morphism $G \to U$. Then the morphisms $g \to h$ in $\mathcal{I}_{[U/_pR]}$ over T correspond exactly to morphisms $r' : T \to R'$ with $s'(r') = g$ and $t'(r') = h$. Moreover, the composition corresponds to the rule explained in (3). Thus the lemma is proved. □

06PC **Lemma 78.26.2.** *Assume $B \to S$ and (U, R, s, t, c) as in Definition 78.20.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see*

Definition 78.16.2. There is a canonical 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{S}_G & \longrightarrow & \mathcal{S}_U \\ \downarrow & & \downarrow \\ \mathcal{I}_{[U/R]} & \longrightarrow & [U/R] \end{array}$$

of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. By Lemma 78.25.3 it suffices to prove that the morphism $s' : R' \to G$ of Lemma 78.26.1 isomorphic to the base change of s by the structure morphism $G \to U$. This base change property is clear from the construction of s'. □

78.27. Gerbes and quotient stacks

06PD In this section we relate quotient stacks to the discussion Stacks, Section 8.11 and especially gerbes as defined in Stacks, Definition 8.11.4. The stacks in groupoids occurring in this section are generally speaking not algebraic stacks!

06PE **Lemma 78.27.1.** *Notation and assumption as in Lemma 78.21.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

turns $[U/R]$ into a gerbe over $[U'/R']$ if $f : U \to U'$ and $R \to R'|_U$ are surjective maps of fppf sheaves. Here $R'|_U$ is the restriction of R' to U via $f : U \to U'$.

Proof. We will verify that Stacks, Lemma 8.11.3 properties (2) (a) and (2) (b) hold. Property (2)(a) holds because $U \to U'$ is a surjective map of sheaves (use Lemma 78.24.1 to see that objects in $[U'/R']$ locally come from U'). To prove (2)(b) let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/'R]_T$. Condition (2)(b) requires us to check the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

on $(Sch/T)_{fppf}$ is surjective. To see this we may work fppf locally on T and assume that come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 78.22.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R' = T \times_{(a,b),U \times_B U} R'|_U.$$

Hence the assumption that $R \to R'|_U$ is a surjective map of fppf sheaves on $(Sch/S)_{fppf}$ implies the desired surjectivity. □

06PF **Lemma 78.27.2.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. The morphism*

$$[B/G] \longrightarrow \mathcal{S}_B$$

(Lemma 78.20.2) turns $[B/G]$ into a gerbe over B.

Proof. Immediate from Lemma 78.27.1 as the morphisms $B \to B$ and $B \times_B G \to B$ are surjective as morphisms of sheaves. □

78.28. Quotient stacks and change of big site

04WW We suggest skipping this section on a first reading. Pullbacks of stacks are defined in Stacks, Section 8.12.

04WX **Lemma 78.28.1.** *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $B, U, R \in Sh((Sch/S)_{fppf})$ be algebraic spaces, and let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : Sch_{fppf} \to Sch'_{fppf}$. Then we have a canonical equivalence*

$$[f^{-1}U/f^{-1}R] \longrightarrow f^{-1}[U/R]$$

of stacks in groupoids over $(Sch'/S)_{fppf}$.

Proof. Note that $f^{-1}B, f^{-1}U, f^{-1}R \in Sh((Sch'/S)_{fppf})$ are algebraic spaces by Spaces, Lemma 65.15.1 and hence $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Thus the statement makes sense.

The category $u_p[U/_pR]$ is the localization of the category $u_{pp}[U/_pR]$ at right multiplicative system I of morphisms. An object of $u_{pp}[U/_pR]$ is a triple

$$(T', \phi : T' \to T, x)$$

where $T' \in \mathrm{Ob}((Sch'/S)_{fppf})$, $T \in \mathrm{Ob}((Sch/S)_{fppf})$, ϕ is a morphism of schemes over S, and $x : T \to U$ is a morphism of sheaves on $(Sch/S)_{fppf}$. Note that the morphism of schemes $\phi : T' \to T$ is the same thing as a morphism $\phi : T' \to u(T)$, and since $u(T)$ represents $f^{-1}T$ it is the same thing as a morphism $T' \to f^{-1}T$. Moreover, as f^{-1} on algebraic spaces is fully faithful, see Spaces, Lemma 65.15.2, we may think of x as a morphism $x : f^{-1}T \to f^{-1}U$ as well. From now on we will make such identifications without further mention. A morphism

$$(a, a', \alpha) : (T'_1, \phi_1 : T'_1 \to T_1, x_1) \longrightarrow (T'_2, \phi_2 : T'_2 \to T_2, x_2)$$

of $u_{pp}[U/_pR]$ is a commutative diagram

$$\begin{array}{ccccc} & & & & U \\ & & & \overset{x_1}{\nearrow} & \uparrow{\scriptstyle s} \\ T'_1 & \xrightarrow[\phi_1]{} & T_1 & \xrightarrow[\alpha]{} & R \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle t} \\ T'_2 & \xrightarrow{\phi_2} & T_2 & \xrightarrow{x_2} & U \end{array}$$

and such a morphism is an element of I if and only if $T'_1 = T'_2$ and $a' = \mathrm{id}$. We define a functor

$$u_{pp}[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

by the rules

$$(T', \phi : T' \to T, x) \longmapsto (x \circ \phi : T' \to f^{-1}U)$$

on objects and

$$(a, a', \alpha) \longmapsto (\alpha \circ \phi_1 : T'_1 \to f^{-1}R)$$

on morphisms as above. It is clear that elements of I are transformed into isomorphisms as $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Hence this functor factors in a canonical way through a functor

$$u_p[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

Applying stackification we obtain a functor of stacks

$$f^{-1}[U/R] \longrightarrow [f^{-1}U/f^{-1}R]$$

over $(Sch'/S)_{fppf}$, as by Stacks, Lemma 8.12.11 the stack $f^{-1}[U/R]$ is the stackification of $u_p[U/_pR]$.

At this point we have a morphism of stacks, and to verify that it is an equivalence it suffices to show that it is fully faithful and that objects are locally in the essential image, see Stacks, Lemmas 8.4.7 and 8.4.8. The statement on objects holds as $f^{-1}R$ admits a surjective étale morphism $f^{-1}W \to f^{-1}R$ for some object W of $(Sch/S)_{fppf}$. To show that the functor is "full", it suffices to show that morphisms are locally in the image of the functor which holds as $f^{-1}U$ admits a surjective étale morphism $f^{-1}W \to f^{-1}U$ for some object W of $(Sch/S)_{fppf}$. We omit the proof that the functor is faithful. □

78.29. Separation conditions

0453 This really means conditions on the morphism $j : R \to U \times_B U$ when given a groupoid in algebraic spaces (U, R, s, t, c) over B. As in the previous section we first formulate the corresponding diagram.

0454 **Lemma 78.29.1.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s,U} U & \longrightarrow & U \\ \downarrow{\scriptstyle \Delta_{R/U \times_B U}} & & \downarrow & & \downarrow \\ R \times_{(U \times_B U)} R & \xrightarrow{(f,g) \mapsto (f, f^{-1} \circ g)} & R \times_{s,U} G & \longrightarrow & G \end{array}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

0455 **Lemma 78.29.2.** *Let $B \to S$ be as in Section 78.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is separated,*
   (b) *$G \to U$ is separated, and*
   (c) *$e : U \to G$ is a closed immersion.*
(2) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is locally separated,*
   (b) *$G \to U$ is locally separated, and*
   (c) *$e : U \to G$ is an immersion.*
(3) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is quasi-separated,*

(b) $G \to U$ *is quasi-separated, and*
(c) $e : U \to G$ *is quasi-compact.*

Proof. The group algebraic space $G \to U$ is the base change of $R \to U \times_B U$ by the diagonal morphism $U \to U \times_B U$, see Lemma 78.16.1. Hence if j is separated (resp. locally separated, resp. quasi-separated), then $G \to U$ is separated (resp. locally separated, resp. quasi-separated). See Morphisms of Spaces, Lemma 67.4.4. Thus (a) $\Rightarrow$ (b) in (1), (2), and (3).

Conversely, if $G \to U$ is separated (resp. locally separated, resp. quasi-separated), then the morphism $e : U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. an immersion, resp. quasi-compact), see Morphisms of Spaces, Lemma 67.4.7. Thus (b) $\Rightarrow$ (c) in (1), (2), and (3).

If e is a closed immersion (resp. an immersion, resp. quasi-compact) then by the result of Lemma 78.29.1 (and Spaces, Lemma 65.12.3, and Morphisms of Spaces, Lemma 67.8.4) we see that $\Delta_{R/U \times_B U}$ is a closed immersion (resp. an immersion, resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in (1), (2), and (3). □

78.30. Other chapters

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CHAPTER 79

More on Groupoids in Spaces

04P4

79.1. Introduction

04P5 This chapter is devoted to advanced topics on groupoids in algebraic spaces. Even though the results are stated in terms of groupoids in algebraic spaces, the reader should keep in mind the 2-cartesian diagram

04P6 (79.1.0.1)

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 78.20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97**] by Keel and Mori.

79.2. Notation

04P7 We continue to abide by the conventions and notation introduced in Groupoids in Spaces, Section 78.3.

79.3. Useful diagrams

04P8 We briefly restate the results of Groupoids in Spaces, Lemmas 78.11.4 and 78.11.5 for easy reference in this chapter. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

04P9 (79.3.0.1)

$$\begin{array}{ccccc} & & U & & \\ & \overset{t}{\nearrow} & & \overset{t}{\nwarrow} & \\ R & \xleftarrow[\mathrm{pr}_0]{} & R \times_{s,U,t} R & \xrightarrow[c]{} & R \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} \\ U & \xleftarrow{t} & R & \xrightarrow{s} & U \end{array}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

0451 (79.3.0.2)

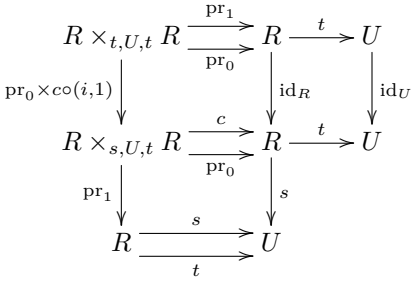

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

79.4. Local structure

0CK9 Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over S. Let $\overline{u}$ be a geometric point of U. In this section we explain what kind of structure we obtain on the local rings (Properties of Spaces, Definition 66.22.2)

$$A = \mathcal{O}_{U,\overline{u}} \quad \text{and} \quad B = \mathcal{O}_{R,e(\overline{u})}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a commutative diagram

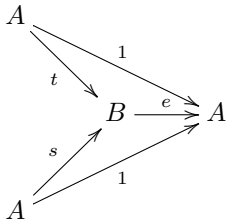

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)(\overline{u})}$$

Then we have

$$C = (B \otimes_{s,A,t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

because the localization $(B \otimes_{s,A,t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$ has separably closed residue field. Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e, e) : C \longrightarrow A$$

The composition law $c : R \times_{s,U,t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

0CKA **Lemma 79.4.1.** *The map $I/I^2 \to J/J^2$ induced by c is the composition*

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another henselian local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$ by Algebra, Lemma 10.155.6 for example. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B,\ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B,\ b_1 \otimes b_2 \mapsto t(e(b_1))b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

The statement on i follows from the statement on c and the fact that $c \circ (1, i) = e \circ t$. Some details omitted. □

79.5. Groupoid of sections

0CKB Suppose we have a groupoid $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$. Then we can construct a monoid Γ whose elements are maps $\delta : \mathrm{Ob} \to \mathrm{Arrows}$ with $s \circ \delta = \mathrm{id}_{\mathrm{Ob}}$ and composition given by

$$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

In other words, an element of Γ is a rule δ which prescribes an arrow emanating from every object and composition is the natural thing. For example

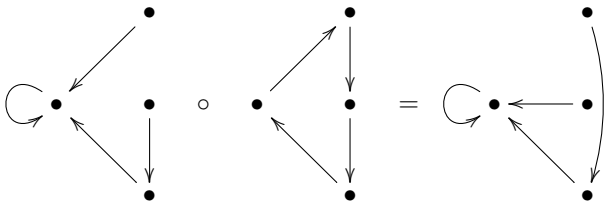

with obvious notation

The same procedure can be applied to a groupoid in algebraic spaces (U, R, s, t, c, e, i) over a scheme S. Namely, as elements of Γ we take the set

$$\Gamma = \{\delta : U \to R \mid s \circ \delta = \mathrm{id}_U\}$$

and composition $\circ : \Gamma \times \Gamma \to \Gamma$ is given by the rule above

0CKC (79.5.0.1) $$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

The identity is given by $e \in \Gamma$. The groupoid Γ is not a group in general because there may be elements $\delta \in \Gamma$ which do not have an inverse. Namely, it is clear that $\delta \in \Gamma$ will have an inverse if and only if $t \circ \delta$ is an automorphism of U and in this case $\delta^{-1} = i \circ \delta \circ (t \circ \delta)^{-1}$.

For later use we discuss what happens with the subgroupoid Γ_0 of Γ of sections which are infinitesimally close to the identity e. More precisely, suppose given an R-invariant closed subspace $U_0 \subset U$ such that U is a first order thickening

of U_0. Denote $R_0 = s^{-1}(U_0) = t^{-1}(U_0)$ and let $(U_0, R_0, s_0, t_0, c_0, e_0, i_0)$ be the corresponding groupoid in algebraic spaces. Set

$$\Gamma_0 = \{\delta \in \Gamma \mid \delta|_{U_0} = e_0\}$$

If s and t are flat, then every element in Γ_0 is invertible. This follows because $t \circ \delta$ will be a morphism $U \to U$ inducing the identity on $\mathcal{O}_{U_0}$ and on $\mathcal{C}_{U_0/U}$ (Lemma 79.5.1) and we conclude because we have a short exact sequence $0 \to \mathcal{C}_{U_0/U} \to \mathcal{O}_U \to \mathcal{O}_{U_0} \to 0$.

0CKD **Lemma 79.5.1.** *In the situation discussed in this section, let $\delta \in \Gamma_0$ and $f = t \circ \delta : U \to U$. If s, t are flat, then the canonical map $\mathcal{C}_{U_0/U} \to \mathcal{C}_{U_0/U}$ induced by f (More on Morphisms of Spaces, Lemma 76.5.3) is the identity map.*

Proof. To see this we extend the bottom of the diagram (79.3.0.2) as follows

$$\begin{array}{ccccccc}
Y & \longrightarrow & R \times_{s,U,t} R & \overset{c}{\underset{\mathrm{pr}_0}{\rightrightarrows}} & R & \xrightarrow{t} & U \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} & & \\
U & \xrightarrow[\delta]{} & R & \overset{s}{\underset{t}{\rightrightarrows}} & U & &
\end{array}$$

where the left square is cartesian and this is our definition of Y; we will not need to know more about Y. There is a similar diagram with similar properties obtained by base change to U_0 everywhere. We are trying to show that $\mathrm{id}_U = s \circ \delta$ and $f = t \circ \delta$ induce the same maps on conormal sheaves. Since s is flat and surjective, it suffices to prove the same thing for the two compositions $a, b : Y \to R$ along the top row. Observe that $a_0 = b_0$ and that one of a and b is an isomorphism as we know that $s \circ \delta$ is an isomorphism. Therefore the two morphisms $a, b : Y \to R$ are morphisms between algebraic spaces flat over U (via the morphism $t : R \to U$ and the morphism $t \circ a = t \circ b : Y \to U$). This implies what we want. Namely, by the compatibility with compositions in More on Morphisms of Spaces, Lemma 76.5.4 we conclude that both maps $a_0^*\mathcal{C}_{R_0/R} \to \mathcal{C}_{Y_0/Y}$ fit into a commutative diagram

$$\begin{array}{ccc}
a_0^*\mathcal{C}_{R_0/R} & \longrightarrow & \mathcal{C}_{Y_0/Y} \\
\uparrow & & \uparrow \\
a_0^*t_0^*\mathcal{C}_{U_0/U} & = & (t_0 \circ a_0)^*\mathcal{C}_{U_0/U}
\end{array}$$

whose vertical arrows are isomorphisms by More on Morphisms of Spaces, Lemma 76.18.1. Thus the lemma holds. □

Let us identify the group Γ_0. Applying the discussion in More on Morphisms of Spaces, Remarks 76.17.3 and 76.17.7 to the diagram

$$\begin{array}{ccc}
(U_0 \subset U) & \overset{(e_0,\delta)}{\dashrightarrow} & (R_0 \subset R) \\
& {\scriptstyle (\mathrm{id}_{U_0},\mathrm{id}_U)} \searrow \quad \swarrow {\scriptstyle (s_0,s)} & \\
& (U_0 \subset U) &
\end{array}$$

we see that $\delta = \theta \cdot e$ for a unique $\mathcal{O}_{U_0}$-linear map $\theta : e_0^*\Omega_{R_0/U_0} \to \mathcal{C}_{U_0/U}$. Thus we get a bijection

0CKE (79.5.1.1) $$\mathrm{Hom}_{\mathcal{O}_{U_0}}(e_0^*\Omega_{R_0/U_0}, \mathcal{C}_{U_0/U}) \longrightarrow \Gamma_0$$

by applying More on Morphisms of Spaces, Lemma 76.17.5.

0CKF **Lemma 79.5.2.** *The bijection (79.5.1.1) is an isomorphism of groups.*

Proof. Let $\delta_1, \delta_2 \in \Gamma_0$ correspond to θ_1, θ_2 as above and the composition $\delta = \delta_1 \circ \delta_2$ in Γ_0 correspond to θ. We have to show that $\theta = \theta_1 + \theta_2$. Recall (More on Morphisms of Spaces, Lemma 76.17.2) that $\theta_1, \theta_2, \theta$ correspond to derivations $D_1, D_2, D : e_0^{-1}\mathcal{O}_{R_0} \to \mathcal{C}_{U_0/U}$ given by $D_1 = \theta_1 \circ \mathrm{d}_{R_0/U_0}$ and so on. It suffices to check that $D = D_1 + D_2$.

We may check equality on stalks. Let $\overline{u}$ be a geometric point of U and let us use the local rings A, B, C introduced in Section 79.4. The morphisms δ_i correspond to ring maps $\delta_i : B \to A$. Let $K \subset A$ be the ideal of square zero such that $A/K = \mathcal{O}_{U_0,\overline{u}}$. In other words, K is the stalk of $\mathcal{C}_{U_0/U}$ at $\overline{u}$. The fact that $\delta_i \in \Gamma_0$ means exactly that $\delta_i(I) \subset K$. The derivation D_i is just the map $\delta_i - e : B \to A$. Since $B = s(A) \oplus I$ we see that D_i is determined by its restriction to I and that this is just given by $\delta_i|_I$. Moreover D_i and hence δ_i annihilates I^2 because $I = \mathrm{Ker}(I)$.

To finish the proof we observe that δ corresponds to the composition

$$B \to C = (B \otimes_{s,A,t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B} \to A$$

where the first arrow is c and the second arrow is determined by the rule $b_1 \otimes b_2 \mapsto \delta_2(t(\delta_1(b_1)))\delta_2(b_2)$ as follows from (79.5.0.1). By Lemma 79.4.1 we see that an element ζ of I maps to $\zeta \otimes 1 + 1 \otimes \zeta$ plus higher order terms. Hence we conclude that

$$D(\zeta) = (\delta_2 \circ t)(D_1(\zeta)) + D_2(\zeta)$$

However, by Lemma 79.5.1 the action of $\delta_2 \circ t$ on $K = \mathcal{C}_{U_0/U,\overline{u}}$ is the identity and we win. □

79.6. Properties of groupoids

044Y This section is the analogue of More on Groupoids, Section 40.6. The reader is strongly encouraged to read that section first.

The following lemma is the analogue of More on Groupoids, Lemma 40.6.4.

044Z **Lemma 79.6.1.** *Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target (Descent on Spaces, Definition 74.10.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant (Groupoids in Spaces, Definition 78.18.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 74.10.3. In Diagram (79.3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent on Spaces, Lemma 74.10.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

06R4 **Lemma 79.6.2.** *Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be its stabilizer group algebraic space. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic*

spaces which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids in Spaces, Definition 78.18.1).

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 74.10.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism of algebraic spaces over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent on Spaces, Lemma 74.10.3. □

79.7. Comparing fibres

04PA This section is the analogue of More on Groupoids, Section 40.7. The reader is strongly encouraged to read that section first.

0452 **Lemma 79.7.1.** *Let $B \to S$ be as in Section 79.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let K be a field and let $r, r' : \mathrm{Spec}(K) \to R$ be morphisms such that $t \circ r = t \circ r' : \mathrm{Spec}(K) \to U$. Set $u = s \circ r$, $u' = s \circ r'$ and denote $F_u = \mathrm{Spec}(K) \times_{u,U,s} R$ and $F_{u'} = \mathrm{Spec}(K) \times_{u',U,s} R$ the fibre products. Then $F_u \cong F_{u'}$ as algebraic spaces over K.*

Proof. We use the properties and the existence of Diagram (79.3.0.1). There exists a morphism $\xi : \mathrm{Spec}(K) \to R \times_{s,U,t} R$ with $\mathrm{pr}_0 \circ \xi = r$ and $c \circ \xi = r'$. Let $\tilde{r} = \mathrm{pr}_1 \circ \xi : \mathrm{Spec}(K) \to R$. Then looking at the bottom two squares of Diagram (79.3.0.1) we see that both F_u and $F_{u'}$ are identified with the algebraic space $\mathrm{Spec}(K) \times_{\tilde{r},R,\mathrm{pr}_1} (R \times_{s,U,t} R)$. □

Actually, in the situation of the lemma the morphisms of pairs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally isomorphic in the τ-topology, provided $\{s : R \to U\}$ is a τ-covering. We will insert a precise statement here if needed.

79.8. Restricting groupoids

04RM In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

04RN (79.8.0.1)

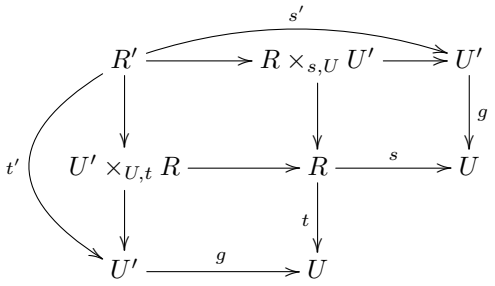

of a restriction. See Groupoids in Spaces, Lemma 78.17.1.

04RP **Lemma 79.8.1.** *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

(1) *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
(2) *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
(3) *If s, t are flat and g is flat, then s', t' are flat.*
(4) *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms of Spaces, Lemmas 67.23.2 and 67.23.3. Hence (1) is clear from Diagram (79.8.0.1). For the other cases, see Morphisms of Spaces, Lemmas 67.28.2, 67.28.3, 67.30.3, and 67.30.4. □

79.9. Properties of groups over fields and groupoids on fields

06DW The reader is advised to first look at the corresponding sections for groupoid schemes, see Groupoids, Section 39.7 and More on Groupoids, Section 40.10.

06DX **Situation 79.9.1.** Here S is a scheme, k is a field over S, and (G, m) is a group algebraic space over $\mathrm{Spec}(k)$.

06DY **Situation 79.9.2.** Here S is a scheme, B is an algebraic space, and (U, R, s, t, c) is a groupoid in algebraic spaces over B with $U = \mathrm{Spec}(k)$ for some field k.

Note that in Situation 79.9.1 we obtain a groupoid in algebraic spaces

06DZ (79.9.2.1) $$(\mathrm{Spec}(k), G, p, p, m)$$

where $p : G \to \mathrm{Spec}(k)$ is the structure morphism of G, see Groupoids in Spaces, Lemma 78.15.1. This is a situation as in Situation 79.9.2. We will use this without further mention in the rest of this section.

06E0 **Lemma 79.9.3.** *In Situation 79.9.2 the composition morphism $c : R \times_{s,U,t} R \to R$ is flat and universally open. In Situation 79.9.1 the group law $m : G \times_k G \to G$ is flat and universally open.*

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (79.3.0.2). The projection is flat as a base change of the flat morphism t and open by Morphisms of Spaces, Lemma 67.6.6. The second assertion follows immediately from the first because m matches c in (79.9.2.1). □

Note that the following lemma applies in particular when working with either quasi-separated or locally separated algebraic spaces (Decent Spaces, Lemma 68.15.2).

08BH **Lemma 79.9.4.** *In Situation 79.9.2 assume R is a decent space. Then R is a separated algebraic space. In Situation 79.9.1 assume that G is a decent algebraic space. Then G is separated algebraic space.*

Proof. We first prove the second assertion. By Groupoids in Spaces, Lemma 78.6.1 we have to show that $e : S \to G$ is a closed immersion. This follows from Decent Spaces, Lemma 68.14.5.

Next, we prove the first assertion. To do this we may replace B by S. By the paragraph above the stabilizer group scheme $G \to U$ is separated. By Groupoids in Spaces, Lemma 78.29.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 26.17). Hence R is separated, see Morphisms of Spaces, Lemma 67.4.9. □

06E1 **Lemma 79.9.5.** *In Situation 79.9.2. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

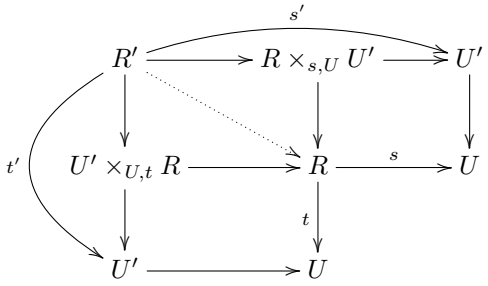

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 29.23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms of Spaces, Lemmas 67.5.5, 67.5.4, 67.6.4, 67.20.5, 67.20.4, 67.30.4, and 67.30.3. □

06E2 **Lemma 79.9.6.** *In Situation 79.9.2. For any point $r \in |R|$ there exist*

(1) *a field extension k'/k with k' algebraically closed,*
(2) *a point $r' : \mathrm{Spec}(k') \to R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s' \circ r', t' \circ r' : \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ are automorphisms.*

Proof. Let's represent r by a morphism $r : \mathrm{Spec}(K) \to R$ for some field K. To prove the lemma we have to find an algebraically closed field k' and a commutative diagram

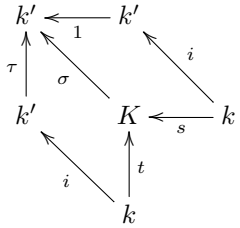

where $s, t : k \to K$ are the field maps coming from $s \circ r$ and $t \circ r$. In the proof of More on Groupoids, Lemma 40.10.5 it is shown how to construct such a diagram. □

06E3 **Lemma 79.9.7.** *In Situation 79.9.2. If $r : \mathrm{Spec}(k) \to R$ is a morphism such that $s \circ r, t \circ r$ are automorphisms of $\mathrm{Spec}(k)$, then the map*

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

is an automorphism $R \to R$ which maps e to r.

Proof. Proof is identical to the proof of More on Groupoids, Lemma 40.10.6. □

06E4 **Lemma 79.9.8.** *In Situation 79.9.2 the algebraic space R is geometrically unibranch. In Situation 79.9.1 the algebraic space G is geometrically unibranch.*

Proof. Let $r \in |R|$. We have to show that R is geometrically unibranch at r. Combining Lemma 79.9.5 with Descent on Spaces, Lemma 74.9.1 we see that it suffices to prove this in case k is algebraically closed and r comes from a morphism $r : \mathrm{Spec}(k) \to R$ such that $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. By Lemma 79.9.7 we reduce to the case that $r = e$ is the identity of R and k is algebraically closed.

Assume $r = e$ and k is algebraically closed. Let $A = \mathcal{O}_{R,e}$ be the étale local ring of R at e and let $C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)}$ be the étale local ring of $R \times_{s,U,t} R$ at (e, e). By More on Algebra, Lemma 15.108.4 the minimal prime ideals $\mathfrak{q}$ of C correspond 1-to-1 to pairs of minimal primes $\mathfrak{p}, \mathfrak{p}' \subset A$. On the other hand, the composition law induces a flat ring map

$$\begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ & & \uparrow \\ & & A \otimes_{s^\sharp,k,t^\sharp} A \end{array} \qquad \begin{array}{c} \mathfrak{q} \\ \top \\ \mathfrak{p} \otimes A + A \otimes \mathfrak{p}' \end{array}$$

Note that $(c^\sharp)^{-1}(\mathfrak{q})$ contains both $\mathfrak{p}$ and $\mathfrak{p}'$ as the diagrams

$$\begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ \uparrow & & \uparrow \\ A \otimes_{s^\sharp,k} k & \xleftarrow{1 \otimes e^\sharp} & A \otimes_{s^\sharp,k,t^\sharp} A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ \uparrow & & \uparrow \\ k \otimes_{k,t^\sharp} A & \xleftarrow{e^\sharp \otimes 1} & A \otimes_{s^\sharp,k,t^\sharp} A \end{array}$$

commute by (79.3.0.1). Since $c^\sharp$ is flat (as c is a flat morphism by Lemma 79.9.3), we see that $(c^\sharp)^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence $\mathfrak{p} = (c^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}'$. □

In the following lemma we use dimension of algebraic spaces (at a point) as defined in Properties of Spaces, Section 66.9. We also use the dimension of the local ring defined in Properties of Spaces, Section 66.10 and transcendence degree of points, see Morphisms of Spaces, Section 67.33.

06FD **Lemma 79.9.9.** *In Situation 79.9.2 assume s, t are locally of finite type. For all $r \in |R|$*

(1) $\dim(R) = \dim_r(R)$,
(2) *the transcendence degree of r over $\mathrm{Spec}(k)$ via s equals the transcendence degree of r over $\mathrm{Spec}(k)$ via t, and*
(3) *if the transcendence degree mentioned in (2) is 0, then $\dim(R) = \dim(\mathcal{O}_{R,\overline{r}})$.*

Proof. Let $r \in |R|$. Denote $\mathrm{trdeg}(r/_s k)$ the transcendence degree of r over $\mathrm{Spec}(k)$ via s. Choose an étale morphism $\varphi : V \to R$ where V is a scheme and $v \in V$ mapping to r. Using the definitions mentioned above the lemma we see that

$$\dim_r(R) = \dim_v(V) = \dim(\mathcal{O}_{V,v}) + \mathrm{trdeg}_{s(k)}(\kappa(v)) = \dim(\mathcal{O}_{R,\overline{r}}) + \mathrm{trdeg}(r/_s k)$$

and similarly for t (the second equality by Morphisms, Lemma 29.28.1). Hence we see that $\mathrm{trdeg}(r/_s k) = \mathrm{trdeg}(r/_t k)$, i.e., (2) holds.

Let k'/k be a field extension. Note that the restriction R' of R to $\mathrm{Spec}(k')$ (see Lemma 79.9.5) is obtained from R by two base changes by morphisms of fields. Thus Morphisms of Spaces, Lemma 67.34.3 shows the dimension of R at a point is unchanged by this operation. Hence in order to prove (1) we may assume, by Lemma 79.9.6, that r is represented by a morphism $r : \mathrm{Spec}(k) \to R$ such that both $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. In this case there exists an automorphism $R \to R$ which maps r to e (Lemma 79.9.7). Hence we see that $\dim_r(R) = \dim_e(R)$ for any r. By definition this means that $\dim_r(R) = \dim(R)$.

Part (3) is a formal consequence of the results obtained in the discussion above. □

06FE **Lemma 79.9.10.** *In Situation 79.9.1 assume G locally of finite type. For all $g \in |G|$*

(1) $\dim(G) = \dim_g(G)$,
(2) *if the transcendence degree of g over k is 0, then* $\dim(G) = \dim(\mathcal{O}_{G,\overline{g}})$.

Proof. Immediate from Lemma 79.9.9 via (79.9.2.1). □

06FF **Lemma 79.9.11.** *In Situation 79.9.2 assume s, t are locally of finite type. Let $G = \mathrm{Spec}(k) \times_{\Delta, \mathrm{Spec}(k) \times_B \mathrm{Spec}(k), t \times s} R$ be the stabilizer group algebraic space. Then we have* $\dim(R) = \dim(G)$.

Proof. Since G and R are equidimensional (see Lemmas 79.9.9 and 79.9.10) it suffices to prove that $\dim_e(R) = \dim_e(G)$. Let V be an affine scheme, $v \in V$, and let $\varphi : V \to R$ be an étale morphism of schemes such that $\varphi(v) = e$. Note that V is a Noetherian scheme as $s \circ \varphi$ is locally of finite type as a composition of morphisms locally of finite type and as V is quasi-compact (use Morphisms of Spaces, Lemmas 67.23.2, 67.39.8, and 67.28.5 and Morphisms, Lemma 29.15.6). Hence V is locally connected (see Properties, Lemma 28.5.5 and Topology, Lemma 5.9.6). Thus we may replace V by the connected component containing v (it is still affine as it is an open and closed subscheme of V). Set $T = V_{red}$ equal to the reduction of V. Consider the two morphisms $a, b : T \to \mathrm{Spec}(k)$ given by $a = s \circ \varphi|_T$ and $b = t \circ \varphi|_T$. Note that a, b induce the same field map $k \to \kappa(v)$ because $\varphi(v) = e$! Let $k_a \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $a^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. Similarly, let $k_b \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $b^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. By Varieties, Proposition 33.31.1 we see that $k_a = k_b$. Thus we obtain the following commutative diagram

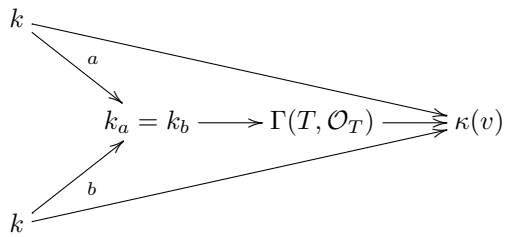

As discussed above the long arrows are equal. Since $k_a = k_b \to \kappa(v)$ is injective we conclude that the two morphisms a and b agree. Hence $T \to R$ factors through G. It follows that $R_{red} = G_{red}$ in an open neighbourhood of e which certainly implies that $\dim_e(R) = \dim_e(G)$. □

79.10. Group algebraic spaces over fields

0B8D There exists a nonseparated group algebraic space over a field, namely $\mathbf{G}_a/\mathbf{Z}$ over a field of characteristic zero, see Examples, Section 110.50. In fact any group scheme over a field is separated (Lemma 79.9.4) hence every nonseparated group algebraic space over a field is nonrepresentable. On the other hand, a group algebraic space over a field is separated as soon as it is decent, see Lemma 79.9.4. In this section we will show that a separated group algebraic space over a field is representable, i.e., a scheme.

0B8E **Lemma 79.10.1.** *Let k be a field with algebraic closure $\overline{k}$. Let G be a group algebraic space over k which is separated*[1]*. Then $G_{\overline{k}}$ is a scheme.*

Proof. By Spaces over Fields, Lemma 72.10.2 it suffices to show that G_K is a scheme for some field extension K/k. Denote $G'_K \subset G_K$ the schematic locus of G_K as in Properties of Spaces, Lemma 66.13.1. By Properties of Spaces, Proposition 66.13.3 we see that $G'_K \subset G_K$ is dense open, in particular not empty. Choose a scheme U and a surjective étale morphism $U \to G$. By Varieties, Lemma 33.14.2 if K is an algebraically closed field of large enough transcendence degree, then U_K is a Jacobson scheme and every closed point of U_K is K-rational. Hence G'_K has a K-rational point and it suffices to show that every K-rational point of G_K is in G'_K. If $g \in G_K(K)$ is a K-rational point and $g' \in G'_K(K)$ a K-rational point in the schematic locus, then we see that g is in the image of G'_K under the automorphism

$$G_K \longrightarrow G_K, \quad h \longmapsto g(g')^{-1}h$$

of G_K. Since automorphisms of G_K as an algebraic space preserve G'_K, we conclude that $g \in G'_K$ as desired. □

0B8F **Lemma 79.10.2.** *Let k be a field. Let G be a group algebraic space over k. If G is separated and locally of finite type over k, then G is a scheme.*

Proof. This follows from Lemma 79.10.1, Groupoids, Lemma 39.8.6, and Spaces over Fields, Lemma 72.10.7. □

0B8G **Proposition 79.10.3.** *Let k be a field. Let G be a group algebraic space over k. If G is separated, then G is a scheme.*

Proof. This lemma generalizes Lemma 79.10.2 (which covers all cases one cares about in practice). The proof is very similar to the proof of Spaces over Fields, Lemma 72.10.7 used in the proof of Lemma 79.10.2 and we encourage the reader to read that proof first.

By Lemma 79.10.1 the base change $G_{\overline{k}}$ is a scheme. Let K/k be a purely transcendental extension of very large transcendence degree. By Spaces over Fields, Lemma 72.10.5 it suffices to show that G_K is a scheme. Let K^{perf} be the perfect closure of K. By Spaces over Fields, Lemma 72.10.1 it suffices to show that $G_{K^{perf}}$

[1] It is enough to assume G is decent, e.g., locally separated or quasi-separated by Lemma 79.9.4.

is a scheme. Let $K \subset K^{perf} \subset \overline{K}$ be the algebraic closure of K. We may choose an embedding $\overline{k} \to \overline{K}$ over k, so that $G_{\overline{K}}$ is the base change of the scheme $G_{\overline{k}}$ by $\overline{k} \to \overline{K}$. By Varieties, Lemma 33.14.2 we see that $G_{\overline{K}}$ is a Jacobson scheme all of whose closed points have residue field $\overline{K}$.

Since $G_{\overline{K}} \to G_{K^{perf}}$ is surjective, it suffices to show that the image $g \in |G_{K^{perf}}|$ of an arbitrary closed point of $G_{\overline{K}}$ is in the schematic locus of G_K. In particular, we may represent g by a morphism $g : \mathrm{Spec}(L) \to G_{K^{perf}}$ where L/K^{perf} is separable algebraic (for example we can take $L = \overline{K}$). Thus the scheme

$$\begin{aligned} T &= \mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{\overline{K}} \\ &= \mathrm{Spec}(L) \times_{\mathrm{Spec}(K^{perf})} \mathrm{Spec}(\overline{K}) \\ &= \mathrm{Spec}(L \otimes_{K^{perf}} \overline{K}) \end{aligned}$$

is the spectrum of a $\overline{K}$-algebra which is a filtered colimit of algebras which are finite products of copies of $\overline{K}$. Thus by Groupoids, Lemma 39.7.13 we can find an affine open $W \subset G_{\overline{K}}$ containing the image of $g_{\overline{K}} : T \to G_{\overline{K}}$.

Choose a quasi-compact open $V \subset G_{K^{perf}}$ containing the image of W. By Spaces over Fields, Lemma 72.10.2 we see that $V_{K'}$ is a scheme for some finite extension K'/K^{perf}. After enlarging K' we may assume that there exists an affine open $U' \subset V_{K'} \subset G_{K'}$ whose base change to $\overline{K}$ recovers W (use that $V_{\overline{K}}$ is the limit of the schemes $V_{K''}$ for $K' \subset K'' \subset \overline{K}$ finite and use Limits, Lemmas 32.4.11 and 32.4.13). We may assume that K'/K^{perf} is a Galois extension (take the normal closure Fields, Lemma 9.16.3 and use that K^{perf} is perfect). Set $H = \mathrm{Gal}(K'/K^{perf})$. By construction the H-invariant closed subscheme $\mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{K'}$ is contained in U'. By Spaces over Fields, Lemmas 72.10.3 and 72.10.4 we conclude. □

79.11. No rational curves on groups

0AEK In this section we prove that there are no nonconstant morphisms from $\mathbf{P}^1$ to a group algebraic space locally of finite type over a field.

0AEL **Lemma 79.11.1.** *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ and $g : X \to Z$ be morphisms of algebraic spaces over B. Assume*

(1) *$Y \to B$ is separated,*
(2) *g is surjective, flat, and locally of finite presentation,*
(3) *there is a scheme theoretically dense open $V \subset Z$ such that $f|_{g^{-1}(V)} : g^{-1}(V) \to Y$ factors through V.*

Then f factors through g.

Proof. Set $R = X \times_Z X$. By (2) we see that $Z = X/R$ as sheaves. Also (2) implies that the inverse image of V in R is scheme theoretically dense in R (Morphisms of Spaces, Lemma 67.30.11). The we see that the two compositions $R \to X \to Y$ are equal by Morphisms of Spaces, Lemma 67.17.8. The lemma follows. □

0AEM **Lemma 79.11.2.** *Let k be a field. Let $n \geq 1$ and let $(\mathbf{P}^1_k)^n$ be the n-fold self product over $\mathrm{Spec}(k)$. Let $f : (\mathbf{P}^1_k)^n \to Z$ be a morphism of algebraic spaces over k. If Z is separated of finite type over k, then f factors as*

$$(\mathbf{P}^1_k)^n \xrightarrow{projection} (\mathbf{P}^1_k)^m \xrightarrow{finite} Z.$$

Proof. We may assume k is algebraically closed (details omitted); we only do this so we may argue using rational points, but the reader can work around this if she/he so desires. In the proof products are over k. The automorphism group algebraic space of $(\mathbf{P}^1_k)^n$ contains $G = (\mathrm{GL}_{2,k})^n$. If $C \subset (\mathbf{P}^1_k)^n$ is a closed subvariety (in particular irreducible over k) which is mapped to a point, then we can apply More on Morphisms of Spaces, Lemma 76.35.3 to the morphism

$$G \times C \to G \times Z, \quad (g, c) \mapsto (g, f(g \cdot c))$$

over G. Hence $g(C)$ is mapped to a point for $g \in G(k)$ lying in a Zariski open $U \subset G$. Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are k-valued points of $(\mathbf{P}^1_k)^n$. Let $I \subset \{1, \ldots, n\}$ be the set of indices i such that $x_i = y_i$. Then

$$\{g(x) \mid g(y) = y,\ g \in U(k)\}$$

is Zariski dense in the fibre of the projection $\pi_I : (\mathbf{P}^1_k)^n \to \prod_{i \in I} \mathbf{P}^1_k$ (exercise). Hence if $x, y \in C(k)$ are distinct, we conclude that f maps the whole fibre of π_I containing x, y to a single point. Moreover, the $U(k)$-orbit of C meets a Zariski open set of fibres of π_I. By Lemma 79.11.1 the morphism f factors through π_I. After repeating this process finitely many times we reach the stage where all fibres of f over k points are finite. In this case f is finite by More on Morphisms of Spaces, Lemma 76.35.2 and the fact that k points are dense in Z (Spaces over Fields, Lemma 72.16.2). □

0AEN **Lemma 79.11.3.** *Let k be a field. Let G be a separated group algebraic space locally of finite type over k. There does not exist a nonconstant morphism $f : \mathbf{P}^1_k \to G$ over* $\mathrm{Spec}(k)$.

Proof. Assume f is nonconstant. Consider the morphisms

$$\mathbf{P}^1_k \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathbf{P}^1_k \longrightarrow G, \quad (t_1, \ldots, t_n) \longmapsto f(g_1) \ldots f(g_n)$$

where on the right hand side we use multiplication in the group. By Lemma 79.11.2 and the assumption that f is nonconstant this morphism is finite onto its image. Hence $\dim(G) \geq n$ for all n, which is impossible by Lemma 79.9.10 and the fact that G is locally of finite type over k. □

79.12. The finite part of a morphism

04PB Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. For an algebraic space or a scheme T over S consider pairs (a, Z) where

04PC (79.12.0.1)
$$\begin{aligned} &a : T \to Y \text{ is a morphism over } S, \\ &Z \subset T \times_Y X \text{ is an open subspace} \\ &\text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite.} \end{aligned}$$

Suppose $h : T' \to T$ is a morphism of algebraic spaces over S and (a, Z) is a pair as in (79.12.0.1) over T. Set $a' = a \circ h$ and $Z' = (h \times \mathrm{id}_X)^{-1}(Z) = T' \times_T Z$. Then (a', Z') is a pair as in (79.12.0.1) over T'. This follows as finite morphisms are preserved under base change, see Morphisms of Spaces, Lemma 67.45.5. Thus we obtain a functor

04PD (79.12.0.2)
$$\begin{array}{rccc} (X/Y)_{fin} : & (Sch/S)^{opp} & \longrightarrow & Sets \\ & T & \longmapsto & \{(a, Z) \text{ as above}\} \end{array}$$

For applications we are mainly interested in this functor $(X/Y)_{fin}$ when f is separated and locally of finite type. To get an idea of what this is all about, take a look at Remark 79.12.6.

04PE **Lemma 79.12.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then we have*

(1) *The presheaf $(X/Y)_{fin}$ satisfies the sheaf condition for the fppf topology.*
(2) *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, (X/Y)_{fin}) = \{(a, Z) \text{ satisfying 79.12.0.1}\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering (by algebraic spaces). Let $s_i = (a_i, Z_i)$ be pairs over T_i satisfying 79.12.0.1 such that we have $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. First, this implies in particular that a_i and a_j define the same morphism $T_i \times_T T_j \to Y$. By Descent on Spaces, Lemma 74.7.2 we deduce that there exists a unique morphism $a : T \to Y$ such that a_i equals the composition $T_i \to T \to Y$. Second, this implies that $Z_i \subset T_i \times_Y X$ are open subspaces whose inverse images in $(T_i \times_T T_j) \times_Y X$ are equal. Since $\{T_i \times_Y X \to T \times_Y X\}$ is an fppf covering we deduce that there exists a unique open subspace $Z \subset T \times_Y X$ which restricts back to Z_i over T_i, see Descent on Spaces, Lemma 74.7.1. We claim that the projection $Z \to T$ is finite. This follows as being finite is local for the fpqc topology, see Descent on Spaces, Lemma 74.11.23.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair satisfying conditions 79.12.0.1.

Let $v : T \to (X/Y)_{fin}$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in (X/Y)_{fin}(U)$ corresponds to a pair (a_U, Z_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, Z_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, Z) over T.

Conversely, let (a, Z) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, Z)|_U$ gives rise to a transformation of functors $v : h_U \to (X/Y)_{fin}$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, Z)|_U$ and $t^*(a, Z)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $(X/Y)_{fin}$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to (X/Y)_{fin}$.

We omit the verification that the two constructions above are mutually inverse. □

04PF **Lemma 79.12.2.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & X \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

of algebraic spaces over S. If j is an open immersion, then there is a canonical injective map of sheaves $j : (X'/Y)_{fin} \to (X/Y)_{fin}$.

Proof. If (a, Z) is a pair over T for X'/Y, then $(a, j(Z))$ is a pair over T for X/Y. □

04PG **Lemma 79.12.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $X' \subset X$ be the maximal open subspace over which f is locally quasi-finite, see Morphisms of Spaces, Lemma 67.34.7. Then $(X/Y)_{fin} = (X'/Y)_{fin}$.*

Proof. Lemma 79.12.2 gives us an injective map $(X'/Y)_{fin} \to (X/Y)_{fin}$. Morphisms of Spaces, Lemma 67.34.7 assures us that formation of X' commutes with base change. Hence everything comes down to proving that if $Z \subset X$ is an open subspace such that $f|_Z : Z \to Y$ is finite, then $Z \subset X'$. This is true because a finite morphism is locally quasi-finite, see Morphisms of Spaces, Lemma 67.45.8. □

04PH **Lemma 79.12.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let T be an algebraic space over S, and let (a, Z) be a pair as in 79.12.0.1. If f is separated, then Z is closed in $T \times_Y X$.*

Proof. A finite morphism of algebraic spaces is universally closed by Morphisms of Spaces, Lemma 67.45.9. Since f is separated so is the morphism $T \times_Y X \to T$, see Morphisms of Spaces, Lemma 67.4.4. Thus the closedness of Z follows from Morphisms of Spaces, Lemma 67.40.6. □

04PI **Remark 79.12.5.** Let $f : X \to Y$ be a separated morphism of algebraic spaces. The sheaf $(X/Y)_{fin}$ comes with a natural map $(X/Y)_{fin} \to Y$ by mapping the pair $(a, Z) \in (X/Y)_{fin}(T)$ to the element $a \in Y(T)$. We can use Lemma 79.12.4 to define operations

$$\star_i : (X/Y)_{fin} \times_Y (X/Y)_{fin} \longrightarrow (X/Y)_{fin}$$

by the rules

$$\begin{aligned}
\star_1 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \cup Z_2) \\
\star_2 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \cap Z_2) \\
\star_3 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \setminus Z_2) \\
\star_4 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_2 \setminus Z_1).
\end{aligned}$$

The reason this works is that $Z_1 \cap Z_2$ is both open and closed inside Z_1 and Z_2 (which also implies that $Z_1 \cup Z_2$ is the disjoint union of the other three pieces). Thus we can think of $(X/Y)_{fin}$ as an $\mathbf{F}_2$-algebra (without unit) over Y with multiplication given by $ss' = \star_2(s, s')$, and addition given by

$$s + s' = \star_1(\star_3(s, s'), \star_4(s, s'))$$

which boils down to taking the symmetric difference. Note that in this sheaf of algebras $0 = (1_Y, \emptyset)$ and that indeed $s + s = 0$ for any local section s. If $f : X \to Y$ is finite, then this algebra has a unit namely $1 = (1_Y, X)$ and $\star_3(s, s') = s(1 + s')$, and $\star_4(s, s') = (1 + s)s'$.

04PJ **Remark 79.12.6.** Let $f : X \to Y$ be a separated, locally quasi-finite morphism of schemes. In this case the sheaf $(X/Y)_{fin}$ is closely related to the sheaf $f_!\mathbf{F}_2$ (insert future reference here) on $Y_{étale}$. Namely, if $V \to Y$ is étale, and $s \in \Gamma(V, f_!\mathbf{F}_2)$, then $s \in \Gamma(V \times_Y X, \mathbf{F}_2)$ is a section with proper support $Z = \mathrm{Supp}(s)$ over V. Since f is also locally quasi-finite we see that the projection $Z \to V$ is actually finite. Since

the support of a section of a constant abelian sheaf is open we see that the pair $(V \to Y, \mathrm{Supp}(s))$ satisfies 79.12.0.1. In fact, $f_!\mathbf{F}_2 \cong (X/Y)_{fin}|_{Y_{étale}}$ in this case which also explains the $\mathbf{F}_2$-algebra structure introduced in Remark 79.12.5.

04PK **Lemma 79.12.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The diagonal of $(X/Y)_{fin} \to Y$*

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times_Y (X/Y)_{fin}$$

is representable (by schemes) and an open immersion and the "absolute" diagonal

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times (X/Y)_{fin}$$

is representable (by schemes).

Proof. The second statement follows from the first as the absolute diagonal is the composition of the relative diagonal and a base change of the diagonal of Y (which is representable by schemes), see Spaces, Section 65.3. To prove the first assertion we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 79.12.0.1 there is an open subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h(T') \subset V \Leftrightarrow \Big(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\Big)$$

Let us construct V. Note that $Z_1 \cap Z_2$ is open in Z_1 and in Z_2. Since $\mathrm{pr}_0|_{Z_i} : Z_i \to T$ is finite, hence proper (see Morphisms of Spaces, Lemma 67.45.9) we see that

$$E = \mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2}(Z_2 \setminus Z_1 \cap Z_2))$$

is closed in T. Now it is clear that $V = T \setminus E$ works. □

04QE **Lemma 79.12.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Suppose that U is a scheme, $U \to Y$ is an étale morphism and $Z \subset U \times_Y X$ is an open subspace finite over U. Then the induced morphism $U \to (X/Y)_{fin}$ is étale.*

Proof. This is formal from the description of the diagonal in Lemma 79.12.7 but we write it out since it is an important step in the development of the theory. We have to check that for any scheme T over S and a morphism $T \to (X/Y)_{fin}$ the projection map

$$T \times_{(X/Y)_{fin}} U \longrightarrow T$$

is étale. Note that

$$T \times_{(X/Y)_{fin}} U = (X/Y)_{fin} \times_{((X/Y)_{fin} \times_Y (X/Y)_{fin})} (T \times_Y U)$$

Applying the result of Lemma 79.12.7 we see that $T \times_{(X/Y)_{fin}} U$ is represented by an open subscheme of $T \times_Y U$. As the projection $T \times_Y U \to T$ is étale by Morphisms of Spaces, Lemma 67.39.4 we conclude. □

04QF **Lemma 79.12.9.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a fibre product square of algebraic spaces over S. Then

$$\begin{array}{ccc} (X'/Y')_{fin} & \longrightarrow & (X/Y)_{fin} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a fibre product square of sheaves on $(Sch/S)_{fppf}$.

Proof. It follows immediately from the definitions that the sheaf $(X'/Y')_{fin}$ is equal to the sheaf $Y' \times_Y (X/Y)_{fin}$. □

04QG **Lemma 79.12.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated and locally quasi-finite, then there exists a scheme U étale over Y and a surjective étale morphism $U \to (X/Y)_{fin}$ over Y.*

Proof. Note that the assertion makes sense by the result of Lemma 79.12.7 on the diagonal of $(X/Y)_{fin}$, see Spaces, Lemma 65.5.10. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. By Lemma 79.12.9 the morphism $(V \times_Y X/V)_{fin} \to (X/Y)_{fin}$ is a base change of the map $V \to Y$ and hence is surjective and étale, see Spaces, Lemma 65.5.5. Hence it suffices to prove the lemma for $(V \times_Y X/V)_{fin}$. (Here we implicitly use that the composition of representable, surjective, and étale transformations of functors is again representable, surjective, and étale, see Spaces, Lemmas 65.3.2 and 65.5.4, and Morphisms, Lemmas 29.9.2 and 29.36.3.) Note that the properties of being separated and locally quasi-finite are preserved under base change, see Morphisms of Spaces, Lemmas 67.4.4 and 67.27.4. Hence $V \times_Y X \to V$ is separated and locally quasi-finite as well, and by Morphisms of Spaces, Proposition 67.50.2 we see that $V \times_Y X$ is a scheme as well. Thus we may assume that $f : X \to Y$ is a separated and locally quasi-finite morphism of schemes.

Pick a point $y \in Y$. Pick $x_1, \ldots, x_n \in X$ points lying over y. Pick an étale neighbourhood $a : (U, u) \to (Y, y)$ and a decomposition

$$U \times_S X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_j} V_{i,j}$$

as in More on Morphisms, Lemma 37.41.5. Pick any subset

$$I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}.$$

Given these choices we obtain a pair (a, Z) with $Z = \bigcup_{(i,j) \in I} V_{i,j}$ which satisfies conditions 79.12.0.1. In other words we obtain a morphism $U \to (X/Y)_{fin}$. The construction of this morphism depends on all the things we picked above, so we should really write

$$U(y, n, x_1, \ldots, x_n, a, I) \longrightarrow (X/Y)_{fin}$$

This morphism is étale by Lemma 79.12.8.

Claim: The disjoint union of all of these is surjective onto $(X/Y)_{fin}$. It is clear that if the claim holds, then the lemma is true.

To show surjectivity we have to show the following (see Spaces, Remark 65.5.2): Given a scheme T over S, a point $t \in T$, and a map $T \to (X/Y)_{fin}$ we can find a

datum $(y, n, x_1, \ldots, x_n, a, I)$ as above such that t is in the image of the projection map

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T \longrightarrow T.$$

To prove this we may clearly replace T by $\mathrm{Spec}(\overline{\kappa(t)})$ and $T \to (X/Y)_{fin}$ by the composition $\mathrm{Spec}(\overline{\kappa(t)}) \to T \to (X/Y)_{fin}$. In other words, we may assume that T is the spectrum of an algebraically closed field.

Let $T = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field k. The morphism $T \to (X/Y)_{fin}$ is given by a pair $(T \to Y, Z)$ satisfying conditions 79.12.0.1. Here is a picture:

$$\begin{array}{ccc} & & Z \longrightarrow X \\ & & \downarrow \quad \downarrow \\ \mathrm{Spec}(k) & = & T \longrightarrow Y \end{array}$$

Let $y \in Y$ be the image point of $T \to Y$. Since Z is finite over k it has finitely many points. Thus there exist finitely many points $x_1, \ldots, x_n \in X$ such that the image of Z in X is contained in $\{x_1, \ldots, x_n\}$. Choose $a : (U, u) \to (Y, y)$ adapted to y and $x_1, \ldots, x_n$ as above, which gives the diagram

$$\begin{array}{ccc} W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y. \end{array}$$

Since k is algebraically closed and $\kappa(y) \subset \kappa(u)$ is finite separable we may factor the morphism $T = \mathrm{Spec}(k) \to Y$ through the morphism $u = \mathrm{Spec}(\kappa(u)) \to \mathrm{Spec}(\kappa(y)) = y \subset Y$. With this choice we obtain the commutative diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & U & \longrightarrow & Y \end{array}$$

We know that the image of the left upper arrow ends up in $\coprod V_{i,j}$. Recall also that Z is an open subscheme of $\mathrm{Spec}(k) \times_Y X$ by definition of $(X/Y)_{fin}$ and that the right hand square is a fibre product square. Thus we see that

$$Z \subset \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_j} \mathrm{Spec}(k) \times_U V_{i,j}$$

is an open subscheme. By construction (see More on Morphisms, Lemma 37.41.5) each $V_{i,j}$ has a unique point $v_{i,j}$ lying over u with purely inseparable residue field extension $\kappa(v_{i,j})/\kappa(u)$. Hence each scheme $\mathrm{Spec}(k) \times_U V_{i,j}$ has exactly one point. Thus we see that

$$Z = \coprod\nolimits_{(i,j) \in I} \mathrm{Spec}(k) \times_U V_{i,j}$$

for a unique subset $I \subset \{(i, j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$. Unwinding the definitions this shows that

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T$$

with I as found above is nonempty as desired. $\square$

04QH **Proposition 79.12.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated and locally of finite type. Then $(X/Y)_{fin}$ is an algebraic space. Moreover, the morphism $(X/Y)_{fin} \to Y$ is étale.*

Proof. By Lemma 79.12.3 we may replace X by the open subscheme which is locally quasi-finite over Y. Hence we may assume that f is separated and locally quasi-finite. We will check the three conditions of Spaces, Definition 65.6.1. Condition (1) follows from Lemma 79.12.1. Condition (2) follows from Lemma 79.12.7. Finally, condition (3) follows from Lemma 79.12.10. Thus $(X/Y)_{fin}$ is an algebraic space. Moreover, that lemma shows that there exists a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & (X/Y)_{fin} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

with horizontal arrow surjective and étale and south-east arrow étale. By Properties of Spaces, Lemma 66.16.3 this implies that the south-west arrow is étale as well. □

04QI **Remark 79.12.12.** The condition that f be separated cannot be dropped from Proposition 79.12.11. An example is to take X the affine line with zero doubled, see Schemes, Example 26.14.3, $Y = \mathbf{A}^1_k$ the affine line, and $X \to Y$ the obvious map. Recall that over $0 \in Y$ there are two points 0_1 and 0_2 in X. Thus $(X/Y)_{fin}$ has four points over 0, namely $\emptyset, \{0_1\}, \{0_2\}, \{0_1, 0_2\}$. Of these four points only three can be lifted to an open subscheme of $U \times_Y X$ finite over U for $U \to Y$ étale, namely $\emptyset, \{0_1\}, \{0_2\}$. This shows that $(X/Y)_{fin}$ if representable by an algebraic space is not étale over Y. Similar arguments show that $(X/Y)_{fin}$ is really not an algebraic space. Details omitted.

04QJ **Remark 79.12.13.** Let $Y = \mathbf{A}^1_{\mathbf{R}}$ be the affine line over the real numbers, and let $X = \mathrm{Spec}(\mathbf{C})$ mapping to the $\mathbf{R}$-rational point 0 in Y. In this case the morphism $f : X \to Y$ is finite, but it is not the case that $(X/Y)_{fin}$ is a scheme. Namely, one can show that in this case the algebraic space $(X/Y)_{fin}$ is isomorphic to the algebraic space of Spaces, Example 65.14.2 associated to the extension $\mathbf{R} \subset \mathbf{C}$. Thus it is really necessary to leave the category of schemes in order to represent the sheaf $(X/Y)_{fin}$, even when f is a finite morphism.

04RI **Lemma 79.12.14.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated, flat, and locally of finite presentation. In this case*

(1) *$(X/Y)_{fin} \to Y$ is separated, representable, and étale, and*
(2) *if Y is a scheme, then $(X/Y)_{fin}$ is (representable by) a scheme.*

Proof. Since f is in particular separated and locally of finite type (see Morphisms of Spaces, Lemma 67.28.5) we see that $(X/Y)_{fin}$ is an algebraic space by Proposition 79.12.11. To prove that $(X/Y)_{fin} \to Y$ is separated we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 79.12.0.1 there is a closed subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h \text{ factors through } V \Leftrightarrow \Big(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\Big)$$

In the proof of Lemma 79.12.7 we have seen that $V = T' \setminus E$ is an open subscheme of T' with closed complement

$$E = \mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2}(Z_2 \setminus Z_1 \cap Z_2)).$$

Thus everything comes down to showing that E is also open. By Lemma 79.12.4 we see that Z_1 and Z_2 are closed in $T' \times_Y X$. Hence $Z_1 \setminus Z_1 \cap Z_2$ is open in Z_1. As f is flat and locally of finite presentation, so is $\mathrm{pr}_0|_{Z_1}$. This is true as Z_1 is an open subspace of the base change $T' \times_Y X$, and Morphisms of Spaces, Lemmas 67.28.3 and Lemmas 67.30.4. Hence $\mathrm{pr}_0|_{Z_1}$ is open, see Morphisms of Spaces, Lemma 67.30.6. Thus $\mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2))$ is open and it follows that E is open as desired.

We have already seen that $(X/Y)_{fin} \to Y$ is étale, see Proposition 79.12.11. Hence now we know it is locally quasi-finite (see Morphisms of Spaces, Lemma 67.39.5) and separated, hence representable by Morphisms of Spaces, Lemma 67.51.1. The final assertion is clear (if you like you can use Morphisms of Spaces, Proposition 67.50.2). □

Variant: Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. For an algebraic space or a scheme T over S consider pairs (a, Z) where

04RQ (79.12.14.1)
$$\begin{array}{c} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace} \\ \text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite and} \\ (1_T, \sigma \circ a) : T \to T \times_Y X \text{ factors through } Z. \end{array}$$

We will denote $(X/Y, \sigma)_{fin}$ the subfunctor of $(X/Y)_{fin}$ parametrizing these pairs.

04RR **Lemma 79.12.15.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. Consider the transformation of functors*

$$t : (X/Y, \sigma)_{fin} \longrightarrow (X/Y)_{fin}.$$

defined above. Then

1. *t is representable by open immersions,*
2. *if f is separated, then t is representable by open and closed immersions,*
3. *if $(X/Y)_{fin}$ is an algebraic space, then $(X/Y, \sigma)_{fin}$ is an algebraic space and an open subspace of $(X/Y)_{fin}$, and*
4. *if $(X/Y)_{fin}$ is a scheme, then $(X/Y, \sigma)_{fin}$ is an open subscheme of it.*

Proof. Omitted. Hint: Given a pair (a, Z) over T as in (79.12.0.1) the inverse image of Z by $(1_T, \sigma \circ a) : T \to T \times_Y X$ is the open subscheme of T we are looking for. □

79.13. Finite collections of arrows

04RS Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 39.13 this corresponds to a septuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$.

Using this data we can make another groupoid $\mathcal{C}_{fin}$ as follows:

1. An object of $\mathcal{C}_{fin}$ consists of a finite subset $Z \subset \mathrm{Arrows}$ with the following properties:
   (a) $s(Z) = \{u\}$ is a singleton, and

(b) $e(u) \in Z$.

(2) A morphism of $\mathcal{C}_{fin}$ consists of a pair (Z, z), where Z is an object of $\mathcal{C}_{fin}$ and $z \in Z$.
(3) The source of (Z, z) is Z.
(4) The target of (Z, z) is $t(Z, z) = \{z' \circ z^{-1}; z' \in Z\}$.
(5) Given (Z_1, z_1), (Z_2, z_2) such that $s(Z_1, z_1) = t(Z_2, z_2)$ the composition $(Z_1, z_1) \circ (Z_2, z_2)$ is $(Z_2, z_1 \circ z_2)$.

We omit the verification that this defines a groupoid. Pictorially an object of $\mathcal{C}_{fin}$ can be viewed as a diagram

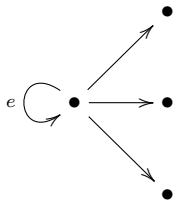

To make a morphism of $\mathcal{C}_{fin}$ you pick one of the arrows and you precompose the other arrows by its inverse. For example if we pick the middle horizontal arrow then the target is the picture

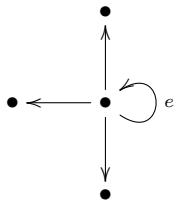

Note that the cardinalities of $s(Z, z)$ and $t(Z, z)$ are equal. So $\mathcal{C}_{fin}$ is really a countable disjoint union of groupoids.

79.14. The finite part of a groupoid

04RT In this section we are going to use the idea explained in Section 79.13 to take the finite part of a groupoid in algebraic spaces.

Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assumption: The morphisms s, t are separated and locally of finite type. This notation and assumption will we be fixed throughout this section.

Denote R_s the algebraic space R seen as an algebraic space over U via s. Let $U' = (R_s/U, e)_{fin}$. Since s is separated and locally of finite type, by Proposition 79.12.11 and Lemma 79.12.15, we see that U' is an algebraic space endowed with an étale morphism $g : U' \to U$. Moreover, by Lemma 79.12.1 there exists a universal open subspace $Z_{univ} \subset R \times_{s,U,g} U'$ which is finite over U' and such that $(1_{U'}, e \circ g) : U' \to R \times_{s,U,g} U'$ factors through Z_{univ}. Moreover, by Lemma 79.12.4 the open

subspace Z_{univ} is also closed in $R \times_{s,U',g} U$. Picture so far:

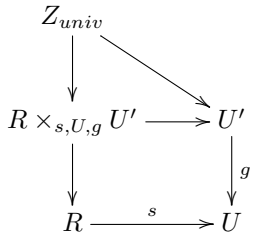

Let T be a scheme over B. We see that a T-valued point of Z_{univ} may be viewed as a triple (u, Z, z) where

(1) $u : T \to U$ is a T-valued point of U,
(2) $Z \subset R \times_{s,U,u} T$ is an open and closed subspace finite over T such that $(e \circ u, 1_T)$ factors through it, and
(3) $z : T \to R$ is a T-valued point of R with $s \circ z = u$ and such that $(z, 1_T)$ factors through Z.

Having said this, it is morally clear from the discussion in Section 79.13 that we can turn (Z_{univ}, U') into a groupoid in algebraic spaces over B. To make sure will define the morphisms s', t', c', e', i' one by one using the functorial point of view. (Please don't read this before reading and understanding the simple construction in Section 79.13.)

The morphism $s' : Z_{univ} \to U'$ corresponds to the rule

$$s' : (u, Z, z) \mapsto (u, Z).$$

The morphism $t' : Z_{univ} \to U'$ is given by the rule

$$t' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z)).$$

The entry $c(Z, i \circ z)$ makes sense as the map $c(-, i \circ z) : R \times_{s,U,u} T \to R \times_{s,U,t \circ z} T$ is an isomorphism with inverse $c(-, z)$. The morphism $e' : U' \to Z_{univ}$ is given by the rule

$$e' : (u, Z) \mapsto (u, Z, (e \circ u, 1_T)).$$

Note that this makes sense by the requirement that $(e \circ u, 1_T)$ factors through Z. The morphism $i' : Z_{univ} \to Z_{univ}$ is given by the rule

$$i' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z), i \circ z).$$

Finally, composition is defined by the rule

$$c' : ((u_1, Z_1, z_1), (u_2, Z_2, z_2)) \mapsto (u_2, Z_2, z_1 \circ z_2).$$

We omit the verification that the axioms of a groupoid in algebraic spaces hold for $(U', Z_{univ}, s', t', c', e', i')$.

A final piece of information is that there is a canonical morphism of groupoids

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

Namely, the morphism $U' \to U$ is the morphism $g : U' \to U$ which is defined by the rule $(u, Z) \mapsto u$. The morphism $Z_{univ} \to R$ is defined by the rule $(u, Z, z) \mapsto z$. This finishes the construction. Let us summarize our findings as follows.

04RU **Lemma 79.14.1.** *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assume the morphisms s, t are separated and locally of finite type. There exists a canonical morphism*

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

of groupoids in algebraic spaces over B where

(1) *$g : U' \to U$ is identified with $(R_s/U, e)_{fin} \to U$, and*
(2) *$Z_{univ} \subset R \times_{s,U,g} U'$ is the universal open (and closed) subspace finite over U' which contains the base change of the unit e.*

Proof. See discussion above. □

79.15. Étale localization of groupoid schemes

04RJ In this section we prove results similar to [**KM97**, Proposition 4.2]. We try to be a bit more general, and we try to avoid using Hilbert schemes by using the finite part of a morphism instead. The goal is to "split" a groupoid in algebraic spaces over a point after étale localization. Here is the definition (very similar to [**KM97**, Definition 4.1]).

04RK **Definition 79.15.1.** Let S be a scheme. Let B be an algebraic space over S Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $u \in |U|$ be a point.

(1) We say R is *strongly split over u* if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $\{r \in |R| : s(r) = u, t(r) = u\} \subset |P|$.

   The choice of such a P will be called a *strong splitting of R over u*.
(2) We say R is *split over u* if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $\{g \in |G| : g \text{ maps to } u\} \subset |P|$ where $G \to U$ is the stabilizer.

   The choice of such a P will be called a *splitting of R over u*.
(3) We say R is *quasi-split over u* if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $e(u) \in |P|$[2].

   The choice of such a P will be called a *quasi-splitting of R over u*.

Note the similarity of the conditions on P to the conditions on pairs in (79.12.0.1). In particular, if s, t are separated, then P is also closed in R (see Lemma 79.12.4).

Suppose we start with a groupoid in algebraic spaces (U, R, s, t, c) over B and a point $u \in |U|$. Since the goal is to split the groupoid after étale localization we may as well replace U by an affine scheme (what we mean is that this is harmless for any possible application). Moreover, the additional hypotheses we are going to have to impose will force R to be a scheme at least in a neighbourhood of $\{r \in |R| : s(r) = u, t(r) = u\}$ or $e(u)$. This is why we start with a groupoid scheme as described below. However, our technique of proof leads us outside of the category

[2]This condition is implied by (a).

of schemes, which is why we have formulated a splitting for the case of groupoids in algebraic spaces above. On the other hand, we know of no applications but the case where the morphisms s, t are also flat and of finite presentation, in which case we end up back in the category of schemes.

04RL **Situation 79.15.2** (Strong splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{r \in R : s(r) = u, t(r) = u\}$ is finite, and
(4) s is quasi-finite at each point of the set in (3).

Note that assumptions (3) and (4) are implied by the assumption that the fibre $s^{-1}(\{u\})$ is finite, see Morphisms, Lemma 29.20.7.

0DTB **Situation 79.15.3** (Splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{g \in G : g \text{ maps to } u\}$ is finite where $G \to U$ is the stabilizer, and
(4) s is quasi-finite at each point of the set in (3).

04RV **Situation 79.15.4** (Quasi-splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type, and
(3) s is quasi-finite at $e(u)$.

For our application to the existence theorems for algebraic spaces the case of quasi-splittings is sufficient. Moreover, the quasi-splitting case will allow us to prove an étale local structure theorem for quasi-DM stacks. The splitting case will be used to prove a version of the Keel-Mori theorem. The strong splitting case applies to give an étale local structure theorem for quasi-DM algebraic stacks with quasi-compact diagonal.

03FM **Lemma 79.15.5** (Existence of strong splitting)**.** *In Situation 79.15.2 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The set $\{r \in R : s(r) = u, t(r) = u\}$ is finite by assumption

and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each of its elements by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $\{r \in R : s(r) = u, t(r) = u\}$. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{r' \in |R'| : s'(r') = u', t'(r') = u'\}$ is contained in $|Z_{univ}|$. Pick any point r' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to R'$. Denote $z : \mathrm{Spec}(k) \to R$ the composition of z' with the map $R' \to R$. Clearly, z defines an element of the set $\{r \in R : s(r) = u, t(r) = u\}$. Also, the compositions $s \circ z, t \circ z : \mathrm{Spec}(k) \to U$ factor through u, so we may think of $s \circ z, t \circ z$ as a morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(u))$. Then $z' = (z, u' \circ t \circ z, u' \circ s \circ u)$ as morphisms into $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Consider the triple

$$(s \circ z, Z_u \times_{\mathrm{Spec}(\kappa(u)), s \circ z} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point r' by the relationship between z and z' mentioned above. This finishes the proof. □

0DTC **Lemma 79.15.6** (Existence of splitting)**.** *In Situation 79.15.3 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. Let $G_u \subset F_u$ be the scheme theoretic fibre of $G \to U$ over u. By assumption G_u is finite and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each point of G_u by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is G_u. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{g' \in |G'| : g' \text{ maps to } u'\}$ is contained in $|Z_{univ}|$. Pick any point g' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to G'$. Denote $z : \mathrm{Spec}(k) \to G$ the composition of z' with the map $G' \to G$. Clearly,

z defines a point of G_u. In fact, let us write $\tilde{u} : \mathrm{Spec}(k) \to u \to U$ for the corresponding map to u or U. Consider the triple

$$(\tilde{u}, Z_u \times_{u,\tilde{u}} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point z' (because the image in R is z). This finishes the proof. □

04RW **Lemma 79.15.7** (Existence of quasi-splitting)**.** *In Situation 79.15.4 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 79.14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The morphism $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at $e(u)$ by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $e(u)$. Hence by the construction of U' in Section 79.14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'. To finish the proof we have to show that $e'(u') \in Z_{univ}$ which is clear. □

Finally, when we add additional assumptions we obtain schemes.

04RX **Lemma 79.15.8.** *In Situation 79.15.2 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 79.15.5 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

0DTD **Lemma 79.15.9.** *In Situation 79.15.3 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 79.15.6 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

04RY **Lemma 79.15.10.** *In Situation 79.15.4 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 79.15.7 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 79.12.14 and 79.12.15. □

In fact we can obtain affine schemes by applying an earlier result on finite locally free groupoids.

04RZ **Lemma 79.15.11.** *In Situation 79.15.2 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. Let $U' \to U$ and $u' \in U'$ be the separated étale morphism of schemes we found in Lemma 79.15.8. Let $P \subset R'$ be the strong splitting of R' over u'. By More on Groupoids, Lemma 40.9.1 the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation. They are finite by assumption. Hence s', t' are finite locally free, see Morphisms, Lemma 29.48.2. In particular $t(s^{-1}(u'))$ is a finite set of points $\{u'_1, u'_2, \ldots, u'_n\}$ of U'. Choose a quasi-compact open $W \subset U'$ containing each u'_i. As U is affine the morphism $W \to U$ is quasi-compact (see Schemes, Lemma 26.19.2). The morphism $W \to U$ is also locally quasi-finite (see Morphisms, Lemma 29.36.6) and separated. Hence by More on Morphisms, Lemma 37.43.2 (a version of Zariski's Main Theorem) we conclude that W is quasi-affine. By Properties, Lemma 28.29.5 we see that $\{u'_1, \ldots, u'_n\}$ are contained in an affine open of U'. Thus we may apply Groupoids, Lemma 39.24.1 to conclude that there exists an affine P-invariant open $U'' \subset U'$ which contains u'.

To finish the proof denote $R'' = R|_{U''}$ the restriction of R to U''. This is the same as the restriction of R' to U''. As $P \subset R'$ is an open and closed subscheme, so is $P|_{U''} \subset R''$. By construction the open subscheme $U'' \subset U'$ is P-invariant which means that $P|_{U''} = (s'|_P)^{-1}(U'') = (t'|_P)^{-1}(U'')$ (see discussion in Groupoids, Section 39.19) so the restrictions of s'' and t'' to $P|_{U''}$ are still finite. The sub groupoid scheme $P|_{U''}$ is still a strong splitting of R'' over u''; above we verified (a), (b) and (c) holds as $\{r' \in R' : t'(r') = u', s'(r') = u'\} = \{r'' \in R'' : t''(r'') = u', s''(r'') = u'\}$ trivially. The lemma is proved. □

0DTE **Lemma 79.15.12.** *In Situation 79.15.3 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. The proof of this lemma is literally the same as the proof of Lemma 79.15.11 except that "strong splitting" needs to be replaced by "splitting" (2 times) and that the reference to Lemma 79.15.8 needs to be replaced by a reference to Lemma 79.15.9. □

04S0 **Lemma 79.15.13.** *In Situation 79.15.4 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme*

U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.

Proof. The proof of this lemma is literally the same as the proof of Lemma 79.15.11 except that "strong splitting" needs to be replaced by "quasi-splitting" (2 times) and that the reference to Lemma 79.15.8 needs to be replaced by a reference to Lemma 79.15.10. □

79.16. Other chapters

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(2) Conventions
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(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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(9) Fields
(10) Commutative Algebra
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Schemes

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Algebraic Spaces

(65) Algebraic Spaces
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(71) Divisors on Algebraic Spaces
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(78) Groupoids in Algebraic Spaces
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CHAPTER 80

Bootstrap

046A

80.1. Introduction

046B In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see [**KM97**].

80.2. Conventions

046C The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

80.3. Morphisms representable by algebraic spaces

02YP Here we define the notion of one presheaf being relatively representable by algebraic spaces over another, and we prove some properties of this notion.

02YQ **Definition 80.3.1.** Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is *representable by algebraic spaces* if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi,G} F$ is an algebraic space.

Here is a sanity check.

03BN **Lemma 80.3.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.*

Proof. This is formal. It relies on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 65.7.3. □

03Y0 **Lemma 80.3.3.** *Let S be a scheme. Let*

$$\begin{array}{ccc} G' \times_G F & \longrightarrow & F \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ G' & \longrightarrow & G \end{array}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces so is a'.

Proof. Omitted. Hint: This is formal. □

02YR **Lemma 80.3.4.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.*

Proof. (Same as the proof of Spaces, Lemma 65.3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. □

05LA **Lemma 80.3.5.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.*

Proof. (Same as the proof of Spaces, Lemma 65.3.6.) Let U be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space V and a morphism $V \to U$ representing the fibre product $U \times_{\xi',G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $U \times_{\xi',G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space. □

The proof of Lemma 80.3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 65.11.3 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.

02YS **Lemma 80.3.6.** *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.*

Proof. We have seen in Lemma 80.3.4 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above R is a scheme and the projections $t, s : R \to W$ are étale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an étale equivalence relation and $F = W/R$. Hence F is an algebraic space by Spaces, Theorem 65.10.5. □

03XY **Lemma 80.3.7.** *Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Suppose $a : F \to G$ is representable by algebraic spaces. If X is an algebraic space over S, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.*

Proof. By Lemma 80.3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 80.3.6. □

03Y1 **Lemma 80.3.8.** *Let S be a scheme. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a and b are representable by algebraic spaces, so is $b \circ a$.

Proof. Let T be a scheme over S, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 80.3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well. □

046D **Lemma 80.3.9.** *Let S be a scheme. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 80.3.8 and 80.3.3. □

0AMN **Lemma 80.3.10.** *Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)^{opp}_{fppf} \to Sets$. Assume*

(1) *$\Delta : G \to G \times_H G$ is representable by algebraic spaces, and*
(2) *$b \circ a : F \to H$ is representable by algebraic spaces.*

Then a is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in G(U)$. Then

$$U \times_{\xi, G, a} F = (U \times_{b(\xi), H, b \circ a} F) \times_{(\xi, a), (G \times_H G), \Delta} G$$

Hence the result using Lemma 80.3.7. □

07WE **Lemma 80.3.11.** *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

(1) *F is a sheaf for the Zariski topology on $(Sch/S)_{fppf}$,*
(2) *there exists an index set I and subfunctors $F_i \subset F$ such that*
   (a) *each F_i is an fppf sheaf,*
   (b) *each $F_i \to F$ is representable by algebraic spaces,*
   (c) *$\coprod F_i \to F$ becomes surjective after fppf sheafification.*

Then F is an fppf sheaf.

Proof. Let $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and let $s \in F(T)$. By (2)(c) there exists an fppf covering $\{T_j \to T\}$ such that $s|_{T_j}$ is a section of $F_{\alpha(j)}$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 29.25.10. By (2)(b) we see $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j \to W_j$ is a monomorphism of algebraic spaces through which T_j factors. Since $\{T_j \to W_j\}$ is an fppf covering, we conclude that $F_{\alpha(j)} \times_{F,s|_{W_j}} W_j = W_j$, in other words $s|_{W_j} \in F_{\alpha(j)}(W_j)$. Hence we conclude that $\coprod F_i \to F$ is surjective for the Zariski topology.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$. Let $s, s' \in F(T)$ with $s|_{T_j} = s'|_{T_j}$ for all j. We want to show that s, s' are equal. As F is a Zariski sheaf by (1) we may work Zariski locally on T. By the result of the previous paragraph we may assume there exist i such that $s \in F_i(T)$. Then we see that $s'|_{T_j}$ is a section of F_i. By (2)(b) we see $F_i \times_{F,s'} T \to T$ is a monomorphism of algebraic spaces through which all of the T_j factor. Hence we conclude that $s' \in F_i(T)$. Since F_i is a sheaf for the fppf topology we conclude that $s = s'$.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$ and let $s_j \in F(T_j)$ such that $s_j|_{T_j \times_T T_{j'}} = s_{j'}|_{T_j \times_T T_{j'}}$. By assumption (2)(c) we may refine the covering and assume that $s_j \in F_{\alpha(j)}(T_j)$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 29.25.10. Then $\{T_j \to W_j\}$ is an fppf covering. Since $F_{\alpha(j)}$ is a sub presheaf of F we see that the two restrictions of s_j to $T_j \times_{W_j} T_j$ agree as elements of $F_{\alpha(j)}(T_j \times_{W_j} T_j)$. Hence, the sheaf condition for $F_{\alpha(j)}$ implies there exists a $s'_j \in F_{\alpha(j)}(W_j)$ whose restriction to T_j is s_j. For a pair of indices j and j' the sections $s'_j|_{W_j \cap W_{j'}}$ and $s'_{j'}|_{W_j \cap W_{j'}}$ of F agree by the result of the previous paragraph. This finishes the proof by the fact that F is a Zariski sheaf. □

80.4. Properties of maps of presheaves representable by algebraic spaces

046E Here is the definition that makes this work.

03XZ **Definition 80.4.1.** Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 74.10.1.

In this case we say that a has *property* $\mathcal{P}$ if for every scheme U and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies[1] for example to the properties of being "surjective", "quasi-compact", "étale", "smooth", "flat", "separated", "(locally) of finite type", "(locally) quasi-finite", "(locally) of finite presentation", "affine", "proper", and "a closed immersion". In other words, a is *surjective* (resp. *quasi-compact*, *étale*, *smooth*, *flat*, *separated*, *(locally) of finite type*, *(locally) quasi-finite*, *(locally) of finite presentation*, *proper*, *a closed immersion*) if for every scheme T and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_{\xi,G} F \to T$ is surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion).

Next, we check consistency with the already existing notions. By Lemma 80.3.2 any morphism between algebraic spaces over S is representable by algebraic spaces. And by Morphisms of Spaces, Lemma 67.5.3 (resp. 67.8.8, 67.39.2, 67.37.4, 67.30.5, 67.4.12, 67.23.4, 67.27.6, 67.28.4, 67.20.3, 67.40.2, 67.12.1) the definition of surjective (resp. quasi-compact, étale, smooth, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, affine, proper, closed immersion) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemmas follow.

046F **Lemma 80.4.2.** *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 80.4.1. Let*

$$\begin{array}{ccc} G' \times_G F & \longrightarrow & F \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ G' & \longrightarrow & G \end{array}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces and has $\mathcal{P}$ so does a'.

Proof. Omitted. Hint: This is formal. □

046G **Lemma 80.4.3.** *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 80.4.1, and assume $\mathcal{P}$ is stable under composition. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a, b are representable by algebraic spaces and has $\mathcal{P}$ so does $b \circ a$.

Proof. Omitted. Hint: See Lemma 80.3.8 and use stability under composition. □

046H **Lemma 80.4.4.** *Let S be a scheme. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Let $\mathcal{P}$ be a property as in Definition 80.4.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 80.3.9. Proof omitted. □

[1] Being preserved under base change holds by Morphisms of Spaces, Lemmas 67.5.5, 67.8.4, 67.39.4, 67.37.3, 67.30.4, 67.4.4, 67.23.3, 67.27.4, 67.28.3, 67.20.5, 67.40.3, and Spaces, Lemma 65.12.3. Being fppf local on the base holds by Descent on Spaces, Lemmas 74.11.6, 74.11.1, 74.11.28, 74.11.26, 74.11.13, 74.11.18, 74.11.11, 74.11.24, 74.11.10, 74.11.16, 74.11.19, and 74.11.17.

0AM1 **Lemma 80.4.5.** *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a transformation of functors representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 80.4.1. Suppose that for any morphism $f : X \to Y$ of algebraic spaces over S we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$, then a has property $\mathcal{P}'$.*

Proof. Formal. □

04S1 **Lemma 80.4.6.** *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be sheaves. Let $a : F \to G$ be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is an algebraic space and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let $U \to T'$ be a surjective étale morphism, where U is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes $U \to T$ is flat. Similarly for "locally of finite presentation". The morphism $U \to T$ is surjective also, see Morphisms of Spaces, Lemma 67.5.3. Hence we see that $\{U \to T\}$ is an fppf covering such that $g|_U \in G(U)$ comes from an element of $F(U)$, namely the map $U \to T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.11.1. □

80.5. Bootstrapping the diagonal

046I In this section we prove that the diagonal of a sheaf F on $(Sch/S)_{fppf}$ is representable as soon as there exists an "fppf cover" of F by a scheme or by an algebraic space, see Lemma 80.5.3.

03Y2 **Lemma 80.5.1.** *Let S be a scheme. If F is a presheaf on $(Sch/S)_{fppf}$. The following are equivalent:*

1. *$\Delta_F : F \to F \times F$ is representable by algebraic spaces,*
2. *for every scheme T any map $T \to F$ is representable by algebraic spaces, and*
3. *for every algebraic space X any map $X \to F$ is representable by algebraic spaces.*

Proof. Assume (1). Let $X \to F$ be as in (3). Let T be a scheme, and let $T \to F$ be a morphism. Then we have

$$T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F$$

which is an algebraic space by Lemma 80.3.7 and (1). Hence $X \to F$ is representable, i.e., (3) holds. The implication (3) $\Rightarrow$ (2) is trivial. Assume (2). Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Then

$$F \times_{\Delta_F, F \times F} T = (T \times_{a, F, b} T) \times_{T \times T, \Delta_T} T$$

which is an algebraic space by assumption. Hence Δ_F is representable by algebraic spaces, i.e., (1) holds. □

In particular if F is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where X is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 80.4.1.

Before we actually do the bootstrap we prove a fun lemma.

046J **Lemma 80.5.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} E & \xrightarrow{a} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H & \xrightarrow{b} & G \end{array}$$

be a cartesian diagram of sheaves on $(Sch/S)_{fppf}$, so $E = H \times_G F$. If

(1) *g is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and*
(2) *a is representable by algebraic spaces, separated, and locally quasi-finite*

then b is representable (by schemes) as well as separated and locally quasi-finite.

Proof. Let T be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is a scheme, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to T and assume that G is a scheme. In this case F is an algebraic space. Let U be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 67.39.7 and 67.39.8. By Lemma 80.3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of a is still separated and locally quasi-finite (by Lemma 80.4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 67.50.2 we conclude that E is a scheme also. By Descent, Lemma 35.39.1 the fact that $E = H \times_G F$ means that we get a descent datum on E relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 37.57.1 this descent datum is effective. By Descent, Lemma 35.39.1 again this implies that H is a scheme. By Descent, Lemmas 35.23.6 and 35.23.26 it now follows that b is separated and locally quasi-finite. □

Here is the result that the section title refers to.

046K **Lemma 80.5.3.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*
(2) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.*

Then Δ_F is representable (by schemes).

Proof. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $U \to X$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 67.39.7 and 67.39.8. By Lemma 80.4.3 the composition $U \to F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that $R = U \times_F U$ is an algebraic space, see Lemma 80.3.7. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 67.10.2). Since $U \to F$ is locally of finite presentation, both

morphisms $R \to U$ are locally of finite presentation, see Lemma 80.4.2. Hence $R \to U \times_S U$ is locally of finite type (use Morphisms of Spaces, Lemmas 67.28.5 and 67.23.6). Altogether this means that $R \to U \times_S U$ is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 67.27.10.

Now we are ready to prove that Δ_F is representable. Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Set

$$T' = (U \times_S U) \times_{F \times F} T.$$

Note that $U \times_S U \to F \times F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 80.4.4. Hence T' is an algebraic space, and the projection morphism $T' \to T$ is surjective, flat, and locally of finite presentation. Consider $Z = T \times_{F \times F} F$ (this is a sheaf) and

$$Z' = T' \times_{U \times_S U} R = T' \times_T Z.$$

We see that Z' is an algebraic space, and $Z' \to T'$ is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that R is an algebraic space and that the morphism $R \to U \times_S U$ has those properties. Hence we may apply Lemma 80.5.2 to the diagram

$$\begin{array}{ccc} Z' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

and we conclude. □

Here is a variant of the result above.

0AHV **Lemma 80.5.4.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let X be a scheme and let $X \to F$ be representable by algebraic spaces and locally quasi-finite. Then $X \to F$ is representable (by schemes).*

Proof. Let T be a scheme and let $T \to F$ be a morphism. We have to show that the algebraic space $X \times_F T$ is representable by a scheme. Consider the morphism

$$X \times_F T \longrightarrow X \times_{\mathrm{Spec}(\mathbf{Z})} T$$

Since $X \times_F T \to T$ is locally quasi-finite, so is the displayed arrow (Morphisms of Spaces, Lemma 67.27.8). On the other hand, the displayed arrow is a monomorphism and hence separated (Morphisms of Spaces, Lemma 67.10.3). Thus $X \times_F T$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. □

80.6. Bootstrap

03XV We warn the reader right away that the result of this section will be superseded by the stronger Theorem 80.10.1. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 65.6 we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has

an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves F which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma 94.13.2.

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem 80.10.1).

03Y3 **Theorem 80.6.1.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*
(2) *the diagonal morphism $F \to F \times F$ is representable by algebraic spaces, and*
(3) *there exists an algebraic space X and a map $X \to F$ which is surjective, and étale.*

or assume that

(a) *the presheaf F is a sheaf, and*
(b) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, and étale.*

Then F is an algebraic space.

Proof. We will use the remarks directly below Definition 80.4.1 without further mention.

Assume (1), (2), and (3) and let $X \to F$ be as in (3). By Lemma 80.5.1 the morphism $X \to F$ is representable by algebraic spaces. Thus we see that (a) and (b) hold.

Assume (a) and (b) and let $X \to F$ be as in (b). Let $U \to X$ be a surjective étale morphism from a scheme towards X. By Lemma 80.3.8 the transformation $U \to F$ is representable by algebraic spaces, surjective, and étale. Hence to prove that F is an algebraic space boils down to proving that Δ_F is representable (Spaces, Definition 65.6.1). This follows immediately from Lemma 80.5.3. On the other hand we can circumvent this lemma and show directly F is an algebraic space as in the next paragraph.

Namely, let U be a scheme and let $U \to F$ be representable by algebraic spaces, surjective, and étale. Consider the fibre product $R = U \times_F U$. Both projections $R \to U$ are representable by algebraic spaces, surjective, and étale (Lemma 80.4.2). In particular R is an algebraic space by Lemma 80.3.6. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Since $R \to U$ is étale, we see that $R \to U$ is locally quasi-finite, see Morphisms of Spaces, Lemma 67.39.5. We conclude that also $R \to U \times_S U$ is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.8. Hence Morphisms of Spaces, Proposition 67.50.2 applies and R is a scheme. By

Lemma 80.4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$. We conclude that F is an algebraic space by Spaces, Theorem 65.10.5. □

80.7. Finding opens

04S2 First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 65.10.2 to quotient sheaves associated to groupoids.

046M **Lemma 80.7.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism. Assume*

(1) *the composition*

$$U' \times_{g,U,t} R \xrightarrow{pr_1} R \xrightarrow{s} U, \quad U' \times_{g,U,t} R \xrightarrow{h} U$$

has an open image $W \subset U$, and

(2) *the resulting map $h : U' \times_{g,U,t} R \to W$ defines a surjection of sheaves in the fppf topology.*

Let $R' = R|_{U'}$ be the restriction of R to U'. Then the map of quotient sheaves

$$U'/R' \to U/R$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that W is an R-invariant open subscheme of U. This is true because the set of points of W is the set of points of U which are equivalent in the sense of Groupoids, Lemma 39.3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 39.13.2). Also $g : U' \to U$ factors through W. Let $R|_W$ be the restriction of R to W. Then it follows that R' is also the restriction of $R|_W$ to U'. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \longrightarrow W/R|_W \longrightarrow U/R$$

By Groupoids, Lemma 39.20.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case g is the immersion of an R-invariant open into U.

Assume $U' \subset U$ is an R-invariant open and g is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 39.20.5 or 39.20.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi,F} F'$ is representable by an open subscheme of T. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = a_i^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ \mathrm{pr}_0$ and $a_j \circ \mathrm{pr}_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ \mathrm{pr}_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ \mathrm{pr}_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 39.20.4. Since U' is R-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}^{-1}(V_i \times_T T_j) = f_{ijk}^{-1}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 35.13.6 there exists an open subscheme $V \subset T$ such that $f_i^{-1}(V) = V_i$. We claim that V represents $T \times_{\xi,F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of F' we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through V. We can find a fppf covering $\{T'_j \to T'\}_{j \in J}$ and morphisms $b_j : T'_j \to U'$ such that $\xi|_{T'_j}$ is the image via $U' \to U/R$ of b_j. Clearly, it is enough to show that the compositions $T'_j \to T$ factor through V. Hence we may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through V_i. Over the scheme $T' \times_T T_i$ the restriction of ξ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ \mathrm{pr}_1$, and $b \circ \mathrm{pr}_0$, the second of which factors through the R-invariant open U'. Hence by Groupoids, Lemma 39.20.4 there exists a covering $\{h_k : Z_k \to T' \times_T T_i\}$ and morphisms $r_k : Z_k \to R$ such that $a_i \circ \mathrm{pr}_1 \circ h_k = s \circ r_k$ and $b \circ \mathrm{pr}_0 \circ h_k = t \circ r_k$. As U' is an R-invariant open the fact that b has image in U' then implies that each $a_i \circ \mathrm{pr}_1 \circ h_k$ has image in U'. It follows from this that $T' \times_T T_i \to T_i$ has image in V_i by definition of V_i which concludes the proof. □

80.8. Slicing equivalence relations

046L In this section we explain how to "improve" a given equivalence relation by slicing. This is not a kind of "étale slicing" that you may be used to but a much coarser kind of slicing.

0489 **Lemma 80.8.1.** *Let S be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over S. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over S, and an isomorphism*

$$U'/R' \longrightarrow U/R$$

induced by a morphism $U' \to U$ which maps R' into R such that $s', t' : R \to U$ are flat, locally of finite presentation and locally quasi-finite.

Proof. We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 39.3.2, 39.13.3, and 39.18.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 40.8.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open over which $s : R \to U$ is Cohen-Macaulay, and denote R' the restriction of R to U'. By the lemma cited above we see that

$$t^{-1}(U') = U' \times_{g,U,t} R \xrightarrow{\mathrm{pr}_1} R \xrightarrow{s} U, \quad U' \times_{g,U,t} R \xrightarrow{h} U$$

is surjective. Since h is flat and locally of finite presentation, we see that $\{h\}$ is a fppf covering. Hence by Groupoids, Lemma 39.20.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of U' we see that s', t' are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume s, t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 40.12.4 there exists an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) h is flat, locally of finite presentation and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

Here we have used the notation introduced in More on Groupoids, Situation 40.12.1.

Step 3. For each point $u \in U$ which is of finite type choose a $g_u : U'_u \to U$ as in Step 2 and denote R'_u the restriction of R to U'_u. Denote $h_u = s \circ \mathrm{pr}_1 : U'_u \times_{g_u, U, t} R \to U$. Set $U' = \coprod_{u \in U} U'_u$, and $g = \coprod g_u$. Let R' be the restriction of R to U' as above. We claim that the pair (U', g) works[2]. Note that

$$\begin{aligned} R' &= \coprod\nolimits_{u_1, u_2 \in U} (U'_{u_1} \times_{g_{u_1}, U, t} R) \times_R (R \times_{s, U, g_{u_2}} U'_{u_2}) \\ &= \coprod\nolimits_{u_1, u_2 \in U} (U'_{u_1} \times_{g_{u_1}, U, t} R) \times_{h_{u_1}, U, g_{u_2}} U'_{u_2} \end{aligned}$$

Hence the projection $s' : R' \to U' = \coprod U'_{u_2}$ is flat, locally of finite presentation and locally quasi-finite as a base change of $\coprod h_{u_1}$. Finally, by construction the morphism $h : U' \times_{g, U, t} R \to U$ is equal to $\coprod h_u$ hence its image contains all points of finite type of U. Since each h_u is flat and locally of finite presentation we conclude that h is flat and locally of finite presentation. In particular, the image of h is open (see Morphisms, Lemma 29.25.10) and since the set of points of finite type is dense (see Morphisms, Lemma 29.16.7) we conclude that the image of h is U. This implies that $\{h\}$ is an fppf covering. By Groupoids, Lemma 39.20.6 this means that $U'/R' \to U/R$ is an isomorphism. This finishes the proof of the lemma. □

80.9. Quotient by a subgroupoid

04S3 We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of "plain" groupoids before embarking on the scheme theoretic version.

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 39.13 this corresponds to a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we are given a subset $P \subset \mathrm{Arrows}$ such that $(\mathrm{Ob}, P, s|_P, t|_P, c|_P)$ is also a groupoid and such that there are no nontrivial automorphisms in P. Then we can construct the quotient groupoid $(\overline{\mathrm{Ob}}, \overline{\mathrm{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ as follows:

(1) $\overline{\mathrm{Ob}} = \mathrm{Ob}/P$ is the set of P-isomorphism classes,
(2) $\overline{\mathrm{Arrows}} = P \backslash \mathrm{Arrows} / P$ is the set of arrows in $\mathcal{C}$ up to pre-composing and post-composing by arrows of P,

[2]Here we should check that U' is not too large, i.e., that it is isomorphic to an object of the category Sch_{fppf}, see Section 80.2. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 3.9. Note that each U'_u has size at most the size of U and that the cardinality of the index set is at most the cardinality of $|U|$ which is bounded by the size of U. Hence U' is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9 part (6).

(3) the source and target maps $\overline{s}, \overline{t} : P\backslash\text{Arrows}/P \to \text{Ob}/P$ are induced by s, t,
(4) composition is defined by the rule $\overline{c}(\overline{a}, \overline{b}) = \overline{c(a, b)}$ which is well defined.

In fact, it turns out that the original groupoid $(\text{Ob}, \text{Arrows}, s, t, c)$ is canonically isomorphic to the restriction (see discussion in Groupoids, Section 39.18) of the groupoid $(\overline{\text{Ob}}, \overline{\text{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ via the quotient map $g : \text{Ob} \to \overline{\text{Ob}}$. Recall that this means that

$$\text{Arrows} = \text{Ob} \times_{g, \overline{\text{Ob}}, \overline{t}} \overline{\text{Arrows}} \times_{\overline{s}, \overline{\text{Ob}}, g} \text{Ob}$$

which holds as P has no nontrivial automorphisms. We omit the details.

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

04S4 **Lemma 80.9.1.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $P \to R$ be monomorphism of schemes. Assume that*

(1) *$(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid scheme,*
(2) *$s|_P, t|_P : P \to U$ are finite locally free,*
(3) *$j|_P : P \to U \times_S U$ is a monomorphism.*
(4) *U is affine, and*
(5) *$j : R \to U \times_S U$ is separated and locally quasi-finite,*

Then U/P is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, R is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 39.23.9 imply the affine scheme $\overline{U}$ representing U/P exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = \text{pr}_0$ and $s|_P = \text{pr}_1$, and such that composition is equal to $\text{pr}_{02} : U \times_{\overline{U}} U \times_{\overline{U}} U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 65.5.7 and Morphisms, Lemmas 29.48.4 and 29.48.3). To get $\overline{R}$ we are going to descend the scheme R via the finite locally free morphism $U \times_S U \to \overline{U} \times_S \overline{U}$. Namely, note that

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = P \times_S P$$

by the above. Thus giving a descent datum (see Descent, Definition 35.34.1) for $R/U \times_S U/\overline{U} \times_S \overline{U}$ consists of an isomorphism

$$\varphi : R \times_{(U \times_S U), t \times t} (P \times_S P) \longrightarrow (P \times_S P) \times_{s \times s, (U \times_S U)} R$$

over $P \times_S P$ satisfying a cocycle condition. We define φ on T-valued points by the rule

$$\varphi : (r, (p, p')) \longmapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a T-valued point of the source of φ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), r') \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = (P \times_S P) \times_{s \times s, (U \times_S U), t \times t} (P \times_S P)$$

By explicit calculation we see that

$$\begin{array}{llll} \varphi_{02} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), (p_1 \circ p_2)^{-1} \circ r \circ (p_1' \circ p_2')) \\ \varphi_{01} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), p_1^{-1} \circ r \circ p_1', (p_2, p_2')) \\ \varphi_{12} & ((p_1, p_1'), r, (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), p_2^{-1} \circ r \circ p_2') \end{array}$$

(with obvious notation) which implies what we want. As j is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 37.57.1 to get a scheme $\overline{R} \to \overline{U} \times_S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$$

which identifies the descent datum φ with the canonical descent datum on $\overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$, see Descent, Definition 35.34.10.

Since $U \times_S U \to \overline{U} \times_S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$. Our choice of φ implies that given T-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \longmapsto \overline{R(T)} = P(T) \backslash R(T) / P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P \backslash R / P)$ exactly as in the discussion of the "plain" groupoid case. It follows from this that (U, R, s, t, c) is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. □

80.10. Final bootstrap

04S5 The following result goes quite a bit beyond the earlier results.

04S6 **Theorem 80.10.1.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Any one of the following conditions implies that F is an algebraic space:*

1. *$F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
2. *$F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
3. *F is a sheaf and there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation,*
4. *F is a sheaf and there exists a scheme U and a morphism $U \to F$ which is representable by algebraic spaces or schemes, surjective, flat and locally of finite presentation,*
5. *F is a sheaf, Δ_F is representable by algebraic spaces, and there exists an algebraic space U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or*
6. *F is a sheaf, Δ_F is representable, and there exists a scheme U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.*

Proof. Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 80.5.3 we see that (3) implies (5). In case (5)

we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 80.4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let (U, R, s, t, c) be a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme U' and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of R to U'. By Groupoids in Spaces, Lemma 78.19.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 79.8.1) this reduces us to the case where U is a scheme. As j is an equivalence relation we see that j is a monomorphism. As $s : R \to U$ is locally of finite presentation we see that $j : R \to U \times_S U$ is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. By Morphisms of Spaces, Lemma 67.27.10 we see that j is locally quasi-finite and separated. Hence if U is a scheme, then R is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus we reduce to proving the theorem in case (2).

Assume $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. By Lemma 80.8.1 we reduce to that case where s, t are flat, locally of finite presentation, and locally quasi-finite. Let $U = \bigcup_{i \in I} U_i$ be an affine open covering (with index set I of cardinality $\leq$ than the size of U to avoid set theoretic problems later – most readers can safely ignore this remark). Let $(U_i, R_i, s_i, t_i, c_i)$ be the restriction of R to U_i. It is clear that s_i, t_i are still flat, locally of finite presentation, and locally quasi-finite as R_i is the open subscheme $s^{-1}(U_i) \cap t^{-1}(U_i)$ of R and s_i, t_i are the restrictions of s, t to this open. By Lemma 80.7.1 (or the simpler Spaces, Lemma 65.10.2) the map $U_i/R_i \to U/R$ is representable by open immersions. Hence if we can show that $F_i = U_i/R_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 65.8.4. As $U = \bigcup U_i$ is an open covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 65.8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite and j is an equivalence.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, such that s, t are flat, locally of finite presentation, and locally quasi-finite, and j is an equivalence relation. Choose $u \in U$. We apply More on Groupoids in Spaces, Lemma 79.15.13 to $u \in U, R, s, t, c$. We obtain an affine scheme U', an étale morphism $g : U' \to U$, a point $u' \in U'$ with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ is quasi-split over u'. Note that the image $g(U')$ is open as g is étale and contains u. Hence, repeatedly applying the lemma, we can find finitely many points $u_i \in U$, $i = 1, \ldots, n$, affine schemes U'_i, étale morphisms $g_i : U'_i \to U$, points $u'_i \in U'_i$ with $g(u'_i) = u_i$ such that (a) each restriction R'_i is quasi-split over some point in U'_i and (b) $U = \bigcup_{i=1,\ldots,n} g_i(U'_i)$. Now we rerun the last part of the argument in the preceding paragraph: Using Lemma 80.7.1 (or the simpler Spaces, Lemma 65.10.2) the map $U'_i/R'_i \to U/R$ is representable by open immersions. If we can show that $F_i = U'_i/R'_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces,

Lemma 65.8.4. As $\{g_i : U_i' \to U\}$ is an étale covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 65.8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite, j is an equivalence, and R is quasi-split over u for some $u \in U$.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and R is quasi-split over u. Let $P \subset R$ be a quasi-splitting of R over u. By Lemma 80.9.1 we see that (U, R, s, t, c) is the restriction of a groupoid $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ by a surjective finite locally free morphism $U \to \overline{U}$ such that $P = U \times_{\overline{U}} U$. Note that s admits a factorization

$$R = U \times_{\overline{U}, \overline{t}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\mathrm{pr}_{23}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\mathrm{pr}_2} U$$

The map pr_2 is the base change of $\overline{s}$, and the map pr_{23} is a base change of the surjective finite locally free map $U \to \overline{U}$. Since s is flat, locally of finite presentation, and locally quasi-finite and since pr_{23} is surjective finite locally free (as a base change of such), we conclude that pr_2 is flat, locally of finite presentation, and locally quasi-finite by Descent, Lemmas 35.27.1 and 35.28.1 and Morphisms, Lemma 29.20.18. Since pr_2 is the base change of the morphism $\overline{s}$ by $U \to \overline{U}$ and $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}$ is flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 35.23.17, 35.23.13, and 35.23.26. The same goes for $\overline{t}$. Consider the commutative diagram

$$\begin{array}{ccccc} U \times_{\overline{U}} U & = & P & \longrightarrow & R \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{U} & \xrightarrow{\overline{e}} & \overline{R} \end{array}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R (by definition of a quasi-splitting) we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 35.23.18. An application of Groupoids, Lemma 39.20.5 shows that $U/R = \overline{U}/\overline{R}$. Hence we have reduced to the case where (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and $e : U \to R$ is an open immersion!

But of course, if e is an open immersion and s, t are flat and locally of finite presentation then the morphisms t, s are étale. For example you can see this by applying More on Groupoids, Lemma 40.4.1 which shows that $\Omega_{R/U} = 0$ which in turn implies that $s, t : R \to U$ is G-unramified (see Morphisms, Lemma 29.35.2), which in turn implies that s, t are étale (see Morphisms, Lemma 29.36.16). And if s, t are étale then finally U/R is an algebraic space by Spaces, Theorem 65.10.5. □

80.11. Applications

04SJ As a first application we obtain the following fundamental fact:

A sheaf which is fppf locally an algebraic space is an algebraic space.

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that $F|_{(Sch/S_i)_{fppf}}$ is an algebraic space, see Spaces, Lemma 65.16.4. Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

04SK **Lemma 80.11.1.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let $\{S_i \to S\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *$\coprod_{i\in I} F_i$ is an algebraic space (see Spaces, Lemma 65.8.4).*

Then F is an algebraic space.

Proof. Consider the morphism $\coprod F_i \to F$. This is the base change of $\coprod S_i \to S$ via $F \to S$. Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma 80.4.2. Thus Theorem 80.10.1 applies to show that F is an algebraic space. □

Here is a special case of Lemma 80.11.1 where we do not need to worry about set theoretical issues.

04U0 **Lemma 80.11.2.** *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let $\{S_i \to S\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *the morphisms $F_i \to S_i$ are of finite type.*

Then F is an algebraic space.

Proof. We will use Lemma 80.11.1 above. To do this we will show that the assumption that F_i is of finite type over S_i to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of S a bit). We suggest the reader skip the rest of the proof.

If $S'_i \to S_i$ is a morphism of schemes then

$$h_{S'_i} \times F = h_{S'_i} \times_{h_{S_i}} h_{S_i} \times F = h_{S'_i} \times_{h_{S_i}} F_i$$

is an algebraic space of finite type over S'_i, see Spaces, Lemma 65.7.3 and Morphisms of Spaces, Lemma 67.23.3. Thus we may refine the given covering. After doing this we may assume: (a) each S_i is affine, and (b) the cardinality of I is at most the cardinality of the set of points of S. (Since to cover all of S it is enough that each point is in the image of $S_i \to S$ for some i.)

Since each S_i is affine and each F_i of finite type over S_i we conclude that F_i is quasi-compact. Hence by Properties of Spaces, Lemma 66.6.3 we can find an affine $U_i \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U_i \to F_i$. The fact that $F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma 3.9.7 we conclude that $\mathrm{size}(U_i) \leq \mathrm{size}(S)$. Since also $|I| \leq \mathrm{size}(S)$ we conclude that $\coprod_{i\in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5 and the construction of Sch. This implies that $\coprod F_i$ is an algebraic space by Spaces, Lemma 65.8.4 and we win. □

As a second application we obtain

Any fppf descent datum for algebraic spaces is effective.

This holds modulo set theoretical difficulties; as an example result we offer the following lemma.

0ADV **Lemma 80.11.3.** *Let S be a scheme. Let $\{X_i \to X\}_{i\in I}$ be an fppf covering of algebraic spaces over S.*

(1) *If I is countable[3], then any descent datum for algebraic spaces relative to $\{X_i \to X\}$ is effective.*
(2) *Any descent datum (Y_i, φ_{ij}) relative to $\{X_i \to X\}_{i\in I}$ (Descent on Spaces, Definition 74.22.3) with $Y_i \to X_i$ of finite type is effective.*

Proof. Proof of (1). By Descent on Spaces, Lemma 74.23.1 this translates into the statement that an fppf sheaf F endowed with a map $F \to X$ is an algebraic space provided that each $F \times_X X_i$ is an algebraic space. The restriction on the cardinality of I implies that coproducts of algebraic spaces indexed by I are algebraic spaces, see Spaces, Lemma 65.8.4 and Sets, Lemma 3.9.9. The morphism

$$\coprod F \times_X X_i \longrightarrow F$$

is representable by algebraic spaces (as the base change of $\coprod X_i \to X$, see Lemma 80.3.3), and surjective, flat, and locally of finite presentation (as the base change of $\coprod X_i \to X$, see Lemma 80.4.2). Hence part (1) follows from Theorem 80.10.1.

Proof of (2). First we apply Descent on Spaces, Lemma 74.23.1 to obtain an fppf sheaf F endowed with a map $F \to X$ such that $F \times_X X_i = Y_i$ for all $i \in I$. Our goal is to show that F is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $F' = U \times_X F \to F$ is representable, surjective, and étale as the base change of $U \to X$. By Theorem 80.10.1 it suffices to show that $F' = U \times_X F$ is an algebraic space. We may choose an fppf covering $\{U_j \to U\}_{j\in J}$ where U_j is a scheme refining the fppf covering $\{X_i \times_X U \to U\}_{i\in I}$, see Topologies on Spaces, Lemma 73.7.4. Thus we get a map $a : J \to I$ and for each j a morphism $U_j \to X_{a(j)}$ over X. Then we see that $U_j \times_U F' = U_j \times_{X_{a(j)}} Y_{a(j)}$ is of finite type over U_j. Hence F' is an algebraic space by Lemma 80.11.2. □

Here is a different type of application.

0AMP **Lemma 80.11.4.** *Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)^{opp}_{fppf} \to Sets$. Assume*

(1) *F, G, H are sheaves,*
(2) *$a : F \to G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective, and*
(3) *$b \circ a : F \to H$ is representable by algebraic spaces.*

Then b is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in H(U)$. We have to show that $U \times_{\xi,H} G$ is an algebraic space. On the other hand, we know that $U \times_{\xi,H} F$ is an algebraic space and that $U \times_{\xi,H} F \to U \times_{\xi,H} G$ is representable by algebraic spaces, flat,

[3]The restriction on countablility can be ignored by those who do not care about set theoretical issues. We can allow larger index sets here if we can bound the size of the algebraic spaces which we are descending. See for example Lemma 80.11.2.

locally of finite presentation, and surjective as a base change of the morphism a (see Lemma 80.4.2). Thus the result follows from Theorem 80.10.1. □

04TB **Lemma 80.11.5.** *Assume $B \to S$ and (U, R, s, t, c) are as in Groupoids in Spaces, Definition 78.20.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ is an algebraic space.*

Proof. By Groupoids in Spaces, Lemma 78.22.3 there exists an fppf covering $\{T_i \to T\}_{i\in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is an algebraic space for each i. By Spaces, Lemma 65.16.4 this means that each $F_i = h_{S_i} \times Isom(x, y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\coprod F_i$ is an algebraic space of Lemma 80.11.1 above to conclude. To do this we use Spaces, Lemma 65.8.4 which requires showing that I and the F_i are not "too large". We suggest the reader skip the rest of the proof.

Choose $U' \in \mathrm{Ob}(Sch/S)_{fppf}$ and a surjective étale morphism $U' \to U$. Let R' be the restriction of R to U'. Since $[U/R] = [U'/R']$ we may, after replacing U by U', assume that U is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each F_i is an algebraic space. Hence we may assume that each T_i is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\mathrm{size}(T_i) \leq \mathrm{size}(T)$, see Sets, Lemma 3.9.7. We may also assume that the cardinality of the index set I is at most the cardinality of the set of points of T since to get a covering it suffices to check that each point of T is in the image. Hence $|I| \leq \mathrm{size}(T)$. Choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in Spaces, Lemma 78.22.3 we showed that F_i is representable by $T_i \times_{(y_i,x_i),U\times_B U} R$ for some $x_i, y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i,x_i),U\times_B U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma 3.9.6 we see that

$$\mathrm{size}(V_i) \leq \max\{\mathrm{size}(T_i), \mathrm{size}(W)\} \leq \max\{\mathrm{size}(T), \mathrm{size}(W)\}$$

Hence, by Sets, Lemma 3.9.5 we conclude that

$$\mathrm{size}(\coprod\nolimits_{i\in I} V_i) \leq \max\{|I|, \mathrm{size}(T), \mathrm{size}(W)\}.$$

Hence we conclude by our construction of Sch that $\coprod_{i\in I} V_i$ is isomorphic to an object V of $(Sch/S)_{fppf}$. This verifies the hypothesis of Spaces, Lemma 65.8.4 and we win. □

06PG **Lemma 80.11.6.** *Let S be a scheme. Consider an algebraic space F of the form $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.*

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 78.19.5 and the fact that j is a monomorphism we see that $R = U \times_F U$. Choose a scheme W and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_{U,t} R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 67.30.4 and 67.28.3. Hence by Descent on Spaces, Lemmas 74.11.13 and 74.11.10 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 65.5.2. □

06PH **Lemma 80.11.7.** *Let S be a scheme. Let $X \to B$ be a morphism of algebraic spaces over S. Let G be a group algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. If*

(1) *a is a free action, and*
(2) *$G \to B$ is flat and locally of finite presentation,*

then X/G (see Groupoids in Spaces, Definition 78.19.1) is an algebraic space, the morphism $X \to X/G$ is surjective, flat, and locally of finite presentation, and X is an fppf G-torsor over X/G.

Proof. The fact that X/G is an algebraic space is immediate from Theorem 80.10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s, t : G \times_B X \to X$ are flat and locally of finite presentation (clear for s as a base change of $G \to B$ and by symmetry using the inverse it follows for t) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 78.8.3 as the action is free. The morphism $X \to X/G$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. To see that $X \to X/G$ is an fppf G-torsor (Groupoids in Spaces, Definition 78.9.3) we have to show that $G \times_S X \to X \times_{X/G} X$ is an isomorphism and that $X \to X/G$ fppf locally has sections. The second part is clear from the properties of $X \to X/G$ already shown. The map $G \times_S X \to X \times_{X/G} X$ is injective (as a map of fppf sheaves) as the action is free. Finally, the map is also surjective as a map of sheaves by Groupoids in Spaces, Lemma 78.19.5. This finishes the proof. □

04U1 **Lemma 80.11.8.** *Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Let G be a group algebraic space over S, and denote $G_i = G_{S_i}$ the base changes. Suppose given*

(1) *for each $i \in I$ an fppf G_i-torsor X_i over S_i, and*
(2) *for each $i, j \in I$ a $G_{S_i \times_S S_j}$-equivariant isomorphism $\varphi_{ij} : X_i \times_S S_j \to S_i \times_S X_j$ satisfying the cocycle condition over every $S_i \times_S S_j \times_S S_j$.*

Then there exists an fppf G-torsor X over S whose base change to S_i is isomorphic to X_i such that we recover the descent datum φ_{ij}.

Proof. We may think of X_i as a sheaf on $(Sch/S_i)_{fppf}$, see Spaces, Section 65.16. By Sites, Section 7.26 the descent datum (X_i, φ_{ij}) is effective in the sense that there exists a unique sheaf X on $(Sch/S)_{fppf}$ which recovers the algebraic spaces X_i after restricting back to $(Sch/S_i)_{fppf}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 80.11.1 we see that X is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$. Now we have to show that a is an action and that X is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 78.9.3). These may be checked fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_{S_i} X_i \to X_i$. Hence the lemma is true.

We suggest the reader skip the rest of the proof, which is purely set theoretical. Pick coverings $\{S_{ij} \to S_j\}_{j \in J_i}$ of $(Sch/S)_{fppf}$ which trivialize the G_i torsors X_i

(possible by assumption, and Topologies, Lemma 34.7.7 part (1)). Then $\{S_{ij} \to S\}_{i\in I, j\in J_i}$ is a covering of $(Sch/S)_{fppf}$ and hence we may assume that each X_i is the trivial torsor! Of course we may also refine the covering further, hence we may assume that each S_i is affine and that the index set I has cardinality bounded by the cardinality of the set of points of S. Choose $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to G$. Then we see that $U_i = U \times_S S_i$ comes with an étale surjective morphism to $X_i \cong G_i$. By Sets, Lemma 3.9.6 we see $\mathrm{size}(U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S_i)\}$. By Sets, Lemma 3.9.7 we have $\mathrm{size}(S_i) \leq \mathrm{size}(S)$. Hence we see that $\mathrm{size}(U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S)\}$ for all $i \in I$. Together with the bound on $|I|$ we found above we conclude from Sets, Lemma 3.9.5 that $\mathrm{size}(\coprod U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S)\}$. Hence Spaces, Lemma 65.8.4 applies to show that $\coprod X_i$ is an algebraic space which is what we had to prove. □

80.12. Algebraic spaces in the étale topology

076L Let S be a scheme. Instead of working with sheaves over the big fppf site $(Sch/S)_{fppf}$ we could work with sheaves over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Spaces, Sections 65.3 and 65.5 makes sense for sheaves over $(Sch/S)_{étale}$. Thus we get a second notion of algebraic spaces by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Spaces, Definition 65.6.1 since a sheaf in the fppf topology is certainly a sheaf in the étale topology. However, the notions are equivalent as is shown by the following lemma.

076M **Lemma 80.12.1.** *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *the diagonal $\Delta : F \to F \times F$ is representable, and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and étale.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 65.6.1 are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and étale. Thus all we have to prove is that an étale sheaf F with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_F U$. By (2) the presheaf R is representable by a scheme and by (3) the projections $R \to U$ are étale. Thus $j : R \to U \times_S U$ is an étale equivalence relation. Moreover $U \to F$ identifies F as the quotient of U by R for the étale topology: (a) if $T \to F$ is a morphism, then $\{T \times_F U \to T\}$ is an étale covering, hence $U \to F$ is a surjection of sheaves for the étale topology, (b) if $a, b : T \to U$ map to the same section of F, then $(a, b) : T \to R$ hence a and b have the same image in the quotient of U by R for the étale topology. Next, let U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 65.10.5. Thus we have morphisms (transformations of functors)

$$U \to F \to U/R.$$

By the aforementioned Spaces, Theorem 65.10.5 the composition is representable, surjective, and étale. Hence for any scheme T and morphism $T \to U/R$ the fibre product $V = T \times_{U/R} U$ is a scheme surjective and étale over T. In other words, $\{V \to U\}$ is an étale covering. This proves that $U \to U/R$ is surjective as a map of sheaves in the étale topology. It follows that $F \to U/R$ is surjective as a map of sheaves in the étale topology. On the other hand, the map $F \to U/R$ is injective (as a map of presheaves) since $R = U \times_{U/R} U$ again by Spaces, Theorem 65.10.5. It follows that $F \to U/R$ is an isomorphism of étale sheaves, see Sites, Lemma 7.11.2 which concludes the proof. □

There is also an analogue of Spaces, Lemma 65.11.1.

0BH4 **Lemma 80.12.2.** *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and étale.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are étale morphisms of algebraic spaces as $U \to F$ is assumed étale. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 80.10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 78.19.5 and since j is a monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is étale, and we conclude that $U \to F'$ is étale by Descent on Spaces, Lemma 74.11.28. Thus $U \to F'$ is surjective as a map of étale sheaves. This means that F' is equal to the quotient sheaf U/R in the étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. □

In fact, it suffices to have a smooth cover by a scheme and it suffices to assume the diagonal is representable by algebraic spaces.

07WF **Lemma 80.12.3.** *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α.*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *the diagonal $\Delta : F \to F \times F$ is representable by algebraic spaces, and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and smooth.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. The proof mirrors the proof of Lemma 80.12.1. Let $R = U \times_F U$. By (2) the presheaf R is an algebraic space and by (3) the projections $R \to U$ are smooth and surjective. Denote (U, R, s, t, c) the groupoid associated to the equivalence relation $j : R \to U \times_S U$ (see Groupoids in Spaces, Lemma 78.11.3). By Theorem 80.10.1 we see that $X = U/R$ (quotient in the fppf-topology) is an algebraic space. Using that the smooth topology and the étale topology have the same sheaves (by More on Morphisms, Lemma 37.38.7) we see the map $U \to F$ identifies F as the quotient of U by R for the smooth topology (details omitted). Thus we have morphisms (transformations of functors)

$$U \to F \to X.$$

By Lemma 80.11.6 we see that $U \to X$ is surjective, flat and locally of finite presentation. By Groupoids in Spaces, Lemma 78.19.5 (and the fact that j is a monomorphism) we have $R = U \times_X U$. By Descent on Spaces, Lemma 74.11.26 we conclude that $U \to X$ is smooth and surjective (as the projections $R \to U$ are smooth and surjective and $\{U \to X\}$ is an fppf covering). Hence for any scheme T and morphism $T \to X$ the fibre product $T \times_X U$ is an algebraic space surjective and smooth over T. Choose a scheme V and a surjective étale morphism $V \to T \times_X U$. Then $\{V \to T\}$ is a smooth covering such that $V \to T \to X$ lifts to a morphism $V \to U$. This proves that $U \to X$ is surjective as a map of sheaves in the smooth topology. It follows that $F \to X$ is surjective as a map of sheaves in the smooth topology. On the other hand, the map $F \to X$ is injective (as a map of presheaves) since $R = U \times_X U$. It follows that $F \to X$ is an isomorphism of smooth (= étale) sheaves, see Sites, Lemma 7.11.2 which concludes the proof. □

Finally, here is the analogue of Spaces, Lemma 65.11.1 with a smooth morphism covering the space.

0GE0 **Lemma 80.12.4.** *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and smooth.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 65.6.1.

Proof. The proof is identical to the proof of Lemma 80.12.2. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are smooth morphisms of algebraic spaces as $U \to F$ is assumed smooth. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 80.10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 80.11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 78.19.5 and since j is a monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is smooth, and we conclude that $U \to F'$ is smooth by Descent on Spaces, Lemma 74.11.26. Thus $U \to F'$ is surjective as a map of étale sheaves (as the smooth topology is equal to the étale topology by More on Morphisms, Lemma 37.38.7). This means that F' is equal to the quotient sheaf U/R in the

étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves (as the smooth topology is the same as the étale topology) and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. □

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CHAPTER 81

Pushouts of Algebraic Spaces

0AHT

81.1. Introduction

0AHU The goal of this chapter is to discuss pushouts in the category of algebraic spaces. This can be done with varying assumptions. A fairly general pushout construction is given in [**TT13**]: one of the morphisms is affine and the other is a closed immersion. We discuss a particular case of this in Section 81.6 where we assume one of the morphisms is affine and the other is a thickening, a situation that often comes up in deformation theory.

In Sections 81.10 and 81.11 we discuss diagrams

$$\begin{array}{ccc} f^{-1}(X \setminus Z) & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X \setminus Z & \longrightarrow & X \end{array}$$

where f is a quasi-compact and quasi-separated morphism of algebraic spaces, $Z \to X$ is a closed immersion of finite presentation, the map $f^{-1}(Z) \to Z$ is an isomorphism, and f is flat along $f^{-1}(Z)$. In this situation we glue quasi-coherent modules on $X \setminus Z$ and Y (in Section 81.10) to quasi-coherent modules on X and we glue algebraic spaces over $X \setminus Z$ and Y (in Section 81.11) to algebraic spaces over X.

In Section 81.13 we discuss how proper birational morphisms of Noetherian algebraic spaces give rise to coequalizer diagrams in algebraic spaces in some sense.

In Section 81.14 we use the construction of elementary distinguished squares in Section 81.9 to prove Nagata's theorem on compactifications in the setting of algebraic spaces.

81.2. Conventions

0GFM The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

81.3. Colimits of algebraic spaces

0GFN We briefly discuss colimits of algebraic spaces. Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram (see Categories, Section 4.14). For each i we

may consider the small étale site $X_{i,étale}$ whose objects are schemes étale over X_i, see Properties of Spaces, Section 66.18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,étale} \to X_{i,étale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,étale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \to X_i$ over $\mathcal{I}$ such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to T$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to T$ is equal to f_i. Then we get a functor $T_{étale} \to \lim X_{i,étale}$. With this notation in hand we can formulate our lemma.

07SX **Lemma 81.3.1.** *Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes over S as above. Assume that*

1. *$X = \operatorname{colim} X_i$ exists in the category of schemes,*
2. *$\coprod X_i \to X$ is surjective,*
3. *if $U \to X$ is étale and $U_i = X_i \times_X U$, then $U = \operatorname{colim} U_i$ in the category of schemes, and*
4. *every object $(U_i \to X_i)$ of $\lim X_{i,étale}$ with $U_i \to X_i$ separated is in the essential image of the functor $X_{étale} \to \lim X_{i,étale}$.*

Then $X = \operatorname{colim} X_i$ in the category of algebraic spaces over S also.

Proof. Let Z be an algebraic space over S. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,étale}$ with $U_i \to X_i$ separated. By assumption (4) we can find an étale morphism $U \to X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \to W$ such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_{X_i} U_i$ by (3) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. □

We can check that a cocone is a colimit (fpqc) locally on the cocone.

0GFQ **Lemma 81.3.2.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 4.14.5). If there exists a fpqc covering $\{U_a \to X\}_{a \in A}$ such that*

(1) *for all $a \in A$ we have $U_a = \operatorname{colim} X_i \times_X U_a$ in the category of algebraic spaces over B, and*
(2) *for all $a, b \in A$ we have $U_a \times_X U_b = \operatorname{colim} X_i \times_X U_a \times_X U_b$ in the category of algebraic spaces over B,*

then $X = \operatorname{colim} X_i$ in the category of algebraic spaces over B.

Proof. Namely, for an algebraic space Y over B a morphism $X \to Y$ over B is the same thing as a collection of morphism $U_a \to Y$ which agree on the overlaps $U_a \times_X U_b$ for all $a, b \in A$, see Descent on Spaces, Lemma 74.7.2. □

We are going to find a common partial generalization of Lemmas 81.3.1 and 81.3.2 which can in particular be used to reduce a colimit construction to a subcategory of the category of all algebraic spaces.

Let S be a scheme and let B be an algebraic space over S. Let $\mathcal{I}$ be an index category and let $i \mapsto X_i$ be a diagram in the category of algebraic spaces over B, see Categories, Section 4.14. For each i we may consider the small étale site $X_{i,spaces,étale}$ whose objects are algebraic spaces étale over X_i, see Properties of Spaces, Section 66.18. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,spaces,étale} \to X_{i,spaces,étale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,spaces,étale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a diagram $i \mapsto (U_i \to X_i)$ in the category of arrows of algebraic spaces over B such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to Z$ is a family of morphisms of algebraic spaces over B such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. Then we get a functor $Z_{spaces,étale} \to \lim X_{i,spaces,étale}$. With this notation in hand we can formulate our next lemma.

0GHL **Lemma 81.3.3.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Let $(X, X_i \to X)$ be a cocone for the diagram in the category of algebraic spaces over B (Categories, Remark 4.14.5). Assume that*

(1) *the base change functor $X_{spaces,étale} \to \lim X_{i,spaces,étale}$, sending U to $U_i = X_i \times_X U$ is an equivalence,*
(2) *given*
   (a) *B' affine and étale over B,*
   (b) *Z an affine scheme over B',*

(c) $U \to X \times_B B'$ *an étale morphism of algebraic spaces with* U *affine,*
(d) $f_i : U_i \to Z$ *a cocone over* B' *of the diagram* $i \mapsto U_i = U \times_X X_i$,
there exists a unique morphism $f : U \to Z$ *over* B' *such that* f_i *equals the composition* $U_i \to U \to Z$.

Then $X = \operatorname{colim} X_i$ *in the category of all algebraic spaces over* B.

Proof. In this paragraph we reduce to the case where B is an affine scheme. Let $B' \to B$ be an étale morphism of algebraic spaces. Observe that conditions (1) and (2) are preserved if we replace B, X_i, X by B', $X_i \times_B B'$, $X \times_B B'$. Let $\{B_a \to B\}_{a \in A}$ be an étale covering with B_a affine, see Properties of Spaces, Lemma 66.6.1. For $a \in A$ denote X_a, $X_{a,i}$ the base changes of X and the diagram to B_a. For $a, b \in A$ denote $X_{a,b}$ and $X_{a,b,i}$ the base changes of X and the diagram to $B_a \times_B B_b$. By Lemma 81.3.2 it suffices to prove that $X_a = \operatorname{colim} X_{a,i}$ and $X_{a,b} = \operatorname{colim} X_{a,b,i}$. This reduces us to the case where $B = B_a$ (an affine scheme) or $B = B_a \times_B B_b$ (a separated scheme). Repeating the argument once more, we conclude that we may assume B is an affine scheme (this uses that the intersection of affine opens in a separated scheme is affine).

Assume B is an affine scheme. Let Z be an algebraic space over B. We have to show

$$\operatorname{Mor}_B(X, Z) \longrightarrow \lim \operatorname{Mor}_B(X_i, Z)$$

is a bijection.

Proof of injectivity. Let $f, g : X \to Z$ be morphisms such that the compositions $f_i, g_i : X_i \to Z$ are the same for all i. Choose an affine scheme Z' and an étale morphism $Z' \to Z$. By Properties of Spaces, Lemma 66.6.1 we know we can cover Z by such affines. Set $U = X \times_{f,Z} Z'$ and $U' = X \times_{g,Z} Z'$ and denote $p : U \to X$ and $p' : U' \to X$ the projections. Since $f_i = g_i$ for all i, we see that

$$U_i = X_i \times_{f_i,Z} Z' = X_i \times_{g_i,Z} Z' = U'_i$$

compatible with transition morphisms. By (1) there is a unique isomorphism $\epsilon : U \to U'$ as algebraic spaces over X, i.e., with $p = p' \circ \epsilon$ which is compatible with the displayed identifications. Choose an étale covering $\{h_a : U_a \to U\}$ with U_a affine. By (2) we see that $f \circ p \circ h_a = g \circ p' \circ \epsilon \circ h_a = g \circ p \circ h_a$. Since $\{h_a : U_a \to U\}$ is an étale covering we conclude $f \circ p = g \circ p$. Since the collection of morphisms $p : U \to X$ we obtain in this manner is an étale covering, we conclude that $f = g$.

Proof of surjectivity. Let $f_i : X_i \to Z$ be an element of the right hand side of the displayed arrow in the first paragraph of the proof. It suffices to find an étale covering $\{U_c \to X\}_{c \in C}$ such that the families $f_{c,i} \in \lim_i \operatorname{Mor}_B(X_i \times_X U_c, Z)$ come from morphisms $f_c : U_c \to Z$. Namely, by the uniqueness proved above the morphisms f_c will agree on $U_c \times_X U_b$ and hence will descend to give the desired morphism $f : X \to Z$. To find our covering, we first choose an étale covering $\{g_a : Z_a \to Z\}_{a \in A}$ where each Z_a is affine. Then we let $U_{a,i} = X_i \times_{f_i,Z} Z_a$. By (1) we find $U_{a,i} = X_i \times_X U_a$ for some algebraic spaces U_a étale over X. Then we choose étale coverings $\{U_{a,b} \to U_a\}_{b \in B_a}$ with $U_{a,b}$ affine and we consider the morphisms

$$U_{a,b,i} = X_i \times_X U_{a,b} \to X_i \times_X U_a = X_i \times_{f_i,Z} Z_a \to Z_a$$

By (2) we obtain morphisms $f_{a,b} : U_{a,b} \to Z_a$ compatible with these morphisms. Setting $C = \coprod_{a \in A} B_a$ and for $c \in C$ corresponding to $b \in B_a$ setting $U_c = U_{a,b}$ and $f_c = g_a \circ f_{a,b} : U_c \to Z$ we conclude. □

Here is an application of these ideas to reduce the general case to the case of separated algebraic spaces.

0GFP **Lemma 81.3.4.** *Let S be a scheme. Let B be an algebraic space over S. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of algebraic spaces over B. Assume that*

(1) *each X_i is separated over B,*
(2) *$X = \operatorname{colim} X_i$ exists in the category of algebraic spaces separated over B,*
(3) *$\coprod X_i \to X$ is surjective,*
(4) *if $U \to X$ is an étale separated morphism of algebraic spaces and $U_i = X_i \times_X U$, then $U = \operatorname{colim} U_i$ in the category of algebraic spaces separated over B, and*
(5) *every object $(U_i \to X_i)$ of $\lim X_{i,spaces,étale}$ with $U_i \to X_i$ separated is of the form $U_i = X_i \times_X U$ for some étale separated morphism of algebraic spaces $U \to X$.*

Then $X = \operatorname{colim} X_i$ in the category of all algebraic spaces over B.

Proof. We encourage the reader to look instead at Lemma 81.3.3 and its proof.

Let Z be an algebraic space over B. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ over B such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. We may assume that W is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated and that W is separated over B. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,spaces,étale}$ with $U_i \to X_i$ separated. By assumption (5) we can find a separated étale morphism $U \to X$ of algebraic spaces and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (4) there exists a morphism $h : U \to W$ over B such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_S W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the algebraic spaces $U_i \times_{X_i} U_i$ in the category of algebraic spaces separated over B by (4) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. □

81.4. Descending étale sheaves

0GFR This section is the analogue for algebraic spaces of Étale Cohomology, Section 59.104.

In order to conveniently express our results we need some notation. Let S be a scheme. Let $\mathcal{U} = \{f_i : X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target. A *descent datum* for étale sheaves with respect to $\mathcal{U}$ is a family $((\mathcal{F}_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ where

(1) $\mathcal{F}_i$ is in $Sh(X_{i,étale})$, and
(2) $\varphi_{ij} : \mathrm{pr}_{0,small}^{-1} \mathcal{F}_i \longrightarrow \mathrm{pr}_{1,small}^{-1} \mathcal{F}_j$ is an isomorphism in $Sh((X_i \times_X X_j)_{étale})$

such that the *cocycle condition* holds: the diagrams

$$\begin{array}{ccc} \mathrm{pr}_{0,small}^{-1}\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{01,small}^{-1}\varphi_{ij}} & \mathrm{pr}_{1,small}^{-1}\mathcal{F}_j \\ & {\scriptstyle \mathrm{pr}_{02,small}^{-1}\varphi_{ik}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12,small}^{-1}\varphi_{jk}} & \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F}_k & \end{array}$$

commute in $Sh((X_i \times_X X_j \times_X X_k)_{étale})$. There is an obvious notion of *morphisms of descent data* and we obtain a category of descent data. A descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is called *effective* if there exist a $\mathcal{F}$ in $Sh(X_{étale})$ and isomorphisms $\varphi_i : f_{i,small}^{-1}\mathcal{F} \to \mathcal{F}_i$ in $Sh(X_{i,étale})$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \mathrm{pr}_{1,small}^{-1}(\varphi_j) \circ \mathrm{pr}_{0,small}^{-1}(\varphi_i^{-1})$$

Another way to say this is the following. Given an object $\mathcal{F}$ of $Sh(X_{étale})$ we obtain the *canonical descent datum* $(f_{i,small}^{-1}\mathcal{F}_i, c_{ij})$ where c_{ij} is the canonical isomorphism

$$c_{ij} : \mathrm{pr}_{0,small}^{-1} f_{i,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1} f_{j,small}^{-1}\mathcal{F}$$

The descent datum $((\mathcal{F}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ is effective if and only if it is isomorphic to the canonical descent datum associated to some $\mathcal{F}$ in $Sh(X_{étale})$.

If the family consists of a single morphism $\{X \to Y\}$, then we think of a descent datum as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is an object of $Sh(X_{étale})$ and φ is an isomorphism

$$\mathrm{pr}_{0,small}^{-1}\mathcal{F} \longrightarrow \mathrm{pr}_{1,small}^{-1}\mathcal{F}$$

in $Sh((X \times_Y X)_{étale})$ such that the cocycle condition holds:

$$\begin{array}{ccc} \mathrm{pr}_{0,small}^{-1}\mathcal{F} & \xrightarrow{\mathrm{pr}_{01,small}^{-1}\varphi} & \mathrm{pr}_{1,small}^{-1}\mathcal{F} \\ & {\scriptstyle \mathrm{pr}_{02,small}^{-1}\varphi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12,small}^{-1}\varphi} & \\ & \mathrm{pr}_{2,small}^{-1}\mathcal{F} & \end{array}$$

commutes in $Sh((X \times_Y X \times_Y X)_{étale})$. There is a notion of morphisms of descent data and effectivity exactly as before.

0GFS **Lemma 81.4.1.** *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an étale covering of algebraic spaces. The functor*

$$Sh(X_{étale}) \longrightarrow \textit{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. In Properties of Spaces, Section 66.18 we have defined a site $X_{spaces,étale}$ whose objects are algebraic spaces étale over X with étale coverings. Moreover, we have identifications $Sh(X_{étale}) = Sh(X_{spaces,étale})$ compatible with morphisms of algebraic spaces, i.e., compatible with pushforward and pullback. Hence the statement of the lemma follows from the much more general discussion in Sites, Section 7.26. □

0GFT **Lemma 81.4.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of algebraic spaces. If for each $i \in I$ the functor*

$$Sh(Y_{i,étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \to Y_i\}$$

is an equivalence of categories and for each $i, j \in I$ the functor

$$Sh((Y_i \times_Y Y_j)_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \times_Y Y_i \times_Y Y_j \to Y_i \times_Y Y_j\}$$

is an equivalence of categories, then

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 81.4.1 and the definitions. □

0GFU **Lemma 81.4.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable (by schemes) and f has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation Then*

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Each of the properties of morphisms of algebraic spaces mentioned in the statement of the lemma is preserved by arbitrary base change, see the lists in Spaces, Section 65.4. Thus we can apply Lemma 81.4.2 to see that we can work étale locally on Y. In this way we reduce to the case where Y is a scheme; some details omitted. In this case X is also a scheme and the result follows from Étale Cohomology, Lemma 59.104.2, 59.104.3, or 59.104.5. □

0GFV **Lemma 81.4.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\pi : X' \to X$ be a morphism of algebraic spaces. Assume*

(1) *$f \circ \pi$ is representable (by schemes),*
(2) *$f \circ \pi$ has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.*

Then

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Formal consequence of Lemma 81.4.3 and Stacks, Lemma 8.3.7. □

0GFW **Lemma 81.4.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which has one of the following properties: surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation. Then the functor*

$$Sh(Y_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

is an equivalence of categories.

Proof. Observe that the base change of a proper surjective morphism is proper and surjective, see Morphisms of Spaces, Lemmas 67.40.3 and 67.5.5. Hence by Lemma 81.4.2 we may work étale locally on Y. Hence we reduce to Y being an affine scheme; some details omitted.

Assume Y is affine. By Lemma 81.4.4 it suffices to find a morphism $X' \to X$ where X' is a scheme such that $X' \to Y$ is surjective and integral, surjective and proper, or surjective and flat and locally of finite presentation.

In case $X \to Y$ is integral and surjective, we can take $X = X'$ as an integral morphism is representable.

If f is proper and surjective, then the algebraic space X is quasi-compact and separated, see Morphisms of Spaces, Section 67.8 and Lemma 67.4.9. Choose a scheme X' and a surjective finite morphism $X' \to X$, see Limits of Spaces, Proposition 70.16.1. Then $X' \to Y$ is surjective and proper.

Finally, if $X \to Y$ is surjective and flat and locally of finite presentation then we can take an affine étale covering $\{U_i \to X\}$ and set X' equal to the disjoint $\coprod U_i$. □

0GFX **Lemma 81.4.6.** *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fppf covering of algebraic spaces over S. The functor*

$$Sh(X_{étale}) \longrightarrow \textit{descent data for étale sheaves wrt } \{f_i : X_i \to X\}$$

is an equivalence of categories.

Proof. We have Lemma 81.4.5 for the morphism $f : \coprod X_i \to X$. Then a formal argument shows that descent data for f are the same thing as descent data for the covering, compare with Descent, Lemma 35.34.5. Details omitted. □

0GFY **Lemma 81.4.7.** *Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$. Picture*

$$\begin{array}{ccc} E & \xrightarrow[j]{} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Sh(Y_{étale}) \longrightarrow Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale})$$

is an equivalence of categories.

Proof. Observe that $X = Y' \coprod Z \to Y$ is a proper surjective morphism. Thus it suffice to construct an equivalence of categories

$$Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale}) \longrightarrow \text{descent data for étale sheaves wrt } \{X \to Y\}$$

compatible with pullback functors from Y because then we can use Lemma 81.4.5 to conclude. Thus let $(\mathcal{G}', \mathcal{G}, \alpha)$ be an object of $Sh(Y'_{étale}) \times_{Sh(E_{étale})} Sh(Z_{étale})$ with notation as in Categories, Example 4.31.3. Then we can consider the sheaf $\mathcal{F}$ on X defined by taking $\mathcal{G}'$ on the summand Y' and $\mathcal{G}$ on the summand Z. We have

$$X \times_Y X = Y' \times_Y Y' \amalg Y' \times_Y Z \amalg Z \times_Y Y' \amalg Z \times_Y Z = Y' \times_Y Y' \amalg E \amalg E \amalg Z$$

The isomorphisms of the two pullbacks of $\mathcal{F}$ to this algebraic space are obvious over the summands E, E, Z. The interesting part of the proof is to find an isomorphism

$\mathrm{pr}_{0,small}^{-1}\mathcal{G}' \to \mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ over $Y' \times_Y Y'$ satisfying the cocycle condition. However, our assumption that $Y' \to Y$ is an isomorphism over $Y \setminus Z$ implies that

$$h : Y \coprod E \times_Z E \longrightarrow Y' \times_Y Y'$$

is a surjective proper morphism. (It is in fact a finite morphism as it is the disjoint union of two closed immersions.) Hence it suffices to construct an isomorphism of the pullbacks of $\mathrm{pr}_{0,small}^{-1}\mathcal{G}'$and $\mathrm{pr}_{1,small}^{-1}\mathcal{G}'$ by h_{small} satisfying a certain cocycle condition. For the diagonal, it is clear how to do this. And for the pullback to $E \times_Z E$ we use that both sheaves pull back to the pullback of $\mathcal{G}$ by the morphism $E \times_Z E \to Z$. We omit the details. □

81.5. Descending étale morphisms of algebraic spaces

0GFZ In this section we combine the glueing results for étale sheaves given in Section 81.4 with the flexibility of algebraic spaces to get some descent statements for étale morphisms of algebraic spaces.

0GG0 **Lemma 81.5.1.** *Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Any descent datum $(U/X, \varphi)$ relative to f (Descent on Spaces, Definition 74.22.1) with U étale over X is effective (Descent on Spaces, Definition 74.22.10). More precisely, there exists an étale morphism $V \to Y$ of algebraic spaces whose corresponding canonical descent datum is isomorphic to $(U/X, \varphi)$.*

Proof. Recall that U gives rise to a representable sheaf $\mathcal{F} = h_U$ in $Sh(X_{spaces,étale}) = Sh(X_{étale})$, see Properties of Spaces, Section 66.18. The descent datum on U relative to f exactly gives a descent datum $(\mathcal{F}, \varphi)$ for étale sheaves with respect to $\{X \to Y\}$. By Lemma 81.4.5 this descent datum is effective. Let $\mathcal{G}$ be the corresponding sheaf on $Y_{étale}$. By Properties of Spaces, Lemma 66.27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{G}$; we omit the verification of the set theoretic condition[1]. The given isomorphism $\mathcal{F} \to f_{small}^{-1}\mathcal{G}$ corresponds to an isomorphism $U \to V \times_Y X$ compatible with the descent datum. □

0GG1 **Lemma 81.5.2.** *Let S be a scheme. Let $f : Y' \to Y$ be a proper morphism of algebraic spaces over S. Let $i : Z \to Y$ be a closed immersion. Set $E = Z \times_Y Y'$. Picture*

$$\begin{array}{ccc} E & \xrightarrow[j]{} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

If f is an isomorphism over $Y \setminus Z$, then the functor

$$Y_{spaces,étale} \longrightarrow Y'_{spaces,étale} \times_{E_{spaces,étale}} Z_{spaces,étale}$$

is an equivalence of categories.

Proof. Let $(V' \to Y', W \to Z, \alpha)$ be an object of the right hand side. Recall that V', resp. W gives rise to a representable sheaf $\mathcal{G}' = h_{V'}$ in $Sh(Y'_{spaces,étale}) = Sh(Y'_{étale})$, resp. $\mathcal{G} = h_W$ in $Sh(Z_{spaces,étale}) = Sh(Z_{étale})$, see Properties of Spaces, Section 66.18. The isomorphism $\alpha : V' \times_{Y'} E \to W \times_Z E$ determines an isomorphism $j_{small}^{-1}\mathcal{G}' \to g_{small}^{-1}\mathcal{G}$ of sheaves on E. By Lemma 81.4.7 we obtain a unique sheaf $\mathcal{F}$

[1] It follows from the fact that $\mathcal{F}$ satisfies the corresponding condition.

on Y pulling pack to $\mathcal{G}'$ and $\mathcal{G}$ compatibly with the isomorphism. By Properties of Spaces, Lemma 66.27.3 we obtain an étale morphism $V \to Y$ of algebraic spaces corresponding to $\mathcal{F}$; we omit the verification of the set theoretic condition[2]. The given isomorphism $\mathcal{G}' \to f_{small}^{-1}\mathcal{F}$ and $\mathcal{G} \to i_{small}^{-1}\mathcal{F}$ corresponds to isomorphisms $V' \to V \times_Y Y'$ and $W \to V \times_Y Z$ compatible with α as desired. □

81.6. Pushouts along thickenings and affine morphisms

07SW This section is analogue of More on Morphisms, Section 37.14.

07SY **Lemma 81.6.1.** *Let S be a scheme. Let $X \to X'$ be a thickening of schemes over S and let $X \to Y$ be an affine morphism of schemes over S. Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 37.14.3). Then Y' is also a pushout in the category of algebraic spaces over S.*

Proof. This is an immediate consequence of Lemma 81.3.1 and More on Morphisms, Lemmas 37.14.3, 37.14.4, and 37.14.6. □

07VX **Lemma 81.6.2.** *Let S be a scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Then there exists a pushout*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y \amalg_X X' \end{array}$$

in the category of algebraic spaces over S. Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $Y_{étale} = (Y')_{étale}$.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. This is a scheme affine over V with a surjective étale morphism $U \to X$. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a $U' \to X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \to U'$ is a thickening too. Apply More on Morphisms, Lemma 37.14.3 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U' \times_{X'} U' \\ \downarrow & & \downarrow \\ V \times_Y V & \longrightarrow & R' \end{array}$$

in the category of schemes. Consider the morphisms

$$U \times_X U \to U \to V', \quad U' \times_{X'} U' \to U' \to V', \quad V \times_Y V \to V \to V'$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \to V'$ which is étale by More on Morphisms, Lemma 37.14.6. Similarly, we obtain $s' : R' \to V'$ étale. The morphism $j' = (t', s') : R' \to V' \times_S V'$

[2]It follows from the fact that $\mathcal{G}$ and $\mathcal{G}'$ satisfies the corresponding condition.

is unramified (as t' is étale) and a monomorphism when restricted to the closed subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that j' is a monomorphism too. Finally, j' is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s',V',t'} R' \to R'$ (details omitted). At this point we set $Y' = V'/R'$, see Spaces, Theorem 65.10.5.

We have morphisms $X' = U'/U' \times_{X'} U' \to V'/R' = Y'$ and $Y = V/V \times_Y V \to V'/R' = Y'$. By construction these fit into the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

Since $Y \to Y'$ is a thickening we have $Y_{étale} = (Y')_{étale}$, see More on Morphisms of Spaces, Lemma 76.9.6. The commutativity of the diagram gives a map of sheaves

$$\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

on this set. By More on Morphisms, Lemma 37.14.3 this map is an isomorphism when we restrict to the scheme V', hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let Z be an algebraic space and let $a' : X' \to Z$ and $b : Y \to Z$ be morphisms of algebraic spaces. By Lemma 81.6.1 we obtain a unique morphism $h : V' \to Z$ fitting into the commutative diagrams

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{a'} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{b} & Z \end{array}$$

The uniqueness shows that $h \circ t' = h \circ s'$. Hence h factors uniquely as $V' \to Y' \to Z$ and we win. □

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.31.3.

07VY **Lemma 81.6.3.** *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 81.6.2). Base change gives a functor*

$$F : (Spaces/Y') \longrightarrow (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$. The functor F has a left adjoint

$$G : (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \longrightarrow (Spaces/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over S. The functor G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Proof. The proof is completely formal. Since the morphisms $X \to X'$ and $X \to Y$ are representable it is clear that F sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$.

Let us construct G. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X =$

U is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms of Spaces, Lemma 67.20.5. Hence we can apply Lemma 81.6.2 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

If (V, U', φ) is an object of $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma 37.14.3. By Lemma 81.6.1 this is also a pushout in the category of schemes, hence G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Let us prove that G is a left adjoint to F. Let Z be an algebraic space over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. □

07VZ **Lemma 81.6.4.** *Let S be a scheme. Let*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume that A, B, C, D and A, B, E, F form cartesian squares and that $B \to D$ is surjective étale. Then C, D, E, F is a cartesian square.

Proof. This is formal. □

07W0 **Lemma 81.6.5.** *In the situation of Lemma 81.6.3 the functor $F \circ G$ is isomorphic to the identity functor.*

Proof. We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma 37.14.4.

Choose a scheme Y_1 and a surjective étale morphism $Y_1 \to Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over Y_1 with a surjective étale morphism $X_1 \to X$. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a $X_1' \to X'$ surjective étale with $X_1 = X_1' \times_{X'} X$. In particular the morphism of schemes $X_1 \to X_1'$ is a thickening too. Apply More on Morphisms, Lemma 37.14.3 to obtain a pushout $Y_1' = Y_1 \amalg_{X_1} X_1'$ in the category of schemes. In the proof of Lemma 81.6.2 we

constructed Y' as a quotient of an étale equivalence relation on Y'_1 such that we get a commutative diagram

07W1 (81.6.5.1)

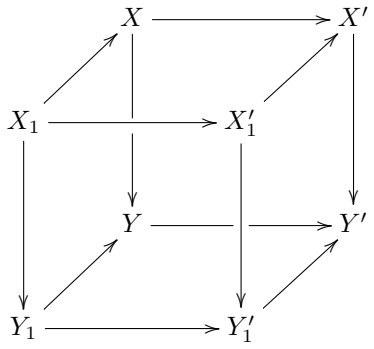

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote F_1, G_1 the functors constructed in More on Morphisms, Lemma 37.14.4 for the front square. Then the diagram of categories

$$\begin{array}{ccc} (Sch/Y'_1) & \underset{F_1}{\overset{G_1}{\leftarrow\!\!\rightarrow}} & (Sch/Y_1) \times_{(Sch/Y'_1)} (Sch/X'_1) \\ \downarrow & & \downarrow \\ (Spaces/Y') & \underset{F}{\overset{G}{\leftarrow\!\!\rightarrow}} & (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \end{array}$$

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 81.6.1.

Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg_U U'$. Choose a scheme V_1 and a surjective étale morphism $V_1 \to Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \longrightarrow (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By More on Morphisms of Spaces, Lemma 76.9.6 there exists a thickening $U_1 \to U'_1$ and a surjective étale morphism $U'_1 \to X'_1 \times_{X'} U'$ whose base change to $X_1 \times_X U$ is the displayed morphism. At this point (V_1, U'_1, φ_1) is an object of $(Sch/Y_1) \times_{(Sch/Y'_1)} (Sch/X'_1)$. In the proof of Lemma 81.6.2 we constructed $G(V, U', \varphi) = V \amalg_U U'$ as a quotient of an étale equivalence relation on

$G_1(V_1, U_1', \varphi_1) = V_1 \amalg_{U_1} U_1'$ such that we get a commutative diagram

07W2 (81.6.5.2)

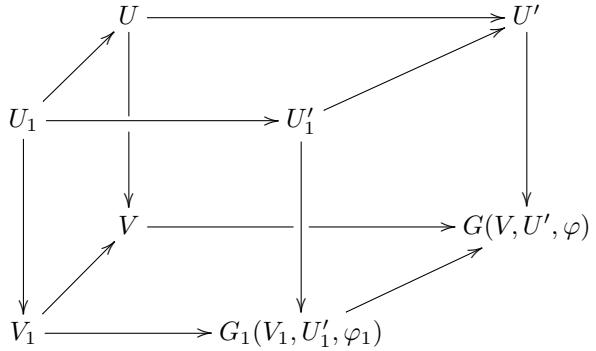

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. In particular

$$G_1(V_1, U_1', \varphi_1) \to G(V, U', \varphi)$$

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. We know $(V_1, U_1', \varphi_1) \to F_1(G_1(V_1, U_1', \varphi_1))$ is an isomorphism by More on Morphisms, Lemma 37.14.4. Recall that F and F_1 are given by base change. Using the properties of (81.6.5.2) and Lemma 81.6.4 we see that $V \to G(V, U', \varphi) \times_{Y'} Y$ and $U' \to G(V, U', \varphi) \times_{Y'} X'$ are isomorphisms, i.e., $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. □

08KV **Lemma 81.6.6.** *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 81.6.2). Let $V' \to Y'$ be a morphism of algebraic spaces over S. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between*

1. *quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and*
2. *the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where*
   (a) *$\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,*
   (b) *$\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X, and*
   (c) *$\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.*

The equivalence maps $\mathcal{G}'$ to $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

(a) *$\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.*
(b) *if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines étale over V' and Y' we recover the equivalence of More on Algebra, Lemma 15.7.5. Details omitted.

Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 66.30) to the case where V' and Y' are affine in which case the result follows from More on Algebra, Lemmas 15.7.4 and 15.7.6. □

07W3 **Lemma 81.6.7.** *In the situation of Lemma 81.6.5. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then*

(1) *$V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,*
(2) *$V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,*
(3) *$V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,*
(4) *$V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,*
(5) *$V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and*
(6) *add more here as needed.*

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of spaces flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Choose a diagram (81.6.5.1) as in the proof of Lemma 81.6.5.

Proof of (1) – (5). Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Construct a diagram (81.6.5.2) as in the proof of Lemma 81.6.5. Then the base change of $G(V, U', \varphi) \to Y'$ to Y'_1 is $G_1(V_1, U'_1, \varphi_1) \to Y'_1$. Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 37.14.6 for schemes.

Suppose that $W' \to Y'$ is flat. Choose a scheme W'_1 and a surjective étale morphism $W'_1 \to Y'_1 \times_{Y'} W'$. Observe that $W'_1 \to W'$ is surjective étale as a composition of surjective étale morphisms. We know that $G_1(F_1(W'_1)) \to W'_1$ is an isomorphism by More on Morphisms, Lemma 37.14.6 applied to W'_1 over Y'_1 and the front of the diagram (with functors G_1 and F_1 as in the proof of Lemma 81.6.5). Then the construction of $G(F(W'))$ (as a pushout, i.e., as constructed in Lemma 81.6.2) shows that $G_1(F_1(W'_1)) \to G(F(W))$ is surjective étale. Whereupon we conclude that $G(F(W)) \to W$ is étale, see for example Properties of Spaces, Lemma 66.16.3. But $G(F(W)) \to W$ is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism. □

81.7. Pushouts along closed immersions and integral morphisms

0GG2 This section is analogue of More on Morphisms, Section 37.67.

0EDP **Lemma 81.7.1.** *In More on Morphisms, Situation 37.67.1 let $Y \amalg_Z X$ be the pushout in the category of schemes (More on Morphisms, Proposition 37.67.3). Then $Y \amalg_Z X$ is also a pushout in the category of algebraic spaces over S.*

Proof. This is a consequence of Lemma 81.3.1, the proposition mentioned in the lemma and More on Morphisms, Lemmas 37.67.6 and 37.67.7. Conditions (1) and (2) of Lemma 81.3.1 follow immediately. To see (3) and (4) note that an étale morphism is locally quasi-finite and use that the equivalence of categories of More on Morphisms, Lemma 37.67.7 is constructed using the pushout construction of More on Morphisms, Lemmas 37.67.6. Minor details omitted. □

81.8. Pushouts and derived categories

0DL6 In this section we discuss the behaviour of the derived category of modules under pushouts.

0DL7 **Lemma 81.8.1.** *Let S be a scheme. Consider a pushout*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y' \end{array}$$

in the category of algebraic spaces over S as in Lemma 81.6.2. Assume i is a thickening. Then the essential image of the functor[3]

$$D(\mathcal{O}_{Y'}) \longrightarrow D(\mathcal{O}_Y) \times_{D(\mathcal{O}_X)} D(\mathcal{O}_{X'})$$

contains every triple (M, K', α) where $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent.

Proof. Let (M, K', α) be an object of the target of the functor of the lemma. Here $\alpha : Lf^*M \to Li^*K'$ is an isomorphism which is adjoint to a map $\beta : M \to Rf_*Li^*K'$. Thus we obtain maps

$$Rj_*M \xrightarrow{Rj_*\beta} Rj_*Rf_*Li^*K' = Rf'_*Ri_*Li^*K' \leftarrow Rf'_*K'$$

where the arrow pointing left comes from $K' \to Ri_*Li^*K'$. Choose a distinguished triangle

$$M' \to Rj_*M \oplus Rf'_*K' \to Rj_*Rf_*Li^*K' \to M'[1]$$

in $D(\mathcal{O}_{Y'})$. The first arrow defines canonical maps $Lj^*M' \to M$ and $L(f')^*M' \to K'$ compatible with α. Thus it suffices to show that the maps $Lj^*M' \to M$ and $L(f')^*M' \to K$ are isomorphisms. This we may check étale locally on Y', hence we may assume Y' is étale.

Assume Y' affine and $M \in D(\mathcal{O}_Y)$ and $K' \in D(\mathcal{O}_{X'})$ are pseudo-coherent. Say our pushout corresponds to the fibre product

$$\begin{array}{ccc} B & \longleftarrow & B' \\ \uparrow & & \uparrow \\ A & \longleftarrow & A' \end{array}$$

of rings where $B' \to B$ is surjective with locally nilpotent kernel I (and hence $A' \to A$ is surjective with locally nilpotent kernel I as well). The assumption on M and K' imply that M comes from a pseudo-coherent object of $D(A)$ and K' comes from a pseudo-coherent object of $D(B')$, see Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 and Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2. Moreover, pushforward and derived pullback agree with the corresponding operations on derived categories of modules, see Derived Categories of Spaces, Remark 75.6.3 and Derived Categories of Schemes, Lemmas 36.3.7 and 36.3.8. This reduces us to the statement formulated in the next paragraph. (To be sure these references show the object M' lies $D_{QCoh}(\mathcal{O}_{Y'})$ as this is a triangulated subcategory of $D(\mathcal{O}_{Y'})$.)

[3] All functors given by derived pullback.

Given a diagram of rings as above and a triple (M, K', α) where $M \in D(A)$, $K' \in D(B')$ are pseudo-coherent and $\alpha : M \otimes^{\mathbf{L}}_A B \to K' \otimes^{\mathbf{L}}_{B'} B$ is an isomorphism suppose we have distinguished triangle

$$M' \to M \oplus K' \to K' \otimes^{\mathbf{L}}_{B'} B \to M'[1]$$

in $D(A')$. Goal: show that the induced maps $M' \otimes^{\mathbf{L}}_{A'} A \to M$ and $M' \otimes^{\mathbf{L}}_{A'} B' \to K'$ are isomorphisms. To do this, choose a bounded above complex $E^\bullet$ of finite free A-modules representing M. Since (B', I) is a henselian pair (More on Algebra, Lemma 15.11.2) with $B = B'/I$ we may apply More on Algebra, Lemma 15.76.9 to see that there exists a bounded above complex $P^\bullet$ of free B'-modules such that α is represented by an isomorphism $E^\bullet \otimes_A B \cong P^\bullet \otimes_{B'} B$. Then we can consider the short exact sequence

$$0 \to L^\bullet \to E^\bullet \oplus P^\bullet \to P^\bullet \otimes_{B'} B \to 0$$

of complexes of B'-modules. More on Algebra, Lemma 15.6.9 implies $L^\bullet$ is a bounded above complex of finite projective A'-modules (in fact it is rather easy to show directly that L^n is finite free in our case) and that we have $L^\bullet \otimes_{A'} A = E^\bullet$ and $L^\bullet \otimes_{A'} B' = P^\bullet$. The short exact sequence gives a distinguished triangle

$$L^\bullet \to M \oplus K' \to K' \otimes^{\mathbf{L}}_{B'} B \to (L^\bullet)[1]$$

in $D(A')$ (Derived Categories, Section 13.12) which is isomorphic to the given distinguished triangle by general properties of triangulated categories (Derived Categories, Section 13.4). In other words, $L^\bullet$ represents M' compatibly with the given maps. Thus the maps $M' \otimes^{\mathbf{L}}_{A'} A \to M$ and $M' \otimes^{\mathbf{L}}_{A'} B' \to K'$ are isomorphisms because we just saw that the corresponding thing is true for $L^\bullet$. □

81.9. Constructing elementary distinguished squares

0DVH Elementary distinguished squares were defined in Derived Categories of Spaces, Section 75.9.

0DVI **Lemma 81.9.1.** *Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square. Then*

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \longrightarrow & W \end{array}$$

is a pushout in the category of algebraic spaces over S.

Proof. Observe that $U \amalg V \to W$ is a surjective étale morphism. The fibre product

$$(U \amalg V) \times_W (U \amalg V)$$

is the disjoint union of four pieces, namely $U = U \times_W U$, $U \times_W V$, $V \times_W U$, and $V \times_W V$. There is a surjective étale morphism

$$V \amalg (U \times_W V) \times_U (U \times_W V) \longrightarrow V \times_W V$$

because f induces an isomorphism over $W \setminus U$ (part of the definition of being an elementary distinguished square). Let B be an algebraic space over S and let $g : V \to B$ and $h : U \to B$ be morphisms over S which agree after restricting to $U \times_W V$. Then the description of $(U \amalg V) \times_W (U \amalg V)$ given above shows that

$h \amalg g : U \amalg V \to B$ equalizes the two projections. Since B is a sheaf for the étale topology we obtain a unique factorization of $h \amalg g$ through W as desired. □

0DVJ **Lemma 81.9.2.** *Let S be a scheme. Let V, U be algebraic spaces over S. Let $V' \subset V$ be an open subspace and let $f' : V' \to U$ be a separated étale morphism of algebraic spaces over S. Then there exists a pushout*

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \longrightarrow & W \end{array}$$

in the category of algebraic spaces over S and moreover $(U \subset W, f : V \to W)$ is an elementary distinguished square.

Proof. We are going to construct W as the quotient of an étale equivalence relation R on $U \amalg V$. Such a quotient is an algebraic space for example by Bootstrap, Theorem 80.10.1. Moreover, the proof of Lemma 81.9.1 tells us to take

$$R = U \amalg V' \amalg V' \amalg V \amalg (V' \times_U V' \setminus \Delta_{V'/U}(V'))$$

Since we assumed $V' \to U$ is separated, the image of $\Delta_{V'/U}$ is closed and hence the complement is an open subspace. The morphism $j : R \to (U \amalg V) \times_S (U \amalg V)$ is given by

$$u,\ v',\ v',\ v,\ (v'_1, v'_2) \mapsto (u, u),\ (f'(v'), v'),\ (v', f'(v')),\ (v, v),\ (v'_1, v'_2)$$

with obvious notation. It is immediately verified that this is a monomorphism, an equivalence relation, and that the induced morphisms $s, t : R \to U \amalg V$ are étale. Let $W = (U \amalg V)/R$ be the quotient algebraic space. We obtain a commutative diagram as in the statement of the lemma. To finish the proof it suffices to show that this diagram is an elementary distinguished square, since then Lemma 81.9.1 implies that it is a pushout. Thus we have to show that $U \to W$ is open and that f is étale and is an isomorphism over $W \setminus U$. This follows from the choice of R; we omit the details. □

81.10. Formal glueing of quasi-coherent modules

0AEP This section is the analogue of More on Algebra, Section 15.90. In the case of morphisms of schemes, the result can be found in the paper by Joyet [**Joy96**]; this is a good place to start reading. For a discussion of applications to descent problems for stacks, see the paper by Moret-Bailly [**MB96**]. In the case of an affine morphism of schemes there is a statement in the appendix of the paper [**FR70**] but one needs to add the hypothesis that the closed subscheme is cut out by a finitely generated ideal (as in the paper by Joyet) since otherwise the result does not hold. A generalization of this material to (higher) derived categories with potential applications to nonflat situations can be found in [**Bha16**, Section 5].

We start with a lemma on abelian sheaves supported on closed subsets.

0AEQ **Lemma 81.10.1.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. We have*

$$(Rf_*Q)_{\overline{x}} = Q_{\overline{y}}$$

in $D(Ab)$ for any object Q of $D(Y_{étale})$ supported on $|f^{-1}Z|$.

Proof. Consider the commutative diagram of algebraic spaces

$$\begin{array}{ccc} f^{-1}Z & \xrightarrow{i'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

By Cohomology of Spaces, Lemma 69.9.4 we can write $Q = Ri'_*K'$ for some object K' of $D(f^{-1}Z_{étale})$. By Morphisms of Spaces, Lemma 67.53.7 we have $K' = (f')^{-1}K$ with $K = Rf'_*K'$. Then we have $Rf_*Q = Rf_*Ri'_*K' = Ri_*Rf'_*K' = Ri_*K$. Let $\overline{z}$ be the geometric point of Z corresponding to $\overline{x}$ and let $\overline{z}'$ be the geometric point of $f^{-1}Z$ corresponding to $\overline{y}$. We obtain the result of the lemma as follows

$$Q_{\overline{y}} = (Ri'_*K')_{\overline{y}} = K'_{\overline{z}'} = (f')^{-1}K_{\overline{z}'} = K_{\overline{z}} = Ri_*K_{\overline{x}} = Rf_*Q_{\overline{x}}$$

The middle equality holds because of the description of the stalk of a pullback given in Properties of Spaces, Lemma 66.19.9. □

0AER **Lemma 81.10.2.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = f(\overline{y})$. Let $\mathcal{G}$ be an abelian sheaf on Y. Then the map of two term complexes*

$$(f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}}) \longrightarrow (\mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}})$$

induces an isomorphism on kernels and an injection on cokernels. Here $V = Y \setminus f^{-1}Z$ and $j' : V \to Y$ is the inclusion.

Proof. Choose a distinguished triangle

$$\mathcal{G} \to Rj'_*\mathcal{G}|_V \to Q \to \mathcal{G}[1]$$

n $D(Y_{étale})$. The cohomology sheaves of Q are supported on $|f^{-1}Z|$. We apply Rf_* and we obtain

$$Rf_*\mathcal{G} \to Rf_*Rj'_*\mathcal{G}|_V \to Rf_*Q \to Rf_*\mathcal{G}[1]$$

Taking stalks at $\overline{x}$ we obtain an exact sequence

$$0 \to (R^{-1}f_*Q)_{\overline{x}} \to f_*\mathcal{G}_{\overline{x}} \to (f \circ j')_*(\mathcal{G}|_V)_{\overline{x}} \to (R^0f_*Q)_{\overline{x}}$$

We can compare this with the exact sequence

$$0 \to H^{-1}(Q)_{\overline{y}} \to \mathcal{G}_{\overline{y}} \to j'_*(\mathcal{G}|_V)_{\overline{y}} \to H^0(Q)_{\overline{y}}$$

Thus we see that the lemma follows because $Q_{\overline{y}} = Rf_*Q_{\overline{x}}$ by Lemma 81.10.1. □

0AES **Lemma 81.10.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\overline{x}$ be a geometric point of X and let $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$ be the canonical morphism. For a quasi-coherent module $\mathcal{G}$ on Y we have*

$$f_*\mathcal{G}_{\overline{x}} = \Gamma(Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), p^*\mathcal{F})$$

where $p : Y \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to Y$ is the projection.

Proof. Observe that $f_*\mathcal{G}_{\overline{x}} = \Gamma(\mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), h^*f_*\mathcal{G})$ where $h : \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) \to X$. Hence the result is true because h is flat so that Cohomology of Spaces, Lemma 69.11.2 applies. □

0AET **Lemma 81.10.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $i : Z \to X$ be a closed immersion of finite presentation. Let $Q \in D_{QCoh}(\mathcal{O}_X)$ be supported on $|Z|$. Let $\overline{x}$ be a geometric point of X and let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf of Z. Then the cohomology modules $H^n(Q_{\overline{x}})$ are $I_{\overline{x}}$-power torsion (see More on Algebra, Definition 15.89.1).*

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that $\overline{x}$ lifts to a geometric point $\overline{u}$ of U. Then we can replace X by U, Z by $U \times_X Z$, Q by the restriction $Q|_U$, and $\overline{x}$ by $\overline{u}$. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal defining Z. Since $i : Z \to X$ is of finite presentation, the ideal $I = (f_1, \ldots, f_r)$ is finitely generated. The object Q comes from a complex of A-modules $M^\bullet$, see Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Since the cohomology sheaves of Q are supported on Z we see that the localization $M^\bullet_f$ is acyclic for each $f \in I$. Take $x \in H^p(M^\bullet)$. By the above we can find n_i such that $f_i^{n_i} x = 0$ in $H^p(M^\bullet)$ for each i. Then with $n = \sum n_i$ we see that I^n annihilates x. Thus $H^p(M^\bullet)$ is I-power torsion. Since the ring map $A \to \mathcal{O}_{X,\overline{x}}$ is flat and since $I_{\overline{x}} = I\mathcal{O}_{X,\overline{x}}$ we conclude. □

0AEU **Lemma 81.10.5.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume $f^{-1}Z \to Z$ is an isomorphism and that f is flat in every point of $f^{-1}Z$. For any Q in $D_{QCoh}(\mathcal{O}_Y)$ supported on $|f^{-1}Z|$ we have $Lf^*Rf_*Q = Q$.*

Proof. We show the canonical map $Lf^*Rf_*Q \to Q$ is an isomorphism by checking on stalks at $\overline{y}$. If $\overline{y}$ is not in $f^{-1}Z$, then both sides are zero and the result is true. Assume the image $\overline{x}$ of $\overline{y}$ is in Z. By Lemma 81.10.1 we have $Rf_*Q_{\overline{x}} = Q_{\overline{y}}$ and since f is flat at $\overline{y}$ we see that

$$(Lf^*Rf_*Q)_{\overline{y}} = (Rf_*Q)_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} = Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$$

Thus we have to check that the canonical map

$$Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} \longrightarrow Q_{\overline{y}}$$

is an isomorphism in the derived category. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$ be the stalk of the ideal sheaf defining Z. Since $Z \to X$ is locally of finite presentation this ideal is finitely generated and the cohomology groups of $Q_{\overline{y}}$ are $I_{\overline{y}} = I_{\overline{x}}\mathcal{O}_{Y,\overline{y}}$-power torsion by Lemma 81.10.4 applied to Q on Y. It follows that they are also $I_{\overline{x}}$-power torsion. The ring map $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is flat and induces an isomorphism after dividing by $I_{\overline{x}}$ and $I_{\overline{y}}$ because we assumed that $f^{-1}Z \to Z$ is an isomorphism. Hence we see that the cohomology modules of $Q_{\overline{y}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}}$ are equal to the cohomology modules of $Q_{\overline{y}}$ by More on Algebra, Lemma 15.90.2 which finishes the proof. □

0AEV **Situation 81.10.6.** Here S is a base scheme, $f : Y \to X$ is a quasi-compact and quasi-separated morphism of algebraic spaces over S, and $Z \to X$ is a closed immersion of finite presentation. We assume that $f^{-1}(Z) \to Z$ is an isomorphism and that f is flat in every point $x \in |f^{-1}Z|$. We set $U = X \setminus Z$ and $V = Y \setminus f^{-1}(Z)$. Picture

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ {\scriptstyle f|_V}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

In Situation 81.10.6 we define $QCoh(Y \to X, Z)$ as the category of triples $(\mathcal{H}, \mathcal{G}, \varphi)$ where $\mathcal{H}$ is a quasi-coherent sheaf of $\mathcal{O}_U$-modules, $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, and $\varphi : f^*\mathcal{H} \to \mathcal{G}|_V$ is an isomorphism of $\mathcal{O}_V$-modules. There is a canonical functor

0AEW (81.10.6.1) $$QCoh(\mathcal{O}_X) \longrightarrow QCoh(Y \to X, Z)$$

which maps $\mathcal{F}$ to the system $(\mathcal{F}|_U, f^*\mathcal{F}, can)$. By analogy with the proof given in the affine case, we construct a functor in the opposite direction. To an object $(\mathcal{H}, \mathcal{G}, \varphi)$ we assign the $\mathcal{O}_X$-module

0AEX (81.10.6.2) $$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)$$

Observe that j and j' are quasi-compact morphisms as $Z \to X$ is of finite presentation. Hence f_*, j_*, and $(f \circ j')_*$ transform quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 67.11.2). Thus the module (81.10.6.2) is quasi-coherent.

0AEY **Lemma 81.10.7.** *In Situation 81.10.6. The functor (81.10.6.2) is right adjoint to the functor (81.10.6.1).*

Proof. This follows easily from the adjointness of f^* to f_* and j^* to j_*. Details omitted. □

0AEZ **Lemma 81.10.8.** *In Situation 81.10.6. Let $X' \to X$ be a flat morphism of algebraic spaces. Set $Z' = X' \times_X Z$ and $Y' = X' \times_X Y$. The pullbacks $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{X'})$ and $QCoh(Y \to X, Z) \to QCoh(Y' \to X', Z')$ are compatible with the functors (81.10.6.2) and 81.10.6.1).*

Proof. This is true because pullback commutes with pullback and because flat pullback commutes with pushforward along quasi-compact and quasi-separated morphisms, see Cohomology of Spaces, Lemma 69.11.2. □

0AF0 **Proposition 81.10.9.** *In Situation 81.10.6 the functor (81.10.6.1) is an equivalence with quasi-inverse given by (81.10.6.2).*

Proof. We first treat the special case where X and Y are affine schemes and where the morphism f is flat. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Then f corresponds to a flat ring map $R \to S$. Moreover, $Z \subset X$ is cut out by a finitely generated ideal $I \subset R$. Choose generators $f_1, \ldots, f_t \in I$. By the description of quasi-coherent modules in terms of modules (Schemes, Section 26.7), we see that the category $QCoh(Y \to X, Z)$ is canonically equivalent to the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ of More on Algebra, Remark 15.90.10 such that the functors (81.10.6.1) and (81.10.6.2) correspond to the functors Can and H^0. Hence the result follows from More on Algebra, Proposition 15.90.16 in this case.

We return to the general case. Let $\mathcal{F}$ be a quasi-coherent module on X. We will show that

$$\alpha : \mathcal{F} \longrightarrow \mathrm{Ker}\,(j_*\mathcal{F}|_U \oplus f_*f^*\mathcal{F} \to (f \circ j')_*f^*\mathcal{F}|_V)$$

is an isomorphism. Let $(\mathcal{H}, \mathcal{G}, \varphi)$ be an object of $QCoh(Y \to X, Z)$. We will show that

$$\beta : f^*\,\mathrm{Ker}\,(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{G}$$

and

$$\gamma : j^*\,\mathrm{Ker}\,(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{H}$$

are isomorphisms. To see these statements are true it suffices to look at stalks. Let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ of X.

Fix an object $(\mathcal{H}, \mathcal{G}, \varphi)$ of $QCoh(Y \to X, Z)$. By Lemma 81.10.2 and a diagram chase (omitted) the canonical map

$$\mathrm{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)_{\overline{x}} \longrightarrow \mathrm{Ker}(j_*\mathcal{H}_{\overline{x}} \oplus \mathcal{G}_{\overline{y}} \to j'_*\mathcal{G}_{\overline{y}})$$

is an isomorphism.

In particular, if $\overline{y}$ is a geometric point of V, then we see that $j'_*\mathcal{G}_{\overline{y}} = \mathcal{G}_{\overline{y}}$ and hence that this kernel is equal to $\mathcal{H}_{\overline{x}}$. This easily implies that $\alpha_{\overline{x}}$, $\beta_{\overline{x}}$, and $\beta_{\overline{y}}$ are isomorphisms in this case.

Next, assume that $\overline{y}$ is a point of $f^{-1}Z$. Let $I_{\overline{x}} \subset \mathcal{O}_{X,\overline{x}}$, resp. $I_{\overline{y}} \subset \mathcal{O}_{Y,\overline{y}}$ be the stalk of the ideal cutting out Z, resp. $f^{-1}Z$. Then $I_{\overline{x}}$ is a finitely generated ideal, $I_{\overline{y}} = I_{\overline{x}}\mathcal{O}_{Y,\overline{y}}$, and $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}$ is a flat local homomorphism inducing an isomorphism $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}} = \mathcal{O}_{Y,\overline{y}}/I_{\overline{y}}$. At this point we can bootstrap using the diagram of categories

$$\begin{array}{ccc} & \overset{(81.10.6.2)}{\curvearrowleft} & \\ QCoh(\mathcal{O}_X) & \xrightarrow[(81.10.6.1)]{} & QCoh(Y \to X, Z) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathcal{O}_{X,\overline{x}}} & \xrightarrow{\mathrm{Can}} & \mathrm{Glue}(\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}, f_1, \ldots, f_t) \\ & \underset{H^0}{\curvearrowleft} & \end{array}$$

Namely, as in the first paragraph of the proof we identify

$$\mathrm{Glue}(\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{Y,\overline{y}}, f_1, \ldots, f_t) = QCoh(\mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}), V(I_{\overline{x}}))$$

The right vertical functor is given by pullback, and it is clear that the inner square is commutative. Our computation of the stalk of the kernel in the third paragraph of the proof combined with Lemma 81.10.3 implies that the outer square (using the curved arrows) commutes. Thus we conclude using the case of a flat morphism of affine schemes which we handled in the first paragraph of the proof. □

0AFJ **Lemma 81.10.10.** *In Situation 81.10.6 the functor Rf_* induces an equivalence between $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ and $D_{QCoh,|Z|}(\mathcal{O}_X)$ with quasi-inverse given by Lf^*.*

Proof. Since f is quasi-compact and quasi-separated we see that Rf_* defines a functor from $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ to $D_{QCoh,|Z|}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 75.6.1. By Derived Categories of Spaces, Lemma 75.5.5 we see that Lf^* maps $D_{QCoh,|Z|}(\mathcal{O}_X)$ into $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. In Lemma 81.10.5 we have seen that $Lf^*Rf_*Q = Q$ for Q in $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. By the dual of Derived Categories, Lemma 13.7.2 to finish the proof it suffices to show that $Lf^*K = 0$ implies $K = 0$ for K in $D_{QCoh,|Z|}(\mathcal{O}_X)$. This follows from the fact that f is flat at all points of $f^{-1}Z$ and the fact that $f^{-1}Z \to Z$ is surjective. □

0AF1 **Lemma 81.10.11.** *In Situation 81.10.6 there exists an fpqc covering $\{X_i \to X\}_{i \in I}$ refining the family $\{U \to X, Y \to X\}$.*

Proof. For the definition and general properties of fpqc coverings we refer to Topologies, Section 34.9. In particular, we can first choose an étale covering $\{X_i \to X\}$ with X_i affine and by base changing Y, Z, and U to each X_i we reduce

to the case where X is affine. In this case U is quasi-compact and hence a finite union $U = U_1 \cup \ldots \cup U_n$ of affine opens. Then Z is quasi-compact hence also $f^{-1}Z$ is quasi-compact. Thus we can choose an affine scheme W and an étale morphism $h : W \to Y$ such that $h^{-1}f^{-1}Z \to f^{-1}Z$ is surjective. Say $W = \mathrm{Spec}(B)$ and $h^{-1}f^{-1}Z = V(J)$ where $J \subset B$ is an ideal of finite type. By Pro-étale Cohomology, Lemma 61.5.1 there exists a localization $B \to B'$ such that points of $\mathrm{Spec}(B')$ correspond exactly to points of $W = \mathrm{Spec}(B)$ specializing to $h^{-1}f^{-1}Z = V(J)$. It follows that the composition $\mathrm{Spec}(B') \to \mathrm{Spec}(B) = W \to Y \to X$ is flat as by assumption $f : Y \to X$ is flat at all the points of $f^{-1}Z$. Then $\{\mathrm{Spec}(B') \to X, U_1 \to X, \ldots, U_n \to X\}$ is an fpqc covering by Topologies, Lemma 34.9.2. □

81.11. Formal glueing of algebraic spaces

0AF2 In Situation 81.10.6 we consider the category $\mathit{Spaces}(Y \to X, Z)$ of commutative diagrams of algebraic spaces over S of the form

$$\begin{array}{ccccc} U' & \longleftarrow & V' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & Y \end{array}$$

where both squares are cartesian. There is a canonical functor

0AF3 (81.11.0.1) $$\mathit{Spaces}/X \longrightarrow \mathit{Spaces}(Y \to X, Z)$$

which maps $X' \to X$ to the morphisms $U \times_X X' \leftarrow V \times_X X' \to Y \times_X X'$.

0AF4 **Lemma 81.11.1.** *In Situation 81.10.6 the functor (81.11.0.1) restricts to an equivalence*

(1) *from the category of algebraic spaces affine over X to the full subcategory of $\mathit{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine,*
(2) *from the category of closed immersions $X' \to X$ to the full subcategory of $\mathit{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ closed immersions, and*
(3) *same statement as in (2) for finite morphisms.*

Proof. The category of algebraic spaces affine over X is equivalent to the category of quasi-coherent sheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras. The full subcategory of $\mathit{Spaces}(Y \to X, Z)$ consisting of $(U' \leftarrow V' \to Y')$ with $U' \to U$, $V' \to V$, and $Y' \to Y$ affine is equivalent to the category of algebra objects of $\mathit{QCoh}(Y \to X, Z)$. In both cases this follows from Morphisms of Spaces, Lemma 67.20.7 with quasi-inverse given by the relative spectrum construction (Morphisms of Spaces, Definition 67.20.8) which commutes with arbitrary base change. Thus part (1) of the lemma follows from Proposition 81.10.9.

Fully faithfulness in part (2) follows from part (1). For essential surjectivity, we reduce by part (1) to proving that $X' \to X$ is a closed immersion if and only if both $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are closed immersions. By Lemma 81.10.11 $\{U \to X, Y \to X\}$ can be refined by an fpqc covering. Hence the result follows from Descent on Spaces, Lemma 74.11.17.

For (3) use the argument proving (2) and Descent on Spaces, Lemma 74.11.23. □

0AF5 **Lemma 81.11.2.** *In Situation 81.10.6 the functor (81.11.0.1) reflects isomorphisms.*

Proof. By a formal argument with base change, this reduces to the following question: A morphism $a : X' \to X$ of algebraic spaces such that $U \times_X X' \to U$ and $Y \times_X X' \to Y$ are isomorphisms, is an isomorphism. The family $\{U \to X, Y \to X\}$ can be refined by an fpqc covering by Lemma 81.10.11. Hence the result follows from Descent on Spaces, Lemma 74.11.15. □

0AF6 **Lemma 81.11.3.** *In Situation 81.10.6 the functor (81.11.0.1) is fully faithful on algebraic spaces separated over X. More precisely, it induces a bijection*

$$\mathrm{Mor}_X(X'_1, X'_2) \longrightarrow \mathrm{Mor}_{Spaces(Y \to X, Z)}(F(X'_1), F(X'_2))$$

whenever $X'_2 \to X$ is separated.

Proof. Since $X'_2 \to X$ is separated, the graph $i : X'_1 \to X'_1 \times_X X'_2$ of a morphism $X'_1 \to X'_2$ over X is a closed immersion, see Morphisms of Spaces, Lemma 67.4.6. Moreover a closed immersion $i : T \to X'_1 \times_X X'_2$ is the graph of a morphism if and only if $\mathrm{pr}_1 \circ i$ is an isomorphism. The same is true for

(1) the graph of a morphism $U \times_X X'_1 \to U \times_X X'_2$ over U,
(2) the graph of a morphism $V \times_X X'_1 \to V \times_X X'_2$ over V, and
(3) the graph of a morphism $Y \times_X X'_1 \to Y \times_X X'_2$ over Y.

Moreover, if morphisms as in (1), (2), (3) fit together to form a morphism in the category $Spaces(Y \to X, Z)$, then these graphs fit together to give an object of $Spaces(Y \times_X (X'_1 \times_X X'_2) \to X'_1 \times_X X'_2, Z \times_X (X'_1 \times_X X'_2))$ whose triple of morphisms are closed immersions. The proof is finished by applying Lemmas 81.11.1 and 81.11.2. □

81.12. Glueing and the Beauville-Laszlo theorem

0F9M Let $R \to R'$ be a ring homomorphism and let $f \in R$ be an element such that

$$0 \to R \to R_f \oplus R' \to R'_f \to 0$$

is a short exact sequence. This implies that $R/f^nR \cong R'/f^nR'$ for all n and $(R \to R', f)$ is a glueing pair in the sense of More on Algebra, Section 15.91. Set $X = \mathrm{Spec}(R)$, $U = \mathrm{Spec}(R_f)$, $X' = \mathrm{Spec}(R')$ and $U' = \mathrm{Spec}(R'_f)$. Picture

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In this situation we can consider the category $Spaces(U \leftarrow U' \to X')$ whose objects are commutative diagrams

$$\begin{array}{ccccc} V & \longleftarrow & V' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & U' & \longrightarrow & X' \end{array}$$

of algebraic spaces with both squares cartesian and whose morphism are defined in the obvious manner. An object of this category will be denoted (V, V', Y') with arrows suppressed from the notation. There is a functor

0F9N (81.12.0.1) $$\textit{Spaces}/X \longrightarrow \textit{Spaces}(U \leftarrow U' \rightarrow X')$$

given by base change: $Y \mapsto (U \times_X Y, U' \times_X Y, X' \times_X Y)$.

We have seen in More on Algebra, Section 15.91 that not every R-module M can be recovered from its gluing data. Similarly, the functor (81.12.0.1) won't be fully faithful on the category of all spaces over X. In order to single out a suitable subcategory of algebraic spaces over X we need a lemma.

0F9P **Lemma 81.12.1.** *Let $(R \to R', f)$ be a glueing pair, see above. Let Y be an algebraic space over X. The following are equivalent*

(1) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ with Y_i affine and $\Gamma(Y_i, \mathcal{O}_{Y_i})$ glueable as an R-module,*
(2) *for every étale morphism $W \to Y$ with W affine $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.*

Proof. It is immediate that (2) implies (1). Assume $\{Y_i \to Y\}$ is as in (1) and let $W \to Y$ be as in (2). Then $\{Y_i \times_Y W \to W\}_{i \in I}$ is an étale covering, which we may refine by an étale covering $\{W_j \to W\}_{j=1,\ldots,m}$ with W_j affine (Topologies, Lemma 34.4.4). Thus to finish the proof it suffices to show the following three algebraic statements:

(1) if $R \to A \to B$ are ring maps with $A \to B$ étale and A glueable as an R-module, then B is glueable as an R-module,
(2) finite products of glueable R-modules are glueable,
(3) if $R \to A \to B$ are ring maps with $A \to B$ faithfully étale and B glueable as an R-module, then A is glueable as an R-module.

Namely, the first of these will imply that $\Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, the second will imply that $\prod \Gamma(W_j, \mathcal{O}_{W_j})$ is a glueable R-module, and the third will imply that $\Gamma(W, \mathcal{O}_W)$ is a glueable R-module.

Consider an étale R-algebra homomorphism $A \to B$. Set $A' = A \otimes_R R'$ and $B' = B \otimes_R R' = A' \otimes_A B$. Statements (1) and (3) then follow from the following facts: (a) A, resp. B is glueable if and only if the sequence

$$0 \to A \to A_f \oplus A' \to A'_f \to 0, \quad \text{resp.} \quad 0 \to B \to B_f \oplus B' \to B'_f \to 0,$$

is exact, (b) the second sequence is equal to the functor $- \otimes_A B$ applied to the first and (c) (faithful) flatness of $A \to B$. We omit the proof of (2). □

Let $(R \to R', f)$ be a glueing pair, see above. We will say an algebraic space Y over $X = \operatorname{Spec}(R)$ is *glueable for* $(R \to R', f)$ if the equivalent conditions of Lemma 81.12.1 are satisfied.

0F9Q **Lemma 81.12.2.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) restricts to an equivalence between the category of affine Y/X which are glueable for $(R \to R', f)$ and the full subcategory of objects (V, V', Y') of $\textit{Spaces}(U \leftarrow U' \rightarrow X')$ with V, V', Y' affine.*

Proof. Let (V, V', Y') be an object of $\textit{Spaces}(U \leftarrow U' \rightarrow X')$ with V, V', Y' affine. Write $V = \operatorname{Spec}(A_1)$ and $Y' = \operatorname{Spec}(A')$. By our definition of the category

Spaces$(U \leftarrow U' \to X')$ we find that V' is the spectrum of $A_1 \otimes_{R_f} R'_f = A_1 \otimes_R R'$ and the spectrum of A'_f. Hence we get an isomorphism $\varphi : A'_f \to A_1 \otimes_R R'$ of R'_f-algebras. By More on Algebra, Theorem 15.91.16 there exists a unique glueable R-module A and isomorphisms $A_f \to A_1$ and $A \otimes_R R' \to A'$ of modules compatible with φ. Since the sequence

$$0 \to A \to A_1 \oplus A' \to A'_f \to 0$$

is short exact, the multiplications on A_1 and A' define a unique R-algebra structure on A such that the maps $A \to A_1$ and $A \to A'$ are ring homomorphisms. We omit the verification that this construction defines a quasi-inverse to the functor (81.12.0.1) restricted to the subcategories mentioned in the statement of the lemma. □

0F9R **Lemma 81.12.3.** *Let P be one of the following properties of morphisms: "finite", "closed immersion", "flat", "finite type", "flat and finite presentation", "étale". Under the equivalence of Lemma 81.12.2 the morphisms having P correspond to morphisms of triples whose components have P.*

Proof. Let P' be one of the following properties of homomorphisms of rings: "finite", "surjective", "flat", "finite type", "flat and of finite presentation", "étale". Translated into algebra, the statement means the following: If $A \to B$ is an R-algebra homomorphism and A and B are glueable for $(R \to R', f)$, then $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ have P' if and only if $A \to B$ has P'.

By More on Algebra, Lemmas 15.91.4 and 15.91.18 the algebraic statement is true for P' equal to "finite" or "flat".

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are surjective, then $N = B/A$ is an R-module with $N_f = 0$ and $N \otimes_R R' = 0$ and hence vanishes by More on Algebra, Lemma 15.91.2. Thus $A \to B$ is surjective.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are finite type, then we can choose an A-algebra homomorphism $A[x_1, \ldots, x_n] \to B$ such that $A_f[x_1, \ldots, x_n] \to B_f$ and $(A \otimes_R R')[x_1, \ldots, x_n] \to B \otimes_R R'$ are surjective (small detail omitted). We conclude that $A[x_1, \ldots, x_n] \to B$ is surjective by the previous result. Thus $A \to B$ is of finite type.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are flat and of finite presentation, then we know that $A \to B$ is flat and of finite type by what we have already shown. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and denote I the kernel. By flatness of B over A we see that I_f is the kernel of $A_f[x_1, \ldots, x_n] \to B_f$ and $I \otimes_R R'$ is the kernel of $A \otimes_R R'[x_1, \ldots, x_n] \to B \otimes_R R'$. Thus I_f is a finite $A_f[x_1, \ldots, x_n]$-module and $I \otimes_R R'$ is a finite $(A \otimes_R R')[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 15.91.4 applied to I viewed as a module over $A[x_1, \ldots, x_n]$ we conclude that I is a finitely generated ideal and we conclude $A \to B$ is flat and of finite presentation.

If $A_f \to B_f$ and $A \otimes_R R' \to B \otimes_R R'$ are étale, then we know that $A \to B$ is flat and of finite presentation by what we have already shown. Since the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are isomorphic to fibres of $\mathrm{Spec}(B_f) \to \mathrm{Spec}(A_f)$ or $\mathrm{Spec}(B/fB) \to \mathrm{Spec}(A/fA)$, we conclude that $A \to B$ is unramified, see Morphisms, Lemmas 29.35.11 and 29.35.12. We conclude that $A \to B$ is étale by Morphisms, Lemma 29.36.16 for example. □

0F9S **Lemma 81.12.4.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) is faithful on the full subcategory of algebraic spaces Y/X glueable for $(R \to R', f)$.*

Proof. Let $f, g : Y \to Z$ be two morphisms of algebraic spaces over X with Y and Z glueable for $(R \to R', f)$ such that f and g are mapped to the same morphism in the category *Spaces*$(U \leftarrow U' \to X')$. We have to show the equalizer $E \to Y$ of f and g is an isomorphism. Working étale locally on Y we may assume Y is an affine scheme. Then E is a scheme and the morphism $E \to Y$ is a monomorphism and locally quasi-finite, see Morphisms of Spaces, Lemma 67.4.1. Moreover, the base change of $E \to Y$ to U and to X' is an isomorphism. As Y is the disjoint union of the affine open $V = U \times_X Y$ and the affine closed $V(f) \times_X Y$, we conclude E is the disjoint union of their isomorphic inverse images. It follows in particular that E is quasi-compact. By Zariski's main theorem (More on Morphisms, Lemma 37.43.3) we conclude that E is quasi-affine. Set $B = \Gamma(E, \mathcal{O}_E)$ and $A = \Gamma(Y, \mathcal{O}_Y)$ so that we have an R-algebra homomorphism $A \to B$. Since $E \to Y$ becomes an isomorphism after base change to U and X' we obtain ring maps $B \to A_f$ and $B \to A \otimes_R R'$ agreeing as maps into $A \otimes_R R'_f$. Since A is glueable for $(R \to R', f)$ we get a ring map $B \to A$ which is left inverse to the map $A \to B$. The corresponding morphism $Y = \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ maps into the open subscheme $E \subset \mathrm{Spec}(B)$ pointwise because this is true after base change to U and X'. Hence we get a morphism $Y \to E$ over Y. Since $E \to Y$ is a monomorhism we conclude $Y \to E$ is an isomorphism as desired. □

0F9T **Lemma 81.12.5.** *Let $(R \to R', f)$ be a glueing pair, see above. The functor (81.12.0.1) is fully faithful on the full subcategory of algebraic spaces Y/X which are (a) glueable for $(R \to R', f)$ and (b) have affine diagonal $Y \to Y \times_X Y$.*

Proof. Let Y, Z be two algebraic spaces over X which are both glueable for $(R \to R', f)$ and assume the diagonal of Z is affine. Let $a : U \times_X Y \to U \times_X Z$ over U and $b : X' \times_X Y \to X' \times_X Z$ over X' be two morphisms of algebraic spaces which induce the same morphism $c : U' \times_X Y \to U' \times_X Z$ over U'. We want to construct a morphism $f : Y \to Z$ over X which produces the morphisms a, b on base change to U, X'. By the faithfulness of Lemma 81.12.4, it suffices to construct the morphism f étale locally on Y (details omitted). Thus we may and do assume Y is affine.

Let $y \in |Y|$ be a point. If y maps into the open $U \subset X$, then $U \times_X Y$ is an open of Y on which the morphism f is defined (we can just take a). Thus we may assume y maps into the closed subset $V(f)$ of X. Since $R/fR = R'/fR'$ there is a unique point $y' \in |X' \times_X Y|$ mapping to y. Denote $z' = b(y') \in |X' \times_X Z|$ and $z \in |Z|$ the images of y'. Choose an étale neighbourhood $(W, w) \to (Z, z)$ with W affine. Observe that

$$(U \times_X W) \times_{U \times_X Z, a} (U \times_X Y), \quad (U' \times_X W) \times_{U' \times_X Z, c} (U' \times_X Y),$$

and

$$(X' \times_X W) \times_{X' \times_X Z, b} (X' \times_X Y)$$

form an object of *Spaces*$(U \leftarrow U' \to X')$ with affine parts (this is where we use that Z has affine diagonal). Hence by Lemma 81.12.2 there exists a unique affine scheme V glueable for $(R \to R', f)$ such that

$$(U \times_X V, U' \times_X V, X' \times_X V)$$

is the triple displayed above. By fully faithfulness for the affine case (Lemma 81.12.2) we get a unique morphisms $V \to W$ and $V \to Y$ agreeing with the first and second projection morphisms over U and X' in the construction above. By Lemma 81.12.3 the morphism $V \to Y$ is étale. To finish the proof, it suffices to show that there is a point $v \in |V|$ mapping to y (because then f is defined on an étale neighbourhood of y, namely V). There is a unique point $w' \in |X' \times_X W|$ mapping to w. By uniqueness w' is mapped to z' under the map $|X' \times_X W| \to |X' \times_X Z|$. Then we consider the cartesian diagram

$$\begin{array}{ccc} X' \times_X V & \longrightarrow & X' \times_X W \\ \downarrow & & \downarrow \\ X' \times_X Y & \longrightarrow & X' \times_X Z \end{array}$$

to see that there is a point $v' \in |X' \times_X V|$ mapping to y' and w', see Properties of Spaces, Lemma 66.4.3. Of course the image v of v' in $|V|$ maps to y and the proof is complete. □

0F9U **Lemma 81.12.6.** *Let $(R \to R', f)$ be a glueing pair, see above. Any object (V, V', Y') of Spaces$(U \leftarrow U' \to X')$ with V, V', Y' quasi-affine is isomorphic to the image under the functor (81.12.0.1) of a separated algebraic space Y over X.*

Proof. Choose n', $T' \to Y'$ and n_1, $T_1 \to V$ as in Properties, Lemma 28.18.6. Picture

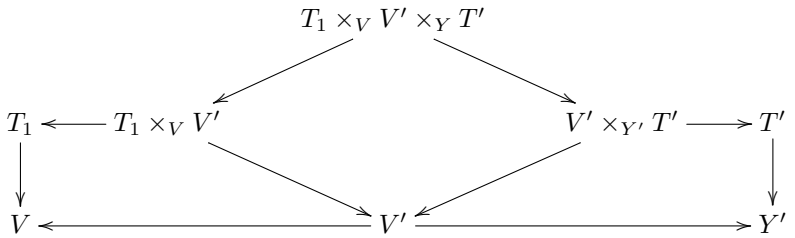

Observe that $T_1 \times_V V'$ and $V' \times_{Y'} T'$ are affine (namely the morphisms $V' \to V$ and $V' \to Y'$ are affine as base changes of the affine morphisms $U' \to U$ and $U' \to X'$). By construction we see that

$$\mathbf{A}^{n'}_{T_1 \times_V V'} \cong T_1 \times_V V' \times_{Y'} T' \cong \mathbf{A}^{n_1}_{V' \times_{Y'} T'}$$

In other words, the affine schemes $\mathbf{A}^{n'}_{T_1}$ and $\mathbf{A}^{n_1}_{T'}$ are part of a triple making an affine object of *Spaces*$(U \leftarrow U' \to X')$. By Lemma 81.12.2 there exists a morphism of affine schemes $T \to X$ and isomorphisms $U \times_X T \cong \mathbf{A}^{n'}_{T_1}$ and $X' \times_X T \cong \mathbf{A}^{n_1}_{T'}$ compatible with the isomorphisms displayed above. These isomorphisms produce morphisms

$$U \times_X T \longrightarrow V \quad \text{and} \quad X' \times_X T \longrightarrow Y'$$

satisfying the property of Properties, Lemma 28.18.6 with $n = n' + n_1$ and moreover define a morphism from the triple $(U \times_X T, U' \times_X T, X' \times_X T)$ to our triple (V, V', Y') in the category *Spaces*$(U \leftarrow U' \to X')$.

By Lemma 81.12.2 there is an affine scheme W whose image in *Spaces*$(U \leftarrow U' \to X')$ is isomorphic to the triple

$$((U \times_X T) \times_V (U \times_X T), (U' \times_X T) \times_{V'} (U' \times_X T), (X' \times_X T) \times_{Y'} (X' \times_X T))$$

By fully faithfulness of this construction, we obtain two maps $p_0, p_1 : W \to T$ whose base changes to U, U', X' are the projection morphisms. By Lemma 81.12.3 the morphisms p_0, p_1 are flat and of finite presentation and the morphism $(p_0, p_1) : W \to T \times_X T$ is a closed immersion. In fact, $W \to T \times_X T$ is an equivalence relation: by the lemmas used above we may check symmetry, reflexivity, and transitivity after base change to U and X', where these are obvious (details omitted). Thus the quotient sheaf

$$Y = T/W$$

is an algebraic space for example by Bootstrap, Theorem 80.10.1. Since it is clear that Y/X is sent to the triple (V, V', Y'). The base change of the diagonal $\Delta : Y \to Y \times_X Y$ by the quasi-compact surjective flat morphism $T \times_X T \to Y \times_X Y$ is the closed immersion $W \to T \times_X T$. Thus Δ is a closed immersion by Descent on Spaces, Lemma 74.11.17. Thus the algebraic space Y is separated and the proof is complete. □

81.13. Coequalizers and glueing

0AGF Let X be a Noetherian algebraic space and $Z \to X$ a closed subspace. Let $X' \to X$ be the blowing up in Z. In this section we show that X can be recovered from X', Z_n and glueing data where Z_n is the nth infinitesimal neighbourhood of Z in X.

0AGG **Lemma 81.13.1.** *Let S be a scheme. Let*

$$g : Y \longrightarrow X$$

be a morphism of algebraic spaces over S. Assume X is locally Noetherian, and g is proper. Let $R = Y \times_X Y$ with projection morphisms $t, s : R \to Y$. There exists a coequalizer X' of $s, t : R \to Y$ in the category of algebraic spaces over S. Moreover

1. *The morphism $X' \to X$ is finite.*
2. *The morphism $Y \to X'$ is proper.*
3. *The morphism $Y \to X'$ is surjective.*
4. *The morphism $X' \to X$ is universally injective.*
5. *If g is surjective, the morphism $X' \to X$ is a universal homeomorphism.*

Proof. Denote $h : R \to X$ denote the composition of either s or t with g. Then h is proper by Morphisms of Spaces, Lemmas 67.40.3 and 67.40.4. The sheaves

$$g_*\mathcal{O}_Y \quad \text{and} \quad h_*\mathcal{O}_R$$

are coherent $\mathcal{O}_X$-algebras by Cohomology of Spaces, Lemma 69.20.2. The X-morphisms s, t induce $\mathcal{O}_X$-algebra maps $s^\sharp, t^\sharp$ from the first to the second. Set

$$\mathcal{A} = \text{Equalizer}\left(s^\sharp, t^\sharp : g_*\mathcal{O}_Y \longrightarrow h_*\mathcal{O}_R\right)$$

Then $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra and we can define

$$X' = \underline{\text{Spec}}_X(\mathcal{A})$$

as in Morphisms of Spaces, Definition 67.20.8. By Morphisms of Spaces, Remark 67.20.9 and functoriality of the $\underline{\text{Spec}}$ construction there is a factorization

$$Y \longrightarrow X' \longrightarrow X$$

and the morphism $g' : Y \to X'$ equalizes s and t.

Before we show that X' is the coequalizer of s and t, we show that $Y \to X'$ and $X' \to X$ have the desired properties. Since $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module it

is clear that $X' \to X$ is a finite morphism of algebraic spaces. This proves (1). The morphism $Y \to X'$ is proper by Morphisms of Spaces, Lemma 67.40.6. This proves (2). Denote $Y \to Y' \to X$ with $Y' = \underline{\mathrm{Spec}}_X(g_*\mathcal{O}_Y)$ the Stein factorization of g, see More on Morphisms of Spaces, Theorem 76.36.4. Of course we obtain morphisms $Y \to Y' \to X' \to X$ fitting with the morphisms studied above. Since $\mathcal{O}_{X'} \subset g_*\mathcal{O}_Y$ is a finite extension we see that $Y' \to X'$ is finite and surjective. Some details omitted; hint: use Algebra, Lemma 10.36.17 and reduce to the affine case by étale localization. Since $Y \to Y'$ is surjective (with geometrically connected fibres) we conclude that $Y \to X'$ is surjective. This proves (3). To show that $X' \to X$ is universally injective, we have to show that $X' \to X' \times_X X'$ is surjective, see Morphisms of Spaces, Definition 67.19.3 and Lemma 67.19.2. Since $Y \to X'$ is surjective (see above) and since base changes and compositions of surjective morphisms are surjective by Morphisms of Spaces, Lemmas 67.5.5 and 67.5.4 we see that $Y \times_X Y \to X' \times_X X'$ is surjective. However, since $Y \to X'$ equalizes s and t, we see that $Y \times_X Y \to X' \times_X X'$ factors through $X' \to X' \times_X X'$ and we conclude this latter map is surjective. This proves (4). Finally, if g is surjective, then since g factors through $X' \to X$ we see that $X' \to X$ is surjective. Since a surjective, universally injective, finite morphism is a universal homeomorphism (because it is universally bijective and universally closed), this proves (5).

In the rest of the proof we show that $Y \to X'$ is the coequalizer of s and t in the category of algebraic spaces over S. Observe that X' is locally Noetherian (Morphisms of Spaces, Lemma 67.23.5). Moreover, observe that $Y \times_{X'} Y \to Y \times_X Y$ is an isomorphism as $Y \to X'$ equalizes s and t (this is a categorical statement). Hence in order to prove the statement that $Y \to X'$ is the coequalizer of s and t, we may and do assume $X = X'$. In other words, $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$.

Let $X_1 \to X$ be a flat morphism of algebraic spaces over S with X_1 locally Noetherian. Denote $g_1 : Y_1 \to X_1$, $h_1 : R_1 \to X_1$ and $s_1, t_1 : R_1 \to Y_1$ the base changes of g, h, s, t to X_1. Of course g_1 is proper and $R_1 = Y_1 \times_{X_1} Y_1$. Since we have flat base change for pushforward of quasi-coherent modules, Cohomology of Spaces, Lemma 69.11.2, we see that $\mathcal{O}_{X_1}$ is the equalizer of the maps $s_1^\sharp, t_1^\sharp : g_{1,*}\mathcal{O}_{Y_1} \to h_{1,*}\mathcal{O}_{R_1}$. Hence all the assumptions we have are preserved by this base change.

At this point we are going to check conditions (1) and (2) of Lemma 81.3.3. Condition (1) follows from Lemma 81.5.1 and the fact that g is proper and surjective (because $X = X'$). To check condition (2), by the remarks on base change above, we reduce to the statement discussed and proved in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ is an affine scheme, $X = X'$ is an affine scheme, and Z is an affine scheme over S. We have to show that

$$\mathrm{Mor}_S(X, Z) \longrightarrow \mathrm{Equalizer}(s, t : \mathrm{Mor}_S(Y, Z) \to \mathrm{Mor}_S(R, Z))$$

is bijective. However, this is clear from the fact that $X = X'$ which implies $\mathcal{O}_X$ is the equalizer of the maps $s^\sharp, t^\sharp : g_*\mathcal{O}_Y \to h_*\mathcal{O}_R$ which in turn implies

$$\Gamma(X, \mathcal{O}_X) = \mathrm{Equalizer}\left(s^\sharp, t^\sharp : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(R, \mathcal{O}_R)\right)$$

Namely, we have

$$\mathrm{Mor}_S(X, Z) = \mathrm{Hom}_A(\Gamma(Z, \mathcal{O}_Z), \Gamma(X, \mathcal{O}_X))$$

and similarly for Y and R, see Properties of Spaces, Lemma 66.33.1. □

We will work in the following situation.

0AGH **Situation 81.13.2.** Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $Z \to X$ be a closed immersion and let $U \subset X$ be the complementary open subspace. Finally, let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism.

0AGI **Lemma 81.13.3.** *In Situation 81.13.2 let $Y = X' \amalg Z$ and $R = Y \times_X Y$ with projections $t, s : R \to Y$. There exists a coequalizer X_1 of $s, t : R \to Y$ in the category of algebraic spaces over S. The morphism $X_1 \to X$ is a finite universal homeomorphism, an isomorphism over U, and $Z \to X$ lifts to X_1.*

Proof. Existence of X_1 and the fact that $X_1 \to X$ is a finite universal homeomorphism is a special case of Lemma 81.13.1. The formation of X_1 commutes with étale localization on X (see proof of Lemma 81.13.1). Thus the morphism $X_1 \to X$ is an isomorphism over U. It is immediate from the construction that $Z \to X$ lifts to X_1. □

In Situation 81.13.2 for $n \geq 1$ let $Z_n \subset X$ be the nth order infinitesimal neighbourhood of Z in X, i.e., the closed subscheme defined by the nth power of the sheaf of ideals cutting out Z. Consider $Y_n = X' \amalg Z_n$ and $R_n = Y_n \times_X Y_n$ and the coequalizer

$$R_n \rightrightarrows Y_n \longrightarrow X_n \longrightarrow X$$

as in Lemma 81.13.3. The maps $Y_n \to Y_{n+1}$ and $R_n \to R_{n+1}$ induce morphisms

0AGJ (81.13.3.1) $$X_1 \to X_2 \to X_3 \to \ldots \to X$$

Each of these morphisms is a universal homeomorphism as the morphisms $X_n \to X$ are universal homeomorphisms.

0AGK **Lemma 81.13.4.** *In Situation 81.13.2 assume X quasi-compact. In (81.13.3.1) for all n large enough, there exists an m such that $X_n \to X_{n+m}$ factors through a closed immersion $X \to X_{n+m}$.*

Proof. Let's look a bit more closely at the construction of X_n and how it changes as we increase n. We have $X_n = \underline{\mathrm{Spec}}(\mathcal{A}_n)$ where $\mathcal{A}_n$ is the equalizer of $s_n^\sharp$ and $t_n^\sharp$ going from $g_{n,*}\mathcal{O}_{Y_n}$ to $h_{n,*}\mathcal{O}_{R_n}$. Here $g_n : Y_n = X' \amalg Z_n \to X$ and $h_n : R_n = Y_n \times_X Y_n \to X$ are the given morphisms. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals corresponding to Z. Then

$$g_{n,*}\mathcal{O}_{Y_n} = f_*\mathcal{O}_{X'} \times \mathcal{O}_X/\mathcal{I}^n$$

Similarly, we have a decomposition

$$R_n = X' \times_X X' \amalg X' \times_X Z_n \amalg Z_n \times_X X' \amalg Z_n \times_X Z_n$$

As $Z_n \to X$ is a monomorphism, we see that $X' \times_X Z_n = Z_n \times_X X'$ and that this identification is compatible with the two morphisms to X, with the two morphisms to X', and with the two morphisms to Z_n. Denote $f_n : X' \times_X Z_n \to X$ the morphism to X. Denote

$$\mathcal{A} = \mathrm{Equalizer}(\ f_*\mathcal{O}_{X'} \rightrightarrows (f \times f)_*\mathcal{O}_{X' \times_X X'}\ )$$

By the remarks above we find that

$$\mathcal{A}_n = \mathrm{Equalizer}(\ \mathcal{A} \times \mathcal{O}_X/\mathcal{I}^n \rightrightarrows f_{n,*}\mathcal{O}_{X' \times_X Z_n}\ )$$

We have canonical maps

$$\mathcal{O}_X \to \ldots \to \mathcal{A}_3 \to \mathcal{A}_2 \to \mathcal{A}_1$$

of coherent $\mathcal{O}_X$-algebras. The statement of the lemma means that for n large enough there exists an $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is isomorphic to $\mathcal{O}_X$. This we may check étale locally on X. Hence by Properties of Spaces, Lemma 66.6.3 we may assume X is an affine Noetherian scheme.

Since $X_n \to X$ is an isomorphism over U we see that the kernel of $\mathcal{O}_X \to \mathcal{A}_n$ is supported on $|Z|$. Since X is Noetherian, the sequence of kernels $\mathcal{J}_n = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{A}_n)$ stabilizes (Cohomology of Spaces, Lemma 69.13.1). Say $\mathcal{J}_{n_0} = \mathcal{J}_{n_0+1} = \ldots = \mathcal{J}$. By Cohomology of Spaces, Lemma 69.13.2 we find that $\mathcal{I}^t\mathcal{J} = 0$ for some $t \geq 0$. On the other hand, there is an $\mathcal{O}_X$-algebra map $\mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n$ and hence $\mathcal{J} \subset \mathcal{I}^n$ for all n. By Artin-Rees (Cohomology of Spaces, Lemma 69.13.3) we find that $\mathcal{J} \cap \mathcal{I}^n \subset \mathcal{I}^{n-c}\mathcal{J}$ for some $c \geq 0$ and all $n \gg 0$. We conclude that $\mathcal{J} = 0$.

Pick $n \geq n_0$ as in the previous paragraph. Then $\mathcal{O}_X \to \mathcal{A}_n$ is injective. Hence it now suffices to find $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \to \mathcal{A}_n$ is equal to the image of $\mathcal{O}_X$. Observe that $\mathcal{A}_n$ sits in a short exact sequence

$$0 \to \mathrm{Ker}(\mathcal{A} \to f_{n,*}\mathcal{O}_{X' \times_X Z_n}) \to \mathcal{A}_n \to \mathcal{O}_X/\mathcal{I}^n \to 0$$

and similarly for $\mathcal{A}_{n+m}$. Hence it suffices to show

$$\mathrm{Ker}(\mathcal{A} \to f_{n+m,*}\mathcal{O}_{X' \times_X Z_{n+m}}) \subset \mathrm{Im}(\mathcal{I}^n \to \mathcal{A})$$

for some $m \geq 0$. To do this we may work étale locally on X and since X is Noetherian we may assume that X is a Noetherian affine scheme. Say $X = \mathrm{Spec}(R)$ and $\mathcal{I}$ corresponds to the ideal $I \subset R$. Let $\mathcal{A} = \widetilde{A}$ for a finite R-algebra A. Let $f_*\mathcal{O}_{X'} = \widetilde{B}$ for a finite R-algebra B. Then $R \to A \subset B$ and these maps become isomorphisms on inverting any element of I.

Note that $f_{n,*}\mathcal{O}_{X' \times_X Z_n}$ is equal to $f_*(\mathcal{O}_{X'}/I^n\mathcal{O}_{X'})$ in the notation used in Cohomology of Spaces, Section 69.22. By Cohomology of Spaces, Lemma 69.22.4 we see that there exists a $c \geq 0$ such that

$$\mathrm{Ker}(B \to \Gamma(X, f_*(\mathcal{O}_{X'}/I^{n+m+c}\mathcal{O}_{X'}))$$

is contained in $I^{n+m}B$. On the other hand, as $R \to B$ is finite and an isomorphism after inverting any element of I we see that $I^{n+m}B \subset \mathrm{Im}(I^n \to B)$ for m large enough (can be chosen independent of n). This finishes the proof as $A \subset B$. □

0AGL **Remark 81.13.5.** The meaning of Lemma 81.13.4 is the system $X_1 \to X_2 \to X_3 \to \ldots$ is essentially constant with value X. See Categories, Definition 4.22.1.

81.14. Compactifications

0F44 This section is the analogue of More on Flatness, Section 38.33. The theorem in this section is the main theorem in [**CLO12**].

Let B be a quasi-compact and quasi-separated algebraic space over some base scheme S. We will say an algebraic space X over B *has a compactification over* B or *is compactifyable over* B if there exists a quasi-compact open immersion $X \to \overline{X}$ into an algebraic space $\overline{X}$ proper over B. If X has a compactification over B, then $X \to B$ is separated and of finite type. The main theorem of this section is that the converse is true as well.

0F45 **Lemma 81.14.1.** *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If $(U \subset X, f : V \to X)$ is an elementary distinguished square such that $U \to Y$ and $V \to Y$ are separated and $U \times_X V \to U \times_Y V$ is closed, then $X \to Y$ is separated.*

Proof. We have to check that $\Delta : X \to X \times_Y X$ is a closed immersion. There is an étale covering of $X \times_Y X$ given by the four parts $U \times_Y U$, $U \times_Y V$, $V \times_Y U$, and $V \times_Y V$. Observe that $(U \times_Y U) \times_{(X \times_Y X), \Delta} X = U$, $(U \times_Y V) \times_{(X \times_Y X), \Delta} X = U \times_X V$, $(V \times_Y U) \times_{(X \times_Y X), \Delta} X = V \times_X U$, and $(V \times_Y V) \times_{(X \times_Y X), \Delta} X = V$. Thus the assumptions of the lemma exactly tell us that Δ is a closed immersion. □

0F46 **Lemma 81.14.2.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open.*

(1) *If $Z_1, Z_2 \subset X$ are closed subspaces of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.*
(2) *If $T_1, T_2 \subset |U|$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.*

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms of Spaces, Lemma 67.28.12. Denote $Z \subset X$ the closed subspace cut out by the product $\mathcal{I}_1\mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors on Spaces, Lemma 71.19.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors on Spaces, Definition 71.19.1 and Morphisms of Spaces, Lemma 67.28.2). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors on Spaces, Lemma 71.18.3. Thus (1) is proved.

Proof of (2). By Limits of Spaces, Lemma 70.14.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Limits of Spaces, Lemma 70.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subspaces $Z_i = V(\mathcal{I}_i)$ to conclude. □

0F47 **Lemma 81.14.3.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset |V|$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in $|Y'|$.*

Proof. Let $T' \subset |V|$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in $|V|$ and $T \cap T' = \emptyset$. Since $|V|$ is a spectral topological space (Properties of Spaces, Lemma 66.15.2) we can find constructible closed subsets T_c, T'_c of $|V|$ with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of $|V|$ containing T' not meeting T and set $T_c = |V| \setminus W$, then choose a quasi-compact open W' of $|V|$ containing T_c not meeting T' and set

$T'_c = |V| \setminus W'$). By Lemma 81.14.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in $|Y|$. Let Y_0 be the open subspace of Y corresponding to the open $|Y| \setminus \overline{T}'_c$ and set $V_0 = V \cap Y_0$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see More on Morphisms of Spaces, Lemma 76.39.4. By Divisors on Spaces, Lemma 71.19.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. □

0F48 **Lemma 81.14.4.** *Let S be a scheme. Consider a diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \longleftarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle f|_U}\downarrow & & \downarrow \\ Y & \longleftarrow & V & \longleftarrow & B \end{array}$$

of quasi-compact and quasi-separated algebraic spaces over S. Assume

(1) *f is proper,*
(2) *V is a quasi-compact open of Y, $U = f^{-1}(V)$,*
(3) *$B \subset V$ and $A \subset U$ are closed subspaces,*
(4) *$f|_A : A \to B$ is an isomorphism, and f is étale at every point of A.*

Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ satisfies: for every geometric point $\overline{a}$ of the closure of $|A|$ in $|X'|$ there exists a quotient $\mathcal{O}_{X',\overline{a}} \to \mathcal{O}$ such that $\mathcal{O}_{Y',f'(\overline{a})} \to \mathcal{O}$ is finite flat.

As you can see from the proof, more is true, but the statement is already long enough and this will be sufficient later on.

Proof. Let $T' \subset |U|$ be the complement of the maximal open on which $f|_U$ is étale. Then T' is closed in $|U|$ and disjoint from $|A|$. Since $|U|$ is a spectral topological space (Properties of Spaces, Lemma 66.15.2) we can find constructible closed subsets T_c, T'_c of $|U|$ with $|A| \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (see proof of Lemma 81.14.3). By Lemma 81.14.2 there is a U-admissible blowing up $X_1 \to X$ such that T_c and T'_c have disjoint closures in $|X_1|$. Let $X_{1,0}$ be the open subspace of X_1 corresponding to the open $|X_1| \setminus \overline{T}'_c$ and set $U_0 = U \cap X_{1,0}$. Observe that the scheme theoretic image $\overline{A}_1 \subset X_1$ of A is contained in $X_{1,0}$ by construction.

After replacing Y by a V-admissible blowing up and taking strict transforms, we may assume $X_{1,0} \to Y$ is flat, quasi-finite, and of finite presentation, see More on Morphisms of Spaces, Lemmas 76.39.1 and 76.37.3. Consider the commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \qquad \text{and the diagram} \qquad \begin{array}{ccc} \overline{A}_1 & \longrightarrow & \overline{A} \\ & \searrow \quad \swarrow & \\ & \overline{B} & \end{array}$$

of scheme theoretic images. The morphism $\overline{A}_1 \to \overline{A}$ is surjective because it is proper and hence the scheme theoretic image of $\overline{A}_1 \to \overline{A}$ must be equal to $\overline{A}$ and then we can use Morphisms of Spaces, Lemma 67.40.8. The statement on étale local rings follows by choosing a lift of the geometric point $\overline{a}$ to a geometric point

$\overline{a}_1$ of $\overline{A}_1$ and setting $\mathcal{O} = \mathcal{O}_{X_1,\overline{a}_1}$. Namely, since $X_1 \to Y$ is flat and quasi-finite on $X_{1,0} \supset \overline{A}_1$, the map $\mathcal{O}_{Y',f'(\overline{a})} \to \mathcal{O}_{X_1,\overline{a}_1}$ is finite flat, see Algebra, Lemmas 10.156.3 and 10.153.3. □

0F49 **Lemma 81.14.5.** *Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. Let $U \subset X$ be an open subspace. Let $V \to X \times_B Y$ be a quasi-compact morphism whose composition with the first projection maps into U. Let $Z \subset X \times_B Y$ be the scheme theoretic image of $V \to X \times_B Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_B Y$ is the strict transform of Z with respect to the blowing up.*

Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_B Y$ which is a closed immersion (as Z' is a closed subspace of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subspace such that $V \to Z'$ factors through Z''. Since both $V \to X \times_B Y$ and $V \to X' \times_B Y$ are quasi-compact (for the latter this follows from Morphisms of Spaces, Lemma 67.8.9 and the fact that $X' \times_B Y \to X \times_B Y$ is separated as a base change of a proper morphism), by Morphisms of Spaces, Lemma 67.16.3 we see that $Z \cap (U \times_B Y) = Z'' \cap (U \times_B Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. □

0F4A **Lemma 81.14.6.** *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let U be an algebraic space of finite type and separated over B. Let $V \to U$ be an étale morphism. If V has a compactification $V \subset Y$ over B, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.*

Proof. Consider the scheme theoretic image $Z \subset Y \times_B U$ of the "diagonal" morphism $V \to Y \times_B U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 81.14.5. Hence by More on Morphisms of Spaces, Lemma 76.39.4 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_B U \to U$ is proper and $V' \cong Z$ is a closed subspace of $Y \times_B U$. □

The following lemma is formulated for finite type separated algebraic spaces over a finite type algebraic space over $\mathbf{Z}$. The version for quasi-compact and quasi-separated algebraic spaces is true as well (with essentially the same proof), but will be trivially implied by the main theorem in this section. We strongly urge the reader to read the proof of this lemma in the case of schemes first.

0F4B **Lemma 81.14.7.** *Let B be an algebraic space of finite type over $\mathbf{Z}$. Let U be an algebraic space of finite type and separated over B. Let $(U_2 \subset U, f : U_1 \to U)$ be an elementary distinguished square. Assume U_1 and U_2 have compactifications over B and $U_1 \times_U U_2 \to U$ has dense image. Then U has a compactification over B.*

Proof. Choose a compactification $U_i \subset X_i$ over B for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and

a proper morphism $\psi_i : V_i \to U$ extending $U_i \to U$, see Lemma 81.14.6. Picture

$$\begin{array}{ccccc} U_i & \longrightarrow & V_i & \longrightarrow & X_i \\ \downarrow & \swarrow_{\psi_i} & & & \\ U & & & & \end{array}$$

Denote $Z_1 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus |U_2|$. Recall that $f^{-1}Z_1$ is a closed subspace of U_1 mapping isomorphically to Z_1. Denote $Z_2 \subset U$ the reduced closed subspace corresponding to the closed subset $|U| \setminus \mathrm{Im}(|f|) = |U_2| \setminus \mathrm{Im}(|U_1 \times_U U_2| \to |U_2|)$. Thus we have

$$U = U_2 \amalg Z_1 = Z_2 \amalg \mathrm{Im}(f) = Z_2 \amalg \mathrm{Im}(U_1 \times_U U_2 \to U_2) \amalg Z_1$$

set theoretically. Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_2 is an isomorphism over an open neighbourhood of Z_2. Observe that $Z_{1,1} = \psi_1^{-1}Z_1 = f^{-1}Z_1 \amalg T$ for some closed subspace $T \subset V_1$ disjoint from $f^{-1}Z_1$ and furthermore ψ_1 is étale along $f^{-1}Z_1$. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subspaces of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subspaces of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the scheme theoretic images of $Z_{i,i}$ and $Z_{i,j}$ in X_i. We recall that $|Z_{i,j}|$ is dense in $|\overline{Z}_{i,j}|$, see Morphisms of Spaces, Lemma 67.17.7. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 81.14.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup. Hence we may replace X_1 by another V_1-admissible blowup and assume $|\overline{Z}_{1,1}|$ is the disjoint union of the closures of $|T|$ and $|f^{-1}Z_1|$ in $|X_1|$.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_B X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_B V_2$ (Morphisms of Spaces, Lemma 67.4.5) and the open immersion $V_1 \times_B V_2 \to X_1 \times_B X_2$. Let $X_{12} \subset X_1 \times_B X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_B X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over B. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 81.14.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subspace

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}Z_{1,2} = (p_2|_{V_{12}})^{-1}Z_{2,2} = \psi^{-1}Z_2 \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 81.14.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict transform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $|Z_{1,2}|$ in $|X_1'|$. After replacing X_1 by X_1' and X_{12} by X_{12}' we may assume that p_1 is an isomorphism over an open neighbourhood of $|\overline{Z}_{1,2}|$.

The result of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_B \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_B X_2$. Namely, the inverse image $p_1^{-1}\overline{Z}_{1,2}$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $|Z_{12,2}|$ is dense in $|p_1^{-1}\overline{Z}_{1,2}|$. Thus p_2 maps $|p_1^{-1}\overline{Z}_{1,2}|$ into $|\overline{Z}_{2,2}|$. Since $|\overline{Z}_{2,2}| \cap |\overline{Z}_{2,1}| = \emptyset$ we conclude.

It turns out that we need to do one additional blowing up before we can conclude the argument. Namely, let $V_2 \subset W_2 \subset X_2$ be the open subspace with underlying topological space

$$|W_2| = |V_2| \cup (|X_2| \setminus |\overline{Z}_{2,1}|) = |X_2| \setminus \big(|\overline{Z}_{2,1}| \setminus |Z_{2,1}|\big)$$

Since $p_2(p_1^{-1}\overline{Z}_{1,2})$ is contained in W_2 (see above) we see that replacing X_2 by a W_2-admissible blowup and X_{21} by the corresponding strict transform will preserve the property of p_1 being an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$. Since $\overline{Z}_{2,1} \cap W_2 = \overline{Z}_{2,1} \cap V_2 = Z_{2,1}$ we see that $Z_{2,1}$ is a closed subspace of W_2 and V_2. Observe that $V_{12} = V_1 \times_U V_2 = p_1^{-1}(V_1) = p_2^{-1}(V_2)$ as open subspaces of X_{12} as it is the largest open subspace of X_{12} over which the morphism $\psi : V_{12} \to U$ extends; details omitted[4]. We have the following equalities of closed subspaces of V_{12}:

$$p_2^{-1}Z_{2,1} = p_2^{-1}\psi_2^{-1}Z_1 = p_1^{-1}\psi_1^{-1}Z_1 = p_1^{-1}Z_{1,1} = p_1^{-1}f^{-1}Z_1 \amalg p_1^{-1}T$$

Here and below we use the slight abuse of notation of writing p_2 in stead of the restriction of p_2 to V_{12}, etc. Since $p_2^{-1}(Z_{2,1})$ is a closed subspace of $p_2^{-1}(W_2)$ as $Z_{2,1}$ is a closed subspace of W_2 we conclude that also $p_1^{-1}f^{-1}Z_1$ is a closed subspace of $p_2^{-1}(W_2)$. Finally, the morphism $p_2 : X_{12} \to X_2$ is étale at points of $p_1^{-1}f^{-1}Z_1$ as ψ_1 is étale along $f^{-1}Z_1$ and $V_{12} = V_1 \times_U V_2$. Thus we may apply Lemma 81.14.4 to the morphism $p_2 : X_{12} \to X_2$, the open W_2, the closed subspace $Z_{2,1} \subset W_2$, and the closed subspace $p_1^{-1}f^{-1}Z_1 \subset p_2^{-1}(W_2)$. Hence after replacing X_2 by a W_2-admissible blowup and X_{12} by the corresponding strict transform, we obtain for every geometric point $\overline{y}$ of the closure of $|p_1^{-1}f^{-1}Z_1|$ a local ring map $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ such that $\mathcal{O}_{X_2,p_2(\overline{y})} \to \mathcal{O}$ is finite flat.

Consider the algebraic space

$$W_2 = U \coprod\nolimits_{U_2} (X_2 \setminus \overline{Z}_{2,1}),$$

and with $T \subset V_1$ as in the first paragraph the algebraic space

$$W_1 = U \coprod\nolimits_{U_1} (X_1 \setminus \overline{Z}_{1,2} \cup \overline{T}),$$

obtained by pushout, see Lemma 81.9.2. Let us apply Lemma 81.14.1 to see that $W_i \to B$ is separated. First, $U \to B$ and $X_i \to B$ are separated. Let us check the quasi-compact immersion $U_i \to U \times_B (X_i \setminus \overline{Z}_{i,j})$ is closed using the valuative criterion, see Morphisms of Spaces, Lemma 67.42.1. Choose a valuation ring A over B with fraction field K and compatible morphisms $(u, x_i) : \mathrm{Spec}(A) \to U \times_B X_i$ and $u_i : \mathrm{Spec}(K) \to U_i$. Since ψ_i is proper, we can find a unique $v_i : \mathrm{Spec}(A) \to V_i$ compatible with u and u_i. Since X_i is proper over B we see that $x_i = v_i$. If v_i does not factor through $U_i \subset V_i$, then we conclude that x_i maps the closed point

[4]Namely, $V_1 \times_U V_2$ is proper over U so if ψ extends to a larger open of X_{12}, then $V_1 \times_U V_2$ would be closed in this open by Morphisms of Spaces, Lemma 67.40.6. Then we get equality as $V_{12} \subset X_{12}$ is dense.

of $\mathrm{Spec}(A)$ into $Z_{i,j}$ or T when $i = 1$. This finishes the proof because we removed $\overline{Z}_{i,j}$ and $\overline{T}$ in the construction of W_i.

On the other hand, for any valuation ring A over B with fraction field K and any morphism

$$\gamma : \mathrm{Spec}(K) \to \mathrm{Im}(U_1 \times_U U_2 \to U)$$

over B, we claim that after replacing A by an extension of valuation rings, there is an i and an extension of γ to a morphism $h_i : \mathrm{Spec}(A) \to W_i$. Namely, we first extend γ to a morphism $g_2 : \mathrm{Spec}(A) \to X_2$ using the valuative criterion of properness. If the image of g_2 does not meet $\overline{Z}_{2,1}$, then we obtain our morphism into W_2. Otherwise, denote $\overline{z} \in \overline{Z}_{2,1}$ a geometric point lying over the image of the closed point under g_2. We may lift this to a geometric point $\overline{y}$ of X_{12} in the closure of $|p_1^{-1}f^{-1}Z_1|$ because the map of spaces $|p_1^{-1}f^{-1}Z_1| \to |\overline{Z}_{2,1}|$ is closed with image containing the dense open $|Z_{2,1}|$. After replacing A by its strict henselization (More on Algebra, Lemma 15.124.6) we get the following diagram

$$\begin{array}{ccccc} A & \dashrightarrow & & & A' \\ \uparrow & & & & \uparrow \\ \mathcal{O}_{X_2,\overline{z}} & \longrightarrow & \mathcal{O}_{X_{12},\overline{y}} & \longrightarrow & \mathcal{O} \end{array}$$

where $\mathcal{O}_{X_{12},\overline{y}} \to \mathcal{O}$ is the map we found in the 5th paragraph of the proof. Since the horizontal composition is finite and flat we can find an extension of valuation rings A'/A and dotted arrow making the diagram commute. After replacing A by A' this means that we obtain a lift $g_{12} : \mathrm{Spec}(A) \to X_{12}$ whose closed point maps into the closure of $|p_1^{-1}f^{-1}Z_1|$. Then $g_1 = p_1 \circ g_{12} : \mathrm{Spec}(A) \to X_1$ is a morphism whose closed point maps into the closure of $|f^{-1}Z_1|$. Since the closure of $|f^{-1}Z_1|$ is disjoint from the closure of $|T|$ and contained in $|\overline{Z}_{1,1}|$ which is disjoint from $|\overline{Z}_{1,2}|$ we conclude that g_1 defines a morphism $h_1 : \mathrm{Spec}(A) \to W_1$ as desired.

Consider a diagram

$$\begin{array}{ccccc} W_1' & \longrightarrow & W & \longleftarrow & W_2' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ W_1 & \longleftarrow & U & \longrightarrow & W_2 \end{array}$$

as in More on Morphisms of Spaces, Lemma 76.40.1. By the previous paragraph for every solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\gamma} & W \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & B \end{array}$$

where $\mathrm{Im}(\gamma) \subset \mathrm{Im}(U_1 \times_U U_2 \to U)$ there is an i and an extension $h_i : \mathrm{Spec}(A) \to W_i$ of γ after possibly replacing A by an extension of valuation rings. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \mathrm{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists after possibly extending A. Since W is separated over B, we see that the choice of extension isn't needed and the arrow is unique as well, see Morphisms of Spaces, Lemmas 67.41.5 and 67.43.1. Then finally the existence of the dotted arrow implies that $W \to B$ is universally

closed by Morphisms of Spaces, Lemma 67.42.5. As $W \to B$ is already of finite type and separated, we win. □

0F4C **Lemma 81.14.8.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $U \subset X$ be a proper dense open subspace. Then there exists an affine scheme V and an étale morphism $V \to X$ such that*

(1) *the open subspace $W = U \cup \mathrm{Im}(V \to X)$ is strictly larger than U,*
(2) *$(U \subset W, V \to W)$ is a distinguished square, and*
(3) *$U \times_W V \to U$ has dense image.*

Proof. Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 68.8.6. Let p be the smallest integer such that $U_p \not\subset U$ (this is possible as $U \neq X$). Choose an affine open $V \subset V_p$ such that the étale morphism $f_p|_V : V \to X$ does not factor through U. Consider the open $W = U \cup \mathrm{Im}(V \to X)$ and the reduced closed subspace $Z \subset W$ with $|Z| = |W| \setminus |U|$. Then $f^{-1}Z \to Z$ is an isomorphism because we have the corresponding property for the morphism f_p, see the lemma cited above. Thus $(U \subset W, f : V \to W)$ is a distinguished square. It may not be true that the open $I = \mathrm{Im}(U \times_W V \to U)$ is dense in U. The algebraic space $U' \subset U$ whose underlying set is $|U| \setminus \overline{|I|}$ is Noetherian and hence we can find a dense open subscheme $U'' \subset U'$, see for example Properties of Spaces, Proposition 66.13.3. Then we can find a dense open affine $U''' \subset U''$, see Properties, Lemmas 28.5.7 and 28.29.1. After we replace f by $V \amalg U''' \to X$ everything is clear. □

0F4D **Theorem 81.14.9.** *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let $X \to B$ be a separated, finite type morphism. Then X has a compactification over B.* [**CLO12**]

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. First, we may replace S by $\mathrm{Spec}(\mathbf{Z})$. See Spaces, Section 65.16 and Properties of Spaces, Definition 66.3.1. There exists a closed immersion $X \to X'$ with $X' \to B$ of finite presentation and separated. See Limits of Spaces, Proposition 70.11.7. If we find a compactification of X' over B, then taking the scheme theoretic closure of X in this will give a compactification of X over B. Thus we may assume $X \to B$ is separated and of finite presentation. We may write $B = \lim B_i$ as a directed limit of a system of Noetherian algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$ with affine transition morphisms. See Limits of Spaces, Proposition 70.8.1. We can choose an i and a morphism $X_i \to B_i$ of finite presentation whose base change to B is $X \to B$, see Limits of Spaces, Lemma 70.7.1. After increasing i we may assume $X_i \to B_i$ is separated, see Limits of Spaces, Lemma 70.6.9. If we can find a compactification of X_i over B_i, then the base change of this to B will be a compactification of X over B. This reduces us to the case discussed in the next paragraph.

Assume B is of finite type over $\mathbf{Z}$ in addition to being quasi-compact and quasi-separated. Let $U \to X$ be an étale morphism of algebraic spaces such that U has a compactification Y over $\mathrm{Spec}(\mathbf{Z})$. The morphism

$$U \longrightarrow B \times_{\mathrm{Spec}(\mathbf{Z})} Y$$

is separated and quasi-finite by Morphisms of Spaces, Lemma 67.27.10 (the displayed morphism factors into an immersion hence is a monomorphism). Hence by Zariski's main theorem (More on Morphisms of Spaces, Lemma 76.34.3) there is an open immersion of U into an algebraic space Y' finite over $B \times_{\mathrm{Spec}(\mathbf{Z})} Y$. Then $Y' \to B$ is proper as the composition $Y' \to B \times_{\mathrm{Spec}(\mathbf{Z})} Y \to B$ of two proper morphisms (use Morphisms of Spaces, Lemmas 67.45.9, 67.40.4, and 67.40.3). We conclude that U has a compactification over B.

There is a dense open subspace $U \subset X$ which is a scheme. (Properties of Spaces, Proposition 66.13.3). In fact, we may choose U to be an affine scheme (Properties, Lemmas 28.5.7 and 28.29.1). Thus U has a compactification over $\mathrm{Spec}(\mathbf{Z})$; this is easily shown directly but also follows from the theorem for schemes, see More on Flatness, Theorem 38.33.8. By the previous paragraph U has a compactification over B. By Noetherian induction we can find a maximal dense open subspace $U \subset X$ which has a compactification over B. We will show that the assumption that $U \neq X$ leads to a contradiction. Namely, by Lemma 81.14.8 we can find a strictly larger open $U \subset W \subset X$ and a distinguished square $(U \subset W, f : V \to W)$ with V affine and $U \times_W V$ dense image in U. Since V is affine, as before it has a compactification over B. Hence Lemma 81.14.7 applies to show that W has a compactification over B which is the desired contradiction. □

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Part 5

Topics in Geometry

CHAPTER 82

Chow Groups of Spaces

0EDQ

82.1. Introduction

0EDR In this chapter we first discuss Chow groups of algebraic spaces. Having defined these, we define Chern classes of vector bundles as operators on these chow groups. The strategy will be entirely the same as the strategy in the case of schemes. Therefore we urge the reader to take a look at the introduction (Chow Homology, Section 42.1) of the corresponding chapter for schemes.

Some related papers: [**EG98**] and [**Kre99**].

82.2. Setup

0EDS We first fix the category of algebraic spaces we will be working with. Please keep in mind throughout this chapter that "decent + locally Noetherian" is the same as "quasi-separated + locally Noetherian" according to Decent Spaces, Lemma 68.14.1.

0EDT **Situation 82.2.1.** Here S is a scheme and B is an algebraic space over S. We assume B is quasi-separated, locally Noetherian, and universally catenary (Decent Spaces, Definition 68.25.4). Moreover, we assume given a dimension function $\delta : |B| \longrightarrow \mathbf{Z}$. We say X/B is *good* if X is an algebraic space over B whose structure morphism $f : X \to B$ is quasi-separated and locally of finite type. In this case we define

$$\delta = \delta_{X/B} : |X| \longrightarrow \mathbf{Z}$$

as the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$ (Morphisms of Spaces, Definition 67.33.1). This is a dimension function by More on Morphisms of Spaces, Lemma 76.32.2.

A special case is when $S = B$ is a scheme and (S, δ) is as in Chow Homology, Situation 42.7.1. Thus B might be the spectrum of a field (Chow Homology, Example 42.7.2) or $B = \mathrm{Spec}(\mathbf{Z})$ (Chow Homology, Example 42.7.3).

Many lemma, proposition, theorems, definitions on algebraic spaces are easier in the setting of Situation 82.2.1 because the algebraic spaces we are working with are quasi-separated (and thus a fortiori decent) and locally Noetherian. We will sprinkle this chapter with remarks such as the following to point this out.

0EDU **Remark 82.2.2.** In Situation 82.2.1 if X/B is good, then $|X|$ is a sober topological space. See Properties of Spaces, Lemma 66.15.1 or Decent Spaces, Proposition 68.12.4. We will use this without further mention to choose generic points of irreducible closed subsets of $|X|$.

0EDV **Remark 82.2.3.** In Situation 82.2.1 if X/B is good, then X is integral (Spaces over Fields, Definition 72.4.1) if and only if X is reduced and $|X|$ is irreducible. Moreover, for any point $\xi \in |X|$ there is a unique integral closed subspace $Z \subset X$ such that ξ is the generic point of the closed subset $|Z| \subset |X|$, see Spaces over Fields, Lemma 72.4.7.

If B is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

0EDW **Lemma 82.2.4.** *In Situation 82.2.1 assume B is Jacobson and that $\delta(b) = 0$ for every closed point b of $|B|$. Let X/B be good. If $Z \subset X$ is an integral closed subspace with generic point $\xi \in |Z|$, then the following integers are the same:*

(1) $\delta(\xi) = \delta_{X/B}(\xi)$,
(2) $\dim(|Z|)$,
(3) *codim*$(\{z\}, |Z|)$ *for $z \in |Z|$ closed,*
(4) *the dimension of the local ring of Z at z for $z \in |Z|$ closed, and*
(5) $\dim(\mathcal{O}_{Z,\overline{z}})$ *for $z \in |Z|$ closed.*

Proof. Let X, Z, ξ be as in the lemma. Since X is locally of finite type over B we see that X is Jacobson, see Decent Spaces, Lemma 68.23.1. Hence $X_{\text{ft-pts}} \subset |X|$ is the set of closed points by Decent Spaces, Lemma 68.23.3. Given a chain $T_0 \supset \ldots \supset T_e$ of irreducible closed subsets of $|Z|$ we have $T_e \cap X_{\text{ft-pts}}$ nonempty by Morphisms of Spaces, Lemma 67.25.6. Thus we can always assume such a chain ends with $T_e = \{z\}$ for some $z \in |Z|$ closed. It follows that $\dim(Z) = \sup_z \operatorname{codim}(\{z\}, |Z|)$ where z runs over the closed points of $|Z|$. We have $\operatorname{codim}(\{z\}, Z) = \delta(\xi) - \delta(z)$ by Topology, Lemma 5.20.2. By Morphisms of Spaces, Lemma 67.25.4 the image of z is a finite type point of B, i.e., a closed point of $|B|$. By Morphisms of Spaces, Lemma 67.33.4 the transcendence degree of z/b is 0. We conclude that $\delta(z) = \delta(b) = 0$ by assumption. Thus we obtain equality

$$\dim(|Z|) = \operatorname{codim}(\{z\}, Z) = \delta(\xi)$$

for all $z \in |Z|$ closed. Finally, we have that $\operatorname{codim}(\{z\}, Z)$ is equal to the dimension of the local ring of Z at z by Decent Spaces, Lemma 68.20.2 which in turn is equal to $\dim(\mathcal{O}_{Z,\overline{z}})$ by Properties of Spaces, Lemma 66.22.4. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. This motivates the following definition.

0EDX **Definition 82.2.5.** In Situation 82.2.1 for any good X/B and any irreducible closed subset $T \subset |X|$ we define

$$\dim_\delta(T) = \delta(\xi)$$

where $\xi \in T$ is the generic point of T. We will call this the *δ-dimension of T*. If $T \subset |X|$ is any closed subset, then we define $\dim_\delta(T)$ as the supremum of the δ-dimensions of the irreducible components of T. If Z is a closed subspace of X, then we set $\dim_\delta(Z) = \dim_\delta(|Z|)$.

Of course this just means that $\dim_\delta(T) = \sup\{\delta(t) \mid t \in T\}$.

82.3. Cycles

0EDY This is the analogue of Chow Homology, Section 42.8

Since we are not assuming our spaces are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

0EDZ **Definition 82.3.1.** In Situation 82.2.1 let X/B be good. Let $k \in \mathbf{Z}$.

(1) A *cycle on* X is a formal sum
$$\alpha = \sum n_Z [Z]$$
where the sum is over integral closed subspaces $Z \subset X$, each $n_Z \in \mathbf{Z}$, and $\{|Z|; n_Z \neq 0\}$ is a locally finite collection of subsets of $|X|$ (Topology, Definition 5.28.4).
(2) A *k-cycle* on X is a cycle
$$\alpha = \sum n_Z [Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.
(3) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subspaces (Remark 82.2.3) of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

82.4. Multiplicities

0EE0 A section with a few simple results on lengths and multiplicities.

0EE1 **Lemma 82.4.1.** *Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

(1) *$\text{length}_{\mathcal{O}_{X,\overline{x}}} \mathcal{F}_{\overline{x}} = d$*
(2) *for some étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$*
(3) *for any étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$*

Proof. Let $U \to X$ and $u \in U$ be as in (2) or (3). Then we know that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ and that
$$\mathcal{F}_{\overline{x}} = (\mathcal{F}|_U)_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$
See Properties of Spaces, Lemmas 66.22.1 and 66.29.4. Thus the equality of the lengths follows from Algebra, Lemma 10.52.13 the fact that $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is flat and the fact that $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_u \mathcal{O}_{X,\overline{x}}$ is equal to the residue field of $\mathcal{O}_{X,\overline{x}}$. These facts about strict henselizations can be found in More on Algebra, Lemma 15.45.1. □

0EE2 **Definition 82.4.2.** Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. We say $\mathcal{F}$ *has length* d *at* x if the equivalent conditions of Lemma 82.4.1 are satisfied.

0EE3 **Lemma 82.4.3.** *Let S be a scheme. Let $i : Y \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $y \in |Y|$ with image $x \in |X|$. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

(1) *$\mathcal{G}$ has length d at y, and*
(2) *$i_*\mathcal{G}$ has length d at x.*

Proof. Choose an étale morphism $f : U \to X$ with U a scheme and $u \in U$ mapping to x. Set $V = Y \times_X U$. Denote $g : V \to Y$ and $j : V \to U$ the projections. Then $j : V \to U$ is a closed immersion and there is a unique point $v \in V$ mapping to $y \in |Y|$ and $u \in U$ (use Properties of Spaces, Lemma 66.4.3 and Spaces, Lemma 65.12.3). We have $j_*(\mathcal{G}|_V) = (i_*\mathcal{G})|_U$ as modules on the scheme V and j_* the "usual" pushforward of modules for the morphism of schemes j, see discussion surrounding Cohomology of Spaces, Equation (69.3.0.1). In this way we reduce to the case of schemes: if $i : Y \to X$ is a closed immersion of schemes, then

$$(i_*\mathcal{G})_x = \mathcal{G}_y$$

as modules over $\mathcal{O}_{X,x}$ where the module structure on the right hand side is given by the surjection $i_y^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. Thus equality by Algebra, Lemma 10.52.5. □

0EE4 **Lemma 82.4.4.** *Let S be a scheme and let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $x \in |X|$. The following are equivalent*

(1) *for some étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}|_U)$,*
(2) *for any étale morphism $U \to X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}|_U)$,*
(3) *the length of $\mathcal{F}$ at x is finite and nonzero.*

If X is decent (equivalently quasi-separated) then these are also equivalent to

(4) *x is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F})$.*

Proof. Assume $f : U \to X$ is an étale morphism with U a scheme and $u \in U$ maps to x. Then $\mathcal{F}|_U = f^*\mathcal{F}$ is a coherent $\mathcal{O}_U$-module on the locally Noetherian scheme U and in particular $(\mathcal{F}|_U)_u$ is a finite $\mathcal{O}_{U,u}$-module, see Cohomology of Spaces, Lemma 69.12.2 and Cohomology of Schemes, Lemma 30.9.1. Recall that the support of $\mathcal{F}|_U$ is a closed subset of U (Morphisms, Lemma 29.5.3) and that the support of $(\mathcal{F}|_U)_u$ is the pullback of the support of $\mathcal{F}|_U$ by the morphism $\mathrm{Spec}(\mathcal{O}_{U,u}) \to U$. Thus u is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}|_U)$ if and only if the support of $(\mathcal{F}|_U)_u$ is equal to the maximal ideal of $\mathcal{O}_{U,u}$. Now the equivalence of (1), (2), (3) follows from by Algebra, Lemma 10.62.3.

If X is decent we choose an étale morphism $f : U \to X$ and a point $u \in U$ mapping to x. The support of $\mathcal{F}$ pulls back to the support of $\mathcal{F}|_U$, see Morphisms of Spaces, Lemma 67.15.2. Also, specializations $x' \rightsquigarrow x$ in $|X|$ lift to specializations $u' \rightsquigarrow u$ in U and any nontrivial specialization $u' \rightsquigarrow u$ in U maps to a nontrivial specialization $f(u') \rightsquigarrow f(u)$ in $|X|$, see Decent Spaces, Lemmas 68.12.2 and 68.12.1. Using that $|X|$ and U are sober topological spaces (Decent Spaces, Proposition 68.12.4 and Schemes, Lemma 26.11.1) we conclude x is a generic point of the support of $\mathcal{F}$ if

and only if u is a generic point of the support of $\mathcal{F}|_U$. We conclude (4) is equivalent to (1).

The parenthetical statement follows from Decent Spaces, Lemma 68.14.1. □

0EE6 **Lemma 82.4.5.** *In Situation 82.2.1 let X/B be good. Let $T \subset |X|$ be a closed subset and $t \in T$. If $\dim_\delta(T) \leq k$ and $\delta(t) = k$, then t is a generic point of an irreducible component of T.*

Proof. We know t is contained in an irreducible component $T' \subset T$. Let $t' \in T'$ be the generic point. Then $k \geq \delta(t') \geq \delta(t)$. Since δ is a dimension function we see that $t = t'$. □

82.5. Cycle associated to a closed subspace

0EE7 This section is the analogue of Chow Homology, Section 42.9.

0EE8 **Remark 82.5.1.** In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. By Remarks 82.2.2 and 82.2.3 there are 1-to-1 correspondences between

(1) irreducible components T of $|Y|$,
(2) generic points of irreducible components of $|Y|$, and
(3) integral closed subspaces $Z \subset Y$ with the property that $|Z|$ is an irreducible component of $|Y|$.

In this chapter we will call Z as in (3) an *irreducible component of Y* and we will call $\xi \in |Z|$ its *generic point*.

0EE9 **Definition 82.5.2.** In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace.

(1) For an irreducible component $Z \subset Y$ with generic point ξ the length of $\mathcal{O}_Y$ at ξ (Definition 82.4.2) is called the *multiplicity of Z in Y*. By Lemma 82.4.4 applied to $\mathcal{O}_Y$ on Y this is a positive integer.
(2) Assume $\dim_\delta(Y) \leq k$. The *k-cycle associated to Y* is
$$[Y]_k = \sum m_{Z,Y}[Z]$$
where the sum is over the irreducible components Z of Y of δ-dimension k and $m_{Z,Y}$ is the multiplicity of Z in Y. This is a k-cycle by Spaces over Fields, Lemma 72.6.1.

It is important to note that we only define $[Y]_k$ if the δ-dimension of Y does not exceed k. In other words, by convention, if we write $[Y]_k$ then this implies that $\dim_\delta(Y) \leq k$.

82.6. Cycle associated to a coherent sheaf

0EEA This is the analogue of Chow Homology, Section 42.10.

0EEB **Definition 82.6.1.** In Situation 82.2.1 let X/B be good. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For an integral closed subspace $Z \subset X$ with generic point ξ such that $|Z|$ is an irreducible component of $\mathrm{Supp}(\mathcal{F})$ the length of $\mathcal{F}$ at ξ (Definition 82.4.2) is called the *multiplicity of Z in $\mathcal{F}$*. By Lemma 82.4.4 this is a positive integer.

(2) Assume $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is

$$[\mathcal{F}]_k = \sum m_{Z,\mathcal{F}}[Z]$$

where the sum is over the integral closed subspaces $Z \subset X$ corresponding to irreducible components of $\mathrm{Supp}(\mathcal{F})$ of δ-dimension k and $m_{Z,\mathcal{F}}$ is the multiplicity of Z in $\mathcal{F}$. This is a k-cycle by Spaces over Fields, Lemma 72.6.1.

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\mathrm{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$.

0EEC **Lemma 82.6.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with $\dim_\delta(\mathit{Supp}(\mathcal{F})) \leq k$. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. Let $\xi \in |Z|$ be the generic point. Then the coefficient of Z in $[\mathcal{F}]_k$ is the length of $\mathcal{F}$ at ξ.*

Proof. Observe that $|Z|$ is an irreducible component of $\mathrm{Supp}(\mathcal{F})$ if and only if $\xi \in \mathrm{Supp}(\mathcal{F})$, see Lemma 82.4.5. Moreover, the length of $\mathcal{F}$ at ξ is zero if $\xi \notin \mathrm{Supp}(\mathcal{F})$. Combining this with Definition 82.6.1 we conclude. □

0EED **Lemma 82.6.3.** *In Situation 82.2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. If $\dim_\delta(Y) \leq k$, then $[Y]_k = [i_*\mathcal{O}_Y]_k$ where $i : Y \to X$ is the inclusion morphism.*

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. If $Z \not\subset Y$ the Z has coefficient zero in both $[Y]_k$ and $[i_*\mathcal{O}_Y]_k$. If $Z \subset Y$, then the generic point of Z may be viewed as a point $y \in |Y|$ whose image $x \in |X|$. Then the coefficient of Z in $[Y]_k$ is the length of $\mathcal{O}_Y$ at y and the coefficient of Z in $[i_*\mathcal{O}_Y]_k$ is the length of $i_*\mathcal{O}_Y$ at x. Thus the equality of the coefficients follows from Lemma 82.4.3. □

0EEE **Lemma 82.6.4.** *In Situation 82.2.1 let X/B be good. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. It suffices to show that the coefficients of Z in $[\mathcal{G}]_k$, $[\mathcal{F}]_k$, and $[\mathcal{H}]_k$ satisfy the corresponding additivity. By Lemma 82.6.2 it suffices to show

$$\text{the length of } \mathcal{G} \text{ at } x = \text{the length of } \mathcal{F} \text{ at } x + \text{the length of } \mathcal{H} \text{ at } x$$

for any $x \in |X|$. Looking at Definition 82.4.2 this follows immediately from additivity of lengths, see Algebra, Lemma 10.52.3. □

82.7. Preparation for proper pushforward

0EEF This section is the analogue of Chow Homology, Section 42.11.

0EEG **Lemma 82.7.1.** *In Situation 82.2.1 let $X, Y/B$ be good and let $f : X \to Y$ be a morphism over B. If $Z \subset X$ is an integral closed subspace, then there exists a unique integral closed subspace $Z' \subset Y$ such that there is a commutative diagram*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z' & \longrightarrow & Y \end{array}$$

with $Z \to Z'$ dominant. If f is proper, then $Z \to Z'$ is proper and surjective.

Proof. Let $\xi \in |Z|$ be the generic point. Let $Z' \subset Y$ be the integral closed subspace whose generic point is $\xi' = f(\xi)$, see Remark 82.2.3. Since $\xi \in |f^{-1}(Z')| = |f|^{-1}(|Z'|)$ by Properties of Spaces, Lemma 66.4.3 and since Z is the reduced with $|Z| = \overline{\{\xi\}}$ we see that $Z \subset f^{-1}(Z')$ as closed subspaces of X (see Properties of Spaces, Lemma 66.12.4). Thus we obtain our morphism $Z \to Z'$. This morphism is dominant as the generic point of Z maps to the generic point of Z'. Uniqueness of Z' is clear. If f is proper, then $Z \to Y$ is proper as a composition of proper morphisms (Morphisms of Spaces, Lemmas 67.40.3 and 67.40.5). Then we conclude that $Z \to Z'$ is proper by Morphisms of Spaces, Lemma 67.40.6. Surjectivity then follows as the image of a proper morphism is closed. □

0ENW **Remark 82.7.2.** In Situation 82.2.1 let X/B be good. Every $x \in |X|$ can be represented by a (unique) monomorphism $\mathrm{Spec}(k) \to X$ where k is a field, see Decent Spaces, Lemma 68.11.1. Then k is the *residue field of* x and is denoted $\kappa(x)$. Recall that X has a dense open subscheme $U \subset X$ (Properties of Spaces, Proposition 66.13.3). If $x \in U$, then $\kappa(x)$ agrees with the residue field of x on U as a scheme. See Decent Spaces, Section 68.11.

0ENX **Remark 82.7.3.** In Situation 82.2.1 let X/B be good. Assume X is integral. In this case the *function field* $R(X)$ of X is defined and is equal to the residue field of X at its generic point. See Spaces over Fields, Definition 72.4.3. Combining this with Remark 82.2.3 we find that for any $x \in X$ the residue field $\kappa(x)$ is the function field of the unique integral closed subspace $Z \subset X$ whose generic point is x.

0ENY **Lemma 82.7.4.** *In Situation 82.2.1 let $X, Y/B$ be good and let $f : X \to Y$ be a morphism over B. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either f factors through a proper closed subspace of Y, or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.*

Proof. By Lemma 82.7.1 there is a unique integral closed subspace $Z \subset Y$ such that f factors through a dominant morphism $X \to Z$. Then $Z = Y$ if and only if $\dim_\delta(Z) = \dim_\delta(Y)$. On the other hand, by our construction of dimension functions (see Situation 82.2.1) we have $\dim_\delta(X) = \dim_\delta(Z) + r$ where r the transcendence degree of the extension $R(X)/R(Z)$. Combining this with Spaces over Fields, Lemma 72.5.1 the lemma follows. □

0ENZ **Lemma 82.7.5.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is quasi-compact, and $\{T_i\}_{i \in I}$ is a locally finite collection of closed subsets of $|X|$. Then $\{\overline{|f|(T_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of $|Y|$.*

Proof. Let $V \subset |Y|$ be a quasi-compact open subset. Then $|f|^{-1}(V) \subset |X|$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.3. Hence the set $\{i \in I : T_i \cap |f|^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I : |f|(T_i) \cap V \neq \emptyset\} = \{i \in I : \overline{|f|(T_i)} \cap V \neq \emptyset\}$$

the lemma is proved. □

82.8. Proper pushforward

0EP0 This section is the analogue of Chow Homology, Section 42.12.

0EP1 **Definition 82.8.1.** In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = k$. Let $Z' \subset Y$ be the image of Z as in Lemma 82.7.1. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(Z') < k, \\ \deg(Z/Z')[Z'] & \text{if} \quad \dim_\delta(Z') = k. \end{cases}$$

The degree of Z over Z' is defined and finite if $\dim_\delta(Z') = \dim_\delta(Z)$ by Lemma 82.7.4 and Spaces over Fields, Definition 72.5.2.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 82.7.5 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category whose object are good algebraic spaces over B and whose morphisms are proper morphisms over B.

0EP2 **Lemma 82.8.2.** *In Situation 82.2.1 let $X, Y, Z/B$ be good. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms over B. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subspace of dimension k. Consider the integral closed subspaces $W' \subset Y$ and $W'' \subset Z$ we get by applying Lemma 82.7.1 to f and W and then to g and W'. Then $W \to W'$ and $W' \to W''$ are surjective and proper. We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see $W \to W'$ and $W' \to W''$ both satisfy the hypotheses of Lemma 82.7.4. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Spaces over Fields, Lemma 72.5.3 to conclude. □

0EP3 **Lemma 82.8.3.** *In Situation 82.2.1 let $f : X \to Y$ be a proper morphism of good algebraic spaces over B.*

(1) *Let $Z \subset X$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then*

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Spaces, Lemma 69.20.2.

Proof. Part (1) follows from (2) and Lemma 82.6.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Cohomology of Spaces, Lemma 69.12.7 there exists a closed immersion $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$, see Morphisms of Spaces, Definition 67.16.2. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f|_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{i'} & Y \end{array}$$

of algebraic spaces over B. Observe that $f|_Z$ is surjective (follows from Morphisms of Spaces, Lemma 67.16.3 and the fact that $|f|$ is closed) and proper (follows from Morphisms of Spaces, Lemmas 67.40.3, 67.40.5, and 67.40.6). We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion follows from Lemma 82.4.3 and the definitions. Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and surjective.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and surjective. For every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subspace Z is actually an irreducible component of Y of δ-dimension k (see Lemma 82.4.5). Pick such an integral closed subspace $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 82.4.5). By Spaces over Fields, Lemma 72.3.2 there exists an open subspace $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Since η is a generic point of an irreducible component of $|Y|$ we may assume V is an affine scheme, see Properties of Spaces, Proposition 66.13.3. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite. In particular X and Y are schemes and we reduce to the corresponding result for schemes, see Chow Homology, Lemma 42.12.4 (applied with $S = Y$). □

82.9. Preparation for flat pullback

0EP4 This section is the analogue of Chow Homology, Section 42.13.

Recall that a morphism of algebraic spaces is said to have relative dimension r if étale locally on the source and the target we get a morphism of schemes which has relative dimension r. The precise definition is equivalent, but in fact slightly different, see Morphisms of Spaces, Definition 67.33.2.

0EP5 **Lemma 82.9.1.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is flat of relative dimension r. For any closed subset $T \subset |Y|$ we have*

$$\dim_\delta(|f|^{-1}(T)) = \dim_\delta(T) + r.$$

provided $|f|^{-1}(T)$ is nonempty. If $Z \subset Y$ is an integral closed subscheme and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. Since the δ-dimension of a closed subset is the supremum of the δ-dimensions of the irreducible components, it suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms of Spaces, Lemma 67.30.6 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Thus Z' dominates $Z = Y$. Finally, we see that ξ and η are in the schematic locus of X and Y by Properties of Spaces, Proposition 66.13.3. Since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0 (we may use usual local rings as ξ and η are in the schematic loci of X and Y). Thus by Morphisms of Spaces, Lemma 67.34.1 (and the definition of morphisms of given relative dimension) we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

0EP6 **Lemma 82.9.2.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume $\{T_i\}_{i\in I}$ is a locally finite collection of closed subsets of $|Y|$. Then $\{|f|^{-1}(T_i)\}_{i\in I}$ is a locally finite collection of closed subsets of X.*

Proof. Let $U \subset |X|$ be a quasi-compact open subset. Since the image $|f|(U) \subset |Y|$ is a quasi-compact subset there exists a quasi-compact open $V \subset |Y|$ such that $|f|(U) \subset V$. Note that

$$\{i \in I : |f|^{-1}(T_i) \cap U \neq \emptyset\} \subset \{i \in I : T_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

82.10. Flat pullback

0EP7 This section is the analogue of Chow Homology, Section 42.14.

Let S be a scheme and let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be a closed subspace. In this chapter we will sometimes use the terminology *scheme theoretic inverse image* for the inverse image $f^{-1}(Z)$ of Z constructed in Morphisms of Spaces, Definition 67.13.2. The scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

If $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y, then $f^{-1}(\mathcal{I})\mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to $f^{-1}(Z)$ in X.

0EP8 **Definition 82.10.1.** In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a morphism over B. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subspace of δ-dimension k. We define $f^*[Z]$ to be the $(k+r)$-cycle on X associated to the scheme theoretic inverse image
$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$
This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 82.9.1.
(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum
$$f^*\alpha = \sum n_i f^*[Z_i]$$
where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 82.9.2.
(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps
$$j^* : Z_k(X) \longrightarrow Z_k(U)$$
are all *surjective*. The reason is that given any integral closed subspace $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subspace of X (see Properties of Spaces, Definition 66.12.5). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

0EP9 **Lemma 82.10.2.** *In Situation 82.2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \to X$ the inclusion morphism. For every $k \in \mathbf{Z}$ the sequence*
$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$
is an exact complex of abelian groups.

Proof. Surjectivity of j^* we saw above. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. Such a basis element maps either to the basis element $[Z \cap U]$ of $Z_k(U)$ or to zero if $Z \subset Y$. Hence the lemma is clear in this case. The general case is similar and the proof is omitted. □

0EPY **Lemma 82.10.3.** *In Situation 82.2.1 let $f : X \to Y$ be an étale morphism of good algebraic spaces over B. If $Z \subset Y$ is an integral closed subspace, then $f^*[Z] = \sum[Z']$ where the sum is over the irreducible components (Remark 82.5.1) of $f^{-1}(Z)$.*

Proof. The meaning of the lemma is that the coefficient of $[Z']$ is 1. This follows from the fact that $f^{-1}(Z)$ is a reduced algebraic space because it is étale over the integral algebraic space Z. □

0EPA **Lemma 82.10.4.** *In Situation 82.2.1 let $X, Y, Z/B$ be good. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s over B. Then $g \circ f$ is flat of relative dimension $r + s$ and*
$$f^* \circ g^* = (g \circ f)^*$$
as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r+s$ by Morphisms of Spaces, Lemmas 67.34.2 and 67.30.3. Suppose that

(1) $A \subset Z$ is a closed integral subspace of δ-dimension k,
(2) $A' \subset Y$ is a closed integral subspace of δ-dimension $k+s$ with $A' \subset g^{-1}(A)$, and
(3) $A'' \subset Y$ is a closed integral subspace of δ-dimension $k+s+r$ with $A'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[A'']$ in $(g \circ f)^*[A]$ agrees with the coefficient m of $[A'']$ in $f^*(g^*[A])$. We may choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where U, V, W are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$, $w \in W$ such that $u \mapsto v \mapsto w$ and such that u, v, w map to the generic points of A'', A', A. (Details omitted.) Then we have flat local ring homorphisms $\mathcal{O}_{W,w} \to \mathcal{O}_{V,v}$, $\mathcal{O}_{V,v} \to \mathcal{O}_{U,u}$, and repeatedly using Lemma 82.4.1 we find

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_w\mathcal{O}_{U,u})$$

and

$$m = \text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v})\text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

Hence the equality follows from Algebra, Lemma 10.52.14. □

0EPB **Lemma 82.10.5.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 82.6.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Spaces, Lemma 69.12.2 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules on Sites, Lemma 18.23.4), hence $f^*\mathcal{F}$ is coherent (Cohomology of Spaces, Lemma 69.12.2 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subspace of δ-dimension k, and let $W' \subset X$ be an integral closed subspace of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. We may choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$ such that $u \mapsto v$ and such that u, v map to the generic points of W', W. (Details omitted.) Consider the stalk $M = (\mathcal{F}|_V)_v$ as an $\mathcal{O}_{V,v}$-module. (Note that

M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 82.4.4.) We have $(f^*\mathcal{F}|_U)_u = \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M$. Thus we see that

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M) \quad \text{and} \quad m = \text{length}_{\mathcal{O}_{V,v}}(M)\text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

Thus the equality follows from Algebra, Lemma 10.52.13. □

82.11. Push and pull

0EPC This section is the analogue of Chow Homology, Section 42.14.

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

0EPD **Lemma 82.11.1.** *In Situation 82.2.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of good algebraic spaces over B. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms of Spaces, Lemma 67.40.3. The assertion that g' is flat of relative dimension r follows from Morphisms of Spaces, Lemmas 67.34.3 and 67.30.4. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subspace $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 82.6.3. By Lemmas 82.8.3 and 82.10.5 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^* f_* \mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Spaces, Lemma 69.11.2. □

0EPE **Lemma 82.11.2.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms of Spaces, Definition 67.46.2). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms of Spaces, Lemma 67.46.6, and a finite morphism is proper by Morphisms of Spaces, Lemma 67.45.9. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms of Spaces, Lemma 67.46.5) we see that $f_* f^* \mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Thus clearly $f_* f^* \mathcal{O}_Z$ has length d at the generic point of Z. Hence

$$f_* f^*[Z] = f_* f^*[\mathcal{O}_Z]_k = [f_* f^* \mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 82.10.5 and 82.8.3. □

82.12. Preparation for principal divisors

0EPF This section is the analogue of Chow Homology, Section 42.16. Some of the material in this section partially overlaps with the discussion in Spaces over Fields, Section 72.6.

0EPZ **Lemma 82.12.1.** *In Situation 82.2.1 let X/B be good. Assume X is integral.*

(1) *If $Z \subset X$ is an integral closed subspace, then the following are equivalent:*
   (a) *Z is a prime divisor,*
   (b) *$|Z|$ has codimension 1 in $|X|$, and*
   (c) *$\dim_\delta(Z) = \dim_\delta(X) - 1$.*
(2) *If Z is an irreducible component of an effective Cartier divisor on X, then $\dim_\delta(Z) = \dim_\delta(X) - 1$.*

Proof. Part (1) follows from the definition of a prime divisor (Spaces over Fields, Definition 72.6.2), Decent Spaces, Lemma 68.20.2, and the definition of a dimension function (Topology, Definition 5.20.1).

Let $D \subset X$ be an effective Cartier divisor. Let $Z \subset D$ be an irreducible component and let $\xi \in |Z|$ be the generic point. Choose an étale neighbourhood $(U, u) \to (X, \xi)$ where $U = \mathrm{Spec}(A)$ and $D \times_X U$ is cut out by a nonzerodivisor $f \in A$, see Divisors on Spaces, Lemma 71.6.2. Then u is a generic point of $V(f)$ by Decent Spaces, Lemma 68.20.1. Hence $\mathcal{O}_{U,u}$ has dimension 1 by Krull's Hauptidealsatz (Algebra, Lemma 10.60.11). Thus ξ is a codimension 1 point on X and Z is a prime divisor as desired. □

82.13. Principal divisors

0EQ0 This section is the analogue of Chow Homology, Section 42.17. The following definition is the analogue of Spaces over Fields, Definition 72.6.7 in our current setup.

0EQ1 **Definition 82.13.1.** In Situation 82.2.1 let X/B be good. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$-cycle

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

defined in Spaces over Fields, Definition 72.6.7. This makes sense because prime divisors have δ-dimension $n-1$ by Lemma 82.12.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\mathrm{div}_X(fg) = \mathrm{div}_X(f) + \mathrm{div}_X(g)$$

in $Z_{n-1}(X)$. See Spaces over Fields, Lemma 72.6.8. The following lemma will allow us to reduce statements about principal divisors to the case of schemes.

0EQ2 **Lemma 82.13.2.** *In Situation 82.2.1 let $f : X \to Y$ be an étale morphism of good algebraic spaces over B. Assume Y is integral. Let $g \in R(Y)^*$. As cycles on X we have*

$$f^*(div_Y(g)) = \sum\nolimits_{X'} (X' \to X)_* div_{X'}(g \circ f|_{X'})$$

where the sum is over the irreducible components of X (Remark 82.5.1).

Proof. The map $|X| \to |Y|$ is open. The set of irreducible components of $|X|$ is locally finite in $|X|$. We conclude that $f|_{X'} : X' \to Y$ is dominant for every irreducible component $X' \subset X$. Thus $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 67.47), hence $\mathrm{div}_{X'}(g \circ f|_{X'})$ is defined. Moreover, the sum is locally finite and we find that the right hand side indeed is a cycle on X. The left hand side is defined by Definition 82.10.1 and the fact that an étale morphism is flat of relative dimension 0.

Since f is étale we see that $\delta_X(x) = \delta_y(f(x))$ for all $x \in |X|$. Thus if $\dim_\delta(Y) = n$, then $\dim_\delta(X') = n$ for every irreducible component X' of X (since generic points of X map to the generic point of Y, see above). Thus both left and right hand side are $(n-1)$-cycles.

Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = n - 1$. To prove the equality, we need to show that the coefficients of Z are the same. Let $Z' \subset Y$ be the integral closed subspace constructed in Lemma 82.7.1. Then $\dim_\delta(Z') = n - 1$ too. Let $\xi \in |Z|$ be the generic point. Then $\xi' = f(\xi) \in |Z'|$ is the generic point. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}^h_{X,\xi}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}^h_{Y,\xi'}) & \longrightarrow & Y \end{array}$$

of Decent Spaces, Remark 68.11.11. We have to be slightly careful as the reduced Noetherian local rings $\mathcal{O}^h_{X,\xi}$ and $\mathcal{O}^h_{Y,\xi'}$ need not be domains. Thus we work with total rings of fractions $Q(-)$ rather than fraction fields. By definition, to get the coefficient of Z' in $\mathrm{div}_Y(g)$ we write the image of g in $Q(\mathcal{O}^h_{Y,\xi'})$ as a/b with $a, b \in \mathcal{O}^h_{Y,\xi'}$ nonzerodivisors and we take

$$\mathrm{ord}_{Z'}(g) = \mathrm{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) - \mathrm{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/b\mathcal{O}^h_{Y,\xi'})$$

Observe that the coefficient of Z in $f^*\mathrm{div}_Y(G)$ is the same integer, see Lemma 82.10.3. Suppose that $\xi \in X'$. Then we can consider the maps

$$\mathcal{O}^h_{Y,\xi'} \to \mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{X',\xi}$$

The first arrow is flat and the second arrow is a surjective map of reduced local Noetherian rings of dimension 1. Therefore both these maps send nonzerodivisors to nonzerodivisors and we conclude the coefficient of Z' in $\mathrm{div}_{X'}(g \circ f|_{X'})$ is

$$\mathrm{ord}_Z(g \circ f|_{X'}) = \mathrm{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi}) - \mathrm{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{X',\xi}/b\mathcal{O}^h_{X',\xi})$$

by the same prescription as above. Thus it suffices to show

$$\mathrm{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) = \sum\nolimits_{\xi \in |X'|} \mathrm{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi})$$

First, since the ring map $\mathcal{O}^h_{Y,\xi'} \to \mathcal{O}^h_{X,\xi}$ is flat and unramified, we have

$$\mathrm{length}_{\mathcal{O}^h_{Y,\xi'}}(\mathcal{O}^h_{Y,\xi'}/a\mathcal{O}^h_{Y,\xi'}) = \mathrm{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi})$$

by Algebra, Lemma 10.52.13. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the nonmaximal primes of $\mathcal{O}^h_{X,\xi}$ and set $R_j = \mathcal{O}^h_{X,\xi}/\mathfrak{q}_j$. For X' as above, denote $J(X') \subset \{1, \ldots, t\}$ the set of indices such that $\mathfrak{q}_j$ corresponds to a point of X', i.e., such that under the surjection

$\mathcal{O}^h_{X,\xi} \to \mathcal{O}_{X',\xi}$ the prime $\mathfrak{q}_j$ corresponds to a prime of $\mathcal{O}_{X',\xi}$. By Chow Homology, Lemma 42.3.2 we get

$$\text{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/a\mathcal{O}^h_{X,\xi}) = \sum\nolimits_j \text{length}_{R_j}(R_j/aR_j)$$

and

$$\text{length}_{\mathcal{O}^h_{X',\xi}}(\mathcal{O}^h_{X',\xi}/a\mathcal{O}^h_{X',\xi}) = \sum\nolimits_{j\in J(X')} \text{length}_{R_j}(R_j/aR_j)$$

Thus the result of the lemma holds because $\{1,\ldots,t\}$ is the disjoint union of the sets $J(X')$: each point of codimension 0 on X lies on a unique X'. □

82.14. Principal divisors and pushforward

0EQ3 This section is the analogue of Chow Homology, Section 42.18.

0EQ4 **Lemma 82.14.1.** *In Situation 82.2.1 let $X, Y/B$ be good. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_* div(f) = div(g)$.

Proof. We are going to deduce this from the case of schemes by étale localization. Let $Z \subset Y$ be an integral closed subspace of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. Apply Spaces over Fields, Lemma 72.3.2 to the morphism $p : X \to Y$ and the generic point $\xi \in |Z|$. We find that we may replace Y by an open subspace containing ξ and assume that $p : X \to Y$ is finite. Pick an étale neighbourhood $(V, v) \to (Y, \xi)$ where V is an affine scheme. By Lemma 82.10.3 it suffices to prove the equality of cycles after pulling back to V. Set $U = V \times_Y X$ and consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{a} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{b} & Y \end{array}$$

Let $V_j \subset V$, $j = 1,\ldots,m$ be the irreducible components of V. For each i, let $U_{j,i}$, $i = 1,\ldots,n_j$ be the irreducible components of U dominating V_j. Denote $p'_{j,i} : U_{j,i} \to V_j$ the restriction of $p' : U \to V$. By the case of schemes (Chow Homology, Lemma 42.18.1) we see that

$$p'_{j,i,*}\text{div}_{U_{j,i}}(f_{j,i}) = \text{div}_{V_j}(g_{j,i})$$

where $f_{j,i}$ is the restriction of f to $U_{j,i}$ and $g_{j,i}$ is the norm of $f_{j,i}$ along the finite extension $R(U_{j,i})/R(V_j)$. We have

$$\begin{aligned} b^*p_*\text{div}_X(f) &= p'_*a^*\text{div}_X(f) \\ &= p'_*\left(\sum\nolimits_{j,i}(U_{j,i} \to U)_*\text{div}_{U_{j,i}}(f_{j,i})\right) \\ &= \sum\nolimits_{j,i}(V_j \to V)_*p'_{j,i,*}\text{div}_{U_{j,i}}(f_{j,i}) \\ &= \sum\nolimits_j(V_j \to V)_*\left(\sum\nolimits_i \text{div}_{V_j}(g_{j,i})\right) \\ &= \sum\nolimits_j(V_j \to V)_*\text{div}_{V_j}(\prod\nolimits_i g_{j,i}) \end{aligned}$$

by Lemmas 82.11.1, 82.13.2, and 82.8.2. To finish the proof, using Lemma 82.13.2 again, it suffices to show that

$$g \circ b|_{V_j} = \prod\nolimits_i g_{j,i}$$

as elements of the function field of V_j. In terms of fields this is the following statement: let L/K be a finite extension. Let M/K be a finite separable extension. Write $M \otimes_K L = \prod M_i$. Then for $t \in L$ with images $t_i \in M_i$ the image of $\mathrm{Norm}_{L/K}(t)$ in M is $\prod \mathrm{Norm}_{M_i/M}(t_i)$. We omit the proof. □

82.15. Rational equivalence

0EQ5 This section is the analogue of Chow Homology, Section 42.19. In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena (see examples in the chapter on schemes). However, if we do not allow these then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

0EQ6 **Definition 82.15.1.** In Situation 82.2.1 let X/B be good. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subspaces with $\dim_\delta(W_j) = k+1$, and any $f_j \in R(W_j)^*$ we may consider

$$\sum (i_j)_* \mathrm{div}(f_j) \in Z_k(X)$$

where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.

(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.

(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.

(4) We define

$$\mathrm{CH}_k(X) = Z_k(X)/\sim_{rat}$$

to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles modulo rational equivalence on X*.

There are many other interesting equivalence relations. Rational equivalence is the coarsest of them all. A very simple but important lemma is the following.

0EQ7 **Lemma 82.15.2.** *In Situation 82.2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \to X$ the inclusion morphism. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*

$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j\in J}$ be a locally finite collection of integral closed subspaces of U of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha-\beta)|_U = \sum (i_j)_*\mathrm{div}(f_j)$ as in the definition. Let $W'_j \subset X$ be the corresponding integral closed subspace of X, i.e., having the same generic point as W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W'_j \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W'_j\}$ is also locally finite. Since $R(W_j) = R(W'_j)$ we see that

$$\alpha - \beta - \sum (i'_j)_*\mathrm{div}(f_j)$$

is a cycle on X whose restriction to U is zero. The lemma follows by applying Lemma 82.10.2. □

0EQ8 **Remark 82.15.3.** In Situation 82.2.1 let X/B be good. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j\in J_i}$ but the union $\{W_{i,j}\}_{i\in I, j\in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i\in I}$ of $|X|$ such that α_i, β_i are supported on T_i and such that $\alpha_i \sim_{rat} \beta_i$ "on" T_i. More precisely, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j\in J_i}$ consist of integral closed subspaces $W_{i,j}$ with $|W_{i,j}| \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i\in I, j\in J_i}$ is automatically locally finite in this case.

82.16. Rational equivalence and push and pull

0EQ9 This section is the analogue of Chow Homology, Section 42.20. In this section we show that flat pullback and proper pushforward commute with rational equivalence.

0EQA **Lemma 82.16.1.** *In Situation 82.2.1 let $X, Y/B$ be good. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then for $g \in R(Y)^*$ we have*

$$f^* div_Y(g) = \sum m_{X',X}(X' \to X)_* div_{X'}(g \circ f|_{X'})$$

as $(k+r-1)$-cycles on X where the sum is over the irreducible components X' of X and $m_{X',X}$ is the multiplicity of X' in X.

Proof. Observe that any irreducible component of X dominates Y (Lemma 82.9.1) and hence the composition $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 67.47). We will reduce this to the case of schemes. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Picture

$$\begin{array}{ccc} U & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{b} & Y \end{array}$$

Since a is surjective and étale it follows from Lemma 82.10.3 that it suffices to prove the equality of cycles after pulling back by a. We can use Lemma 82.13.2 to write

$$b^*\mathrm{div}_Y(g) = \sum (V' \to V)_*\mathrm{div}_{V'}(g \circ b|_{V'})$$

where the sum is over the irreducible components V' of V. Using Lemma 82.11.1 we find

$$h^*b^*\mathrm{div}_Y(g) = \sum (V' \times_V U \to U)_*(h')^*\mathrm{div}_{V'}(g \circ b|_{V'})$$

where $h' : V' \times_V U \to V'$ is the projection. We may apply the lemma in the case of schemes (Chow Homology, Lemma 42.20.1) to the morphism $h' : V' \times_V U \to V'$ to see that we have

$$(h')^*\mathrm{div}_{V'}(g \circ b|_{V'}) = \sum m_{U',V' \times_V U}(U' \to V' \times_V U)_*\mathrm{div}_{U'}(g \circ b|_{V'} \circ h'|_{U'})$$

where the sum is over the irreducible components U' of $V' \times_V U$. Each U' occurring in this sum is an irreducible component of U and conversely every irreducible component U' of U is an irreducible component of $V' \times_V U$ for a unique irreducible component $V' \subset V$. Given an irreducible component $U' \subset U$, denote $\overline{a(U')} \subset X$ the "image" in X (Lemma 82.7.1); this is an irreducible component of X for example by Lemma 82.9.1. The muplticity $m_{U',V' \times_V U}$ is equal to the multiplicity $m_{\overline{a(U')},X}$. This follows from the equality $h^*a^*[Y] = b^*f^*[Y]$ (Lemma 82.10.4), the definitions, and Lemma 82.10.3. Combining all of what we just said we obtain

$$a^*f^*\mathrm{div}_Y(g) = h^*b^*\mathrm{div}_Y(g) = \sum m_{\overline{a(U')},X}(U' \to U)_*\mathrm{div}_{U'}(g \circ (f \circ a)|_{U'})$$

Next, we analyze what happens with the right hand side of the formula in the statement of the lemma if we pullback by a. First, we use Lemma 82.11.1 to get

$$a^* \sum m_{X',X}(X' \to X)_*\mathrm{div}_{X'}(g \circ f|_{X'}) = \sum m_{X',X}(X' \times_X U \to U)_*(a')^*\mathrm{div}_{X'}(g \circ f|_{X'})$$

where $a' : X' \times_X U \to X'$ is the projection. By Lemma 82.13.2 we get

$$(a')^*\mathrm{div}_{X'}(g \circ f|_{X'}) = \sum (U' \to X' \times_X U)_*\mathrm{div}_{U'}(g \circ (f \circ a)|_{U'})$$

where the sum is over the irreducible components U' of $X' \times_X U$. These U' are irreducible components of U and in fact are exactly the irreducible components of U such that $\overline{a(U')} = X'$. Comparing with what we obtained above we conclude. □

0EQB **Lemma 82.16.2.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k+r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{|i_j|(|W_j|)\}_{j \in J}$ is locally finite in $|Y|$. Then we have to show that

$$f^*(\sum i_{j,*}\mathrm{div}(g_j)) = \sum f^*i_{j,*}\mathrm{div}(g_j)$$

is rationally equivalent to zero on X. The sum on the right makes sense by Lemma 82.9.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \to W_j$ the first projection. By Lemma 82.11.1 we can write the sum above as

$$\sum i'_{j,*}(f_j^* \mathrm{div}(g_j))$$

By Lemma 82.16.1 we see that each $f_j^*\mathrm{div}(g_j)$ is rationally equivalent to zero on W'_j. Hence each $i'_{j,*}(f_j^*\mathrm{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 82.15.3. □

0EQC **Lemma 82.16.3.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j\in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\mathrm{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\mathrm{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subspace which is the image of $p \circ i_j$, see Lemma 82.7.1. The collection $\{W'_j\}$ is locally finite in Y by Lemma 82.7.5. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\mathrm{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\mathrm{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subspace $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ W' & \xrightarrow{i'} & Y \end{array}$$

Note that $p_* i_* \mathrm{div}(f) = i'_*(p')_*\mathrm{div}(f)$ by Lemma 82.8.2. As explained above we have to show that $(p')_*\mathrm{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\mathrm{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\mathrm{div}(f) = 0$ in this case. Since $(p')_*\mathrm{div}(f)$ is a k-cycle, we see that $(p')_*\mathrm{div}(f) = n[W']$ for some $n \in \mathbf{Z}$. In order to prove that $n = 0$ we may replace W' by a nonempty open subspace. In particular,

we may and do assume that W' is a scheme. Let $\eta \in W'$ be the generic point. Let $K = \kappa(\eta) = R(W')$ be the function field. Consider the base change diagram

$$\begin{array}{ccc} W_\eta & \longrightarrow & W \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathrm{Spec}(K) & \xrightarrow{\eta} & W' \end{array}$$

Observe that c is proper. Also $|W_\eta|$ has dimension 1: use Decent Spaces, Lemma 68.18.6 to identify $|W_\eta|$ as the subspace of $|W|$ of points mapping to η and note that since $\dim_\delta(W) = k+1$ and $\delta(\eta) = k$ points of W_η must have δ-value either k or $k+1$. Thus the local rings have dimension ≤ 1 by Decent Spaces, Lemma 68.20.2. By Spaces over Fields, Lemma 72.9.3 we find that W_η is a scheme. Since $\mathrm{Spec}(K)$ is the limit of the nonempty affine open subschemes of W' we conclude that we may assume that W is a scheme by Limits of Spaces, Lemma 70.5.11. Then finally by the case of schemes (Chow Homology, Lemma 42.20.3) we find that $n = 0$.

The case $\dim_\delta(W') = k+1$. In this case Lemma 82.14.1 applies, and we see that indeed $p'_*\mathrm{div}(f) = \mathrm{div}(g)$ for some $g \in R(W')^*$ as desired. □

82.17. The divisor associated to an invertible sheaf

0EQD This section is the analogue of Chow Homology, Section 42.24. The following definition is the analogue of Spaces over Fields, Definition 72.7.4 in our current setup.

0EQE **Definition 82.17.1.** In Situation 82.2.1 let X/B be good. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to* s is the $(n-1)$-cycle

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z]$$

defined in Spaces over Fields, Definition 72.7.4. This makes sense because Weil divisors have δ-dimension $n-1$ by Lemma 82.12.1.

(2) We define *Weil divisor associated to* $\mathcal{L}$ as

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_{\mathcal{L}}(s) \in \mathrm{CH}_{n-1}(X)$$

where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Spaces over Fields, Lemma 72.7.3.

The zero scheme of a nonzero section is an effective Cartier divisor whose Weil divisor class computes the Weil divisor associated to the invertible module.

0EQF **Lemma 82.17.2.** *In Situation 82.2.1 let X/B be good. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*

$$div_{\mathcal{L}}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $\mathrm{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ-dimension $n-1$. Let $\xi \in |Z|$ be its generic point. To prove the first equality we compare the coefficients of Z on both sides. Choose an elementary étale neighbourhood $(U, u) \to (X, \xi)$, see Decent Spaces, Section 68.11 and recall that $\mathcal{O}^h_{X,\xi} = \mathcal{O}^h_{U,u}$ in this case. After replacing U by an open neighbourhood of u we may assume there is a trivializing section s_U of $\mathcal{L}|_U$. Write $s|_U = f s_U$ for some $f \in \Gamma(U, \mathcal{O}_U)$. Then $Z \times_X U$ is equal to $V(f)$ as a closed subscheme of U, see Divisors on Spaces, Definition 71.7.6. As in Spaces over Fields, Section 72.7 denote $\mathcal{L}_\xi$ the pullback of $\mathcal{L}$ under the canonical morphism $c_\xi : \mathrm{Spec}(\mathcal{O}^h_{X,\xi}) \to X$. Denote s_ξ the pullback of s_U; it is a trivialization of $\mathcal{L}_\xi$. Then we see that $c_\xi^*(s) = f s_\xi$. The coefficient of Z in $[Z(s)]_{n-1}$ is by definition

$$\mathrm{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/f\mathcal{O}_{U,u})$$

Since $\mathcal{O}_{U,u} \to \mathcal{O}^h_{X,\xi}$ is flat and identifies residue fields this is equal to

$$\mathrm{length}_{\mathcal{O}^h_{X,\xi}}(\mathcal{O}^h_{X,\xi}/f\mathcal{O}^h_{X,\xi})$$

by Algebra, Lemma 10.52.13. This final quantity is equal to $\mathrm{ord}_{Z,\mathcal{L}}(s)$ by Spaces over Fields, Definition 72.7.1, i.e., to the coefficient of Z in $\mathrm{div}_{\mathcal{L}}(s)$ as desired. □

0EQG **Lemma 82.17.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The morphism*

$$q : T = \underline{\mathrm{Spec}}\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right) \longrightarrow X$$

has the following properties:

1. *q is surjective, smooth, affine, of relative dimension 1,*
2. *there is an isomorphism $\alpha : q^*\mathcal{L} \cong \mathcal{O}_T$,*
3. *formation of $(q : T \to X, \alpha)$ commutes with base change,*
4. *$q^* : Z_k(X) \to Z_{k+1}(T)$ is injective,*
5. *if $Z \subset X$ is an integral closed subspace, then $q^{-1}(Z) \subset T$ is an integral closed subspace,*
6. *if $Z \subset X$ is a closed subspace of X of δ-dimension $\leq k$, then $q^{-1}(Z)$ is a closed subspace of T of δ-dimension $\leq k+1$ and $q^*[Z]_k = [q^{-1}(Z)]_{k+1}$,*
7. *if $\xi' \in |T|$ is the generic point of the fibre of $|T| \to |X|$ over ξ, then the ring map $\mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{T,\xi'}$ is flat, we have $\mathfrak{m}^h_{\xi'} = \mathfrak{m}^h_\xi \mathcal{O}^h_{T,\xi'}$, and the residue field extension is purely transcendental of transcendence degree 1, and*
8. *add more here as needed.*

Proof. Let $U \to X$ be an étale morphism such that $\mathcal{L}|_U$ is trivial. Then $T \times_X U \to U$ is isomorphic to the projection morphism $\mathbf{G}_m \times U \to U$, where $\mathbf{G}_m$ is the multipliciative group scheme, see Groupoids, Example 39.5.1. Thus (1) is clear.

To see (2) observe that $q_*q^*\mathcal{L} = \bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n+1}$. Thus there is an obvious isomorphism $q_*q^*\mathcal{L} \to q_*\mathcal{O}_T$ of $q_*\mathcal{O}_T$-modules. By Morphisms of Spaces, Lemma 67.20.10 this determines an isomorphism $q^*\mathcal{L} \to \mathcal{O}_T$.

Part (3) holds because forming the relative spectrum commutes with arbitrary base change and the same thing is clearly true for the isomorphism α.

Part (4) follows immediately from (1) and the definitions.

Part (5) follows from the fact that if Z is an integral algebraic space, then $\mathbf{G}_m \times Z$ is an integral algebraic space.

Part (6) follows from the fact that lengths are preserved: if $(A, \mathfrak{m})$ is a local ring and $B = A[x]_{\mathfrak{m}A[x]}$ and if M is an A-module, then $\text{length}_A(M) = \text{length}_B(M \otimes_A B)$. This implies that if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and $\xi \in |X|$ with $\xi' \in |T|$ the generic point of the fibre over ξ, then the length of $\mathcal{F}$ at ξ is the same as the length of $q^*\mathcal{F}$ at ξ'. Tracing through the definitions this gives (6) and more.

The map in part (7) comes from Decent Spaces, Remark 68.11.11. However, in our case we have

$$\text{Spec}(\mathcal{O}^h_{X,\xi}) \times_X T = \mathbf{G}_m \times \text{Spec}(\mathcal{O}^h_{X,\xi}) = \text{Spec}(\mathcal{O}^h_{X,\xi}[t, t^{-1}])$$

and ξ' corresponds to the generic point of the special fibre of this over $\text{Spec}(\mathcal{O}^h_{X,\xi})$. Thus $\mathcal{O}^h_{T,\xi'}$ is the henselization of the localization of $\mathcal{O}^h_{X,\xi}[t, t^{-1}]$ at the corresponding prime. Part (7) follows from this and some commutative algebra; details omitted. □

0EQH **Lemma 82.17.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume X is integral. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $q : T \to X$ be the morphism of Lemma 82.17.3. Then*

$$q^* div_{\mathcal{L}}(s) = div_T(q^*(s))$$

where we view the pullback $q^*(s)$ as a nonzero meromorphic function on T using the isomorphism $q^*\mathcal{L} \to \mathcal{O}_T$

Proof. Observe that $\text{div}_T(q^*(s)) = \text{div}_{\mathcal{O}_T}(q^*(s))$ by the compatibility between the constructions given in Spaces over Fields, Sections 72.6 and 72.7. We will show the agreement with $\text{div}_{\mathcal{O}_T}(q^*(s))$ in this proof. We will use all the properties of $q : T \to X$ stated in Lemma 82.17.3 without further mention. Let $Z \subset T$ be a prime divisor. Then either $Z \to X$ is dominant or $Z = q^{-1}(Z')$ for some prime divisor $Z' \subset X$. If $Z \to X$ is dominant, then the coefficient of Z in either side of the equality of the lemma is zero. Thus we may assume $Z = q^{-1}(Z')$ where $Z' \subset X$ is a prime divisor. Let $\xi' \in |Z'|$ and $\xi \in |Z|$ be the generic points. Then we obtain a commutative diagram

$$\begin{array}{ccc} \text{Spec}(\mathcal{O}^h_{T,\xi}) & \xrightarrow{c_\xi} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle q} \\ \text{Spec}(\mathcal{O}^h_{X,\xi'}) & \xrightarrow{c_{\xi'}} & X \end{array}$$

see Decent Spaces, Remark 68.11.11. Choose a trivialization $s_{\xi'}$ of $\mathcal{L}_{\xi'} = c^*_{\xi'}\mathcal{L}$. Then we can use the pullback s_ξ of $s_{\xi'}$ via h as our trivialization of $\mathcal{L}_\xi = c^*_\xi q^*\mathcal{L}$. Write $s/s_{\xi'} = a/b$ for $a, b \in \mathcal{O}_{X,\xi'}$ nonzerodivisors. By definition the coefficient of Z' in $\text{div}_{\mathcal{L}}(s)$ is

$$\text{length}_{\mathcal{O}^h_{X,\xi'}}(\mathcal{O}^h_{X,\xi'}/a\mathcal{O}^h_{X,\xi'}) - \text{length}_{\mathcal{O}^h_{X,\xi'}}(\mathcal{O}^h_{X,\xi'}/b\mathcal{O}^h_{X,\xi'})$$

Since $Z = q^{-1}(Z')$, this is also the coefficient of Z in $q^*\text{div}_{\mathcal{L}}(s)$. Since $\mathcal{O}^h_{X,\xi'} \to \mathcal{O}^h_{T,\xi}$ is flat the elements a, b map to nonzerodivisors in $\mathcal{O}^h_{T,\xi}$. Thus $q^*(s)/s_\xi = a/b$ in the total quotient ring of $\mathcal{O}^h_{T,\xi}$. By definition the coefficient of Z in $\text{div}_T(q^*(s))$ is

$$\text{length}_{\mathcal{O}^h_{T,\xi}}(\mathcal{O}^h_{T,\xi}/a\mathcal{O}^h_{T,\xi}) - \text{length}_{\mathcal{O}^h_{T,\xi}}(\mathcal{O}^h_{T,\xi}/b\mathcal{O}^h_{T,\xi})$$

The proof is finished because these lengths are the same as before by Algebra, Lemma 10.52.13 and the fact that $\mathfrak{m}^h_\xi = \mathfrak{m}^h_{\xi'}\mathcal{O}^h_{T,\xi}$ shown in Lemma 82.17.3. □

82.18. Intersecting with an invertible sheaf

0EQI This section is the analogue of Chow Homology, Section 42.25. In this section we study the following construction.

0EQJ **Definition 82.18.1.** In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to \mathrm{CH}_k(X)$$

called *intersection with the first Chern class of* $\mathcal{L}$.

(1) Given an integral closed subspace $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 82.17.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subspaces of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subspaces on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subspaces of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, often more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

0EQK **Lemma 82.18.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Spaces over Fields, Lemma 72.7.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subspace $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. □

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on an algebraic space X, see Divisors on Spaces, Definition 71.7.6.

0EQL **Lemma 82.18.3.** *In Situation 82.2.1 let Y/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $s \in \Gamma(Y, \mathcal{L})$ be a regular section and assume $\dim_\delta(Y) \leq k+1$. Write $[Y]_{k+1} = \sum n_i[Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then*

0EQM (82.18.3.1) $$[Z(s)]_k = \sum n_i [Z(s_i)]_k$$

as k-cycles on Y.

Proof. Let $\varphi : V \to Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the equality after pulling back by φ, see Lemma 82.10.3. That same lemma tells us that $\varphi^*[Y_i] = [\varphi^{-1}(Y_i)] = \sum [V_{i,j}]$ where $V_{i,j}$ are the irreducible components of V lying over Y_i. Hence if we first apply the case of schemes (Chow Homology, Lemma 42.25.3) to $\varphi^* s_i$ on $Y_i \times_Y V$ we find that $\varphi^*[Z(s_i)]_k = [Z(\varphi^* s_i)] = \sum [Z(s_{i,j})]_k$ where $s_{i,j}$ is the pullback of s to $V_{i,j}$. Applying the case of schemes to $\varphi^* s$ we get

$$\varphi^*[Z(s)]_k = [Z(\varphi^* s)]_k = \sum n_i [Z(s_{i,j})]_k$$

by our remark on multiplicities above. Combining all of the above the proof is complete. □

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 31.14.8.

0EQN **Lemma 82.18.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y \subset X$ be a closed subscheme with $\dim_\delta(Y) \leq k+1$ and let $s \in \Gamma(Y, \mathcal{L}|_Y)$ be a regular section. Then*

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\mathrm{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 82.17.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 82.18.3. □

82.19. Intersecting with an invertible sheaf and push and pull

0EQP This section is the analogue of Chow Homology, Section 42.26. In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

0EQQ **Lemma 82.19.1.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of $\mathcal{L}$. Then we have*

$$f^* \mathit{div}_{\mathcal{L}}(s) = \sum n_i \mathit{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ-dimension $n+r$, the section $s_i = f|^*_{X_i}(s)$ is the pullback of s, and $n_i = m_{X_i,X}$ is the multiplicity of X_i in X.

Proof. Using sleight of hand we will deduce this from Lemma 82.16.1. (An alternative is to redo the proof of that lemma in the setting of meromorphic sections of invertible modules.) Namely, let $q : T \to Y$ be the morphism of Lemma 82.17.3 constructed using $\mathcal{L}$ on Y. We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{array}{ccc} T' & \xrightarrow{q'} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{q} & Y \end{array}$$

Then $q' : T' \to X$ is the morphism constructed using $f^*\mathcal{L}$ on X. Then it suffices to prove

$$(q')^* f^* \mathrm{div}_{\mathcal{L}}(s) = \sum n_i (q')^* \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

Observe that $T'_i = q^{-1}(X_i)$ are the irreducible components of T' and that n_i is the multiplicity of T'_i in T'. The left hand side is equal to

$$h^* q^* \mathrm{div}_{\mathcal{L}}(s) = h^* \mathrm{div}_T(q^*(s))$$

by Lemma 82.17.4 (and Lemma 82.10.4). On the other hand, denoting $q'_i : T'_i \to X_i$ the restriction of q' we find that Lemma 82.17.4 also tells us the right hand side is equal to

$$\sum n_i \mathrm{div}_{T_i}((q'_i)^*(s_i))$$

In these two formulas the expressions $q^*(s)$ and $(q'_i)^*(s_i)$ represent the rational functions corresponding to the pulled back meromorphic sections of $q^*\mathcal{L}$ and $(q'_i)^* f^* \mathcal{L}|_{X_i}$ via the isomorphism $\alpha : q^*\mathcal{L} \to \mathcal{O}_T$ and its pullbacks to spaces over T. With this convention it is clear that $(q'_i)^*(s_i)$ is the composition of the rational function $q^*(s)$ on T and the morphism $h|_{T'_i} : T'_i \to T$. Thus Lemma 82.16.1 exactly says that

$$h^* \mathrm{div}_T(q^*(s)) = \sum n_i \mathrm{div}_{T_i}((q'_i)^*(s_i))$$

as desired. □

0EQR **Lemma 82.19.2.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $\mathrm{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i [W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\mathrm{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subspace $f^{-1}(W_i)$ of X. By the discussion in Remark 82.15.3 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subspace of δ-dimension k. Consider the closed subspace $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{array}{ccc} W' & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & Y \end{array}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ-dimension $k+r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i[W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. By Lemma 82.19.1 we have the following equality of cycles

$$h^*\mathrm{div}_{\mathcal{L}|_W}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\mathrm{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\mathrm{CH}_{k+r-1}(X)$. □

0EQS **Lemma 82.19.3.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Then*

$$f_*\left(div_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)] div_{\mathcal{L}}(s).$$

as cycles on Y. In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. Proof of the first equaltion. Let $q : T \to Y$ be the morphism of Lemma 82.17.3 constructed using $\mathcal{L}$ on Y. We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{array}{ccc} T' & \xrightarrow{q'} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{q} & Y \end{array}$$

Then $q' : T' \to X$ is the morphism constructed using $f^*\mathcal{L}$ on X. It suffices to prove the equality after pulling back to T'. The left hand side pulls back to

$$\begin{aligned} q^*f_*\left(\mathrm{div}_{f^*\mathcal{L}}(f^*s)\right) &= h_*(q')^*\mathrm{div}_{f^*\mathcal{L}}(f^*s) \\ &= h_*\mathrm{div}_{(q')^*f^*\mathcal{L}}((q')^*f^*s) \\ &= h_*\mathrm{div}_{h^*q^*\mathcal{L}}(h^*q^*s) \end{aligned}$$

The first equality by Lemma 82.11.1. The second by Lemma 82.19.1 using that T' is integral. The third because the displayed diagram commutes. The right hand side pulls back to

$$[R(X) : R(Y)]q^*\mathrm{div}_{\mathcal{L}}(s) = [R(T') : R(T)]\mathrm{div}_{q^*\mathcal{L}}(q^*s)$$

This follows from Lemma 82.19.1, the fact that T is integral, and the equality $[R(T') : R(T)] = [R(X) : R(Y)]$ whose proof we omit (it follows from Lemma 82.11.1 but that would be a silly way to prove the equality). Thus it suffices to prove the lemma for $h : T' \to T$, the invertible module $q^{\mathcal{L}}$ and the section q^*s. Since $q^*\mathcal{L} = \mathcal{O}_T$ we reduce to the case where $\mathcal{L} \cong \mathcal{O}$ discussed in the next paragraph.

Assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$. Using the embedding $R(Y) \subset R(X)$ we may think of g as a rational on X and we

are simply trying to prove

$$f_*(\mathrm{div}_X(g)) = [R(X) : R(Y)]\mathrm{div}_Y(g).$$

Comparing with the result of Lemma 82.14.1 we see this true since $\mathrm{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

0EQT **Lemma 82.19.4.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\mathrm{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subspace $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the "image" of W_i as in Lemma 82.7.1. Consider the diagram

$$\begin{array}{ccc} X' = \coprod W_i & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' = \coprod W_i' & \xrightarrow{q'} & Y. \end{array}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 82.19.3. □

82.20. The key formula

0EQU This section is the analogue of Chow Homology, Section 42.27. We strongly urge the reader to read the proof in that case first.

In Situation 82.2.1 let X/B be good. Assume X is integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s be a nonzero meromorphic section of $\mathcal{L}$ and let t be a nonzero meromorphic section of $\mathcal{N}$. Let $Z \subset X$ be a prime divisor with generic point $\xi \in |Z|$. Consider the morphism

$$c_\xi : \mathrm{Spec}(\mathcal{O}^h_{X,\xi}) \longrightarrow X$$

used in Spaces over Fields, Section 72.7. We denote $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$ the pullbacks of $\mathcal{L}$ and $\mathcal{N}$ by c_ξ; we often think of $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$ as the rank 1 free $\mathcal{O}^h_{X,\xi}$-modules they give rise to. Note that the pullback of s, resp. t is a regular meromorphic section of $\mathcal{L}_\xi$, resp. $\mathcal{N}_\xi$.

Let $Z_i \subset X$, $i \in I$ be a locally finite set of prime divisors with the following property: If $Z \notin \{Z_i\}$, then s is a generator for $\mathcal{L}_\xi$ and t is a generator for $\mathcal{N}_\xi$. Such a set exists by Spaces over Fields, Lemma 72.7.2. Then

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z_i,\mathcal{L}}(s)[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i. Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

with f_i, g_i invertible elements of the total ring of fractions $Q(\mathcal{O}^h_{X,\xi_i})$. We abbreviate $B_i = \mathcal{O}^h_{X,\xi_i}$. Let us denote

$$\mathrm{ord}_{B_i} : Q(B_i)^* \longrightarrow \mathbf{Z}, \quad a/b \longmapsto \mathrm{length}_{B_i}(B_i/aB_i) - \mathrm{length}_{B_i}(B_i/bB_i)$$

In other words, we temporarily extend Algebra, Definition 10.121.2 to these reduced Noetherian local rings of dimension 1. Then by definition

$$\mathrm{ord}_{Z_i,\mathcal{L}}(s) = \mathrm{ord}_{B_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$$

Since ξ_i is the generic point of Z_i we see that the residue field $\kappa(\xi_i)$ is the function field of Z_i. Moreover $\kappa(\xi_i)$ is the residue field of B_i, see Decent Spaces, Lemma 68.11.10. Since t_i is a generator of $\mathcal{N}_{\xi_i}$ we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes_{B_i} \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{B_i}(f_i)(Z_i \to X)_* \mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i)(Z_i \to X)_* \mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\mathrm{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^* = R(Z_i)^*$$

see Chow Homology, Section 42.5.

0EQV **Lemma 82.20.1** (Key formula)**.** *In the situation above the cycle*

$$\sum (Z_i \to X)_* \left(ord_{B_i}(f_i) div_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - ord_{B_i}(g_i) div_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right)$$

is equal to the cycle

$$\sum (Z_i \to X)_* div(\partial_{B_i}(f_i, g_i))$$

Proof. The strategy of the proof will be: first reduce to the case where $\mathcal{L}$ and $\mathcal{N}$ are trivial invertible modules, then change our choices of local trivializations, and then finally use étale localization to reduce to the case of schemes[1].

First step. Let $q : T \to X$ be the morphism constructed in Lemma 82.17.3. We will use all properties stated in that lemma without further mention. In particular, it suffices to show that the cycles are equal after pulling back by q. Denote s' and t' the pullbacks of s and t to meromorphic sections of $q^*\mathcal{L}$ and $q^*\mathcal{N}$. Denote $Z'_i = q^{-1}(Z_i)$, denote $\xi'_i \in |Z'_i|$ the generic point, denote $B'_i = \mathcal{O}^h_{T,\xi'_i}$, denote $\mathcal{L}_{\xi'_i}$

[1] It is possible that a shorter proof can be given by immediately applying étale localization.

and $\mathcal{N}_{\xi_i'}$ the pullbacks of $\mathcal{L}$ and $\mathcal{N}$ to $\mathrm{Spec}(B_i')$. Recall that we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(B_i') & \xrightarrow[c_{\xi_i'}]{} & T \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(B_i) & \xrightarrow{c_{\xi_i}} & X \end{array}$$

see Decent Spaces, Remark 68.11.11. Denote s_i' and t_i' the pullbacks of s_i and t_i which are generators of $\mathcal{L}_{\xi_i'}$ and $\mathcal{N}_{\xi_i'}$. Then we have

$$s' = f_i' s_i' \quad \text{and} \quad t' = g_i' t_i'$$

where f_i' and g_i' are the images of f_i, g_i under the map $Q(B_i) \to Q(B_i')$ induced by $B_i \to B_i'$. By Algebra, Lemma 10.52.13 we have

$$\mathrm{ord}_{B_i}(f_i) = \mathrm{ord}_{B_i'}(f_i') \quad \text{and} \quad \mathrm{ord}_{B_i}(g_i) = \mathrm{ord}_{B_i'}(g_i')$$

By Lemma 82.19.1 applied to $q : Z_i' \to Z_i$ we have

$$q^*\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) = \mathrm{div}_{q^*\mathcal{N}|_{Z_i'}}(t_i'|_{Z_i'}) \quad \text{and} \quad q^*\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = \mathrm{div}_{q^*\mathcal{L}|_{Z_i'}}(s_i'|_{Z_i'})$$

This already shows that the first cycle in the statement of the lemma pulls back to the corresponding cycle for s', t', Z_i', s_i', t_i'. To see the same is true for the second, note that by Chow Homology, Lemma 42.5.4 we have

$$\partial_{B_i}(f_i, g_i) \mapsto \partial_{B_i'}(f_i', g_i') \quad \text{via} \quad \kappa(\xi_i) \to \kappa(\xi_i')$$

Hence the same lemma as before shows that

$$q^*\mathrm{div}(\partial_{B_i}(f_i, g_i)) = \mathrm{div}(\partial_{B_i'}(f_i', g_i'))$$

Since $q^*\mathcal{L} \cong \mathcal{O}_T$ we find that it suffices to prove the equality in case $\mathcal{L}$ is trivial. Exchanging the roles of $\mathcal{L}$ and $\mathcal{N}$ we see that we may similarly assume $\mathcal{N}$ is trivial. This finishes the proof of the first step.

Second step. Assume $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$. Denote 1 the trivializing section of $\mathcal{L}$. Then $s_i = u \cdot 1$ for some unit $u \in B_i$. Let us examine what happens if we replace s_i by 1. Then f_i gets replaced by uf_i. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$\mathrm{ord}_{B_i}(g_i)\mathrm{div}(u|_{Z_i})$$

where $u|_{Z_i}$ is the image of u in the residue field by Spaces over Fields, Lemma 72.7.3 and in the second expression we add

$$\mathrm{div}(\partial_{B_i}(u, g_i))$$

by bi-linearity of the tame symbol. These terms agree by the property of the tame symbol given in Chow Homology, Equation (6).

Let $Y \subset X$ be an integral closed subspace with $\dim_\delta(Y) = n - 2$. To show that the coefficients of Y of the two cycles of the lemma is the same, we may do a replacement of s_i by 1 as in the previous paragraph. In exactly the same way one shows that we may do a replacement of t_i by 1. Since there are only a finite number of Z_i such that $Y \subset Z_i$ we may assume $s_i = 1$ and $t_i = 1$ for all these Z_i.

Third step. Here we prove the coefficients of Y in the cycles of the lemma agree for an integral closed subspace Y with $\dim_\delta(Y) = n - 2$ such that moreover $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ and $s_i = 1$ and $t_i = 1$ for all Z_i such that $Y \subset Z_i$. After replacing X

by a smaller open subspace we may in fact assume that s_i and t_i are equal to 1 for all i. In this case the first cycle is zero. Our task is to show that the coefficient of Y in the second cycle is zero as well.

First, since $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ we may and do think of s, t as rational functions f, g on X. Since s_i and t_i are equal to 1 we find that f_i, resp. g_i is the image of f, resp. g in $Q(B_i)$ for all i. Let $\zeta \in |Y|$ be the generic point. Choose an étale neighbourhood

$$(U, u) \longrightarrow (X, \zeta)$$

and denote $Y' = \overline{\{u\}} \subset U$. Since an étale morphism is flat, we can pullback f and g to regular meromorphic functions on U which we will also denote f and g. For every prime divisor $Y \subset Z \subset X$ the scheme $Z \times_X U$ is a union of prime divisors of U. Conversely, given a prime divisor $Y' \subset Z' \subset U$, there is a prime divisor $Y \subset Z \subset X$ such that Z' is a component of $Z \times_X U$. Given such a pair (Z, Z') the ring map

$$\mathcal{O}^h_{X,\xi} \to \mathcal{O}^h_{U,\xi'}$$

is étale (in fact it is finite étale). Hence we find that

$$\partial_{\mathcal{O}^h_{X,\xi}}(f, g) \mapsto \partial_{\mathcal{O}^h_{U,\xi'}}(f, g) \quad \text{via} \quad \kappa(\xi) \to \kappa(\xi')$$

by Chow Homology, Lemma 42.5.4. Thus Lemma 82.13.2 applies to show

$$(Z \times_X U \to Z)^* \text{div}_Z(\partial_{\mathcal{O}^h_{X,\xi}}(f, g)) = \sum\nolimits_{Z' \subset Z \times_X U} \text{div}_{Z'}(\partial_{\mathcal{O}^h_{U,\xi'}}(f, g))$$

Since flat pullback commutes with pushforward along closed immersions (Lemma 82.11.1) we see that it suffices to prove that the coefficient of Y' in

$$\sum\nolimits_{Z' \subset U} (Z' \to U)_* \text{div}_{Z'}(\partial_{\mathcal{O}^h_{U,\xi'}}(f, g))$$

is zero.

Let $A = \mathcal{O}_{U,u}$. Then $f, g \in Q(A)^*$. Thus we can write $f = a/b$ and $g = c/d$ with $a, b, c, d \in A$ nonzerodivisors. The coefficient of Y' in the expression above is

$$\sum\nolimits_{\mathfrak{q} \subset A \text{ height } 1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(f, g))$$

By bilinearity of ∂_A it suffices to prove

$$\sum\nolimits_{\mathfrak{q} \subset A \text{ height } 1} \text{ord}_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(a, c))$$

is zero and similarly for the other pairs (a, d), (b, c), and (b, d). This is true by Chow Homology, Lemma 42.6.2. □

82.21. Intersecting with an invertible sheaf and rational equivalence

0EQW This section is the analogue of Chow Homology, Section 42.28. Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

0EQX **Lemma 82.21.1.** *In Situation 82.2.1 let X/B be good. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_\mathcal{L}(s)$ and $\beta = div_\mathcal{N}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\mathrm{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 82.20.1 and the discussion preceding it. □

0EQY **Lemma 82.21.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 82.18.1 is zero. By Definition 82.15.1 there exists a locally finite family $\{W_j\}$ of integral closed subspaces with $\dim_\delta(W_j) = k+2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By Lemma 82.19.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 82.21.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 82.18.2 we see that the right hand side is zero in $\mathrm{CH}_k(X)$ as desired. □

In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be invertible on X. We will denote

$$c_1(\mathcal{L})^s \cap - : \mathrm{CH}_{k+s}(X) \to \mathrm{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 82.21.2. We will denote $c_1(\mathcal{L}^s \cap -$ the s-fold iterate of this operation for all $s \geq 0$.

0EQZ **Lemma 82.21.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in \mathrm{CH}_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\mathrm{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subspaces $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 82.19.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 82.21.1 and the definitions. □

82.22. Intersecting with effective Cartier divisors

0ER0 This section is the analogue of Chow Homology, Section 42.29. Please read the introduction of that section we motivation.

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a global section, see Divisors on Spaces, Lemma 71.7.8. If D corresponds to $(\mathcal{L}, s)$, then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

0ER1 **Definition 82.22.1.** In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s)$ be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the vanishing locus of s, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a (refined) *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D).$$

by the following rules:

(1) Given an integral closed subspace $W \subset X$ with $\dim_\delta(W) = k+1$ we define
   (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
   (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

(3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

0ER2 **Remark 82.22.2.** Let S, B, X, $\mathcal{L}$, s, $i : D \to X$ be as in Definition 82.22.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \to Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$. The possibility to do this will be useful later on.

0ER3 **Remark 82.22.3.** Let $f : X' \to X$ be a morphism of good algebraic spaces over B as in Situation 82.2.1. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

0ER4 **Lemma 82.22.4.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Let α be a $(k+1)$-cycle on X. Then $i_* i^* \alpha = c_1(\mathcal{L}) \cap \alpha$ in $\mathrm{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.*

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subspaces with $\dim_\delta(W_j) = k$. Since D is the vanishing locus of s we see that $D \cap W_j$ is the vanishing locus of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 82.18.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j])$$

in $\mathrm{CH}_k(X)$ by Definition 82.18.1. The right hand side matches (termwise) the pushforward of the class $i^*\alpha$ on D from Definition 82.22.1. Hence we win. □

0ER5 **Lemma 82.22.5.** *In Situation 82.2.1. Let $f : X' \to X$ be a proper morphism of good algebraic spaces over B. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 82.22.3. For any $(k+1)$-cycle α' on X' we have $i^*f_*\alpha' = g_*(i')^*\alpha'$ in $\mathrm{CH}_k(D)$ (this makes sense as f_* is defined on the level of cycles).

Proof. Suppose $\alpha = [W']$ for some integral closed subspace $W' \subset X'$. Let $W \subset X$ be the "image" of W' as in Lemma 82.7.1. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 82.19.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\mathrm{CH}_k(W)$ by the second assertion of Lemma 82.19.3. By Remark 82.15.3 the result follows for general α'. □

0ER6 **Lemma 82.22.6.** *In Situation 82.2.1. Let $f : X' \to X$ be a flat morphism of relative dimension r of good algebraic spaces over B. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

as in Remark 82.22.3. For any $(k+1)$-cycle α on X we have $(i')^*f^*\alpha = g^*i^*\alpha'$ in $\mathrm{CH}_{k+r}(D)$ (this makes sense as f^* is defined on the level of cycles).

Proof. Suppose $\alpha = [W]$ for some integral closed subspace $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subspaces of D'. Hence the equality holds as cycles, see Lemma 82.10.5. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 82.19.2. By Remark 82.15.3 the result follows for general α'. □

0ER7 **Lemma 82.22.7.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*([Z]_{k+1}) = [D \cap Z]_k$.*

Proof. The assumption means that $s|_Z$ is a regular section of $\mathcal{L}|_Z$. Thus $D \cap Z = Z(s)$ and we get

$$[D \cap Z]_k = \sum n_i [Z(s_i)]_k$$

as cycles where $s_i = s|_{Z_i}$, the Z_i are the irreducible components of δ-dimension $k+1$, and $[Z]_{k+1} = \sum n_i [Z_i]$. See Lemma 82.18.3. We have $D \cap Z_i = Z(s_i)$. Comparing with the definition of the Gysin map we conclude. □

82.23. Gysin homomorphisms

0ER8 This section is the analogue of Chow Homology, Section 42.30. In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence.

0ER9 **Lemma 82.23.1.** *In Situation 82.2.1 let X/B be good. Assume X integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module and let t be a nonzero meromorphic section of $\mathcal{N}$. Then $i^* div_{\mathcal{N}}(t) = c_1(\mathcal{N}) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$.*

Proof. Write $\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$ for some integral closed subspaces $Z_i \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i, see Spaces over Fields, Lemmas 72.6.1, 72.6.6, and 72.7.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 82.20 to our current situation. For each i let $\xi_i \in |Z_i|$ be its generic point. Let $B_i = \mathcal{O}^h_{X,\xi_i}$. For each i we pick generators s_i of $\mathcal{L}_{\xi_i}$ and t_i of $\mathcal{N}_{\xi_i}$ over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of s_i. We claim that

$$i^*\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\mathrm{div}_{\mathcal{N}}(t)$ and the fact that $\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 82.17.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 82.20.1) gives the equality

$$\sum \Big(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})\Big) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^*\mathrm{div}_{\mathcal{N}}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D. This finishes the proof. □

0ERA **Lemma 82.23.2.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 82.22.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subspaces $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\mathrm{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{array}{ccc} D' & \xrightarrow[i']{} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ D & \xrightarrow{i} & X \end{array}$$

of Remark 82.22.3. Since $X' \to X$ is proper we see that $i^*p_* = q_*(i')^*$ by Lemma 82.22.5. As we know that q_* factors through rational equivalence (Lemma 82.16.3), it suffices to prove the result for $\alpha' = \sum \mathrm{div}_{W_j}(f_j)$ on X'. Clearly this reduces us to the case where X is integral and $\alpha = \mathrm{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^*\alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 82.21.2. If $D \neq X$, then we see that $i^*\mathrm{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\mathrm{CH}_k(D)$ by Lemma 82.23.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map. □

0ERB **Lemma 82.23.3.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$ in $\mathrm{CH}_{k-2}(D)$ for all $\alpha \in \mathrm{CH}_k(Z)$.*

Proof. With exactly the same proof as in Lemma 82.23.2 this follows from Lemmas 82.19.4, 82.21.3, and 82.23.1. □

0ERC **Lemma 82.23.4.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ and $(\mathcal{L}', s', i' : D' \to X)$ be two triples as in Definition 82.22.1. Then the diagram*

$$\begin{array}{ccc} \mathrm{CH}_k(X) & \xrightarrow[i^*]{} & \mathrm{CH}_{k-1}(D) \\ {\scriptstyle (i')^*}\downarrow & & \downarrow \\ \mathrm{CH}_{k-1}(D') & \longrightarrow & \mathrm{CH}_{k-2}(D \cap D') \end{array}$$

commutes where each of the maps is a Gysin map.

Proof. Denote $j : D \cap D' \to D$ and $j' : D \cap D' \to D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'}$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^*i^*\alpha = j^*(i')^*\alpha$ for all $\alpha \in \mathrm{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k. We will prove the equality in case $\alpha = [W]$. The general case will then follow from the observation in Remark 82.15.3 (and the specific shape of our rational equivalence produced below). We will deduce the equality for $\alpha = [W]$ from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\mathrm{div}_{\mathcal{L}|_W}(\sigma) = \sum \mathrm{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\mathrm{div}_{\mathcal{L}'|_W}(\sigma') = \sum \mathrm{ord}_{Z_i,\mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 82.20. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i, let $\xi_i \in |Z_i|$ be the generic point. As in Section 82.20 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}^h_{W,\xi_i}$ and which is equal to the image of s, resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i\sigma_i$ and $\sigma' = f'_i\sigma'_i$ so that $n_i = \mathrm{ord}_{B_i}(f_i)$ and $n'_i = \mathrm{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^*i^*[W] = \sum \mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 82.20.1) now gives the equality

$$\sum \Big(\mathrm{ord}_{B_i}(f_i)\mathrm{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \mathrm{ord}_{B_i}(f'_i)\mathrm{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})\Big) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^*i^*[W]$ and $j^*(i')^*[W]$ on $D \cap D' \cap W$. □

82.24. Relative effective Cartier divisors

0ERD This section is the analogue of Chow Homology, Section 42.31. Relative effective Cartier divisors are defined in Divisors on Spaces, Section 71.9. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

0ERE **Lemma 82.24.1.** *In Situation 82.2.1. Let $X, Y/B$ be good. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be a relative effective Cartier divisor (Divisors on Spaces, Definition 71.9.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \mathrm{CH}_{k+1}(Y)$ we have*

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $\mathrm{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $\mathrm{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subspace of δ-dimension $k+1$. By Divisors on Spaces, Lemma 71.9.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 82.22.7 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as algebraic spaces. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^*p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j[W_j]$ is a general $k+1$ cycle, then we get $i^*\alpha = \sum m_j i^*p^*[W_j] = \sum m_j(p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 82.22.4. □

82.25. Affine bundles

0ERF This section is the analogue of Chow Homology, Section 42.32. For an affine bundle the pullback map is surjective on Chow groups.

0ERG **Lemma 82.25.1.** *In Situation 82.2.1 let $X, Y/B$ be good. Let $f : X \to Y$ be a quasi-compact flat morphism over B of relative dimension r. Assume that for every $y \in Y$ we have $X_y \cong \mathbf{A}^r_{\kappa(y)}$. Then $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in X. Let $Z_j \subset Y$ be the integral closed subspace such that we obtain a dominant morphism $W_j \to Z_j$ as in Lemma 82.7.1. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection Z_j of closures of images is a locally finite collection of integral closed subspaces of Y.

Consider the fibre product diagrams

$$\begin{array}{ccc} f^{-1}(Z_j) & \longrightarrow & X \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f} \\ Z_j & \longrightarrow & Y \end{array}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 82.15.3). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $V \subset Y$. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 82.10.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(f^{-1}(V))$ of Lemma 82.15.2 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

In particular, by replacing Y by a suitable open we may assume Y is a scheme with generic point η. The isomorphism $Y_\eta \cong \mathbf{A}^r_\eta$ extends to an isomorphism over a nonempty open $V \subset Y$, see Limits of Spaces, Lemma 70.7.1. This reduces us to the case of schemes which is Chow Homology, Lemma 42.32.1. □

0ERH **Lemma 82.25.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let*

$$p : L = \underline{\mathrm{Spec}}(Sym^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 82.25.1. Let $o : X \to L$ be the zero section of $L \to X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \to \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier

divisor on L. Hence by Lemma 82.24.1 we see that $o^* \circ p^* = \mathrm{id}$ and we conclude that p^* is injective too. $\square$

82.26. Bivariant intersection theory

0ERI This section is the analogue of Chow Homology, Section 42.33. In order to intelligently talk about higher Chern classes of vector bundles we introduce the following notion, following [**FM81**]. It follows from [**Ful98**, Theorem 17.1] that our definition agrees with that of [**Ful98**] modulo the caveat that we are working in different settings.

0ERJ **Definition 82.26.1.** In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B. Let $p \in \mathbf{Z}$. A *bivariant class c of degree p for f* is given by a rule which assigns to every morphism $Y' \to Y$ of good algebraic spaces over B and every k a map

Similar to [**Ful98**, Definition 17.1]

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

(1) if $Y'' \to Y'$ is a proper morphism, then $c \cap (Y'' \to Y')_* \alpha'' = (X'' \to X')_*(c \cap \alpha'')$ for all α'' on Y'',
(2) if $Y'' \to Y'$ a morphism of good algebraic spaces over B which is flat of relative dimension r, then $c \cap (Y'' \to Y')^* \alpha' = (X'' \to X')^*(c \cap \alpha')$ for all α' on Y',
(3) if $(\mathcal{L}', s', i' : D' \to Y')$ is as in Definition 82.22.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X', then we have $c \cap (i')^* \alpha' = (j')^*(c \cap \alpha')$ for all α' on Y'.

The collection of all bivariant classes of degree p for f is denoted $A^p(X \to Y)$.

In Situation 82.2.1 let $X \to Y$ and $Y \to Z$ be morphisms of good algebraic spaces over B. Let $p \in \mathbf{Z}$. It is clear that $A^p(X \to Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \to Y) \times A^q(Y \to Z) \to A^{p+q}(X \to Z)$$

which is associative. We will be most interested in $A^p(X) = A^p(X \to X)$, which will always mean the bivariant cohomology classes for id_X. Namely, that is where Chern classes will live.

0ERK **Definition 82.26.2.** In Situation 82.2.1 let X/B be good. The *Chow cohomology* of X is the graded $\mathbf{Z}$-algebra $A^*(X)$ whose degree p component is $A^p(X \to X)$.

Warning: It is not clear that the $\mathbf{Z}$-algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

0ERL **Remark 82.26.3.** In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B. Then there is a canonical $\mathbf{Z}$-algebra map $A^*(Y) \to A^*(X)$. Namely, given $c \in A^p(Y)$ and $X' \to X$, then we can let f^*c be defined by the map $c \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-p}(X')$ which is given by thinking of X' as an algebraic space over Y.

0ERM **Lemma 82.26.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the rule that to $f : X' \to X$ assigns $c_1(f^*\mathcal{L}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(X')$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 82.21.2, 82.19.4, 82.19.2, and 82.23.3. $\square$

0ERN **Lemma 82.26.5.** *In Situation 82.2.1 let $f : X \to Y$ be a morphism of good algebraic spaces over B which is flat of relative dimension r. Then the rule that to $Y' \to Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.*

Proof. This follows from Lemmas 82.16.2, 82.10.4, 82.11.1, and 82.22.6. □

0ERP **Lemma 82.26.6.** *In Situation 82.2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 82.22.1. Then the rule that to $f : X' \to X$ assigns $(i')^* : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 82.23.2, 82.22.5, 82.22.6, and 82.23.4. □

0ERQ **Lemma 82.26.7.** *In Situation 82.2.1 let $f : X \to Y$ and $g : Y \to Z$ be morphisms of good algebraic spaces over B. Let $c \in A^p(X \to Z)$ and assume f is proper. Then the rule that to $X' \to X$ assigns $\alpha \longmapsto f_*(c \cap \alpha)$ is a bivariant class of degree p.*

Proof. This follows from Lemmas 82.8.2, 82.11.1, and 82.22.5. □

Here we see that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

0ERR **Lemma 82.26.8.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $c_1(\mathcal{L}) \in A^1(X)$ commutes with every element $c \in A^p(X)$.*

Proof. Let $p : L \to X$ be as in Lemma 82.25.2 and let $o : X \to L$ be the zero section. Observe that $p^*\mathcal{L}^{\otimes -1}$ has a canonical section whose vanishing locus is exactly the effective Cartier divisor $o(X)$. Let $\alpha \in \mathrm{CH}_k(X)$. Then we see that

$$p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) = c_1(p^*\mathcal{L}^{\otimes -1}) \cap p^*\alpha = o_*o^*p^*\alpha$$

by Lemmas 82.19.2 and 82.24.1. Since c is a bivariant class we have

$$\begin{aligned} p^*(c \cap c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) \\ &= c \cap o_*o^*p^*\alpha \\ &= o_*o^*p^*(c \cap \alpha) \\ &= p^*(c_1(\mathcal{L}^{\otimes -1}) \cap c \cap \alpha) \end{aligned}$$

(last equality by the above applied to $c \cap \alpha$). Since p^* is injective by a lemma cited above we get that $c_1(\mathcal{L}^{\otimes -1})$ is in the center of $A^*(X)$. This proves the lemma. □

Here a criterion for when a bivariant class is zero.

0ERS **Lemma 82.26.9.** *In Situation 82.2.1 let X/B be good. Let $c \in A^p(X)$. Then c is zero if and only if $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X.*

Proof. The if direction is clear. For the converse, assume that $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X. Let $X' \to X$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha = \sum n_i[Y_i]$ with $Y_i \subset X'$ a locally finite collection of integral closed subschemes of δ-dimension k. Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y_i]$ on $X'' = \coprod Y_i$ under the proper morphism $X'' \to X'$. By the properties of bivariant classes it suffices to prove that $c \cap \alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(Y_i)$ as follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \to \mathrm{CH}_{k-p}(Y_i)$ are given by flat pullback. Since capping with c commutes with flat

pullback, we see that it suffices to show that $c \cap [Y_i]$ is zero in $\mathrm{CH}_{k-p}(Y_i)$ which is true by assumption. □

82.27. Projective space bundle formula

0ERT In Situation 82.2.1 let X/B be good. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

0ERU **Lemma 82.27.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in \mathrm{CH}_k(X)$ the element*

$$\pi_*\left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in \mathrm{CH}_{k+r-1-s}(X)$$

is 0 if $s < r-1$ and is equal to α when $s = r-1$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ-dimension k. We will prove the lemma for $\alpha = [Z]$. We omit the argument deducing the general case from this special case; hint: argue as in Remark 82.15.3.

Let $P_Z = P \times_X Z$ be the base change; of course $\pi_Z : P_Z \to Z$ is the projective bundle associated to $\mathcal{E}|_Z$ and $\mathcal{O}_P(1)$ pulls back to the corresponding invertible module on P_Z. Since $c_1(\mathcal{O}_P(1)) \cap -$, and π^* are bivariant classes by Lemmas 82.26.4 and 82.26.5 we see that

$$\pi_*\left(c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]\right) = (Z \to X)_*\pi_{Z,*}\left(c_1(\mathcal{O}_{P_Z}(1))^s \cap \pi_Z^*[Z]\right)$$

Hence it suffices to prove the lemma in case X is integral and $\alpha = [X]$.

Assume X is integral, $\dim_\delta(X) = k$, and $\alpha = [X]$. Note that $\pi^*[X] = [P]$ as P is integral of δ-dimension $r-1$. If $s < r-1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap [P]$ a $(k+r-1-s)$-cycle. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap [P]) = n[X]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by a dense open. Thus we may assume X is a scheme and the result follows from Chow Homology, Lemma 42.36.1. □

0ERV **Lemma 82.27.2** (Projective space bundle formula)**.** *Let (S, δ) be as in Situation 82.2.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} \mathrm{CH}_{k+i}(X) \longrightarrow \mathrm{CH}_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 82.27.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

To prove the map is surjective, we will argue exactly as in the proof of Lemma 82.25.1 to reduce to the case of schemes. We urge the reader to skip the proof.

Let $\beta \in \mathrm{CH}_{k+r-1}(P)$. Write $\beta = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in P. Let $Z_j \subset X$ be the "image" of W_j as in Lemma 82.7.1. For any quasi-compact open $U \subset X$ we see that $\pi^{-1}(U) \cap W_j$ is nonempty only for finitely many j. Hence the collection Z_j of images is a locally finite collection of integral closed subspaces of X.

Consider the fibre product diagrams

$$\begin{array}{ccc} P_j & \longrightarrow & P \\ {\scriptstyle \pi_j}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z_j & \longrightarrow & X \end{array}$$

Suppose that $[W_j] \in Z_{k+r-1}(P_j)$ is rationally equivalent to

$$\pi_j^*\alpha_{j,0} + c_1(\mathcal{O}(1)) \cap \pi_j^*\alpha_{j,1} + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi_j^*\alpha_{j,r-1}$$

for some $(k+i)$-cycle $\alpha_{j,i} \in \mathrm{CH}_{k+i}(Z_j)$. Then $\alpha_i = \sum m_j\beta_{j,i}$ will be a $(k+i)$-cycle on X and

$$\pi^*\alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

will be rationally equivalent to β (see Remark 82.15.3). This reduces us to the case X integral, and $\alpha = [W]$ for some integral closed subscheme of P dominating X. In particular we may assume that $d = \dim_\delta(X) < \infty$.

Hence we can use induction on $d = \dim_\delta(X)$. If $d < k$, then $\mathrm{CH}_{k+r-1}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $U \subset X$. Suppose that we can show that

$$\beta|_{\pi^{-1}(U)} = \pi^*\alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

for some $\alpha_i \in Z_{k+i}(U)$. By Lemma 82.10.2 we see that $\alpha_i = \alpha_i'|_U$ for some $\alpha_i' \in Z_{k+i}(X)$. By the exact sequences $\mathrm{CH}_{k+i}(\pi^{-1}(X \setminus U)) \to \mathrm{CH}_{k+i}(P) \to \mathrm{CH}_{k+i}(\pi^{-1}(U))$ of Lemma 82.15.2 we see that

$$\beta - \left(\pi^*\alpha_0' + c_1(\mathcal{O}(1)) \cap \pi^*\alpha_1' + \ldots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^*\alpha_{r-1}'\right)$$

comes from a cycle $\beta' \in \mathrm{CH}_{k+r}(\pi^{-1}(X \setminus U))$. Since $\dim_\delta(X \setminus U) < d$ we win by induction on d.

In particular, by replacing X by a suitable open we may assume X is a scheme and we have reduced our problem to Chow Homology, Lemma 42.36.2. $\square$

0ERW **Lemma 82.27.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{\mathrm{Spec}}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is an isomorphism for all k.

Proof. (For the case of linebundles, see Lemma 82.25.2.) For surjectivity see Lemma 82.25.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors on Spaces, Lemma 71.7.8). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 82.15.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 82.27.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 82.24.1 and 82.19.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 82.19.4 unless all α and all β_i are zero. □

82.28. The Chern classes of a vector bundle

0ERX This section is the analogue of Chow Homology, Sections 42.37 and 42.38. However, contrary to what is done there, we directly define the Chern classes of a vector bundle as bivariant classes. This saves a considerable amount of work.

0ERY **Lemma 82.28.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$. For every morphism $X' \to X$ of good algebraic spaces over B there are unique maps*

$$c_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad i = 0, \ldots, r$$

such that for $\alpha \in \mathrm{CH}_k(X')$ we have $c_0(\mathcal{E}) \cap \alpha = \alpha$ and

$$\sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^* \left(c_{r-i}(\mathcal{E}) \cap \alpha\right) = 0$$

where $\pi' : P' \to X'$ is the base change of π. Moreover, these maps define a bivariant class $c_i(\mathcal{E})$ of degree i on X.

Proof. Uniqueness and existence of the maps $c_i(\mathcal{E}) \cap -$ follows immediately from Lemma 82.27.2 and the given description of $c_0(\mathcal{E})$. For every $i \in \mathbf{Z}$ the rule which to every morphism $X' \to X$ of good algebraic spaces over B assigns the map

$$t_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad \alpha \longmapsto \pi'_*(c_1(\mathcal{O}_{P'}(1))^{r-1+i} \cap (\pi')^*\alpha)$$

is a bivariant class[2] by Lemmas 82.26.4, 82.26.5, and 82.26.7. By Lemma 82.27.1 we have $t_i(\mathcal{E}) = 0$ for $i < 0$ and $t_0(\mathcal{E}) = 1$. Applying pushforward to the equation in the statement of the lemma we find from Lemma 82.27.1 that

$$(-1)^r t_1(\mathcal{E}) + (-1)^{r-1} c_1(\mathcal{E}) = 0$$

In particular we find that $c_1(\mathcal{E})$ is a bivariant class. If we multiply the equation in the statement of the lemma by $c_1(\mathcal{O}_{P'}(1))$ and push the result forward to X' we find

$$(-1)^r t_2(\mathcal{E}) + (-1)^{r-1} t_1(\mathcal{E}) \cap c_1(\mathcal{E}) + (-1)^{r-2} c_2(\mathcal{E}) = 0$$

As before we conclude that $c_2(\mathcal{E})$ is a bivariant class. And so on. □

0ERZ **Definition 82.28.2.** In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. For $i = 0, \ldots, r$ the *ith Chern class of* $\mathcal{E}$ is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 82.28.1. The *total Chern class of* $\mathcal{E}$ is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X.

For convenience we often set $c_i(\mathcal{E}) = 0$ for $i > r$ and $i < 0$. By definition we have $c_0(\mathcal{E}) = 1 \in A^0(X)$. Here is a sanity check.

0ES0 **Lemma 82.28.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class of $\mathcal{L}$ on X of Definition 82.28.2 is equal to the bivariant class of Lemma 82.26.4.*

Proof. Namely, in this case $P = \mathbf{P}(\mathcal{L}) = X$ with $\mathcal{O}_P(1) = \mathcal{L}$ by our normalization of the projective bundle, see Section 82.27. Hence the equation in Lemma 82.28.1 reads

$$(-1)^0 c_1(\mathcal{L})^0 \cap c_1^{new}(\mathcal{L}) \cap \alpha + (-1)^1 c_1(\mathcal{L})^1 \cap c_0^{new}(\mathcal{L}) \cap \alpha = 0$$

where $c_i^{new}(\mathcal{L})$ is as in Definition 82.28.2. Since $c_0^{new}(\mathcal{L}) = 1$ and $c_1(\mathcal{L})^0 = 1$ we conclude. □

Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

0ES1 **Lemma 82.28.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then $c_j(\mathcal{L}) \in A^j(X)$ commutes with every element $c \in A^p(X)$. In particular, if $\mathcal{F}$ is a second locally free $\mathcal{O}_X$-module on X of rank s, then*

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. Let $X' \to X$ be a morphism of good algebraic spaces over B. Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 82.28.1 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi' : P' \to X', \mathcal{O}_{P'}(1))$ associated to $(X' \to X)^*\mathcal{E}$. Applying $c \cap -$ and using Lemma 82.26.8 and the properties of bivariant classes we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap \pi^*(c \cap \alpha_{r-i}) = 0$$

[2] Up to signs these are the Segre classes of $\mathcal{E}$.

in the Chow group of P'. Hence we see that $c \cap \alpha_j$ is equal to $c_j(\mathcal{E}) \cap (c \cap \alpha)$ by the uniqueness in Lemma 82.28.1. This proves the lemma. □

0ES2 **Remark 82.28.5.** In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. If the rank of $\mathcal{E}$ is not constant then we can still define the Chern classes of $\mathcal{E}$. Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

where $X_r \subset X$ is the open and closed subspace where the rank of $\mathcal{E}$ is r. If $X' \to X$ is a morphism of good algebraic spaces over B, then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod\nolimits_{r \geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then we simply define $c_i(\mathcal{E})$ to be the bivariant class which preserves these direct product decompositions and acts by the already defined operations $c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_i(\mathcal{E})$ is nonzero for infinitely many i.

82.29. Polynomial relations among Chern classes

0ES3 In Situation 82.2.1 let X/B be good. Let $\mathcal{E}_i$ be a finite collection of finite locally free $\mathcal{O}_X$-modules. By Lemma 82.28.4 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) $\mathbf{Z}$-subalgebra of the Chow cohomology $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \to X$ of good algebraic spaces over B. By Lemma 82.26.9 this is equivalent to the requirement that given any morphism $f : Y \to X$ where Y is an integral algebraic space locally of finite type over X the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 82.18.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

0ES4 **Lemma 82.29.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have*

0ES5 (82.29.1.1)
$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.39.1 replacing the lemmas used there by Lemmas 82.26.9 and 82.28.1. □

82.30. Additivity of Chern classes

0ES6 This section is the analogue of Chow Homology, Section 42.40.

0ES7 **Lemma 82.30.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.1 replacing the lemmas used there by Lemmas 82.26.9, 82.24.1, 82.19.4, and 82.28.1. □

0ES8 **Lemma 82.30.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.2 replacing the lemmas used there by Lemmas 82.30.1 and 82.29.1. □

0ES9 **Lemma 82.30.3.** *In Situation 82.2.1 let X/B be good. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 42.40.3 replacing the lemmas used there by Lemmas 82.26.9, 82.30.2, and 82.28.1. □

0ESA **Lemma 82.30.4.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules. Let $\mathcal{E}$ be a locally free rank $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r} (1 + x_i)$$

in $A^*(X)$.

Proof. Apply Lemma 82.30.2 and induction. □

82.31. The splitting principle

0ESB This section is the analogue of Chow Homology, Section 42.40.

0ESC **Lemma 82.31.1.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a projective flat morphism $\pi : P \to X$ of relative dimension d such that*

(1) *for any morphism $f : Y \to X$ of good algebraic spaces over B the map $\pi_Y^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
(2) *each $\pi^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.*

Proof. We prove this by induction on the integer $r = \sum r_i$. If $r = 0$ we can take $\pi = \mathrm{id}_X$. If $r_i = 1$ for all i, then we can also take $\pi = \mathrm{id}_X$. Assume that $r_{i_0} > 1$ for some i_0. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}_{i_0}$. The canonical map $\pi^*\mathcal{E}_{i_0} \to \mathcal{O}_P(1)$ is surjective and hence its kernel $\mathcal{E}'_{i_0}$ is finite locally free of rank $r_{i_0} - 1$. Observe that π_Y^* is injective for any morphism $f : Y \to X$ of good algebraic spaces over B, see Lemma 82.27.2. Thus it suffices to prove the lemma for P and the locally free sheaves $\pi^*\mathcal{E}_i$. However, because we have the subbundle $\mathcal{E}_{i_0} \subset \pi^*\mathcal{E}_{i_0}$ with invertible quotient, it now suffices to prove the lemma for the collection $\{\mathcal{E}_i\}_{i \neq i_0} \cup \{\mathcal{E}'_{i_0}\}$. This decreases r by 1 and we win by induction hypothesis. □

Rather than explaining what the splitting principle says, let us use it in the proof of some lemmas.

0ESD **Lemma 82.31.2.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^i(X)$.

Proof. Choose a morphism $\pi : P \to X$ as in Lemma 82.31.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of $\mathcal{E}$ and $\mathcal{E}^\vee$ after pulling back to P. Thus we may assume there exist invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_1^\perp \subset \mathcal{E}_2^\perp \subset \ldots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 82.18.2. By Lemma 82.30.4 we have

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod\nolimits_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

0ESE **Lemma 82.31.3.** *In Situation 82.2.1 let X/B be good. Let $\mathcal{E}$ and $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-modules of ranks r and s. Then we have*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = r c_1(\mathcal{F}) + s c_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2 c_2(\mathcal{F}) + r s c_1(\mathcal{F}) c_1(\mathcal{E}) + s^2 c_2(\mathcal{E})$$

and so on (see proof).

Proof. Arguing exactly as in the proof of Lemma 82.31.2 we may assume we have invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ $\mathcal{N}_i$, $i = 1, \ldots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \ldots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \ldots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \ldots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 82.30.4 we have

$$c(\mathcal{E}) = \prod(1 + x_i), \quad c(\mathcal{F}) = \prod(1 + y_j), \quad \text{and} \quad c(\mathcal{F}) = \prod(1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

82.32. Degrees of zero cycles

0ESF This section is the analogue of Chow Homology, Section 42.41. We start with defining the degree of a zero cycle on a proper algebraic space over a field.

0ESG **Definition 82.32.1.** Let k be a field. Let $p : X \to \mathrm{Spec}(k)$ be a proper morphism of algebraic spaces. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(\mathrm{Spec}(k)) \longrightarrow \mathbf{Z}$$

(Lemma 82.16.3) composed with the natural isomorphism $\mathrm{CH}_0(\mathrm{Spec}(k)) \to \mathbf{Z}$ which maps $[\mathrm{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

0ESH **Lemma 82.32.2.** *Let k be a field. Let X be a proper algebraic space over k. Let $\alpha = \sum n_i[Z_i]$ be in $Z_0(X)$. Then*

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \to \mathrm{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 82.8.1). □

0ESI **Lemma 82.32.3.** *Let k be a field. Let X be a proper algebraic space over k. Let $Z \subset X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_1) \cap [Z]_d)$$

where the left hand side is defined in Spaces over Fields, Definition 72.18.3.

Proof. Let $Z_i \subset Z$, $i = 1, \ldots, t$ be the irreducible components of dimension d. Let m_i be the multiplicity of Z_i in Z. Then $[Z]_d = \sum m_i[Z_i]$ and $c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by Spaces over Fields, Lemma 72.18.2 it suffices to prove the lemma in case $Z = X$ is a proper integral algebraic space over k.

By Chow's lemma there exists a proper morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is a scheme. See More on Morphisms of Spaces, Lemma 76.40.5. Then X' is a proper scheme over k. After replacing

X' by the scheme theoretic closure of $f^{-1}(U)$ we may assume that X' is integral. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot X') = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Spaces over Fields, Lemma 72.18.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \ldots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 82.19.4. Thus we may replace X by X' and assume that X is a proper scheme over k. This case was proven in Chow Homology, Lemma 42.41.4. □

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CHAPTER 83

Quotients of Groupoids

048A

83.1. Introduction

048B This chapter is devoted to generalities concerning groupoids and their quotients (as far as they exist). There is a lot of literature on this subject, see for example [**MFK94**], [**Ses72**], [**Kol97**], [**KM97**], [**Kol08**] and many more.

83.2. Conventions and notation

048C In this chapter the conventions and notation are those introduced in Groupoids in Spaces, Sections 78.2 and 78.3.

83.3. Invariant morphisms

048D

048E **Definition 83.3.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. We say a morphism $\phi : U \to X$ of algebraic spaces over B is *R-invariant* if the diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \phi} \\ U & \xrightarrow{\phi} & X \end{array}$$

is commutative. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G on U over B as in Groupoids in Spaces, Lemma 78.15.1, then we say that ϕ is *G-invariant*.

In other words, a morphism $U \to X$ is R-invariant if it equalizes s and t. We can reformulate this in terms of associated quotient sheaves as follows.

048F **Lemma 83.3.2.** *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. A morphism of algebraic spaces $\phi : U \to X$ is R-invariant if and only if it factors as $U \to U/R \to X$.*

Proof. This is clear from the definition of the quotient sheaf in Groupoids in Spaces, Section 78.19. □

048G **Lemma 83.3.3.** *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $X' \to X$ be any morphism of algebraic spaces.*

(1) *Setting $U' = X' \times_X U$, $R' = X' \times_X R$ we obtain a pre-relation $j' : R' \to U' \times_B U'$.*

(2) *If j is a relation, then j' is a relation.*
(3) *If j is a pre-equivalence relation, then j' is a pre-equivalence relation.*
(4) *If j is an equivalence relation, then j' is an equivalence relation.*
(5) *If j comes from a groupoid in algebraic spaces (U, R, s, t, c) over B, then*
   (a) *(U, R, s, t, c) is a groupoid in algebraic spaces over X, and*
   (b) *j' comes from the base change (U', R', s', t', c') of this groupoid to X', see Groupoids in Spaces, Lemma 78.11.6.*
(6) *If j comes from the action of a group algebraic space G/B on U as in Groupoids in Spaces, Lemma 78.15.1 then j' comes from the induced action of G on U'.*

Proof. Omitted. Hint: Functorial point of view combined with the picture:

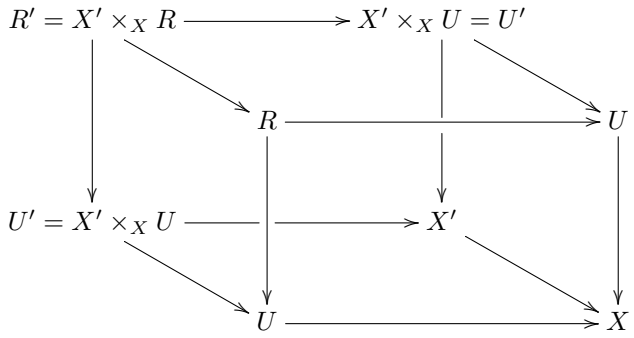

□

048H **Definition 83.3.4.** In the situation of Lemma 83.3.3 we call $j' : R' \to U' \times_B U'$ the *base change* of the pre-relation j to X'. We say it is a *flat base change* if $X' \to X$ is a flat morphism of algebraic spaces.

This kind of base change interacts well with taking quotient sheaves and quotient stacks.

0DTF **Lemma 83.3.5.** *In the situation of Lemma 83.3.3 there is an isomorphism of sheaves*

$$U'/R' = X' \times_X U/R$$

For the construction of quotient sheaves, see Groupoids in Spaces, Section 78.19.

Proof. Since $U \to X$ is R-invariant, it is clear that the map $U \to X$ factors through the quotient sheaf U/R. Recall that by definition

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category Sh of sheaves of sets on $(Sch/S)_{fppf}$. In fact, this is a coequalizer diagram in the comma category Sh/X. Since the base change functor $X' \times_X - : Sh/X \to Sh/X'$ is exact (true in any topos), we conclude. □

0DTG **Lemma 83.3.6.** *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $U \to X$ be an R-invariant*

morphism of algebraic spaces over B. Let $g : X' \to X$ be a morphism of algebraic spaces over B and let (U', R', s', t', c') be the base change as in Lemma 83.3.3. Then

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{S}_{X'} & \longrightarrow & \mathcal{S}_X \end{array}$$

is a 2-fibre product of stacks in groupoids over $(Sch/S)_{fppf}$. For the construction of quotient stacks and the morphisms in this diagram, see Groupoids in Spaces, Section 78.20.

Proof. We will prove this by using the explicit description of the quotient stacks given in Groupoids in Spaces, Lemma 78.24.1. However, we strongly urge the reader to find their own proof. First, we may view (U, R, s, t, c) as a groupoid in algebraic spaces over X, hence we obtain a map $f : [U/R] \to \mathcal{S}_X$, see Groupoids in Spaces, Lemma 78.20.2. Similarly, we have $f' : [U'/R'] \to X'$.

An object of the 2-fibre product $\mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$ over a scheme T over S is the same as a morphism $x' : T \to X'$ and an object y of $[U/R]$ over T such that such that the composition $g \circ x'$ is equal to $f(y)$. This makes sense because objects of $\mathcal{S}_X$ over T are morphisms $T \to X$. By Groupoids in Spaces, Lemma 78.24.1 we may assume y is given by a $[U/R]$-descent datum (u_i, r_{ij}) relative to an fppf covering $\{T_i \to T\}$. The agreement of $g \circ x' = f(y)$ means that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow[u_i]{} & U \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} X' \xrightarrow{g} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} T_i \times_T T_j & \xrightarrow[r_{ij}]{} & R \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} X' \xrightarrow{g} & X \end{array}$$

are commutative.

On the other hand, an object y' of $[U'/R']$ over a scheme T over S by Groupoids in Spaces, Lemma 78.24.1 is given by a $[U'/R']$-descent datum (u'_i, r'_{ij}) relative to an fppf covering $\{T_i \to T\}$. Setting $f'(y') = x' : T \to X'$ we see that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow[u'_i]{} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} & X' \end{array} \quad \text{and} \quad \begin{array}{ccc} T_i \times_T T_j & \xrightarrow[r'_{ij}]{} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{x'} & X' \end{array}$$

are commutative.

With this notation in place, we define a functor

$$[U'/R'] \longrightarrow \mathcal{S}_{X'} \times_{\mathcal{S}_X} [U/R]$$

by sending $y' = (u'_i, r'_{ij})$ as above to the object $(x', (u_i, r_{ij}))$ where $x' = f'(y')$, where u_i is the composition $T_i \to U' \to U$, and where r_{ij} is the composition $T_i \times_T T_j \to R' \to R$. Conversely, given an object $(x', (u_i, r_{ij})$ of the right hand side we can send this to the object $((x', u_i), (x', r_{ij}))$ of the left hand side. We omit the discussion of what to do with morphisms (works in exactly the same manner). $\square$

83.4. Categorical quotients

048I This is the most basic kind of quotient one can consider.

048J **Definition 83.4.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in algebraic spaces over B.

(1) We say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient* if it is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ of algebraic spaces over B there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(2) Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B. Assume U, R are objects of $\mathcal{C}$. In this situation we say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient in* $\mathcal{C}$ if $X \in \mathrm{Ob}(\mathcal{C})$, and ϕ is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ with $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(3) If $B = S$ and $\mathcal{C}$ is the category of schemes over S, then we say $U \to X$ is a *categorical quotient in the category of schemes*, or simply a *categorical quotient in schemes*.

We often single out a category $\mathcal{C}$ of algebraic spaces over B by some separation axiom, see Example 83.4.3 for some standard cases. Note that $\phi : U \to X$ is a categorical quotient if and only if $U \to X$ is a coequalizer for the morphisms $t, s : R \to U$ in the category. Hence we immediately deduce the following lemma.

048K **Lemma 83.4.2.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation in algebraic spaces over B. If a categorical quotient in the category of algebraic spaces over B exists, then it is unique up to unique isomorphism. Similarly for categorical quotients in full subcategories of Spaces/B.*

Proof. See Categories, Section 4.11. □

049V **Example 83.4.3.** Let S be a scheme, and let B be an algebraic space over S. Here are some standard examples of categories $\mathcal{C}$ that we often come up when applying Definition 83.4.1:

(1) $\mathcal{C}$ is the category of all algebraic spaces over B,
(2) B is separated and $\mathcal{C}$ is the category of all separated algebraic spaces over B,
(3) B is quasi-separated and $\mathcal{C}$ is the category of all quasi-separated algebraic spaces over B,
(4) B is locally separated and $\mathcal{C}$ is the category of all locally separated algebraic spaces over B,
(5) B is decent and $\mathcal{C}$ is the category of all decent algebraic spaces over B, and
(6) $S = B$ and $\mathcal{C}$ is the category of schemes over S.

In this case, if $\phi : U \to X$ is a categorical quotient then we say $U \to X$ is (1) a *categorical quotient*, (2) a *categorical quotient in separated algebraic spaces*, (3) a *categorical quotient in quasi-separated algebraic spaces*, (4) a *categorical quotient in locally separated algebraic spaces*, (5) a *categorical quotient in decent algebraic spaces*, (6) a *categorical quotient in schemes*.

048L **Definition 83.4.4.** Let S be a scheme, and let B be an algebraic space over S. Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B closed under fibre products. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in $\mathcal{C}$, and let $U \to X$ be an R-invariant morphism with $X \in \mathrm{Ob}(\mathcal{C})$.

(1) We say $U \to X$ is a *universal categorical quotient* in $\mathcal{C}$ if for every morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.
(2) We say $U \to X$ is a *uniform categorical quotient* in $\mathcal{C}$ if for every flat morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the base change $j' : R' \to U'$ of j.

049W **Lemma 83.4.5.** *In the situation of Definition 83.4.1. If $\phi : U \to X$ is a categorical quotient and U is reduced, then X is reduced. The same holds for categorical quotients in a category of spaces $\mathcal{C}$ listed in Example 83.4.3.*

Proof. Let X_{red} be the reduction of the algebraic space X. Since U is reduced the morphism $\phi : U \to X$ factors through $i : X_{red} \to X$ (Properties of Spaces, Lemma 66.12.4). Denote this morphism by $\phi_{red} : U \to X_{red}$. Since $\phi \circ s = \phi \circ t$ we see that also $\phi_{red} \circ s = \phi_{red} \circ t$ (as $i : X_{red} \to X$ is a monomorphism). Hence by the universal property of ϕ there exists a morphism $\chi : X \to X_{red}$ such that $\phi_{red} = \phi \circ \chi$. By uniqueness we see that $i \circ \chi = \mathrm{id}_X$ and $\chi \circ i = \mathrm{id}_{X_{red}}$. Hence i is an isomorphism and X is reduced.

To show that this argument works in a category $\mathcal{C}$ one just needs to show that the reduction of an object of $\mathcal{C}$ is an object of $\mathcal{C}$. We omit the verification that this holds for each of the standard examples. □

83.5. Quotients as orbit spaces

048M Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. If j is a pre-equivalence relation, then loosely speaking the "orbits" of R on U are the subsets $t(s^{-1}(\{u\}))$ of U. However, if j is just a pre-relation, then we need to take the equivalence relation generated by R.

048N **Definition 83.5.1.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. If $u \in |U|$, then the *orbit*, or more precisely the *R-orbit* of u is

$$O_u = \left\{ u' \in |U| \ : \begin{array}{c} \exists n \geq 1,\ \exists u_0, \ldots, u_n \in |U| \text{ such that } u_0 = u \text{ and } u_n = u' \\ \text{and for all } i \in \{0, \ldots, n-1\} \text{ either } u_i = u_{i+1} \text{ or} \\ \exists r \in |R|,\ s(r) = u_i, t(r) = u_{i+1} \text{ or} \\ \exists r \in |R|,\ t(r) = u_i, s(r) = u_{i+1} \end{array} \right\}$$

It is clear that these are the equivalence classes of an equivalence relation, i.e., we have $u' \in O_u$ if and only if $u \in O_{u'}$. The following lemma is a reformulation of Groupoids in Spaces, Lemma 78.4.4.

048O **Lemma 83.5.2.** *Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-equivalence relation of algebraic spaces over B. Then*

$$O_u = \{u' \in |U| \text{ such that } \exists r \in |R|,\ s(r) = u,\ t(r) = u'\}.$$

Proof. By the aforementioned Groupoids in Spaces, Lemma 78.4.4 we see that the orbits O_u as defined in the lemma give a disjoint union decomposition of $|U|$. Thus we see they are equal to the orbits as defined in Definition 83.5.1. □

048P **Lemma 83.5.3.** *In the situation of Definition 83.5.1. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then $|\phi| : |U| \to |X|$ is constant on the orbits.*

Proof. To see this we just have to show that $\phi(u) = \phi(u')$ for all $u, u' \in |U|$ such that there exists an $r \in |R|$ such that $s(r) = u$ and $t(r) = u'$. And this is clear since ϕ equalizes s and t. □

There are several problems with considering the orbits $O_u \subset |U|$ as a tool for singling out properties of quotient maps. One issue is the following. Suppose that $\mathrm{Spec}(k) \to B$ is a geometric point of B. Consider the canonical map

$$U(k) \longrightarrow |U|.$$

Then it is usually not the case that the equivalence classes of the equivalence relation generated by $j(R(k)) \subset U(k) \times U(k)$ are the inverse images of the orbits $O_u \subset |U|$. A silly example is to take $S = B = \mathrm{Spec}(\mathbf{Z})$, $U = R = \mathrm{Spec}(k)$ with $s = t = \mathrm{id}_k$. Then $|U| = |R|$ is a single point but $U(k)/R(k)$ is enormous. A more interesting example is to take $S = B = \mathrm{Spec}(\mathbf{Q})$, choose some of number fields $K \subset L$, and set $U = \mathrm{Spec}(L)$ and $R = \mathrm{Spec}(L \otimes_K L)$ with obvious maps $s, t : R \to U$. In this case $|U|$ still has just one point, but the quotient

$$U(k)/R(k) = \mathrm{Hom}(K, k)$$

consists of more than one element. We conclude from both examples that if $U \to X$ is an R-invariant map and if we want it to "separate orbits" we get a much stronger and interesting notion by considering the induced maps $U(k) \to X(k)$ and ask that those maps separate orbits.

There is an issue with this too. Namely, suppose that $S = B = \mathrm{Spec}(\mathbf{R})$, $U = \mathrm{Spec}(\mathbf{C})$, and $R = \mathrm{Spec}(\mathbf{C}) \amalg \mathrm{Spec}(K)$ for some field extension $\sigma : \mathbf{C} \to K$. Let the maps s, t be given by the identity on the component $\mathrm{Spec}(\mathbf{C})$, but by $\sigma, \sigma \circ \tau$ on the second component where τ is complex conjugation. If K is a nontrivial extension of $\mathbf{C}$, then the two points $1, \tau \in U(\mathbf{C})$ are not equivalent under $j(R(\mathbf{C}))$. But after choosing an extension $\mathbf{C} \subset \Omega$ of sufficiently large cardinality (for example larger than the cardinality of K) then the images of $1, \tau \in U(\mathbf{C})$ in $U(\Omega)$ do become equivalent! It seems intuitively clear that this happens either because $s, t : R \to U$ are not locally of finite type or because the cardinality of the field k is not large enough.

Keeping this in mind we make the following definition.

048Q **Definition 83.5.4.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.

(1) We say $\overline{u}, \overline{u}' \in U(k)$ are *weakly R-equivalent* if they are in the same equivalence class for the equivalence relation generated by the relation $j(R(k)) \subset U(k) \times U(k)$.
(2) We say $\overline{u}, \overline{u}' \in U(k)$ are *R-equivalent* if for some overfield $k \subset \Omega$ the images in $U(\Omega)$ are weakly R-equivalent.
(3) The *weak orbit*, or more precisely the *weak R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are weakly R-equivalent to $\overline{u}$.
(4) The *orbit*, or more precisely the *R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are R-equivalent to $\overline{u}$.

It turns out that in good cases orbits and weak orbits agree, see Lemma 83.5.7. The following lemma illustrates the difference in the special case of a pre-equivalence relation.

048R **Lemma 83.5.5.** *Let S be a scheme, and let B be an algebraic space over S. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. In this case the weak orbit of $\overline{u} \in U(k)$ is simply*

$$\{\overline{u}' \in U(k) \text{ such that } \exists \overline{r} \in R(k),\ s(\overline{r}) = \overline{u},\ t(\overline{r}) = \overline{u}'\}$$

and the orbit of $\overline{u} \in U(k)$ is

$$\{\overline{u}' \in U(k) : \exists \text{ field extension } K/k,\ \exists\ r \in R(K),\ s(r) = \overline{u},\ t(r) = \overline{u}'\}$$

Proof. This is true because by definition of a pre-equivalence relation the image $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation. □

Let us describe the recipe for turning any pre-relation into a pre-equivalence relation. We will use the morphisms

048S (83.5.5.1)
$$\begin{array}{lcccccc} j_{diag} & : & U & \longrightarrow & U \times_B U, & u & \longmapsto & (u, u) \\ j_{flip} & : & R & \longrightarrow & U \times_B U, & r & \longmapsto & (s(r), t(r)) \\ j_{comp} & : & R \times_{s,U,t} R & \longrightarrow & U \times_B U, & (r, r') & \longmapsto & (t(r), s(r')) \end{array}$$

We define $j_1 = (t_1, s_1) : R_1 \to U \times_B U$ to be the morphism

$$j \amalg j_{diag} \amalg j_{flip} : R \amalg U \amalg R \longrightarrow U \times_B U$$

with notation as in Equation (83.5.5.1). For $n > 1$ we set

$$j_n = (t_n, s_n) : R_n = R_1 \times_{s_1,U,t_{n-1}} R_{n-1} \longrightarrow U \times_B U$$

where t_n comes from t_1 precomposed with projection onto R_1 and s_n comes from s_{n-1} precomposed with projection onto R_{n-1}. Finally, we denote

$$j_\infty = (t_\infty, s_\infty) : R_\infty = \coprod\nolimits_{n \geq 1} R_n \longrightarrow U \times_B U.$$

048T **Lemma 83.5.6.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Then $j_\infty : R_\infty \to U \times_B U$ is a pre-equivalence relation over B. Moreover*

1. *$\phi : U \to X$ is R-invariant if and only if it is R_∞-invariant,*
2. *the canonical map of quotient sheaves $U/R \to U/R_\infty$ (see Groupoids in Spaces, Section 78.19) is an isomorphism,*
3. *weak R-orbits agree with weak R_∞-orbits,*
4. *R-orbits agree with R_∞-orbits,*
5. *if s, t are locally of finite type, then s_∞, t_∞ are locally of finite type,*
6. *add more here as needed.*

Proof. Omitted. Hint for (5): Any property of s, t which is stable under composition and stable under base change, and Zariski local on the source will be inherited by s_∞, t_∞. □

048U **Lemma 83.5.7.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.*

1. *If $s, t : R \to U$ are locally of finite type then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*

(2) *If k has sufficiently large cardinality then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*

Proof. We first prove (1). Assume s, t locally of finite type. By Lemma 83.5.6 we may assume that R is a pre-equivalence relation. Let k be an algebraically closed field over B. Suppose $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent. Then for some extension field Ω/k there exists a point $\overline{r} \in R(\Omega)$ mapping to $(\overline{u}, \overline{u}') \in (U \times_B U)(\Omega)$, see Lemma 83.5.5. Hence

$$Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$$

is nonempty. As s is locally of finite type we see that also j is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. This implies Z is a nonempty algebraic space locally of finite type over the algebraically closed field k (use Morphisms of Spaces, Lemma 67.23.3). Thus Z has a k-valued point, see Morphisms of Spaces, Lemma 67.24.1. Hence we conclude there exists a $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$, and we conclude that $\overline{u}, \overline{u}'$ are R-equivalent as desired.

The proof of part (2) is the same, except that it uses Morphisms of Spaces, Lemma 67.24.2 instead of Morphisms of Spaces, Lemma 67.24.1. This shows that the assertion holds as soon as $|k| > \lambda(R)$ with $\lambda(R)$ as introduced just above Morphisms of Spaces, Lemma 67.24.1. □

In the following definition we use the terminology "k is a field over B" to mean that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to B$.

048V **Definition 83.5.8.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say $\phi : U \to X$ is *set-theoretically R-invariant* if and only if the map $U(k) \to X(k)$ equalizes the two maps $s, t : R(k) \to U(k)$ for every algebraically closed field k over B.
(2) We say $\phi : U \to X$ *separates orbits*, or *separates R-orbits* if it is set-theoretically R-invariant and $\phi(\overline{u}) = \phi(\overline{u}')$ in $X(k)$ implies that $\overline{u}, \overline{u}' \in U(k)$ are in the same orbit for every algebraically closed field k over B.

In Example 83.5.12 we show that being set-theoretically invariant is "too weak" a notion in the category of algebraic spaces. A more geometric reformulation of what it means to be set-theoretically invariant or to separate orbits is in Lemma 83.5.17.

048W **Lemma 83.5.9.** *In the situation of Definition 83.5.8. A morphism $\phi : U \to X$ is set-theoretically R-invariant if and only if for any algebraically closed field k over B the map $U(k) \to X(k)$ is constant on orbits.*

Proof. This is true because the condition is supposed to hold for all algebraically closed fields over B. □

048X **Lemma 83.5.10.** *In the situation of Definition 83.5.8. An invariant morphism is set-theoretically invariant.*

Proof. This is immediate from the definitions. □

048Y **Lemma 83.5.11.** *In the situation of Definition 83.5.8. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Assume*

(1) *ϕ is set-theoretically R-invariant,*
(2) *R is reduced, and*

(3) *X is locally separated over B.*

Then ϕ is R-invariant.

Proof. Consider the equalizer

$$Z = R \times_{(\phi,\phi)\circ j, X\times_B X, \Delta_{X/B}} X$$

algebraic space. Then $Z \to R$ is an immersion by assumption (3). By assumption (1) $|Z| \to |R|$ is surjective. This implies that $Z \to R$ is a bijective closed immersion (use Schemes, Lemma 26.10.4) and by assumption (2) we conclude that $Z = R$. □

048Z **Example 83.5.12.** There exist reduced quasi-separated algebraic spaces X, Y and a pair of morphisms $a, b : Y \to X$ which agree on all k-valued points but are not equal. To get an example take $Y = \mathrm{Spec}(k[[x]])$ and

$$X = \mathbf{A}^1_k \Big/ \big(\Delta \amalg \{(x, -x) \mid x \neq 0\}\big)$$

the algebraic space of Spaces, Example 65.14.1. The two morphisms $a, b : Y \to X$ come from the two maps $x \mapsto x$ and $x \mapsto -x$ from Y to $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$. On the generic point the two maps are the same because on the open part $x \neq 0$ of the space X the functions x and $-x$ are equal. On the closed point the maps are obviously the same. It is also true that $a \neq b$. This implies that Lemma 83.5.11 does not hold with assumption (3) replaced by the assumption that X be quasi-separated. Namely, consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{1} & Y \\ {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{a} & X \end{array}$$

then the composition $a \circ (-1) = b$. Hence we can set $R = Y$, $U = Y$, $s = 1$, $t = -1$, $\phi = a$ to get an example of a set-theoretically invariant morphism which is not invariant.

The example above is instructive because the map $Y \to X$ even separates orbits. It shows that in the category of algebraic spaces there are simply too many set-theoretically invariant morphisms lying around. Next, let us define what it means for R to be a set-theoretic equivalence relation, while remembering that we need to allow for field extensions to make this work correctly.

0490 **Definition 83.5.13.** Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say j is a *set-theoretic pre-equivalence relation* if for all algebraically closed fields k over B the relation $\sim_R$ on $U(k)$ defined by

$$\overline{u} \sim_R \overline{u}' \Leftrightarrow \begin{array}{c} \exists \text{ field extension } K/k,\ \exists\ r \in R(K), \\ s(r) = \overline{u},\ t(r) = \overline{u}' \end{array}$$

is an equivalence relation.

(2) We say j is a *set-theoretic equivalence relation* if j is universally injective and a set-theoretic pre-equivalence relation.

Let us reformulate this in more geometric terms.

0491 **Lemma 83.5.14.** *In the situation of Definition 83.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic pre-equivalence relation.*
(2) *The subset $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (83.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

If s, t are locally of finite type these are also equivalent to

(4) *For every algebraically closed field k over B the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

Proof. Assume (2). Let k be an algebraically closed field over B. We are going to show that $\sim_R$ is an equivalence relation. Suppose that $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2$ are k-valued points of U. Suppose that $(\overline{u}_1, \overline{u}_2)$ is the image of a K-valued point $r \in R(K)$. Consider the solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_2, \overline{u}_1)} & & \downarrow \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{flip}} & R \end{array}$$

We also denote $r \in |R|$ the image of r. By assumption the image of $|j_{flip}|$ is contained in the image of $|j|$, in other words there exists a $r' \in |R|$ such that $|j|(r') = |j_{flip}|(r)$. But note that $(\overline{u}_2, \overline{u}_1)$ is in the equivalence class that defines $|j|(r')$ (by the commutativity of the solid part of the diagram). This means there exists a field extension K'/k and a morphism $r' : \mathrm{Spec}(K) \to R$ (abusively denoted r' as well) with $j \circ r' = (\overline{u}_2, \overline{u}_1) \circ i$ where $i : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is the obvious map. In other words the dotted part of the diagram commutes. This proves that $\sim_R$ is a symmetric relation on $U(k)$. In the similar way, using that the image of $|j_{diag}|$ is contained in the image of $|j|$ we see that $\sim_R$ is reflexive (details omitted).

To show that $\sim_R$ is transitive assume given $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2, 3$ and field extensions K_i/k and points $r_i : \mathrm{Spec}(K_i) \to R$, $i = 1, 2$ such that $j(r_1) = (\overline{u}_1, \overline{u}_2)$ and $j(r_1) = (\overline{u}_2, \overline{u}_3)$. Then we may choose a commutative diagram of fields

$$\begin{array}{ccc} K & \longleftarrow & K_2 \\ \uparrow & & \uparrow \\ K_1 & \longleftarrow & k \end{array}$$

and we may think of $r_1, r_2 \in R(K)$. We consider the commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_1, \overline{u}_3)} & & \downarrow{\scriptstyle (r_1, r_2)} \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{comp}} & R \times_{s,U,t} R \end{array}$$

By exactly the same reasoning as in the first part of the proof, but this time using that $|j_{comp}|((r_1, r_2))$ is in the image of $|j|$, we conclude that a field K' and dotted arrows exist making the diagram commute. This proves that $\sim_R$ is transitive and concludes the proof that (2) implies (1).

Assume (1) and let k be an algebraically closed field over B whose cardinality is larger than $\lambda(R)$, see Morphisms of Spaces, Lemma 67.24.2. Suppose that $\overline{u} \sim_R \overline{u}'$

with $\overline{u}, \overline{u}' \in U(k)$. By assumption there exists a point in $|R|$ mapping to $(\overline{u}, \overline{u}') \in |U \times_B U|$. Hence by Morphisms of Spaces, Lemma 67.24.2 we conclude there exists an $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$. In this way we see that (1) implies (3).

Assume (3). Let us show that $\mathrm{Im}(|j_{comp}|) \subset \mathrm{Im}(|j|)$. Pick any point $c \in |R \times_{s,U,t} R|$. We may represent this by a morphism $\overline{c} : \mathrm{Spec}(k) \to R \times_{s,U,t} R$, with k over B having sufficiently large cardinality. By assumption we see that $j_{comp}(\overline{c}) \in U(k) \times U(k) = (U \times_B U)(k)$ is also the image $j(\overline{r})$ for some $\overline{r} \in R(k)$. Hence $j_{comp}(c) = j(r)$ in $|U \times_B U|$ as desired (with $r \in |R|$ the equivalence class of $\overline{r}$). The same argument shows also that $\mathrm{Im}(|j_{diag}|) \subset \mathrm{Im}(|j|)$ and $\mathrm{Im}(|j_{flip}|) \subset \mathrm{Im}(|j|)$ (details omitted). In this way we see that (3) implies (2). At this point we have shown that (1), (2) and (3) are all equivalent.

It is clear that (4) implies (3) (without any assumptions on s, t). To finish the proof of the lemma we show that (1) implies (4) if s, t are locally of finite type. Namely, let k be an algebraically closed field over B. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption the algebraic space $Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. On the other hand, since $j = (t, s)$ is locally of finite type the morphism $Z \to \mathrm{Spec}(k)$ is locally of finite type as well (use Morphisms of Spaces, Lemmas 67.23.6 and 67.23.3). Hence Z has a k point by Morphisms of Spaces, Lemma 67.24.1 and we conclude that $(\overline{u}, \overline{u}') \in j(R(k))$ as desired. This finishes the proof of the lemma. □

049X **Lemma 83.5.15.** *In the situation of Definition 83.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic equivalence relation.*
(2) *The morphism j is universally injective and $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (83.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

If j is decent, or locally separated, or quasi-separated these are also equivalent to

(4) *For every algebraically closed field k over B the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from Lemma 83.5.14 and the definitions. The same lemma shows that (3) implies j is a set-theoretic pre-equivalence relation. But of course condition (3) also implies that j is universally injective, see Morphisms of Spaces, Lemma 67.19.2, so that j is indeed a set-theoretic equivalence relation. At this point we know that (1), (2), (3) are all equivalent.

Condition (4) implies (3) without any further hypotheses on j. Assume j is decent, or locally separated, or quasi-separated and the equivalent conditions (1), (2), (3) hold. By More on Morphisms of Spaces, Lemma 76.3.4 we see that j is radicial. Let k be any algebraically closed field over B. Let $\overline{u}, \overline{u}' \in U(k)$ with $\overline{u} \sim_R \overline{u}'$. We see that $R \times_{U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. Hence, as j is radicial, its reduction is the spectrum of a field purely inseparable over k. As $k = \overline{k}$ we see that it is the spectrum of k. Whence a point $\overline{r} \in R(k)$ with $t(\overline{r}) = \overline{u}$ and $s(\overline{r}) = \overline{u}'$ as desired. □

0492 **Lemma 83.5.16.** *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.*

(1) *If j is a pre-equivalence relation, then j is a set-theoretic pre-equivalence relation. This holds in particular when j comes from a groupoid in algebraic spaces, or from an action of a group algebraic space on U.*
(2) *If j is an equivalence relation, then j is a set-theoretic equivalence relation.*

Proof. Omitted. □

049Y **Lemma 83.5.17.** *Let $B \to S$ be as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Consider the diagram*

$$\begin{array}{ccc} (U \times_X U) \times_{(U \times_B U)} R & \xrightarrow{p} & R \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U \times_X U & \xrightarrow{c} & U \times_B U \end{array}$$

Then we have:

(1) *The morphism ϕ is set-theoretically invariant if and only if p is surjective.*
(2) *If j is a set-theoretic pre-equivalence relation then ϕ separates orbits if and only if p and q are surjective.*
(3) *If p and q are surjective, then j is a set-theoretic pre-equivalence relation (and ϕ separates orbits).*
(4) *If ϕ is R-invariant and j is a set-theoretic pre-equivalence relation, then ϕ separates orbits if and only if the induced morphism $R \to U \times_X U$ is surjective.*

Proof. Assume ϕ is set-theoretically invariant. This means that for any algebraically closed field k over B and any $\overline{r} \in R(k)$ we have $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. Hence $((\phi(t(\overline{r})), \phi(s(\overline{r}))), \overline{r})$ defines a point in the fibre product mapping to $\overline{r}$ via p. This shows that p is surjective. Conversely, assume p is surjective. Pick $\overline{r} \in R(k)$. As p is surjective, we can find a field extension K/k and a K-valued point $\tilde{r}$ of the fibre product with $p(\tilde{r}) = \overline{r}$. Then $q(\tilde{r}) \in U \times_X U$ maps to $(t(\overline{r}), s(\overline{r}))$ in $U \times_B U$ and we conclude that $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. This proves that ϕ is set-theoretically invariant.

The proofs of (2), (3), and (4) are omitted. Hint: Assume k is an algebraically closed field over B of large cardinality. Consider the associated diagram of sets

$$\begin{array}{ccc} (U(k) \times_{X(k)} U(k)) \times_{U(k) \times U(k)} R(k) & \xrightarrow{p} & R(k) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U(k) \times_{X(k)} U(k) & \xrightarrow{c} & U(k) \times U(k) \end{array}$$

By the lemmas above the equivalences posed in (2), (3), and (4) become set-theoretic questions related to the diagram we just displayed, using that surjectivity translates into surjectivity on k-valued points by Morphisms of Spaces, Lemma 67.24.2. □

Because we have seen above that the notion of a set-theoretically invariant morphism is a rather weak one in the category of algebraic spaces, we define an orbit space for a pre-relation as follows.

0493 **Definition 83.5.18.** Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a pre-relation. We say $\phi : U \to X$ is an *orbit space for* R if

(1) ϕ is R-invariant,
(2) ϕ separates R-orbits, and
(3) ϕ is surjective.

The definition of separating R-orbits involves a discussion of points with values in algebraically closed fields. But as we've seen in many cases this just corresponds to the surjectivity of certain canonically associated morphisms of algebraic spaces. We summarize some of the discussion above in the following characterization of orbit spaces.

049Z **Lemma 83.5.19.** *Let $B \to S$ as in Section 83.2. Let $j : R \to U \times_B U$ be a set-theoretic pre-equivalence relation. A morphism $\phi : U \to X$ is an orbit space for R if and only if*

(1) *$\phi \circ s = \phi \circ t$, i.e., ϕ is invariant,*
(2) *the induced morphism $(t, s) : R \to U \times_X U$ is surjective, and*
(3) *the morphism $\phi : U \to X$ is surjective.*

This characterization applies for example if j is a pre-equivalence relation, or comes from a groupoid in algebraic spaces over B, or comes from the action of a group algebraic space over B on U.

Proof. Follows immediately from Lemma 83.5.17 part (4). □

In the following lemma it is (probably) not good enough to assume just that the morphisms s, t are locally of finite type. The reason is that it may happen that some map $\phi : U \to X$ is an orbit space, yet is not locally of finite type. In that case $U(k) \to X(k)$ may not be surjective for all algebraically closed fields k over B.

04A0 **Lemma 83.5.20.** *Let $B \to S$ as in Section 83.2. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. Assume R, U are locally of finite type over B. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then ϕ is an orbit space for R if and only if the natural map*

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \longrightarrow X(k)$$

is bijective for all algebraically closed fields k over B.

Proof. Note that since U, R are locally of finite type over B all of the morphisms s, t, j, ϕ are locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. We will also use without further mention Morphisms of Spaces, Lemma 67.24.1. Assume ϕ is an orbit space. Let k be any algebraically closed field over B. Let $\overline{x} \in X(k)$. Consider $U \times_{\phi, X, \overline{x}} \mathrm{Spec}(k)$. This is a nonempty algebraic space which is locally of finite type over k. Hence it has a k-valued point. This shows the displayed map of the lemma is surjective. Suppose that $\overline{u}, \overline{u}' \in U(k)$ map to the same element of $X(k)$. By Definition 83.5.8 this means that $\overline{u}, \overline{u}'$ are in the same R-orbit. By Lemma 83.5.7 this means that they are equivalent under the equivalence relation generated by $j(R(k))$. Thus the displayed morphism is injective.

Conversely, assume the displayed map is bijective for all algebraically closed fields k over B. This condition clearly implies that ϕ is surjective. We have already assumed that ϕ is R-invariant. Finally, the injectivity of all the displayed maps implies that ϕ separates orbits. Hence ϕ is an orbit space. □

83.6. Coarse quotients

04A1 We only add this here so that we can later say that coarse quotients correspond to coarse moduli spaces (or moduli schemes).

04A2 **Definition 83.6.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *coarse quotient* if

(1) ϕ is a categorical quotient, and
(2) ϕ is an orbit space.

If $S = B$, U, R are all schemes, then we say a morphism of schemes $\phi : U \to X$ is a *coarse quotient in schemes* if

(1) ϕ is a categorical quotient in schemes, and
(2) ϕ is an orbit space.

In many situations the algebraic spaces R and U are locally of finite type over B and the orbit space condition simply means that

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \cong X(k)$$

for all algebraically closed fields k. See Lemma 83.5.20. If j is also a (set-theoretic) pre-equivalence relation, then the condition is simply equivalent to $U(k)/j(R(k)) \to X(k)$ being bijective for all algebraically closed fields k.

83.7. Topological properties

04A3 Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. We say a subset $T \subset |U|$ is *R-invariant* if $s^{-1}(T) = t^{-1}(T)$ as subsets of $|R|$. Note that if T is closed, then it may not be the case that the corresponding reduced closed subspace of U is R-invariant (as in Groupoids in Spaces, Definition 78.18.1) because the pullbacks $s^{-1}(T)$, $t^{-1}(T)$ may not be reduced. Here are some conditions that we can consider for an invariant morphism $\phi : U \to X$.

04A4 **Definition 83.7.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B.

04A5 (1) The morphism ϕ is submersive.
04A6 (2) For any R-invariant closed subset $Z \subset |U|$ the image $\phi(Z)$ is closed in $|X|$.
04A7 (3) Condition (2) holds and for any pair of R-invariant closed subsets $Z_1, Z_2 \subset |U|$ we have
$$\phi(Z_1 \cap Z_2) = \phi(Z_1) \cap \phi(Z_2)$$
04A8 (4) The morphism $(t, s) : R \to U \times_X U$ is universally submersive.

For each of these properties we can also require them to hold after any flat base change, or after any base change, see Definition 83.3.4. In this case we say condition (1), (2), (3), or (4) holds *uniformly* or *universally*.

83.8. Invariant functions

04A9 In some cases it is convenient to pin down the structure sheaf of a quotient by requiring any invariant function to be a local section of the structure sheaf of the quotient.

04AA **Definition 83.8.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism. Denote $\phi' = \phi \circ s = \phi \circ t : R \to X$.

(1) We denote $(\phi_*\mathcal{O}_U)^R$ the $\mathcal{O}_X$-sub-algebra of $\phi_*\mathcal{O}_U$ which is the equalizer of the two maps

$$\phi_*\mathcal{O}_U \xrightarrow[\phi_* t^\sharp]{\phi_* s^\sharp} \phi'_*\mathcal{O}_R$$

on $X_{étale}$. We sometimes call this the *sheaf of R-invariant functions on X*.

(2) We say *the functions on X are the R-invariant functions on U* if the natural map $\mathcal{O}_X \to (\phi_*\mathcal{O}_U)^R$ is an isomorphism.

Of course we can require this property holds after any (flat or any) base change, leading to a (uniform or) universal notion. This condition is often thrown in with other conditions in order to obtain a (more) unique quotient. And of course a good deal of motivation for the whole subject comes from the following special case: $U = \mathrm{Spec}(A)$ is an affine scheme over a field $S = B = \mathrm{Spec}(k)$ and where $R = G \times U$, with G an affine group scheme over k. In this case you have the option of taking for the quotient:

$$X = \mathrm{Spec}(A^G)$$

so that at least the condition of the definition above is satisfied. Even though this is a nice thing you can do it is often not the right quotient; for example if $U = \mathrm{GL}_{n,k}$ and G is the group of upper triangular matrices, then the above gives $X = \mathrm{Spec}(k)$, whereas a much better quotient (namely the flag variety) exists.

83.9. Good quotients

04AB Especially when taking quotients by group actions the following definition is useful.

04AC **Definition 83.9.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *good quotient* if

(1) ϕ is invariant,
(2) ϕ is affine,
(3) ϕ is surjective,
(4) condition (3) holds universally, and
(5) the functions on X are the R-invariant functions on U.

In [**Ses72**] Seshadri gives almost the same definition, except that instead of (4) he simply requires the condition (3) to hold – he does not require it to hold universally.

83.10. Geometric quotients

04AD This is Mumford's definition of a geometric quotient (at least the definition from the first edition of GIT; as far as we can tell later editions changed "universally submersive" to "submersive").

04AE **Definition 83.10.1.** Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *geometric quotient* if

(1) ϕ is an orbit space,
(2) condition (1) holds universally, i.e., ϕ is universally submersive, and
(3) the functions on X are the R-invariant functions on U.

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CHAPTER 84

More on Cohomology of Spaces

0DFR

84.1. Introduction

0DFS In this chapter continues the discussion started in Cohomology of Spaces, Section 69.1. One can also view this chapter as the analogue for algebraic spaces of the chapter on étale cohomology for schemes, see Étale Cohomology, Section 59.1.

In fact, we intend this chapter to be mainly a translation of the results already proved for schemes into the language of algebraic spaces. Some of our results can be found in [**Knu71**].

84.2. Conventions

0DFT The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

84.3. Transporting results from schemes

0DFU In this section we explain briefly how results for schemes imply results for (representable) algebraic spaces and (representable) morphisms of algebraic spaces. For quasi-coherent modules more is true (because étale cohomology of a quasi-coherent module over a scheme agrees with Zariski cohomology) and this has already been discussed in Cohomology of Spaces, Section 69.3.

Let S be a scheme. Let X be an algebraic space over S. Now suppose that X is representable by the scheme X_0 (awkward but temporary notation; we usually just say "X is a scheme"). In this case X and X_0 have the same small étale sites:

$$X_{étale} = (X_0)_{étale}$$

This is pointed out in Properties of Spaces, Section 66.18. Moreover, if $f : X \to Y$ is a morphism of representable algebraic spaces over S and if $f_0 : X_0 \to Y_0$ is a morphism of schemes representing f, then the induced morphisms of small étale topoi agree:

$$\begin{array}{ccc} Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \\ \| & & \| \\ Sh((X_0)_{étale}) & \xrightarrow{(f_0)_{small}} & Sh((Y_0)_{étale}) \end{array}$$

See Properties of Spaces, Lemma 66.18.8 and Topologies, Lemma 34.4.17.

Thus there is absolutely no difference between étale cohomology of a scheme and the étale cohomology of the corresponding algebraic space. Similarly for higher direct images along morphisms of schemes. In fact, if $f : X \to Y$ is a morphism of algebraic spaces over S which is representable (by schemes), then the higher direct images $R^i f_* \mathcal{F}$ of a sheaf $\mathcal{F}$ on $X_{étale}$ can be computed étale locally on Y (Cohomology on Sites, Lemma 21.7.4) hence this often reduces computations and proofs to the case where Y and X are schemes.

We will use the above without further mention in this chapter. For other topologies the same thing is true; we state it explicitly as a lemma for cohomology here.

0DFV **Lemma 84.3.1.** *Let S be a scheme. Let $\tau \in \{étale, fppf, ph\}$ (add more here). The inclusion functor*

$$(Sch/S)_\tau \longrightarrow (Spaces/S)_\tau$$

is a special cocontinuous functor (Sites, Definition 7.29.2) and hence identifies topoi.

Proof. The conditions of Sites, Lemma 7.29.1 are immediately verified as our functor is fully faithful and as every algebraic space has an étale covering by schemes. □

84.4. Proper base change

0DFW The proper base change theorem for algebraic spaces follows from the proper base change theorem for schemes and Chow's lemma with a little bit of work.

0DFX **Lemma 84.4.1.** *Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is injective with image the equalizer of the two maps $f_* f^{-1}\mathcal{F} \to g_* g^{-1}\mathcal{F}$ where g is the structure morphism $g : Y \times_X Y \to X$.*

Proof. For any surjective morphism $f : Y \to X$ of algebraic spaces over S, the map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is injective. Namely, if $\overline{x}$ is a geometric point of X, then we choose a geometric point $\overline{y}$ of Y lying over $\overline{x}$ and we consider

$$\mathcal{F}_{\overline{x}} \to (f_* f^{-1}\mathcal{F})_{\overline{x}} \to (f^{-1}\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}}$$

See Properties of Spaces, Lemma 66.19.9 for the last equality.

The second statement is local on X in the étale topology, hence we may and do assume Y is an affine scheme.

Choose a surjective proper morphism $Z \to Y$ where Z is a scheme, see Cohomology of Spaces, Lemma 69.18.1. The result for $Z \to X$ implies the result for $Y \to X$. Since $Z \to X$ is a surjective proper morphism of schemes and hence a ph covering (Topologies, Lemma 34.8.6) the result for $Z \to X$ follows from Étale Cohomology, Lemma 59.102.1 (in fact it is in some sense equivalent to this lemma). □

0DFY **Lemma 84.4.2.** *Let (A, I) be a henselian pair. Let X be an algebraic space over A such that the structure morphism $f : X \to \mathrm{Spec}(A)$ is proper. Let $i : X_0 \to X$ be the inclusion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, i^{-1}\mathcal{F})$.*

Proof. Choose a surjective proper morphism $Y \to X$ where Y is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Consider the diagram

$$\begin{array}{ccccc} \Gamma(X_0, \mathcal{F}_0) & \longrightarrow & \Gamma(Y_0, \mathcal{G}_0) & \rightrightarrows & \Gamma((Y \times_X Y)_0, \mathcal{H}_0) \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(Y, \mathcal{G}) & \rightrightarrows & \Gamma(Y \times_X Y, \mathcal{H}) \end{array}$$

Here $\mathcal{G}$, resp. $\mathcal{H}$ is the pullbackf or $\mathcal{F}$ to Y, resp. $Y \times_X Y$ and the index 0 indicates base change to $\mathrm{Spec}(A/I)$. By the case of schemes (Étale Cohomology, Lemma 59.91.2) we see that the middle and right vertical arrows are bijective. By Lemma 84.4.1 it follows that the left one is too. □

0DFZ **Lemma 84.4.3.** *Let A be a henselian local ring. Let X be an algebraic space over A such that $f : X \to \mathrm{Spec}(A)$ is a proper morphism. Let $X_0 \subset X$ be the fibre of f over the closed point. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(X_0, \mathcal{F}|_{X_0})$.*

Proof. This is a special case of Lemma 84.4.2. □

0DG0 **Lemma 84.4.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y' \to Y$ be a morphisms of algebraic spaces over S. Assume f is proper. Set $X' = Y' \times_Y X$ with projections $f' : X' \to Y'$ and $g' : X' \to X$. Let $\mathcal{F}$ be any sheaf on $X_{étale}$. Then $g^{-1} f_* \mathcal{F} = f'_* (g')^{-1} \mathcal{F}$.*

Proof. The question is étale local on Y'. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then we may replace Y' by V' and Y by V. Hence we may assume Y and Y' are schemes. Then we may work Zariski locally on Y and Y' and hence we may assume Y and Y' are affine schemes.

Assume Y and Y' are affine schemes. Choose a surjective proper morphism $h_1 : X_1 \to X$ where X_1 is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Set $X_2 = X_1 \times_X X_1$ and denote $h_2 : X_2 \to X$ the structure morphism. Observe this is a scheme. By the case of schemes (Étale Cohomology, Lemma 59.91.5) we know the lemma is true for the cartesian diagrams

$$\begin{array}{ccc} X'_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X'_2 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

and the sheaves $\mathcal{F}_i = (X_i \to X)^{-1}\mathcal{F}$. By Lemma 84.4.1 we have an exact sequence $0 \to \mathcal{F} \to h_{1,*}\mathcal{F}_1 \to h_{2,*}\mathcal{F}_2$ and similarly for $(g')^{-1}\mathcal{F}$ because $X'_2 = X'_1 \times_{X'} X'_1$. Hence we conlude that the lemma is true (some details omitted). □

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\overline{x} : \mathrm{Spec}(k) \to S$ be a geometric point. The fibre of f at $\overline{x}$ is the algebraic space $Y_{\overline{x}} = \mathrm{Spec}(k) \times_{\overline{x}, X} Y$ over $\mathrm{Spec}(k)$. If $\mathcal{F}$ is a sheaf on $Y_{étale}$, then denote $\mathcal{F}_{\overline{x}} = p^{-1}\mathcal{F}$ the pullback of $\mathcal{F}$ to $(Y_{\overline{x}})_{étale}$. Here $p : Y_{\overline{x}} \to Y$ is the projection. In the following we will consider the set $\Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$.

0DG1 **Lemma 84.4.5.** *Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Let $\overline{x} \to X$ be a geometric point. For any sheaf $\mathcal{F}$ on $Y_{étale}$ the canonical map*

$$(f_*\mathcal{F})_{\overline{x}} \longrightarrow \Gamma(Y_{\overline{x}}, \mathcal{F}_{\overline{x}})$$

is bijective.

Proof. This is a special case of Lemma 84.4.4. □

0DG2 **Theorem 84.4.6.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then the base change map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. This proof repeats a few of the arguments given in the proof of the proper base change theorem for schemes. See Étale Cohomology, Section 59.91 for more details.

The statement is étale local on Y' and Y, hence we may assume both Y and Y' are affine schemes. Observe that this in particular proves the theorem in case f is representable (we will use this below).

For every $n \geq 1$ let $\mathcal{F}[n]$ be the subsheaf of sections of $\mathcal{F}$ annihilated by n. Then $\mathcal{F} = \operatorname{colim} \mathcal{F}[n]$. By Cohomology of Spaces, Lemma 69.5.2 the functors $g^{-1}R^pf_*$ and $R^pf'_*(g')^{-1}$ commute with filtered colimits. Hence it suffices to prove the theorem if $\mathcal{F}$ is killed by n.

Let $\mathcal{F} \to \mathcal{I}^\bullet$ be a resolution by injective sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules. Observe that $g^{-1}f_*\mathcal{I}^\bullet = f'_*(g')^{-1}\mathcal{I}^\bullet$ by Lemma 84.4.4. Applying Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we conclude it suffices to prove $R^pf'_*(g')^{-1}\mathcal{I}^m = 0$ for $p > 0$ and $m \in \mathbf{Z}$.

Choose a surjective proper morphism $h : Z \to X$ where Z is a scheme, see Cohomology of Spaces, Lemma 69.18.1. Choose an injective map $h^{-1}\mathcal{I}^m \to \mathcal{J}$ where $\mathcal{J}$ is an injective sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $Z_{étale}$. Since h is surjective the map $\mathcal{I}^m \to h_*\mathcal{J}$ is injective (see Lemma 84.4.1). Since $\mathcal{I}^m$ is injective we see that $\mathcal{I}^m$ is a direct summand of $h_*\mathcal{J}$. Thus it suffices to prove the desired vanishing for $h_*\mathcal{J}$.

Denote h' the base change by g and denote $g'' : Z' \to Z$ the projection. There is a spectral sequence

$$E_2^{p,q} = R^pf'_*R^qh'_*(g'')^{-1}\mathcal{J}$$

converging to $R^{p+q}(f' \circ h')_*(g'')^{-1}\mathcal{J}$. Since h and $f \circ h$ are representable (by schemes) we know the result we want holds for them. Thus in the spectral sequence we see that $E_2^{p,q} = 0$ for $q > 0$ and $R^{p+q}(f' \circ h')_*(g'')^{-1}\mathcal{J} = 0$ for $p + q > 0$. It follows that $E_2^{p,0} = 0$ for $p > 0$. Now

$$E_2^{p,0} = R^pf'_*h'_*(g'')^{-1}\mathcal{J} = R^pf'_*(g')^{-1}h_*\mathcal{J}$$

by Lemma 84.4.4. This finishes the proof. □

0DG3 **Lemma 84.4.7.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Assume f is proper. Let $E \in D^+(X_{étale})$ have torsion cohomology sheaves. Then the base change map $g^{-1}Rf_*E \to Rf'_*(g')^{-1}E$ is an isomorphism.

Proof. This is a simple consequence of the proper base change theorem (Theorem 84.4.6) using the spectral sequences

$$E_2^{p,q} = R^p f_* H^q(E) \quad \text{and} \quad E'^{p,q}_2 = R^p f'_*(g')^{-1} H^q(E)$$

converging to $R^n f_* E$ and $R^n f'_*(g')^{-1}E$. The spectral sequences are constructed in Derived Categories, Lemma 13.21.3. Some details omitted. □

0DG4 **Lemma 84.4.8.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces. Let $\overline{y} \to Y$ be a geometric point.*

1. *For a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $(R^n f_* \mathcal{F})_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.*
2. *For $E \in D^+(X_{étale})$ with torsion cohomology sheaves we have $(R^n f_* E)_{\overline{y}} = H^n_{étale}(X_{\overline{y}}, E_{\overline{y}})$.*

Proof. In the statement, $\mathcal{F}_{\overline{y}}$ denotes the pullback of $\mathcal{F}$ to $X_{\overline{y}} = \overline{y} \times_Y X$. Since pulling back by $\overline{y} \to Y$ produces the stalk of $\mathcal{F}$, the first statement of the lemma is a special case of Theorem 84.4.6. The second one is a special case of Lemma 84.4.7. □

0DG5 **Lemma 84.4.9.** *Let k'/k be an extension of separably closed fields. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a torsion abelian sheaf on X. Then the map $H^q_{étale}(X, \mathcal{F}) \to H^q_{étale}(X_{k'}, \mathcal{F}|_{X_{k'}})$ is an isomorphism for $q \geq 0$.*

Proof. This is a special case of Theorem 84.4.6. □

84.5. Comparing big and small topoi

0DG6 Let S be a scheme and let X be an algebraic space over S. In Topologies on Spaces, Lemma 73.4.8 we have introduced comparison morphisms $\pi_X : (\mathit{Spaces}/X)_{étale} \to X_{spaces,étale}$ and $i_X : \mathit{Sh}(X_{étale}) \to \mathit{Sh}((\mathit{Spaces}/X)_{étale})$ with $\pi_X \circ i_X = \mathrm{id}$ as morphisms of topoi and $\pi_{X,*} = i_X^{-1}$. More generally, if $f : Y \to X$ is an object of $(\mathit{Spaces}/X)_{étale}$, then there is a morphism $i_f : \mathit{Sh}(Y_{étale}) \to \mathit{Sh}((\mathit{Spaces}/X)_{étale})$ such that $f_{small} = \pi_X \circ i_f$, see Topologies on Spaces, Lemmas 73.4.7 and 73.4.11. In Topologies on Spaces, Remark 73.4.14 we have extended these to a morphism of ringed sites

$$\pi_X : ((\mathit{Spaces}/X)_{étale}, \mathcal{O}) \to (X_{spaces,étale}, \mathcal{O}_X)$$

and morphisms of ringed topoi

$$i_X : (\mathit{Sh}(X_{étale}), \mathcal{O}_X) \to (\mathit{Sh}((\mathit{Spaces}/X)_{étale}), \mathcal{O})$$

and

$$i_f : (\mathit{Sh}(Y_{étale}), \mathcal{O}_Y) \to (\mathit{Sh}((\mathit{Spaces}/X)_{étale}, \mathcal{O}))$$

Note that the restriction $i_X^{-1} = \pi_{X,*}$ (see Topologies, Definition 34.4.15) transforms $\mathcal{O}$ into $\mathcal{O}_X$. Similarly, i_f^{-1} transforms $\mathcal{O}$ into $\mathcal{O}_Y$. See Topologies on Spaces, Remark 73.4.14. Hence $i_X^*\mathcal{F} = i_X^{-1}\mathcal{F}$ and $i_f^*\mathcal{F} = i_f^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Spaces/X)_{étale}$. In particular i_X^* and i_f^* are exact functors. The functor i_X^* is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{X_{étale}}$ (and this does not conflict with the notation in Topologies on Spaces, Definition 73.4.9).

0DG7 **Lemma 84.5.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then $\pi_X^{-1}\mathcal{F}$ is given by the rule*

$$(\pi_X^{-1}\mathcal{F})(Y) = \Gamma(Y_{étale}, f_{small}^{-1}\mathcal{F})$$

for $f : Y \to X$ in $(Spaces/X)_{étale}$. Moreover, $\pi_Y^{-1}\mathcal{F}$ satisfies the sheaf condition with respect to smooth, syntomic, fppf, fpqc, and ph coverings.

Proof. Since pullback is transitive and $f_{small} = \pi_X \circ i_f$ (see above) we see that $i_f^{-1}\pi_X^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$. This shows that π_X^{-1} has the description given in the lemma.

To prove that $\pi_X^{-1}\mathcal{F}$ is a sheaf for the ph topology it suffices by Topologies on Spaces, Lemma 73.8.7 to show that for a surjective proper morphism $V \to U$ of algebraic spaces over X we have $(\pi_X^{-1}\mathcal{F})(U)$ is the equalizer of the two maps $(\pi_X^{-1}\mathcal{F})(V) \to (\pi_X^{-1}\mathcal{F})(V \times_U V)$. This we have seen in Lemma 84.4.1.

The case of smooth, syntomic, fppf coverings follows from the case of ph coverings by Topologies on Spaces, Lemma 73.8.2.

Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an fpqc covering of algebraic spaces over X. Let $s_i \in (\pi_X^{-1}\mathcal{F})(U_i)$ be sections which agree over $U_i \times_U U_j$. We have to prove there exists a unique $s \in (\pi_X^{-1}\mathcal{F})(U)$ restricting to s_i over U_i. Case I: U and U_i are schemes. This case follows from Étale Cohomology, Lemma 59.39.2. Case II: U is a scheme. Here we choose surjective étale morphisms $T_i \to U_i$ where T_i is a scheme. Then $\mathcal{T} = \{T_i \to U\}$ is an fpqc covering by schemes and by case I the result holds for $\mathcal{T}$. We omit the verification that this implies the result for $\mathcal{U}$. Case III: general case. Let $W \to U$ be a surjective étale morphism, where W is a scheme. Then $\mathcal{W} = \{U_i \times_U W \to W\}$ is an fpqc covering (by algebraic spaces) of the scheme W. By case II the result hold for $\mathcal{W}$. We omit the verification that this implies the result for $\mathcal{U}$. □

0DG8 **Lemma 84.5.2.** *Let S be a scheme. Let $Y \to X$ be a morphism of $(Spaces/S)_{étale}$.*

1. *If $\mathcal{I}$ is injective in $Ab((Spaces/X)_{étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}$ is injective in $Ab(Y_{étale})$,*
   (b) *$\mathcal{I}|_{X_{étale}}$ is injective in $Ab(X_{étale})$,*
2. *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab((Spaces/X)_{étale})$, then*
   (a) *$i_f^{-1}\mathcal{I}^\bullet$ is a K-injective complex in $Ab(Y_{étale})$,*
   (b) *$\mathcal{I}^\bullet|_{X_{étale}}$ is a K-injective complex in $Ab(X_{étale})$,*

The corresponding statements for modules do not hold.

Proof. Parts (1)(b) and (2)(b) follow formally from the fact that the restriction functor $\pi_{X,*} = i_X^{-1}$ is a right adjoint of the exact functor π_X^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9.

Parts (1)(a) and (2)(a) can be seen in two ways. First proof: We can use that i_f^{-1} is a right adjoint of the exact functor $i_{f,!}$. This functor is constructed in Topologies,

Lemma 34.4.13 for sheaves of sets and for abelian sheaves in Modules on Sites, Lemma 18.16.2. It is shown in Modules on Sites, Lemma 18.16.3 that it is exact. Second proof. We can use that $i_f = i_Y \circ f_{big}$ as is shown in Topologies, Lemma 34.4.17. Since f_{big} is a localization, we see that pullback by it preserves injectives and K-injectives, see Cohomology on Sites, Lemmas 21.7.1 and 21.20.1. Then we apply the already proved parts (1)(b) and (2)(b) to the functor i_Y^{-1} to conclude.

To see a counter example for the case of modules we refer to Étale Cohomology, Lemma 59.99.1. □

Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The commutative diagram of Topologies on Spaces, Lemma 73.4.11 (3) leads to a commutative diagram of ringed sites

$$\begin{array}{ccc} (Y_{spaces,étale}, \mathcal{O}_Y) & \xleftarrow{\pi_Y} & ((Spaces/Y)_{étale}, \mathcal{O}) \\ \downarrow{\scriptstyle f_{spaces,étale}} & & \downarrow{\scriptstyle f_{big}} \\ (X_{spaces,étale}, \mathcal{O}_X) & \xleftarrow{\pi_X} & ((Spaces/X)_{étale}, \mathcal{O}) \end{array}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_X^\sharp$, and $\pi_Y^\sharp$. In particular this means that

0DG9 (84.5.2.1) $$(f_{big,*}\mathcal{F})|_{X_{étale}} = f_{small,*}(\mathcal{F}|_{Y_{étale}})$$

for any sheaf $\mathcal{F}$ on $(Spaces/Y)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (84.5.2.1) is an isomorphism of $\mathcal{O}_X$-modules on $X_{étale}$.

0DGA **Lemma 84.5.3.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.*

1. *For K in $D((Spaces/Y)_{étale})$ we have $(Rf_{big,*}K)|_{X_{étale}} = Rf_{small,*}(K|_{Y_{étale}})$ in $D(X_{étale})$.*
2. *For K in $D((Spaces/Y)_{étale}, \mathcal{O})$ we have $(Rf_{big,*}K)|_{X_{étale}} = Rf_{small,*}(K|_{Y_{étale}})$ in $D(Mod(X_{étale}, \mathcal{O}_X))$.*

More generally, let $g : X' \to X$ be an object of $(Spaces/X)_{étale}$. Consider the fibre product

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

Then

3. *For K in $D((Spaces/Y)_{étale})$ we have $i_g^{-1}(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^{-1}K)$ in $D(X'_{étale})$.*
4. *For K in $D((Spaces/Y)_{étale}, \mathcal{O})$ we have $i_g^*(Rf_{big,*}K) = Rf'_{small,*}(i_{g'}^*K)$ in $D(Mod(X'_{étale}, \mathcal{O}_{X'}))$.*
5. *For K in $D((Spaces/Y)_{étale})$ we have $g_{big}^{-1}(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^{-1}K)$ in $D((Spaces/X')_{étale})$.*
6. *For K in $D((Spaces/Y)_{étale}, \mathcal{O})$ we have $g_{big}^*(Rf_{big,*}K) = Rf'_{big,*}((g'_{big})^*K)$ in $D(Mod(X'_{étale}, \mathcal{O}_{X'}))$.*

Proof. Part (1) follows from Lemma 84.5.2 and (84.5.2.1) on choosing a K-injective complex of abelian sheaves representing K.

Part (3) follows from Lemma 84.5.2 and Topologies, Lemma 34.4.19 on choosing a K-injective complex of abelian sheaves representing K.

Part (5) is Cohomology on Sites, Lemma 21.21.1.

Part (6) is Cohomology on Sites, Lemma 21.21.2.

Part (2) can be proved as follows. Above we have seen that $\pi_X \circ f_{big} = f_{small} \circ \pi_Y$ as morphisms of ringed sites. Hence we obtain $R\pi_{X,*} \circ Rf_{big,*} = Rf_{small,*} \circ R\pi_{Y,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{X,*}$ and $\pi_{Y,*}$ are exact, we conclude.

Part (4) follows from part (6) and part (2) applied to $f' : Y' \to X'$. □

Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{H}$ be an abelian sheaf on $(Spaces/X)_{étale}$. Recall that $H^n_{étale}(U, \mathcal{H})$ denotes the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{étale}$.

0DGB **Lemma 84.5.4.** *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then*

(1) *For K in $D(X_{étale})$ we have $H^n_{étale}(X, \pi_X^{-1}K) = H^n(X_{étale}, K)$.*
(2) *For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H^n_{étale}(X, L\pi_X^*K) = H^n(X_{étale}, K)$.*
(3) *For K in $D(X_{étale})$ we have $H^n_{étale}(Y, \pi_X^{-1}K) = H^n(Y_{étale}, f_{small}^{-1}K)$.*
(4) *For K in $D(X_{étale}, \mathcal{O}_X)$ we have $H^n_{étale}(Y, L\pi_X^*K) = H^n(Y_{étale}, Lf_{small}^*K)$.*
(5) *For M in $D((Spaces/X)_{étale})$ we have $H^n_{étale}(Y, M) = H^n(Y_{étale}, i_f^{-1}M)$.*
(6) *For M in $D((Spaces/X)_{étale}, \mathcal{O})$ we have $H^n_{étale}(Y, M) = H^n(Y_{étale}, i_f^*M)$.*

Proof. To prove (5) represent M by a K-injective complex of abelian sheaves and apply Lemma 84.5.2 and work out the definitions. Part (3) follows from this as $i_f^{-1}\pi_X^{-1} = f_{small}^{-1}$. Part (1) is a special case of (3).

Part (6) follows from the very general Cohomology on Sites, Lemma 21.37.5. Then part (4) follows because $Lf_{small}^* = i_f^* \circ L\pi_X^*$. Part (2) is a special case of (4). □

0DGC **Lemma 84.5.5.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D(X_{étale})$ the map*

$$K \longrightarrow R\pi_{X,*}\pi_X^{-1}K$$

is an isomorphism where $\pi_X : Sh((Spaces/X)_{étale}) \to Sh(X_{étale})$ is as above.

Proof. This is true because both π_X^{-1} and $\pi_{X,*} = i_X^{-1}$ are exact functors and the composition $\pi_{X,*} \circ \pi_X^{-1}$ is the identity functor. □

0DGD **Lemma 84.5.6.** *Let S be a scheme. Let $f : Y \to X$ be a proper morphism of algebraic spaces over S. Then we have*

(1) *$\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$ as functors $Sh(Y_{étale}) \to Sh((Spaces/X)_{étale})$,*
(2) *$\pi_X^{-1}Rf_{small,*}K = Rf_{big,*}\pi_Y^{-1}K$ for K in $D^+(Y_{étale})$ whose cohomology sheaves are torsion, and*
(3) *$\pi_X^{-1}Rf_{small,*}K = Rf_{big,*}\pi_Y^{-1}K$ for all K in $D(Y_{étale})$ if f is finite.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf on $Y_{étale}$. Let $g : X' \to X$ be an object of $(Spaces/X)_{étale}$. Consider the fibre product

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

Then we have

$$(f_{big,*}\pi_Y^{-1}\mathcal{F})(X') = (\pi_Y^{-1}\mathcal{F})(Y') = ((g'_{small})^{-1}\mathcal{F})(Y') = (f'_{small,*}(g'_{small})^{-1}\mathcal{F})(X')$$

the second equality by Lemma 84.5.1. On the other hand

$$(\pi_X^{-1}f_{small,*}\mathcal{F})(X') = (g_{small}^{-1}f_{small,*}\mathcal{F})(X')$$

again by Lemma 84.5.1. Hence by proper base change for sheaves of sets (Lemma 84.4.4) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_X^{-1}f_{small,*}\mathcal{F} = f_{big,*}\pi_Y^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_X^{-1} \circ f_{small,*} = f_{big,*} \circ \pi_Y^{-1}$.

Proof of (2). There is a canonical base change map $\pi_X^{-1}Rf_{small,*}K \to Rf_{big,*}\pi_Y^{-1}K$ for any K in $D(Y_{étale})$, see Cohomology on Sites, Remark 21.19.3. To prove it is an isomorphism, it suffices to prove the pull back of the base change map by $i_g : Sh(X'_{étale}) \to Sh((Sch/X)_{étale})$ is an isomorphism for any object $g : X' \to X$ of $(Sch/X)_{étale}$. Let T', g', f' be as in the previous paragraph. The pullback of the base change map is

$$\begin{aligned} g_{small}^{-1}Rf_{small,*}K &= i_g^{-1}\pi_X^{-1}Rf_{small,*}K \\ &\to i_g^{-1}Rf_{big,*}\pi_Y^{-1}K \\ &= Rf'_{small,*}(i_{g'}^{-1}\pi_Y^{-1}K) \\ &= Rf'_{small,*}((g'_{small})^{-1}K) \end{aligned}$$

where we have used $\pi_X \circ i_g = g_{small}$, $\pi_Y \circ i_{g'} = g'_{small}$, and Lemma 84.5.3. This map is an isomorphism by the proper base change theorem (Lemma 84.4.7) provided K is bounded below and the cohomology sheaves of K are torsion.

Proof of (3). If f is finite, then the functors $f_{small,*}$ and $f_{big,*}$ are exact. This follows from Cohomology of Spaces, Lemma 69.4.1 for f_{small}. Since any base change f' of f is finite too, we conclude from Lemma 84.5.3 part (3) that $f_{big,*}$ is exact too (as the higher derived functors are zero). Thus this case follows from part (1). □

84.6. Comparing fppf and étale topologies

0DGE This section is the analogue of Étale Cohomology, Section 59.100.

Let S be a scheme. Let X be an algebraic space over S. On the category $Spaces/X$ we consider the fppf and étale topologies. The identity functor $(Spaces/X)_{étale} \to (Spaces/X)_{fppf}$ is continuous and defines a morphism of sites

$$\epsilon_X : (Spaces/X)_{fppf} \longrightarrow (Spaces/X)_{étale}$$

by an application of Sites, Proposition 7.14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the fppf sheafification. Consider the morphism of sites

$$\pi_X : (Spaces/X)_{étale} \longrightarrow X_{spaces,étale}$$

comparing big and small étale sites, see Section 84.5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (Spaces/X)_{fppf} \longrightarrow X_{spaces,étale}$$

If $\mathcal{H}$ is an abelian sheaf on $(Spaces/X)_{fppf}$, then we will write $H^n_{fppf}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{fppf}$.

0DGF **Lemma 84.6.1.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *For $\mathcal{F} \in Sh(X_{étale})$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$.*
(2) *For $\mathcal{F} \in Ab(X_{étale})$ we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*

Proof. We have $a_X^{-1}\mathcal{F} = \epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$. By Lemma 84.5.1 the étale sheaf $\pi_X^{-1}\mathcal{F}$ is a sheaf for the fppf topology and therefore is equal to $a_X^{-1}\mathcal{F}$ (as pulling back by ϵ_X is given by fppf sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 84.5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 59.100.6. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(Spaces/X)_{fppf}$ the higher direct image $R^p\epsilon_{X,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{fppf}(U, \mathcal{H})$ on $(Spaces/X)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Since every object of $(Spaces/X)_{étale}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H^p_{fppf}(U, a_X^{-1}\mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$\begin{aligned} H^p_{fppf}(U, a_X^{-1}\mathcal{F}) &= H^p((Spaces/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{Spaces/U}) \\ &= H^p((Sch/U)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U}) \end{aligned}$$

where the second identification is Lemma 84.3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 21.7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 59.100.1) we conclude that the sheaf $(a_X^{-1}\mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{fppf}, (a_X^{-1}\mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 59.100.6 (the precise statement one should use here is that V_n holds for all n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. □

The hard work done in the case of schemes now tells us that étale and fppf cohomology agree for sheaves coming from the small étale site.

0DGG **Lemma 84.6.2.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{étale})$ the maps*

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad \text{and} \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{étale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is an immediate reduction to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 84.6.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 84.6.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 84.5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DGH **Lemma 84.6.3.** *Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{étale})$ as above:*

(1) *$H^q(X_{étale}, \mathcal{F}) = H^q_{fppf}(X, a_X^{-1}\mathcal{F})$ for an abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(2) *$H^q(X_{étale}, K) = H^q_{fppf}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$.*

Example: if A is an abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{fppf}(X, \underline{A})$.

Proof. This follows from Lemma 84.6.2 by Cohomology on Sites, Remark 21.14.4. □

0DGI **Lemma 84.6.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh((Spaces/X)_{fppf}) & \xrightarrow{f_{big,fppf}} & Sh((Spaces/Y)_{fppf}) \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh((Spaces/X)_{étale}) & \xrightarrow{f_{big,étale}} & Sh((Spaces/Y)_{étale}) \end{array}$$

and

$$\begin{array}{ccc} Sh((Spaces/X)_{fppf}) & \xrightarrow{f_{big,fppf}} & Sh((Spaces/Y)_{fppf}) \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 73.7 and Section 84.5. □

0DGJ **Lemma 84.6.5.** *In Lemma 84.6.4 if f is proper, then we have*

(1) *$a_Y^{-1} \circ f_{small,*} = f_{big,fppf,*} \circ a_X^{-1}$, and*
(2) *$a_Y^{-1}(Rf_{small,*}K) = Rf_{big,fppf,*}(a_X^{-1}K)$ for K in $D^+(X_{étale})$ with torsion cohomology sheaves.*

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 84.5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 84.6.1 shows that $\epsilon_{Y,*} \circ$

$a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,fppf,*} a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*} a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,fppf,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*} a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 84.5.6.

To see (2) we use that

$$\begin{aligned} R\epsilon_{Y,*} R f_{big,fppf,*} a_X^{-1} K &= R f_{big,étale,*} R\epsilon_{X,*} a_X^{-1} K \\ &= R f_{big,étale,*}\pi_X^{-1} K \\ &= \pi_Y^{-1} R f_{small,*} K \\ &= R\epsilon_{Y,*} a_Y^{-1} R f_{small,*} K \end{aligned}$$

The first equality by the commutative diagram in Lemma 84.6.4 and Cohomology on Sites, Lemma 21.19.2. Then second equality is Lemma 84.6.2. The third is Lemma 84.5.6 part (2). The fourth is Lemma 84.6.2 again. Thus the base change map $a_Y^{-1}(R f_{small,*} K) \to R f_{big,fppf,*}(a_X^{-1} K)$ induces an isomorphism

$$R\epsilon_{Y,*} a_Y^{-1} R f_{small,*} K \to R\epsilon_{Y,*} R f_{big,fppf,*} a_X^{-1} K$$

The proof is finished by the following remark: a map $\alpha : a_Y^{-1} L \to M$ with L in $D^+(Y_{étale})$ and M in $D^+((Spaces/Y)_{fppf})$ such that $R\epsilon_{Y,*}\alpha$ is an isomorphism, is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*} H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by Lemma 84.6.1 because $H^i(M) = a_Y^{-1} H^i(L)$ in this range. Hence $\epsilon_{Y,*} H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*} M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*} H^{i_0+1}(M) = \pi_Y^{-1} H^{i_0+1}(L) = \epsilon_{Y,*} a_Y^{-1} H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

0DGK **Lemma 84.6.6.** *In Lemma 84.6.4 if f is finite, then $a_Y^{-1}(R f_{small,*} K) = R f_{big,fppf,*}(a_X^{-1} K)$ for K in $D^+(X_{étale})$.*

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the base change map is an isomorphism after restricting to V. Hence we may assume that Y is a scheme. As the morphism is finite, hence representable, we conclude that we may assume both X and Y are schemes. In this case the result follows from the case of schemes (Étale Cohomology, Lemma 59.100.6 part (2)) using the comparison of topoi discussed in Section 84.3 and in particular given in Lemma 84.3.1. Some details omitted. □

0DGL **Lemma 84.6.7.** *In Lemma 84.6.4 assume f is flat, locally of finite presentation, and surjective. Then the functor*

$$Sh(Y_{étale}) \longrightarrow \left\{ (\mathcal{G}, \mathcal{H}, \alpha) \middle| \begin{array}{l} \mathcal{G} \in Sh(X_{étale}),\ \mathcal{H} \in Sh((Sch/Y)_{fppf}), \\ \alpha : a_X^{-1}\mathcal{G} \to f_{big,fppf}^{-1}\mathcal{H} \text{ an isomorphism} \end{array} \right\}$$

sending $\mathcal{F}$ to $(f_{small}^{-1}\mathcal{F}, a_Y^{-1}\mathcal{F}, can)$ is an equivalence.

Proof. The functor a_X^{-1} is fully faithful (as $a_{X,*}a_X^{-1} = \text{id}$ by Lemma 84.6.1). Hence the forgetful functor $(\mathcal{G}, \mathcal{H}, \alpha) \mapsto \mathcal{H}$ identifies the category of triples with a full subcategory of $Sh((Sch/Y)_{fppf})$. Moreover, the functor a_Y^{-1} is fully faithful, hence the functor in the lemma is fully faithful as well.

Suppose that we have an étale covering $\{Y_i \to Y\}$. Let $f_i : X_i \to Y_i$ be the base change of f. Denote $f_{ij} = f_i \times f_j : X_i \times_X X_j \to Y_i \times_Y Y_j$. Claim: if the lemma is true for f_i and f_{ij} for all i, j, then the lemma is true for f. To see this, note that the given étale covering determines an étale covering of the final object in each of the four sites $Y_{étale}, X_{étale}, (Sch/Y)_{fppf}, (Sch/X)_{fppf}$. Thus the category of sheaves is equivalent to the category of glueing data for this covering (Sites, Lemma 7.26.5) in each of the four cases. A huge commutative diagram of categories then finishes the proof of the claim. We omit the details. The claim shows that we may work étale locally on Y. In particular, we may assume Y is a scheme.

Assume Y is a scheme. Choose a scheme X' and a surjective étale morphism $s : X' \to X$. Set $f' = f \circ s : X' \to Y$ and observe that f' is surjective, locally of finite presentation, and flat. Claim: if the lemma is true for f', then it is true for f. Namely, given a triple $(\mathcal{G}, \mathcal{H}, \alpha)$ for f, we can pullback by s to get a triple $(s_{small}^{-1}\mathcal{G}, \mathcal{H}, s_{big,fppf}^{-1}\alpha)$ for f'. A solution for this triple gives a sheaf $\mathcal{F}$ on $Y_{étale}$ with $a_Y^{-1}\mathcal{F} = \mathcal{H}$. By the first paragraph of the proof this means the triple is in the essential image. This reduces us to the case where both X and Y are schemes. This case follows from Étale Cohomology, Lemma 59.100.4 via the discussion in Section 84.3 and in particular Lemma 84.3.1. □

84.7. Comparing fppf and étale topologies: modules

0DGM We continue the discussion in Section 84.6 but in this section we briefly discuss what happens for sheaves of modules.

Let S be a scheme. Let X be an algebraic space over S. The morphisms of sites ϵ_X, π_X, and their composition a_X introduced in Section 84.6 have natural enhancements to morphisms of ringed sites. The first is written as

$$\epsilon_X : ((Spaces/X)_{fppf}, \mathcal{O}) \longrightarrow ((Spaces/X)_{étale}, \mathcal{O})$$

Note that we can use the same symbol for the structure sheaf as indeed the sheaves have the same underlying presheaf. The second is

$$\pi_X : ((Spaces/X)_{étale}, \mathcal{O}) \longrightarrow (X_{étale}, \mathcal{O}_X)$$

The third is the morphism

$$a_X : ((Spaces/X)_{fppf}, \mathcal{O}) \longrightarrow (X_{étale}, \mathcal{O}_X)$$

Let us review what we already know about quasi-coherent modules on these sites.

0DGN **Lemma 84.7.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *The rule*

$$\mathcal{F}^a : (Spaces/X)_{étale} \longrightarrow Ab, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^*\mathcal{F})$$

satisfies the sheaf condition for fpqc and a fortiori fppf and étale coverings,

(2) $\mathcal{F}^a = \pi_X^*\mathcal{F}$ *on* $(Spaces/X)_{étale}$,

(3) $\mathcal{F}^a = a_X^*\mathcal{F}$ *on* $(Spaces/X)_{fppf}$,

(4) *the rule* $\mathcal{F} \mapsto \mathcal{F}^a$ *defines an equivalence between quasi-coherent* $\mathcal{O}_X$*-modules and quasi-coherent modules on* $((Spaces/X)_{étale}, \mathcal{O})$,
(5) *the rule* $\mathcal{F} \mapsto \mathcal{F}^a$ *defines an equivalence between quasi-coherent* $\mathcal{O}_X$*-modules and quasi-coherent modules on* $((Spaces/X)_{fppf}, \mathcal{O})$,
(6) *we have* $\epsilon_{X,*}a_X^*\mathcal{F} = \pi_X^*\mathcal{F}$ *and* $a_{X,*}a_X^*\mathcal{F} = \mathcal{F}$,
(7) *we have* $R^i\epsilon_{X,*}(a_X^*\mathcal{F}) = 0$ *and* $R^i a_{X,*}(a_X^*\mathcal{F}) = 0$ *for* $i > 0$.

Proof. Part (1) is a consequence of fppf descent of quasi-coherent modules. Namely, suppose that $\{f_i : U_i \to U\}$ is an fpqc covering in $(Spaces/X)_{étale}$. Denote $g : U \to X$ the structure morphism. Suppose that we have a family of sections $s_i \in \Gamma(U_i, f_i^*g^*\mathcal{F})$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$. We have to find the correspond section $s \in \Gamma(U, g^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_U = \mathcal{O}_{U_i} \to f_i^*g^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_U$ and $g^*\mathcal{F}$ on U. Hence by Descent on Spaces, Proposition 74.4.1 we see that we may (uniquely) descend these to a map $\mathcal{O}_U \to g^*\mathcal{F}$ which gives us our section s.

We will deduce (2) – (7) from the corresponding statement for schemes. Choose an étale covering $\{X_i \to X\}_{i \in I}$ where each X_i is a scheme. Observe that $X_i \times_X X_j$ is a scheme too. This covering induces a covering of the final object in each of the three sites $(Spaces/X)_{fppf}$, $(Spaces/X)_{étale}$, and $X_{étale}$. Hence we see that the category of sheaves on these sites are equivalent to descent data for these coverings, see Sites, Lemma 7.26.5. Parts (2), (3) are local (because we have the glueing statement). Being quasi-coherent is a local property, hence parts (4), (5) are local. Clearly (6) and (7) are local. It follows that it suffices to prove parts (2) – (7) of the lemma when X is a scheme.

Assume X is a scheme. The embeddings $(Sch/X)_{étale} \subset (Spaces/X)_{étale}$ and $(Sch/X)_{fppf} \subset (Spaces/X)_{fppf}$ determine equivalences of ringed topoi by Lemma 84.3.1. We conclude that (2) – (7) follows from the case of schemes. Étale Cohomology, Lemma 59.101.1. To transport the property of being quasi-coherent via this equivalence use that being quasi-coherent is an intrinsic property of modules as explained in Modules on Sites, Section 18.23. Some minor details omitted. □

0DGP **Lemma 84.7.2.** *Let* S *be a scheme. Let* X *be an algebraic space over* S*. For* $\mathcal{F}$ *a quasi-coherent* $\mathcal{O}_X$*-module the maps*

$$\pi_X^*\mathcal{F} \longrightarrow R\epsilon_{X,*}(a_X^*\mathcal{F}) \quad \text{and} \quad \mathcal{F} \longrightarrow Ra_{X,*}(a_X^*\mathcal{F})$$

are isomorphisms.

Proof. This is an immediate consequence of parts (6) and (7) of Lemma 84.7.1. □

0DGQ **Lemma 84.7.3.** *Let* S *be a scheme. Let* X *be an algebraic space over* S*. Let* $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ *be a complex of quasi-coherent* $\mathcal{O}_X$*-modules. Set*

$$\mathcal{H}_{étale} = \mathrm{Ker}(\pi_X^*\mathcal{F}_2 \to \pi_X^*\mathcal{F}_3)/\mathrm{Im}(\pi_X^*\mathcal{F}_1 \to \pi_X^*\mathcal{F}_2)$$

on $(Spaces/X)_{étale}$ *and set*

$$\mathcal{H}_{fppf} = \mathrm{Ker}(a_X^*\mathcal{F}_2 \to a_X^*\mathcal{F}_3)/\mathrm{Im}(a_X^*\mathcal{F}_1 \to a_X^*\mathcal{F}_2)$$

on $(Spaces/X)_{fppf}$*. Then* $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$ *and*

$$H^p_{étale}(U, \mathcal{H}_{étale}) = H^p_{fppf}(U, \mathcal{H}_{fppf}) = 0$$

for $p > 0$ *and any affine object* U *of* $(Spaces/X)_{étale}$.

More is true, namely the collection of modules on $(Spaces/X)_{fppf}$ which fppf locally look like those in the lemma are called adquate modules. They form a weak Serre subcategory of the category of all $\mathcal{O}$-modules and their cohomology is studied in Adequate Modules, Section 46.5.

Proof. For any object $f : U \to X$ of $(Spaces/X)_{étale}$ consider the restriction $\mathcal{H}_{étale}|_{U_{étale}}$ of $\mathcal{H}_{étale}$ to $U_{étale}$ via the functor $i_f^* = i_f^{-1}$ discussed in Section 84.5. The sheaf $\mathcal{H}_{étale}|_{U_{étale}}$ is equal to the homology of complex $f^*\mathcal{F}_\bullet$ in degree 1. This is true because $i_f \circ \pi_X = f$ as morphisms of ringed sites $U_{étale} \to X_{étale}$. In particular we see that $\mathcal{H}_{étale}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. Next, let $g : V \to U$ be a flat morphism in $(Spaces/X)_{étale}$. Since

$$i_{f\circ g}^* \circ \pi_X^* = (f \circ g)^* = g^* \circ f^*$$

as morphisms of sites $V_{étale} \to X_{étale}$ and since g is flat hence g^* is exact, we obtain

$$\mathcal{H}_{étale}|_{V_{étale}} = g^* \left(\mathcal{H}_{étale}|_{U_{étale}}\right)$$

With these preparations we are ready to prove the lemma.

Let $\mathcal{U} = \{g_i : U_i \to U\}_{i\in I}$ be an fppf covering with $f : U \to X$ as above. The sheaf property holds for $\mathcal{H}_{étale}$ and the covering $\mathcal{U}$ by (1) of Lemma 84.7.1 applied to $\mathcal{H}_{étale}|_{U_{étale}}$ and the above. Therefore we see that $\mathcal{H}_{étale}$ is already an fppf sheaf and this means that $\mathcal{H}_{fppf}$ is equal to $\mathcal{H}_{étale}$ as a presheaf. In particular $\mathcal{H}_{étale} = \epsilon_{X,*}\mathcal{H}_{fppf}$.

Finally, to prove the vanishing, we use Cohomology on Sites, Lemma 21.10.9. We let $\mathcal{B}$ be the affine objects of $(Spaces/X)_{fppf}$ and we let Cov be the set of finite fppf coverings $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ with U, U_i affine. We have

$$\check{H}^p(\mathcal{U}, \mathcal{H}_{étale}) = \check{H}^p(\mathcal{U}, (\mathcal{H}_{étale}|_{U_{étale}})^a)$$

because the values of $\mathcal{H}_{étale}$ on the affine schemes $U_{i_0} \times_U \ldots \times_U U_{i_p}$ flat over U agree with the values of the pullback of the quasi-coherent module $\mathcal{H}_{étale}|_{U_{étale}}$ by the first paragraph. Hence we obtain vanishing by Descent, Lemma 35.9.2. This finishes the proof. □

0DGR **Lemma 84.7.4.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D_{QCoh}(\mathcal{O}_X)$ the maps*

$$L\pi_X^* K \longrightarrow R\epsilon_{X,*}(La_X^* K) \quad \text{and} \quad K \longrightarrow Ra_{X,*}(La_X^* K)$$

are isomorphisms. Here $a_X : Sh((Spaces/X)_{fppf}) \to Sh(X_{étale})$ is as above.

Proof. The question is étale local on X hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$. Then we have $D_{QCoh}(\mathcal{O}_X) = D(A)$ by Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5. Hence we can choose an K-flat complex of A-modules $K^\bullet$ whose corresponding complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules represents K. We claim that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}_X$-modules.

Proof of the claim. By Derived Categories of Schemes, Lemma 36.3.6 we see that $\widetilde{K}^\bullet$ is K-flat on the scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. Next, note that $\mathcal{K}^\bullet = \epsilon^*\widetilde{K}^\bullet$ where ϵ is as in Derived Categories of Spaces, Lemma 75.4.2 whence $\mathcal{K}^\bullet$ is K-flat by Cohomology on Sites, Lemma 21.18.7 and the fact that the étale site of a scheme has enough points (Étale Cohomology, Remarks 59.29.11).

By the claim we see that $La_X^*K = a_X^*\mathcal{K}^\bullet$ and $L\pi_X^*K = \pi_X^*\mathcal{K}^\bullet$. Since the first part of the proof shows that the pullback $a_X^*\mathcal{K}^n$ of the quasi-coherent module is acyclic for $\epsilon_{X,*}$, resp. $a_{X,*}$, surely the proof is done by Leray's acyclicity lemma? Actually..., no because Leray's acyclicity lemma only applies to bounded below complexes. However, in the next paragraph we will show the result does follow from the bounded below case because our complex is the derived limit of bounded below complexes of quasi-coherent modules.

The cohomology sheaves of $\pi_X^*\mathcal{K}^\bullet$ and $a_X^*\mathcal{K}^\bullet$ have vanishing higher cohomology groups over affine objects of $(Spaces/X)_{étale}$ by Lemma 84.7.3. Therefore we have

$$L\pi_X^*K = R\lim \tau_{\geq -n}(L\pi_X^*K) \quad \text{and} \quad La_X^*K = R\lim \tau_{\geq -n}(La_X^*K)$$

by Cohomology on Sites, Lemma 21.23.10.

Proof of $L\pi_X^*K = R\epsilon_{X,*}(La_X^*\mathcal{F})$. By the above we have

$$R\epsilon_{X,*}La_X^*K = R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^*K))$$

by Cohomology on Sites, Lemma 21.23.3. Note that $\tau_{\geq -n}(La_X^*K)$ is represented by $\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)$ which may not be the same as $a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)$. But clearly the systems

$$\{\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)\}_{n\geq 1} \quad \text{and} \quad \{a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)\}_{n\geq 1}$$

are isomorphic as pro-systems. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and the first part of the lemma we see that

$$R\epsilon_{X,*}(a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)) = \pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)$$

Then we can use that the systems

$$\{\tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet)\}_{n\geq 1} \quad \text{and} \quad \{\pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)\}_{n\geq 1}$$

are isomorphic as pro-systems. Finally, we put everything together as follows

$$\begin{aligned}
R\epsilon_{X,*}La_X^*K &= R\epsilon_{X,*}(R\lim \tau_{\geq -n}(La_X^*K)) \\
&= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(La_X^*K)) \\
&= R\lim R\epsilon_{X,*}(\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)) \\
&= R\lim R\epsilon_{X,*}(a_X^*(\tau_{\geq -n}\mathcal{K}^\bullet)) \\
&= R\lim \pi_X^*(\tau_{\geq -n}\mathcal{K}^\bullet) \\
&= R\lim \tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet) \\
&= R\lim \tau_{\geq -n}(L\pi_X^*K) \\
&= L\pi_X^*K
\end{aligned}$$

Here in equalities four and six we have used that isomorphic pro-systems have the same $R\lim$ (small detail omitted). You can avoid this step by using more about cohomology of the terms of the complex $\tau_{\geq -n}a_X^*\mathcal{K}^\bullet$ proved in Lemma 84.7.3 as this will prove directly that $R\epsilon_{X,*}(\tau_{\geq -n}(a_X^*\mathcal{K}^\bullet)) = \tau_{\geq -n}(\pi_X^*\mathcal{K}^\bullet)$.

The equality $K = Ra_{X,*}(La_X^*\mathcal{F})$ is proved in exactly the same way using in the final step that $K = R\lim \tau_{\geq -n}K$ by Derived Categories of Spaces, Lemma 75.5.7. □

84.8. Comparing ph and étale topologies

0DGS This section is the analogue of Étale Cohomology, Section 59.102.

Let S be a scheme. Let X be an algebraic space over S. On the category $Spaces/X$ we consider the ph and étale topologies. The identity functor $(Spaces/X)_{étale} \to (Spaces/X)_{ph}$ is continuous as every étale covering is a ph covering by Topologies on Spaces, Lemma 73.8.2. Hence it defines a morphism of sites

$$\epsilon_X : (Spaces/X)_{ph} \longrightarrow (Spaces/X)_{étale}$$

by an application of Sites, Proposition 7.14.7. Please note that $\epsilon_{X,*}$ is the identity functor on underlying presheaves and that ϵ_X^{-1} associates to an étale sheaf the ph sheafification. Consider the morphism of sites

$$\pi_X : (Spaces/X)_{étale} \longrightarrow X_{spaces,étale}$$

comparing big and small étale sites, see Section 84.5. The composition determines a morphism of sites

$$a_X = \pi_X \circ \epsilon_X : (Spaces/X)_{ph} \longrightarrow X_{spaces,étale}$$

If $\mathcal{H}$ is an abelian sheaf on $(Spaces/X)_{ph}$, then we will write $H^n_{ph}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of $(Spaces/X)_{ph}$.

0DGT **Lemma 84.8.1.** *Let S be a scheme. Let X be an algebraic space over S.*

(1) *For $\mathcal{F} \in Sh(X_{étale})$ we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$.*
(2) *For $\mathcal{F} \in Ab(X_{étale})$ torsion we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*

Proof. We have $a_X^{-1}\mathcal{F} = \epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$. By Lemma 84.5.1 the étale sheaf $\pi_X^{-1}\mathcal{F}$ is a sheaf for the ph topology and therefore is equal to $a_X^{-1}\mathcal{F}$ (as pulling back by ϵ_X is given by ph sheafification). Recall moreover that $\epsilon_{X,*}$ is the identity on underlying presheaves. Now part (1) is immediate from the explicit description of π_X^{-1} in Lemma 84.5.1.

We will prove part (2) by reducing it to the case of schemes – see part (1) of Étale Cohomology, Lemma 59.102.5. This will "clearly work" as every algebraic space is étale locally a scheme. The details are given below but we urge the reader to skip the proof.

For an abelian sheaf $\mathcal{H}$ on $(Spaces/X)_{ph}$ the higher direct image $R^p\epsilon_{X,*}\mathcal{H}$ is the sheaf associated to the presheaf $U \mapsto H^p_{ph}(U, \mathcal{H})$ on $(Spaces/X)_{étale}$. See Cohomology on Sites, Lemma 21.7.4. Since every object of $(Spaces/X)_{étale}$ has a covering by schemes, it suffices to prove that given U/X a scheme and $\xi \in H^p_{ph}(U, a_X^{-1}\mathcal{F})$ we can find an étale covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. We have

$$\begin{aligned} H^p_{ph}(U, a_X^{-1}\mathcal{F}) &= H^p((Spaces/U)_{ph}, (a_X^{-1}\mathcal{F})|_{Spaces/U}) \\ &= H^p((Sch/U)_{ph}, (a_X^{-1}\mathcal{F})|_{Sch/U}) \end{aligned}$$

where the second identification is Lemma 84.3.1 and the first is a general fact about restriction (Cohomology on Sites, Lemma 21.7.1). Looking at the first paragraph and the corresponding result in the case of schemes (Étale Cohomology, Lemma 59.102.1) we conclude that the sheaf $(a_X^{-1}\mathcal{F})|_{Sch/U}$ matches the pullback by the "schemes version of a_U". Therefore we can find an étale covering $\{U_i \to U\}$ such that our class dies in $H^p((Sch/U_i)_{ph}, (a_X^{-1}\mathcal{F})|_{Sch/U_i})$ for each i, see Étale Cohomology, Lemma 59.102.5 (the precise statement one should use here is that V_n holds

for all n which is the statement of part (2) for the case of schemes). Transporting back (using the same formulas as above but now for U_i) we conclude ξ restricts to zero over U_i as desired. □

The hard work done in the case of schemes now tells us that étale and ph cohomology agree for torsion abelian sheaves coming from the small étale site.

0DGU **Lemma 84.8.2.** *Let S be a scheme. Let X be an algebraic space over S. For $K \in D^+(X_{étale})$ with torsion cohomology sheaves the maps*

$$\pi_X^{-1}K \longrightarrow R\epsilon_{X,*}a_X^{-1}K \quad \text{and} \quad K \longrightarrow Ra_{X,*}a_X^{-1}K$$

are isomorphisms with $a_X : Sh((Spaces/X)_{ph}) \to Sh(X_{étale})$ as above.

Proof. We only prove the second statement; the first is easier and proved in exactly the same manner. There is a reduction to the case where K is given by a single torsion abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$ of torsion abelian sheaves. This is possible by Cohomology on Sites, Lemma 21.19.8. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$ where $\mathcal{F}$ is a torsion abelian sheaf.

By Lemma 84.8.1 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7). By Lemma 84.8.1 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$. We have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and by Lemma 84.5.5 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

0DGV **Lemma 84.8.3.** *Let S be a scheme and let X be an algebraic space over S. With $a_X : Sh((Spaces/X)_{ph}) \to Sh(X_{étale})$ as above:*

1. *$H^q(X_{étale}, \mathcal{F}) = H^q_{ph}(X, a_X^{-1}\mathcal{F})$ for a torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
2. *$H^q(X_{étale}, K) = H^q_{ph}(X, a_X^{-1}K)$ for $K \in D^+(X_{étale})$ with torsion cohomology sheaves*

Example: if A is a torsion abelian group, then $H^q_{étale}(X, \underline{A}) = H^q_{ph}(X, \underline{A})$.

Proof. This follows from Lemma 84.8.2 by Cohomology on Sites, Remark 21.14.4. □

0DGW **Lemma 84.8.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh((Spaces/X)_{ph}) & \xrightarrow{f_{big,ph}} & Sh((Spaces/Y)_{ph}) \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ Sh((Spaces/X)_{étale}) & \xrightarrow{f_{big,étale}} & Sh((Spaces/Y)_{étale}) \end{array}$$

and

$$\begin{array}{ccc} Sh((Spaces/X)_{ph}) & \xrightarrow{f_{big,ph}} & Sh((Spaces/Y)_{ph}) \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ Sh(X_{étale}) & \xrightarrow{f_{small}} & Sh(Y_{étale}) \end{array}$$

with $a_X = \pi_X \circ \epsilon_X$ and $a_Y = \pi_X \circ \epsilon_X$.

Proof. This follows immediately from working out the definitions of the morphisms involved, see Topologies on Spaces, Section 73.8 and Section 84.5. □

0DGX **Lemma 84.8.5.** *In Lemma 84.8.4 if f is proper, then we have*

(1) $a_Y^{-1} \circ f_{small,*} = f_{big,ph,*} \circ a_X^{-1}$, *and*
(2) $a_Y^{-1}(Rf_{small,*}K) = Rf_{big,ph,*}(a_X^{-1}K)$ *for K in $D^+(X_{étale})$ with torsion cohomology sheaves.*

Proof. Proof of (1). You can prove this by repeating the proof of Lemma 84.5.6 part (1); we will instead deduce the result from this. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. Lemma 84.8.1 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_{small,*}\mathcal{F} \to f_{big,ph,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \pi_Y^{-1} f_{small,*}\mathcal{F} &= \epsilon_{Y,*}a_Y^{-1} f_{small,*}\mathcal{F} \\ &\to \epsilon_{Y,*} f_{big,ph,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} \\ &= f_{big,étale,*}\pi_X^{-1}\mathcal{F} \end{aligned}$$

is an isomorphism. This is part (1) of Lemma 84.5.6.

To see (2) we use that

$$\begin{aligned} R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K &= Rf_{big,étale,*}R\epsilon_{X,*}a_X^{-1}K \\ &= Rf_{big,étale,*}\pi_X^{-1}K \\ &= \pi_Y^{-1}Rf_{small,*}K \\ &= R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \end{aligned}$$

The first equality by the commutative diagram in Lemma 84.8.4 and Cohomology on Sites, Lemma 21.19.2. Then second equality is Lemma 84.8.2. The third is Lemma 84.5.6 part (2). The fourth is Lemma 84.8.2 again. Thus the base change map $a_Y^{-1}(Rf_{small,*}K) \to Rf_{big,ph,*}(a_X^{-1}K)$ induces an isomorphism

$$R\epsilon_{Y,*}a_Y^{-1}Rf_{small,*}K \to R\epsilon_{Y,*}Rf_{big,ph,*}a_X^{-1}K$$

The proof is finished by the following remark: consider a map $\alpha : a_Y^{-1}L \to M$ with L in $D^+(Y_{étale})$ having torsion cohomology sheaves and M in $D^+((Spaces/Y)_{ph})$. If $R\epsilon_{Y,*}\alpha$ is an isomorphism, then α is an isomorphism. Namely, we show by induction on i that $H^i(\alpha)$ is an isomorphism. This is true for all sufficiently small i. If it holds for $i \leq i_0$, then we see that $R^j\epsilon_{Y,*}H^i(M) = 0$ for $j > 0$ and $i \leq i_0$ by Lemma 84.8.1 because $H^i(M) = a_Y^{-1}H^i(L)$ in this range. Hence $\epsilon_{Y,*}H^{i_0+1}(M) = H^{i_0+1}(R\epsilon_{Y,*}M)$ by a spectral sequence argument. Thus $\epsilon_{Y,*}H^{i_0+1}(M) = \pi_Y^{-1}H^{i_0+1}(L) = \epsilon_{Y,*}a_Y^{-1}H^{i_0+1}(L)$. This implies $H^{i_0+1}(\alpha)$ is an isomorphism (because $\epsilon_{Y,*}$ reflects isomorphisms as it is the identity on underlying presheaves) as desired. □

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CHAPTER 85

Simplicial Spaces

09VI

85.1. Introduction

09VJ This chapter develops some theory concerning simplicial topological spaces, simplicial ringed spaces, simplicial schemes, and simplicial algebraic spaces. The theory of simplicial spaces sometimes allows one to prove local to global principles which appear difficult to prove in other ways. Some example applications can be found in the papers [**Fal03**], [**Kie72**], and [**Del74**].

We assume throughout that the reader is familiar with the basic concepts and results of the chapter Simplicial Methods, see Simplicial, Section 14.1. In particular, we continue to write X and not $X_\bullet$ for a simplicial object.

85.2. Simplicial topological spaces

09VK A *simplicial space* is a simplicial object in the category of topological spaces where morphisms are continuous maps of topological spaces. (We will use "simplicial algebraic space" to refer to simplicial objects in the category of algebraic spaces.) We may picture a simplicial space X as follows

$$X_2 \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} X_1 \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} X_0$$

Here there are two morphisms $d_0^1, d_1^1 : X_1 \to X_0$ and a single morphism $s_0^0 : X_0 \to X_1$, etc. It is important to keep in mind that $d_i^n : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s_j^n : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

Let X be a simplicial space. We associate a site X_{Zar}[1] to X as follows.

(1) An object of X_{Zar} is an open U of X_n for some n,
(2) a morphism $U \to V$ of X_{Zar} is given by a $\varphi : [m] \to [n]$ where n, m are such that $U \subset X_n$, $V \subset X_m$ and φ is such that $X(\varphi)(U) \subset V$, and
(3) a covering $\{U_i \to U\}$ in X_{Zar} means that $U, U_i \subset X_n$ are open, the maps $U_i \to U$ are given by $\mathrm{id} : [n] \to [n]$, and $U = \bigcup U_i$.

Note that in particular, if $U \to V$ is a morphism of X_{Zar} given by φ, then $X(\varphi) : X_n \to X_m$ does in fact induce a continuous map $U \to V$ of topological spaces.
It is clear that the above is a special case of a construction that associates to any diagram of topological spaces a site. We formulate the obligatory lemma.

09VL **Lemma 85.2.1.** *Let X be a simplicial space. Then X_{Zar} as defined above is a site.*

Proof. Omitted. □

[1] This notation is similar to the notation in Sites, Example 7.6.4 and Topologies, Definition 34.3.7.

Let X be a simplicial space. Let $\mathcal{F}$ be a sheaf on X_{Zar}. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the opens of X_n defines a sheaf $\mathcal{F}_n$ on the topological space X_n. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ for pairs $U \subset X_n$, $V \subset X_m$ with $X(\varphi)(U) \subset V$, define an $X(\varphi)$-map $\mathcal{F}(\varphi) : \mathcal{F}_m \to \mathcal{F}_n$, see Sheaves, Definition 6.21.7. Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$\mathcal{F}(\varphi) \circ \mathcal{F}(\psi) = \mathcal{F}(\varphi \circ \psi)$$

(LHS uses composition of f-maps, see Sheaves, Definition 6.21.9). Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \mathrm{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on X_{Zar}.

09VM **Lemma 85.2.2.** *Let X be a simplicial space. There is an equivalence of categories between*

(1) *$Sh(X_{Zar})$, and*
(2) *category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

Proof. See discussion above. □

09VN **Lemma 85.2.3.** *Let $f : Y \to X$ be a morphism of simplicial spaces. Then the functor $u : X_{Zar} \to Y_{Zar}$ which associates to the open $U \subset X_n$ the open $f_n^{-1}(U) \subset Y_n$ defines a morphism of sites $f_{Zar} : Y_{Zar} \to X_{Zar}$.*

Proof. It is clear that u is a continuous functor. Hence we obtain functors $f_{Zar,*} = u^s$ and $f_{Zar}^{-1} = u_s$, see Sites, Section 7.14. To see that we obtain a morphism of sites we have to show that u_s is exact. We will use Sites, Lemma 7.14.6 to see this. Let $V \subset Y_n$ be an open subset. The category $\mathcal{I}_V^u$ (see Sites, Section 7.5) consists of pairs (U, φ) where $\varphi : [m] \to [n]$ and $U \subset X_m$ open such that $Y(\varphi)(V) \subset f_m^{-1}(U)$. Moreover, a morphism $(U, \varphi) \to (U', \varphi')$ is given by a $\psi : [m'] \to [m]$ such that $X(\psi)(U) \subset U'$ and $\varphi \circ \psi = \varphi'$. It is our task to show that $\mathcal{I}_V^u$ is cofiltered.

We verify the conditions of Categories, Definition 4.20.1. Condition (1) holds because $(X_n, \mathrm{id}_{[n]})$ is an object. Let (U, φ) be an object. The condition $Y(\varphi)(V) \subset f_m^{-1}(U)$ is equivalent to $V \subset f_n^{-1}(X(\varphi)^{-1}(U))$. Hence we obtain a morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by setting $\psi = \varphi$. Moreover, given a pair of objects of the form $(U, \mathrm{id}_{[n]})$ and $(U', \mathrm{id}_{[n]})$ we see there exists an object, namely $(U \cap U', \mathrm{id}_{[n]})$, which maps to both of them. Thus condition (2) holds. To verify condition (3) suppose given two morphisms $a, a' : (U, \varphi) \to (U', \varphi')$ given by $\psi, \psi' : [m'] \to [m]$. Then precomposing with the morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by φ equalizes a, a' because $\varphi \circ \psi = \varphi' = \varphi \circ \psi'$. This finishes the proof. □

09VP **Lemma 85.2.4.** *Let $f : Y \to X$ be a morphism of simplicial spaces. In terms of the description of sheaves in Lemma 85.2.2 the morphism f_{Zar} of Lemma 85.2.3 can be described as follows.*

(1) *If $\mathcal{G}$ is a sheaf on Y, then $(f_{Zar,*}\mathcal{G})_n = f_{n,*}\mathcal{G}_n$.*
(2) *If $\mathcal{F}$ is a sheaf on X, then $(f_{Zar}^{-1}\mathcal{F})_n = f_n^{-1}\mathcal{F}_n$.*

Proof. The first part is immediate from the definitions. For the second part, note that in the proof of Lemma 85.2.3 we have shown that for a $V \subset Y_n$ open the category $(\mathcal{I}_V^u)^{opp}$ contains as a cofinal subcategory the category of opens $U \subset X_n$ with $f_n^{-1}(U) \supset V$ and morphisms given by inclusions. Hence we see that the restriction of $u_p\mathcal{F}$ to opens of Y_n is the presheaf $f_{n,p}\mathcal{F}_n$ as defined in Sheaves, Lemma 6.21.3. Since $f_{Zar}^{-1}\mathcal{F} = u_s\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$ and since sheafification uses

only coverings and since coverings in Y_{Zar} use only inclusions between opens on the same Y_n, the result follows from the fact that $f_n^{-1}\mathcal{F}_n$ is (correspondingly) the sheafification of $f_{n,p}\mathcal{F}_n$, see Sheaves, Section 6.21. □

Let X be a topological space. In Sites, Example 7.6.4 we denoted X_{Zar} the site consisting of opens of X with inclusions as morphisms and coverings given by open coverings. We identify the topos $Sh(X_{Zar})$ with the category of sheaves on X.

09W0 **Lemma 85.2.5.** *Let X be a simplicial space. The functor $X_{n,Zar} \to X_{Zar}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g_n : Sh(X_n) \to Sh(X_{Zar})$ satisfies*

(1) *g_n^{-1} associates to the sheaf $\mathcal{F}$ on X the sheaf $\mathcal{F}_n$ on X_n,*
(2) *$g_n^{-1} : Sh(X_{Zar}) \to Sh(X_n)$ has a left adjoint $g_{n!}^{Sh}$,*
(3) *$g_{n!}^{Sh}$ commutes with finite connected limits,*
(4) *$g_n^{-1} : Ab(X_{Zar}) \to Ab(X_n)$ has a left adjoint $g_{n!}$, and*
(5) *$g_{n!}$ is exact.*

Proof. Besides the properties of our functor mentioned in the statement, the category $X_{n,Zar}$ has fibre products and equalizers and the functor commutes with them (beware that X_{Zar} does not have all fibre products). Hence the lemma follows from the discussion in Sites, Sections 7.20 and 7.21 and Modules on Sites, Section 18.16. More precisely, Sites, Lemmas 7.21.1, 7.21.5, and 7.21.6 and Modules on Sites, Lemmas 18.16.2 and 18.16.3. □

09W1 **Lemma 85.2.6.** *Let X be a simplicial space. If $\mathcal{I}$ is an injective abelian sheaf on X_{Zar}, then $\mathcal{I}_n$ is an injective abelian sheaf on X_n.*

Proof. This follows from Homology, Lemma 12.29.1 and Lemma 85.2.5. □

09W2 **Lemma 85.2.7.** *Let $f : Y \to X$ be a morphism of simplicial spaces. Then*

$$\begin{array}{ccc} Sh(Y_n) & \xrightarrow[f_n]{} & Sh(X_n) \\ \downarrow & & \downarrow \\ Sh(Y_{Zar}) & \xrightarrow{f_{Zar}} & Sh(X_{Zar}) \end{array}$$

is a commutative diagram of topoi.

Proof. Direct from the description of pullback functors in Lemmas 85.2.4 and 85.2.5. □

09W4 **Lemma 85.2.8.** *Let Y be a simplicial space and let $a : Y \to X$ be an augmentation (Simplicial, Definition 14.20.1). Let $a_n : Y_n \to X$ be the corresponding morphisms of topological spaces. There is a canonical morphism of topoi*

$$a : Sh(Y_{Zar}) \to Sh(X)$$

with the following properties:

(1) *$a^{-1}\mathcal{F}$ is the sheaf restricting to $a_n^{-1}\mathcal{F}$ on Y_n,*
(2) *$a_m \circ Y(\varphi) = a_n$ for all $\varphi : [m] \to [n]$,*
(3) *$a \circ g_n = a_n$ as morphisms of topoi with g_n as in Lemma 85.2.5,*
(4) *$a_*\mathcal{G}$ for $\mathcal{G} \in Sh(Y_{Zar})$ is the equalizer of the two maps $a_{0,*}\mathcal{G}_0 \to a_{1,*}\mathcal{G}_1$.*

Proof. Part (2) holds for augmentations of simplicial objects in any category. Thus $Y(\varphi)^{-1}a_m^{-1}\mathcal{F} = a_n^{-1}\mathcal{F}$ which defines an $Y(\varphi)$-map from $a_m^{-1}\mathcal{F}$ to $a_n^{-1}\mathcal{F}$. Thus we can use (1) as the definition of $a^{-1}\mathcal{F}$ (using Lemma 85.2.2) and (4) as the definition of a_*. If this defines a morphism of topoi then part (3) follows because we'll have $g_n^{-1} \circ a^{-1} = a_n^{-1}$ by construction. To check a is a morphism of topoi we have to show that a^{-1} is left adjoint to a_* and we have to show that a^{-1} is exact. The last fact is immediate from the exactness of the functors a_n^{-1}.

Let $\mathcal{F}$ be an object of $Sh(X)$ and let $\mathcal{G}$ be an object of $Sh(Y_{Zar})$. Given $\beta : a^{-1}\mathcal{F} \to \mathcal{G}$ we can look at the components $\beta_n : a_n^{-1}\mathcal{F} \to \mathcal{G}_n$. These maps are adjoint to maps $\beta_n : \mathcal{F} \to a_{n,*}\mathcal{G}_n$. Compatibility with the simplicial structure shows that β_0 maps into $a_*\mathcal{G}$. Conversely, suppose given a map $\alpha : \mathcal{F} \to a_*\mathcal{G}$. For any n choose a $\varphi : [0] \to [n]$. Then we can look at the composition

$$\mathcal{F} \xrightarrow{\alpha} a_*\mathcal{G} \to a_{0,*}\mathcal{G}_0 \xrightarrow{\mathcal{G}(\varphi)} a_{n,*}\mathcal{G}_n$$

These are adjoint to maps $a_n^{-1}\mathcal{F} \to \mathcal{G}_n$ which define a morphism of sheaves $a^{-1}\mathcal{F} \to \mathcal{G}$. We omit the proof that the constructions given above define mutually inverse bijections

$$\mathrm{Mor}_{Sh(Y_{Zar})}(a^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, a_*\mathcal{G})$$

This finishes the proof. An interesting observation is here that this morphism of topoi does not correspond to any obvious geometric functor between the sites defining the topoi. □

09W5 **Lemma 85.2.9.** *Let X be a simplicial topological space. The complex of abelian presheaves on X_{Zar}*

$$\ldots \to \mathbf{Z}_{X_2} \to \mathbf{Z}_{X_1} \to \mathbf{Z}_{X_0}$$

with boundary $\sum(-1)^i d_i^n$ is a resolution of the constant presheaf $\mathbf{Z}$.

Proof. Let $U \subset X_m$ be an object of X_{Zar}. Then the value of the complex above on U is the complex of abelian groups

$$\ldots \to \mathbf{Z}[\mathrm{Mor}_\Delta([2], [m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([1], [m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([0], [m])]$$

In other words, this is the complex associated to the free abelian group on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian groups" is a functor, we see that the complex above is homotopy equivalent to the free abelian group on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

09W6 **Lemma 85.2.10.** *Let X be a simplicial topological space. Let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X_{Zar}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \mathcal{I}^q(X_p)$$

and first differential given by the alternating sum along the maps d_i^{p+1}-maps $\mathcal{I}_p^q \to \mathcal{I}_{p+1}^q$, see Lemma 85.2.2. Note that

$$A^{p,q} = \Gamma(X_p, \mathcal{I}_p^q) = \mathrm{Mor}_{PSh}(h_{X_p}, \mathcal{I}^q) = \mathrm{Mor}_{PAb}(\mathbf{Z}_{X_p}, \mathcal{I}^q)$$

Hence it follows from Lemma 85.2.9 and Cohomology on Sites, Lemma 21.10.1 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(X_{Zar}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 85.2.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on X_p (Lemma 85.2.6). Hence the cohomology of the columns computes the groups $H^q(X_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. □

0D84 **Lemma 85.2.11.** *Let X be a simplicial space and let $a : X \to Y$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X_{Zar}. Then $R^n a_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^n((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Proof. This is the analogue of Cohomology, Lemma 20.7.3 or of Cohomology on Sites, Lemma 21.7.4 and we strongly encourage the reader to skip the proof. Choosing an injective resolution of $\mathcal{F}$ on X_{Zar} and using the definitions we see that it suffices to show: (1) the restriction of an injective abelian sheaf on X_{Zar} to $(X \times_Y V)_{Zar}$ is an injective abelian sheaf and (2) $a_*\mathcal{F}$ is equal to the rule

$$V \longmapsto H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Part (2) follows from the following facts

(2a) $a_*\mathcal{F}$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$ by Lemma 85.2.8,
(2b) $a_{0,*}\mathcal{F}_0(V) = H^0(a_0^{-1}(V), \mathcal{F}_0)$ and $a_{1,*}\mathcal{F}_1(V) = H^0(a_1^{-1}(V), \mathcal{F}_1)$,
(2c) $X_0 \times_Y V = a_0^{-1}(V)$ and $X_1 \times_Y V = a_1^{-1}(V)$,
(2d) $H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$ is the equalizer of the two maps $H^0(X_0 \times_Y V, \mathcal{F}_0) \to H^0(X_1 \times_Y V, \mathcal{F}_1)$ for example by Lemma 85.2.10.

Part (1) follows after one defines an exact left adjoint $j_! : Ab((X \times_Y V)_{Zar}) \to Ab(X_{Zar})$ (extension by zero) to restriction $Ab(X_{Zar}) \to Ab((X \times_Y V)_{Zar})$ and using Homology, Lemma 12.29.1. □

Let X be a topological space. Denote $X_\bullet$ the constant simplicial topological space with value X. By Lemma 85.2.2 a sheaf on $X_{\bullet,Zar}$ is the same thing as a cosimplicial object in the category of sheaves on X.

09W3 **Lemma 85.2.12.** *Let X be a topological space. Let $X_\bullet$ be the constant simplicial topological space with value X. The functor*

$$X_{\bullet,Zar} \longrightarrow X_{Zar}, \quad U \longmapsto U$$

is continuous and cocontinuous and defines a morphism of topoi $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ as well as a left adjoint $g_!$ to g^{-1}. We have

(1) *g^{-1} associates to a sheaf on X the constant cosimplicial sheaf on X,*
(2) *$g_!$ associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the sheaf $\mathcal{F}_0$, and*
(3) *g_* associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the equalizer of the two maps $\mathcal{F}_0 \to \mathcal{F}_1$.*

Proof. The statements about the functor are straightforward to verify. The existence of g and $g_!$ follow from Sites, Lemmas 7.21.1 and 7.21.5. The description of g^{-1} is immediate from Sites, Lemma 7.21.5. The description of g_* and $g_!$ follows as the functors given are right and left adjoint to g^{-1}. □

85.3. Simplicial sites and topoi

09WB It seems natural to define a *simplicial site* as a simplicial object in the (big) category whose objects are sites and whose morphisms are morphisms of sites. See Sites, Definitions 7.6.2 and 7.14.1 with composition of morphisms as in Sites, Lemma 7.14.4. But here are some variants one might want to consider: (a) we could work with cocontinuous functors (see Sites, Sections 7.20 and 7.21) between sites instead, (b) we could work in a suitable 2-category of sites where one introduces the notion of a 2-morphism between morphisms of sites, (c) we could work in a 2-category constructed out of cocontinuous functors. Instead of picking one of these variants as a definition we will simply develop theory as needed.

Certainly a *simplicial topos* should probably be defined as a pseudo-functor from Δ^{opp} into the 2-category of topoi. See Categories, Definition 4.29.5 and Sites, Section 7.15 and 7.36. We will try to avoid working with such a beast if possible.

Case A. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. This morphism is given by a continuous functor in the opposite direction which we will denote $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

09WC **Lemma 85.3.1.** *Let $\mathcal{C}$ be a simplicial object in the category of sites. With notation as above we construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{total}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : U \to u_\varphi(V)$ of $\mathcal{C}_n$, and*
3. *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{total}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, u_\varphi(g) \circ f)$. This uses that $u_\varphi \circ u_\psi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,u_\varphi(U),u_\varphi(f_i)} u_\varphi(U_i)$$

in the category $\mathcal{C}_{total}$. This makes sense as by our definition of morphisms of sites, the required fibre products in $\mathcal{C}_m$ exist since u_φ transforms coverings into coverings. The same reasoning implies the claim (details omitted). Thus we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. □

Case B. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor denoted $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. The associated morphism of topoi is denoted $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

09WD **Lemma 85.3.2.** *Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. With notation as above, assume the functors $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ have property P of Sites, Remark 7.20.5. Then we can construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*

(2) *a morphism* $(\varphi, f) : U \to V$ *of* $\mathcal{C}_{total}$ *is given by a map* $\varphi : [m] \to [n]$ *with* $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ *and a morphism* $f : u_\varphi(U) \to V$ *of* $\mathcal{C}_m$, *and*
(3) *a covering* $\{(id, f_i) : U_i \to U\}$ *in* $\mathcal{C}_{total}$ *is given by an* n *and a covering* $\{f_i : U_i \to U\}$ *of* $\mathcal{C}_n$.

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, g \circ u_\psi(f))$. This uses that $u_\psi \circ u_\varphi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,U,f_i} U_i$$

in the category $\mathcal{C}_{total}$ where the right hand side is the object of $\mathcal{C}_m$ defined in Sites, Remark 7.20.5 which exists by property P. Compatibility of this type of fibre product with compositions of functors implies the claim (details omitted). Since the family $\{W \times_{g,U,f_i} U_i \to W\}$ is a covering of $\mathcal{C}_m$ by property P we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. □

09WE **Situation 85.3.3.** Here we have one of the following two cases:

(A) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. For every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ given by a continuous functor $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.
(B) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors having property P of Sites, Remark 7.20.5. For every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ which induces a morphism of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

As usual we will denote f_φ^{-1} and $f_{\varphi,*}$ the pullback and pushforward. We let $\mathcal{C}_{total}$ denote the site defined in Lemma 85.3.1 (case A) or Lemma 85.3.2 (case B).

Let $\mathcal{C}$ be as in Situation 85.3.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}_{total}$. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the objects of $\mathcal{C}_n$ defines a sheaf $\mathcal{F}_n$ on the site $\mathcal{C}_n$. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ along the morphisms $(\varphi, f) : U \to V$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$ and $V \in \mathrm{Ob}(\mathcal{C}_m)$ define an element $\mathcal{F}(\varphi)$ of

$$\mathrm{Mor}_{Sh(\mathcal{C}_m)}(\mathcal{F}_m, f_{\varphi,*}\mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_n)}(f_\varphi^{-1}\mathcal{F}_m, \mathcal{F}_n)$$

Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ the diagrams

$$\begin{array}{ccc} \mathcal{F}_l & \xrightarrow{\mathcal{F}(\varphi\circ\psi)} & f_{\varphi\circ\psi,*}\mathcal{F}_n \\ & {\scriptstyle \mathcal{F}(\psi)} \searrow \quad \nearrow {\scriptstyle f_{\psi,*}\mathcal{F}(\varphi)} & \\ & f_{\psi,*}\mathcal{F}_m & \end{array} \quad \text{and} \quad \begin{array}{ccc} f_{\varphi\circ\psi}^{-1}\mathcal{F}_l & \xrightarrow{\mathcal{F}(\varphi\circ\psi)} & \mathcal{F}_n \\ & {\scriptstyle f_\varphi^{-1}\mathcal{F}(\psi)} \searrow \quad \nearrow {\scriptstyle \mathcal{F}(\varphi)} & \\ & f_\varphi^{-1}\mathcal{F}_m & \end{array}$$

commute. Clearly, the converse statement is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi\in \mathrm{Arrows}(\Delta)})$ satisfying the commutativity constraints above, then we obtain a sheaf on $\mathcal{C}_{total}$.

09WF **Lemma 85.3.4.** *In Situation 85.3.3 there is an equivalence of categories between*

(1) $Sh(\mathcal{C}_{total})$, *and*

(2) *the category of systems* $(\mathcal{F}_n, \mathcal{F}(\varphi))$ *described above.*

In particular, the topos $Sh(\mathcal{C}_{total})$ *only depends on the topoi* $Sh(\mathcal{C}_n)$ *and the morphisms of topoi* f_φ*.*

Proof. See discussion above. □

09WG **Lemma 85.3.5.** *In Situation 85.3.3 the functor* $\mathcal{C}_n \to \mathcal{C}_{total}$, $U \mapsto U$ *is continuous and cocontinuous. The associated morphism of topoi* $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ *satisfies*

(1) g_n^{-1} *associates to the sheaf* $\mathcal{F}$ *on* $\mathcal{C}_{total}$ *the sheaf* $\mathcal{F}_n$ *on* $\mathcal{C}_n$,
(2) $g_n^{-1} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ *has a left adjoint* $g_{n!}^{Sh}$,
(3) *for* $\mathcal{G}$ *in* $Sh(\mathcal{C}_n)$ *the restriction of* $g_{n!}^{Sh}\mathcal{G}$ *to* $\mathcal{C}_m$ *is* $\coprod_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$,
(4) $g_{n!}^{Sh}$ *commutes with finite connected limits,*
(5) $g_n^{-1} : Ab(\mathcal{C}_{total}) \to Ab(\mathcal{C}_n)$ *has a left adjoint* $g_{n!}$,
(6) *for* $\mathcal{G}$ *in* $Ab(\mathcal{C}_n)$ *the restriction of* $g_{n!}\mathcal{G}$ *to* $\mathcal{C}_m$ *is* $\bigoplus_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$*, and*
(7) $g_{n!}$ *is exact.*

Proof. Case A. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 85.3.1). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 85.3.1). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

Case B. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 85.3.2). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 85.3.2). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

At this point part (1) and the existence of $g_{n!}^{Sh}$ and $g_{n!}$ in cases A and B follows from Sites, Lemmas 7.21.1 and 7.21.5 and Modules on Sites, Lemma 18.16.2.

Proof of (3). Let $\mathcal{G}$ be a sheaf on $\mathcal{C}_n$. Consider the sheaf $\mathcal{H}$ on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\mathcal{H}_m = \coprod_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

given in part (3) of the statement of the lemma. Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by the identifications

$$f_\psi^{-1} f_\varphi^{-1}\mathcal{G} \to f_{\psi\circ\varphi}^{-1}\mathcal{G}$$

Observe that given a map $\alpha : \mathcal{H} \to \mathcal{F}$ of sheaves on $\mathcal{C}_{total}$ we obtain a map $\mathcal{G} \to \mathcal{F}_n$ corresponding to the restriction of α_n to the component $\mathcal{G}$ in $\mathcal{H}_n$. Conversely, given a map $\beta : \mathcal{G} \to \mathcal{F}_n$ of sheaves on $\mathcal{C}_n$ we can define $\alpha : \mathcal{H} \to \mathcal{F}$ by letting α_m be the map which on components

$$f_\varphi^{-1}\mathcal{G} \to \mathcal{F}_m$$

uses the maps adjoint to $\mathcal{F}(\varphi) \circ f_\varphi^{-1}\beta$. We omit the arguments showing these two constructions give mutually inverse maps

$$\mathrm{Mor}_{Sh(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

Thus $\mathcal{H} = g_{n!}^{Sh}\mathcal{G}$ as desired.

Proof of (4). If $\mathcal{G}$ is an abelian sheaf on $\mathcal{C}_n$, then we proceed in exactly the same ammner as above, except that we define $\mathcal{H}$ is the abelian sheaf on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

with transition maps defined exactly as above. The bijection

$$\mathrm{Mor}_{Ab(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{Ab(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

is proved exactly as above. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired.

The exactness properties of $g_{n!}^{Sh}$ and $g_{n!}$ follow from formulas given for these functors. □

09WH **Lemma 85.3.6.** *In Situation 85.3.3. If $\mathcal{I}$ is injective in $Ab(\mathcal{C}_{total})$, then $\mathcal{I}_n$ is injective in $Ab(\mathcal{C}_n)$. If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab(\mathcal{C}_{total})$, then $\mathcal{I}_n^\bullet$ is K-injective in $Ab(\mathcal{C}_n)$.*

Proof. The first statement follows from Homology, Lemma 12.29.1 and Lemma 85.3.5. The second statement from Derived Categories, Lemma 13.31.9 and Lemma 85.3.5. □

85.4. Augmentations of simplicial sites

0D93 We continue in the fashion described in Section 85.3 working out the meaning of augmentations in cases A and B treated in that section.

0D6Z **Remark 85.4.1.** In Situation 85.3.3 an *augmentation a_0 towards a site $\mathcal{D}$* will mean

(A) $a_0 : \mathcal{C}_0 \to \mathcal{D}$ is a morphism of sites given by a continuous functor $u_0 : \mathcal{D} \to \mathcal{C}_0$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_\varphi \circ u_0 = u_\psi \circ u_0$.
(B) $a_0 : Sh(\mathcal{C}_0) \to Sh(\mathcal{D})$ is a morphism of topoi given by a cocontinuous functor $u_0 : \mathcal{C}_0 \to \mathcal{D}$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_0 \circ u_\varphi = u_0 \circ u_\psi$.

0D70 **Lemma 85.4.2.** *In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. Then a_0 induces*

(1) *a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$ for all $n \geq 0$,*
(2) *a morphism of topoi $a : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$*

such that

(1) *for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$,*
(2) *if $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ is as in Lemma 85.3.5, then $a \circ g_n = a_n$, and*
(3) *$a_*\mathcal{F}$ for $\mathcal{F} \in Sh(\mathcal{C}_{total})$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$.*

Proof. Case A. Let $u_n : \mathcal{D} \to \mathcal{C}_n$ be the common value of the functors $u_\varphi \circ u_0$ for $\varphi : [0] \to [n]$. Then u_n corresponds to a morphism of sites $a_n : \mathcal{C}_n \to \mathcal{D}$, see Sites, Lemma 7.14.4. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Case B. Let $u_n : \mathcal{C}_n \to \mathcal{D}$ be the common value of the functors $u_0 \circ u_\varphi$ for $\varphi : [0] \to [n]$. Then u_n is cocontinuous and hence defines a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$, see Sites, Lemma 7.21.2. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Consider the functor $a^{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{C}_{total})$ which to a sheaf of sets $\mathcal{G}$ associates the sheaf $\mathcal{F} = a^{-1}\mathcal{G}$ whose components are $a_n^{-1}\mathcal{G}$ and whose transition maps $\mathcal{F}(\varphi)$ are the identifications

$$f_\varphi^{-1}\mathcal{F}_m = f_\varphi^{-1}a_m^{-1}\mathcal{G} = a_n^{-1}\mathcal{G} = \mathcal{F}_n$$

for $\varphi : [m] \to [n]$, see the description of $Sh(\mathcal{C}_{total})$ in Lemma 85.3.4. Since the functors a_n^{-1} are exact, a^{-1} is an exact functor. Finally, for $a_* : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$ we take the functor which to a sheaf $\mathcal{F}$ on $Sh(\mathcal{D})$ associates

$$a_*\mathcal{F} = \text{Equalizer}(a_{0,*}\mathcal{F}_0 \rightrightarrows a_{1,*}\mathcal{F}_1)$$

Here the two maps come from the two maps $\varphi : [0] \to [1]$ via

$$a_{0,*}\mathcal{F}_0 \to a_{0,*}f_{\varphi,*}f_\varphi^{-1}\mathcal{F}_0 \xrightarrow{\mathcal{F}(\varphi)} a_{0,*}f_{\varphi,*}\mathcal{F}_1 = a_{1,*}\mathcal{F}_1$$

where the first arrow comes from $1 \to f_{\varphi,*}f_\varphi^{-1}$. Let $\mathcal{G}_\bullet$ denote the constant coosimplicial sheaf with value $\mathcal{G}$ and let $a_{\bullet,*}\mathcal{F}$ denote the cosimplicial sheaf having $a_{n,*}\mathcal{F}_n$ in degree n. By the usual adjuntion for the morphisms of topoi a_n we see that a map $a^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map

$$\mathcal{G}_\bullet \longrightarrow a_{\bullet,*}\mathcal{F}$$

of cosimplicial sheaves. By the dual to Simplicial, Lemma 14.20.2 this is the same thing as a map $\mathcal{G} \to a_*\mathcal{F}$. Thus a^{-1} and a_* are adjoint functors and we obtain our morphism of topoi a^2. The equalities $a \circ g_n = f_n$ follow immediately from the definitions. □

85.5. Morphisms of simplicial sites

0D94 We continue in the fashion described in Section 85.3 working out the meaning of morphisms of simplicial sites in cases A and B treated in that section.

0D95 **Remark 85.5.1.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. A *morphism h between simplicial sites* will mean

(A) Morphisms of sites $h_n : \mathcal{C}_n \to \mathcal{C}'_n$ such that $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of sites for all $\varphi : [m] \to [n]$.
(B) Cocontinuous functors $v_n : \mathcal{C}_n \to \mathcal{C}'_n$ inducing morphisms of topoi $h_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}'_n)$ such that $u'_\varphi \circ v_n = v_m \circ u_\varphi$ as functors for all $\varphi : [m] \to [n]$.

In both cases we have $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of topoi, see Sites, Lemma 7.21.2 for case B and Sites, Definition 7.14.5 for case A.

0D96 **Lemma 85.5.2.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let h be a morphism between simplicial sites as in Remark 85.5.1. Then we obtain a morphism of topoi*

$$h_{total} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$$

[2]In case B the morphism a corresponds to the cocontinuous functor $\mathcal{C}_{total} \to \mathcal{D}$ sending U in $\mathcal{C}_n$ to $u_n(U)$.

and commutative diagrams

$$\begin{array}{ccc} Sh(\mathcal{C}_n) & \xrightarrow[h_n]{} & Sh(\mathcal{C}'_n) \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g'_n} \\ Sh(\mathcal{C}_{total}) & \xrightarrow{h_{total}} & Sh(\mathcal{C}'_{total}) \end{array}$$

Moreover, we have $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$.

Proof. Case A. Say h_n corresponds to the continuous functor $v_n : \mathcal{C}'_n \to \mathcal{C}_n$. Then we can define a functor $v_{total} : \mathcal{C}'_{total} \to \mathcal{C}_{total}$ by using v_n in degree n. This is clearly a continuous functor (see definition of coverings in Lemma 85.3.1). Let $h_{total}^{-1} = v_{total,s} : Sh(\mathcal{C}'_{total}) \to Sh(\mathcal{C}_{total})$ and $h_{total,*} = v_{total}^s = v_{total}^p : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$ be the adjoint pair of functors constructed and studied in Sites, Sections 7.13 and 7.14. To see that h_{total} is a morphism of topoi we still have to verify that h_{total}^{-1} is exact. We first observe that $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$; this is immediate by computing sections over an object U of $\mathcal{C}'_n$. Thus, if we think of a sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ as a system $(\mathcal{F}_n, \mathcal{F}(\varphi))$ as in Lemma 85.3.4, then $h_{total,*}\mathcal{F}$ corresponds to the system $(h_{n,*}\mathcal{F}_n, h_{n,*}\mathcal{F}(\varphi))$. Clearly, the functor $(\mathcal{F}'_n, \mathcal{F}'(\varphi)) \to (h_n^{-1}\mathcal{F}'_n, h_n^{-1}\mathcal{F}'(\varphi))$ is its left adjoint. By uniqueness of adjoints, we conclude that h_{total}^{-1} is given by this rule on systems. In particular, h_{total}^{-1} is exact (by the description of sheaves on $\mathcal{C}_{total}$ given in the lemma and the exactness of the functors h_n^{-1}) and we have our morphism of topoi. Finally, we obtain $g_n^{-1} \circ h_{total}^{-1} = h_n^{-1} \circ (g'_n)^{-1}$ as well, which proves that the displayed diagram of the lemma commutes.

Case B. Here we have a functor $v_{total} : \mathcal{C}_{total} \to \mathcal{C}'_{total}$ by using v_n in degree n. This is clearly a cocontinuous functor (see definition of coverings in Lemma 85.3.2). Let h_{total} be the morphism of topoi associated to v_{total}. The commutativity of the displayed diagram of the lemma follows immediately from Sites, Lemma 7.21.2. Taking left adjoints the final equality of the lemma becomes

$$h_{total}^{-1} \circ (g'_n)_!^{Sh} = g_{n!}^{Sh} \circ h_n^{-1}$$

This follows immediately from the explicit description of the functors $(g'_n)_!^{Sh}$ and $g_{n!}^{Sh}$ in Lemma 85.3.5, the fact that $h_n^{-1} \circ (f'_\varphi)^{-1} = f_\varphi^{-1} \circ h_m^{-1}$ for $\varphi : [m] \to [n]$, and the fact that we already know h_{total}^{-1} commutes with restrictions to the degree n parts of the simplicial sites. □

0D97 **Lemma 85.5.3.** *With notation and hypotheses as in Lemma 85.5.2. For* $K \in D(\mathcal{C}_{total})$ *we have* $(g'_n)^{-1} R h_{total,*} K = R h_{n,*} g_n^{-1} K$.

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex on $\mathcal{C}_{total}$ representing K. Then $g_n^{-1}K$ is represented by $g_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ which is K-injective by Lemma 85.3.6. We have $(g'_n)^{-1} h_{total,*} \mathcal{I}^\bullet = h_{n,*} g_n^{-1} \mathcal{I}_n^\bullet$ by Lemma 85.5.2 which gives the desired equality. □

0D98 **Remark 85.5.4.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let a_0, resp. a'_0 be an augmentation towards a site $\mathcal{D}$, resp. $\mathcal{D}'$ as in Remark 85.4.1. Let h be a morphism between simplicial sites as in Remark 85.5.1. We say a morphism of topoi $h_{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is *compatible with* h, a_0, a'_0 if

(A) h_{-1} comes from a morphism of sites $h_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of sites.

(B) h_{-1} comes from a cocontinuous functor $v_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $u'_0 \circ v_0 = v_{-1} \circ u_0$ as functors.

In both cases we have $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of topoi, see Sites, Lemma 7.21.2 for case B and Sites, Definition 7.14.5 for case A.

0D99 **Lemma 85.5.5.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi, \mathcal{D}, a_0, \mathcal{C}'_n, f'_\varphi, u'_\varphi, \mathcal{D}', a'_0$, and h_n, $n \geq -1$ be as in Remark 85.5.4. Then we obtain a commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{C}_{total}) & \xrightarrow{h_{total}} & Sh(\mathcal{C}'_{total}) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a'} \\ Sh(\mathcal{D}) & \xrightarrow{h_{-1}} & Sh(\mathcal{D}') \end{array}$$

Proof. The morphism h is defined in Lemma 85.5.2. The morphisms a and a' are defined in Lemma 85.4.2. Thus the only thing is to prove the commutativity of the diagram. To do this, we prove that $a^{-1} \circ h_{-1}^{-1} = h_{total}^{-1} \circ (a')^{-1}$. By the commutative diagrams of Lemma 85.5.2 and the description of $Sh(\mathcal{C}_{total})$ and $Sh(\mathcal{C}'_{total})$ in terms of components in Lemma 85.3.4, it suffices to show that

$$\begin{array}{ccc} Sh(\mathcal{C}_n) & \xrightarrow{h_n} & Sh(\mathcal{C}'_n) \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle a'_n} \\ Sh(\mathcal{D}) & \xrightarrow{h_{-1}} & Sh(\mathcal{D}') \end{array}$$

commutes for all n. This follows from the case for $n = 0$ (which is an assumption in Remark 85.5.4) and for $n > 0$ we pick $\varphi : [0] \to [n]$ and then the required commutativity follows from the case $n = 0$ and the relations $a_n = a_0 \circ f_\varphi$ and $a'_n = a'_0 \circ f'_\varphi$ as well as the commutation relations $f'_\varphi \circ h_n = h_0 \circ f_\varphi$. □

85.6. Ringed simplicial sites

0D71 Let us endow our simplicial topos with a sheaf of rings.

0D72 **Lemma 85.6.1.** *In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a canonical morphism of ringed topoi $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ agreeing with the morphism g_n of Lemma 85.3.5 on underlying topoi. The functor $g_n^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_n)$ has a left adjoint $g_{n!}$. For $\mathcal{G}$ in $Mod(\mathcal{O}_n)$-modules the restriction of $g_{n!}\mathcal{G}$ to $\mathcal{C}_m$ is*

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

where $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ is the morphism of ringed topoi agreeing with the previously defined f_φ on topoi and using the map $\mathcal{O}(\varphi) : f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ on sheaves of rings.

Proof. By Lemma 85.3.5 we have $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ and hence we obtain our morphism of ringed topoi. By Modules on Sites, Lemma 18.41.1 we obtain the adjoint $g_{n!}$. To prove the formula for $g_{n!}$ we first define a sheaf of $\mathcal{O}$-modules $\mathcal{H}$ on $\mathcal{C}_{total}$ with degree m component the $\mathcal{O}_m$-module

$$\mathcal{H}_m = \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by

$$f_\psi^{-1} f_\varphi^* \mathcal{G} \to f_\psi^* f_\varphi^* \mathcal{G} \to f_{\psi\circ\varphi}^* \mathcal{G}$$

Since this map $f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is $\mathcal{O}(\psi) : f_\psi^{-1}\mathcal{O}_m \to \mathcal{O}_{m'}$-semi-linear, this indeed does define an $\mathcal{O}$-module (use Lemma 85.3.4). Then one proves directly that

$$\mathrm{Mor}_{\mathcal{O}_n}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{\mathcal{O}}(\mathcal{H}, \mathcal{F})$$

proceeding as in the proof of Lemma 85.3.5. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired. □

0D73 **Lemma 85.6.2.** *In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $\mathcal{I}$ is injective in $Mod(\mathcal{O})$, then $\mathcal{I}_n$ is a totally acyclic sheaf on $\mathcal{C}_n$.*

Proof. This follows from Cohomology on Sites, Lemma 21.37.4 applied to the inclusion functor $\mathcal{C}_n \to \mathcal{C}_{total}$ and its properties proven in Lemma 85.3.5. □

0D74 **Lemma 85.6.3.** *With assumptions as in Lemma 85.6.1 the functor $g_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O})$ is exact if the maps $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$.*

Proof. Recall that $g_{n!}\mathcal{G}$ is the $\mathcal{O}$-module whose degree m part is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^* \mathcal{G}$$

Here the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ uses the map $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ of the statement of the lemma. If these maps are flat, then f_φ^* is exact (Modules on Sites, Lemma 18.31.2). By definition of the site $\mathcal{C}_{total}$ we see that these functors have the desired exactness properties and we conclude. □

0D75 **Lemma 85.6.4.** *In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for all $\varphi : [n] \to [m]$. If $\mathcal{I}$ is injective in $Mod(\mathcal{O})$, then $\mathcal{I}_n$ is injective in $Mod(\mathcal{O}_n)$.*

Proof. This follows from Homology, Lemma 12.29.1 and Lemma 85.6.3. □

85.7. Morphisms of ringed simplicial sites

0DGY We continue the discussion of Section 85.5.

0DGZ **Remark 85.7.1.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. We will say that $(h, h^\sharp)$ is a *morphism between ringed simplicial sites* if h is a morphism between simplicial sites as in Remark 85.5.1 and $h^\sharp : h_{total}^{-1}\mathcal{O}' \to \mathcal{O}$ or equivalently $h^\sharp : \mathcal{O}' \to h_{total,*}\mathcal{O}$ is a homomorphism of sheaves of rings.

0DH0 **Lemma 85.7.2.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 85.3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. Let $(h, h^\sharp)$ be a morphism between simplicial sites as in Remark 85.7.1. Then we obtain a morphism of ringed topoi*

$$h_{total} : (Sh(\mathcal{C}_{total}), \mathcal{O}) \to (Sh(\mathcal{C}'_{total}), \mathcal{O}')$$

and commutative diagrams

$$\begin{array}{ccc} (Sh(\mathcal{C}_n), \mathcal{O}_n) & \xrightarrow[h_n]{} & (Sh(\mathcal{C}'_n), \mathcal{O}'_n) \\ \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g'_n} \\ (Sh(\mathcal{C}_{total}), \mathcal{O}) & \xrightarrow{h_{total}} & (Sh(\mathcal{C}'_{total}), \mathcal{O}') \end{array}$$

of ringed topoi where g_n and g'_n are as in Lemma 85.6.1. Moreover, we have $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ as functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}'_n)$.

Proof. Follows from Lemma 85.5.2 and 85.6.1 by keeping track of the sheaves of rings. A small point is that in order to define h_n as a morphism of ringed topoi we set $h_n^\sharp = g_n^{-1}h^\sharp : g_n^{-1}h_{total}^{-1}\mathcal{O}' \to g_n^{-1}\mathcal{O}$ which makes sense because $g_n^{-1}h_{total}^{-1}\mathcal{O}' = h_n^{-1}(g'_n)^{-1}\mathcal{O}' = h_n^{-1}\mathcal{O}'_n$ and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$. Note that $g_n^*\mathcal{F} = g_n^{-1}\mathcal{F}$ for a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ and similarly for g'_n and this helps explain why $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ follows from the corresponding statement of Lemma 85.5.2. □

0DH1 **Lemma 85.7.3.** *With notation and hypotheses as in Lemma 85.7.2. For $K \in D(\mathcal{O})$ we have $(g'_n)^* Rh_{total,*}K = Rh_{n,*}g_n^*K$.*

Proof. Recall that $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by the construction in Lemma 85.6.1. In particular g_n^* is exact and Lg_n^* is given by applying g_n^* to any representative complex of modules. Similarly for g'_n. There is a canonical base change map $(g'_n)^* Rh_{total,*}K \to Rh_{n,*}g_n^*K$, see Cohomology on Sites, Remark 21.19.3. By Cohomology on Sites, Lemma 21.20.7 the image of this in $D(\mathcal{C}'_n)$ is the map $(g'_n)^{-1}Rh_{total,*}K_{ab} \to Rh_{n,*}g_n^{-1}K_{ab}$ where K_{ab} is the image of K in $D(\mathcal{C}_{total})$. This we proved to be an isomorphism in Lemma 85.5.3 and the result follows. □

85.8. Cohomology on simplicial sites

0D76 Let $\mathcal{C}$ be as in Situation 85.3.3. In statement of the following lemmas we will let $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ be the morphism of topoi of Lemma 85.3.5. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}_n) & \xrightarrow{f_\varphi} & Sh(\mathcal{C}_m) \\ & {\scriptstyle g_n} \searrow \quad \swarrow {\scriptstyle g_m} & \\ & Sh(\mathcal{C}_{total}) & \end{array}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 7.36.

09WI **Lemma 85.8.1.** *In Situation 85.3.3 and with notation as above there is a complex*

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

of abelian sheaves on $\mathcal{C}_{total}$ which forms a resolution of the constant sheaf with value $\mathbf{Z}$ on $\mathcal{C}_{total}$.

Proof. We will use the description of the functors $g_{n!}$ in Lemma 85.3.5 without further mention. As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}\mathbf{Z} \to g_{n-1!}\mathbf{Z}$ is the adjoint to the map $\mathbf{Z} \to \bigoplus_{[n-1]\to[n]} \mathbf{Z} = g_n^{-1}g_{n-1!}\mathbf{Z}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free abelian sheaf on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the

free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

0D77 **Lemma 85.8.2.** *In Situation 85.3.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. Observe that $\mathrm{Hom}(\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\mathrm{Hom}(g_{n!}\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 85.8.1 and the fact that $\mathrm{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. □

09WJ **Lemma 85.8.3.** *In Situation 85.3.3. For K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

where the horizontal arrows come from Lemma 85.8.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 85.3.6). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. □

0H0V **Remark 85.8.4.** Assumptions and notation as in Lemma 85.8.3 except we do not require K in $D(\mathcal{C}_{total})$ to be bounded below. We claim there is a natural spectral sequence in this case also. Namely, suppose that $\mathcal{I}^\bullet$ is a K-injective complex of sheaves on $\mathcal{C}_{total}$ with injective terms representing K. We have

$$\begin{aligned} R\Gamma(\mathcal{C}_{total}, K) &= R\,\mathrm{Hom}(\mathbf{Z}, K) \\ &= R\,\mathrm{Hom}(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, K) \\ &= \Gamma(\mathcal{C}_{total}, \mathcal{H}om^\bullet(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, \mathcal{I}^\bullet)) \\ &= \mathrm{Tot}_\pi(A^{\bullet,\bullet}) \end{aligned}$$

where $A^{\bullet,\bullet}$ is the double complex with terms $A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$ and Tot_π denotes the product totalization of this double complex. Namely, the first equality holds in any site. The second equality holds by Lemma 85.8.1. The third equality holds because $\mathcal{I}^\bullet$ is K-injective, see Cohomology on Sites, Sections 21.34 and 21.35. The final equality holds by the construction of $\mathcal{H}om^\bullet$ and the fact that $\mathrm{Hom}(g_{p!}\mathbf{Z}, \mathcal{I}^q) = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$. Then we get our spectral sequence by viewing $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ as a filtered complex with $F^i\mathrm{Tot}_\pi^n(A^{\bullet,\bullet}) = \prod_{p+q=n,\ p \geq i} A^{p,q}$. The spectral sequence we obtain behaves like the spectral sequence $({}'E_r, {}'d_r)_{r \geq 0}$ in Homology, Section 12.25 (where the case of the direct sum totalization is discussed) except for regularity, boundedness, convergence, and abutment issues. In particular we obtain $E_1^{p,q} = H^q(\mathcal{C}_p, K_p)$ as in Lemma 85.8.3.

0H0W **Lemma 85.8.5.** *In Situation 85.3.3. Let K be an object of $D(\mathcal{C}_{total})$.*

(1) *If $H^{-p}(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $H^0(\mathcal{C}_{total}, K) = 0$.*
(2) *If $R\Gamma(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $R\Gamma(\mathcal{C}_{total}, K) = 0$.*

Proof. With notation as in Remark 85.8.4 we see that $R\Gamma(\mathcal{C}_{total}, K)$ is represented by $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$. The assumption in (1) tells us that $H^{-p}(A^{p,\bullet}) = 0$. Thus the vanishing in (1) follows from More on Algebra, Lemma 15.104.1. Part (2) follows from part (1) and taking shifts. □

0DBZ **Lemma 85.8.6.** *Let $\mathcal{C}$ be as in Situation 85.3.3. Let $U \in \mathrm{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in Ab(\mathcal{C}_{total})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^{-1}\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.*

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 85.3.1 in case (A) and Lemma 85.3.2 in case (B). The equality then follows from Lemma 85.3.6 and the definition of cohomology. □

85.9. Cohomology and augmentations of simplicial sites

0D9A Consider a simplicial site $\mathcal{C}$ as in Situation 85.3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. By Lemma 85.4.2 we obtain a morphism of topoi

$$a : Sh(\mathcal{C}_{total}) \longrightarrow Sh(\mathcal{D})$$

and morphisms of topoi $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ as in Lemma 85.3.5. The compositions $a \circ g_n$ are denoted $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$. Furthermore, the simplicial structure gives morphisms of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$ such that $a_n \circ f_\varphi = a_m$ for all $\varphi : [m] \to [n]$.

0D78 **Lemma 85.9.1.** *In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ there is an exact complex*

$$\ldots \to g_{2!}(a_2^{-1}\mathcal{G}) \to g_{1!}(a_1^{-1}\mathcal{G}) \to g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G} \to 0$$

of abelian sheaves on $\mathcal{C}_{total}$.

Proof. We encourage the reader to read the proof of Lemma 85.8.1 first. We will use Lemma 85.4.2 and the description of the functors $g_{n!}$ in Lemma 85.3.5 without further mention. In particular $g_{n!}(a_n^{-1}\mathcal{G})$ is the sheaf on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1} a_n^{-1}\mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^{-1}\mathcal{G}$$

As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}(a_n^{-1}\mathcal{G}) \to g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ is the adjoint to the map $a_n^{-1}\mathcal{G} \to \bigoplus_{[n-1]\to[n]} a_n^{-1}\mathcal{G} = g_n^{-1} g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. The map $g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G}$ is adjoint to the identity map of $a_0^{-1}\mathcal{G}$. Then g_m^{-1} applied to the chain complex in degrees $\ldots, 2, 1, 0$ gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} a_m^{-1}\mathcal{G}$$

on $\mathcal{C}_m$. This is equal to $a_m^{-1}\mathcal{G}$ tensored over the constant sheaf $\mathbf{Z}$ with the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

discussed in the proof of Lemma 85.8.1. There we have seen that this complex is homotopy equivalent to $\mathbf{Z}$ placed in degree 0 which finishes the proof. □

0D79 **Lemma 85.9.2.** *In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

on $\mathcal{D}$ which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}(-, \mathcal{F})$ to the exact complex of Lemma 85.9.1 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective abelian sheaf, then the complex is exact because $\mathrm{Hom}(-, \mathcal{F})$ is exact. □

0D7A **Lemma 85.9.3.** *In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. For any K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = R^q a_{p,*} K_p, \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $R^{p+q}a_*K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 85.9.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 85.3.6). Hence the cohomology of the columns computes $R^q a_{p,*} K_p$. We conclude by applying Homology, Lemmas 12.25.3 and 12.25.4. □

85.10. Cohomology on ringed simplicial sites

0D7B This section is the analogue of Section 85.8 for sheaves of modules.

In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. In statement of the following lemmas we will let $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ be the morphism of ringed topoi of Lemma 85.6.1. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}_n), \mathcal{O}_n) & \xrightarrow{\quad f_\varphi \quad} & (Sh(\mathcal{C}_m), \mathcal{O}_m) \\ & {\scriptstyle g_n} \searrow \quad \swarrow {\scriptstyle g_m} & \\ & (Sh(\mathcal{C}_{total}), \mathcal{O}) & \end{array}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 7.36.

0D9B **Lemma 85.10.1.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a complex*

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

of $\mathcal{O}$-modules which forms a resolution of $\mathcal{O}$. Here $g_{n!}$ is as in Lemma 85.6.1.

Proof. We will use the description of $g_{n!}$ given in Lemma 85.3.5. As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}\mathcal{O}_n \to g_{n-1!}\mathcal{O}_{n-1}$ is the adjoint to the map $\mathcal{O}_n \to \bigoplus_{[n-1]\to[n]} \mathcal{O}_n = g_n^* g_{n-1!}\mathcal{O}_{n-1}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathcal{O}_m$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free $\mathcal{O}_m$-module on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 14.26.4 and Lemma 14.27.2). This complex is acyclic in positive degrees and equal to $\mathcal{O}_m$ in degree 0. □

0D9C **Lemma 85.10.2.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a canonical complex*

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is an injective $\mathcal{O}$-module.

Proof. Observe that $\mathrm{Hom}(\mathcal{O}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\mathrm{Hom}(g_{n!}\mathcal{O}_n, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 85.10.1 and the fact that $\mathrm{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. □

0D7E **Lemma 85.10.3.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. For K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

where the horizontal arrows come from Lemma 85.10.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. Observe that $\Gamma(\mathcal{D}, -) = \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ on $Mod(\mathcal{O}_\mathcal{D})$. Hence the lemma says rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. Thus the total complex associated to the double complex computes $R\Gamma(\mathcal{C}_{total}, K)$ by Homology, Lemma 12.25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are totally acyclic on $\mathcal{C}_p$ (Lemma 85.6.2). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$ by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Cohomology on Sites, Lemma 21.14.3. We conclude by applying Homology, Lemma 12.25.3. □

0DH2 **Lemma 85.10.4.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings. Let $U \in \mathrm{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in Mod(\mathcal{O})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^*\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.*

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 85.3.1 in case (A) and Lemma 85.3.2 in case (B). In both cases the functor $\mathcal{C}_n \to \mathcal{C}$ is continuous and cocontinuous, see Lemma 85.3.5, and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by definition. Hence the result is a special case of Cohomology on Sites, Lemma 21.37.5. □

85.11. Cohomology and augmentations of ringed simplicial sites

0D9D This section is the analogue of Section 85.9 for sheaves of modules.

Consider a simplicial site $\mathcal{C}$ as in Situation 85.3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Suppose we are given a morphism

$$a^\sharp : \mathcal{O}_\mathcal{D} \longrightarrow a_*\mathcal{O}$$

where a is as in Lemma 85.4.2. Consequently, we obtain a morphism of ringed topoi

$$a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

We will think of $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ as a morphism of ringed topoi as in Lemma 85.6.1, then taking the composition $a_n = a \circ g_n$ (Lemma 85.4.2) as morphisms of ringed topoi we obtain

$$a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Using the transition maps $f_\varphi^{-1}\mathcal{O}_m \to \mathcal{O}_n$ we obtain morphisms of ringed topoi

$$f_\varphi : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_m), \mathcal{O}_m)$$

such that $a_m \circ f_\varphi = a_n$ as morphisms of ringed topoi for all $\varphi : [m] \to [n]$.

0DH3 **Lemma 85.11.1.** *With notation as above. The morphism $a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat if and only if $a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat for $n \geq 0$.*

Proof. Since $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ is flat, we see that if a is flat, then $a_n = a \circ g_n$ is flat as a composition. Conversely, suppose that a_n is flat for all n. We have to check that $\mathcal{O}$ is flat as a sheaf of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. We have to show that

$$\mathcal{F} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O} \to \mathcal{G} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}$$

is injective. We can check this on $\mathcal{C}_n$, i.e., after applying g_n^{-1}. Since $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ we obtain

$$g_n^{-1}\mathcal{F} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n \to g_n^{-1}\mathcal{G} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n$$

which is injective because $g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D} = a_n^{-1}\mathcal{O}_\mathcal{D}$ and we assume a_n was flat. □

0D7C **Lemma 85.11.2.** *With notation as above. For a $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ there is an exact complex*

$$\ldots \to g_{2!}(a_2^*\mathcal{G}) \to g_{1!}(a_1^*\mathcal{G}) \to g_{0!}(a_0^*\mathcal{G}) \to a^*\mathcal{G} \to 0$$

of sheaves of $\mathcal{O}$-modules on $\mathcal{C}_{total}$. Here $g_{n!}$ is as in Lemma 85.6.1.

Proof. Observe that $a^*\mathcal{G}$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module $a_m^*\mathcal{G}$. The description of the functors $g_{n!}$ on modules in Lemma 85.6.1 shows that $g_{n!}(a_n^*\mathcal{G})$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^* a_n^* \mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^* \mathcal{G}$$

The rest of the proof is exactly the same as the proof of Lemma 85.9.1, replacing $a_m^{-1}\mathcal{G}$ by $a_m^*\mathcal{G}$. □

0D7D **Lemma 85.11.3.** *With notation as above. For an $\mathcal{O}$-module $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

of $\mathcal{O}_\mathcal{D}$-modules which is exact in degrees $-1, 0$. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact in all degrees and remains exact on applying the functor $\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, -)$ for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any $\mathcal{O}_\mathcal{D}$-modules $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ to the exact complex of Lemma 85.11.2 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact because $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is exact. □

0D7F **Lemma 85.11.4.** *With notation as above for any K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ in $Mod(\mathcal{O}_\mathcal{D})$ with*

$$E_1^{p,q} = R^q a_{p,*} K_p$$

converging to $R^{p+q}a_*K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 85.11.3 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The lemma says rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. Thus the total complex associated to the double complex computes Ra_*K by Homology, Lemma 12.25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 85.3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are totally acyclic on $\mathcal{C}_p$ (Lemma 85.6.2). Hence the cohomology of the columns are the sheaves $R^q a_{p,*}K_p$ by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Cohomology on Sites, Lemma 21.14.3. We conclude by applying Homology, Lemma 12.25.3. □

85.12. Cartesian sheaves and modules

0D7G Here is the definition.

07TF **Definition 85.12.1.** In Situation 85.3.3.

(1) A sheaf $\mathcal{F}$ of sets or of abelian groups on $\mathcal{C}_{total}$ is *cartesian* if the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(2) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is *cartesian* if the maps $f_\varphi^*\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(3) An object K of $D(\mathcal{C}_{total})$ is *cartesian* if the maps $f_\varphi^{-1}K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(4) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then an object K of $D(\mathcal{O})$ is *cartesian* if the maps $Lf_\varphi^*K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.

Of course there is a general notion of a cartesian section of a fibred category and the above are merely examples of this. The property on pullbacks needs only be checked for the degeneracies.

07TG **Lemma 85.12.2.** *In Situation 85.3.3.*

(1) *A sheaf $\mathcal{F}$ of sets or abelian groups is cartesian if and only if the maps $(f_{\delta_j^n})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*
(2) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if the maps $(f_{\delta_j^n})^{-1}K_{n-1} \to K_n$ are isomorphisms.*
(3) *If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is cartesian if and only if the maps $(f_{\delta_j^n})^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*
(4) *If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ an object K of $D(\mathcal{O})$ is cartesian if and only if the maps $L(f_{\delta_j^n})^*K_{n-1} \to K_n$ are isomorphisms.*
(5) *Add more here.*

Proof. In each case the key is that the pullback functors compose to pullback functor; for part (4) see Cohomology on Sites, Lemma 21.18.3. We show how the argument works in case (1) and omit the proof in the other cases. The category Δ is generated by the morphisms the morphisms δ_j^n and σ_j^n, see Simplicial, Lemma 14.2.2. Hence we only need to check the maps $(f_{\delta_j^n})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ and $(f_{\sigma_j^n})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$ are isomorphisms, see Simplicial, Lemma 14.3.2 for notation. Since $\sigma_j^n \circ \delta_j^{n+1} = \mathrm{id}_{[n]}$ the composition

$$\mathcal{F}_n = (f_{\sigma_j^n})^{-1}(f_{\delta_j^{n+1}})^{-1}\mathcal{F}_n \to (f_{\sigma_j^n})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$$

is the identity. Thus the result for δ_j^{n+1} implies the result for σ_j^n. □

0D7H **Lemma 85.12.3.** *In Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 85.4.1.*

(1) *The pullback $a^{-1}\mathcal{G}$ of a sheaf of sets or abelian groups on $\mathcal{D}$ is cartesian.*
(2) *The pullback $a^{-1}K$ of an object K of $D(\mathcal{D})$ is cartesian.*

Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{O}_\mathcal{D}$ a sheaf of rings on $\mathcal{D}$ and $a^\sharp : \mathcal{O}_\mathcal{D} \to a_*\mathcal{O}$ a morphism as in Section 85.11.

(3) *The pullback $a^*\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{D}$-modules is cartesian.*
(4) *The derived pullback La^*K of an object K of $D(\mathcal{O}_\mathcal{D})$ is cartesian.*

Proof. This follows immediately from the identities $a_m \circ f_\varphi = a_n$ for all $\varphi : [m] \to [n]$. See Lemma 85.4.2 and the discussion in Section 85.11. □

0D7I **Lemma 85.12.4.** *In Situation 85.3.3. The category of cartesian sheaves of sets (resp. abelian groups) is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a sheaf of sets (resp. abelian groups) on $\mathcal{C}_0$ and*

$$\alpha : (f_{\delta_1^1})^{-1}\mathcal{F} \longrightarrow (f_{\delta_0^1})^{-1}\mathcal{F}$$

is an isomorphism of sheaves of sets (resp. abelian groups) on $\mathcal{C}_1$ such that $(f_{\delta_1^2})^{-1}\alpha = (f_{\delta_0^2})^{-1}\alpha \circ (f_{\delta_2^2})^{-1}\alpha$ as maps of sheaves on $\mathcal{C}_2$.

Proof. We abbreviate $d_j^n = f_{\delta_j^n} : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{n-1})$. The condition on α in the statement of the lemma makes sense because

$$d_1^1 \circ d_2^2 = d_1^1 \circ d_1^2, \quad d_1^1 \circ d_0^2 = d_0^1 \circ d_2^2, \quad d_0^1 \circ d_0^2 = d_0^1 \circ d_1^2$$

as morphisms of topoi $Sh(\mathcal{C}_2) \to Sh(\mathcal{C}_0)$, see Simplicial, Remark 14.3.3. Hence we can picture these maps as follows

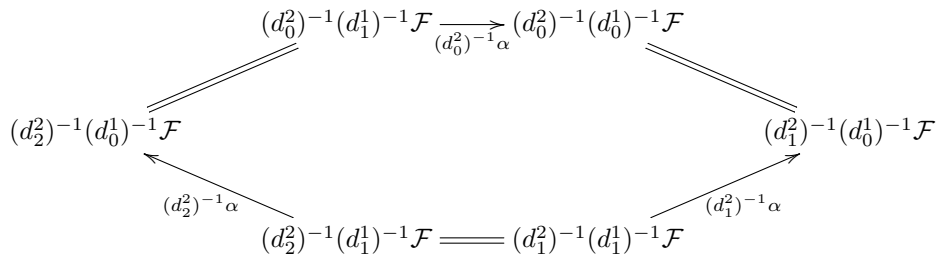

and the condition signifies the diagram is commutative. It is clear that given a cartesian sheaf $\mathcal{G}$ of sets (resp. abelian groups) on $\mathcal{C}_{total}$ we can set $\mathcal{F} = \mathcal{G}_0$ and α equal to the composition

$$(d_1^1)^{-1}\mathcal{G}_0 \to \mathcal{G}_1 \leftarrow (d_1^0)^{-1}\mathcal{G}_0$$

where the arrows are invertible as $\mathcal{G}$ is cartesian. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair $(\mathcal{F}, \alpha)$ as in the lemma we set $\mathcal{G}_n = (f_{\tau_n^n})^{-1}\mathcal{F}$. Given $\varphi : [n] \to [m]$ we define $\mathcal{G}(\varphi) : (f_\varphi)^{-1}\mathcal{G}_n \to \mathcal{G}_m$ using

$$\begin{array}{ccccccc}
(f_\varphi)^{-1}\mathcal{G}_n & = & (f_\varphi)^{-1}(f_{\tau_n^n})^{-1}\mathcal{F} & = & (f_{\tau^m_{\varphi(n)}})^{-1}\mathcal{F} & = & (f_{\tau^m_{\varphi(n)m}})^{-1}(d_1^1)^{-1}\mathcal{F} \\
 & & & & & & \Big\downarrow (f_{\tau^m_{\varphi(n)m}})^{-1}\alpha \\
\mathcal{G}_m & = & (f_{\tau_m^m})^{-1}\mathcal{F} & = & (f_{\tau^m_{\varphi(n)m}})^{-1}(d_0^1)^{-1}\mathcal{F} & &
\end{array}$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly and hence give rise to a sheaf on $\mathcal{C}_{total}$, see Lemma 85.3.4. We also omit the verification that the two functors are quasi-inverse to each other. □

07TH **Lemma 85.12.5.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. The category of cartesian $\mathcal{O}$-modules is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a $\mathcal{O}_0$-module and*

$$\alpha : (f_{\delta_1^1})^*\mathcal{F} \longrightarrow (f_{\delta_0^1})^*\mathcal{F}$$

is an isomorphism of $\mathcal{O}_1$-modules such that $(f_{\delta_1^2})^*\alpha = (f_{\delta_0^2})^*\alpha \circ (f_{\delta_2^2})^*\alpha$ as $\mathcal{O}_2$-module maps.

Proof. The proof is identical to the proof of Lemma 85.12.4 with pullback of sheaves of abelian groups replaced by pullback of modules. □

0D7J **Lemma 85.12.6.** *In Situation 85.3.3.*

(1) *The full subcategory of cartesian abelian sheaves forms a weak Serre subcategory of $Ab(\mathcal{C}_{total})$. Colimits of systems of cartesian abelian sheaves are cartesian.*

(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms*

$$f_{\delta^n_j} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$$

are flat. The full subcategory of cartesian $\mathcal{O}$-modules forms a weak Serre subcategory of $Mod(\mathcal{O})$. Colimits of systems of cartesian $\mathcal{O}$-modules are cartesian.

Proof. To see we obtain a weak Serre subcategory in (1) we check the conditions listed in Homology, Lemma 12.10.3. First, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a map between cartesian abelian sheaves, then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are cartesian too because the restriction functors $Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ and the functors f_φ^{-1} are exact. Similarly, if

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$$

is a short exact sequence of abelian sheaves on $\mathcal{C}_{total}$ with $\mathcal{F}$ and $\mathcal{G}$ cartesian, then it follows that $\mathcal{H}$ is cartesian from the 5-lemma. To see the property of colimits, use that colimits commute with pullback as pullback is a left adjoint. In the case of modules we argue in the same manner, using the exactness of flat pullback (Modules on Sites, Lemma 18.31.2) and the fact that it suffices to check the condition for $f_{\delta^n_j}$, see Lemma 85.12.2. □

0D7K **Remark 85.12.7** (Warning)**.** Lemma 85.12.6 notwithstanding, it can happen that the category of cartesian $\mathcal{O}$-modules is abelian without being a Serre subcategory of $Mod(\mathcal{O})$. Namely, suppose that we only know that $f_{\delta^1_1}$ and $f_{\delta^1_0}$ are flat. Then it follows easily from Lemma 85.12.5 that the category of cartesian $\mathcal{O}$-modules is abelian. But if $f_{\delta^2_0}$ is not flat (for example), there is no reason for the inclusion functor from the category of cartesian $\mathcal{O}$-modules to all $\mathcal{O}$-modules to be exact.

0D7L **Lemma 85.12.8.** *In Situation 85.3.3.*

(1) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if $H^q(K)$ is a cartesian abelian sheaf for all q.*
(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_{\delta^n_j} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$ are flat. Then an object K of $D(\mathcal{O})$ is cartesian if and only if $H^q(K)$ is a cartesian $\mathcal{O}$-module for all q.*

Proof. Part (1) is true because the pullback functors $(f_\varphi)^{-1}$ are exact. Part (2) follows from the characterization in Lemma 85.12.2 and the fact that $L(f_{\delta^n_j})^* = (f_{\delta^n_j})^*$ by flatness. □

0D9E **Lemma 85.12.9.** *In Situation 85.3.3.*

(1) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only the canonical map*

$$g_{n!}K_n \longrightarrow g_{n!}\mathbf{Z} \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

is an isomorphism for all n.
(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$. Then an object K of $D(\mathcal{O})$ is cartesian if and only if the canonical map*

$$g_{n!}K_n \longrightarrow g_{n!}\mathcal{O}_n \otimes^{\mathbf{L}}_{\mathcal{O}} K$$

is an isomorphism for all n.

Proof. Proof of (1). Since $g_{n!}$ is exact, it induces a functor on derived categories adjoint to g_n^{-1}. The map is the adjoint of the map $K_n \to (g_n^{-1}g_{n!}\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathbf{Z}} K_n$ corresponding to $\mathbf{Z} \to g_n^{-1}g_{n!}\mathbf{Z}$ which in turn is adjoint to $\mathrm{id} : g_{n!}\mathbf{Z} \to g_{n!}\mathbf{Z}$. Using the description of $g_{n!}$ given in Lemma 85.3.5 we see that the restriction to $\mathcal{C}_m$ of this map is

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

Thus the statement is clear.

Proof of (2). Since $g_{n!}$ is exact (Lemma 85.6.3), it induces a functor on derived categories adjoint to g_n^* (also exact). The map is the adjoint of the map $K_n \to (g_n^*g_{n!}\mathcal{O}_n) \otimes^{\mathbf{L}}_{\mathcal{O}_n} K_n$ corresponding to $\mathcal{O}_n \to g_n^*g_{n!}\mathcal{O}_n$ which in turn is adjoint to $\mathrm{id} : g_{n!}\mathcal{O}_n \to g_{n!}\mathcal{O}_n$. Using the description of $g_{n!}$ given in Lemma 85.6.1 we see that the restriction to $\mathcal{C}_m$ of this map is

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{O}_n \otimes_{\mathcal{O}_m} K_m = \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

Thus the statement is clear. □

0D7M **Lemma 85.12.10.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent in the sense of Modules on Sites, Definition 18.23.1 if and only if $\mathcal{F}$ is cartesian and $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n.*

Proof. Assume $\mathcal{F}$ is quasi-coherent. Since pullbacks of quasi-coherent modules are quasi-coherent (Modules on Sites, Lemma 18.23.4) we see that $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n. To show that $\mathcal{F}$ is cartesian, let U be an object of $\mathcal{C}_n$ for some n. Let us view U as an object of $\mathcal{C}_{total}$. Because $\mathcal{F}$ is quasi-coherent there exists a covering $\{U_i \to U\}$ and for each i a presentation

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

Observe that $\{U_i \to U\}$ is a covering of $\mathcal{C}_n$ by the construction of the site $\mathcal{C}_{total}$. Next, let V be an object of $\mathcal{C}_m$ for some m and let $V \to U$ be a morphism of $\mathcal{C}_{total}$ lying over $\varphi : [n] \to [m]$. The fibre products $V_i = V \times_U U_i$ exist and we get an induced covering $\{V_i \to V\}$ in $\mathcal{C}_m$. Restricting the presentation above to the sites $\mathcal{C}_n/U_i$ and $\mathcal{C}_m/V_i$ we obtain presentations

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

and

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \mathcal{F}_m|_{\mathcal{C}_m/V_i} \to 0$$

These presentations are compatible with the map $\mathcal{F}(\varphi) : f_\varphi^*\mathcal{F}_n \to \mathcal{F}_m$ (as this map is defined using the restriction maps of $\mathcal{F}$ along morphisms of $\mathcal{C}_{total}$ lying over φ). We conclude that $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V_i}$ is an isomorphism. As $\{V_i \to V\}$ is a covering we conclude $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V}$ is an isomorphism. Since V and U were arbitrary this proves that $\mathcal{F}$ is cartesian. (In case A use Sites, Lemma 7.14.10.)

Conversely, assume $\mathcal{F}_n$ is quasi-coherent for all n and that $\mathcal{F}$ is cartesian. Then for any n and object U of $\mathcal{C}_n$ we can choose a covering $\{U_i \to U\}$ of $\mathcal{C}_n$ and for each i a presentation

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

Pulling back to $\mathcal{C}_{total}/U_i$ we obtain complexes

$$\bigoplus\nolimits_{j\in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k\in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

of modules on $\mathcal{C}_{total}/U_i$. Then the property that $\mathcal{F}$ is cartesian implies that this is exact. We omit the details. □

85.13. Simplicial systems of the derived category

0D9F In this section we are going to prove a special case of [**BBD82**, Proposition 3.2.9] in the setting of derived categories of abelian sheaves. The case of modules is discussed in Section 85.14.

0D9G **Definition 85.13.1.** In Situation 85.3.3. A *simplicial system of the derived category* consists of the following data

(1) for every n an object K_n of $D(\mathcal{C}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : f_\varphi^{-1}K_m \to K_n$ in $D(\mathcal{C}_n)$

subject to the condition that

$$K_{\varphi\circ\psi} = K_\varphi \circ f_\varphi^{-1}K_\psi : f_{\varphi\circ\psi}^{-1}K_l = f_\varphi^{-1}f_\psi^{-1}K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is *cartesian* if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a *morphism of simplicial systems of the derived category*.

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category comes from an object of $D(\mathcal{C}_{total})$ under certain hypotheses.

0D9H **Lemma 85.13.2.** *In Situation 85.3.3. If $K \in D(\mathcal{C}_{total})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category. If K is cartesian, so is the system.*

Proof. This is obvious. □

0GME **Lemma 85.13.3.** *In Situation 85.3.3 suppose given $K_0 \in D(\mathcal{C}_0)$ and an isomorphism*

$$\alpha : f_{\delta_1^1}^{-1}K_0 \longrightarrow f_{\delta_0^1}^{-1}K_0$$

satisfying the cocycle condition. Set $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and set $K_n = f_{\tau_n^n}^{-1}K_0$. Then the K_n form a cartesian simplicial system of the derived category.

Proof. Please compare with Lemma 85.12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : f_\varphi^{-1}K_n \to K_m$ using

$$\begin{array}{ccccccc} f_\varphi^{-1}K_n & = & f_\varphi^{-1}f_{\tau_n^n}^{-1}K_0 & = & f_{\tau_{\varphi(n)}^m}^{-1}K_0 & = & f_{\tau_{\varphi(n)m}^m}^{-1}f_{\delta_1^1}^{-1}K_0 \\ & & & & & & \Big\downarrow{\scriptstyle f_{\tau_{\varphi(n)m}^m}^{-1}\alpha} \\ K_m & = & f_{\tau_m^m}^{-1}K_0 & = & f_{\tau_{\varphi(n)m}^m}^{-1}K_0 & = & f_{\tau_{\varphi(n)m}^m}^{-1}f_{\delta_0^1}^{-1}K_0 \end{array}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial system in the derived category. □

0D9I **Lemma 85.13.4.** *In Situation 85.3.3. Let K be an object of $D(\mathcal{C}_{total})$. Set*

$$X_n = (g_{n!}\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathbf{Z}} K \quad \text{and} \quad Y_n = (g_{n!}\mathbf{Z} \to \ldots \to g_{0!}\mathbf{Z})[-n] \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

as objects of $D(\mathcal{C}_{total})$ where the maps are as in Lemma 85.8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

(1) *the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system (Derived Categories, Definition 13.41.1) for $\ldots \to X_2 \to X_1 \to X_0$,*
(2) *$K = hocolim Y_n[n]$ in $D(\mathcal{C}_{total})$.*

Proof. First, if $K = \mathbf{Z}$, then this is the construction of Derived Categories, Example 13.41.2 applied to the complex

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

in $Ab(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathbf{Z}$ in $D(\mathcal{C}_{total})$ by Lemma 85.8.1. The general case follows from this, the fact that the exact functor $- \otimes^{\mathbf{L}}_{\mathbf{Z}} K$ sends Postnikov systems to Postnikov systems, and that $- \otimes^{\mathbf{L}}_{\mathbf{Z}} K$ commutes with homotopy colimits. □

0D9J **Lemma 85.13.5.** *In Situation 85.3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume*

(1) *K is cartesian,*
(2) *$\mathrm{Hom}(K_i[i], K'_i) = 0$ for $i > 0$, and*
(3) *$\mathrm{Hom}(K_i[i+1], K'_i) = 0$ for $i \geq 0$.*

Then any map $K \to K'$ which induces the zero map $K_0 \to K'_0$ is zero.

Proof. Consider the objects X_n and the Postnikov system Y_n associated to K in Lemma 85.13.4. As $K = \mathrm{hocolim} Y_n[n]$ the map $K \to K'$ induces a compatible family of morphisms $Y_n[n] \to K'$. By (1) and Lemma 85.12.9 we have $X_n = g_{n!}K_n$. Since $Y_0 = X_0$ we find that $K_0 \to K'_0$ being zero implies $Y_0 \to K'$ is zero. Suppose we've shown that the map $Y_n[n] \to K'$ is zero for some $n \geq 0$. From the distinguished triangle

$$Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1] \to Y_n[n+1]$$

we get an exact sequence

$$\mathrm{Hom}(X_{n+1}[n+1], K') \to \mathrm{Hom}(Y_{n+1}[n+1], K') \to \mathrm{Hom}(Y_n[n], K')$$

As $X_{n+1}[n+1] = g_{n+1!}K_{n+1}[n+1]$ the first group is equal to

$$\mathrm{Hom}(K_{n+1}[n+1], K'_{n+1})$$

which is zero by assumption (2). By induction we conclude all the maps $Y_n[n] \to K'$ are zero. Consider the defining distinguished triangle

$$\bigoplus Y_n[n] \to \bigoplus Y_n[n] \to K \to (\bigoplus Y_n[n])[1]$$

for the homotopy colimit. Arguing as above, we find that it suffices to show that

$$\mathrm{Hom}((\bigoplus Y_n[n])[1], K') = \prod \mathrm{Hom}(Y_n[n+1], K')$$

is zero for all $n \geq 0$. To see this, arguing as above, it suffices to show that

$$\mathrm{Hom}(K_n[n+1], K'_n) = 0$$

for all $n \geq 0$ which follows from condition (3). □

0D9K **Lemma 85.13.6.** *In Situation 85.3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume*

(1) *K is cartesian,*
(2) *$\mathrm{Hom}(K_i[i-1], K'_i) = 0$ for $i > 1$.*

Then any map $\{K_n \to K'_n\}$ between the associated simplicial systems of K and K' comes from a map $K \to K'$ in $D(\mathcal{C}_{total})$.

Proof. Let $\{K_n \to K'_n\}_{n \geq 0}$ be a morphism of simplicial systems of the derived category. Consider the objects X_n and Postnikov system Y_n associated to K of Lemma 85.13.4. By (1) and Lemma 85.12.9 we have $X_n = g_{n!}K_n$. In particular, the map $K_0 \to K'_0$ induces a morphism $X_0 \to K'$. Since $\{K_n \to K'_n\}$ is a morphism of systems, a computation (omitted) shows that the composition

$$X_1 \to X_0 \to K'$$

is zero. As $Y_0 = X_0$ and as Y_1 fits into a distinguished triangle

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

we conclude that there exists a morphism $Y_1[1] \to K'$ whose composition with $X_0 = Y_0 \to Y_1[1]$ is the morphism $X_0 \to K'$ given above. Suppose given a map $Y_n[n] \to K'$ for $n \geq 1$. From the distinguished triangle

$$X_{n+1}[n] \to Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1]$$

we get an exact sequence

$$\mathrm{Hom}(Y_{n+1}[n+1], K') \to \mathrm{Hom}(Y_n[n], K') \to \mathrm{Hom}(X_{n+1}[n], K')$$

As $X_{n+1}[n] = g_{n+1!}K_{n+1}[n]$ the last group is equal to

$$\mathrm{Hom}(K_{n+1}[n], K'_{n+1})$$

which is zero by assumption (2). By induction we get a system of maps $Y_n[n] \to K'$ compatible with transition maps and reducing to the given map on Y_0. This produces a map

$$\gamma : K = \mathrm{hocolim} Y_n[n] \longrightarrow K'$$

This map in any case has the property that the diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & K \\ & \searrow & \downarrow \gamma \\ & & K' \end{array}$$

is commutative. Restricting to $\mathcal{C}_0$ we deduce that the map $\gamma_0 : K_0 \to K'_0$ is the same as the first map $K_0 \to K'_0$ of the morphism of simplicial systems. Since K is cartesian, this easily gives that $\{\gamma_n\}$ is the map of simplicial systems we started out with. □

0D9L **Lemma 85.13.7.** *In Situation 85.3.3. Let (K_n, K_φ) be a simplicial system of the derived category. Assume*

(1) *(K_n, K_φ) is cartesian,*
(2) *$\mathrm{Hom}(K_i[t], K_i) = 0$ for $i \geq 0$ and $t > 0$.*

Then there exists a cartesian object K of $D(\mathcal{C}_{total})$ whose associated simplicial system is isomorphic to (K_n, K_φ).

Proof. Set $X_n = g_{n!}K_n$ in $D(\mathcal{C}_{total})$. For each $n \geq 1$ we have

$$\mathrm{Hom}(X_n, X_{n-1}) = \mathrm{Hom}(K_n, g_n^{-1}g_{n-1!}K_{n-1}) = \bigoplus\nolimits_{\varphi:[n-1]\to[n]} \mathrm{Hom}(K_n, f_\varphi^{-1}K_{n-1})$$

Thus we get a map $X_n \to X_{n-1}$ corresponding to the alternating sum of the maps $K_\varphi^{-1} : K_n \to f_\varphi^{-1}K_{n-1}$ where φ runs over $\delta_0^n, \ldots, \delta_n^n$. We can do this because K_φ is invertible by assumption (1). Please observe the similarity with the definition of the maps in the proof of Lemma 85.8.1. We obtain a complex

$$\ldots \to X_2 \to X_1 \to X_0$$

in $D(\mathcal{C}_{total})$. We omit the computation which shows that the compositions are zero. By Derived Categories, Lemma 13.41.6 if we have

$$\mathrm{Hom}(X_i[i-j-2], X_j) = 0 \text{ for } i > j+2$$

then we can extend this complex to a Postnikov system. The group is equal to

$$\mathrm{Hom}(K_i[i-j-2], g_i^{-1}g_{j!}K_j)$$

Again using that (K_n, K_φ) is cartesian we see that $g_i^{-1}g_{j!}K_j$ is isomorphic to a finite direct sum of copies of K_i. Hence the group vanishes by assumption (2). Let the Postnikov system be given by $Y_0 = X_0$ and distinguished sequences $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ for $n \geq 1$. We set

$$K = \mathrm{hocolim} Y_n[n]$$

To finish the proof we have to show that $g_m^{-1}K$ is isomorphic to K_m for all m compatible with the maps K_φ. Observe that

$$g_m^{-1}K = \mathrm{hocolim} g_m^{-1}Y_n[n]$$

and that $g_m^{-1}Y_n[n]$ is a Postnikov system for $g_m^{-1}X_n$. Consider the isomorphisms

$$g_m^{-1}X_n = \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}K_n \xrightarrow{\bigoplus K_\varphi} \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

These maps define an isomorphism of complexes

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & g_m^{-1}X_2 & \longrightarrow & g_m^{-1}X_1 & \longrightarrow & g_m^{-1}X_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & \bigoplus_{\varphi:[2]\to[m]} K_m & \longrightarrow & \bigoplus_{\varphi:[1]\to[m]} K_m & \longrightarrow & \bigoplus_{\varphi:[0]\to[m]} K_m
\end{array}$$

in $D(\mathcal{C}_m)$ where the arrows in the bottom row are as in the proof of Lemma 85.8.1. The squares commute by our choice of the arrows of the complex $\ldots \to X_2 \to X_1 \to X_0$; we omit the computation. The bottom row complex has a postnikov tower given by

$$Y'_{m,n} = \left(\bigoplus\nolimits_{\varphi:[n]\to[m]} \mathbf{Z} \to \ldots \to \bigoplus\nolimits_{\varphi:[0]\to[m]} \mathbf{Z}\right)[-n] \otimes^{\mathbf{L}}_{\mathbf{Z}} K_m$$

and $\mathrm{hocolim} Y'_{m,n} = K_m$ (please compare with the proof of Lemma 85.13.4 and Derived Categories, Example 13.41.2). Applying the second part of Derived Categories, Lemma 13.41.6 the vertical maps in the big diagram extend to an isomorphism of Postnikov systems provided we have

$$\mathrm{Hom}(g_m^{-1}X_i[i-j-1], \bigoplus\nolimits_{\varphi:[j]\to[m]} K_m) = 0 \text{ for } i > j+1$$

The is true if $\mathrm{Hom}(K_m[i-j-1], K_m) = 0$ for $i > j+1$ which holds by assumption (2). Choose an isomorphism given by $\gamma_{m,n} : g_m^{-1}Y_n \to Y'_{m,n}$ of Postnikov systems in $D(\mathcal{C}_m)$. By uniqueness of homotopy colimits, we can find an isomorphism

$$g_m^{-1}K = \mathrm{hocolim} g_m^{-1}Y_n[n] \xrightarrow{\gamma_m} \mathrm{hocolim} Y'_{m,n} = K_m$$

compatible with $\gamma_{m,n}$.

We still have to prove that the maps γ_m fit into commutative diagrams

$$\begin{array}{ccc} f_\varphi^{-1}g_m^{-1}K & \xrightarrow{K(\varphi)} & g_n^{-1}K \\ \downarrow{\scriptstyle f_\varphi^{-1}\gamma_m} & & \downarrow{\scriptstyle \gamma_n} \\ f_\varphi^{-1}K_m & \xrightarrow{K_\varphi} & K_n \end{array}$$

for every $\varphi : [m] \to [n]$. Consider the diagram

$$\begin{array}{ccccccc} f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} f_\psi^{-1}K_0) & = & f_\varphi^{-1}g_m^{-1}X_0 & \xrightarrow{X_0(\varphi)} & g_n^{-1}X_0 & = & \bigoplus_{\chi:[0]\to[n]} f_\chi^{-1}K_0 \\ \downarrow{\scriptstyle f_\varphi^{-1}(\bigoplus K_\psi)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \bigoplus K_\chi} \\ f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} K_m) & & f_\varphi^{-1}g_m^{-1}K & \xrightarrow{K(\varphi)} & g_n^{-1}K & & \bigoplus_{\chi:[0]\to[n]} K_n \\ \| & & \downarrow{\scriptstyle f_\varphi^{-1}\gamma_m} & & \downarrow{\scriptstyle \gamma_n} & & \| \\ f_\varphi^{-1}Y'_{0,m} & \longrightarrow & f_\varphi^{-1}K_m & \xrightarrow{K_\varphi} & K_n & \longleftarrow & Y'_{0,n} \end{array}$$

The top middle square is commutative as $X_0 \to K$ is a morphism of simplicial objects. The left, resp. the right rectangles are commutative as γ_m, resp. γ_n is compatible with $\gamma_{0,m}$, resp. $\gamma_{0,n}$ which are the arrows $\bigoplus K_\psi$ and $\bigoplus K_\chi$ in the diagram. Going around the outer rectangle of the diagram is commutative as (K_n, K_φ) is a simplical system and the map $X_0(\varphi)$ is given by the obvious identifications $f_\varphi^{-1}f_\psi^{-1}K_0 = f_{\varphi\circ\psi}^{-1}K_0$. Note that the arrow $\bigoplus_\psi K_m \to Y'_{0,m} \to K_m$ induces an isomorphism on any of the direct summands (because of our explicit construction of the Postnikov systems $Y'_{i,j}$ above). Hence, if we take a direct summand of the upper left and corner, then this maps isomorphically to $f_\varphi^{-1}g_m^{-1}K$ as γ_m is an isomorphism. Working out what the above says, but looking only at this direct summand we conclude the lower middle square commutes as we well. This concludes the proof. □

85.14. Simplicial systems of the derived category: modules

0D9M In this section we are going to prove a special case of [**BBD82**, Proposition 3.2.9] in the setting of derived categories of $\mathcal{O}$-modules. The (slightly) easier case of abelian sheaves is discussed in Section 85.13.

0D9N **Definition 85.14.1.** In Situation 85.3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. A *simplicial system of the derived category of modules* consists of the following data

(1) for every n an object K_n of $D(\mathcal{O}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : Lf_\varphi^* K_m \to K_n$ in $D(\mathcal{O}_n)$

subject to the condition that

$$K_{\varphi\circ\psi} = K_\varphi \circ Lf_\varphi^* K_\psi : Lf_{\varphi\circ\psi}^* K_l = Lf_\varphi^* Lf_\psi^* K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is *cartesian* if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a *morphism of simplicial systems of the derived category of modules*.

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category of modules comes from an object of $D(\mathcal{O})$ under certain hypotheses.

0D9P **Lemma 85.14.2.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K \in D(\mathcal{O})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category of modules. If K is cartesian, so is the system.*

Proof. This is immediate from the definitions. □

0GMF **Lemma 85.14.3.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Suppose given $K_0 \in D(\mathcal{O}_0)$ and an isomorphism*

$$\alpha : L(f_{\delta_1^1})^* K_0 \longrightarrow L(f_{\delta_0^1})^* K_0$$

satisfying the cocycle condition. Set $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf_{\tau_n^n}^* K_0$. The objects K_n form the members of a cartesian simplicial system of the derived category of modules.

Proof. Please compare with Lemmas 85.13.3 and 85.12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : L(f_\varphi)^* K_n \to K_m$ using

$$\begin{array}{ccccccc} L(f_\varphi)^* K_n & = & L(f_\varphi)^* L(f_{\tau_n^n})^* K_0 & = & L(f_{\tau^m_{\varphi(n)}})^* K_0 & = & L(f_{\tau^m_{\varphi(n)m}})^* L(f_{\delta_1^1})^* K_0 \\ & & & & & & \Big\downarrow {\scriptstyle L(f_{\tau^m_{\varphi(n)m}})^*\alpha} \\ K_m & = & L(f_{\tau_m^m})^* K_0 & = & L(f_{\tau^m_{\varphi(n)m}})^* L(f_{\delta_0^1})^* K_0 & & \end{array}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial systems of the derived category of modules. □

0D9Q **Lemma 85.14.4.** *In Situation 85.3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let K be an object of $D(\mathcal{C}_{total})$. Set*

$$X_n = (g_{n!}\mathcal{O}_n) \otimes^{\mathbf{L}}_{\mathcal{O}} K \quad \text{and} \quad Y_n = (g_{n!}\mathcal{O}_n \to \ldots \to g_{0!}\mathcal{O}_0)[-n] \otimes^{\mathbf{L}}_{\mathcal{O}} K$$

as objects of $D(\mathcal{O})$ where the maps are as in Lemma 85.8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

(1) *the distinguished triangles* $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ *define a Postnikov system (Derived Categories, Definition 13.41.1) for* $\ldots \to X_2 \to X_1 \to X_0$,
(2) $K = hocolim Y_n[n]$ *in* $D(\mathcal{O})$.

Proof. First, if $K = \mathcal{O}$, then this is the construction of Derived Categories, Example 13.41.2 applied to the complex

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

in $Ab(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathcal{O}$ in $D(\mathcal{C}_{total})$ by Lemma 85.10.1. The general case follows from this, the fact that the exact functor $- \otimes^{\mathbf{L}}_{\mathcal{O}} K$ sends Postnikov systems to Postnikov systems, and that $- \otimes^{\mathbf{L}}_{\mathcal{O}} K$ commutes with homotopy colimits. □

0D9R **Lemma 85.14.5.** *In Situation 85.3.3 let* $\mathcal{O}$ *be a sheaf of rings on* $\mathcal{C}_{total}$. *If* $K, K' \in D(\mathcal{O})$. *Assume*

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ *is flat for* $\varphi : [m] \to [n]$,
(2) K *is cartesian,*
(3) $\mathrm{Hom}(K_i[i], K'_i) = 0$ *for* $i > 0$, *and*
(4) $\mathrm{Hom}(K_i[i+1], K'_i) = 0$ *for* $i \geq 0$.

Then any map $K \to K'$ *which induces the zero map* $K_0 \to K'_0$ *is zero.*

Proof. The proof is exactly the same as the proof of Lemma 85.13.5 except using Lemma 85.14.4 instead of Lemma 85.13.4. □

0D9S **Lemma 85.14.6.** *In Situation 85.3.3 let* $\mathcal{O}$ *be a sheaf of rings on* $\mathcal{C}_{total}$. *If* $K, K' \in D(\mathcal{O})$. *Assume*

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ *is flat for* $\varphi : [m] \to [n]$,
(2) K *is cartesian,*
(3) $\mathrm{Hom}(K_i[i-1], K'_i) = 0$ *for* $i > 1$.

Then any map $\{K_n \to K'_n\}$ *between the associated simplicial systems of* K *and* K' *comes from a map* $K \to K'$ *in* $D(\mathcal{O})$.

Proof. The proof is exactly the same as the proof of Lemma 85.13.6 except using Lemma 85.14.4 instead of Lemma 85.13.4. □

0D9T **Lemma 85.14.7.** *In Situation 85.3.3 let* $\mathcal{O}$ *be a sheaf of rings on* $\mathcal{C}_{total}$. *Let* (K_n, K_φ) *be a simplicial system of the derived category of modules. Assume*

(1) $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ *is flat for* $\varphi : [m] \to [n]$,
(2) (K_n, K_φ) *is cartesian,*
(3) $\mathrm{Hom}(K_i[t], K_i) = 0$ *for* $i \geq 0$ *and* $t > 0$.

Then there exists a cartesian object K *of* $D(\mathcal{O})$ *whose associated simplicial system is isomorphic to* (K_n, K_φ).

Proof. The proof is exactly the same as the proof of Lemma 85.13.7 with the following changes

(1) use $g_n^* = Lg_n^*$ everywhere instead of g_n^{-1},
(2) use $f_\varphi^* = Lf_\varphi^*$ everywhere instead of f_φ^{-1},
(3) refer to Lemma 85.10.1 instead of Lemma 85.8.1,
(4) in the construction of $Y'_{m,n}$ use $\mathcal{O}_m$ instead of $\mathbf{Z}$,

(5) compare with the proof of Lemma 85.14.4 rather than the proof of Lemma 85.13.4.

This ends the proof. □

85.15. The site associated to a semi-representable object

09WK Let $\mathcal{C}$ be a site. Recall that a *semi-representable object* of $\mathcal{C}$ is simply a family $\{U_i\}_{i\in I}$ of objects of $\mathcal{C}$. A *morphism* $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ *of semi-representable objects* is given by a map $\alpha : I \to J$ and for every $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$. The category of semi-representable objects of $\mathcal{C}$ is denoted $\mathrm{SR}(\mathcal{C})$. See Hypercoverings, Definition 25.2.1 and the enclosing section for more information.

For a semi-representable object $K = \{U_i\}_{i\in I}$ of $\mathcal{C}$ we let

$$\mathcal{C}/K = \coprod\nolimits_{i\in I} \mathcal{C}/U_i$$

be the disjoint union of the localizations of $\mathcal{C}$ at U_i. There is a natural structure of a site on this category, with coverings inherited from the localizations $\mathcal{C}/U_i$. The site $\mathcal{C}/K$ is called the *localization of* $\mathcal{C}$ *at* K. Observe that a sheaf on $\mathcal{C}/K$ is the same thing as a family of sheaves $\mathcal{F}_i$ on $\mathcal{C}/U_i$, i.e.,

$$Sh(\mathcal{C}/K) = \prod\nolimits_{i\in I} Sh(\mathcal{C}/U_i)$$

This is occasionally useful to understand what is going on.

Let $\mathcal{C}$ be a site. Let $K = \{U_i\}_{i\in I}$ be an object of $\mathrm{SR}(\mathcal{C})$. There is a continuous and cocontinuous localization functor $j : \mathcal{C}/K \to \mathcal{C}$ which is the product of the localization functors $j_i : \mathcal{C}/V_i \to \mathcal{C}$. We obtain functors $j_!$, j^{-1}, j_* exactly as in Sites, Section 7.25. In terms of the product decomposition $Sh(\mathcal{C}/K) = \prod_{i\in I} Sh(\mathcal{C}/U_i)$ we have

$$\begin{array}{ccccc} j_! & : & (\mathcal{F}_i)_{i\in I} & \longmapsto & \coprod j_{i,!}\mathcal{F}_i \\ j^{-1} & : & \mathcal{G} & \longmapsto & (j_i^{-1}\mathcal{G})_{i\in I} \\ j_* & : & (\mathcal{F}_i)_{i\in I} & \longmapsto & \prod j_{i,*}\mathcal{F}_i \end{array}$$

as the reader easily verifies.

Let $f : K \to L$ be a morphism of $\mathrm{SR}(\mathcal{C})$. Then we obtain a continuous and cocontinuous functor

$$v : \mathcal{C}/K \longrightarrow \mathcal{C}/L$$

by applying the construction of Sites, Lemma 7.25.8 to the components. More precisely, suppose $f = (\alpha, f_i)$ where $K = \{U_i\}_{i\in I}$, $L = \{V_j\}_{j\in J}$, $\alpha : I \to J$, and $f_i : U_i \to V_{\alpha(i)}$. Then the functor v maps the component $\mathcal{C}/U_i$ into the component $\mathcal{C}/V_{\alpha(i)}$ via the construction of the aforementioned lemma. In particular we obtain a morphism

$$f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$$

of topoi. In terms of the product decompositions $Sh(\mathcal{C}/K) = \prod_{i\in I} Sh(\mathcal{C}/U_i)$ and $Sh(\mathcal{C}/L) = \prod_{j\in J} Sh(\mathcal{C}/V_j)$ the reader verifies that

$$\begin{array}{ccccc} f_! & : & (\mathcal{F}_i)_{i\in I} & \longmapsto & (\coprod_{i\in I,\alpha(i)=j} f_{i,!}\mathcal{F}_i)_{j\in J} \\ f^{-1} & : & (\mathcal{G}_j)_{j\in J} & \longmapsto & (f_i^{-1}\mathcal{G}_{\alpha(i)})_{i\in I} \\ f_* & : & (\mathcal{F}_i)_{i\in I} & \longmapsto & (\prod_{i\in I,\alpha(i)=j} f_{i,*}\mathcal{F}_i)_{j\in J} \end{array}$$

where $f_i : Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C}/V_{\alpha(i)})$ is the morphism associated to the localization functor $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ corresponding to $f_i : U_i \to V_{\alpha(i)}$.

0D85 **Lemma 85.15.1.** *Let $\mathcal{C}$ be a site.*

(1) *For K in $SR(\mathcal{C})$ the functor $j : \mathcal{C}/K \to \mathcal{C}$ is continuous, cocontinuous, and has property P of Sites, Remark 7.20.5.*
(2) *For $f : K \to L$ in $SR(\mathcal{C})$ the functor $v : \mathcal{C}/K \to \mathcal{C}/L$ (see above) is continuous, cocontinuous, and has property P of Sites, Remark 7.20.5.*

Proof. Proof of (2). In the notation of the discussion preceding the lemma, the localization functors $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ are continuous and cocontinuous by Sites, Section 7.25 and satisfy P by Sites, Remark 7.25.11. It is formal to deduce v is continuous and cocontinuous and has P. We omit the details. We also omit the proof of (1). □

0D86 **Lemma 85.15.2.** *Let $\mathcal{C}$ be a site and K in $SR(\mathcal{C})$. For $\mathcal{F}$ in $Sh(\mathcal{C})$ we have*

$$j_*j^{-1}\mathcal{F} = \mathcal{H}om(F(K)^\#, \mathcal{F})$$

where F is as in Hypercoverings, Definition 25.2.2.

Proof. Say $K = \{U_i\}_{i\in I}$. Using the description of the functors j^{-1} and j_* given above we see that

$$j_*j^{-1}\mathcal{F} = \prod\nolimits_{i\in I} j_{i,*}(\mathcal{F}|_{\mathcal{C}/U_i}) = \prod\nolimits_{i\in I} \mathcal{H}om(h^\#_{U_i}, \mathcal{F})$$

The second equality by Sites, Lemma 7.26.3. Since $F(K) = \coprod h_{U_i}$ in $PSh(\mathcal{C}$, we have $F(K)^\# = \coprod h^\#_{U_i}$ in $Sh(\mathcal{C})$ and since $\mathcal{H}om(-, \mathcal{F})$ turns coproducts into products (immediate from the construction in Sites, Section 7.26), we conclude. □

0D87 **Lemma 85.15.3.** *Let $\mathcal{C}$ be a site.*

(1) *For K in $SR(\mathcal{C})$ the functor $j_!$ gives an equivalence $Sh(\mathcal{C}/K) \to Sh(\mathcal{C})/F(K)^\#$ where F is as in Hypercoverings, Definition 25.2.2.*
(2) *The functor $j^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{C}/K)$ corresponds via the identification of (1) with $\mathcal{F} \mapsto (\mathcal{F} \times F(K)^\# \to F(K)^\#)$.*
(3) *For $f : K \to L$ in $SR(\mathcal{C})$ the functor f^{-1} corresponds via the identifications of (1) to the functor $Sh(\mathcal{C})/F(L)^\# \to Sh(\mathcal{C})/F(K)^\#$, $(\mathcal{G} \to F(L)^\#) \mapsto (\mathcal{G} \times_{F(L)^\#} F(K)^\# \to F(K)^\#)$.*

Proof. Observe that if $K = \{U_i\}_{i\in I}$ then the category $Sh(\mathcal{C}/K)$ decomposes as the product of the categories $Sh(\mathcal{C}/U_i)$. Observe that $F(K)^\# = \coprod_{i\in I} h^\#_{U_i}$ (coproduct in sheaves). Hence $Sh(\mathcal{C})/F(K)^\#$ is the product of the categories $Sh(\mathcal{C})/h^\#_{U_i}$. Thus (1) and (2) follow from the corresponding statements for each i, see Sites, Lemmas 7.25.4 and 7.25.7. Similarly, if $L = \{V_j\}_{j\in J}$ and f is given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$, then we can apply Sites, Lemma 7.25.9 to each of the re-localization morphisms $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ to get (3). □

0D88 **Lemma 85.15.4.** *Let $\mathcal{C}$ be a site. For K in $SR(\mathcal{C})$ the functor j^{-1} sends injective abelian sheaves to injective abelian sheaves. Similarly, the functor j^{-1} sends K-injective complexes of abelian sheaves to K-injective complexes of abelian sheaves.*

Proof. The first statement is the natural generalization of Cohomology on Sites, Lemma 21.7.1 to semi-representable objects. In fact, it follows from this lemma by the product decomposition of $Sh(\mathcal{C}/K)$ and the description of the functor j^{-1} given above. The second statement is the natural generalization of Cohomology on Sites, Lemma 21.20.1 and follows from it by the product decomposition of the topos.

Alternative: since j induces a localization of topoi by Lemma 85.15.3 part (1) it also follows immediately from Cohomology on Sites, Lemmas 21.7.1 and 21.20.1 by enlarging the site; compare with the proof of Cohomology on Sites, Lemma 21.13.3 in the case of injective sheaves. □

0D89 **Remark 85.15.5** (Variant for over an object)**.** Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. The category $\mathrm{SR}(\mathcal{C}, X)$ of *semi-representable objects over* X is defined by the formula $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. See Hypercoverings, Definition 25.2.1. Thus we may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

(1) a site $\mathcal{C}/K$ for K in $\mathrm{SR}(\mathcal{C}, X)$,
(2) a decomposition $Sh(\mathcal{C}/K) = \prod Sh(\mathcal{C}/U_i)$ if $K = \{U_i/X\}$,
(3) a localization functor $j : \mathcal{C}/K \to \mathcal{C}/X$,
(4) a morphism $f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$ for $f : K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$.

All results of this section hold in this situation by replacing $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9U **Remark 85.15.6** (Ringed variant)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. In this case, for any semi-representable object K of $\mathcal{C}$ the site $\mathcal{C}/K$ is a ringed site with sheaf of rings $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$. The constructions above give

(1) a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\mathrm{SR}(\mathcal{C})$,
(2) a decomposition $Mod(\mathcal{O}_K) = \prod Mod(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,
(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\mathrm{SR}(\mathcal{C})$.

Many of the results above hold in this setting. For example, the functor j^* has an exact left adjoint

$$j_! : Mod(\mathcal{O}_K) \to Mod(\mathcal{O}_\mathcal{C}),$$

which in terms of the product decomposition given in (2) sends $(\mathcal{F}_i)_{i\in I}$ to $\bigoplus j_{i,!}\mathcal{F}_i$. Similarly, given $f : K \to L$ as above, the functor f^* has an exact left adjoint $f_! : Mod(\mathcal{O}_K) \to Mod(\mathcal{O}_L)$. Thus the functors j^* and f^* are exact, i.e., j and f are flat morphisms of ringed topoi (also follows from the equalities $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_K = f^{-1}\mathcal{O}_L$).

0D9V **Remark 85.15.7** (Ringed variant over an object)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $X \in \mathrm{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/U}$. Then we can combine the constructions given in Remarks 85.15.5 and 85.15.6 to get

(1) a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\mathrm{SR}(\mathcal{C}, X)$,
(2) a decomposition $Mod(\mathcal{O}_K) = \prod Mod(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,
(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$.

Of course all of the results mentioned in Remark 85.15.6 hold in this setting as well.

85.16. The site associate to a simplicial semi-representable object

0D8A Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\mathrm{SR}(\mathcal{C})$. As usual, set $K_n = K([n])$ and denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. By the construction in Section 85.15 we obtain a simplicial object $n \mapsto \mathcal{C}/K_n$ in the category whose objects are sites and whose morphisms are cocontinuous functors.

In other words, we get a gadget as in Case B of Section 85.3. The functors satisfy property P by Lemma 85.15.1. Hence we may apply Lemma 85.3.2 to obtain a site $(\mathcal{C}/K)_{total}$.

We can describe the site $(\mathcal{C}/K)_{total}$ explicitly as follows. Say $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. Then we have

(1) an object of $(\mathcal{C}/K)_{total}$ corresponds to an object $(U/U_{n,i})$ of $\mathcal{C}/U_{n,i}$ for some n and some $i \in I_n$,
(2) a morphism between $U/U_{n,i}$ and $V/U_{m,j}$ is a pair (φ, f) where $\varphi : [m] \to [n]$, $j = \alpha(\varphi)(i)$, and $f : U \to V$ is a morphism of $\mathcal{C}$ such that

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ U_{n,i} & \xrightarrow{f_{\varphi,i}} & U_{m,j} \end{array}$$

is commutative, and
(3) coverings of the object $U/U_{n,i}$ are constructed by starting with a covering $\{f_j : U_j \to U\}$ in $\mathcal{C}$ and letting $\{(\mathrm{id}, f_j) : U_j/U_{n,i} \to U/U_{n,i}\}$ be a covering in $(\mathcal{C}/K)_{total}$.

All of our general theory developed for simplicial sites applies to $(\mathcal{C}/K)_{total}$. Observe that the obvious forgetful functor

$$j_{total} : (\mathcal{C}/K)_{total} \longrightarrow \mathcal{C}$$

is continuous and cocontinuous. It turns out that the associated morphism of topoi comes from an (obvious) augmentation.

0D8B **Lemma 85.16.1.** *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. The localization functor $j_0 : \mathcal{C}/K_0 \to \mathcal{C}$ defines an augmentation $a_0 : Sh(\mathcal{C}/K_0) \to Sh(\mathcal{C})$, as in case (B) of Remark 85.4.1. The corresponding morphisms of topoi*

$$a_n : Sh(\mathcal{C}/K_n) \longrightarrow Sh(\mathcal{C}), \quad a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$

of Lemma 85.4.2 are equal to the morphisms of topoi associated to the continuous and cocontinuous localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}$ and $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}$.

Proof. This is immediate from working through the definitions. See in particular the footnote in the proof of Lemma 85.4.2 for the relationship between a and j_{total}. □

09WM **Lemma 85.16.2.** *With assumption and notation as in Lemma 85.16.1 we have the following properties:*

(1) *there is a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ left adjoint to $a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$,*
(2) *there is a functor $a_! : Ab((\mathcal{C}/K)_{total}) \to Ab(\mathcal{C})$ left adjoint to $a^{-1} : Ab(\mathcal{C}) \to Ab((\mathcal{C}/K)_{total})$,*
(3) *the functor a^{-1} associates to $\mathcal{F}$ in $Sh(\mathcal{C})$ the sheaf on $(\mathcal{C}/K)_{total}$ which in degree n is equal to $a_n^{-1}\mathcal{F}$,*
(4) *the functor a_* associates to $\mathcal{G}$ in $Ab((\mathcal{C}/K)_{total})$ the equalizer of the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$,*

Proof. Parts (3) and (4) hold for any augmentation of a simplicial site, see Lemma 85.4.2. Parts (1) and (2) follow as j_{total} is continuous and cocontinuous. The functor $a_!^{Sh}$ is constructed in Sites, Lemma 7.21.5 and the functor $a_!$ is constructed in Modules on Sites, Lemma 18.16.2. □

0DC0 **Lemma 85.16.3.** *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. Let $U/U_{n,i}$ be an object of $\mathcal{C}/K_n$. Let $\mathcal{F} \in Ab((\mathcal{C}/K)_{total})$. Then*

$$H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}_{n,i})$$

where

(1) *on the left hand side U is viewed as an object of $\mathcal{C}_{total}$, and*
(2) *on the right hand side $\mathcal{F}_{n,i}$ is the ith component of the sheaf $\mathcal{F}_n$ on $\mathcal{C}/K_n$ in the decomposition $Sh(\mathcal{C}/K_n) = \prod Sh(\mathcal{C}/U_{n,i})$ of Section 85.15.*

Proof. This follows immediately from Lemma 85.8.6 and the product decompositions of Section 85.15. □

0D8C **Remark 85.16.4** (Variant for over an object)**.** Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. Recall that we have a category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$ of semi-representable objects over X, see Remark 85.15.5. We may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

(1) a site $(\mathcal{C}/K)_{total}$ for a simplicial K object of $\mathrm{SR}(\mathcal{C}, X)$,
(2) a localization functor $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}/X$,
(3) localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}/X$,
(4) a morphism of topoi $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$,
(5) morphisms of topoi $a_n : Sh(\mathcal{C}/K_n) \to Sh(\mathcal{C}/X)$,
(6) a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$ left adjoint to a^{-1}, and
(7) a functor $a_! : Ab((\mathcal{C}/K)_{total}) \to Ab(\mathcal{C}/X)$ left adjoint to a^{-1}.

All of the results of this section hold in this setting. To prove this one replaces the site $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9W **Remark 85.16.5** (Ringed variant)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Given a simplicial semi-representable object K of $\mathcal{C}$ we set $\mathcal{O} = a^{-1}\mathcal{O}_\mathcal{C}$, where a is as in Lemmas 85.16.1 and 85.16.2. The constructions above, keeping track of the sheaves of rings as in Remark 85.15.6, give

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\mathrm{SR}(\mathcal{C})$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(4) a functor $a_! : Mod(\mathcal{O}) \to Mod(\mathcal{O}_\mathcal{C})$ left adjoint to a^*.

The functor $a_!$ exists (but in general is not exact) because $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ and we can replace the use of Modules on Sites, Lemma 18.16.2 in the proof of Lemma 85.16.2 by Modules on Sites, Lemma 18.41.1. As discussed in Remark 85.15.6 there are exact functors $a_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O}_\mathcal{C})$ left adjoint to a_n^*. Consequently, the morphisms a and a_n are flat. Remark 85.15.6 implies the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/K_m), \mathcal{O}_m)$ for $\varphi : [m] \to [n]$ is flat and there exists an exact functor $f_{\varphi!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O}_m)$ left adjoint to f_φ^*. This in turn implies that for the flat morphism of ringed topoi $g_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh((\mathcal{C}/K)_{total}), \mathcal{O})$ the functor $g_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O})$ left adjoint to g_n^* is exact, see Lemma 85.6.3.

0D9X **Remark 85.16.6** (Ringed variant over an object)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let $X \in \mathrm{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. Then we can combine the constructions given in Remarks 85.16.4 and 85.16.5 to get

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\mathrm{SR}(\mathcal{C}, X)$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,
(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,
(4) a functor $a_! : Mod(\mathcal{O}) \to Mod(\mathcal{O}_X)$ left adjoint to a^*.

Of course, all the results mentioned in Remark 85.16.5 hold in this setting as well.

85.17. Cohomological descent for hypercoverings

0D8D Let $\mathcal{C}$ be a site. In this section we assume $\mathcal{C}$ has equalizers and fibre products. We let K be a hypercovering as defined in Hypercoverings, Definition 25.6.1. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$

of Section 85.16.

0D8E **Lemma 85.17.1.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then*

(1) *$a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
(2) *$a^{-1} : Ab(\mathcal{C}) \to Ab((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

In both cases a_* provides the quasi-inverse functor.

Proof. The case of abelian sheaves follows immediately from the case of sheaves of sets as the functor a^{-1} commutes with products. In the rest of the proof we work with sheaves of sets. Observe that $a^{-1}\mathcal{F}$ is cartesian for $\mathcal{F}$ in $Sh(\mathcal{C})$ by Lemma 85.12.3. It suffices to show that the adjunction map $\mathcal{F} \to a_*a^{-1}\mathcal{F}$ is an isomorphism $\mathcal{F}$ in $Sh(\mathcal{C})$ and that for a cartesian sheaf $\mathcal{G}$ on $(\mathcal{C}/K)_{total}$ the adjunction map $a^{-1}a_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.

Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Recall that $a_*a^{-1}\mathcal{F}$ is the equalizer of the two maps $a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F}$, see Lemma 85.16.2. By Lemma 85.15.2

$$a_{0,*}a_0^{-1}\mathcal{F} = \mathcal{H}om(F(K_0)^\#, \mathcal{F}) \quad \text{and} \quad a_{1,*}a_1^{-1}\mathcal{F} = \mathcal{H}om(F(K_1)^\#, \mathcal{F})$$

On the other hand, we know that

$$F(K_1)^\# \rightrightarrows F(K_0)^\# \longrightarrow \text{final object } * \text{ of } Sh(\mathcal{C})$$

is a coequalizer diagram in sheaves of sets by definition of a hypercovering. Thus it suffices to prove that $\mathcal{H}om(-, \mathcal{F})$ transforms coequalizers into equalizers which is immediate from the construction in Sites, Section 7.26.

Let $\mathcal{G}$ be a cartesian sheaf on $(\mathcal{C}/K)_{total}$. We will show that $\mathcal{G} = a^{-1}\mathcal{F}$ for some sheaf $\mathcal{F}$ on $\mathcal{C}$. This will finish the proof because then $a^{-1}a_*\mathcal{G} = a^{-1}a_*a^{-1}\mathcal{F} = a^{-1}\mathcal{F} = \mathcal{G}$ by the result of the previous paragraph. Set $\mathcal{K}_n = F(K_n)^\#$ for $n \geq 0$. Then we have maps of sheaves

$$\mathcal{K}_2 \rightrightarrows \mathcal{K}_1 \rightrightarrows \mathcal{K}_0$$

coming from the fact that K is a simplicial semi-representable object. The fact that K is a hypercovering means that

$$\mathcal{K}_1 \to \mathcal{K}_0 \times \mathcal{K}_0 \quad \text{and} \quad \mathcal{K}_2 \to \Big(\text{cosk}_1(\ \mathcal{K}_1 \lll \mathcal{K}_0\ )\Big)_2$$

are surjective maps of sheaves. Using the description of cartesian sheaves on $(\mathcal{C}/K)_{total}$ given in Lemma 85.12.4 and using the description of $Sh(\mathcal{C}/K_n)$ in Lemma 85.15.3 we find that our problem can be entirely formulated[3] in terms of

(1) the topos $Sh(\mathcal{C})$, and
(2) the simplicial object $\mathcal{K}$ in $Sh(\mathcal{C})$ whose terms are $\mathcal{K}_n$.

Thus, after replacing $\mathcal{C}$ by a different site $\mathcal{C}'$ as in Sites, Lemma 7.29.5, we may assume $\mathcal{C}$ has all finite limits, the topology on $\mathcal{C}$ is subcanonical, a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}$ is a covering if and only if $\coprod h_{V_j} \to V$ is surjective, and there exists a simplicial object U of $\mathcal{C}$ such that $\mathcal{K}_n = h_{U_n}$ as simplicial sheaves. Working backwards through the equivalences we may assume $K_n = \{U_n\}$ for all n.

Let X be the final object of $\mathcal{C}$. Then $\{U_0 \to X\}$ is a covering, $\{U_1 \to U_0 \times U_0\}$ is a covering, and $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering. Let us use $d_i^n : U_n \to U_{n-1}$ and $s_j^n : U_n \to U_{n+1}$ the morphisms corresponding to δ_i^n and σ_j^n as in Simplicial, Definition 14.2.1. By abuse of notation, given a morphism $c : V \to W$ of $\mathcal{C}$ we denote the morphism of topoi $c : Sh(\mathcal{C}/V) \to Sh(\mathcal{C}/W)$ by the same letter. Now $\mathcal{G}$ is given by a sheaf $\mathcal{G}_0$ on $\mathcal{C}/U_0$ and an isomorphism $\alpha : (d_1^1)^{-1}\mathcal{G}_0 \to (d_0^1)^{-1}\mathcal{G}_0$ satisfying the cocycle condition on $\mathcal{C}/U_2$ formulated in Lemma 85.12.4. Since $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering, the corresponding pullback functor on sheaves is faithful (small detail omitted). Hence we may replace U by $\text{cosk}_1\text{sk}_1 U$, because this replaces U_2 by $(\text{cosk}_1\text{sk}_1 U)_2$ and leaves U_1 and U_0 unchanged. Then

$$(d_0^2, d_1^2, d_2^2) : U_2 \to U_1 \times U_1 \times U_1$$

is a monomorphism whose its image on T-valued points is described in Simplicial, Lemma 14.19.6. In particular, there is a morphism c fitting into a commutative diagram

$$\begin{array}{ccc} U_1 \times_{(d_1^1, d_0^1), U_0 \times U_0, (d_1^1, d_0^1)} U_1 & \xrightarrow[c]{} & U_2 \\ \downarrow & & \downarrow \\ U_1 \times U_1 & \xrightarrow{(\text{pr}_1, \text{pr}_2, s_0^0 \circ d_1^1 \circ \text{pr}_1)} & U_1 \times U_1 \times U_1 \end{array}$$

as going around the other way defines a point of U_2. Pulling back the cocycle condition for α on U_2 translates into the condition that the pullbacks of α via the projections to $U_1 \times_{(d_1^1, d_0^1), U_0 \times U_0, (d_1^1, d_0^1)} U_1$ are the same as the pullback of α via $s_0^0 \circ d_1^1 \circ \text{pr}_1$ is the identity map (namely, the pullback of α by s_0^0 is the identity). By Sites, Lemma 7.26.1 this means that α comes from an isomorphism

$$\alpha' : \text{pr}_1^{-1}\mathcal{G}_0 \to \text{pr}_2^{-1}\mathcal{G}_0$$

of sheaves on $\mathcal{C}/U_0 \times U_0$. Then finally, the morphism $U_2 \to U_0 \times U_0 \times U_0$ is surjective on associated sheaves as is easily seen using the surjectivity of $U_1 \to U_0 \times U_0$ and

[3] Even though it does not matter what the precise formulation is, we spell it out: the problem is to show that given an object $\mathcal{G}_0/\mathcal{K}_0$ of $Sh(\mathcal{C})/\mathcal{K}_0$ and an isomorphism

$$\alpha : \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta_1^1)} \mathcal{K}_1 \to \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta_0^1)} \mathcal{K}_1$$

over $\mathcal{K}_1$ satisfying a cocycle condition in $Sh(\mathcal{C})/\mathcal{K}_2$, there exists $\mathcal{F}$ in $Sh(\mathcal{C})$ and an isomorphism $\mathcal{F} \times \mathcal{K}_0 \to \mathcal{G}_0$ over $\mathcal{K}_0$ compatible with α.

the description of U_2 given above. Therefore α' satisfies the cocycle condition on $U_0 \times U_0 \times U_0$. The proof is finished by an application of Sites, Lemma 7.26.5 to the covering $\{U_0 \to X\}$. □

0D8F **Lemma 85.17.2.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. The Čech complex of Lemma 85.9.2 associated to $a^{-1}\mathcal{F}$*

$$a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F} \to a_{2,*}a_2^{-1}\mathcal{F} \to \ldots$$

is equal to the complex $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{F})$. Here $s(\mathbf{Z}^\#_{F(K)})$ is as in Hypercoverings, Definition 25.4.1.

Proof. By Lemma 85.15.2 we have

$$a_{n,*}a_n^{-1}\mathcal{F} = \mathcal{H}om'(F(K_n)^\#, \mathcal{F})$$

where $\mathcal{H}om'$ is as in Sites, Section 7.26. The boundary maps in the complex of Lemma 85.9.2 come from the simplicial structure. Thus the equality of complexes comes from the canonical identifications $\mathcal{H}om'(\mathcal{G}, \mathcal{F}) = \mathcal{H}om(\mathbf{Z}_\mathcal{G}, \mathcal{F})$ for $\mathcal{G}$ in $Sh(\mathcal{C})$. □

0D8G **Lemma 85.17.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. For $E \in D(\mathcal{C})$ the map*

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. First, let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the spectral sequence of Lemma 85.9.3 for the sheaf $a^{-1}\mathcal{I}$ degenerates as $(a^{-1}\mathcal{I})_p = a_p^{-1}\mathcal{I}$ is injective by Lemma 85.15.4. Thus the complex

$$a_{0,*}a_0^{-1}\mathcal{I} \to a_{1,*}a_1^{-1}\mathcal{I} \to a_{2,*}a_2^{-1}\mathcal{I} \to \ldots$$

computes $Ra_*a^{-1}\mathcal{I}$. By Lemma 85.17.2 this is equal to the complex $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{I})$. Because K is a hypercovering, we see that $s(\mathbf{Z}^\#_{F(K)})$ is exact in degrees > 0 by Hypercoverings, Lemma 25.4.4 applied to the simplicial presheaf $F(K)$. Since $\mathcal{I}$ is injective, the functor $\mathcal{H}om(-, \mathcal{I})$ is exact and we conclude that $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{I})$ is exact in positive degrees. We conclude that $R^pa_*a^{-1}\mathcal{I} = 0$ for $p > 0$. On the other hand, we have $\mathcal{I} = a_*a^{-1}\mathcal{I}$ by Lemma 85.17.1.

Bounded case. Let $E \in D^+(\mathcal{C})$. Choose a bounded below complex $\mathcal{I}^\bullet$ of injectives representing E. By the result of the first paragraph and Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) $Ra_*a^{-1}\mathcal{I}^\bullet$ is computed by the complex $a_*a^{-1}\mathcal{I}^\bullet = \mathcal{I}^\bullet$ and we conclude the lemma is true in this case.

Unbounded case. We urge the reader to skip this, since the argument is the same as above, except that we use explicit representation by double complexes to get around convergence issues. Let $E \in D(\mathcal{C})$. To show the map $E \to Ra_*a^{-1}E$ is an isomorphism, it suffices to show for every object U of $\mathcal{C}$ that

$$R\Gamma(U, E) = R\Gamma(U, Ra_*a^{-1}E)$$

We will compute both sides and show the map $E \to Ra_*a^{-1}E$ induces an isomorphism. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. Choose a quasi-isomorphism $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ for some K-injective complex $\mathcal{J}^\bullet$ on $(\mathcal{C}/K)_{total}$. We

have

$$R\Gamma(U, E) = R\operatorname{Hom}(\mathbf{Z}_U^\#, E)$$

and

$$R\Gamma(U, Ra_*a^{-1}E) = R\operatorname{Hom}(\mathbf{Z}_U^\#, Ra_*a^{-1}E) = R\operatorname{Hom}(a^{-1}\mathbf{Z}_U^\#, a^{-1}E)$$

By Lemma 85.9.1 we have a quasi-isomorphism

$$\Big(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}_U^\#) \to g_{1!}(a_1^{-1}\mathbf{Z}_U^\#) \to g_{0!}(a_0^{-1}\mathbf{Z}_U^\#)\Big) \longrightarrow a^{-1}\mathbf{Z}_U^\#$$

Hence $R\operatorname{Hom}(a^{-1}\mathbf{Z}_U^\#, a^{-1}E)$ is equal to

$$R\Gamma((\mathcal{C}/K)_{total}, R\mathcal{H}om(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}_U^\#) \to g_{1!}(a_1^{-1}\mathbf{Z}_U^\#) \to g_{0!}(a_0^{-1}\mathbf{Z}_U^\#), \mathcal{J}^\bullet))$$

By the construction in Cohomology on Sites, Section 21.35 and since $\mathcal{J}^\bullet$ is K-injective, we see that this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(g_{p!}(a_p^{-1}\mathbf{Z}_U^\#), \mathcal{J}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^q)$$

See Cohomology on Sites, Lemmas 21.34.6 and 21.35.1 for more information. Thus we find that $R\Gamma(U, Ra_*a^{-1}E)$ is computed by the product total complex $\operatorname{Tot}_\pi(B^{\bullet,\bullet})$ with $B^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^q)$. For the other side we argue similarly. First we note that

$$s(\mathbf{Z}_{F(K)}^\#) \longrightarrow \mathbf{Z}$$

is a quasi-isomorphism of complexes on $\mathcal{C}$ by Hypercoverings, Lemma 25.4.4. Since $\mathbf{Z}_U^\#$ is a flat sheaf of $\mathbf{Z}$-modules we see that

$$s(\mathbf{Z}_{F(K)}^\#) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\# \longrightarrow \mathbf{Z}_U^\#$$

is a quasi-isomorphism. Therefore $R\operatorname{Hom}(\mathbf{Z}_U^\#, E)$ is equal to

$$R\Gamma(\mathcal{C}, R\mathcal{H}om(s(\mathbf{Z}_{F(K)}^\#) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\#, \mathcal{I}^\bullet))$$

By the construction of $R\mathcal{H}om$ and since $\mathcal{I}^\bullet$ is K-injective, this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(\mathbf{Z}_{K_p}^\# \otimes_{\mathbf{Z}} \mathbf{Z}_U^\#, \mathcal{I}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^q)$$

The equality of terms follows from the fact that $\mathbf{Z}_{K_p}^\# \otimes_{\mathbf{Z}} \mathbf{Z}_U^\# = a_{p!}a_p^{-1}\mathbf{Z}_U^\#$ by Modules on Sites, Remark 18.27.10. Thus we find that $R\Gamma(U, E)$ is computed by the product total complex $\operatorname{Tot}_\pi(A^{\bullet,\bullet})$ with $A^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^q)$.

Since $\mathcal{I}^\bullet$ is K-injective we see that $a_p^{-1}\mathcal{I}^\bullet$ is K-injective, see Lemma 85.15.4. Since $\mathcal{J}^\bullet$ is K-injective we see that $g_p^{-1}\mathcal{J}^\bullet$ is K-injective, see Lemma 85.3.6. Both represent the object $a_p^{-1}E$. Hence for every $p \geq 0$ the map of complexes

$$A^{p,\bullet} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}\mathcal{I}^\bullet) \longrightarrow \operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, g_p^{-1}\mathcal{J}^\bullet) = B^{p,\bullet}$$

induced by g_p^{-1} applied to the given map $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ is a quasi-isomorphisms as these complexes both compute

$$R\operatorname{Hom}(a_p^{-1}\mathbf{Z}_U^\#, a_p^{-1}E)$$

By More on Algebra, Lemma 15.104.2 we conclude that the right vertical arrow in the commutative diagram

$$\begin{array}{ccc} R\Gamma(U, E) & \longrightarrow & \mathrm{Tot}_\pi(A^{\bullet,\bullet}) \\ \downarrow & & \downarrow \\ R\Gamma(U, Ra_*a^{-1}E) & \longrightarrow & \mathrm{Tot}_\pi(B^{\bullet,\bullet}) \end{array}$$

is a quasi-isomorphism. Since we saw above that the horizontal arrows are quasi-isomorphisms, so is the left vertical arrow. □

0D8H **Lemma 85.17.4.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C})$.

Proof. This follows from Lemma 85.17.3 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0D8I **Lemma 85.17.5.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{A} \subset Ab((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.17.1 and 85.17.3 and Cohomology on Sites, Lemma 21.28.5 □

We urge the reader to skip the following remark.

09X6 **Remark 85.17.6.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets on $\mathcal{C}$. If $\mathcal{C}$ has equalizers and fibre products, then we've defined the notion of a hypercovering of $\mathcal{G}$ in Hypercoverings, Definition 25.6.1. We claim that all the results in this section have a valid counterpart in this setting. To see this, define the localization $\mathcal{C}/\mathcal{G}$ of $\mathcal{C}$ at $\mathcal{G}$ exactly as in Sites, Lemma 7.30.3 (which is stated only for sheaves; the topos $Sh(\mathcal{C}/\mathcal{G})$ is equal to the localization of the topos $Sh(\mathcal{C})$ at the sheaf $\mathcal{G}^\#$). Then the reader easily shows that the site $\mathcal{C}/\mathcal{G}$ has fibre products and equalizers and that a hypercovering of $\mathcal{G}$ in $\mathcal{C}$ is the same thing as a hypercovering for the site $\mathcal{C}/\mathcal{G}$. Hence replacing the site $\mathcal{C}$ by $\mathcal{C}/\mathcal{G}$ in the lemmas on hypercoverings above we obtain proofs of the corresponding results for hypercoverings of $\mathcal{G}$. Example: for a hypercovering K of $\mathcal{G}$ we have

$$R\Gamma(\mathcal{C}/\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D^+(\mathcal{C}/\mathcal{G})$ where $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/\mathcal{G})$ is the canonical augmentation. This is Lemma 85.17.4. Let $R\Gamma(\mathcal{G}, -) : D(\mathcal{C}) \to D(Ab)$ be defined as the derived functor of the functor $H^0(\mathcal{G}, -) = H^0(\mathcal{G}^\#, -)$ discussed in Hypercoverings, Section 25.6 and Cohomology on Sites, Section 21.13. We have

$$R\Gamma(\mathcal{G}, E) = R\Gamma(\mathcal{C}/\mathcal{G}, j^{-1}E)$$

by the analogue of Cohomology on Sites, Lemma 21.7.1 for the localization fuctor $j : \mathcal{C}/\mathcal{G} \to \mathcal{C}$. Putting everything together we obtain

$$R\Gamma(\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}j^{-1}E) = R\Gamma((\mathcal{C}/K)_{total}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$ where $g : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ is the composition of a and j.

85.18. Cohomological descent for hypercoverings: modules

0D9Y Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering as defined in Hypercoverings, Definition 25.6.1. We will study cohomological descent for the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$$

of Remark 85.16.5.

0D9Z **Lemma 85.18.1.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. With notation as above*

$$a^* : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $a^* = a^{-1}$. Hence the lemma follows immediately from Lemma 85.17.1. □

0DA0 **Lemma 85.18.2.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. For $E \in D(\mathcal{O}_\mathcal{C})$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $La^* = a^* = a^{-1}$. Moreover Ra_* agrees with Ra_* on abelian sheaves, see Cohomology on Sites, Lemma 21.20.7. Hence the lemma follows immediately from Lemma 85.17.3. □

0DA1 **Lemma 85.18.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This follows from Lemma 85.18.2 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4 or by Cohomology on Sites, Lemma 21.20.5. □

0DA2 **Lemma 85.18.4.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_\mathcal{C}) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6 (the required hypotheses hold by the discussion in Remark 85.16.5). The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.18.1 and 85.18.2 and Cohomology on Sites, Lemma 21.28.5. □

85.19. Cohomological descent for hypercoverings of an object

0D8J In this section we assume $\mathcal{C}$ has fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 25.3.3. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

of Remark 85.16.4. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$[4] by Hypercoverings, Lemma 25.3.9. This means that every single result proved for hypercoverings in Section 85.17 has an immediate analogue in the situation in this section.

0D8K **Lemma 85.19.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then*

(1) *$a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
(2) *$a^{-1} : Ab(\mathcal{C}/X) \to Ab((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

In both cases a_* provides the quasi-inverse functor.

Proof. Via Remarks 85.15.5 and 85.16.4 and the discussion in the introduction to this section this follows from Lemma 85.17.1. □

0D8L **Lemma 85.19.2.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. For $E \in D(\mathcal{C}/X)$ the map*

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. Via Remarks 85.15.5 and 85.16.4 and the discussion in the introduction to this section this follows from Lemma 85.17.3. □

09X7 **Lemma 85.19.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. Via Remarks 85.15.5 and 85.16.4 this follows from Lemma 85.17.4. □

0D8M **Lemma 85.19.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Let $\mathcal{A} \subset Ab((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 85.15.5 and 85.16.4 this follows from Lemma 85.17.5. □

[4] The converse may not be the case, i.e., if K is a simplicial object of $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$ which defines a hypercovering for the site $\mathcal{C}/X$ as in Hypercoverings, Definition 25.6.1, then it may not be true that K defines a hypercovering of X. For example, if $K_0 = \{U_{0,i}\}_{i \in I_0}$ then the latter condition guarantees $\{U_{0,i} \to X\}$ is a covering of $\mathcal{C}$ whereas the former condition only requires $\coprod h^\#_{U_{0,i}} \to h^\#_X$ to be a surjective map of sheaves.

85.20. Cohomological descent for hypercoverings of an object: modules

0DA3 In this section we assume $\mathcal{C}$ has fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 25.3.3. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Set $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. We will study the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

of Remark 85.16.6. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$. Therefore the results in this section are immediate consequences of the corresponding results in Section 85.18.

0DA4 **Lemma 85.20.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. With notation as above*

$$a^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.1. □

0DA5 **Lemma 85.20.2.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. For $E \in D(\mathcal{O}_X)$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.2. □

0DA6 **Lemma 85.20.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.3. □

0DA7 **Lemma 85.20.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 85.15.7 and 85.16.6 and the discussion in the introduction to this section this follows from Lemma 85.18.4. □

85.21. Hypercovering by a simplicial object of the site

09X8 Let $\mathcal{C}$ be a site with fibre products and let $X \in \mathrm{Ob}(\mathcal{C})$. In this section we elucidate the results of Section 85.19 in the case that our hypercovering is given by a simplicial object of the site. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$ and $f_\varphi : U_n \to U_m$ the morphism $f_\varphi = U(\varphi)$ corresponding to $\varphi : [m] \to [n]$. Assume we have an augmentation

$$a : U \to X$$

From this we obtain a simplicial site $(\mathcal{C}/U)_{total}$ and an augmentation morphism

$$a : Sh((\mathcal{C}/U)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

Namely, from U we obtain a simiplical object K of $\mathrm{SR}(\mathcal{C}, X)$ with degree n part $K_n = \{U_n \to X\}$ and we can apply the constructions in Remark 85.16.4. More precisely, an object of the site $(\mathcal{C}/U)_{total}$ is given by a V/U_n and a morphism $(\varphi, f) : V/U_n \to W/U_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : V \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \downarrow \\ U_n & \xrightarrow{f_\varphi} & U_m \end{array}$$

is commutative. The morphism of topoi a is given by the cocontinuous functor $V/U_n \mapsto V/X$. That's all folks!

In this section we will say the augmentation $a : U \to X$ is a *hypercovering of X in $\mathcal{C}$* if the following hold

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

This is equivalent to the condition that K (as above) is a hypercovering of X, see Hypercoverings, Example 25.3.5.

0DA8 **Lemma 85.21.1.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then*

(1) *$a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
(2) *$a^{-1} : Ab(\mathcal{C}/X) \to Ab((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

In both cases a_* provides the quasi-inverse functor.

Proof. This is a special case of Lemma 85.19.1. □

0D8N **Lemma 85.21.2.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. For $E \in D(\mathcal{C}/X)$ the map*

$$E \longrightarrow Ra_* a^{-1} E$$

is an isomorphism.

Proof. This is a special case of Lemma 85.19.2. □

09X9 **Lemma 85.21.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. This is a special case of Lemma 85.19.3. □

0DA9 **Lemma 85.21.4.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Let $\mathcal{A} \subset Ab((\mathcal{C}/U)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/U)_{total})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 85.19.4 □

09WL **Lemma 85.21.5.** *Let U be a simplicial object of a site $\mathcal{C}$ with fibre products.*

1. *$\mathcal{C}/U$ has the structure of a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites,*
2. *the construction of Lemma 85.3.1 applied to the structure in (1) reproduces the site $(\mathcal{C}/U)_{total}$ above,*
3. *if $a : U \to X$ is an augmentation, then $a_0 : \mathcal{C}/U_0 \to \mathcal{C}/X$ is an augmentation as in Remark 85.4.1 part (A) and gives the same morphism of topoi $a : Sh((\mathcal{C}/U)_{total}) \to Sh(\mathcal{C}/X)$ as the one above.*

Proof. Given a morphism of objects $V \to W$ of $\mathcal{C}$ the localization morphism $j : \mathcal{C}/V \to \mathcal{C}/W$ is a left adjoint to the base change functor $\mathcal{C}/W \to \mathcal{C}/V$. The base change functor is continuous and induces the same morphism of topoi as j. See Sites, Lemma 7.27.3. This proves (1).

Part (2) holds because a morphism $V/U_n \to W/U_m$ of the category constructed in Lemma 85.3.1 is a morphism $V \to W \times_{U_m, f_\varphi} U_n$ over U_n which is the same thing as a morphism $f : V \to W$ over the morphism $f_\varphi : U_n \to U_m$, i.e., the same thing as a morphism in the category $(\mathcal{C}/U)_{total}$ defined above. Equality of sets of coverings is immediate from the definition.

We omit the proof of (3). □

85.22. Hypercovering by a simplicial object of the site: modules

0DAA Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined in Section 85.21. In this section we study the augmentation

$$a : (Sh((\mathcal{C}/U)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

we obtain by thinking of U as a simiplical semi-representable object of $\mathcal{C}/X$ whose degree n part is the singleton element $\{U_n/X\}$ and applying the constructions in Remark 85.16.6. Thus all the results in this section are immediate consequences of the corresponding results in Section 85.20.

0DAB **Lemma 85.22.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. With notation as above*

$$a^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. This is a special case of Lemma 85.20.1. □

0DAC **Lemma 85.22.2.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. For $E \in D(\mathcal{O}_X)$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. This is a special case of Lemma 85.20.2. □

0DAD **Lemma 85.22.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This is a special case of Lemma 85.20.3. □

0DAE **Lemma 85.22.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 85.20.4. □

85.23. Unbounded cohomological descent for hypercoverings

0DC1 In this section we discuss unbounded cohomological descent. The results themselves will be immediate consequences of our results on bounded cohomological descent in the previous sections and Cohomology on Sites, Lemmas 21.28.6 and/or 21.28.7; the real work lies in setting up notation and choosing appropriate assumptions. Our discussion is motivated by the discussion in [**LO08a**] although the details are a good bit different.

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given for every object U of $\mathcal{C}$ a weak Serre subcategory $\mathcal{A}_U \subset Mod(\mathcal{O}_U)$ satisfying the following properties

0DC2 (1) given a morphism $U \to V$ of $\mathcal{C}$ the restriction functor $Mod(\mathcal{O}_V) \to Mod(\mathcal{O}_U)$ sends $\mathcal{A}_V$ into $\mathcal{A}_U$,

0DC3 (2) given a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ an object $\mathcal{F}$ of $Mod(\mathcal{O}_U)$ is in $\mathcal{A}_U$ if and only if the restriction of $\mathcal{F}$ to $\mathcal{C}/U_i$ is in $\mathcal{A}_{U_i}$ for all $i \in I$.

0DC4 (3) there exists a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that
   (a) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and

(b) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \mathrm{Cov}_V,\ p > d_V,\ \text{and } \mathcal{F} \in \mathrm{Ob}(\mathcal{A}_V)$$

Note that we require this to be true for $\mathcal{F}$ in $\mathcal{A}_V$ and not just for "global" objects (and thus it is stronger than the condition imposed in Cohomology on Sites, Situation 21.25.1). In this situation, there is a weak Serre subcategory $\mathcal{A} \subset \mathit{Mod}(\mathcal{O}_\mathcal{C})$ consisting of objects whose restriction to $\mathcal{C}/U$ is in $\mathcal{A}_U$ for all $U \in \mathrm{Ob}(\mathcal{C})$. Moreover, there are derived categories $D_\mathcal{A}(\mathcal{O}_\mathcal{C})$ and $D_{\mathcal{A}_U}(\mathcal{O}_U)$ and the restriction functors send these into each other.

0DC5 **Example 85.23.1.** Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{C} = X_{spaces,étale}$ be the étale site on the category of algebraic spaces étale over X, see Properties of Spaces, Definition 66.18.2. Denote $\mathcal{O}_\mathcal{C}$ the structure sheaf, i.e., the sheaf given by the rule $U \mapsto \Gamma(U, \mathcal{O}_U)$. Denote $\mathcal{A}_U$ the category of quasi-coherent $\mathcal{O}_U$-modules. Let $\mathcal{B} = \mathrm{Ob}(\mathcal{C})$ and for $V \in \mathcal{B}$ set $d_V = 0$ and let Cov_V denote the coverings $\{V_i \to V\}$ with V_i affine for all i. Then the assumptions (1), (2), (3) are satisfied. See Properties of Spaces, Lemmas 66.29.2 and 66.29.7 for properties (1) and (2) and the vanishing in (3) follows from Cohomology of Schemes, Lemma 30.2.2 and the discussion in Cohomology of Spaces, Section 69.3.

0DC6 **Example 85.23.2.** Let S be one of the following types of schemes

(1) the spectrum of a finite field,
(2) the spectrum of a separably closed field,
(3) the spectrum of a strictly henselian Noetherian local ring,
(4) the spectrum of a henselian Noetherian local ring with finite residue field,
(5) add more here.

Let Λ be a finite ring whose order is invertible on S. Let $\mathcal{C} \subset (Sch/S)_{étale}$ be the full subcategory consisting of schemes locally of finite type over S endowed with the étale topology. Let $\mathcal{O}_\mathcal{C} = \underline{\Lambda}$ be the constant sheaf. Set $\mathcal{A}_U = \mathit{Mod}(\mathcal{O}_U)$, in other words, we consider all étale sheaves of Λ-modules. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be the set of quasi-compact objects. For $V \in \mathcal{B}$ set

$$d_V = 1 + 2\dim(S) + \sup_{v \in V}(\mathrm{trdeg}_{\kappa(s)}(\kappa(v)) + 2\dim \mathcal{O}_{V,v})$$

and let Cov_V denote the étale coverings $\{V_i \to V\}$ with V_i quasi-compact for all i. Our choice of bound d_V comes from Gabber's theorem on cohomological dimension. To see that condition (3) holds with this choice, use [**ILO14**, Exposé VIII-A, Corollary 1.2 and Lemma 2.2] plus elementary arguments on cohomological dimensions of fields. We add 1 to the formula because our list contains cases where we allow S to have finite residue field. We will come back to this example later (insert future reference).

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset \mathit{Mod}(\mathcal{O}_U)$ satisfying condition (1). Then

(1) given a semi-representable object $K = \{U_i\}_{i \in I}$ we get a weak Serre subcategory $\mathcal{A}_K \subset \mathit{Mod}(\mathcal{O}_K)$ by taking $\prod \mathcal{A}_{U_i} \subset \prod \mathit{Mod}(\mathcal{O}_{U_i}) = \mathit{Mod}(\mathcal{O}_K)$, and
(2) given a morphism of semi-representable objects $f : K \to L$ the pullback map $f^* : \mathit{Mod}(\mathcal{O}_L) \to \mathit{Mod}(\mathcal{O}_L)$ sends $\mathcal{A}_L$ into $\mathcal{A}_K$.

See Remark 85.15.6 for notation and explanation. In particular, given a simplicial semi-representable object K it is unambiguous to say what it means for an object $\mathcal{F}$ of $Mod(\mathcal{O})$ as in Remark 85.16.5 to have restrictions $\mathcal{F}_n$ in $\mathcal{A}_{K_n}$ for all n.

0DC7 **Lemma 85.23.3.** *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset Mod(\mathcal{O}_U)$ satisfying conditions (1), (2), and (3) above. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 85.16.5. Let $\mathcal{A}_{total} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules $\mathcal{F}$ whose restriction $\mathcal{F}_n$ is in $\mathcal{A}_{K_n}$ for all n (as defined above). Then the functor La^* defines an equivalence*

$$D_\mathcal{A}(\mathcal{O}_\mathcal{C}) \longrightarrow D_{\mathcal{A}_{total}}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. The cartesian $\mathcal{O}$-modules form a weak Serre subcategory by Lemma 85.12.6 (the required hypotheses hold by the discussion in Remark 85.16.5). Since the restriction functor $g_n^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_n)$ are exact, it follows that $\mathcal{A}_{total}$ is a weak Serre subcategory.

Let us show that $a^* : \mathcal{A} \to \mathcal{A}_{total}$ is an equivalence of categories with inverse given by La_*. We already know that $La_*a^*\mathcal{F} = \mathcal{F}$ by the bounded version (Lemma 85.18.4). It is clear that $a^*\mathcal{F}$ is in $\mathcal{A}_{total}$ for $\mathcal{F}$ in $\mathcal{A}$. Conversely, assume that $\mathcal{G} \in \mathcal{A}_{total}$. Because $\mathcal{G}$ is cartesian we see that $\mathcal{G} = a^*\mathcal{F}$ for some $\mathcal{O}_\mathcal{C}$-module $\mathcal{F}$ by Lemma 85.18.1. We want to show that $\mathcal{F}$ is in $\mathcal{A}$. Take $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that the restriction of $\mathcal{F}$ to $\mathcal{C}/U$ is in $\mathcal{A}_U$. As usual, write $K_0 = \{U_{0,i}\}_{i \in I_0}$. Since K is a hypercovering, the map $\coprod_{i \in I_0} h_{U_{0,i}} \to *$ becomes surjective after sheafification. This implies there is a covering $\{U_j \to U\}_{j \in J}$ and a map $\tau : J \to I_0$ and for each $j \in J$ a morphism $\varphi_j : U_j \to U_{0,\tau(j)}$. Since $\mathcal{G}_0 = a_0^*\mathcal{F}$ we find that the restriction of $\mathcal{F}$ to $\mathcal{C}/U_j$ is equal to the restriction of the $\tau(j)$th component of $\mathcal{G}_0$ to $\mathcal{C}/U_j$ via the morphism $\varphi_j : U_j \to U_{0,\tau(i)}$. Hence by (1) we find that $\mathcal{F}|_{\mathcal{C}/U_j}$ is in $\mathcal{A}_{U_j}$ and in turn by (2) we find that $\mathcal{F}|_{\mathcal{C}/U}$ is in $\mathcal{A}_U$.

In particular the statement of the lemma makes sense. The lemma now follows from Cohomology on Sites, Lemma 21.28.6. Assumption (1) is clear (see Remark 85.16.5). Assumptions (2) and (3) we proved in the preceding paragraph. Assumption (4) is immediate from (3). For assumption (5) let $\mathcal{B}_{total}$ be the set of objects $U/U_{n,i}$ of the site $(\mathcal{C}/K)_{total}$ such that $U \in \mathcal{B}$ where $\mathcal{B}$ is as in (3). Here we use the description of the site $(\mathcal{C}/K)_{total}$ given in Section 85.16. Moreover, we set $\mathrm{Cov}_{U/U_{n,i}}$ equal to Cov_U and $d_{U/U_{n,i}}$ equal d_U where Cov_U and d_U are given to us by (3). Then we claim that condition (5) holds with these choices. This follows immediately from Lemma 85.16.3 and the fact that $\mathcal{F} \in \mathcal{A}_{total}$ implies $\mathcal{F}_n \in \mathcal{A}_{K_n}$ and hence $\mathcal{F}_{n,i} \in \mathcal{A}_{U_{n,i}}$. (The reader who worries about the difference between cohomology of abelian sheaves versus cohomology of sheaves of modules may consult Cohomology on Sites, Lemma 21.12.4.) □

85.24. Glueing complexes

0DC8 This section is the continuation of Cohomology, Section 20.45. The goal is to prove a slight generalization of [**BBD82**, Theorem 3.2.4]. Our method will be a tiny bit different in that we use the material from Sections 85.13 and 85.14. We will also reprove the unbounded version as it is proved in [**LO08a**].

Advice to the reader: We suggest the reader first look at the statement of Lemma 85.24.5 as well as the second proof of this lemma.

Here is the situation we are interested in.

0DC9 **Situation 85.24.1.** Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. We are given

(1) a category $\mathcal{B}$ and a functor $u : \mathcal{B} \to \mathcal{C}$,
(2) an object E_U in $D(\mathcal{O}_{u(U)})$ for $U \in \mathrm{Ob}(\mathcal{B})$,
(3) an isomorphism $\rho_a : E_U|_{\mathcal{C}/u(V)} \to E_V$ in $D(\mathcal{O}_{u(V)})$ for $a : V \to U$ in $\mathcal{B}$

such that whenever we have composable arrows $b : W \to V$ and $a : V \to U$ of $\mathcal{B}$, then $\rho_{a \circ b} = \rho_b \circ \rho_a|_{\mathcal{C}/u(W)}$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a full subcategory such that every object of $\mathcal{C}$ has a covering whose members are objects of $\mathcal{B}$ (this is the case considered in [**BBD82**]). For us it is important to allow cases where this is not the case; the main alternative case is where we have a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ and $\mathcal{B}$ is a full subcategory of $\mathcal{D}$ such that every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$.

In Situation 85.24.1 a *solution* will be a pair (E, ρ_U) where E is an object of $D(\mathcal{O}_\mathcal{C})$ and $\rho_U : E|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \mathrm{Ob}(\mathcal{B})$ are isomorphisms such that we have $\rho_a \circ \rho_U|_{\mathcal{C}/u(V)} = \rho_V$ for $a : V \to U$ in $\mathcal{B}$.

0DCA **Lemma 85.24.2.** *In Situation 85.24.1. Assume negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero. Let L be a simplicial object of $SR(\mathcal{B})$. Consider the simplicial object $K = u(L)$ of $SR(\mathcal{C})$ and let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 85.16.5. There exists a cartesian object E of $D(\mathcal{O})$ such that writing $L_n = \{U_{n,i}\}_{i \in I_n}$ the restriction of E to $D(\mathcal{O}_{\mathcal{C}/u(U_{n,i})})$ is $E_{U_{n,i}}$ compatibly (see proof for details). Moreover, E is unique up to unique isomorphism.*

Proof. Recall that $Sh(\mathcal{C}/K_n) = \prod_{i \in I_n} Sh(\mathcal{C}/u(U_{n,i}))$ and similarly for the categories of modules. This product decomposition is also inherited by the derived categories of sheaves of modules. Moreover, this product decomposition is compatible with the morphisms in the simplicial semi-representable object K. See Section 85.15. Hence we can set $E_n = \prod_{i \in I_n} E_{U_{n,i}}$ ("formal" product) in $D(\mathcal{O}_n)$. Taking (formal) products of the maps ρ_a of Situation 85.24.1 we obtain isomorphisms $E_\varphi : f_\varphi^* E_n \to E_m$. The assumption about compostions of the maps ρ_a immediately implies that (E_n, E_φ) defines a simplicial system of the derived category of modules as in Definition 85.14.1. The vanishing of negative exts assumed in the lemma implies that $\mathrm{Hom}(E_n[t], E_n) = 0$ for $n \geq 0$ and $t > 0$. Thus by Lemma 85.14.7 we obtain E. Uniqueness up to unique isomorphism follows from Lemmas 85.14.5 and 85.14.6. □

0DCB **Lemma 85.24.3** (BBD glueing lemma)**.** *In Situation 85.24.1. Assume*

(1) *$\mathcal{C}$ has equalizers and fibre products,*
(2) *there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that*
   (a) *$\mathcal{D}$ has equalizers and fibre products and u commutes with them,*
   (b) *$\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,*
   (c) *every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,*
(3) *for all U in $\mathcal{B}$ all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and*
(4) *there exists a $t \in \mathbf{Z}$ such that $H^i(E_U) = 0$ for $i < t$ and $U \in \mathrm{Ob}(\mathcal{B})$.*

Then there exists a solution unique up to unique isomorphism.

Proof. By Hypercoverings, Lemma 25.12.3 there exists a hypercovering L for the site $\mathcal{D}$ such that $L_n = \{U_{n,i}\}_{i \in I_n}$ with $U_{i,n} \in \mathrm{Ob}(\mathcal{B})$. Set $K = u(L)$. Apply Lemma 85.24.2 to get a cartesian object E of $D(\mathcal{O})$ on the site $(\mathcal{C}/K)_{total}$ restricting to $E_{U_{n,i}}$ on $\mathcal{C}/u(U_{n,i})$ compatibly. The assumption on t implies that $E \in D^+(\mathcal{O})$. By Hypercoverings, Lemma 25.12.4 we see that K is a hypercovering too. By Lemma 85.18.4 we find that $E = a^*F$ for some F in $D^+(\mathcal{O}_\mathcal{C})$.

To prove that F is a solution we will use the construction of L_0 and L_1 given in the proof of Hypercoverings, Lemma 25.12.3. (This is a bit inelegant but there does not seem to be a completely straightforward way around it.)

Namely, we have $I_0 = \mathrm{Ob}(\mathcal{B})$ and so $L_0 = \{U\}_{U \in \mathrm{Ob}(\mathcal{B})}$. Hence the isomorphism $a^*F \to E$ restricted to the components $\mathcal{C}/u(U)$ of $\mathcal{C}/K_0$ defines isomorphisms $\rho_U : F|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \mathrm{Ob}(\mathcal{B})$ by our choice of E.

To prove that ρ_U satisfy the requirement of compatibility with the maps ρ_a of Situation 85.24.1 we use that I_1 contains the set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

and that for $i = (U, V, W, a, b)$ in Ω we have $U_{1,i} = U$. Moreover, the component maps $f_{\delta^1_0,i}$ and $f_{\delta^1_1,i}$ of the two morphisms $K_1 \to K_0$ are the morphisms

$$a : U \to V \quad \text{and} \quad b : U \to V$$

Hence the compatibility mentioned in Lemma 85.24.2 gives that

$$\rho_a \circ \rho_V|_{\mathcal{C}/u(U)} = \rho_U \quad \text{and} \quad \rho_b \circ \rho_W|_{\mathcal{C}/u(U)} = \rho_U$$

Taking $i = (U, V, U, a, \mathrm{id}_U) \in \Omega$ for example, we find that we have the desired compatibility. The uniqueness of F follows from the uniqueness of E in the previous lemma (small detail omitted). □

0DCC **Lemma 85.24.4** (Unbounded BBD glueing lemma)**.** *In Situation 85.24.1. Assume*

(1) *$\mathcal{C}$ has equalizers and fibre products,*
(2) *there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that*
   (a) *$\mathcal{D}$ has equalizers and fibre products and u commutes with them,*
   (b) *$\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,*
   (c) *every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,*
(3) *all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and*
(4) *there exist weak Serre subcategories $\mathcal{A}_U \subset Mod(\mathcal{O}_U)$ for all $U \in \mathrm{Ob}(\mathcal{C})$ satisfying conditions (1), (2), and (3),*
(5) *$E_U \in D_{\mathcal{A}_U}(\mathcal{O}_U)$.*

Then there exists a solution unique up to unique isomorphism.

Proof. The proof is **exactly** the same as the proof of Lemma 85.24.3. The only change is that E is an object of $D_{\mathcal{A}_{total}}(\mathcal{O})$ and hence we use Lemma 85.23.3 to obtain F with $E = a^*F$ instead of Lemma 85.18.4. □

Here is an example application of the general theory above.

0GMG **Lemma 85.24.5.** *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume $\mathcal{C}$ has fibre products. Let $\{U_i \to X\}_{i\in I}$ be a covering in $\mathcal{C}$. For $i \in I$ let E_i be an object of $D(\mathcal{O}_{U_i})$ and for $i, j \in I$ let*

$$\rho_{ij} : E_i|_{\mathcal{C}/U_{ij}} \longrightarrow E_j|_{\mathcal{C}/U_{ij}}$$

be an isomorphism in $D(\mathcal{O}_{U_{ij}})$ where $U_{ij} = U_i \times_X U_j$. Assume

1. *the ρ_{ij} satisfy the cocycle condition on $U_i \times_X U_j \times_X U_k$ for all $i, j, k \in I$,*
2. *$\mathcal{E}xt^p_{\mathcal{O}_{U_i}}(E_i, E_i) = 0$ for all $p < 0$ and $i \in I$, and*
3. *there exists a $t \in \mathbf{Z}$ such that $H^p(E_i) = 0$ for $p < t$ and all $i \in I$.*

Then there exists a unique pair (E, ρ_i) where E is an object of $D(\mathcal{O}_X)$ and $\rho_i : E|_{U_i} \to E_i$ are isomorphisms in $D(\mathcal{O}_{U_i})$ compatible with the ρ_{ij}.

Email of Martin Olsson dated Sep 9, 2021.

First proof. In this proof we deduce the lemma from the very general Lemma 85.24.3. We urge the reader to look at the second proof in stead.

We may replace $\mathcal{C}$ with $\mathcal{C}/X$. Thus we may and do assume X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of $U \in \mathrm{Ob}(\mathcal{C})$ such that there exists an $i(U) \in I$ and a morphism $a_U : U \to U_{i(U)}$. We denote $E_U = a_U^* E_{i(U)}$ in $D(\mathcal{O}_U)$ the pullback (restriction) of E_i via a_U. Given a morphism $a : U \to U'$ of $\mathcal{B}$ we obtain a morphism $(a_{U'} \circ a, a_U) : U \to U_{i(U')} \times_X U_{i(U)} = U_{i(U')i(U)}$ and hence an isomorphism

$$\rho_a : a^* E_{U'} = a^* a_{U'}^* E_{i(U')} \xrightarrow{(a_{U'} \circ a, a_U)^* \rho_{i(U')i(U)}} a_U^* E_{i(U)} = E_U$$

in $D(\mathcal{O}_U)$. The data $\mathcal{B}, E_U, \rho_a$ are as in Situation 85.24.1; the isomorphisms ρ_a satisfy the cocycle condition exactly because of condition (1) in the statement of the lemma (details omitted).

We are going to apply Lemma 85.24.3 with $\mathcal{B}$, E_U, ρ_a as above and with $\mathcal{D} = \mathcal{C}$ and $f : \mathcal{C} \to \mathcal{D}$ the identity morphism. Assumptions (1) and (2)(a) of Lemma 85.24.3 we have seen above. Assumption (2)(b) of Lemma 85.24.3 is clear. Assumption (2)(c) of Lemma 85.24.3 holds because $\{U_i \to X\}$ is a covering[5]. Assumption (3) of Lemma 85.24.3 holds because we have assumed the vanishing of all negative Ext sheaves of E_i which certainly implies that for any object U lying over U_i the negative self-Exts of $E_i|_U$ are zero. Assumption (4) of Lemma 85.24.3 holds because we have assumed the cohomology sheaves of each E_i are zero to the left of t.

We obtain a unique solution (E, ρ_U). Setting $\rho_i = \rho_{U_i}$ the lemma follows. □

Second proof. We sketch a more direct proof. Denote K the Čech hypercovering of X associated to the covering $\{U_i \to X\}_{i\in I}$, see Hypercoverings, Example 25.3.4. Thus for example $K_0 = \{U_i \to X\}_{i\in I}$ and $K_1 = \{U_i \times_X U_j \to X\}_{i,j\in I}$ and so on. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$, a, a_n be as in Remark 85.16.6. The objects E_i determine an object M_0 in $D(\mathcal{O}_0) = \prod D(\mathcal{O}_{U_i})$. Similarly, the isomorphisms ρ_{ij} determine an isomorphism

$$\alpha : L(f_{\delta^1_1})^* M_0 \longrightarrow L(f_{\delta^1_0})^* M_0$$

in $D(\mathcal{O}_1)$ satisfying the cocycle condition. By Lemma 85.14.3 we obtain a cartesian simplicial system (M_n) of the derived category. By the assumed vanishing of the

[5] In fact, it would suffice if the map $\coprod_{i\in I} h_{U_i} \to h_X$ becomes surjective on sheafification and the lemma holds in this case with the same proof.

negative Ext sheaves we see that the objects M_n have vanishing negative self-exts. Thus we find a cartesian object M of $D(\mathcal{O})$ whose associated simplicial system is isomorphic to (M_n) by Lemma 85.14.7. Since the cohomology sheaves of M are zero in degrees $< t$ we see that by Lemma 85.20.4 we have $M = La^*E$ for some E in $D(\mathcal{O}_X)$. The isomorphism $La^*E \to M$ restricted to $\mathcal{C}/U_i$ produces the isomorphisms ρ_i. We omit the verification of the compatibility with the isomorphisms ρ_{ij}. □

85.25. Proper hypercoverings in topology

09XA Let's work in the category LC of Hausdorff and locally quasi-compact topological spaces and continuous maps, see Cohomology on Sites, Section 21.31. Let X be an object of LC and let U be a simplicial object of LC. Assume we have an augmentation

$$a : U \to X$$

We say that U is a *proper hypercovering* of X if

(1) $U_0 \to X$ is a proper surjective map,
(2) $U_1 \to U_0 \times_X U_0$ is a proper surjective map,
(3) $U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}$ is a proper surjective map for $n \geq 1$.

The category LC has all finite limits, hence the coskeleta used in the formulation above exist.

Principle: Proper hypercoverings can be used to compute cohomology.

A key idea behind the proof of the principle is to find a topology on LC which is stronger than the usual one such that (a) a surjective proper map defines a covering, and (b) cohomology of usual sheaves with respect to this stronger topology agrees with the usual cohomology. Properties (a) and (b) hold for the qc topology, see Cohomology on Sites, Section 21.31. Once we have (a) and (b) we deduce the principle via the earlier work done in this chapter.

0DAF **Lemma 85.25.1.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{array}{ccc} Sh((LC_{qc}/U)_{total}) & \xrightarrow[h]{} & Sh(U_{Zar}) \\ \downarrow{\scriptstyle a_{qc}} & & \downarrow{\scriptstyle a} \\ Sh(LC_{qc}/X) & \xrightarrow{h_{-1}} & Sh(X) \end{array}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Lemma 85.2.8.

Proof. Write $Sh(X) = Sh(X_{Zar})$. Observe that both $(LC_{qc}/U)_{total}$ and U_{Zar} fall into case A of Situation 85.3.3. This is immediate from the construction of U_{Zar} in Section 85.2 and it follows from Lemma 85.21.5 for $(LC_{qc}/U)_{total}$. Next, consider the functors $U_{n,Zar} \to LC_{qc}/U_n$, $U \mapsto U/U_n$ and $X_{Zar} \to LC_{qc}/X$, $U \mapsto U/X$. We have seen that these define morphisms of sites in Cohomology on Sites, Section 21.31. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. □

0DAG **Lemma 85.25.2.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then*

$$a^{-1} : Sh(X) \to Sh(U_{Zar}) \quad \text{and} \quad a^{-1} : Ab(X) \to Ab(U_{Zar})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.25.1. By Cohomology on Sites, Lemma 21.31.6 the functor $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in LC_{qc} (as defined in Section 85.21) by Cohomology on Sites, Lemma 21.31.4. By Lemma 85.21.1 we see that a_{qc}^{-1} is fully faithful with quasi-inverse $a_{qc,*}$ and with essential image the cartesian sheaves on $(LC_{qc}/U)_{total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on U_{Zar}. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on LC_{qc}/U. Hence $h^{-1}\mathcal{G} = a_{qc}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on LC_{qc}/X. We compute

$$\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\mathrm{Eq}(\ a_{0,*}\mathcal{G}_0 \rightrightarrows a_{1,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ (h_{-1})^{-1}a_{0,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{qc,0,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{qc,1,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{qc,0,*}a_{qc,0}^{-1}\mathcal{H} \rightrightarrows a_{qc,1,*}a_{qc,1}^{-1}\mathcal{H}\ ) \\
&= a_{qc,*}a_{qc}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}$$

Here the first equality follows from Lemma 85.2.8, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from Cohomology on Sites, Lemma 21.31.8 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 85.4.2, and the sixth we've seen above. Since $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. □

09XS **Lemma 85.25.3.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then for $K \in D^+(X)$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. Consider the diagram of Lemma 85.25.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_n)$ by Cohomology on Sites, Lemma 21.31.11. Hence

Rh_*h^{-1} is the identity functor on $D^+(U_{Zar})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{qc,*}a_{qc}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 (U is a hypercovering of X in LC_{qc} by Cohomology on Sites, Lemma 21.31.4), and the last equality by the already used Cohomology on Sites, Lemma 21.31.11. □

09XC **Lemma 85.25.4.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If U is a proper hypercovering of X, then*

$$R\Gamma(X, K) = R\Gamma(U_{Zar}, a^{-1}K)$$

for $K \in D^+(X)$ where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. This follows from Lemma 85.25.3 because $R\Gamma(U_{Zar}, -) = R\Gamma(X, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DAH **Lemma 85.25.5.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{Zar})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X) \longrightarrow D^+_{\mathcal{A}}(U_{Zar})$$

with quasi-inverse Ra_* where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 85.2.8.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.25.2 and 85.25.3 and Cohomology on Sites, Lemma 21.28.5. □

09XB **Lemma 85.25.6.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_n$ be the pullback to U_n. If U is a proper hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.25.4 and 85.2.10. □

85.26. Simplicial schemes

09XT A *simplicial scheme* is a simplicial object in the category of schemes, see Simplicial, Definition 14.3.1. Recall that a simplicial scheme looks like

$$X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0$$

Here there are two morphisms $d_0^1, d_1^1 : X_1 \to X_0$ and a single morphism $s_0^0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d_0^1 \circ s_0^0 = \mathrm{id}_{X_0} = d_1^1 \circ s_0^0$, see Simplicial, Lemma 14.3.2. It is useful to think of $d_i^n : X_n \to X_{n-1}$ as the "projection forgetting the ith coordinate" and to think of $s_j^n : X_n \to X_{n+1}$ as the "diagonal map repeating the jth coordinate".

A *morphism of simplicial schemes* $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of schemes, see Simplicial, Definition 14.3.1. Thus h consists of morphisms of schemes $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d_j^n = d_j^n \circ h_n$ and $h_{n+1} \circ s_j^n = s_j^n \circ h_n$ whenever this makes sense.

An *augmentation* of a simplicial scheme X is a morphism of schemes $a_0 : X_0 \to S$ such that $a_0 \circ d_0^1 = a_0 \circ d_1^1$. See Simplicial, Section 14.20.

Let X be a simplicial scheme. The construction of Section 85.2 applied to the underlying simplicial topological space gives a site X_{Zar}. On the other hand, for every n we have the small Zariski site $X_{n,Zar}$ (Topologies, Definition 34.3.7) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites $f_\varphi = X(\varphi)_{small} : X_{n,Zar} \to X_{m,Zar}$, associated to the morphism of schemes $X(\varphi) : X_n \to X_m$ (Topologies, Lemma 34.3.17). This gives a simplicial object $\mathcal{C}$ in the category of sites. In Lemma 85.3.1 we constructed an associated site $\mathcal{C}_{total}$. Assigning to an open immersion its image defines an equivalence $\mathcal{C}_{total} \to X_{Zar}$ which identifies sheaves, i.e., $Sh(\mathcal{C}_{total}) = Sh(X_{Zar})$. The difference between $\mathcal{C}_{total}$ and X_{Zar} is similar to the difference between the small Zariski site S_{Zar} and the underlying topological space of S. We will silently identify these sites in what follows.

Let X_{Zar} be the site associated to a simplicial scheme X. There is a sheaf of rings $\mathcal{O}$ on X_{Zar} whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This follows from Lemma 85.2.2 or from Lemma 85.3.4. We will say $\mathcal{O}$ *is the structure sheaf of the simplicial scheme* X. At this point all the material developed for simplicial (ringed) sites applies, see Sections 85.3, 85.4, 85.5, 85.6, 85.8, 85.9, 85.10, 85.11, 85.12, 85.13, and 85.14.

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a *quasi-coherent $\mathcal{O}$-module on X_{Zar}*, see Modules on Sites, Definition 18.23.1. However, a quasi-coherent $\mathcal{O}$-module on X_{Zar} is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 85.12.10.

Let $h : X \to Y$ be a morphism of simplicial schemes. Either by Lemma 85.2.3 or by (the proof of) Lemma 85.5.2 we obtain a morphism of sites $h_{Zar} : X_{Zar} \to Y_{Zar}$. Recall that h_{Zar}^{-1} and $h_{Zar,*}$ have a simple description in terms of the components, see Lemma 85.2.4 or Lemma 85.5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{Zar}^\sharp : h_{Zar,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on Y_{Zar} given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed sites

$$h_{Zar} : (X_{Zar}, \mathcal{O}_X) \longrightarrow (Y_{Zar}, \mathcal{O}_Y)$$

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. Let S be a scheme and let $a_0 : X_0 \to S$ be an augmentation of X. Either by Lemma 85.2.8 or by Lemma 85.4.2 we obtain a corresponding morphism of topoi $a : Sh(X_{Zar}) \to Sh(S)$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on X_{Zar} with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_S \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_S \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_S \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 85.11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{Zar}), \mathcal{O}) \longrightarrow (Sh(S), \mathcal{O}_S)$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial schemes X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{array}{ccc} X_0 & \xrightarrow[h_0]{} & Y_0 \\ {\scriptstyle a_0}\downarrow & & \downarrow{\scriptstyle b_0} \\ X_{-1} & \xrightarrow{h_{-1}} & Y_{-1} \end{array}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{array}{ccc} (Sh(X_{Zar}), \mathcal{O}_X) & \xrightarrow[h_{Zar}]{} & (Sh(Y_{Zar}), \mathcal{O}_Y) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) & \xrightarrow{h_{-1}} & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{array}$$

85.27. Descent in terms of simplicial schemes

0248 Cartesian morphisms are defined as follows.

0249 **Definition 85.27.1.** Let $a : Y \to X$ be a morphism of simplicial schemes. We say a is *cartesian*, or that Y *is cartesian over* X, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{array}{ccc} Y_m & \xrightarrow[a]{} & X_m \\ {\scriptstyle Y(\varphi)}\downarrow & & \downarrow{\scriptstyle X(\varphi)} \\ Y_n & \xrightarrow{a} & X_n \end{array}$$

is a fibre square in the category of schemes.

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian simplicial schemes over a fixed simplicial scheme. In this lemma we denote $f^* : Sch/X \to Sch/Y$ the base change functor associated to a morphism of schemes $f : Y \to X$.

07TC **Lemma 85.27.2.** *Let X be a simplicial scheme. The category of simplicial schemes cartesian over X is equivalent to the category of pairs (V, φ) where V is a scheme over X_0 and*

$$\varphi : V \times_{X_0, d_1^1} X_1 \longrightarrow X_1 \times_{d_0^1, X_0} V$$

is an isomorphism over X_1 such that $(s_0^0)^*\varphi = id_V$ and such that

$$(d_1^2)^*\varphi = (d_0^2)^*\varphi \circ (d_2^2)^*\varphi$$

as morphisms of schemes over X_2.

Proof. The statement of the displayed equality makes sense because $d_1^1 \circ d_2^2 = d_1^1 \circ d_1^2$, $d_1^1 \circ d_0^2 = d_0^1 \circ d_2^2$, and $d_0^1 \circ d_0^2 = d_0^1 \circ d_1^2$ as morphisms $X_2 \to X_0$, see

Simplicial, Remark 14.3.3 hence we can picture these maps as follows

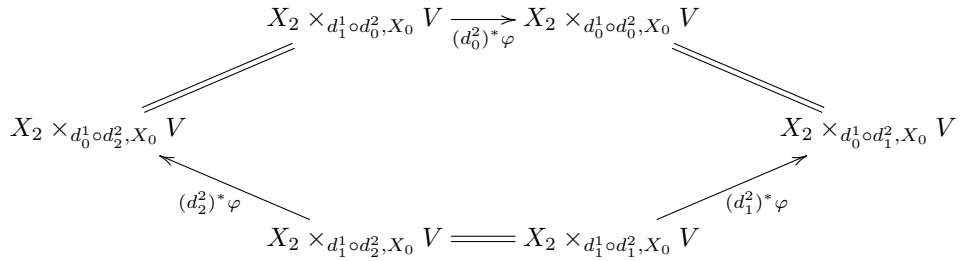

and the condition signifies the diagram is commutative. It is clear that given a simplicial scheme Y cartesian over X we can set $V = Y_0$ and φ equal to the composition

$$V \times_{X_0, d_1^1} X_1 = Y_0 \times_{X_0, d_1^1} X_1 = Y_1 = X_1 \times_{X_0, d_0^1} Y_0 = X_1 \times_{X_0, d_0^1} V$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 35.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair (V, φ) as in the lemma we set $Y_n = X_n \times_{X(\tau_n^n), X_0} V$. Then given $\beta : [n] \to [m]$ we define $V(\beta) : Y_m \to Y_n$ as the pullback by $X(\tau_{\beta(n)m}^m)$ of the map φ postcomposed by the projection $X_m \times_{X(\beta), X_n} Y_n \to Y_n$. This makes sense because

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d_1^1, X_0} V = X_m \times_{X(\tau_m^m), X_0} V = Y_m$$

and

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d_0^1, X_0} V = X_m \times_{X(\tau^m_{\beta(n)}), X_0} V = X_m \times_{X(\beta), X_n} Y_n.$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. □

024A **Definition 85.27.3.** Let $f : X \to S$ be a morphism of schemes. The *simplicial scheme associated to* f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 14.3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d_0^1 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d_1^1 is the other projection. The morphism s_0^0 is the diagonal morphism $X \to X \times_S X$.

024B **Lemma 85.27.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes. Set $V = Y_0$ considered as a scheme over X. The morphisms $d_0^1, d_1^1 : Y_1 \to Y_0$ and the morphism $\pi_1 : Y_1 \to X \times_S X$ induce isomorphisms*

$$V \times_S X \xleftarrow{(d_1^1, pr_1 \circ \pi_1)} Y_1 \xrightarrow{(pr_0 \circ \pi_1, d_0^1)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. This is a special case of (part of) Lemma 85.27.2 as the displayed equation of that lemma is equivalent to the cocycle condition of Descent, Definition 35.34.1. □

024C **Lemma 85.27.5.** *Let $f : X \to S$ be a morphism of schemes. The construction*

$$\begin{array}{c}\text{category of cartesian}\\ \text{schemes over } (X/S)_\bullet\end{array} \longrightarrow \begin{array}{c}\text{category of descent data}\\ \text{relative to } X/S\end{array}$$

of Lemma 85.27.4 is an equivalence of categories.

Proof. The functor from left to right is given in Lemma 85.27.4. Hence this is a special case of Lemma 85.27.2. □

We may reinterpret the pullback of Descent, Lemma 35.34.6 as follows. Suppose given a morphism of simplicial schemes $f : X' \to X$ and a cartesian morphism of simplicial schemes $Y \to X$. Then the fibre product (viewed as a "pullback")

$$f^*Y = Y \times_X X'$$

of simplicial schemes is a simplicial scheme cartesian over X'. Suppose given a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

We claim that the "pullback" $f_\bullet^*$ along the morphism $f_\bullet : (X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 85.27.5 with the pullback defined in terms of descent data in the aforementioned Descent, Lemma 35.34.6.

85.28. Quasi-coherent modules on simplicial schemes

07TE

07TI **Lemma 85.28.1.** *Let $f : V \to U$ be a morphism of simplicial schemes. Given a quasi-coherent module $\mathcal{F}$ on U_{Zar} the pullback $f^*\mathcal{F}$ is a quasi-coherent module on V_{Zar}.*

Proof. Recall that $\mathcal{F}$ is cartesian with $\mathcal{F}_n$ quasi-coherent, see Lemma 85.12.10. By Lemma 85.2.4 we see that $(f^*\mathcal{F})_n = f_n^*\mathcal{F}_n$ (some details omitted). Hence $(f^*\mathcal{F})_n$ is quasi-coherent. The same fact and the cartesian property for $\mathcal{F}$ imply the cartesian property for $f^*\mathcal{F}$. Thus $\mathcal{F}$ is quasi-coherent by Lemma 85.12.10 again. □

07TJ **Lemma 85.28.2.** *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d_j^n : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. For a quasi-coherent module $\mathcal{G}$ on V_{Zar} the pushforward $f_*\mathcal{G}$ is a quasi-coherent module on U_{Zar}.*

Proof. If $\mathcal{F} = f_*\mathcal{G}$, then $\mathcal{F}_n = f_{n,*}\mathcal{G}_n$ by Lemma 85.2.4. The maps $\mathcal{F}(\varphi)$ are defined using the base change maps, see Cohomology, Section 20.17. The sheaves $\mathcal{F}_n$ are quasi-coherent by Schemes, Lemma 26.24.1 and the fact that $\mathcal{G}_n$ is quasi-coherent by Lemma 85.12.10. The base change maps along the degeneracies d_j^n are isomorphisms by Cohomology of Schemes, Lemma 30.5.2 and the fact that $\mathcal{G}$ is cartesian by Lemma 85.12.10. Hence $\mathcal{F}$ is cartesian by Lemma 85.12.2. Thus $\mathcal{F}$ is quasi-coherent by Lemma 85.12.10. □

07TK **Lemma 85.28.3.** *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. Then f^* and f_* form an adjoint pair of functors between the categories of quasi-coherent modules on U_{Zar} and V_{Zar}.*

Proof. We have seen in Lemmas 85.28.1 and 85.28.2 that the statement makes sense. The adjointness property follows immediately from the fact that each f_n^* is adjoint to $f_{n,*}$. □

07TL **Lemma 85.28.4.** *Let $f : X \to S$ be a morphism of schemes which has a section[6]. Let $(X/S)_\bullet$ be the simplicial scheme associated to $X \to S$, see Definition 85.27.3. Then pullback defines an equivalence between the category of quasi-coherent $\mathcal{O}_S$-modules and the category of quasi-coherent modules on $((X/S)_\bullet)_{Zar}$.*

Proof. Let $\sigma : S \to X$ be a section of f. Let $(\mathcal{F}, \alpha)$ be a pair as in Lemma 85.12.5. Set $\mathcal{G} = \sigma^*\mathcal{F}$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow[(\sigma\circ f,1)]{} & X \times_S X & \xrightarrow[\mathrm{pr}_1]{} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_0} & & \\ S & \xrightarrow{\sigma} & X & & \end{array}$$

Note that $\mathrm{pr}_0 = d^1_1$ and $\mathrm{pr}_1 = d^1_0$. Hence we see that $(\sigma \circ f, 1)^*\alpha$ defines an isomorphism

$$f^*\mathcal{G} = (\sigma \circ f, 1)^*\mathrm{pr}_0^*\mathcal{F} \longrightarrow (\sigma \circ f, 1)^*\mathrm{pr}_1^*\mathcal{F} = \mathcal{F}$$

We omit the verification that this isomorphism is compatible with α and the canonical isomorphism $\mathrm{pr}_0^*f^*\mathcal{G} \to \mathrm{pr}_1^*f^*\mathcal{G}$. □

85.29. Groupoids and simplicial schemes

07TM Given a groupoid in schemes we can build a simplicial scheme. It will turn out that the category of quasi-coherent sheaves on a groupoid is equivalent to the category of cartesian quasi-coherent sheaves on the associated simplicial scheme.

07TN **Lemma 85.29.1.** *Let (U, R, s, t, c, e, i) be a groupoid scheme over S. There exists a simplicial scheme X over S with the following properties*

(1) *$X_0 = U$, $X_1 = R$, $X_2 = R \times_{s,U,t} R$,*
(2) *$s^0_0 = e : X_0 \to X_1$,*
(3) *$d^1_0 = s : X_1 \to X_0$, $d^1_1 = t : X_1 \to X_0$,*
(4) *$s^1_0 = (e \circ t, 1) : X_1 \to X_2$, $s^1_1 = (1, e \circ t) : X_1 \to X_2$,*
(5) *$d^2_0 = pr_1 : X_2 \to X_1$, $d^2_1 = c : X_2 \to X_1$, $d^2_2 = pr_0$, and*
(6) *$X = cosk_2 sk_2 X$.*

For all n we have $X_n = R \times_{s,U,t} \ldots \times_{s,U,t} R$ with n factors. The map $d^n_j : X_n \to X_{n-1}$ is given on functors of points by

$$(r_1, \ldots, r_n) \longmapsto (r_1, \ldots, c(r_j, r_{j+1}), \ldots, r_n)$$

for $1 \leq j \leq n-1$ whereas $d^n_0(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$ and $d^n_n(r_1, \ldots, r_n) = (r_1, \ldots, r_{n-1})$.

[6] In fact, it would be enough to assume that f has fpqc locally on S a section, since we have descent of quasi-coherent modules by Descent, Section 35.5.

Proof. We only have to verify that the rules prescribed in (1), (2), (3), (4), (5) define a 2-truncated simplicial scheme U' over S, since then (6) allows us to set $X = \mathrm{cosk}_2 U'$, see Simplicial, Lemma 14.19.2. Using the functor of points approach, all we have to verify is that if $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$ is a groupoid, then

$$\begin{array}{ccc} \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} & & \\ {\scriptstyle \mathrm{pr}_1}\downarrow \; \uparrow{\scriptstyle 1,e} & {\scriptstyle c}\downarrow \quad \uparrow{\scriptstyle e,1} & \downarrow{\scriptstyle \mathrm{pr}_0} \\ \mathrm{Arrows} & & \\ {\scriptstyle s}\downarrow \quad \uparrow{\scriptstyle e} & \downarrow{\scriptstyle t} & \\ \mathrm{Ob} & & \end{array}$$

is a 2-truncated simplicial set. We omit the details.

Finally, the description of X_n for $n > 2$ follows by induction from the description of X_0, X_1, X_2, and Simplicial, Remark 14.19.9 and Lemma 14.19.6. Alternately, one shows that cosk_2 applied to the 2-truncated simplicial set displayed above gives a simplicial set whose nth term equals $\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \ldots \times_{s,\mathrm{Ob},t} \mathrm{Arrows}$ with n factors and degeneracy maps as given in the lemma. Some details omitted. □

07TP **Lemma 85.29.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let X be the simplicial scheme over S constructed in Lemma 85.29.1. Then the category of quasi-coherent modules on (U, R, s, t, c) is equivalent to the category of quasi-coherent modules on X_{Zar}.*

Proof. This is clear from Lemmas 85.12.10 and 85.12.5 and Groupoids, Definition 39.14.1. □

In the following lemma we will use the concept of a cartesian morphism $V \to U$ of simplicial schemes as defined in Definition 85.27.1.

07TQ **Lemma 85.29.3.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Let X be the simplicial scheme over S constructed in Lemma 85.29.1. Let $(R/U)_\bullet$ be the simplicial scheme associated to $s : R \to U$, see Definition 85.27.3. There exists a cartesian morphism $t_\bullet : (R/U)_\bullet \to X$ of simplicial schemes with low degree morphisms given by*

$$\begin{array}{ccccc} R \times_{s,U,s} R \times_{s,U,s} R & \overset{pr_{12}}{\underset{pr_{01}}{\overset{pr_{02}}{\rightrightarrows}}} & R \times_{s,U,s} R & \overset{pr_1}{\underset{pr_0}{\rightrightarrows}} & R \\ \downarrow{\scriptstyle (r_0,r_1,r_2)\mapsto(r_0\circ r_1^{-1}, r_1\circ r_2^{-1})} & & \downarrow{\scriptstyle (r_0,r_1)\mapsto r_0\circ r_1^{-1}} & & \downarrow{\scriptstyle t} \\ R \times_{s,U,t} R & \overset{pr_1}{\underset{pr_0}{\overset{c}{\rightrightarrows}}} & R & \overset{s}{\underset{t}{\rightrightarrows}} & U \end{array}$$

Proof. For arbitrary n we define $(R/U)_\bullet \to X_n$ by the rule

$$(r_0, \ldots, r_n) \longrightarrow (r_0 \circ r_1^{-1}, \ldots, r_{n-1} \circ r_n^{-1})$$

Compatibility with degeneracy maps is clear from the description of the degeneracies in Lemma 85.29.1. We omit the verification that the maps respect the morphisms s_j^n. Groupoids, Lemma 39.13.5 (with the roles of s and t reversed)

shows that the two right squares are cartesian. In exactly the same manner one shows all the other squares are cartesian too. Hence the morphism is cartesian. □

85.30. Descent data give equivalence relations

024D In Section 85.27 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

024E **Lemma 85.30.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes, see Definitions 85.27.1 and 85.27.3. Then the morphism*

$$j = (d_1^1, d_0^1) : Y_1 \to Y_0 \times_S Y_0$$

defines an equivalence relation on Y_0 over S, see Groupoids, Definition 39.3.1.

Proof. Note that j is a monomorphism. Namely the composition $Y_1 \to Y_0 \times_S Y_0 \to Y_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d_2^2, d_0^2) : Y_2 \to Y_1 \times_{d_0^1, Y_0, d_1^1} Y_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 14.3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S Y_0 \times_S X$ because π is cartesian. Details omitted.

As in Groupoids, Definition 39.3.1 we denote $t = \mathrm{pr}_0 \circ j = d_1^1$ and $s = \mathrm{pr}_1 \circ j = d_0^1$. The isomorphism above, combined with the morphism $d_1^2 : Y_2 \to Y_1$ give us a composition morphism

$$c : Y_1 \times_{s, Y_0, t} Y_1 \longrightarrow Y_1$$

over $Y_0 \times_S Y_0$. This immediately implies that for any scheme T/S the relation $Y_1(T) \subset Y_0(T) \times Y_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d_2^2, d_1^2) : Y_2 \to Y_1 \times_{d_1^1, Y_0, d_1^1} Y_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 14.3.3. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = Y_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in Y_0(T)$ be synonymous with $(a, b) \in Y_1(T)$. The isomorphism (d_2^2, d_1^2) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

85.31. An example case

024F In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

024G **Lemma 85.31.1.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. For $y \in Y_0$ a point define*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d_1^1(y_1) = y, d_0^1(y_1) = y'\}$$

as a subset of Y_0. Then $y \in T_y$ and $T_y \cap T_{y'} \neq \emptyset \Rightarrow T_y = T_{y'}$.

Proof. Combine Lemma 85.30.1 and Groupoids, Lemma 39.3.4. □

024H **Lemma 85.31.2.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. Let $y \in Y_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d_1^1(y_1) = y, d_0^1(y_1) = y'\}$$

is a quasi-compact subset of Y_0.

Proof. Let F_y be the scheme theoretic fibre of $d_1^1 : Y_1 \to Y_0$ at y. Then we see that T_y is the image of the morphism

$$\begin{array}{ccccc} F_y & \longrightarrow & Y_1 & \xrightarrow{d_0^1} & Y_0 \\ \downarrow & & \downarrow{\scriptstyle d_1^1} & & \\ y & \longrightarrow & Y_0 & & \end{array}$$

Note that F_y is quasi-compact. This proves the lemma. □

024I **Lemma 85.31.3.** *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. Let $Y \to (X/S)_\bullet$ be the cartesian morphism of simplicial schemes corresponding to (V, φ) by Lemma 85.27.5. Observe that $Y_0 = V$. Write $V = \coprod_{i \in I} \mathrm{Spec}(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $\mathrm{Spec}(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 85.31.1 above. By Lemmas 85.31.1 and 85.31.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\bar{i} \in \overline{I}} V_{\bar{i}}$ with $V_{\bar{i}} = \coprod_{i \in \bar{i}} \mathrm{Spec}(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\bar{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\bar{i}}$. In other words, we get descent data $(V_{\bar{i}}, \varphi_{\bar{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\bar{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\bar{i}} \to X$ is affine, see Morphisms, Lemma 29.11.14. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\bar{i}}, \varphi_{\bar{i}})$ are effective by Descent, Lemma 35.37.1. □

To be sure, the lemma above has very limited applicability!

85.32. Simplicial algebraic spaces

0DE7 Let S be a scheme. A *simplicial algebraic space* is a simplicial object in the category of algebraic spaces over S, see Simplicial, Definition 14.3.1. Recall that a simplicial algebraic space looks like

$$X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0$$

Here there are two morphisms $d_0^1, d_1^1 : X_1 \to X_0$ and a single morphism $s_0^0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d_0^1 \circ s_0^0 = \mathrm{id}_{X_0} = d_1^1 \circ s_0^0$, see Simplicial, Lemma 14.3.2. It is useful to think of $d_i^n : X_n \to X_{n-1}$ as the "projection forgetting the ith coordinate" and to think of $s_j^n : X_n \to X_{n+1}$ as the "diagonal map repeating the jth coordinate".

A *morphism of simplicial algebraic spaces* $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of algebraic spaces over S, see Simplicial, Definition 14.3.1. Thus h consists of morphisms of algebraic spaces $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d_j^n = d_j^n \circ h_n$ and $h_{n+1} \circ s_j^n = s_j^n \circ h_n$ whenever this makes sense.

An *augmentation* $a : X \to X_{-1}$ of a simplicial algebraic space X is given by a morphism of algebraic spaces $a_0 : X_0 \to X_{-1}$ such that $a_0 \circ d_0^1 = a_0 \circ d_1^1$. See Simplicial, Section 14.20. In this situation we always indicate $a_n : X_n \to X_{-1}$ the induced morphisms for $n \geq 0$.

Let X be a simplicial algebraic space. For every n we have the site $X_{n,spaces,étale}$ (Properties of Spaces, Definition 66.18.2) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites

$$f_\varphi = X(\varphi)_{spaces,étale} : X_{n,spaces,étale} \to X_{m,spaces,étale},$$

associated to the morphism of algebraic spaces $X(\varphi) : X_n \to X_m$ (Properties of Spaces, Lemma 66.18.8). This gives a simplicial object in the category of sites. In Lemma 85.3.1 we constructed an associated site which we denote $X_{spaces,étale}$. An object of the site $X_{spaces,étale}$ is a an algebraic space U étale over X_n for some n and a morphism $(\varphi, f) : U/X_n \to V/X_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : U \to V$ of algebraic spaces such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_\varphi} & X_m \end{array}$$

is commutative. Consider the full subcategories

$$X_{affine,étale} \subset X_{étale} \subset X_{spaces,étale}$$

whose objects are U/X_n with U affine, respectively a scheme. Endowing these categories with their natural topologies (see Properties of Spaces, Lemma 66.18.6, Definition 66.18.1, and Lemma 66.18.3) these inclusion functors define equivalences of topoi

$$Sh(X_{affine,étale}) = Sh(X_{étale}) = Sh(X_{spaces,étale})$$

In the following we will silently identify these topoi. We will say that $X_{étale}$ is the *small étale site of* X and its topos is the *small étale topos of* X.

Let $X_{étale}$ be the small étale site of a simplicial algebraic space X. There is a sheaf of rings $\mathcal{O}$ on $X_{étale}$ whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This

follows from Lemma 85.3.4. We will say $\mathcal{O}$ *is the structure sheaf of the simplicial algebraic space* X. At this point all the material developed for simplicial (ringed) sites applies, see Sections 85.3, 85.4, 85.5, 85.6, 85.8, 85.9, 85.10, 85.11, 85.12, 85.13, and 85.14.

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a *quasi-coherent $\mathcal{O}$-module on $X_{étale}$*, see Modules on Sites, Definition 18.23.1. However, a quasi-coherent $\mathcal{O}$-module on $X_{étale}$ is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 85.12.10.

Let $h : X \to Y$ be a morphism of simplicial algebraic spaces over S. By Lemma 85.5.2 applied to the morphisms of sites $(h_n)_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ (Properties of Spaces, Lemma 66.18.8) we obtain a morphism of small étale topoi $h_{étale} : Sh(X_{étale}) \to Sh(Y_{étale})$. Recall that $h_{étale}^{-1}$ and $h_{étale,*}$ have a simple description in terms of the components, see Lemma 85.5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{étale}^\sharp : h_{étale,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on $Y_{étale}$ given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed topoi

$$h_{étale} : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. Let X_{-1} be an algebraic space over S and let $a_0 : X_0 \to X_{-1}$ be an augmentation of X. By Lemma 85.4.2 applied to the morphism of sites $(a_0)_{spaces,étale} : X_{0,spaces,étale} \to X_{-1,spaces,étale}$ we obtain a corresponding morphism of topoi $a : Sh(X_{étale}) \to Sh(X_{-1,étale})$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on $X_{étale}$ with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_{X_{-1}} \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 85.11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{étale}), \mathcal{O}) \longrightarrow (Sh(X_{-1}), \mathcal{O}_{X_{-1}})$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial algebraic spaces X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{array}{ccc} X_0 & \xrightarrow[h_0]{} & Y_0 \\ {\scriptstyle a_0}\downarrow & & \downarrow{\scriptstyle b_0} \\ X_{-1} & \xrightarrow{h_{-1}} & Y_{-1} \end{array}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{array}{ccc} (Sh(X_{étale}), \mathcal{O}_X) & \xrightarrow[h_{étale}]{} & (Sh(Y_{étale}), \mathcal{O}_Y) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) & \xrightarrow{h_{-1}} & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{array}$$

85.33. Fppf hypercoverings of algebraic spaces

0DH4 This section is the analogue of Section 85.25 for the case of algebraic spaces and fppf hypercoverings. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 84.6 with references to Étale Cohomology, Section 59.100.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 85.32. We say that U is an *fppf hypercovering* of X if

(1) $U_0 \to X$ is flat, locally of finite presentation, and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is flat, locally of finite presentation, and surjective,
(3) $U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}$ is flat, locally of finite presentation, and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

Principle: Fppf hypercoverings can be used to compute étale cohomology.

The key idea behind the proof of the principle is to compare the fppf and étale topologies on the category $Spaces/S$. Namely, the fppf topology is stronger than the étale topology and we have (a) a flat, locally finitely presented, surjective map defines an fppf covering, and (b) fppf cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 84.6.

0DH5 **Lemma 85.33.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{array}{ccc} Sh((Spaces/U)_{fppf,total}) & \xrightarrow{h} & Sh(U_{étale}) \\ \downarrow{\scriptstyle a_{fppf}} & & \downarrow{\scriptstyle a} \\ Sh((Spaces/X)_{fppf}) & \xrightarrow{h_{-1}} & Sh(X_{étale}) \end{array}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Section 85.32.

Proof. The notation $(Spaces/U)_{fppf,total}$ indicates that we are using the construction of Section 85.21 for the site $(Spaces/S)_{fppf}$ and the simplicial object U of this site[7]. We will use the sites $X_{spaces,étale}$ and $U_{spaces,étale}$ for the topoi on the right hand side; this is permissible see discussion in Section 85.32.

Observe that both $(Spaces/U)_{fppf,total}$ and $U_{spaces,étale}$ fall into case A of Situation 85.3.3. This is immediate from the construction of $U_{étale}$ in Section 85.32 and it follows from Lemma 85.21.5 for $(Spaces/U)_{fppf,total}$. Next, consider the functors $U_{n,spaces,étale} \to (Spaces/U_n)_{fppf}$, $U \mapsto U/U_n$ and $X_{spaces,étale} \to (Spaces/X)_{fppf}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 84.6 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and

[7] We could also use the étale topology and this would be denoted $(Spaces/U)_{étale,total}$.

$a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. □

0DH6 **Lemma 85.33.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$a^{-1} : Sh(X_{étale}) \to Sh(U_{étale}) \quad and \quad a^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.33.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.6. Thus it follows from More on Cohomology of Spaces, Lemma 84.6.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(Spaces/S)_{fppf}$ as defined in Section 85.21. By Lemma 85.21.1 we see that a_{fppf}^{-1} is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves on $(Spaces/U)_{fppf,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{étale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(Spaces/U)_{fppf,total}$. Hence $h^{-1}\mathcal{G} = a_{fppf}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(Spaces/X)_{fppf}$. In particular we find that $h_0^{-1}\mathcal{G}_0 = (a_{0,big,fppf})^{-1}\mathcal{H}$. Recalling that $h_0 = a_{U_0}$ and that $U_0 \to X$ is flat, locally of finite presentation, and surjective, we find from More on Cohomology of Spaces, Lemma 84.6.7 that there exists a sheaf $\mathcal{F}$ on $X_{étale}$ and isomorphism $\mathcal{H} = (h_{-1})^{-1}\mathcal{F}$. Since $a_{fppf}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}\mathcal{F}$. By fully faithfulness of h^{-1} we conclude that $a^{-1}\mathcal{F} \cong \mathcal{G}$.

Fix an isomorphism $\theta : a^{-1}\mathcal{F} \to \mathcal{G}$. To finish the proof we have to show $\mathcal{G} = a^{-1}a_*\mathcal{G}$ (in order to show that the quasi-inverse is given by a_*; everything else has been proven above). Because a^{-1} is fully faithful we have $\mathrm{id} \cong a_*a^{-1}$ by Categories, Lemma 4.24.4. Thus $\mathcal{F} \cong a_*a^{-1}\mathcal{F}$ and $a_*\theta : a_*a^{-1}\mathcal{F} \to a_*\mathcal{G}$ combine to an isomorphism $\mathcal{F} \to a_*\mathcal{G}$. Pulling back by a and precomposing by θ^{-1} we find the desired isomorphism. □

0DH7 **Lemma 85.33.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $K \in D^+(X_{étale})$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.33.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,étale})$ by More on Cohomology of Spaces, Lemma 84.6.2.

Hence Rh_*h^{-1} is the identity functor on $D^+(U_{étale})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 as U is a hypercovering of X in $(Spaces/S)_{fppf}$, and the last equality by the already used More on Cohomology of Spaces, Lemma 84.6.2. □

0DH8 **Lemma 85.33.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$R\Gamma(X_{étale}, K) = R\Gamma(U_{étale}, a^{-1}K)$$

for $K \in D^+(X_{étale})$. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.33.3 because $R\Gamma(U_{étale}, -) = R\Gamma(X_{étale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DH9 **Lemma 85.33.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{étale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is an fppf hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X_{étale}) \longrightarrow D^+_{\mathcal{A}}(U_{étale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.33.2 and 85.33.3 and Cohomology on Sites, Lemma 21.28.5. □

0DHA **Lemma 85.33.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,étale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{étale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.33.4 and 85.8.3. □

85.34. Fppf hypercoverings of algebraic spaces: modules

0DHB We continue the discussion of (cohomological) descent for fppf hypercoverings started in Section 85.33 but in this section we discuss what happens for sheaves of modules. We mainly discuss quasi-coherent modules and it turns out that we can do unbounded cohomological descent for those.

0DHC **Lemma 85.34.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{array}{ccc} (Sh((Spaces/U)_{fppf,total}), \mathcal{O}_{big,total}) & \xrightarrow{h} & (Sh(U_{étale}), \mathcal{O}_U) \\ \downarrow{\scriptstyle a_{fppf}} & & \downarrow{\scriptstyle a} \\ (Sh((Spaces/X)_{fppf}), \mathcal{O}_{big}) & \xrightarrow{h_{-1}} & (Sh(X_{étale}), \mathcal{O}_X) \end{array}$$

of ringed topoi where the left vertical arrow is defined in Section 85.22 and the right vertical arrow is defined in Section 85.32.

Proof. For the underlying diagram of topoi we refer to the discussion in the proof of Lemma 85.33.1. The sheaf $\mathcal{O}_U$ is the structure sheaf of the simplicial algebraic space U as defined in Section 85.32. The sheaf $\mathcal{O}_X$ is the usual structure sheaf of the algebraic space X. The sheaves of rings $\mathcal{O}_{big,total}$ and $\mathcal{O}_{big}$ come from the structure sheaf on $(Spaces/S)_{fppf}$ in the manner explained in Section 85.22 which also constructs a_{fppf} as a morphism of ringed topoi. The component morphisms $h_n = a_{U_n}$ and $h_{-1} = a_X$ are morphisms of ringed topoi by More on Cohomology of Spaces, Section 84.7. Finally, since the continuous functor $u : U_{spaces,étale} \to (Spaces/U)_{fppf,total}$ used to define h[8] is given by $V/U_n \mapsto V/U_n$ we see that $h_*\mathcal{O}_{big,total} = \mathcal{O}_U$ which is how we endow h with the structure of a morphism of ringed simplicial sites as in Remark 85.7.1. Then we obtain h as a morphism of ringed topoi by Lemma 85.7.2. Please observe that the morphisms h_n indeed agree with the morphisms a_{U_n} described above. We omit the verification that the diagram is commutative (as a diagram of ringed topoi – we already know it is commutative as a diagram of topoi). □

0DHD **Lemma 85.34.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$a^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$$

is an equivalence fully faithful with quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.34.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.7. Thus it follows from More on Cohomology of Spaces, Lemma 84.7.1 that

$$(h_{-1})^* : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_{big})$$

is an equivalence with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. Recall that $QCoh(\mathcal{O}_U)$ and $QCoh(\mathcal{O}_{big,total})$ consist of cartesian modules whose components are quasi-coherent, see Lemma 85.12.10. Since the functors h^* and h_* of Lemma 85.7.2 agree with the functors h_n^* and $h_{n,*}$ on components we conclude that

$$h^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

is an equivalence with quasi-inverse h_*. Observe that U is a hypercovering of X in $(Spaces/S)_{fppf}$ as defined in Section 85.21. By Lemma 85.22.1 we see that

[8] This happened in the proof of Lemma 85.33.1 via an application of Lemma 85.5.5.

a_{fppf}^* is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves of $\mathcal{O}_{fppf,total}$-modules. Thus, by the description of $QCoh(\mathcal{O}_{big})$ and $QCoh(\mathcal{O}_{big,total})$ of Lemma 85.12.10, we get an equivalence

$$a_{fppf}^* : QCoh(\mathcal{O}_{big}) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

with quasi-inverse given by $a_{fppf,*}$. A formal argument (chasing around the diagram) now shows that a^* is fully faithful on $QCoh(\mathcal{O}_X)$ and has image contained in $QCoh(\mathcal{O}_U)$.

Finally, suppose that $\mathcal{G}$ is in $QCoh(\mathcal{O}_U)$. Then $h^*\mathcal{G}$ is in $QCoh(\mathcal{O}_{big,total})$. Hence $h^*\mathcal{G} = a_{fppf}^*\mathcal{H}$ with $\mathcal{H} = a_{fppf,*}h^*\mathcal{G}$ in $QCoh(\mathcal{O}_{big})$ (see above). In turn we see that $\mathcal{H} = (h_{-1})^*\mathcal{F}$ with $\mathcal{F} = h_{-1,*}\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. Going around the diagram we deduce that $h^*\mathcal{G} \cong h^*a^*\mathcal{F}$. By fully faithfulness of h^* we conclude that $a^*\mathcal{F} \cong \mathcal{G}$. Since $\mathcal{F} = h_{-1,*}a_{fppf,*}h^*\mathcal{G} = a_*h_*h^*\mathcal{G} = a_*\mathcal{G}$ we also obtain the statement that the quasi-inverse is given by a_*. □

0DHE **Lemma 85.34.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module the map*

$$\mathcal{F} \to Ra_*(a^*\mathcal{F})$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.33.1. Let $\mathcal{F}_n = a_n^*\mathcal{F}$ be the nth component of $a^*\mathcal{F}$. This is a quasi-coherent $\mathcal{O}_{U_n}$-module. Then $\mathcal{F}_n = Rh_{n,*}h_n^*\mathcal{F}_n$ by More on Cohomology of Spaces, Lemma 84.7.2. Hence $a^*\mathcal{F} = Rh_*h^*a^*\mathcal{F}$ by Lemma 85.7.3. We have

$$\begin{aligned} Ra_*(a^*\mathcal{F}) &= Ra_*Rh_*h^*a^*\mathcal{F} \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^*(h_{-1})^*\mathcal{F} \\ &= Rh_{-1,*}(h_{-1})^*\mathcal{F} \\ &= \mathcal{F} \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.22.2 as U is a hypercovering of X in $(Spaces/S)_{fppf}$ and $La_{fppf}^* = a_{fppf}^*$ as a_{fppf} is flat (namely $a_{fppf}^{-1}\mathcal{O}_{big} = \mathcal{O}_{big,total}$, see Remark 85.16.5), and the last equality by the already used More on Cohomology of Spaces, Lemma 84.7.2. □

0DHF **Lemma 85.34.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Assume $a : U \to X$ is an fppf hypercovering of X. Then $QCoh(\mathcal{O}_U)$ is a weak Serre subcategory of $Mod(\mathcal{O}_U)$ and*

$$a^* : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_U)$$

is an equivalence of categories with quasi-inverse given by Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. First observe that the maps $a_n : U_n \to X$ and $d_i^n : U_n \to U_{n-1}$ are flat, locally of finite presentation, and surjective by Hypercoverings, Remark 25.8.4.

Recall that an $\mathcal{O}_U$-module $\mathcal{F}$ is quasi-coherent if and only if it is cartesian and $\mathcal{F}_n$ is quasi-coherent for all n. See Lemma 85.12.10. By Lemma 85.12.6 (and flatness of the maps $d_i^n : U_n \to U_{n-1}$ shown above) the cartesian modules for a weak Serre subcategory of $Mod(\mathcal{O}_U)$. On the other hand $QCoh(\mathcal{O}_{U_n}) \subset Mod(\mathcal{O}_{U_n})$ is a weak Serre subcategory for each n (Properties of Spaces, Lemma 66.29.7). Combined we see that $QCoh(\mathcal{O}_U) \subset Mod(\mathcal{O}_U)$ is a weak Serre subcategory.

To finish the proof we check the conditions (1) – (5) of Cohomology on Sites, Lemma 21.28.6 one by one.

Ad (1). This holds since a_n flat (seen above) implies a is flat by Lemma 85.11.1.

Ad (2). This is the content of Lemma 85.34.2.

Ad (3). This is the content of Lemma 85.34.3.

Ad (4). Recall that we can use either the site $U_{étale}$ or $U_{spaces,étale}$ to define the small étale topos $Sh(U_{étale})$, see Section 85.32. The assumption of Cohomology on Sites, Situation 21.25.1 holds for the triple $(U_{spaces,étale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$ and by the same reasoning for the triple $(U_{étale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$. Namely, take

$$\mathcal{B} \subset \mathrm{Ob}(U_{étale}) \subset \mathrm{Ob}(U_{spaces,étale})$$

to be the set of affine objects. For $V/U_n \in \mathcal{B}$ take $d_{V/U_n} = 0$ and take Cov_{V/U_n} to be the set of étale coverings $\{V_i \to V\}$ with V_i affine. Then we get the desired vanishing because for $\mathcal{F} \in QCoh(\mathcal{O}_U)$ and any $V/U_n \in \mathcal{B}$ we have

$$H^p(V/U_n, \mathcal{F}) = H^p(V, \mathcal{F}_n)$$

by Lemma 85.10.4. Here on the right hand side we have the cohomology of the quasi-coherent sheaf $\mathcal{F}_n$ on U_n over the affine object V of $U_{n,étale}$. This vanishes for $p > 0$ by the discussion in Cohomology of Spaces, Section 69.3 and Cohomology of Schemes, Lemma 30.2.2.

Ad (5). Follows by taking $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ the set of affine objects and the references given above. □

0DHG **Lemma 85.34.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$R\Gamma(X_{étale}, K) = R\Gamma(U_{étale}, a^*K)$$

for $K \in D_{QCoh}(\mathcal{O}_X)$. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.34.4 because $R\Gamma(U_{étale}, -) = R\Gamma(X_{étale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DHH **Lemma 85.34.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}_n$ be the pullback to $U_{n,étale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{étale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.34.5 and 85.10.3. □

85.35. Fppf descent of complexes

0DL8 In this section we pull some of the previously shown results together for fppf coverings of algebraic spaces and derived categories of quasi-coherent modules.

0DL9 **Lemma 85.35.1.** *Let X be an algebraic space over a scheme S. Let $K, E \in D_{QCoh}(\mathcal{O}_X)$. Let $a : U \to X$ be an fppf hypercovering. Assume that for all $n \geq 0$ we have*

$$\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(La_n^*K, La_n^*E) = 0 \text{ for } i < 0$$

Then we have

(1) $\mathrm{Ext}^i_{\mathcal{O}_X}(K, E) = 0$ *for* $i < 0$, *and*
(2) *there is an exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(K, E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(La_0^*K, La_0^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(La_1^*K, La_1^*E)$$

Proof. Write $K_n = La_n^*K$ and $E_n = La_n^*E$. Then these are the simplicial systems of the derived category of modules (Definition 85.14.1) associated to La^*K and La^*E (Lemma 85.14.2) where $a : U_{étale} \to X_{étale}$ is as in Section 85.32. Let us prove (2) first. By Lemma 85.34.4 we have

$$\mathrm{Hom}_{\mathcal{O}_X}(K, E) = \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E)$$

Thus the sequence looks like this:

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(K_0, E_0) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(K_1, E_1)$$

The first arrow is injective by Lemma 85.14.5. The image of this arrow is the kernel of the second by Lemma 85.14.6. This finishes the proof of (2). Part (1) follows by applying part (2) with $K[i]$ and E for $i > 0$. □

0DLA **Lemma 85.35.2.** *Let X be an algebraic space over a scheme S. Let $a : U \to X$ be an fppf hypercovering. Suppose given $K_0 \in D_{QCoh}(U_0)$ and an isomorphism*

$$\alpha : L(f_{\delta^1_1})^*K_0 \longrightarrow L(f_{\delta^1_0})^*K_0$$

satisfying the cocycle condition on U_1. Set $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf_{\tau_n^n}^*K_0$. Assume $\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(K_n, K_n) = 0$ for $i < 0$. Then there exists an object $K \in D_{QCoh}(\mathcal{O}_X)$ and an isomorphism $La_0^*K \to K$ compatible with α.

Proof. The objects K_n form the members of a simplicial system of the derived category of modules by Lemma 85.14.3. Then we obtain an object $K' \in D_{QCoh}(\mathcal{O}_{U_{étale}})$ such that (K_n, K_φ) is the system deduced from K', see Lemma 85.14.7. Finally, we apply Lemma 85.34.4 to see that $K' = La^*K$ for some $K \in D_{QCoh}(\mathcal{O}_X)$ as desired. □

85.36. Proper hypercoverings of algebraic spaces

0DHI This section is the analogue of Section 85.25 for the case of algebraic spaces. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 84.8 with references to Étale Cohomology, Section 59.102.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 85.32. We say that U is a *proper hypercovering* of X if

(1) $U_0 \to X$ is proper and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is proper and surjective,
(3) $U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}$ is proper and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

Principle: Proper hypercoverings can be used to compute étale cohomology.

The key idea behind the proof of the principle is to compare the ph and étale topologies on the category $Spaces/S$. Namely, the ph topology is stronger than the étale topology and we have (a) a proper surjective map defines a ph covering, and (b) ph cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 84.8.

All results in this section generalize to the case where $U \to X$ is merely a "ph hypercovering", meaning a hypercovering of X in the site $(Spaces/S)_{ph}$ as defined in Section 85.21. If we ever need this, we will precisely formulate and prove this here.

0DHJ **Lemma 85.36.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{array}{ccc} Sh((Spaces/U)_{ph,total}) & \xrightarrow{h} & Sh(U_{étale}) \\ \downarrow{\scriptstyle a_{ph}} & & \downarrow{\scriptstyle a} \\ Sh((Spaces/X)_{ph}) & \xrightarrow{h_{-1}} & Sh(X_{étale}) \end{array}$$

where the left vertical arrow is defined in Section 85.21 and the right vertical arrow is defined in Section 85.32.

Proof. The notation $(Spaces/U)_{ph,total}$ indicates that we are using the construction of Section 85.21 for the site $(Spaces/S)_{ph}$ and the simplicial object U of this site[9]. We will use the sites $X_{spaces,étale}$ and $U_{spaces,étale}$ for the topoi on the right hand side; this is permissible see discussion in Section 85.32.

Observe that both $(Spaces/U)_{ph,total}$ and $U_{spaces,étale}$ fall into case A of Situation 85.3.3. This is immediate from the construction of $U_{étale}$ in Section 85.32 and it follows from Lemma 85.21.5 for $(Spaces/U)_{ph,total}$. Next, consider the functors $U_{n,spaces,étale} \to (Spaces/U_n)_{ph}$, $U \mapsto U/U_n$ and $X_{spaces,étale} \to (Spaces/X)_{ph}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 84.8 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and $a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 85.5.4 and we may apply Lemma 85.5.5 to conclude. □

0DHK **Lemma 85.36.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then*

$$a^{-1} : Sh(X_{étale}) \to Sh(U_{étale}) \quad \text{and} \quad a^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$$

[9]To distinguish from $(Spaces/U)_{fppf,total}$ defined using the fppf topology in Section 85.33.

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 85.36.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 84.8. Thus it follows from More on Cohomology of Spaces, Lemma 84.8.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 85.5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(Spaces/S)_{ph}$ as defined in Section 85.21 since a surjective proper morphism gives a ph covering by Topologies on Spaces, Lemma 73.8.3. By Lemma 85.21.1 we see that a_{ph}^{-1} is fully faithful with quasi-inverse $a_{ph,*}$ and with essential image the cartesian sheaves on $(Spaces/U)_{ph,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{étale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(Spaces/U)_{ph,total}$. Hence $h^{-1}\mathcal{G} = a_{ph}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(Spaces/X)_{ph}$. We compute using somewhat pedantic notation

$$\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\mathrm{Eq}(\ a_{0,small,*}\mathcal{G}_0 \rightrightarrows a_{1,small,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ (h_{-1})^{-1}a_{0,small,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,small,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{0,big,ph,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{1,big,ph,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{0,big,ph,*}(a_{0,big,ph})^{-1}\mathcal{H} \rightrightarrows a_{1,big,ph,*}(a_{1,big,ph})^{-1}\mathcal{H}\ ) \\
&= a_{ph,*}a_{ph}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}$$

Here the first equality follows from Lemma 85.4.2, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from More on Cohomology of Spaces, Lemma 84.8.5 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 85.4.2, and the sixth we've seen above. Since $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. □

0DHL **Lemma 85.36.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then for $K \in D^+(X_{étale})$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Consider the diagram of Lemma 85.36.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,étale})$ by More on Cohomology of Spaces, Lemma 84.8.2.

Hence Rh_*h^{-1} is the identity functor on $D^+(U_{étale})$ by Lemma 85.5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{ph,*}a_{ph}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 85.25.1, the third equality by Lemma 85.21.2 as U is a hypercovering of X in $(Spaces/S)_{ph}$ by Topologies on Spaces, Lemma 73.8.3, and the last equality by the already used More on Cohomology of Spaces, Lemma 84.8.2. □

0DHM **Lemma 85.36.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then*

$$R\Gamma(X_{étale}, K) = R\Gamma(U_{étale}, a^{-1}K)$$

for $K \in D^+(X_{étale})$. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. This follows from Lemma 85.36.3 because $R\Gamma(U_{étale}, -) = R\Gamma(X_{étale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 21.14.4. □

0DHN **Lemma 85.36.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{étale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X_{étale}) \longrightarrow D^+_{\mathcal{A}}(U_{étale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 85.32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 85.12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 85.36.2 and 85.36.3 and Cohomology on Sites, Lemma 21.28.5. □

0DHP **Lemma 85.36.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,étale}$. If U is a ph hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{étale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 85.36.4 and 85.8.3. □

85.37. Other chapters

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CHAPTER 86

Duality for Spaces

0E4V

86.1. Introduction

0E4W This chapter is the analogue of the corresponding chapter for schemes, see Duality for Schemes, Section 48.1. The development is similar to the development in the papers [**Nee96**], [**LN07**], [**Lip09**], and [**Nee14**].

86.2. Dualizing complexes on algebraic spaces

0E4X Let U be a locally Noetherian scheme. Let $\mathcal{O}_{étale}$ be the structure sheaf of U on the small étale site of U. We will say an object $K \in D_{QCoh}(\mathcal{O}_{étale})$ is a dualizing complex on U if $K = \epsilon^*(\omega_U^\bullet)$ for some dualizing complex $\omega_U^\bullet$ in the sense of Duality for Schemes, Section 48.2. Here $\epsilon^* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_{étale})$ is the equivalence of Derived Categories of Spaces, Lemma 75.4.2. Most of the properties of $\omega_U^\bullet$ studied in Duality for Schemes, Section 48.2 are inherited by K via the discussion in Derived Categories of Spaces, Sections 75.4 and 75.13.

We define a dualizing complex on a locally Noetherian algebraic space to be a complex which étale locally comes from a dualizing complex on the corresponding scheme.

0E4Y **Lemma 86.2.1.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *For every étale morphism $U \to X$ where U is a scheme the restriction $K|_U$ is a dualizing complex for U (as discussed above).*
2. *There exists a surjective étale morphism $U \to X$ where U is a scheme such that $K|_U$ is a dualizing complex for U.*

Proof. Assume $U \to X$ is surjective étale where U is a scheme. Let $V \to X$ be an étale morphism where V is a scheme. Then

$$U \leftarrow U \times_X V \to V$$

are étale morphisms of schemes with the arrow to V surjective. Hence we can use Duality for Schemes, Lemma 48.26.1 to see that if $K|_U$ is a dualizing complex for U, then $K|_V$ is a dualizing complex for V. □

0E4Z **Definition 86.2.2.** Let S be a scheme. Let X be a locally Noetherian algebraic space over S. An object K of $D_{QCoh}(\mathcal{O}_X)$ is called a *dualizing complex* if K satisfies the equivalent conditions of Lemma 86.2.1.

0E50 **Lemma 86.2.3.** *Let A be a Noetherian ring and let $X = \mathrm{Spec}(A)$. Let $\mathcal{O}_{étale}$ be the structure sheaf of X on the small étale site of X. Let K, L be objects of $D(A)$. If $K \in D_{Coh}(A)$ and L has finite injective dimension, then*

$$\epsilon^* R\widetilde{\mathrm{Hom}_A(K, L)} = R\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L})$$

in $D(\mathcal{O}_{étale})$ where $\epsilon : (X_{étale}, \mathcal{O}_{étale}) \to (X, \mathcal{O}_X)$ is as in Derived Categories of Spaces, Section 75.4.

Proof. By Duality for Schemes, Lemma 48.2.3 we have a canonical isomorphism

$$R\widetilde{\mathrm{Hom}_A(K, L)} = R\,\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

in $D(\mathcal{O}_X)$. There is a canonical map

$$\epsilon^* R\,\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) \longrightarrow R\,\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L})$$

in $D(\mathcal{O}_{étale})$, see Cohomology on Sites, Remark 21.35.11. We will show the left and right hand side of this arrow have isomorphic cohomology sheaves, but we will omit the verification that the isomorphism is given by this arrow.

We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. Recall that the cohomology sheaves of $R\,\mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\widetilde{K}, \epsilon^*\widetilde{L}))$ are the sheafifications of the presheaf sending U étale over X to the ith ext group between the restrictions of $\epsilon^*\widetilde{K}$ and $\epsilon^*\widetilde{L}$ to $U_{étale}$. See Cohomology on Sites, Lemma 21.35.1. If $U = \mathrm{Spec}(B)$ is affine, then this ext group is equal to $\mathrm{Ext}^i_B(K \otimes_A B, L \otimes_A B)$ by the equivalence of Derived Categories of Spaces, Lemma 75.4.2 and Derived Categories of Schemes, Lemma 36.3.5 (this also uses the compatibilities detailed in Derived Categories of Spaces, Remark 75.6.3). Since $A \to B$ is étale, we see that $I \otimes_A B$ is an injective B-module by Dualizing Complexes, Lemma 47.26.4. Hence we see that

$$\begin{aligned}
\mathrm{Ext}^n_B(K \otimes_A B, I \otimes_A B) &= \mathrm{Hom}_B(H^{-n}(K \otimes_A B), I \otimes_A B) \\
&= \mathrm{Hom}_{A_f}(H^{-n}(K) \otimes_A B, I \otimes_A B) \\
&= \mathrm{Hom}_A(H^{-n}(K), I) \otimes_A B \\
&= \mathrm{Ext}^n_A(K, I) \otimes_A B
\end{aligned}$$

The penultimate equality because $H^{-n}(K)$ is a finite A-module, see More on Algebra, Lemma 15.66.4. Therefore the cohomology sheaves of the left and right hand side of the equality in the lemma are the same. □

0E51 **Lemma 86.2.4.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let K be a dualizing complex on X. Then K is an object of $D_{Coh}(\mathcal{O}_X)$ and $D = R\,\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $id \to D \circ D$. If X is quasi-compact, then D exchanges $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and induces an equivalence $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$.

Proof. Let $U \to X$ be an étale morphism with U affine. Say $U = \mathrm{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 86.2.1 and

Duality for Schemes, Lemma 48.2.1. By Lemma 86.2.3 the diagram

$$\begin{array}{ccc} D_{Coh}(A) & \longrightarrow & D_{Coh}(\mathcal{O}_{étale}) \\ {\scriptstyle R\mathrm{Hom}_A(-,\omega_A^\bullet)}\downarrow & & \downarrow{\scriptstyle R\mathcal{H}om_{\mathcal{O}_{étale}}(-,K|_U)} \\ D_{Coh}(A) & \longrightarrow & D(\mathcal{O}_{étale}) \end{array}$$

commutes where $\mathcal{O}_{étale}$ is the structure sheaf of the small étale site of U. Since formation of $R\mathcal{H}om$ commutes with restriction, we conclude that D sends $D_{Coh}(\mathcal{O}_X)$ into $D_{Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(Cohomology on Sites, Lemma 21.35.5) is an isomorphism for all L in $D_{Coh}(\mathcal{O}_X)$ because this is true over all U as above by Dualizing Complexes, Lemma 47.15.3. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 47.15.3. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Recall that an object L of $D(\mathcal{O})$ is *invertible* if it is an invertible object for the symmetric monoidal structure on $D(\mathcal{O}_X)$ given by derived tensor product. In Cohomology on Sites, Lemma 21.49.2 we we have seen this means L is perfect and if $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}$ of U in $\mathcal{C}$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i.

Let S be a scheme and let X be an algebraic space over S. If L in $D(\mathcal{O}_X)$ is invertible, then there is a disjoint union decomposition $X = \coprod_{n\in\mathbf{Z}} X_n$ such that $L|_{X_n}$ is an invertible module sitting in degree n. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$-modules (with zero differentials) representing L.

0E52 **Lemma 86.2.5.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. If K and K' are dualizing complexes on X, then K' is isomorphic to $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for some invertible object L of $D(\mathcal{O}_X)$.*

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true. Use Lemma 86.2.3 and Dualizing Complexes, Lemma 47.15.5 and its proof. The evaluation map $L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K'$ is an isomorphism for the same reason. □

0E53 **Lemma 86.2.6.** *Let S be a scheme. Let X be a locally Noetherian quasi-separated algebraic space over S. Let $\omega_X^\bullet$ be a dualizing complex on X. Then X the function $|X| \to \mathbf{Z}$ defined by*

$$x \longmapsto \delta(x) \text{ such that } \omega^\bullet_{X,\overline{x}}[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,\overline{x}}$$

is a dimension function on $|X|$.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Let $\omega_U^\bullet$ be the dualizing complex on U associated to $\omega_X^\bullet|_U$. If $u \in U$ maps to $x \in |X|$, then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$. By Dualizing Complexes, Lemma 47.22.1 we see that if $\omega^\bullet$ is a normalized dualizing complex for $\mathcal{O}_{U,u}$, then $\omega^\bullet \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$ is a normalized dualizing complex for $\mathcal{O}_{X,\overline{x}}$. Hence we see that the dimension function

$U \to \mathbf{Z}$ of Duality for Schemes, Lemma 48.2.7 for the scheme U and the complex $\omega_U^\bullet$ is equal to the composition of $U \to |X|$ with δ. Using the specializations in $|X|$ lift to specializations in U and that nontrivial specializations in U map to nontrivial specializations in X (Decent Spaces, Lemmas 68.12.2 and 68.12.1) an easy topological argument shows that δ is a dimension function on $|X|$. □

86.3. Right adjoint of pushforward

0E54 This is the analogue of Duality for Schemes, Section 48.3.

0E55 **Lemma 86.3.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact algebraic spaces over S. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.*

This is almost the same as [**Nee96**, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 13.38.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Spaces, Lemma 75.5.3. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Spaces, Theorem 75.15.4. Since X and Y are quasi-compact and quasi-separated, so is f, see Morphisms of Spaces, Lemmas 67.4.10 and 67.8.9. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Spaces, Lemma 75.6.2. This finishes the proof. □

0E56 **Lemma 86.3.2.** *Notation and assumptions as in Lemma 86.3.1. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_*. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.*

Proof. By Derived Categories of Spaces, Lemma 75.6.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that $H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. □

Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact algebraic spaces over S. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0E57 (86.3.2.1) $$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology on Sites, Remark 21.35.10 and the second arrow is the counit $Rf_*a(K) \to K$ of the adjunction.

0E58 **Lemma 86.3.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$. Let $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$. Then the map (86.3.2.1)*

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

becomes an isomorphism after applying the functor $DQ_Y : D(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_Y)$ discussed in Derived Categories of Spaces, Section 75.19.

Proof. The statement makes sense as DQ_Y exists by Derived Categories of Spaces, Lemma 75.19.1. Since DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$ to prove the lemma we have to show that for any $M \in D_{QCoh}(\mathcal{O}_Y)$ the map (86.3.2.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

To see this we use the following string of equalities

$$\begin{aligned}
\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\
&= \mathrm{Hom}_X(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, a(K)) \\
&= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L), K) \\
&= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L, K) \\
&= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))
\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by Cohomology on Sites, Lemma 21.35.2. The third equality by construction of a. The fourth equality by Derived Categories of Spaces, Lemma 75.20.1 (this is the important step). The fifth by Cohomology on Sites, Lemma 21.35.2. □

0GG3 **Example 86.3.4.** The statement of Lemma 86.3.3 is not true without applying the "coherator" DQ_Y. See Duality for Schemes, Example 48.3.7.

0GG4 **Remark 86.3.5.** In the situation of Lemma 86.3.3 we have

$$DQ_Y(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = Rf_*DQ_X(R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

by Derived Categories of Spaces, Lemma 75.19.2. Thus if $R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \in D_{QCoh}(\mathcal{O}_X)$, then we can "erase" the DQ_Y on the left hand side of the arrow. On the other hand, if we know that $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \in D_{QCoh}(\mathcal{O}_Y)$, then we can "erase" the DQ_Y from the right hand side of the arrow. If both are true then we see that (86.3.2.1) is an isomorphism. Combining this with Derived Categories of Spaces, Lemma 75.13.10 we see that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism if

(1) L and Rf_*L are perfect, or
(2) K is bounded below and L and Rf_*L are pseudo-coherent.

For (2) we use that $a(K)$ is bounded below if K is bounded below, see Lemma 86.3.2.

0GG5 **Example 86.3.6.** Let S be a scheme. Let $f : X \to Y$ be a proper morphism of Noetherian algebraic spaces over S, $L \in D^-_{Coh}(X)$ and $K \in D^+_{QCoh}(\mathcal{O}_Y)$. Then the map $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \to R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism. Namely, the complexes L and Rf_*L are pseudo-coherent by Derived Categories of Spaces, Lemmas 75.13.7 and 75.8.1 and the discussion in Remark 86.3.5 applies.

0E59 **Lemma 86.3.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-separated and quasi-compact algebraic spaces over S. For all $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ (86.3.2.1) induces an isomorphism $R\mathrm{Hom}_X(L, a(K)) \to R\mathrm{Hom}_Y(Rf_*L, K)$ of global derived homs.*

Proof. By construction (Cohomology on Sites, Section 21.36) the complexes

$$R\operatorname{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\operatorname{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_X}(Rf_*L, a(K)))$$

Thus the lemma is a consequence of Lemma 86.3.3. Namely, a map $E \to E'$ in $D(\mathcal{O}_Y)$ which induces an isomorphism $DQ_Y(E) \to DQ_Y(E')$ induces a quasi-isomorphism $R\Gamma(Y, E) \to R\Gamma(Y, E')$. Indeed we have $H^i(Y, E) = \operatorname{Ext}^i_Y(\mathcal{O}_Y, E) = \operatorname{Hom}(\mathcal{O}_Y[-i], E) = \operatorname{Hom}(\mathcal{O}_Y[-i], DQ_Y(E))$ because $\mathcal{O}_Y[-i]$ is in $D_{QCoh}(\mathcal{O}_Y)$ and DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_Y)$. □

86.4. Right adjoint of pushforward and base change, I

0E5A Let us define the base change map between right adjoints of pushforward. Let S be a scheme. Consider a cartesian diagram

0E5B (86.4.0.1)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where Y' and X are **Tor independent** over Y. Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 86.3.1). The base change map of Cohomology on Sites, Remark 21.19.3 gives a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0E5C **Lemma 86.4.1.** *In diagram (86.4.0.1) the map $a \circ Rg_* \leftarrow Rg'_* \circ a'$ is an isomorphism.*

Proof. The base change map $Lg^* \circ Rf_*K \to Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Spaces, Lemma 75.20.4 (this uses the assumption of Tor independence). Thus the corresponding transformation between adjoint functors is an isomorphism as well. □

Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0E5D (86.4.1.1) $L(g')^* \circ a \to L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \to a' \circ Lg^*$

The first arrow comes from the adjunction map $\mathrm{id} \to Rg_*Lg^*$ and the last arrow from the adjunction map $L(g')^*Rg'_* \to \mathrm{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 86.4.1. Alternatively, we can think of (86.4.1.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \to a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned}
\mathrm{Hom}_X(M, a(K)) &= \mathrm{Hom}_Y(Rf_*M, K) \\
&\to \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\
&= \mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\
&\leftarrow \mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\
&= \mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\
&= \mathrm{Hom}_X(M, Rg'_*a'(Lg^*K))
\end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Spaces, Lemma 75.20.4) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology on Sites, Remarks 21.19.4 and 21.19.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0E5E **Lemma 86.4.2.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc}
X' & \xrightarrow{k} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{l} & Y \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
Z' & \xrightarrow{m} & Z
\end{array}$$

of quasi-compact and quasi-separated algebraic spaces over S where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (86.4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_*. Let a, b, and c be the right adjoints of Lemma 86.3.1 for f, g, and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \to k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \to a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \to a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $l^* \circ f_* \to f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \to l_* \circ l^*$ and $k^* \circ k_* \to 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \to l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \to b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \to k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \to c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

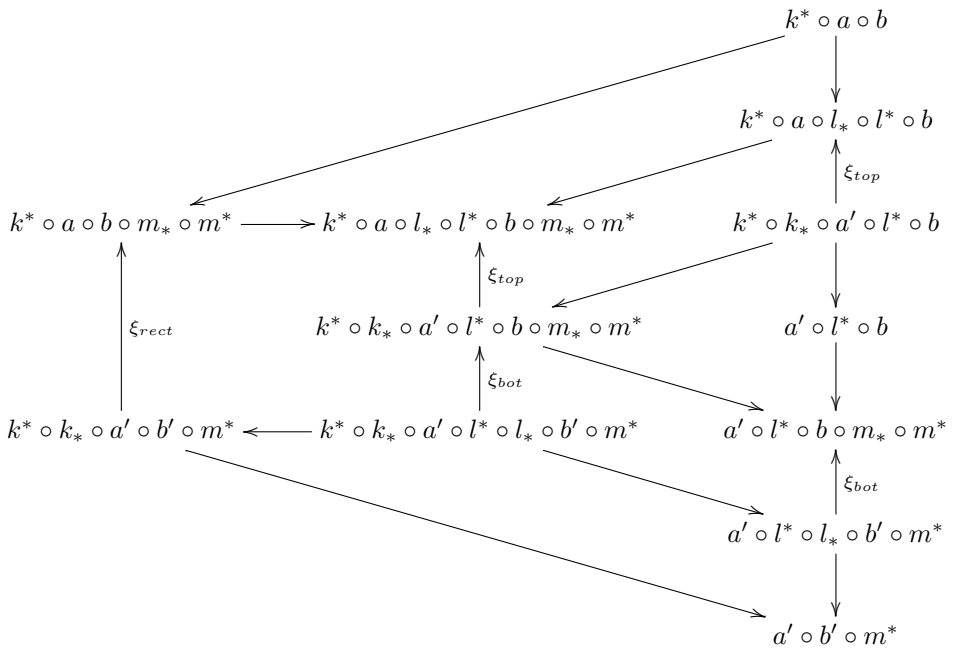

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show the diagram

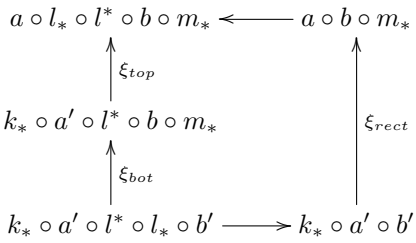

becomes commutative if we invert the arrows ξ_{top}, ξ_{bot}, and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

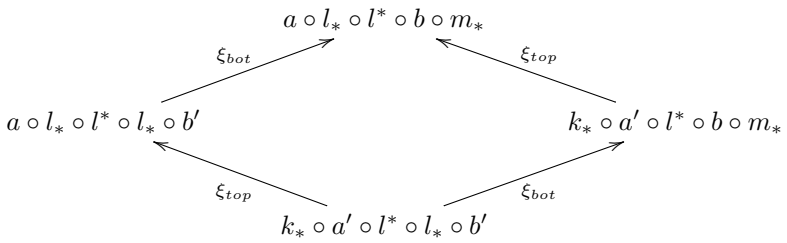

commutes by Categories, Lemma 4.28.2. Since the diagrams

$$\begin{array}{ccc} a \circ l_* \circ l^* \circ b \circ m_* & \longleftarrow & a \circ b \circ m \\ \uparrow & & \uparrow \\ a \circ l_* \circ l^* \circ l_* \circ b' & \longleftarrow & a \circ l_* \circ b' \end{array} \quad \text{and} \quad \begin{array}{ccc} a \circ l_* \circ l^* \circ l_* \circ b' & \longrightarrow & a \circ l_* \circ b' \\ \uparrow & & \uparrow \\ k_* \circ a' \circ l^* \circ l_* \circ b' & \longrightarrow & k_* \circ a' \circ b' \end{array}$$

commute (see references cited) and since the composition of $l_* \to l_* \circ l^* \circ l_* \to l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology on Sites, Remark 21.19.4 and the proof is complete. □

0E5F **Lemma 86.4.3.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccccc} X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h'} & Y' & \xrightarrow{h} & Y \end{array}$$

of quasi-compact and quasi-separated algebraic spaces over S where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (86.4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_*. Let a, a', and a'' be the right adjoints of Lemma 86.3.1 for f, f', and f''. The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \to g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \to a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \to a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow "dual" to the base change map $h^* \circ f_* \to f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \to h_* \circ h^*$ and $g^* \circ g_* \to 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \to (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \to a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \to k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \to a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

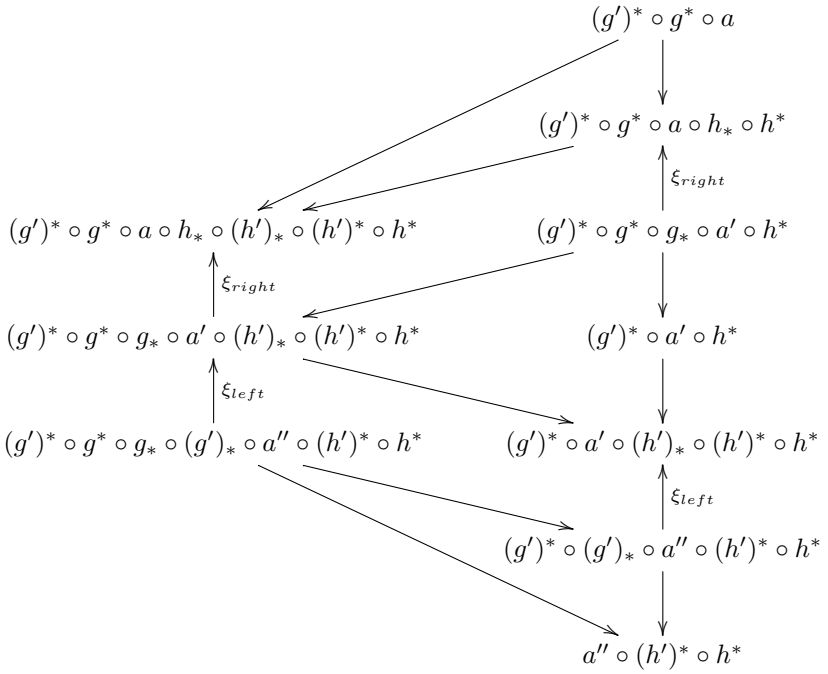

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect}. All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect}. This is the statement dual to Cohomology, Remark 20.28.5 and the proof is complete. □

0E5G **Remark 86.4.4.** Let S be a scheme. Consider a commutative diagram

$$\begin{array}{ccccc}
X'' & \xrightarrow[k']{} & X' & \xrightarrow[k]{} & X \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\
{\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z
\end{array}$$

of quasi-compact and quasi-separated algebraic spaces over S where all squares are cartesian and where (f, l), (g, m), (f', l'), (g', m') are Tor independent pairs of maps. Let a, a', a'', b, b', b'' be the right adjoints of Lemma 86.3.1 for f, f', f'', g, g', g''. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{array}{cc} A & B \\ C & D \end{array}$$

Then the maps (86.4.1.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\gamma_A : (k')^* \circ a' \to a'' \circ (l')^* \quad \gamma_B : k^* \circ a \to a' \circ l^*$$
$$\gamma_C : (l')^* \circ b' \to b'' \circ (m')^* \quad \gamma_D : l^* \circ b \to b' \circ m^*$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\gamma_{A+B} : (k \circ k')^* \circ a \to a'' \circ (l \circ l')^*$$
$$\gamma_{C+D} : (l \circ l')^* \circ b \to b'' \circ (m \circ m')^*$$
$$\gamma_{A+C} : (k')^* \circ (a' \circ b') \to (a'' \circ b'') \circ (m')^*$$
$$\gamma_{A+C} : k^* \circ (a \circ b) \to (a' \circ b') \circ m^*$$

By Lemma 86.4.3 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 86.4.2 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \mathrm{id}_{l^*}) \circ (\mathrm{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 4.28 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 86.4.2 and 86.4.3 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{array}{ccc} (k')^* \circ a' \circ l^* \circ b & \xrightarrow[\gamma_D]{} & (k')^* \circ a' \circ b' \circ m^* \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_A} \\ a'' \circ (l')^* \circ l^* \circ b & \xrightarrow{\gamma_D} & a'' \circ (l')^* \circ b' \circ m^* \end{array}$$

commutes. This is true by Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1.

86.5. Right adjoint of pushforward and base change, II

0E5H In this section we prove that the base change map of Section 86.4 is an isomorphism in some cases.

0E5I **Lemma 86.5.1.** *In diagram (86.4.0.1) assume in addition $g : Y' \to Y$ is a morphism of affine schemes and $f : X \to Y$ is proper. Then the base change map (86.4.1.1) induces an isomorphism*

$$L(g')^* a(K) \longrightarrow a'(Lg^* K)$$

in the following cases

(1) *for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is flat of finite presentation,*
(2) *for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is perfect and Y Noetherian,*

(3) *for $K \in D^+_{QCoh}(\mathcal{O}_X)$ if g has finite Tor dimension and Y Noetherian.*

Proof. Write $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Theorem 75.15.4. Then $L(g')^*M$ is a generator for $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Spaces, Remark 75.15.5. Hence it suffices to show that (86.4.1.1) induces an isomorphism

0E5J (86.5.1.1) $$R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology on Sites, Section 21.36, as this will imply the cone of $L(g')^*a(K) \to a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (86.5.1.1).

The left hand side. Because M is perfect, the canonical map

$$R\mathrm{Hom}_X(M, a(K)) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Spaces, Lemma 75.20.5. We can combine this with the isomorphism $R\mathrm{Hom}_Y(Rf_*M, K) = R\mathrm{Hom}_X(M, a(K))$ of Lemma 86.3.7 to get that the left hand side equals $R\mathrm{Hom}_Y(Rf_*M, K) \otimes^{\mathbf{L}}_A A'$.

The right hand side. Here we first use the isomorphism

$$R\mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 86.3.7. Since f and g are Tor independent the base change map $Lg^*Rf_*M \to Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Spaces, Lemma 75.20.4. Hence we may rewrite this as $R\mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{QCoh}(\mathcal{O}_Y)$ (Derived Categories of Spaces, Lemma 75.6.1) we have a canonical map

$$\beta : R\mathrm{Hom}_Y(Rf_*M, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (15.100.1.1) where we have used Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.8 to translate back and forth into algebra.

(1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Spaces, Lemma 75.25.4 and β is an isomorphism by More on Algebra, Lemma 15.100.2 part (1).
(2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms of Spaces, Lemma 76.47.5 and β is an isomorphism as before.
(3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Spaces, Lemmas 75.8.1 and 75.13.7) and β is an isomorphism by More on Algebra, Lemma 15.100.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (86.5.1.1) given in the second and third paragraph are in fact given by (86.5.1.1). To make our formulas manageable we will use $(-,-)_X = R\mathrm{Hom}_X(-,-)$, use $-\otimes A'$

in stead of $- \otimes^{\mathbf{L}}_A A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$\begin{array}{ccccc}
((g')^*M, (g')^*a(K))_{X'} & \xleftarrow{\alpha} & (M, a(K))_X \otimes A' & = & (f_*M, K)_Y \otimes A' \\
\downarrow & & \downarrow & & \downarrow \\
((g')^*M, (g')^*a(g_*g^*K))_{X'} & \xleftarrow{\alpha} & (M, a(g_*g^*K))_X \otimes A' & = & (f_*M, g_*g^*K)_Y \otimes A' \\
\uparrow & & \uparrow & \overset{\mu'}{\swarrow} & \\
((g')^*M, (g')^*g'_*a'(g^*K))_{X'} & \xleftarrow{\alpha} & (M, g'_*a'(g^*K))_X \otimes A' & & (f_*M, K) \otimes A' \\
\downarrow & \overset{\mu}{\searrow} & & & \downarrow \beta \\
((g')^*M, a'(g^*K))_{X'} & = & (f'_*(g')^*M, g^*K)_{Y'} & \longrightarrow & (g^*f_*M, g^*K)_{Y'}
\end{array}$$

The arrows labeled α are the maps from Derived Categories of Spaces, Lemma 75.20.5 for the diagram with corners X', X, Y', Y. The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \to a \circ g_*$ of Lemma 86.4.1 and therefore the middle square is commutative. Going down the left hand side is (86.5.1.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \to E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \to E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_*a(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Spaces, Remark 75.20.6.

Observe that

$$\begin{array}{ccccc}
(M, a(g_*g^*K))_X & = & (f_*M, g_*g^*K)_Y & = & (g^*f_*M, g^*K)_{Y'} \\
\uparrow & & & & \uparrow \\
(M, g'_*a'(g^*K))_X & = & ((g')^*M, a'(g^*K))_{X'} & = & (f'_*(g')^*M, g^*K)_{Y'}
\end{array}$$

is commutative by the very definition of the transformation $g'_* \circ a' \to a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \to (f_*M, g_*g^*K)_X \otimes A'$ and μ'. To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \to (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{array}{ccc}
R\operatorname{Hom}_A(E, K) & \xrightarrow[\text{induced by } \beta]{} & R\operatorname{Hom}_{A'}(E \otimes^{\mathbf{L}}_A A', K \otimes^{\mathbf{L}}_A A') \\
& \searrow \quad \nearrow & \\
& R\operatorname{Hom}_A(E, K \otimes^{\mathbf{L}}_A A') &
\end{array}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 15.100 the proof is complete. $\square$

86.6. Right adjoint of pushforward and trace maps

0E5K Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 86.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\mathrm{Tr}_f : Rf_* \circ a \longrightarrow \mathrm{id}$$

The corresponding map $\mathrm{Tr}_{f,K} : Rf_*a(K) \longrightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the *trace map*. This is the map which has the property that the bijection

$$\mathrm{Hom}_X(L, a(K)) \longrightarrow \mathrm{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \mathrm{Tr}_{f,K} \circ Rf_*\varphi$$

The canonical map (86.3.2.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

comes about by composition with $\mathrm{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0E5L **Lemma 86.6.1** (Trace map and base change)**.** *Suppose we have a diagram (86.4.0.1). Then the maps $1 \star Tr_f : Lg^* \circ Rf_* \circ a \to Lg^*$ and $Tr_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \to Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ (Cohomology on Sites, Remark 21.19.3) and $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ (86.4.1.1). More precisely, the diagram*

$$\begin{array}{ccc} Lg^* \circ Rf_* \circ a & \xrightarrow[1\star Tr_f]{} & Lg^* \\ {\scriptstyle \beta\star 1}\downarrow & & \uparrow{\scriptstyle Tr_{f'}\star 1} \\ Rf'_* \circ L(g')^* \circ a & \xrightarrow{1\star\alpha} & Rf'_* \circ a' \circ Lg^* \end{array}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 86.4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \to g^* \circ f_* \circ a \circ g_* \circ g^* \to g^* \circ g_* \circ g^* \to g^*$$

where the first arrow is the unit for (g^*, g_*), the second arrow is Tr_f, and the third arrow is the counit for (g^*, g_*). This is a simple consequence of the fact that the composition $g^* \to g^* \circ g_* \circ g^* \to g^*$ of unit and counit is the identity. Consider the

diagram

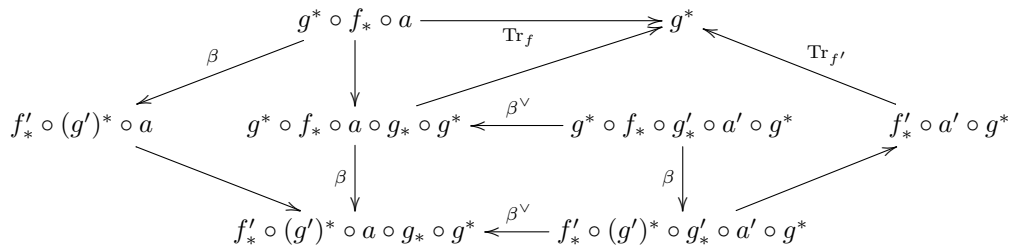

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By Categories, Lemma 4.24.8 the square

$$\begin{array}{ccc} g^* \circ f_* \circ g'_* \circ a' & \xrightarrow[\beta]{} & f'_* \circ (g')^* \circ g'_* \circ a' \\ \downarrow{\scriptstyle \beta^\vee} & & \downarrow \\ g^* \circ f_* \circ a \circ g_* & \longrightarrow & \mathrm{id} \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\mathrm{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. □

Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 86.3.1. By Categories, Section 4.24 we obtain a transformation of functors

$$\eta_f : \mathrm{id} \to a \circ Rf_*$$

which is called the unit of the adjunction.

0E5M **Lemma 86.6.2.** *Suppose we have a diagram (86.4.0.1). Then the maps* $1 \star \eta_f : L(g')^* \to L(g')^* \circ a \circ Rf_*$ *and* $\eta_{f'} \star 1 : L(g')^* \to a' \circ Rf'_* \circ L(g')^*$ *agree via the base change maps* $\beta : Lg^* \circ Rf_* \to Rf'_* \circ L(g')^*$ *(Cohomology on Sites, Remark 21.19.3) and* $\alpha : L(g')^* \circ a \to a' \circ Lg^*$ *(86.4.1.1). More precisely, the diagram*

$$\begin{array}{ccc} L(g')^* & \xrightarrow[1\star\eta_f]{} & L(g')^* \circ a \circ Rf_* \\ \downarrow{\scriptstyle \eta_{f'}\star 1} & & \downarrow{\scriptstyle \alpha} \\ a' \circ Rf'_* \circ L(g')^* & \xleftarrow{\beta} & a' \circ Lg^* \circ Rf_* \end{array}$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 86.6.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \to f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \to a \circ g_*$ which is the adjoint of β, see Lemma 86.4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \to (g')^* \circ g'_* \circ (g')^* \to (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \to a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \to (g')^* \circ (g')_* \circ (g')^* \to (g')^*$ of unit and counit is the identity. Consider the diagram

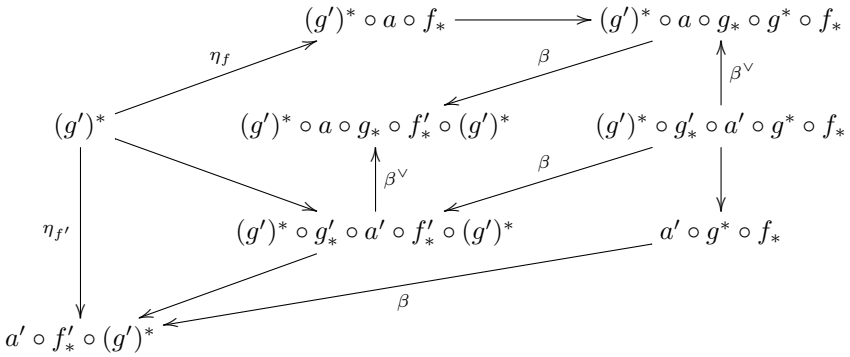

In this diagram the two squares commute Categories, Lemma 4.28.2 or more simply the discussion preceding Categories, Definition 4.28.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 4.24.8 the square

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & g'_* \circ a' \circ g^* \circ f_* \\ \downarrow & & \downarrow \beta \\ g'_* \circ a' \circ g^* \circ f_* & \xrightarrow{\beta^\vee} & a \circ g_* \circ f'_* \circ (g')^* \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. □

86.7. Right adjoint of pushforward and pullback

0E5N Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Let a be the right adjoint of pushforward as in Lemma 86.3.1. For $K, L \in D_{QCoh}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \longrightarrow a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(a(L)) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L$$

(equality by Derived Categories of Spaces, Lemma 75.20.1) for which we use the trace map $Rf_*a(L) \to L$. When $L = \mathcal{O}_Y$ we obtain a map

0E5P (86.7.0.1) $$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0E5Q **Lemma 86.7.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. The map $Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L) \to a(K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (86.7.0.1) is an isomorphism if K is perfect.*

Proof. Let $K^\vee$ be the "dual" to K, see Cohomology on Sites, Lemma 21.48.4. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(L))\end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology on Sites, Lemma 21.50.1) or the more general Derived Categories of Spaces, Lemma 75.20.1. Hence the result by the Yoneda lemma. □

0E5R **Lemma 86.7.2.** *Suppose we have a diagram (86.4.0.1). Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram*

$$\begin{array}{ccc} L(g')^*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) & \longrightarrow & L(g')^*a(K) \\ \downarrow & & \downarrow \\ L(f')^*Lg^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} a'(\mathcal{O}_{Y'}) & \longrightarrow & a'(Lg^*K) \end{array}$$

commutes where the horizontal arrows are the maps (86.7.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology on Sites, Remark 21.19.3 and (86.4.1.1).

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^*, etc, and we will write $\otimes$ for $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, etc. Let us write (86.7.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\to a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\to a(K \otimes \mathcal{O}_Y) \\ &\to a(K) \end{aligned}$$

Here the first arrow is the unit η_f, the second arrow is a applied to Cohomology on Sites, Equation (21.50.0.1) which is an isomorphism by Derived Categories of Spaces, Lemma 75.20.1, the third arrow is a applied to $\mathrm{id}_K \otimes \mathrm{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 86.6.2 and 86.6.1. For the arrow using Cohomology on Sites, Equation (21.50.0.1) this is Cohomology on Sites, Remark 21.50.2. For the multiplication map it is clear. This finishes the proof. □

86.8. Right adjoint of pushforward for proper flat morphisms

0E5S For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated algebraic spaces the right adjoint of pushforward enjoys some remarkable properties.

0E5T **Lemma 86.8.1.** *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 86.3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Spaces, Lemma 75.25.4 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i))\end{aligned}$$

because a perfect object is compact (Derived Categories of Spaces, Proposition 75.16.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Spaces, Theorem 75.15.4) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

0E5U **Lemma 86.8.2.** *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. The map (86.7.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 86.8.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (86.7.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Spaces, Theorem 75.17.3) generated by a single perfect object (Derived Categories of Spaces, Theorem 75.15.4) we can argue as in More on Algebra, Remark 15.59.11 to see that it suffices to prove (86.7.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 86.7.1. □

0E5V **Lemma 86.8.3.** *Let Y be an affine scheme. Let $f : X \to Y$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 86.3.1. Then*

1. *$a(\mathcal{O}_Y)$ is a Y-perfect object of $D(\mathcal{O}_X)$,*
2. *$Rf_*a(\mathcal{O}_Y)$ has vanishing cohomology sheaves in positive degrees,*
3. *$\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ is an isomorphism.*

Proof. For a perfect object E of $D(\mathcal{O}_X)$ we have

$$\begin{aligned}Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega^\bullet_{X/Y}) &= Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) \\ &= R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y) \\ &= (Rf_*E^\vee)^\vee\end{aligned}$$

For the first equality, see Cohomology on Sites, Lemma 21.48.4. For the second equality, see Lemma 86.3.3, Remark 86.3.5, and Derived Categories of Spaces, Lemma 75.25.4. The third equality is the definition of the dual. In particular these references also show that the outcome is a perfect object of $D(\mathcal{O}_Y)$. We conclude that $\omega^\bullet_{X/Y}$ is Y-perfect by More on Morphisms of Spaces, Lemma 76.52.14. This proves (1).

Let M be an object of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned}\mathrm{Hom}_Y(M, Rf_*a(\mathcal{O}_Y)) &= \mathrm{Hom}_X(Lf^*M, a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*\mathcal{O}_Y, \mathcal{O}_Y)\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.19.1. The second equality by construction of a. The third equality by Derived Categories of Spaces, Lemma 75.20.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N, see Derived Categories of Spaces, Lemma 75.25.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 15.75.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*a(\mathcal{O}_Y)) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y)) &= \mathrm{Hom}_X(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y), a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, a(\mathcal{O}_Y))), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\ &= R\Gamma(Y, Rf_*E^\vee) \\ &= \mathrm{Hom}_X(E, \mathcal{O}_X)\end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 21.35.2. The second equality is the definition of a. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology on Sites, Lemma 21.48.4. The fourth equality follows from the equality $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega^\bullet_{X/Y}) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ shown in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology on Sites, Lemma 21.48.4) and the fact that Rf_*E is perfect by Derived Categories of Spaces, Lemma 75.25.4 The last equality is Leray for f. This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Spaces, Lemma 75.13.10. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E. Since the perfect objects generate (Derived Categories of Spaces, Theorem 75.15.4) we conclude that α is an isomorphism. □

86.9. Relative dualizing complexes for proper flat morphisms

0E5W Motivated by Duality for Schemes, Sections 48.12 and 48.28 and the material in Section 86.8 we make the following definition.

0E5X **Definition 86.9.1.** Let S be a scheme. Let $f : X \to Y$ be a proper, flat morphism of algebraic spaces over S which is of finite presentation. A *relative dualizing complex* for X/Y is a pair $(\omega^\bullet_{X/Y}, \tau)$ consisting of a Y-perfect object $\omega^\bullet_{X/Y}$ of $D(\mathcal{O}_X)$ and a map

$$\tau : Rf_*\omega^\bullet_{X/Y} \longrightarrow \mathcal{O}_Y$$

such that for any cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where Y' is an affine scheme the pair $(L(g')^*\omega^\bullet_{X/Y}, Lg^*\tau)$ is isomorphic to the pair $(a'(\mathcal{O}_{Y'}), \mathrm{Tr}_{f',\mathcal{O}_{Y'}})$ studied in Sections 86.3, 86.4, 86.5, 86.6, 86.7, and 86.8.

There are several remarks we should make here.

(1) In Definition 86.9.1 one may drop the assumption that $\omega^\bullet_{X/Y}$ is Y-perfect. Namely, running Y' through the members of an étale covering of Y by affines, we see from Lemma 86.8.3 that the restrictions of $\omega^\bullet_{X/Y}$ to the members of an étale covering of X are Y-perfect, which implies $\omega^\bullet_{X/Y}$ is Y-perfect, see More on Morphisms of Spaces, Section 76.52.

(2) Consider a relative dualizing complex $(\omega^\bullet_{X/Y}, \tau)$ and a cartesian square as in Definition 86.9.1. We are going to think of the existence of the isomorphism $(L(g')^*\omega^\bullet_{X/Y}, Lg^*\tau) \cong (a'(\mathcal{O}_{Y'}), \mathrm{Tr}_{f',\mathcal{O}_{Y'}})$ as follows: it says that for any $M' \in D_{QCoh}(\mathcal{O}_{X'})$ the map

$$\mathrm{Hom}_{X'}(M', L(g')^*\omega^\bullet_{X/Y}) \longrightarrow \mathrm{Hom}_{Y'}(Rf'_*M', \mathcal{O}_{Y'}), \quad \varphi' \longmapsto Lg^*\tau \circ Rf'_*\varphi'$$

is an isomorphism. This follows from the definition of a' and the discussion in Section 86.6. In particular, the Yoneda lemma guarantees that the isomorphism is unique.

(3) If Y is affine itself, then a relative dualizing complex $(\omega^\bullet_{X/Y}, \tau)$ exists and is canonically isomorphic to $(a(\mathcal{O}_Y), \mathrm{Tr}_{f,\mathcal{O}_Y})$ where a is the right adjoint for Rf_* as in Lemma 86.3.1 and Tr_f is as in Section 86.6. Namely, given a diagram as in the definition we get an isomorphism $L(g')^*a(\mathcal{O}_Y) \to a'(\mathcal{O}_{Y'})$ by Lemma 86.5.1 which is compatible with trace maps by Lemma 86.6.1.

This produces exactly enough information to glue the locally given relative dualizing complexes to global ones. We suggest the reader skip the proofs of the following lemmas.

0E5Y **Lemma 86.9.2.** *Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. If $(\omega^\bullet_{X/Y}, \tau)$ is a relative dualizing complex, then $\mathcal{O}_X \to R\mathcal{H}om_{\mathcal{O}_X}(\omega^\bullet_{X/Y}, \omega^\bullet_{X/Y})$ is an isomorphism and $Rf_*\omega^\bullet_{X/Y}$ has vanishing cohomology sheaves in positive degrees.*

Proof. It suffices to prove this after base change to an affine scheme étale over Y in which case it follows from Lemma 86.8.3. □

0E5Z **Lemma 86.9.3.** *Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. If $(\omega^\bullet_j, \tau_j)$, $j = 1, 2$ are two relative dualizing complexes on X/Y, then there is a unique isomorphism $(\omega^\bullet_1, \tau_1) \to (\omega^\bullet_2, \tau_2)$.*

Proof. Consider $g : Y' \to Y$ étale with Y' an affine scheme and denote $X' = Y' \times_Y X$ the base change. By Definition 86.9.1 and the discussion following, there is a unique isomorphism $\iota : (\omega^\bullet_1|_{X'}, \tau_1|_{Y'}) \to (\omega^\bullet_2|_{X'}, \tau_2|_{Y'})$. If $Y'' \to Y'$ is a further

étale morphism of affines and $X'' = Y'' \times_Y X$, then $\iota|_{X''}$ is the unique isomorphism $(\omega_1^\bullet|_{X''}, \tau_1|_{Y''}) \to (\omega_2^\bullet|_{X''}, \tau_2|_{Y''})$ (by uniqueness). Also we have

$$\mathrm{Ext}^p_{X'}(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'}) = 0, \quad p < 0$$

because $\mathcal{O}_{X'} \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_1^\bullet|_{X'}) \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'})$ by Lemma 86.9.2.

Choose a étale hypercovering $b : V \to Y$ such that each $V_n = \coprod_{i \in I_n} Y_{n,i}$ with $Y_{n,i}$ affine. This is possible by Hypercoverings, Lemma 25.12.2 and Remark 25.12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote $X_{n,i} = Y_{n,i} \times_Y X$ and $U_n = V_n \times_Y X$ so that we obtain an étale hypercovering $a : U \to X$ (Hypercoverings, Lemma 25.12.4) with $U_n = \coprod X_{n,i}$. The assumptions of Simplicial Spaces, Lemma 85.35.1 are satisfied for $a : U \to X$ and the complexes $\omega_1^\bullet$ and $\omega_2^\bullet$. Hence we obtain a unique morphism $\iota : \omega_1^\bullet \to \omega_2^\bullet$ whose restriction to $X_{0,i}$ is the unique isomorphism $(\omega_1^\bullet|_{X_{0,i}}, \tau_1|_{Y_{0,i}}) \to (\omega_2^\bullet|_{X_{0,i}}, \tau_2|_{Y_{0,i}})$ We still have to see that the diagram

$$\begin{array}{ccc} Rf_*\omega_1^\bullet & \xrightarrow{Rf_*\iota} & Rf_*\omega_1^\bullet \\ & {\scriptstyle\tau_1}\searrow \quad \swarrow{\scriptstyle\tau_2} & \\ & \mathcal{O}_Y & \end{array}$$

is commutative. However, we know that $Rf_*\omega_1^\bullet$ and $Rf_*\omega_2^\bullet$ have vanishing cohomology sheaves in positive degrees (Lemma 86.9.2) thus this commutativity may be proved after restricting to the affines $Y_{0,i}$ where it holds by construction. □

0E60 **Lemma 86.9.4.** *Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. Let $(\omega^\bullet, \tau)$ be a pair consisting of a Y-perfect object of $D(\mathcal{O}_X)$ and a map $\tau : Rf_*\omega^\bullet \to \mathcal{O}_Y$. Assume we have cartesian diagrams*

$$\begin{array}{ccc} X_i & \xrightarrow{g_i'} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ Y_i & \xrightarrow{g_i} & Y \end{array}$$

with Y_i affine such that $\{g_i : Y_i \to Y\}$ is an étale covering and isomorphisms of pairs $(\omega^\bullet|_{X_i}, \tau|_{Y_i}) \to (a_i(\mathcal{O}_{Y_i}), Tr_{f_i, \mathcal{O}_{Y_i}})$ as in Definition 86.9.1. Then $(\omega^\bullet, \tau)$ is a relative dualizing complex for X over Y.

Proof. Let $g : Y' \to Y$ and X', f', g', a' be as in Definition 86.9.1. Set $((\omega')^\bullet, \tau') = (L(g')^*\omega^\bullet, Lg^*\tau)$. We can find a finite étale covering $\{Y'_j \to Y'\}$ by affines which refines $\{Y_i \times_Y Y' \to Y'\}$ (Topologies, Lemma 34.4.4). Thus for each j there is an i_j and a morphism $k_j : Y'_j \to Y_{i_j}$ over Y. Consider the fibre products

$$\begin{array}{ccc} X'_j & \xrightarrow{h_j'} & X' \\ {\scriptstyle f'_j}\downarrow & & \downarrow{\scriptstyle f'} \\ Y'_j & \xrightarrow{h_j} & Y' \end{array}$$

Denote $k'_j : X'_j \to X_{i_j}$ the induced morphism (base change of k_j by f_{i_j}). Restricting the given isomorphisms to Y'_j via the morphism k'_j we get isomorphisms of pairs

$((\omega')^\bullet|_{X'_j}, \tau'|_{Y'_j}) \to (a_j(\mathcal{O}_{Y'_j}), \mathrm{Tr}_{f'_j, \mathcal{O}_{Y'_j}})$. After replacing $f : X \to Y$ by $f' : X' \to Y'$ we reduce to the problem solved in the next paragraph.

Assume Y is affine. Problem: show $(\omega^\bullet, \tau)$ is isomorphic to $(\omega^\bullet_{X/Y}, \mathrm{Tr}) = (a(\mathcal{O}_Y), \mathrm{Tr}_{f, \mathcal{O}_Y})$. We may assume our covering $\{Y_i \to Y\}$ is given by a single surjective étale morphism $\{g : Y' \to Y\}$ of affines. Namely, we can first replace $\{g_i : Y_i \to Y\}$ by a finite subcovering, and then we can set $g = \coprod g_i : Y' = \coprod Y_i \to Y$; some details omitted. Set $X' = Y' \times_Y X$ with maps f', g' as in Definition 86.9.1. Then all we're given is that we have an isomorphism

$$(\omega^\bullet|_{X'}, \tau|_{Y'}) \to (a'(\mathcal{O}_{Y'}), \mathrm{Tr}_{f', \mathcal{O}_{Y'}})$$

Since $(\omega^\bullet_{X/Y}, \mathrm{Tr})$ is a relative dualizing complex (see discussion following Definition 86.9.1) there is a unique isomorphism

$$(\omega^\bullet_{X/Y}|_{X'}, \mathrm{Tr}|_{Y'}) \to (a'(\mathcal{O}_{Y'}), \mathrm{Tr}_{f', \mathcal{O}_{Y'}})$$

Uniqueness by Lemma 86.9.3 for example. Combining the displayed isomorphisms we find an isomorphism

$$\alpha : (\omega^\bullet|_{X'}, \tau|_{Y'}) \to (\omega^\bullet_{X/Y}|_{X'}, \mathrm{Tr}|_{Y'})$$

Set $Y'' = Y' \times_Y Y'$ and $X'' = Y'' \times_Y X$ the two pullbacks of α to X'' have to be the same by uniqueness again. Since we have vanishing negative self exts for $\omega^\bullet_{X'/Y'}$ over X' (Lemma 86.9.2) and since this remains true after pulling back by any projection $Y' \times_Y \ldots \times_Y Y' \to Y'$ (small detail omitted – compare with the proof of Lemma 86.9.3), we find that α descends to an isomorphism $\omega^\bullet \to \omega^\bullet_{X/Y}$ over X by Simplicial Spaces, Lemma 85.35.1. □

0E61 **Lemma 86.9.5.** *Let S be a scheme. Let $X \to Y$ be a proper, flat morphism of algebraic spaces which is of finite presentation. There exists a relative dualizing complex $(\omega^\bullet_{X/Y}, \tau)$.*

Proof. Choose a étale hypercovering $b : V \to Y$ such that each $V_n = \coprod_{i \in I_n} Y_{n,i}$ with $Y_{n,i}$ affine. This is possible by Hypercoverings, Lemma 25.12.2 and Remark 25.12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote $X_{n,i} = Y_{n,i} \times_Y X$ and $U_n = V_n \times_Y X$ so that we obtain an étale hypercovering $a : U \to X$ (Hypercoverings, Lemma 25.12.4) with $U_n = \coprod X_{n,i}$. For each n, i there exists a relative dualizing complex $(\omega^\bullet_{n,i}, \tau_{n,i})$ on $X_{n,i}/Y_{n,i}$. See discussion following Definition 86.9.1. For $\varphi : [m] \to [n]$ and $i \in I_n$ consider the morphisms $g_{\varphi,i} : Y_{n,i} \to Y_{m,\alpha(\varphi)}$ and $g'_{\varphi,i} : X_{n,i} \to X_{m,\alpha(\varphi)}$ which are part of the structure of the given hypercoverings (Hypercoverings, Section 25.12). Then we have a unique isomorphisms

$$\iota_{n,i,\varphi} : (L(g'_{n,i})^* \omega^\bullet_{n,i}, Lg^*_{n,i} \tau_{n,i}) \longrightarrow (\omega^\bullet_{m,\alpha(\varphi)(i)}, \tau_{m,\alpha(\varphi)(i)})$$

of pairs, see discussion following Definition 86.9.1. Observe that $\omega^\bullet_{n,i}$ has vanishing negative self exts on $X_{n,i}$ by Lemma 86.9.2. Denote $(\omega^\bullet_n, \tau_n)$ the pair on U_n/V_n constructed using the pairs $(\omega^\bullet_{n,i}, \tau_{n,i})$ for $i \in I_n$. For $\varphi : [m] \to [n]$ and $i \in I_n$ consider the morphisms $g_\varphi : V_n \to V_m$ and $g'_\varphi : U_n \to U_m$ which are part of the structure of the simplicial algebraic spaces V and U. Then we have unique isomorphisms

$$\iota_\varphi : (L(g'_\varphi)^* \omega^\bullet_n, Lg^*_\varphi \tau_n) \longrightarrow (\omega^\bullet_m, \tau_m)$$

of pairs constructed from the isomorphisms on the pieces. The uniqueness guarantees that these isomorphisms satisfy the transitivity condition as formulated in Simplicial Spaces, Definition 85.14.1. The assumptions of Simplicial Spaces, Lemma 85.35.2 are satisfied for $a : U \to X$, the complexes $\omega_n^\bullet$ and the isomorphisms ι_φ[1]. Thus we obtain an object $\omega^\bullet$ of $D_{QCoh}(\mathcal{O}_X)$ together with an isomorphism $\iota_0 : \omega^\bullet|_{U_0} \to \omega_0^\bullet$ compatible with the two isomorphisms $\iota_{\delta_0^1}$ and $\iota_{\delta_1^1}$. Finally, we apply Simplicial Spaces, Lemma 85.35.1 to find a unique morphism

$$\tau : Rf_*\omega^\bullet \longrightarrow \mathcal{O}_Y$$

whose restriction to V_0 agrees with τ_0; some details omitted – compare with the end of the proof of Lemma 86.9.3 for example to see why we have the required vanishing of negative exts. By Lemma 86.9.4 the pair $(\omega^\bullet, \tau)$ is a relative dualizing complex and the proof is complete. □

0E6C **Lemma 86.9.6.** *Let S be a scheme. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of algebraic spaces over S. Assume $X \to Y$ is proper, flat, and of finite presentation. Let $(\omega_{X/Y}^\bullet, \tau)$ be a relative dualizing complex for f. Then $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau)$ is a relative dualizing complex for f'.

Proof. Observe that $L(g')^*\omega_{X/Y}^\bullet$ is Y'-perfect by More on Morphisms of Spaces, Lemma 76.52.6. The other condition of Definition 86.9.1 holds by transitivity of fibre products. □

86.10. Comparison with the case of schemes

0E6D We should add a lot more in this section.

0E6E **Lemma 86.10.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over S. Assume X and Y are representable and let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* from Lemma 86.3.1. Let $a_0 : D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_{X_0})$ be the right adjoint of Rf_* from Duality for Schemes, Lemma 48.3.1. Then*

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal{\text{Derived Categories of Spaces, Lemma 75.4.2}} & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle a_0}\uparrow & & \uparrow{\scriptstyle a} \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal{\text{Derived Categories of Spaces, Lemma 75.4.2}} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

is commutative.

Proof. Follows from uniqueness of adjoints and the compatibilities of Derived Categories of Spaces, Remark 75.6.3. □

[1]This lemma uses only $\omega_0^\bullet$ and the two maps $\delta_1^1, \delta_0^1 : [1] \to [0]$. The reader can skip the first few lines of the proof of the referenced lemma because here we actually are already given a simplicial system of the derived category of modules.

86.11. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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(15) More on Algebra
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(21) Cohomology on Sites
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CHAPTER 87

Formal Algebraic Spaces

0AHW

87.1. Introduction

0AHX Formal schemes were introduced in [**DG67**]. A more general version of formal schemes was introduced in [**McQ02**] and another in [**Yas09**]. Formal algebraic spaces were introduced in [**Knu71**]. Related material and much besides can be found in [**Abb10**] and [**FK**]. This chapter introduces the notion of formal algebraic spaces we will work with. Our definition is general enough to allow most classes of formal schemes/spaces in the literature as full subcategories.

Although we do discuss the comparison of some of these alternative theories with ours, we do not always give full details when it is not necessary for the logical development of the theory.

Besides introducing formal algebraic spaces, we also prove a few very basic properties and we discuss a few types of morphisms.

87.2. Formal schemes à la EGA

0AHY In this section we review the construction of formal schemes in [**DG67**]. This notion, although very useful in algebraic geometry, may not always be the correct one to consider. Perhaps it is better to say that in the setup of the theory a number of choices are made, where for different purposes others might work better. And indeed in the literature one can find many different closely related theories adapted to the problem the authors may want to consider. Still, one of the major advantages of the theory as sketched here is that one gets to work with definite geometric objects.

Before we start we should point out an issue with the sheaf condition for sheaves of topological rings or more generally sheaves of topological spaces. Namely, the big categories

(1) category of topological spaces,
(2) category of topological groups,
(3) category of topological rings,
(4) category of topological modules over a given topological ring,

endowed with their natural forgetful functors to *Sets* are not examples of types of algebraic structures as defined in Sheaves, Section 6.15. Thus we cannot blithely apply to them the machinery developed in that chapter. On the other hand, each of the categories listed above has limits and equalizers and the forgetful functor to sets, groups, rings, modules commutes with them (see Topology, Lemmas 5.14.1, 5.30.3, 5.30.8, and 5.30.11). Thus we can define the notion of a sheaf as in Sheaves, Definition 6.9.1 and the underlying presheaf of sets, groups, rings, or modules is a

sheaf. The key difference is that for an open covering $U = \bigcup_{i\in I} U_i$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

has to be an equalizer diagram in the category of topological spaces, topological groups, topological rings, topological modules, i.e., that the first map identifies $\mathcal{F}(U)$ with a subspace of $\prod_{i\in I} \mathcal{F}(U_i)$ which is endowed with the product topology.

The stalk $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ of topological spaces, topological groups, topological rings, or topological modules at a point $x \in X$ is defined as the colimit over open neighbourhoods

$$\mathcal{F}_x = \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

in the corresponding category. This is the same as taking the colimit on the level of sets, groups, rings, or modules (see Topology, Lemmas 5.29.1, 5.30.6, 5.30.9, and 5.30.12) but comes equipped with a topology. Warning: the topology one gets depends on which category one is working with, see Examples, Section 110.78. One can sheafify presheaves of topological spaces, topological groups, topological rings, or topological modules and taking stalks commutes with this operation, see Remark 87.2.4.

Let $f : X \to Y$ be a continuous map of topological spaces. There is a functor f_* from the category of sheaves of topological spaces, topological groups, topological rings, topological modules, to the corresponding category of sheaves on Y which is defined by setting $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ as usual. (We delay discussing the pullback in this setting till later.) We define the notion of an f-map $\xi : \mathcal{G} \to \mathcal{F}$ between a sheaf of topological spaces $\mathcal{G}$ on Y and a sheaf of topological spaces $\mathcal{F}$ on X in exactly the same manner as in Sheaves, Definition 6.21.7 with the additional constraint that $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ be continuous for every open $V \subset Y$. We have

$$\{f\text{-maps from } \mathcal{G} \text{ to } \mathcal{F}\} = \operatorname{Mor}_{Sh(Y,Top)}(\mathcal{G}, f_*\mathcal{F})$$

as in Sheaves, Lemma 6.21.8. Similarly for sheaves of topological groups, topological rings, topological modules. Finally, let $\xi : \mathcal{G} \to \mathcal{F}$ be an f-map as above. Then given $x \in X$ with image $y = f(x)$ there is a continuous map

$$\xi_x : \mathcal{G}_y \longrightarrow \mathcal{F}_x$$

of stalks defined in exactly the same manner as in the discussion following Sheaves, Definition 6.21.9.

Using the discussion above, we can define a category $LTRS$ of "locally topologically ringed spaces". An object is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of topological rings $\mathcal{O}_X$ whose stalks $\mathcal{O}_{X,x}$ are local rings (if one forgets about the topology). A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $LTRS$ is a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map of topological spaces and $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is an f-map such that for every $x \in X$ the induced map

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism of local rings (forgetting about the topologies). The composition works in exactly the same manner as composition of morphisms of locally ringed spaces.

Assume now that the topological space X has a basis consisting of quasi-compact opens. Given a sheaf $\mathcal{F}$ of sets, groups, rings, modules over a ring, one can endow $\mathcal{F}$

with the structure of a sheaf of topological spaces, topological groups, topological rings, topological modules. Namely, if $U \subset X$ is quasi-compact open, we endow $\mathcal{F}(U)$ with the discrete topology. If $U \subset X$ is arbitrary, then we choose an open covering $U = \bigcup_{i \in I} U_i$ by quasi-compact opens and we endow $\mathcal{F}(U)$ with the induced topology from $\prod_{i \in I} \mathcal{F}(U_i)$ (as we should do according to our discussion above). The reader may verify (omitted) that we obtain a sheaf of topological spaces, topological groups, topological rings, topological modules in this fashion. Let us say that a sheaf of topological spaces, topological groups, topological rings, topological modules is *pseudo-discrete* if the topology on $\mathcal{F}(U)$ is discrete for every quasi-compact open $U \subset X$. Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological spaces (similarly for groups, rings, modules).

Grothendieck and Dieudonné first define formal affine schemes. These correspond to admissible topological rings A, see More on Algebra, Definition 15.36.1. Namely, given A one considers a fundamental system I_λ of ideals of definition for the ring A. (In any admissible topological ring the family of all ideals of definition forms a fundamental system.) For each λ we can consider the scheme $\mathrm{Spec}(A/I_\lambda)$. For $I_\lambda \subset I_\mu$ the induced morphism

$$\mathrm{Spec}(A/I_\mu) \to \mathrm{Spec}(A/I_\lambda)$$

is a thickening because $I_\mu^n \subset I_\lambda$ for some n. Another way to see this, is to notice that the image of each of the maps

$$\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A)$$

is a homeomorphism onto the set of open prime ideals of A. This motivates the definition

$$\mathrm{Spf}(A) = \{\text{open prime ideals } \mathfrak{p} \subset A\}$$

endowed with the topology coming from $\mathrm{Spec}(A)$. For each λ we can consider the structure sheaf $\mathcal{O}_{\mathrm{Spec}(A/I_\lambda)}$ as a sheaf on $\mathrm{Spf}(A)$. Let $\mathcal{O}_\lambda$ be the corresponding pseudo-discrete sheaf of topological rings, see above. Then we set

$$\mathcal{O}_{\mathrm{Spf}(A)} = \lim \mathcal{O}_\lambda$$

where the limit is taken in the category of sheaves of topological rings. The pair $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$ is called the *formal spectrum* of A.

At this point one should check several things. The first is that the stalks $\mathcal{O}_{\mathrm{Spf}(A),x}$ are local rings (forgetting about the topology). The second is that given $f \in A$, for the corresponding open $D(f) \cap \mathrm{Spf}(A)$ we have

$$\Gamma(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} = \lim (A/I_\lambda)_f$$

as topological rings where I_λ is a fundamental system of ideals of definition as above. Moreover, the ring $A_{\{f\}}$ is admissible too and $(\mathrm{Spf}(A_f), \mathcal{O}_{\mathrm{Spf}(A_{\{f\}})})$ is isomorphic to $(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}|_{D(f) \cap \mathrm{Spf}(A)})$. Finally, given a pair of admissible topological rings A, B we have

0AHZ (87.2.0.1) $$\mathrm{Mor}_{LTRS}((\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}), (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})) = \mathrm{Hom}_{cont}(A, B)$$

where $LTRS$ is the category of "locally topologically ringed spaces" as defined above.

Having said this, in [**DG67**] a *formal scheme* is defined as a pair $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ where $\mathfrak{X}$ is a topological space and $\mathcal{O}_\mathfrak{X}$ is a sheaf of topological rings such that every point

has an open neighbourhood isomorphic (in $LTRS$) to an affine formal scheme. A *morphism of formal schemes* $f : (\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \to (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$ is a morphism in the category $LTRS$.

Let A be a ring endowed with the discrete topology. Then A is admissible and the formal scheme $\mathrm{Spf}(A)$ is equal to $\mathrm{Spec}(A)$. The structure sheaf $\mathcal{O}_{\mathrm{Spf}(A)}$ is the pseudo-discrete sheaf of topological rings associated to $\mathcal{O}_{\mathrm{Spec}(A)}$, in other words, its underlying sheaf of rings is equal to $\mathcal{O}_{\mathrm{Spec}(A)}$ and the ring $\mathcal{O}_{\mathrm{Spf}(A)}(U) = \mathcal{O}_{\mathrm{Spec}(A)}(U)$ over a quasi-compact open U has the discrete topology, but not in general. Thus we can associate to every affine scheme a formal affine scheme. In exactly the same manner we can start with a general scheme $(X, \mathcal{O}_X)$ and associate to it $(X, \mathcal{O}'_X)$ where $\mathcal{O}'_X$ is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is $\mathcal{O}_X$. This construction is compatible with morphisms and defines a functor

0AI0 (87.2.0.2) $$\textit{Schemes} \longrightarrow \textit{Formal Schemes}$$

It follows in a straightforward manner from (87.2.0.1) that this functor is fully faithful.

Let $\mathfrak{X}$ be a formal scheme. Let us define the *size* of the formal scheme by the formula $\mathrm{size}(\mathfrak{X}) = \max(\aleph_0, \kappa_1, \kappa_2)$ where κ_1 is the cardinality of the formal affine opens of $\mathfrak{X}$ and κ_2 is the supremum of the cardinalities of $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$ where $\mathfrak{U} \subset \mathfrak{X}$ is such a formal affine open.

0AI1 **Lemma 87.2.1.** *Choose a category of schemes Sch_α as in Sets, Lemma 3.9.2. Given a formal scheme $\mathfrak{X}$ let*

$$h_\mathfrak{X} : (Sch_\alpha)^{opp} \longrightarrow Sets, \quad h_\mathfrak{X}(S) = \mathrm{Mor}_{Formal\ Schemes}(S, \mathfrak{X})$$

be its functor of points. Then we have

$$\mathrm{Mor}_{Formal\ Schemes}(\mathfrak{X}, \mathfrak{Y}) = \mathrm{Mor}_{PSh(Sch_\alpha)}(h_\mathfrak{X}, h_\mathfrak{Y})$$

provided the size of $\mathfrak{X}$ is not too large.

Proof. First we observe that $h_\mathfrak{X}$ satisfies the sheaf property for the Zariski topology for any formal scheme $\mathfrak{X}$ (see Schemes, Definition 26.15.3). This follows from the local nature of morphisms in the category of formal schemes. Also, for an open immersion $\mathfrak{V} \to \mathfrak{W}$ of formal schemes, the corresponding transformation of functors $h_\mathfrak{V} \to h_\mathfrak{W}$ is injective and representable by open immersions (see Schemes, Definition 26.15.3). Choose an open covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$ of a formal scheme by affine formal schemes $\mathfrak{U}_i$. Then the collection of functors $h_{\mathfrak{U}_i}$ covers $h_\mathfrak{X}$ (see Schemes, Definition 26.15.3). Finally, note that

$$h_{\mathfrak{U}_i} \times_{h_\mathfrak{X}} h_{\mathfrak{U}_j} = h_{\mathfrak{U}_i \cap \mathfrak{U}_j}$$

Hence in order to give a map $h_\mathfrak{X} \to h_\mathfrak{Y}$ is equivalent to giving a family of maps $h_{\mathfrak{U}_i} \to h_\mathfrak{Y}$ which agree on overlaps. Thus we can reduce the bijectivity (resp. injectivity) of the map of the lemma to bijectivity (resp. injectivity) for the pairs $(\mathfrak{U}_i, \mathfrak{Y})$ and injectivity (resp. nothing) for $(\mathfrak{U}_i \cap \mathfrak{U}_j, \mathfrak{Y})$. In this way we reduce to the case where $\mathfrak{X}$ is an affine formal scheme. Say $\mathfrak{X} = \mathrm{Spf}(A)$ for some admissible topological ring A. Also, choose a fundamental system of ideals of definition $I_\lambda \subset A$.

We can also localize on $\mathfrak{Y}$. Namely, suppose that $\mathfrak{V} \subset \mathfrak{Y}$ is an open formal subscheme and $\varphi : h_\mathfrak{X} \to h_\mathfrak{Y}$. Then

$$h_\mathfrak{V} \times_{h_\mathfrak{Y}, \varphi} h_\mathfrak{X} \to h_\mathfrak{X}$$

is representable by open immersions. Pulling back to $\mathrm{Spec}(A/I_\lambda)$ for all λ we find an open subscheme $U_\lambda \subset \mathrm{Spec}(A/I_\lambda)$. However, for $I_\lambda \subset I_\mu$ the morphism $\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A/I_\mu)$ pulls back U_μ to U_λ. Thus these glue to give an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$. A straightforward argument (omitted) shows that

$$h_{\mathfrak{U}} = h_{\mathfrak{V}} \times_{h_{\mathfrak{Y}}} h_{\mathfrak{X}}$$

In this way we see that given an open covering $\mathfrak{Y} = \bigcup \mathfrak{V}_j$ and a transformation of functors $\varphi : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ we obtain a corresponding open covering of $\mathfrak{X}$. Since $\mathfrak{X}$ is affine, we can refine this covering by a finite open covering $\mathfrak{X} = \mathfrak{U}_1 \cup \ldots \cup \mathfrak{U}_n$ by affine formal subschemes. In other words, for each i there is a j and a map $\varphi_i : h_{\mathfrak{U}_i} \to h_{\mathfrak{V}_j}$ such that

$$\begin{array}{ccc} h_{\mathfrak{U}_i} & \xrightarrow{\varphi_i} & h_{\mathfrak{V}_j} \\ \downarrow & & \downarrow \\ h_{\mathfrak{X}} & \xrightarrow{\varphi} & h_{\mathfrak{Y}} \end{array}$$

commutes. With a few additional arguments (which we omit) this implies that it suffices to prove the bijectivity of the lemma in case both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes.

Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes. Say $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Let $\varphi : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ be a transformation of functors. Let $I_\lambda \subset A$ be a fundamental system of ideals of definition. The canonical inclusion morphism $i_\lambda : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{X}$ maps to a morphism $\varphi(i_\lambda) : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{Y}$. By (87.2.0.1) this corresponds to a continuous map $\chi_\lambda : B \to A/I_\lambda$. Since φ is a transformation of functors it follows that for $I_\lambda \subset I_\mu$ the composition $B \to A/I_\lambda \to A/I_\mu$ is equal to χ_μ. In other words we obtain a ring map

$$\chi = \lim \chi_\lambda : B \longrightarrow \lim A/I_\lambda = A$$

This is a continuous homomorphism because the inverse image of I_λ is open for all λ (as A/I_λ has the discrete topology and χ_λ is continuous). Thus we obtain a morphism $\mathrm{Spf}(\chi) : \mathfrak{X} \to \mathfrak{Y}$ by (87.2.0.1). We omit the verification that this construction is the inverse to the map of the lemma in this case.

Set theoretic remarks. To make this work on the given category of schemes Sch_α we just have to make sure all the schemes used in the proof above are isomorphic to objects of Sch_α. In fact, a careful analysis shows that it suffices if the schemes $\mathrm{Spec}(A/I_\lambda)$ occurring above are isomorphic to objects of Sch_α. For this it certainly suffices to assume the size of $\mathfrak{X}$ is at most the size of a scheme contained in Sch_α. □

0AI2 **Lemma 87.2.2.** *Let $\mathfrak{X}$ be a formal scheme. The functor of points $h_{\mathfrak{X}}$ (see Lemma 87.2.1) satisfies the sheaf condition for fpqc coverings.*

Proof. Topologies, Lemma 34.9.14 reduces us to the case of a Zariski covering and a covering $\{\mathrm{Spec}(S) \to \mathrm{Spec}(R)\}$ with $R \to S$ faithfully flat. We observed in the proof of Lemma 87.2.1 that $h_{\mathfrak{X}}$ satisfies the sheaf condition for Zariski coverings.

Suppose that $R \to S$ is a faithfully flat ring map. Denote $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the corresponding morphism of schemes. It is surjective and flat. Let $f : \mathrm{Spec}(S) \to \mathfrak{X}$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as maps $\mathrm{Spec}(S \otimes_R S) \to \mathfrak{X}$. By Descent, Lemma 35.13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$

for some (set theoretic) map $g : \mathrm{Spec}(R) \to \mathfrak{X}$. By Morphisms, Lemma 29.25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in \mathrm{Spec}(R)$. Choose $\mathfrak{U} \subset \mathfrak{X}$ an affine formal open subscheme containing $g(y)$. Say $\mathfrak{U} = \mathrm{Spf}(A)$ for some admissible topological ring A. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(\mathfrak{U})$. The restriction of f to $\pi^{-1}(D(r))$ into $\mathfrak{U}$ corresponds to a continuous ring map $A \to S_r$ by (87.2.0.1). The two induced ring maps $A \to S_r \otimes_{R_r} S_r = (S \otimes_R S)_r$ are equal by assumption on f. Note that $R_r \to S_r$ is faithfully flat. By Descent, Lemma 35.3.6 the equalizer of the two arrows $S_r \to S_r \otimes_{R_r} S_r$ is R_r. We conclude that $A \to S_r$ factors uniquely through a map $A \to R_r$ which is also continuous as it has the same (open) kernel as the map $A \to S_r$. This map in turn gives a morphism $D(r) \to \mathfrak{U}$ by (87.2.0.1).

What have we proved so far? We have shown that for any $y \in \mathrm{Spec}(R)$ there exists a standard affine open $y \in D(r) \subset \mathrm{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to \mathfrak{X}$ factors uniquely though some morphism $D(r) \to \mathfrak{X}$. We omit the verification that these morphisms glue to the desired morphism $\mathrm{Spec}(R) \to \mathfrak{X}$. □

0AI3 **Remark 87.2.3** (McQuillan's variant)**.** There is a variant of the construction of formal schemes due to McQuillan, see [**McQ02**]. He suggests a slight weakening of the condition of admissibility. Namely, recall that an admissible topological ring is a complete (and separated by our conventions) topological ring A which is linearly topologized such that there exists an ideal of definition: an open ideal I such that any neighbourhood of 0 contains I^n for some $n \geq 1$. McQuillan works with what we will call *weakly admissible* topological rings. A weakly admissible topological ring A is a complete (and separated by our conventions) topological ring which is linearly topologized such that there exists an *weak ideal of definition*: an open ideal I such that for all $f \in I$ we have $f^n \to 0$ for $n \to \infty$. Similarly to the admissible case, if I is a weak ideal of definition and $J \subset A$ is an open ideal, then $I \cap J$ is a weak ideal of definition. Thus the weak ideals of definition form a fundamental system of open neighbourhoods of 0 and one can proceed along much the same route as above to define a larger category of formal schemes based on this notion. The analogues of Lemmas 87.2.1 and 87.2.2 still hold in this setting (with the same proof).

0AI4 **Remark 87.2.4** (Sheafification of presheaves of topological spaces)**.** In this remark we briefly discuss sheafification of presheaves of topological spaces. The exact same arguments work for presheaves of topological abelian groups, topological rings, and topological modules (over a given topological ring). In order to do this in the correct generality let us work over a site $\mathcal{C}$. The reader who is interested in the case of (pre)sheaves over a topological space X should think of objects of $\mathcal{C}$ as the opens of X, of morphisms of $\mathcal{C}$ as inclusions of opens, and of coverings in $\mathcal{C}$ as coverings in X, see Sites, Example 7.6.4. Denote $Sh(\mathcal{C}, Top)$ the category of sheaves of topological spaces on $\mathcal{C}$ and denote $PSh(\mathcal{C}, Top)$ the category of presheaves of topological spaces on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of topological spaces on $\mathcal{C}$. The sheafification $\mathcal{F}^\#$ should satisfy the formula [**Gra65**]

$$\mathrm{Mor}_{PSh(\mathcal{C}, Top)}(\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C}, Top)}(\mathcal{F}^\#, \mathcal{G})$$

functorially in $\mathcal{G}$ from $Sh(\mathcal{C}, Top)$. In other words, we are trying to construct the left adjoint to the inclusion functor $Sh(\mathcal{C}, Top) \to PSh(\mathcal{C}, Top)$. We first claim that

$Sh(\mathcal{C}, Top)$ has limits and that the inclusion functor commutes with them. Namely, given a category $\mathcal{I}$ and a functor $i \mapsto \mathcal{G}_i$ into $Sh(\mathcal{C}, Top)$ we simply define

$$(\lim \mathcal{G}_i)(U) = \lim \mathcal{G}_i(U)$$

where we take the limit in the category of topological spaces (Topology, Lemma 5.14.1). This defines a sheaf because limits commute with limits (Categories, Lemma 4.14.10) and in particular products and equalizers (which are the operations used in the sheaf axiom). Finally, a morphism of presheaves from $\mathcal{F} \to \lim \mathcal{G}_i$ is clearly the same thing as a compatible system of morphisms $\mathcal{F} \to \mathcal{G}_i$. In other words, the object $\lim \mathcal{G}_i$ is the limit in the category of presheaves of topological spaces and a fortiori in the category of sheaves of topological spaces. Our second claim is that any morphism of presheaves $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ an object of $Sh(\mathcal{C}, Top)$ factors through a subsheaf $\mathcal{G}' \subset \mathcal{G}$ whose size is bounded. Here we define the *size* $|\mathcal{H}|$ of a sheaf of topological spaces $\mathcal{H}$ to be the cardinal $\sup_{U \in \mathrm{Ob}(\mathcal{C})} |\mathcal{H}(U)|$. To prove our claim we let

$$\mathcal{G}'(U) = \left\{ \begin{array}{c|l} s \in \mathcal{G}(U) & \begin{array}{l}\text{there exists a covering } \{U_i \to U\}_{i \in I} \\ \text{such that } s|_{U_i} \in \mathrm{Im}(\mathcal{F}(U_i) \to \mathcal{G}(U_i))\end{array} \end{array} \right\}$$

We endow $\mathcal{G}'(U)$ with the induced topology. Then $\mathcal{G}'$ is a sheaf of topological spaces (details omitted) and $\mathcal{G}' \to \mathcal{G}$ is a morphism through which the given map $\mathcal{F} \to \mathcal{G}$ factors. Moreover, the size of $\mathcal{G}'$ is bounded by some cardinal κ depending only on $\mathcal{C}$ and the presheaf $\mathcal{F}$ (hint: use that coverings in $\mathcal{C}$ form a set by our conventions). Putting everything together we see that the assumptions of Categories, Theorem 4.25.3 are satisfied and we obtain sheafification as the left adjoint of the inclusion functor from sheaves to presheaves. Finally, let p be a point of the site $\mathcal{C}$ given by a functor $u : \mathcal{C} \to Sets$, see Sites, Definition 7.32.2. For a topological space M the presheaf defined by the rule

$$U \mapsto \mathrm{Map}(u(U), M) = \prod\nolimits_{x \in u(U)} M$$

endowed with the product topology is a sheaf of topological spaces. Hence the exact same argument as given in the proof of Sites, Lemma 7.32.5 shows that $\mathcal{F}_p = \mathcal{F}_p^\#$, in other words, sheafification commutes with taking stalks at a point.

87.3. Conventions and notation

0AI5 The conventions from now on will be similar to the conventions in Properties of Spaces, Section 66.2. Thus from now on the standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site. For topological rings A we assume only that all discrete quotients have this property (but usually we assume more, compare with Remark 87.11.5).

Let S be a scheme and let X be a "space" over S, i.e., a sheaf on $(Sch/S)_{fppf}$. In this chapter we will write $X \times_S X$ for the product of X with itself in the category of sheaves on $(Sch/S)_{fppf}$ instead of $X \times X$. Moreover, if X and Y are "spaces" then we say "let $f : X \to Y$ be a morphism" to indicate that f is a natural transformation of functors, i.e., a map of sheaves on $(Sch/S)_{fppf}$. Similarly, if U is a scheme over S and X is a "space" over S, then we say "let $f : U \to X$ be a morphism" or "let $g : X \to U$ be a morphism" to indicate that f or g is a map of sheaves $h_U \to X$ or $X \to h_U$ where h_U is as in Categories, Example 4.3.4.

87.4. Topological rings and modules

0AMQ This section is a continuation of More on Algebra, Section 15.36. Let R be a topological ring and let M be a linearly topologized R-module. When we say "*let M_λ be a fundamental system of open submodules*" we will mean that each M_λ is an open submodule and that any neighbourhood of 0 contains one of the M_λ. In other words, this means that M_λ is a fundamental system of neighbourhoods of 0 in M consisting of submodules. Similarly, if R is a linearly topologized ring, then we say "*let I_λ be a fundamental system of open ideals*" to mean that I_λ is a fundamental system of neighbourhoods of 0 in R consisting of ideals.

0AMR **Example 87.4.1.** Let R be a linearly topologized ring and let M be a linearly topologized R-module. Let I_λ be a fundamental system of open ideals in R and let M_μ be a fundamental system of open submodules of M. The continuity of $+ : M \times M \to M$ is automatic and the continuity of $R \times M \to M$ signifies

$$\forall f, x, \mu \ \exists \lambda, \nu, \ (f + I_\lambda)(x + M_\nu) \subset fx + M_\mu$$

Since $fM_\nu + I_\lambda M_\nu \subset M_\mu$ if $M_\nu \subset M_\mu$ we see that the condition is equivalent to

$$\forall x, \mu \ \exists \lambda \ I_\lambda x \subset M_\mu$$

However, it need not be the case that given μ there is a λ such that $I_\lambda M \subset M_\mu$. For example, consider $R = k[[t]]$ with the t-adic topology and $M = \bigoplus_{n \in \mathbf{N}} R$ with fundamental system of open submodules given by

$$M_m = \bigoplus\nolimits_{n \in \mathbf{N}} t^{nm} R$$

Since every $x \in M$ has finitely many nonzero coordinates we see that, given m and x there exists a k such that $t^k x \in M_m$. Thus M is a linearly topologized R-module, but it isn't true that given m there is a k such that $t^k M \subset M_m$. On the other hand, if $R \to S$ is a continuous map of linearly topologized rings, then the corresponding statement does hold, i.e., for every open ideal $J \subset S$ there exists an open ideal $I \subset R$ such that $IS \subset J$ (as the reader can easily deduce from continuity of the map $R \to S$).

0AMS **Lemma 87.4.2.** *Let R be a topological ring. Let M be a linearly topologized R-module and let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Let $N \subset M$ be a submodule. The closure of N is $\bigcap_{\lambda \in \Lambda}(N + M_\lambda)$.*

Proof. Since each $N + M_\lambda$ is open, it is also closed. Hence the intersection is closed. If $x \in M$ is not in the closure of N, then $(x + M_\lambda) \cap N = 0$ for some λ. Hence $x \notin N + M_\lambda$. This proves the lemma. □

Unless otherwise mentioned we endow submodules and quotient modules with the induced topology. Let M be a linearly topologized module over a topological ring R, and let $0 \to N \to M \to Q \to 0$ be a short exact sequence of R-modules. If M_λ is a fundamental system of open submodules of M, then $N \cap M_\lambda$ is a fundamental system of open submodules of N. If $\pi : M \to Q$ is the quotient map, then $\pi(M_\lambda)$ is a fundamental system of open submodules of Q. In particular these induced topologies are linear topologies.

0ARZ **Lemma 87.4.3.** *Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Then*

(1) *$0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ is exact, and*

(2) $N^\wedge$ *is the closure of the image of* $N \to M^\wedge$.

Proof. Let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Then $N \cap M_\lambda$ is a fundamental system of open submodules of N and $M_\lambda + N/N$ is a fundamental system of open submodules of M/N. Thus we see that (1) follows from the exactness of the sequences

$$0 \to N/N \cap M_\lambda \to M/M_\lambda \to M/(M_\lambda + N) \to 0$$

and the fact that taking limits commutes with limits. The second statement follows from this and the fact that $N \to N^\wedge$ has dense image and that the kernel of $M^\wedge \to (M/N)^\wedge$ is closed. □

0AMT **Lemma 87.4.4.** *Let R be a topological ring. Let M be a complete, linearly topologized R-module. Let $N \subset M$ be a closed submodule. If M has a countable fundamental system of neighbourhoods of 0, then M/N is complete and the map $M \to M/N$ is open.*

Proof. Let M_n, $n \in \mathbf{N}$ be a fundamental system of open submodules of M. We may assume $M_{n+1} \subset M_n$ for all n. The system $(M_n + N)/N$ is a fundamental system in M/N. Hence we have to show that $M/N = \lim M/(M_n + N)$. Consider the short exact sequences

$$0 \to N/N \cap M_n \to M/M_n \to M/(M_n + N) \to 0$$

Since the transition maps of the system $\{N/N \cap M_n\}$ are surjective we see that $M = \lim M/M_n$ (by completeness of M) surjects onto $\lim M/(M_n + N)$ by Algebra, Lemma 10.86.4. As N is closed we see that the kernel of $M \to \lim M/(M_n + N)$ is N (see Lemma 87.4.2). Finally, $M \to M/N$ is open by definition of the quotient topology. □

0AS0 **Lemma 87.4.5.** *Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Assume M has a countable fundamental system of neighbourhoods of 0. Then* [**Mat86**, Theorem 8.1]

(1) $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge \to 0$ *is exact,*
(2) $N^\wedge$ *is the closure of the image of* $N \to M^\wedge$*,*
(3) $M^\wedge \to (M/N)^\wedge$ *is open.*

Proof. We have $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ is exact and statement (2) by Lemma 87.4.3. This produces a canonical map $c : M^\wedge/N^\wedge \to (M/N)^\wedge$. The module $M^\wedge/N^\wedge$ is complete and $M^\wedge \to M^\wedge/N^\wedge$ is open by Lemma 87.4.4. By the universal property of completion we obtain a canonical map $b : (M/N)^\wedge \to M^\wedge/N^\wedge$. Then b and c are mutually inverse as they are on a dense subset. □

0F1S **Lemma 87.4.6.** *Let R be a topological ring. Let M be a topological R-module. Let $I \subset R$ be a finitely generated ideal. Assume M has an open submodule whose topology is I-adic. Then $M^\wedge$ has an open submodule whose topology is I-adic and we have $M^\wedge/I^nM^\wedge = M/I^nM$ for all $n \geq 1$.*

Proof. Let $M' \subset M$ be an open submodule whose topology is I-adic. Then $\{I^nM'\}_{n\geq 1}$ is a fundamental system of open submodules of M. Thus $M^\wedge = \lim M/I^nM'$ contains $(M')^\wedge = \lim M'/I^nM'$ as an open submodule and the topology on $(M')^\wedge$ is I-adic by Algebra, Lemma 10.96.3. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Observe that the surjection $(f_1, \ldots, f_r)$:

$M^{\oplus r} \to I^n M$ is continuous and open by our description of the topology on M above. By Lemma 87.4.5 applied to this surjection and to the short exact sequence $0 \to I^n M \to M \to M/I^n M \to 0$ we conclude that

$$(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \longrightarrow M^\wedge$$

surjects onto the kernel of the surjection $M^\wedge \to M/I^n M$. Since $f_1, \ldots, f_r$ generate I^n we conclude. □

0AMU **Definition 87.4.7.** Let R be a topological ring. Let M and N be linearly topologized R-modules. The *tensor product* of M and N is the (usual) tensor product $M \otimes_R N$ endowed with the linear topology defined by declaring

$$\mathrm{Im}(M_\mu \otimes_R N + M \otimes_R N_\nu \longrightarrow M \otimes_R N)$$

to be a fundamental system of open submodules, where $M_\mu \subset M$ and $N_\nu \subset N$ run through fundamental systems of open submodules in M and N. The *completed tensor product*

$$M \widehat{\otimes}_R N = \lim M \otimes_R N/(M_\mu \otimes_R N + M \otimes_R N_\nu) = \lim M/M_\mu \otimes_R N/N_\nu$$

is the completion of the tensor product.

Observe that the topology on R is immaterial for the construction of the tensor product or the completed tensor product. If $R \to A$ and $R \to B$ are continuous maps of linearly topologized rings, then the construction above gives a tensor product $A \otimes_R B$ and a completed tensor product $A \widehat{\otimes}_R B$.

We record here the notions introduced in Remark 87.2.3.

0AMV **Definition 87.4.8.** Let A be a linearly topologized ring.

(1) An element $f \in A$ is called *topologically nilpotent* if $f^n \to 0$ as $n \to \infty$.
(2) A *weak ideal of definition* for A is an open ideal $I \subset A$ consisting entirely of topologically nilpotent elements.
(3) We say A is *weakly pre-admissible* if A has a weak ideal of definition.
(4) We say A is *weakly admissible* if A is weakly pre-admissible and complete[1].

Given a weak ideal of definition I in a linearly topologized ring A and an open ideal J the intersection $I \cap J$ is a weak ideal of definition. Hence if there is one weak ideal of definition, then there is a fundamental system of open ideals consisting of weak ideals of definition. In particular, given a weakly admissible topological ring A then $A = \lim A/I_\lambda$ where $\{I_\lambda\}$ is a fundamental system of weak ideals of definition.

0DCZ **Lemma 87.4.9.** *Let A be a weakly admissible topological ring. Let $I \subset A$ be a weak ideal of definition. Then (A, I) is a henselian pair.*

Proof. Let $A \to A'$ be an étale ring map and let $\sigma : A' \to A/I$ be an A-algebra map. By More on Algebra, Lemma 15.11.6 it suffices to lift σ to an A-algebra map $A' \to A$. To do this, as A is complete, it suffices to find, for every open ideal $J \subset I$, a unique A-algebra map $A' \to A/J$ lifting σ. Since I is a weak ideal of definition, the ideal I/J is locally nilpotent. We conclude by More on Algebra, Lemma 15.11.2. □

[1] By our conventions this includes separated.

0AMW **Lemma 87.4.10.** *Let B be a linearly topologized ring. The set of topologically nilpotent elements of B is a closed, radical ideal of B. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings.*

(1) *If $f \in A$ is topologically nilpotent, then $\varphi(f)$ is topologically nilpotent.*
(2) *If $I \subset A$ consists of topologically nilpotent elements, then the closure of $\varphi(I)B$ consists of topologically nilpotent elements.*

Proof. Let $\mathfrak{b} \subset B$ be the set of topologically nilpotent elements. We omit the proof of the fact that $\mathfrak{b}$ is a radical ideal (good exercise in the definitions). Let g be an element of the closure of $\mathfrak{b}$. Our goal is to show that g is topologically nilpotent. Let $J \subset B$ be an open ideal. We have to show $g^e \in J$ for some $e \geq 1$. We have $g \in \mathfrak{b} + J$ by Lemma 87.4.2. Hence $g = f + h$ for some $f \in \mathfrak{b}$ and $h \in J$. Pick $m \geq 1$ such that $f^m \in J$. Then $g^{m+1} \in J$ as desired.

Let $\varphi : A \to B$ be as in the statement of the lemma. Assertion (1) is clear and assertion (2) follows from this and the fact that $\mathfrak{b}$ is a closed ideal. □

0AMZ **Lemma 87.4.11.** *Let $A \to B$ be a continuous map of linearly topologized rings. Let $I \subset A$ be an ideal. The closure of IB is the kernel of $B \to B\widehat{\otimes}_A A/I$.*

Proof. Let J_μ be a fundamental system of open ideals of B. The closure of IB is $\bigcap(IB + J_\lambda)$ by Lemma 87.4.2. Let I_μ be a fundamental system of open ideals in A. Then

$$B\widehat{\otimes}_A A/I = \lim(B/J_\lambda \otimes_A A/(I_\mu + I)) = \lim B/(J_\lambda + I_\mu B + IB)$$

Since $A \to B$ is continuous, for every λ there is a μ such that $I_\mu B \subset J_\lambda$, see discussion in Example 87.4.1. Hence the limit can be written as $\lim B/(J_\lambda + IB)$ and the result is clear. □

0GB4 **Lemma 87.4.12.** *Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings.*

(1) *If A and C are weakly pre-admissible, then $A\widehat{\otimes}_B C$ is weakly admissible.*
(2) *If A and C are pre-admissible, then $A\widehat{\otimes}_B C$ is admissible.*
(3) *If A and C have a countable fundamental system of open ideals, then $A\widehat{\otimes}_B C$ has a countable fundamental system of open ideals.*
(4) *If A and C are pre-adic and have finitely generated ideals of definition, then $A\widehat{\otimes}_B C$ is adic and has a finitely generated ideal of definition.*
(5) *If A and C are pre-adic Noetherian rings and $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the ideals of topologically nilpotent elements, then $A\widehat{\otimes}_B C$ is adic Noetherian.*

Proof. Let $I_\lambda \subset A$, $\lambda \in \Lambda$ and $J_\mu \subset C$, $\mu \in M$ be fundamental systems of open ideals, then by definition

$$A\widehat{\otimes}_B C = \lim\nolimits_{\lambda,\mu} A/I_\lambda \otimes_B C/J_\mu$$

with the limit topology. Thus a fundamental system of open ideals is given by the kernels $K_{\lambda,\mu}$ of the maps $A\widehat{\otimes}_B C \to A/I_\lambda \otimes_B C/J_\mu$. Note that $K_{\lambda,\mu}$ is the closure of the ideal $I_\lambda(A\widehat{\otimes}_B C)+J_\mu(A\widehat{\otimes}_B C)$. Finally, we have a ring homomorphism $\tau : A \otimes_B C \to A\widehat{\otimes}_B C$ with dense image.

Proof of (1). If I_λ and J_μ consist of topologically nilpotent elements, then so does $K_{\lambda,\mu}$ by Lemma 87.4.10. Hence $A\widehat{\otimes}_B C$ is weakly admissible by definition.

Proof of (2). Assume for some λ_0 and μ_0 the ideals $I = I_{\lambda_0} \subset A$ and $J_{\mu_0} \subset C$ are ideals of definition. Thus for every λ there exists an n such that $I^n \subset I_\lambda$. For every μ there exists an m such that $J^m \subset J_\mu$. Then

$$\big(I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)\big)^{n+m} \subset I_\lambda(A\widehat{\otimes}_B C) + J_\mu(A\widehat{\otimes}_B C)$$

It follows that the open ideal $K = K_{\lambda_0,\mu_0}$ satisfies $K^{n+m} \subset K_{\lambda,\mu}$. Hence K is an ideal of definition of $A\widehat{\otimes}_B C$ and $A\widehat{\otimes}_B C$ is admissible by definition.

Proof of (3). If Λ and M are countable, so is $\Lambda \times M$.

Proof of (4). Assume $\Lambda = \mathbf{N}$ and $M = \mathbf{N}$ and we have finitely generated ideals $I \subset A$ and $J \subset C$ such that $I_n = I^n$ and $J_n = J^n$. Then

$$I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)$$

is a finitely generated ideal and it is easily seen that $A\widehat{\otimes}_B C$ is the completion of $A \otimes_B C$ with respect to this ideal. Hence (4) follows from Algebra, Lemma 10.96.3.

Proof of (5). Let $\mathfrak{c} \subset C$ be the ideal of topologically nilpotent elements. Since A and C are adic Noetherian, we see that $\mathfrak{a}$ and $\mathfrak{c}$ are ideals of definition (details omitted). From part (4) we already know that $A\widehat{\otimes}_B C$ is adic and that $\mathfrak{a}(A\widehat{\otimes}_B C) + \mathfrak{c}(A\widehat{\otimes}_B C)$ is a finitely generated ideal of definition. Since

$$A\widehat{\otimes}_B C/\left(\mathfrak{a}(A\widehat{\otimes}_B C) + \mathfrak{c}(A\widehat{\otimes}_B C)\right) = A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$$

is Noetherian as a finite type algebra over the Noetherian ring $C/\mathfrak{c}$ we conclude by Algebra, Lemma 10.97.5. □

87.5. Taut ring maps

0GX1 It turns out to be convenient to have a name for the following property of continuous maps between linearly topologized rings.

0AMX **Definition 87.5.1.** Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. We say φ is *taut*[2] if for every open ideal $I \subset A$ the closure of the ideal $\varphi(I)B$ is open and these closures form a fundamental system of open ideals.

If $\varphi : A \to B$ is a continuous map of linearly topologized rings and I_λ a fundamental system of open ideals of A, then φ is taut if and only if the closures of $I_\lambda B$ are open and form a fundamental system of open ideals in A.

0AMY **Lemma 87.5.2.** *Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings. The following are equivalent*

(1) *φ is taut,*
(2) *for every weak ideal of definition $I \subset A$ the closure of $\varphi(I)B$ is a weak ideal of definition of B and these form a fundamental system of weak ideals of definition of B.*

Proof. The remarks following Definition 87.5.1 show that (2) implies (1). Conversely, assume φ is taut. If $I \subset A$ is a weak ideal of definition, then the closure of $\varphi(I)B$ is open by definition of tautness and consists of topologically nilpotent elements by Lemma 87.4.10. Hence the closure of $\varphi(I)B$ is a weak ideal of definition.

[2]This is nonstandard notation. The definition generalizes to modules, by saying a linearly topologized A-module M is A-taut if for every open ideal $I \subset A$ the closure of IM in M is open and these closures form a fundamental system of neighbourhoods of 0 in M.

Furthermore, by definition of tautness these ideals form a fundamental system of open ideals and we see that (2) is true. □

0GX2 **Lemma 87.5.3.** *Let A be a linearly topologized ring. The map $A \to A^\wedge$ from A to its completion is taut.*

Proof. Let I_λ be a fundamental system of open ideals of A. Recall that $A^\wedge = \lim A/I_\lambda$ with the limit topology, which means that the kernels $J_\lambda = \mathrm{Ker}(A^\wedge \to A/I_\lambda)$ form a fundamental system of open ideals of $A^\wedge$. Since J_λ is the closure of $I_\lambda A^\wedge$ (compare with Lemma 87.4.11) we conclude. □

0GX3 **Lemma 87.5.4.** *Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ and $B \to C$ are taut, then $A \to C$ is taut.*

Proof. Omitted. Hint: if $I \subset A$ is an ideal and J is the closure of IB, then the closure of JC is equal to the closure of IC. □

0GX4 **Lemma 87.5.5.** *Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to C$ is taut, then $B \to C$ is taut.*

Proof. Let $J \subset B$ be an open ideal with inverse image $I \subset A$. Then the closure of JC contains the closure of IC. Hence this closure is open as $A \to C$ is taut. Let I_λ be a fundamental system of open ideals of A. Let K_λ be the closure of $I_\lambda C$. Since $A \to C$ is taut, these form a fundamental system of open ideals of C. Denote $J_\lambda \subset B$ the inverse image of K_λ. Then the closure of $J_\lambda C$ is K_λ. Hence we see that the closures of the ideals JC, where J runs over the open ideals of B form a fundamental system of open ideals of C. □

0GX5 **Lemma 87.5.6.** *Let $A \to B$ and $A \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ is taut, then $C \to B\widehat{\otimes}_A C$ is taut.*

Proof. Let $K \subset C$ be an open ideal. Choose any open ideal $I \subset A$ whose image in C is contained in J. By assumption the closure J of IB is open. Since $A \to B$ is taut we see that $B\widehat{\otimes}_A C$ is the limit of the rings $B/J \otimes_{A/I} C/K$ over all choices of K and I, i.e, the ideals $J(B\widehat{\otimes}_A C) + K(B\widehat{\otimes}_A C)$ form a fundamental system of open ideals. Now, since $B \to B\widehat{\otimes}_A C$ is continuous we see that J maps into the closure of $K(B\widehat{\otimes}_A C)$ (as I maps into K). Hence this closure is equal to $J(B\widehat{\otimes}_A C) + K(B\widehat{\otimes}_A C)$ and the proof is complete. □

0GX6 **Lemma 87.5.7.** *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A has a countable fundamental system of open ideals, then B has a countable fundamental system of open ideals.*

Proof. Immediate from the definitions. □

0GX7 **Lemma 87.5.8.** *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-admissible, then B is weakly pre-admissible.*

Proof. Let $I \subset A$ be a weak ideal of definition. Then the closure J of IB is open and consists of topologically nilpotent elements by Lemma 87.4.10. Hence J is a weak ideal of definition of B. □

0GX8 **Lemma 87.5.9.** *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is pre-admissible, then B is pre-admissible.*

Proof. Let $I \subset A$ be an ideal of definition. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Then the closure J of IB is open and the closures J_λ of $I_\lambda B$ are open and form a fundamental system of open ideals of B. For every λ there is an n such that $I^n \subset I_\lambda$. Observe that J^n is contained in the closure of $I^n B$. Thus $J^n \subset J_\lambda$ and we conclude J is an ideal of definition. □

0APT **Lemma 87.5.10.** *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. Assume*

(1) *φ is taut and has dense image,*
(2) *A is complete and has a countable fundamental system of open ideals, and*
(3) *B is separated.*

Then φ is surjective and open, B is complete, and $B = A/K$ for some closed ideal $K \subset A$.

Proof. By the open mapping lemma (More on Algebra, Lemma 15.36.5) combined with tautness of φ, we see the map φ is open. Since the image of φ is dense, we see that φ is surjective. The kernel K of φ is closed as φ is continuous. It follows that $B = A/K$ is complete, see for example Lemma 87.4.4. □

87.6. Adic ring maps

0GX9 Let us make the following definition.

0GBR **Definition 87.6.1.** Let A and B be pre-adic topological rings. A ring homomorphism $\varphi : A \to B$ is *adic*[3] if there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology.

If $\varphi : A \to B$ is an adic homomorphism of pre-adic rings, then φ is continuous and the topology on B is the I-adic topology for every ideal of definition I of A.

0GXA **Lemma 87.6.2.** *Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to B$ and $B \to C$ are adic, then $A \to C$ is adic.*

Proof. Omitted. □

0GXB **Lemma 87.6.3.** *Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to C$ is adic, then $B \to C$ is adic.*

Proof. Choose an ideal of definition I of A. As $A \to C$ is adic, we see that IC is an ideal of definition of C. As $B \to C$ is continuous, we can find an ideal of definition $J \subset B$ mapping into IC. As $A \to B$ is continuous the inverse image $I' \subset I$ of J in I is an ideal of definition of A too. Hence $I'C \subset JC \subset IC$ is sandwiched between two ideals of definition, hence is an ideal of definition itself. □

0GXC **Lemma 87.6.4.** *Let $\varphi : A \to B$ be a continuous homomorphism between pre-adic topological rings. If φ is adic, then φ is taut.*

Proof. Immediate from the definitions. □

The next lemma says two things

(1) the property of being adic ascents along taut maps of complete linearly topologized rings, and

[3] This may be nonstandard terminology.

(2) the properties "φ is taut" and "φ is adic" are equivalent for continuous maps $\varphi : A \to B$ between adic rings if A has a finitely generated ideal of definition.

Because of (2) we can say that "tautness" generalizes "adicness" to continuous ring maps between arbitrary linearly topologized rings. See also Section 87.23.

0APU **Lemma 87.6.5.** *Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. If φ is taut, A is pre-adic and has a finitely generated ideal of definition, and B is complete, then B is adic and has a finitely generated ideal of definition and the ring map φ is adic.*

Proof. Choose a finitely generated ideal of definition I of A. Let J_n be the closure of $\varphi(I^n)B$ in B. Since B is complete we have $B = \lim B/J_n$. Let $B' = \lim B/I^nB$ be the I-adic completion of B. By Algebra, Lemma 10.96.3, the I-adic topology on B' is complete and $B'/I^nB' = B/I^nB$. Thus the ring map $B' \to B$ is continuous and has dense image as $B' \to B/I^nB \to B/J_n$ is surjective for all n. Finally, the map $B' \to B$ is taut because $(I^nB')B = I^nB$ and $A \to B$ is taut. By Lemma 87.5.10 we see that $B' \to B$ is open and surjective. Thus the topology on B is the I-adic topology and the proof is complete. □

87.7. Weakly adic rings

0GXD We suggest the reader skip this section. The following is a natural generalization of adic rings.

0GXE **Definition 87.7.1.** Let A be a linearly topologized ring. [**GR04**, Definition 8.3.8]

(1) We say A is *weakly pre-adic*[4] if there exists an ideal $I \subset A$ such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals.
(2) We say A is *weakly adic* if A is weakly pre-adic and complete[5].

For complete linearly topologized rings we have the following implications

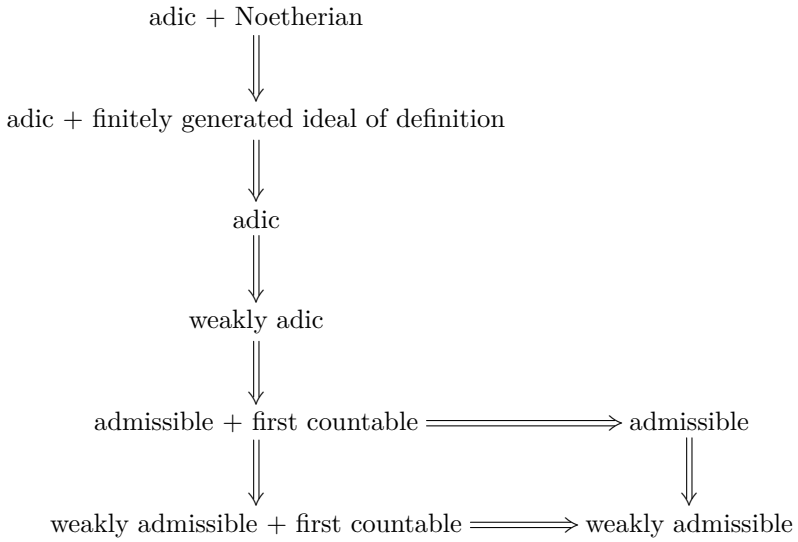

[4] In [**GR04**] the authors say A is *c-adic*.
[5] By our conventions this includes separated.

where "first countable" means that our topological ring has a countable fundamental system of open ideals. There is a similar diagram of implications for noncomplete linearly topologized rings (i.e., using the notions of pre-adic, weakly pre-adic, pre-admissible, and weakly pre-admissible). Contrary to what happens with pre-adic rings the completion of a weakly pre-adic ring is weakly adic as the following lemma characterizing weakly pre-adic rings shows.

0GXF **Lemma 87.7.2.** *Let A be a linearly topologized ring. The following are equivalent*

(1) *A is weakly pre-adic,*
(2) *there exists a taut continuous ring map $A' \to A$ where A' is a pre-adic topological ring, and*
(3) *A is pre-admissible and there exists an ideal of definition I such that the closure of I^n is open for all $n \geq 1$, and*
(4) *A is pre-admissible and for every ideal of definition I the closure of I^n is open for all $n \geq 1$.*

The completion of a weakly pre-adic ring is weakly adic. If A is weakly adic, then A is admissible and has a countable fundamental system of open ideals.

Proof. Assume (1). Choose an ideal I such that the closure of I^n is open for all n and such that these closures form a fundamental system of open ideals. Denote $A' = A$ endowed with the I-adic topology. Then $A' \to A$ is taut by definition and we see that (2) holds.

Assume (2). Let $I' \subset A'$ be an ideal of definition. Denote I the closure of $I'A$. Tautness of $A' \to A$ means that the closures I_n of $(I')^n A$ are open and form a fundamental system of open ideals. Thus $I = I_1$ is open and the closures of I^n are equal to I_n and hence open and form a fundamental system of open ideals. Thus certainly I is an ideal of definition such that the closure of I^n is open for all n. Hence (3) holds.

If $I \subset A$ is as in (3), then I is an ideal as in Definition 87.7.1 and we see that (1) holds. Also, if $I' \subset A$ is any other ideal of definition, then I' is open (see More on Algebra, Definition 15.36.1) and hence contains I^n for some $n \geq 1$. Thus $(I')^m$ contains I^{nm} for all $m \geq 1$ and we conclude that the closures of $(I')^m$ are open for all m. In this way we see that (3) implies (4). The implication (4) $\Rightarrow$ (3) is trivial.

Let A be weakly pre-adic. Choose $A' \to A$ as in (2). By Lemmas 87.5.3 and 87.5.4 the composition $A' \to A^\wedge$ is taut. Hence $A^\wedge$ is weakly pre-adic by the equivalence of (2) and (1). Since the completion of a linearly topologized ring A is complete (More on Algebra, Section 15.36) we see that $A^\wedge$ is weakly adic.

Let A be weakly adic. Then A is complete and and pre-admissible by (1) $\Rightarrow$ (3) and hence A is admissible. Of course by definition A has a countable fundamental system of open ideals. □

We give two criteria that guarantee that a weakly adic ring is adic and has a finitely generated ideal of definition.

0GXG **Lemma 87.7.3.** *Let A be a complete linearly topologized ring. Let $I \subset A$ be a finitely generated ideal such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals. Then A is adic and has a finitely generated ideal of definition.*

Proof. Denote A' the ring A endowed with the I-adic topology. The assumptions tells us that $A' \to A$ is taut. We conclude by Lemma 87.6.5 (to be sure, this lemma also tells us that I is an ideal of definition). □

0GXH **Lemma 87.7.4.** *Let A be a weakly adic topological ring. Let I be an ideal of definition such that I/I_2 is a finitely generated module where I_2 is the closure of I^2. Then A is adic and has a finitely generated ideal of definition.*

Proof. We use the characterization of Lemma 87.7.2 without further mention. Choose $f_1, \ldots, f_r \in I$ which map to generators of I/I_2. Set $I' = (f_1, \ldots, f_r)$. We have $I' + I_2 = I$. Then I_2 is the closure of $I^2 = (I' + I_2)^2 \subset I' + I_3$ where I_3 is the closure of I^3. Hence $I' + I_3 = I$. Continuing in this fashion we see that $I' + I_n = I$ for all $n \geq 2$ where I_n is the closure of I^n. In other words, the closure of I' in A is I. Hence the closure of $(I')^n$ is I_n. Thus the closures of $(I')^n$ are a fundamental system of open ideals of A. We conclude by Lemma 87.7.3. □

A key feature of the property "weakly pre-adic" is that it ascents along taut ring homomorphisms of linearly topologized rings.

0GXI **Lemma 87.7.5.** *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-adic, then B is weakly pre-adic.*

Proof. Let $I \subset A$ be an ideal such that the closure I_n of I^n is open and these closures define a fundamental system of open ideals. Then the closure of $I^n B$ is equal to the closure of $I_n B$. Since φ is taut, these closures are open and form a fundamental system of open ideals of B. Hence B is weakly pre-adic. □

0GXJ **Lemma 87.7.6.** *Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If A and C are weakly pre-adic, then $A \widehat{\otimes}_B C$ is weakly adic.*

Proof. We will use the characterization of Lemma 87.7.2 without further mention. By Lemma 87.4.12 we know that $A \widehat{\otimes}_B C$ is admissible. Moreover, the proof of that lemma shows that the closure $K \subset A \widehat{\otimes}_B C$ is an ideal of definition, when $I \subset A$ and $J \subset C$ of $I(A \widehat{\otimes}_B C) + J(A \widehat{\otimes}_B C)$ are ideals of definition. Then it suffices to show that the closure of K^n is open for all $n \geq 1$. Since the ideal K^n contains $I^n(A \widehat{\otimes}_B C) + J^n(A \widehat{\otimes}_B C)$, since the closure of I^n in A is open, and since the closure of J^n in C is open, we see that the closure of K^n is open in $A \widehat{\otimes}_B C$. □

87.8. Descending properties

0GXK In this section we consider the following situation

(1) $\varphi : A \to B$ is a continuous map of linearly topologized topological rings,
(2) φ is taut, and
(3) for every open ideal $I \subset A$ if $J \subset B$ denotes the closure of IB, then the map $A/I \to B/J$ is faithfully flat.

We are going to show that properties of B are inherited by A in this situation.

0GXL **Lemma 87.8.1.** *In the situation above, if B has a countable fundamental system of open ideals, then A has a countable fundamental system of open ideals.*

Proof. Choose a fundamental system $B \supset J_1 \supset J_2 \supset \ldots$ of open ideals. By tautness of φ, for every n we can find an open ideal I_n such that $J_n \supset I_n B$. We claim that I_n is a fundamental system of open ideals of A. Namely, suppose that

$I \subset A$ is open. As φ is taut, the closure of IB is open and hence contains J_n for some n large enough. Hence $I_nB \subset IB$. Let J be the closure of IB in B. Since $A/I \to B/J$ is faithfully flat, it is injective. Hence, since $I_n \to A/I \to B/J$ is zero as $I_nB \subset IB \subset J$, we conclude that $I_n \to A/I$ is zero. Hence $I_n \subset I$ and we win. □

0GXM **Lemma 87.8.2.** *In the situation above, if B is weakly pre-admissible, then A is weakly pre-admissible.*

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. To show that I is a weak ideal of definition we have to show that any $f \in I$ is topologically nilpotent. Let $I' \subset A$ be an open ideal. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $f^n \in I'$ it suffices to show that $\varphi(f)^n \in J'$. This holds for $n \gg 0$ since $\varphi(f) \in J$, the ideal J is a weak ideal of definition of B, and J' is open in B. □

0GXN **Lemma 87.8.3.** *In the situation above, if B is pre-admissible, then A is pre-admissible.*

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. Let $I' \subset A$ be an open ideal. To show that I is an ideal of definition we have to show that $I^n \subset I'$ for $n \gg 0$. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $I^n \subset I'$ it suffices to show that $\varphi(I)^n \subset J'$. This holds for $n \gg 0$ since $\varphi(I) \subset J$, the ideal J is an ideal of definition of B, and J' is open in B. □

0GXP **Lemma 87.8.4.** *In the situation above, if B is weakly pre-adic, then A is weakly pre-adic.*

Proof. We will use the characterization of weakly pre-adic rings given in Lemma 87.7.2 without further mention. By Lemma 87.8.3 the topological ring A is pre-admissible. Let $I \subset A$ be an ideal of definition. Fix $n \geq 1$. To prove the lemma we have to show that the closure of I^n is open. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Denote $J \subset B$, resp. $J_\lambda \subset B$ the closure of IB, resp. $I_\lambda B$. Since B is weakly pre-adic, the closure of J^n is open. Hence there exists a λ such that

$$J_\lambda \subset \bigcap\nolimits_\mu (J^n + J_\mu)$$

because the right hand side is the closure of J^n by Lemma 87.4.2. This means that the image of J_λ in B/J_μ is contained in the image of J^n in B/J_μ. Observe that the image of J^n in B/J_μ is equal to the image of I^nB in B/J_μ (since every element of J is congruent to an element of IB modulo J_μ). Since $A/I_\mu \to B/J_\mu$ is faithfully flat and since $I_\lambda B \subset J_\lambda$, we conclude that the image of I_λ in A/I_μ is contained in the image of I^n. We conclude that I_λ is contained in the closure of I^n and the proof is complete. □

0GXQ **Lemma 87.8.5.** *In the situation above, if B is adic and has a finitely generated ideal of definition and A is complete, then A is adic and has a finitely generated ideal of definition.*

Proof. We already know that A is weakly adic and a fortiori admissible by Lemma 87.8.4 (and Lemma 87.7.2 to see that adic rings are weakly adic). Let $I \subset A$ be an ideal of definition. Let $J \subset B$ be a finitely generated ideal of definition. Since the

closure of IB is open, we can find an $n > 0$ such that J^n is contained in the closure of IB. Thus after replacing J by J^n we may assume J is a finitely generated ideal of definition contained in the closure of IB. By Lemma 87.4.2 this certainly implies that

$$J \subset IB + J^2$$

Consider the finitely generated A-module $M = (J + IB)/IB$. The displayed equation shows that $JM = M$. By Lemma 87.4.9 (for example) we see that J is contained in the Jacobson radical of B. Hence by Nakayama's lemma, more precisely part (2) of Algebra, Lemma 10.20.1, we conclude $M = 0$. Thus $J \subset IB$.

Since J is finitely generated, we can find a finitely generated ideal $I' \subset I$ such that $J \subset I'B$. Since $A \to B$ is continuous, $J \subset B$ is open, and I is an ideal of definition, we can find an $n > 0$ such that $I^n B \subset J$. Let $J_{n+1} \subset B$ be the closure of $I^{n+1}B$. We have

$$I^n \cdot (B/J_{n+1}) \subset J \cdot (B/J_{n+1}) \subset I' \cdot (B/J_{n+1})$$

Since $A/I^{n+1} \to B/J_{n+1}$ is faithfully flat, this implies $I^n \cdot (A/I^{n+1}) \subset I' \cdot (A/I^{n+1})$ which in turn means

$$I^n \subset I' + I^{n+1}$$

This implies $I^n \subset I' + I^{n+k}$ for all $k \geq 1$ which in turn implies that $I^{nm} \subset (I')^m + I^{nm+k}$ for all $k, m \geq 1$. This implies that the closure of $(I')^m$ contains I^{nm}. Since the closure of I^{nm} is open as A is weakly adic, we conclude that the closure $(I')^m$ is open for all m. Since these closures form a fundamental system of open ideals of A (as the same thing is true for the closures of I^n) we conclude by Lemma 87.7.3. □

87.9. Affine formal algebraic spaces

0AI6 In this section we introduce affine formal algebraic spaces. These will in fact be the same as what are called affine formal schemes in [**BD**]. However, we will call them affine formal algebraic spaces, in order to prevent confusion with the notion of an affine formal scheme as defined in [**DG67**].

Recall that a thickening of schemes is a closed immersion which induces a surjection on underlying topological spaces, see More on Morphisms, Definition 37.2.1.

0AI7 **Definition 87.9.1.** Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is an *affine formal algebraic space* if there exist

(1) a directed set Λ,
(2) a system $(X_\lambda, f_{\lambda\mu})$ over Λ in $(Sch/S)_{fppf}$ where
   (a) each X_λ is affine,
   (b) each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening,

such that

$$X \cong \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$$

as fppf sheaves and X satisfies a set theoretic condition (see Remark 87.11.5). A *morphism of affine formal algebraic spaces* over S is a map of sheaves.

Observe that the system $(X_\lambda, f_{\lambda\mu})$ is not part of the data. Suppose that U is a quasi-compact scheme over S. Since the transition maps are monomorphisms, we see that

$$X(U) = \operatorname{colim} X_\lambda(U)$$

by Sites, Lemma 7.17.7. Thus the fppf sheafification inherent in the colimit of the definition is a Zariski sheafification which does not do anything for quasi-compact schemes.

0AI8 **Lemma 87.9.2.** *Let S be a scheme. If X is an affine formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable and a closed immersion.*

Proof. Suppose given $U \to X$ and $V \to X$ where U, V are schemes over S. Let us show that $U \times_X V$ is representable. Write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. The discussion above shows that Zariski locally on U and V the morphisms factors through some X_λ. In this case $U \times_X V = U \times_{X_\lambda} V$ which is a scheme. Thus the diagonal is representable, see Spaces, Lemma 65.5.10. Given $(a, b) : W \to X \times_S X$ where W is a scheme over S consider the map $X \times_{\Delta, X \times_S X, (a,b)} W \to W$. As before locally on W the morphisms a and b map into the affine scheme X_λ for some λ and then we get the morphism $X_\lambda \times_{\Delta_\lambda, X_\lambda \times_S X_\lambda, (a,b)} W \to W$. This is the base change of $\Delta_\lambda : X_\lambda \to X_\lambda \times_S X_\lambda$ which is a closed immersion as $X_\lambda \to S$ is separated (because X_λ is affine). Thus $X \to X \times_S X$ is a closed immersion. □

A morphism of schemes $X \to X'$ is a thickening if it is a closed immersion and induces a surjection on underlying sets of points, see (More on Morphisms, Definition 37.2.1). Hence the property of being a thickening is preserved under arbitrary base change and fpqc local on the target, see Spaces, Section 65.4. Thus Spaces, Definition 65.5.1 applies to "thickening" and we know what it means for a representable transformation $F \to G$ of presheaves on $(Sch/S)_{fppf}$ to be a thickening. We observe that this does not clash with our definition (More on Morphisms of Spaces, Definition 76.9.1) of thickenings in case F and G are algebraic spaces.

0AI9 **Lemma 87.9.3.** *Let $X_\lambda, \lambda \in \Lambda$ and $X = \operatorname{colim} X_\lambda$ be as in Definition 87.9.1. Then $X_\lambda \to X$ is representable and a thickening.*

Proof. The statement makes sense by the discussion in Spaces, Section 65.3 and 65.5. By Lemma 87.9.2 the morphisms $X_\lambda \to X$ are representable. Given $U \to X$ where U is a scheme, then the discussion following Definition 87.9.1 shows that Zariski locally on U the morphism factors through some X_μ with $\lambda \leq \mu$. In this case $U \times_X X_\lambda = U \times_{X_\mu} X_\lambda$ so that $U \times_X X_\lambda \to U$ is a base change of the thickening $X_\lambda \to X_\mu$. □

0AIA **Lemma 87.9.4.** *Let $X_\lambda, \lambda \in \Lambda$ and $X = \operatorname{colim} X_\lambda$ be as in Definition 87.9.1. If Y is a quasi-compact algebraic space over S, then any morphism $Y \to X$ factors through an X_λ.*

Proof. Choose an affine scheme V and a surjective étale morphism $V \to Y$. The composition $V \to Y \to X$ factors through X_λ for some λ by the discussion following Definition 87.9.1. Since $V \to Y$ is a surjection of sheaves, we conclude. □

0AIB **Lemma 87.9.5.** *Let S be a scheme. Let X be a sheaf on $(Sch/S)_{fppf}$. Then X is an affine formal algebraic space if and only if the following hold*

1. *any morphism $U \to X$ where U is an affine scheme over S factors through a morphism $T \to X$ which is representable and a thickening with T an affine scheme over S, and*
2. *a set theoretic condition as in Remark 87.11.5.*

Proof. It follows from Lemmas 87.9.3 and 87.9.4 that an affine formal algebraic space satisfies (1) and (2). In order to prove the converse we may assume X is not empty. Let Λ be the category of representable morphisms $T \to X$ which are thickenings where T is an affine scheme over S. This category is directed. Since X is not empty, Λ contains at least one object. If $T \to X$ and $T' \to X$ are in Λ, then we can factor $T \amalg T' \to X$ through $T'' \to X$ in Λ. Between any two objects of Λ there is a unique arrow or none. Thus Λ is a directed set and by assumption $X = \operatorname{colim}_{T \to X \text{ in } \Lambda} T$. To finish the proof we need to show that any arrow $T \to T'$ in Λ is a thickening. This is true because $T' \to X$ is a monomorphism of sheaves, so that $T = T \times_{T'} T' = T \times_X T'$ and hence the morphism $T \to T'$ equals the projection $T \times_X T' \to T'$ which is a thickening because $T \to X$ is a thickening. □

For a general affine formal algebraic space X there is no guarantee that X has enough functions to separate points (for example). See Examples, Section 110.75. To characterize those that do we offer the following lemma.

0AIC **Lemma 87.9.6.** *Let S be a scheme. Let X be an fppf sheaf on $(Sch/S)_{fppf}$ which satisfies the set theoretic condition of Remark 87.11.5. The following are equivalent:*

1. *there exists a weakly admissible topological ring A over S (see Remark 87.2.3) such that $X = \operatorname{colim}_{I \subset A \text{ weak ideal of definition}} \operatorname{Spec}(A/I)$,*
2. *X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \operatorname{Spec}(A)$ such that for a closed immersion $T \to X$ with T an affine scheme the composition $T \to \operatorname{Spec}(A)$ is a closed immersion,*
3. *X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \operatorname{Spec}(A)$ such that for a closed immersion $T \to X$ with T a scheme the composition $T \to \operatorname{Spec}(A)$ is a closed immersion,*
4. *X is an affine formal algebraic space and for some choice of $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective,*
5. *X is an affine formal algebraic space and for any choice of $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective.*

Moreover, the weakly admissible topological ring is $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ endowed with its limit topology and the weak ideals of definition classify exactly the morphisms $T \to X$ which are representable and thickenings.

Proof. It is clear that (5) implies (4).

Assume (4) for $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Set $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. Let $T \to X$ be a closed immersion with T a scheme (note that $T \to X$ is representable by Lemma 87.9.2). Since $X_\lambda \to X$ is a thickening, so is $X_\lambda \times_X T \to T$. On the other hand, $X_\lambda \times_X T \to X_\lambda$ is a closed immersion, hence $X_\lambda \times_X T$ is affine. Hence T is affine by Limits, Proposition 32.11.2. Then $T \to X$ factors through X_λ for some λ by Lemma 87.9.4. Thus $A \to \Gamma(X_\lambda, \mathcal{O}) \to \Gamma(T, \mathcal{O})$ is surjective. In this way we see that (3) holds.

It is clear that (3) implies (2).

Assume (2) for A and $X \to \operatorname{Spec}(A)$. Write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Then $A_\lambda = \Gamma(X_\lambda, \mathcal{O})$ is a quotient of A by assumption (2). Hence $A^\wedge = \lim A_\lambda$ is a complete topological ring, see discussion in More on Algebra, Section 15.36. The maps $A^\wedge \to A_\lambda$ are surjective as $A \to A_\lambda$ is. We claim that for any λ the kernel

$I_\lambda \subset A^\wedge$ of $A^\wedge \to A_\lambda$ is a weak ideal of definition. Namely, it is open by definition of the limit topology. If $f \in I_\lambda$, then for any $\mu \in \Lambda$ the image of f in A_μ is zero in all the residue fields of the points of X_μ. Hence it is a nilpotent element of A_μ. Hence some power $f^n \in I_\mu$. Thus $f^n \to 0$ as $n \to 0$. Thus $A^\wedge$ is weakly admissible. Finally, suppose that $I \subset A^\wedge$ is a weak ideal of definition. Then $I \subset A^\wedge$ is open and hence there exists some λ such that $I \supset I_\lambda$. Thus we obtain a morphism $\mathrm{Spec}(A^\wedge/I) \to \mathrm{Spec}(A_\lambda) \to X$. Then it follows that $X = \mathrm{colim}\, \mathrm{Spec}(A^\wedge/I)$ where now the colimit is over all weak ideals of definition. Thus (1) holds.

Assume (1). In this case it is clear that X is an affine formal algebraic space. Let $X = \mathrm{colim}\, X_\lambda$ be any presentation as in Definition 87.9.1. For each λ we can find a weak ideal of definition $I \subset A$ such that $X_\lambda \to X$ factors through $\mathrm{Spec}(A/I) \to X$, see Lemma 87.9.4. Then $X_\lambda = \mathrm{Spec}(A/I_\lambda)$ with $I \subset I_\lambda$. Conversely, for any weak ideal of definition $I \subset A$ the morphism $\mathrm{Spec}(A/I) \to X$ factors through X_λ for some λ, i.e., $I_\lambda \subset I$. It follows that each I_λ is a weak ideal of definition and that they form a cofinal subset of the set of weak ideals of definition. Hence $A = \lim A/I = \lim A/I_\lambda$ and we see that (5) is true and moreover that $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. □

With this lemma in hand we can make the following definition.

0AID **Definition 87.9.7.** Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is *McQuillan* if X satisfies the equivalent conditions of Lemma 87.9.6. Let A be the weakly admissible topological ring associated to X. We say

(1) X is *classical* if X is McQuillan and A is admissible (More on Algebra, Definition 15.36.1),
(2) X is *weakly adic* if X is McQuillan and A is weakly adic (Definition 87.7.1),
(3) X is *adic* if X is McQuillan and A is adic (More on Algebra, Definition 15.36.1),
(4) X is *adic** if X is McQuillan, A is adic, and A has a finitely generated ideal of definition, and
(5) X is *Noetherian* if X is McQuillan and A is both Noetherian and adic.

In [**FK**] they use the terminology "of finite ideal type" for the property that an adic topological ring A contains a finitely generated ideal of definition. Given an affine formal algebraic space X here are the implications among the notions introduced in the definition:

$$\begin{array}{ccccc} X \text{ Noetherian} & \Longrightarrow & X \text{ adic*} & \Longrightarrow & X \text{ adic} \\ & & & \Longleftarrow & \\ X \text{ weakly adic} & \Longrightarrow & X \text{ classical} & \Longrightarrow & X \text{ McQuillan} \end{array}$$

See discussion in Section 87.7 and for a precise statement see Lemma 87.10.3.

0AIE **Remark 87.9.8.** The classical affine formal algebraic spaces correspond to the affine formal schemes considered in EGA ([**DG67**]). To explain this we assume our base scheme is $\mathrm{Spec}(\mathbf{Z})$. Let $\mathfrak{X} = \mathrm{Spf}(A)$ be an affine formal scheme. Let $h_\mathfrak{X}$ be its functor of points as in Lemma 87.2.1. Then $h_\mathfrak{X} = \mathrm{colim}\, h_{\mathrm{Spec}(A/I)}$ where the colimit is over the collection of ideals of definition of the admissible topological ring A. This follows from (87.2.0.1) when evaluating on affine schemes and it suffices to check on affine schemes as both sides are fppf sheaves, see Lemma 87.2.2. Thus $h_\mathfrak{X}$ is an affine formal algebraic space. In fact, it is a classical affine formal algebraic

space by Definition 87.9.7. Thus Lemma 87.2.1 tells us the category of affine formal schemes is equivalent to the category of classical affine formal algebraic spaces.

Having made the connection with affine formal schemes above, it seems natural to make the following definition.

0AIF **Definition 87.9.9.** Let S be a scheme. Let A be a weakly admissible topological ring over S, see Definition 87.4.8[6]. The *formal spectrum* of A is the affine formal algebraic space

$$\mathrm{Spf}(A) = \mathrm{colim}\, \mathrm{Spec}(A/I)$$

where the colimit is over the set of weak ideals of definition of A and taken in the category $Sh((Sch/S)_{fppf})$.

Such a formal spectrum is McQuillan by construction and conversely every McQuillan affine formal algebraic space is isomorphic to a formal spectrum. To be sure, in our theory there exist affine formal algebraic spaces which are not the formal spectrum of any weakly admissible topological ring. Following [**Yas09**] we could introduce S-pro-rings to be pro-objects in the category of S-algebras, see Categories, Remark 4.22.5. Then every affine formal algebraic space over S would be the formal spectrum of such an S-pro-ring. We will not do this and instead we will work directly with the corresponding affine formal algebraic spaces.

The construction of the formal spectrum is functorial. To explain this let $\varphi : B \to A$ be a continuous map of weakly admissible topological rings over S. Then

$$\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$$

is the unique morphism of affine formal algebraic spaces such that the diagrams

$$\begin{array}{ccc} \mathrm{Spec}(B/J) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(B) & \longrightarrow & \mathrm{Spf}(A) \end{array}$$

commute for all weak ideals of definition $I \subset A$ and $J \subset B$ with $\varphi(I) \subset J$. Since continuity of φ implies that for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ with the required property, we see that the required commutativities uniquely determine and define $\mathrm{Spf}(\varphi)$.

0AN0 **Lemma 87.9.10.** *Let S be a scheme. Let A, B be weakly admissible topological rings over S. Any morphism $f : Spf(B) \to Spf(A)$ of affine formal algebraic spaces over S is equal to $Spf(f^\sharp)$ for a unique continuous S-algebra map $f^\sharp : A \to B$.*

Proof. Let $f : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ be as in the lemma. Let $J \subset B$ be a weak ideal of definition. By Lemma 87.9.4 there exists a weak ideal of definition $I \subset A$ such that $\mathrm{Spec}(B/J) \to \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ factors through $\mathrm{Spec}(A/I)$. By Schemes, Lemma 26.6.4 we obtain an S-algebra map $A/I \to B/J$. These maps are compatible for varying J and define the map $f^\sharp : A \to B$. This map is continuous because for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ such that $f^\sharp(I) \subset J$. The equality $f = \mathrm{Spf}(f^\sharp)$ holds by our choice of the ring maps $A/I \to B/J$ which make up $f^\sharp$. □

[6] See More on Algebra, Definition 15.36.1 for the classical case and see Remark 87.2.3 for a discussion of differences.

0AIG **Lemma 87.9.11.** *Let S be a scheme. Let $f : X \to Y$ be a map of presheaves on $(Sch/S)_{fppf}$. If X is an affine formal algebraic space and f is representable by algebraic spaces and locally quasi-finite, then f is representable (by schemes).*

Proof. Let T be a scheme over S and $T \to Y$ a map. We have to show that the algebraic space $X \times_Y T$ is a scheme. Write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Let $W \subset X \times_Y T$ be a quasi-compact open subspace. The restriction of the projection $X \times_Y T \to X$ to W factors through X_λ for some λ. Then

$$W \to X_\lambda \times_S T$$

is a monomorphism (hence separated) and locally quasi-finite (because $W \to X \times_Y T \to T$ is locally quasi-finite by our assumption on $X \to Y$, see Morphisms of Spaces, Lemma 67.27.8). Hence W is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus $X \times_Y T$ is a scheme by Properties of Spaces, Lemma 66.13.1. □

87.10. Countably indexed affine formal algebraic spaces

0AIH These are the affine formal algebraic spaces as in the following lemma.

0AII **Lemma 87.10.1.** *Let S be a scheme. Let X be an affine formal algebraic space over S. The following are equivalent*

(1) *there exists a system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S such that $X = \operatorname{colim} X_n$,*
(2) *there exists a choice $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 such that Λ is countable.*

Proof. This follows from the observation that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. See proof of Algebra, Lemma 10.86.3. □

0AIJ **Definition 87.10.2.** Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is *countably indexed* if the equivalent conditions of Lemma 87.10.1 are satisfied.

In the language of [**BD**] this is expressed by saying that X is an $\aleph_0$-ind scheme.

0AIK **Lemma 87.10.3.** *Let X be an affine formal algebraic space over a scheme S.*

(1) *If X is Noetherian, then X is adic*.*
(2) *If X is adic*, then X is adic.*
(3) *If X is adic, then X is weakly adic.*
(4) *If X is weakly adic, then X is classical.*
(5) *If X is weakly adic, then X is countably indexed.*
(6) *If X is countably indexed, then X is McQuillan.*

Proof. Statements (1), (2), (3), and (4) follow by writing $X = \operatorname{Spf}(A)$ and where A is a weakly admissible (hence complete) linearly topologized ring and using the implications between the various types of such rings discussed in Section 87.7.

Proof of (5). By definition there exists a weakly adic topological ring A such that $X = \operatorname{colim} \operatorname{Spec}(A/I)$ where the colimit is over the ideals of definition of A. As A is weakly adic, there exits in particular a countable fundamental system I_λ of open ideals, see Definition 87.7.1. Then $X = \operatorname{colim} \operatorname{Spec}(A/I_n)$ by definition of $\operatorname{Spf}(A)$. Thus X is countably indexed.

Proof of (6). Write $X = \operatorname{colim} X_n$ for some system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S. Then

$$A = \lim \Gamma(X_n, \mathcal{O}_{X_n})$$

surjects onto each $\Gamma(X_n, \mathcal{O}_{X_n})$ because the transition maps are surjections as the morphisms $X_n \to X_{n+1}$ are closed immersions. Hence X is McQuillan. □

0AN1 **Lemma 87.10.4.** *Let S be a scheme. Let X be a presheaf on $(Sch/S)_{fppf}$. The following are equivalent*

(1) *X is a countably indexed affine formal algebraic space,*
(2) *$X = Spf(A)$ where A is a weakly admissible topological S-algebra which has a countable fundamental system of neighbourhoods of 0,*
(3) *$X = Spf(A)$ where A is a weakly admissible topological S-algebra which has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition,*
(4) *$X = Spf(A)$ where A is a complete topological S-algebra with a fundamental system of open neighbourhoods of 0 given by a countable sequence $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of ideals such that I_n/I_{n+1} is locally nilpotent, and*
(5) *$X = Spf(A)$ where $A = \lim B/J_n$ with the limit topology where $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ is a sequence of ideals in an S-algebra B with J_n/J_{n+1} locally nilpotent.*

Proof. Assume (1). By Lemma 87.10.3 we can write $X = \operatorname{Spf}(A)$ where A is a weakly admissible topological S-algebra. For any presentation $X = \operatorname{colim} X_n$ as in Lemma 87.10.1 part (1) we see that $A = \lim A_n$ with $X_n = \operatorname{Spec}(A_n)$ and $A_n = A/I_n$ for some weak ideal of definition $I_n \subset A$. This follows from the final statement of Lemma 87.9.6 which moreover implies that $\{I_n\}$ is a fundamental system of open neighbourhoods of 0. Thus we have a sequence

$$A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$$

of weak ideals of definition with $A = \lim A/I_n$. In this way we see that condition (1) implies each of the conditions (2) – (5).

Assume (5). First note that the limit topology on $A = \lim B/J_n$ is a linearly topologized, complete topology, see More on Algebra, Section 15.36. If $f \in A$ maps to zero in B/J_1, then some power maps to zero in B/J_2 as its image in J_1/J_2 is nilpotent, then a further power maps to zero in J_2/J_3, etc, etc. In this way we see the open ideal $\operatorname{Ker}(A \to B/J_1)$ is a weak ideal of definition. Thus A is weakly admissible. In this way we see that (5) implies (2).

It is clear that (4) is a special case of (5) by taking $B = A$. It is clear that (3) is a special case of (2).

Assume A is as in (2). Let E_n be a countable fundamental system of neighbourhoods of 0 in A. Since A is a weakly admissible topological ring we can find open ideals $I_n \subset E_n$. We can also choose a weak ideal of definition $J \subset A$. Then $J \cap I_n$ is a fundamental system of weak ideals of definition of A and we get $X = \operatorname{Spf}(A) = \operatorname{colim} \operatorname{Spec}(A/(J \cap I_n))$ which shows that X is a countably indexed affine formal algebraic space. □

0AKM **Lemma 87.10.5.** *Let S be a scheme. Let X be an affine formal algebraic space. The following are equivalent*

(1) *X is Noetherian,*
(2) *X is adic* and for every closed immersion $T \to X$ with T a scheme, T is Noetherian,*
(3) *X is adic* and for some choice of $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the schemes X_λ are Noetherian, and*
(4) *X is weakly adic and for some choice $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1 the schemes X_λ are Noetherian.*

Proof. Assume X is Noetherian. Then $X = \operatorname{Spf}(A)$ where A is a Noetherian adic ring. Let $T \to X$ be a closed immersion where T is a scheme. By Lemma 87.9.6 we see that T is affine and that $T \to \operatorname{Spec}(A)$ is a closed immersion. Since A is Noetherian, we see that T is Noetherian. In this way we see that (1) $\Rightarrow$ (2).

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are immediate (see Lemma 87.10.3).

To prove (3) $\Rightarrow$ (1) write $X = \operatorname{Spf}(A)$ for some adic ring A with finitely generated ideal of definition I. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Since I is open, we can find a λ such that $I_\lambda \subset I$. Then A/I is Noetherian and we conclude that A is Noetherian by Algebra, Lemma 10.97.5.

To prove (4) $\Rightarrow$ (3) write $X = \operatorname{Spf}(A)$ for some weakly adic ring A. Then A is admissible and has an ideal of definition I and the closure I_2 of I^2 is open, see Lemma 87.7.2. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Choose a λ such that $I_\lambda \subset I_2$. Then A/I_2 is Noetherian as a quotient of A/I_λ. Hence I/I_2 is a finite A-module. Hence A is an adic ring with a finitely generated ideal of definition by Lemma 87.7.4. Thus X is adic* and (3) holds. □

87.11. Formal algebraic spaces

0AIL We take a break from our habit of introducing new concepts first for rings, then for schemes, and then for algebraic spaces, by introducing formal algebraic spaces without first introducing formal schemes. The general idea will be that a formal algebraic space is a sheaf in the fppf topology which étale locally is an affine formal scheme in the sense of [**BD**]. Related material can be found in [**Yas09**].

In the definition of a formal algebraic space we are going to borrow some terminology from Bootstrap, Sections 80.3 and 80.4.

0AIM **Definition 87.11.1.** Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is a *formal algebraic space* if there exist a family of maps $\{X_i \to X\}_{i \in I}$ of sheaves such that

(1) X_i is an affine formal algebraic space,
(2) $X_i \to X$ is representable by algebraic spaces and étale,
(3) $\coprod X_i \to X$ is surjective as a map of sheaves

and X satisfies a set theoretic condition (see Remark 87.11.5). A *morphism of formal algebraic spaces* over S is a map of sheaves.

Discussion. Sanity check: an affine formal algebraic space is a formal algebraic space. In the situation of the definition the morphisms $X_i \to X$ are representable

(by schemes), see Lemma 87.9.11. By Bootstrap, Lemma 80.4.6 we could instead of asking $\coprod X_i \to X$ to be surjective as a map of sheaves, require that it be surjective (which makes sense because it is representable).

Our notion of a formal algebraic space is **very general**. In fact, even affine formal algebraic spaces as defined above are very nasty objects.

0AIP **Lemma 87.11.2.** *Let S be a scheme. If X is a formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable, a monomorphism, locally quasi-finite, locally of finite type, and separated.*

Proof. Suppose given $U \to X$ and $V \to X$ with U, V schemes over S. Then $U \times_X V$ is a sheaf. Choose $\{X_i \to X\}$ as in Definition 87.11.1. For every i the morphism

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) = (U \times_X V) \times_X X_i \to U \times_X V$$

is representable and étale as a base change of $X_i \to X$ and its source is a scheme (use Lemmas 87.9.2 and 87.9.11). These maps are jointly surjective hence $U \times_X V$ is an algebraic space by Bootstrap, Theorem 80.10.1. The morphism $U \times_X V \to U \times_S V$ is a monomorphism. It is also locally quasi-finite, because on precomposing with the morphism displayed above we obtain the composition

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) \to (U \times_X X_i) \times_S (V \times_X X_i) \to U \times_S V$$

which is locally quasi-finite as a composition of a closed immersion (Lemma 87.9.2) and an étale morphism, see Descent on Spaces, Lemma 74.19.2. Hence we conclude that $U \times_X V$ is a scheme by Morphisms of Spaces, Proposition 67.50.2. Thus Δ is representable, see Spaces, Lemma 65.5.10.

In fact, since we've shown above that the morphisms of schemes $U \times_X V \to U \times_S V$ are always monomorphisms and locally quasi-finite we conclude that $\Delta : X \to X \times_S X$ is a monomorphism and locally quasi-finite, see Spaces, Lemma 65.5.11. Then we can use the principle of Spaces, Lemma 65.5.8 to see that Δ is separated and locally of finite type. Namely, a monomorphism of schemes is separated (Schemes, Lemma 26.23.3) and a locally quasi-finite morphism of schemes is locally of finite type (follows from the definition in Morphisms, Section 29.20). □

0AIQ **Lemma 87.11.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism from an algebraic space over S to a formal algebraic space over S. Then f is representable by algebraic spaces.*

Proof. Let $Z \to Y$ be a morphism where Z is a scheme over S. We have to show that $X \times_Y Z$ is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \times_Y Z \to X \times_Y Z$ is representable surjective étale (Spaces, Lemma 65.5.5) and $U \times_Y Z$ is a scheme by Lemma 87.11.2. Hence the result by Bootstrap, Theorem 80.10.1. □

0AIR **Remark 87.11.4.** Modulo set theoretic issues the category of formal schemes à la EGA (see Section 87.2) is equivalent to a full subcategory of the category of formal algebraic spaces. To explain this we assume our base scheme is $\mathrm{Spec}(\mathbf{Z})$. By Lemma 87.2.2 the functor of points $h_{\mathfrak{X}}$ associated to a formal scheme $\mathfrak{X}$ is a sheaf in the fppf topology. By Lemma 87.2.1 the assignment $\mathfrak{X} \mapsto h_{\mathfrak{X}}$ is a fully faithful embedding of the category of formal schemes into the category of fppf sheaves. Given a formal scheme $\mathfrak{X}$ we choose an open covering $\mathfrak{X} = \bigcup \mathfrak{X}_i$ with $\mathfrak{X}_i$ affine formal schemes.

Then $h_{\mathfrak{X}_i}$ is an affine formal algebraic space by Remark 87.9.8. The morphisms $h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}$ are representable and open immersions. Thus $\{h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}\}$ is a family as in Definition 87.11.1 and we see that $h_{\mathfrak{X}}$ is a formal algebraic space.

0AIS **Remark 87.11.5.** Let S be a scheme and let $(Sch/S)_{fppf}$ be a big fppf site as in Topologies, Definition 34.7.8. As our set theoretic condition on X in Definitions 87.9.1 and 87.11.1 we take: there exist objects U, R of $(Sch/S)_{fppf}$, a morphism $U \to X$ which is a surjection of fppf sheaves, and a morphism $R \to U \times_X U$ which is a surjection of fppf sheaves. In other words, we require our sheaf to be a coequalizer of two maps between representable sheaves. Here are some observations which imply this notion behaves reasonably well:

(1) Suppose $X = \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$ and the system satisfies conditions (1) and (2) of Definition 87.9.1. Then $U = \coprod_{\lambda \in \Lambda} X_\lambda \to X$ is a surjection of fppf sheaves. Moreover, $U \times_X U$ is a closed subscheme of $U \times_S U$ by Lemma 87.9.2. Hence if U is representable by an object of $(Sch/S)_{fppf}$ then $U \times_S U$ is too (see Sets, Lemma 3.9.9) and the set theoretic condition is satisfied. This is always the case if Λ is countable, see Sets, Lemma 3.9.9.

(2) Sanity check. Let $\{X_i \to X\}_{i \in I}$ be as in Definition 87.11.1 (with the set theoretic condition as formulated above) and assume that each X_i is actually an affine scheme. Then X is an algebraic space. Namely, if we choose a larger big fppf site $(Sch'/S)_{fppf}$ such that $U' = \coprod X_i$ and $R' = \coprod X_i \times_X X_j$ are representable by objects in it, then $X' = U'/R'$ will be an object of the category of algebraic spaces for this choice. Then an application of Spaces, Lemma 65.15.2 shows that X is an algebraic space for $(Sch/S)_{fppf}$.

(3) Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves satisfying conditions (1), (2), (3) of Definition 87.11.1. For each i we can pick $U_i \in \mathrm{Ob}((Sch/S)_{fppf})$ and $U_i \to X_i$ which is a surjection of sheaves. Thus if I is not too large (for example countable) then $U = \coprod U_i \to X$ is a surjection of sheaves and U is representable by an object of $(Sch/S)_{fppf}$. To get $R \in \mathrm{Ob}((Sch/S)_{fppf})$ surjecting onto $U \times_X U$ it suffices to assume the diagonal $\Delta : X \to X \times_S X$ is not too wild, for example this always works if the diagonal of X is quasi-compact, i.e., X is quasi-separated.

87.12. The reduction

0GB5 All formal algebraic spaces have an underlying reduced algebraic space as the following lemma demonstrates.

0AIN **Lemma 87.12.1.** *Let S be a scheme. Let X be a formal algebraic space over S. There exists a reduced algebraic space X_{red} and a representable morphism $X_{red} \to X$ which is a thickening. A morphism $U \to X$ with U a reduced algebraic space factors uniquely through X_{red}.*

Proof. First assume that X is an affine formal algebraic space. Say $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Since the transition morphisms are thickenings, the affine schemes X_λ all have isomorphic reductions X_{red}. The morphism $X_{red} \to X$ is representable and a thickening by Lemma 87.9.3 and the fact that compositions of thickenings are thickenings. We omit the verification of the universal property (use Schemes, Definition 26.12.5, Schemes, Lemma 26.12.7, Properties of Spaces, Definition 66.12.5, and Properties of Spaces, Lemma 66.12.4).

Let X and $\{X_i \to X\}_{i\in I}$ be as in Definition 87.11.1. For each i let $X_{i,red} \to X_i$ be the reduction as constructed above. For $i, j \in I$ the projection $X_{i,red} \times_X X_j \to X_{i,red}$ is an étale (by assumption) morphism of schemes (by Lemma 87.9.11). Hence $X_{i,red} \times_X X_j$ is reduced (see Descent, Lemma 35.18.1). Thus the projection $X_{i,red} \times_X X_j \to X_j$ factors through $X_{j,red}$ by the universal property. We conclude that

$$R_{ij} = X_{i,red} \times_X X_j = X_{i,red} \times_X X_{j,red} = X_i \times_X X_{j,red}$$

because the morphisms $X_{i,red} \to X_i$ are injections of sheaves. Set $U = \coprod X_{i,red}$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X_{red} = U/R$ is an algebraic space by Spaces, Theorem 65.10.5. By construction the diagram

$$\begin{array}{ccc} \coprod X_{i,red} & \longrightarrow & \coprod X_i \\ \downarrow & & \downarrow \\ X_{red} & \longrightarrow & X \end{array}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a thickening we conclude that $X_{red} \to X$ is representable by Bootstrap, Lemma 80.5.2 (to verify the assumptions of the lemma use that a surjective étale morphism is surjective, flat, and locally of finite presentation and use that thickenings are separated and locally quasi-finite). Then we can use Spaces, Lemma 65.5.6 to conclude that $X_{red} \to X$ is a thickening (use that being a thickening is equivalent to being a surjective closed immersion).

Finally, suppose that $U \to X$ is a morphism with U a reduced algebraic space over S. Then each $X_i \times_X U$ is étale over U and therefore reduced (by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7). Then $X_i \times_X U \to X_i$ factors through $X_{i,red}$. Hence $U \to X$ factors through X_{red} because $\{X_i \times_X U \to U\}$ is an étale covering. □

0GB6 **Example 87.12.2.** Let A be a weakly admissible topological ring. In this case we have

$$\mathrm{Spf}(A)_{red} = \mathrm{Spec}(A/\mathfrak{a})$$

where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements. Namely, $\mathfrak{a}$ is a radical ideal (Lemma 87.4.10) which is open because A is weakly admissible.

0GB7 **Lemma 87.12.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces and smooth (for example étale). Then $X_{red} = X \times_Y Y_{red}$.*

Proof. (The étale case follows directly from the construction of the underlying reduced algebraic space in the proof of Lemma 87.12.1.) Assume f is smooth. Observe that $X \times_Y Y_{red} \to Y_{red}$ is a smooth morphism of algebraic spaces. Hence $X \times_Y Y_{red}$ is a reduced algebraic space by Descent on Spaces, Lemma 74.9.5. Then the universal property of reduction shows that the canonical morphism $X_{red} \to X \times_Y Y_{red}$ is an isomorphism. □

0GB8 **Lemma 87.12.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is surjective in the sense of Bootstrap, Definition 80.4.1 if and only if $f_{red} : X_{red} \to Y_{red}$ is a surjective morphism of algebraic spaces.*

Proof. Omitted. □

87.13. Colimits of algebraic spaces along thickenings

0AIT A special type of formal algebraic space is one which can globally be written as a cofiltered colimit of algebraic spaces along thickenings as in the following lemma. We will see later (in Section 87.18) that any quasi-compact and quasi-separated formal algebraic space is such a global colimit.

0AIU **Lemma 87.13.1.** *Let S be a scheme. Suppose given a directed set Λ and a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening. Then $X = \operatorname{colim}_{\lambda\in\Lambda} X_\lambda$ is a formal algebraic space over S.*

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose an étale covering $\{X_{i,\lambda} \to X_\lambda\}$ where X_i is an affine scheme over S, see Properties of Spaces, Lemma 66.6.1. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{array}{ccc} X_{i,\lambda} & \longrightarrow & X_{i,\mu} \\ \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X_\mu \end{array}$$

with étale vertical arrows, see More on Morphisms of Spaces, Theorem 76.8.1 (this also uses that a thickening is a surjective closed immersion which satisfies the conditions of the theorem). Moreover, these diagrams are unique up to unique isomorphism and hence $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu$. The morphisms $X_{i,\mu} \to X_{i,\mu'}$ is a thickening as a base change of a thickening. Each $X_{i,\mu}$ is an affine scheme by Limits of Spaces, Proposition 70.15.2 and the fact that $X_{i,\lambda}$ is affine. Set $X_i = \operatorname{colim}_{\mu\geq\lambda} X_{i,\mu}$. Then X_i is an affine formal algebraic space. The morphism $X_i \to X$ is étale because given an affine scheme U any $U \to X$ factors through X_μ for some $\mu \geq \lambda$ (details omitted). In this way we see that X is a formal algebraic space. □

Let S be a scheme. Let X be a formal algebraic space over S. How does one prove or check that X is a global colimit as in Lemma 87.13.1? To do this we look for maps $i : Z \to X$ where Z is an algebraic space over S and i is surjective and a closed immersion, in other words, i is a thickening. This makes sense as i is representable by algebraic spaces (Lemma 87.11.3) and we can use Bootstrap, Definition 80.4.1 as before.

0CB8 **Example 87.13.2.** Let $(A, \mathfrak{m}, \kappa)$ be a valuation ring, which is (π)-adically complete for some nonzero $\pi \in \mathfrak{m}$. Assume also that $\mathfrak{m}$ is not finitely generated. An example is $A = \mathcal{O}_{\mathbf{C}_p}$ and $\pi = p$ where $\mathcal{O}_{\mathbf{C}_p}$ is the ring of integers of the field of p-adic complex numbers $\mathbf{C}_p$ (this is the completion of the algebraic closure of $\mathbf{Q}_p$). Another example is

$$A = \left\{ \sum\nolimits_{\alpha\in\mathbf{Q},\ \alpha\geq 0} a_\alpha t^\alpha \middle| \begin{array}{c} a_\alpha \in \kappa \text{ and for all } n \text{ there are only a} \\ \text{finite number of nonzero } a_\alpha \text{ with } \alpha \leq n \end{array} \right\}$$

and $\pi = t$. Then $X = \mathrm{Spf}(A)$ is an affine formal algebraic space and $\mathrm{Spec}(\kappa) \to X$ is a thickening which corresponds to the weak ideal of definition $\mathfrak{m} \subset A$ which is however not an ideal of definition.

0AIV **Remark 87.13.3** (Weak ideals of definition)**.** Let $\mathfrak{X}$ be a formal scheme in the sense of McQuillan, see Remark 87.2.3. An *weak ideal of definition* for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\mathfrak{X}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_\mathfrak{X}(\mathfrak{U})$ is a weak ideal of definition of the weakly admissible topological ring $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. There is a one-to-one correspondence

$$\{\text{weak ideals of definition for } \mathfrak{X}\} \leftrightarrow \{\text{thickenings } i : Z \to h_\mathfrak{X} \text{ as above}\}$$

This correspondence associates to $\mathcal{I}$ the scheme $Z = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathcal{I})$ together with the obvious morphism to $\mathfrak{X}$. A *fundamental system of weak ideals of definition* is a collection of weak ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_\mathfrak{X})$$

form a fundamental system of weak ideals of definition of the weakly admissible topological ring A. It suffices to check on the members of an affine open covering. We conclude that the formal algebraic space $h_\mathfrak{X}$ associated to the McQuillan formal scheme $\mathfrak{X}$ is a colimit of schemes as in Lemma 87.13.1 if and only if there exists a fundamental system of weak ideals of definition for $\mathfrak{X}$.

0AIW **Remark 87.13.4** (Ideals of definition)**.** Let $\mathfrak{X}$ be a formal scheme à la EGA. An *ideal of definition* for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\mathfrak{X}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_\mathfrak{X}(\mathfrak{U})$ is an ideal of definition of the admissible topological ring $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. We do **not** get the same correspondence between ideals of definition and thickenings $Z \to h_\mathfrak{X}$ as in Remark 87.13.3; an example is given in Example 87.13.2. A *fundamental system of ideals of definition* is a collection of ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_\mathfrak{X})$$

form a fundamental system of ideals of definition of the admissible topological ring A. It suffices to check on the members of an affine open covering. Suppose that $\mathfrak{X}$ is quasi-compact and that $\{\mathcal{I}_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of weak ideals of definition. If A is an admissible topological ring then all sufficiently small open ideals are ideals of definition (namely any open ideal contained in an ideal of definition is an ideal of definition). Thus since we only need to check on the finitely many members of an affine open covering we see that $\mathcal{I}_\lambda$ is an ideal of definition for λ sufficiently large. Using the discussion in Remark 87.13.3 we conclude that the formal algebraic space $h_\mathfrak{X}$ associated to the quasi-compact formal scheme $\mathfrak{X}$ à la EGA is a colimit of schemes as in Lemma 87.13.1 if and only if there exists a fundamental system of ideals of definition for $\mathfrak{X}$.

87.14. Completion along a closed subset

0AIX Our notion of a formal algebraic space is well adapted to taking the completion along a closed subset.

0AIY **Lemma 87.14.1.** *Let S be a scheme. Let X be an affine scheme over S. Let $T \subset |X|$ be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a McQuillan affine formal algebraic space.

Proof. Say $X = \mathrm{Spec}(A)$ and T corresponds to the radical ideal $I \subset A$. Let $U = \mathrm{Spec}(B)$ be an affine scheme over S and let $f : U \to X$ be an element of $F(U)$. Then f corresponds to a ring map $\varphi : A \to B$ such that every prime of B contains $\varphi(I)B$. Thus every element of $\varphi(I)$ is nilpotent in B, see Algebra, Lemma 10.17.2. Setting $J = \mathrm{Ker}(\varphi)$ we conclude that I/J is a locally nilpotent ideal in A/J. Equivalently, $V(J) = V(I) = T$. In other words, the functor of the lemma equals $\mathrm{colim}\,\mathrm{Spec}(A/J)$ where the colimit is over the collection of ideals J with $V(J) = T$. Thus our functor is an affine formal algebraic space. It is McQuillan (Definition 87.9.7) because the maps $A \to A/J$ are surjective and hence $A^\wedge = \lim A/J \to A/J$ is surjective, see Lemma 87.9.6. □

0AIZ **Lemma 87.14.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. Denote F the functor. Let $\{U_i \to U\}$ be an fppf covering. Then $\coprod |U_i| \to |U|$ is surjective. Since X is an fppf sheaf, it follows that F is an fppf sheaf.

Let $\{g_i : X_i \to X\}$ be an étale covering such that X_i is affine for all i, see Properties of Spaces, Lemma 66.6.1. The morphisms $F \times_X X_i \to F$ are étale (see Spaces, Lemma 65.5.5) and the map $\coprod F \times_X X_i \to F$ is a surjection of sheaves. Thus it suffices to prove that $F \times_X X_i$ is an affine formal algebraic space. A U-valued point of $F \times_X X_i$ is a morphism $U \to X_i$ whose image is contained in the closed subset $g_i^{-1}(T) \subset |X_i|$. Thus this follows from Lemma 87.14.1. □

0AMC **Definition 87.14.3.** Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The formal algebraic space of Lemma 87.14.2 is called the *completion of X along T*.

In [**DG67**, Chapter I, Section 10.8] the notation $X_{/T}$ is used to denote the completion and we will occasionally use this notation as well. Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $T \subset |X|$ and $T' \subset |X'|$ are closed subsets such that $|f|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

0APV **Lemma 87.14.4.** *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $T \subset |X|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X'|$. Then*

$$\begin{array}{ccc} X'_{/T'} & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ X_{/T} & \longrightarrow & X \end{array}$$

is a cartesian diagram of sheaves. In particular, the morphism $X'_{/T'} \to X_{/T}$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $|Y|$ maps into T. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $|Y \times_X X'|$ maps into T'. Hence the functor $Y \times_{X_{/T}} X'_{/T'}$ is represented by $Y \times_X X'$ and we see that the lemma holds. □

0GB9 **Lemma 87.14.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X corresponding to T.*

Proof. It follows from Lemma 87.12.1, Properties of Spaces, Definition 66.12.5 (which uses Properties of Spaces, Lemma 66.12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. □

0GBA **Lemma 87.14.6.** *Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.*

1. *If $X \setminus T$ is quasi-compact, i.e., T is constructible, then $X_{/T}$ is adic*.*
2. *If $T = V(I)$ for some finitely generated ideal $I \subset A$, then $X_{/T} = Spf(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.*
3. *If X is Noetherian, then $X_{/T}$ is Noetherian.*

Proof. By Algebra, Lemma 10.29.1 if (1) holds, then we can find an ideal $I \subset A$ as in (2). If (3) holds then we can find an ideal $I \subset A$ as in (2). Moreover, completions of Noetherian rings are Noetherian by Algebra, Lemma 10.97.6. All in all we see that it suffices to prove (2).

Proof of (2). Let $I = (f_1, \ldots, f_r) \subset A$ cut out T. If $Z = \mathrm{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp(f_i)$ is nilpotent in B for each i. Thus I^n maps to zero in B for some n. Hence we see that $X_{/T} = \mathrm{colim}\, \mathrm{Spec}(A/I^n) = \mathrm{Spf}(A^\wedge)$. □

The following lemma is due to Ofer Gabber.

0APW **Lemma 87.14.7.** *Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subscheme.* Email by Ofer Gabber of September 11, 2014.

1. *If the formal completion $X_{/T}$ is countably indexed and there exist countably many $f_1, f_2, f_3, \ldots \in A$ such that $T = V(f_1, f_2, f_3, \ldots)$, then $X_{/T}$ is adic*.*
2. *The conclusion of (1) is wrong if we omit the assumption that T can be cut out by countably many functions in X.*

Proof. The assumption that $X_{/T}$ is countably indexed means that there exists a sequence of ideals

$$A \supset J_1 \supset J_2 \supset J_3 \supset \ldots$$

with $V(J_n) = T$ such that every ideal $J \subset A$ with $V(J) = T$ there exists an n such that $J \supset J_n$.

To construct an example for (2) let ω_1 be the first uncountable ordinal. Let k be a field and let A be the k-algebra generated by x_α, $\alpha \in \omega_1$ and $y_{\alpha\beta}$ with $\alpha \in \beta \in \omega_1$

subject to the relations $x_\alpha = y_{\alpha\beta}x_\beta$. Let $T = V(x_\alpha)$. Let $J_n = (x_\alpha^n)$. If $J \subset A$ is an ideal such that $V(J) = T$, then $x_\alpha^{n_\alpha} \in J$ for some $n_\alpha \geq 1$. One of the sets $\{\alpha \mid n_\alpha = n\}$ must be unbounded in ω_1. Then the relations imply that $J_n \subset J$.

To see that (2) holds it now suffices to show that $A^\wedge = \lim A/J_n$ is not a ring complete with respect to a finitely generated ideal. For $\gamma \in \omega_1$ let A_γ be the quotient of A by the ideal generated by x_α, $\alpha \in \gamma$ and $y_{\alpha\beta}$, $\alpha \in \gamma$. As A/J_1 is reduced, every topologically nilpotent element f of $\lim A/J_n$ is in $J_1^\wedge = \lim J_1/J_n$. This means f is an infinite series involving only a countable number of generators. Hence f dies in $A_\gamma^\wedge = \lim A_\gamma/J_nA_\gamma$ for some γ. Note that $A^\wedge \to A_\gamma^\wedge$ is continuous and open by Lemma 87.4.5. If the topology on $A^\wedge$ was I-adic for some finitely generated ideal $I \subset A^\wedge$, then I would go to zero in some $A_\gamma^\wedge$. This would mean that $A_\gamma^\wedge$ is discrete, which is not the case as there is a surjective continuous and open (by Lemma 87.4.5) map $A_\gamma^\wedge \to k[[t]]$ given by $x_\alpha \mapsto t$, $y_{\alpha\beta} \mapsto 1$ for $\gamma = \alpha$ or $\gamma \in \alpha$.

Before we prove (1) we first prove the following: If $I \subset A^\wedge$ is a finitely generated ideal whose closure $\bar{I}$ is open, then $I = \bar{I}$. Since $V(J_n^2) = T$ there exists an m such that $J_n^2 \supset J_m$. Thus, we may assume that $J_n^2 \supset J_{n+1}$ for all n by passing to a subsequence. Set $J_n^\wedge = \lim_{k \geq n} J_n/J_k \subset A^\wedge$. Since the closure $\bar{I} = \bigcap(I + J_n^\wedge)$ (Lemma 87.4.2) is open we see that there exists an m such that $I + J_n^\wedge \supset J_m^\wedge$ for all $n \geq m$. Fix such an m. We have

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_{n-1}^\wedge(I + J_{n+1}^\wedge) \supset J_{n-1}^\wedge J_m^\wedge$$

for all $n \geq m + 1$. Namely, the first inclusion is trivial and the second was shown above. Because $J_{n-1}J_m \supset J_{n-1}^2 \supset J_n$ these inclusions show that the image of J_n in $A^\wedge$ is contained in the ideal $J_{n-1}^\wedge I + J_{n+1}^\wedge$. Because this ideal is open we conclude that

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_n^\wedge.$$

Say $I = (g_1, \ldots, g_t)$. Pick $f \in J_{m+1}^\wedge$. Using the last displayed inclusion, valid for all $n \geq m + 1$, we can write by induction on $c \geq 0$

$$f = \sum f_{i,c}g_i \mod J_{m+1+c}^\wedge$$

with $f_{i,c} \in J_m^\wedge$ and $f_{i,c} \equiv f_{i,c-1} \bmod J_{m+c}^\wedge$. It follows that $IJ_m^\wedge \supset J_{m+1}^\wedge$. Combined with $I + J_{m+1}^\wedge \supset J_m^\wedge$ we conclude that I is open.

Proof of (1). Assume $T = V(f_1, f_2, f_3, \ldots)$. Let $I_m \subset A^\wedge$ be the ideal generated by $f_1, \ldots, f_m$. We distinguish two cases.

Case I: For some m the closure of I_m is open. Then I_m is open by the result of the previous paragraph. For any n we have $(J_n)^2 \supset J_{n+1}$ by design, so the closure of $(J_n^\wedge)^2$ contains $J_{n+1}^\wedge$ and thus is open. Taking n large, it follows that the closure of the product of any two open ideals in $A^\wedge$ is open. Let us prove I_m^k is open for $k \geq 1$ by induction on k. The case $k = 1$ is our hypothesis on m in Case I. For $k > 1$, suppose I_m^{k-1} is open. Then $I_m^k = I_m^{k-1} \cdot I_m$ is the product of two open ideals and hence has open closure. But then since I_m^k is finitely generated it follows that I_m^k is open by the previous paragraph (applied to $I = I_m^k$), so we can continue the induction on k. As each element of I_m is topologically nilpotent, we conclude that I_m is an ideal of definition which proves that $A^\wedge$ is adic with a finitely generated ideal of definition, i.e., $X_{/T}$ is adic*.

Case II. For all m the closure $\bar{I}_m$ of I_m is not open. Then the topology on $A^\wedge/\bar{I}_m$ is not discrete. This means we can pick $\phi(m) \geq m$ such that

$$\mathrm{Im}(J_{\phi(m)} \to A/(f_1, \ldots, f_m)) \neq \mathrm{Im}(J_{\phi(m)+1} \to A/(f_1, \ldots, f_m))$$

To see this we have used that $A^\wedge/(\bar{I}_m + J_n^\wedge) = A/((f_1, \ldots, f_m) + J_n)$. Choose exponents $e_i > 0$ such that $f_i^{e_i} \in J_{\phi(m)+1}$ for $0 < m < i$. Let $J = (f_1^{e_1}, f_2^{e_2}, f_3^{e_3}, \ldots)$. Then $V(J) = T$. We claim that $J \not\supset J_n$ for all n which is a contradiction proving Case II does not occur. Namely, the image of J in $A/(f_1, \ldots, f_m)$ is contained in the image of $J_{\phi(m)+1}$ which is properly contained in the image of J_m. □

87.15. Fibre products

0AJ0 Obligatory section about fibre products of formal algebraic spaces.

0AJ1 **Lemma 87.15.1.** *Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves on $(Sch/S)_{fppf}$. Assume (a) X_i is a formal algebraic space over S, (b) $X_i \to X$ is representable by algebraic spaces and étale, and (c) $\coprod X_i \to X$ is a surjection of sheaves. Then X is a formal algebraic space over S.*

Proof. For each i pick $\{X_{ij} \to X_i\}_{j \in J_i}$ as in Definition 87.11.1. Then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a family as in Definition 87.11.1 for X. □

0AJ2 **Lemma 87.15.2.** *Let S be a scheme. Let X, Y be formal algebraic spaces over S and let Z be a sheaf whose diagonal is representable by algebraic spaces. Let $X \to Z$ and $Y \to Z$ be maps of sheaves. Then $X \times_Z Y$ is a formal algebraic space.*

Proof. Choose $\{X_i \to X\}$ and $\{Y_j \to Y\}$ as in Definition 87.11.1. Then $\{X_i \times_Z Y_j \to X \times_Z Y\}$ is a family of maps which are representable by algebraic spaces and étale. Thus Lemma 87.15.1 tells us it suffices to show that $X \times_Z Y$ is a formal algebraic space when X and Y are affine formal algebraic spaces.

Assume X and Y are affine formal algebraic spaces. Write $X = \mathrm{colim}\, X_\lambda$ and $Y = \mathrm{colim}\, Y_\mu$ as in Definition 87.9.1. Then $X \times_Z Y = \mathrm{colim}\, X_\lambda \times_Z Y_\mu$. Each $X_\lambda \times_Z Y_\mu$ is an algebraic space. For $\lambda \leq \lambda'$ and $\mu \leq \mu'$ the morphism

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times_Z Y_{\mu'} \to X_{\lambda'} \times_Z Y_{\mu'}$$

is a thickening as a composition of base changes of thickenings. Thus we conclude by applying Lemma 87.13.1. □

0AJ3 **Lemma 87.15.3.** *Let S be a scheme. The category of formal algebraic spaces over S has fibre products.*

Proof. Special case of Lemma 87.15.2 because formal algebraic spaces have representable diagonals, see Lemma 87.11.2. □

0CB9 **Lemma 87.15.4.** *Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Then $(X \times_Z Y)_{red} = (X_{red} \times_{Z_{red}} Y_{red})_{red}$.*

Proof. This follows from the universal property of the reduction in Lemma 87.12.1. □

We have already proved the following lemma (without knowing that fibre products exist).

0AN2 **Lemma 87.15.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The diagonal morphism $\Delta : X \to X \times_Y X$ is representable (by schemes), a monomorphism, locally quasi-finite, locally of finite type, and separated.*

Proof. Let T be a scheme and let $T \to X \times_Y X$ be a morphism. Then

$$T \times_{(X \times_Y X)} X = T \times_{(X \times_S X)} X$$

Hence the result follows immediately from Lemma 87.11.2. □

87.16. Separation axioms for formal algebraic spaces

0AJ4 This section is about "absolute" separation conditions on formal algebraic spaces. We will discuss separation conditions for morphisms of formal algebraic spaces later.

0AJ5 **Lemma 87.16.1.** *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

1. *the reduction of X (Lemma 87.12.1) is a quasi-separated algebraic space,*
2. *for $U \to X$, $V \to X$ with U, V quasi-compact schemes the fibre product $U \times_X V$ is quasi-compact,*
3. *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is quasi-compact.*

Proof. Observe that $U \times_X V$ is a scheme by Lemma 87.11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. From this the equivalence of (1) and (2) is clear, keeping in mind the analogous lemma for algebraic spaces, see Properties of Spaces, Lemma 66.3.3. We omit the proof of the equivalence of (2) and (3). □

0AJ6 **Lemma 87.16.2.** *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

1. *the reduction of X (Lemma 87.12.1) is a separated algebraic space,*
2. *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is affine and*
$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$
is surjective.

Proof. If (2) holds, then X_{red} is a separated algebraic space by applying Properties of Spaces, Lemma 66.3.3 to morphisms $U \to X_{red}$ and $V \to X_{red}$ with U, V affine and using that $U \times_{X_{red}} V = U \times_X V$.

Assume (1). Let $U \to X$ and $V \to X$ be as in (2). Observe that $U \times_X V$ is a scheme by Lemma 87.11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. It follows that $(U \times_X V)_{red} = (U_{red} \times_{X_{red}} V_{red})_{red}$. In particular, we see that $(U \times_X V)_{red}$ is an affine scheme and that

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}((U \times_X V)_{red})$$

is surjective, see Properties of Spaces, Lemma 66.3.3. Then $U \times_X V$ is affine by Limits of Spaces, Proposition 70.15.2. On the other hand, the morphism $U \times_X V \to U \times V$ of affine schemes is the composition

$$U \times_X V = X \times_{(X \times_S X)} (U \times_S V) \to U \times_S V \to U \times V$$

The first morphism is a monomorphism and locally of finite type (Lemma 87.11.2). The second morphism is an immersion (Schemes, Lemma 26.21.9). Hence the composition is a monomorphism which is locally of finite type. On the other hand, the composition is integral as the map on underlying reduced affine schemes is a closed immersion by the above and hence universally closed (use Morphisms, Lemma 29.44.7). Thus the ring map

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

is an epimorphism which is integral of finite type hence finite hence surjective (use Morphisms, Lemma 29.44.4 and Algebra, Lemma 10.107.6). □

0AJ7 **Definition 87.16.3.** Let S be a scheme. Let X be a formal algebraic space over S. We say

(1) X is *quasi-separated* if the equivalent conditions of Lemma 87.16.1 are satisfied.
(2) X is *separated* if the equivalent conditions of Lemma 87.16.2 are satisfied.

The following lemma implies in particular that the completed tensor product of weakly admissible topological rings is a weakly admissible topological ring.

0AN3 **Lemma 87.16.4.** *Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Assume Z separated.*

(1) *If X and Y are affine formal algebraic spaces, then so is $X \times_Z Y$.*
(2) *If X and Y are McQuillan affine formal algebraic spaces, then so is $X \times_Z Y$.*
(3) *If X, Y, and Z are McQuillan affine formal algebraic spaces corresponding to the weakly admissible topological S-algebras A, B, and C, then $X \times_Z Y$ corresponds to $A \widehat{\otimes}_C B$.*

Proof. Write $X = \operatorname{colim} X_\lambda$ and $Y = \operatorname{colim} Y_\mu$ as in Definition 87.9.1. Then $X \times_Z Y = \operatorname{colim} X_\lambda \times_Z Y_\mu$. Since Z is separated the fibre products are affine, hence we see that (1) holds. Assume X and Y corresponds to the weakly admissible topological S-algebras A and B and $X_\lambda = \operatorname{Spec}(A/I_\lambda)$ and $Y_\mu = \operatorname{Spec}(B/J_\mu)$. Then

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times Y_\mu \to \operatorname{Spec}(A \otimes B)$$

is a closed immersion. Thus one of the conditions of Lemma 87.9.6 holds and we conclude that $X \times_Z Y$ is McQuillan. If also Z is McQuillan corresponding to C, then

$$X_\lambda \times_Z Y_\mu = \operatorname{Spec}(A/I_\lambda \otimes_C B/J_\mu)$$

hence we see that the weakly admissible topological ring corresponding to $X \times_Z Y$ is the completed tensor product (see Definition 87.4.7). □

0APX **Lemma 87.16.5.** *Let S be a scheme. Let X be a formal algebraic space over S. Let $U \to X$ be a morphism where U is a separated algebraic space over S. Then $U \to X$ is separated.*

Proof. The statement makes sense because $U \to X$ is representable by algebraic spaces (Lemma 87.11.3). Let T be a scheme and $T \to X$ a morphism. We have to show that $U \times_X T \to T$ is separated. Since $U \times_X T \to U \times_S T$ is a monomorphism, it suffices to show that $U \times_S T \to T$ is separated. As this is the base change of $U \to S$ this follows. We used in the argument above: Morphisms of Spaces, Lemmas 67.4.4, 67.4.8, 67.10.3, and 67.4.11. □

87.17. Quasi-compact formal algebraic spaces

0AJ8 Here is the characterization of quasi-compact formal algebraic spaces.

0AJ9 **Lemma 87.17.1.** *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

(1) *the reduction of X (Lemma 87.12.1) is a quasi-compact algebraic space,*
(2) *we can find $\{X_i \to X\}_{i \in I}$ as in Definition 87.11.1 with I finite,*
(3) *there exists a morphism $Y \to X$ representable by algebraic spaces which is étale and surjective and where Y is an affine formal algebraic space.*

Proof. Omitted. □

0AJA **Definition 87.17.2.** Let S be a scheme. Let X be a formal algebraic space over S. We say X is *quasi-compact* if the equivalent conditions of Lemma 87.17.1 are satisfied.

0AJB **Lemma 87.17.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *the induced map $f_{red} : X_{red} \to Y_{red}$ between reductions (Lemma 87.12.1) is a quasi-compact morphism of algebraic spaces,*
(2) *for every quasi-compact scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space,*
(3) *for every affine scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space, and*
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j$ is a quasi-compact formal algebraic space.*

Proof. Omitted. □

0AJC **Definition 87.17.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. We say f is *quasi-compact* if the equivalent conditions of Lemma 87.17.3 are satisfied.

This agrees with the already existing notion when the morphism is representable by algebraic spaces (and in particular when it is representable).

0AM2 **Lemma 87.17.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is quasi-compact in the sense of Definition 87.17.4 if and only if f is quasi-compact in the sense of Bootstrap, Definition 80.4.1.*

Proof. This is immediate from the definitions and Lemma 87.17.3. □

87.18. Quasi-compact and quasi-separated formal algebraic spaces

0AJD The following result is due to Yasuda, see [**Yas09**, Proposition 3.32].

0AJE **Lemma 87.18.1.** *Let S be a scheme. Let X be a quasi-compact and quasi-separated formal algebraic space over S. Then $X = \operatorname{colim} X_\lambda$ for a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over a directed set Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening.* [**Yas09**, Proposition 3.32]

Proof. By Lemma 87.17.1 we may choose an affine formal algebraic space Y and a representable surjective étale morphism $Y \to X$. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 87.9.1.

Pick $\lambda \in \Lambda$. Then $Y_\lambda \times_X Y$ is a scheme by Lemma 87.9.11. The reduction (Lemma 87.12.1) of $Y_\lambda \times_X Y$ is equal to the reduction of $Y_{red} \times_{X_{red}} Y_{red}$ which is quasi-compact as X is quasi-separated and Y_{red} is affine. Therefore $Y_\lambda \times_X Y$ is a quasi-compact scheme. Hence there exists a $\mu \geq \lambda$ such that $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y$ factors through Y_μ, see Lemma 87.9.4. Let Z_λ be the scheme theoretic image of the morphism $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y_\mu$. This is independent of the choice of μ and we can and will think of $Z_\lambda \subset Y$ as the scheme theoretic image of the morphism $\mathrm{pr}_2 : Y_\lambda \times_X Y \to Y$. Observe that Z_λ is also equal to the scheme theoretic image of the morphism $\mathrm{pr}_1 : Y \times_X Y_\lambda \to Y$ since this is isomorphic to the morphism used to define Z_λ. We claim that $Z_\lambda \times_X Y = Y \times_X Z_\lambda$ as subfunctors of $Y \times_X Y$. Namely, since $Y \to X$ is étale we see that $Z_\lambda \times_X Y$ is the scheme theoretic image of the morphism

$$\mathrm{pr}_{13} = \mathrm{pr}_1 \times \mathrm{id}_Y : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

by Morphisms of Spaces, Lemma 67.16.3. By the same token, $Y \times_X Z_\lambda$ is the scheme theoretic image of the morphism

$$\mathrm{pr}_{13} = \mathrm{id}_Y \times \mathrm{pr}_2 : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

The claim follows. Then $R_\lambda = Z_\lambda \times_X Y = Y \times_X Z_\lambda$ together with the morphism $R_\lambda \to Z_\lambda \times_S Z_\lambda$ defines an étale equivalence relation. In this way we obtain an algebraic space $X_\lambda = Z_\lambda / R_\lambda$. By construction the diagram

$$\begin{array}{ccc} Z_\lambda & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X \end{array}$$

is cartesian (because X is the coequalizer of the two projections $R = Y \times_X Y \to Y$, because $Z_\lambda \subset Y$ is R-invariant, and because R_λ is the restriction of R to Z_λ). Hence $X_\lambda \to X$ is representable and a closed immersion, see Spaces, Lemma 65.11.5. On the other hand, since $Y_\lambda \subset Z_\lambda$ we see that $(X_\lambda)_{red} = X_{red}$, in other words, $X_\lambda \to X$ is a thickening. Finally, we claim that

$$X = \operatorname{colim} X_\lambda$$

We have $Y \times_X X_\lambda = Z_\lambda \supset Y_\lambda$. Every morphism $T \to X$ where T is a scheme over S lifts étale locally to a morphism into Y which lifts étale locally into a morphism into some Y_λ. Hence $T \to X$ lifts étale locally on T to a morphism into X_λ. This finishes the proof. □

0AJF **Remark 87.18.2.** In this remark we translate the statement and proof of Lemma 87.18.1 into the language of formal schemes à la EGA. Looking at Remark 87.13.4 we see that the lemma can be translated as follows

(*) Every quasi-compact and quasi-separated formal scheme has a fundamental system of ideals of definition.

To prove this we first use the induction principle (reformulated for quasi-compact and quasi-separated formal schemes) of Cohomology of Schemes, Lemma 30.4.1 to reduce to the following situation: $\mathfrak{X} = \mathfrak{U} \cup \mathfrak{V}$ with $\mathfrak{U}$, $\mathfrak{V}$ open formal subschemes, with $\mathfrak{V}$ affine, and the result is true for $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{U} \cap \mathfrak{V}$. Pick any ideals of definition $\mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. By our assumption that we have a fundamental system of ideals of definition on $\mathfrak{U}$ and $\mathfrak{V}$ and because $\mathfrak{U} \cap \mathfrak{V}$ is quasi-compact, we can find ideals of definition $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ such that

$$\mathcal{I}'|_{\mathfrak{U}\cap\mathfrak{V}} \subset \mathcal{J}|_{\mathfrak{U}\cap\mathfrak{V}} \quad \text{and} \quad \mathcal{J}'|_{\mathfrak{U}\cap\mathfrak{V}} \subset \mathcal{I}|_{\mathfrak{U}\cap\mathfrak{V}}$$

Let $U \to U' \to \mathfrak{U}$ and $V \to V' \to \mathfrak{V}$ be the closed immersions determined by the ideals of definition $\mathcal{I}' \subset \mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J}' \subset \mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. Let $\mathfrak{U} \cap V$ denote the open subscheme of V whose underlying topological space is that of $\mathfrak{U} \cap \mathfrak{V}$. By our choice of $\mathcal{I}'$ there is a factorization $\mathfrak{U} \cap V \to U'$. We define similarly $U \cap \mathfrak{V}$ which factors through V'. Then we consider

$$Z_U = \text{scheme theoretic image of } U \amalg (\mathfrak{U} \cap V) \longrightarrow U'$$

and

$$Z_V = \text{scheme theoretic image of } (U \cap \mathfrak{V}) \amalg V \longrightarrow V'$$

Since taking scheme theoretic images of quasi-compact morphisms commutes with restriction to opens (Morphisms, Lemma 29.6.3) we see that $Z_U \cap \mathfrak{V} = \mathfrak{U} \cap Z_V$. Thus Z_U and Z_V glue to a scheme Z which comes equipped with a morphism $Z \to \mathfrak{X}$. Analogous to the discussion in Remark 87.13.3 we see that Z corresponds to a weak ideal of definition $\mathcal{I}_Z \subset \mathcal{O}_\mathfrak{X}$. Note that $Z_U \subset U'$ and that $Z_V \subset V'$. Thus the collection of all $\mathcal{I}_Z$ constructed in this manner forms a fundamental system of weak ideals of definition. Hence a subfamily gives a fundamental system of ideals of definition, see Remark 87.13.4.

0DE8 **Lemma 87.18.3.** *Let S be a scheme. Let X be a formal algebraic space over S. Then X is an affine formal algebraic space if and only if its reduction X_{red} (Lemma 87.12.1) is affine.*

Proof. By Lemmas 87.16.1 and 87.17.1 and Definitions 87.16.3 and 87.17.2 we see that X is quasi-compact and quasi-separated. By Yasuda's lemma (Lemma 87.18.1) we can write $X = \operatorname{colim} X_\lambda$ as a filtered colimit of thickenings of algebraic spaces. However, each X_λ is affine by Limits of Spaces, Lemma 70.15.3 because $(X_\lambda)_{red} = X_{red}$. Hence X is an affine formal algebraic space by definition. □

87.19. Morphisms representable by algebraic spaces

0AJG Let $f : X \to Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces. For these types of morphisms we have a lot of theory at our disposal, thanks to the work done in the chapters on algebraic spaces.

0APY **Lemma 87.19.1.** *The composition of morphisms representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

Proof. See Bootstrap, Lemma 80.3.8. □

0APZ **Lemma 87.19.2.** *A base change of a morphism representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

Proof. See Bootstrap, Lemma 80.3.3. □

0AQ0 **Lemma 87.19.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is representable by algebraic spaces, then $f : X \to Y$ is representable by algebraic spaces.*

Proof. Note that the diagonal of $Y \to Z$ is representable by Lemma 87.15.5. Thus $X \to Y$ is representable by algebraic spaces by Bootstrap, Lemma 80.3.10. □

The property of being representable by algebraic spaces is local on the source and the target.

0AN4 **Lemma 87.19.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:*

(1) *the morphism f is representable by algebraic spaces,*
(2) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces, $U \to X$ is surjective étale, and $U \to V$ is representable by algebraic spaces,
(3) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are formal algebraic spaces and the vertical arrows are representable by algebraic spaces, the morphism $U \to V$ is representable by algebraic spaces,
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that $X_{ji} \to Y_j$ is representable by algebraic spaces for each j and i,*
(5) *there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces, such that $X_i \to Y_i$ is representable by algebraic spaces, and*
(6) *add more here.*

Proof. It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, (2) implies (1) by Bootstrap, Lemma 80.11.4 applied to $U \to X \to Y$.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is representable by algebraic spaces (as the composition $U \to X \to Y$, see Bootstrap, Lemma

80.3.8) and factors through V. Thus $U \to V$ is representable by algebraic spaces by Lemma 87.19.3.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 87.15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. Then $X_{ij} \to Y_j$ is representable by algebraic spaces by (3) and we see that (4) is true. This concludes the proof. □

0AJH **Lemma 87.19.5.** *Let S be a scheme. Let Y be an affine formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.*

Proof. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 87.9.1. For each λ the fibre product $X \times_Y Y_\lambda$ is an algebraic space. Hence $X = \operatorname{colim} X \times_Y Y_\lambda$ is a formal algebraic space by Lemma 87.13.1. □

0AJI **Lemma 87.19.6.** *Let S be a scheme. Let Y be a formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.*

Proof. Let $\{Y_i \to Y\}$ be as in Definition 87.11.1. Then $X \times_Y Y_i \to X$ is a family of morphisms representable by algebraic spaces, étale, and jointly surjective. Thus it suffices to show that $X \times_Y Y_i$ is a formal algebraic space, see Lemma 87.15.1. This follows from Lemma 87.19.5. □

0AKN **Lemma 87.19.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then f is representable (by schemes) and affine.*

Proof. We will show that f is affine; it will then follow that f is representable and affine by Morphisms of Spaces, Lemma 67.20.3. Write $Y = \operatorname{colim} Y_\mu$ and $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Let $T \to Y$ be a morphism where T is a scheme over S. We have to show that $X \times_Y T \to T$ is affine, see Bootstrap, Definition 80.4.1. To do this we may assume that T is affine and we have to prove that $X \times_Y T$ is affine. In this case $T \to Y$ factors through $Y_\mu \to Y$ for some μ, see Lemma 87.9.4. Since f is quasi-compact we see that $X \times_Y T$ is quasi-compact (Lemma 87.17.3). Hence $X \times_Y T \to X$ factors through X_λ for some λ. Similarly $X_\lambda \to Y$ factors through Y_μ after increasing μ. Then $X \times_Y T = X_\lambda \times_{Y_\mu} T$. We conclude as fibre products of affine schemes are affine. □

0AN5 **Lemma 87.19.8.** *Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent*

(1) *$Spf(\varphi) : Spf(B) \to Spf(A)$ is representable by algebraic spaces,*
(2) *$Spf(\varphi) : Spf(B) \to Spf(A)$ is representable (by schemes),*
(3) *φ is taut, see Definition 87.5.1.*

Proof. Parts (1) and (2) are equivalent by Lemma 87.19.7.

Assume the equivalent conditions (1) and (2) hold. If $I \subset A$ is a weak ideal of definition, then $\mathrm{Spec}(A/I) \to \mathrm{Spf}(A)$ is representable and a thickening (this is clear from the construction of the formal spectrum but it also follows from Lemma 87.9.6). Then $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) \to \mathrm{Spf}(B)$ is representable and a thickening as a base change. Hence by Lemma 87.9.6 there is a weak ideal of definition $J(I) \subset B$ such that $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) = \mathrm{Spec}(B/J(I))$ as subfunctors of $\mathrm{Spf}(B)$. We obtain a cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(B/J(I)) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(B) & \longrightarrow & \mathrm{Spf}(A) \end{array}$$

By Lemma 87.16.4 we see that $B/J(I) = B\widehat{\otimes}_A A/I$. It follows that $J(I)$ is the closure of the ideal $\varphi(I)B$, see Lemma 87.4.11. Since $\mathrm{Spf}(A) = \mathrm{colim}\, \mathrm{Spec}(A/I)$ with I as above, we find that $\mathrm{Spf}(B) = \mathrm{colim}\, \mathrm{Spec}(B/J(I))$. Thus the ideals $J(I)$ form a fundamental system of weak ideals of definition (see Lemma 87.9.6). Hence (3) holds.

Assume (3) holds. We are essentially just going to reverse the arguments given in the previous paragraph. Let $I \subset A$ be a weak ideal of definition. By Lemma 87.16.4 we get a cartesian diagram

$$\begin{array}{ccc} \mathrm{Spf}(B\widehat{\otimes}_A A/I) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(B) & \longrightarrow & \mathrm{Spf}(A) \end{array}$$

If $J(I)$ is the closure of IB, then $J(I)$ is open in B by tautness of φ. Hence if J is open in B and $J \subset J(B)$, then $B/J \otimes_A A/I = B/(IB + J) = B/J(I)$ because $J(I) = \bigcap_{J \subset B \text{ open}} (IB + J)$ by Lemma 87.4.2. Hence the limit defining the completed tensor product collapses to give $B\widehat{\otimes}_A A/I = B/J(I)$. Thus $\mathrm{Spf}(B\widehat{\otimes}_A A/I) = \mathrm{Spec}(B/J(I))$. This proves that $\mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spec}(A/I)$ is representable for every weak ideal of definition $I \subset A$. Since every morphism $T \to \mathrm{Spf}(A)$ with T quasi-compact factors through $\mathrm{Spec}(A/I)$ for some weak ideal of definition I (Lemma 87.9.4) we conclude that $\mathrm{Spf}(\varphi)$ is representable, i.e., (2) holds. This finishes the proof. □

0AKP **Lemma 87.19.9.** *Let S be a scheme. Let Y be an affine formal algebraic space. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable and affine. Then*

(1) *X is an affine formal algebraic space,*
(2) *if Y is countably indexed, then X is countably indexed,*
(3) *if Y is countably indexed and classical, then X is countably indexed and classical,*
(4) *if Y is weakly adic, then X is weakly adic,*
(5) *if Y is adic*, then X is adic*, and*
(6) *if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.*

Proof. Proof of (1). Write $Y = \operatorname{colim}_{\lambda\in\Lambda} Y_\lambda$ as in Definition 87.9.1. Since f is representable and affine, the fibre products $X_\lambda = Y_\lambda \times_Y X$ are affine. And $X = \operatorname{colim} Y_\lambda \times_Y X$. Thus X is an affine formal algebraic space.

Proof of (2). If Y is countably indexed, then in the argument above we may assume Λ is countable. Then we immediately see that X is countably indexed too.

Proof of (3), (4), and (5). In each of these cases the assumptions imply that Y is a countably indexed affine formal algebraic space (Lemma 87.10.3) and hence X is too by (2). Thus we may write $X = \operatorname{Spf}(A)$ and $Y = \operatorname{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 87.10.4. By Lemma 87.9.10 the morphism f corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 87.19.8 that φ is taut. We conclude that (3) follows from Lemma 87.5.9, (4) follows from Lemma 87.7.5, and (5) follows from Lemma 87.6.5.

Proof of (6). Combining (3) with Lemma 87.10.3 we see that X is adic*. Thus we can use the criterion of Lemma 87.10.5. First, it tells us the affine schemes Y_λ are Noetherian. Then $X_\lambda \to Y_\lambda$ is of finite type, hence X_λ is Noetherian too (Morphisms, Lemma 29.15.6). Then the criterion tells us X is Noetherian and the proof is complete. □

0AKQ **Lemma 87.19.10.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then*

(1) *if Y is countably indexed, then X is countably indexed,*
(2) *if Y is countably indexed and classical, then X is countably indexed and classical,*
(3) *if Y is weakly adic, then X is weakly adic,*
(4) *if Y is adic*, then X is adic*, and*
(5) *if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.*

Proof. Combine Lemmas 87.19.7 and 87.19.9. □

0AN6 **Example 87.19.11.** Let B be a weakly admissible topological ring. Let $B \to A$ be a ring map (no topology). Then we can consider

$$A^\wedge = \lim A/JA$$

where the limit is over all weak ideals of definition J of B. Then $A^\wedge$ (endowed with the limit topology) is a complete linearly topologized ring. The (open) kernel I of the surjection $A^\wedge \to A/JA$ is the closure of $JA^\wedge$, see Lemma 87.4.2. By Lemma 87.4.10 we see that I consists of topologically nilpotent elements. Thus I is a weak ideal of definition of $A^\wedge$ and we conclude $A^\wedge$ is a weakly admissible topological ring. Thus $\varphi : B \to A^\wedge$ is taut map of weakly admissible topological rings and

$$\operatorname{Spf}(A^\wedge) \longrightarrow \operatorname{Spf}(B)$$

is a special case of the phenomenon studied in Lemma 87.19.8.

0AN7 **Remark 87.19.12** (Warning)**.** The discussion in Lemmas 87.19.8, 87.19.9, and 87.19.10 is sharp in the following two senses:

(1) If A and B are weakly admissible rings and $\varphi : A \to B$ is a continuous map, then $\operatorname{Spf}(\varphi) : \operatorname{Spf}(B) \to \operatorname{Spf}(A)$ is in general not representable.
(2) If $f : Y \to X$ is a representable morphism of affine formal algebraic spaces and $X = \operatorname{Spf}(A)$ is McQuillan, then it does not follow that Y is McQuillan.

An example for (1) is to take $A = k$ a field (with discrete topology) and $B = k[[t]]$ with the t-adic topology. An example for (2) is given in Examples, Section 110.75.

The warning above notwithstanding, we do have the following result.

0AN8 **Lemma 87.19.13.** *Let S be a scheme. Let Y be a McQuillan affine formal algebraic space over S, i.e., $Y = Spf(B)$ for some weakly admissible topological S-algebra B. Then there is an equivalence of categories between*

(1) *the category of morphisms $f : X \to Y$ of affine formal algebraic spaces which are representable by algebraic spaces and étale, and*
(2) *the category of topological B-algebras of the form $A^\wedge$ where A is an étale B-algebra and $A^\wedge = \lim A/JA$ with $J \subset B$ running over the weak ideals of definition of B.*

The equivalence is given by sending $A^\wedge$ to $X = Spf(A^\wedge)$. In particular, any X as in (1) is McQuillan.

Proof. Let A be an étale B-algebra. Then $B/J \to A/JA$ is étale for every open ideal $J \subset B$. Hence the morphism $\mathrm{Spf}(A^\wedge) \to Y$ is representable and étale. The functor Spf is fully faithful by Lemma 87.9.10. To finish the proof we will show in the next paragraph that any $X \to Y$ as in (1) is in the essential image.

Choose a weak ideal of definition $J_0 \subset B$. Set $Y_0 = \mathrm{Spec}(B/J_0)$ and $X_0 = Y_0 \times_Y X$. Then $X_0 \to Y_0$ is an étale morphism of affine schemes (see Lemma 87.19.7). Say $X_0 = \mathrm{Spec}(A_0)$. By Algebra, Lemma 10.143.10 we can find an étale algebra map $B \to A$ such that $A_0 \cong A/J_0A$. Consider an ideal of definition $J \subset J_0$. As above we may write $\mathrm{Spec}(B/J) \times_Y X = \mathrm{Spec}(\bar{A})$ for some étale ring map $B/J \to \bar{A}$. Then both $B/J \to \bar{A}$ and $B/J \to A/JA$ are étale ring maps lifting the étale ring map $B/J_0 \to A_0$. By More on Algebra, Lemma 15.11.2 there is a unique B/J-algebra isomorphism $\varphi_J : A/JA \to \bar{A}$ lifting the identification modulo J_0. Since the maps φ_J are unique they are compatible for varying J. Thus

$$X = \mathrm{colim}\, \mathrm{Spec}(B/J) \times_Y X = \mathrm{colim}\, \mathrm{Spec}(A/JA) = \mathrm{Spf}(A)$$

and we see that the lemma holds. □

0AN9 **Lemma 87.19.14.** *With notation and assumptions as in Lemma 87.19.13 let $f : X \to Y$ correspond to $B \to A^\wedge$. The following are equivalent*

(1) *$f : X \to Y$ is surjective,*
(2) *$B \to A$ is faithfully flat,*
(3) *for every weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat, and*
(4) *for some weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat.*

Proof. Let $J \subset B$ be a weak ideal of definition. As every element of J is topologically nilpotent, we see that every element of $1 + J$ is a unit. It follows that J is contained in the Jacobson radical of B (Algebra, Lemma 10.19.1). Hence a flat ring map $B \to A$ is faithfully flat if and only if $B/J \to A/JA$ is faithfully flat (Algebra, Lemma 10.39.16). In this way we see that (2) – (4) are equivalent. If (1) holds, then for every weak ideal of definition $J \subset B$ the morphism $\mathrm{Spec}(A/JA) = \mathrm{Spec}(B/J) \times_Y X \to \mathrm{Spec}(B/J)$ is surjective which implies (3). Conversely, assume (3). A morphism $T \to Y$ with T quasi-compact factors

through $\mathrm{Spec}(B/J)$ for some ideal of definition J of B (Lemma 87.9.4). Hence $X \times_Y T = \mathrm{Spec}(A/JA) \times_{\mathrm{Spec}(B/J)} T \to T$ is surjective as a base change of the surjective morphism $\mathrm{Spec}(A/JA) \to \mathrm{Spec}(B/J)$. Thus (1) holds. □

87.20. Types of formal algebraic spaces

0AKR In this section we define "locally Noetherian", "locally adic*", "locally weakly adic", "locally countably indexed and classical", and "locally countably indexed" formal algebraic spaces. The types "locally adic", "locally classical", and "locally McQuillan" are missing as we do not know how to prove the analogue of the following lemmas for those cases (it would suffice to prove the analogue of these lemmas for étale coverings between affine formal algebraic spaces).

0AKS **Lemma 87.20.1.** *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed if and only if Y is countably indexed.*

Proof. Assume X is countably indexed. We write $X = \mathrm{colim}\, X_n$ as in Lemma 87.10.1. Write $Y = \mathrm{colim}\, Y_\lambda$ as in Definition 87.9.1. For every n we can pick a λ_n such that $X_n \to Y$ factors through Y_{λ_n}, see Lemma 87.9.4. On the other hand, for every λ the scheme $Y_\lambda \times_Y X$ is affine (Lemma 87.19.7) and hence $Y_\lambda \times_Y X \to X$ factors through X_n for some n (Lemma 87.9.4). Picture

$$\begin{array}{ccccc} Y_\lambda \times_Y X & \longrightarrow & X_n & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y_\lambda & \dashrightarrow & Y_{\lambda_n} & \longrightarrow & Y \end{array}$$

If we can show the dotted arrow exists, then we conclude that $Y = \mathrm{colim}\, Y_{\lambda_n}$ and Y is countably indexed. To do this we pick a μ with $\mu \geq \lambda$ and $\mu \geq \lambda_n$. Thus both $Y_\lambda \to Y$ and $Y_{\lambda_n} \to Y$ factor through $Y_\mu \to Y$. Say $Y_\mu = \mathrm{Spec}(B_\mu)$, the closed subscheme Y_λ corresponds to $J \subset B_\mu$, and the closed subscheme Y_{λ_n} corresponds to $J' \subset B_\mu$. We are trying to show that $J' \subset J$. By the diagram above we know $J'A_\mu \subset JA_\mu$ where $Y_\mu \times_Y X = \mathrm{Spec}(A_\mu)$. Since $X \to Y$ is surjective and flat the morphism $Y_\lambda \times_Y X \to Y_\lambda$ is a faithfully flat morphism of affine schemes, hence $B_\mu \to A_\mu$ is faithfully flat. Thus $J' \subset J$ as desired.

Assume Y is countably indexed. Then X is countably indexed by Lemma 87.19.10. □

0GXR **Lemma 87.20.2.** *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed and classical if and only if Y is countably indexed and classical.*

Proof. We have already seen the implication in one direction in Lemma 87.19.10. For the other direction, note that by Lemma 87.20.1 we may assume both X and Y are countably indexed. Thus $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 87.10.4. By Lemma 87.9.10 the morphism $X \to Y$ corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 87.19.8 that φ is taut. Let $J \subset B$ be an open ideal and let $I \subset A$ be the closure of JA. By Lemmas 87.16.4 and 87.4.11 we see that $\mathrm{Spec}(B/J) \times_Y X = \mathrm{Spec}(A/I)$. Hence $B/J \to A/I$ is faithfully flat (since $X \to Y$

is surjective and flat). This means that $\varphi : B \to A$ is as in Section 87.8 (with the roles of A and B swapped). We conclude that the lemma holds by Lemma 87.8.2. □

0GXS **Lemma 87.20.3.** *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is weakly adic if and only if Y is weakly adic.*

Proof. The proof is exactly the same as the proof of Lemma 87.20.2 except that at the end we use Lemma 87.8.4. □

0AKT **Lemma 87.20.4.** *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is adic* if and only if Y is adic*.*

Proof. The proof is exactly the same as the proof of Lemma 87.20.2 except that at the end we use Lemma 87.8.5. □

0AKW **Lemma 87.20.5.** *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and (locally) of finite type. Then X is Noetherian if and only if Y is Noetherian.*

Proof. Observe that a Noetherian affine formal algebraic space is adic*, see Lemma 87.10.3. Thus by Lemma 87.20.4 we may assume that both X and Y are adic*. We will use the criterion of Lemma 87.10.5 to see that the lemma holds. Namely, write $Y = \operatorname{colim} Y_n$ as in Lemma 87.10.1. For each n set $X_n = Y_n \times_Y X$. Then X_n is an affine scheme (Lemma 87.19.7) and $X = \operatorname{colim} X_n$. Each of the morphisms $X_n \to Y_n$ is faithfully flat and of finite type. Thus the lemma follows from the fact that in this situation X_n is Noetherian if and only if Y_n is Noetherian, see Algebra, Lemma 10.164.1 (to go down) and Algebra, Lemma 10.31.1 (to go up). □

0AKX **Lemma 87.20.6.** *Let S be a scheme. Let*

$$P \in \left\{ \begin{array}{c} \text{countably indexed,} \\ \text{countably indexed and classical,} \\ \text{weakly adic, adic}*\text{, Noetherian} \end{array} \right\}$$

Let X be a formal algebraic space over S. The following are equivalent

1. *if Y is an affine formal algebraic space and $f : Y \to X$ is representable by algebraic spaces and étale, then Y has property P,*
2. *for some $\{X_i \to X\}_{i \in I}$ as in Definition 87.11.1 each X_i has property P.*

Proof. It is clear that (1) implies (2). Assume (2) and let $Y \to X$ be as in (1). Since the fibre products $X_i \times_X Y$ are formal algebraic spaces (Lemma 87.15.2) we can pick coverings $\{X_{ij} \to X_i \times_X Y\}$ as in Definition 87.11.1. Since Y is quasi-compact, there exist $(i_1, j_1), \ldots, (i_n, j_n)$ such that

$$X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n} \longrightarrow Y$$

is surjective and étale. Then $X_{i_k j_k} \to X_{i_k}$ is representable by algebraic spaces and étale hence $X_{i_k j_k}$ has property P by Lemma 87.19.10. Then $X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n}$ is an affine formal algebraic space with property P (small detail omitted on finite disjoint unions of affine formal algebraic spaces). Hence we conclude by applying one of Lemmas 87.20.1, 87.20.2, 87.20.3, 87.20.4, and 87.20.5. □

The previous lemma clears the way for the following definition.

0AKY **Definition 87.20.7.** Let S be a scheme. Let X be a formal algebraic space over S. We say X is *locally countably indexed*, *locally countably indexed and classical*, *locally weakly adic*, *locally adic**, or *locally Noetherian* if the equivalent conditions of Lemma 87.20.6 hold for the corresponding property.

The formal completion of a locally Noetherian algebraic space along a closed subset is a locally Noetherian formal algebraic space.

0AQ1 **Lemma 87.20.8.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.*

(1) *If $X \setminus T \to X$ is quasi-compact, then $X_{/T}$ is locally adic*.*
(2) *If X is locally Noetherian, then $X_{/T}$ is locally Noetherian.*

Proof. Choose a surjective étale morphism $U \to X$ with $U = \coprod U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. Let $T_i \subset U_i$ be the inverse image of T. We have $X_{/T} \times_X U_i = (U_i)_{/T_i}$ (Lemma 87.14.4). Hence $\{(U_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 87.11.1. Moreover, if $X \setminus T \to X$ is quasi-compact, so is $U_i \setminus T_i \to U_i$ and if X is locally Noetherian, so is U_i. Thus the lemma follows from the affine case which is Lemma 87.14.6. □

0CKX **Remark 87.20.9** (Warning)**.** Suppose $X = \mathrm{Spec}(A)$ and $T \subset X$ is the zero locus of a finitely generated ideal $I \subset A$. Let $J = \sqrt{I}$ be the radical of I. Then from the definitions we see that $X_{/T} = \mathrm{Spf}(A^\wedge)$ where $A^\wedge = \lim A/I^n$ is the I-adic completion of A. On the other hand, the map $A^\wedge \to \lim A/J^n$ from the I-adic completion to the J-adic completion can fail to be a ring isomorphism. As an example let

$$A = \bigcup\nolimits_{n \geq 1} \mathbf{C}[t^{1/n}]$$

and $I = (t)$. Then $J = \mathfrak{m}$ is the maximal ideal of the valuation ring A and $J^2 = J$. Thus the J-adic completion of A is $\mathbf{C}$ whereas the I-adic completion is the valuation ring described in Example 87.13.2 (but in particular it is easy to see that $A \subset A^\wedge$).

0GBB **Lemma 87.20.10.** *Let S be a scheme. Let $X \to Y$ and $Z \to Y$ be morphisms of formal algebraic space over S. Then*

(1) *If X and Z are locally countably indexed, then $X \times_Y Z$ is locally countably indexed.*
(2) *If X and Z are locally countably indexed and classical, then $X \times_Y Z$ is locally countably indexed and classical.*
(3) *If X and Z are weakly adic, then $X \times_Y Z$ is weakly adic.*
(4) *If X and Z are locally adic*, then $X \times_Y Z$ is locally adic*.*
(5) *If X and Z are locally Noetherian and $X_{red} \to Y_{red}$ is locally of finite type, then $X \times_Y Z$ is locally Noetherian.*

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 87.11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 87.11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space by Lemma 87.16.4. Hence

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 87.11.1. Thus it suffices to prove (1), (2), (3), and (4) in case X, Y, and Z are affine formal algebraic spaces.

Assume X and Z are countably indexed. Say $X = \operatorname{colim} X_n$ and $Z = \operatorname{colim} Z_m$ as in Lemma 87.10.1. Write $Y = \operatorname{colim}_{\lambda\in\Lambda} Y_\lambda$ as in Definition 87.9.1. For each n and m we can find $\lambda_{n,m} \in \Lambda$ such that $X_n \to Y$ and $Z_m \to Y$ factor through $Y_{\lambda_{n,m}}$ (for example see Lemma 87.9.4). Pick $\lambda_0 \in \Lambda$. By induction for $t \geq 1$ pick an element $\lambda_t \in \Lambda$ such that $\lambda_t \geq \lambda_{n,m}$ for all $1 \leq n, m \leq t$ and $\lambda_t \geq \lambda_{t-1}$. Set $Y' = \operatorname{colim} Y_{\lambda_t}$. Then $Y' \to Y$ is a monomorphism such that $X \to Y$ and $Z \to Y$ factor through Y'. Hence we may replace Y by Y', i.e., we may assume that Y is countably indexed.

Assume X, Y, and Z are countably indexed. By Lemma 87.10.4 we can write $X = \operatorname{Spf}(A)$, $Y = \operatorname{Spf}(B)$, $Z = \operatorname{Spf}(C)$ for some weakly admissible topological rings A, B, and C. The morphsms $X \to Y$ and $Z \to Y$ are given by continuous ring maps $B \to A$ and $B \to C$, see Lemma 87.9.10. By Lemma 87.16.4 we see that $X \times_Y Z = \operatorname{Spf}(A\widehat{\otimes}_B C)$ and that $A\widehat{\otimes}_B C$ is a weakly admissible topological ring. In particular, we see that $X \times_Y Z$ is countably indexed by Lemma 87.4.12 part (3). This proves (1).

Proof of (2). In this case X and Z are countably indexed and hence the arguments above show that $X \times_Y Z$ is the formal spectrum of $A\widehat{\otimes}_B C$ where A and C are admissible. Then $A\widehat{\otimes}_B C$ is admissible by Lemma 87.4.12 part (2).

Proof of (3). As before we conclude that $X \times_Y Z$ is the formal spectrum of $A\widehat{\otimes}_B C$ where A and C are weakly adic. Then $A\widehat{\otimes}_B C$ is weakly adic by Lemma 87.7.6.

Proof of (4). Arguing as above, this follows from Lemma 87.4.12 part (4).

Proof of (5). To deduce case (5) from Lemma 87.4.12 part (5) we need to show the hypotheses match. Namely, with notation as in the first parapgrah of the proof, if $X_{red} \to Y_{red}$ is locally of finite type, then $(X_{ji})_{red} \to (Y_j)_{red}$ is locally of finite type. This follows from Morphisms of Spaces, Lemma 67.23.4 and the fact that in the commutative diagram

$$\begin{array}{ccc} (X_{ji})_{red} & \longrightarrow & (Y_j)_{red} \\ \downarrow & & \downarrow \\ X_{red} & \longrightarrow & Y_{red} \end{array}$$

the vertical morphisms are étale. Namely, we have $(X_{ji})_{red} = X_{ij} \times_X X_{red}$ and $(Y_j)_{red} = Y_j \times_Y Y_{red}$ by Lemma 87.12.3. Thus as above we reduce to the case where X, Y, Z are affine formal algebraic spaces, X, Z are Noetherian, and $X_{red} \to Y_{red}$ is of finite type. Next, in the second paragraph of the proof we replaced Y by Y' but by construction $Y_{red} = Y'_{red}$, hence the finite type assumption is preserved by this replacement. Then we see that X, Y, Z correspond to A, B, C and $X \times_Y Z$ to $A\widehat{\otimes}_B C$ with A, C Noetherian adic. Finally, taking the reduction corresponds to dividing by the ideal of topologically nilpotent elements (Example 87.12.2) hence the fact that $X_{red} \to Y_{red}$ is of finite type does indeed mean that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type and the proof is complete. □

0GHM **Lemma 87.20.11.** *Let S be a scheme. Let X be a locally Noetherian formal algebraic space over S. Then $X = \operatorname{colim} X_n$ for a system $X_1 \to X_2 \to X_3 \to \ldots$ of finite order thickenings of locally Noetherian algebraic spaces over S where $X_1 = X_{red}$ and X_n is the nth infinitesimal neighbourhood of X_1 in X_m for all $m \geq n$.*

Proof. We only sketch the proof and omit some of the details. Set $X_1 = X_{red}$. Define $X_n \subset X$ as the subfunctor defined by the rule: a morphism $f : T \to X$ where T is a scheme factors through X_n if and only if the nth power of the ideal sheaf of the closed immersion $X_1 \times_X T \to T$ is zero. Then $X_n \subset X$ is a subsheaf as vanishing of quasi-coherent modules can be checked fppf locally. We claim that $X_n \to X$ is representable by schemes, a closed immersion, and that $X = \operatorname{colim} X_n$ (as fppf sheaves). To check this we may work étale locally on X. Hence we may assume $X = \mathrm{Spf}(A)$ is a Noetherian affine formal algebraic space. Then $X_1 = \mathrm{Spec}(A/\mathfrak{a})$ where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements of the Noetherian adic topological ring A. Then $X_n = \mathrm{Spec}(A/\mathfrak{a}^n)$ and we obtain what we want. □

87.21. Morphisms and continuous ring maps

0ANA In this section we denote $WAdm$ the category of weakly admissible topological rings and continuous ring homomorphisms. We define full subcategories

$$WAdm \supset WAdm^{count} \supset WAdm^{cic} \supset WAdm^{weakly\ adic} \supset WAdm^{adic*} \supset WAdm^{Noeth}$$

whose objects are

(1) $WAdm^{count}$: those weakly admissible topological rings A which have a countable fundamental system of open ideals,
(2) $WAdm^{cic}$: the admissible topological rings A which have a countable fundamental system of open ideals,
(3) $WAdm^{weakly\ adic}$: the weakly adic topological rings (Section 87.7),
(4) $WAdm^{adic*}$: the adic topological rings which have a finitely generated ideal of definition, and
(5) $WAdm^{Noeth}$: the adic topological rings which are Noetherian.

Clearly, the formal spectra of these types of rings are the basic building blocks of locally countably indexed, locally countably indexed and classical, locally weakly adic, locally adic*, and locally Noetherian formal algebraic spaces.

We briefly review the relationship between morphisms of countably indexed, affine formal algebraic spaces and morphisms of $WAdm^{count}$. Let S be a scheme. Let X and Y be countably indexed, affine formal algebraic spaces. Write $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ topological S-algebras A and B in $WAdm^{count}$, see Lemma 87.10.4. By Lemma 87.9.10 there is a 1-to-1 correspondence between morphisms $f : X \to Y$ and continuous maps

$$\varphi : B \longrightarrow A$$

of topological S-algebras. The relationship is given by $f \mapsto f^\sharp$ and $\varphi \mapsto \mathrm{Spf}(\varphi)$.

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces. Consider a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces and $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. By Definition 87.20.7 (and hence via Lemma 87.20.6) we see that U and V are countably indexed affine formal algebraic spaces. By the

discussion in the previous paragraph we see that $U \to V$ is isomorphic to $\mathrm{Spf}(\varphi)$ for some continuous map

$$\varphi : B \longrightarrow A$$

of topological S-algebras in $WAdm^{count}$.

0ANB **Lemma 87.21.1.** *Let $A \in \mathrm{Ob}(WAdm)$. Let $A \to A'$ be a ring map (no topology). Let $(A')^\wedge = \lim_{I \subset A \ w.i.d} A'/IA'$ be the object of $WAdm$ constructed in Example 87.19.11.*

(1) *If A is in $WAdm^{count}$, so is $(A')^\wedge$.*
(2) *If A is in $WAdm^{cic}$, so is $(A')^\wedge$.*
(3) *If A is in $WAdm^{weakly\ adic}$, so is $(A')^\wedge$.*
(4) *If A is in $WAdm^{adic*}$, so is $(A')^\wedge$.*
(5) *If A is in $WAdm^{Noeth}$ and A' is Noetherian, then $(A')^\wedge$ is in $WAdm^{Noeth}$.*

Proof. Recall that $A \to (A')^\wedge$ is taut, see discussion in Example 87.19.11. Hence statements (1), (2), (3), and (4) follow from Lemmas 87.5.7, 87.5.9, 87.7.5, and 87.6.5. Finally, assume that A is Noetherian and adic. By (4) we know that $(A')^\wedge$ is adic. By Algebra, Lemma 10.97.6 we see that $(A')^\wedge$ is Noetherian. Hence (5) holds. □

0CBA **Situation 87.21.2.** Let P be a property of morphisms of $WAdm^{count}$. Consider commutative diagrams

0ANC (87.21.2.1)
$$\begin{array}{ccc} A & \longrightarrow & (A')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ B & \longrightarrow & (B')^\wedge \end{array}$$

satisfying the following conditions

(1) A and B are objects of $WAdm^{count}$,
(2) $A \to A'$ and $B \to B'$ are étale ring maps,
(3) $(A')^\wedge = \lim A'/IA'$, resp. $(B')^\wedge = \lim B'/JB'$ where $I \subset A$, resp. $J \subset B$ runs through the weakly admissible ideals of definition of A, resp. B,
(4) $\varphi : B \to A$ and $\varphi' : (B')^\wedge \to (A')^\wedge$ are continuous.

By Lemma 87.21.1 the topological rings $(A')^\wedge$ and $(B')^\wedge$ are objects of $WAdm^{count}$. We say P is a *local property* if the following axioms hold:

0AND (1) for any diagram (87.21.2.1) we have $P(\varphi) \Rightarrow P(\varphi')$,
0ANE (2) for any diagram (87.21.2.1) with $A \to A'$ faithfully flat we have $P(\varphi') \Rightarrow P(\varphi)$,
0ANF (3) if $P(B \to A_i)$ for $i = 1, \ldots, n$, then $P(B \to \prod_{i=1,\ldots,n} A_i)$.

Axiom (3) makes sense as $WAdm^{count}$ has finite products.

0ANG **Lemma 87.21.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. Let P be a local property of morphisms of $WAdm^{count}$. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of $WAdm^{count}$ with property P,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of $WAdm^{count}$ with property P, and*

(3) *there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of $WAdm^{count}$ with property P.*

Proof. It is clear that (1) implies (2) and that (2) implies (3). Assume $\{X_i \to X\}$ and $X_i \to Y_i \to Y$ as in (3) and let a diagram as in (1) be given. Since $Y_i \times_Y V$ is a formal algebraic space (Lemma 87.15.2) we may pick coverings $\{Y_{ij} \to Y_i \times_Y V\}$ as in Definition 87.11.1. For each (i, j) we may similarly choose coverings $\{X_{ijk} \to Y_{ij} \times_{Y_i} X_i \times_X U\}$ as in Definition 87.11.1. Since U is quasi-compact we can choose $(i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)$ such that

$$X_{i_1 j_1 k_1} \amalg \ldots \amalg X_{i_n j_n k_n} \longrightarrow U$$

is surjective. For $s = 1, \ldots, n$ consider the commutative diagram

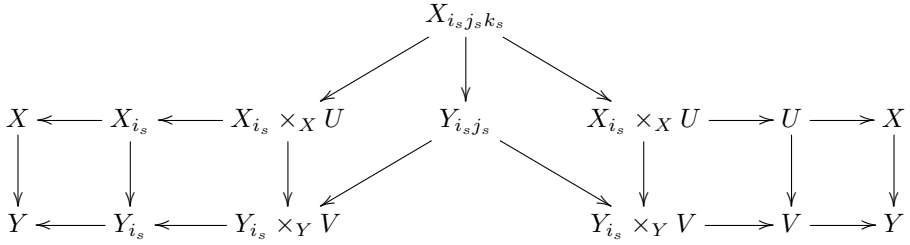

Let us say that P holds for a morphism of countably indexed affine formal algebraic spaces if it holds for the corresponding morphism of $WAdm^{count}$. Observe that the maps $X_{i_s j_s k_s} \to X_{i_s}$, $Y_{i_s j_s} \to Y_{i_s}$ are given by completions of étale ring maps, see Lemma 87.19.13. Hence we see that $P(X_{i_s} \to Y_{i_s})$ implies $P(X_{i_s j_s k_s} \to Y_{i_s j_s})$ by axiom (1). Observe that the maps $Y_{i_s j_s} \to V$ are given by completions of étale rings maps (same lemma as before). By axiom (2) applied to the diagram

$$\begin{array}{ccc} X_{i_s j_s k_s} & = & X_{i_s j_s k_s} \\ \downarrow & & \downarrow \\ Y_{i_s j_s} & \longrightarrow & V \end{array}$$

(this is permissible as identities are faithfully flat ring maps) we conclude that $P(X_{i_s j_s k_s} \to V)$ holds. By axiom (3) we find that $P(\coprod_{s=1,\ldots,n} X_{i_s j_s k_s} \to V)$ holds. Since the morphism $\coprod X_{i_s j_s k_s} \to U$ is surjective by construction, the corresponding

morphism of $WAdm^{count}$ is the completion of a faithfully flat étale ring map, see Lemma 87.19.14. One more application of axiom (2) (with $B' = B$) implies that $P(U \to V)$ is true as desired. □

0ANH **Remark 87.21.4** (Variant for adic-star)**.** Let P be a property of morphisms of $WAdm^{adic*}$. We say P is a *local property* if axioms (1), (2), (3) of Situation 87.21.2 hold for morphisms of $WAdm^{adic*}$. In exactly the same way we obtain a variant of Lemma 87.21.3 for morphisms between locally adic* formal algebraic spaces over S.

0ANI **Remark 87.21.5** (Variant for Noetherian)**.** Let P be a property of morphisms of $WAdm^{Noeth}$. We say P is a *local property* if axioms (1), (2), (3), of Situation 87.21.2 hold for morphisms of $WAdm^{Noeth}$. In exactly the same way we obtain a variant of Lemma 87.21.3 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBC **Situation 87.21.6.** Let P be a local property of morphisms of $WAdm^{count}$, see Situation 87.21.2. We say P is *stable under base change* if given $B \to A$ and $B \to C$ in $WAdm^{count}$ we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of $WAdm^{count}$ by Lemma 87.4.12.

0GBD **Lemma 87.21.7.** *Let S be a scheme. Let P be a local property of morphisms of $WAdm^{count}$ which is stable under base change. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally countably indexed formal algebraic spaces over S. If f satisfies the equivalent conditions of Lemma 87.21.3 then so does $pr_2 : X \times_Y Z \to Z$.*

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 87.11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 87.11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space which is countably indexed, see Lemma 87.20.10. Then we see that

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 87.11.1. Moreover, the morphisms $X_{ji} \times_{Y_j} Z_{jk} \to Z$ factor through Z_{jk}. By assumption we know that $X_{ji} \to Y_j$ corresponds to a morphism $B_j \to A_{ji}$ of WAdm^{count} having property P. The morphisms $Z_{jk} \to Y_j$ correspond to morphisms $B_j \to C_{jk}$ in WAdm^{count}. Since $X_{ji} \times_{Y_j} Z_{jk} = \mathrm{Spf}(A_{ji}\widehat{\otimes}_{B_j} C_{jk})$ by Lemma 87.16.4 we see that it suffices to show that $C_{jk} \to A_{ji}\widehat{\otimes}_{B_j} C_{jk}$ has property P which is exactly what the condition that P is stable under base change guarantees. □

0GBE **Remark 87.21.8** (Variant for adic-star)**.** Let P be a local property of morphisms of $WAdm^{adic*}$, see Remark 87.21.4. We say P is *stable under base change* if given $B \to A$ and $B \to C$ in $WAdm^{adic*}$ we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of $WAdm^{adic*}$ by Lemma 87.4.12. In exactly the same way we obtain a variant of Lemma 87.21.7 for morphisms between locally adic* formal algebraic spaces over S.

0GBF **Remark 87.21.9** (Variant for Noetherian)**.** Let P be a local property of morphisms of $WAdm^{Noeth}$, see Remark 87.21.5. We say P is *stable under base change* if given $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ the property $P(B \to A)$ implies both

that $A\widehat{\otimes}_B C$ is adic Noetherian[7] and that $P(C \to A\widehat{\otimes}_B C)$. In exactly the same way we obtain a variant of Lemma 87.21.7 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBG **Remark 87.21.10** (Another variant for Noetherian)**.** Let P and Q be local properties of morphisms of $WAdm^{Noeth}$, see Remark 87.21.5. We say P *is stable under base change by* Q if given $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ satisfying $P(B \to A)$ and $Q(B \to C)$, then $A\widehat{\otimes}_B C$ is adic Noetherian and $P(C \to A\widehat{\otimes}_B C)$ holds. Arguing exactly as in the proof of Lemma 87.21.7 we obtain the following statement: given morphisms $f : X \to Y$ and $g : Y \to Z$ of locally Noetherian formal algebraic spaces over S such that

(1) the equivalent conditions of Lemma 87.21.3 hold for f and P,
(2) the equivalent conditions of Lemma 87.21.3 hold for g and Q,

then the equivalent conditions of Lemma 87.21.3 hold for $\mathrm{pr}_2 : X \times_Y Z \to Z$ and P.

0GBH **Situation 87.21.11.** Let P be a local property of morphisms of $WAdm^{count}$, see Situation 87.21.2. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in $WAdm^{count}$ we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$.

0GBI **Lemma 87.21.12.** *Let S be a scheme. Let P be a local property of morphisms of $WAdm^{count}$ which is stable under composition. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally countably indexed formal algebraic spaces over S. If f and g satisfies the equivalent conditions of Lemma 87.21.3 then so does $g \circ f : X \to Z$.*

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 87.11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 87.11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 87.11.1. If f and g satisfies the equivalent conditions of Lemma 87.21.3 then $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count} having property P. Hence the compositions do too and we conclude. □

0GBJ **Remark 87.21.13** (Variant for adic-star)**.** Let P be a local property of morphisms of $WAdm^{adic*}$, see Remark 87.21.4. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in $WAdm^{adic*}$ we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally adic* formal algebraic spaces over S.

0GBK **Remark 87.21.14** (Variant for Noetherian)**.** Let P be a local property of morphisms of $WAdm^{Noeth}$, see Remark 87.21.5. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in $WAdm^{Noeth}$ we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

0GBL **Situation 87.21.15.** Let P be a local property of morphisms of $WAdm^{count}$, see Situation 87.21.2. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in $WAdm^{count}$ we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(B \to A)$.

0GBM **Lemma 87.21.16.** *Let S be a scheme. Let P be a local property of morphisms of $WAdm^{count}$ which has the cancellation property. Let $f : X \to Y$ and $g : Y \to Z$ be*

[7] See Lemma 87.4.12 for a criterion.

morphisms of locally countably indexed formal algebraic spaces over S. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 87.21.3 then so does $f : X \to Y$.

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 87.11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 87.11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 87.11.1. Let $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count}. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 87.21.3 then $C_k \to B_{kj}$ and $C_k \to A_{kji}$ satisfy P. Hence $B_{kj} \to A_{kji}$ does too and we conclude. □

0GBN **Remark 87.21.17** (Variant for adic-star)**.** Let P be a local property of morphisms of $WAdm^{adic*}$, see Remark 87.21.4. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in $WAdm^{adic*}$ we have $P(C \to A) \wedge P(C \to B) \Rightarrow P(B \to A)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally adic* formal algebraic spaces over S.

0GBP **Remark 87.21.18** (Variant for Noetherian)**.** Let P be a local property of morphisms of $WAdm^{Noeth}$, see Remark 87.21.5. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in $WAdm^{Noeth}$ we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(C \to B)$. In exactly the same way we obtain a variant of Lemma 87.21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

87.22. Taut ring maps and representability by algebraic spaces

0GBQ In this section we briefly show that morphisms between locally countably index formal algebraic spaces correspond étale locally to taut continuous ring homomorphisms between weakly admissible topological rings having countable fundamental systems of open ideals. In fact, this is rather clear from Lemma 87.19.8 and we encourage the reader to skip this section.

0ANJ **Lemma 87.22.1.** *Let $B \to A$ be an arrow of $WAdm^{count}$. The following are equivalent*

(a) *$B \to A$ is taut (Definition 87.5.1),*
(b) *for $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ a fundamental system of weak ideals of definitions there exist a commutative diagram*

$$\begin{array}{ccccccccc} A & \longrightarrow & \ldots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 \\ \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & \longrightarrow & \ldots & \longrightarrow & B/J_3 & \longrightarrow & B/J_2 & \longrightarrow & B/J_1 \end{array}$$

such that $A_{n+1}/J_nA_{n+1} = A_n$ and $A = \lim A_n$ as topological ring.

Moreover, these equivalent conditions define a local property, i.e., they satisfy axioms (1), (2), (3).

Proof. The equivalence of (a) and (b) is immediate. Below we will give an algebraic proof of the axioms, but it turns out we've already proven them. Namely, using Lemma 87.19.8 the equivalent conditions (a) and (b) translate to saying the corresponding morphism of affine formal algebraic spaces is representable by algebraic spaces. Since this condition is "étale local on the source and target" by Lemma 87.19.4 we immediately get axioms (1), (2), and (3).

Direct algebraic proof of (1), (2), (3). Let a diagram (87.21.2.1) as in Situation 87.21.2 be given. By Example 87.19.11 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a) and (b).

Assume (a) and (b) hold for φ. Let $J \subset B$ be a weak ideal of definition. Then the closure of JA, resp. $J(B')^\wedge$ is a weak ideal of definition $I \subset A$, resp. $J' \subset (B')^\wedge$. Then the closure of $I(A')^\wedge$ is a weak ideal of definition $I' \subset (A')^\wedge$. A topological argument shows that I' is also the closure of $J(A')^\wedge$ and of $J'(A')^\wedge$. Finally, as J runs over a fundamental system of weak ideals of definition of B so do the ideals I and I' in A and $(A')^\wedge$. It follows that (a) holds for φ'. This proves (1).

Assume $A \to A'$ is faithfully flat and that (a) and (b) hold for φ'. Let $J \subset B$ be a weak ideal of definition. Using (a) and (b) for the maps $B \to (B')^\wedge \to (A')^\wedge$ we find that the closure I' of $J(A')^\wedge$ is a weak ideal of definition. In particular, I' is open and hence the inverse image of I' in A is open. Now we have (explanation below)

$$\begin{aligned} A \cap I' &= A \cap \bigcap (J(A')^\wedge + \mathrm{Ker}((A')^\wedge \to A'/I_0A')) \\ &= A \cap \bigcap \mathrm{Ker}((A')^\wedge \to A'/JA' + I_0A') \\ &= \bigcap (JA + I_0) \end{aligned}$$

which is the closure of JA by Lemma 87.4.2. The intersections are over weak ideals of definition $I_0 \subset A$. The first equality because a fundamental system of neighbourhoods of 0 in $(A')^\wedge$ are the kernels of the maps $(A')^\wedge \to A'/I_0A'$. The second equality is trivial. The third equality because $A \to A'$ is faithfully flat, see Algebra, Lemma 10.82.11. Thus the closure of JA is open. By Lemma 87.4.10 the closure of JA is a weak ideal of definition of A. Finally, given a weak ideal of definition $I \subset A$ we can find J such that $J(A')^\wedge$ is contained in the closure of $I(A')^\wedge$ by property (a) for $B \to (B')^\wedge$ and φ'. Thus we see that (a) holds for φ. This proves (2).

We omit the proof of (3). □

0ANK **Lemma 87.22.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a taut map $B \to A$ of WAdm^{count},

(2) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a taut ring map in WAdm^{count},*

(3) *there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic*

space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a taut ring map in $WAdm^{count}$, and

(4) *f is representable by algebraic spaces.*

Proof. The property of a map in $WAdm^{count}$ being "taut" is a local property by Lemma 87.22.1. Thus Lemma 87.21.3 exactly tells us that (1), (2), and (3) are equivalent. On the other hand, by Lemma 87.19.8 being "taut" on maps in $WAdm^{count}$ corresponds exactly to being "representable by algebraic spaces" for the corresponding morphisms of countably indexed affine formal algebraic spaces. Thus the implication (1) $\Rightarrow$ (2) of Lemma 87.19.4 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) $\Rightarrow$ (1) of Lemma 87.19.4 shows that (2) implies (4) of the current lemma. □

87.23. Adic morphisms

0AQ2 This section matches the occasionally used notion of an "adic morphism" $f : X \to Y$ of locally adic* formal algebraic spaces X and Y on the one hand with representability of f by algebraic spaces and on the other hand with our notion of taut continuous ring homomorphisms. First we recall that tautness is equivalent to adicness for adic rings with finitely generated ideal of definition.

0GBS **Lemma 87.23.1.** *Let A and B be pre-adic topological rings. Let $\varphi : A \to B$ be a continuous ring homomorphism.*

(1) *If φ is adic, then φ is taut.*
(2) *If B is complete, A has a finitely generated ideal of definition, and φ is taut, then φ is adic.*

In particular the conditions "φ is adic" and "φ is taut" are equivalent on the category $WAdm^{adic*}$.

Proof. Part (1) is Lemma 87.6.4. Part (2) is Lemma 87.6.5. The final statement is a consequence of (1) and (2). □

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. By Lemma 87.22.2 the following are equivalent

(1) f is representable by algebraic spaces (in other words, the equivalent conditions of Lemma 87.19.4 hold),
(2) for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an adic[8] map in $WAdm^{adic*}$.

In this situation we will say that *f is an adic morphism* (the formal definition is below). This notion/terminology will only be defined/used for morphisms between formal algebraic spaces which are locally adic* since otherwise we don't have the equivalence between (1) and (2) above.

[8] Equivalently taut by Lemma 87.23.1.

0AQ3 **Definition 87.23.2.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume X and Y are locally adic*. We say f is an *adic morphism* if f is representable by algebraic spaces. See discussion above.

87.24. Morphisms of finite type

0AM3 Due to how things are setup in the Stacks project, the following is really the correct thing to do and stronger notions should have a different name.

0AM4 **Definition 87.24.1.** Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S.

(1) We say f is *locally of finite type* if f is representable by algebraic spaces and is locally of finite type in the sense of Bootstrap, Definition 80.4.1.
(2) We say f is of *finite type* if f is locally of finite type and quasi-compact (Definition 87.17.4).

We will discuss the relationship between finite type morphisms of certain formal algebraic spaces and continuous ring maps $A \to B$ which are topologically of finite type in Section 87.29.

0AJJ **Lemma 87.24.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is of finite type,*
(2) *f is representable by algebraic spaces and is of finite type in the sense of Bootstrap, Definition 80.4.1.*

Proof. This follows from Bootstrap, Lemma 80.4.5, the implication "quasi-compact + locally of finite type $\Rightarrow$ finite type" for morphisms of algebraic spaces, and Lemma 87.17.5. □

0AQ4 **Lemma 87.24.3.** *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. See Bootstrap, Lemma 80.4.3 and use Morphisms of Spaces, Lemma 67.23.2. □

0AQ5 **Lemma 87.24.4.** *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. See Bootstrap, Lemma 80.4.2 and use Morphisms of Spaces, Lemma 67.23.3. □

0AQ6 **Lemma 87.24.5.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.*

Proof. By Lemma 87.19.3 we see that f is representable by algebraic spaces. Let T be a scheme and let $T \to Z$ be a morphism. Then we can apply Morphisms of Spaces, Lemma 67.23.6 to the morphisms $T \times_Z X \to T \times_Z Y \to T$ of algebraic spaces to conclude. □

Being locally of finite type is local on the source and the target.

0ANL **Lemma 87.24.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:*

(1) *the morphism f is locally of finite type,*
(2) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces and étale, $U \to X$ is surjective, and $U \to V$ is locally of finite type,
(3) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are formal algebraic spaces and vertical arrows representable by algebraic spaces and étale, the morphism $U \to V$ is locally of finite type,
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that $X_{ji} \to Y_j$ is locally of finite type for each j and i,*
(5) *there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, such that $X_i \to Y_i$ is locally of finite type, and*
(6) *add more here.*

Proof. In each of the 5 cases the morphism $f : X \to Y$ is representable by algebraic spaces, see Lemma 87.19.4. We will use this below without further mention.

It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, assume given a diagram as in (2). Let T be a scheme and let $T \to Y$ be a morphism. Then we can consider

$$\begin{array}{ccc} U \times_Y T & \longrightarrow & V \times_Y T \\ \downarrow & & \downarrow \\ X \times_Y T & \longrightarrow & T \end{array}$$

The vertical arrows are étale and the top horizontal arrow is locally of finite type as base changes of such morphisms. Hence by Morphisms of Spaces, Lemma 67.23.4 we conclude that $X \times_Y T \to T$ is locally of finite type. In other words (1) holds.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is locally of finite type (as the composition $U \to X \to Y$, see Bootstrap, Lemma 80.4.3). Let T be a scheme and let $T \to V$ be a morphism. Then the projection $T \times_V U \to T$ factors as

$$T \times_V U = (T \times_Y U) \times_{(V \times_Y V)} V \to T \times_Y U \to T$$

The second arrow is locally of finite type (as a base change of the composition $U \to X \to Y$) and the first is the base change of the diagonal $V \to V \times_Y V$ which is locally of finite type by Lemma 87.15.5.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 87.15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1. Then $X_{ij} \to Y_j$ is locally of finite type by (3) and we see that (4) is true. This concludes the proof. □

0ANM **Example 87.24.7.** Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $A \to A'$ be a finite type ring map. Then

$$(A')^\wedge = \lim_{I \subset A \text{ w.i.d.}} A'/IA'$$

is a weakly admissible ring and the corresponding morphism $\mathrm{Spf}((A')^\wedge) \to \mathrm{Spf}(A)$ is representable, see Example 87.19.11. If $T \to \mathrm{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\mathrm{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 87.9.4). Then $T \times_{\mathrm{Spf}(A)} \mathrm{Spf}((A')^\wedge)$ is equal to $T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(A'/IA')$ and we see that $\mathrm{Spf}((A')^\wedge) \to \mathrm{Spf}(A)$ is of finite type.

0AQ7 **Lemma 87.24.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. If Y is locally Noetherian and f locally of finite type, then X is locally Noetherian.*

Proof. Pick $\{Y_j \to Y\}$ and $\{X_{ij} \to Y_j \times_Y X\}$ as in Lemma 87.24.6. Then it follows from Lemma 87.19.9 that each X_{ij} is Noetherian. This proves the lemma. □

0AQ8 **Lemma 87.24.9.** *Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. If Z is locally Noetherian and f locally of finite type, then $Z \times_Y X$ is locally Noetherian.*

Proof. The morphism $Z \times_Y X \to Z$ is locally of finite type by Lemma 87.24.4. Hence this follows from Lemma 87.24.8. □

87.25. Surjective morphisms

0GHN By Lemma 87.12.4 the following definition does not clash with the already existing definitions for morphisms of algebraic spaces or morphisms of formal algebraic spaces which are representable by algebraic spaces.

0GHP **Definition 87.25.1.** Let S be a scheme. A morphism $f : X \to Y$ of formal algebraic spaces over S is said to be *surjective* if it induces a surjective morphism $X_{red} \to Y_{red}$ on underlying reduced algebraic spaces.

0GHQ **Lemma 87.25.2.** *The composition of two surjective morphisms is a surjective morphism.*

Proof. Omitted. □

0GHR **Lemma 87.25.3.** *A base change of a surjective morphism is a surjective morphism.*

Proof. Omitted. □

0GHS **Lemma 87.25.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is surjective,*
(2) *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,*
(3) *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,*
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a surjective morphism of formal algebraic spaces,*
(5) *there exists a surjective morphism $Z \to Y$ of formal algebraic spaces such that $X \times_Y Z \to Z$ is surjective, and*
(6) *add more here.*

Proof. Omitted. □

87.26. Monomorphisms

0AQA Here is the definition.

0AQB **Definition 87.26.1.** Let S be a scheme. A morphism of formal algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves.

An example is the following. Let X be an algebraic space and let $T \subset |X|$ be a closed subset. Then the morphism $X_{/T} \to X$ from the formal completion of X along T to X is a monomorphism. In particular, monomorphisms of formal algebraic spaces are in general not representable.

0GHT **Lemma 87.26.2.** *The composition of two monomorphisms is a monomorphism.*

Proof. Omitted. □

0GHU **Lemma 87.26.3.** *A base change of a monomorphism is a monomorphism.*

Proof. Omitted. □

0GHV **Lemma 87.26.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is a monomorphism,*
(2) *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,*
(3) *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,*
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a monomorphism of formal algebraic spaces, and*
(5) *there exists a family of morphisms $\{Y_j \to Y\}$ such that $\coprod Y_j \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ such that each $X \times_Y Y_j \to Y_j$ is a monomorphism for all j,*
(6) *there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a monomorphism, and*
(7) *add more here.*

Proof. Omitted. □

87.27. Closed immersions

0ANN Here is the definition.

0ANP **Definition 87.27.1.** Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is a *closed immersion* if f is representable by algebraic spaces and a closed immersion in the sense of Bootstrap, Definition 80.4.1.

Please skip the initial the obligatory lemmas when reading this section.

0GHW **Lemma 87.27.2.** *The composition of two closed immersions is a closed immersion.*

Proof. Omitted. □

0GHX **Lemma 87.27.3.** *A base change of a closed immersion is a closed immersion.*

Proof. Omitted. □

0GHY **Lemma 87.27.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

1. *f is a closed immersion,*
2. *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,*
3. *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,*
4. *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that each $X \times_Y Y_j \to Y_j$ is a closed immersion, and*
5. *there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a closed immersion, and*
6. *add more here.*

Proof. Omitted. □

0ANQ **Lemma 87.27.5.** *Let S be a scheme. Let X be a McQuillan affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a McQuillan affine formal algebraic space and f corresponds to a continuous homomorphism $A \to B$ of weakly admissible topological S-algebras which is taut, has closed kernel, and has dense image.*

Proof. Write $X = \mathrm{Spf}(A)$ where A is a weakly admissible topological ring. Let I_λ be a fundamental system of weakly admissible ideals of definition in A. Then $Y \times_X \mathrm{Spec}(A/I_\lambda)$ is a closed subscheme of $\mathrm{Spec}(A/I_\lambda)$ and hence affine (Definition 87.27.1). Say $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is surjective. Hence the projections

$$B = \lim B_\lambda \longrightarrow B_\lambda$$

are surjective as the compositions $A \to B \to B_\lambda$ are surjective. It follows that Y is McQuillan by Lemma 87.9.6. The ring map $A \to B$ is taut by Lemma 87.19.8. The kernel is closed because B is complete and $A \to B$ is continuous. Finally, as $A \to B_\lambda$ is surjective for all λ we see that the image of A in B is dense. □

Even though we have the result above, in general we do not know how closed immersions behave when the target is a McQuillan affine formal algebraic space, see Remark 87.29.4.

0ANR **Example 87.27.6.** Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $K \subset A$ be a closed ideal. Setting

$$B = (A/K)^\wedge = \lim_{I \subset A \ w.i.d.} A/(I + K)$$

the morphism $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is representable, see Example 87.19.11. If $T \to \mathrm{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\mathrm{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 87.9.4). Then $T \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B)$ is equal to $T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(A/(K + I))$ and we see that $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a closed immersion. The kernel of $A \to B$ is K as K is closed, but beware that in general the ring map $A \to B = (A/K)^\wedge$ need not be surjective.

0GHZ **Lemma 87.27.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume*

(1) *f is representable by algebraic spaces,*
(2) *f is a monomorphism,*
(3) *the inclusion $Y_{red} \to Y$ factors through f, and*
(4) *f is locally of finite type or Y is locally Noetherian.*

Then f is a closed immersion.

Proof. Assumptions (2) and (3) imply that $X_{red} = X \times_Y Y_{red} = Y_{red}$. We will use this without further mention.

If $Y' \to Y$ is an étale morphism of formal algebraic spaces over S, then the base change $f' : X \times_Y Y' \to Y'$ satisfies conditions (1) – (4). Hence by Lemma 87.27.4 we may assume Y is an affine formal algebraic space.

Say $Y = \mathrm{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 87.9.1. Then $X_\lambda = X \times_Y Y_\lambda$ is an algebraic space endowed with a monomorphism $f_\lambda : X_\lambda \to Y_\lambda$ which induces an isomorphism $X_{\lambda,red} \to Y_{\lambda,red}$. Thus X_λ is an affine scheme by Limits of Spaces, Proposition 70.15.2 (as $X_{\lambda,red} \to X_\lambda$ is surjective and integral). To finish the proof it suffices to show that $X_\lambda \to Y_\lambda$ is a closed immersion which we will do in the next paragraph.

Let $X \to Y$ be a monomorphism of affine schemes such that $X_{red} = X \times_Y Y_{red} = Y_{red}$. In general, this does not imply that $X \to Y$ is a closed immersion, see Examples, Section 110.36. However, under our assumption (4) we know that in the previous parapgrah either $X_\lambda \to Y_\lambda$ is of finite type or Y_λ is Noetherian. This means that $X \to Y$ corresponds to a ring map $R \to A$ such that $R/I \to A/IA$ is an isomorphism where $I \subset R$ is the nil radical (ie., the maximal locally nilpotent ideal of R) and either $R \to A$ is of finite type or R is Noetherian. In the first case $R \to A$ is surjective by Algebra, Lemma 10.126.9 and in the second case I is finitely generated, hence nilpotent, hence $R \to A$ is surjective by Nakayama's lemma, see Algebra, Lemma 10.20.1 part (11). □

87.28. Restricted power series

0AKZ Let A be a topological ring complete with respect to a linear topology (More on Algebra, Definition 15.36.1). Let I_λ be a fundamental system of open ideals. Let $r \geq 0$ be an integer. In this setting one often denotes

$$A\{x_1, \ldots, x_r\} = \lim_\lambda A/I_\lambda[x_1, \ldots, x_r] = \lim_\lambda (A[x_1, \ldots, x_r]/I_\lambda A[x_1, \ldots, x_r])$$

endowed with the limit topology. In other words, this is the completion of the polynomial ring with respect to the ideals I_λ. We can think of elements of $A\{x_1, \ldots, x_r\}$

as power series

$$f = \sum\nolimits_{E=(e_1,\ldots,e_r)} a_E x_1^{e_1} \ldots x_r^{e_r}$$

in $x_1, \ldots, x_r$ with coefficients $a_E \in A$ which tend to zero in the topology of A. In other words, for any λ all but a finite number of a_E are in I_λ. For this reason elements of $A\{x_1, \ldots, x_r\}$ are sometimes called *restricted power series*. Sometimes this ring is denoted $A\langle x_1, \ldots, x_r\rangle$; we will refrain from using this notation.

0AJM **Remark 87.28.1** (Universal property restricted power series)**.** Let $A \to C$ be a continuous map of complete linearly topologized rings. Then any A-algebra map $A[x_1, \ldots x_r] \to C$ extends uniquely to a continuous map $A\{x_1, \ldots, x_r\} \to C$ on restricted power series. [**DG67**, Chapter 0, 7.5.3]

0AL0 **Remark 87.28.2.** Let A be a ring and let $I \subset A$ be an ideal. If A is I-adically complete, then the I-adic completion $A[x_1, \ldots, x_r]^\wedge$ of $A[x_1, \ldots, x_r]$ is the restricted power series ring over A as a ring. However, it is not clear that $A[x_1, \ldots, x_r]^\wedge$ is I-adically complete. We think of the topology on $A\{x_1, \ldots, x_r\}$ as the limit topology (which is always complete) whereas we often think of the topology on $A[x_1, \ldots, x_r]^\wedge$ as the I-adic topology (not always complete). If I is finitely generated, then $A\{x_1, \ldots, x_r\} = A[x_1, \ldots, x_r]^\wedge$ as topological rings, see Algebra, Lemma 10.96.3.

87.29. Algebras topologically of finite type

0ALL Here is our definition. This definition is not generally agreed upon. Many authors impose further conditions, often because they are only interested in specific types of rings and not the most general case.

0ANS **Definition 87.29.1.** Let $A \to B$ be a continuous map of topological rings (More on Algebra, Definition 15.36.1). We say B is *topologically of finite type over A* if there exists an A-algebra map $A[x_1, \ldots, x_n] \to B$ whose image is dense in B.

If A is a complete, linearly topologized ring, then the restricted power series ring $A\{x_1, \ldots, x_r\}$ is topologically of finite type over A. If k is a field, then the power series ring $k[[x_1, \ldots, x_r]]$ is topologically of finite type over k.

For continuous taut maps of weakly admissible topological rings, being topologically of finite type corresponds exactly to morphisms of finite type between the associated affine formal algebraic spaces.

0ANT **Lemma 87.29.2.** *Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent*

1. *$Spf(\varphi) : Y = Spf(B) \to Spf(A) = X$ is of finite type,*
2. *φ is taut and B is topologically of finite type over A.*

Proof. We can use Lemma 87.19.8 to relate tautness of φ to representability of $\mathrm{Spf}(\varphi)$. We will use this without further mention below. It follows that $X = \operatorname{colim} \operatorname{Spec}(A/I)$ and $Y = \operatorname{colim} \operatorname{Spec}(B/J(I))$ where $I \subset A$ runs over the weak ideals of definition of A and $J(I)$ is the closure of IB in B.

Assume (2). Choose a ring map $A[x_1, \ldots, x_r] \to B$ whose image is dense. Then $A[x_1, \ldots, x_r] \to B \to B/J(I)$ has dense image too which means that it is surjective. Therefore $B/J(I)$ is of finite type over A/I. Let $T \to X$ be a morphism with T a quasi-compact scheme. Then $T \to X$ factors through $\operatorname{Spec}(A/I)$ for some I

(Lemma 87.9.4). Then $T \times_X Y = T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(B/J(I))$, see proof of Lemma 87.19.8. Hence $T \times_Y X \to T$ is of finite type as the base change of the morphism $\mathrm{Spec}(B/J(I)) \to \mathrm{Spec}(A/I)$ which is of finite type. Thus (1) is true.

Assume (1). Pick any $I \subset A$ as above. Since $\mathrm{Spec}(A/I) \times_X Y = \mathrm{Spec}(B/J(I))$ we see that $A/I \to B/J(I)$ is of finite type. Choose $b_1, \ldots, b_r \in B$ mapping to generators of $B/J(I)$ over A/I. We claim that the image of the ring map $A[x_1, \ldots, x_r] \to B$ sending x_i to b_i is dense. To prove this, let $I' \subset I$ be a second weak ideal of definition. Then we have

$$B/(J(I') + IB) = B/J(I)$$

because $J(I)$ is the closure of IB and because $J(I')$ is open. Hence we may apply Algebra, Lemma 10.126.9 to see that $(A/I')[x_1, \ldots, x_r] \to B/J(I')$ is surjective. Thus (2) is true, concluding the proof. □

Let A be a topological ring complete with respect to a linear topology. Let (I_λ) be a fundamental system of open ideals. Let $\mathcal{C}$ be the category of inverse systems (B_λ) where

(1) B_λ is a finite type A/I_λ-algebra, and
(2) $B_\mu \to B_\lambda$ is an A/I_μ-algebra homomorphism which induces an isomorphism $B_\mu/I_\lambda B_\mu \to B_\lambda$.

Morphisms in $\mathcal{C}$ are given by compatible systems of homomorphisms.

0AL1 **Lemma 87.29.3.** *Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan and let A be the weakly admissible topological ring associated to X. Then there is an anti-equivalence of categories between*

(1) *the category $\mathcal{C}$ introduced above, and*
(2) *the category of maps $Y \to X$ of finite type of affine formal algebraic spaces.*

Proof. Let (I_λ) be a fundamental system of weakly admissible ideals of definition in A. Consider Y as in (2). Then $Y \times_X \mathrm{Spec}(A/I_\lambda)$ is affine (Definition 87.24.1 and Lemma 87.19.7). Say $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is of finite type because $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A/I_\lambda)$ is of finite type (by Definition 87.24.1). Then (B_λ) is an object of $\mathcal{C}$.

Conversely, given an object (B_λ) of $\mathcal{C}$ we can set $Y = \mathrm{colim}\, \mathrm{Spec}(B_\lambda)$. This is an affine formal algebraic space. We claim that

$$Y \times_X \mathrm{Spec}(A/I_\lambda) = (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$$

To show this it suffices we get the same values if we evaluate on a quasi-compact scheme U. A morphism $U \to (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda)$ comes from a morphism $U \to \mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda)$ for some $\mu \geq \lambda$ (use Lemma 87.9.4 two times). Since $\mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$ by our second assumption on objects of $\mathcal{C}$ this proves what we want. Using this we can show the morphism $Y \to X$ is of finite type. Namely, we note that for any morphism $U \to X$ with U a quasi-compact scheme, we get a factorization $U \to \mathrm{Spec}(A/I_\lambda) \to X$ for some λ (see lemma cited above). Hence

$$Y \times_X U = Y \times_X \mathrm{Spec}(A/I_\lambda)) \times_{\mathrm{Spec}(A/I_\lambda)} U = \mathrm{Spec}(B_\lambda) \times_{\mathrm{Spec}(A/I_\lambda)} U$$

is a scheme of finite type over U as desired. Thus the construction $(B_\lambda) \mapsto \mathrm{colim}\, \mathrm{Spec}(B_\lambda)$ does give a functor from category (1) to category (2).

To finish the proof we show that the above constructions define quasi-inverse functors between the categories (1) and (2). In one direction you have to show that

$$(\operatorname{colim}_\mu \operatorname{Spec}(B_\mu)) \times_X \operatorname{Spec}(A/I_\lambda) = \operatorname{Spec}(B_\lambda)$$

for any object (B_λ) in the category $\mathcal{C}$. This we proved above. For the other direction you have to show that

$$Y = \operatorname{colim}(Y \times_X \operatorname{Spec}(A/I_\lambda))$$

given Y in the category (2). Again this is true by evaluating on quasi-compact test objects and because $X = \operatorname{colim} \operatorname{Spec}(A/I_\lambda)$. □

0AJK **Remark 87.29.4.** Let A be a weakly admissible topological ring and let (I_λ) be a fundamental system of weak ideals of definition. Let $X = \operatorname{Spf}(A)$, in other words, X is a McQuillan affine formal algebraic space. Let $f : Y \to X$ be a morphism of affine formal algebraic spaces. In general it will not be true that Y is McQuillan. More specifically, we can ask the following questions:

(1) Assume that $f : Y \to X$ is a closed immersion. Then Y is McQuillan and f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings which is taut, whose kernel $K \subset A$ is a closed ideal, and whose image $\varphi(A)$ is dense in B, see Lemma 87.27.5. What conditions on A guarantee that $B = (A/K)^\wedge$ as in Example 87.27.6?
(2) What conditions on A guarantee that closed immersions $f : Y \to X$ correspond to quotients A/K of A by closed ideals, in other words, the corresponding continuous map φ is surjective and open?
(3) Suppose that $f : Y \to X$ is of finite type. Then we get $Y = \operatorname{colim} \operatorname{Spec}(B_\lambda)$ where (B_λ) is an object of $\mathcal{C}$ by Lemma 87.29.3. In this case it is true that there exists a fixed integer r such that B_λ is generated by r elements over A/I_λ for all λ (the argument is essentially already given in the proof of (1) $\Rightarrow$ (2) in Lemma 87.29.2). However, it is not clear that the projections $\lim B_\lambda \to B_\lambda$ are surjective, i.e., it is not clear that Y is McQuillan. Is there an example where Y is not McQuillan?
(4) Suppose that $f : Y \to X$ is of finite type and Y is McQuillan. Then f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings. In fact φ is taut and B is topologically of finite type over A, see Lemma 87.29.2. In other words, f factors as
$$Y \longrightarrow \mathbf{A}^r_X \longrightarrow X$$
where the first arrow is a closed immersion of McQuillan affine formal algebraic spaces. However, then questions (1) and (2) are in force for $Y \to \mathbf{A}^r_X$.

Below we will answer these questions when X is countably indexed, i.e., when A has a countable fundamental system of open ideals. If you have answers to these questions in greater generality, or if you have counter examples, please email stacks.project@gmail.com.

0AQI **Lemma 87.29.5.** *Let S be a scheme. Let X be a countably indexed affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a countably indexed affine formal algebraic space and f corresponds to $A \to A/K$ where A is an object of $WAdm^{count}$ (Section 87.21) and $K \subset A$ is a closed ideal.*

Proof. By Lemma 87.10.4 we see that $X = \mathrm{Spf}(A)$ where A is an object of $WAdm^{count}$. Since a closed immersion is representable and affine, we conclude by Lemma 87.19.9 that Y is an affine formal algebraic space and countably index. Thus applying Lemma 87.10.4 again we see that $Y = \mathrm{Spf}(B)$ with B an object of $WAdm^{count}$. By Lemma 87.27.5 we conclude that f is given by a morphism $A \to B$ of $WAdm^{count}$ which is taut and has dense image. To finish the proof we apply Lemma 87.5.10. □

0ANU **Lemma 87.29.6.** *Let $B \to A$ be an arrow of $WAdm^{count}$, see Section 87.21. The following are equivalent*

(a) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for every weak ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA,*
(b) *$B \to A$ is taut and $B/J_\lambda \to A/I_\lambda$ is of finite type for a cofinal system (J_λ) of weak ideals of definition of B where $I_\lambda \subset A$ is the closure of $J_\lambda A$,*
(c) *$B \to A$ is taut and A is topologically of finite type over B,*
(d) *A is isomorphic as a topological B-algebra to a quotient of $B\{x_1, \ldots, x_n\}$ by a closed ideal.*

Moreover, these equivalent conditions define a local property, i.e., they satisfy Axioms (1), (2), (3).

Proof. The implications (a) ⇒ (b), (c) ⇒ (a), (d) ⇒ (c) are straightforward from the definitions. Assume (b) holds and let $J \subset B$ and $I \subset A$ be as in (a). Choose a commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & \ldots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 \\ \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & \longrightarrow & \ldots & \longrightarrow & B/J_3 & \longrightarrow & B/J_2 & \longrightarrow & B/J_1 \end{array}$$

such that $A_{n+1}/J_nA_{n+1} = A_n$ and such that $A = \lim A_n$ as in Lemma 87.22.1. For every m there exists a λ such that $J_\lambda \subset J_m$. Since $B/J_\lambda \to A/I_\lambda$ is of finite type, this implies that $B/J_m \to A/I_m$ is of finite type. Let $\alpha_1, \ldots, \alpha_n \in A_1$ be generators of A_1 over B/J_1. Since A is a countable limit of a system with surjective transition maps, we can find $a_1, \ldots, a_n \in A$ mapping to $\alpha_1, \ldots, \alpha_n$ in A_1. By Remark 87.28.1 we find a continuous map $B\{x_1, \ldots, x_n\} \to A$ mapping x_i to a_i. This map induces surjections $(B/J_m)[x_1, \ldots, x_n] \to A_m$ by Algebra, Lemma 10.126.9. For $m \geq 1$ we obtain a short exact sequence

$$0 \to K_m \to (B/J_m)[x_1, \ldots, x_n] \to A_m \to 0$$

The induced transition maps $K_{m+1} \to K_m$ are surjective because $A_{m+1}/J_mA_{m+1} = A_m$. Hence the inverse limit of these short exact sequences is exact, see Algebra, Lemma 10.86.4. Since $B\{x_1, \ldots, x_n\} = \lim(B/J_m)[x_1, \ldots, x_n]$ and $A = \lim A_m$ we conclude that $B\{x_1, \ldots, x_n\} \to A$ is surjective and open. As A is complete the kernel is a closed ideal. In this way we see that (a), (b), (c), and (d) are equivalent.

Let a diagram (87.21.2.1) as in Situation 87.21.2 be given. By Example 87.24.7 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a), (b), (c), and (d). Moreover, by Lemma 87.22.1 in order to prove Axioms (1) and (2) we may assume both $B \to A$ and $(B')^\wedge \to (A')^\wedge$ are taut. Now pick a weak ideal of definition $J \subset B$. Let

$J' \subset (B')^\wedge$, $I \subset A$, $I' \subset (A')^\wedge$ be the closure of $J(B')^\wedge$, JA, $J(A')^\wedge$. By what was said above, it suffices to consider the commutative diagram

$$\begin{array}{ccc} A/I & \longrightarrow & (A')^\wedge/I' \\ {\scriptstyle\overline{\varphi}}\uparrow & & \uparrow{\scriptstyle\overline{\varphi}'} \\ B/J & \longrightarrow & (B')^\wedge/J' \end{array}$$

and to show (1) $\overline{\varphi}$ finite type $\Rightarrow$ $\overline{\varphi}'$ finite type, and (2) if $A \to A'$ is faithfully flat, then $\overline{\varphi}'$ finite type $\Rightarrow$ $\overline{\varphi}$ finite type. Note that $(B')^\wedge/J' = B'/JB'$ and $(A')^\wedge/I' = A'/IA'$ by the construction of the topologies on $(B')^\wedge$ and $(A')^\wedge$. In particular the horizontal maps in the diagram are étale. Part (1) now follows from Algebra, Lemma 10.6.2 and part (2) from Descent, Lemma 35.14.2 as the ring map $A/I \to (A')^\wedge/I' = A'/IA'$ is faithfully flat and étale.

We omit the proof of Axiom (3). □

0CB6 **Lemma 87.29.7.** *In Lemma 87.29.6 if B is admissible (for example adic), then the equivalent conditions (a) – (d) are also equivalent to*

(e) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for some ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA.*

Proof. It is enough to show that (e) implies (a). Let $J' \subset B$ be a weak ideal of definition and let $I' \subset A$ be the closure of $J'A$. We have to show that $B/J' \to A/I'$ is of finite type. If the corresponding statement holds for the smaller weak ideal of definition $J'' = J' \cap J$, then it holds for J'. Thus we may assume $J' \subset J$. As J is an ideal of definition (and not just a weak ideal of definition), we get $J^n \subset J'$ for some $n \geq 1$. Thus we can consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I' & \longrightarrow & A/I' & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & J/J' & \longrightarrow & B/J' & \longrightarrow & B/J & \longrightarrow & 0 \end{array}$$

with exact rows. Since $I' \subset A$ is open and since I is the closure of JA we see that $I/I' = (J/J') \cdot A/I'$. Because J/J' is a nilpotent ideal and as $B/J \to A/I$ is of finite type, we conclude from Algebra, Lemma 10.126.8 that A/I' is of finite type over B/J' as desired. □

0ANV **Lemma 87.29.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces. Assume Y countably indexed. The following are equivalent*

(1) *f is locally of finite type,*
(2) *f is of finite type,*
(3) *f corresponds to a morphism $B \to A$ of $WAdm^{count}$ (Section 87.21) satisfying the equivalent conditions of Lemma 87.29.6.*

Proof. Since X and Y are affine it is clear that conditions (1) and (2) are equivalent. In cases (1) and (2) the morphism f is representable by algebraic spaces by definition, hence affine by Lemma 87.19.7. Thus if (1) or (2) holds we see that X is countably indexed by Lemma 87.19.9. Write $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ for topological S-algebras A and B in $WAdm^{count}$, see Lemma 87.10.4. By Lemma

87.9.10 we see that f corresponds to a continuous map $B \to A$. Hence now the result follows from Lemma 87.29.2. □

0ANW **Lemma 87.29.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of $WAdm^{count}$ which is taut and topologically of finite type,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of $WAdm^{count}$ which is taut and topologically of finite type,*

(3) *there exist a covering $\{X_i \to X\}$ as in Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of $WAdm^{count}$ which is, taut and topologically of finite type, and*

(4) *f is locally of finite type.*

Proof. By Lemma 87.29.6 the property $P(\varphi) =$"φ is taut and topologically of finite type" is local on WAdm^{count}. Hence by Lemma 87.21.3 we see that conditions (1), (2), and (3) are equivalent. On the other hand, by Lemma 87.29.8 the condition P on morphisms of $WAdm^{count}$ corresponds exactly to morphisms of countably indexed, affine formal algebraic spaces being locally of finite type. Thus the implication (1) ⇒ (3) of Lemma 87.24.6 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) ⇒ (1) of Lemma 87.24.6 shows that (2) implies (4) of the current lemma. □

87.30. Separation axioms for morphisms

0ARM This section is the analogue of Morphisms of Spaces, Section 67.4 for morphisms of formal algebraic spaces.

0ARN **Definition 87.30.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.

(2) We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

Since $\Delta_{X/Y}$ is representable (by schemes) by Lemma 87.15.5 we can test this by considering morphisms $T \to X \times_Y X$ from affine schemes T and checking whether

$$E = T \times_{X \times_Y X} X \longrightarrow T$$

is quasi-compact or a closed immersion, see Lemma 87.17.5 or Definition 87.27.1. Note that the scheme E is the equalizer of two morphisms $a, b : T \to X$ which agree as morphisms into Y and that $E \to T$ is a monomorphism and locally of finite type.

0ARP **Lemma 87.30.2.** *All of the separation axioms listed in Definition 87.30.1 are stable under base change.*

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of formal algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. Each of the properties of the diagonal used in Definition 87.30.1 is stable under base change. Hence the lemma is true. □

0ARQ **Lemma 87.30.3.** *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of formal algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

(1) *i is representable (by schemes), locally of finite type, locally quasi-finite, separated, and a monomorphism,*
(2) *if $Z \to T$ is separated, then i is a closed immersion, and*
(3) *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow[i]{} & X \times_T Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Delta_{Z/T}} & Z \times_T Z \end{array}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 87.15.5. □

0ARR **Lemma 87.30.4.** *All of the separation axioms listed in Definition 87.30.1 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 87.30.3 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$. □

0ARS **Lemma 87.30.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 87.30.1. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Definition 87.16.3),*
(5) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that the base change $Y_j \times_Y X \to Y_j$ has $\mathcal{P}$ for all j.*

Proof. We will repeatedly use Lemma 87.30.2 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Assume (3) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Then

$$U \times_{Z \times_Y X} V = (Z \times_Y X) \times_{\Delta, (Z \times_Y X) \times_Z (Z \times_Y X)} (U \times_Z V)$$

and we see that this is quasi-compact if $\mathcal{P}$ ="quasi-separated" or an affine scheme equipped with a closed immersion into $U \times_Z V$ if $\mathcal{P}$ ="separated". Thus (4) holds.

Assume (4) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Reading the argument above backwards, we see that $U \times_{Z \times_Y X} V \to U \times_Z V$ is quasi-compact if $\mathcal{P}$ ="quasi-separated" or a closed immersion if $\mathcal{P}$ ="separated". Since we can choose U and V as above such that U varies through an étale covering of $Z \times_Y X$, we find that the corresponding morphisms

$$U \times_Z V \to (Z \times_Y X) \times_Z (Z \times_Y X)$$

form an étale covering by affines. Hence we conclude that $\Delta : (Z \times_Y X) \to (Z \times_Y X) \times_Z (Z \times_Y X)$ is quasi-compact, resp. a closed immersion. Thus (3) holds.

Let us prove that (3) implies (5). Assume (3) and let $\{Y_j \to Y\}$ be as in Definition 87.11.1. We have to show that the morphisms

$$\Delta_j : Y_j \times_Y X \longrightarrow (Y_j \times_Y X) \times_{Y_j} (Y_j \times_Y X) = Y_j \times_Y X \times_Y X$$

has the corresponding property (i.e., is quasi-compact or a closed immersion). Write $Y_j = \operatorname{colim} Y_{j,\lambda}$ as in Definition 87.9.1. Replacing Y_j by $Y_{j,\lambda}$ in the formula above, we have the property by our assumption that (3) holds. Since the displayed arrow is the colimit of the arrows $\Delta_{j,\lambda}$ and since we can test whether Δ_j has the corresponding property by testing after base change by affine schemes mapping into $Y_j \times_Y X \times_Y X$, we conclude by Lemma 87.9.4.

Let us prove that (5) implies (1). Let $\{Y_j \to Y\}$ be as in (5). Then we have the fibre product diagram

$$\begin{array}{ccc} \coprod Y_j \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod Y_j \times_Y X \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

By assumption the left vertical arrow is quasi-compact or a closed immersion. It follows from Spaces, Lemma 65.5.6 that also the right vertical arrow is quasi-compact or a closed immersion. □

87.31. Proper morphisms

0AM5 Here is the definition we will use.

0AM6 **Definition 87.31.1.** Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is *proper* if f is representable by algebraic spaces and is proper in the sense of Bootstrap, Definition 80.4.1.

It follows from the definitions that a proper morphism is of finite type.

0ART **Lemma 87.31.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is proper,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is an algebraic space proper over Z,*
(5) *there exists a covering $\{Y_j \to Y\}$ as in Definition 87.11.1 such that the base change $Y_j \times_Y X \to Y_j$ is proper for all j.*

Proof. Omitted. □

0GBT **Lemma 87.31.3.** *Proper morphisms of formal algebraic spaces are preserved by base change.*

Proof. This is an immediate consequence of Lemma 87.31.2 and transitivity of base change. □

87.32. Formal algebraic spaces and fpqc coverings

0AQC This section is the analogue of Properties of Spaces, Section 66.17. Please read that section first.

0AQD **Lemma 87.32.1.** *Let S be a scheme. Let X be a formal algebraic space over S. Then X satisfies the sheaf property for the fpqc topology.*

Proof. The proof is **identical** to the proof of Properties of Spaces, Proposition 66.17.1. Since X is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines $f : T' \to T$ we have: $X(T)$ is the equalizer of the two maps $X(T') \to X(T' \times_T T')$. See Topologies, Lemma 34.9.14.

Let $a, b : T \to X$ be two morphisms such that $a \circ f = b \circ f$. We have to show $a = b$. Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Lemma 87.11.2 the morphism $\Delta_{X/S}$ is a representable monomorphism. Hence $E \to T$ is a monomorphism of schemes. Our assumption that $a \circ f = b \circ f$ implies that $T' \to T$ factors (uniquely) through E. Consider the commutative diagram

$$\begin{array}{ccc} T' \times_T E & \longrightarrow & E \\ \uparrow \downarrow & \nearrow & \downarrow \\ T' & \longrightarrow & T \end{array}$$

Since the projection $T' \times_T E \to T'$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $E \to T$ is an isomorphism by Descent, Lemma 35.23.19. This means $a = b$ as desired.

Next, let $c : T' \to X$ be a morphism such that the two compositions $T' \times_T T' \to T' \to X$ are the same. We have to find a morphism $a : T \to X$ whose composition with $T' \to T$ is c. Choose a formal affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |X_{red}|$ contains the image of $|c| : |T'| \to |X_{red}|$.

This is possible by Definition 87.11.1, Properties of Spaces, Lemma 66.4.6, the fact that a finite union of formal affine algebraic spaces is a formal affine algebraic space, and the fact that $|T'|$ is quasi-compact (small argument omitted). The morphism $U \to X$ is representable by schemes (Lemma 87.9.11) and separated (Lemma 87.16.5). Thus

$$V = U \times_{X,c} T' \longrightarrow T'$$

is an étale and separated morphism of schemes. It is also surjective by our choice of $U \to X$ (if you do not want to argue this you can replace U by a disjoint union of formal affine algebraic spaces so that $U \to X$ is surjective everything else still works as well). The fact that $c \circ \mathrm{pr}_0 = c \circ \mathrm{pr}_1$ means that we obtain a descent datum on $V/T'/T$ (Descent, Definition 35.34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c\circ\mathrm{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c\circ\mathrm{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism $V \to T'$ is ind-quasi-affine by More on Morphisms, Lemma 37.66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 29.36.6). By More on Groupoids, Lemma 40.15.3 the descent datum is effective. Say $W \to T$ is a morphism such that there is an isomorphism $\alpha : T' \times_T W \to V$ compatible with the given descent datum on V and the canonical descent datum on $T' \times_T W$. Then $W \to T$ is surjective and étale (Descent, Lemmas 35.23.7 and 35.23.31). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions $b' \circ (\mathrm{pr}_0, 1), b' \circ (\mathrm{pr}_1, 1) : (T' \times_T T') \times_T W \to T' \times_T W \to U$ agree by our choice of α and the corresponding property of c (computation omitted). Hence b' descends to a morphism $b : W \to U$ by Descent, Lemma 35.13.7. The diagram

$$\begin{array}{ccccc} T' \times_T W & \longrightarrow & W & \xrightarrow[b]{} & U \\ \downarrow & & & & \downarrow \\ T' & & \xrightarrow{\quad c \quad} & & X \end{array}$$

is commutative. What this means is that we have proved the existence of a étale locally on T, i.e., we have an $a' : W \to X$. However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have $\mathrm{pr}_0^* a' = \mathrm{pr}_1^* a'$ as elements of $X(W \times_T W)$. Since X is an étale sheaf we find a unique $a \in X(T)$ restricting to a' on W. □

87.33. Maps out of affine formal schemes

0AQE We prove a few results that will be useful later. In the paper [**Bha16**] the reader can find very general results of a similar nature.

0AQF **Lemma 87.33.1.** *Let S be a scheme. Let A be a weakly admissible topological S-algebra. Let X be an affine scheme over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(A), X)$$

is bijective.

Proof. If X is affine, say $X = \mathrm{Spec}(B)$, then we see from Lemma 87.9.10 that morphisms $\mathrm{Spf}(A) \to \mathrm{Spec}(B)$ correspond to continuous S-algebra maps $B \to A$ where B has the discrete topology. These are just S-algebra maps, which correspond to morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. □

0AQG **Lemma 87.33.2.** *Let S be a scheme. Let A be a weakly admissible topological S-algebra such that A/I is a local ring for some weak ideal of definition $I \subset A$. Let X be a scheme over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(A), X)$$

is bijective.

Proof. Let $\varphi : \mathrm{Spf}(A) \to X$ be a morphism. Since $\mathrm{Spec}(A/I)$ is local we see that φ maps $\mathrm{Spec}(A/I)$ into an affine open $U \subset X$. However, this then implies that $\mathrm{Spec}(A/J)$ maps into U for every ideal of definition J. Hence we may apply Lemma 87.33.1 to see that φ comes from a morphism $\mathrm{Spec}(A) \to X$. This proves surjectivity of the map. We omit the proof of injectivity. □

0AQH **Lemma 87.33.3.** *Let S be a scheme. Let R be a complete local Noetherian S-algebra. Let X be an algebraic space over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(R), X)$$

is bijective.

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. We have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above.

Injectivity: Let $x, x' : \mathrm{Spec}(R) \to X$ be two morphisms mapping to the same element in the right hand side. Consider the fibre product

$$T = \mathrm{Spec}(R) \times_{(x,x'), X \times_S X, \Delta} X$$

Then T is a scheme and $T \to \mathrm{Spec}(R)$ is locally of finite type, monomorphism, separated, and locally quasi-finite, see Morphisms of Spaces, Lemma 67.4.1. In particular T is locally Noetherian, see Morphisms, Lemma 29.15.6. Let $t \in T$ be the unique point mapping to the closed point of $\mathrm{Spec}(R)$ which exists as x and x' agree over $R/\mathfrak{m}$. Then $R \to \mathcal{O}_{T,t}$ is a local ring map of Noetherian rings such that $R/\mathfrak{m}^n \to \mathcal{O}_{T,t}/\mathfrak{m}^n\mathcal{O}_{T,t}$ is an isomorphism for all n (because x and x' agree over $\mathrm{Spec}(R/\mathfrak{m}^n)$ for all n). Since $\mathcal{O}_{T,t}$ maps injectively into its completion (see Algebra, Lemma 10.51.4) we conclude that $R = \mathcal{O}_{T,t}$. Hence x and x' agree over R.

Surjectivity: Let (x_n) be an element of the right hand side. Choose a scheme U and a surjective étale morphism $U \to X$. Denote $x_0 : \mathrm{Spec}(k) \to X$ the morphism induced on the residue field $k = R/\mathfrak{m}$. The morphism of schemes $U \times_{X,x_0} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale. Thus $U \times_{X,x_0} \mathrm{Spec}(k)$ is a nonempty disjoint union of spectra of finite separable field extensions of k, see Morphisms, Lemma 29.36.7. Hence we can find a finite separable field extension k'/k and a

k'-point $u_0 : \mathrm{Spec}(k') \to U$ such that

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow[u_0]{} & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x_0} & X \end{array}$$

commutes. Let $R \subset R'$ be the finite étale extension of Noetherian complete local rings which induces k'/k on residue fields (see Algebra, Lemmas 10.153.7 and 10.153.9). Denote x'_n the restriction of x_n to $\mathrm{Spec}(R'/\mathfrak{m}^n R')$. By More on Morphisms of Spaces, Lemma 76.16.8 we can find an element $(u'_n) \in \lim \mathrm{Mor}_S(\mathrm{Spec}(R'/\mathfrak{m}^n R'), U)$ mapping to (x'_n). By Lemma 87.33.2 the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote $x' : \mathrm{Spec}(R') \to X$ the composition. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Hence the diagram

$$\begin{array}{ccc} \mathrm{Spec}(R' \otimes_R R') & \longrightarrow & \mathrm{Spec}(R') \\ \downarrow & & \downarrow{\scriptstyle x'} \\ \mathrm{Spec}(R') & \xrightarrow{x'} & X \end{array}$$

is commutative by the injectivity shown above and the fact that x'_n is the restriction of x_n which is defined over $R/\mathfrak{m}^n$. Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to a morphism $x : \mathrm{Spec}(R) \to X$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. □

0GBU **Lemma 87.33.4.** *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is quasi-compact. Let R be a complete local Noetherian S-algebra. Then an adic morphism $p : Spf(R) \to X_{/T}$ corresponds to a unique morphism $g : \mathrm{Spec}(R) \to X$ such that $g^{-1}(T) = \{\mathfrak{m}_R\}$.*

Proof. The statement makes sense because $X_{/T}$ is adic* by Lemma 87.20.8 (and hence we're allowed to use the terminology adic for morphisms, see Definition 87.23.2). Let p be given. By Lemma 87.33.3 we get a unique morphism $g : \mathrm{Spec}(R) \to X$ corresponding to the composition $\mathrm{Spf}(R) \to X_{/T} \to X$. Let $Z \subset X$ be the reduced induced closed subspace structure on T. The incusion morphism $Z \to X$ corresponds to a morphism $Z \to X_{/T}$. Since p is adic it is representable by algebraic spaces and we find

$$\mathrm{Spf}(R) \times_{X_{/T}} Z = \mathrm{Spf}(R) \times_X Z$$

is an algebraic space endowed with a closed immersion to $\mathrm{Spf}(R)$. (Equality holds because $X_{/T} \to X$ is a monomorphism.) Thus this fibre product is equal to $\mathrm{Spec}(R/J)$ for some ideal $J \subset R$ which contains $\mathfrak{m}_R^{n_0}$ for some $n_0 \geq 1$. This implies that $\mathrm{Spec}(R) \times_X Z$ is a closed subscheme of $\mathrm{Spec}(R)$, say $\mathrm{Spec}(R) \times_X Z = \mathrm{Spec}(R/I)$, whose intersection with $\mathrm{Spec}(R/\mathfrak{m}_R^n)$ for $n \geq n_0$ is equal to $\mathrm{Spec}(R/J)$. In algebraic terms this says $I + \mathfrak{m}_R^n = J + \mathfrak{m}_R^n = J$ for all $n \geq n_0$. By Krull's intersection theorem this implies $I = J$ and we conclude. □

87.34. The small étale site of a formal algebraic space

0DE9

The motivation for the following definition comes from classical formal schemes: the underlying topological space of a formal scheme $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is the underlying topological space of the reduction $\mathfrak{X}_{red}$.

An important remark is the following. Suppose that X is an algebraic space with reduction X_{red} (Properties of Spaces, Definition 66.12.5). Then we have

$$X_{spaces,étale} = X_{red,spaces,étale}, \quad X_{étale} = X_{red,étale}, \quad X_{affine,étale} = X_{red,affine,étale}$$

by More on Morphisms of Spaces, Theorem 76.8.1 and Lemma 76.8.2. Therefore the following definition does not conflict with the already existing notion in case our formal algebraic space happens to be an algebraic space.

0DEA **Definition 87.34.1.** Let S be a scheme. Let X be a formal algebraic space with reduction X_{red} (Lemma 87.12.1).

(1) The *small étale site* $X_{étale}$ of X is the site $X_{red,étale}$ of Properties of Spaces, Definition 66.18.1.
(2) The site $X_{spaces,étale}$ is the site $X_{red,spaces,étale}$ of Properties of Spaces, Definition 66.18.2.
(3) The site $X_{affine,étale}$ is the site $X_{red,affine,étale}$ of Properties of Spaces, Lemma 66.18.6.

In Lemma 87.34.6 we will see that $X_{spaces,étale}$ can be described by in terms of morphisms of formal algebraic spaces which are representable by algebraic spaces and étale. By Properties of Spaces, Lemmas 66.18.3 and 66.18.6 we have identifications

0DEB (87.34.1.1) $$Sh(X_{étale}) = Sh(X_{spaces,étale}) = Sh(X_{affine,étale})$$

We will call this the *(small) étale topos* of X.

0DEC **Lemma 87.34.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S.*

(1) *There is a continuous functor $Y_{spaces,étale} \to X_{spaces,étale}$ which induces a morphism of sites*
$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$
(2) *The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 7.14.5).*
(3) *The morphism of topoi associated to $f_{spaces,étale}$ induces, via (87.34.1.1), a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.*

Proof. The only point here is that f induces a morphism of reductions $X_{red} \to Y_{red}$ by Lemma 87.12.1. Hence this lemma is immediate from the corresponding lemma for morphisms of algebraic spaces (Properties of Spaces, Lemma 66.18.8). □

If the morphism of formal algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

0DED **Lemma 87.34.3.** *Let S be a scheme, and let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume f is representable by algebraic spaces and étale. In this case there is a cocontinuous functor $j : X_{étale} \to Y_{étale}$. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 7.21.1. Moreover, j is continuous as well, hence Sites, Lemma 7.21.5 applies.*

Proof. This will follow immediately from the case of algebraic spaces (Properties of Spaces, Lemma 66.18.11) if we can show that the induced morphism $X_{red} \to Y_{red}$ is étale. Observe that $X \times_Y Y_{red}$ is an algebraic space, étale over the reduced algebraic space Y_{red}, and hence reduced itself (by our definition of reduced algebraic spaces in Properties of Spaces, Section 66.7. Hence $X_{red} = X \times_Y Y_{red}$ as desired. □

0DEE **Lemma 87.34.4.** *Let S be a scheme. Let X be an affine formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that*

(1) *U is an affine formal algebraic space,*
(2) *φ is representable by algebraic spaces and étale.*

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Observe that for $\varphi : U \to X$ in $\mathcal{C}$ the morphism φ is representable (by schemes) and affine, see Lemma 87.19.7. Recall that $X_{affine,étale} = X_{red,affine,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{affine,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an affine scheme.

To finish the proof we will construct a quasi-inverse. Namely, write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. For each λ we have $X_{red} \subset X_\lambda$ is a thickening. Thus for every λ we have an equivalence

$$X_{red,affine,étale} = X_{\lambda,affine,étale}$$

for example by More on Algebra, Lemma 15.11.2. Hence if $U_{red} \to X_{red}$ is an étale morphism with U_{red} affine, then we obtain a system of étale morphisms $U_\lambda \to X_\lambda$ of affine schemes compatible with the transition morphisms in the system defining X. Hence we can take

$$U = \operatorname{colim} U_\lambda$$

as our affine formal algebraic space over X. The construction gives that $U \times_X X_\lambda = U_\lambda$. This shows that $U \to X$ is representable and étale. We omit the verification that the constructions are mutually inverse to each other. □

0DEF **Lemma 87.34.5.** *Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan, i.e., equal to $Spf(A)$ for some weakly admissible topological S-algebra A. Then $(X_{affine,étale})^{opp}$ is equivalent to the category whose*

(1) *objects are A-algebras of the form $B^\wedge = \lim B/JB$ where $A \to B$ is an étale ring map and J runs over the weak ideals of definition of A, and*
(2) *morphisms are continuous A-algebra homomorphisms.*

Proof. Combine Lemmas 87.34.4 and 87.19.13. □

0DEG **Lemma 87.34.6.** *Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{spaces,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that φ is representable by algebraic spaces and étale.*

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Recall that $X_{spaces,étale} = X_{red,spaces,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{spaces,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an algebraic space étale over X_{red}.

To finish the proof we will construct a quasi-inverse. Choose an object $\psi : V \to X_{red}$ of $X_{red,spaces,étale}$. Consider the functor $U_{V,\psi} : (Sch/S)_{fppf} \to Sets$ given by

$$U_{V,\psi}(T) = \{(a, b) \mid a : T \to X,\ b : T \times_{a,X} X_{red} \to V,\ \psi \circ b = a|_{T \times_{a,X} X_{red}}\}$$

We claim that the transformation $U_{V,\psi} \to X$, $(a, b) \mapsto a$ defines an object of the category $\mathcal{C}$. First, let's prove that $U_{V,\psi}$ is a formal algebraic space. Observe that $U_{V,\psi}$ is a sheaf for the fppf topology (some details omitted). Next, suppose that $X_i \to X$ is an étale covering by affine formal algebraic spaces as in Definition 87.11.1. Set $V_i = V \times_{X_{red}} X_{i,red}$ and denote $\psi_i : V_i \to X_{i,red}$ the projection. Then we have

$$U_{V,\psi} \times_X X_i = U_{V_i,\psi_i}$$

by a formal argument because $X_{i,red} = X_i \times_X X_{red}$ (as $X_i \to X$ is representable by algebraic spaces and étale). Hence it suffices to show that U_{V_i,ψ_i} is an affine formal algebraic space, because then we will have a covering $U_{V_i,\psi_i} \to U_{V,\psi}$ as in Definition 87.11.1. On the other hand, we have seen in the proof of Lemma 87.34.3 that $\psi_i : V_i \to X_i$ is the base change of a representable and étale morphism $U_i \to X_i$ of affine formal algebraic spaces. Then it is not hard to see that $U_i = U_{V_i,\psi_i}$ as desired.

We omit the verification that $U_{V,\psi} \to X$ is representable by algebraic spaces and étale. Thus we obtain our functor $(V, \psi) \mapsto (U_{V,\psi} \to X)$ in the other direction. We omit the verification that the constructions are mutually inverse to each other. □

0DEH **Lemma 87.34.7.** *Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that*

(1) *U is an affine formal algebraic space,*
(2) *φ is representable by algebraic spaces and étale.*

Proof. This follows by combining Lemmas 87.34.6 and 87.18.3. □

87.35. The structure sheaf

0DEI As in Section 87.34 the motivation for the discussion in this section comes from classical formal schemes. These are defined as pairs $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ where $\mathcal{O}_\mathfrak{X}$ is a sheaf of topological rings.

Let X be a formal algebraic space. A *structure sheaf for X* is a sheaf of topological rings $\mathcal{O}_X$ on the étale site $X_{étale}$ (which we defined in Section 87.34) such that

$$\mathcal{O}_X(U_{red}) = \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$$

as topological rings whenever

(1) $\varphi : U \to X$ is a morphism of formal algebraic spaces,
(2) U is an affine formal algebraic space,
(3) φ is representable by algebraic spaces and étale,
(4) $U_{red} \to X_{red}$ is the corresponding affine object of $X_{étale}$, see Lemma 87.34.7,
(5) $U = \operatorname{colim} U_\lambda$ is a colimit representation for U as in Definition 87.9.1.

Structure sheaves exist but may behave in unexpected manner.

0DEJ **Lemma 87.35.1.** *Every formal algebraic space has a structure sheaf.*

Proof. Let S be a scheme. Let X be a formal algebraic space over S. By (87.34.1.1) it suffices to construct $\mathcal{O}_X$ as a sheaf of topological rings on $X_{affine,étale}$. Denote $\mathcal{C}$ the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that U is an affine formal algebraic space and φ is representable by algebraic spaces and étale. By Lemma 87.34.7 the functor $U \mapsto U_{red}$ is an equivalence of categories $\mathcal{C} \to X_{affine,étale}$. Hence by the rule given above the lemma, we already have $\mathcal{O}_X$ as a presheaf of topological rings on $X_{affine,étale}$. Thus it suffices to check the sheaf condition.

By definition of $X_{affine,étale}$ a covering corresponds to a finite family $\{g_i : U_i \to U\}_{i=1,\ldots,n}$ of morphisms of $\mathcal{C}$ such that $\{U_{i,red} \to U_{red}\}$ is an étale covering. The morphisms g_i are representably by algebraic spaces (Lemma 87.19.3) hence affine (Lemma 87.19.7). Then g_i is étale (follows formally from Properties of Spaces, Lemma 66.16.6 as U_i and U are étale over X in the sense of Bootstrap, Section 80.4). Finally, write $U = \operatorname{colim} U_\lambda$ as in Definition 87.9.1.

With these preparations out of the way, we can prove the sheaf property as follows. For each λ we set $U_{i,\lambda} = U_i \times_U U_\lambda$ and $U_{ij,\lambda} = (U_i \times_U U_j) \times_U U_\lambda$. By the above, these are affine schemes, $\{U_{i,\lambda} \to U_\lambda\}$ is an étale covering, and $U_{ij,\lambda} = U_{i,\lambda} \times_{U_\lambda} U_{j,\lambda}$. Also we have $U_i = \operatorname{colim} U_{i,\lambda}$ and $U_i \times_U U_j = \operatorname{colim} U_{ij,\lambda}$. For each λ we have an exact sequence

$$0 \to \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

as we have the sheaf condition for the structure sheaf on U_λ and the étale topology (see Étale Cohomology, Proposition 59.17.1). Since limits commute with limits, the inverse limit of these exact sequences is an exact sequence

$$0 \to \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \lim \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \lim \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

This means that

$$0 \to \mathcal{O}_X(U_{red}) \to \prod\nolimits_i \mathcal{O}_X(U_{i,red}) \to \prod\nolimits_{i,j} \mathcal{O}_X((U_i \times_U U_j)_{red})$$

is exact as a sequence of abelian groups. We still have to show that the limit topology on $\mathcal{O}_X(U_{red})$ is induced from the limit topology on $\prod_{i=1,\ldots,n} \mathcal{O}_X(U_{i,red})$. The kernels of the maps $\mathcal{O}_X(U_{red}) \to \mathcal{O}_{U_\lambda}(U_\lambda)$ form a fundamental system of open submodules. Since these are the inverse images of the kernels of the maps $\prod_i \mathcal{O}_X(U_{i,red}) \to \prod_i \mathcal{O}_{U_{i,\lambda}}(U_{i,\lambda})$ and these kernels form a fundamental system of open submodules of $\mathcal{O}_X(U_{red}) \to \mathcal{O}_{U_\lambda}(U_\lambda)$ we win. □

0DEK **Remark 87.35.2.** The structure sheaf does not always have "enough sections". In Examples, Section 110.75 we have seen that there exist affine formal algebraic spaces which aren't McQuillan and there are even examples whose points are not separated by regular functions.

In the next lemma we prove that the structure sheaf on a countably indexed affine formal scheme has vanishing higher cohomology. For non-countably indexed ones, presumably this generally doesn't hold.

0DEL **Lemma 87.35.3.** *If X is a countably indexed affine formal algebraic space, then we have $H^n(X_{étale}, \mathcal{O}_X) = 0$ for $n > 0$.*

Proof. We may work with $X_{affine,étale}$ as this gives the same topos. We will apply Cohomology on Sites, Lemma 21.10.9 to show we have vanishing. Since $X_{affine,étale}$ has finite disjoint unions, this reduces us to the Čech complex of a covering given by a single arrow $\{U_{red} \to V_{red}\}$ in $X_{affine,étale} = X_{red,affine,étale}$ (see Étale Cohomology, Lemma 59.22.1). Thus we have to show that

$$0 \to \mathcal{O}_X(V_{red}) \to \mathcal{O}_X(U_{red}) \to \mathcal{O}_X(U_{red} \times_{V_{red}} U_{red}) \to \ldots$$

is exact. We will do this below in the case $V_{red} = X_{red}$. The general case is proven in exactly the same way.

Recall that $X = \mathrm{Spf}(A)$ where A is a weakly admissible topological ring having a countable fundamental system of weak ideals of definition. We have seen in Lemmas 87.34.4 and 87.34.5 that the object U_{red} in $X_{affine,étale}$ corresponds to a morphism $U \to X$ of affine formal algebraic spaces which is representable by algebraic space and étale and $U = \mathrm{Spf}(B^\wedge)$ where B is an étale A-algebra. By our rule for the structure sheaf we see

$$\mathcal{O}_X(U_{red}) = B^\wedge$$

We recall that $B^\wedge = \lim B/JB$ where the limit is over weak ideals of definition $J \subset A$. Working through the definitions we obtain

$$\mathcal{O}_X(U_{red} \times_{X_{red}} U_{red}) = (B \otimes_A B)^\wedge$$

and so on. Since $U \to X$ is a covering the map $A \to B$ is faithfully flat, see Lemma 87.19.14. Hence the complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

is **universally** exact, see Descent, Lemma 35.3.6. Our goal is to show that

$$H^n(0 \to A^\wedge \to B^\wedge \to (B \otimes_A B)^\wedge \to (B \otimes_A B \otimes_A B)^\wedge \to \ldots)$$

is zero for $n > 0$. To see what is going on, let's split our exact complex (before completion) into short exact sequences

$$0 \to A \to B \to M_1 \to 0, \quad 0 \to M_i \to B^{\otimes_A i+1} \to M_{i+1} \to 0$$

By what we said above, these are universally exact short exact sequences. Hence $JM_i = M_i \cap J(B^{\otimes_A i+1})$ for every ideal J of A. In particular, the topology on M_i as a submodule of $B^{\otimes_A i+1}$ is the same as the topology on M_i as a quotient module of $B^{\otimes_A i}$. Therefore, since there exists a countable fundamental system of weak ideals of definition in A, the sequences

$$0 \to A^\wedge \to B^\wedge \to M_1^\wedge \to 0, \quad 0 \to M_i^\wedge \to (B^{\otimes_A i+1})^\wedge \to M_{i+1}^\wedge \to 0$$

remain exact by Lemma 87.4.5. This proves the lemma. □

0DEM **Remark 87.35.4.** Even if the structure sheaf has good properties, this does not mean there is a good theory of quasi-coherent modules. For example, in Examples, Section 110.13 we have seen that for almost any Noetherian affine formal algebraic spaces the most natural notion of a quasi-coherent module leads to a category of modules which is not abelian.

87.36. Colimits of formal algebraic spaces

0GVL In this section we generalize the result of Section 87.13 to the case of systems of morphisms of formal algebraic spaces. We remark that in the lemmas below the condition "$f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$" can be reformulated as "$f_{\lambda\mu}$ is representable and a thickening".

0GVM **Lemma 87.36.1.** *Let S be a scheme. Suppose given a directed set Λ and a system of affine formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \operatorname{colim}_{\lambda\in\Lambda} X_\lambda$ is an affine formal algebraic space over S.*

Proof. We may write $X_\lambda = \operatorname{colim}_{\omega\in\Omega_\lambda} X_{\lambda,\omega}$ as the colimit of affine schemes over a directed set Ω_λ such that the transition morphisms $X_{\lambda,\omega} \to X_{\lambda,\omega'}$ are thickenings. For each $\lambda, \mu \in \Lambda$ and $\omega \in \Omega_\lambda$, with $\mu \geq \lambda$ there exists an $\omega' \in \Omega_\mu$ such that the morphism $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$, see Lemma 87.9.4. Then the morphism $X_{\lambda,\omega} \to X_{\mu,\omega'}$ is a closed immersion inducing an isomorphism on reductions and hence a thickening. Set $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ and say $(\lambda, \omega) \leq (\mu, \omega')$ if and only if $\lambda \leq \mu$ and $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$. It follows from the above that Ω is a directed set and that $X = \operatorname{colim}_{\lambda\in\Lambda} X_\lambda = \operatorname{colim}_{(\lambda,\omega)\in\Omega} X_{\lambda,\omega}$. This finishes the proof. □

0GVN **Lemma 87.36.2.** *Let S be a scheme. Suppose given a directed set Λ and a system of formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \operatorname{colim}_{\lambda\in\Lambda} X_\lambda$ is a formal algebraic space over S.*

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose a covering $\{X_{i,\lambda} \to X_\lambda\}$ as in Definition 87.11.1. In particular, we see that $\{X_{i,\lambda,red} \to X_{\lambda,red}\}$ is an étale covering by affine schemes. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{array}{ccc} X_{i,\lambda} & \longrightarrow & X_{i,\mu} \\ \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X_\mu \end{array}$$

with étale vertical arrows. Namely, the étale morphism $X_{i,\lambda,red} \to X_{\lambda,red} = X_{\mu,red}$ corresponds to an étale morphism $X_{i,\mu} \to X_\mu$ of formal algebraic spaces with $X_{i,\mu}$ an affine formal algebraic space, see Lemma 87.34.4. The same lemma implies the base change of $X_{i,\mu}$ to X_λ agrees with $X_{i,\lambda}$. It also follows that $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu \geq \lambda$. Set $X_i = \operatorname{colim} X_{i,\mu}$. Then $X_{i,\mu} = X_i \times_X X_\mu$ (as functors). Since any morphism $T \to X = \operatorname{colim} X_\mu$ from an affine (or quasi-compact) scheme T maps into X_μ for some μ, we see conclude that $\operatorname{colim} X_{i,\mu} \to \operatorname{colim} X_\mu$ is étale. Thus, if we can show that $\operatorname{colim} X_{i,\mu}$ is an affine formal algebraic space, then the lemma holds. Note that the morphisms $X_{i,\mu} \to X_{i,\mu'}$ are closed immersions as a base change of the closed immersion $X_\mu \to X_{\mu'}$. Finally, the morphism $X_{i,\mu,red} \to X_{i,\mu',red}$ is an isomorphism as $X_{\mu,red} \to X_{\mu',red}$ is an isomorphism. Hence this reduces us to the case discussed in Lemma 87.36.1. □

87.37. Recompletion

0GVP

In this section we define the completion of a formal algebraic space along a closed subset of its reduction. It is the natural generalization of Section 87.14.

0GVQ **Lemma 87.37.1.** *Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor*

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X : f(|U|) \subset T\}$$

is an affine formal algebraic space.

Proof. Write $X = \operatorname{colim} X_\lambda$ as in Definition 87.9.1. Then $X_{\lambda,red} = X_{red}$ and we may and do view T as a closed subset of $|X_\lambda| = |X_{\lambda,red}|$. By Lemma 87.14.1 for each λ the completion $(X_\lambda)_{/T}$ is an affine formal algebraic space. The transition morphisms $(X_\lambda)_{/T} \to (X_\mu)_{/T}$ are closed immersions as base changes of the transition morphisms $X_\lambda \to X_\mu$, see Lemma 87.14.4. Also the morphisms $((X_\lambda)_{/T})_{red} \to ((X_\mu)_{/T})_{red}$ are isomorphisms by Lemma 87.14.5. Since $X_{/T} = \operatorname{colim}(X_\lambda)_{/T}$ we conclude by Lemma 87.36.1. □

0GVR **Lemma 87.37.2.** *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor*

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. The functor $X_{/T}$ is an fppf sheaf since if $\{U_i \to U\}$ is an fppf covering, then $\coprod |U_i| \to |U|$ is surjective.

Choose a covering $\{g_i : X_i \to X\}_{i \in I}$ as in Definition 87.11.1. The morphisms $X_i \times_X X_{/T} \to X_{/T}$ are étale (see Spaces, Lemma 65.5.5) and the map $\coprod X_i \times_X X_{/T} \to X_{/T}$ is a surjection of sheaves. Thus it suffices to prove that $X_{/T} \times_X X_i$ is an affine formal algebraic space. A U-valued point of $X_i \times_X X_{/T}$ is a morphism $U \to X_i$ whose image is contained in the closed subset $|g_{i,red}|^{-1}(T) \subset |X_{i,red}|$. Thus this follows from Lemma 87.37.1. □

0GVS **Definition 87.37.3.** Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The formal algebraic space $X_{/T}$ of Lemma 87.14.2 is called the *completion of X along T*.

Let $f : X \to X'$ be a morphism of formal algebraic spaces over a scheme S. Suppose that $T \subset |X_{red}|$ and $T' \subset |X'_{red}|$ are closed subsets such that $|f_{red}|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

0GVT **Lemma 87.37.4.** *Let S be a scheme. Let $f : X' \to X$ be a morphism of formal algebraic spaces over S. Let $T \subset |X_{red}|$ be a closed subset and let $T' = |f_{red}|^{-1}(T) \subset |X'_{red}|$. Then*

$$\begin{array}{ccc} X'_{/T'} & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ X_{/T} & \longrightarrow & X \end{array}$$

is a cartesian diagram of formal algebraic spaces over S.

Proof. Namely, observe that the horizontal arrows are monomorphisms by construction. Thus it suffices to show that a morphism $g : U \to X'$ from a scheme U defines a point of $X'_{/T}$ if and only if $f \circ g$ defines a point of $X_{/T}$. In other words, we have to show that $g(U)$ is contained in $T' \subset |X'_{red}|$ if and only if $(f \circ g)(U)$ is contained in $T \subset |X_{red}|$. This follows immediately from our choice of T' as the inverse image of T. □

0GVU **Lemma 87.37.5.** *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X_{red} corresponding to T.*

Proof. It follows from Lemma 87.12.1, Properties of Spaces, Definition 66.12.5 (which uses Properties of Spaces, Lemma 66.12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. □

0GVV **Lemma 87.37.6.** *Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then*

1. *$X_{/T}$ is an affine formal algebraic space,*
2. *if X is McQuillan, then $X_{/T}$ is McQuillan,*
3. *if $|X_{red}| \setminus T$ is quasi-compact and X is countably indexed, then $X_{/T}$ is countably indexed,*
4. *if $|X_{red}| \setminus T$ is quasi-compact and X is adic*, then $X_{/T}$ is adic*,*
5. *if X is Noetherian, then $X_{/T}$ is Noetherian.*

Proof. Part (1) is Lemma 87.37.1. If X is McQuillan, then $X = \mathrm{Spf}(A)$ for some weakly admissible topological ring A. Then $X_{/T} \to X \to \mathrm{Spec}(A)$ satisfies property (2) of Lemma 87.9.6 and hence $X_{/T}$ is McQuillan, see Definition 87.9.7.

Assume X and T are as in (3). Then $X = \mathrm{Spf}(A)$ where A has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition, see Lemma 87.10.4. By Algebra, Lemma 10.29.1 we can find a finitely generated ideal $\overline{J} = (\overline{f}_1, \ldots, \overline{f}_r) \subset A/I_1$ such that T is cut out by $\overline{J}$ inside $\mathrm{Spec}(A/I_1) = |X_{red}|$. Choose $f_i \in A$ lifting $\overline{f}_i$. If $Z = \mathrm{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp : A \to B$ factors through A/I_n for some n and $g^\sharp(f_i)$ is nilpotent in B for each i. Thus $J_{m,n} = (f_1, \ldots, f_r)^m + I_n$ maps to zero in B for some $n, m \geq 1$. It follows that $X_{/T}$ is the formal spectrum of $\lim_{n,m} A/J_{m,n}$ and hence countably indexed. This proves (3).

Proof of (4). Here the argument is the same as in (3). However, here we may choose $I_n = I^n$ for some finitely generated ideal $I \subset A$. Then it is clear that $X_{/T}$ is the formal spectrum of $\lim A/J^n$ where $J = (f_1, \ldots, f_r) + I$. Some details omitted.

Proof of (5). In this case X_{red} is the spectrum of a Noetherian ring and hence the assumption that $|X_{red}| \setminus T$ is quasi-compact is satisfied. Thus as in the proof of (4) we see that $X_{/T}$ is the spectrum of $\lim A/J^n$ which is a Noetherian adic topological ring, see Algebra, Lemma 10.97.6. □

0GVW **Lemma 87.37.7.** *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then*

(1) *if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally countably indexed, then $X_{/T}$ is locally countably indexed,*
(2) *if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally adic*, then $X_{/T}$ is locally adic*, and*
(3) *if X is locally Noetherian, then $X_{/T}$ is locally Noetherian.*

Proof. Choose a covering $\{X_i \to X\}$ as in Definition 87.11.1. Let $T_i \subset X_{i,red}$ be the inverse image of T. We have $X_i \times_X X_{/T} = (X_i)_{/T_i}$ (Lemma 87.37.4). Hence $\{(X_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 87.11.1. Moreover, if $X_{red} \setminus T \to X_{red}$ is quasi-compact, so is $X_{i,red} \setminus T_i \to X_{i,red}$ and if X is locally countably indexed, or locally adic*, pr locally Noetherian, the is X_i is countably index, or adic*, or Noetherian. Thus the lemma follows from the affine case which is Lemma 87.37.6. □

87.38. Completion along a closed subspace

0GXT This section is the analgue of Section 87.14 for completions with respect to a closed subspace.

0GXU **Definition 87.38.1.** Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace and denote $Z_n \subset X$ the nth order infinitesimal neighbourhood. The formal algebraic space

$$X^\wedge_Z = \operatorname{colim} Z_n$$

(see Lemma 87.36.2) is called the *completion of X along Z*.

For example, if $X = \operatorname{Spec}(A)$ and Z is cut out by the ideal $I \subset A$, then we have

$$X^\wedge_Z = \operatorname{colim} \operatorname{Spec}(A/I^n) = \operatorname{Spf}(B)$$

where $B = \lim A/I^n$ with the limit topology. Note that B is a weakly adic topological ring but in general not adic.

Returning to the general case, if $T = |Z|$ then there is a canonical morphism $X^\wedge_Z \to X_{/T}$ comparing the completions along Z and T (Section 87.14) which need not be an isomorphism (see Lemma 87.38.6).

Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $Z \subset X$ and $Z' \subset X'$ are closed subspaces such that $f|_Z$ maps Z into Z' inducing a morphism $Z \to Z'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X^\wedge_Z \longrightarrow (X')^\wedge_{Z'}$$

between the completions.

0GXV **Lemma 87.38.2.** *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace and let $Z' = f^{-1}(Z) = X' \times_X Z$. Then*

$$\begin{array}{ccc} (X')^\wedge_{Z'} & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ X^\wedge_Z & \longrightarrow & X \end{array}$$

is a cartesian diagram of sheaves. In particular, the morphism $(X')^\wedge_{Z'} \to X^\wedge_Z$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $Y \to X$ factors through Z. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $Y \times_X X' \to X'$ factors through Z'. Since $Z'_n = X' \times_X Z_n$ for all $n \geq 1$ the same is true for the infinitesimal neighbourhoods. Hence the cartesian square of functors follows from the formulas $X^\wedge_Z = \operatorname{colim} Z_n$ and $(X')^\wedge_{Z'} = \operatorname{colim} Z'_n$. □

0GXW **Lemma 87.38.3.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The reduction $(X^\wedge_Z)_{red}$ of the completion $X^\wedge_Z$ of X along Z is Z_{red}.*

Proof. Omitted. □

0GXX **Lemma 87.38.4.** *Let S be a scheme. Let $X = \operatorname{Spec}(A)$ be an affine scheme over S. Let $Z \subset X$ be a closed subscheme corresponding to the ideal $I \subset A$. Then*

1. *The affine formal algebraic space $X^\wedge_Z$ is weakly adic.*
2. *If I is finitely generated, then $X^\wedge_Z = Spf(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.*
3. *If $Z \to X$ is of finite presentation, i.e., I is finitely generated, then $X^\wedge_Z$ is adic*.*
4. *If X is Noetherian, then $X^\wedge_Z$ is Noetherian.*

Proof. Omitted. □

0GXY **Lemma 87.38.5.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $X^\wedge_Z$ be the formal completion of X along Z.*

1. *The formal algebraic space $X^\wedge_Z$ is locally weakly adic.*
2. *If $Z \to X$ is of finite presentation, then $X^\wedge_Z$ is locally adic*.*
3. *If X is locally Noetherian, then X_Z is locally Noetherian.*

Proof. Omitted. □

0H8N **Lemma 87.38.6.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace and set $T = |Z|$. The canonical morphism $c : X^\wedge_Z \to X_{/T}$ is an isomorphism if $Z \to X$ is of finite presentation, but not in general.*

Proof. Both constructions commute with étale localization, hence it suffices to prove this when X is affine. Say $X = \operatorname{Spec}(A)$ and Z corresponds to the ideal $I \subset A$. If I is finitely generated then both $X^\wedge_Z$ and $X_{/T}$ are equal to $\operatorname{Spf}(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A, see Lemmas 87.14.6 and 87.38.4. If $A = \mathbf{Z}[x_1, x_2, \ldots]$ and $I = (x_1, x_2, \ldots)$ then $\operatorname{Spec}(A/(x_n^n, n \geq 1)) \to X$ factors through $X_{/T}$ but not through $X^\wedge_Z$ and hence c is not an isomorphism. □

87.39. Other chapters

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CHAPTER 88

Algebraization of Formal Spaces

0AM7

88.1. Introduction

0AM8 The main goal of this chapter is to prove Artin's theorem on dilatations, see Theorem 88.29.1; the result on contractions will be discussed in Artin's Axioms, Section 98.27. Both results use some material on formal algebraic spaces, hence in the middle part of this chapter, we continue the discussion of formal algebraic spaces from the previous chapter, see Formal Spaces, Section 87.1. The first part of this chapter is dedicated to algebraic preliminaries, mostly dealing with algebraization of rig-étale algebras.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. In the first part of this chapter (Sections 88.2 – 88.10) we discuss the category of I-adically complete algebras B topologically of finite type over a Noetherian ring A. It is shown that $B = A\{x_1, \ldots, x_n\}/J$ for some (closed) ideal J in the restricted power series ring (where A is endowed with the I-adic topology). We show there is a good notion of a naive cotangent complex $NL^\wedge_{B/A}$. If some power of I annihilates $NL^\wedge_{B/A}$, then we say B is a rig-étale algebra over (A, I); there is a similar notion of rig-smooth algebras. If A is a G-ring, then we show, using Popescu's theorem, that any rig-smooth algebra B over (A, I) is the completion of a finite type A-algebra; informally we say that we can "algebraize" B. However, the main result of the first part is that any rig-étale algebra B over (A, I) can be algebraized (without assuming A is a G-ring), see Lemma 88.10.2. For pointers to the literature on this type of algebraization, see Remark 88.7.3. General references for the first part are [**DG67**], [**Abb10**], and [**FK**].

In the second part of this chapter (Sections 88.12 – 88.24) we talk about types of morphisms of formal algebraic spaces in a reasonable level of generality (mostly for locally Noetherian formal algebraic spaces). The most interesting of these is the notion of a "formal modification" in the last section. We carefully check that our definition agrees with Artin's definition in [**Art70**].

Finally, in the third and last part of this chapter (Sections 88.25 – 88.30) we prove the main theorem and we give a few applications. In fact, we deduce Artin's theorem from a stronger result, namely, Theorem 88.27.4. This theorem says very roughly: if $f : \mathfrak{X} \to \mathfrak{X}'$ is a rig-étale morphism and $\mathfrak{X}'$ is the formal completion of a locally Noetherian algebraic space, then so is $\mathfrak{X}$. In Artin's work the morphism f is assumed proper and rig-surjective.

88.2. Two categories

0AL2

Let A be a ring and let $I \subset A$ be an ideal. In this section $^\wedge$ will mean I-adic completion. Set $A_n = A/I^n$ so that the I-adic completion of A is $A^\wedge = \lim A_n$. Let $\mathcal{C}$ be the category

0AL3 (88.2.0.1)
$$\mathcal{C} = \left\{ \begin{array}{c} \text{inverse systems } \ldots \to B_3 \to B_2 \to B_1 \\ \text{where } B_n \text{ is a finite type } A_n\text{-algebra,} \\ B_{n+1} \to B_n \text{ is an } A_{n+1}\text{-algebra map} \\ \text{which induces } B_{n+1}/I^nB_{n+1} \cong B_n \end{array} \right\}$$

Morphisms in $\mathcal{C}$ are given by systems of homomorphisms. Let $\mathcal{C}'$ be the category

0AL4 (88.2.0.2)
$$\mathcal{C}' = \left\{ \begin{array}{c} A\text{-algebras } B \text{ which are } I\text{-adically complete} \\ \text{such that } B/IB \text{ is of finite type over } A/I \end{array} \right\}$$

Morphisms in $\mathcal{C}'$ are A-algebra maps. There is a functor

0AJN (88.2.0.3)
$$\mathcal{C}' \longrightarrow \mathcal{C}, \quad B \longmapsto (B/I^nB)$$

Indeed, since B/IB is of finite type over A/I the ring maps $A_n = A/I^n \to B/I^nB$ are of finite type by Algebra, Lemma 10.126.8.

0AJP **Lemma 88.2.1.** *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor*
$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (B_n) \longmapsto B = \lim B_n$$
is a quasi-inverse to (88.2.0.3). The completions $A[x_1, \ldots, x_r]^\wedge$ are in $\mathcal{C}'$ and any object of $\mathcal{C}'$ is of the form
$$B = A[x_1, \ldots, x_r]^\wedge/J$$
for some ideal $J \subset A[x_1, \ldots, x_r]^\wedge$.

Proof. Let (B_n) be an object of $\mathcal{C}$. By Algebra, Lemma 10.98.2 we see that $B = \lim B_n$ is I-adically complete and $B/I^nB = B_n$. Hence we see that B is an object of $\mathcal{C}'$ and that we can recover the object (B_n) by taking the quotients. Conversely, if B is an object of $\mathcal{C}'$, then $B = \lim B/I^nB$ by assumption. Thus $B \mapsto (B/I^nB)$ is a quasi-inverse to the functor of the lemma.

Since $A[x_1, \ldots, x_r]^\wedge = \lim A_n[x_1, \ldots, x_r]$ it is an object of $\mathcal{C}'$ by the first statement of the lemma. Finally, let B be an object of $\mathcal{C}'$. Choose $b_1, \ldots, b_r \in B$ whose images in B/IB generate B/IB as an algebra over A/I. Since B is I-adically complete, the A-algebra map $A[x_1, \ldots, x_r] \to B$, $x_i \mapsto b_i$ extends to an A-algebra map $A[x_1, \ldots, x_r]^\wedge \to B$. To finish the proof we have to show this map is surjective which follows from Algebra, Lemma 10.96.1 as our map $A[x_1, \ldots, x_r] \to B$ is surjective modulo I and as $B = B^\wedge$. □

We warn the reader that, in case A is not Noetherian, the quotient of an object of $\mathcal{C}'$ may not be an object of $\mathcal{C}'$. See Examples, Lemma 110.8.1. Next we show this does not happen when A is Noetherian.

0AJQ **Lemma 88.2.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Then* [**GD60**, Proposition 7.5.5]

(1) *every object of the category $\mathcal{C}'$ (88.2.0.2) is Noetherian,*
(2) *if $B \in \mathrm{Ob}(\mathcal{C}')$ and $J \subset B$ is an ideal, then B/J is an object of $\mathcal{C}'$,*
(3) *for a finite type A-algebra C the I-adic completion $C^\wedge$ is in $\mathcal{C}'$,*
(4) *in particular the completion $A[x_1, \ldots, x_r]^\wedge$ is in $\mathcal{C}'$.*

Proof. Part (4) follows from Algebra, Lemma 10.97.6 as $A[x_1, \ldots, x_r]$ is Noetherian (Algebra, Lemma 10.31.1). To see (1) by Lemma 88.2.1 we reduce to the case of the completion of the polynomial ring which we just proved. Part (2) follows from Algebra, Lemma 10.97.1 which tells us that ever finite B-module is IB-adically complete. Part (3) follows in the same manner as part (4). □

0AL5 **Remark 88.2.3** (Base change)**.** Let $\varphi : A_1 \to A_2$ be a ring map and let $I_i \subset A_i$ be ideals such that $\varphi(I_1^c) \subset I_2$ for some $c \geq 1$. This induces ring maps $A_{1,cn} = A_1/I_1^{cn} \to A_2/I_2^n = A_{2,n}$ for all $n \geq 1$. Let $\mathcal{C}_i$ be the category (88.2.0.1) for (A_i, I_i). There is a base change functor

0AJZ (88.2.3.1) $$\mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad (B_n) \longmapsto (B_{cn} \otimes_{A_{1,cn}} A_{2,n})$$

Let $\mathcal{C}_i'$ be the category (88.2.0.2) for (A_i, I_i). If I_2 is finitely generated, then there is a base change functor

0AK0 (88.2.3.2) $$\mathcal{C}_1' \longrightarrow \mathcal{C}_2', \quad B \longmapsto (B \otimes_{A_1} A_2)^\wedge$$

because in this case the completion is complete (Algebra, Lemma 10.96.3). If both I_1 and I_2 are finitely generated, then the two base change functors agree via the functors (88.2.0.3) which are equivalences by Lemma 88.2.1.

0AL6 **Remark 88.2.4** (Base change by closed immersion)**.** Let A be a Noetherian ring and $I \subset A$ an ideal. Let $\mathfrak{a} \subset A$ be an ideal. Denote $\bar{A} = A/\mathfrak{a}$. Let $\bar{I} \subset \bar{A}$ be an ideal such that $I^c\bar{A} \subset \bar{I}$ and $\bar{I}^d \subset I\bar{A}$ for some $c, d \geq 1$. In this case the base change functor (88.2.3.2) for (A, I) to $(\bar{A}, \bar{I})$ is given by $B \mapsto \bar{B} = B/\mathfrak{a}B$. Namely, we have

0AK1 (88.2.4.1) $$\bar{B} = (B \otimes_A \bar{A})^\wedge = (B/\mathfrak{a}B)^\wedge = B/\mathfrak{a}B$$

the last equality because any finite B-module is I-adically complete by Algebra, Lemma 10.97.1 and if annihilated by $\mathfrak{a}$ also $\bar{I}$-adically complete by Algebra, Lemma 10.96.9.

88.3. A naive cotangent complex

0AJL Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an A-algebra which is I-adically complete such that $A/I \to B/IB$ is of finite type, i.e., an object of (88.2.0.2). By Lemma 88.2.2 we can write

$$B = A[x_1, \ldots, x_r]^\wedge/J$$

for some finitely generated ideal J. For a choice of presentation as above we define the *naive cotangent complex* in this setting by the formula

0AJR (88.3.0.1) $$NL^\wedge_{B/A} = (J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i)$$

with terms sitting in degrees -1 and 0 where the map sends the residue class of $g \in J$ to the differential $\mathrm{d}g = \sum(\partial g/\partial x_i)\mathrm{d}x_i$. Here the partial derivative is taken by thinking of g as a power series. The following lemma shows that $NL^\wedge_{B/A}$ is well defined up to homotopy.

0GAE **Lemma 88.3.1.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). The naive cotangent complex $NL^\wedge_{B/A}$ is well defined in $K(B)$.*

Proof. The lemma signifies that given a second presentation $B = A[y_1, \ldots, y_s]^\wedge/K$ the complexes of B-modules

$$(J/J^2 \to B\mathrm{d}x_i) \quad \text{and} \quad (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

are homotopy equivalent. To see this, we can argue exactly as in the proof of Algebra, Lemma 10.134.2.

Step 1. If we choose $g_i(y_1, \ldots, y_s) \in A[y_1, \ldots, y_s]^\wedge$ mapping to the image of x_i in B, then we obtain a (unique) continuous A-algebra homomorphism

$$A[x_1, \ldots, x_r]^\wedge \to A[y_1, \ldots, y_s]^\wedge, \quad x_i \mapsto g_i(y_1, \ldots, y_s)$$

compatible with the given surjections to B. Such a map is called a morphism of presentations. It induces a map from J into K and hence induces a B-module map $J/J^2 \to K/K^2$. Sending $\mathrm{d}x_i$ to $\sum(\partial g_i/\partial y_j)\mathrm{d}y_j$ we obtain a map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

Of course we can do the same thing with the roles of the two presentations exchanged to get a map of complexes in the other direction.

Step 2. The construction above is compatible with compositions of morphsms of presentations. Hence to finish the proof it suffices to show: given $g_i(x_1, \ldots, x_r) \in A[x_1, \ldots, x_n]^\wedge$ mapping to the image of x_i in B, the induced map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (J/J^2 \to \bigoplus B\mathrm{d}x_i)$$

is homotopic to the identity map. To see this consider the map $h : \bigoplus B\mathrm{d}x_i \to J/J^2$ given by the rule $\mathrm{d}x_i \mapsto g_i(x_1, \ldots, x_n) - x_i$ and compute. □

0GAF **Lemma 88.3.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $A \to B$ be a finite type ring map. Choose a presentation $\alpha : A[x_1, \ldots, x_n] \to B$. Then $NL^\wedge_{B^\wedge/A} = \lim NL(\alpha) \otimes_B B^\wedge$ as complexes and $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes^{\mathbf{L}}_B B^\wedge$ in $D(B^\wedge)$.*

Proof. The statement makes sense as $B^\wedge$ is an object of (88.2.0.2) by Lemma 88.2.2. Let $J = \mathrm{Ker}(\alpha)$. The functor of taking I-adic completion is exact on finite modules over $A[x_1, \ldots, x_n]$ and agrees with the functor $M \mapsto M \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]^\wedge$, see Algebra, Lemmas 10.97.1 and 10.97.2. Moreover, the ring maps $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]^\wedge$ and $B \to B^\wedge$ are flat. Hence $B^\wedge = A[x_1, \ldots, x_n]^\wedge/J^\wedge$ and

$$(J/J^2) \otimes_B B^\wedge = (J/J^2)^\wedge = J^\wedge/(J^\wedge)^2$$

Since $NL(\alpha) = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$, see Algebra, Section 10.134, we conclude the complex $NL^\wedge_{B^\wedge/A}$ is equal to $NL(\alpha) \otimes_B B^\wedge$. The final statement follows as $NL_{B/A}$ is homotopy equivalent to $NL(\alpha)$ and because the ring map $B \to B^\wedge$ is flat (so derived base change along $B \to B^\wedge$ is just base change). □

0AJS **Lemma 88.3.3.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Then*

1. *the pro-objects $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$ of $D(B)$ are strictly isomorphic (see proof for elucidation),*
2. *$NL^\wedge_{B/A} = R\lim NL_{B_n/A_n}$ in $D(B)$.*

Here B_n and A_n are as in Section 88.2.

Proof. The statement means the following: for every n we have a well defined complex NL_{B_n/A_n} of B_n-modules and we have transition maps $NL_{B_{n+1}/A_{n+1}} \to NL_{B_n/A_n}$. See Algebra, Section 10.134. Thus we can consider

$$\ldots \to NL_{B_3/A_3} \to NL_{B_2/A_2} \to NL_{B_1/A_1}$$

as an inverse system of complexes of B-modules and a fortiori as an inverse system in $D(B)$. Furthermore $R\lim NL_{B_n/A_n}$ is a homotopy limit of this inverse system, see Derived Categories, Section 13.34.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. This defines presentations

$$B_n = B/I^nB = A_n[x_1, \ldots, x_r]/J_n$$

where

$$J_n = JA_n[x_1, \ldots, x_r] = J/(J \cap I^nA[x_1, \ldots, x_r]^\wedge)$$

The two term complex $J_n/J_n^2 \longrightarrow \bigoplus B_n \mathrm{d}x_i$ represents NL_{B_n/A_n}, see Algebra, Section 10.134. By Artin-Rees (Algebra, Lemma 10.51.2) in the Noetherian ring $A[x_1, \ldots, x_r]^\wedge$ (Lemma 88.2.2) we find a $c \geq 0$ such that we have canonical surjections

$$J/I^nJ \to J_n \to J/I^{n-c}J \to J_{n-c}, \quad n \geq c$$

for all $n \geq c$. A moment's thought shows that these maps are compatible with differentials and we obtain maps of complexes

$$NL^\wedge_{B/A} \otimes_B B/I^nB \to NL_{B_n/A_n} \to NL^\wedge_{B/A} \otimes_B B/I^{n-c}B \to NL_{B_{n-c}/A_{n-c}}$$

compatible with the transition maps of the inverse systems $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$. This proves part (1) of the lemma.

By part (1) and since pro-isomorphic systems have the same $R\lim$ in order to prove (2) it suffices to show that $NL^\wedge_{B/A}$ is equal to $R\lim NL^\wedge_{B/A} \otimes_B B/I^nB$. However, $NL^\wedge_{B/A}$ is a two term complex $M^\bullet$ of finite B-modules which are I-adically complete for example by Algebra, Lemma 10.97.1. Hence $M^\bullet = \lim M^\bullet/I^nM^\bullet = R\lim M^\bullet/I^nM^\bullet$, see More on Algebra, Lemma 15.88.1 and Remark 15.88.6. □

0GAG **Lemma 88.3.4.** *Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. Then there is a canonical map*

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}.

Proof. Choose a presentation $B_1 = A_1[x_1, \ldots, x_r]^\wedge/J_1$. Since $A_2/I_2^n[x_1, \ldots, x_r] = A_1/I_1^{cn}[x_1, \ldots, x_r] \otimes_{A_1/I_1^{cn}} A_2/I_2^n$ we have

$$A_2[x_1, \ldots, x_r]^\wedge = (A_1[x_1, \ldots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use I_2-adic completion on both sides (but of course I_1-adic completion for $A_1[x_1, \ldots, x_r]^\wedge$). Set $J_2 = J_1A_2[x_1, \ldots, x_r]^\wedge$. Arguing similarly we get the

presentation

$$\begin{aligned} B_2 &= (B_1 \otimes_{A_1} A_2)^\wedge \\ &= \lim \frac{A_1/I_1^{cn}[x_1, \ldots, x_r]}{J_1(A_1/I_1^{cn}[x_1, \ldots, x_r])} \otimes_{A_1/I_1^{cn}} A_2/I_2^n \\ &= \lim \frac{A_2/I_2^n[x_1, \ldots, x_r]}{J_2(A_2/I_2^n[x_1, \ldots, x_r])} \\ &= A_2[x_1, \ldots, x_r]^\wedge/J_2 \end{aligned}$$

for B_2 over A_2. As a consequence obtain a commutative diagram

$$\begin{tikzcd} NL^\wedge_{B_1/A_1} : \arrow[d] & J_1/J_1^2 \arrow[r, "\mathrm{d}"'] \arrow[d] & \bigoplus B_1 \mathrm{d}x_i \arrow[d] \\ NL^\wedge_{B_2/A_2} : & J_2/J_2^2 \arrow[r, "\mathrm{d}"] & \bigoplus B_2 \mathrm{d}x_i \end{tikzcd}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \to J_2/J_2^2$ is surjective because J_2 is generated by the image of J_1. This determines the arrow displayed in the lemma. We omit the proof that this arrow is well defined up to homotopy (i.e., independent of the choice of the presentations up to homotopy). The statement about the induced map on cohomology modules follows easily from the discussion (details omitted). □

0ALM **Lemma 88.3.5.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $B \to C$ be morphism of (88.2.0.2). Then there is an exact sequence*

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] & \end{tikzcd}$$

See proof for elucidation.

Proof. Observe that taking the tensor product $NL^\wedge_{B/A} \otimes_B C$ makes sense as $NL^\wedge_{B/A}$ is well defined up to homotopy by Lemma 88.3.1. Also, (B, IB) is pair where B is a Noetherian ring (Lemma 88.2.2) and C is in the corresponding category (88.2.0.2). Thus all the terms in the 6-term sequence are (well) defined.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a presentation $C = B[y_1, \ldots, y_s]^\wedge/J'$. Combinging these presentations gives a presentation

$$C = A[x_1, \ldots, x_r, y_1, \ldots, y_s]^\wedge/K$$

Then the reader verifies that we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus C\mathrm{d}x_i \arrow[r] & \bigoplus C\mathrm{d}x_i \oplus \bigoplus C\mathrm{d}y_j \arrow[r] & \bigoplus C\mathrm{d}y_j \arrow[r] & 0 \\ & J/J^2 \otimes_B C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J'/(J')^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining $NL^\wedge_{B/A}$ with id_C. The lemma follows by applying the snake lemma (Algebra, Lemma 10.4.1). □

0AQJ **Lemma 88.3.6.** *With assumptions as in Lemma 88.3.5 assume that $B/I^nB \to C/I^nC$ is a local complete intersection homomorphism for all n. Then $H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$ is injective.*

Proof. For each $n \geq 1$ we set $A_n = A/I^n$, $B_n = B/I^nB$, and $C_n = C/I^nC$. We have

$$\begin{aligned} H^{-1}(NL^\wedge_{B/A} \otimes_B C) &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B C_n) \\ &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B B_n \otimes_{B_n} C_n) \\ &= \lim H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \end{aligned}$$

The first equality follows from More on Algebra, Lemma 15.101.1 and the fact that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ is a finite C-module and hence I-adically complete for example by Algebra, Lemma 10.97.1. The second equality is trivial. The third holds by Lemma 88.3.3. The maps $H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \to H^{-1}(NL_{C_n/A_n})$ are injective by More on Algebra, Lemma 15.33.6. The proof is finished because we also have $H^{-1}(NL^\wedge_{C/A}) = \lim H^{-1}(NL_{C_n/A_n})$ similarly to the above. □

88.4. Rig-smooth algebras

0GAH As motivation for the following definition, please take a look at More on Algebra, Remark 15.85.2.

0GAI **Definition 88.4.1.** Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). We say B is *rig-smooth over* (A, I) if there exists an integer $c \geq 0$ such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for every B-module N.

Let us work out what this means.

0GAJ **Lemma 88.4.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 88.2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\,dx_i)$ be its naive cotangent complex (88.3.0.1). The following are equivalent*

1. *B is rig-smooth over (A, I),*
2. *the object $NL^\wedge_{B/A}$ of $D(B)$ satisfies the equivalent conditions (1) – (6) of More on Algebra, Lemma 15.85.10 with respect to the ideal IB,*
3. *there exists a $c \geq 0$ such that for all $a \in I^c$ there is a map $h : \bigoplus B\,dx_i \to J/J^2$ such that $a : J/J^2 \to J/J^2$ is equal to $h \circ d$,*
4. *there exist $b_1, \ldots, b_s \in B$ such that $V(b_1, \ldots, b_s) \subset V(IB)$ and such that for every $l = 1, \ldots, s$ there exist $m \geq 0$, $f_1, \ldots, f_m \in J$, and subset $T \subset \{1, \ldots, n\}$ with $|T| = m$ such that*
   (a) *$\det_{i \in T, j \leq m}(\partial f_j/\partial x_i)$ divides b_l in B, and*
   (b) *$b_l J \subset (f_1, \ldots, f_m) + J^2$.*

Proof. The equivalence of (1), (2), and (3) is immediate from More on Algebra, Lemma 15.85.10.

Assume $b_1, \ldots, b_s$ are as in (4). Since B is Noetherian the inclusion $V(b_1, \ldots, b_s) \subset V(IB)$ implies $I^cB \subset (b_1, \ldots, b_s)$ for some $c \geq 0$ (for example by Algebra, Lemma 10.62.4). Pick $1 \leq l \leq s$ and $m \geq 0$ and $f_1, \ldots, f_m \in J$ and $T \subset \{1, \ldots, n\}$ with $|T| = m$ satisfying (4)(a) and (b). Then if we invert b_l we see that

$$NL^\wedge_{B/A} \otimes_B B_{b_l} = \Big(\bigoplus\nolimits_{j \leq m} B_{b_l} f_j \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B_{b_l} \mathrm{d}x_i \Big)$$

and moreover the arrow is isomorphic to the inclusion of the direct summand $\bigoplus_{i\in T} B_{b_l}\mathrm{d}x_i$. We conclude that $H^{-1}(NL^\wedge_{B/A})$ is b_l-power torsion and that $H^0(NL^\wedge_{B/A})$ becomes finite free after inverting b_l. Combined with the inclusion $I^cB \subset (b_1,\ldots,b_s)$ we see that $H^{-1}(NL^\wedge_{B/A})$ is IB-power torsion. Hence we see that condition (4) of More on Algebra, Lemma 15.85.10 holds. In this way we see that (4) implies (2).

Assume the equivalent conditions (1), (2), and (3) hold. We will prove that (4) holds, but we strongly urge the reader to convince themselves of this. The complex $NL^\wedge_{B/A}$ determines an object of $D^b_{Coh}(\mathrm{Spec}(B))$ whose restriction to the Zariski open $U = \mathrm{Spec}(B) \setminus V(IB)$ is a finite locally free module $\mathcal{E}$ placed in degree 0 (this follows for example from the the fourth equivalent condition in More on Algebra, Lemma 15.85.10). Choose generators $f_1,\ldots,f_M$ for J. This determines an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,M} \mathcal{O}_U \cdot f_j \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \cdot \mathrm{d}x_i \to \mathcal{E} \to 0$$

Let $U = \bigcup_{l=1,\ldots,s} U_l$ be a finite affine open covering such that $\mathcal{E}|_{U_l}$ is free of rank $r_l = n - m_l$ for some integer $n \geq m_l \geq 0$. After replacing each U_l by an affine open covering we may assume there exists a subset $T_l \subset \{1,\ldots,n\}$ such that the elements $\mathrm{d}x_i$, $i \in \{1,\ldots,n\} \setminus T_l$ map to a basis for $\mathcal{E}|_{U_l}$. Repeating the argument, we may assume there exists a subset $T_l' \subset \{1,\ldots,M\}$ of cardinality m_l such that f_j, $j \in T_l'$ map to a basis of the kernel of $\mathcal{O}_{U_l} \cdot \mathrm{d}x_i \to \mathcal{E}|_{U_l}$. Finally, since the open covering $U = \bigcup U_l$ may be refined by a open covering by standard opens (Algebra, Lemma 10.17.2) we may assume $U_l = D(g_l)$ for some $g_l \in B$. In particular we have $V(g_1,\ldots,g_s) = V(IB)$. A linear algebra argument using our choices above shows that $\det_{i\in T_l, j\in T_l'}(\partial f_j/\partial x_i)$ maps to an invertible element of B_{b_l}. Similarly, the vanishing of cohomology of $NL^\wedge_{B/A}$ in degree -1 over U_l shows that $J/J^2 + (f_j; j \in T')$ is annihilated by a power of b_l. After replacing each g_l by a suitable power we obtain conditions (4)(a) and (4)(b) of the lemma. Some details omitted. □

0GAK **Lemma 88.4.3.** *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.*

(1) *If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$, then $B^\wedge$ is rig-smooth over (A, I).*
(2) *If $B^\wedge$ is rig-smooth over (A, I), then there exists $g \in 1 + IB$ such that $\mathrm{Spec}(B_g)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$.*

Proof. We will use Lemma 88.4.2 without further mention.

Assume (1). Recall that formation of $NL_{B/A}$ commutes with localization, see Algebra, Lemma 10.134.13. Hence by the very definition of smooth ring maps (in terms of the naive cotangent complex being quasi-isomorphic to a finite projective module placed in degree 0), we see that $NL_{B/A}$ satisfies the fourth equivalent condition of More on Algebra, Lemma 15.85.10 with respect to the ideal IB (small detail omitted). Since $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ by Lemma 88.3.2 we conclude (2) holds by More on Algebra, Lemma 15.85.7.

Assume (2). Choose a presentation $B = A[x_1,\ldots,x_n]/J$, set $N = J/J^2$, and consider the element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, J/J^2)$ determined by the identity map on J/J^2. Using again that $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ we find that our assumption implies the image

$$\xi \otimes 1 \in \mathrm{Ext}^1_{B^\wedge}(NL_{B/A} \otimes_B B^\wedge, N \otimes_B B^\wedge) = \mathrm{Ext}^1_{B^\wedge}(NL_{B/A}, N) \otimes_B B^\wedge$$

is annihilated by I^c for some integer $c \geq 0$. The equality holds for example by More on Algebra, Lemma 15.100.2 (but can also easily be deduced from the much simpler More on Algebra, Lemma 15.66.4). Thus $M = I^cB\xi \subset \mathrm{Ext}^1_B(NL_{B/A}, N)$ is a finite submodule which maps to zero in $\mathrm{Ext}^1_B(NL_{B/A}, N) \otimes_B B^\wedge$. Since $B \to B^\wedge$ is flat this means that $M \otimes_B B^\wedge$ is zero. By Nakayama's lemma (Algebra, Lemma 10.20.1) this means that $M = I^cB\xi$ is annihilated by an element of the form $g = 1 + x$ with $x \in IB$. This implies that for every $b \in I^cB$ there is a B-linear dotted arrow making the diagram commute

$$\begin{array}{ccc} J/J^2 & \longrightarrow & \bigoplus B\mathrm{d}x_i \\ \downarrow{\scriptstyle b} & & \vdots{\scriptstyle h} \\ J/J^2 & \longrightarrow & (J/J^2)_g \end{array}$$

Thus $(NL_{B/A})_{gb}$ is quasi-isomorphic to a finite projective module; small detail omitted. Since $(NL_{B/A})_{gb} = NL_{B_{gb}/A}$ in $D(B_{gb})$ this shows that B_{gb} is smooth over $\mathrm{Spec}(A)$. As this holds for all $b \in I^cB$ we conclude that $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$ as desired. □

0GAL **Lemma 88.4.4.** *Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. Let $f_1 \in B_1$ with image $f_2 \in B_2$. If $\mathrm{Ext}^1_{B_1}(NL^\wedge_{B_1/A_1}, N_1)$ is annihilated by f_1 for every B_1-module N_1, then $\mathrm{Ext}^1_{B_2}(NL^\wedge_{B_2/A_2}, N_2)$ is annihilated by f_2 for every B_2-module N_2.*

Proof. By Lemma 88.3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemmas 15.85.6, 15.85.7, and 15.85.9 the last two applied with the principal ideals $(f_1) \subset B_1$ and $(f_2) \subset B_2$. □

0GAM **Lemma 88.4.5.** *Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1A_2) = V(I_2)$. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 88.2.3. If B_1 is rig-smooth over (A_1, I_1), then B_2 is rig-smooth over (A_2, I_2).*

Proof. Follows from Lemma 88.4.4 and Definition 88.4.1 and the fact that I_2^c is contained in I_1A_2 for some $c \geq 0$ as A_2 is Noetherian. □

88.5. Deformations of ring homomorphisms

0GAN Some work on lifting ring homomorphisms from rig-smooth algebras.

0AK3 **Remark 88.5.1** (Linear approximation)**.** Let A be a ring and $I \subset A$ be a finitely generated ideal. Let C be an I-adically complete A-algebra. Let $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ be a continuous A-algebra map. Suppose given $\delta_i \in C$, $i = 1, \ldots, r$. Then we can consider

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) + \delta_i$$

see Formal Spaces, Remark 87.28.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g/\partial x_i)\delta_i + \xi$$

with error term $\xi \in (\delta_i\delta_j)$. This follows by writing g as a power series and working term by term. Convergence is automatic as the coefficients of g tend to zero. Details omitted.

0GAP **Remark 88.5.2** (Lifting maps)**.** Let A be a Noetherian ring and $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Let C be an I-adically complete A-algebra. Let $\psi_n : B \to C/I^nC$ be an A-algebra homomorphism. The obstruction to lifting ψ_n to an A-algebra homomorphism into $C/I^{2n}C$ is an element

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

as we will explain. Namely, choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a lift $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ of ψ_n. Since $\psi(J) \subset I^nC$ we get $\psi(J^2) \subset I^{2n}C$ and hence we get a B-linear homomorphism

$$o(\psi) : J/J^2 \longrightarrow I^nC/I^{2n}C, \quad g \longmapsto \psi(g)$$

which of course extends to a C-linear map $J/J^2 \otimes_B C \to I^nC/I^{2n}C$. Since $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$ we get $o(\psi_n)$ as the image of $o(\psi)$ by the identification

$$\begin{aligned}&\mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)\\ &= \mathrm{Coker}\left(\mathrm{Hom}_B(\bigoplus B\mathrm{d}x_i, I^nC/I^{2n}C) \to \mathrm{Hom}_B(J/J^2, I^nC/I^{2n}C)\right)\end{aligned}$$

See More on Algebra, Lemma 15.85.4 part (1) for the equality.

Suppose that $o(\psi_n)$ maps to zero in $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C)$ for some integer n' with $n > n' > n/2$. We claim that this means we can find an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ which agrees with ψ_n as maps into $C/I^{n'}C$. The extreme case $n' = n$ explains why we previously said $o(\psi_n)$ is the obstruction to lifting ψ_n to $C/I^{2n}C$. Proof of the claim: the hypothesis that $o(\psi_n)$ maps to zero tells us we can find a B-module map

$$h : \bigoplus B\mathrm{d}x_i \longrightarrow I^{n'}C/I^{2n'}C$$

such that $o(\psi)$ and $h \circ \mathrm{d}$ agree as maps into $I^{n'}C/I^{2n'}C$. Say $h(\mathrm{d}x_i) = \delta_i \bmod I^{2n'}C$ for some $\delta_i \in I^{n'}C$. Then we look at the map

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) - \delta_i$$

A computation with power series shows that $\psi'(J) \subset I^{2n'}C$. Namely, for $g \in J$ we get

$$\psi'(g) \equiv \psi(g) - \sum \psi(\partial g/\partial x_i)\delta_i \equiv o(\psi)(g) - (h \circ \mathrm{d})(g) \equiv 0 \bmod I^{2n'}C$$

See Remark 88.5.1 for the first equality. Hence ψ' induces an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ as desired.

0GAQ **Lemma 88.5.3.** *Assume given the following data*

(1) *an integer $c \geq 0$,*
(2) *an ideal I of a Noetherian ring A,*
(3) *B in (88.2.0.2) for (A, I) such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for any B-module N,*
(4) *a Noetherian I-adically complete A-algebra C; denote $d = d(Gr_I(C))$ and $q_0 = q(Gr_I(C))$ the integers found in Local Cohomology, Section 51.22,*
(5) *an integer $n \geq \max(q_0 + (d+1)c, 2(d+1)c + 1)$, and*

(6) *an A-algebra homomorphism $\psi_n : B \to C/I^nC$.*

Then there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{n-(d+1)c} = \varphi \bmod I^{n-(d+1)c}$.

Proof. Consider the obstruction class

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

of Remark 88.5.2. For any C/I^nC-module N we have

$$\begin{aligned}\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N) &= \mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes^{\mathbf{L}}_B C/I^nC, N) \\ &= \mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)\end{aligned}$$

The first equality by More on Algebra, Lemma 15.100.1 and the second one by More on Algebra, Lemma 15.85.6. In particular, we see that $\mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)$ is annihilated by I^cC for all C/I^nC-modules N. It follows that we may apply Local Cohomology, Lemma 51.22.7 to see that $o(\psi_n)$ maps to zero in

$$\mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, I^{n'}C/I^{2n'}C) = \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C) =$$

where $n' = n - (d+1)c$. By the discussion in Remark 88.5.2 we obtain a map

$$\psi'_{2n'} : B \to C/I^{2n'}C$$

which agrees with ψ_n modulo $I^{n'}$. Observe that $2n' > n$ because $n \geq 2(d+1)c+1$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/I^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $I^{n_i - tc}$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod I^{n_i-(d+1)c} : B \to C/I^{n_i-(d+1)c}C$ and the lemma follows. □

We suggest the reader skip ahead to the next section. Namely, the following two lemmas are consequences of the result above if the algebra C in them is assumed Noetherian.

0AK6 **Lemma 88.5.4.** *Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Assume given an integer $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra such that a is a nonzerodivisor on C. Let $n > 2c$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c}C = \varphi \bmod a^{n-c}C$.*

Proof. Consider the obstruction class

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^nC/a^{2n}C)$$

of Remark 88.5.2. Since a is a nonzerodivisor on C the map $a^c : a^nC/a^{2n}C \to a^nC/a^{2n}C$ is isomorphic to the map $a^nC/a^{2n}C \to a^{n-c}C/a^{2n-c}C$ in the category of C-modules. Hence by our assumption on $NL^\wedge_{B/A}$ we conclude that the class $o(\psi_n)$ maps to zero in

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-c}C)$$

and a fortiori in

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-2c}C)$$

By the discussion in Remark 88.5.2 we obtain a map

$$\psi_{2n-2c} : B \to C/a^{2n-2c}C$$

which agrees with ψ_n modulo $a^{n-c}C$. Observe that $2n - 2c > n$ because $n > 2c$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/a^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $a^{n_i-c}C$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod a^{n_i-c}C : B \to C/a^{n_i-c}C$ and the lemma follows. □

0AK7 **Lemma 88.5.5.** *Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Assume given an integer $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra. Assume given an integer $d \geq 0$ such that $C[a^\infty] \cap a^d C = 0$. Let $n > \max(2c, c + d)$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

If C is Noetherian we have $C[a^\infty] = C[a^e]$ for some $e \geq 0$. By Artin-Rees (Algebra, Lemma 10.51.2) there exists an integer f such that $a^nC \cap C[a^\infty] \subset a^{n-f}C[a^\infty]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if C is an object of (88.2.0.2) by Lemma 88.2.2.

Proof. Let $C \to C'$ be the quotient of C by $C[a^\infty]$. The A-algebra C' is I-adically complete by Algebra, Lemma 10.96.10 and the fact that $\bigcap(C[a^\infty] + a^nC) = C[a^\infty]$ because for $n \geq d$ the sum $C[a^\infty] + a^nC$ is direct. For $m \geq d$ the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[a^\infty] & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C[a^\infty] & \longrightarrow & C/a^mC & \longrightarrow & C'/a^mC' & \longrightarrow & 0 \end{array}$$

has exact rows. Thus C is the fibre product of C' and C/a^mC over C'/a^mC' for all $m \geq d$. By Lemma 88.5.4 we can choose a homomorphism $\varphi' : B \to C'$ such that φ' and ψ_n agree as maps into $C'/a^{n-c}C'$. We obtain a homomorphism $(\varphi', \psi_n \bmod a^{n-c}C) : B \to C' \times_{C'/a^{n-c}C'} C/a^{n-c}C$. Since $n - c \geq d$ this is the same thing as a homomorphism $\varphi : B \to C$. This finishes the proof. □

88.6. Algebraization of rig-smooth algebras over G-rings

0ALU If the base ring A is a Noetherian G-ring, then we can prove [**Elk73**, III Theorem 7] for arbitrary rig-smooth algebras with respect to any ideal $I \subset A$ (not necessarily principal).

0GAR **Lemma 88.6.1.** *Let I be an ideal of a Noetherian ring A. Let $r \geq 0$ and write $P = A[x_1, \ldots, x_r]$ the I-adic completion. Consider a resolution*

$$P^{\oplus t} \xrightarrow{K} P^{\oplus m} \xrightarrow{g_1,\ldots,g_m} P \to B \to 0$$

of a quotient of P. Assume B is rig-smooth over (A, I). Then there exists an integer n such that for any complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1,\ldots,g'_m} P$$

with $g_i - g'_i \in I^n P$ and $K - K' \in I^n Mat(m \times t, P)$ there exists an isomorphism $B \to B'$ of A-algebras where $B' = P/(g'_1, \ldots, g'_m)$.

Proof. (A) By Definition 88.4.1 we can choose a $c \geq 0$ such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for all B-modules N.

(B) By More on Algebra, Lemmas 15.4.1 and 15.4.2 there exists a constant $c_1 = c(g_1, \ldots, g_m, K)$ such that for $n \geq c_1 + 1$ the complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1,\ldots,g'_m} P \to B' \to 0$$

is exact and $\mathrm{Gr}_I(B) \cong \mathrm{Gr}_I(B')$.

(C) Let $d_0 = d(\mathrm{Gr}_I(B))$ and $q_0 = q(\mathrm{Gr}_I(B))$ be the integers found in Local Cohomology, Section 51.22.

We claim that $n = \max(c_1 + 1, q_0 + (d_0 + 1)c, 2(d_0 + 1)c + 1)$ works where c is as in (A), c_1 is as in (B), and q_0, d_0 are as in (C).

Let $g'_1, \ldots, g'_m$ and K' be as in the lemma. Since $g_i = g'_i \in I^n P$ we obtain a canonical A-algebra homomorphism

$$\psi_n : B \longrightarrow B'/I^n B'$$

which induces an isomorphism $B/I^n B \to B'/I^n B'$. Since $\mathrm{Gr}_I(B) \cong \mathrm{Gr}_I(B')$ we have $d_0 = d(\mathrm{Gr}_I(B'))$ and $q_0 = q(\mathrm{Gr}_I(B'))$ and since $n \geq \max(q_0 + (1 + d_0)c, 2(d_0 + 1)c + 1)$ we may apply Lemma 88.5.3 to find an A-algebra homomorphism

$$\varphi : B \longrightarrow B'$$

such that $\varphi \bmod I^{n-(d_0+1)c}B' = \psi_n \bmod I^{n-(d_0+1)c}B'$. Since $n - (d_0 + 1)c > 0$ we see that φ is an A-algebra homomorphism which modulo I induces the isomorphism $B/IB \to B'/IB'$ we found above. The rest of the proof shows that these facts force φ to be an isomorphism; we suggest the reader find their own proof of this.

Namely, it follows that φ is surjective for example by applying Algebra, Lemma 10.96.1 part (1) using the fact that B and B' are complete. Thus φ induces a surjection $\mathrm{Gr}_I(B) \to \mathrm{Gr}_I(B')$ which has to be an isomorphism because the source and target are isomorphic Noetherian rings, see Algebra, Lemma 10.31.10 (of course you can show φ induces the isomorphism we found above but that would need a tiny argument). Thus φ induces injective maps $I^e B/I^{e+1}B \to I^e B'/I^{e+1}B'$ for all $e \geq 0$. This implies φ is injective since for any $b \in B$ there exists an $e \geq 0$ such that $b \in I^e B$, $b \notin I^{e+1}B$ by Krull's intersection theorem (Algebra, Lemma 10.51.4). This finishes the proof. □

0GAS **Lemma 88.6.2.** *Let I be an ideal of a Noetherian ring A. Let C^h be the henselization of a finite type A-algebra C with respect to the ideal IC. Let $J \subset C^h$ be an ideal. Then there exists a finite type A-algebra B such that $B^\wedge \cong (C^h/J)^\wedge$.*

Proof. By More on Algebra, Lemma 15.12.4 the ring C^h is Noetherian. Say $J = (g_1, \ldots, g_m)$. The ring C^h is a filtered colimit of étale C algebras C' such that $C/IC \to C'/IC'$ is an isomorphism (see proof of More on Algebra, Lemma 15.12.1).

Pick an C' such that $g_1, \ldots, g_m$ are the images of $g'_1, \ldots, g'_m \in C'$. Setting $B = C'/(g'_1, \ldots, g'_m)$ we get a finite type A-algebra. Of course (C, IC) and $C', IC')$ have the same henselizations and the same completions. It follows easily from this that $B^\wedge = (C^h/J)^\wedge$. □

0GAT **Proposition 88.6.3.** *Let I be an ideal of a Noetherian G-ring A. Let B be an object of (88.2.0.2). If B is rig-smooth over (A, I), then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$ of A-algebras.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Write $P = A[x_1, \ldots, x_r]^\wedge$. Choose generators $g_1, \ldots, g_m \in J$. Choose generators $k_1, \ldots, k_t$ of the module of relations between $g_1, \ldots, g_m$, i.e., such that

$$P^{\oplus t} \xrightarrow{k_1, \ldots, k_t} P^{\oplus m} \xrightarrow{g_1, \ldots, g_m} P \to B \to 0$$

is a resolution. Write $k_i = (k_{i1}, \ldots, k_{im})$ so that we have

0AKB (88.6.3.1) $$\sum\nolimits_j k_{ij} g_j = 0$$

for $i = 1, \ldots, t$. Denote $K = (k_{ij})$ the $m \times t$-matrix with entries k_{ij}.

Let $A[x_1, \ldots, x_r]^h$ be the henselization of the pair $(A[x_1, \ldots, x_r], IA[x_1, \ldots, x_r])$, see More on Algebra, Lemma 15.12.1. We may and do think of $A[x_1, \ldots, x_r]^h$ as a subring of $P = A[x_1, \ldots, x_r]^\wedge$, see More on Algebra, Lemma 15.12.4. Since A is a Noetherian G-ring, so is $A[x_1, \ldots, x_r]$, see More on Algebra, Proposition 15.50.10. Hence we have approximation for the map $A[x_1, \ldots, x_r]^h \to A[x_1, \ldots, x_r]^\wedge = P$ with respect to the ideal generated by I, see Smoothing Ring Maps, Lemma 16.14.1. Choose a large enough integer n as in Lemma 88.6.1. By the approximation property we may choose $g'_1, \ldots, g'_m \in A[x_1, \ldots, x_r]^h$ and a matrix $K' = (k'_{ij}) \in \mathrm{Mat}(m \times t, A[x_1, \ldots, x_r]^h)$ such that $\sum_j k'_{ij} g'_j = 0$ in $A[x_1, \ldots, x_r]^h$ and such that $g_i - g'_i \in I^n P$ and $K - K' \in I^n \mathrm{Mat}(m \times t, P)$. By our choice of n we conclude that there is an isomorphism

$$B \to P/(g'_1, \ldots, g'_m) = \big(A[x_1, \ldots, x_r]^h/(g'_1, \ldots, g'_m)\big)^\wedge$$

This finishes the proof by Lemma 88.6.2. □

The following lemma isn't true in general if A is not a G-ring but just Noetherian. Namely, if $(A, \mathfrak{m})$ is local and $I = \mathfrak{m}$, then the lemma is equivalent to Artin approximation for A^h (as in Smoothing Ring Maps, Theorem 16.13.1) which does not hold for every Noetherian local ring.

0AK4 **Lemma 88.6.4.** *Let A be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let B, C be finite type A-algebras. For any A-algebra map $\varphi : B^\wedge \to C^\wedge$ of I-adic completions and any $N \geq 1$ there exist*

1. *an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$,*
2. *an A-algebra map $\varphi : B \to C'$*

such that φ and ψ agree modulo I^N into $C^\wedge = (C')^\wedge$.

Proof. The statement of the lemma makes sense as $C \to C'$ is flat (Algebra, Lemma 10.143.3) hence induces an isomorphism $C/I^nC \to C'/I^nC'$ for all n (More on Algebra, Lemma 15.90.2) and hence an isomorphism on completions. Let C^h be the henselization of the pair (C, IC), see More on Algebra, Lemma 15.12.1.

Then C^h is the filtered colimit of the algebras C' and the maps $C \to C' \to C^h$ induce isomorphism on completions (More on Algebra, Lemma 15.12.4). Thus it suffices to prove there exists an A-algebra map $B \to C^h$ which is congruent to ψ modulo I^N. Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The ring map ψ corresponds to elements $\hat{c}_1, \ldots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \ldots, \hat{c}_n) = 0$ for $j = 1, \ldots, m$. Namely, as A is a Noetherian G-ring, so is C, see More on Algebra, Proposition 15.50.10. Thus Smoothing Ring Maps, Lemma 16.14.1 applies to give elements $c_1, \ldots, c_n \in C^h$ such that $f_j(c_1, \ldots, c_n) = 0$ for $j = 1, \ldots, m$ and such that $\hat{c}_i - c_i \in I^N C^h$. This determines the map $B \to C^h$ as desired. □

88.7. Algebraization of rig-smooth algebras

0GAU It turns out that if the rig-smooth algebra has a specific presentation, then it is straightforward to algebraize it. Please also see Remark 88.7.3 for a discussion.

0GAV **Lemma 88.7.1.** *Let A be a ring. Let $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ and set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Assume $m \leq n$ and set $g = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$. Then*

(1) *g annihilates $\mathrm{Ext}^1_B(NL_{B/A}, N)$ for every B-module N,*
(2) *if $n = m$, then multiplication by g on $NL_{B/A}$ is 0 in $D(B)$.*

Proof. Let T be the $m \times m$ matrix with entries $\partial f_j/\partial x_i$ for $1 \leq i, j \leq n$. Let $K \in D(B)$ be represented by the complex $T : B^{\oplus m} \to B^{\oplus m}$ with terms sitting in degrees -1 and 0. By More on Algebra, Lemmas 15.85.12 we have $g : K \to K$ is zero in $D(B)$. Set $J = (f_1, \ldots, f_m)$. Recall that $NL_{B/A}$ is homotopy equivalent to $J/J^2 \to \bigoplus_{i=1,\ldots,n} B\mathrm{d}x_i$, see Algebra, Section 10.134. Denote L the complex $J/J^2 \to \bigoplus_{i=1,\ldots,m} B\mathrm{d}x_i$ to that we have the quotient map $NL_{B/A} \to L$. We also have a surjective map of complexes $K \to L$ by sending the jth basis element in the term $B^{\oplus m}$ in degree -1 to the class of f_j in J/J^2. Picture

$$NL_{B/A} \to L \leftarrow K$$

From More on Algebra, Lemma 15.85.8 we conclude that multiplication by g on L is 0 in $D(B)$. On the other hand, the distinguished triangle $B^{\oplus n-m}[0] \to NL_{B/A} \to L$ shows that $\mathrm{Ext}^1_B(L, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$ is surjective for every B-module N and hence annihilated by g. This proves part (1). If $n = m$ then $NL_{B/A} = L$ and we see that (2) holds. □

0GAW **Lemma 88.7.2.** *Let I be an ideal of a Noetherian ring A. Let B be an object of (88.2.0.2). Let $B = A[x_1, \ldots, x_r]^\wedge/J$ be a presentation. Assume there exists an element $b \in B$, $0 \leq m \leq r$, and $f_1, \ldots, f_m \in J$ such that*

(1) *$V(b) \subset V(IB)$ in $\mathrm{Spec}(B)$,*
(2) *the image of $\Delta = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ in B divides b, and*
(3) *$bJ \subset (f_1, \ldots, f_m) + J^2$.*

Then there exists a finite type A-algebra C and an A-algebra isomorphism $B \cong C^\wedge$.

Proof. The conditions imply that B is rig-smooth over (A, I), see Lemma 88.4.2. Write $b'\Delta = b$ in B for some $b' \in B$. Say $I = (a_1, \ldots, a_t)$. Since $V(b) \subset V(IB)$ there exists an integer $c \geq 0$ such that $I^c B \subset bB$. Write $bb_i = a_i^c$ in B for some $b_i \in B$.

Choose an integer $n \gg 0$ (we will see later how large). Choose polynomials $f'_1, \ldots, f'_m \in A[x_1, \ldots, x_r]$ such that $f_i - f'_i \in I^n A[x_1, \ldots, x_r]^\wedge$. We set $\Delta' =$

$\det_{1\leq i,j\leq m}(\partial f'_j/\partial x_i)$ and we consider the finite type A-algebra

$$C = A[x_1, \ldots, x_r, z_1, \ldots, z_t]/(f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c)$$

We will apply Lemma 88.7.1 to C. We compute

$$\det \begin{pmatrix} \text{matrix of partials of} \\ f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c \\ \text{with respect to the variables} \\ x_1, \ldots, x_m, z_1, \ldots, z_t \end{pmatrix} = (\Delta')^{t+1}$$

Hence we see that $\mathrm{Ext}^1_C(NL_{C/A}, N)$ is annihilated by $(\Delta')^{t+1}$ for all C-modules N. Since a_i^c is divisible by Δ' in C we see that $a_i^{(t+1)c}$ annihilates these Ext^1's also. Thus I^{c_1} annihilates $\mathrm{Ext}^1_C(NL_{C/A}, N)$ for all C-modules N where $c_1 = 1+t((t+1)c-1)$. The exact value of c_1 doesn't matter for the rest of the argument; what matters is that it is independent of n.

Since $NL^\wedge_{C^\wedge/A} = NL_{C/A} \otimes_C C^\wedge$ by Lemma 88.3.2 we conclude that multiplication by I^{c_1} is zero on $\mathrm{Ext}^1_{C^\wedge}(NL^\wedge_{C^\wedge/A}, N)$ for any $C^\wedge$-module N as well, see More on Algebra, Lemmas 15.85.7 and 15.85.6. In particular $C^\wedge$ is rig-smooth over (A, I).

Observe that we have a surjective A-algebra homomorphism

$$\psi_n : C \longrightarrow B/I^nB$$

sending the class of x_i to the class of x_i and sending the class of z_i to the class of b_ib'. This works because of our choices of b' and b_i in the first paragraph of the proof.

Let $d = d(\mathrm{Gr}_I(B))$ and $q_0 = q(\mathrm{Gr}_I(B))$ be the integers found in Local Cohomology, Section 51.22. By Lemma 88.5.3 if we take $n \geq \max(q_0 + (d+1)c_1, 2(d+1)c_1 + 1)$ we can find a homomorphism $\varphi : C^\wedge \to B$ of A-algebras which is congruent to ψ_n modulo $I^{n-(d+1)c_1}B$.

Since $\varphi : C^\wedge \to B$ is surjective modulo I we see that it is surjective (for example use Algebra, Lemma 10.96.1). To finish the proof it suffices to show that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of I, see More on Algebra, Lemma 15.109.4.

Since φ is surjective we see that $NL^\wedge_{B/C^\wedge}$ has cohomology modules $H^0(NL^\wedge_{B/C^\wedge}) = 0$ and $H^{-1}(NL^\wedge_{B/C^\wedge}) = \mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$. We have an exact sequence

$$H^{-1}(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^{-1}(NL^\wedge_{B/A}) \to H^{-1}(NL^\wedge_{B/C^\wedge}) \to H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A}) \to 0$$

by Lemma 88.3.5. The first two modules are annihilated by a power of I as B and $C^\wedge$ are rig-smooth over (A, I). Hence it suffices to show that the kernel of the surjective map $H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A})$ is annihilated by a power of I. For this it suffices to show that it is annihilated by a power of b. In other words, it suffices to show that

$$H^0(NL^\wedge_{C^\wedge/A}) \otimes_{C^\wedge} B[1/b] \longrightarrow H^0(NL^\wedge_{B/A}) \otimes_B B[1/b]$$

is an isomorphism. However, both are free $B[1/b]$ modules of rank $r-m$ with basis $\mathrm{d}x_{m+1}, \ldots, \mathrm{d}x_r$ and we conclude the proof. $\square$

0GAX **Remark 88.7.3.** Let I be an ideal of a Noetherian ring A. Let B be an object of (88.2.0.2) which is rig-smooth over (A, I). It is shown in [**GZ24**, Theorem 1.2] that B is isomorphic to the I-adic completion of a finite type A-algebra. This result supercedes the following list of partial results:

(1) If A is a G-ring, then the result follows from Proposition 88.6.3.
(2) If B is rig-étale over (A, I), then the result follows from Lemma 88.10.2.
(3) If I is principal, then the result follows from [**Elk73**, III Theorem 7].

88.8. Rig-étale algebras

0ALP In view of our definition of rig-smooth algebras (Definition 88.4.1), the following definition should not come as a surprise.

0GAY **Definition 88.8.1.** Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). We say B is *rig-étale over* (A, I) if there exists an integer $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$.

Condition (7) in the next lemma is one of the conditions used in [**Art70**] to define formal modifications. We have added it to the list of conditions to facilitate comparison with our conditions later on.

0AJU **Lemma 88.8.2.** *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 88.2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B dx_i)$ be its naive cotangent complex (88.3.0.1). The following are equivalent*

(1) *B is rig-étale over (A, I),*
0AJV (2) *there exists a $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,*
0AJW (3) *there exits a $c \geq 0$ such that $H^i(NL^\wedge_{B/A})$, $i = -1, 0$ is annihilated by I^c,*
0AJX (4) *there exists a $c \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihilated by I^c for all $n \geq 1$ where $A_n = A/I^n$ and $B_n = B/I^nB$,*
0GAZ (5) *for every $a \in I$ there exists a $c \geq 0$ such that*
   (a) *a^c annihilates $H^0(NL^\wedge_{B/A})$, and*
   (b) *there exist $f_1, \ldots, f_r \in J$ such that $a^cJ \subset (f_1, \ldots, f_r) + J^2$.*
0GB0 (6) *for every $a \in I$ there exist $f_1, \ldots, f_r \in J$ and $c \geq 0$ such that*
   (a) *$\det_{1 \leq i,j \leq r}(\partial f_j/\partial x_i)$ divides a^c in B, and*
   (b) *$a^cJ \subset (f_1, \ldots, f_r) + J^2$.*
0AJY (7) *choosing generators $f_1, \ldots, f_t$ for J we have*
   (a) *the Jacobian ideal of B over A, namely the ideal in B generated by the $r \times r$ minors of the matrx $(\partial f_j/\partial x_i)_{1 \leq i \leq r, 1 \leq j \leq t}$, contains the ideal I^cB for some c, and*
   (b) *the Cramer ideal of B over A, namely the ideal in B generated by the image in B of the rth Fitting ideal of J as an $A[x_1, \ldots, x_r]^\wedge$-module, contains I^cB for some c.*

Proof. The equivalence of (1) and (2) is a restatement of Definition 88.8.1.

The equivalence of (2) and (3) follows from More on Algebra, Lemma 15.85.11.

The equivalence of (3) and (4) follows from the fact that the systems $\{NL_{B_n/A_n}\}$ and $NL^\wedge_{B/A} \otimes_B B_n$ are strictly isomorphic, see Lemma 88.3.3. Some details omitted.

Assume (2). Let $a \in I$. Let c be such that multiplication by a^c is zero on $NL^\wedge_{B/A}$. By More on Algebra, Lemma 15.85.4 part (1) there exists a map $\alpha : \bigoplus B\mathrm{d}x_i \to J/J^2$ such that $\mathrm{d} \circ \alpha$ and $\alpha \circ \mathrm{d}$ are both multiplication by a^c. Let $f_i \in J$ be an element whose class modulo J^2 is equal to $\alpha(\mathrm{d}x_i)$. A simple calculation gives that (6)(a), (b) hold.

We omit the verification that (6) implies (5); it is just a statement on two term complexes over B of the form $M \to B^{\oplus r}$.

Assume (5) holds. Say $I = (a_1, \ldots, a_t)$. Let $c_i \geq 0$ be the integer such that (5)(a), (b) hold for $a_i^{c_i}$. Then we see that $I^{\sum c_i}$ annihilates $H^0(NL^\wedge_{B/A})$. Let $f_{i,1}, \ldots, f_{i,r} \in J$ be as in (5)(b) for a_i. Consider the composition

$$B^{\oplus r} \to J/J^2 \to \bigoplus B\mathrm{d}x_i$$

where the jth basis vector is mapped to the class of $f_{i,j}$ in J/J^2. By (5)(a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma 10.16.4). Thus the kernel of $B^{\oplus r} \to \bigoplus B\mathrm{d}x_i$ is a_i-power torsion, and hence $H^{-1}(NL^\wedge_{B/A}) = \mathrm{Ker}(J/J^2 \to \bigoplus B\mathrm{d}x_i)$ is a_i-power torsion. Since B is Noetherian (Lemma 88.2.2), all modules including $H^{-1}(NL^\wedge_{B/A})$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ for some $d_i \geq 0$. It follows that $I^{\sum d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ and we see that (3) holds.

Thus conditions (2), (3), (4), (5), and (6) are equivalent. Thus it remains to show that these conditions are equivalent with (7). Observe that the Cramer ideal $\mathrm{Fit}_r(J)B$ is equal to $\mathrm{Fit}_r(J/J^2)$ as $J/J^2 = J \otimes_{A[x_1,\ldots,x_r]^\wedge} B$, see More on Algebra, Lemma 15.8.4 part (3). Also, observe that the Jacobian ideal is just $\mathrm{Fit}_0(H^0(NL^\wedge_{B/A}))$. Thus we see that the equivalence of (3) and (7) is a purely algebraic question which we discuss in the next paragraph.

Let R be a Noetherian ring and let $I \subset R$ be an ideal. Let $M \xrightarrow{d} R^{\oplus r}$ be a two term complex. We have to show that the following are equivalent

(A) the cohomology of $M \to R^{\oplus r}$ is annihilated by a power of I, and
(B) the ideals $\mathrm{Fit}_r(M)$ and $\mathrm{Fit}_0(\mathrm{Coker}(d))$ contain a power of I.

Since R is Noetherian, we can reformulate part (2) as an inclusion of the corresponding closed subschemes, see Algebra, Lemmas 10.17.2 and 10.32.5. On the other hand, over the complement of $V(\mathrm{Fit}_0(\mathrm{Coker}(d)))$ the cokernel of d vanishes and over the complement of $V(\mathrm{Fit}_r(M))$ the module M is locally generated by r elements, see More on Algebra, Lemma 15.8.7. Thus (B) is equivalent to

(C) away from $V(I)$ the cokernel of d vanishes and the module M is locally generated by $\leq r$ elements.

Of course this is equivalent to the condition that $M \to R^{\oplus r}$ has vanishing cohomology over $\mathrm{Spec}(R) \setminus V(I)$ which in turn is equivalent to (A). This finishes the proof. □

0GB1 **Lemma 88.8.3.** *Let A be a Noetherian ring and let I be an ideal. Let B be an object of (88.2.0.2). If B is rig-étale over (A, I), then B is rig-smooth over (A, I).*

Proof. Immediate from Definitions 88.4.1 and 88.8.1. □

0ALQ **Lemma 88.8.4.** *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.*

(1) *If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$, then $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2.*
(2) *If $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2, then there exists $g \in 1 + IB$ such that $\mathrm{Spec}(B_g)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$.*

Proof. Assume $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2. The naive cotangent complex $NL_{B/A}$ is a complex of finite type B-modules and hence H^{-1} and H^0 are finite B-modules. Completion is an exact functor on finite B-modules (Algebra, Lemma 10.97.2) and $NL^\wedge_{B^\wedge/A}$ is the completion of the complex $NL_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that H^{-1}/I^nH^{-1} and H^0/I^nH^0 are annihilated by I^c for all n. By Nakayama's lemma (Algebra, Lemma 10.20.1) this means that I^cH^{-1} and I^cH^0 are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting g (which does not change the quotients B/I^nB) we see that $NL_{B/A}$ has cohomology annihilated by I^c. Thus $A \to B$ is étale at any prime of B not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 10.143.1.

Conversely, assume that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by a^c is zero $NL_{B/A}$. Since $NL^\wedge_{B^\wedge/A}$ is the derived completion of $NL_{B/A}$ (see Lemma 88.3.3) it follows that $B^\wedge$ satisfies the equivalent conditions of Lemma 88.8.2. □

0AK2 **Lemma 88.8.5.** *Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 88.2.3 with A_1 and A_2 Noetherian. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. If multiplication by $f_1 \in B_1$ on $NL^\wedge_{B_1/A_1}$ is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on $NL^\wedge_{B_2/A_2}$ is zero in $D(B_2)$.*

Proof. By Lemma 88.3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemma 15.85.8. □

0GB2 **Lemma 88.8.6.** *Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1A_2) = V(I_2)$. Let B_1 be in (88.2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 88.2.3. If B_1 is rig-étale over (A_1, I_1), then B_2 is rig-étale over (A_2, I_2).*

Proof. Follows from Lemma 88.8.5 and Definition 88.8.1 and the fact that $I_2^c \subset I_1A_2$ for some $c \geq 0$ as A_2 is Noetherian. □

0AKJ **Lemma 88.8.7.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be a finite type A-algebra such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$. Let C be a Noetherian A-algebra. Then any A-algebra map $B^\wedge \to C^\wedge$ of I-adic completions comes from a unique A-algebra map*

$$B \longrightarrow C^h$$

where C^h is the henselization of the pair (C, IC) as in More on Algebra, Lemma 15.12.1. Moreover, any A-algebra homomorphism $B \to C^h$ factors through some étale C-algebra C' such that $C/IC \to C'/IC'$ is an isomorphism.

Proof. Uniqueness follows from the fact that C^h is a subring of $C^\wedge$, see for example More on Algebra, Lemma 15.12.4. The final assertion follows from the fact that C^h is the filtered colimit of these C-algebras C', see proof of More on Algebra, Lemma 15.12.1. Having said this we now turn to the proof of existence.

Let $\varphi : B^\wedge \to C^\wedge$ be the given map. This defines a section

$$\sigma : (B \otimes_A C)^\wedge \longrightarrow C^\wedge$$

of the completion of the map $C \to B \otimes_A C$. We may replace (A, I, B, C, φ) by $(C, IC, B \otimes_A C, C, \sigma)$. In this way we see that we may assume that $A = C$.

Proof of existence in the case $A = C$. In this case the map $\varphi : B^\wedge \to A^\wedge$ is necessarily surjective. By Lemmas 88.8.4 and 88.3.5 we see that the cohomology groups of $NL^\wedge_{A^\wedge/_\varphi B^\wedge}$ are annihilated by a power of I. Since φ is surjective, this implies that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of I. Hence $\varphi : B^\wedge \to A^\wedge$ is the completion of a finite type B-algebra $B \to D$, see More on Algebra, Lemma 15.109.4. Hence $A \to D$ is a finite type algebra map which induces an isomorphism $A^\wedge \to D^\wedge$. By Lemma 88.8.4 we may replace D by a localization and assume that $A \to D$ is étale away from $V(I)$. Since $A^\wedge \to D^\wedge$ is an isomorphism, we see that $\mathrm{Spec}(D) \to \mathrm{Spec}(A)$ is also étale in a neighbourhood of $V(ID)$ (for example by More on Morphisms, Lemma 37.12.3). Thus $\mathrm{Spec}(D) \to \mathrm{Spec}(A)$ is étale. Therefore D maps to A^h and the lemma is proved. □

88.9. A pushout argument

0AK8 The only goal in this section is to prove the following lemma which will play a key role in algebraization of rig-étale algebras. We will use a bit of the theory of algebraic spaces to prove this lemma; an earlier version of this chapter gave a (much longer) proof using algebra and a bit of deformation theory that the interested reader can find in the history of the Stacks project.

0ALT **Lemma 88.9.1.** *Let A be a Noetherian ring and $I \subset A$ an ideal. Let $J \subset A$ be a nilpotent ideal. Consider a commutative diagram*

$$\begin{array}{ccccc} C & \longrightarrow & C_0 & = & C/JC \\ \uparrow & & \uparrow & & \\ & & B_0 & & \\ & & \uparrow & & \\ A & \longrightarrow & A_0 & = & A/J \end{array}$$

whose vertical arrows are of finite type such that

(1) $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is étale over* $\mathrm{Spec}(A) \setminus V(I)$,
(2) $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ *is étale over* $\mathrm{Spec}(A_0) \setminus V(IA_0)$, *and*
(3) $B_0 \to C_0$ *is étale and induces an isomorphism* $B_0/IB_0 = C_0/IC_0$.

Then we can fill in the diagram above to a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & C/JC \\ \uparrow & & \uparrow \\ B & \longrightarrow & B_0 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A/J \end{array}$$

with $A \to B$ *of finite type,* $B/JB = B_0$, $B \to C$ *étale, and* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *étale over* $\mathrm{Spec}(A) \setminus V(I)$.

Proof. Set $X = \mathrm{Spec}(A)$, $X_0 = \mathrm{Spec}(A_0)$, $Y_0 = \mathrm{Spec}(B_0)$, $Z = \mathrm{Spec}(C)$, $Z_0 = \mathrm{Spec}(C_0)$. Furthermore, denote $U \subset X$, $U_0 \subset X_0$, $V_0 \subset Y_0$, $W \subset Z$, $W_0 \subset Z_0$ the complement of the vanishing set of I. Here is a picture to help visualize the situation:

$$\begin{array}{ccc} Z & \longleftarrow & Z_0 \\ \downarrow & & \downarrow \\ & & Y_0 \\ & & \downarrow \\ X & \longleftarrow & X_0 \end{array} \qquad \begin{array}{ccc} W & \longleftarrow & W_0 \\ \downarrow & & \downarrow \\ & & V_0 \\ & & \downarrow \\ U & \longleftarrow & U_0 \end{array}$$

The conditions in the lemma guarantee that

$$\begin{array}{ccc} W_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & Y_0 \end{array}$$

is an elementary distinguished square, see Derived Categories of Spaces, Definition 75.9.1. In addition we know that $W_0 \to U_0$ and $V_0 \to U_0$ are étale. The morphism $X_0 \subset X$ is a finite order thickening as J is assumed nilpotent. By the topological invariance of the étale site we can find a unique étale morphism $V \to X$ of schemes with $V_0 = V \times_X X_0$ and we can lift the given morphism $W_0 \to V_0$ to a unique morphism $W \to V$ over X. See Étale Morphisms, Theorem 41.15.2. Since $W_0 \to V_0$ is separated, the morphism $W \to V$ is separated too, see for example More on Morphisms, Lemma 37.10.3. By Pushouts of Spaces, Lemma 81.9.2 we can construct an elementary distinguished square

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

in the category of algebraic spaces over X. Since the base change of an elementary distinguished square is an elementary distinguished square (Derived Categories of

Spaces, Lemma 75.9.2) we see that

$$\begin{array}{ccc} W_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & Y \times_X X_0 \end{array}$$

is an elementary distinguished square. It follows that there is a unique isomorphism $Y \times_X X_0 = Y_0$ compatible with the two squares involving these spaces because elementary distinguished squares are pushouts (Pushouts of Spaces, Lemma 81.9.1). It follows that Y is affine by Limits of Spaces, Proposition 70.15.2. Write $Y = \mathrm{Spec}(B)$. It is clear that B fits into the desired diagram and satisfies all the properties required of it. □

88.10. Algebraization of rig-étale algebras

0AK5 The main goal is to prove algebraization for rig-étale algebras when the underlying Noetherian ring A is not assumed to be a G-ring and when the ideal $I \subset A$ is arbitrary – not necessarily principal. We first prove the principal ideal case and then use the result of Section 88.9 to finish the proof.

0ALS **Lemma 88.10.1.** *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (88.2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.* The rig-étale case of [**Elk73**, III Theorem 7]

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. By Lemma 88.8.2 part (6) we can find $c \geq 0$ and $f_1, \ldots, f_r \in J$ such that $\det_{1 \leq i,j \leq r}(\partial f_j/\partial x_i)$ divides a^c in B and $a^c J \subset (f_1, \ldots, f_r) + J^2$. Hence Lemma 88.7.2 applies. This finishes the proof, but we'd like to point out that in this case the use of Lemma 88.5.3 can be replaced by the much easier Lemma 88.5.5. □

0AKA **Lemma 88.10.2.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an object of (88.2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.*

Proof. We prove this lemma by induction on the number of generators of I. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 88.10.1. Assume $t > 1$.

For any $m \geq 1$ set $\bar{A}_m = A/(a_t^m)$. Consider the ideal $\bar{I}_m = (\bar{a}_1, \ldots, \bar{a}_{t-1})$ in $\bar{A}_m$. Observe that $V(I\bar{A}_m) = V(\bar{I}_m)$. Let $B_m = B/(a_t^m)$ be the base change of B for the map $(A, I) \to (\bar{A}_m, \bar{I}_m)$, see Remark 88.2.4. By Lemma 88.8.6 we find that B_m is rig-étale over $(\bar{A}_m, \bar{I}_m)$.

By induction hypothesis (on t) we can find a finite type $\bar{A}_m$-algebra C_m and a map $C_m \to B_m$ which induces an isomorphism $C_m^\wedge \cong B_m$ where the completion is with respect to $\bar{I}_m$. By Lemma 88.8.4 we may assume that $\mathrm{Spec}(C_m) \to \mathrm{Spec}(\bar{A}_m)$ is étale over $\mathrm{Spec}(\bar{A}_m) \setminus V(\bar{I}_m)$.

We claim that we may choose $A_m \to C_m \to B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_t^{m-1}) \to C_{m-1}$ compatible with the given A-algebra structure and the maps to $B_{m-1} = B_m/(a_t^{m-1})$. Namely, first fix a choice of $A_1 \to C_1 \to B_1$. Suppose we have found $C_{m-1} \to C_{m-2} \to \ldots \to C_1$ with the desired properties. Note that $C_m/(a_t^{m-1})$ is étale over $\mathrm{Spec}(\bar{A}_{m-1}) \setminus V(\bar{I}_{m-1})$.

Hence by Lemma 88.8.7 there exists an étale extension $C_{m-1} \to C'_{m-1}$ which induces an isomorphism modulo $\bar{I}_{m-1}$ and an $\bar{A}_{m-1}$-algebra map $C_m/(a_t^{m-1}) \to C'_{m-1}$ inducing the isomorphism $B_m/(a_t^{m-1}) \to B_{m-1}$ on completions. Note that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale over the complement of $V(\bar{I}_{m-1})$ by Morphisms, Lemma 29.36.18 and over $V(\bar{I}_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 37.12.3). Thus $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 41.15.2) there exists an étale ring map $C_m \to C'_m$ such that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is isomorphic to $C_m/(a_t^{m-1}) \to C'_m/(a_t^{m-1})$. Observe that the $\bar{I}_m$-adic completion of C'_m is equal to the $\bar{I}_m$-adic completion of C_m, i.e., to B_m (details omitted). We apply Lemma 88.9.1 to the diagram

$$\begin{array}{ccc} C'_m & \longrightarrow & C'_m/(a_t^{m-1}) \\ \uparrow & \nearrow & \uparrow \\ C''_m & \dashrightarrow & C_{m-1} \\ & \nwarrow & \uparrow \\ \bar{A}_m & \longrightarrow & \bar{A}_{m-1} \end{array}$$

to see that there exists a "lift" of C''_m of C_{m-1} to an algebra over $\bar{A}_m$ with all the desired properties.

By construction (C_m) is an object of the category (88.2.0.1) for the principal ideal (a_t). Thus the inverse limit $B' = \lim C_m$ is an (a_t)-adically complete A-algebra such that B'/a_tB' is of finite type over $A/(a_t)$, see Lemma 88.2.1. By construction we have $C_m = B'/(a_t^m)$, the I-adic completion of C_m is B_m, and the I-adic completion of B' is isomorphic to B. For each m the complex NL_{C_m/A_m} has finite cohomology modules supported on $V(IC_m) \subset \mathrm{Spec}(C_m)$ by construction. Hence these modules are I-adically complete (as these modules are annihilated by a power of I). Since $NL^\wedge_{B_m/A_m}$ is the I-adic completion of NL_{C_m/A_m}, see Lemma 88.3.2, it follows that the map $NL_{C_m/A_m} \to NL_{B_m/A_m}$ induces an isomorphism on cohomologies. Thus because B is rig étale over A it follows that there is a fixed power of I that annihilates the cohomologies of NL_{C_m/A_m} for all m, see Lemma 88.8.2. It follows that B' is rig étale over $(A, (a_t))$ by the same lemma. Hence finally, we may apply Lemma 88.10.1 to B' over $(A, (a_t))$ to finish the proof. □

0AKG **Lemma 88.10.3.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra with $A/I \to B/IB$ of finite type. The equivalent conditions of Lemma 88.8.2 are also equivalent to*

0AKH (8) *there exists a finite type A-algebra C such that $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus V(I)$ and such that $B \cong C^\wedge$.*

Proof. Combine Lemmas 88.8.2, 88.10.2, and 88.8.4. Small detail omitted. □

88.11. Finite type morphisms

0GBV In Formal Spaces, Section 87.24 we have defined finite type morphisms of formal algebraic spaces. In this section we study the corresponding types of continuous ring

maps of adic topological rings which have a finitely generated ideal of definition. We strongly suggest the reader skip this section.

0GBW **Lemma 88.11.1.** *Let A and B be adic topological rings which have a finitely generated ideal of definition. Let $\varphi : A \to B$ be a continuous ring homomorphism. The following are equivalent:*

(1) *φ is adic and B is topologically of finite type over A,*
(2) *φ is taut and B is topologically of finite type over A,*
(3) *there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and there exist an ideal of definition $I' \subset A$ such that $A/I' \to B/I'B$ is of finite type,*
(4) *for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and $A/I \to B/IB$ is of finite type,*
(5) *there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and B is in the category (88.2.0.2),*
(6) *for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and B is in the category (88.2.0.2),*
(7) *B as a topological A-algebra is the quotient of $A\{x_1, \ldots, x_r\}$ by a closed ideal,*
(8) *B as a topological A-algebra is the quotient of $A[x_1, \ldots, x_r]^\wedge$ by a closed ideal where $A[x_1, \ldots, x_r]^\wedge$ is the completion of $A[x_1, \ldots, x_r]$ with respect to some ideal of definition of A, and*
(9) *add more here.*

Moreover, these equivalent conditions define a local property of morphisms of WAdm^{adic*} as defined in Formal Spaces, Remark 87.21.4.

Proof. Taut ring homomorphisms are defined in Formal Spaces, Definition 87.5.1. Adic ring homomorphisms are defined in Formal Spaces, Definition 87.6.1. The lemma follows from a combination of Formal Spaces, Lemmas 87.29.6, 87.29.7, and 87.23.1. We omit the details. To be sure, there is no difference between the topological rings $A[x_1, \ldots, x_n]^\wedge$ and $A\{x_1, \ldots, x_r\}$, see Formal Spaces, Remark 87.28.2. □

0GBX **Remark 88.11.2.** Let $A \to B$ be an arrow of WAdm^{adic*} which is adic and topologically of finite type (see Lemma 88.11.1). Write $B = A\{x_1, \ldots, x_r\}/J$. Then we can set[1]

$$NL^\wedge_{B/A} = \left(J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i\right)$$

Exactly as in the proof of Lemma 88.3.1 the reader can show that this complex of B-modules is well defined up to (unique isomorphism) in the homotopy category $K(B)$. Now, if A is Noetherian and $I \subset A$ is an ideal of definition, then this construction reproduces the naive cotangent complex of B over (A, I) defined by Equation (88.3.0.1) in Section 88.3 simply because $A[x_1, \ldots, x_n]^\wedge$ agrees with $A\{x_1, \ldots, x_r\}$ by Formal Spaces, Remark 87.28.2. In particular, we find that, still when A is an adic Noetherian topological ring, the object $NL^\wedge_{B/A}$ is independent of the choice of the ideal of definition $I \subset A$.

[1] In fact, this construction works for arrows of WAdm^{count} satisfying the equivalent conditions of Formal Spaces, Lemma 87.29.6.

0GBY **Lemma 88.11.3.** *Consider the property P on arrows of $WAdm^{adic*}$ defined in Lemma 88.11.1. Then P is stable under base change as defined in Formal Spaces, Remark 87.21.8.*

Proof. The statement makes sense by Lemma 88.11.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{adic*}$ and that as a topological B-algebra we have $A = B\{x_1, \ldots, x_r\}/J$ for some closed ideal J. Then $A \widehat{\otimes}_B C$ is isomorphic to the quotient of $C\{x_1, \ldots, x_r\}/J'$ where J' is the closure of $JC\{x_1, \ldots, x_r\}$. Some details omitted. □

0GBZ **Lemma 88.11.4.** *Consider the property P on arrows of $WAdm^{adic*}$ defined in Lemma 88.11.1. Then P is stable under composition as defined in Formal Spaces, Remark 87.21.13.*

Proof. The statement makes sense by Lemma 88.11.1. The easiest way to prove it is true is to show that (a) compositions of adic ring maps between adic topological rings are adic and (b) that compositions of continuous ring maps preserves the property of being topologically of finite type. We omit the details. □

The following lemma says that morphisms of adic* formal algebraic spaces are locally of finite type if and only if they are étale locally given by the types of maps of topological rings described in Lemma 88.11.1.

0GC0 **Lemma 88.11.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of $WAdm^{adic*}$ which is adic and topologically of finite type,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to an arrow of $WAdm^{adic*}$ which is adic and topologically of finite type,*

(3) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to an arrow of $WAdm^{adic*}$ which is adic and topologically of finite type, and*

(4) *f is locally of finite type.*

Proof. Immediate consequence of the equivalence of (1) and (2) in Lemma 88.11.1 and Formal Spaces, Lemma 87.29.9. □

88.12. Finite type on reductions

0GC1 In this section we talk a little bit about morphisms $X \to Y$ of locally countably indexed formal algebraic spaces such that $X_{red} \to Y_{red}$ is locally of finite type.

We will translate this into an algebraic condition. To understand this algebraic condition it pays to keep in mind the following:

- If A is a weakly admissible topological ring, then the set $\mathfrak{a} \subset A$ of topological nilpotent elements is an open, radical ideal and $\mathrm{Spf}(A)_{red} = \mathrm{Spec}(A/\mathfrak{a})$.

See Formal Spaces, Definition 87.4.8, Lemma 87.4.10, and Example 87.12.2.

0GC2 **Lemma 88.12.1.** *For an arrow $\varphi : A \to B$ in $WAdm^{count}$ consider the property $P(\varphi) =$"the induced ring homomorphism $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type" where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the ideals of topologically nilpotent elements. Then P is a local property as defined in Formal Spaces, Situation 87.21.2.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & (B')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & (A')^\wedge \end{array}$$

as in Formal Spaces, Situation 87.21.2. Taking Spf of this diagram we obtain

$$\begin{array}{ccc} \mathrm{Spf}(B) & \longleftarrow & \mathrm{Spf}((B')^\wedge) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(A) & \longleftarrow & \mathrm{Spf}((A')^\wedge) \end{array}$$

of affine formal algebraic spaces whose horizontal arrows are representable by algebraic spaces and étale by Formal Spaces, Lemma 87.19.13. Hence we obtain a commutative diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spf}(B)_{red} & \xleftarrow[g]{} & \mathrm{Spf}((B')^\wedge)_{red} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathrm{Spf}(A)_{red} & \longleftarrow & \mathrm{Spf}((A')^\wedge)_{red} \end{array}$$

whose horizontal arrows are étale by Formal Spaces, Lemma 87.12.3. By Formal Spaces, Example 87.12.2 and Lemma 87.19.14 conditions (1), (2), and (3) of Formal Spaces, Situation 87.21.2 translate into the following statements

(1) if f is locally of finite type, then f' is locally of finite type,
(2) if f' is locally of finite type and g is surjective, then f is locally of finite type, and
(3) if $T_i \to S$, $i = 1, \ldots, n$ are locally of finite type, then $\coprod_{i=1,\ldots,n} T_i \to S$ is locally of finite type.

Properties (1) and (2) follow from the fact that being locally of finite type is local on the source and target in the étale topology, see discussion in Morphisms of Spaces, Section 67.23. Property (3) is a straightforward consequence of the definition. □

0GC3 **Lemma 88.12.2.** *Consider the property P on arrows of $WAdm^{count}$ defined in Lemma 88.12.1. Then P is stable under base change (Formal Spaces, Situation 87.21.6).*

Proof. The statement makes sense by Lemma 88.12.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{count}$ such that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type where $\mathfrak{b} \subset B$ and $\mathfrak{a} \subset A$ are the ideals of topologically nilpotent elements. Since A and B are weakly admissible, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are open. Let $\mathfrak{c} \subset C$ be the (open) ideal of topologically nilpotent elements. Then we find a surjection $A\widehat{\otimes}_B C \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ whose kernel is a weak ideal of definition and hence consists of topologically nilpotent elements (please compare with the proof of Formal Spaces, Lemma 87.4.12). Since already $C/\mathfrak{c} \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ is of finite type as a base change of $B/\mathfrak{b} \to A/\mathfrak{a}$ we conclude. □

0GC4 **Lemma 88.12.3.** *Consider the property P on arrows of $WAdm^{count}$ defined in Lemma 88.12.1. Then P is stable under composition (Formal Spaces, Situation 87.21.11).*

Proof. Omitted. Hint: compositions of finite type ring maps are of finite type. □

0GC5 **Lemma 88.12.4.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{count}$. If φ is taut and topologically of finite type, then φ satisfies the condition defined in Lemma 88.12.1.*

Proof. This is an easy consequence of the definitions. □

0GC6 **Lemma 88.12.5.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ satisfying the condition defined in Lemma 88.12.1. Then $A \to B$ is topologically of finite type.*

Proof. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Choose $b_1, \ldots, b_r \in B$ which map to generators of $B/\mathfrak{b}$ over A. Choose generators $b_{r+1}, \ldots, b_s$ of the ideal $\mathfrak{b}$. We claim that the image of

$$\varphi : A[x_1, \ldots, x_s] \longrightarrow B, \quad x_i \longmapsto b_i$$

has dense image. Namely, if $b \in \mathfrak{b}^n$ for some $n \geq 0$, then we can write $b = \sum b_E b_{r+1}^{e_{r+1}} \ldots b_s^{e_s}$ for multiindices $E = (e_{r+1}, \ldots, e_s)$ with $|E| = \sum e_i = n$ and $b_E \in B$. Next, we can write $b_E = f_E(b_1, \ldots, b_r) + b'_E$ with $b'_E \in \mathfrak{b}$ and $f_E \in A[x_1, \ldots, x_r]$. Combined we obtain $b \in \mathrm{Im}(\varphi) + \mathfrak{b}^{n+1}$. By induction we see that $B = \mathrm{Im}(\varphi) + \mathfrak{b}^n$ for all $n \geq 0$ which mplies what we want as $\mathfrak{b}$ is an ideal of definition of B. □

0GG6 **Lemma 88.12.6.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$. If φ is adic the following are equivalent*

(1) *φ satisfies the condition defined in Lemma 88.12.1 and*
(2) *φ satisfies the condition defined in Lemma 88.11.1.*

Proof. Omitted. Hint: For the proof of (1) $\Rightarrow$ (2) use Lemma 88.12.5. □

0GC7 **Lemma 88.12.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of $WAdm^{count}$ satisfying the property defined in Lemma 88.12.1,

(2) *there exists a covering* $\{Y_j \to Y\}$ *as in Formal Spaces, Definition 87.11.1 and for each* j *a covering* $\{X_{ji} \to Y_j \times_Y X\}$ *as in Formal Spaces, Definition 87.11.1 such that each* $X_{ji} \to Y_j$ *corresponds to an arrow of* $WAdm^{count}$ *satisfying the property defined in Lemma 88.12.1,*
(3) *there exist a covering* $\{X_i \to X\}$ *as in Formal Spaces, Definition 87.11.1 and for each* i *a factorization* $X_i \to Y_i \to Y$ *where* Y_i *is an affine formal algebraic space,* $Y_i \to Y$ *is representable by algebraic spaces and étale, and* $X_i \to Y_i$ *corresponds to an arrow of* $WAdm^{count}$ *satisfying the property defined in Lemma 88.12.1, and*
(4) *the morphism* $f_{red} : X_{red} \to Y_{red}$ *is locally of finite type.*

Proof. The equivalence of (1), (2), and (3) follows from Lemma 88.12.1 and an application of Formal Spaces, Lemma 87.21.3. Let Y_j and X_{ji} be as in (2). Then

- The families $\{Y_{j,red} \to Y_{red}\}$ and $\{X_{ji,red} \to X_{red}\}$ are étale coverings by affine schemes. This follows from the discussion in the proof of Formal Spaces, Lemma 87.12.1 or directly from Formal Spaces, Lemma 87.12.3.
- If $X_{ji} \to Y_j$ corresponds to the morphism $B_j \to A_{ji}$ of $WAdm^{count}$, then $X_{ji,red} \to Y_{j,red}$ corresponds to the ring map $B_j/\mathfrak{b}_j \to A_{ji}/\mathfrak{a}_{ji}$ where $\mathfrak{b}_j$ and $\mathfrak{a}_{ji}$ are the ideals of topologically nilpotent elements. This follows from Formal Spaces, Example 87.12.2. Hence $X_{ji,red} \to Y_{j,red}$ is locally of finite type if and only if $B_j \to A_{ji}$ satisfies the property defined in Lemma 88.12.1.

The equivalence of (2) and (4) follows from these remarks because being locally of finite type is a property of morphisms of algebraic spaces which is étale local on source and target, see discussion in Morphisms of Spaces, Section 67.23. □

88.13. Flat morphisms

0GC8 In this section we define flat morphisms of locally Noetherian formal algebraic spaces.

0GC9 **Lemma 88.13.1.** *The property* $P(\varphi) =$*"*φ *is flat" on arrows of* $WAdm^{Noeth}$ *is a local property as defined in Formal Spaces, Remark 87.21.5.*

Proof. Let us recall what the statement signifies. First, $WAdm^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & (B')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & (A')^\wedge \end{array}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is flat $\Rightarrow \varphi'$ is flat,
(2) if $B \to B'$ faithfully flat, then φ' is flat $\Rightarrow \varphi$ is flat, and

(3) if $A \to B_i$ is flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is flat.

We will use without further mention that completions of Noetherian rings are flat (Algebra, Lemma 10.97.2). Since of course $A \to A'$ and $B \to B'$ are flat, we see in particular that the horizontal arrows in the diagram are flat.

Proof of (1). If φ is flat, then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Hence $A' \to (B')^\wedge$ is flat by More on Flatness, Lemma 38.2.3. Hence we see that $(A')^\wedge \to (B')^\wedge$ is flat by applying More on Algebra, Lemma 15.27.5 with $R = A'$, with ideal $I(A')$, and with $M = (B')^\wedge = M^\wedge$.

Proof of (2). Assume φ' is flat and $B \to B'$ is faithfully flat. Then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Also we see that $B \to (B')^\wedge$ is faithfully flat by Formal Spaces, Lemma 87.19.14. Hence by Algebra, Lemma 10.39.9 we find that $\varphi : A \to B$ is flat.

Proof of (3). Omitted. □

0GCA **Lemma 88.13.2.** *Denote P the property of arrows of $WAdm^{Noeth}$ defined in Lemma 88.13.1. Denote Q the property defined in Lemma 88.12.1 viewed as a property of arrows of $WAdm^{Noeth}$. Denote R the property defined in Lemma 88.11.1 viewed as a property of arrows of $WAdm^{Noeth}$. Then*

(1) *P is stable under base change by Q (Formal Spaces, Remark 87.21.10), and*
(2) *$P + R$ is stable under base change (Formal Spaces, Remark 87.21.9).*

Proof. The statement makes sense as each of the properties P, Q, and R is a local property of morphisms of $WAdm^{Noeth}$. Let $\varphi : B \to A$ and $\psi : B \to C$ be morphisms of $WAdm^{Noeth}$. If either $Q(\varphi)$ or $Q(\psi)$ then we see that $A \widehat{\otimes}_B C$ is Noetherian by Formal Spaces, Lemma 87.4.12. Since R implies Q (Lemma 88.12.4), we find that this holds in both cases (1) and (2). This is the first thing we have to check. It remains to show that $C \to A \widehat{\otimes}_B C$ is flat.

Proof of (1). Fix ideals of definition $I \subset A$ and $J \subset B$. By Lemma 88.12.5 the ring map $B \to C$ is topologically of finite type. Hence $B \to C/J^n$ is of finite type for all $n \geq 1$. Hence $A \otimes_B C/J^n$ is Noetherian as a ring (because it is of finite type over A and A is Noetherian). Thus the I-adic completion $A \widehat{\otimes}_B C/J^n$ of $A \otimes_B C/J^n$ is flat over C/J^n because $C/J^n \to A \otimes_B C/J^n$ is flat as a base change of $B \to A$ and because $A \otimes_B C/J^n \to A \widehat{\otimes}_B C/J^n$ is flat by Algebra, Lemma 10.97.2 Observe that $A \widehat{\otimes}_B C/J^n = (A \widehat{\otimes}_B C)/J^n (A \widehat{\otimes}_B C)$; details omitted. We conclude that $M = A \widehat{\otimes}_B C$ is a C-module which is complete with respect to the J-adic topology such that $M/J^n M$ is flat over C/J^n for all $n \geq 1$. This implies that M is flat over C by More on Algebra, Lemma 15.27.4.

Proof of (2). In this case $B \to A$ is adic and hence we have just $A \widehat{\otimes}_B C = \lim A \otimes_B C/J^n$. The rings $A \otimes_B C/J^n$ are Noetherian by an application of Formal Spaces, Lemma 87.4.12 with C replaced by C/J^n. We conclude in the same manner as before. □

0GCB **Lemma 88.13.3.** *Denote P the property of arrows of $WAdm^{Noeth}$ defined in Lemma 88.13.1. Then P is stable under composition (Formal Spaces, Remark 87.21.14).*

Proof. This is true because compositions of flat ring maps are flat. □

0GCC **Definition 88.13.4.** Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *flat* if for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on the source and target.

0GCD **Lemma 88.13.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is flat,*
(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map in WAdm^{Noeth},
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a flat map in WAdm^{Noeth}, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a flat map in WAdm^{Noeth}.*

Proof. The equivalence of (1) and (2) is Definition 88.13.4. The equivalence of (2), (3), and (4) follows from the fact that being flat is a local property of arrows of WAdm^{Noeth} by Lemma 88.13.1 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GCE **Lemma 88.13.6.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S.*

(1) *If f is flat and $g_{red} : Z_{red} \to Y_{red}$ is locally of finite type, then the base change $X \times_Y Z \to Z$ is flat.*
(2) *If f is flat and locally of finite type, then the base change $X \times_Y Z \to Z$ is flat and locally of finite type.*

Proof. Part (1) follows from a combination of Formal Spaces, Remark 87.21.10, Lemma 88.13.2 part (1), Lemma 88.13.5, and Lemma 88.12.7.

Part (2) follows from a combination of Formal Spaces, Remark 87.21.9, Lemma 88.13.2 part (2), Lemma 88.13.5, and Lemma 88.11.5. □

0GCF **Lemma 88.13.7.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are flat, then so is $g \circ f$.*

Proof. Combine Formal Spaces, Remark 87.21.14 and Lemma 88.13.3. □

0GCG **Lemma 88.13.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphisms of locally Noetherian formal algebraic spaces over S. If f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 80.4.1, then f is flat in the sense of Definition 88.13.4.*

Proof. This is a sanity check whose proof should be trivial but isn't quite. We urge the reader to skip the proof. Assume f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 80.4.1. Consider a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. Then the morphism $U \to V$ corresponds to a taut map $B \to A$ of $WAdm^{Noeth}$ by Formal Spaces, Lemma 87.22.2. Observe that this means $B \to A$ is adic (Formal Spaces, Lemma 87.23.1) and in particular for any ideal of definition $J \subset B$ the topology on A is the J-adic topology and the diagrams

$$\begin{array}{ccc} \mathrm{Spec}(A/J^nA) & \longrightarrow & \mathrm{Spec}(B/J^n) \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

are cartesian.

Let $T \to V$ is a morphism where T is a scheme. Then

$$\begin{aligned} X \times_Y T \to T \text{ is flat} &\Rightarrow U \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to V \times_Y T \text{ is flat} \\ &\Rightarrow U \times_V T \to T \text{ is flat} \end{aligned}$$

The first statement is the assumption on f. The first implication because $U \to X$ is étale and hence flat and compositions of flat morphisms of algebraic spaces are flat. The second impliciation because $U \times_Y T = U \times_V V \times_Y T$. The third implication by More on Flatness, Lemma 38.2.3. The fourth implication because we can pullback by the morphism $T \to V \times_Y T$. We conclude that $U \to V$ is flat in the sense of Bootstrap, Definition 80.4.1. In terms of the continuous ring map $B \to A$ this means the ring maps $B/J^n \to A/J^nA$ are flat (see diagram above).

Finally, we can conclude that $B \to A$ is flat for example by More on Algebra, Lemma 15.27.4. □

88.14. Rig-closed points

0GG7 We develop just enough theory to be able to use this for testing rig-flatness in a later section. The reader can find more theory in [**BL93**] who discuss (among other things) the case of locally Noetherian formal schemes.

0GG8 **Lemma 88.14.1.** *Let A be a Noetherian adic topological ring. Let $\mathfrak{q} \subset A$ be a prime ideal. The following are equivalent*

(1) *for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$ and $\mathfrak{q}$ is maximal with respect to this property,*
(2) *for some ideal of definition $I \subset A$ the prime $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(A) \setminus V(I)$,*
(3) *for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$ and $\mathfrak{q}$ is maximal with respect to this property,*
(4) *for any ideal of definition $I \subset A$ the prime $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(A) \setminus V(I)$,*
(5) *$\dim(A/\mathfrak{q}) = 1$ and for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$,*
(6) *$\dim(A/\mathfrak{q}) = 1$ and for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak{q}$,*
(7) *$\dim(A/\mathfrak{q}) = 1$ and the induced topology on $A/\mathfrak{q}$ is nontrivial,*
(8) *$A/\mathfrak{q}$ is a 1-dimensional Noetherian complete local domain whose maximal ideal is the radical of the image of any ideal of definition of A, and*
(9) *add more here.*

Proof. It is clear that (1) and (2) are equivalent and for the same reason that (3) and (4) are equivalent. Since $V(I)$ is independent of the choice of the ideal of definition I of A, we see that (2) and (4) are equivalent.

Assume the equivalent conditions (1) – (4) hold. If $\dim(A/\mathfrak{q}) > 1$ we can choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m} \subset A$ such that $\dim((A/\mathfrak{q})_\mathfrak{m}) > 1$. Then $\mathrm{Spec}((A/\mathfrak{q})_\mathfrak{m}) - V(I(A/\mathfrak{q})_\mathfrak{m})$ would be infinite by Algebra, Lemma 10.61.1. This contradicts the fact that $\mathfrak{q}$ is closed in $\mathrm{Spec}(A) \setminus V(I)$. Hence we see that (6) holds. Trivially (6) implies (5).

Conversely, assume (5) holds. Let $I \subset A$ be an ideal of definition. Since $A/\mathfrak{q}$ is complete with respect to $I(A/\mathfrak{q})$ (for example by Algebra, Lemma 10.97.1) we see that all closed points of $\mathrm{Spec}(A/\mathfrak{q})$ are contained in $V(IA/\mathfrak{q})$ by Algebra, Lemma 10.96.6. Since $\dim(A/\mathfrak{q}) = 1$ and since $I \not\subset \mathfrak{q}$ we conclude two things: (a) $V(IA/\mathfrak{q})$ must contain all points distinct from the generic point of $\mathrm{Spec}(A/\mathfrak{q})$, and (b) $V(IA/\mathfrak{q})$ must be a (finite) discrete set. From (a) we see that $\mathfrak{q}$ is a closed point of $\mathrm{Spec}(A) \setminus V(I)$ and we conclude that (2) holds.

Continuing to assume (5) we see that the finite discrete space $V(IA/\mathfrak{q})$ must be a singleton by More on Algebra, Lemma 15.11.16 for example (and the fact that complete pairs are henselian pairs, see More on Algebra, Lemma 15.11.4). Hence we see that (8) is true. Conversely, it is clear that (8) implies (5).

At this point we know that (1) – (6) and (8) are equivalent. We omit the verification that these are also equivalent to (7). □

In order to comfortably talk about such primes we introduce the following nonstandard notation.

0GG9 **Definition 88.14.2.** Let A be a Noetherian adic topological ring. Let $\mathfrak{q} \subset A$ be a prime ideal. We say $\mathfrak{q}$ is *rig-closed* if the equivalent conditions of Lemma 88.14.1 are satisfied.

We will need a few lemmas which essentially tell us there are plenty of rig-closed primes even in a relative setting.

0GGA **Lemma 88.14.3.** *Let $\varphi : A \to B$ in $WAdm^{Noeth}$. Denote $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ the ideals of topologically nilpotent elements. Assume $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. The residue field κ of the local ring $B/\mathfrak{q}$ is a finite type $A/\mathfrak{a}$-algebra.*

Proof. Let $\mathfrak{q} \subset \mathfrak{m} \subset B$ be the unique maximal ideal containing $\mathfrak{q}$. Then $\mathfrak{b} \subset \mathfrak{m}$. Hence $A/\mathfrak{a} \to B/\mathfrak{b} \to B/\mathfrak{m} = \kappa$ is of finite type. □

0GGB **Lemma 88.14.4.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. Let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$. Let $\mathfrak{a} \subset A$ be the ideal of topologically nilpotent elements. The following are equivalent*

(1) *the residue field κ of $B/\mathfrak{q}$ is finite over $A/\mathfrak{a}$,*
(2) *$\mathfrak{p} \subset A$ is rig-closed,*
(3) *$A/\mathfrak{p} \subset B/\mathfrak{q}$ is a finite extension of rings.*

Proof. Assume (1). Recall that $B/\mathfrak{q}$ is a Noetherian local ring of dimension 1 whose topology is the adic topology coming from the maximal ideal. Since φ is adic, we see that $A \to B/\mathfrak{q}$ is adic. Hence $\varphi(\mathfrak{a})$ is a nonzero ideal in $B/\mathfrak{q}$. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ has finite length. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ is finite as an $A/\mathfrak{a}$-module by our assumption. Thus $B/\mathfrak{q}$ is finite over A by Algebra, Lemma 10.96.12. Thus (3) holds.

Assume (3). Then $\mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective by Algebra, Lemma 10.36.17. This implies (2).

Assume (2). Denote κ' the residue field of $A/\mathfrak{p}$. By Lemma 88.14.3 (and Lemma 88.12.4) the extension κ/κ' is finitely generated as an algebra. By the Hilbert Nullstellensatz (Algebra, Lemma 10.34.2) we see that κ/κ' is a finite extension. Hence we see that (1) holds. □

0GGC **Lemma 88.14.5.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. If A/I is Jacobson for some ideal of definition $I \subset A$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$ is rig-closed.*

Proof. By Lemma 88.14.3 (combined with Lemma 88.12.4) the residue field κ of $B/\mathfrak{q}$ is of finite type over $A/\mathfrak{a}$. Since $A/\mathfrak{a}$ is Jacobson, we see that κ is finite over $A/\mathfrak{a}$ by Algebra, Lemma 10.35.18. We conclude by Lemma 88.14.4. □

0GGD **Lemma 88.14.6.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{p} \subset A$ be rig-closed. Let $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ be the ideals of topologically nilpotent elements. If φ is flat, then the following are equivalent*

(1) *the maximal ideal of $A/\mathfrak{p}$ is in the image of $\mathrm{Spec}(B/\mathfrak{b}) \to \mathrm{Spec}(A/\mathfrak{a})$,*
(2) *there exists a rig-closed prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.*

and if so then φ, $\mathfrak{p}$, and $\mathfrak{q}$ satisfy the conclusions of Lemma 88.14.4.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). To prove the existence of $\mathfrak{q}$ we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ (some details omitted). Thus we may assume $(A, \mathfrak{m}, \kappa)$ is a local complete 1-dimensional Noetherian ring, $\mathfrak{m} = \mathfrak{a}$, and $\mathfrak{p} = (0)$. Condition (1) just says that $B_0 = B \otimes_A \kappa = B/\mathfrak{m}B = B/\mathfrak{a}B$ is nonzero. Note that B_0 is of finite type over κ. Hence we can use induction on $\dim(B_0)$. If $\dim(B_0) = 0$, then any minimal prime $\mathfrak{q} \subset B$ will do (flatness of $A \to B$ insures that $\mathfrak{q}$ will lie over $\mathfrak{p} = (0)$). If $\dim(B_0) > 0$ then we can find an element $b \in B$ which maps to an element $b_0 \in B_0$ which is a nonzerodivisor and a nonunit, see Algebra, Lemma 10.63.20. By Algebra, Lemma 10.99.2 the ring $B' = B/bB$ is flat over A. Since $B'_0 = B' \otimes_A \kappa = B_0/(b_0)$ is not zero, we may apply the induction hypothesis to B' and conclude. The final statement of the lemma is clear from Lemma 88.14.4. □

We introduce some notation.

0GGE **Definition 88.14.7.** Let A be an adic topological ring which has a finitely generated ideal of definition. Let $f \in A$. The *completed principal localization* $A_{\{f\}}$ of A is the completion of $A_f = A[1/f]$ of the principal localization of A at f with respect to any ideal of definition of A.

To be sure, if f is topologically nilpotent, then $A_{\{f\}}$ is the zero ring.

0GGF **Lemma 88.14.8.** *Let A be an adic Noetherian topological ring. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $f \in A$ be an element mapping to a unit in $A/\mathfrak{p}$. Then*

$$\mathfrak{p}A_{\{f\}} = \mathfrak{p}(A_f)^\wedge = \mathfrak{p} \otimes_A (A_f)^\wedge = (\mathfrak{p}_f)^\wedge$$

is a prime ideal with quotient

$$A/\mathfrak{p} = (A/\mathfrak{p}) \otimes_A (A_f)^\wedge = (A_f)^\wedge/\mathfrak{p}(A_f)^\wedge = A_{\{f\}}/\mathfrak{p}A_{\{f\}}$$

Proof. Since A_f is Noetherian the ring map $A \to A_f \to (A_f)^\wedge$ is flat. For any finite A-module M we see that $M \otimes_A (A_f)^\wedge$ is the completion of M_f. If f is a unit on M, then $M_f = M$ is already complete. See discussion in Algebra, Section 10.97. From these observations the results follow easily. □

0GGG **Lemma 88.14.9.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. There exists an $f \in A$ which maps to a unit in $B/\mathfrak{q}$ such that we obtain a diagram*

$$\begin{tikzcd} B \arrow[r] & B_{\{f\}} \\ A \arrow[u, "\varphi"] \arrow[r] & A_{\{f\}} \arrow[u, "\varphi_{\{f\}}"'] \end{tikzcd} \quad \textit{with primes} \quad \begin{tikzcd} \mathfrak{q} \arrow[r, dash] \arrow[d, dash] & \mathfrak{q}' \arrow[r, equal] \arrow[d, dash] & \mathfrak{q}B_{\{f\}} \\ \mathfrak{p} \arrow[r, dash] & \mathfrak{p}' & \end{tikzcd}$$

such that $\mathfrak{p}'$ is rig-closed, i.e., the map $A_{\{f\}} \to B_{\{f\}}$ and the prime ideals $\mathfrak{q}'$ and $\mathfrak{p}'$ satisfy the equivalent conditions of Lemma 88.14.4.

Proof. Please see Lemma 88.14.8 for the description of $\mathfrak{q}'$. The only assertion the lemma makes is that for a suitable choice of f the prime ideal $\mathfrak{p}'$ has the property $\dim((A_f)^\wedge/\mathfrak{p}') = 1$. By Lemma 88.14.4 this in turn just means that the residue field κ of $B/\mathfrak{q} = (B_f)^\wedge/\mathfrak{q}'$ is finite over $(A_f)^\wedge/\mathfrak{a}' = (A/\mathfrak{a})_f$. By Lemma 88.14.3 we know that $A/\mathfrak{a} \to \kappa$ is a finite type algebra homomorphism. By the Hilbert Nullstellensatz in the form of Algebra, Lemma 10.34.2 we can find an $f \in A$ which maps to a unit in κ such that κ is finite over A_f. This finishes the proof. □

0GGH **Lemma 88.14.10.** *Let A be a Noetherian adic topological ring. Denote $A\{x_1, \ldots, x_n\}$ the restricted power series over A. Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be a prime ideal. Set $\mathfrak{q}' = A[x_1, \ldots, x_n] \cap \mathfrak{q}$ and $\mathfrak{p} = A \cap \mathfrak{q}$. If $\mathfrak{q}$ and $\mathfrak{p}$ are rig-closed, then the map*

$$A[x_1, \ldots, x_n]_{\mathfrak{q}'} \to A\{x_1, \ldots, x_n\}_{\mathfrak{q}}$$

defines an isomorphism on completions with respect to their maximal ideals.

Proof. By Lemma 88.14.4 the ring map $A/\mathfrak{p} \to A\{x_1, \ldots, x_n\}/\mathfrak{q}$ is finite. For every $m \geq 1$ the module $\mathfrak{q}^m/\mathfrak{q}^{m+1}$ is finite over A as it is a finite $A\{x_1, \ldots, x_n\}/\mathfrak{q}$-module. Hence $A\{x_1, \ldots x_n\}/\mathfrak{q}^m$ is a finite A-module. Hence $A[x_1, \ldots, x_n] \to A\{x_1, \ldots, x_n\}/\mathfrak{q}^m$ is surjective (as the image is dense and an A-submodule). It follows in a straightforward manner that $A[x_1, \ldots, x_n]/(\mathfrak{q}')^m \to A\{x_1, \ldots, x_n\}/\mathfrak{q}^m$ is an isomorphism for all m. From this the lemma easily follows. Hint: Pick a topologically nilpotent $g \in A$ which is not contained in $\mathfrak{p}$. Then the map of completions is the map

$$\lim_m (A[x_1, \ldots, x_n]/(\mathfrak{q}')^m)_g \longrightarrow (A\{x_1, \ldots, x_n\}/\mathfrak{q}^m)_g$$

Some details omitted. □

0GGI **Lemma 88.14.11.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$. Assume φ is adic, topologically of finite type, flat, and $A/I \to B/IB$ is étale for some (resp. any) ideal of definition $I \subset A$. Let $\mathfrak{q} \subset B$ be rig-closed such that $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. Then $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$.*

Proof. Let κ be the residue field of the 1-dimensional complete local ring $A/\mathfrak{p}$. Since $A/I \to B/IB$ is étale, we see that $B \otimes_A \kappa$ is a finite product of finite separable extensions of κ, see Algebra, Lemma 10.143.4. One of these is the residue field of $B/\mathfrak{q}$. By Algebra, Lemma 10.96.12 we see that $B/\mathfrak{p}B$ is a finite $A/\mathfrak{p}$-algebra. It is also flat. Combining the above we see that $A/\mathfrak{p} \to B/\mathfrak{p}B$ is finite étale, see Algebra, Lemma 10.143.7. Hence $B/\mathfrak{p}B$ is reduced, which implies the statement of the lemma (details omitted). □

0GGJ **Lemma 88.14.12.** *Let A be an adic Noetherian topological ring. Let $\mathfrak{p} \subset A$ be a rig-closed prime. For any $n \geq 1$ the ring map*

$$A/\mathfrak{p} \longrightarrow A\{x_1, \ldots, x_n\} \otimes_A A/\mathfrak{p} = A/\mathfrak{p}\{x_1, \ldots, x_n\}$$

is regular. In particular, the algebra $A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$.

Proof. We will use some fact on regular ring maps the reader can find in More on Algebra, Section 15.41. Since $A/\mathfrak{p}$ is a complete local Noetherian ring it is excellent (More on Algebra, Proposition 15.52.3). Hence $A/\mathfrak{p}[x_1, \ldots, x_n]$ is excellent (by the same reference). Hence $A/\mathfrak{p}[x_1, \ldots, x_n] \to A/\mathfrak{p}\{x_1, \ldots, x_n\}$ is a regular ring homomorphism by More on Algebra, Lemma 15.50.14. Of course $A/\mathfrak{p} \to A/\mathfrak{p}[x_1, \ldots, x_n]$ is smooth and hence regular. Since the composition of regular ring maps is regular the proof is complete. □

88.15. Rig-flat homomorphisms

0GGK In this section we define rig-flat homomorphisms of adic Noetherian topological rings.

0GGL **Lemma 88.15.1.** *Let $\varphi : A \to B$ be a morphism in $WAdm^{adic*}$ (Formal Spaces, Section 87.21). Assume φ is adic. The following are equivalent:*

(1) *B_f is flat over A for all topologically nilpotent $f \in A$,*
(2) *B_g is flat over A for all topologically nilpotent $g \in B$,*
(3) *$B_{\mathfrak{q}}$ is flat over A for all primes $\mathfrak{q} \subset B$ which do not contain an ideal of definition,*
(4) *$B_{\mathfrak{q}}$ is flat over A for every rig-closed prime $\mathfrak{q} \subset B$, and*
(5) *add more here.*

Proof. Follows from the definitions and Algebra, Lemma 10.39.18. □

0GGM **Definition 88.15.2.** Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $WAdm^{Noeth}$. We say φ is *naively rig-flat* if φ is adic, topologically of finite type, and satisfies the equivalent conditions of Lemma 88.15.1.

The example below shows that this notion does not "localize".

0GGN **Example 88.15.3.** By Examples, Lemma 110.18.1 there exists a local Noetherian 2-dimensional domain $(A, \mathfrak{m})$ complete with respect to a principal ideal $I = (a)$ and an element $f \in \mathfrak{m}$, $f \notin I$ with the following property: the ring $A_{\{f\}}[1/a]$ is nonreduced. Here $A_{\{f\}}$ is the I-adic completion $(A_f)^\wedge$ of the principal localization A_f. To be sure the ring $A_{\{f\}}[1/a]$ is nonzero. Let $B = A_{\{f\}}/\mathrm{nil}(A_{\{f\}})$ be the quotient by its nilradical. Observe that $A \to B$ is adic and topologically of finite type. In fact, B is a quotient of $A\{x\} = A[x]^\wedge$ by the map sending x to the image of $1/f$ in B. Every prime $\mathfrak{q}$ of B not containing a must lie over $(0) \subset A^2$. Hence $B_{\mathfrak{q}}$ is flat over A as it is a module over the fraction field of A. Thus $A \to B$ is naively rig-flat. On the other hand, the map

$$A_{\{f\}} \longrightarrow B_{\{f\}} = (B_f)^\wedge = B = A_{\{f\}}/\mathrm{nil}(A_{\{f\}})$$

is not flat after inverting a because we get the nontrivial surjection $A_{\{f\}}[1/a] \to A_{\{f\}}[1/a]/\mathrm{nil}(A_{\{f\}}[1/a])$. Hence $A_{\{f\}} \to B^\wedge_{\{f\}}$ is not naively rig-flat!

It turns out that it is easy to work around this problem by using the following definition.

0GGP **Definition 88.15.4.** Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $WAdm^{Noeth}$. We say φ is *rig-flat* if φ is adic, topologically of finite type, and for all $f \in A$ the induced map

$$A_{\{f\}} \longrightarrow B_{\{f\}}$$

is naively rig-flat (Definition 88.15.2).

Setting $f = 1$ in the definition above we see that rig-flatness implies naive rig-flatness. The example shows the converse is false. However, in many situations we don't need to worry about the difference between rig-flatness and its naive version as the next lemma shows.

[2]Namely, we can find $\mathfrak{q} \subset \mathfrak{q}' \subset B$ with $a \in \mathfrak{q}'$ because B is a-adically complete. Then $\mathfrak{p}' = A \cap \mathfrak{q}'$ contains a but not f hence is a height 1 prime. Then $\mathfrak{p} = A \cap \mathfrak{q}$ must be strictly contained in $\mathfrak{p}'$ as $a \notin \mathfrak{p}$. Since $\dim(A) = 2$ we see that $\mathfrak{p} = (0)$.

0GGQ **Lemma 88.15.5.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$. If A/I is Jacobson for some (equivalently any) ideal of definition $I \subset A$ and φ is naively rig-flat, then φ is rig-flat.*

Proof. Assume φ is naively rig-flat. We first state some obvious consequences of the assumptions. Namely, let $f \in A$. Then $A, B, A_{\{f\}}, B_{\{f\}}$ are Noetherian adic topological rings. The maps $A \to A_{\{f\}} \to B_{\{f\}}$ and $A \to B \to B_{\{f\}}$ are adic and topologically of finite type. The ring maps $A \to A_{\{f\}}$ and $B \to B_{\{f\}}$ are flat as compositions of $A \to A_f$ and $B \to B_f$ and the completion maps which are flat by Algebra, Lemma 10.97.2. The quotients of each of the rings $A, B, A_{\{f\}}, B_{\{f\}}$ by I is of finite type over A/I and hence Jacobson too (Algebra, Proposition 10.35.19).

Let $\mathfrak{q}' \subset B_{\{f\}}$ be rig-closed. It suffices to prove that $(B_{\{f\}})_{\mathfrak{q}'}$ is flat over $A_{\{f\}}$, see Lemma 88.15.1. By Lemma 88.14.5 the primes $\mathfrak{q} \subset B$ and $\mathfrak{p}' \subset A_{\{f\}}$ and $\mathfrak{p} \subset A$ lying under $\mathfrak{q}'$ are rig-closed. We are going to apply Algebra, Lemma 10.100.2 to the diagram

$$\begin{array}{ccc} B_{\mathfrak{q}} & \longrightarrow & (B_{\{f\}})_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ A_{\mathfrak{p}} & \longrightarrow & (A_{\{f\}})_{\mathfrak{p}'} \end{array}$$

with $M = B_{\mathfrak{q}}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak{p}$ generates the maximal ideal of $(A_{\{f\}})_{\mathfrak{p}'}$. This follows from Lemma 88.14.8; here we use that $\mathfrak{p}$ and $\mathfrak{p}'$ are rig-closed to see that f maps to a unit of $A/\mathfrak{p}$ (this is the only step in the proof that fails without the Jacobson assumption). Namely, this tells us that $A/\mathfrak{p} \to A_{\{f\}}/\mathfrak{p}'$ is a finite inclusion of local rings (Lemma 88.14.4) and f maps to a unit in the second one. □

0GGR **Lemma 88.15.6.** *Let $\varphi : A \to B$ and $A \to C$ be arrows of $WAdm^{Noeth}$. Assume φ is rig-flat and $A \to C$ adic and topologically of finite type. Then $C \to B\widehat{\otimes}_A C$ is rig-flat.*

Proof. Assume φ is rig-flat. Let $f \in C$ be an element. We have to show that $C_{\{f\}} \to B\widehat{\otimes}_A C_{\{f\}}$ is naively rig-flat. Since we can replace C by $C_{\{f\}}$ we it suffices to show that $C \to B\widehat{\otimes}_A C$ is naively rig-flat.

If $A \to C$ is surjective or more generally if C is finite as an A-module, then $B \otimes_A C = B\widehat{\otimes}_A C$ as a finite module over a complete Noetherian ring is complete, see Algebra, Lemma 10.97.1. By the usual base change for flatness (Algebra, Lemma 10.39.7) we see that naive rig-flatness of φ implies naive rig-flatness for $C \to B \times_A C$ in this case.

In the general case, we can factor $A \to C$ as $A \to A\{x_1, \ldots, x_n\} \to C$ where $A\{x_1, \ldots, x_n\}$ is the restricted power series ring and $A\{x_1, \ldots, x_n\} \to C$ is surjective. Thus it suffices to show $C \to B\widehat{\otimes}_A B$ is naively rig-flat in case $C = A\{x_1, \ldots, x_n\}$. Since $A\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_{n-1}\}\{x_n\}$ by induction on n we reduce to the case discussed in the next paragraph.

Here $C = A\{x\}$. Note that $B\widehat{\otimes}_A C = B\{x\}$. We have to show that $A\{x\} \to B\{x\}$ is naively rig-flat. Let $\mathfrak{q} \subset B\{x\}$ be a rig-closed prime ideal. We have to show that $B\{x\}_{\mathfrak{q}}$ is flat over $A\{x\}$. Set $\mathfrak{p} = A \cap \mathfrak{q}$. By Lemma 88.14.9 we can find an $f \in A$ such that f maps to a unit in $B\{x\}/\mathfrak{q}$ and such that the prime ideal $\mathfrak{p}'$ in $A_{\{f\}}$

induced is rig-closed. Below we will use that $A_{\{f\}}\{x\} = A\{x\}_{\{f\}}$ and similarly for B; details omitted. Consider the diagram

$$\begin{array}{ccc} (B\{x\})_{\mathfrak{q}} & \longrightarrow & (B_{\{f\}}\{x\})_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ A\{x\} & \longrightarrow & A_{\{f\}}\{x\} \end{array}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B_{\{f\}}\{x\}$ is the completion of a localization of the Noetherian ring $B\{x\}$. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. This reduces us to the case discussed in the next paragraph.

Here $C = A\{x\}$, we have a rig-closed prime ideal $\mathfrak{q} \subset B\{x\}$ such that $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. This implies, via Lemma 88.14.4, that the intermediate primes $B \cap \mathfrak{q}$ and $A\{x\} \cap \mathfrak{q}$ are rig-closed as well. Consider the diagram

$$\begin{array}{ccc} (B[x])_{B[x]\cap\mathfrak{q}} & \longrightarrow & (B\{x\})_{\mathfrak{q}} \\ \uparrow & & \uparrow \\ (A[x])_{A[x]\cap\mathfrak{q}} & \longrightarrow & (A\{x\})_{A\{x\}\cap\mathfrak{q}} \end{array}$$

of local homomorphisms of Noetherian local rings. By Lemma 88.14.10 the horizontal arrows define isomorphisms on completions. We already know that the left vertical arrow is flat (as $A \to B$ is naively rig-flat and hence $A[x] \to B[x]$ is flat away from the closed locus defined by an ideal of definition). Hence we finally conclude by More on Algebra, Lemma 15.43.8. □

0GGS **Lemma 88.15.7.** *Consider a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & A' \end{array}$$

in $WAdm^{Noeth}$ with all arrows adic and topologically of finite type. Assume $A \to A'$ and $B \to B'$ are flat. Let $I \subset A$ be an ideal of definition. If φ is rig-flat and $A/I \to A'/IA'$ is étale, then φ' is rig-flat.

Proof. Given $f \in A'$ the assumptions of the lemma remain true for the digram

$$\begin{array}{ccc} B & \longrightarrow & (B')_{\{f\}} \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ A & \longrightarrow & (A')_{\{f\}} \end{array}$$

Hence it suffices to prove that φ' is naively rig-flat.

Take a rig-closed prime ideal $\mathfrak{q}' \subset B'$. We have to show that $(B')_{\mathfrak{q}'}$ is flat over A'. We can choose an $f \in A$ which maps to a unit of $B'/\mathfrak{q}'$ such that the induced prime

ideal $\mathfrak{p}''$ of $A_{\{f\}}$ is rig-closed, see Lemma 88.14.9. To be precise, here $\mathfrak{q}'' = \mathfrak{q}'B'_{\{f\}}$ and $\mathfrak{p}'' = A_{\{f\}} \cap \mathfrak{q}''$. Consider the diagram

$$\begin{array}{ccc} B'_{\mathfrak{q}'} & \longrightarrow & (B'_{\{f\}})_{\mathfrak{q}''} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_{\{f\}} \end{array}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B'_{\{f\}}$ is the completion of a localization of the Noetherian ring B'_f. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. Finally, all the assumptions of the lemma remain true for the diagram

$$\begin{array}{ccc} B_{\{f\}} & \longrightarrow & B'_{\{f\}} \\ \uparrow & & \uparrow \\ A_{\{f\}} & \longrightarrow & A'_{\{f\}} \end{array}$$

This reduces us to the case discussed in the next paragraph.

Take a rig-closed prime ideal $\mathfrak{q}' \subset B'$ and assume $\mathfrak{p} = A \cap \mathfrak{q}'$ is rig-closed as well. This implies also the primes $\mathfrak{q} = B \cap \mathfrak{q}'$ and $\mathfrak{p}' = A' \cap \mathfrak{q}'$ are rig-closed, see Lemma 88.14.4. We are going to apply Algebra, Lemma 10.100.2 to the diagram

$$\begin{array}{ccc} B_{\mathfrak{q}} & \longrightarrow & B'_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ A_{\mathfrak{p}} & \longrightarrow & A'_{\mathfrak{p}'} \end{array}$$

with $M = B_{\mathfrak{q}}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak{p}$ generates the maximal ideal of $A'_{\mathfrak{p}'}$. This follows from Lemma 88.14.11. □

0GGT **Lemma 88.15.8.** *Consider a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & A' \end{array}$$

in WAdm^{Noeth} with all arrows adic and topologically of finite type. Assume $A \to A'$ flat and $B \to B'$ faithfully flat. If φ' is rig-flat, then φ is rig-flat.

Proof. Given $f \in A$ the assumptions of the lemma remain true for the digram

$$\begin{array}{ccc} B_{\{f\}} & \longrightarrow & (B')_{\{f\}} \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ A_{\{f\}} & \longrightarrow & (A')_{\{f\}} \end{array}$$

(To check the condition on faithful flatness: faithful flatness of $B \to B'$ is equivalent to $B \to B'$ being flat and $\mathrm{Spec}(B'/IB') \to \mathrm{Spec}(B/IB)$ being surjective for some

ideal of definition $I \subset A$.) Hence it suffices to prove that φ is naively rig-flat. However, we know that φ' is naively rig-flat and that $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is surjective. From this the result follows immediately. □

Finally, we can show that rig-flatness is a local property.

0GGU **Lemma 88.15.9.** *The property $P(\varphi) =$"φ is rig-flat" on arrows of $WAdm^{Noeth}$ is a local property as defined in Formal Spaces, Remark 87.21.4.*

Proof. Let us recall what the statement signifies. First, $WAdm^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & (B')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & (A')^\wedge \end{array}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-flat $\Rightarrow \varphi'$ is rig-flat,
(2) if $B \to B'$ faithfully flat, then φ' is rig-flat $\Rightarrow \varphi$ is rig-flat, and
(3) if $A \to B_i$ is rig-flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-flat.

Being adic and topologically of finite type satisfies conditions (1), (2), and (3), see Lemma 88.11.1. Thus in verifying (1), (2), and (3) for the property "rig-flat" we may already assume our ring maps are all adic and topologically of finite type. Then (1) and (2) follow from Lemmas 88.15.7 and 88.15.8. We omit the trivial proof of (3). □

0GGV **Lemma 88.15.10.** *The property $P(\varphi) =$"φ is rig-flat" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 87.21.14.*

Proof. The statement makes sense by Lemma 88.15.9. To see that it is true assume we have rig-flat morphisms $A \to B$ and $B \to C$ in $WAdm^{Noeth}$. Then $A \to C$ is adic and topologically of finite type by Lemma 88.11.4. To finish the proof we have to show that for all $f \in A$ the map $A_{\{f\}} \to C_{\{f\}}$ is naively rig-flat. Since $A_{\{f\}} \to B_{\{f\}}$ and $B_{\{f\}} \to C_{\{f\}}$ are naively rig-flat, it suffices to show that compositions of naively rig-flat maps are naively rig-flat. This is a consequence of Algebra, Lemma 10.39.4. □

88.16. Rig-flat morphisms

0GGW In this section we use the work done in Section 88.15 to define rig-flat morphisms of locally Noetherian algebraic spaces.

0GGX **Definition 88.16.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-flat* if for every

commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GGY **Lemma 88.16.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-flat,*
(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map in WAdm^{Noeth},
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-flat map in WAdm^{Noeth}, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-flat map in WAdm^{Noeth}.*

Proof. The equivalence of (1) and (2) is Definition 88.16.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-flat is a local property of arrows of WAdm^{Noeth} by Lemma 88.15.9 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GGZ **Lemma 88.16.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-flat and g is locally of finite type, then the base change $X \times_Y Z \to Z$ is rig-flat.*

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.15.6. □

0GH0 **Lemma 88.16.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-flat, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.15.10. □

88.17. Rig-smooth homomorphisms

0GCH In this section we prove some properties of rig-smooth homomorphisms of adic Noetherian topological rings which are needed to introduce rig-smooth morpisms of locally Noetherian formal algebraic spaces.

0GCI **Lemma 88.17.1.** *Let $A \to B$ be a morphism in $WAdm^{Noeth}$ (Formal Spaces, Section 87.21). The following are equivalent:*

(a) *$A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-smooth over (A, I), and*
(b) *$A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-smooth over (A, I).*

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-smooth over (A, I) if and only if B is rig-smooth over (A, I'). This follows from Definition 88.4.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 88.11.2. □

0GCJ **Definition 88.17.2.** Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $WAdm^{Noeth}$. We say φ is *rig-smooth* if the equivalent conditions of Lemma 88.17.1 hold.

This defines a local property.

0GCK **Lemma 88.17.3.** *The property $P(\varphi) =$ "φ is rig-smooth" on arrows of $WAdm^{Noeth}$ is a local property as defined in Formal Spaces, Remark 87.21.5.*

Proof. Let us recall what the statement signifies. First, $WAdm^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & (B')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & (A')^\wedge \end{array}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-smooth $\Rightarrow \varphi'$ is rig-smooth,
(2) if $B \to B'$ faithfully flat, then φ' is rig-smooth $\Rightarrow \varphi$ is rig-smooth, and
(3) if $A \to B_i$ is rig-smooth for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-smooth.

The equivalent conditions of Lemma 88.11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-smooth" we may already assume our ring maps satisfy the equivalent conditions of Lemma 88.11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (88.2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$ and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 88.3.2. Applying Lemma 88.3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 10.143.2). Hence we may apply Lemmas 88.3.5 and 88.3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-smooth. Then there exists a $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c for every B-module N. By More on Algebra, Lemmas 15.85.6 and 15.85.7 this property is preserved under base change by $B \to (B')^\wedge$. Hence $\mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N)$ is annihilated by $I^c(A')^\wedge$ for all $(B')^\wedge$-modules N which tells us that φ' is rig-smooth. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-smooth. Then there exists a $c \geq 0$ such that $\mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N')$ is annihilated by $I^c(B')^\wedge$ for every $(B')^\wedge$-module N'. The composition $B \to B' \to (B')^\wedge$ is flat (Algebra, Lemma 10.97.2) hence for any B-module N we have

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N) \otimes_B (B')^\wedge = \mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{B/A} \otimes_B (B')^\wedge, N \otimes_B (B')^\wedge)$$

by More on Algebra, Lemma 15.100.2 part (3) (minor details omitted). Thus we see that this module is annihilated by I^c. However, $B \to (B')^\wedge$ is actually faithfully flat by our assumption that $B \to B'$ is faithfully flat (Formal Spaces, Lemma 87.19.14). Thus we conclude that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c. Hence φ is rig-smooth. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. □

0GCL **Lemma 88.17.4.** *Consider the properties $P(\varphi)$ = "φ is rig-smooth" and $Q(\varphi)$ = "φ is adic" on arrows of $WAdm^{Noeth}$. Then P is stable under base change by Q as defined in Formal Spaces, Remark 87.21.10.*

Proof. The statement makes sense by Lemma 88.17.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ and that $B \to A$ is rig-smooth and $B \to C$ is adic (Formal Spaces, Definition 87.6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-smooth over (C, IC) by Lemma 88.4.5. □

0GCM **Lemma 88.17.5.** *The property $P(\varphi) =$ "φ is rig-smooth" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 87.21.14.*

Proof. We strongly urge the reader to find their own proof and not read the proof that follows. The statement makes sense by Lemma 88.17.1. To see that it is true assume we have rig-smooth morphisms $A \to B$ and $B \to C$ in $WAdm^{Noeth}$. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 88.3.5 we obtain an exact sequence

$$\begin{array}{ccccccc} C \otimes_B H^0(NL^\wedge_{B/A}) & \longrightarrow & H^0(NL^\wedge_{C/A}) & \longrightarrow & H^0(NL^\wedge_{C/B}) & \longrightarrow & 0 \\ & \nwarrow & & \nwarrow & & & \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) & \longrightarrow & H^{-1}(NL^\wedge_{C/A}) & \longrightarrow & H^{-1}(NL^\wedge_{C/B}) & & \end{array}$$

Observe that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a power of I; this follows from Lemma 88.4.2 part (2) combined with More on Algebra, Lemmas 15.85.6 and 15.85.7 (to deal with the base change by $B \to C$). Hence $H^{-1}(NL^\wedge_{C/A})$ is annihilated by a power of I. Next, by the characterization of rig-smooth algebras in Lemma 88.4.2 part (2) which in turn refers to More on Algebra, Lemma 15.85.10 part (5) we can choose $f_1, \ldots, f_s \in IB$ and $g_1, \ldots, g_t \in IC$ such that $V(f_1, \ldots, f_s) = V(IB)$ and $V(g_1, \ldots, g_t) = V(IC)$ and such that $H^0(NL^\wedge_{B/A})_{f_i}$ is a finite projective B_{f_i}-module and $H^0(NL^\wedge_{C/B})_{g_j}$ is a finite projective C_{g_j}-module. Since the cohomologies in degree -1 vanish upon localization at $f_i g_j$ we get a short exact sequence

$$0 \to (C \otimes_B H^0(NL^\wedge_{B/A}))_{f_i g_j} \to H^0(NL^\wedge_{C/A})_{f_i g_j} \to H^0(NL^\wedge_{C/B})_{f_i g_j} \to 0$$

and we conclude that $H^0(NL^\wedge_{C/A})_{f_i g_j}$ is a finite projective $C_{f_i g_j}$-module as an extension of same. Thus by the criterion in Lemma 88.4.2 part (2) and via that the criterion in More on Algebra, Lemma 15.85.10 part (4) we conclude that C is rig-smooth over (A, I). □

The following lemma can be interpreted as saying that a rig-smooth homomorphism is "rig-syntomic" or "rig-flat+rig-lci".

0GH1 **Lemma 88.17.6.** *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$. If φ is rig-smooth, then φ is rig-flat, and for any presentation $B = A\{x_1, \ldots, x_n\}/J$ and prime $J \subset \mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ not containing an ideal of definition the ideal $J_\mathfrak{q} \subset A\{x_1, \ldots, x_n\}_\mathfrak{q}$ is generated by a regular sequence.*

Proof. Let $f \in A$. To prove that φ is rig-flat we have to show that $\varphi_{\{f\}} : A_{\{f\}} \to B_{\{f\}}$ is naively rig-flat. Now either by viewing $\varphi_{\{f\}}$ as a base change of φ and using Lemma 88.17.4 or by using the fact that being rig-smooth is a local property (Lemma 88.17.3) we see that $\varphi_{\{f\}}$ is rig-smooth. Hence it suffices to show that φ is naively rig-flat.

Choose a presentation $B = A\{x_1, \ldots, x_n\}/J$. In order to check the second part of the lemma it suffices to check $J_\mathfrak{q} \subset A\{x_1, \ldots, x_n\}_\mathfrak{q}$ is generated by a regular sequence for $J \subset \mathfrak{q}$ for $\mathfrak{q}$ maximal with respect to not containing an ideal of definition, see Algebra, Lemma 10.68.6 (which shows that the set of primes in $V(J)$ where there is a regular sequence generating J is open). In other words, we may

assume $\mathfrak{q}$ is rig-closed in $A\{x_1, \ldots, x_n\}$. And to check that B is naively rig-flat, it also suffices to check that the corresponding localizations $B_{\mathfrak{q}}$ are flat over A.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$. By Lemma 88.14.9 we may choose an $f \in A$ mapping to a unit in $A\{x_1, \ldots, x_n\}/\mathfrak{q}$ and such that the prime ideal $\mathfrak{p}'$ in $A_{\{f\}}$ induced is rig-closed. Below we will use that $A_{\{f\}}\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_n\}_{\{f\}}$; details omitted. Consider the diagram

$$\begin{array}{ccc} A\{x_1, \ldots, x_n\}_{\mathfrak{q}}/J_{\mathfrak{q}} & \longrightarrow & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ A\{x_1, \ldots, x_n\}_{\mathfrak{q}} & \longrightarrow & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_{\{f\}} \end{array}$$

The middle horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $A_{\{f\}}\{x_1, \ldots, x_n\}$ is the completion of a localization of the Noetherian ring $A\{x_1, \ldots, x_n\}$. Similarly the bottom horizontal arrow is flat. Hence to show that $J_{\mathfrak{q}}$ is generated by a regular sequence and that $A \to A\{x_1, \ldots, x_n\}_{\mathfrak{q}}/J_{\mathfrak{q}}$ is flat, it suffices to prove the same things for $JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$ and $A_{\{f\}} \to A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$. See Algebra, Lemma 10.68.5 or More on Algebra, Lemma 15.32.4 for the statement on regular sequences. Finally, we have already seen that $A_{\{f\}} \to B_{\{f\}}$ is rig-smooth. This reduces us to the case discussed in the next paragraph.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$ such that moreover $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. By the characterization of rig-smooth algebras given in Lemma 88.4.2 after reordering the variables $x_1, \ldots, x_n$ we can find $m \geq 0$ and $f_1, \ldots, f_m \in J$ such that

(1) $J_{\mathfrak{q}}$ is generated by $f_1, \ldots, f_m$, and
(2) $\det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ maps to a unit in $A\{x_1, \ldots, x_n\}_{\mathfrak{q}}$.

By Lemma 88.14.12 the fibre ring

$$F = A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak{p})$$

is regular. Observe that the A-derivations $\partial/\partial x_i$ extend (uniquely) to derivations $D_i : F \to F$. By More on Algebra, Lemma 15.48.3 we see that $f_1, \ldots, f_m$ map to a regular sequence in $F_{\mathfrak{q}}$. By flatness of $A \to A\{x_1, \ldots, x_n\}$ and Algebra, Lemma 10.99.3 this shows that $f_1, \ldots, f_m$ map to a regular sequence in $A\{x_1, \ldots, x_m\}_{\mathfrak{q}}$ and the quotient by these elements is flat over A. This finishes the proof. □

0GH2 **Lemma 88.17.7.** *Let $A \to B \to C$ be arrows in $WAdm^{Noeth}$ which are adic and topologically of finite type. If $B \to C$ is rig-smooth, then the kernel of the map*

$$H^{-1}(NL^{\wedge}_{B/A} \otimes_B C) \to H^{-1}(NL^{\wedge}_{C/A})$$

(see Lemma 88.3.5) is annihilated by an ideal of definition.

Proof. Let $\overline{\mathfrak{q}} \subset C$ be a prime ideal which does not contain an ideal of definition. Since the modules in question are finite it suffices to show that

$$H^{-1}(NL^{\wedge}_{B/A} \otimes_B C)_{\overline{\mathfrak{q}}} \to H^{-1}(NL^{\wedge}_{C/A})_{\overline{\mathfrak{q}}}$$

is injective. As in the proof of Lemma 88.3.5 choose presentations $B = A\{x_1, \ldots, x_r\}/J$, $C = B\{y_1, \ldots, y_s\}/J'$, and $C = A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}/K$. Looking at the diagram in the proof of Lemma 88.3.5 we see that it suffices to show that $J/J^2 \otimes_B C \to K/K^2$ is injective after localization at the prime ideal $\mathfrak{q} \subset A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ corresponding to $\overline{\mathfrak{q}}$. Please compare with More on Algebra, Lemma 15.33.6 and its proof. This is the same as asking $J/KJ \to K/K^2$ to be injective after localization at $\mathfrak{q}$. Equivalently, we have to show that $J_{\mathfrak{q}} \cap K^2_{\mathfrak{q}} = (KJ)_{\mathfrak{q}}$. By Lemma 88.17.6 we know that $(K/J)_{\mathfrak{q}} = J'_{\mathfrak{q}}$ is generated by a regular sequence. Hence the desired intersection property follows from More on Algebra, Lemma 15.32.5 (and the fact that an ideal generated by a regular sequence is H_1-regular, see More on Algebra, Section 15.32). □

88.18. Rig-smooth morphisms

0GCN In this section we use the work done in Section 88.17 to define rig-smooth morphisms of locally Noetherian algebraic spaces.

0GCP **Definition 88.18.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-smooth* if for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GCQ **Lemma 88.18.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-smooth,*
(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map in $WAdm^{Noeth}$,
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-smooth map in $WAdm^{Noeth}$, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-smooth map in $WAdm^{Noeth}$.*

Proof. The equivalence of (1) and (2) is Definition 88.18.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-smooth is a local property of arrows of WAdm^{Noeth} by Lemma 88.17.3 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

0GCR **Lemma 88.18.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-smooth and g is adic, then the base change $X \times_Y Z \to Z$ is rig-smooth.*

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.17.4. □

0GCS **Lemma 88.18.4.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-smooth, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.17.5. □

0GH3 **Lemma 88.18.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. If f is rig-smooth, then f is rig-flat.*

Proof. Follows immediately from Lemma 88.17.6 and the definitions. □

88.19. Rig-étale homomorphisms

0GCT In this section we prove some properties of rig-étale homomorphisms of adic Noetherian topological rings which are needed to introduce rig-étale morphisms of locally Noetherian algebraic spaces.

0GCU **Lemma 88.19.1.** *Let $A \to B$ be a morphism in $WAdm^{Noeth}$ (Formal Spaces, Section 87.21). The following are equivalent:*

(a) *$A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-étale over (A, I), and*
(b) *$A \to B$ satisfies the equivalent conditions of Lemma 88.11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-étale over (A, I).*

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-étale over (A, I) if and only if B is rig-étale over (A, I'). This follows from Definition 88.8.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 88.11.2. □

0GCV **Definition 88.19.2.** Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $WAdm^{Noeth}$. We say φ is *rig-etale* if the equivalent conditions of Lemma 88.19.1 hold.

This defines a local property.

0AQL **Lemma 88.19.3.** *The property $P(\varphi) =$"φ is rig-étale" on arrows of $WAdm^{Noeth}$ is a local property as defined in Formal Spaces, Remark 87.21.5.*

Proof. This proof is exactly the same as the proof of Lemma 88.17.3. Let us recall what the statement signifies. First, $WAdm^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & (B')^\wedge \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A & \longrightarrow & (A')^\wedge \end{array}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 87.21.1. We have to show

(1) φ is rig-étale $\Rightarrow \varphi'$ is rig-étale,
(2) if $B \to B'$ faithfully flat, then φ' is rig-étale $\Rightarrow \varphi$ is rig-étale, and
(3) if $A \to B_i$ is rig-étale for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-étale.

The equivalent conditions of Lemma 88.11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-étale" we may already assume our ring maps satisfy the equivalent conditions of Lemma 88.11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (88.2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$ and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 88.3.2. Applying Lemma 88.3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 10.143.2). Hence we may apply Lemmas 88.3.5 and 88.3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{B/A}$ in $D(B)$. This property is preserved under base change by $B \to (B')^\wedge$, see More on Algebra, Lemmas 15.85.6. By the isomorphism above we find that φ' is rig-étale. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{(B')^\wedge/(A')^\wedge}$ in $D((B')^\wedge)$. By the isomorphism above we see that a^c annihilates the cohomology

modules of $NL^\wedge_{B/A} \otimes_B (B')^\wedge$. The composition $B \to (B')^\wedge$ is faithfully flat by our assumption that $B \to B'$ is faithfully flat, see Formal Spaces, Lemma 87.19.14. Hence the cohomology modules of $NL^\wedge_{B/A}$ are annihilated by I^c. It follows from Lemma 88.8.2 that φ is rig-étale. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. □

0GCW **Lemma 88.19.4.** *Consider the properties $P(\varphi)$ = "φ is rig-étale" and $Q(\varphi)$ = "φ is adic" on arrows of $WAdm^{Noeth}$. Then P is stable under base change by Q as defined in Formal Spaces, Remark 87.21.10.*

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ and that $B \to A$ is rig-étale and $B \to C$ is adic (Formal Spaces, Definition 87.6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-étale over (C, IC) by Lemma 88.8.6. □

0GCX **Lemma 88.19.5.** *The property $P(\varphi)$ = "φ is rig-étale" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 87.21.14.*

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have rig-étale morphisms $A \to B$ and $B \to C$ in $WAdm^{Noeth}$. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 88.3.5 we obtain an exact sequence

$$C \otimes_B H^0(NL^\wedge_{B/A}) \longrightarrow H^0(NL^\wedge_{C/A}) \longrightarrow H^0(NL^\wedge_{C/B}) \longrightarrow 0$$

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C) \longrightarrow H^{-1}(NL^\wedge_{C/A}) \longrightarrow H^{-1}(NL^\wedge_{C/B})$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/B}$ in $D(C)$. Then of course multiplication by a^c is zero on $NL^\wedge_{B/A} \otimes_B C$ in $D(C)$ too. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/B})$, $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$, and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/A})$ and $H^{-1}(NL^\wedge_{C/A})$. It follows from Lemma 88.8.2 that C is rig-étale over (A, I) as desired. □

0GCY **Lemma 88.19.6.** *The property $P(\varphi)$ = "φ is rig-étale" on arrows of $WAdm^{Noeth}$ has the cancellation property as defined in Formal Spaces, Remark 87.21.18.*

Proof. The statement makes sense by Lemma 88.19.1. To see that it is true assume we have maps $A \to B$ and $B \to C$ in $WAdm^{Noeth}$ with $A \to C$ and $A \to B$ rig-étale. We have to show that $B \to C$ is rig-étale. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 88.3.5

we obtain an exact sequence

$$C \otimes_B H^0(NL^\wedge_{B/A}) \longrightarrow H^0(NL^\wedge_{C/A}) \longrightarrow H^0(NL^\wedge_{C/B}) \longrightarrow 0$$

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C) \longrightarrow H^{-1}(NL^\wedge_{C/A}) \longrightarrow H^{-1}(NL^\wedge_{C/B})$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/A}$ in $D(C)$. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/A})$, and $H^{-1}(NL^\wedge_{C/A})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/B})$ and $H^{-1}(NL^\wedge_{C/B})$. It follows from Lemma 88.8.2 that C is rig-étale over (B, IB) as desired. □

88.20. Rig-étale morphisms

0AQK In this section we use the work done in Section 88.19 to define rig-étale morphisms of locally Noetherian algebraic spaces.

0AQM **Definition 88.20.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-étale* if for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

0GCZ **Lemma 88.20.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-étale,*
(2) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map in $WAdm^{Noeth}$,
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 87.11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-étale map in $WAdm^{Noeth}$, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 87.11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-étale map in $WAdm^{Noeth}$.*

Proof. The equivalence of (1) and (2) is Definition 88.20.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-étale is a local property of arrows of WAdm^{Noeth} by Lemma 88.19.3 and an application of the variant of Formal Spaces, Lemma 87.21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 87.21.5. □

To be sure, a rig-étale morphism is locally of finite type.

0AQN **Lemma 88.20.3.** *A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.*

Proof. The property P in Lemma 88.19.3 implies the equivalent conditions (a), (b), (c), and (d) in Formal Spaces, Lemma 87.29.6. Hence this follows from Formal Spaces, Lemma 87.29.9. □

0GD0 **Lemma 88.20.4.** *A rig-étale morphism of locally Noetherian formal algebraic spaces is rig-smooth.*

Proof. Follows from the definitions and Lemma 88.8.3. □

0GD1 **Lemma 88.20.5.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-étale and g is adic, then the base change $X \times_Y Z \to Z$ is rig-étale.*

Proof. By Formal Spaces, Remark 87.21.10 and the discussion in Formal Spaces, Section 87.23, this follows from Lemma 88.19.4. □

0GD2 **Lemma 88.20.6.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-étale, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 87.21.14 this follows from Lemma 88.19.5. □

0GD3 **Lemma 88.20.7.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be a morphism of locally Noetherian formal algebraic spaces over S. If $g \circ f$ and g are rig-étale, then so is f.*

Proof. By Formal Spaces, Remark 87.21.18 this follows from Lemma 88.19.6. □

0GH4 **Lemma 88.20.8.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-étale and g is an adic monomorphism, then f is rig-étale.*

Proof. Use Lemma 88.20.5 and that f is the base change of $g \circ f$ by g. □

0GD4 **Lemma 88.20.9.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

(1) *f is rig-smooth,*
(2) *f is rig-étale,*
(3) *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $WAdm^{Noeth}$ whose kernel J has the following property: $I(J/J^2) = 0$ for some ideal of definition I of B.*

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $WAdm^{Noeth}$ by Formal Spaces, Lemma 87.29.5.

We have (2) $\Rightarrow$ (1) by Lemma 88.20.4.

Proof of (3) $\Rightarrow$ (2). Assume (3). By Lemma 88.20.2 it suffices to show that the ring maps $B \to A$ occurring in (3) are rig-étale in the sense of Definition 88.19.2. Let I be as in (3). The naive cotangent complex $NL^\wedge_{A/B}$ of A over (B, I) is the complex of A-modules given by putting J/J^2 in degree -1. Hence A is rig-étale over (B, I) by Definition 88.8.1.

Assume (1) and let V and $B \to A$ be as in (3). By Definition 88.18.1 we see that $B \to A$ is rig-smooth. Choose any ideal of definition $I \subset B$. Then A is rig-smooth over (B, I). As above the complex $NL^\wedge_{A/B}$ is given by putting J/J^2 in degree -1. Hence by Lemma 88.4.2 we see that J/J^2 is annihilated by a power I^n for some $n \geq 1$. Since B is adic, we see that I^n is an ideal of definition of B and the proof is complete. □

88.21. Rig-surjective morphisms

0AQP For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [**Art70**] can be used. See Remark 88.21.2 for a discussion of what to do in more general cases.

0AQQ **Definition 88.21.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume that X and Y are locally Noetherian and that f is locally of finite type. We say f is *rig-surjective* if for every solid diagram

$$\begin{array}{ccc} \mathrm{Spf}(R') & \dashrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spf}(R) & \xrightarrow{p} & Y \end{array}$$

where R is a complete discrete valuation ring and where p is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\mathrm{Spf}(R') \to X$ making the displayed diagram commute.

We will see in the lemmas below that this notion behaves reasonably well in the context of locally Noetherian formal algebraic spaces and morphisms which are locally of finite type. In the next remark we discuss options for modifying this definition to a wider class of morphisms of formal algebraic spaces.

0AQZ **Remark 88.21.2.** The condition as formulated in Definition 88.21.1 is not right even for morphisms of finite type of locally adic* formal algebraic spaces. For example, if $A = (\bigcup_{n\geq 1} k[t^{1/n}])^\wedge$ where the completion is the t-adic completion, then there are no adic morphisms $\mathrm{Spf}(R) \to \mathrm{Spf}(A)$ where R is a complete discrete valuation ring. Thus any morphism $X \to \mathrm{Spf}(A)$ would be rig-surjective, but since A is a domain and $t \in A$ is not zero, we want to think of A as having at least one "rig-point", and we do not want to allow $X = \emptyset$. To cover this particular case, one can consider adic morphisms

$$\mathrm{Spf}(R) \longrightarrow Y$$

where R is a valuation ring complete with respect to a principal ideal J whose radical is $\mathfrak{m}_R = \sqrt{J}$. In this case the value group of R can be embedded into $(\mathbf{R}, +)$ and one obtains the point of view used by Berkovich in defining an analytic space associated to Y, see [**Ber90**]. Another approach is championed by Huber. In his theory, one drops the hypothesis that $\mathrm{Spec}(R/J)$ is a singleton, see [**Hub93a**].

0AQR **Lemma 88.21.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. Then if f and g are rig-surjective, so is $g \circ f$.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma 87.24.3). □

0AQS **Lemma 88.21.4.** *Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. If f is rig-surjective, then the base change $Z \times_Y X \to Z$ is too.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas 87.24.9 and 87.24.4). □

0GH5 **Lemma 88.21.5.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of finite type of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-surjective and g is a monomorphism, then f is rig-surjective.*

Proof. Use Lemma 88.21.4 and that f is the base change of $g \circ f$ by g. □

0AQT **Lemma 88.21.6.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z locally Noetherian and f and g locally of finite type. If $g \circ f : X \to Z$ is rig-surjective, so is $g : Y \to Z$.*

Proof. Immediate from the definition. □

0AQU **Lemma 88.21.7.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Then f is rig-surjective.*

Proof. Let $p : \mathrm{Spf}(R) \to Y$ be an adic morphism where R is a complete discrete valuation ring. Let $Z = \mathrm{Spf}(R) \times_Y X$. Then $Z \to \mathrm{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence Z is nonempty. Pick a nonempty affine formal algebraic space V and an étale morphism $V \to Z$ (possible by our definitions). Then $V \to \mathrm{Spf}(R)$ corresponds to $R \to A^\wedge$ where $R \to A$ is an étale ring map, see Formal Spaces, Lemma 87.19.13. Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal $\mathfrak{m}$ of A lying over $\mathfrak{m}_R$. Then $A_\mathfrak{m}$ is a discrete valuation ring (More on Algebra, Lemma 15.44.4). Then $R' = A^\wedge_\mathfrak{m}$ is a complete discrete valuation ring (More on Algebra, Lemma 15.43.5). Applying Formal Spaces, Lemma 87.9.10. we find the desired morphism $\mathrm{Spf}(R') \to V \to Z \to X$. □

The upshot of the lemmas above is that we may check whether $f : X \to Y$ is rig-surjective, étale locally on Y.

0AQV **Lemma 88.21.8.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. Let $\{g_i : Y_i \to Y\}$ be a family of morphisms of formal algebraic spaces which are representable by*

algebraic spaces and étale such that $\coprod g_i$ is surjective. Then f is rig-surjective if and only if each $f_i : X \times_Y Y_i \to Y_i$ is rig-surjective.

Proof. Namely, if f is rig-surjective, so is any base change (Lemma 88.21.4). Conversely, if all f_i are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \to \coprod Y_i$. By Lemma 88.21.7 the morphism $\coprod g_i : \coprod Y_i \to Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \to Y$ is rig-surjective (Lemma 88.21.3). Since this morphism factors through $X \to Y$ we see that $X \to Y$ is rig-surjective by Lemma 88.21.6. □

0AQX **Lemma 88.21.9.** *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. If $A/I^n \to B/I^nB$ is of finite type and flat for all n and faithfully flat for $n = 1$, then $Spf(B) \to Spf(A)$ is rig-surjective.*

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 87.9.10. Let $\varphi : A \to R$ be a continuous map into a complete discrete valuation ring A. This implies that $\varphi(I) \subset \mathfrak{m}_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \to R'$ in the case that φ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B\widehat{\otimes}_A R$, see Remark 88.2.3 for example. Then C is an $\mathfrak{m}_R$-adically complete R-algebra such that $C/\mathfrak{m}_R^n C$ is of finite type and flat over $R/\mathfrak{m}_R^n$ and such that $C/\mathfrak{m}_R C$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over $\mathfrak{m}_R$. By flatness (which implies going down) we see that $\mathrm{Spec}(C_\mathfrak{m}) \setminus V(\mathfrak{m}_R C_\mathfrak{m})$ is a nonempty open. Hence We can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(C_\mathfrak{m}) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \notin V(IC_\mathfrak{m})$, see Properties, Lemma 28.6.4. Then $C/\mathfrak{q}$ is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with R' a discrete valuation ring (Algebra, Lemma 10.119.13). By construction $\mathfrak{m}_R R' \subset \mathfrak{m}_{R'}$ and we see that $C \to R'$ extends to a continuous map $C \to (R')^\wedge$ (in fact we can pick R' such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{array}{ccccc} (R')^\wedge & \longleftarrow & C & \longleftarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ R & \longleftarrow & R & \longleftarrow & A \end{array}$$

which gives the desired lift. □

0AQY **Lemma 88.21.10.** *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. Assume that*

1. *the I-torsion in A is 0,*
2. *$A/I^n \to B/I^nB$ is flat and of finite type for all n.*

Then $Spf(B) \to Spf(A)$ is rig-surjective if and only if $A/I \to B/IB$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 88.21.9. To prove the converse we will use without further mention that the vanishing of I-torsion is equivalent to the vanishing of I-power torsion (More on Algebra, Lemma 15.89.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 87.9.10.

Assume $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \mathrm{Spec}(A_\mathfrak{m}) \setminus V(I_\mathfrak{m})$ of $\mathrm{Spec}(A_\mathfrak{m})$ is nonempty as the $I_\mathfrak{m}$-torsion of $A_\mathfrak{m}$ is zero (use Algebra, Lemma 10.62.4). Thus we can find a prime $\mathfrak{q} \subset A_\mathfrak{m}$ which defines a point of U (i.e., $IA_\mathfrak{m} \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\mathrm{Spec}(A_\mathfrak{m}) \setminus \{\mathfrak{m}\}$, see Properties, Lemma 28.6.4. Then $A_\mathfrak{m}/\mathfrak{q}$ is a dimension 1 local domain. Thus we can find an injective local homomorphism of local rings $A_\mathfrak{m}/\mathfrak{q} \subset R$ where R is a discrete valuation ring (Algebra, Lemma 10.119.13). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \to R$ extends to a continuous map $A \to R^\wedge$. Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{array}{ccc} R' & \longleftarrow & B \\ \uparrow & & \uparrow \\ R^\wedge & \longleftarrow & A \end{array}$$

Thus we find a prime ideal of B lying over $\mathfrak{m}$. It follows that $\mathrm{Spec}(B/IB) \to \mathrm{Spec}(A/I)$ is surjective, whence $A/I \to B/IB$ is faithfully flat (Algebra, Lemma 10.39.16). □

0AR0 **Lemma 88.21.11.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume X and Y are locally Noetherian, f locally of finite type, and f a monomorphism. Then f is rig surjective if and only if every adic morphism $Spf(R) \to Y$ where R is a complete discrete valuation ring factors through X.*

Proof. One direction is trivial. For the other, suppose that $\mathrm{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\mathrm{Spf}(R') \to \mathrm{Spf}(R) \to X$ factoring through Y. Then $\mathrm{Spec}(R'/\mathfrak{m}_R^n R') \to \mathrm{Spec}(R/\mathfrak{m}_R^n)$ is surjective and flat, hence the morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to X$ factor through X as X satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 87.32.1. In other words, $\mathrm{Spf}(R) \to Y$ factors through X. □

0GD5 **Lemma 88.21.12.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

1. *f is rig-surjective, and*
2. *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $WAdm^{Noeth}$ whose kernel J has the following property: $IJ^n = 0$ for some ideal of definition I of B and some $n \geq 1$.*

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $WAdm^{Noeth}$ by Formal Spaces, Lemma 87.29.5.

Assume (1). By Lemma 88.21.4 we see that $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ is rig-surjective. Let $I \subset B$ be an ideal of definition. Since B is adic, $I^m \subset B$ is an ideal of definition for all $m \geq 1$. If $I^m J^n \neq 0$ for all $n, m \geq 1$, then IJ is not nilpotent, hence $V(IJ) \neq \mathrm{Spec}(B)$. Thus we can find a prime ideal $\mathfrak{p} \subset B$ with $\mathfrak{p} \notin V(I) \cup V(J)$. Observe that $I(B/\mathfrak{p}) \neq B/\mathfrak{p}$ hence we can find a maximal ideal $\mathfrak{p} + I \subset \mathfrak{m} \subset B$. By Algebra, Lemma 10.119.13 we can find a discrete valuation ring R and an injective

local ring homomorphism $(B/\mathfrak{p})_{\mathfrak{m}} \to R$. Clearly, the ring map $B \to R$ cannot factor through $A = B/J$. According to Lemma 88.21.11 this contradicts the fact that $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ is rig-surjective. Hence for some n, m we do have $I^n J^m = 0$ which shows that (2) holds.

Assume (2). By Lemma 88.21.8 it suffices to show that $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ is rig-surjective. Pick an ideal of definition $I \subset B$ and an integer n such that $IJ^n = 0$. Consider a ring map $B \to R$ where R is a discrete valuation ring and the image of I is nonzero. Since R is a domain, we conclude the image of J in R is zero. Hence $B \to R$ factors through the surjection $B \to A$ and we are done by definition of rig-surjective morphisms. □

0GD6 **Lemma 88.21.13.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

(1) *f is rig-smooth and rig-surjective,*
(2) *f is rig-étale and rig-surjective, and*
(3) *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $WAdm^{Noeth}$ whose kernel J has the following property: $IJ = 0$ for some ideal of definition I of B.*

Proof. Let I and J be ideals of a ring B such that $IJ^n = 0$ and $I(J/J^2) = 0$. Then $I^n J = 0$ (proof omitted). Hence this lemma follows from a trivial combination of Lemmas 88.20.9 and 88.21.12. □

0GH6 **Lemma 88.21.14.** *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. Assume*

(1) *g is locally of finite type,*
(2) *f is rig-smooth (resp. rig-étale) and rig-surjective,*
(3) *$g \circ f$ is rig-smooth (resp. rig-étale)*

then g is rig-smooth (resp. rig-étale).

Proof. We will prove this in the rig-smooth case and indicate the necessary changes to prove the rig-étale case at the end of the proof. Consider a commutative diagram

$$\begin{array}{ccccc} X \times_Y V & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with V and W affine formal algebraic spaces, $V \to Y$ and $W \to Z$ representable by algebraic spaces and étale. We have to show that $V \to W$ corresponds to a rig-smooth map of adic Noetherian topological rings, see Definition 88.18.1. We may write $V = \mathrm{Spf}(B)$ and $W = \mathrm{Spf}(C)$ and that $V \to W$ corresponds to an adic ring map $C \to B$ which is topologically of finite type, see Lemma 88.11.5.

We will use below without further mention that $X \times_Y V \to V$ is rig-smooth and rig-surjective, see Lemmas 88.18.3 and 88.21.4. Also, the composition $X \times_Y V \to V \to W$ is rig-smooth since $g \circ f$ is rig-smooth.

Let $I \subset C$ be an ideal of definition. The module Assume $C \to B$ is not rig-smooth to get a contradiction. This means that there exists a prime ideal $\mathfrak{q} \subset B$ not containing IB such that either $H^{-1}(NL^\wedge_{B/C})_\mathfrak{p}$ is nonzero or $H^0(NL^\wedge_{B/C})_\mathfrak{p}$ is not a finite free $B_\mathfrak{q}$-module. See Lemma 88.4.2; some details omitted. We may choose a maximal ideal $IB + \mathfrak{q} \subset \mathfrak{m}$. By Algebra, Lemma 10.119.13 we can find a complete discrete valuation ring R and an injective local ring homomorphism $(B/\mathfrak{q})_\mathfrak{m} \to R$.

After replacing R by an extension, we may assume given a lift $\mathrm{Spf}(R) \to X \times_Y V$ of the adic morphism $\mathrm{Spf}(R) \to V = \mathrm{Spf}(B)$. Choose an étale covering $\{\mathrm{Spf}(A_i) \to X \times_Y V\}$ as in Formal Spaces, Definition 87.11.1. By Lemma 88.21.7 we may assume $\mathrm{Spf}(R) \to X \times_Y V$ lifts to a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(A_i)$ for some i (this might require replacing R by another extension). Set $A = A_i$. Consider the ring maps

$$C \to B \to A \to R$$

Let $\mathfrak{p} \subset A$ be the kernel of the map $A \to R$ and note that $\mathfrak{p}$ lies over $\mathfrak{q}$. We know that $C \to A$ and $B \to A$ are rig-smooth. In particular the ring map $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat by Lemma 88.17.6. Consider the associated exact sequence

$$\begin{aligned} H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p} &\longrightarrow H^0(NL^\wedge_{A/C})_\mathfrak{p} \longrightarrow H^0(NL^\wedge_{A/B})_\mathfrak{p} \longrightarrow 0 \\ 0 \longrightarrow H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} &\longrightarrow H^{-1}(NL^\wedge_{A/C})_\mathfrak{p} \longrightarrow H^{-1}(NL^\wedge_{A/B})_\mathfrak{p} \end{aligned}$$

of Lemmas 88.3.5 and 88.17.7. Given the rig-smoothness of $C \to A$ and $B \to A$ we conclude that $H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} = 0$ and that $H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p}$ is finite free as a kernel of a surjection of finite free $A_\mathfrak{p}$-modules. Since $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat and hence faithfully flat, this implies that $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q} = 0$ and that $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ is finite free which is the contradiction we were looking for.

In the rig-étale case one argues in exactly the same manner but the conclusion obtained is that both $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q}$ and $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ are zero. □

88.22. Formal algebraic spaces over cdvrs

0GD7 In this section we will use the following terminology: if A is a weakly admissible topological ring, then we say "*X is a formal algebraic space over A*" to mean that X is a formal algebraic space which comes equipped with a morphism $p : X \to \mathrm{Spf}(A)$ of formal algebraic spaces. In this situation we will call p the *structure morphism*.

0GD8 **Lemma 88.22.1.** *Let X be a locally Noetherian formal algebraic space over a complete discrete valuation ring A. Then there exists a closed immersion $X' \to X$ of formal algebraic spaces such that X' is flat over A and such that any morphism $Y \to X$ of locally Noetherian formal algebraic spaces with Y flat over A factors through X'.*

Proof. Let $\pi \in A$ be the uniformizer. Recall that an A-module is flat if and only if the π-power torsion is 0.

First assume that X is an affine formal algebraic space. Then $X = \mathrm{Spf}(B)$ with B an adic Noetherian A-algebra. In this case we set $X' = \mathrm{Spf}(B')$ where $B' = B/\pi$-power torsion. It is clear that X' is flat over A and that $X' \to X$ is a closed immersion. Let $g : Y \to X$ be a morphism of locally Noetherian formal algebraic

spaces with Y flat over A. Choose a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 87.11.1. Then $Y_j = \mathrm{Spf}(C_j)$ with C_j flat over A. Hence the morphism $Y_j \to X$, which correspond to a continuous R-algebra map $B \to C_j$, factors through X' as clearly $B \to C_j$ kills the π-power torsion. Since $\{Y_j \to Y\}$ is a covering and since $X' \to X$ is a monomorphism, we conclude that g factors through X'.

Let X and $\{X_i \to X\}_{i \in I}$ be as in Formal Spaces, Definition 87.11.1. For each i let $X'_i \to X_i$ be the flat part as constructed above. For $i, j \in I$ the projection $X'_i \times_X X_j \to X'_i$ is an étale (by assumption) morphism of schemes (by Formal Spaces, Lemma 87.9.11). Hence $X'_i \times_X X_j$ is flat over A as morphisms representable by algebraic spaces and étale are flat (Lemma 88.13.8). Thus the projection $X'_i \times_X X_j \to X_j$ factors through X'_j by the universal property. We conclude that

$$R_{ij} = X'_i \times_X X_j = X'_i \times_X X'_j = X_i \times_X X'_j$$

because the morphisms $X'_i \to X_i$ are injections of sheaves. Set $U = \coprod X'_i$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X' = U/R$ is an algebraic space by Spaces, Theorem 65.10.5. By construction the diagram

$$\begin{array}{ccc} \coprod X'_i & \longrightarrow & \coprod X_i \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a closed immersion we conclude that $X' \to X$ is representable by Bootstrap, Lemma 80.5.2. Then we can use Spaces, Lemma 65.5.6 to conclude that $X' \to X$ is a closed immersion.

Finally, suppose that $Y \to X$ is a morphism with Y a locally Noetherian formal algebraic space flat over A. Then each $X_i \times_X Y$ is étale over Y and therefore flat over A (see above). Then $X_i \times_X Y \to X_i$ factors through X'_i. Hence $Y \to X$ factors through X' because $\{X_i \times_X Y \to Y\}$ is an étale covering. □

0GD9 **Lemma 88.22.2.** *Let X be a locally Noetherian formal algebraic space which is locally of finite type over a complete discrete valuation ring A. Let $X' \subset X$ be as in Lemma 88.22.1. If $X \to X \times_{Spf(A)} X$ is rig-étale and rig-surjective, then $X' = Spf(A)$ or $X' = \emptyset$.*

Proof. (Aside: the diagonal is always locally of finite type by Formal Spaces, Lemma 87.15.5 and $X \times_{\mathrm{Spf}(A)} X$ is locally Noetherian by Formal Spaces, Lemmas 87.24.4 and 87.24.8. Thus imposing the conditions on the diagonal morphism makes sense.) The diagram

$$\begin{array}{ccc} X' & \longrightarrow & X' \times_{\mathrm{Spf}(A)} X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_{\mathrm{Spf}(A)} X \end{array}$$

is cartesian. Hence $X' \to X' \times_{\mathrm{Spf}(A)} X'$ is rig-étale and rig-surjective by Lemma 88.21.4. Choose an affine formal algebraic space U and a morphism $U \to X'$ which is representable by algebraic spaces and étale. Then $U = \mathrm{Spf}(B)$ where B is an

adic Noetherian topological ring which is a flat A-algebra, whose topology is the π-adic topology where $\pi \in A$ is a uniformizer, and such that $A/\pi^n A \to B/\pi^n B$ is of finite type for each n. For later use, we remark that this in particular implies: if $B \neq 0$, then the map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a surjection of sheaves (please recall that we are using the fppf topology as always). Repeating the argument above, we see that

$$W = U \times_{X'} U = X' \times_{X' \times_{\mathrm{Spf}(A)} X'} (U \times_{\mathrm{Spf}(A)} U) \longrightarrow U \times_{\mathrm{Spf}(A)} U$$

is a closed immersion and rig-étale and rig-surjective. We have $U \times_{\mathrm{Spf}(A)} U = \mathrm{Spf}(B \widehat{\otimes}_A B)$ by Formal Spaces, Lemma 87.16.4. Then $B \widehat{\otimes}_A B$ is a flat A-algebra as the π-adic completion of the flat A-algebra $B \otimes_A B$. Hence $W = U \times_{\mathrm{Spf}(A)} U$ by Lemma 88.21.13. In other words, we have $U \times_{X'} U = U \times_{\mathrm{Spf}(A)} U$ which in turn means that the image of $U \to X'$ (as a map of sheaves) maps injectively to $\mathrm{Spf}(A)$. Choose a covering $\{U_i \to X'\}$ as in Formal Spaces, Definition 87.11.1. In particular $\coprod U_i \to X'$ is a surjection of sheaves. By applying the above to $U_i \coprod \coprod U_j \to X'$ (using the fact that $U_i \amalg U_j$ is an affine formal algebraic space as well) we see that $X' \to \mathrm{Spf}(A)$ is an injective map of fppf sheaves. Since X' is flat over A, either X' is empty (if U_i is empty for all i) or the map is an isomorphism (if U_i is nonempty for some i when we have seen that $U_i \to \mathrm{Spf}(A)$ is a surjective map of sheaves) and the proof is complete. □

0GDA **Lemma 88.22.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume*

(1) *X and Y are locally Noetherian,*
(2) *f locally of finite type,*
(3) *$\Delta_f : X \to X \times_Y X$ is rig-étale and rig-surjective.*

Then f is rig surjective if and only if every adic morphism $Spf(R) \to Y$ where R is a complete discrete valuation ring lifts to a morphism $Spf(R) \to X$.

Proof. One direction is trivial. For the other, suppose that $\mathrm{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\mathrm{Spf}(R') \to \mathrm{Spf}(R) \to X$ factoring through Y. Consider the fibre product diagram

$$\begin{array}{ccccc} \mathrm{Spf}(R') & \longrightarrow & \mathrm{Spf}(R) \times_Y X & \longrightarrow & X \\ & \searrow & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ & & \mathrm{Spf}(R) & \longrightarrow & Y \end{array}$$

The morphism p is locally of finite type as a base change of f, see Formal Spaces, Lemma 87.24.4. The diagonal morphism Δ_p is the base change of Δ_f and hence is rig-étale and rig-surjective. By Lemma 88.22.2 the flat locus of $\mathrm{Spf}(R) \times_Y X$ over R is either $\emptyset$ or equal to $\mathrm{Spf}(R)$. However, since $\mathrm{Spf}(R')$ factors through it we conclude it is not empty and hence we get a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R) \times_Y X \to X$ as desired. □

88.23. The completion functor

0GDB In this section we consider the following situation. First we fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix an algebraic space X and a

closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

In this situation, given an algebraic space X' over X, i.e., an algebraic space X' endowed with a morphism $f : X' \to X$, then we denote $T' \subset |X'|$ the inverse image of T and we let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Picture

$$U' = f^{-1}U \qquad \begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & X \end{array} \qquad \begin{array}{ccccc} |U'| & \longrightarrow & |X'| & \longleftarrow & T' \\ \downarrow & & \downarrow{\scriptstyle |f|} & & \downarrow \\ |U| & \longrightarrow & |X| & \longleftarrow & T \end{array} \qquad T' = |f|^{-1}T$$

We will relate properties of f to properties of the induced morphism

$$f_{/T} : X'_{/T'} \longrightarrow X_{/T}$$

of formal completions. As indicated in the displayed formula, we will denote this morphism $f_{/T}$. We have already seen that $f_{/T}$ is representable by algebraic spaces in Formal Spaces, Lemma 87.14.4. In fact, as the proof of that lemma shows, the diagram

$$\begin{array}{ccc} X'_{/T'} & \longrightarrow & X' \\ {\scriptstyle f_{/T}}\downarrow & & \downarrow{\scriptstyle f} \\ X_{/T} & \longrightarrow & X \end{array}$$

is cartesian. Please keep this fact in mind whilst reading the lemmas stated and proved below.

0AQ9 **Lemma 88.23.1.** *In the situation above. If f is locally of finite type, then $f_{/T}$ is locally of finite type.*

Proof. (Finite type morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.24.) Namely, suppose that $Z \to X$ is a morphism from a scheme into X such that $|Z|$ maps into T. From the cartesian square above we see that $Z \times_X X'$ is an algebraic space representing $Z \times_{X_{/T}} X'_{/T'}$. Since $Z \times_X X' \to Z$ is locally of finite type by Morphisms of Spaces, Lemma 67.23.3 we conclude. □

0GI0 **Lemma 88.23.2.** *In the situation above. If f is étale, then $f_{/T}$ is étale.*

Proof. By the same argument as in the proof of Lemma 88.23.1 this follows from Morphisms of Spaces, Lemma 67.39.4. □

0GDC **Lemma 88.23.3.** *In the situation above. If f is a closed immersion, then $f_{/T}$ is a closed immersion.*

Proof. (Closed immersions of formal algebraic spaces are discussed in Formal Spaces, Section 87.27.) By the same argument as in the proof of Lemma 88.23.1 this follows from Spaces, Lemma 65.12.3. □

0GDD **Lemma 88.23.4.** *In the situation above. If f is proper, then $f_{/T}$ is proper.*

Proof. (Proper morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.31.) By the same argument as in the proof of Lemma 88.23.1 this follows from Morphisms of Spaces, Lemma 67.40.3. □

0GDE **Lemma 88.23.5.** *In the situation above. If f is quasi-compact, then $f_{/T}$ is quasi-compact.*

Proof. (Quasi-compact morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.17.) We have to show that $(X'_{/T'})_{red} \to (X_{/T})_{red}$ is a quasi-compact morphism of algebraic spaces. By Formal Spaces, Lemma 87.14.5 this is the morphism $Z' \to Z$ where $Z' \subset X'$, resp. $Z \subset X$ is the reduced induced algebraic space structure on T', resp. T. It follows that $Z' \to f^{-1}Z = Z \times_X X'$ is a thickening (a closed immersion defining an isomorphism on underlying topological spaces). Since $Z \times_X X' \to Z$ is quasi-compact as a base change of f (Morphisms of Spaces, Lemma 67.8.4) we conclude that $Z' \to Z$ is too by More on Morphisms of Spaces, Lemma 76.10.1. □

0GDF **Remark 88.23.6.** In the situation above consider the diagonal morphisms $\Delta_f : X' \to X' \times_X X'$ and $\Delta_{f_{/T}} : X'_{/T'} \to X'_{/T'} \times_{X_{/T}} X'_{/T'}$. It is easy to see that

$$X'_{/T'} \times_{X_{/T}} X'_{/T'} = (X' \times_X X')_{/T''}$$

as subfunctors of $X' \times_X X'$ where $T'' \subset |X' \times_X X'|$ is the inverse image of T. Hence we see that $\Delta_{f_{/T}} = (\Delta_f)_{/T''}$. We will use this below to show that properties of Δ_f are inherited by $\Delta_{f_{/T}}$.

0GDG **Lemma 88.23.7.** *In the situation above. If f is (quasi-)separated, then $f_{/T}$ is too.*

Proof. (Separation conditions on morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 87.30.) We have to show that if Δ_f is quasi-compact, resp. a closed immersion, then the same is true for $\Delta_{f_{/T}}$. This follows from the discussion in Remark 88.23.6 and Lemmas 88.23.5 and 88.23.3. □

0GDH **Lemma 88.23.8.** *In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is smooth, then $f_{/T}$ is rig-smooth.*

Proof. The strategy of the proof is this: reduce to the case where X and X' are affine, translate the affine case into algebra, and finally apply Lemma 88.4.3. We urge the reader to skip the details.

Choose a surjective étale morphism $W \to X$ with $W = \coprod W_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 66.6.1. For each i choose a surjective étale morphism $W'_i \to W_i \times_X X'$ where $W'_i = \coprod W'_{ij}$ is a disjoint union of affines. In particular $\coprod W'_{ij} \to X'$ is surjective and étale. Denote $f_{ij} : W_{ij} \to W_i$ the given morphism. Denote $T_i \subset W_i$ and $T'_{ij} \subset W_{ij}$ the inverse images of T. Since taking the completion along the inverse image of T produces cartesian diagrams (see above) we have $(W_i)_{/T_i} = W_i \times_X X_{/T}$ and similarly $(W'_{ij})_{/T'_{ij}} = W'_{ij} \times_{X'} X'_{/T'}$. Moreover, recall that $(W_i)_{/T_i}$ and $(W'_{ij})_{/T'_{ij}}$ are affine formal algebraic spaces. Hence $\{(W'_{ij})_{/T'_{ij}} \to X'_{/T'}\}$ is a covering as in Formal Spaces, Definition 87.11.1. By Lemma 88.18.2 we see that it suffices to prove that

$$(W'_{ij})_{/T'_{ij}} \longrightarrow (W_i)_{/T_i}$$

is rig-smooth. Observe that $W'_{ij} \to W_i$ is locally of finite type and induces a smooth morphism $W'_{ij} \backslash T'_{ij} \to W_i \backslash T_i$ (as this is true for f and these properties of morphisms are étale local on the source and target). Observe that W_i is locally Noetherian

(as X is locally Noetherian and this property is étale local on the algebraic space). Hence it suffices to prove the lemma when X and X' are affine schemes.

Assume $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ are affine schemes. Since X is Noetherian, we see that A is Noetherian. The morphism f is given by a ring map $A \to A'$ of finite type. Let $I \subset A$ be an ideal cutting out T. Then IA' cuts out T'. Also $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus T$. Let $A^\wedge$ and $(A')^\wedge$ be the I-adic completions. We have $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X'_{/T'} = \mathrm{Spf}((A')^\wedge)$, see proof of Formal Spaces, Lemma 87.20.8. By Lemma 88.4.3 we see that $(A')^\wedge$ is rig-smooth over $(A.I)$ which in turn means that $A^\wedge \to (A')^\wedge$ is rig-smooth which finally implies that $X'_{/T'} \to X_{/T}$ is rig smooth by Lemma 88.18.2. □

0AR2 **Lemma 88.23.9.** *In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is étale, then $f_{/T}$ is rig-étale.*

Proof. The proof is exactly the same as the proof of Lemma 88.23.8 except with Lemmas 88.4.3 and 88.18.2 replaced by Lemmas 88.8.4 and 88.20.2 □

0AQW **Lemma 88.23.10.** *In the situation above. If X is locally Noetherian, f is proper, and $U' \to U$ is surjective, then $f_{/T}$ is rig-surjective.*

Proof. (The statement makes sense by Lemma 88.23.1 and Formal Spaces, Lemma 87.20.8.) Let R be a complete discrete valuation ring with fraction field K. Let $p : \mathrm{Spf}(R) \to X_{/T}$ be an adic morphism of formal algebraic spaces. By Formal Spaces, Lemma 87.33.4 the composition $\mathrm{Spf}(R) \to X_{/T} \to X$ corresponds to a morphism $q : \mathrm{Spec}(R) \to X$ which maps $\mathrm{Spec}(K)$ into U. Since $U' \to U$ is proper and surjective we see that $\mathrm{Spec}(K) \times_U U'$ is nonempty and proper over K. Hence we can choose a field extension K'/K and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & U' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \end{array}$$

Let $R' \subset K'$ be a discrete valuation ring dominating R with fraction field K', see Algebra, Lemma 10.119.13. Since $\mathrm{Spec}(K) \to X$ extends to $\mathrm{Spec}(R) \to X$ we see by the valuative criterion of properness (Morphisms of Spaces, Lemma 67.44.1) that we can extend our K'-point of U' to a morphism $\mathrm{Spec}(R') \to X'$ over $\mathrm{Spec}(R) \to X$. It follows that the inverse image of T' in $\mathrm{Spec}(R')$ is the closed point and we find an adic morphism $\mathrm{Spf}((R')^\wedge) \to X'_{/T'}$ lifting p as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 15.43.5). □

0GDI **Lemma 88.23.11.** *In the situation above. If X is locally Noetherian, f is separated and locally of finite type, and $U' \to U$ is a monomorphism, then $\Delta_{f_{/T}}$ is rig-surjective.*

Proof. The diagonal $\Delta_f : X' \to X' \times_X X'$ is a closed immersion and the restriction $U' \to U' \times_U U'$ of Δ_f is surjective. Hence the lemma follows from the discussion in Remark 88.23.6 and Lemma 88.23.10. □

88.24. Formal modifications

0GDJ In this section we define and study Artin's notion of a formal modification of locally Noetherian formal algebraic spaces. First, here is the definition.

0GDK **Definition 88.24.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is a *formal modification* if

(1) f is a proper morphism (Formal Spaces, Definition 87.31.1),
(2) f is rig-étale,
(3) f is rig-surjective,
(4) $\Delta_f : X \to X \times_Y X$ is rig-surjective.

A typical example is given in Lemma 88.24.3 and indeed we will later show that every formal modification is "formal locally" of this type, see Lemma 88.29.2. Let us compare these conditions with those in Artin's paper.

0GDL **Remark 88.24.2.** In [**Art70**, Definition 1.7] a formal modification is defined as a proper morphism $f : X \to Y$ of locally Noetherian formal algebraic spaces satisfying the following three conditions[3]

(i) the Cramer and Jacobian ideal of f each contain an ideal of definition of X,
(ii) the ideal defining the diagonal map $\Delta : X \to X \times_Y X$ is annihilated by an ideal of definition of $X \times_Y X$, and
(iii) any adic morphism $\mathrm{Spf}(R) \to Y$ lifts to $\mathrm{Spf}(R) \to X$ whenever R is a complete discrete valuation ring.

Let us compare these to our list of conditions above.

Ad (i). Property (i) agrees with our condition that f be a rig-étale morphism: this follows from Lemma 88.8.2 part (7).

Ad (ii). Assume f is rig-étale. Then $\Delta_f : X \to X \times_Y X$ is rig-étale as a morphism of locally Noetherian formal algebraic spaces which are rig-étale over X (via id_X for the first one and via pr_1 for the second one). See Lemmas 88.20.5 and 88.20.7. Hence property (ii) agrees with our condition that Δ_f be rig-surjective by Lemma 88.21.13.

Ad (iii). Property (iii) does not quite agree with our notion of a rig-surjective morphism, as Artin requires all adic morphisms $\mathrm{Spf}(R) \to Y$ to lift to morphisms into X whereas our notion of rig-surjective only asserts the existence of a lift after replacing R by an extension. However, since we already have that Δ_f is rig-étale and rig-surjective by (i) and (ii), these conditions are equivalent by Lemma 88.22.3.

0GDM **Lemma 88.24.3.** *Let S, $f : X' \to X$, $T \subset |X|$, $U \subset X$, $T' \subset |X'|$, and $U' \subset X'$ be as in Section 88.23. If X is locally Noetherian, f is proper, and $U' \to U$ is an isomorphism, then $f_{/T} : X'_{/T'} \to X_{/T}$ is a formal modification.*

Proof. By Formal Spaces, Lemmas 87.20.8 the source and target of the arrow are locally Noetherian formal algebraic spaces. The other conditions follow from Lemmas 88.23.4, 88.23.9, 88.23.10, and 88.23.11. □

[3] We will not completely translate these conditions into the language developed in the Stacks project. We hope nonetheless the discussion here will be useful to the reader.

0GDN **Lemma 88.24.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S which is a formal modification. Then for any adic morphism $Y' \to Y$ of locally Noetherian formal algebraic spaces, the base change $f' : X \times_Y Y' \to Y'$ is a formal modification.*

Proof. The morphism f' is proper by Formal Spaces, Lemma 87.31.3. The morphism f' is rig-etale by Lemma 88.20.5. Then morphism f' is rig-surjective by Lemma 88.21.4. Set $X' = X \times'_Y$. The morphism $\Delta_{f'}$ is the base change of Δ_f by the adic morphism $X' \times_{Y'} X' \to X \times_Y X$. Hence $\Delta_{f'}$ is rig-surjective by Lemma 88.21.4. □

88.25. Completions and morphisms, I

0GDP In this section we put some preliminary results on completions which we will use in the proof of Theorem 88.27.4. Although the lemmas stated and proved here are not trivial (some are based on our work on algebraization of rig-étale algebras), we still suggest the reader skip this section on a first reading.

0AR4 **Lemma 88.25.1.** *Let $T \subset X$ be a closed subset of a Noetherian affine scheme X. Let W be a Noetherian affine formal algebraic space. Let $g : W \to X_{/T}$ be a rig-étale morphism. Then there exists an affine scheme X' and a finite type morphism $f : X' \to X$ étale over $X \setminus T$ such that there is an isomorphism $X'_{/f^{-1}T} \cong W$ compatible with $f_{/T}$ and g. Moreover, if $W \to X_{/T}$ is étale, then $X' \to X$ is étale.*

Proof. The existence of X' is a restatement of Lemma 88.10.3. The final statement follows from More on Morphisms, Lemma 37.12.3. □

0AR3 **Lemma 88.25.2.** *Assume we have*

(1) *Noetherian affine schemes X, X', and Y,*
(2) *a closed subset $T \subset |X|$,*
(3) *a morphism $f : X' \to X$ locally of finite type and étale over $X \setminus T$,*
(4) *a morphism $h : Y \to X$,*
(5) *a morphism $\alpha : Y_{/T} \to X'_{/T}$ over $X_{/T}$ (see proof for notation).*

Then there exists an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $a : Y' \to X'$ over X such that $\alpha = a_{/T} \circ b^{-1}_{/T}$.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism of schemes $g : V \to X$ associates the morphism $g_{/T} : V_{/g^{-1}T} \to X_{/T}$ of formal algebraic spaces; it is a functor from the category of schemes over X to the category of formal algebraic spaces over $X_{/T}$, see Section 88.23. Having said this, the lemma is just a reformulation of Lemma 88.8.7. □

0AR6 **Lemma 88.25.3.** *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that*

(1) *X is locally Noetherian,*
(2) *g is a monomorphism and locally of finite type,*
(3) *$f|_{X \setminus T} : X \setminus T \to Y$ factors through g, and*
(4) *$f_{/T} : X_{/T} \to Y$ factors through g,*

then f factors through g.

Proof. Consider the fibre product $E = X \times_Y Z \to X$. By assumption the open immersion $X \setminus T \to X$ factors through E and any morphism $\varphi : X' \to X$ with $|\varphi|(|X'|) \subset T$ factors through E as well, see Formal Spaces, Section 87.14. By More on Morphisms of Spaces, Lemma 76.20.3 this implies that $E \to X$ is étale at every point of E mapping to a point of T. Hence $E \to X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma 67.51.2). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$. □

0GI1 **Lemma 88.25.4.** *Let S be a scheme. Let X, W be algebraic spaces over S with X locally Noetherian. Let $T \subset |X|$ be a closed subset. Let $a, b : X \to W$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$ and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to W$. Then $a = b$.*

Proof. Let E be the equalizer of a and b. Then E is an algebraic space and $E \to X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma 67.4.1. Our assumptions imply we can apply Lemma 88.25.3 to the two morphisms $f = \mathrm{id} : X \to X$ and $g : E \to X$ and the closed subset T of $|X|$. □

0AR7 **Lemma 88.25.5.** *Let S be a scheme. Let X, Y be locally Noetherian algebraic spaces over S. Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \to Y$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to Y_{/T'}$. Then $a = b$.*

Proof. Consequence of the more general Lemma 88.25.4. □

0AR8 **Lemma 88.25.6.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $s, t : R \to U$ be two morphisms of algebraic spaces over X. Assume*

1. *R, U are locally of finite type over X,*
2. *the base change of s and t to $X \setminus T$ is an étale equivalence relation, and*
3. *the formal completion $(t_{/T}, s_{/T}) : R_{/T} \to U_{/T} \times_{X_{/T}} U_{/T}$ is an equivalence relation too (see proof for notation).*

Then $(t, s) : R \to U \times_X U$ is an étale equivalence relation.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 88.23. The morphisms $s, t : R \to U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \to U$ are étale, we see that s and t are étale for example by More on Morphisms, Lemma 37.12.3. Applying Lemma 88.25.3 to the morphisms $\mathrm{id} : R \times_{U \times_X U} R \to R \times_{U \times_X U} R$ and $\Delta : R \to R \times_{U \times_X U} R$ we conclude that (t, s) is a monomorphism. Applying it again to $(t \circ \mathrm{pr}_0, s \circ \mathrm{pr}_1) : R \times_{s,U,t} R \to U \times_X U$ and $(t, s) : R \to U \times_X U$ we find that "transitivity" holds. We omit the proof of the other two axioms of an equivalence relation. □

0AR9 **Lemma 88.25.7.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S and let $T \subset |X|$ be a closed subset. Let $f : X' \to X$ be a morphism of algebraic spaces which is locally of finite type and étale outside of T. There exists a factorization*

$$X' \longrightarrow X'' \longrightarrow X$$

of f with the following properties: $X'' \to X$ is locally of finite type, $X'' \to X$ is an isomorphism over $X \setminus T$, and $X'_{/T} \to X''_{/T}$ is an isomorphism (see proof for notation).

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 88.23. We will also use the notion $U \subset X$ and $U' \subset X'$ to denote the open subspaces with $|U| = |X| \setminus T$ and $U' = |X'| \setminus f^{-1}T$ introduced in Section 88.23.

After replacing X' by $X' \amalg U$ we may and do assume the image of $X' \to X$ contains U. Let

$$R = X' \amalg_{U'} (U' \times_U U')$$

be the pushout of $U' \to X'$ and the diagonal morphism $U' \to U' \times_U U' = U' \times_X U'$. Since $U' \to X$ is étale, this diagonal is an open immersion and we see that R is an algebraic space (this follows for example from Spaces, Lemma 65.8.5). The two projections $U' \times_U U' \to U'$ extend to R and we obtain two étale morphisms $s, t : R \to X'$. Checking on each piece separately we find that R is an étale equivalence relation on X'. Set $X'' = X'/R$ which is an algebraic space by Bootstrap, Theorem 80.10.1. By construction have the factorization as in the lemma and the morphism $X'' \to X$ is locally of finite type (as this can be checked étale locally, i.e., on X'). Since $U' \to U$ is a surjective étale morphism and since $s^{-1}(U') = t^{-1}(U') = U' \times_U U'$ we see that $U'' = U \times_X X'' \to U$ is an isomorphism. Finally, we have to show the morphism $X' \to X''$ induces an isomorphism $X'_{/T} \to X''_{/T}$. To see this, note that the formal completion of R along the inverse image of T is equal to the formal completion of X' along the inverse image of T by our choice of R! By our construction of the formal completion in Formal Spaces, Section 87.14 we have $X''_{/T} = (X'_{/T})/(R_{/T})$ as sheaves. Since $X'_{/T} = R_{/T}$ we conclude that $X'_{/T} = X''_{/T}$ and this finishes the proof. □

88.26. Rig glueing of morphisms

0GI2 Let X, W be algebraic spaces with X Noetherian. Let $Z \subset X$ be a closed subspace with open complement U. The proposition below says roughly speaking that

$$\{\text{morphisms } X \to W\} = \{\text{compatible morphisms } U \to W \text{ and } X_{/Z} \to W\}$$

where compatibility of $a : U \to W$ and $b : X_{/Z} \to W$ means that a and b define the same "morphism of rig-spaces". To introduce the category of "rig-spaces" requires a lot of work, but we don't need to do so in order to state precisely what the condition means in this case.

0GI3 **Proposition 88.26.1.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset with complementary open subspace $U \subset X$. Let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism. For any algebraic space W over S the map*

$$\mathrm{Mor}_S(X, W) \longrightarrow \mathrm{Mor}_S(X', W) \times_{\mathrm{Mor}_S(X'_{/T}, W)} \mathrm{Mor}_S(X_{/T}, W)$$

is bijective.

Proof. Let $w' : X' \to W$ and $\hat{w} : X_{/T} \to W$ be morphisms which determine the same morphism $X'_{/T} \to W$ by composition with $X'_{/T} \to X$ and $X'_{/T} \to X_{/T}$. We

have to prove there exists a unique morphism $w : X \to W$ whose composition with $X' \to X$ and $X_{/T} \to X$ recovers w' and $\hat{w}$. The uniqueness is immediate from Lemma 88.25.4.

The assumptions on T and f are preserved by base change by any étale morphism $X_1 \to X$ of algebraic spaces. Since formal algebraic spaces are sheaves for the étale topology and since we already have the uniqueness, it suffices to prove existence after replacing X by the members of an étale covering. Thus we may assume X is an affine Noetherian scheme.

Assume X is an affine Noetherian scheme. We will construct the morphism $w : X \to W$ using the material in Pushouts of Spaces, Section 81.13. It makes sense to read a little bit of the material in that section before continuing the read the proof.

Set $X'' = X' \times_X X'$ and consider the two morphisms $a = w' \circ \mathrm{pr}_1 : X'' \to W$ and $b = w' \circ \mathrm{pr}_2 : X'' \to W$. Then we see that a and b agree over the open U and that $a_{/T}$ and $b_{a/T}$ agree (as these are both equal to the composition $X''_{/T} \to X_{/T} \to W$ where the second arrow is $\hat{w}$). Thus by Lemma 88.25.4 we see $a = b$.

Denote $Z \subset X$ the reduced induced closed subscheme structure on T. For $n \geq 1$ denote $Z_n \subset X$ the nth infinitesimal neighbourhood of Z. Denote $w_n = \hat{w}|_{Z_n} : Z_n \to W$ so that we have $\hat{w} = \operatorname{colim} w_n$ on $X_{/T} = \operatorname{colim} Z_n$. Set $Y_n = X' \amalg Z_n$. Consider the two projections

$$s_n, t_n : R_n = Y_n \times_X Y_n \longrightarrow Y_n$$

Let $Y_n \to X_n \to X$ be the coequalizer of s_n and t_n as in Pushouts of Spaces, Section 81.13 (in particular this coequalizer exists, has good properties, etc, see Pushouts of Spaces, Lemma 81.13.1). By the result $a = b$ of the previous parapgraph and the agreement of w' and $\hat{w}$ over $X'_{/T}$ we see that the morphism

$$w' \amalg w_n : Y_n \longrightarrow W$$

equalizes the morphisms s_n and t_n. Hence we see that for all $n \geq 1$ there is a morphism $w^n : X_n \to W$ compatible with w' and w_n. Moreover, for $m \geq 1$ the composition

$$X_n \to X_{n+m} \xrightarrow{w^{n+m}} W$$

is equal to w^n by construction (as the corresponding statement holds for $w' \amalg w_{n+m}$ and $w' \amalg w_n$). By Pushouts of Spaces, Lemma 81.13.4 and Remark 81.13.5 the system of algebraic spaces X_n is essentially constant with value X and we conclude. □

88.27. Algebraization of rig-étale morphisms

0AR1 In this section we prove a generalization of the result on dilatations from the paper of Artin [**Art70**].

The notation in this section will agree with the notation in Section 88.23 except our algebraic spaces and formal algebraic spaces will be locally Noetherian.

Thus, we first fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix a locally Noetherian algebraic space X and a closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

Given a morphism of algebraic spaces $f : X' \to X$, we will use the notation $U' = f^{-1}U$, $T' = |f|^{-1}(T)$, and $f_{/T} : X'_{/T'} \to X_{/T}$ as in Section 88.23. We will sometimes write $X'_{/T}$ in stead of $X'_{/T'}$ and more generally for a morphism $a : X' \to X''$ of algebraic spaces over X we will denote $a_{/T} : X'_{/T} \to X''_{/T}$ the induced morphism of formal algebraic spaces obtained by completing the morphism a along the inverse images of T in X' and X''.

Given this setup we will consider the functor
(88.27.0.1)

0AR5
$$\left\{\begin{array}{c}\text{morphisms of algebraic spaces}\\ f : X' \to X \text{ which are locally}\\ \text{of finite type and such that}\\ U' \to U \text{ is an isomorphism}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{morphisms } g : W \to X_{/T}\\ \text{of formal algebraic spaces}\\ \text{with } W \text{ locally Noetherian}\\ \text{and } g \text{ rig-étale}\end{array}\right\}$$

sending $f : X' \to X$ to $f_{/T} : X'_{/T'} \to X_{/T}$. This makes sense because $f_{/T}$ is rig-étale by Lemma 88.23.9.

0GDQ **Lemma 88.27.1.** *In the situation above, let $X_1 \to X$ be a morphism of algebraic spaces with X_1 locally Noetherian. Denote $T_1 \subset |X_1|$ the inverse image of T and $U_1 \subset X_1$ the inverse image of U. We denote*

(1) *$\mathcal{C}_{X,T}$ the category whose objects are morphisms of algebraic spaces $f : X' \to X$ which are locally of finite type and such that $U' = f^{-1}U \to U$ is an isomorphism,*
(2) *$\mathcal{C}_{X_1,T_1}$ the category whose objects are morphisms of algebraic spaces $f_1 : X'_1 \to X_1$ which are locally of finite type and such that $f_1^{-1}U_1 \to U_1$ is an isomorphism,*
(3) *$\mathcal{C}_{X_{/T}}$ the category whose objects are morphisms $g : W \to X_{/T}$ of formal algebraic spaces with W locally Noetherian and g rig-étale,*
(4) *$\mathcal{C}_{X_{1,/T_1}}$ the category whose objects are morphisms $g_1 : W_1 \to X_{1,/T_1}$ of formal algebraic spaces with W_1 locally Noetherian and g_1 rig-étale.*

Then the diagram

$$\begin{array}{ccc} \mathcal{C}_{X,T} & \longrightarrow & \mathcal{C}_{X_{/T}} \\ \downarrow & & \downarrow \\ \mathcal{C}_{X_1,T_1} & \longrightarrow & \mathcal{C}_{X_{1,/T_1}} \end{array}$$

is commutative where the horizontal arrows are given by (88.27.0.1) and the vertical arrows by base change along $X_1 \to X$ and along $X_{1,/T_1} \to X_{/T}$.

Proof. This follows immediately from the fact that the completion functor $(h : Y \to X) \mapsto Y_{/T} = Y_{/|h|^{-1}T}$ on the category of algebraic spaces over X commutes with fibre products. □

0GDR **Lemma 88.27.2.** *In the situation above. Let $f : X' \to X$ be a morphism of algebraic spaces which is locally of finite type and an isomorphism over U. Let $g : Y \to X$ be a morphism with Y locally Noetherian. Then completion defines a bijection*

$$\mathrm{Mor}_X(Y, X') \longrightarrow \mathrm{Mor}_{X_{/T}}(Y_{/T}, X'_{/T})$$

In particular, the functor (88.27.0.1) is fully faithful.

Proof. Let $a, b : Y \to X'$ be morphisms over X such that $a_{/T} = b_{/T}$. Then we see that a and b agree over the open subspace $g^{-1}U$ and after completion along $g^{-1}T$. Hence $a = b$ by Lemma 88.25.5. In other words, the completion map is always injective.

Let $\alpha : Y_{/T} \to X'_{/T}$ be a morphism of formal algebraic spaces over $X_{/T}$. We have to prove there exists a morphism $a : Y \to X'$ over X such that $\alpha = a_{/T}$. The proof proceeds by a standard but cumbersome reduction to the affine case and then applying Lemma 88.25.2.

Let $\{h_i : Y_i \to Y\}$ be an étale covering of algebraic spaces. If we can find for each i a morphism $a_i : Y_i \to X'$ over X whose completion $(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}$ is equal to $\alpha \circ (h_i)_{/T}$, then we get a morphism $a : Y \to X'$ with $\alpha = a_{/T}$. Namely, we first observe that $(a_i)_{/T} \circ \mathrm{pr}_1 = (a_j)_{/T} \circ \mathrm{pr}_2$ as morphisms $(Y_i \times_Y Y_j)_{/T} \to X'_{/T}$ by the agreement with α (this uses that completion ${}_{/T}$ commutes with fibre products). By the injectivity already proven this shows that $a_i \circ \mathrm{pr}_1 = a_j \circ \mathrm{pr}_2$ as morphisms $Y_i \times_Y Y_j \to X'$. Since X' is an fppf sheaf this means that the collection of morphisms a_i descends to a morphism $a : Y \to X'$. We have $\alpha = a_{/T}$ because $\{(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}\}$ is an étale covering.

By the result of the previous paragraph, to prove existence, we may assume that Y is affine and that $g : Y \to X$ factors as $g_1 : Y \to X_1$ and an étale morphism $X_1 \to X$ with X_1 affine. Then we can consider $T_1 \subset |X_1|$ the inverse image of T and we can set $X'_1 = X' \times_X X_1$ with projection $f_1 : X'_1 \to X_1$ and

$$\alpha_1 = (\alpha, (g_1)_{/T_1}) : Y_{/T_1} = Y_{/T} \longrightarrow X'_{/T} \times_{X_{/T}} (X_1)_{/T_1} = (X'_1)_{/T_1}$$

We conclude that it suffices to prove the existence for α_1 over X_1, in other words, we may replace $X, T, X', Y, f, g, \alpha$ by $X_1, T_1, X'_1, Y, g_1, \alpha_1$. This reduces us to the case described in the next paragraph.

Assume Y and X are affine. Recall that $(Y_{/T})_{red}$ is an affine scheme (isomorphic to the reduced induced scheme structure on $g^{-1}T \subset Y$, see Formal Spaces, Lemma 87.14.5). Hence $\alpha_{red} : (Y_{/T})_{red} \to (X'_{/T})_{red}$ has quasi-compact image E in $f^{-1}T$ (this is the underlying topological space of $(X'_{/T})_{red}$ by the same lemma as above). Thus we can find an affine scheme V and an étale morpism $h : V \to X'$ such that the image of h contains E. Choose a solid cartesian diagram

$$\begin{array}{ccccc} Y'_{/T} & \dashrightarrow & W & \longrightarrow & V_{/T} \\ & \searrow & \downarrow & & \downarrow{\scriptstyle h_{/T}} \\ & & Y_{/T} & \xrightarrow{\alpha} & X'_{/T} \end{array}$$

By construction, the morphism $W \to Y_{/T}$ is representable by algebraic spaces, étale, and surjective (surjectivity can be seen by looking at the reductions, see Formal Spaces, Lemma 87.12.4). By Lemma 88.25.1 we can write $W = Y'_{/T}$ for $Y' \to Y$ étale and Y' affine. This gives the dotted arrows in the diagram. Since $W \to Y_{/T}$ is surjective, we see that the image of $Y' \to Y$ contains $g^{-1}T$. Hence $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$ is an étale covering. As f is an isomorphism over U we have a (unique) morphism $Y \setminus g^{-1}T \to X'$ over X agreeing with α on completions (as the completion of $Y \setminus g^{-1}T$ is empty). Thus it suffices to prove the existence for Y' which reduces us to the case studied in the next paragraph.

By the result of the previous paragraph, we may assume that Y is affine and that α factors as $Y_{/T} \to V_{/T} \to X'_{/T}$ where V is an affine scheme étale over X'. We may still replace Y by the members of an affine étale covering. By Lemma 88.25.2 we may find an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $c : Y' \to V$ such that $c_{/T} \circ b_{/T}^{-1}$ is the given morphism $Y_{/T} \to V_{/T}$. Setting $a' : Y' \to X'$ equal to the composition of c and $V \to X'$ we find that $a'_{/T} = \alpha \circ b_{/T}$, in other words, we have existence for Y' and $\alpha \circ b_{/T}$. Then we are done by replacing considering once more the étale covering $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$. □

0ARA **Lemma 88.27.3.** *In the situation above. Assume X is affine. Then the functor (88.27.0.1) is an equivalence.*

Before we prove this lemma let us discuss an example. Suppose that $S = \mathrm{Spec}(k)$, $X = \mathbf{A}^1_k$, and $T = \{0\}$. Then $X_{/T} = \mathrm{Spf}(k[[x]])$. Let $W = \mathrm{Spf}(k[[x]] \times k[[x]])$. Then the corresponding $f : X' \to X$ is the affine line with zero doubled mapping to the affine line (Schemes, Example 26.14.3). Moreover, this is the output of the construction in Lemma 88.25.7 starting with $X \amalg X$ over X.

Proof. We already know the functor is fully faithful, see Lemma 88.27.2. Essential surjectivity. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image in a number of steps.

Step 1: W is an affine formal algebraic space. Then we can find $U \to X$ of finite type and étale over $X \setminus T$ such that $U_{/T}$ is isomorphic to W, see Lemma 88.25.1. Thus we see that W is in the essential image by Lemma 88.25.7.

Step 2: W is separated. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 87.11.1. By Step 1 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Say $W_i = (X'_i)_{/T}$ and $W_i \times_W W_j = (X'_{ij})_{/T}$. By fully faithfulness we obtain morphisms $t_{ij} : X'_{ij} \to X'_i$ and $s_{ij} : X'_{ij} \to X'_j$ matching the projections $W_i \times_W W_j \to W_i$ and $W_i \times_W W_j \to W_j$. Consider the structure

$$R = \coprod X'_{ij}, \quad V = \coprod X'_i, \quad s = \coprod s_{ij}, \quad t = \coprod t_{ij}$$

(We can't use the letter U as it has already been used.) Applying Lemma 88.25.6 we find that $(t, s) : R \to V \times_X V$ defines an étale equivalence relation on V over X. Thus we can take the quotient $X' = V/R$ and it is an algebraic space, see Bootstrap, Theorem 80.10.1. Since completion commutes with fibre products and taking quotient sheaves, we find that $X'_{/T} \cong W$ as formal algebraic spaces over $X_{/T}$.

Step 3: W is general. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 87.11.1. The formal algebraic spaces W_i and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that W is in the essential image as well. This concludes the proof. □

0ARB **Theorem 88.27.4.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $U \subset X$ be the open subspace with*

$|U| = |X| \setminus T$. *The completion functor (88.27.0.1)*

$$\left\{\begin{array}{c}\text{morphisms of algebraic spaces}\\ f : X' \to X \text{ which are locally}\\ \text{of finite type and such that}\\ f^{-1}U \to U \text{ is an isomorphism}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{morphisms } g : W \to X_{/T}\\ \text{of formal algebraic spaces}\\ \text{with } W \text{ locally Noetherian}\\ \text{and } g \text{ rig-étale}\end{array}\right\}$$

sending $f : X' \to X$ *to* $f_{/T} : X'_{/T'} \to X_{/T}$ *is an equivalence.*

Proof. The functor is fully faithful by Lemma 88.27.2. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image to finish the proof.

Choose an étale covering $\{X_i \to X\}$ with X_i affine for all i. Denote $U_i \subset X_i$ the inverse image of U and denote $T_i \subset X_i$ the inverse image of T. Recall that $(X_i)_{/T_i} = (X_i)_{/T} = (X_i \times_X X)_{/T}$ and $W_i = X_i \times_X W = (X_i)_{/T} \times_{X_{/T}} W$, see Lemma 88.27.1. Observe that we obtain isomorphisms

$$\alpha_{ij} : W_i \times_{X_{/T}} (X_j)_{/T} \longrightarrow (X_i)_{/T} \times_{X_{/T}} W_j$$

satisfying a suitable cocycle condition. By Lemma 88.27.3 applied to $X_i, T_i, U_i, W_i \to (X_i)_{/T}$ there exists a morphism $X'_i \to X_i$ of algebraic spaces which is locally of finite type and an isomorphism over U_i and an isomorphism $\beta_i : (X'_i)_{/T} \cong W_i$ over $(X_i)_{/T}$. By fully faithfullness we find an isomorphism

$$a_{ij} : X'_i \times_X X_j \longrightarrow X_i \times_X X'_j$$

over $X_i \times_X X_j$ such that $\alpha_{ij} = \beta_j|_{X_i \times_X X_j} \circ (a_{ij})_{/T} \circ \beta_i^{-1}|_{X_i \times_X X_j}$. By fully faithfulness again (this time over $X_i \times_X X_j \times_X X_k$) we see that these morphisms a_{ij} satisfy the same cocycle condition as satisfied by the α_{ij}. In other words, we obtain a descent datum (as in Descent on Spaces, Definition 74.22.3) (X'_i, a_{ij}) relative to the family $\{X_i \to X\}$. By Bootstrap, Lemma 80.11.3, this descent datum is effective. Thus we find a morphism $f : X' \to X$ of algebraic spaces and isomorphisms $h_i : X' \times_X X_i \to X'_i$ over X_i such that $a_{ij} = h_j|_{X_i \times_X X_j} \circ h_i^{-1}|_{X_i \times_X X_j}$. The reader can check that the ensuing isomorphisms

$$(X' \times_X X_i)_{/T} \xrightarrow{\beta_i \circ (h_i)_{/T}} W_i$$

over X_i glue to an isomorphism $X'_{/T} \to W$ over $X_{/T}$; some details omitted. □

88.28. Completions and morphisms, II

0GDS To obtain Artin's theorem on dilatations, we need to match formal modifications with actual modifications in the correspondence given by Theorem 88.27.4. We urge the reader to skip this section.

0ARU **Lemma 88.28.1.** *With assumptions and notation as in Theorem 88.27.4 let* $f : X' \to X$ *correspond to* $g : W \to X_{/T}$. *Then* f *is quasi-compact if and only if* g *is quasi-compact.*

Proof. If f is quasi-compact, then g is quasi-compact by Lemma 88.23.5. Conversely, assume g is quasi-compact. Choose an étale covering $\{X_i \to X\}$ with X_i affine. It suffices to prove that the base change $X' \times_X X_i \to X_i$ is quasi-compact, see Morphisms of Spaces, Lemma 67.8.8. By Formal Spaces, Lemma 87.17.3 the base changes $W_i \times_{X_{/T}} (X_i)_{/T} \to (X_i)_{/T}$ are quasi-compact. By Lemma 88.27.1 we reduce to the case described in the next paragraph.

Assume X is affine and $g : W \to X_{/T}$ quasi-compact. We have to show that X' is quasi-compact. Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $V_{/T} \to X'_{/T} = W$ is a surjective étale morphism. Since W is quasi-compact, then we can find a finite subset $J' \subset J$ such that $\coprod_{j \in J'} (V_j)_{/T} \to W$ is surjective. Then it follows that

$$U \amalg \coprod\nolimits_{j \in J'} V_j \longrightarrow X'$$

is surjective (and hence X' is quasi-compact). Namely, we have $|X'| = |U| \amalg |W_{red}|$ as $X'_{/T} = W$. □

0ARV **Lemma 88.28.2.** *With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is quasi-separated if and only if g is so.*

Proof. If f is quasi-separated, then g is quasi-separated by Lemma 88.23.7. Conversely, assume g is quasi-separated. We have to show that f is quasi-separated. Exactly as in the proof of Lemma 88.28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 67.4.12 and Formal Spaces, Lemma 87.30.5. Thus we may and do assume X is affine.

Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. To show that X' is quasi-separated, it suffices to show that $V_j \times_{X'} V_{j'}$ is quasi-compact for all $j, j' \in J$. Since W is quasi-separated the fibre products $(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{X'_{/T}} (V_{j'})_{/T}$ are quasi-compact for all $j, j' \in J$. Since X is Noetherian affine and $U' \to U$ is an isomorphism, we see that

$$(V_j \times_{X'} V_{j'}) \times_X U = (V_j \times_X V_{j'}) \times_X U$$

is quasi-compact. Hence we conclude by the equality

$$|V_j \times_{X'} V_{j'}| = |(V_j \times_{X'} V_{j'}) \times_X U| \amalg |(V_j \times_{X'} V_{j'})_{/T,red}|$$

and the fact that a formal algebraic space is quasi-compact if and only if its associated reduced algebraic space is so. □

0ARW **Lemma 88.28.3.** *With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is separated $\Leftrightarrow$ g is separated and $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective.*

Proof. If f is separated, then g is separated and Δ_g is rig-surjective by Lemmas 88.23.7 and 88.23.11. Assume g is separated and Δ_g is rig-surjective. Exactly as in the proof of Lemma 88.28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 67.4.4 (locality on the base of being separated for morphisms of algebraic spaces), Formal Spaces, Lemma 87.30.2 (being separated for morphisms of formal algebraic spaces is preserved by base change), and Lemma 88.21.4 (being rig-surjective is preserved by base change). Thus we may and do assume X is affine. Furthermore, we already know that $f : X' \to X$ is quasi-separated by Lemma 88.28.2.

By Cohomology of Spaces, Lemma 69.19.1 and Remark 69.19.3 it suffices to show that given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X' \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{p} & X' \times_X X' \end{array}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute (as this will give the uniqueness part of the valuative criterion). Let $h : \mathrm{Spec}(R) \to X$ be the composition of p with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' = X' \times_X U \to U$ is an isomorphism. Case II: h maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is separated. Namely, if Z denotes the reduced induced closed subspace structure on T, then h factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is separated by assumption (and for example Formal Spaces, Lemma 87.30.5) which implies we get the lifting property by Cohomology of Spaces, Lemma 69.19.1 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow (X' \times_X X')_{/T} = W \times_{X_{/T}} W$$

is an adic morphism (by Formal Spaces, Lemma 87.14.4 and Definition 87.23.2). Hence our assumption that $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective implies we can lift $p_{/T}$ to a morphism $\mathrm{Spf}(R) \to W = X'_{/T}$, see Lemma 88.21.11. Algebraizing the composition $\mathrm{Spf}(R) \to X'$ using Formal Spaces, Lemma 87.33.3 we find a morphism $\mathrm{Spec}(R) \to X'$ lifting p as desired. □

0ARX **Lemma 88.28.4.** *With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is proper if and only if g is a formal modification (Definition 88.24.1).*

Proof. If f is proper, then g is a formal modification by Lemma 88.24.3. Assume g is a formal modification. By Lemmas 88.28.1 and 88.28.3 we see that f is quasi-compact and separated.

By Cohomology of Spaces, Lemma 69.19.2 and Remark 69.19.3 it suffices to show that given any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X' \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(R) & \xrightarrow{p} & X \end{array}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute. There are three cases: Case I: $p(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' \to U$ is an isomorphism. Case II: p maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is proper. Namely,

if Z denotes the reduced induced closed subspace structure on T, then p factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 69.19.2 applied to the displayed arrow. Case III: $p(\mathrm{Spec}(K))$ is not in T but p maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow X'_{/T} = W$$

is an adic morphism (by Formal Spaces, Lemma 87.14.4 and Definition 87.23.2). Hence our assumption that $g : W \to X_{/T}$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $\mathrm{Spf}(R') \to W = X'_{/T}$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\mathrm{Spf}(R') \to X'$ using Formal Spaces, Lemma 87.33.3 we find a morphism $\mathrm{Spec}(R') \to X'$ lifting p as desired. □

0GI4 **Lemma 88.28.5.** *With assumptions and notation as in Theorem 88.27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is étale if and only if g is étale.*

Proof. If f is étale, then g is étale by Lemma 88.23.2. Conversely, assume g is étale. Since f is an isomorphism over U we see that f is étale over U. Thus it suffices to prove that f is étale at any point of X' lying over T. Denote $Z \subset X$ the reduced closed subspace whose underlying topological space is $|Z| = T \subset |X|$, see Properties of Spaces, Definition 66.12.5. Letting $Z_n \subset X$ be the nth infinitesimal neighbourhood we have $X_{/T} = \mathrm{colim}\, Z_n$. Since $X'_{/T} = W \to X_{/T}$ we conclude that $f^{-1}(Z_n) = X' \times_X Z_n \to Z_n$ is étale by the assumed étaleness of g. By More on Morphisms of Spaces, Lemma 76.20.3 we conclude that f is étale at points lying over T. □

88.29. Artin's theorem on dilatations

0GDT In this section we use a different font for formal algebraic spaces to stress the similarity of the statements with the corresponding statements in [**Art70**]. Here is the first main theorem of this chapter.

0GDU **Theorem 88.29.1.** *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $\mathfrak{X} = X_{/T}$ be the formal completion of X along T. Let* [**Art70**, Theorem 3.2]

$$\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$$

be a formal modification (Definition 88.24.1). Then there exists a unique proper morphism $f : X' \to X$ which is an isomorphism over the complement of T in X whose completion $f_{/T}$ recovers $\mathfrak{f}$.

Proof. This follows from Theorem 88.27.4 and Lemma 88.28.4. □

Here is the characterization of formal modifications as promised in Section 88.24.

0GDV **Lemma 88.29.2.** *Let S be a scheme. Let $\mathfrak{X}' \to \mathfrak{X}$ be a formal modification (Definition 88.24.1) of locally Noetherian formal algebraic spaces over S. Given*

(1) *any adic Noetherian topological ring A,*
(2) *any adic morphism $Spf(A) \longrightarrow \mathfrak{X}$*

there exists a proper morphism $X \to \mathrm{Spec}(A)$ of algebraic spaces and an isomorphism

$$Spf(A) \times_{\mathfrak{X}} \mathfrak{X}' \longrightarrow X_{/Z}$$

over $Spf(A)$ of the base change of $\mathfrak{X}$ with the formal completion of X along the "closed fibre" $Z = X \times_{\mathrm{Spec}(A)} Spf(A)_{red}$ of X over A.

Proof. The morphism $\mathrm{Spf}(A) \times_{\mathfrak{X}} \mathfrak{X}' \to \mathrm{Spf}(A)$ is a formal modification by Lemma 88.24.4. Hence this follows from Theorem 88.29.1. □

88.30. Application to modifications

0AS1 Let A be a Noetherian ring and let $I \subset A$ be an ideal. We set $X = \mathrm{Spec}(A)$ and $U = X \setminus V(I)$. In this section we will consider the category

0AS2 (88.30.0.1)
$$\left\{ f : X' \longrightarrow X \;\middle|\; \begin{array}{c} X' \text{ is an algebraic space} \\ f \text{ is locally of finite type} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X'/X to X''/X will be a morphism of algebraic spaces $X' \to X''$ over X.

Let $A \to B$ be a homomorphism of Noetherian rings and let $J \subset B$ be an ideal such that $J = \sqrt{IB}$. Then base change along the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives a functor from the category (88.30.0.1) for A to the category (88.30.0.1) for B.

0AE5 **Lemma 88.30.1.** *Let $A \to B$ be a ring homomorphism of Noetherian rings inducing an isomorphism on I-adic completions for some ideal $I \subset A$ (for example if B is the I-adic completion of A). Then base change defines an equivalence of categories between the category (88.30.0.1) for (A, I) with the category (88.30.0.1) for (B, IB).*

Proof. Set $X = \mathrm{Spec}(A)$ and $T = V(I)$. Set $X_1 = \mathrm{Spec}(B)$ and $T_1 = V(IB)$. By Theorem 88.27.4 (in fact we only need the affine case treated in Lemma 88.27.3) the category (88.30.0.1) for X and T is equivalent to the category of rig-étale morphisms $W \to X_{/T}$ of locally Noetherian formal algebraic spaces. Similarly, the category (88.30.0.1) for X_1 and T_1 is equivalent to the category of rig-étale morphisms $W_1 \to X_{1,/T_1}$ of locally Noetherian formal algebraic spaces. Since $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X_{1,/T_1} = \mathrm{Spf}(B^\wedge)$ (Formal Spaces, Lemma 87.14.6) we see that these categories are equivalent by our assumption that $A^\wedge \to B^\wedge$ is an isomorphism. We omit the verification that this equivalence is given by base change. □

0BH5 **Lemma 88.30.2.** *Notation and assumptions as in Lemma 88.30.1. Let $f : X' \to \mathrm{Spec}(A)$ correspond to $g : Y' \to \mathrm{Spec}(B)$ via the equivalence. Then f is quasi-compact, quasi-separated, separated, proper, finite, and add more here if and only if g is so.*

Proof. You can deduce this for the statements quasi-compact, quasi-separated, separated, and proper by using Lemmas 88.28.1 88.28.2, 88.28.3, 88.28.2, and 88.28.4 to translate the corresponding property into a property of the formal completion and using the argument of the proof of Lemma 88.30.1. However, there is a direct argument using fpqc descent as follows. First, you can reduce to proving the lemma for $A \to A^\wedge$ and $B \to B^\wedge$ since $A^\wedge \to B^\wedge$ is an isomorphism. Then note that $\{U \to \mathrm{Spec}(A), \mathrm{Spec}(A^\wedge) \to \mathrm{Spec}(A)\}$ is an fpqc covering with $U = \mathrm{Spec}(A) \setminus V(I)$

as before. The base change of f by $U \to \mathrm{Spec}(A)$ is id_U by definition of our category (88.30.0.1). Let P be a property of morphisms of algebraic spaces which is fpqc local on the base (Descent on Spaces, Definition 74.10.1) such that P holds for identity morphisms. Then we see that P holds for f if and only if P holds for g. This applies to P equal to quasi-compact, quasi-separated, separated, proper, and finite by Descent on Spaces, Lemmas 74.11.1, 74.11.2, 74.11.18, 74.11.19, and 74.11.23. □

0AF7 **Lemma 88.30.3.** *Let $A \to B$ be a local map of local Noetherian rings such that*

(1) *$A \to B$ is flat,*
(2) *$\mathfrak{m}_B = \mathfrak{m}_A B$, and*
(3) *$\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$*

Then the base change functor from the category (88.30.0.1) for $(A, \mathfrak{m}_A)$ to the category (88.30.0.1) for $(B, \mathfrak{m}_B)$ is an equivalence.

Proof. The conditions signify that $A \to B$ induces an isomorphism on completions, see More on Algebra, Lemma 15.43.9. Hence this lemma is a special case of Lemma 88.30.1. □

0AE6 **Lemma 88.30.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f : X \to S$ be an object of (88.30.0.1) such that f is proper. Then there exists a U-admissible blowup $S' \to S$ which dominates X.*

Proof. Special case of More on Morphisms of Spaces, Lemma 76.39.5. □

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CHAPTER 89

Resolution of Surfaces Revisited

0BH6

89.1. Introduction

0BH7 This chapter discusses resolution of singularities of Noetherian algebraic spaces of dimension 2. We have already discussed resolution of surfaces for schemes following Lipman [**Lip78**] in an earlier chapter. See Resolution of Surfaces, Section 54.1. Most of the results in this chapter are straightforward consequences of the results on schemes.

Unless specifically mentioned otherwise all geometric objects in this chapter will be algebraic spaces. Thus if we say "let $f : X \to Y$ be a modification" then this means that f is a morphism as in Spaces over Fields, Definition 72.8.1. Similarly for proper morphism, etc, etc.

89.2. Modifications

0BH8 Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. We set $S = \mathrm{Spec}(A)$ and $U = S \setminus \{\mathfrak{m}\}$. In this section we will consider the category

0AE2 (89.2.0.1)
$$\left\{ f : X \longrightarrow S \;\middle|\; \begin{array}{c} X \text{ is an algebraic space} \\ f \text{ is a proper morphism} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over S. In Algebraization of Formal Spaces, Section 88.30 we have seen that this category only depends on the completion of A and we have proven some elementary properties of objects in this category. In this section we specifically study cases where $\dim(A) \leq 2$ or where the dimension of the closed fibre is at most 1.

0AE3 **Lemma 89.2.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a 2-dimensional Noetherian local domain such that $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is a normal scheme. Then any modification $f : X \to \mathrm{Spec}(A)$ is a morphism as in (89.2.0.1).*

Proof. Let $f : X \to S$ be a modification. We have to show that $f^{-1}(U) \to U$ is an isomorphism. Since every closed point u of U has codimension 1, this follows from Spaces over Fields, Lemma 72.3.3. □

0AGM **Lemma 89.2.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $g : X \to Y$ be a morphism in the category (89.2.0.1). If the induced morphism $X_\kappa \to Y_\kappa$ of special fibres is a closed immersion, then g is a closed immersion.*

Proof. This is a special case of More on Morphisms of Spaces, Lemma 76.49.3. □

0AYJ **Lemma 89.2.3.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension ≥ 1. Let $f : X \to \mathrm{Spec}(A)$ be a morphism of algebraic spaces. Assume at least one of the following conditions is satisfied*

(1) *f is a modification (Spaces over Fields, Definition 72.8.1),*
(2) *f is an alteration (Spaces over Fields, Definition 72.8.3),*
(3) *f is locally of finite type, quasi-separated, X is integral, and there is exactly one point of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$,*
(4) *f is locally of finite type, X is decent, and the points of $|X|$ mapping to the generic point of $\mathrm{Spec}(A)$ are the generic points of irreducible components of $|X|$,*
(5) *add more here.*

Then $\dim(X_\kappa) \leq \dim(A) - 1$.

Proof. Cases (1), (2), and (3) are special cases of (4). Choose an affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. The ring map $A \to B$ is of finite type. We have to show that $\dim(U_\kappa) \leq \dim(A) - 1$. Since X is decent, the generic points of irreducible components of U are the points lying over generic points of irreducible components of $|X|$, see Decent Spaces, Lemma 68.20.1. Hence the fibre of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over (0) is the (finite) set of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. Thus $A_f \to B_f$ is finite for some nonzero $f \in A$ (Algebra, Lemma 10.122.11). We conclude $\kappa(\mathfrak{q}_i)$ is a finite extension of the fraction field of A. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. Then

$$\dim(B_\mathfrak{q}) = \max \dim((B/\mathfrak{q}_i)_\mathfrak{q}) \leq \dim(A)$$

the inequality by the dimension formula for $A \subset B/\mathfrak{q}_i$, see Algebra, Lemma 10.113.1. However, the dimension of $B_\mathfrak{q}/\mathfrak{m}B_\mathfrak{q}$ (which is the local ring of U_κ at the corresponding point) is at least one less because the minimal primes $\mathfrak{q}_i$ are not in $V(\mathfrak{m})$. We conclude by Properties, Lemma 28.10.2. □

0AGN **Lemma 89.2.4.** *If $(A, \mathfrak{m}, \kappa)$ is a complete Noetherian local domain of dimension 2, then every modification of $\mathrm{Spec}(A)$ is projective over A.*

Proof. By More on Morphisms of Spaces, Lemma 76.43.6 it suffices to show that the special fibre of any modification X of $\mathrm{Spec}(A)$ has dimension ≤ 1. This follows from Lemma 89.2.3. □

89.3. Strategy

0BH9 Let S be a scheme. Let X be a decent algebraic space over S. Let $x_1, \ldots, x_n \in |X|$ be pairwise distinct closed points. For each i we pick an elementary étale neighbourhood $(U_i, u_i) \to (X, x_i)$ as in Decent Spaces, Lemma 68.11.4. This means that U_i is an affine scheme, $U_i \to X$ is étale, u_i is the unique point of U_i lying over x_i, and $\mathrm{Spec}(\kappa(u_i)) \to X$ is a monomorphism representing x_i. After shrinking U_i we may and do assume that for $j \neq i$ there does not exist a point of U_i mapping to x_j. Observe that $u_i \in U_i$ is a closed point.

Denote $\mathcal{C}_{X,\{x_1,\ldots,x_n\}}$ the category of morphisms of algebraic spaces $f : Y \to X$ which induce an isomorphism $f^{-1}(X \setminus \{x_1, \ldots, x_n\}) \to X \setminus \{x_1, \ldots, x_n\}$. For each i denote $\mathcal{C}_{U_i,u_i}$ the category of morphisms of algebraic spaces $g_i : Y_i \to U_i$ which induce an isomorphism $g_i^{-1}(U_i \setminus \{u_i\}) \to U_i \setminus \{u_i\}$. Base change defines an functor

0BHA (89.3.0.1) $$F : \mathcal{C}_{X,\{x_1,\ldots,x_n\}} \longrightarrow \mathcal{C}_{U_1,u_1} \times \ldots \times \mathcal{C}_{U_n,u_n}$$

To reduce at least some of the problems in this chapter to the case of schemes we have the following lemma.

0BHB **Lemma 89.3.1.** *The functor F (89.3.0.1) is an equivalence.*

Proof. For $n = 1$ this is Limits of Spaces, Lemma 70.19.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to U_i$ are objects of $\mathcal{C}_{U_i,u_i}$. Then by the case $n = 1$ we can find $f'_i : Y'_i \to X$ which are isomorphisms over $X \setminus \{x_i\}$ and whose base change to U_i is f_i. Then we can set

$$f : Y = Y'_1 \times_X \ldots \times_X Y'_n \to X$$

This is an object of $\mathcal{C}_{X,\{x_1,\ldots,x_n\}}$ whose base change by $U_i \to X$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. □

0BHC **Lemma 89.3.2.** *Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then f is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, if and only if g_i is so for $i = 1, \ldots, n$.*

Proof. Follows from Limits of Spaces, Lemma 70.19.2. □

0BHD **Lemma 89.3.3.** *Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1). If $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F, then $Y_{x_i} \cong (Y_i)_{u_i}$ as algebraic spaces.*

Proof. This is clear because $u_i \to x_i$ is an isomorphism. □

89.4. Dominating by quadratic transformations

0AHG We define the blowup of a space at a point only if X is decent.

0BHE **Definition 89.4.1.** Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 68.14.6 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$. The *blowing up $X' \to X$ of X at x* means the blowing up of X in the closed subspace $Z = i(\mathrm{Spec}(k)) \subset X$.

In this generality the blowing up of X at x is not necessarily proper. However, if X is locally Noetherian, then it follows from Divisors on Spaces, Lemma 71.17.11 that the blowing up is proper. Recall that a locally Noetherian algebraic space is Noetherian if and only if it is quasi-compact and quasi-separated. Moreover, for a locally Noetherian algebraic space, being quasi-separated is equivalent to being decent (Decent Spaces, Lemma 68.14.1).

0BHF **Lemma 89.4.2.** *Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Then there exists a factorization*

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. A blowing up is a representable morphism. Hence in either case we inductively see that $Z_j \to X$ or $Z_{i,j} \to U_i$ is representable. Whence each Z_j or $Z_{i,j}$ is a decent algebraic space by Decent Spaces, Lemma 68.6.5. This shows that the assertions make sense (since blowing up is only defined for decent spaces). To prove the equivalence, let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$. The first morphism $Z_1 \to X$ is given by blowing up one of the x_i, say x_1. Applying F to $Z_1 \to X$ we find a blowup $Z_{1,1} \to U_1$ at u_1 is the blowing up at u_1 and otherwise $Z_{i,0} = U_i$ for $i > 1$. In the next step, we either blow up one of the x_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to X$ over x_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{x_1} \cong (Z_{1,1})_{u_1}$ (Lemma 89.3.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$ we construct the sequence of blowing ups of X in exactly the same manner. □

0BHG **Lemma 89.4.3.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence*

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{j+1} \to X_j$ is the blowing up of X_j at a closed point x_j lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 89.3. For each i the restriction $\mathcal{I}_i = \mathcal{I}|_{U_i} \subset \mathcal{O}_{U_i}$ is a quasi-coherent sheaf of ideals supported at u_i. The local ring of U_i at u_i is regular and has dimension 2. Thus we may apply Resolution of Surfaces, Lemma 54.4.1 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of blowing ups in closed points lying over u_i such that $\mathcal{I}_i\mathcal{O}_{X_{i,m_i}}$ is invertible. By Lemma 89.4.2 we find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. Namely, we know this is true over $X \setminus \{x_1, \ldots, x_n\}$ and in an étale neighbourhood of the fibre of each x_i it is true by construction. □

0BHH **Lemma 89.4.4.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $f : Y \to X$ be a proper morphism of algebraic spaces which is an isomorphism over $U = X \setminus T$. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Morphisms of Spaces, Lemma 76.39.5 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 89.4.3 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors on Spaces, Lemma 71.17.5) we find the desired factorization. □

89.5. Dominating by normalized blowups

0BHI In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

0BHJ **Definition 89.5.1.** Let S be a scheme. Let X be a decent algebraic space over S satisfying the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1. Let $x \in |X|$ be a closed point. The *normalized blowup of X at x* is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X at x (Definition 89.4.1) and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the algebraic space X' satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1 by Divisors on Spaces, Lemma 71.17.8. See Morphisms of Spaces, Definition 67.49.6 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that an algebraic space is Nagata if it has an étale covering by affines which are spectra of Nagata rings (Properties of Spaces, Definition 66.7.2 and Remark 66.7.3 and Properties, Definition 28.13.1).

0BHK **Lemma 89.5.2.** *In Definition 89.5.1 if X is Nagata, then the normalized blowing up of X at x is a normal Nagata algebraic space proper over X.*

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors on Spaces, Lemma 71.17.11). Thus X' is Nagata (Morphisms of Spaces, Lemma 67.26.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms of Spaces, Lemma 67.49.9) and we conclude that $X'' \to X$ is proper as well (Morphisms of Spaces, Lemmas 67.45.9 and 67.40.4). It follows that the normalized blowing up is a normal (Morphisms of Spaces, Lemma 67.49.8) Nagata algebraic space. □

Here is the analogue of Lemma 89.4.2 for normalized blowups.

0BHL **Lemma 89.5.3.** *Let $X, x_i, U_i \to X, u_i$ be as in (89.3.0.1) and assume $f : Y \to X$ corresponds to $g_i : Y_i \to U_i$ under F. Assume X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1. Then there exists a factorization*

$$Y = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = X$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = U_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over u_i.

Proof. This follows by the exact same argument as used to prove Lemma 89.4.2. □

A Nagata algebraic space is locally Noetherian.

0BHM **Lemma 89.5.4.** *Let S be a scheme. Let X be a Noetherian Nagata algebraic space over S with $\dim(X) = 2$. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} X_n & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow & & & & & & & & \downarrow \\ Y & & & & & & & & X \end{array}$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. Although one can prove this lemma directly for algebraic spaces, we will continue the approach used above to reduce it to the case of schemes.

We will use that Noetherian algebraic spaces are quasi-separated and hence points have well defined residue fields (for example by Decent Spaces, Lemma 68.11.4). We will use the results of Morphisms of Spaces, Sections 67.26, 67.35, and 67.49 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Spaces over Fields, Lemma 72.3.3. Hence we see that there is a finite set of closed points $T \subset |X|$ such that f is an isomorphism over $X \setminus T$. By More on Morphisms of Spaces, Lemma 76.39.5 there exists an $X \setminus T$-admissible blowup $Y' \to X$ which dominates Y. After replacing Y by the normalization of Y' we see that we may assume that $Y \to X$ is representable.

Say $T = \{x_1, \ldots, x_r\}$. Pick elementary étale neighbourhoods $(U_i, u_i) \to (X, x_i)$ as in Section 89.3. For each i the morphism $Y_i = Y \times_X U_i \to U_i$ is a proper birational morphism which is an isomorphism over $U_i \setminus \{u_i\}$. Thus we may apply Resolution of Surfaces, Lemma 54.5.3 to find a sequence

$$X_{i,m_i} \to X_{i,m_i-1} \to \ldots \to X_1 \to X_{i,0} = U_i$$

of normalized blowing ups in closed points lying over u_i such that X_{i,m_i} dominates Y_i. By Lemma 89.5.3 we find a sequence of normalized blowing ups

$$X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X$$

as in the statement of the lemma whose base change to our U_i produces the given sequences. It follows that X_m dominates Y by the equivalence of categories of Lemma 89.3.1. □

89.6. Base change to the completion

0BHN The following simple lemma will turn out to be a useful tool in what follows.

0BHP **Lemma 89.6.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak{m}$. Let X be a decent algebraic space over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak{m}$-adic completion of A. For a point $q \in |Y|$ with image $p \in |X|$ lying over the closed point of $\mathrm{Spec}(A)$ the map $\mathcal{O}^h_{X,p} \to \mathcal{O}^h_{Y,q}$ of henselian local rings induces an isomorphism on completions.*

Proof. This follows immediately from the case of schemes by choosing an elementary étale neighbourhood $(U, u) \to (X, p)$ as in Decent Spaces, Lemma 68.11.4, setting $V = U \times_X Y = U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ and $v = (u, q)$. The case of schemes is Resolution of Surfaces, Lemma 54.11.1. □

0BHQ **Lemma 89.6.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type with X a decent algebraic space. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in |Y|$ with image $x \in |X|$. Then*

1. *if $\mathcal{O}^h_{Y,y}$ is regular, then $\mathcal{O}^h_{X,x}$ is regular,*
2. *if y is in the closed fibre, then $\mathcal{O}^h_{Y,y}$ is regular $\Leftrightarrow$ $\mathcal{O}^h_{X,x}$ is regular, and*
3. *If X is proper over A, then X is regular if and only if Y is regular.*

Proof. By étale localization the first two statements follow immediately from the counter part to this lemma for schemes, see Resolution of Surfaces, Lemma 54.11.2. For part (3), since $Y \to X$ is surjective (as $A \to A^\wedge$ is faithfully flat) we see that Y regular implies X regular by part (1). Conversely, if X is regular, then the henselian local rings of Y are regular for all points of the special fibre. Let $y \in |Y|$ be a general point. Since $|Y| \to |\mathrm{Spec}(A^\wedge)|$ is closed in the proper case, we can find a specialization $y \rightsquigarrow y_0$ with y_0 in the closed fibre. Choose an elementary étale neighbourhood $(V, v_0) \to (Y, y_0)$ as in Decent Spaces, Lemma 68.11.4. Since Y is decent we can lift $y \rightsquigarrow y_0$ to a specialization $v \rightsquigarrow v_0$ in V (Decent Spaces, Lemma 68.12.2). Then we conclude that $\mathcal{O}_{V,v}$ is a localization of $\mathcal{O}_{V,v_0}$ hence regular and the proof is complete. □

0BHR **Lemma 89.6.3.** *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let X be an algebraic space over A. Assume*

1. *A is analytically unramified (Algebra, Definition 10.162.9),*
2. *X is locally of finite type over A,*
3. *$X \to \mathrm{Spec}(A)$ is étale at every point of codimension 0 in X.*

Then the normalization of X is finite over X.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \to \mathrm{Spec}(A)$ satisfies the assumptions and hence the conclusions of Resolution of Surfaces, Lemma 54.11.5. □

89.7. Implied properties

0BHS In this section we prove that for a Noetherian integral algebraic space the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian algebraic spaces.

0BHT **Lemma 89.7.1.** *Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.*

Proof. By étale localization, it suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 69.20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 66.25.4) we see that C is normal domain (Spaces over Fields, Lemma 72.4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Spaces over Fields, Definition 72.8.3. After shrinking V we may assume that $f^{-1}V \to V$ is flat (Morphisms of Spaces, Proposition 67.32.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 10.164.4. □

0BHU **Lemma 89.7.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \mathrm{Spec}(A)$ with X regular. Then*

1. *there exists a nonzero $f \in A$ such that A_f is regular,*
2. *the integral closure B of A in its fraction field is finite over A,*
3. *the $\mathfrak{m}$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and*
4. *the generic formal fibre of A is regular.*

Proof. Parts (1) and (2) follow from Lemma 89.7.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Spaces, Lemma 69.20.2 we see that C is a finite A-module. As X is normal (Properties of Spaces, Lemma 66.25.4) we see that C is normal domain (Spaces over Fields, Lemma 72.4.6). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Resolution of Surfaces, Lemma 54.13.2 in order to prove (3) it suffices to show that the $\mathfrak{m}$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 10.97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak{m}$. There are finitely many of these and these are the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak{m}_i}$ and X by $X_i = X \times_{\mathrm{Spec}(C)} \mathrm{Spec}(C_{\mathfrak{m}_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal{O}) = C_{\mathfrak{m}_i}$ by Cohomology of Spaces, Lemma 69.11.2.)

Here A is a Noetherian local normal domain and $f : X \to \mathrm{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 89.6.2 $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Spaces, Lemma 69.11.2. We conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties of Spaces, Lemma 66.25.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for separated algebraic spaces). Thus $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Spaces over Fields, Lemma 72.4.6. This proves (3).

Proof of (4). Let $\eta \in \mathrm{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is regular by Lemma 89.6.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \mathrm{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 10.164.4. □

89.8. Resolution

0BHV Here is a definition.

0BHW **Definition 89.8.1.** Let S be a scheme. Let Y be a Noetherian integral algebraic space over S. A *resolution of singularities* of X is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

0BHX **Definition 89.8.2.** Let S be a scheme. Let Y be a 2-dimensional Noetherian integral algebraic space over S. We say Y has a *resolution of singularities by normalized blowups* if there exists a sequence

$$Y_n \to X_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well. We finally come to the main theorem of this chapter.

0BHY **Theorem 89.8.3.** *Let S be a scheme. Let Y be a two dimensional integral Noetherian algebraic space over S. The following are equivalent*

(1) *there exists an alteration $X \to Y$ with X regular,*
(2) *there exists a resolution of singularities of Y,*
(3) *Y has a resolution of singularities by normalized blowups,*
(4) *the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m \in |Y|$, and for each i the completion of the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is normal.*

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 89.7.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms of Spaces, Lemma 67.49.8. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Spaces over Fields, Lemma 72.3.2. Then f is flat over V by Algebra, Lemma 10.128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 10.157.4). Then V is regular by Algebra, Lemma 10.164.4. As Y^ν is Noetherian we conclude that $Y^\nu \setminus V = \{y_1, \ldots, y_m\}$ is finite. For each i let $\mathcal{O}^h_{Y^\nu, y_i}$ be the henselian local ring. Then $X \times_Y \mathrm{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$ is a regular alteration of $\mathrm{Spec}(\mathcal{O}^h_{Y^\nu, y_i})$ (some details omitted). By Lemma 89.7.2 the completion of $\mathcal{O}^h_{Y^\nu, y_i}$ is normal. In this way we see that (1) $\Rightarrow$ (4).

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in |Y|$ be the singular points. Choose a collection of elementary étale neighbourhoods $(V_i, v_i) \to (Y, y_i)$ as in Section 89.3. For each i the henselian local ring $\mathcal{O}^h_{Y^\nu, y_i}$ is the henselization of $\mathcal{O}_{V_i, v_i}$. Hence these rings have

isomorphic completions. Thus by the result for schemes (Resolution of Surfaces, Theorem 54.14.5) we see that there exist finite sequences of normalized blowups

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to V_i$$

blowing up only in points lying over v_i such that X_{i,n_i} is regular. By Lemma 89.5.3 there is a sequence of normalized blowing ups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. □

89.9. Examples

0AE8 Some examples related to the results earlier in this chapter.

0AE9 **Example 89.9.1.** Let k be a field. The ring $A = k[x,y,z]/(x^r + y^s + z^t)$ is a UFD for r, s, t pairwise coprime integers. Namely, since $x^r + y^s + z^t$ is irreducible A is a domain. The element z is a prime element, i.e., generates a prime ideal in A. On the other hand, if $t = 1 + ers$ for some e, then [**Sam68**, 4(c)]

$$A[1/z] \cong k[x', y', 1/z]$$

where $x' = x/z^{es}$, $y' = y/z^{er}$ and $z = (x')^r + (y')^s$. Thus $A[1/z]$ is a localization of a polynomial ring and hence a UFD. It follows from an argument of Nagata that A is a UFD. See Algebra, Lemma 10.120.7. A similar argument can be given if t is not congruent to 1 modulo rs.

0AEA **Example 89.9.2.** The ring $A = \mathbf{C}[[x,y,z]]/(x^r + y^s + z^t)$ is not a UFD when $1 < r < s < t$ are pairwise coprime integers and not equal to $2, 3, 5$. For example consider the special case $A = \mathbf{C}[[x,y,z]]/(x^2 + y^5 + z^7)$. Consider the maps See [**Bri68**] and [**Lip69**] for nonvanishing of local Picard groups in general.

$$\psi_\zeta : \mathbf{C}[[x,y,z]]/(x^2 + y^5 + z^7) \to \mathbf{C}[[t]]$$

given by

$$x \mapsto t^7, \quad y \mapsto t^3, \quad z \mapsto -\zeta t^2(1+t)^{1/7}$$

where ζ is a 7th root of unity. The kernel $\mathfrak{p}_\zeta$ of ψ_ζ is a height one prime, hence if A is a UFD, then it is principal, say given by $f_\zeta \in \mathbf{C}[[x,y,z]]$. Note that $V(x^3 - y^7) = \bigcup V(\mathfrak{p}_\zeta)$ and $A/(x^3 - y^7)$ is reduced away from the closed point. Hence, still assuming A is a UFD, we would obtain

$$\prod\nolimits_\zeta f_\zeta = u(x^3 - y^7) + a(x^2 + y^5 + z^7) \quad \text{in} \quad \mathbf{C}[[x,y,z]]$$

for some unit $u \in \mathbf{C}[[x,y,z]]$ and some element $a \in \mathbf{C}[[x,y,z]]$. After scaling by a constant we may assume $u(0,0,0) = 1$. Note that the left hand side vanishes to order 7. Hence $a = -x \bmod \mathfrak{m}^2$. But then we get a term xy^5 on the right hand side which does not occur on the left hand side. A contradiction.

0AEB **Example 89.9.3.** There exists an excellent 2-dimensional Noetherian local ring and a modification $X \to S = \operatorname{Spec}(A)$ which is not a scheme. We sketch a construction. Let X be a normal surface over $\mathbf{C}$ with a unique singular point $x \in X$. Assume that there exists a resolution $\pi : X' \to X$ such that the exceptional fibre $C = \pi^{-1}(x)_{red}$ is a smooth projective curve. Furthermore, assume there exists a point $c \in C$ such that if $\mathcal{O}_C(nc)$ is in the image of $\operatorname{Pic}(X') \to \operatorname{Pic}(C)$, then $n = 0$. Then we let $X'' \to X'$ be the blowing up in the nonsingular point c. Let $C' \subset X''$ be the strict transform of C and let $E \subset X''$ be the exceptional fibre. By Artin's

results ([**Art70**]; use for example [**Mum61**] to see that the normal bundle of C' is negative) we can blow down the curve C' in X'' to obtain an algebraic space X'''. Picture

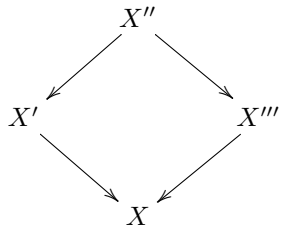

We claim that X''' is not a scheme. This provides us with our example because X''' is a scheme if and only if the base change of X''' to $A = \mathcal{O}_{X,x}$ is a scheme (details omitted). If X''' where a scheme, then the image of C' in X''' would have an affine neighbourhood. The complement of this neighbourhood would be an effective Cartier divisor on X''' (because X''' is nonsingular apart from 1 point). This effective Cartier divisor would correspond to an effective Cartier divisor on X'' meeting E and avoiding C'. Taking the image in X' we obtain an effective Cartier divisor meeting C (set theoretically) in c. This is impossible as no multiple of c is the restriction of a Cartier divisor by assumption.

To finish we have to find such a singular surface X. We can just take X to be the affine surface given by

$$x^3 + y^3 + z^3 + x^4 + y^4 + z^4 = 0$$

in $\mathbf{A}^3_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[x, y, z])$ and singular point $(0, 0, 0)$. Then $(0, 0, 0)$ is the only singular point. Blowing up X in the maximal ideal corresponding to $(0, 0, 0)$ we find three charts each isomorphic to the smooth affine surface

$$1 + s^3 + t^3 + x(1 + s^4 + t^4) = 0$$

which is nonsingular with exceptional divisor C given by $x = 0$. The reader will recognize C as an elliptic curve. Finally, the surface X is rational as projection from $(0, 0, 0)$ shows, or because in the equation for the blowup we can solve for x. Finally, the Picard group of a nonsingular rational surface is countable, whereas the Picard group of an elliptic curve over the complex numbers is uncountable. Hence we can find a closed point c as indicated.

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Part 6

Deformation Theory

CHAPTER 90

Formal Deformation Theory

06G7

90.1. Introduction

06G8 This chapter develops formal deformation theory in a form applicable later in the Stacks project, closely following Rim [**GRR72**, Exposee VI] and Schlessinger [**Sch68**]. We strongly encourage the reader new to this topic to read the paper by Schlessinger first, as it is sufficiently general for most applications, and Schlessinger's results are indeed used in most papers that use this kind of formal deformation theory.

Let Λ be a complete Noetherian local ring with residue field k, and let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field k. Given a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ such that $F(k)$ is a one element set, Schlessinger's paper introduced conditions (H1)-(H4) such that:

(1) F has a "hull" if and only if (H1)-(H3) hold.
(2) F is prorepresentable if and only if (H1)-(H4) hold.

The purpose of this chapter is to generalize these results in two ways exactly as is done in Rim's paper:

(A) The functor F is replaced by a category $\mathcal{F}$ cofibered in groupoids over $\mathcal{C}_\Lambda$, see Section 90.3.
(B) We let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map to a field. The category $\mathcal{C}_\Lambda$ is the category of Artinian local Λ-algebras A endowed with a given identification $A/\mathfrak{m}_A = k$.

The analogue of the condition that $F(k)$ is a one element set is that $\mathcal{F}(k)$ is the trivial groupoid. If $\mathcal{F}$ satisfies this condition then we say it is a *predeformation category*, but in general we do not make this assumption. Rim's paper [**GRR72**, Exposee VI] is the original source for the results in this document. We also mention the useful paper [**TV13**], which discusses deformation theory with groupoids but in less generality than we do here.

An important role is played by the "completion" $\widehat{\mathcal{C}}_\Lambda$ of the category $\mathcal{C}_\Lambda$. An object of $\widehat{\mathcal{C}}_\Lambda$ is a Noetherian complete local Λ-algebra R whose residue field is identified with k, see Section 90.4. On the one hand $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is a strictly full subcategory and on the other hand $\widehat{\mathcal{C}}_\Lambda$ is a full subcategory of the category of pro-objects of $\mathcal{C}_\Lambda$. A functor $\mathcal{C}_\Lambda \to \mathit{Sets}$ is *prorepresentable* if it is isomorphic to the restriction of a representable functor $\underline{R} = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, -)$ to $\mathcal{C}_\Lambda$ where $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Categories cofibred in groupoids are dual to categories fibred in groupoids; we introduce them in Section 90.5. A *smooth* morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ is one that satisfies the infinitesimal lifting criterion for objects, see Section 90.8. This is analogous to the definition of a formally smooth ring map, see

Algebra, Definition 10.138.1 and is exactly dual to the notion in Criteria for Representability, Section 97.6. This is an important notion as we eventually want to prove that certain kinds of categories cofibred in groupoids have a smooth prorepresentable presentation, much like the characterization of algebraic stacks in Algebraic Stacks, Sections 94.16 and 94.17. A *versal formal object* of a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$ is an object $\xi \in \widehat{\mathcal{F}}(R)$ of the completion such that the associated morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth.

In Section 90.10, we define conditions (S1) and (S2) on $\mathcal{F}$ generalizing Schlessinger's (H1) and (H2). The analogue of Schlessinger's (H3)—the condition that $\mathcal{F}$ has finite dimensional tangent space—is not given a name. A key step in the development of the theory is the existence of versal formal objects for predeformation categories satisfying (S1), (S2) and (H3), see Lemma 90.13.4. Schlessinger's notion of a *hull* for a functor $F : \mathcal{C}_\Lambda \to Sets$ is, in our terminology, a versal formal object $\xi \in \widehat{F}(R)$ such that the induced map of tangent spaces $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism. In the literature a hull is often called a "miniversal" object. We do not do so, and here is why. It can happen that a functor has a versal formal object without having a hull. Moreover, we show in Section 90.14 that if a predeformation category has a versal formal object, then it always has a *minimal* one (as defined in Definition 90.14.4) which is unique up to isomorphism, see Lemma 90.14.5. But it can happen that the minimal versal formal object does not induce an isomorphism on tangent spaces! (See Examples 90.15.3 and 90.15.8.)

Keeping in mind the differences pointed out above, Theorem 90.15.5 is the direct generalization of (1) above: it recovers Schlessinger's result in the case that $\mathcal{F}$ is a functor and it characterizes minimal versal formal objects, in the presence of conditions (S1) and (S2), in terms of the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ on tangent spaces.

In Section 90.16, we define Rim's condition (RS) on $\mathcal{F}$ generalizing Schlessinger's (H4). A *deformation category* is defined as a predeformation category satisfying (RS). The analogue to prorepresentable functors are the categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a *presentation by a smooth prorepresentable groupoid in functors* on $\mathcal{C}_\Lambda$, see Definitions 90.21.1, 90.22.1, and 90.23.1. This notion of a presentation takes into account the groupoid structure of the fibers of $\mathcal{F}$. In Theorem 90.26.4 we prove that $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid in functors if and only if $\mathcal{F}$ has a finite dimensional tangent space and finite dimensional infinitesimal automorphism space. This is the generalization of (2) above: it reduces to Schlessinger's result in the case that $\mathcal{F}$ is a functor. There is a final Section 90.27 where we discuss how to use minimal versal formal objects to produce a (unique up to isomorphism) minimal presentation by a smooth prorepresentable groupoid in functors.

We also find the following conceptual explanation for Schlessinger's conditions. If a predeformation category $\mathcal{F}$ satisfies (RS), then the associated functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$ satisfies (H1) and (H2) (Lemmas 90.16.6 and 90.10.5). Conversely, if a functor $F : \mathcal{C}_\Lambda \to Sets$ arises naturally as the functor of isomorphism classes of a category $\mathcal{F}$ cofibered in groupoids, then it seems to happen in practice that an argument showing F satisfies (H1) and (H2) will also show $\mathcal{F}$ satisfies (RS). Examples are discussed in Deformation Problems, Section 93.1. Moreover, if $\mathcal{F}$ satisfies (RS), then condition (H4) for $\overline{\mathcal{F}}$ has a simple interpretation in terms of extending automorphisms of objects of $\mathcal{F}$ (Lemma 90.16.7). These observations

suggest that (RS) should be regarded as the fundamental deformation theoretic glueing condition.

90.2. Notation and Conventions

06G9 A ring is commutative with 1. The maximal ideal of a local ring A is denoted by $\mathfrak{m}_A$. The set of positive integers is denoted by $\mathbf{N} = \{1, 2, 3, \ldots\}$. If U is an object of a category $\mathcal{C}$, we denote by $\underline{U}$ the functor $\mathrm{Mor}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathit{Sets}$, see Remarks 90.5.2 (12). Warning: this may conflict with the notation in other chapters where we sometimes use $\underline{U}$ to denote $h_U(-) = \mathrm{Mor}_{\mathcal{C}}(-, U)$.

Throughout this chapter Λ is a Noetherian ring and $\Lambda \to k$ is a finite ring map from Λ to a field. The kernel of this map is denoted $\mathfrak{m}_\Lambda$ and the image $k' \subset k$. It turns out that $\mathfrak{m}_\Lambda$ is a maximal ideal, $k' = \Lambda/\mathfrak{m}_\Lambda$ is a field, and the extension k/k' is finite. See discussion surrounding (90.3.3.1).

90.3. The base category

06GB Motivation. An important application of formal deformation theory is to criteria for representability by algebraic spaces. Suppose given a locally Noetherian base S and a functor $F : (\mathit{Sch}/S)^{opp}_{fppf} \to \mathit{Sets}$. Let k be a finite type field over S, i.e., we are given a finite type morphism $\mathrm{Spec}(k) \to S$. One of Artin's criteria is that for any element $x \in F(\mathrm{Spec}(k))$ the predeformation functor associated to the triple (S, k, x) should be prorepresentable. By Morphisms, Lemma 29.16.1 the condition that k is of finite type over S means that there exists an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra. This motivates why we work throughout this chapter with a base category as follows.

06GC **Definition 90.3.1.** Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathcal{C}_\Lambda$ to be the category with

(1) objects are pairs (A, φ) where A is an Artinian local Λ-algebra and where $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

We say we are in the *classical case* if Λ is a Noetherian complete local ring and k is its residue field.

Note that if $\Lambda \to k$ is surjective and if A is an Artinian local Λ-algebra, then the identification φ, if it exists, is unique. Moreover, in this case any Λ-algebra map $A \to B$ is going to be compatible with the identifications. Hence in this case $\mathcal{C}_\Lambda$ is just the category of local Artinian Λ-algebras whose residue field "is" k. By abuse of notation we also denote objects of $\mathcal{C}_\Lambda$ simply A in the general case. Moreover, we will often write $A/\mathfrak{m} = k$, i.e., we will pretend all rings in $\mathcal{C}_\Lambda$ have residue field k (since all ring maps in $\mathcal{C}_\Lambda$ are compatible with the given identifications this should never cause any problems). Throughout the rest of this chapter the base ring Λ and the field k are fixed. The category $\mathcal{C}_\Lambda$ will be the base category for the cofibered categories considered below.

06GD **Definition 90.3.2.** Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is a *small extension* if it is surjective and $\mathrm{Ker}(f)$ is a nonzero principal ideal which is annihilated by $\mathfrak{m}_B$.

By the following lemma we can often reduce arguments involving surjective ring maps in $\mathcal{C}_\Lambda$ to the case of small extensions.

06GE **Lemma 90.3.3.** *Let $f : B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.*

Proof. Let I be the kernel of f. The maximal ideal $\mathfrak{m}_B$ is nilpotent since B is Artinian, say $\mathfrak{m}_B^n = 0$. Hence we get a factorization

$$B = B/I\mathfrak{m}_B^{n-1} \to B/I\mathfrak{m}_B^{n-2} \to \ldots \to B/I \cong A$$

of f into a composition of surjective maps whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map, i.e. when I is annihilated by $\mathfrak{m}_B$. In this case I is a k-vector space, which has finite dimension, see Algebra, Lemma 10.53.6. Take a basis $x_1, \ldots, x_n$ of I as a k-vector space to get a factorization

$$B \to B/(x_1) \to \ldots \to B/(x_1, \ldots, x_n) \cong A$$

of f into a composition of small extensions. □

The next lemma says that we can compute the length of a module over a local Λ-algebra with residue field k in terms of the length over Λ. To explain the notation in the statement, let $k' \subset k$ be the image of our fixed finite ring map $\Lambda \to k$. Note that $k' \subset k$ is a finite extension of rings. Hence k' is a field and k/k' is a finite extension of fields, see Algebra, Lemma 10.36.18. Moreover, as $\Lambda \to k'$ is surjective we see that its kernel is a maximal ideal $\mathfrak{m}_\Lambda$. Thus

06S2 (90.3.3.1) $$[k : k'] = [k : \Lambda/\mathfrak{m}_\Lambda] < \infty$$

and in the classical case we have $k = k'$. The notation $k' = \Lambda/\mathfrak{m}_\Lambda$ will be fixed throughout this chapter.

06GG **Lemma 90.3.4.** *Let A be a local Λ-algebra with residue field k. Let M be an A-module. Then $[k : k']\text{length}_A(M) = \text{length}_\Lambda(M)$. In the classical case we have $\text{length}_A(M) = \text{length}_\Lambda(M)$.*

Proof. If M is a simple A-module then $M \cong k$ as an A-module, see Algebra, Lemma 10.52.10. In this case $\text{length}_A(M) = 1$ and $\text{length}_\Lambda(M) = [k' : k]$, see Algebra, Lemma 10.52.6. If $\text{length}_A(M)$ is finite, then the result follows on choosing a filtration of M by A-submodules with simple quotients using additivity, see Algebra, Lemma 10.52.3. If $\text{length}_A(M)$ is infinite, the result follows from the obvious inequality $\text{length}_A(M) \leq \text{length}_\Lambda(M)$. □

06S3 **Lemma 90.3.5.** *Let $A \to B$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and*
(3) *$\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.*

Proof. For any ring map $f : A \to B$ in $\mathcal{C}_\Lambda$ we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ for example because $\mathfrak{m}_A$, $\mathfrak{m}_B$ is the set of nilpotent elements of A, B. Suppose f is surjective. Let $y \in \mathfrak{m}_B$. Choose $x \in A$ with $f(x) = y$. Since f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ we see that $x \in \mathfrak{m}_A$. Hence the induced map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. In this way we see that (1) implies (2).

It is clear that (2) implies (3). The map $A \to B$ gives rise to a canonical commutative diagram

$$\begin{array}{ccccccc} \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) & \longrightarrow & 0 \end{array}$$

with exact rows. Hence if (3) holds, then so does (2).

Assume (2). To show that $A \to B$ is surjective it suffices by Nakayama's lemma (Algebra, Lemma 10.20.1) to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. (Note that $\mathfrak{m}_A$ is a nilpotent ideal.) As $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. Applying Nakayama's lemma once more we see that it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective which is what we assumed. □

If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then the map $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is the map on relative cotangent spaces. Here is a formal definition.

06GY **Definition 90.3.6.** Let $R \to S$ be a local homomorphism of local rings. The *relative cotangent space*[1] of R over S is the $S/\mathfrak{m}_S$-vector space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

If $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are two ring maps, then the fiber product $A_1 \times_A A_2$ is the subring of $A_1 \times A_2$ consisting of elements whose two projections to A are equal. Throughout this chapter we will be considering conditions involving such a fiber product when f_1 and f_2 are in $\mathcal{C}_\Lambda$. It isn't always the case that the fibre product is an object of $\mathcal{C}_\Lambda$.

06S4 **Example 90.3.7.** Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Let $A = k[\epsilon] = k[x]/(x^2)$. Then A is an object of $\mathcal{C}_\Lambda$. Suppose that $D : k \to k$ is a derivation of k over Λ, for example $D = \partial/\partial x_i$. Then the map

$$f_D : k \longrightarrow k[\epsilon], \quad a \mapsto a + D(a)\epsilon$$

is a morphism of $\mathcal{C}_\Lambda$. Set $A_1 = A_2 = k$ and set $f_1 = f_{\partial/\partial x_1}$ and $f_2(a) = a$. Then $A_1 \times_A A_2 = \{a \in k \mid \partial/\partial x_1(a) = 0\}$ which does not surject onto k. Hence the fibre product isn't an object of $\mathcal{C}_\Lambda$.

It turns out that this problem can only occur if the residue field extension k/k' (90.3.3.1) is inseparable and neither f_1 nor f_2 is surjective.

06GH **Lemma 90.3.8.** *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$. Then:*

(1) *If f_1 or f_2 is surjective, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*
(2) *If f_2 is a small extension, then so is $A_1 \times_A A_2 \to A_1$.*
(3) *If the field extension k/k' is separable, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*

Proof. The ring $A_1 \times_A A_2$ is a Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_2$. It is a local ring with unique maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \mathrm{Ker}(A_1 \times_A A_2 \longrightarrow k)$$

[1] Caution: We will see later that in our general setting the tangent space of an object $A \in \mathcal{C}_\Lambda$ over Λ should not be defined simply as the k-linear dual of the relative cotangent space. In fact, the correct definition of the relative cotangent space is $\Omega_{S/R} \otimes_S S/\mathfrak{m}_S$.

A ring is Artinian if and only if it has finite length as a module over itself, see Algebra, Lemma 10.53.6. Since A_1 and A_2 are Artinian, Lemma 90.3.4 implies $\text{length}_\Lambda(A_1)$ and $\text{length}_\Lambda(A_2)$, and hence $\text{length}_\Lambda(A_1 \times A_2)$, are all finite. As $A_1 \times_A A_2 \subset A_1 \times A_2$ is a Λ-submodule, this implies $\text{length}_{A_1 \times_A A_2}(A_1 \times_A A_2) \leq \text{length}_\Lambda(A_1 \times_A A_2)$ is finite. So $A_1 \times_A A_2$ is Artinian. Thus the only thing that is keeping $A_1 \times_A A_2$ from being an object of $\mathcal{C}_\Lambda$ is the possibility that its residue field maps to a proper subfield of k via the map $A_1 \times_A A_2 \to A \to A/\mathfrak{m}_A = k$ above.

Proof of (1). If f_2 is surjective, then the projection $A_1 \times_A A_2 \to A_1$ is surjective. Hence the composition $A_1 \times_A A_2 \to A_1 \to A_1/\mathfrak{m}_{A_1} = k$ is surjective and we conclude that $A_1 \times_A A_2$ is an object of $\mathcal{C}_\Lambda$.

Proof of (2). If f_2 is a small extension then $A_2 \to A$ and $A_1 \times_A A_2 \to A_1$ are both surjective with the same kernel. Hence the kernel of $A_1 \times_A A_2 \to A_1$ is a 1-dimensional k-vector space and we see that $A_1 \times_A A_2 \to A_1$ is a small extension.

Proof of (3). Choose $\overline{x} \in k$ such that $k = k'(\overline{x})$ (see Fields, Lemma 9.19.1). Let $P'(T) \in k'[T]$ be the minimal polynomial of $\overline{x}$ over k'. Since k/k' is separable we see that $\mathrm{d}P/\mathrm{d}T(\overline{x}) \neq 0$. Choose a monic $P \in \Lambda[T]$ which maps to P' under the surjective map $\Lambda[T] \to k'[T]$. Because A, A_1, A_2 are henselian, see Algebra, Lemma 10.153.10, we can find $x, x_1, x_2 \in A, A_1, A_2$ with $P(x) = 0, P(x_1) = 0, P(x_2) = 0$ and such that the image of x, x_1, x_2 in k is $\overline{x}$. Then $(x_1, x_2) \in A_1 \times_A A_2$ because x_1, x_2 map to $x \in A$ by uniqueness, see Algebra, Lemma 10.153.2. Hence the residue field of $A_1 \times_A A_2$ contains a generator of k over k' and we win. □

Next we define essential surjections in $\mathcal{C}_\Lambda$. A necessary and sufficient condition for a surjection in $\mathcal{C}_\Lambda$ to be essential is given in Lemma 90.3.12.

06GF **Definition 90.3.9.** Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is an *essential surjection* if it has the following properties:

(1) f is surjective.
(2) If $g : C \to B$ is a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective, then g is surjective.

Using Lemma 90.3.5, we can characterize essential surjections in $\mathcal{C}_\Lambda$ as follows.

06S5 **Lemma 90.3.10.** *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is an essential surjection,*
(2) *the map $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is an essential surjection, and*
(3) *the map $B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an essential surjection.*

Proof. Assume (3). Let $C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $C \to A$ is surjective. Then $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective too. We conclude that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective by our assumption. Hence $C \to B$ is surjective by applying Lemma 90.3.5 (2 times).

Assume (1). Let $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ be a morphism of $\mathcal{C}_\Lambda$ such that $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective. Set $C' = C \times_{B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)} B$ which is an object of $\mathcal{C}_\Lambda$ by Lemma 90.3.8. Note that $C' \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is still surjective, hence $C' \to A$ is surjective by Lemma 90.3.5. Thus $C' \to B$ is surjective by our assumption. This implies that $C' \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective, which implies by the construction of C' that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

In the first paragraph we proved (3) $\Rightarrow$ (1) and in the second paragraph we proved (1) $\Rightarrow$ (3). The equivalence of (2) and (3) is a special case of the equivalence of (1) and (3), hence we are done. $\square$

To analyze essential surjections in $\mathcal{C}_\Lambda$ a bit more we introduce some notation. Suppose that A is an object of $\mathcal{C}_\Lambda$ or more generally any Λ-algebra equipped with a Λ-algebra surjection $A \to k$. There is a canonical exact sequence

06S6 (90.3.10.1) $$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0$$

see Algebra, Lemma 10.131.9. Note that $\Omega_{k/\Lambda} = \Omega_{k/k'}$ with k' as in (90.3.3.1). Let $H_1(L_{k/\Lambda})$ be the first homology module of the naive cotangent complex of k over Λ, see Algebra, Definition 10.134.1. Then we can extend (90.3.10.1) to the exact sequence

06S7 (90.3.10.2) $$H_1(L_{k/\Lambda}) \to \mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0,$$

see Algebra, Lemma 10.134.4. If $B \to A$ is a ring map in $\mathcal{C}_\Lambda$ or more generally a map of Λ-algebras equipped with Λ-algebra surjections onto k, then we obtain a commutative diagram

06S8 (90.3.10.3)
$$\begin{array}{ccccccc}
H_1(L_{k/\Lambda}) & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \xrightarrow[\mathrm{d}_B]{} & \Omega_{B/\Lambda} \otimes_B k & \longrightarrow & \Omega_{k/\Lambda} \longrightarrow 0 \\
\| & & \downarrow & & \downarrow & & \| \\
H_1(L_{k/\Lambda}) & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \xrightarrow{\mathrm{d}_A} & \Omega_{A/\Lambda} \otimes_A k & \longrightarrow & \Omega_{k/\Lambda} \longrightarrow 0
\end{array}$$

with exact rows.

06S9 **Lemma 90.3.11.** *There is a canonical map*

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}).$$

If $k' \subset k$ is separable (for example if the characteristic of k is zero), then this map induces an isomorphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. If $k = k'$ (for example in the classical case), then $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 = H_1(L_{k/\Lambda})$. The composition

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$$

comes from the canonical map $\mathfrak{m}_\Lambda \to \mathfrak{m}_A$.

Proof. Note that $H_1(L_{k'/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$ as $\Lambda \to k'$ is surjective with kernel $\mathfrak{m}_\Lambda$. The map arises from functoriality of the naive cotangent complex. If $k' \subset k$ is separable, then $k' \to k$ is an étale ring map, see Algebra, Lemma 10.143.4. Thus its naive cotangent complex has trivial homology groups, see Algebra, Definition 10.143.1. Then Algebra, Lemma 10.134.4 applied to the ring maps $\Lambda \to k' \to k$ implies that $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. We omit the proof of the final statement. $\square$

06H0 **Lemma 90.3.12.** *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Notation as in (90.3.10.3).*

(1) *The equivalent conditions of Lemma 90.3.5 characterizing when f is surjective are also equivalent to*
   (a) $\mathrm{Im}(d_B) \to \mathrm{Im}(d_A)$ *is surjective, and*
   (b) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective.*

(2) *The following are equivalent*
   (a) *f is an essential surjection (see Lemma 90.3.10),*
   (b) *the map $\mathrm{Im}(d_B) \to \mathrm{Im}(d_A)$ is an isomorphism, and*

(c) *the map* $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ *is an isomorphism.*

(3) *If* k/k' *is separable, then* f *is an essential surjection if and only if the map* $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ *is an isomorphism.*

(4) *If* f *is a small extension, then* f *is not essential if and only if* f *has a section* $s : A \to B$ *in* $\mathcal{C}_\Lambda$ *with* $f \circ s = id_A$.

Proof. Proof of (1). It follows from (90.3.10.3) that (1)(a) and (1)(b) are equivalent. Also, if $A \to B$ is surjective, then (1)(a) and (1)(b) hold. Assume (1)(a). Since the kernel of d_A is the image of $H_1(L_{k/\Lambda})$ which also maps to $\mathfrak{m}_B/\mathfrak{m}_B^2$ we conclude that $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Hence $B \to A$ is surjective by Lemma 90.3.5. This finishes the proof of (1).

Proof of (2). The equivalence of (2)(b) and (2)(c) is immediate from (90.3.10.3).

Assume (2)(b). Let $g : C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective. We conclude that $\mathfrak{m}_C/\mathfrak{m}_C^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective by Lemma 90.3.5. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective and by the assumption we see that $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_B)$ is surjective. It follows that $C \to B$ is surjective by (1).

Assume (2)(a). Then f is surjective and we see that $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective. Let K be the kernel. Note that $K = \mathrm{d}_B(\mathrm{Ker}(\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2))$ by (90.3.10.3). Choose a splitting

$$\Omega_{B/\Lambda} \otimes_B k = \Omega_{A/\Lambda} \otimes_A k \oplus K$$

of k-vector space. The map $\mathrm{d} : B \to \Omega_{B/\Lambda}$ induces via the projection onto K a map $D : B \to K$. Set $C = \{b \in B \mid D(b) = 0\}$. The Leibniz rule shows that this is a Λ-subalgebra of B. Let $\overline{x} \in k$. Choose $x \in B$ mapping to $\overline{x}$. If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ such that $D(y) = D(x)$. Hence $x - y \in C$ is an element which maps to $\overline{x}$. Thus $C \to k$ is surjective and C is an object of $\mathcal{C}_\Lambda$. Similarly, pick $\omega \in \mathrm{Im}(\mathrm{d}_A)$. We can find $x \in \mathfrak{m}_B$ such that $\mathrm{d}_B(x)$ maps to ω by (1). If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ which maps to zero in $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $D(y) = D(x)$. Hence $z = x - y$ is an element of $\mathfrak{m}_C$ whose image $\mathrm{d}_C(z) \in \Omega_{C/k} \otimes_C k$ maps to ω. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective. We conclude that $C \to A$ is surjective by (1). Hence $C \to B$ is surjective by assumption. Hence $D = 0$, i.e., $K = 0$, i.e., (2)(c) holds. This finishes the proof of (2).

Proof of (3). If k'/k is separable, then $H_1(L_{k/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k$, see Lemma 90.3.11. Hence $\mathrm{Im}(\mathrm{d}_A) = \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ and similarly for B. Thus (3) follows from (2).

Proof of (4). A section s of f is not surjective (by definition a small extension has nontrivial kernel), hence f is not essentially surjective. Conversely, assume f is a small extension but not an essential surjection. Choose a ring map $C \to B$ in $\mathcal{C}_\Lambda$ which is not surjective, such that $C \to A$ is surjective. Let $C' \subset B$ be the image of $C \to B$. Then $C' \neq B$ but C' surjects onto A. Since $f : B \to A$ is a small extension, $\mathrm{length}_C(B) = \mathrm{length}_C(A) + 1$. Thus $\mathrm{length}_C(C') \leq \mathrm{length}_C(A)$ since C' is a proper subring of B. But $C' \to A$ is surjective, so in fact we must have $\mathrm{length}_C(C') = \mathrm{length}_C(A)$ and $C' \to A$ is an isomorphism which gives us our section. □

06SA **Example 90.3.13.** Let $\Lambda = k[[x]]$ be the power series ring in 1 variable over k. Set $A = k$ and $B = \Lambda/(x^2)$. Then $B \to A$ is an essential surjection by Lemma

90.3.12 because it is a small extension and the map $B \to A$ does not have a right inverse (in the category $\mathcal{C}_\Lambda$). But the map

$$k \cong \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 = 0$$

is not an isomorphism. Thus in Lemma 90.3.12 (3) it is necessary to consider the map of relative cotangent spaces $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$.

90.4. The completed base category

06GV The following "completion" of the category $\mathcal{C}_\Lambda$ will serve as the base category of the completion of a category cofibered in groupoids over $\mathcal{C}_\Lambda$ (Section 90.7).

06GW **Definition 90.4.1.** Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\widehat{\mathcal{C}}_\Lambda$ to be the category with

(1) objects are pairs (R, φ) where R is a Noetherian complete local Λ-algebra and where $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

As in the discussion following Definition 90.3.1 we will usually denote an object of $\widehat{\mathcal{C}}_\Lambda$ simply R, with the identification $R/\mathfrak{m}_R = k$ understood. In this section we discuss some basic properties of objects and morphisms of the category $\widehat{\mathcal{C}}_\Lambda$ paralleling our discussion of the category $\mathcal{C}_\Lambda$ in the previous section.

Our first observation is that any object $A \in \mathcal{C}_\Lambda$ is an object of $\widehat{\mathcal{C}}_\Lambda$ as an Artinian local ring is always Noetherian and complete with respect to its maximal ideal (which is after all a nilpotent ideal). Moreover, it is clear from the definitions that $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is the strictly full subcategory consisting of all Artinian rings. As it turns out, conversely every object of $\widehat{\mathcal{C}}_\Lambda$ is a limit of objects of $\mathcal{C}_\Lambda$.

Suppose that R is an object of $\widehat{\mathcal{C}}_\Lambda$. Consider the rings $R_n = R/\mathfrak{m}_R^n$ for $n \in \mathbf{N}$. These are Noetherian local rings with a unique nilpotent prime ideal, hence Artinian, see Algebra, Proposition 10.60.7. The ring maps

$$\ldots \to R_{n+1} \to R_n \to \ldots \to R_2 \to R_1 = k$$

are all surjective. Completeness of R by definition means that $R = \lim R_n$. If $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$ then we obtain a system of ring maps $f_n : R_n \to S_n$ whose limit is the given map.

06GZ **Lemma 90.4.2.** *Let $f : R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *the map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective, and*
(3) *the map $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to \mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$ is surjective.*

Proof. Note that for $n \geq 2$ we have the equality of relative cotangent spaces

$$\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) = \mathfrak{m}_{R_n}/(\mathfrak{m}_\Lambda R_n + \mathfrak{m}_{R_n}^2)$$

and similarly for S. Hence by Lemma 90.3.5 we see that $R_n \to S_n$ is surjective for all n. Now let K_n be the kernel of $R_n \to S_n$. Then the sequences

$$0 \to K_n \to R_n \to S_n \to 0$$

form an exact sequence of directed inverse systems. The system (K_n) is Mittag-Leffler since each K_n is Artinian. Hence by Algebra, Lemma 10.86.4 taking limits preserves exactness. So $\lim R_n \to \lim S_n$ is surjective, i.e., f is surjective. □

06SB **Lemma 90.4.3.** *The category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.*

Proof. Let $R \to S_1$ and $R \to S_2$ be morphisms of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = S_1 \otimes_R S_2$. This ring has a finitely generated maximal ideal $\mathfrak{m} = \mathfrak{m}_{S_1} \otimes S_2 + S_1 \otimes \mathfrak{m}_{S_2}$ with residue field k. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ whose residue field is identified with k, see Algebra, Lemma 10.97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $S_1 \to C^\wedge$ and $S_2 \to C^\wedge$ turn $C^\wedge$ into a pushout over R in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

We will not need the following lemma.

06H1 **Lemma 90.4.4.** *The category $\widehat{\mathcal{C}}_\Lambda$ admits coproducts of pairs of objects.*

Proof. Let R and S be objects of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = R \otimes_\Lambda S$. There is a canonical surjective map $C \to R \otimes_\Lambda S \to k \otimes_\Lambda k \to k$ where the last map is the multiplication map. The kernel of $C \to k$ is a maximal ideal $\mathfrak{m}$. Note that $\mathfrak{m}$ is generated by $\mathfrak{m}_R C$, $\mathfrak{m}_S C$ and finitely many elements of C which map to generators of the kernel of $k \otimes_\Lambda k \to k$. Hence $\mathfrak{m}$ is a finitely generated ideal. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ with residue field k, see Algebra, Lemma 10.97.5. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $R \to C^\wedge$ and $S \to C^\wedge$ turn $C^\wedge$ into a coproduct in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

An empty coproduct in a category is an initial object of the category. In the classical case $\widehat{\mathcal{C}}_\Lambda$ has an initial object, namely Λ itself. More generally, if $k' = k$, then the completion $\Lambda^\wedge$ of Λ with respect to $\mathfrak{m}_\Lambda$ is an initial object. More generally still, if $k' \subset k$ is separable, then $\widehat{\mathcal{C}}_\Lambda$ has an initial object too. Namely, choose a monic polynomial $P \in \Lambda[T]$ such that $k \cong k'[T]/(P')$ where $p' \in k'[T]$ is the image of P. Then $R = \Lambda^\wedge[T]/(P)$ is an initial object, see proof of Lemma 90.3.8.

If R is an initial object as above, then we have $\mathcal{C}_\Lambda = \mathcal{C}_R$ and $\widehat{\mathcal{C}}_\Lambda = \widehat{\mathcal{C}}_R$ which effectively brings the whole discussion in this chapter back to the classical case. But, if $k' \subset k$ is inseparable, then an initial object does not exist.

06SC **Lemma 90.4.5.** *Let S be an object of $\widehat{\mathcal{C}}_\Lambda$. Then $\dim_k Der_\Lambda(S, k) < \infty$.*

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_S$ map to a k-basis for the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$. Choose $y_1, \ldots, y_m \in S$ whose images in k generate k over k'. We claim that $\dim_k \mathrm{Der}_\Lambda(S, k) \leq n+m$. To see this it suffices to prove that if $D(x_i) = 0$ and $D(y_j) = 0$, then $D = 0$. Let $a \in S$. We can find a polynomial $P = \sum \lambda_J y^J$ with $\lambda_J \in \Lambda$ whose image in k is the same as the image of a in k. Then we see that $D(a - P) = D(a) - D(P) = D(a)$ by our assumption that $D(y_j) = 0$ for all j. Thus we may assume $a \in \mathfrak{m}_S$. Write $a = \sum a_i x_i$ with $a_i \in S$. By the Leibniz rule

$$D(a) = \sum x_i D(a_i) + \sum a_i D(x_i) = \sum x_i D(a_i)$$

as we assumed $D(x_i) = 0$. We have $\sum x_i D(a_i) = 0$ as multiplication by x_i is zero on k. □

06SD **Lemma 90.4.6.** *Let $f : R \to S$ be a morphism of $\widehat{\mathcal{C}}_\Lambda$. If $Der_\Lambda(S, k) \to Der_\Lambda(R, k)$ is injective, then f is surjective.*

Proof. If f is not surjective, then $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero by Lemma 90.4.2. Then also $Q = S/(f(R)+\mathfrak{m}_R S+\mathfrak{m}_S^2)$ is nonzero. Note that Q is a $k = R/\mathfrak{m}_R$-vector space via f. We turn Q into an S-module via $S \to k$. The quotient map $D : S \to Q$ is an R-derivation: if $a_1, a_2 \in S$, we can write $a_1 = f(b_1) + a_1'$ and $a_2 = f(b_2) + a_2'$ for some $b_1, b_2 \in R$ and $a_1', a_2' \in \mathfrak{m}_S$. Then b_i and a_i have the same image in k for $i = 1, 2$ and

$$\begin{aligned} a_1a_2 &= (f(b_1) + a_1')(f(b_2) + a_2') \\ &= f(b_1)a_2' + f(b_2)a_1' \\ &= f(b_1)(f(b_2) + a_2') + f(b_2)(f(b_1) + a_1') \\ &= f(b_1)a_2 + f(b_2)a_1 \end{aligned}$$

in Q which proves the Leibniz rule. Hence $D : S \to Q$ is a Λ-derivation which is zero on composing with $R \to S$. Since $Q \neq 0$ there also exist derivations $D : S \to k$ which are zero on composing with $R \to S$, i.e., $\mathrm{Der}_\Lambda(S, k) \to \mathrm{Der}_\Lambda(R, k)$ is not injective. □

06SE **Lemma 90.4.7.** *Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. Let (J_n) be a decreasing sequence of ideals such that $\mathfrak{m}_R^n \subset J_n$. Set $J = \bigcap J_n$. Then the sequence (J_n/J) defines the $\mathfrak{m}_{R/J}$-adic topology on R/J.*

Proof. It is clear that $\mathfrak{m}_{R/J}^n \subset J_n/J$. Thus it suffices to show that for every n there exists an N such that $J_N/J \subset \mathfrak{m}_{R/J}^n$. This is equivalent to $J_N \subset \mathfrak{m}_R^n + J$. For each n the ring $R/\mathfrak{m}_R^n$ is Artinian, hence there exists a N_n such that

$$J_{N_n} + \mathfrak{m}_R^n = J_{N_n+1} + \mathfrak{m}_R^n = \ldots$$

Set $E_n = (J_{N_n} + \mathfrak{m}_R^n)/\mathfrak{m}_R^n$. Set $E = \lim E_n \subset \lim R/\mathfrak{m}_R^n = R$. Note that $E \subset J$ as for any $f \in E$ and any m we have $f \in J_m + \mathfrak{m}_R^n$ for all $n \gg 0$, so $f \in J_m$ by Krull's intersection theorem, see Algebra, Lemma 10.51.4. Since the transition maps $E_n \to E_{n-1}$ are all surjective, we see that J surjects onto E_n. Hence for $N = N_n$ works. □

06SF **Lemma 90.4.8.** *Let $\ldots \to A_3 \to A_2 \to A_1$ be a sequence of surjective ring maps in $\mathcal{C}_\Lambda$. If $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ is bounded, then $S = \lim A_n$ is an object in $\widehat{\mathcal{C}}_\Lambda$ and the ideals $I_n = \mathrm{Ker}(S \to A_n)$ define the $\mathfrak{m}_S$-adic topology on S.*

Proof. We will use freely that the maps $S \to A_n$ are surjective for all n. Note that the maps $\mathfrak{m}_{A_{n+1}}/\mathfrak{m}_{A_{n+1}}^2 \to \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2$ are surjective, see Lemma 90.4.2. Hence for n sufficiently large the dimension $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ stabilizes to an integer, say r. Thus we can find $x_1, \ldots, x_r \in \mathfrak{m}_S$ whose images in A_n generate $\mathfrak{m}_{A_n}$. Moreover, pick $y_1, \ldots, y_t \in S$ whose images in k generate k over Λ. Then we get a ring map $P = \Lambda[z_1, \ldots, z_{r+t}] \to S$, $z_i \mapsto x_i$ and $z_{r+j} \mapsto y_j$ such that the composition $P \to S \to A_n$ is surjective for all n. Let $\mathfrak{m} \subset P$ be the kernel of $P \to k$. Let $R = P^\wedge$ be the $\mathfrak{m}$-adic completion of P; this is an object of $\widehat{\mathcal{C}}_\Lambda$. Since we still have the compatible system of (surjective) maps $R \to A_n$ we get a map $R \to S$. Set $J_n = \mathrm{Ker}(R \to A_n)$. Set $J = \bigcap J_n$. By Lemma 90.4.7 we see that $R/J = \lim R/J_n = \lim A_n = S$ and that the ideals $J_n/J = I_n$ define the $\mathfrak{m}$-adic topology.

(Note that for each n we have $\mathfrak{m}_R^{N_n} \subset J_n$ for some N_n and not necessarily $N_n = n$, so a renumbering of the ideals J_n may be necessary before applying the lemma.) $\square$

06SG **Lemma 90.4.9.** *Let $R', R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Suppose that $R = R' \oplus I$ for some ideal I of R. Let $x_1, \ldots, x_r \in I$ map to a basis of $I/\mathfrak{m}_R I$. Set $S = R'[[X_1, \ldots, X_r]]$ and consider the R'-algebra map $S \to R$ mapping X_i to x_i. Assume that for every $n \gg 0$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ has a left inverse in $\mathcal{C}_\Lambda$. Then $S \to R$ is an isomorphism.*

Proof. As $R = R' \oplus I$ we have

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus I/\mathfrak{m}_R I$$

and similarly

$$\mathfrak{m}_S/\mathfrak{m}_S^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus \bigoplus kX_i$$

Hence for $n > 1$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ induces an isomorphism on cotangent spaces. Thus a left inverse $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ is surjective by Lemma 90.4.2. Since h_n is injective as a left inverse it is an isomorphism. Thus the canonical surjections $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ are all isomorphisms and we win. $\square$

90.5. Categories cofibered in groupoids

06GA In developing the theory we work with categories *cofibered* in groupoids. We assume as known the definition and basic properties of categories *fibered* in groupoids, see Categories, Section 4.35.

06GJ **Definition 90.5.1.** Let $\mathcal{C}$ be a category. A *category cofibered in groupoids over $\mathcal{C}$* is a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}^{opp}$ is a category fibered in groupoids over $\mathcal{C}^{opp}$ via $p^{opp} : \mathcal{F}^{opp} \to \mathcal{C}^{opp}$.

Explicitly, $p : \mathcal{F} \to \mathcal{C}$ is cofibered in groupoids if the following two conditions hold:

(1) For every morphism $f : U \to V$ in $\mathcal{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathcal{F}$ lying over f.
(2) For every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathcal{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathcal{F}$ lying over f such that $b = c \circ a$.

06GK **Remarks 90.5.2.** Everything about categories fibered in groupoids translates directly to the cofibered setting. The following remarks are meant to fix notation. Let $\mathcal{C}$ be a category.

(1) We often omit the functor $p : \mathcal{F} \to \mathcal{C}$ from the notation.
(2) The fiber category over an object U in $\mathcal{C}$ is denoted by $\mathcal{F}(U)$. Its objects are those of $\mathcal{F}$ lying over U and its morphisms are those of $\mathcal{F}$ lying over id_U. If x, y are objects of $\mathcal{F}(U)$, we sometimes write $\mathrm{Mor}_U(x, y)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, y)$.
(3) The fibre categories $\mathcal{F}(U)$ are groupoids, see Categories, Lemma 4.35.2. Hence the morphisms in $\mathcal{F}(U)$ are all isomorphisms. We sometimes write $\mathrm{Aut}_U(x)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, x)$.

06SH (4) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$, let $f : U \to V$ be a morphism in $\mathcal{C}$, and let $x \in \mathrm{Ob}(\mathcal{F}(U))$. A *pushforward* of x along f is a morphism $x \to y$ of $\mathcal{F}$ lying over f. A pushforward is unique up to unique isomorphism (see the discussion following Categories, Definition 4.33.1). We sometimes write $x \to f_* x$ for "the" pushforward of x along f.

(5) A *choice of pushforwards for* $\mathcal{F}$ is the choice of a pushforward of x along f for every pair (x, f) as above. We can make such a choice of pushforwards for $\mathcal{F}$ by the axiom of choice.

(6) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$. Given a choice of pushforwards for $\mathcal{F}$, there is an associated pseudo-functor $\mathcal{C} \to \mathit{Groupoids}$. We will never use this construction so we give no details.

06GL (7) A morphism of categories cofibered in groupoids over $\mathcal{C}$ is a functor commuting with the projections to $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{F}'$ are categories cofibered in groupoids over $\mathcal{C}$, we denote the morphisms from $\mathcal{F}$ to $\mathcal{F}'$ by $\mathrm{Mor}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$.

06GM (8) Categories cofibered in groupoids form a $(2,1)$-category $\mathrm{Cof}(\mathcal{C})$. Its 1-morphisms are the morphisms described in (7). If $p : \mathcal{F} \to C$ and $p' : \mathcal{F}' \to \mathcal{C}$ are categories cofibered in groupoids and $\varphi, \psi : \mathcal{F} \to \mathcal{F}'$ are 1-morphisms, then a 2-morphism $t : \varphi \to \psi$ is a morphism of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{F})$.

06GN (9) Let $F : \mathcal{C} \to \mathit{Groupoids}$ be a functor. There is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}$ associated to F as follows. An object of $\mathcal{F}$ is a pair (U, x) where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(F(U))$. A morphism $(U, x) \to (V, y)$ is a pair (f, a) where $f \in \mathrm{Mor}_{\mathcal{C}}(U, V)$ and $a \in \mathrm{Mor}_{F(V)}(F(f)(x), y)$. The functor $\mathcal{F} \to \mathcal{C}$ sends (U, x) to U. See Categories, Section 4.37.

07W5 (10) Let $\mathcal{F}$ be cofibered in groupoids over $\mathcal{C}$. For $U \in \mathrm{Ob}(\mathcal{C})$ set $\overline{\mathcal{F}}(U)$ equal to the set of isomorphisms classes of the category $\mathcal{F}(U)$. If $f : U \to V$ is a morphism of $\mathcal{C}$, then we obtain a map of sets $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ by mapping the isomorphism class of x to the isomorphism class of a pushforward f_*x of x see (4). Then $\overline{\mathcal{F}} : \mathcal{C} \to \mathit{Sets}$ is a functor. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of cofibered categories, we denote by $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ the associated morphism of functors.

06GP (11) Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor. We can think of a set as a discrete category, i.e., as a groupoid with only identity morphisms. Then the construction (9) associates to F a category cofibered in sets. This defines a fully faithful embedding of the category of functors $\mathcal{C} \to \mathit{Sets}$ to the category of categories cofibered in groupoids over $\mathcal{C}$. We identify the category of functors with its image under this embedding. Hence if $F : \mathcal{C} \to \mathit{Sets}$ is a functor, we denote the associated category cofibered in sets also by F; and if $\varphi : F \to G$ is a morphism of functors, we denote still by φ the corresponding morphism of categories cofibered in sets, and vice-versa. See Categories, Section 4.38.

06GQ (12) Let U be an object of $\mathcal{C}$. We write $\underline{U}$ for the functor $\mathrm{Mor}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathit{Sets}$. This defines a fully faithful embedding of $\mathcal{C}^{opp}$ into the category of functors $\mathcal{C} \to \mathit{Sets}$. Hence, if $f : U \to V$ is a morphism, we are justified in denoting still by f the induced morphism $\underline{V} \to \underline{U}$, and vice-versa.

06SI (13) Fiber products of categories cofibered in groupoids: If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$, then a construction of their 2-fiber product is given by the construction for their 2-fiber product as categories over $\mathcal{C}_\Lambda$, as described in Categories, Lemma 4.32.3.

0DZJ (14) Products of categories cofibered in groupoids: If $\mathcal{F}$ and $\mathcal{G}$ are categories cofibered in groupoids over $\mathcal{C}_\Lambda$ then their product is defined to be the 2-fiber product $\mathcal{F} \times_{\mathcal{C}_\Lambda} \mathcal{G}$ as described in Categories, Lemma 4.32.3.

06GR (15) Restricting the base category: Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The restriction $\mathcal{F}|_{\mathcal{C}'}$ is the full subcategory of $\mathcal{F}$ whose objects lie over objects of $\mathcal{C}'$. It is a category cofibered in groupoids via the functor $p|_{\mathcal{C}'} : \mathcal{F}|_{\mathcal{C}'} \to \mathcal{C}'$.

90.6. Prorepresentable functors and predeformation categories

06GI Our basic goal is to understand categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}_\Lambda}$. Since $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}_\Lambda}$ we can restrict categories cofibred in groupoids over $\widehat{\mathcal{C}_\Lambda}$ to $\mathcal{C}_\Lambda$, see Remarks 90.5.2 (15). In particular we can do this with functors, in particular with representable functors. The functors on $\mathcal{C}_\Lambda$ one obtains in this way are called prorepresentable functors.

06GX **Definition 90.6.1.** Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. We say F is *prorepresentable* if there exists an isomorphism $F \cong \underline{R}|_{\mathcal{C}_\Lambda}$ of functors for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$.

Note that if $F : \mathcal{C}_\Lambda \to Sets$ is prorepresentable by $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$, then

$$F(k) = \mathrm{Mor}_{\widehat{\mathcal{C}_\Lambda}}(R, k) = \{*\}$$

is a singleton. The categories cofibered in groupoids over $\mathcal{C}_\Lambda$ that are arise in deformation theory will often satisfy an analogous condition.

06GS **Definition 90.6.2.** A *predeformation category* $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to a category with a single object and a single morphism, i.e., $\mathcal{F}(k)$ contains at least one object and there is a unique morphism between any two objects. A *morphism of predeformation categories* is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

A feature of a predeformation category is the following. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then every object of $\mathcal{F}$ comes equipped with a unique morphism to x_0. Namely, if x is an object of $\mathcal{F}$ over A, then we can choose a pushforward $x \to q_*x$ where $q : A \to k$ is the quotient map. There is a unique isomorphism $q_*x \to x_0$ and the composition $x \to q_*x \to x_0$ is the desired morphism.

06GT **Remark 90.6.3.** We say that a functor $F : \mathcal{C}_\Lambda \to Sets$ is a *predeformation functor* if the associated cofibered set is a predeformation category, i.e. if $F(k)$ is a one element set. Thus if $\mathcal{F}$ is a predeformation category, then $\overline{\mathcal{F}}$ is a predeformation functor.

06GU **Remark 90.6.4.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. We denote by $\mathcal{F}_x$ the category of objects over x. An object of $\mathcal{F}_x$ is an arrow $y \to x$. A morphism $(y \to x) \to (z \to x)$ in $\mathcal{F}_x$ is a commutative diagram

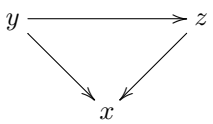

There is a forgetful functor $\mathcal{F}_x \to \mathcal{F}$. We define the functor $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ as the composition $\mathcal{F}_x \to \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$. Then $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ is a predeformation category (proof omitted). In this way we can pass from an arbitrary category cofibered in groupoids over $\mathcal{C}_\Lambda$ to a predeformation category at any $x \in \mathrm{Ob}(\mathcal{F}(k))$.

90.7. Formal objects and completion categories

06H2 In this section we discuss how to go between categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ and vice versa.

06H3 **Definition 90.7.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The *category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$* is the category with the following objects and morphisms.

(1) A *formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$* consists of an object R of $\widehat{\mathcal{C}}_\Lambda$, and a collection indexed by $n \in \mathbf{N}$ of objects ξ_n of $\mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.
(2) Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. A *morphism $a : \xi \to \eta$ of formal objects* consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, such that for every n the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow[f_n]{} & \xi_n \\ {\scriptstyle a_{n+1}}\downarrow & & \downarrow{\scriptstyle a_n} \\ \eta_{n+1} & \xrightarrow[g_n]{} & \eta_n \end{array}$$

commutes.

The category of formal objects comes with a functor $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ which sends an object (R, ξ_n, f_n) to R and a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ to the map $R \to S$.

06H4 **Lemma 90.7.2.** *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Then $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is a category cofibered in groupoids.*

Proof. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Let (R, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}$. For each n choose a pushforward $\xi_n \to \eta_n$ of ξ_n along $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. For each n there exists a unique morphism $g_n : \eta_{n+1} \to \eta_n$ in $\mathcal{F}$ lying over $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ such that

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow[f_n]{} & \xi_n \\ \downarrow & & \downarrow \\ \eta_{n+1} & \xrightarrow[g_n]{} & \eta_n \end{array}$$

commutes (by the first axiom of a category cofibred in groupoids). Hence we obtain a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ lying over $R \to S$, i.e., the first axiom of a category cofibred in groupoids holds for $\widehat{\mathcal{F}}$. To see the second axiom suppose that we have morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ and $b : (R, \xi_n, f_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ and a morphism $c_0 : S \to T$ in $\widehat{\mathcal{C}}_\Lambda$ such that $c_0 \circ a_0 = b_0$. By the second axiom of a category cofibred in groupoids for $\mathcal{F}$ we obtain unique maps $c_n : \eta_n \to \theta_n$ lying over $S/\mathfrak{m}_S^n \to T/\mathfrak{m}_T^n$ such that $c_n \circ a_n = b_n$. Setting $c = (c_n)_{n \geq 0}$ gives the desired morphism $c : (S, \eta_n, g_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ (we omit the verification that $h_n \circ c_{n+1} = c_n \circ g_n$). □

06H5 **Definition 90.7.3.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The category cofibered in groupoids $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is called the *completion of $\mathcal{F}$*.

If $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$, we have defined $\widehat{\mathcal{F}}(R)$ for $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ in terms of the filtration of R by powers of its maximal ideal. But suppose $\mathcal{I} = (I_n)$ is a filtration of R by ideals inducing the $\mathfrak{m}_R$-adic topology. We define $\widehat{\mathcal{F}}_\mathcal{I}(R)$ to be the category with the following objects and morphisms:

(1) An object is a collection $(\xi_n, f_n)_{n\in\mathbf{N}}$ of objects ξ_n of $\mathcal{F}(R/I_n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projections $R/I_{n+1} \to R/I_n$.
(2) A morphism $a : (\xi_n, f_n) \to (\eta_n, g_n)$ consists of a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}(R/I_n)$, such that for every n the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ {\scriptstyle a_{n+1}}\downarrow & & \downarrow{\scriptstyle a_n} \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes.

06H6 **Lemma 90.7.4.** *In the situation above, $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is equivalent to the category $\widehat{\mathcal{F}}(R)$.*

Proof. An equivalence $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ can be defined as follows. For each n, let $m(n)$ be the least m that $I_m \subset \mathfrak{m}_R^n$. Given an object (ξ_n, f_n) of $\widehat{\mathcal{F}}_\mathcal{I}(R)$, let η_n be the pushforward of $\xi_{m(n)}$ along $R/I_{m(n)} \to R/\mathfrak{m}_R^n$. Let $g_n : \eta_{n+1} \to \eta_n$ be the unique morphism of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ such that

$$\begin{array}{ccc} \xi_{m(n+1)} & \xrightarrow{f_{m(n)} \circ \ldots \circ f_{m(n+1)-1}} & \xi_{m(n)} \\ \downarrow & & \downarrow \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes (existence and uniqueness is guaranteed by the axioms of a cofibred category). The functor $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ sends (ξ_n, f_n) to (R, η_n, g_n). We omit the verification that this is indeed an equivalence of categories. □

06H7 **Remark 90.7.5.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Suppose that for each $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ we are given a filtration $\mathcal{I}_R$ of R by ideals. If $\mathcal{I}_R$ induces the $\mathfrak{m}_R$-adic topology on R for all R, then one can define a category $\widehat{\mathcal{F}}_\mathcal{I}$ by mimicking the definition of $\widehat{\mathcal{F}}$. This category comes equipped with a morphism $\widehat{p_\mathcal{I}} : \widehat{\mathcal{F}}_\mathcal{I} \to \widehat{\mathcal{C}}_\Lambda$ making it into a category cofibered in groupoids such that $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is isomorphic to $\widehat{\mathcal{F}}_{\mathcal{I}_R}(R)$ as defined above. The categories cofibered in groupoids $\widehat{\mathcal{F}}_\mathcal{I}$ and $\widehat{\mathcal{F}}$ are equivalent, by using over an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ the equivalence of Lemma 90.7.4.

06H8 **Remark 90.7.6.** Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a functor. Identifying functors with cofibered sets, the completion of F is the functor $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ given by $\widehat{F}(S) = \lim F(S/\mathfrak{m}_S^n)$. This agrees with the definition in Schlessinger's paper [**Sch68**].

06SJ **Remark 90.7.7.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. We claim that there is a canonical equivalence

$$can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F}.$$

Namely, let $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ and let (A, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda}(A)$. Since A is Artinian there is a minimal $m \in \mathbf{N}$ such that $\mathfrak{m}_A^m = 0$. Then *can* sends (A, ξ_n, f_n)

to ξ_m. This functor is an equivalence of categories cofibered in groupoids by Categories, Lemma 4.35.9 because it is an equivalence on all fibre categories by Lemma 90.7.4 and the fact that the $\mathfrak{m}_A$-adic topology on a local Artinian ring A comes from the zero ideal. We will frequently identify $\mathcal{F}$ with a full subcategory of $\widehat{\mathcal{F}}$ via a quasi-inverse to the functor *can*.

06H9 **Remark 90.7.8.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then there is an induced morphism $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories cofibered in groupoids over $\widehat{\mathcal{C}}_\Lambda$. It sends an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ to $(R, \varphi(\xi_n), \varphi(f_n))$, and it sends a morphism $(a_0 : R \to S, a_n : \xi_n \to \eta_n)$ between objects ξ and η of $\widehat{\mathcal{F}}$ to $(a_0 : R \to S, \varphi(a_n) : \varphi(\xi_n) \to \varphi(\eta_n))$. Finally, if $t : \varphi \to \varphi'$ is a 2-morphism between 1-morphisms $\varphi, \varphi' : \mathcal{F} \to \mathcal{G}$ of categories cofibred in groupoids, then we obtain a 2-morphism $\widehat{t} : \widehat{\varphi} \to \widehat{\varphi}'$. Namely, for $\xi = (R, \xi_n, f_n)$ as above we set $\widehat{t}_\xi = (t_{\varphi(\xi_n)})$. Hence completion defines a functor between 2-categories

$$\widehat{\ } : \mathrm{Cof}(\mathcal{C}_\Lambda) \longrightarrow \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda)$$

from the 2-category of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to the 2-category of categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$.

06HA **Remark 90.7.9.** We claim the completion functor of Remark 90.7.8 and the restriction functor $|_{\mathcal{C}_\Lambda} : \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{Cof}(\mathcal{C}_\Lambda)$ of Remarks 90.5.2 (15) are "2-adjoint" in the following precise sense. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Cof}(\mathcal{C}_\Lambda))$ and let $\mathcal{G} \in \mathrm{Ob}(\mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda))$. Then there is an equivalence of categories

$$\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$$

To describe this equivalence, we define canonical morphisms $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ and $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ as follows

(1) Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda))$ and let ξ be an object of the fiber category $\mathcal{G}(R)$. Choose a pushforward $\xi \to \xi_n$ of ξ to $R/\mathfrak{m}_R^n$ for each $n \in \mathbf{N}$, and let $f_n : \xi_{n+1} \to \xi_n$ be the induced morphism. Then $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ sends ξ to (R, ξ_n, f_n).
(2) This is the equivalence $can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 90.7.7.

Having said this, the equivalence $\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \to \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$ sends a morphism $\varphi : \mathcal{G}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to

$$\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{F}}$$

There is a quasi-inverse $\Psi : \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}}) \to \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F})$ to Φ which sends $\psi : \mathcal{G} \to \widehat{\mathcal{F}}$ to

$$\mathcal{G}|_{\mathcal{C}_\Lambda} \xrightarrow{\psi|_{\mathcal{C}_\Lambda}} \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}.$$

We omit the verification that Φ and Ψ are quasi-inverse. We also do not address functoriality of Φ (because it would lead into 3-category territory which we want to avoid at all cost).

06HB **Remark 90.7.10.** For a category $\mathcal{C}$ we denote by $\mathrm{CofSet}(\mathcal{C})$ the category of cofibered sets over $\mathcal{C}$. It is a 1-category isomorphic the category of functors $\mathcal{C} \to \mathit{Sets}$. See Remarks 90.5.2 (11). The completion and restriction functors restrict to functors $\widehat{\ } : \mathrm{CofSet}(\mathcal{C}_\Lambda) \to \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda)$ and $|_{\mathcal{C}_\Lambda} : \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{CofSet}(\mathcal{C}_\Lambda)$ which we denote by the same symbols. As functors on the categories of cofibered sets,

completion and restriction are adjoints in the usual 1-categorical sense: the same construction as in Remark 90.7.9 defines a functorial bijection

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(G|_{\mathcal{C}_\Lambda}, F) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(G, \widehat{F})$$

for $F \in \mathrm{Ob}(\mathrm{CofSet}(\mathcal{C}_\Lambda))$ and $G \in \mathrm{Ob}(\mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda))$. Again the map $\widehat{F}|_{\mathcal{C}_\Lambda} \to F$ is an isomorphism.

06HE **Remark 90.7.11.** Let $G : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ be a functor that commutes with limits. Then the map $G \to \widehat{G|_{\mathcal{C}_\Lambda}}$ described in Remark 90.7.9 is an isomorphism. Indeed, if S is an object of $\widehat{\mathcal{C}}_\Lambda$, then we have canonical bijections

$$\widehat{G|_{\mathcal{C}_\Lambda}}(S) = \lim_n G(S/\mathfrak{m}_S^n) = G(\lim_n S/\mathfrak{m}_S^n) = G(S).$$

In particular, if R is an object of $\widehat{\mathcal{C}}_\Lambda$ then $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ because the representable functor $\underline{R}$ commutes with limits by definition of limits.

06HC **Remark 90.7.12.** Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. It defines a functor $\underline{R} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ as described in Remarks 90.5.2 (12). As usual we identify this functor with the associated cofibered set. If $\mathcal{F}$ is a cofibered category over $\mathcal{C}_\Lambda$, then there is an equivalence of categories

06SK (90.7.12.1) $$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \widehat{\mathcal{F}}(R).$$

It is given by the composition

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \xrightarrow{\Phi} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R)$$

where Φ is as in Remark 90.7.9 and the second equivalence comes from the 2-Yoneda lemma (the cofibered analogue of Categories, Lemma 4.41.2). Explicitly, the equivalence sends a morphism $\varphi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to the formal object $(R, \varphi(R \to R/\mathfrak{m}_R^n), \varphi(f_n))$ in $\widehat{\mathcal{F}}(R)$, where $f_n : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the projection.

Assume a choice of pushforwards for $\mathcal{F}$ has been made. Given any $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we construct an explicit $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ which maps to ξ under (90.7.12.1). Namely, say $\xi = (R, \xi_n, f_n)$. An object α in $\underline{R}|_{\mathcal{C}_\Lambda}$ is the same thing as a morphism $\alpha : R \to A$ of $\widehat{\mathcal{C}}_\Lambda$ with A Artinian. Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. Then α factors through a unique $\alpha_m : R/\mathfrak{m}_R^m \to A$ and we can set $\underline{\xi}(\alpha) = \alpha_{m,*}\xi_m$. We omit the description of $\underline{\xi}$ on morphisms and we omit the proof that $\underline{\xi}$ maps to ξ via (90.7.12.1).

Assume a choice of pushforwards for $\widehat{\mathcal{F}}$ has been made. In this case the proof of Categories, Lemma 4.41.2 gives an explicit quasi-inverse

$$\iota : \widehat{\mathcal{F}}(R) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}})$$

to the 2-Yoneda equivalence which takes ξ to the morphism $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ sending $f \in \underline{R}(S) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(R, S)$ to $f_*\xi$. A quasi-inverse to (90.7.12.1) is then

$$\widehat{\mathcal{F}}(R) \xrightarrow{\iota} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\Psi} \mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F})$$

where Ψ is as in Remark 90.7.9. Given $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we have $\Psi(\iota(\xi)) \cong \underline{\xi}$ where $\underline{\xi}$ is as in the previous paragraph, because both are mapped to ξ under the equivalence of categories (90.7.12.1). Using $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ (see Remark 90.7.11) and unwinding the definitions of Φ and Ψ we conclude that $\iota(\xi)$ is isomorphic to the completion of $\underline{\xi}$.

06SL **Remark 90.7.13.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. Let $a = (a_n) : \xi \to \eta$ be a morphism of formal objects, i.e., a morphism of $\widehat{\mathcal{F}}$. Let $f = \widehat{p}(a) = a_0 : R \to S$ be the projection of a in $\widehat{\mathcal{C}}_\Lambda$. Then we obtain a 2-commutative diagram

$$\begin{array}{ccc} \underline{R}|_{\mathcal{C}_\Lambda} & \xleftarrow{\quad f \quad} & \underline{S}|_{\mathcal{C}_\Lambda} \\ & {\scriptstyle \underline{\xi}} \searrow \quad \swarrow {\scriptstyle \underline{\eta}} & \\ & \mathcal{F} & \end{array}$$

where $\underline{\xi}$ and $\underline{\eta}$ are the morphisms constructed in Remark 90.7.12. To see this let $\alpha : S \to A$ be an object of $\underline{S}|_{\mathcal{C}_\Lambda}$ (see loc. cit.). Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. We get a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{m}_R^m & & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_m} & \searrow{\scriptstyle \beta_m} & \\ S & \longrightarrow & S/\mathfrak{m}_S^m & \xrightarrow{\alpha_m} & A \end{array}$$

such that the bottom arrows compose to give α. Then $\underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$ and $\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m$. The morphism $a_m : \xi_m \to \eta_m$ lies over f_m hence we obtain a canonical morphism

$$\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m \longrightarrow \underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$$

lying over id_A such that

$$\begin{array}{ccc} \xi_m & \longrightarrow & \beta_{m,*}\xi_m \\ \downarrow{\scriptstyle a_m} & & \downarrow \\ \eta_m & \longrightarrow & \alpha_{m,*}\eta_m \end{array}$$

commutes by the axioms of a category cofibred in groupoids. This defines a transformation of functors $\underline{\xi} \circ f \to \underline{\eta}$ which witnesses the 2-commutativity of the first diagram of this remark.

06HD **Remark 90.7.14.** According to Remark 90.7.12, giving a formal object ξ of $\mathcal{F}$ is equivalent to giving a prorepresentable functor $U : \mathcal{C}_\Lambda \to \mathit{Sets}$ and a morphism $U \to \mathcal{F}$.

90.8. Smooth morphisms

06HF In this section we discuss smooth morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HG **Definition 90.8.1.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say φ is *smooth* if it satisfies the following condition: Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B)), x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. Then there exists $x' \in \mathrm{Ob}(\mathcal{F}(B))$, a morphism $x' \to x$ lying over $B \to A$, and a morphism $\varphi(x') \to y$ lying over

$\mathrm{id} : B \to B$, such that the diagram

$$\begin{array}{ccc} \varphi(x') & \longrightarrow & y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes.

06HH **Lemma 90.8.2.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then φ is smooth if the condition in Definition 90.8.1 is assumed to hold only for small extensions $B \to A$.*

Proof. Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B))$, $x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. By Lemma 90.3.3 we can factor $B \to A$ into small extensions $B = B_n \to B_{n-1} \to \ldots \to B_0 = A$. We argue by induction on n. If $n = 1$ the result is true by assumption. If $n > 1$, then denote $f : B = B_n \to B_{n-1}$ and denote $g : B_{n-1} \to B_0 = A$. Choose a pushforward $y \to f_*y$ of y along f, so that the morphism $y \to \varphi(x)$ factors as $y \to f_*y \to \varphi(x)$. By the induction hypothesis we can find $x_{n-1} \to x$ lying over $g : B_{n-1} \to A$ and $a : \varphi(x_{n-1}) \to f_*y$ lying over $\mathrm{id} : B_{n-1} \to B_{n-1}$ such that

$$\begin{array}{ccc} \varphi(x_{n-1}) & \xrightarrow[a]{} & f_*y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes. We can apply the assumption to the composition $y \to \varphi(x_{n-1})$ of $y \to f_*y$ with $a^{-1} : f_*y \to \varphi(x_{n-1})$. We obtain $x_n \to x_{n-1}$ lying over $B_n \to B_{n-1}$ and $\varphi(x_n) \to y$ lying over $\mathrm{id} : B_n \to B_n$ so that the diagram

$$\begin{array}{ccc} \varphi(x_n) & \longrightarrow & y \\ \downarrow & & \downarrow \\ \varphi(x_{n-1}) & \xrightarrow{a} & f_*y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes. Then the composition $x_n \to x_{n-1} \to x$ and $\varphi(x_n) \to y$ are the morphisms required by the definition of smoothness. □

06HI **Remark 90.8.3.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Choices of pushforwards along $B \to A$ for objects in the fiber categories $\mathcal{F}(B)$ and $\mathcal{G}(B)$ determine functors $\mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(B) \to \mathcal{G}(A)$ fitting into a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}(B) & \xrightarrow{\varphi} & \mathcal{G}(B) \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\varphi} & \mathcal{G}(A). \end{array}$$

Hence there is an induced functor $\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$. Unwinding the definitions shows that $\varphi : \mathcal{F} \to \mathcal{G}$ is smooth if and only if this induced functor is essentially surjective whenever $B \to A$ is surjective (or equivalently, by Lemma 90.8.2, whenever $B \to A$ is a small extension).

06HJ **Remark 90.8.4.** The characterization of smooth morphisms in Remark 90.8.3 is analogous to Schlessinger's notion of a smooth morphism of functors, cf. [**Sch68**, Definition 2.2.]. In fact, when $\mathcal{F}$ and $\mathcal{G}$ are cofibered in sets then our notion is equivalent to Schlessinger's. Namely, in this case let $F, G : \mathcal{C}_\Lambda \to \mathit{Sets}$ be the corresponding functors, see Remarks 90.5.2 (11). Then $F \to G$ is smooth if and only if for every surjection of rings $B \to A$ in $\mathcal{C}_\Lambda$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

06HK **Remark 90.8.5.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then the morphism $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth. Namely, suppose that $f : B \to A$ is a ring map in $\mathcal{C}_\Lambda$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$ and let $\overline{y} \in \overline{\mathcal{F}}(B)$ be the isomorphism class of $y \in \mathrm{Ob}(\mathcal{F}(B))$ such that $\overline{f_* y} = \overline{x}$. Then we simply take $x' = y$, the implied morphism $x' = y \to x$ over $B \to A$, and the equality $\overline{x'} = \overline{y}$ as the solution to the problem posed in Definition 90.8.1.

If $R \to S$ is a ring map $\widehat{\mathcal{C}}_\Lambda$, then there is an induced morphism $\underline{S} \to \underline{R}$ between the functors $\underline{S}, \underline{R} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$. In this situation, smoothness of the restriction $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is a familiar notion:

06HL **Lemma 90.8.6.** *Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Then the induced morphism $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth if and only if S is a power series ring over R.*

Proof. Assume S is a power series ring over R. Say $S = R[[x_1, \ldots, x_n]]$. Smoothness of $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ means the following (see Remark 90.8.4): Given a surjective ring map $B \to A$ in $\mathcal{C}_\Lambda$, a ring map $R \to B$, a ring map $S \to A$ such that the solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

is commutative then a dotted arrow exists making the diagram commute. (Note the similarity with Algebra, Definition 10.138.1.) To construct the dotted arrow choose elements $b_i \in B$ whose images in A are equal to the images of x_i in A. Note that $b_i \in \mathfrak{m}_B$ as x_i maps to an element of $\mathfrak{m}_A$. Hence there is a unique R-algebra map $R[[x_1, \ldots, x_n]] \to B$ which maps x_i to b_i and which can serve as our dotted arrow.

Conversely, assume $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Let $x_1, \ldots, x_n \in S$ be elements whose images form a basis in the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ of S over R. Set $T = R[[X_1, \ldots, X_n]]$. Note that both

$$S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \cong R/\mathfrak{m}_R[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/(\mathfrak{m}_R T + \mathfrak{m}_T^2) \cong R/\mathfrak{m}_R[X_1, \ldots, X_n]/(X_i X_j).$$

Let $S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the local R-algebra isomorphism given by mapping the class of x_i to the class of X_i. Let $f_1 : S \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be

the composition $S \to S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$. The assumption that $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means we can lift f_1 to a map $f_2 : S \to T/\mathfrak{m}_T^2$, then to a map $f_3 : S \to T/\mathfrak{m}_T^3$, and so on, for all $n \geq 1$. Thus we get an induced map $f : S \to T = \lim T/\mathfrak{m}_T^n$ of local R-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to \mathfrak{m}_T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ of relative cotangent spaces. Hence f is surjective by Lemma 90.4.2 (where we think of f as a map in $\widehat{\mathcal{C}}_R$). Choose preimages $y_i \in S$ of $X_i \in T$ under f. As T is a power series ring over R there exists a local R-algebra homomorphism $s : T \to S$ mapping X_i to y_i. By construction $f \circ s = \mathrm{id}$. Then s is injective. But s induces an isomorphism on relative cotangent spaces since f does, so it is also surjective by Lemma 90.4.2 again. Hence s and f are isomorphisms. □

Smooth morphisms satisfy the following functorial properties.

06HM **Lemma 90.8.7.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.*

(1) *If φ and ψ are smooth, then $\psi \circ \varphi$ is smooth.*
(2) *If φ is essentially surjective and $\psi \circ \varphi$ is smooth, then ψ is smooth.*
(3) *If $\mathcal{G}' \to \mathcal{G}$ is a morphism of categories cofibered in groupoids and φ is smooth, then $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}'$ is smooth.*

Proof. Statements (1) and (2) follow immediately from the definitions. Proof of (3) omitted. Hints: use the formulation of smoothness given in Remark 90.8.3 and use that $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}'$ is the 2-fibre product, see Remarks 90.5.2 (13). □

06HN **Lemma 90.8.8.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ is essentially surjective. Then $\varphi : \mathcal{F} \to \mathcal{G}$ and $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ are essentially surjective.*

Proof. Let y be an object of $\mathcal{G}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. Let $y \to y_0$ be a pushforward of y along $A \to k$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ there exist an object x_0 of $\mathcal{F}$ lying over k and an isomorphism $y_0 \to \varphi(x_0)$. Smoothness of φ implies there exists an object x of $\mathcal{F}$ over A whose image $\varphi(x)$ is isomorphic to y. Thus $\varphi : \mathcal{F} \to \mathcal{G}$ is essentially surjective.

Let $\eta = (R, \eta_n, g_n)$ be an object of $\widehat{\mathcal{G}}$. We construct an object ξ of $\widehat{\mathcal{F}}$ with an isomorphism $\eta \to \varphi(\xi)$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$, there exists a morphism $\eta_1 \to \varphi(\xi_1)$ in $\mathcal{G}(k)$ for some $\xi_1 \in \mathrm{Ob}(\mathcal{F}(k))$. The morphism $\eta_2 \xrightarrow{g_1} \eta_1 \to \varphi(\xi_1)$ lies over the surjective ring map $R/\mathfrak{m}_R^2 \to k$, hence by smoothness of φ there exists $\xi_2 \in \mathrm{Ob}(\mathcal{F}(R/\mathfrak{m}_R^2))$, a morphism $f_1 : \xi_2 \to \xi_1$ lying over $R/\mathfrak{m}_R^2 \to k$, and a morphism $\eta_2 \to \varphi(\xi_2)$ such that

$$\begin{array}{ccc} \varphi(\xi_2) & \xrightarrow{\varphi(f_1)} & \varphi(\xi_1) \\ \uparrow & & \uparrow \\ \eta_2 & \xrightarrow{g_1} & \eta_1 \end{array}$$

commutes. Continuing in this way we construct an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ and a morphism $\eta \to \varphi(\xi) = (R, \varphi(\xi_n), \varphi(f_n))$ in $\widehat{\mathcal{G}}(R)$. □

Later we are interested in producing smooth morphisms from prorepresentable functors to predeformation categories $\mathcal{F}$. By the discussion in Remark 90.7.12 these

morphisms correspond to certain formal objects of $\mathcal{F}$. More precisely, these are the so-called versal formal objects of $\mathcal{F}$.

06HR **Definition 90.8.9.** Let $\mathcal{F}$ be a category cofibered in groupoids. Let ξ be a formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We say ξ is *versal* if the corresponding morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 90.7.12 is smooth.

06HS **Remark 90.8.10.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let ξ be a formal object of $\mathcal{F}$. It follows from the definition of smoothness that versality of ξ is equivalent to the following condition: If

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

is a diagram in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \to A$ of Artinian rings (we may assume it is a small extension), then there exists a morphism $\xi \to y$ such that

$$\begin{array}{ccc} & \nearrow & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

commutes. In particular, the condition that ξ be versal does not depend on the choices of pushforwards made in the construction of $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ in Remark 90.7.12.

06HT **Lemma 90.8.11.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$. For any formal object η of $\widehat{\mathcal{F}}$, there exists a morphism $\xi \to \eta$.*

Proof. By assumption the morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth. Then $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ is the completion of $\underline{\xi}$, see Remark 90.7.12. By Lemma 90.8.8 there exists an object f of $\underline{R}$ such that $\iota(\xi)(f) = \eta$. Then f is a ring map $f : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$. And $\iota(\xi)(f) = \eta$ means that $f_*\xi \cong \eta$ which means exactly that there is a morphism $\xi \to \eta$ lying over f. □

90.9. Smooth or unobstructed categories

0DYK Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We can consider $\mathcal{C}_\Lambda$ as a category cofibered in groupoids over $\mathcal{C}_\Lambda$ using the identity functor. In this way $p : \mathcal{F} \longrightarrow \mathcal{C}_\Lambda$ becomes a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

06HP **Definition 90.9.1.** Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say $\mathcal{F}$ is *smooth* or *unobstructed* if its structure morphism p is smooth in the sense of Definition 90.8.1.

This is the "absolute" notion of smoothness for a category cofibered in groupoids over $\mathcal{C}_\Lambda$, although it would be more correct to say that $\mathcal{F}$ is smooth over Λ. One has to be careful with the phrase "$\mathcal{F}$ *is unobstructed*": it may happen that $\mathcal{F}$ has an obstruction theory with nonvanishing obstruction spaces even though $\mathcal{F}$ is smooth.

06HQ **Remark 90.9.2.** Suppose $\mathcal{F}$ is a predeformation category admitting a smooth morphism $\varphi : \mathcal{U} \to \mathcal{F}$ from a predeformation category $\mathcal{U}$. Then by Lemma 90.8.8 φ is essentially surjective, so by Lemma 90.8.7 $p : \mathcal{F} \to \mathcal{C}_\Lambda$ is smooth if and only if the composition $\mathcal{U} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$ is smooth, i.e. $\mathcal{F}$ is smooth if and only if $\mathcal{U}$ is smooth.

0DYL **Lemma 90.9.3.** *Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent*

(1) *$\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth,*
(2) *$\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology,*
(3) *$\Lambda \to R$ is flat and $R \otimes_\Lambda k'$ is geometrically regular over k', and*
(4) *$\Lambda \to R$ is flat and $k' \to R \otimes_\Lambda k'$ is formally smooth in the $\mathfrak{m}_R$-adic topology.*

In the classical case, these are also equivalent to

(5) *R is isomorphic to $\Lambda[[x_1, \ldots, x_n]]$ for some n.*

Proof. Smoothness of $p : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{C}_\Lambda$ means that given $B \to A$ surjective in $\mathcal{C}_\Lambda$ and given $R \to A$ we can find the dotted arrow in the diagram

$$\begin{array}{ccc} R & \longrightarrow & A \\ \uparrow & \nearrow & \uparrow \\ \Lambda & \longrightarrow & B \end{array}$$

This is certainly true if $\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology, see More on Algebra, Definitions 15.37.3 and 15.37.1. Conversely, if this holds, then we see that $\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology by More on Algebra, Lemma 15.38.1. Thus (1) and (2) are equivalent.

The equivalence of (2), (3), and (4) is More on Algebra, Proposition 15.40.5. The equivalence with (5) follows for example from Lemma 90.8.6 and the fact that $\mathcal{C}_\Lambda$ is the same as $\underline{\Lambda}|_{\mathcal{C}_\Lambda}$ in the classical case. □

0DZK **Lemma 90.9.4.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. The following are equivalent*

(1) *$\mathcal{F}$ is unobstructed, and*
(2) *$\Lambda \to R$ is formally smooth in the $\mathfrak{m}_R$-adic topology.*

In the classical case these are also equivalent to

(3) *$R \cong \Lambda[[x_1, \ldots, x_n]]$ for some n.*

Proof. If (1) holds, i.e., if $\mathcal{F}$ is unobstructed, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \xrightarrow{\xi} \mathcal{F} \to \mathcal{C}_\Lambda$$

is smooth, see Lemma 90.8.7. Hence we see that (2) holds by Lemma 90.9.3. Conversely, if (2) holds, then the composition is smooth and moreover the first arrow is essentially surjective by Lemma 90.8.11. Hence we find that the second arrow is smooth by Lemma 90.8.7 which means that $\mathcal{F}$ is unobstructed by definition. The equivalence with (3) in the classical case follows from Lemma 90.9.3. □

06SM **Lemma 90.9.5.** *There exists an $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that the equivalent conditions of Lemma 90.9.3 hold and moreover $H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$.*

Proof. In the classical case we choose $R = \Lambda$. More generally, if the residue field extension k/k' is separable, then there exists a unique finite étale extension $\Lambda^\wedge \to R$ (Algebra, Lemmas 10.153.9 and 10.153.7) of the completion $\Lambda^\wedge$ of Λ inducing the extension k/k' on residue fields.

In the general case we proceed as follows. Choose a smooth Λ-algebra P and a Λ-algebra surjection $P \to k$. (For example, let P be a polynomial algebra.) Denote $\mathfrak{m}_P$ the kernel of $P \to k$. The Jacobi-Zariski sequence, see (90.3.10.2) and Algebra, Lemma 10.134.4, is an exact sequence

$$0 \to H_1(NL_{k/\Lambda}) \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{P/\Lambda} \otimes_P k \to \Omega_{k/\Lambda} \to 0$$

We have the 0 on the left because P/k is smooth, hence $NL_{P/\Lambda}$ is quasi-isomorphic to a finite projective module placed in degree 0, hence $H_1(NL_{P/\Lambda} \otimes_P k) = 0$. Suppose $f \in \mathfrak{m}_P$ maps to a nonzero element of $\Omega_{P/\Lambda} \otimes_P k$. Setting $P' = P/(f)$ we have a Λ-algebra surjection $P' \to k$. Observe that P' is smooth at $\mathfrak{m}_{P'}$: this follows from More on Morphisms, Lemma 37.38.1. Thus after replacing P by a principal localization of P', we see that $\dim(\mathfrak{m}_P/\mathfrak{m}_P^2)$ decreases. Repeating finitely many times, we may assume the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{P/\Lambda} \otimes_P k$ is zero so that the exact sequence breaks into isomorphisms $H_1(L_{k/\Lambda}) = \mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{k/\Lambda}$.

Let R be the $\mathfrak{m}_P$-adic completion of P. Then R is an object of $\widehat{\mathcal{C}}_\Lambda$. Namely, it is a complete local Noetherian ring (see Algebra, Lemma 10.97.6) and its residue field is identified with k. We claim that R works.

First observe that the map $P \to R$ induces isomorphisms $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{P/\Lambda} \otimes_P k = \Omega_{R/\Lambda} \otimes_R k$. This is true because both $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $\Omega_{P/\Lambda} \otimes_P k$ only depend on the Λ-algebra $P/\mathfrak{m}_P^2$, see Algebra, Lemma 10.131.11, the same holds for R and we have $P/\mathfrak{m}_P^2 = R/\mathfrak{m}_R^2$. Using the functoriality of the Jacobi-Zariski sequence (90.3.10.3) we deduce that $H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$ as the same is true for P.

Finally, since $\Lambda \to P$ is smooth we see that $\Lambda \to P$ is formally smooth by Algebra, Proposition 10.138.13. Then $\Lambda \to P$ is formally smooth for the $\mathfrak{m}_P$-adic topology by More on Algebra, Lemma 15.37.2. This property is inherited by the completion R by More on Algebra, Lemma 15.37.4 and the proof is complete. In fact, it turns out that whenever $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, then R is isomorphic to a completion of a smooth algebra over Λ, but we won't use this. □

06SN **Example 90.9.6.** Here is a more explicit example of an R as in Lemma 90.9.5. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Then we can take

$$R = \Lambda[x_1, \ldots, x_n]^\wedge_{(x_1^p - t_1, \ldots, x_n^p - t_n)}$$

We cannot do "better" in this example, i.e., we cannot approximate $\mathcal{C}_\Lambda$ by a smaller smooth object of $\widehat{\mathcal{C}}_\Lambda$ (one can argue that the dimension of R has to be at least n since the map $\Omega_{R/\Lambda} \otimes_R k \to \Omega_{k/\Lambda}$ is surjective). We will discuss this phenomenon later in more detail.

90.10. Schlessinger's conditions

06HV In the following we often consider fibre products $A_1 \times_A A_2$ of rings in the category $\mathcal{C}_\Lambda$. We have seen in Example 90.3.7 that such a fibre product may not always be

an object of $\mathcal{C}_\Lambda$. However, in virtually all cases below one of the two maps $A_i \to A$ is surjective and $A_1 \times_A A_2$ will be an object of $\mathcal{C}_\Lambda$ by Lemma 90.3.8. We will use this result without further mention.

We denote by $k[\epsilon]$ the ring of dual numbers over k. More generally, for a k-vector space V, we denote by $k[V]$ the k-algebra whose underlying vector space is $k \oplus V$ and whose multiplication is given by $(a, v) \cdot (a', v') = (aa', av' + a'v)$. When $V = k$, $k[V]$ is the ring of dual numbers over k. For any finite dimensional k-vector space V the ring $k[V]$ is in $\mathcal{C}_\Lambda$.

06HW **Definition 90.10.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We define *conditions (S1) and (S2)* on $\mathcal{F}$ as follows:

(S1) Every diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective can be completed to a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

(S2) The condition of (S1) holds for diagrams in $\mathcal{F}$ lying over a diagram in $\mathcal{C}_\Lambda$ of the form

$$\begin{array}{ccc} & & k[\epsilon] \\ & & \downarrow \\ A & \longrightarrow & k. \end{array}$$

Moreover, if we have two commutative diagrams in $\mathcal{F}$

$$\begin{array}{ccc} y & \xrightarrow{c} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow{c'} & x_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

then there exists a morphism $b : y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$ such that $a = a' \circ b$.

We can partly explain the meaning of conditions (S1) and (S2) in terms of fibre categories. Suppose that $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. Denote $p_i : A_1 \times_A A_2 \to A_i$ the projection maps. Assume a choice of pushforwards for $\mathcal{F}$ has been made. Then the commutative diagram of rings translates into a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}(A_1 \times_A A_2) & \xrightarrow[p_{2,*}]{} & \mathcal{F}(A_2) \\ {\scriptstyle p_{1,*}}\downarrow & & \downarrow{\scriptstyle f_{2,*}} \\ \mathcal{F}(A_1) & \xrightarrow{f_{1,*}} & \mathcal{F}(A) \end{array}$$

of fibre categories whence a functor

06SP (90.10.1.1) $$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

into the 2-fibre product of categories. Condition (S1) requires that this functor be essentially surjective. The first part of condition (S2) requires that this functor be a essentially surjective if f_2 equals the map $k[\epsilon] \to k$. Moreover in this case, the second part of (S2) implies that two objects which become isomorphic in the target are isomorphic in the source (but it is *not* equivalent to this statement). The advantage of stating the conditions as in the definition is that no choices have to be made.

06HX **Lemma 90.10.2.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (S1) if the condition of (S1) is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Proof omitted. Hints: apply Lemma 90.3.3 and use induction similar to the proof of Lemma 90.8.2. □

06HY **Remark 90.10.3.** When $\mathcal{F}$ is cofibered in sets, conditions (S1) and (S2) are exactly conditions (H1) and (H2) from Schlessinger's paper [**Sch68**]. Namely, for a functor $F : \mathcal{C}_\Lambda \to Sets$, conditions (S1) and (S2) state:

(S1) If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.

(S2) If $A \to k$ is a map in $\mathcal{C}_\Lambda$, then the induced map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ is bijective.

The injectivity of the map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ comes from the second part of condition (S2) and the fact that morphisms are identities.

06HZ **Lemma 90.10.4.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}$ satisfies (S2), then the condition of (S2) also holds when $k[\epsilon]$ is replaced by $k[V]$ for any finite dimensional k-vector space V.*

Proof. In the case that $\mathcal{F}$ is cofibred in sets, i.e., corresponds to a functor $F : \mathcal{C}_\Lambda \to Sets$ this follows from the description of (S2) for F in Remark 90.10.3 and the fact that $k[V] \cong k[\epsilon] \times_k \ldots \times_k k[\epsilon]$ with $\dim_k V$ factors. The case of functors is what we will use in the rest of this chapter.

We prove the general case by induction on $\dim(V)$. If $\dim(V) = 1$, then $k[V] \cong k[\epsilon]$ and the result holds by assumption. If $\dim(V) > 1$ we write $V = V' \oplus k\epsilon$. Pick a diagram

$$\begin{array}{ccc} x_V & & k[V] \\ \downarrow & \text{lying over} & \downarrow \\ x \longrightarrow x_0 & & A \longrightarrow k \end{array}$$

Choose a morphism $x_V \to x_{V'}$ lying over $k[V] \to k[V']$ and a morphism $x_V \to x_\epsilon$ lying over $k[V] \to k[\epsilon]$. Note that the morphism $x_V \to x_0$ factors as $x_V \to x_{V'} \to x_0$ and as $x_V \to x_\epsilon \to x_0$. By induction hypothesis we can find a diagram

$$\begin{array}{ccc} y' \longrightarrow x_{V'} & & A \times_k k[V'] \longrightarrow k[V'] \\ \downarrow \quad \downarrow & \text{lying over} & \downarrow \quad \downarrow \\ x \longrightarrow x_0 & & A \longrightarrow k \end{array}$$

This gives us a commutative diagram

$$\begin{array}{ccc} & & x_\epsilon \\ & & \downarrow \\ y' & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & k[\epsilon] \\ & & \downarrow \\ A \times_k k[V'] & \longrightarrow & k \end{array}$$

Hence by (S2) we get a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y' & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} (A \times_k k[V']) \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A \times_k k[V'] & \longrightarrow & k \end{array}$$

Note that $(A \times_k k[V']) \times_k k[\epsilon] = A \times_k k[V' \oplus k\epsilon] = A \times_k k[V]$. We claim that y fits into the correct commutative diagram. To see this we let $y \to y_V$ be a morphism lying over $A \times_k k[V] \to k[V]$. We can factor the morphisms $y \to y' \to x_{V'}$ and $y \to x_\epsilon$ through the morphism $y \to y_V$ (by the axioms of categories cofibred in groupoids). Hence we see that both y_V and x_V fit into commutative diagrams

$$\begin{array}{ccc} y_V & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x_{V'} & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} x_V & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x_{V'} & \longrightarrow & x_0 \end{array}$$

and hence by the second part of (S2) there exists an isomorphism $y_V \to x_V$ compatible with $y_V \to x_{V'}$ and $x_V \to x_{V'}$ and in particular compatible with the maps to x_0. The composition $y \to y_V \to x_V$ then fits into the required commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V] & \longrightarrow & k[V] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

In this way we see that the first part of $(S2)$ holds with $k[\epsilon]$ replaced by $k[V]$.

To prove the second part suppose given two commutative diagrams

$$\begin{array}{ccc} y & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V] & \longrightarrow & k[V] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

We will use the morphisms $x_V \to x_{V'} \to x_0$ and $x_V \to x_\epsilon \to x_0$ introduced in the first paragraph of the proof. Choose morphisms $y \to y_{V'}$ and $y' \to y'_{V'}$ lying over $A \times_k k[V] \to A \times_k k[V']$. The axioms of a cofibred category imply we can find commutative diagrams

$$\begin{array}{ccc} y_{V'} & \longrightarrow & x_{V'} \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y'_{V'} & \longrightarrow & x_{V'} \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V'] & \longrightarrow & k[V'] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

By induction hypothesis we obtain an isomorphism $b : y_{V'} \to y'_{V'}$ compatible with the morphisms $y_{V'} \to x$ and $y'_{V'} \to x$, in particular compatible with the morphisms to x_0. Then we have commutative diagrams

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y'_{V'} & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y'_{V'} & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

where the morphism $y \to y'_{V'}$ is the composition $y \to y_{V'} \xrightarrow{b} y'_{V'}$ and where the morphisms $y \to x_\epsilon$ and $y' \to x_\epsilon$ are the compositions of the maps $y \to x_V$ and $y' \to x_V$ with the morphism $x_V \to x_\epsilon$. Then the second part of (S2) guarantees the existence of an isomorphism $y \to y'$ compatible with the maps to $y'_{V'}$, in particular compatible with the maps to x (because b was compatible with the maps to x). □

06I0 **Lemma 90.10.5.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\overline{\mathcal{F}}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\overline{\mathcal{F}}$ provided at least one of the following conditions is satisfied*
   (a) *$\mathcal{F}$ is a predeformation category,*
   (b) *the category $\mathcal{F}(k)$ is a set or a setoid, or*
   (c) *for any morphism $x_\epsilon \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$ the pushforward map $Aut_{k[\epsilon]}(x_\epsilon) \to Aut_k(x_0)$ is surjective.*

Proof. Assume $\mathcal{F}$ has (S1). Suppose we have ring maps $f_i : A_i \to A$ in $\mathcal{C}_\Lambda$ with f_2 surjective. Let $x_i \in \mathcal{F}(A_i)$ such that the pushforwards $f_{1,*}(x_1)$ and $f_{2,*}(x_2)$ are isomorphic. Then we can denote x an object of $\mathcal{F}$ over A isomorphic to both of these and we obtain a diagram as in (S1). Hence we find an object y of $\mathcal{F}$ over $A_1 \times_A A_2$ whose pushforward to A_1, resp. A_2 is isomorphic to x_1, resp. x_2. In this way we see that (S1) holds for $\overline{\mathcal{F}}$.

Assume $\mathcal{F}$ has (S2). The first part of (S2) for $\overline{\mathcal{F}}$ follows as in the argument above. The second part of (S2) for $\overline{\mathcal{F}}$ signifies that the map

$$\overline{\mathcal{F}}(A \times_k k[\epsilon]) \to \overline{\mathcal{F}}(A) \times_{\overline{\mathcal{F}}(k)} \overline{\mathcal{F}}(k[\epsilon])$$

is injective for any ring A in $\mathcal{C}_\Lambda$. Suppose that $y, y' \in \mathcal{F}(A \times_k k[\epsilon])$. Using the axioms of cofibred categories we can choose commutative diagrams

$$\begin{array}{ccc} y & \xrightarrow{c} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow{c'} & x'_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e'} \\ x' & \xrightarrow{d'} & x'_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

Assume that there exist isomorphisms $\alpha : x \to x'$ in $\mathcal{F}(A)$ and $\beta : x_\epsilon \to x'_\epsilon$ in $\mathcal{F}(k[\epsilon])$. This also means there exists an isomorphism $\gamma : x_0 \to x'_0$ compatible with α. To prove (S2) for $\overline{\mathcal{F}}$ we have to show that there exists an isomorphism $y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$. By (S2) for $\mathcal{F}$ such a morphism will exist if we can choose the

isomorphisms α and β and γ such that

$$\begin{array}{ccccc} x & \longrightarrow & x_0 & \xleftarrow{e} & x_\epsilon \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ x' & \longrightarrow & x'_0 & \xleftarrow{e'} & x'_\epsilon \end{array}$$

is commutative (because then we can replace x by x' and x_ϵ by x'_ϵ in the previous displayed diagram). The left hand square commutes by our choice of γ. We can factor $e' \circ \beta$ as $\gamma' \circ e$ for some second map $\gamma' : x_0 \to x'_0$. Now the question is whether we can arrange it so that $\gamma = \gamma'$? This is clear if $\mathcal{F}(k)$ is a set, or a setoid. Moreover, if $\mathrm{Aut}_{k[\epsilon]}(x_\epsilon) \to \mathrm{Aut}_k(x_0)$ is surjective, then we can adjust the choice of β by precomposing with an automorphism of x_ϵ whose image is $\gamma^{-1} \circ \gamma'$ to make things work. □

06SQ **Lemma 90.10.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 90.6.4.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\mathcal{F}_{x_0}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$.*

Proof. Any diagram as in Definition 90.10.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of condition (S1) or (S2) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

06IS **Lemma 90.10.7.** *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Consider a diagram of $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \textit{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k. \end{array}$$

in $\mathcal{C}_\Lambda$. Assume $\mathcal{F}$ satisfies (S2). Then there exists a morphism $s : x \to y$ with $a \circ s = id_x$ if and only if there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$.

Proof. The "only if" direction is clear. Conversely, assume there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$. Note that $p(s_\epsilon) : A \to k[\epsilon]$ is a ring map compatible with the map $A \to k$. Hence we obtain

$$\sigma = (\mathrm{id}_A, p(s_\epsilon)) : A \to A \times_k k[\epsilon].$$

Choose a pushforward $x \to \sigma_* x$. By construction we can factor s_ϵ as $x \to \sigma_* x \to x_\epsilon$. Moreover, as σ is a section of $A \times_k k[\epsilon] \to A$, we get a morphism $\sigma_* x \to x$ such that $x \to \sigma_* x \to x$ is id_x. Because $e \circ s_\epsilon = d$ we find that the diagram

$$\begin{array}{ccc} \sigma_* x & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array}$$

is commutative. Hence by (S2) we obtain a morphism $\sigma_* x \to y$ such that $\sigma_* x \to y \to x$ is the given map $\sigma_* x \to x$. The solution to the problem is now to take $a : x \to y$ equal to the composition $x \to \sigma_* x \to y$. □

06IT **Lemma 90.10.8.** *Consider a commutative diagram in a predeformation category $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow{\scriptstyle a_2} \\ x_1 & \xrightarrow{a_1} & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{f_1} & A \end{array}$$

in $\mathcal{C}_\Lambda$ where $f_2 : A_2 \to A$ is a small extension. Assume there is a map $h : A_1 \to A_2$ such that $f_2 = f_1 \circ h$. Let $I = \mathrm{Ker}(f_2)$. Consider the ring map

$$g : A_1 \times_A A_2 \longrightarrow k[I] = k \oplus I, \quad (u, v) \longmapsto \overline{u} \oplus (v - h(u))$$

Choose a pushforward $y \to g_* y$. Assume $\mathcal{F}$ satisfies (S2). If there exists a morphism $x_1 \to g_* y$, then there exists a morphism $b : x_1 \to x_2$ such that $a_1 = a_2 \circ b$.

Proof. Note that $\mathrm{id}_{A_1} \times g : A_1 \times_A A_2 \to A_1 \times_k k[I]$ is an isomorphism and that $k[I] \cong k[\epsilon]$. Hence we have a diagram

$$\begin{array}{ccc} y & \longrightarrow & g_* y \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & k. \end{array}$$

where x_0 is an object of $\mathcal{F}$ lying over k (every object of $\mathcal{F}$ has a unique morphism to x_0, see discussion following Definition 90.6.2). If we have a morphism $x_1 \to g_* y$ then Lemma 90.10.7 provides us with a section $s : x_1 \to y$ of the map $y \to x_1$. Composing this with the map $y \to x_2$ we obtain $b : x_1 \to x_2$ which has the property that $a_1 = a_2 \circ b$ because the diagram of the lemma commutes and because s is a section. □

90.11. Tangent spaces of functors

06I2 Let R be a ring. We write Mod_R for the category of R-modules and Mod_R^{fg} for the category of finitely generated R-modules.

06I3 **Definition 90.11.1.** Let $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$, resp. $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a functor. We say that L is *R-linear* if for every pair of objects M, N of Mod_R^{fg}, resp. Mod_R the map

$$L : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(L(M), L(N))$$

is a map of R-modules.

06I4 **Remark 90.11.2.** One can define the notion of an R-linearity for any functor between categories enriched over Mod_R. We made the definition specifically for functors $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ and $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ because these are the cases that we have needed so far.

06I5 **Remark 90.11.3.** If $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ is an R-linear functor, then L preserves finite products and sends the zero module to the zero module, see Homology, Lemma 12.3.7. On the other hand, if a functor $\mathrm{Mod}_R^{fg} \to \mathit{Sets}$ preserves finite products and sends the zero module to a one element set, then it has a unique lift to a R-linear functor, see Lemma 90.11.4.

06I6 **Lemma 90.11.4.** *Let $L : Mod_R^{fg} \to Sets$, resp. $L : Mod_R \to Sets$ be a functor. Suppose $L(0)$ is a one element set and L preserves finite products. Then there exists a unique R-linear functor $\widetilde{L} : Mod_R^{fg} \to Mod_R$, resp. $\widetilde{L} : Mod_R \to Mod_R$, such that*

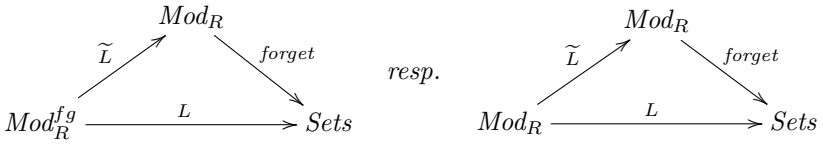

commutes.

Proof. We only prove this in case $L : \mathrm{Mod}_R^{fg} \to Sets$. Let M be a finitely generated R-module. We define $\widetilde{L}(M)$ to be the set $L(M)$ with the following R-module structure.

Multiplication: If $r \in R$, multiplication by r on $L(M)$ is defined to be the map $L(M) \to L(M)$ induced by the multiplication map $r\cdot : M \to M$.

Addition: The sum map $M \times M \to M : (m_1, m_2) \mapsto m_1 + m_2$ induces a map $L(M \times M) \to L(M)$. By assumption $L(M \times M)$ is canonically isomorphic to $L(M) \times L(M)$. Addition on $L(M)$ is defined by the map $L(M) \times L(M) \cong L(M \times M) \to L(M)$.

Zero: There is a unique map $0 \to M$. The zero element of $L(M)$ is the image of $L(0) \to L(M)$.

We omit the verification that this defines an R-module $\widetilde{L}(M)$, the unique such that is R-linearly functorial in M. □

06I7 **Lemma 90.11.5.** *Let $L_1, L_2 : Mod_R^{fg} \to Sets$ be functors that take 0 to a one element set and preserve finite products. Let $t : L_1 \to L_2$ be a morphism of functors. Then t induces a morphism $\widetilde{t} : \widetilde{L}_1 \to \widetilde{L}_2$ between the functors guaranteed by Lemma 90.11.4, which is given simply by $\widetilde{t}_M = t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ for each $M \in \mathrm{Ob}(Mod_R^{fg})$. In other words, $t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ is a map of R-modules.*

Proof. Omitted. □

In the case $R = K$ is a field, a K-linear functor $L : \mathrm{Mod}_K^{fg} \to \mathrm{Mod}_K$ is determined by its value $L(K)$.

06I8 **Lemma 90.11.6.** *Let K be a field. Let $L : Mod_K^{fg} \to Mod_K$ be a K-linear functor. Then L is isomorphic to the functor $L(K) \otimes_K - : Mod_K^{fg} \to Mod_K$.*

Proof. For $V \in \mathrm{Ob}(\mathrm{Mod}_K^{fg})$, the isomorphism $L(K) \otimes_K V \to L(V)$ is given on pure tensors by $x \otimes v \mapsto L(f_v)(x)$, where $f_v : K \to V$ is the K-linear map sending $1 \mapsto v$. When $V = K$, this is the isomorphism $L(K) \otimes_K K \to L(K)$ given by multiplication by K. For general V, it is an isomorphism by the case $V = K$ and the fact that L commutes with finite products (Remark 90.11.3). □

For a ring R and an R-module M, let $R[M]$ be the R-algebra whose underlying R-module is $R \oplus M$ and whose multiplication is given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. When $M = R$ this is the ring of dual numbers over R, which we denote by $R[\epsilon]$.

Now let S be a ring and assume R is an S-algebra. Then the assignment $M \mapsto R[M]$ determines a functor $\mathrm{Mod}_R \to S\text{-Alg}/R$, where $S\text{-Alg}/R$ denotes the category of

S-algebras over R. Note that S-Alg/R admits finite products: if $A_1 \to R$ and $A_2 \to R$ are two objects, then $A_1 \times_R A_2$ is a product.

06I9 **Lemma 90.11.7.** *Let R be an S-algebra. Then the functor $Mod_R \to S$-Alg/R described above preserves finite products.*

Proof. This is merely the statement that if M and N are R-modules, then the map $R[M \times N] \to R[M] \times_R R[N]$ is an isomorphism in S-Alg/R. □

06IA **Lemma 90.11.8.** *Let R be an S-algebra, and let $\mathcal{C}$ be a strictly full subcategory of S-Alg/R containing $R[M]$ for all $M \in \mathrm{Ob}(Mod_R^{fg})$. Let $F : \mathcal{C} \to Sets$ be a functor. Suppose that $F(R)$ is a one element set and that for any $M, N \in \mathrm{Ob}(Mod_R^{fg})$, the induced map*

$$F(R[M] \times_R R[N]) \to F(R[M]) \times F(R[N])$$

is a bijection. Then $F(R[M])$ has a natural R-module structure for any $M \in \mathrm{Ob}(Mod_R^{fg})$.

Proof. Note that $R \cong R[0]$ and $R[M] \times_R R[N] \cong R[M \times N]$ hence R and $R[M] \times_R R[N]$ are objects of $\mathcal{C}$ by our assumptions on $\mathcal{C}$. Thus the conditions on F make sense. The functor $\mathrm{Mod}_R \to S$-Alg/R of Lemma 90.11.7 restricts to a functor $\mathrm{Mod}_R^{fg} \to \mathcal{C}$ by the assumption on $\mathcal{C}$. Let L be the composition $\mathrm{Mod}_R^{fg} \to \mathcal{C} \to Sets$, i.e., $L(M) = F(R[M])$. Then L preserves finite products by Lemma 90.11.7 and the assumption on F. Hence Lemma 90.11.4 shows that $L(M) = F(R[M])$ has a natural R-module structure for any $M \in \mathrm{Ob}(\mathrm{Mod}_R^{fg})$. □

06IB **Definition 90.11.9.** Let $\mathcal{C}$ be a category as in Lemma 90.11.8. Let $F : \mathcal{C} \to Sets$ be a functor such that $F(R)$ is a one element set. The *tangent space TF of F* is $F(R[\epsilon])$.

When $F : \mathcal{C} \to Sets$ satisfies the hypotheses of Lemma 90.11.8, the tangent space TF has a natural R-module structure.

06SR **Example 90.11.10.** Since $\mathcal{C}_\Lambda$ contains all $k[V]$ for finite dimensional vector spaces V we see that Definition 90.11.9 applies with $S = \Lambda$, $R = k$, $\mathcal{C} = \mathcal{C}_\Lambda$, and $F : \mathcal{C}_\Lambda \to Sets$ a predeformation functor. The tangent space is $TF = F(k[\epsilon])$.

06IC **Example 90.11.11.** Let us work out the tangent space of Example 90.11.10 when $F : \mathcal{C}_\Lambda \to Sets$ is a prorepresentable functor, say $F = \underline{S}|_{\mathcal{C}_\Lambda}$ for $S \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Then F commutes with arbitrary limits and thus satisfies the hypotheses of Lemma 90.11.8. We compute

$$TF = F(k[\epsilon]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[\epsilon]) = \mathrm{Der}_\Lambda(S, k)$$

and more generally for a finite dimensional k-vector space V we have

$$F(k[V]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[V]) = \mathrm{Der}_\Lambda(S, V).$$

Explicitly, a Λ-algebra map $f : S \to k[V]$ compatible with the augmentations $q : S \to k$ and $k[V] \to k$ corresponds to the derivation D defined by $s \mapsto f(s) - q(s)$. Conversely, a Λ-derivation $D : S \to V$ corresponds to $f : S \to k[V]$ in $\mathcal{C}_\Lambda$ defined by the rule $f(s) = q(s) + D(s)$. Since these identifications are functorial we see that the k-vector spaces structures on TF and $\mathrm{Der}_\Lambda(S, k)$ correspond (see Lemma 90.11.5). It follows that $\dim_k TF$ is finite by Lemma 90.4.5.

06SS **Example 90.11.12.** The computation of Example 90.11.11 simplifies in the classical case. Namely, in this case the tangent space of the functor $F = \underline{S}|_{\mathcal{C}_\Lambda}$ is simply the relative cotangent space of S over Λ, in a formula $TF = T_{S/\Lambda}$. In fact, this works more generally when the field extension k/k' is separable. See Exercises, Exercise 111.35.2.

06ID **Lemma 90.11.13.** *Let $F, G : \mathcal{C} \to Sets$ be functors satisfying the hypotheses of Lemma 90.11.8. Let $t : F \to G$ be a morphism of functors. For any $M \in \mathrm{Ob}(Mod_R^{fg})$, the map $t_{R[M]} : F(R[M]) \to G(R[M])$ is a map of R-modules, where $F(R[M])$ and $G(R[M])$ are given the R-module structure from Lemma 90.11.8. In particular, $t_{R[\epsilon]} : TF \to TG$ is a map of R-modules.*

Proof. Follows from Lemma 90.11.5. □

06ST **Example 90.11.14.** Suppose that $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$. Set $F = \underline{R}|_{\mathcal{C}_\Lambda}$ and $G = \underline{S}|_{\mathcal{C}_\Lambda}$. The ring map f induces a transformation of functors $G \to F$. By Lemma 90.11.13 we get a k-linear map $TG \to TF$. This is the map

$$TG = \mathrm{Der}_\Lambda(S, k) \longrightarrow \mathrm{Der}_\Lambda(R, k) = TF$$

as follows from the canonical identifications $F(k[V]) = \mathrm{Der}_\Lambda(R, V)$ and $G(k[V]) = \mathrm{Der}_\Lambda(S, V)$ of Example 90.11.11 and the rule for computing the map on tangent spaces.

06IE **Lemma 90.11.15.** *Let $F : \mathcal{C} \to Sets$ be a functor satisfying the hypotheses of Lemma 90.11.8. Assume $R = K$ is a field. Then $F(K[V]) \cong TF \otimes_K V$ for any finite dimensional K-vector space V.*

Proof. Follows from Lemma 90.11.6. □

90.12. Tangent spaces of predeformation categories

06I1 We will define tangent spaces of predeformation functors using the general Definition 90.11.9. We have spelled this out in Example 90.11.10. It applies to predeformation categories by looking at the associated functor of isomorphism classes.

06IG **Definition 90.12.1.** Let $\mathcal{F}$ be a predeformation category. The *tangent space $T\mathcal{F}$ of $\mathcal{F}$* is the set $\overline{\mathcal{F}}(k[\epsilon])$ of isomorphism classes of objects in the fiber category $\mathcal{F}(k[\epsilon])$.

Thus $T\mathcal{F}$ is nothing but the tangent space of the associated functor $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$. It has a natural vector space structure when $\mathcal{F}$ satisfies (S2), or, in fact, as long as $\overline{\mathcal{F}}$ does.

06IH **Lemma 90.12.2.** *Let $\mathcal{F}$ be a predeformation category such that $\overline{\mathcal{F}}$ satisfies (S2)[2]. Then $T\mathcal{F}$ has a natural k-vector space structure. For any finite dimensional vector space V we have $\overline{\mathcal{F}}(k[V]) = T\mathcal{F} \otimes_k V$ functorially in V.*

Proof. Let us write $F = \overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$. This is a predeformation functor and F satisfies (S2). By Lemma 90.10.4 (and the translation of Remark 90.10.3) we see that

$$F(A \times_k k[V]) \longrightarrow F(A) \times F(k[V])$$

is a bijection for every finite dimensional vector space V and every $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. In particular, if $A = k[W]$ then we see that $F(k[W] \times_k k[V]) = F(k[W]) \times F(k[V])$.

[2]For example if $\mathcal{F}$ satisfies (S2), see Lemma 90.10.5.

In other words, the hypotheses of Lemma 90.11.8 hold and we see that $TF = T\mathcal{F}$ has a natural k-vector space structure. The final assertion follows from Lemma 90.11.15. □

A morphism of predeformation categories induces a map on tangent spaces.

06II **Definition 90.12.3.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. The *differential* $d\varphi : T\mathcal{F} \to T\mathcal{G}$ *of* φ is the map obtained by evaluating the morphism of functors $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ at $A = k[\epsilon]$.

06IJ **Lemma 90.12.4.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of predeformation categories. Assume $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ both satisfy (S2). Then $d\varphi : T\mathcal{F} \to T\mathcal{G}$ is k-linear.*

Proof. In the proof of Lemma 90.12.2 we have seen that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ satisfy the hypotheses of Lemma 90.11.8. Hence the lemma follows from Lemma 90.11.13. □

06IK **Remark 90.12.5.** We can globalize the notions of tangent space and differential to arbitrary categories cofibered in groupoids as follows. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. As in Remark 90.6.4, we get a predeformation category $\mathcal{F}_x$. We define

$$T_x\mathcal{F} = T\mathcal{F}_x$$

to be the *tangent space of* $\mathcal{F}$ *at* x. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(k))$, then there is an induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_{\varphi(x)}$. We define the *differential* $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ *of* φ *at* x to be the map $d\varphi_x : T\mathcal{F}_x \to T\mathcal{G}_{\varphi(x)}$. If both $\mathcal{F}$ and $\mathcal{G}$ satisfy (S2) then all of these tangent spaces have a natural k-vector space structure and all the differentials $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ are k-linear (use Lemmas 90.10.6 and 90.12.4).

The following observations are uninteresting in the classical case or when k/k' is a separable field extension, because then $\mathrm{Der}_\Lambda(k, k)$ and $\mathrm{Der}_\Lambda(k, V)$ are zero. There is a canonical identification

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[\epsilon]) = \mathrm{Der}_\Lambda(k, k).$$

Namely, for $D \in \mathrm{Der}_\Lambda(k, k)$ let $f_D : k \to k[\epsilon]$ be the map $a \mapsto a + D(a)\epsilon$. More generally, given a finite dimensional vector space V over k we have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[V]) = \mathrm{Der}_\Lambda(k, V)$$

and we will use the same notation f_D for the map associated to the derivation D. We also have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k[W], k[V]) = \mathrm{Hom}_k(V, W) \oplus \mathrm{Der}_\Lambda(k, V)$$

where (φ, D) corresponds to the map $f_{\varphi,D} : a + w \mapsto a + \varphi(w) + D(a)$. We will sometimes write $f_{1,D} : a+v \to a+v+D(a)$ for the automorphism of $k[V]$ determined by the derivation $D : k \to V$. Note that $f_{1,D} \circ f_{1,D'} = f_{1,D+D'}$.

Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. By the above there is a canonical map

$$\gamma_V : \mathrm{Der}_\Lambda(k, V) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $D \mapsto f_{D,*}(x_0)$. Moreover, there is an action

$$a_V : \mathrm{Der}_\Lambda(k, V) \times \overline{\mathcal{F}}(k[V]) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $(D, x) \mapsto f_{1,D,*}(x)$. These two maps are compatible, i.e., $f_{1,D,*}f_{D',*}x_0 = f_{D+D',*}x_0$ as follows from a computation of the compositions of these maps. Note that the maps γ_V and a_V are independent of the choice of x_0 as there is a unique x_0 up to isomorphism.

06SU **Lemma 90.12.6.** *Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. If $\overline{\mathcal{F}}$ has (S2) then the maps γ_V are k-linear and we have $a_V(D, x) = x + \gamma_V(D)$.*

Proof. In the proof of Lemma 90.12.2 we have seen that the functor $V \mapsto \overline{\mathcal{F}}(k[V])$ transforms 0 to a singleton and products to products. The same is true of the functor $V \mapsto \mathrm{Der}_\Lambda(k, V)$. Hence γ_V is linear by Lemma 90.11.5. Let $D : k \to V$ be a Λ-derivation. Set $D_1 : k \to V^{\oplus 2}$ equal to $a \mapsto (D(a), 0)$. Then

$$\begin{array}{ccc} k[V \times V] & \xrightarrow{+} & k[V] \\ \downarrow{\scriptstyle f_{1,D_1}} & & \downarrow{\scriptstyle f_{1,D}} \\ k[V \times V] & \xrightarrow{+} & k[V] \end{array}$$

commutes. Unwinding the definitions and using that $\overline{F}(V \times V) = \overline{F}(V) \times \overline{F}(V)$ this means that $a_D(x_1) + x_2 = a_D(x_1 + x_2)$ for all $x_1, x_2 \in \overline{F}(V)$. Thus it suffices to show that $a_V(D, 0) = 0 + \gamma_V(D)$ where $0 \in \overline{F}(V)$ is the zero vector. By definition this is the element $f_{0,*}(x_0)$. Since $f_D = f_{1,D} \circ f_0$ the desired result follows. □

A special case of the constructions above are the map

06SV (90.12.6.1) $$\gamma : \mathrm{Der}_\Lambda(k, k) \longrightarrow T\mathcal{F}$$

and the action

06SW (90.12.6.2) $$a : \mathrm{Der}_\Lambda(k, k) \times T\mathcal{F} \longrightarrow T\mathcal{F}$$

defined for any predeformation category $\mathcal{F}$. Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of predeformation categories, then we get commutative diagrams

$$\begin{array}{ccc} \mathrm{Der}_\Lambda(k, k) & \xrightarrow{\gamma} & T\mathcal{F} \\ & \searrow^{\gamma} & \downarrow{\scriptstyle d\varphi} \\ & & T\mathcal{G} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Der}_\Lambda(k, k) \times T\mathcal{F} & \xrightarrow{a} & T\mathcal{F} \\ \downarrow{\scriptstyle 1 \times d\varphi} & & \downarrow{\scriptstyle d\varphi} \\ \mathrm{Der}_\Lambda(k, k) \times T\mathcal{G} & \xrightarrow{a} & T\mathcal{G} \end{array}$$

90.13. Versal formal objects

06SX The existence of a versal formal object forces $\mathcal{F}$ to have property (S1).

06SY **Lemma 90.13.1.** *Let $\mathcal{F}$ be a predeformation category. Assume $\mathcal{F}$ has a versal formal object. Then $\mathcal{F}$ satisfies (S1).*

Proof. Let ξ be a versal formal object of $\mathcal{F}$. Let

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

be a diagram in $\mathcal{F}$ such that $x_2 \to x$ lies over a surjective ring map. Since the natural morphism $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \xrightarrow{\sim} \mathcal{F}$ is an equivalence (see Remark 90.7.7), we can consider this

diagram also as a diagram in $\widehat{\mathcal{F}}$. By Lemma 90.8.11 there exists a morphism $\xi \to x_1$, so by Remark 90.8.10 we also get a morphism $\xi \to x_2$ making the diagram

$$\begin{array}{ccc} \xi & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

commute. If $x_1 \to x$ and $x_2 \to x$ lie above ring maps $A_1 \to A$ and $A_2 \to A$ then taking the pushforward of ξ to $A_1 \times_A A_2$ gives an object y as required by (S1). $\square$

In the case that our cofibred category satisfies (S1) and (S2) we can characterize the versal formal objects as follows.

06IU **Lemma 90.13.2.** *Let $\mathcal{F}$ be a predeformation category satisfying (S1) and (S2). Let ξ be a formal object of $\mathcal{F}$ corresponding to $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$, see Remark 90.7.12. Then ξ is versal if and only if the following two conditions hold:*

(1) *the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to T\mathcal{F}$ on tangent spaces is surjective, and*

(2) *given a diagram in $\widehat{\mathcal{F}}$*

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & B \\ & & \downarrow f \\ R & \longrightarrow & A \end{array}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $B \to A$ a small extension of Artinian rings, then there exists a ring map $R \to B$ such that

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow f \\ R & \longrightarrow & A \end{array}$$

commutes.

Proof. If ξ is versal then (1) holds by Lemma 90.8.8 and (2) holds by Remark 90.8.10. Assume (1) and (2) hold. By Remark 90.8.10 we must show that given a diagram in $\widehat{\mathcal{F}}$ as in (2), there exists $\xi \to y$ such that

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

commutes. Let $b : R \to B$ be the map guaranteed by (2). Denote $y' = b_*\xi$ and choose a factorization $\xi \to y' \to x$ lying over $R \to B \to A$ of the given morphism $\xi \to x$. By (S1) we obtain a commutative diagram

$$\begin{array}{ccc} z & \longrightarrow & y \\ \downarrow & & \downarrow \\ y' & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B \times_A B & \longrightarrow & B \\ \downarrow & & \downarrow f \\ B & \xrightarrow{f} & A. \end{array}$$

Set $I = \mathrm{Ker}(f)$. Let $\overline{g} : B \times_A B \to k[I]$ be the ring map $(u, v) \mapsto \overline{u} \oplus (v - u)$, cf. Lemma 90.10.8. By (1) there exists a morphism $\xi \to \overline{g}_* z$ which lies over a

ring map $i : R \to k[\epsilon]$. Choose an Artinian quotient $b_1 : R \to B_1$ such that both $b : R \to B$ and $i : R \to k[\epsilon]$ factor through $R \to B_1$, i.e., giving $h : B_1 \to B$ and $i' : B_1 \to k[\epsilon]$. Choose a pushforward $y_1 = b_{1,*}\xi$, a factorization $\xi \to y_1 \to y'$ lying over $R \to B_1 \to B$ of $\xi \to y'$, and a factorization $\xi \to y_1 \to \overline{g}_* z$ lying over $R \to B_1 \to k[\epsilon]$ of $\xi \to \overline{g}_* z$. Applying (S1) once more we obtain

$$\begin{array}{ccccc} z_1 & \longrightarrow & z & \longrightarrow & y \\ \downarrow & & \downarrow & & \downarrow \\ y_1 & \longrightarrow & y' & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccccc} B_1 \times_A B & \longrightarrow & B \times_A B & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow f \\ B_1 & \longrightarrow & B & \longrightarrow & A. \end{array}$$

Note that the map $g : B_1 \times_A B \to k[I]$ of Lemma 90.10.8 (defined using h) is the composition of $B_1 \times_A B \to B \times_A B$ and the map $\overline{g}$ above. By construction there exists a morphism $y_1 \to g_* z_1 \cong \overline{g}_* z$! Hence Lemma 90.10.8 applies (to the outer rectangles in the diagrams above) to give a morphism $y_1 \to y$ and precomposing with $\xi \to y_1$ gives the desired morphism $\xi \to y$. □

If $\mathcal{F}$ has property (S1) then the "largest quotient where a lift exists" exists. Here is a precise statement.

06SZ **Lemma 90.13.3.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let $B \to A$ be a surjection in $\mathcal{C}_\Lambda$ with kernel I annihilated by $\mathfrak{m}_B$. Let $x \in \mathcal{F}(A)$. The set of ideals*

$$\mathcal{J} = \{J \subset I \mid \text{there exists an } y \to x \text{ lying over } B/J \to A\}$$

has a smallest element.

Proof. Note that $\mathcal{J}$ is nonempty as $I \in \mathcal{J}$. Also, if $J \in \mathcal{J}$ and $J \subset J' \subset I$ then $J' \in \mathcal{J}$ because we can pushforward the object y to an object y' over B/J'. Let J and K be elements of the displayed set. We claim that $J \cap K \in \mathcal{J}$ which will prove the lemma. Since I is a k-vector space we can find an ideal $J \subset J' \subset I$ such that $J \cap K = J' \cap K$ and such that $J' + K = I$. By the above we may replace J by J' and assume that $J + K = I$. In this case

$$A/(J \cap K) = A/J \times_{A/I} A/K.$$

Hence the existence of an element $z \in \mathcal{F}(A/(J \cap K))$ mapping to x follows, via (S1), from the existence of the elements we have assumed exist over A/J and A/K. □

We will improve on the following result later.

06IW **Lemma 90.13.4.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume the following conditions hold:*

(1) *$\mathcal{F}$ is a predeformation category.*
(2) *$\mathcal{F}$ satisfies (S1).*
(3) *$\mathcal{F}$ satisfies (S2).*
(4) *$\dim_k T\mathcal{F}$ is finite.*

Then $\mathcal{F}$ has a versal formal object.

Proof. Assume (1), (2), (3), and (4) hold. Choose an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. See Lemma 90.9.5. Let $r = \dim_k T\mathcal{F}$ and put $S = R[[X_1, \ldots, X_r]]$.

We are going to inductively construct for $n \geq 2$ pairs $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ where $J_n \subset S$ is an decreasing sequence of ideals and $f_{n-1} : \xi_n \to \xi_{n-1}$ is a morphism of $\mathcal{F}$ lying over the projection $S/J_n \to S/J_{n-1}$.

Step 1. Let $J_1 = \mathfrak{m}_S$. Let ξ_1 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over $k = S/J_1 = S/\mathfrak{m}_S$

Step 2. Let $J_2 = \mathfrak{m}_S^2 + \mathfrak{m}_R S$. Then $S/J_2 = k[V]$ with $V = kX_1 \oplus \ldots \oplus kX_r$ By (S2) for $\overline{\mathcal{F}}$ we get a bijection

$$\overline{\mathcal{F}}(S/J_2) \longrightarrow T\mathcal{F} \otimes_k V,$$

see Lemmas 90.10.5 and 90.12.2. Choose a basis $\theta_1, \ldots, \theta_r$ for $T\mathcal{F}$ and set $\xi_2 = \sum \theta_i \otimes X_i \in \mathrm{Ob}(\mathcal{F}(S/J_2))$. The point of this choice is that

$$d\xi_2 : \mathrm{Mor}_{\mathcal{C}_\Lambda}(S/J_2, k[\epsilon]) \longrightarrow T\mathcal{F}$$

is surjective. Let $f_1 : \xi_2 \to \xi_1$ be the unique morphism.

Induction step. Assume $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ has been constructed for some $n \geq 2$. There is a minimal element J_{n+1} of the set of ideals $J \subset S$ satisfying: (a) $\mathfrak{m}_S J_n \subset J \subset J_n$ and (b) there exists a morphism $\xi_{n+1} \to \xi_n$ lying over $S/J \to S/J_n$, see Lemma 90.13.3. Let $f_n : \xi_{n+1} \to \xi_n$ be any morphism of $\mathcal{F}$ lying over $S/J_{n+1} \to S/J_n$.

Set $J = \bigcap J_n$. Set $\overline{S} = S/J$. Set $\overline{J}_n = J_n/J$. By Lemma 90.4.7 the sequence of ideals $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology on $\overline{S}$. Since (ξ_n, f_n) is an object of $\widehat{\mathcal{F}}_{\mathcal{I}}(\overline{S})$, where $\mathcal{I}$ is the filtration $(\overline{J}_n)$ of $\overline{S}$, we see that (ξ_n, f_n) induces an object ξ of $\widehat{\mathcal{F}}(\overline{S})$. see Lemma 90.7.4.

We prove ξ is versal. For versality it suffices to check conditions (1) and (2) of Lemma 90.13.2. Condition (1) follows from our choice of ξ_2 in Step 2 above. Suppose given a diagram in $\widehat{\mathcal{F}}$

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & B \\ & & \downarrow f \\ \overline{S} & \longrightarrow & A \end{array}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $f : B \to A$ a small extension of Artinian rings. We have to show there is a map $\overline{S} \to B$ fitting into the diagram on the right. Choose n such that $\overline{S} \to A$ factors through $\overline{S} \to S/J_n$. This is possible as the sequence $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology as we saw above. The pushforward of ξ along $\overline{S} \to S/J_n$ is ξ_n. We may factor $\xi \to x$ as $\xi \to \xi_n \to x$ hence we get a diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi_n & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & B \\ & & \downarrow f \\ S/J_n & \longrightarrow & A. \end{array}$$

To check condition (2) of Lemma 90.13.2 it suffices to complete the diagram

$$\begin{array}{ccc} S/J_{n+1} & \dashrightarrow & B \\ \downarrow & & \downarrow f \\ S/J_n & \longrightarrow & A \end{array}$$

or equivalently, to complete the diagram

$$\begin{array}{ccc} & & S/J_n \times_A B \\ & \nearrow & \downarrow p_1 \\ S/J_{n+1} & \longrightarrow & S/J_n. \end{array}$$

If p_1 has a section we are done. If not, by Lemma 90.3.8 (2) p_1 is a small extension, so by Lemma 90.3.12 (4) p_1 is an essential surjection. Recall that $S = R[[X_1, \ldots, X_r]]$ and that we chose R such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Hence there exists a map $h : R \to B$ lifting the map $R \to S \to S/J_n \to A$. By the universal property of a power series ring there is an R-algebra map $h : S = R[[X_1, \ldots, X_r]] \to B$ lifting the given map $S \to S/J_n \to A$. This induces a map $g : S \to S/J_n \times_A B$ making the solid square in the diagram

$$\begin{array}{ccc} S & \xrightarrow[g]{} & S/J_n \times_A B \\ \downarrow & \nearrow & \downarrow p_1 \\ S/J_{n+1} & \longrightarrow & S/J_n \end{array}$$

commute. Then g is a surjection since p_1 is an essential surjection. We claim the ideal $K = \mathrm{Ker}(g)$ of S satisfies conditions (a) and (b) of the construction of J_{n+1} in the induction step above. Namely, $K \subset J_n$ is clear and $\mathfrak{m}_S J_n \subset K$ as p_1 is a small extension; this proves (a). By (S1) applied to

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi_n & \longrightarrow & x, \end{array}$$

there exists a lifting of ξ_n to $S/K \cong S/J_n \times_A B$, so (b) holds. Since J_{n+1} was the minimal ideal with properties (a) and (b) this implies $J_{n+1} \subset K$. Thus the desired map $S/J_{n+1} \to S/K \cong S/J_n \times_A B$ exists. □

0D3G **Remark 90.13.5.** Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a predeformation functor satisfying (S1) and (S2). The condition $\dim_k TF < \infty$ is precisely condition (H3) from Schlessinger's paper. Recall that (S1) and (S2) correspond to conditions (H1) and (H2), see Remark 90.10.3. Thus Lemma 90.13.4 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a versal formal object}$$

for predeformation functors. We will make the link with hulls in Remark 90.15.6.

90.14. Minimal versal formal objects

06T0 We do a little bit of work to try and understand (non)uniqueness of versal formal objects. It turns out that if a predeformation category has a versal formal object, then it has a minimal versal formal object and any two such are isomorphic. Moreover, all versal formal objects are "more or less" the same up to replacing the base ring by a power series extension.

Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. For every object x of $\mathcal{F}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ consider the category $\mathcal{S}_x$ with objects

$$\mathrm{Ob}(\mathcal{S}_x) = \{x' \to x \mid x' \to x \text{ lies over } A' \subset A\}$$

and morphisms are morphisms over x. For every $y \to x$ in $\mathcal{F}$ lying over $f : B \to A$ in $\mathcal{C}_\Lambda$ there is a functor $f_* : \mathcal{S}_y \to \mathcal{S}_x$ defined as follows: Given $y' \to y$ lying over $B' \subset B$ set $A' = f(B')$ and let $y' \to x'$ be over $B' \to f(B')$ be the pushforward of y'. By the axioms of a category cofibred in groupoids we obtain a unique morphism $x' \to x$ lying over $f(B') \to A$ such that

$$\begin{array}{ccc} y' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array}$$

commutes. Then $x' \to x$ is an object of $\mathcal{S}_x$. We say an object $x' \to x$ of $\mathcal{S}_x$ is *minimal* if any morphism $(x'_1 \to x) \to (x' \to x)$ in $\mathcal{S}_x$ is an isomorphism, i.e., x' and x'_1 are defined over the same subring of A. Since A has finite length as a Λ-module we see that minimal objects always exist.

06T1 **Lemma 90.14.1.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1).*

(1) *For $y \to x$ in $\mathcal{F}$ a minimal object in $\mathcal{S}_y$ maps to a minimal object of $\mathcal{S}_x$.*
(2) *For $y \to x$ in $\mathcal{F}$ lying over a surjection $f : B \to A$ in $\mathcal{C}_\Lambda$ every minimal object of $\mathcal{S}_x$ is the image of a minimal object of $\mathcal{S}_y$.*

Proof. Proof of (1). Say $y \to x$ lies over $f : B \to A$. Let $y' \to y$ lying over $B' \subset B$ be a minimal object of $\mathcal{S}_y$. Let

$$\begin{array}{ccc} y' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B' & \longrightarrow & f(B') \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

be as in the construction of f_* above. Suppose that $(x'' \to x) \to (x' \to x)$ is a morphism of $\mathcal{S}_x$ with $x'' \to x'$ lying over $A'' \subset f(B')$. By (S1) there exists $y'' \to y'$ lying over $B' \times_{f(B')} A'' \to B'$. Since $y' \to y$ is minimal we conclude that $B' \times_{f(B')} A'' \to B'$ is an isomorphism, which implies that $A'' = f(B')$, i.e., $x' \to x$ is minimal.

Proof of (2). Suppose $f : B \to A$ is surjective and $y \to x$ lies over f. Let $x' \to x$ be a minimal object of $\mathcal{S}_x$ lying over $A' \subset A$. By (S1) there exists $y' \to y$ lying over $B' = f^{-1}(A') = B \times_A A' \to B$ whose image in $\mathcal{S}_x$ is $x' \to x$. So $f_*(y' \to y) = x' \to x$. Choose a morphism $(y'' \to y) \to (y' \to y)$ in $\mathcal{S}_y$ with $y'' \to y$ a minimal object (this is possible by the remark on lengths above the lemma). Then $f_*(y'' \to y)$ is an object of $\mathcal{S}_x$ which maps to $x' \to x$ (by functoriality of f_*) hence is isomorphic to $x' \to x$ by minimality of $x' \to x$. □

06T2 **Lemma 90.14.2.** *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. There exists a morphism $\xi' \to \xi$ lying over $R' \subset R$ with the following minimality properties*

(1) *for every* $f : R \to A$ *with* $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ *the pushforwards*

$$\begin{array}{ccc} \xi' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} R' & \longrightarrow & f(R') \\ \downarrow & & \downarrow \\ R & \longrightarrow & A \end{array}$$

produce a minimal object $x' \to x$ *of* $\mathcal{S}_x$*, and*

(2) *for any morphism of formal objects* $\xi'' \to \xi'$ *the corresponding morphism* $R'' \to R'$ *is surjective.*

Proof. Write $\xi = (R, \xi_n, f_n)$. Set $R'_1 = k$ and $\xi'_1 = \xi_1$. Suppose that we have constructed minimal objects $\xi'_m \to \xi_m$ of $\mathcal{S}_{\xi_m}$ lying over $R'_m \subset R/\mathfrak{m}_R^m$ for $m \leq n$ and morphisms $f'_m : \xi'_{m+1} \to \xi'_m$ compatible with f_m for $m \leq n-1$. By Lemma 90.14.1 (2) there exists a minimal object $\xi'_{n+1} \to \xi_{n+1}$ lying over $R'_{n+1} \subset R/\mathfrak{m}_R^{n+1}$ whose image is $\xi'_n \to \xi_n$ over $R'_n \subset R/\mathfrak{m}_R^n$. This produces the commutative diagram

$$\begin{array}{ccc} \xi'_{n+1} & \xrightarrow[f'_n]{} & \xi'_n \\ \downarrow & & \downarrow \\ \xi_{n+1} & \xrightarrow{f_n} & \xi_n \end{array}$$

by construction. Moreover the ring map $R'_{n+1} \to R'_n$ is surjective. Set $R' = \lim_n R'_n$. Then $R' \to R$ is injective.

However, it isn't a priori clear that R' is Noetherian. To prove this we use that ξ is versal. Namely, versality implies that there exists a morphism $\xi \to \xi'_n$ in $\widehat{\mathcal{F}}$, see Lemma 90.8.11. The corresponding map $R \to R'_n$ has to be surjective (as $\xi'_n \to \xi_n$ is minimal in $\mathcal{S}_{\xi_n}$). Thus the dimensions of the cotangent spaces are bounded and Lemma 90.4.8 implies R' is Noetherian, i.e., an object of $\widehat{\mathcal{C}}_\Lambda$. By Lemma 90.7.4 (plus the result on filtrations of Lemma 90.4.8) the sequence of elements ξ'_n defines a formal object ξ' over R' and we have a map $\xi' \to \xi$.

By construction (1) holds for $R \to R/\mathfrak{m}_R^n$ for each n. Since each $R \to A$ as in (1) factors through $R \to R/\mathfrak{m}_R^n \to A$ we see that (1) for $x' \to x$ over $f(R) \subset A$ follows from the minimality of $\xi'_n \to \xi_n$ over $R'_n \to R/\mathfrak{m}_R^n$ by Lemma 90.14.1 (1).

If $R'' \to R'$ as in (2) is not surjective, then $R'' \to R' \to R'_n$ would not be surjective for some n and $\xi'_n \to \xi_n$ wouldn't be minimal, a contradiction. This contradiction proves (2). □

06T3 **Lemma 90.14.3.** *Let* $\mathcal{F}$ *be a category cofibred in groupoids over* $\mathcal{C}_\Lambda$ *which has (S1). Let* ξ *be a versal formal object of* $\mathcal{F}$ *lying over* R*. Let* $\xi' \to \xi$ *be a morphism of formal objects lying over* $R' \subset R$ *as constructed in Lemma 90.14.2. Then*

$$R \cong R'[[x_1, \ldots, x_r]]$$

is a power series ring over R'*. Moreover,* ξ' *is a versal formal object too.*

Proof. By Lemma 90.8.11 there exists a morphism $\xi \to \xi'$. By Lemma 90.14.2 the corresponding map $f : R \to R'$ induces a surjection $f|_{R'} : R' \to R'$. This is an isomorphism by Algebra, Lemma 10.31.10. Hence $I = \mathrm{Ker}(f)$ is an ideal of R such that $R = R' \oplus I$. Let $x_1, \ldots, x_n \in I$ be elements which form a basis for $I/\mathfrak{m}_R I$. Consider the map $S = R'[[X_1, \ldots, X_r]] \to R$ mapping X_i to x_i. For every $n \geq 1$

we get a surjection of Artinian R'-algebras $B = S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n = A$. Denote $y \in \mathrm{Ob}(\mathcal{F}(B)$, resp. $x \in \mathrm{Ob}(\mathcal{F}(A))$ the pushforward of ξ' along $R' \to S \to B$, resp. $R' \to S \to A$. Note that x is also the pushforward of ξ along $R \to A$ as ξ is the pushforward of ξ' along $R' \to R$. Thus we have a solid diagram

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & S/\mathfrak{m}_S^n \\ & \nearrow & \downarrow \\ R & \longrightarrow & R/\mathfrak{m}_R^n \end{array}$$

Because ξ is versal, using Remark 90.8.10 we obtain the dotted arrows fitting into these diagrams. In particular, the maps $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ have sections $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. It follows from Lemma 90.4.9 that $S \to R$ is an isomorphism.

As ξ is a pushforward of ξ' along $R' \to R$ we obtain from Remark 90.7.13 a commutative diagram

$$\begin{array}{ccc} \underline{R}|_{\mathcal{C}_\Lambda} & \longrightarrow & \underline{R'}|_{\mathcal{C}_\Lambda} \\ & \searrow^{\underline{\xi}} \quad \swarrow_{\underline{\xi'}} & \\ & \mathcal{F} & \end{array}$$

Since $R' \to R$ has a left inverse (namely $R \to R/I = R'$) we see that $\underline{R}|_{\mathcal{C}_\Lambda} \to \underline{R'}|_{\mathcal{C}_\Lambda}$ is essentially surjective. Hence by Lemma 90.8.7 we see that $\underline{\xi'}$ is smooth, i.e., ξ' is a versal formal object. □

Motivated by the preceding lemmas we make the following definition.

06T4 **Definition 90.14.4.** Let $\mathcal{F}$ be a predeformation category. We say a versal formal object ξ of $\mathcal{F}$ is *minimal*[3] if for any morphism of formal objects $\xi' \to \xi$ the underlying map on rings is surjective. Sometimes a minimal versal formal object is called *miniversal*.

The work in this section shows this definition is reasonable. First of all, the existence of a versal formal object implies that $\mathcal{F}$ has (S1). Then the preceding lemmas show there exists a minimal versal formal object. Finally, any two minimal versal formal objects are isomorphic. Here is a summary of our results (with detailed proofs).

06T5 **Lemma 90.14.5.** *Let $\mathcal{F}$ be a predeformation category which has a versal formal object. Then*

(1) *$\mathcal{F}$ has a minimal versal formal object,*
(2) *minimal versal objects are unique up to isomorphism, and*
(3) *any versal object is the pushforward of a minimal versal object along a power series ring extension.*

Proof. Suppose $\mathcal{F}$ has a versal formal object ξ over R. Then it satisfies (S1), see Lemma 90.13.1. Let $\xi' \to \xi$ over $R' \subset R$ be any of the morphisms constructed in Lemma 90.14.2. By Lemma 90.14.3 we see that ξ' is versal, hence it is a minimal versal formal object (by construction). This proves (1). Also, $R \cong R'[[x_1, \ldots, x_n]]$ which proves (3).

[3]This may be nonstandard terminology. Many authors tie this notion in with properties of tangent spaces. We will make the link in Section 90.15.

Suppose that ξ_i/R_i are two minimal versal formal objects. By Lemma 90.8.11 there exist morphisms $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$. The corresponding ring maps $f : R_1 \to R_2$ and $g : R_2 \to R_1$ are surjective by minimality. Hence the compositions $g \circ f : R_1 \to R_1$ and $f \circ g : R_2 \to R_2$ are isomorphisms by Algebra, Lemma 10.31.10. Thus f and g are isomorphisms whence the maps $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$ are isomorphisms (because $\widehat{\mathcal{F}}$ is cofibred in groupoids by Lemma 90.7.2). This proves (2) and finishes the proof of the lemma. □

90.15. Miniversal formal objects and tangent spaces

06IL The general notion of minimality introduced in Definition 90.14.4 can sometimes be deduced from the behaviour on tangent spaces. Let ξ be a formal object of the predeformation category $\mathcal{F}$ and let $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ be the corresponding morphism. Then we can consider the following the condition

06IM (90.15.0.1) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective}$$

and the condition

06T6 (90.15.0.2) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective on } \mathrm{Der}_\Lambda(k, k)\text{-orbits.}$$

Here we are using the identification $T\underline{R}|_{\mathcal{C}_\Lambda} = \mathrm{Der}_\Lambda(R, k)$ of Example 90.11.11 and the action (90.12.6.2) of derivations on the tangent spaces. If $k' \subset k$ is separable, then $\mathrm{Der}_\Lambda(k, k) = 0$ and the two conditions are equivalent. It turns out that, in the presence of condition (S2) a versal formal object is minimal if and only if $\underline{\xi}$ satisfies (90.15.0.2). Moreover, if $\underline{\xi}$ satisfies (90.15.0.1), then $\mathcal{F}$ satisfies (S2).

06IR **Lemma 90.15.1.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (90.15.0.2) holds. Then ξ is a minimal versal formal object. In particular, such ξ are unique up to isomorphism.*

Proof. If ξ is not minimal, then there exists a morphism $\xi' \to \xi$ lying over $R' \to R$ such that $R = R'[[x_1, \ldots, x_n]]$ with $n > 0$, see Lemma 90.14.5. Thus $d\underline{\xi}$ factors as

$$\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R', k) \to T\mathcal{F}$$

and we see that (90.15.0.2) cannot hold because $D : f \mapsto \partial/\partial x_1(f) \bmod \mathfrak{m}_R$ is an element of the kernel of the first arrow which is not in the image of $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$. □

06IV **Lemma 90.15.2.** *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (90.15.0.1) holds. Then*

(1) *$\mathcal{F}$ satisfies (S1).*
(2) *$\mathcal{F}$ satisfies (S2).*
(3) *$\dim_k T\mathcal{F}$ is finite.*

Proof. Condition (S1) holds by Lemma 90.13.1. The first part of (S2) holds since (S1) holds. Let

$$\begin{array}{ccc} y & \xrightarrow{c} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow{c'} & x_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

be diagrams as in the second part of (S2). As above we can find morphisms $b : \xi \to y$ and $b' : \xi \to y'$ such that

$$\begin{array}{ccc} \xi & \xrightarrow{b'} & y' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a'} \\ y & \xrightarrow{a} & x \end{array}$$

commutes. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ denote the structure morphism. Say $\widehat{p}(\xi) = R$, i.e., ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We see that the pushforward of ξ via $p(c) \circ p(b)$ is x_ϵ and that the pushforward of ξ via $p(c') \circ p(b')$ is x_ϵ. Since ξ satisfies (90.15.0.1), we see that $p(c) \circ p(b) = p(c') \circ p(b')$ as maps $R \to k[\epsilon]$. Hence $p(b) = p(b')$ as maps from $R \to A \times_k k[\epsilon]$. Thus we see that y and y' are isomorphic to the pushforward of ξ along this map and we get a unique morphism $y \to y'$ over $A \times_k k[\epsilon]$ compatible with b and b' as desired.

Finally, by Example 90.11.11 we see $\dim_k T\mathcal{F} = \dim_k T\underline{R}|_{\mathcal{C}_\Lambda}$ is finite. □

06T7 **Example 90.15.3.** There exist predeformation categories which have a versal formal object satisfying (90.15.0.2) but which do not satisfy (S2). A quick example is to take $F = \underline{k[\epsilon]}/G$ where $G \subset \mathrm{Aut}_{\mathcal{C}_\Lambda}(k[\epsilon])$ is a finite nontrivial subgroup. Namely, the map $\underline{k[\epsilon]} \to F$ is smooth, but the tangent space of F does not have a natural k-vector space structure (as it is a quotient of a k-vector space by a finite group).

06T8 **Lemma 90.15.4.** *Let $\mathcal{F}$ be a predeformation category satisfying (S2) which has a versal formal object. Then its minimal versal formal object satisfies (90.15.0.2).*

Proof. Let ξ be a minimal versal formal object for $\mathcal{F}$, see Lemma 90.14.5. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. In order to parse (90.15.0.2) we point out that $T\mathcal{F}$ has a natural k-vector space structure (see Lemma 90.12.2), that $d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$ is linear (see Lemma 90.12.4), and that the action of $\mathrm{Der}_\Lambda(k, k)$ is given by addition (see Lemma 90.12.6). Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k) & & \\ & & \uparrow & & \\ K & \longrightarrow & \mathrm{Der}_\Lambda(R, k) & \xrightarrow{d\underline{\xi}} & T\mathcal{F} \\ & & \uparrow & \nearrow & \\ & & \mathrm{Der}_\Lambda(k, k) & & \end{array}$$

The vector space K is the kernel of $d\underline{\xi}$. Note that the middle column is exact in the middle as it is dual to the sequence (90.3.10.1). If (90.15.0.2) fails, then we can find a nonzero element $D \in K$ which does not map to zero in $\mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k)$. This means there exists an $t \in \mathfrak{m}_R$ such that $D(t) = 1$. Set $R' = \{a \in R \mid D(a) = 0\}$. As D is a derivation this is a subring of R. Since $D(t) = 1$ we see that $R' \to k$ is surjective (compare with the proof of Lemma 90.3.12). Note that $\mathfrak{m}_{R'} = \mathrm{Ker}(D : \mathfrak{m}_R \to k)$ is an ideal of R and $\mathfrak{m}_R^2 \subset \mathfrak{m}_{R'}$. Hence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_R^2 + k\bar{t}$$

which implies that the map

$$R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \to R/\mathfrak{m}_R^2$$

sending ϵ to $\bar{t}$ is an isomorphism. In particular there is a map $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$.

Let $\xi \to y$ be a morphism lying over $R \to R/\mathfrak{m}_R^2$. Let $y \to x$ be a morphism lying over $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$. Let $y \to x_\epsilon$ be a morphism lying over $R/\mathfrak{m}_R^2 \to k[\epsilon]$. Let x_0 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over k. By the axioms of a category cofibred in groupoids we obtain a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} R'/\mathfrak{m}_R^2 \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ R'/\mathfrak{m}_R^2 & \longrightarrow & k. \end{array}$$

Because $D \in K$ we see that x_ϵ is isomorphic to $0 \in \mathcal{F}(k[\epsilon])$, i.e., x_ϵ is the pushforward of x_0 via $k \to k[\epsilon], a \mapsto a$. Hence by Lemma 90.10.7 we see that there exists a morphism $x \to y$. Since $\text{length}_\Lambda(R'/\mathfrak{m}_R^2) < \text{length}_\Lambda(R/\mathfrak{m}_R^2)$ the corresponding ring map $R'/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2$ is not surjective. This contradicts the minimality of ξ/R, see part (1) of Lemma 90.14.2. This contradiction shows that such a D cannot exist, hence we win. □

06IX **Theorem 90.15.5.** *Let $\mathcal{F}$ be a predeformation category. Consider the following conditions*

(1) *$\mathcal{F}$ has a minimal versal formal object satisfying (90.15.0.1),*
(2) *$\mathcal{F}$ has a minimal versal formal object satisfying (90.15.0.2),*
(3) *the following conditions hold:*
   (a) *$\mathcal{F}$ satisfies (S1).*
   (b) *$\mathcal{F}$ satisfies (S2).*
   (c) *$\dim_k T\mathcal{F}$ is finite.*

We always have

$$(1) \Rightarrow (3) \Rightarrow (2).$$

If $k' \subset k$ is separable, then all three are equivalent.

Proof. Lemma 90.15.2 shows that (1) ⇒ (3). Lemmas 90.13.4 and 90.15.4 show that (3) ⇒ (2). If $k' \subset k$ is separable then $\text{Der}_\Lambda(k, k) = 0$ and we see that (90.15.0.1) = (90.15.0.2), i.e., (1) is the same as (2).

An alternative proof of (3) ⇒ (1) in the classical case is to add a few words to the proof of Lemma 90.13.4 to see that one can right away construct a versal object which satisfies (90.15.0.1) in this case. This avoids the use of Lemma 90.13.4 in the classical case. Details omitted. □

06IY **Remark 90.15.6.** Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a predeformation functor satisfying (S1) and (S2) and $\dim_k TF < \infty$. Recall that these conditions correspond to the conditions (H1), (H2), and (H3) from Schlessinger's paper, see Remark 90.13.5. Now, in the classical case (or if $k' \subset k$ is separable) following Schlessinger we introduce the notion of a hull: a *hull* is a versal formal object $\xi \in \widehat{F}(R)$ such that $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to TF$ is an isomorphism, i.e., (90.15.0.1) holds. Thus Theorem 90.15.5 tells us

$$(H1) + (H2) + (H3) \Rightarrow \text{ there exists a hull}$$

in the classical case. In other words, our theorem recovers Schlessinger's theorem on the existence of hulls.

06IZ **Remark 90.15.7.** Let $\mathcal{F}$ be a predeformation category. Recall that $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth, see Remark 90.8.5. Hence if $\xi \in \widehat{\mathcal{F}}(R)$ is a versal formal object, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F} \longrightarrow \overline{\mathcal{F}}$$

is smooth (Lemma 90.8.7) and we conclude that the image $\overline{\xi}$ of ξ in $\overline{\mathcal{F}}$ is a versal formal object. If (90.15.0.1) holds, then $\overline{\xi}$ induces an isomorphism $T\underline{R}|_{\mathcal{C}_\Lambda} \to T\overline{\mathcal{F}}$ because $\mathcal{F} \to \overline{\mathcal{F}}$ identifies tangent spaces. Hence in this case $\overline{\xi}$ is a hull for $\overline{\mathcal{F}}$, see Remark 90.15.6. By Theorem 90.15.5 we can always find such a ξ if $k' \subset k$ is separable and $\mathcal{F}$ is a predeformation category satisfying (S1), (S2), and $\dim_k T\mathcal{F} < \infty$.

06T9 **Example 90.15.8.** In Lemma 90.9.5 we constructed objects $R \in \widehat{\mathcal{C}}_\Lambda$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth and such that

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_R/\mathfrak{m}_R^2 \quad \text{and} \quad \Omega_{R/\Lambda} \otimes_R k = \Omega_{k/\Lambda}$$

Let us reinterpret this using the theorem above. Namely, consider $\mathcal{F} = \mathcal{C}_\Lambda$ as a category cofibred in groupoids over itself (using the identity functor). Then $\mathcal{F}$ is a predeformation category, satisfies (S1) and (S2), and we have $T\mathcal{F} = 0$. Thus $\mathcal{F}$ satisfies condition (3) of Theorem 90.15.5. The theorem implies that (2) holds, i.e., we can find a minimal versal formal object $\xi \in \widehat{\mathcal{F}}(S)$ over some $S \in \widehat{\mathcal{C}}_\Lambda$ satisfying (90.15.0.2). Lemma 90.9.3 shows that $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology (because $\xi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F} = \mathcal{C}_\Lambda$ is smooth). Now condition (90.15.0.2) tells us that $\mathrm{Der}_\Lambda(S, k) \to 0$ is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. This means the injection $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(S, k)$ is also surjective. In other words, we have $\Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$. Since $\Lambda \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology, we can apply More on Algebra, Lemma 15.40.4 to conclude the exact sequence (90.3.10.2) turns into a pair of identifications

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_S/\mathfrak{m}_S^2 \quad \text{and} \quad \Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}$$

Reading the argument backwards, we find that the R constructed in Lemma 90.9.5 carries a minimal versal object. By the uniqueness of minimal versal objects (Lemma 90.14.5) we also conclude $R \cong S$, i.e., the two constructions give the same answer.

90.16. Rim-Schlessinger conditions and deformation categories

06J1 There is a very natural property of categories fibred in groupoids over $\mathcal{C}_\Lambda$ which is easy to check in practice and which implies Schlessinger's properties (S1) and (S2) we have introduced earlier.

06J2 **Definition 90.16.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We say that $\mathcal{F}$ satisfies *condition (RS)* if for every diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram

$$\begin{array}{ccc} x_1 \times_x x_2 & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lies over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

06J3 **Lemma 90.16.2.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Given a commutative diagram in $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \textit{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

with $A_2 \to A$ surjective, then it is a fiber square.

Proof. Since $\mathcal{F}$ satisfies (RS), there exists a fiber product diagram

$$\begin{array}{ccc} x_1 \times_x x_2 & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

The induced map $y \to x_1 \times_x x_2$ lies over $\mathrm{id} : A_1 \times_A A_1 \to A_1 \times_A A_1$, hence it is an isomorphism. □

06J4 **Lemma 90.16.3.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (RS) if the condition in Definition 90.16.1 is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Apply Lemma 90.3.3. The proof is similar to that of Lemma 90.8.2. □

06J5 **Lemma 90.16.4.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *$\mathcal{F}$ satisfies (RS),*
(2) *the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ see (90.10.1.1) is an equivalence of categories whenever $A_2 \to A$ is surjective, and*
(3) *same as in (2) whenever $A_2 \to A$ is a small extension.*

Proof. Assume (1). By Lemma 90.16.2 we see that every object of $\mathcal{F}(A_1 \times_A A_2)$ is of the form $x_1 \times_x x_2$. Moreover

$$\mathrm{Mor}_{A_1 \times_A A_2}(x_1 \times_x x_2, y_1 \times_y y_2) = \mathrm{Mor}_{A_1}(x_1, y_1) \times_{\mathrm{Mor}_A(x,y)} \mathrm{Mor}_{A_2}(x_2, y_2).$$

Hence we see that $\mathcal{F}(A_1 \times_A A_2)$ is a 2-fibre product of $\mathcal{F}(A_1)$ with $\mathcal{F}(A_2)$ over $\mathcal{F}(A)$ by Categories, Remark 4.31.5. In other words, we see that (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate.

Assume (3). Let $q_1 : A_1 \to A$ and $q_2 : A_2 \to A$ be given with q_2 a small extension. We will use the description of the 2-fibre product $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ from Categories, Remark 4.31.5. Hence let $y \in \mathcal{F}(A_1 \times_A A_2)$ correspond to

$(x_1, x_2, x, a_1 : x_1 \to x, a_2 : x_2 \to x)$. Let z be an object of $\mathcal{F}$ lying over C. Then

$$\begin{aligned}\mathrm{Mor}_{\mathcal{F}}(z, y) &= \{(f, \alpha) \mid f : C \to A_1 \times_A A_2, \alpha : f_* z \to y\} \\ &= \{(f_1, f_2, \alpha_1, \alpha_2) \mid f_i : C \to A_i,\ \alpha_i : f_{i,*} z \to x_i, \\ &\qquad q_1 \circ f_1 = q_2 \circ f_2,\ q_{1,*}\alpha_1 = q_{2,*}\alpha_2\} \\ &= \mathrm{Mor}_{\mathcal{F}}(z, x_1) \times_{\mathrm{Mor}_{\mathcal{F}}(z,x)} \mathrm{Mor}_{\mathcal{F}}(z, x_2)\end{aligned}$$

whence y is a fibre product of x_1 and x_2 over x. Thus we see that $\mathcal{F}$ satisfies (RS) in case $A_2 \to A$ is a small extension. Hence (RS) holds by Lemma 90.16.3. □

06J6 **Remark 90.16.5.** When $\mathcal{F}$ is cofibered in sets, condition (RS) is exactly condition (H4) from Schlessinger's paper [**Sch68**, Theorem 2.11]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$, condition (RS) states: If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is bijective.

06J7 **Lemma 90.16.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The condition (RS) for $\mathcal{F}$ implies both (S1) and (S2) for $\mathcal{F}$.*

Proof. Using the reformulation of Lemma 90.16.4 and the explanation of (S1) following Definition 90.10.1 it is immediate that (RS) implies (S1). This proves the first part of (S2). The second part of (S2) follows because Lemma 90.16.2 tells us that $y = x_1 \times_{d, x_0, e} x_2 = y'$ if y, y' are as in the second part of the definition of (S2) in Definition 90.10.1. (In fact the morphism $y \to y'$ is compatible with both a, a' and c, c'!) □

The following lemma is the analogue of Lemma 90.10.5. Recall that if $\mathcal{F}$ is a category cofibred in groupoids over $\mathcal{C}_\Lambda$ and x is an object of $\mathcal{F}$ lying over A, then we denote $\mathrm{Aut}_A(x) = \mathrm{Mor}_A(x, x) = \mathrm{Mor}_{\mathcal{F}(A)}(x, x)$. If $x' \to x$ is a morphism of $\mathcal{F}$ lying over $A' \to A$ then there is a well defined map of groups $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$.

06J8 **Lemma 90.16.7.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). The following conditions are equivalent:*

1. *$\overline{\mathcal{F}}$ satisfies (RS).*
2. *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. The induced map of sets of isomorphism classes*
$$\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$
is injective.
3. *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a surjective ring map $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*
4. *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a small extension $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*

Proof. We prove that (1) is equivalent to (2) and (2) is equivalent to (3). The equivalence of (3) and (4) follows from Lemma 90.3.3.

Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. By Remark 90.16.5 we see $\overline{\mathcal{F}}$ satisfies (RS) if and only if the map

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$

is bijective for any such f_1, f_2. This map is at least surjective since that is the condition of (S1) and $\overline{\mathcal{F}}$ satisfies (S1) by Lemmas 90.16.6 and 90.10.5. Moreover, this map factors as

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \longrightarrow \overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \longrightarrow \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2),$$

where the first map is a bijection since

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence by (RS) for $\mathcal{F}$. Hence (1) is equivalent to (2).

Assume (2) holds. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map $f : A' \to A$. Let $a \in \mathrm{Aut}_A(x)$. The objects

$$(x', x', a : x \to x),\ (x', x', \mathrm{id} : x \to x)$$

of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ have the same image in $\overline{\mathcal{F}}(A') \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A')$. By (2) there exists maps $b_1, b_2 : x' \to x'$ such that

$$\begin{array}{ccc} x & \xrightarrow[a]{} & x \\ {\scriptstyle f_*b_1}\downarrow & & \downarrow{\scriptstyle f_*b_2} \\ x & \xrightarrow{\mathrm{id}} & x \end{array}$$

commutes. Hence $b_2^{-1} \circ b_1 \in \mathrm{Aut}_{A'}(x')$ has image $a \in \mathrm{Aut}_A(x)$. Hence (3) holds.

Assume (3) holds. Suppose

$$(x_1, x_2, a : (f_1)_*x_1 \to (f_2)_*x_2),\ (x_1', x_2', a' : (f_1)_*x_1' \to (f_2)_*x_2')$$

are objects of $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ with the same image in $\overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$. Then there are morphisms $b_1 : x_1 \to x_1'$ in $\mathcal{F}(A_1)$ and $b_2 : x_2 \to x_2'$ in $\mathcal{F}(A_2)$. By (3) we can modify b_2 by an automorphism of x_2 over A_2 so that the diagram

$$\begin{array}{ccc} (f_1)_*x_1 & \xrightarrow[a]{} & (f_2)_*x_2 \\ {\scriptstyle (f_1)_*b_1}\downarrow & & \downarrow{\scriptstyle (f_2)_*b_2} \\ (f_1)_*x_1' & \xrightarrow{a'} & (f_2)_*x_2'. \end{array}$$

commutes. This proves $(x_1, x_2, a) \cong (x_1', x_2', a')$ in $\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)}$. Hence (2) holds. □

Finally we define the notion of a deformation category.

06J9 **Definition 90.16.8.** A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS). A morphism of deformation categories is a morphism of categories over $\mathcal{C}_\Lambda$.

06JA **Remark 90.16.9.** We say that a functor $F : \mathcal{C}_\Lambda \to Sets$ is a *deformation functor* if the associated cofibered set is a deformation category, i.e. if $F(k)$ is a one element set and F satisfies (RS). If $\mathcal{F}$ is a deformation category, then $\overline{\mathcal{F}}$ is a predeformation functor but not necessarily a deformation functor, as Lemma 90.16.7 shows.

06JB **Example 90.16.10.** A prorepresentable functor F is a deformation functor. Namely, suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ and $F(A) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, A)$. There is a unique morphism $R \to k$, so $F(k)$ is a one element set. Since

$$\mathrm{Hom}_\Lambda(R, A_1 \times_A A_2) = \mathrm{Hom}_\Lambda(R, A_1) \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A_2)$$

the same is true for maps in $\widehat{\mathcal{C}}_\Lambda$ and we see that F has (RS).

The following is one of our typical remarks on passing from a category cofibered in groupoids to the predeformation category at a point over k: it says that this process preserves (RS).

06JC **Lemma 90.16.11.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 90.6.4. If $\mathcal{F}$ satisfies (RS), then so does $\mathcal{F}_{x_0}$. In particular, $\mathcal{F}_{x_0}$ is a deformation category.*

Proof. Any diagram as in Definition 90.16.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of (RS) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

The following lemma is the analogue of the fact that 2-fibre products of algebraic stacks are algebraic stacks.

06L4 **Lemma 90.16.12.** *Let*

$$\begin{array}{ccc} \mathcal{H} \times_{\mathcal{F}} \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{H} & \xrightarrow{f} & \mathcal{F} \end{array}$$

be 2-fibre product of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ all satisfy (RS), then $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ satisfies (RS).

Proof. If A is an object of $\mathcal{C}_\Lambda$, then an object of the fiber category of $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ over A is a triple (u, v, a) where $u \in \mathcal{H}(A)$, $v \in \mathcal{G}(A)$, and $a : f(u) \to g(v)$ is a morphism in $\mathcal{F}(A)$. Consider a diagram in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$

$$\begin{array}{ccc} & & (u_2, v_2, a_2) \\ & & \downarrow \\ (u_1, v_1, a_1) & \longrightarrow & (u, v, a) \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective. Since $\mathcal{H}$ and $\mathcal{G}$ satisfy (RS), there are fiber products $u_1 \times_u u_2$ and $v_1 \times_v v_2$ lying over $A_1 \times_A A_2$. Since $\mathcal{F}$ satisfies (RS), Lemma 90.16.2 shows

$$\begin{array}{ccc} f(u_1 \times_u u_2) & \longrightarrow & f(u_2) \\ \downarrow & & \downarrow \\ f(u_1) & \longrightarrow & f(u) \end{array} \quad \text{and} \quad \begin{array}{ccc} g(v_1 \times_v v_2) & \longrightarrow & g(v_2) \\ \downarrow & & \downarrow \\ g(v_1) & \longrightarrow & g(v) \end{array}$$

are both fiber squares in $\mathcal{F}$. Thus we can view $a_1 \times_a a_2$ as a morphism from $f(u_1 \times_u u_2)$ to $g(v_1 \times_v v_2)$ over $A_1 \times_A A_2$. It follows that

$$\begin{array}{ccc} (u_1 \times_u u_2, v_1 \times_v v_2, a_1 \times_a a_2) & \longrightarrow & (u_2, v_2, a_2) \\ \downarrow & & \downarrow \\ (u_1, v_1, a_1) & \longrightarrow & (u, v, a) \end{array}$$

is a fiber square in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ as desired. □

90.17. Lifts of objects

06JD The content of this section is that the tangent space has a principal homogeneous action on the set of lifts along a small extension in the case of a deformation category.

06JE **Definition 90.17.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $f : A' \to A$ be a map in $\mathcal{C}_\Lambda$. Let $x \in \mathcal{F}(A)$. The category $Lift(x, f)$ of lifts of x along f is the category with the following objects and morphisms.

(1) Objects: A *lift of x along f* is a morphism $x' \to x$ lying over f.
(2) Morphisms: A *morphism of lifts* from $a_1 : x'_1 \to x$ to $a_2 : x'_2 \to x$ is a morphism $b : x'_1 \to x'_2$ in $\mathcal{F}(A')$ such that $a_2 = a_1 \circ b$.

The set $\text{Lift}(x, f)$ of lifts of x along f is the set of isomorphism classes of $Lift(x, f)$.

06JF **Remark 90.17.2.** When the map $f : A' \to A$ is clear from the context, we may write $Lift(x, A')$ and $\text{Lift}(x, A')$ in place of $Lift(x, f)$ and $\text{Lift}(x, f)$.

06JG **Remark 90.17.3.** Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \text{Ob}(\mathcal{F}(k))$. Let V be a finite dimensional vector space. Then $\text{Lift}(x_0, k[V])$ is the set of isomorphism classes of $\mathcal{F}_{x_0}(k[V])$ where $\mathcal{F}_{x_0}$ is the predeformation category of objects in $\mathcal{F}$ lying over x_0, see Remark 90.6.4. Hence if $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$ (see Lemma 90.10.6) and by Lemma 90.12.2 we see that

$$\text{Lift}(x_0, k[V]) = T\mathcal{F}_{x_0} \otimes_k V$$

as k-vector spaces.

06JH **Remark 90.17.4.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let

$$\begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

be a fibre square in $\mathcal{C}_\Lambda$ such that either $A_1 \to A$ or $A_2 \to A$ is surjective. Let $x \in \text{Ob}(\mathcal{F}(A))$. Given lifts $x_1 \to x$ and $x_2 \to x$ of x to A_1 and A_2, we get by (RS) a lift $x_1 \times_x x_2 \to x$ of x to $A_1 \times_A A_2$. Conversely, by Lemma 90.16.2 any lift of x to $A_1 \times_A A_2$ is of this form. Hence a bijection

$$\text{Lift}(x, A_1) \times \text{Lift}(x, A_2) \longrightarrow \text{Lift}(x, A_1 \times_A A_2).$$

Similarly, if $x_1 \to x$ is a fixed lifting of x to A_1, then there is a bijection

$$\text{Lift}(x_1, A_1 \times_A A_2) \longrightarrow \text{Lift}(x, A_2).$$

Now let

$$\begin{array}{ccccc} A'_1 \times_A A_2 & \longrightarrow & A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ A'_1 & \longrightarrow & A_1 & \longrightarrow & A \end{array}$$

be a composition of fibre squares in $\mathcal{C}_\Lambda$ with both $A'_1 \to A_1$ and $A_1 \to A$ surjective. Let $x_1 \to x$ be a morphism lying over $A_1 \to A$. Then by the above we have

bijections

$$\begin{aligned}\text{Lift}(x_1, A'_1 \times_A A_2) &= \text{Lift}(x_1, A'_1) \times \text{Lift}(x_1, A_1 \times_A A_2) \\ &= \text{Lift}(x_1, A'_1) \times \text{Lift}(x, A_2).\end{aligned}$$

06JI **Lemma 90.17.5.** *Let $\mathcal{F}$ be a deformation category. Let $A' \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$ whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \text{Ob}(\mathcal{F}(A))$. If Lift$(x, A')$ is nonempty, then there is a free and transitive action of $T\mathcal{F} \otimes_k I$ on Lift(x, A').*

Proof. Consider the ring map $g : A' \times_A A' \to k[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$ (compare with Lemma 90.10.8). There is an isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections of $A' \times_A A'$ and $A' \times_k k[I]$ to A. Thus there is a bijection

06TA (90.17.5.1) $$\text{Lift}(x, A' \times_A A') \longrightarrow \text{Lift}(x, A' \times_k k[I])$$

By Remark 90.17.4 there is a bijection

06TB (90.17.5.2) $$\text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow \text{Lift}(x, A' \times_A A')$$

There is a commutative diagram

$$\begin{array}{ccccc} A' \times_k k[I] & \longrightarrow & A \times_k k[I] & \longrightarrow & k[I] \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & A & \longrightarrow & k. \end{array}$$

Thus if we choose a pushforward $x \to x_0$ of x along $A \to k$, we obtain by the end of Remark 90.17.4 a bijection

06TC (90.17.5.3) $$\text{Lift}(x, A' \times_k k[I]) \longrightarrow \text{Lift}(x, A') \times \text{Lift}(x_0, k[I])$$

Composing (90.17.5.2), (90.17.5.1), and (90.17.5.3) we get a bijection

$$\Phi : \text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow \text{Lift}(x, A') \times \text{Lift}(x_0, k[I]).$$

This bijection commutes with the projections on the first factors. By Remark 90.17.3 we see that $\text{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. If pr_2 is the second projection of $\text{Lift}(x, A') \times \text{Lift}(x, A')$, then we get a map

$$a = \text{pr}_2 \circ \Phi^{-1} : \text{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \longrightarrow \text{Lift}(x, A').$$

Unwinding all the above we see that $a(x' \to x, \theta)$ is the unique lift $x'' \to x$ such that $g_*(x', x'') = \theta$ in $\text{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. To see this is an action of $T\mathcal{F} \otimes_k I$ on $\text{Lift}(x, A')$ we have to show the following: if x', x'', x''' are lifts of x and $g_*(x', x'') = \theta$, $g_*(x'', x''') = \theta'$, then $g_*(x', x''') = \theta + \theta'$. This follows from the commutative diagram

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow{(a_1,a_2,a_3) \mapsto (g(a_1,a_2), g(a_2,a_3))} & k[I] \times_k k[I] = k[I \times I] \\ & \searrow^{(a_1,a_2,a_3) \mapsto g(a_1,a_3)} & \downarrow + \\ & & k[I] \end{array}$$

The action is free and transitive because Φ is bijective. □

06JJ **Remark 90.17.6.** The action of Lemma 90.17.5 is functorial. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories. Let $A' \to A$ be a surjective ring map whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. In this situation φ induces the vertical arrows in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) & \longrightarrow & \mathrm{Lift}(x, A') \\ \downarrow{\scriptstyle (\varphi, d\varphi \otimes \mathrm{id}_I)} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Lift}(\varphi(x), A') \times (T\mathcal{G} \otimes_k I) & \longrightarrow & \mathrm{Lift}(\varphi(x), A') \end{array}$$

The commutativity follows as each of the maps (90.17.5.2), (90.17.5.1), and (90.17.5.3) of the proof of Lemma 90.17.5 gives rise to a similar commutative diagram.

90.18. Schlessinger's theorem on prorepresentable functors

06JK We deduce Schlessinger's theorem characterizing prorepresentable functors on $\mathcal{C}_\Lambda$.

06JL **Lemma 90.18.1.** *Let $F, G : \mathcal{C}_\Lambda \to Sets$ be deformation functors. Let $\varphi : F \to G$ be a smooth morphism which induces an isomorphism $d\varphi : TF \to TG$ of tangent spaces. Then φ is an isomorphism.*

Proof. We prove $F(A) \to G(A)$ is a bijection for all $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ by induction on $\mathrm{length}_A(A)$. For $A = k$ the statement follows from the assumption that F and G are deformation functors. Suppose that the statement holds for rings of length less than n and let A' be a ring of length n. Choose a small extension $f : A' \to A$. We have a commutative diagram

$$\begin{array}{ccc} F(A') & \longrightarrow & G(A') \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow{\sim} & G(A) \end{array}$$

where the map $F(A) \to G(A)$ is a bijection. By smoothness of $F \to G$, $F(A') \to G(A')$ is surjective (Lemma 90.8.8). Thus we can check bijectivity by checking it on fibers $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty. These fibers are precisely $\mathrm{Lift}(x, A')$ and $\mathrm{Lift}(\varphi(x), A')$ and by assumption we have an isomorphism $d\varphi \otimes \mathrm{id} : TF \otimes_k \mathrm{Ker}(f) \to TG \otimes_k \mathrm{Ker}(f)$. Thus, by Lemma 90.17.5 and Remark 90.17.6, for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty the map $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ is a map of sets commuting with free transitive actions by $TF \otimes_k \mathrm{Ker}(f)$. Hence it is bijective. □

Note that in case $k' \subset k$ is separable condition (c) in the theorem below is empty.

06JM **Theorem 90.18.2.** *Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. Then F is prorepresentable if and only if (a) F is a deformation functor, (b) $\dim_k TF$ is finite, and (c) $\gamma : Der_\Lambda(k, k) \to TF$ is injective.*

Proof. Assume F is prorepresentable by $R \in \widehat{\mathcal{C}}_\Lambda$. We see F is a deformation functor by Example 90.16.10. We see $\dim_k TF$ is finite by Example 90.11.11. Finally, $\mathrm{Der}_\Lambda(k, k) \to TF$ is identified with $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$ by Example 90.11.14 which is injective because $R \to k$ is surjective.

Conversely, assume (a), (b), and (c) hold. By Lemma 90.16.6 we see that (S1) and (S2) hold. Hence by Theorem 90.15.5 there exists a minimal versal formal object ξ of F such that (90.15.0.2) holds. Say ξ lies over R. The map

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$$

is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. Since the action of $\mathrm{Der}_\Lambda(k, k)$ on the left hand side is free by (c) and Lemma 90.12.6 we see that the map is bijective. Thus we see that $\underline{\xi}$ is an isomorphism by Lemma 90.18.1. □

90.19. Infinitesimal automorphisms

06JN Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Given a morphism $x' \to x$ in $\mathcal{F}$ lying over $A' \to A$, there is an induced homomorphism

$$\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x).$$

Lemma 90.16.7 says that the cokernel of this homomorphism determines whether condition (RS) on $\mathcal{F}$ passes to $\overline{\mathcal{F}}$. In this section we study the kernel of this homomorphism. We will see that it also gives a measure of how far $\mathcal{F}$ is from $\overline{\mathcal{F}}$.

06JP **Definition 90.19.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over $A' \to A$. The kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}(\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x))$$

is the *group of infinitesimal automorphisms of x' over x.*

06JQ **Definition 90.19.2.** Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Assume a choice of pushforward $x_0 \to x_0'$ of x_0 along the map $k \to k[\epsilon], a \mapsto a$ has been made. Then there is a unique map $x_0' \to x_0$ such that $x_0 \to x_0' \to x_0$ is the identity on x_0. Then

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}(x_0'/x_0)$$

is the *group of infinitesimal automorphisms of x_0*

06JR **Remark 90.19.3.** Up to canonical isomorphism $\mathrm{Inf}_{x_0}(\mathcal{F})$ does not depend on the choice of pushforward $x_0 \to x_0'$ because any two pushforwards are canonically isomorphic. Moreover, if $y_0 \in \mathcal{F}(k)$ and $x_0 \cong y_0$ in $\mathcal{F}(k)$, then $\mathrm{Inf}_{x_0}(\mathcal{F}) \cong \mathrm{Inf}_{y_0}(\mathcal{F})$ where the isomorphism depends (only) on the choice of an isomorphism $x_0 \to y_0$. In particular, $\mathrm{Aut}_k(x_0)$ acts on $\mathrm{Inf}_{x_0}(\mathcal{F})$.

06JS **Remark 90.19.4.** Assume $\mathcal{F}$ is a predeformation category. Then

(1) for $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ the automorphism group $\mathrm{Aut}_k(x_0)$ is trivial and hence $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Aut}_{k[\epsilon]}(x_0')$, and
(2) for $x_0, y_0 \in \mathrm{Ob}(\mathcal{F}(k))$ there is a unique isomorphism $x_0 \to y_0$ and hence a canonical identification $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Inf}_{y_0}(\mathcal{F})$.

Since $\mathcal{F}(k)$ is nonempty, choosing $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and setting

$$\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

we get a well defined *group of infinitesimal automorphisms of $\mathcal{F}$*. With this notation we have $\mathrm{Inf}(\mathcal{F}_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$. Please compare with the equality $T\mathcal{F}_{x_0} = T_{x_0}\mathcal{F}$ in Remark 90.12.5.

We will see that $\mathrm{Inf}_{x_0}(\mathcal{F})$ has a natural k-vector space structure when $\mathcal{F}$ satisfies (RS). At the same time, we will see that if $\mathcal{F}$ satisfies (RS), then the infinitesimal automorphisms $\mathrm{Inf}(x'/x)$ of a morphism $x' \to x$ lying over a small extension are governed by $\mathrm{Inf}_{x_0}(\mathcal{F})$, where x_0 is a pushforward of x to $\mathcal{F}(k)$. In order to do this, we introduce the automorphism functor for any object $x \in \mathrm{Ob}(\mathcal{F})$ as follows.

06JT **Definition 90.19.5.** Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids over an arbitrary base category $\mathcal{C}$. Assume a choice of pushforwards has been made. Let $x \in \mathrm{Ob}(\mathcal{F})$ and let $U = p(x)$. Let $U/\mathcal{C}$ denote the category of objects under U. The *automorphism functor of* x is the functor $Aut(x) : U/\mathcal{C} \to Sets$ sending an object $f : U \to V$ to $\mathrm{Aut}_V(f_*x)$ and sending a morphism

$$\begin{array}{ccc} V' & \longrightarrow & V \\ & {\scriptstyle f'} \nwarrow \quad \nearrow {\scriptstyle f} & \\ & U & \end{array}$$

to the homomorphism $\mathrm{Aut}_{V'}(f'_*x) \to \mathrm{Aut}_V(f_*x)$ coming from the unique morphism $f'_*x \to f_*x$ lying over $V' \to V$ and compatible with $x \to f'_*x$ and $x \to f_*x$.

We will be concerned with the automorphism functors of objects in a category cofibered in groupoids $\mathcal{F}$ over $\mathcal{C}_\Lambda$. If $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$, then the category $A/\mathcal{C}_\Lambda$ is nothing but the category $\mathcal{C}_A$, i.e. the category defined in Section 90.3 where we take $\Lambda = A$ and $k = A/\mathfrak{m}_A$. Hence the automorphism functor of an object $x \in \mathrm{Ob}(\mathcal{F}(A))$ is a functor $Aut(x) : \mathcal{C}_A \to Sets$.

The following lemma could be deduced from Lemma 90.16.12 by thinking about the "inertia" of a category cofibred in groupoids, see for example Stacks, Section 8.7 and Categories, Section 4.34. However, it is easier to see it directly.

06JU **Lemma 90.19.6.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Then $Aut(x) : \mathcal{C}_A \to Sets$ satisfies (RS).*

Proof. It follows that $Aut(x)$ satisfies (RS) from the fully faithfulness of the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ in Lemma 90.16.4. □

06JV **Lemma 90.19.7.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$.*

1. *$T_{id_{x_0}} Aut(x)$ has a natural k-vector space structure such that addition agrees with composition in $T_{id_{x_0}} Aut(x)$. In particular, composition in $T_{id_{x_0}} Aut(x)$ is commutative.*
2. *There is a canonical isomorphism $T_{id_{x_0}} Aut(x) \to T_{id_{x_0}} Aut(x_0)$ of k-vector spaces.*

Proof. We apply Remark 90.6.4 to the functor $Aut(x) : \mathcal{C}_A \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x)_{\mathrm{id}_{x_0}}$. By Lemmas 90.19.6 and 90.16.11 F is a deformation functor. By definition $T_{\mathrm{id}_{x_0}} Aut(x) = TF = F(k[\epsilon])$ which has a natural k-vector space structure specified by Lemma 90.11.8.

Addition is defined as the composition

$$F(k[\epsilon]) \times F(k[\epsilon]) \longrightarrow F(k[\epsilon] \times_k k[\epsilon]) \longrightarrow F(k[\epsilon])$$

where the first map is the inverse of the bijection guaranteed by (RS) and the second is induced by the k-algebra map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ which maps $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then $F(A) \to F(B)$ is a homomorphism where $F(A) = Aut(x)_{\mathrm{id}_{x_0}}(A)$ and $F(B) = Aut(x)_{\mathrm{id}_{x_0}}(B)$ are groups under composition. We conclude that $+ : F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is a homomorphism where $F(k[\epsilon])$ is regarded as a group under composition. With $\mathrm{id} \in F(k[\epsilon])$ the unit element we see that $+(v, \mathrm{id}) = +(\mathrm{id}, v) = v$ for any $v \in F(k[\epsilon])$ because (id, v) is the pushforward of v along the ring map $k[\epsilon] \to k[\epsilon] \times_k k[\epsilon]$ with $\epsilon \mapsto (\epsilon, 0)$. In general, given a group G with multiplication $\circ$ and $+ : G \times G \to G$ is a homomorphism such that $+(g, 1) = +(1, g) = g$, where 1 is the identity of G, then $+ = \circ$. This shows addition in the k-vector space structure on $F(k[\epsilon])$ agrees with composition.

Finally, (2) is a matter of unwinding the definitions. Namely $T_{\mathrm{id}_{x_0}} Aut(x)$ is the set of automorphisms α of the pushforward of x along $A \to k \to k[\epsilon]$ which are trivial modulo ϵ. On the other hand $T_{\mathrm{id}_{x_0}} Aut(x_0)$ is the set of automorphisms of the pushforward of x_0 along $k \to k[\epsilon]$ which are trivial modulo ϵ. Since x_0 is the pushforward of x along $A \to k$ the result is clear. □

06JW **Remark 90.19.8.** We point out some basic relationships between infinitesimal automorphism groups, liftings, and tangent spaces to automorphism functors. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism lying over a ring map $A' \to A$. Then from the definitions we have an equality

$$\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$$

where the liftings are of id_x as an object of $Aut(x')$. If $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and x_0' is the pushforward to $\mathcal{F}(k[\epsilon])$, then applying this to $x_0' \to x_0$ we get

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Lift}(\mathrm{id}_{x_0}, k[\epsilon]) = T_{\mathrm{id}_{x_0}} Aut(x_0),$$

the last equality following directly from the definitions.

06JX **Lemma 90.19.9.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F})$ is equal as a set to $T_{id_{x_0}} Aut(x_0)$, and so has a natural k-vector space structure such that addition agrees with composition of automorphisms.*

Proof. The equality of sets is as in the end of Remark 90.19.8 and the statement about the vector space structure follows from Lemma 90.19.7. □

07W6 **Lemma 90.19.10.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then φ induces a k-linear map $Inf_{x_0}(\mathcal{F}) \to Inf_{\varphi(x_0)}(\mathcal{G})$.*

Proof. It is clear that φ induces a morphism from $Aut(x_0) \to Aut(\varphi(x_0))$ which maps the identity to the identity. Hence this follows from the result for tangent spaces, see Lemma 90.12.4. □

06JY **Lemma 90.19.11.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism lying over a surjective ring map $A' \to A$ with kernel I annihilated by $\mathfrak{m}_{A'}$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then $Inf(x'/x)$ has a free and transitive action by $T_{id_{x_0}} Aut(x') \otimes_k I = Inf_{x_0}(\mathcal{F}) \otimes_k I$.*

Proof. This is just the analogue of Lemma 90.17.5 in the setting of automorphism sheaves. To be precise, we apply Remark 90.6.4 to the functor $Aut(x') : \mathcal{C}_{A'} \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x')_{\mathrm{id}_{x_0}}$. By Lemmas 90.19.6 and 90.16.11 F is a deformation functor. Hence Lemma 90.17.5 gives a free and transitive action of $TF \otimes_k I$ on $\mathrm{Lift}(\mathrm{id}_x, A')$, because as $\mathrm{Lift}(\mathrm{id}_x, A')$ is a group it is always nonempty. Note that we have equalities of vector spaces

$$TF = T_{\mathrm{id}_{x_0}} Aut(x') \otimes_k I = \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k I$$

by Lemma 90.19.7. The equality $\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$ of Remark 90.19.8 finishes the proof. □

06JZ **Lemma 90.19.12.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. If $Inf_{x_0}(\mathcal{F}) = 0$ then $Inf(x'/x) = 0$.*

Proof. Follows from Lemmas 90.3.3 and 90.19.11. □

06K0 **Lemma 90.19.13.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F}) = 0$ if and only if the natural morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ of categories cofibered in groupoids is an equivalence.*

Proof. The morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ is an equivalence if and only if $\mathcal{F}_{x_0}$ is fibered in setoids, cf. Categories, Section 4.39 (a setoid is by definition a groupoid in which the only automorphism of any object is the identity). We prove that $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if this condition holds for $\mathcal{F}_{x_0}$. Obviously if $\mathcal{F}_{x_0}$ is fibered in setoids then $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Conversely assume $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Let A be an object of $\mathcal{C}_\Lambda$. Then by Lemma 90.19.12, $\mathrm{Inf}(x/x_0) = 0$ for any object $x \to x_0$ of $\mathcal{F}_{x_0}(A)$. Since by definition $\mathrm{Inf}(x/x_0)$ equals the group of automorphisms of $x \to x_0$ in $\mathcal{F}_{x_0}(A)$, this proves $\mathcal{F}_{x_0}(A)$ is a setoid. □

90.20. Applications

0DYM We collect some results on deformation categories we will use later.

06L5 **Lemma 90.20.1.** *Let $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{F}$ be 1-morphisms of deformation categories. Then*

(1) *$\mathcal{W} = \mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ is a deformation category, and*
(2) *we have a 6-term exact sequence of vector spaces*

$$0 \to Inf(\mathcal{W}) \to Inf(\mathcal{H}) \oplus Inf(\mathcal{G}) \to Inf(\mathcal{F}) \to T\mathcal{W} \to T\mathcal{H} \oplus T\mathcal{G} \to T\mathcal{F}$$

Proof. Part (1) follows from Lemma 90.16.12 and the fact that $\mathcal{W}(k)$ is the fibre product of two setoids with a unique isomorphism class over a setoid with a unique isomorphism class.

Part (2). Let $w_0 \in \mathrm{Ob}(\mathcal{W}(k))$ and let x_0, y_0, z_0 be the image of w_0 in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. Then $\mathrm{Inf}(\mathcal{W}) = \mathrm{Inf}_{w_0}(\mathcal{W})$ and similarly for $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{F}$, see Remark 90.19.4. We apply Lemmas 90.12.4 and 90.19.10 to get all the linear maps except for the "boundary map" $\delta : \mathrm{Inf}_{x_0}(\mathcal{F}) \to T\mathcal{W}$. We will insert suitable signs later.

Construction of δ. Choose a pushforward $w_0 \to w_0'$ along $k \to k[\epsilon]$. Denote x_0', y_0', z_0' the images of w_0' in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. In particular we obtain isomorphisms $b' : f(y_0') \to x_0'$ and $c' : x_0' \to g(z_0')$. Denote $b : f(y_0) \to x_0$ and $c : x_0 \to g(z_0)$ the pushforwards along $k[\epsilon] \to k$. Observe that this means $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ and $w_0 =$

$(k, y_0, z_0, c \circ b)$ in terms of the explicit form of the fibre product of categories, see Remarks 90.5.2 (13). Given $\alpha : x_0' \to x_0'$ we set $\delta(\alpha) = (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ which is indeed an object of $\mathcal{W}$ over $k[\epsilon]$ and comes with a morphism $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b') \to w_0$ over $k[\epsilon] \to k$ as α pushes forward to the identity over k. More generally, for any k-vector space V we can define a map

$$\text{Lift}(\text{id}_{x_0}, k[V]) \longrightarrow \text{Lift}(w_0, k[V])$$

using exactly the same formulae. This construction is functorial in the vector space V (details omitted). Hence δ is k-linear by an application of Lemma 90.11.5.

Having constructed these maps it is straightforward to show the sequence is exact. Injectivity of the first map comes from the fact that $f \times g : \mathcal{W} \to \mathcal{H} \times \mathcal{G}$ is faithful. If $(\beta, \gamma) \in \text{Inf}_{y_0}(\mathcal{H}) \oplus \text{Inf}_{z_0}(\mathcal{G})$ map to the same element of $\text{Inf}_{x_0}(\mathcal{F})$ then (β, γ) defines an automorphism of $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ whence exactness at the second spot. If α as above gives the trivial deformation $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ of w_0, then the isomorphism $w_0' = (k[\epsilon], y_0', z_0', c' \circ b') \to (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ produces a pair (β, γ) which is a preimage of α. If $w = (k[\epsilon], y, z, \phi)$ is a deformation of w_0 such that $y_0' \cong y$ and $z \cong z_0'$ then the map

$$f(y_0') \to f(y) \xrightarrow{\phi} g(z) \to g(z_0')$$

is an α which maps to w under δ. Finally, if y and z are deformations of y_0 and z_0 and there exists an isomorphism $\phi : f(y) \to g(z)$ of deformations of $f(y_0) = x_0 = g(z_0)$ then we get a preimage $w = (k[\epsilon], y, z, \phi)$ of (x, y) in $T\mathcal{W}$. This finishes the proof. □

0DYN **Lemma 90.20.2.** *Let $\mathcal{H}_1 \to \mathcal{G}$, $\mathcal{H}_2 \to \mathcal{G}$, and $\mathcal{G} \to \mathcal{F}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume*

(1) *$\mathcal{F}$ and $\mathcal{G}$ are deformation categories,*
(2) *$T\mathcal{G} \to T\mathcal{F}$ is injective, and*
(3) *$Inf(\mathcal{G}) \to Inf(\mathcal{F})$ is surjective.*

Then $\mathcal{H}_1 \times_{\mathcal{G}} \mathcal{H}_2 \to \mathcal{H}_1 \times_{\mathcal{F}} \mathcal{H}_2$ is smooth.

Proof. Denote $p_i : \mathcal{H}_i \to \mathcal{G}$ and $q : \mathcal{G} \to \mathcal{F}$ be the given maps. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$. An object of $\mathcal{H}_1 \times_{\mathcal{F}} \mathcal{H}_2$ over A' is a triple (x_1', x_2', a') where x_i' is an object of $\mathcal{H}_i$ over A' and $a' : q(p_1(x_1')) \to q(p_2(x_2'))$ is a morphism of the fibre category of $\mathcal{F}$ over A'. By pushforward along $A' \to A$ we get (x_1, x_2, a). Lifting this to an object of $\mathcal{H}_1 \times_{\mathcal{G}} \mathcal{H}_2$ over A means finding a morphism $b : p_1(x_1) \to p_2(x_2)$ over A with $q(b) = a$. Thus we have to show that we can lift b to a morphism $b' : p_1(x_1') \to p_2(x_2')$ whose image under q is a'.

Observe that we can think of

$$p_1(x_1') \to p_1(x_1) \xrightarrow{b} p_2(x_2) \quad \text{and} \quad p_2(x_2') \to p_2(x_2)$$

as two objects of $Lift(p_2(x_2), A' \to A)$. The functor q sends these objects to the two objects

$$q(p_1(x_1')) \to q(p_1(x_1)) \xrightarrow{b} q(p_2(x_2)) \quad \text{and} \quad q(p_2(x_2')) \to q(p_2(x_2))$$

of $Lift(q(p_2(x_2)), A' \to A)$ which are isomorphic using the map $a' : q(p_1(x_1')) \to q(p_2(x_2'))$. On the other hand, the functor

$$q : Lift(p_2(x_2), A' \to A) \to Lift(q(p_2(x_2)), A' \to A)$$

defines an injection on isomorphism classes by Lemma 90.17.5 and our assumption on tangent spaces. Thus we see that there is a morphism $b' : p_1(x'_1) \to p_2(x'_2)$ whose pushforward to A is b. However, we may need to adjust our choice of b' to achieve $q(b') = a'$. For this it suffices to see that $q : \mathrm{Inf}(p_2(x'_2)/p_2(x_2)) \to \mathrm{Inf}(q(p_2(x'_2))/q(p_2(x_2)))$ is surjective. This follows from our assumption on infinitesimal automorphisms and Lemma 90.19.11. □

0DYP **Lemma 90.20.3.** *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of deformation categories. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ with image $y_0 \in \mathrm{Ob}(\mathcal{G}(k))$. If*

(1) *the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and*
(2) *for every small extension $A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$ if there is a lift of y to A', then there is a lift of x to A',*

then $\mathcal{F} \to \mathcal{G}$ is smooth (and vice versa).

Proof. Let $A' \to A$ be a small extension. Let $x \in \mathcal{F}(A)$. Let $y' \to f(x)$ be a morphism in $\mathcal{G}$ over $A' \to A$. Consider the functor $\mathrm{Lift}(A', x) \to \mathrm{Lift}(A', f(x))$ induced by f. We have to show that there exists an object $x' \to x$ of $\mathrm{Lift}(A', x)$ mapping to $y' \to f(x)$, see Lemma 90.8.2. By condition (2) we know that $\mathrm{Lift}(A', x)$ is not the empty category. By condition (2) and Lemma 90.17.5 we conlude that the map on isomorphism classes is surjective as desired. □

0E3R **Lemma 90.20.4.** *Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be maps of categories cofibred in groupoids over $\mathcal{C}_\Lambda$. If*

(1) *$\mathcal{F}$, $\mathcal{G}$ are deformation categories*
(2) *the map $T\mathcal{F} \to T\mathcal{G}$ is surjective, and*
(3) *$\mathcal{F} \to \mathcal{H}$ is smooth.*

Then $\mathcal{F} \to \mathcal{G}$ is smooth.

Proof. Let $A' \to A$ be a small extension in $\mathcal{C}_\Lambda$ and let $x \in \mathcal{F}(A)$ with image $y \in \mathcal{G}(A)$. Assume there is a lift $y' \in \mathcal{G}(A')$. According to Lemma 90.20.3 all we have to do is check that x has a lift too. Take the image $z' \in \mathcal{H}(A')$ of y'. Since $\mathcal{F} \to \mathcal{H}$ is smooth, there is an $x' \in \mathcal{F}(A')$ mapping to both $x \in \mathcal{F}(A)$ and $z' \in \mathcal{H}(A')$, see Definition 90.8.1. This finishes the proof. □

90.21. Groupoids in functors on an arbitrary category

06K2 We begin with generalities on groupoids in functors on an arbitrary category. In the next section we will pass to the category $\mathcal{C}_\Lambda$. For clarity we shall sometimes refer to an ordinary groupoid, i.e., a category whose morphisms are all isomorphisms, as a groupoid category.

06K3 **Definition 90.21.1.** Let $\mathcal{C}$ be a category. The *category of groupoids in functors on $\mathcal{C}$* is the category with the following objects and morphisms.

(1) Objects: A *groupoid in functors on $\mathcal{C}$* is a quintuple (U, R, s, t, c) where $U, R : \mathcal{C} \to \mathit{Sets}$ are functors and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms with the following property: For any object T of $\mathcal{C}$, the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category.

(2) Morphisms: A *morphism* $(U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in functors on* $\mathcal{C}$ consists of morphisms $U \to U'$ and $R \to R'$ with the following property: For any object T of $\mathcal{C}$, the induced maps $U(T) \to U'(T)$ and $R(T) \to R'(T)$ define a functor between groupoid categories
$$(U(T), R(T), s, t, c) \to (U'(T), R'(T), s', t', c').$$

06K4 **Remark 90.21.2.** A groupoid in functors on $\mathcal{C}$ amounts to the data of a functor $\mathcal{C} \to \mathit{Groupoids}$, and a morphism of groupoids in functors on $\mathcal{C}$ amounts to a morphism of the corresponding functors $\mathcal{C} \to \mathit{Groupoids}$ (where *Groupoids* is regarded as a 1-category). However, for our purposes it is more convenient to use the terminology of groupoids in functors. In fact, thinking of a groupoid in functors as the corresponding functor $\mathcal{C} \to \mathit{Groupoids}$, or equivalently as the category cofibered in groupoids associated to that functor, can lead to confusion (Remark 90.23.2).

06K5 **Remark 90.21.3.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. There are unique morphisms $e : U \to R$ and $i : R \to R$ such that for every object T of $\mathcal{C}$, $e : U(T) \to R(T)$ sends $x \in U(T)$ to the identity morphism on x and $i : R(T) \to R(T)$ sends $a \in U(T)$ to the inverse of a in the groupoid category $(U(T), R(T), s, t, c)$. We will sometimes refer to s, t, c, e, and i as "source", "target", "composition", "identity", and "inverse".

06K6 **Definition 90.21.4.** Let $\mathcal{C}$ be a category. A groupoid in functors on $\mathcal{C}$ is *representable* if it is isomorphic to one of the form $(\underline{U}, \underline{R}, s, t, c)$ where U and R are objects of $\mathcal{C}$ and the pushout $R \amalg_{s,U,t} R$ exists.

06K7 **Remark 90.21.5.** Hence a representable groupoid in functors on $\mathcal{C}$ is given by objects U and R of $\mathcal{C}$ and morphisms $s, t : U \to R$ and $c : R \to R \amalg_{s,U,t} R$ such that $(\underline{U}, \underline{R}, s, t, c)$ satisfies the condition of Definition 90.21.1. The reason for requiring the existence of the pushout $R \amalg_{s,U,t} R$ is so that the composition morphism c is defined at the level of morphisms in $\mathcal{C}$. This requirement will always be satisfied below when we consider representable groupoids in functors on $\widehat{\mathcal{C}_\Lambda}$, since by Lemma 90.4.3 the category $\widehat{\mathcal{C}_\Lambda}$ admits pushouts.

06K8 **Remark 90.21.6.** We will say "*let* $(\underline{U}, \underline{R}, s, t, c)$ *be a groupoid in functors on* $\mathcal{C}$" to mean that we have a representable groupoid in functors. Thus this means that U and R are objects of $\mathcal{C}$, there are morphisms $s, t : U \to R$, the pushout $R \amalg_{s,U,t} R$ exists, there is a morphism $c : R \to R \amalg_{s,U,t} R$, and $(\underline{U}, \underline{R}, s, t, c)$ is a groupoid in functors on $\mathcal{C}$.

We introduce notation for restriction of groupoids in functors. This will be relevant below in situations where we restrict from $\widehat{\mathcal{C}_\Lambda}$ to $\mathcal{C}_\Lambda$.

06K9 **Definition 90.21.7.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. The *restriction* $(U, R, s, t, c)|_{\mathcal{C}'}$ *of* (U, R, s, t, c) *to* $\mathcal{C}'$ is the groupoid in functors on $\mathcal{C}'$ given by $(U|_{\mathcal{C}'}, R|_{\mathcal{C}'}, s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'})$.

06KA **Remark 90.21.8.** In the situation of Definition 90.21.7, we often denote $s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'}$ simply by s, t, c.

06KB **Definition 90.21.9.** Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$.

(1) The assignment $T \mapsto (U(T), R(T), s, t, c)$ determines a functor $\mathcal{C} \to \mathit{Groupoids}$. The *quotient category cofibered in groupoids* $[U/R] \to \mathcal{C}$ is the category cofibered in groupoids over $\mathcal{C}$ associated to this functor (as in Remarks 90.5.2 (9)).

(2) The *quotient morphism* $U \to [U/R]$ is the morphism of categories cofibered in groupoids over $\mathcal{C}$ defined by the rules
   (a) $x \in U(T)$ maps to the object $(T, x) \in \mathrm{Ob}([U/R](T))$, and
   (b) $x \in U(T)$ and $f : T \to T'$ give rise to the morphism $(f, \mathrm{id}_{U(f)(x)}) : (T, x) \to (T, U(f)(x))$ lying over $f : T \to T'$.

90.22. Groupoids in functors on the base category

06KC In this section we discuss groupoids in functors on $\mathcal{C}_\Lambda$. Our eventual goal is to show that prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$ serve as "presentations" for well-behaved deformation categories in the same way that smooth groupoids in algebraic spaces serve as presentations for algebraic stacks, cf. Algebraic Stacks, Section 94.16.

06KD **Definition 90.22.1.** A groupoid in functors on $\mathcal{C}_\Lambda$ is *prorepresentable* if it is isomorphic to $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ for some representable groupoid in functors $(\underline{R_0}, \underline{R_1}, s, t, c)$ on the category $\widehat{\mathcal{C}_\Lambda}$.

Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Taking completions, we get a quintuple $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$. By Remark 90.7.10 completion as a functor on $\mathrm{CofSet}(\mathcal{C}_\Lambda)$ is a right adjoint, so it commutes with limits. In particular, there is a canonical isomorphism

$$\widehat{R \times_{s,U,t} R} \longrightarrow \widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R},$$

so $\widehat{c}$ can be regarded as a functor $\widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R} \to \widehat{R}$. Then $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ is a groupoid in functors on $\widehat{\mathcal{C}_\Lambda}$, with identity and inverse morphisms being the completions of those of (U, R, s, t, c).

06KE **Definition 90.22.2.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The *completion* $(U, R, s, t, c)^\wedge$ *of* (U, R, s, t, c) is the groupoid in functors $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ on $\widehat{\mathcal{C}_\Lambda}$ described above.

06KF **Remark 90.22.3.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Then there is a canonical isomorphism $(U, R, s, t, c)^\wedge|_{\mathcal{C}_\Lambda} \cong (U, R, s, t, c)$, see Remark 90.7.7. On the other hand, let (U, R, s, t, c) be a groupoid in functors on $\widehat{\mathcal{C}_\Lambda}$ such that $U, R : \widehat{\mathcal{C}_\Lambda} \to \mathit{Sets}$ both commute with limits, e.g. if U, R are representable. Then there is a canonical isomorphism $((U, R, s, t, c)|_{\mathcal{C}_\Lambda})^\wedge \cong (U, R, s, t, c)$. This follows from Remark 90.7.11.

06KG **Lemma 90.22.4.** *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is prorepresentable if and only if its completion is representable as a groupoid in functors on $\widehat{\mathcal{C}_\Lambda}$.*
(2) *(U, R, s, t, c) is prorepresentable if and only if U and R are prorepresentable.*

Proof. Part (1) follows from Remark 90.22.3. For (2), the "only if" direction is clear from the definition of a prorepresentable groupoid in functors. Conversely, assume U and R are prorepresentable, say $U \cong \underline{R_0}|_{\mathcal{C}_\Lambda}$ and $R \cong \underline{R_1}|_{\mathcal{C}_\Lambda}$ for objects R_0 and R_1 of $\widehat{\mathcal{C}_\Lambda}$. Since $\underline{R_0} \cong \widehat{\underline{R_0}|_{\mathcal{C}_\Lambda}}$ and $\underline{R_1} \cong \widehat{\underline{R_1}|_{\mathcal{C}_\Lambda}}$ by Remark 90.7.11 we see that the completion $(U, R, s, t, c)^\wedge$ is a groupoid in functors of the form $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$. By Lemma 90.4.3 the pushout $\underline{R_1} \times_{\widehat{s},\underline{R_1},\widehat{t}} \underline{R_1}$ exists. Hence $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$ is a

representable groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$. Finally, the restriction $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ gives back (U, R, s, t, c) by Remark 90.22.3 hence (U, R, s, t, c) is prorepresentable by definition. □

90.23. Smooth groupoids in functors on the base category

06KH The notion of smoothness for groupoids in functors on $\mathcal{C}_\Lambda$ is defined as follows.

06KI **Definition 90.23.1.** Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. We say (U, R, s, t, c) is *smooth* if $s, t : R \to U$ are smooth.

06KJ **Remark 90.23.2.** We note that this terminology is potentially confusing: if (U, R, s, t, c) is a smooth groupoid in functors, then the quotient $[U/R]$ need not be a smooth category cofibred in groupoids as defined in Definition 90.9.1. However smoothness of (U, R, s, t, c) does imply (in fact is equivalent to) smoothness of the quotient morphism $U \to [U/R]$ as we shall see in Lemma 90.23.4.

06KK **Remark 90.23.3.** Let $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ be a prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Then $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ is smooth if and only if R_1 is a power series over R_0 via both s and t. This follows from Lemma 90.8.6.

06KL **Lemma 90.23.4.** *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The following are equivalent:*

1. *The groupoid in functors (U, R, s, t, c) is smooth.*
2. *The morphism $s : R \to U$ is smooth.*
3. *The morphism $t : R \to U$ is smooth.*
4. *The quotient morphism $U \to [U/R]$ is smooth.*

Proof. Statement (2) is equivalent to (3) since the inverse $i : R \to R$ of (U, R, s, t, c) is an isomorphism and $t = s \circ i$. By definition (1) is equivalent to (2) and (3) together, hence it is equivalent to either of them individually.

Finally we prove (2) is equivalent to (4). Unwinding the definitions:

(2) Smoothness of $s : R \to U$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$, $a \in R(A)$, and $y \in U(B)$ such that $s(a) = U(f)(y)$, then there exists $a' \in R(B)$ such that $R(f)(a') = a$ and $s(a') = y$.

(4) Smoothness of $U \to [U/R]$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$ and $(f, a) : (B, y) \to (A, x)$ is a morphism of $[U/R]$, then there exists $x' \in U(B)$ and $b \in R(B)$ with $s(b) = x'$, $t(b) = y$ such that $c(a, R(f)(b)) = e(x)$. Here $e : U \to R$ denotes the identity and the notation (f, a) is as in Remarks 90.5.2 (9); in particular $a \in R(A)$ with $s(a) = U(f)(y)$ and $t(a) = x$.

If (4) holds and f, a, y as in (2) are given, let $x = t(a)$ so that we have a morphism $(f, a) : (B, y) \to (A, x)$. Then (4) produces x' and b, and $a' = i(b)$ satisfies the requirements of (2). Conversely, assume (2) holds and let $(f, a) : (B, y) \to (A, x)$ as in (4) be given. Then (2) produces $a' \in R(B)$, and $x' = t(a')$ and $b = i(a')$ satisfy the requirements of (4). □

90.24. Deformation categories as quotients of groupoids in functors

06KS

We discuss conditions on a groupoid in functors on $\mathcal{C}_\Lambda$ which guarantee that the quotient is a deformation category, and we calculate the tangent and infinitesimal automorphism spaces of such a quotient.

06KT **Lemma 90.24.1.** *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R satisfy (RS). Then $[U/R]$ satisfies (RS).*

Proof. Let

$$\begin{array}{ccc} & & (A_2, x_2) \\ & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

be a diagram in $[U/R]$ such that $f_2 : A_2 \to A$ is surjective. The notation is as in Remarks 90.5.2 (9). Hence $f_1 : A_1 \to A$, $f_2 : A_2 \to A$ are maps in $\mathcal{C}_\Lambda$, $x \in U(A)$, $x_1 \in U(A_1)$, $x_2 \in U(A_2)$, and $a_1, a_2 \in R(A)$ with $s(a_1) = U(f_1)(x_1)$, $t(a_1) = x$ and $s(a_2) = U(f_2)(x_2)$, $t(a_2) = x$. We construct a fiber product lying over $A_1 \times_A A_2$ for this diagram in $[U/R]$ as follows.

Let $a = c(i(a_1), a_2)$, where $i : R \to R$ is the inverse morphism. Then $a \in R(A)$, $x_2 \in U(A_2)$ and $s(a) = U(f_2)(x_2)$. Hence an element $(a, x_2) \in R(A) \times_{s, U(A), U(f_2)} U(A_2)$. By smoothness of $s : R \to U$ there is an element $\widetilde{a} \in R(A_2)$ with $R(f_2)(\widetilde{a}) = a$ and $s(\widetilde{a}) = x_2$. In particular $U(f_2)(t(\widetilde{a})) = t(a) = U(f_1)(x_1)$. Thus x_1 and $t(\widetilde{a})$ define an element

$$(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2).$$

By the assumption that U satisfies (RS), we have an identification $U(A_1) \times_{U(A)} U(A_2) = U(A_1 \times_A A_2)$. Let us denote $x_1 \times t(\widetilde{a}) \in U(A_1 \times_A A_2)$ the element corresponding to $(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2)$. Let p_1, p_2 be the projections of $A_1 \times_A A_2$. We claim

$$\begin{array}{ccc} (A_1 \times_A A_2, x_1 \times t(\widetilde{a})) & \xrightarrow[(p_2, i(\widetilde{a}))]{} & (A_2, x_2) \\ {\scriptstyle (p_1, e(x_1))}\downarrow & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

is a fiber square in $[U/R]$. (Note $e : U \to R$ denotes the identity.)

The diagram is commutative because $c(a_2, R(f_2)(i(\widetilde{a}))) = c(a_2, i(a)) = a_1$. To check it is a fiber square, let

$$\begin{array}{ccc} (B, z) & \xrightarrow[(g_2, b_2)]{} & (A_2, x_2) \\ {\scriptstyle (g_1, b_1)}\downarrow & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

be a commutative diagram in $[U/R]$. We will show there is a unique morphism $(g, b) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ compatible with the morphisms to (A_1, x_1) and (A_2, x_2). We must take $g = (g_1, g_2) : B \to A_1 \times_A A_2$. Since by assumption R satisfies (RS), we have an identification $R(A_1 \times_A A_2) = R(A_1) \times_{R(A)} R(A_2)$. Hence we can write $b = (b_1', b_2')$ for some $b_1' \in R(A_1)$, $b_2' \in R(A_2)$ which agree in $R(A)$. Then $((g_1, g_2), (b_1', b_2')) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ will commute

with the projections if and only if $b'_1 = b_1$ and $b'_2 = c(\widetilde{a}, b_2)$ proving unicity and existence. □

06KU **Lemma 90.24.2.** *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R are deformation functors. Then:*

(1) *The quotient $[U/R]$ is a deformation category.*
(2) *The tangent space of $[U/R]$ is*

$$T[U/R] = \text{Coker}(ds - dt : TR \to TU).$$

(3) *The space of infinitesimal automorphisms of $[U/R]$ is*

$$Inf([U/R]) = \text{Ker}(ds \oplus dt : TR \to TU \oplus TU).$$

Proof. Since U and R are deformation functors $[U/R]$ is a predeformation category. Since (RS) holds for deformation functors by definition we see that (RS) holds for [U/R] by Lemma 90.24.1. Hence $[U/R]$ is a deformation category. Statements (2) and (3) follow directly from the definitions. □

90.25. Presentations of categories cofibered in groupoids

06KW A presentation is defined as follows.

06KX **Definition 90.25.1.** Let $\mathcal{F}$ be a category cofibered in groupoids over a category $\mathcal{C}$. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}$. A *presentation of $\mathcal{F}$ by (U, R, s, t, c)* is an equivalence $\varphi : [U/R] \to \mathcal{F}$ of categories cofibered in groupoids over $\mathcal{C}$.

The following two general lemmas will be used to get presentations.

06KY **Lemma 90.25.2.** *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Define R, s, t, c as follows:*

(1) *$R : \mathcal{C} \to Sets$ is the functor $U \times_{f,\mathcal{F},f} U$.*
(2) *$t, s : R \to U$ are the first and second projections, respectively.*
(3) *$c : R \times_{s,U,t} R \to R$ is the morphism given by projection onto the first and last factors of $U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$ under the canonical isomorphism $R \times_{s,U,t} R \to U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$.*

Then (U, R, s, t, c) is a groupoid in functors on $\mathcal{C}$.

Proof. Omitted. □

06KZ **Lemma 90.25.3.** *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. Then there is a natural morphism $[f] : [U/R] \to \mathcal{F}$ such that:*

(1) *$[f] : [U/R] \to \mathcal{F}$ is fully faithful.*
(2) *$[f] : [U/R] \to \mathcal{F}$ is an equivalence if and only if $f : U \to \mathcal{F}$ is essentially surjective.*

Proof. Omitted. □

90.26. Presentations of deformation categories

06L0 According to the next lemma, a smooth morphism from a predeformation functor to a predeformation category $\mathcal{F}$ gives rise to a presentation of $\mathcal{F}$ by a smooth groupoid in functors.

06L1 **Lemma 90.26.1.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $U : \mathcal{C}_\Lambda \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a smooth morphism of categories cofibered in groupoids. Then:*

(1) *If (U, R, s, t, c) is the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2, then (U, R, s, t, c) is smooth.*
(2) *If $f : U(k) \to \mathcal{F}(k)$ is essentially surjective, then the morphism $[f] : [U/R] \to \mathcal{F}$ of Lemma 90.25.3 is an equivalence.*

Proof. From the construction of Lemma 90.25.2 we have a commutative diagram

$$\begin{array}{ccc} R = U \times_{f,\mathcal{F},f} U & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{f} & \mathcal{F} \end{array}$$

where t, s are the first and second projections. So t, s are smooth by Lemma 90.8.7. Hence (1) holds.

If the assumption of (2) holds, then by Lemma 90.8.8 the morphism $f : U \to \mathcal{F}$ is essentially surjective. Hence by Lemma 90.25.3 the morphism $[f] : [U/R] \to \mathcal{F}$ is an equivalence. □

06L6 **Lemma 90.26.2.** *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a deformation functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Then $U \times_{f,\mathcal{F},f} U$ is a deformation functor with tangent space fitting into an exact sequence of k-vector spaces*

$$0 \to Inf(\mathcal{F}) \to T(U \times_{f,\mathcal{F},f} U) \to TU \oplus TU$$

Proof. Follows from Lemma 90.20.1 and the fact that $\mathrm{Inf}(U) = (0)$. □

06L7 **Lemma 90.26.3.** *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. If $\dim_k Inf(\mathcal{F}) < \infty$, then (U, R, s, t, c) is prorepresentable.*

Proof. Note that U is a deformation functor by Example 90.16.10. By Lemma 90.26.2 we see that $R = U \times_{f,\mathcal{F},f} U$ is a deformation functor whose tangent space $TR = T(U \times_{f,\mathcal{F},f} U)$ sits in an exact sequence $0 \to \mathrm{Inf}(\mathcal{F}) \to TR \to TU \oplus TU$. Since we have assumed the first space has finite dimension and since TU has finite dimension by Example 90.11.11 we see that $\dim TR < \infty$. The map $\gamma : \mathrm{Der}_\Lambda(k, k) \to TR$ see (90.12.6.1) is injective because its composition with $TR \to TU$ is injective by Theorem 90.18.2 for the prorepresentable functor U. Thus R is prorepresentable by Theorem 90.18.2. It follows from Lemma 90.22.4 that (U, R, s, t, c) is prorepresentable. □

06L8 **Theorem 90.26.4.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ if and only if the following conditions hold:*

(1) *$\mathcal{F}$ is a deformation category.*
(2) *$\dim_k T\mathcal{F}$ is finite.*
(3) *$\dim_k Inf(\mathcal{F})$ is finite.*

Proof. Recall that a prorepresentable functor is a deformation functor, see Example 90.16.10. Thus if $\mathcal{F}$ is equivalent to a smooth prorepresentable groupoid in functors, then conditions (1), (2), and (3) follow from Lemma 90.24.2 (1), (2), and (3).

Conversely, assume conditions (1), (2), and (3) hold. Condition (1) implies that (S1) and (S2) are satisfied, see Lemma 90.16.6. By Lemma 90.13.4 there exists a versal formal object ξ. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 90.25.2 from the map $\underline{\xi}$. By Lemma 90.26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 90.26.3 we see that (U, R, s, t, c) is prorepresentable. Hence $[U/R] \to \mathcal{F}$ is the desired presentation of $\mathcal{F}$. □

90.27. Remarks regarding minimality

06TD The main theorem of this chapter is Theorem 90.26.4 above. It describes completely those categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors. In this section we briefly discuss how the minimality discussed in Sections 90.14 and 90.15 can be used to obtain a "minimal" smooth prorepresentable presentation.

06KM **Definition 90.27.1.** Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) We say (U, R, s, t, c) is *normalized* if the groupoid $(U(k[\epsilon]), R(k[\epsilon]), s, t, c)$ is totally disconnected, i.e., there are no morphisms between distinct objects.
(2) We say (U, R, s, t, c) is *minimal* if the $U \to [U/R]$ is given by a minimal versal formal object of $[U/R]$.

The difference between the two notions is related to the difference between conditions (90.15.0.1) and (90.15.0.2) and disappears when $k' \subset k$ is separable. Also a normalized smooth prorepresentable groupoid in functors is minimal as the following lemma shows. Here is a precise statement.

06KN **Lemma 90.27.2.** *Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is normalized if and only if the morphism $U \to [U/R]$ induces an isomorphism on tangent spaces, and*
(2) *(U, R, s, t, c) is minimal if and only if the kernel of $TU \to T[U/R]$ is contained in the image of $Der_\Lambda(k, k) \to TU$.*

Proof. Part (1) follows immediately from the definitions. To see part (2) set $\mathcal{F} = [U/R]$. Since $\mathcal{F}$ has a presentation it is a deformation category, see Theorem 90.26.4. In particular it satisfies (RS), (S1), and (S2), see Lemma 90.16.6. Recall that minimal versal formal objects are unique up to isomorphism, see Lemma 90.14.5.

By Theorem 90.15.5 a minimal versal object induces a map $\xi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ satisfying (90.15.0.2). Since $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ over $\mathcal{F}$ we see that $TU \to T\mathcal{F} = T[U/R]$ satisfies the property as stated in the lemma. □

The quotient of a minimal prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ does not admit autoequivalences which are not automorphisms. To prove this, we first note the following lemma.

06KP **Lemma 90.27.3.** *Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $\varphi : U \to U$ be a morphism such that $d\varphi : TU \to TU$ is an isomorphism. Then φ is an isomorphism.*

Proof. If $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$, then completing φ gives a morphism $\underline{R} \to \underline{R}$. If $f : R \to R$ is the corresponding morphism in $\widehat{\mathcal{C}}_\Lambda$, then f induces an isomorphism $\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R, k)$, see Example 90.11.14. In particular f is a surjection by Lemma 90.4.6. As a surjective endomorphism of a Noetherian ring is an isomorphism (see Algebra, Lemma 10.31.10) we conclude f, hence $\underline{R} \to \underline{R}$, hence $\varphi : U \to U$ is an isomorphism. □

06KQ **Lemma 90.27.4.** *Let (U, R, s, t, c) be a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U/R]$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. A morphism $\varphi : [U/R] \to [U/R]$ is the same thing as a morphism $\varphi : (U, R, s, t, c) \to (U, R, s, t, c)$ of groupoids in functors over $\mathcal{C}_\Lambda$ as defined in Definition 90.21.1. Denote $\phi : U \to U$ and $\psi : R \to R$ the corresponding morphisms. Because the diagram

$$\begin{array}{ccc} & \mathrm{Der}_\Lambda(k,k) & \\ {\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle\gamma} \\ TU & \xrightarrow[d\phi]{} & TU \\ \downarrow & & \downarrow \\ T[U/R] & \xrightarrow{d\varphi} & T[U/R] \end{array}$$

is commutative, since $d\varphi$ is bijective, and since we have the characterization of minimality in Lemma 90.27.2 we conclude that $d\phi$ is injective (hence bijective by dimension reasons). Thus $\phi : U \to U$ is an isomorphism by Lemma 90.27.3. We can use a similar argument, using the exact sequence

$$0 \to \mathrm{Inf}([U/R]) \to TR \to TU \oplus TU$$

of Lemma 90.26.2 to prove that $\psi : R \to R$ is an isomorphism. But is also a consequence of the fact that $R = U \times_{[U/R]} U$ and that φ and ϕ are isomorphisms. □

06KR **Lemma 90.27.5.** *Let (U, R, s, t, c) and (U', R', s', t', c') be minimal smooth prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U'/R']$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. Let $\psi : [U'/R'] \to [U/R]$ be a quasi-inverse to φ. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms by Lemma 90.27.4, hence φ and ψ are isomorphisms. □

The following lemma summarizes some of the things we have seen earlier in this chapter.

06L2 **Lemma 90.27.6.** *Let $\mathcal{F}$ be a deformation category such that $\dim_k T\mathcal{F} < \infty$ and $\dim_k Inf(\mathcal{F}) < \infty$. Then there exists a minimal versal formal object ξ of $\mathcal{F}$. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Let $U = \underline{R}|_{\mathcal{C}_\Lambda}$. Let $f = \underline{\xi} : U \to \mathcal{F}$ be the associated morphism. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 90.25.2. Then (U, R, s, t, c) is a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ and there is an equivalence $[U/R] \to \mathcal{F}$.*

Proof. As $\mathcal{F}$ is a deformation category it satisfies (S1) and (S2), see Lemma 90.16.6. By Lemma 90.13.4 there exists a versal formal object. By Lemma 90.14.5 there exists a minimal versal formal object ξ/R as in the statement of the lemma. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 90.25.2 from the map $\underline{\xi}$. By Lemma 90.26.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 90.26.3 we see that (U, R, s, t, c) is prorepresentable. Finally, (U, R, s, t, c) is minimal because $U \to [U/R] = \mathcal{F}$ corresponds to the minimal versal formal object ξ. □

Presentations by minimal prorepresentable groupoids in functors satisfy the following uniqueness property.

06L3 **Lemma 90.27.7.** *Let $\mathcal{F}$ be category cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume there exist presentations of $\mathcal{F}$ by minimal smooth prorepresentable groupoids in functors (U, R, s, t, c) and (U', R', s', t', c'). Then (U, R, s, t, c) and (U', R', s', t', c') are isomorphic.*

Proof. Follows from Lemma 90.27.5 and the observation that a morphism $[U/R] \to [U'/R']$ is the same thing as a morphism of groupoids in functors (by our explicit construction of $[U/R]$ in Definition 90.21.9). □

In summary we have proved the following theorem.

06TE **Theorem 90.27.8.** *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Consider the following conditions*

1. *$\mathcal{F}$ admits a presentation by a normalized smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
2. *$\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
3. *$\mathcal{F}$ admits a presentation by a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, and*
4. *$\mathcal{F}$ satisfies the following conditions*
   (a) *$\mathcal{F}$ is a deformation category.*
   (b) *$\dim_k T\mathcal{F}$ is finite.*
   (c) *$\dim_k Inf(\mathcal{F})$ is finite.*

Then (2), (3), (4) are equivalent and are implied by (1). If $k' \subset k$ is separable, then (1), (2), (3), (4) are all equivalent. Furthermore, the minimal smooth prorepresentable groupoids in functors which provide a presentation of $\mathcal{F}$ are unique up to isomorphism.

Proof. We see that (1) implies (3) and is equivalent to (3) if $k' \subset k$ is separable from Lemma 90.27.2. It is clear that (3) implies (2). We see that (2) implies (4) by

Theorem 90.26.4. We see that (4) implies (3) by Lemma 90.27.6. This proves all the implications. The final uniqueness statement follows from Lemma 90.27.7. □

90.28. Uniqueness of versal rings

0DQA Given R, S in $\widehat{\mathcal{C}}_\Lambda$ we say maps $f, g : R \to S$ are *formally homotopic* if there exists an $r \geq 0$ and maps $h : R \to R[[t_1, \ldots, t_r]]$ and $k : R[[t_1, \ldots, t_r]] \to S$ in $\widehat{\mathcal{C}}_\Lambda$ such that for all $a \in R$ we have

(1) $h(a) \bmod (t_1, \ldots, t_r) = a$,
(2) $f(a) = k(a)$,
(3) $g(a) = k(h(a))$.

We will say (r, h, k) is a *formal homotopy* between f and g.

0DQB **Lemma 90.28.1.** *Being formally homotopic is an equivalence relation on sets of morphisms in $\widehat{\mathcal{C}}_\Lambda$.*

Proof. Suppose we have any $r \geq 1$ and two maps $h_1, h_2 : R \to R[[t_1, \ldots, t_r]]$ such that $h_1(a) \bmod (t_1, \ldots, t_r) = h_2(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$ and a map $k : R[[t_1, \ldots, t_r]] \to S$. Then we claim $k \circ h_1$ is formally homotopic to $k \circ h_2$. The symmetric inherent in this claim will show that our notion of formally homotopic is symmetric. Namely, the map

$$\Psi : R[[t_1, \ldots, t_r]] \longrightarrow R[[t_1, \ldots, t_r]], \quad \sum a_I t^I \longmapsto \sum h_1(a_I) t^I$$

is an isomorphism. Set $h(a) = \Psi^{-1}(h_2(a))$ for $a \in R$ and $k' = k \circ \Psi$, then we see that (r, h, k') is a formal homotopy between $k \circ h_1$ and $k \circ h_2$, proving the claim

Say we have three maps $f_1, f_2, f_3 : R \to S$ as above and a formal homotopy (r_1, h_1, k_1) between f_1 and f_2 and a formal homotopy (r_2, h_2, k_2) between f_3 and f_2 (!). After relabeling the coordinates we may assume $h_2 : R \to R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]]$ and $k_2 : R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \to S$. By choosing a suitable isomorphism

$$R[[t_1, \ldots, t_{r_1+r_2}]] \longrightarrow R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] \widehat{\otimes}_{h_2, R, h_1} R[[t_1, \ldots, t_{r_1}]]$$

we may fit these maps into a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow[h_1]{} & R[[t_1, \ldots, t_{r_1}]] \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_2'} \\ R[[t_{r_1+1}, \ldots, t_{r_1+r_2}]] & \xrightarrow{h_1'} & R[[t_1, \ldots, t_{r_1+r_2}]] \end{array}$$

with $h_2'(t_i) = t_i$ for $1 \leq i \leq r_1$ and $h_1'(t_i) = t_i$ for $r_1 + 1 \leq i \leq r_2$. Some details omitted. Since this diagram is a pushout in the category $\widehat{\mathcal{C}}_\Lambda$ (see proof of Lemma 90.4.3) and since $k_1 \circ h_1 = f_2 = k_2 \circ h_2$ we conclude there exists a map

$$k : R[[t_1, \ldots, t_{r_1+r_2}]] \to S$$

with $k_1 = k \circ h_2'$ and $k_2 = k \circ h_1'$. Denote $h = h_1' \circ h_2 = h_2' \circ h_1$. Then we have

(1) $k(h_1'(a)) = k_2(a) = f_3(a)$, and
(2) $k(h_2'(a)) = k_1(a) = f_1(a)$.

By the claim in the first paragraph of the proof this shows that f_1 and f_3 are formally homotopic. □

0DQC **Lemma 90.28.2.** *In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $g : S \to S'$ is a morphism, then $g \circ f_1$ and $g \circ f_2$ are formally homotopic.*

Proof. Namely, if (r, h, k) is a formal homotopy between f_1 and f_2, then $(r, h, g \circ k)$ is a formal homotopy between $g \circ f_1$ and $g \circ f_2$. □

0DQD **Lemma 90.28.3.** *Let $\mathcal{F}$ be a deformation category over $\mathcal{C}_\Lambda$ with $\dim_k T\mathcal{F} < \infty$ and $\dim_k Inf(\mathcal{F}) < \infty$. Let ξ be a versal formal object lying over R. Let η be a formal object lying over S. Then any two maps*

$$f, g : R \to S$$

such that $f_*\xi \cong \eta \cong g_*\xi$ are formally homotopic.

Proof. By Theorem 90.26.4 and its proof, $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid

$$(\underline{R}, \underline{R_1}, s, t, c, e, i)|_{\mathcal{C}_\Lambda}$$

in functors on $\mathcal{C}_\lambda$ such that $\mathcal{F}$. Then the maps $s : R \to R_1$ and $t : R \to R_1$ are formally smooth ring maps and $e : R_1 \to R$ is a section. In particular, we can choose an isomorphism $R_1 = R[[t_1, \ldots, t_r]]$ for some $r \geq 0$ such that s is the embedding $R \subset R[[t_1, \ldots, t_r]]$ and t corresponds to a map $h : R \to R[[t_1, \ldots, t_r]]$ with $h(a) \bmod (t_1, \ldots, t_r) = a$ for all $a \in R$. The existence of the isomorphism $\alpha : f_*\xi \to g_*\xi$ means exactly that there is a map $k : R_1 \to S$ such that $f = k \circ s$ and $g = k \circ t$. This exactly means that (r, h, k) is a formal homotopy between f and g. □

0DQE **Lemma 90.28.4.** *In the category $\widehat{\mathcal{C}}_\Lambda$, if $f_1, f_2 : R \to S$ are formally homotopic and $\mathfrak{p} \subset R$ is a minimal prime ideal, then $f_1(\mathfrak{p})S = f_2(\mathfrak{p})S$ as ideals.*

Proof. Suppose (r, h, k) is a formal homotopy between f_1 and f_2. We claim that $\mathfrak{p}R[[t_1, \ldots, t_r]] = h(\mathfrak{p})R[[t_1, \ldots, t_r]]$. The claim implies the lemma by further composing with k. To prove the claim, observe that the map $\mathfrak{p} \mapsto \mathfrak{p}R[[t_1, \ldots, t_r]]$ is a bijection between the minimal prime ideals of R and the minimal prime ideals of $R[[t_1, \ldots, t_r]]$. Finally, $h(\mathfrak{p})R[[t_1, \ldots, t_r]]$ is a minimal prime as h is flat, and hence of the form $\mathfrak{q}R[[t_1, \ldots, t_r]]$ for some minimal prime $\mathfrak{q} \subset R$ by what we just said. But since $h \bmod (t_1, \ldots, t_r) = \mathrm{id}_R$ by definition of a formal homotopy, we conclude that $\mathfrak{q} = \mathfrak{p}$ as desired. □

90.29. Change of residue field

07W7 In this section we quickly discuss what happens if we replace the residue field k by a finite extension. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Throughout this whole chapter we have used $\mathcal{C}_\Lambda$ to denote the category of Artinian local Λ-algebras whose residue field is identified with k, see Definition 90.3.1. However, since in this section we will discuss what happen when we change k we will instead use the notation $\mathcal{C}_{\Lambda,k}$ to indicate the dependence on k.

07W8 **Situation 90.29.1.** Let Λ be a Noetherian ring and let $\Lambda \to k \to l$ be a finite ring maps where k and l are fields. Thus l/k is a finite extensions of fields. A typical object of $\mathcal{C}_{\Lambda,l}$ will be denoted B and a typical object of $\mathcal{C}_{\Lambda,k}$ will be denoted A. We define

07W9 (90.29.1.1) $$\mathcal{C}_{\Lambda,l} \longrightarrow \mathcal{C}_{\Lambda,k}, \quad B \longmapsto B \times_l k$$

Given a category cofibred in groupoids $p : \mathcal{F} \to \mathcal{C}_{\Lambda,k}$ we obtain an associated category cofibred in groupoids

$$p_{l/k} : \mathcal{F}_{l/k} \longrightarrow \mathcal{C}_{\Lambda,l}$$

by setting $\mathcal{F}_{l/k}(B) = \mathcal{F}(B \times_l k)$.

The functor (90.29.1.1) makes sense: because $B \times_l k \subset B$ we have

$$\begin{aligned}[k : k'] \operatorname{length}_{B \times_l k}(B \times_l k) &= \operatorname{length}_\Lambda(B \times_l k) \\ &\leq \operatorname{length}_\Lambda(B) \\ &= [l : k'] \operatorname{length}_B(B) < \infty\end{aligned}$$

(see Lemma 90.3.4) hence $B \times_l k$ is Artinian (see Algebra, Lemma 10.53.6). Thus $B \times_l k$ is an Artinian local ring with residue field k. Note that (90.29.1.1) commutes with fibre products

$$(B_1 \times_B B_2) \times_l k = (B_1 \times_l k) \times_{(B \times_l k)} (B_2 \times_l k)$$

and transforms surjective ring maps into surjective ring maps. We use the "expensive" notation $\mathcal{F}_{l/k}$ to prevent confusion with the construction of Remark 90.6.4. Here are some elementary observations.

07WA **Lemma 90.29.2.** *With notation and assumptions as in Situation 90.29.1.*

(1) *We have* $\overline{\mathcal{F}_{l/k}} = (\overline{\mathcal{F}})_{l/k}$.
(2) *If* $\mathcal{F}$ *is a predeformation category, then* $\mathcal{F}_{l/k}$ *is a predeformation category.*
(3) *If* $\mathcal{F}$ *satisfies (S1), then* $\mathcal{F}_{l/k}$ *satisfies (S1).*
(4) *If* $\mathcal{F}$ *satisfies (S2), then* $\mathcal{F}_{l/k}$ *satisfies (S2).*
(5) *If* $\mathcal{F}$ *satisfies (RS), then* $\mathcal{F}_{l/k}$ *satisfies (RS).*

Proof. Part (1) is immediate from the definitions.

Since $\mathcal{F}_{l/k}(l) = \mathcal{F}(k)$ part (2) follows from the definition, see Definition 90.6.2.

Part (3) follows as the functor (90.29.1.1) commutes with fibre products and transforms surjective maps into surjective maps, see Definition 90.10.1.

Part (4). To see this consider a diagram

$$\begin{array}{ccc} & & l[\epsilon] \\ & & \downarrow \\ B & \longrightarrow & l \end{array}$$

in $\mathcal{C}_{\Lambda,l}$ as in Definition 90.10.1. Applying the functor (90.29.1.1) we obtain

$$\begin{array}{ccc} & & k[l\epsilon] \\ & & \downarrow \\ B \times_l k & \longrightarrow & k \end{array}$$

where $l\epsilon$ denotes the finite dimensional k-vector space $l\epsilon \subset l[\epsilon]$. According to Lemma 90.10.4 the condition of (S2) for $\mathcal{F}$ also holds for this diagram. Hence (S2) holds for $\mathcal{F}_{l/k}$.

Part (5) follows from the characterization of (RS) in Lemma 90.16.4 part (2) and the fact that (90.29.1.1) commutes with fibre products. □

The following lemma applies in particular when $\mathcal{F}$ satisfies (S2) and is a predeformation category, see Lemma 90.10.5.

07WB **Lemma 90.29.3.** *With notation and assumptions as in Situation 90.29.1. Assume $\mathcal{F}$ is a predeformation category and $\overline{\mathcal{F}}$ satisfies (S2). Then there is a canonical l-vector space isomorphism*

$$T\mathcal{F} \otimes_k l \longrightarrow T\mathcal{F}_{l/k}$$

of tangent spaces.

Proof. By Lemma 90.29.2 we may replace $\mathcal{F}$ by $\overline{\mathcal{F}}$. Moreover we see that $T\mathcal{F}$, resp. $T\mathcal{F}_{l/k}$ has a canonical k-vector space structure, resp. l-vector space structure, see Lemma 90.12.2. Then

$$T\mathcal{F}_{l/k} = \mathcal{F}_{l/k}(l[\epsilon]) = \mathcal{F}(k[l\epsilon]) = T\mathcal{F} \otimes_k l$$

the last equality by Lemma 90.12.2. More generally, given a finite dimensional l-vector space V we have

$$\mathcal{F}_{l/k}(l[V]) = \mathcal{F}(k[V_k]) = T\mathcal{F} \otimes_k V_k$$

where V_k denotes V seen as a k-vector space. We conclude that the functors $V \mapsto \mathcal{F}_{l/k}(l[V])$ and $V \mapsto T\mathcal{F} \otimes_k V_k$ are canonically identified as functors to the category of sets. By Lemma 90.11.4 we see there is at most one way to turn either functor into an l-linear functor. Hence the isomorphisms are compatible with the l-vector space structures and we win. □

07WC **Lemma 90.29.4.** *With notation and assumptions as in Situation 90.29.1. Assume $\mathcal{F}$ is a deformation category. Then there is a canonical l-vector space isomorphism*

$$Inf(\mathcal{F}) \otimes_k l \longrightarrow Inf(\mathcal{F}_{l/k})$$

of infinitesimal automorphism spaces.

Proof. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and denote $x_{l,0}$ the corresponding object of $\mathcal{F}_{l/k}$ over l. Recall that $\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_{x_0}(\mathcal{F})$ and $\mathrm{Inf}(\mathcal{F}_{l/k}) = \mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$, see Remark 90.19.4. Recall that the vector space structure on $\mathrm{Inf}_{x_0}(\mathcal{F})$ comes from identifying it with the tangent space of the functor $Aut(x_0)$ which is defined on the category $\mathcal{C}_{k,k}$ of Artinian local k-algebras with residue field k. Similarly, $\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$ is the tangent space of $Aut(x_{l,0})$ which is defined on the category $\mathcal{C}_{l,l}$ of Artinian local l-algebras with residue field l. Unwinding the definitions we see that $Aut(x_{l,0})$ is the restriction of $Aut(x_0)_{l/k}$ (which lives on $\mathcal{C}_{k,l}$) to $\mathcal{C}_{l,l}$. Since there is no difference between the tangent space of $Aut(x_0)_{l/k}$ seen as a functor on $\mathcal{C}_{k,l}$ or $\mathcal{C}_{l,l}$, the lemma follows from Lemma 90.29.3 and the fact that $Aut(x_0)$ satisfies (RS) by Lemma 90.19.6 (whence we have (S2) by Lemma 90.16.6). □

07WD **Lemma 90.29.5.** *With notation and assumptions as in Situation 90.29.1. If $\mathcal{F} \to \mathcal{G}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,k}$, then $\mathcal{F}_{l/k} \to \mathcal{G}_{l/k}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$.*

Proof. This follows immediately from the definitions and the fact that (90.29.1.1) preserves surjections. □

There are many more things you can say about the relationship between $\mathcal{F}$ and $\mathcal{F}_{l/k}$ (in particular about the relationship between versal deformations) and we will add these here as needed.

0DQF **Lemma 90.29.6.** *With notation and assumptions as in Situation 90.29.1. Let ξ be a versal formal object for $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_{\Lambda,k})$. Then there exist*

(1) *an* $S \in \mathrm{Ob}(\widehat{\mathcal{C}}_{\Lambda,l})$ *and a local* Λ*-algebra homomorphism* $R \to S$ *which is formally smooth in the* $\mathfrak{m}_S$*-adic topology and induces the given field extension* l/k *on residue fields, and*
(2) *a versal formal object of* $\mathcal{F}_{l/k}$ *lying over* S*.*

Proof. Construction of S. Choose a surjection $R[x_1, \ldots, x_n] \to l$ of R-algebras. The kernel is a maximal ideal $\mathfrak{m}$. Set S equal to the $\mathfrak{m}$-adic completion of the Noetherian ring $R[x_1, \ldots, x_n]$. Then S is in $\widehat{\mathcal{C}}_{\Lambda,l}$ by Algebra, Lemma 10.97.6. The map $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by More on Algebra, Lemmas 15.37.2 and 15.37.4 and the fact that $R \to R[x_1, \ldots, x_n]$ is formally smooth. (Compare with the proof Lemma 90.9.5.)

Since ξ is versal, the transformation $\underline{\xi} : \underline{R}|_{\mathcal{C}_{\Lambda,k}} \to \mathcal{F}$ is smooth. By Lemma 90.29.5 the induced map

$$(\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k} \longrightarrow \mathcal{F}_{l/k}$$

is smooth. Thus it suffices to construct a smooth morphism $\underline{S}|_{\mathcal{C}_{\Lambda,l}} \to (\underline{R}|_{\mathcal{C}_{\Lambda,k}})_{l/k}$. To give such a map means for every object B of $\mathcal{C}_{\Lambda,l}$ a map of sets

$$\mathrm{Mor}_{\widehat{\mathcal{C}}_{\Lambda,l}}(S, B) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_{\Lambda,k}}(R, B \times_l k)$$

functorial in B. Given an element $\varphi : S \to B$ on the left hand side we send it to the composition $R \to S \to B$ whose image is contained in the sub Λ-algebra $B \times_l k$. Smoothness of the map means that given a surjection $B' \to B$ and a commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & B & = & B \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & B' \times_l k & \longrightarrow & B' \end{array}$$

we have to find a ring map $S \to B'$ fitting into the outer rectangle. The existence of this map is guaranteed as we chose $R \to S$ to be formally smooth in the $\mathfrak{m}_S$-adic topology, see More on Algebra, Lemma 15.37.5. □

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CHAPTER 91

Deformation Theory

08KW

91.1. Introduction

08KX The goal of this chapter is to give a (relatively) gentle introduction to deformation theory of modules, morphisms, etc. In this chapter we deal with those results that can be proven using the naive cotangent complex. In the chapter on the cotangent complex we will extend these results a little bit. The advanced reader may wish to consult the treatise by Illusie on this subject, see [**Ill72**].

91.2. Deformations of rings and the naive cotangent complex

08S3 In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08S4 (91.2.0.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (91.2.0.1).

08S5 **Lemma 91.2.1.** *Given a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & N_2 & \longrightarrow & B'_2 & \longrightarrow & B_2 & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle c_2} & & \uparrow & & \uparrow & & \\ & & 0 & \longrightarrow & I_2 & \longrightarrow & A'_2 & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \longrightarrow & B_1 & \longrightarrow & 0 & & \\ & & \uparrow{\scriptstyle c_1} & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & & & \\ 0 & \longrightarrow & I_1 & \longrightarrow & A'_1 & \longrightarrow & A_1 & \longrightarrow & 0 & & \end{array}$$

with front and back solutions to (91.2.0.1) we have

(1) *There exist a canonical element in* $\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2)$ *whose vanishing is a necessary and sufficient condition for the existence of a ring map* $B_1' \to B_2'$ *fitting into the diagram.*
(2) *If there exists a map* $B_1' \to B_2'$ *fitting into the diagram the set of all such maps is a principal homogeneous space under* $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.

Proof. Let $E = B_1$ viewed as a set. Consider the surjection $A_1[E] \to B_1$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B_1/A_1} = (J/J^2 \to \Omega_{A_1[E]/A_1} \otimes_{A_1[E]} B_1)$$

in Algebra, Section 10.134. Since $\Omega_{A_1[E]/A_1} \otimes B_1$ is a free B_1-module we have

$$\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2) = \frac{\mathrm{Hom}_{B_1}(J/J^2, N_2)}{\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1} \otimes B_1, N_2)}$$

We will construct an obstruction in the module on the right. Let $J' = \mathrm{Ker}(A_1'[E] \to B_1)$. Note that there is a surjection $J' \to J$ whose kernel is $I_1 A_1'[E]$. For every $e \in E$ denote $x_e \in A_1[E]$ the corresponding variable. Choose a lift $y_e \in B_1'$ of the image of x_e in B_1 and a lift $z_e \in B_2'$ of the image of x_e in B_2. These choices determine A_1'-algebra maps

$$A_1'[E] \to B_1' \quad \text{and} \quad A_1'[E] \to B_2'$$

The first of these gives a map $J' \to N_1$, $f' \mapsto f'(y_e)$ and the second gives a map $J' \to N_2$, $f' \mapsto f'(z_e)$. A calculation shows that these maps annihilate $(J')^2$. Because the left square of the diagram (involving c_1 and c_2) commutes we see that these maps agree on $I_1 A_1'[E]$ as maps into N_2. Observe that B_1' is the pushout of $J' \to A_1'[E]$ and $J' \to N_1$. Thus, if the maps $J' \to N_1 \to N_2$ and $J' \to N_2$ agree, then we obtain a map $B_1' \to B_2'$ fitting into the diagram. Thus we let the obstruction be the class of the map

$$J/J^2 \to N_2, \quad f \mapsto f'(z_e) - \nu(f'(y_e))$$

where $\nu : N_1 \to N_2$ is the given map and where $f' \in J'$ is a lift of f. This is well defined by our remarks above. Note that we have the freedom to modify our choices of z_e into $z_e + \delta_{2,e}$ and y_e into $y_e + \delta_{1,e}$ for some $\delta_{i,e} \in N_i$. This will modify the map above into

$$f \mapsto f'(z_e + \delta_{2,e}) - \nu(f'(y_e + \delta_{1,e})) = f'(z_e) - \nu(f'(z_e)) + \sum (\delta_{2,e} - \nu(\delta_{1,e})) \frac{\partial f}{\partial x_e}$$

This means exactly that we are modifying the map $J/J^2 \to N_2$ by the composition $J/J^2 \to \Omega_{A_1[E]/A_1} \otimes B_1 \to N_2$ where the second map sends $\mathrm{d}x_e$ to $\delta_{2,e} - \nu(\delta_{1,e})$. Thus our obstruction is well defined and is zero if and only if a lift exists.

Part (2) comes from the observation that given two maps $\varphi, \psi : B_1' \to B_2'$ fitting into the diagram, then $\varphi - \psi$ factors through a map $D : B_1 \to N_2$ which is an A_1-derivation:

$$\begin{aligned} D(fg) &= \varphi(f'g') - \psi(f'g') \\ &= \varphi(f')\varphi(g') - \psi(f')\psi(g') \\ &= (\varphi(f') - \psi(f'))\varphi(g') + \psi(f')(\varphi(g') - \psi(g')) \\ &= gD(f) + fD(g) \end{aligned}$$

Thus D corresponds to a unique B_1-linear map $\Omega_{B_1/A_1} \to N_2$. Conversely, given such a linear map we get a derivation D and given a ring map $\psi : B_1' \to B_2'$ fitting into the diagram the map $\psi + D$ is another ring map fitting into the diagram. $\square$

08S7 **Lemma 91.2.2.** *If there exists a solution to (91.2.0.1), then the set of isomorphism classes of solutions is principal homogeneous under* $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. We observe right away that given two solutions B_1' and B_2' to (91.2.0.1) we obtain by Lemma 91.2.1 an obstruction element $o(B_1', B_2') \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ to the existence of a map $B_1' \to B_2'$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution B' and an element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ we can find a second solution B'_ξ such that $o(B', B'_\xi) = \xi$.

Let $E = B$ viewed as a set. Consider the surjection $A[E] \to B$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B/A} = (J/J^2 \to \Omega_{A[E]/A} \otimes_{A[E]} B)$$

in Algebra, Section 10.134. Since $\Omega_{A[E]/A} \otimes B$ is a free B-module we have

$$\mathrm{Ext}^1_B(NL_{B/A}, N) = \frac{\mathrm{Hom}_B(J/J^2, N)}{\mathrm{Hom}_B(\Omega_{A[E]/A} \otimes B, N)}$$

Thus we may represent ξ as the class of a morphism $\delta : J/J^2 \to N$.

For every $e \in E$ denote $x_e \in A[E]$ the corresponding variable. Choose a lift $y_e \in B'$ of the image of x_e in B. These choices determine an A'-algebra map $\varphi : A'[E] \to B'$. Let $J' = \mathrm{Ker}(A'[E] \to B)$. Observe that φ induces a map $\varphi|_{J'} : J' \to N$ and that B' is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{J'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & J' & \longrightarrow & A'[E] & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Let $\psi : J' \to N$ be the sum of the map $\varphi|_{J'}$ and the composition

$$J' \to J'/(J')^2 \to J/J^2 \xrightarrow{\delta} N.$$

Then the pushout along ψ is an other ring extension B'_ξ fitting into a diagram as above. A calculation shows that $o(B', B'_\xi) = \xi$ as desired. $\square$

0GPT **Lemma 91.2.3.** *Let A be a ring. Let B be an A-algebra. Let N be a B-module. The set of isomorphism classes of extensions of A-algebras*

$$0 \to N \to B' \to B \to 0$$

where N is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. To prove this we apply the previous results to the case where (91.2.0.1) is given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 \end{array}$$

Thus our lemma follows from Lemma 91.2.2 and the fact that there exists a solution, namely $N \oplus B$. (See remark below for a direct construction of the bijection.) $\square$

0GPU **Remark 91.2.4.** Let $A \to B$ and N be as in Lemma 91.2.3. Let $\alpha : P \to B$ be a presentation of B over A, see Algebra, Section 10.134. With $J = \mathrm{Ker}(\alpha)$ the naive cotangent complex $NL(\alpha)$ associated to α is the complex $J/J^2 \to \Omega_{P/A} \otimes_P B$. We have

$$\mathrm{Ext}^1_B(NL(\alpha), N) = \mathrm{Coker}\left(\mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, N) \to \mathrm{Hom}_B(J/J^2, N)\right)$$

because $\Omega_{P/A}$ is a free module. Consider a extension $0 \to N \to B' \to B \to 0$ as in the lemma. Since P is a polynomial algebra over A we can lift α to an A-algebra map $\alpha' : P' \to B'$. Then $\alpha'|_J : J \to N$ factors as $J \to J/J^2 \to N$ as N has square zero in B'. The lemma sends our extension to the class of this map $J/J^2 \to N$ in the displayed cokernel.

0GPV **Lemma 91.2.5.** *Given ring maps $A \to B \to C$, a B-module M, a C-module N, a B-linear map $c : M \to N$, and extensions of A-algebras with square zero kernels*

(a) $0 \to M \to B' \to B \to 0$ *corresponding to* $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, *and*
(b) $0 \to N \to C' \to C \to 0$ *corresponding to* $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$.

See Lemma 91.2.3. Then there is an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_B(NL_{B/A}, N)$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

using the map $M \to N$ and

$$\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$$

where the first arrows uses the restriction map $D(C) \to D(B)$ and the second arrow uses the canonical map of complexes $NL_{B/A} \to NL_{C/A}$. The statement of the lemma can be deduced from Lemma 91.2.1 applied to the diagram

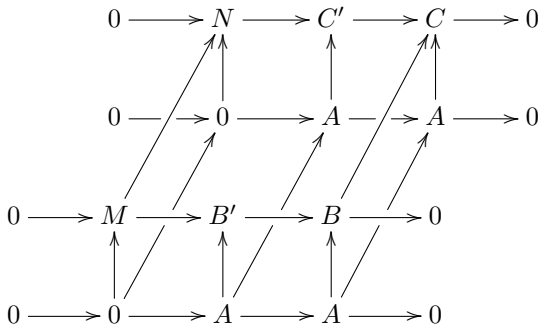

and a compatibility between the constructions in the proofs of Lemmas 91.2.3 and 91.2.1 whose statement and proof we omit. (See remark below for a direct argument.) $\square$

0GPW **Remark 91.2.6.** Let $A \to B \to C$, M, N, $c : M \to N$, $0 \to M \to B' \to B \to 0$, $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, M)$, $0 \to N \to C' \to C \to 0$, and $\zeta \in \mathrm{Ext}^1_C(NL_{C/A}, N)$ be as in

Lemma 91.2.5. Using pushout along $c : M \to N$ we can construct an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & B'_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

by setting $B'_1 = (N \times B')/M$ where M is antidiagonally embedded. Using pullback along $B \to C$ we can construct an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & B'_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

by setting $B'_2 = C' \times_C B$ (fibre product of rings). A simple diagram chase tells us that there exists an A-algebra map $B' \to C'$ compatible with $B \to C$ and c if and only if B'_1 is isomorphic to B'_2 as A-algebra extensions of B by N. Thus to see Lemma 91.2.5 is true, it suffices to show that B'_1 corresponds via the bijection of Lemma 91.2.3 to the image of ξ by the map $\mathrm{Ext}^1_B(NL_{B/A}, M) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$ and that B'_2 correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$. The first of these two statements is immediate from the construction of the class in Remark 91.2.4. For the second, choose a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow[\beta]{} & C \\ {\scriptstyle \varphi}\uparrow & & \uparrow \\ P & \xrightarrow{\alpha} & B \end{array}$$

of A-algebras, such that α is a presentation of B over A and β is a presentation of C over A. See Remark 91.2.4 and references therein. Set $J = \mathrm{Ker}(\alpha)$ and $K = \mathrm{Ker}(\beta)$. The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ and in particular $\bar{\varphi} : J/J^2 \to K/K^2$. Choose A-algebra homomorphism $\beta' : Q \to C'$ which is a lift of β. Then $\alpha' = (\beta' \circ \varphi, \alpha) : P \to B'_2 = C' \times_C B$ is a lift of α. With these choices the composition of the map $K/K^2 \to N$ induced by β' and the map $\bar{\varphi} : J/J^2 \to K/K^2$ is the restriction of α' to J/J^2. Unwinding the constructions of our classes in Remark 91.2.4 this indeed shows that B'_2 correspond to the image of ζ by the map $\mathrm{Ext}^1_C(NL_{C/A}, N) \to \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0GPX **Lemma 91.2.7.** *Let $0 \to I \to A' \to A \to 0$, $A \to B$, and $c : I \to N$ be as in (91.2.0.1). Denote $\xi \in \mathrm{Ext}^1_A(NL_{A/A'}, I)$ the element corresponding to the extension A' of A by I via Lemma 91.2.3. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$\mathrm{Ext}^1_B(NL_{B/A'}, N) \to \mathrm{Ext}^1_A(NL_{A/A'}, N)$$

over the image of ξ.

Proof. By Lemma 91.2.3 applied to $A' \to B$ and the B-module N we see that elements ζ of $\mathrm{Ext}^1_B(NL_{B/A'}, N)$ parametrize extensions $0 \to N \to B' \to B \to 0$ of A'-algebras. By Lemma 91.2.5 applied to $A' \to A \to B$ and $c : I \to N$ we see that there is an A'-algebra map $A' \to B'$ compatible with c and $A \to B$ if and only if ζ maps to ξ. Of course this is the same thing as saying B' is a solution of (91.2.0.1). □

0GPY **Remark 91.2.8.** Observe that in the situation of Lemma 91.2.7 we have

$$\mathrm{Ext}^1_A(NL_{A/A'}, N) = \mathrm{Ext}^1_B(NL_{A/A'} \otimes^{\mathbf{L}}_A B, N) = \mathrm{Ext}^1_B(NL_{A/A'} \otimes_A B, N)$$

The first equality by More on Algebra, Lemma 15.60.3 and the second by More on Algebra, Lemma 15.86.1. We have maps of complexes

$$NL_{A/A'} \otimes_A B \to NL_{B/A'} \to NL_{B/A}$$

which is close to being a distinguished triangle, see Algebra, Lemma 10.134.4. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_B(NL_{B/A}, N)$ would be the obstruction to the existence of a solution to (91.2.0.1).

If our ring map $A \to B$ is a local complete intersection, then there is a solution. This is a kind of lifting result; observe that for syntomic ring maps we have proved a rather strong lifting result in Smoothing Ring Maps, Proposition 16.3.2.

08S6 **Lemma 91.2.9.** *If $A \to B$ is a local complete intersection ring map, then there exists a solution to (91.2.0.1).*

First proof. Write $B = A[x_1, \ldots, x_n]/J$. By More on Algebra, Definition 15.33.2 the ideal J is Koszul-regular. This implies J is H_1-regular and quasi-regular, see More on Algebra, Section 15.32. Let $J' \subset A'[x_1, \ldots, x_n]$ be the inverse image of J. Denote $I[x_1, \ldots, x_n]$ the kernel of $A'[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$. By More on Algebra, Lemma 15.32.5 we have $I[x_1, \ldots, x_n] \cap (J')^2 = J'I[x_1, \ldots, x_n] = JI[x_1, \ldots, x_n]$. Hence we obtain a short exact sequence

$$0 \to I \otimes_A B \to J'/(J')^2 \to J/J^2 \to 0$$

Since J/J^2 is projective (More on Algebra, Lemma 15.32.3) we can choose a splitting of this sequence

$$J'/(J')^2 = I \otimes_A B \oplus J/J^2$$

Let $(J')^2 \subset J'' \subset J'$ be the elements which map to the second summand in the decomposition above. Then

$$0 \to I \otimes_A B \to A'[x_1, \ldots, x_n]/J'' \to B \to 0$$

is a solution to (91.2.0.1) with $N = I \otimes_A B$. The general case is obtained by doing a pushout along the given map $I \otimes_A B \to N$. □

Second proof. Please read Remark 91.2.8 before reading this proof. By More on Algebra, Lemma 15.33.6 the maps $NL_{A/A'} \otimes_A B \to NL_{B/A'} \to NL_{B/A}$ do form a distinguished triangle in $D(B)$. Hence it suffices to show that $\mathrm{Ext}^2_B(NL_{B/A}, N)$ vanishes. By More on Algebra, Lemma 15.86.4 the complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$. This implies our Ext^2 vanishes for example by More on Algebra, Lemma 15.77.1 part (1). □

91.3. Thickenings of ringed spaces

08KY In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ on a ringed space $(X', \mathcal{O}_{X'})$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent. Compare with Algebra, Item 29.

(2) A *thickening* of ringed spaces is a morphism $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that
   (a) i induces a homeomorphism $X \to X'$,

(b) the map $i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X$ is surjective, and
(c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.

(3) A *first order thickening* of ringed spaces is a thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define *morphisms of thickenings*, *morphisms of thickenings over a base ringed space*, etc.

If $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a thickening of ringed spaces then we identify the underlying topological spaces and think of $\mathcal{O}_X$, $\mathcal{O}_{X'}$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $X = X'$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. By Modules, Lemma 17.13.4 the category of $\mathcal{O}_X$-modules is equivalent to the category of $\mathcal{O}_{X'}$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}_X$-module.

08KZ **Situation 91.3.1.** A morphism of thickenings (f, f') is given by a commutative diagram

08L0 (91.3.1.1)
$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow[i]{} & (X', \mathcal{O}_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \mathcal{O}_S) & \xrightarrow{t} & (S', \mathcal{O}_{S'}) \end{array}$$

of ringed spaces whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. As $f = f'$ on underlying topological spaces we will identify the (topological) pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{S'} \to \mathcal{O}_{X'}$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}_{X'}$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

08L1 **Definition 91.3.2.** In Situation 91.3.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

The following lemma in particular shows that a morphism $(f, f') : (X \subset X') \to (S \subset S')$ of thickenings of schemes is strict if and only if $X = S \times_{S'} X'$.

08L2 **Lemma 91.3.3.** *In Situation 91.3.1 the morphism (f, f') is a strict morphism of thickenings if and only if (91.3.1.1) is cartesian in the category of ringed spaces.*

Proof. Omitted. □

91.4. Modules on first order thickenings of ringed spaces

08L3 In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We will freely use the notation introduced in Section 91.3, in particular we will identify the underlying topological spaces. In this section we consider short exact sequences

08L4 (91.4.0.1)
$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}_X$-modules and $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module. In this situation we have a canonical $\mathcal{O}_X$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08L5 **Lemma 91.4.1.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \textit{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}_{X'}$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_{\mathcal{F}'}]{} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative.

(2) *The set of $\mathcal{O}_{X'}$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ by Modules, Lemma 17.13.4. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. □

08L6 **Lemma 91.4.2.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \textit{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_{\mathcal{F}'}]{} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

as desired. □

08L7 **Lemma 91.4.3.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}_X$-modules, see Modules, Lemma 17.13.4. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}_{X'}$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. □

08L8 **Lemma 91.4.4.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}_X$-module, then there does exist a sequence (91.4.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}_{X'}$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules, Lemma 17.17.7). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{array}{ccc} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \hookrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\ \downarrow & & \downarrow \gamma \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c} & \mathcal{K} \end{array}$$

a diagram of $\mathcal{O}_X$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}_X$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}_X$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}_X$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 91.4.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

08L9 **Remark 91.4.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. A first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is said to be *trivial* if there exists a morphism of ringed spaces $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

08LA (91.4.5.1) $$\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I}$$

as sheaves of algebras on X by using $\pi^\sharp$ to split the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(X, \mathcal{O}_X)$ is equivalent to the category of $\mathcal{O}_X$-modules.

08LB **Remark 91.4.6.** Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a trivial first order thickening of ringed spaces and let $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}_X$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (91.4.5.1) associated to π and the map c to define the $\mathcal{O}_{X'}$-module structure and obtain an extension (91.4.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (91.4.0.1) we can use $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$ to think of $\mathcal{F}'$ as an $\mathcal{O}_X$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$. Lemma 91.4.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (91.4.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08LC **Remark 91.4.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be a morphism of first order thickenings of $(X, \mathcal{O}_X)$. Picture

$$\begin{array}{ccc} & (X, \mathcal{O}_X) & \\ \swarrow & & \searrow \\ (X'_1, \mathcal{O}_{X'_1}) & \xrightarrow{h} & (X'_2, \mathcal{O}_{X'_2}) \end{array}$$

Observe that $h^\sharp : \mathcal{O}_{X'_2} \to \mathcal{O}_{X'_1}$ in particular induces an $\mathcal{O}_X$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{ \begin{array}{c} \mathcal{F}'_2 \text{ as in (91.4.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{F}'_1 \text{ as in (91.4.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}$$

Namely, thinking of all sheaves $\mathcal{O}_X$, $\mathcal{O}_{X'_i}$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on X, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

We omit the construction of the $\mathcal{O}_{X'_1}$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08LD **Remark 91.4.8.** Let $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $\mathcal{I}_i$, and $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be as in Remark 91.4.7. Assume that we are given trivializations $\pi_i : X'_i \to X$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickening of $(X, \mathcal{O}_X)$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. In this situation the construction of Remark 91.4.6 induces a commutative diagram

$$\begin{array}{ccc} \{\mathcal{F}'_2 \text{ as in (91.4.0.1) with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_2) \\ \downarrow & & \downarrow \\ \{\mathcal{F}'_1 \text{ as in (91.4.0.1) with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \end{array}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 91.4.7.

08LE **Remark 91.4.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. We define a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}_X$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(X'_1, \mathcal{O}_{X'_1}) \to (X'_3, \mathcal{O}_{X'_3})$ factors through $(X, \mathcal{O}_X) \to (X'_3, \mathcal{O}_{X'_3})$, i.e., the first order thickening $(X'_1, \mathcal{O}_{X'_1})$ of $(X, \mathcal{O}_X)$ is trivial and comes with a canonical trivialization $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$.

We say a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}_X$-modules.

08LF **Remark 91.4.10.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$(X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'}) \to (X_3', \mathcal{O}_{X_3'})$$

be a complex first order thickenings of $(X, \mathcal{O}_X)$, see Remark 91.4.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_3 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_3]{} & \mathcal{K}_3 \\ \downarrow & & \downarrow \\ \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \end{array}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (91.4.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_{X_2'}$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 91.4.7 to obtain an extension $\mathcal{F}_1'$ on X_1' (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 91.4.6 using the canonical splitting $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ of Remark 91.4.9 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 91.4.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}_1'}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}_X$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}_2' = j'_* \mathcal{F}_2'$ equal to the pushout of $\mathcal{F}_2'$ by j' so that $c_{\mathcal{E}_2'} = j' \circ c_2$. Picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Set $\mathcal{E}_3' = \mathcal{E}_2'$ but viewed as an $\mathcal{O}_{X_3'}$-module via $\mathcal{O}_{X_3'} \to \mathcal{O}_{X_2'}$. Then $c_{\mathcal{E}_3'} = j \circ c_3$. The proof of Lemma 91.4.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}_3'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}_1' = \mathcal{F}_2'/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}_1'}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}_2'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}_X$-modules using $\pi^\sharp$ where $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}'_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}'_3/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

which is $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structures given above).

91.5. Infinitesimal deformations of modules on ringed spaces

08LG Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We freely use the notation introduced in Section 91.3. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}_X$-module structure. Thus the sequence above is an extension as in (91.4.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}_{X'}$ we have $i^*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathcal{I}\mathcal{O}_{X'} = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

08LH **Lemma 91.5.1.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.*

Proof. This is a special case of Lemma 91.4.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}_X$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

08LI **Lemma 91.5.2.** *Let (f, f') be a morphism of first order thickenings of ringed spaces as in Situation 91.3.1. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over S and that (f, f') is a strict morphism of thickenings (Definition 91.3.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over S', and*
(2) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). We may check these conditions at stalks. Let $x \in X \subset X'$ be a point with image $s = f(x) \in S \subset S'$. Set $A' = \mathcal{O}_{S',s}$, $B' = \mathcal{O}_{X',x}$, $A = \mathcal{O}_{S,s}$, and $B = \mathcal{O}_{X,x}$. Then $A = A'/J$ and $B = B'/I$ for some square zero ideals. Since (f, f') is a strict morphism of thickenings we have $I = JB'$. Let $M' = \mathcal{F}'_x$ and $M = \mathcal{F}_x$. Then M' is a B'-module and M is a B-module. Since $\mathcal{F} = i^*\mathcal{F}'$ we see that the kernel of the surjection $M' \to M$ is $IM' = JM'$. Thus we have a short exact sequence

$$0 \to JM' \to M' \to M \to 0$$

Using Sheaves, Lemma 6.26.4 and Modules, Lemma 17.16.1 to identify stalks of pullbacks and tensor products we see that the stalk at x of the canonical map of the lemma is the map

$$(J \otimes_A B) \otimes_B M = J \otimes_A M = J \otimes_{A'} M' \longrightarrow JM'$$

The assumption that $\mathcal{F}$ is flat over S signifies that M is a flat A-module.

Assume (1). Flatness implies $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Lemma 10.75.8. This means $J \otimes_{A'} M' \to M'$ is injective by Algebra, Remark 10.75.9. Hence $J \otimes_A M \to JM'$ is an isomorphism.

Assume (2). Then $J \otimes_{A'} M' \to M'$ is injective. Hence $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Remark 10.75.9. Hence M' is flat over A' by Algebra, Lemma 10.99.8. □

08LJ **Lemma 91.5.3.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{G}'$ is flat over S' and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Proof. Combine Lemmas 91.5.1 and 91.5.2. □

08LK **Lemma 91.5.4.** *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 91.5.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology, Lemma 20.28.1). □

08LL **Lemma 91.5.5.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over S' and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 91.5.4 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J}$ by Lemma 91.5.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{O}_X)$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 91.5.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. □

Second proof. We can apply Lemma 91.4.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G}$ by Lemma 91.5.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

08LM **Lemma 91.5.6.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism by Lemma 91.5.2. Apply Lemma 91.4.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $c = 1$. By Lemma 91.5.2 the corresponding extensions $\mathcal{F}'$ are all flat over S'. □

08LN **Lemma 91.5.7.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

(1) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an isomorphism, and*
(2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 91.4.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}_{X'}$-modules flat over S' of Lemma 91.5.2 and Lemma 91.4.4. □

91.6. Application to flat modules on flat thickenings of ringed spaces

08VQ Consider a commutative diagram

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow[i]{} & (X', \mathcal{O}_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \mathcal{O}_S) & \xrightarrow{t} & (S', \mathcal{O}_{S'}) \end{array}$$

of ringed spaces whose horizontal arrows are first order thickenings as in Situation 91.3.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over S.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 91.5.2 to $\mathcal{O}_{X'}$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VR **Lemma 91.6.1.** *In the situation above.*

(1) *There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 91.4.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\mathrm{id}_\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 91.5.7 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 91.5.6. Part (3) follows from Lemma 91.5.3. □

08VS **Situation 91.6.2.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a commutative diagram

$$\begin{array}{ccccc} (X'_1, \mathcal{O}'_1) & \xrightarrow[h]{} & (X'_2, \mathcal{O}'_2) & \longrightarrow & (X'_3, \mathcal{O}'_3) \\ {\scriptstyle f'_1}\downarrow & & {\scriptstyle f'_2}\downarrow & & {\scriptstyle f'_3}\downarrow \\ (S'_1, \mathcal{O}_{S'_1}) & \longrightarrow & (S'_2, \mathcal{O}_{S'_2}) & \longrightarrow & (S'_3, \mathcal{O}_{S'_3}) \end{array}$$

where (a) the top row is a short exact sequence of first order thickenings of X, (b) the lower row is a short exact sequence of first order thickenings of S, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over S'_2 and set $\mathcal{F} = \mathcal{F}'_2|_X$. Let $\pi : X'_1 \to X$ be the canonical splitting (Remark 91.4.9).

08VT **Lemma 91.6.3.** *In Situation 91.6.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over S'_1 restricting to $\mathcal{F}$ on X. Their difference (Lemma 91.6.1) is an element θ of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F})$ whose boundary in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F})$ equals the obstruction (Lemma 91.6.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over S'_3.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on X and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence flatness by Lemma 91.5.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 91.6.2 and the fact that $\mathcal{F}$ is flat over S. The statement on the obstruction class is a direct translation of the result of Remark 91.4.10 to this particular situation. □

91.7. Deformations of ringed spaces and the naive cotangent complex

08U6 In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topological spaces of S and S'. Moreover we assume given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules (Sheaves, Definition 6.21.7 and Section 6.26). In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08U7 (91.7.0.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

(where the vertical arrows are f-maps) and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in (91.3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (91.7.0.1).

08U8 **Lemma 91.7.1.** *Assume given a commutative diagram of morphisms of ringed spaces*

08U9 (91.7.1.1)
$$\begin{array}{ccccc} & & (X_2, \mathcal{O}_{X_2}) & \xrightarrow[i_2]{} & (X'_2, \mathcal{O}_{X'_2}) \\ & \swarrow^{g} & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f'_2} \\ & & (S_2, \mathcal{O}_{S_2}) & \xrightarrow{t_2} & (S'_2, \mathcal{O}_{S'_2}) \\ (X_1, \mathcal{O}_{X_1}) & \xrightarrow[i_1]{} & (X'_1, \mathcal{O}_{X'_1}) & \swarrow & \\ \downarrow{\scriptstyle f_1} & \swarrow & \downarrow{\scriptstyle f'_1} & & \\ (S_1, \mathcal{O}_{S_1}) & \xrightarrow{t_1} & (S'_1, \mathcal{O}_{S'_1}) & & \end{array}$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \mathrm{Ker}(i_j^\sharp)$ *and assume given a* g*-map* $\nu : \mathcal{G}_1 \to \mathcal{G}_2$ *of modules giving rise to the commutative diagram*

08UA (91.7.1.2)

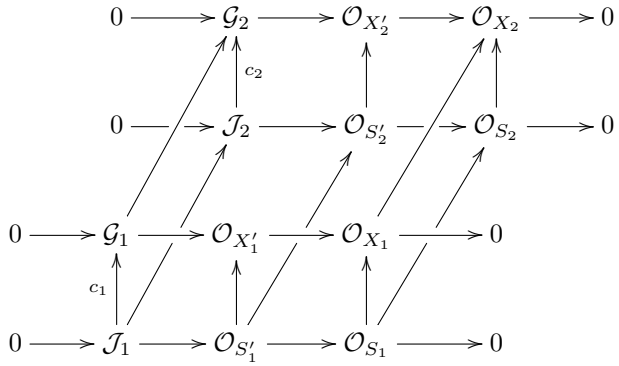

with front and back solutions to (91.7.0.1).

(1) *There exist a canonical element in* $\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$ *whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed spaces* $X_2' \to X_1'$ *fitting into (91.7.1.1) compatibly with* ν*.*

(2) *If there exists a morphism* $X_2' \to X_1'$ *fitting into (91.7.1.1) compatibly with* ν *the set of all such morphisms is a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_{X_1}}(\Omega_{X_1/S_1}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_{X_2}}(g^*\Omega_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2).$$

Proof. The naive cotangent complex NL_{X_1/S_1} is defined in Modules, Definition 17.31.6. The equalities in the last statement of the lemma follow from the fact that g^* is adjoint to g_*, the fact that $H^0(NL_{X_1/S_1}) = \Omega_{X_1/S_1}$ (by construction of the naive cotangent complex) and the fact that Lg^* is the left derived functor of g^*. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topological spaces of all ringed spaces in the lemma is the same.

To do this, observe that $g^{-1} NL_{X_1/S_1}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{S_1} \to g^{-1}\mathcal{O}_{X_1}$, see Modules, Lemma 17.31.3. Moreover, the degree 0 term of NL_{X_1/S_1} is a flat $\mathcal{O}_{X_1}$-module, hence the canonical map

$$Lg^* NL_{X_1/S_1} \longrightarrow g^{-1} NL_{X_1/S_1} \otimes_{g^{-1}\mathcal{O}_{X_1}} \mathcal{O}_{X_2}$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(g^{-1} NL_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(NL_{g^{-1}\mathcal{O}_{X_1}/g^{-1}f_1^{-1}\mathcal{O}_{S_1}}, \mathcal{G}_2)$$

The set of morphism of ringed spaces $X_2' \to X_1'$ fitting into (91.7.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{S_1'}$-algebras $g^{-1}\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (91.7.1.2) of sheaves on X and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (91.7.1.2) of sheaves on a space X and we are

looking for homomorphisms of sheaves of rings $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_{X_1}}(NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_{X'_1} \times_{\mathcal{O}_{X_2}} \mathcal{O}_{X'_2}$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_{X_1}$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}$, see Modules, Lemma 17.31.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X_1}) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{O}_{X_1})$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{S'_1}$-algebras

$$a : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_1} \quad \text{and} \quad b : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_2}$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (91.7.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_{X_1}$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $\nu(a(f')) - b(f')$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{S_1}[\mathcal{E}]/\mathcal{O}_{S_1}} \otimes_{\mathcal{O}_{S_1}[\mathcal{E}]} \mathcal{O}_{X_1}$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (91.7.1.2). In this case consider the map $\mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_{X_1}$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_2}$$

restricted to $\mathcal{I}'$ agrees with $\nu \circ a : \mathcal{I}' \to \mathcal{G}_2$. Since $\mathcal{O}_{X'_1}$ is the pushout of $\mathcal{I}' \to \mathcal{O}_{S'_1}[\mathcal{E}]$ and $\mathcal{I}' \to \mathcal{G}_1$ the maps $b + \theta \circ d$ and a define a map $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (91.7.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 91.2.1. □

08UB **Lemma 91.7.2.** *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules, Lemma 17.31.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. □

08UC **Lemma 91.7.3.** *If there exists a solution to (91.7.0.1), then the set of isomorphism classes of solutions is principal homogeneous under* $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.

Proof. We observe right away that given two solutions X'_1 and X'_2 to (91.7.0.1) we obtain by Lemma 91.7.1 an obstruction element $o(X'_1, X'_2) \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ to the existence of a map $X'_1 \to X'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution X' and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ we can find a second solution X'_ξ such that $o(X', X'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}_X$ as in Lemma 91.7.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X)$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}_X} \mathcal{O}_{X'}$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}_{X'}$.

These choices determine an $f^{-1}\mathcal{O}_{S'}$-algebra map $\varphi : f^{-1}\mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_{X'}$. Let $\mathcal{I}' = \mathrm{Ker}(f^{-1}\mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_X)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}_{X'}$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{\mathcal{I}'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{S'}[\mathcal{E}] & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}_{X'_\xi}$ fitting into a diagram as above. A calculation (omitted) shows that $o(X', X'_\xi) = \xi$ as desired. □

0GPZ **Lemma 91.7.4.** *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a $\mathcal{O}_X$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_S$-algebras*

$$0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $\mathcal{G}$ is an ideal of square zero[1] is canonically bijective to $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.

Proof. To prove this we apply the previous results to the case where (91.7.0.1) is given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{\mathrm{id}} & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

Thus our lemma follows from Lemma 91.7.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}_X$. (See remark below for a direct construction of the bijection.) □

0GQ0 **Remark 91.7.5.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $\mathcal{G}$ be as in Lemma 91.7.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & & \\ {\scriptstyle \alpha'}\downarrow & \searrow^{\alpha} & \\ \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \end{array}$$

such that $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}_{X'}$.) Then

$$NL_{X/S} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X\right)$$

See Modules, Section 17.31 and in particular Lemma 17.31.2. Of course α' determines a map $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_{X'}$ which in turn determines a map

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}$$

which in turn determines the element of $\mathrm{Ext}^1_{\mathcal{O}_X}(NL(\alpha), \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ corresponding to $\mathcal{O}_{X'}$ by the bijection of the lemma.

0GQ1 **Lemma 91.7.6.** *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be morphisms of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. Let $c : \mathcal{F} \to \mathcal{G}$ be a g-map. Finally, consider*

(a) *$0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ an extension of $f^{-1}\mathcal{O}_S$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, and*

(b) *$0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_S$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$.*

See Lemma 91.7.4. Then there is an S-morphism $g' : Y' \to X'$ compatible with g and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$.

[1] In other words, the set of isomorphism classes of first order thickenings $i : X \to X'$ over S endowed with an isomorphism $\mathcal{G} \to \mathrm{Ker}(i^\sharp)$ of $\mathcal{O}_X$-modules.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, Lg^*\mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

using the map $Lg^*\mathcal{F} \to g^*\mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

using the map $Lg^* NL_{X/S} \to NL_{Y/S}$. The statement of the lemma can be deduced from Lemma 91.7.1 applied to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 & &
\end{array}$$

and a compatibility between the constructions in the proofs of Lemmas 91.7.4 and 91.7.1 whose statement and proof we omit. (See remark below for a direct argument.) □

0GQ2 **Remark 91.7.7.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, $\mathcal{F}$, $\mathcal{G}$, $c : \mathcal{F} \to \mathcal{G}$, $0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$, $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$, and $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(NL_{Y/S}, \mathcal{G})$ be as in Lemma 91.7.6. Using pushout along $c : g^{-1}\mathcal{F} \to \mathcal{G}$ we can construct an extension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}'_1 & \longrightarrow & g^{-1}\mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle c} & & \uparrow & & \| & & \\
0 & \longrightarrow & g^{-1}\mathcal{F} & \longrightarrow & g^{-1}\mathcal{O}_{X'} & \longrightarrow & g^{-1}\mathcal{O}_X & \longrightarrow & 0
\end{array}$$

Using pullback along $g^\sharp : g^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ we can construct an extension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}'_2 & \longrightarrow & g^{-1}\mathcal{O}_X & \longrightarrow & 0
\end{array}$$

A diagram chase tells us that there exists an S-morphism $Y' \to X'$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as $g^{-1}f^{-1}\mathcal{O}_S$-algebra extensions of $g^{-1}\mathcal{O}_X$ by $\mathcal{G}$. By Lemma 91.7.4 these extensions are classified by the LHS of

$$\mathrm{Ext}^1_{g^{-1}\mathcal{O}_X}(NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_Y}(Lg^* NL_{X/S}, \mathcal{G})$$

Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S} = g^{-1} NL_{X/S} \quad \text{and} \quad Lg^* NL_{X/S} = g^{-1} NL_{X/S} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules, Lemma 17.31.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 91.7.6 is true,

it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 91.7.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & & \\ {\scriptstyle\beta'}\downarrow & \searrow^{\beta} & \\ \mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_Y \end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_Y$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \xleftarrow{\varphi} & \mathcal{E}' & & \\ {\scriptstyle\beta'}\downarrow & & {\scriptstyle\alpha'}\downarrow & \searrow^{\alpha} & \\ \mathcal{O}_{Y'} & \longleftarrow & \mathcal{O}'_2 & \longrightarrow & g^{-1}\mathcal{O}_X \end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_S[\mathcal{E}'] \to g^{-1}\mathcal{O}_X$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 17.31) and in particular $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{Y/S}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_S}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^* NL_{X/S} \to NL_{Y/S}$ used in the statement of Lemma 91.7.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 91.7.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ3 **Lemma 91.7.8.** *Let $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (91.7.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/S'}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{S'}$ of $\mathcal{O}_S$ by $\mathcal{J}$ via Lemma 91.7.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 91.7.4 applied to $X \to S'$ and the $\mathcal{O}_X$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ of $f^{-1}\mathcal{O}_{S'}$-algebras. By Lemma 91.7.6 applied to $X \to S \to S'$ and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an S'-morphism $X' \to S'$ compatible with c and $f : X \to S$ if and only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}_{X'}$ is a solution of (91.7.0.1). □

0GQ4 **Remark 91.7.9.** In the situation of Lemma 91.7.8 we have maps of complexes

$$Lf^* NL_{S'/S} \to NL_{X/S'} \to NL_{X/S}$$

These maps are closed to forming a distinguished triangle, see Modules, Lemma 17.31.7. If it were a distinguished triangle we would conclude that the image of ξ in $\mathrm{Ext}^2_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ would be the obstruction to the existence of a solution to (91.7.0.1).

91.8. Deformations of schemes

0D13 In this section we spell out what the results in Section 91.7 mean for deformations of schemes.

0D14 **Lemma 91.8.1.** *Let $S \subset S'$ be a first order thickening of schemes. Let $f : X \to S$ be a flat morphism of schemes. If there exists a flat morphism $f' : X' \to S'$ of schemes and an isomorphism $a : X \to X' \times_{S'} S$ over S, then*

(1) *the set of isomorphism classes of pairs $(f' : X' \to S', a)$ is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$, and*
(2) *the set of automorphisms of $\varphi : X' \to X'$ over S' which reduce to the identity on $X' \times_{S'} S$ is $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/S}, f^*\mathcal{C}_{S/S'})$.*

Proof. First we observe that thickenings of schemes as defined in More on Morphisms, Section 37.2 are the same things as morphisms of schemes which are thickenings in the sense of Section 91.3. We may think of X as a closed subscheme of X' so that $(f, f') : (X \subset X') \to (S \subset S')$ is a morphism of first order thickenings. Then we see from More on Morphisms, Lemma 37.10.1 (or from the more general Lemma 91.5.2) that the ideal sheaf of X in X' is equal to $f^*\mathcal{C}_{S/S'}$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathcal{C}_{S/S'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{C}_{S/S'} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

where the vertical arrows are f-maps; please compare with (91.7.0.1). Thus part (1) follows from Lemma 91.7.3 and part (2) from part (2) of Lemma 91.7.1. (Note that $NL_{X/S}$ as defined for a morphism of schemes in More on Morphisms, Section 37.13 agrees with $NL_{X/S}$ as used in Section 91.7.) □

91.9. Thickenings of ringed topoi

08M6 This section is the analogue of Section 91.3 for ringed topoi. In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}'$ on a ringed topos $(Sh(\mathcal{D}), \mathcal{O}')$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent.
(2) A *thickening* of ringed topoi is a morphism $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that
   (a) i_* is an equivalence $Sh(\mathcal{C}) \to Sh(\mathcal{D})$,
   (b) the map $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective, and
   (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
(3) A *first order thickening* of ringed topoi is a thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define *morphisms of thickenings of ringed topoi*, *morphisms of thickenings of ringed topoi over a base ringed topos*, etc.

If $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a thickening of ringed topoi then we identify the underlying topoi and think of $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $\mathcal{C}$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

of $\mathcal{O}'$-modules. By Modules on Sites, Lemma 18.25.1 the category of $\mathcal{O}$-modules is equivalent to the category of $\mathcal{O}'$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}$-module.

08M7 **Situation 91.9.1.** A morphism of thickenings of ringed topoi (f, f') is given by a commutative diagram

08M8 (91.9.1.1)
$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{i} & (Sh(\mathcal{D}), \mathcal{O}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) & \xrightarrow{t} & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{array}$$

of ringed topoi whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. As $f = f'$ on underlying topoi we will identify the pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{\mathcal{B}'} \to \mathcal{O}'$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}'$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

08M9 **Definition 91.9.2.** In Situation 91.9.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

91.10. Modules on first order thickenings of ringed topoi

08MA In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (Sh(\mathcal{C}, \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We will freely use the notation introduced in Section 91.9, in particular we will identify the underlying topological topoi. In this section we consider short exact sequences

08MB (91.10.0.1)
$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}$-modules and $\mathcal{F}'$ is an $\mathcal{O}'$-module. In this situation we have a canonical $\mathcal{O}$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

08MC **Lemma 91.10.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}'$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_{\mathcal{F}'}} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative.

(2) *The set of $\mathcal{O}'$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$.

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$ by Modules on Sites, Lemma 18.25.1. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. □

08MD **Lemma 91.10.2.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (91.10.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_{\mathcal{F}'}]{} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \varphi'}\downarrow & & {\scriptstyle \varphi}\downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

as desired. □

08ME **Lemma 91.10.3.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}$-modules, see Modules on Sites, Lemma 18.25.1. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}'$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. □

08MF **Lemma 91.10.4.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}$-module, then there does exist a sequence (91.10.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}'$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules on Sites, Lemma 18.28.8). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{array}{ccc} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \hookrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\ \downarrow & & \downarrow \gamma \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c} & \mathcal{K} \end{array}$$

a diagram of $\mathcal{O}$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 91.10.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. □

08MG **Remark 91.10.5.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. A first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is said to be *trivial* if there exists a morphism of ringed topoi $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

08MH (91.10.5.1) $$\mathcal{O}' = \mathcal{O} \oplus \mathcal{I}$$

as sheaves of algebras on $\mathcal{C}$ by using $\pi^\sharp$ to split the surjection $\mathcal{O}' \to \mathcal{O}$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to the category of $\mathcal{O}$-modules.

08MI **Remark 91.10.6.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a trivial first order thickening of ringed topoi and let $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (91.10.5.1) associated to π and the map c to define the $\mathcal{O}'$-module structure and obtain an extension (91.10.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any

extension $\mathcal{F}'$ as in (91.10.0.1) we can use $\pi^\sharp : \mathcal{O} \to \mathcal{O}'$ to think of $\mathcal{F}'$ as an $\mathcal{O}$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$. Lemma 91.10.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (91.10.0.1) with } c = c_{\mathcal{F}'}\end{array}\right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

08MJ **Remark 91.10.7.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Picture

$$\begin{array}{ccc} & (Sh(\mathcal{C}), \mathcal{O}) & \\ \swarrow & & \searrow \\ (Sh(\mathcal{D}_1), \mathcal{O}'_1) & \xrightarrow{h} & (Sh(\mathcal{D}_2), \mathcal{O}'_2) \end{array}$$

Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{\begin{array}{c}\mathcal{F}'_2 \text{ as in (91.10.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\mathcal{F}'_1 \text{ as in (91.10.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\end{array}\right\}$$

Namely, thinking of all sheaves $\mathcal{O}$, $\mathcal{O}'_i$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on $\mathcal{C}$, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

We omit the construction of the $\mathcal{O}'_1$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

08MK **Remark 91.10.8.** Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i), \mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be as in Remark 91.10.7. Assume that we are given trivializations $\pi_i : (Sh(\mathcal{D}_i), \mathcal{O}'_i) \to (Sh(\mathcal{C}), \mathcal{O})$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. In this situation the construction of Remark 91.10.6 induces a commutative diagram

$$\begin{array}{ccc} \{\mathcal{F}_2' \text{ as in (91.10.0.1) with } c_2 = c_{\mathcal{F}_2'} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \\ \downarrow & & \downarrow \\ \{\mathcal{F}_1' \text{ as in (91.10.0.1) with } c_1 = c_{\mathcal{F}_1'} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \end{array}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 91.10.7.

0CYC **Remark 91.10.9.** Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}_i')$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2')$ be as in Remark 91.10.7. Observe that $h^\sharp : \mathcal{O}_2' \to \mathcal{O}_1'$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then we **claim** the map

$$\mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$$

sends $o(\mathcal{F}, \mathcal{K}_2, c_2)$ to $o(\mathcal{F}, \mathcal{K}_1, c_1)$.

To prove this claim choose an embedding $j_2 : \mathcal{K}_2 \to \mathcal{K}_2'$ where $\mathcal{K}_2'$ is an injective $\mathcal{O}$-module. As in the proof of Lemma 91.10.4 we can choose an extension of $\mathcal{O}_2$-modules

$$0 \to \mathcal{K}_2' \to \mathcal{E}_2 \to \mathcal{F} \to 0$$

such that $c_{\mathcal{E}_2} = j_2 \circ c_2$. The proof of Lemma 91.10.4 constructs $o(\mathcal{F}, \mathcal{K}_2, c_2)$ as the Yoneda extension class (in the sense of Derived Categories, Section 13.27) of the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_2 \to \mathcal{K}_2' \to \mathcal{E}_2/\mathcal{K}_2 \to \mathcal{F} \to 0$$

Let $\mathcal{K}_1'$ be the cokernel of $\mathcal{K}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2'$. There is an injection $j_1 : \mathcal{K}_1 \to \mathcal{K}_1'$ and a map $\mathcal{K}_2' \to \mathcal{K}_1'$ forming a commutative square. We form the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_1' & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

There is a canonical $\mathcal{O}_1$-module structure on $\mathcal{E}_1$ and for this structure we have $c_{\mathcal{E}_1} = j_1 \circ c_1$ (this uses the commutativity of the diagram involving c_1 and c_2 above). The procedure of Lemma 91.10.4 tells us that $o(\mathcal{F}, \mathcal{K}_1, c_1)$ is the Yoneda extension class of the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_1 \to \mathcal{K}_1' \to \mathcal{E}_1/\mathcal{K}_1 \to \mathcal{F} \to 0$$

Since we have maps of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2/\mathcal{K}_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_2' & \longrightarrow & \mathcal{E}_2/\mathcal{K}_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

we conclude that the claim is true.

08ML **Remark 91.10.10.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. We define a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$ factors through $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$, i.e., the first order thickening $(Sh(\mathcal{D}_1), \mathcal{O}_1')$ of $(Sh(\mathcal{C}), \mathcal{O})$ is trivial and comes with a canonical trivialization $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$.

We say a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}$-modules.

08MM **Remark 91.10.11.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

be a complex first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, see Remark 91.10.10. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_3 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_3} & \mathcal{K}_3 \\ \downarrow & & \downarrow \\ \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \end{array}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (91.10.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_2'$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 91.10.7 to obtain an extension $\mathcal{F}_1'$ of $\mathcal{O}_1'$-modules (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 91.10.6 using the canonical splitting $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ of Remark 91.10.10 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 91.10.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}'_1}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}'_2 = j'_*\mathcal{F}'_2$ equal to the pushout of $\mathcal{F}'_2$ by j' so that $c_{\mathcal{E}'_2} = j' \circ c_2$. Picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Set $\mathcal{E}'_3 = \mathcal{E}'_2$ but viewed as an $\mathcal{O}'_3$-module via $\mathcal{O}'_3 \to \mathcal{O}'_2$. Then $c_{\mathcal{E}'_3} = j \circ c_3$. The proof of Lemma 91.10.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}'_3/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}'_1 = \mathcal{F}'_2/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}'_1}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}'_2/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}$-modules using $\pi^\sharp$ where $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}'_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}'_3/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

which is $\mathcal{O}$-linear (with the $\mathcal{O}$-module structures given above).

91.11. Infinitesimal deformations of modules on ringed topoi

08MN Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We freely use the notation introduced in Section 91.9. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}'$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}$-module structure. Thus the sequence above is an extension as in (91.10.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}'$ we have $i^*\mathcal{O}' = \mathcal{O}$ and $\mathcal{I}\mathcal{O}' = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

08MP **Lemma 91.11.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.

Proof. This is a special case of Lemma 91.10.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}'$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

08MQ **Lemma 91.11.2.** *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 91.9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$ and that (f, f') is a strict morphism of thickenings (Definition 91.9.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$, and*
(2) *the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). By definition flatness over $\mathcal{O}_\mathcal{B}$ means flatness over $f^{-1}\mathcal{O}_\mathcal{B}$. Similarly for flatness over $f^{-1}\mathcal{O}_{\mathcal{B}'}$. Note that the strictness of (f, f') and the assumption that $\mathcal{F} = i^*\mathcal{F}'$ imply that

$$\mathcal{F} = \mathcal{F}'/(f^{-1}\mathcal{J})\mathcal{F}'$$

as sheaves on $\mathcal{C}$. Moreover, observe that $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}} \mathcal{F}$. Hence the equivalence of (1) and (2) follows from Modules on Sites, Lemma 18.28.15. □

08VU **Lemma 91.11.3.** *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 91.9.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. Then the following are equivalent*

(1) *$\mathcal{F}'$ is an $\mathcal{O}'$-module of finite presentation, and*
(2) *$\mathcal{F}$ is an $\mathcal{O}$-module of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Modules on Sites, Lemma 18.23.4. For the converse, assume $\mathcal{F}$ of finite presentation. We may and do assume that $\mathcal{C} = \mathcal{C}'$. By Lemma 91.11.2 we have a short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0$$

Let U be an object of $\mathcal{C}$ such that $\mathcal{F}|_U$ has a presentation

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

After replacing U by the members of a covering we may assume the map $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ lifts to a map $(\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U$. The induced map $\mathcal{I}^{\oplus n} \to \mathcal{I} \otimes \mathcal{F}$ is surjective

by right exactness of $\otimes$. Thus after replacing U by the members of a covering we can find a lift $(\mathcal{O}'|_U)^{\oplus m} \to (\mathcal{O}'|_U)^{\oplus n}$ of the given map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$ such that

$$(\mathcal{O}'_U)^{\oplus m} \to (\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U \to 0$$

is a complex. Using right exactness of $\otimes$ once more it is seen that this complex is exact. □

08MR **Lemma 91.11.4.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{G}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Proof. Combine Lemmas 91.11.1 and 91.11.2. □

08MS **Lemma 91.11.5.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 91.11.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology on Sites, Lemma 21.19.1). □

08MT **Lemma 91.11.6.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 91.11.5 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J}$ by Lemma 91.11.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}'}(\mathcal{F}', \mathcal{O})$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 91.11.2. Thus $\tau_{\geq -1} Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\tau_{\geq -1} Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. □

Second proof. We can apply Lemma 91.10.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G}$ by Lemma 91.11.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

08MU **Lemma 91.11.7.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

is an isomorphism by Lemma 91.11.2. Apply Lemma 91.10.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $c = 1$. By Lemma 91.11.2 the corresponding extensions $\mathcal{F}'$ are all flat over $\mathcal{O}_{\mathcal{B}'}$. □

08MV **Lemma 91.11.8.** *Let (f, f') be a morphism of first order thickenings as in Situation 91.9.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

(1) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ is an isomorphism, and*
(2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$ of Lemma 91.10.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}'$-modules flat over $\mathcal{O}_{\mathcal{B}'}$ of Lemma 91.11.2 and Lemma 91.10.4. □

91.12. Application to flat modules on flat thickenings of ringed topoi

08VV Consider a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow[i]{} & (Sh(\mathcal{D}), \mathcal{O}') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) & \xrightarrow{t} & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{array}$$

of ringed topoi whose horizontal arrows are first order thickenings as in Situation 91.9.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 91.11.2 to $\mathcal{O}'$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

08VW **Lemma 91.12.1.** *In the situation above.*

(1) *There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$ of Lemma 91.10.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 91.11.8 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 91.11.7. Part (3) follows from Lemma 91.11.4. □

0CYD **Situation 91.12.2.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ be a morphism of ringed topoi. Consider a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow{h} & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) \\ \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f'_2} \\ (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) \end{array}$$

where h is a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, the lower horizontal arrow is a morphism of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, each f'_i restricts to f, both pairs (f, f'_i) are strict morphisms of thickenings, and both f'_i are flat. Finally, let $\mathcal{F}$ be an $\mathcal{O}$-module flat over $\mathcal{O}_{\mathcal{B}}$.

0CYE **Lemma 91.12.3.** *In Situation 91.12.2 the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F}, 1)$ maps to the obstruction class $o(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}, 1)$ under the canonical map*

$$\mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F}) \to \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F})$$

Proof. Follows from Remark 91.10.9. □

08VX **Situation 91.12.4.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ be a morphism of ringed topoi. Consider a commutative diagram

$$\begin{array}{ccccc} (Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow{h} & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) & \longrightarrow & (Sh(\mathcal{C}'_3), \mathcal{O}'_3) \\ \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f'_2} & & \downarrow{\scriptstyle f'_3} \\ (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) & \longrightarrow & (Sh(\mathcal{B}'_3), \mathcal{O}_{\mathcal{B}'_3}) \end{array}$$

where (a) the top row is a short exact sequence of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, (b) the lower row is a short exact sequence of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over $\mathcal{O}_{\mathcal{B}'_2}$ and set $\mathcal{F} = \mathcal{F}'_2 \otimes \mathcal{O}$. Let $\pi : (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ be the canonical splitting (Remark 91.10.10).

08VY **Lemma 91.12.5.** *In Situation 91.12.4 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over $\mathcal{O}_{\mathcal{B}'_1}$ restricting to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$. Their difference (Lemma 91.12.1) is an element θ of $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F})$ whose boundary in $\mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F})$ equals the obstruction (Lemma 91.12.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over $\mathcal{O}_{\mathcal{B}'_3}$.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$ and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}$. Hence flatness by Lemma 91.11.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 91.12.4 and the fact that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$. The statement on the obstruction class is a direct translation of the result of Remark 91.10.11 to this particular situation. □

91.13. Deformations of ringed topoi and the naive cotangent complex

08UE In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topoi of $\mathcal{B}$ and $\mathcal{B}'$. Moreover we assume given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08UF (91.13.0.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & {\scriptstyle c}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}} & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in (91.9.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (91.13.0.1).

08UG **Lemma 91.13.1.** *Assume given a commutative diagram of morphisms ringed topoi*

08UH (91.13.1.1)
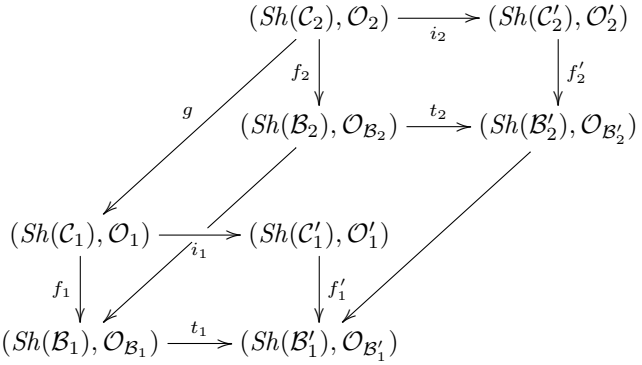

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = \mathrm{Ker}(i_j^\sharp)$ and assume given a map of $g^{-1}\mathcal{O}_1$-modules $\nu : g^{-1}\mathcal{G}_1 \to \mathcal{G}_2$ giving rise to the commutative

diagram

08UI (91.13.1.2)

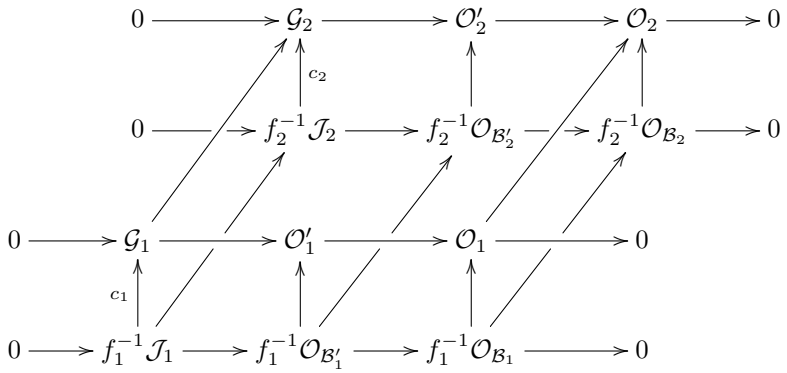

with front and back solutions to (91.13.0.1). (The north-north-west arrows are maps on $\mathcal{C}_2$ after applying g^{-1} to the source.)

(1) *There exist a canonical element in* $\mathrm{Ext}^1_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$ *whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed topoi* $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ *fitting into (91.13.1.1) compatibly with* ν.

(2) *If there exists a morphism* $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ *fitting into (91.13.1.1) compatibly with* ν *the set of all such morphisms is a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_1}(\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_2}(g^*\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^0_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2).$$

Proof. The proof of this lemma is identical to the proof of Lemma 91.7.1. We urge the reader to read that proof instead of this one. We will identify the underlying topoi for every thickening in sight (we have already used this convention in the statement). The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topos of all ringed topoi in the lemma is the same.

To do this, observe that $g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1} \to g^{-1}\mathcal{O}_1$, see Modules on Sites, Lemma 18.33.5. Moreover, the degree 0 term of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is a flat $\mathcal{O}_1$-module, hence the canonical map

$$Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \longrightarrow g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \otimes_{g^{-1}\mathcal{O}_1} \mathcal{O}_2$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(NL_{g^{-1}\mathcal{O}_1/g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$$

The set of morphism of ringed topoi $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ fitting into (91.13.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}'_1}$-algebras $g^{-1}\mathcal{O}'_1 \to \mathcal{O}'_2$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (91.13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (91.13.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \mathrm{id}$ on underlying topoi) and we are looking for homomorphisms of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_1}(NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}'_1 \times_{\mathcal{O}_2} \mathcal{O}'_2$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_1$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$, see Modules on Sites, Lemma 18.35.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}_1) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{O}_1)$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{\mathcal{B}'_2}$-algebras

$$a : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_1 \quad \text{and} \quad b : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_2$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (91.13.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_1$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_1}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]/\mathcal{O}_{\mathcal{B}_1}} \otimes_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]} \mathcal{O}_1$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (91.13.1.2). In this case consider the map $\mathcal{O}'_1[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_1$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \longrightarrow \mathcal{O}'_2$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (91.13.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 91.2.1. □

08UJ **Lemma 91.13.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules on Sites, Lemma 18.35.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. □

08UK **Lemma 91.13.3.** *If there exists a solution to (91.13.0.1), then the set of isomorphism classes of solutions is principal homogeneous under* $\mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.

Proof. We observe right away that given two solutions $\mathcal{O}'_1$ and $\mathcal{O}'_2$ to (91.13.0.1) we obtain by Lemma 91.13.1 an obstruction element $o(\mathcal{O}'_1, \mathcal{O}'_2) \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ to the existence of a map $\mathcal{O}'_1 \to \mathcal{O}'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution $\mathcal{O}'$ and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ we can find a second solution $\mathcal{O}'_\xi$ such that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}$ as in Lemma 91.13.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}] \to \mathcal{O}$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]/f^{-1}\mathcal{O}_{\mathcal{B}}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]} \mathcal{O})$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}} \mathcal{O}'$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}'$.

These choices determine an $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebra map $\varphi : \mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \to \mathcal{O}'$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}'$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & {\scriptstyle \varphi|_{\mathcal{I}'}}\uparrow & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'}[\mathcal{E}] & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}'_\xi$ fitting into a diagram as above. A calculation (omitted) shows that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$ as desired. $\square$

0GQ5 **Lemma 91.13.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Let $\mathcal{G}$ be an $\mathcal{O}$-module. The set of isomorphism classes of extensions of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras*

$$0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$$

where $\mathcal{G}$ is an ideal of square zero[2] is canonically bijective to $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.

Proof. To prove this we apply the previous results to the case where (91.13.0.1) is given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & f^{-1}\mathcal{O}_\mathcal{B} & \xrightarrow{\mathrm{id}} & f^{-1}\mathcal{O}_\mathcal{B} & \longrightarrow & 0 \end{array}$$

Thus our lemma follows from Lemma 91.13.3 and the fact that there exists a solution, namely $\mathcal{G} \oplus \mathcal{O}$. (See remark below for a direct construction of the bijection.) $\square$

0GQ6 **Remark 91.13.5.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathcal{G}$ be as in Lemma 91.13.4. Consider an extension $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ as in the lemma. We can choose a sheaf of sets $\mathcal{E}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & & \\ {\scriptstyle \alpha'}\downarrow & \searrow^{\alpha} & \\ \mathcal{O}' & \longrightarrow & \mathcal{O} \end{array}$$

such that $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}$ is surjective with kernel $\mathcal{J}$. (For example you can take any sheaf of sets surjecting onto $\mathcal{O}'$.) Then

$$NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}} \cong NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]/f^{-1}\mathcal{O}_\mathcal{B}} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]} \mathcal{O}\right)$$

See Modules on Sites, Section 18.35 and in particular Lemma 18.35.2. Of course α' determines a map $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}'$ which in turn determines a map

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{G}$$

which in turn determines the element of $\mathrm{Ext}^1_\mathcal{O}(NL(\alpha), \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ corresponding to $\mathcal{O}'$ by the bijection of the lemma.

0GQ7 **Lemma 91.13.6.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be a $\mathcal{O}_\mathcal{C}$-module. Let $\mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-module. Let $c : g^*\mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_\mathcal{D}$-linear map. Finally, consider*

(a) *$0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_\mathcal{C} \to 0$ an extension of $f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{F})$, and*

(b) *$0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_\mathcal{D} \to 0$ an extension of $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}$-algebras corresponding to $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{D}}(NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.*

[2] In other words, the set of isomorphism classes of first order thickenings $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}')$ over $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ endowed with an isomorphism $\mathcal{G} \to \mathrm{Ker}(i^\sharp)$ of $\mathcal{O}$-modules.

See Lemma 91.13.4. Then there is a morphism

$$g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$$

of ringed topoi over $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ compatible with g and c if and only if ξ and ζ map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.

Proof. The stament makes sense as we have the maps

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, Lg^*\mathcal{F}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

using the map $Lg^*\mathcal{F} \to g^*\mathcal{F} \xrightarrow{c} \mathcal{G}$ and

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{Y}}}(NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{Y}}}(Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

using the map $Lg^* NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}} \to NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}$. The statement of the lemma can be deduced from Lemma 91.13.1 applied to the diagram

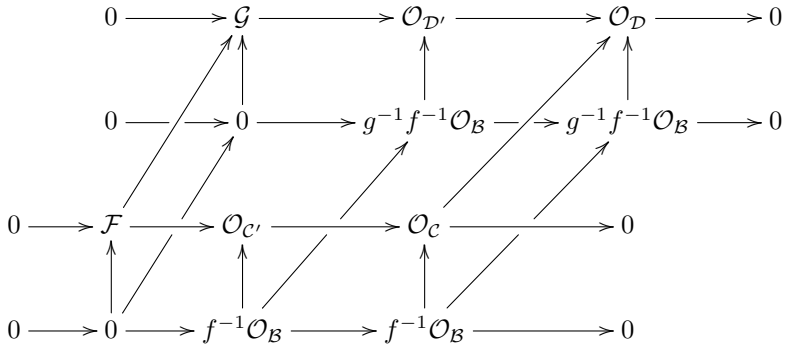

and a compatibility between the constructions in the proofs of Lemmas 91.13.4 and 91.13.1 whose statement and proof we omit. (See remark below for a direct argument.) □

0GQ8 **Remark 91.13.7.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, $g : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \to (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$, $\mathcal{F}, \mathcal{G}, c : g^*\mathcal{F} \to \mathcal{G}, 0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}} \to 0, \xi \in \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(NL_{\mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{F})$, $0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_{\mathcal{D}} \to 0$, and $\zeta \in \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{D}}}(NL_{\mathcal{O}_{\mathcal{D}}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ be as in Lemma 91.13.6. Using pushout along $c : g^{-1}\mathcal{F} \to \mathcal{G}$ we can construct an extension

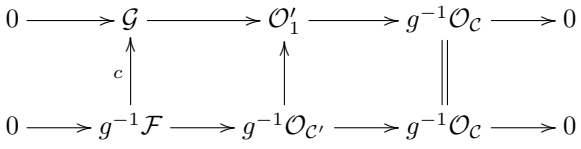

Using pullback along $g^\sharp : g^{-1}\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$ we can construct an extension

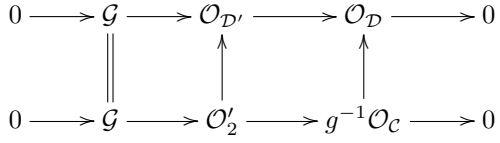

A diagram chase tells us that there exists a morphism $g' : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}'}) \to (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}'})$ over $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ compatible with g and c if and only if $\mathcal{O}'_1$ is isomorphic to $\mathcal{O}'_2$ as

$g^{-1}f^{-1}\mathcal{O}_\mathcal{B}$-algebra extensions of $g^{-1}\mathcal{O}_\mathcal{C}$ by $\mathcal{G}$. By Lemma 91.13.4 these extensions are classified by the LHS of

$$\mathrm{Ext}^1_{g^{-1}\mathcal{O}_\mathcal{C}}(NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_\mathcal{D}}(Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$$

Here the equality comes from tensor-hom adjunction and the equalities

$$NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}} = g^{-1} NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \quad \text{and} \quad Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} = g^{-1} NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_X} \mathcal{O}_Y$$

For the first of these see Modules on Sites, Lemma 18.35.3; the second follows from the definition of derived pullback. Thus, in order to see that Lemma 91.13.6 is true, it suffices to show that $\mathcal{O}'_1$ corresponds to the image of ξ and that $\mathcal{O}'_2$ correspond to the image of ζ. The correspondence between ξ and $\mathcal{O}'_1$ is immediate from the construction of the class ξ in Remark 91.13.5. For the correspondence between ζ and $\mathcal{O}'_2$, we first choose a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & & \\ {\scriptstyle\beta'}\downarrow & \searrow^{\beta} & \\ \mathcal{O}_{\mathcal{D}'} & \longrightarrow & \mathcal{O}_\mathcal{D} \end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}_\mathcal{D}$ is surjective with kernel $\mathcal{K}$. Next choose a commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \xleftarrow{\varphi} & \mathcal{E}' & & \\ {\scriptstyle\beta'}\downarrow & & {\scriptstyle\alpha'}\downarrow & \searrow^{\alpha} & \\ \mathcal{O}_{\mathcal{D}'} & \longleftarrow & \mathcal{O}'_2 & \longrightarrow & g^{-1}\mathcal{O}_\mathcal{C} \end{array}$$

such that $g^{-1}f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}'] \to g^{-1}\mathcal{O}_\mathcal{C}$ is surjective with kernel $\mathcal{J}$. (For example just take $\mathcal{E}' = \mathcal{E} \amalg \mathcal{O}'_2$ as a sheaf of sets.) The map φ induces a map of complexes $NL(\alpha) \to NL(\beta)$ (notation as in Modules, Section 17.31) and in particular $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. Then $NL(\alpha) \cong NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ and $NL(\beta) \cong NL_{g^{-1}\mathcal{O}_\mathcal{C}/g^{-1}f^{-1}\mathcal{O}_\mathcal{B}}$ and the map of complexes $NL(\alpha) \to NL(\beta)$ represents the map $Lg^* NL_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{B}} \to NL_{\mathcal{O}_\mathcal{D}/\mathcal{O}_\mathcal{B}}$ used in the statement of Lemma 91.13.6 (see first part of its proof). Now ζ corresponds to the class of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β', see Remark 91.13.5. Similarly, the extension $\mathcal{O}'_2$ corresponds to the map $\mathcal{J}/\mathcal{J}^2 \to \mathcal{G}$ induced by α'. The commutative diagram above shows that this map is the composition of the map $\mathcal{K}/\mathcal{K}^2 \to \mathcal{G}$ induced by β' with the map $\bar{\varphi} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{K}/\mathcal{K}^2$. This proves the compatibility we were looking for.

0GQ9 **Lemma 91.13.8.** *Let $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$, $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, $\mathcal{G}$, and $c : \mathcal{J} \to \mathcal{G}$ be as in (91.13.0.1). Denote $\xi \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{J})$ the element corresponding to the extension $\mathcal{O}_{\mathcal{B}'}$ of $\mathcal{O}_\mathcal{B}$ by $\mathcal{J}$ via Lemma 91.13.4. The set of isomorphism classes of solutions is canonically bijective to the fibre of*

$$\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_\mathcal{O}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

over the image of ξ.

Proof. By Lemma 91.13.4 applied to $t \circ f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ and the $\mathcal{O}$-module $\mathcal{G}$ we see that elements ζ of $\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$ parametrize extensions $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ of $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebras. By Lemma 91.13.6 applied to

$$(Sh(\mathcal{C}), \mathcal{O}) \xrightarrow{f} (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \xrightarrow{t} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

and $c : \mathcal{J} \to \mathcal{G}$ we see that there is an morphism

$$f' : (Sh(\mathcal{C}), \mathcal{O}') \longrightarrow (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$$

over $(Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ compatible with c and f if and only if ζ maps to ξ. Of course this is the same thing as saying $\mathcal{O}'$ is a solution of (91.13.0.1). □

91.14. Deformations of algebraic spaces

0D15 In this section we spell out what the results in Section 91.13 mean for deformations of algebraic spaces.

0D16 **Lemma 91.14.1.** *Let S be a scheme. Let $i : Z \to Z'$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *i is a thickening of algebraic spaces as defined in More on Morphisms of Spaces, Section 76.9, and*
(2) *the associated morphism $i_{small} : (Sh(Z_{étale}), \mathcal{O}_Z) \to (Sh(Z'_{étale}), \mathcal{O}_{Z'})$ of ringed topoi (Properties of Spaces, Lemma 66.21.3) is a thickening in the sense of Section 91.9.*

Proof. We stress that this is not a triviality.

Assume (1). By More on Morphisms of Spaces, Lemma 76.9.6 the morphism i induces an equivalence of small étale sites and in particular of topoi. Of course $i^\sharp$ is surjective with locally nilpotent kernel by definition of thickenings.

Assume (2). (This direction is less important and more of a curiosity.) For any étale morphism $Y' \to Z'$ we see that $Y = Z \times_{Z'} Y'$ has the same étale topos as Y'. In particular, Y' is quasi-compact if and only if Y is quasi-compact because being quasi-compact is a topos theoretic notion (Sites, Lemma 7.17.3). Having said this we see that Y' is quasi-compact and quasi-separated if and only if Y is quasi-compact and quasi-separated (because you can characterize Y' being quasi-separated by saying that for all Y'_1, Y'_2 quasi-compact algebraic spaces étale over Y' we have that $Y'_1 \times_{Y'} Y'_2$ is quasi-compact). Take Y' affine. Then the algebraic space Y is quasi-compact and quasi-separated. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we have $H^q(Y, \mathcal{F}) = H^q(Y', (Y \to Y')_*\mathcal{F})$ because the étale topoi are the same. Then $H^q(Y', (Y \to Y')_*\mathcal{F}) = 0$ because the pushforward is quasi-coherent (Morphisms of Spaces, Lemma 67.11.2) and Y is affine. It follows that Y' is affine by Cohomology of Spaces, Proposition 69.16.7 (there surely is a proof of this direction of the lemma avoiding this reference). Hence i is an affine morphism. In the affine case it follows easily from the conditions in Section 91.9 that i is a thickening of algebraic spaces. □

0D17 **Lemma 91.14.2.** *Let S be a scheme. Let $Y \subset Y'$ be a first order thickening of algebraic spaces over S. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. If there exists a flat morphism $f' : X' \to Y'$ of algebraic spaces over S and an isomorphsm $a : X \to X' \times_{Y'} Y$ over Y, then*

(1) *the set of isomorphism classes of pairs $(f' : X' \to Y', a)$ is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$, and*
(2) *the set of automorphisms of $\varphi : X' \to X'$ over Y' which reduce to the identity on $X' \times_{Y'} Y$ is $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/Y}, f^*\mathcal{C}_{Y/Y'})$.*

Proof. We will apply the material on deformations of ringed topoi to the small étale topoi of the algebraic spaces in the lemma. We may think of X as a closed subspace of X' so that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of first order thickenings. By Lemma 91.14.1 this translates into a morphism of thickenings of ringed topoi. Then we see from More on Morphisms of Spaces, Lemma 76.18.1 (or from the more general Lemma 91.11.2) that the ideal sheaf of X in X' is equal to $f^*\mathcal{C}_{Y'/Y}$ and this is in fact equivalent to flatness of X' over Y'. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathcal{C}_{Y/Y'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}_{small}\mathcal{C}_{Y/Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_{Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

Please compare with (91.13.0.1). Observe that automorphisms φ as in (2) give automorphisms $\varphi^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting in the diagram above. Conversely, an automorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ fitting into the diagram of sheaves above is equal to $\varphi^\sharp$ for some automorphism φ as in (2) by More on Morphisms of Spaces, Lemma 76.9.2. Finally, by More on Morphisms of Spaces, Lemma 76.9.7 if we find another sheaf of rings $\mathcal{A}$ on $X_{étale}$ fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathcal{C}_{Y/Y'} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}_{small}\mathcal{C}_{Y/Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_{Y'} & \longrightarrow & f^{-1}_{small}\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

then there exists a first order thickening $X \subset X''$ with $\mathcal{O}_{X''} = \mathcal{A}$ and applying More on Morphisms of Spaces, Lemma 76.9.2 once more, we obtain a morphism $(f, f'') : (X \subset X'') \to (Y \subset Y')$ with all the desired properties. Thus part (1) follows from Lemma 91.13.3 and part (2) from part (2) of Lemma 91.13.1. (Note that $NL_{X/Y}$ as defined for a morphism of algebraic spaces in More on Morphisms of Spaces, Section 76.21 agrees with $NL_{X/Y}$ as used in Section 91.13.) □

Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Consider the functor

$$F : \left\{ \begin{array}{c} \text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\ 0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\ \text{where } \mathcal{F} \text{ is an ideal of square zero} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{extensions of } f^{-1}\mathcal{O}_B \text{ algebras} \\ 0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O}_X \to 0 \\ \text{where } \mathcal{G} \text{ is an ideal of square zero} \end{array} \right\}$$

given by pushout.

0D3P **Lemma 91.14.3.** *In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 75.19) is an isomorphism. Then the functor F is an equivalence of categories.*

Proof. Recall that $NL_{X/B}$ is an object of $D_{QCoh}(\mathcal{O}_X)$, see More on Morphisms of Spaces, Lemma 76.21.4. Hence our assumption implies the maps

$$\mathrm{Ext}^i_X(NL_{X/B}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_X(NL_{X/B}, \mathcal{G})$$

are isomorphisms for all i. This implies our functor is fully faithful by Lemma 91.13.1. On the other hand, the functor is essentially surjective by Lemma 91.13.3 because we have the solutions $\mathcal{O}_X \oplus \mathcal{F}$ and $\mathcal{O}_X \oplus \mathcal{G}$ in both categories. □

Let S be a scheme. Let $B \subset B'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{J}$ which we view either as a quasi-coherent $\mathcal{O}_B$-module or as a quasi-coherent sheaf of ideals on B', see More on Morphisms of Spaces, Section 76.9. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules (not necessarily quasi-coherent). Let $c : f^{-1}\mathcal{J} \to \mathcal{F}$ be a map of $f^{-1}\mathcal{O}_B$-modules and denote $c' : f^{-1}\mathcal{J} \to \mathcal{G}$ the composition. Consider the functor

$$FT : \{\text{solutions to (91.13.0.1) for } \mathcal{F} \text{ and } c\} \longrightarrow \{\text{solutions to (91.13.0.1) for } \mathcal{G} \text{ and } c'\}$$

given by pushout.

0D3Q **Lemma 91.14.4.** *In the situation above assume that X is quasi-compact and quasi-separated and that $DQ_X(\mathcal{F}) \to DQ_X(\mathcal{G})$ (Derived Categories of Spaces, Section 75.19) is an isomorphism. Then the functor FT is an equivalence of categories.*

Proof. A solution of (91.13.0.1) for $\mathcal{F}$ in particular gives an extension of $f^{-1}\mathcal{O}_{B'}$-algebras

$$0 \to \mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X \to 0$$

where $\mathcal{F}$ is an ideal of square zero. Similarly for $\mathcal{G}$. Moreover, given such an extension, we obtain a map $c_{\mathcal{O}'} : f^{-1}\mathcal{J} \to \mathcal{F}$. Thus we are looking at the full subcategory of such extensions of $f^{-1}\mathcal{O}_{B'}$-algebras with $c = c_{\mathcal{O}'}$. Clearly, if $\mathcal{O}'' = F(\mathcal{O}')$ where F is the equivalence of Lemma 91.14.3 (applied to $X \to B'$ this time), then $c_{\mathcal{O}''}$ is the composition of $c_{\mathcal{O}'}$ and the map $\mathcal{F} \to \mathcal{G}$. This proves the lemma. □

91.15. Deformations of complexes

0DYQ This section is a warmup for the next one. We will use as much as possible the material in the chapters on commutative algebra.

0DYR **Lemma 91.15.1.** *Let $R' \to R$ be a surjection of rings whose kernel is an ideal I of square zero. For every $K \in D^-(R)$ there is a canonical map*

$$\omega(K) : K \longrightarrow K \otimes^{\mathbf{L}}_R I[2]$$

in $D(R)$ with the following properties

1. *$\omega(K) = 0$ if and only if there exists $K' \in D(R')$ with $K' \otimes^{\mathbf{L}}_{R'} R = K$,*
2. *given $K \to L$ in $D^-(R)$ the diagram*

$$\begin{array}{ccc} K & \xrightarrow{\omega(K)} & K \otimes^{\mathbf{L}}_R I[2] \\ \downarrow & & \downarrow \\ L & \xrightarrow{\omega(L)} & L \otimes^{\mathbf{L}}_R I[2] \end{array}$$

   commutes, and
3. *formation of $\omega(K)$ is compatible with ring maps $R' \to S'$ (see proof for a precise statement).*

Proof. Choose a bounded above complex $K^\bullet$ of free R-modules representing K. Then we can choose free R'-modules $(K')^n$ lifting K^n. We can choose R'-module maps $(d')^n_K : (K')^n \to (K')^{n+1}$ lifting the differentials $d^n_K : K^n \to K^{n+1}$ of $K^\bullet$. Although the compositions

$$(d')^{n+1}_K \circ (d')^n_K : (K')^n \to (K')^{n+2}$$

may not be zero, they do factor as

$$(K')^n \to K^n \xrightarrow{\omega^n_K} K^{n+2} \otimes_R I = I(K')^{n+2} \to (K')^{n+2}$$

because $d^{n+1} \circ d^n = 0$. A calculation shows that ω^n_K defines a map of complexes. This map of complexes defines $\omega(K)$.

Let us prove this construction is compatible with a map of complexes $\alpha^\bullet : K^\bullet \to L^\bullet$ of bounded above free R-modules and given choices of lifts $(K')^n, (L')^n, (d')^n_K, (d')^n_L$. Namely, choose $(\alpha')^n : (K')^n \to (L')^n$ lifting the components $\alpha^n : K^n \to L^n$. As before we get a factorization

$$(K')^n \to K^n \xrightarrow{h^n} L^{n+1} \otimes_R I = I(L')^{n+1} \to (L')^{n+2}$$

of $(d')^n_L \circ (\alpha')^n - (\alpha')^{n+1} \circ (d')^n_K$. Then it is an pleasant calculation to show that

$$\omega^n_L \circ \alpha^n = (d^{n+1}_L \otimes \mathrm{id}_I) \circ h^n + h^{n+1} \circ d^n_K + (\alpha^{n+2} \otimes \mathrm{id}_I) \circ \omega^n_K$$

This proves the commutativity of the diagram in (2) of the lemma in this particular case. Using this for two different choices of bounded above free complexes representing K, we find that $\omega(K)$ is well defined! And of course (2) holds in general as well.

If K lifts to K' in $D^-(R')$, then we can represent K' by a bounded above complex of free R'-modules and we see immediately that $\omega(K) = 0$. Conversely, going back to our choices $K^\bullet$, $(K')^n$, $(d')^n_K$, if $\omega(K) = 0$, then we can find $g^n : K^n \to K^{n+1} \otimes_R I$ with

$$\omega^n = (d^{n+1}_K \otimes \mathrm{id}_I) \circ g^n + g^{n+1} \circ d^n_K$$

This means that with differentials $(d')^n_K - g^n : (K')^n \to (K')^{n+1}$ we obtain a complex of free R'-modules lifting $K^\bullet$. This proves (1).

Finally, part (3) means the following: Let $R' \to S'$ be a map of rings. Set $S = S' \otimes_{R'} R$ and denote $J = IS' \subset S'$ the square zero kernel of $S' \to S$. Then given $K \in D^-(R)$ the statement is that we get a commutative diagram

$$\begin{array}{ccc} K \otimes^{\mathbf{L}}_R S & \xrightarrow[\omega(K)\otimes\mathrm{id}]{} & (K \otimes^{\mathbf{L}}_R I[2]) \otimes^{\mathbf{L}}_R S \\ \downarrow & & \downarrow \\ K \otimes^{\mathbf{L}}_R S & \xrightarrow{\omega(K \otimes^{\mathbf{L}}_R S)} & (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S J[2] \end{array}$$

Here the right vertical arrow comes from

$$(K \otimes^{\mathbf{L}}_R I[2]) \otimes^{\mathbf{L}}_R S = (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (I \otimes^{\mathbf{L}}_R S)[2] \longrightarrow (K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S J[2]$$

Choose $K^\bullet$, $(K')^n$, and $(d')^n_K$ as above. Then we can use $K^\bullet \otimes_R S$, $(K')^n \otimes_{R'} S'$, and $(d')^n_K \otimes \mathrm{id}_{S'}$ for the construction of $\omega(K \otimes^{\mathbf{L}}_R S)$. With these choices commutativity is immediately verified on the level of maps of complexes. $\square$

91.16. Deformations of complexes on ringed topoi

0DIS This material is taken from [**Lie06a**].

The material in this section works in the setting of a first order thickening of ringed topoi as defined in Section 91.9. However, in order to simplify the notation we will assume the underlying sites $\mathcal{C}$ and $\mathcal{D}$ are the same. Moreover, the surjective homomorphism $\mathcal{O}' \to \mathcal{O}$ of sheaves of rings will be denoted $\mathcal{O} \to \mathcal{O}_0$ as is perhaps more customary in the literature.

0DIT **Lemma 91.16.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data*

(1) *flat $\mathcal{O}$-modules $\mathcal{G}^n$,*
(2) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{G}^{n+1}$,*
(3) *a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,*
(4) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{K}_0^n$*

such that

(a) *$H^n(\mathcal{K}_0^\bullet) = 0$ for $n \gg 0$,*
(b) *$\mathcal{G}^n = 0$ for $n \gg 0$,*
(c) *with $\mathcal{G}_0^n = \mathcal{G}^n \otimes_{\mathcal{O}} \mathcal{O}_0$ the induced maps determine a complex $\mathcal{G}_0^\bullet$ and a map of complexes $\mathcal{G}_0^\bullet \to \mathcal{K}_0^\bullet$.*

Then there exist

(i) *flat $\mathcal{O}$-modules $\mathcal{F}^n$,*
(ii) *maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{F}^{n+1}$,*
(iii) *maps of $\mathcal{O}$-modules $\mathcal{F}^n \to \mathcal{K}_0^n$,*
(iv) *maps of $\mathcal{O}$-modules $\mathcal{G}^n \to \mathcal{F}^n$,*

such that $\mathcal{F}^n = 0$ for $n \gg 0$, such that the diagrams

$$\begin{array}{ccc} \mathcal{G}^n & \longrightarrow & \mathcal{G}^{n+1} \\ \downarrow & & \downarrow \\ \mathcal{F}^n & \longrightarrow & \mathcal{F}^{n+1} \end{array}$$

commute for all n, such that the composition $\mathcal{G}^n \to \mathcal{F}^n \to \mathcal{K}_0^n$ is the given map $\mathcal{G}^n \to \mathcal{K}_0^n$, and such that with $\mathcal{F}_0^n = \mathcal{F}^n \otimes_{\mathcal{O}} \mathcal{O}_0$ we obtain a complex $\mathcal{F}_0^\bullet$ and map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ which is a quasi-isomorphism.

Proof. We will prove by descending induction on e that we can find $\mathcal{F}^n$, $\mathcal{G}^n \to \mathcal{F}^n$, and $\mathcal{F}^n \to \mathcal{F}^{n+1}$ for $n \geq e$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathcal{G}^{e-1} & \longrightarrow & \mathcal{G}^e & \longrightarrow & \mathcal{G}^{e+1} & \longrightarrow & \ldots \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{F}^e & \longrightarrow & \mathcal{F}^{e+1} & \longrightarrow & \ldots \\ & & & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathcal{K}_0^{e-1} & \longrightarrow & \mathcal{K}_0^e & \longrightarrow & \mathcal{K}_0^{e+1} & \longrightarrow & \ldots \end{array}$$

such that $\mathcal{F}_0^\bullet$ is a complex, the induced map $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n > e$ and a surjection for $n = e$. For $e \gg 0$ this is true because we can take $\mathcal{F}^n = 0$ for $n \geq e$ in that case by assumptions (a) and (b).

Induction step. We have to construct $\mathcal{F}^{e-1}$ and the maps $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$, $\mathcal{F}^{e-1} \to \mathcal{F}^e$, and $\mathcal{F}^{e-1} \to \mathcal{K}_0^{e-1}$. We will choose $\mathcal{F}^{e-1} = A \oplus B \oplus C$ as a direct sum of three pieces.

For the first we take $A = \mathcal{G}^{e-1}$ and we choose our map $\mathcal{G}^{e-1} \to \mathcal{F}^{e-1}$ to be the inclusion of the first summand. The maps $A \to \mathcal{K}_0^{e-1}$ and $A \to \mathcal{F}^e$ will be the obvious ones.

To choose B we consider the surjection (by induction hypothesis)

$$\gamma : \mathrm{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1}) \longrightarrow \mathrm{Ker}(\mathcal{K}_0^e \to \mathcal{K}_0^{e+1}) / \mathrm{Im}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)$$

We can choose a set I, for each $i \in I$ an object U_i of $\mathcal{C}$, and sections $s_i \in \mathcal{F}^e(U_i)$, $t_i \in \mathcal{K}_0^{e-1}(U_i)$ such that

(1) s_i maps to a section of $\mathrm{Ker}(\gamma) \subset \mathrm{Ker}(\mathcal{F}_0^e \to \mathcal{F}_0^{e+1})$,
(2) s_i and t_i map to the same section of $\mathcal{K}_0^e$,
(3) the sections s_i generate $\mathrm{Ker}(\gamma)$ as an $\mathcal{O}_0$-module.

We omit giving the full justification for this; one uses that $\mathcal{F}^e \to \mathcal{F}_0^e$ is a surjective maps of sheaves of sets. Then we set to put

$$B = \bigoplus\nolimits_{i \in I} j_{U_i!}\mathcal{O}_{U_i}$$

and define the maps $B \to \mathcal{F}^e$ and $B \to \mathcal{K}_0^{e-1}$ by using s_i and t_i to determine where to send the summand $j_{U_i!}\mathcal{O}_{U_i}$.

With $\mathcal{F}^{e-1} = A \oplus B$ and maps as above, this produces a diagram as above for $e-1$ such that $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ induces an isomorphism on H^n for $n \geq e$. To get the map to be surjective on H^{e-1} we choose the summand C as follows. Choose a set J, for each $j \in J$ an object U_j of $\mathcal{C}$ and a section t_j of $\mathrm{Ker}(\mathcal{K}_0^{e-1} \to \mathcal{K}_0^e)$ over U_j such that these sections generate this kernel over $\mathcal{O}_0$. Then we put

$$C = \bigoplus\nolimits_{j \in J} j_{U_j!}\mathcal{O}_{U_j}$$

and the zero map $C \to \mathcal{F}^e$ and the map $C \to \mathcal{K}_0^{e-1}$ by using s_j to determine where to the summand $j_{U_j!}\mathcal{O}_{U_j}$. This finishes the induction step by taking $\mathcal{F}^{e-1} = A \oplus B \oplus C$ and maps as indicated. □

0DIU **Lemma 91.16.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ there is a canonical map*

$$\omega(K_0) : K_0 \longrightarrow K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}[2]$$

in $D(\mathcal{O}_0)$ such that for any map $K_0 \to L_0$ in $D^-(\mathcal{O}_0)$ the diagram

$$\begin{array}{ccc} K_0 & \xrightarrow{\omega(K_0)} & (K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})[2] \\ \downarrow & & \downarrow \\ L_0 & \xrightarrow{\omega(L_0)} & (L_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})[2] \end{array}$$

commutes.

Proof. Represent K_0 by any complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules. Apply Lemma 91.16.1 with $\mathcal{G}^n = 0$ for all n. Denote $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ the maps produced by the lemma. Then we see that $d \circ d : \mathcal{F}^n \to \mathcal{F}^{n+2}$ is zero modulo $\mathcal{I}$. Since $\mathcal{F}^n$ is flat, we see that $\mathcal{I}\mathcal{F}^n = \mathcal{F}^n \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{F}_0^n \otimes_{\mathcal{O}_0} \mathcal{I}$. Hence we obtain a canonical map of complexes

$$d \circ d : \mathcal{F}_0^\bullet \longrightarrow (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

Since $\mathcal{F}_0^\bullet$ is a bounded above complex of flat $\mathcal{O}_0$-modules, it is K-flat and may be used to compute derived tensor product. Moreover, the map of complexes $\mathcal{F}_0^\bullet \to \mathcal{K}_0^\bullet$ is a quasi-isomorphism by construction. Therefore the source and target of the map just constructed represent K_0 and $K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}[2]$ and we obtain our map $\omega(K_0)$.

Let us show that this procedure is compatible with maps of complexes. Namely, let $\mathcal{L}_0^\bullet$ represent another object of $D^-(\mathcal{O}_0)$ and suppose that

$$\mathcal{K}_0^\bullet \longrightarrow \mathcal{L}_0^\bullet$$

is a map of complexes. Apply Lemma 91.16.1 for the complex $\mathcal{L}_0^\bullet$, the flat modules $\mathcal{F}^n$, the maps $\mathcal{F}^n \to \mathcal{F}^{n+1}$, and the compositions $\mathcal{F}^n \to \mathcal{K}_0^n \to \mathcal{L}_0^n$ (we apologize for the reversal of letters used). We obtain flat modules $\mathcal{G}^n$, maps $\mathcal{F}^n \to \mathcal{G}^n$, maps $\mathcal{G}^n \to \mathcal{G}^{n+1}$, and maps $\mathcal{G}^n \to \mathcal{L}_0^n$ with all properties as in the lemma. Then it is clear that

$$\begin{array}{ccc} \mathcal{F}_0^\bullet & \longrightarrow & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \\ \downarrow & & \downarrow \\ \mathcal{G}_0^\bullet & \longrightarrow & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \end{array}$$

is a commutative diagram of complexes.

To see that $\omega(K_0)$ is well defined, suppose that we have two complexes $\mathcal{K}_0^\bullet$ and $(\mathcal{K}'_0)^\bullet$ of $\mathcal{O}_0$-modules representing K_0 and two systems $(\mathcal{F}^n, d : \mathcal{F}^n \to \mathcal{F}^{n+1}, \mathcal{F}^n \to \mathcal{K}_0^n)$ and $((\mathcal{F}')^n, d : (\mathcal{F}')^n \to (\mathcal{F}')^{n+1}, (\mathcal{F}')^n \to \mathcal{K}_0^n)$ as above. Then we can choose a complex $(\mathcal{K}''_0)^\bullet$ and quasi-isomorphisms $\mathcal{K}_0^\bullet \to (\mathcal{K}''_0)^\bullet$ and $(\mathcal{K}'_0)^\bullet \to (\mathcal{K}''_0)^\bullet$ realizing the fact that both complexes represent K_0 in the derived category. Next, we apply the result of the previous paragraph to

$$(\mathcal{K}_0)^\bullet \oplus (\mathcal{K}'_0)^\bullet \longrightarrow (\mathcal{K}''_0)^\bullet$$

This produces a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_0^\bullet \oplus (\mathcal{F}'_0)^\bullet & \longrightarrow & (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \oplus ((\mathcal{F}'_0)^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \\ \downarrow & & \downarrow \\ \mathcal{G}_0^\bullet & \longrightarrow & (\mathcal{G}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2] \end{array}$$

Since the vertical arrows give quasi-isomorphisms on the summands we conclude the desired commutativity in $D(\mathcal{O}_0)$.

Having established well-definedness, the statement on compatibility with maps is a consequence of the result in the second paragraph. □

0DIV **Lemma 91.16.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : K \to L$ be a map of $D^-(\mathcal{O})$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $n \in \mathbf{Z}$.*

(1) *If $H^i(\alpha)$ is an isomorphism for $i \geq n$, then $H^i(\alpha \otimes^{\mathbf{L}}_{\mathcal{O}} id_{\mathcal{F}})$ is an isomorphism for $i \geq n$.*

(2) *If $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, then $H^i(\alpha \otimes^{\mathbf{L}}_{\mathcal{O}} id_{\mathcal{F}})$ is an isomorphism for $i > n$ and surjective for $i = n$.*

Proof. Choose a distinguished triangle

$$K \to L \to C \to K[1]$$

In case (2) we see that $H^i(C) = 0$ for $i \geq n$. Hence $H^i(C \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) = 0$ for $i \geq n$ by (the dual of) Derived Categories, Lemma 13.16.1. This in turn shows that $H^i(\alpha \otimes^{\mathbf{L}}_{\mathcal{O}} \mathrm{id}_{\mathcal{F}})$ is an isomorphism for $i > n$ and surjective for $i = n$. In case (1) we moreover see that $H^{n-1}(L) \to H^{n-1}(C)$ is surjective. Considering the diagram

$$\begin{array}{ccc} H^{n-1}(L) \otimes_{\mathcal{O}} \mathcal{F} & \longrightarrow & H^{n-1}(C) \otimes_{\mathcal{O}} \mathcal{F} \\ \downarrow & & \| \\ H^{n-1}(L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) & \longrightarrow & H^{n-1}(C \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) \end{array}$$

we conclude the lower horizontal arrow is surjective. Combined with what was said before this implies that $H^n(\alpha \otimes^{\mathbf{L}}_{\mathcal{O}} \mathrm{id}_{\mathcal{F}})$ is an isomorphism. □

0DIW **Lemma 91.16.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. For every object K_0 in $D^-(\mathcal{O}_0)$ the following are equivalent*

(1) *the class $\omega(K_0) \in \mathrm{Ext}^2_{\mathcal{O}_0}(K_0, K_0 \otimes_{\mathcal{O}_0} \mathcal{I})$ constructed in Lemma 91.16.2 is zero,*
(2) *there exists $K \in D^-(\mathcal{O})$ with $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 = K_0$ in $D(\mathcal{O}_0)$.*

Proof. Let K be as in (2). Then we can represent K by a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules. Then $\mathcal{F}_0^\bullet = \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$ represents K_0 in $D(\mathcal{O}_0)$. Since $d_{\mathcal{F}^\bullet} \circ d_{\mathcal{F}^\bullet} = 0$ as $\mathcal{F}^\bullet$ is a complex, we see from the very construction of $\omega(K_0)$ that it is zero.

Assume (1). Let $\mathcal{F}^n$, $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ be as in the construction of $\omega(K_0)$. The nullity of $\omega(K_0)$ implies that the map

$$\omega = d \circ d : \mathcal{F}_0^\bullet \longrightarrow (\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I})[2]$$

is zero in $D(\mathcal{O}_0)$. By definition of the derived category as the localization of the homotopy category of complexes of $\mathcal{O}_0$-modules, there exists a quasi-isomorphism $\alpha : \mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$ such that there exist $\mathcal{O}_0$-modules maps $h^n : \mathcal{G}_0^n \to \mathcal{F}_0^{n+1} \otimes_{\mathcal{O}} \mathcal{I}$ with

$$\omega \circ \alpha = d_{\mathcal{F}_0^\bullet \otimes \mathcal{I}} \circ h + h \circ d_{\mathcal{G}_0^\bullet}$$

We set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{G}_0^n$$

and we define

$$d' : \mathcal{H}^n \longrightarrow \mathcal{H}^{n+1}, \quad (f^n, g_0^n) \longmapsto (d(f^n) - h^n(g_0^n), d(g_0^n))$$

with obvious notation using that $\mathcal{F}_0^{n+1} \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. Then one checks $d' \circ d' = 0$ by our choice of h^n and definition of ω. Hence $\mathcal{H}^\bullet$ defines an object in $D(\mathcal{O})$. On the other hand, there is a short exact sequence of complexes of $\mathcal{O}$-modules

$$0 \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}_0^\bullet \to 0$$

We still have to show that $\mathcal{H}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ is isomorphic to K_0. Choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{H}^\bullet$ where $\mathcal{E}^\bullet$ is a bounded above complex of flat $\mathcal{O}$-modules. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I} & \longrightarrow & \mathcal{E}^\bullet & \longrightarrow & \mathcal{E}_0^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} & \longrightarrow & \mathcal{H}^\bullet & \longrightarrow & \mathcal{G}_0^\bullet & \longrightarrow & 0 \end{array}$$

We claim that δ is a quasi-isomorphism. Since $H^i(\delta)$ is an isomorphism for $i \gg 0$, we can use descending induction on n such that $H^i(\delta)$ is an isomorphism for $i \geq n$. Observe that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}$ represents $\mathcal{E}_0^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}$, that $\mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}$ represents $\mathcal{G}_0^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I}$, and that $\beta = \delta \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathrm{id}_{\mathcal{I}}$ as maps in $D(\mathcal{O}_0)$. This is true because $\beta = (\alpha \otimes \mathrm{id}_{\mathcal{I}}) \circ (\delta \otimes \mathrm{id}_{\mathcal{I}})$. Suppose that $H^i(\delta)$ is an isomorphism in degrees $\geq n$. Then the same is true for β by what we just said and Lemma 91.16.3. Then we can look at the diagram

$$\begin{array}{ccccccccccc} H^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}) & \longrightarrow & H^{n-1}(\mathcal{E}^\bullet) & \longrightarrow & H^{n-1}(\mathcal{E}_0^\bullet) & \longrightarrow & H^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{I}) & \longrightarrow & H^n(\mathcal{E}^\bullet) \\ \downarrow{\scriptstyle H^{n-1}(\beta)} & & \downarrow & & \downarrow{\scriptstyle H^{n-1}(\delta)} & & \downarrow{\scriptstyle H^n(\beta)} & & \downarrow \\ H^{n-1}(\mathcal{F}_0^\bullet \otimes_{\mathcal{O}} \mathcal{I}) & \longrightarrow & H^{n-1}(\mathcal{H}^\bullet) & \longrightarrow & H^{n-1}(\mathcal{G}_0^\bullet) & \longrightarrow & H^n(\mathcal{F}_0^\bullet \otimes_{\mathcal{O}} \mathcal{I}) & \longrightarrow & H^n(\mathcal{H}^\bullet) \end{array}$$

Using Homology, Lemma 12.5.19 we see that $H^{n-1}(\delta)$ is surjective. This in turn implies that $H^{n-1}(\beta)$ is surjective by Lemma 91.16.3. Using Homology, Lemma 12.5.19 again we see that $H^{n-1}(\delta)$ is an isomorphism. The claim holds by induction, so δ is a quasi-isomorphism which is what we wanted to show. □

0DIX **Lemma 91.16.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings. Assume given the following data*

(1) *a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$,*
(2) *a complex $\mathcal{K}_0^\bullet$ of $\mathcal{O}_0$-modules,*
(3) *a quasi-isomorphism $\mathcal{K}_0^\bullet \to \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$,*

Then there exist a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ such that the map of complexes $\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0 \to \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$ factors through $\mathcal{K}_0^\bullet$ in the homotopy category of complexes of $\mathcal{O}_0$-modules.

Proof. Set $\mathcal{F}_0^\bullet = \mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}_0$. By Derived Categories, Lemma 13.9.8 there exists a factorization

$$\mathcal{K}_0^\bullet \to \mathcal{L}_0^\bullet \to \mathcal{F}_0^\bullet$$

of the given map such that the first arrow has an inverse up to homotopy and the second arrow is termwise split surjective. Hence we may assume that $\mathcal{K}_0^\bullet \to \mathcal{F}_0^\bullet$ is termwise surjective. In that case we take

$$\mathcal{G}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{K}_0^n$$

and everything is clear. □

0DIY **Lemma 91.16.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K, L \in D^-(\mathcal{O})$. Set $K_0 = K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ and $L_0 = L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ in $D^-(\mathcal{O}_0)$. Given $\alpha_0 : K_0 \to L_0$ in $D(\mathcal{O}_0)$ there is a canonical element*

$$o(\alpha_0) \in \mathrm{Ext}^1_{\mathcal{O}_0}(K_0, L_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

whose vanishing is necessary and sufficient for the existence of a map $\alpha : K \to L$ in $D(\mathcal{O})$ with $\alpha_0 = \alpha \otimes^{\mathbf{L}}_{\mathcal{O}} id$.

Proof. Finding $\alpha : K \to L$ lifing α_0 is the same as finding $\alpha : K \to L$ such that the composition $K \xrightarrow{\alpha} L \to L_0$ is equal to the composition $K \to K_0 \xrightarrow{\alpha_0} L_0$. The short exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}_0 \to 0$ gives rise to a canonical distinguished triangle

$$L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I} \to L \to L_0 \to (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

in $D(\mathcal{O})$. By Derived Categories, Lemma 13.4.2 the composition

$$K \to K_0 \xrightarrow{\alpha_0} L_0 \to (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

is zero if and only if we can find $\alpha : K \to L$ lifting α_0. The composition is an element in

$$\mathrm{Hom}_{D(\mathcal{O})}(K, (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]) = \mathrm{Hom}_{D(\mathcal{O}_0)}(K_0, (L \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]) = \mathrm{Ext}^1_{\mathcal{O}_0}(K_0, L_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

by adjunction. □

0DIZ **Lemma 91.16.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}_0$ be a surjection of sheaves of rings whose kernel is an ideal sheaf $\mathcal{I}$ of square zero. Let $K_0 \in D^-(\mathcal{O})$. A lift of K_0 is a pair (K, α_0) consisting of an object K in $D^-(\mathcal{O})$ and an isomorphism $\alpha_0 : K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 \to K_0$ in $D(\mathcal{O}_0)$.*

(1) *Given a lift (K, α) the group of automorphism of the pair is canonically the cokernel of a map*

$$\mathrm{Ext}^{-1}_{\mathcal{O}_0}(K_0, K_0) \longrightarrow \mathrm{Hom}_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

(2) *If there is a lift, then the set of isomorphism classes of lifts is principal homogenenous under $\mathrm{Ext}^1_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$.*

Proof. An automorphism of (K, α) is a map $\varphi : K \to K$ in $D(\mathcal{O})$ with $\varphi \otimes_{\mathcal{O}} \mathrm{id}_{\mathcal{O}_0} = \mathrm{id}$. This is the same thing as saying that

$$K \xrightarrow{\varphi - \mathrm{id}} K \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$$

is zero. We conclude the group of automorphisms is the cokernel of a map

$$\mathrm{Hom}_{\mathcal{O}}(K, K_0[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(K, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$$

by the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I} \to K \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 \to (K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I})[1]$$

in $D(\mathcal{O})$ and Derived Categories, Lemma 13.4.2. To translate into the groups in the lemma use adjunction of the restriction functor $D(\mathcal{O}_0) \to D(\mathcal{O})$ and $- \otimes_{\mathcal{O}} \mathcal{O}_0 : D(\mathcal{O}) \to D(\mathcal{O}_0)$. This proves (1).

Proof of (2). Assume that $K_0 = K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0$ in $D(\mathcal{O})$. By Lemma 91.16.6 the map sending a lift (K', α_0) to the obstruction $o(\alpha_0)$ to lifting α_0 defines a canonical injective map from the set of isomomorphism classes of pairs to $\mathrm{Ext}^1_{\mathcal{O}_0}(K_0, K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})$. To finish the proof we show that it is surjective. Pick $\xi : K_0 \to (K_0 \otimes^{\mathbf{L}}_{\mathcal{O}_0} \mathcal{I})[1]$ in the Ext^1 of the lemma. Choose a bounded above complex $\mathcal{F}^\bullet$ of flat $\mathcal{O}$-modules representing K. The map ξ can be represented as $t \circ s^{-1}$ where $s : \mathcal{K}^\bullet_0 \to \mathcal{F}^\bullet_0$ is a quasi-isomorphism and $t : \mathcal{K}^\bullet_0 \to \mathcal{F}^\bullet_0 \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ is a map of complexes. By Lemma 91.16.5 we can assume there exists a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ of complexes of $\mathcal{O}$-modules such that $\mathcal{G}^\bullet_0 \to \mathcal{F}^\bullet_0$ factors through s up to homotopy. We may and do

replace $\mathcal{G}^\bullet$ by a bounded above complex of flat $\mathcal{O}$-modules (by picking a qis from such to $\mathcal{G}^\bullet$ and replacing). Then we see that ξ is represented by a map of complexes $t : \mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I}[1]$ and the quasi-isomorphism $\mathcal{G}_0^\bullet \to \mathcal{F}_0^\bullet$. Set

$$\mathcal{H}^n = \mathcal{F}^n \times_{\mathcal{F}_0^n} \mathcal{G}_0^n$$

with differentials

$$\mathcal{H}^n \to \mathcal{H}^{n+1}, \quad (f^n, g_0^n) \mapsto (d(f^n) + t(g_0^n), d(g_0^n))$$

This makes sense as $\mathcal{F}_0^{n+1} \otimes_{\mathcal{O}_0} \mathcal{I} = \mathcal{F}^{n+1} \otimes_{\mathcal{O}} \mathcal{I} = \mathcal{I}\mathcal{F}^{n+1} \subset \mathcal{F}^{n+1}$. We omit the computation that shows that $\mathcal{H}^\bullet$ is a complex of $\mathcal{O}$-modules. By construction there is a short exact sequence

$$0 \to \mathcal{F}_0^\bullet \otimes_{\mathcal{O}_0} \mathcal{I} \to \mathcal{H}^\bullet \to \mathcal{G}_0^\bullet \to 0$$

of complexes of $\mathcal{O}$-modules. Exactly as in the proof of Lemma 91.16.4 one shows that this sequence induces an isomorphism $\alpha_0 : \mathcal{H}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}_0 \to \mathcal{G}_0^\bullet$ in $D(\mathcal{O}_0)$. In other words, we have produced a pair $(\mathcal{H}^\bullet, \alpha_0)$. We omit the verification that $o(\alpha_0) = \xi$; hint: $o(\alpha_0)$ can be computed explicitly in this case as we have maps $\mathcal{H}^n \to \mathcal{F}^n$ (not compatible with differentials) lifting the components of α_0. This finishes the proof. □

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CHAPTER 92

The Cotangent Complex

08P5

92.1. Introduction

08P6 The goal of this chapter is to construct the cotangent complex of a ring map, of a morphism of schemes, and of a morphism of algebraic spaces. Some references are the notes [**Qui**], the paper [**Qui70**], and the books [**And67**] and [**Ill72**].

92.2. Advice for the reader

08UM In writing this chapter we have tried to minimize the use of simplicial techniques. We view the choice of a *resolution* $P_\bullet$ of a ring B over a ring A as a tool to calculating the *homology* of abelian sheaves on the category $\mathcal{C}_{B/A}$, see Remark 92.5.5. This is similar to the role played by a "good cover" to compute cohomology using the Čech complex. To read a bit on homology on categories, please visit Cohomology on Sites, Section 21.39. The derived lower shriek functor $L\pi_!$ is to homology what $R\Gamma(\mathcal{C}_{B/A}, -)$ is to cohomology. The category $\mathcal{C}_{B/A}$, studied in Section 92.4, is the opposite of the category of factorizations $A \to P \to B$ where P is a polynomial algebra over A. This category comes with maps of sheaves of rings

$$\underline{A} \longrightarrow \mathcal{O} \longrightarrow \underline{B}$$

where over the object $U = (P \to B)$ we have $\mathcal{O}(U) = P$. It turns out that we obtain the cotangent complex of B over A as

$$L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{B})$$

see Lemma 92.4.3. We have consistently tried to use this point of view to prove the basic properties of cotangent complexes of ring maps. In particular, all of the results can be proven without relying on the existence of standard resolutions, although we have not done so. The theory is quite satisfactory, except that perhaps the proof of the fundamental triangle (Proposition 92.7.4) uses just a little bit more theory on derived lower shriek functors. To provide the reader with an alternative, we give a rather complete sketch of an approach to this result based on simple properties of standard resolutions in Remarks 92.7.5 and 92.7.6.

Our approach to the cotangent complex for morphisms of ringed topoi, morphisms of schemes, morphisms of algebraic spaces, etc is to deduce as much as possible from the case of "plain ring maps" discussed above.

92.3. The cotangent complex of a ring map

08PL Let A be a ring. Let Alg_A be the category of A-algebras. Consider the pair of adjoint functors (U, V) where $V : Alg_A \to Sets$ is the forgetful functor and $U : Sets \to Alg_A$ assigns to a set E the polynomial algebra $A[E]$ on E over A. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(Alg_A, Alg_A)$ constructed in Simplicial, Section 14.34.

Consider an A-algebra B. Denote $P_\bullet = X_\bullet(B)$ the resulting simplicial A-algebra. Recall that $P_0 = A[B]$, $P_1 = A[A[B]]$, and so on. In particular each term P_n is a polynomial A-algebra. Recall also that there is an augmentation

$$\epsilon : P_\bullet \longrightarrow B$$

where we view B as a constant simplicial A-algebra.

08PM **Definition 92.3.1.** Let $A \to B$ be a ring map. The *standard resolution of B over A* is the augmentation $\epsilon : P_\bullet \to B$ with terms

$$P_0 = A[B], \quad P_1 = A[A[B]], \quad \ldots$$

and maps as constructed above.

It will turn out that we can use the standard resolution to compute left derived functors in certain settings.

08PN **Definition 92.3.2.** The *cotangent complex* $L_{B/A}$ of a ring map $A \to B$ is the complex of B-modules associated to the simplicial B-module

$$\Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

where $\epsilon : P_\bullet \to B$ is the standard resolution of B over A.

In Simplicial, Section 14.23 we associate a chain complex to a simplicial module, but here we work with cochain complexes. Thus the term $L^{-n}_{B/A}$ in degree $-n$ is the B-module $\Omega_{P_n/A} \otimes_{P_n, \epsilon_n} B$ and $L^m_{B/A} = 0$ for $m > 0$.

08PP **Remark 92.3.3.** Let $A \to B$ be a ring map. Let $\mathcal{A}$ be the category of arrows $\psi : C \to B$ of A-algebras and let $\mathcal{S}$ be the category of maps $E \to B$ where E is a set. There are adjoint functors $V : \mathcal{A} \to \mathcal{S}$ (the forgetful functor) and $U : \mathcal{S} \to \mathcal{A}$ which sends $E \to B$ to $A[E] \to B$. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ constructed in Simplicial, Section 14.34. The diagram

$$\begin{array}{ccc} \mathcal{A} & \rightleftarrows & \mathcal{S} \\ \downarrow & & \downarrow \\ Alg_A & \rightleftarrows & Sets \end{array}$$

commutes. It follows that $X_\bullet(\mathrm{id}_B : B \to B)$ is equal to the standard resolution of B over A.

08S9 **Lemma 92.3.4.** *Let $A_i \to B_i$ be a system of ring maps over a directed index set I. Then* $\operatorname{colim} L_{B_i/A_i} = L_{\operatorname{colim} B_i / \operatorname{colim} A_i}$.

Proof. This is true because the forgetful functor $V : A\text{-}Alg \to Sets$ and its adjoint $U : Sets \to A\text{-}Alg$ commute with filtered colimits. Moreover, the functor $B/A \mapsto \Omega_{B/A}$ does as well (Algebra, Lemma 10.131.5). □

92.4. Simplicial resolutions and derived lower shriek

08PQ Let $A \to B$ be a ring map. Consider the category whose objects are A-algebra maps $\alpha : P \to B$ where P is a polynomial algebra over A (in some set[1] of variables) and whose morphisms $s : (\alpha : P \to B) \to (\alpha' : P' \to B)$ are A-algebra homomorphisms $s : P \to P'$ with $\alpha' \circ s = \alpha$. Let $\mathcal{C} = \mathcal{C}_{B/A}$ denote the **opposite** of this category.

[1] It suffices to consider sets of cardinality at most the cardinality of B.

The reason for taking the opposite is that we want to think of objects (P, α) as corresponding to the diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

We endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6), i.e., we endow $\mathcal{C}$ with the structure of a site where coverings are given by identities so that all presheaves are sheaves. Moreover, we endow $\mathcal{C}$ with two sheaves of rings. The first is the sheaf $\mathcal{O}$ which sends to object (P, α) to P. Then second is the constant sheaf B, which we will denote $\underline{B}$. We obtain the following diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}), \underline{B}) & \xrightarrow[i]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ {\scriptstyle \pi}\downarrow & & \\ (Sh(*), B) & & \end{array} \tag{92.4.0.1}$$

08PR

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{B}$ is the obvious map. The map π is as in Cohomology on Sites, Example 21.39.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{B})$ left adjoint to $Ri_* = i_* : D(\underline{B}) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{B}) \longrightarrow D(B)$ left adjoint to $\pi^* = \pi^{-1} : D(B) \to D(\underline{B})$.

08PS **Lemma 92.4.1.** *With notation as above let $P_\bullet$ be a simplicial A-algebra endowed with an augmentation $\epsilon : P_\bullet \to B$. Assume each P_n is a polynomial algebra over A and ϵ is a trivial Kan fibration on underlying simplicial sets. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. We will use the criterion of Cohomology on Sites, Lemma 21.39.7 to prove this. Given an object $U = (Q, \beta)$ of $\mathcal{C}$ we have to show that

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}((P_\bullet, \epsilon), (Q, \beta))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). We see that

$$S_\bullet = \mathrm{Mor}_{Sets}((E, \beta|_E), (P_\bullet, \epsilon))$$

Let $*$ be the constant simplicial set on a singleton. For $b \in B$ let $F_{b,\bullet}$ be the simplicial set defined by the cartesian diagram

$$\begin{array}{ccc} F_{b,\bullet} & \longrightarrow & P_\bullet \\ \downarrow & & {\scriptstyle \epsilon}\downarrow \\ * & \xrightarrow{b} & B \end{array}$$

With this notation $S_\bullet = \prod_{e \in E} F_{\beta(e),\bullet}$. Since we assumed ϵ is a trivial Kan fibration we see that $F_{b,\bullet} \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.30.3). Thus $S_\bullet \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.30.6). Therefore $S_\bullet$ is homotopy equivalent to $*$ (Simplicial, Lemma 14.30.8). $\square$

In particular, we can use the standard resolution of B over A to compute derived lower shriek.

08PT **Lemma 92.4.2.** *Let $A \to B$ be a ring map. Let $\epsilon : P_\bullet \to B$ be the standard resolution of B over A. Let π be as in (92.4.0.1). Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

First proof. We will apply Lemma 92.4.1. Since the terms P_n are polynomial algebras we see the first assumption of that lemma is satisfied. The second assumption is proved as follows. By Simplicial, Lemma 14.34.3 the map ϵ is a homotopy equivalence of underlying simplicial sets. By Simplicial, Lemma 14.31.9 this implies ϵ induces a quasi-isomorphism of associated complexes of abelian groups. By Simplicial, Lemma 14.31.8 this implies that ϵ is a trivial Kan fibration of underlying simplicial sets. □

Second proof. We will use the criterion of Cohomology on Sites, Lemma 21.39.7. Let $U = (Q, \beta)$ be an object of $\mathcal{C}$. We have to show that

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}((P_\bullet, \epsilon), (Q, \beta))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). Using the notation of Remark 92.3.3 we see that

$$S_\bullet = \mathrm{Mor}_{\mathcal{S}}((E \to B), i(P_\bullet \to B))$$

By Simplicial, Lemma 14.34.3 the map $i(P_\bullet \to B) \to i(B \to B)$ is a homotopy equivalence in $\mathcal{S}$. Hence $S_\bullet$ is homotopy equivalent to

$$\mathrm{Mor}_{\mathcal{S}}((E \to B), (B \to B)) = \{*\}$$

as desired. □

08PU **Lemma 92.4.3.** *Let $A \to B$ be a ring map. Let π and i be as in (92.4.0.1). There is a canonical isomorphism*

$$L_{B/A} = L\pi_!(Li^*\Omega_{\mathcal{O}/A}) = L\pi_!(i^*\Omega_{\mathcal{O}/A}) = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B})$$

in $D(B)$.

Proof. For an object $\alpha : P \to B$ of the category $\mathcal{C}$ the module $\Omega_{P/A}$ is a free P-module. Thus $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module. Hence $Li^*\Omega_{\mathcal{O}/A} = i^*\Omega_{\mathcal{O}/A}$ is the sheaf of $\underline{B}$-modules which associates to $\alpha : P \to A$ the B-module $\Omega_{P/A} \otimes_{P,\alpha} B$. By Lemma 92.4.2 we see that the right hand side is computed by the value of this sheaf on the standard resolution which is our definition of the left hand side (Definition 92.3.2). □

08QE **Lemma 92.4.4.** *If $A \to B$ is a ring map, then $L\pi_!(\pi^{-1}M) = M$ with π as in (92.4.0.1).*

Proof. This follows from Lemma 92.4.1 which tells us $L\pi_!(\pi^{-1}M)$ is computed by $(\pi^{-1}M)(P_\bullet, \epsilon)$ which is the constant simplicial object on M. □

08QF **Lemma 92.4.5.** *If $A \to B$ is a ring map, then $H^0(L_{B/A}) = \Omega_{B/A}$.*

Proof. We will prove this by a direct calculation. We will use the identification of Lemma 92.4.3. There is clearly a map from $\Omega_{\mathcal{O}/A} \otimes \underline{B}$ to the constant sheaf with value $\Omega_{B/A}$. Thus this map induces a map

$$H^0(L_{B/A}) = H^0(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) = \pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B}) \to \Omega_{B/A}$$

By choosing an object $P \to B$ of $\mathcal{C}_{B/A}$ with $P \to B$ surjective we see that this map is surjective (by Algebra, Lemma 10.131.6). To show that it is injective, suppose that $P \to B$ is an object of $\mathcal{C}_{B/A}$ and that $\xi \in \Omega_{P/A} \otimes_P B$ is an element which maps to zero in $\Omega_{B/A}$. We first choose factorization $P \to P' \to B$ such that $P' \to B$ is surjective and P' is a polynomial algebra over A. We may replace P by P'. If $B = P/I$, then the kernel $\Omega_{P/A} \otimes_P B \to \Omega_{B/A}$ is the image of I/I^2 (Algebra, Lemma 10.131.9). Say ξ is the image of $f \in I$. Then we consider the two maps $a, b : P' = P[x] \to P$, the first of which maps x to 0 and the second of which maps x to f (in both cases $P[x] \to B$ maps x to zero). We see that ξ and 0 are the image of $\mathrm{d}x \otimes 1$ in $\Omega_{P'/A} \otimes_{P'} B$. Thus ξ and 0 have the same image in the colimit (see Cohomology on Sites, Example 21.39.1) $\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})$ as desired. □

08QG **Lemma 92.4.6.** *If B is a polynomial algebra over the ring A, then with π as in (92.4.0.1) we have that $\pi_!$ is exact and $\pi_!\mathcal{F} = \mathcal{F}(B \to B)$.*

Proof. This follows from Lemma 92.4.1 which tells us the constant simplicial algebra on B can be used to compute $L\pi_!$. □

08QH **Lemma 92.4.7.** *If B is a polynomial algebra over the ring A, then $L_{B/A}$ is quasi-isomorphic to $\Omega_{B/A}[0]$.*

Proof. Immediate from Lemmas 92.4.3 and 92.4.6. □

92.5. Constructing a resolution

08PV In the Noetherian finite type case we can construct a "small" simplicial resolution for finite type ring maps.

08PW **Lemma 92.5.1.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. Let $n \geq 0$ and let $P_\bullet$ be a simplicial object of $\mathcal{A}$ such that*

1. *$P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,*
2. *P_k is finite type over A for $k \leq n$,*
3. *$P_\bullet = cosk_n sk_n P_\bullet$ as simplicial objects of $\mathcal{A}$.*

Then P_{n+1} is a finite type A-algebra.

Proof. Although the proof we give of this lemma is straightforward, it is a bit messy. To clarify the idea we explain what happens for low n before giving the proof in general. For example, if $n = 0$, then (3) means that $P_1 = P_0 \times_B P_0$. Since the ring map $P_0 \to B$ is surjective, this is of finite type over A by More on Algebra, Lemma 15.5.1.

If $n = 1$, then (3) means that

$$P_2 = \{(f_0, f_1, f_2) \in P_1^3 \mid d_0 f_0 = d_0 f_1,\ d_1 f_0 = d_0 f_2,\ d_1 f_1 = d_1 f_2\}$$

where the equalities take place in P_0. Observe that the triple

$$(d_0 f_0, d_1 f_0, d_1 f_1) = (d_0 f_1, d_0 f_2, d_1 f_2)$$

is an element of the fibre product $P_0 \times_B P_0 \times_B P_0$ over B because the maps $d_i : P_1 \to P_0$ are morphisms over B. Thus we get a map

$$\psi : P_2 \longrightarrow P_0 \times_B P_0 \times_B P_0$$

The fibre of ψ over an element $(g_0, g_1, g_2) \in P_0 \times_B P_0 \times_B P_0$ is the set of triples (f_0, f_1, f_2) of 1-simplices with $(d_0, d_1)(f_0) = (g_0, g_1)$, $(d_0, d_1)(f_1) = (g_0, g_2)$, and $(d_0, d_1)(f_2) = (g_1, g_2)$. As $P_\bullet \to B$ is a trivial Kan fibration the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus we see that P_2 fits into the cartesian diagram

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1^3 \\ \downarrow & & \downarrow \\ P_0 \times_B P_0 \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^3 \end{array}$$

By More on Algebra, Lemma 15.5.2 we conclude. The general case is similar, but requires a bit more notation.

The case $n > 1$. By Simplicial, Lemma 14.19.14 the condition $P_\bullet = \mathrm{cosk}_n \mathrm{sk}_n P_\bullet$ implies the same thing is true in the category of simplicial A-algebras and hence in the category of sets (as the forgetful functor from A-algebras to sets commutes with limits). Thus

$$P_{n+1} = \mathrm{Mor}(\Delta[n+1], P_\bullet) = \mathrm{Mor}(\mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n P_\bullet)$$

by Simplicial, Lemma 14.11.3 and Equation (14.19.0.1). We will prove by induction on $1 \leq k < m \leq n+1$ that the ring

$$Q_{k,m} = \mathrm{Mor}(\mathrm{sk}_k \Delta[m], \mathrm{sk}_k P_\bullet)$$

is of finite type over A. The case $k = 1$, $1 < m \leq n+1$ is entirely similar to the discussion above in the case $n = 1$. Namely, there is a cartesian diagram

$$\begin{array}{ccc} Q_{1,m} & \longrightarrow & P_1^N \\ \downarrow & & \downarrow \\ P_0 \times_B \ldots \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^N \end{array}$$

where $N = \binom{m+1}{2}$. We conclude as before.

Let $1 \leq k_0 \leq n$ and assume $Q_{k,m}$ is of finite type over A for all $1 \leq k \leq k_0$ and $k < m \leq n+1$. For $k_0 + 1 < m \leq n+1$ we claim there is a cartesian square

$$\begin{array}{ccc} Q_{k_0+1,m} & \longrightarrow & P_{k_0+1}^N \\ \downarrow & & \downarrow \\ Q_{k_0,m} & \longrightarrow & Q_{k_0,k_0+1}^N \end{array}$$

where N is the number of nondegenerate (k_0+1)-simplices of $\Delta[m]$. Namely, to see this is true, think of an element of $Q_{k_0+1,m}$ as a function f from the (k_0+1)-skeleton of $\Delta[m]$ to $P_\bullet$. We can restrict f to the k_0-skeleton which gives the left vertical map of the diagram. We can also restrict to each nondegenerate (k_0+1)-simplex which gives the top horizontal arrow. Moreover, to give such an f is the same thing as giving its restriction to k_0-skeleton and to each nondegenerate (k_0+1)-face,

provided these agree on the overlap, and this is exactly the content of the diagram. Moreover, the fact that $P_\bullet \to B$ is a trivial Kan fibration implies that the map

$$P_{k_0} \to Q_{k_0,k_0+1} = \mathrm{Mor}(\partial\Delta[k_0+1], P_\bullet)$$

is surjective as every map $\partial\Delta[k_0+1] \to B$ can be extended to $\Delta[k_0+1] \to B$ for $k_0 \geq 1$ (small argument about constant simplicial sets omitted). Since by induction hypothesis the rings $Q_{k_0,m}$, Q_{k_0,k_0+1} are finite type A-algebras, so is $Q_{k_0+1,m}$ by More on Algebra, Lemma 15.5.2 once more. □

08PX **Proposition 92.5.2.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. There exists a simplicial A-algebra $P_\bullet$ with an augmentation $\epsilon : P_\bullet \to B$ such that each P_n is a polynomial algebra of finite type over A and such that ϵ is a trivial Kan fibration of simplicial sets.*

Proof. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. In this proof our simplicial objects and skeleton and coskeleton functors will be taken in this category.

Choose a polynomial algebra P_0 of finite type over A and a surjection $P_0 \to B$. As a first approximation we take $P_\bullet = \mathrm{cosk}_0(P_0)$. In other words, $P_\bullet$ is the simplicial A-algebra with terms $P_n = P_0 \times_A \ldots \times_A P_0$. (In the final paragraph of the proof this simplicial object will be denoted $P^0_\bullet$.) By Simplicial, Lemma 14.32.3 the map $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets. Also, observe that $P_\bullet = \mathrm{cosk}_0\mathrm{sk}_0 P_\bullet$.

Suppose for some $n \geq 0$ we have constructed $P_\bullet$ (in the final paragraph of the proof this will be $P^n_\bullet$) such that

(a) $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,
(b) P_k is a finitely generated polynomial algebra for $0 \leq k \leq n$, and
(c) $P_\bullet = \mathrm{cosk}_n\mathrm{sk}_n P_\bullet$

By Lemma 92.5.1 we can find a finitely generated polynomial algebra Q over A and a surjection $Q \to P_{n+1}$. Since P_n is a polynomial algebra the A-algebra maps $s_i : P_n \to P_{n+1}$ lift to maps $s'_i : P_n \to Q$. Set $d'_j : Q \to P_n$ equal to the composition of $Q \to P_{n+1}$ and $d_j : P_{n+1} \to P_n$. We obtain a truncated simplicial object $P'_\bullet$ of $\mathcal{A}$ by setting $P'_k = P_k$ for $k \leq n$ and $P'_{n+1} = Q$ and morphisms $d'_i = d_i$ and $s'_i = s_i$ in degrees $k \leq n-1$ and using the morphisms d'_j and s'_i in degree n. Extend this to a full simplicial object $P'_\bullet$ of $\mathcal{A}$ using cosk_{n+1}. By functoriality of the coskeleton functors there is a morphism $P'_\bullet \to P_\bullet$ of simplicial objects extending the given morphism of $(n+1)$-truncated simplicial objects. (This morphism will be denoted $P^{n+1}_\bullet \to P^n_\bullet$ in the final paragraph of the proof.)

Note that conditions (b) and (c) are satisfied for $P'_\bullet$ with n replaced by $n+1$. We claim the map $P'_\bullet \to P_\bullet$ satisfies assumptions (1), (2), (3), and (4) of Simplicial, Lemmas 14.32.1 with $n+1$ instead of n. Conditions (1) and (2) hold by construction. By Simplicial, Lemma 14.19.14 we see that we have $P_\bullet = \mathrm{cosk}_{n+1}\mathrm{sk}_{n+1}P_\bullet$ and $P'_\bullet = \mathrm{cosk}_{n+1}\mathrm{sk}_{n+1}P'_\bullet$ not only in $\mathcal{A}$ but also in the category of A-algebras, whence in the category of sets (as the forgetful functor from A-algebras to sets commutes with all limits). This proves (3) and (4). Thus the lemma applies and $P'_\bullet \to P_\bullet$ is a trivial Kan fibration. By Simplicial, Lemma 14.30.4 we conclude that $P'_\bullet \to B$ is a trivial Kan fibration and (a) holds as well.

To finish the proof we take the inverse limit $P_\bullet = \lim P^n_\bullet$ of the sequence of simplicial algebras

$$\ldots \to P^2_\bullet \to P^1_\bullet \to P^0_\bullet$$

constructed above. The map $P_\bullet \to B$ is a trivial Kan fibration by Simplicial, Lemma 14.30.5. However, the construction above stabilizes in each degree to a fixed finitely generated polynomial algebra as desired. □

08PY **Lemma 92.5.3.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let π, $\underline{B}$ be as in (92.4.0.1). If $\mathcal{F}$ is an $\underline{B}$-module such that $\mathcal{F}(P, \alpha)$ is a finite B-module for all $\alpha : P = A[x_1, \ldots, x_n] \to B$, then the cohomology modules of $L\pi_!(\mathcal{F})$ are finite B-modules.*

Proof. By Lemma 92.4.1 and Proposition 92.5.2 we can compute $L\pi_!(\mathcal{F})$ by a complex constructed out of the values of $\mathcal{F}$ on finite type polynomial algebras. □

08PZ **Lemma 92.5.4.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Then $H^n(L_{B/A})$ is a finite B-module for all $n \in \mathbf{Z}$.*

Proof. Apply Lemmas 92.4.3 and 92.5.3. □

08QI **Remark 92.5.5** (Resolutions)**.** Let $A \to B$ be any ring map. Let us call an augmented simplicial A-algebra $\epsilon : P_\bullet \to B$ a *resolution of B over A* if each P_n is a polynomial algebra and ϵ is a trivial Kan fibration of simplicial sets. If $P_\bullet \to B$ is an augmentation of a simplicial A-algebra with each P_n a polynomial algebra surjecting onto B, then the following are equivalent

(1) $\epsilon : P_\bullet \to B$ is a resolution of B over A,
(2) $\epsilon : P_\bullet \to B$ is a quasi-isomorphism on associated complexes,
(3) $\epsilon : P_\bullet \to B$ induces a homotopy equivalence of simplicial sets.

To see this use Simplicial, Lemmas 14.30.8, 14.31.9, and 14.31.8. A resolution $P_\bullet$ of B over A gives a cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$ as in Cohomology on Sites, Lemma 21.39.7 and it follows that

$$L\pi_!\mathcal{F} = \mathcal{F}(P_\bullet)$$

functorially in $\mathcal{F}$, see Lemma 92.4.1. The (formal part of the) proof of Proposition 92.5.2 shows that resolutions exist. We also have seen in the first proof of Lemma 92.4.2 that the standard resolution of B over A is a resolution (so that this terminology doesn't lead to a conflict). However, the argument in the proof of Proposition 92.5.2 shows the existence of resolutions without appealing to the simplicial computations in Simplicial, Section 14.34. Moreover, for *any* choice of resolution we have a canonical isomorphism

$$L_{B/A} = \Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

in $D(B)$ by Lemma 92.4.3. The freedom to choose an arbitrary resolution can be quite useful.

08QJ **Lemma 92.5.6.** *Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (92.4.0.1). For any $\mathcal{O}$-module $\mathcal{F}$ we have*

$$L\pi_!(\mathcal{F}) = L\pi_!(Li^*\mathcal{F}) = L\pi_!(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$$

in $D(Ab)$.

Proof. It suffices to verify the assumptions of Cohomology on Sites, Lemma 21.39.12 hold for $\mathcal{O} \to \underline{B}$ on $\mathcal{C}_{B/A}$. We will use the results of Remark 92.5.5 without further mention. Choose a resolution $P_\bullet$ of B over A to get a suitable cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$. Since $P_\bullet \to B$ induces a quasi-isomorphism on associated complexes of abelian groups we see that $L\pi_!\mathcal{O} = B$. On the other hand $L\pi_!\underline{B}$ is computed by

$\underline{B}(U_\bullet) = B$. This verifies the second assumption of Cohomology on Sites, Lemma 21.39.12 and we are done with the proof. □

08QK **Lemma 92.5.7.** *Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (92.4.0.1). We have*

$$L\pi_!(\mathcal{O}) = L\pi_!(\underline{B}) = B \quad \text{and} \quad L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = L\pi_!(\Omega_{\mathcal{O}/A})$$

in $D(Ab)$.

Proof. This is just an application of Lemma 92.5.6 (and the first equality on the right is Lemma 92.4.3). □

Here is a special case of the fundamental triangle that is easy to prove.

08SA **Lemma 92.5.8.** *Let $A \to B \to C$ be ring maps. If B is a polynomial algebra over A, then there is a distinguished triangle $L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$ in $D(C)$.*

Proof. We will use the observations of Remark 92.5.5 without further mention. Choose a resolution $\epsilon : P_\bullet \to C$ of C over B (for example the standard resolution). Since B is a polynomial algebra over A we see that $P_\bullet$ is also a resolution of C over A. Hence $L_{C/A}$ is computed by $\Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} C$ and $L_{C/B}$ is computed by $\Omega_{P_\bullet/B} \otimes_{P_\bullet,\epsilon} C$. Since for each n we have the short exact sequence $0 \to \Omega_{B/A} \otimes_B P_n \to \Omega_{P_n/A} \to \Omega_{P_n/B} \to 0$ (Algebra, Lemma 10.138.9) and since $L_{B/A} = \Omega_{B/A}[0]$ (Lemma 92.4.7) we obtain the result. □

09D4 **Example 92.5.9.** Let $A \to B$ be a ring map. In this example we will construct an "explicit" resolution $P_\bullet$ of B over A of length 2. To do this we follow the procedure of the proof of Proposition 92.5.2, see also the discussion in Remark 92.5.5.

We choose a surjection $P_0 = A[u_i] \to B$ where u_i is a set of variables. Choose generators $f_t \in P_0$, $t \in T$ of the ideal $\mathrm{Ker}(P_0 \to B)$. We choose $P_1 = A[u_i, x_t]$ with face maps d_0 and d_1 the unique A-algebra maps with $d_j(u_i) = u_i$ and $d_0(x_t) = 0$ and $d_1(x_t) = f_t$. The map $s_0 : P_0 \to P_1$ is the unique A-algebra map with $s_0(u_i) = u_i$. It is clear that

$$P_1 \xrightarrow{d_0 - d_1} P_0 \to B \to 0$$

is exact, in particular the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus, if $P_\bullet$ denotes the 1-truncated simplicial A-algebra given by P_0, P_1, d_0, d_1, and s_0, then the augmentation $\mathrm{cosk}_1(P_\bullet) \to B$ is a trivial Kan fibration. The next step of the procedure in the proof of Proposition 92.5.2 is to choose a polynomial algebra P_2 and a surjection

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

Recall that

$$\mathrm{cosk}_1(P_\bullet)_2 = \{(g_0, g_1, g_2) \in P_1^3 \mid d_0(g_0) = d_0(g_1), d_1(g_0) = d_0(g_2), d_1(g_1) = d_1(g_2)\}$$

Thinking of $g_i \in P_1$ as a polynomial in x_t the conditions are

$$g_0(0) = g_1(0), \quad g_0(f_t) = g_2(0), \quad g_1(f_t) = g_2(f_t)$$

Thus $\mathrm{cosk}_1(P_\bullet)_2$ contains the elements $y_t = (x_t, x_t, f_t)$ and $z_t = (0, x_t, x_t)$. Every element G in $\mathrm{cosk}_1(P_\bullet)_2$ is of the form $G = H + (0, 0, g)$ where H is in the image of $A[u_i, y_t, z_t] \to \mathrm{cosk}_1(P_\bullet)_2$. Here $g \in P_1$ is a polynomial with vanishing constant term such that $g(f_t) = 0$ in P_0. Observe that

(1) $g = x_t x_{t'} - f_t x_{t'}$ and
(2) $g = \sum r_t x_t$ with $r_t \in P_0$ if $\sum r_t f_t = 0$ in P_0

are elements of P_1 of the desired form. Let

$$Rel = \mathrm{Ker}(\bigoplus\nolimits_{t \in T} P_0 \longrightarrow P_0), \quad (r_t) \longmapsto \sum r_t f_t$$

We set $P_2 = A[u_i, y_t, z_t, v_r, w_{t,t'}]$ where $r = (r_t) \in Rel$, with map

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

given by $y_t \mapsto (x_t, x_t, f_t)$, $z_t \mapsto (0, x_t, x_t)$, $v_r \mapsto (0, 0, \sum r_t x_t)$, and $w_{t,t'} \mapsto (0, 0, x_t x_{t'} - f_t x_{t'})$. A calculation (omitted) shows that this map is surjective. Our choice of the map displayed above determines the maps $d_0, d_1, d_2 : P_2 \to P_1$. Finally, the procedure in the proof of Proposition 92.5.2 tells us to choose the maps $s_0, s_1 : P_1 \to P_2$ lifting the two maps $P_1 \to \mathrm{cosk}_1(P_\bullet)_2$. It is clear that we can take s_i to be the unique A-algebra maps determined by $s_0(x_t) = y_t$ and $s_1(x_t) = z_t$.

92.6. Functoriality

08QL In this section we consider a commutative square

08QM (92.6.0.1)
$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of ring maps. We claim there is a canonical B-linear map of complexes

$$L_{B/A} \longrightarrow L_{B'/A'}$$

associated to this diagram. Namely, if $P_\bullet \to B$ is the standard resolution of B over A and $P'_\bullet \to B'$ is the standard resolution of B' over A', then there is a canonical map $P_\bullet \to P'_\bullet$ of simplicial A-algebras compatible with the augmentations $P_\bullet \to B$ and $P'_\bullet \to B'$. This can be seen in terms of the construction of standard resolutions in Simplicial, Section 14.34 but in the special case at hand it probably suffices to say simply that the maps

$$P_0 = A[B] \longrightarrow A'[B'] = P'_0, \quad P_1 = A[A[B]] \longrightarrow A'[A'[B']] = P'_1,$$

and so on are given by the given maps $A \to A'$ and $B \to B'$. The desired map $L_{B/A} \to L_{B'/A'}$ then comes from the associated maps $\Omega_{P_n/A} \to \Omega_{P'_n/A'}$.

Another description of the functoriality map can be given as follows. Let $\mathcal{C} = \mathcal{C}_{B/A}$ and $\mathcal{C}' = \mathcal{C}'_{B'/A}$ be the categories considered in Section 92.4. There is a functor

$$u : \mathcal{C} \longrightarrow \mathcal{C}', \quad (P, \alpha) \longmapsto (P \otimes_A A', c \circ (\alpha \otimes 1))$$

where $c : B \otimes_A A' \to B'$ is the obvious map. As discussed in Cohomology on Sites, Example 21.39.3 we obtain a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ and a commutative diagram of maps of ringed topoi

08QN (92.6.0.2)
$$\begin{array}{ccccc} (Sh(\mathcal{C}'), \underline{B}) & \xleftarrow{h} & (Sh(\mathcal{C}'), \underline{B'}) & \xleftarrow{g} & (Sh(\mathcal{C}), \underline{B'}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ (Sh(*), B) & \xleftarrow{f} & (Sh(*), B') & \longleftarrow & (Sh(*), B') \end{array}$$

Here h is the identity on underlying topoi and given by the ring map $B \to B'$ on sheaves of rings. By Cohomology on Sites, Remark 21.38.7 given $\mathcal{F}$ on $\mathcal{C}$ and $\mathcal{F}'$ on $\mathcal{C}'$ and a transformation $t : \mathcal{F} \to g^{-1}\mathcal{F}'$ we obtain a canonical map $L\pi_!(\mathcal{F}) \to L\pi'_!(\mathcal{F}')$. If we apply this to the sheaves

$$\mathcal{F} : (P, \alpha) \mapsto \Omega_{P/A} \otimes_P B, \quad \mathcal{F}' : (P', \alpha') \mapsto \Omega_{P'/A'} \otimes_{P'} B',$$

and the transformation t given by the canonical maps

$$\Omega_{P/A} \otimes_P B \longrightarrow \Omega_{P \otimes_A A'/A'} \otimes_{P \otimes_A A'} B'$$

to get a canonical map

$$L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \longrightarrow L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$$

By Lemma 92.4.3 this gives $L_{B/A} \to L_{B'/A'}$. We omit the verification that this map agrees with the map defined above in terms of simplicial resolutions.

08QP **Lemma 92.6.1.** *Assume (92.6.0.1) induces a quasi-isomorphism $B \otimes^{\mathbf{L}}_A A' = B'$. Then, with notation as in (92.6.0.2) and $\mathcal{F}' \in Ab(\mathcal{C}')$, we have $L\pi_!(g^{-1}\mathcal{F}') = L\pi'_!(\mathcal{F}')$.*

Proof. We use the results of Remark 92.5.5 without further mention. We will apply Cohomology on Sites, Lemma 21.39.8. Let $P_\bullet \to B$ be a resolution. If we can show that $u(P_\bullet) = P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism, then we are done. The complex of A-modules $s(P_\bullet)$ associated to $P_\bullet$ (viewed as a simplicial A-module) is a free A-module resolution of B. Namely, P_n is a free A-module and $s(P_\bullet) \to B$ is a quasi-isomorphism. Thus $B \otimes^{\mathbf{L}}_A A'$ is computed by $s(P_\bullet) \otimes_A A' = s(P_\bullet \otimes_A A')$. Therefore the assumption of the lemma signifies that $\epsilon' : P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism. □

The following lemma in particular applies when $A \to A'$ is flat and $B' = B \otimes_A A'$ (flat base change).

08QQ **Lemma 92.6.2.** *If (92.6.0.1) induces a quasi-isomorphism $B \otimes^{\mathbf{L}}_A A' = B'$, then the functoriality map induces an isomorphism*

$$L_{B/A} \otimes^{\mathbf{L}}_B B' \longrightarrow L_{B'/A'}$$

Proof. We will use the notation introduced in Equation (92.6.0.2). We have

$$L_{B/A} \otimes^{\mathbf{L}}_B B' = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \otimes^{\mathbf{L}}_B B' = L\pi_!(Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}))$$

the first equality by Lemma 92.4.3 and the second by Cohomology on Sites, Lemma 21.39.6. Since $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module, we see that $\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ is a flat $\underline{B}$-module. Thus $Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B'}$ which is equal to $g^{-1}(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$ by inspection. we conclude by Lemma 92.6.1 and the fact that $L_{B'/A'}$ is computed by $L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$. □

08SB **Remark 92.6.3.** Suppose that we are given a square (92.6.0.1) such that there exists an arrow $\kappa : B \to A'$ making the diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ \uparrow & \nearrow_{\kappa} & \uparrow \\ A & \xrightarrow{\alpha} & A' \end{array}$$

In this case we claim the functoriality map $P_\bullet \to P'_\bullet$ is homotopic to the composition $P_\bullet \to B \to A' \to P'_\bullet$. Namely, using κ the functoriality map factors as

$$P_\bullet \to P_{A'/A',\bullet} \to P'_\bullet$$

where $P_{A'/A',\bullet}$ is the standard resolution of A' over A'. Since A' is the polynomial algebra on the empty set over A' we see from Simplicial, Lemma 14.34.3 that the augmentation $\epsilon_{A'/A'} : P_{A'/A',\bullet} \to A'$ is a homotopy equivalence of simplicial rings. Observe that the homotopy inverse map $c : A' \to P_{A'/A',\bullet}$ constructed in the proof of that lemma is just the structure morphism, hence we conclude what we want because the two compositions

$$P_\bullet \longrightarrow P_{A'/A',\bullet} \underset{c \circ \epsilon_{A'/A'}}{\overset{\mathrm{id}}{\rightrightarrows}} P_{A'/A',\bullet} \longrightarrow P'_\bullet$$

are the two maps discussed above and these are homotopic (Simplicial, Remark 14.26.5). Since the second map $P_\bullet \to P'_\bullet$ induces the zero map $\Omega_{P_\bullet/A} \to \Omega_{P'_\bullet/A'}$ we conclude that the functoriality map $L_{B/A} \to L_{B'/A'}$ is homotopic to zero in this case.

08SC **Lemma 92.6.4.** *Let $A \to B$ and $A \to C$ be ring maps. Then the map $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ is an isomorphism in $D(B \times C)$.*

Proof. Although this lemma can be deduced from the fundamental triangle we will give a direct and elementary proof of this now. Factor the ring map $A \to B \times C$ as $A \to A[x] \to B \times C$ where $x \mapsto (1, 0)$. By Lemma 92.5.8 we have a distinguished triangle

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C) \to L_{B\times C/A} \to L_{B\times C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C)[1]$$

in $D(B \times C)$. Similarly we have the distinguished triangles

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B \to L_{B/A} \to L_{B/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B[1]$$
$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C \to L_{C/A} \to L_{C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C[1]$$

Thus it suffices to prove the result for $B\times C$ over $A[x]$. Note that $A[x] \to A[x, x^{-1}]$ is flat, that $(B\times C)\otimes_{A[x]} A[x, x^{-1}] = B\otimes_{A[x]} A[x, x^{-1}]$, and that $C\otimes_{A[x]} A[x, x^{-1}] = 0$. Thus by base change (Lemma 92.6.2) the map $L_{B\times C/A[x]} \to L_{B/A[x]} \oplus L_{C/A[x]}$ becomes an isomorphism after inverting x. In the same way one shows that the map becomes an isomorphism after inverting $x - 1$. This proves the lemma. □

92.7. The fundamental triangle

08QR In this section we consider a sequence of ring maps $A \to B \to C$. It is our goal to show that this triangle gives rise to a distinguished triangle

08QS (92.7.0.1) $$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

in $D(C)$. This will be proved in Proposition 92.7.4. For an alternative approach see Remark 92.7.5.

Consider the category $\mathcal{C}_{C/B/A}$ which is the **opposite** of the category whose objects are $(P \to B, Q \to C)$ where

(1) P is a polynomial algebra over A,
(2) $P \to B$ is an A-algebra homomorphism,
(3) Q is a polynomial algebra over P, and

(4) $Q \to C$ is a P-algebra-homomorphism.

We take the opposite as we want to think of $(P \to B, Q \to C)$ as corresponding to the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(C) & \longrightarrow & \mathrm{Spec}(Q) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\ \downarrow & \swarrow & \\ \mathrm{Spec}(A) & & \end{array}$$

Let $\mathcal{C}_{B/A}$, $\mathcal{C}_{C/A}$, $\mathcal{C}_{C/B}$ be the categories considered in Section 92.4. There are functors

$$\begin{aligned} u_1 : \mathcal{C}_{C/B/A} &\to \mathcal{C}_{B/A}, \quad (P \to B, Q \to C) \mapsto (P \to B) \\ u_2 : \mathcal{C}_{C/B/A} &\to \mathcal{C}_{C/A}, \quad (P \to B, Q \to C) \mapsto (Q \to C) \\ u_3 : \mathcal{C}_{C/B/A} &\to \mathcal{C}_{C/B}, \quad (P \to B, Q \to C) \mapsto (Q \otimes_P B \to C) \end{aligned}$$

These functors induce corresponding morphisms of topoi g_i. Let us denote $\mathcal{O}_i = g_i^{-1}\mathcal{O}$ so that we get morphisms of ringed topoi

$$\begin{aligned} g_1 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_1) \longrightarrow (Sh(\mathcal{C}_{B/A}), \mathcal{O}) \\ g_2 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_2) \longrightarrow (Sh(\mathcal{C}_{C/A}), \mathcal{O}) \\ g_3 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_3) \longrightarrow (Sh(\mathcal{C}_{C/B}), \mathcal{O}) \end{aligned} \tag{92.7.0.2}$$

08QT

Let us denote $\pi : Sh(\mathcal{C}_{C/B/A}) \to Sh(*)$, $\pi_1 : Sh(\mathcal{C}_{B/A}) \to Sh(*)$, $\pi_2 : Sh(\mathcal{C}_{C/A}) \to Sh(*)$, and $\pi_3 : Sh(\mathcal{C}_{C/B}) \to Sh(*)$, so that $\pi = \pi_i \circ g_i$. We will obtain our distinguished triangle from the identification of the cotangent complex in Lemma 92.4.3 and the following lemmas.

08QU **Lemma 92.7.1.** *With notation as in (92.7.0.2) set*

$$\begin{aligned} \Omega_1 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B} \text{ on } \mathcal{C}_{B/A} \\ \Omega_2 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/A} \\ \Omega_3 &= \Omega_{\mathcal{O}/B} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/B} \end{aligned}$$

Then we have a canonical short exact sequence of sheaves of $\underline{C}$-modules

$$0 \to g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C} \to g_2^{-1}\Omega_2 \to g_3^{-1}\Omega_3 \to 0$$

on $\mathcal{C}_{C/B/A}$.

Proof. Recall that g_i^{-1} is gotten by simply precomposing with u_i. Given an object $U = (P \to B, Q \to C)$ we have a split short exact sequence

$$0 \to \Omega_{P/A} \otimes Q \to \Omega_{Q/A} \to \Omega_{Q/P} \to 0$$

for example by Algebra, Lemma 10.138.9. Tensoring with C over Q we obtain a short exact sequence

$$0 \to \Omega_{P/A} \otimes C \to \Omega_{Q/A} \otimes C \to \Omega_{Q/P} \otimes C \to 0$$

We have $\Omega_{P/A} \otimes C = \Omega_{P/A} \otimes B \otimes C$ whence this is the value of $g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}$ on U. The module $\Omega_{Q/A} \otimes C$ is the value of $g_2^{-1}\Omega_2$ on U. We have $\Omega_{Q/P} \otimes C = \Omega_{Q \otimes_P B/B} \otimes C$ by Algebra, Lemma 10.131.12 hence this is the value of $g_3^{-1}\Omega_3$ on U. Thus the

short exact sequence of the lemma comes from assigning to U the last displayed short exact sequence. □

08QV **Lemma 92.7.2.** *With notation as in (92.7.0.2) suppose that C is a polynomial algebra over B. Then $L\pi_!(g_3^{-1}\mathcal{F}) = L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$*

Proof. Write $C = B[E]$ for some set E. Choose a resolution $P_\bullet \to B$ of B over A. For every n consider the object $U_n = (P_n \to B, P_n[E] \to C)$ of $\mathcal{C}_{C/B/A}$. Then $U_\bullet$ is a cosimplicial object of $\mathcal{C}_{C/B/A}$. Note that $u_3(U_\bullet)$ is the constant cosimplicial object of $\mathcal{C}_{C/B}$ with value $(C \to C)$. We will prove that the object $U_\bullet$ of $\mathcal{C}_{C/B/A}$ satisfies the hypotheses of Cohomology on Sites, Lemma 21.39.7. This implies the lemma as it shows that $L\pi_!(g_3^{-1}\mathcal{F})$ is computed by the constant simplicial abelian group $\mathcal{F}(C \to C)$ which is the value of $L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ by Lemma 92.4.6.

Let $U = (\beta : P \to B, \gamma : Q \to C)$ be an object of $\mathcal{C}_{C/B/A}$. We may write $P = A[S]$ and $Q = A[S \amalg T]$ by the definition of our category $\mathcal{C}_{C/B/A}$. We have to show that

$$\mathrm{Mor}_{\mathcal{C}_{C/B/A}}(U_\bullet, U)$$

is homotopy equivalent to a singleton simplicial set $*$. Observe that this simplicial set is the product

$$\prod\nolimits_{s\in S} F_s \times \prod\nolimits_{t\in T} F'_t$$

where F_s is the corresponding simplicial set for $U_s = (A[\{s\}] \to B, A[\{s\}] \to C)$ and F'_t is the corresponding simplicial set for $U_t = (A \to B, A[\{t\}] \to C)$. Namely, the object U is the product $\prod U_s \times \prod U_t$ in $\mathcal{C}_{C/B/A}$. It suffices each F_s and F'_t is homotopy equivalent to $*$, see Simplicial, Lemma 14.26.10. The case of F_s follows as $P_\bullet \to B$ is a trivial Kan fibration (as a resolution) and F_s is the fibre of this map over $\beta(s)$. (Use Simplicial, Lemmas 14.30.3 and 14.30.8). The case of F'_t is more interesting. Here we are saying that the fibre of

$$P_\bullet[E] \longrightarrow C = B[E]$$

over $\gamma(t) \in C$ is homotopy equivalent to a point. In fact we will show this map is a trivial Kan fibration. Namely, $P_\bullet \to B$ is a trivial can fibration. For any ring R we have

$$R[E] = \mathrm{colim}_{\Sigma\subset \mathrm{Map}(E,\mathbf{Z}_{\geq 0}) \text{ finite}} \prod\nolimits_{I\in\Sigma} R$$

(filtered colimit). Thus the displayed map of simplicial sets is a filtered colimit of trivial Kan fibrations, whence a trivial Kan fibration by Simplicial, Lemma 14.30.7. □

08QW **Lemma 92.7.3.** *With notation as in (92.7.0.2) we have $Lg_{i,!} \circ g_i^{-1} = id$ for $i = 1, 2, 3$ and hence also $L\pi_! \circ g_i^{-1} = L\pi_{i,!}$ for $i = 1, 2, 3$.*

Proof. Proof for $i = 1$. We claim the functor $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{B/A}$ Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(P' \to B) \to (P \to B)$ of $\mathcal{C}_{B/A}$. Recall that this means we have an A-algebra homomorphism $P \to P'$ compatible with maps to B. Then we set $Q' = Q \otimes_P P'$ with induced map to C and the morphism

$$(P' \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

in $\mathcal{C}_{C/B/A}$ (note reversal arrows again) is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{B/A}$. Moreover, observe that the fibre category of u_1 over $P \to B$ is the category $\mathcal{C}_{C/P}$.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{B/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.40.2. Thus $L_n g_{1,!} g_1^{-1}\mathcal{F}$ is the (pre)sheaf which assigns to $U \in \mathrm{Ob}(\mathcal{C}_{B/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Lemma 92.4.4 via our identifications of fibre categories above.

The case $i = 2$. We claim $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{C/A}$ is a fibred category. Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(Q' \to C) \to (Q \to C)$ of $\mathcal{C}_{C/A}$. Recall that this means we have a B-algebra homomorphism $Q \to Q'$ compatible with maps to C. Then

$$(P \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{C/A}$. Note that the fibre category of u_2 over $Q \to C$ has an final (beware reversal arrows) object, namely, $(A \to B, Q \to C)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{C/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.40.2. Thus $L_n g_{2,!} g_2^{-1}\mathcal{F}$ is the (pre)sheaf which assigns to $U \in \mathrm{Ob}(\mathcal{C}_{C/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Cohomology on Sites, Lemma 21.39.5 because the fibre categories all have final objects.

The case $i = 3$. In this case we will apply Cohomology on Sites, Lemma 21.40.3 to $u = u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}$ and $\mathcal{F}' = g_3^{-1}\mathcal{F}$ for some abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$. Suppose $U = (\overline{Q} \to C)$ is an object of $\mathcal{C}_{C/B}$. Then $\mathcal{I}_U = \mathcal{C}_{\overline{Q}/B/A}$ (again beware of reversal of arrows). The sheaf $\mathcal{F}'_U$ is given by the rule $(P \to B, Q \to \overline{Q}) \mapsto \mathcal{F}(Q \otimes_P B \to C)$. In other words, this sheaf is the pullback of a sheaf on $\mathcal{C}_{\overline{Q}/C}$ via the morphism $Sh(\mathcal{C}_{\overline{Q}/B/A}) \to Sh(\mathcal{C}_{\overline{Q}/B})$. Thus Lemma 92.7.2 shows that $H_n(\mathcal{I}_U, \mathcal{F}'_U) = 0$ for $n > 0$ and equal to $\mathcal{F}(\overline{Q} \to C)$ for $n = 0$. The aforementioned Cohomology on Sites, Lemma 21.40.3 implies that $Lg_{3,!}(g_3^{-1}\mathcal{F}) = \mathcal{F}$ and the proof is done. □

08QX **Proposition 92.7.4.** *Let $A \to B \to C$ be ring maps. There is a canonical distinguished triangle*

$$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

in $D(C)$.

Proof. Consider the short exact sequence of sheaves of Lemma 92.7.1 and apply the derived functor $L\pi_!$ to obtain a distinguished triangle

$$L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}) \to L\pi_!(g_2^{-1}\Omega_2) \to L\pi_!(g_3^{-1}\Omega_3) \to L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C})[1]$$

in $D(C)$. Using Lemmas 92.7.3 and 92.4.3 we see that the second and third terms agree with $L_{C/A}$ and $L_{C/B}$ and the first one equals

$$L\pi_{1,!}(\Omega_1 \otimes_{\underline{B}} \underline{C}) = L\pi_{1,!}(\Omega_1) \otimes^{\mathbf{L}}_B C = L_{B/A} \otimes^{\mathbf{L}}_B C$$

The first equality by Cohomology on Sites, Lemma 21.39.6 (and flatness of Ω_1 as a sheaf of modules over $\underline{B}$) and the second by Lemma 92.4.3. □

08SD **Remark 92.7.5.** We sketch an alternative, perhaps simpler, proof of the existence of the fundamental triangle. Let $A \to B \to C$ be ring maps and assume that $B \to C$

is injective. Let $P_\bullet \to B$ be the standard resolution of B over A and let $Q_\bullet \to C$ be the standard resolution of C over B. Picture

$$\begin{array}{lccccc} P_\bullet : & A[A[A[B]]] & \rightleftarrows & A[A[B]] & \rightleftarrows & A[B] \longrightarrow B \\ & \downarrow & & \downarrow & & \downarrow \\ Q_\bullet : & A[A[A[C]]] & \rightleftarrows & A[A[C]] & \rightleftarrows & A[C] \longrightarrow C \end{array}$$

Observe that since $B \to C$ is injective, the ring Q_n is a polynomial algebra over P_n for all n. Hence we obtain a cosimplicial object in $\mathcal{C}_{C/B/A}$ (beware reversal arrows). Now set $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B$. The key to the proof of Proposition 92.7.4 is to show that $\overline{Q}_\bullet$ is a resolution of C over B. This follows from Cohomology on Sites, Lemma 21.39.12 applied to $\mathcal{C} = \Delta$, $\mathcal{O} = P_\bullet$, $\mathcal{O}' = B$, and $\mathcal{F} = Q_\bullet$ (this uses that Q_n is flat over P_n; see Cohomology on Sites, Remark 21.39.11 to relate simplicial modules to sheaves). The key fact implies that the distinguished triangle of Proposition 92.7.4 is the distinguished triangle associated to the short exact sequence of simplicial C-modules

$$0 \to \Omega_{P_\bullet/A} \otimes_{P_\bullet} C \to \Omega_{Q_\bullet/A} \otimes_{Q_\bullet} C \to \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C \to 0$$

which is deduced from the short exact sequences $0 \to \Omega_{P_n/A} \otimes_{P_n} Q_n \to \Omega_{Q_n/A} \to \Omega_{Q_n/P_n} \to 0$ of Algebra, Lemma 10.138.9. Namely, by Remark 92.5.5 and the key fact the complex on the right hand side represents $L_{C/B}$ in $D(C)$.

If $B \to C$ is not injective, then we can use the above to get a fundamental triangle for $A \to B \to B \times C$. Since $L_{B\times C/B} \to L_{B/B} \oplus L_{C/B}$ and $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ are quasi-isomorphism in $D(B \times C)$ (Lemma 92.6.4) this induces the desired distinguished triangle in $D(C)$ by tensoring with the flat ring map $B \times C \to C$.

08SE **Remark 92.7.6.** Let $A \to B \to C$ be ring maps with $B \to C$ injective. Recall the notation $P_\bullet$, $Q_\bullet$, $\overline{Q}_\bullet$ of Remark 92.7.5. Let $R_\bullet$ be the standard resolution of C over B. In this remark we explain how to get the canonical identification of $\Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$ with $L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C$. Let $S_\bullet \to B$ be the standard resolution of B over B. Note that the functoriality map $S_\bullet \to R_\bullet$ identifies R_n as a polynomial algebra over S_n because $B \to C$ is injective. For example in degree 0 we have the map $B[B] \to B[C]$, in degree 1 the map $B[B[B]] \to B[B[C]]$, and so on. Thus $\overline{R}_\bullet = R_\bullet \otimes_{S_\bullet} B$ is a simplicial polynomial algebra over B as well and it follows (as in Remark 92.7.5) from Cohomology on Sites, Lemma 21.39.12 that $\overline{R}_\bullet \to C$ is a resolution. Since we have a commutative diagram

$$\begin{array}{ccccc} Q_\bullet & \longrightarrow & R_\bullet & & \\ \uparrow & & \uparrow & & \\ P_\bullet & \longrightarrow & S_\bullet & \longrightarrow & B \end{array}$$

we obtain a canonical map $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B \to \overline{R}_\bullet$. Thus the maps

$$L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C \longrightarrow \Omega_{\overline{R}_\bullet/B} \otimes_{\overline{R}_\bullet} C \longleftarrow \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$$

are quasi-isomorphisms (Remark 92.5.5) and composing one with the inverse of the other gives the desired identification.

92.8. Localization and étale ring maps

08QY In this section we study what happens if we localize our rings. Let $A \to A' \to B$ be ring maps such that $B = B \otimes^{\mathbf{L}}_A A'$. This happens for example if $A' = S^{-1}A$ is the localization of A at a multiplicative subset $S \subset A$. In this case for an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}_{B/A'}$ the homology of $g^{-1}\mathcal{F}'$ over $\mathcal{C}_{B/A}$ agrees with the homology of $\mathcal{F}'$ over $\mathcal{C}_{B/A'}$, see Lemma 92.6.1 for a precise statement.

08QZ **Lemma 92.8.1.** *Let $A \to A' \to B$ be ring maps such that $B = B \otimes^{\mathbf{L}}_A A'$. Then $L_{B/A} = L_{B/A'}$ in $D(B)$.*

Proof. According to the discussion above (i.e., using Lemma 92.6.1) and Lemma 92.4.3 we have to show that the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A} \otimes_P B$ on $\mathcal{C}_{B/A}$ is the pullback of the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A'} \otimes_P B$. The pullback functor g^{-1} is given by precomposing with the functor $u : \mathcal{C}_{B/A} \to \mathcal{C}_{B/A'}$, $(P \to B) \mapsto (P \otimes_A A' \to B)$. Thus we have to show that

$$\Omega_{P/A} \otimes_P B = \Omega_{P \otimes_A A'/A'} \otimes_{(P \otimes_A A')} B$$

By Algebra, Lemma 10.131.12 the right hand side is equal to

$$(\Omega_{P/A} \otimes_A A') \otimes_{(P \otimes_A A')} B$$

Since P is a polynomial algebra over A the module $\Omega_{P/A}$ is free and the equality is obvious. □

08R0 **Lemma 92.8.2.** *Let $A \to B$ be a ring map such that $B = B \otimes^{\mathbf{L}}_A B$. Then $L_{B/A} = 0$ in $D(B)$.*

Proof. This is true because $L_{B/A} = L_{B/B} = 0$ by Lemmas 92.8.1 and 92.4.7. □

08R1 **Lemma 92.8.3.** *Let $A \to B$ be a ring map such that $Tor^A_i(B, B) = 0$ for $i > 0$ and such that $L_{B/B \otimes_A B} = 0$. Then $L_{B/A} = 0$ in $D(B)$.*

Proof. By Lemma 92.6.2 we see that $L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) = L_{B \otimes_A B/B}$. Now we use the distinguished triangle (92.7.0.1)

$$L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B \to L_{B/B} \to L_{B/B \otimes_A B} \to L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B[1]$$

associated to the ring maps $B \to B \otimes_A B \to B$ and the vanishing of $L_{B/B}$ (Lemma 92.4.7) and $L_{B/B \otimes_A B}$ (assumed) to see that

$$0 = L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B = L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) \otimes^{\mathbf{L}}_{(B \otimes_A B)} B = L_{B/A}$$

as desired. □

08R2 **Lemma 92.8.4.** *The cotangent complex $L_{B/A}$ is zero in each of the following cases:*

1. *$A \to B$ and $B \otimes_A B \to B$ are flat, i.e., $A \to B$ is weakly étale (More on Algebra, Definition 15.105.1),*
2. *$A \to B$ is a flat epimorphism of rings,*
3. *$B = S^{-1}A$ for some multiplicative subset $S \subset A$,*
4. *$A \to B$ is unramified and flat,*
5. *$A \to B$ is étale,*
6. *$A \to B$ is a filtered colimit of ring maps for which the cotangent complex vanishes,*
7. *B is a henselization of a local ring of A,*

(8) *B is a strict henselization of a local ring of A, and*
(9) *add more here.*

Proof. In case (1) we may apply Lemma 92.8.2 to the surjective flat ring map $B \otimes_A B \to B$ to conclude that $L_{B/B\otimes_A B} = 0$ and then we use Lemma 92.8.3 to conclude. The cases (2) – (5) are each special cases of (1). Part (6) follows from Lemma 92.3.4. Parts (7) and (8) follows from the fact that (strict) henselizations are filtered colimits of étale ring extensions of A, see Algebra, Lemmas 10.155.7 and 10.155.11. □

08R3 **Lemma 92.8.5.** *Let $A \to B \to C$ be ring maps such that $L_{C/B} = 0$. Then $L_{C/A} = L_{B/A} \otimes^{\mathbf{L}}_B C$.*

Proof. This is a trivial consequence of the distinguished triangle (92.7.0.1). □

08SF **Lemma 92.8.6.** *Let $A \to B$ be ring maps and $S \subset A$, $T \subset B$ multiplicative subsets such that S maps into T. Then $L_{T^{-1}B/S^{-1}A} = L_{B/A} \otimes_B T^{-1}B$ in $D(T^{-1}B)$.*

Proof. Lemma 92.8.5 shows that $L_{T^{-1}B/A} = L_{B/A} \otimes_B T^{-1}B$ and Lemma 92.8.1 shows that $L_{T^{-1}B/A} = L_{T^{-1}B/S^{-1}A}$. □

08UN **Lemma 92.8.7.** *Let $A \to B$ be a local ring homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced maps of henselizations, resp. strict henselizations. Then*

$$L_{B^h/A^h} = L_{B^h/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^h \quad \text{resp.} \quad L_{B^{sh}/A^{sh}} = L_{B^{sh}/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^{sh}$$

in $D(B^h)$, resp. $D(B^{sh})$.

Proof. The complexes $L_{A^h/A}$, $L_{A^{sh}/A}$, $L_{B^h/B}$, and $L_{B^{sh}/B}$ are all zero by Lemma 92.8.4. Using the fundamental distinguished triangle (92.7.0.1) for $A \to B \to B^h$ we obtain $L_{B^h/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^h$. Using the fundamental triangle for $A \to A^h \to B^h$ we obtain $L_{B^h/A^h} = L_{B^h/A}$. Similarly for strict henselizations. □

92.9. Smooth ring maps

08R4 Let $C \to B$ be a surjection of rings with kernel I. Let us call such a ring map "weakly quasi-regular" if I/I^2 is a flat B-module and $\mathrm{Tor}^C_*(B, B)$ is the exterior algebra on I/I^2. The generalization to "smooth ring maps" of what is done in Lemma 92.8.4 for "étale ring maps" is to look at flat ring maps $A \to B$ such that the multiplication map $B \otimes_A B \to B$ is weakly quasi-regular. For the moment we just stick to smooth ring maps.

08R5 **Lemma 92.9.1.** *If $A \to B$ is a smooth ring map, then $L_{B/A} = \Omega_{B/A}[0]$.*

Proof. We have the agreement in cohomological degree 0 by Lemma 92.4.5. Thus it suffices to prove the other cohomology groups are zero. It suffices to prove this locally on $\mathrm{Spec}(B)$ as $L_{B_g/A} = (L_{B/A})_g$ for $g \in B$ by Lemma 92.8.5. Thus we may assume that $A \to B$ is standard smooth (Algebra, Lemma 10.137.9), i.e., that we can factor $A \to B$ as $A \to A[x_1, \ldots, x_n] \to B$ with $A[x_1, \ldots, x_n] \to B$ étale. In this case Lemmas 92.8.4 and Lemma 92.8.5 show that $L_{B/A} = L_{A[x_1,\ldots,x_n]/A} \otimes B$ whence the conclusion by Lemma 92.4.7. □

92.10. Positive characteristic

0G5X In this section we fix a prime number p. If A is a ring with $p = 0$ in A, then $F_A : A \to A$ denotes the Frobenius endomorphism $a \mapsto a^p$.

0G5Y **Lemma 92.10.1.** *Let $A \to B$ be a ring map with $p = 0$ in A. Let $P_\bullet$ be the standard resolution of B over A. The map $P_\bullet \to P_\bullet$ induced by the diagram*

$$\begin{array}{ccc} B & \xrightarrow{F_B} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{F_A} & A \end{array}$$

discussed in Section 92.6 is homotopic to the Frobenius endomorphism $P_\bullet \to P_\bullet$ given by Frobenius on each P_n.

Proof. Let $\mathcal{A}$ be the category of $\mathbf{F}_p$-algebra maps $A \to B$. Let $\mathcal{S}$ be the category of pairs (A, E) where A is an $\mathbf{F}_p$-algebra and E is a set. Consider the adjoint functors

$$V : \mathcal{A} \to \mathcal{S}, \quad (A \to B) \mapsto (A, B)$$

and

$$U : \mathcal{S} \to \mathcal{A}, \quad (A, E) \mapsto (A \to A[E])$$

Let X be the simplicial object in in the category of functors from $\mathcal{A}$ to $\mathcal{A}$ constructed in Simplicial, Section 14.34. It is clear that $P_\bullet = X(A \to B)$ because if we fix A then.

Set $Y = U \circ V$. Recall that X is constructed from Y and certain maps and has terms $X_n = Y \circ \ldots \circ Y$ with $n + 1$ terms; the construction is given in Simplicial, Example 14.33.1 and please see proof of Simplicial, Lemma 14.34.2 for details.

Let $f : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$ be the Frobenius endomorphism of the identity functor. In other words, we set $f_{A \to B} = (F_A, F_B) : (A \to B) \to (A \to B)$. Then our two maps on $X(A \to B)$ are given by the natural transformations $f \star 1_X$ and $1_X \star f$. Details omitted. Thus we conclude by Simplicial, Lemma 14.33.6. □

0G5Z **Lemma 92.10.2.** *Let p be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in A. The map $L_{B/A} \to L_{B/A}$ of Section 92.6 induced by the Frobenius maps F_A and F_B is homotopic to zero.*

Proof. Let $P_\bullet$ be the standard resolution of B over A. By Lemma 92.10.1 the map $P_\bullet \to P_\bullet$ induced by F_A and F_B is homotopic to the map $F_{P_\bullet} : P_\bullet \to P_\bullet$ given by Frobenius on each term. Hence we obtain what we want as clearly $F_{P_\bullet}$ induces the zero map $\Omega_{P_n/A} \to \Omega_{P_n/A}$ (since the derivative of a pth power is zero). □

0G60 **Lemma 92.10.3.** *Let p be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in A. If A and B are perfect, then $L_{B/A}$ is zero in $D(B)$.*

Proof. The map $(F_A, F_B) : (A \to B) \to (A \to B)$ is an isomorphism hence induces an isomorphism on $L_{B/A}$ and on the other hand induces zero on $L_{B/A}$ by Lemma 92.10.2. □

92.11. Comparison with the naive cotangent complex

08R6 The naive cotangent complex was introduced in Algebra, Section 10.134.

08R7 **Remark 92.11.1.** Let $A \to B$ be a ring map. Working on $\mathcal{C}_{B/A}$ as in Section 92.4 let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Note that $L\pi_!(\mathcal{J}) = 0$ by Lemma 92.5.7. Set $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ so that $L_{B/A} = L\pi_!(\Omega)$ by Lemma 92.4.3. It follows that $L\pi_!(\mathcal{J} \to \Omega) = L\pi_!(\Omega) = L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

08R8 (92.11.1.1) $$(J \to \Omega_{P/A} \otimes_P B) \longrightarrow L_{B/A}$$

where $J = \mathrm{Ker}(P \to B)$ in $D(A)$, see Cohomology on Sites, Remark 21.39.4. Continuing in this manner, note that $L\pi_!(\mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = L\pi_!(\mathcal{J}) = 0$ by Lemma 92.5.6. Since $\mathrm{Tor}^{\mathcal{O}}_0(\mathcal{J}, \underline{B}) = \mathcal{J}/\mathcal{J}^2$ the spectral sequence

$$H_p(\mathcal{C}_{B/A}, \mathrm{Tor}^{\mathcal{O}}_q(\mathcal{J}, \underline{B})) \Rightarrow H_{p+q}(\mathcal{C}_{B/A}, \mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = 0$$

(dual of Derived Categories, Lemma 13.21.3) implies that $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$ and $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$. It follows that the complex of $\underline{B}$-modules $\mathcal{J}/\mathcal{J}^2 \to \Omega$ satisfies $\tau_{\geq -1} L\pi_!(\mathcal{J}/\mathcal{J}^2 \to \Omega) = \tau_{\geq -1} L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

08R9 (92.11.1.2) $$(J/J^2 \to \Omega_{P/A} \otimes_P B) \longrightarrow \tau_{\geq -1} L_{B/A}$$

in $D(B)$, see Cohomology on Sites, Remark 21.39.4.

The first case is where we have a surjection of rings.

08RA **Lemma 92.11.2.** *Let $A \to B$ be a surjective ring map with kernel I. Then $H^0(L_{B/A}) = 0$ and $H^{-1}(L_{B/A}) = I/I^2$. This isomorphism comes from the map (92.11.1.2) for the object $(A \to B)$ of $\mathcal{C}_{B/A}$.*

Proof. We will show below (using the surjectivity of $A \to B$) that there exists a short exact sequence

$$0 \to \pi^{-1}(I/I^2) \to \mathcal{J}/\mathcal{J}^2 \to \Omega \to 0$$

of sheaves on $\mathcal{C}_{B/A}$. Taking $L\pi_!$ and the associated long exact sequence of homology, and using the vanishing of $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ and $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ shown in Remark 92.11.1 we obtain what we want using Lemma 92.4.4.

What is left is to verify the local statement mentioned above. For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. The value on U of the short sequence of sheaves above is the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{P/A} \otimes_P B \to 0$$

Verification omitted (hint: the only tricky point is that $IP \cap J^2 = IJ$; which follows for example from More on Algebra, Lemma 15.30.9). □

08RB **Lemma 92.11.3.** *Let $A \to B$ be a ring map. Then $\tau_{\geq -1} L_{B/A}$ is canonically quasi-isomorphic to the naive cotangent complex.*

Proof. Consider $P = A[B] \to B$ with kernel I. The naive cotangent complex $NL_{B/A}$ of B over A is the complex $I/I^2 \to \Omega_{P/A} \otimes_P B$, see Algebra, Definition 10.134.1. Observe that in (92.11.1.2) we have already constructed a canonical map

$$c : NL_{B/A} \longrightarrow \tau_{\geq -1} L_{B/A}$$

Consider the distinguished triangle (92.7.0.1)

$$L_{P/A} \otimes^{\mathbf{L}}_P B \to L_{B/A} \to L_{B/P} \to (L_{P/A} \otimes^{\mathbf{L}}_P B)[1]$$

associated to the ring maps $A \to A[B] \to B$. We know that $L_{P/A} = \Omega_{P/A}[0] = NL_{P/A}$ in $D(P)$ (Lemma 92.4.7 and Algebra, Lemma 10.134.3) and that $\tau_{\geq -1} L_{B/P} = I/I^2[1] = NL_{B/P}$ in $D(B)$ (Lemma 92.11.2 and Algebra, Lemma 10.134.6). To show c is a quasi-isomorphism it suffices by Algebra, Lemma 10.134.4 and the long exact cohomology sequence associated to the distinguished triangle to show that the maps $L_{P/A} \to L_{B/A} \to L_{B/P}$ are compatible on cohomology groups with the corresponding maps $NL_{P/A} \to NL_{B/A} \to NL_{B/P}$ of the naive cotangent complex. We omit the verification. □

08UP **Remark 92.11.4.** We can make the comparison map of Lemma 92.11.3 explicit in the following way. Let $P_\bullet$ be the standard resolution of B over A. Let $I = \mathrm{Ker}(A[B] \to B)$. Recall that $P_0 = A[B]$. The map of the lemma is given by the commutative diagram

$$\begin{array}{ccccccc} L_{B/A} & \ldots \longrightarrow & \Omega_{P_2/A} \otimes_{P_2} B & \longrightarrow & \Omega_{P_1/A} \otimes_{P_1} B & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ NL_{B/A} & \ldots \longrightarrow & 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B \end{array}$$

We construct the downward arrow with target I/I^2 by sending $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in I/I^2. Here $d_i : P_1 \to P_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps P_1 into $I = \mathrm{Ker}(P_0 \to B)$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{P_1/A} \otimes_{P_1} B \to \Omega_{P_0/A} \otimes_{P_0} B$ as this differential maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{P_2/A} \otimes_{P_2} B \to \Omega_{P_1/A} \otimes_{P_1} B$ maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes. We omit the verification that this is the same as the map of Lemma 92.11.3.

09D5 **Remark 92.11.5.** Adopt notation as in Remark 92.11.1. The arguments given there show that the differential

$$H_2(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \longrightarrow H_0(\mathcal{C}_{B/A}, \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{J}, \underline{B}))$$

of the spectral sequence is an isomorphism. Let $\mathcal{C}'_{B/A}$ denote the full subcategory of $\mathcal{C}_{B/A}$ consisting of surjective maps $P \to B$. The agreement of the cotangent complex with the naive cotangent complex (Lemma 92.11.3) shows that we have an exact sequence of sheaves

$$0 \to \underline{H_1(L_{B/A})} \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\mathrm{d}} \Omega \to \underline{H_2(L_{B/A})} \to 0$$

on $\mathcal{C}'_{B/A}$. It follows that $\mathrm{Ker}(d)$ and $\mathrm{Coker}(d)$ on the whole category $\mathcal{C}_{B/A}$ have vanishing higher homology groups, since these are computed by the homology groups

of constant simplicial abelian groups by Lemma 92.4.1. Hence we conclude that

$$H_n(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \to H_n(L_{B/A})$$

is an isomorphism for all $n \geq 2$. Combined with the remark above we obtain the formula $H_2(L_{B/A}) = H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$.

92.12. A spectral sequence of Quillen

08RC In this section we discuss a spectral sequence relating derived tensor product to the cotangent complex.

08RD **Lemma 92.12.1.** *Notation and assumptions as in Cohomology on Sites, Example 21.39.1. Assume $\mathcal{C}$ has a cosimplicial object as in Cohomology on Sites, Lemma 21.39.7. Let $\mathcal{F}$ be a flat $\underline{B}$-module such that $H_0(\mathcal{C}, \mathcal{F}) = 0$. Then $H_l(\mathcal{C}, Sym^k_{\underline{B}}(\mathcal{F})) = 0$ for $l < k$.*

Proof. We drop the subscript $\underline{B}$ from tensor products, wedge powers, and symmetric powers. We will prove the lemma by induction on k. The cases $k = 0, 1$ follow from the assumptions. If $k > 1$ consider the exact complex

$$\ldots \to \wedge^2\mathcal{F} \otimes \mathrm{Sym}^{k-2}\mathcal{F} \to \mathcal{F} \otimes \mathrm{Sym}^{k-1}\mathcal{F} \to \mathrm{Sym}^k\mathcal{F} \to 0$$

with differentials as in the Koszul complex. If we think of this as a resolution of $\mathrm{Sym}^k\mathcal{F}$, then this gives a first quadrant spectral sequence

$$E_1^{p,q} = H_p(\mathcal{C}, \wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) \Rightarrow H_{p+q}(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$$

By Cohomology on Sites, Lemma 21.39.10 we have

$$L\pi_!(\wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) = L\pi_!(\wedge^{q+1}\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathrm{Sym}^{k-q-1}\mathcal{F}))$$

It follows (from the construction of derived tensor products) that the induction hypothesis combined with the vanishing of $H_0(\mathcal{C}, \wedge^{q+1}(\mathcal{F})) = 0$ will prove what we want. This is true because $\wedge^{q+1}(\mathcal{F})$ is a quotient of $\mathcal{F}^{\otimes q+1}$ and $H_0(\mathcal{C}, \mathcal{F}^{\otimes q+1})$ is a quotient of $H_0(\mathcal{C}, \mathcal{F})^{\otimes q+1}$ which is zero. □

08SG **Remark 92.12.2.** In the situation of Lemma 92.12.1 one can show that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F})) = \wedge^k_B(H_1(\mathcal{C}, \mathcal{F}))$. Namely, it can be deduced from the proof that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$ is the S_k-coinvariants of

$$H^{-k}(L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B \ldots \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F})) = H_1(\mathcal{C}, \mathcal{F})^{\otimes k}$$

Thus our claim is that this action is given by the usual action of S_k on the tensor product multiplied by the sign character. To prove this one has to work through the sign conventions in the definition of the total complex associated to a multicomplex. We omit the verification.

08RE **Lemma 92.12.3.** *Let A be a ring. Let $P = A[E]$ be a polynomial ring. Set $I = (e; e \in E) \subset P$. The maps $Tor_i^P(A, I^{n+1}) \to Tor_i^P(A, I^n)$ are zero for all i and n.*

Proof. Denote $x_e \in P$ the variable corresponding to $e \in E$. A free resolution of A over P is given by the Koszul complex $K_\bullet$ on the x_e. Here K_i has basis given by wedges $e_1 \wedge \ldots \wedge e_i$, $e_1, \ldots, e_i \in E$ and $d(e) = x_e$. Thus $K_\bullet \otimes_P I^n = I^n K_\bullet$ computes $\mathrm{Tor}_i^P(A, I^n)$. Observe that everything is graded with $\deg(x_e) = 1$, $\deg(e) = 1$, and $\deg(a) = 0$ for $a \in A$. Suppose $\xi \in I^{n+1}K_i$ is a cocycle homogeneous of degree m. Note that $m \geq i + 1 + n$. Then $\xi = \mathrm{d}\eta$ for some $\eta \in K_{i+1}$ as $K_\bullet$ is exact in degrees

> 0. (The case $i = 0$ is left to the reader.) Now $\deg(\eta) = m \geq i + 1 + n$. Hence writing η in terms of the basis we see the coordinates are in I^n. Thus ξ maps to zero in the homology of $I^n K_\bullet$ as desired. $\square$

08RF **Theorem 92.12.4** (Quillen spectral sequence)**.** *Let $A \to B$ be a surjective ring map. Consider the sheaf $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ of $\underline{B}$-modules on $\mathcal{C}_{B/A}$, see Section 92.4. Then there is a spectral sequence with E_1-page*

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, Sym^p_{\underline{B}}(\Omega)) \Rightarrow Tor^A_{-p-q}(B, B)$$

with d_r of bidegree $(r, -r+1)$. Moreover, $H_i(\mathcal{C}_{B/A}, Sym^k_{\underline{B}}(\Omega)) = 0$ for $i < k$.

Proof. Let $I \subset A$ be the kernel of $A \to B$. Let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Then $I\mathcal{O} \subset \mathcal{J}$. Set $\mathcal{K} = \mathcal{J}/I\mathcal{O}$ and $\overline{\mathcal{O}} = \mathcal{O}/I\mathcal{O}$.

For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. Moreover $\overline{\mathcal{O}}(U) = B[E]$ and $K = \mathcal{K}(U) = (e; e \in E)$ is the ideal generated by the variables in the polynomial ring $B[E]$. In particular it is clear that

$$K/K^2 \xrightarrow{\mathrm{d}} \Omega_{P/A} \otimes_P B$$

is a bijection. In other words, $\Omega = \mathcal{K}/\mathcal{K}^2$ and $\mathrm{Sym}^k_B(\Omega) = \mathcal{K}^k/\mathcal{K}^{k+1}$. Note that $\pi_!(\Omega) = \Omega_{B/A} = 0$ (Lemma 92.4.5) as $A \to B$ is surjective (Algebra, Lemma 10.131.4). By Lemma 92.12.1 we conclude that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^k/\mathcal{K}^{k+1}) = H_i(\mathcal{C}_{B/A}, \mathrm{Sym}^k_{\underline{B}}(\Omega)) = 0$$

for $i < k$. This proves the final statement of the theorem.

The approach to the theorem is to note that

$$B \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O}) \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O} \otimes^{\mathbf{L}}_{\underline{A}} \underline{B}) = L\pi_!(\overline{\mathcal{O}})$$

The first equality by Lemma 92.5.7, the second equality by Cohomology on Sites, Lemma 21.39.6, and the third equality as $\mathcal{O}$ is flat over $\underline{A}$. The sheaf $\overline{\mathcal{O}}$ has a filtration

$$\ldots \subset \mathcal{K}^3 \subset \mathcal{K}^2 \subset \mathcal{K} \subset \overline{\mathcal{O}}$$

This induces a filtration F on a complex C representing $L\pi_!(\overline{\mathcal{O}})$ with F^pC representing $L\pi_!(\mathcal{K}^p)$ (construction of C and F omitted). Consider the spectral sequence of Homology, Section 12.24 associated to (C, F). It has E_1-page

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathcal{K}^p/\mathcal{K}^{p+1}) \quad \Rightarrow \quad H_{-p-q}(\mathcal{C}_{B/A}, \overline{\mathcal{O}}) = \mathrm{Tor}^A_{-p-q}(B, B)$$

and differentials $E_r^{p,q} \to E_r^{p+r,q-r+1}$. To show convergence we will show that for every k there exists a c such that $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < k$ and $n > c$[2].

Given $k \geq 0$ set $c = k^2$. We claim that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$$

is zero for $i < k$ and all $n \geq 0$. Note that $\mathcal{K}^n/\mathcal{K}^{n+c}$ has a finite filtration whose successive quotients $\mathcal{K}^m/\mathcal{K}^{m+1}$, $n \leq m < n + c$ have $H_i(\mathcal{C}_{B/A}, \mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for $i < n$ (see above). Hence the claim implies $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) = 0$ for $i < k$ and all $n \geq k$ which is what we need to show.

[2] A posteriori the "correct" vanishing $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < n$ can be concluded.

Proof of the claim. Recall that for any $\mathcal{O}$-module $\mathcal{F}$ the map $\mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} B$ induces an isomorphism on applying $L\pi_!$, see Lemma 92.5.6. Consider the map

$$\mathcal{K}^{n+k} \otimes^{\mathbf{L}}_{\mathcal{O}} B \longrightarrow \mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$$

We claim that this map induces the zero map on cohomology sheaves in degrees $0, -1, \ldots, -k+1$. If this second claim holds, then the k-fold composition

$$\mathcal{K}^{n+c} \otimes^{\mathbf{L}}_{\mathcal{O}} B \longrightarrow \mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$$

factors through $\tau_{\leq -k}\mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$ hence induces zero on $H_i(\mathcal{C}_{B/A}, -) = L_i\pi_!(-)$ for $i < k$, see Derived Categories, Lemma 13.12.5. By the remark above this means the same thing is true for $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$ which proves the (first) claim.

Proof of the second claim. The statement is local, hence we may work over an object $U = (P \to B)$ as above. We have to show the maps

$$\mathrm{Tor}^P_i(B, K^{n+k}) \to \mathrm{Tor}^P_i(B, K^n)$$

are zero for $i < k$. There is a spectral sequence

$$\mathrm{Tor}^P_a(P/IP, \mathrm{Tor}^{P/IP}_b(B, K^n)) \Rightarrow \mathrm{Tor}^P_{a+b}(B, K^n),$$

see More on Algebra, Example 15.62.2. Thus it suffices to prove the maps

$$\mathrm{Tor}^{P/IP}_i(B, K^{n+1}) \to \mathrm{Tor}^{P/IP}_i(B, K^n)$$

are zero for all i. This is Lemma 92.12.3. □

08RG **Remark 92.12.5.** In the situation of Theorem 92.12.4 let $I = \mathrm{Ker}(A \to B)$. Then $H^{-1}(L_{B/A}) = H_1(\mathcal{C}_{B/A}, \Omega) = I/I^2$, see Lemma 92.11.2. Hence $H_k(\mathcal{C}_{B/A}, \mathrm{Sym}^k(\Omega)) = \wedge^k_B(I/I^2)$ by Remark 92.12.2. Thus the E_1-page looks like

$$\begin{array}{llll}
B & & & \\
0 & & & \\
0 & I/I^2 & & \\
0 & H^{-2}(L_{B/A}) & & \\
0 & H^{-3}(L_{B/A}) & \wedge^2(I/I^2) & \\
0 & H^{-4}(L_{B/A}) & H_3(\mathcal{C}_{B/A}, \mathrm{Sym}^2(\Omega)) & \\
0 & H^{-5}(L_{B/A}) & H_4(\mathcal{C}_{B/A}, \mathrm{Sym}^2(\Omega)) & \wedge^3(I/I^2)
\end{array}$$

with horizontal differential. Thus we obtain edge maps $\mathrm{Tor}^A_i(B, B) \to H^{-i}(L_{B/A})$, $i > 0$ and $\wedge^i_B(I/I^2) \to \mathrm{Tor}^A_i(B, B)$. Finally, we have $\mathrm{Tor}^A_1(B, B) = I/I^2$ and there is a five term exact sequence

$$\mathrm{Tor}^A_3(B, B) \to H^{-3}(L_{B/A}) \to \wedge^2_B(I/I^2) \to \mathrm{Tor}^A_2(B, B) \to H^{-2}(L_{B/A}) \to 0$$

of low degree terms.

09D6 **Remark 92.12.6.** Let $A \to B$ be a ring map. Let $P_\bullet$ be a resolution of B over A (Remark 92.5.5). Set $J_n = \mathrm{Ker}(P_n \to B)$. Note that

$$\mathrm{Tor}^{P_n}_2(B, B) = \mathrm{Tor}^{P_n}_1(J_n, B) = \mathrm{Ker}(J_n \otimes_{P_n} J_n \to J_n^2).$$

Hence $H_2(L_{B/A})$ is canonically equal to

$$\mathrm{Coker}(\mathrm{Tor}^{P_1}_2(B, B) \to \mathrm{Tor}^{P_0}_2(B, B))$$

by Remark 92.11.5. To make this more explicit we choose P_2, P_1, P_0 as in Example 92.5.9. We claim that

$$\mathrm{Tor}_2^{P_1}(B,B) = \wedge^2(\bigoplus\nolimits_{t\in T} B) \ \oplus \ \bigoplus\nolimits_{t\in T} J_0 \ \oplus \ \mathrm{Tor}_2^{P_0}(B,B)$$

Namely, the basis elements $x_t \wedge x_{t'}$ of the first summand corresponds to the element $x_t \otimes x_{t'} - x_{t'} \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. For $f \in J_0$ the element $x_t \otimes f$ of the second summand corresponds to the element $x_t \otimes s_0(f) - s_0(f) \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. Finally, the map $\mathrm{Tor}_2^{P_0}(B,B) \to \mathrm{Tor}_2^{P_1}(B,B)$ is given by s_0. The map $d_0 - d_1 : \mathrm{Tor}_2^{P_1}(B,B) \to \mathrm{Tor}_2^{P_0}(B,B)$ is zero on the last summand, maps $x_t \otimes f$ to $f \otimes f_t - f_t \otimes f$, and maps $x_t \wedge x_{t'}$ to $f_t \otimes f_{t'} - f_{t'} \otimes f_t$. All in all we conclude that there is an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B,B) \to H^{-2}(L_{B/A}) \to 0$$

In this way we obtain a direct proof of a consequence of Quillen's spectral sequence discussed in Remark 92.12.5.

92.13. Comparison with Lichtenbaum-Schlessinger

09AM Let $A \to B$ be a ring map. In [**LS67**] there is a fairly explicit determination of $\tau_{\geq -2}L_{B/A}$ which is often used in calculations of versal deformation spaces of singularities. The construction follows. Choose a polynomial algebra P over A and a surjection $P \to B$ with kernel I. Choose generators f_t, $t \in T$ for I which induces a surjection $F = \bigoplus_{t\in T} P \to I$ with F a free P-module. Let $Rel \subset F$ be the kernel of $F \to I$, in other words Rel is the set of relations among the f_t. Let $TrivRel \subset Rel$ be the submodule of trivial relations, i.e., the submodule of Rel generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$. Consider the complex of B-modules

09CD (92.13.0.1) $$Rel/TrivRel \longrightarrow F \otimes_P B \longrightarrow \Omega_{P/A} \otimes_P B$$

where the last term is placed in degree 0. The first map is the obvious one and the second map sends the basis element corresponding to $t \in T$ to $\mathrm{d}f_t \otimes 1$.

09CE **Definition 92.13.1.** Let $A \to B$ be a ring map. Let M be a (B,B)-bimodule over A. An *A-biderivation* is an A-linear map $\lambda : B \to M$ such that $\lambda(xy) = x\lambda(y) + \lambda(x)y$.

For a polynomial algebra the biderivations are easy to describe.

09CF **Lemma 92.13.2.** *Let $P = A[S]$ be a polynomial ring over A. Let M be a (P,P)-bimodule over A. Given $m_s \in M$ for $s \in S$, there exists a unique A-biderivation $\lambda : P \to M$ mapping s to m_s for $s \in S$.*

Proof. We set

$$\lambda(s_1 \ldots s_t) = \sum s_1 \ldots s_{i-1} m_{s_i} s_{i+1} \ldots s_t$$

in M. Extending by A-linearity we obtain a biderivation. □

Here is the comparison statement. The reader may also read about this in [**And74a**, page 206, Proposition 12] or in the paper [**DRGV92**] which extends the complex (92.13.0.1) by one term and the comparison to $\tau_{\geq -3}$.

09CG **Lemma 92.13.3.** *In the situation above denote L the complex (92.13.0.1). There is a canonical map $L_{B/A} \to L$ in $D(B)$ which induces an isomorphism $\tau_{\geq -2}L_{B/A} \to L$ in $D(B)$.*

Proof. Let $P_\bullet \to B$ be a resolution of B over A (Remark 92.5.5). We will identify $L_{B/A}$ with $\Omega_{P_\bullet/A} \otimes B$. To construct the map we make some choices.

Choose an A-algebra map $\psi : P_0 \to P$ compatible with the given maps $P_0 \to B$ and $P \to B$.

Write $P_1 = A[S]$ for some set S. For $s \in S$ we may write

$$\psi(d_0(s) - d_1(s)) = \sum p_{s,t} f_t$$

for some $p_{s,t} \in P$. Think of $F = \bigoplus_{t \in T} P$ as a (P_1, P_1)-bimodule via the maps $(\psi \circ d_0, \psi \circ d_1)$. By Lemma 92.13.2 we obtain a unique A-biderivation $\lambda : P_1 \to F$ mapping s to the vector with coordinates $p_{s,t}$. By construction the composition

$$P_1 \longrightarrow F \longrightarrow P$$

sends $f \in P_1$ to $\psi(d_0(f) - d_1(f))$ because the map $f \mapsto \psi(d_0(f) - d_1(f))$ is an A-biderivation agreeing with the composition on generators.

For $g \in P_2$ we claim that $\lambda(d_0(g) - d_1(g) + d_2(g))$ is an element of Rel. Namely, by the last remark of the previous paragraph the image of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in P is

$$\psi((d_0 - d_1)(d_0(g) - d_1(g) + d_2(g)))$$

which is zero by Simplicial, Section 14.23).

The choice of ψ determines a map

$$\mathrm{d}\psi \otimes 1 : \Omega_{P_0/A} \otimes B \longrightarrow \Omega_{P/A} \otimes B$$

Composing λ with the map $F \to F \otimes B$ gives a usual A-derivation as the two P_1-module structures on $F \otimes B$ agree. Thus λ determines a map

$$\overline{\lambda} : \Omega_{P_1/A} \otimes B \longrightarrow F \otimes B$$

Finally, We obtain a B-linear map

$$q : \Omega_{P_2/A} \otimes B \longrightarrow Rel/TrivRel$$

by mapping $\mathrm{d}g$ to the class of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in the quotient.

The diagram

$$\begin{array}{ccccccc}
\Omega_{P_3/A} \otimes B & \longrightarrow & \Omega_{P_2/A} \otimes B & \longrightarrow & \Omega_{P_1/A} \otimes B & \longrightarrow & \Omega_{P_0/A} \otimes B \\
\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \overline{\lambda}} & & \downarrow{\scriptstyle \mathrm{d}\psi \otimes 1} \\
0 & \longrightarrow & Rel/TrivRel & \longrightarrow & F \otimes B & \longrightarrow & \Omega_{P/A} \otimes B
\end{array}$$

commutes (calculation omitted) and we obtain the map of the lemma. By Remark 92.11.4 and Lemma 92.11.3 we see that this map induces isomorphisms $H_1(L_{B/A}) \to H_1(L)$ and $H_0(L_{B/A}) \to H_0(L)$.

It remains to see that our map $L_{B/A} \to L$ induces an isomorphism $H_2(L_{B/A}) \to H_2(L)$. Choose a resolution of B over A with $P_0 = P = A[u_i]$ and then P_1 and P_2 as in Example 92.5.9. In Remark 92.12.6 we have constructed an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

where $P_0 = P$ and $J_0 = \mathrm{Ker}(P \to B) = I$. Calculating the Tor group using the short exact sequences $0 \to I \to P \to B \to 0$ and $0 \to Rel \to F \to I \to 0$ we find that $\mathrm{Tor}_2^P(B, B) = \mathrm{Ker}(Rel \otimes B \to F \otimes B)$. The image of the map $\wedge^2_B(I/I^2) \to$

$\mathrm{Tor}_2^P(B,B)$ under this identification is exactly the image of $TrivRel \otimes B$. Thus we see that $H_2(L_{B/A}) \cong H_2(L)$.

Finally, we have to check that our map $L_{B/A} \to L$ actually induces this isomorphism. We will use the notation and results discussed in Example 92.5.9 and Remarks 92.12.6 and 92.11.5 without further mention. Pick an element ξ of $\mathrm{Tor}_2^{P_0}(B,B) = \mathrm{Ker}(I \otimes_P I \to I^2)$. Write $\xi = \sum h_{t',t} f_{t'} \otimes f_t$ for some $h_{t',t} \in P$. Tracing through the exact sequences above we find that ξ corresponds to the image in $Rel \otimes B$ of the element $r \in Rel \subset F = \bigoplus_{t \in T} P$ with tth coordinate $r_t = \sum_{t' \in T} h_{t',t} f_{t'}$. On the other hand, ξ corresponds to the element of $H_2(L_{B/A}) = H_2(\Omega)$ which is the image via $\mathrm{d} : H_2(\mathcal{J}/\mathcal{J}^2) \to H_2(\Omega)$ of the boundary of ξ under the 2-extension

$$0 \to \mathrm{Tor}_2^{\mathcal{O}}(\underline{B}, \underline{B}) \to \mathcal{J} \otimes_{\mathcal{O}} \mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$$

We compute the successive transgressions of our element. First we have

$$\xi = (d_0 - d_1)(-\sum s_0(h_{t',t} f_{t'}) \otimes x_t)$$

and next we have

$$\sum s_0(h_{t',t} f_{t'}) x_t = d_0(v_r) - d_1(v_r) + d_2(v_r)$$

by our choice of the variables v in Example 92.5.9. We may choose our map λ above such that $\lambda(u_i) = 0$ and $\lambda(x_t) = -e_t$ where $e_t \in F$ denotes the basis vector corresponding to $t \in T$. Hence the construction of our map q above sends $\mathrm{d}v_r$ to

$$\lambda(\sum s_0(h_{t',t} f_{t'}) x_t) = \sum\nolimits_t \left(\sum\nolimits_{t'} h_{t',t} f_{t'} \right) e_t$$

matching the image of ξ in $Rel \otimes B$ (the two minus signs we found above cancel out). This agreement finishes the proof. □

09D7 **Remark 92.13.4** (Functoriality of the Lichtenbaum-Schlessinger complex)**.** Consider a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

of ring maps. Choose a factorization

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & P & \longrightarrow & B \end{array}$$

with P a polynomial algebra over A and P' a polynomial algebra over A'. Choose generators f_t, $t \in T$ for $\mathrm{Ker}(P \to B)$. For $t \in T$ denote f'_t the image of f_t in P'. Choose $f'_s \in P'$ such that the elements f'_t for $t \in T' = T \amalg S$ generate the kernel of $P' \to B'$. Set $F = \bigoplus_{t \in T} P$ and $F' = \bigoplus_{t' \in T'} P'$. Let $Rel = \mathrm{Ker}(F \to P)$ and $Rel' = \mathrm{Ker}(F' \to P')$ where the maps are given by multiplication by f_t, resp. f'_t on the coordinates. Finally, set $TrivRel$, resp. $TrivRel'$ equal to the submodule of Rel, resp. $TrivRel$ generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$ for

$t, t' \in T$, resp. T'. Having made these choices we obtain a canonical commutative diagram

$$\begin{array}{ccccc} L': & Rel'/TrivRel' & \longrightarrow & F' \otimes_{P'} B' & \longrightarrow & \Omega_{P'/A'} \otimes_{P'} B' \\ \uparrow & \uparrow & & \uparrow & & \uparrow \\ L: & Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \end{array}$$

Moreover, tracing through the choices made in the proof of Lemma 92.13.3 the reader sees that one obtains a commutative diagram

$$\begin{array}{ccc} L_{B'/A'} & \longrightarrow & L' \\ \uparrow & & \uparrow \\ L_{B/A} & \longrightarrow & L \end{array}$$

92.14. The cotangent complex of a local complete intersection

08SH If $A \to B$ is a local complete intersection map, then $L_{B/A}$ is a perfect complex. The key to proving this is the following lemma.

08SI **Lemma 92.14.1.** *Let $A = \mathbf{Z}[x_1, \ldots, x_n] \to B = \mathbf{Z}$ be the ring map which sends x_i to 0 for $i = 1, \ldots, n$. Let $I = (x_1, \ldots, x_n) \subset A$. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. There are several ways to prove this. For example one can explicitly construct a resolution of B over A and compute. We will use (92.7.0.1). Namely, consider the distinguished triangle

$$L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1,\ldots,x_n]} \mathbf{Z} \to L_{\mathbf{Z}/\mathbf{Z}} \to L_{\mathbf{Z}/\mathbf{Z}[x_1,\ldots,x_n]} \to L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1,\ldots,x_n]} \mathbf{Z}[1]$$

The complex $L_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}}$ is quasi-isomorphic to $\Omega_{\mathbf{Z}[x_1,\ldots,x_n]/\mathbf{Z}}$ by Lemma 92.4.7. The complex $L_{\mathbf{Z}/\mathbf{Z}}$ is zero in $D(\mathbf{Z})$ by Lemma 92.8.4. Thus we see that $L_{B/A}$ has only one nonzero cohomology group which is as described in the lemma by Lemma 92.11.2. □

08SJ **Lemma 92.14.2.** *Let $A \to B$ be a surjective ring map whose kernel I is generated by a Koszul-regular sequence (for example a regular sequence). Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. Let $f_1, \ldots, f_r \in I$ be a Koszul regular sequence generating I. Consider the ring map $\mathbf{Z}[x_1, \ldots, x_r] \to A$ sending x_i to f_i. Since $x_1, \ldots, x_r$ is a regular sequence in $\mathbf{Z}[x_1, \ldots, x_r]$ we see that the Koszul complex on $x_1, \ldots, x_r$ is a free resolution of $\mathbf{Z} = \mathbf{Z}[x_1, \ldots, x_r]/(x_1, \ldots, x_r)$ over $\mathbf{Z}[x_1, \ldots, x_r]$ (see More on Algebra, Lemma 15.30.2). Thus the assumption that $f_1, \ldots, f_r$ is Koszul regular exactly means that $B = A \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]} \mathbf{Z}$. Hence $L_{B/A} = L_{\mathbf{Z}/\mathbf{Z}[x_1,\ldots,x_r]} \otimes^{\mathbf{L}}_{\mathbf{Z}} B$ by Lemmas 92.6.2 and 92.14.1. □

08SK **Lemma 92.14.3.** *Let $A \to B$ be a surjective ring map whose kernel I is Koszul. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. Locally on $\mathrm{Spec}(A)$ the ideal I is generated by a Koszul regular sequence, see More on Algebra, Definition 15.32.1. Hence this follows from Lemma 92.6.2. □

08SL **Proposition 92.14.4.** *Let $A \to B$ be a local complete intersection map. Then $L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$.*

Proof. Choose a surjection $P = A[x_1, \ldots, x_n] \to B$ with kernel J. By Lemma 92.11.3 we see that $J/J^2 \to \bigoplus B\mathrm{d}x_i$ is quasi-isomorphic to $\tau_{\geq -1}L_{B/A}$. Note that J/J^2 is finite projective (More on Algebra, Lemma 15.32.3), hence $\tau_{\geq -1}L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$. Thus it suffices to show that $H^i(L_{B/A}) = 0$ for $i \notin [-1, 0]$. This follows from (92.7.0.1)

$$L_{P/A} \otimes^{\mathbf{L}}_P B \to L_{B/A} \to L_{B/P} \to L_{P/A} \otimes^{\mathbf{L}}_P B[1]$$

and Lemma 92.14.3 to see that $H^i(L_{B/P})$ is zero unless $i \in \{-1, 0\}$. (We also use Lemma 92.4.7 for the term on the left.) □

92.15. Tensor products and the cotangent complex

09D8 Let R be a ring and let A, B be R-algebras. In this section we discuss $L_{A\otimes_R B/R}$. Most of the information we want is contained in the following diagram (92.15.0.1)

09D9
$$\begin{array}{ccccc}
L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) & \longrightarrow & L_{A\otimes_R B/B} & \longrightarrow & E \\
\| & & \uparrow & & \uparrow \\
L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) & \longrightarrow & L_{A\otimes_R B/R} & \longrightarrow & L_{A\otimes_R B/A} \\
 & & \uparrow & & \uparrow \\
 & & L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) & = & L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B)
\end{array}$$

Explanation: The middle row is the fundamental triangle (92.7.0.1) for the ring maps $R \to A \to A \otimes_R B$. The middle column is the fundamental triangle (92.7.0.1) for the ring maps $R \to B \to A \otimes_R B$. Next, E is an object of $D(A \otimes_R B)$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.9.1) of the map of complexes

$$L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) \oplus L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) \longrightarrow L_{A\otimes_R B/R}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

09DA **Lemma 92.15.1.** *If A and B are Tor independent R-algebras, then the object E in (92.15.0.1) is zero. In this case we have*

$$L_{A\otimes_R B/R} = L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) \oplus L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B)$$

which is represented by the complex $L_{A/R} \otimes_R B \oplus L_{B/R} \otimes_R A$ of $A \otimes_R B$-modules.

Proof. The first two statements are immediate from Lemma 92.6.2. The last statement follows as $L_{A/R}$ is a complex of free A-modules, hence $L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B)$ is represented by $L_{A/R} \otimes_A (A \otimes_R B) = L_{A/R} \otimes_R B$ □

In general we can say this about the object E.

09DB **Lemma 92.15.2.** *Let R be a ring and let A, B be R-algebras. The object E in (92.15.0.1) satisfies*

$$H^i(E) = \begin{cases} 0 & \text{if} \quad i \geq -1 \\ Tor_1^R(A, B) & \text{if} \quad i = -2 \end{cases}$$

Proof. We use the description of E as the cone on $L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) \to L_{A \otimes_R B/A}$. By Lemma 92.13.3 the canonical truncations $\tau_{\geq -2} L_{B/R}$ and $\tau_{\geq -2} L_{A \otimes_R B/A}$ are computed by the Lichtenbaum-Schlessinger complex (92.13.0.1). These isomorphisms are compatible with functoriality (Remark 92.13.4). Thus in this proof we work with the Lichtenbaum-Schlessinger complexes.

Choose a polynomial algebra P over R and a surjection $P \to B$. Choose generators $f_t \in P$, $t \in T$ of the kernel of this surjection. Let $Rel \subset F = \bigoplus_{t \in T} P$ be the kernel of the map $F \to P$ which maps the basis vector corresponding to t to f_t. Set $P_A = A \otimes_R P$ and $F_A = A \otimes_R F = P_A \otimes_P F$. Let Rel_A be the kernel of the map $F_A \to P_A$. Using the exact sequence

$$0 \to Rel \to F \to P \to B \to 0$$

and standard short exact sequences for Tor we obtain an exact sequence

$$A \otimes_R Rel \to Rel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

Note that $P_A \to A \otimes_R B$ is a surjection whose kernel is generated by the elements $1 \otimes f_t$ in P_A. Denote $TrivRel_A \subset Rel_A$ the P_A-submodule generated by the elements $(\ldots, 1 \otimes f_{t'}, 0, \ldots, 0, -1 \otimes f_t \otimes 1, 0, \ldots)$. Since $TrivRel \otimes_R A \to TrivRel_A$ is surjective, we find a canonical exact sequence

$$A \otimes_R (Rel/TrivRel) \to Rel_A/TrivRel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

The map of Lichtenbaum-Schlessinger complexes is given by the diagram

$$\begin{array}{ccccc} Rel_A/TrivRel_A & \longrightarrow & F_A \otimes_{P_A} (A \otimes_R B) & \longrightarrow & \Omega_{P_A/A \otimes_R B} \otimes_{P_A} (A \otimes_R B) \\ \uparrow{\scriptstyle -2} & & \uparrow{\scriptstyle -1} & & \uparrow{\scriptstyle 0} \\ Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \end{array}$$

Note that vertical maps -1 and -0 induce an isomorphism after applying the functor $A \otimes_R - = P_A \otimes_P -$ to the source and the vertical map -2 gives exactly the map whose cokernel is the desired Tor module as we saw above. □

92.16. Deformations of ring maps and the cotangent complex

08SM This section is the continuation of Deformation Theory, Section 91.2 which we urge the reader to read first. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

08SN (92.16.0.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle c}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (92.16.0.1).

08SP **Lemma 92.16.1.** *In the situation above we have*

(1) *There is a canonical element $\xi \in \mathrm{Ext}^2_B(L_{B/A}, N)$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.16.0.1).*
(2) *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_B(L_{B/A}, N)$.*
(3) *Given a solution B', the set of automorphisms of B' fitting into (92.16.0.1) is canonically isomorphic to $\mathrm{Ext}^0_B(L_{B/A}, N)$.*

Proof. Via the identifications $NL_{B/A} = \tau_{\geq -1} L_{B/A}$ (Lemma 92.11.3) and $H^0(L_{B/A}) = \Omega_{B/A}$ (Lemma 92.4.5) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.2.1 and 91.2.2.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 91.2.8 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 91.2.7 and Remark 91.2.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_A(NL_{A/A'}, N) = \mathrm{Ext}^1_B(NL_{A/A'} \otimes^{\mathbf{L}}_A B, N) = \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

(for the equalities see Deformation Theory, Remark 91.2.8 and use that $NL_{A'/A} = \tau_{\geq -1} L_{A'/A}$) such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_B(NL_{B/A'}, N) = \mathrm{Ext}^1_B(L_{B/A'}, N) \longrightarrow \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

The distinguished triangle (92.7.0.1) for $A' \to A \to B$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_B(L_{B/A'}, N) \to \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N) \to \mathrm{Ext}^2_B(L_{B/A}, N) \to \ldots$$

Hence taking ξ the image of ξ' works. □

92.17. The Atiyah class of a module

09DC Let $A \to B$ be a ring map. Let M be a B-module. Let $P \to B$ be an object of $\mathcal{C}_{B/A}$ (Section 92.4). Consider the extension of principal parts

$$0 \to \Omega_{P/A} \otimes_P M \to P^1_{P/A}(M) \to M \to 0$$

see Algebra, Lemma 10.133.6. This sequence is functorial in P by Algebra, Remark 10.133.7. Thus we obtain a short exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M} \to P^1_{\mathcal{O}/\underline{A}}(M) \to \underline{M} \to 0$$

on $\mathcal{C}_{B/A}$. We have $L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M}) = L_{B/A} \otimes_B M = L_{B/A} \otimes^{\mathbf{L}}_B M$ by Lemma 92.4.2 and the flatness of the terms of $L_{B/A}$. We have $L\pi_!(\underline{M}) = M$ by Lemma 92.4.4. Thus a distinguished triangle

09DD (92.17.0.1) $$L_{B/A} \otimes^{\mathbf{L}}_B M \to L\pi_!\left(P^1_{\mathcal{O}/\underline{A}}(M)\right) \to M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$$

in $D(B)$. Here we use Cohomology on Sites, Remark 21.39.13 to get a distinguished triangle in $D(B)$ and not just in $D(A)$.

09DE **Definition 92.17.1.** Let $A \to B$ be a ring map. Let M be a B-module. The map $M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$ in (92.17.0.1) is called the *Atiyah class* of M.

92.18. The cotangent complex

08UQ In this section we discuss the cotangent complex of a map of sheaves of rings on a site. In later sections we specialize this to obtain the cotangent complex of a morphism of ringed topoi, a morphism of ringed spaces, a morphism of schemes, a morphism of algebraic space, etc.

Let $\mathcal{C}$ be a site and let $Sh(\mathcal{C})$ denote the associated topos. Let $\mathcal{A}$ denote a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A}$-Alg be the category of $\mathcal{A}$-algebras. Consider the pair of adjoint functors (U, V) where $V : \mathcal{A}\text{-}Alg \to Sh(\mathcal{C})$ is the forgetful functor and $U : Sh(\mathcal{C}) \to \mathcal{A}\text{-}Alg$ assigns to a sheaf of sets $\mathcal{E}$ the polynomial algebra $\mathcal{A}[\mathcal{E}]$ on $\mathcal{E}$ over $\mathcal{A}$. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{A}\text{-}Alg, \mathcal{A}\text{-}Alg)$ constructed in Simplicial, Section 14.34.

Now assume that $\mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings. Then $\mathcal{B}$ is an object of the category $\mathcal{A}$-Alg. Denote $\mathcal{P}_\bullet = X_\bullet(\mathcal{B})$ the resulting simplicial $\mathcal{A}$-algebra. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$, $\mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]]$, and so on. Recall also that there is an augmentation

$$\epsilon : \mathcal{P}_\bullet \longrightarrow \mathcal{B}$$

where we view $\mathcal{B}$ as a constant simplicial $\mathcal{A}$-algebra.

08SR **Definition 92.18.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *standard resolution of $\mathcal{B}$ over $\mathcal{A}$* is the augmentation $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ with terms

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}], \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]], \quad \ldots$$

and maps as constructed above.

With this definition in hand the cotangent complex of a map of sheaves of rings is defined as follows. We will use the module of differentials as defined in Modules on Sites, Section 18.33.

08SS **Definition 92.18.2.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *cotangent complex* $L_{\mathcal{B}/\mathcal{A}}$ is the complex of $\mathcal{B}$-modules associated to the simplicial module

$$\Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet, \epsilon} \mathcal{B}$$

where $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. We usually think of $L_{\mathcal{B}/\mathcal{A}}$ as an object of $D(\mathcal{B})$.

These constructions satisfy a functoriality similar to that discussed in Section 92.6. Namely, given a commutative diagram

08ST (92.18.2.1)
$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$L_{\mathcal{B}/\mathcal{A}} \longrightarrow L_{\mathcal{B}'/\mathcal{A}'}$$

constructed as follows. If $\mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$ is the standard resolution of $\mathcal{B}'$ over $\mathcal{A}'$, then there is a canonical map

$\mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ of simplicial $\mathcal{A}$-algebras compatible with the augmentations $\mathcal{P}_\bullet \to \mathcal{B}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$. The maps

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{A}'[\mathcal{B}'] = \mathcal{P}'_0, \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]] \longrightarrow \mathcal{A}'[\mathcal{A}'[\mathcal{B}']] = \mathcal{P}'_1$$

and so on are given by the given maps $\mathcal{A} \to \mathcal{A}'$ and $\mathcal{B} \to \mathcal{B}'$. The desired map $L_{\mathcal{B}/\mathcal{A}} \to L_{\mathcal{B}'/\mathcal{A}'}$ then comes from the associated maps on sheaves of differentials.

08SV **Lemma 92.18.3.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $f^{-1}L_{\mathcal{B}/\mathcal{A}} = L_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. The diagram

$$\begin{array}{ccc} \mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{C}) \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ f^{-1}\mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{D}) \end{array}$$

commutes. □

08SW **Lemma 92.18.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^i(L_{\mathcal{B}/\mathcal{A}})$ is the sheaf associated to the presheaf $U \mapsto H^i(L_{\mathcal{B}(U)/\mathcal{A}(U)})$.*

Proof. Let $\mathcal{C}'$ be the site we get by endowing $\mathcal{C}$ with the chaotic topology (presheaves are sheaves). There is a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ where f_* is the inclusion of sheaves into presheaves and f^{-1} is sheafification. By Lemma 92.18.3 it suffices to prove the result for $\mathcal{C}'$, i.e., in case $\mathcal{C}$ has the chaotic topology.

If $\mathcal{C}$ carries the chaotic topology, then $L_{\mathcal{B}/\mathcal{A}}(U)$ is equal to $L_{\mathcal{B}(U)/\mathcal{A}(U)}$ because

$$\begin{array}{ccc} \mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{C}) \\ {\scriptstyle \text{sections over } U}\downarrow & & \downarrow{\scriptstyle \text{sections over } U} \\ \mathcal{A}(U)\text{-}Alg & \rightleftarrows & Sets \end{array}$$

commutes. □

08SX **Remark 92.18.5.** It is clear from the proof of Lemma 92.18.4 that for any $U \in \mathrm{Ob}(\mathcal{C})$ there is a canonical map $L_{\mathcal{B}(U)/\mathcal{A}(U)} \to L_{\mathcal{B}/\mathcal{A}}(U)$ of complexes of $\mathcal{B}(U)$-modules. Moreover, these maps are compatible with restriction maps and the complex $L_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the rule $U \mapsto L_{\mathcal{B}(U)/\mathcal{A}(U)}$.

08UR **Lemma 92.18.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^0(L_{\mathcal{B}/\mathcal{A}}) = \Omega_{\mathcal{B}/\mathcal{A}}$.*

Proof. Follows from Lemmas 92.18.4 and 92.4.5 and Modules on Sites, Lemma 18.33.4. □

08SY **Lemma 92.18.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be homomorphisms of sheaves of rings on $\mathcal{C}$. Then*

$$L_{\mathcal{B}\times\mathcal{B}'/\mathcal{A}} \longrightarrow L_{\mathcal{B}/\mathcal{A}} \oplus L_{\mathcal{B}'/\mathcal{A}}$$

is an isomorphism in $D(\mathcal{B} \times \mathcal{B}')$.

Proof. By Lemma 92.18.4 it suffices to prove this for ring maps. In the case of rings this is Lemma 92.6.4. □

The fundamental triangle for the cotangent complex of sheaves of rings is an easy consequence of the result for homomorphisms of rings.

08SZ **Lemma 92.18.8.** *Let $\mathcal{D}$ be a site. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be homomorphisms of sheaves of rings on $\mathcal{D}$. There is a canonical distinguished triangle*

$$L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{C} \to L_{\mathcal{C}/\mathcal{A}} \to L_{\mathcal{C}/\mathcal{B}} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{C}[1]$$

in $D(\mathcal{C})$.

Proof. We will use the method described in Remarks 92.7.5 and 92.7.6 to construct the triangle; we will freely use the results mentioned there. As in those remarks we first construct the triangle in case $\mathcal{B} \to \mathcal{C}$ is an injective map of sheaves of rings. In this case we set

(1) $\mathcal{P}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$,
(2) $\mathcal{Q}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{A}$,
(3) $\mathcal{R}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{B}$,
(4) $\mathcal{S}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{B}$,
(5) $\overline{\mathcal{Q}}_\bullet = \mathcal{Q}_\bullet \otimes_{\mathcal{P}_\bullet} \mathcal{B}$, and
(6) $\overline{\mathcal{R}}_\bullet = \mathcal{R}_\bullet \otimes_{\mathcal{S}_\bullet} \mathcal{B}$.

The distinguished triangle is the distinguished triangle associated to the short exact sequence of simplicial $\mathcal{C}$-modules

$$0 \to \Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet} \mathcal{C} \to \Omega_{\mathcal{Q}_\bullet/\mathcal{A}} \otimes_{\mathcal{Q}_\bullet} \mathcal{C} \to \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C} \to 0$$

The first two terms are equal to the first two terms of the triangle of the statement of the lemma. The identification of the last term with $L_{\mathcal{C}/\mathcal{B}}$ uses the quasi-isomorphisms of complexes

$$L_{\mathcal{C}/\mathcal{B}} = \Omega_{\mathcal{R}_\bullet/\mathcal{B}} \otimes_{\mathcal{R}_\bullet} \mathcal{C} \longrightarrow \Omega_{\overline{\mathcal{R}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{R}}_\bullet} \mathcal{C} \longleftarrow \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C}$$

All the constructions used above can first be done on the level of presheaves and then sheafified. Hence to prove sequences are exact, or that map are quasi-isomorphisms it suffices to prove the corresponding statement for the ring maps $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ which are known. This finishes the proof in the case that $\mathcal{B} \to \mathcal{C}$ is injective.

In general, we reduce to the case where $\mathcal{B} \to \mathcal{C}$ is injective by replacing $\mathcal{C}$ by $\mathcal{B} \times \mathcal{C}$ if necessary. This is possible by the argument given in Remark 92.7.5 by Lemma 92.18.7. □

08T0 **Lemma 92.18.9.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. If p is a point of $\mathcal{C}$, then $(L_{\mathcal{B}/\mathcal{A}})_p = L_{\mathcal{B}_p/\mathcal{A}_p}$.*

Proof. This is a special case of Lemma 92.18.3. □

For the construction of the naive cotangent complex and its properties we refer to Modules on Sites, Section 18.35.

08US **Lemma 92.18.10.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. There is a canonical map $L_{\mathcal{B}/\mathcal{A}} \to NL_{\mathcal{B}/\mathcal{A}}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1} L_{\mathcal{B}/\mathcal{A}}$.*

Proof. Let $\mathcal{P}_\bullet$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{A}[\mathcal{B}] \to \mathcal{B})$. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$. The map of the lemma is given by the commutative diagram

$$\begin{array}{ccccccccc} L_{\mathcal{B}/\mathcal{A}} & & \ldots & \longrightarrow & \Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ NL_{\mathcal{B}/\mathcal{A}} & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \end{array}$$

We construct the downward arrow with target $\mathcal{I}/\mathcal{I}^2$ by sending a local section $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in $\mathcal{I}/\mathcal{I}^2$. Here $d_i : \mathcal{P}_1 \to \mathcal{P}_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps $\mathcal{P}_1$ into $\mathcal{I} = \mathrm{Ker}(\mathcal{P}_0 \to \mathcal{B})$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} \to \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B}$ as this differential maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} \to \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B}$ maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes.

To see that our map induces an isomorphism on the cohomology sheaves H^0 and H^{-1} we argue as follows. Let $\mathcal{C}'$ be the site with the same underlying category as $\mathcal{C}$ but endowed with the chaotic topology. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be the morphism of topoi whose pullback functor is sheafification. Let $\mathcal{A}' \to \mathcal{B}'$ be the given map, but thought of as a map of sheaves of rings on $\mathcal{C}'$. The construction above gives a map $L_{\mathcal{B}'/\mathcal{A}'} \to NL_{\mathcal{B}'/\mathcal{A}'}$ on $\mathcal{C}'$ whose value over any object U of $\mathcal{C}'$ is just the map

$$L_{\mathcal{B}(U)/\mathcal{A}(U)} \to NL_{\mathcal{B}(U)/\mathcal{A}(U)}$$

of Remark 92.11.4 which induces an isomorphism on H^0 and H^{-1}. Since $f^{-1} L_{\mathcal{B}'/\mathcal{A}'} = L_{\mathcal{B}/\mathcal{A}}$ (Lemma 92.18.3) and $f^{-1}\, NL_{\mathcal{B}'/\mathcal{A}'} = NL_{\mathcal{B}/\mathcal{A}}$ (Modules on Sites, Lemma 18.35.3) the lemma is proved. □

92.19. The Atiyah class of a sheaf of modules

09DF Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Let $\mathcal{P}_\bullet \to \mathcal{B}$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ (Section 92.18). For every $n \geq 0$ consider the extension of principal parts

09DG (92.19.0.1) $$0 \to \Omega_{\mathcal{P}_n/\mathcal{A}} \otimes_{\mathcal{P}_n} \mathcal{F} \to \mathcal{P}^1_{\mathcal{P}_n/\mathcal{A}}(\mathcal{F}) \to \mathcal{F} \to 0$$

see Modules on Sites, Lemma 18.34.6. The functoriality of this construction (Modules on Sites, Remark 18.34.7) tells us (92.19.0.1) is the degree n part of a short exact sequence of simplicial $\mathcal{P}_\bullet$-modules (Cohomology on Sites, Section 21.41). Using the functor $L\pi_! : D(\mathcal{P}_\bullet) \to D(\mathcal{B})$ of Cohomology on Sites, Remark 21.41.3 (here we use that $\mathcal{P}_\bullet \to \mathcal{A}$ is a resolution) we obtain a distinguished triangle

09DH (92.19.0.2) $$L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F} \to L\pi_! \left(\mathcal{P}^1_{\mathcal{P}_\bullet/\mathcal{A}}(\mathcal{F}) \right) \to \mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1]$$

in $D(\mathcal{B})$.

09DI **Definition 92.19.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. The map $\mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1]$ in (92.19.0.2) is called the *Atiyah class* of $\mathcal{F}$.

92.20. The cotangent complex of a morphism of ringed spaces

08UT The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

08UU **Definition 92.20.1.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. The *cotangent complex* L_f of f is $L_f = L_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$. We will also use the notation $L_f = L_{X/S} = L_{\mathcal{O}_X/\mathcal{O}_S}$.

More precisely, this means that we consider the cotangent complex (Definition 92.18.2) of the homomorphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ of sheaves of rings on the site associated to the topological space X (Sites, Example 7.6.4).

08UV **Lemma 92.20.2.** *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Then $H^0(L_{X/S}) = \Omega_{X/S}$.*

Proof. Special case of Lemma 92.18.6. □

08T4 **Lemma 92.20.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then there is a canonical distinguished triangle*

$$Lf^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to Lf^*L_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_Z = f^{-1}g^{-1}\mathcal{O}_Z$. By Lemma 92.18.3 we have $f^{-1}L_{Y/Z} = L_{f^{-1}\mathcal{O}_Y/h^{-1}\mathcal{O}_Z}$ and this is a complex of flat $f^{-1}\mathcal{O}_Y$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 92.18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_Z$, $\mathcal{B} = f^{-1}\mathcal{O}_Y$, and $\mathcal{C} = \mathcal{O}_X$. □

08UW **Lemma 92.20.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical map $L_{X/Y} \to NL_{X/Y}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_{X/Y}$.*

Proof. Special case of Lemma 92.18.10. □

92.21. Deformations of ringed spaces and the cotangent complex

08UX This section is the continuation of Deformation Theory, Section 91.7 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules. We ask whether we can find the question mark fitting into the following diagram

08UY (92.21.0.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in Deformation Theory, Equation (91.3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (92.21.0.1).

08UZ **Lemma 92.21.1.** *In the situation above we have*

(1) *There is a canonical element* $\xi \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$ *whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.21.0.1).*
(2) *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under* $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.
(3) *Given a solution* X'*, the set of automorphisms of* X' *fitting into (92.21.0.1) is canonically isomorphic to* $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.

Proof. Via the identifications $NL_{X/S} = \tau_{\geq -1} L_{X/S}$ (Lemma 92.20.4) and $H^0(L_{X/S}) = \Omega_{X/S}$ (Lemma 92.20.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.7.1 and 91.7.3.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 91.7.9 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 91.7.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G})$$

The distinguished triangle of Lemma 92.20.3 for $X \to S \to S'$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G}) \to \ldots$$

Hence taking ξ the image of ξ' works. □

92.22. The cotangent complex of a morphism of ringed topoi

08SQ The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

08SU **Definition 92.22.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The *cotangent complex* L_f of f is $L_f = L_{\mathcal{O}_\mathcal{C}/f^{-1}\mathcal{O}_\mathcal{D}}$. We sometimes write $L_f = L_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{D}}$.

This definition applies to many situations, but it doesn't always produce the thing one expects. For example, if $f : X \to Y$ is a morphism of schemes, then f induces a morphism of big étale sites $f_{big} : (Sch/X)_{étale} \to (Sch/Y)_{étale}$ which is a morphism of ringed topoi (Descent, Remark 35.8.4). However, $L_{f_{big}} = 0$ since $(f_{big})^\sharp$ is an isomorphism. On the other hand, if we take L_f where we think of f as a morphism between the underlying Zariski ringed topoi, then L_f does agree with the cotangent complex $L_{X/Y}$ (as defined below) whose zeroth cohomology sheaf is $\Omega_{X/Y}$.

08V0 **Lemma 92.22.2.** *Let* $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ *be a morphism of ringed topoi. Then* $H^0(L_f) = \Omega_f$.

Proof. Special case of Lemma 92.18.6. □

08V1 **Lemma 92.22.3.** *Let* $f : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ *and* $g : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ *be morphisms of ringed topoi. Then there is a canonical distinguished triangle*

$$Lf^* L_g \to L_{g \circ f} \to L_f \to Lf^* L_g[1]$$

in $D(\mathcal{O}_1)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_3 = f^{-1}g^{-1}\mathcal{O}_3$. By Lemma 92.18.3 we have $f^{-1}L_g = L_{f^{-1}\mathcal{O}_2/h^{-1}\mathcal{O}_3}$ and this is a complex of flat $f^{-1}\mathcal{O}_2$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 92.18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_3$, $\mathcal{B} = f^{-1}\mathcal{O}_2$, and $\mathcal{C} = \mathcal{O}_1$. □

08V2 **Lemma 92.22.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. There is a canonical map $L_f \to NL_f$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_f$.*

Proof. Special case of Lemma 92.18.10. □

92.23. Deformations of ringed topoi and the cotangent complex

08V3 This section is the continuation of Deformation Theory, Section 91.13 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_\mathcal{B}$-modules. We ask whether we can find the question mark fitting into the following diagram

08V4 (92.23.0.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & {\scriptstyle c}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_\mathcal{B} & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in Deformation Theory, Equation (91.9.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (92.23.0.1).

08V5 **Lemma 92.23.1.** *In the situation above we have*

1. *There is a canonical element $\xi \in \mathrm{Ext}^2_\mathcal{O}(L_f, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (92.23.0.1).*
2. *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_\mathcal{O}(L_f, \mathcal{G})$.*
3. *Given a solution X', the set of automorphisms of X' fitting into (92.23.0.1) is canonically isomorphic to $\mathrm{Ext}^0_\mathcal{O}(L_f, \mathcal{G})$.*

Proof. Via the identifications $NL_f = \tau_{\geq -1}L_f$ (Lemma 92.22.4) and $H^0(L_f) = \Omega_f$ (Lemma 92.22.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 91.13.1 and 91.13.3.

Proof of (1). To match notation with Deformation Theory, Section 91.13 we will write $NL_f = NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}$ and $L_f = L_{\mathcal{O}/\mathcal{O}_\mathcal{B}}$ and similarly for the morphisms t and $t \circ f$. By Deformation Theory, Lemma 91.13.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_\mathcal{O}(Lf^*\, NL_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(Lf^* L_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_\mathcal{O}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_\mathcal{O}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_\mathcal{O}(Lf^* L_{\mathcal{O}_\mathcal{B}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

The distinguished triangle of Lemma 92.22.3 for f and t gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(Lf^* L_{\mathcal{O}_{\mathcal{B}}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

Hence taking ξ the image of ξ' works. □

92.24. The cotangent complex of a morphism of schemes

08T1 As promised above we define the cotangent complex of a morphism of schemes as follows.

08T2 **Definition 92.24.1.** Let $f : X \to Y$ be a morphism of schemes. The *cotangent complex* $L_{X/Y}$ *of* X *over* Y is the cotangent complex of f as a morphism of ringed spaces (Definition 92.20.1).

In particular, the results of Section 92.20 apply to cotangent complexes of morphisms of schemes. The next lemma shows this definition is compatible with the definition for ring maps and it also implies that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

08T3 **Lemma 92.24.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $U = \mathrm{Spec}(B) \subset X$ and $V = \mathrm{Spec}(A) \subset Y$ be affine opens such that $f(U) \subset V$. There is a canonical map*

$$\widetilde{L_{B/A}} \longrightarrow L_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$. This map is compatible with restricting to smaller affine opens of X and Y.

Proof. By Remark 92.18.5 there is a canonical map of complexes $L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to L_{X/Y}(U)$ of $B = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $A \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $L_{B/A} \to L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of B-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 26.7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows immediately from Lemmas 92.18.9 and 92.8.6 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(B)$ as $B_{\mathfrak{p}}$ and $A_{\mathfrak{q}}$ where $\mathfrak{q} = A \cap \mathfrak{p}$; references used are Schemes, Lemma 26.5.4 and Sheaves, Lemma 6.21.5). □

08V6 **Lemma 92.24.3.** *Let Λ be a ring. Let X be a scheme over Λ. Then*

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on X.

Proof. Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda) \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 92.20.3 it suffices to show that $L_q = 0$. To see this it suffices to show for $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ that

$$(L_q)_{\mathfrak{p}} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda),\mathfrak{p}}/\Lambda} = L_{\Lambda_{\mathfrak{p}}/\Lambda}$$

(Lemma 92.18.9) is zero which follows from Lemma 92.8.4. □

92.25. The cotangent complex of a scheme over a ring

08V7 Let Λ be a ring and let X be a scheme over Λ. Write $L_{X/\operatorname{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 92.24.3. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 92.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over X_{Zar} and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be a scheme over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

08V8 (92.25.0.1)
$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\Lambda) & \longleftarrow & \mathbf{A} \end{array}$$

of schemes where

(1) U is an open subscheme of X,
(2) there exists an isomorphism $\mathbf{A} = \operatorname{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\operatorname{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that X_{Zar} denotes the small Zariski site X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{Zar},\ (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 92.4.

08V9 **Lemma 92.25.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over X_{Zar}.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of X_{Zar} consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over X_{Zar}. □

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from X_{Zar} (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{Zar})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

08VA (92.25.1.1)
$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow[i]{} & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{Zar}), \mathcal{O}_X) & & \end{array}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

08VB **Remark 92.25.2.** In the situation above, for every $U \subset X$ open let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 92.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.40.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

08T9 **Lemma 92.25.3.** *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 92.25.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 92.25.2. Now Remark 92.18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on X. This is what we want by Lemma 92.24.3. □

92.26. The cotangent complex of a morphism of algebraic spaces

08VC We define the cotangent complex of a morphism of algebraic spaces using the associated morphism between the small étale sites.

08VD **Definition 92.26.1.** Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *cotangent complex* $L_{X/Y}$ *of* X *over* Y is the cotangent complex of the morphism of ringed topoi f_{small} between the small étale sites of X and Y (see Properties of Spaces, Lemma 66.21.3 and Definition 92.22.1).

In particular, the results of Section 92.22 apply to cotangent complexes of morphisms of algebraic spaces. The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

08VE **Lemma 92.26.2.** *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow[g]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $L_{X/Y}|_{U_{étale}} = L_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the cotangent complex commutes with pullback (Lemma 92.18.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ (Properties of Spaces, Lemma 66.26.1). Tracing through the definitions we conclude that $L_{X/Y}|_{U_{étale}} = L_{U/V}$. □

08VF **Lemma 92.26.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $L_{X/Y} = \epsilon^* L_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 75.4 and L_{X_0/Y_0} is as in Definition 92.24.1.*

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0} \to f_{small}^{-1}\mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1}\mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow[\epsilon]{} & X_{étale} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{étale} \end{array}$$

see Derived Categories of Spaces, Remark 75.6.3. Thus we obtain a canonical map

$$\epsilon^{-1}L_{X_0/Y_0} = \epsilon^{-1}L_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = L_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \longrightarrow L_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = L_{X/Y}$$

by the functoriality discussed in Section 92.18 and Lemma 92.18.3. To see that the induced map $\epsilon^* L_{X_0/Y_0} \to L_{X/Y}$ is an isomorphism we may check on stalks at geometric points (Properties of Spaces, Theorem 66.19.12). We will use Lemma 92.18.9 to compute the stalks. Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$L_{X/Y,\overline{x}} = L_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

and

$$(\epsilon^* L_{X_0/Y_0})_{\overline{x}} = L_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = L_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 7.34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 18.36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 66.22.1). Thus the result follows from Lemma 92.8.7. □

08VG **Lemma 92.26.4.** *Let Λ be a ring. Let X be an algebraic space over Λ. Then*

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on $X_{étale}$.

Proof. Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda)_{étale} \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 92.22.3 it suffices to show that $L_q = 0$. To see this it suffices to show (Properties of Spaces, Theorem 66.19.12) for a geometric point $\overline{t} : \mathrm{Spec}(k) \to \mathrm{Spec}(\Lambda)$ that

$$(L_q)_{\overline{t}} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda)_{étale}, \overline{t}}/\Lambda}$$

(Lemma 92.18.9) is zero. Since $\mathcal{O}_{\mathrm{Spec}(\Lambda)_{étale}, \overline{t}}$ is a strict henselization of a local ring of Λ (Properties of Spaces, Lemma 66.22.1) this follows from Lemma 92.8.4. □

92.27. The cotangent complex of an algebraic space over a ring

08VH Let Λ be a ring and let X be an algebraic space over Λ. Write $L_{X/\mathrm{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 92.26.4. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 92.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over $X_{étale}$ and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be an algebraic space over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

08VI (92.27.0.1)
$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\Lambda) & \longleftarrow & \mathbf{A} \end{array}$$

of schemes where

(1) U is a scheme,
(2) $U \to X$ is étale,
(3) there exists an isomorphism $\mathbf{A} = \mathrm{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\mathrm{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that $X_{étale}$ denotes the small étale site of X whose objects are schemes étale over X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{étale}, \quad (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 92.4.

08VJ **Lemma 92.27.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over $X_{étale}$.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of $X_{étale}$ consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over $X_{étale}$. □

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from $X_{étale}$ (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{étale})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.

(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

08VK (92.27.1.1)
$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow{i} & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{étale}), \mathcal{O}_X) & & \end{array}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

08VL **Remark 92.27.2.** In the situation above, for every object $U \to X$ of $X_{étale}$ let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 92.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.40.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

08VM **Lemma 92.27.3.** *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 92.27.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 92.27.2. Now Remark 92.18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on $X_{étale}$. This is what we want by Lemma 92.26.4. □

92.28. Fibre products of algebraic spaces and the cotangent complex

09DJ Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. Consider the fibre product $X \times_B Y$ with projection morphisms $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$. In this section we discuss $L_{X \times_B Y/B}$. Most of the information we want is contained in the following diagram

09DK (92.28.0.1)
$$\begin{array}{ccccc} Lp^*L_{X/B} & \longrightarrow & L_{X\times_B Y/Y} & \longrightarrow & E \\ \| & & \uparrow & & \uparrow \\ Lp^*L_{X/B} & \longrightarrow & L_{X\times_B Y/B} & \longrightarrow & L_{X\times_B Y/X} \\ & & \uparrow & & \uparrow \\ & & Lq^*L_{Y/B} & = & Lq^*L_{Y/B} \end{array}$$

Explanation: The middle row is the fundamental triangle of Lemma 92.22.3 for the morphisms $X \times_B Y \to X \to B$. The middle column is the fundamental triangle for the morphisms $X \times_B Y \to Y \to B$. Next, E is an object of $D(\mathcal{O}_{X \times_B Y})$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.9.1) of the map of complexes

$$Lp^*L_{X/B} \oplus Lq^*L_{Y/B} \longrightarrow L_{X\times_B Y/B}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

09DL **Lemma 92.28.1.** *In the situation above, if X and Y are Tor independent over B, then the object E in (92.28.0.1) is zero. In this case we have*

$$L_{X\times_B Y/B} = Lp^*L_{X/B} \oplus Lq^*L_{Y/B}$$

Proof. Choose a scheme W and a surjective étale morphism $W \to B$. Choose a scheme U and a surjective étale morphism $U \to X \times_B W$. Choose a scheme V and a surjective étale morphism $V \to Y \times_B W$. Then $U \times_W V \to X \times_B Y$ is surjective étale too. Hence it suffices to prove that the restriction of E to $U \times_W V$ is zero. By Lemma 92.26.3 and Derived Categories of Spaces, Lemma 75.20.3 this reduces us to the case of schemes. Taking suitable affine opens we reduce to the case of affine schemes. Using Lemma 92.24.2 we reduce to the case of a tensor product of rings, i.e., to Lemma 92.15.1. □

In general we can say the following about the object E.

09DM **Lemma 92.28.2.** *Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. The object E in (92.28.0.1) satisfies $H^i(E) = 0$ for $i = 0, -1$ and for a geometric point $(\overline{x}, \overline{y}) : \mathrm{Spec}(k) \to X \times_B Y$ we have*

$$H^{-2}(E)_{(\overline{x},\overline{y})} = Tor_1^R(A, B) \otimes_{A \otimes_R B} C$$

where $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{Y,\overline{y}}$, and $C = \mathcal{O}_{X\times_B Y,(\overline{x},\overline{y})}$.

Proof. The formation of the cotangent complex commutes with taking stalks and pullbacks, see Lemmas 92.18.9 and 92.18.3. Note that C is a henselization of $A \otimes_R B$. $L_{C/R} = L_{A \otimes_R B/R} \otimes_{A \otimes_R B} C$ by the results of Section 92.8. Thus the stalk of E at our geometric point is the cone of the map $L_{A/R} \otimes C \to L_{A \otimes_R B/R} \otimes C$. Therefore the results of the lemma follow from the case of rings, i.e., Lemma 92.15.2. □

92.29. Other chapters

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CHAPTER 93

Deformation Problems

0DVK

93.1. Introduction

0DVL The goal of this chapter is to work out examples of the general theory developed in the chapters Formal Deformation Theory, Deformation Theory, The Cotangent Complex.

Section 3 of the paper [**Sch68**] by Schlessinger discusses some examples as well.

93.2. Examples of deformation problems

06LA List of things that should go here:

(1) Deformations of schemes:
   (a) The Rim-Schlessinger condition.
   (b) Computing the tangent space.
   (c) Computing the infinitesimal deformations.
   (d) The deformation category of an affine hypersurface.
(2) Deformations of sheaves (for example fix X/S, a finite type point s of S, and a quasi-coherent sheaf $\mathcal{F}_s$ over X_s).
(3) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
(4) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
(5) Add more here.

93.3. General outline

0DVM This section lays out the procedure for discussing the next few examples.

Step I. For each section we fix a Noetherian ring Λ and we fix a finite ring map $\Lambda \to k$ where k is a field. As usual we let $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$ be our base category, see Formal Deformation Theory, Definition 90.3.1.

Step II. In each section we define a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$. Occasionally we will consider instead a functor $F : \mathcal{C}_\Lambda \to Sets$.

Step III. We explain to what extent $\mathcal{F}$ satisfies the Rim-Schlesssinger condition (RS) discussed in Formal Deformation Theory, Section 90.16. Similarly, we may discuss to what extent our $\mathcal{F}$ satisfies (S1) and (S2) or to what extent F satisfies the corresponding Schlessinger's conditions (H1) and (H2). See Formal Deformation Theory, Section 90.10.

Step IV. Let x_0 be an object of $\mathcal{F}(k)$, in other words an object of $\mathcal{F}$ over k. In this chapter we will use the notation

$$\mathcal{D}ef_{x_0} = \mathcal{F}_{x_0}$$

to denote the predeformation category constructed in Formal Deformation Theory, Remark 90.6.4. If $\mathcal{F}$ satisfies (RS), then $\mathcal{D}ef_{x_0}$ is a deformation category (Formal Deformation Theory, Lemma 90.16.11) and satisfies (S1) and (S2) (Formal Deformation Theory, Lemma 90.16.6). If (S1) and (S2) are satisfied, then an important question is whether the tangent space

$$T\mathcal{D}ef_{x_0} = T_{x_0}\mathcal{F} = T\mathcal{F}_{x_0}$$

(see Formal Deformation Theory, Remark 90.12.5 and Definition 90.12.1) is finite dimensional. Namely, this insures that $\mathcal{D}ef_{x_0}$ has a versal formal object (Formal Deformation Theory, Lemma 90.13.4).

Step V. If $\mathcal{F}$ passes Step IV, then the next question is whether the k-vector space

$$\mathrm{Inf}(\mathcal{D}ef_{x_0}) = \mathrm{Inf}_{x_0}(\mathcal{F})$$

of infinitesimal automorphisms of x_0 is finite dimensional. Namely, if true, this implies that $\mathcal{D}ef_{x_0}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Theorem 90.26.4.

93.4. Finite projective modules

0DVN This section is just a warmup. Of course finite projective modules should not have any "moduli".

0D3I **Example 93.4.1** (Finite projective modules)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, M) consisting of an object A of $\mathcal{C}_\Lambda$ and a finite projective A-module M, and
(2) a morphism $(f, g) : (B, N) \to (A, M)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V, let $x_0 = (k, V)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_V = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{matrix}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod\nolimits_{n\geq 0}\{*\}$$

Although this means that the deformation theory of $\mathcal{F}$ is essentially trivial, we still work through the steps outlined in Section 93.3 to provide an easy example.

0DVP **Lemma 93.4.2.** *Example 93.4.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_V$ is a deformation category for any finite dimensional vector space V over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Thus we have to show that the category of finite projective modules over $A_1 \times_A A_2$ is equivalent to the fibre product of the categories of finite projective modules over

A_1 and A_2 over the category of finite projective modules over A. This is a special case of More on Algebra, Lemma 15.6.9. We recall that the inverse functor sends the triple (M_1, M_2, φ) where M_1 is a finite projective A_1-module, M_2 is a finite projective A_2-module, and $\varphi : M_1 \otimes_{A_1} A \to M_2 \otimes_{A_2} A$ is an isomorphism of A-module, to the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. □

0DVQ **Lemma 93.4.3.** *In Example 93.4.1 let V be a finite dimensional k-vector space. Then*

$$T\mathcal{D}ef_V = (0) \quad \text{and} \quad Inf(\mathcal{D}ef_V) = End_k(V)$$

are finite dimensional.

Proof. With $\mathcal{F}$ as in Example 93.4.1 set $x_0 = (k, V) \in \mathrm{Ob}(\mathcal{F}(k))$. Recall that $T\mathcal{D}ef_V = T_{x_0}\mathcal{F}$ is the set of isomorphism classes of pairs (x, α) consisting of an object x of $\mathcal{F}$ over the dual numbers $k[\epsilon]$ and a morphism $\alpha : x \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$.

Up to isomorphism, there is a unique pair (M, α) consisting of a finite projective module M over $k[\epsilon]$ and $k[\epsilon]$-linear map $\alpha : M \to V$ which induces an isomorphism $M \otimes_{k[\epsilon]} k \to V$. For example, if $V = k^{\oplus n}$, then we take $M = k[\epsilon]^{\oplus n}$ with the obvious map α.

Similarly, $\mathrm{Inf}(\mathcal{D}ef_V) = \mathrm{Inf}_{x_0}(\mathcal{F})$ is the set of automorphisms of the trivial deformation x_0' of x_0 over $k[\epsilon]$. See Formal Deformation Theory, Definition 90.19.2 for details.

Given (M, α) as in the second paragraph, we see that an element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ is an automorphism $\gamma : M \to M$ with $\gamma \bmod \epsilon = \mathrm{id}$. Then we can write $\gamma = \mathrm{id}_M + \epsilon\psi$ where $\psi : M/\epsilon M \to M/\epsilon M$ is k-linear. Using α we can think of ψ as an element of $\mathrm{End}_k(V)$ and this finishes the proof. □

93.5. Representations of a group

0DVR The deformation theory of representations can be very interesting.

0D3J **Example 93.5.1** (Representations of a group)**.** Let Γ be a group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$, and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{matrix}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{matrix} = \coprod_{n \geq 0} \begin{matrix}\mathrm{GL}_n(A)\text{-conjugacy classes of} \\ \text{homomorphisms } \rho : \Gamma \to \mathrm{GL}_n(A)\end{matrix}$$

This is already more interesting than the discussion in Section 93.4.

0DVS **Lemma 93.5.2.** *Example 93.5.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{V,\rho_0}$ is a deformation category for any finite dimensional representation $\rho_0 : \Gamma \to GL_k(V)$.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, M_1, \rho_1), (A_2, M_2, \rho_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then, as seen in the proof of Lemma 93.4.2, we can consider the finite projective $A_1 \times_A A_2$-module $M_1 \times_\varphi M_2$. Since φ is compatible with the given actions we obtain

$$\rho_1 \times \rho_2 : \Gamma \longrightarrow \mathrm{GL}_{A_1 \times_A A_2}(M_1 \times_\varphi M_2)$$

Then $(M_1 \times_\varphi M_2, \rho_1 \times \rho_2)$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor. □

0DVT **Lemma 93.5.3.** *In Example 93.5.1 let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Then*

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V) = H^1(\Gamma, End_k(V)) \quad \text{and} \quad Inf(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, End_k(V))$$

Thus $Inf(\mathcal{D}ef_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}ef_{V,\rho_0}$ is finite dimensional if Γ is finitely generated.

Proof. We first deal with the infinitesimal automorphisms. Let $M = V \otimes_k k[\epsilon]$ with induced action $\rho'_0 : \Gamma \to \mathrm{GL}_n(M)$. Then an infinitesimal automorphism, i.e., an element of $\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0})$, is given by an automorphism $\gamma = \mathrm{id} + \epsilon\psi : M \to M$ as in the proof of Lemma 93.4.3, where moreover ψ has to commute with the action of Γ (given by ρ_0). Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, \mathrm{End}_k(V))$$

as predicted in the lemma.

Next, let $(k[\epsilon], M, \rho)$ be an object of $\mathcal{F}$ over $k[\epsilon]$ and let $\alpha : M \to V$ be a Γ-equivariant map inducing an isomorphism $M/\epsilon M \to V$. Since M is free as a $k[\epsilon]$-module we obtain an extension of Γ-modules

$$0 \to V \to M \xrightarrow{\alpha} V \to 0$$

We omit the detailed construction of the map on the left. Conversely, if we have an extension of Γ-modules as above, then we can use this to make a $k[\epsilon]$-module structure on M and get an object of $\mathcal{F}(k[\epsilon])$ together with a map α as above. It follows that

$$T\mathcal{D}ef_{V,\rho_0} = \mathrm{Ext}^1_{k[\Gamma]}(V, V)$$

as predicted in the lemma. This is equal to $H^1(\Gamma, \mathrm{End}_k(V))$ by Étale Cohomology, Lemma 59.57.4.

The statement on dimensions follows from Étale Cohomology, Lemma 59.57.5. □

In Example 93.5.1 if Γ is finitely generated and (V, ρ_0) is a finite dimensional representation of Γ over k, then $\mathcal{D}ef_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.5.2 and 93.5.3 and the general discussion in Section 93.3.

0ET1 **Lemma 93.5.4.** *In Example 93.5.1 assume Γ finitely generated. Let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, End_k(V)) = k$, then F is prorepresentable.

Proof. The existence of a hull follows from Lemmas 93.5.2 and 93.5.3 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $H^0(\Gamma, \mathrm{End}_k(V)) = k$. To see that F is prorepresentable it suffices to show that F is a deformation functor, see Formal Deformation Theory, Theorem 90.18.2. In other words, we have to show F satisfies (RS). For this we can use the criterion of Formal Deformation Theory, Lemma 90.16.7. The required surjectivity of automorphism groups will follow if we show that

$$A \cdot \mathrm{id}_M = \mathrm{End}_{A[\Gamma]}(M)$$

for any object (A, M, ρ) of $\mathcal{F}$ such that $M \otimes_A k$ is isomorphic to V as a representation of Γ. Since the left hand side is contained in the right hand side, it suffices to show $\mathrm{length}_A \mathrm{End}_{A[\Gamma]}(M) \leq \mathrm{length}_A A$. Choose pairwise distinct ideals $(0) = I_n \subset \ldots \subset I_1 \subset A$ with $n = \mathrm{length}(A)$. By correspondingly filtering M, we see that it suffices to prove $\mathrm{Hom}_{A[\Gamma]}(M, I_t M/I_{t+1}M)$ has length 1. Since $I_t M/I_{t+1}M \cong M \otimes_A k$ and since any $A[\Gamma]$-module map $M \to M \otimes_A k$ factors uniquely through the quotient map $M \to M \otimes_A k$ to give an element of

$$\mathrm{End}_{A[\Gamma]}(M \otimes_A k) = \mathrm{End}_{k[\Gamma]}(V) = k$$

we conclude. □

93.6. Continuous representations

0DVU A very interesting thing one can do is to take an infinite Galois group and study the deformation theory of its representations, see [**Maz89**].

0D3K **Example 93.6.1** (Representations of a topological group)**.** Let Γ be a topological group. Let $\mathcal{F}$ be the category defined as follows

(1) an object is a triple (A, M, ρ) consisting of an object A of $\mathcal{C}_\Lambda$, a finite projective A-module M, and a continuous homomorphism $\rho : \Gamma \to \mathrm{GL}_A(M)$ where $\mathrm{GL}_A(M)$ is given the discrete topology[1], and
(2) a morphism $(f, g) : (B, N, \tau) \to (A, M, \rho)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : N \to M$ which is f-linear and Γ-equivariant and induces an isomorpism $N \otimes_{B,f} A \cong M$.

[1] An alternative would be to require the A-module M with G-action given by ρ is an A-G-module as defined in Étale Cohomology, Definition 59.57.1. However, since M is a finite A-module, this is equivalent.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, M, ρ) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a finite dimensional k-vector space V and a continuous representation $\rho_0 : \Gamma \to \mathrm{GL}_k(V)$, let $x_0 = (k, V, \rho_0)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{V,\rho_0} = \mathcal{F}_{x_0}$$

Since every finite projective module over a local ring is finite free (Algebra, Lemma 10.78.2) we see that

$$\begin{array}{c}\text{isomorphism classes} \\ \text{of objects of } \mathcal{F}(A)\end{array} = \coprod_{n \geq 0} \begin{array}{c}\mathrm{GL}_n(A)\text{-conjugacy classes of} \\ \text{continuous homomorphisms } \rho : \Gamma \to \mathrm{GL}_n(A)\end{array}$$

0DVV **Lemma 93.6.2.** *Example 93.6.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{V,\rho_0}$ is a deformation category for any finite dimensional continuous representation $\rho_0 : \Gamma \to GL_k(V)$.*

Proof. The proof is exactly the same as the proof of Lemma 93.5.2. □

0DVW **Lemma 93.6.3.** *In Example 93.6.1 let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional continuous representation. Then*

$$T\mathcal{D}ef_{V,\rho_0} = H^1(\Gamma, End_k(V)) \quad \text{and} \quad Inf(\mathcal{D}ef_{V,\rho_0}) = H^0(\Gamma, End_k(V))$$

Thus $Inf(\mathcal{D}ef_{V,\rho_0})$ is always finite dimensional and $T\mathcal{D}ef_{V,\rho_0}$ is finite dimensional if Γ is topologically finitely generated.

Proof. The proof is exactly the same as the proof of Lemma 93.5.3. □

In Example 93.6.1 if Γ is topologically finitely generated and (V, ρ_0) is a finite dimensional continuous representation of Γ over k, then $\mathcal{D}ef_{V,\rho_0}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.6.2 and 93.6.3 and the general discussion in Section 93.3.

0ET2 **Lemma 93.6.4.** *In Example 93.6.1 assume Γ is topologically finitely generated. Let $\rho_0 : \Gamma \to GL_k(V)$ be a finite dimensional representation. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{V,\rho_0}(A))/\cong$$

of isomorphism classes of objects has a hull. If $H^0(\Gamma, End_k(V)) = k$, then F is prorepresentable.

Proof. The proof is exactly the same as the proof of Lemma 93.5.4. □

93.7. Graded algebras

0DVX We will use the example in this section in the proof that the stack of polarized proper schemes is an algebraic stack. For this reason we will consider commutative graded algebras whose homogeneous parts are finite projective modules (sometimes called "locally finite").

0D3L **Example 93.7.1** (Graded algebras)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a graded A-algebra P such that P_d is a finite projective A-module for all $d \geq 0$, and

(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a graded k-algebra P with $\dim_k(P_d) < \infty$ for all $d \geq 0$, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DVY **Lemma 93.7.2.** *Example 93.7.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any graded k-algebra P.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories.

Consider an object

$$((A_1, P_1), (A_2, P_2), (\mathrm{id}_A, \varphi))$$

of the category $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$. Then we consider $P_1 \times_\varphi P_2$. Since $\varphi : P_1 \otimes_{A_1} A \to P_2 \otimes_{A_2} A$ is an isomorphism of graded algebras, we see that the graded pieces of $P_1 \times_\varphi P_2$ are finite projective $A_1 \times_A A_2$-modules, see proof of Lemma 93.4.2. Thus $P_1 \times_\varphi P_2$ is an object of $\mathcal{F}(A_1 \times_A A_2)$. This construction determines a quasi-inverse to our functor and the proof is complete. □

0DVZ **Lemma 93.7.3.** *In Example 93.7.1 let P be a graded k-algebra. Then*

$$T\mathcal{D}ef_P \quad \text{and} \quad \mathit{Inf}(\mathcal{D}ef_P) = \mathit{Der}_k(P, P)$$

are finite dimensional if P is finitely generated over k.

Proof. We first deal with the infinitesimal automorphisms. Let $Q = P \otimes_k k[\epsilon]$. Then an element of $\mathrm{Inf}(\mathcal{D}ef_P)$ is given by an automorphism $\gamma = \mathrm{id} + \epsilon\delta : Q \to Q$ as above where now $\delta : P \to P$. The fact that γ is graded implies that δ is homogeneous of degree 0. The fact that γ is k-linear implies that δ is k-linear. The fact that γ is multiplicative implies that δ is a k-derivation. Conversely, given a k-derivation $\delta : P \to P$ homogeneous of degree 0, we obtain an automorphism $\gamma = \mathrm{id} + \epsilon\delta$ as above. Thus we see that

$$\mathrm{Inf}(\mathcal{D}ef_P) = \mathrm{Der}_k(P, P)$$

as predicted in the lemma. Clearly, if P is generated in degrees P_i, $0 \leq i \leq N$, then δ is determined by the linear maps $\delta_i : P_i \to P_i$ for $0 \leq i \leq N$ and we see that

$$\dim_k \mathrm{Der}_k(P, P) < \infty$$

as desired.

To finish the proof of the lemma we show that there is a finite dimensional deformation space. To do this we choose a presentation

$$k[X_1, \ldots, X_n]/(F_1, \ldots, F_m) \longrightarrow P$$

of graded k-algebras where $\deg(X_i) = d_i$ and F_j is homogeneous of degree e_j. Let Q be any graded $k[\epsilon]$-algebra finite free in each degree which comes with an isomorphsm $\alpha : Q/\epsilon Q \to P$ so that (Q, α) defines an element of $T\mathcal{D}ef_P$. Choose a

homogeneous element $q_i \in Q$ of degree d_i mapping to the image of X_i in P. Then we obtain

$$k[\epsilon][X_1, \ldots, X_n] \longrightarrow Q, \quad X_i \longmapsto q_i$$

and since $P = Q/\epsilon Q$ this map is surjective by Nakayama's lemma. A small diagram chase shows we can choose homogeneous elements $F_{\epsilon,j} \in k[\epsilon][X_1, \ldots, X_n]$ of degree e_j mapping to zero in Q and mapping to F_j in $k[X_1, \ldots, X_n]$. Then

$$k[\epsilon][X_1, \ldots, X_n]/(F_{\epsilon,1}, \ldots, F_{\epsilon,m}) \longrightarrow Q$$

is a presentation of Q by flatness of Q over $k[\epsilon]$. Write

$$F_{\epsilon,j} = F_j + \epsilon G_j$$

There is some ambiguity in the vector $(G_1, \ldots, G_m)$. First, using different choices of $F_{\epsilon,j}$ we can modify G_j by an arbitrary element of degree e_j in the kernel of $k[X_1, \ldots, X_n] \to P$. Hence, instead of $(G_1, \ldots, G_m)$, we remember the element

$$(g_1, \ldots, g_m) \in P_{e_1} \oplus \ldots \oplus P_{e_m}$$

where g_j is the image of G_j in P_{e_j}. Moreover, if we change our choice of q_i into $q_i + \epsilon p_i$ with p_i of degree d_i then a computation (omitted) shows that g_j changes into

$$g_j^{new} = g_j - \sum\nolimits_{i=1}^{n} p_i \partial F_j/\partial X_i$$

We conclude that the isomorphism class of Q is determined by the image of the vector $(G_1, \ldots, G_m)$ in the k-vector space

$$W = \mathrm{Coker}(P_{d_1} \oplus \ldots \oplus P_{d_n} \xrightarrow{(\frac{\partial F_j}{\partial X_i})} P_{e_1} \oplus \ldots \oplus P_{e_m})$$

In this way we see that we obtain an injection

$$T\mathcal{D}ef_P \longrightarrow W$$

Since W visibly has finite dimension, we conclude that the lemma is true. □

In Example 93.7.1 if P is a finitely generated graded k-algebra, then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.7.2 and 93.7.3 and the general discussion in Section 93.3.

0ET3 **Lemma 93.7.4.** *In Example 93.7.1 assume P is a finitely generated graded k-algebra. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow \mathit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 93.7.2 and 93.7.3 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7. □

93.8. Rings

0DY0 The deformation theory of rings is the same as the deformation theory of affine schemes. For rings and schemes when we talk about deformations it means we are thinking about *flat* deformations.

0DY1 **Example 93.8.1** (Rings)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, P) consisting of an object A of $\mathcal{C}_\Lambda$ and a flat A-algebra P, and
(2) a morphism $(f, g) : (B, Q) \to (A, P)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a map $g : Q \to P$ which is f-linear and induces an isomorpism $Q \otimes_{B,f} A \cong P$.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, P) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a k-algebra P, let $x_0 = (k, P)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_P = \mathcal{F}_{x_0}$$

0DY2 **Lemma 93.8.2.** *Example 93.8.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_P$ is a deformation category for any k-algebra P.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. This is a special case of More on Algebra, Lemma 15.7.7. □

0DY3 **Lemma 93.8.3.** *In Example 93.8.1 let P be a k-algebra. Then*

$$T\mathcal{D}ef_P = Ext^1_P(NL_{P/k}, P) \quad \text{and} \quad Inf(\mathcal{D}ef_P) = Der_k(P, P)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_P)$ is the set of automorphisms of the trivial deformation $P[\epsilon] = P \otimes_k k[\epsilon]$ of P to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.2.1 this is equal to $\mathrm{Hom}_P(\Omega_{P/k}, P)$ which in turn is equal to $\mathrm{Der}_k(P, P)$ by Algebra, Lemma 10.131.3.

Recall that $T\mathcal{D}ef_P$ is the set of isomorphism classes of flat deformations Q of P to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_P(k[\epsilon])$. Recall that a $k[\epsilon]$-algebra Q with $Q/\epsilon Q = P$ is flat over $k[\epsilon]$ if and only if

$$0 \to P \xrightarrow{\epsilon} Q \to P \to 0$$

is exact. This is proven in More on Morphisms, Lemma 37.10.1 and more generally in Deformation Theory, Lemma 91.5.2. Thus we may apply Deformation Theory, Lemma 91.2.2 to see that the set of isomorphism classes of such deformations is equal to $\mathrm{Ext}^1_P(NL_{P/k}, P)$. □

0DZL **Lemma 93.8.4.** *In Example 93.8.1 let P be a smooth k-algebra. Then $T\mathcal{D}ef_P = (0)$.*

Proof. By Lemma 93.8.3 we have to show $\mathrm{Ext}^1_P(NL_{P/k}, P) = (0)$. Since $k \to P$ is smooth $NL_{P/k}$ is quasi-isomorphic to the complex consisting of a finite projective P-module placed in degree 0. □

0DY4 **Lemma 93.8.5.** *In Lemma 93.8.3 if P is a finite type k-algebra, then*

(1) *$Inf(\mathcal{D}ef_P)$ is finite dimensional if and only if $\dim(P) = 0$, and*

(2) *$T\mathcal{D}ef_P$ is finite dimensional if* $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ *is smooth except at a finite number of points.*

Proof. Proof of (1). We view $\mathrm{Der}_k(P, P)$ as a P-module. If it has finite dimension over k, then it has finite length as a P-module, hence it is supported in finitely many closed points of $\mathrm{Spec}(P)$ (Algebra, Lemma 10.52.11). Since $\mathrm{Der}_k(P, P) = \mathrm{Hom}_P(\Omega_{P/k}, P)$ we see that $\mathrm{Der}_k(P, P)_\mathfrak{p} = \mathrm{Der}_k(P_\mathfrak{p}, P_\mathfrak{p})$ for any prime $\mathfrak{p} \subset P$ (this uses Algebra, Lemmas 10.131.8, 10.131.15, and 10.10.2). Let $\mathfrak{p}$ be a minimal prime ideal of P corresponding to an irreducible component of dimension $d > 0$. Then $P_\mathfrak{p}$ is an Artinian local ring essentially of finite type over k with residue field and $\Omega_{P_\mathfrak{p}/k}$ is nonzero for example by Algebra, Lemma 10.140.3. Any nonzero finite module over an Artinian local ring has both a sub and a quotient module isomorphic to the residue field. Thus we find that $\mathrm{Der}_k(P_\mathfrak{p}, P_\mathfrak{p}) = \mathrm{Hom}_{P_\mathfrak{p}}(\Omega_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ is nonzero too. Combining all of the above we find that (1) is true.

Proof of (2). For a prime $\mathfrak{p}$ of P we will use that $NL_{P_\mathfrak{p}/k} = (NL_{P/k})_\mathfrak{p}$ (Algebra, Lemma 10.134.13) and we will use that $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = \mathrm{Ext}^1_{P_\mathfrak{p}}(NL_{P_\mathfrak{p}/k}, P_\mathfrak{p})$ (More on Algebra, Lemma 15.66.4). Given a prime $\mathfrak{p} \subset P$ then $k \to P$ is smooth at $\mathfrak{p}$ if and only if $(NL_{P/k})_\mathfrak{p}$ is quasi-isomorphic to a finite projective module placed in degree 0 (this follows immediately from the definition of a smooth ring map but it also follows from the stronger Algebra, Lemma 10.137.11).

Assume that P is smooth over k at all but finitely many primes. Then these "bad" primes are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset P$ by Algebra, Lemma 10.61.3 and the fact that the "bad" primes form a closed subset of $\mathrm{Spec}(P)$. For $\mathfrak{p} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ we have $\mathrm{Ext}^1_P(NL_{P/k}, P)_\mathfrak{p} = 0$ by the results above. Thus $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite P-module whose support is contained in $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. By Algebra, Proposition 10.63.6 for example, we find that the dimension over k of $\mathrm{Ext}^1_P(NL_{P/k}, P)$ is a finite integer combination of $\dim_k \kappa(\mathfrak{m}_i)$ and hence finite by the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1). □

In Example 93.8.1, let P be a finite type k-algebra. Then $\mathcal{D}ef_P$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ if and only if $\dim(P) = 0$. Furthermore, $\mathcal{D}ef_P$ has a versal formal object if $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ has finitely many singular points. This follows from Lemmas 93.8.2 and 93.8.5 and the general discussion in Section 93.3.

0ET4 **Lemma 93.8.6.** *In Example 93.8.1 assume P is a finite type k-algebra such that $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at a finite number of points. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow \mathit{Sets}, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_P(A))/\cong$$

of isomorphism classes of objects has a hull.

Proof. This follows immediately from Lemmas 93.8.2 and 93.8.5 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7. □

0DYS **Lemma 93.8.7.** *In Example 93.8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

of deformation categories.

Proof. Given a deformation of P we can take the localization of it to get a deformation of the localization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Let $S_Q \subset Q$ be the inverse image of S. Then Hence $(A, S_Q^{-1}Q) \to (k, S^{-1}P)$ is the desired object of $\mathcal{D}ef_{S^{-1}P}$. □

0DYT **Lemma 93.8.8.** *In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

of deformation categories.

Proof. Given a deformation of P we can take the henselization of it to get a deformation of the henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Denote $J_Q \subset Q$ the inverse image of J in Q. Let (Q^h, J_Q^h) be the henselization of the pair (Q, J_Q). Recall that $Q \to Q^h$ is flat (More on Algebra, Lemma 15.12.2) and hence Q^h is flat over A. By More on Algebra, Lemma 15.12.7 we see that the map $Q^h \to P^h$ induces an isomorphism $Q^h \otimes_A k = Q^h \otimes_Q P = P^h$. Hence $(A, Q^h) \to (k, P^h)$ is the desired object of $\mathcal{D}ef_{P^h}$. □

0DYU **Lemma 93.8.9.** *In Example 93.8.1 let P be a k-algebra. Assume P is a local ring and let P^{sh} be a strict henselization of P. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^{sh}}$$

of deformation categories.

Proof. Given a deformation of P we can take the strict henselization of it to get a deformation of the strict henselization; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Since the kernel of the surjection $Q \to P$ is nilpotent, we find that Q is a local ring with the same residue field as P. Let Q^{sh} be the strict henselization of Q. Recall that $Q \to Q^{sh}$ is flat (More on Algebra, Lemma 15.45.1) and hence Q^{sh} is flat over A. By Algebra, Lemma 10.156.4 we see that the map $Q^{sh} \to P^{sh}$ induces an isomorphism $Q^{sh} \otimes_A k = Q^{sh} \otimes_Q P = P^{sh}$. Hence $(A, Q^{sh}) \to (k, P^{sh})$ is the desired object of $\mathcal{D}ef_{P^{sh}}$. □

0DYV **Lemma 93.8.10.** *In Example 93.8.1 let P be a k-algebra. Assume P Noetherian and let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. There is a natural functor*

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

of deformation categories.

Proof. Given a deformation of P we can take the completion of it to get a deformation of the completion; this is clear and we encourage the reader to skip the proof. More precisely, let $(A, Q) \to (k, P)$ be a morphism in $\mathcal{F}$, i.e., an object of $\mathcal{D}ef_P$. Observe that Q is a Noetherian ring: the kernel of the surjective ring map $Q \to P$ is nilpotent and finitely generated and P is Noetherian; apply Algebra, Lemma 10.97.5. Denote $J_Q \subset Q$ the inverse image of J in Q. Let $Q^\wedge$ be the J_Q-adic completion of Q. Recall that $Q \to Q^\wedge$ is flat (Algebra, Lemma 10.97.2)

and hence $Q^\wedge$ is flat over A. The induced map $Q^\wedge \to P^\wedge$ induces an isomorphism $Q^\wedge \otimes_A k = Q^\wedge \otimes_Q P = P^\wedge$ by Algebra, Lemma 10.97.1 for example. Hence $(A, Q^\wedge) \to (k, P^\wedge)$ is the desired object of $\mathcal{D}ef_{P^\wedge}$. □

0DY5 **Lemma 93.8.11.** *In Lemma 93.8.3 if $P = k[[x_1, \ldots, x_n]]/(f)$ for some nonzero $f \in (x_1, \ldots, x_n)^2$, then*

(1) *$Inf(\mathcal{D}ef_P)$ is finite dimensional if and only if $n = 1$, and*
(2) *$T\mathcal{D}ef_P$ is finite dimensional if*

$$\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$$

Proof. Proof of (1). Consider the derivations $\partial/\partial x_i$ of $k[[x_1, \ldots, x_n]]$ over k. Write $f_i = \partial f/\partial x_i$. The derivation

$$\theta = \sum h_i \partial/\partial x_i$$

of $k[[x_1, \ldots, x_n]]$ induces a derivation of $P = k[[x_1, \ldots, x_n]]/(f)$ if and only if $\sum h_i f_i \in (f)$. Moreover, the induced derivation of P is zero if and only if $h_i \in (f)$ for $i = 1, \ldots, n$. Thus we find

$$\mathrm{Ker}((f_1, \ldots, f_n) : P^{\oplus n} \longrightarrow P) \subset \mathrm{Der}_k(P, P)$$

The left hand side is a finite dimensional k-vector space only if $n = 1$; we omit the proof. We also leave it to the reader to see that the right hand side has finite dimension if $n = 1$. This proves (1).

Proof of (2). Let Q be a flat deformation of P over $k[\epsilon]$ as in the proof of Lemma 93.8.3. Choose lifts $q_i \in Q$ of the image of x_i in P. Then Q is a complete local ring with maximal ideal generated by $q_1, \ldots, q_n$ and ϵ (small argument omitted). Thus we get a surjection

$$k[\epsilon][[x_1, \ldots, x_n]] \longrightarrow Q, \quad x_i \longmapsto q_i$$

Choose an element of the form $f + \epsilon g \in k[\epsilon][[x_1, \ldots, x_n]]$ mapping to zero in Q. Observe that g is well defined modulo (f). Since Q is flat over $k[\epsilon]$ we get

$$Q = k[\epsilon][[x_1, \ldots, x_n]]/(f + \epsilon g)$$

Finally, if we changing the choice of q_i amounts to changing the coordinates x_i into $x_i + \epsilon h_i$ for some $h_i \in k[[x_1, \ldots, x_n]]$. Then $f + \epsilon g$ changes into $f + \epsilon(g + \sum h_i f_i)$ where $f_i = \partial f/\partial x_i$. Thus we see that the isomorphism class of the deformation Q is determined by an element of

$$k[[x_1, \ldots, x_n]]/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

This has finite dimension over k if and only if its support is the closed point of $k[[x_1, \ldots, x_n]]$ if and only if $\sqrt{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)} = (x_1, \ldots, x_n)$. □

93.9. Schemes

0DY6 The deformation theory of schemes.

0DY7 **Example 93.9.1** (Schemes)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and a scheme X flat over A, and

(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow[g]{} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(B) \end{array}$$

is a cartesian commutative diagram of schemes.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given a scheme X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0DY8 **Lemma 93.9.2.** *Example 93.9.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any scheme X over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 37.14.3. Thus the lemma is a special case of More on Morphisms, Lemma 37.14.6. □

0DY9 **Lemma 93.9.3.** *In Example 93.9.1 let X be a scheme over k. Then*

$$\mathit{Inf}(\mathcal{D}ef_X) = \mathit{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathit{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathit{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.8.1 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms, Lemma 37.13.3. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from Morphisms, Lemma 29.32.2.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 91.8.1. □

0DYA **Lemma 93.9.4.** *In Lemma 93.9.3 if X is proper over k, then $\mathit{Inf}(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.*

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms, Lemma 37.13.4 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in

degrees 0 and -1. By Derived Categories of Schemes, Lemma 36.11.7 the displayed Ext-groups are finite k-vector spaces and the proof is complete. $\square$

In Example 93.9.1 if X is a proper scheme over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.9.2 and 93.9.4 and the general discussion in Section 93.3.

0ET5 **Lemma 93.9.5.** *In Example 93.9.1 assume X is a proper k-scheme. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.9.2 and 93.9.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. $\square$

0DYW **Lemma 93.9.6.** *In Example 93.9.1 let X be a scheme over k. Let $U \subset X$ be an open subscheme. There is a natural functor*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_U$$

of deformation categories.

Proof. Given a deformation of X we can take the corresponding open of it to get a deformation of U. We omit the details. $\square$

0DYX **Lemma 93.9.7.** *In Example 93.9.1 let $X = \mathrm{Spec}(P)$ be an affine scheme over k. With $\mathcal{D}ef_P$ as in Example 93.8.1 there is a natural equivalence*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_P$$

of deformation categories.

Proof. The functor sends (A, Y) to $\Gamma(Y, \mathcal{O}_Y)$. This works because any deformation of X is affine by More on Morphisms, Lemma 37.2.3. $\square$

0DZM **Lemma 93.9.8.** *In Example 93.9.1 let X be a scheme over k Let $p \in X$ be a point. With $\mathcal{D}ef_{\mathcal{O}_{X,p}}$ as in Example 93.8.1 there is a natural functor*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{\mathcal{O}_{X,p}}$$

of deformation categories.

Proof. Choose an affine open $U = \mathrm{Spec}(P) \subset X$ containing p. Then $\mathcal{O}_{X,p}$ is a localization of P. We combine the functors from Lemmas 93.9.6, 93.9.7, and 93.8.7. $\square$

0DYY **Situation 93.9.9.** Let $\Lambda \to k$ be as in Section 93.3. Let X be a scheme over k which has an affine open covering $X = U_1 \cup U_2$ with $U_{12} = U_1 \cap U_2$ affine too. Write $U_1 = \mathrm{Spec}(P_1)$, $U_2 = \mathrm{Spec}(P_2)$ and $U_{12} = \mathrm{Spec}(P_{12})$. Let $\mathcal{D}ef_X$, $\mathcal{D}ef_{U_1}$, $\mathcal{D}ef_{U_2}$, and $\mathcal{D}ef_{U_{12}}$ be as in Example 93.9.1 and let $\mathcal{D}ef_{P_1}$, $\mathcal{D}ef_{P_2}$, and $\mathcal{D}ef_{P_{12}}$ be as in Example 93.8.1.

0DYZ **Lemma 93.9.10.** *In Situation 93.9.9 there is an equivalence*

$$\mathcal{D}ef_X = \mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2}$$

of deformation categories, see Examples 93.9.1 and 93.8.1.

Proof. It suffices to show that the functors of Lemma 93.9.6 define an equivalence

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times_{\mathcal{D}ef_{U_{12}}} \mathcal{D}ef_{U_2}$$

because then we can apply Lemma 93.9.7 to translate into rings. To do this we construct a quasi-inverse. Denote $F_i : \mathcal{D}ef_{U_i} \to \mathcal{D}ef_{U_{12}}$ the functor of Lemma 93.9.6. An object of the RHS is given by an A in $\mathcal{C}_\Lambda$, objects $(A, V_1) \to (k, U_1)$ and $(A, V_2) \to (k, U_2)$, and a morphism

$$g : F_1(A, V_1) \to F_2(A, V_2)$$

Now $F_i(A, V_i) = (A, V_{i,3-i})$ where $V_{i,3-i} \subset V_i$ is the open subscheme whose base change to k is $U_{12} \subset U_i$. The morphism g defines an isomorphism $V_{1,2} \to V_{2,1}$ of schemes over A compatible with $\mathrm{id} : U_{12} \to U_{12}$ over k. Thus $(\{1, 2\}, V_i, V_{i,3-i}, g, g^{-1})$ is a glueing data as in Schemes, Section 26.14. Let Y be the glueing, see Schemes, Lemma 26.14.1. Then Y is a scheme over A and the compatibilities mentioned above show that there is a canonical isomorphism $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X$. Thus $(A, Y) \to (k, X)$ is an object of $\mathcal{D}ef_X$. We omit the verification that this construction is a functor and is quasi-inverse to the given one. □

93.10. Morphisms of Schemes

0E3S The deformation theory of morphisms of schemes. Of course this is just an example of deformations of diagrams of schemes.

0E3T **Example 93.10.1** (Morphisms of schemes)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair $(A, X \to Y)$ consisting of an object A of $\mathcal{C}_\Lambda$ and a morphism $X \to Y$ of schemes over A with both X and Y flat over A, and
(2) a morphism $(f, g, h) : (A', X' \to Y') \to (A, X \to Y)$ consists of a morphism $f : A' \to A$ in $\mathcal{C}_\Lambda$ together with morphisms of schemes $g : X \to X'$ and $h : Y \to Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow[g]{} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow[h]{} & Y' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(A') \end{array}$$

is a commutative diagram of schemes where both squares are cartesian.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends $(A, X \to Y)$ to A and (f, g, h) to f. It is clear that p is cofibred in groupoids. Given a morphism of schemes $X \to Y$ over k, let $x_0 = (k, X \to Y)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_{X \to Y} = \mathcal{F}_{x_0}$$

0E3U **Lemma 93.10.2.** *Example 93.10.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_{X\to Y}$ is a deformation category for any morphism of schemes $X \to Y$ over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in More on Morphisms, Lemma 37.14.3. Thus the lemma follows immediately from More on Morphisms, Lemma 37.14.6 as this describes the category of schemes flat over $A_1 \times_A A_2$ as the fibre product of the category of schemes flat over A_1 with the category of schemes flat over A_2 over the category of schemes flat over A. □

0E3V **Lemma 93.10.3.** *In Example 93.9.1 let $f : X \to Y$ be a morphism of schemes over k. There is a canonical exact sequence of k-vector spaces*

$$0 \longrightarrow Inf(\mathcal{D}ef_{X\to Y}) \longrightarrow Inf(\mathcal{D}ef_X \times \mathcal{D}ef_Y) \longrightarrow Der_k(\mathcal{O}_Y, f_*\mathcal{O}_X)$$
$$T\mathcal{D}ef_{X\to Y} \longrightarrow T(\mathcal{D}ef_X \times \mathcal{D}ef_Y) \longrightarrow Ext^1_{\mathcal{O}_X}(Lf^* NL_{Y/k}, \mathcal{O}_X)$$

Proof. The obvious map of deformation categories $\mathcal{D}ef_{X\to Y} \to \mathcal{D}ef_X \times \mathcal{D}ef_Y$ gives two of the arrows in the exact sequence of the lemma. Recall that $\mathrm{Inf}(\mathcal{D}ef_{X\to Y})$ is the set of automorphisms of the trivial deformation

$$f' : X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \xrightarrow{f\times \mathrm{id}} Y' = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

of $X \to Y$ to $k[\epsilon]$ equal to the identity modulo ϵ. This is clearly the same thing as pairs $(\alpha, \beta) \in \mathrm{Inf}(\mathcal{D}ef_X \times \mathcal{D}ef_Y)$ of infinitesimal automorphisms of X and Y compatible with f', i.e., such that $f' \circ \alpha = \beta \circ f'$. By Deformation Theory, Lemma 91.7.1 for an arbitrary pair (α, β) the difference between the morphism $f' : X' \to Y'$ and the morphism $\beta^{-1} \circ f' \circ \alpha : X' \to Y'$ defines an element in

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/k}, f_*\mathcal{O}_X)$$

Equality by More on Morphisms, Lemma 37.13.3. This defines the last top horizontal arrow and shows exactness in the first two places. For the map

$$\mathrm{Der}_k(\mathcal{O}_Y, f_*\mathcal{O}_X) \to T\mathcal{D}ef_{X\to Y}$$

we interpret elements of the source as morphisms $f_\epsilon : X' \to Y'$ over $\mathrm{Spec}(k[\epsilon])$ equal to f modulo ϵ using Deformation Theory, Lemma 91.7.1. We send f_ϵ to the isomorphism class of $(f_\epsilon : X' \to Y')$ in $T\mathcal{D}ef_{X\to Y}$. Note that $(f_\epsilon : X' \to Y')$ is isomorphic to the trivial deformation $(f' : X' \to Y')$ exactly when $f_\epsilon = \beta^{-1} \circ f \circ \alpha$ for some pair (α, β) which implies exactness in the third spot. Clearly, if some first order deformation $(f_\epsilon : X_\epsilon \to Y_\epsilon)$ maps to zero in $T(\mathcal{D}ef_X \times \mathcal{D}ef_Y)$, then we can choose isomorphisms $X' \to X_\epsilon$ and $Y' \to Y_\epsilon$ and we conclude we are in the image

of the south-west arrow. Therefore we have exactness at the fourth spot. Finally, given two first order deformations X_ϵ, Y_ϵ of X, Y there is an obstruction in

$$ob(X_\epsilon, Y_\epsilon) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{Y/k}, \mathcal{O}_X)$$

which vanishes if and only if $f : X \to Y$ lifts to $X_\epsilon \to Y_\epsilon$, see Deformation Theory, Lemma 91.7.1. This finishes the proof. □

0E3W **Lemma 93.10.4.** *In Lemma 93.10.3 if X and Y are both proper over k, then $Inf(\mathcal{D}ef_{X\to Y})$ and $T\mathcal{D}ef_{X\to Y}$ are finite dimensional.*

Proof. Omitted. Hint: argue as in Lemma 93.9.4 and use the exact sequence of the lemma. □

In Example 93.10.1 if $X \to Y$ is a morphism of proper schemes over k, then $\mathcal{D}ef_{X\to Y}$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.10.2 and 93.10.4 and the general discussion in Section 93.3.

0ET6 **Lemma 93.10.5.** *In Example 93.10.1 assume $X \to Y$ is a morphism of proper k-schemes. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_{X\to Y}(A))/\cong$$

of isomorphism classes of objects has a hull. If $Der_k(\mathcal{O}_X, \mathcal{O}_X) = Der_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.10.2 and 93.10.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_Y, \mathcal{O}_Y) = 0$. Then the exact sequence of Lemma 93.10.3 combined with Lemma 93.9.3 shows that $\mathrm{Inf}(\mathcal{D}ef_{X\to Y}) = 0$. Then $\mathcal{D}ef_{X\to Y}$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}ef_{X\to Y}$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. □

0E3X **Lemma 93.10.6.** *In Example 93.9.1 let $f : X \to Y$ be a morphism of schemes over k. If $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^1 f_*\mathcal{O}_X = 0$, then the morphism of deformation categories*

$$\mathcal{D}ef_{X\to Y} \to \mathcal{D}ef_X$$

is an equivalence.

This is discussed in [**Vak06**, Section 5.3] and [**Ran89**, Theorem 3.3].

Proof. We construct a quasi-inverse to the forgetful functor of the lemma. Namely, suppose that (A, U) is an object of $\mathcal{D}ef_X$. The given map $X \to U$ is a finite order thickening and we can use it to identify the underlying topological spaces of U and X, see More on Morphisms, Section 37.2. Thus we may and do think of $\mathcal{O}_U$ as a sheaf of A-algebras on X; moreover the fact that $U \to \mathrm{Spec}(A)$ is flat, means that $\mathcal{O}_U$ is flat as a sheaf of A-modules. In particular, we have a filtration

$$0 = \mathfrak{m}_A^n \mathcal{O}_U \subset \mathfrak{m}_A^{n-1} \mathcal{O}_U \subset \ldots \subset \mathfrak{m}_A^2 \mathcal{O}_U \subset \mathfrak{m}_A \mathcal{O}_U \subset \mathcal{O}_U$$

with subquotients equal to $\mathcal{O}_X \otimes_k \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ by flatness, see More on Morphisms, Lemma 37.10.1 or the more general Deformation Theory, Lemma 91.5.2. Set

$$\mathcal{O}_V = f_*\mathcal{O}_U$$

viewed as sheaf of A-algebras on Y. Since $R^1 f_* \mathcal{O}_X = 0$ we find by the description above that $R^1 f_*(\mathfrak{m}_A^i \mathcal{O}_U / \mathfrak{m}_A^{i+1} \mathcal{O}_U) = 0$ for all i. This implies that the sequences

$$0 \to (f_* \mathcal{O}_X) \otimes_k \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \to f_*(\mathcal{O}_U / \mathfrak{m}_A^{i+1} \mathcal{O}_U) \to f_*(\mathcal{O}_U / \mathfrak{m}_A^i \mathcal{O}_U) \to 0$$

are exact for all i. Reading the references given above backwards (and using induction) we find that $\mathcal{O}_V$ is a flat sheaf of A-algebras with $\mathcal{O}_V / \mathfrak{m}_A \mathcal{O}_V = \mathcal{O}_Y$. Using More on Morphisms, Lemma 37.2.2 we find that $(Y, \mathcal{O}_V)$ is a scheme, call it V. The equality $\mathcal{O}_V = f_* \mathcal{O}_U$ defines a morphism of ringed spaces $U \to V$ which is easily seen to be a morphism of schemes. This finishes the proof by the flatness already established. □

93.11. Algebraic spaces

0E3Y The deformation theory of algebraic spaces.

0E3Z **Example 93.11.1** (Algebraic spaces)**.** Let $\mathcal{F}$ be the category defined as follows

(1) an object is a pair (A, X) consisting of an object A of $\mathcal{C}_\Lambda$ and an algebraic space X flat over A, and
(2) a morphism $(f, g) : (B, Y) \to (A, X)$ consists of a morphism $f : B \to A$ in $\mathcal{C}_\Lambda$ together with a morphism $g : X \to Y$ of algebraic spaces over Λ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(B) \end{array}$$

is a cartesian commutative diagram of algebraic spaces.

The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ sends (A, X) to A and (f, g) to f. It is clear that p is cofibred in groupoids. Given an algebraic space X over k, let $x_0 = (k, X)$ be the corresponding object of $\mathcal{F}(k)$. We set

$$\mathcal{D}ef_X = \mathcal{F}_{x_0}$$

0E40 **Lemma 93.11.2.** *Example 93.11.1 satisfies the Rim-Schlessinger condition (RS). In particular, $\mathcal{D}ef_X$ is a deformation category for any algebraic space X over k.*

Proof. Let $A_1 \to A$ and $A_2 \to A$ be morphisms of $\mathcal{C}_\Lambda$. Assume $A_2 \to A$ is surjective. According to Formal Deformation Theory, Lemma 90.16.4 it suffices to show that the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ is an equivalence of categories. Observe that

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_1) & \longrightarrow & \mathrm{Spec}(A_1 \times_A A_2) \end{array}$$

is a pushout diagram as in Pushouts of Spaces, Lemma 81.6.2. Thus the lemma is a special case of Pushouts of Spaces, Lemma 81.6.7. □

0E41 **Lemma 93.11.3.** *In Example 93.11.1 let X be an algebraic space over k. Then*

$$\mathit{Inf}(\mathcal{D}ef_X) = \mathit{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathit{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

and

$$T\mathcal{D}ef_X = \mathit{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

Proof. Recall that $\mathrm{Inf}(\mathcal{D}ef_X)$ is the set of automorphisms of the trivial deformation $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$ of X to $k[\epsilon]$ equal to the identity modulo ϵ. By Deformation Theory, Lemma 91.14.2 this is equal to $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$. The equality $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Lemma 76.21.4. The equality $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ follows from More on Morphisms of Spaces, Definition 76.7.2 and Modules on Sites, Definition 18.33.3.

Recall that $T_{x_0}\mathcal{D}ef_X$ is the set of isomorphism classes of flat deformations X' of X to $k[\epsilon]$, more precisely, the set of isomorphism classes of $\mathcal{D}ef_X(k[\epsilon])$. Thus the second statement of the lemma follows from Deformation Theory, Lemma 91.14.2. □

0E42 **Lemma 93.11.4.** *In Lemma 93.11.3 if X is proper over k, then $Inf(\mathcal{D}ef_X)$ and $T\mathcal{D}ef_X$ are finite dimensional.*

Proof. By the lemma we have to show $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ and $\mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$ are finite dimensional. By More on Morphisms of Spaces, Lemma 76.21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Spaces, Lemma 75.8.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. □

In Example 93.11.1 if X is a proper algebraic space over k, then $\mathcal{D}ef_X$ admits a presentation by a smooth prorepresentable groupoid in functors over $\mathcal{C}_\Lambda$ and a fortiori has a (minimal) versal formal object. This follows from Lemmas 93.11.2 and 93.11.4 and the general discussion in Section 93.3.

0ET7 **Lemma 93.11.5.** *In Example 93.11.1 assume X is a proper algebraic space over k. Assume Λ is a complete local ring with residue field k (the classical case). Then the functor*

$$F : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \mathrm{Ob}(\mathcal{D}ef_X(A))/\cong$$

of isomorphism classes of objects has a hull. If $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$, then F is prorepresentable.

Proof. The existence of a hull follows immediately from Lemmas 93.11.2 and 93.11.4 and Formal Deformation Theory, Lemma 90.16.6 and Remark 90.15.7.

Assume $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. Then $\mathcal{D}ef_X$ and F are equivalent by Formal Deformation Theory, Lemma 90.19.13. Hence F is a deformation functor (because $\mathcal{D}ef_X$ is a deformation category) with finite tangent space and we can apply Formal Deformation Theory, Theorem 90.18.2. □

93.12. Deformations of completions

0DZ0 In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZ1 **Lemma 93.12.1.** *Let $A' \to A$ be a surjection of rings with nilpotent kernel. Let $A' \to P'$ be a flat ring map. Set $P = P' \otimes_{A'} A$. Let M be an A-flat P-module. Then the following are equivalent*

1. *there is an A'-flat P'-module M' with $M' \otimes_{P'} P = M$, and*
2. *there is an object $K' \in D^-(P')$ with $K' \otimes^{\mathbf{L}}_{P'} P = M$.*

Proof. Suppose that M' is as in (1). Then

$$M = M' \otimes_{P'} P' = M' \otimes_{A'} A = M' \otimes^{\mathbf{L}}_{A} A' = M' \otimes^{\mathbf{L}}_{P'} P$$

The first two equalities are clear, the third holds because M' is flat over A', and the fourth holds by More on Algebra, Lemma 15.61.2. Thus (2) holds. Conversely, suppose K' is as in (2). We may and do assume M is nonzero. Let t be the largest integer such that $H^t(K')$ is nonzero (exists because M is nonzero). Then $H^t(K') \otimes_{P'} P = H^t(K' \otimes^{\mathbf{L}}_{P'} P)$ is zero if $t > 0$. Since the kernel of $P' \to P$ is nilpotent this implies $H^t(K') = 0$ by Nakayama's lemma a contradiction. Hence $t = 0$ (the case $t < 0$ is absurd as well). Then $M' = H^0(K')$ is a P'-module such that $M = M' \otimes_{P'} P$ and the spectral sequence for Tor gives an injective map

$$\mathrm{Tor}_1^{P'}(M', P) \to H^{-1}(M' \otimes^{\mathbf{L}}_{P'} P) = 0$$

By the reference on derived base change above $0 = \mathrm{Tor}_1^{P'}(M', P) = \mathrm{Tor}_1^{A'}(M', A)$. We conclude that M' is A'-flat by Algebra, Lemma 10.99.8. □

0DZ2 **Lemma 93.12.2.** *Consider a commutative diagram of Noetherian rings*

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & Q' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & P & \longrightarrow & Q \end{array}$$

with cartesian squares, with flat horizontal arrows, and with surjective vertical arrows whose kernels are nilpotent. Let $J' \subset P'$ be an ideal such that $P'/J' = Q'/J'Q'$. Let M be an A-flat P-module. Assume for all $g \in J'$ there exists an A'-flat $(P')_g$-module lifting M_g. Then the following are equivalent

(1) *M has an A'-flat lift to a P'-module, and*
(2) *$M \otimes_P Q$ has an A'-flat lift to a Q'-module.*

Proof. Let $I = \mathrm{Ker}(A' \to A)$. By induction on the integer $n > 1$ such that $I^n = 0$ we reduce to the case where I is an ideal of square zero; details omitted. We translate the condition of liftability of M into the problem of finding an object of $D^-(P')$ as in Lemma 93.12.1. The obstruction to doing this is the element

$$\omega(M) \in \mathrm{Ext}^2_P(M, M \otimes^{\mathbf{L}}_P IP) = \mathrm{Ext}^2_P(M, M \otimes_P IP)$$

constructed in Deformation Theory, Lemma 91.15.1. The equality in the displayed formula holds as $M \otimes^{\mathbf{L}}_P IP = M \otimes_P IP$ since M and P are A-flat[2]. The obstruction for lifting $M \otimes_P Q$ is similarly the element

$$\omega(M \otimes_P Q) \in \mathrm{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ)$$

which is the image of $\omega(M)$ by the functoriality of the construction $\omega(-)$ of Deformation Theory, Lemma 91.15.1. By More on Algebra, Lemma 15.100.2 we have

$$\mathrm{Ext}^2_Q(M \otimes_P Q, (M \otimes_P Q) \otimes_Q IQ) = \mathrm{Ext}^2_P(M, M \otimes_P IP) \otimes_P Q$$

here we use that P is Noetherian and M finite. Our assumption on $P' \to Q'$ guarantees that for an P-module E the map $E \to E \otimes_P Q$ is bijective on J'-power torsion, see More on Algebra, Lemma 15.90.3. Thus we conclude that it suffices to

[2]Choose a resolution $F_\bullet \to I$ by free A-modules. Since $A \to P$ is flat, $P \otimes_A F_\bullet$ is a free resolution of IP. Hence $M \otimes^{\mathbf{L}}_P IP$ is represented by $M \otimes_P P \otimes_A F_\bullet = M \otimes_A F_\bullet$. This only has cohomology in degree 0 as M is A-flat.

show $\omega(M)$ is J'-power torsion. In other words, it suffices to show that $\omega(M)$ dies in

$$\mathrm{Ext}^2_P(M, M \otimes_P IP)_g = \mathrm{Ext}^2_{P_g}(M_g, M_g \otimes_{P_g} IP_g)$$

for all $g \in J'$. Howeover, by the compatibility of formation of $\omega(M)$ with base change again, we conclude that this is true as M_g is assumed to have a lift (of course you have to use the whole string of equivalences again). □

0DZ3 **Lemma 93.12.3.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak{b} \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is syntomic on the complement of $V(\mathfrak{b})$. Then B has a flat lift to A' if and only if the $\mathfrak{b}$-adic completion $B^\wedge$ has a flat lift to A'.*

Proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak{p} \subset P$ be the inverse image of $\mathfrak{b}$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak{p}' \subset P'$ the inverse image of $\mathfrak{p}$. (Of course $\mathfrak{p}$ and $\mathfrak{p}'$ do not designate prime ideals here.) We will denote $P^\wedge$ and $(P')^\wedge$ the respective completions.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^\wedge$ and $\mathfrak{c} = \mathfrak{b}C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is complete with respect to the inverse image $\mathfrak{c}'$ of $\mathfrak{c}$ (Algebra, Lemma 10.97.10). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through completions to give surjections $P^\wedge \to C$ and $(P')^\wedge \to C'$ (for the second again using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^\wedge$-module structure lifting C as an A-flat $P^\wedge$-module. Conversely, if we can lift C to a $(P')^\wedge$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^\wedge/\tilde{J}$ (using Nakayama again) and setting $C' = (P')^\wedge/\tilde{J}$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^\wedge$ is a flat ring map which induces an isomorphism $P'/\mathfrak{p}' = (P')^\wedge/\mathfrak{p}'(P')^\wedge$. We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak{p}'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

Notation. Let $A \to B$ be a ring map. Let N be a B-module. We denote $\mathrm{Exal}_A(B, N)$ the set of isomorphism classes of extensions

$$0 \to N \to C \to B \to 0$$

of A-algebras such that N is an ideal of square zero in C. Given a second such $0 \to N \to C' \to B \to 0$ an isomorphism is a A-algebra isomorpism $C \to C'$ such

that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes. The assignment $N \mapsto \mathrm{Exal}_A(B, N)$ is a functor which transforms products into products. Hence this is an additive functor and $\mathrm{Exal}_A(B, N)$ has a natural B-module structure. In fact, by Deformation Theory, Lemma 91.2.2 we have $\mathrm{Exal}_A(B, N) = \mathrm{Ext}^1_B(NL_{B/A}, N)$.

0DZ4 **Lemma 93.12.4.** *Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(B^\wedge, N^\wedge)$$

Here $B^\wedge$ and $N^\wedge$ are the J-adic completions.

Proof. The map is given by completion: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the completion $C^\wedge$ of C with respect to the inverse image of J. Compare with the proof of Lemma 93.8.10.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. By More on Algebra, Lemma 15.103.1 we conclude that

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \longrightarrow \lim_n \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

is an isomorphism for any finite B-module N.

Injectivity of the map. Suppose that $0 \to N \to C \to B \to 0$ is in $\mathrm{Exal}_k(B, N)$ and maps to zero in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Choose a splitting $C^\wedge = B^\wedge \oplus N^\wedge$. Then the induced map $C \to C^\wedge \to N^\wedge$ gives maps $C \to N/J^nN$ for all n. Hence we see that our element is in the kernel of the maps

$$\mathrm{Ext}^1_B(NL_{B/k}, N) \to \mathrm{Ext}^1_B(NL_{B/k}, N/J^nN)$$

for all n. By the previous paragraph we conclude that our element is zero.

Surjectivity of the map. Let $0 \to N^\wedge \to C' \to B^\wedge \to 0$ be an element of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. Pulling back by $B \to B^\wedge$ we get an element $0 \to N^\wedge \to C'' \to B \to 0$ in $\mathrm{Exal}_k(B, N^\wedge)$. we have

$$\mathrm{Ext}^1_B(NL_{B/k}, N^\wedge) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B B^\wedge = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality as $N^\wedge = N \otimes_B B^\wedge$ (Algebra, Lemma 10.97.1) and More on Algebra, Lemma 15.66.3. The second equality because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion (see above), $B \to B^\wedge$ is flat and induces an isomorphism $B/J \to B^\wedge/JB^\wedge$, and More on Algebra, Lemma 15.90.3. Thus we can find a $C \in \mathrm{Exal}_k(B, N)$ mapping to C'' in $\mathrm{Exal}_k(B, N^\wedge)$. Thus

$$0 \to N^\wedge \to C' \to B^\wedge \to 0 \quad \text{and} \quad 0 \to N^\wedge \to C^\wedge \to B^\wedge \to 0$$

are two elements of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ mapping to the same element of $\mathrm{Exal}_k(B, N^\wedge)$. Taking the difference we get an element $0 \to N^\wedge \to C' \to B^\wedge \to 0$ of $\mathrm{Exal}_k(B^\wedge, N^\wedge)$ whose image in $\mathrm{Exal}_k(B, N^\wedge)$ is zero. This means there exists

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^\wedge & \longrightarrow & C' & \longrightarrow & B^\wedge & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \sigma} & \nearrow & & & \\ & & & & B & & & & \end{array}$$

Let $J' \subset C'$ be the inverse image of $JB^\wedge \subset B^\wedge$. To finish the proof it suffices to note that σ is continuous for the J-adic topology on B and the J'-adic topology on C' and that C' is J'-adically complete by Algebra, Lemma 10.97.10 (here we also use that C' is Noetherian; small detail omitted). Namely, this means that σ factors through the completion $B^\wedge$ and $C' = 0$ in $\mathrm{Exal}_k(B^\wedge, N^\wedge)$. □

0DZ5 **Lemma 93.12.5.** *In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote $P^\wedge$ the J-adic completion. If*

(1) *$k \to P$ is of finite type, and*
(2) *$\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$.*

then the functor between deformation categories of Lemma 93.8.10

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^\wedge}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^\wedge}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.12.3 and the isomorphism on tangent spaces is the special case of Lemma 93.12.4 where $N = B$. □

93.13. Deformations of localizations

0DZ6 In this section we compare the deformation problem posed by an algebra and its localization at a multiplicative subset. We first discuss "liftability".

0DZ7 **Lemma 93.13.1.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $S \subset B$ be a multiplicative subset such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic at $\mathfrak{q}$, then $S \cap \mathfrak{q} = \emptyset$. Then B has a flat lift to A' if and only if $S^{-1}B$ has a flat lift to A'.*

Proof. This proof is the same as the proof of Lemma 93.12.3 but easier. We suggest the reader to skip the proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $S_P \subset P$ be the inverse image of S. Set $P' = A'[x_1, \ldots, x_n]$ and denote $S_{P'} \subset P'$ the inverse image of S_P.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = S^{-1}B$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Elements of C' which map to invertible elements of C are invertible. We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. By the remark above these maps pass through localizations to give surjections $S_P^{-1}P \to C$ and $S_{P'}^{-1}P' \to C'$ (for the second use Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $S_{P'}^{-1}P'$-module structure lifting C as an A-flat $S_P^{-1}P$-module. Conversely, if we can lift C to a $S_{P'}^{-1}P'$-module N' flat over A', then N' is a cyclic module $N' \cong S_{P'}^{-1}P'/\tilde{J}$ (using Nakayama again) and setting $C' = S_{P'}^{-1}P'/\tilde{J}$ we find a flat lift of C as an algebra.

The syntomic locus of a morphism of schemes is open by definition. Let $J_B \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not syntomic. Denote $J_P \subset P$ and $J_{P'} \subset P'$ the corresponding ideals. Observe that $P' \to S_{P'}^{-1}P'$ is a flat ring map which induces an isomorphism $P'/J_{P'} = S_{P'}^{-1}P'/J_{P'}S_{P'}^{-1}P'$ by our assumption on S in the lemma, namely, the assumption in the lemma is exactly that $B/J_B = S^{-1}(B/J_B)$. We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in J_B$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

0DZ8 **Lemma 93.13.2.** *Let k be a field. Let B be a finite type k-algebra. Let $S \subset B$ be a multiplicative subset ideal such that if $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is not smooth at $\mathfrak{q}$ then $S \cap \mathfrak{q} = \emptyset$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(S^{-1}B, S^{-1}N)$$

Proof. This proof is the same as the proof of Lemma 93.12.4 but easier. We suggest the reader to skip the proof. The map is given by localization: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the localization $S_C^{-1}C$ of C with respect to the inverse image $S_C \subset C$ of S. Compare with the proof of Lemma 93.8.7.

The smooth locus of a morphism of schemes is open by definition. Let $J \subset B$ be an ideal cutting out the set of points in $\mathrm{Spec}(B)$ where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not smooth. Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\mathrm{Ext}^1_{S^{-1}B}(NL_{S^{-1}B/k}, S^{-1}N) = \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B S^{-1}B = \mathrm{Ext}^1_B(NL_{B/k}, N)$$

The first equality by More on Algebra, Lemma 15.100.2 and Algebra, Lemma 10.134.13. The second because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion and elements of S act invertibly on J-power torsion modules. This concludes the proof by the description of $\mathrm{Exal}_A(B, N)$ as $\mathrm{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 93.12.4. □

0DZ9 **Lemma 93.13.3.** *In Example 93.8.1 let P be a k-algebra. Let $S \subset P$ be a multiplicative subset. If*

(1) $k \to P$ *is of finite type, and*
(2) $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ *is smooth at all points of* $V(g)$ *for all* $g \in S$.

then the functor between deformation categories of Lemma 93.8.7

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{S^{-1}P}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{S^{-1}P}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.13.1 and the isomorphism on tangent spaces is the special case of Lemma 93.13.2 where $N = B$. □

93.14. Deformations of henselizations

0DZA In this section we compare the deformation problem posed by an algebra and its completion. We first discuss "liftability".

0DZB **Lemma 93.14.1.** *Let $A' \to A$ be a surjective map of Noetherian rings with nilpotent kernel. Let $A \to B$ be a finite type flat ring map. Let $\mathfrak{b} \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is syntomic on the complement of $V(\mathfrak{b})$. Let $(B^h, \mathfrak{b}^h)$ be the henselization of the pair $(B, \mathfrak{b})$. Then B has a flat lift to A' if and only if B^h has a flat lift to A'.*

First proof. This proof is a cheat. Namely, if B has a flat lift B', then taking the henselization $(B')^h$ we obtain a flat lift of B^h (compare with the proof of Lemma 93.8.8). Conversely, suppose that C' is an A'-flat lift of $(B')^h$. Then let $\mathfrak{c}' \subset C'$ be the inverse image of the ideal $\mathfrak{b}^h$. Then the completion $(C')^\wedge$ of C' with respect to $\mathfrak{c}'$ is a lift of $B^\wedge$ (details omitted). Hence we see that B has a flat lift by Lemma 93.12.3. □

Second proof. Choose an A-algebra surjection $P = A[x_1, \ldots, x_n] \to B$. Let $\mathfrak{p} \subset P$ be the inverse image of $\mathfrak{b}$. Set $P' = A'[x_1, \ldots, x_n]$ and denote $\mathfrak{p}' \subset P'$ the inverse image of $\mathfrak{p}$. (Of course $\mathfrak{p}$ and $\mathfrak{p}'$ do not designate prime ideals here.) We will denote P^h and $(P')^h$ the respective henselizations. We will use that taking henselizations is functorial and that the henselization of a quotient is the corresponding quotient of the henselization, see More on Algebra, Lemmas 15.11.16 and 15.12.7.

Suppose $A' \to B'$ is a flat lift of $A \to B$, in other words, $A' \to B'$ is flat and there is an A-algebra isomorphism $B = B' \otimes_{A'} A$. Then we can choose an A'-algebra map $P' \to B'$ lifting the given surjection $P \to B$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we find that B' is a quotient of P'. In particular, we find that we can endow B' with an A'-flat P'-module structure lifting B as an A-flat P-module. Conversely, if we can lift B to a P'-module M' flat over A', then M' is a cyclic module $M' \cong P'/J'$ (using Nakayama again) and setting $B' = P'/J'$ we find a flat lift of B as an algebra.

Set $C = B^h$ and $\mathfrak{c} = \mathfrak{b}C$. Suppose that $A' \to C'$ is a flat lift of $A \to C$. Then C' is henselian with respect to the inverse image $\mathfrak{c}'$ of $\mathfrak{c}$ (by More on Algebra, Lemma 15.11.9 and the fact that the kernel of $C' \to C$ is nilpotent). We choose an A'-algebra map $P' \to C'$ lifting the A-algebra map $P \to C$. These maps pass through henselizations to give surjections $P^h \to C$ and $(P')^h \to C'$ (for the second again

using Nakayama's lemma). In particular, we find that we can endow C' with an A'-flat $(P')^h$-module structure lifting C as an A-flat P^h-module. Conversely, if we can lift C to a $(P')^h$-module N' flat over A', then N' is a cyclic module $N' \cong (P')^h/\tilde{J}$ (using Nakayama again) and setting $C' = (P')^h/\tilde{J}$ we find a flat lift of C as an algebra.

Observe that $P' \to (P')^h$ is a flat ring map which induces an isomorphism $P'/\mathfrak{p}' = (P')^h/\mathfrak{p}'(P')^h$ (More on Algebra, Lemma 15.12.2). We conclude that our lemma is a consequence of Lemma 93.12.2 provided we can show that B_g lifts to an A'-flat P'_g-module for $g \in \mathfrak{p}'$. However, the ring map $A \to B_g$ is syntomic and hence lifts to an A'-flat algebra B' by Smoothing Ring Maps, Proposition 16.3.2. Since $A' \to P'_g$ is smooth, we can lift $P_g \to B_g$ to a surjective map $P'_g \to B'$ as before and we get what we want. □

0DZC **Lemma 93.14.2.** *Let k be a field. Let B be a finite type k-algebra. Let $J \subset B$ be an ideal such that $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$. Let N be a finite B-module. Then there is a canonical bijection*

$$Exal_k(B, N) \to Exal_k(B^h, N^h)$$

Here (B^h, J^h) is the henselization of (B, J) and $N^h = N \otimes_B B^h$.

Proof. This proof is the same as the proof of Lemma 93.12.4 but easier. We suggest the reader to skip the proof. The map is given by henselization: given $0 \to N \to C \to B \to 0$ in $\mathrm{Exal}_k(B, N)$ we send it to the henselization C^h of C with respect to the inverse image $J_C \subset C$ of J. Compare with the proof of Lemma 93.8.8.

Since $k \to B$ is of finite presentation the complex $NL_{B/k}$ can be represented by a complex $N^{-1} \to N^0$ where N^i is a finite B-module, see Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. As B is Noetherian, this means that $NL_{B/k}$ is pseudo-coherent. For $g \in J$ the k-algebra B_g is smooth and hence $(NL_{B/k})_g = NL_{B_g/k}$ is quasi-isomorphic to a finite projective B-module sitting in degree 0. Thus $\mathrm{Ext}^i_B(NL_{B/k}, N)_g = 0$ for $i \geq 1$ and any B-module N. Finally, we have

$$\begin{aligned} \mathrm{Ext}^1_{B^h}(NL_{B^h/k}, N^h) &= \mathrm{Ext}^1_{B^h}(NL_{B/k} \otimes_B B^h, N \otimes_B B^h) \\ &= \mathrm{Ext}^1_B(NL_{B/k}, N) \otimes_B B^h \\ &= \mathrm{Ext}^1_B(NL_{B/k}, N) \end{aligned}$$

The first equality by More on Algebra, Lemma 15.33.8 (or rather its analogue for henselizations of pairs). The second by More on Algebra, Lemma 15.100.2. The third because $\mathrm{Ext}^1_B(NL_{B/k}, N)$ is J-power torsion, the map $B \to B^h$ is flat and induces an isomorphism $B/J \to B^h/JB^h$ (More on Algebra, Lemma 15.12.2), and More on Algebra, Lemma 15.90.3. This concludes the proof by the description of $\mathrm{Exal}_A(B, N)$ as $\mathrm{Ext}^1_B(NL_{B/A}, N)$ given just above Lemma 93.12.4. □

0DZD **Lemma 93.14.3.** *In Example 93.8.1 let P be a k-algebra. Let $J \subset P$ be an ideal. Denote (P^h, J^h) the henselization of the pair (P, J). If*

(1) *$k \to P$ is of finite type, and*
(2) *$\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth on the complement of $V(J)$,*

then the functor between deformation categories of Lemma 93.8.8

$$\mathcal{D}ef_P \longrightarrow \mathcal{D}ef_{P^h}$$

is smooth and induces an isomorphism on tangent spaces.

Proof. We know that $\mathcal{D}ef_P$ and $\mathcal{D}ef_{P^h}$ are deformation categories by Lemma 93.8.2. Thus it suffices to check our functor identifies tangent spaces and a correspondence between liftability, see Formal Deformation Theory, Lemma 90.20.3. The property on liftability is proven in Lemma 93.14.1 and the isomorphism on tangent spaces is the special case of Lemma 93.14.2 where $N = B$. □

93.15. Application to isolated singularities

0DZE We apply the discussion above to study the deformation theory of a finite type algebra with finitely many singular points.

0DZF **Lemma 93.15.1.** *In Example 93.8.1 let P be a k-algebra. Assume that $k \to P$ is of finite type and that $\mathrm{Spec}(P) \to \mathrm{Spec}(k)$ is smooth except at the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of P. Let $P_{\mathfrak{m}_i}$, $P^h_{\mathfrak{m}_i}$, $P^\wedge_{\mathfrak{m}_i}$ be the local ring, henselization, completion. Then the maps of deformation categories*

$$\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^h_{\mathfrak{m}_i}} \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$$

are smooth and induce isomorphisms on their finite dimensional tangent spaces.

Proof. The tangent space is finite dimensional by Lemma 93.8.5. The functors between the categories are constructed in Lemmas 93.8.7, 93.8.8, and 93.8.10 (we omit some verifications of the form: the completion of the henselization is the completion).

Set $J = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ and apply Lemma 93.12.5 to get that $\mathcal{D}ef_P \to \mathcal{D}ef_{P^\wedge}$ is smooth and induces an isomorphism on tangent spaces where $P^\wedge$ is the J-adic completion of P. However, since $P^\wedge = \prod P^\wedge_{\mathfrak{m}_i}$ we see that the map $\mathcal{D}ef_P \to \prod \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induces an isomorphism on tangent spaces.

Let (P^h, J^h) be the henselization of the pair (P, J). Then $P^h = \prod P^h_{\mathfrak{m}_i}$ (look at idempotents and use More on Algebra, Lemma 15.11.6). Hence we can apply Lemma 93.14.3 to conclude as in the case of completion.

To get the final case it suffices to show that $\mathcal{D}ef_{P_{\mathfrak{m}_i}} \to \mathcal{D}ef_{P^\wedge_{\mathfrak{m}_i}}$ is smooth and induce isomorphisms on tangent spaces for each i separately. To do this, we may replace P by a principal localization whose only singular point is a maximal ideal $\mathfrak{m}$ (corresponding to $\mathfrak{m}_i$ in the original P). Then we can apply Lemma 93.13.3 with multiplicative subset $S = P \setminus \mathfrak{m}$ to conclude. Minor details omitted. □

93.16. Unobstructed deformation problems

0DZG Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibred in groupoids. Recall that we say $\mathcal{F}$ is *smooth* or *unobstructed* if p is smooth. This means that given a surjection $\varphi : A' \to A$ in $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(A))$ there exists a morphism $f : x' \to x$ in $\mathcal{F}$ with $p(f) = \varphi$. See Formal Deformation Theory, Section 90.9. In this section we give some geometrically meaningful examples.

0DZH **Lemma 93.16.1.** *In Example 93.8.1 let P be a local complete intersection over k (Algebra, Definition 10.135.1). Then $\mathcal{D}ef_P$ is unobstructed.*

Proof. Let $(A, Q) \to (k, P)$ be an object of $\mathcal{D}ef_P$. Then we see that $A \to Q$ is a syntomic ring map by Algebra, Definition 10.136.1. Hence for any surjection $A' \to A$ in $\mathcal{C}_\Lambda$ we see that there is a morphism $(A', Q') \to (A, Q)$ lifting $A' \to A$ by Smoothing Ring Maps, Proposition 16.3.2. This proves the lemma. □

0DZN **Lemma 93.16.2.** *In Situation 93.9.9 if $U_{12} \to \mathrm{Spec}(k)$ is smooth, then the morphism*

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_1} \times \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2}$$

is smooth. If in addition U_1 is a local complete intersection over k, then

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2} = \mathcal{D}ef_{P_2}$$

is smooth.

Proof. The equality signs hold by Lemma 93.9.7. Let us think of $\mathcal{C}_\Lambda$ as a deformation category over $\mathcal{C}_\Lambda$ as in Formal Deformation Theory, Section 90.9. Then

$$\mathcal{D}ef_{P_1} \times \mathcal{D}ef_{P_2} = \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2},$$

see Formal Deformation Theory, Remarks 90.5.2 (14). Using Lemma 93.9.10 the first statement is that the functor

$$\mathcal{D}ef_{P_1} \times_{\mathcal{D}ef_{P_{12}}} \mathcal{D}ef_{P_2} \longrightarrow \mathcal{D}ef_{P_1} \times_{\mathcal{C}_\Lambda} \mathcal{D}ef_{P_2}$$

is smooth. This follows from Formal Deformation Theory, Lemma 90.20.2 as long as we can show that $T\mathcal{D}ef_{P_{12}} = (0)$. This vanishing follows from Lemma 93.8.4 as P_{12} is smooth over k. For the second statement it suffices to show that $\mathcal{D}ef_{P_1} \to \mathcal{C}_\Lambda$ is smooth, see Formal Deformation Theory, Lemma 90.8.7. In other words, we have to show $\mathcal{D}ef_{P_1}$ is unobstructed, which is Lemma 93.16.1. □

0DZP **Lemma 93.16.3.** *In Example 93.9.1 let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *$X \to \mathrm{Spec}(k)$ is smooth except at the closed points $p_1, \ldots, p_n \in X$.*

Let $\mathcal{O}_{X,p_1}$, $\mathcal{O}^h_{X,p_1}$, $\mathcal{O}^\wedge_{X,p_1}$ be the local ring, henselization, completion. Consider the maps of deformation categories

$$\mathcal{D}ef_X \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^h_{X,p_i}} \longrightarrow \prod \mathcal{D}ef_{\mathcal{O}^\wedge_{X,p_i}}$$

The first arrow is smooth and the second and third arrows are smooth and induce isomorphisms on tangent spaces.

Proof. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 33.43.3 and Properties, Lemma 28.29.5. Then $X \setminus U_2$ is finite and we can choose an affine open $U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 93.16.2. Applying Lemmas 93.9.7 and 93.15.1 we win. □

0DZQ **Lemma 93.16.4.** *In Example 93.9.1 let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *X is a local complete intersection over k, and*
(3) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then $\mathcal{D}ef_X$ is unobstructed.

Proof. Let $p_1, \ldots, p_n \in X$ be the points where $X \to \mathrm{Spec}(k)$ isn't smooth. Choose an affine open $U_2 \subset X$ containing $p_1, \ldots, p_n$ and the generic point of every irreducible component of X. This is possible by Varieties, Lemma 33.43.3 and Properties, Lemma 28.29.5. Then $X \setminus U_2$ is finite and we can choose an affine open $U_1 \subset X \setminus \{p_1, \ldots, p_n\}$ such that $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$. Then U_1 and U_{12} are smooth affine schemes over k. We conclude that

$$\mathcal{D}ef_X \longrightarrow \mathcal{D}ef_{U_2}$$

is smooth by Lemma 93.16.2. Applying Lemmas 93.9.7 and 93.16.1 we win. □

93.17. Smoothings

0E7S Suppose given a finite type scheme or algebraic space X over a field k. It is often useful to find a flat morphism of finite type $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Such a thing is called a smoothing of X. In this section we will find a smoothing for 1-dimensional separated X which have isolated local complete intersection singularities.

0E7T **Lemma 93.17.1.** *Let k be a field. Set $S = \mathrm{Spec}(k[[t]])$ and $S_n = \mathrm{Spec}(k[t]/(t^n))$. Let $Y \to S$ be a proper, flat morphism of schemes whose special fibre X is Cohen-Macaulay and equidimensional of dimension d. Denote $X_n = Y \times_S S_n$. If for some $n \geq 1$ the dth Fitting ideal of Ω_{X_n/S_n} contains t^{n-1}, then the generic fibre of $Y \to S$ is smooth.*

Proof. By More on Morphisms, Lemma 37.22.7 we see that $Y \to S$ is a Cohen-Macaulay morphism. By Morphisms, Lemma 29.29.4 we see that $Y \to S$ has relative dimension d. By Divisors, Lemma 31.10.3 the dth Fitting ideal $\mathcal{I} \subset \mathcal{O}_Y$ of $\Omega_{Y/S}$ cuts out the singular locus of the morphism $Y \to S$. In other words, $V(\mathcal{I}) \subset Y$ is the closed subset of points where $Y \to S$ is not smooth. By Divisors, Lemma 31.10.1 formation of this Fitting ideal commutes with base change. By assumption we see that t^{n-1} is a section of $\mathcal{I} + t^n\mathcal{O}_Y$. Thus for every $x \in X = V(t) \subset Y$ we conclude that $t^{n-1} \in \mathcal{I}_x$ where $\mathcal{I}_x$ is the stalk at x. This implies that $V(\mathcal{I}) \subset V(t)$ in an open neighbourhood of X in Y. Since $Y \to S$ is proper, this implies $V(\mathcal{I}) \subset V(t)$ as desired. □

0E7U **Lemma 93.17.2.** *Let k be a field. Let $1 \leq c \leq n$ be integers. Let $f_1, \ldots, f_c \in k[x_1, \ldots x_n]$ be elements. Let a_{ij}, $0 \leq i \leq n$, $1 \leq j \leq c$ be variables. Consider*

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[a_{ij}][x_1, \ldots, x_n]$$

Denote $Y \subset \mathbf{A}_k^{n+c(n+1)}$ the closed subscheme cut out by $g_1, \ldots, g_c$. Denote $\pi : Y \to \mathbf{A}_k^{c(n+1)}$ the projection onto the affine space with variables a_{ij}. Then there is a nonempty Zariski open of $\mathbf{A}_k^{c(n+1)}$ over which π is smooth.

Proof. Recall that the set of points where π is smooth is open. Thus the complement, i.e., the singular locus, is closed. By Chevalley's theorem (in the form of Morphisms, Lemma 29.22.2) the image of the singular locus is constructible. Hence if the generic point of $\mathbf{A}_k^{c(n+1)}$ is not in the image of the singular locus, then the lemma follows (by Topology, Lemma 5.15.15 for example). Thus we have to show there is no point $y \in Y$ where π is not smooth mapping to the generic point of

$\mathbf{A}_k^{c(n+1)}$. Consider the matrix of partial derivatives

$$(\frac{\partial g_j}{\partial x_i}) = (\frac{\partial f_j}{\partial x_i} + a_{ij})$$

The image of this matrix in $\kappa(y)$ must have rank $< c$ since otherwise π would be smooth at y, see discussion in Smoothing Ring Maps, Section 16.2. Thus we can find $\lambda_1, \ldots, \lambda_c \in \kappa(y)$ not all zero such that the vector $(\lambda_1, \ldots, \lambda_c)$ is in the kernel of this matrix. After renumbering we may assume $\lambda_1 \neq 0$. Dividing by λ_1 we may assume our vector has the form $(1, \lambda_2, \ldots, \lambda_c)$. Then we obtain

$$a_{i1} = -\frac{\partial f_j}{\partial x_1} - \sum\nolimits_{j=2,\ldots,c} \lambda_j (\frac{\partial f_j}{\partial x_i} + a_{ij})$$

in $\kappa(y)$ for $i = 1, \ldots, n$. Moreover, since $y \in Y$ we also have

$$a_{0j} = -f_j - a_{1j}x_1 - \ldots - a_{nj}x_n$$

in $\kappa(y)$. This means that the subfield of $\kappa(y)$ generated by a_{ij} is contained in the subfield of $\kappa(y)$ generated by the images of $x_1, \ldots, x_n, \lambda_2, \ldots, \lambda_c$, and a_{ij} except for a_{i1} and a_{0j}. We count and we see that the transcendence degree of this is at most $c(n+1) - 1$. Hence y cannot map to the generic point as desired. □

0E7V **Lemma 93.17.3.** *Let k be a field. Let A be a global complete intersection over k. There exists a flat finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that $B[1/t]$ is smooth over $k((t))$.*

Proof. Write $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Algebra, Definition 10.135.1. We are going to choose $a_{ij} \in (t) \subset k[[t]]$ and set

$$g_j = f_j + a_{0j} + a_{1j}x_1 + \ldots + a_{nj}x_n \in k[[t]][x_1, \ldots, x_n]$$

After doing this we take $B = k[[t]][x_1, \ldots, x_n]/(g_1, \ldots, g_c)$. We claim that $k[[t]] \to B$ is flat at every prime ideal lying over (t). Namely, the elements $f_1, \ldots, f_c$ form a regular sequence in the local ring at any prime ideal $\mathfrak{p}$ of $k[x_1, \ldots, x_n]$ containing $f_1, \ldots, f_c$ (Algebra, Lemma 10.135.4). Thus $g_1, \ldots, g_c$ is locally a lift of a regular sequence and we can apply Algebra, Lemma 10.99.3. Flatness at primes lying over $(0) \subset k[[t]]$ is automatic because $k((t)) = k[[t]]_{(0)}$ is a field. Thus B is flat over $k[[t]]$.

All that remains is to show that for suitable choices of a_{ij} the generic fibre $B_{(0)}$ is smooth over $k((t))$. For this we have to show that we can choose our a_{ij} so that the induced morphism

$$(a_{ij}) : \mathrm{Spec}(k[[t]]) \longrightarrow \mathbf{A}_k^{c(n+1)}$$

maps into the nonempty Zariski open of Lemma 93.17.2. This is clear because there is no nonzero polynomial in the a_{ij} which vanishes on $(t)^{\oplus c(n+1)}$. (We leave this as an exercise to the reader.) □

0E7W **Lemma 93.17.4.** *Let k be a field. Let A be a finite dimensional k-algebra which is a local complete intersection over k. Then there is a finite flat $k[[t]]$-algebra B with $B/tB \cong A$ and $B[1/t]$ étale over $k((t))$.*

Proof. Since A is Artinian (Algebra, Lemma 10.53.2), we can write A as a product of local Artinian rings (Algebra, Lemma 10.53.6). Thus it suffices to prove the lemma if A is local (this uses that being a local complete intersection is preserved under taking principal localizations, see Algebra, Lemma 10.135.2). In this case A

is a global complete intersection. Consider the algebra B constructed in Lemma 93.17.3. Then $k[[t]] \to B$ is quasi-finite at the unique prime of B lying over (t) (Algebra, Definition 10.122.3). Observe that $k[[t]]$ is a henselian local ring (Algebra, Lemma 10.153.9). Thus $B = B' \times C$ where B' is finite over $k[[t]]$ and C has no prime lying over (t), see Algebra, Lemma 10.153.3. Then B' is the ring we are looking for (recall that étale is the same thing as smooth of relative dimension 0). □

0E7X **Lemma 93.17.5.** *Let k be a field. Let A be a k-algebra. Assume*

(1) *A is a local ring essentially of finite type over k,*
(2) *A is a complete intersection over k (Algebra, Definition 10.135.5).*

Set $d = \dim(A) + trdeg_k(\kappa)$ where κ is the residue field of A. Then there exists an integer n and a flat, essentially of finite type ring map $k[[t]] \to B$ with $B/tB \cong A$ such that t^n is in the dth Fitting ideal of $\Omega_{B/k[[t]]}$.

Proof. By Algebra, Lemma 10.135.7 we can write A as the localization at a prime $\mathfrak{p}$ of a global complete intersection P over k. Observe that $\dim(P) = d$ by Algebra, Lemma 10.116.3. By Lemma 93.17.3 we can find a flat, finite type ring map $k[[t]] \to Q$ such that $P \cong Q/tQ$ and such that $k((t)) \to Q[1/t]$ is smooth. It follows from the construction of Q in the lemma that $k[[t]] \to Q$ is a relative global complete intersection of relative dimension d; alternatively, Algebra, Lemma 10.136.15 tells us that Q or a suitable principal localization of Q is such a global complete intersection. Hence by Divisors, Lemma 31.10.3 the dth Fitting ideal $I \subset Q$ of $\Omega_{Q/k[[t]]}$ cuts out the singular locus of $\mathrm{Spec}(Q) \to \mathrm{Spec}(k[[t]])$. Thus $t^n \in I$ for some n. Let $\mathfrak{q} \subset Q$ be the inverse image of $\mathfrak{p}$. Set $B = Q_{\mathfrak{q}}$. The lemma is proved. □

0E7Y **Lemma 93.17.6.** *Let X be a scheme over a field k. Assume*

(1) *X is proper over k,*
(2) *X is a local complete intersection over k,*
(3) *X has dimension ≤ 1, and*
(4) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. Observe that X is Cohen-Macaulay, see Algebra, Lemma 10.135.3. Thus $X = X' \amalg X''$ with $\dim(X') = 0$ and X'' equidimensional of dimension 1, see Morphisms, Lemma 29.29.4. Since X' is finite over k (Varieties, Lemma 33.20.2) we can find $Y' \to \mathrm{Spec}(k[[t]])$ with special fibre X' and generic fibre smooth by Lemma 93.17.4. Thus it suffices to prove the lemma for X''. After replacing X by X'' we have X is Cohen-Macaulay and equidimensional of dimension 1.

We are going to use deformation theory for the situation $\Lambda = k \to k$. Let $p_1, \ldots, p_r \in X$ be the closed singular points of X, i.e., the points where $X \to \mathrm{Spec}(k)$ isn't smooth. For each i we pick an integer n_i and a flat, essentially of finite type ring map

$$k[[t]] \longrightarrow B_i$$

with $B_i/tB_i \cong \mathcal{O}_{X,p_i}$ such that t^{n_i} is in the 1st Fitting ideal of $\Omega_{B_i/k[[t]]}$. This is possible by Lemma 93.17.5. Observe that the system (B_i/t^nB_i) defines a formal object of $\mathcal{D}ef_{\mathcal{O}_{X,p_i}}$ over $k[[t]]$. By Lemma 93.16.3 the map

$$\mathcal{D}ef_X \longrightarrow \prod\nolimits_{i=1,\ldots,r} \mathcal{D}ef_{\mathcal{O}_{X,p_i}}$$

is a smooth map between deformation categories. Hence by Formal Deformation Theory, Lemma 90.8.8 there exists a formal object (X_n) in $\mathcal{D}ef_X$ mapping to the formal object $\prod_i (B_i/t^n)$ by the arrow above. By More on Morphisms of Spaces, Lemma 76.43.5 there exists a projective scheme Y over $k[[t]]$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k[t]/(t^n)) \cong X_n$. By More on Morphisms, Lemma 37.12.4 we see that $Y \to \mathrm{Spec}(k[[t]])$ is flat. Since X is Cohen-Macaulay and equidimensional of dimension 1 we may apply Lemma 93.17.1 to check Y has smooth generic fibre[3]. Choose n strictly larger than the maximum of the integers n_i found above. If we can show t^{n-1} is in the first Fitting ideal of Ω_{X_n/S_n} with $S_n = \mathrm{Spec}(k[t]/(t^n))$, then the proof is done. To do this it suffices to prove this is true in each of the local rings of X_n at closed points p. However, if p corresponds to a smooth point for $X \to \mathrm{Spec}(k)$, then $\Omega_{X_n/S_n,p}$ is free of rank 1 and the first Fitting ideal is equal to the local ring. If $p = p_i$ for some i, then

$$\Omega_{X_n/S_n,p_i} = \Omega_{(B_i/t^nB_i)/(k[t]/(t^n))} = \Omega_{B_i/k[[t]]}/t^n\Omega_{B_i/k[[t]]}$$

Since taking Fitting ideals commutes with base change (with already used this but in this algebraic setting it follows from More on Algebra, Lemma 15.8.4), and since $n - 1 \geq n_i$ we see that t^{n-1} is in the Fitting ideal of this module over B_i/t^nB_i as desired. □

0E7Z **Lemma 93.17.7.** *Let k be a field and let X be a scheme over k. Assume*

(1) *X is separated, finite type over k and $\dim(X) \leq 1$,*
(2) *X is a local complete intersection over k, and*
(3) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Then there exists a flat, separated, finite type morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X.

Proof. If X is reduced, then we can choose an embedding $X \subset \overline{X}$ as in Varieties, Lemma 33.43.6. Writing $X = \overline{X} \setminus \{x_1, \ldots, x_n\}$ we see that $\mathcal{O}_{\overline{X},x_i}$ is a discrete valuation ring and hence in particular a local complete intersection (Algebra, Definition 10.135.5). Thus $\overline{X}$ is a local complete intersection over k because this holds over the open X and at the points x_i by Algebra, Lemma 10.135.7. Thus we may apply Lemma 93.17.6 to find a projective flat morphism $\overline{Y} \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is $\overline{X}$. Then we remove $x_1, \ldots, x_n$ from $\overline{Y}$ to obtain Y.

In the general case, write $X = X' \amalg X''$ where with $\dim(X') = 0$ and X'' equidimensional of dimension 1. Then X'' is reduced and the first paragraph applies to it. On the other hand, X' can be dealt with as in the proof of Lemma 93.17.6. Some details omitted. □

93.18. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields

[3]Warning: in general it is **not** true that the local ring of Y at the point p_i is isomorphic to B_i. We only know that this is true after dividing by t^n on both sides!

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Part 7

Algebraic Stacks

CHAPTER 94

Algebraic Stacks

026K

94.1. Introduction

026L This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [**LMB00**].

This chapter is not an introduction to algebraic stacks. For an informal discussion of algebraic stacks, please take a look at Introducing Algebraic Stacks, Section 105.1.

94.2. Conventions

026M The conventions we use in this chapter are the same as those in the chapter on algebraic spaces. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We discuss what changes if you change the big fppf site in Section 94.18.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 35.13.7. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

94.3. Notation

0400 We use the letters S, T, U, V, X, Y to indicate schemes. We use the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to indicate categories (fibred, fibred in groupoids, stacks, ...) over $(Sch/S)_{fppf}$. We use small case letters f, g for functors such as $f : \mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We use capital F, G, H for algebraic spaces over S, and more generally for presheaves of sets on $(Sch/S)_{fppf}$. (In future chapters we will revert to using also X, Y, etc for algebraic spaces.)

The reason for these choices is that we want to clearly distinguish between the different types of objects in this chapter, to build the foundations.

94.4. Representable categories fibred in groupoids

02ZQ Let S be a scheme contained in Sch_{fppf}. The basic object of study in this chapter will be a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$, see Categories, Definition 4.35.1. We will often simply say "let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$" to indicate this situation. A 1-morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ will be a 1-morphism in the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Categories, Definition 4.35.6. It is simply a functor $\mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We recall this is really a $(2,1)$-category and that all 2-fibre products exist.

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be *representable* if there exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 4.40.1. We will sometimes say that $\mathcal{X}$ is *representable by a scheme* to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

04SR (94.4.0.1) $$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{Sch/S}(U, V)$$

see Categories, Lemma 4.40.3. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $U \to V$. Conversely, given a morphism of schemes $U \to V$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $U \to V$ and which is unique up to unique 2-isomorphism.

94.5. The 2-Yoneda lemma

04SS Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We will frequently use the 2-Yoneda lemma, see Categories, Lemma 4.41.2. Technically it says that there is an equivalence of categories

$$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}((Sch/U)_{fppf}, \mathcal{X}) \longrightarrow \mathcal{X}_U, \quad f \longmapsto f(U/U).$$

It says that 1-morphisms $(Sch/U)_{fppf} \to \mathcal{X}$ correspond to objects x of the fibre category $\mathcal{X}_U$. Namely, given a 1-morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$ we obtain the object $x = f(U/U) \in \mathrm{Ob}(\mathcal{X}_U)$. Conversely, given a choice of pullbacks for $\mathcal{X}$ as in Categories, Definition 4.33.6, and an object x of $\mathcal{X}_U$, we obtain a functor $(Sch/U)_{fppf} \to \mathcal{X}$ defined by the rule

$$(\varphi : V \to U) \longmapsto \varphi^* x$$

on objects. By abuse of notation we use $x : (Sch/U)_{fppf} \to \mathcal{X}$ to indicate this functor. It indeed has the property that $x(U/U) = x$ and moreover, given any other functor f with $f(U/U) = x$ there exists a unique 2-isomorphism $x \to f$. In other words the functor x is well determined by the object x up to unique 2-isomorphism.

We will use this without further mention in the following.

94.6. Representable morphisms of categories fibred in groupoids

04ST Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *representable* 1*-morphism*, see Categories, Definition 4.42.3. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable. Choose a representing object V_y and an equivalence

$$(Sch/V_y)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}.$$

The projection $(Sch/V_y)_{fppf} \to (Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{Y} \to (Sch/U)_{fppf}$ comes from a morphism of schemes $f_y : V_y \to U$, see Section 94.4. We represent this by the diagram

0401 (94.6.0.1)
$$\begin{array}{ccccc} V_y & \rightsquigarrow & (Sch/V_y)_{fppf} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f_y}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ U & \rightsquigarrow & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where the squiggly arrows represent the 2-Yoneda embedding. Here are some lemmas about this notion that work in great generality (namely, they work for categories fibred in groupoids over any base category which has fibre products).

02ZR **Lemma 94.6.1.** *Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then the* 1*-morphism induced by f*

$$(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$$

is a representable 1*-morphism.*

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

0456 **Lemma 94.6.2.** *Let S be an object of Sch_{fppf}. Consider a* 2*-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1*-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable if and only if f' is representable.*

Proof. Omitted. □

02ZS **Lemma 94.6.3.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be representable* 1*-morphisms. Then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a representable 1*-morphism.*

Proof. This is entirely formal and works in any category. □

02ZT **Lemma 94.6.4.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Then the base change f' is a representable 1-morphism.

Proof. This is entirely formal and works in any category. □

02ZU **Lemma 94.6.5.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be representable 1-morphisms. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a representable 1-morphism.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 94.6.3 and 94.6.4. □

94.7. Split categories fibred in groupoids

04SU Let S be a scheme contained in Sch_{fppf}. Recall that given a "presheaf of groupoids"

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Groupoids$$

we get a category fibred in groupoids $\mathcal{S}_F$ over $(Sch/S)_{fppf}$, see Categories, Example 4.37.1. Any category fibred in groupoids isomorphic (!) to one of these is called a *split category fibred in groupoids*. Any category fibred in groupoids is equivalent to a split one.

If F is a presheaf of sets then $\mathcal{S}_F$ is fibred in sets, see Categories, Definition 4.38.2, and Categories, Example 4.38.5. The rule $F \mapsto \mathcal{S}_F$ is in some sense fully faithful on presheaves, see Categories, Lemma 4.38.6. If F, G are presheaves, then

$$\mathcal{S}_{F \times G} = \mathcal{S}_F \times_{(Sch/S)_{fppf}} \mathcal{S}_G$$

and if $F \to H$ and $G \to H$ are maps of presheaves of sets, then

$$\mathcal{S}_{F \times_H G} = \mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$$

where the right hand sides are 2-fibre products. This is immediate from the definitions as the fibre categories of $\mathcal{S}_F, \mathcal{S}_G, \mathcal{S}_H$ have only identity morphisms.

An even more special case is where $F = h_X$ is a representable presheaf. In this case we have $\mathcal{S}_{h_X} = (Sch/X)_{fppf}$, see Categories, Example 4.38.7.

We will use the notation $\mathcal{S}_F$ without further mention in the following.

94.8. Categories fibred in groupoids representable by algebraic spaces

02ZV A slightly weaker notion than being representable is the notion of being representable by algebraic spaces which we discuss in this section. This discussion might have been avoided had we worked with some category $Spaces_{fppf}$ of algebraic spaces instead of the category Sch_{fppf}. However, it seems to us natural to consider the category of schemes as the natural collection of "test objects" over which the fibre categories of an algebraic stack are defined.

In analogy with Categories, Definitions 4.40.1 we make the following definition.

04SV **Definition 94.8.1.** Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over* S if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

We continue our abuse of notation in suppressing the equivalence j whenever we encounter such a situation. It follows formally from the above that if $\mathcal{X}$ is representable (by a scheme), then it is representable by an algebraic space. Here is the analogue of Categories, Lemma 4.40.2.

02ZX **Lemma 94.8.2.** *Let S be a scheme contained in Sch_{fppf}. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Then $\mathcal{X}$ is representable by an algebraic space over S if and only if the following conditions are satisfied:*

(1) *$\mathcal{X}$ is fibred in setoids*[1]*, and*
(2) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$ is an algebraic space.*

Proof. Omitted, but see Categories, Lemma 4.40.2. □

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

04SW (94.8.2.1) $$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{Sch/S}(F, G)$$

see Categories, Lemma 4.39.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, given a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

94.9. Morphisms representable by algebraic spaces

04SX In analogy with Categories, Definition 4.42.3 we make the following definition.

02ZW **Definition 94.9.1.** Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

[1] This means that it is fibred in groupoids and objects in the fibre categories have no nontrivial automorphisms, see Categories, Definition 4.39.2.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic space over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 65.16. Here is the diagram

0402 (94.9.1.1)
$$\begin{array}{ccccc} F_y & \rightsquigarrow & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} & \xrightarrow{\mathrm{pr}_1} & \mathcal{X} \\ {\scriptstyle f_y}\downarrow & & {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle f} \\ U & \rightsquigarrow & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Here is the analogue of Categories, Lemma 4.42.5.

02ZY **Lemma 94.9.2.** *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are necessary and sufficient conditions for f to be representable by algebraic spaces:*

(1) *for each scheme U/S the functor $f_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
(2) *for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf*
$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$
is an algebraic space over U.

Here we have made a choice of pullbacks for $\mathcal{Y}$.

Proof. This follows from the description of fibre categories of the 2-fibre products $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ in Categories, Lemma 4.42.1 combined with Lemma 94.8.2. □

Here are some lemmas about this notion that work in great generality.

0457 **Lemma 94.9.3.** *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*
$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$
of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable by algebraic spaces if and only if f' is representable by algebraic spaces.

Proof. Omitted. □

02ZZ **Lemma 94.9.4.** *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.*

Proof. Omitted. This relies only on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 65.7.3. □

0458 **Lemma 94.9.5.** *Let S be an object of Sch_{fppf}. Let $a : F \to G$ be a map of presheaves of sets on $(Sch/S)_{fppf}$. Denote $a' : \mathcal{S}_F \to \mathcal{S}_G$ the associated map of categories fibred in sets. Then a is representable by algebraic spaces (see Bootstrap, Definition 80.3.1) if and only if a' is representable by algebraic spaces.*

Proof. Omitted. □

04SY **Lemma 94.9.6.** *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a is representable by algebraic spaces (see Bootstrap, Definition 80.3.1) if and only if f is representable by algebraic spaces.*

Proof. Omitted. Hint: Combine Lemmas 94.9.3 and 94.9.5. □

0302 **Lemma 94.9.7.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Then the base change f' is a 1-morphism representable by algebraic spaces.

Proof. This is formal. □

0300 **Lemma 94.9.8.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume*

(1) *f is representable by algebraic spaces, and*
(2) *$\mathcal{Z}$ is representable by an algebraic space over S.*

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

Proof. This is a reformulation of Bootstrap, Lemma 80.3.6. First note that $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is fibred in setoids over $(Sch/S)_{fppf}$. Hence it is equivalent to $\mathcal{S}_F$ for some presheaf F on $(Sch/S)_{fppf}$, see Categories, Lemma 4.39.5. Moreover, let G be an algebraic space which represents $\mathcal{Z}$. The 1-morphism $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces by Lemma 94.9.7. And $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $F \to G$ by Categories, Lemma 4.39.6. Then $F \to G$ is representable by algebraic spaces by Lemma 94.9.6. Hence Bootstrap, Lemma 80.3.6 implies that F is an algebraic space as desired. □

Let S, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, f, g be as in Lemma 94.9.8. Let F and G be algebraic spaces over S such that F represents $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ and G represents $\mathcal{Z}$. The 1-morphism $f' : \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $f' : F \to G$ of algebraic spaces by (94.8.2.1). Thus we have the following diagram

0403 (94.9.8.1)
$$\begin{array}{ccccc} F & \rightsquigarrow & \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ G & \rightsquigarrow & \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects.

0301 **Lemma 94.9.9.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

Proof. This follows from Lemma 94.9.8. Details omitted. □

0303 **Lemma 94.9.10.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 94.9.9 and 94.9.7. □

0CKY **Lemma 94.9.11.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces and $\mathcal{Y}$ is a stack in groupoids, then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack in groupoids.* Lemma in an email of Matthew Emerton dated June 15, 2016

Proof. The property of a morphism being representable by algebraic spaces is preserved under base-change (Lemma 94.9.8), and so, passing to the base-change $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over $\mathcal{Y}$, we may reduce to the case of a morphism of categories fibred in groupoids $\mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces, and whose target is a stack in groupoids; our goal is then to prove that $\mathcal{X}$ is also a stack in groupoids. This follows from Stacks, Lemma 8.6.11 whose assumptions are satisfied as a result of Lemma 94.9.2. □

94.10. Properties of morphisms representable by algebraic spaces

03YJ Here is the definition that makes this work.

03YK **Definition 94.10.1.** Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 74.10.1.

In this case we say that f has *property* $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (94.9.1.1), has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the target. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

0459 **Lemma 94.10.2.** *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 94.10.1. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences and f (or equivalently f') is representable by algebraic spaces. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.

Proof. Note that this makes sense by Lemma 94.9.3. Proof omitted. □

Here is a sanity check.

045A **Lemma 94.10.3.** *Let S be a scheme contained in Sch_{fppf}. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be as in Definition 94.10.1. Assume a is representable by algebraic spaces. Then $a : F \to G$ has property $\mathcal{P}$ (see Bootstrap, Definition 80.4.1) if and only if the corresponding morphism $\mathcal{S}_F \to \mathcal{S}_G$ of categories fibred in groupoids has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 94.9.5. Proof omitted. □

04TC **Lemma 94.10.4.** *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a has $\mathcal{P}$ if and only if f has $\mathcal{P}$.*

Proof. The lemma makes sense by Lemma 94.9.6. The lemma follows on combining Lemmas 94.10.2 and 94.10.3. □

045B **Lemma 94.10.5.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.*

Proof. Note that the lemma makes sense by Lemma 94.9.9. Proof omitted. □

045C **Lemma 94.10.6.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow[g']{} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 94.9.7. Proof omitted. □

045D **Lemma 94.10.7.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 94.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow[g']{} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Assume that for every scheme U and object x of $\mathcal{Y}_U$, there exists an fppf covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of the functor $g : \mathcal{Z}_{U_i} \to \mathcal{Y}_{U_i}$. In this case, if f' has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Proof omitted. Hint: Compare with the proof of Spaces, Lemma 65.5.6. □

045E **Lemma 94.10.8.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 94.10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.*

Proof. The lemma makes sense by Lemma 94.9.10. Proof omitted. □

045F **Lemma 94.10.9.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 94.10.1. Suppose that for any morphism of algebraic spaces $a : F \to G$ we have $\mathcal{P}(a) \Rightarrow \mathcal{P}'(a)$. If f has property $\mathcal{P}$ then f has property $\mathcal{P}'$.*

Proof. Formal. □

05UK **Lemma 94.10.10.** *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism (see Definition 94.10.1 and Descent on Spaces, Lemma 74.11.30). Then j is fully faithful on fibre categories.*

Proof. We have seen in Lemma 94.9.2 that j is faithful on fibre categories. Consider a scheme U, two objects u, v of $\mathcal{X}_U$, and an isomorphism $t : j(u) \to j(v)$ in $\mathcal{Y}_U$. We have to construct an isomorphism in $\mathcal{X}_U$ between u and v. By the 2-Yoneda lemma (see Section 94.5) we think of u, v as 1-morphisms $u, v : (Sch/U)_{fppf} \to \mathcal{X}$ and we consider the 2-fibre product

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X}.$$

By assumption this is representable by an algebraic space $F_{j \circ v}$, over U and the morphism $F_{j \circ v} \to U$ is a monomorphism. But since $(1_U, v, 1_{j(v)})$ gives a 1-morphism of $(Sch/U)_{fppf}$ into the displayed 2-fibre product, we see that $F_{j \circ v} = U$ (here we use that if $V \to U$ is a monomorphism of algebraic spaces which has a section, then $V = U$). Therefore the 1-morphism projecting to the first coordinate

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X} \to (Sch/U)_{fppf}$$

is an equivalence of fibre categories. Since $(1_U, u, t)$ and $(1_U, v, 1_{j(v)})$ give two objects in $((Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X})_U$ which have the same first coordinate, there

must be a 2-morphism between them in the 2-fibre product. This is by definition a morphism $\tilde{t} : u \to v$ such that $j(\tilde{t}) = t$. □

Here is a characterization of those categories fibred in groupoids for which the diagonal is representable by algebraic spaces.

045G **Lemma 94.10.11.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent:*

1. *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
2. *for every scheme U over S, and any $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the sheaf $Isom(x, y)$ is an algebraic space over U,*
3. *for every scheme U over S, and any $x \in \mathrm{Ob}(\mathcal{X}_U)$ the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is representable by algebraic spaces,*
4. *for every pair of schemes T_1, T_2 over S, and any $x_i \in \mathrm{Ob}(\mathcal{X}_{T_i})$, $i = 1, 2$ the 2-fibre product $(Sch/T_1)_{fppf} \times_{x_1, \mathcal{X}, x_2} (Sch/T_2)_{fppf}$ is representable by an algebraic space,*
5. *for every representable category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
6. *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of representable categories fibred in groupoids over $(Sch/S)_{fppf}$ and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$, $i = 1, 2$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space,*
7. *for every category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
8. *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ which are representable by algebraic spaces, and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.2.5 and the definitions. Let us prove the equivalence of (1) and (3). Write $\mathcal{C} = (Sch/S)_{fppf}$ for the base category. We will use some of the observations of the proof of the similar Categories, Lemma 4.42.6. We will use the symbol $\cong$ to mean "equivalence of categories fibred in groupoids over $\mathcal{C} = (Sch/S)_{fppf}$". Assume (1). Suppose given U and x as in (3). For any scheme V and $y \in \mathrm{Ob}(\mathcal{X}_V)$ we see (compare reference above) that

$$\mathcal{C}/U \times_{x, \mathcal{X}, y} \mathcal{C}/V \cong (\mathcal{C}/U \times_S V) \times_{(x,y), \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

which is representable by an algebraic space by assumption. Conversely, assume (3). Consider any scheme U over S and a pair (x, x') of objects of $\mathcal{X}$ over U. We have to show that $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x,x')} U$ is representable by an algebraic space. This is clear because (compare reference above)

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x,x')} \mathcal{C}/U \cong (\mathcal{C}/U \times_{x, \mathcal{X}, x'} \mathcal{C}/U) \times_{\mathcal{C}/U \times_S U, \Delta} \mathcal{C}/U$$

and the right hand side is representable by an algebraic space by assumption and the fact that the category of algebraic spaces over S has fibre products and contains U and S.

The equivalences (3) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6), and (7) $\Leftrightarrow$ (8) are formal. The equivalences (3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) follow from Lemma 94.9.3. Assume (3), and let $\mathcal{U} \to \mathcal{X}$ be as in (7). To prove (7) we have to show that for every scheme V and 1-morphism $y : (Sch/V)_{fppf} \to \mathcal{X}$ the 2-fibre product $(Sch/V)_{fppf} \times_{y, \mathcal{X}} \mathcal{U}$ is representable by

an algebraic space. Property (3) tells us that y is representable by algebraic spaces hence Lemma 94.9.8 implies what we want. Finally, (7) directly implies (3). □

In the situation of the lemma, for any 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ as in the lemma, it makes sense to say that x has property $\mathcal{P}$, for any property as in Definition 94.10.1. In particular this holds for $\mathcal{P}$ = "surjective", $\mathcal{P}$ = "smooth", and $\mathcal{P}$ = "étale", see Descent on Spaces, Lemmas 74.11.6, 74.11.26, and 74.11.28. We will use these three cases in the definitions of algebraic stacks below.

94.11. Stacks in groupoids

0304 Let S be a scheme contained in Sch_{fppf}. Recall that a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ over $(Sch/S)_{fppf}$ is said to be a *stack in groupoids* (see Stacks, Definition 8.5.1) if and only if

(1) $p : \mathcal{X} \to (Sch/S)_{fppf}$ is fibred in groupoids over $(Sch/S)_{fppf}$,
(2) for all $U \in \mathrm{Ob}((Sch/S)_{fppf})$, for all $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $(Sch/U)_{fppf}$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $(Sch/S)_{fppf}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

For examples see Examples of Stacks, Section 95.9 ff.

94.12. Algebraic stacks

026N Here is the definition of an algebraic stack. We remark that condition (2) implies we can make sense out of the condition in part (3) that $(Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective, see discussion following Lemma 94.10.11.

026O **Definition 94.12.1.** Let S be a base scheme contained in Sch_{fppf}. An *algebraic stack over* S is a category

$$p : \mathcal{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$ with the following properties:

(1) The category $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$.
(2) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.
(3) There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$ which is surjective and smooth[2].

There are some differences with other definitions found in the literature.

The first is that we require $\mathcal{X}$ to be a stack in groupoids in the fppf topology, whereas in many references the étale topology is used. It somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting 2-category of algebraic stacks ends up being the same. This is explained in Criteria for Representability, Section 97.19.

The second is that we only require the diagonal map of $\mathcal{X}$ to be representable by algebraic spaces, whereas in most references some other conditions are imposed. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of $\mathcal{X}$ be representable by algebraic spaces, and simply add an additional hypothesis wherever this is necessary. It has the added benefit

[2]In future chapters we will denote this simply $U \to \mathcal{X}$ as is customary in the literature. Another good alternative would be to formulate this condition as the existence of a representable category fibred in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$.

that any algebraic space (as defined in Spaces, Definition 65.6.1) gives rise to an algebraic stack.

The third is that in some papers it is required that there exists a scheme U and a surjective and étale morphism $U \to \mathcal{X}$. In the groundbreaking paper [**DM69**] where algebraic stacks were first introduced Deligne and Mumford used this definition and showed that the moduli stack of stable genus $g > 1$ curves is an algebraic stack which has an étale covering by a scheme. Michael Artin, see [**Art74**], realized that many natural results on algebraic stacks generalize to the case where one only assume a smooth covering by a scheme. Hence our choice above. To distinguish the two cases one sees the terms "Deligne-Mumford stack" and "Artin stack" used in the literature. We will reserve the term "Artin stack" for later use (insert future reference here), and continue to use "algebraic stack", but we will use "Deligne-Mumford stack" to indicate those algebraic stacks which have an étale covering by a scheme.

03YO **Definition 94.12.2.** Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. We say $\mathcal{X}$ is a *Deligne-Mumford stack* if there exists a scheme U and a surjective étale morphism $(Sch/U)_{fppf} \to \mathcal{X}$.

We will compare our notion of a Deligne-Mumford stack with the notion as defined in the paper by Deligne and Mumford later (see insert future reference here).

The category of algebraic stacks over S forms a 2-category. Here is the precise definition.

03YP **Definition 94.12.3.** Let S be a scheme contained in Sch_{fppf}. The 2-*category of algebraic stacks over* S is the sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ (see Categories, Definition 4.35.6) defined as follows:

(1) Its objects are those categories fibred in groupoids over $(Sch/S)_{fppf}$ which are algebraic stacks over S.
(2) Its 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ are any functors of categories over $(Sch/S)_{fppf}$, as in Categories, Definition 4.32.1.
(3) Its 2-morphisms are transformations between functors over $(Sch/S)_{fppf}$, as in Categories, Definition 4.32.1.

In other words this 2-category is the full sub 2-category of $Cat/(Sch/S)_{fppf}$ whose objects are algebraic stacks. Note that every 2-morphism is automatically an isomorphism. Hence this is actually a $(2,1)$-category and not just a 2-category.

We will see later (insert future reference here) that this 2-category has 2-fibre products.

Similar to the remark above the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$. It turns out that it is closed under equivalences. Here is the precise statement.

03YQ **Lemma 94.12.4.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are equivalent as categories over $(Sch/S)_{fppf}$. Then $\mathcal{X}$ is an algebraic stack if and only if $\mathcal{Y}$ is an algebraic stack. Similarly, $\mathcal{X}$ is a Deligne-Mumford stack if and only if $\mathcal{Y}$ is a Deligne-Mumford stack.*

Proof. Assume $\mathcal{X}$ is an algebraic stack (resp. a Deligne-Mumford stack). By Stacks, Lemma 8.5.4 this implies that $\mathcal{Y}$ is a stack in groupoids over Sch_{fppf}.

Choose an equivalence $f : \mathcal{X} \to \mathcal{Y}$ over Sch_{fppf}. This gives a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ {\scriptstyle \Delta_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{Y}}} \\ \mathcal{X} \times \mathcal{X} & \xrightarrow{f \times f} & \mathcal{Y} \times \mathcal{Y} \end{array}$$

whose horizontal arrows are equivalences. This implies that $\Delta_{\mathcal{Y}}$ is representable by algebraic spaces according to Lemma 94.9.3. Finally, let U be a scheme over S, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a 1-morphism which is surjective and smooth (resp. étale). Considering the diagram

$$\begin{array}{ccc} (Sch/U)_{fppf} & \xrightarrow{\mathrm{id}} & (Sch/U)_{fppf} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle f \circ x} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

and applying Lemma 94.10.2 we conclude that $f \circ x$ is surjective and smooth (resp. étale) as desired. □

94.13. Algebraic stacks and algebraic spaces

03YR In this section we discuss some simple criteria which imply that an algebraic stack is an algebraic space. The main result is that this happens exactly when objects of fibre categories have no nontrivial automorphisms. This is not a triviality! Before we come to this we first do a sanity check.

03YS **Lemma 94.13.1.** *Let S be a scheme contained in Sch_{fppf}.*

(1) *A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which is representable by an algebraic space is a Deligne-Mumford stack.*
(2) *If F is an algebraic space over S, then the associated category fibred in groupoids $p : \mathcal{S}_F \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*
(3) *If $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $(Sch/X)_{fppf} \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*

Proof. It is clear that (2) implies (3). Parts (1) and (2) are equivalent by Lemma 94.12.4. Hence it suffices to prove (2). First, we note that $\mathcal{S}_F$ is stack in sets since F is a sheaf (Stacks, Lemma 8.6.3). A fortiori it is a stack in groupoids. Second the diagonal morphism $\mathcal{S}_F \to \mathcal{S}_F \times \mathcal{S}_F$ is the same as the morphism $\mathcal{S}_F \to \mathcal{S}_{F \times F}$ which comes from the diagonal of F. Hence this is representable by algebraic spaces according to Lemma 94.9.4. Actually it is even representable (by schemes), as the diagonal of an algebraic space is representable, but we do not need this. Let U be a scheme and let $h_U \to F$ be a surjective étale morphism. We may think of this as a surjective étale morphism of algebraic spaces. Hence by Lemma 94.10.3 the corresponding 1-morphism $(Sch/U)_{fppf} \to \mathcal{S}_F$ is surjective and étale. □

The following result says that a Deligne-Mumford stack whose inertia is trivial "is" an algebraic space. This lemma will be obsoleted by the stronger Proposition 94.13.3 below which says that this holds more generally for algebraic stacks...

045H **Lemma 94.13.2.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) $\mathcal{X}$ *is a Deligne-Mumford stack and is a stack in setoids,*
(2) $\mathcal{X}$ *is a Deligne-Mumford stack such that the canonical* 1*-morphism* $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ *is an equivalence, and*
(3) $\mathcal{X}$ *is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 94.13.1. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists a sheaf F on $(Sch/S)_{fppf}$ and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$. By Lemma 94.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective étale morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 80.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 94.10.4 we conclude that $h_U \to F$ is surjective and étale. Finally, we apply Bootstrap, Theorem 80.6.1 to see that F is an algebraic space. □

04SZ **Proposition 94.13.3.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) $\mathcal{X}$ *is a stack in setoids,*
(2) *the canonical* 1*-morphism* $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ *is an equivalence, and*
(3) $\mathcal{X}$ *is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 94.13.2. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ where F is a sheaf on $(Sch/S)_{fppf}$. By Lemma 94.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective smooth morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 80.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 94.10.4 we conclude that $h_U \to F$ is surjective and smooth. In particular it is surjective, flat and locally of finite presentation (by Lemma 94.10.9 and the fact that a smooth morphism of algebraic spaces is flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 67.37.5 and 67.37.7). Finally, we apply Bootstrap, Theorem 80.10.1 to see that F is an algebraic space. □

94.14. 2-Fibre products of algebraic stacks

04TD The 2-category of algebraic stacks has products and 2-fibre products. The first lemma is really a special case of Lemma 94.14.3 but its proof is slightly easier.

04TE **Lemma 94.14.1.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Then $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is an algebraic stack, and is a product in the* 2*-category of algebraic stacks over S.*

Proof. An object of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T is just a pair (x, y) where x is an object of $\mathcal{X}_T$ and y is an object of $\mathcal{Y}_T$. Hence it is immediate from the definitions that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is a stack in groupoids. If (x, y) and (x', y') are two objects of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T, then

$$Isom((x, y), (x', y')) = Isom(x, x') \times Isom(y, y').$$

Hence it follows from the equivalences in Lemma 94.10.11 and the fact that the category of algebraic spaces has products that the diagonal of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is representable by algebraic spaces. Finally, suppose that $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Note that

$$(Sch/U \times_S V)_{fppf} = (Sch/U)_{fppf} \times_{(Sch/S)_{fppf}} (Sch/V)_{fppf}.$$

The object $(\mathrm{pr}_U^* x, \mathrm{pr}_V^* y)$ of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over $(Sch/U \times_S V)_{fppf}$ thus defines a 1-morphism

$$(Sch/U \times_S V)_{fppf} \longrightarrow \mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$$

which is the composition of base changes of x and y, hence is surjective and smooth, see Lemmas 94.10.6 and 94.10.5. We conclude that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is indeed an algebraic stack. We omit the verification that it really is a product. □

04TF **Lemma 94.14.2.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{Z}$ be a stack in groupoids over $(Sch/S)_{fppf}$ whose diagonal is representable by algebraic spaces. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack.*

Proof. We have to check conditions (1), (2), and (3) of Definition 94.12.1. The first condition follows from Stacks, Lemma 8.5.6.

The second condition we have to check is that the *Isom*-sheaves are representable by algebraic spaces. To do this, suppose that T is a scheme over S, and u, v are objects of $(\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y})_T$. By our construction of 2-fibre products (which goes all the way back to Categories, Lemma 4.32.3) we may write $u = (x, y, \alpha)$ and $v = (x', y', \alpha')$. Here $\alpha : f(x) \to g(y)$ and similarly for α'. Then it is clear that

$$\begin{array}{ccc} Isom(u, v) & \longrightarrow & Isom(y, y') \\ \downarrow & & \downarrow{\scriptstyle \phi \mapsto g(\phi) \circ \alpha} \\ Isom(x, x') & \xrightarrow{\psi \mapsto \alpha' \circ f(\psi)} & Isom(f(x), g(y')) \end{array}$$

is a cartesian diagram of sheaves on $(Sch/T)_{fppf}$. Since by assumption the sheaves $Isom(y, y')$, $Isom(x, x')$, $Isom(f(x), g(y'))$ are algebraic spaces (see Lemma 94.10.11) we see that $Isom(u, v)$ is an algebraic space.

Let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Consider the morphism

$$(Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf} \longrightarrow \mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}.$$

As the diagonal of $\mathcal{Z}$ is representable by algebraic spaces the source of this arrow is representable by an algebraic space F, see Lemma 94.10.11. Moreover, the morphism is the composition of base changes of x and y, hence surjective and smooth, see Lemmas 94.10.6 and 94.10.5. Choosing a scheme W and a surjective étale morphism $W \to F$ we see that the composition of the displayed 1-morphism with the corresponding 1-morphism

$$(Sch/W)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf}$$

is surjective and smooth which proves the last condition. □

04T2 **Lemma 94.14.3.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of algebraic stacks. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack. It is also the 2-fibre product in the 2-category of algebraic stacks over $(Sch/S)_{fppf}$.*

Proof. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack follows from the stronger Lemma 94.14.2. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is a 2-fibre product in the 2-category of algebraic stacks over S follows formally from the fact that the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of stacks in groupoids over $(Sch/S)_{fppf}$. □

94.15. Algebraic stacks, overhauled

04T0 Some basic results on algebraic stacks.

04T1 **Lemma 94.15.1.** *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. Let $V \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $y : (Sch/V)_{fppf} \to \mathcal{Y}$ be surjective and smooth. Then there exists an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 2-commutative diagram*

$$\begin{array}{ccc} (Sch/U)_{fppf} & \xrightarrow{a} & (Sch/V)_{fppf} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

with x surjective and smooth.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective smooth 1-morphism $z : (Sch/W)_{fppf} \to \mathcal{X}$. As $\mathcal{Y}$ is an algebraic stack we may choose an equivalence

$$j : \mathcal{S}_F \longrightarrow (Sch/W)_{fppf} \times_{f \circ z, \mathcal{Y}, y} (Sch/V)_{fppf}$$

where F is an algebraic space. By Lemma 94.10.6 the morphism $\mathcal{S}_F \to (Sch/W)_{fppf}$ is surjective and smooth as a base change of y. Hence by Lemma 94.10.5 we see that $\mathcal{S}_F \to \mathcal{X}$ is surjective and smooth. Choose an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$. Then applying Lemma 94.10.5 once more we obtain the desired properties. □

This lemma is a generalization of Proposition 94.13.3.

04Y5 **Lemma 94.15.2.** *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. The following are equivalent:*

1. *for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ the functor $f : \mathcal{X}_U \to \mathcal{Y}_U$ is faithful,*
2. *the functor f is faithful, and*
3. *f is representable by algebraic spaces.*

Proof. Parts (1) and (2) are equivalent by general properties of 1-morphisms of categories fibred in groupoids, see Categories, Lemma 4.35.9. We see that (3) implies (2) by Lemma 94.9.2. Finally, assume (2). Let U be a scheme. Let $y \in \mathrm{Ob}(\mathcal{Y}_U)$. We have to prove that

$$\mathcal{W} = (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

is representable by an algebraic space over U. Since $(Sch/U)_{fppf}$ is an algebraic stack we see from Lemma 94.14.3 that $\mathcal{W}$ is an algebraic stack. On the other hand the explicit description of objects of $\mathcal{W}$ as triples $(V, x, \alpha : y(V) \to f(x))$ and the

fact that f is faithful, shows that the fibre categories of $\mathcal{W}$ are setoids. Hence Proposition 94.13.3 guarantees that $\mathcal{W}$ is representable by an algebraic space. □

05UL **Lemma 94.15.3.** *Let S be a scheme contained in Sch_{fppf}. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{U}$ is representable by an algebraic space, and*
(2) *u is representable by algebraic spaces, surjective and smooth,*

then $\mathcal{X}$ is an algebraic stack over S.

Proof. We have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, see Definition 94.12.1. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Lemma 94.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

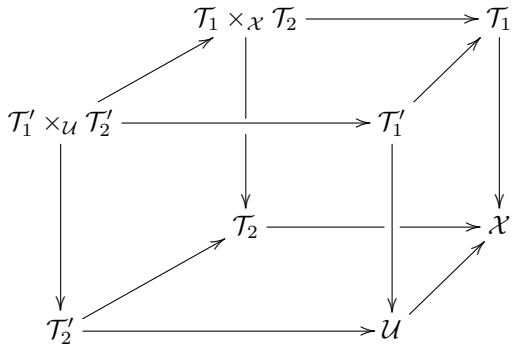

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{array}{ccc} \mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X} \end{array}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 94.9.6, 94.9.7, 94.10.4, and 94.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 80.10.1 and we win. □

An application of Lemma 94.15.3 is that something which is an algebraic space over an algebraic stack is an algebraic stack. This is the analogue of Bootstrap, Lemma 80.3.6. Actually, it suffices to assume the morphism $\mathcal{X} \to \mathcal{Y}$ is "algebraic", as we will see in Criteria for Representability, Lemma 97.8.2.

05UM **Lemma 94.15.4.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that*

(1) *$\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, and*
(2) *$\mathcal{Y}$ is an algebraic stack over S.*

Then $\mathcal{X}$ is an algebraic stack over S.

Proof. Let $\mathcal{V} \to \mathcal{Y}$ be a surjective smooth 1-morphism from a representable stack in groupoids to $\mathcal{Y}$. This exists by Definition 94.12.1. Then the 2-fibre product $\mathcal{U} = \mathcal{V} \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space by Lemma 94.9.8. The 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 94.9.7 and 94.10.6. By Lemma 94.15.3 we conclude that $\mathcal{X}$ is an algebraic stack. □

05UN **Lemma 94.15.5.** *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces. Then, if $\mathcal{Y}$ is a stack in groupoids (resp. an algebraic stack), so is $\mathcal{X}$.*

Removing the hypothesis that j is a monomorphism was observed in an email from Matthew Emerton dates June 15, 2016

Proof. The statement on algebraic stacks will follow from the statement on stacks in groupoids by Lemma 94.15.4. If j is representable by algebraic spaces, then j is faithful on fibre categories and for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf

$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$

is an algebraic space over U. See Lemma 94.9.2. In particular this presheaf is a sheaf and the conclusion follows from Stacks, Lemma 8.6.11. □

94.16. From an algebraic stack to a presentation

04T3 Given an algebraic stack over S we obtain a groupoid in algebraic spaces over S whose associated quotient stack is the algebraic stack.

Recall that if (U, R, s, t, c) is a groupoid in algebraic spaces over S then $[U/R]$ denotes the quotient stack associated to this datum, see Groupoids in Spaces, Definition 78.20.1. In general $[U/R]$ is **not** an algebraic stack. In particular the stack $[U/R]$ occurring in the following lemma is in general not algebraic.

04T4 **Lemma 94.16.1.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{U}$ be an algebraic stack over S which is representable by an algebraic space. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism. Then*

(1) *the 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space,*
(2) *there is a canonical equivalence*
$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R},$$
(3) *the projection pr_{02} induces via (2) a 1-morphism*
$$pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \longrightarrow \mathcal{R}$$
(4) *let U, R be the algebraic spaces representing $\mathcal{U}, \mathcal{R}$ and $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ are the morphisms corresponding to the 1-morphisms $pr_0, pr_1 : \mathcal{R} \to \mathcal{U}$ and $pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \to \mathcal{R}$ above, then the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over S,*
(5) *the morphism f induces a canonical 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ of stacks in groupoids over $(Sch/S)_{fppf}$, and*

(6) *the* 1*-morphism* $f_{can} : [U/R] \to \mathcal{X}$ *is fully faithful.*

Proof. Proof of (1). By definition $\Delta_{\mathcal{X}}$ is representable by algebraic spaces so Lemma 94.10.11 applies to show that $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence the result follows from Lemma 94.9.8.

Let T be a scheme over S. By construction of the 2-fibre product (see Categories, Lemma 4.32.3) we see that the objects of the fibre category $\mathcal{R}_T$ are triples (a, b, α) where $a, b \in \mathrm{Ob}(\mathcal{U}_T)$ and $\alpha : f(a) \to f(b)$ is a morphism in the fibre category $\mathcal{X}_T$.

Proof of (2). The equivalence comes from repeatedly applying Categories, Lemmas 4.31.8 and 4.31.10. Let us identify $\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ with $(\mathcal{U} \times_{\mathcal{X}} \mathcal{U}) \times_{\mathcal{X}} \mathcal{U}$. If T is a scheme over S, then on fibre categories over T this equivalence maps the object $((a, b, \alpha), c, \beta)$ on the left hand side to the object $((a, b, \alpha), (b, c, \beta))$ of the right hand side.

Proof of (3). The 1-morphism pr_{02} is constructed in the proof of Categories, Lemma 4.31.9. In terms of the description of objects of the fibre category above we see that $((a, b, \alpha), (b, c, \beta))$ maps to $(a, c, \beta \circ \alpha)$.

Unfortunately, this is *not compatible* with our conventions on groupoids where we always have $j = (t, s) : R \to U$, and we "think" of a T-valued point r of R as a morphism $r : s(r) \to t(r)$. However, this does not affect the proof of (4), since the opposite of a groupoid is a groupoid. But in the proof of (5) it is responsible for the inverses in the displayed formula below.

Proof of (4). Recall that the sheaf U is isomorphic to the sheaf $T \mapsto \mathrm{Ob}(\mathcal{U}_T)/\cong$, and similarly for R, see Lemma 94.8.2. It follows from Categories, Lemma 4.39.8 that this description is compatible with 2-fibre products so we get a similar matching of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ and $R \times_{s,U,t} R$. The morphisms $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ we get from the general equality (94.8.2.1). Explicitly these maps are the transformations of functors that come from letting pr_0, pr_0, pr_{02} act on isomorphism classes of objects of fibre categories. Hence to show that we obtain a groupoid in algebraic spaces it suffices to show that for every scheme T over S the structure

$$(\mathrm{Ob}(\mathcal{U}_T)/\cong, \mathrm{Ob}(\mathcal{R}_T)/\cong, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is a groupoid which is clear from our description of objects of $\mathcal{R}_T$ above.

Proof of (5). We will eventually apply Groupoids in Spaces, Lemma 78.23.2 to obtain the functor $[U/R] \to \mathcal{X}$. Consider the 1-morphism $f : \mathcal{U} \to \mathcal{X}$. We have a 2-arrow $\tau : f \circ \mathrm{pr}_1 \to f \circ \mathrm{pr}_0$ by definition of $\mathcal{R}$ as the 2-fibre product. Namely, on an object (a, b, α) of $\mathcal{R}$ over T it is the map $\alpha^{-1} : b \to a$. We claim that

$$\tau \circ \mathrm{id}_{\mathrm{pr}_{02}} = (\tau \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\tau \star \mathrm{id}_{\mathrm{pr}_1}).$$

This identity says that given an object $((a, b, \alpha), (b, c, \beta))$ of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ over T, then the composition of

$$c \xrightarrow{\beta^{-1}} b \xrightarrow{\alpha^{-1}} a$$

is the same as the arrow $(\beta \circ \alpha)^{-1} : a \to c$. This is clearly true, hence the claim holds. In this way we see that all the assumption of Groupoids in Spaces, Lemma 78.23.2 are satisfied for the structure $(\mathcal{U}, \mathcal{R}, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ and the 1-morphism f and the 2-morphism τ. Except, to apply the lemma we need to prove this holds for the structure $(\mathcal{S}_U, \mathcal{S}_R, s, t, c)$ with suitable morphisms.

Now there should be some general abstract nonsense argument which transfer these data between the two, but it seems to be quite long. Instead, we use the following trick. Pick a quasi-inverse $j^{-1} : \mathcal{S}_U \to \mathcal{U}$ of the canonical equivalence $j : \mathcal{U} \to \mathcal{S}_U$ which comes from $U(T) = \mathrm{Ob}(\mathcal{U}_T)/\cong$. This just means that for every scheme T/S and every object $a \in \mathcal{U}_T$ we have picked out a particular element of its isomorphism class, namely $j^{-1}(j(a))$. Using j^{-1} we may therefore see $\mathcal{S}_U$ as a subcategory of $\mathcal{U}$. Having chosen this subcategory we can consider those objects (a, b, α) of $\mathcal{R}_T$ such that a, b are objects of $(\mathcal{S}_U)_T$, i.e., such that $j^{-1}(j(a)) = a$ and $j^{-1}(j(b)) = b$. Then it is clear that this forms a subcategory of $\mathcal{R}$ which maps isomorphically to $\mathcal{S}_R$ via the canonical equivalence $\mathcal{R} \to \mathcal{S}_R$. Moreover, this is clearly compatible with forming the 2-fibre product $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$. Hence we see that we may simply restrict f to $\mathcal{S}_U$ and restrict τ to a transformation between functors $\mathcal{S}_R \to \mathcal{X}$. Hence it is clear that the displayed equality of Groupoids in Spaces, Lemma 78.23.2 holds since it holds even as an equality of transformations of functors $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R} \to \mathcal{X}$ before restricting to the subcategory $\mathcal{S}_{R \times_{s,U,t} R}$.

This proves that Groupoids in Spaces, Lemma 78.23.2 applies and we get our desired morphism of stacks $f_{can} : [U/R] \to \mathcal{X}$. We briefly spell out how f_{can} is defined in this special case. On an object a of $\mathcal{S}_U$ over T we have $f_{can}(a) = f(a)$, where we think of $\mathcal{S}_U \subset \mathcal{U}$ by the chosen embedding above. If a, b are objects of $\mathcal{S}_U$ over T, then a morphism $\varphi : a \to b$ in $[U/R]$ is by definition an object of the form $\varphi = (b, a, \alpha)$ of $\mathcal{R}$ over T. (Note switch.) And the rule in the proof of Groupoids in Spaces, Lemma 78.23.2 is that

04TG (94.16.1.1) $$f_{can}(\varphi) = \Big(f(a) \xrightarrow{\alpha^{-1}} f(b) \Big).$$

Proof of (6). Both $[U/R]$ and $\mathcal{X}$ are stacks. Hence given a scheme T/S and objects a, b of $[U/R]$ over T we obtain a transformation of fppf sheaves

$$Isom(a, b) \longrightarrow Isom(f_{can}(a), f_{can}(b))$$

on $(Sch/T)_{fppf}$. We have to show that this is an isomorphism. We may work fppf locally on T, hence we may assume that a, b come from morphisms $a, b : T \to U$. By the embedding $\mathcal{S}_U \subset \mathcal{U}$ above we may also think of a, b as objects of $\mathcal{U}$ over T. In Groupoids in Spaces, Lemma 78.22.1 we have seen that the left hand sheaf is represented by the algebraic space

$$R \times_{(t,s), U \times_S U, (b,a)} T$$

over T. On the other hand, the right hand side is by Stacks, Lemma 8.2.5 equal to the sheaf associated to the following stack in setoids:

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f \circ b, f \circ a)} T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f,f)} (\mathcal{U} \times \mathcal{U}) \times_{\mathcal{U} \times \mathcal{U}, (b,a)} T = \mathcal{R} \times_{(\mathrm{pr}_0, \mathrm{pr}_1), \mathcal{U} \times \mathcal{U}, (b,a)} T$$

which is representable by the fibre product displayed above. At this point we have shown that the two *Isom*-sheaves are isomorphic. Our 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ induces this isomorphism on *Isom*-sheaves by Equation (94.16.1.1). □

We can use the previous very abstract lemma to produce presentations.

04T5 **Lemma 94.16.2.** *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let U be an algebraic space over S. Let $f : \mathcal{S}_U \to \mathcal{X}$ be a surjective smooth morphism. Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ be the result of applying Lemma 94.16.1 to U and f. Then*

(1) *the morphisms s, t are smooth, and*
(2) *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence.*

Proof. The morphisms s, t are smooth by Lemmas 94.10.2 and 94.10.3. As the 1-morphism f is smooth and surjective it is clear that given any scheme T and any object $a \in \mathrm{Ob}(\mathcal{X}_T)$ there exists a smooth and surjective morphism $T' \to T$ such that $a|'_T$ comes from an object of $[U/R]_{T'}$. Since $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful, we deduce that $[U/R] \to \mathcal{X}$ is essentially surjective as descent data on objects are effective on both sides, see Stacks, Lemma 8.4.8. □

04WY **Remark 94.16.3.** If the morphism $f : \mathcal{S}_U \to \mathcal{X}$ of Lemma 94.16.2 is only assumed surjective, flat and locally of finite presentation, then it will still be the case that $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence. In this case the morphisms s, t will be flat and locally of finite presentation, but of course not smooth in general.

Lemma 94.16.2 suggests the following definitions.

04TH **Definition 94.16.4.** Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. We say (U, R, s, t, c) is a *smooth groupoid*[3] if $s, t : R \to U$ are smooth morphisms of algebraic spaces.

04TI **Definition 94.16.5.** Let $\mathcal{X}$ be an algebraic stack over S. A *presentation* of $\mathcal{X}$ is given by a smooth groupoid (U, R, s, t, c) in algebraic spaces over S, and an equivalence $f : [U/R] \to \mathcal{X}$.

We have seen above that every algebraic stack has a presentation. Our next task is to show that every smooth groupoid in algebraic spaces over S gives rise to an algebraic stack.

94.17. The algebraic stack associated to a smooth groupoid

04TJ In this section we start with a smooth groupoid in algebraic spaces and we show that the associated quotient stack is an algebraic stack.

04WZ **Lemma 94.17.1.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Then the diagonal of $[U/R]$ is representable by algebraic spaces.*

Proof. It suffices to show that the *Isom*-sheaves are algebraic spaces, see Lemma 94.10.11. This follows from Bootstrap, Lemma 80.11.5. □

04X0 **Lemma 94.17.2.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the morphism $\mathcal{S}_U \to [U/R]$ is smooth and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Lemmas 94.17.1 and 94.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective and smooth.

[3] This terminology might be a bit confusing: it does not imply that $[U/R]$ is smooth over anything.

Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.31.10 and the second equality by Groupoids in Spaces, Lemma 78.22.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is smooth as the base change of the smooth morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the main result of this section.

04TK **Theorem 94.17.3.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the three conditions of Definition 94.12.1. By construction we have that $[U/R]$ is a stack in groupoids which is the first condition.

The second condition follows from the stronger Lemma 94.17.1.

Finally, we have to show there exists a scheme W over S and a surjective smooth 1-morphism $(Sch/W)_{fppf} \longrightarrow \mathcal{X}$. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to U$. Note that this gives a surjective étale morphism $\mathcal{S}_W \to \mathcal{S}_U$ of categories fibred in sets, see Lemma 94.10.3. Of course then $\mathcal{S}_W \to \mathcal{S}_U$ is also surjective and smooth, see Lemma 94.10.9. Hence $\mathcal{S}_W \to \mathcal{S}_U \to [U/R]$ is surjective and smooth by a combination of Lemmas 94.17.2 and 94.10.5. □

94.18. Change of big site

04X1 In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. We encourage the reader to skip this section.

Pullbacks of stacks is defined in Stacks, Section 8.12.

04X2 **Lemma 94.18.1.** *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 34.12. Let S be an object of Sch_{fppf}. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : (Sch/S)_{fppf} \to (Sch'/S)_{fppf}$. Let $\mathcal{X}$ be a stack in groupoids over $(Sch/S)_{fppf}$.*

1. *if $\mathcal{X}$ is representable by some $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $f^{-1}\mathcal{X}$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$,*
2. *if $\mathcal{X}$ is representable by $F \in Sh((Sch/S)_{fppf})$ which is an algebraic space, then $f^{-1}\mathcal{X}$ is representable by the algebraic space $f^{-1}F$,*

(3) *if $\mathcal{X}$ is an algebraic stack, then $f^{-1}\mathcal{X}$ is an algebraic stack, and*
(4) *if $\mathcal{X}$ is a Deligne-Mumford stack, then $f^{-1}\mathcal{X}$ is a Deligne-Mumford stack too.*

Proof. Let us prove (3). By Lemma 94.16.2 we may write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c). By Groupoids in Spaces, Lemma 78.28.1 we see that $f^{-1}[U/R] = [f^{-1}U/f^{-1}R]$. Of course $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a smooth groupoid in algebraic spaces too. Hence (3) is proved.

Now the other cases (1), (2), (4) each mean that $\mathcal{X}$ has a presentation $[U/R]$ of a particular kind, and hence translate into the same kind of presentation for $f^{-1}\mathcal{X} = [f^{-1}U/f^{-1}R]$. Whence the lemma is proved. □

It is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

04X3 **Lemma 94.18.2.** *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Algebraic-Stacks/S the 2-category of algebraic stacks over S defined using Sch_{fppf}. Similarly, denote Algebraic-Stacks$'$/S the 2-category of algebraic stacks over S defined using Sch'_{fppf}. The rule $\mathcal{X} \mapsto f^{-1}\mathcal{X}$ of Lemma 94.18.1 defines a functor of 2-categories*

$$\textit{Algebraic-Stacks}/S \longrightarrow \textit{Algebraic-Stacks}'/S$$

which defines equivalences of morphism categories

$$\mathrm{Mor}_{\textit{Algebraic-Stacks}/S}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Mor}_{\textit{Algebraic-Stacks}'/S}(f^{-1}\mathcal{X}, f^{-1}\mathcal{Y})$$

for every objects $\mathcal{X}, \mathcal{Y}$ of Algebraic-Stacks/S. An object $\mathcal{X}'$ of Algebraic-Stacks$'$/S is equivalence to $f^{-1}\mathcal{X}$ for some $\mathcal{X}$ in Algebraic-Stacks/S if and only if it has a presentation $\mathcal{X} = [U'/R']$ with U', R' isomorphic to $f^{-1}U$, $f^{-1}R$ for some $U, R \in$ Spaces/S.

Proof. The statement on morphism categories is a consequence of the more general Stacks, Lemma 8.12.12. The characterization of the "essential image" follows from the description of f^{-1} in the proof of Lemma 94.18.1. □

94.19. Change of base scheme

04X4 In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S'.

04X5 **Lemma 94.19.1.** *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The constructions A and B of Stacks, Section 8.13 above give isomorphisms of 2-categories*

$$\left\{\begin{array}{c}\text{2-category of algebraic}\\ \text{stacks } \mathcal{X} \text{ over } S\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{2-category of pairs } (\mathcal{X}', f) \text{ consisting of an}\\ \text{algebraic stack } \mathcal{X}' \text{ over } S' \text{ and a morphism}\\ f : \mathcal{X}' \to (Sch/S)_{fppf} \text{ of algebraic stacks over } S'\end{array}\right\}$$

Proof. The statement makes sense as the functor $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ is the localization functor associated to the object S/S' of $(Sch/S')_{fppf}$. By Stacks, Lemma 8.13.2 the only thing to show is that the constructions A and B preserve

the subcategories of algebraic stacks. For example, if $\mathcal{X} = [U/R]$ then construction A applied to $\mathcal{X}$ just produces $\mathcal{X}' = \mathcal{X}$. Conversely, if $\mathcal{X}' = [U'/R']$ the morphism p induces morphisms of algebraic spaces $U' \to S$ and $R' \to S$, and then $\mathcal{X} = [U'/R']$ but now viewed as a stack over S. Hence the lemma is clear. □

04X6 **Definition 94.19.2.** Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. If $p : \mathcal{X} \to (Sch/S)_{fppf}$ is an algebraic stack over S, then $\mathcal{X}$ *viewed as an algebraic stack over* S' is the algebraic stack

$$\mathcal{X} \longrightarrow (Sch/S')_{fppf}$$

gotten by applying construction A of Lemma 94.19.1 to $\mathcal{X}$.

Conversely, what if we start with an algebraic stack $\mathcal{X}'$ over S' and we want to get an algebraic stack over S? Well, then we consider the 2-fibre product

$$\mathcal{X}'_S = (Sch/S)_{fppf} \times_{(Sch/S')_{fppf}} \mathcal{X}'$$

which is an algebraic stack over S' according to Lemma 94.14.3. Moreover, it comes equipped with a natural 1-morphism $p : \mathcal{X}'_S \to (Sch/S)_{fppf}$ and hence by Lemma 94.19.1 it corresponds in a canonical way to an algebraic stack over S.

04X7 **Definition 94.19.3.** Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let $\mathcal{X}'$ be an algebraic stack over S'. The *change of base of* $\mathcal{X}'$ is the algebraic stack $\mathcal{X}'_S$ over S described above.

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CHAPTER 95

Examples of Stacks

04SL

95.1. Introduction

04SM This is a discussion of examples of stacks in algebraic geometry. Some of them are algebraic stacks, some are not. We will discuss which are algebraic stacks in a later chapter. This means that in this chapter we mainly worry about the descent conditions. See [**Vis04**] for example.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

95.2. Notation

04SN In this chapter we fix a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

95.3. Examples of stacks

04SQ We first give some important examples of stacks over $(Sch/S)_{fppf}$.

95.4. Quasi-coherent sheaves

03YL We define a category $\mathcal{QC}oh$ as follows:

(1) An object of $\mathcal{QC}oh$ is a pair $(X, \mathcal{F})$, where X/S is an object of $(Sch/S)_{fppf}$, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and
(2) a morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair consisting of a morphism $f : Y \to X$ of schemes over S and an f-map (see Sheaves, Section 6.26) $\varphi : \mathcal{F} \to \mathcal{G}$.
(3) The composition of morphisms

$$(Z, \mathcal{H}) \xrightarrow{(g,\psi)} (Y, \mathcal{G}) \xrightarrow{(f,\phi)} (X, \mathcal{F})$$

is $(f \circ g, \psi \circ \phi)$ where $\psi \circ \phi$ is the composition of f-maps.

Thus $\mathcal{QC}oh$ is a category and

$$p : \mathcal{QC}oh \to (Sch/S)_{fppf}, \quad (X, \mathcal{F}) \mapsto X$$

is a functor. Note that the fibre category of $\mathcal{QC}oh$ over a scheme X is the opposite of the category $QCoh(\mathcal{O}_X)$ of quasi-coherent $\mathcal{O}_X$-modules. We remark for later use

that given $(X, \mathcal{F}), (Y, \mathcal{G}) \in \mathrm{Ob}(\mathcal{QC}oh)$ we have

04U2 (95.4.0.1) $$\mathrm{Mor}_{\mathcal{QC}oh}((Y, \mathcal{G}), (X, \mathcal{F})) = \coprod_{f \in \mathrm{Mor}_S(Y,X)} \mathrm{Mor}_{QCoh(\mathcal{O}_Y)}(f^*\mathcal{F}, \mathcal{G})$$

See the discussion on f-maps of modules in Sheaves, Section 6.26.

The category $\mathcal{QC}oh$ is not a stack over $(Sch/S)_{fppf}$ because its collection of objects is a proper class. On the other hand we will see that it does satisfy all the axioms of a stack. We will get around the set theoretical issue in Section 95.5.

04U3 **Lemma 95.4.1.** *A morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of $\mathcal{QC}oh$ is strongly cartesian if and only if the map φ induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$.*

Proof. Let $(X, \mathcal{F}) \in \mathrm{Ob}(\mathcal{QC}oh)$. Let $f : Y \to X$ be a morphism of $(Sch/S)_{fppf}$. Note that there is a canonical f-map $c : \mathcal{F} \to f^*\mathcal{F}$ and hence we get a morphism $(f, c) : (Y, f^*\mathcal{F}) \to (X, \mathcal{F})$. We claim that (f, c) is strongly cartesian. Namely, for any object $(Z, \mathcal{H})$ of $\mathcal{QC}oh$ we have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{QC}oh}((Z, \mathcal{H}), (Y, f^*\mathcal{F})) &= \coprod_{g \in \mathrm{Mor}_S(Z,Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}(g^*f^*\mathcal{F}, \mathcal{H}) \\
&= \coprod_{g \in \mathrm{Mor}_S(Z,Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}((f \circ g)^*\mathcal{F}, \mathcal{H}) \\
&= \mathrm{Mor}_{\mathcal{QC}oh}((Z, \mathcal{H}), (X, \mathcal{F})) \times_{\mathrm{Mor}_S(Z,X)} \mathrm{Mor}_S(Z, Y)
\end{aligned}$$

where we have used Equation (95.4.0.1) twice. This proves that the condition of Categories, Definition 4.33.1 holds for (f, c), and hence our claim is true. Now by Categories, Lemma 4.33.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the "if" part of the lemma. For the converse, note that given $(X, \mathcal{F})$ and $f : Y \to X$, if there exists a strongly cartesian morphism lifting f with target $(X, \mathcal{F})$ then it has to be isomorphic to (f, c) (see discussion following Categories, Definition 4.33.1). Hence the "only if" part of the lemma holds. □

03YM **Lemma 95.4.2.** *The functor $p : \mathcal{QC}oh \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. It is clear from Lemma 95.4.1 that $\mathcal{QC}oh$ is a fibred category over $(Sch/S)_{fppf}$. Given covering $\mathcal{U} = \{X_i \to X\}_{i \in I}$ of $(Sch/S)_{fppf}$ the functor

$$QCoh(\mathcal{O}_X) \longrightarrow DD(\mathcal{U})$$

is fully faithful and essentially surjective, see Descent, Proposition 35.5.2. Hence Stacks, Lemma 8.4.2 applies to show that $\mathcal{QC}oh$ satisfies all the axioms of a stack. □

95.5. The stack of finitely generated quasi-coherent sheaves

0404 It turns out that we can get a stack of quasi-coherent sheaves if we only consider finite type quasi-coherent modules. Let us denote

$$p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$$

the full subcategory of $\mathcal{QC}oh$ over $(Sch/S)_{fppf}$ consisting of pairs $(T, \mathcal{F})$ such that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_T$-module of finite type.

04U4 **Lemma 95.5.1.** *The functor $p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. We will verify assumptions (1), (2), (3) of Stacks, Lemma 8.4.3 to prove this. By Lemma 95.4.1 a morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is strongly cartesian if and only if it induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$. By Modules, Lemma 17.9.2 the pullback of a finite type $\mathcal{O}_X$-module is of finite type. Hence assumption (1) of Stacks, Lemma 8.4.3 holds. Assumption (2) holds trivially. Finally, to prove assumption (3) we have to show: If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\{f_i : X_i \to X\}$ is an fppf covering such that each $f_i^*\mathcal{F}$ is of finite type, then $\mathcal{F}$ is of finite type. Considering the restriction of $\mathcal{F}$ to an affine open of X this reduces to the following algebra statement: Suppose that $R \to S$ is a finitely presented, faithfully flat ring map and M an R-module. If $M \otimes_R S$ is a finitely generated S-module, then M is a finitely generated R-module. A stronger form of the algebra fact can be found in Algebra, Lemma 10.83.2. □

04U5 **Lemma 95.5.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *The category of finite type $\mathcal{O}_X$-modules has a set of isomorphism classes.*
(2) *The category of finite type quasi-coherent $\mathcal{O}_X$-modules has a set of isomorphism classes.*

Proof. Part (2) follows from part (1) as the category in (2) is a full subcategory of the category in (1). Consider any open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Denote $j_i : U_i \to X$ the inclusion maps. Consider any map $r : I \to \mathbf{N}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module whose restriction to U_i is generated by at most $r(i)$ sections from $\mathcal{F}(U_i)$, then $\mathcal{F}$ is a quotient of the sheaf

$$\mathcal{H}_{\mathcal{U},r} = \bigoplus\nolimits_{i \in I} j_{i,!}\mathcal{O}_{U_i}^{\oplus r(i)}$$

By definition, if $\mathcal{F}$ is of finite type, then there exists some open covering with $\mathcal{U}$ whose index set is $I = X$ such that this condition is true. Hence it suffices to show that there is a set of possible choices for $\mathcal{U}$ (obvious), a set of possible choices for $r : I \to \mathbf{N}$ (obvious), and a set of possible quotient modules of $\mathcal{H}_{\mathcal{U},r}$ for each $\mathcal{U}$ and r. In other words, it suffices to show that given an $\mathcal{O}_X$-module $\mathcal{H}$ there is at most a set of isomorphism classes of quotients. This last assertion becomes obvious by thinking of the kernels of a quotient map $\mathcal{H} \to \mathcal{F}$ as being parametrized by a subset of the power set of $\prod_{U \subset X \text{ open}} \mathcal{H}(U)$. □

04U6 **Lemma 95.5.3.** *There exists a subcategory $\mathcal{QC}oh_{fg,small} \subset \mathcal{QC}oh_{fg}$ with the following properties:*

(1) *the inclusion functor $\mathcal{QC}oh_{fg,small} \to \mathcal{QC}oh_{fg}$ is fully faithful and essentially surjective, and*
(2) *the functor $p_{fg,small} : \mathcal{QC}oh_{fg,small} \to (Sch/S)_{fppf}$ turns $\mathcal{QC}oh_{fg,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 95.5.1 and 95.5.2 that $p_{fg} : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1 as well as the additional condition (4) of Stacks, Remark 8.4.9. Hence we obtain $\mathcal{QC}oh_{fg,small}$ from the discussion in that remark. □

We will often perform the replacement

$$\mathcal{QC}oh_{fg} \rightsquigarrow \mathcal{QC}oh_{fg,small}$$

without further remarking on it, and by abuse of notation we will simply denote $\mathcal{QC}oh_{fg}$ this replacement.

04U7 **Remark 95.5.4.** Note that the whole discussion in this section works if we want to consider those quasi-coherent sheaves which are locally generated by at most κ sections, for some infinite cardinal κ, e.g., $\kappa = \aleph_0$.

95.6. Finite étale covers

0BLY We define a category $F\acute{E}t$ as follows:

(1) An object of $F\acute{E}t$ is a finite étale morphism $Y \to X$ of schemes (by our conventions this means a finite étale morphism in $(Sch/S)_{fppf}$),
(2) A morphism $(b, a) : (Y \to X) \to (Y' \to X')$ of $F\acute{E}t$ is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow[b]{} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow[a]{} & X' \end{array}$$

in the category of schemes.

Thus $F\acute{E}t$ is a category and

$$p : F\acute{E}t \to (Sch/S)_{fppf}, \quad (Y \to X) \mapsto X$$

is a functor. Note that the fibre category of $F\acute{E}t$ over a scheme X is just the category $F\acute{E}t_X$ studied in Fundamental Groups, Section 58.5.

0BLZ **Lemma 95.6.1.** *The functor*

$$p : F\acute{E}t \longrightarrow (Sch/S)_{fppf}$$

defines a stack over $(Sch/S)_{fppf}$.

Proof. Fppf descent for finite étale morphisms follows from Descent, Lemmas 35.37.1, 35.23.25, and 35.23.31. Details omitted. □

95.7. Algebraic spaces

04SP We define a category $\mathcal{S}paces$ as follows:

(1) An object of $\mathcal{S}paces$ is a morphism $X \to U$ of algebraic spaces over S, where U is representable by an object of $(Sch/S)_{fppf}$, and
(2) a morphism $(f, g) : (X \to U) \to (Y \to V)$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[f]{} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & V \end{array}$$

of morphisms of algebraic spaces over S.

Thus $\mathcal{S}paces$ is a category and

$$p : \mathcal{S}paces \to (Sch/S)_{fppf}, \quad (X \to U) \mapsto U$$

is a functor. Note that the fibre category of $\mathcal{S}paces$ over a scheme U is just the category $Spaces/U$ of algebraic spaces over U (see Topologies on Spaces, Section 73.2). Hence we sometimes think of an object of $\mathcal{S}paces$ as a pair X/U consisting of

a scheme U and an algebraic space X over U. We remark for later use that given $(X/U), (Y/V) \in \mathrm{Ob}(\mathit{Spaces})$ we have

04U8 (95.7.0.1) $$\mathrm{Mor}_{\mathit{Spaces}}(X/U, Y/V) = \coprod_{g \in \mathrm{Mor}_S(U,V)} \mathrm{Mor}_{\mathit{Spaces}/U}(X, U \times_{g,V} Y)$$

The category Spaces is almost, but not quite a stack over $(\mathit{Sch}/S)_{fppf}$. The problem is a set theoretical issue as we will explain below.

04U9 **Lemma 95.7.1.** *A morphism $(f, g) : X/U \to Y/V$ of Spaces is strongly cartesian if and only if the map f induces an isomorphism $X \to U \times_{g,V} Y$.*

Proof. Let $Y/V \in \mathrm{Ob}(\mathit{Spaces})$. Let $g : U \to V$ be a morphism of $(\mathit{Sch}/S)_{fppf}$. Note that the projection $p : U \times_{g,V} Y \to Y$ gives rise a morphism $(p, g) : U \times_{g,V} Y/U \to Y/V$ of Spaces. We claim that (p, g) is strongly cartesian. Namely, for any object Z/W of Spaces we have

$$\begin{aligned}
\mathrm{Mor}_{\mathit{Spaces}}(Z/W, U \times_{g,V} Y/U) &= \coprod_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{\mathit{Spaces}/W}(Z, W \times_{h,U} U \times_{g,V} Y) \\
&= \coprod_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{\mathit{Spaces}/W}(Z, W \times_{g \circ h,V} Y) \\
&= \mathrm{Mor}_{\mathit{Spaces}}(Z/W, Y/V) \times_{\mathrm{Mor}_S(W,V)} \mathrm{Mor}_S(W, U)
\end{aligned}$$

where we have used Equation (95.7.0.1) twice. This proves that the condition of Categories, Definition 4.33.1 holds for (p, g), and hence our claim is true. Now by Categories, Lemma 4.33.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the "if" part of the lemma. For the converse, note that given Y/V and $g : U \to V$, if there exists a strongly cartesian morphism lifting g with target Y/V then it has to be isomorphic to (p, g) (see discussion following Categories, Definition 4.33.1). Hence the "only if" part of the lemma holds. □

04UA **Lemma 95.7.2.** *The functor $p : \mathit{Spaces} \to (\mathit{Sch}/S)_{fppf}$ satisfies conditions (1) and (2) of Stacks, Definition 8.4.1.*

Proof. It is follows from Lemma 95.7.1 that Spaces is a fibred category over $(\mathit{Sch}/S)_{fppf}$ which proves (1). Suppose that $\{U_i \to U\}_{i \in I}$ is a covering of $(\mathit{Sch}/S)_{fppf}$. Suppose that X, Y are algebraic spaces over U. Finally, suppose that $\varphi_i : X_{U_i} \to Y_{U_i}$ are morphisms of Spaces/U_i such that φ_i and φ_j restrict to the same morphisms $X_{U_i \times_U U_j} \to Y_{U_i \times_U U_j}$ of algebraic spaces over $U_i \times_U U_j$. To prove (2) we have to show that there exists a unique morphism $\varphi : X \to Y$ over U whose base change to U_i is equal to φ_i. As a morphism from X to Y is the same thing as a map of sheaves this follows directly from Sites, Lemma 7.26.1. □

04UB **Remark 95.7.3.** Ignoring set theoretical difficulties[1] Spaces also satisfies descent for objects and hence is a stack. Namely, we have to show that given

(1) an fppf covering $\{U_i \to U\}_{i \in I}$,
(2) for each $i \in I$ an algebraic space X_i/U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

[1] The difficulty is not that Spaces is a proper class, since by our definition of an algebraic space over S there is only a set worth of isomorphism classes of algebraic spaces over S. It is rather that arbitrary disjoint unions of algebraic spaces may end up being too large, hence lie outside of our chosen "partial universe" of sets.

there exists an algebraic space X/U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. First, note that by Sites, Lemma 7.26.4 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 80.11.1 we see that X is an algebraic space (if we ignore the set theoretic condition of that lemma). We will use this argument in the next section to show that if we consider only algebraic spaces of finite type, then we obtain a stack.

95.8. The stack of finite type algebraic spaces

04UC It turns out that we can get a stack of spaces if we only consider spaces of finite type. Let us denote

$$p_{ft} : \mathcal{S}paces_{ft} \to (Sch/S)_{fppf}$$

the full subcategory of $\mathcal{S}paces$ over $(Sch/S)_{fppf}$ consisting of pairs X/U such that $X \to U$ is a morphism of finite type.

04UD **Lemma 95.8.1.** *The functor $p_{ft} : \mathcal{S}paces_{ft} \to (Sch/S)_{fppf}$ satisfies the conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. We are going to write this out in ridiculous detail (which may make it hard to see what is going on).

We have seen in Lemma 95.7.1 that a morphism $(f, g) : X/U \to Y/V$ of $\mathcal{S}paces$ is strongly cartesian if the induced morphism $f : X \to U \times_V Y$ is an isomorphism. Note that if $Y \to V$ is of finite type then also $U \times_V Y \to U$ is of finite type, see Morphisms of Spaces, Lemma 67.23.3. So if $(f, g) : X/U \to Y/V$ of $\mathcal{S}paces$ is strongly cartesian in $\mathcal{S}paces$ and Y/V is an object of $\mathcal{S}paces_{ft}$ then automatically also X/U is an object of $\mathcal{S}paces_{ft}$, and of course (f, g) is also strongly cartesian in $\mathcal{S}paces_{ft}$. In this way we conclude that $\mathcal{S}paces_{ft}$ is a fibred category over $(Sch/S)_{fppf}$. This proves (1).

The argument above also shows that the inclusion functor $\mathcal{S}paces_{ft} \to \mathcal{S}paces$ transforms strongly cartesian morphisms into strongly cartesian morphisms. In other words $\mathcal{S}paces_{ft} \to \mathcal{S}paces$ is a 1-morphism of fibred categories over $(Sch/S)_{fppf}$.

Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Let X, Y be algebraic spaces of finite type over U. By Stacks, Lemma 8.2.3 we obtain a map of presheaves

$$Mor_{\mathcal{S}paces_{ft}}(X, Y) \longrightarrow Mor_{\mathcal{S}paces}(X, Y)$$

which is an isomorphism as $\mathcal{S}paces_{ft}$ is a full subcategory of $\mathcal{S}paces$. Hence the left hand side is a sheaf, because in Lemma 95.7.2 we showed the right hand side is a sheaf. This proves (2).

To prove condition (3) of Stacks, Definition 8.4.1 we have to show the following: Given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,
(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. This follows from Bootstrap, Lemma 80.11.3 part (2). By Descent on Spaces, Lemma 74.11.10 we see that $X \to U$ is of finite type which concludes the proof. □

04UE **Lemma 95.8.2.** *There exists a subcategory $\mathcal{S}paces_{ft,small} \subset \mathcal{S}paces_{ft}$ with the following properties:*

(1) *the inclusion functor $\mathcal{S}paces_{ft,small} \to \mathcal{S}paces_{ft}$ is fully faithful and essentially surjective, and*
(2) *the functor $p_{ft,small} : \mathcal{S}paces_{ft,small} \to (Sch/S)_{fppf}$ turns $\mathcal{S}paces_{ft,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 95.8.1 that $p_{ft} : \mathcal{S}paces_{ft} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1. The additional condition (4) of Stacks, Remark 8.4.9 holds because every algebraic space X over S is of the form U/R for $U, R \in \mathrm{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 65.9.1. Thus there is only a set worth of isomorphism classes of objects. Hence we obtain $\mathcal{S}paces_{ft,small}$ from the discussion in that remark. □

We will often perform the replacement

$$\mathcal{S}paces_{ft} \rightsquigarrow \mathcal{S}paces_{ft,small}$$

without further remarking on it, and by abuse of notation we will simply denote $\mathcal{S}paces_{ft}$ this replacement.

04UF **Remark 95.8.3.** Note that the whole discussion in this section works if we want to consider those algebraic spaces X/U which are locally of finite type such that the inverse image in X of an affine open of U can be covered by countably many affines. If needed we can also introduce the notion of a morphism of κ-type (meaning some bound on the number of generators of ring extensions and some bound on the cardinality of the affines over a given affine in the base) where κ is a cardinal, and then we can produce a stack

$$\mathcal{S}paces_\kappa \longrightarrow (Sch/S)_{fppf}$$

in exactly the same manner as above (provided we make sure that Sch is large enough depending on κ).

95.9. Examples of stacks in groupoids

04UG The examples above are examples of stacks which are not stacks in groupoids. In the rest of this chapter we give algebraic geometric examples of stacks in groupoids.

95.10. The stack associated to a sheaf

0305 Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a presheaf. We obtain a category fibred in sets

$$p_F : \mathcal{S}_F \to (Sch/S)_{fppf},$$

see Categories, Example 4.38.5. This is a stack in sets if and only if F is a sheaf, see Stacks, Lemma 8.6.3.

95.11. The stack in groupoids of finitely generated quasi-coherent sheaves

03YN Let $p : \mathcal{QC}oh_{fg} \to (Sch/S)_{fppf}$ be the stack introduced in Section 95.5 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : \mathcal{QC}oh'_{fg} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.35.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $\mathcal{QC}oh'_{fg}$ has the

same objects as $\mathcal{QC}oh_{fg}$ but the morphisms are pairs $(f, g) : (U, \mathcal{F}) \to (U', \mathcal{F}')$ where g is an isomorphism $g : f^*\mathcal{F}' \to \mathcal{F}$.

95.12. The stack in groupoids of finite type algebraic spaces

04UH Let $p : \mathcal{S}paces_{ft} \to (Sch/S)_{fppf}$ be the stack introduced in Section 95.8 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : \mathcal{S}paces'_{ft} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.35.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $\mathcal{S}paces'_{ft}$ has the same objects as $\mathcal{S}paces_{ft}$, i.e., finite type morphisms $X \to U$ where X is an algebraic space over S and U is a scheme over S. But the morphisms $(f, g) : X/U \to Y/V$ are now commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & V \end{array}$$

which are cartesian.

95.13. Quotient stacks

04UI Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. In this case the quotient stack

$$[U/R] \longrightarrow (Sch/S)_{fppf}$$

is a stack in groupoids by construction, see Groupoids in Spaces, Definition 78.20.1. It is even the case that the *Isom*-sheaves are representable by algebraic spaces, see Bootstrap, Lemma 80.11.5. These quotient stacks are of fundamental importance to the theory of algebraic stacks.

A special case of the construction above is the quotient stack

$$[X/G] \longrightarrow (Sch/S)_{fppf}$$

associated to a datum $(B, G/B, m, X/B, a)$. Here

(1) B is an algebraic space over S,
(2) (G, m) is a group algebraic space over B,
(3) X is an algebraic space over B, and
(4) $a : G \times_B X \to X$ is an action of G on X over B.

Namely, by Groupoids in Spaces, Definition 78.20.1 the stack in groupoids $[X/G]$ is the quotient stack $[X/G \times_B X]$ given above. It behooves us to spell out what the category $[X/G]$ really looks like. We will do this in Section 95.15.

95.14. Classifying torsors

036Z We want to carefully explain a number of variants of what it could mean to study the stack of torsors for a group algebraic space G or a sheaf of groups $\mathcal{G}$.

04UJ **95.14.1. Torsors for a sheaf of groups.** Let $\mathcal{G}$ be a sheaf of groups on $(Sch/S)_{fppf}$. For $U \in \mathrm{Ob}((Sch/S)_{fppf})$ we denote $\mathcal{G}|_U$ the restriction of $\mathcal{G}$ to $(Sch/U)_{fppf}$. We define a category $\mathcal{G}$-*Torsors* as follows:

(1) An object of $\mathcal{G}$-*Torsors* is a pair $(U, \mathcal{F})$ where U is an object of $(Sch/S)_{fppf}$ and $\mathcal{F}$ is a $\mathcal{G}|_U$-torsor, see Cohomology on Sites, Definition 21.4.1.
(2) A morphism $(U, \mathcal{F}) \to (V, \mathcal{H})$ is given by a pair (f, α), where $f : U \to V$ is a morphism of schemes over S, and $\alpha : f^{-1}\mathcal{H} \to \mathcal{F}$ is an isomorphism of $\mathcal{G}|_U$-torsors.

Thus $\mathcal{G}$-*Torsors* is a category and

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}$-*Torsors* over U is the category of $\mathcal{G}|_U$-torsors which is a groupoid.

04UK **Lemma 95.14.2.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a $\mathcal{G}|_{U_i}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $\mathcal{G}|_{U_i \times_U U_j}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.26 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $a : \mathcal{G}|_U \times \mathcal{F} \to \mathcal{F}$. Now we have to show that a is an action and that $\mathcal{F}$ becomes a $\mathcal{G}|_U$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}$-*Torsors*. Some details omitted. □

04UL **95.14.3. Variant on torsors for a sheaf.** The construction of Subsection 95.14.1 can be generalized slightly. Namely, let $\mathcal{G} \to \mathcal{B}$ be a map of sheaves on $(Sch/S)_{fppf}$ and let

$$m : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \longrightarrow \mathcal{G}$$

be a group law on $\mathcal{G}/\mathcal{B}$. In other words, the pair $(\mathcal{G}, m)$ is a group object of the topos $Sh((Sch/S)_{fppf})/\mathcal{B}$. See Sites, Section 7.30 for information regarding localizations of topoi. In this setting we can define a category $\mathcal{G}/\mathcal{B}$-*Torsors* as follows (where we use the Yoneda embedding to think of schemes as sheaves):

(1) An object of $\mathcal{G}/\mathcal{B}$-*Torsors* is a triple $(U, b, \mathcal{F})$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to \mathcal{B}$ is a section of $\mathcal{B}$ over U, and
   (c) $\mathcal{F}$ is a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor over U.
(2) A morphism $(U, b, \mathcal{F}) \to (U', b', \mathcal{F}')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over S such that $b = b' \circ f$, and $g : f^{-1}\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors.

Thus $\mathcal{G}/\mathcal{B}$-*Torsors* is a category and

$$p : \mathcal{G}/\mathcal{B}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}/\mathcal{B}$-*Torsors* over U is the disjoint union over $b : U \to \mathcal{B}$ of the categories of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors, hence is a groupoid.

In the special case $\mathcal{B} = S$ we recover the category $\mathcal{G}$-*Torsors* introduced in Subsection 95.14.1.

04UM **Lemma 95.14.4.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : \mathcal{G}/\mathcal{B}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. This proof is a repeat of the proof of Lemma 95.14.2. The reader is encouraged to read that proof first since the notation is less cumbersome. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a pair $(b_i, \mathcal{F}_i)$ consisting of a morphism $b_i : U_i \to \mathcal{B}$ and a $U_i \times_{b_i,\mathcal{B}} \mathcal{G}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ we have $b_i|_{U_i \times_U U_j} = b_j|_{U_i \times_U U_j}$ and we are given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $(U_i \times_U U_j) \times_{\mathcal{B}} \mathcal{G}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.26 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the sheaf axiom for $\mathcal{B}$ the morphisms b_i come from a unique morphism $b : U \to \mathcal{B}$. By the equivalence of categories in Sites, Lemma 7.26.5 the action maps $(U_i \times_{b_i,\mathcal{B}} \mathcal{G}) \times_{U_i} \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $(U \times_{b,\mathcal{B}} \mathcal{G}) \times \mathcal{F} \to \mathcal{F}$. Now we have to show that this is an action and that $\mathcal{F}$ becomes a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions on the $\mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}/\mathcal{B}$-*Torsors*. Some details omitted. □

04UN **95.14.5. Principal homogeneous spaces.** Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category G-*Principal* as follows:

(1) An object of G-*Principal* is a triple (U, b, X) where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S, and
   (c) X is a principal homogeneous G_U-space over U where $G_U = U \times_{b,B} G$.
   See Groupoids in Spaces, Definition 78.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of principal homogeneous G_U-spaces.

Thus G-*Principal* is a category and

$$p : G\text{-}Principal \longrightarrow (Sch/S)_{fppf}, \quad (U, b, X) \longmapsto U$$

is a functor. Note that the fibre category of G-*Principal* over U is the disjoint union over $b : U \to B$ of the categories of principal homogeneous $U \times_{b,B} G$-spaces, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is a principal homogeneous G_U-space over U. Moreover, morphisms

are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow[g]{} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G-equivariant.

04UP **Remark 95.14.6.** We conjecture that up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : G\text{-}Principal \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$. This would follow if one could show that given

(1) a covering $\{U_i \to U\}_{i\in I}$ of $(Sch/S)_{fppf}$,
(2) an group algebraic space H over U,
(3) for every i a principal homogeneous H_{U_i}-space X_i over U_i, and
(4) H-equivariant isomorphisms $\varphi_{ij} : X_{i,U_i\times_U U_j} \to X_{j,U_i\times_U U_j}$ satisfying the cocycle condition,

there exists a principal homogeneous H-space X over U which recovers (X_i, φ_{ij}). The technique of the proof of Bootstrap, Lemma 80.11.8 reduces this to a set theoretical question, so the reader who ignores set theoretical questions will "know" that the result is true. In `https://math.columbia.edu/~dejong/wordpress/?p=591` there is a suggestion as to how to approach this problem.

04UQ **95.14.7. Variant on principal homogeneous spaces.** Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory $G\text{-}Principal\text{-}Schemes \subset G\text{-}Principal$ whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is a principal homogeneous G-space over U which is representable, i.e., a scheme.

It is in general not the case that $G\text{-}Principal\text{-}Schemes$ is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist principal homogeneous spaces which are not schemes, hence descent for objects will not be satisfied in general.

04UR **95.14.8. Torsors in fppf topology.** Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category $G\text{-}Torsors$ as follows:

(1) An object of $G\text{-}Torsors$ is a triple (U, b, X) where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism, and
   (c) X is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$.
   See Groupoids in Spaces, Definition 78.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of G_U-torsors.

Thus $G\text{-}Torsors$ is a category and

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, a, X) \longmapsto U$$

is a functor. Note that the fibre category of $G\text{-}Torsors$ over U is the disjoint union over $b : U \to B$ of the categories of fppf $U \times_{b,B} G$-torsors, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is an fppf G_U-torsor over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G-equivariant.

04US **Lemma 95.14.9.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects, which is Bootstrap, Lemma 80.11.8. We omit the proof of axioms (1) and (2) of Stacks, Definition 8.5.1. □

04UT **Lemma 95.14.10.** *Let B be an algebraic space over S. Let G be a group algebraic space over B. Denote $\mathcal{G}$, resp. $\mathcal{B}$ the algebraic space G, resp. B seen as a sheaf on $(Sch/S)_{fppf}$. The functor*

$$G\text{-}Torsors \longrightarrow \mathcal{G}/\mathcal{B}\text{-}Torsors$$

which associates to a triple (U, b, X) the triple $(U, b, \mathcal{X})$ where $\mathcal{X}$ is X viewed as a sheaf is an equivalence of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We will use the result of Stacks, Lemma 8.4.8 to prove this. The functor is fully faithful since the category of algebraic spaces over S is a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. Moreover, all objects (on both sides) are locally trivial torsors so condition (2) of the lemma referenced above holds. Hence the functor is an equivalence. □

04UU **95.14.11. Variant on torsors in fppf topology.** Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory $G\text{-}Torsors\text{-}Schemes \subset G\text{-}Torsors$ whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is an fppf G-torsor over U which is representable, i.e., a scheme.

It is in general not the case that $G\text{-}Torsors\text{-}Schemes$ is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist fppf G-torsors which are not schemes, hence descent for objects will not be satisfied in general.

95.15. Quotients by group actions

04UV At this point we have introduced enough notation that we can work out in more detail what the stacks $[X/G]$ of Section 95.13 look like.

04WL **Situation 95.15.1.** Here

(1) S is a scheme contained in Sch_{fppf},
(2) B is an algebraic space over S,
(3) (G, m) is a group algebraic space over B,
(4) $\pi : X \to B$ is an algebraic space over B, and
(5) $a : G \times_B X \to X$ is an action of G on X over B.

In this situation we construct a category $[[X/G]]$[2] as follows:

(1) An object of $[[X/G]]$ consists of a quadruple $(U, b, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S,
   (c) P is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$, and
   (d) $\varphi : P \to X$ is a G-equivariant morphism fitting into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ U & \xrightarrow{b} & B \end{array}$$

(2) A morphism of $[[X/G]]$ is a pair $(f, g) : (U, b, P, \varphi) \to (U', b', P', \varphi')$ where $f : U \to U'$ is a morphism of schemes over B and $g : P \to P'$ is a G-equivariant morphism over f which induces an isomorphism $P \cong U \times_{f,U'} P'$, and has the property that $\varphi = \varphi' \circ g$. In other words (f, g) fits into the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{g} & P' & & \\ \downarrow & \searrow^{\varphi} & \downarrow & \searrow^{\varphi'} & \\ U & \xrightarrow{f} & U' & & X \\ & \searrow_{b} & & \searrow_{b'} & \downarrow \\ & & & & B \end{array}$$

Thus $[[X/G]]$ is a category and

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}, \quad (U, b, P, \varphi) \longmapsto U$$

is a functor. Note that the fibre category of $[[X/G]]$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of fppf $U \times_{b,B} G$-torsors P endowed with a G-equivariant morphism to X. Hence the fibre categories of $[[X/G]]$ are groupoids.

Note that the functor

$$[[X/G]] \longrightarrow G\text{-}Torsors, \quad (U, b, P, \varphi) \longmapsto (U, b, P)$$

is a 1-morphism of categories over $(Sch/S)_{fppf}$.

0370 **Lemma 95.15.2.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Suppose that $\{U_i \to U\}_{i \in I}$ is a covering in $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, P_i, \varphi_i)$ be objects of $[[X/G]]$ over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This in particular implies that we get a descent datum on the triples (U_i, b_i, P_i) for the stack in groupoids $G\text{-}Torsors$ by applying the functor $[[X/G]] \to G\text{-}Torsors$ above. We have seen that $G\text{-}Torsors$ is a stack in groupoids

[2]The notation $[[X/G]]$ with double brackets serves to distinguish this category from the stack $[X/G]$ introduced earlier. In Proposition 95.15.3 we show that the two are canonically equivalent. Afterwards we will use the notation $[X/G]$ to indicate either.

(Lemma 95.14.9). Hence we may assume that $b_i = b|_{U_i}$ for some morphism $b : U \to B$, and that $P_i = U_i \times_U P$ for some fppf $G_U = U \times_{b,B} G$-torsor P over U. The morphisms φ_i are compatible with the canonical descent datum on the restrictions $U_i \times_U P$ and hence define a morphism $\varphi : P \to X$. (For example you can use Sites, Lemma 7.26.5 or you can use Descent on Spaces, Lemma 74.7.2 to get φ.) This proves descent for objects. We omit the proof of axioms (1) and (2) of Stacks, Definition 8.5.1. □

04WM **Proposition 95.15.3.** *In Situation 95.15.1 there exists a canonical equivalence*

$$[X/G] \longrightarrow [[X/G]]$$

of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We write this out in detail, to make sure that all the definitions work out in exactly the correct manner. Recall that $[X/G]$ is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$, see Groupoids in Spaces, Definition 78.20.1. This means that $[X/G]$ is the stackification of the category fibred in groupoids $[X/_pG]$ associated to the functor

$$(Sch/S)_{fppf} \longrightarrow Groupoids, \quad U \longmapsto (X(U), G(U) \times_{B(U)} X(U), s, t, c)$$

where $s(g, x) = x$, $t(g, x) = a(g, x)$, and $c((g, x), (g', x')) = (m(g, g'), x')$. By the construction of Categories, Example 4.37.1 an object of $[X/_pG]$ is a pair (U, x) with $x \in X(U)$ and a morphism $(f, g) : (U, x) \to (U', x')$ of $[X/_pG]$ is given by a morphism of schemes $f : U \to U'$ and an element $g \in G(U)$ such that $a(g, x) = x' \circ f$. Hence we can define a 1-morphism of stacks in groupoids

$$F_p : [X/_pG] \longrightarrow [[X/G]]$$

by the following rules: On objects we set

$$F_p(U, x) = (U, \pi \circ x, G \times_{B,\pi\circ x} U, a \circ (\mathrm{id}_G \times x))$$

This makes sense because the diagram

$$\begin{array}{ccccc} G \times_{B,\pi\circ x} U & \xrightarrow[\mathrm{id}_G \times x]{} & G \times_{B,\pi} X & \xrightarrow[a]{} & X \\ \downarrow & & & & \downarrow \pi \\ U & & \xrightarrow{\qquad\qquad \pi\circ x \qquad\qquad} & & B \end{array}$$

commutes, and the two horizontal arrows are G-equivariant if we think of the fibre products as trivial G-torsors over U, resp. X. On morphisms $(f, g) : (U, x) \to (U', x')$ we set $F_p(f, g) = (f, R_{g^{-1}})$ where $R_{g^{-1}}$ denotes right translation by the inverse of g. More precisely, the morphism $F_p(f, g) : F_p(U, x) \to F_p(U', x')$ is given by the cartesian diagram

$$\begin{array}{ccc} G \times_{B,\pi\circ x} U & \xrightarrow[R_{g^{-1}}]{} & G \times_{B,\pi\circ x'} U' \\ \downarrow & & \downarrow \\ U & \xrightarrow{\qquad f \qquad} & U' \end{array}$$

where $R_{g^{-1}}$ on T-valued points is given by

$$R_{g^{-1}}(g', u) = (m(g', i(g(u))), f(u))$$

To see that this works we have to verify that

$$a \circ (\mathrm{id}_G \times x) = a \circ (\mathrm{id}_G \times x') \circ R_{g^{-1}}$$

which is true because the right hand side applied to the T-valued point (g', u) gives the desired equality

$$\begin{aligned} a((\mathrm{id}_G \times x')(m(g', i(g(u))), f(u))) &= a(m(g', i(g(u))), x'(f(u))) \\ &= a(g', a(i(g(u)), x'(f(u)))) \\ &= a(g', x(u)) \end{aligned}$$

because $a(g, x) = x' \circ f$ and hence $a(i(g), x' \circ f) = x$.

By the universal property of stackification from Stacks, Lemma 8.9.2 we obtain a canonical extension $F : [X/G] \to [[X/G]]$ of the 1-morphism F_p above. We first prove that F is fully faithful. To do this, since both source and target are stacks in groupoids, it suffices to prove that the *Isom*-sheaves are identified under F. Pick a scheme U and objects ξ, ξ' of $[X/G]$ over U. We want to show that

$$F : Isom_{[X/G]}(\xi, \xi') \longrightarrow Isom_{[[X/G]]}(F(\xi), F(\xi'))$$

is an isomorphism of sheaves. To do this it suffices to work locally on U, and hence we may assume that ξ, ξ' come from objects (U, x), (U, x') of $[X/_pG]$ over U; this follows directly from the construction of the stackification, and it is also worked out in detail in Groupoids in Spaces, Section 78.24. Either by directly using the description of morphisms in $[X/_pG]$ above, or using Groupoids in Spaces, Lemma 78.22.1 we see that in this case

$$Isom_{[X/G]}(\xi, \xi') = U \times_{(x,x'),X \times_S X,(s,t)} (G \times_B X)$$

A T-valued point of this fibre product corresponds to a pair (u, g) with $u \in U(T)$, and $g \in G(T)$ such that $a(g, x \circ u) = x' \circ u$. (Note that this implies $\pi \circ x \circ u = \pi \circ x' \circ u$.) On the other hand, a T-valued point of $Isom_{[[X/G]]}(F(\xi), F(\xi'))$ by definition corresponds to a morphism $u : T \to U$ such that $\pi \circ x \circ u = \pi \circ x' \circ u : T \to B$ and an isomorphism

$$R : G \times_{B,\pi \circ x \circ u} T \longrightarrow G \times_{B,\pi \circ x' \circ u} T$$

of trivial G_T-torsors compatible with the given maps to X. Since the torsors are trivial we see that $R = R_{g^{-1}}$ (right multiplication) by some $g \in G(T)$. Compatibility with the maps $a \circ (1_G, x \circ u), a \circ (1_G, x' \circ u) : G \times_B T \to X$ is equivalent to the condition that $a(g, x \circ u) = x' \circ u$. Hence we obtain the desired equality of *Isom*-sheaves.

Now that we know that F is fully faithful we see that Stacks, Lemma 8.4.8 applies. Thus to show that F is an equivalence it suffices to show that objects of $[[X/G]]$ are fppf locally in the essential image of F. This is clear as fppf torsors are fppf locally trivial, and hence we win. □

0CQJ **Lemma 95.15.4.** *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then the stacks in groupoids*

$$[B/G], \quad [[B/G]], \quad G\text{-}Torsors, \quad \mathcal{G}/\mathcal{B}\text{-}Torsors$$

are all canonically equivalent. If $G \to B$ is flat and locally of finite presentation, then these are also equivalent to G-Principal.

Proof. The equivalence G-*Torsors* $\to \mathcal{G}/\mathcal{B}$-*Torsors* is given in Lemma 95.14.10. The equivalence $[B/G] \to [[B/G]]$ is given in Proposition 95.15.3. Unwinding the definition of $[[B/G]]$ given in Section 95.15 we see that $[[B//G]] = G$-*Torsors*.

Finally, assume $G \to B$ is flat and locally of finite presentation. To show that the natural functor $G\text{-}Torsors \to G\text{-}Principal$ is an equivalence it suffices to show that for a scheme U over B a principal homogeneous G_U-space $X \to U$ is fppf locally trivial. By our definition of principal homogeneous spaces (Groupoids in Spaces, Definition 78.9.3) there exists an fpqc covering $\{U_i \to U\}$ such that $U_i \times_U X \cong G \times_B U_i$ as algebraic spaces over U_i. This implies that $X \to U$ is surjective, flat, and locally of finite presentation, see Descent on Spaces, Lemmas 74.11.6, 74.11.13, and 74.11.10. Choose a scheme W and a surjective étale morphism $W \to X$. Then it follows from what we just said that $\{W \to U\}$ is an fppf covering such that $X_W \to W$ has a section. Hence X is an fppf G_U-torsor. □

0371 **Remark 95.15.5.** Let S be a scheme. Let G be an abstract group. Let X be an algebraic space over S. Let $G \to \mathrm{Aut}_S(X)$ be a group homomorphism. In this setting we can define $[[X/G]]$ similarly to the above as follows:

(1) An object of $[[X/G]]$ consists of a triple $(U, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) P is a sheaf on $(Sch/U)_{fppf}$ which comes with an action of G that turns it into a torsor under the constant sheaf with value G, and
   (c) $\varphi : P \to X$ is a G-equivariant map of sheaves.
(2) A morphism $(f, g) : (U, P, \varphi) \to (U', P', \varphi')$ is given by a morphism of schemes $f : T \to T'$ and a G-equivariant isomorphism $g : P \to f^{-1}P'$ such that $\varphi = \varphi' \circ g$.

In exactly the same manner as above we obtain a functor

$$[[X/G]] \longrightarrow (Sch/S)_{fppf}$$

which turns $[[X/G]]$ into a stack in groupoids over $(Sch/S)_{fppf}$. The constant sheaf $\underline{G}$ is (provided the cardinality of G is not too large) representable by G_S on $(Sch/S)_{fppf}$ and this version of $[[X/G]]$ is equivalent to the stack $[[X/G_S]]$ introduced above.

95.16. The Picard stack

0372 In this section we introduce the Picard stack in complete generality. In the chapter on Quot and Hilb we will show that it is an algebraic stack under suitable hypotheses, see Quot, Section 99.10.

Let S be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over S. We define a category $\mathcal{P}ic_{X/B}$ as follows:

(1) An object is a triple $(U, b, \mathcal{L})$, where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S, and
   (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b,B} X$.
(2) A morphism $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \to U'$ over B and an isomorphism $g : f^*\mathcal{L}' \to \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \to (U'', b'', \mathcal{L}'')$ is given by $(f' \circ f, g \circ f^*(g'))$. Thus we get a category $\mathcal{P}ic_{X/B}$ and

$$p : \mathcal{P}ic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{L}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{P}ic_{X/B}$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of the categories of invertible sheaves on $X_U = U \times_{b,B} X$. Hence the fibre categories are groupoids.

04WN **Lemma 95.16.1.** *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$\mathcal{P}ic_{X/B} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, \mathcal{L}_i)$ be an object of $\mathcal{P}ic_{X/B}$ lying over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This implies in particular that the morphisms b_i are the restrictions of a morphism $b : U \to B$. Write $X_U = U \times_{b,B} X$ and $X_i = U_i \times_{b_i,B} X = U_i \times_U U \times_{b,B} X = U_i \times_U X_U$. Observe that $\mathcal{L}_i$ is an invertible $\mathcal{O}_{X_i}$-module. Note that $\{X_i \to X_U\}$ forms an fppf covering as well. Moreover, the descent datum φ_{ij} translates into a descent datum on the invertible sheaves $\mathcal{L}_i$ relative to the fppf covering $\{X_i \to X_U\}$. Hence by Descent on Spaces, Proposition 74.4.1 we obtain a unique invertible sheaf $\mathcal{L}$ on X_U which recovers $\mathcal{L}_i$ and the descent data over X_i. The triple $(U, b, \mathcal{L})$ is therefore the object of $\mathcal{P}ic_{X/B}$ over U we were looking for. Details omitted. □

95.17. Examples of inertia stacks

0373 Here are some examples of inertia stacks.

0374 **Example 95.17.1.** Let S be a scheme. Let G be a commutative group. Let $X \to S$ be a scheme over S. Let $a : G \times X \to X$ be an action of G on X. For $g \in G$ we denote $g : X \to X$ the corresponding automorphism. In this case the inertia stack of $[X/G]$ (see Remark 95.15.5) is given by

$$I_{[X/G]} = \coprod\nolimits_{g \in G} [X^g/G],$$

where, given an element g of G, the symbol X^g denotes the scheme $X^g = \{x \in X \mid g(x) = x\}$. In a formula X^g is really the fibre product

$$X^g = X \times_{(1,1),X \times_S X,(g,1)} X.$$

Indeed, for any S-scheme T, a T-point on the inertia stack of $[X/G]$ consists of a triple $(P/T, \phi, \alpha)$ consisting of an fppf G-torsor $P \to T$ together with a G-equivariant morphism $\phi : P \to X$, together with an automorphism α of P over T such that $\phi \circ \alpha = \phi$. Since G is a sheaf of *commutative* groups, α is, locally in the fppf topology over T, given by multiplication by some element g of G. The condition that $\phi \circ \alpha = \phi$ means that ϕ factors through the inclusion of X^g in X, i.e., ϕ is obtained by composing that inclusion with a morphism $P \to X^g$. The above discussion allows us to define a morphism of fibred categories $I_{[X/G]} \to \coprod_{g \in G} [X^g/G]$ given on T-points by the discussion above. We omit showing that this is an equivalence.

0375 **Example 95.17.2.** Let $f : X \to S$ be a morphism of schemes. Assume that for any $T \to S$ the base change $f_T : X_T \to T$ has the property that the map $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. (This implies that f is *cohomologically flat in dimension* 0 (insert future reference here) but is stronger.) Consider the Picard stack $\mathcal{P}ic_{X/S}$, see Section 95.16. The points of its inertia stack over an S-scheme T consist of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X_T and α is an automorphism of that line

bundle. I.e., we can think of α as an element of $H^0(X_T, \mathcal{O}_{X_T})^\times = H^0(T, \mathcal{O}_T^*)$ by our condition. Note that $H^0(T, \mathcal{O}_T^*) = \mathbf{G}_{m,S}(T)$, see Groupoids, Example 39.5.1. Hence the inertia stack of $\mathcal{P}ic_{X/S}$ is

$$I_{\mathcal{P}ic_{X/S}} = \mathbf{G}_{m,S} \times_S \mathcal{P}ic_{X/S}.$$

as a stack over $(Sch/S)_{fppf}$.

95.18. Finite Hilbert stacks

05WA We formulate this in somewhat greater generality than is perhaps strictly needed. Fix a 1-morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. For each integer $d \geq 1$ consider a category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ defined as follows:

(1) An object (U, Z, y, x, α) where U, Z are objects of in $(Sch/S)_{fppf}$ and Z is a finite locally free of degree d over U, where $y \in \mathrm{Ob}(\mathcal{Y}_U)$, $x \in \mathrm{Ob}(\mathcal{X}_Z)$ and $\alpha : y|_Z \to F(x)$ is an isomorphism[3].

(2) A morphism $(U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is given by a morphism of schemes $f : U \to U'$, a morphism of schemes $g : Z \to Z'$ which induces an isomorphism $Z \to Z' \times_{U'} U$, and isomorphisms $b : y \to f^*y'$, $a : x \to g^*x'$ inducing a commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ {\scriptstyle b|_Z}\downarrow & & \downarrow{\scriptstyle F(a)} \\ f^*y'|_Z & \xrightarrow{\alpha'} & F(g^*x') \end{array}$$

It is clear from the definitions that there is a canonical forgetful functor

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow (Sch/S)_{fppf}$$

which assigns to the quintuple (U, Z, y, x, α) the scheme U and to the morphism $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ the morphism $f : U \to U'$.

05WB **Lemma 95.18.1.** *The category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ endowed with the functor p above defines a stack in groupoids over $(Sch/S)_{fppf}$.*

---

[3]This means the data gives rise, via the 2-Yoneda lemma (Categories, Lemma 4.41.2), to a 2-commutative diagram

$$\begin{array}{ccc} (Sch/Z)_{fppf} & \xrightarrow{x} & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle F} \\ (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. Alternatively, we may picture α as a 2-morphism

$$(Sch/Z)_{fppf} \xrightarrow[F\circ x]{\overset{y\circ(Z\to U)}{\Downarrow \alpha}} \mathcal{Y}.$$

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, Z_i, y_i, x_i, \alpha_i)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ lying over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. First, observe that φ_{ij} induces a descent datum $(Z_i/U_i, \varphi_{ij})$ which is effective by Descent, Lemma 35.37.1 This produces a scheme Z/U which is finite locally free of degree d by Descent, Lemma 35.23.32. From now on we identify Z_i with $Z \times_U U_i$. Next, the objects y_i in the fibre categories $\mathcal{Y}_{U_i}$ descend to an object y in $\mathcal{Y}_U$ because $\mathcal{Y}$ is a stack in groupoids. Similarly the objects x_i in the fibre categories $\mathcal{X}_{Z_i}$ descend to an object x in $\mathcal{X}_Z$ because $\mathcal{X}$ is a stack in groupoids. Finally, the given isomorphisms

$$\alpha_i : (y|_Z)_{Z_i} = y_i|_{Z_i} \longrightarrow F(x_i) = F(x|_{Z_i})$$

glue to a morphism $\alpha : y|_Z \to F(x)$ as the $\mathcal{Y}$ is a stack and hence $Isom_{\mathcal{Y}}(y|_Z, F(x))$ is a sheaf. Details omitted. □

05WC **Definition 95.18.2.** We will denote $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ the *degree d finite Hilbert stack of $\mathcal{X}$ over $\mathcal{Y}$* constructed above. If $\mathcal{Y} = S$ we write $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y} = S$ we denote it $\mathcal{H}_d$.

Note that given $F : \mathcal{X} \to \mathcal{Y}$ as above we have the following natural 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$:

05WD (95.18.2.1)
$$\begin{array}{ccccc} \mathcal{H}_d(\mathcal{X}) & \longleftarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) & \longrightarrow & \mathcal{Y} \\ & \searrow & \downarrow & & \\ & & \mathcal{H}_d & & \end{array}$$

Each of the arrows is given by a "forgetful functor".

05XV **Lemma 95.18.3.** *The 1-morphism $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful.*

Proof. To check that $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful it suffices to prove that it is faithful on fibre categories. Suppose that $\xi = (U, Z, y, x, \alpha)$ and $\xi' = (U, Z', y', x', \alpha')$ are two objects of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over the scheme U. Let $(g, b, a), (g', b', a') : \xi \to \xi'$ be two morphisms in the fibre category of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. The image of these morphisms in $\mathcal{H}_d(\mathcal{X})$ agree if and only if $g = g'$ and $a = a'$. Then the commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ {\scriptstyle b|_Z,\ b'|_Z}\downarrow & & \downarrow{\scriptstyle F(a)=F(a')} \\ y'|_Z & \xrightarrow{\alpha'} & F(g^*x') = F((g')^*x') \end{array}$$

implies that $b|_Z = b'|_Z$. Since $Z \to U$ is finite locally free of degree d we see $\{Z \to U\}$ is an fppf covering, hence $b = b'$. □

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CHAPTER 96

Sheaves on Algebraic Stacks

06TF

96.1. Introduction

06TG There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes $\mathcal{O}$-modules whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [**Ols07b**]. We will return to this in Cohomology of Stacks, Section 103.1.

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site $\mathcal{X}_{smooth}$ defined below.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

96.2. Conventions

06TH The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 34.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

96.3. Presheaves

06TI

In this section we define presheaves on categories fibred in groupoids over $(Sch/S)_{fppf}$, but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

06TJ **Definition 96.3.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *presheaf on* $\mathcal{X}$ is a presheaf on the underlying category of $\mathcal{X}$.
(2) A *morphism of presheaves on* $\mathcal{X}$ is a morphism of presheaves on the underlying category of $\mathcal{X}$.

We denote $PSh(\mathcal{X})$ the category of presheaves on $\mathcal{X}$.

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian presheaves*, i.e., presheaves of abelian groups, is denoted $PAb(\mathcal{X})$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Recall that this means just that f is a functor over $(Sch/S)_{fppf}$. The material in Sites, Section 7.19 provides us with a pair of adjoint functors[1]

06TK (96.3.1.1) $$f^p : PSh(\mathcal{Y}) \longrightarrow PSh(\mathcal{X}) \quad \text{and} \quad {}_pf : PSh(\mathcal{X}) \longrightarrow PSh(\mathcal{Y}).$$

The adjointness is

$$\mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F})$$

where $\mathcal{F} \in \mathrm{Ob}(PSh(\mathcal{X}))$ and $\mathcal{G} \in \mathrm{Ob}(PSh(\mathcal{Y}))$. We call $f^p\mathcal{G}$ the *pullback* of $\mathcal{G}$. It follows from the definitions that

$$f^p\mathcal{G}(x) = \mathcal{G}(f(x))$$

for any $x \in \mathrm{Ob}(\mathcal{X})$. The presheaf ${}_pf\mathcal{F}$ is called the *pushforward* of $\mathcal{F}$. It is described by the formula

$$({}_pf\mathcal{F})(y) = \lim_{f(x) \to y} \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

06TL **Lemma 96.3.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Then $(g \circ f)^p = f^p \circ g^p$ and there is a canonical isomorphism ${}_p(g \circ f) \to {}_pg \circ {}_pf$ compatible with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$.*

Proof. Let $\mathcal{H}$ be a presheaf on $\mathcal{Z}$. Then $(g \circ f)^p\mathcal{H} = f^p(g^p\mathcal{H})$ is given by the equalities

$$(g \circ f)^p\mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p\mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation ${}_p(g \circ f) \to {}_pg \circ {}_pf$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. If z is an object of $\mathcal{Z}$ then we get a category $\mathcal{J}$ of quadruples $(x, f(x) \to y, y, g(y) \to z)$ and a category $\mathcal{I}$ of pairs $(x, g(f(x)) \to z)$. There is a canonical functor $\mathcal{J} \to \mathcal{I}$

[1] These functors will be denoted f^{-1} and f_* after Lemma 96.4.4 has been proved.

sending the object $(x, \alpha : f(x) \to y, y, \beta : g(y) \to z)$ to $(x, \beta \circ f(\alpha) : g(f(x)) \to z)$. This gives the arrow in

$$\begin{aligned}
({}_p(g \circ f)\mathcal{F})(z) &= \lim_{g(f(x)) \to z} \mathcal{F}(x) \\
&= \lim_{\mathcal{I}} \mathcal{F} \\
&\to \lim_{\mathcal{J}} \mathcal{F} \\
&= \lim_{g(y) \to z} \left(\lim_{f(x) \to y} \mathcal{F}(x) \right) \\
&= ({}_pg \circ {}_pf\mathcal{F})(x)
\end{aligned}$$

by Categories, Lemma 4.14.9. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define ${}_p(g \circ f) \cong {}_pg \circ {}_pf$ as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$ means that given presheaves $\mathcal{H}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mor}_{PSh(\mathcal{X})}(f^pg^p\mathcal{H}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(g^p\mathcal{H}, {}_pf\mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{H}, {}_pg{}_pf\mathcal{F}) \\
\| & & & & \uparrow \\
\mathrm{Mor}_{PSh(\mathcal{X})}((g \circ f)^p\mathcal{G}, \mathcal{F}) & & = & & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f)\mathcal{F})
\end{array}$$

Proof omitted. □

06TM **Lemma 96.3.3.** *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $t : f \to g$ be a 2-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assigned to t there are canonical isomorphisms of functors*

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_pt : {}_pf \longrightarrow {}_pg$$

which compatible with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ and with vertical and horizontal composition of 2-morphisms.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then $t^p : g^p\mathcal{G} \to f^p\mathcal{G}$ is given by the family of maps

$$g^p\mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p\mathcal{G}(x)$$

parametrized by $x \in \mathrm{Ob}(\mathcal{X})$. This makes sense as $t_x : f(x) \to g(x)$ and $\mathcal{G}$ is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation ${}_pt$ for $y \in \mathrm{Ob}(\mathcal{Y})$ define ${}^f_y\mathcal{I}$, resp. ${}^g_y\mathcal{I}$ to be the category of pairs $(x, \psi : f(x) \to y)$, resp. $(x, \psi : g(x) \to y)$, see Sites, Section 7.19. Note that t defines a functor ${}_yt : {}^g_y\mathcal{I} \to {}^f_y\mathcal{I}$ given by the rule

$$(x, g(x) \to y) \longmapsto (x, f(x) \xrightarrow{t_x} g(x) \to y).$$

Note that for $\mathcal{F}$ a presheaf on $\mathcal{X}$ the composition of ${}_yt$ with $\mathcal{F} : {}^f_y\mathcal{I}^{opp} \to Sets$, $(x, f(x) \to y) \mapsto \mathcal{F}(x)$ is equal to $\mathcal{F} : {}^g_y\mathcal{I}^{opp} \to Sets$. Hence by Categories, Lemma 4.14.9 we get for every $y \in \mathrm{Ob}(\mathcal{Y})$ a canonical map

$$({}_pf\mathcal{F})(y) = \lim_{{}^f_y\mathcal{I}} \mathcal{F} \longrightarrow \lim_{{}^g_y\mathcal{I}} \mathcal{F} = ({}_pg\mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define ${}_pt$ as the unique map compatible

with the adjointness properties of the pairs $(f^p, {}_pf)$ and $(g^p, {}_pg)$ (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ means that given presheaves $\mathcal{G}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F}) \\ {\scriptstyle -\circ t^p}\downarrow & & \downarrow{\scriptstyle {}_pt\circ -} \\ \mathrm{Mor}_{PSh(\mathcal{X})}(g^p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pg\mathcal{F}) \end{array}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 7.19.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for t^p is straightforward and one can conclude that this holds for the ${}_pt$ maps using compatibility with adjointness. □

96.4. Sheaves

06TN We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site Sch_{fppf} as in Topologies, Definition 34.7.6 and denote its underlying category Sch_α. Besides being the underlying category of a fppf site, the category Sch_α can also can serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 34.11.1. We denote these sites Sch_{Zar}, $Sch_{étale}$, Sch_{smooth}, and $Sch_{syntomic}$. In this situation, since we have defined the big Zariski site $(Sch/S)_{Zar}$ of S, the big étale site $(Sch/S)_{étale}$ of S, the big smooth site $(Sch/S)_{smooth}$ of S, the big syntomic site $(Sch/S)_{syntomic}$ of S, and the big fppf site $(Sch/S)_{fppf}$ of S as the localizations (see Sites, Section 7.25) Sch_{Zar}/S, $Sch_{étale}/S$, Sch_{smooth}/S, $Sch_{syntomic}/S$, and Sch_{fppf}/S of these (absolute) big sites we see that all of these have the same underlying category, namely Sch_α/S.

It follows that if we have a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids, then $\mathcal{X}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 8.10.2.

06TP **Definition 96.4.1.** Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) The *associated Zariski site*, denoted $\mathcal{X}_{Zar}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{Zar}$.
(2) The *associated étale site*, denoted $\mathcal{X}_{étale}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{étale}$.
(3) The *associated smooth site*, denoted $\mathcal{X}_{smooth}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{smooth}$.
(4) The *associated syntomic site*, denoted $\mathcal{X}_{syntomic}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{syntomic}$.
(5) The *associated fppf site*, denoted $\mathcal{X}_{fppf}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{fppf}$.

This definition makes sense by the discussion above. If $\mathcal{X}$ is an algebraic stack, the literature calls $\mathcal{X}_{fppf}$ (or a site equivalent to it) the *big fppf site* of $\mathcal{X}$ and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

06TQ **Remark 96.4.2.** We only use this notation when the symbol $\mathcal{X}$ refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted $X_{étale}$ (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on $\mathcal{X}$, i.e., define the corresponding topoi.

06TR **Definition 96.4.3.** Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) We say $\mathcal{F}$ is a *Zariski sheaf*, or a *sheaf for the Zariski topology* if $\mathcal{F}$ is a sheaf on the associated Zariski site $\mathcal{X}_{Zar}$.
(2) We say $\mathcal{F}$ is an *étale sheaf*, or a *sheaf for the étale topology* if $\mathcal{F}$ is a sheaf on the associated étale site $\mathcal{X}_{étale}$.
(3) We say $\mathcal{F}$ is a *smooth sheaf*, or a *sheaf for the smooth topology* if $\mathcal{F}$ is a sheaf on the associated smooth site $\mathcal{X}_{smooth}$.
(4) We say $\mathcal{F}$ is a *syntomic sheaf*, or a *sheaf for the syntomic topology* if $\mathcal{F}$ is a sheaf on the associated syntomic site $\mathcal{X}_{syntomic}$.
(5) We say $\mathcal{F}$ is an *fppf sheaf*, or a *sheaf*, or a *sheaf for the fppf topology* if $\mathcal{F}$ is a sheaf on the associated fppf site $\mathcal{X}_{fppf}$.

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves $Sh(\mathcal{X}_{Zar})$, $Sh(\mathcal{X}_{étale})$, $Sh(\mathcal{X}_{smooth})$, $Sh(\mathcal{X}_{syntomic})$, and $Sh(\mathcal{X}_{fppf})$.

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian sheaves*, i.e., sheaves of abelian groups, is denoted $Ab(\mathcal{X}_{fppf})$ and similarly for the other topologies. If $\mathcal{X}$ is an algebraic stack, then $Sh(\mathcal{X}_{fppf})$ is equivalent (modulo set theoretical problems) to what in the literature would be termed the *category of sheaves on the big fppf site of* $\mathcal{X}$. Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{étale}) \subset Sh(\mathcal{X}_{Zar}) \subset PSh(\mathcal{X})$$

We sometimes write $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$ and $Ab(\mathcal{X}_{fppf}) = Ab(\mathcal{X})$ in accordance with our terminology that a sheaf on $\mathcal{X}$ is an fppf sheaf on $\mathcal{X}$.

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

06TS **Lemma 96.4.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functors ${}_pf$ and f^p of (96.3.1.1) transform τ sheaves into τ sheaves and define a morphism of topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$.*

Proof. This follows immediately from Stacks, Lemma 8.10.3. □

In other words, pushforward and pullback of presheaves as defined in Section 96.3 also produces *pushforward* and *pullback* of τ-sheaves. Having said all of the above we see that we can write $f^p = f^{-1}$ and ${}_pf = f_*$ without any possibility of confusion.

06TT **Definition 96.4.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the *associated morphism of fppf topoi* constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 7.44 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

96.5. Computing pushforward

06W5 Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $y \in \mathrm{Ob}(\mathcal{Y})$. We can compute $f_*\mathcal{F}(y)$ in the following way. Suppose that y lies over the scheme V and using the 2-Yoneda lemma think of y as a 1-morphism. Consider the projection

$$\mathrm{pr} : (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

06W6 (96.5.0.1) $$f_*\mathcal{F}(y) = \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Namely, objects of the 2-fibre product are triples $(h : U \to V, x, f(x) \to h^*y)$. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $\alpha : f(x) \to y$ of $\mathcal{Y}$. Since $f_*\mathcal{F}(y) = \lim_{f(x)\to y} \mathcal{F}(x)$ by definition the equality follows.

As a consequence we have the following "base change" result for pushforwards. This result is trivial and hinges on the fact that we are using "big" sites.

075B **Lemma 96.5.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of categories fibred in groupoids over S. Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Given an object y' of $\mathcal{Y}'$ over V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (96.5.0.1) a bijection $g^{-1}f_*\mathcal{F}(y') \to f'_*(g')^{-1}\mathcal{F}(y')$. We omit the verification that this is compatible with restriction mappings. □

In the case of a representable morphism of categories fibred in groupoids this formula (96.5.0.1) simplifies. We suggest the reader skip the rest of this section.

06W7 **Lemma 96.5.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

(1) *f is representable, and*
(2) *for every $y \in \mathrm{Ob}(\mathcal{Y})$ the functor $\mathcal{X}^{opp} \to Sets$, $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ is representable.*

Proof. According to the discussion in Algebraic Stacks, Section 94.6 we see that f is representable if and only if for every $y \in \mathrm{Ob}(\mathcal{Y})$ lying over U the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable, i.e., of the form $(Sch/V_y)_{fppf}$ for some scheme V_y over U. Objects in this 2-fibre products are triples $(h : V \to U, x, \alpha : f(x) \to h^*y)$ where α lies over id_V. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $f(x) \to y$. Hence the 2-fibre product is representable by V_y and $f(x_y) \to y$ where x_y is an object of $\mathcal{X}$ over V_y if and only if the functor in (2) is representable by x_y with universal object a map $f(x_y) \to y$. □

Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. Assume f is representable. For every $y \in \mathrm{Ob}(\mathcal{Y})$ we choose an object $u(y) \in \mathrm{Ob}(\mathcal{X})$ representing the functor $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ of Lemma 96.5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms $f(u(y)) \to y$ by construction. For every morphism $\beta : y' \to y$ in $\mathcal{Y}$ we obtain a unique morphism $u(\beta) : u(y') \to u(y)$ in $\mathcal{X}$ such that the diagram

$$\begin{array}{ccc} f(u(y')) & \xrightarrow{f(u(\beta))} & f(u(y)) \\ \downarrow & & \downarrow \\ y' & \longrightarrow & y \end{array}$$

commutes. In other words, $u : \mathcal{Y} \to \mathcal{X}$ is a functor. In fact, we can say a little bit more. Namely, suppose that $V' = q(y')$, $V = q(y)$, $U' = p(u(y'))$ and $U = p(u(y))$. Then

$$\begin{array}{ccc} U' & \xrightarrow{p(u(\beta))} & U \\ \downarrow & & \downarrow \\ V' & \xrightarrow{q(\beta)} & V \end{array}$$

is a fibre product square. This is true because $U' \to U$ represents the base change $(Sch/V')_{fppf} \times_{y',\mathcal{Y}} \mathcal{X} \to (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ of $V' \to V$.

06W8 **Lemma 96.5.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Then the functor $u : \mathcal{Y}_\tau \to \mathcal{X}_\tau$ is continuous and defines a morphism of sites $\mathcal{X}_\tau \to \mathcal{Y}_\tau$ which induces the same morphism of topoi $Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ as the*

morphism f constructed in Lemma 96.4.4. Moreover, $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ for any presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Let $\{y_i \to y\}$ be a τ-covering in $\mathcal{Y}$. By definition this simply means that $\{q(y_i) \to q(y)\}$ is a τ-covering of schemes. By the final remark above the lemma we see that $\{p(u(y_i)) \to p(u(y))\}$ is the base change of the τ-covering $\{q(y_i) \to q(y)\}$ by $p(u(y)) \to q(y)$, hence is itself a τ-covering by the axioms of a site. Hence $\{u(y_i) \to u(y)\}$ is a τ-covering of $\mathcal{X}$. This proves that u is continuous.

Let's use the notation u_p, u_s, u^p, u^s of Sites, Sections 7.5 and 7.13. If we can show the final assertion of the lemma, then we see that $f_* = u^p = u^s$ (by continuity of u seen above) and hence by adjointness $f^{-1} = u_s$ which will prove u_s is exact, hence that u determines a morphism of sites, and the equality will be clear as well. To see that $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ note that by definition

$$f_*\mathcal{F}(y) = ({}_pf\mathcal{F})(y) = \lim\nolimits_{f(x)\to y} \mathcal{F}(x).$$

Since $u(y)$ is a final object in the category the limit is taken over we conclude. □

96.6. The structure sheaf

06TU Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ has a final object, namely, $\mathrm{id} : (Sch/S)_{fppf} \to (Sch/S)_{fppf}$ and p is a 1-morphism from $\mathcal{X}$ to this final object. Hence any presheaf $\mathcal{G}$ on $(Sch/S)_{fppf}$ gives a presheaf $p^{-1}\mathcal{G}$ on $\mathcal{X}$ defined by the rule $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$. Moreover, the discussion in Section 96.4 shows that $p^{-1}\mathcal{G}$ is a τ sheaf whenever $\mathcal{G}$ is a τ-sheaf.

Recall that the site $(Sch/S)_{fppf}$ is a ringed site with structure sheaf $\mathcal{O}$ defined by the rule

$$(Sch/S)^{opp} \longrightarrow Rings, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 35.8.2.

06TV **Definition 96.6.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *structure sheaf of* $\mathcal{X}$ is the sheaf of rings $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$.

For an object x of $\mathcal{X}$ lying over U we have $\mathcal{O}_\mathcal{X}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$. Needless to say $\mathcal{O}_\mathcal{X}$ is also a Zariski, étale, smooth, and syntomic sheaf, and hence each of the sites $\mathcal{X}_{Zar}$, $\mathcal{X}_{étale}$, $\mathcal{X}_{smooth}$, $\mathcal{X}_{syntomic}$, and $\mathcal{X}_{fppf}$ is a ringed site. This construction is functorial as well.

06TW **Lemma 96.6.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a canonical identification $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ which turns $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ into a morphism of ringed topoi.*

Proof. Denote $p : \mathcal{X} \to (Sch/S)_{fppf}$ and $q : \mathcal{Y} \to (Sch/S)_{fppf}$ the structural functors. Then $p = q \circ f$, hence $p^{-1} = f^{-1} \circ q^{-1}$ by Lemma 96.3.2. Since $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$ and $\mathcal{O}_\mathcal{Y} = q^{-1}\mathcal{O}$ the result follows. □

06TX **Remark 96.6.3.** In the situation of Lemma 96.6.2 the morphism of ringed topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ is flat as is clear from the equality $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same

thing happens when taking a closed immersion $i : X \to Y$ of schemes: in this case the associated morphism of big τ-sites $i : (Sch/X)_\tau \to (Sch/Y)_\tau$ also is flat.

96.7. Sheaves of modules

06WA Since we have a structure sheaf we have modules.

06WB **Definition 96.7.1.** Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) A *presheaf of modules on* $\mathcal{X}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules. The category of presheaves of modules is denoted $PMod(\mathcal{O}_\mathcal{X})$.
(2) We say a presheaf of modules $\mathcal{F}$ is an *$\mathcal{O}_\mathcal{X}$-module*, or more precisely a *sheaf of $\mathcal{O}_\mathcal{X}$-modules* if $\mathcal{F}$ is an fppf sheaf. The category of $\mathcal{O}_\mathcal{X}$-modules is denoted $Mod(\mathcal{O}_\mathcal{X})$.

These (pre)sheaves of modules occur in the literature as *(pre)sheaves of $\mathcal{O}_\mathcal{X}$-modules on the big fppf site of $\mathcal{X}$*. We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski, étale, smooth, or syntomic topologies (without necessarily being sheaves). If need be these will be denoted $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and similarly for the other topologies.

Next, we address functoriality – first for presheaves of modules. Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. The functors f^{-1}, f_* on abelian presheaves extend to functors

(96.7.1.1)

06WD $$f^{-1} : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : PMod(\mathcal{O}_\mathcal{X}) \longrightarrow PMod(\mathcal{O}_\mathcal{Y})$$

This is immediate for f^{-1} because $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$ which is a module over $\mathcal{O}_\mathcal{Y}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_\mathcal{X}(x)$. Alternatively it follows because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ and because f^{-1} commutes with limits (on presheaves). Since f_* is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend f_* to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 18.12.1. We claim that the functors (96.7.1.1) form an adjoint pair of functors:

$$\mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

As $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ this follows from Modules on Sites, Lemma 18.12.3 by endowing $\mathcal{X}$ and $\mathcal{Y}$ with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by f also the induced morphism of ringed topoi, see Lemma 96.6.2 (for the fppf topologies right now). Note that the functors f^{-1} and f_* of (96.7.1.1) preserve the subcategories of sheaves of modules, see Lemma 96.4.4. Hence it follows immediately that

06WE (96.7.1.2) $$f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y})$$

form an adjoint pair of functors:

$$\mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that $f^* = f^{-1}$ where f^* is as defined in Modules on Sites, Section 18.13 for the morphism of ringed topoi f above. Of course we could have seen this directly because $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ and because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$.

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

96.8. Representable categories

076N In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

075I **Lemma 96.8.1.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over (Sch/S). Assume $\mathcal{X}$ is representable by a scheme X. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ there is a canonical equivalence*

$$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) = ((Sch/X)_\tau, \mathcal{O}_X)$$

of ringed sites.

Proof. This follows by choosing an equivalence $(Sch/X)_\tau \to \mathcal{X}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ and using the functoriality of the construction $\mathcal{X} \rightsquigarrow \mathcal{X}_\tau$. □

075J **Lemma 96.8.2.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by schemes X, Y. Let $f : X \to Y$ be the morphism of schemes corresponding to f. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ the morphism of ringed topoi $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \to (Sh(\mathcal{Y}_\tau), \mathcal{O}_\mathcal{Y})$ agrees with the morphism of ringed topoi $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \to (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$ via the identifications of Lemma 96.8.1.*

Proof. Follows by unwinding the definitions. □

96.9. Restriction

075C A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

06W0 **Lemma 96.9.1.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}_\tau/x \to (Sch/U)_\tau$.*

Proof. Special case of Stacks, Lemma 8.10.4. □

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

06W1 **Definition 96.9.2.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) The *pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$* is the restriction $\mathcal{F}|_{(\mathcal{X}/x)}$ viewed as a presheaf on $(Sch/U)_{fppf}$ via the equivalence $\mathcal{X}/x \to (Sch/U)_{fppf}$ of Lemma 96.9.1.
(2) The *restriction of $\mathcal{F}$ to $U_{étale}$* is $x^{-1}\mathcal{F}|_{U_{étale}}$, abusively written $\mathcal{F}|_{U_{étale}}$.

This notation makes sense because to the object x the 2-Yoneda lemma, see Algebraic Stacks, Section 94.5 associates a 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}/x$ which is quasi-inverse to $p : \mathcal{X}/x \to (Sch/U)_{fppf}$. Hence $x^{-1}\mathcal{F}$ truly is the pullback of $\mathcal{F}$ via this 1-morphism. In particular, by the material above, if $\mathcal{F}$ is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then $x^{-1}\mathcal{F}$ is a sheaf on $(Sch/U)_{fppf}$ (or on $(Sch/U)_{Zar}$, $(Sch/U)_{étale}$, $(Sch/U)_{smooth}$, $(Sch/U)_{syntomic}$).

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\varphi : x \to y$ be a morphism of $\mathcal{X}$ lying over the morphism of schemes $a : U \to V$. Recall that a induces a morphism of small étale sites $a_{small} : U_{étale} \to V_{étale}$, see Étale Cohomology, Section 59.34. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $\mathcal{F}|_{U_{étale}}$ and $\mathcal{F}|_{V_{étale}}$ be the restrictions of $\mathcal{F}$ via x and y. There is a natural *comparison* map

06W2 (96.9.2.1) $$c_\varphi : \mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{étale}})$$

of presheaves on $U_{étale}$. Namely, if $V' \to V$ is étale, set $U' = V' \times_V U$ and define c_φ on sections over V' via

$$\begin{array}{ccccc} a_{small,*}(\mathcal{F}|_{U_{étale}})(V') & = & \mathcal{F}|_{U_{étale}}(U') & = & \mathcal{F}(x') \\ {\scriptstyle c_\varphi}\uparrow & & & & \uparrow{\scriptstyle \mathcal{F}(\varphi')} \\ \mathcal{F}|_{V_{étale}}(V') & = & = & = & \mathcal{F}(y') \end{array}$$

Here $\varphi' : x' \to y'$ is a morphism of $\mathcal{X}$ fitting into a commutative diagram

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ y' & \longrightarrow & y \end{array} \quad \text{lying over} \quad \begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle a} \\ V' & \longrightarrow & V \end{array}$$

The existence and uniqueness of φ' follow from the axioms of a category fibred in groupoids. We omit the verification that c_φ so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in $\mathcal{F}$. In case $\mathcal{F}$ is a sheaf for the étale topology we obtain a *comparison* map

06W3 (96.9.2.2) $$c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}}$$

which is also denoted c_φ as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

075D **Lemma 96.9.3.** *Let $\mathcal{F}$ be an étale sheaf on $\mathcal{X} \to (Sch/S)_{fppf}$.*

(1) *If $\varphi : x \to y$ and $\psi : y \to z$ are morphisms of $\mathcal{X}$ lying over $a : U \to V$ and $b : V \to W$, then the composition*

$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{étale}}$$

is equal to $c_{\psi\circ\varphi}$ via the identification

$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{étale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})).$$

(2) *If $\varphi : x \to y$ lies over an étale morphism of schemes $a : U \to V$, then (96.9.2.2) is an isomorphism.*

(3) *Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ and y is an object of $\mathcal{Y}$ lying over the scheme U with image $x = f(y)$. Then there is a canonical identification $f^{-1}\mathcal{F}|_{U_{étale}} = \mathcal{F}|_{U_{étale}}$.*

(4) *Moreover, given $\psi : y' \to y$ in $\mathcal{Y}$ lying over $a : U' \to U$ the comparison map $c_\psi : a_{small}^{-1}(f^{-1}\mathcal{F}|_{U_{étale}}) \to f^{-1}\mathcal{F}|_{U'_{étale}}$ is equal to the comparison map $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{étale}} \to \mathcal{F}|_{U'_{étale}}$ via the identifications in (3).*

Proof. The verification of these properties is omitted. □

Next, we turn to the restriction of (pre)sheaves of modules.

06W9 **Lemma 96.9.4.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The equivalence of Lemma 96.9.1 extends to an equivalence of ringed sites $(\mathcal{X}_\tau/x, \mathcal{O}_\mathcal{X}|_x) \to ((Sch/U)_\tau, \mathcal{O})$.*

Proof. This is immediate from the construction of the structure sheaves. □

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a (pre)sheaf of modules on $\mathcal{X}$ as in Definition 96.7.1. Let x be an object of $\mathcal{X}$ lying over U. Then Lemma 96.9.4 guarantees that the restriction $x^{-1}\mathcal{F}$ is a (pre)sheaf of modules on $(Sch/U)_{fppf}$. We will sometimes write $x^*\mathcal{F} = x^{-1}\mathcal{F}$ in this case. Similarly, if $\mathcal{F}$ is a sheaf for the Zariski, étale, smooth, or syntomic topology, then $x^{-1}\mathcal{F}$ is as well. Moreover, the restriction $\mathcal{F}|_{U_{étale}} = x^{-1}\mathcal{F}|_{U_{étale}}$ to U is a presheaf of $\mathcal{O}_{U_{étale}}$-modules. If $\mathcal{F}$ is a sheaf for the étale topology, then $\mathcal{F}|_{U_{étale}}$ is a sheaf of modules. Moreover, if $\varphi : x \to y$ is a morphism of $\mathcal{X}$ lying over $a : U \to V$ then the comparison map (96.9.2.2) is compatible with $a_{small}^\sharp$ (see Descent, Remark 35.8.4) and induces a *comparison* map

06WC (96.9.4.1) $$c_\varphi : a_{small}^*(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}}$$

of $\mathcal{O}_{U_{étale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 96.9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

06W4 **Lemma 96.9.5.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The site $\mathcal{X}_\tau$ has enough points.*

Proof. By Sites, Lemma 7.38.5 we have to show that there exists a family of objects x of $\mathcal{X}$ such that $\mathcal{X}_\tau/x$ has enough points and such that the sheaves $h_x^\#$ cover the final object of the category of sheaves. By Lemma 96.9.1 and Étale Cohomology, Lemma 59.30.1 we see that $\mathcal{X}_\tau/x$ has enough points for every object x and we win. □

96.10. Restriction to algebraic spaces

076P In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 34.4.14 for algebraic spaces.

073M **Lemma 96.10.1.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space F. Then there exists a continuous and cocontinuous functor $F_{étale} \to \mathcal{X}_{étale}$ which induces a morphism of ringed sites*

$$\pi_F : (\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow (F_{étale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{étale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X})$$

such that $\pi_F \circ i_F = id$. *Moreover* $\pi_{F,*} = i_F^{-1}$.

Proof. Choose an equivalence $j : \mathcal{S}_F \to \mathcal{X}$, see Algebraic Stacks, Sections 94.7 and 94.8. An object of $F_{étale}$ is a scheme U together with an étale morphism $\varphi : U \to F$. Then φ is an object of $\mathcal{S}_F$ over U. Hence $j(\varphi)$ is an object of $\mathcal{X}$ over U. In this way j induces a functor $u : F_{étale} \to \mathcal{X}$. It is clear that u is continuous and cocontinuous for the étale topology on $\mathcal{X}$. Since j is an equivalence, the functor u is fully faithful. Also, fibre products and equalizers exist in $F_{étale}$ and u commutes with them because these are computed on the level of underlying schemes in $F_{étale}$. Thus Sites, Lemmas 7.21.5, 7.21.6, and 7.21.7 apply. In particular u defines a morphism of topoi $i_F : Sh(F_{étale}) \to Sh(\mathcal{X}_{étale})$ and there exists a left adjoint $i_{F,!}$ of i_F^{-1} which commutes with fibre products and equalizers.

We claim that $i_{F,!}$ is exact. If this is true, then we can define π_F by the rules $\pi_F^{-1} = i_{F,!}$ and $\pi_{F,*} = i_F^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_{F,!}$ is right exact and preserves fibre products. Hence it suffices to show that $i_{F,!}* = *$ where $*$ indicates the final object in the category of sheaves of sets. Let U be a scheme and let $\varphi : U \to F$ be surjective and étale. Set $R = U \times_F U$. Then

$$h_R \rightrightarrows h_U \longrightarrow *$$

is a coequalizer diagram in $Sh(F_{étale})$. Using the right exactness of $i_{F,!}$, using $i_{F,!} = (u_p)^\#$, and using Sites, Lemma 7.5.6 we see that

$$h_{u(R)} \rightrightarrows h_{u(U)} \longrightarrow i_{F,!}*$$

is a coequalizer diagram in $Sh(\mathcal{X}_{étale})$. Using that j is an equivalence and that $F = U/R$ it follows that the coequalizer in $Sh(\mathcal{X}_{étale})$ of the two maps $h_{u(R)} \to h_{u(U)}$ is $*$. We omit the proof that these morphisms are compatible with structure sheaves. □

0GQA **Remark 96.10.2.** The constructions in Lemma 96.10.1 are compatible with étale localization. Here is a precise formulation. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G, and that the induced morphism $f : F \to G$ of algebraic spaces is étale. Denote $f_{small} : F_{étale} \to G_{étale}$ the corresponding morphism of ringed topoi. Then

$$\begin{array}{ccc}
(Sh(F_{étale}), \mathcal{O}_F) & \xrightarrow{f_{small}} & (Sh(G_{étale}), \mathcal{O}_G) \\
\downarrow{\scriptstyle i_F} & & \downarrow{\scriptstyle i_G} \\
(Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X}) & \xrightarrow{f} & (Sh(\mathcal{Y}_{étale}), \mathcal{O}_\mathcal{Y}) \\
\downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_G} \\
(Sh(F_{étale}), \mathcal{O}_F) & \xrightarrow{f_{small}} & (Sh(G_{étale}), \mathcal{O}_G)
\end{array}$$

is a commutative diagram of ringed topoi. We omit the details.

Assume $\mathcal{X}$ is an algebraic stack represented by the algebraic space F. Let $j : \mathcal{S}_F \to \mathcal{X}$ be an equivalence and denote $u : F_{étale} \to \mathcal{X}_{étale}$ the functor of the proof of Lemma 96.10.1 above. Given a sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of i_F^{-1} as a *restriction functor* similarly to Definition 96.9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

075K (96.10.2.1) $$\mathcal{F}|_{F_{étale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F}$$

in this situation.

073N **Lemma 96.10.3.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces, and $f_{small} : F_{étale} \to G_{étale}$ the corresponding morphism of ringed topoi. Then*

$$\begin{array}{ccc} (Sh(\mathcal{X}_{étale}), \mathcal{O}_{\mathcal{X}}) & \xrightarrow{f} & (Sh(\mathcal{Y}_{étale}), \mathcal{O}_{\mathcal{Y}}) \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_G} \\ (Sh(F_{étale}), \mathcal{O}_F) & \xrightarrow{f_{small}} & (Sh(G_{étale}), \mathcal{O}_G) \end{array}$$

is a commutative diagram of ringed topoi.

Proof. This is similar to Topologies, Lemma 34.4.17 (3) but there is a small snag due to the fact that $F \to G$ may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences $j : \mathcal{S}_F \to \mathcal{X}$ and $j' : \mathcal{S}_G \to \mathcal{Y}$ which induce functors $u : F_{étale} \to \mathcal{X}$ and $u' : G_{étale} \to \mathcal{Y}$ as in the proof of Lemma 96.10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over Sch_{fppf} (see discussion in Section 96.3) we may assume that $\mathcal{X} = \mathcal{S}_F$ and $\mathcal{Y} = \mathcal{S}_G$ and that $f : \mathcal{S}_F \to \mathcal{S}_G$ is the functor associated to the morphism $f : F \to G$. Correspondingly we will omit u and u' from the notation, i.e., given an object $U \to F$ of $F_{étale}$ we denote U/F the corresponding object of $\mathcal{X}$. Similarly for G.

Let $\mathcal{G}$ be a sheaf on $\mathcal{X}_{étale}$. To prove (2) we compute $\pi_{G,*}f_*\mathcal{G}$ and $f_{small,*}\pi_{F,*}\mathcal{G}$. To do this let $V \to G$ be an object of $G_{étale}$. Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\Big((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)$$

see (96.5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space $V \times_G F$. And $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf on $\mathcal{S}_{V \times_G F}$ for the étale topology.

In particular, if $V \times_G F$ is representable, i.e., if it is a scheme, then $\pi_{G,*}f_*\mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$ and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme U and a surjective étale morphism $U \to V \times_G F$. Set $R = U \times_{V \times_G F} U$. Then $U/V \times_G F$ and $R/V \times_G F$ are objects of the fibre product

category above. Since $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf for the étale topology on $\mathcal{S}_{V\times_G F}$ the diagram

$$\Gamma\Big((Sch/V)_{fppf}\times_{\mathcal{Y}}\mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)\longrightarrow \mathrm{pr}^{-1}\mathcal{G}(U/V\times_G F)\rightrightarrows \mathrm{pr}^{-1}\mathcal{G}(R/V\times_G F)$$

is an equalizer diagram. Note that $\mathrm{pr}^{-1}\mathcal{G}(U/V\times_G F)=\mathcal{G}(U/F)$ and $\mathrm{pr}^{-1}\mathcal{G}(R/V\times_G F)=\mathcal{G}(R/F)$ by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 66.18 (especially, Properties of Spaces, Remark 66.18.4 and Lemma 66.18.8) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V)\longrightarrow \pi_{F,*}\mathcal{G}(U/F)\rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have $\pi_{F,*}\mathcal{G}(U/F)=\mathcal{G}(U/F)$ and $\pi_{F,*}\mathcal{G}(U/F)=\mathcal{G}(U/F)$ we obtain a canonical identification $f_{small,*}\pi_{F,*}\mathcal{G}(V)=\pi_{G,*}f_*\mathcal{G}(V)$. We omit the proof that this is compatible with restriction mappings and that it is functorial in $\mathcal{G}$. □

Let $f:\mathcal{X}\to\mathcal{Y}$ and $f:F\to G$ be as in the second part of the lemma above. A consequence of the lemma, using (96.10.2.1), is that

075M (96.10.3.1) $$(f_*\mathcal{F})|_{G_{étale}}=f_{small,*}(\mathcal{F}|_{F_{étale}})$$

for any sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (96.10.3.1) is an isomorphism of $\mathcal{O}_G$-modules on $G_{étale}$.

Finally, suppose that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "a"] \arrow[dr, "f"'] & \mathcal{V} \arrow[d, "g"] \arrow[dl, phantom, "\swarrow\varphi"] \\ & \mathcal{X} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$, that $\mathcal{F}$ is a sheaf on $\mathcal{X}_{étale}$, and that $\mathcal{U},\mathcal{V}$ are representable by algebraic spaces U,V. Then we obtain a comparison map

076Q (96.10.3.2) $$c_\varphi : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{étale}})\longrightarrow f^{-1}\mathcal{F}|_{U_{étale}}$$

where $a:U\to V$ denotes the morphism of algebraic spaces corresponding to a. This is the analogue of (96.9.2.2). We define c_φ as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{étale}}\longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{étale}})=(a_*f^{-1}\mathcal{F})|_{V_{étale}}$$

(equality by (96.10.3.1)) which is the restriction to V (96.10.2.1) of the map

$$g^{-1}\mathcal{F}\to a_*a^{-1}g^{-1}\mathcal{F}=a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules c_φ induces a *comparison* map

076R (96.10.3.3) $$c_\varphi : a_{small}^{*}(g^{*}\mathcal{F}|_{V_{étale}})\longrightarrow f^{*}\mathcal{F}|_{U_{étale}}$$

of $\mathcal{O}_{U_{étale}}$-modules. This is the analogue of (96.9.4.1). Note that the properties (1), (2), (3), and (4) of Lemma 96.9.3 hold in this setting as well.

96.11. Quasi-coherent modules

06WF At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

06WG **Definition 96.11.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. A *quasi-coherent module on $\mathcal{X}$*, or a *quasi-coherent $\mathcal{O}_\mathcal{X}$-module* is a quasi-coherent module on the ringed site $(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ as in Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted $QCoh(\mathcal{O}_\mathcal{X})$.

If $\mathcal{X}$ is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that $QCoh(\mathcal{O}_\mathcal{X})$ is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [**Ols07b**, Definition 6.1] in Cohomology on Stacks, Lemma 96.12.2. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

06WH **Lemma 96.11.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{O}_\mathcal{X})$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. □

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 96.12.2 for a characterization in terms of restrictions.

06WI **Lemma 96.11.3.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if $x^*\mathcal{F}$ is a quasi-coherent sheaf on $(Sch/U)_{fppf}$ for every object x of $\mathcal{X}$ with $U = p(x)$.*

Proof. By Lemma 96.11.2 the condition is necessary. Conversely, since $x^*\mathcal{F}$ is just the restriction to $\mathcal{X}_{fppf}/x$ we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18.18). □

0EM8 **Lemma 96.11.4.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of modules on $\mathcal{X}$. The following are equivalent*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 18.23.1,*
2. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 18.23.1, and*
3. *$\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$ in the sense of Definition 96.11.1.*

Proof. Assume either (1), (2), or (3) holds. Let x be an object of $\mathcal{X}$ lying over the scheme U. Recall that $x^*\mathcal{F} = x^{-1}\mathcal{F}$ is just the restriction to $\mathcal{X}/x = (Sch/U)_\tau$ where $\tau = fppf$, $\tau = étale$, or $\tau = Zar$, see Section 96.9. By the definition of quasi-coherent modules on a ringed site this restriction is quasi-coherent provided $\mathcal{F}$ is. By Descent, Proposition 35.8.9 we see that $x^*\mathcal{F}$ is the sheaf associated to a quasi-coherent $\mathcal{O}_U$-module and is therefore a quasi-coherent module in the fppf,

étale, and Zariski topology; here we also use Descent, Lemma 35.8.1 and Definition 35.8.2. Since this holds for every object x of $\mathcal{X}$, we see that $\mathcal{F}$ is a sheaf in any of the three topologies. Moreover, we find that $\mathcal{F}$ is quasi-coherent in any of the three topologies directly from the definition of being quasi-coherent and the fact that x is an arbitrary object of $\mathcal{X}$. □

96.12. Locally quasi-coherent modules

0GQB Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

06WJ **Definition 96.12.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We say $\mathcal{F}$ is *locally quasi-coherent*[2] if $\mathcal{F}$ is a sheaf for the étale topology and for every object x of $\mathcal{X}$ the restriction $x^*\mathcal{F}|_{U_{étale}}$ is a quasi-coherent sheaf. Here $U = p(x)$.

We use $LQCoh(\mathcal{O}_\mathcal{X})$ to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$\begin{array}{ccc} QCoh(\mathcal{O}_\mathcal{X}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{X}) \\ \downarrow & & \downarrow \\ LQCoh(\mathcal{O}_\mathcal{X}) & \longrightarrow & Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \end{array}$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counter part for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are "cartesian", i.e., satisfy the second condition of the lemma below.

06WK **Lemma 96.12.2.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if the following two conditions hold*

(1) *$\mathcal{F}$ is locally quasi-coherent, and*
(2) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ lying over $f : U \to V$ the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (96.9.4.1) is an isomorphism.*

Proof. Assume $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F}$ is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of $\mathcal{F}$ to a ringed topos is quasi-coherent, hence the restrictions $x^*\mathcal{F}|_{U_{étale}}$ are quasi-coherent. This proves $\mathcal{F}$ is locally quasi-coherent. Let y be an object of $\mathcal{X}$ with $V = p(y)$. We have seen that $\mathcal{X}/y = (Sch/V)_{fppf}$. By Descent, Proposition 35.8.9 it follows that $y^*\mathcal{F}$ is the quasi-coherent module associated to a (usual) quasi-coherent module $\mathcal{F}_V$ on the scheme V. Hence certainly the comparison maps (96.9.4.1) are isomorphisms.

Conversely, suppose that $\mathcal{F}$ satisfies (1) and (2). Let y be an object of $\mathcal{X}$ with $V = p(y)$. Denote $\mathcal{F}_V$ the quasi-coherent module on the scheme V corresponding to the restriction $y^*\mathcal{F}|_{V_{étale}}$ which is quasi-coherent by assumption (1), see Descent, Proposition 35.8.9. Condition (2) now signifies that the restrictions $x^*\mathcal{F}|_{U_{étale}}$ for x over y are each isomorphic to the (étale sheaf associated to the) pullback of $\mathcal{F}_V$

[2] This is nonstandard notation.

via the corresponding morphism of schemes $U \to V$. Hence $y^*\mathcal{F}$ is the sheaf on $(Sch/V)_{fppf}$ associated to $\mathcal{F}_V$. Hence it is quasi-coherent (by Descent, Proposition 35.8.9 again) and we see that $\mathcal{F}$ is quasi-coherent on $\mathcal{X}$ by Lemma 96.11.3. □

06WL **Lemma 96.12.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ preserves locally quasi-coherent sheaves.*

Proof. Let $\mathcal{G}$ be locally quasi-coherent on $\mathcal{Y}$. Choose an object x of $\mathcal{X}$ lying over the scheme U. The restriction $x^*f^*\mathcal{G}|_{U_{étale}}$ equals $(f \circ x)^*\mathcal{G}|_{U_{étale}}$ hence is a quasi-coherent sheaf by assumption on $\mathcal{G}$. □

06WM **Lemma 96.12.4.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
2. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ is abelian with kernels and cokernels computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, in other words the inclusion functor is exact.*
3. *Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ if two out of three are locally quasi-coherent so is the third.*
4. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation on $\mathcal{X}_{étale}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*

Proof. In the arguments below x denotes an arbitrary object of $\mathcal{X}$ lying over the scheme U. To show that an object $\mathcal{H}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is in $LQCoh(\mathcal{O}_\mathcal{X})$ we will show that the restriction $x^*\mathcal{H}|_{U_{étale}} = \mathcal{H}|_{U_{étale}}$ is a quasi-coherent object of $Mod(U_{étale}, \mathcal{O}_U)$.

Proof of (1). Let $\mathcal{I} \to LQCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram. Consider the object $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. The pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = \operatorname{colim}_i x^*\mathcal{F}_i$. Similarly we have $x^*\mathcal{F}|_{U_{étale}} = \operatorname{colim}_i x^*\mathcal{F}_i|_{U_{étale}}$. Now by assumption each $x^*\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent. Hence $\operatorname{colim}_i x^*\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent by Descent, Lemma 35.10.3. Thus $x^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent as desired.

Proof of (2). It follows from (1) that cokernels exist in $LQCoh(\mathcal{O}_\mathcal{X})$ and agree with the cokernels computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $LQCoh(\mathcal{O}_\mathcal{X})$ and let $\mathcal{K} = \operatorname{Ker}(\varphi)$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. If we can show that $\mathcal{K}$ is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence $\mathcal{K}|_{U_{étale}}$ is the kernel of the map $\mathcal{F}|_{U_{étale}} \to \mathcal{G}|_{U_{étale}}$, i.e., is the kernel of a map of quasi-coherent sheaves on $U_{étale}$ whence quasi-coherent by Descent, Lemma 35.10.3. This proves (2).

Proof of (3). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Since we are using the étale topology, the restriction $0 \to \mathcal{F}_1|_{U_{étale}} \to \mathcal{F}_2|_{U_{étale}} \to \mathcal{F}_3|_{U_{étale}} \to 0$ is a short exact sequence too. Hence (3) follows from the corresponding statement in Descent, Lemma 35.10.3.

Proof of (4). Let $\mathcal{F}$ and $\mathcal{G}$ be in $LQCoh(\mathcal{O}_\mathcal{X})$. Since restriction to $U_{étale}$ is given by pullback along the morphism of ringed topoi $U_{étale} \to (Sch/U)_{étale} \to \mathcal{X}_{étale}$ we see that the restriction of the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ to $U_{étale}$ is equal to

$\mathcal{F}|_{U_{étale}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{étale}}$, see Modules on Sites, Lemma 18.26.2. Since $\mathcal{F}|_{U_{étale}}$ and $\mathcal{G}|_{U_{étale}}$ are quasi-coherent, so is their tensor product, see Descent, Lemma 35.10.3.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{G}$ be in $LQCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation. Since $(Sch/U)_{étale} = \mathcal{X}_{étale}/x$ is a localization of $\mathcal{X}_{étale}$ at an object we see that the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $(Sch/U)_{étale}$ is equal to

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}|_{(Sch/U)_{étale}}}(\mathcal{F}|_{(Sch/U)_{étale}}, \mathcal{G}|_{(Sch/U)_{étale}})$$

by Modules on Sites, Lemma 18.27.2. The morphism of ringed topoi $(U_{étale}, \mathcal{O}_U) \to ((Sch/U)_{étale}, \mathcal{O})$ is flat as the pullback of $\mathcal{O}$ is $\mathcal{O}_U$. Hence the pullback of $\mathcal{H}$ by this morphism is equal to $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$ by Modules on Sites, Lemma 18.31.4. In other words, the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $U_{étale}$ is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$. Since $\mathcal{F}|_{U_{étale}}$ and $\mathcal{G}|_{U_{étale}}$ are quasi-coherent, so is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$, see Descent, Lemma 35.10.3. We conclude as before. □

In the generality discussed here the category of quasi-coherent sheaves is not abelian. See Examples, Section 110.13. Here is what we can prove without any further work.

06WN **Lemma 96.12.5.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the categories $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{O}_\mathcal{X})$, and $LQCoh(\mathcal{O}_\mathcal{X})$.*
2. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor products $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ computed in $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
3. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*

Proof. Let x be an arbitrary object of $\mathcal{X}$ lying over the scheme U. Let $\tau \in \{Zariski, étale, fppf\}$. To show that an object $\mathcal{H}$ of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ is in $QCoh(\mathcal{O}_\mathcal{X})$ it suffices show that the restriction $x^*\mathcal{H}$ (Section 96.9) is a quasi-coherent object of $Mod((Sch/U)_\tau, \mathcal{O})$. See Lemmas 96.11.3 and 96.11.4. Similarly for being finite locally free. Recall that $(Sch/U)_\tau = \mathcal{X}_\tau/x$ is a localization of $\mathcal{X}_\tau$ at an object. Hence restriction commutes with colimits, tensor products, and forming internal hom (see Modules on Sites, Lemmas 18.14.3, 18.26.2, and 18.27.2). This reduces the lemma to Descent, Lemma 35.10.6. □

96.13. Stackification and sheaves

06WP It turns out that the category of sheaves on a category fibred in groupoids only "knows about" the stackification.

06WQ **Lemma 96.13.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then the morphism of topoi $f : Sh(\mathcal{X}_{fppf}) \to Sh(\mathcal{Y}_{fppf})$ is an equivalence.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. We claim that $f : \mathcal{X} \to \mathcal{Y}$ is a special cocontinuous functor, see Sites, Definition 7.29.2 which will prove the lemma. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. By Stacks, Lemma 8.8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 7.29.1 hold. □

06WR **Lemma 96.13.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then f^* induces equivalences $Mod(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{Y})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. The first assertion is clear from Lemma 96.13.1 and $\mathcal{O}_\mathcal{X} = f^{-1}\mathcal{O}_\mathcal{Y}$. Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 96.11.2. Hence it suffices to show that if $f^*\mathcal{G}$ is quasi-coherent, then $\mathcal{G}$ is. To see this, let y be an object of $\mathcal{Y}$. Translating the condition that $\mathcal{Y}$ is the stackification of $\mathcal{X}$ we see there exists an fppf covering $\{y_i \to y\}$ in $\mathcal{Y}$ such that $y_i \cong f(x_i)$ for some x_i object of $\mathcal{X}$. Say x_i and y_i lie over the scheme U_i. Then $f^*\mathcal{G}$ being quasi-coherent, means that $x_i^* f^*\mathcal{G}$ is quasi-coherent. Since $x_i^* f^*\mathcal{G}$ is isomorphic to $y_i^*\mathcal{G}$ (as sheaves on $(Sch/U_i)_{fppf}$ we see that $y_i^*\mathcal{G}$ is quasi-coherent. It follows from Modules on Sites, Lemma 18.23.3 that the restriction of $\mathcal{G}$ to $\mathcal{Y}/y$ is quasi-coherent. Hence $\mathcal{G}$ is quasi-coherent by Lemma 96.11.3. □

96.14. Quasi-coherent sheaves and presentations

06WS Let us first match quasi-coherent sheaves with our previously defined notions for schemes and algebraic spaces.

0GQC **Lemma 96.14.1.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids which is representable by an algebraic space F. If $\mathcal{F}$ is in $LQCoh(\mathcal{O}_\mathcal{X})$ then the restriction $\mathcal{F}|_{F_{étale}}$ (96.10.2.1) is quasi-coherent.*

Proof. Let U be a scheme étale over F. Then $\mathcal{F}|_{U_{étale}} = (\mathcal{F}|_{F_{étale}})|_{U_{étale}}$. This is clear but see also Remark 96.10.2. Thus the assertion follows from the definitions. □

0GQD **Lemma 96.14.2.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids which is representable by an algebraic space F. The functor (96.10.2.1) defines an equivalence*

$$QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_F), \quad \mathcal{F} \longmapsto \mathcal{F}|_{F_{étale}}$$

with quasi-inverse given by $\mathcal{G} \mapsto \pi_F^*\mathcal{G}$. This equivalence is compatible with pullback for morphisms between categories fibred in groupoids representable by algebraic spaces.

Proof. By Lemma 96.11.4 we may work with the étale topology. We will use the notation and results of Lemma 96.10.1 without further mention. Recall that the restriction functor $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to Mod(F_{étale}, \mathcal{O}_F)$, $\mathcal{F} \mapsto \mathcal{F}|_{F_{étale}}$ is given by i_F^*. By Lemma 96.14.1 or by Modules on Sites, Lemma 18.23.4 we see that $\mathcal{F}|_{F_{étale}}$ is quasi-coherent if $\mathcal{F}$ is quasi-coherent. Hence we get a functor as indicated in the statement of the lemma and we get a functor π_F^* in the opposite direction. Since $\pi_F \circ i_F = \mathrm{id}$ we see that $i_F^*\pi_F^*\mathcal{G} = \mathcal{G}$.

For $\mathcal{F}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ there is a canonical map $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$, namely the map adjoint to the identification $\mathcal{F}|_{F_{étale}} = \pi_{F,*}\mathcal{F}$. We will show that this map is an isomorphism if $\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$. Choose a scheme U and a surjective étale morphism $U \to F$. Denote $x : U \to \mathcal{X}$ the corresponding object of $\mathcal{X}$ over U. It suffices to show that $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$ is an isomorphism after restricting to $\mathcal{X}_{étale}/x = (Sch/U)_{étale}$. Since $U \to F$ is étale, it follows from Remark 96.10.2 that

$$\pi_F^*(\mathcal{F}|_{F_{étale}})|_{\mathcal{X}_{étale}/x} = \pi_U^*(\mathcal{F}|_{U_{étale}})$$

and that the restriction of the map $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$ to $\mathcal{X}_{étale}/x = (Sch/U)_{étale}$ is equal to the corresponding map $\pi_U^*(\mathcal{F}|_{U_{étale}}) \to \mathcal{F}|_{(Sch/U)_{étale}}$. Since we have seen the result is true for schemes in Descent, Section 35.8[3] we conclude.

Compatibility with pullbacks follows from the fact that the quasi-inverse is given by π_F^* and the commutative diagram of ringed topoi in Lemma 96.10.3. □

In Groupoids in Spaces, Definition 78.12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

06WT **Proposition 96.14.3.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Let $\mathcal{X} = [U/R]$ be the quotient stack. The category of quasi-coherent modules on $\mathcal{X}$ is equivalent to the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_\mathcal{X}) \longleftrightarrow QCoh(U, R, s, t, c).$$

where $QCoh(U, R, s, t, c)$ denotes the category of quasi-coherent modules on the groupoid (U, R, s, t, c).

Let $\mathcal{F}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. Denote $\mathcal{U}$, $\mathcal{R}$ the categories fibred in groupoids corresponding to U and R. Denote x the (defining) object of $\mathcal{X}$ over U. Recall that we have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow[s]{} & \mathcal{U} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle x} \\ \mathcal{U} & \xrightarrow{x} & \mathcal{X} \end{array}$$

See Groupoids in Spaces, Lemma 78.20.3. By Lemma 96.3.3 the 2-arrow inherent in the diagram induces an isomorphism $\alpha : t^*x^*\mathcal{F} \to s^*x^*\mathcal{F}$ which satisfies the cocycle condition over $\mathcal{R} \times_{s,\mathcal{U},t} \mathcal{R}$; this is a consequence of Groupoids in Spaces, Lemma 78.23.1. Thus if we set $\mathcal{G} = x^*\mathcal{F}|_{U_{étale}}$ then the equivalence of categories in Lemma 96.14.2 (used several times compatibly with pullbacks) gives an isomorphism $\alpha : t_{small}^*\mathcal{G} \to s_{small}^*\mathcal{G}$ satisfying the cocycle condition on $R \times_{s,U,t} R$, i.e., $(\mathcal{G}, \alpha)$ is an object of $QCoh(U, R, s, t, c)$. The rule $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ is our functor from left to right.

Construction of the functor in the other direction. Let $(\mathcal{G}, \alpha)$ be an object of $QCoh(U, R, s, t, c)$. According to Lemma 96.13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 78.20.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to construct a quasi-coherent module $\mathcal{F}$ on $[U/_pR]$.

Recall that an object $x = (T, u)$ of $[U/_pR]$ is given by a scheme T and a morphism $u : T \to U$. A morphism $(T, u) \to (T', u')$ is given by a pair (f, r) where $f : T \to T'$ and $r : T \to R$ with $s \circ r = u$ and $t \circ r = u' \circ f$. Let us call a *special morphism* any

[3]Namely, if U is a scheme and $\mathcal{F}$ is quasi-coherent on $(Sch/U)_{étale}$, then $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent module $\mathcal{H}$ on the scheme U by Descent, Proposition 35.8.9. In other words, $\mathcal{F} = (\mathrm{id}_{étale,Zar})^*\mathcal{H}$ by Descent, Remark 35.8.6 with notation as in Descent, Lemma 35.8.5. Then we have $\mathrm{id}_{étale,Zar} = \pi_U \circ \mathrm{id}_{small,étale,Zar}$ and hence we see that $\mathcal{F} = \pi_U^*\mathcal{G}$ where $\mathcal{G} = (\mathrm{id}_{small,étale,Zar})^*\mathcal{H}$ is quasi-coherent. Then $\pi_U^*i_U^*\mathcal{F} = \pi_U^*i_U^*\pi_U^*\mathcal{G} = \pi_U^*\mathcal{G} = \mathcal{F}$ as desired.

morphism of the form $(f, e \circ u' \circ f) : (T, u' \circ f) \to (T', u')$. The category of (T, u) with special morphisms is just the category of schemes over U.

With this notation in place, given an object (T, u) of $[U/_pR]$, we set

$$\mathcal{F}(T, u) := \Gamma(T, u^*_{small}\mathcal{G}).$$

Given a morphism $(f, r) : (T, u) \to (T', u')$ we get a map

$$\begin{aligned}
\mathcal{F}(T', u') &= \Gamma(T', (u')^*_{small}\mathcal{G}) \\
&\to \Gamma(T, f^*_{small}(u')^*_{small}\mathcal{G}) = \Gamma(T, (u' \circ f)^*_{small}\mathcal{G}) \\
&= \Gamma(T, (t \circ r)^*_{small}\mathcal{G}) = \Gamma(T, r^*_{small}t^*_{small}\mathcal{G}) \\
&\to \Gamma(T, r^*_{small}s^*_{small}\mathcal{G}) = \Gamma(T, (s \circ r)^*_{small}\mathcal{G}) \\
&= \Gamma(T, u^*_{small}\mathcal{G}) \\
&= \mathcal{F}(T, u)
\end{aligned}$$

where the first arrow is pullback along f and the second arrow is α. Note that if (f, r) is a special morphism, then this map is just pullback along f as $e^*_{small}\alpha = \mathrm{id}$ by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that $\mathcal{F}$ is a presheaf of modules (details omitted). We see that the restriction of $\mathcal{F}$ to $(Sch/T)_{fppf}$ is quasi-coherent by the simple description of the restriction maps of $\mathcal{F}$ in case of a special morphism. Hence $\mathcal{F}$ is a sheaf on $[U/_pR]$ and quasi-coherent (Lemma 96.11.3).

We omit the verification that the functors constructed above are quasi-inverse to each other. □

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 26.7.1. We will see later (Criteria for Representability, Theorem 97.17.2) that the assumptions on the groupoid imply that $\mathcal{X}$ is an algebraic stack.

076S **Lemma 96.14.4.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $\mathcal{X} = [U/R]$ be the quotient stack. Denote x the object of $\mathcal{X}$ over U. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, and let $\mathcal{H}$ be any object of $Mod(\mathcal{O}_\mathcal{X})$. The map*

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{H}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(x^*\mathcal{F}|_{U_{étale}}, x^*\mathcal{H}|_{U_{étale}}), \quad \phi \longmapsto x^*\phi|_{U_{étale}}$$

is injective and its image consists of exactly those $\varphi : x^*\mathcal{F}|_{U_{étale}} \to x^*\mathcal{H}|_{U_{étale}}$ which give rise to a commutative diagram

$$\begin{array}{ccccc}
s^*_{small}(x^*\mathcal{F}|_{U_{étale}}) & \longrightarrow & (x \circ s)^*\mathcal{F}|_{R_{étale}} = (x \circ t)^*\mathcal{F}|_{R_{étale}} & \longleftarrow & t^*_{small}(x^*\mathcal{F}|_{U_{étale}}) \\
\downarrow{\scriptstyle s^*_{small}\varphi} & & & & \downarrow{\scriptstyle t^*_{small}\varphi} \\
s^*_{small}(x^*\mathcal{H}|_{U_{étale}}) & \longrightarrow & (x \circ s)^*\mathcal{H}|_{R_{étale}} = (x \circ t)^*\mathcal{H}|_{R_{étale}} & \longleftarrow & t^*_{small}(x^*\mathcal{H}|_{U_{étale}})
\end{array}$$

of modules on $R_{étale}$ where the horizontal arrows are the comparison maps (96.10.3.3).

Proof. According to Lemma 96.13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 78.20.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf $\mathcal{O}$-modules. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$. By Proposition 96.14.3 and its proof there exists a quasi-coherent module $(\mathcal{G}, \alpha)$ on (U, R, s, t, c) such that $\mathcal{F}$ is given by the rule $\mathcal{F}(T, u) = \Gamma(T, u^*\mathcal{G})$.

In particular $x^*\mathcal{F}|_{U_{étale}} = \mathcal{G}$ and it is clear that the map of the statement of the lemma is injective. Moreover, given a map $\varphi : \mathcal{G} \to x^*\mathcal{H}|_{U_{étale}}$ and given any object $y = (T, u)$ of $[U/_pR]$ we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^*\mathcal{G}) \xrightarrow{u^*_{small}\varphi} \Gamma(T, u^*_{small}x^*\mathcal{H}|_{U_{étale}}) \to \Gamma(T, y^*\mathcal{H}|_{T_{étale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (96.9.4.1) for the sheaf $\mathcal{H}$. This assignment is compatible with the restriction mappings of the sheaves $\mathcal{F}$ and $\mathcal{G}$ for morphisms of $[U/_pR]$ if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of $\mathcal{F}$ are made explicit in terms of $(\mathcal{G}, \alpha)$ in the proof of Proposition 96.14.3. □

96.15. Quasi-coherent sheaves on algebraic stacks

06WU Let $\mathcal{X}$ be an algebraic stack over S. By Algebraic Stacks, Lemma 94.16.2 we can find an equivalence $[U/R] \to \mathcal{X}$ where (U, R, s, t, c) is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 96.14.3. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 78.12.6 we see that $QCoh(\mathcal{O}_\mathcal{X})$ is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

is in general **not** exact. However, exactly the same thing happens for schemes: for most schemes X the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow Mod((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 35.10.2. Parenthetically, the example in the proof of Descent, Lemma 35.10.2 shows that in general the strictly full embedding $QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh(\mathcal{O}_\mathcal{X})$ isn't exact either.

We collect all the results obtained so far in a single statement.

06WV **Lemma 96.15.1.** *Let $\mathcal{X}$ be an algebraic stack over S.*

1. *If $[U/R] \to \mathcal{X}$ is a presentation of $\mathcal{X}$ then there is a canonical equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$.*
2. *The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian.*
3. *The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is right exact but* **not** *exact in general.*
4. *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
6. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
7. *Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ quasi-coherent, then $\mathcal{F}_2$ is quasi-coherent.*

Proof. Properties (4), (5), and (6) were proven in Lemma 96.12.5. Part (1) is Proposition 96.14.3. Part (2) follows from part (1) and Groupoids in Spaces, Lemma 78.12.6 as discussed above. Right exactness of the inclusion functor in (3) follows from (4); please compare with Homology, Lemma 12.7.2. For the nonexactness of the inclusion functor in part (3) see Descent, Lemma 35.10.2. To see (7) observe that it suffices to check the restriction of $\mathcal{F}_2$ to the big site of a scheme is quasi-coherent (Lemma 96.11.3), hence this follows from the corresponding part of Descent, Lemma 35.10.2. □

Next we construct the coherator for modules on an algebraic stack.

0781 **Proposition 96.15.2.** *Let $\mathcal{X}$ be an algebraic stack over S.*

(1) *The category $QCoh(\mathcal{O}_\mathcal{X})$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_\mathcal{X})$ has enough injectives and all limits.*
(2) *The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ has a right adjoint*[4]

$$Q : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 28.23.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 66.32.2. We advise the reader to read either of those proofs first.

Part (1) means $QCoh(\mathcal{O}_\mathcal{X})$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 96.15.1 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

Choose a presentation $\mathcal{X} = [U/R]$ so that (U, R, s, t, c) is a smooth groupoid in algebraic spaces and in particular s and t are flat morphisms of algebraic spaces. By Lemma 96.15.1 above we have $QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$. By Groupoids in Spaces, Lemma 78.14.2 there exists a set T and a family $(\mathcal{F}_t)_{t\in T}$ of quasi-coherent sheaves on $\mathcal{X}$ such that every quasi-coherent sheaf on $\mathcal{X}$ is the directed colimit of its subsheaves which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ we consider the functor

$$QCoh(\mathcal{O}_\mathcal{X})^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_\mathcal{X}(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is fully faithful. □

[4]This functor is sometimes called the *coherator*.

96.16. Cohomology

075E Let S be a scheme and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. For any $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the categories $Ab(\mathcal{X}_\tau)$ and $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ have enough injectives, see Injectives, Theorems 19.7.4 and 19.8.4. Thus we can use the machinery of Cohomology on Sites, Section 21.2 to define the cohomology groups

$$H^p(\mathcal{X}_\tau, \mathcal{F}) = H^p_\tau(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_\tau(x, \mathcal{F})$$

for any $x \in \mathrm{Ob}(\mathcal{X})$ and any object $\mathcal{F}$ of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then we obtain the higher direct images $R^i f_* \mathcal{F}$ in $Ab(\mathcal{Y}_\tau)$ or $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$. Of course, as explained in Cohomology on Sites, Section 21.3 there are also derived versions of $H^p(-)$ and $R^i f_*$.

075F **Lemma 96.16.1.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ be an object lying over the scheme U. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then*

$$H^p_\tau(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if $\tau = étale$, then we also have

$$H^p_{étale}(x, \mathcal{F}) = H^p(U_{étale}, \mathcal{F}|_{U_{étale}}).$$

Proof. The first statement follows from Cohomology on Sites, Lemma 21.7.1 and the equivalence of Lemma 96.9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 59.20.3. □

96.17. Injective sheaves

06WW The pushforward of an injective abelian sheaf or module is injective.

06WX **Lemma 96.17.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$.*

1. *$f_*\mathcal{I}$ is injective in $Ab(\mathcal{Y}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_\tau)$, and*
2. *$f_*\mathcal{I}$ is injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ for $\mathcal{I}$ injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. This follows formally from the fact that f^{-1} is an exact left adjoint of f_*, see Homology, Lemma 12.29.1. □

In the rest of this section we prove that pullback f^{-1} has a left adjoint $f_!$ on abelian sheaves and modules. If f is representable (by schemes or by algebraic spaces), then it will turn out that $f_!$ is exact and f^{-1} will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

06WY **Lemma 96.17.2.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $\mathcal{X}$ has fibre products.*
2. *If the Isom-presheaves of $\mathcal{X}$ are representable by algebraic spaces, then $\mathcal{X}$ has equalizers.*
3. *If $\mathcal{X}$ is an algebraic stack (or more generally a quotient stack), then $\mathcal{X}$ has equalizers.*

Proof. Part (1) follows Categories, Lemma 4.35.15 as $(Sch/S)_{fppf}$ has fibre products.

Let $a, b : x \to y$ be morphisms of $\mathcal{X}$. Set $U = p(x)$ and $V = p(y)$. The category of schemes has equalizers hence we can let $W \to U$ be the equalizer of $p(a)$ and $p(b)$. Denote $c : z \to x$ a morphism of $\mathcal{X}$ lying over $W \to U$. The equalizer of a and b, if it exists, is the equalizer of $a \circ c$ and $b \circ c$. Thus we may assume that $p(a) = p(b) = f : U \to V$. As $\mathcal{X}$ is fibred in groupoids, there exists a unique automorphism $i : x \to x$ in the fibre category of $\mathcal{X}$ over U such that $a \circ i = b$. Again the equalizer of a and b is the equalizer of id_x and i. Recall that the $Isom_{\mathcal{X}}(x)$ is the presheaf on $(Sch/U)_{fppf}$ which to T/U associates the set of automorphisms of $x|_T$ in the fibre category of $\mathcal{X}$ over T, see Stacks, Definition 8.2.2. If $Isom_{\mathcal{X}}(x)$ is representable by an algebraic space $G \to U$, then we see that id_x and i define morphisms $e, i : U \to G$ over U. Set $M = U \times_{e,G,i} U$, which by Morphisms of Spaces, Lemma 67.4.7 is a scheme. Then it is clear that $x|_M \to x$ is the equalizer of the maps id_x and i in $\mathcal{X}$. This proves (2).

If $\mathcal{X} = [U/R]$ for some groupoid in algebraic spaces (U, R, s, t, c) over S, then the hypothesis of (2) holds by Bootstrap, Lemma 80.11.5. If $\mathcal{X}$ is an algebraic stack, then we can choose a presentation $[U/R] \cong \mathcal{X}$ by Algebraic Stacks, Lemma 94.16.2. □

06WZ **Lemma 96.17.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *The functor f transforms fibre products into fibre products.*
(2) *If f is faithful, then f transforms equalizers into equalizers.*

Proof. By Categories, Lemma 4.35.15 we see that a fibre product in $\mathcal{X}$ is any commutative square lying over a fibre product diagram in $(Sch/S)_{fppf}$. Similarly for $\mathcal{Y}$. Hence (1) is clear.

Let $x \to x'$ be the equalizer of two morphisms $a, b : x' \to x''$ in $\mathcal{X}$. We will show that $f(x) \to f(x')$ is the equalizer of $f(a)$ and $f(b)$. Let $y \to f(x')$ be a morphism of $\mathcal{Y}$ equalizing $f(a)$ and $f(b)$. Say x, x', x'' lie over the schemes U, U', U'' and y lies over V. Denote $h : V \to U'$ the image of $y \to f(x')$ in the category of schemes. The morphism $y \to f(x')$ is isomorphic to $f(h^*x') \to f(x')$ by the axioms of fibred categories. Hence, as f is faithful, we see that $h^*x' \to x'$ equalizes a and b. Thus we obtain a unique morphism $h^*x' \to x$ whose image $y = f(h^*x') \to f(x)$ is the desired morphism in $\mathcal{Y}$. □

06X0 **Lemma 96.17.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be faithful 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ is faithful, and*
(2) *if $\mathcal{X}, \mathcal{Z}$ have equalizers, so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. We think of objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ as quadruples (U, x, z, α) where $\alpha : f(x) \to g(z)$ is an isomorphism over U, see Categories, Lemma 4.32.3. A morphism $(U, x, z, \alpha) \to (U', x', z', \alpha')$ is a pair of morphisms $a : x \to x'$ and $b : z \to z'$ compatible with α and α'. Thus it is clear that if f and g are faithful, so is the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$. Now, suppose that $(a, b), (a', b') : (U, x, z, \alpha) \to (U', x', z', \alpha')$ are two morphisms of the 2-fibre product. Then consider the equalizer $x'' \to x$ of a and a' and the equalizer $z'' \to z$ of b and b'. Since f commutes with equalizers (by Lemma 96.17.3)

we see that $f(x'') \to f(x)$ is the equalizer of $f(a)$ and $f(a')$. Similarly, $g(z'') \to g(z)$ is the equalizer of $g(b)$ and $g(b')$. Picture

$$\begin{array}{ccccc} f(x'') & \longrightarrow & f(x) & \overset{f(a)}{\underset{f(a')}{\rightrightarrows}} & f(x') \\ {\scriptstyle \alpha''}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ g(z'') & \longrightarrow & g(z) & \overset{g(b)}{\underset{g(b')}{\rightrightarrows}} & g(z') \end{array}$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of α'' in the category of schemes is the identity of its source. On the other hand, the existence of α'' means that we can assume that x'' and z'' are defined over the same scheme and that the morphisms $x'' \to x$ and $z'' \to z$ have the same image in the category of schemes. Redoing the diagram above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted. □

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that f is faithful.

06X1 **Lemma 96.17.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Ab(\mathcal{X}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{Y}_\tau)$.*

Proof. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. Hence by Modules on Sites, Lemma 18.16.2 the functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. To see (1) we apply Modules on Sites, Lemma 18.16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 96.17.2 and 96.17.3 above. Part (2) follows from this formally, see Homology, Lemma 12.29.1. □

06X2 **Lemma 96.17.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^* : Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ has a left adjoint $f_! : Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \to Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ which agrees with the functor $f_!$ of Lemma 96.17.5 on underlying abelian sheaves. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ for $\mathcal{I}$ injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. Recall that f is a continuous and cocontinuous functor of sites and that $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$. Hence Modules on Sites, Lemma 18.41.1 implies f^* has a left adjoint $f_!^{Mod}$. Let x be an object of $\mathcal{X}$ lying over the scheme U. Then f induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to $(Sch/U)_\tau$, see Lemma 96.9.4. Modules on Sites, Remark 18.41.2 shows that $f_!$ agrees with the functor on abelian sheaves.

Assume now that $\mathcal{X}$ has equalizers and that f is faithful. Lemma 96.17.5 tells us that $f_!$ is exact. Finally, Homology, Lemma 12.29.1 implies the statement on pullbacks of injective modules. □

96.18. The Čech complex

06X3 To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

06X4 (96.18.0.1)
$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & \mathcal{X} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & (Sch/S)_{fppf} & \end{array}$$

We are going to think about $\mathcal{U}$ as a "covering" of $\mathcal{X}$. Hence we want to consider the simplicial object

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \;\substack{\longrightarrow\\ \longrightarrow}\; \mathcal{U}$$

in the category of categories fibred in groupoids over $(Sch/S)_{fppf}$. However, since this is a $(2,1)$-category and not a category, we should say explicitly what we mean. Namely, we let $\mathcal{U}_n$ be the category with objects $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ where $\alpha_i : f(u_i) \to x$ is an isomorphism in $\mathcal{X}$. We denote $f_n : \mathcal{U}_n \to \mathcal{X}$ the 1-morphism which assigns to $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ the object x. Note that $\mathcal{U}_0 = \mathcal{U}$ and $f_0 = f$. Given a map $\varphi : [m] \to [n]$ we consider the 1-morphism $\mathcal{U}_\varphi : \mathcal{U}_n \longrightarrow \mathcal{U}_n$ given by

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_{\varphi(0)}, \ldots, u_{\varphi(m)}, x, \alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(m)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over $\mathcal{X}$. We denote $\mathcal{U}_\bullet$ this simplicial object. If $\mathcal{F}$ is a presheaf of sets on $\mathcal{X}$, then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \;\substack{\longrightarrow}\; \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \;\substack{\longrightarrow\\ \longrightarrow}\; \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If $\mathcal{F}$ is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let $\mathcal{U} \to \mathcal{X}$ be as above and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. The *Čech complex* associated to the situation is denoted $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$. It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 14.25. It has terms

$$\check{\mathcal{C}}^n(\mathcal{U} \to \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta_i^{n+1} : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where δ_i^{n+1} corresponds to the map $[n] \to [n+1]$ omitting the index i. Note that the map $\Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$ is in the kernel of the differential d^0. Hence we define the *extended Čech complex* to be the complex

$$\ldots \to 0 \to \Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \to \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \to \ldots$$

with $\Gamma(\mathcal{X}, \mathcal{F})$ placed in degree -1. The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

06X5 **Lemma 96.18.1.** *Generalities on Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow[h]{} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

is 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, e^{-1}\mathcal{F})$$

(2) *if h and e are equivalences, then the map of (1) is an isomorphism,*
(3) *if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated Čech complexes are isomorphic.*

Proof. In the situation of (1) let $t : f \circ h \to e \circ g$ be a 2-morphism. The map on complexes is given in degree n by pullback along the 1-morphisms $\mathcal{V}_n \to \mathcal{U}_n$ given by the rule

$$(v_0, \ldots, v_n, y, \beta_0, \ldots, \beta_n) \longmapsto (h(v_0), \ldots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \ldots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2). □

06X6 **Lemma 96.18.2.** *If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $id_\mathcal{X}$ then the extended Čech complex is homotopic to zero.*

Proof. Set $\mathcal{U}' = \mathcal{U} \times_\mathcal{X} \mathcal{X}$ equal to the fibre product as described in Categories, Lemma 4.32.3. Set $f' : \mathcal{U}' \to \mathcal{X}$ equal to the second projection. Then $\mathcal{U} \to \mathcal{U}'$, $u \mapsto (u, f(x), 1)$ is an equivalence over $\mathcal{X}$, hence we may replace $(\mathcal{U}, f)$ by $(\mathcal{U}', f')$ by Lemma 96.18.1. The advantage of this is that now f' has a section s' such that $f' \circ s' = \mathrm{id}_\mathcal{X}$ on the nose. Namely, if $t : s \circ f \to \mathrm{id}_\mathcal{X}$ is a 2-isomorphism then we can set $s'(x) = (s(x), x, t_x)$. Thus we may assume that $f \circ s = \mathrm{id}_\mathcal{X}$.

In the case that $f \circ s = \mathrm{id}_\mathcal{X}$ the result follows from general principles. We give the homotopy explicitly. Namely, for $n \geq 0$ define $s_n : \mathcal{U}_n \to \mathcal{U}_{n+1}$ to be the 1-morphism defined by the rule on objects

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_0, \ldots, u_n, s(x), x, \alpha_0, \ldots, \alpha_n, \mathrm{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along s_n. We also set $s_{-1} = s$ and $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F})$ equal to pullback along s_{-1}. Then the family of maps $\{h^n\}_{n \geq 0}$ is a homotopy between 1 and 0 on the extended Čech complex. □

96.19. The relative Čech complex

06X7 Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ as in (96.18.0.1). Consider the associated simplicial object $\mathcal{U}_\bullet$ and the maps $f_n : \mathcal{U}_n \to \mathcal{X}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Finally, suppose that $\mathcal{F}$ is a sheaf (of sets) on $\mathcal{X}_\tau$. Then

$$f_{0,*}f_0^{-1}\mathcal{F} \rightrightarrows f_{1,*}f_1^{-1}\mathcal{F} \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} f_{2,*}f_2^{-1}\mathcal{F}$$

is a cosimplicial sheaf on $\mathcal{X}_\tau$ where we use the pullback maps introduced in Sites, Section 7.45. If $\mathcal{F}$ is an abelian sheaf, then $f_{n,*}f_n^{-1}\mathcal{F}$ form a cosimplicial abelian sheaf on $\mathcal{X}_\tau$. The associated complex (see Simplicial, Section 14.25)

$$\ldots \to 0 \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is called the *relative Čech complex* associated to the situation. We will denote this complex $\mathcal{K}^\bullet(f, \mathcal{F})$. The *extended relative Čech complex* is the complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

with $\mathcal{F}$ in degree -1. The extended relative Čech complex is acyclic if and only if the map $\mathcal{F}[0] \to \mathcal{K}^\bullet(f, \mathcal{F})$ is a quasi-isomorphism of complexes of sheaves.

06X8 **Remark 96.19.1.** We can define the complex $\mathcal{K}^\bullet(f, \mathcal{F})$ also if $\mathcal{F}$ is a presheaf, only we cannot use the reference to Sites, Section 7.45 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathcal{X} & \end{array}$$

of categories fibred in groupoids over $(Sch/S)_{fppf}$ and a presheaf $\mathcal{G}$ on $\mathcal{U}$ we can define the pullback map $f_*\mathcal{G} \to g_*h^{-1}\mathcal{G}$ as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map $\mathcal{G} \to h_*h^{-1}\mathcal{G}$. This works because in our situation the functors h_* and h^{-1} are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 96.3 and 96.4.

06X9 **Lemma 96.19.2.** *Generalities on relative Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

is a 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$.

(2) *if h and e are equivalences, then the map of (1) is an isomorphism,*

(3) *if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated relative Čech complexes are isomorphic,*

Proof. Literally the same as the proof of Lemma 96.18.1 using the pullback maps of Remark 96.19.1. □

06XA **Lemma 96.19.3.** *If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $id_{\mathcal{X}}$ then the extended relative Čech complex is homotopic to zero.*

Proof. Literally the same as the proof of Lemma 96.18.2. □

06XB **Remark 96.19.4.** Let us "compute" the value of the relative Čech complex on an object x of $\mathcal{X}$. Say $p(x) = U$. Consider the 2-fibre product diagram (which serves to introduce the notation $g : \mathcal{V} \to \mathcal{Y}$)

$$\begin{array}{ccccc} \mathcal{V} & = & (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} & \longrightarrow & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & = & (Sch/U)_{fppf} & \xrightarrow{x} & \mathcal{X} \end{array}$$

Note that the morphism $\mathcal{V}_n \to \mathcal{U}_n$ of the proof of Lemma 96.18.1 induces an equivalence $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}_n$. Hence we see from (96.5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object x of $\mathcal{X}$ is the Čech complex of the base change of f to $\mathcal{X}/x \cong (Sch/U)_{fppf}$. This implies for example that Lemma 96.18.2 implies Lemma 96.19.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

06XC **Lemma 96.19.5.** *Let*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow[h]{} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$ and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. Then the map $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ of Lemma 96.19.2 is an isomorphism of complexes of abelian presheaves.

Proof. Let y be an object of $\mathcal{Y}$ lying over the scheme T. Set $x = e(y)$. We are going to show that the map induces an isomorphism on sections over y. Note that

$$\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \to (Sch/T)_{fppf}, x^{-1}\mathcal{F})$$

by Remark 96.19.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 96.19.4. Note that $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$ and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over y is an isomorphism by Lemma 96.18.1. □

Exactness can be checked on a "covering".

06XD **Lemma 96.19.6.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

be a complex in $Ab(\mathcal{X}_\tau)$. Assume that

(1) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$, and*
(2) *$f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ is exact.*

Then the sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact.

Proof. Let x be an object of $\mathcal{X}$ lying over the scheme T. Consider the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T)_\tau$. It suffices to show this sequence is exact. By assumption there exists a τ-covering $\{T_i \to T\}$ such that $x|_{T_i}$ is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$ over T_i and moreover the sequence $u_i^{-1}f^{-1}\mathcal{F} \to u_i^{-1}f^{-1}\mathcal{G} \to u_i^{-1}f^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T_i)_\tau$ is exact. Since $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$ we conclude that the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ become exact after localizing at each of the members of a covering, hence the sequence is exact. □

06XE **Proposition 96.19.7.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. If*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$, and*
(2) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*

then the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is exact in $Ab(\mathcal{X}_\tau)$.

Proof. By Lemma 96.19.6 it suffices to check exactness after pulling back to $\mathcal{U}$. By Lemma 96.19.5 the pullback of the extended relative Čech complex is isomorphic to the extend relative Čech complex for the morphism $\mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{U}$ and an abelian sheaf on $\mathcal{U}_\tau$. Since there is a section $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \to \mathcal{U} \times_\mathcal{X} \mathcal{U}$ exactness follows from Lemma 96.19.3. □

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

06XF **Lemma 96.19.8.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
(2) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
(3) *the category $\mathcal{U}$ has equalizers, and*
(4) *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian groups

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. Before we start the proof we make some remarks. By Lemma 96.17.4 (and induction) all of the categories fibred in groupoids $\mathcal{U}_p$ have equalizers and all of the morphisms $f_p : \mathcal{U}_p \to \mathcal{X}$ are faithful. Let $\mathcal{I}$ be an injective object of $Ab(\mathcal{X}_\tau)$. By Lemma 96.17.5 we see $f_p^{-1}\mathcal{I}$ is an injective object of $Ab((\mathcal{U}_p)_\tau)$. Hence $f_{p,*}f_p^{-1}\mathcal{I}$ is

an injective object of $Ab(\mathcal{X}_\tau)$ by Lemma 96.17.1. Hence Proposition 96.19.7 shows that the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{I} \to f_{0,*}f_0^{-1}\mathcal{I} \to f_{1,*}f_1^{-1}\mathcal{I} \to f_{2,*}f_2^{-1}\mathcal{I} \to \ldots$$

is an exact complex in $Ab(\mathcal{X}_\tau)$ all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I})$ is quasi-isomorphic to $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$.

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 12.25)

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I}^\bullet)$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The discussion above shows that Homology, Lemma 12.25.4 applies which shows that $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$. Hence the other spectral sequence is as indicated in the lemma. □

To be sure there is a version for modules as well.

06XG **Lemma 96.19.9.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of $\Gamma(\mathcal{O}_\mathcal{X})$-modules

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. The proof of this lemma is identical to the proof of Lemma 96.19.8 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 96.17.6 instead of Lemma 96.17.5. □

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

06XH **Lemma 96.19.10.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

1. *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object y of $\mathcal{Y}$ there exists an fppf covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*
2. *Assume that f is representable by algebraic spaces, surjective, and smooth. Then for any object y of $\mathcal{Y}$ there exists an étale covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*

Proof. Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective, flat, and locally of finite presentation. Choose a scheme U and a surjective étale

morphism $U \to W$. Then $U \to V$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.39.7, 67.39.8, 67.5.4, 67.28.2, and 67.30.3). Hence $\{U \to V\}$ is an fppf covering. Denote x the object of $\mathcal{X}$ over U corresponding to the 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$. Then $\{f(x) \to y\}$ is the desired fppf covering of $\mathcal{Y}$.

Proof of (2). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective and smooth. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective and smooth (see Morphisms of Spaces, Lemmas 67.39.6, 67.5.4, and 67.37.2). Hence $\{U \to V\}$ is a smooth covering. By More on Morphisms, Lemma 37.38.7 there exists an étale covering $\{V_i \to V\}$ such that each $V_i \to V$ factors through U. Denote x_i the object of $\mathcal{X}$ over V_i corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \to (Sch/U)_{fppf} \to \mathcal{X}.$$

Then $\{f(x_i) \to y\}$ is the desired étale covering of $\mathcal{Y}$. □

072D **Lemma 96.19.11.** *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian sheaves on $\mathcal{Y}_\tau$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. Note that the assumptions on $f : \mathcal{U} \to \mathcal{X}$ and $\mathcal{F}$ are identical to those in Lemma 96.19.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \to g_*\mathcal{I} \to (g \circ f_0)_* f_0^{-1}\mathcal{I} \to (g \circ f_1)_* f_1^{-1}\mathcal{I} \to \ldots$$

is exact if $\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$. Having said this, consider the two spectral sequences of Homology, Section 12.25 associated to the double complex $\mathcal{C}^{\bullet,\bullet}$ with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_* \mathcal{I}^q$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The first spectral sequence implies, via Homology, Lemma 12.25.4, that $g_*\mathcal{I}^\bullet$ is quasi-isomorphic to the total complex associated to $\mathcal{C}^{\bullet,\bullet}$. Since $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$ (see Lemma 96.17.5) the second spectral sequence has terms $E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F}$ as in the statement of the lemma. □

072E **Lemma 96.19.12.** *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*

(2) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
(3) *the category $\mathcal{U}$ has equalizers, and*
(4) *the functor f is faithful.*

Then there is a first quadrant spectral sequence in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. The proof is identical to the proof of Lemma 96.19.11 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 96.17.6 instead of Lemma 96.17.5. □

96.20. Cohomology on algebraic stacks

06XI Let $\mathcal{X}$ be an algebraic stack over S. In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on $\mathcal{X}$. In fact, we have constructed a site $\mathcal{X}_\tau$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a notion of an abelian sheaf $\mathcal{F}$ on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the *derived global sections* or *total cohomology*

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{étale}(\mathcal{X}, \mathcal{F}), \ldots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith cohomology group of $\mathcal{F}$ is the ith cohomology of the total cohomology. We will denote this

$$H^i_{Zar}(\mathcal{X}, \mathcal{F}), H^i_{étale}(\mathcal{X}, \mathcal{F}), \ldots, H^i_{fppf}(\mathcal{X}, \mathcal{F}).$$

It will turn out that $H^i_{étale} = H^i_{smooth}$ because of More on Morphisms, Lemma 37.38.7.

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute its total cohomology, resp. cohomology groups; the end result is quasi-isomorphic, resp. isomorphic to the cohomology of $\mathcal{F}$ viewed as a sheaf of abelian groups by the very general Cohomology on Sites, Lemma 21.12.4.

So far our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are $(p+1)$-factors. Also, recall that a sheaf on $\mathcal{X}$ is a sheaf for the fppf topology. Note that if $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

06XJ **Proposition 96.20.1.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks.*

(1) *Let $\mathcal{F}$ be an abelian étale sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence*

$$E_1^{p,q} = H^q_{étale}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{étale}(\mathcal{X}, \mathcal{F})$$

(2) *Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence*

$$E_1^{p,q} = H^q_{fppf}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{fppf}(\mathcal{X}, \mathcal{F})$$

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 96.19.8. The 1-morphism f is faithful by Algebraic Stacks, Lemma 94.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 96.17.2. To see (2) apply Lemma 96.19.10. Condition (1) is satisfied by fiat. □

96.21. Higher direct images and algebraic stacks

072F Let $g : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. In the sections above we have constructed a morphism of ringed topoi $g : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. In the chapter on cohomology of sites we have explained how to define higher direct images. Hence the *total direct image* $Rg_*\mathcal{F}$ is defined as $g_*\mathcal{I}^\bullet$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith higher direct image $R^ig_*\mathcal{F}$ is the ith cohomology of the total direct image. Important: it matters which topology τ is used here!

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute total direct image and higher direct images.

So far our only tool to compute the higher direct images of g_* is the result on Čech complexes proved above. This requires the choice of a "covering" $f : \mathcal{U} \to \mathcal{X}$. If $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

072G **Proposition 96.21.1.** *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of algebraic stacks.*

(1) *Assume that f is representable by algebraic spaces, surjective and smooth.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X}_{étale})$ then there is a spectral sequence*
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y}_{étale})$ with higher direct images computed in the étale topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y})$.

(2) *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X})$ then there is a spectral sequence*
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y})$ with higher direct images computed in the fppf topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
   $$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{O}_\mathcal{Y})$.

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 96.19.11 and Lemma 96.19.12. The 1-morphism f is faithful by Algebraic Stacks, Lemma 94.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 96.17.2. To see (2) apply Lemma 96.19.10. Condition (1) is satisfied by fiat in all four cases. □

Here is a description of higher direct images for a morphism of algebraic stacks.

075G **Lemma 96.21.2.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks[5] over S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ pr^{-1}\mathcal{F}\Big)$$

Here y is an object of $\mathcal{Y}$ lying over the scheme V.

Proof. Choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. By the formula for pushforward (96.5.0.1) we see that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{array}{c} \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i+1}\Big) \end{array}$$

Since pr^{-1} is exact, it suffices to show that pr^{-1} preserves injectives. This follows from Lemmas 96.17.5 and 96.17.6 as well as the fact that pr is a representable morphism of algebraic stacks (so that pr is faithful by Algebraic Stacks, Lemma 94.15.2 and that $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ has equalizers by Lemma 96.17.2). □

Here is a trivial base change result.

075H **Lemma 96.21.3.** *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} & \xrightarrow[g']{} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of algebraic stacks over S. Then the base change map is an isomorphism

$$g^{-1} R f_* \mathcal{F} \longrightarrow R f'_* (g')^{-1} \mathcal{F}$$

functorial for $\mathcal{F}$ in $Ab(\mathcal{X}_\tau)$ or $\mathcal{F}$ in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. The isomorphism $g^{-1} f_* \mathcal{F} = f'_* (g')^{-1} \mathcal{F}$ is Lemma 96.5.1 (and it holds for arbitrary presheaves). For the total direct images, there is a base change map because the morphisms g and g' are flat, see Cohomology on Sites, Section 21.15.

[5] This result should hold for any 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

To see that this map is a quasi-isomorphism we can use that for an object y' of $\mathcal{Y}'$ over a scheme V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

We conclude that the induced map $g^{-1}R^if_*\mathcal{F} \to R^if'_*(g')^{-1}\mathcal{F}$ is an isomorphism by Lemma 96.21.2. □

96.22. Comparison

073L In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

075L **Lemma 96.22.1.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S representable by the algebraic space F.*

(1) *If $\mathcal{I}$ injective in $Ab(\mathcal{X}_{étale})$, then $\mathcal{I}|_{F_{étale}}$ is injective in $Ab(F_{étale})$,*
(2) *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab(\mathcal{X}_{étale})$, then $\mathcal{I}^\bullet|_{F_{étale}}$ is a K-injective complex in $Ab(F_{étale})$.*

The same does not hold for modules.

Proof. This follows formally from the fact that the restriction functor $\pi_{F,*} = i_F^{-1}$ (see Lemma 96.10.1) is right adjoint to the exact functor π_F^{-1}, see Homology, Lemma 12.29.1 and Derived Categories, Lemma 13.31.9. To see that the lemma does not hold for modules, we refer the reader to Étale Cohomology, Lemma 59.99.1. □

075N **Lemma 96.22.2.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces.*

(1) *For any $\mathcal{F} \in Ab(\mathcal{X}_{étale})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(G_{étale})$.
(2) *For any object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(\mathcal{O}_G)$.

Proof. Part (1) follows immediately from Lemma 96.22.1 and (96.10.3.1) on choosing an injective resolution of $\mathcal{F}$.

Part (2) can be proved as follows. In Lemma 96.10.3 we have seen that $\pi_G \circ f = f_{small} \circ \pi_F$ as morphisms of ringed sites. Hence we obtain $R\pi_{G,*} \circ Rf_* = Rf_{small,*} \circ R\pi_{F,*}$ by Cohomology on Sites, Lemma 21.19.2. Since the restriction functors $\pi_{F,*}$ and $\pi_{G,*}$ are exact, we conclude. □

075P **Lemma 96.22.3.** *Let S be a scheme. Consider a 2-fibre product square*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

of algebraic stacks over S. Assume that f is representable by algebraic spaces and that $\mathcal{Y}'$ is representable by an algebraic space G'. Then $\mathcal{X}'$ is representable by an

algebraic space F' and denoting $f' : F' \to G'$ the induced morphism of algebraic spaces we have

$$g^{-1}(Rf_*\mathcal{F})|_{G'_{étale}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{étale}})$$

for any $\mathcal{F}$ in $Ab(\mathcal{X}_{étale})$ or in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$

Proof. Follows formally on combining Lemmas 96.21.3 and 96.22.2. □

96.23. Change of topology

075Q Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of $\mathcal{X}$ lying over flat morphisms.

076T **Lemma 96.23.1.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Assume*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (96.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is a sheaf for the fppf topology.

Proof. Let $\{x_i \to x\}$ be an fppf covering of $\mathcal{X}$ lying over the fppf covering $\{f_i : U_i \to U\}$ of schemes over S. By assumption the restriction $\mathcal{G} = \mathcal{F}|_{U_{étale}}$ is quasi-coherent and the comparison maps $f^*_{i,small}\mathcal{G} \to \mathcal{F}|_{U_{i,étale}}$ are isomorphisms. Hence the sheaf condition for $\mathcal{F}$ and the covering $\{x_i \to x\}$ is equivalent to the sheaf condition for $\mathcal{G}^a$ on $(Sch/U)_{fppf}$ and the covering $\{U_i \to U\}$ which holds by Descent, Lemma 35.8.1. □

075R **Lemma 96.23.2.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf $\mathcal{O}_\mathcal{X}$-module such that*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation, the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (96.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module and we have the following

(1) *If $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ is the comparison morphism, then $R\epsilon_*\mathcal{F} = \epsilon_*\mathcal{F}$.*
(2) *The cohomology groups $H^p_{fppf}(\mathcal{X}, \mathcal{F})$ are equal to the cohomology groups computed in the étale topology on $\mathcal{X}$. Similarly for the cohomology groups $H^p_{fppf}(x, \mathcal{F})$ and the derived versions $R\Gamma(\mathcal{X}, \mathcal{F})$ and $R\Gamma(x, \mathcal{F})$.*
(3) *If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ then $R^if_*\mathcal{F}$ is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived pullback.*

Proof. The assertion that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module follows from Lemma 96.23.1. Note that ϵ is a morphism of sites given by the identity functor on $\mathcal{X}$. The sheaf $R^p\epsilon_*\mathcal{F}$ is therefore the sheaf associated to the presheaf $x \mapsto H^p_{fppf}(x, \mathcal{F})$, see Cohomology on Sites, Lemma 21.7.4. To prove (1) it suffices to show that $H^p_{fppf}(x, \mathcal{F}) = 0$ for $p > 0$ whenever x lies over an affine scheme U. By Lemma 96.16.1 we have $H^p_{fppf}(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1}\mathcal{F})$. Combining Descent, Lemma 35.12.4 with

Cohomology of Schemes, Lemma 30.2.2 we see that these cohomology groups are zero.

We have seen above that $\epsilon_*\mathcal{F}$ and $\mathcal{F}$ are the sheaves on $\mathcal{X}_{étale}$ and $\mathcal{X}_{fppf}$ corresponding to the same presheaf on $\mathcal{X}$ (and this is true more generally for any sheaf in the fppf topology on $\mathcal{X}$). We often abusively identify $\mathcal{F}$ and $\epsilon_*\mathcal{F}$ and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 21.14.6.

Finally we prove (3). The sheaf $R^if_*\mathcal{F}$ (resp. $Rf_{étale,*}\mathcal{F}$) is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

where τ is $fppf$ (resp. $étale$), see Lemma 96.21.2. Note that $\mathrm{pr}^{-1}\mathcal{F}$ satisfies properties (a) and (b) also (by Lemmas 96.12.3 and 96.9.3), hence these two presheaves are equal by (2). This immediately implies (3). □

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

07AK **Lemma 96.23.3.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\tau = étale$ (resp. $\tau = fppf$). Let $\mathcal{X}' \subset \mathcal{X}$ be a full subcategory with the following properties*

(1) *if $x \to x'$ is a morphism of $\mathcal{X}$ which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and $x' \in \mathrm{Ob}(\mathcal{X}')$, then $x \in \mathrm{Ob}(\mathcal{X}')$, and*
(2) *there exists an object $x \in \mathrm{Ob}(\mathcal{X}')$ lying over a scheme U such that the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective.*

We get a site $\mathcal{X}'_\tau$ by declaring a covering of $\mathcal{X}'$ to be any family of morphisms $\{x_i \to x\}$ in $\mathcal{X}'$ which is a covering in $\mathcal{X}_\tau$. Then the inclusion functor $\mathcal{X}' \to \mathcal{X}_\tau$ is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi

$$g : Sh(\mathcal{X}'_\tau) \longrightarrow Sh(\mathcal{X}_\tau)$$

and $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$ for all $p \geq 0$ and all $\mathcal{F} \in Ab(\mathcal{X}_\tau)$.

Proof. Note that assumption (1) implies that if $\{x_i \to x\}$ is a covering of $\mathcal{X}_\tau$ and $x \in \mathrm{Ob}(\mathcal{X}')$, then we have $x_i \in \mathrm{Ob}(\mathcal{X}')$. Hence we see that $\mathcal{X}' \to \mathcal{X}$ is continuous and cocontinuous as the coverings of objects of $\mathcal{X}'_\tau$ agree with their coverings seen as objects of $\mathcal{X}_\tau$. We obtain the morphism g and the functor g^{-1} is identified with the restriction functor, see Sites, Lemma 7.21.5.

In particular, if $\{x_i \to x\}$ is a covering in $\mathcal{X}'_\tau$, then for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ then

$$\check{H}^p(\{x_i \to x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \to x\}, \mathcal{F})$$

Thus if $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{X}_\tau$ then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 21.10.2). Hence $H^p(x, g^{-1}\mathcal{I}) = 0$ for all objects x of $\mathcal{X}'$ (Cohomology on Sites, Lemma 21.10.9). In other words injective abelian sheaves on $\mathcal{X}_\tau$ are right acyclic for the functor $H^0(x, g^{-1}-)$. It follows that $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$ for all $\mathcal{F} \in Ab(\mathcal{X})$ and all $x \in \mathrm{Ob}(\mathcal{X}')$.

Choose an object $x \in \mathcal{X}'$ lying over a scheme U as in assumption (2). In particular $\mathcal{X}/x \to \mathcal{X}$ is a morphism of algebraic stacks which representable by algebraic spaces,

surjective, and smooth. (Note that $\mathcal{X}/x$ is equivalent to $(Sch/U)_{fppf}$, see Lemma 96.9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in $Sh(\mathcal{X}_\tau)$ is surjective. Namely, for any object x' of $\mathcal{X}$ there exists a τ-covering $\{x'_i \to x'\}$ such that there exist morphisms $x'_i \to x$, see Lemma 96.19.10. Since g is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in $Sh(\mathcal{X}'_\tau)$ is surjective also. Let $h_{x,n}$ be the $(n+1)$-fold product $h_x \times \ldots \times h_x$. Then we have spectral sequences

07AL (96.23.3.1) $$E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

and

07AM (96.23.3.2) $$E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F})$$

see Cohomology on Sites, Lemma 21.13.2.

Case I: $\mathcal{X}$ has a final object x which is also an object of $\mathcal{X}'$. This case follows immediately from the discussion in the second paragraph above.

Case II: $\mathcal{X}$ is representable by an algebraic space F. In this case the sheaves $h_{x,n}$ are representable by an object x_n in $\mathcal{X}$. (Namely, if $\mathcal{S}_F = \mathcal{X}$ and $x : U \to F$ is the given object, then $h_{x,n}$ is representable by the object $U \times_F \ldots \times_F U \to F$ of $\mathcal{S}_F$.) It follows that $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$. The morphisms $x_n \to x$ lie over smooth morphisms of schemes, hence $x_n \in \mathcal{X}'$ for all n. Hence $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$. Thus in the two spectral sequences (96.23.3.1) and (96.23.3.2) above the $E_1^{p,q}$ terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III: $\mathcal{X}$ is an algebraic stack. We claim that in this case the cohomology groups $H^q(h_{x,p}, \mathcal{F})$ and $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$ agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category $\mathcal{X}/h_{x,n}$, see Sites, Lemma 7.30.3. Since $h_{x,n}$ is the $(n+1)$-fold product of h_x an object of this category is an $(n+2)$-tuple $(y, s_0, \ldots, s_n)$ where y is an object of $\mathcal{X}$ and each $s_i : y \to x$ is a morphism of $\mathcal{X}$. This is a category over $(Sch/S)_{fppf}$. There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over $(Sch/S)_{fppf}$. Namely, if $x : (Sch/U)_{fppf} \to \mathcal{X}$ also denotes the 1-morphism associated with x and $p : \mathcal{X} \to (Sch/S)_{fppf}$ the structure functor, then we can think of $(y, s_0, \ldots, s_n)$ as $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ where y is an object of $\mathcal{X}$, $f_i : p(y) \to p(x)$ is a morphism of schemes, and $\alpha_i : y \to x(f_i)$ an isomorphism. The category of $2n+3$-tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ is an incarnation of the $(n+1)$-fold fibred product $\mathcal{U}_n$ of algebraic stacks displayed above, as we discussed in Section 96.18. By Cohomology on Sites, Lemma 21.13.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the "primed" analogue of this. Namely, $\mathcal{X}'/h_{x,n}$ corresponds, via the equivalence above to the full subcategory $\mathcal{U}'_n \subset \mathcal{U}_n$ consisting of those tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ with $y \in \mathcal{X}'$. Hence certainly property (1) of the statement of the lemma holds for the inclusion $\mathcal{U}'_n \subset \mathcal{U}_n$. To see property (2) choose an object $\xi = (y, s_0, \ldots, s_n)$ which lies over a scheme W such that $(Sch/W)_{fppf} \to$

$\mathcal{U}_n$ is smooth and surjective (this is possible as $\mathcal{U}_n$ is an algebraic stack). Then $(Sch/W)_{fppf} \to \mathcal{U}_n \to (Sch/U)_{fppf}$ is smooth as a composition of base changes of the morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 94.10.6 and 94.10.5. Thus axiom (1) for $\mathcal{X}$ implies that y is an object of $\mathcal{X}'$ whence ξ is an object of $\mathcal{U}'_n$. Using again

$$H^p(\mathcal{U}'_n, \mathcal{F}|_{\mathcal{U}'_n}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for $\mathcal{U}'_n \subset \mathcal{U}_n$ to conclude. □

96.24. Restricting to affines

0H08 In this section, given a category $\mathcal{X}$ fibred in groupoids over $(Sch/S)_{fppf}$ we will consider the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ consisting of objects x lying over affine schemes U. We will see how, for any topology τ finer than the Zariski topology, the category of sheaves on $\mathcal{X}$ and $\mathcal{X}_{affine,\tau}$ agree.

0H09 **Definition 96.24.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *associated affine site* is the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that U is affine. The topology on $\mathcal{X}_{affine}$ will be the chaotic one, i.e., such that sheaves on $\mathcal{X}_{affine}$ are the same as presheaves.

Thus the functor $p : \mathcal{X} \to (Sch/S)_{fppf}$ restricts to a functor

$$p : \mathcal{X}_{affine} \longrightarrow (Aff/S)_{fppf}$$

where the notation on the right hand side is the one introduced in Topologies, Definition 34.7.8. It is clear that $\mathcal{X}_{affine}$ is fibred in groupoids over $(Aff/S)_{fppf}$. It follows that $\mathcal{X}_{affine}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology from $(Aff/S)_{Zar}$, $(Aff/S)_{étale}$, $(Aff/S)_{smooth}$, $(Aff/S)_{syntomic}$, and $(Aff/S)_{fppf}$, see Stacks, Definition 8.10.2.

0H0A **Definition 96.24.2.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The *associated affine Zariski site* $\mathcal{X}_{affine,Zar}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{Zar}$.
(2) The *associated affine étale site* $\mathcal{X}_{affine,étale}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{étale}$.
(3) The *associated affine smooth site* $\mathcal{X}_{affine,smooth}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{smooth}$.
(4) The *associated affine syntomic site* $\mathcal{X}_{affine,syntomic}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{syntomic}$.
(5) The *associated affine fppf site* $\mathcal{X}_{affine,fppf}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{fppf}$.

This definition makes sense by the discussion above. For each $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ a family of morphisms $\{x_i \to x\}_{i\in I}$ with fixed target in $\mathcal{X}_{affine}$ is a covering in $\mathcal{X}_{affine,\tau}$ if and only if the family of morphisms $\{p(x_i) \to p(x)\}_{i\in I}$ of affine schemes is a standard τ-covering as defined in Topologies, Definitions 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5.

0H0B **Lemma 96.24.3.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The functor $\mathcal{X}_{affine,\tau} \to \mathcal{X}_\tau$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(\mathcal{X}_{affine,\tau})$ to $Sh(\mathcal{X}_\tau)$.*

Proof. Omitted. Hint: the proof is exactly the same as the proof of Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11. □

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let us denote $\mathcal{O}$ the restriction of $\mathcal{O}_\mathcal{X}$ to $\mathcal{X}_{affine}$. Then $\mathcal{O}$ is a sheaf in the Zariski, étale, smooth, syntomic, and fppf topologies on $\mathcal{X}_{affine}$. Furthermore, the equivalence of topoi of Lemma 96.24.3 extends to an equivalence

0H0C (96.24.3.1) $$(Sh(\mathcal{X}_{affine,\tau}), \mathcal{O}) \longrightarrow (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X})$$

of ringed topoi for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.

96.25. Quasi-coherent modules and affines

0H0D Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. In Section 96.24 we have associated to this a ringed site $(\mathcal{X}_{affine}, \mathcal{O})$.

0H0E **Lemma 96.25.1.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent*

1. *for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,*
2. *$\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1,*
3. *$\mathcal{F}$ is a sheaf for the Zariski topology on $\mathcal{X}_{affine}$ and a quasi-coherent module on $(\mathcal{X}_{affine,Zar}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1,*
4. *same as in (3) for the étale topology,*
5. *same as in (3) for the smooth topology,*
6. *same as in (3) for the syntomic topology,*
7. *same as in (3) for the fppf topology, and*
8. *$\mathcal{F}$ corresponds to a quasi-coherent module on $\mathcal{X}$ via the equivalence (96.24.3.1).*

Proof. To make sense out of part (2), recall that $\mathcal{X}_{affine}$ is a site gotten by endowing the category $\mathcal{X}_{affine}$ with the chaotic topology (Definition 96.24.1) and hence a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is the same thing as a presheaf of $\mathcal{O}$-modules. Conditions (1) and (2) are equivalent by Modules on Sites, Lemma 18.24.3. Observe that for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the presheaf $\mathcal{F}$ is a τ-sheaf if and only if for all $x \in \mathrm{Ob}(\mathcal{X}_{affine})$ the restriction to $\mathcal{X}_{affine}/x$ is a τ-sheaf. Set $U = p(x)$. Similarly to the discussion in Section 96.9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,étale}/x \to (Aff/U)_{étale}$ of sites. In this way we see that the equivalence of (1) with (3) – (7) follows from Descent, Lemma 35.11.1 applied to each of these sites. The equivalence of (8) and (7) is immediate from the fact that "being quasi-coherent" is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18.18 □

0H0F **Lemma 96.25.2.** *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent*

1. *for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ such that $p(x) \to p(x')$ is an étale morphism (of affine schemes), the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,*
2. *$\mathcal{F}$ is a sheaf for the étale topology on $\mathcal{X}_{affine}$ and for every object x of $\mathcal{X}_{affine}$ the restriction $x^*\mathcal{F}|_{U_{affine,étale}}$ is quasi-coherent where $U = p(x)$,*

(3) *$\mathcal{F}$ corresponds to a locally quasi-coherent module on $\mathcal{X}$ via the equivalence (96.24.3.1) for the étale topology.*

Proof. To make sense out of condition (2), recall that $U_{affine,étale}$ is the full subcategory of $U_{étale}$ consisting of affine objects, see Topologies, Definition 34.4.8. Similarly to the discussion in Section 96.9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,étale}/x \to (Aff/U)_{étale}$ of sites. Then $x^*\mathcal{F}$ is the sheaf of modules on $(Aff/U)_{étale}$ corresponding to the restriction $\mathcal{F}|_{\mathcal{X}_{affine,étale}/x}$. Finally, using the continuous and cocontinuous inclusion functor $U_{affine,étale} \to (Aff/U)_{étale}$ we can further restrict and obtain $x^*\mathcal{F}|_{U_{affine,étale}}$.

The equivalence of (1) and (2) follows from the remarks above and Descent, Lemma 35.11.2 applied to the restriction of $\mathcal{F}$ to $U_{affine,étale}$ for every object x of $\mathcal{X}$ lying over an affine scheme U. The equivalence of (2) and (3) is immediate from the definitions and the fact that quasi-coherent modules on $U_{affine,étale}$ and $U_{étale}$ correspond (again by Descent, Lemma 35.11.2 for example). □

96.26. Quasi-coherent objects in the derived category

0H0G Algebraic geometers have contemplated invariants for non-representable functors X (valued in sets or groupoids) on Sch/S for decades. For instance, before the notion of a stack was invented, Mumford defined [**Mum65**] the Picard groupoid $\mathrm{Pic}(X)$ for the moduli functor X of elliptic curves as the 2-limit $Pic(U)$ over the category of all schemes U equipped with a map to X (i.e., with a family of elliptic curves). Similarly, Beilinson-Drinfeld defined [**BD**] the category $QCoh(X)$ for an ind-scheme $X = \operatorname{colim} X_i$ as the 2-limit $\lim QCoh(X_i)$. This strategy is sufficient for defining 1-categorical invariants like $QCoh(-)$, but inadequate for derived categorical ones (such as the quasi-coherent derived category) as 2-limits of triangulated categories are poorly behaved. With the advent of higher categorical technology and derived algebraic geometry, this problem can be resolved gracefully: one can define the quasi-coherent derived ∞-category $\mathcal{D}_{qc}(X)$ of the functor X as the limit $\lim \mathcal{D}_{qc}(U)$, where U ranges over all derived affines over X (see [**Lur04**]).

The goal of this section is to attach a triangulated category $QC(X)$ to a functor X (valued in sets or groupoids) as above. In fact, the construction works for any category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids (not just split ones). In good cases, the category $QC(\mathcal{X})$ can be shown to agree with the homotopy category of $\mathcal{D}_{qc}(\mathcal{X})$, though it is outside the scope of this document to explain this comparison. The salient features of the construction are:

(a) $QC(\mathcal{X})$ is a full subcategory of $D(\mathcal{X}_{affine}, \mathcal{O})$ by construction,
(b) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_X)$ when $\mathcal{X}$ is representable by the algebraic space X,
(c) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_\mathcal{X})$ when $\mathcal{X}$ is an algebraic stack,
(d) when $X = \mathrm{Spf}(A)$ is an affine formal algebraic space attached to a noetherian ring A equipped with the I-adic topology for an ideal I, the triangulated category $QC(X)$ agrees with the full subcategory $D_{comp}(A, I) \subset D(A)$ of derived complete objects.

These results are proven in Proposition 96.26.4, Derived Categories of Stacks, Proposition 104.8.4, and Proposition 96.26.5.

As a motivation for the precise definition of $QC(\mathcal{X})$ we point the reader to the characterization, in Lemma 96.25.1, of quasi-coherent modules on $\mathcal{X}$ as presheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$ which satisfy a kind of base change property.

0H0H **Definition 96.26.1.** Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{O}$ be the sheaf of rings on $\mathcal{X}_{affine}$ introduced in Section 96.24. We define the *triangulated category of quasi-coherent objects in the derived category* by the formula

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

where the right hand side is as defined in Cohomology on Sites, Definition 21.43.1.

Note that this makes sense as $\mathcal{X}_{affine}$ is a category and is viewed as a site by endowing it with the chaotic topology and $\mathcal{O}$ is a sheaf of rings on this category, exactly as required in Cohomology on Sites, Definition 21.43.1.

The relationship of this definition with the category of quasi-coherent modules on $\mathcal{X}$ is not so clear in general! For example, suppose that M is an object of $QC(\mathcal{X})$. Then the cohomology sheaves $H^i(M)$ of M are (pre)sheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$, but in general they are not quasi-coherent. The last nonvanishing cohomology sheaf is quasi-coherent however.

0H0I **Lemma 96.26.2.** *In the situation of Definition 96.26.1 suppose that M is an object of $QC(\mathcal{X})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$, see Lemma 96.25.1.*

Proof. Special case of Cohomology on Sites, Lemma 21.43.3. □

0H0J **Lemma 96.26.3.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparison morphism $\epsilon : \mathcal{X}_{affine,étale} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 21.43.12.*

Proof. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \mathrm{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,étale}/x = (Aff/U)_{étale}$ compatible with structure sheaves; see discussion in Section 96.9. Thus it suffices to show: given an affine scheme $U = \mathrm{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(Aff/U)_{étale}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This follows from a combination of: Derived Categories of Schemes, Lemma 36.3.5 (total cohomology of complexes of modules over affines in the Zariski topology), Derived Categories of Spaces, Remark 75.6.3 (agreement between total cohomology in small Zariski and étale topologies for quasi-coherent complexes of modules), and Étale Cohomology, Lemma 59.99.3 (to see that the étale cohomology of a complex of modules on the big étale site of a scheme may be computed after restricting to the small étale site). □

If we apply the definition in case our category fibred in groupoids $\mathcal{X}$ is representable by an algebraic space X, then we recover $D_{QCoh}(\mathcal{O}_X)$. We will later state and prove the analogous result for algebraic stacks (insert future reference here).

0H0K **Proposition 96.26.4.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space X. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Denote X_{affine} the category of affine schemes étale over X endowed with the chaotic topology and its structure sheaf $\mathcal{O}_X$, see Derived Categories of Spaces, Section 75.30. The functor $u : X_{étale} \to \mathcal{X}_{étale}$ of Lemma 96.10.1 gives rise to a functor $X_{affine} \to \mathcal{X}_{affine}$. This is compatible with structure sheaves and produces a functor

$$G : QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O}) \longrightarrow QC(X_{affine}, \mathcal{O}_X)$$

See Cohomology on Sites, Lemma 21.43.10. By Derived Categories of Spaces, Lemma 75.30.1 the triangulated category $QC(X_{affine}, \mathcal{O}_X)$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. Hence it suffices to prove that G is an equivalence.

Consider the flat comparison morphisms $\epsilon_{\mathcal{X}} : \mathcal{X}_{affine,étale} \to \mathcal{X}_{affine}$ and $\epsilon_X : X_{affine,étale} \to X_{affine}$ of ringed sites. Lemma 96.26.3 and (the proof of) Derived Categories of Spaces, Lemma 75.30.1 show that the functors $\epsilon_{\mathcal{X}}^*$ and ϵ_X^* identify $QC(\mathcal{X}_{affine}, \mathcal{O})$ and $QC(X_{affine}, \mathcal{O}_X)$ with subcategories $Q_{\mathcal{X}} \subset D(\mathcal{X}_{affine,étale}, \mathcal{O})$ and $Q_X \subset D(X_{affine,étale}, \mathcal{O}_X)$. With these identifications the functor G in the first paragraph is induced by the functor

$$Li_X^* = R\pi_{X,*} : D(\mathcal{X}_{affine,étale}, \mathcal{O}) \longrightarrow D(X_{affine,étale}, \mathcal{O}_X)$$

where i_X and π_X are the morphisms from Lemma 96.10.1 but with the étale sites replaced by the corresponding affine ones. The reader can show that this replacement is permissible either by reproving the lemma for the affine sites directly or by using the equivalences of topoi $Sh(\mathcal{X}_{affine,étale}) = Sh(\mathcal{X}_{étale})$ and $Sh(X_{affine,étale}) = Sh(X_{étale})$. The lemma also tells us Li_X^* has a left adjoint

$$L\pi_X^* : D(X_{affine,étale}, \mathcal{O}_X) \longrightarrow D(\mathcal{X}_{affine,étale}, \mathcal{O})$$

and moreover we have $Li_X^* \circ L\pi_X^* = \mathrm{id}$ since $\pi_X \circ i_X$ is the identity. Thus it suffices to show that (a) $L\pi_X^*$ sends Q_X into $Q_{\mathcal{X}}$ and (b) the kernel of Li_X^* is 0. See Derived Categories, Lemma 13.7.2.

Proof of (a). By Derived Categories of Spaces, Lemma 75.30.1 we have $Q_X = D_{QCoh}(X_{affine,étale}, \mathcal{O}_X)$. Let K be an object of Q_X. Let x be an object of $\mathcal{X}_{affine,étale}$ lying over the affine scheme $U = p(x)$. Denote $f : U \to X$ the morphism corresponding to x. Then we see that

$$R\Gamma(x, L\pi_X^* K) = R\Gamma(U, Lf^* K)$$

This follows from transitivity of pullbacks; see discussion in Section 96.10. Next, suppose that $x \to x'$ is a morphism of $\mathcal{X}_{affine,étale}$ lying over the morphism $h : U \to U'$ of affine schemes. As before denote $f : U \to X$ and $f' : U' \to X$ the morphisms corresponding to x and x' so that we have $f = f' \circ h$. Then

$$\begin{aligned} R\Gamma(x, L\pi_X^* K) &= R\Gamma(U, Lf^* K) \\ &= R\Gamma(U, Lh^* L(f')^* K) \\ &= R\Gamma(U', L(f')^* K) \otimes^{\mathbf{L}}_{\mathcal{O}(U')} \mathcal{O}(U) \\ &= R\Gamma(x', L\pi_X^* K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \end{aligned}$$

and hence we have (a) by the footnote in the statement of Cohomology on Sites, Lemma 21.43.12. The third equality is Derived Categories of Schemes, Lemma 36.3.8.

Proof of (b). Let M be an object of $Q_{\mathcal{X}}$ such that $Li_X^* M = 0$. Let x' be an object of $\mathcal{X}_{affine,étale}$ lying over the affine scheme $U' = p(x')$ and assume that

the corresponding morphism $f' : U' \to X$ is étale. Then $f' : U' \to X$ is an object of $X_{affine,étale}$ and the condition $Li_X^* M = 0$ implies that $M|_{U'_{étale}} = 0$. In particular, we see that $R\Gamma(x', M) = 0$. However, for an arbitrary object x of the site $\mathcal{X}_{affine,étale}$ there exists a covering $\{x_i \to x\}$ such that for each i there is a morphism $x_i \to x'_i$ with x'_i corresponding to an object of $X_{affine,étale}$. Now since M is in $Q_{\mathcal{X}}$ we have

$$R\Gamma(x_i, M) = R\Gamma(x'_i, M) \otimes^{\mathbf{L}}_{\mathcal{O}(x'_i)} \mathcal{O}(x_i) = 0$$

and we conclude that M is zero as desired. □

To show that the construction produces an interesting category in another case, let us state and prove a characterization of $QC(\mathrm{Spf}(A))$ for the formal spectrum of a Noetherian adic ring A.

0H0L **Proposition 96.26.5.** *Let S be a scheme. Let $X = Spf(A)$ where A is an an adic Noetherian topological S-algebra with ideal of definition I, see More on Algebra, Definition 15.36.1 and Formal Spaces, Definition 87.9.9. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ the be category fibred in sets associated to the functor X, see Categories, Example 4.38.5. Then $QC(\mathcal{X})$ is canonically equivalent to the category $D_{comp}(A, I)$ of objects of $D(A)$ which are derived complete with respect to I.*

Proof. Recall that $X = \operatorname{colim} \mathrm{Spec}(A/I^n)$ as an fppf sheaf. An object of $\mathcal{X}_{affine}$ is the same thing as an affine scheme $U = \mathrm{Spec}(R)$ with a given morphism $f : U \to X$. By Formal Spaces, Lemma 87.9.4 there exists an $n \geq 1$ such that f factors through the monomorphism $\mathrm{Spec}(A/I^n) \to X$. Consider the full subcategory $\mathcal{C} \subset \mathcal{X}_{affine}$ consisting of the objects $\mathrm{Spec}(A/I^n) \to X$. By the remarks just made and Differential Graded Sheaves, Lemma 24.34.1 restriction to $\mathcal{C}$ is an exact equivalence $QC(\mathcal{X}) \to QC(\mathcal{C}, \mathcal{O}|_{\mathcal{C}})$. For simplicity, let us assume that $I^n \neq I^{n+1}$ for all $n \geq 1$. Then $(\mathcal{C}, \mathcal{O}|_{\mathcal{C}})$ is isomorphic as a ringed site to the ringed site $(\mathbf{N}, (A/I^n))$, see Differential Graded Sheaves, Section 24.35. Hence we conclude by Differential Graded Sheaves, Proposition 24.35.4. □

The following lemma will be used in comparing $QC(\mathcal{X})$ to $D_{QCoh}(\mathcal{O}_{\mathcal{X}})$ when $\mathcal{X}$ is an algebraic stack.

0H0X **Lemma 96.26.6.** *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparison morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 21.43.12.*

Proof. The proof is exactly the same as the proof of Lemma 96.26.3. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \mathrm{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,étale}/x = (Aff/U)_{étale}$ compatible with structure sheaves; see discussion in Section 96.9. Thus it suffices to show: given an affine scheme $U = \mathrm{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(Aff/U)_{fppf}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This is Étale Cohomology, Lemma 59.101.3. □

96.27. Other chapters

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CHAPTER 97

Criteria for Representability

05XE

97.1. Introduction

05XF The purpose of this chapter is to find criteria guaranteeing that a stack in groupoids over the category of schemes with the fppf topology is an algebraic stack. Historically, this often involved proving that certain functors were representable, see Grothendieck's lectures [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This explains the title of this chapter. Another important source of this material comes from the work of Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

97.2. Conventions

05XG The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2.

97.3. What we already know

05XH The analogue of this chapter for algebraic spaces is the chapter entitled "Bootstrap", see Bootstrap, Section 80.1. That chapter already contains some representability results. Moreover, some of the preliminary material treated there we already have worked out in the chapter on algebraic stacks. Here is a list:

(1) We discuss morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 80.3. In Algebraic Stacks, Section 94.9 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(2) We discuss properties of morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 80.4. In Algebraic Stacks, Section 94.10 we discuss properties of 1-morphisms of categories fibred in groupoids representable by algebraic spaces.
(3) We proved that if F is a sheaf whose diagonal is representable by algebraic spaces and which has an étale covering by an algebraic space, then F is an algebraic space, see Bootstrap, Theorem 80.6.1. (This is a weak version of the result in the next item on the list.)

05XI (4) We proved that if F is a sheaf and if there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat, and locally of finite presentation, then F is an algebraic space, see Bootstrap, Theorem 80.10.1.

(5) We have also proved the "smooth" analogue of (4) for algebraic stacks: If $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ and if there exists a stack in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space and a 1-morphism $u : \mathcal{U} \to \mathcal{X}$ which is representable by algebraic spaces, surjective, and smooth then $\mathcal{X}$ is an algebraic stack, see Algebraic Stacks, Lemma 94.15.3.

Our first task now is to prove the analogue of (4) for algebraic stacks in general; it is Theorem 97.16.1.

97.4. Morphisms of stacks in groupoids

05XJ This section is preliminary and should be skipped on a first reading.

05XK **Lemma 97.4.1.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are representable by algebraic spaces and étale so is $\mathcal{X} \to \mathcal{Y}$.*

Proof. Let $\mathcal{U}$ be a representable category fibred in groupoids over S. Let $f : \mathcal{U} \to \mathcal{Y}$ be a 1-morphism. We have to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U}$ is representable by an algebraic space and étale over $\mathcal{U}$. Consider the composition $h : \mathcal{U} \to \mathcal{Z}$. Then

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{U} \longrightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$$

is a 1-morphism between categories fibres in groupoids which are both representable by algebraic spaces and both étale over $\mathcal{U}$. Hence by Properties of Spaces, Lemma 66.16.6 this is represented by an étale morphism of algebraic spaces. Finally, we obtain the result we want as the morphism f induces a morphism $\mathcal{U} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$ and we have

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} = (\mathcal{X} \times_{\mathcal{Z}} \mathcal{U}) \times_{(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{U})} \mathcal{U}.$$

□

05XL **Lemma 97.4.2.** *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks in groupoids over $(Sch/S)_{fppf}$. Suppose that $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are 1-morphisms. If*

(1) *$\mathcal{Y}$, $\mathcal{Z}$ are representable by algebraic spaces Y, Z over S,*
(2) *the associated morphism of algebraic spaces $Y \to Z$ is surjective, flat and locally of finite presentation, and*
(3) *$\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X}$ is a stack in setoids,*

then $\mathcal{X}$ is a stack in setoids.

Proof. This is a special case of Stacks, Lemma 8.6.10. □

The following lemma is the analogue of Algebraic Stacks, Lemma 94.15.3 and will be superseded by the stronger Theorem 97.16.1.

05XW **Lemma 97.4.3.** *Let S be a scheme. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{U}$ is representable by an algebraic space, and*
(2) *u is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

Proof. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Algebraic Stacks, Lemma 94.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

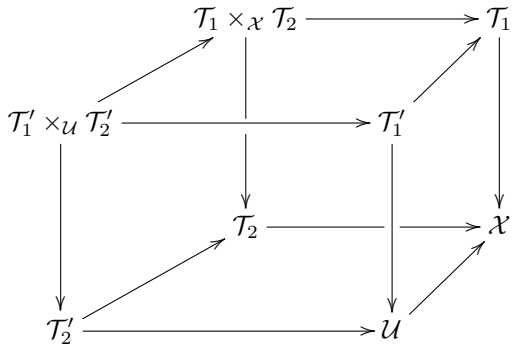

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{array}{ccc} \mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X} \end{array}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, flat, locally of finite presentation and surjective, see Algebraic Stacks, Lemmas 94.9.6, 94.9.7, 94.10.4, and 94.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 80.10.1 and we win. □

07WG **Lemma 97.4.4.** *Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

1. *$\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,*
2. *for every 1-morphism $\mathcal{V} \to \mathcal{X} \times \mathcal{X}$ with $\mathcal{V}$ representable (by a scheme) the fibre product $\mathcal{Y} = \mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X}} \mathcal{V}$ has diagonal representable by algebraic spaces.*

Proof. Although this is a bit of a brain twister, it is completely formal. Namely, recall that $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} = \mathcal{I}_{\mathcal{X}}$ is the inertia of $\mathcal{X}$ and that Δ_Δ is the identity section of $\mathcal{I}_{\mathcal{X}}$, see Categories, Section 4.34. Thus condition (1) says the following: Given a scheme V, an object x of $\mathcal{X}$ over V, and a morphism $\alpha : x \to x$ of $\mathcal{X}_V$ the condition "$\alpha = \mathrm{id}_x$" defines an algebraic space over V. (In other words, there exists a monomorphism of algebraic spaces $W \to V$ such that a morphism of schemes $f : T \to V$ factors through W if and only if $f^*\alpha = \mathrm{id}_{f^*x}$.)

On the other hand, let V be a scheme and let x, y be objects of $\mathcal{X}$ over V. Then (x, y) define a morphism $\mathcal{V} = (Sch/V)_{fppf} \to \mathcal{X} \times \mathcal{X}$. Next, let $h : V' \to V$ be a morphism of schemes and let $\alpha : h^*x \to h^*y$ and $\beta : h^*x \to h^*y$ be morphisms of $\mathcal{X}_{V'}$. Then (α, β) define a morphism $\mathcal{V}' = (Sch/V)_{fppf} \to \mathcal{Y} \times \mathcal{Y}$. Condition (2) now says that (with any choices as above) the condition "$\alpha = \beta$" defines an algebraic space over V.

To see the equivalence, given (α, β) as in (2) we see that (1) implies that "$\alpha^{-1} \circ \beta = \mathrm{id}_{h^*x}$" defines an algebraic space. The implication (2) $\Rightarrow$ (1) follows by taking $h = \mathrm{id}_V$ and $\beta = \mathrm{id}_x$. □

97.5. Limit preserving on objects

06CT Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *limit preserving on objects* if the following condition holds: Given any data consisting of

(1) an affine scheme $U = \lim_{i \in I} U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i,
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y_i|_U$,

then there exists an $i' \geq i$, an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$, an isomorphism $\beta : x_{i'}|_U \to x$, and an isomorphism $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that

06CU (97.5.0.1)
$$\begin{array}{ccc} p(x_{i'}|_U) & \xrightarrow{\gamma_{i'}|_U} & (y_i|_{U_{i'}})|_U \\ {\scriptstyle p(\beta)}\downarrow & & \| \\ p(x) & \xrightarrow{\gamma} & y_i|_U \end{array}$$

commutes. In this situation we say that "$(i', x_{i'}, \beta, \gamma_{i'})$ is a *solution* to the problem posed by our data (1), (2), (3), (4)". The motivation for this definition comes from Limits of Spaces, Lemma 70.3.2.

06CV **Lemma 97.5.1.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z_i|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z_i|_U$. Set $y_i = q(z_i)$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y_i|_U$. As p is limit preserving on objects there exists an $i' \geq i$ and an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$ as well as isomorphisms $\beta : x_{i'}|_U \to x$ and $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that (97.5.0.1) commutes. Then we consider the object $w_{i'} = (U_{i'}, x_{i'}, z_i|_{U_{i'}}, \gamma_{i'})$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over $U_{i'}$ and define isomorphisms

$$w_{i'}|_U = (U, x_{i'}|_U, z_i|_U, \gamma_{i'}|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w_{i'}) = z_i|_{U_{i'}} \xrightarrow{\mathrm{id}} z_i|_{U_{i'}}.$$

These combine to give a solution to the problem. □

06CW **Lemma 97.5.2.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z_i|_U$ be an isomorphism. As q is limit preserving on objects there exist an $i' \geq i$, an object $y_{i'}$ of $\mathcal{Y}$ over $U_{i'}$, an isomorphism $\beta : y_{i'}|_U \to p(x)$, and an isomorphism $\gamma_{i'} : q(y_{i'}) \to z_i|_{U_{i'}}$ such that (97.5.0.1) is commutative. As p is limit preserving on objects there exist an $i'' \geq i'$, an object $x_{i''}$ of $\mathcal{X}$ over $U_{i''}$, an isomorphism $\beta' : x_{i''}|_U \to x$, and an isomorphism $\gamma'_{i''} : p(x_{i''}) \to y_{i'}|_{U_{i''}}$ such that (97.5.0.1) is commutative. The solution is to take $x_{i''}$ over $U_{i''}$ with isomorphism

$$q(p(x_{i''})) \xrightarrow{q(\gamma'_{i''})} q(y_{i'})|_{U_{i''}} \xrightarrow{\gamma_{i'}|_{U_{i''}}} z_i|_{U_{i''}}$$

and isomorphism $\beta' : x_{i''}|_U \to x$. We omit the verification that (97.5.0.1) is commutative. □

06CX **Lemma 97.5.3.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

1. *p is limit preserving on objects, and*
2. *p is locally of finite presentation (see Algebraic Stacks, Definition 94.10.1).*

Proof. Assume (2). Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let y_i be an object of $\mathcal{Y}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y_i|_U$ be an isomorphism. Let X_{y_i} denote an algebraic space over U_i representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U_i, x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y_i}$ over U_i. By Limits of Spaces, Proposition 70.3.10 there exists an $i' \geq i$ and a morphism $f_{i'} : U_{i'} \to X_{y_i}$ such that f_ξ is the composition of $f_{i'}$ and the projection morphism $U \to U_{i'}$. Also, the 2-Yoneda lemma tells us that $f_{i'}$ corresponds to an object $\xi_{i'} = (U_{i'}, U_{i'} \to U_i, x_{i'}, \alpha)$ of the displayed 2-fibre product over $U_{i'}$ whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x_{i'}|U \to x$. Note that $\alpha : y_i|_{U_{i'}} \to p(x_{i'})$. Hence we see that taking $x_{i'}$, the isomorphism $\gamma : x_{i'}|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x_{i'}) \to y_i|_{U_{i'}}$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \to T$ is locally of finite presentation. To do this we will use the criterion in Limits of Spaces, Remark 70.3.11. Consider an affine scheme $U = \lim_{i \in I} U_i$ written as the directed limit of affine schemes over T. Pick any $i \in I$ and set $y_i = y|_{U_i}$. Also denote i' an element of I which is bigger than or equal to i. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y_i|_U$. Similarly for morphisms

$U_{i'} \to X_y$ over T. Hence (1) guarantees that the canonical map

$$\operatorname{colim}_{i' \geq i} X_y(U_{i'}) \longrightarrow X_y(U)$$

is surjective in this situation. It follows from Limits of Spaces, Lemma 70.3.12 that $X_y \to T$ is locally of finite presentation. □

06CY **Lemma 97.5.4.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume p is representable by algebraic spaces and an open immersion. Then p is limit preserving on objects.*

Proof. This follows from Lemma 97.5.3 and (via the general principle Algebraic Stacks, Lemma 94.10.9) from the fact that an open immersion of algebraic spaces is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.11. □

Let S be a scheme. In the following lemma we need the notion of the *size* of an algebraic space X over S. Namely, given a cardinal κ we will say X has $\text{size}(X) \leq \kappa$ if and only if there exists a scheme U with $\text{size}(U) \leq \kappa$ (see Sets, Section 3.9) and a surjective étale morphism $U \to X$.

07WH **Lemma 97.5.5.** *Let S be a scheme. Let $\kappa = size(T)$ for some $T \in \text{Ob}((Sch/S)_{fppf})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ such that*

(1) *$\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
(2) *for an affine scheme V locally of finite presentation over S and $y \in \text{Ob}(\mathcal{Y}_V)$ the fibre product $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space of size $\leq \kappa$[1],*
(3) *$\mathcal{X}$ and $\mathcal{Y}$ are stacks for the Zariski topology.*

Then f is representable by algebraic spaces.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to prove $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space.

Case I: V is affine and maps into an affine open $\text{Spec}(\Lambda) \subset S$. Then we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\text{Spec}(\Lambda)$, see Algebra, Lemma 10.127.2. Then y comes from an object y_i over V_i for some i by assumption (1). By assumption (3) the fibre product $(Sch/V_i)_{fppf} \times_{y_i,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space Z_i. Then $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by $Z \times_{V_i} V$.

Case II: V is general. Choose an affine open covering $V = \bigcup_{i \in I} V_i$ such that each V_i maps into an affine open of S. We first claim that $\mathcal{Z} = (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is a stack in setoids for the Zariski topology. Namely, it is a stack in groupoids for the Zariski topology by Stacks, Lemma 8.5.6. Then suppose that z is an object of $\mathcal{Z}$ over a scheme T. Denote $g : T \to V$ the morphism corresponding to the projection of z in $(Sch/V)_{fppf}$. Consider the Zariski sheaf $I = Isom_{\mathcal{Z}}(z, z)$. By Case I we see that $I|_{g^{-1}(V_i)} = *$ (the singleton sheaf). Hence $\mathcal{I} = *$. Thus $\mathcal{Z}$ is fibred in setoids. To finish the proof we have to show that the Zariski sheaf $Z : T \mapsto \text{Ob}(\mathcal{Z}_T)/\cong$ is an algebraic space, see Algebraic Stacks, Lemma 94.8.2. There is a map $p : Z \to V$ (transformation of functors) and by Case I we know that $Z_i = p^{-1}(V_i)$ is an algebraic space. The morphisms $Z_i \to Z$ are representable by open immersions and $\coprod Z_i \to Z$ is surjective (in the Zariski topology). Hence Z is

[1]The condition on size can be dropped by those ignoring set theoretic issues.

a sheaf for the fppf topology by Bootstrap, Lemma 80.3.11. Thus Spaces, Lemma 65.8.5 applies and we conclude that Z is an algebraic space[2]. □

07WI **Lemma 97.5.6.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces as in Algebraic Stacks, Definition 94.10.1. If*

(1) *f is representable by algebraic spaces,*
(2) *$\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
(3) *for an affine scheme V locally of finite presentation over S and $y \in \mathcal{Y}_V$ the resulting morphism of algebraic spaces $f_y : F_y \to V$, see Algebraic Stacks, Equation (94.9.1.1), has property $\mathcal{P}$.*

Then f has property $\mathcal{P}$.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to show that $F_y \to V$ has property $\mathcal{P}$. Since $\mathcal{P}$ is fppf local on the base we may assume that V is an affine scheme which maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$. Thus we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\mathrm{Spec}(\Lambda)$, see Algebra, Lemma 10.127.2. Then y comes from an object y_i over V_i for some i by assumption (2). By assumption (3) the morphism $F_{y_i} \to V_i$ has property $\mathcal{P}$. As $\mathcal{P}$ is stable under arbitrary base change and since $F_y = F_{y_i} \times_{V_i} V$ we conclude that $F_y \to V$ has property $\mathcal{P}$ as desired. □

97.6. Formally smooth on objects

06CZ Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *formally smooth on objects* if the following condition holds: Given any data consisting of

(1) a first order thickening $U \subset U'$ of affine schemes over S,
(2) an object y' of $\mathcal{Y}$ over U',
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y'|_U$,

then there exists an object x' of $\mathcal{X}$ over U' with an isomorphism $\beta : x'|_U \to x$ and an isomorphism $\gamma' : p(x') \to y'$ such that

06D0 (97.6.0.1)
$$\begin{array}{ccc} p(x'|_U) & \xrightarrow{\gamma'|_U} & y'|_U \\ {\scriptstyle p(\beta)}\downarrow & & \| \\ p(x) & \xrightarrow{\gamma} & y'|_U \end{array}$$

commutes. In this situation we say that "(x', β, γ') is a *solution* to the problem posed by our data (1), (2), (3), (4)".

06D1 **Lemma 97.6.1.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is formally smooth on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

[2]To see that the set theoretic condition of that lemma is satisfied we argue as follows: First choose the open covering such that $|I| \leq \mathrm{size}(V)$. Next, choose schemes U_i of size $\leq \max(\kappa, \mathrm{size}(V))$ and surjective étale morphisms $U_i \to Z_i$; we can do this by assumption (2) and Sets, Lemma 3.9.6 (details omitted). Then Sets, Lemma 3.9.9 implies that $\coprod U_i$ is an object of $(Sch/S)_{fppf}$. Hence $\coprod Z_i$ is an algebraic space by Spaces, Lemma 65.8.4.

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z'|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z|_U$. Set $y' = q(z')$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y'|_U$. As p is formally smooth on objects there exists an object x' of $\mathcal{X}$ over U' as well as isomorphisms $\beta : x'|_U \to x$ and $\gamma' : p(x') \to y'$ such that (97.6.0.1) commutes. Then we consider the object $w = (U', x', z', \gamma')$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U' and define isomorphisms

$$w'|_U = (U, x'|_U, z'|_U, \gamma'|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w') = z' \xrightarrow{\mathrm{id}} z'.$$

These combine to give a solution to the problem. □

06D2 **Lemma 97.6.2.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are formally smooth on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z'|_U$ be an isomorphism. As q is formally smooth on objects there exist an object y' of $\mathcal{Y}$ over U', an isomorphism $\beta : y'|_U \to p(x)$, and an isomorphism $\gamma' : q(y') \to z'$ such that (97.6.0.1) is commutative. As p is formally smooth on objects there exist an object x' of $\mathcal{X}$ over U', an isomorphism $\beta' : x'|_U \to x$, and an isomorphism $\gamma'' : p(x') \to y'$ such that (97.6.0.1) is commutative. The solution is to take x' over U' with isomorphism

$$q(p(x')) \xrightarrow{q(\gamma'')} q(y') \xrightarrow{\gamma'} z'$$

and isomorphism $\beta' : x'|_U \to x$. We omit the verification that (97.6.0.1) is commutative. □

Note that the class of formally smooth morphisms of algebraic spaces is stable under arbitrary base change and local on the target in the fpqc topology, see More on Morphisms of Spaces, Lemma 76.19.3 and 76.19.11. Hence condition (2) in the lemma below makes sense.

06D3 **Lemma 97.6.3.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is formally smooth on objects, and*
(2) *p is formally smooth (see Algebraic Stacks, Definition 94.10.1).*

Proof. Assume (2). Let $U \subset U'$ be a first order thickening of affine schemes over S, let y' be an object of $\mathcal{Y}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y'|_U$ be an isomorphism. Let $X_{y'}$ denote an algebraic space over U' representing the 2-fibre product

$$(Sch/U')_{fppf} \times_{y', \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U', x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y'}$ over U'. As

$X_{y'} \to U'$ is formally smooth by assumption there exists a morphism $f' : U' \to X_{y'}$ such that f_ξ is the composition of f' and the morphism $U \to U'$. Also, the 2-Yoneda lemma tells us that f' corresponds to an object $\xi' = (U', U' \to U', x', \alpha)$ of the displayed 2-fibre product over U' whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x'|U \to x$. Note that $\alpha : y' \to p(x')$. Hence we see that taking x', the isomorphism $\gamma : x'|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x') \to y'$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is formally smooth. Hence it suffices to show that given a first order thickening $U \subset U'$ of affine schemes over T, then $X_y(U') \to X_y(U')$ is surjective (morphisms in the category of algebraic spaces over T). Set $y' = y|_{U'}$. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y'|_U$. Similarly for morphisms $U' \to X_y$ over T. Hence (1) guarantees the surjectivity of $X_y(U') \to X_y(U')$ in this situation and we win. □

97.7. Surjective on objects

06D4 Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *surjective on objects* if the following condition holds: Given any data consisting of

(1) a field k over S, and
(2) an object y of $\mathcal{Y}$ over $\mathrm{Spec}(k)$,

then there exists an extension K/k of fields over S, an object x of $\mathcal{X}$ over $\mathrm{Spec}(K)$ such that $p(x) \cong y|_{\mathrm{Spec}(K)}$.

06D5 **Lemma 97.7.1.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is surjective on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As p is surjective on objects there exists an extension K/k and an object x of $\mathcal{X}$ over K and an isomorphism $\alpha : p(x) \to q(z)|_{\mathrm{Spec}(K)}$. Then $w = (\mathrm{Spec}(K), x, z|_{\mathrm{Spec}(K)}, \alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over K with $p'(w) = z|_{\mathrm{Spec}(K)}$. □

06D6 **Lemma 97.7.2.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are surjective on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As q is surjective on objects there exists a field extension K/k and an object y of $\mathcal{Y}$ over K such that $q(y) \cong x|_{\mathrm{Spec}(K)}$. As p is surjective on objects there exists a field extension L/K and an object x of $\mathcal{X}$ over L such that $p(x) \cong y|_{\mathrm{Spec}(L)}$. Then the field extension L/k and the object x of $\mathcal{X}$ over L satisfy $q(p(x)) \cong z|_{\mathrm{Spec}(L)}$ as desired. □

06D7 **Lemma 97.7.3.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is surjective on objects, and*
(2) *p is surjective (see Algebraic Stacks, Definition 94.10.1).*

Proof. Assume (2). Let k be a field and let y be an object of $\mathcal{Y}$ over k. Let X_y denote an algebraic space over k representing the 2-fibre product

$$(Sch/\operatorname{Spec}(k))_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}.$$

As we've assumed that p is surjective we see that X_y is not empty. Hence we can find a field extension K/k and a K-valued point x of X_y. Via the 2-Yoneda lemma this corresponds to an object x of $\mathcal{X}$ over K together with an isomorphism $p(x) \cong y|_{\operatorname{Spec}(K)}$ and we see that (1) holds.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is surjective. By Morphisms of Spaces, Definition 67.5.2 we have to show that $|X_y| \to |T|$ is surjective. This means exactly that given a field k over T and a morphism $t : \operatorname{Spec}(k) \to T$ there exists a field extension K/k and a morphism $x : \operatorname{Spec}(K) \to X_y$ such that

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{x} & X_y \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \xrightarrow{t} & T \end{array}$$

commutes. By the 2-Yoneda lemma this means exactly that we have to find $k \subset K$ and an object x of $\mathcal{X}$ over K such that $p(x) \cong t^*y|_{\operatorname{Spec}(K)}$. Hence (1) guarantees that this is the case and we win. □

97.8. Algebraic morphisms

05XX The following notion is occasionally useful.

06CF **Definition 97.8.1.** Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We say that F is *algebraic* if for every scheme T and every object ξ of $\mathcal{Y}$ over T the 2-fibre product

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is an algebraic stack over S.

With this terminology in place we have the following result that generalizes Algebraic Stacks, Lemma 94.15.4.

05XY **Lemma 97.8.2.** *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{Y}$ is an algebraic stack, and*
(2) *F is algebraic (see above),*

then $\mathcal{X}$ is an algebraic stack.

Proof. By assumption (1) there exists a scheme T and an object ξ of $\mathcal{Y}$ over T such that the corresponding 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$ is smooth an surjective. Then $\mathcal{U} = (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$ is an algebraic stack by assumption (2). Choose a scheme U and a surjective smooth 1-morphism $(Sch/U)_{fppf} \to \mathcal{U}$. The projection $\mathcal{U} \longrightarrow \mathcal{X}$ is, as the base change of the morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$,

surjective and smooth, see Algebraic Stacks, Lemma 94.10.6. Then the composition $(Sch/U)_{fppf} \to \mathcal{U} \to \mathcal{X}$ is surjective and smooth as a composition of surjective and smooth morphisms, see Algebraic Stacks, Lemma 94.10.5. Hence $\mathcal{X}$ is an algebraic stack by Algebraic Stacks, Lemma 94.15.3. □

06CG **Lemma 97.8.3.** *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$ is an algebraic stack and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then F is algebraic.*

Proof. Choose a representable stack in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$. Let T be a scheme and let ξ be an object of $\mathcal{Y}$ over T. The morphism of 2-fibre products

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{U} \longrightarrow (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is representable by algebraic spaces, surjective, and smooth as a base change of $\mathcal{U} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6. By our condition on the diagonal of $\mathcal{Y}$ we see that the source of this morphism is representable by an algebraic space, see Algebraic Stacks, Lemma 94.10.11. Hence the target is an algebraic stack by Algebraic Stacks, Lemma 94.15.3. □

0D3R **Lemma 97.8.4.** *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If F is algebraic and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Assume F is algebraic and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Take a scheme U over S and two objects x_1, x_2 of $\mathcal{X}$ over U. We have to show that $Isom(x_1, x_2)$ is an algebraic space over U, see Algebraic Stacks, Lemma 94.10.11. Set $y_i = F(x_i)$. We have a morphism of sheaves of sets

$$f : Isom(x_1, x_2) \to Isom(y_1, y_2)$$

and the target is an algebraic space by assumption. Thus it suffices to show that f is representable by algebraic spaces, see Bootstrap, Lemma 80.3.6. Thus we can choose a scheme V over U and an isomorphism $\beta : y_{1,V} \to y_{2,V}$ and we have to show the functor

$$(Sch/V)_{fppf} \to Sets, \quad T/V \mapsto \{\alpha : x_{1,T} \to x_{2,T} \text{ in } \mathcal{X}_T \mid F(\alpha) = \beta|_T\}$$

is an algebraic space. Consider the objects $z_1 = (V, x_{1,V}, \mathrm{id})$ and $z_2 = (V, x_{2,V}, \beta)$ of

$$\mathcal{Z} = (Sch/V)_{fppf} \times_{y_{1,V},\mathcal{Y}} \mathcal{X}$$

Then it is straightforward to verify that the functor above is equal to $Isom(z_1, z_2)$ on $(Sch/V)_{fppf}$. Hence this is an algebraic space by our assumption that F is algebraic (and the definition of algebraic stacks). □

97.9. Spaces of sections

05XZ Given morphisms $W \to Z \to U$ we can consider the functor that associates to a scheme U' over U the set of sections $\sigma : Z_{U'} \to W_{U'}$ of the base change $W_{U'} \to Z_{U'}$ of the morphism $W \to Z$. In this section we prove some preliminary lemmas on this functor.

05XQ **Lemma 97.9.1.** *Let $Z \to U$ be a finite morphism of schemes. Let W be an algebraic space and let $W \to Z$ be a surjective étale morphism. Then there exists a surjective étale morphism $U' \to U$ and a section*

$$\sigma : Z_{U'} \to W_{U'}$$

of the morphism $W_{U'} \to Z_{U'}$.

Proof. We may choose a separated scheme W' and a surjective étale morphism $W' \to W$. Hence after replacing W by W' we may assume that W is a separated scheme. Write $f : W \to Z$ and $\pi : Z \to U$. Note that $f \circ \pi : W \to U$ is separated as W is separated (see Schemes, Lemma 26.21.13). Let $u \in U$ be a point. Clearly it suffices to find an étale neighbourhood (U', u') of (U, u) such that a section σ exists over U'. Let $z_1, \ldots, z_r$ be the points of Z lying above u. For each i choose a point $w_i \in W$ which maps to z_i. We may pick an étale neighbourhood $(U', u') \to (U, u)$ such that the conclusions of More on Morphisms, Lemma 37.41.5 hold for both $Z \to U$ and the points $z_1, \ldots, z_r$ and $W \to U$ and the points $w_1, \ldots, w_r$. Hence, after replacing (U, u) by (U', u') and relabeling, we may assume that all the field extensions $\kappa(z_i)/\kappa(u)$ and $\kappa(w_i)/\kappa(u)$ are purely inseparable, and moreover that there exist disjoint union decompositions

$$Z = V_1 \amalg \ldots \amalg V_r \amalg A, \quad W = W_1 \amalg \ldots \amalg W_r \amalg B$$

by open and closed subschemes with $z_i \in V_i$, $w_i \in W_i$ and $V_i \to U$, $W_i \to U$ finite. After replacing U by $U \setminus \pi(A)$ we may assume that $A = \emptyset$, i.e., $Z = V_1 \amalg \ldots \amalg V_r$. After replacing W_i by $W_i \cap f^{-1}(V_i)$ and B by $B \cup \bigcup W_i \cap f^{-1}(Z \setminus V_i)$ we may assume that f maps W_i into V_i. Then $f_i = f|_{W_i} : W_i \to V_i$ is a morphism of schemes finite over U, hence finite (see Morphisms, Lemma 29.44.14). It is also étale (by assumption), $f_i^{-1}(\{z_i\}) = w_i$, and induces an isomorphism of residue fields $\kappa(z_i) = \kappa(w_i)$ (because both are purely inseparable extensions of $\kappa(u)$ and $\kappa(w_i)/\kappa(z_i)$ is separable as f is étale). Hence by Étale Morphisms, Lemma 41.14.2 we see that f_i is an isomorphism in a neighbourhood V_i' of z_i. Since $\pi : Z \to U$ is closed, after shrinking U, we may assume that $W_i \to V_i$ is an isomorphism. This proves the lemma. □

05XR **Lemma 97.9.2.** *Let $Z \to U$ be a finite locally free morphism of schemes. Let W be an algebraic space and let $W \to Z$ be an étale morphism. Then the functor*

$$F : (Sch/U)^{opp}_{fppf} \longrightarrow Sets,$$

defined by the rule

$$U' \longmapsto F(U') = \{\sigma : Z_{U'} \to W_{U'} \text{ section of } W_{U'} \to Z_{U'}\}$$

is an algebraic space and the morphism $F \to U$ is étale.

Proof. Assume first that $W \to Z$ is also separated. Let U' be a scheme over U and let $\sigma \in F(U')$. By Morphisms of Spaces, Lemma 67.4.7 the morphism σ is a closed immersion. Moreover, σ is étale by Properties of Spaces, Lemma 66.16.6. Hence σ is also an open immersion, see Morphisms of Spaces, Lemma 67.51.2. In other words, $Z_\sigma = \sigma(Z_{U'}) \subset W_{U'}$ is an open subspace such that the morphism $Z_\sigma \to Z_{U'}$ is an isomorphism. In particular, the morphism $Z_\sigma \to U'$ is finite. Hence we obtain a transformation of functors

$$F \longrightarrow (W/U)_{fin}, \quad \sigma \longmapsto (U' \to U, Z_\sigma)$$

where $(W/U)_{fin}$ is the finite part of the morphism $W \to U$ introduced in More on Groupoids in Spaces, Section 79.12. It is clear that this transformation of functors is injective (since we can recover σ from Z_σ as the inverse of the isomorphism $Z_\sigma \to Z_{U'}$). By More on Groupoids in Spaces, Proposition 79.12.11 we know that $(W/U)_{fin}$ is an algebraic space étale over U. Hence to finish the proof in this case it suffices to show that $F \to (W/U)_{fin}$ is representable and an open immersion. To see this suppose that we are given a morphism of schemes $U' \to U$ and an open subspace $Z' \subset W_{U'}$ such that $Z' \to U'$ is finite. Then it suffices to show that there exists an open subscheme $U'' \subset U'$ such that a morphism $T \to U'$ factors through U'' if and only if $Z' \times_{U'} T$ maps isomorphically to $Z \times_{U'} T$. This follows from More on Morphisms of Spaces, Lemma 76.49.6 (here we use that $Z \to B$ is flat and locally of finite presentation as well as finite). Hence we have proved the lemma in case $W \to Z$ is separated as well as étale.

In the general case we choose a separated scheme W' and a surjective étale morphism $W' \to W$. Note that the morphisms $W' \to W$ and $W \to Z$ are separated as their source is separated. Denote F' the functor associated to $W' \to Z \to U$ as in the lemma. In the first paragraph of the proof we showed that F' is representable by an algebraic space étale over U. By Lemma 97.9.1 the map of functors $F' \to F$ is surjective for the étale topology on Sch/U. Moreover, if U' and $\sigma : Z_{U'} \to W_{U'}$ define a point $\xi \in F(U')$, then the fibre product

$$F'' = F' \times_{F,\xi} U'$$

is the functor on Sch/U' associated to the morphisms

$$W'_{U'} \times_{W_{U'},\sigma} Z_{U'} \to Z_{U'} \to U'.$$

Since the first morphism is separated as a base change of a separated morphism, we see that F'' is an algebraic space étale over U' by the result of the first paragraph. It follows that $F' \to F$ is a surjective étale transformation of functors, which is representable by algebraic spaces. Hence F is an algebraic space by Bootstrap, Theorem 80.10.1. Since $F' \to F$ is an étale surjective morphism of algebraic spaces it follows that $F \to U$ is étale because $F' \to U$ is étale. □

97.10. Relative morphisms

05Y0 We continue the discussion started in More on Morphisms, Section 37.68.

Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to T \times_{a,B} X$ is a morphism over T. Picture

05Y1 (97.10.0.1)

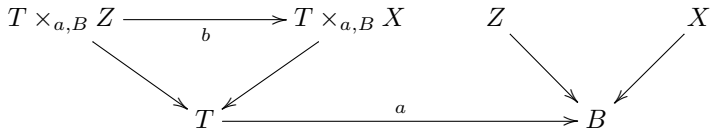

Of course, we can also think of b as a morphism $b : T \times_{a,B} Z \to X$ such that

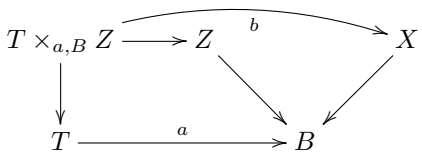

commutes. In this situation we can define a functor

05Y2 (97.10.0.2) $$Mor_B(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

05Y3 **Lemma 97.10.1.** *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Then*

(1) *$Mor_B(Z, X)$ is a sheaf on $(Sch/S)_{fppf}$.*
(2) *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Mor_B(Z, X)) = \{(a, b) \text{ as in } (97.10.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 73.7). Suppose that $(a_i, b_i) \in Mor_B(Z, X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 74.7.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i, B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to T \times_{a,B} X$ such that b_i is the composition of $T_i \times_{a_i, B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in Mor_B(Z, X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (97.10.0.1).

Let $v : T \to Mor_B(Z, X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in Mor_B(Z, X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to Mor_B(Z, X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $Mor_B(Z, X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to Mor_B(Z, X)$.

We omit the verification that the two constructions above are mutually inverse. □

05Y4 **Lemma 97.10.2.** *Let S be a scheme. Let $Z \to B$, $X \to B$, and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Mor_{B'}(Z', X') = B' \times_B Mor_B(Z, X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 97.10.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

05Y5 **Lemma 97.10.3.** *Let S be a scheme. Let $Z \to B$ and $X' \to X \to B$ be morphisms of algebraic spaces over S. Assume*

(1) *$X' \to X$ is étale, and*
(2) *$Z \to B$ is finite locally free.*

Then $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Mor_B(Z, X') \to Mor_B(Z, X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $Mor_B(Z, X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, Mor_B(Z,X)} Mor_B(Z, X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 97.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 97.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

05Y7 **Proposition 97.10.4.** *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Mor_B(Z, X)$ is an algebraic space.*

Proof. Choose a scheme $B' = \coprod B'_i$ which is a disjoint union of affine schemes B'_i and an étale surjective morphism $B' \to B$. We may also assume that $B'_i \times_B Z$ is the spectrum of a ring which is finite free as a $\Gamma(B'_i, \mathcal{O}_{B'_i})$-module. By Lemma 97.10.2 and Spaces, Lemma 65.5.5 the morphism $Mor_{B'}(Z', X') \to Mor_B(Z, X)$ is surjective étale. Hence by Bootstrap, Theorem 80.10.1 it suffices to prove the proposition when $B = B'$ is a disjoint union of affine schemes B'_i so that each $B'_i \times_B Z$ is finite free over B'_i. Then it actually suffices to prove the result for the restriction to each B'_i. Thus we may assume that B is affine and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module.

Choose a scheme X' which is a disjoint union of affine schemes and a surjective étale morphism $X' \to X$. By Lemma 97.10.3 the morphism $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces, étale, and surjective. Hence by Bootstrap, Theorem 80.10.1 it suffices to prove the proposition when X is a disjoint union of affine schemes. This reduces us to the case discussed in the next paragraph.

Assume $X = \coprod_{i \in I} X_i$ is a disjoint union of affine schemes, B is affine, and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. For any finite subset $E \subset I$ set

$$F_E = Mor_B(Z, \coprod\nolimits_{i \in E} X_i).$$

By More on Morphisms, Lemma 37.68.1 we see that F_E is an algebraic space. Consider the morphism

$$\coprod\nolimits_{E \subset I \text{ finite}} F_E \longrightarrow Mor_B(Z, X)$$

Each of the morphisms $F_E \to Mor_B(Z, X)$ is an open immersion, because it is simply the locus parametrizing pairs (a, b) where b maps into the open subscheme $\coprod_{i \in E} X_i$ of X. Moreover, if T is quasi-compact, then for any pair (a, b) the image

of b is contained in $\coprod_{i\in E} X_i$ for some $E \subset I$ finite. Hence the displayed arrow is in fact an open covering and we win[3] by Spaces, Lemma 65.8.5. □

97.11. Restriction of scalars

05Y8 Suppose $X \to Z \to B$ are morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to X$ is a morphism over Z. Picture

05Y9 (97.11.0.1)

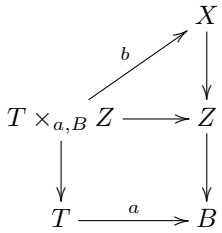

In this situation we can define a functor

05YA (97.11.0.2) $$\mathrm{Res}_{Z/B}(X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

05YB **Lemma 97.11.1.** *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. Then*

(1) *$Res_{Z/B}(X)$ is a sheaf on $(Sch/S)_{fppf}$.*
(2) *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Res_{Z/B}(X)) = \{(a, b) \text{ as in } (97.11.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 73.7). Suppose that $(a_i, b_i) \in \mathrm{Res}_{Z/B}(X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 74.7.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in \mathrm{Res}_{Z/B}(X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (97.11.0.1).

Let $v : T \to \mathrm{Res}_{Z/B}(X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in \mathrm{Res}_{Z/B}(X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

[3]Modulo some set theoretic arguments. Namely, we have to show that $\coprod F_E$ is an algebraic space. This follows because $|I| \leq \mathrm{size}(X)$ and $\mathrm{size}(F_E) \leq \mathrm{size}(X)$ as follows from the explicit description of F_E in the proof of More on Morphisms, Lemma 37.68.1. Some details omitted.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to \mathrm{Res}_{Z/B}(X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 65.9.1 and since $\mathrm{Res}_{Z/B}(X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to \mathrm{Res}_{Z/B}(X)$.

We omit the verification that the two constructions above are mutually inverse. □

Of course the sheaf $\mathrm{Res}_{Z/B}(X)$ comes with a natural transformation of functors $\mathrm{Res}_{Z/B}(X) \to B$. We will use this without further mention in the following.

05YC **Lemma 97.11.2.** *Let S be a scheme. Let $X \to Z \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Res_{Z'/B'}(X') = B' \times_B Res_{Z/B}(X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 97.11.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

05YD **Lemma 97.11.3.** *Let S be a scheme. Let $X' \to X \to Z \to B$ be morphisms of algebraic spaces over S. Assume*

(1) *$X' \to X$ is étale, and*
(2) *$Z \to B$ is finite locally free.*

Then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $\mathrm{Res}_{Z/B}(X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, \mathrm{Res}_{Z/B}(X)} \mathrm{Res}_{Z/B}(X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 97.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 97.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

At this point we can use the lemmas above to prove that $\mathrm{Res}_{Z/B}(X)$ is an algebraic space whenever $Z \to B$ is finite locally free in almost exactly the same way as in the proof that $Mor_B(Z, X)$ is an algebraic spaces, see Proposition 97.10.4. Instead we will directly deduce this result from the following lemma and the fact that $Mor_B(Z, X)$ is an algebraic space.

05YE **Lemma 97.11.4.** *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. The following diagram*

$$\begin{array}{ccc} Mor_B(Z, X) & \longrightarrow & Mor_B(Z, Z) \\ \uparrow & & \uparrow{\scriptstyle id_Z} \\ Res_{Z/B}(X) & \longrightarrow & B \end{array}$$

is a cartesian diagram of sheaves on $(Sch/S)_{fppf}$.

Proof. Omitted. Hint: Exercise in the functorial point of view in algebraic geometry. □

05YF **Proposition 97.11.5.** *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Res_{Z/B}(X)$ is an algebraic space.*

Proof. By Proposition 97.10.4 the functors $Mor_B(Z, X)$ and $Mor_B(Z, Z)$ are algebraic spaces. Hence this follows from the cartesian diagram of Lemma 97.11.4 and the fact that fibre products of algebraic spaces exist and are given by the fibre product in the underlying category of sheaves of sets (see Spaces, Lemma 65.7.2). □

97.12. Finite Hilbert stacks

05XM In this section we prove some results concerning the finite Hilbert stacks $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ introduced in Examples of Stacks, Section 95.18.

05XN **Lemma 97.12.1.** *Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow[G]{} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{H} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(Sch/S)_{fppf}$ with a given 2-isomorphism $\gamma : H \circ F' \to F \circ G$. In this situation we obtain a canonical 1-morphism $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. This morphism is compatible with the forgetful 1-morphisms of Examples of Stacks, Equation (95.18.2.1).

Proof. We map the object (U, Z, y', x', α') to the object $(U, Z, H(y'), G(x'), \gamma \star \mathrm{id}_H \star \alpha')$ where $\star$ denotes horizontal composition of 2-morphisms, see Categories, Definition 4.28.1. To a morphism $(f, g, b, a) : (U_1, Z_1, y'_1, x'_1, \alpha'_1) \to (U_2, Z_2, y'_2, x'_2, \alpha'_2)$ we assign $(f, g, H(b), G(a))$. We omit the verification that this defines a functor between categories over $(Sch/S)_{fppf}$. □

05XP **Lemma 97.12.2.** *In the situation of Lemma 97.12.1 assume that the given square is 2-cartesian. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') & \longrightarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

is 2-cartesian.

Proof. We get a 2-commutative diagram by Lemma 97.12.1 and hence we get a 1-morphism (i.e., a functor)

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

We indicate why this functor is essentially surjective. Namely, an object of the category on the right hand side is given by a scheme U over S, an object y' of $\mathcal{Y}'_U$, an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U and an isomorphism $H(y') \to y$ in $\mathcal{Y}_U$. The assumption means exactly that there exists an object x' of $\mathcal{X}'_Z$ such that there exist isomorphisms $G(x') \cong x$ and $\alpha' : y'|_Z \to F'(x')$ compatible with α. Then we see that (U, Z, y', x', α') is an object of $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}')$ over U. Details omitted. □

05YG **Lemma 97.12.3.** *In the situation of Lemma 97.12.1 assume*

(1) $\mathcal{Y}' = \mathcal{Y}$ *and* $H = id_{\mathcal{Y}}$,
(2) G *is representable by algebraic spaces and étale.*

Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If G is also surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

05XT (97.12.3.1) $$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space étale over U. An object of this over U' corresponds to an object x' in the fibre category of $\mathcal{X}'$ over $Z_{U'}$ such that $G(x') \cong x|_{Z_{U'}}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W such that the projection $W \to Z$ is étale. Then (97.12.3.1) is representable by the algebraic space F parametrizing sections of $W \to Z$ over U introduced in Lemma 97.9.2. Since $F \to U$ is étale we conclude that $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. Finally, if $\mathcal{X}' \to \mathcal{X}$ is surjective also, then $W \to Z$ is surjective, and hence $F \to U$ is surjective by Lemma 97.9.1. Thus in this case $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is also surjective. □

05XS **Lemma 97.12.4.** *In the situation of Lemma 97.12.1. Assume that G, H are representable by algebraic spaces and étale. Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If also H is surjective and the induced functor $\mathcal{X}' \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.*

Proof. Set $\mathcal{X}'' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 97.4.1 the 1-morphism $\mathcal{X}' \to \mathcal{X}''$ is representable by algebraic spaces and étale (in particular the condition in the second statement of the lemma that $\mathcal{X}' \to \mathcal{X}''$ be surjective makes sense). We obtain a 2-commutative diagram

$$\begin{array}{ccccc} \mathcal{X}' & \longrightarrow & \mathcal{X}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

It follows from Lemma 97.12.2 that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}')$ is the base change of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ by $\mathcal{Y}' \to \mathcal{Y}$. In particular we see that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale, see Algebraic Stacks, Lemma 94.10.6. Moreover, it is also surjective if H is. Hence if we can show that the result holds for the left square

in the diagram, then we're done. In this way we reduce to the case where $\mathcal{Y}' = \mathcal{Y}$ which is the content of Lemma 97.12.3. □

05YH **Lemma 97.12.5.** *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Then*

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

see Examples of Stacks, Equation (95.18.2.1) is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, p, x, 1)$ be an object of $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/S)$ over U. Here p is just the structure morphism of U. The fifth component 1 exists and is unique since everything is over S. Also, let y be an object of $\mathcal{Y}$ over U. We have to show the 2-fibre product

05YI (97.12.5.1) $$(Sch/U)_{fppf} \times_{\xi \times y, \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by an algebraic space. To explain why this is so we introduce

$$I = Isom_{\mathcal{Y}}(y|_Z, F(x))$$

which is an algebraic space over Z by assumption. Let $a : U' \to U$ be a scheme over U. What does it mean to give an object of the fibre category of (97.12.5.1) over U'? Well, it means that we have an object $\xi' = (U', Z', y', x', \alpha')$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' and isomorphisms $(U', Z', p', x', 1) \cong (U, Z, p, x, 1)|_{U'}$ and $y' \cong y|_{U'}$. Thus ξ' is isomorphic to $(U', U' \times_{a,U} Z, a^*y, x|_{U' \times_{a,U} Z}, \alpha)$ for some morphism

$$\alpha : a^*y|_{U' \times_{a,U} Z} \longrightarrow F(x|_{U' \times_{a,U} Z})$$

in the fibre category of $\mathcal{Y}$ over $U' \times_{a,U} Z$. Hence we can view α as a morphism $b : U' \times_{a,U} Z \to I$. In this way we see that (97.12.5.1) is representable by $\mathrm{Res}_{Z/U}(I)$ which is an algebraic space by Proposition 97.11.5. □

The following lemma is a (partial) generalization of Lemma 97.12.3.

05YJ **Lemma 97.12.6.** *Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{X}' \to \mathcal{X}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If G is representable by algebraic spaces, then the 1-morphism*

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

05YK (97.12.6.1) $$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space over U. An object of this over $a : U' \to U$ corresponds to an object x' of $\mathcal{X}'$ over $U' \times_{a,U} Z$ such that $G(x') \cong x|_{U' \times_{a,U} Z}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W over Z. It follows that (97.12.6.1) is representable by $\mathrm{Res}_{Z/U}(W)$, which is an algebraic space by Proposition 97.11.5. □

06CH **Lemma 97.12.7.** *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces and locally of finite presentation. Then*

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is limit preserving on objects.

Proof. This means we have to show the following: Given

(1) an affine scheme $U = \lim_i U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i, and
(3) an object $\Xi = (U, Z, y, x, \alpha)$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U such that $y = y_i|_U$,

then there exists an $i' \geq i$ and an object $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over $U_{i'}$ with $\Xi_{i'}|_U = \Xi$ and $y_{i'} = y_i|_{U_{i'}}$. Namely, the last two equalities will take care of the commutativity of (97.5.0.1).

Let $X_{y_i} \to U_i$ be an algebraic space representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}.$$

Note that $X_{y_i} \to U_i$ is locally of finite presentation by our assumption on F. Write Ξ. It is clear that $\xi = (Z, Z \to U_i, x, \alpha)$ is an object of the 2-fibre product displayed above, hence ξ gives rise to a morphism $f_\xi : Z \to X_{y_i}$ of algebraic spaces over U_i (since X_{y_i} is the functor of isomorphisms classes of objects of $(Sch/U_i)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 94.8.2). By Limits, Lemmas 32.10.1 and 32.8.8 there exists an $i' \geq i$ and a finite locally free morphism $Z_{i'} \to U_{i'}$ of degree d whose base change to U is Z. By Limits of Spaces, Proposition 70.3.10 we may, after replacing i' by a bigger index, assume there exists a morphism $f_{i'} : Z_{i'} \to X_{y_i}$ such that

$$\begin{array}{ccccc} Z & \xrightarrow{} & Z_{i'} & \xrightarrow[f_{i'}]{} & X_{y_i} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U_{i'} & \longrightarrow & U_i \end{array} \qquad (f_\xi : Z \to X_{y_i})$$

is commutative. We set $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ where

(1) $y_{i'}$ is the object of $\mathcal{Y}$ over $U_{i'}$ which is the pullback of y_i to $U_{i'}$,
(2) $x_{i'}$ is the object of $\mathcal{X}$ over $Z_{i'}$ corresponding via the 2-Yoneda lemma to the 1-morphism

$$(Sch/Z_{i'})_{fppf} \to \mathcal{S}_{X_{y_i}} \to (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} \to \mathcal{X}$$

where the middle arrow is the equivalence which defines X_{y_i} (notation as in Algebraic Stacks, Sections 94.8 and 94.7).
(3) $\alpha_{i'} : y_{i'}|_{Z_{i'}} \to F(x_{i'})$ is the isomorphism coming from the 2-commutativity of the diagram

$$\begin{array}{ccccc} (Sch/Z_{i'})_{fppf} & \longrightarrow & (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow F \\ & & (Sch/U_{i'})_{fppf} & \longrightarrow & \mathcal{Y} \end{array}$$

Recall that $f_\xi : Z \to X_{y_i}$ was the morphism corresponding to the object $\xi = (Z, Z \to U_i, x, \alpha)$ of $(Sch/U_i)_{fppf} \times_{y_i,\mathcal{Y},F} \mathcal{X}$ over Z. By construction $f_{i'}$ is the morphism corresponding to the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$. As $f_\xi = f_{i'} \circ (Z \to Z_{i'})$ we see that the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$ pulls back to ξ over Z. Thus $x_{i'}$ pulls back to x and $\alpha_{i'}$ pulls back to α. This means that $\Xi_{i'}$ pulls back to Ξ over U and we win. □

97.13. The finite Hilbert stack of a point

05YL Let $d \geq 1$ be an integer. In Examples of Stacks, Definition 95.18.2 we defined a stack in groupoids $\mathcal{H}_d$. In this section we prove that $\mathcal{H}_d$ is an algebraic stack. We will throughout assume that $S = \mathrm{Spec}(\mathbf{Z})$. The general case will follow from this by base change. Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $\pi : Z \to T$ of degree d. Instead of classifying these directly we first study the quasi-coherent sheaves of algebras $\pi_*\mathcal{O}_Z$.

Let R be a ring. Let us temporarily make the following definition: A *free d-dimensional algebra over R* is given by a commutative R-algebra structure m on $R^{\oplus d}$ such that $e_1 = (1, 0, \ldots, 0)$ is a unit[4]. We think of m as an R-linear map

$$m : R^{\oplus d} \otimes_R R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $m(e_1, x) = m(x, e_1) = x$ and such that m defines a commutative and associative ring structure. If we write $m(e_i, e_j) = \sum a_{ij}^k e_k$ then we see this boils down to the conditions

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m = \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k = a_{ji}^k & \forall i, j, k \\ a_{i1}^j = \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} is the Kronecker δ-function. OK, so let's define

$$R_{univ} = \mathbf{Z}[a_{ij}^k]/J$$

where the ideal J is the ideal generated by the relations displayed above. Denote

$$m_{univ} : R_{univ}^{\oplus d} \otimes_{R_{univ}} R_{univ}^{\oplus d} \longrightarrow R_{univ}^{\oplus d}$$

the free d-dimensional algebra m over R_{univ} whose structure constants are the classes of a_{ij}^k modulo J. Then it is clear that given any free d-dimensional algebra m over a ring R there exists a unique $\mathbf{Z}$-algebra homomorphism $\psi : R_{univ} \to R$ such that $\psi_* m_{univ} = m$ (this means that m is what you get by applying the base change functor $- \otimes_{R_{univ}} R$ to m_{univ}). In other words, setting $X = \mathrm{Spec}(R_{univ})$ we obtain a canonical identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } R\}$$

for varying $T = \mathrm{Spec}(R)$. By Zariski localization we obtain the following seemingly more general identification

05YM (97.13.0.1) $$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } \Gamma(T, \mathcal{O}_T)\}$$

for any scheme T.

[4]It may be better to think of this as a pair consisting of a multiplication map $m : R^{\oplus d} \otimes_R R^{\oplus d} \to R^{\oplus d}$ and a ring map $\psi : R \to R^{\oplus d}$ satisfying a bunch of axioms.

Next we talk a little bit about *isomorphisms of free d-dimensional R-algebras*. Namely, suppose that m, m' are two free d-dimensional algebras over a ring R. An *isomorphism from m to m'* is given by an invertible R-linear map

$$\varphi : R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $\varphi(e_1) = e_1$ and such that

$$m \circ \varphi \otimes \varphi = \varphi \circ m'.$$

Note that we can compose these so that the collection of free d-dimensional algebras over R becomes a category. In this way we obtain a functor

05YN (97.13.0.2) $$FA_d : Sch_{fppf}^{opp} \longrightarrow Groupoids$$

from the category of schemes to groupoids: to a scheme T we associate the set of free d-dimensional algebras over $\Gamma(T, \mathcal{O}_T)$ endowed with the structure of a category using the notion of isomorphisms just defined.

The above suggests we consider the functor G in groups which associates to any scheme T the group

$$G(T) = \{g \in \mathrm{GL}_d(\Gamma(T, \mathcal{O}_T)) \mid g(e_1) = e_1\}$$

It is clear that $G \subset \mathrm{GL}_d$ (see Groupoids, Example 39.5.4) is the closed subgroup scheme cut out by the equations $x_{11} = 1$ and $x_{i1} = 0$ for $i > 1$. Hence G is a smooth affine group scheme over $\mathrm{Spec}(\mathbf{Z})$. Consider the action

$$a : G \times_{\mathrm{Spec}(\mathbf{Z})} X \longrightarrow X$$

which associates to a T-valued point (g, m) with $T = \mathrm{Spec}(R)$ on the left hand side the free d-dimensional algebra over R given by

$$a(g, m) = g^{-1} \circ m \circ g \otimes g.$$

Note that this means that g defines an isomorphism $m \to a(g, m)$ of d-dimensional free R-algebras. We omit the verification that a indeed defines an action of the group scheme G on the scheme X.

05YP **Lemma 97.13.1.** *The functor in groupoids FA_d defined in (97.13.0.2) is isomorphic (!) to the functor in groupoids which associates to a scheme T the category with*

1. *set of objects is $X(T)$,*
2. *set of morphisms is $G(T) \times X(T)$,*
3. *$s : G(T) \times X(T) \to X(T)$ is the projection map,*
4. *$t : G(T) \times X(T) \to X(T)$ is $a(T)$, and*
5. *composition $G(T) \times X(T) \times_{s,X(T),t} G(T) \times X(T) \to G(T) \times X(T)$ is given by $((g, m), (g', m')) \mapsto (gg', m')$.*

Proof. We have seen the rule on objects in (97.13.0.1). We have also seen above that $g \in G(T)$ can be viewed as a morphism from m to $a(g, m)$ for any free d-dimensional algebra m. Conversely, any morphism $m \to m'$ is given by an invertible linear map φ which corresponds to an element $g \in G(T)$ such that $m' = a(g, m)$. □

In fact the groupoid $(X, G \times X, s, t, c)$ described in the lemma above is the groupoid associated to the action $a : G \times X \to X$ as defined in Groupoids, Lemma 39.16.1. Since G is smooth over $\mathrm{Spec}(\mathbf{Z})$ we see that the two morphisms $s, t : G \times X \to X$ are smooth: by symmetry it suffices to prove that one of them is, and s is the base

change of $G \to \mathrm{Spec}(\mathbf{Z})$. Hence $(G \times X, X, s, t, c)$ is a smooth groupoid scheme, and the quotient stack $[X/G]$ is an algebraic stack by Algebraic Stacks, Theorem 94.17.3.

05YQ **Proposition 97.13.2.** *The stack $\mathcal{H}_d$ is equivalent to the quotient stack $[X/G]$ described above. In particular $\mathcal{H}_d$ is an algebraic stack.*

Proof. Note that by Groupoids in Spaces, Definition 78.20.1 the quotient stack $[X/G]$ is the stackification of the category fibred in groupoids associated to the "presheaf in groupoids" which associates to a scheme T the groupoid

$$(X(T), G(T) \times X(T), s, t, c).$$

Since this "presheaf in groupoids" is isomorphic to FA_d by Lemma 97.13.1 it suffices to prove that the $\mathcal{H}_d$ is the stackification of (the category fibred in groupoids associated to the "presheaf in groupoids") FA_d. To do this we first define a functor

$$\mathrm{Spec} : FA_d \longrightarrow \mathcal{H}_d$$

Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $Z \to T$ of degree d. Thus given a scheme T and a free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra m we may assign to this the object

$$Z = \underline{\mathrm{Spec}}_T(\mathcal{A})$$

of $\mathcal{H}_{d,T}$ where $\mathcal{A} = \mathcal{O}_T^{\oplus d}$ endowed with a $\mathcal{O}_T$-algebra structure via m. Moreover, if m' is a second such free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra and if $\varphi : m \to m'$ is an isomorphism of these, then the induced $\mathcal{O}_T$-linear map $\varphi : \mathcal{O}_T^{\oplus d} \to \mathcal{O}_T^{\oplus d}$ induces an isomorphism

$$\varphi : \mathcal{A}' \longrightarrow \mathcal{A}$$

of quasi-coherent $\mathcal{O}_T$-algebras. Hence

$$\underline{\mathrm{Spec}}_T(\varphi) : \underline{\mathrm{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\mathrm{Spec}}_T(\mathcal{A}')$$

is a morphism in the fibre category $\mathcal{H}_{d,T}$. We omit the verification that this construction is compatible with base change so we get indeed a functor $\mathrm{Spec} : FA_d \to \mathcal{H}_d$ as claimed above.

To show that $\mathrm{Spec} : FA_d \to \mathcal{H}_d$ induces an equivalence between the stackification of FA_d and $\mathcal{H}_d$ it suffices to check that

(1) $Isom(m, m') = Isom(\mathrm{Spec}(m), \mathrm{Spec}(m'))$ for any $m, m' \in FA_d(T)$.
(2) for any scheme T and any object $Z \to T$ of $\mathcal{H}_{d,T}$ there exists a covering $\{T_i \to T\}$ such that $Z|_{T_i}$ is isomorphic to $\mathrm{Spec}(m)$ for some $m \in FA_d(T_i)$, and

see Stacks, Lemma 8.9.1. The first statement follows from the observation that any isomorphism

$$\underline{\mathrm{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\mathrm{Spec}}_T(\mathcal{A}')$$

is necessarily given by a global invertible matrix g when $\mathcal{A} = \mathcal{A}' = \mathcal{O}_T^{\oplus d}$ as modules. To prove the second statement let $\pi : Z \to T$ be a finite locally free morphism of degree d. Then $\mathcal{A}$ is a locally free sheaf $\mathcal{O}_T$-modules of rank d. Consider the element $1 \in \Gamma(T, \mathcal{A})$. This element is nonzero in $\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ for every $t \in T$ since the scheme $Z_t = \mathrm{Spec}(\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t))$ is nonempty being of degree $d > 0$ over $\kappa(t)$. Thus $1 : \mathcal{O}_T \to \mathcal{A}$ can locally be used as the first basis element (for example you can use Algebra, Lemma 10.79.4 parts (1) and (2) to see this). Thus, after localizing

on T we may assume that there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{O}_T^{\oplus d}$ such that $1 \in \Gamma(\mathcal{A})$ corresponds to the first basis element. In this situation the multiplication map $\mathcal{A} \otimes_{\mathcal{O}_T} \mathcal{A} \to \mathcal{A}$ translates via φ into a free d-dimensional algebra m over $\Gamma(T, \mathcal{O}_T)$. This finishes the proof. □

97.14. Finite Hilbert stacks of spaces

05YR The finite Hilbert stack of an algebraic space is an algebraic stack.

05YS **Lemma 97.14.1.** *Let S be a scheme. Let X be an algebraic space over S. Then $\mathcal{H}_d(X)$ is an algebraic stack.*

Proof. The 1-morphism

$$\mathcal{H}_d(X) \longrightarrow \mathcal{H}_d$$

is representable by algebraic spaces according to Lemma 97.12.6. The stack $\mathcal{H}_d$ is an algebraic stack according to Proposition 97.13.2. Hence $\mathcal{H}_d(X)$ is an algebraic stack by Algebraic Stacks, Lemma 94.15.4. □

This lemma allows us to bootstrap.

06CI **Lemma 97.14.2.** *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$ such that*

1. *$\mathcal{X}$ is representable by an algebraic space, and*
2. *F is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*

Then $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an algebraic stack.

Proof. Choose a representable stack in groupoids $\mathcal{U}$ over S and a 1-morphism $f : \mathcal{U} \to \mathcal{H}_d(\mathcal{X})$ which is representable by algebraic spaces, smooth, and surjective. This is possible because $\mathcal{H}_d(\mathcal{X})$ is an algebraic stack by Lemma 97.14.1. Consider the 2-fibre product

$$\mathcal{W} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}.$$

Since $\mathcal{U}$ is representable (in particular a stack in setoids) it follows from Examples of Stacks, Lemma 95.18.3 and Stacks, Lemma 8.6.7 that $\mathcal{W}$ is a stack in setoids. The 1-morphism $\mathcal{W} \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, smooth, and surjective as a base change of the morphism f (see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6). Thus, if we can show that $\mathcal{W}$ is representable by an algebraic space, then the lemma follows from Algebraic Stacks, Lemma 94.15.3.

The diagonal of $\mathcal{Y}$ is representable by algebraic spaces according to Lemma 97.4.3. We may apply Lemma 97.12.5 to see that the 1-morphism

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

is representable by algebraic spaces. Consider the 2-fibre product

$$\mathcal{V} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}).$$

The projection morphism $\mathcal{V} \to \mathcal{U} \times \mathcal{X}$ is representable by algebraic spaces as a base change of the last displayed morphism. Hence $\mathcal{V}$ is an algebraic space (see

Bootstrap, Lemma 80.3.6 or Algebraic Stacks, Lemma 94.9.8). The 1-morphism $\mathcal{V} \to \mathcal{U}$ fits into the following 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array}$$

because

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X})\times\mathcal{Y}),f\times F} (\mathcal{U} \times \mathcal{X}) = (\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}),f} \mathcal{U}) \times_{\mathcal{Y},F} \mathcal{X}.$$

Hence $\mathcal{V} \to \mathcal{W}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation as a base change of F. It follows that the same thing is true for the corresponding sheaves of sets associated to $\mathcal{V}$ and $\mathcal{W}$, see Algebraic Stacks, Lemma 94.10.4. Thus we conclude that the sheaf associated to $\mathcal{W}$ is an algebraic space by Bootstrap, Theorem 80.10.1. □

97.15. LCI locus in the Hilbert stack

06CJ Please consult Examples of Stacks, Section 95.18 for notation. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that F is representable by algebraic spaces. Fix $d \geq 1$. Consider an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. There is an induced 1-morphism

$$(Sch/Z)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$$

(by the universal property of 2-fibre products) which is representable by a morphism of algebraic spaces over U. Namely, since F is representable by algebraic spaces, we may choose an algebraic space X_y over U which represents the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$. Since $\alpha : y|_Z \to F(x)$ is an isomorphism we see that $\xi = (Z, Z \to U, x, \alpha)$ is an object of the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$ over Z. Hence ξ gives rise to a morphism $x_\alpha : Z \to X_y$ of algebraic spaces over U as X_y is the functor of isomorphisms classes of objects of $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$, see Algebraic Stacks, Lemma 94.8.2. Here is a picture

(97.15.0.1)

06CK

$$\begin{array}{ccc} Z & \xrightarrow{x_\alpha} & X_y \\ & \searrow & \downarrow \\ & & U \end{array} \qquad \begin{array}{ccccc} (Sch/Z)_{fppf} & \xrightarrow{x,\alpha} & (Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow F \\ & & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

We remark that if $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is a morphism between objects of $\mathcal{H}_d$, then the morphism $x'_{\alpha'} : Z' \to X'_{y'}$ is the base change of the morphism x_α by the morphism $g : U' \to U$ (details omitted).

Now assume moreover that F is flat and locally of finite presentation. In this situation we define a full subcategory

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

consisting of those objects (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ such that the corresponding morphism $x_\alpha : Z \to X_y$ is unramified and a local complete intersection morphism (see Morphisms of Spaces, Definition 67.38.1 and More on Morphisms of Spaces, Definition 76.48.1 for definitions).

06CL **Lemma 97.15.1.** *Let S be a scheme. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids and the inclusion functor*

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable and an open immersion.

Proof. Let $\Xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d$. It follows from the remark following (97.15.0.1) that the pullback of Ξ by $U' \to U$ belongs to $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ if and only if the base change of x_α is unramified and a local complete intersection morphism. Note that $Z \to U$ is finite locally free (hence flat, locally of finite presentation and universally closed) and that $X_y \to U$ is flat and locally of finite presentation by our assumption on F. Then More on Morphisms of Spaces, Lemmas 76.49.1 and 76.49.7 imply exists an open subscheme $W \subset U$ such that a morphism $U' \to U$ factors through W if and only if the base change of x_α via $U' \to U$ is unramified and a local complete intersection morphism. This implies that

$$(Sch/U)_{fppf} \times_{\Xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is representable by W. Hence the final statement of the lemma holds. The first statement (that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids) follows from this and Algebraic Stacks, Lemma 94.15.5. □

Local complete intersection morphisms are "locally unobstructed". This holds in much greater generality than the special case that we need in this chapter here.

06D8 **Lemma 97.15.2.** *Let $U \subset U'$ be a first order thickening of affine schemes. Let X' be an algebraic space flat over U'. Set $X = U \times_{U'} X'$. Let $Z \to U$ be finite locally free of degree d. Finally, let $f : Z \to X$ be unramified and a local complete intersection morphism. Then there exists a commutative diagram*

$$\begin{array}{ccc} (Z \subset Z') & \xrightarrow[(f,f')]{} & (X \subset X') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of algebraic spaces over U' such that $Z' \to U'$ is finite locally free of degree d and $Z = U \times_{U'} Z'$.

Proof. By More on Morphisms of Spaces, Lemma 76.48.12 the conormal sheaf $\mathcal{C}_{Z/X}$ of the unramified morphism $Z \to X$ is a finite locally free $\mathcal{O}_Z$-module and by More on Morphisms of Spaces, Lemma 76.48.13 we have an exact sequence

$$0 \to i^*\mathcal{C}_{X/X'} \to \mathcal{C}_{Z/X'} \to \mathcal{C}_{Z/X} \to 0$$

of conormal sheaves. Since Z is affine this sequence is split. Choose a splitting

$$\mathcal{C}_{Z/X'} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z \subset Z''$ be the universal first order thickening of Z over X' (see More on Morphisms of Spaces, Section 76.15). Denote $\mathcal{I} \subset \mathcal{O}_{Z''}$ the quasi-coherent sheaf of ideals corresponding to $Z \subset Z''$. By definition we have $\mathcal{C}_{Z/X'}$ is $\mathcal{I}$ viewed as a sheaf on Z. Hence the splitting above determines a splitting

$$\mathcal{I} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z' \subset Z''$ be the closed subscheme cut out by $\mathcal{C}_{Z/X} \subset \mathcal{I}$ viewed as a quasi-coherent sheaf of ideals on Z''. It is clear that Z' is a first order thickening of Z and that we obtain a commutative diagram of first order thickenings as in the statement of the lemma.

Since $X' \to U'$ is flat and since $X = U \times_{U'} X'$ we see that $\mathcal{C}_{X/X'}$ is the pullback of $\mathcal{C}_{U/U'}$ to X, see More on Morphisms of Spaces, Lemma 76.18.1. Note that by construction $\mathcal{C}_{Z/Z'} = i^*\mathcal{C}_{X/X'}$ hence we conclude that $\mathcal{C}_{Z/Z'}$ is isomorphic to the pullback of $\mathcal{C}_{U/U'}$ to Z. Applying More on Morphisms of Spaces, Lemma 76.18.1 once again (or its analogue for schemes, see More on Morphisms, Lemma 37.10.1) we conclude that $Z' \to U'$ is flat and that $Z = U \times_{U'} Z'$. Finally, More on Morphisms, Lemma 37.10.3 shows that $Z' \to U'$ is finite locally free of degree d. □

06D9 **Lemma 97.15.3.** *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then*

$$p : \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is formally smooth on objects.

Proof. We have to show the following: Given

(1) an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over an affine scheme U,
(2) a first order thickening $U \subset U'$, and
(3) an object y' of $\mathcal{Y}$ over U' such that $y'|_U = y$,

then there exists an object $(U', Z', y', x', \alpha')$ of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. Namely, the last two equalities will take care of the commutativity of (97.6.0.1).

Consider the morphism $x_\alpha : Z \to X_y$ constructed in Equation (97.15.0.1). Denote similarly $X'_{y'}$ the algebraic space over U' representing the 2-fibre product $(Sch/U')_{fppf} \times_{y',\mathcal{Y},F} \mathcal{X}$. By assumption the morphism $X'_{y'} \to U'$ is flat (and locally of finite presentation). As $y'|_U = y$ we see that $X_y = U \times_{U'} X'_{y'}$. Hence we may apply Lemma 97.15.2 to find $Z' \to U'$ finite locally free of degree d with $Z = U \times_{U'} Z'$ and with $Z' \to X'_{y'}$ extending x_α. By construction the morphism $Z' \to X'_{y'}$ corresponds to a pair (x', α'). It is clear that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. As we've seen in Lemma 97.15.1 that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an "open substack" it follows that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ as desired. □

06DA **Lemma 97.15.4.** *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then*

$$\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is surjective on objects.

Proof. It suffices to prove the following: For any field k and object y of $\mathcal{Y}$ over $\mathrm{Spec}(k)$ there exists an integer $d \geq 1$ and an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ with $U = \mathrm{Spec}(k)$. Namely, in this case we see that p is surjective on objects in the strong sense that an extension of the field is not needed.

Denote X_y the algebraic space over $U = \mathrm{Spec}(k)$ representing the 2-fibre product $(Sch/U')_{fppf} \times_{y',\mathcal{Y},F} \mathcal{X}$. By assumption the morphism $X_y \to \mathrm{Spec}(k)$ is surjective

and locally of finite presentation (and flat). In particular X_y is nonempty. Choose a nonempty affine scheme V and an étale morphism $V \to X_y$. Note that $V \to \mathrm{Spec}(k)$ is (flat), surjective, and locally of finite presentation (by Morphisms of Spaces, Definition 67.28.1). Pick a closed point $v \in V$ where $V \to \mathrm{Spec}(k)$ is Cohen-Macaulay (i.e., V is Cohen-Macaulay at v), see More on Morphisms, Lemma 37.22.7. Applying More on Morphisms, Lemma 37.23.4 we find a regular immersion $Z \to V$ with $Z = \{v\}$. This implies $Z \to V$ is a closed immersion. Moreover, it follows that $Z \to \mathrm{Spec}(k)$ is finite (for example by Algebra, Lemma 10.122.1). Hence $Z \to \mathrm{Spec}(k)$ is finite locally free of some degree d. Now $Z \to X_y$ is unramified as the composition of a closed immersion followed by an étale morphism (see Morphisms of Spaces, Lemmas 67.38.3, 67.39.10, and 67.38.8). Finally, $Z \to X_y$ is a local complete intersection morphism as a composition of a regular immersion of schemes and an étale morphism of algebraic spaces (see More on Morphisms, Lemma 37.62.9 and Morphisms of Spaces, Lemmas 67.39.6 and 67.37.8 and More on Morphisms of Spaces, Lemmas 76.48.6 and 76.48.5). The morphism $Z \to X_y$ corresponds to an object x of $\mathcal{X}$ over Z together with an isomorphism $\alpha : y|_Z \to F(x)$. We obtain an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. By what was said above about the morphism $Z \to X_y$ we see that it actually is an object of the subcategory $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ and we win. □

97.16. Bootstrapping algebraic stacks

06DB The following theorem is one of the main results of this chapter.

06DC **Theorem 97.16.1.** *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{X}$ is representable by an algebraic space, and*
(2) *F is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\mathcal{Y}$ is an algebraic stack.

Proof. By Lemma 97.4.3 we see that the diagonal of $\mathcal{Y}$ is representable by algebraic spaces. Hence we only need to verify the existence of a 1-morphism $f : \mathcal{V} \to \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$ with $\mathcal{V}$ representable and f surjective and smooth. By Lemma 97.14.2 we know that

$$\coprod_{d \geq 1} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack. It follows from Lemma 97.15.1 and Algebraic Stacks, Lemma 94.15.5 that

$$\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack as well. Choose a representable stack in groupoids $\mathcal{V}$ over $(Sch/S)_{fppf}$ and a surjective and smooth 1-morphism

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}).$$

We claim that the composition

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is smooth and surjective which finishes the proof of the theorem. In fact, the smoothness will be a consequence of Lemmas 97.12.7 and 97.15.3 and the surjectivity a consequence of Lemma 97.15.4. We spell out the details in the following paragraph.

By construction $\mathcal{V} \to \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, surjective, and smooth (and hence also locally of finite presentation and formally smooth by the general principle Algebraic Stacks, Lemma 94.10.9 and More on Morphisms of Spaces, Lemma 76.19.6). Applying Lemmas 97.5.3, 97.6.3, and 97.7.3 we see that $\mathcal{V} \to \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is limit preserving on objects, formally smooth on objects, and surjective on objects. The 1-morphism $\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ is

(1) limit preserving on objects: this is Lemma 97.12.7 for $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ and we combine it with Lemmas 97.15.1, 97.5.4, and 97.5.2 to get it for $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$,
(2) formally smooth on objects by Lemma 97.15.3, and
(3) surjective on objects by Lemma 97.15.4.

Using Lemmas 97.5.2, 97.6.2, and 97.7.2 we conclude that the composition $\mathcal{V} \to \mathcal{Y}$ is limit preserving on objects, formally smooth on objects, and surjective on objects. Using Lemmas 97.5.3, 97.6.3, and 97.7.3 we see that $\mathcal{V} \to \mathcal{Y}$ is locally of finite presentation, formally smooth, and surjective. Finally, using (via the general principle Algebraic Stacks, Lemma 94.10.9) the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 76.19.6) we see that $\mathcal{V} \to \mathcal{Y}$ is smooth and we win. □

97.17. Applications

06FG Our first task is to show that the quotient stack $[U/R]$ associated to a "flat and locally finitely presented groupoid" is an algebraic stack. See Groupoids in Spaces, Definition 78.20.1 for the definition of the quotient stack. The following lemma is preliminary and is the analogue of Algebraic Stacks, Lemma 94.17.2.

06FH **Lemma 97.17.1.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the morphism $\mathcal{S}_U \to [U/R]$ is flat, locally of finite presentation, and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective, flat, and locally of finite presentation. We already know that the left hand side is representable by an algebraic space F, see Algebraic Stacks, Lemmas 94.17.1 and 94.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective, locally of finite presentation, and flat. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.31.10 and the second equality by Groupoids in Spaces, Lemma 78.22.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is flat and locally of finite presentation as the base change of the flat locally finitely presented morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the first main result of this section.

06FI **Theorem 97.17.2.** *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the two conditions of Theorem 97.16.1 for the morphism

$$(Sch/U)_{fppf} \longrightarrow [U/R].$$

The first is trivial (as U is an algebraic space). The second is Lemma 97.17.1. □

97.18. When is a quotient stack algebraic?

06PI In Groupoids in Spaces, Section 78.20 we have defined the quotient stack $[U/R]$ associated to a groupoid (U, R, s, t, c) in algebraic spaces. Note that $[U/R]$ is a stack in groupoids whose diagonal is representable by algebraic spaces (see Bootstrap, Lemma 80.11.5 and Algebraic Stacks, Lemma 94.10.11) and such that there exists an algebraic space U and a 1-morphism $(Sch/U)_{fppf} \to [U/R]$ which is an "fppf surjection" in the sense that it induces a map on presheaves of isomorphism classes of objects which becomes surjective after sheafification. However, it is not the case that $[U/R]$ is an algebraic stack in general. This is not a contradiction with Theorem 97.16.1 as the 1-morphism $(Sch/U)_{fppf} \to [U/R]$ may not be flat and locally of finite presentation.

The easiest way to make examples of non-algebraic quotient stacks is to look at quotients of the form $[S/G]$ where S is a scheme and G is a group scheme over S acting trivially on S. Namely, we will see below (Lemma 97.18.3) that if $[S/G]$ is algebraic, then $G \to S$ has to be flat and locally of finite presentation. An explicit example can be found in Examples, Section 110.53.

06PJ **Lemma 97.18.1.** *Let S be a scheme and let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack $[U/R]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $g : U' \to U$ such that*

1. *the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves, and*
2. *the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via g.*

Proof. First, assume that $g : U' \to U$ satisfies (1) and (2). Property (1) implies that $[U'/R'] \to [U/R]$ is an equivalence, see Groupoids in Spaces, Lemma 78.25.2. By Theorem 97.17.2 the quotient stack $[U'/R']$ is an algebraic stack. Hence $[U/R]$ is an algebraic stack too, see Algebraic Stacks, Lemma 94.12.4.

Conversely, assume that $[U/R]$ is an algebraic stack. We may choose a scheme W and a surjective smooth 1-morphism

$$f : (Sch/W)_{fppf} \longrightarrow [U/R].$$

By the 2-Yoneda lemma (Algebraic Stacks, Section 94.5) this corresponds to an object ξ of $[U/R]$ over W. By the description of $[U/R]$ in Groupoids in Spaces, Lemma 78.24.1 we can find a surjective, flat, locally finitely presented morphism $b : U' \to W$ of schemes such that $\xi' = b^*\xi$ corresponds to a morphism $g : U' \to U$. Note that the 1-morphism

$$f' : (Sch/U')_{fppf} \longrightarrow [U/R].$$

corresponding to ξ' is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Lemma 94.10.5. Hence $(Sch/U')_{fppf} \times_{[U/R]} (Sch/U')_{fppf}$ which is represented by the algebraic space

$$Isom_{[U/R]}(\mathrm{pr}_0^*\xi', \mathrm{pr}_1^*\xi') = (U' \times_S U') \times_{(g\circ\mathrm{pr}_0, g\circ\mathrm{pr}_1), U\times_S U} R = R'$$

(see Groupoids in Spaces, Lemma 78.22.1 for the first equality; the second is the definition of restriction) is flat and locally of finite presentation over U' via both s' and t' (by base change, see Algebraic Stacks, Lemma 94.10.6). By this description of R' and by Algebraic Stacks, Lemma 94.16.1 we obtain a canonical fully faithful 1-morphism $[U'/R'] \to [U/R]$. This 1-morphism is essentially surjective because f' is flat, locally of finite presentation, and surjective (see Stacks, Lemma 8.4.8); another way to prove this is to use Algebraic Stacks, Remark 94.16.3. Finally, we can use Groupoids in Spaces, Lemma 78.25.2 to conclude that the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves. □

06PK **Lemma 97.18.2.** *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. The quotient stack $[X/G]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $\varphi : X' \to X$ such that*

(1) *$G \times_B X' \to X$, $(g, x') \mapsto a(g, \varphi(x'))$ is a surjection of sheaves, and*
(2) *the two projections $X'' \to X'$ of the algebraic space X'' given by the rule*

$$T \longmapsto \{(x'_1, g, x'_2) \in (X' \times_B G \times_B X')(T) \mid \varphi(x'_1) = a(g, \varphi(x'_2))\}$$

are flat and locally of finite presentation.

Proof. This lemma is a special case of Lemma 97.18.1. Namely, the quotient stack $[X/G]$ is by Groupoids in Spaces, Definition 78.20.1 equal to the quotient stack $[X/G \times_B X]$ of the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ associated to the group action in Groupoids in Spaces, Lemma 78.15.1. There is one small observation that is needed to get condition (1). Namely, the morphism $s : G \times_B X \to X$ is the second projection and the morphism $t : G \times_B X \to X$ is the action morphism a. Hence the morphism $h : U' \times_{g,U,t} R \to R \xrightarrow{s} U$ from Lemma 97.18.1 corresponds to the morphism

$$X' \times_{\varphi,X,a} (G \times_B X) \xrightarrow{\mathrm{pr}_1} X$$

in the current setting. However, because of the symmetry given by the inverse of G this morphism is isomorphic to the morphism

$$(G \times_B X) \times_{\mathrm{pr}_1,X,\varphi} X' \xrightarrow{a} X$$

of the statement of the lemma. Details omitted. □

06PL **Lemma 97.18.3.** *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. Then the quotient stack $[B/G]$ is an algebraic stack if and only if G is flat and locally of finite presentation over B.*

Proof. If G is flat and locally of finite presentation over B, then $[B/G]$ is an algebraic stack by Theorem 97.17.2.

Conversely, assume that $[B/G]$ is an algebraic stack. By Lemma 97.18.2 and because the action is trivial, we see there exists an algebraic space B' and a morphism $B' \to B$ such that (1) $B' \to B$ is a surjection of sheaves and (2) the projections

$$B' \times_B G \times_B B' \to B'$$

are flat and locally of finite presentation. Note that the base change $B' \times_B G \times_B B' \to G \times_B B'$ of $B' \to B$ is a surjection of sheaves also. Thus it follows from Descent on Spaces, Lemma 74.8.1 that the projection $G \times_B B' \to B'$ is flat and locally of finite presentation. By (1) we can find an fppf covering $\{B_i \to B\}$ such that $B_i \to B$ factors through $B' \to B$. Hence $G \times_B B_i \to B_i$ is flat and locally of finite presentation by base change. By Descent on Spaces, Lemmas 74.11.13 and 74.11.10 we conclude that $G \to B$ is flat and locally of finite presentation. □

Later we will see that the quotient stack of a smooth S-space by a group algebraic space G is smooth, even when G is not smooth (Morphisms of Stacks, Lemma 101.33.7).

97.19. Algebraic stacks in the étale topology

076U Let S be a scheme. Instead of working with stacks in groupoids over the big fppf site $(Sch/S)_{fppf}$ we could work with stacks in groupoids over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Stacks, Sections 94.4, 94.5, 94.6, 94.7, 94.8, 94.9, 94.10, and 94.11 makes sense for categories fibred in groupoids over $(Sch/S)_{étale}$. Thus we get a second notion of an algebraic stack by working in the étale topology. This notion is (a priori) weaker than the notion introduced in Algebraic Stacks, Definition 94.12.1 since a stack in the fppf topology is certainly a stack in the étale topology. However, the notions are equivalent as is shown by the following lemma.

076V **Lemma 97.19.1.** *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Sheaves on Stacks, Section 96.4 and Topologies, Remark 34.11.1). Let S be an object of Sch_α. Let*

$$p : \mathcal{X} \to Sch_\alpha/S$$

be a category fibred in groupoids with the following properties:

1. *$\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{étale}$,*
2. *the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces[5], and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and a 1-morphism $(Sch/U)_{étale} \to \mathcal{X}$ which is surjective and smooth.*

Then $\mathcal{X}$ is an algebraic stack in the sense of Algebraic Stacks, Definition 94.12.1.

[5]Here we can either mean sheaves in the étale topology whose diagonal is representable and which have an étale surjective covering by a scheme or algebraic spaces as defined in Algebraic Spaces, Definition 65.6.1. Namely, by Bootstrap, Lemma 80.12.1 there is no difference.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Stacks, Definition 94.12.1 are independent of the topology. This is true because these properties involve only the notion of a 2-fibre product of categories fibred in groupoids, 1- and 2-morphisms of categories fibred in groupoids, the notion of a 1-morphism of categories fibred in groupoids representable by algebraic spaces, and what it means for such a 1-morphism to be surjective and smooth. Thus all we have to prove is that an étale stack in groupoids $\mathcal{X}$ with properties (2) and (3) is also an fppf stack in groupoids.

Using (2) let R be an algebraic space representing

$$(Sch_\alpha/U) \times_{\mathcal{X}} (Sch_\alpha/U)$$

By (3) the projections $s, t : R \to U$ are smooth. Exactly as in the proof of Algebraic Stacks, Lemma 94.16.1 there exists a groupoid in spaces (U, R, s, t, c) and a canonical fully faithful 1-morphism $[U/R]_{étale} \to \mathcal{X}$ where $[U/R]_{étale}$ is the étale stackification of presheaf in groupoids

$$T \longmapsto (U(T), R(T), s(T), t(T), c(T))$$

Claim: If $V \to T$ is a surjective smooth morphism from an algebraic space V to a scheme T, then there exists an étale covering $\{T_i \to T\}$ refining the covering $\{V \to T\}$. This follows from More on Morphisms, Lemma 37.38.7 or the more general Sheaves on Stacks, Lemma 96.19.10. Using the claim and arguing exactly as in Algebraic Stacks, Lemma 94.16.2 it follows that $[U/R]_{étale} \to \mathcal{X}$ is an equivalence.

Next, let $[U/R]$ denote the quotient stack in the fppf topology which is an algebraic stack by Algebraic Stacks, Theorem 94.17.3. Thus we have 1-morphisms

$$U \to [U/R]_{étale} \to [U/R].$$

Both $U \to [U/R]_{étale} \cong \mathcal{X}$ and $U \to [U/R]$ are surjective and smooth (the first by assumption and the second by the theorem) and in both cases the fibre product $U \times_{\mathcal{X}} U$ and $U \times_{[U/R]} U$ is representable by R. Hence the 1-morphism $[U/R]_{étale} \to [U/R]$ is fully faithful (since morphisms in the quotient stacks are given by morphisms into R, see Groupoids in Spaces, Section 78.24).

Finally, for any scheme T and morphism $t : T \to [U/R]$ the fibre product $V = T \times_{U/R} U$ is an algebraic space surjective and smooth over T. By the claim above there exists an étale covering $\{T_i \to T\}_{i \in I}$ and morphisms $T_i \to V$ over T. This proves that the object t of $[U/R]$ over T comes étale locally from U. We conclude that $[U/R]_{étale} \to [U/R]$ is an equivalence of stacks in groupoids over $(Sch/S)_{étale}$ by Stacks, Lemma 8.4.8. This concludes the proof. □

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CHAPTER 98

Artin's Axioms

07SZ

98.1. Introduction

07T0 In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [**Art69b**], [**Art70**], [**Art74**].

Some of the notation, conventions, and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 99.2 for an explanation.

Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let x_0 be an object of $\mathcal{X}$ over a field k of finite type over S. Throughout this chapter an important role is played by the predeformation category (see Formal Deformation Theory, Definition 90.6.2)

$$\mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \{\text{Artinian local } S\text{-algebras with residue field } k\}$$

associated to x_0 over k. We introduce the Rim-Schlessinger condition (RS) for $\mathcal{X}$ and show it guarantees that $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category, i.e., $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (RS) itself. We discuss how $\mathcal{F}_{\mathcal{X},k,x_0}$ changes if one replaces k by a finite extension and we discuss tangent spaces.

Next, we discuss formal objects $\xi = (\xi_n)$ of $\mathcal{X}$ which are inverse systems of objects lying over the quotients $R/\mathfrak{m}^n$ where R is a Noetherian complete local S-algebra whose residue field is of finite type over S. This is the same thing as having a formal object in $\mathcal{F}_{\mathcal{X},k,x_0}$ for some x_0 and k. A formal object is called effective when there is an object of $\mathcal{X}$ over R which gives rise to the inverse system. A formal object of $\mathcal{X}$ is called versal if it gives rise to a versal formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$. Finally, given a finite type S-scheme U, an object x of $\mathcal{X}$ over U, and a closed point $u_0 \in U$ we say x is versal at u_0 if the induced formal object over the complete local ring $\mathcal{O}^\wedge_{U,u_0}$ is versal.

Having worked through this material we can state Artin's celebrated theorem: our $\mathcal{X}$ is an algebraic stack if the following are true

(1) $\mathcal{O}_{S,s}$ is a G-ring for all $s \in S$,
(2) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,
(3) $\mathcal{X}$ is a stack for the étale topology,
(4) $\mathcal{X}$ is limit preserving,
(5) $\mathcal{X}$ satisfies (RS),
(6) tangent spaces and spaces of infinitesimal automorphisms of the deformation categories $\mathcal{F}_{\mathcal{X},k,x_0}$ are finite dimensional,
(7) formal objects are effective,

(8) $\mathcal{X}$ satisfies openness of versality.

This is Lemma 98.17.1; see also Proposition 98.17.2 for a slight improvement. There is an analogous proposition characterizing which functors $F : (Sch/S)^{opp}_{fppf} \to Sets$ are algebraic spaces, see Section 98.16.

Here is a rough outline of the proof of Artin's theorem. First we show that there are plenty of versal formal objects using (RS) and the finite dimensionality of tangent and aut spaces, see for example Formal Deformation Theory, Lemma 90.27.6. These formal objects are effective by assumption. Effective formal objects can be "approximated" by objects x over finite type S-schemes U, see Lemma 98.10.1. This approximation uses the local rings of S are G-rings and that $\mathcal{X}$ is limit preserving; it is perhaps the most difficult part of the proof relying as it does on general Néron desingularization to approximate formal solutions of algebraic equations over a Noetherian local G-ring by solutions in the henselization. Next openness of versality implies we may (after shrinking U) assume x is versal at every closed point of U. Having done all of this we show that $U \to \mathcal{X}$ is a smooth morphism. Taking sufficiently many $U \to \mathcal{X}$ we show that we obtain a "smooth atlas" for $\mathcal{X}$ which shows that $\mathcal{X}$ is an algebraic stack.

In checking Artin's axioms for a given category $\mathcal{X}$ fibred in groupoids, the most difficult step is often to verify openness of versality. For the discussion that follows, assume that $\mathcal{X}/S$ already satisfies the other conditions listed above. In this chapter we offer two methods that will allow the reader to prove $\mathcal{X}$ satisfies openness of versality:

(1) The first is to assume a stronger Rim-Schlessinger condition, called (RS*) and to assume a stronger version of formal effectiveness, essentially requiring objects over inverse systems of thickenings to be effective. It turns out that under these assumptions, openness of versality comes for free, see Lemma 98.20.3. Please observe that here we are using in an essential manner that $\mathcal{X}$ is defined on that category of all schemes over S, not just the category of Noetherian schemes!
(2) The second, following Artin, is to require $\mathcal{X}$ to come equipped with an obstruction theory. If said obstruction theory "commutes with products" in a suitable sense, then $\mathcal{X}$ satisfies openness of versality, see Lemma 98.22.2.

Obstruction theories can be axiomatized in many different ways and indeed many variants (often adapted to specific moduli stacks) can be found in the literature. We explain a variant using the derived category (which often arises naturally from deformation theory computations done in the literature) in Lemma 98.24.4.

In Section 98.26 we discuss what needs to be modified to make things work for functors defined on the category $(Noetherian/S)_{étale}$ of locally Noetherian schemes over S.

In the final section of this chapter as an application of Artin's axioms we prove Artin's theorem on the existence of contractions, see Section 98.27. The theorem says roughly that given an algebraic space X' separated of finite type over S, a closed subset $T' \subset |X'|$, and a formal modification

$$\mathfrak{f} : X'_{/T'} \longrightarrow \mathfrak{X}$$

where $\mathfrak{X}$ is a Noetherian formal algebraic space over S, there exists a proper morphism $f : X' \to X$ which "realizes the contraction". By this we mean that there exists an identification $\mathfrak{X} = X_{/T}$ such that $\mathfrak{f} = f_{/T'} : X'_{/T'} \to X_{/T}$ where $T = f(T')$ and moreover f is an isomorphism over $X \setminus T$. The proof proceeds by defining a functor F on the category of locally Noetherian schemes over S and proving Artin's axioms for F. Amusingly, in this application of Artin's axioms, openness of versality is not the hardest thing to prove, instead the proof that F is limit preserving requires a lot of work and preliminary results.

98.2. Conventions

07T1 The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 94.2. In this chapter the base scheme S will often be locally Noetherian (although we will always reiterate this condition when stating results).

98.3. Predeformation categories

07T2 Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let k be a field and let $\mathrm{Spec}(k) \to S$ be a morphism of finite type (see Morphisms, Lemma 29.16.1). We will sometimes simply say that *k is a field of finite type over S*. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Given S, $\mathcal{X}$, k, and x_0 we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 90.6.2. The construction will resemble the construction of Formal Deformation Theory, Remark 90.6.4.

First, by Morphisms, Lemma 29.16.1 we may pick an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through $\mathrm{Spec}(\Lambda)$ and the associated ring map $\Lambda \to k$ is finite. This provides us with the category $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Definition 90.3.1. The category $\mathcal{C}_\Lambda$, up to canonical equivalence, does not depend on the choice of the affine open $\mathrm{Spec}(\Lambda)$ of S. Namely, $\mathcal{C}_\Lambda$ is equivalent to the opposite of the category of factorizations

07T3 (98.3.0.1) $$\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to S$$

of the structure morphism such that A is an Artinian local ring and such that $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ corresponds to a ring map $A \to k$ which identifies k with the residue field of A.

We let $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ be the category whose

(1) objects are morphisms $x_0 \to x$ of $\mathcal{X}$ where $p(x) = \mathrm{Spec}(A)$ with A an Artinian local ring and $p(x_0) \to p(x) \to S$ a factorization as in (98.3.0.1), and
(2) morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams

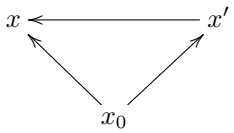

in $\mathcal{X}$. (Note the reversal of arrows.)

If $x_0 \to x$ is an object of $\mathcal{F}$ then writing $p(x) = \mathrm{Spec}(A)$ we obtain an object A of $\mathcal{C}_\Lambda$. We often say that $x_0 \to x$ or x lies over A. A morphism of $\mathcal{F}$ between objects $x_0 \to x$ lying over A and $x_0 \to x'$ lying over A' corresponds to a morphism $x' \to x$ of $\mathcal{X}$, hence a morphism $p(x' \to x) : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ which in turn corresponds to a ring map $A \to A'$. As $\mathcal{X}$ is a category over the category of schemes over S we see that $A \to A'$ is Λ-algebra homomorphism. Thus we obtain a functor

07T4 (98.3.0.2) $$p : \mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{C}_\Lambda.$$

We will use the notation $\mathcal{F}(A)$ to denote the fibre category over an object A of $\mathcal{C}_\Lambda$. An object of $\mathcal{F}(A)$ is simply a morphism $x_0 \to x$ of $\mathcal{X}$ such that x lies over $\mathrm{Spec}(A)$ and $x_0 \to x$ lies over $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$.

07T5 **Lemma 98.3.1.** *The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ defined above is a predeformation category.*

Proof. We have to show that $\mathcal{F}$ is (a) cofibred in groupoids over $\mathcal{C}_\Lambda$ and (b) that $\mathcal{F}(k)$ is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of $\mathcal{F}$ over $\mathcal{C}_\Lambda$ are groupoids as the fibre categories of $\mathcal{X}$ are groupoids. Let $A \to A'$ be a morphism of $\mathcal{C}_\Lambda$ and let $x_0 \to x$ be an object of $\mathcal{F}(A)$. Because $\mathcal{X}$ is fibred in groupoids, we can find a morphism $x' \to x$ lying over $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. Since the composition $A \to A' \to k$ is equal the given map $A \to k$ we see (by uniqueness of pullbacks up to isomorphism) that the pullback via $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ of x' is x_0, i.e., that there exists a morphism $x_0 \to x'$ lying over $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ compatible with $x_0 \to x$ and $x' \to x$. This proves that $\mathcal{F}$ has pushforwards. We conclude by (the dual of) Categories, Lemma 4.35.2.

Proof of (b). If $A = k$, then $\mathrm{Spec}(k) = \mathrm{Spec}(A)$ and since $\mathcal{X}$ is fibred in groupoids over $(Sch/S)_{fppf}$ we see that given any object $x_0 \to x$ in $\mathcal{F}(k)$ the morphism $x_0 \to x$ is an isomorphism. Hence every object of $\mathcal{F}(k)$ is isomorphic to $x_0 \to x_0$. Clearly the only self morphism of $x_0 \to x_0$ in $\mathcal{F}$ is the identity. □

Let S be a locally Noetherian base scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism between categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k is a field of finite type over S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Set $y_0 = F(x_0)$ which is an object of $\mathcal{Y}$ lying over $\mathrm{Spec}(k)$. Then F induces a functor

07WJ (98.3.1.1) $$F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0}$$

of categories cofibred over $\mathcal{C}_\Lambda$. Namely, to the object $x_0 \to x$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ we associate the object $F(x_0) \to F(x)$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

07WK **Lemma 98.3.2.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume either*

1. *F is formally smooth on objects (Criteria for Representability, Section 97.6),*
2. *F is representable by algebraic spaces and formally smooth, or*
3. *F is representable by algebraic spaces and smooth.*

Then for every finite type field k over S and object x_0 of $\mathcal{X}$ over k the functor (98.3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 90.8.1.

Proof. Case (1) is a matter of unwinding the definitions. Assumption (2) implies (1) by Criteria for Representability, Lemma 97.6.3. Assumption (3) implies (2) by More on Morphisms of Spaces, Lemma 76.19.6 and the principle of Algebraic Stacks, Lemma 94.10.9. □

07WL **Lemma 98.3.3.** *Let S be a locally Noetherian scheme. Let*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a finite type field over S and w_0 an object of $\mathcal{W}$ over k. Let x_0, z_0, y_0 be the images of w_0 under the morphisms in the diagram. Then

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{W},k,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},k,z_0} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{X},k,x_0} & \longrightarrow & \mathcal{F}_{\mathcal{Y},k,y_0} \end{array}$$

is a fibre product of predeformation categories.

Proof. This is a matter of unwinding the definitions. Details omitted. □

98.4. Pushouts and stacks

07WM In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when Y, X', X, Y' are algebraic spaces, see (insert future reference here).

07WN **Lemma 98.4.1.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Let $\mathcal{Z}$ be an algebraic stack over S. Then the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

Proof. Let y' be an object of left hand side. The sheaf $Isom(y', y')$ on the category of schemes over Y' is representable by an algebraic space I over Y', see Algebraic Stacks, Lemma 94.10.11. We conclude that the functor of the lemma is fully faithful as Y' is the pushout in the category of algebraic spaces as well as the category of schemes, see Pushouts of Spaces, Lemma 81.6.1.

Let (y, x', f) be an object of the right hand side. Here $f : y|_X \to x'|_X$ is an isomorphism. To finish the proof we have to construct an object y' of $\mathcal{Z}_{Y'}$ whose restrictions to Y and X' agree with y and x' in a manner compatible with f. In

fact, it suffices to construct y' fppf locally on Y', see Stacks, Lemma 8.4.8. Choose a representable algebraic stack $\mathcal{W}$ and a surjective smooth morphism $\mathcal{W} \to \mathcal{Z}$. Then

$$(Sch/Y)_{fppf} \times_{y,\mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x',\mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces V and U' smooth over Y and X'. The isomorphism f induces an isomorphism $\varphi : V \times_Y X \to U' \times_{X'} X$ over X. By Pushouts of Spaces, Lemmas 81.6.2 and 81.6.7 we see that the pushout $V' = V \amalg_{V \times_Y X} U'$ is an algebraic space smooth over Y' whose base change to Y and X' recovers V and U' in a manner compatible with φ.

Let W be the algebraic space representing $\mathcal{W}$. The projections $V \to W$ and $U' \to W$ agree as morphisms over $V \times_Y X \cong U' \times_{X'} X$ hence the universal property of the pushout determines a morphism of algebraic spaces $V' \to W$. Choose a scheme Y'_1 and a surjective étale morphism $Y'_1 \to V'$. Set $Y_1 = Y \times_{Y'} Y'_1$, $X'_1 = X' \times_{Y'} Y'_1$, $X_1 = X \times_{Y'} Y'_1$. The composition

$$(Sch/Y'_1) \to (Sch/V') \to (Sch/W) = \mathcal{W} \to \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object y'_1 of $\mathcal{Z}$ over Y'_1 whose restriction to Y_1 and X'_1 agrees with $y|_{Y_1}$ and $x'|_{X'_1}$ in a manner compatible with $f|_{X_1}$. Thus we have constructed our desired object smooth locally over Y' and we win. □

98.5. The Rim-Schlessinger condition

06L9 The motivation for the following definition comes from Lemma 98.4.1 and Formal Deformation Theory, Definition 90.16.1 and Lemma 90.16.4.

07WP **Definition 98.5.1.** Let S be a locally Noetherian scheme. Let $\mathcal{Z}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{Z}$ satisfies *condition (RS)* if for every pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

in the category of schemes over S where

(1) X, X', Y, Y' are spectra of local Artinian rings,
(2) X, X', Y, Y' are of finite type over S, and
(3) $X \to X'$ (and hence $Y \to Y'$) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If A is an Artinian local ring with residue field k, then any morphism $\mathrm{Spec}(A) \to S$ is affine and of finite type if and only if the induced morphism $\mathrm{Spec}(k) \to S$ is of finite type, see Morphisms, Lemmas 29.11.14 and 29.16.2.

07WQ **Lemma 98.5.2.** *Let $\mathcal{X}$ be an algebraic stack over a locally Noetherian base S. Then $\mathcal{X}$ satisfies (RS).*

Proof. Immediate from the definitions and Lemma 98.4.1. □

07WR **Lemma 98.5.3.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Let

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

be a diagram as in Definition 98.5.1. We have to show that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_Y \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_X} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

07WS (98.5.3.1) $$\mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}).$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \to \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \to \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \to \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (98.5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then $(x_Y, x_{Y'}, \alpha)$ is isomorphic to the image of an object $x_{Y'}$ in $\mathcal{X}_{Y'}$ and $(z_Y, z_{Y'}, \beta)$ is isomorphic to the image of an object $z_{Y'}$ of $\mathcal{Z}_{Y'}$. The pair of morphisms $(\phi_Y, \phi_{X'})$ corresponds to a morphism ψ between the images of $x_{Y'}$ and $z_{Y'}$ in $\mathcal{Y}_{Y'}$. Then $(x_{Y'}, z_{Y'}, \psi)$ is an object of the left hand side of (98.5.3.1) mapping to the given object of the right hand side. This proves that (98.5.3.1) is essentially surjective. We omit the proof that it is fully faithful. □

98.6. Deformation categories

07WT We match the notation introduced above with the notation from the chapter "Formal Deformation Theory".

07WU **Lemma 98.6.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS). For any field k of finite type over S and any object x_0 of $\mathcal{X}$ lying over k the predeformation category $p : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{C}_\Lambda$ (98.3.0.2) is a deformation category, see Formal Deformation Theory, Definition 90.16.8.*

Proof. Set $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$. Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 90.16.4. Set $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A_2)$, $Y = \mathrm{Spec}(A_1)$ and $Y' = \mathrm{Spec}(A_1 \times_A A_2)$. Note that $Y' = Y \amalg_X X'$ in the category of schemes, see More on Morphisms, Lemma 37.14.3. We know that in the diagram of functors of fibre categories

$$\begin{array}{ccc} \mathcal{X}_{Y'} & \longrightarrow & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathrm{Spec}(k)} & = & \mathcal{X}_{\mathrm{Spec}(k)} \end{array}$$

the top horizontal arrow is an equivalence by Definition 98.5.1. Since $\mathcal{F}(B)$ is the category of objects of $\mathcal{X}_{\mathrm{Spec}(B)}$ with an identification with x_0 over k we win. $\square$

07WV **Remark 98.6.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and x_0 an object of $\mathcal{X}$ over k. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be as in (98.3.0.2). If $\mathcal{F}$ is a deformation category, i.e., if $\mathcal{F}$ satisfies the Rim-Schlessinger condition (RS), then we see that $\mathcal{F}$ satisfies Schlessinger's conditions (S1) and (S2) by Formal Deformation Theory, Lemma 90.16.6. Let $\overline{\mathcal{F}}$ be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 90.5.2 (10). Then $\overline{\mathcal{F}}$ satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 90.10.5. This holds in particular in the situation of Lemma 98.6.1.

98.7. Change of field

07WW This section is the analogue of Formal Deformation Theory, Section 90.29. As pointed out there, to discuss what happens under change of field we need to write $\mathcal{C}_{\Lambda,k}$ instead of $\mathcal{C}_\Lambda$. In the following lemma we use the notation $\mathcal{F}_{l/k}$ introduced in Formal Deformation Theory, Situation 90.29.1.

07WX **Lemma 98.7.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let l/k be a finite extension. Let x_0 be an object of $\mathcal{F}$ lying over $\mathrm{Spec}(k)$. Denote $x_{l,0}$ the restriction of x_0 to $\mathrm{Spec}(l)$. Then there is a canonical functor*

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$. If $\mathcal{X}$ satisfies (RS), then this functor is an equivalence.

Proof. Consider a factorization

$$\mathrm{Spec}(l) \to \mathrm{Spec}(B) \to S$$

as in (98.3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 90.29.1. Thus an object of this is a morphism $x_0 \to x$ of $\mathcal{X}$ lying over the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(B \times_l k)$. Choosing pullback functor for $\mathcal{X}$ we can associate to $x_0 \to x$ the morphism $x_{l,0} \to x_B$ where x_B is the restriction of x to $\mathrm{Spec}(B)$ (via the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B \times_l k)$ coming from $B \times_l k \subset B$). This construction is functorial in B and compatible with morphisms.

Next, assume $\mathcal{X}$ satisfies (RS). Consider the diagrams

$$\begin{array}{ccc} l & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & B \times_l k \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(l) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(B \times_l k) \end{array}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 37.14.3.

These schemes are all of finite type over S (see remarks following Definition 98.5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(B\times_l k)} \longrightarrow \mathcal{X}_{\mathrm{Spec}(k)} \times_{\mathcal{X}_{\mathrm{Spec}(l)}} \mathcal{X}_{\mathrm{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 4.35.9. □

98.8. Tangent spaces

07WY Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over k. In Formal Deformation Theory, Section 90.12 we have defined the *tangent space*

07WZ (98.8.0.1)
$$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{\begin{matrix}\text{isomorphism classes of morphisms} \\ x_0 \to x \text{ over } \mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])\end{matrix}\right\}$$

of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. In Formal Deformation Theory, Section 90.19 we have defined

07X0 (98.8.0.2)
$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathrm{Spec}(k[\epsilon])}(x_0') \to \mathrm{Aut}_{\mathrm{Spec}(k)}(x_0)\right)$$

where x_0' is the pullback of x_0 to $\mathrm{Spec}(k[\epsilon])$. If $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS), then $T\mathcal{F}_{\mathcal{X},k,x_0}$ comes equipped with a natural k-vector space structure by Formal Deformation Theory, Lemma 90.12.2 (assumptions hold by Lemma 98.6.1 and Remark 98.6.2). Moreover, Formal Deformation Theory, Lemma 90.19.9 shows that $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ has a natural k-vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if $\mathcal{X}$ is locally of finite type over S (see Morphisms of Stacks, Section 101.17).

07X1 **Lemma 98.8.1.** *Let S be a locally Noetherian scheme. Assume*

(1) *$\mathcal{X}$ is an algebraic stack,*
(2) *U is a scheme locally of finite type over S, and*
(3) *$(Sch/U)_{fppf} \to \mathcal{X}$ is a smooth surjective morphism.*

Then, for any $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ as in Section 98.3 the tangent space $T\mathcal{F}$ and infinitesimal automorphism space $\mathit{Inf}(\mathcal{F})$ have finite dimension over k.

Proof. Let us write $\mathcal{U} = (Sch/U)_{fppf}$. By our definition of algebraic stacks the 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence in particular the 2-fibre product

$$\mathcal{U}_{x_0} = (Sch/\mathrm{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$

is representable by an algebraic space U_{x_0} over $\mathrm{Spec}(k)$. Then $U_{x_0} \to \mathrm{Spec}(k)$ is smooth and surjective (in particular U_{x_0} is nonempty). By Spaces over Fields, Lemma 72.16.2 we can find a finite extension l/k and a point $\mathrm{Spec}(l) \to U_{x_0}$ over k. We have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

by Lemma 98.7.1 and the fact that $\mathcal{X}$ satisfies (RS). Thus we see that

$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \mathrm{Inf}(\mathcal{F}) \otimes_k l \cong \mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$

by Formal Deformation Theory, Lemmas 90.29.3 and 90.29.4 (these are applicable by Lemmas 98.5.2 and 98.6.1 and Remark 98.6.2). Hence it suffices to prove that $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$ have finite dimension over l. Note that $x_{l,0}$ comes from a point u_0 of $\mathcal{U}$ over l.

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$. Let r_0 be the l-valued point $(u_0, u_0, \mathrm{id}_{x_0})$ of $\mathcal{R}$. Combining Lemma 98.3.3 and Formal Deformation Theory, Lemma 90.26.2 we see that

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},l,x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R},l,r_0}$$

Note that $\mathcal{R}$ is an algebraic stack, see Algebraic Stacks, Lemma 94.14.2. Also, $\mathcal{R}$ is representable by an algebraic space R smooth over U (via either projection, see Algebraic Stacks, Lemma 94.16.2). Hence, choose an scheme U' and a surjective étale morphism $U' \to R$ we see that U' is smooth over U, hence locally of finite type over S. As $(Sch/U')_{fppf} \to \mathcal{R}$ is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (98.3.1.1)

$$\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth by Lemma 98.3.2. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is l-linear (by Formal Deformation Theory, Lemma 90.12.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by Formal Deformation Theory, Lemma 90.8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack $\mathcal{U}$ at the point u_0 is finite dimensional. Let $\mathrm{Spec}(R) \subset U$ be an affine open such that $u_0 : \mathrm{Spec}(l) \to U$ factors through $\mathrm{Spec}(R)$ and such that $\mathrm{Spec}(R) \to S$ factors through $\mathrm{Spec}(\Lambda) \subset S$. Let $\mathfrak{m}_R \subset R$ be the kernel of the Λ-algebra map $\varphi_0 : R \to l$ corresponding to u_0. Note that R, being of finite type over the Noetherian ring Λ, is a Noetherian ring. Hence $\mathfrak{m}_R = (f_1, \ldots, f_n)$ is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U},l,u_0} = \{\varphi : R \to l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on $f_1, \ldots, f_n$ hence the dimension is at most n and we win. Some details omitted. □

07X2 **Lemma 98.8.2.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS). Let k be a field of finite type over S and let w_0 be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over k. Denote x_0, y_0, z_0 the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w_0. Then there is a 6-term exact sequence*

$$0 \longrightarrow \mathit{Inf}(\mathcal{F}_{\mathcal{W},k,w_0}) \longrightarrow \mathit{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) \oplus \mathit{Inf}(\mathcal{F}_{\mathcal{Z},k,z_0}) \longrightarrow \mathit{Inf}(\mathcal{F}_{\mathcal{Y},k,y_0})$$

$$T\mathcal{F}_{\mathcal{W},k,w_0} \longrightarrow T\mathcal{F}_{\mathcal{X},k,x_0} \oplus T\mathcal{F}_{\mathcal{Z},k,z_0} \longrightarrow T\mathcal{F}_{\mathcal{Y},k,y_0}$$

of k-vector spaces.

Proof. By Lemma 98.5.3 we see that $\mathcal{W}$ satisfies (RS) and hence the lemma makes sense. To see the lemma is true, apply Lemmas 98.3.3 and 98.6.1 and Formal Deformation Theory, Lemma 90.20.1. □

98.9. Formal objects

07X3 In this section we transfer some of the notions already defined in the chapter "Formal Deformation Theory" to the current setting. In the following we will say "R is an S-algebra" to indicate that R is a ring endowed with a morphism of schemes $\mathrm{Spec}(R) \to S$.

07X4 **Definition 98.9.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *formal object* $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ consists of a Noetherian complete local S-algebra R, objects ξ_n of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}_R^n)$, and morphisms $f_n : \xi_n \to \xi_{n+1}$ of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ such that $R/\mathfrak{m}$ is a field of finite type over S.

(2) A *morphism of formal objects* $a : \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n : \xi_n \to \eta_n$ such that for every n the diagram

$$\begin{array}{ccc} \xi_n & \xrightarrow[f_n]{} & \xi_{n+1} \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle a_{n+1}} \\ \eta_n & \xrightarrow[g_n]{} & \eta_{n+1} \end{array}$$

is commutative. Applying the functor p we obtain a compatible collection of morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(T/\mathfrak{m}_T^n)$ and hence a morphism $a_0 : \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ over S. We say that *a lies over a_0*.

Thus we obtain a category of formal objects of $\mathcal{X}$.

0CXH **Remark 98.9.2.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. The formal object ξ defines a formal object ξ of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. This follows immediately from Definition 98.9.1 above, Formal Deformation Theory, Definition 90.7.1, and our construction of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ in Section 98.3.

If $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then F induces a functor between categories of formal objects as well.

07X5 **Lemma 98.9.3.** *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\eta = (R, \eta_n, g_n)$ be a formal object of $\mathcal{Y}$ and let ξ_1 be an object of $\mathcal{X}$ with $F(\xi_1) \cong \eta_1$. If F is formally smooth on objects (see Criteria for Representability, Section 97.6), then there exists a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ such that $F(\xi) \cong \eta$.*

Proof. Note that each of the morphisms $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ is a first order thickening of affine schemes over S. Hence the assumption on F means that we can successively lift ξ_1 to objects $\xi_2, \xi_3, \ldots$ of $\mathcal{X}$ endowed with compatible isomorphisms $\eta_n|_{\mathrm{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$ and $F(\eta_n) \cong \xi_n$. □

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Suppose that x is an object of $\mathcal{X}$ over R, where R is a Noetherian complete local S-algebra with residue field of finite type over S. Then we can consider the system of restrictions $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ endowed with the natural morphisms $\xi_1 \to \xi_2 \to \ldots$ coming from transitivity of restriction. Thus $\xi = (R, \xi_n, \xi_n \to \xi_{n+1})$ is a formal object of $\mathcal{X}$. This construction is functorial in the object x. Thus we obtain a functor

(98.9.3.1)

07X6
$$\left\{\begin{array}{c}\text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \mathrm{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S\end{array}\right\} \longrightarrow \left\{\text{formal objects of } \mathcal{X}\right\}$$

To be precise the left hand side is the full subcategory of $\mathcal{X}$ consisting of objects as indicated and the right hand side is the category of formal objects of $\mathcal{X}$ as in Definition 98.9.1.

07X7 **Definition 98.9.4.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is called *effective* if it is in the essential image of the functor (98.9.3.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

07X8 **Lemma 98.9.5.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be an algebraic stack over S. The functor (98.9.3.1) is an equivalence.*

Proof. Case I: $\mathcal{X}$ is representable (by a scheme). Say $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X over S. Unwinding the definitions we have to prove the following: Given a Noetherian complete local S-algebra R with $R/\mathfrak{m}$ of finite type over S we have

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective. This follows from Formal Spaces, Lemma 87.33.2.

Case II. $\mathcal{X}$ is representable by an algebraic space. Say $\mathcal{X}$ is representable by X. Again we have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for R as above. This is Formal Spaces, Lemma 87.33.3.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (98.9.3.1) are categories fibred in groupoids over the category of affine schemes over S which are spectra of Noetherian complete local rings with residue field of finite type over S. We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let R be a Noetherian complete local S-algebra with $k = R/\mathfrak{m}$ of finite type over S. Let x, x' be objects of $\mathcal{X}$ over R. As $\mathcal{X}$ is an algebraic stack $Isom(x, x')$ is representable by an algebraic space I over $\mathrm{Spec}(R)$, see Algebraic Stacks, Lemma 94.10.11. Applying Case II to I over $\mathrm{Spec}(R)$ implies immediately that (98.9.3.1) is fully faithful on fibre categories over $\mathrm{Spec}(R)$. Hence the functor is fully faithful by Categories, Lemma 4.35.9.

Essential surjectivity. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$. Choose a scheme U over S and a surjective smooth morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$. For every n

consider the fibre product

$$(Sch/\operatorname{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space V_n surjective and smooth over $\operatorname{Spec}(R/\mathfrak{m}^n)$. The morphisms $f_n : \xi_n \to \xi_{n+1}$ induce cartesian squares

$$\begin{array}{ccc} V_{n+1} & \longleftarrow & V_n \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R/\mathfrak{m}^{n+1}) & \longleftarrow & \operatorname{Spec}(R/\mathfrak{m}^n) \end{array}$$

of algebraic spaces. By Spaces over Fields, Lemma 72.16.2 we can find a finite separable extension k'/k and a point $v'_1 : \operatorname{Spec}(k') \to V_1$ over k. Let $R \subset R'$ be the finite étale extension whose residue field extension is k'/k (exists and is unique by Algebra, Lemmas 10.153.7 and 10.153.9). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 76.19.6) applied to $V_n \to \operatorname{Spec}(R/\mathfrak{m}^n)$ for $n = 2, 3, 4, \ldots$ we can successively find morphisms $v'_n : \operatorname{Spec}(R'/(\mathfrak{m}')^n) \to V_n$ over $\operatorname{Spec}(R/\mathfrak{m}^n)$ fitting into commutative diagrams

$$\begin{array}{ccc} \operatorname{Spec}(R'/(\mathfrak{m}')^{n+1}) & \longleftarrow & \operatorname{Spec}(R'/(\mathfrak{m}')^n) \\ {\scriptstyle v'_{n+1}}\downarrow & & \downarrow{\scriptstyle v'_n} \\ V_{n+1} & \longleftarrow & V_n \end{array}$$

Composing with the projection morphisms $V_n \to U$ we obtain a compatible system of morphisms $u'_n : \operatorname{Spec}(R'/(\mathfrak{m}')^n) \to U$. By Case I the family (u'_n) comes from a unique morphism $u' : \operatorname{Spec}(R') \to U$. Denote x' the object of $\mathcal{X}$ over $\operatorname{Spec}(R')$ we get by applying the 1-morphism f to u'. By construction, there exists a morphism of formal objects

$$(98.9.3.1)(x') = (R', x'|_{\operatorname{Spec}(R'/(\mathfrak{m}')^n)}, \ldots) \longrightarrow (R, \xi_n, f_n)$$

lying over $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote $p_0, p_1 : \operatorname{Spec}(R' \otimes_R R') \to \operatorname{Spec}(R')$ the two projections. By the fully faithfulness shown above there exists a canonical isomorphism $\varphi : p_0^* x' \to p_1^* x'$ because we have such isomorphisms over $\operatorname{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$. We omit the proof that the isomorphism φ satisfies the cocycle condition (see Stacks, Definition 8.3.1). Since $\{\operatorname{Spec}(R') \to \operatorname{Spec}(R)\}$ is an fppf covering we conclude that x' descends to an object x of $\mathcal{X}$ over $\operatorname{Spec}(R)$. We omit the proof that x_n is the restriction of x to $\operatorname{Spec}(R/\mathfrak{m}^n)$. □

07X9 **Lemma 98.9.6.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If the functor (98.9.3.1) is an equivalence for $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, then it is an equivalence for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The left and the right hand side of (98.9.3.1) for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ are simply the 2-fibre products of the left and the right hand side of (98.9.3.1) for $\mathcal{X}$, $\mathcal{Z}$ over $\mathcal{Y}$. Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 4.31.7. □

98.10. Approximation

07XA A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object x of the stack over a Noetherian complete local ring, you can find an object x_A over an algebraic ring which is "close to" x. Here an algebraic ring means a finite type S-algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the Stacks project. First, in Criteria for Representability, Section 97.5 we introduced *limit preserving on objects* for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 15.50.1 we defined the notion of a *G-ring*. Let S be a locally Noetherian scheme. Let A be an S-algebra. We say that A is *of finite type over S* or is a *finite type S-algebra* if $\mathrm{Spec}(A) \to S$ is of finite type. In this case A is a Noetherian ring. Finally, given a ring A and ideal I we denote $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$.

07XB **Lemma 98.10.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ where R is a Noetherian complete local ring with residue field k of finite type over S. Let $s \in S$ be the image of $\mathrm{Spec}(k) \to S$. Assume that (a) $\mathcal{O}_{S,s}$ is a G-ring and (b) p is limit preserving on objects. Then for every integer $N \geq 1$ there exist*

(1) *a finite type S-algebra A,*
(2) *a maximal ideal $\mathfrak{m}_A \subset A$,*
(3) *an object x_A of $\mathcal{X}$ over $\mathrm{Spec}(A)$,*
(4) *an S-isomorphism $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$,*
(5) *an isomorphism $x|_{\mathrm{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\mathrm{Spec}(A/\mathfrak{m}_A^N)}$ compatible with (4), and*
(6) *an isomorphism $Gr_{\mathfrak{m}_R}(R) \cong Gr_{\mathfrak{m}_A}(A)$ of graded k-algebras.*

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra, see Morphisms, Lemma 29.16.1. We may and do replace S by $\mathrm{Spec}(\Lambda)$.

We may write R as a directed colimit $R = \mathrm{colim}\, C_j$ where each C_j is a finite type Λ-algebra (see Algebra, Lemma 10.127.2). By assumption (b) the object x is isomorphic to the restriction of an object over one of the C_j. Hence we may choose a finite type Λ-algebra C, a Λ-algebra map $C \to R$, and an object x_C of $\mathcal{X}$ over $\mathrm{Spec}(C)$ such that $x = x_C|_{\mathrm{Spec}(R)}$. The choice of C is a bookkeeping device and could be avoided. For later use, let us write $C = \Lambda[y_1, \ldots, y_u]/(f_1, \ldots, f_v)$ and we denote $\overline{a}_i \in R$ the image of y_i under the map $C \to R$. Set $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Choose a Λ-algebra surjection $\Lambda[x_1, \ldots, x_s] \to k$ and denote $\mathfrak{m}'$ the kernel. By the universal property of polynomial rings we may lift this to a Λ-algebra map $\Lambda[x_1, \ldots, x_s] \to R$. We add some variables (i.e., we increase s a bit) mapping to generators of $\mathfrak{m}_R$. Having done this we see that $\Lambda[x_1, \ldots, x_s] \to R/\mathfrak{m}_R^2$ is surjective. Then we see that

07XC (98.10.1.1) $$P = \Lambda[x_1, \ldots, x_s]^\wedge_{\mathfrak{m}'} \longrightarrow R$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 90.4.2.

Choose lifts $a_i \in P$ of $\overline{a}_i$ we found above. Choose generators $b_1, \ldots, b_r \in P$ for the kernel of (98.10.1.1). Choose $c_{ji} \in P$ such that

$$f_j(a_1, \ldots, a_u) = \sum c_{ji} b_i$$

in P which is possible by the choices made so far. Choose generators

$$k_1, \ldots, k_t \in \mathrm{Ker}(P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P)$$

and write $k_i = (k_{i1}, \ldots, k_{ir})$ and $K = (k_{ij})$ so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \to R \to 0$$

is an exact sequence of P-modules. In particular we have $\sum k_{ij} b_j = 0$. After possibly increasing N we may assume $N - 1$ works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 15.4 for terminology).

By assumption $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$ is a G-ring. Hence by More on Algebra, Proposition 15.50.10 the ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$ is a G-ring. Hence by Smoothing Ring Maps, Theorem 16.13.2 there exist an étale ring map

$$\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \to B,$$

a maximal ideal $\mathfrak{m}_B$ of B lying over $\mathfrak{m}'$, and elements $a'_i, b'_i, c'_{ij}, k'_{ij} \in B'$ such that

(1) $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$ which implies that $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$ and P is identified with the completion of B at $\mathfrak{m}_B$, see remark preceding Smoothing Ring Maps, Theorem 16.13.2,
(2) $a_i - a'_i, b_i - b'_i, c_{ij} - c'_{ij}, k_{ij} - k'_{ij} \in (\mathfrak{m}')^N P$, and
(3) $f_j(a'_1, \ldots, a'_u) = \sum c'_{ji} b'_i$ and $\sum k'_{ij} b'_j = 0$.

Set $A = B/(b'_1, \ldots, b'_r)$ and denote $\mathfrak{m}_A$ the image of $\mathfrak{m}_B$ in A. (Note that A is essentially of finite type over Λ; at the end of the proof we will show how to obtain an A which is of finite type over Λ.) There is a ring map $C \to A$ sending $y_i \mapsto a'_i$ because the a'_i satisfy the desired equations modulo $(b'_1, \ldots, b'_r)$. Note that $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ as quotients of $P = B^\wedge$ by property (2) above. Set $x_A = x_C|_{\mathrm{Spec}(A)}$. Since the maps

$$C \to A \to A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \to R \to R/\mathfrak{m}_R^N$$

are equal we see that x_A and x agree modulo $\mathfrak{m}_R^N$ via the isomorphism $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$. At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \quad \text{and} \quad P^{\oplus t} \xrightarrow{K'} P^{\oplus r} \xrightarrow{(b'_1, \ldots, b'_r)} P$$

are two complexes of P-modules which are congruent modulo $(\mathfrak{m}')^N$ with the first one being exact. By our choice of N above we see from More on Algebra, Lemma 15.4.2 that $R = P/(b_1, \ldots, b_r)$ and $P/(b'_1, \ldots, b'_r) = B^\wedge/(b'_1, \ldots, b'_r) = A^\wedge$ have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that A is essentially of finite type over S and not yet of finite type. The construction above gives $A = B/(b'_1, \ldots, b'_r)$ and $\mathfrak{m}_A \subset A$ with B étale over $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Hence A is of finite type over the Noetherian ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$. Thus we can write $A = (A_0)_{\mathfrak{m}'}$ for some finite type $\Lambda[x_1, \ldots, x_n]$ algebra A_0. Then $A = \mathrm{colim}(A_0)_f$ where $f \in \Lambda[x_1, \ldots, x_n] \setminus \mathfrak{m}'$, see Algebra, Lemma 10.9.9. Because $p : \mathcal{X} \to (Sch/S)_{fppf}$ is

limit preserving on objects, we see that x_A comes from some object $x_{(A_0)_f}$ over $\mathrm{Spec}((A_0)_f)$ for an f as above. After replacing A by $(A_0)_f$ and x_A by $x_{(A_0)_f}$ and $\mathfrak{m}_A$ by $(A_0)_f \cap \mathfrak{m}_A$ the proof is finished. □

98.11. Limit preserving

07XK The morphism $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, as defined in Criteria for Representability, Section 97.5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 110.54 shows that this isn't equivalent to being limit preserving.

07XL **Definition 98.11.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ is *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have an equivalence

$$\mathrm{colim}\, \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects x, y of $\mathcal{X}$ over T_i we should have

$$\mathrm{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \mathrm{colim}_{i' \geq i} \mathrm{Mor}_{\mathcal{X}_{T_{i'}}}(x|_{T_{i'}}, y|_{T_{i'}})$$

and second every object of $\mathcal{X}_T$ is isomorphic to the restriction of an object over T_i for some i. Note that the first condition means that the presheaves $Isom_{\mathcal{X}}(x, y)$ (see Stacks, Definition 8.2.2) are limit preserving.

07XM **Lemma 98.11.2.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

1. *If $\mathcal{X} \to (Sch/S)_{fppf}$ and $\mathcal{Z} \to (Sch/S)_{fppf}$ are limit preserving on objects and $\mathcal{Y}$ is limit preserving, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to (Sch/S)_{fppf}$ is limit preserving on objects.*
2. *If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are limit preserving, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Proof of (1). Let $T = \lim_{i \in I} T_i$ be the directed limit of affine schemes T_i over S. We will prove that the functor $\mathrm{colim}\, \mathcal{X}_{T_i} \to \mathcal{X}_T$ is essentially surjective. Recall that an object of the fibre product over T is a quadruple (T, x, z, α) where x is an object of $\mathcal{X}$ lying over T, z is an object of $\mathcal{Z}$ lying over T, and $\alpha : p(x) \to q(z)$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By assumption on $\mathcal{X}$ and $\mathcal{Z}$ we can find an i and objects x_i and z_i over T_i such that $x_i|_T \cong T$ and $z_i|_T \cong z$. Then α corresponds to an isomorphism $p(x_i)|_T \to q(z_i)|_T$ which comes from an isomorphism $\alpha_{i'} : p(x_i)|_{T_{i'}} \to q(z_i)|_{T_{i'}}$ by our assumption on $\mathcal{Y}$. After replacing i by i', x_i by $x_i|_{T_{i'}}$, and z_i by $z_i|_{T_{i'}}$ we see that $(T_i, x_i, z_i, \alpha_i)$ is an object of the fibre product over T_i which restricts to an object isomorphic to (T, x, z, α) over T as desired.

We omit the arguments showing that $\mathrm{colim}\, \mathcal{X}_{T_i} \to \mathcal{X}_T$ is fully faithful in (2). □

07XN **Lemma 98.11.3.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Then the following are equivalent*

1. *$\mathcal{X}$ is a stack in setoids and $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
2. *$\mathcal{X}$ is a stack in setoids and limit preserving,*
3. *$\mathcal{X}$ is representable by an algebraic space locally of finite presentation.*

Proof. Under each of the three assumptions $\mathcal{X}$ is representable by an algebraic space X over S, see Algebraic Stacks, Proposition 94.13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 70.3.10. □

0CXI **Lemma 98.11.4.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and $\mathcal{X}$ is limit preserving. Then Δ is locally of finite type.*

Proof. We apply Criteria for Representability, Lemma 97.5.6. Let V be an affine scheme V locally of finite presentation over S and let θ be an object of $\mathcal{X} \times \mathcal{X}$ over V. Let F_θ be an algebraic space representing $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \theta} (Sch/V)_{fppf}$ and let $f_\theta : F_\theta \to V$ be the canonical morphism (see Algebraic Stacks, Section 94.9). It suffices to show that $F_\theta \to V$ has the corresponding properties. By Lemmas 98.11.2 and 98.11.3 we see that $F_\theta \to S$ is locally of finite presentation. It follows that $F_\theta \to V$ is locally of finite type by Morphisms of Spaces, Lemma 67.23.6. □

98.12. Versality

07XD In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack $\mathcal{X}$ is an algebraic stack, we need to find smooth 1-morphisms from schemes towards $\mathcal{X}$. Since we are not going to assume a priori that $\mathcal{X}$ has a representable diagonal, we cannot even speak about smooth morphisms towards $\mathcal{X}$. Instead, borrowing terminology from deformation theory, we will introduce versal objects.

0CXJ **Definition 98.12.1.** Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object. Set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. We will say that ξ is *versal* if ξ as a formal object of $\mathcal{F}_{\mathcal{X}, k, x_0}$ (Remark 98.9.2) is versal in the sense of Formal Deformation Theory, Definition 90.8.9.

We briefly spell out what this means. With notation as in the definition, suppose given morphisms $\xi_1 = x_0 \to y \to z$ of $\mathcal{X}$ lying over closed immersions $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are Artinian local rings with residue field k. Suppose given an $n \geq 1$ and a commutative diagram

$$\begin{array}{ccc} y & & \mathrm{Spec}(A) \\ \uparrow & \text{lying over} & \uparrow \\ \xi_1 & & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \xi_n & & \mathrm{Spec}(R/\mathfrak{m}^n) \end{array}$$

Versality means that for any data as above there exists an $m \geq n$ and a commutative diagram

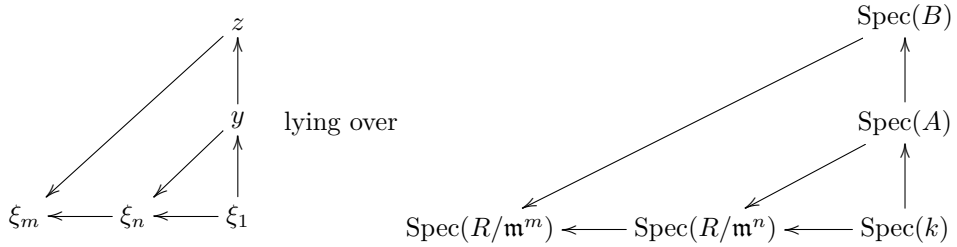

Please compare with Formal Deformation Theory, Remark 90.8.10.

Let S be a locally Noetherian scheme. Let U be a scheme over S with structure morphism $U \to S$ locally of finite type. Let $u_0 \in U$ be a finite type point of U, see Morphisms, Definition 29.16.3. Set $k = \kappa(u_0)$. Note that the composition $\mathrm{Spec}(k) \to S$ is also of finite type, see Morphisms, Lemma 29.15.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ which lies over U. Denote x_0 the pullback of x by u_0. By the 2-Yoneda lemma x corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 94.5. We obtain a morphism of predeformation categories

07XE (98.12.1.1) $$\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0},$$

over $\mathcal{C}_\Lambda$ see (98.3.1.1).

07XF **Definition 98.12.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ lying over U. Let u_0 be finite type point of U. We say x is *versal* at u_0 if the morphism $\hat{x}$ (98.12.1.1) is smooth, see Formal Deformation Theory, Definition 90.8.1.

This definition matches our notion of versality for formal objects of $\mathcal{X}$.

0CXK **Lemma 98.12.3.** *With notation as in Definition 98.12.2. Let $R = \mathcal{O}^\wedge_{U,u_0}$. Let ξ be the formal object of $\mathcal{X}$ over R associated to $x|_{\mathrm{Spec}(R)}$, see (98.9.3.1). Then*

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

Proof. Observe that $\mathcal{O}_{U,u_0}$ is a Noetherian local S-algebra with residue field k. Hence $R = \mathcal{O}^\wedge_{U,u_0}$ is an object of $\mathcal{C}^\wedge_\Lambda$, see Formal Deformation Theory, Definition 90.4.1. Recall that ξ is versal if $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth and x is versal at u_0 if $\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0},$$

see Formal Deformation Theory, Remark 90.7.12 for notation. Namely, given an Artinian local S-algebra A with residue field identified with k we have

$$\mathrm{Mor}_{\mathcal{C}^\wedge_\Lambda}(R, A) = \{\varphi \in \mathrm{Mor}_S(\mathrm{Spec}(A), U) \mid \varphi|_{\mathrm{Spec}(k)} = u_0\}$$

Unwinding the definitions the reader verifies that the resulting map

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X},k,x_0},$$

is equal to $\underline{\xi}$ and we see that the lemma is true. □

Here is a sanity check.

0CXL **Lemma 98.12.4.** *Let S be a locally Noetherian scheme. Let $f : U \to V$ be a morphism of schemes locally of finite type over S. Let $u_0 \in U$ be a finite type point. The following are equivalent*

(1) *f is smooth at u_0,*
(2) *f viewed as an object of $(Sch/V)_{fppf}$ over U is versal at u_0.*

Proof. This is a restatement of More on Morphisms, Lemma 37.12.1. □

It turns out that this notion is well behaved with respect to field extensions.

07XG **Lemma 98.12.5.** *Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 98.12.2. Let l be a field and let $u_{l,0} : \mathrm{Spec}(l) \to U$ be a morphism with image u_0 such that $l/k = \kappa(u_0)$ is finite. Set $x_{l,0} = x_0|_{\mathrm{Spec}(l)}$. If $\mathcal{X}$ satisfies (RS) and x is versal at u_0, then*

$$\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth.

Proof. Note that $(Sch/U)_{fppf}$ satisfies (RS) by Lemma 98.5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf},k,u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}$$

associated to $\hat{x}$, see Lemma 98.7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 90.29.5. □

The following lemma is another sanity check. It more or less signifies that if x is versal at u_0 as in Definition 98.12.2, then x viewed as a morphism from U to $\mathcal{X}$ is smooth whenever we make a base change by a scheme.

0CXM **Lemma 98.12.6.** *Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 98.12.2. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *Δ is locally of finite type (for example if $\mathcal{X}$ is limit preserving), and*
(3) *$\mathcal{X}$ has (RS).*

Let V be a scheme locally of finite type over S and let y be an object of $\mathcal{X}$ over V. Form the 2-fibre product

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & (Sch/U)_{fppf} \\ \downarrow & & \downarrow{\scriptstyle x} \\ (Sch/V)_{fppf} & \xrightarrow{y} & \mathcal{X} \end{array}$$

Let Z be the algebraic space representing $\mathcal{Z}$ and let $z_0 \in |Z|$ be a finite type point lying over u_0. If x is versal at u_0, then the morphism $Z \to V$ is smooth at z_0.

Proof. (The parenthetical remark in the statement holds by Lemma 98.11.4.) Observe that Z exists by assumption (1) and Algebraic Stacks, Lemma 94.10.11. By assumption (2) we see that $Z \to V \times_S U$ is locally of finite type. Choose a scheme W, a closed point $w_0 \in W$, and an étale morphism $W \to Z$ mapping w_0 to z_0, see Morphisms of Spaces, Definition 67.25.2. Then W is locally of finite type over S and w_0 is a finite type point of W. Let $l = \kappa(z_0)$. Denote $z_{l,0}$, $v_{l,0}$, $u_{l,0}$, and $x_{l,0}$ the objects of $\mathcal{Z}$, $(Sch/V)_{fppf}$, $(Sch/U)_{fppf}$, and $\mathcal{X}$ over $\mathrm{Spec}(l)$ obtained by pullback to $\mathrm{Spec}(l) = w_0$. Consider

$$\begin{array}{ccccc} \mathcal{F}_{(Sch/W)_{fppf},l,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},l,z_{l,0}} & \longrightarrow & \mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}} & \longrightarrow & \mathcal{F}_{\mathcal{X},l,x_{l,0}} \end{array}$$

By Lemma 98.3.3 the square is a fibre product of predeformation categories. By Lemma 98.12.5 we see that the right vertical arrow is smooth. By Formal Deformation Theory, Lemma 90.8.7 the left vertical arrow is smooth. By Lemma 98.3.2 we see that the left horizontal arrow is smooth. We conclude that the map

$$\mathcal{F}_{(Sch/W)_{fppf},l,w_0} \to \mathcal{F}_{(Sch/V)_{fppf},l,v_{l,0}}$$

is smooth by Formal Deformation Theory, Lemma 90.8.7. Thus we conclude that $W \to V$ is smooth at w_0 by More on Morphisms, Lemma 37.12.1. This exactly means that $Z \to V$ is smooth at z_0 and the proof is complete. □

We restate the approximation result in terms of versal objects.

07XH **Lemma 98.12.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$ with ξ_1 lying over $\mathrm{Spec}(k) \to S$ with image $s \in S$. Assume*

(1) *ξ is versal,*
(2) *ξ is effective,*
(3) *$\mathcal{O}_{S,s}$ is a G-ring, and*
(4) *$p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects.*

Then there exist a morphism of finite type $U \to S$, a finite type point $u_0 \in U$ with residue field k, and an object x of $\mathcal{X}$ over U such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}_{u_0}^n)} \cong \xi_n$.

Proof. Choose an object x_R of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ whose associated formal object is ξ. Let $N = 2$ and apply Lemma 98.10.1. We obtain $A, \mathfrak{m}_A, x_A, \ldots$. Let $\eta = (A^\wedge, \eta_n, g_n)$ be the formal object associated to $x_A|_{\mathrm{Spec}(A^\wedge)}$. We have a diagram

$$\begin{array}{ccc} \eta & & A^\wedge \\ \downarrow & \text{lying over} & \downarrow \\ \xi \longrightarrow \xi_2 = \eta_2 & & R \longrightarrow R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{array}$$

The versality of ξ means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms $\xi \to \eta_3$, $\xi \to \eta_4$, and so on by Formal Deformation Theory, Remark 90.8.10. The corresponding ring map $R \to A^\wedge$ is surjective by Formal Deformation Theory, Lemma 90.4.2. On the other hand, we have $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ for all n by construction. Hence $R/\mathfrak{m}_R^n$

and $A/\mathfrak{m}_A^n$ have the same (finite) length as Λ-modules by additivity of length and Formal Deformation Theory, Lemma 90.3.4. It follows that $R/\mathfrak{m}_R^n \to A/\mathfrak{m}_A^n$ is an isomorphism for all n, hence $R \to A^\wedge$ is an isomorphism. Thus η is isomorphic to a versal object, hence versal itself. By Lemma 98.12.3 we conclude that x_A is versal at the point u_0 of $U = \mathrm{Spec}(A)$ corresponding to $\mathfrak{m}_A$. □

07XI **Example 98.12.8.** In this example we show that the local ring $\mathcal{O}_{S,s}$ has to be a G-ring in order for the result of Lemma 98.12.7 to be true. Namely, let Λ be a Noetherian ring and let $\mathfrak{m}$ be a maximal ideal of Λ. Set $R = \Lambda^\wedge_\mathfrak{m}$. Let $\Lambda \to C \to R$ be a factorization with C of finite type over Λ. Set $S = \mathrm{Spec}(\Lambda)$, $U = S \setminus \{\mathfrak{m}\}$, and $S' = U \amalg \mathrm{Spec}(C)$. Consider the functor $F : (Sch/S)^{opp}_{fppf} \to Sets$ defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \to S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let $\mathcal{X} = \mathcal{S}_F$ is the category fibred in sets associated to F, see Algebraic Stacks, Section 94.7. Then $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects and there exists an effective, versal formal object ξ over R. Hence if the conclusion of Lemma 98.12.7 holds for $\mathcal{X}$, then there exists a finite type ring map $\Lambda \to A$ and a maximal ideal $\mathfrak{m}_A$ lying over $\mathfrak{m}$ such that

(1) $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$,
(2) $\Lambda \to A$ and $\mathfrak{m}_A$ satisfy condition (4) of Algebra, Lemma 10.141.2, and
(3) there exists a Λ-algebra map $C \to A$.

Thus $\Lambda \to A$ is smooth at $\mathfrak{m}_A$ by the lemma cited. Slicing A we may assume that $\Lambda \to A$ is étale at $\mathfrak{m}_A$, see for example More on Morphisms, Lemma 37.38.5 or argue directly. Write $C = \Lambda[y_1, \ldots, y_n]/(f_1, \ldots, f_m)$. Then $C \to R$ corresponds to a solution in R of the system of equations $f_1 = \ldots = f_m = 0$, see Smoothing Ring Maps, Section 16.13. Thus if the conclusion of Lemma 98.12.7 holds for every $\mathcal{X}$ as above, then a system of equations which has a solution in R has a solution in the henselization of $\Lambda_\mathfrak{m}$. In other words, the approximation property holds for $\Lambda^h_\mathfrak{m}$. This implies that $\Lambda^h_\mathfrak{m}$ is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 16.1) which in turn implies that $\Lambda_\mathfrak{m}$ is a G-ring.

98.13. Openness of versality

07XP Next, we come to openness of versality.

07XQ **Definition 98.13.1.** Let S be a locally Noetherian scheme.

(1) Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *openness of versality* if given a scheme U locally of finite type over S, an object x of $\mathcal{X}$ over U, and a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.
(2) Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f satisfies *openness of versality* if given a scheme U locally of finite type over S, an object y of $\mathcal{Y}$ over U, openness of versality holds for $(Sch/U)_{fppf} \times_\mathcal{Y} \mathcal{X}$.

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

07XR **Example 98.13.2.** Let k be a field and set $\Lambda = k[s,t]$. Consider the functor $F : \Lambda\text{-algebras} \longrightarrow \mathit{Sets}$ defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \ldots, f_n \in A \text{ such that} \\ & A = (s, t, f_1, \ldots, f_n) \text{ and } f_i s = 0 \ \forall i \\ \emptyset & \text{else} \end{cases}$$

Geometrically $F(A) = *$ means there exists a quasi-compact open neighbourhood W of $V(s,t) \subset \mathrm{Spec}(A)$ such that $s|_W = 0$. Let $\mathcal{X} \subset (\mathit{Sch}/\mathrm{Spec}(\Lambda))_{fppf}$ be the full subcategory consisting of schemes T which have an affine open covering $T = \bigcup \mathrm{Spec}(A_j)$ with $F(A_j) = *$ for all j. Then $\mathcal{X}$ satisfies [0], [1], [2], [3], and [4] but not [5]. Namely, over $U = \mathrm{Spec}(k[s,t]/(s))$ there exists an object x which is versal at $u_0 = (s,t)$ but not at any other point. Details omitted.

Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. Consider the following property

07XS (98.13.2.1) for all fields k of finite type over S and all $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$ the map $\mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{F}_{\mathcal{Y},k,f(x_0)}$ of predeformation categories is smooth

We formulate some lemmas around this concept. First we link it with (openness of) versality.

07XT **Lemma 98.13.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ over U. Assume that x is versal at every finite type point of U and that $\mathcal{X}$ satisfies (RS). Then $x : (\mathit{Sch}/U)_{fppf} \to \mathcal{X}$ satisfies (98.13.2.1).*

Proof. Let $\mathrm{Spec}(l) \to U$ be a morphism with l of finite type over S. Then the image $u_0 \in U$ is a finite type point of U and $l/\kappa(u_0)$ is a finite extension, see discussion in Morphisms, Section 29.16. Hence we see that $\mathcal{F}_{(\mathit{Sch}/U)_{fppf},l,u_{l,0}} \to \mathcal{F}_{\mathcal{X},l,x_{l,0}}$ is smooth by Lemma 98.12.5. □

07XU **Lemma 98.13.4.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be composable 1-morphisms of categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. If f and g satisfy (98.13.2.1) so does $g \circ f$.*

Proof. This follows formally from Formal Deformation Theory, Lemma 90.8.7. □

07XV **Lemma 98.13.5.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. If f satisfies (98.13.2.1) so does the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$.*

Proof. Follows immediately from Lemma 98.3.3 and Formal Deformation Theory, Lemma 90.8.7. □

07XW **Lemma 98.13.6.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. If f is formally smooth on objects, then f satisfies (98.13.2.1). If f is representable by algebraic spaces and smooth, then f satisfies (98.13.2.1).*

Proof. A reformulation of Lemma 98.3.2. □

07XX **Lemma 98.13.7.** *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. Assume*

(1) *f is representable by algebraic spaces,*

(2) f *satisfies (98.13.2.1),*
(3) $\mathcal{X} \to (Sch/S)_{fppf}$ *is limit preserving on objects, and*
(4) $\mathcal{Y}$ *is limit preserving.*

Then f is smooth.

Proof. The key ingredient of the proof is More on Morphisms, Lemma 37.12.1 which (almost) says that a morphism of schemes of finite type over S satisfying (98.13.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let V be a scheme over S and let y be an object of $\mathcal{Y}$ over V. Let Z be an algebraic space representing the 2-fibre product $\mathcal{Z} = \mathcal{X} \times_{f,\mathcal{X},y} (Sch/V)_{fppf}$. We have to show that the projection morphism $Z \to V$ is smooth, see Algebraic Stacks, Definition 94.10.1. In fact, it suffices to do this when V is an affine scheme locally of finite presentation over S, see Criteria for Representability, Lemma 97.5.6. Then $(Sch/V)_{fppf}$ is limit preserving by Lemma 98.11.3. Hence $Z \to S$ is locally of finite presentation by Lemmas 98.11.2 and 98.11.3. Choose a scheme W and a surjective étale morphism $W \to Z$. Then W is locally of finite presentation over S.

Since f satisfies (98.13.2.1) we see that so does $\mathcal{Z} \to (Sch/V)_{fppf}$, see Lemma 98.13.5. Next, we see that $(Sch/W)_{fppf} \to \mathcal{Z}$ satisfies (98.13.2.1) by Lemma 98.13.6. Thus the composition

$$(Sch/W)_{fppf} \to \mathcal{Z} \to (Sch/V)_{fppf}$$

satisfies (98.13.2.1) by Lemma 98.13.4. More on Morphisms, Lemma 37.12.1 shows that the composition $W \to Z \to V$ is smooth at every finite type point w_0 of W. Since the smooth locus is open we conclude that $W \to V$ is a smooth morphism of schemes by Morphisms, Lemma 29.16.7. Thus we conclude that $Z \to V$ is a smooth morphism of algebraic spaces by definition. □

The lemma below is how we will use openness of versality.

07XY **Lemma 98.13.8.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let k be a finite type field over S and let x_0 be an object of $\mathcal{X}$ over* $\mathrm{Spec}(k)$ *with image $s \in S$. Assume*

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is representable by algebraic spaces,*
(2) $\mathcal{X}$ *satisfies axioms [1], [2], [3] (see Section 98.14),*
(3) *every formal object of $\mathcal{X}$ is effective,*
(4) *openness of versality holds for $\mathcal{X}$, and*
(5) $\mathcal{O}_{S,s}$ *is a G-ring.*

Then there exist a morphism of finite type $U \to S$ and an object x of $\mathcal{X}$ over U such that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth and such that there exists a finite type point $u_0 \in U$ whose residue field is k and such that $x|_{u_0} \cong x_0$.

Proof. By axiom [2], Lemma 98.6.1, and Remark 98.6.2 we see that $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 90.13.4 that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a versal formal object ξ. Assumption (3) says ξ is effective. By axiom [1] and Lemma 98.12.7 there exists a morphism of finite type $U \to S$, an object x of $\mathcal{X}$ over U, and

a finite type point u_0 of U with residue field k such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(k)} \cong x_0$. By openness of versality we may shrink U and assume that x is versal at every finite type point of U. We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 98.13.3 x satisfies (98.13.2.1) whereupon Lemma 98.13.7 finishes the proof. □

98.14. Axioms

07XJ Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Here are the axioms we will consider on $\mathcal{X}$.

- [-1] a set theoretic condition[1] to be ignored by readers who are not interested in set theoretical issues,
- [0] $\mathcal{X}$ is a stack in groupoids for the étale topology,
- [1] $\mathcal{X}$ is limit preserving,
- [2] $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS),
- [3] the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for every k and x_0, see (98.8.0.1) and (98.8.0.2),
- [4] the functor (98.9.3.1) is an equivalence,
- [5] $\mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfy openness of versality.

98.15. Axioms for functors

07XZ Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Denote $\mathcal{X} = \mathcal{S}_F$ the category fibred in sets associated to F, see Algebraic Stacks, Section 94.7. In this section we provide a translation between the material above as it applies to $\mathcal{X}$, to statements about F.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let k be a field of finite type over S. Let $x_0 \in F(\mathrm{Spec}(k))$. The associated predeformation category (98.3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \{x \in F(\mathrm{Spec}(A)) \mid x|_{\mathrm{Spec}(k)} = x_0\}.$$

Recall that we do not distinguish between categories cofibred in sets over $\mathcal{C}_\Lambda$ and functor $\mathcal{C}_\Lambda \to Sets$, see Formal Deformation Theory, Remarks 90.5.2 (11). Given a transformation of functors $a : F \to G$, setting $y_0 = a(x_0)$ we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (98.3.1.1). Lemma 98.3.2 tells us that if $a : F \to G$ is formally smooth (in the sense of More on Morphisms of Spaces, Definition 76.13.1), then $F_{k,x_0} \longrightarrow G_{k,y_0}$ is smooth as in Formal Deformation Theory, Remark 90.8.4.

Lemma 98.4.1 says that if $Y' = Y \amalg_X X'$ in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, then the map

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that F is an algebraic space. We say a general functor F satisfies the *Rim-Schlessinger condition* or we say F *satisfies (RS)* if given any

[1] The condition is the following: the supremum of all the cardinalities $|\mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})/\cong|$ and $|\mathrm{Arrows}(\mathcal{X}_{\mathrm{Spec}(k)})|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

pushout $Y' = Y \amalg_X X'$ where Y, X, X' are spectra of Artinian local rings of finite type over S, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 98.6.1 says that given a functor F which satisfies (RS), then all F_{k,x_0} are deformation functors as in Formal Deformation Theory, Definition 90.16.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 90.16.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\mathrm{Spec}(k[\epsilon])) \mid x|_{\mathrm{Spec}(k)} = x_0\}$$

has the structure of a k-vector space by Formal Deformation Theory, Lemma 90.12.2.

Lemma 98.8.1 says that an algebraic space F locally of finite type over S gives rise to deformation functors F_{k,x_0} with finite dimensional tangent spaces TF_{k,x_0}.

A *formal object*[2] $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\mathrm{Spec}(R/\mathfrak{m}^n))$ such that $\xi_{n+1}|_{\mathrm{Spec}(R/\mathfrak{m}^n)} = \xi_n$. A formal object ξ defines a formal object ξ of $F_{R/\mathfrak{m},\xi_1}$. We say ξ is *versal* if and only if it is versal in the sense of Formal Deformation Theory, Definition 90.8.9. A formal object $\xi = (R, \xi_n)$ is called *effective* if there exists an $x \in F(\mathrm{Spec}(R))$ such that $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ for all $n \geq 1$. Lemma 98.9.5 says that if F is an algebraic space, then every formal object is effective.

Let U be a scheme locally of finite type over S and let $x \in F(U)$. Let $u_0 \in U$ be a finite type point. We say that x is versal at u_0 if and only if $\xi = (\mathcal{O}^\wedge_{U,u_0}, x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}^n_{u_0})})$ is a versal formal object in the sense described above.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Here are the axioms we will consider on F.

- [-1] a set theoretic condition[3] to be ignored by readers who are not interested in set theoretical issues,
- [0] F is a sheaf for the étale topology,
- [1] F is limit preserving,
- [2] F satisfies the Rim-Schlessinger condition (RS),
- [3] every tangent space TF_{k,x_0} is finite dimensional,
- [4] every formal object is effective,
- [5] F satisfies openness of versality.

Here *limit preserving* is the notion defined in Limits of Spaces, Definition 70.3.1 and *openness of versality* means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

[2] This is what Artin calls a formal deformation.

[3] The condition is the following: the supremum of all the cardinalities $|F(\mathrm{Spec}(k))|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

98.16. Algebraic spaces

07Y0 The following is our first main result on algebraic spaces.

07Y1 **Proposition 98.16.1.** *Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *$\Delta : F \to F \times F$ is representable by algebraic spaces,*
(2) *F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see Section 98.15), and*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then F is an algebraic space.

Proof. Lemma 98.13.8 applies to F. Using this we choose, for every finite type field k over S and $x_0 \in F(\mathrm{Spec}(k))$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $U_{k,x_0} \to F$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$U = \coprod\nolimits_{k,x_0} U_{k,x_0} \longrightarrow F$$

is smooth[4]. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 80.12.3 (this is where we use axiom [0]). By Criteria for Representability, Lemma 97.5.6 it suffices to show that $U \times_F V \to V$ is surjective for those $V \to F$ where V is an affine scheme locally of finite presentation over S. Since $U \times_F V \to V$ is smooth the image is open. Hence it suffices to show that the image of $U \times_F V \to V$ contains all finite type points of V, see Morphisms, Lemma 29.16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote x_0 the composition $\mathrm{Spec}(k) \xrightarrow{v_0} V \to F$. Then $(u_{k,x_0}, v_0) : \mathrm{Spec}(k) \to U \times_F V$ is a point mapping to v_0 and we win. □

07Y2 **Lemma 98.16.2.** *Let S be a locally Noetherian scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)^{opp}_{fppf} \to Sets$. Assume that*

(1) *a is injective,*
(2) *F satisfies axioms [0], [1], [2], [4], and [5],*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S,*
(4) *G is an algebraic space locally of finite type over S,*

Then F is an algebraic space.

Proof. By Lemma 98.8.1 the functor G satisfies [3]. As $F \to G$ is injective, we conclude that F also satisfies [3]. Moreover, as $F \to G$ is injective, we see that given schemes U, V and morphisms $U \to F$ and $V \to F$, then $U \times_F V = U \times_G V$. Hence $\Delta : F \to F \times F$ is representable (by schemes) as this holds for G by assumption. Thus Proposition 98.16.1 applies[5]. □

98.17. Algebraic stacks

07Y3 Proposition 98.17.2 is our first main result on algebraic stacks.

07Y4 **Lemma 98.17.1.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume that*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*

[4]Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

[5]The set theoretic condition [-1] holds for F as it holds for G. Details omitted.

(2) $\mathcal{X}$ *satisfies axioms [-1], [0], [1], [2], [3] (see Section 98.14),*
(3) *every formal object of* $\mathcal{X}$ *is effective,*
(4) $\mathcal{X}$ *satisfies openness of versality, and*
(5) $\mathcal{O}_{S,s}$ *is a G-ring for all finite type points* s *of* S.

Then $\mathcal{X}$ *is an algebraic stack.*

Proof. Lemma 98.13.8 applies to $\mathcal{X}$. Using this we choose, for every finite type field k over S and every isomorphism class of object $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $(Sch/U_{k,x_0})_{fppf} \to \mathcal{X}$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$(Sch/U)_{fppf} \to \mathcal{X}, \quad \text{with} \quad U = \coprod\nolimits_{k,x_0} U_{k,x_0}$$

is smooth[6]. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 97.19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 97.5.6 it suffices to show that $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf} \to (Sch/V)_{fppf}$ is surjective for those $y : (Sch/V)_{fppf} \to \mathcal{X}$ where V is an affine scheme locally of finite presentation over S. By assumption (1) the fibre product $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf}$ is representable by an algebraic space W. Then $W \to V$ is smooth, hence the image is open. Hence it suffices to show that the image of $W \to V$ contains all finite type points of V, see Morphisms, Lemma 29.16.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote $x_0 = y|_{\mathrm{Spec}(k)}$ the pullback of y by v_0. Then (u_{k,x_0}, v_0) will give a morphism $\mathrm{Spec}(k) \to W$ whose composition with $W \to V$ is v_0 and we win. □

07Y5 **Proposition 98.17.2.** *Let* S *be a locally Noetherian scheme. Let* $p : \mathcal{X} \to (Sch/S)_{fppf}$ *be a category fibred in groupoids. Assume that*

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ *is representable by algebraic spaces,*
(2) $\mathcal{X}$ *satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 98.14),*
(3) $\mathcal{O}_{S,s}$ *is a G-ring for all finite type points* s *of* S.

Then $\mathcal{X}$ *is an algebraic stack.*

Proof. We first prove that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},y} (Sch/V)_{fppf}$$

is representable by an algebraic space for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V, see Criteria for Representability, Lemma 97.5.5[7]. Observe that $\mathcal{Y}$ is fibred in setoids (Stacks, Lemma 8.2.5) and let $Y : (Sch/S)^{opp}_{fppf} \to Sets$, $T \mapsto \mathrm{Ob}(\mathcal{Y}_T)/\cong$ be the functor of isomorphism classes. We will apply Proposition 98.16.1 to see that Y is an algebraic space.

Note that $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ (and hence also $Y \to Y \times Y$) is representable by algebraic spaces by condition (1) and Criteria for Representability, Lemma 97.4.4. Observe that Y is a sheaf for the étale topology by Stacks, Lemmas 8.6.3 and 8.6.7,

[6]Set theoretical remark: This coproduct is (isomorphic to) an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

[7]The set theoretic condition in Criteria for Representability, Lemma 97.5.5 will hold: the size of the algebraic space Y representing $\mathcal{Y}$ is suitably bounded. Namely, $Y \to S$ will be locally of finite type and Y will satisfy axiom [-1]. Details omitted.

i.e., axiom [0] holds. Also Y is limit preserving by Lemma 98.11.2, i.e., we have [1]. Note that Y has (RS), i.e., axiom [2] holds, by Lemmas 98.5.2 and 98.5.3. Axiom [3] for Y follows from Lemmas 98.8.1 and 98.8.2. Axiom [4] follows from Lemmas 98.9.5 and 98.9.6. Axiom [5] for Y follows directly from openness of versality for $\Delta_{\mathcal{X}}$ which is part of axiom [5] for $\mathcal{X}$. Thus all the assumptions of Proposition 98.16.1 are satisfied and Y is an algebraic space.

At this point it follows from Lemma 98.17.1 that $\mathcal{X}$ is an algebraic stack. □

98.18. Strong Rim-Schlessinger

0CXN In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

07Y8 **Definition 98.18.1.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *condition (RS*)* if given a fibre product diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' = A \times_B B' & \longrightarrow & A \end{array}$$

of S-algebras, with $B' \to B$ surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

We make some observations: with $A \to B \leftarrow B'$ as in Definition 98.18.1

(1) we have $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$ in the category of schemes, see More on Morphisms, Lemma 37.14.3, and
(2) if $\mathcal{X}$ is an algebraic stack, then $\mathcal{X}$ satisfies (RS*) by Lemma 98.18.2.

If S is locally Noetherian, then

(3) if A, B, B' are of finite type over S and B is finite over A, then A' is of finite type over S[8], and
(4) if $\mathcal{X}$ satisfies (RS*), then $\mathcal{X}$ satisfies (RS) because (RS) covers exactly those cases of (RS*) where A, B, B' are Artinian local.

0CXP **Lemma 98.18.2.** *Let $\mathcal{X}$ be an algebraic stack over a base S. Then $\mathcal{X}$ satisfies (RS*).*

Proof. This is implied by Lemma 98.4.1, see remarks following Definition 98.18.1. □

0CXQ **Lemma 98.18.3.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS*), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The proof is exactly the same as the proof of Lemma 98.5.3. □

[8] If $\mathrm{Spec}(A)$ maps into an affine open of S this follows from More on Algebra, Lemma 15.5.1. The general case follows using More on Algebra, Lemma 15.5.3.

98.19. Versality and generalizations

0G2I We prove that versality is preserved under generalizations for stacks which have (RS*) and are limit preserving. We suggest skipping this section on a first reading.

0G2J **Lemma 98.19.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u \in U$ be a finite type point such that x is not versal at u. Then there exists a morphism $x \to y$ of $\mathcal{X}$ lying over $U \to T$ satisfying*

(1) *the morphism $U \to T$ is a first order thickening,*
(2) *we have a short exact sequence*

$$0 \to \kappa(u) \to \mathcal{O}_T \to \mathcal{O}_U \to 0$$

(3) *there does* **not** *exist a pair (W, α) consisting of an open neighbourhood $W \subset T$ of u and a morphism $\beta : y|_W \to x$ such that the composition*

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to y} y|_W \xrightarrow{\beta} x$$

is the canonical morphism $x|_{U \cap W} \to x$.

Proof. Let $R = \mathcal{O}^\wedge_{U,u}$. Let $k = \kappa(u)$ be the residue field of R. Let ξ be the formal object of $\mathcal{X}$ over R associated to x. Since x is not versal at u, we see that ξ is not versal, see Lemma 98.12.3. By the discussion following Definition 98.12.1 this means we can find morphisms $\xi_1 \to x_A \to x_B$ of $\mathcal{X}$ lying over closed immersions $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ where A, B are Artinian local rings with residue field k, an $n \geq 1$ and a commutative diagram

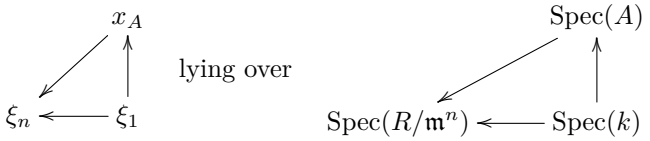

such that there does **not** exist an $m \geq n$ and a commutative diagram

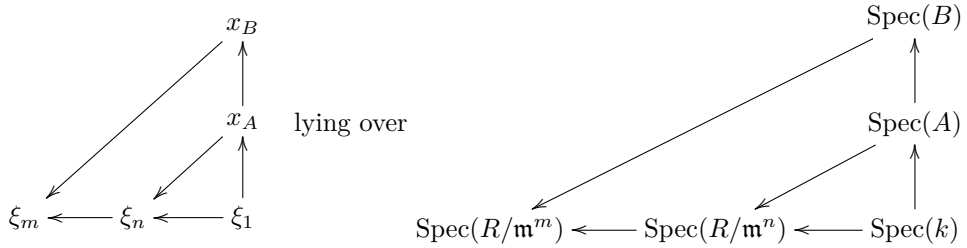

We may moreover assume that $B \to A$ is a small extension, i.e., that the kernel I of the surjection $B \to A$ is isomorphic to k as an A-module. This follows from Formal Deformation Theory, Remark 90.8.10. Then we simply define

$$T = U \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$$

By property (RS*) we find y over T whose restriction to $\mathrm{Spec}(B)$ is x_B and whose restriction to U is x (this gives the arrow $x \to y$ lying over $U \to T$). To finish the proof we verify conditions (1), (2), and (3).

By the construction of the pushout we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \Gamma(T, \mathcal{O}_T) & \longrightarrow & \Gamma(U, \mathcal{O}_U) & \longrightarrow & 0 \end{array}$$

with exact rows. This immediately proves (1) and (2). To finish the proof we will argue by contradiction. Assume we have a pair (W, β) as in (3). Since $\mathrm{Spec}(B) \to T$ factors through W we get the morphism

$$x_B \to y|_W \xrightarrow{\beta} x$$

Since B is Artinian local with residue field $k = \kappa(u)$ we see that $x_B \to x$ lies over a morphism $\mathrm{Spec}(B) \to U$ which factors through $\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{m}_u^m)$ for some $m \geq n$. In other words, $x_B \to x$ factors through ξ_m giving a map $x_B \to \xi_m$. The compatibility condition on the morphism α in condition (3) translates into the condition that

$$\begin{array}{ccc} x_B & \longleftarrow & x_A \\ \downarrow & & \downarrow \\ \xi_m & \longleftarrow & \xi_n \end{array}$$

is commutative. This gives the contradiction we were looking for. □

0G2K **Lemma 98.19.2.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*
(3) *$\mathcal{X}$ is limit preserving.*

Let x be an object of $\mathcal{X}$ over a scheme U of finite type over S. Let $u \rightsquigarrow u_0$ be a specialization of finite type points of U such that x is versal at u_0. Then x is versal at u.

Proof. After shrinking U we may assume U is affine and U maps into an affine open $\mathrm{Spec}(\Lambda)$ of S. If x is not versal at u then we may pick $x \to y$ lying over $U \to T$ as in Lemma 98.19.1. Write $U = \mathrm{Spec}(R_0)$ and $T = \mathrm{Spec}(R)$. The morphism $U \to T$ corresponds to a surjective ring map $R \to R_0$ whose kernel is an ideal of square zero. By assumption (3) we get that y comes from an object x' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to y \to x' \text{ lying over } U \to T \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x,\mathcal{X},x'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 94.10.11. By construction of 2-fibre products, a V-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : V \to U$, $a' : V \to U'$ and a morphism $\alpha : a^*x \to (a')^*x'$. We obtain a commutative

diagram

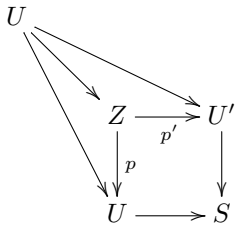

The morphism $i : U \to Z$ comes the isomorphism $x \to x'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 98.12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Since $u \rightsquigarrow u_0$ the point u is in this open. Condition (3) of Lemma 98.19.1 is clearly preserved by shrinking U (all of the schemes U, T, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$ such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of Spaces, Lemma 76.19.6). Pulling back the universal morphism $p^*x \to (p')^*x'$ by i' we obtain a morphism

$$x' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to x' \to x$$

is the identity. Recall that we have $y \to x'$ lying over the morphism $T \to U'$. Composing we get a morphism $y \to x$ whose existence contradicts condition (3) of Lemma 98.19.1. This contradiction finishes the proof. □

98.20. Strong formal effectiveness

0CXR In this section we demonstrate how a strong version of effectiveness of formal objects implies openness of versality. The proof of [**Bha16**, Theorem 1.1] shows that quasi-compact and quasi-separated algebraic spaces satisfy the strong formal effectiveness discussed in Remark 98.20.2. In addition, the theory we develop is nonempty: we use it later to show openness of versality for the stack of coherent sheaves and for moduli of complexes, see Quot, Theorems 99.6.1 and 99.16.12.

0G2S **Lemma 98.20.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u_n \in U$, $n \geq 1$ be finite type points such that (a) there are no specializations $u_n \rightsquigarrow u_m$ for $n \neq m$, and (b) x is not versal at u_n for all n. Then there exist morphisms*

$$x \to x_1 \to x_2 \to \ldots \quad \text{in } \mathcal{X} \text{ lying over} \quad U \to U_1 \to U_2 \to \ldots$$

over S such that

(1) *for each n the morphism $U \to U_n$ is a first order thickening,*
(2) *for each n we have a short exact sequence*

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0$$

with $U_0 = U$ for $n = 1$,

(3) *for each n there does* **not** *exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \to x$ such that the composition*

$$x|_{U\cap W} \xrightarrow{\text{restriction of } x\to x_n} x_n|_W \xrightarrow{\alpha} x$$

is the canonical morphism $x|_{U\cap W} \to x$.

Proof. Since there are no specializations among the points u_n (and in particular the u_n are pairwise distinct), for every n we can find an open $U' \subset U$ such that $u_n \in U'$ and $u_i \notin U'$ for $i = 1, \ldots, n-1$. By Lemma 98.19.1 for each $n \geq 1$ we can find

$$x \to y_n \quad \text{in } \mathcal{X} \text{ lying over } \quad U \to T_n$$

such that

(1) the morphism $U \to T_n$ is a first order thickening,
(2) we have a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{T_n} \to \mathcal{O}_U \to 0$$

(3) there does **not** exist a pair (W, α) consisting of an open neighbourhood $W \subset T_n$ of u_n and a morphism $\beta : y_n|_W \to x$ such that the composition

$$x|_{U\cap W} \xrightarrow{\text{restriction of } x\to y_n} y_n|_W \xrightarrow{\beta} x$$

is the canonical morphism $x|_{U\cap W} \to x$.

Thus we can define inductively

$$U_1 = T_1, \quad U_{n+1} = U_n \amalg_U T_{n+1}$$

Setting $x_1 = y_1$ and using (RS*) we find inductively x_{n+1} over U_{n+1} restricting to x_n over U_n and y_{n+1} over T_{n+1}. Property (1) for $U \to U_n$ follows from the construction of the pushout in More on Morphisms, Lemma 37.14.3. Property (2) for U_n similarly follows from property (2) for T_n by the construction of the pushout. After shrinking to an open neighbourhood U' of u_n as discussed above, property (3) for (U_n, x_n) follows from property (3) for (T_n, y_n) simply because the corresponding open subschemes of T_n and U_n are isomorphic. Some details omitted. □

0CXT **Remark 98.20.2** (Strong effectiveness)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume we have

(1) an affine open $\mathrm{Spec}(\Lambda) \subset S$,
(2) an inverse system (R_n) of Λ-algebras with surjective transition maps whose kernels are locally nilpotent,
(3) a system (ξ_n) of objects of $\mathcal{X}$ lying over the system $(\mathrm{Spec}(R_n))$.

In this situation, set $R = \lim R_n$. We say that (ξ_n) is *effective* if there exists an object ξ of $\mathcal{X}$ over $\mathrm{Spec}(R)$ whose restriction to $\mathrm{Spec}(R_n)$ gives the system (ξ_n).

It is not the case that every algebraic stack $\mathcal{X}$ over S satisfies a strong effectiveness axiom of the form: every system (ξ_n) as in Remark 98.20.2 is effective. An example is given in Examples, Section 110.73.

0CXU **Lemma 98.20.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*

(3) $\mathcal{X}$ *is limit preserving,*
(4) *systems* (ξ_n) *as in Remark 98.20.2 where* $\mathrm{Ker}(R_m \to R_n)$ *is an ideal of square zero for all* $m \geq n$ *are effective.*

Then $\mathcal{X}$ *satisfies openness of versality.*

Proof. Choose a scheme U locally of finite type over S, a finite type point u_0 of U, and an object x of $\mathcal{X}$ over U such that x is versal at u_0. After shrinking U we may assume U is affine and U maps into an affine open $\mathrm{Spec}(\Lambda)$ of S. Let $E \subset U$ be the set of finite type points u such that x is not versal at u. By Lemma 98.19.2 if $u \in E$ then u_0 is not a specialization of u. If openness of versality does not hold, then u_0 is in the closure $\overline{E}$ of E. By Properties, Lemma 28.5.13 we may choose a countable subset $E' \subset E$ with the same closure as E. By Properties, Lemma 28.5.12 we may assume there are no specializations among the points of E'. Observe that E' has to be (countably) infinite as u_0 isn't the specialization of any point of E' as pointed out above. Thus we can write $E' = \{u_1, u_2, u_3, \ldots\}$, there are no specializations among the u_i, and u_0 is in the closure of E'.

Choose $x \to x_1 \to x_2 \to \ldots$ lying over $U \to U_1 \to U_2 \to \ldots$ as in Lemma 98.20.1. Write $U_n = \mathrm{Spec}(R_n)$ and $U = \mathrm{Spec}(R_0)$. Set $R = \lim R_n$. Observe that $R \to R_0$ is surjective with kernel an ideal of square zero. By assumption (4) we get ξ over $\mathrm{Spec}(R)$ whose base change to R_n is x_n. By assumption (3) we get that ξ comes from an object ξ' over $U' = \mathrm{Spec}(R')$ for some finite type Λ-subalgebra $R' \subset R$. After increasing R' we may and do assume that $R' \to R_0$ is surjective, so that $U \subset U'$ is a first order thickening. Thus we now have

$$x \to x_1 \to x_2 \to \ldots \to \xi' \text{ lying over } U \to U_1 \to U_2 \to \ldots \to U'$$

By assumption (1) there is an algebraic space Z over S representing

$$(Sch/U)_{fppf} \times_{x,\mathcal{X},\xi'} (Sch/U')_{fppf}$$

see Algebraic Stacks, Lemma 94.10.11. By construction of 2-fibre products, a T-valued point of Z corresponds to a triple (a, a', α) consisting of morphisms $a : T \to U$, $a' : T \to U'$ and a morphism $\alpha : a^*x \to (a')^*\xi'$. We obtain a commutative diagram

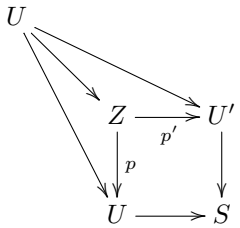

The morphism $i : U \to Z$ comes the isomorphism $x \to \xi'|_U$. Let $z_0 = i(u_0) \in Z$. By Lemma 98.12.6 we see that $Z \to U'$ is smooth at z_0. After replacing U by an affine open neighbourhood of u_0, replacing U' by the corresponding open, and replacing Z by the intersection of the inverse images of these opens by p and p', we reach the situation where $Z \to U'$ is smooth along $i(U)$. Note that this also involves replacing u_n by a subsequence, namely by those indices such that u_n is in the open. Moreover, condition (3) of Lemma 98.20.1 is clearly preserved by shrinking U (all of the schemes U, U_n, U' have the same underlying topological space). Since $U \to U'$ is a first order thickening of affine schemes, we can choose a morphism $i' : U' \to Z$ such that $p' \circ i' = \mathrm{id}_{U'}$ and whose restriction to U is i (More on Morphisms of

Spaces, Lemma 76.19.6). Pulling back the universal morphism $p^*x \to (p')^*\xi'$ by i' we obtain a morphism

$$\xi' \to x$$

lying over $p \circ i' : U' \to U$ such that the composition

$$x \to \xi' \to x$$

is the identity. Recall that we have $x_1 \to \xi'$ lying over the morphism $U_1 \to U'$. Composing we get a morphism $x_1 \to x$ whose existence contradicts condition (3) of Lemma 98.20.1. This contradiction finishes the proof. □

0CXV **Remark 98.20.4.** There is a way to deduce openness of versality of the diagonal of an category fibred in groupoids from a strong formal effectiveness axiom. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume

(1) $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
(2) $\mathcal{X}$ has (RS*),
(3) $\mathcal{X}$ is limit preserving,
(4) given an inverse system (R_n) of S-algebras as in Remark 98.20.2 where $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$ the functor

$$\mathcal{X}_{\mathrm{Spec}(\lim R_n)} \longrightarrow \lim_n \mathcal{X}_{\mathrm{Spec}(R_n)}$$

is fully faithful.

Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfies openness of versality. This follows by applying Lemma 98.20.3 to fibre products of the form $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},y} (Sch/V)_{fppf}$ for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V. If we ever need this, we will change this remark into a lemma and provide a detailed proof.

98.21. Infinitesimal deformations

07Y6 In this section we discuss a generalization of the notion of the tangent space introduced in Section 98.8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 90.11, 90.17, and 90.19.

Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Given a homomorphism $A' \to A$ of S-algebras and an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we write $Lift(x, A')$ for the category of lifts of x to $\mathrm{Spec}(A')$. An object of $Lift(x, A')$ is a morphism $x \to x'$ of $\mathcal{X}$ lying over $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ and morphisms of $Lift(x, A')$ are defined as commutative diagrams. The set of isomorphism classes of $Lift(x, A')$ is denoted $\mathrm{Lift}(x, A')$. See Formal Deformation Theory, Definition 90.17.1 and Remark 90.17.2. If $A' \to A$ is surjective with locally nilpotent kernel we call an element x' of $\mathrm{Lift}(x, A')$ a *(infinitesimal) deformation* of x. In this case the *group of infinitesimal automorphisms of x' over x* is the kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A')}}(x') \to \mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A)}}(x)\right)$$

Note that an element of $\mathrm{Inf}(x'/x)$ is the same thing as a lift of id_x over $\mathrm{Spec}(A')$ for (the category fibred in sets associated to) $Aut_{\mathcal{X}}(x')$. Compare with Formal Deformation Theory, Definition 90.19.1 and Formal Deformation Theory, Remark 90.19.8.

If M is an A-module we denote $A[M]$ the A-algebra whose underlying A-module is $A \oplus M$ and whose multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + a'm)$. When $M = A$ this is the ring of dual numbers over A, which we denote $A[\epsilon]$ as is customary. There is an A-algebra map $A[M] \to A$. The pullback of x to $\mathrm{Spec}(A[M])$ is called the *trivial deformation* of x to $\mathrm{Spec}(A[M])$.

07Y7 **Lemma 98.21.1.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let*

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

be a commutative diagram of S-algebras. Let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$, let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(B)$, and let $\phi : f(x)|_{\mathrm{Spec}(B)} \to y$ be a morphism of $\mathcal{Y}$ over $\mathrm{Spec}(B)$. Then there is a canonical functor

$$Lift(x, A') \longrightarrow Lift(y, B')$$

of categories of lifts induced by f and ϕ. The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.

Proof. This lemma proves itself. □

Let S be a base scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We define a category whose objects are pairs $(x, A' \to A)$ where

(1) $A' \to A$ is a surjection of S-algebras whose kernel is an ideal of square zero,
(2) x is an object of $\mathcal{X}$ lying over $\mathrm{Spec}(A)$.

A morphism $(y, B' \to B) \to (x, A' \to A)$ is given by a commutative diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

of S-algebras together with a morphism $x|_{\mathrm{Spec}(B)} \to y$ over $\mathrm{Spec}(B)$. Let us call this the category of *deformation situations*.

07Y9 **Lemma 98.21.2.** *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra and let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$.*

(1) *There exists an A-linear functor $Inf_x : Mod_A \to Mod_A$ such that given a deformation situation $(x, A' \to A)$ and a lift x' there is an isomorphism $Inf_x(I) \to Inf(x'/x)$ where $I = \mathrm{Ker}(A' \to A)$.*
(2) *There exists an A-linear functor $T_x : Mod_A \to Mod_A$ such that*
   (a) *given M in Mod_A there is a bijection $T_x(M) \to Lift(x, A[M])$,*
   (b) *given a deformation situation $(x, A' \to A)$ there is an action*

$$T_x(I) \times Lift(x, A') \to Lift(x, A')$$

where $I = \mathrm{Ker}(A' \to A)$. It is simply transitive if $Lift(x, A') \neq \emptyset$.

Proof. We define Inf_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathit{Sets}, \quad M \longrightarrow \mathrm{Inf}(x'_M/x) = \mathrm{Lift}(\mathrm{id}_x, A[M])$$

mapping M to the group of infinitesimal automorphisms of the trivial deformation x'_M of x to $\mathrm{Spec}(A[M])$ or equivalently the group of lifts of id_x in $Aut_{\mathcal{X}}(x'_M)$. We define T_x as the functor

$$\mathrm{Mod}_A \longrightarrow \mathit{Sets}, \quad M \longrightarrow \mathrm{Lift}(x, A[M])$$

of isomorphism classes of infinitesimal deformations of x to $\mathrm{Spec}(A[M])$. We apply Formal Deformation Theory, Lemma 90.11.4 to Inf_x and T_x. This lemma is applicable, since (RS*) tells us that

$$\mathit{Lift}(x, A[M \times N]) = \mathit{Lift}(x, A[M]) \times \mathit{Lift}(x, A[N])$$

as categories (and trivial deformations match up too).

Let $(x, A' \to A)$ be a deformation situation. Consider the ring map $g : A' \times_A A' \to A[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$. There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections to A. Thus applying (RS*) twice we find equivalences of categories

$$\begin{aligned} \mathit{Lift}(x, A') \times \mathit{Lift}(x, A') &= \mathit{Lift}(x, A' \times_A A') \\ &= \mathit{Lift}(x, A' \times_A A[I]) \\ &= \mathit{Lift}(x, A') \times \mathit{Lift}(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain "difference maps"

$$\mathrm{Inf}(x'/x) \times \mathrm{Inf}(x'/x) \longrightarrow \mathrm{Inf}_x(I) \quad \text{and} \quad \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule "$(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$" because

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow[(a_1,a_2,a_3) \mapsto (g(a_1,a_2), g(a_2,a_3))]{} & A[I] \times_A A[I] = A[I \times I] \\ & \searrow^{(a_1,a_2,a_3) \mapsto g(a_1,a_3)} & \downarrow + \\ & & A[I] \end{array}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided $\mathrm{Inf}(x'/x)$, resp. $\mathrm{Lift}(x, A')$ is nonempty. Note that $\mathrm{Inf}(x'/x)$ is always nonempty as it is a group. □

07YA **Remark 98.21.3** (Functoriality)**.** Assumptions and notation as in Lemma 98.21.2. Suppose $A \to B$ is a ring map and $y = x|_{\mathrm{Spec}(B)}$. Let $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$ and let $M \to N$ an A-linear map. Then there are canonical maps $\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N)$ and $T_x(M) \to T_y(N)$ simply because there is a pullback functor

$$\mathit{Lift}(x, A[M]) \to \mathit{Lift}(y, B[N])$$

coming from the ring map $A[M] \to B[N]$. Similarly, given a morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ we obtain a pullback functor $\mathit{Lift}(x, A') \to \mathit{Lift}(y, B')$. Since the construction of the action, the addition, and the scalar multiplication on Inf_x and T_x use only morphisms in the categories of

lifts (see proof of Formal Deformation Theory, Lemma 90.11.4) we see that the constructions above are functorial. In other words we obtain A-linear maps

$$\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N) \quad \text{and} \quad T_x(M) \to T_y(N)$$

such that the diagrams

$$\begin{array}{ccc} \mathrm{Inf}_y(J) & \longrightarrow & \mathrm{Inf}(y'/y) \\ \uparrow & & \uparrow \\ \mathrm{Inf}_x(I) & \longrightarrow & \mathrm{Inf}(x'/x) \end{array} \quad \text{and} \quad \begin{array}{ccc} T_y(J) \times \mathrm{Lift}(y, B') & \longrightarrow & \mathrm{Lift}(y, B') \\ \uparrow & & \uparrow \\ T_x(I) \times \mathrm{Lift}(x, A') & \longrightarrow & \mathrm{Lift}(x, A') \end{array}$$

commute. Here $I = \mathrm{Ker}(A' \to A)$, $J = \mathrm{Ker}(B' \to B)$, x' is a lift of x to A' (which may not always exist) and $y' = x'|_{\mathrm{Spec}(B')}$.

07YB **Remark 98.21.4** (Automorphisms)**.** Assumptions and notation as in Lemma 98.21.2. Let x', x'' be lifts of x to A'. Then we have a composition map

$$\mathrm{Inf}(x'/x) \times \mathrm{Mor}_{Lift(x,A')}(x', x'') \times \mathrm{Inf}(x''/x) \longrightarrow \mathrm{Mor}_{Lift(x,A')}(x', x'').$$

Since $Lift(x, A')$ is a groupoid, if $\mathrm{Mor}_{Lift(x,A')}(x', x'')$ is nonempty, then this defines a simply transitive left action of $\mathrm{Inf}(x'/x)$ on $\mathrm{Mor}_{Lift(x,A')}(x', x'')$ and a simply transitive right action by $\mathrm{Inf}(x''/x)$. Now the lemma says that $\mathrm{Inf}(x'/x) = \mathrm{Inf}_x(I) = \mathrm{Inf}(x''/x)$. We claim that the two actions described above agree via these identifications. Namely, either $x' \not\cong x''$ in which the claim is clear, or $x' \cong x''$ and in that case we may assume that $x'' = x'$ in which case the result follows from the fact that $\mathrm{Inf}(x'/x)$ is commutative. In particular, we obtain a well defined action

$$\mathrm{Inf}_x(I) \times \mathrm{Mor}_{Lift(x,A')}(x', x'') \longrightarrow \mathrm{Mor}_{Lift(x,A')}(x', x'')$$

which is simply transitive as soon as $\mathrm{Mor}_{Lift(x,A')}(x', x'')$ is nonempty.

07YE **Remark 98.21.5.** Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let A be an S-algebra. There is a notion of a *short exact sequence*

$$(x, A'_1 \to A) \to (x, A'_2 \to A) \to (x, A'_3 \to A)$$

of deformation situations: we ask the corresponding maps between the kernels $I_i = \mathrm{Ker}(A'_i \to A)$ give a short exact sequence

$$0 \to I_3 \to I_2 \to I_1 \to 0$$

of A-modules. Note that in this case the map $A'_3 \to A'_1$ factors through A, hence there is a canonical isomorphism $A'_1 = A[I_1]$.

0DNN **Lemma 98.21.6.** *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS*). Let A be an S-algebra and let w be an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over A. Denote x, y, z the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w. For any A-module M there is a 6-term exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & Inf_w(M) & \longrightarrow & Inf_x(M) \oplus Inf_z(M) & \longrightarrow & Inf_y(M) \\ & & & & & \searrow & \\ & & T_w(M) & \longrightarrow & T_x(M) \oplus T_z(M) & \longrightarrow & T_y(M) \end{array}$$

of A-modules.

Proof. By Lemma 98.18.3 we see that $\mathcal{W}$ satisfies (RS*) and hence $T_w(M)$ and $\mathrm{Inf}_w(M)$ are defined. The horizontal arrows are defined using the functoriality of Lemma 98.21.1.

Definition of the "boundary" map $\delta : \mathrm{Inf}_y(M) \to T_w(M)$. Choose isomorphisms $p(x) \to y$ and $y \to q(z)$ such that $w = (x, z, p(x) \to y \to q(z))$ in the description of the 2-fibre product of Categories, Lemma 4.35.7 and more precisely Categories, Lemma 4.32.3. Let x', y', z', w' denote the trivial deformation of x, y, z, w over $A[M]$. By pullback we get isomorphisms $y' \to p(x')$ and $q(z') \to y'$. An element $\alpha \in \mathrm{Inf}_y(M)$ is the same thing as an automorphism $\alpha : y' \to y'$ over $A[M]$ which restricts to the identity on y over A. Thus setting

$$\delta(\alpha) = (x', z', p(x') \to y' \xrightarrow{\alpha} y' \to q(z'))$$

we obtain an object of $T_w(M)$. This is a map of A-modules by Formal Deformation Theory, Lemma 90.11.5.

The rest of the proof is exactly the same as the proof of Formal Deformation Theory, Lemma 90.20.1. □

0D18 **Remark 98.21.7** (Compatibility with previous tangent spaces)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ has (RS*). Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over $\mathrm{Spec}(k)$. Then we have equalities of k-vector spaces

$$T\mathcal{F}_{\mathcal{X},k,x_0} = T_{x_0}(k) \quad \text{and} \quad \mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Inf}_{x_0}(k)$$

where the spaces on the left hand side of the equality signs are given in (98.8.0.1) and (98.8.0.2) and the spaces on the right hand side are given by Lemma 98.21.2.

07YC **Remark 98.21.8** (Canonical element)**.** Assumptions and notation as in Lemma 98.21.2. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(A) \to S$ corresponds to a ring map $\Lambda \to A$. Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, \mathrm{d}_{A/\Lambda}(a))$$

Pulling back x along the corresponding morphism $\mathrm{Spec}(A[\Omega_{A/\Lambda}]) \to \mathrm{Spec}(A)$ we obtain a deformation x_{can} of x over $A[\Omega_{A/\Lambda}]$. We call this the *canonical element*

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that Λ is Noetherian and $\Lambda \to A$ is of finite type. Let $k = \kappa(\mathfrak{p})$ be a residue field at a finite type point u_0 of $U = \mathrm{Spec}(A)$. Let $x_0 = x|_{u_0}$. By (RS*) and the fact that $A[k] = A \times_k k[k]$ the space $T_x(k)$ is the tangent space to the deformation functor $\mathcal{F}_{\mathcal{X},k,x_0}$. Via

$$T\mathcal{F}_{U,k,u_0} = \mathrm{Der}_\Lambda(A, k) = \mathrm{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 90.11.11) and functoriality of T_x the canonical element produces the map on tangent spaces induced by the object x over U. Namely, $\theta \in T\mathcal{F}_{U,k,u_0}$ maps to $T_x(\theta)(x_{can})$ in $T_x(k) = T\mathcal{F}_{\mathcal{X},k,x_0}$.

07YD **Remark 98.21.9** (Canonical automorphism)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open and let x, y be objects of $\mathcal{X}$ over $\mathrm{Spec}(A)$. Further, let $A \to B$ be

a ring map and let $\alpha : x|_{\mathrm{Spec}(B)} \to y|_{\mathrm{Spec}(B)}$ be a morphism of $\mathcal{X}$ over $\mathrm{Spec}(B)$. Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, \mathrm{d}_{B/A}(b))$$

Pulling back α along the corresponding morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ we obtain a morphism α_{can} between the pullbacks of x and y over $B[\Omega_{B/A}]$. On the other hand, we can pullback α by the morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ corresponding to the injection of B into the first summand of $B[\Omega_{B/A}]$. By the discussion of Remark 98.21.4 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\mathrm{Spec}(B[\Omega_{B/A}])} \in \mathrm{Inf}_{x|_{\mathrm{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the *canonical automorphism*. It depends on all the ingredients A, x, y, $A \to B$ and α.

98.22. Obstruction theories

07YF In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 98.21.2 and Remark 98.21.3.

07YG **Definition 98.22.1.** Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. An *obstruction theory* is given by the following data

(1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ an A-linear functor
$$\mathcal{O}_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$$
of *obstruction modules*,
(2) for (x, A) as in (1), a ring map $A \to B$, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$, and an A-linear map $M \to N$ an induced A-linear map $\mathcal{O}_x(M) \to \mathcal{O}_y(N)$ where $y = x|_{\mathrm{Spec}(B)}$, and
(3) for every deformation situation $(x, A' \to A)$ an *obstruction* element $o_x(A') \in \mathcal{O}_x(I)$ where $I = \mathrm{Ker}(A' \to A)$.

These data are subject to the following conditions

(i) the functoriality maps turn the obstruction modules into a functor from the category of triples (x, A, M) to sets,
(ii) for every morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ the element $o_x(A')$ maps to $o_y(B')$, and
(iii) we have
$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$
for every deformation situation $(x, A' \to A)$.

This last condition explains the terminology. The module $\mathcal{O}_x(A')$ is called the *obstruction module*. The element $o_x(A')$ is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over S.

One of the main reasons for introducing obstruction theories is to check openness of versality. An example of this type of result is Lemma 98.22.2 below. The initial idea

to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [**Fle81**], Hall [**Hal17**], Hall and Rydh [**HR12**], Olsson [**Ols06a**], Olsson and Starr [**OS03**], and Lieblich [**Lie06a**] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

0CYF **Lemma 98.22.2.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume* This is [**Hal17**, Theorem 4.4]

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ has (RS*),*
(3) *$\mathcal{X}$ is limit preserving,*
(4) *there exists an obstruction theory*[9]*,*
(5) *for an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ and A-modules M_n, $n \geq 1$ we have*
   (a) *$T_x(\prod M_n) = \prod T_x(M_n)$,*
   (b) *$\mathcal{O}_x(\prod M_n) \to \prod \mathcal{O}_x(M_n)$ is injective.*

Then $\mathcal{X}$ satisfies openness of versality.

Proof. We prove this by verifying condition (4) of Lemma 98.20.3. Let (ξ_n) and (R_n) be as in Remark 98.20.2 such that $\mathrm{Ker}(R_m \to R_n)$ is an ideal of square zero for all $m \geq n$. Set $A = R_1$ and $x = \xi_1$. Denote $M_n = \mathrm{Ker}(R_n \to R_1)$. Then M_n is an A-module. Set $R = \lim R_n$. Let

$$\tilde{R} = \{(r_1, r_2, r_3 \ldots) \in \prod R_n \text{ such that all have the same image in } A\}$$

Then $\tilde{R} \to A$ is surjective with kernel $M = \prod M_n$. There is a map $R \to \tilde{R}$ and a map $\tilde{R} \to A[M]$, $(r_1, r_2, r_3, \ldots) \mapsto (r_1, r_2 - r_1, r_3 - r_2, \ldots)$. Together these give a short exact sequence

$$(x, R \to A) \to (x, \tilde{R} \to A) \to (x, A[M])$$

of deformation situations, see Remark 98.21.5. The associated sequence of kernels $0 \to \lim M_n \to M \to M \to 0$ is the canonical sequence computing the limit of the system of modules (M_n).

Let $o_x(\tilde{R}) \in \mathcal{O}_x(M)$ be the obstruction element. Since we have the lifts ξ_n we see that $o_x(\tilde{R})$ maps to zero in $\mathcal{O}_x(M_n)$. By assumption (5)(b) we see that $o_x(\tilde{R}) = 0$. Choose a lift $\tilde{\xi}$ of x to $\mathrm{Spec}(\tilde{R})$. Let $\tilde{\xi}_n$ be the restriction of $\tilde{\xi}$ to $\mathrm{Spec}(R_n)$. There exists elements $t_n \in T_x(M_n)$ such that $t_n \cdot \tilde{\xi}_n = \xi_n$ by Lemma 98.21.2 part (2)(b). By assumption (5)(a) we can find $t \in T_x(M)$ mapping to t_n in $T_x(M_n)$. After replacing $\tilde{\xi}$ by $t \cdot \tilde{\xi}$ we find that $\tilde{\xi}$ restricts to ξ_n over $\mathrm{Spec}(R_n)$ for all n. In particular, since ξ_{n+1} restricts to ξ_n over $\mathrm{Spec}(R_n)$, the restriction $\overline{\xi}$ of $\tilde{\xi}$ to $\mathrm{Spec}(A[M])$ has the property that it restricts to the trivial deformation over $\mathrm{Spec}(A[M_n])$ for all n. Hence by assumption (5)(a) we find that $\overline{\xi}$ is the trivial deformation of x. By axiom (RS*) applied to $R = \tilde{R} \times_{A[M]} A$ this implies that $\tilde{\xi}$ is the pullback of a deformation ξ of x over R. This finishes the proof. □

[9] Analyzing the proof the reader sees that in fact it suffices to check the functoriality (ii) of obstruction classes in Definition 98.22.1 for maps $(y, B' \to B) \to (x, A' \to A)$ with $B = A$ and $y = x$.

07YH **Example 98.22.3.** Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Let $W \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_W$-module flat over S. Consider the functor

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$

where $W_T = T \times_S W$ is the base change and $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ to W_T. If $T = \mathrm{Spec}(A)$ we will write $W_T = W_A$, etc. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be the category fibred in groupoids associated to F. Then $\mathcal{X}$ has an obstruction theory. Namely,

(1) given A over Λ and $x \in H^0(W_A, \mathcal{F}_A)$ we set $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$,
(2) given a deformation situation $(x, A' \to A)$ we let $o_x(A') \in \mathcal{O}_x(A)$ be the image of x under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \to \mathcal{F}_A \otimes_A I \to \mathcal{F}_{A'} \to \mathcal{F}_A \to 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that W_A and $W_{A'}$ have the same underlying topological space) and the explanation for why flatness of $\mathcal{F}$ over S implies that the sequence above is short exact.

07YI **Example 98.22.4** (Key example)**.** Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Say $\mathcal{X} = (Sch/X)_{fppf}$ with $X = \mathrm{Spec}(R)$ and $R = \Lambda[x_1, \ldots, x_n]/J$. The naive cotangent complex $NL_{R/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R\mathrm{d}x_i,$$

see Algebra, Lemma 10.134.2. Consider a deformation situation $(x, A' \to A)$. Denote I the kernel of $A' \to A$. The object x corresponds to $(a_1, \ldots, a_n)$ with $a_i \in A$ such that $f(a_1, \ldots, a_n) = 0$ in A for all $f \in J$. Set

$$\begin{aligned} \mathcal{O}_x(A') &= \mathrm{Hom}_R(J/J^2, I)/\mathrm{Hom}_R(R^{\oplus n}, I) \\ &= \mathrm{Ext}^1_R(NL_{R/\Lambda}, I) \\ &= \mathrm{Ext}^1_A(NL_{R/\Lambda} \otimes_R A, I). \end{aligned}$$

Choose lifts $a'_i \in A'$ of a_i in A. Then $o_x(A')$ is the class of the map $J/J^2 \to I$ defined by sending $f \in J$ to $f(a'_1, \ldots, a'_n) \in I$. We omit the verification that $o_x(A')$ is independent of choices. It is clear that if $o_x(A') = 0$ then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that $o_x(A')$ can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'} = I[1]$$

in $D(A)$, see Algebra, Lemma 10.134.6 for the last identification.

98.23. Naive obstruction theories

07YJ The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let $(x, A' \to A)$ be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme S. The key Example 98.22.4 suggests that any obstruction theory should be closely related to maps in $D(A)$ with target the naive cotangent complex of A. Working this out we find a criterion for versality in Lemma 98.23.3 which leads to a criterion for openness of

versality in Lemma 98.23.4. We introduce a notion of a naive obstruction theory in Definition 98.23.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 10.134. In particular, if $A' \to A$ is a surjection of Λ-algebras with square zero kernel I, then there are maps

$$NL_{A'/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 10.134.5). This doesn't form a distinguished triangle in general as we are using the naive cotangent complex and not the full one. There is a homotopy equivalence $NL_{A/A'} \to I[1]$ (the complex consisting of I placed in degree -1, see Algebra, Lemma 10.134.6). Finally, note that there is a canonical map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$.

07YK **Lemma 98.23.1.** *Let $A \to k$ be a ring map with k a field. Let $E \in D^-(A)$. Then $\mathrm{Ext}^i_A(E, k) = \mathrm{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$.*

Proof. Omitted. Hint: Replace E by a bounded above complex of free A-modules and compute both sides. □

07YL **Lemma 98.23.2.** *Let $\Lambda \to A \to k$ be finite type ring maps of Noetherian rings with $k = \kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Let $\xi : E \to NL_{A/\Lambda}$ be morphism of $D^-(A)$ such that $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is not surjective. Then there exists a surjection $A' \to A$ of Λ-algebras such that*

(a) *$I = \mathrm{Ker}(A' \to A)$ has square zero and is isomorphic to k as an A-module,*
(b) *$\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$, and*
(c) *$E \to NL_{A/A'}$ is zero.*

Proof. Let $f \in A$, $f \notin \mathfrak{p}$. Suppose that $A'' \to A_f$ satisfies (a), (b), (c) for the induced map $E \otimes_A A_f \to NL_{A_f/\Lambda}$, see Algebra, Lemma 10.134.13. Then we can set $A' = A'' \times_{A_f} A$ and get a solution. Namely, it is clear that $A' \to A$ satisfies (a) because $\mathrm{Ker}(A' \to A) = \mathrm{Ker}(A'' \to A) = I$. Pick $f'' \in A''$ lifting f. Then the localization of A' at (f'', f) is isomorphic to A'' (for example by More on Algebra, Lemma 15.5.3). Thus (b) and (c) are clear for A' too. In this way we see that we may replace A by the localization A_f (finitely many times). In particular (after such a replacement) we may assume that $\mathfrak{p}$ is a maximal ideal of A, see Morphisms, Lemma 29.16.1.

Choose a presentation $A = \Lambda[x_1, \ldots, x_n]/J$. Then $NL_{A/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i,$$

see Algebra, Lemma 10.134.2. After localizing if necessary (using Nakayama's lemma) we can choose generators $f_1, \ldots, f_m$ of J such that $f_j \otimes 1$ form a basis for $J/J^2 \otimes_A k$. Moreover, after renumbering, we can assume that the images of $\mathrm{d}f_1, \ldots, \mathrm{d}f_r$ form a basis for the image of $J/J^2 \otimes k \to \bigoplus k\mathrm{d}x_i$ and that $\mathrm{d}f_{r+1}, \ldots, \mathrm{d}f_m$ map to zero in $\bigoplus k\mathrm{d}x_i$. With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis $f_{r+1} \otimes 1, \ldots, f_m \otimes 1$. Changing basis once again we may assume that the image of $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is contained in the k-span of $f_{r+1} \otimes 1, \ldots, f_{m-1} \otimes 1$. Set

$$A' = \Lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_{m-1}, \mathfrak{p}f_m)$$

By construction $A' \to A$ satisfies (a). Since $\mathrm{d}f_m$ maps to zero in $\bigoplus k\mathrm{d}x_i$ we see that (b) holds. Finally, by construction the induced map $E \to NL_{A/A'} = I[1]$ induces the zero map $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to I \otimes_A k$. By Lemma 98.23.1 we see that the composition is zero. □

The following lemma is our key technical result.

07YM **Lemma 98.23.3.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero, and*
(ii) *there is an isomorphism of functors $T_x(-) \to \mathrm{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 98.21.8).*

Let $u_0 \in U$ be a finite type point with residue field $k = \kappa(u_0)$. Consider the following statements

(1) *x is versal at u_0, and*
(2) *$\xi : E \to NL_{A/\Lambda}$ induces a surjection $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ and an injection $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$.*

Then we always have (2) ⇒ (1) and we have (1) ⇒ (2) if u_0 is a closed point.

Proof. Let $\mathfrak{p} = \mathrm{Ker}(A \to k)$ be the prime corresponding to u_0.

Assume that x versal at u_0 and that u_0 is a closed point of U. If $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is not surjective, then let $A' \to A$ be an extension with kernel I as in Lemma 98.23.2. Because u_0 is a closed point, we see that I is a finite A-module, hence that A' is a finite type Λ-algebra (this fails if u_0 is not closed). In particular A' is Noetherian. By property (c) for A' and (i) for ξ we see that x lifts to an object x' over A'. Let $\mathfrak{p}' \subset A'$ be kernel of the surjective map to k. By Artin-Rees (Algebra, Lemma 10.51.2) there exists an $n > 1$ such that $(\mathfrak{p}')^n \cap I = 0$. Then we see that

$$B' = A'/(\mathfrak{p}')^n \longrightarrow A/\mathfrak{p}^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 90.3.12. On the other hand, as x is versal at u_0 and as $x'|_{\mathrm{Spec}(B')}$ is a lift of $x|_{\mathrm{Spec}(B)}$, there exists an integer $m \geq n$ and a map $q : A/\mathfrak{p}^m \to B'$ such that the composition $A/\mathfrak{p}^m \to B' \to B$ is the quotient map. Since the maximal ideal of B' has nth power equal to zero, this q factors through B which contradicts the fact that $B' \to B$ is an essential surjection. This contradiction shows that $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is surjective.

Assume that x versal at u_0. By Lemma 98.23.1 the map $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is dual to the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$. Note that

$$\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) = \mathrm{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \mathrm{Ext}^0_A(E, k)$$

Condition (ii) assures us the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$ sends a tangent vector θ to U at u_0 to the corresponding infinitesimal deformation of x_0, see Remark 98.21.8. Hence if x is versal, then this map is surjective, see Formal Deformation Theory, Lemma 90.13.2. Hence $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is injective. This finishes the proof of (1) ⇒ (2) in case u_0 is a closed point.

For the rest of the proof assume $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ is surjective and $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ injective. Set $R = A^\wedge_{\mathfrak{p}}$ and let η be the formal object over R associated to $x|_{\mathrm{Spec}(R)}$. The map $d\underline{\eta}$ on tangent spaces is surjective because it is identified with the dual of the injective map $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ (see previous paragraph). According to Formal Deformation Theory, Lemma 90.13.2 it suffices to prove the following: Let $C' \to C$ be a small extension of finite type Artinian local Λ-algebras with residue field k. Let $R \to C$ be a Λ-algebra map compatible with identifications of residue fields. Let $y = x|_{\mathrm{Spec}(C)}$ and let y' be a lift of y to C'. To show: we can lift the Λ-algebra map $R \to C$ to $R \to C'$.

Observe that it suffices to lift the Λ-algebra map $A \to C$. Let $I = \mathrm{Ker}(C' \to C)$. Note that I is a 1-dimensional k-vector space. The obstruction *ob* to lifting $A \to C$ is an element of $\mathrm{Ext}^1_A(NL_{A/\Lambda}, I)$, see Example 98.22.4. By Lemma 98.23.1 and our assumption the map ξ induces an injection

$$\mathrm{Ext}^1_A(NL_{A/\Lambda}, I) \longrightarrow \mathrm{Ext}^1_A(E, I)$$

By the construction of *ob* and (i) the image of *ob* in $\mathrm{Ext}^1_A(E, I)$ is the obstruction to lifting x to $A \times_C C'$. By (RS*) the fact that y/C lifts to y'/C' implies that x lifts to $A \times_C C'$. Hence $ob = 0$ and we are done. □

The key lemma above allows us to conclude that we have openness of versality in some cases.

07YN **Lemma 98.23.4.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero,*
(ii) *there is an isomorphism of functors $T_x(-) \to \mathrm{Ext}^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 98.21.8),*
(iii) *the cohomology groups of E are finite A-modules.*

If x is versal at a closed point $u_0 \in U$, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

Proof. Let C be the cone of ξ so that we have a distinguished triangle

$$E \to NL_{A/\Lambda} \to C \to E[1]$$

in $D^-(A)$. By Lemma 98.23.3 the assumption that x is versal at u_0 implies that $H^{-1}(C \otimes^{\mathbf{L}} k) = 0$. By More on Algebra, Lemma 15.77.4 there exists an $f \in A$ not contained in the prime corresponding to u_0 such that $H^{-1}(C \otimes^{\mathbf{L}}_A M) = 0$ for any A_f-module M. Using Lemma 98.23.3 again we see that we have versality for all finite type points of the open $D(f) \subset U$. □

The technical lemmas above suggest the following definition.

07YP **Definition 98.23.5.** Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ satisfies (RS*). A *naive obstruction theory* is given by the following data

07YQ (1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$ and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we are given an object $E_x \in D^-(A)$ and a map $\xi_x : E \to NL_{A/\Lambda}$,

07YR (2) given (x, A) as in (1) there are transformations of functors

$$\mathrm{Inf}_x(-) \to \mathrm{Ext}^{-1}_A(E_x, -) \quad \text{and} \quad T_x(-) \to \mathrm{Ext}^0_A(E_x, -)$$

07YS (3) for (x, A) as in (1) and a ring map $A \to B$ setting $y = x|_{\mathrm{Spec}(B)}$ there is a functoriality map $E_x \to E_y$ in $D(A)$.

These data are subject to the following conditions

(i) in the situation of (3) the diagram

$$\begin{array}{ccc} E_y & \xrightarrow[\xi_y]{} & NL_{B/\Lambda} \\ \uparrow & & \uparrow \\ E_x & \xrightarrow{\xi_x} & NL_{A/\Lambda} \end{array}$$

is commutative in $D(A)$,

(ii) given (x, A) as in (1) and $A \to B \to C$ setting $y = x|_{\mathrm{Spec}(B)}$ and $z = x|_{\mathrm{Spec}(C)}$ the composition of the functoriality maps $E_x \to E_y$ and $E_y \to E_z$ is the functoriality map $E_x \to E_z$,

(iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 98.21.3,

(iv) the composition $E_x \to NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ corresponds to the canonical element of $T_x(\Omega_{A/\Lambda}) = \mathrm{Ext}^0(E_x, \Omega_{A/\Lambda})$, see Remark 98.21.8,

(v) given a deformation situation $(x, A' \to A)$ with $I = \mathrm{Ker}(A' \to A)$ the composition $E_x \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero in

$$\mathrm{Hom}_A(E_x, NL_{A/\Lambda}) = \mathrm{Ext}^0_A(E_x, NL_{A/A'}) = \mathrm{Ext}^1_A(E_x, I)$$

if and only if x lifts to A'.

Thus we see in particular that we obtain an obstruction theory as in Section 98.22 by setting $\mathcal{O}_x(-) = \mathrm{Ext}^1_A(E_x, -)$.

07YT **Lemma 98.23.6.** *Let S and $\mathcal{X}$ be as in Definition 98.23.5 and let $\mathcal{X}$ be endowed with a naive obstruction theory. Let $A \to B$ and $y \to x$ be as in (3). Let k be a B-algebra which is a field. Then the functoriality map $E_x \to E_y$ induces bijections*

$$H^i(E_x \otimes^{\mathbf{L}}_A k) \to H^i(E_y \otimes^{\mathbf{L}}_B k)$$

for $i = 0, 1$.

Proof. Let $z = x|_{\mathrm{Spec}(k)}$. Then (RS*) implies that

$$\mathit{Lift}(x, A[k]) = \mathit{Lift}(z, k[k]) \quad \text{and} \quad \mathit{Lift}(y, B[k]) = \mathit{Lift}(z, k[k])$$

because $A[k] = A \times_k k[k]$ and $B[k] = B \times_k k[k]$. Hence the properties of a naive obstruction theory imply that the functoriality map $E_x \to E_y$ induces bijections $\mathrm{Ext}^i_A(E_x, k) \to \mathrm{Ext}^i_B(E_y, k)$ for $i = -1, 0$. By Lemma 98.23.1 our maps $H^i(E_x \otimes^{\mathbf{L}}_A k) \to H^i(E_y \otimes^{\mathbf{L}}_B k)$, $i = 0, 1$ induce isomorphisms on dual vector spaces hence are isomorphisms. □

07YU **Lemma 98.23.7.** *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)^{opp}_{fppf}$ be a category fibred in groupoids. Assume that $\mathcal{X}$ satisfies (RS*) and that $\mathcal{X}$ has a naive obstruction theory. Then openness of versality holds for $\mathcal{X}$ provided the complexes E_x of Definition 98.23.5 have finitely generated cohomology groups for pairs (A, x) where A is of finite type over S.*

Proof. Let U be a scheme locally of finite type over S, let x be an object of $\mathcal{X}$ over U, and let u_0 be a finite type point of U such that x is versal at u_0. We may first shrink U to an affine scheme such that u_0 is a closed point and such that $U \to S$ maps into an affine open $\mathrm{Spec}(\Lambda)$. Say $U = \mathrm{Spec}(A)$. Let $\xi_x : E_x \to NL_{A/\Lambda}$ be the obstruction map. At this point we may apply Lemma 98.23.4 to conclude. □

98.24. A dual notion

07YV Let $(x, A' \to A)$ be a deformation situation for a given category $\mathcal{X}$ fibred in groupoids over a locally Noetherian scheme S. Assume $\mathcal{X}$ has an obstruction theory, see Definition 98.22.1. In practice one often has a complex $K^\bullet$ of A-modules and isomorphisms of functors

$$\mathrm{Inf}_x(-) \to H^0(K^\bullet \otimes^{\mathbf{L}}_A -), \quad T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -), \quad \mathcal{O}_x(-) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

07YW **Example 98.24.1.** Let $\Lambda, S, W, \mathcal{F}$ be as in Example 98.22.3. Assume that $W \to S$ is proper and $\mathcal{F}$ coherent. By Cohomology of Schemes, Remark 30.22.2 there exists a finite complex of finite projective Λ-modules $N^\bullet$ which universally computes the cohomology of $\mathcal{F}$. In particular the obstruction spaces from Example 98.22.3 are $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$. Hence with $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$ we see that $\mathcal{O}_x(M) = H^2(K^\bullet \otimes^{\mathbf{L}}_A M)$.

07YX **Situation 98.24.2.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ has (RS*) so that we can speak of the functor $T_x(-)$, see Lemma 98.21.2. Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Assume we are given

(1) a complex of A-modules $K^\bullet$,
(2) a transformation of functors $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$,
(3) for every deformation situation $(x, A' \to A)$ with kernel $I = \mathrm{Ker}(A' \to A)$ an element $o_x(A') \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$

satisfying the following (minimal) conditions

(i) the transformation $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$ is an isomorphism,
(ii) given a morphism $(x, A'' \to A) \to (x, A' \to A)$ of deformation situations the element $o_x(A')$ maps to the element $o_x(A'')$ via the map $H^2(K^\bullet \otimes^{\mathbf{L}}_A I) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A I')$ where $I' = \mathrm{Ker}(A'' \to A)$, and
(iii) x lifts to an object over $\mathrm{Spec}(A')$ if and only if $o_x(A') = 0$.

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 98.24.2 an important role will be played by $K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda}$. Suppose we are given an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$. Then (1) for any surjection $A' \to A$ of Λ-algebras with kernel I of square zero the canonical map $NL_{A/\Lambda} \to NL_{A/A'} = I[1]$

sends ξ to an element $\xi_{A'} \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$ and (2) the map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ sends ξ to an element ξ_{can} of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \Omega_{A/\Lambda})$.

07YY **Lemma 98.24.3.** *In Situation 98.24.2. Assume furthermore that*

(iv) *given a short exact sequence of deformation situations as in Remark 98.21.5 and a lift $x'_2 \in Lift(x, A'_2)$ then $o_x(A'_3) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I_3)$ equals $\partial\theta$ where $\theta \in H^1(K^\bullet \otimes^{\mathbf{L}}_A I_1)$ is the element corresponding to $x'_2|_{\mathrm{Spec}(A'_1)}$ via $A'_1 = A[I_1]$ and the given map $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.*

In this case there exists an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$ such that

(1) *for every deformation situation $(x, A' \to A)$ we have $\xi_{A'} = o_x(A')$, and*
(2) *ξ_{can} matches the canonical element of Remark 98.21.8 via the given transformation $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.*

Proof. Choose a $\alpha : \Lambda[x_1, \ldots, x_n] \to A$ with kernel J. Write $P = \Lambda[x_1, \ldots, x_n]$. In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus A\mathrm{d}x_i)$$

which is permissible by Algebra, Lemma 10.134.2 and More on Algebra, Lemma 15.58.2. Consider the element $o_x(P/J^2) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A J/J^2)$ and consider the quotient

$$C = (P/J^2 \times \bigoplus A\mathrm{d}x_i)/(J/J^2)$$

where J/J^2 is embedded diagonally. Note that $C \to A$ is a surjection with kernel $\bigoplus A\mathrm{d}x_i$. Moreover there is a section $A \to C$ to $C \to A$ given by mapping the class of $f \in P$ to the class of $(f, \mathrm{d}f)$ in the pushout. For later use, denote x_C the pullback of x along the corresponding morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Thus we see that $o_x(C) = 0$. We conclude that $o_x(P/J^2)$ maps to zero in $H^2(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. It follows that there exists some element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL(\alpha))$ mapping to $o_x(P/J^2)$.

Note that for any deformation situation $(x, A' \to A)$ there exists a Λ-algebra map $P/J^2 \to A'$ compatible with the augmentations to A. Hence the element ξ satisfies the first property of the lemma by construction and property (ii) of Situation 98.24.2.

Note that our choice of ξ was well defined up to the choice of an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. We will show that after modifying ξ by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let $C' \subset C$ be the image of P/J^2 under the Λ-algebra map $P/J^2 \to C$ (inclusion of first factor). Observe that $\mathrm{Ker}(C' \to A) = \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i)$. Set $\overline{C} = A[\Omega_{A/\Lambda}]$. The map $P/J^2 \times \bigoplus A\mathrm{d}x_i \to \overline{C}$, $(f, \sum f_i \mathrm{d}x_i) \mapsto (f \bmod J, \sum f_i \mathrm{d}x_i)$ factors through a surjective map $C \to \overline{C}$. Then

$$(x, \overline{C} \to A) \to (x, C \to A) \to (x, C' \to A)$$

is a short exact sequence of deformation situations. The associated splitting $\overline{C} = A[\Omega_{A/\Lambda}]$ (from Remark 98.21.5) equals the given splitting above. Moreover, the section $A \to C$ composed with the map $C \to \overline{C}$ is the map $(1, \mathrm{d}) : A \to A[\Omega_{A/\Lambda}]$ of Remark 98.21.8. Thus x_C restricts to the canonical element x_{can} of $T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}])$. By condition (iv) we conclude that $o_x(P/J^2)$ maps to ∂x_{can} in

$$H^1(K^\bullet \otimes^{\mathbf{L}}_A \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i))$$

By construction ξ maps to $o_x(P/J^2)$. It follows that x_{can} and ξ_{can} map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$ as desired. □

07YZ **Lemma 98.24.4.** *In Situation 98.24.2 assume that (iv) of Lemma 98.24.3 holds and that $K^\bullet$ is a perfect object of $D(A)$. In this case, if x is versal at a closed point $u_0 \in U$ then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.*

Proof. We may assume that $K^\bullet$ is a finite complex of finite projective A-modules. Thus the derived tensor product with $K^\bullet$ is the same as simply tensoring with $K^\bullet$. Let $E^\bullet$ be the dual perfect complex to $K^\bullet$, see More on Algebra, Lemma 15.75.15. (So $E^n = \mathrm{Hom}_A(K^{-n}, A)$ with differentials the transpose of the differentials of $K^\bullet$.) Let $E \in D^-(A)$ denote the object represented by the complex $E^\bullet[-1]$. Let $\xi \in H^1(\mathrm{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$ be the element constructed in Lemma 98.24.3 and denote $\xi : E = E^\bullet[-1] \to NL_{A/\Lambda}$ the corresponding map (loc.cit.). We claim that the pair (E, ξ) satisfies all the assumptions of Lemma 98.23.4 which finishes the proof.

Namely, assumption (i) of Lemma 98.23.4 follows from conclusion (1) of Lemma 98.24.3 and the fact that $H^2(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^1(E, -)$ by loc.cit. Assumption (ii) of Lemma 98.23.4 follows from conclusion (2) of Lemma 98.24.3 and the fact that $H^1(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^0(E, -)$ by loc.cit. Assumption (iii) of Lemma 98.23.4 is clear. □

98.25. Limit preserving functors on Noetherian schemes

0GE1 It is sometimes convenient to consider functors or stacks defined only on the full subcategory of (locally) Noetherian schemes. In this section we discuss this in the case of algebraic spaces.

Let S be a locally Noetherian scheme. Let us be a bit pedantic in order to line up our categories correctly; people who are ignoring set theoretical issues can just replace the sets of schemes we choose by the collection of all schemes in what follows. As in Topologies, Remark 34.11.1 we choose a category Sch_α of schemes containing S such that we obtain big sites $(Sch/S)_{Zar}$, $(Sch/S)_{étale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, and $(Sch/S)_{fppf}$ all with the same underlying category Sch_α/S. Denote

$$Noetherian_\alpha \subset Sch_\alpha$$

the full subcategory consisting of locally Noetherian schemes. This determines a full subcategory

$$Noetherian_\alpha/S \subset Sch_\alpha/S$$

For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have

(1) if $f : X \to Y$ is a morphism of Sch_α/S with Y in $Noetherian_\alpha/S$ and f locally of finite type, then X is in $Noetherian_\alpha/S$,
(2) for morphisms $f : X \to Y$ and $g : Z \to Y$ of $Noetherian_\alpha/S$ with f locally of finite type the fibre product $X \times_Y Z$ in $Noetherian_\alpha/S$ exists and agrees with the fibre product in Sch_α/S,
(3) if $\{X_i \to X\}_{i \in I}$ is a covering of $(Sch/S)_\tau$ and X is in $Noetherian_\alpha/S$, then the objects X_i are in $Noetherian_\alpha/S$

(4) the category $Noetherian_\alpha/S$ endowed with the set of coverings of $(Sch/S)_\tau$ whose objects are in $Noetherian_\alpha/S$ is a site we will denote $(Noetherian/S)_\tau$,
(5) the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ is fully faithful, continuous, and cocontinuous.

By Sites, Lemmas 7.21.1 and 7.21.5 we obtain a morphism of topoi

$$g_\tau : Sh((Noetherian/S)_\tau) \longrightarrow Sh((Sch/S)_\tau)$$

whose pullback functor is the restriction of sheaves along the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$.

0GE2 **Remark 98.25.1** (Warning)**.** The site $(Noetherian/S)_\tau$ does not have fibre products. Hence we have to be careful in working with sheaves. For example, the continuous inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ does not define a morphism of sites. See Examples, Section 110.60 for an example in case $\tau = fppf$.

Let $F : (Noetherian/S)_\tau^{opp} \to Sets$ be a functor. We say F is *limit preserving* if for any directed limit of affine schemes $X = \lim X_i$ of $(Noetherian/S)_\tau$ we have $F(X) = \operatorname{colim} F(X_i)$.

0GE3 **Lemma 98.25.2.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Restricting along the inclusion functor $(Noetherian/S)_\tau \to (Sch/S)_\tau$ defines an equivalence of categories between*

(1) *the category of limit preserving sheaves on $(Sch/S)_\tau$ and*
(2) *the category of limit preserving sheaves on $(Noetherian/S)_\tau$*

Proof. Let $F : (Noetherian/S)_\tau^{opp} \to Sets$ be a functor which is both limit preserving and a sheaf. By Topologies, Lemmas 34.13.1 and 34.13.3 there exists a unique functor $F' : (Sch/S)_\tau^{opp} \to Sets$ which is limit preserving, a sheaf, and restricts to F. In fact, the construction of F' in Topologies, Lemma 34.13.1 is functorial in F and this construction is a quasi-inverse to restriction. Some details omitted. □

0GE4 **Lemma 98.25.3.** *Let X be an object of $(Noetherian/S)_\tau$. If the functor of points $h_X : (Noetherian/S)_\tau^{opp} \to Sets$ is limit preserving, then X is locally of finite presentation over S.*

Proof. Let $V \subset X$ be an affine open subscheme which maps into an affine open $U \subset S$. We may write $V = \lim V_i$ as a directed limit of affine schemes V_i of finite presentation over U, see Algebra, Lemma 10.127.2. By assumption, the arrow $V \to X$ factors as $V \to V_i \to X$ for some i. After increasing i we may assume $V_i \to X$ factors through V as the inverse image of $V \subset X$ in V_i eventually becomes equal to V_i by Limits, Lemma 32.4.11. Then the identity morphism $V \to V$ factors through V_i for some i in the category of schemes over U. Thus $V \to U$ is of finite presentation; the corresponding algebra fact is that if B is an A-algebra such that $\mathrm{id} : B \to B$ factors through a finitely presented A-algebra, then B is of finite presentation over A (nice exercise). Hence X is locally of finite presentation over S. □

The following lemma has a variant for transformations representable by algebraic spaces.

0GE5 **Lemma 98.25.4.** *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $F', G' : (Sch/S)_\tau^{opp} \to Sets$ be limit preserving and sheaves. Let $a' : F' \to G'$ be a transformation of functors. Denote $a : F \to G$ the restriction of $a' : F' \to G'$ to $(Noetherian/S)_\tau$. The following are equivalent*

(1) *a' is representable (as a transformation of functors, see Categories, Definition 4.6.4), and*
(2) *for every object V of $(Noetherian/S)_\tau$ and every map $V \to G$ the fibre product $F \times_G V : (Noetherian/S)_\tau^{opp} \to Sets$ is a representable functor, and*
(3) *same as in (2) but only for V affine finite type over S mapping into an affine open of S.*

Proof. Assume (1). By Limits of Spaces, Lemma 70.3.4 the transformation a' is limit preserving[10]. Take $\xi : V \to G$ as in (2). Denote $V' = V$ but viewed as an object of $(Sch/S)_\tau$. Since G is the restriction of G' to $(Noetherian/S)_\tau$ we see that $\xi \in G(V)$ corresponds to $\xi' \in G'(V')$. By assumption $V' \times_{\xi', G'} F'$ is representable by a scheme U'. The morphism of schemes $U' \to V'$ corresponding to the projection $V' \times_{\xi', G'} F' \to V'$ is locally of finite presentation by Limits of Spaces, Lemma 70.3.5 and Limits, Proposition 32.6.1. Hence U' is a locally Noetherian scheme and therefore U' is isomorphic to an object U of $(Noetherian/S)_\tau$. Then U represents $F \times_G V$ as desired.

The implication (2) $\Rightarrow$ (3) is immediate. Assume (3). We will prove (1). Let T be an object of $(Sch/S)_\tau$ and let $T \to G'$ be a morphism. We have to show the functor $F' \times_{G'} T$ is representable by a scheme X over T. Let $\mathcal{B}$ be the set of affine opens of T which map into an affine open of S. This is a basis for the topology of T. Below we will show that for $W \in \mathcal{B}$ the fibre product $F' \times_{G'} W$ is representable by a scheme X_W over W. If $W_1 \subset W_2$ in $\mathcal{B}$, then we obtain an isomorphism $X_{W_1} \to X_{W_2} \times_{W_2} W_1$ because both X_{W_1} and $X_{W_2} \times_{W_2} W_1$ represent the functor $F' \times_{G'} W_1$. These isomorphisms are canonical and satisfy the cocycle condition mentioned in Constructions, Lemma 27.2.1. Hence we can glue the schemes X_W to a scheme X over T. Compatibility of the glueing maps with the maps $X_W \to F'$ provide us with a map $X \to F'$. The resulting map $X \to F' \times_{G'} T$ is an isomorphism as we may check this locally on T (as source and target of this arrow are sheaves for the Zariski topology).

Let W be an affine scheme which maps into an affine open $U \subset S$. Let $W \to G'$ be a map. Still assuming (3) we have to show that $F' \times_{G'} W$ is representable by a scheme. We may write $W = \lim V_i'$ as a directed limit of affine schemes V_i' of finite presentation over U, see Algebra, Lemma 10.127.2. Since V_i' is of finite type over an Noetherian scheme, we see that V_i' is a Noetherian scheme. Denote $V_i = V_i'$ but viewed as an object of $(Noetherian/S)_\tau$. As G' is limit preserving can choose an i and a map $V_i' \to G'$ such that $W \to G'$ is the composition $W \to V_i' \to G'$. Since G is the restriction of G' to $(Noetherian/S)_\tau$ the morphism $V_i' \to G'$ is the same thing as a morphism $V_i \to G$ (see above). By assumption (3) the functor $F \times_G V_i$ is representable by an object X_i of $(Noetherian/S)_\tau$. The functor $F \times_G V_i$ is limit preserving as it is the restriction of $F' \times_{G'} V_i'$ and this functor is limit preserving by Limits of Spaces, Lemma 70.3.6, the assumption that F' and G'

[10]This makes sense even if $\tau \neq fppf$ as the underlying category of $(Sch/S)_\tau$ equals the underlying category of $(Sch/S)_{fppf}$ and the statement doesn't refer to the topology.

are limit preserving, and Limits, Remark 32.6.2 which tells us that the functor of points of V_i' is limit preserving. By Lemma 98.25.3 we conclude that X_i is locally of finite presentation over S. Denote $X_i' = X_i$ but viewed as an object of $(Sch/S)_\tau$. Then we see that $F' \times_{G'} V_i'$ and the functors of points $h_{X_i'}$ are both extensions of $h_{X_i} : (Noetherian/S)_\tau^{opp} \to Sets$ to limit preserving sheaves on $(Sch/S)_\tau$. By the equivalence of categories of Lemma 98.25.2 we deduce that X_i' represents $F' \times_{G'} V_i'$. Then finally

$$F' \times_{G'} W = F' \times_{G'} V_i' \times_{V_i'} W = X_i' \times_{V_i'} W$$

is representable as desired. □

98.26. Algebraic spaces in the Noetherian setting

0GE6 Let S be a locally Noetherian scheme. Let $(Noetherian/S)_{étale} \subset (Sch/S)_{étale}$ denote the site studied in Section 98.25. Let $F : (Noetherian/S)_{étale}^{opp} \to Sets$ be a functor, i.e., F is a presheaf on $(Noetherian/S)_{étale}$. In this setting all the axioms [-1], [0], [1], [2], [3], [4], [5] of Section 98.15 make sense. We will review them one by one and make sure the reader knows exactly what we mean.

Axiom [-1]. This is a set theoretic condition to be ignored by readers who are not interested in set theoretic questions. It makes sense for F since it concerns the evaluation of F on spectra of fields of finite type over S which are objects of $(Noetherian/S)_{étale}$.

Axiom [0]. This is the axiom that F is a sheaf on $(Noetherian/S)_{étale}^{opp}$, i.e., satisfies the sheaf condition for étale coverings.

Axiom [1]. This is the axiom that F is limit preserving as defined in Section 98.25: for any directed limit of affine schemes $X = \lim X_i$ of $(Noetherian/S)_{étale}$ we have $F(X) = \operatorname{colim} F(X_i)$.

Axiom [2]. This is the axiom that F satisfies the Rim-Schlessinger condition (RS). Looking at the definition of condition (RS) in Definition 98.5.1 and the discussion in Section 98.15 we see that this means: given any pushout $Y' = Y \amalg_X X'$ of schemes of finite type over S where Y, X, X' are spectra of Artinian local rings, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. This condition makes sense as the schemes X, X', Y, and Y' are in $(\text{Noetherian}/S)_{étale}$ since they are of finite type over S.

Axiom [3]. This is the axiom that every tangent space TF_{k,x_0} is finite dimensional. This makes sense as the tangent spaces TF_{k,x_0} are constructed from evaluations of F at $\operatorname{Spec}(k)$ and $\operatorname{Spec}(k[\epsilon])$ with k a field of finite type over S and hence are obtained by evaluating at objects of the category $(Noetherian/S)_{étale}$.

Axiom [4]. This is axiom that the every formal object is effective. Looking at the discussion in Sections 98.9 and 98.15 we see that this involves evaluating our functor at Noetherian schemes only and hence this condition makes sense for F.

Axiom [5]. This is the axiom stating that F satisfies openness of versality. Recall that this means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'. As before, verifying this only involves evaluating our functor at Noetherian schemes.

0GE7 **Proposition 98.26.1.** *Let S be a locally Noetherian scheme. Let $F : (Noetherian/S)^{opp}_{étale} \to Sets$ be a functor. Assume that*

(1) *$\Delta : F \to F \times F$ is representable (as a transformation of functors, see Categories, Definition 4.6.4),*
(2) *F satisfies axioms [-1], [0], [1], [2], [3], [4], [5] (see above), and*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then there exists a unique algebraic space $F' : (Sch/S)^{opp}_{fppf} \to Sets$ whose restriction to $(Noetherian/S)_{étale}$ is F (see proof for elucidation).

Proof. Recall that the sites $(Sch/S)_{fppf}$ and $(Sch/S)_{étale}$ have the same underlying category, see discussion in Section 98.25. Similarly the sites $(Noetherian/S)_{étale}$ and $(Noetherian/S)_{fppf}$ have the same underlying categories. By axioms [0] and [1] the functor F is a sheaf and limit preserving. Let $F' : (Sch/S)^{opp}_{étale} \to Sets$ be the unique extension of F which is a sheaf (for the étale topology) and which is limit preserving, see Lemma 98.25.2. Then F' satisfies axioms [0] and [1] as given in Section 98.15. By Lemma 98.25.4 we see that $\Delta' : F' \to F' \times F'$ is representable (by schemes). On the other hand, it is immediately clear that F' satisfies axioms [-1], [2], [3], [4], [5] of Section 98.15 as each of these involves only evaluating F' at objects of $(Noetherian/S)_{étale}$ and we've assumed the corresponding conditions for F. Whence F' is an algebraic space by Proposition 98.16.1. □

98.27. Artin's theorem on contractions

0GH7 In this section we will freely use the language of formal algebraic spaces, see Formal Spaces, Section 87.1. Artin's theorem on contractions is one of the two main theorems of Artin's paper [**Art70**]; the first one is his theorem on dilatations which we stated and proved in Algebraization of Formal Spaces, Section 88.29.

0GH8 **Situation 98.27.1.** Let S be a locally Noetherian scheme. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 88.24.1. Recall that $X'_{/T'}$ denotes the formal completion of X' along T', see Formal Spaces, Section 87.14.

In the situation above our goal is to prove that there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that

(1) T' is the inverse image of T by $|f| : |X'| \to |X|$,
(2) $f : X' \to X$ maps U' isomorphically to an open subspace U of X, and
(3) $g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.

Let us say that $(f : X' \to X, T, a)$ is a *solution*.

We will follow Artin's strategy by constructing a functor F on the category of locally Noetherian schemes over S, showing that F is an algebraic space using Proposition 98.26.1, and proving that setting $X = F$ works.

0GH9 **Remark 98.27.2.** In particular, we cannot prove that the desired result is true for every Situation 98.27.1 because we will need to assume the local rings of S are G-rings. If you can prove the result in general or if you have a counter example, please let us know at stacks.project@gmail.com.

In Situation 98.27.1 let V be a locally Noetherian scheme over S. The value of our functor F on V will be all triples

$$(Z, u' : V \setminus Z \to U', \hat{x} : V_{/Z} \to W)$$

satisfying the following conditions

(1) $Z \subset V$ is a closed subset,
(2) $u' : V \setminus Z \to U'$ is a morphism over S,
(3) $\hat{x} : V_{/Z} \to W$ is an adic morphism of formal algebraic spaces over S,
(4) u' and $\hat{x}$ are compatible (see below).

The compatibility condition is the following: pulling back the formal modification g we obtain a formal modification

$$X'_{/T'} \times_{g,W,\hat{x}} V_{/Z} \longrightarrow V_{/Z}$$

See Algebraization of Formal Spaces, Lemma 88.24.4. By the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 88.29.1), there is a unique proper morphism $V' \to V$ of algebraic spaces which is an isomorphism over $V \setminus Z$ such that $V'_{/Z} \to V_{/Z}$ is isomorphic to the displayed arrow. In other words, for some morphism $\hat{x}' : V'_{/Z} \to X'_{/T'}$ we have a cartesian diagram

$$\begin{array}{ccc} V'_{/Z} & \longrightarrow & V_{/Z} \\ {\scriptstyle \hat{x}'}\downarrow & & \downarrow{\scriptstyle \hat{x}} \\ X'_{/T'} & \xrightarrow{g} & W \end{array}$$

of formal algebraic spaces. We will think of $V \setminus Z$ as an open subspace of V' without further mention. The compatibility condition is that there should be a morphism $x' : V' \to X'$ restricting to u' and $\hat{x}$ over $V \setminus Z \subset V'$ and $V'_{/Z}$. In other words, such that the diagram

$$\begin{array}{ccccccc} V \setminus Z & \longrightarrow & V' & \longleftarrow & V'_{/Z} & \longrightarrow & V_{/Z} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle \hat{x}'} & & \downarrow{\scriptstyle \hat{x}} \\ U' & \longrightarrow & X' & \longleftarrow & X'_{/T'} & \xrightarrow{g} & W \end{array}$$

is commutative. Observe that by Algebraization of Formal Spaces, Lemma 88.25.5 the morphism x' is unique if it exists. We will indicate this situation by saying "*$V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$*".

0GID **Remark 98.27.3.** In Situation 98.27.1 let V be a locally Noetherian scheme over S. Let $(Z, u', \hat{x})$ be a triple satisfying (1), (2), and (3) above. We want to explain a way to think about the compatibility condition (4). It will not be mathematically precise as we are going use a fictitious category An_S of analytic spaces over S and a fictitious analytification functor

$$\left\{\begin{array}{c}\text{locally Noetherian formal} \\ \text{algebraic spaces over } S\end{array}\right\} \longrightarrow An_S, \qquad Y \longmapsto Y^{an}$$

For example if $Y = \mathrm{Spf}(k[[t]])$ over $S = \mathrm{Spec}(k)$, then Y^{an} should be thought of as an open unit disc. If $Y = \mathrm{Spec}(k)$, then Y^{an} is a single point. The category An_S should have open and closed immersions and we should be able to take the open complement of a closed. Given Y the morphism $Y_{red} \to Y$ should induces a closed immersion $Y^{an}_{red} \to Y^{an}$. We set $Y^{rig} = Y^{an} \setminus Y^{an}_{red}$ equal to its open complement. If Y is an algebraic space and if $Z \subset Y$ is closed, then the morphism $Y_{/Z} \to Y$ should induce an open immersion $Y^{an}_{/Z} \to Y^{an}$ which in turn should induce an open immersion

$$can : (Y_{/Z})^{rig} \longrightarrow (Y \setminus Z)^{an}$$

Also, given a formal modification $g : Y' \to Y$ of locally Noetherian formal algebraic spaces, the induced morphism $g^{rig} : (Y')^{rig} \to Y^{rig}$ should be an isomorphism. Given An_S and the analytification functor, we can consider the requirement that

$$\begin{array}{ccc} (V_{/Z})^{rig} & \xrightarrow{can} & (V \setminus Z)^{an} \\ {\scriptstyle (g^{rig})^{-1} \circ \hat{x}^{an}} \downarrow & & \downarrow {\scriptstyle (u')^{an}} \\ (X'_{/T'})^{rig} & \xrightarrow{can} & (X' \setminus T')^{an} \end{array}$$

commutes. This makes sense as $g^{rig} : (X'_{T'})^{rig} \to W^{rig}$ is an isomorphism and $U' = X' \setminus T'$. Finally, under some assumptions of faithfulness of the analytification functor, this requirement will be equivalent to the compatibility condition formulated above. We hope this will motivate the reader to think of the compatibility of u' and $\hat{x}$ as the requirement that some maps be equal, rather than asking for the existence of a certain commutative diagram.

0GHA **Lemma 98.27.4.** *In Situation 98.27.1 the rule F that sends a locally Noetherian scheme V over S to the set of triples $(Z, u', \hat{x})$ satisfying the compatibility condition and which sends a a morphism $\varphi : V_2 \to V_1$ of locally Noetherian schemes over S to the map*

$$F(\varphi) : F(V_1) \longrightarrow F(V_2)$$

sending an element $(Z_1, u'_1, \hat{x}_1)$ of $F(V_1)$ to $(Z_2, u'_2, \hat{x}_2)$ in $F(V_2)$ given by

1. *$Z_2 \subset V_2$ is the inverse image of Z_1 by φ,*
2. *u'_2 is the composition of u'_1 and $\varphi|_{V_2 \setminus Z_2} : V_2 \setminus Z_2 \to V_1 \setminus Z_1$,*
3. *$\hat{x}_2$ is the composition of $\hat{x}_1$ and $\varphi_{/Z_2} : V_{2,/Z_2} \to V_{1,/Z_1}$*

is a contravariant functor.

Proof. To see the compatibility condition between u'_2 and $\hat{x}_2$, let $V'_1 \to V_1$, $\hat{x}'_1$, and x'_1 witness the compatibility between u'_1 and $\hat{x}_1$. Set $V'_2 = V_2 \times_{V_1} V'_1$, set $\hat{x}'_2$ equal to the composition of $\hat{x}'_1$ and $V'_{2,/Z_2} \to V'_{1,/Z_1}$, and set x'_2 equal to the composition of x'_1 and $V'_2 \to V'_1$. Then $V'_2 \to V_2$, $\hat{x}'_2$, and x'_2 witness the compatibility between u'_2 and $\hat{x}_2$. We omit the detailed verification. □

0GHB **Lemma 98.27.5.** *In Situation 98.27.1 if there exists a solution $(f : X' \to X, T, a)$ then there is a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S.*

Proof. Let V be a locally Noetherian scheme over S. Let $x : V \to X$ be a morphism over S. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

1. $Z \subset V$ is the inverse image of T by x,
2. $u' : V \setminus Z \to U' = U$ is the restriction of x to $V \setminus Z$,

(3) $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X_{/T}$ with the isomorphism $a : X_{/T} \to W$.

This triple satisfies the compatibility condition because we can take $V' = V \times_{x,X} X'$, we can take $\hat{x}'$ the completion of the projection $x' : V' \to X'$.

Conversely, suppose given an element $(Z, u', \hat{x})$ of $F(V)$. We claim there is a unique morphism $x : V \to X$ compatible with u' and $\hat{x}$. Namely, let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Then Algebraization of Formal Spaces, Proposition 88.26.1 is exactly the result we need to find a unique morphism $x : V \to X$ agreeing with $\hat{x}$ over $V_{/Z}$ and with x' over V' (and a fortiori agreeing with u' over $V \setminus Z$).

We omit the verification that the two constructions above define inverse bijections between their respective domains. □

0GHC **Lemma 98.27.6.** *In Situation 98.27.1 if there exists an algebraic space X locally of finite type over S and a functorial bijection $F(V) = \mathrm{Mor}_S(V, X)$ on the category of locally Noetherian schemes V over S, then X is a solution.*

Proof. We have to construct a proper morphism $f : X' \to X$, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ with properties (1), (2), (3) listed just below Situation 98.27.1.

The discussion in this proof is a bit pedantic because we want to carefully match the underlying categories. In this paragraph we explain how the adventurous reader can proceed less timidly. Namely, the reader may extend our definition of the functor F to all locally Noetherian algebraic spaces over S. Doing so the reader may then conclude that F and X agree as functors on the category of these algebraic spaces, i.e., X represents F. Then one considers the universal object $(T, u', \hat{x})$ in $F(X)$. Then the reader will find that for the triple $X'' \to X$, $\hat{x}'$, x' witnessing the compatibility between u' and $\hat{x}$ the morphism $x' : X'' \to X'$ is an isomorphism and this will produce $f : X' \to X$ by inverting x'. Finally, we already have $T \subset |X|$ and the reader may show that $\hat{x}$ is an isomorphism which can served as the last ingredient namely a.

Denote $h_X(-) = \mathrm{Mor}_S(-, X)$ the functor of points of X restricted to the category $(\mathit{Noetherian}/S)_{\mathit{étale}}$ of Section 98.25. By Limits of Spaces, Remark 70.3.11 the algebraic spaces X and X' are limit preserving. Hence so are the restrictions h_X and $h_{X'}$. To construct f it therefore suffices to construct a transformation $h_{X'} \to h_X = F$, see Lemma 98.25.2. Thus let $V \to S$ be an object of $(\mathit{Noetherian}/S)_{\mathit{étale}}$ and let $\tilde{x} : V \to X'$ be in $h_{X'}(V)$. Then we get an element $(Z, u', \hat{x})$ in $F(V)$ as follows

(1) $Z \subset V$ is the inverse image of T' by $\tilde{x}$,
(2) $u' : V \setminus Z \to U'$ is the restriction of $\tilde{x}$ to $V \setminus Z$,
(3) $\hat{x} : V_{/Z} \to W$ is the composition of $x_{/Z} : V_{/Z} \to X'_{/T'}$ with $g : X'_{/T'} \to W$.

This triple satisfies the compatibility condition: first we always obtain $V' \to V$ and $\hat{x}' : V'_{/Z'} \to X'_{/T'}$ for free (see discussion preceding Lemma 98.27.4). Then we just define $x' : V' \to X'$ to be the composition of $V' \to V$ and the morphism $\tilde{x} : V \to X'$. We omit the verification that this works.

If $\xi : V \to X$ is an étale morphism where V is a scheme, then we obtain $\xi = (Z, u', \hat{x}) \in F(V) = h_X(V) = X(V)$. Of course, if $\varphi : V' \to V$ is a further étale

morphism of schemes, then $(Z, u', \hat{x})$ pulled back to $F(V')$ corresponds to $\xi \circ \varphi$. The closed subset $T \subset |X|$ is just defined as the closed subset such that $\xi : V \to X$ for $\xi = (Z, u', \hat{x})$ pulls T back to Z

Consider Noetherian schemes V over S and a morphism $\xi : V \to X$ corresponding to $(Z, u', \hat{x})$ as above. Then we see that $\xi(V)$ is set theoretically contained in T if and only if $V = Z$ (as topological spaces). Hence we see that $X_{/T}$ agrees with W as a functor. This produces the isomorphism $a : X_{/T} \to W$. (We've omitted a small detail here which is that for the locally Noetherian formal algebraic spaces $X_{/T}$ and W it suffices to check one gets an isomorphism after evaluating on locally Noetherian schemes over S.)

We omit the proof of conditions (1), (2), and (3). □

0GHD **Remark 98.27.7.** In Situation 98.27.1. Let V be a locally Noetherian scheme over S. Let $(Z_i, u'_i, \hat{x}_i) \in F(V)$ for $i = 1, 2$. Let $V'_i \to V$, $\hat{x}'_i$ and x'_i witness the compatibility between u'_i and $\hat{x}_i$ for $i = 1, 2$.

Set $V' = V'_1 \times_V V'_2$. Let $E' \to V'$ denote the equalizer of the morphisms

$$V' \to V'_1 \xrightarrow{x'_1} X' \quad \text{and} \quad V' \to V'_2 \xrightarrow{x'_2} X'$$

Set $Z = Z_1 \cap Z_2$. Let $E_W \to V_{/Z}$ be the equalizer of the morphisms

$$V_{/Z} \to V_{/Z_1} \xrightarrow{\hat{x}_1} W \quad \text{and} \quad V_{/Z} \to V_{/Z_2} \xrightarrow{\hat{x}_2} W$$

Observe that $E' \to V$ is separated and locally of finite type and that E_W is a locally Noetherian formal algebraic space separated over V. The compatibilities between the various morphisms involved show that

(1) $\mathrm{Im}(E' \to V) \cap (Z_1 \cup Z_2)$ is contained in $Z = Z_1 \cap Z_2$,
(2) the morphism $E' \times_V (V \setminus Z) \to V \setminus Z$ is a monomorphism and is equal to the equalizer of the restrictions of u'_1 and u'_2 to $V \setminus (Z_1 \cup Z_2)$,
(3) the morphism $E'_{/Z} \to V_{/Z}$ factors through E_W and the diagram

$$\begin{array}{ccc} E'_{/Z} & \longrightarrow & X'_{/T'} \\ \downarrow & & \downarrow g \\ E_W & \longrightarrow & W \end{array}$$

is cartesian. In particular, the morphism $E'_{/Z} \to E_W$ is a formal modification as the base change of g,
(4) E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ is a triple as in Situation 98.27.1 with base scheme the locally Noetherian scheme V,
(5) given a morphism $\varphi : A \to V$ of locally Noetherian schemes, the following are equivalent
   (a) $(Z_1, u'_1, \hat{x}_1)$ and $(Z_2, u'_2, \hat{x}_2)$ restrict to the same element of $F(A)$,
   (b) $A \setminus \varphi^{-1}(Z) \to V \setminus Z$ factors through $E' \times_V (V \setminus Z)$ and $A_{/\varphi^{-1}(Z)} \to V_{/Z}$ factors through E_W.

We conclude, using Lemmas 98.27.5 and 98.27.6, that if there is a solution $E \to V$ for the triple in (4), then E represents $F \times_{\Delta, F \times F} V$ on the category of locally Noetherian schemes over V.

0GHE **Lemma 98.27.8.** *In Situation 98.27.1 assume given a closed subset $Z \subset S$ such that*

(1) *the inverse image of Z in X' is T',*
(2) *$U' \to S \setminus Z$ is a closed immersion,*
(3) *$W \to S_{/Z}$ is a closed immersion.*

Then there exists a solution $(f : X' \to X, T, a)$ and moreover $X \to S$ is a closed immersion.

Proof. Suppose we have a closed subscheme $X \subset S$ such that $X \cap (S \setminus Z) = U'$ and $X_{/Z} = W$. Then X represents the functor F (some details omitted) and hence is a solution. To find X is clearly a local question on S. In this way we reduce to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the radical ideal cutting out Z. Write $I = (f_1, \ldots, f_r)$. By assumption we are given

(1) the closed immersion $U' \to S \setminus Z$ determines ideals $J_i \subset A[1/f_i]$ such that J_i and J_j generate the same ideal in $A[1/f_if_j]$,
(2) the closed immersion $W \to S_{/Z}$ is the map $\mathrm{Spf}(A^\wedge/J') \to \mathrm{Spf}(A^\wedge)$ for some ideal $J' \subset A^\wedge$ in the I-adic completion $A^\wedge$ of A.

To finish the proof we need to find an ideal $J \subset A$ such that $J_i = J[1/f_i]$ and $J' = JA^\wedge$. By More on Algebra, Proposition 15.90.16 it suffices to show that J_i and J' generate the same ideal in $A^\wedge[1/f_i]$ for all i.

Recall that $A' = H^0(X', \mathcal{O})$ is a finite A-algebra whose formation commutes with flat base change (Cohomology of Spaces, Lemmas 69.20.3 and 69.11.2). Denote $J'' = \mathrm{Ker}(A \to A')$[11]. We have $J_i = J''A[1/f_i]$ as follows from base change to the spectrum of $A[1/f_i]$. Observe that we have a commutative diagram

$$\begin{array}{ccccc} X' & \longleftarrow & X'_{/T'} \times_{S_{/Z}} \mathrm{Spf}(A^\wedge) & = & X'_{/T'} \times_W \mathrm{Spf}(A^\wedge/J') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spf}(A^\wedge) & \longleftarrow & \mathrm{Spf}(A^\wedge/J') \end{array}$$

The middle vertical arrow is the completion of the left vertical arrow along the obvious closed subsets. By the theorem on formal functions we have

$$(A')^\wedge = \Gamma(X' \times_S \mathrm{Spec}(A^\wedge), \mathcal{O}) = \lim H^0(X' \times_S \mathrm{Spec}(A/I^n), \mathcal{O})$$

See Cohomology of Spaces, Theorem 69.22.5. From the diagram we conclude that J' maps to zero in $(A')^\wedge$. Hence $J' \subset J''A^\wedge$. Consider the arrows

$$X'_{/T'} \to \mathrm{Spf}(A^\wedge/J''A^\wedge) \to \mathrm{Spf}(A^\wedge/J') = W$$

We know the composition g is a formal modification (in particular rig-étale and rig-surjective) and the second arrow is a closed immersion (in particular an adic monomorphism). Hence $X'_{/T'} \to \mathrm{Spf}(A^\wedge/J''A^\wedge)$ is rig-surjective and rig-étale, see Algebraization of Formal Spaces, Lemmas 88.21.5 and 88.20.8. Applying Algebraization of Formal Spaces, Lemmas 88.21.14 and 88.21.6 we conclude that $\mathrm{Spf}(A^\wedge/J''A^\wedge) \to W$ is rig-étale and rig-surjective. By Algebraization of Formal Spaces, Lemma 88.21.13 we conclude that $I^nJ''A^\wedge \subset J'$ for some $n > 0$. It follows

[11] Contrary to what the reader may expect, the ideals J and J'' won't agree in general.

that $J''A^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ and we deduce $J_iA^\wedge[1/f_i] = J'A^\wedge[1/f_i]$ for all i as desired. □

0GHF **Lemma 98.27.9.** *In Situation 98.27.1 assume $X' \to S$ and $W \to S$ are separated. Then the diagonal $\Delta : F \to F \times F$ is representable by closed immersions.*

Proof. Combine Lemma 98.27.8 with the discussion in Remark 98.27.7. □

0GHG **Lemma 98.27.10.** *In Situation 98.27.1 the functor F satisfies the sheaf property for all étale coverings of locally Noetherian schemes over S.*

Proof. Omitted. Hint: morphisms may be defined étale locally. □

0GI5 **Lemma 98.27.11.** *In Situation 98.27.1 the functor F is limit preserving: for any directed limit $V = \lim V_\lambda$ of Noetherian affine schemes over S we have $F(V) = \operatorname{colim} F(V_\lambda)$.*

Proof. This is an absurdly long proof. Much of it consists of standard arguments on limits and étale localization. We urge the reader to skip ahead to the last part of the proof where something interesting happens.

Let $V = \lim_{\lambda\in\Lambda} V_i$ be a directed limit of schemes over S with V and V_λ Noetherian and with affine transition morphisms. See Limits, Section 32.2 for material on limits of schemes. We will prove that $\operatorname{colim} F(V_\lambda) \to F(V)$ is bijective.

Proof of injectivity: notation. Let $\lambda \in \Lambda$ and $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$ be elements which restrict to the same element of $F(V)$. Write $\xi_{\lambda,1} = (Z_{\lambda,1}, u'_{\lambda,1}, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_{\lambda,2}, u'_{\lambda,2}, \hat{x}_{\lambda,2})$.

Proof of injectivity: agreement of $Z_{\lambda,i}$. Since $Z_{\lambda,1}$ and $Z_{\lambda,2}$ restrict to the same closed subset of V, we may after increasing i assume $Z_{\lambda,1} = Z_{\lambda,2}$, see Limits, Lemma 32.4.2 and Topology, Lemma 5.14.2. Let us denote the common value $Z_\lambda \subset V_\lambda$, for $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the inverse image in V_μ and and denote Z the inverse image in V. We will use below that $Z = \lim_{\mu\geq\lambda} Z_\mu$ as schemes if we view Z and Z_μ as reduced closed subschemes.

Proof of injectivity: agreement of $u'_{\lambda,i}$. Since U' is locally of finite type over S and since the restrictions of $u'_{\lambda,1}$ and $u'_{\lambda,2}$ to $V \setminus Z$ are the same, we may after increasing λ assume $u'_{\lambda,1} = u'_{\lambda,2}$, see Limits, Proposition 32.6.1. Let us denote the common value u'_λ and denote u' the restriction to $V \setminus Z$.

Proof of injectivity: restatement. At this point we have $\xi_{\lambda,1} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,1})$ and $\xi_{\lambda,2} = (Z_\lambda, u'_\lambda, \hat{x}_{\lambda,2})$. The main problem we face in this part of the proof is to show that the morphisms $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$ become the same after increasing λ.

Proof of injectivity: agreement of $\hat{x}_{\lambda,i}|_{Z_\lambda}$. Consider the morpisms $\hat{x}_{\lambda,1}|_{Z_\lambda}, \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$. These morphisms restrict to the same morphism $Z \to W_{red}$. Since W_{red} is a scheme locally of finite type over S we see using Limits, Proposition 32.6.1 that after replacing λ by a bigger index we may assume $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda} : Z_\lambda \to W_{red}$.

Proof of injectivity: end. Next, we are going to apply the discussion in Remark 98.27.7 to V_λ and the two elements $\xi_{\lambda,1}, \xi_{\lambda,2} \in F(V_\lambda)$. This gives us

(1) $e_\lambda : E'_\lambda \to V_\lambda$ separated and locally of finite type,
(2) $e_\lambda^{-1}(V_\lambda \setminus Z_\lambda) \to V_\lambda \setminus Z_\lambda$ is an isomorphism,
(3) a monomorphism $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ which is the equalizer of $\hat{x}_{\lambda,1}$ and $\hat{x}_{\lambda,2}$,

(4) a formal modification $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$

Assertion (2) holds by assertion (2) in Remark 98.27.7 and the preparatory work we did above getting $u'_{\lambda,1} = u'_{\lambda,2} = u'_\lambda$. Since $Z_\lambda = (V_{\lambda,/Z_\lambda})_{red}$ factors through $E_{W,\lambda}$ because $\hat{x}_{\lambda,1}|_{Z_\lambda} = \hat{x}_{\lambda,2}|_{Z_\lambda}$ we see from Formal Spaces, Lemma 87.27.7 that $E_{W,\lambda} \to V_{\lambda,/Z_\lambda}$ is a closed immersion. Then we see from assertion (4) in Remark 98.27.7 and Lemma 98.27.8 applied to the triple E'_λ, $e_\lambda^{-1}(Z_\lambda)$, $E'_{\lambda,/Z_\lambda} \to E_{W,\lambda}$ over V_λ that there exists a closed immersion $E_\lambda \to V_\lambda$ which is a solution for this triple. Next we use assertion (5) in Remark 98.27.7 which combined with Lemma 98.27.5 says that E_λ is the "equalizer" of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$. In particular, we see that $V \to V_\lambda$ factors through E_λ. Then using Limits, Proposition 32.6.1 once more we find $\mu \geq \lambda$ such that $V_\mu \to V_\lambda$ factors through E_λ and hence the pullbacks of $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$ to V_μ are the same as desired.

Proof of surjectivity: statement. Let $\xi = (Z, u', \hat{x})$ be an element of $F(V)$. We have to find a $\lambda \in \Lambda$ and an element $\xi_\lambda \in F(V_\lambda)$ restricting to ξ.

Proof of surjectivity: the question is étale local. By the unicity proved in the previous part of the proof and by the sheaf property of F in Lemma 98.27.10, the problem is local on V in the étale topology. More precisely, let $v \in V$. We claim it suffices to find an étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ and some λ, some an étale morphism $\tilde{V}_\lambda \to V_\lambda$, and some element $\tilde{\xi}_\lambda \in F(\tilde{V}_\lambda)$ such that $\tilde{V} = \tilde{V}_\lambda \times_{V_\lambda} V$ and $\xi|_U = \tilde{\xi}_\lambda|_U$. We omit a detailed proof of this claim[12].

Proof of surjectivity: rephrasing the problem. Recall that any étale morphism $(\tilde{V}, \tilde{v}) \to (V, v)$ with $\tilde{V}$ affine is the base change of an étale morphism $\tilde{V}_\lambda \to V_\lambda$ with $\tilde{V}_\lambda$ affine for some λ, see for example Topologies, Lemma 34.13.2. Given $\tilde{V}_\lambda$ we have $\tilde{V} = \lim_{\mu \geq \lambda} \tilde{V}_\lambda \times_{V_\lambda} V_\mu$. Hence given $(\tilde{V}, \tilde{v}) \to (V, v)$ étale with $\tilde{V}$ affine, we may replace (V, v) by $(\tilde{V}, \tilde{v})$ and ξ by the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: reduce to base being affine. In particular, suppose $\tilde{S} \subset S$ is an affine open subscheme such that $v \in V$ maps to a point of $\tilde{S}$. Then we may according to the previous paragraph, replace V by $\tilde{V} = \tilde{S} \times_S V$. Of course, if we do this, it suffices to solve the problem for the functor F restricted to the category of locally Noetherian schemes over $\tilde{S}$. This functor is of course the functor associated to the whole situation base changed to $\tilde{S}$. Thus we may and do assume $S = \mathrm{Spec}(R)$ is a Noetherian affine scheme for the rest of the proof.

Proof of surjectivity: easy case. If $v \in V \setminus Z$, then we can take $\tilde{V} = V \setminus Z$. This descends to an open subscheme $\tilde{V}_\lambda \subset V_\lambda$ for some λ by Limits, Lemma 32.4.11. Next, after increasing λ we may assume there is a morphism $u'_\lambda : \tilde{V}_\lambda \to U'$ restricting to u'. Taking $\tilde{\xi}_\lambda = (\emptyset, u'_\lambda, \emptyset)$ gives the desired element of $F(\tilde{V}_\lambda)$.

Proof of surjectivity: hard case and reduction to affine W. The most difficult case comes from considering $v \in Z \subset V$. We claim that we can reduce this to the case where W is an affine formal scheme; we urge the reader to skip this argument[13]. Namely, we can choose an étale morphism $\tilde{W} \to W$ where $\tilde{W}$ is an affine formal

[12] To prove this one assembles a collection of the morphisms $\tilde{V} \to V$ into a finite étale covering and shows that the corresponding morphisms $\tilde{V}_\lambda \to V_\lambda$ form an étale covering as well (after increasing λ). Next one uses the injectivity to see that the elements $\tilde{\xi}_\lambda$ glue (after increasing λ) and one uses the sheaf property for F to descend these elements to an element of $F(V_\lambda)$.

[13] Artin's approach to the proof of this lemma is to work around this and consequently he can avoid proving the injectivity first. Namely, Artin consistently works with a finite affine étale

algebraic space such that the image of v by $\hat{x} : V_{/Z} \to W$ is in the image of $\tilde{W} \to W$ (on reductions). Then the morphisms

$$p : \tilde{W} \times_{W,g} X'_{/T'} \longrightarrow X'_{/T'}$$

and

$$q : \tilde{W} \times_{W,\hat{x}} V_{/Z} \to V_{/Z}$$

are étale morphisms of locally Noetherian formal algebraic spaces. By (an easy case of) Algebraization of Formal Spaces, Theorem 88.27.4 there exists a morphism $\tilde{X}' \to X'$ of algebraic spaces which is locally of finite type, is an isomorphism over U', and such that $\tilde{X}'_{/T'} \to X'_{/T'}$ is isomorphic to p. By Algebraization of Formal Spaces, Lemma 88.28.5 the morphism $\tilde{X}' \to X'$ is étale. Denote $\tilde{T}' \subset |\tilde{X}'|$ the inverse image of T'. Denote $\tilde{U}' \subset \tilde{X}'$ the complementary open subspace. Denote $\tilde{g}' : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$ the formal modification which is the base change of g by $\tilde{W} \to W$. Then we see that

$$\tilde{X}',\ \tilde{T}',\ \tilde{U}',\ \tilde{W},\ \tilde{g} : \tilde{X}'_{/\tilde{T}'} \to \tilde{W}$$

is another example of Situation 98.27.1. Denote $\tilde{F}$ the functor constructed from this triple. There is a transformation of functors

$$\tilde{F} \longrightarrow F$$

constructed using the morphisms $\tilde{X}' \to X'$ and $\tilde{W} \to W$ in the obvious manner; details omitted.

Proof of surjectivity: hard case and reduction to affine W, part 2. By the same theorem as used above, there exists a morphism $\tilde{V} \to V$ of algebraic spaces which is locally of finite type, is an isomorphism over $V \setminus Z$ and such that $\tilde{V}_{/Z} \to V_{/Z}$ is isomorphic to q. Denote $\tilde{Z} \subset \tilde{V}$ the inverse image of Z. By Algebraization of Formal Spaces, Lemmas 88.28.5 and 88.28.3 the morphism $\tilde{V} \to V$ is étale and separated. In particular $\tilde{V}$ is a (locally Noetherian) scheme, see for example Morphisms of Spaces, Proposition 67.50.2. We have the morphism u' which we may view as a morphism

$$\tilde{u}' : \tilde{V} \setminus \tilde{Z} \longrightarrow \tilde{U}'$$

where $\tilde{U}' \subset \tilde{X}'$ is the open mapping isomorphically to U'. We have a morphism

$$\tilde{\hat{x}} : \tilde{V}_{/\tilde{Z}} = \tilde{W} \times_{W,\hat{x}} V_{/Z} \longrightarrow \tilde{W}$$

Namely, here we just use the projection. Thus we have the triple

$$\tilde{\xi} = (\tilde{Z}, \tilde{u}', \tilde{\hat{x}}) \in \tilde{F}(\tilde{V})$$

We omit proving the compatibility condition; hints: if $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$, then one sets $\tilde{V}' = V' \times_V \tilde{V}$ which comes with morphsms $\tilde{\hat{x}}'$ and $\tilde{x}'$ and show this works. The image of $\tilde{\xi}$ under the transformation $\tilde{F} \to F$ is the restriction of ξ to $\tilde{V}$.

Proof of surjectivity: hard case and reduction to affine W, part 3. By our choice of $\tilde{W} \to W$, there is an affine open $\tilde{V}_{open} \subset \tilde{V}$ (we're running out of notation) whose image in V contains our chosen point $v \in V$. Now by the case studied in the next paragraph and the remarks made earlier, we can descend $\tilde{\xi}|_{\tilde{V}_{open}}$ to some element $\tilde{\xi}_\lambda$ of $\tilde{F}$ over $\tilde{V}_{\lambda,open}$ for some étale morphism $\tilde{V}_{\lambda,open} \to V_\lambda$ whose base change to

coverings of all spaces in sight keeping track of the maps between them during the proof. In hindsight that might be preferable to what we do here.

V is $\tilde{V}_{open}$. Applying the transformation of functors $\tilde{F} \to F$ we obtain the element of $F(\tilde{V}_{\lambda,open})$ we were looking for. This reduces us to the case discussed in the next paragraph.

Proof of surjectivity: the case of an affine W. We have $v \in Z \subset V$ and W is an affine formal algebraic space. Recall that

$$\xi = (Z, u', \hat{x}) \in F(V)$$

We may still replace V by an étale neighbourhood of v. In particular we may and do assume V and V_λ are affine.

Proof of surjectivity: descending Z. We can find a λ and a closed subscheme $Z_\lambda \subset V_\lambda$ such that Z is the base change of Z_λ to V. See Limits, Lemma 32.10.1. Warning: we don't know (and in general it won't be true) that Z_λ is a reduced closed subscheme of V_λ. For $\mu \geq \lambda$ denote $Z_\mu \subset V_\mu$ the scheme theoretic inverse image in V_μ. We will use below that $Z = \lim_{\mu \geq \lambda} Z_\mu$ as schemes.

Proof of surjectivity: descending u'. Since U' is locally of finite type over S we may assume after increasing λ that there exists a morphism $u'_\lambda : V_\lambda \setminus Z_\lambda \to U'$ whose restriction to $V \setminus Z$ is u'. See Limits, Proposition 32.6.1. For $\mu \geq \lambda$ we will denote u'_μ the restriction of u'_λ to $V_\mu \setminus Z_\mu$.

Proof of surjectivity: descending a witness. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. Using the same references as above we may assume (after increasing λ) that there exists a morphism $V'_\lambda \to V_\lambda$ of finite type whose base change to V is $V' \to V$. After increasing λ we may assume $V'_\lambda \to V_\lambda$ is proper (Limits, Lemma 32.13.1). Next, we may assume $V'_\lambda \to V_\lambda$ is an isomorphism over $V_\lambda \setminus Z_\lambda$ (Limits, Lemma 32.8.11). Next, we may assume there is a morphism $x'_\lambda : V'_\lambda \to X'$ whose restriction to V' is x'. Increasing λ again we may assume x'_λ agrees with u'_λ over $V_\lambda \setminus Z_\lambda$. For $\mu \geq \lambda$ we denote V'_μ and x'_μ the base change of V'_λ and the restriction of x'_λ.

Proof of surjectivity: algebra. Write $W = \mathrm{Spf}(B)$, $V = \mathrm{Spec}(A)$, and for $\mu \geq \lambda$ write $V_\mu = \mathrm{Spec}(A_\mu)$. Denote $I_\mu \subset A_\mu$ and $I \subset A$ the ideals cutting out Z_μ and Z. Then $I_\lambda A_\mu = I_\mu$ and $I_\lambda A = I$. The morphism $\hat{x}$ determines and is determined by a continuous ring map

$$(\hat{x})^\sharp : B \longrightarrow A^\wedge$$

where $A^\wedge$ is the I-adic completion of A. To finish the proof we have to show that this map descends to a map into $A^\wedge_\mu$ for some sufficiently large μ where $A^\wedge_\mu$ is the I_μ-adic completion of A_μ. This is a nontrivial fact; Artin writes in his paper [**Art70**]: "Since the data (3.5) involve I-adic completions, which do not commute with direct limits, the verification is somewhat delicate. It is an algebraic analogue of a convergence proof in analysis."

Proof of surjectivity: algebra, more rings. Let us denote

$$C_\mu = \Gamma(V'_\mu, \mathcal{O}) \quad \text{and} \quad C = \Gamma(V', \mathcal{O})$$

Observe that $A \to C$ and $A_\mu \to C_\mu$ are finite ring maps as $V' \to V$ and $V'_\mu \to V_\mu$ are proper morphisms, see Cohomology of Spaces, Lemma 69.20.3. Since $V = \lim V_\mu$ and $V' = \lim V'_\mu$ we have

$$A = \mathrm{colim}\, A_\mu \quad \text{and} \quad C = \mathrm{colim}\, C_\mu$$

by Limits, Lemma 32.4.7[14]. For an element $a \in I$, resp. $a \in I_\mu$ the maps $A_a \to C_a$, resp. $(A_\mu)_a \to (C_\mu)_a$ are isomorphisms by flat base change (Cohomology of Spaces, Lemma 69.11.2). Hence the kernel and cokernel of $A \to C$ is supported on $V(I)$ and similarly for $A_\mu \to C_\mu$. We conclude the kernel and cokernel of $A \to C$ are annihilated by a power of I and the kernel and cokernel of $A_\mu \to C_\mu$ are annihilated by a power of I_μ, see Algebra, Lemma 10.62.4.

Proof of surjectivity: algebra, more ring maps. Denote $Z_n \subset V$ the nth infinitesimal neighbourhood of Z and denote $Z_{\mu,n} \subset V_\mu$ the nth infinitesimal neighbourhood of Z_μ. By the theorem on formal functions (Cohomology of Spaces, Theorem 69.22.5) we have

$$C^\wedge = \lim_n H^0(V' \times_V Z_n, \mathcal{O}) \quad \text{and} \quad C_\mu^\wedge = \lim_n H^0(V'_\mu \times_{V_\mu} Z_{\mu,n}, \mathcal{O})$$

where $C^\wedge$ and $C_\mu^\wedge$ are the completion with respect to I and I_μ. Combining the completion of the morphism $x'_\mu : V'_\mu \to X'$ with the morphism $g : X'_{/T'} \to W$ we obtain

$$g \circ x'_{\mu,/Z_\mu} : V'_{\mu,/Z_\mu} = \operatorname{colim} V'_\mu \times_{V_\mu} Z_{\mu,n} \longrightarrow W$$

and hence by the description of the completion $C_\mu^\wedge$ above we obtain a continuous ring homomorphism

$$(g \circ x'_{\mu,/Z_\mu})^\sharp : B \longrightarrow C_\mu^\wedge$$

The fact that $V' \to V$, $\hat{x}'$, x' witnesses the compatibility between u' and $\hat{x}$ implies the commutativity of the following diagram

$$\begin{array}{ccc} C_\mu^\wedge & \longrightarrow & C^\wedge \\ {\scriptstyle (g \circ x'_{\mu,/Z_\mu})^\sharp}\uparrow & & \uparrow \\ B & \xrightarrow{(\hat{x})^\sharp} & A^\wedge \end{array}$$

Proof of surjectivity: more algebra arguments. Recall that the finite A-modules $\operatorname{Ker}(A \to C)$ and $\operatorname{Coker}(A \to C)$ are annihilated by a power of I and similarly the finite A_μ-modules $\operatorname{Ker}(A_\mu \to C_\mu)$ and $\operatorname{Coker}(A_\mu \to C_\mu)$ are annihilated by a power of I_μ. This implies that these modules are equal to their completions. Since I-adic completion on the category of finite A-modules is exact (see Algebra, Section 10.97) it follows that we have

$$\operatorname{Coker}(A^\wedge \to C^\wedge) = \operatorname{Coker}(A \to C)$$

and similarly for kernels and for the maps $A_\mu \to C_\mu$. Of course we also have

$$\operatorname{Ker}(A \to C) = \operatorname{colim} \operatorname{Ker}(A_\mu \to C_\mu) \quad \text{and} \quad \operatorname{Coker}(A \to C) = \operatorname{colim} \operatorname{Coker}(A_\mu \to C_\mu)$$

Recall that $S = \operatorname{Spec}(R)$ is affine. All of the ring maps above are R-algebra homomorphisms as all of the morphisms are morphisms over S. By Algebraization of Formal Spaces, Lemma 88.12.5 we see that B is topologically of finite type over R. Say B is topologically generated by $b_1, \ldots, b_n$. Pick some μ (for example λ) and consider the elements

$$\text{images of } (g \circ x'_{\mu,/Z_\mu})^\sharp(b_1), \ldots, (g \circ x'_{\mu,/Z_\mu})^\sharp(b_n) \text{ in } \operatorname{Coker}(A_\mu \to C_\mu)$$

The image of these elements in $\operatorname{Coker}(\alpha)$ are zero by the commutativity of the square above. Since $\operatorname{Coker}(A \to C) = \operatorname{colim} \operatorname{Coker}(A_\mu \to C_\mu)$ and these cokernels are equal to their completions we see that after increasing μ we may assume these

[14] We don't know that $C_\mu = C_\lambda \otimes_{A_\lambda} A_\mu$ as the various morphisms aren't flat.

images are all zero. This means that the continuous homomorphism $(g \circ x'_{\mu,/Z_\mu})^\sharp$ has image contained in $\mathrm{Im}(A_\mu \to C_\mu)$. Choose elements $a_{\mu,j} \in (A_\mu)^\wedge$ mapping to $(g \circ x'_{\mu,/Z_\mu})^\sharp(b_1)$ in $(C_\mu)^\wedge$. Then $a_{\mu,j} \in A_\mu^\wedge$ and $(\hat{x})^\sharp(b_j) \in A^\wedge$ map to the same element of $C^\wedge$ by the commutativity of the square above. Since $\mathrm{Ker}(A \to C) = \mathrm{colim}\, \mathrm{Ker}(A_\mu \to C_\mu)$ and these kernels are equal to their completions, we may after increasing μ adjust our choices of $a_{\mu,j}$ such that the image of $a_{\mu,j}$ in $A^\wedge$ is equal to $(\hat{x})^\sharp(b_j)$.

Proof of surjectivity: final algebra arguments. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Let $J \subset R[x_1, \ldots, x_n]$ be the ideal consisting of those $h(x_1, \ldots, x_n)$ such that $h(b_1, \ldots, b_n) \in \mathfrak{b}$. Then we get a continuous surjection of topological R-algebras

$$\Phi : R[x_1, \ldots, x_n]^\wedge \longrightarrow B, \quad x_j \longmapsto b_j$$

where the completion on the left hand side is with respect to J. Since $R[x_1, \ldots, x_n]$ is Noetherian we can choose generators $h_1, \ldots, h_m$ for J. By the commutativity of the square above we see that $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is an element of $A_\mu^\wedge$ whose image in $A^\wedge$ is contained in $IA^\wedge$. Namely, the ring map $(\hat{x})^\sharp$ is continuous and $IA^\wedge$ is the ideal of topological nilpotent elements of $A^\wedge$ because $A^\wedge/IA^\wedge = A/I$ is reduced. (See Algebra, Section 10.97 for results on completion in Noetherian rings.) Since $A/I = \mathrm{colim}\, A_\mu/I_\mu$ we conclude that after increasing μ we may assume $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ is in $I_\mu A_\mu^\wedge$. In particular the elements $h_j(a_{\mu,1}, \ldots, a_{\mu,n})$ of $A_\mu^\wedge$ are topologically nilpotent in $A_\mu^\wedge$. Thus we obtain a continuous R-algebra homomorphism

$$\Psi : R[x_1, \ldots, x_n]^\wedge \longrightarrow A_\mu^\wedge, \quad x_j \longmapsto a_{\mu,j}$$

In order to conclude what we want, we need to see if $\mathrm{Ker}(\Phi)$ is annihilated by Ψ. This may not be true, but we can achieve this after increasing μ. Indeed, since $R[x_1, \ldots, x_n]^\wedge$ is Noetherian, we can choose generators $g_1, \ldots, g_l$ of the ideal $\mathrm{Ker}(\Phi)$. Then we see that

$$\Psi(g_1), \ldots, \Psi(g_l) \in \mathrm{Ker}(A_\mu^\wedge \to C_\mu^\wedge) = \mathrm{Ker}(A_\mu \to C_\mu)$$

map to zero in $\mathrm{Ker}(A \to C) = \mathrm{colim}\, \mathrm{Ker}(A_\mu \to C_\mu)$. Hence increasing μ as before we get the desired result.

Proof of surjectivity: mopping up. The continuous ring homomorphism $B \to (A_\mu)^\wedge$ constructed above determines a morphism $\hat{x}_\mu : V_{\mu,/Z_\mu} \to W$. The compatibility of $\hat{x}_\mu$ and u'_μ follows from the fact that the ring map $B \to (A_\mu)^\wedge$ is by construction compatible with the ring map $A_\mu \to C_\mu$. In fact, the compatibility will be witnessed by the proper morphism $V'_\mu \to V_\mu$ and the morphisms x'_μ and $\hat{x}'_\mu = x'_{\mu,/Z_\mu}$ we used in the construction. This finishes the proof. □

0GI6 **Lemma 98.27.12.** *In Situation 98.27.1 the functor F satisfies the Rim-Schlessinger condition (RS).*

Proof. Recall that the condition only involves the evaluation $F(V)$ of the functor F on schemes V over S which are spectra of Artinian local rings and the restriction maps $F(V_2) \to F(V_1)$ for morphisms $V_1 \to V_2$ of schemes over S which are spectra of Artinian local rings. Thus let V/S be the spetruim of an Artinian local ring. If $\xi = (Z, u', \hat{x}) \in F(V)$ then either $Z = \emptyset$ or $Z = V$ (set theoretically). In the first case we see that $\hat{x}$ is a morphism from the empty formal algebraic space into W.

In the second case we see that u' is a morphism from the empty scheme into X' and we see that $\hat{x} : V \to W$ is a morphism into W. We conclude that

$$F(V) = U'(V) \amalg W(V)$$

and moreover for $V_1 \to V_2$ as above the induced map $F(V_2) \to F(V_1)$ is compatible with this decomposition. Hence it suffices to prove that both U' and W satisfy the Rim-Schlessinger condition. For U' this follows from Lemma 98.5.2. To see that it is true for W, we write $W = \operatorname{colim} W_n$ as in Formal Spaces, Lemma 87.20.11. Say $V = \operatorname{Spec}(A)$ with $(A, \mathfrak{m})$ an Artinian local ring. Pick $n \geq 1$ such that $\mathfrak{m}^n = 0$. Then we have $W(V) = W_n(V)$. Hence we see that the Rim-Schlessinger condition for W follows from the Rim-Schlessinger condition for W_n for all n (which in turn follows from Lemma 98.5.2). □

0GI7 **Lemma 98.27.13.** *In Situation 98.27.1 the tangent spaces of the functor F are finite dimensional.*

Proof. In the proof of Lemma 98.27.12 we have seen that $F(V) = U'(V) \amalg W(V)$ if V is the spectrum of an Artinian local ring. The tangent spaces are computed entirely from evaluations of F on such schemes over S. Hence it suffices to prove that the tangent spaces of the functors U' and W are finite dimensional. For U' this follows from Lemma 98.8.1. Write $W = \operatorname{colim} W_n$ as in the proof of Lemma 98.27.12. Then we see that the tangent spaces of W are equal to the tangent spaces of W_2, as to get at the tangent space we only need to evaluate W on spectra of Artinian local rings $(A, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$. Then again we see that the tangent spaces of W_2 have finite dimension by Lemma 98.8.1. □

0GI8 **Lemma 98.27.14.** *In Situation 98.27.1 assume $X' \to S$ is separated. Then every formal object for F is effective.*

Proof. A formal object $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\operatorname{Spec}(R/\mathfrak{m}^n))$ for all n such that $\xi_{n+1}|_{\operatorname{Spec}(R/\mathfrak{m}^n)} = \xi_n$. By the discussion in the proof of Lemma 98.27.12 we see that either ξ is a formal object of U' or a formal object of W. In the first case we see that ξ is effective by Lemma 98.9.5. The second case is the interesting case. Set $V = \operatorname{Spec}(R)$. We will construct an element $(Z, u', \hat{x}) \in F(V)$ whose image in $F(\operatorname{Spec}(R/\mathfrak{m}^n))$ is ξ_n for all $n \geq 1$.

We may view the collection of elements ξ_n as a morphism

$$\xi : \operatorname{Spf}(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Observe that ξ is *not* an adic morphism in general. To fix this, let $I \subset R$ be the ideal corresponding to the formal closed subspace

$$\operatorname{Spf}(R) \times_{\xi, W} W_{red} \subset \operatorname{Spf}(R)$$

Note that $I \subset \mathfrak{m}_R$. Set $Z = V(I) \subset V = \operatorname{Spec}(R)$. Since R is $\mathfrak{m}_R$-adically complete it is a fortiori I-adically complete (Algebra, Lemma 10.96.8). Moreover, we claim that for each $n \geq 1$ the morphism

$$\xi|_{\operatorname{Spf}(R/I^n)} : \operatorname{Spf}(R/I^n) \longrightarrow W$$

actually comes from a morphism

$$\xi'_n : \operatorname{Spec}(R/I^n) \longrightarrow W$$

Namely, this follows from writing $W = \operatorname{colim} W_n$ as in the proof of Lemma 98.27.12, noticing that $\xi|_{\operatorname{Spf}(R/I^n)}$ maps into W_n, and applying Formal Spaces, Lemma 87.33.3 to algebraize this to a morphism $\operatorname{Spec}(R/I^n) \to W_n$ as desired. Let us denote $\operatorname{Spf}'(R) = V_{/Z}$ the formal spectrum of R endowed with the I-adic topology – equivalently the formal completion of V along Z. Using the morphisms ξ'_n we obtain an adic morphism

$$\hat{x} = (\xi'_n) : \operatorname{Spf}'(R) \longrightarrow W$$

of locally Noetherian formal algebraic spaces over S. Consider the base change

$$\operatorname{Spf}'(R) \times_{\hat{x},W,g} X'_{/T'} \longrightarrow \operatorname{Spf}'(R)$$

This is a formal modification by Algebraization of Formal Spaces, Lemma 88.24.4. Hence by the main theorem on dilatations (Algebraization of Formal Spaces, Theorem 88.29.1) we obtain a proper morphism

$$V' \longrightarrow V = \operatorname{Spec}(R)$$

which is an isomorphism over $\operatorname{Spec}(R) \setminus V(I)$ and whose completion recovers the formal modification above, in other words

$$V' \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R/I^n) = \operatorname{Spec}(R/I^n) \times_{\xi'_n,W,g} X'_{/T'}$$

This in particular tells us we have a compatible system of morphisms

$$V' \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R/I^n) \longrightarrow X' \times_S \operatorname{Spec}(R/I^n)$$

Hence by Grothendieck's algebraization theorem (in the form of More on Morphisms of Spaces, Lemma 76.43.3) we obtain a morphism

$$x' : V' \to X'$$

over S recovering the morphisms displayed above. Then finally setting $u' : V \setminus Z \to X'$ the restriction of x' to $V \setminus Z \subset V'$ gives the third component of our desired element $(Z, u', \hat{x}) \in F(V)$. □

0GI9 **Lemma 98.27.15.** *Let S be a locally Noetherian scheme. Let V be a scheme locally of finite type over S. Let $Z \subset V$ be closed. Let W be a locally Noetherian formal algebraic space over S such that W_{red} is locally of finite type over S. Let $g : V_{/Z} \to W$ be an adic morphism of formal algebraic spaces over S. Let $v \in V$ be a closed point such that g is versal at v (as in Section 98.15). Then after replacing V by an open neighbourhood of v the morphism g is smooth (see proof).*

Proof. Since g is adic it is representable by algebraic spaces (Formal Spaces, Section 87.23). Thus by saying g is smooth we mean that g should be smooth in the sense of Bootstrap, Definition 80.4.1.

Write $W = \operatorname{colim} W_n$ as in Formal Spaces, Lemma 87.20.11. Set $V_n = V_{/Z} \times_{\hat{x},W} W_n$. Then V_n is a closed subscheme with underlying set Z. Smoothness of $V \to W$ is equivalent to the smoothness of all the morphisms $V_n \to W_n$ (this holds because any morphism $T \to W$ with T a quasi-compact scheme factors through W_n for some n). We know that the morphism $V_n \to W_n$ is smooth at v by Lemma 98.12.6[15]. Of course this means that given any n we can shrink V such that $V_n \to W_n$ is smooth. The problem is to find an open which works for all n at the same time.

[15]The lemma applies since the diagonal of W is representable by algebraic spaces and locally of finite type, see Formal Spaces, Lemma 87.15.5 and we have seen that W has (RS) in the proof of Lemma 98.27.12.

The question is local on V, hence we may assume $S = \mathrm{Spec}(R)$ and $V = \mathrm{Spec}(A)$ are affine.

In this paragraph we reduce to the case where W is an affine formal algebraic space. Choose an affine formal scheme W' and an étale morphism $W' \to W$ such that the image of v in W_{red} is in the image of $W'_{red} \to W_{red}$. Then $V_{/Z} \times_{g,W} W' \to V_{/Z}$ is an adic étale morphism of formal algebraic spaces over S and $V_{/Z} \times_{g,W} W'$ is an affine formal algebraic space. By Algebraization of Formal Spaces, Lemma 88.25.1 there exists an étale morphism $\varphi : V' \to V$ of affine schemes such that the completion of V' along $Z' = \varphi^{-1}(Z)$ is isomorphic to $V_{/Z} \times_{g,W} W'$ over $V_{/Z}$. Observe that v is the image of some $v' \in V'$. Since smoothness is preserved under base change we see that $V'_n \to W'_n$ is smooth for all n. In the next paragraph we show that after replacing V' by an open neighbourhood of v' the morphisms $V'_n \to W'_n$ are smooth for all n. Then, after we replace V by the open image of $V' \to V$, we obtain that $V_n \to W_n$ is smooth by étale descent of smoothness. Some details omitted.

Assume $S = \mathrm{Spec}(R)$, $V = \mathrm{Spec}(A)$, $Z = V(I)$, and $W = \mathrm{Spf}(B)$. Let v correspond to the maximal ideal $I \subset \mathfrak{m} \subset A$. We are given an adic continuous R-algebra homomorphism

$$B \longrightarrow A^\wedge$$

Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements (this is the maximal ideal of definition of the Noetherian adic topological ring B). Observe that $\mathfrak{b}A^\wedge$ and $IA^\wedge$ are both ideals of definition of the Noetherian adic ring $A^\wedge$. Also, $\mathfrak{m}A^\wedge$ is a maximal ideal of $A^\wedge$ containing both $\mathfrak{b}A^\wedge$ and $IA^\wedge$. We are given that

$$B_n = B/\mathfrak{b}^n \to A^\wedge/\mathfrak{b}^n A^\wedge = A_n$$

is smooth at $\mathfrak{m}$ for all n. By the discussion above we may and do assume that $B_1 \to A_1$ is a smooth ring map. Denote $\mathfrak{m}_1 \subset A_1$ the maximal ideal corresponding to $\mathfrak{m}$. Since smoothness implies flatness, we see that: for all $n \geq 1$ the map

$$\mathfrak{b}^n/\mathfrak{b}^{n+1} \otimes_{B_1} (A_1)_{\mathfrak{m}_1} \longrightarrow \left(\mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1}A^\wedge\right)_{\mathfrak{m}_1}$$

is an isomorphism (see Algebra, Lemma 10.99.9). Consider the Rees algebra

$$B' = \bigoplus\nolimits_{n\geq 0} \mathfrak{b}^n/\mathfrak{b}^{n+1}$$

which is a finite type graded algebra over the Noetherian ring B_1 and the Rees algebra

$$A' = \bigoplus\nolimits_{n\geq 0} \mathfrak{b}^n A^\wedge/\mathfrak{b}^{n+1}A^\wedge$$

which is a a finite type graded algebra over the Noetherian ring A_1. Consider the homomorphism of graded A_1-algebras

$$\Psi : B' \otimes_{B_1} A_1 \longrightarrow A'$$

By the above this map is an isomorphism after localizing at the maximal ideal $\mathfrak{m}_1$ of A_1. Hence $\mathrm{Ker}(\Psi)$, resp. $\mathrm{Coker}(\Psi)$ is a finite module over $B' \otimes_{B_1} A_1$, resp. A' whose localization at $\mathfrak{m}_1$ is zero. It follows that after replacing A_1 (and correspondingly A) by a principal localization we may assume Ψ is an isomorphism. (This is the key step of the proof.) Then working backwards we see that $B_n \to A_n$ is flat, see Algebra, Lemma 10.99.9. Hence $A_n \to B_n$ is smooth (as a flat ring map with smooth fibres, see Algebra, Lemma 10.137.16) and the proof is complete. $\square$

0GIA **Lemma 98.27.16.** *In Situation 98.27.1 the functor F satisfies openness of versality.*

Proof. We have to show the following. Given a scheme V locally of finite type over S, given $\xi \in F(V)$, and given a finite type point $v_0 \in V$ such that ξ is versal at v_0, after replacing V by an open neighbourhood of v_0 we have that ξ is versal at every finite type point of V. Write $\xi = (Z, u', \hat{x})$.

First case: $v_0 \notin Z$. Then we can first replace V by $V \setminus Z$. Hence we see that $\xi = (\emptyset, u', \emptyset)$ and the morphism $u' : V \to X'$ is versal at v_0. By More on Morphisms of Spaces, Lemma 76.20.1 this means that $u' : V \to X'$ is smooth at v_0. Since the set of a points where a morphism is smooth is open, we can after shrinking V assume u' is smooth. Then the same lemma tells us that ξ is versal at every point as desired.

Second case: $v_0 \in Z$. Write $W = \operatorname{colim} W_n$ as in Formal Spaces, Lemma 87.20.11. By Lemma 98.27.15 we may assume $\hat{x} : V_{/Z} \to W$ is a smooth morphism of formal algebraic spaces. It follows immediately that $\xi = (Z, u', \hat{x})$ is versal at all finite type points of Z. Let $V' \to V$, $\hat{x}'$, and x' witness the compatibility between u' and $\hat{x}$. We see that $\hat{x}' : V'_{/Z} \to X'_{/T'}$ is smooth as a base change of $\hat{x}$. Since $\hat{x}'$ is the completion of $x' : V' \to X'$ this implies that $x' : V' \to X'$ is smooth at all points of $(V' \to V)^{-1}(Z) = |x'|^{-1}(T') \subset |V'|$ by the already used More on Morphisms of Spaces, Lemma 76.20.1. Since the set of smooth points of a morphism is open, we see that the closed set of points $B \subset |V'|$ where x' is not smooth does not meet $(V' \to V)^{-1}(Z)$. Since $V' \to V$ is proper and hence closed, we see that $(V' \to V)(B) \subset V$ is a closed subset not meeting Z. Hence after shrinking V we may assume $B = \emptyset$, i.e., x' is smooth. By the discussion in the previous paragraph this exactly means that ξ is versal at all finite type points of V not contained in Z and the proof is complete. □

Here is the final result.

0GIB **Theorem 98.27.17.** *Let S be a locally Noetherian scheme such that $\mathcal{O}_{S,s}$ is a G-ring for all finite type points $s \in S$. Let X' be an algebraic space locally of finite type over S. Let $T' \subset |X'|$ be a closed subset. Let W be a locally Noetherian formal algebraic space over S with W_{red} locally of finite type over S. Finally, we let* [**Art70**, Theorem 3.1]

$$g : X'_{/T'} \longrightarrow W$$

be a formal modification, see Algebraization of Formal Spaces, Definition 88.24.1. If X' and W are separated[16] *over S, then there exists a proper morphism $f : X' \to X$ of algebraic spaces over S, a closed subset $T \subset |X|$, and an isomorphism $a : X_{/T} \to W$ of formal algebraic spaces such that*

1. *T' is the inverse image of T by $|f| : |X'| \to |X|$,*
2. *$f : X' \to X$ maps $X' \setminus T'$ isomorphically to $X \setminus T$, and*
3. *$g = a \circ f_{/T}$ where $f_{/T} : X'_{/T'} \to X_{/T}$ is the induced morphism.*

In other words, $(f : X' \to X, T, a)$ is a solution as defined earlier in this section.

Proof. Let F be the functor constructed using X', T', W, g in this section. By Lemma 98.27.6 it suffices to show that F corresponds to an algebraic space X locally of finite type over S. In order to do this, we will apply Proposition 98.26.1.

[16] See Remark 98.27.18.

Namely, by Lemma 98.27.9 the diagonal of F is representable by closed immersions and by Lemmas 98.27.10, 98.27.11, 98.27.12, 98.27.13, 98.27.14, and 98.27.16 we have axioms [0], [1], [2], [3], [4], and [5]. □

0GIC **Remark 98.27.18.** The proof of Theorem 98.27.17 uses that X' and W are separated over S in two places. First, the proof uses this in showing $\Delta : F \to F \times F$ is representable by algebraic spaces. This use of the assumption can be entirely avoided by proving that Δ is representable by applying the theorem in the separated case to the triples E', $(E' \to V)^{-1}Z$, and $E'_{/Z} \to E_W$ found in Remark 98.27.7 (this is the usual bootstrap procedure for the diagonal). Thus the proof of Lemma 98.27.14 is the only place in our proof of Theorem 98.27.17 where we really need to use that $X' \to S$ is separated. The reader checks that we use the assumption only to obtain the morphism $x' : V' \to X'$. The existence of x' can be shown, using results in the literature, if $X' \to S$ is quasi-separated, see More on Morphisms of Spaces, Remark 76.43.4. We conclude the theorem holds as stated with "separated" replaced by "quasi-separated". If we ever need this we will precisely state and carefully prove this here.

98.28. Other chapters

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CHAPTER 99

Quot and Hilbert Spaces

05X4

99.1. Introduction

05X5 As initially conceived, the purpose of this chapter was to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. This material, in the setting of schemes, is covered in Grothendieck's lectures in the séminair Bourbaki, see [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. For projective schemes the Quot and Hilbert schemes live inside Grassmannians of spaces of sections of suitable very ample invertible sheaves, and this provides a method of construction for these schemes. Our approach is different: we use Artin's axioms to prove Quot and Hilb are algebraic spaces.

Upon further consideration, it turned out to be more convenient for the development of theory in the Stacks project, to start the discussion with the stack $\mathcal{C}oh_{X/B}$ of coherent sheaves (with proper support over the base) as introduced in [**Lie06b**]. For us $f : X \to B$ is a morphism of algebraic spaces satisfying suitable technical conditions, although this can be generalized (see below). Given modules $\mathcal{F}$ and $\mathcal{G}$ on X, under suitably hypotheses, the functor $T/B \mapsto \mathrm{Hom}_{X_T}(\mathcal{F}_T, \mathcal{G}_T)$ is an algebraic space $Hom(\mathcal{F}, \mathcal{G})$ over B. See Section 99.3. The subfunctor $Isom(\mathcal{F}, \mathcal{G})$ of isomorphisms is shown to be an algebraic space in Section 99.4. This is used in the next sections to show the diagonal of the stack $\mathcal{C}oh_{X/B}$ is representable. We prove $\mathcal{C}oh_{X/B}$ is an algebraic stack in Section 99.5 when $X \to B$ is flat and in Section 99.6 in general. Please see the introduction of this section for pointers to the literature.

Having proved this, it is rather straightforward to prove that $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, and $\mathrm{Pic}_{X/B}$ are algebraic spaces and that $\mathcal{P}ic_{X/B}$ is an algebraic stack. See Sections 99.8, 99.9, 99.11, and 99.10.

In the usual manner we deduce that the functor $Mor_B(Z, X)$ of relative morphisms is an algebraic space (under suitable hypotheses) in Section 99.12.

In Section 99.13 we prove that the stack in groupoids

$$\mathcal{S}paces'_{fp,flat,proper}$$

parametrizing flat families of proper algebraic spaces satisfies all of Artin's axioms (including openness of versality) except for formal effectiveness. We've chosen the very awkward notation for this stack intentionally, because the reader should be careful in using its properties.

In Section 99.14 we prove that the stack $\mathcal{P}olarized$ parametrizing flat families of polarized proper algebraic spaces is an algebraic stack. Because of our work on flat

families of proper algebraic spaces, this comes down to proving formal effectiveness for polarized schemes which is often known as Grothendieck's algebraization theorem.

In Section 99.15 we prove that the stack $\mathcal{C}urves$ parametrizing families of curves is algebraic.

In Section 99.16 we study moduli of complexes on a proper morphism and we obtain an algebraic stack $\mathcal{C}omplexes_{X/B}$. The idea of the statement and the proof are taken from [**Lie06a**].

What is not in this chapter? There is almost no discussion of the properties the resulting moduli spaces and moduli stacks possess (beyond their algebraicity); to read about this we refer to Moduli Stacks, Section 108.1. In most of the results discussed, we can generalize the constructions by considering a morphism $\mathcal{X} \to \mathcal{B}$ of algebraic stacks instead of a morphism $X \to B$ of algebraic space. We will discuss this (insert future reference here). In the case of Hilbert spaces there is a more general notion of "Hilbert stacks" which we will discuss in a separate chapter, see (insert future reference here).

99.2. Conventions

05X6 We have intentionally placed this chapter, as well as the chapters "Examples of Stacks", "Sheaves on Algebraic Stacks", "Criteria for Representability", and "Artin's Axioms" before the general development of the theory of algebraic stacks. The reason for this is that starting with the next chapter (see Properties of Stacks, Section 100.2) we will no longer distinguish between a scheme and the algebraic stack it gives rise to. Thus our language will become more flexible and easier for a human to parse, but also less precise. These first few chapters, including the initial chapter "Algebraic Stacks", lay the groundwork that later allow us to ignore some of the very technical distinctions between different ways of thinking about algebraic stacks. But especially in the chapters "Artin's Axioms" and "Criteria of Representability" we need to be very precise about what objects exactly we are working with, as we are trying to show that certain constructions produce algebraic stacks or algebraic spaces.

Unfortunately, this means that some of the notation, conventions and terminology is awkward and may seem backwards to the more experienced reader. We hope the reader will forgive us!

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

99.3. The Hom functor

08JS In this section we study the functor of homomorphisms defined below.

08JT **Situation 99.3.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $\mathcal{F}_T$ and $\mathcal{G}_T$ the base changes of $\mathcal{F}$ and $\mathcal{G}$ to T, in other words, the pullbacks via the projection morphism $X_T = X \times_B T \to X$. We consider the functor

08JU (99.3.1.1) $$Hom(\mathcal{F}, \mathcal{G}) : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

In Situation 99.3.1 we sometimes think of $Hom(\mathcal{F}, \mathcal{G})$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $Hom(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $Hom(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module map where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

08JV **Lemma 99.3.2.** *In Situation 99.3.1 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = u_{T_i}$ and $\mathcal{G}_i = \mathcal{G}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3. Thus a family of maps $u_i : \mathcal{F}_i \to \mathcal{G}_i$ such that u_i and u_j restrict to the same map on $X_{T_i \times_T T_j}$ comes from a unique map $u : \mathcal{F}_T \to \mathcal{G}_T$ by descent (Descent on Spaces, Proposition 74.4.1). □

Sanity check: *Hom* sheaf plays the same role among algebraic spaces over S.

0D3S **Lemma 99.3.3.** *In Situation 99.3.1. Let T be an algebraic space over S. We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Hom(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \to B, u : \mathcal{F}_T \to \mathcal{G}_T\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B,h} T$.

Proof. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $t, s : R \to U$.

Let $v : T \to Hom(\mathcal{F}, \mathcal{G})$ be a natural transformation. Then $v(p)$ corresponds to a pair (h_U, u_U) over U. As v is a transformation of functors we see that the pullbacks of (h_U, u_U) by s and t agree. Since $T = U/R$ (Spaces, Lemma 65.9.1), we obtain a morphism $h : T \to B$ such that $h_U = h \circ p$. Then $\mathcal{F}_U$ is the pullback of $\mathcal{F}_T$ to X_U and similarly for $\mathcal{G}_U$. Hence u_U descends to a $\mathcal{O}_{X_T}$-module map $u : \mathcal{F}_T \to \mathcal{G}_T$ by Descent on Spaces, Proposition 74.4.1.

Conversely, let (h, u) be a pair over T. Then we get a natural transformation $v : T \to Hom(\mathcal{F}, \mathcal{G})$ by sending a morphism $a : T' \to T$ where T' is a scheme to $(h \circ a, a^*u)$. We omit the verification that the construction of this and the previous paragraph are mutually inverse. □

08JW **Remark 99.3.4.** In Situation 99.3.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$, $\mathcal{G}'$ the pullback of $\mathcal{F}$, $\mathcal{G}$ to X'. Then we obtain a functor $Hom(\mathcal{F}', \mathcal{G}') : (Sch/B')^{opp} \to Sets$ associated to the base change $f' : X' \to B'$. For a scheme T over B' it is clear that we have

$$Hom(\mathcal{F}', \mathcal{G}')(T) = Hom(\mathcal{F}, \mathcal{G})(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

08K3 **Lemma 99.3.5.** *In Situation 99.3.1 let $\{X_i \to X\}_{i\in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. Similarly define $\mathcal{G}_i$ and $\mathcal{G}_{ijk}$. For every scheme T over B the diagram*

$$Hom(\mathcal{F}, \mathcal{G})(T) \longrightarrow \prod_i Hom(\mathcal{F}_i, \mathcal{G}_i)(T) \xrightarrow[pr_1^*]{pr_0^*} \prod_{i,j,k} Hom(\mathcal{F}_{ijk}, \mathcal{G}_{ijk})(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $u_i : \mathcal{F}_{i,T} \to \mathcal{G}_{i,T}$ be an element in the equalizer of pr_0^* and pr_1^*. Since the base change of an fppf covering is an fppf covering (Topologies on Spaces, Lemma 73.7.3) we see that $\{X_{i,T} \to X_T\}_{i\in I}$ and $\{X_{ijk,T} \to X_{i,T} \times_{X_T} X_{j,T}\}$ are fppf coverings. Applying Descent on Spaces, Proposition 74.4.1 we first conclude that u_i and u_j restrict to the same morphism over $X_{i,T} \times_{X_T} X_{j,T}$, whereupon a second application shows that there is a unique morphism $u : \mathcal{F}_T \to \mathcal{G}_T$ restricting to u_i for each i. This finishes the proof. □

08K4 **Lemma 99.3.6.** *In Situation 99.3.1. If $\mathcal{F}$ is of finite presentation and f is quasi-compact and quasi-separated, then $Hom(\mathcal{F}, \mathcal{G})$ is limit preserving.*

Proof. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine B-schemes. We have to show that

$$Hom(\mathcal{F}, \mathcal{G})(T) = \operatorname{colim} Hom(\mathcal{F}, \mathcal{G})(T_i)$$

Pick $0 \in I$. We may replace B by T_0, X by X_{T_0}, $\mathcal{F}$ by $\mathcal{F}_{T_0}$, $\mathcal{G}$ by $\mathcal{G}_{T_0}$, and I by $\{i \in I \mid i \geq 0\}$. See Remark 99.3.4. Thus we may assume $B = \operatorname{Spec}(R)$ is affine.

When B is affine, then X is quasi-compact and quasi-separated. Choose a surjective étale morphism $U \to X$ where U is an affine scheme (Properties of Spaces, Lemma 66.6.3). Since X is quasi-separated, the scheme $U \times_X U$ is quasi-compact and we may choose a surjective étale morphism $V \to U \times_X U$ where V is an affine scheme. Applying Lemma 99.3.5 we see that $Hom(\mathcal{F}, \mathcal{G})$ is the equalizer of two maps between

$$Hom(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{and} \quad Hom(\mathcal{F}|_V, \mathcal{G}|_V)$$

This reduces us to the case that X is affine.

In the affine case the statement of the lemma reduces to the following problem: Given a ring map $R \to A$, two A-modules M, N and a directed system of R-algebras $C = \operatorname{colim} C_i$. When is it true that the map

$$\operatorname{colim} \operatorname{Hom}_{A\otimes_R C_i}(M \otimes_R C_i, N \otimes_R C_i) \longrightarrow \operatorname{Hom}_{A\otimes_R C}(M \otimes_R C, N \otimes_R C)$$

is bijective? By Algebra, Lemma 10.127.5 this holds if $M \otimes_R C$ is of finite presentation over $A \otimes_R C$, i.e., when M is of finite presentation over A. □

08K5 **Lemma 99.3.7.** *Let S be a scheme. Let B be an algebraic space over S. Let $i : X' \to X$ be a closed immersion of algebraic spaces over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\mathcal{G}'$ be a quasi-coherent $\mathcal{O}_{X'}$-module. Then*

$$Hom(\mathcal{F}, i_*\mathcal{G}') = Hom(i^*\mathcal{F}, \mathcal{G}')$$

as functors on (Sch/B).

Proof. Let $g : T \to B$ be a morphism where T is a scheme. Denote $i_T : X'_T \to X_T$ the base change of i. Denote $h : X_T \to X$ and $h' : X'_T \to X'$ the projections. Observe that $(h')^* i^* \mathcal{F} = i_T^* h^* \mathcal{F}$. As a closed immersion is affine (Morphisms of Spaces, Lemma 67.20.6) we have $h^* i_* \mathcal{G} = i_{T,*}(h')^* \mathcal{G}$ by Cohomology of Spaces, Lemma 69.11.1. Thus we have

$$\begin{aligned} Hom(\mathcal{F}, i_*\mathcal{G}')(T) &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, h^* i_* \mathcal{G}') \\ &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, i_{T,*}(h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}(i_T^* h^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}((h')^* i^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= Hom(i^*\mathcal{F}, \mathcal{G}')(T) \end{aligned}$$

as desired. The middle equality follows from the adjointness of the functors $i_{T,*}$ and i_T^*. □

08JX **Lemma 99.3.8.** *Let S be a scheme. Let B be an algebraic space over S. Let K be a pseudo-coherent object of $D(\mathcal{O}_B)$.*

(1) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaf $H^{-1}(Lg^*K)$ is zero, then the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, H^0(Lg^*K))$$

is an algebraic space affine and of finite presentation over B.

(2) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaves $H^i(Lg^*K)$ are zero for $i < 0$, then K is perfect, K locally has tor amplitude in $[0, b]$, and the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, Lg^*K)$$

is an algebraic space affine and of finite presentation over B.

Proof. Under the assumptions of (2) we have $H^0(T, Lg^*K) = H^0(T, H^0(Lg^*K))$. Let us prove that the rule $T \mapsto H^0(T, H^0(Lg^*K))$ satisfies the sheaf property for the fppf topology. To do this assume we have an fppf covering $\{h_i : T_i \to T\}$ of a scheme $g : T \to B$ over B. Set $g_i = g \circ h_i$. Note that since h_i is flat, we have $Lh_i^* = h_i^*$ and h_i^* commutes with taking cohomology. Hence

$$H^0(T_i, H^0(Lg_i^*K)) = H^0(T_i, H^0(h_i^* Lg^*K)) = H^0(T, h_i^* H^0(Lg^*K))$$

Similarly for the pullback to $T_i \times_T T_j$. Since Lg^*K is a pseudo-coherent complex on T (Cohomology on Sites, Lemma 21.45.3) the cohomology sheaf $\mathcal{F} = H^0(Lg^*K)$ is quasi-coherent (Derived Categories of Spaces, Lemma 75.13.6). Hence by Descent on Spaces, Proposition 74.4.1 we see that

$$H^0(T, \mathcal{F}) = \mathrm{Ker}(\prod H^0(T_i, h_i^*\mathcal{F}) \to \prod H^0(T_i \times_T T_j, (T_i \times_T T_j \to T)^*\mathcal{F}))$$

In this way we see that the rules in (1) and (2) satisfy the sheaf property for fppf coverings. This means we may apply Bootstrap, Lemma 80.11.2 to see it suffices to prove the representability étale locally on B. Moreover, we may check whether the end result is affine and of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. Hence we may assume that B is an affine scheme.

Assume $B = \mathrm{Spec}(A)$ is an affine scheme. By the results of Derived Categories of Spaces, Lemmas 75.13.6, 75.4.2, and 75.13.2 we deduce that in the rest of the proof we may think of K as a perfect object of the derived category of complexes of modules on B in the Zariski topology. By Derived Categories of Schemes, Lemmas 36.10.1, 36.3.5, and 36.10.2 we can find a pseudo-coherent complex $M^\bullet$ of A-modules such that K is the corresponding object of $D(\mathcal{O}_B)$. Our assumption on pullbacks implies that $M^\bullet \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p})$ has vanishing H^{-1} for all primes $\mathfrak{p} \subset A$. By More on Algebra, Lemma 15.77.4 we can write

$$M^\bullet = \tau_{\geq 0} M^\bullet \oplus \tau_{\leq -1} M^\bullet$$

with $\tau_{\geq 0} M^\bullet$ perfect with Tor amplitude in $[0, b]$ for some $b \geq 0$ (here we also have used More on Algebra, Lemmas 15.75.12 and 15.67.16). Note that in case (2) we also see that $\tau_{\leq -1} M^\bullet = 0$ in $D(A)$ whence $M^\bullet$ and K are perfect with tor amplitude in $[0, b]$. For any B-scheme $g : T \to B$ we have

$$H^0(T, H^0(Lg^* K)) = H^0(T, H^0(Lg^* \tau_{\geq 0} K))$$

(by the dual of Derived Categories, Lemma 13.16.1) hence we may replace K by $\tau_{\geq 0} K$ and correspondingly $M^\bullet$ by $\tau_{\geq 0} M^\bullet$. In other words, we may assume $M^\bullet$ has tor amplitude in $[0, b]$.

Assume $M^\bullet$ has tor amplitude in $[0, b]$. We may assume $M^\bullet$ is a bounded above complex of finite free A-modules (by our definition of pseudo-coherent complexes, see More on Algebra, Definition 15.65.1 and the discussion following the definition). By More on Algebra, Lemma 15.67.2 we see that $M = \mathrm{Coker}(M^{-1} \to M^0)$ is flat. By Algebra, Lemma 10.78.2 we see that M is finite locally free. Hence $M^\bullet$ is quasi-isomorphic to

$$M \to M^1 \to M^2 \to \ldots \to M^d \to 0 \ldots$$

Note that this is a K-flat complex (Cohomology, Lemma 20.26.9), hence derived pullback of K via a morphism $T \to B$ is computed by the complex

$$g^* \widetilde{M} \to g^* \widetilde{M^1} \to \ldots$$

Thus it suffices to show that the functor

$$(g : T \to B) \longmapsto \mathrm{Ker}(\Gamma(T, g^* \widetilde{M}) \to \Gamma(T, g^* (\widetilde{M^1}))$$

is representable by an affine scheme of finite presentation over B.

We may still replace B by the members of an affine open covering in order to prove this last statement. Hence we may assume that M is finite free (recall that M^1 is finite free to begin with). Write $M = A^{\oplus n}$ and $M^1 = A^{\oplus m}$. Let the map $M \to M^1$ be given by the $m \times n$ matrix (a_{ij}) with coefficients in A. Then $\widetilde{M} = \mathcal{O}_B^{\oplus n}$ and $\widetilde{M^1} = \mathcal{O}_B^{\oplus m}$. Thus the functor above is equal to the functor

$$(g : T \to B) \longmapsto \{(f_1, \ldots, f_n) \in \Gamma(T, \mathcal{O}_T) \mid \sum g^\sharp(a_{ij}) f_i = 0,\ j = 1, \ldots, m\}$$

Clearly this is representable by the affine scheme

$$\mathrm{Spec}\left(A[x_1, \ldots, x_n]/(\sum a_{ij} x_i; j = 1, \ldots, m)\right)$$

and the lemma has been proved. $\square$

The functor $Hom(\mathcal{F}, \mathcal{G})$ is representable in a number of situations. All of our results will be based on the following basic case. The proof of this lemma as given below is in some sense the natural generalization to the proof of [**DG67**, III, Cor 7.7.8].

08JY **Lemma 99.3.9.** *In Situation 99.3.1 assume that*

(1) *B is a Noetherian algebraic space,*
(2) *f is locally of finite type and quasi-separated,*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
(4) *$\mathcal{G}$ is a finite type $\mathcal{O}_X$-module, flat over B, with support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine and of finite presentation over B.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to \mathcal{F}$ by a perfect complex P of the triple $(X, \mathcal{F}, -1)$, see Derived Categories of Spaces, Definition 75.14.1 and Theorem 75.14.7). Then the induced map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(P, \mathcal{G})$$

is an isomorphism because $P \to \mathcal{F}$ induces an isomorphism $H^0(P) \to \mathcal{F}$ and $H^i(P) = 0$ for $i > 0$. Moreover, for any morphism $g : T \to B$ denote $h : X_T = T \times_B X \to X$ the projection and set $P_T = Lh^*P$. Then it is equally true that

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T)$$

is an isomorphism, as $P_T = Lh^*P \to Lh^*\mathcal{F} \to \mathcal{F}_T$ induces an isomorphism $H^0(P_T) \to \mathcal{F}_T$ (because h^* is right exact and $H^i(P) = 0$ for $i > 0$). Thus it suffices to prove the result for the functor

$$T \longmapsto \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T).$$

By the Leray spectral sequence (see Cohomology on Sites, Remark 21.14.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T) = H^0(X_T, R\mathcal{H}om(P_T, \mathcal{G}_T)) = H^0(T, Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T))$$

where $f_T : X_T \to T$ is the base change of f. By Derived Categories of Spaces, Lemma 75.21.5 we have

$$Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T) = Lg^*Rf_*R\mathcal{H}om(P, \mathcal{G}).$$

By Derived Categories of Spaces, Lemma 75.22.3 the object $K = Rf_*R\mathcal{H}om(P, \mathcal{G})$ of $D(\mathcal{O}_B)$ is perfect. This means we can apply Lemma 99.3.8 as long as we can prove that the cohomology sheaf $H^i(Lg^*K)$ is 0 for all $i < 0$ and $g : T \to B$ as above. This is clear from the last displayed formula as the cohomology sheaves of $Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T)$ are zero in negative degrees due to the fact that $R\mathcal{H}om(P_T, \mathcal{G}_T)$ has vanishing cohomology sheaves in negative degrees as P_T is perfect with vanishing cohomology sheaves in positive degrees. □

Here is a cheap consequence of Lemma 99.3.9.

08K6 **Proposition 99.3.10.** *In Situation 99.3.1 assume that*

(1) *f is of finite presentation, and*
(2) *$\mathcal{G}$ is a finitely presented $\mathcal{O}_X$-module, flat over B, with support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $Hom(\mathcal{F}, \mathcal{G})$ is of finite presentation over B.

Proof. By Lemma 99.3.2 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for fppf coverings. This mean we may[1] apply Bootstrap, Lemma 80.11.1 to check the representability étale locally on B. Moreover, we may check whether the end result is affine or of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. Hence we may assume that B is an affine scheme.

Assume B is an affine scheme. As f is of finite presentation, it follows X is quasi-compact and quasi-separated. Thus we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of $\mathcal{O}_X$-modules of finite presentation (Limits of Spaces, Lemma 70.9.1). It is clear that

$$Hom(\mathcal{F}, \mathcal{G}) = \lim Hom(\mathcal{F}_i, \mathcal{G})$$

Hence if we can show that each $Hom(\mathcal{F}_i, \mathcal{G})$ is representable by an affine scheme, then we see that the same thing holds for $Hom(\mathcal{F}, \mathcal{G})$. Use the material in Limits, Section 32.2 and Limits of Spaces, Section 70.4. Thus we may assume that $\mathcal{F}$ is of finite presentation.

Say $B = \operatorname{Spec}(R)$. Write $R = \operatorname{colim} R_i$ with each R_i a finite type $\mathbf{Z}$-algebra. Set $B_i = \operatorname{Spec}(R_i)$. By the results of Limits of Spaces, Lemmas 70.7.1 and 70.7.2 we can find an i, a morphism of algebraic spaces $X_i \to B_i$, and finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{F}_i$ and $\mathcal{G}_i$ such that the base change of $(X_i, \mathcal{F}_i, \mathcal{G}_i)$ to B recovers $(X, \mathcal{F}, \mathcal{G})$. By Limits of Spaces, Lemma 70.6.12 we may, after increasing i, assume that $\mathcal{G}_i$ is flat over B_i. By Limits of Spaces, Lemma 70.12.3 we may similarly assume the scheme theoretic support of $\mathcal{G}_i$ is proper over B_i. At this point we can apply Lemma 99.3.9 to see that $H_i = Hom(\mathcal{F}_i, \mathcal{G}_i)$ is an algebraic space affine of finite presentation over B_i. Pulling back to B (using Remark 99.3.4) we see that $H_i \times_{B_i} B = Hom(\mathcal{F}, \mathcal{G})$ and we win. □

99.4. The Isom functor

08K7 In Situation 99.3.1 we can consider the subfunctor

$$Isom(\mathcal{F}, \mathcal{G}) \subset Hom(\mathcal{F}, \mathcal{G})$$

whose value on a scheme T over B is the set of *invertible* $\mathcal{O}_{X_T}$-homomorphisms $u : \mathcal{F}_T \to \mathcal{G}_T$.

We sometimes think of $Isom(\mathcal{F}, \mathcal{G})$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $Isom(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $Isom(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module isomorphism where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $Isom(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

08K8 **Lemma 99.4.1.** *In Situation 99.3.1 the functor $Isom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

[1] We omit the verification of the set theoretical condition (3) of the referenced lemma.

Proof. We have already seen that $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property. Hence it remains to show the following: Given an fpqc covering $\{T_i \to T\}_{i \in I}$ of schemes over B and an $\mathcal{O}_{X_T}$-linear map $u : \mathcal{F}_T \to \mathcal{G}_T$ such that u_{T_i} is an isomorphism for all i, then u is an isomorphism. Since $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 73.9.3, this follows from Descent on Spaces, Proposition 74.4.1. □

Sanity check: *Isom* sheaf plays the same role among algebraic spaces over S.

0D3T **Lemma 99.4.2.** *In Situation 99.3.1. Let T be an algebraic space over S. We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Isom(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \to B, u : \mathcal{F}_T \to \mathcal{G}_T \text{ isomorphism}\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B,h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the equality in Lemma 99.3.3. We omit the verification that these subsets correspond under the identification given in the proof of that lemma. □

08K9 **Proposition 99.4.3.** *In Situation 99.3.1 assume that*

(1) *f is of finite presentation, and*
(2) *$\mathcal{F}$ and $\mathcal{G}$ are finitely presented $\mathcal{O}_X$-modules, flat over B, with support proper over B.*

Then the functor $Isom(\mathcal{F}, \mathcal{G})$ is an algebraic space affine of finite presentation over B.

Proof. We will use the abbreviations $H = Hom(\mathcal{F}, \mathcal{G})$, $I = Hom(\mathcal{F}, \mathcal{F})$, $H' = Hom(\mathcal{G}, \mathcal{F})$, and $I' = Hom(\mathcal{G}, \mathcal{G})$. By Proposition 99.3.10 the functors H, I, H', I' are algebraic spaces and the morphisms $H \to B$, $I \to B$, $H' \to B$, and $I' \to B$ are affine and of finite presentation. The composition of maps gives a morphism

$$c : H' \times_B H \longrightarrow I \times_B I', \quad (u', u) \longmapsto (u \circ u', u' \circ u)$$

of algebraic spaces over B. Since $I \times_B I' \to B$ is separated, the section $\sigma : B \to I \times_B I'$ corresponding to $(\mathrm{id}_{\mathcal{F}}, \mathrm{id}_{\mathcal{G}})$ is a closed immersion (Morphisms of Spaces, Lemma 67.4.7). Moreover, σ is of finite presentation (Morphisms of Spaces, Lemma 67.28.9). Hence

$$Isom(\mathcal{F}, \mathcal{G}) = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

is an algebraic space affine of finite presentation over B as well. Some details omitted. □

99.5. The stack of coherent sheaves

08KA In this section we prove that the stack of coherent sheaves on X/B is algebraic under suitable hypotheses. This is a special case of [**Lie06b**, Theorem 2.1.1] which treats the case of the stack of coherent sheaves on an Artin stack over a base.

08KB **Situation 99.5.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. We denote $\mathcal{C}oh_{X/B}$ the category whose objects are triples $(T, g, \mathcal{F})$ where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$

(3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_T}$-module of finite presentation, flat over T, with support proper over T.

A morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : (h')^*\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $\mathcal{O}_{X_T}$-modules where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $\mathcal{C}oh_{X/B}$ is a category and the rule

$$p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (T, g, \mathcal{F}) \longmapsto T$$

is a functor. For a scheme T over S we denote $\mathcal{C}oh_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

08W5 **Lemma 99.5.2.** *In Situation 99.5.1 the functor $p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 4.35.1. Given an object $(T', g', \mathcal{F}')$ of $\mathcal{C}oh_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $\mathcal{F} = (h')^*\mathcal{F}'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ of $\mathcal{C}oh_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, \mathcal{F}_1) \to (T, g, \mathcal{F}) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, \mathcal{F}_2) \to (T, g, \mathcal{F})$$

of $\mathcal{C}oh_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$(h')^*\mathcal{F}_2 \xrightarrow{(h')^*\varphi_2^{-1}} (h')^*(h_2)^*\mathcal{F} = (h_1)^*\mathcal{F} \xrightarrow{\varphi_1} \mathcal{F}_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, \mathcal{F}_1) \to (T_2, g_2, \mathcal{F}_2)$ that witnesses the truth of condition (2). □

08W6 **Lemma 99.5.3.** *In Situation 99.5.1. Denote $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, \mathcal{F})$ and $y = (T, h, \mathcal{G})$ of $\mathcal{X}$ over a scheme T. We have to show that $Isom_{\mathcal{X}}(x, y)$ is an algebraic space over T, see Algebraic Stacks, Lemma 94.10.11. If for $a : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ a = h \circ a$. Hence there is a transformation of presheaves

$$Isom_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, h)$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and the sheaves $\mathcal{F}$ and $\mathcal{G}$ by their pullbacks. Thus we may assume $g = h$. In this case we have $Isom_{\mathcal{X}}(x, y) = Isom(\mathcal{F}, \mathcal{G})$ and the result follows from Proposition 99.4.3. □

08KC **Lemma 99.5.4.** *In Situation 99.5.1 the functor $p : \mathcal{C}oh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.*

Proof. To prove that $\mathcal{C}oh_{X/B}$ is a stack in groupoids, we have to show that the presheaves *Isom* are sheaves and that descent data are effective. The statement on *Isom* follows from Lemma 99.5.3, see Algebraic Stacks, Lemma 94.10.11. Let us prove the statement on descent data. Suppose that $\{a_i : T_i \to T\}$ is an fppf covering

of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $\mathcal{C}oh_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, \mathcal{F}_i)$. Denote $\mathrm{pr}_0 : T_i \times_T T_j \to T_i$ and $\mathrm{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} = \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \mathrm{pr}_0 = g_j \circ \mathrm{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 74.7.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of $(T_i, g_i, \mathcal{F}_i)$ by $\mathrm{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \mathrm{pr}_0, \mathrm{pr}_i^* \mathcal{F}_i)$. Hence a descent datum for $\{T_i \to T\}$ in $\mathcal{C}oh_{X/B}$ is given by the objects $(T_i, g \circ a_i, \mathcal{F}_i)$ and for each pair i, j an isomorphism of $\mathcal{O}_{X_{ij}}$-modules

$$\varphi_{ij} : \mathrm{pr}_i^* \mathcal{F}_i \longrightarrow \mathrm{pr}_j^* \mathcal{F}_j$$

satisfying the cocycle condition over (the pullback of X to) $T_i \times_T T_j \times_T T_k$. Ok, and now we simply use that $\{X_i \to X_T\}$ is an fppf covering so that we can view $(\mathcal{F}_i, \varphi_{ij})$ as a descent datum for this covering. By Descent on Spaces, Proposition 74.4.1 this descent datum is effective and we obtain a quasi-coherent sheaf $\mathcal{F}$ over X_T restricting to $\mathcal{F}_i$ on X_i. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{F}$ is flat over T and Descent on Spaces, Lemma 74.6.2 guarantees that $\mathcal{F}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. Finally, by Descent on Spaces, Lemma 74.11.19 we see that the scheme theoretic support of $\mathcal{F}$ is proper over T as we've assumed the scheme theoretic support of $\mathcal{F}_i$ is proper over T_i (note that taking scheme theoretic support commutes with flat base change by Morphisms of Spaces, Lemma 67.30.10). In this way we obtain our desired object over T. □

08LP **Remark 99.5.5.** In Situation 99.5.1 the rule $(T, g, \mathcal{F}) \mapsto (T, g)$ defines a 1-morphism

$$\mathcal{C}oh_{X/B} \longrightarrow \mathcal{S}_B$$

of stacks in groupoids (see Lemma 99.5.4, Algebraic Stacks, Section 94.7, and Examples of Stacks, Section 95.10). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $\mathcal{C}oh_{X'/B'} \to (Sch/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{array}{ccc} \mathcal{C}oh_{X'/B'} & \longrightarrow & \mathcal{C}oh_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{S}_{B'} & \longrightarrow & \mathcal{S}_B \end{array} \quad \text{or in another notation} \quad \begin{array}{ccc} \mathcal{C}oh_{X'/B'} & \longrightarrow & \mathcal{C}oh_{X/B} \\ \downarrow & & \downarrow \\ Sch/B' & \longrightarrow & Sch/B \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

08KD **Lemma 99.5.6.** *In Situation 99.5.1 assume that $B \to S$ is locally of finite presentation. Then $p : \mathcal{C}oh_{X/B} \to (Sch/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an

object $(T, h, \mathcal{F})$ of $\mathcal{C}oh_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{array}{ccc} \operatorname{colim} \mathcal{C}oh_{X/B,T_i} & \longrightarrow & \mathcal{C}oh_{X/B,T} \\ \downarrow & & \downarrow \\ \operatorname{colim} B(T_i) & \longrightarrow & B(T) \end{array}$$

We have to show the top horizontal arrow is an equivalence. Since we have assumed that B is locally of finite presentation over S we see from Limits of Spaces, Remark 70.3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$ and that the algebraic spaces X_i and X_T are quasi-separated and quasi-compact (as they are of finite presentation over the affines T_i and T). By Limits of Spaces, Lemma 70.7.2 we see that

$$\operatorname{colim} FP(X_i) = FP(X_T).$$

where $FP(W)$ is short hand for the category of finitely presented $\mathcal{O}_W$-modules. The results of Limits of Spaces, Lemmas 70.6.12 and 70.12.3 tell us the same thing is true if we replace $FP(X_i)$ and $FP(X_T)$ by the full subcategory of objects flat over T_i and T with scheme theoretic support proper over T_i and T. This proves the lemma. □

08LQ **Lemma 99.5.7.** *In Situation 99.5.1. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{C}oh_{X/B,Y'} \longrightarrow \mathcal{C}oh_{X/B,Y} \times_{\mathcal{C}oh_{X/B,Z}} \mathcal{C}oh_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 81.6.1. Thus using the commutative diagram

$$\begin{array}{ccc} \mathcal{C}oh_{X/B,Y'} & \longrightarrow & \mathcal{C}oh_{X/B,Y} \times_{\mathcal{C}oh_{X/B,Z}} \mathcal{C}oh_{X/B,Z'} \\ \downarrow & & \downarrow \\ B(Y') & \longrightarrow & B(Y) \times_{B(Z)} B(Z') \end{array}$$

we see that we may assume that Y' is a scheme over B'. By Remark 99.5.5 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$. In this case the statement follows from Pushouts of Spaces, Lemma 81.6.6. □

08W7 **Lemma 99.5.8.** *Let*

$$\begin{array}{ccc} X & \xrightarrow[i]{} & X' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

be a cartesian square of algebraic spaces where $T \to T'$ is a first order thickening. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module flat over T'. Set $\mathcal{F} = i^*\mathcal{F}'$. The following are equivalent

(1) *$\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{X'}$-module of finite presentation,*
(2) *$\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module of finite presentation,*
(3) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation,*
(4) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation,*

Proof. Recall that a finitely presented module is quasi-coherent hence the equivalence of (1) and (2) and (3) and (4). The equivalence of (2) and (4) is a special case of Deformation Theory, Lemma 91.11.3. □

08W8 **Lemma 99.5.9.** *In Situation 99.5.1 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, \mathcal{G}_0)$ be an object of $\mathcal{X} = \mathcal{C}oh_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $Inf(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 98.8) are finite dimensional.*

Proof. Observe that by Lemma 99.5.7 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 98.21. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 98.6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0,B} X$ and denote $\mathcal{X}_0 = \mathcal{C}oh_{X_0/k}$. In Remark 99.5.5 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(Sch/S)_{fppf}$. Thus by Artin's Axioms, Lemma 98.8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 98.8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion $k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Unwinding the definitions and using Lemma 99.5.8 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of lifts of $\mathcal{G}_0$ to a flat module on X_ϵ. By Deformation Theory, Lemma 91.12.1 we conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 91.12.1 once more we see that

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ has support proper over $\mathrm{Spec}(k)$. Namely, X_0 is of finite presentation over $\mathrm{Spec}(k)$, hence Noetherian. Since $\mathcal{G}_0$ is of finite presentation it is a coherent $\mathcal{O}_{X_0}$-module. Thus we may apply Derived Categories of Spaces, Lemma 75.8.4 to conclude the desired finiteness. □

08W9 **Lemma 99.5.10.** *In Situation 99.5.1 assume that S is a locally Noetherian scheme and that $f : X \to B$ is separated. Let $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then the functor Artin's Axioms, Equation (98.9.3.1) is an equivalence.*

Proof. Let A be an S-algebra which is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field k is of finite type over S. We have to show that the category of objects over A is equivalent to the category of formal objects over A. Since we know this holds for the category $\mathcal{S}_B$ fibred in sets associated to B by Artin's Axioms, Lemma 98.9.5, it suffices to prove this for those objects lying over a given morphism $\mathrm{Spec}(A) \to B$.

Set $X_A = \mathrm{Spec}(A) \times_B X$ and $X_n = \mathrm{Spec}(A/\mathfrak{m}^n) \times_B X$. By Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 76.42.11) we see that the category of coherent modules $\mathcal{F}$ on X_A with support proper over $\mathrm{Spec}(A)$ is equivalent to the category of systems $(\mathcal{F}_n)$ of coherent modules $\mathcal{F}_n$ on X_n with support proper over $\mathrm{Spec}(A/\mathfrak{m}^n)$. The equivalence sends $\mathcal{F}$ to the system $(\mathcal{F} \otimes_A A/\mathfrak{m}^n)$. See discussion in More on Morphisms of Spaces, Remark 76.42.12. To finish the proof of the lemma, it suffices to show that $\mathcal{F}$ is flat over A if and only if all $\mathcal{F} \otimes_A A/\mathfrak{m}^n$ are flat over $A/\mathfrak{m}^n$. This follows from More on Morphisms of Spaces, Lemma 76.24.3. □

08WA **Lemma 99.5.11.** *In Situation 99.5.1 assume that S is a locally Noetherian scheme, $S = B$, and $f : X \to B$ is flat. Let $\mathcal{X} = \mathcal{C}oh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 98.13.1).*

First proof. This proof is based on the criterion of Artin's Axioms, Lemma 98.24.4. Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 29.16.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

According to Deformation Theory, Lemma 91.12.1 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

and given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

This uses that for any $A' \to A$ as above the base change $X_{A'} = \mathrm{Spec}(A') \times_B X$ is flat over A'. Moreover, the construction of the obstruction class is functorial in the surjection $A' \to A$ (for fixed A) by Deformation Theory, Lemma 91.12.3. Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 98.24.2. Finally, condition (iv) of Artin's Axioms, Lemma 98.24.3 holds by Deformation Theory, Lemma 91.12.5. Thus Artin's Axioms, Lemma 98.24.4 does indeed apply and the lemma is proved. □

Second proof. This proof is based on Artin's Axioms, Lemma 98.22.2. Conditions (1), (2), and (3) of that lemma correspond to Lemmas 99.5.3, 99.5.7, and 99.5.6.

We have constructed an obstruction theory in the chapter on deformation theory. Namely, given an S-algebra A and an object x of $\mathcal{C}oh_{X/B}$ over $\mathrm{Spec}(A)$ given by $\mathcal{F}$ on X_A we set $\mathcal{O}_x(M) = \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ and if $A' \to A$ is a surjection with kernel I, then as obstruction element we take the element

$$o_x(A') = o(\mathcal{F}, \mathcal{F} \otimes_A I, 1) \in \mathcal{O}_x(I) = \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

of Deformation Theory, Lemma 91.12.1. All properties of an obstruction theory as defined in Artin's Axioms, Definition 98.22.1 follow from this lemma except for functoriality of obstruction classes as formulated in condition (ii) of the definition. But as stated in the footnote to assumption (4) of Artin's Axioms, Lemma 98.22.2 it suffices to check functoriality of obstruction classes for a fixed A which follows from Deformation Theory, Lemma 91.12.3. Deformation Theory, Lemma 91.12.1 also tells us that $T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for any A-module M.

To finish the proof it suffices to show that $T_x(\prod M_n) = \prod T_x(M_n)$ and $\mathcal{O}_x(\prod M_n) = \prod \mathcal{O}_x(M)$. Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i = 1, 2$. A straightforward argument shows that

$$H^i(K \otimes^{\mathbf{L}}_A \prod M_n) = \prod H^i(K \otimes^{\mathbf{L}}_A M_n)$$

whenever K is a pseudo-coherent object of $D(A)$. In fact, this property (for all i) characterizes pseudo-coherent complexes, see More on Algebra, Lemma 15.66.5. □

08WC **Theorem 99.5.12** (Algebraicity of the stack of coherent sheaves; flat case)**.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation, separated, and flat*[2]*. Then $\mathcal{C}oh_{X/B}$ is an algebraic stack over S.*

Proof. Set $\mathcal{X} = \mathcal{C}oh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.5.4 and 99.5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathcal{C}oh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.5.5). By the material in Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth

[2]This assumption is not necessary. See Section 99.6.

morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is of finite presentation, separated, and flat and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1, 70.6.9, and 70.6.12. If we show that $\mathcal{C}oh_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.5.5 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. By Lemma 99.5.11 we have that $\mathcal{X}$ satisfies openness of versality, hence (4) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.5.4, 99.5.6, 99.5.7, 99.5.9. Condition (3) follows from Lemma 99.5.10. Finally, condition (1) is Lemma 99.5.3. This finishes the proof of the theorem. □

99.6. The stack of coherent sheaves in the non-flat case

08WB In Theorem 99.5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we give a different proof which avoids the flatness assumption and avoids checking openness of versality by using the results in Flatness on Spaces, Section 77.12 and Artin's Axioms, Section 98.20.

For a different approach to this problem the reader may wish to consult [**Art69b**] and follow the method discussed in the papers [**OS03**], [**Lie06b**], [**Ols05**], [**HR13**], [**HR10**], [**Ryd11**]. Some of these papers deal with the more general case of the stack of coherent sheaves on an algebraic stack over an algebraic stack and others deal with similar problems in the case of Hilbert stacks or Quot functors. Our strategy will be to show algebraicity of some cases of Hilbert stacks and Quot functors as a consequence of the algebraicity of the stack of coherent sheaves.

09DS **Theorem 99.6.1** (Algebraicity of the stack of coherent sheaves; general case)**.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and separated. Then $\mathcal{C}oh_{X/B}$ is an algebraic stack over S.*

Proof. Only the last step of the proof is different from the proof in the flat case, but we repeat all the arguments here to make sure everything works.

Set $\mathcal{X} = \mathcal{C}oh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.5.4 and 99.5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathcal{C}oh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.5.5). By the material in

Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \operatorname{Spec}(\Lambda)$. Write $\Lambda = \operatorname{colim} \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \operatorname{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. If we show that $\mathcal{C}oh_{X_i/\operatorname{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.5.5 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \operatorname{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.5.4, 99.5.6, 99.5.7, 99.5.9. Condition (3) is Lemma 99.5.10. Condition (1) is Lemma 99.5.3.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3. This is Flatness on Spaces, Theorem 77.12.8 and the proof is complete. □

99.7. The functor of quotients

082L In this section we discuss some generalities regarding the functor $Q_{\mathcal{F}/X/B}$ defined below. The notation $\mathrm{Quot}_{\mathcal{F}/X/B}$ is reserved for a subfunctor of $\mathrm{Q}_{\mathcal{F}/X/B}$. We urge the reader to skip this section on a first reading.

082M **Situation 99.7.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_B T \to X$. Given such a T we set

$$\mathrm{Q}_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{c} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a} \\ \text{quasi-coherent } \mathcal{O}_{X_T}\text{-module flat over } T \end{array} \right\}$$

We identify quotients if they have the same kernel. Suppose that $T' \to T$ is a morphism of schemes over B and $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. Then the

pullback $\mathcal{Q}' = (X_{T'} \to X_T)^*\mathcal{Q}$ is a quasi-coherent $\mathcal{O}_{X_{T'}}$-module flat over T' by Morphisms of Spaces, Lemma 67.31.3. Thus we obtain a functor

082N (99.7.1.1) $$\mathrm{Q}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

This is the *functor of quotients of* $\mathcal{F}/X/B$. We define a subfunctor

0CZL (99.7.1.2) $$\mathrm{Q}^{fp}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

which assigns to T the subset of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$ consisting of those quotients $\mathcal{F}_T \to \mathcal{Q}$ such that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. This is a subfunctor by Properties of Spaces, Section 66.30.

In Situation 99.7.1 we sometimes think of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Q}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$ and $\mathcal{Q}$ is a T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ of finite presentation on $X_T = X \times_{B,h} T$. In particular, when we say that $\mathrm{Q}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space. Similar remarks apply to $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$.

08IT **Remark 99.7.2.** In Situation 99.7.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. Thus we have the functor $Q_{\mathcal{F}'/X'/B'}$ on the category of schemes over B'. For a scheme T over B' it is clear that we have

$$Q_{\mathcal{F}'/X'/B'}(T) = Q_{\mathcal{F}/X/B}(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. Similar remarks apply to $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$. These trivial remarks will occasionally be useful to change the base algebraic space.

08IU **Remark 99.7.3.** Let S be a scheme, X an algebraic space over S, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering and for each $i, j \in I$ we are given an fpqc covering $\{X_{ijk} \to X_i \times_X X_j\}$. In this situation we have a bijection

$$\left\{\begin{array}{c}\text{quotients } \mathcal{F} \to \mathcal{Q} \text{ where} \\ \mathcal{Q} \text{ is a quasi-coherent}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{families of quotients } f_i^*\mathcal{F} \to \mathcal{Q}_i \text{ where} \\ \mathcal{Q}_i \text{ is quasi-coherent and } \mathcal{Q}_i \text{ and } \mathcal{Q}_j \\ \text{restrict to the same quotient on } X_{ijk}\end{array}\right\}$$

Namely, let $(f_i^*\mathcal{F} \to \mathcal{Q}_i)_{i\in I}$ be an element of the right hand side. Then since $\{X_{ijk} \to X_i \times_X X_j\}$ is an fpqc covering we see that the pullbacks of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ restrict to the same quotient of the pullback of $\mathcal{F}$ to $X_i \times_X X_j$ (by fully faithfulness in Descent on Spaces, Proposition 74.4.1). Hence we obtain a descent datum for quasi-coherent modules with respect to $\{X_i \to X\}_{i\in I}$. By Descent on Spaces, Proposition 74.4.1 we find a map of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{Q}$ whose restriction to X_i recovers the given maps $f_i^*\mathcal{F} \to \mathcal{Q}_i$. Since the family of morphisms $\{X_i \to X\}$ is jointly surjective and flat, for every point $x \in |X|$ there exists an i and a point $x_i \in |X_i|$ mapping to x. Note that the induced map on local rings $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is faithfully flat, see Morphisms of Spaces, Section 67.30. Thus we see that $\mathcal{F} \to \mathcal{Q}$ is surjective.

082P **Lemma 99.7.4.** *In Situation 99.7.1. The functors $Q_{\mathcal{F}/X/B}$ and $Q^{fp}_{\mathcal{F}/X/B}$ satisfy the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = \mathcal{F}_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T (Topologies on Spaces, Lemma 73.9.3) and that $X_{T_i\times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $\mathcal{F}_i \to \mathcal{Q}_i$ is a collection of elements of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i)$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ restrict to the same element of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i \times_T T_{i'})$. By Remark 99.7.3 we obtain a surjective map of quasi-coherent $\mathcal{O}_{X_T}$-modules $\mathcal{F}_T \to \mathcal{Q}$ whose restriction to X_i recovers the given quotients. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{Q}$ is flat over T. Finally, in the case of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$, i.e., if $\mathcal{Q}_i$ are of finite presentation, then Descent on Spaces, Lemma 74.6.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. □

Sanity check: $\mathrm{Q}_{\mathcal{F}/X/B}$, $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ play the same role among algebraic spaces over S.

0D3U **Lemma 99.7.5.** *In Situation 99.7.1. Let T be an algebraic space over S. We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Q_{\mathcal{F}/X/B}) = \left\{ \begin{array}{l} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is quasi-coherent and flat over } T \end{array} \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$. Similarly, we have

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Q^{fp}_{\mathcal{F}/X/B}) = \left\{ \begin{array}{l} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is of finite presentation and flat over } T \end{array} \right\}$$

Proof. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $t, s : R \to U$.

Let $v : T \to \mathrm{Q}_{\mathcal{F}/X/B}$ be a natural transformation. Then $v(p)$ corresponds to a pair $(h_U, \mathcal{F}_U \to \mathcal{Q}_U)$ over U. As v is a transformation of functors we see that the pullbacks of $(h_U, \mathcal{F}_U \to \mathcal{Q}_U)$ by s and t agree. Since $T = U/R$ (Spaces, Lemma 65.9.1), we obtain a morphism $h : T \to B$ such that $h_U = h \circ p$. By Descent on Spaces, Proposition 74.4.1 the quotient $\mathcal{Q}_U$ descends to a quotient $\mathcal{F}_T \to \mathcal{Q}$ over X_T. Since $U \to T$ is surjective and flat, it follows from Morphisms of Spaces, Lemma 67.31.5 that $\mathcal{Q}$ is flat over T.

Conversely, let $(h, \mathcal{F}_T \to \mathcal{Q})$ be a pair over T. Then we get a natural transformation $v : T \to \mathrm{Q}_{\mathcal{F}/X/B}$ by sending a morphism $a : T' \to T$ where T' is a scheme to $(h \circ a, \mathcal{F}_{T'} \to a^*\mathcal{Q})$. We omit the verification that the construction of this and the previous paragraph are mutually inverse.

In the case of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$ we add: given a morphism $h : T \to B$, a quasi-coherent sheaf on X_T is of finite presentation as an $\mathcal{O}_{X_T}$-module if and only if the pullback to X_U is of finite presentation as an $\mathcal{O}_{X_U}$-module. This follows from the fact that $X_U \to X_T$ is surjective and étale and Descent on Spaces, Lemma 74.6.2. □

08IV **Lemma 99.7.6.** *In Situation 99.7.1 let $\{X_i \to X\}_{i\in I}$ be an fpqc covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fpqc covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. For every scheme T over B the diagram*

$$Q_{\mathcal{F}/X/B}(T) \longrightarrow \prod\nolimits_i Q_{\mathcal{F}_i/X_i/B}(T) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod\nolimits_{i,j,k} Q_{\mathcal{F}_{ijk}/X_{ijk}/B}(T)$$

presents the first arrow as the equalizer of the other two. The same is true for the functor $Q^{fp}_{\mathcal{F}/X/B}$.

Proof. Let $\mathcal{F}_{i,T} \to \mathcal{Q}_i$ be an element in the equalizer of pr_0^* and pr_1^*. By Remark 99.7.3 we obtain a surjection $\mathcal{F}_T \to \mathcal{Q}$ of quasi-coherent $\mathcal{O}_{X_T}$-modules whose restriction to $X_{i,T}$ recovers $\mathcal{F}_i \to \mathcal{Q}_i$. By Morphisms of Spaces, Lemma 67.31.5 we see that $\mathcal{Q}$ is flat over T as desired. In the case of the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}$, i.e., if $\mathcal{Q}_i$ is of finite presentation, then $\mathcal{Q}$ is of finite presentation too by Descent on Spaces, Lemma 74.6.2. □

082Q **Lemma 99.7.7.** *In Situation 99.7.1 assume also that (a) f is quasi-compact and quasi-separated and (b) $\mathcal{F}$ is of finite presentation. Then the functor $Q^{fp}_{\mathcal{F}/X/B}$ is limit preserving in the following sense: If $T = \lim T_i$ is a directed limit of affine schemes over B, then $Q^{fp}_{\mathcal{F}/X/B}(T) = \mathrm{colim}\, Q^{fp}_{\mathcal{F}/X/B}(T_i)$.*

Proof. Let $T = \lim T_i$ be as in the statement of the lemma. Choose $i_0 \in I$ and replace I by $\{i \in I \mid i \geq i_0\}$. We may set $B = S = T_{i_0}$ and we may replace X by X_{T_0} and $\mathcal{F}$ by the pullback to X_{T_0}. Then $X_T = \lim X_{T_i}$, see Limits of Spaces, Lemma 70.4.1. Let $\mathcal{F}_T \to \mathcal{Q}$ be an element of $\mathrm{Q}^{fp}_{\mathcal{F}/X/B}(T)$. By Limits of Spaces, Lemma 70.7.2 there exists an i and a map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ of $\mathcal{O}_{X_{T_i}}$-modules of finite presentation whose pullback to X_T is the given quotient map.

We still have to check that, after possibly increasing i, the map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ is surjective and $\mathcal{Q}_i$ is flat over T_i. To do this, choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 66.6.3). We may check surjectivity and flatness over T_i after pulling back to the étale cover $U_{T_i} \to X_{T_i}$ (by definition). This reduces us to the case where $X = \mathrm{Spec}(B_0)$ is an affine scheme of finite presentation over $B = S = T_0 = \mathrm{Spec}(A_0)$. Writing $T_i = \mathrm{Spec}(A_i)$, then $T = \mathrm{Spec}(A)$ with $A = \mathrm{colim}\, A_i$ we have reached the following algebra problem. Let $M_i \to N_i$ be a map of finitely presented $B_0 \otimes_{A_0} A_i$-modules such that $M_i \otimes_{A_i} A \to N_i \otimes_{A_i} A$ is surjective and $N_i \otimes_{A_i} A$ is flat over A. Show that for some $i' \geq i$ $M_i \otimes_{A_i} A_{i'} \to N_i \otimes_{A_i} A_{i'}$ is surjective and $N_i \otimes_{A_i} A_{i'}$ is flat over A. The first follows from Algebra, Lemma 10.127.5 and the second from Algebra, Lemma 10.168.1. □

08IW **Lemma 99.7.8.** *In Situation 99.7.1. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over B where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the natural map

$$Q_{\mathcal{F}/X/B}(Y') \longrightarrow Q_{\mathcal{F}/X/B}(Y) \times_{Q_{\mathcal{F}/X/B}(Z)} Q_{\mathcal{F}/X/B}(Z')$$

is bijective. If $X \to B$ is locally of finite presentation, then the same thing is true for $Q^{fp}_{\mathcal{F}/X/B}$.

Proof. Let us construct an inverse map. Namely, suppose we have $\mathcal{F}_Y \to \mathcal{A}$, $\mathcal{F}_{Z'} \to \mathcal{B}'$, and an isomorphism $\mathcal{A}|_{X_Z} \to \mathcal{B}'|_{X_Z}$ compatible with the given surjections. Then we apply Pushouts of Spaces, Lemma 81.6.6 to get a quasi-coherent module $\mathcal{A}'$ on $X_{Y'}$ flat over Y'. Since this sheaf is constructed as a fibre product (see proof of

cited lemma) there is a canonical map $\mathcal{F}_{Y'} \to \mathcal{A}'$. That this map is surjective can be seen because it factors as

$$\begin{array}{c} \mathcal{F}_{Y'} \\ \downarrow \\ (X_Y \to X_{Y'})_*\mathcal{F}_Y \times_{(X_Z \to X_{Y'})_*\mathcal{F}_Z} (X_{Z'} \to X_{Y'})_*\mathcal{F}_{Z'} \\ \downarrow \\ \mathcal{A}' = (X_Y \to X_{Y'})_*\mathcal{A} \times_{(X_Z \to X_{Y'})_*\mathcal{A}|_{X_Z}} (X_{Z'} \to X_{Y'})_*\mathcal{B}' \end{array}$$

and the first arrow is surjective by More on Algebra, Lemma 15.6.5 and the second by More on Algebra, Lemma 15.6.6.

In the case of $Q^{fp}_{\mathcal{F}/X/B}$ all we have to show is that the construction above produces a finitely presented module. This is explained in More on Algebra, Remark 15.7.8 in the commutative algebra setting. The current case of modules over algebraic spaces follows from this by étale localization. □

0CZU **Remark 99.7.9** (Obstructions for quotients)**.** In Situation 99.7.1 **assume** that $\mathcal{F}$ is flat over B. Let $T \subset T'$ be an first order thickening of schemes over B with ideal sheaf $\mathcal{J}$. Then $X_T \subset X_{T'}$ is a first order thickening of algebraic spaces whose ideal sheaf $\mathcal{I}$ is a quotient of $f_T^*\mathcal{J}$. We will think of sheaves on $X_{T'}$, resp. T' as sheaves on X_T, resp. T using the fundamental equivalence described in More on Morphisms of Spaces, Section 76.9. Let

$$0 \to \mathcal{K} \to \mathcal{F}_T \to \mathcal{Q} \to 0$$

define an element x of $Q_{\mathcal{F}/X/B}(T)$. Since $\mathcal{F}_{T'}$ is flat over T' we have a short exact sequence

$$0 \to f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T \xrightarrow{i} \mathcal{F}_{T'} \xrightarrow{\pi} \mathcal{F}_T \to 0$$

and we have $f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T = \mathcal{I} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T$, see Deformation Theory, Lemma 91.11.2. Let us use the abbreviation $f_T^*\mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_T} \mathcal{J}$ for an $\mathcal{O}_{X_T}$-module $\mathcal{G}$. Since $\mathcal{Q}$ is flat over T, we obtain a short exact sequence

$$0 \to \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \to\to 0$$

Combining the above we obtain an canonical extension

$$0 \to \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \to \pi^{-1}(\mathcal{K})/i(\mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J}) \to \mathcal{K} \to 0$$

of $\mathcal{O}_{X_T}$-modules. This defines a canonical class

$$o_x(T') \in \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

If $o_x(T')$ is zero, then we obtain a splitting of the short exact sequence defining it, in other words, we obtain a $\mathcal{O}_{X_{T'}}$-submodule $\mathcal{K}' \subset \pi^{-1}(\mathcal{K})$ sitting in a short exact sequence $0 \to \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \to \mathcal{K}' \to \mathcal{K} \to 0$. Then it follows from the lemma reference above that $\mathcal{Q}' = \mathcal{F}_{T'}/\mathcal{K}'$ is a lift of x to an element of $Q_{\mathcal{F}/X/B}(T')$. Conversely, the reader sees that the existence of a lift implies that $o_x(T')$ is zero. Moreover, if $x \in Q^{fp}_{\mathcal{F}/X/B}(T)$, then automatically $x' \in Q^{fp}_{\mathcal{F}/X/B}(T')$ by Deformation Theory, Lemma 91.11.3. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

0CZV **Remark 99.7.10** (Deformations of quotients)**.** In Situation 99.7.1 **assume** that $\mathcal{F}$ is flat over B. We continue the discussion of Remark 99.7.9. Assume $o_x(T') = 0$. Then we claim that the set of lifts $x' \in Q_{\mathcal{F}/X/B}(T')$ is a principal homogeneous space under the group

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

Namely, given any $\mathcal{F}_{T'} \to \mathcal{Q}'$ flat over T' lifting the quotient $\mathcal{Q}$ we obtain a commutative diagram with exact rows and columns

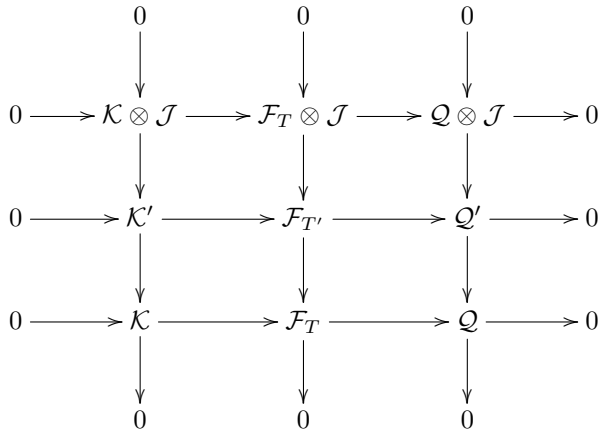

(to see this use the observations made in the previous remark). Given a map $\varphi : \mathcal{K} \to \mathcal{Q} \otimes \mathcal{J}$ we can consider the subsheaf $\mathcal{K}'_\varphi \subset \mathcal{F}_{T'}$ consisting of those local sections s whose image in $\mathcal{F}_T$ is a local section k of $\mathcal{K}$ and whose image in $\mathcal{Q}'$ is the local section $\varphi(k)$ of $\mathcal{Q} \otimes \mathcal{J}$. Then set $\mathcal{Q}'_\varphi = \mathcal{F}_{T'}/\mathcal{K}'_\varphi$. Conversely, any second lift of x corresponds to one of the qotients constructed in this manner. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

99.8. The Quot functor

09TQ In this section we prove the Quot functor is an algebraic space.

09TR **Situation 99.8.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Quot}_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{c} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \\ \text{with support proper over } T \end{array} \right\}$$

By Derived Categories of Spaces, Lemma 75.7.8 this is a subfunctor of the functor $Q^{fp}_{\mathcal{F}/X/B}$ we discussed in Section 99.7. Thus we obtain a functor

09TS (99.8.1.1) $$\mathrm{Quot}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

This is the *Quot functor* associated to $\mathcal{F}/X/B$.

In Situation 99.8.1 we sometimes think of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h is a morphism $h : T \to B$ and Q is a finitely presented, T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ on $X_T = X \times_{B,h} T$ with support proper over T. In particular, when we say that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

09TT **Lemma 99.8.2.** *In Situation 99.8.1. The functor* $\mathrm{Quot}_{\mathcal{F}/X/B}$ *satisfies the sheaf property for the fpqc topology.*

Proof. In Lemma 99.7.4 we have seen that the functor $\mathrm{Q}^{fp}_{\mathcal{F}/X/S}$ is a sheaf. Recall that for a scheme T over S the subset $\mathrm{Quot}_{\mathcal{F}/X/S}(T) \subset \mathrm{Q}_{\mathcal{F}/X/S}(T)$ picks out those quotients whose support is proper over T. This defines a subsheaf by the result of Descent on Spaces, Lemma 74.11.19 combined with Morphisms of Spaces, Lemma 67.30.10 which shows that taking scheme theoretic support commutes with flat base change. □

Sanity check: $\mathrm{Quot}_{\mathcal{F}/X/B}$ plays the same role among algebraic spaces over S.

0D3V **Lemma 99.8.3.** *In Situation 99.8.1. Let T be an algebraic space over S. We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Quot}_{\mathcal{F}/X/B}) = \left\{ \begin{array}{c} (h, \mathcal{F}_T \to \mathcal{Q}) \text{ where } h : T \to B \text{ and} \\ \mathcal{Q} \text{ is of finite presentation and} \\ \text{flat over } T \text{ with support proper over } T \end{array} \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the second equality in Lemma 99.7.5. To see that these subsets correspond under the identification given in the proof of that lemma it suffices to show: given $h : T \to B$, a surjective étale morphism $U \to T$, a finite type quasi-coherent $\mathcal{O}_{X_T}$-module $\mathcal{Q}$ the following are equivalent

(1) the scheme theoretic support of $\mathcal{Q}$ is proper over T, and
(2) the scheme theoretic support of $(X_U \to X_T)^*\mathcal{Q}$ is proper over U.

This follows from Descent on Spaces, Lemma 74.11.19 combined with Morphisms of Spaces, Lemma 67.30.10 which shows that taking scheme theoretic support commutes with flat base change. □

09TU **Proposition 99.8.4.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If f is of finite presentation and separated, then* $\mathrm{Quot}_{\mathcal{F}/X/B}$ *is an algebraic space. If $\mathcal{F}$ is of finite presentation, then* $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ *is locally of finite presentation.*

Proof. By Lemma 99.8.2 we have that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is a sheaf in the fppf topology. Let $Quot_{\mathcal{F}/X/B}$ be the stack in groupoids corresponding to $\mathrm{Quot}_{\mathcal{F}/X/S}$, see Algebraic Stacks, Section 94.7. By Algebraic Stacks, Proposition 94.13.3 it suffices to show that $Quot_{\mathcal{F}/X/B}$ is an algebraic stack. Consider the 1-morphism of stacks in groupoids

$$Quot_{\mathcal{F}/X/S} \longrightarrow \mathcal{C}oh_{X/B}$$

on $(Sch/S)_{fppf}$ which associates to the quotient $\mathcal{F}_T \to \mathcal{Q}$ the module $\mathcal{Q}$. By Theorem 99.6.1 we know that $\mathcal{C}oh_{X/B}$ is an algebraic stack. By Algebraic Stacks,

Lemma 94.15.4 it suffices to show that this 1-morphism is representable by algebraic spaces.

Let T be a scheme over S and let the object $(h, \mathcal{G})$ of $\mathcal{C}oh_{X/B}$ over T correspond to a 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{C}oh_{X/B}$. The 2-fibre product

$$\mathcal{Z} = (Sch/T)_{fppf} \times_{\xi, \mathcal{C}oh_{X/B}} Quot_{\mathcal{F}/X/S}$$

is a stack in setoids, see Stacks, Lemma 8.6.7. The corresponding sheaf of sets (i.e., functor, see Stacks, Lemmas 8.6.7 and 8.6.2) assigns to a scheme T'/T the set of surjections $u : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$ of quasi-coherent modules on $X_{T'}$. Thus we see that $\mathcal{Z}$ is representable by an open subspace (by Flatness on Spaces, Lemma 77.9.3) of the algebraic space $Hom(\mathcal{F}_T, \mathcal{G})$ from Proposition 99.3.10. □

0CZW **Remark 99.8.5** (Quot via Artin's axioms)**.** Let S be a Noetherian scheme all of whose local rings are G-rings. Let X be an algebraic space over S whose structure morphism $f : X \to S$ is of finite presentation and separated. Let $\mathcal{F}$ be a finitely presented quasi-coherent sheaf on X flat over S. In this remark we sketch how one can use Artin's axioms to prove that $\mathrm{Quot}_{\mathcal{F}/X/S}$ is an algebraic space locally of finite presentation over S and avoid using the algebraicity of the stack of coherent sheaves as was done in the proof of Proposition 99.8.4.

We check the conditions listed in Artin's Axioms, Proposition 98.16.1. Representability of the diagonal of $\mathrm{Quot}_{\mathcal{F}/X/S}$ can be seen as follows: suppose we have two quotients $\mathcal{F}_T \to \mathcal{Q}_i$, $i = 1, 2$. Denote $\mathcal{K}_1$ the kernel of the first one. Then we have to show that the locus of T over which $u : \mathcal{K}_1 \to \mathcal{Q}_2$ becomes zero is representable. This follows for example from Flatness on Spaces, Lemma 77.8.6 or from a discussion of the *Hom* sheaf earlier in this chapter. Axioms [0] (sheaf), [1] (limits), [2] (Rim-Schlessinger) follow from Lemmas 99.8.2, 99.7.7, and 99.7.8 (plus some extra work to deal with the properness condition). Axiom [3] (finite dimensionality of tangent spaces) follows from the description of the infinitesimal deformations in Remark 99.7.10 and finiteness of cohomology of coherent sheaves on proper algebraic spaces over fields (Cohomology of Spaces, Lemma 69.20.2). Axiom [4] (effectiveness of formal objects) follows from Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 76.42.11). As usual, the trickiest to verify is axiom [5] (openness of versality). One can for example use the obstruction theory described in Remark 99.7.9 and the description of deformations in Remark 99.7.10 to do this using the criterion in Artin's Axioms, Lemma 98.22.2. Please compare with the second proof of Lemma 99.5.11.

99.9. The Hilbert functor

0CZX In this section we prove the Hilb functor is an algebraic space.

0CZY **Situation 99.9.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. For any scheme T over B we will denote X_T the base change of X to T. Given such a T we set

$$\mathrm{Hilb}_{X/B}(T) = \left\{ \begin{array}{c} \text{closed subspaces } Z \subset X_T \text{ such that } Z \to T \\ \text{is of finite presentation, flat, and proper} \end{array} \right\}$$

Since base change preserves the required properties (Spaces, Lemma 65.12.3 and Morphisms of Spaces, Lemmas 67.28.3, 67.30.4, and 67.40.3) we obtain a functor

0CZZ (99.9.1.1) $$\mathrm{Hilb}_{X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

This is the *Hilbert functor* associated to X/B.

In Situation 99.9.1 we sometimes think of $\mathrm{Hilb}_{X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Hilb}_{X/S} \to B$. Namely, if T is a scheme over S, then an element of $\mathrm{Hilb}_{X/B}(T)$ is a pair (h, Z) where h is a morphism $h : T \to B$ and $Z \subset X_T = X \times_{B,h} T$ is a closed subscheme, flat, proper, and of finite presentation over T. In particular, when we say that $\mathrm{Hilb}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Of course the Hilbert functor is just a special case of the Quot functor.

0D00 **Lemma 99.9.2.** *In Situation 99.9.1 we have* $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$.

Proof. Let T be a scheme over B. Given an element $Z \in \mathrm{Hilb}_{X/B}(T)$ we can consider the quotient $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$ where $i : Z \to X_T$ is the inclusion morphism. Note that $i_*\mathcal{O}_Z$ is quasi-coherent. Since $Z \to T$ and $X_T \to T$ are of finite presentation, we see that i is of finite presentation (Morphisms of Spaces, Lemma 67.28.9), hence $i_*\mathcal{O}_Z$ is an $\mathcal{O}_{X_T}$-module of finite presentation (Descent on Spaces, Lemma 74.6.7). Since $Z \to T$ is proper we see that $i_*\mathcal{O}_Z$ has support proper over T (as defined in Derived Categories of Spaces, Section 75.7). Since $\mathcal{O}_Z$ is flat over T and i is affine, we see that $i_*\mathcal{O}_Z$ is flat over T (small argument omitted). Hence $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$ is an element of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$.

Conversely, given an element $\mathcal{O}_{X_T} \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$, we can consider the closed immersion $i : Z \to X_T$ corresponding to the quasi-coherent ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X_T} \to \mathcal{Q})$ (Morphisms of Spaces, Lemma 67.13.1). By construction of Z we see that $\mathcal{Q} = i_*\mathcal{O}_Z$. Then we can read the arguments given above backwards to see that Z defines an element of $\mathrm{Hilb}_{X/B}(T)$. For example, $\mathcal{I}$ is quasi-coherent of finite type (Modules on Sites, Lemma 18.24.2) hence $i : Z \to X_T$ is of finite presentation (Morphisms of Spaces, Lemma 67.28.12) hence $Z \to T$ is of finite presentation (Morphisms of Spaces, Lemma 67.28.2). Properness of $Z \to T$ follows from the discussion in Derived Categories of Spaces, Section 75.7. Flatness of $Z \to T$ follows from flatness of $\mathcal{Q}$ over T.

We omit the (immediate) verification that the two constructions given above are mutually inverse. □

Sanity check: $\mathrm{Hilb}_{X/B}$ sheaf plays the same role among algebraic spaces over S.

0D3W **Lemma 99.9.3.** *In Situation 99.9.1. Let T be an algebraic space over S. We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Hilb}_{X/B}) = \left\{ \begin{array}{c} (h, Z) \text{ where } h : T \to B,\ Z \subset X_T \\ \text{finite presentation, flat, proper over } T \end{array} \right\}$$

where $X_T = X \times_{B,h} T$.

Proof. By Lemma 99.9.2 we have $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Thus we can apply Lemma 99.8.3 to see that the left hand side is bijective with the set of surjections $\mathcal{O}_{X_T} \to \mathcal{Q}$ which are finitely presented, flat over T, and have support proper over T. Arguing exactly as in the proof of Lemma 99.9.2 we see that such quotients correspond exactly to the closed immersions $Z \to X_T$ such that $Z \to T$ is proper, flat, and of finite presentation. □

0D01 **Proposition 99.9.4.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. If f is of finite presentation and separated, then* $\mathrm{Hilb}_{X/B}$ *is an algebraic space locally of finite presentation over B.*

Proof. Immediate consequence of Lemma 99.9.2 and Proposition 99.8.4. □

99.10. The Picard stack

0D02 The Picard stack for a morphism of algebraic spaces was introduced in Examples of Stacks, Section 95.16. We will deduce it is an open substack of the stack of coherent sheaves (in good cases) from the following lemma.

0D03 **Lemma 99.10.1.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S which is flat, of finite presentation, and proper. The natural map*

$$\mathcal{P}ic_{X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

is representable by open immersions.

Proof. Observe that the map simply sends a triple $(T, g, \mathcal{L})$ as in Examples of Stacks, Section 95.16 to the same triple $(T, g, \mathcal{L})$ but where now we view this as a triple of the kind described in Situation 99.5.1. This works because the invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ is certainly a finitely presented $\mathcal{O}_{X_T}$-module, it is flat over T because $X_T \to T$ is flat, and the support is proper over T as $X_T \to T$ is proper (Morphisms of Spaces, Lemmas 67.30.4 and 67.40.3). Thus the statement makes sense.

Having said this, it is clear that the content of the lemma is the following: given an object $(T, g, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ there is an open subscheme $U \subset T$ such that for a morphism of schemes $T' \to T$ the following are equivalent

(a) $T' \to T$ factors through U,
(b) the pullback $\mathcal{F}_{T'}$ of $\mathcal{F}$ by $X_{T'} \to X_T$ is invertible.

Let $W \subset |X_T|$ be the set of points $x \in |X_T|$ such that $\mathcal{F}$ is locally free in a neighbourhood of x. By More on Morphisms of Spaces, Lemma 76.23.8. W is open and formation of W commutes with arbitrary base change. Clearly, if $T' \to T$ satisfies (b), then $|X_{T'}| \to |X_T|$ maps into W. Hence we may replace T by the open $T \setminus f_T(|X_T| \setminus W)$ in order to construct U. After doing so we reach the situation where $\mathcal{F}$ is finite locally free. In this case we get a disjoint union decomposition $X_T = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$ into open and closed subspaces such that the restriction of $\mathcal{F}$ is locally free of rank i on X_i. Then clearly

$$U = T \setminus f_T(|X_0| \cup |X_2| \cup |X_3| \cup \ldots)$$

works. (Note that if we assume that T is quasi-compact, then X_T is quasi-compact hence only a finite number of X_i are nonempty and so U is indeed open.) □

0D04 **Proposition 99.10.2.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. If f is flat, of finite presentation, and proper, then $\mathcal{P}ic_{X/B}$ is an algebraic stack.*

Proof. Immediate consequence of Lemma 99.10.1, Algebraic Stacks, Lemma 94.15.4 and either Theorem 99.5.12 or Theorem 99.6.1 □

99.11. The Picard functor

0D24 In this section we revisit the Picard functor discussed in Picard Schemes of Curves, Section 44.4. The discussion will be more general as we want to study the Picard functor of a morphism of algebraic spaces as in the section on the Picard stack, see Section 99.10.

Let S be a scheme and let X be an algebraic space over S. An invertible sheaf on X is an invertible $\mathcal{O}_X$-module on $X_{étale}$, see Modules on Sites, Definition 18.32.1. The group of isomorphism classes of invertible modules is denoted $\mathrm{Pic}(X)$, see Modules on Sites, Definition 18.32.6. Given a morphism $f : X \to Y$ of algebraic spaces over S pullback defines a group homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. The assignment $X \rightsquigarrow \mathrm{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

0D25 **Situation 99.11.1.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. We define

$$\mathrm{Pic}_{X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

as the fppf sheafification of the functor which to a scheme T over B associates the group $\mathrm{Pic}(X_T)$.

In Situation 99.11.1 we sometimes think of $\mathrm{Pic}_{X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Pic}_{X/B} \to B$. In this point of view, we define $\mathrm{Pic}_{X/B}$ to be the fppf sheafification of the functor

$$T/S \longmapsto \{(h, \mathcal{L}) \mid h : T \to B,\ \mathcal{L} \in \mathrm{Pic}(X \times_{B,h} T)\}$$

In particular, when we say that $\mathrm{Pic}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

An often used remark is that if T is a scheme over B, then $\mathrm{Pic}_{X_T/T}$ is the restriction of $\mathrm{Pic}_{X/B}$ to $(Sch/T)_{fppf}$.

0D26 **Lemma 99.11.2.** *In Situation 99.11.1 the functor* $\mathrm{Pic}_{X/B}$ *is the sheafification of the functor* $T \mapsto \mathrm{Ob}(\mathcal{P}ic_{X/B,T})/\cong$.

Proof. Since the fibre category $\mathcal{P}ic_{X/B,T}$ of the Picard stack $\mathcal{P}ic_{X/B}$ over T is the category of invertible sheaves on X_T (see Section 99.10 and Examples of Stacks, Section 95.16) this is immediate from the definitions. □

It turns out to be nontrivial to see what the value of $\mathrm{Pic}_{X/B}$ is on schemes T over B. Here is a lemma that helps with this task.

0D27 **Lemma 99.11.3.** *In Situation 99.11.1. If* $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ *is an isomorphism for all schemes* T *over* B*, then*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/B}(T)$$

is an exact sequence for all T.

Proof. We may replace B by T and X by X_T and assume that $B = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_B$-module. If $f^*\mathcal{N} \cong \mathcal{O}_X$, then we see that $f_*f^*\mathcal{N} \cong f_*\mathcal{O}_X \cong \mathcal{O}_B$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \to f_*f^*\mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_B \to f_*f^*\mathcal{O}_B$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_B$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is in the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/B}(B)$. Then there exists an fppf covering $\{B_i \to B\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{B_i}. Choose a trivializing section s_i. Then $\mathrm{pr}_0^*s_i$ and $\mathrm{pr}_1^*s_j$ are

both trivialising sections of $\mathcal{L}$ over $X_{B_i \times_B B_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_B B_j}, \mathcal{O}^*_{X_{B_i \times_B B_j}}) = \Gamma(B_i \times_B B_j, \mathcal{O}^*_{B_i \times_N B_j})$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $B_i \times_B B_j \times_B B_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{B_i \to B\}$. By Descent, Proposition 35.5.2 there is an invertible $\mathcal{O}_B$-module $\mathcal{N}$ with trivializations over B_i whose associated descent datum is $\{f_{ij}\}$. (The proposition applies because B is a scheme by the replacement performed at the start of the proof.) Then $f^*\mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful. □

0D28 **Lemma 99.11.4.** *In Situation 99.11.1 let $\sigma : B \to X$ be a section. Assume that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B. Then*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/B}(T) \to 0$$

is a split exact sequence with splitting given by $\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T)$.

Proof. Denote $K(T) = \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$. Since σ is a section of f we see that $\mathrm{Pic}(X_T)$ is the direct sum of $\mathrm{Pic}(T)$ and $K(T)$. Thus by Lemma 99.11.3 we see that $K(T) \subset \mathrm{Pic}_{X/B}(T)$ for all T. Moreover, it is clear from the construction that $\mathrm{Pic}_{X/B}$ is the sheafification of the presheaf K. To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \to T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j. Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i$ for all i. Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma^*_{T_i \times_T T_j} \varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \to \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij}, then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some section f_{ijk} of the structure sheaf of $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X, we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \to X\}$. By Descent on Spaces, Proposition 74.4.1 there is an invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}$ and an isomorphism $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. □

In Situation 99.11.1 let $\sigma : B \to X$ be a section. We denote $\mathcal{P}ic_{X/B,\sigma}$ the category defined as follows:

(1) An object is a quadruple $(T, h, \mathcal{L}, \alpha)$, where $(T, h, \mathcal{L})$ is an object of $\mathcal{P}ic_{X/B}$ over T and $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ is an isomorphism.
(2) A morphism $(g, \varphi) : (T, h, \mathcal{L}, \alpha) \to (T', h', \mathcal{L}', \alpha')$ is given by a morphism of schemes $g : T \to T'$ with $h = h' \circ g$ and an isomorphism $\varphi : (g')^*\mathcal{L}' \to \mathcal{L}$ such that $\sigma_T^*\varphi \circ g^*\alpha' = \alpha$. Here $g' : X_{T'} \to X_T$ is the base change of g.

There is a natural faithful forgetful functor

$$\mathcal{P}ic_{X/B,\sigma} \longrightarrow \mathcal{P}ic_{X/B}$$

In this way we view $\mathcal{P}ic_{X/B,\sigma}$ as a category over $(Sch/S)_{fppf}$.

0D29 **Lemma 99.11.5.** *In Situation 99.11.1 let $\sigma : B \to X$ be a section. Then $\mathcal{P}ic_{X/B,\sigma}$ as defined above is a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. We already know that $\mathcal{P}ic_{X/B}$ is a stack in groupoids over $(Sch/S)_{fppf}$ by Examples of Stacks, Lemma 95.16.1. Let us show descent for objects for $\mathcal{P}ic_{X/B,\sigma}$. Let $\{T_i \to T\}$ be an fppf covering and let $\xi_i = (T_i, h_i, \mathcal{L}_i, \alpha_i)$ be an object of $\mathcal{P}ic_{X/B,\sigma}$ lying over T_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. Applying the result for $\mathcal{P}ic_{X/B}$ we see that we may assume we have an object $(T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T which pulls back to ξ_i for all i. Then we get

$$\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^*\mathcal{L}_i = (T_i \to T)^*\sigma_T^*\mathcal{L}$$

Since the maps φ_{ij} are compatible with the α_i we see that α_i and α_j pullback to the same map on $T_i \times_T T_j$. By descent of quasi-coherent sheaves (Descent, Proposition 35.5.2, we see that the α_i are the restriction of a single map $\alpha : \mathcal{O}_T \to \sigma_T^*\mathcal{L}$ as desired. We omit the proof of descent for morphisms. □

0D2A **Lemma 99.11.6.** *In Situation 99.11.1 let $\sigma : B \to X$ be a section. The morphism $\mathcal{P}ic_{X/B,\sigma} \to \mathcal{P}ic_{X/B}$ is representable, surjective, and smooth.*

Proof. Let T be a scheme and let $(Sch/T)_{fppf} \to \mathcal{P}ic_{X/B}$ be given by the object $\xi = (T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T. We have to show that

$$(Sch/T)_{fppf} \times_{\xi, \mathcal{P}ic_{X/B}} \mathcal{P}ic_{X/B,\sigma}$$

is representable by a scheme V and that the corresponding morphism $V \to T$ is surjective and smooth. See Algebraic Stacks, Sections 94.6, 94.9, and 94.10. The forgetful functor $\mathcal{P}ic_{X/B,\sigma} \to \mathcal{P}ic_{X/B}$ is faithful on fibre categories and for T'/T the set of isomorphism classes is the set of isomorphisms

$$\alpha' : \mathcal{O}_{T'} \longrightarrow (T' \to T)^*\sigma_T^*\mathcal{L}$$

See Algebraic Stacks, Lemma 94.9.2. We know this functor is representable by an affine scheme U of finite presentation over T by Proposition 99.4.3 (applied to $\mathrm{id} : T \to T$ and $\mathcal{O}_T$ and $\sigma^*\mathcal{L}$). Working Zariski locally on T we may assume that $\sigma_T^*\mathcal{L}$ is isomorphic to $\mathcal{O}_T$ and then we see that our functor is representable by $\mathbf{G}_m \times T$ over T. Hence $U \to T$ Zariski locally on T looks like the projection $\mathbf{G}_m \times T \to T$ which is indeed smooth and surjective. □

0D2B **Lemma 99.11.7.** *In Situation 99.11.1 let $\sigma : B \to X$ be a section. If $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B, then $\mathcal{P}ic_{X/B,\sigma} \to (Sch/S)_{fppf}$ is fibred in setoids with set of isomorphism classes over T given by*

$$\coprod\nolimits_{h:T\to B} \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X \times_{B,h} T) \to \mathrm{Pic}(T))$$

Proof. If $\xi = (T, h, \mathcal{L}, \alpha)$ is an object of $\mathcal{P}ic_{X/B,\sigma}$ over T, then an automorphism φ of ξ is given by multiplication with an invertible global section u of the structure sheaf of X_T such that moreover $\sigma_T^*u = 1$. Then $u = 1$ by our assumption that $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Hence $\mathcal{P}ic_{X/B,\sigma}$ is fibred in setoids over $(Sch/S)_{fppf}$. Given T and $h : T \to B$ the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ is the same as the set of isomorphism classes of $\mathcal{L}$ with $\sigma_T^*\mathcal{L} \cong \mathcal{O}_T$ (isomorphism not specified).

This is clear because any two choices of α differ by a global unit on T and this is the same thing as a global unit on X_T. □

0D2C **Proposition 99.11.8.** *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that*

(1) *f is flat, of finite presentation, and proper, and*
(2) *$\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over B.*

Then $\mathrm{Pic}_{X/B}$ is an algebraic space.

In the situation of the proposition the algebraic stack $\mathcal{P}ic_{X/B}$ is a gerbe over the algebraic space $\mathrm{Pic}_{X/B}$. After developing the general theory of gerbes, this provides a shorter proof of the proposition (but using more general theory).

Proof. There exists a surjective, flat, finitely presented morphism $B' \to B$ of algebraic spaces such that the base change $X' = X \times_B B'$ over B' has a section: namely, we can take $B' = X$. Observe that $\mathrm{Pic}_{X'/B'} = B' \times_B \mathrm{Pic}_{X/B}$. Hence $\mathrm{Pic}_{X'/B'} \to \mathrm{Pic}_{X/B}$ is representable by algebraic spaces, surjective, flat, and finitely presented. Hence, if we can show that $\mathrm{Pic}_{X'/B'}$ is an algebraic space, then it follows that $\mathrm{Pic}_{X/B}$ is an algebraic space by Bootstrap, Theorem 80.10.1. In this way we reduce to the case described in the next paragraph.

In addition to the assumptions of the proposition, assume that we have a section $\sigma : B \to X$. By Proposition 99.10.2 we see that $\mathcal{P}ic_{X/B}$ is an algebraic stack. By Lemma 99.11.6 and Algebraic Stacks, Lemma 94.15.4 we see that $\mathcal{P}ic_{X/B,\sigma}$ is an algebraic stack. By Lemma 99.11.7 and Algebraic Stacks, Lemma 94.8.2 we see that $T \mapsto \mathrm{Ker}(\sigma_T^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$ is an algebraic space. By Lemma 99.11.4 this functor is the same as $\mathrm{Pic}_{X/B}$. □

0D2D **Lemma 99.11.9.** *With assumptions and notation as in Proposition 99.11.8. Then the diagonal $\mathrm{Pic}_{X/B} \to \mathrm{Pic}_{X/B} \times_B \mathrm{Pic}_{X/B}$ is representable by immersions. In other words, $\mathrm{Pic}_{X/B} \to B$ is locally separated.*

Proof. Let T be a scheme over B and let $s, t \in \mathrm{Pic}_{X/B}(T)$. We want to show that there exists a locally closed subscheme $Z \subset T$ such that $s|_Z = t|_Z$ and such that a morphism $T' \to T$ factors through Z if and only if $s|_{T'} = t|_{T'}$.

We first reduce the general problem to the case where s and t come from invertible modules on X_T. We suggest the reader skip this step. Choose an fppf covering $\{T_i \to T\}_{i \in I}$ such that $s|_{T_i}$ and $t|_{T_i}$ come from $\mathrm{Pic}(X_{T_i})$ for all i. Suppose that we can show the result for all the pairs $s|_{T_i}, t|_{T_i}$. Then we obtain locally closed subschemes $Z_i \subset T_i$ with the desired universal property. It follows that Z_i and Z_j have the same scheme theoretic inverse image in $T_i \times_T T_j$. This determines a descend datum on Z_i/T_i. Since $Z_i \to T_i$ is locally quasi-finite, it follows from More on Morphisms, Lemma 37.57.1 that we obtain a locally quasi-finite morphism $Z \to T$ recovering $Z_i \to T_i$ by base change. Then $Z \to T$ is an immersion by Descent, Lemma 35.24.1. Finally, because $\mathrm{Pic}_{X/B}$ is an fppf sheaf, we conclude that $s|_Z = t|_Z$ and that Z satisfies the universal property mentioned above.

Assume s and t come from invertible modules $\mathcal{V}$, $\mathcal{W}$ on X_T. Set $\mathcal{L} = \mathcal{V} \otimes \mathcal{W}^{\otimes -1}$ We are looking for a locally closed subscheme Z of T such that $T' \to T$ factors through Z if and only if $\mathcal{L}_{X_{T'}}$ is the pullback of an invertible sheaf on T', see Lemma 99.11.3. Hence the existence of Z follows from More on Morphisms of Spaces, Lemma 76.53.1. □

99.12. Relative morphisms

0D19 We continue the discussion from Criteria for Representability, Section 97.10. In that section, starting with a scheme S and morphisms of algebraic spaces $Z \to B$ and $X \to B$ over S we constructed a functor

$$Mor_B(Z, X) : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longmapsto \{f : Z_T \to X_T\}$$

We sometimes think of $Mor_B(Z, X)$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $Mor_B(Z, X) \to B$. Namely, if T is a scheme over S, then an element of $Mor_B(Z, X)(T)$ is a pair (f, h) where h is a morphism $h : T \to B$ and $f : Z \times_{B,h} T \to X \times_{B,h} T$ is a morphism of algebraic spaces over T. In particular, when we say that $Mor_B(Z, X)$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

0D1A **Lemma 99.12.1.** *Let S be a scheme. Consider morphisms of algebraic spaces $Z \to B$ and $X \to B$ over S. If $X \to B$ is separated and $Z \to B$ is of finite presentation, flat, and proper, then there is a natural injective transformation of functors*

$$Mor_B(Z, X) \longrightarrow \mathrm{Hilb}_{Z \times_B X/B}$$

which maps a morphism $f : Z_T \to X_T$ to its graph.

Proof. Given a scheme T over B and a morphism $f_T : Z_T \to X_T$ over T, the graph of f is the morphism $\Gamma_f = (\mathrm{id}, f) : Z_T \to Z_T \times_T X_T = (Z \times_B X)_T$. Recall that being separated, flat, proper, or finite presentation are properties of morphisms of algebraic spaces which are stable under base change (Morphisms of Spaces, Lemmas 67.4.4, 67.30.4, 67.40.3, and 67.28.3). Hence Γ_f is a closed immersion by Morphisms of Spaces, Lemma 67.4.6. Moreover, $\Gamma_f(Z_T)$ is flat, proper, and of finite presentation over T. Thus $\Gamma_f(Z_T)$ defines an element of $\mathrm{Hilb}_{Z \times_B X/B}(T)$. To show the transformation is injective it suffices to show that two morphisms with the same graph are the same. This is true because if $Y \subset (Z \times_B X)_T$ is the graph of a morphism f, then we can recover f by using the inverse of $\mathrm{pr}_1|_Y : Y \to Z_T$ composed with $\mathrm{pr}_2|_Y$. □

0D1B **Lemma 99.12.2.** *Assumption and notation as in Lemma 99.12.1. The transformation $Mor_B(Z, X) \longrightarrow \mathrm{Hilb}_{Z \times_B X/B}$ is representable by open immersions.*

Proof. Let T be a scheme over B and let $Y \subset (Z \times_B X)_T$ be an element of $\mathrm{Hilb}_{Z \times_B X/B}(T)$. Then we see that Y is the graph of a morphism $Z_T \to X_T$ over T if and only if $k = \mathrm{pr}_1|_Y : Y \to Z_T$ is an isomorphism. By More on Morphisms of Spaces, Lemma 76.49.6 there exists an open subscheme $V \subset T$ such that for any morphism of schemes $T' \to T$ we have $k_{T'} : Y_{T'} \to Z_{T'}$ is an isomorphism if and only if $T' \to T$ factors through V. This proves the lemma. □

0D1C **Proposition 99.12.3.** *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Assume $X \to B$ is of finite presentation and separated and $Z \to B$ is of finite presentation, flat, and proper. Then $Mor_B(Z, X)$ is an algebraic space locally of finite presentation over B.*

Proof. Immediate consequence of Lemma 99.12.2 and Proposition 99.9.4. □

99.13. The stack of algebraic spaces

0D1D This section continuous the discussion started in Examples of Stacks, Sections 95.7, 95.8, and 95.12. Working over $\mathbf{Z}$, the discussion therein shows that we have a stack in groupoids

$$p'_{ft} : \mathcal{S}paces'_{ft} \longrightarrow Sch_{fppf}$$

parametrizing (nonflat) families of finite type algebraic spaces. More precisely, an object[3] of $\mathcal{S}paces'_{ft}$ is a finite type morphism $X \to S$ from an algebraic space X to a scheme S and a morphism $(X' \to S') \to (X \to S)$ is given by a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces and $g : S' \to S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian in the category of algebraic spaces. The functor p'_{ft} sends $(X \to S)$ to S and sends (f, g) to g. We define a full subcategory

$$\mathcal{S}paces'_{fp,flat,proper} \subset \mathcal{S}paces'_{ft}$$

consisting of objects $X \to S$ of $\mathcal{S}paces'_{ft}$ such that $X \to S$ is of finite presentation, flat, and proper. We denote

$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$

the restriction of the functor p'_{ft} to the indicated subcategory. We first review the results already obtained in the references listed above, and then we start adding further results.

0D1E **Lemma 99.13.1.** *The category $\mathcal{S}paces'_{ft}$ is fibred in groupoids over Sch_{fppf}. The same is true for $\mathcal{S}paces'_{fp,flat,proper}$.*

Proof. We have seen this in Examples of Stacks, Section 95.12 for the case of $\mathcal{S}paces'_{ft}$ and this easily implies the result for the other case. However, let us also prove this directly by checking conditions (1) and (2) of Categories, Definition 4.35.1.

Condition (1). Let $X \to S$ be an object of $\mathcal{S}paces'_{ft}$ and let $S' \to S$ be a morphism of schemes. Then we set $X' = S' \times_S X$. Note that $X' \to S'$ is of finite type by Morphisms of Spaces, Lemma 67.23.3. to obtain a morphism $(X' \to S') \to (X \to S)$ lying over $S' \to S$. Argue similarly for the other case using Morphisms of Spaces, Lemmas 67.28.3, 67.30.4, and 67.40.3.

Condition (2). Consider morphisms $(f, g) : (X' \to S') \to (X \to S)$ and $(a, b) : (Y \to T) \to (X \to S)$ of $\mathcal{S}paces'_{ft}$. Given a morphism $h : T \to S'$ with $g \circ h = b$ we have to show there is a unique morphism $(k, h) : (Y \to T) \to (X' \to S')$ of $\mathcal{S}paces'_{ft}$ such that $(f, g) \circ (k, h) = (a, b)$. This is clear from the fact that $X' = S' \times_S X$. The same therefore works for any full subcategory of $\mathcal{S}paces'_{ft}$ satisfying (1). □

[3] We always perform a replacement as in Examples of Stacks, Lemma 95.8.2.

0D1F **Lemma 99.13.2.** *The diagonal*

$$\Delta : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow \mathcal{S}paces'_{fp,flat,proper} \times \mathcal{S}paces'_{fp,flat,proper}$$

is representable by algebraic spaces.

Proof. We will use criterion (2) of Algebraic Stacks, Lemma 94.10.11. Let S be a scheme and let X and Y be algebraic spaces of finite presentation over S, flat over S, and proper over S. We have to show that the functor

$$Isom_S(X, Y) : (Sch/S)_{fppf} \longrightarrow Sets, \quad T \longmapsto \{f : X_T \to Y_T \text{ isomorphism}\}$$

is an algebraic space. An elementary argument shows that $Isom_S(X, Y)$ sits in a fibre product

$$\begin{array}{ccc} Isom_S(X,Y) & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} \\ Mor_S(X,Y) \times Mor_S(Y,X) & \longrightarrow & Mor_S(X,X) \times Mor_S(Y,Y) \end{array}$$

The bottom arrow sends (φ, ψ) to $(\psi \circ \varphi, \varphi \circ \psi)$. By Proposition 99.12.3 the functors on the bottom row are algebraic spaces over S. Hence the result follows from the fact that the category of algebraic spaces over S has fibre products. □

0D1G **Lemma 99.13.3.** *The category $\mathcal{S}paces'_{ft}$ is a stack in groupoids over Sch_{fppf}. The same is true for $\mathcal{S}paces'_{fp,flat,proper}$.*

Proof. The reason this lemma holds is the slogan: any fppf descent datum for algebraic spaces is effective, see Bootstrap, Section 80.11. More precisely, the lemma for $\mathcal{S}paces'_{ft}$ follows from Examples of Stacks, Lemma 95.8.1 as we saw in Examples of Stacks, Section 95.12. However, let us review the proof. We need to check conditions (1), (2), and (3) of Stacks, Definition 8.5.1.

Property (1) we have seen in Lemma 99.13.1.

Property (2) follows from Lemma 99.13.2 in the case of $\mathcal{S}paces'_{fp,flat,proper}$. In the case of $\mathcal{S}paces'_{ft}$ it follows from Examples of Stacks, Lemma 95.7.2 (and this is really the "correct" reference).

Condition (3) for $\mathcal{S}paces'_{ft}$ is checked as follows. Suppose given

(1) an fppf covering $\{U_i \to U\}_{i \in I}$ in Sch_{fppf},
(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$.

We have to show there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. This follows from Bootstrap, Lemma 80.11.3 part (2). By Descent on Spaces, Lemma 74.11.11 we see that $X \to U$ is of finite type. In the case of $\mathcal{S}paces'_{fp,flat,proper}$ one additionally uses Descent on Spaces, Lemma 74.11.12, 74.11.13, and 74.11.19 in the last step. □

Sanity check: the stacks $\mathcal{S}paces'_{ft}$ and $\mathcal{S}paces'_{fp,flat,proper}$ play the same role among algebraic spaces.

0E93 **Lemma 99.13.4.** *Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 94.7, 94.8, and 94.13). We have an equivalence of categories*

$$\left\{\begin{array}{c}\text{morphisms of algebraic spaces} \\ X \to T \text{ of finite type}\end{array}\right\} \longrightarrow \mathrm{Mor}_{Cat/Sch_{fppf}}(\mathcal{S}_T, \mathcal{S}paces'_{ft})$$

and an equivalence of categories

$$\left\{\begin{array}{l}\text{morphisms of algebraic spaces } X \to T \\ \text{of finite presentation, flat, and proper}\end{array}\right\} \longrightarrow \mathrm{Mor}_{Cat/Sch_{fppf}}(\mathcal{S}_T, \mathcal{S}paces'_{fp,flat,proper})$$

Proof. We are going to deduce this lemma from the fact that it holds for schemes (essentially by construction of the stacks) and the fact that fppf descent data for algebraic spaces over algerbaic spaces are effective. We strongly encourage the reader to skip the proof.

The construction from left to right in either arrow is straightforward: given $X \to T$ of finite type the functor $\mathcal{S}_T \to \mathcal{S}paces'_{ft}$ assigns to U/T the base change $X_U \to U$. We will explain how to construct a quasi-inverse.

If T is a scheme, then there is a quasi-inverse by the 2-Yoneda lemma, see Categories, Lemma 4.41.2. Let $p : U \to T$ be a surjective étale morphism where U is a scheme. Let $R = U \times_T U$ with projections $s, t : R \to U$. Observe that we obtain morphisms

$$\mathcal{S}_{U \times_T U \times_T U} \begin{array}{c}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{array} \mathcal{S}_R \begin{array}{c}\longrightarrow \\ \longrightarrow\end{array} \mathcal{S}_U \longrightarrow \mathcal{S}_T$$

satisfying various compatibilities (on the nose).

Let $G : \mathcal{S}_T \to \mathcal{S}paces'_{ft}$ be a functor over Sch_{fppf}. The restriction of G to $\mathcal{S}_U$ via the map displayed above corresponds to a finite type morphism $X_U \to U$ of algebraic spaces via the 2-Yoneda lemma. Since $p \circ s = p \circ t$ we see that $R \times_{s,U} X_U$ and $R \times_{t,U} X_U$ both correspond to the restriction of G to $\mathcal{S}_R$. Thus we obtain a canonical isomorphism $\varphi : X_U \times_{U,t} R \to R \times_{s,U} X_U$ over R. This isomorphism satisfies the cocycle condition by the various compatibilities of the diagram given above. Thus a descent datum which is effective by Bootstrap, Lemma 80.11.3 part (2). In other words, we obtain an object $X \to T$ of the right hand side category. We omit checking the construction $G \rightsquigarrow X$ is functorial and that it is quasi-inverse to the other construction. In the case of $\mathcal{S}paces'_{fp,flat,proper}$ one additionally uses Descent on Spaces, Lemma 74.11.12, 74.11.13, and 74.11.19 in the last step to see that $X \to T$ is of finite presentation, flat, and proper. □

0D1H **Remark 99.13.5.** Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let B-$\mathcal{S}paces'_{ft}$ be the category consisting of pairs $(X \to S, h : S \to B)$ where $X \to S$ is an object of $\mathcal{S}paces'_{ft}$ and $h : S \to B$ is a morphism. A morphism $(X' \to S', h') \to (X \to S, h)$ in B-$\mathcal{S}paces'_{ft}$ is a morphism (f, g) in $\mathcal{S}paces'_{ft}$ such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-}\mathcal{S}paces'_{ft} & \longrightarrow & \mathcal{S}paces'_{ft} \\ \downarrow & & \downarrow \\ (Sch/B)_{fppf} & \longrightarrow & Sch_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case $\mathcal{S}paces'_{ft}$ to the case of families over a given base algebraic space. Of course, a similar construction works for B-$\mathcal{S}paces'_{fp,flat,proper}$

0D1I **Lemma 99.13.6.** *The stack $p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \to \mathcal{S}ch_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).*

Proof. Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. By Limits of Spaces, Lemma 70.7.1 the category of algebraic spaces of finite presentation over T is the colimit of the categories of algebraic spaces of finite presentation over T_i. To finish the proof use that flatness and properness descends through the limit, see Limits of Spaces, Lemmas 70.6.12 and 70.6.13. □

0D1J **Lemma 99.13.7.** *Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{S}paces'_{fp,flat,proper,S'} \downarrow \mathcal{S}paces'_{fp,flat,proper,S} \times_{\mathcal{S}paces'_{fp,flat,proper,T}} \mathcal{S}paces'_{fp,flat,proper,T'}$$

is an equivalence.

Proof. The functor is an equivalence if we drop "proper" from the list of conditions and replace "of finite presentation" by "locally of finite presentation", see Pushouts of Spaces, Lemma 81.6.7. Thus it suffices to show that given a morphism $X' \to S'$ of an algebraic space to S' which is flat and locally of finite presentation, then $X' \to S'$ is proper if and only if $S \times_{S'} X' \to S$ and $T' \times_{S'} X' \to T'$ are proper. One implication follows from the fact that properness is preserved under base change (Morphisms of Spaces, Lemma 67.40.3) and the other from the fact that properness of $S \times_{S'} X' \to S$ implies properness of $X' \to S'$ by More on Morphisms of Spaces, Lemma 76.10.2. □

0D1K **Lemma 99.13.8.** *Let k be a field and let $x = (X \to \mathrm{Spec}(k))$ be an object of $\mathcal{X} = \mathcal{S}paces'_{fp,flat,proper}$ over $\mathrm{Spec}(k)$.*

1. *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $Inf(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and*
2. *in general the vector spaces $T_x(k)$ and $Inf_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.*

Proof. The discussion in Artin's Axioms, Section 98.8 only applies to fields of finite type over the base scheme $\mathrm{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 99.13.7 and we may apply Artin's Axioms, Lemma 98.21.2 to get the vector spaces $T_x(k)$ and $\mathrm{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in (1) by Artin's Axioms, Remark 98.21.7. With this out of the way we can start the proof. Observe that the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]) = \mathrm{Spec}(k[k])$ has conormal module k. Hence the formula

in Deformation Theory, Lemma 91.14.2 describing infinitesimal deformations of X and infinitesimal automorphisms of X become

$$T_x(k) = \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X) \quad \text{and} \quad \mathrm{Inf}_x(k) = \mathrm{Ext}^0_{\mathcal{O}_X}(NL_{X/k}, \mathcal{O}_X)$$

By More on Morphisms of Spaces, Lemma 76.21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1. By Derived Categories of Spaces, Lemma 75.8.4 the displayed Ext-groups are finite k-vector spaces and the proof is complete. □

Beware that openness of versality (as proved in the next lemma) is a bit strange because our stack does not satisfy formal effectiveness, see Examples, Section 110.71. Later we will apply the openness of versality to suitable substacks of $\mathcal{S}paces'_{fp,flat,proper}$ which do satisfy formal effectiveness to conclude that these stacks are algebraic.

0D3X **Lemma 99.13.9.** *The stack in groupoids $\mathcal{X} = \mathcal{S}paces'_{fp,flat,proper}$ satisfies openness of versality over* $\mathrm{Spec}(\mathbf{Z})$. *Similarly, after base change (Remark 99.13.5) openness of versality holds over any Noetherian base scheme S.*

Proof. For the "usual" proof of this fact, please see the discussion in the remark following this proof. We will prove this using Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving, see Lemmas 99.13.2, 99.13.7, and 99.13.6. Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3.

Let (R_n) be an inverse system of rings such that $R_n \to R_m$ is surjective with square zero kernel for all $n \geq m$. Let $X_n \to \mathrm{Spec}(R_n)$ be a finitely presented, flat, proper morphism where X_n is an algebraic space and let $X_{n+1} \to X_n$ be a morphism over $\mathrm{Spec}(R_{n+1})$ inducing an isomorphism $X_n = X_{n+1} \times_{\mathrm{Spec}(R_{n+1})} \mathrm{Spec}(R_n)$. We have to find a flat, proper, finitely presented morphism $X \to \mathrm{Spec}(\lim R_n)$ whose source is an algebraic space such that X_n is the base change of X for all n.

Let $I_n = \mathrm{Ker}(R_n \to R_1)$. We may think of $(X_1 \subset X_n) \to (\mathrm{Spec}(R_1) \subset \mathrm{Spec}(R_n))$ as a morphism of first order thickenings. (Please read some of the material on thickenings of algebraic spaces in More on Morphisms of Spaces, Section 76.9 before continuing.) The structure sheaf of X_n is an extension

$$0 \to \mathcal{O}_{X_1} \otimes_{R_1} I_n \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_1} \to 0$$

over $0 \to I_n \to R_n \to R_1$, see More on Morphisms of Spaces, Lemma 76.18.1. Let's consider the extension

$$0 \to \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n \to \lim \mathcal{O}_{X_n} \to \mathcal{O}_{X_1} \to 0$$

over $0 \to \lim I_n \to \lim R_n \to R_1 \to 0$. The displayed sequence is exact as the $R^1 \lim$ of the system of kernels is zero by Derived Categories of Spaces, Lemma 75.5.4. Observe that the map

$$\mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \longrightarrow \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n$$

induces an isomorphism upon applying the functor DQ_X, see Derived Categories of Spaces, Lemma 75.25.6. Hence we obtain a unique extension

$$0 \to \mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \to \mathcal{O}' \to \mathcal{O}_{X_1} \to 0$$

over $0 \to \lim I_n \to \lim R_n \to R_1 \to 0$ by the equivalence of categories of Deformation Theory, Lemma 91.14.4. The sheaf $\mathcal{O}'$ determines a first order thickening of algebraic spaces $X_1 \subset X$ over $\mathrm{Spec}(R_1) \subset \mathrm{Spec}(\lim R_n)$ by More on Morphisms of Spaces, Lemma 76.9.7. Observe that $X \to \mathrm{Spec}(\lim R_n)$ is flat by the already used More on Morphisms of Spaces, Lemma 76.18.1. By More on Morphisms of Spaces, Lemma 76.18.3 we see that $X \to \mathrm{Spec}(\lim R_n)$ is proper and of finite presentation. This finishes the proof. □

0D1P **Remark 99.13.10.** Lemma 99.13.9 can also be shown using either Artin's Axioms, Lemma 98.24.4 (as in the first proof of Lemma 99.5.11), or using an obstruction theory as in Artin's Axioms, Lemma 98.22.2 (as in the second proof of Lemma 99.5.11). In both cases one uses the deformation and obstruction theory developed in Cotangent, Section 92.23 to translate the needed properties of deformations and obstructions into Ext-groups to which Derived Categories of Spaces, Lemma 75.23.3 can be applied. The second method (using an obstruction theory and therefore using the full cotangent complex) is perhaps the "standard" method used in most references.

99.14. The stack of polarized proper schemes

0D1L To study the stack of polarized proper schemes it suffices to work over $\mathbf{Z}$ as we can later pullback to any scheme or algebraic space we want (see Remark 99.14.5).

0D1M **Situation 99.14.1.** We define a category $\mathcal{P}olarized$ as follows. Objects are pairs $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, and of finite presentation, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S (Morphisms, Definition 29.37.1).

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner (see Examples of Stacks, Sections 95.7 and 95.4). The forgetful functor

$$p : \mathcal{P}olarized \longrightarrow Sch_{fppf}, \quad (X \to S, \mathcal{L}) \longmapsto S$$

is how we view $\mathcal{P}olarized$ as a category over Sch_{fppf} (see Section 99.2 for notation).

In the previous section we have done a substantial amount of work on the stack $\mathcal{S}paces'_{fp,flat,proper}$ of finitely presented, flat, proper algebraic spaces. To use this material we consider the forgetful functor

0D3Y (99.14.1.1) $$\mathcal{P}olarized \longrightarrow \mathcal{S}paces'_{fp,flat,proper}, \quad (X \to S, \mathcal{L}) \longmapsto (X \to S)$$

This functor will be a useful tool in what follows. Observe that if $(X \to S)$ is in the essential image of (99.14.1.1), then X and S are schemes.

0D3Z **Lemma 99.14.2.** *The category $\mathcal{P}olarized$ is fibred in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$. The category $\mathcal{P}olarized$ is fibred in groupoids over Sch_{fppf}.*

Proof. We check conditions (1) and (2) of Categories, Definition 4.35.1.

Condition (1). Let $(X \to S, \mathcal{L})$ be an object of $\mathcal{P}olarized$ and let $(X' \to S') \to (X \to S)$ be a morphism of $\mathcal{S}paces'_{fp,flat,proper}$. Then we let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to X'. Observe that X, S, S' are schemes, hence X' is a scheme as well (as the fibre product of schemes). Then $\mathcal{L}'$ is ample on X'/S' by Morphisms, Lemma 29.37.9. In this way we obtain a morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ lying over $(X' \to S') \to (X \to S)$.

Condition (2). Consider morphisms $(f, g, \varphi) : (X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ and $(a, b, \psi) : (Y \to T, \mathcal{N}) \to (X \to S, \mathcal{L})$ of $\mathcal{P}olarized$. Given a morphism $(k, h) : (Y \to T) \to (X' \to S')$ of $\mathcal{S}paces'_{fp,flat,proper}$ with $(f, g) \circ (k, h) = (a, b)$ we have to show there is a unique morphism $(k, h, \chi) : (Y \to T, \mathcal{N}) \to (X' \to S', \mathcal{L}')$ of $\mathcal{P}olarized$ such that $(f, g, \varphi) \circ (k, h, \chi) = (a, b, \psi)$. We can just take

$$\chi = \psi \circ (k^*\varphi)^{-1}$$

This proves condition (2). A composition of functors defining fibred categories defines a fibred category, see Categories, Lemma 4.33.12. This we see that $\mathcal{P}olarized$ is fibred in groupoids over Sch_{fppf} (strictly speaking we should check the fibre categories are groupoids and apply Categories, Lemma 4.35.2). □

0D40 **Lemma 99.14.3.** *The category $\mathcal{P}olarized$ is a stack in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$ (endowed with the inherited topology, see Stacks, Definition 8.10.2). The category $\mathcal{P}olarized$ is a stack in groupoids over Sch_{fppf}.*

Proof. We prove $\mathcal{P}olarized$ is a stack in groupoids over $\mathcal{S}paces'_{fp,flat,proper}$ by checking conditions (1), (2), and (3) of Stacks, Definition 8.5.1. We have already seen (1) in Lemma 99.14.2.

A covering of $\mathcal{S}paces'_{fp,flat,proper}$ comes about in the following manner: Let $X \to S$ be an object of $\mathcal{S}paces'_{fp,flat,proper}$. Suppose that $\{S_i \to S\}_{i \in I}$ is a covering of Sch_{fppf}. Set $X_i = S_i \times_S X$. Then $\{(X_i \to S_i) \to (X \to S)\}_{i \in I}$ is a covering of $\mathcal{S}paces'_{fp,flat,proper}$ and every covering of $\mathcal{S}paces'_{fp,flat,proper}$ is isomorphic to one of these. Set $S_{ij} = S_i \times_S S_j$ and $X_{ij} = S_{ij} \times_S X$ so that $(X_{ij} \to S_{ij}) = (X_i \to S_i) \times_{(X \to S)} (X_j \to S_j)$. Next, suppose that $\mathcal{L}, \mathcal{N}$ are ample invertible sheaves on X/S so that $(X \to S, \mathcal{L})$ and $(X \to S, \mathcal{N})$ are two objects of $\mathcal{P}olarized$ over the object $(X \to S)$. To check descent for morphisms, we assume we have morphisms $(\mathrm{id}, \mathrm{id}, \varphi_i)$ from $(X_i \to S_i, \mathcal{L}|_{X_i})$ to $(X_i \to S_i, \mathcal{N}|_{X_i})$ whose base changes to morphisms from $(X_{ij} \to S_{ij}, \mathcal{L}|_{X_{ij}})$ to $(X_{ij} \to S_{ij}, \mathcal{N}|_{X_{ij}})$ agree. Then $\varphi_i : \mathcal{L}|_{X_i} \to \mathcal{N}|_{X_i}$ are isomorphisms of invertible modules over X_i such that φ_i and φ_j restrict to the same isomorphisms over X_{ij}. By descent for quasi-coherent sheaves (Descent on Spaces, Proposition 74.4.1) we obtain a unique isomorphism $\varphi : \mathcal{L} \to \mathcal{N}$ whose restriction to X_i recovers φ_i.

Decent for objects is proved in exactly the same manner. Namely, suppose that $\{(X_i \to S_i) \to (X \to S)\}_{i \in I}$ is a covering of $\mathcal{S}paces'_{fp,flat,proper}$ as above. Suppose we have objects $(X_i \to S_i, \mathcal{L}_i)$ of $\mathcal{P}olarized$ lying over $(X_i \to S_i)$ and a descent datum

$$(\mathrm{id}, \mathrm{id}, \varphi_{ij}) : (X_{ij} \to S_{ij}, \mathcal{L}_i|_{X_{ij}}) \to (X_{ij} \to S_{ij}, \mathcal{L}_j|_{X_{ij}})$$

satisfying the obvious cocycle condition over $(X_{ijk} \to S_{ijk})$ for every triple of indices. Then by descent for quasi-coherent sheaves (Descent on Spaces, Proposition 74.4.1) we obtain a unique invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $\mathcal{L}|_{X_i} \to \mathcal{L}_i$ recovering the descent datum φ_{ij}. To show that $(X \to S, \mathcal{L})$ is an object of *Polarized* we have to prove that $\mathcal{L}$ is ample. This follows from Descent on Spaces, Lemma 74.13.1.

Since we already have seen that $\mathcal{S}paces'_{fp,flat,proper}$ is a stack in groupoids over Sch_{fppf} (Lemma 99.13.3) it now follows formally that *Polarized* is a stack in groupoids over Sch_{fppf}. See Stacks, Lemma 8.10.6. □

Sanity check: the stack *Polarized* plays the same role among algebraic spaces.

0E94 **Lemma 99.14.4.** *Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 94.7, 94.8, and 94.13). We have an equivalence of categories*

$$\left\{\begin{array}{l}(X \to T, \mathcal{L}) \text{ where } X \to T \text{ is a morphism} \\ \text{of algebraic spaces, is proper, flat, and of} \\ \text{finite presentation and } \mathcal{L} \text{ ample on } X/T\end{array}\right\} \longrightarrow \mathrm{Mor}_{Cat/Sch_{fppf}}(\mathcal{S}_T, \mathcal{P}olarized)$$

Proof. Omitted. Hints: Argue exactly as in the proof of Lemma 99.13.4 and use Descent on Spaces, Proposition 74.4.1 to descent the invertible sheaf in the construction of the quasi-inverse functor. The relative ampleness property descends by Descent on Spaces, Lemma 74.13.1. □

0D1N **Remark 99.14.5.** Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let *$B$-Polarized* be the category consisting of triples $(X \to S, \mathcal{L}, h : S \to B)$ where $(X \to S, \mathcal{L})$ is an object of *Polarized* and $h : S \to B$ is a morphism. A morphism $(X' \to S', \mathcal{L}', h') \to (X \to S, \mathcal{L}, h)$ in *B-Polarized* is a morphism (f, g, φ) in *Polarized* such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-}Polarized & \longrightarrow & Polarized \\ \downarrow & & \downarrow \\ (Sch/B)_{fppf} & \longrightarrow & Sch_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case *Polarized* to the case of families over a given base algebraic space.

0D41 **Lemma 99.14.6.** *The functor (99.14.1.1) defines a 1-morphism*

$$\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$$

of stacks in groupoids over Sch_{fppf} which is algebraic in the sense of Criteria for Representability, Definition 97.8.1.

Proof. By Lemmas 99.13.3 and 99.14.3 the statement makes sense. To prove it, we choose a scheme S and an object $\xi = (X \to S)$ of $\mathcal{S}paces'_{fp,flat,proper}$ over S. We have to show that

$$\mathcal{X} = (Sch/S)_{fppf} \times_{\xi, \mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

is an algebraic stack over S. Observe that an object of $\mathcal{X}$ is given by a pair $(T/S, \mathcal{L})$ where T is a scheme over S and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$-module which is

ample on X_T/T. Morphisms are defined in the obvious manner. In particular, we see immediately that we have an inclusion

$$\mathcal{X} \subset \mathcal{P}ic_{X/S}$$

of categories over $(Sch/S)_{fppf}$, inducing equality on morphism sets. Since $\mathcal{P}ic_{X/S}$ is an algebraic stack by Proposition 99.10.2 it suffices to show that the inclusion above is representable by open immersions. This is exactly the content of Descent on Spaces, Lemma 74.13.2. □

0D42 **Lemma 99.14.7.** *The diagonal*

$$\Delta : \mathcal{P}olarized \longrightarrow \mathcal{P}olarized \times \mathcal{P}olarized$$

is representable by algebraic spaces.

Proof. This is a formal consequence of Lemmas 99.14.6 and 99.13.2. See Criteria for Representability, Lemma 97.8.4. □

0D43 **Lemma 99.14.8.** *The stack in groupoids $\mathcal{P}olarized$ is limit preserving (Artin's Axioms, Definition 98.11.1).*

Proof. Let I be a directed set and let $(A_i, \varphi_{ii'})$ be a system of rings over I. Set $S = \mathrm{Spec}(A)$ and $S_i = \mathrm{Spec}(A_i)$. We have to show that on fibre categories we have

$$\mathcal{P}olarized_S = \mathrm{colim}\, \mathcal{P}olarized_{S_i}$$

We know that the category of schemes of finite presentation over S is the colimit of the category of schemes of finite presentation over S_i, see Limits, Lemma 32.10.1. Moreover, given $X_i \to S_i$ of finite presentation, with limit $X \to S$, then the category of invertible $\mathcal{O}_X$-modules $\mathcal{L}$ is the colimit of the categories of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$, see Limits, Lemma 32.10.2 and 32.10.3. If $X \to S$ is proper and flat, then for sufficiently large i the morphism $X_i \to S_i$ is proper and flat too, see Limits, Lemmas 32.13.1 and 32.8.7. Finally, if $\mathcal{L}$ is ample on X then $\mathcal{L}_i$ is ample on X_i for i sufficiently large, see Limits, Lemma 32.4.15. Putting everything together finishes the proof. □

0D44 **Lemma 99.14.9.** *In Situation 99.5.1. Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{P}olarized_{S'} \longrightarrow \mathcal{P}olarized_S \times_{\mathcal{P}olarized_T} \mathcal{P}olarized_{T'}$$

is an equivalence.

Proof. By More on Morphisms, Lemma 37.14.6 there is an equivalence

$$\textit{flat-lfp}_{S'} \longrightarrow \textit{flat-lfp}_S \times_{\textit{flat-lfp}_T} \textit{flat-lfp}_{T'}$$

where $\textit{flat-lfp}_S$ signifies the category of schemes flat and locally of finite presentation over S. Let X'/S' on the left hand side correspond to the triple $(X/S, Y'/T', \varphi)$

on the right hand side. Set $Y = T \times_{T'} Y'$ which is isomorphic with $T \times_S X$ via φ. Then More on Morphisms, Lemma 37.14.5 shows that we have an equivalence

$$QCoh\text{-}flat_{X'/S'} \longrightarrow QCoh\text{-}flat_{X/S} \times_{QCoh\text{-}flat_{Y/T}} QCoh\text{-}flat_{Y'/T'}$$

where $QCoh\text{-}flat_{X/S}$ signifies the category of quasi-coherent $\mathcal{O}_X$-modules flat over S. Since $X \to S$, $Y \to T$, $X' \to S'$, $Y' \to T'$ are flat, this will in particular apply to invertible modules to give an equivalence of categories

$$Pic(X') \longrightarrow Pic(X) \times_{Pic(Y)} Pic(Y')$$

where $Pic(X)$ signifies the category of invertible $\mathcal{O}_X$-modules. There is a small point here: one has to show that if an object $\mathcal{F}'$ of $QCoh\text{-}flat_{X'/S'}$ pulls back to invertible modules on X and Y', then $\mathcal{F}'$ is an invertible $\mathcal{O}_{X'}$-module. It follows from the cited lemma that $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module of finite presentation. By More on Morphisms, Lemma 37.16.7 it suffices to check the restriction of $\mathcal{F}'$ to fibres of $X' \to S'$ is invertible. But the fibres of $X' \to S'$ are the same as the fibres of $X \to S$ and hence these restrictions are invertible.

Having said the above we obtain an equivalence of categories if we drop the assumption (for the category of objects over S) that $X \to S$ be proper and the assumption that $\mathcal{L}$ be ample. Now it is clear that if $X' \to S'$ is proper, then $X \to S$ and $Y' \to T'$ are proper (Morphisms, Lemma 29.41.5). Conversely, if $X \to S$ and $Y' \to T'$ are proper, then $X' \to S'$ is proper by More on Morphisms, Lemma 37.3.3. Similarly, if $\mathcal{L}'$ is ample on X'/S', then $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T' (Morphisms, Lemma 29.37.9). Finally, if $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T', then $\mathcal{L}'$ is ample on X'/S' by More on Morphisms, Lemma 37.3.2. □

0D4S **Lemma 99.14.10.** *Let k be a field and let $x = (X \to \mathrm{Spec}(k), \mathcal{L})$ be an object of $\mathcal{X} = \mathcal{P}olarized$ over $\mathrm{Spec}(k)$.*

1. *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $Inf(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and*
2. *in general the vector spaces $T_x(k)$ and $Inf_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.*

Proof. The discussion in Artin's Axioms, Section 98.8 only applies to fields of finite type over the base scheme $\mathrm{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 99.14.9 and we may apply Artin's Axioms, Lemma 98.21.2 to get the vector spaces $T_x(k)$ and $\mathrm{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in part (1) by Artin's Axioms, Remark 98.21.7. With this out of the way we can start the proof.

One proof is to use an argument as in the proof of Lemma 99.13.8; this would require us to develop a deformation theory for pairs consisting of a scheme and a quasi-coherent module. Another proof would be the use the result from Lemma 99.13.8, the algebraicity of $\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$, and a computation of the deformation space of an invertible module. However, what we will do instead is to translate the question into a deformation question on graded k-algebras and deduce the result that way.

Let $\mathcal{C}_k$ be the category of Artinian local k-algebras A with residue field k. We get a predeformation category $p : \mathcal{F} \to \mathcal{C}_k$ from our object x of $\mathcal{X}$ over k, see Artin's Axioms, Section 98.3. Thus $\mathcal{F}(A)$ is the category of triples $(X_A, \mathcal{L}_A, \alpha)$, where $(X_A, \mathcal{L}_A)$ is an object of $\mathcal{P}olarized$ over A and α is an isomorphism $(X_A, \mathcal{L}_A) \times_{\mathrm{Spec}(A)}$

$\mathrm{Spec}(k) \cong (X, \mathcal{L})$. On the other hand, let $q : \mathcal{G} \to \mathcal{C}_k$ be the category cofibred in groupoids defined in Deformation Problems, Example 93.7.1. Choose $d_0 \gg 0$ (we'll see below how large). Let P be the graded k-algebra

$$P = k \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X, \mathcal{L}^{\otimes d})$$

Then $y = (k, P)$ is an object of $\mathcal{G}(k)$. Let $\mathcal{G}_y$ be the predeformation category of Formal Deformation Theory, Remark 90.6.4. Given $(X_A, \mathcal{F}_A, \alpha)$ as above we set

$$Q = A \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X_A, \mathcal{L}_A^{\otimes d})$$

The isomorphism α induces a map $\beta : Q \to P$. By deformation theory of projective schemes (More on Morphisms, Lemma 37.10.6) we obtain a 1-morphism

$$\mathcal{F} \longrightarrow \mathcal{G}_y, \quad (X_A, \mathcal{F}_A, \alpha) \longmapsto (Q, \beta : Q \to P)$$

of categories cofibred in groupoids over $\mathcal{C}_k$. In fact, this functor is an equivalence with quasi-inverse given by $Q \mapsto \underline{\mathrm{Proj}}_A(Q)$. Namely, the scheme $X_A = \underline{\mathrm{Proj}}_A(Q)$ is flat over A by Divisors, Lemma 31.30.6. Set $\mathcal{L}_A = \mathcal{O}_{X_A}(1)$; this is flat over A by the same lemma. We get an isomorphism $(X_A, \mathcal{L}_A) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = (X, \mathcal{L})$ from β. Then we can deduce all the desired properties of the pair $(X_A, \mathcal{L}_A)$ from the corresponding properties of $(X, \mathcal{L})$ using the techniques in More on Morphisms, Sections 37.3 and 37.10. Some details omitted.

In conclusion, we see that $T\mathcal{F} = T\mathcal{G}_y = T_y\mathcal{G}$ and $\mathrm{Inf}(\mathcal{F}) = \mathrm{Inf}_y(\mathcal{G})$. These vector spaces are finite dimensional by Deformation Problems, Lemma 93.7.3 and the proof is complete. □

0D4T **Lemma 99.14.11** (Strong formal effectiveness for polarized schemes)**.** *Let (R_n) be an inverse system of rings with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Set $S_n = \mathrm{Spec}(R_n)$ and $S = \mathrm{Spec}(R)$. Consider a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

(1) *$X_n \to S_n$ is proper, flat, of finite presentation, and*
(2) *$\mathcal{L}_1$ is ample on X_1*

then there exists a morphism of schemes $X \to S$ proper, flat, and of finite presentation and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Choose d_0 for $X_1 \to S_1$ and $\mathcal{L}_1$ as in More on Morphisms, Lemma 37.10.6. For any $n \geq 1$ set

$$A_n = R_n \oplus \bigoplus\nolimits_{d \geq d_0} H^0(X_n, \mathcal{L}_n^{\otimes d})$$

By the lemma each A_n is a finitely presented graded R_n-algebra whose homogeneous parts $(A_n)_d$ are finite projective R_n-modules such that $X_n = \mathrm{Proj}(A_n)$ and $\mathcal{L}_n = \mathcal{O}_{\mathrm{Proj}(A_n)}(1)$. The lemma also guarantees that the maps

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \ldots$$

induce isomorphisms $A_n = A_m \otimes_{R_m} R_n$ for $n \leq m$. We set

$$B = \bigoplus\nolimits_{d \geq 0} B_d \quad \text{with} \quad B_d = \lim_n (A_n)_d$$

By More on Algebra, Lemma 15.13.4 we see that B_d is a finite projective R-module and that $B \otimes_R R_n = A_n$. Thus the scheme

$$X = \text{Proj}(B) \quad \text{and} \quad \mathcal{L} = \mathcal{O}_X(1)$$

is flat over S and $\mathcal{L}$ is a quasi-coherent $\mathcal{O}_X$-module flat over S, see Divisors, Lemma 31.30.6. Because formation of Proj commutes with base change (Constructions, Lemma 27.11.6) we obtain canonical isomorphisms

$$X \times_S S_n = X_n \quad \text{and} \quad \mathcal{L}|_{X_n} \cong \mathcal{L}_n$$

compatible with the transition maps of the system. Thus we may think of $X_1 \subset X$ as a closed subscheme. Below we will show that B is of finite presentation over R. By Divisors, Lemmas 31.30.4 and 31.30.7 this implies that $X \to S$ is of finite presentation and proper and that $\mathcal{L} = \mathcal{O}_X(1)$ is of finite presentation as an $\mathcal{O}_X$-module. Since the restriction of $\mathcal{L}$ to the base change $X_1 \to S_1$ is invertible, we see from More on Morphisms, Lemma 37.16.8 that $\mathcal{L}$ is invertible on an open neighbourhood of X_1 in X. Since $X \to S$ is closed and since $\text{Ker}(R \to R_1)$ is contained in the Jacobson radical (More on Algebra, Lemma 15.11.3) we see that any open neighbourhood of X_1 in X is equal to X. Thus $\mathcal{L}$ is invertible. Finally, the set of points in S where $\mathcal{L}$ is ample on the fibre is open in S (More on Morphisms, Lemma 37.50.3) and contains S_1 hence equals S. Thus $X \to S$ and $\mathcal{L}$ have all the properties required of them in the statement of the lemma.

We prove the claim above. Choose a presentation $A_1 = R_1[X_1, \ldots, X_s]/(F_1, \ldots, F_t)$ where X_i are variables having degrees d_i and F_j are homogeneous polynomials in X_i of degree e_j. Then we can choose a map

$$\Psi : R[X_1, \ldots, X_s] \longrightarrow B$$

lifting the map $R_1[X_1, \ldots, X_s] \to A_1$. Since each B_d is finite projective over R we conclude from Nakayama's lemma (Algebra, Lemma 10.20.1 using again that $\text{Ker}(R \to R_1)$ is contained in the Jacobson radical of R) that Ψ is surjective. Since $- \otimes_R R_1$ is right exact we can find $G_1, \ldots, G_t \in \text{Ker}(\Psi)$ mapping to $F_1, \ldots, F_t$ in $R_1[X_1, \ldots, X_s]$. Observe that $\text{Ker}(\Psi)_d$ is a finite projective R-module for all $d \geq 0$ as the kernel of the surjection $R[X_1, \ldots, X_s]_d \to B_d$ of finite projective R-modules. We conclude from Nakayama's lemma once more that $\text{Ker}(\Psi)$ is generated by $G_1, \ldots, G_t$. □

0D4U **Lemma 99.14.12.** *Consider the stack Polarized over the base scheme* $\text{Spec}(\mathbf{Z})$. *Then every formal object is effective.*

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 98.9. From the definitions we see the lemma follows immediately from the more general Lemma 99.14.11. □

0D4V **Lemma 99.14.13.** *The stack in groupoids Polarized satisfies openness of versality over* $\text{Spec}(\mathbf{Z})$. *Similarly, after base change (Remark 99.14.5) openness of versality holds over any Noetherian base scheme S.*

Proof. This follows from Artin's Axioms, Lemma 98.20.3 and Lemmas 99.14.7, 99.14.9, 99.14.8, and 99.14.11. For the "usual" proof of this fact, please see the discussion in the remark following this proof. □

0D4W **Remark 99.14.14.** Lemma 99.14.13 can also be shown using an obstruction theory as in Artin's Axioms, Lemma 98.22.2 (as in the second proof of Lemma 99.5.11). To do this one has to generalize the deformation and obstruction theory developed in Cotangent, Section 92.23 to the case of pairs of algebraic spaces and quasi-coherent modules. Another possibility is to use that the 1-morphism $\mathcal{P}olarized \to \mathcal{S}paces'_{fp,flat,proper}$ is algebraic (Lemma 99.14.6) and the fact that we know openness of versality for the target (Lemma 99.13.9 and Remark 99.13.10).

0D4X **Theorem 99.14.15** (Algebraicity of the stack of polarized schemes)**.** *The stack $\mathcal{P}olarized$ (Situation 99.14.1) is algebraic. In fact, for any algebraic space B the stack B-Polarized (Remark 99.14.5) is algebraic.*

Proof. The absolute case follows from Artin's Axioms, Lemma 98.17.1 and Lemmas 99.14.7, 99.14.9, 99.14.8, 99.14.12, and 99.14.13. The case over B follows from this, the description of B-*Polarized* as a 2-fibre product in Remark 99.14.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 94.14.3. □

99.15. The stack of curves

0D4Y In this section we prove the stack of curves is algebraic. For a further discussion of moduli of curves, we refer the reader to Moduli of Curves, Section 109.1.

A curve in the Stacks project is a variety of dimension 1. However, when we speak of families of curves, we often allow the fibres to be reducible and/or nonreduced. In this section, the stack of curves will "parametrize proper schemes of dimension ≤ 1". However, it turns out that in order to get the correct notion of a family we need to allow the total space of our family to be an algebraic space. This leads to the following definition.

0D4Z **Situation 99.15.1.** We define a category $\mathcal{C}urves$ as follows:

(1) Objects are *families of curves*. More precisely, an object is a morphism $f : X \to S$ where the base S is a scheme, the *total space* X is an algebraic space, and f is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 67.33.2).

(2) A morphism $(X' \to S') \to (X \to S)$ between objects is given by a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces and $g : S' \to S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian.

The forgetful functor

$$p : \mathcal{C}urves \longrightarrow Sch_{fppf}, \quad (X \to S) \longmapsto S$$

is how we view $\mathcal{C}urves$ as a category over Sch_{fppf} (see Section 99.2 for notation).

It follows from Spaces over Fields, Lemma 72.9.3 and more generally More on Morphisms of Spaces, Lemma 76.43.6 that if S is the spectrum of a field, or an Artinian local ring, or a Noetherian complete local ring, then for any family of curves $X \to S$ the total space X is a scheme. On the other hand, there are families of curves over $\mathbf{A}^1_k$ where the total space is not a scheme, see Examples, Section 110.67.

It is clear that

0D50 (99.15.1.1) $$\mathcal{C}urves \subset \mathcal{S}paces'_{fp,flat,proper}$$

and that an object $X \to S$ of $\mathcal{S}paces'_{fp,flat,proper}$ is in $\mathcal{C}urves$ if and only if $X \to S$ has relative dimension ≤ 1. We will use this to verify Artin's axioms for $\mathcal{C}urves$.

0D51 **Lemma 99.15.2.** *The category $\mathcal{C}urves$ is fibred in groupoids over Sch_{fppf}.*

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.1) this reduces to the following statement: Given a morphism

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

in $\mathcal{S}paces'_{fp,flat,proper}$ (recall that this implies in particular the diagram is cartesian) if $X \to S$ has relative dimension ≤ 1, then $X' \to S'$ has relative dimension ≤ 1. This follows from Morphisms of Spaces, Lemma 67.34.3. □

0D52 **Lemma 99.15.3.** *The category $\mathcal{C}urves$ is a stack in groupoids over Sch_{fppf}.*

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.3) this reduces to the following statement: Given an object $X \to S$ of $\mathcal{S}paces'_{fp,flat,proper}$ and an fppf covering $\{S_i \to S\}_{i \in I}$ the following are equivalent:

(1) $X \to S$ has relative dimension ≤ 1, and
(2) for each i the base change $X_i \to S_i$ has relative dimension ≤ 1.

This follows from Morphisms of Spaces, Lemma 67.34.3. □

0D53 **Lemma 99.15.4.** *The diagonal*

$$\Delta : \mathcal{C}urves \longrightarrow \mathcal{C}urves \times \mathcal{C}urves$$

is representable by algebraic spaces.

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.2). □

0D54 **Remark 99.15.5.** Let B be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let B-$\mathcal{C}urves$ be the category consisting of pairs $(X \to S, h : S \to B)$ where $X \to S$ is an object of $\mathcal{C}urves$ and $h : S \to B$ is a morphism. A morphism $(X' \to S', h') \to (X \to S, h)$ in

B-Curves is a morphism (f, g) in *Curves* such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-}\mathcal{C}urves & \longrightarrow & \mathcal{C}urves \\ \downarrow & & \downarrow \\ (Sch/B)_{fppf} & \longrightarrow & Sch_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case *Curves* to the case of families of curves over a given base algebraic space.

0D55 **Lemma 99.15.6.** *The stack $\mathcal{C}urves \to Sch_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).*

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.6) this reduces to the following statement: Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. Let $i \in I$ and let $X_i \to T_i$ be an object of $\mathcal{S}paces'_{fp,flat,proper}$ over T_i. Assume that $T \times_{T_i} X_i \to T$ has relative dimension ≤ 1. Then for some $i' \geq i$ the morphism $T_{i'} \times_{T_i} X_i \to T_i$ has relative dimension ≤ 1. This follows from Limits of Spaces, Lemma 70.6.14. □

0D56 **Lemma 99.15.7.** *Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \to T'$ is a thickening and $T \to S$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathcal{C}urves_{S'} \longrightarrow \mathcal{C}urves_S \times_{\mathcal{C}urves_T} \mathcal{C}urves_{T'}$$

is an equivalence.

Proof. Using the embedding (99.15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 99.13.7) this reduces to the following statement: given a morphism $X' \to S'$ of an algebraic space to S' which is of finite presentation, flat, proper then $X' \to S'$ has relative dimension ≤ 1 if and only if $S \times_{S'} X' \to S$ and $T' \times_{S'} X' \to T'$ have relative dimension ≤ 1. One implication follows from the fact that having relative dimension ≤ 1 is preserved under base change (Morphisms of Spaces, Lemma 67.34.3). The other follows from the fact that having relative dimension ≤ 1 is checked on the fibres and that the fibres of $X' \to S'$ (over points of the scheme S') are the same as the fibres of $S \times_{S'} X' \to S$ since $S \to S'$ is a thickening by More on Morphisms, Lemma 37.14.3. □

0D57 **Lemma 99.15.8.** *Let k be a field and let $x = (X \to \mathrm{Spec}(k))$ be an object of $\mathcal{X} = \mathcal{C}urves$ over $\mathrm{Spec}(k)$.*

1. *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $Inf(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 98.8) are finite dimensional, and*
2. *in general the vector spaces $T_x(k)$ and $Inf_x(k)$ (see Artin's Axioms, Section 98.21) are finite dimensional.*

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathit{Spaces}'_{fp,flat,proper}$ (Lemma 99.13.8). □

0D58 **Lemma 99.15.9.** *Consider the stack Curves over the base scheme* $\mathrm{Spec}(\mathbf{Z})$. *Then every formal object is effective.*

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 98.9. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Let $(X_n \to \mathrm{Spec}(A/\mathfrak{m}^n))$ be a formal object of Curves over A. By More on Morphisms of Spaces, Lemma 76.43.5 there exists a projective morphism $X \to \mathrm{Spec}(A)$ and a compatible system of ismomorphisms $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X_n$. By More on Morphisms, Lemma 37.12.4 we see that $X \to \mathrm{Spec}(A)$ is flat. By More on Morphisms, Lemma 37.30.6 we see that $X \to \mathrm{Spec}(A)$ has relative dimension ≤ 1. This proves the lemma. □

0D59 **Lemma 99.15.10.** *The stack in groupoids $\mathcal{X} = \mathit{Curves}$ satisfies openness of versality over* $\mathrm{Spec}(\mathbf{Z})$. *Similarly, after base change (Remark 99.15.5) openness of versality holds over any Noetherian base scheme S.*

Proof. This is immediate from the fully faithful embedding (99.15.1.1) and the corresponding fact for $\mathit{Spaces}'_{fp,flat,proper}$ (Lemma 99.13.9). □

0D5A **Theorem 99.15.11** (Algebraicity of the stack of curves)**.** *The stack Curves (Situation 99.15.1) is algebraic. In fact, for any algebraic space B the stack B-Curves (Remark 99.15.5) is algebraic.*

See [**dJHS11**, Proposition 3.3, page 8] and [**Smy13**, Appendix B by Jack Hall, Theorem B.1].

Proof. The absolute case follows from Artin's Axioms, Lemma 98.17.1 and Lemmas 99.15.4, 99.15.7, 99.15.6, 99.15.9, and 99.15.10. The case over B follows from this, the description of B-Curves as a 2-fibre product in Remark 99.15.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 94.14.3. □

0D5B **Lemma 99.15.12.** *The* 1*-morphism (99.15.1.1)*

$$\mathit{Curves} \longrightarrow \mathit{Spaces}'_{fp,flat,proper}$$

is representable by open and closed immersions.

Proof. Since (99.15.1.1) is a fully faithful embedding of categories it suffices to show the following: given an object $X \to S$ of $\mathit{Spaces}'_{fp,flat,proper}$ there exists an open and closed subscheme $U \subset S$ such that a morphism $S' \to S$ factors through U if and only if the base change $X' \to S'$ of $X \to S$ has relative dimension ≤ 1. This follows immediately from More on Morphisms of Spaces, Lemma 76.31.5. □

0D5C **Remark 99.15.13.** Consider the 2-fibre product

$$\begin{array}{ccc} \mathit{Curves} \times_{\mathit{Spaces}'_{fp,flat,proper}} \mathit{Polarized} & \longrightarrow & \mathit{Polarized} \\ \downarrow & & \downarrow \\ \mathit{Curves} & \longrightarrow & \mathit{Spaces}'_{fp,flat,proper} \end{array}$$

This fibre product parametrized polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. It turns out that the left vertical arrow

$$\mathit{PolarizedCurves} \longrightarrow \mathit{Curves}$$

is algebraic, smooth, and surjective. Namely, this 1-morphism is algebraic (as base change of the arrow in Lemma 99.14.6), every point is in the image, and there are no obstructions to deforming invertible sheaves on curves (see proof of Lemma 99.15.9). This gives another approach to the algebraicity of $\mathcal{C}urves$. Namely, by Lemma 99.15.12 we see that $PolarizedCurves$ is an open and closed substack of the algebraic stack $\mathcal{P}olarized$ and any stack in groupoids which is the target of a smooth algebraic morphism from an algebraic stack is an algebraic stack.

99.16. Moduli of complexes on a proper morphism

0DLB The title and the material of this section are taken from [**Lie06a**]. Let S be a scheme and let $f : X \to B$ be a proper, flat, finitely presented morphism of algebraic spaces. We will prove that there is an algebraic stack

$$\mathcal{C}omplexes_{X/B}$$

parametrizing "families" of objects of D^b_{Coh} of the fibres with vanishing negative self-exts. More precisely a family is given by a relatively perfect object of the derived category of the total space; this somewhat technical notion is studied in More on Morphisms of Spaces, Section 76.52.

Already if X is a proper algebraic space over a field k we obtain a very interesting algebraic stack. Namely, there is an embedding

$$\mathcal{C}oh_{X/k} \longrightarrow \mathcal{C}omplexes_{X/k}$$

since for any $\mathcal{O}$-module $\mathcal{F}$ (on any ringed topos) we have $\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{F}, \mathcal{F}) = 0$ for $i < 0$. Although this certainly shows our stack is nonempty, the true motivation for the study of $\mathcal{C}omplexes_{X/k}$ is that there are often objects of the derived category $D^b_{Coh}(\mathcal{O}_X)$ with vanishing negative self-exts and nonvanishing cohomology sheaves in more than one degree. For example, X could be derived equivalent to another proper algebraic space Y over k, i.e., we have a k-linear equivalence

$$F : D^b_{Coh}(\mathcal{O}_Y) \longrightarrow D^b_{Coh}(\mathcal{O}_X)$$

There are cases where this happens and F is not given by an isomorphism between X and Y; for example in the case of an abelian variety and its dual. In this situation F induces an isomorphism of algebraic stacks

$$\mathcal{C}omplexes_{Y/k} \longrightarrow \mathcal{C}omplexes_{X/k}$$

(insert future reference here) and in particular the stack of coherent sheaves on Y maps into the stack of complexes on X. Turning this around, if we can understand well enough the geometry of $\mathcal{C}omplexes_{X/k}$, then we can try to use this to study all possible derived equivalent Y.

0DLC **Lemma 99.16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. Let $K, E \in D(\mathcal{O}_X)$. Assume K is pseudo-coherent and E is Y-perfect (More on Morphisms of Spaces, Definition 76.52.1). For a field k and a morphism $y : \mathrm{Spec}(k) \to Y$ denote K_y, E_y the pullback to the fibre X_y.*

(1) *There is an open $W \subset Y$ characterized by the property*

$$y \in |W| \Leftrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) = 0 \text{ for } i < 0.$$

(2) *For any morphism $V \to Y$ factoring through W we have*
$$\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(K_V, E_V) = 0 \quad \text{for} \quad i < 0$$
where X_V is the base change of X and K_V and E_V are the derived pullbacks of K and E to X_V.

(3) *The functor $V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf on $(Spaces/W)_{fppf}$ representable by an algebraic space affine and of finite presentation over W.*

Proof. For any morphism $V \to Y$ the complex K_V is pseudo-coherent (Cohomology on Sites, Lemma 21.45.3) and E_V is V-perfect (More on Morphisms of Spaces, Lemma 76.52.6). Another observation is that given $y : \mathrm{Spec}(k) \to Y$ and a field extension k'/k with $y' : \mathrm{Spec}(k') \to Y$ the induced morphism, we have
$$\mathrm{Ext}^i_{\mathcal{O}_{X_{y'}}}(K_{y'}, E_{y'}) = \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) \otimes_k k'$$
by Derived Categories of Schemes, Lemma 36.22.6. Thus the vanishing in (1) is really a property of the induced point $y \in |Y|$. We will use these two observations without further mention in the proof.

Assume first Y is an affine scheme. Then we may apply More on Morphisms of Spaces, Lemma 76.52.11 and find a pseudo-coherent $L \in D(\mathcal{O}_Y)$ which "universally computes" $Rf_*R\mathcal{H}om(K, E)$ in the sense described in that lemma. Unwinding the definitions, we obtain for a point $y \in Y$ the equality
$$\mathrm{Ext}^i_{\kappa(y)}(L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y), \kappa(y)) = \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y)$$
We conclude that
$$H^i(L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \kappa(y)) = 0 \text{ for } i > 0 \Leftrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X_y}}(K_y, E_y) = 0 \text{ for } i < 0.$$
By Derived Categories of Schemes, Lemma 36.31.1 the set W of $y \in Y$ where this happens defines an open of Y. This open W then satisfies the requirement in (1) for all morphisms from spectra of fields, by the "universality" of L.

Let's go back to Y a general algebraic space. Choose an étale covering $\{V_i \to Y\}$ by affine schemes V_i. Then we see that the subset $W \subset |Y|$ pulls back to the corresponding subset $W_i \subset |V_i|$ for X_{V_i}, K_{V_i}, E_{V_i}. By the previous paragraph we find that W_i is open, hence W is open. This proves (1) in general. Moreover, parts (2) and (3) are entirely formulated in terms of the category $Spaces/W$ and the restrictions X_W, K_W, E_W. This reduces us to the case $W = Y$.

Assume $W = Y$. We claim that for any algebraic space V over Y we have $Rf_{V,*}R\mathcal{H}om(K_V, E_V)$ has vanishing cohomology sheaves in degrees < 0. This will prove (2) because
$$\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(K_V, E_V) = H^i(X_V, R\mathcal{H}om(K_V, E_V)) = H^i(V, Rf_{V,*}R\mathcal{H}om(K_V, E_V))$$
by Cohomology on Sites, Lemmas 21.35.1 and 21.20.5 and the vanishing of the cohomology sheaves implies the cohomology group H^i is zero for $i < 0$ by Derived Categories, Lemma 13.16.1.

To prove the claim, we may work étale locally on V. In particular, we may assume Y is affine and $W = Y$. Let $L \in D(\mathcal{O}_Y)$ be as in the second paragraph of the proof. For an algebraic space V over Y denote L_V the derived pullback of L to V. (An important feature we will use is that L "works" for all algebraic spaces V over Y and not just affine V.) As $W = Y$ we have $H^i(L) = 0$ for $i > 0$ (use More on

Algebra, Lemma 15.76.6 to go from fibres to stalks). Hence $H^i(L_V) = 0$ for $i > 0$. The property defining L is that

$$Rf_{V,*}R\mathcal{H}om(K_V, E_V) = R\mathcal{H}om(L_V, \mathcal{O}_V)$$

Since L_V sits in degrees ≤ 0, we conclude that $R\mathcal{H}om(L_V, \mathcal{O}_V)$ sits in degrees ≥ 0 thereby proving the claim. This finishes the proof of (2).

Assume $W = Y$ but make no assumptions on the algebraic space Y. Since we have (2), we see from Simplicial Spaces, Lemma 85.35.1 that the functor F given by $F(V) = \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf[4] on $(Spaces/Y)_{fppf}$. Thus to prove that F is an algebraic space and that $F \to Y$ is affine and of finite presentation, we may work étale locally on Y; see Bootstrap, Lemma 80.11.2 and Morphisms of Spaces, Lemmas 67.20.3 and 67.28.4. We conclude that it suffices to prove F is an affine algebraic space of finite presentation over Y when Y is an affine scheme. In this case we go back to our pseudo-coherent complex $L \in D(\mathcal{O}_Y)$. Since $H^i(L) = 0$ for $i > 0$, we can represent L by a complex of the form

$$\ldots \to \mathcal{O}_Y^{\oplus m_1} \to \mathcal{O}_Y^{\oplus m_0} \to 0 \to \ldots$$

with the last term in degree 0, see More on Algebra, Lemma 15.65.5. Combining the two displayed formulas earlier in the proof we find that

$$F(V) = \mathrm{Ker}(\mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_0}, \mathcal{O}_V) \to \mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_1}, \mathcal{O}_V))$$

In other words, there is a fibre product diagram

$$\begin{array}{ccc} F & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle 0} \\ \mathbf{A}_Y^{m_0} & \longrightarrow & \mathbf{A}_Y^{m_1} \end{array}$$

which proves what we want. □

0DLD **Lemma 99.16.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. Let $E \in D(\mathcal{O}_X)$. Assume*

(1) *E is S-perfect (More on Morphisms of Spaces, Definition 76.52.1), and*
(2) *for every point $s \in S$ we have*

$$\mathrm{Ext}^i_{\mathcal{O}_{X_s}}(E_s, E_s) = 0 \quad \text{for} \quad i < 0$$

where E_s is the pullback to the fibre X_s.

Then

(a) *(1) and (2) are preserved by arbitrary base change $V \to Y$,*
(b) *$\mathrm{Ext}^i_{\mathcal{O}_{X_V}}(E_V, E_V) = 0$ for $i < 0$ and all V over Y,*
(c) *$V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(E_V, E_V)$ is representable by an algebraic space affine and of finite presentation over Y.*

Here X_V is the base change of X and E_V is the derived pullback of E to X_V.

Proof. Immediate consequence of Lemma 99.16.1. □

[4] To check the sheaf property for a covering $\{V_i \to V\}_{i \in I}$ first consider the Čech fppf hypercovering $a : V_\bullet \to V$ with $V_n = \coprod_{i_0 \ldots i_n} V_{i_0} \times_V \ldots \times_V V_{i_n}$ and then set $U_\bullet = V_\bullet \times_{a,V} X_V$. Then $U_\bullet \to X_V$ is an fppf hypercovering to which we may apply Simplicial Spaces, Lemma 85.35.1.

0DLE **Situation 99.16.3.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume f is proper, flat, and of finite presentation. We denote $\mathit{Complexes}_{X/B}$ the category whose objects are triples (T, g, E) where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$
(3) E is an object of $D(\mathcal{O}_{X_T})$ satisfying conditions (1) and (2) of Lemma 99.16.2.

A morphism $(T, g, E) \to (T', g', E')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : L(h')^*E' \to E$ is an isomorphism of $D(\mathcal{O}_{X_T})$ where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $\mathit{Complexes}_{X/B}$ is a category and the rule

$$p : \mathit{Complexes}_{X/B} \longrightarrow (\mathit{Sch}/S)_{fppf}, \quad (T, g, E) \longmapsto T$$

is a functor. For a scheme T over S we denote $\mathit{Complexes}_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

0DLF **Lemma 99.16.4.** *In Situation 99.16.3 the functor $p : \mathit{Complexes}_{X/B} \longrightarrow (\mathit{Sch}/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 4.35.1. Given an object (T', g', E') of $\mathit{Complexes}_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $E = L(h')^*E'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, E) \to (T', g', E')$ of $\mathit{Complexes}_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, E_1) \to (T, g, E) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, E_2) \to (T, g, E)$$

of $\mathit{Complexes}_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$L(h')^*E_2 \xrightarrow{L(h')^*\varphi_2^{-1}} L(h')^*L(h_2)^*E = L(h_1)^*E \xrightarrow{\varphi_1} E_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, E_1) \to (T_2, g_2, E_2)$ that witnesses the truth of condition (2). □

0DLG **Lemma 99.16.5.** *In Situation 99.16.3. Denote $\mathcal{X} = \mathit{Complexes}_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, E)$ and $y = (T, g', E')$ of $\mathcal{X}$ over a scheme T. We have to show that $\mathit{Isom}_{\mathcal{X}}(x, y)$ is an algebraic space over T, see Algebraic Stacks, Lemma 94.10.11. If for $h : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ h = g' \circ h$. Hence there is a transformation of presheaves

$$\mathit{Isom}_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, g')$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and E and E' by their pullbacks. Thus we may assume $g = g'$.

Assume $g = g'$. After replacing B by T and X by X_T we arrive at the following problem. Given $E, E' \in D(\mathcal{O}_X)$ satisfying conditions (1), (2) of Lemma 99.16.2

we have to show that $Isom(E, E')$ is an algebraic space. Here $Isom(E, E')$ is the functor

$$(Sch/B)^{opp} \to Sets, \quad T \mapsto \{\varphi : E_T \to E'_T \text{ isomorphism in } D(\mathcal{O}_{X_T})\}$$

where E_T and E'_T are the derived pullbacks of E and E' to X_T. Now, let $W \subset B$, resp. $W' \subset B$ be the open subspace of B associated to E, E', resp. to E', E by Lemma 99.16.1. Clearly, if there exists an isomorphism $E_T \to E'_T$ as in the definition of $Isom(E, E')$, then we see that $T \to B$ factors into both W and W' (because we have condition (1) for E and E' and we'll obviously have $E_t \cong E'_t$ so no nonzero maps $E_t[i] \to E_t$ or $E'_t[i] \to E_t$ over the fibre X_t for $i > 0$. Thus we may replace B by the open $W \cap W'$. In this case the functor $H = \mathcal{H}om(E, E')$

$$(Sch/B)^{opp} \to Sets, \quad T \mapsto \mathrm{Hom}_{\mathcal{O}_{X_T}}(E_T, E'_T)$$

is an algebraic space affine and of finite presentation over B by Lemma 99.16.1. The same is true for $H' = \mathcal{H}om(E', E)$, $I = \mathcal{H}om(E, E)$, and $I' = \mathcal{H}om(E', E')$. Therefore we can repeat the argument of the proof of Proposition 99.4.3 to see that

$$Isom(E, E') = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

for some morphisms c and σ. Thus $Isom(E, E')$ is an algebraic space. □

0DLH **Lemma 99.16.6.** *In Situation 99.16.3 the functor $p : \mathcal{C}omplexes_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.*

Proof. To prove that $\mathcal{C}omplexes_{X/B}$ is a stack in groupoids, we have to show that the presheaves *Isom* are sheaves and that descent data are effective. The statement on *Isom* follows from Lemma 99.16.5, see Algebraic Stacks, Lemma 94.10.11. Let us prove the statement on descent data.

Suppose that $\{a_i : T_i \to T\}$ is an fppf covering of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $\mathcal{C}omplexes_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, E_i)$. Denote $\mathrm{pr}_0 : T_i \times_T T_j \to T_i$ and $\mathrm{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} \cong \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \mathrm{pr}_0 = g_j \circ \mathrm{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 74.7.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i, B} X = T_i \times_{a_i, T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of (T_i, g_i, E_i) by $\mathrm{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \mathrm{pr}_0, L\mathrm{pr}_i^* E_i)$. Hence a descent datum for $\{T_i \to T\}$ in $\mathcal{C}omplexes_{X/B}$ is given by the objects $(T_i, g \circ a_i, E_i)$ and for each pair i, j an isomorphism in $D\mathcal{O}_{X_{ij}}$)

$$\varphi_{ij} : L\mathrm{pr}_i^* E_i \longrightarrow L\mathrm{pr}_j^* E_j$$

satisfying the cocycle condition over the pullback of X to $T_i \times_T T_j \times_T T_k$. Using the vanishing of negative Exts provided by (b) of Lemma 99.16.2, we may apply Simplicial Spaces, Lemma 85.35.2 to obtain descent[5] for these complexes. In other words, we find there exists an object E in $D_{QCoh}(\mathcal{O}_{X_T})$ restricting to E_i on X_{T_i} compatible with φ_{ij}. Recall that being T-perfect signifies being pseudo-coherent

[5]To check this, first consider the Čech fppf hypercovering $a : T_\bullet \to T$ with $T_n = \coprod_{i_0 \ldots i_n} T_{i_0} \times_T \ldots \times_T T_{i_n}$ and then set $U_\bullet = T_\bullet \times_{a,T} X_T$. Then $U_\bullet \to X_T$ is an fppf hypercovering to which we may apply Simplicial Spaces, Lemma 85.35.2.

and having locally finite tor dimension over $f^{-1}\mathcal{O}_T$. Thus E is T-perfect by an application of More on Morphisms of Spaces, Lemmas 76.54.1 and 76.54.2. Finally, we have to check condition (2) from Lemma 99.16.2 for E. This immediately follows from the description of the open W in Lemma 99.16.1 and the fact that (2) holds for E_i on X_{T_i}/T_i. □

0DLI **Remark 99.16.7.** In Situation 99.16.3 the rule $(T, g, E) \mapsto (T, g)$ defines a 1-morphism

$$\mathit{Complexes}_{X/B} \longrightarrow \mathcal{S}_B$$

of stacks in groupoids (see Lemma 99.16.6, Algebraic Stacks, Section 94.7, and Examples of Stacks, Section 95.10). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $\mathit{Complexes}_{X'/B'} \to (\mathit{Sch}/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{array}{ccc} \mathit{Complexes}_{X'/B'} & \longrightarrow & \mathit{Complexes}_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{S}_{B'} & \longrightarrow & \mathcal{S}_B \end{array} \quad \text{or in another notation} \quad \begin{array}{ccc} \mathit{Complexes}_{X'/B'} & \longrightarrow & \mathit{Complexes}_{X/B} \\ \downarrow & & \downarrow \\ \mathit{Sch}/B' & \longrightarrow & \mathit{Sch}/B \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

0DLJ **Lemma 99.16.8.** *In Situation 99.16.3 assume that $B \to S$ is locally of finite presentation. Then $p : \mathit{Complexes}_{X/B} \to (\mathit{Sch}/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 98.11.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an object (T, h, E) of $\mathit{Complexes}_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{array}{ccc} \operatorname{colim} \mathit{Complexes}_{X/B,T_i} & \longrightarrow & \mathit{Complexes}_{X/B,T} \\ \downarrow & & \downarrow \\ \operatorname{colim} B(T_i) & \longrightarrow & B(T) \end{array}$$

We have to show the top horizontal arrow is an equivalence. Since we have assume that B is locally of finite presentation over S we see from Limits of Spaces, Remark 70.3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$. By More on Morphisms of Spaces, Lemma 76.52.9 the category of T-perfect objects of $D(\mathcal{O}_{X_T})$ is the colimit of the categories of T_i-perfect objects of $D(\mathcal{O}_{X_{T_i}})$. Thus all we have to prove is that given an T_i-perfect object E_i of $D(\mathcal{O}_{X_{T_i}})$ such that the derived pullback E of E_i to X_T satisfies condition (2) of Lemma 99.16.2, then after increasing i we have that E_i satisfies condition (2) of Lemma 99.16.2. Let $W \subset |T_i|$ be the open constructed in Lemma 99.16.1 for E_i and E_i. By assumption on E we find that $T \to T_i$ factors through T. Hence there is an $i' \geq i$ such that $T_{i'} \to T_i$ factors through W, see Limits, Lemma 32.4.10 Then i' works by construction of W. □

0DLK **Lemma 99.16.9.** *In Situation 99.16.3. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a finite order thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 37.14.3. Then the functor on fibre categories

$$\mathit{Complexes}_{X/B,Y'} \longrightarrow \mathit{Complexes}_{X/B,Y} \times_{\mathit{Complexes}_{X/B,Z}} \mathit{Complexes}_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 81.6.1. Thus using the commutative diagram

$$\begin{array}{ccc} \mathit{Complexes}_{X/B,Y'} & \longrightarrow & \mathit{Complexes}_{X/B,Y} \times_{\mathit{Complexes}_{X/B,Z}} \mathit{Complexes}_{X/B,Z'} \\ \downarrow & & \downarrow \\ B(Y') & \longrightarrow & B(Y) \times_{B(Z)} B(Z') \end{array}$$

we see that we may assume that Y' is a scheme over B'. By Remark 99.16.7 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$.

Assume $B = Y'$. We first prove fully faithfulness of our functor. To do this, let ξ_1, ξ_2 be two objects of $\mathit{Complexes}_{X/B}$ over Y'. Then we have to show that

$$\mathit{Isom}(\xi_1, \xi_2)(Y') \longrightarrow \mathit{Isom}(\xi_1, \xi_2)(Y) \times_{\mathit{Isom}(\xi_1, \xi_2)(Z)} \mathit{Isom}(\xi_1, \xi_2)(Z')$$

is bijective. However, we already know that $\mathit{Isom}(\xi_1, \xi_2)$ is an algebraic space over $B = Y'$. Thus this bijectivity follows from Artin's Axioms, Lemma 98.4.1 (or the aforementioned Pushouts of Spaces, Lemma 81.6.1).

Essential surjectivity. Let $(E_Y, E_{Z'}, \alpha)$ be a triple, where $E_Y \in D(\mathcal{O}_Y)$ and $E_{Z'} \in D(\mathcal{O}_{X_{Z'}})$ are objects such that $(Y, Y \to B, E_Y)$ is an object of $\mathit{Complexes}_{X/B}$ over Y, such that $(Z', Z' \to B, E_{Z'})$ is an object of $\mathit{Complexes}_{X/B}$ over Z', and $\alpha : L(X_Z \to X_Y)^* E_Y \to L(X_Z \to X_{Z'})^* E_{Z'}$ is an isomorphism in $D(\mathcal{O}_{Z'})$. That is to say

$$((Y, Y \to B, E_Y), (Z', Z' \to B, E_{Z'}), \alpha)$$

is an object of the target of the arrow of our lemma. Observe that the diagram

$$\begin{array}{ccc} X_Z & \longrightarrow & X_{Z'} \\ \downarrow & & \downarrow \\ X_Y & \longrightarrow & X_{Y'} \end{array}$$

is a pushout with $X_Z \to X_Y$ affine and $X_Z \to X_{Z'}$ a thickening (see Pushouts of Spaces, Lemma 81.6.7). Hence by Pushouts of Spaces, Lemma 81.8.1 we find an object $E_{Y'} \in D(\mathcal{O}_{X_{Y'}})$ together with isomorphisms $L(X_Y \to X_{Y'})^* E_{Y'} \to E_Y$ and

$L(X_{Z'} \to X_{Y'})^* E_{Y'} \to E_Z$ compatible with α. Clearly, if we show that $E_{Y'}$ is Y'-perfect, then we are done, because property (2) of Lemma 99.16.2 is a property on points (and Y and Y' have the same points). This follows from More on Morphisms of Spaces, Lemma 76.54.4. □

0DLL **Lemma 99.16.10.** *In Situation 99.16.3 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, E_0)$ be an object of $\mathcal{X} = \mathit{Complexes}_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathit{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 98.8) are finite dimensional.*

Proof. Observe that by Lemma 99.16.9 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 98.18. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 98.6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0, B} X$ and denote $\mathcal{X}_0 = \mathit{Complexes}_{X_0/k}$. In Remark 99.16.7 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(\mathit{Sch}/S)_{fppf}$. Thus by Artin's Axioms, Lemma 98.8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 98.8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion $k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Observe that X_0 and X_ϵ give rise to canonically equivalent small étale topoi, see More on Morphisms of Spaces, Section 76.9. By More on Morphisms of Spaces, Lemma 76.54.4 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of isomorphism classes of lifts of E_0 to X_ϵ in the sense of Deformation Theory, Lemma 91.16.7. We conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(E_0, E_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 91.16.7 once more we see that there is a surjection

$$\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) \leftarrow \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(E_0, E_0)$$

of k-vector spaces. As E_0 is pseudo-coherent it lies in $D^-_{Coh}(\mathcal{O}_{X_0})$ by Derived Categories of Spaces, Lemma 75.13.7. Since E_0 locally has finite tor dimension and X_0 is quasi-compact we see $E_0 \in D^b_{Coh}(\mathcal{O}_{X_0})$. Thus the Exts above are finite dimensional k-vector spaces by Derived Categories of Spaces, Lemma 75.8.4. □

0DLM **Lemma 99.16.11.** *In Situation 99.16.3 assume $B = S$ is locally Noetherian. Then strong formal effectiveness in the sense of Artin's Axioms, Remark 98.20.2 holds for $p : \mathit{Complexes}_{X/S} \to (\mathit{Sch}/S)_{fppf}$.*

Proof. Let (R_n) be an inverse system of S-algebras with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Let (ξ_n) be a system of objects of $\mathit{Complexes}_{X/B}$ lying over $(\mathrm{Spec}(R_n))$. We have to show (ξ_n) is effective, i.e., there exists an object ξ of $\mathit{Complexes}_{X/B}$ lying over $\mathrm{Spec}(R)$.

Write $X_R = \mathrm{Spec}(R) \times_S X$ and $X_n = \mathrm{Spec}(R_n) \times_S X$. Of course X_n is the base change of X_R by $R \to R_n$. Since $S = B$, we see that ξ_n corresponds simply to an R_n-perfect object $E_n \in D(\mathcal{O}_{X_n})$ satisfying condition (2) of Lemma 99.16.2. In particular E_n is pseudo-coherent. The isomorphisms $\xi_{n+1}|_{\mathrm{Spec}(R_n)} \cong \xi_n$ correspond to isomorphisms $L(X_n \to X_{n+1})^* E_{n+1} \to E_n$. Therefore by Flatness on Spaces, Theorem 77.13.6 we find a pseudo-coherent object E of $D(\mathcal{O}_{X_R})$ with E_n equal to the derived pullback of E for all n compatible with the transition isomorphisms.

Observe that $(R, \mathrm{Ker}(R \to R_1))$ is a henselian pair, see More on Algebra, Lemma 15.11.3. In particular, $\mathrm{Ker}(R \to R_1)$ is contained in the Jacobson radical of R. Then we may apply More on Morphisms of Spaces, Lemma 76.54.5 to see that E is R-perfect.

Finally, we have to check condition (2) of Lemma 99.16.2. By Lemma 99.16.1 the set of points t of $\mathrm{Spec}(R)$ where the negative self-exts of E_t vanish is an open. Since this condition is true in $V(\mathrm{Ker}(R \to R_1))$ and since $\mathrm{Ker}(R \to R_1)$ is contained in the Jacobson radical of R we conclude it holds for all points. □

Theorem 99.16.12 (Algebraicity of moduli of complexes on a proper morphism)**.**
0DLN *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is proper, flat, and of finite presentation. Then $\mathit{Complexes}_{X/B}$ is an algebraic stack over S.* [**Lie06a**]

Proof. Set $\mathcal{X} = \mathit{Complexes}_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(\mathit{Sch}/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 99.16.6 and 99.16.5). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = \mathit{Complexes}_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 99.16.7). By the material in Algebraic Stacks, Section 94.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 94.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(\mathit{Sch}/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is proper, flat, of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1, 70.6.12, and 70.6.13. If we show that $\mathit{Complexes}_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 99.16.7 and Algebraic Stacks, Section 94.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 98.17.1 to

conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 98.14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 99.16.6, 99.16.8, 99.16.9, 99.16.10. Condition (3) follows from Lemma 99.16.11. Condition (1) is Lemma 99.16.5.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 98.20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 98.20.3. This follows from Lemma 99.16.11 and the proof is complete. □

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CHAPTER 100

Properties of Algebraic Stacks

04X8

100.1. Introduction

04X9 Please see Algebraic Stacks, Section 94.1 for a brief introduction to algebraic stacks, and please read some of that chapter for our foundations of algebraic stacks. The intent is that in that chapter we are careful to distinguish between schemes, algebraic spaces, algebraic stacks, and starting with this chapter we employ the customary abuse of language where all of these concepts are used interchangeably.

The goal of this chapter is to introduce some basic notions and properties of algebraic stacks. A fundamental reference for the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00**].

100.2. Conventions and abuse of language

04XA We choose a big fppf site Sch_{fppf}. All schemes are contained in Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

We also fix a base scheme S, by the conventions above an element of Sch_{fppf}. The reader who is only interested in the absolute case can take $S = \mathrm{Spec}(\mathbf{Z})$.

Here are our conventions regarding algebraic stacks:

(1) When we say *algebraic stack* we will mean an algebraic stacks over S, i.e., a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which satisfies the conditions of Algebraic Stacks, Definition 94.12.1.
(2) We will say $f : \mathcal{X} \to \mathcal{Y}$ is a *morphism of algebraic stacks* to indicate a 1-morphism of algebraic stacks over S, i.e., a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Algebraic Stacks, Definition 94.12.3.
(3) A *2-morphism* $\alpha : f \to g$ will indicate a 2-morphism in the 2-category of algebraic stacks over S, see Algebraic Stacks, Definition 94.12.3.
(4) Given morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ of algebraic stacks we abusively call the 2-fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ the *fibre product*.
(5) We will write $\mathcal{X} \times_S \mathcal{Y}$ for the product of the algebraic stacks $\mathcal{X}$, $\mathcal{Y}$.
(6) We will often abuse notation and say two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* if they are equivalent in this 2-category.

Here are our conventions regarding algebraic spaces.

(1) If we say X is an *algebraic space* then we mean that X is an algebraic space over S, i.e., X is a presheaf on $(Sch/S)_{fppf}$ which satisfies the conditions of Spaces, Definition 65.6.1.
(2) A *morphism of algebraic spaces* $f : X \to Y$ is a morphism of algebraic spaces over S as defined in Spaces, Definition 65.6.3.

(3) We will **not** distinguish between an algebraic space X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 94.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is an algebraic space* to indicate that $\mathcal{X}$ is representable by an algebraic space, see Algebraic Stacks, Definition 94.8.1.
(6) We will use the following notational convention: If we indicate an algebraic stack by a roman capital (such as $X, Y, Z, A, B, \ldots$) then it will be the case that its inertia stack is trivial, and hence it is an algebraic space, see Algebraic Stacks, Proposition 94.13.3.

Here are our conventions regarding schemes.

(1) If we say X is a *scheme* then we mean that X is a scheme over S, i.e., X is an object of $(Sch/S)_{fppf}$.
(2) By a *morphism of schemes* we mean a morphism of schemes over S.
(3) We will **not** distinguish between a scheme X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 94.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from a scheme X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is a scheme* to indicate that $\mathcal{X}$ is representable, see Algebraic Stacks, Section 94.4.

Here are our conventions regarding morphisms of algebraic stacks:

(1) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable*, or *representable by schemes* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is a scheme. See Algebraic Stacks, Section 94.6.
(2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. See Algebraic Stacks, Definition 94.9.1. In this case $Z \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space whenever $Z \to \mathcal{Y}$ is a morphism whose source is an algebraic space, see Algebraic Stacks, Lemma 94.9.8.
(3) We may abuse notation and say that a diagram of algebraic stacks commutes if the diagram is 2-commutative in the 2-category of algebraic stacks.

Note that every morphism $X \to \mathcal{Y}$ from an algebraic space to an algebraic stack is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.11. We will use this basic result without further mention.

100.3. Properties of morphisms representable by algebraic spaces

04XB We will study properties of (arbitrary) morphisms of algebraic stacks in its own chapter. For morphisms representable by algebraic spaces we know what it means to be surjective, smooth, or étale, etc. This applies in particular to morphisms $X \to \mathcal{Y}$ from algebraic spaces to algebraic stacks. In this section, we recall how this works, we list the properties to which this applies, and we prove a few easy lemmas.

Our first lemma says a morphism is representable by algebraic spaces if it is so after a base change by a flat, locally finitely presented, surjective morphism.

04ZP **Lemma 100.3.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. The following are equivalent*

(1) *f is representable by algebraic spaces, and*
(2) *$W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.*

Proof. The implication (1) $\Rightarrow$ (2) is Algebraic Stacks, Lemma 94.9.8. Conversely, let $W \to \mathcal{Y}$ be as in (2). To prove (1) it suffices to show that f is faithful on fibre categories, see Algebraic Stacks, Lemma 94.15.2. Assumption (2) implies in particular that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is faithful. Hence the faithfulness of f follows from Stacks, Lemma 8.6.9. □

Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then we say *f has property P* if and only if for every scheme T and morphism $T \to \mathcal{Y}$ the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ has property P, see Algebraic Stacks, Definition 94.10.1.

It turns out that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces and has property P, then for any morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property P, see Algebraic Stacks, Lemmas 94.9.7 and 94.10.6. If the property P is preserved under compositions, then this holds also in the setting of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemmas 94.9.9 and 94.10.5. Moreover, in this case products $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ of morphisms representable by algebraic spaces having property $\mathcal{P}$ have property $\mathcal{P}$, see Algebraic Stacks, Lemma 94.10.8.

Finally, if we have two properties P, P' of morphisms of algebraic spaces which are fppf local on the target and preserved by arbitrary base change and if $P(f) \Rightarrow P'(f)$ for every morphism f, then the same implication holds for the corresponding property of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemma 94.10.9. We will use this *without further mention* in the following and in the following chapters.

The discussion above applies to each of the following properties of morphisms of algebraic spaces

(1) quasi-compact, see Morphisms of Spaces, Lemma 67.8.4 and Descent on Spaces, Lemma 74.11.1,
(2) quasi-separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.11.2,
(3) universally closed, see Morphisms of Spaces, Lemma 67.9.3 and Descent on Spaces, Lemma 74.11.3,
(4) universally open, see Morphisms of Spaces, Lemma 67.6.3 and Descent on Spaces, Lemma 74.11.4,
(5) universally submersive, see Morphisms of Spaces, Lemma 67.7.3 and Descent on Spaces, Lemma 74.11.5,
(6) universal homeomorphism, see Morphisms of Spaces, Lemma 67.53.4 and Descent on Spaces, Lemma 74.11.8,

(7) surjective, see Morphisms of Spaces, Lemma 67.5.5 and Descent on Spaces, Lemma 74.11.6,
(8) universally injective, see Morphisms of Spaces, Lemma 67.19.5 and Descent on Spaces, Lemma 74.11.7,
(9) locally of finite type, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.9,
(10) locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.10,
(11) finite type, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.11,
(12) finite presentation, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.12,
(13) flat, see Morphisms of Spaces, Lemma 67.30.4 and Descent on Spaces, Lemma 74.11.13,
(14) open immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.11.14,
(15) isomorphism, see Descent on Spaces, Lemma 74.11.15,
(16) affine, see Morphisms of Spaces, Lemma 67.20.5 and Descent on Spaces, Lemma 74.11.16,
(17) closed immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.11.17,
(18) separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.11.18,
(19) proper, see Morphisms of Spaces, Lemma 67.40.3 and Descent on Spaces, Lemma 74.11.19,
(20) quasi-affine, see Morphisms of Spaces, Lemma 67.21.5 and Descent on Spaces, Lemma 74.11.20,
(21) integral, see Morphisms of Spaces, Lemma 67.45.5 and Descent on Spaces, Lemma 74.11.22,
(22) finite, see Morphisms of Spaces, Lemma 67.45.5 and Descent on Spaces, Lemma 74.11.23,
(23) (locally) quasi-finite, see Morphisms of Spaces, Lemma 67.27.4 and Descent on Spaces, Lemma 74.11.24,
(24) syntomic, see Morphisms of Spaces, Lemma 67.36.3 and Descent on Spaces, Lemma 74.11.25,
(25) smooth, see Morphisms of Spaces, Lemma 67.37.3 and Descent on Spaces, Lemma 74.11.26,
(26) unramified, see Morphisms of Spaces, Lemma 67.38.4 and Descent on Spaces, Lemma 74.11.27,
(27) étale, see Morphisms of Spaces, Lemma 67.39.4 and Descent on Spaces, Lemma 74.11.28,
(28) finite locally free, see Morphisms of Spaces, Lemma 67.46.5 and Descent on Spaces, Lemma 74.11.29,
(29) monomorphism, see Morphisms of Spaces, Lemma 67.10.5 and Descent on Spaces, Lemma 74.11.30,
(30) immersion, see Morphisms of Spaces, Section 67.12 and Descent on Spaces, Lemma 74.12.1,

(31) locally separated, see Morphisms of Spaces, Lemma 67.4.4 and Descent on Spaces, Lemma 74.12.2,

04XC **Lemma 100.3.2.** *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. The following are equivalent:*

(1) *f has P,*
(2) *for every algebraic space Z and morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has P.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). Let $Z \to \mathcal{Y}$ be as in (2). Choose a scheme U and a surjective étale morphism $U \to Z$. By assumption the morphism $U \times_{\mathcal{Y}} \mathcal{X} \to U$ has P. But the diagram

$$\begin{array}{ccc} U \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & Z \end{array}$$

is cartesian, hence the right vertical arrow has P as $\{U \to Z\}$ is an fppf covering. □

The following lemma tells us it suffices to check P after a base change by a surjective, flat, locally finitely presented morphism.

04XD **Lemma 100.3.3.** *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. Set $V = W \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } V \to W \text{ has } P).$$

Proof. The implication from left to right follows from Lemma 100.3.2. Assume $V \to W$ has P. Let T be a scheme, and let $T \to \mathcal{Y}$ be a morphism. Consider the commutative diagram

$$\begin{array}{ccccc} T \times_{\mathcal{Y}} \mathcal{X} & \longleftarrow & T \times_{\mathcal{Y}} V & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ T & \longleftarrow & T \times_{\mathcal{Y}} W & \longrightarrow & W \end{array}$$

of algebraic spaces. The squares are cartesian. The bottom left morphism is a surjective, flat morphism which is locally of finite presentation, hence $\{T \times_{\mathcal{Y}} V \to T\}$ is an fppf covering. Hence the fact that the right vertical arrow has property P implies that the left vertical arrow has property P. □

06TY **Lemma 100.3.4.** *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By the discussion above the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V =$

$W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 100.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. □

06M2 **Lemma 100.3.5.** *Let P be a property of morphisms of algebraic spaces as above. Let $\tau \in \{étale, smooth, syntomic, fppf\}$. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks representable by algebraic spaces. Assume*

(1) *$\mathcal{X} \to \mathcal{Y}$ is surjective and étale, smooth, syntomic, or flat and locally of finite presentation,*
(2) *the composition has P, and*
(3) *P is local on the source in the τ topology.*

Then $\mathcal{Y} \to \mathcal{Z}$ has property P.

Proof. Let Z be a scheme and let $Z \to \mathcal{Z}$ be a morphism. Set $X = \mathcal{X} \times_{\mathcal{Z}} Z$, $Y = \mathcal{Y} \times_{\mathcal{Z}} Z$. By (1) $\{X \to Y\}$ is a τ covering of algebraic spaces and by (2) $X \to Z$ has property P. By (3) this implies that $Y \to Z$ has property P and we win. □

04Y6 **Lemma 100.3.6.** *Let $g : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Let $[U/R] \to \mathcal{X}$ be a presentation. Set $U' = U \times_{\mathcal{X}} \mathcal{X}'$, and $R' = R \times_{\mathcal{X}} \mathcal{X}'$. Then there exists a groupoid in algebraic spaces of the form (U', R', s', t', c'), a presentation $[U'/R'] \to \mathcal{X}'$, and the diagram*

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & \mathcal{X}' \\ {\scriptstyle [pr]}\downarrow & & \downarrow{\scriptstyle g} \\ [U/R] & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative where the morphism $[pr]$ comes from a morphism of groupoids $pr : (U', R', s', t', c') \to (U, R, s, t, c)$.

Proof. Since $U \to \mathcal{Y}$ is surjective and smooth, see Algebraic Stacks, Lemma 94.17.2 the base change $U' \to \mathcal{X}'$ is also surjective and smooth. Hence, by Algebraic Stacks, Lemma 94.16.2 it suffices to show that $R' = U' \times_{\mathcal{X}'} U'$ in order to get a smooth groupoid (U', R', s', t', c') and a presentation $[U'/R'] \to \mathcal{X}'$. Using that $R = V \times_{\mathcal{Y}} V$ (see Groupoids in Spaces, Lemma 78.22.2) this follows from

$$R' = U \times_{\mathcal{X}} U \times_{\mathcal{X}} \mathcal{X}' = (U \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}'} (U \times_{\mathcal{X}} \mathcal{X}')$$

see Categories, Lemmas 4.31.8 and 4.31.10. Clearly the projection morphisms $U' \to U$ and $R' \to R$ give the desired morphism of groupoids pr : $(U', R', s', t', c') \to (U, R, s, t, c)$. Hence the morphism [pr] of quotient stacks by Groupoids in Spaces, Lemma 78.21.1.

We still have to show that the diagram 2-commutes. It is clear that the diagram

$$\begin{array}{ccc} U' & \xrightarrow[f']{} & \mathcal{X}' \\ {\scriptstyle \mathrm{pr}_U}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & \mathcal{X} \end{array}$$

2-commutes where $\mathrm{pr}_U : U' \to U$ is the projection. There is a canonical 2-arrow $\tau : f \circ t \to f \circ s$ in $\mathrm{Mor}(R, \mathcal{X})$ coming from $R = U \times_{\mathcal{X}} U$, $t = \mathrm{pr}_0$, and $s = \mathrm{pr}_1$. Using the isomorphism $R' \to U' \times_{\mathcal{X}'} U'$ we get similarly an isomorphism $\tau' : f' \circ t' \to f' \circ s'$.

Note that $g \circ f' \circ t' = f \circ t \circ \mathrm{pr}_R$ and $g \circ f' \circ s' = f \circ s \circ \mathrm{pr}_R$, where $\mathrm{pr}_R : R' \to R$ is the projection. Thus it makes sense to ask if

04Y7 (100.3.6.1) $$\tau \star \mathrm{id}_{\mathrm{pr}_R} = \mathrm{id}_g \star \tau'.$$

Now we make two claims: (1) if Equation (100.3.6.1) holds, then the diagram 2-commutes, and (2) Equation (100.3.6.1) holds. We omit the proof of both claims. Hints: part (1) follows from the construction of $f = f_{can}$ and $f' = f'_{can}$ in Algebraic Stacks, Lemma 94.16.1. Part (2) follows by carefully working through the definitions. □

04ZQ **Remark 100.3.7.** Let $\mathcal{Y}$ be an algebraic stack. Consider the following 2-category:

(1) An object is a morphism $f : \mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces,
(2) a 1-morphism $(g, \beta) : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ consists of a morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and a 2-morphism $\beta : f_1 \to f_2 \circ g$, and
(3) a 2-morphism between $(g, \beta), (g', \beta') : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ is a 2-morphism $\alpha : g \to g'$ such that $(\mathrm{id}_{f_2} \star \alpha) \circ \beta = \beta'$.

Let us denote this 2-category $\mathit{Spaces}/\mathcal{Y}$ by analogy with the notation of Topologies on Spaces, Section 73.2. Now we claim that in this 2-category the morphism categories

$$\mathrm{Mor}_{\mathit{Spaces}/\mathcal{Y}}((f_1 : \mathcal{X}_1 \to \mathcal{Y}), (f_2 : \mathcal{X}_2 \to \mathcal{Y}))$$

are all setoids. Namely, a 2-morphism α is a rule which to each object x_1 of $\mathcal{X}_1$ assigns an isomorphism $\alpha_{x_1} : g(x_1) \longrightarrow g'(x_1)$ in the relevant fibre category of $\mathcal{X}_2$ such that the diagram

$$\begin{array}{ccc} & f_2(x_1) & \\ {\scriptstyle \beta_{x_1}} \swarrow & & \searrow {\scriptstyle \beta'_{x_1}} \\ f_2(g(x_1)) & \xrightarrow{f_2(\alpha_{x_1})} & f_2(g'(x_1)) \end{array}$$

commutes. But since f_2 is faithful (see Algebraic Stacks, Lemma 94.15.2) this means that if α_{x_1} exists, then it is unique! In other words the 2-category $\mathit{Spaces}/\mathcal{Y}$ is very close to being a category. Namely, if we replace 1-morphisms by isomorphism classes of 1-morphisms we obtain a category. We will often perform this replacement without further mention.

100.4. Points of algebraic stacks

04XE Let $\mathcal{X}$ be an algebraic stack. Let K, L be two fields and let $p : \mathrm{Spec}(K) \to \mathcal{X}$ and $q : \mathrm{Spec}(L) \to \mathcal{X}$ be morphisms. We say that p and q are *equivalent* if there exists a field Ω and a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow {\scriptstyle q} \\ \mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X}. \end{array}$$

04XF **Lemma 100.4.1.** *The notion above does indeed define an equivalence relation on morphisms from spectra of fields into the algebraic stack $\mathcal{X}$.*

Proof. It is clear that the relation is reflexive and symmetric. Hence we have to prove that it is transitive. This comes down to the following: Given a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\Omega) & \xrightarrow{b} & \mathrm{Spec}(L) & \xleftarrow{b'} & \mathrm{Spec}(\Omega') \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a'} \\ \mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X} & \xleftarrow{p'} & \mathrm{Spec}(K') \end{array}$$

with both squares 2-commutative we have to show that p is equivalent to p'. By the 2-Yoneda lemma (see Algebraic Stacks, Section 94.5) the morphisms p, p', and q are given by objects x, x', and y in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(K)$, $\mathrm{Spec}(K')$, and $\mathrm{Spec}(L)$. The 2-commutativity of the squares means that there are isomorphisms $\alpha : a^*x \to b^*y$ and $\alpha' : (a')^*x' \to (b')^*y$ in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(\Omega)$ and $\mathrm{Spec}(\Omega')$. Choose any field Ω'' and embeddings $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ agreeing on L. Then we can extend the diagram above to

$$\begin{array}{ccccc} & & \mathrm{Spec}(\Omega'') & & \\ & \swarrow^{c} & \downarrow{\scriptstyle q'} & \searrow^{c'} & \\ \mathrm{Spec}(\Omega) & \xrightarrow{b} & \mathrm{Spec}(L) & \xleftarrow{b'} & \mathrm{Spec}(\Omega') \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a'} \\ \mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X} & \xleftarrow{p'} & \mathrm{Spec}(K') \end{array}$$

with commutative triangles and

$$(q')^*(\alpha')^{-1} \circ (q')^*\alpha : (a \circ c)^*x \longrightarrow (a' \circ c')^*x'$$

is an isomorphism in the fibre category over $\mathrm{Spec}(\Omega'')$. Hence p is equivalent to p' as desired. □

04XG **Definition 100.4.2.** Let $\mathcal{X}$ be an algebraic stack. A *point* of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields into $\mathcal{X}$. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

This agrees with our definition of points of algebraic spaces, see Properties of Spaces, Definition 66.4.1. Moreover, for a scheme we recover the usual notion of points, see Properties of Spaces, Lemma 66.4.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then there is an induced map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ which maps a representative $x : \mathrm{Spec}(K) \to \mathcal{X}$ to the representative $f \circ x : \mathrm{Spec}(K) \to \mathcal{Y}$. This is well defined: namely 2-isomorphic 1-morphisms remain 2-isomorphic after pre- or post-composing by a 1-morphism because you can horizontally pre- or post-compose by the identity of the given 1-morphism. This holds in any (strict) $(2,1)$-category. If

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{W} & \longrightarrow & \mathcal{Z} \end{array}$$

is a 2-commutative diagram of algebraic stacks, then the diagram of sets

$$\begin{array}{ccc} |\mathcal{X}| & \longrightarrow & |\mathcal{Y}| \\ \downarrow & & \downarrow \\ |\mathcal{W}| & \longrightarrow & |\mathcal{Z}| \end{array}$$

is commutative. In particular, if $\mathcal{X} \to \mathcal{Y}$ is an equivalence then $|\mathcal{X}| \to |\mathcal{Y}|$ is a bijection.

04XH **Lemma 100.4.3.** *Let*

$$\begin{array}{ccc} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. Then the map of sets of points

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{Y}|} |\mathcal{X}|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to \mathcal{X}$, $\mathrm{Spec}(L) \to \mathcal{Z}$, then the assumption that they agree as elements of $|\mathcal{Y}|$ means that there is a common extension M/K and M/L such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to \mathcal{X} \to \mathcal{Y}$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to \mathcal{Z} \to \mathcal{Y}$ are 2-isomorphic. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. □

04XI **Lemma 100.4.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *$|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is surjective, and*
(2) *f is surjective (in the sense of Section 100.3).*

Proof. Assume (1). Let $T \to \mathcal{Y}$ be a morphism whose source is a scheme. To prove (2) we have to show that the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is surjective. By Morphisms of Spaces, Definition 67.5.2 this means we have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is surjective. Applying Lemma 100.4.3 we see that this follows from (1).

Conversely, assume (2). Let $y : \mathrm{Spec}(K) \to \mathcal{Y}$ be a morphism from the spectrum of a field into $\mathcal{Y}$. By assumption the morphism $\mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} \to \mathrm{Spec}(K)$ of algebraic spaces is surjective. By Morphisms of Spaces, Definition 67.5.2 this means there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X}$ such that the left square of the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & = & \mathrm{Spec}(K) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

is commutative. This shows that $|X| \to |\mathcal{Y}|$ is surjective. □

Here is a lemma explaining how to compute the set of points in terms of a presentation.

04XJ **Lemma 100.4.5.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X} = [U/R]$ be a presentation of $\mathcal{X}$, see Algebraic Stacks, Definition 94.16.5. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that we have a smooth groupoid (U, R, s, t, c) in algebraic spaces, and an equivalence $f : [U/R] \to \mathcal{X}$. We may assume $\mathcal{X} = [U/R]$. The induced morphism $p : U \to \mathcal{X}$ is smooth and surjective, see Algebraic Stacks, Lemma 94.17.2. Hence $|U| \to |\mathcal{X}|$ is surjective by Lemma 100.4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 78.22.2. Hence Lemma 100.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. □

04XK **Remark 100.4.6.** The result of Lemma 100.4.5 can be generalized as follows. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be a surjective morphism (which makes sense by Section 100.3). Let $R = U \times_{\mathcal{X}} U$, let (U, R, s, t, c) be the groupoid in algebraic spaces, and let $f_{can} : [U/R] \to \mathcal{X}$ be the canonical morphism as constructed in Algebraic Stacks, Lemma 94.16.1. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}| = |U|/|R|$. The proof of Lemma 100.4.5 works without change. (Of course in general $[U/R]$ is not an algebraic stack, and in general f_{can} is not an isomorphism.)

04XL **Lemma 100.4.7.** *There exists a unique topology on the sets of points of algebraic stacks with the following properties:*

1. *for every morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ the map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous, and*
2. *for every morphism $U \to \mathcal{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathcal{X}|$ is continuous and open.*

Proof. Choose a morphism $p : U \to \mathcal{X}$ which is surjective, flat, and locally of finite presentation with U an algebraic space. Such exist by the definition of an algebraic stack, as a smooth morphism is flat and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.37.5 and 67.37.7). We define a topology on $|\mathcal{X}|$ by the rule: $W \subset |\mathcal{X}|$ is open if and only if $|p|^{-1}(W)$ is open in $|U|$. To show that this is independent of the choice of p, let $p' : U' \to \mathcal{X}$ be another morphism which is surjective, flat, locally of finite presentation from an algebraic space to $\mathcal{X}$. Set $U'' = U \times_{\mathcal{X}} U'$ so that we have a 2-commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

As $U \to \mathcal{X}$ and $U' \to \mathcal{X}$ are surjective, flat, locally of finite presentation we see that $U'' \to U'$ and $U'' \to U$ are surjective, flat and locally of finite presentation, see Lemma 100.3.2. Hence the maps $|U''| \to |U'|$ and $|U''| \to |U|$ are continuous, open and surjective, see Morphisms of Spaces, Definition 67.5.2 and Lemma 67.30.6. This clearly implies that our definition is independent of the choice of $p : U \to \mathcal{X}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. By Algebraic Stacks, Lemma 94.15.1 we can find a 2-commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

with surjective smooth vertical arrows. Consider the associated commutative diagram

$$\begin{array}{ccc} |U| & \xrightarrow{|a|} & |V| \\ {\scriptstyle |x|}\downarrow & & \downarrow{\scriptstyle |y|} \\ |\mathcal{X}| & \xrightarrow{|f|} & |\mathcal{Y}| \end{array}$$

of sets. If $W \subset |\mathcal{Y}|$ is open, then by the definition above this means exactly that $|y|^{-1}(W)$ is open in $|V|$. Since $|a|$ is continuous we conclude that $|a|^{-1}|y|^{-1}(W) = |x|^{-1}|f|^{-1}(W)$ is open in $|W|$ which means by definition that $|f|^{-1}(W)$ is open in $|\mathcal{X}|$. Thus $|f|$ is continuous.

Finally, we have to show that if U is an algebraic space, and $U \to \mathcal{X}$ is flat and locally of finite presentation, then $|U| \to |\mathcal{X}|$ is open. Let $V \to \mathcal{X}$ be surjective, flat, and locally of finite presentation with V an algebraic space. Consider the commutative diagram

$$\begin{array}{ccccc} |U \times_{\mathcal{X}} V| & \xrightarrow{e} & |U| \times_{|\mathcal{X}|} |V| & \xrightarrow{d} & |V| \\ & \searrow^{f} & {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle b} \\ & & |U| & \xrightarrow{a} & |\mathcal{X}| \end{array}$$

Now the morphism $U \times_{\mathcal{X}} V \to U$ is surjective, i.e, $f : |U \times_{\mathcal{X}} V| \to |U|$ is surjective. The left top horizontal arrow is surjective, see Lemma 100.4.3. The morphism $U \times_{\mathcal{X}} V \to V$ is flat and locally of finite presentation, hence $d \circ e : |U \times_{\mathcal{X}} V| \to |V|$ is open, see Morphisms of Spaces, Lemma 67.30.6. Pick $W \subset |U|$ open. The properties above imply that $b^{-1}(a(W)) = (d \circ e)(f^{-1}(W))$ is open, which by construction means that $a(W)$ is open as desired. □

04Y8 **Definition 100.4.8.** Let $\mathcal{X}$ be an algebraic stack. The underlying *topological space* of $\mathcal{X}$ is the set of points $|\mathcal{X}|$ endowed with the topology constructed in Lemma 100.4.7.

This definition does not conflict with the already existing topology on $|\mathcal{X}|$ if $\mathcal{X}$ is an algebraic space.

04Y9 **Lemma 100.4.9.** *Let $\mathcal{X}$ be an algebraic stack. Every point of $|\mathcal{X}|$ has a fundamental system of quasi-compact open neighbourhoods. In particular $|\mathcal{X}|$ is locally quasi-compact in the sense of Topology, Definition 5.13.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |\mathcal{X}|$ of topological spaces. Namely, if $U \to \mathcal{X}$ is surjective and smooth, then Lemma 100.4.7 guarantees that $|U| \to |\mathcal{X}|$ is continuous, surjective, and open. □

100.5. Surjective morphisms

04ZR Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. In Section 100.3 we have already defined what it means for f to be surjective. In Lemma 100.4.4 we have seen that this is equivalent to requiring $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ to be surjective. This clears the way for the following definition.

04ZS **Definition 100.5.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *surjective* if the map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ of associated topological spaces is surjective.

Here are some lemmas.

04ZT **Lemma 100.5.2.** *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

04ZU **Lemma 100.5.3.** *The base change of a surjective morphism is surjective.*

Proof. Omitted. Hint: Use Lemma 100.4.3. □

06PM **Lemma 100.5.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Y}' \to \mathcal{Y}$ be a surjective morphism of algebraic stacks. If the base change $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ of f is surjective, then f is surjective.*

Proof. Immediate from Lemma 100.4.3. □

06PN **Lemma 100.5.5.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is surjective so is $\mathcal{Y} \to \mathcal{Z}$.*

Proof. Immediate. □

100.6. Quasi-compact algebraic stacks

04YA The following definition is equivalent with the definition for algebraic spaces by Properties of Spaces, Lemma 66.5.2.

04YB **Definition 100.6.1.** Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *quasi-compact* if and only if $|\mathcal{X}|$ is quasi-compact.

04YC **Lemma 100.6.2.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) *$\mathcal{X}$ is quasi-compact,*
(2) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U an affine scheme,*
(3) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact scheme,*
(4) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact algebraic space, and*
(5) *there exists a surjective morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks such that $\mathcal{U}$ is quasi-compact.*

Proof. We will use Lemma 100.4.4. Suppose $\mathcal{U}$ and $\mathcal{U} \to \mathcal{X}$ are as in (5). Then since $|\mathcal{U}| \to |\mathcal{X}|$ is surjective and continuous we conclude that $|\mathcal{X}|$ is quasi-compact. Thus (5) implies (1). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate. Assume (1), i.e., $\mathcal{X}$ is quasi-compact, i.e., that $|\mathcal{X}|$ is quasi-compact. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then since $|U| \to |\mathcal{X}|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is

surjective (and still smooth). Choose a finite affine open covering $U' = U_1 \cup \ldots \cup U_n$. Then $U_1 \amalg \ldots \amalg U_n \to \mathcal{X}$ is a surjective smooth morphism whose source is an affine scheme (Schemes, Lemma 26.6.8). Hence (2) holds. □

04YD **Lemma 100.6.3.** *A finite disjoint union of quasi-compact algebraic stacks is a quasi-compact algebraic stack.*

Proof. This is clear from the corresponding topological fact. □

100.7. Properties of algebraic stacks defined by properties of schemes

04YE Any smooth local property of schemes gives rise to a corresponding property of algebraic stacks via the following lemma. Note that a property of schemes which is smooth local is also étale local as any étale covering is also a smooth covering. Hence for a smooth local property P of schemes we know what it means to say that an algebraic space has P, see Properties of Spaces, Section 66.7.

04YF **Lemma 100.7.1.** *Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology, see Descent, Definition 35.15.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *for some scheme U and some surjective smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(2) *for every scheme U and every smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(3) *for some algebraic space U and some surjective smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$, and*
(4) *for every algebraic space U and every smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$.*

If $\mathcal{X}$ is a scheme U this is equivalent to $\mathcal{P}(U)$. If $\mathcal{X}$ is an algebraic space X this is equivalent to X having property $\mathcal{P}$.

Proof. Let $U \to \mathcal{X}$ surjective and smooth with U an algebraic space. Let $V \to \mathcal{X}$ be a smooth morphism with V an algebraic space. Choose schemes U' and V' and surjective étale morphisms $U' \to U$ and $V' \to V$. Finally, choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2. Moreover, $W \to V'$ is surjective as $U' \to \mathcal{X}$ is surjective. Hence, we have

$$\mathcal{P}(U) \Leftrightarrow \mathcal{P}(U') \Rightarrow \mathcal{P}(W) \Rightarrow \mathcal{P}(V') \Leftrightarrow \mathcal{P}(V)$$

where the equivalences are by definition of property $\mathcal{P}$ for algebraic spaces, and the two implications come from Descent, Definition 35.15.1. This proves (3) $\Rightarrow$ (4).

The implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (2) are immediate. □

04YG **Definition 100.7.2.** Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology. We say $\mathcal{X}$ *has property* $\mathcal{P}$ if any of the equivalent conditions of Lemma 100.7.1 hold.

04YH **Remark 100.7.3.** Here is a list of properties which are local for the smooth topology (keep in mind that the fpqc, fppf, and syntomic topologies are stronger than the smooth topology):

(1) locally Noetherian, see Descent, Lemma 35.16.1,

(2) Jacobson, see Descent, Lemma 35.16.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 35.17.1,
(4) Cohen-Macaulay, see Descent, Lemma 35.17.2,
(5) reduced, see Descent, Lemma 35.18.1,
(6) normal, see Descent, Lemma 35.18.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 35.18.3,
(8) regular, see Descent, Lemma 35.18.4,
(9) Nagata, see Descent, Lemma 35.18.5.

Any smooth local property of germs of schemes gives rise to a corresponding property of algebraic stacks. Note that a property of germs which is smooth local is also étale local. Hence for a smooth local property of germs of schemes P we know what it means to say that an algebraic space X has property P at $x \in |X|$, see Properties of Spaces, Section 100.7.

04YI **Lemma 100.7.4.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local, see Descent, Definition 35.21.1. The following are equivalent*

(1) *for any smooth morphism $U \to \mathcal{X}$ with U a scheme and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(2) *for some smooth morphism $U \to \mathcal{X}$ with U a scheme and some $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(3) *for any smooth morphism $U \to \mathcal{X}$ with U an algebraic space and $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u, and*
(4) *for some smooth morphism $U \to \mathcal{X}$ with U an algebraic space and some $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u.*

If $\mathcal{X}$ is representable, then this is equivalent to $\mathcal{P}(\mathcal{X}, x)$. If $\mathcal{X}$ is an algebraic space then this is equivalent to $\mathcal{X}$ having property $\mathcal{P}$ at x.

Proof. Let $a : U \to \mathcal{X}$ and $u \in |U|$ as in (3). Let $b : V \to \mathcal{X}$ be another smooth morphism with V an algebraic space and $v \in |V|$ with $b(v) = x$ also. Choose a scheme U', an étale morphism $U' \to U$ and $u' \in U'$ mapping to u. Choose a scheme V', an étale morphism $V' \to V$ and $v' \in V'$ mapping to v. By Lemma 100.4.3 there exists a point $\overline{w} \in |V' \times_{\mathcal{X}} U'|$ mapping to u' and v'. Choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. We may choose a $w \in |W|$ mapping to $\overline{w}$ (see Properties of Spaces, Lemma 66.4.4). Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 67.39.6 and 67.37.2. Hence

$$\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u') \Leftrightarrow \mathcal{P}(W, w) \Leftrightarrow \mathcal{P}(V', v') \Leftrightarrow \mathcal{P}(V, v)$$

The outer two equivalences by Properties of Spaces, Definition 66.7.5 and the other two by what it means to be a smooth local property of germs of schemes. This proves (4) $\Rightarrow$ (3).

The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (1) are immediate. □

04YJ **Definition 100.7.5.** Let $\mathcal{P}$ be a property of germs of schemes which is smooth local. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. We say $\mathcal{X}$ *has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 100.7.4 holds.

100.8. Monomorphisms of algebraic stacks

04ZV We define a monomorphism of algebraic stacks in the following way. We will see in Lemma 100.8.4 that this is compatible with the corresponding 2-category theoretic notion.

04ZW **Definition 100.8.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *monomorphism* if it is representable by algebraic spaces and a monomorphism in the sense of Section 100.3.

First some basic lemmas.

04ZX **Lemma 100.8.2.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a monomorphism. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a monomorphism.*

Proof. This follows from the general discussion in Section 100.3. □

04ZY **Lemma 100.8.3.** *Compositions of monomorphisms of algebraic stacks are monomorphisms.*

Proof. This follows from the general discussion in Section 100.3 and Morphisms of Spaces, Lemma 67.10.4. □

04ZZ **Lemma 100.8.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is a monomorphism,*
(2) *f is fully faithful,*
(3) *the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an equivalence, and*
(4) *there exists an algebraic space W and a surjective, flat morphism $W \to \mathcal{Y}$ which is locally of finite presentation such that $V = \mathcal{X} \times_{\mathcal{Y}} W$ is an algebraic space, and the morphism $V \to W$ is a monomorphism of algebraic spaces.*

Proof. The equivalence of (1) and (4) follows from the general discussion in Section 100.3 and in particular Lemmas 100.3.1 and 100.3.3.

The equivalence of (2) and (3) is Categories, Lemma 4.35.10.

Assume the equivalent conditions (2) and (3). Then f is representable by algebraic spaces according to Algebraic Stacks, Lemma 94.15.2. Moreover, the 2-Yoneda lemma combined with the fully faithfulness implies that for every scheme T the functor

$$\mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, \mathcal{Y})$$

is fully faithful. Hence given a morphism $y : T \to \mathcal{Y}$ there exists up to unique 2-isomorphism at most one morphism $x : T \to \mathcal{X}$ such that $y \cong f \circ x$. In particular, given a morphism of schemes $h : T' \to T$ there exists at most one lift $\tilde{h} : T' \to T \times_{\mathcal{Y}} \mathcal{X}$ of h. Thus $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism of algebraic spaces, which proves that (1) holds.

Finally, assume that (1) holds. Then for any scheme T and morphism $y : T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism. Hence there exists up to unique isomorphism exactly one pair (x, α) where $x : T \to \mathcal{X}$ is a morphism and $\alpha : f \circ x \to y$ is a 2-morphism. Applying the 2-Yoneda lemma this says exactly that f is fully faithful, i.e., that (2) holds. □

0500 **Lemma 100.8.5.** *A monomorphism of algebraic stacks induces an injective map of sets of points.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Suppose that $x_i : \mathrm{Spec}(K_i) \to \mathcal{X}$ be morphisms such that $f \circ x_1$ and $f \circ x_2$ define the same element of $|\mathcal{Y}|$. Applying the definition we find a common extension Ω with corresponding morphisms $c_i : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K_i)$ and a 2-isomorphism $\beta : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. As f is fully faithful, see Lemma 100.8.4, we can lift β to an isomorphism $\alpha : x_1 \circ c_1 \to x_1 \circ c_2$. Hence x_1 and x_2 define the same point of $|\mathcal{X}|$ as desired. □

0CBB **Lemma 100.8.6.** *Let $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{X}'$ is a monomorphism then the canonical diagram*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X}' \times_{\mathcal{Y}} \mathcal{X}' \end{array}$$

is a fibre product square.

Proof. We have $\mathcal{X} = \mathcal{X} \times_{\mathcal{X}'} \mathcal{X}$ by Lemma 100.8.4. Thus the result by applying Categories, Lemma 4.31.13. □

100.9. Immersions of algebraic stacks

04YK Immersions of algebraic stacks are defined as follows.

04YL **Definition 100.9.1.** Immersions.

(1) A morphism of algebraic stacks is called an *open immersion* if it is representable, and an open immersion in the sense of Section 100.3.
(2) A morphism of algebraic stacks is called a *closed immersion* if it is representable, and a closed immersion in the sense of Section 100.3.
(3) A morphism of algebraic stacks is called an *immersion* if it is representable, and an immersion in the sense of Section 100.3.

This is not the most convenient way to think about immersions for us. For us it is a little bit more convenient to think of an immersion as a morphism of algebraic stacks which is representable by algebraic spaces and is an immersion in the sense of Section 100.3. Similarly for closed and open immersions. Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0501 **Lemma 100.9.2.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a (closed, resp. open) immersion. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a (closed, resp. open) immersion.*

Proof. This follows from the general discussion in Section 100.3. □

0502 **Lemma 100.9.3.** *Compositions of immersions of algebraic stacks are immersions. Similarly for closed immersions and open immersions.*

Proof. This follows from the general discussion in Section 100.3 and Spaces, Lemma 65.12.2. □

0503 **Lemma 100.9.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be a surjective, flat morphism which is locally of finite presentation. The following are equivalent:*

(1) f *is an (open, resp. closed) immersion, and*
(2) $V = W \times_{\mathcal{Y}} \mathcal{X}$ *is an algebraic space, and* $V \to W$ *is an (open, resp. closed) immersion.*

Proof. This follows from the general discussion in Section 100.3 and in particular Lemmas 100.3.1 and 100.3.3. □

0504 **Lemma 100.9.5.** *An immersion is a monomorphism.*

Proof. See Morphisms of Spaces, Lemma 67.10.7. □

0H20 **Lemma 100.9.6.** *If $f : \mathcal{X} \to \mathcal{Y}$ is an immersion, then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism onto a locally closed subset. If f is a closed, resp. open immersion, then $|f|$ is closed, resp. open.*

Proof. Omitted. □

The following two lemmas explain how to think about immersions in terms of presentations.

0505 **Lemma 100.9.7.** *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $i : \mathcal{Z} \to [U/R]$ be an immersion. Then there exists an R-invariant locally closed subspace $Z \subset U$ and a presentation $[Z/R_Z] \to \mathcal{Z}$ where R_Z is the restriction of R to Z such that*

$$\begin{array}{ccc} [Z/R_Z] & \longrightarrow & \mathcal{Z} \\ & \searrow \quad \swarrow_{i} & \\ & [U/R] & \end{array}$$

is 2-commutative. If i is a closed (resp. open) immersion then Z is a closed (resp. open) subspace of U.

Proof. By Lemma 100.3.6 we get a commutative diagram

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & \mathcal{Z} \\ & \searrow \quad \swarrow & \\ & [U/R] & \end{array}$$

where $U' = \mathcal{Z} \times_{[U/R]} U$ and $R' = \mathcal{Z} \times_{[U/R]} R$. Since $\mathcal{Z} \to [U/R]$ is an immersion we see that $U' \to U$ is an immersion of algebraic spaces. Let $Z \subset U$ be the locally closed subspace such that $U' \to U$ factors through Z and induces an isomorphism $U' \to Z$. It is clear from the construction of R' that $R' = U' \times_{U,t} R = R \times_{s,U} U'$. This implies that $Z \cong U'$ is R-invariant and that the image of $R' \to R$ identifies R' with the restriction $R_Z = s^{-1}(Z) = t^{-1}(Z)$ of R to Z. Hence the lemma holds. □

04YN **Lemma 100.9.8.** *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $\mathcal{X} = [U/R]$ be the associated algebraic stack, see Algebraic Stacks, Theorem 94.17.3. Let $Z \subset U$ be an R-invariant locally closed subspace. Then*

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks, where R_Z is the restriction of R to Z. If $Z \subset U$ is open (resp. closed) then the morphism is an open (resp. closed) immersion of algebraic stacks.

Proof. Recall that by Groupoids in Spaces, Definition 78.18.1 (see also discussion following the definition) we have $R_Z = s^{-1}(Z) = t^{-1}(Z)$ as locally closed subspaces of R. Hence the two morphisms $R_Z \to Z$ are smooth as base changes of s and t. Hence $(Z, R_Z, s|_{R_Z}, t|_{R_Z}, c|_{R_Z \times_{s,Z,t} R_Z})$ is a smooth groupoid in algebraic spaces, and we see that $[Z/R_Z]$ is an algebraic stack, see Algebraic Stacks, Theorem 94.17.3. The assumptions of Groupoids in Spaces, Lemma 78.25.3 are all satisfied and it follows that we have a 2-fibre square

$$\begin{array}{ccc} Z & \longrightarrow & [Z/R_Z] \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array}$$

It follows from this and Lemma 100.3.1 that $[Z/R_Z] \to [U/R]$ is representable by algebraic spaces, whereupon it follows from Lemma 100.3.3 that the right vertical arrow is an immersion (resp. closed immersion, resp. open immersion) if and only if the left vertical arrow is. □

We can define open, closed, and locally closed substacks as follows.

04YM **Definition 100.9.9.** Let $\mathcal{X}$ be an algebraic stack.

(1) An *open substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an open immersion.
(2) A *closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is a closed immersion.
(3) A *locally closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an immersion.

This definition should be used with caution. Namely, if $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}' \subset \mathcal{X}$ is an open substack, then it is not necessarily the case that the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a *strictly* full subcategory; but this is also the only problem. Here is a formal statement.

0506 **Lemma 100.9.10.** *For any immersion $i : \mathcal{Z} \to \mathcal{X}$ there exists a unique locally closed substack $\mathcal{X}' \subset \mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \to \mathcal{X}'$ followed by the inclusion morphism $\mathcal{X}' \to \mathcal{X}$. If i is a closed (resp. open) immersion, then $\mathcal{X}'$ is a closed (resp. open) substack of $\mathcal{X}$.*

Proof. Omitted. □

0507 **Lemma 100.9.11.** *Let $[U/R] \to \mathcal{X}$ be a presentation of an algebraic stack. There is a canonical bijection*

locally closed substacks $\mathcal{Z}$ of $\mathcal{X}$ $\longrightarrow$ R-invariant locally closed subspaces Z of U

which sends $\mathcal{Z}$ to $U \times_{\mathcal{X}} \mathcal{Z}$. Moreover, a morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Similarly for closed substacks and open substacks.

Proof. By Lemmas 100.9.7 and 100.9.8 we find that the map is a bijection. If $\mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ then of course the base change $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Converse, suppose that $\mathcal{Y} \to \mathcal{X}$ is a morphism such that $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. We will show that for every scheme T and morphism $T \to \mathcal{Y}$,

given by an object y of the fibre category of $\mathcal{Y}$ over T, the object y is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the fibre product $T \times_{\mathcal{X}} U$ is an algebraic space and $T \times_{\mathcal{X}} U \to T$ is a surjective smooth morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} U \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{Y} \times_{\mathcal{X}} U$ and hence through $Z \subset U$. Thus $y|_{T_i}$ is an object of $\mathcal{Z}$ (as Z is the fibre product of U with $\mathcal{Z}$ over $\mathcal{X}$). Since $\mathcal{Z}$ is a strictly full substack, we conclude that y is an object of $\mathcal{Z}$ as desired. □

06FJ **Lemma 100.9.12.** *Let $\mathcal{X}$ be an algebraic stack. The rule $\mathcal{U} \mapsto |\mathcal{U}|$ defines an inclusion preserving bijection between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.2. By Lemma 100.9.11 we see that open substacks correspond to R-invariant open subschemes of U. On the other hand Lemmas 100.4.5 and 100.4.7 guarantee these correspond bijectively to open subsets of $|\mathcal{X}|$. □

05UP **Lemma 100.9.13.** *Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and $U \to \mathcal{X}$ a surjective smooth morphism. For an open immersion $V \hookrightarrow U$, there exists an algebraic stack $\mathcal{Y}$, an open immersion $\mathcal{Y} \to \mathcal{X}$, and a surjective smooth morphism $V \to \mathcal{Y}$.*

Proof. We define a category fibred in groupoids $\mathcal{Y}$ by letting the fiber category $\mathcal{Y}_T$ over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $y \in \mathrm{Ob}(\mathcal{X}_T)$ such that the projection morphism $V \times_{\mathcal{X},y} T \to T$ surjective. Now for any morphism $x : T \to \mathcal{X}$, the 2-fibred product $T \times_{x,\mathcal{X}} \mathcal{Y}$ has fiber category over T' consisting of triples $(f : T' \to T, y \in \mathcal{X}_{T'}, f^*x \simeq y)$ such that $V \times_{\mathcal{X},y} T' \to T'$ is surjective. Note that $T \times_{x,\mathcal{X}} \mathcal{Y}$ is fibered in setoids since $\mathcal{Y} \to \mathcal{X}$ is faithful (see Stacks, Lemma 8.6.7). Now the isomorphism $f^*x \simeq y$ gives the diagram

$$\begin{array}{ccccc} V \times_{\mathcal{X},y} T' & \longrightarrow & V \times_{\mathcal{X},x} T & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ T' & \xrightarrow{f} & T & \xrightarrow{x} & \mathcal{X} \end{array}$$

where both squares are cartesian. The morphism $V \times_{\mathcal{X},x} T \to T$ is smooth by base change, and hence open. Let $T_0 \subset T$ be its image. From the cartesian squares we deduce that $V \times_{\mathcal{X},y} T' \to T'$ is surjective if and only if f lands in T_0. Therefore $T \times_{x,\mathcal{X}} \mathcal{Y}$ is representable by T_0, so the inclusion $\mathcal{Y} \to \mathcal{X}$ is an open immersion. By Algebraic Stacks, Lemma 94.15.5 we conclude that $\mathcal{Y}$ is an algebraic stack. Lastly if we denote the morphism $V \to \mathcal{X}$ by g, we have $V \times_{\mathcal{X}} V \to V$ is surjective (the diagonal gives a section). Hence g is in the image of $\mathcal{Y}_V \to \mathcal{X}_V$, i.e., we obtain a morphism $g' : V \to \mathcal{Y}$ fitting into the commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow{\scriptstyle g'} & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

Since $V \times_{g,\mathcal{X}} \mathcal{Y} \to V$ is a monomorphism, it is in fact an isomorphism since $(1, g')$ defines a section. Therefore $g' : V \to \mathcal{Y}$ is a smooth morphism, as it is the base change of the smooth morphism $g : V \to \mathcal{X}$. It is surjective by our construction of $\mathcal{Y}$ which finishes the proof of the lemma. □

05UQ **Lemma 100.9.14.** *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}_i \subset \mathcal{X}$ a collection of open substacks indexed by $i \in I$. Then there exists an open substack, which we denote $\bigcup_{i\in I} \mathcal{X}_i \subset \mathcal{X}$, such that the $\mathcal{X}_i$ are open substacks covering it.*

Proof. We define a fibred subcategory $\mathcal{X}' = \bigcup_{i\in I} \mathcal{X}_i$ by letting the fiber category over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $x \in \mathrm{Ob}(\mathcal{X}_T)$ such that the morphism $\coprod_{i\in I}(\mathcal{X}_i \times_{\mathcal{X}} T) \to T$ is surjective. Let $x_i \in \mathrm{Ob}((\mathcal{X}_i)_T)$. Then $(x_i, 1)$ gives a section of $\mathcal{X}_i \times_{\mathcal{X}} T \to T$, so we have an isomorphism. Thus $\mathcal{X}_i \subset \mathcal{X}'$ is a full subcategory. Now let $x \in \mathrm{Ob}(\mathcal{X}_T)$. Then $\mathcal{X}_i \times_{\mathcal{X}} T$ is representable by an open subscheme $T_i \subset T$. The 2-fibred product $\mathcal{X}' \times_{\mathcal{X}} T$ has fiber over T' consisting of $(y \in \mathcal{X}_{T'}, f : T' \to T, f^*x \simeq y)$ such that $\coprod(\mathcal{X}_i \times_{\mathcal{X},y} T') \to T'$ is surjective. The isomorphism $f^*x \simeq y$ induces an isomorphism $\mathcal{X}_i \times_{\mathcal{X},y} T' \simeq T_i \times_T T'$. Then the $T_i \times_T T'$ cover T' if and only if f lands in $\bigcup T_i$. Therefore we have a diagram

$$\begin{array}{ccccc} T_i & \longrightarrow & \bigcup T_i & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_i & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

with both squares cartesian. By Algebraic Stacks, Lemma 94.15.5 we conclude that $\mathcal{X}' \subset \mathcal{X}$ is algebraic and an open substack. It is also clear from the cartesian squares above that the morphism $\coprod_{i\in I} \mathcal{X}_i \to \mathcal{X}'$ which finishes the proof of the lemma. □

05UR **Lemma 100.9.15.** *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}' \subset \mathcal{X}$ a quasi-compact open substack. Suppose that we have a collection of open substacks $\mathcal{X}_i \subset \mathcal{X}$ indexed by $i \in I$ such that $\mathcal{X}' \subset \bigcup_{i\in I} \mathcal{X}_i$, where we define the union as in Lemma 100.9.14. Then there exists a finite subset $I' \subset I$ such that $\mathcal{X}' \subset \bigcup_{i\in I'} \mathcal{X}_i$.*

Proof. Since $\mathcal{X}$ is algebraic, there exists a scheme U with a surjective smooth morphism $U \to \mathcal{X}$. Let $U_i \subset U$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} U$ and $U' \subset U$ the open subscheme representing $\mathcal{X}' \times_{\mathcal{X}} U$. By hypothesis, $U' \subset \bigcup_{i\in I} U_i$. From the proof of Lemma 100.6.2, there is a quasi-compact open $V \subset U'$ such that $V \to \mathcal{X}'$ is a surjective smooth morphism. Therefore there exists a finite subset $I' \subset I$ such that $V \subset \bigcup_{i\in I'} U_i$. We claim that $\mathcal{X}' \subset \bigcup_{i\in I'} \mathcal{X}_i$. Take $x \in \mathrm{Ob}(\mathcal{X}'_T)$ for $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Since $\mathcal{X}' \to \mathcal{X}$ is a monomorphism, we have cartesian squares

$$\begin{array}{ccccc} V \times_{\mathcal{X}} T & \longrightarrow & T & = & T \\ \downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ V & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

By base change, $V \times_{\mathcal{X}} T \to T$ is surjective. Therefore $\bigcup_{i\in I'} U_i \times_{\mathcal{X}} T \to T$ is also surjective. Let $T_i \subset T$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} T$. By a formal argument, we have a Cartesian square

$$\begin{array}{ccc} U_i \times_{\mathcal{X}_i} T_i & \longrightarrow & U \times_{\mathcal{X}} T \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T \end{array}$$

where the vertical arrows are surjective by base change. Since $U_i \times_{\mathcal{X}_i} T_i \simeq U_i \times_{\mathcal{X}} T$, we find that $\bigcup_{i \in I'} T_i = T$. Hence x is an object of $(\bigcup_{i \in I'} \mathcal{X}_i)_T$ by definition of the union. Observe that the inclusion $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$ is automatically an open substack. □

05WE **Lemma 100.9.16.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is an algebraic space.*

Then $\mathcal{X}$ is an algebraic space.

Proof. Apply Stacks, Lemma 8.6.10 to the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ and the morphism $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ to see that $\mathcal{X}$ is a stack in setoids. Hence $\mathcal{X}$ is an algebraic space, see Algebraic Stacks, Proposition 94.13.3. □

05WF **Lemma 100.9.17.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is a scheme*

Then $\mathcal{X}$ is a scheme.

Proof. By Lemma 100.9.16 we see that $\mathcal{X}$ is an algebraic space. Since any algebraic space has a largest open subspace which is a scheme, see Properties of Spaces, Lemma 66.13.1 we see that $\mathcal{X}$ is a scheme. □

The following lemma is the analogue of More on Groupoids, Lemma 40.6.1.

06M3 **Lemma 100.9.18.** *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of algebraic spaces. Assume*

(1) *$\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are fppf local on the target and stable under arbitrary base change,*
(2) *smooth $\Rightarrow \mathcal{R}$,*
(3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open subspace $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Assume f has $\mathcal{Q}$. Then

(A) *there exists a largest open substack $\mathcal{X}' \subset \mathcal{X}$ such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$, and*
(B) *if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}'$ is the largest open substack of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ over which the base change $\mathrm{id}_{\mathcal{Z}} \times f$ has property $\mathcal{P}$.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ and let $f' : U \to V$ be the base change of f. The morphism of algebraic spaces $f' : U \to V$ has property $\mathcal{Q}$. Thus we obtain the open $W(\mathcal{P}, f') \subset U$ by assumption (3). Note that $U \times_{\mathcal{X}} U = (V \times_{\mathcal{Y}} V) \times_{\mathcal{Y}} \mathcal{X}$ hence the morphism $f'' : U \times_{\mathcal{X}} U \to V \times_{\mathcal{Y}} V$ is the base change of f via either projection $V \times_{\mathcal{Y}} V \to V$. By our choice of V these projections are smooth, hence have property $\mathcal{R}$ by (2). Thus by (4) we see that the inverse images of $W(\mathcal{P}, f')$ under the two projections $\mathrm{pr}_i : U \times_{\mathcal{X}} U \to U$ agree. In other words, $W(\mathcal{P}, f')$ is an R-invariant subspace of U

(where $R = U \times_{\mathcal{X}} U$). Let $\mathcal{X}'$ be the open substack of $\mathcal{X}$ corresponding to $W(\mathcal{P}, f)$ via Lemma 100.9.7. By construction $W(\mathcal{P}, f') = \mathcal{X}' \times_{\mathcal{Y}} V$ hence $f|_{\mathcal{X}'}$ has property $\mathcal{P}$ by Lemma 100.3.3. Also, $\mathcal{X}'$ is the largest open substack such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$ as the same maximality holds for $W(\mathcal{P}, f)$. This proves (A).

Finally, if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then we set $T = V \times_{\mathcal{Y}} \mathcal{Z}$ and we see that $T \to V$ is a morphism of algebraic spaces having property $\mathcal{R}$. Set $f'_T : T \times_V U \to T$ the base change of f'. By (4) again we see that $W(\mathcal{P}, f'_T)$ is the inverse image of $W(\mathcal{P}, f)$ in $T \times_V U$. This implies (B); some details omitted. □

06M4 **Remark 100.9.19.** Warning: Lemma 100.9.18 should be used with care. For example, it applies to $\mathcal{P}$ ="flat", $\mathcal{Q}$ ="empty", and $\mathcal{R}$ ="flat and locally of finite presentation". But given a morphism of algebraic spaces $f : X \to Y$ the largest open subspace $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

06M5 **Remark 100.9.20.** Notwithstanding the warning in Remark 100.9.19 there are some cases where Lemma 100.9.18 can be used without causing ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

06M6 (1) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="relative dimension $\leq d$". See Morphisms of Spaces, Definition 67.33.2 and Morphisms of Spaces, Lemmas 67.34.4 and 67.34.3.

06M7 (2) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms of Spaces, Lemma 67.34.6. On the other hand, properties (3) and (4) are spelled out in Morphisms of Spaces, Lemma 67.34.7.

06M8 (3) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="unramified". This is Morphisms of Spaces, Lemma 67.38.10.

06M9 (4) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="flat". See More on Morphisms of Spaces, Theorem 76.22.1 and Lemma 76.22.2. Note that here $W(\mathcal{P}, f)$ is always exactly the set of points where the morphism f is flat because we only consider this open when f has $\mathcal{Q}$ (see loc.cit.).

06MA (5) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="étale". This follows on combining (3) and (4) because an unramified morphism which is flat and locally of finite presentation is étale, see Morphisms of Spaces, Lemma 67.39.12.

(6) Add more here as needed (compare with the longer list at More on Groupoids, Remark 40.6.3).

100.10. Reduced algebraic stacks

0508 We have already defined reduced algebraic stacks in Section 100.7.

0509 **Lemma 100.10.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a closed subset. There exists a unique closed substack $\mathcal{Z} \subset \mathcal{X}$ with the following properties: (a) we have $|\mathcal{Z}| = T$, and (b) $\mathcal{Z}$ is reduced.*

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism, where U is an algebraic space. Set $R = U \times_{\mathcal{X}} U$, so that there is a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.2. As usual we denote $s, t : R \to U$ the two smooth

projection morphisms. By Lemma 100.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $|s|^{-1}(T') = |t|^{-1}(T')$. Let $Z \subset U$ be the reduced induced algebraic space structure on T', see Properties of Spaces, Definition 66.12.5. The fibre products $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are closed subspaces of R (Spaces, Lemma 65.12.3). The projections $Z \times_{U,t} R \to Z$ and $R \times_{s,U} Z \to Z$ are smooth by Morphisms of Spaces, Lemma 67.37.3. Thus as Z is reduced, it follows that $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are reduced, see Remark 100.7.3. Since

$$|Z \times_{U,t} R| = |t|^{-1}(T') = |s|^{-1}(T') = R \times_{s,U} Z$$

we conclude from the uniqueness in Properties of Spaces, Lemma 66.12.3 that $Z \times_{U,t} R = R \times_{s,U} Z$. Hence Z is an R-invariant closed subspace of U. By the correspondence of Lemma 100.9.11 we obtain a closed substack $\mathcal{Z} \subset \mathcal{X}$ with $Z = \mathcal{Z} \times_{\mathcal{X}} U$. Then $[Z/R_Z] \to \mathcal{Z}$ is a presentation (Lemma 100.9.7). Then $|\mathcal{Z}| = |Z|/|R_Z| = |T'|/\sim$ is the given closed subset T. We omit the proof of unicity. □

050A **Lemma 100.10.2.** *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}' \subset \mathcal{X}$ is a closed substack, $\mathcal{X}$ is reduced and $|\mathcal{X}'| = |\mathcal{X}|$, then $\mathcal{X}' = \mathcal{X}$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$ with U a scheme. As $\mathcal{X}$ is reduced, we see that U is reduced (by definition of reduced algebraic stacks). By Lemma 100.9.11 $\mathcal{X}'$ corresponds to an R-invariant closed subscheme $Z \subset U$. But now $|Z| \subset |U|$ is the inverse image of $|\mathcal{X}'|$, and hence $|Z| = |U|$. Hence Z is a closed subscheme of U whose underlying sets of points agree. By Schemes, Lemma 26.12.7 the map $\mathrm{id}_U : U \to U$ factors through $Z \to U$, and hence $Z = U$, i.e., $\mathcal{X}' = \mathcal{X}$. □

050B **Lemma 100.10.3.** *Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack Assume $\mathcal{Y}$ is reduced. A morphism $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$.*

Proof. Assume $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$. Consider $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}$. There is an equivalence $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is a closed substack of $\mathcal{Y}$, see Lemmas 100.9.2 and 100.9.10. Using Lemmas 100.4.3, 100.8.5, and 100.9.5 we see that $|\mathcal{Y}'| = |\mathcal{Y}|$. Hence we have reduced the lemma to Lemma 100.10.2. □

050C **Definition 100.10.4.** Let $\mathcal{X}$ be an algebraic stack. Let $Z \subset |\mathcal{X}|$ be a closed subset. An *algebraic stack structure on Z* is given by a closed substack $\mathcal{Z}$ of $\mathcal{X}$ with $|\mathcal{Z}|$ equal to Z. The *reduced induced algebraic stack structure* on Z is the one constructed in Lemma 100.10.1. The *reduction $\mathcal{X}_{red}$ of $\mathcal{X}$* is the reduced induced algebraic stack structure on $|\mathcal{X}|$.

In fact we can use this to define the reduced induced algebraic stack structure on a locally closed subset.

06FK **Remark 100.10.5.** Let X be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|\mathcal{X}|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $\mathcal{U} \subset \mathcal{X}$ be the open substack of X with $|\mathcal{U}| = |\mathcal{X}| \setminus \partial T$, see Lemma 100.9.12. Let $\mathcal{Z}$ be the reduced closed substack of $\mathcal{U}$ with $|\mathcal{Z}| = T$ obtained by taking the reduced induced closed subspace structure, see Definition 100.10.4. By construction $\mathcal{Z} \to \mathcal{U}$ is a closed immersion of algebraic stacks and $\mathcal{U} \to \mathcal{X}$ is an open immersion, hence $\mathcal{Z} \to \mathcal{X}$ is an immersion of algebraic stacks by Lemma 100.9.3. Note that $\mathcal{Z}$ is a

reduced algebraic stack and that $|\mathcal{Z}| = T$ as subsets of $|X|$. We sometimes say $\mathcal{Z}$ is the *reduced induced substack structure* on T.

100.11. Residual gerbes

06ML In the Stacks project we would like to define the *residual gerbe* of an algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ to be a monomorphism of algebraic stacks $m_x : \mathcal{Z}_x \to \mathcal{X}$ where $\mathcal{Z}_x$ is a reduced algebraic stack having a unique point which is mapped by m_x to x. It turns out that there are many issues with this notion; existence is not clear in general and neither is uniqueness. We resolve the uniqueness issue by imposing a slightly stronger condition on the algebraic stacks $\mathcal{Z}_x$. We discuss this in more detail by working through a few simple lemmas regarding reduced algebraic stacks having a unique point.

06MM **Lemma 100.11.1.** *Let $\mathcal{Z}$ be an algebraic stack. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to \mathcal{Z}$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & \mathcal{Z} \end{array}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to \mathcal{Z}$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 67.31.5 (via the discussion in Section 100.3) that $\mathrm{Spec}(k') \to \mathcal{Z}$ is flat. It is clear that it is surjective as by assumption $|\mathcal{Z}|$ is a singleton. □

06MN **Lemma 100.11.2.** *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

1. *$\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton,*
2. *there exists a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field, and*
3. *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to \mathcal{Z}$ is flat (see Morphisms of Spaces, Lemma 67.30.3). It is also surjective as $|\mathcal{Z}|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence regular (see Spaces over Fields, Lemma 72.16.1) and in particular reduced. Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent on Spaces, Lemma 74.9.2). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite type by Morphisms of Spaces, Lemmas 67.23.2 and 67.37.6. It is also flat and surjective by Lemma 100.11.1. Hence (3) holds. □

The following lemma singles out a slightly better class of singleton algebraic stacks than the preceding lemma.

06MP **Lemma 100.11.3.** *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{Z}$ is reduced, locally Noetherian, and $|\mathcal{Z}|$ is a singleton, and*
(2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (2) holds. By Lemma 100.11.2 we see that $\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton. Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence locally Noetherian (see Morphisms of Spaces, Lemma 67.23.5). Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent on Spaces, Lemma 74.9.3). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 29.21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 67.28.2 and 67.37.5. It is also flat and surjective by Lemma 100.11.1. Hence (2) holds. □

06MQ **Lemma 100.11.4.** *Let $\mathcal{Z}' \to \mathcal{Z}$ be a monomorphism of algebraic stacks. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then either $\mathcal{Z}'$ is empty or $\mathcal{Z}' \to \mathcal{Z}$ is an equivalence.*

Proof. We may assume that $\mathcal{Z}'$ is nonempty. In this case the fibre product $T = \mathcal{Z}' \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is nonempty, see Lemma 100.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 67.10.8. We conclude that $\mathrm{Spec}(k) \to \mathcal{Z}$ factors through $\mathcal{Z}'$. Suppose the morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ is given by the object ξ over $\mathrm{Spec}(k)$. We have just seen that ξ is isomorphic to an object ξ' of $\mathcal{Z}'$ over $\mathrm{Spec}(k)$. Since z is surjective, flat, and locally of finite presentation we see that every object of $\mathcal{Z}$ over any scheme is fppf locally isomorphic to a pullback of ξ, hence also to a pullback of ξ'. By descent of objects for stacks in groupoids this implies that $\mathcal{Z}' \to \mathcal{Z}$ is essentially surjective (as well as fully faithful, see Lemma 100.8.4). Hence we win. □

06MR **Lemma 100.11.5.** *Let $\mathcal{Z}$ be an algebraic stack. Assume $\mathcal{Z}$ satisfies the equivalent conditions of Lemma 100.11.2. Then there exists a unique strictly full subcategory $\mathcal{Z}' \subset \mathcal{Z}$ such that $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of*

Lemma 100.11.3. The inclusion morphism $\mathcal{Z}' \to \mathcal{Z}$ is a monomorphism of algebraic stacks.

Proof. The last part is immediate from the first part and Lemma 100.8.4. Pick a field k and a morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ which is surjective, flat, and locally of finite type. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$. The projections $s, t : R \to U$ are locally of finite type. Since U is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 67.28.7). We see that $\mathcal{Z}' = [U/R]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. By Algebraic Stacks, Lemma 94.16.1 we obtain a canonical morphism

$$f : \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.15.2 and a monomorphism, see Lemma 100.8.4. By Criteria for Representability, Lemma 97.17.1 the morphism $U \to \mathcal{Z}'$ is surjective, flat, and locally of finite presentation. Hence $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 100.11.3. By Algebraic Stacks, Lemma 94.12.4 we may replace $\mathcal{Z}'$ by its essential image in $\mathcal{Z}$. Hence we have proved all the assertions of the lemma except for the uniqueness of $\mathcal{Z}' \subset \mathcal{Z}$. Suppose that $\mathcal{Z}'' \subset \mathcal{Z}$ is a second such algebraic stack. Then the projections

$$\mathcal{Z}' \longleftarrow \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' \longrightarrow \mathcal{Z}''$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 100.4.3. Hence the two projections are isomorphisms by Lemma 100.11.4 and we win. □

06MS **Example 100.11.6.** Here is an example where the morphism constructed in Lemma 100.11.5 isn't an isomorphism. This example shows that imposing that residual gerbes are locally Noetherian is necessary in Definition 100.11.8. In fact, the example is even an algebraic space! Let $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group of $\mathbf{Q}$ with the pro-finite topology. Let

$$U = \mathrm{Spec}(\overline{\mathbf{Q}}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\overline{\mathbf{Q}}) = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{Spec}(\overline{\mathbf{Q}})$$

(we omit a precise explanation of the meaning of the last equal sign). Let G denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the discrete topology viewed as a constant group scheme over $\mathrm{Spec}(\overline{\mathbf{Q}})$, see Groupoids, Example 39.5.6. Then G acts freely and transitively on U. Let $X = U/G$, see Spaces, Definition 65.14.4. Then X is a non-noetherian reduced algebraic space with exactly one point. Furthermore, X has a (locally) finite type point:

$$x : \mathrm{Spec}(\overline{\mathbf{Q}}) \longrightarrow U \longrightarrow X$$

Indeed, every point of U is actually closed! As X is an algebraic space over $\overline{\mathbf{Q}}$ it follows that x is a monomorphism. So x is the morphism constructed in Lemma 100.11.5 but x is not an isomorphism. In fact $\mathrm{Spec}(\overline{\mathbf{Q}}) \to X$ is the residual gerbe of X at x.

It will turn out later that under mild assumptions on the algebraic stack $\mathcal{X}$ the equivalent conditions of the following lemma are satisfied for every point $x \in |\mathcal{X}|$ (see Morphisms of Stacks, Section 101.31).

06MT **Lemma 100.11.7.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

(1) *there exists an algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
(2) *there exists a reduced algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
(3) *there exists an algebraic stack $\mathcal{Z}$, a monomorphism $f : \mathcal{Z} \to \mathcal{X}$, and a surjective flat morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field such that $x = f(z)$.*

Moreover, if these conditions hold, then there exists a unique strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ such that $\mathcal{Z}_x$ is a reduced, locally Noetherian algebraic stack and $|\mathcal{Z}_x|$ is a singleton which maps to x via the map $|\mathcal{Z}_x| \to |\mathcal{X}|$.

Proof. If $\mathcal{Z} \to \mathcal{X}$ is as in (1), then $\mathcal{Z}_{red} \to \mathcal{X}$ is as in (2). (See Section 100.10 for the notion of the reduction of an algebraic stack.) Hence (1) implies (2). It is immediate that (2) implies (1). The equivalence of (2) and (3) is immediate from Lemma 100.11.2.

At this point we've seen the equivalence of (1) – (3). Pick a monomorphism $f : \mathcal{Z} \to \mathcal{X}$ as in (2). Note that this implies that f is fully faithful, see Lemma 100.8.4. Denote $\mathcal{Z}' \subset \mathcal{X}$ the essential image of the functor f. Then $f : \mathcal{Z} \to \mathcal{Z}'$ is an equivalence and hence $\mathcal{Z}'$ is an algebraic stack, see Algebraic Stacks, Lemma 94.12.4. Apply Lemma 100.11.5 to get a strictly full subcategory $\mathcal{Z}_x \subset \mathcal{Z}'$ as in the statement of the lemma. This proves all the statements of the lemma except for uniqueness.

In order to prove the uniqueness suppose that $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}'_x \subset \mathcal{X}$ are two strictly full subcategories as in the statement of the lemma. Then the projections

$$\mathcal{Z}'_x \longleftarrow \mathcal{Z}'_x \times_{\mathcal{X}} \mathcal{Z}_x \longrightarrow \mathcal{Z}_x$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 100.4.3. Hence the two projections are isomorphisms by Lemma 100.11.4 and we win. □

Having explained the above we can now make the following definition.

06MU **Definition 100.11.8.** Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

(1) We say the *residual gerbe of $\mathcal{X}$ at x exists* if the equivalent conditions (1), (2), and (3) of Lemma 100.11.7 hold.
(2) If the residual gerbe of $\mathcal{X}$ at x exists, then the *residual gerbe of $\mathcal{X}$ at x*[1] is the strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ constructed in Lemma 100.11.7.

In particular we know that $\mathcal{Z}_x$ (if it exists) is a locally Noetherian, reduced algebraic stack and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow \mathcal{Z}_x.$$

We will see in Morphisms of Stacks, Lemma 101.28.12 that $\mathcal{Z}_x$ is a gerbe. Existence of residual gerbes is discussed in Morphisms of Stacks, Section 101.31.

[1] This clashes with [**LMB00**] in spirit, but not in fact. Namely, in Chapter 11 they associate to any point on any quasi-separated algebraic stack a gerbe (not necessarily algebraic) which they call the residual gerbe. We will see in Morphisms of Stacks, Lemma 101.31.1 that on a quasi-separated algebraic stack every point has a residual gerbe in our sense which is then equivalent to theirs. For more information on this topic see [**Ryd10**, Appendix B].

0H21 **Example 100.11.9.** Let X be a scheme and let $x \in X$ be a point. Then the monomorphism $x \to X$ is the residual gerbe of X at x where we, as usual, identify x with the scheme $x = \mathrm{Spec}(\kappa(x))$. If X is an algebraic space and $x \in |X|$, then the residual gerbe at x (which is called the residual space) always exists, see Decent Spaces, Section 68.13.

The residual gerbe, if it exists, is a regular algebraic stack by the following lemma.

06MV **Lemma 100.11.10.** *A reduced, locally Noetherian algebraic stack $\mathcal{Z}$ such that $|\mathcal{Z}|$ is a singleton is regular.*

Proof. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective, flat, and locally of finite presentation (see Lemma 100.11.3). The algebraic space $T = W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is smooth over k in particular regular, see Spaces over Fields, Lemma 72.16.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent on Spaces, Lemma 74.9.4. By definition this means that $\mathcal{Z}$ is regular. □

06MW **Lemma 100.11.11.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume that the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ exists. Let $f : \mathrm{Spec}(K) \to \mathcal{X}$ be a morphism where K is a field in the equivalence class of x. Then f factors through the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$.*

Proof. Choose a field k and a surjective flat locally finite presentation morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. Set $T = \mathrm{Spec}(K) \times_{\mathcal{X}} \mathcal{Z}_x$. By Lemma 100.4.3 we see that T is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we see that $T \to \mathrm{Spec}(K)$ is a monomorphism. Hence by Morphisms of Spaces, Lemma 67.10.8 we see that $T = \mathrm{Spec}(K)$ which proves the lemma. □

06MX **Lemma 100.11.12.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Let $\mathcal{Z}$ be an algebraic stack satisfying the equivalent conditions of Lemma 100.11.3 and let $\mathcal{Z} \to \mathcal{X}$ be a monomorphism such that the image of $|\mathcal{Z}| \to |\mathcal{X}|$ is x. Then the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z} \to \mathcal{X}$ factors as $\mathcal{Z} \to \mathcal{Z}_x \to \mathcal{X}$ where the first arrow is an equivalence.*

Proof. Let $\mathcal{Z}_x \subset \mathcal{X}$ be the full subcategory corresponding to the essential image of the functor $\mathcal{Z} \to \mathcal{X}$. Then $\mathcal{Z} \to \mathcal{Z}_x$ is an equivalence, hence $\mathcal{Z}_x$ is an algebraic stack, see Algebraic Stacks, Lemma 94.12.4. Since $\mathcal{Z}_x$ inherits all the properties of $\mathcal{Z}$ from this equivalence it is clear from the uniqueness in Lemma 100.11.7 that $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x. □

0DTH **Lemma 100.11.13.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. If the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist, then f induces a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & \mathcal{Z}_x \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{Y} & \longleftarrow & \mathcal{Z}_y \end{array}$$

Proof. Choose a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. The morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ factors through $\mathcal{Z}_y$ by Lemma 100.11.11. Thus $\mathcal{Z}_x \times_{\mathcal{Y}} \mathcal{Z}_y$ is a nonempty substack of $\mathcal{Z}_x$ hence equal to $\mathcal{Z}_x$ by Lemma 100.11.4. □

0DTI **Lemma 100.11.14.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist and that there exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x such that*

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$$

is an isomorphism. Then $\mathcal{Z}_x \to \mathcal{Z}_y$ is an isomorphism.

Proof. Let k'/k be an extension of fields. Then

$$\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k') \times_{\mathcal{Y}} \mathrm{Spec}(k')$$

is the base change of the morphism in the lemma by the faithfully flat morphism $\mathrm{Spec}(k' \otimes k') \to \mathrm{Spec}(k \otimes k)$. Thus the property described in the lemma is independent of the choice of the morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x. Thus we may assume that $\mathrm{Spec}(k) \to \mathcal{Z}_x$ is surjective, flat, and locally of finite presentation. In this situation we have

$$\mathcal{Z}_x = [\mathrm{Spec}(k)/R]$$

with $R = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)$. See proof of Lemma 100.11.5. Since also $R = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$ we conclude that the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ of Lemma 100.11.13 is fully faithful by Algebraic Stacks, Lemma 94.16.1. We conclude for example by Lemma 100.11.12. □

100.12. Dimension of a stack

0AFL We can define the dimension of an algebraic stack $\mathcal{X}$ at a point x, using the notion of dimension of an algebraic space at a point (Properties of Spaces, Definition 66.9.1). In the following lemma the output may be ∞ either because $\mathcal{X}$ is not quasi-compact or because we run into the phenomenon described in Examples, Section 110.16.

0AFM **Lemma 100.12.1.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 94.16.5) where U is a scheme. Let $u \in U$ be a point that maps to x. Let $e : U \to R$ be the "identity" map and let $s : R \to U$ be the "source" map, which is a smooth morphism of algebraic spaces. Let R_u be the fiber of $s : R \to U$ over u. The element*

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u) \in \mathbf{Z} \cup \infty$$

is independent of the choice of presentation and the point u over x.

Proof. Since $R \to U$ is smooth, the scheme R_u is smooth over $\kappa(u)$ and hence has finite dimension. On the other hand, the scheme U is locally Noetherian, but this does not guarantee that $\dim_u(U)$ is finite. Thus the difference is an element of $\mathbf{Z} \cup \{\infty\}$.

Let $[U'/R'] \to \mathcal{X}$ and $u' \in U'$ be a second presentation where U' is a scheme and u' maps to x. Consider the algebraic space $P = U \times_{\mathcal{X}} U'$. By Lemma 100.4.3 there exists a $p \in |P|$ mapping to u and u'. Since $P \to U$ and $P \to U'$ are smooth we see that $\dim_p(P) = \dim_u(U) + \dim_p(P_u)$ and $\dim_p(P) = \dim_{u'}(U') + \dim_p(P_{u'})$, see Morphisms of Spaces, Lemma 67.37.10. Note that

$$R'_{u'} = \mathrm{Spec}(\kappa(u')) \times_{\mathcal{X}} U' \quad \text{and} \quad P_u = \mathrm{Spec}(\kappa(u)) \times_{\mathcal{X}} U'$$

Let us represent $p \in |P|$ by a morphism $\mathrm{Spec}(\Omega) \to P$. Since p maps to both u and u' it induces a 2-morphism between the compositions $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u')) \to \mathcal{X}$ and $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ which in turn defines an isomorphism

$$\mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u'))} R'_{u'} \cong \mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u))} P_u$$

as algebraic spaces over $\mathrm{Spec}(\Omega)$ mapping the Ω-rational point $(1, e'(u'))$ to $(1, p)$ (some details omitted). We conclude that

$$\dim_{e'(u')}(R'_{u'}) = \dim_p(P_u)$$

by Morphisms of Spaces, Lemma 67.34.3. By symmetry we have $\dim_{e(u)}(R_u) = \dim_p(P_{u'})$. Putting everything together we obtain the independence of choices. □

We can use the lemma above to make the following definition.

0AFN **Definition 100.12.2.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 94.16.5) where U is a scheme and let $u \in U$ be a point that maps to x. We define the *dimension of $\mathcal{X}$ at x* to be the element $\dim_x(\mathcal{X}) \in \mathbf{Z} \cup \infty$ such that

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u).$$

with notation as in Lemma 100.12.1.

The dimension of a stack at a point agrees with the usual notion when $\mathcal{X}$ is a scheme (Topology, Definition 5.10.1), or more generally when $\mathcal{X}$ is a locally Noetherian algebraic space (Properties of Spaces, Definition 66.9.1).

0AFP **Definition 100.12.3.** Let S be a scheme. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over S. The *dimension* $\dim(\mathcal{X})$ of $\mathcal{X}$ is defined to be

$$\dim(\mathcal{X}) = \sup_{x \in |\mathcal{X}|} \dim_x(\mathcal{X})$$

This definition of dimension agrees with the usual notion if $\mathcal{X}$ is a scheme (Properties, Lemma 28.10.2) or an algebraic space (Properties of Spaces, Definition 66.9.2).

0AFQ **Remark 100.12.4.** If $\mathcal{X}$ is a nonempty stack of finite type over a field, then $\dim(\mathcal{X})$ is an integer. For an arbitrary locally Noetherian algebraic stack $\mathcal{X}$, $\dim(\mathcal{X})$ is in $Z \cup \{\pm\infty\}$, and $\dim(\mathcal{X}) = -\infty$ if and only if $\mathcal{X}$ is empty.

0AFR **Example 100.12.5.** Let X be a scheme of finite type over a field k, and let G be a group scheme of finite type over k which acts on X. Then the dimension of the quotient stack $[X/G]$ is equal to $\dim(X) - \dim(G)$. In particular, the dimension of the classifying stack $BG = [\mathrm{Spec}(k)/G]$ is $-\dim(G)$. Thus the dimension of an algebraic stack can be a negative integer, in contrast to what happens for schemes or algebraic spaces.

100.13. Local irreducibility

0DQG We have defined the geometric number of branches of a scheme at a point in Properties, Section 28.15 and for an algebraic space at a point in Properties of Spaces, Section 66.23. Let $n \in \mathbf{N}$. For a local ring A set

$$P_n(A) = \text{the number of geometric branches of } A \text{ is } n$$

For a smooth ring map $A \to B$ and a prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$ of A we have

$$P_n(A_\mathfrak{p}) \Leftrightarrow P_n(B_\mathfrak{q})$$

by More on Algebra, Lemma 15.107.8. As in Properties of Spaces, Remark 66.7.6 we may use P_n to define an étale local property $\mathcal{P}_n$ of germs (U, u) of schemes by setting $\mathcal{P}_n(U, u) = P_n(\mathcal{O}_{U,u})$. The corresponding property $\mathcal{P}_n$ of an algebraic space X at a point x (see Properties of Spaces, Definition 66.7.5) is just the property "the number of geometric branches of X at x is n", see Properties of Spaces, Definition 66.23.4. Moreover, the property $\mathcal{P}_n$ is smooth local, see Descent, Definition 35.21.1. This follows either from the equivalence displayed above or More on Morphisms, Lemma 37.36.4. Thus Definition 100.7.5 applies and we obtain a notion for algebraic stacks at a point.

0DQH **Definition 100.13.1.** Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

(1) The *number of geometric branches of $\mathcal{X}$ at x* is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 100.7.4 hold for $\mathcal{P}_n$ defined above, or else ∞.
(2) We say $\mathcal{X}$ is *geometrically unibranch at x* if the number of geometric branches of $\mathcal{X}$ at x is 1.

100.14. Finiteness conditions and points

0DTJ This section is the analogue of Decent Spaces, Section 68.4 for points of algebraic stacks.

0DTK **Lemma 100.14.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

(1) *some morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact, and*
(2) *any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact.*

Proof. Let $\mathrm{Spec}(k) \to \mathcal{X}$ be in the equivalence class of x. Let k'/k be a field extension. Then we have to show that $\mathrm{Spec}(k) \to \mathcal{X}$ is quasi-compact if and only if $\mathrm{Spec}(k') \to \mathcal{X}$ is quasi-compact. This follows from Morphisms of Spaces, Lemma 67.8.6 and the principle of Algebraic Stacks, Lemma 94.10.9. □

Sometimes people say that a point $x \in |\mathcal{X}|$ satisfying the equivalent conditions of Lemma 100.14.1 is a "*quasi-compact point*".

100.15. Other chapters

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(26) Schemes

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CHAPTER 101

Morphisms of Algebraic Stacks

04XM

101.1. Introduction

04XN In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00**].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 100.3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

101.2. Conventions and abuse of language

04XP We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

101.3. Properties of diagonals

04XQ The diagonal of an algebraic stack is closely related to the *Isom*-sheaves, see Algebraic Stacks, Lemma 94.10.11. By the second defining property of an algebraic stack these *Isom*-sheaves are always algebraic spaces.

04XR **Lemma 101.3.1.** *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_{\mathcal{X}}(x, y) \to T$ is locally of finite type.*

Proof. By Algebraic Stacks, Lemma 94.16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 74.11.9 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 78.22.1 we see that in this case $Isom_{\mathcal{X}}(x, y) = T \times_{(y', x'), U \times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition $s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 67.37.5 and 67.28.5) and Morphisms of Spaces, Lemma 67.23.6. □

04YP **Lemma 101.3.2.** *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then*

(1) *$Isom_{\mathcal{X}}(y, y)$ is a group algebraic space over T, and*
(2) *$Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(y, y)$ over T.*

Proof. See Groupoids in Spaces, Definitions 78.5.1 and 78.9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. □

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *diagonal of* f is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

04XS **Lemma 101.3.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *Δ_f is representable by algebraic spaces, and*
(2) *Δ_f is locally of finite type.*

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $Isom_{\mathcal{X}}(x, x')$ and $Isom_{\mathcal{Y}}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $Isom_{\mathcal{Y}}(f(x), f(x')) \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

04XT (101.3.3.1)
$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T & \longrightarrow & Isom_{\mathcal{X}}(x, x') \\ \downarrow & & \downarrow \\ T & \xrightarrow{\alpha} & Isom_{\mathcal{Y}}(f(x), f(x')) \end{array}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (101.3.3.1) is locally of finite type. By Lemma 101.3.1 the algebraic space $Isom_{\mathcal{X}}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (101.3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. We conclude by Morphisms of Spaces, Lemma 67.23.3. □

04YQ **Lemma 101.3.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then*

(1) *Δ_f is representable (by schemes),*
(2) *Δ_f is locally of finite type,*
(3) *Δ_f is a monomorphism,*
(4) *Δ_f is separated, and*
(5) *Δ_f is locally quasi-finite.*

Proof. We have already seen in Lemma 101.3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) – (5) make sense, see Properties of Stacks, Section 100.3. Also Lemma 101.3.3 guarantees (2) holds. Let $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism and contemplate Diagram (101.3.3.1). By Algebraic Stacks, Lemma 94.9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 67.27.10 allows us to conclude that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5).

Finally, Morphisms of Spaces, Proposition 67.50.2 implies that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). □

04YS **Lemma 101.3.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is separated,*
(2) *Δ_f is a closed immersion,*
(3) *Δ_f is proper, or*
(4) *Δ_f is universally closed.*

Proof. The statements "f is separated", "Δ_f is a closed immersion", "Δ_f is universally closed", and "Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 100.3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.31.14 and Properties of Stacks, Lemma 100.3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 67.40.9 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 4.31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. □

04YT **Lemma 101.3.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is quasi-separated,*
(2) *Δ_f is quasi-compact, or*
(3) *Δ_f is of finite type.*

Proof. The statements "f is quasi-separated", "Δ_f is quasi-compact", and "Δ_f is of finite type" refer to the notions defined in Properties of Stacks, Section 100.3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 101.3.4 (and Algebraic Stacks, Lemma 94.10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.31.14 and Properties of Stacks, Lemma 100.3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 4.31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. □

04YU **Lemma 101.3.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is locally separated, and*

(2) Δ_f *is an immersion.*

Proof. The statements "f is locally separated", and "Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 100.3. Proof omitted. Hint: Argue as in the proofs of Lemmas 101.3.5 and 101.3.6. □

101.4. Separation axioms

04YV Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to S$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a "quasi-separated algebraic stack", and we should also require the diagonal itself to be quasi-separated.

What about "separated algebraic stacks"? We have seen in Morphisms of Spaces, Lemma 67.40.9 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[\mathrm{Spec}(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

04YW **Definition 101.4.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *DM* if Δ_f is unramified[1].
(2) We say f is *quasi-DM* if Δ_f is locally quasi-finite[2].
(3) We say f is *separated* if Δ_f is proper.
(4) We say f is *quasi-separated* if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 100.3 to make sense out of imposing conditions on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 101.3.6 and 101.3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 101.6.5.

050D **Definition 101.4.2.** Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 101.21.6. In other words a DM morphism is one whose "fibres" are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the "fibres" $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 101.21.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

(1) We say $\mathcal{X}$ is *DM over* S if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is *quasi-DM over* S if $p : \mathcal{X} \to S$ is quasi-DM.
(3) We say $\mathcal{X}$ is *separated over* S if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is *quasi-separated over* S if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is *DM* if $\mathcal{X}$ is DM[3] over $\mathrm{Spec}(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is *quasi-DM* if $\mathcal{X}$ is quasi-DM over $\mathrm{Spec}(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is *separated* if $\mathcal{X}$ is separated over $\mathrm{Spec}(\mathbf{Z})$.
(8) We say $\mathcal{X}$ is *quasi-separated* if $\mathcal{X}$ is quasi-separated over $\mathrm{Spec}(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 94.19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually suppressed by our abuse of notation introduced in Properties of Stacks, Section 100.2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 101.4.13. We spend some time proving some standard results on these notions.

050E **Lemma 101.4.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is separated, then f is quasi-separated.*
(2) *If f is DM, then f is quasi-DM.*
(3) *If f is representable by algebraic spaces, then f is DM.*

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 67.40.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 67.38.7. Finally (3) follows from Lemma 101.3.4. □

050F **Lemma 101.4.4.** *All of the separation axioms listed in Definition 101.4.1 are stable under base change.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 4.31.14. By the results of Properties of Stacks, Section 100.3 each of the properties of the diagonal used in Definition 101.4.1 is stable under base change. Hence the lemma is true. □

06TZ **Lemma 101.4.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 101.4.1 then so does f.*

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 100.3.4. □

050G **Lemma 101.4.6.** *Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 101.4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 94.19.3.*

[3] Theorem 101.21.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

Proof. Follows immediately from Lemma 101.4.4. □

050H **Lemma 101.4.7.** *Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,*
(3) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,*
(4) *if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,*
(5) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(6) *if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and*
(7) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. The following diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow[i]{} & \mathcal{X} \times_{\mathcal{T}} \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{\Delta_{\mathcal{Z}/\mathcal{T}}} & \mathcal{Z} \times_{\mathcal{T}} \mathcal{Z} \end{array}$$

is a 2-fibre product diagram, see Categories, Lemma 4.31.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 101.3.3, and the material in Properties of Stacks, Section 100.3. □

050I **Lemma 101.4.8.** *Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$ of g. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,*
(3) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(4) *if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and*
(5) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 101.4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. □

050J **Lemma 101.4.9.** *Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $id_{\mathcal{T}}$. Then*

(1) *s is representable by algebraic spaces and locally of finite type,*
(2) *if f is DM, then s is unramified,*
(3) *if f is quasi-DM, then s is locally quasi-finite,*
(4) *if f is separated, then s is proper, and*
(5) *if f is quasi-separated, then s is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 101.4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_{\mathcal{T}} \mathcal{X}$ is 2-isomorphic to s. □

050K **Lemma 101.4.10.** *All of the separation axioms listed in Definition 101.4.1 are stable under composition of morphisms.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 101.4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemmas 67.38.3, 67.27.3, 67.40.4, 67.8.5, and 67.4.8. □

050L **Lemma 101.4.11.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.*

(1) *If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.*
(2) *If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*
(5) *If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.*
(6) *If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.*
(7) *If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.*
(8) *If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.*

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 101.4.10 and Definition 101.4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 101.4.10. Details omitted. □

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 67.4.10.

050M **Lemma 101.4.12.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.*

(1) *If $g \circ f$ is DM then so is f.*
(2) *If $g \circ f$ is quasi-DM then so is f.*
(3) *If $g \circ f$ is separated and Δ_g is separated, then f is separated.*
(4) *If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.*

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 101.3.3 and 101.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 67.38.11 (resp. Morphisms of Spaces, Lemma 67.27.8). This proves (1) and (2).

Proof of (4). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 101.3.3 and 101.4.7.

Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 67.4.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 67.8.9. Thus f is quasi-separated.

Proof of (3). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 101.3.3 and 101.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 67.40.6 which means that f is separated. □

050N **Lemma 101.4.13.** *Let $\mathcal{X}$ be an algebraic stack over the base scheme S.*

(1) *$\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.*
(2) *$\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

(5) *If $\mathcal{X}$ is DM over S, then f is DM.*
(6) *If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.*
(7) *If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.*
(8) *If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.*

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 101.4.12 and Spaces, Definition 65.13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 101.4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 67.38.3 and 67.38.11 (resp. Morphisms of Spaces, Lemmas 67.27.3 and 67.27.8). □

06MB **Lemma 101.4.14.** *Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.*

(1) *If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.*
(2) *If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 67.39.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X} = W \times_{\mathcal{X}} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 100.3.3. By assumption the morphism $\mathrm{pr}_i : W \times_{\mathcal{X}} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 67.38.11 (resp. Morphisms of Spaces, Lemma 67.27.8). □

06MY **Lemma 101.4.15.** *A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.*

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 100.8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 100.9.5. □

06MZ **Lemma 101.4.16.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.*

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 101.4.15. Apply Lemma 101.4.11 to conclude. □

101.5. Inertia stacks

050P The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.34.

050Q **Lemma 101.5.1.** *Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_{\mathcal{X}}$ is an algebraic stack as well. The morphism*

$$\mathcal{I}_{\mathcal{X}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the relative inertia $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic stack and the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 4.34.1 there are equivalences

$$\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product

square

$$\begin{array}{ccc} Isom_{\mathcal{X}}(x,x) & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{x} & \mathcal{X} \end{array}$$

This follows immediately from the definition of $\mathcal{I}_{\mathcal{X}}$. Since $Isom_{\mathcal{X}}(x,x)$ is always an algebraic space locally of finite type over T (see Lemma 101.3.1) we conclude that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$\begin{array}{ccccc} Isom_{\mathcal{X}}(x,x) & \longleftarrow & K & \longrightarrow & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \\ \downarrow & & \downarrow & & \downarrow \\ Isom_{\mathcal{Y}}(f(x),f(x)) & \xleftarrow{e} & T & \xrightarrow{x} & \mathcal{X} \end{array}$$

with both squares 2-fibre products. This follows from Categories, Lemma 4.34.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 67.23.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. □

050R **Remark 101.5.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In Properties of Stacks, Remark 100.3.7 we have seen that the 2-category of morphisms $\mathcal{Z} \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}/\mathcal{Y}$ is a *group object*. Recall that an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in with $f(\alpha) = \mathrm{id}$. The composition

$$c : \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

is given by the rule on objects

$$((x,\alpha),(x',\alpha'),\beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ be its inertia stack. Let T be a scheme and let x be an object of $\mathcal{X}$ over T. Set $y = f(x)$. We have seen in the proof of Lemma 101.5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is an exact sequence of sheaves of groups

0CPJ (101.5.2.1) $$0 \to Isom_{\mathcal{X}/\mathcal{Y}}(x,x) \to Isom_{\mathcal{X}}(x,x) \to Isom_{\mathcal{Y}}(y,y)$$

The group structure on the second and third term is the one defined in Lemma 101.3.2 and the sequence gives a meaning to the first term. Also, there is a canonical cartesian square

$$\begin{array}{ccc} Isom_{\mathcal{X}/\mathcal{Y}}(x,x) & \longrightarrow & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{x} & \mathcal{X} \end{array}$$

In fact, the group structure on $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ discussed in Remark 101.5.2 induces the group structure on $Isom_{\mathcal{X}/\mathcal{Y}}(x,x)$. This allows us to define the sheaf $Isom_{\mathcal{X}/\mathcal{Y}}(x,x)$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

06PP **Definition 101.5.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space.

(1) Let $x : Z \to \mathcal{X}$ be a morphism. We set

$$Isom_{\mathcal{X}/\mathcal{Y}}(x,x) = Z \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

We endow it with the structure of a group algebraic space over Z by pulling back the composition law discussed in Remark 101.5.2. We will sometimes refer to $Isom_{\mathcal{X}/\mathcal{Y}}(x,x)$ as the *relative sheaf of automorphisms of* x.

(2) Let $x_1, x_2 : Z \to \mathcal{X}$ be morphisms. Set $y_i = f \circ x_i$. Let $\alpha : y_1 \to y_2$ be a 2-morphism. Then α determines a morphism $\Delta^\alpha : Z \to Z \times_{y_1,\mathcal{Y},y_2} Z$ and we set

$$Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1,x_2) = (Z \times_{x_1,\mathcal{X},x_2} Z) \times_{Z \times_{y_1,\mathcal{Y},y_2} Z, \Delta^\alpha} Z.$$

We will sometimes refer to $Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1,x_2)$ as the *relative sheaf of isomorphisms from* x_1 *to* x_2.

If $\mathcal{Y} = \mathrm{Spec}(\mathbf{Z})$ or more generally when $\mathcal{Y}$ is an algebraic space, then we use the notation $Isom_{\mathcal{X}}(x,x)$ and $Isom_{\mathcal{X}}(x_1,x_2)$ and we use the terminology *sheaf of automorphisms of* x and *sheaf of isomorphisms from* x_1 *to* x_2.

0CPK **Lemma 101.5.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space and let $x_i : Z \to \mathcal{X}$, $i = 1, 2$ be morphisms. Then*

(1) *$Isom_{\mathcal{X}/\mathcal{Y}}(x_2,x_2)$ is a group algebraic space over Z,*
(2) *there is an exact sequence of groups*

$$0 \to Isom_{\mathcal{X}/\mathcal{Y}}(x_2,x_2) \to Isom_{\mathcal{X}}(x_2,x_2) \to Isom_{\mathcal{Y}}(f \circ x_2, f \circ x_2)$$

(3) *there is a map of algebraic spaces $Isom_{\mathcal{X}}(x_1,x_2) \to Isom_{\mathcal{Y}}(f \circ x_1, f \circ x_2)$ such that for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ we obtain a cartesian diagram*

$$\begin{array}{ccc} Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1,x_2) & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ Isom_{\mathcal{X}}(x_1,x_2) & \longrightarrow & Isom_{\mathcal{Y}}(f \circ x_1, f \circ x_2) \end{array}$$

(4) *for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ the algebraic space $Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1,x_2)$ is a pseudo torsor for $Isom_{\mathcal{X}/\mathcal{Y}}(x_2,x_2)$ over Z.*

Proof. Part (1) follows from Definition 101.5.3. Part (2) comes from the exact sequence (101.5.2.1) étale locally on Z. Part (3) can be seen by unwinding the definitions. Locally on Z in the étale topology part (4) reduces to part (2) of Lemma 101.3.2. □

06PQ **Lemma 101.5.5.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Set $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$. Then both squares in the diagram*

$$\begin{array}{ccccc} \mathcal{I}_{\mathcal{X}'/\mathcal{Y}'} & \longrightarrow & \mathcal{X}' & \xrightarrow{\pi'} & \mathcal{Y}' \\ \downarrow{\scriptstyle \text{Categories, Equation (4.34.2.3)}} & & \downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} & \longrightarrow & \mathcal{X} & \xrightarrow{\pi} & \mathcal{Y} \end{array}$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'/\mathcal{Y}'}$ is defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' with $\pi'(\alpha') = \text{id}$, see Categories, Section 4.34. Suppose that x' lies over the scheme U and maps to the object x of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 4.32.3 we see that $x' = (U, x, y', \beta)$ where y' is an object of $\mathcal{Y}'$ over U and β is an isomorphism $\beta : \pi(x) \to f(y')$ in the fibre category of $\mathcal{Y}$ over U. By the very construction of the 2-fibre product the automorphism α' is a pair (α, γ) where α is an automorphism of x over U and γ is an automorphism of y' over U such that α and γ are compatible via β. The condition $\pi'(\alpha') = \text{id}$ signifies that $\gamma = \text{id}$ whereupon the condition that α, β, γ are compatible is exactly the condition $\pi(\alpha) = \text{id}$, i.e., means exactly that (x, α) is an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. In this way we see that the left square is a fibre product square (some details omitted). □

06R5 **Lemma 101.5.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \end{array}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 100.8.4) and the definition of the inertia in Categories, Section 4.34. Namely, an object of $\mathcal{I}_{\mathcal{X}}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_{\mathcal{X}}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T, $\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorphism over T, i.e., an object of $\mathcal{I}_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over T. □

06PR **Lemma 101.5.7.** *Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then*

$$\begin{array}{ccc} G & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \end{array}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 78.26.2. □

101.6. Higher diagonals

04YX Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the *zeroth diagonal*, we denote $\Delta_{f,1} = \Delta_f$ the *first diagonal*, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the *second diagonal*. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 101.3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 101.3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 4.34.1. Moreover, via this identification the second diagonal becomes the *neutral section*

$$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. By analogy with what happens for groupoids in algebraic spaces (Groupoids in Spaces, Lemma 78.29.2) we have the following equivalences.

0CL0 **Lemma 101.6.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion.*

(2) *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is quasi-separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is quasi-compact.*

(3) *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an immersion.*

(4) *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is unramified,*
   (b) *f is DM.*

(5) *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally quasi-finite,*
   (b) *f is quasi-DM.*

Proof. Proof of (1), (2), and (3). Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X}$. Then $G = U \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic space over U (Lemma 101.5.1). In fact, G is a group algebraic space over U by the group law on relative inertia constructed in Remark 101.5.2. Moreover, $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is surjective and smooth as a base change of $U \to \mathcal{X}$. Finally, the base change of $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$

by $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is the identity $U \to G$ of G/U. Thus the equivalence of (a) and (c) follows from Groupoids in Spaces, Lemma 78.6.1. Since $\Delta_{f,2}$ is the diagonal of Δ_f we have (b) $\Leftrightarrow$ (c) by definition.

Proof of (4) and (5). Recall that (4)(b) means Δ_f is unramified and (5)(b) means that Δ_f is locally quasi-finite. Choose a scheme Z and a morphism $a : Z \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Then $a = (x_1, x_2, \alpha)$ where $x_i : Z \to \mathcal{X}$ and $\alpha : f \circ x_1 \to f \circ x_2$ is a 2-morphism. Recall that

$$\begin{array}{ccc} Isom^{\alpha}_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta_f} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array} \quad \text{and} \quad \begin{array}{ccc} Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2) & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle x_2} \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} & \longrightarrow & \mathcal{X} \end{array}$$

are cartesian squares. By Lemma 101.5.4 the algebraic space $Isom^{\alpha}_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ is a pseudo torsor for $Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ over Z. Thus the equivalences in (4) and (5) follow from Groupoids in Spaces, Lemma 78.9.5. □

04YY **Lemma 101.6.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *the morphism f is representable by algebraic spaces,*
(2) *the second diagonal of f is an isomorphism,*
(3) *the group stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is trivial over $\mathcal{X}$, and*
(4) *for a scheme T and a morphism $x : T \to \mathcal{X}$ the kernel of $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$ is trivial.*

Proof. We first prove the equivalence of (1) and (2). Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 94.15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the equivalence of (1) and (2). To prove the equivalence of (2) and (3), by the discussion above, it suffices to note that a group stack is trivial if and only if its identity section is an isomorphism. Finally, the equivalence of (3) and (4) follows from the definitions: in the proof of Lemma 101.5.1 we have seen that the kernel in (4) corresponds to the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ over T. □

This lemma leads to the following hierarchy for morphisms of algebraic stacks.

0AHJ **Lemma 101.6.3.** *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is*

(1) *a monomorphism if and only if $\Delta_{f,1}$ is an isomorphism, and*
(2) *representable by algebraic spaces if and only if $\Delta_{f,1}$ is a monomorphism.*

Moreover, the second diagonal $\Delta_{f,2}$ is always a monomorphism.

Proof. Recall from Properties of Stacks, Lemma 100.8.4 that a morphism of algebraic stacks is a monomorphism if and only if its diagonal is an isomorphism of stacks. Thus Lemma 101.6.2 can be rephrased as saying that a morphism is representable by algebraic spaces if the diagonal is a monomorphism. In particular, it shows that condition (3) of Lemma 101.3.4 is actually an if and only if, i.e., a morphism of algebraic stacks is representable by algebraic spaces if and only if its diagonal is a monomorphism. □

04YZ **Lemma 101.6.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *$\Delta_{f,1}$ separated $\Leftrightarrow$ $\Delta_{f,2}$ closed immersion $\Leftrightarrow$ $\Delta_{f,2}$ proper $\Leftrightarrow$ $\Delta_{f,2}$ universally closed,*
(2) *$\Delta_{f,1}$ quasi-separated $\Leftrightarrow$ $\Delta_{f,2}$ finite type $\Leftrightarrow$ $\Delta_{f,2}$ quasi-compact, and*
(3) *$\Delta_{f,1}$ locally separated $\Leftrightarrow$ $\Delta_{f,2}$ immersion.*

Proof. Follows from Lemmas 101.3.5, 101.3.6, and 101.3.7 applied to $\Delta_{f,1}$. □

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

04Z0 **Lemma 101.6.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and*
(2) *f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.*
(3) *f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.*
(4) *f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.*

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 101.6.4. By Morphisms of Spaces, Lemma 67.9.7 and Algebraic Stacks, Lemma 94.10.9 we see that $\Delta_{f,1}$ is quasi-compact. Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 101.3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 101.6.4.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 101.3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 101.3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 76.14.8. □

0CPL **Lemma 101.6.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated (resp. quasi-separated, resp. quasi-DM, resp. DM) morphism of algebraic stacks. Then*

(1) *given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism*
$$T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \longrightarrow T_1 \times_{y_1, \mathcal{Y}, y_2} T_2$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified),
(2) *given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism*
$$Isom_{\mathcal{X}}(x_1, x_2) \longrightarrow Isom_{\mathcal{Y}}(y_1, y_2)$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof. Proof of (1). Observe that the diagram
$$\begin{array}{ccc} T_1 \times_{x_1, \mathcal{X}, x_2} T_2 & \longrightarrow & T_1 \times_{y_1, \mathcal{Y}, y_2} T_2 \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is cartesian. Hence this follows from the fact that f is separated (resp. quasi-separated, resp. quasi-DM, resp. DM) if and only if the diagonal is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof of (2). This is true because

$$Isom_{\mathcal{X}}(x_1, x_2) = (T \times_{x_1, \mathcal{X}, x_2} T) \times_{T \times T, \Delta_T} T$$

hence the morphism in (2) is a base change of the morphism in (1). □

101.7. Quasi-compact morphisms

050S Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be quasi-compact. Here is another characterization.

050T **Lemma 101.7.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *f is quasi-compact (as in Properties of Stacks, Section 100.3), and*
(2) *for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 100.6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 100.6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 67.8.8. □

This motivates the following definition.

050U **Definition 101.7.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-compact* if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 101.7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

050V **Lemma 101.7.3.** *The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.*

Proof. Omitted. □

050W **Lemma 101.7.4.** *The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.*

Proof. Omitted. □

0CL1 **Lemma 101.7.5.** *A closed immersion of algebraic stacks is quasi-compact.*

Proof. This follows from the fact that immersions are always representable and the corresponding fact for closed immersion of algebraic spaces. □

050X **Lemma 101.7.6.** *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{Z} & \end{array}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 100.5.3 the morphism $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 100.6.2. □

050Y **Lemma 101.7.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 101.4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 101.4.4). The second map is quasi-compact as it is the base change of f, see Lemma 101.7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 101.7.4. □

075S **Lemma 101.7.8.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.*
(2) *If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.*
(3) *A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 101.7.7. Part (2) follows from (1) and Lemma 101.4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 101.7.3 and 101.4.4. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 101.4.11 and 101.7.4. □

0CL2 **Lemma 101.7.9.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathcal{Y} \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y.

Proof. Choose an affine scheme V and a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ sending v to y. Consider the base change diagram

$$\begin{array}{ccc} V \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & \mathcal{Y} \end{array}$$

Recall that $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V| \times_{|\mathcal{Y}|} |\mathcal{X}|$ is surjective (Properties of Stacks, Lemma 100.4.3). Because $|V| \to |\mathcal{Y}|$ is open (Properties of Stacks, Lemma 100.4.7) we conclude that v is in the closure of the image of $|g|$. Thus it suffices to prove the lemma for the quasi-compact morphism g (Lemma 101.7.3) which we do in the next paragraph.

Assume $\mathcal{Y} = Y$ is an affine scheme. Then $\mathcal{X}$ is quasi-compact as f is quasi-compact (Definition 101.7.2). Choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the image of $W \to Y$. By Morphisms, Lemma 29.6.5 we can choose a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & W & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. Composing with $W \to \mathcal{X}$ we obtain a solution. □

0DTL **Lemma 101.7.10.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact, then f is quasi-compact.*

Proof. Assume $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ a quasi-compact algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_{\mathcal{Y}} \mathcal{Z}$. Since $U \to \mathcal{Z}$ is flat, surjective, and locally of finite presentation and $\mathcal{Z}$ is quasi-compact, we can find a quasi-compact open subscheme $U' \subset U$ such that $U' \to \mathcal{Z}$ is surjective. Then $U' \times_{\mathcal{Y}} \mathcal{X} = U' \times_W (W \times_{\mathcal{Y}} \mathcal{X})$ is quasi-compact by assumption and surjects onto $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact as desired. □

101.8. Noetherian algebraic stacks

050Z We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 100.7.

0510 **Definition 101.8.1.** Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *Noetherian* if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 101.6 if we denote $p : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ the "absolute" structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$$\mathcal{X} \text{ Noetherian} \Leftrightarrow \mathcal{X} \text{ locally Noetherian and } \Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2} \text{ quasi-compact.}$$

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

0CPM **Lemma 101.8.2.** *Let $j : \mathcal{X} \to \mathcal{Y}$ be an immersion of algebraic stacks.*

(1) *If $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian and j is quasi-compact.*
(2) *If $\mathcal{Y}$ is Noetherian, then $\mathcal{X}$ is Noetherian.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Then $U = \mathcal{X} \times_{\mathcal{Y}} V$ is a scheme and $V \to U$ is an immersion, see Properties of Stacks, Definition 100.9.1. Recall that $\mathcal{Y}$ is locally Noetherian if and only if V is locally Noetherian. In this case U is locally Noetherian too (Morphisms, Lemmas 29.15.5 and 29.15.6) and $U \to V$ is quasi-compact (Properties, Lemma 28.5.3). This shows that j is quasi-compact (Lemma 101.7.10) and that $\mathcal{X}$ is locally Noetherian. Finally, if $\mathcal{Y}$ is Noetherian, then we see from the above that $\mathcal{X}$ is quasi-compact and locally Noetherian. To finish the proof observe that j is separated and hence $\mathcal{X}$ is quasi-separated because $\mathcal{Y}$ is so by Lemma 101.4.11. □

0DQI **Lemma 101.8.3.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *If $\mathcal{X}$ is locally Noetherian then $|\mathcal{X}|$ is a locally Noetherian topological space.*
(2) *If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is a Noetherian topological space.*

Proof. Assume $\mathcal{X}$ is locally Noetherian. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. As $\mathcal{X}$ is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 28.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |\mathcal{X}|$ is open and surjective we conclude that $|\mathcal{X}|$ is locally Noetherian by Topology, Lemma 5.9.3. This proves (1). If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is quasi-compact and locally Noetherian. Hence $|\mathcal{X}|$ is Noetherian by Topology, Lemma 5.12.14. □

0GVX **Lemma 101.8.4.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is quasi-sober (Topology, Definition 5.8.6).*

Proof. We have to prove that every irreducible closed subset $T \subset |\mathcal{X}|$ has a generic point. Choose an affine scheme U and a smooth morphism $f : U \to \mathcal{X}$ such that $f^{-1}(T) \subset |U|$ is nonempty. Since U is Noetherian, the closed subset $f^{-1}(T)$ has finitely many irreducible components (Topology, Lemma 5.9.2). Say $f^{-1}(T) = Z_1 \cup \ldots \cup Z_n$ is the decomposition into irreducible components. As f is open, the image of $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ contains a nonempty open subset of T. Since T is irreducible, this means that $f(f^{-1}(T))$ is dense. Since T is irreducible, it follows that $f(Z_i)$ is dense for some i. Then if $\xi_i \in Z_i$ is the generic point we see that $f(\xi_i)$ is a generic point of T. □

101.9. Affine morphisms

0CHP Affine morphisms of algebraic stacks are defined as follows.

0CHQ **Definition 101.9.1.** A morphism of algebraic stacks is said to be *affine* if it is representable and affine in the sense of Properties of Stacks, Section 100.3.

For us it is a little bit more convenient to think of an affine morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces

and affine in the sense of Properties of Stacks, Section 100.3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0CHR **Lemma 101.9.2.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an affine morphism of algebraic stacks.*

Proof. This follows from the discussion in Properties of Stacks, Section 100.3. □

0CHS **Lemma 101.9.3.** *Compositions of affine morphisms of algebraic stacks are affine.*

Proof. This follows from the discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemma 67.20.4. □

0GQE **Lemma 101.9.4.** *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & \mathcal{Z} & \end{array}$$

be a commutative diagram of morphisms of algebraic stacks. If a is affine and Δ_b is affine, then f is affine.

Proof. The base change $\mathrm{pr}_2 : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ of a is affine by Lemma 101.9.2. The morphism $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the base change of $\Delta_b : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ by the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ (see material in Categories, Section 4.31). Hence it is affine by Lemma 101.9.2. The composition $f = \mathrm{pr}_2 \circ (1, f)$ of affine morphisms is affine by Lemma 101.9.3 and the proof is done. □

101.10. Integral and finite morphisms

0CHT Integral and finite morphisms of algebraic stacks are defined as follows.

0CHU **Definition 101.10.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *integral* if f is representable and integral in the sense of Properties of Stacks, Section 100.3.
(2) We say f is *finite* if f is representable and finite in the sense of Properties of Stacks, Section 100.3.

For us it is a little bit more convenient to think of an integral, resp. finite morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces and integral, resp. finite in the sense of Properties of Stacks, Section 100.3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

0CHV **Lemma 101.10.2.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an integral (or finite) morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an integral (or finite) morphism of algebraic stacks.*

Proof. This follows from the discussion in Properties of Stacks, Section 100.3. □

0CHW **Lemma 101.10.3.** *Compositions of integral, resp. finite morphisms of algebraic stacks are integral, resp. finite.*

Proof. This follows from the discussion in Properties of Stacks, Section 100.3 and Morphisms of Spaces, Lemma 67.45.4. □

101.11. Open morphisms

06U0 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally open. Here is another characterization.

06U1 **Lemma 101.11.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally open (as in Properties of Stacks, Section 100.3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is open.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

06U2 **Definition 101.11.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *open* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$
is open, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is open.

06U3 **Lemma 101.11.3.** *The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.*

Proof. This is immediate from the definition. □

06U4 **Lemma 101.11.4.** *The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.*

Proof. Omitted. □

101.12. Submersive morphisms

06U5 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally submersive. Here is another characterization.

0CHX **Lemma 101.12.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally submersive (as in Properties of Stacks, Section 100.3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is submersive.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally submersive, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is submersive. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

06U6 **Definition 101.12.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 5.6.3.
(2) We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CHY **Lemma 101.12.3.** *The base change of a universally submersive morphism of algebraic stacks by any morphism of algebraic stacks is universally submersive.*

Proof. This is immediate from the definition. □

0CHZ **Lemma 101.12.4.** *The composition of a pair of (universally) submersive morphisms of algebraic stacks is (universally) submersive.*

Proof. Omitted. □

[4]This is very different from the notion of a submersion of differential manifolds.

101.13. Universally closed morphisms

0511 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally closed. Here is another characterization.

0512 **Lemma 101.13.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally closed (as in Properties of Stacks, Section 100.3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is closed.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally closed, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

0513 **Definition 101.13.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$
is closed, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is closed.

0514 **Lemma 101.13.3.** *The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.*

Proof. This is immediate from the definition. □

0515 **Lemma 101.13.4.** *The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.*

Proof. Omitted. □

0CL3 **Lemma 101.13.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is universally closed,*
(2) *for every scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_{\mathcal{Y}} \mathcal{X}| \to |Z|$ is closed,*

(3) *for every affine scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_{\mathcal{Y}} \mathcal{X}| \to |Z|$ is closed, and*
(4) *there exists an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective smooth morphism $V \to \mathcal{Y}$. We are going to show that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks. Let $\mathcal{Z} \to V$ be a morphism from an algebraic stack to V. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where $W = \coprod W_i$ is a disjoint union of affine schemes. Then we have the following commutative diagram

$$\begin{array}{ccccccc} \coprod_i |W_i \times_{\mathcal{Y}} \mathcal{X}| & = & |W \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| & = & |\mathcal{Z} \times_V (V \times_{\mathcal{Y}} \mathcal{X})| \\ \downarrow & & \downarrow & & \downarrow & & \swarrow \\ \coprod |W_i| & = & |W| & \longrightarrow & |\mathcal{Z}| & & \end{array}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Stacks, Lemma 100.4.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the diagram

$$\begin{array}{ccccc} |(V \times_{\mathcal{Y}} \mathcal{Z}) \times_V (V \times_{\mathcal{Y}} \mathcal{X})| & = & |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |Z \times_{\mathcal{Y}} \mathcal{X}| \\ & \searrow & \downarrow & & \downarrow \\ & & |V \times_{\mathcal{Y}} \mathcal{Z}| & \longrightarrow & |\mathcal{Z}| \end{array}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic stacks are surjective and smooth (see reference above). It follows that the right vertical arrow is closed. □

101.14. Universally injective morphisms

0CI0 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be universally injective. Here is another characterization.

0CI1 **Lemma 101.14.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally injective (as in Properties of Stacks, Section 100.3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is injective.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$

of algebraic spaces is universally injective, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is injective. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is injective it follows that the right vertical arrow is injective. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

0CI2 **Definition 101.14.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *universally injective* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is injective.

0CI3 **Lemma 101.14.3.** *The base change of a universally injective morphism of algebraic stacks by any morphism of algebraic stacks is universally injective.*

Proof. This is immediate from the definition. □

0CI4 **Lemma 101.14.4.** *The composition of a pair of universally injective morphisms of algebraic stacks is universally injective.*

Proof. Omitted. □

0CPN **Lemma 101.14.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is universally injective,*
(2) *$\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, and*
(3) *for an algebraically closed field, for $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$, and for a 2-arrow $\beta : f \circ x_1 \to f \circ x_2$ there is a 2-arrow $\alpha : x_1 \to x_2$ with $\beta = id_f \star \alpha$.*

Proof. (1) $\Rightarrow$ (2). If f is universally injective, then the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is injective, which implies that $|\Delta|$ is surjective.

(2) $\Rightarrow$ (1). Assume Δ is surjective. Then any base change of Δ is surjective (see Properties of Stacks, Section 100.5). Since the diagonal of a base change of f is a base change of Δ, we see that it suffices to show that $|\mathcal{X}| \to |\mathcal{Y}|$ is injective. If not, then by Properties of Stacks, Lemma 100.4.3 we find that the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is not injective. Of course this means that $|\Delta|$ is not surjective.

(3) $\Rightarrow$ (2). Let $t \in |\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}|$. Then we can represent t by a morphism $t : \mathrm{Spec}(k) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ with k an algebraically closed field. By our construction of 2-fibre products we can represent t by (x_1, x_2, β) where $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ and $\beta : f \circ x_1 \to f \circ x_2$ is a 2-morphism. Then (3) implies that there is a 2-morphism $\alpha : x_1 \to x_2$ mapping to β. This exactly means that $\Delta(x_1) = (x_1, x_1, \mathrm{id})$ is isomorphic to t. Hence (2) holds.

(2) $\Rightarrow$ (3). Let $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ be morphisms with k an algebraically closed field. Let $\beta : f \circ x_1 \to f \circ x_2$ be a 2-morphism. As in the previous paragraph, we obtain a morphism $t = (x_1, x_2, \beta) : \mathrm{Spec}(k) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 101.3.3

$$T = \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, t} \mathrm{Spec}(k)$$

is an algebraic space locally of finite type over $\mathrm{Spec}(k)$. Condition (2) implies that T is nonempty. Then since k is algebraically closed, there is a k-point in T. Unwinding the definitions this means there is a morphism $\alpha : x_1 \to x_2$ in $\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$ such that $\beta = \mathrm{id}_f \star \alpha$. □

0DTM **Lemma 101.14.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a universally injective morphism of algebraic stacks. Let $y : \mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism where k is an algebraically closed field. If y is in the image of $|\mathcal{X}| \to |\mathcal{Y}|$, then there is a morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ with $y = f \circ x$.*

Proof. We first remark this lemma is not a triviality, because the assumption that y is in the image of $|f|$ means only that we can lift y to a morphism into $\mathcal{X}$ after possibly replacing k by an extension field. To prove the lemma we may base change f by y, hence we may assume we have a nonempty algebraic stack $\mathcal{X}$ and a universally injective morphism $\mathcal{X} \to \mathrm{Spec}(k)$ and we want to find a k-valued point of $\mathcal{X}$. We may replace $\mathcal{X}$ by its reduction. We may choose a field k' and a surjective, flat, locally finite type morphism $\mathrm{Spec}(k') \to \mathcal{X}$, see Properties of Stacks, Lemma 100.11.2. Since $\mathcal{X} \to \mathrm{Spec}(k)$ is universally injective, we find that

$$\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \to \mathrm{Spec}(k' \otimes_k k')$$

is surjective as the base change of the surjective morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$ (Lemma 101.14.5). Since k is algebraically closed $k' \otimes_k k'$ is a domain (Algebra, Lemma 10.49.4). Let $\xi \in \mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k')$ be a point mapping to the generic point of $\mathrm{Spec}(k' \otimes_k k')$. Let U be the reduced induced closed subscheme structure on the connected component of $\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k')$ containing ξ. Then the two projections $U \to \mathrm{Spec}(k')$ are locally of finite type, as this was true for the projections $\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ as base changes of the morphism $\mathrm{Spec}(k') \to \mathcal{X}$. Applying Varieties, Proposition 33.31.1 we find that the integral closures of the two images of k' in $\Gamma(U, \mathcal{O}_U)$ are equal. Looking in $\kappa(\xi)$ means that any element of the form $\lambda \otimes 1$ is algebraically dependent on the subfield

$$1 \otimes k' \subset (\text{fraction field of } k' \otimes_k k') \subset \kappa(\xi).$$

Since k is algebraically closed, this is only possible if $k' = k$ and the proof is complete. □

0DTN **Lemma 101.14.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

1. *f is universally injective,*
2. *for every affine scheme Z and any morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is universally injective, and*
3. *add more here.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2) holds. We will show that $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, which implies (1) by Lemma 101.14.5. Consider an affine scheme V and a smooth morphism $V \to \mathcal{Y}$. Since $g : V \times_{\mathcal{Y}} \mathcal{X} \to V$ is universally injective by (2), we see that Δ_g is surjective. However, Δ_g is the base

change of Δ_f by the smooth morphism $V \to \mathcal{Y}$. Since the collection of these morphisms $V \to \mathcal{Y}$ are jointly surjective, we conclude Δ_f is surjective. □

0DTP **Lemma 101.14.8.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is universally injective, then f is universally injective.*

Proof. Observe that the diagonal Δ_g of the morphism $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ is the base change of Δ_f by $W \to \mathcal{Y}$. Hence if Δ_g is surjective, then so is Δ_f by Properties of Stacks, Lemma 100.3.3. Thus the lemma follows from the characterization (2) in Lemma 101.14.5. □

101.15. Universal homeomorphisms

0CI5 Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 100.3 we have defined what it means for f to be a universal homeomorphism. Here is another characterization.

0CI6 **Lemma 101.15.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is a universal homeomorphism (Properties of Stacks, Section 100.3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is a homeomorphism.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is a universal homeomorphism, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is a homeomorphism. By Properties of Stacks, Section 100.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is a homeomorphism it follows that the right vertical arrow is a homeomorphism. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

0CI7 **Definition 101.15.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *universal homeomorphism* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is a homeomorphism.

0CI8 **Lemma 101.15.3.** *The base change of a universal homeomorphism of algebraic stacks by any morphism of algebraic stacks is a universal homeomorphism.*

Proof. This is immediate from the definition. □

0CI9 **Lemma 101.15.4.** *The composition of a pair of universal homeomorphisms of algebraic stacks is a universal homeomorphism.*

Proof. Omitted. □

0DTQ **Lemma 101.15.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is a universal homeomorphism, then f is a universal homeomorphism.*

Proof. Assume $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ is a universal homeomorphism. Then g is universally injective, hence f is universally injective by Lemma 101.14.8. On the other hand, let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ an algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_{\mathcal{Y}} \mathcal{Z}$. Consider the diagram

$$\begin{array}{ccccc} W \times_{\mathcal{Y}} \mathcal{X} & \longleftarrow & U \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\ W & \longleftarrow & U & \longrightarrow & \mathcal{Z} \end{array}$$

The middle vertical arrow induces a homeomorphism on topological space by assumption on g. The morphism $U \to \mathcal{Z}$ and $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ are surjective, flat, and locally of finite presentation hence induce open maps on topological spaces. We conclude that $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is open. Surjectivity is easy to prove; we omit the proof. □

101.16. Types of morphisms smooth local on source-and-target

06FL Given a property of morphisms of algebraic spaces which is *smooth local on the source-and-target*, see Descent on Spaces, Definition 74.20.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

06FM **Lemma 101.16.1.** *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

1. *for any diagram as above such that in addition $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is smooth the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are

representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 100.3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow{\scriptstyle h} & \nwarrow & \downarrow & \overset{(h,h')}{\searrow} & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_{\mathcal{Y}} V' & \longrightarrow & V' \end{array}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$\begin{array}{ccc} U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U \times_{\mathcal{Y}} V' & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V' \end{array}$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 74.20.2 part (2) we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$. Since $\mathcal{P}$ is smooth local on the source by Descent on Spaces, Lemma 74.20.2 part (1) we can precompose by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ and conclude (h, h') has $\mathcal{P}$. Since $V \times_{\mathcal{Y}} V' \to V'$ is smooth we conclude $U \times_{\mathcal{X}} U' \to V'$ has $\mathcal{P}$ by Descent on Spaces, Lemma 74.20.2 part (3). Finally, since $U \times_X U' \to U'$ is surjective and smooth and $\mathcal{P}$ is smooth local on the source (same lemma) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 74.20.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for

any scheme Z and morphism $Z \to \mathcal{Y}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_{\mathcal{Y}} U & \longrightarrow & Z \times_{\mathcal{Y}} V \\ \downarrow & & \downarrow \\ Z \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 74.20.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. □

06FN **Definition 101.16.2.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 101.16.1 hold.

06FP **Remark 101.16.3.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 101.16.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having property $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

06FQ **Remark 101.16.4.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 101.16.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has property $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{array}{ccccc} V' \times_V U & \longrightarrow & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ V' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

06PS **Remark 101.16.5.** Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of

morphisms of algebraic stacks defined in Definition 101.16.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

101.17. Morphisms of finite type

06FR The property "locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.23.3 and Descent on Spaces, Lemma 74.11.9. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06FS **Definition 101.17.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ locally of finite type.
(2) We say f is *of finite type* if it is locally of finite type and quasi-compact.

06FT **Lemma 101.17.2.** *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.23.2. □

06FU **Lemma 101.17.3.** *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.23.3. □

06FV **Lemma 101.17.4.** *An immersion is locally of finite type.*

Proof. Combine Remark 101.16.5 with Morphisms of Spaces, Lemma 67.23.7. □

06R6 **Lemma 101.17.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 29.15.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. □

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

06U7 **Lemma 101.17.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 74.11.9 and we win. □

06U8 **Lemma 101.17.7.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 74.16.2. □

06U9 **Lemma 101.17.8.** *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_{\mathcal{Z}} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 29.15.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. □

101.18. Points of finite type

06FW Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

06FX **Lemma 101.18.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *There exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type and represents x.*

(2) *There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 67.39.8 and 67.28.5). Hence we see (1) holds by Lemma 101.17.2.

Conversely, assume $\mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} \mathrm{Spec}(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} \mathrm{Spec}(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 67.25.3). By Morphisms of Spaces, Lemma 67.25.4 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 29.16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

06FY **Definition 101.18.2.** Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a *finite type point*[5] if the equivalent conditions of Lemma 101.18.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

06FZ **Lemma 101.18.3.** *Let $\mathcal{X}$ be an algebraic stack. We have*

$$\mathcal{X}_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to \mathcal{X} \text{ smooth}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 101.18.1. □

06G0 **Lemma 101.18.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{ft\text{-}pts}) \subset \mathcal{Y}_{ft\text{-}pts}$.*

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 101.17.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. □

06G1 **Lemma 101.18.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{ft\text{-}pts}) = \mathcal{Y}_{ft\text{-}pts}$.*

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 101.18.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 101.18.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $\mathrm{Spec}(k) \to \mathcal{Y}$ (Lemma 101.17.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 101.18.4 which maps to y by construction. □

06G2 **Lemma 101.18.6.** *Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have*

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{ft\text{-}pts}$ is dense in T.

[5] This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 100.10.5. An immersion is locally of finite type, see Lemma 101.17.4. Hence by Lemma 101.18.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 101.18.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

06G3 **Lemma 101.18.7.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *x is a finite type point,*
(2) *there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and*
(3) *the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.*

Proof. (All of the morphisms occurring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 100.3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 101.18.3. Set $u = \operatorname{Spec}(\kappa(u))$ as usual. Set $R = u \times_{\mathcal{X}} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 94.16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_{\mathcal{X}} u \to u \times_{\mathcal{X}} U \to u \times_{\mathcal{X}} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 67.23.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 67.28.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. By Algebraic Stacks, Lemma 94.16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 94.15.2 and a monomorphism, see Properties of Stacks, Lemma 100.8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 100.11.12. By construction the diagram

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is commutative. By Criteria for Representability, Lemma 97.17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$\begin{array}{ccccc} u \times_{\mathcal{X}} U & \longrightarrow & \mathcal{Z} \times_{\mathcal{X}} U & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ u & \longrightarrow & \mathcal{Z} & \xrightarrow{f} & \mathcal{X} \end{array}$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_{\mathcal{X}} U \to U$ is locally of finite type. Moreover, $u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is locally of finite type by Descent on Spaces, Lemma 74.16.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_{\mathcal{X}} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 101.18.4 and Lemma 101.18.6 to see that $\mathcal{Z}_{\text{ft-pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. □

101.19. Automorphism groups

0DTR Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. In this situation we often use the phrase "let G_x/k be the automorphism group algebraic space of x". This just means that

$$G_x = Isom_{\mathcal{X}}(x, x) = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$$

is the group algebraic space of automorphism of x. This is a group algebraic space over $\mathrm{Spec}(k)$. If k'/k is an extension of fields then the automorphism group algebraic space of the induced morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}$ is the base change of G_x to $\mathrm{Spec}(k')$.

0DTS **Lemma 101.19.1.** *In the situation above G_x is a scheme if one of the following holds*

(1) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is quasi-separated*
(2) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is locally separated,*
(3) $\mathcal{X}$ *is quasi-DM,*
(4) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is quasi-separated,*
(5) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is locally separated, or*
(6) $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is locally quasi-finite.*

Proof. Observe that (1) ⇒ (4), (2) ⇒ (5), and (3) ⇒ (6) by Lemma 101.6.1. In case (4) we see that G_x is a quasi-separated algebraic space and in case (5) we see that G_x is a locally separated algebraic space. In both cases G_x is a decent algebraic space (Decent Spaces, Section 68.6 and Lemma 68.15.2). Then G_x is separated by More on Groupoids in Spaces, Lemma 79.9.4 whereupon we conclude that G_x is a scheme by More on Groupoids in Spaces, Proposition 79.10.3. In case (6) we see that $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite and hence G_x is a scheme by Spaces over Fields, Lemma 72.10.8. □

0DTT **Lemma 101.19.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. Let P be a property of algebraic spaces over fields which is invariant under ground field extensions; for example $P(X/k) = X \to \mathrm{Spec}(k)$ is finite. The following are equivalent*

(1) *for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P, and*
(2) *for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P.*

Proof. Omitted. □

0DTU **Remark 101.19.3.** Let P be a property of algebraic spaces over fields which is invariant under ground field extensions. Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, we say the automorphism group of $\mathcal{X}$ at x has P if the equivalent conditions of Lemma 101.19.2 are satisfied. For example, we say *the automorphism group of $\mathcal{X}$ at x is finite*, if $G_x \to \mathrm{Spec}(k)$ is finite whenever $x : \mathrm{Spec}(k) \to \mathcal{X}$ is a representative of x. Similarly for smooth, proper, etc. (There is clearly an abuse of language going on here, but we believe it will not cause confusion or imprecision.)

0DTV **Lemma 101.19.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

(1) *for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism, and*
(2) *for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism.*

Proof. This comes down to the fact that being an isomorphism is fpqc local on the target, see Descent on Spaces, Lemma 74.11.15. Namely, suppose that k'/k is an extension of fields and denote $x' : \mathrm{Spec}(k') \to \mathcal{X}$ the composition and set $y' = f \circ x'$. Then the morphism $G_{x'} \to G_{y'}$ is the base change of $G_x \to G_y$ by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Hence $G_x \to G_y$ is an isomorphism if and only if $G_{x'} \to G_{y'}$ is an isomorphism. Thus we see that the property propagates through the equivalence class if it holds for one. □

0DTW **Remark 101.19.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. To indicate the equivalent conditions of Lemma 101.19.4 are satisfied for f and x in the literature the terminology *f is stabilizer preserving at x* or *f is fixed-point reflecting at x* is used. We prefer to say *f induces an isomorphism between automorphism groups at x and $f(x)$*.

101.20. Presentations and properties of algebraic stacks

0DTX Let (U, R, s, t, c) be a groupoid in algebraic spaces. If $s, t : R \to U$ are flat and locally of finite presentation, then the quotient stack $[U/R]$ is an algebraic stack, see Criteria for Representability, Theorem 97.17.2. In this section we study what properties of (U, R, s, t, c) imply for the algebraic stack $[U/R]$.

0DTY **Lemma 101.20.1.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above).*

(1) *If $R \to U \times U$ is separated, then $\Delta_{\mathcal{X}}$ is separated.*

(2) *If U, R are separated, then $\Delta_{\mathcal{X}}$ is separated.*
(3) *If $R \to U \times U$ is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*
(4) *If $s, t : R \to U$ are locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*
(5) *If $R \to U \times U$ is proper, then $\mathcal{X}$ is separated.*
(6) *If $s, t : R \to U$ are proper and U is separated, then $\mathcal{X}$ is separated.*
(7) *Add more here.*

Proof. Observe that the morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. Hence the same is true for $U \times U \to \mathcal{X} \times \mathcal{X}$. We have the cartesian diagram

$$\begin{array}{ccc} R = U \times_{\mathcal{X}} U & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

(see Groupoids in Spaces, Lemma 78.22.2). Thus we see that $\Delta_{\mathcal{X}}$ has one of the properties listed in Properties of Stacks, Section 100.3 if and only if the morphism $R \to U \times U$ does, see Properties of Stacks, Lemma 100.3.3. This explains why (1), (3), and (5) are true. The condition in (2) implies $R \to U \times U$ is separated hence (2) follows from (1). The condition in (4) implies the condition in (3) hence (4) follows from (3). The condition in (6) implies the condition in (5) by Morphisms of Spaces, Lemma 67.40.6 hence (6) follows from (5). □

0DTZ **Lemma 101.20.2.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above). Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The induced morphism $p : U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Criteria for Representability, Lemma 97.17.1. Hence $|U| \to |\mathcal{X}|$ is surjective by Properties of Stacks, Lemma 100.4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 78.22.2. Hence Properties of Stacks, Lemma 100.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. □

101.21. Special presentations of algebraic stacks

06MC In this section we prove two important theorems. The first is the characterization of quasi-DM stacks $\mathcal{X}$ as the stacks of the form $\mathcal{X} = [U/R]$ with $s, t : R \to U$ locally quasi-finite (as well as flat and locally of finite presentation). The second is the statement that DM algebraic stacks are Deligne-Mumford.

The following lemma gives a criterion for when a "slice" of a presentation is still flat over the algebraic stack.

06MD **Lemma 101.21.1.** *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields k'/k and a morphism $w : \mathrm{Spec}(k') \to W$ such that $\mathrm{Spec}(k') \to W \to \mathcal{X}$ is 2-isomorphic to $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$\begin{array}{ccccc} U & \xleftarrow{\mathrm{pr}_0} & U \times_{\mathcal{X}} W & \xleftarrow{p'} & F' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & W & \longleftarrow & \mathrm{Spec}(k') \end{array}$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 67.30.8. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 10.68.5. Note that $U \times_{\mathcal{X}} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 76.28.1 we see that there exists an open subspace U' of $U \times_{\mathcal{X}} W$ containing $p'(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W)$ is flat and locally of finite presentation over W. Note that $\mathrm{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 100.3.5 implies $\mathrm{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. □

06ME **Lemma 101.21.2.** *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 100.9.20 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 67.34.6. □

A quasi-DM stack has a locally quasi-finite "covering" by a scheme.

06MF **Theorem 101.21.3.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is quasi-DM, and*
(2) *there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 101.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 101.18.1. Denote $u = \mathrm{Spec}(\kappa(u))$ as usual. Consider the following commutative diagram

$$\begin{array}{ccc} u & \longleftarrow & R \\ \downarrow & & \downarrow \\ U & \xleftarrow[p]{} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & u \end{array}$$

with both squares fibre product squares, in particular $R = u \times_{\mathcal{X}} u$. In the proof of Lemma 101.18.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 78.16.2). Note that

$$G = R \times_{(u \times u)} u = (u \times_{\mathcal{X}} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 79.9.11 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. The composition $u \to R \to u$ is equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F; call it z. Consider the maps of étale local rings

$$\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\overline{z}} \longrightarrow \mathcal{O}_{R,\overline{e}}$$

Note that $\mathcal{O}_{R,\overline{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{z}}$ as R is cut out in F by $\mathfrak{m}_u$.

Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{z}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 72.16.1). Hence $\mathcal{O}_{F,\overline{z}}$ is a regular local ring (Properties of Spaces, Lemma 66.25.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3). Let $d = \dim(\mathcal{O}_{F,\overline{z}})$. By Algebra, Lemma 10.104.10 we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 101.21.1 after shrinking U we may assume that $Z = V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_{\mathcal{X}} u$ is a closed subspace of $F = U \times_{\mathcal{X}} u$, that z is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{z}}) = 0.$$

By Morphisms of Spaces, Lemma 67.34.1 it follows that $\dim_z(F_Z) = 0$ because the transcendence degree of z relative to u is zero. Hence it follows from Lemma 101.21.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 100.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 101.18.6 (and Properties of Stacks, Lemma 100.4.4). □

06N0 **Lemma 101.21.4.** *Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism $\mathrm{Spec}(k) \to \mathcal{Z}$.*

Proof. By Properties of Stacks, Lemma 100.11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 94.9.2. Note that by Algebraic Stacks, Remark 94.16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{Z}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_{\mathcal{Z}} U) = U \times_{\mathcal{Z} \times \mathcal{Z}, \Delta_{\mathcal{Z}}} \mathcal{Z}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 79.9.11 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 72.9.1. By Varieties, Lemma 33.20.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = \mathrm{Spec}(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{\mathrm{Spec}(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^\sharp : k \to A$ or $t^\sharp : k \to A$. Note that $s, t : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ are finite (by the lemma referenced above). Since $G \to \mathrm{Spec}(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = \mathrm{Spec}(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$ is universally injective (as P has only one point with residue field k), unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 41.7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 39.23.9 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = \mathrm{Spec}(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (39.23.0.1) that k_0 is a field.

We claim that

06N1 (101.21.4.1) $$\mathrm{Spec}(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{Z}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 100.11.1 that this morphism is surjective. Thus it suffices to show that (101.21.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 80.9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to \overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 78.25.2.

Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 35.23.17, 35.23.13, and 35.23.26. Consider the commutative diagram

$$\begin{array}{ccccc} U \times_{\overline{U}} U & = & P & \longrightarrow & R \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{U} & \xrightarrow{\overline{e}} & \overline{R} \end{array}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 35.23.18.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 40.4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 29.35.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 29.36.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conclude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 100.3.3. □

[6]We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 80.10.1 hence probably should be isolated into its own lemma somewhere.

06N2 **Lemma 101.21.5.** *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to \mathrm{Spec}(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 100.9.20 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to \mathrm{Spec}(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 67.39.12. □

A DM stack is a Deligne-Mumford stack.

06N3 **Theorem 101.21.6.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is DM,*
(2) *$\mathcal{X}$ is Deligne-Mumford, and*
(3) *there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.*

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 94.12.2. The implication (3) $\Rightarrow$ (1) is Lemma 101.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 101.18.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 101.4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 101.21.4 there exists a field k and a surjective étale morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}_x$. In particular the composition $x : \mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 67.23.2 and 67.39.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$\begin{array}{ccc} U & \longleftarrow & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \xleftarrow{x} & \mathrm{Spec}(k) \end{array}$$

in other words $F = U \times_{\mathcal{X},x} \mathrm{Spec}(k)$. By Properties of Stacks, Lemma 100.4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$\mathrm{Spec}(k) \times_{z,\mathcal{Z}_x,z} \mathrm{Spec}(k) = \mathrm{Spec}(k) \times_{x,\mathcal{X},x} \mathrm{Spec}(k)$$

with étale projection maps to $\mathrm{Spec}(k)$ by construction of z. Since

$$F \times_U F = (\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 67.39.11). By Morphisms of Spaces, Lemma 67.51.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 67.38.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{array}{ccccc} U & \longleftarrow & F & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ \mathcal{X} & \xleftarrow{x} & \mathrm{Spec}(k) & & \end{array}$$

Then $V \to \mathrm{Spec}(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 67.37.2 and 67.38.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 33.25.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 33.25.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 29.35.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 10.106.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 101.21.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \longleftarrow V(f_1, \ldots, f_d) \times_U V$$

is étale and $V(f_1, \ldots, f_d) \times_U V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to \mathrm{Spec}(k)$ is unramified at v, see Morphisms, Lemma 29.35.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is unramified at the image of v in $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k)$. Applying Lemma 101.21.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 100.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 101.18.6 (and Properties of Stacks, Lemma 100.4.4). $\square$

Here is a useful corollary which tells us that the "fibres" of a DM morphism of algebraic stacks are Deligne-Mumford.

0CIA **Lemma 101.21.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Then*

(1) *For every DM algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*
(2) *For every algebraic space Z and morphism $Z \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} Z$.*

Proof. Proof of (1). As f is DM we see that the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is DM by Lemma 101.4.4. Since $\mathcal{Z}$ is DM this implies that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is DM by Lemma 101.4.11. Hence there exists a scheme U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, see Theorem 101.21.6. Part (2) is a special case of (1) since an algebraic space (when viewed as an algebraic stack) is DM by Lemma 101.4.3. □

101.22. The Deligne-Mumford locus

0DSL Every algebraic stack has a largest open substack which is a Deligne-Mumford stack; this is more or less clear but we also write out the proof below. Of course this substack may be empty, for example if $X = [\mathrm{Spec}(\mathbf{Z})/\mathbf{G}_{m,\mathbf{Z}}]$. Below we will characterize the points of the DM locus.

0DSM **Lemma 101.22.1.** *Let $\mathcal{X}$ be an algebraic stack. There exist open substacks*

$$\mathcal{X}'' \subset \mathcal{X}' \subset \mathcal{X}$$

such that $\mathcal{X}''$ is DM, $\mathcal{X}'$ is quasi-DM, and such that these are the largest open substacks with these properties.

Proof. All we are really saying here is that if $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{X}$ are open substacks which are DM, then the open substack $\mathcal{W} \subset \mathcal{X}$ with $|\mathcal{W}| = |\mathcal{U}| \cup |\mathcal{V}|$ is DM as well. (Similarly for quasi-DM.) Although this is a cheat, let us use Theorem 101.21.6 to prove this. By that theorem we can choose schemes U and V and surjective étale morphisms $U \to \mathcal{U}$ and $V \to \mathcal{V}$. Then of course $U \amalg V \to \mathcal{W}$ is surjective and étale. The quasi-DM case is proven by exactly the same method using Theorem 101.21.3. □

0DSN **Lemma 101.22.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. Let G_x/k be the automorphism group algebraic space of x. Then*

(1) *x is in the DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is unramified, and*
(2) *x is in the quasi-DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. Proof of (2). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Consider the fibre product

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Recall that G is the automorphism group algebraic space of $U \to \mathcal{X}$. By Groupoids in Spaces, Lemma 78.6.3 there is a maximal open subscheme $U' \subset U$ such that $G_{U'} \to U'$ is locally quasi-finite. Moreover, formation of U' commutes with arbitrary base change. In particular the two inverse images of U' in $R = U \times_{\mathcal{X}} U$ are

the same open subspace of R (since after all the two maps $R \to \mathcal{X}$ are isomorphic and hence have isomorphic automorphism group spaces). Hence U' is the inverse image of an open substack $\mathcal{X}' \subset \mathcal{X}$ by Properties of Stacks, Lemma 100.9.11 and we have a cartesian diagram

$$\begin{array}{ccc} G_{U'} & \longrightarrow & \mathcal{I}_{\mathcal{X}'} \\ \downarrow & & \downarrow \\ U' & \longrightarrow & \mathcal{X}' \end{array}$$

Thus the morphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is locally quasi-finite and we conclude that $\mathcal{X}'$ is quasi-DM by Lemma 101.6.1 part (5). On the other hand, if $\mathcal{W} \subset \mathcal{X}$ is an open substack which is quasi-DM, then the inverse image $W \subset U$ of $\mathcal{W}$ must be contained in U' by our construction of U' since $\mathcal{I}_{\mathcal{W}} = \mathcal{W} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$ is locally quasi-finite over $\mathcal{W}$. Thus $\mathcal{X}'$ is the quasi-DM locus. Finally, choose a field extension K/k and a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow x \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Then we find an isomorphism $G_x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \cong G \times_U \mathrm{Spec}(K)$ of group algebraic spaces over K. Hence G_x is locally quasi-finite over k if and only if $\mathrm{Spec}(K) \to U$ maps into U' (use the commutation of formation of U' and Groupoids in Spaces, Lemma 78.6.3 applied to $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ and G_x to see this). This finishes the proof of (2). The proof of (1) is exactly the same. □

101.23. Locally quasi-finite morphisms

06PT The property "locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 101.16 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 100.3. To find a condition suitable for general morphisms we make the following observation.

06UA **Lemma 101.23.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

(1) *f is locally quasi-finite (as in Properties of Stacks, Section 100.3), and*
(2) *f is locally of finite type and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to \mathrm{Spec}(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 67.27.5. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 100.3.3. The morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $\mathrm{Spec}(k) \to V$ where k is a field

$$\mathrm{Spec}(k) \times_V (V \times_{\mathcal{Y}} \mathcal{X}) = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 67.27.5. □

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 101.4.3. Combined with the lemma above we see that the following definition does not conflict with the already existing notion in the case of morphisms representable by algebraic spaces.

06PU **Definition 101.23.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *locally quasi-finite* if f is quasi-DM, locally of finite type, and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [\mathrm{Spec}(k)/\mathbf{G}_m] \to \mathrm{Spec}(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 101.23.8).

Another justification for the definition above is Lemma 101.23.7 below which characterizes being locally quasi-finite in terms of the existence of suitable "presentations" or "coverings" of $\mathcal{X}$ and $\mathcal{Y}$.

06UB **Lemma 101.23.3.** *A base change of a locally quasi-finite morphism is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 101.4.4 and for locally finite type morphisms in Lemma 101.17.3. It is immediate that the condition on fibres is inherited by a base change. □

06UC **Lemma 101.23.4.** *Let $\mathcal{X} \to \mathrm{Spec}(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. By Lemma 101.17.2 we see that $V \to \mathrm{Spec}(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to \mathrm{Spec}(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \rightsquigarrow v'$ of points of V, see Morphisms, Lemma 29.20.6. In particular $\mathrm{trdeg}_k(\kappa(v)) > \mathrm{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 29.28.7. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_{\mathcal{X}} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 100.3). By Properties of Stacks, Lemma 100.4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 67.33.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. □

06UD **Lemma 101.23.5.** *A composition of a locally quasi-finite morphisms is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 101.4.10 and for locally finite type morphisms in Lemma 101.17.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be

locally quasi-finite. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 101.23.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to \mathrm{Spec}(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 101.4.13. By Theorem 101.21.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 101.23.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 33.20.2. Thus it suffices to show that each

$$|\mathrm{Spec}(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. □

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

06UE **Lemma 101.23.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is quasi-DM,*
(2) *for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and*
(3) *there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite.*

Proof. The implication (2) ⇒ (3) is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 101.4.4. By Lemma 101.4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM by Lemma 101.4.11. Hence we may apply Theorem 101.21.3 to get the morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 101.4.5. By Lemma 101.4.14 we see that $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM. Hence $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM by Lemma 101.4.13 and (1) holds. This finishes the proof of the lemma. □

06UF **Lemma 101.23.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,*
(3) *for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,*

(4) *there exists algebraic spaces U, V, a surjective, flat, and locally of finite presentation morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.*

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (2). By Lemma 101.23.5 the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 101.23.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 101.23.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally of finite type, see Lemma 101.17.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is locally of finite type, see Lemma 101.17.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism. Then $V \times_{\mathcal{Y}} \mathrm{Spec}(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension k'/k and a morphism $\mathrm{Spec}(k') \to V$ such that

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathcal{Y} \end{array}$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V \mathrm{Spec}(k') = U \times_{\mathcal{X} \times_{\mathcal{Y}} V} \mathcal{X}_{k'}$$

we see that $U \times_V \mathrm{Spec}(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$). Hence $|U \times_V \mathrm{Spec}(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V \mathrm{Spec}(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 67.27.5). □

06UG **Lemma 101.23.8.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.*

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 101.23.3. The first arrow is locally quasi-finite by Lemma 101.4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 101.23.5. □

101.24. Quasi-finite morphisms

0G2L We have defined "locally quasi-finite" morphisms of algebraic stacks in Section 101.23 and "quasi-compact" morphisms of algebraic stacks in Section 101.7. Since a morphism of algebraic spaces is by definition quasi-finite if and only if it is both locally quasi-finite and quasi-compact (Morphisms of Spaces, Definition 67.27.1), we may define what it means for a morphism of algebraic stacks to be quasi-finite as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

0G2M **Definition 101.24.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-finite* if f is locally quasi-finite (Definition 101.23.2) and quasi-compact (Definition 101.7.2). [**Ryd08**]

0G2N **Lemma 101.24.2.** *The composition of quasi-finite morphisms is quasi-finite.*

Proof. Combine Lemmas 101.23.5 and 101.7.4. □

0G2P **Lemma 101.24.3.** *A base change of a quasi-finite morphism is quasi-finite.*

Proof. Combine Lemmas 101.23.3 and 101.7.3. □

0G2Q **Lemma 101.24.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-finite and g is quasi-separated and quasi-DM then f is quasi-finite.*

Proof. Combine Lemmas 101.23.8 and 101.7.7. □

101.25. Flat morphisms

06PV The property "being flat" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.30.4 and Descent on Spaces, Lemma 74.11.13. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06PW **Definition 101.25.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *flat* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} = \text{flat}$.

06PX **Lemma 101.25.2.** *The composition of flat morphisms is flat.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.30.3. □

06PY **Lemma 101.25.3.** *A base change of a flat morphism is flat.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.30.4. □

06PZ **Lemma 101.25.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 67.37.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 100.5.2 and Lemma

101.25.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 101.25.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 67.31.3 (2) and we win. □

06Q0 **Lemma 101.25.5.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 67.31.5. □

0DN5 **Lemma 101.25.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let $\mathrm{Spec}(A) \to \mathcal{Y}$ be a morphism where A is a valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|\mathcal{Y}|$ in the image of $|\mathcal{X}| \to |\mathcal{Y}|$, then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathcal{Y} \end{array}$$

where $A \to A'$ is an extension of valuation rings (More on Algebra, Definition 15.124.1).

Proof. The base change $\mathcal{X}_A \to \mathrm{Spec}(A)$ is flat (Lemma 101.25.3) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|\mathcal{X}_A| \to |\mathrm{Spec}(A)|$ (Properties of Stacks, Lemma 100.4.3). Thus we may assume $\mathcal{Y} = \mathrm{Spec}(A)$. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Then we can apply Morphisms of Spaces, Lemma 67.42.4 to the morphism $U \to \mathrm{Spec}(A)$ to conclude. □

101.26. Flat at a point

0CIB We still have to develop the general machinery needed to say what it means for a morphism of algebraic stacks to have a given property at a point. For the moment the following lemma is sufficient.

0CIC **Lemma 101.26.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \quad \text{with points} \quad \begin{array}{c} u \in |U| \\ \downarrow \\ x \in |\mathcal{X}| \end{array}$$

where U and V are algebraic spaces, b is flat, and $(a, h) : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is flat. The following are equivalent

(1) *h is flat at u for one diagram as above,*
(2) *h is flat at u for every diagram as above.*

Proof. Suppose we are given a second diagram U', V', u', a', b', h' as in the lemma. Then we can consider

$$\begin{array}{ccccc} U & \longleftarrow & U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V \times_{\mathcal{Y}} V' & \longrightarrow & V' \end{array}$$

By Properties of Stacks, Lemma 100.4.3 there is a point $u'' \in |U \times_{\mathcal{X}} U'|$ mapping to u and u'. If h is flat at u, then the base change $U \times_V (V \times_{\mathcal{Y}} V') \to V \times_{\mathcal{Y}} V'$ is flat at any point over u, see Morphisms of Spaces, Lemma 67.31.3. On the other hand, the morphism

$$U \times_{\mathcal{X}} U' \to U \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V') = U \times_{\mathcal{Y}} V' = U \times_V (V \times_{\mathcal{Y}} V')$$

is flat as a base change of (a', h'), see Lemma 101.25.3. Composing and using Morphisms of Spaces, Lemma 67.31.4 we conclude that $U \times_{\mathcal{X}} U' \to V \times_{\mathcal{Y}} V'$ is flat at u''. Then we can use composition by the flat map $V \times_{\mathcal{Y}} V' \to V'$ to conclude that $U \times_{\mathcal{X}} U' \to V'$ is flat at u''. Finally, since $U \times_{\mathcal{X}} U' \to U'$ is flat at u'' and u'' maps to u' we conclude that $U' \to V'$ is flat at u' by Morphisms of Spaces, Lemma 67.31.5. □

0CID **Definition 101.26.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$. We say f is *flat at x* if the equivalent conditions of Lemma 101.26.1 hold.

101.27. Morphisms of finite presentation

06Q1 The property "locally of finite presentation" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.28.3 and Descent on Spaces, Lemma 74.11.10. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic stacks to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

06Q2 **Definition 101.27.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite presentation* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ locally of finite presentation.
(2) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

06Q3 **Lemma 101.27.2.** *The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.28.2. □

06Q4 **Lemma 101.27.3.** *A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.28.3. □

06Q5 **Lemma 101.27.4.** *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Combine Remark 101.16.5 with Morphisms of Spaces, Lemma 67.28.5. □

0DQJ **Lemma 101.27.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If $\mathcal{Y}$ is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If $\mathcal{Y}$ is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : \mathcal{X} \to \mathcal{Y}$ locally of finite type and $\mathcal{Y}$ locally Noetherian. This means there exists a diagram as in Lemma 101.16.1 with h locally of finite type and surjective vertical arrow a. By Morphisms of Spaces, Lemma 67.28.7 h is locally of finite presentation. Hence $\mathcal{X} \to \mathcal{Y}$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

06Q6 **Lemma 101.27.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 67.28.9 to the morphisms $U \to V \to W$. □

0CMG **Lemma 101.27.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks with diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism over $\mathcal{X}$ (via the second projection). If f is locally of finite type, then $\mathcal{X}$ is of finite presentation over $\mathcal{X}$ and $\mathrm{pr}_2 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is locally of finite type by Lemma 101.17.3. Thus the first statement holds by Lemma 101.27.6. The second statement follows from the first and the definitions (because f being quasi-separated means by definition that Δ_f is quasi-compact and quasi-separated). □

06Q7 **Lemma 101.27.8.** *An open immersion is locally of finite presentation.*

Proof. In view of Properties of Stacks, Definition 100.9.1 this follows from Morphisms of Spaces, Lemma 67.28.11. □

0CPP **Lemma 101.27.9.** *Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

For the meaning of this statement see Properties of Stacks, Section 100.3.

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2 the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Properties of Stacks, Lemma 100.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. □

0DN6 **Lemma 101.27.10.** *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and fppf local on the target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. Assume $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Observe that $W \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = W \times_{\mathcal{Y}} \mathcal{X}$. Thus by the very definition of what it means for $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ to have $\mathcal{P}$ (see Definition 101.16.2 and Lemma 101.16.1) we see that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has $\mathcal{P}$. On the other hand, $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 67.37.7 and 67.37.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2). Thus we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ has $\mathcal{P}$. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by Lemma 101.16.1 used in the definition of $\mathcal{X}' \to \mathcal{Y}' = W$ having $\mathcal{P}$ we see that $U' \to V'$ has $\mathcal{P}$. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is local in the fppf topology on the target. □

06Q8 **Lemma 101.27.11.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.*

Proof. The property "locally of finite presentation" satisfies the conditions of Lemma 101.27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 74.11.10. □

06Q9 **Lemma 101.27.12.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 74.16.1. □

06QA **Lemma 101.27.13.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.*

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 100.5.3 and Lemmas 101.27.3 and 101.25.3 we see that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_{\mathcal{Y}} U$ smooth and surjective. Then $V \to \mathcal{X} \times_{\mathcal{Y}} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 67.37.7 and 67.37.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 100.5.2 and Lemmas 101.27.2, and 101.25.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. □

07AN **Lemma 101.27.14.** *Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i \in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.*

Proof. Apply Lemma 101.27.13 to the morphism $\coprod_{j \in J} \mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here – due to our setup of things – which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j \in J} \mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. □

06R7 **Lemma 101.27.15.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.*

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 29.25.10. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. □

0DQK **Lemma 101.27.16.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, then f is quasi-compact.*

Proof. We have to show that given $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'$ quasi-compact, we have $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Denote $\mathcal{Z}' = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}'$. Then $|\mathcal{Z}'| \to |\mathcal{Y}'|$ is open, see Lemma 101.27.15. Hence we can find a quasi-compact open substack $\mathcal{W} \subset \mathcal{Z}'$ mapping onto $\mathcal{Y}'$. Because $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, we know that

$$\mathcal{W} \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = \mathcal{W} \times_{\mathcal{Y}} \mathcal{X}$$

is quasi-compact. And the map $\mathcal{W} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, hence we win. Some details omitted. □

0CPQ **Lemma 101.27.17.** *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be composable morphisms of algebraic stacks with composition $h = g \circ f : \mathcal{X} \to \mathcal{Z}$. If f is surjective, flat, locally of finite presentation, and universally injective and if h is separated, then g is separated.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Z}} \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{Y} & \longrightarrow & \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y} \end{array}$$

The square is cartesian. We have to show the bottom horizontal arrow is proper. We already know that it is representable by algebraic spaces and locally of finite type (Lemma 101.3.3). Since the right vertical arrow is surjective, flat, and locally of finite presentation it suffices to show the top right horizontal arrow is proper (Lemma 101.27.9). Since h is separated, the composition of the top horizontal arrows is proper.

Since f is universally injective Δ is surjective (Lemma 101.14.5). Since the composition of Δ with the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is the identity, we see that Δ is universally closed. By Morphisms of Spaces, Lemma 67.9.8 we conclude that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated. Here we use that implications between properties of morphisms of algebraic spaces can be transferred to the same implications between properties of morphisms of algebraic stacks representable by algebraic spaces; this is discussed in Properties of Stacks, Section 100.3. Finally, we use the same principle to conlude that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is proper from Morphisms of Spaces, Lemma 67.40.7. □

101.28. Gerbes

06QB An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 97.18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 101.28.7. In this section we briefly discuss this notion and the corresponding relative notion.

06QC **Definition 101.28.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 8.11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 101.28.8. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

06QD **Lemma 101.28.2.** *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf*

$$(Sch/S)^{opp}_{fppf} \to Sets, \quad U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 101.2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 8.11.3 properties (2)(a) and (2)(b) to see that the map $c^\# : \mathcal{F}^\# \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 7.11.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^\#$ over T as desired. □

06QE **Lemma 101.28.3.** *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 8.11.5. □

06R8 **Lemma 101.28.4.** *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Immediate from Stacks, Lemma 8.11.6. □

06QF **Lemma 101.28.5.** *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 101.27.13 and Stacks, Lemma 8.11.7. □

06QG **Lemma 101.28.6.** *Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = Isom_{\mathcal{X}}(x, x)$, see Definition 101.5.3. If $\mathcal{X}$ is a gerbe over U, then*

(1) *there is a canonical equivalence of stacks in groupoids*
$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$
where $[U/G]$ is the quotient stack for the trivial action of G on U,
(2) *$G \to U$ is flat and locally of finite presentation, and*
(3) *$U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.*

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because
$$\begin{aligned} G &= Isom_{\mathcal{X}}(x, x) \\ &= U \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}} \\ &= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\ &= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\ &= R \times_{U\times U,\Delta_U} U \\ &= R \end{aligned}$$
for the fourth equality use Categories, Lemma 4.31.12. Let $t, s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 94.16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 94.16.1 shows we obtain a canonical fully faithful 1-morphism
$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$
of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 8.4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over U, see property (2)(a) of Stacks, Lemma 8.11.3.

By Criteria for Representability, Lemma 97.18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. □

06QH **Lemma 101.28.7.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
(2) *there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} U \cong [U/G]$ over U.*

Proof. Assume (2). By Lemma 101.28.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 78.27.2.

Assume (1). Any base change of π is a gerbe, see Lemma 101.28.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks,

Lemma 8.11.3 (2)(a). Note that $U = \coprod V_i \to V$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 101.28.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. □

06QI **Lemma 101.28.8.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.*

Proof. By Properties of Stacks, Lemma 100.5.4 and Lemmas 101.25.4 and 101.27.11 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 101.28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 101.28.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 100.5.5 and Lemmas 101.25.5 and 101.27.12. □

06QJ **Proposition 101.28.9.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe, and*
(2) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation.*

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ be a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 101.28.3. Then both squares in

$$\begin{array}{ccccc} \mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_\mathcal{X} & \longrightarrow & \mathcal{X} & \longrightarrow & X \end{array}$$

are fibre product squares, see Lemma 101.5.5. Hence to prove $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 101.25.4 and 101.27.11. Thus we can apply Lemma 101.28.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 101.28.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_\mathcal{X}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 101.25.4 and 101.27.11.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 94.16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 78.16.2. By Lemma 101.5.7 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemmas 101.25.3 and 101.27.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 80.11.7. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally

of finite presentation, see Morphisms of Spaces, Lemmas 67.31.5 and Descent on Spaces, Lemma 74.16.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 78.11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 80.10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 78.27.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 80.4.6 to see this implies the required hypothesis). □

0CPR **Lemma 101.28.10.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which makes $\mathcal{X}$ a gerbe over $\mathcal{Y}$. Then*

(1) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is flat and locally of finite presentation,*
(2) *$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, flat, and locally of finite presentation,*
(3) *given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism*

$$T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \longrightarrow T_1 \times_{y_1, \mathcal{Y}, y_2} T_2$$

is surjective, flat, and locally of finite presentation,
(4) *given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism*

$$Isom_{\mathcal{X}}(x_1, x_2) \longrightarrow Isom_{\mathcal{Y}}(y_1, y_2)$$

is surjective, flat, and locally of finite presentation.

Proof. Proof of (1). Choose a scheme Y and a surjective smooth morphism $Y \to \mathcal{Y}$. Set $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} Y$. By Lemma 101.5.5 we obtain cartesian squares

$$\begin{array}{ccccc} \mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

By Lemmas 101.25.4 and 101.27.11 it suffices to prove that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is flat and locally of finite presentation. This follows from Proposition 101.28.9 (because $\mathcal{X}'$ is a gerbe over Y by Lemma 101.28.3).

Proof of (2). With notation as above, note that we may assume that $\mathcal{X}' = [Y/G]$ for some group algebraic space G flat and locally of finite presentation over Y, see Lemma 101.28.7. The base change of the morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over $\mathcal{Y}$ by the morphism $Y \to \mathcal{Y}$ is the morphism $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_Y \mathcal{X}'$. Hence it suffices to show

that Δ' is surjective, flat, and locally of finite presentation (see Lemmas 101.25.4 and 101.27.11). In other words, we have to show that

$$[Y/G] \longrightarrow [Y/G \times_Y G]$$

is surjective, flat, and locally of finite presentation. This is true because the base change by the surjective, flat, locally finitely presented morphism $Y \to [Y/G \times_Y G]$ is the morphism $G \to Y$.

Proof of (3). Observe that the diagram

$$\begin{array}{ccc} T_1 \times_{x_1,\mathcal{X},x_2} T_2 & \longrightarrow & T_1 \times_{y_1,\mathcal{Y},y_2} T_2 \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is cartesian. Hence (3) follows from (2).

Proof of (4). This is true because

$$Isom_{\mathcal{X}}(x_1, x_2) = (T \times_{x_1,\mathcal{X},x_2} T) \times_{T \times T, \Delta_T} T$$

hence the morphism in (4) is a base change of the morphism in (3). □

0CPS **Proposition 101.28.11.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
(2) *$f : \mathcal{X} \to \mathcal{Y}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are surjective, flat, and locally of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemmas 101.28.8 and 101.28.10.

Assume (2). It suffices to prove (1) for the base change of f by a surjective, flat, and locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$, see Lemma 101.28.5 (note that the base change of the diagonal of f is the diagonal of the base change). Thus we may assume $\mathcal{Y}$ is a scheme Y. In this case $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is a base change of Δ and we conclude that $\mathcal{X}$ is a gerbe by Proposition 101.28.9. We still have to show that $\mathcal{X}$ is a gerbe over Y. Let $\mathcal{X} \to X$ be the morphism of Lemma 101.28.2 turning $\mathcal{X}$ into a gerbe over the algebraic space X classifying isomorphism classes of objects of $\mathcal{X}$. It is clear that $f : \mathcal{X} \to Y$ factors as $\mathcal{X} \to X \to Y$. Since f is surjective, flat, and locally of finite presentation, we conclude that $X \to Y$ is surjective as a map of fppf sheaves (for example use Lemma 101.27.13). On the other hand, $X \to Y$ is injective too: for any scheme T and any two T-valued points x_1, x_2 of X which map to the same point of Y, we can first fppf locally on T lift x_1, x_2 to objects ξ_1, ξ_2 of $\mathcal{X}$ over T and second deduce that ξ_1 and ξ_2 are fppf locally isomorphic by our assumption that $\Delta : \mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is surjective, flat, and locally of finite presentation. Whence $x_1 = x_2$ by construction of X. Thus $X = Y$ and the proof is complete. □

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

06QK **Lemma 101.28.12.** *Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.*

Proof. By Properties of Stacks, Lemma 100.11.3 there exists a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_Z \times_{\mathcal{Z}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 101.5.1 and 101.17.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 67.28.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 101.25.4 and 101.27.11 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$ is a gerbe by Proposition 101.28.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 101.28.8. Hence $\mathrm{Spec}(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 100.5.2 and Lemmas 101.25.2 and 101.27.2. Hence by Properties of Stacks, Lemma 100.11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. □

06R9 **Lemma 101.28.13.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then f is a universal homeomorphism.*

Proof. By Lemma 101.28.3 the assumption on f is preserved under base change. Hence it suffices to show that the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces. Let k be a field and let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(k)$. By Stacks, Lemma 8.11.3 property (2)(a) there exists an fppf covering $\{T_i \to \mathrm{Spec}(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $\mathrm{Spec}(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $\mathrm{Spec}(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 8.11.3 property (2)(b) guarantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 101.28.8 and 101.27.15. □

0DQL **Lemma 101.28.14.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks such that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. If $\Delta_{\mathcal{X}}$ is quasi-compact, so is $\Delta_{\mathcal{Y}}$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \\ & & \downarrow & & \downarrow \\ & & \mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathcal{Y} \end{array}$$

By Proposition 101.28.11 we find that the arrow on the top left is surjective. Since the composition of the top horizontal arrows is quasi-compact, we conclude that the top right arrow is quasi-compact by Lemma 101.7.6. The square is cartesian and the right vertical arrow is surjective, flat, and locally of finite presentation. Thus we conclude by Lemma 101.27.16. □

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

06RA **Lemma 101.28.15.** *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.*

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 68.13.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 101.28.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 101.28.13) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 101.28.8 and 101.27.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 101.17.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to $\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). □

101.29. Stratification by gerbes

06RB The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a "stratification" by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup\nolimits_i |\mathcal{X}_i|$$

is a stratification of the topological space associated to $\mathcal{X}$ and nothing more (in this section). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 100.10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

06RC **Proposition 101.29.1.** *Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.*

Proof. According to Proposition 101.28.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 101.5.5.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 101.5.7 we see that $G \to U$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 101.5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 67.32.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 79.6.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 100.9.11. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 101.25.4 and 101.27.11. □

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 101.31.1.

It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

06RE **Example 101.29.2.** Let k be a field. Take $U = \mathrm{Spec}(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

(1) $\mathcal{X}_0$ is where x_0 is not zero,
(2) $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,
(3) $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
(4) and so on, and
(5) $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 101.29.1 find possibly infinite stratifications by gerbes...!

06RF **Lemma 101.29.3.** *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well-ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that*

(1) *each $\mathcal{U}_i$ is a gerbe,*
(2) *we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,*
(3) *$T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and*
(4) *$|\mathcal{U}_i|$ is open in T_i.*

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 100.10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 101.5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 101.7.3. Therefore Proposition 101.29.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 100.4.9 (we can do this for all T simultaneously by the axiom of choice).

Using transfinite recursion we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap\nolimits_{\alpha < \beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well-ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. □

06RG **Remark 101.29.4.** We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 101.29.3. A natural guess is that the well-ordered set I has *cardinality* at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

101.30. The topological space of an algebraic stack

0DQM In this section we apply the previous results to the topological space $|\mathcal{X}|$ associated to an algebraic stack.

0DQN **Lemma 101.30.1.** *Let $\mathcal{X}$ be a quasi-compact algebraic stack whose diagonal Δ is quasi-compact. Then $|\mathcal{X}|$ is a spectral topological space.*

Proof. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$, see Properties of Stacks, Lemma 100.6.2. Then $|U| \to |\mathcal{X}|$ is continuous, open, and surjective, see Properties of Stacks, Lemma 100.4.7. Hence the quasi-compact opens of $|\mathcal{X}|$ form a basis for the topology. For $W_1, W_2 \subset |\mathcal{X}|$ quasi-compact open, we may choose a quasi-compact opens V_1, V_2 of U mapping to W_1 and W_2. Since Δ is quasi-compact, we see that

$$V_1 \times_{\mathcal{X}} V_2 = (V_1 \times V_2) \times_{\mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

is quasi-compact. Then image of $|V_1 \times_{\mathcal{X}} V_2|$ in $|\mathcal{X}|$ is $W_1 \cap W_2$ by Properties of Stacks, Lemma 100.4.3. Thus $W_1 \cap W_2$ is quasi-compact. To finish the proof, it suffices to show that $|\mathcal{X}|$ is sober, see Topology, Definition 5.23.1.

Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a unique generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 100.10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\Delta_{\mathcal{Z}}$ is quasi-compact: first show that $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact as the composition of $\mathcal{Z} \to \mathcal{X}$ with Δ, then write $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ as the composition of $\Delta_{\mathcal{Z}}$ and $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ and use Lemma 101.7.7 and the fact that $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is separated. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is quasi-compact, Δ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a unique generic point. Since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is a base change of Δ, we see that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is quasi-compact (Lemma 101.7.3). Thus there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe by Proposition 101.29.1. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 101.28.13. In particular, X is quasi-compact. By Lemma 101.28.14 we see that X has quasi-compact diagonal, i.e., X is a quasi-separated algebraic space.

Then $|X|$ is spectral by Properties of Spaces, Lemma 66.15.2 which implies what we want is true. □

0DQP **Lemma 101.30.2.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then $|\mathcal{X}|$ is a spectral topological space.*

Proof. This is a special case of Lemma 101.30.1. □

0DQQ **Lemma 101.30.3.** *Let $\mathcal{X}$ be an algebraic stack whose diagonal is quasi-compact (for example if $\mathcal{X}$ is quasi-separated). Then there is an open covering $|\mathcal{X}| = \bigcup U_i$ with U_i spectral. In particular $|\mathcal{X}|$ is a sober topological space.*

Proof. Immediate consequence of Lemma 101.30.1. □

101.31. Existence of residual gerbes

06UH The definition of a residual gerbe of a point on an algebraic stack is Properties of Stacks, Definition 100.11.8. We have already shown that residual gerbes exist for finite type points (Lemma 101.18.7) and for any point of a gerbe (Lemma 101.28.15). In this section we prove that residual gerbes exist on many algebraic stacks. First, here is the promised application of Proposition 101.29.1.

06RD **Lemma 101.31.1.** *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 100.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 101.5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 101.7.3. Therefore Proposition 101.29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 101.28.15. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. □

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Deligne-Mumford stacks.

06UI **Lemma 101.31.2.** *Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 101.21.3. Set $R = U \times_\mathcal{X} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 94.16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 94.16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \operatorname{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_\mathcal{X} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 72.10.9 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 67.28.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 78.25.1 and Properties of Stacks, Lemma 100.8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 100.8.5. By Properties of Stacks, Lemma 100.4.3 we see that

$$|U'| = |\mathcal{Z} \times_{\mathcal{X}} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 100.11.12. □

0H22 **Lemma 101.31.3.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose an affine scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of the open continuous map $|U| \to |\mathcal{X}|$. We may and do replace $\mathcal{X}$ with the open substack corresponding to the image of $|U| \to |\mathcal{X}|$, see Properties of Stacks, Lemma 100.9.12. Thus we may assume $\mathcal{X} = [U/R]$ for a smooth groupoid (U, R, s, t, c) in algebraic spaces where U is a Noetherian affine scheme, see Algebraic Stacks, Section 94.16.

Let $E \subset |U|$ be the inverse image of $\{x\} \subset |\mathcal{X}|$. Of course $E \neq \emptyset$. Since $|U|$ is a Noetherian topological space, we can choose an element $u \in E$ such that $\overline{\{u\}} \cap E = \{u\}$. As usual, we think of $u = \mathrm{Spec}(\kappa(u))$ as the spectrum of its residue field. Let us write

$$F = u \times_{U,t} R = u \times_{\mathcal{X}} U \quad \text{and} \quad R' = (u \times u) \times_{(U \times U),(t,s)} R = u \times_{\mathcal{X}} u$$

Furthermore, denote $Z = \overline{\{u\}} \subset U$ with the reduced induced scheme structure. Denote $p : F \to U$ the morphism induced by the second projection (using $s : R \to U$ in the first fibre product description of F). Then E is the set theoretic image of p. The morphism $R' \to F$ is a monomorphism which factors through the inverse image $p^{-1}(Z)$ of Z. This inverse image $p^{-1}(Z) \subset F$ is a closed subscheme and the restriction $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ has image set theoretically contained in $\{u\} \subset Z$ by our careful choice of $u \in E$ above. Since $u = \lim W$ where the limit is over the nonempty affine open subschemes of the irreducible reduced scheme Z, we conclude that the morphism $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ factors through the morphism $u \to Z$. Clearly this implies that $R' = p^{-1}(Z)$. In particular the morphism $t' : R' \to u$ is locally of finite presentation as the composition of the closed immersion $p^{-1}(Z) \to F$ of locally Noetherian algebraic spaces with the smooth morphism $\mathrm{pr}_1 : F \to u$; use Morphisms of Spaces, Lemmas 67.23.5, 67.28.12, and 67.28.2. Hence the restriction

(u, R', s', t', c') of (U, R, s, t, c) by $u \to U$ is a groupoid in algebraic spaces where s' and t' are flat and locally of finite presentation. Therefore $\mathcal{Z} = [u/R']$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. As R' is the restriction of R by $u \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 78.25.1 and Properties of Stacks, Lemma 100.8.4. Then $\mathcal{Z}$ is (isomorphic to) the residual gerbe by the material in Properties of Stacks, Section 100.11. □

101.32. Étale local structure

0DU0 In this section we start discussing what we can say about the étale local structure of an algebraic stack.

0DU1 **Lemma 101.32.1.** *Let Y be an algebraic space. Let (U, R, s, t, c) be a groupoid in algebraic spaces over Y. Assume $U \to Y$ is flat and locally of finite presentation and $R \to U \times_Y U$ an open immersion. Then $X = [U/R] = U/R$ is an algebraic space and $X \to Y$ is étale.*

Proof. The quotient stack $[U/R]$ is an algebraic stacks by Criteria for Representability, Theorem 97.17.2. On the other hand, since $R \to U \times U$ is a monomorphism, it is an algebraic space (by our abuse of language and Algebraic Stacks, Proposition 94.13.3) and of course it is equal to the algebraic space U/R (of Bootstrap, Theorem 80.10.1). Since $U \to X$ is surjective, flat, and locally of finite presentation (Bootstrap, Lemma 80.11.6) we conclude that $X \to Y$ is flat and locally of finite presentation by Morphisms of Spaces, Lemma 67.31.5 and Descent on Spaces, Lemma 74.8.2. Finally, consider the cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & U \times_Y U \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_Y X \end{array}$$

Since the right vertical arrow is surjective, flat, and locally of finite presentation (small detail omitted), we find that $X \to X \times_Y X$ is an open immersion as the top horizontal arrow has this property by assumption (use Properties of Stacks, Lemma 100.3.3). Thus $X \to Y$ is unramified by Morphisms of Spaces, Lemma 67.38.9. We conclude by Morphisms of Spaces, Lemma 67.39.12. □

0DU2 **Lemma 101.32.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $P \subset R$ be an open subspace such that $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over S. Then*

$$[U/P] \longrightarrow [U/R]$$

is a morphism of algebraic stacks which is representable by algebraic spaces, surjective, and étale.

Proof. Since $P \subset R$ is open, we see that $s|_P$ and $t|_P$ are flat and locally of finite presentation. Thus $[U/R]$ and $[U/P]$ are algebraic stacks by Criteria for Representability, Theorem 97.17.2. To see that the morphism is representable by algebraic spaces, it suffices to show that $[U/P] \to [U/R]$ is faithful on fibre categories, see Algebraic Stacks, Lemma 94.15.2. This follows immediately from the fact that $P \to R$ is a monomorphism and the explicit description of quotient stacks given in Groupoids in Spaces, Lemma 78.24.1. Having said this, we know what it

means for $[U/P] \to [U/R]$ to be surjective and étale by Algebraic Stacks, Definition 94.10.1. Surjectivity follows for example from Criteria for Representability, Lemma 97.7.3 and the description of objects of quotient stacks (see lemma cited above) over spectra of fields. It remains to prove that our morphism is étale.

To do this it suffices to show that $U \times_{[U/R]} [U/P] \to U$ is étale, see Properties of Stacks, Lemma 100.3.3. By Groupoids in Spaces, Lemma 78.21.2 the fibre product is equal to $[R/P \times_{s,U,t} R]$ with morphism to U induced by $s : R \to U$. The maps $s', t' : P \times_{s,U,t} R \to R$ are given by $s' : (p, r) \mapsto r$ and $t' : (p, r) \mapsto c(p, r)$. Since $P \subset R$ is open we conclude that $(t', s') : P \times_{s,U,t} R \to R \times_{s,U,s} R$ is an open immersion. Thus we may apply Lemma 101.32.1 to conclude. □

0DU3 **Lemma 101.32.3.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Then there exists a morphism of algebraic stacks*

$$\mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

1. *there exists a point $u \in |\mathcal{U}|$ mapping to x,*
2. *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
3. *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. (The parenthetical statement follows from the equivalences in Lemma 101.6.1). Choose an affine scheme U and a flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X}$ such that x is the image of some point $u \in U$. This is possible by Theorem 101.21.3 and the assumption that $\mathcal{X}$ is quasi-DM. Let (U, R, s, t, c) be the groupoid in algebraic spaces we obtain by setting $R = U \times_\mathcal{X} U$, see Algebraic Stacks, Lemma 94.16.1. Let $\mathcal{X}' \subset \mathcal{X}$ be the open substack corresponding to the open image of $|U| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 100.4.7 and 100.9.12). Clearly, we may replace $\mathcal{X}$ by the open substack $\mathcal{X}'$. Thus we may assume $U \to \mathcal{X}$ is surjective and then Algebraic Stacks, Remark 94.16.3 gives $\mathcal{X} = [U/R]$. Observe that $s, t : R \to U$ are flat, locally of finite presentation, and locally quasi-finite. Since $R = U \times U \times_{(\mathcal{X} \times \mathcal{X})} \mathcal{X}$ and since the diagonal of $\mathcal{X}$ is separated, we find that R is separated. Hence $s, t : R \to U$ are separated. It follows that R is a scheme by Morphisms of Spaces, Proposition 67.50.2 applied to $s : R \to U$.

Above we have verified all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13 are satisfied for (U, R, s, t, c) and u. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is quasi-split over u. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a quasi-splitting over u (the point u is irrelevant from now on as can be seen from the footnote in More on Groupoids in Spaces, Definition 79.15.1). Let $P \subset R$ be a quasi-splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

has all the desired properties. □

0DU4 **Lemma 101.32.4.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let*

$x \in |\mathcal{X}|$. Assume the automorphism group of $\mathcal{X}$ at x is finite (Remark 101.19.3). Then there exists a morphism of algebraic stacks

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

(1) *there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between automorphism groups at u and x (Remark 101.19.5),*
(2) *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
(3) *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. Observe that G_x is a group scheme by Lemma 101.19.1. The first part of the proof is **exactly** the same as the first part of the proof of Lemma 101.32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13. Our assumption on G_x implies that G_u is finite over u. Hence all the assumptions of More on Groupoids in Spaces, Lemma 79.15.12 are satisfied. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is split over u. Note that $R' = U' \times_{\mathcal{X}} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a splitting over u. Let $P \subset R$ be a splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. By construction G_u is contained in P, hence this morphism defines an isomorphism on automorphism groups at u as desired. □

0DU5 **Lemma 101.32.5.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Assume x can be represented by a quasi-compact morphism $\mathrm{Spec}(k) \to \mathcal{X}$. Then there exists a morphism of algebraic stacks*

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

(1) *there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between the residual gerbes at u and x,*
(2) *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
(3) *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. The first part of the proof is **exactly** the same as the first part of the proof of Lemma 101.32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 79.15.13. Observe that $u = \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ is quasi-compact, see Properties of Stacks, Lemma 100.14.1. Consider the cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & U \\ \downarrow & & \downarrow \\ u & \xrightarrow{u} & \mathcal{X} \end{array}$$

Since U is an affine scheme and $F \to U$ is quasi-compact, we see that F is quasi-compact. Since $U \to \mathcal{X}$ is locally quasi-finite, we see that $F \to u$ is locally quasi-finite. Hence $F \to u$ is quasi-finite and F is an affine scheme whose underlying topological space is finite discrete (Spaces over Fields, Lemma 72.10.8). Observe that we have a monomorphism $u \times_{\mathcal{X}} u \to F$. In particular the set $\{r \in R : s(r) = u, t(r) = u\}$ which is the image of $|u \times_{\mathcal{X}} u| \to |R|$ is finite. we conclude that all the assumptions of More on Groupoids in Spaces, Lemma 79.15.11 hold.

Thus we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is strongly split over u'. Note that $R' = U' \times_{\mathcal{X}} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a strong splitting over u. Let $P \subset R$ be a strong splitting of R over u. Apply Lemma 101.32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. Since $P \subset R$ is open and contains $\{r \in R : s(r) = u, t(r) = u\}$ by construction we see that $u \times_{\mathcal{U}} u \to u \times_{\mathcal{X}} u$ is an isomorphism. The statement on residual gerbes then follows from Properties of Stacks, Lemma 100.11.14 (we observe that the residual gerbes in question exist by Lemma 101.31.2). □

101.33. Smooth morphisms

075T The property "being smooth" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 74.20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.37.3 and Descent on Spaces, Lemma 74.11.26. Hence, by Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces.

075U **Definition 101.33.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ smooth.

075V **Lemma 101.33.2.** *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 101.16.3 with Morphisms of Spaces, Lemma 67.37.2. □

075W **Lemma 101.33.3.** *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 101.16.4 with Morphisms of Spaces, Lemma 67.37.3. □

0DN7 **Lemma 101.33.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is smooth, then f is smooth.*

Proof. The property "smooth" satisfies the conditions of Lemma 101.27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 74.11.26. □

0DNP **Lemma 101.33.5.** *A smooth morphism of algebraic stacks is locally of finite presentation.*

Proof. Omitted. □

0DZR **Lemma 101.33.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. There is a maximal open substack $\mathcal{U} \subset \mathcal{X}$ such that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ is smooth. Moreover, formation of this open commutes with*

(1) *precomposing by smooth morphisms,*
(2) *base change by morphisms which are flat and locally of finite presentation,*
(3) *base change by flat morphisms provided f is locally of finite presentation.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where U and V are algebraic spaces, the vertical arrows are smooth, and $a : U \to \mathcal{X}$ surjective. There is a maximal open subspace $U' \subset U$ such that $h_{U'} : U' \to V$ is smooth, see Morphisms of Spaces, Lemma 67.37.9. Let $\mathcal{U} \subset \mathcal{X}$ be the open substack corresponding to the image of $|U'| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 100.4.7 and 100.9.12). By the equivalence in Lemma 101.16.1 we find that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ is smooth and that $\mathcal{U}$ is the largest open substack with this property.

Assertion (1) follows from the fact that being smooth is smooth local on the source (this property was used to even define smooth morphisms of algebraic stacks). Assertions (2) and (3) follow from the case of algebraic spaces, see Morphisms of Spaces, Lemma 67.37.9. □

0DLS **Lemma 101.33.7.** *Let $X \to Y$ be a smooth morphism of algebraic spaces. Let G be a group algebraic space over Y which is flat and locally of finite presentation over Y. Let G act on X over Y. Then the quotient stack $[X/G]$ is smooth over Y.*

This holds even if G is not smooth over Y!

Proof. The quotient $[X/G]$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2. The smoothness of $[X/G]$ over Y follows from the fact that smoothness descends under fppf coverings: Choose a surjective smooth morphism $U \to [X/G]$ where U is a scheme. Smoothness of $[X/G]$ over Y is equivalent to smoothness of U over Y. Observe that $U \times_{[X/G]} X$ is smooth over X and hence smooth over Y (because compositions of smooth morphisms are smooth). On the other hand, $U \times_{[X/G]} X \to U$ is locally of finite presentation, flat, and surjective (because it is the base change of $X \to [X/G]$ which has those properties for example by Criteria for Representability, Lemma 97.17.1). Therefore we may apply Descent on Spaces, Lemma 74.8.4. □

0DN8 **Lemma 101.33.8.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective and smooth.*

Proof. We have seen surjectivity in Lemma 101.28.8. By Lemma 101.33.4 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 101.28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U with $G \to U$ flat and locally of finite presentation. Then we win by Lemma 101.33.7. □

101.34. Types of morphisms étale-smooth local on source-and-target

0CIE Given a property of morphisms of algebraic spaces which is *étale-smooth local on the source-and-target*, see Descent on Spaces, Definition 74.21.1 we may use it to define a corresponding property of DM morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

0CIF **Lemma 101.34.1.** *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where U and V are algebraic spaces, $V \to \mathcal{Y}$ is smooth, and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is étale. The following are equivalent

1. *for any diagram as above the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 100.3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. As f is DM there exists a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$, see Lemma 101.21.7. Thus we see that (2) holds. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (h,h')} & & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_{\mathcal{Y}} V' & \longrightarrow & V' \end{array}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that the

squares

$$\begin{array}{ccc} U & \longleftarrow & U \times_{\mathcal{X}} U' \\ \downarrow & & \downarrow \\ V \times_{\mathcal{Y}} \mathcal{X} & \longleftarrow & V \times_{\mathcal{Y}} U' \end{array} \qquad \begin{array}{ccc} U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U \times_{\mathcal{Y}} V' & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V' \end{array}$$

are cartesian, hence the vertical arrows are étale by our assumptions on U', V', a', b', h' and U, V, a, b, h. Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 74.21.2 part (2) we see that the base change $t : U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ of h has $\mathcal{P}$. Since $\mathcal{P}$ is étale local on the source by Descent on Spaces, Lemma 74.21.2 part (1) and $s : U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ is étale, we see the morphism $(h, h') = t \circ s$ has $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} U \times_{\mathcal{X}} U' & \xrightarrow[(h,h')]{} & V \times_{\mathcal{Y}} V' \\ \downarrow & & \downarrow \\ U' & \xrightarrow{h'} & V' \end{array}$$

The left vertical arrow is surjective, the right vertical arrow is smooth, and the induced morphism

$$U \times_{\mathcal{X}} U' \longrightarrow U' \times_{V'} (V \times_{\mathcal{Y}} V') = V \times_{\mathcal{Y}} U'$$

is étale as seen above. Hence by Descent on Spaces, Definition 74.21.1 part (3) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 74.21.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_{\mathcal{Y}} U & \longrightarrow & Z \times_{\mathcal{Y}} V \\ \downarrow & & \downarrow \\ Z \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is surjective, the induced morphism

$$Z \times_{\mathcal{Y}} U \longrightarrow (Z \times_{\mathcal{Y}} \mathcal{X}) \times_Z (Z \times_{\mathcal{Y}} V)$$

is étale, and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 74.21.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. □

0CIG **Definition 101.34.2.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. We say a DM morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 101.16.1 hold.

0CIH **Remark 101.34.3.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under composition. Then the property of DM morphisms of algebraic stacks defined in Definition 101.34.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be DM morphisms

of algebraic stacks having property $\mathcal{P}$. By Lemma 101.4.10 the composition $g \circ f$ is DM. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective étale morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ (Lemma 101.21.7). Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

0CII **Remark 101.34.4.** Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under base change. Then the property of DM morphisms of algebraic stacks defined in Definition 101.34.2 is stable under arbitrary base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks and $g : \mathcal{Y}' \to \mathcal{Y}$ be a morphism of algebraic stacks and assume f has property $\mathcal{P}$. Then the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is a DM morphism by Lemma 101.4.4. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{array}{ccccc} V' \times_V U & \longrightarrow & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ V' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

we see that the left top horizontal arrow is surjective and

$$V' \times_V U \to V' \times_{\mathcal{Y}} (\mathcal{Y}' \times_{\mathcal{Y}'} \mathcal{X}) = V' \times_V (\mathcal{X} \times_{\mathcal{Y}} V)$$

is étale as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

0CIJ **Remark 101.34.5.** Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are étale-smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 101.34.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

101.35. Étale morphisms

0CIK An étale morphism of algebraic stacks should not be defined as a smooth morphism of relative dimension 0. Namely, the morphism

$$[\mathbf{A}^1_k/\mathbf{G}_{m,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth of relative dimension 0 for any choice of action of the group scheme $\mathbf{G}_{m,k}$ on $\mathbf{A}^1_k$. This does not correspond to our usual idea that étale morphisms should identify tangent spaces. The example above isn't quasi-finite, but the morphism

$$\mathcal{X} = [\mathrm{Spec}(k)/\mu_{p,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth and quasi-finite (Section 101.23). However, if the characteristic of k is $p > 0$, then we see that the representable morphism $\mathrm{Spec}(k) \to \mathcal{X}$ isn't étale as the base change $\mu_{p,k} = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a morphism from a nonreduced

scheme to the spectrum of a field. Thus if we define an étale morphism as smooth and locally quasi-finite, then the analogue of Morphisms of Spaces, Lemma 67.39.11 would fail.

Instead, our approach will be to start with the requirements that "étaleness" should be a property preserved under base change and that if $\mathcal{X} \to X$ is an étale morphism from an algebraic stack to a scheme, then $\mathcal{X}$ should be Deligne-Mumford. In other words, we will require étale morphisms to be DM and we will use the material in Section 101.34 to define étale morphisms of algebraic stacks.

In Lemma 101.36.10 we will characterize étale morphisms of algebraic stacks as morphisms which are (a) locally of finite presentation, (b) flat, and (c) have étale diagonal.

The property "étale" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 74.21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.39.4 and Descent on Spaces, Lemma 74.11.28. Hence, by Lemma 101.34.1 above, we may define what it means for a morphism of algebraic spaces to be étale as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 101.4.3.

0CIL **Definition 101.35.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *étale* if f is DM and the equivalent conditions of Lemma 101.34.1 hold with $\mathcal{P} = \textit{étale}$.

We will use without further mention that this agrees with the already existing notion of étale morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

0CIM **Lemma 101.35.2.** *The composition of étale morphisms is étale.*

Proof. Combine Remark 101.34.3 with Morphisms of Spaces, Lemma 67.39.3. □

0CIN **Lemma 101.35.3.** *A base change of an étale morphism is étale.*

Proof. Combine Remark 101.34.4 with Morphisms of Spaces, Lemma 67.39.4. □

0CIP **Lemma 101.35.4.** *An open immersion is étale.*

Proof. Let $j : \mathcal{U} \to \mathcal{X}$ be an open immersion of algebraic stacks. Since j is representable, it is DM by Lemma 101.4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $U = \mathcal{U} \times_{\mathcal{X}} X$ is an open subscheme of X. Hence $U \to X$ is étale (Morphisms, Lemma 29.36.9) and we conclude that j is étale from the definition. □

0CIQ **Lemma 101.35.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is étale,*
(2) *f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is étale,*

(3) *there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_{\mathcal{Y}} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is étale.*

Proof. Assume (1). Then f is DM and since being étale is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (Lemma 101.21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is DM, see Lemma 101.4.5. By Lemma 101.4.12 it suffices to check $W \times_{\mathcal{Y}} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_{\mathcal{Y}} \mathcal{X}$ by (the easy direction of) Theorem 101.21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and étale (Lemma 101.21.7). We have to check that $U \to V$ is étale. It suffices to prove $U \times_{\mathcal{Y}} W \to V \times_{\mathcal{Y}} W$ is étale by Descent on Spaces, Lemma 74.11.28. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_{\mathcal{Y}} W, W, W, T, U \times_{\mathcal{Y}} W, V \times_{\mathcal{Y}} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is étale, and U and V are as before. In this case we know that

$$U \to V \text{ is étale} \Leftrightarrow \mathcal{X} \to Y \text{ is étale} \Leftrightarrow T \to Y \text{ is étale}$$

by the equivalence of properties (1) and (2) of Lemma 101.34.1 and Definition 101.35.1. This finishes the proof. □

0CIR **Lemma 101.35.6.** *Let $\mathcal{X}, \mathcal{Y}$ be algebraic stacks étale over an algebraic stack $\mathcal{Z}$. Any morphism $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{Z}$ is étale.*

Proof. The morphism $\mathcal{X} \to \mathcal{Y}$ is DM by Lemma 101.4.12. Let $W \to \mathcal{Z}$ be a surjective smooth morphism whose source is an algebraic space. Let $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ be a surjective étale morphism whose source is an algebraic space (Lemma 101.21.7). Let $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be a surjective étale morphism whose source is an algebraic space (Lemma 101.21.7). Then

$$U \longrightarrow \mathcal{X} \times_{\mathcal{Z}} W$$

is surjective étale as the composition of $U \to \mathcal{X} \times_{\mathcal{Y}} V$ and the base change of $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ by $\mathcal{X} \times_{\mathcal{Z}} W \to \mathcal{Y} \times_{\mathcal{Z}} W$. Hence it suffices to show that $U \to W$ is étale. Since $U \to W$ and $V \to W$ are étale because $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are étale, this follows from the version of the lemma for algebraic spaces, namely Morphisms of Spaces, Lemma 67.39.11. □

101.36. Unramified morphisms

0CIS For a justification of our choice of definition of unramified morphisms we refer the reader to the discussion in the section on étale morphisms Section 101.35.

In Lemma 101.36.9 we will characterize unramified morphisms of algebraic stacks as morphisms which are locally of finite type and have étale diagonal.

The property "unramified" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 74.21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 67.38.4 and Descent on Spaces, Lemma 74.11.27. Hence, by Lemma 101.34.1 above, we may define what it means for a morphism of algebraic spaces to be unramified as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 100.3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 101.4.3.

0CIT **Definition 101.36.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *unramified* if f is DM and the equivalent conditions of Lemma 101.34.1 hold with $\mathcal{P} =$"unramified".

We will use without further mention that this agrees with the already existing notion of unramified morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

0CIU **Lemma 101.36.2.** *The composition of unramified morphisms is unramified.*

Proof. Combine Remark 101.34.3 with Morphisms of Spaces, Lemma 67.38.3. □

0CIV **Lemma 101.36.3.** *A base change of an unramified morphism is unramified.*

Proof. Combine Remark 101.34.4 with Morphisms of Spaces, Lemma 67.38.4. □

0CIW **Lemma 101.36.4.** *An étale morphism is unramified.*

Proof. Follows from Remark 101.34.5 and Morphisms of Spaces, Lemma 67.39.10. □

0CIX **Lemma 101.36.5.** *An immersion is unramified.*

Proof. Let $j : \mathcal{Z} \to \mathcal{X}$ be an immersion of algebraic stacks. Since j is representable, it is DM by Lemma 101.4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $Z = \mathcal{Z} \times_{\mathcal{X}} X$ is a locally closed subscheme of X. Hence $Z \to X$ is unramified (Morphisms, Lemmas 29.35.7 and 29.35.8) and we conclude that j is unramified from the definition. □

0CIY **Lemma 101.36.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is unramified,*
(2) *f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is unramified,*
(3) *there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_{\mathcal{Y}} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is unramified.*

Proof. Assume (1). Then f is DM and since being unramified is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (Lemma 101.21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is DM, see Lemma 101.4.5. By Lemma 101.4.12 it suffices to check $W \times_{\mathcal{Y}} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_{\mathcal{Y}} \mathcal{X}$ by (the easy direction of) Theorem 101.21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and étale (Lemma 101.21.7). We have to check that $U \to V$ is unramified. It suffices to prove $U \times_{\mathcal{Y}} W \to V \times_{\mathcal{Y}} W$ is unramified by Descent on Spaces, Lemma 74.11.27. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_{\mathcal{Y}} W, W, W, T, U \times_{\mathcal{Y}} W, V \times_{\mathcal{Y}} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is unramified, and U and V are as before. In this case we know that

$$U \to V \text{ is unramified} \Leftrightarrow \mathcal{X} \to Y \text{ is unramified} \Leftrightarrow T \to Y \text{ is unramified}$$

by the equivalence of properties (1) and (2) of Lemma 101.34.1 and Definition 101.36.1. This finishes the proof. □

0H2Z **Lemma 101.36.7.** *An unramified morphism of algebraic stacks is locally quasi-finite.*

Proof. This follows from Lemma 101.36.6 (characterizing unramified morphisms), Lemma 101.23.7 (characterizing locally quasi-finite morphisms), and Morphisms of Spaces, Lemma 67.38.7 (the corresponding result for algebraic spaces). □

0CIZ **Lemma 101.36.8.** *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is unramified and $\mathcal{Y} \to \mathcal{Z}$ is DM, then $\mathcal{X} \to \mathcal{Y}$ is unramified.*

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is unramified. By Lemma 101.4.12 the morphism $\mathcal{X} \to \mathcal{Y}$ is DM. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

with U, V, W algebraic spaces, with $W \to \mathcal{Z}$ surjective smooth, $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ surjective étale, and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ surjective étale (see Lemma 101.21.7). Then also $U \to \mathcal{X} \times_{\mathcal{Z}} W$ is surjective and étale. Hence we know that $U \to W$ is unramified and we have to show that $U \to V$ is unramified. This follows from Morphisms of Spaces, Lemma 67.38.11. □

0CJ0 **Lemma 101.36.9.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is unramified, and*
(2) *f is locally of finite type and its diagonal is étale.*

Proof. Assume f is unramified. Then f is DM hence we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). Since f is unramified the induced morphism $U \to V$ is unramified. Thus $U \to V$ is locally of finite type (Morphisms of Spaces, Lemma 67.38.6) and we conclude that f is locally of finite type. The diagonal

$\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a morphism of algebraic stacks over $\mathcal{Y}$. The base change of Δ by the surjective smooth morphism $V \to \mathcal{Y}$ is the diagonal of the base change of f, i.e., of $\mathcal{X}_V = \mathcal{X} \times_{\mathcal{Y}} V \to V$. In other words, the diagram

$$\begin{array}{ccc} \mathcal{X}_V & \longrightarrow & \mathcal{X}_V \times_V \mathcal{X}_V \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is cartesian. Since the right vertical arrow is surjective and smooth it suffices to show that the top horizontal arrow is étale by Properties of Stacks, Lemma 100.3.4. Consider the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & U \times_V U \\ \downarrow & & \downarrow \\ \mathcal{X}_V & \longrightarrow & \mathcal{X}_V \times_V \mathcal{X}_V \end{array}$$

All arrows are representable by algebraic spaces, the vertical arrows are étale, the left one is surjective, and the top horizontal arrow is an open immersion by Morphisms of Spaces, Lemma 67.38.9. This implies what we want: first we see that $U \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale as a composition of étale morphisms, and then we can use Properties of Stacks, Lemma 100.3.5 to see that $\mathcal{X}_V \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale because being étale (for morphisms of algebraic spaces) is local on the source in the étale topology (Descent on Spaces, Lemma 74.19.1).

Assume f is locally of finite type and that its diagonal is étale. Then f is DM by definition (as étale morphisms of algebraic spaces are unramified). As above this means we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). To finish the proof we have to show that $U \to V$ is unramified. We already know that $U \to V$ is locally of finite type. Arguing as above we find a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & U \times_V U \\ \downarrow & & \downarrow \\ \mathcal{X}_V & \longrightarrow & \mathcal{X}_V \times_V \mathcal{X}_V \end{array}$$

where all arrows are representable by algebraic spaces, the vertical arrows are étale, and the lower horizontal one is étale as a base change of Δ. It follows that $U \to U \times_V U$ is étale for example by Lemma 101.35.6[7]. Thus $U \to U \times_V U$ is an étale monomorphism hence an open immersion (Morphisms of Spaces, Lemma 67.51.2). Then $U \to V$ is unramified by Morphisms of Spaces, Lemma 67.38.9. □

0CJ1 **Lemma 101.36.10.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is étale, and*
(2) *f is locally of finite presentation, flat, and unramified,*
(3) *f is locally of finite presentation, flat, and its diagonal is étale.*

[7] It is quite easy to deduce this directly from Morphisms of Spaces, Lemma 67.39.11.

Proof. The equivalence of (2) and (3) follows immediately from Lemma 101.36.9. Thus in each case the morphism f is DM. Then we can choose Then we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 101.21.7). To finish the proof we have to show that $U \to V$ is étale if and only if it is locally of finite presentation, flat, and unramified. This follows from Morphisms of Spaces, Lemma 67.39.12 (and the more trivial Morphisms of Spaces, Lemmas 67.39.10, 67.39.8, and 67.39.7). □

101.37. Proper morphisms

0CL4 The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic stacks.

0CL5 **Definition 101.37.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *proper* if f is separated, finite type, and universally closed.

This does not conflict with the already existing notion of a proper morphism of algebraic spaces: a morphism of algebraic spaces is proper if and only if it is separated, finite type, and universally closed (Morphisms of Spaces, Definition 67.40.1) and we've already checked the compatibility of these notions in Lemma 101.3.5, Section 101.17, and Lemmas 101.13.1. Similarly, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks which is representable by algebraic spaces then we have defined what it means for f to be proper in Properties of Stacks, Section 100.3. However, the discussion in that section shows that this is equivalent to requiring f to be separated, finite type, and universally closed and the same references as above give the compatibility.

0CL6 **Lemma 101.37.2.** *A base change of a proper morphism is proper.*

Proof. See Lemmas 101.4.4, 101.17.3, and 101.13.3. □

0CL7 **Lemma 101.37.3.** *A composition of proper morphisms is proper.*

Proof. See Lemmas 101.4.10, 101.17.2, and 101.13.4. □

0CL8 **Lemma 101.37.4.** *A closed immersion of algebraic stacks is a proper morphism of algebraic stacks.*

Proof. A closed immersion is by definition representable (Properties of Stacks, Definition 100.9.1). Hence this follows from the discussion in Properties of Stacks, Section 100.3 and the corresponding result for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 67.40.5. □

0CPT **Lemma 101.37.5.** *Consider a commutative diagram*

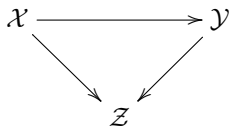

of algebraic stacks.

(1) *If $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is universally closed. In particular, the image of $|\mathcal{X}|$ in $|\mathcal{Y}|$ is closed.*

(2) *If $\mathcal{X} \to \mathcal{Z}$ is proper and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is proper.*

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated. We factor the morphism as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first morphism is proper (Lemma 101.4.8) hence universally closed. The projection $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ is the base change of a universally closed morphism and hence universally closed, see Lemma 101.13.3. Thus $\mathcal{X} \to \mathcal{Y}$ is universally closed as the composition of universally closed morphisms, see Lemma 101.13.4. This proves (1). To deduce (2) combine (1) with Lemmas 101.4.12, 101.7.7, and 101.17.8. □

0CQK **Lemma 101.37.6.** *Let $\mathcal{Z}$ be an algebraic stack. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{Z}$. If $\mathcal{X}$ is universally closed over $\mathcal{Z}$ and f is surjective then $\mathcal{Y}$ is universally closed over $\mathcal{Z}$. In particular, if also $\mathcal{Y}$ is separated and of finite type over $\mathcal{Z}$, then $\mathcal{Y}$ is proper over $\mathcal{Z}$.*

Proof. Assume $\mathcal{X}$ is universally closed and f surjective. Denote $p : \mathcal{X} \to \mathcal{Z}$, $q : \mathcal{Y} \to \mathcal{Z}$ the structure morphisms. Let $\mathcal{Z}' \to \mathcal{Z}$ be a morphism of algebraic stacks. The base change $f' : \mathcal{X}' \to \mathcal{Y}'$ of f by $\mathcal{Z}' \to \mathcal{Z}$ is surjective (Properties of Stacks, Lemma 100.5.3) and the base change $p' : \mathcal{X}' \to \mathcal{Z}'$ of p is closed. If $T \subset |\mathcal{Y}'|$ is closed, then $(f')^{-1}(T) \subset |\mathcal{X}'|$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

101.38. Scheme theoretic image

0CMH Here is the definition.

0CMI **Definition 101.38.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *scheme theoretic image* of f is the smallest closed substack $\mathcal{Z} \subset \mathcal{Y}$ through which f factors[8].

We often denote $f : \mathcal{X} \to \mathcal{Z}$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open substacks of $\mathcal{Y}$. However, if f is quasi-compact then the scheme theoretic image commutes with flat base change (Lemma 101.38.5).

0CMJ **Lemma 101.38.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $g : \mathcal{W} \to \mathcal{X}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then the scheme theoretic image of f exists if and only if the scheme theoretic image of $f \circ g$ exists and if so then these scheme theoretic images are the same.*

Proof. Assume $\mathcal{Z} \subset \mathcal{Y}$ is a closed substack and $f \circ g$ factors through $\mathcal{Z}$. To prove the lemma it suffices to show that f factors through $\mathcal{Z}$. Consider a scheme T and a morphism $T \to \mathcal{X}$ given by an object x of the fibre category of $\mathcal{X}$ over T. We will show that $f(x)$ is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the projection $T \times_{\mathcal{X}} \mathcal{W} \to T$ is a surjective, flat, locally finitely presented morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} \mathcal{W} \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{W}$ and hence $T_i \to \mathcal{Y}$ factors through $\mathcal{Z}$. Thus $f(x)|_{T_i}$ is an object of $\mathcal{Z}$. Since $\mathcal{Z}$ is a strictly full substack, we conclude that $f(x)$ is an object of $\mathcal{Z}$ as desired. □

[8] We will see in Lemma 101.38.3 that the scheme theoretic image always exists.

0CPU **Lemma 101.38.3.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Then the scheme theoretic image of f exists.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Lemma 101.38.2 we may replace $\mathcal{Y}$ by V. Thus it suffices to show that if $X \to \mathcal{X}$ is a morphism from a scheme to an algebraic stack, then the scheme theoretic image exists. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. We have $\mathcal{X} = [U/R]$ by Algebraic Stacks, Lemma 94.16.2. By Properties of Stacks, Lemma 100.9.11 the closed substacks $\mathcal{Z}$ of $\mathcal{X}$ are in 1-to-1 correspondence with R-invariant closed subschemes $Z \subset U$. Let $Z_1 \subset U$ be the scheme theoretic image of $X \times_{\mathcal{X}} U \to U$. Observe that $X \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $X \times_{\mathcal{X}} U \to U$ factors through the corresponding R-invariant closed subscheme Z (details omitted; hint: this follows because $X \times_{\mathcal{X}} U \to X$ is surjective and smooth). Thus we have to show that there exists a smallest R-invariant closed subscheme $Z \subset U$ containing Z_1.

Let $\mathcal{I}_1 \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_1 \subset U$. Let Z_α, $\alpha \in A$ be the set of all R-invariant closed subschemes of U containing Z_1. For $\alpha \in A$, let $\mathcal{I}_\alpha \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_\alpha \subset U$. The containment $Z_1 \subset Z_\alpha$ means $\mathcal{I}_\alpha \subset \mathcal{I}_1$. The R-invariance of Z_α means that

$$s^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R = t^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R$$

as (quasi-coherent) ideal sheaves on (the algebraic space) R. Consider the image

$$\mathcal{I} = \mathrm{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{I}_1\right) = \mathrm{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{O}_X\right)$$

Since direct sums of quasi-coherent sheaves are quasi-coherent and since images of maps between quasi-coherent sheaves are quasi-coherent, we find that $\mathcal{I}$ is quasi-coherent. Since pull back is exact and commutes with direct sums we find

$$s^{-1}\mathcal{I} \cdot \mathcal{O}_R = t^{-1}\mathcal{I} \cdot \mathcal{O}_R$$

Hence $\mathcal{I}$ defines an R-invariant closed subscheme $Z \subset U$ which is contained in every Z_α and contains Z_1 as desired. □

0CPV **Lemma 101.38.4.** *Let*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{f_1} & \mathcal{Y}_1 \\ \downarrow & & \downarrow \\ \mathcal{X}_2 & \xrightarrow{f_2} & \mathcal{Y}_2 \end{array}$$

be a commutative diagram of algebraic stacks. Let $\mathcal{Z}_i \subset \mathcal{Y}_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ induces a morphism $\mathcal{Z}_1 \to \mathcal{Z}_2$ and a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}_1 & \longrightarrow & \mathcal{Z}_1 & \longrightarrow & \mathcal{Y}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_2 & \longrightarrow & \mathcal{Z}_2 & \longrightarrow & \mathcal{Y}_2 \end{array}$$

Proof. The scheme theoretic inverse image of $\mathcal{Z}_2$ in $\mathcal{Y}_1$ is a closed substack of $\mathcal{Y}_1$ through which f_1 factors. Hence $\mathcal{Z}_1$ is contained in this. This proves the lemma. □

0CMK **Lemma 101.38.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then formation of the scheme theoretic image commutes with flat base change.*

Proof. Let $\mathcal{Y}' \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme V' and a surjective smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. We may and do assume that $V = \coprod_{i \in I} V_i$ is a disjoint union of affine schemes and that $V' = \coprod_{i \in I} \coprod_{j \in J_i} V_{i,j}$ is a disjoint union of affine schemes with each $V_{i,j}$ mapping into V_i. Let

(1) $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f,
(2) $\mathcal{Z}' \subset \mathcal{Y}'$ be the scheme theoretic image of the base change of f by $\mathcal{Y}' \to \mathcal{Y}$,
(3) $Z \subset V$ be the scheme theoretic image of the base change of f by $V \to \mathcal{Y}$,
(4) $Z' \subset V'$ be the scheme theoretic image of the base change of f by $V' \to \mathcal{Y}$.

If we can show that (a) $Z = V \times_{\mathcal{Y}} \mathcal{Z}$, (b) $Z' = V' \times_{\mathcal{Y}'} \mathcal{Z}'$, and (c) $Z' = V' \times_V Z$ then the lemma follows: the inclusion $\mathcal{Z}' \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}'$ (Lemma 101.38.4) has to be an isomorphism because after base change by the surjective smooth morphism $V' \to \mathcal{Y}'$ it is.

Proof of (a). Set $R = V \times_{\mathcal{Y}} V$. By Properties of Stacks, Lemma 100.9.11 the rule $\mathcal{Z} \mapsto \mathcal{Z} \times_{\mathcal{Y}} V$ defines a 1-to-1 correspondence between closed substacks of $\mathcal{Y}$ and R-invariant closed subspaces of V. Moreover, $f : \mathcal{X} \to \mathcal{Y}$ factors through $\mathcal{Z}$ if and only if the base change $g : \mathcal{X} \times_{\mathcal{Y}} V \to V$ factors through $\mathcal{Z} \times_{\mathcal{Y}} V$. We claim: the scheme theoretic image $Z \subset V$ of g is R-invariant. The claim implies (a) by what we just said.

For each i the morphism $\mathcal{X} \times_{\mathcal{Y}} V_i \to V_i$ is quasi-compact and hence $\mathcal{X} \times_{\mathcal{Y}} V_i$ is quasi-compact. Thus we can choose an affine scheme W_i and a surjective smooth morphism $W_i \to \mathcal{X} \times_{\mathcal{Y}} V_i$. Observe that $W = \coprod W_i$ is a scheme endowed with a smooth and surjective morphism $W \to \mathcal{X} \times_{\mathcal{Y}} V$ such that the composition $W \to V$ with g is quasi-compact. Let $Z \to V$ be the scheme theoretic image of $W \to V$, see Morphisms, Section 29.6 and Morphisms of Spaces, Section 67.16. It follows from Lemma 101.38.2 that $Z \subset V$ is the scheme theoretic image of g. To show that Z is R-invariant we claim that both

$$\mathrm{pr}_0^{-1}(Z), \mathrm{pr}_1^{-1}(Z) \subset R = V \times_{\mathcal{Y}} V$$

are the scheme theoretic image of $\mathcal{X} \times_{\mathcal{Y}} R \to R$. Namely, we first use Morphisms of Spaces, Lemma 67.30.12 to see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of the composition

$$W \times_{V, \mathrm{pr}_0} R = W \times_{\mathcal{Y}} V \to \mathcal{X} \times_{\mathcal{Y}} R \to R$$

Since the first arrow here is surjective and smooth we see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of $\mathcal{X} \times_{\mathcal{Y}} R \to R$. The same argument applies that $\mathrm{pr}_1^{-1}(Z)$. Hence Z is R-invariant.

Statement (b) is proved in exactly the same way as one proves (a).

Proof of (c). Let $Z_i \subset V_i$ be the scheme theoretic image of $\mathcal{X} \times_{\mathcal{Y}} V_i \to V_i$ and let $Z_{i,j} \subset V_{i,j}$ be the scheme theoretic image of $\mathcal{X} \times_{\mathcal{Y}} V_{i,j} \to V_{i,j}$. Clearly it suffices to show that the inverse image of Z_i in $V_{i,j}$ is $Z_{i,j}$. Above we've seen that Z_i is the scheme theoretic image of $W_i \to V_i$ and by the same token $Z_{i,j}$ is the scheme theoretic image of $W_i \times_{V_i} V_{i,j} \to V_{i,j}$. Hence the equality follows from the case of schemes (Morphisms, Lemma 29.25.16) and the fact that $V_{i,j} \to V_i$ is flat. $\square$

0CML **Lemma 101.38.6.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f. Then $|\mathcal{Z}|$ is the closure of the image of $|f|$.*

Proof. Let $z \in |\mathcal{Z}|$ be a point. Choose an affine scheme V, a point $v \in V$, and a smooth morphism $V \to \mathcal{Y}$ mapping v to z. Then $\mathcal{X} \times_{\mathcal{Y}} V$ is a quasi-compact algebraic stack. Hence we can find an affine scheme W and a surjective smooth morphism $W \to \mathcal{X} \times_{\mathcal{Y}} V$. By Lemma 101.38.5 the scheme theoretic image of $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is $Z = \mathcal{Z} \times_{\mathcal{Y}} V$. Hence the inverse image of $|\mathcal{Z}|$ in $|V|$ is $|Z|$ by Properties of Stacks, Lemma 100.4.3. By Lemma 101.38.2 Z is the scheme theoretic image of $W \to V$. By Morphisms of Spaces, Lemma 67.16.3 we see that the image of $|W| \to |Z|$ is dense. Hence the image of $|\mathcal{X} \times_{\mathcal{Y}} V| \to |Z|$ is dense. Observe that $v \in Z$. Since $|V| \to |\mathcal{Y}|$ is open, a topology argument tells us that z is in the closure of the image of $|f|$ as desired. □

0CPW **Lemma 101.38.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces and separated. Let $\mathcal{V} \subset \mathcal{Y}$ be an open substack such that $\mathcal{V} \to \mathcal{Y}$ is quasi-compact. Let $s : \mathcal{V} \to \mathcal{X}$ be a morphism such that $f \circ s = id_{\mathcal{V}}$. Let $\mathcal{Y}'$ be the scheme theoretic image of s. Then $\mathcal{Y}' \to \mathcal{Y}$ is an isomorphism over $\mathcal{V}$.*

Proof. By Lemma 101.7.7 the morphism $s : \mathcal{V} \to \mathcal{Y}$ is quasi-compact. Hence the construction of the scheme theoretic image $\mathcal{Y}'$ of s commutes with flat base change by Lemma 101.38.5. Thus to prove the lemma we may assume $\mathcal{Y}$ is representable by an algebraic space and we reduce to the case of algebraic spaces which is Morphisms of Spaces, Lemma 67.16.7. □

101.39. Valuative criteria

0CL9 We need to be careful when stating the valuative criterion. Namely, in the formulation we need to speak about commutative diagrams but we are working in a 2-category and we need to make sure the 2-morphisms compose correctly as well!

0CLA **Definition 101.39.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

0CLB (101.39.1.1)
$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where A is a valuation ring with field of fractions K. Let

$$\gamma : y \circ j \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 4.28 and 4.29.) Given (101.39.1.1) and γ a *dotted arrow* is a triple (a, α, β) consisting of a morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ and 2-arrows $\alpha : a \circ j \to x$, $\beta : y \to f \circ a$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_j)$, in other words such that

$$\begin{array}{ccccc} & & f \circ a \circ j & & \\ & {\scriptstyle \beta \star \mathrm{id}_j} \nearrow & & \searrow {\scriptstyle \mathrm{id}_f \star \alpha} & \\ y \circ j & & \xrightarrow{\gamma} & & f \circ x \end{array}$$

is commutative. A *morphism of dotted arrows* $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ such that $\alpha = \alpha' \circ (\theta \star \mathrm{id}_j)$ and $\beta' = (\mathrm{id}_f \star \theta) \circ \beta$.

The preceding definition is a special case of Categories, Definition 4.44.1. The category of dotted arrows depends on γ in general. If $\mathcal{Y}$ is representable by an algebraic space (or if automorphism groups of objects over fields are trivial), then of course there is at most one γ and one does not need to check the commutativity of the triangle. More generally, we have Lemma 101.39.3. The commutativity of the triangle is important in the proof of compatibility with base change, see proof of Lemma 101.39.4.

0CLC **Lemma 101.39.2.** *In the situation of Definition 101.39.1 the category of dotted arrows is a groupoid. If Δ_f is separated, then it is a setoid.*

Proof. Since 2-arrows are invertible it is clear that the category of dotted arrows is a groupoid. Given a dotted arrow (a, α, β) an automorphism of (a, α, β) is a 2-morphism $\theta : a \to a$ satisfying two conditions. The first condition $\beta = (\mathrm{id}_f \star \theta) \circ \beta$ signifies that θ defines a morphism $(a, \theta) : \mathrm{Spec}(A) \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. The second condition $\alpha = \alpha \circ (\theta \star \mathrm{id}_j)$ implies that the restriction of (a, θ) to $\mathrm{Spec}(K)$ is the identity. Picture

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}/\mathcal{Y}} & \xleftarrow{(a\circ j, \mathrm{id})} & \mathrm{Spec}(K) \\ \downarrow & \swarrow_{(a,\theta)} & \downarrow{\scriptstyle j} \\ \mathcal{X} & \xleftarrow{a} & \mathrm{Spec}(A) \end{array}$$

In other words, if $G \to \mathrm{Spec}(A)$ is the group algebraic space we get by pulling back the relative inertia by a, then θ defines a point $\theta \in G(A)$ whose image in $G(K)$ is trivial. Certainly, if the identity $e : \mathrm{Spec}(A) \to G$ is a closed immersion, then this can happen only if θ is the identity. Looking at Lemma 101.6.1 we obtain the result we want. □

0CLD **Lemma 101.39.3.** *In Definition 101.39.1 assume $\mathcal{I}_{\mathcal{Y}} \to \mathcal{Y}$ is proper (for example if $\mathcal{Y}$ is separated or if $\mathcal{Y}$ is separated over an algebraic space). Then the category of dotted arrows is independent (up to noncanonical equivalence) of the choice of γ and the existence of a dotted arrow (for some and hence equivalently all γ) is equivalent to the existence of a diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow_{a} & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

with 2-commutative triangles (without checking the 2-morphisms compose correctly).

Proof. Let $\gamma, \gamma' : y \circ j \longrightarrow f \circ x$ be two 2-morphisms. Then $\gamma^{-1} \circ \gamma'$ is an automorphism of y over $\mathrm{Spec}(K)$. Hence if $Isom_{\mathcal{Y}}(y, y) \to \mathrm{Spec}(A)$ is proper, then by the valuative criterion of properness (Morphisms of Spaces, Lemma 67.44.1) we can find $\delta : y \to y$ whose restriction to $\mathrm{Spec}(K)$ is $\gamma^{-1} \circ \gamma'$. Then we can use δ to define an equivalence between the category of dotted arrows for γ to the category of dotted arrows for γ' by sending (a, α, β) to $(a, \alpha, \beta \circ \delta)$. The final statement is clear. □

0CLE **Lemma 101.39.4.** *Assume given a 2-commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{x'} & \mathcal{X}' & \xrightarrow{q} & \mathcal{X} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y'} & \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

with the right square 2-cartesian. Choose a 2-arrow $\gamma' : y' \circ j \to p \circ x'$. Set $x = q \circ x'$, $y = g \circ y'$ and let $\gamma : y \circ j \to f \circ x$ be the composition of γ' with the 2-arrow implicit in the 2-commutativity of the right square. Then the category of dotted arrows for the left square and γ' is equivalent to the category of dotted arrows for the outer rectangle and γ.

Proof. (We do not know how to prove the analogue of this lemma if instead of the category of dotted arrows we look at the set of isomorphism classes of morphisms producing two 2-commutative triangles as in Lemma 101.39.3; in fact this analogue may very well be wrong.) First proof: this lemma is a special case of Categories, Lemma 4.44.2. Second proof: we are allowed to replace $\mathcal{X}'$ by the 2-fibre product $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as described in Categories, Lemma 4.32.3. Then the object x' becomes the triple $(y' \circ j, x, \gamma)$. Then we can go from a dotted arrow (a, α, β) for the outer rectangle to a dotted arrow (a', α', β') for the left square by taking $a' = (y', a, \beta)$ and $\alpha' = (\mathrm{id}_{y' \circ j}, \alpha)$ and $\beta' = \mathrm{id}_{y'}$. Details omitted. □

0CLF **Lemma 101.39.5.** *Assume given a 2-commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ & & \mathcal{Y} \\ & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(A) & \xrightarrow{z} & \mathcal{Z} \end{array}$$

Choose a 2-arrow $\gamma : z \circ j \to g \circ f \circ x$. Let $\mathcal{C}$ be the category of dotted arrows for the outer rectangle and γ. Let $\mathcal{C}'$ be the category of dotted arrows for the square

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{f \circ x} & \mathcal{Y} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(A) & \xrightarrow{z} & \mathcal{Z} \end{array}$$

and γ. Then $\mathcal{C}$ is equivalent to a category $\mathcal{C}''$ which has the following property: there is a functor $\mathcal{C}'' \to \mathcal{C}'$ which turns $\mathcal{C}''$ into a category fibred in groupoids over $\mathcal{C}'$ and whose fibre categories are categories of dotted arrows for certain squares of the form

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

and some choices of $y \circ j \to f \circ x$.

Proof. This lemma is a special case of Categories, Lemma 4.44.3. □

0CLG **Definition 101.39.6.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the *uniqueness part of the valuative criterion* if for every diagram (101.39.1.1) and γ as in Definition 101.39.1 the category of dotted arrows is either empty or a setoid with exactly one isomorphism class.

0CLH **Lemma 101.39.7.** *The base change of a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.*

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLI **Lemma 101.39.8.** *The composition of morphisms of algebraic stacks which satisfy the uniqueness part of the valuative criterion is another morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.*

Proof. Follows from Lemma 101.39.5 and the definition. □

0CLJ **Lemma 101.39.9.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

1. *f satisfies the uniqueness part of the valuative criterion,*
2. *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ satisfies the uniqueness part of the valuative criterion as a morphism of algebraic spaces.*

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLK **Definition 101.39.10.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the *existence part of the valuative criterion* if for every diagram (101.39.1.1) and γ as in Definition 101.39.1 there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the outer rectangle of the diagram

$$\begin{array}{ccccc}
 & & \overset{x'}{\frown} & & \\
\mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \xrightarrow[x]{} & \mathcal{X} \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\
\mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \\
 & & \underset{y'}{\smile} & &
\end{array}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty.

We have already seen in the case of morphisms of algebraic spaces, that it is necessary to allow extensions of the fractions fields in order to get the correct notion of the valuative criterion. See Morphisms of Spaces, Example 67.41.6. Still, for morphisms between separated algebraic spaces, such an extension is not needed (Morphisms of Spaces, Lemma 67.41.5). However, for morphisms between algebraic stacks, an extension may be needed even if $\mathcal{X}$ and $\mathcal{Y}$ are both separated. For example consider the morphism of algebraic stacks

$$[\mathrm{Spec}(\mathbf{C})/G] \to \mathrm{Spec}(\mathbf{C})$$

over the base scheme $\mathrm{Spec}(\mathbf{C})$ where G is a group of order 2. Both source and target are separated algebraic stacks and the morphism is proper. Whence it satisfies the uniqueness and existence parts of the valuative criterion (see Lemma 101.43.1). But on the other hand, there is a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & [\mathrm{Spec}(\mathbf{C})/G] \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(\mathbf{C}) \end{array}$$

where no dotted arrow exists with $A = \mathbf{C}[[t]]$ and $K = \mathbf{C}((t))$. Namely, the top horizontal arrow is given by a G-torsor over the spectrum of $K = \mathbf{C}((t))$. Since any G-torsor over the strictly henselian local ring $A = \mathbf{C}[[t]]$ is trivial, we see that if a dotted arrow always exists, then every G-torsor over K is trivial. This is not true because $G = \{+1, -1\}$ and by Kummer theory G-torsors over K are classified by $K^*/(K^*)^2$ which is nontrivial.

0CLL **Lemma 101.39.11.** *The base change of a morphism of algebraic stacks which satisfies the existence part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the existence part of the valuative criterion.*

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLM **Lemma 101.39.12.** *The composition of morphisms of algebraic stacks which satisfy the existence part of the valuative criterion is another morphism of algebraic stacks which satisfies the existence part of the valuative criterion.*

Proof. Follows from Lemma 101.39.5 and the definition. □

0CLN **Lemma 101.39.13.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

(1) *f satisfies the existence part of the valuative criterion,*
(2) *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ satisfies the existence part of the valuative criterion as a morphism of algebraic spaces.*

Proof. Follows from Lemma 101.39.4 and the definition. □

0CLP **Lemma 101.39.14.** *A closed immersion of algebraic stacks satisfies both the existence and uniqueness part of the valuative criterion.*

Proof. Omitted. Hint: reduce to the case of a closed immersion of schemes by Lemmas 101.39.9 and 101.39.13. □

101.40. Valuative criterion for second diagonal

0CLQ The converse statement has already been proved in Lemma 101.39.2. The criterion itself is the following.

0CLR **Lemma 101.40.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If Δ_f is quasi-separated and if for every diagram (101.39.1.1) and choice of γ as in Definition 101.39.1 the category of dotted arrows is a setoid, then Δ_f is separated.*

Proof. We are going to write out a detailed proof, but we strongly urge the reader to find their own proof, inspired by reading the argument given in the proof of Lemma 101.39.2.

Assume Δ_f is quasi-separated and for every diagram (101.39.1.1) and choice of γ as in Definition 101.39.1 the category of dotted arrows is a setoid. By Lemma 101.6.1 it suffices to show that $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion. By Lemma 101.6.4 it in fact suffices to show that $e = \Delta_{f,2}$ is universally closed. Either of these lemmas tells us that $e = \Delta_{f,2}$ is quasi-compact by our assumption that Δ_f is quasi-separated.

In this paragraph we will show that e satisfies the existence part of the valuative criterion. Consider a 2-commutative solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow[x]{} & \mathcal{X} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle e} \\ \mathrm{Spec}(A) & \xrightarrow{(a,\theta)} & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \end{array}$$

and let $\alpha : (a, \theta) \circ j \to e \circ x$ be any 2-morphism witnessing the 2-commutativity of the diagram (we use α instead of the letter γ used in Definition 101.39.1). Note that $f \circ \theta = \mathrm{id}$; we will use this below. Observe that $e \circ x = (x, \mathrm{id}_x)$ and $(a, \theta) \circ j = (a \circ j, \theta \star \mathrm{id}_j)$. Thus we see that α is a 2-arrow $\alpha : a \circ j \to x$ compatible with $\theta \star \mathrm{id}_j$ and id_x. Set $y = f \circ x$ and $\beta = \mathrm{id}_{f \circ a}$. Reading the arguments given in the proof of Lemma 101.39.2 backwards, we see that θ is an automorphism of the dotted arrow (a, α, β) with

$$\gamma : y \circ j \to f \circ x \quad \text{equal to} \quad \mathrm{id}_f \star \alpha : f \circ a \circ j \to f \circ x$$

On the other hand, id_a is an automorphism too, hence we conclude $\theta = \mathrm{id}_a$ from the assumption on f. Then we can take as dotted arrow for the displayed diagram above the morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ with 2-morphisms $(a, \mathrm{id}_a) \circ j \to (x, \mathrm{id}_x)$ given by α and $(a, \theta) \to e \circ a$ given by id_a.

By Lemma 101.39.11 any base change of e satisfies the existence part of the valuative criterion. Since e is representable by algebraic spaces, it suffices to show that e is universally closed after a base change by a morphism $I \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ which is surjective and smooth and with I an algebraic space (see Properties of Stacks, Section 100.3). This base change $e' : X' \to I'$ is a quasi-compact morphism of algebraic spaces which satisfies the existence part of the valuative criterion and hence is universally closed by Morphisms of Spaces, Lemma 67.42.1. □

101.41. Valuative criterion for the diagonal

0CLS The result is Lemma 101.41.2. We first state and prove a formal helper lemma.

0E8L **Lemma 101.41.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow[x]{} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle \Delta_f} \\ \mathrm{Spec}(A) & \xrightarrow{(a_1,a_2,\varphi)} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

where A is a valuation ring with field of fractions K. Let $\gamma : (a_1, a_2, \varphi) \circ j \longrightarrow \Delta_f \circ x$ be a 2-morphism witnessing the 2-commutativity of the diagram. Then

(1) *Writing $\gamma = (\alpha_1, \alpha_2)$ with $\alpha_i : a_i \circ j \to x$ we obtain two dotted arrows (a_1, α_1, id) and (a_2, α_2, φ) in the diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{f \circ a_1} & \mathcal{Y} \end{array}$$

(2) *The category of dotted arrows for the original diagram and γ is a setoid whose set of isomorphism classes of objects equal to the set of morphisms $(a_1, \alpha_1, id) \to (a_2, \alpha_2, \varphi)$ in the category of dotted arrows.*

Proof. Since Δ_f is representable by algebraic spaces (hence the diagonal of Δ_f is separated), we see that the category of dotted arrows in the first commutative diagram of the lemma is a setoid by Lemma 101.39.2. All the other statements of the lemma are consequences of 2-diagramatic computations which we omit. □

0CLT **Lemma 101.41.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is quasi-separated. If f satisfies the uniqueness part of the valuative criterion, then f is separated.*

Proof. The assumption on f means that Δ_f is quasi-compact and quasi-separated (Definition 101.4.1). We have to show that Δ_f is proper. Lemma 101.40.1 says that Δ_f is separated. By Lemma 101.3.3 we know that Δ_f is locally of finite type. To finish the proof we have to show that Δ_f is universally closed. A formal argument (see Lemma 101.41.1) shows that the uniqueness part of the valuative criterion implies that we have the existence of a dotted arrow in any solid diagram like so:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle \Delta_f} \\ \mathrm{Spec}(A) & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

Using that this property is preserved by any base change we conclude that any base change by Δ_f by an algebraic space mapping into $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ has the existence part of the valuative criterion and we conclude is universally closed by the valuative criterion for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 67.42.1. □

Here is a converse.

0CLU **Lemma 101.41.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Since f is separated we see that all categories of dotted arrows are setoids by Lemma 101.39.2. Consider a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

and a 2-morphism $\gamma : y \circ j \to f \circ x$ as in Definition 101.39.1. Consider two objects (a, α, β) and (a', β', α') of the category of dotted arrows. To finish the proof we have to show these objects are isomorphic. The isomorphism

$$f \circ a \xrightarrow{\beta^{-1}} y \xrightarrow{\beta'} f \circ a'$$

means that $(a, a', \beta' \circ \beta^{-1})$ is a morphism $\mathrm{Spec}(A) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. On the other hand, α and α' define a 2-arrow

$$(a, a', \beta' \circ \beta^{-1}) \circ j = (a \circ j, a' \circ j, (\beta' \star \mathrm{id}_j) \circ (\beta \star \mathrm{id}_j)^{-1}) \xrightarrow{(\alpha, \alpha')} (x, x, \mathrm{id}) = \Delta_f \circ x$$

Here we use that both (a, α, β) and (a', α', β') are dotted arrows with respect to γ. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow[x]{} & \mathcal{X} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta_f} \\ \mathrm{Spec}(A) & \xrightarrow{(a, a', \beta' \circ \beta^{-1})} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

with 2-commutativity witnessed by (α, α'). Now Δ_f is representable by algebraic spaces (Lemma 101.3.3) and proper as f is separated. Hence by Lemma 101.39.13 and the valuative criterion for properness for algebraic spaces (Morphisms of Spaces, Lemma 67.44.1) we see that there exists a dotted arrow. Unwinding the construction, we see that this means (a, α, β) and (a', α', β') are isomorphic in the category of dotted arrows as desired. □

101.42. Valuative criterion for universal closedness

0CLV Here is a statement.

0CLW **Lemma 101.42.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 101.7.3 and 101.39.11 properties (1) and (2) are preserved under any base change. By Lemma 101.13.5 we only have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is closed, whenever T is an affine scheme mapping into $\mathcal{Y}$. Hence it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|$ is closed. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that $f(T)$ is closed to finish the proof.

Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced algebraic stack structure on T (Properties of Stacks, Definition 100.10.4). Then $i : \mathcal{Z} \to \mathcal{X}$ is a closed immersion and we have to show that the image of $|\mathcal{Z}| \to |Y|$ is closed. Since closed immersions are quasi-compact (Lemma 101.7.5) and satisfies the existence part of the valuative criterion (Lemma 101.39.14) and since compositions of quasi-compact morphisms are quasi-compact (Lemma 101.7.4) and since compositions preserve the property of satisfying the existence part of the valuative criterion (Lemma 101.39.12) we conclude that it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|(|\mathcal{X}|)$ is closed.

Since $\mathcal{X}$ is quasi-compact (being quasi-compact over the affine Y), we can choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 100.6.2). Suppose that $y \in Y$ is in the closure of the image of $U \to Y$ (in other words, in the closure of the image of $|f|$). Then by Morphisms, Lemma 29.6.5 we can find a valuation ring A with fraction field K and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. By assumption we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and the dotted arrow in the following diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow f \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y & = & Y \end{array}$$

Thus y is in the image of $|f|$ and we win. □

Here is a converse.

0CLX **Lemma 101.42.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *f is quasi-separated, and*
(2) *f is universally closed.*

Then f satisfies the existence part of the valuative criterion.

Proof. Consider a solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where A is a valuation ring with field of fractions K and $\gamma : y \circ j \longrightarrow f \circ x$ as in Definition 101.39.1. By Lemma 101.39.4 in order to find a dotted arrow (after possibly replacing K by an extension and A by a valuation ring dominating it) we may replace $\mathcal{Y}$ by $\mathrm{Spec}(A)$ and $\mathcal{X}$ by $\mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$. Of course we use here that being quasi-separated and universally closed are preserved under base change. Thus we reduce to the case discussed in the next paragraph.

Consider a solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & = & \mathrm{Spec}(A) \end{array}$$

where A is a valuation ring with field of fractions K as in Definition 101.39.1. By Lemma 101.7.7 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$

is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |\mathcal{X}|$ in the closure of x mapping to the closed point of $\mathrm{Spec}(A)$. By Lemma 101.7.9 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathcal{X} \end{array}$$

such that the closed point of $\mathrm{Spec}(A')$ maps to $x' \in |\mathcal{X}|$. It follows that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. □

101.43. Valuative criterion for properness

0CLY Here is the statement.

0CLZ **Lemma 101.43.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is of finite type and quasi-separated. Then the following are equivalent*

(1) *f is proper, and*
(2) *f satisfies both the uniqueness and existence parts of the valuative criterion.*

Proof. A proper morphism is the same thing as a separated, finite type, and universally closed morphism. Thus this lemma follows from Lemmas 101.41.2, 101.41.3, 101.42.1, and 101.42.2. □

101.44. Local complete intersection morphisms

0CJ2 The property "being a local complete intersection morphism" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Lemma 74.20.4 and More on Morphisms of Spaces, Lemmas 76.48.9 and 76.48.10. By Lemma 101.16.1 above, we may define what it means for a morphism of algebraic spaces to be a local complete intersection morphism as follows and it agrees with the already existing notion defined in More on Morphisms of Spaces, Section 76.48 when both source and target are algebraic spaces.

0CJ3 **Definition 101.44.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *local complete intersection morphism* or *Koszul* if the equivalent conditions of Lemma 101.16.1 hold with $\mathcal{P} =$ local complete intersection.

0CJ4 **Lemma 101.44.2.** *The composition of local complete intersection morphisms is a local complete intersection.*

Proof. Combine Remark 101.16.3 with More on Morphisms of Spaces, Lemma 76.48.5. □

0CJ5 **Lemma 101.44.3.** *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Argue exactly as in Remark 101.16.4 (but only for flat $\mathcal{Y}' \to \mathcal{Y}$) using More on Morphisms of Spaces, Lemma 76.48.4. □

0CJ6 **Lemma 101.44.4.** *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & \mathcal{Z} & \end{array}$$

be a commutative diagram of morphisms of algebraic stacks. Assume $\mathcal{Y} \to \mathcal{Z}$ is smooth and $\mathcal{X} \to \mathcal{Z}$ is a local complete intersection morphism. Then $f : \mathcal{X} \to \mathcal{Y}$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose a scheme V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 37.62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. □

101.45. Stabilizer preserving morphisms

0DU6 In the literature a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be *stabilizer preserving* or *fixed-point reflecting* if the induced morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{I}_{\mathcal{Y}}$ is an isomorphism. Such a morphism induces an isomorphism between automorphism groups (Remark 101.19.5) in every point of $\mathcal{X}$. In this section we prove some simple lemmas around this concept.

0DU7 **Lemma 101.45.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{I}_{\mathcal{Y}}$ is an isomorphism, then f is representable by algebraic spaces.*

Proof. Immediate from Lemma 101.6.2. □

0DU8 **Remark 101.45.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $U \to \mathcal{X}$ be a morphism whose source is an algebraic space. Let $G \to H$ be the pullback of the morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{I}_{\mathcal{Y}}$ to U. If Δ_f is unramified, étale, etc, so is $G \to H$. This is true because

$$\begin{array}{ccc} U \times_{\mathcal{X}} U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \Delta_f} \\ U \times_{\mathcal{Y}} U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is cartesian and the morphism $G \to H$ is the base change of the left vertical arrow by the diagonal $U \to U \times U$. Compare with the proof of Lemma 101.6.6.

0DU9 **Lemma 101.45.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an unramified morphism of algebraic stacks. The following are equivalent*

1. *$\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{I}_{\mathcal{Y}}$ is an isomorphism, and*
2. *f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5) for all $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{I}_{\mathcal{Y}}$ to U. By Remark 101.45.2 and Lemma 101.36.9 the morphism $G \to H$ is étale. Condition (1) is equivalent to the condition that $G \to H$ is an isomorphism (this follows for example by applying

Properties of Stacks, Lemma 100.3.3). Condition (2) is equivalent to the condition that for every $u \in U$ the morphism $G_u \to H_u$ of fibres is an isomorphism. Thus (1) $\Rightarrow$ (2) is trivial. If (2) holds, then $G \to H$ is a surjective, universally injective, étale morphism of algebraic spaces. Such a morphism is an isomorphism by Morphisms of Spaces, Lemma 67.51.2. □

0DUA **Lemma 101.45.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *f is representable by algebraic spaces and unramified, and*
(2) *$\mathcal{I}_\mathcal{Y} \to \mathcal{Y}$ is proper.*

Then the set of $x \in |\mathcal{X}|$ such that f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5) is open. Letting $\mathcal{U} \subset \mathcal{X}$ be the corresponding open substack, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism.

[**Ryd07a**, Proposition 3.5] and [**Alp10**, Proposition 2.5]

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. By Remark 101.45.2 and Lemma 101.36.9 the morphism $G \to H$ is étale. Since f is representable by algebraic spaces, we see that $G \to H$ is a monomorphism. Hence $G \to H$ is an open immersion, see Morphisms of Spaces, Lemma 67.51.2. By assumption $H \to U$ is proper.

With these preparations out of the way, we can prove the lemma as follows. The inverse image of the subset of $|\mathcal{X}|$ of the lemma is clearly the set of $u \in U$ such that $G_u \to H_u$ is an isomorphism (since after all G_u is an open sub group algebraic space of H_u). This is an open subset because the complement is the image of the closed subset $|H| \setminus |G|$ and $|H| \to |U|$ is closed. By Properties of Stacks, Lemma 100.9.12 we can consider the corresponding open substack $\mathcal{U}$ of $\mathcal{X}$. The final statement of the lemma follows from applying Lemma 101.45.3 to $\mathcal{U} \to \mathcal{Y}$. □

0DUB **Lemma 101.45.5.** *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a cartesian diagram of algebraic stacks.

(1) *Let $x' \in |\mathcal{X}'|$ with image $x \in |\mathcal{X}|$. If f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 101.19.5), then f' induces an isomorphism between automorphism groups at x' and $f(x')$.*
(2) *If $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, then $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{I}_{\mathcal{Y}'}$ is an isomorphism.*

Proof. Omitted. □

0DUC **Lemma 101.45.6.** *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a cartesian diagram of algebraic stacks. If f induces an isomorphism between automorphism groups at points (Remark 101.19.5), then

$$\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}') \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y}') \times \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$$

is injective on isomorphism classes for any field k.

Proof. We have to show that given (y', x) there is at most one x' mapping to it. By our construction of 2-fibre products, a morphism x' is given by a triple (x, y', α) where $\alpha : g \circ y' \to f \circ x$ is a 2-morphism. Now, suppose we have a second such triple (x, y', β). Then α and β differ by a k-valued point ϵ of the automorphism group algebraic space $G_{f(x)}$. Since f induces an isomorphism $G_x \to G_{f(x)}$ by assumption, this means we can lift ϵ to a k-valued point γ of G_x. Then $(\gamma, \mathrm{id}) : (x, y', \alpha) \to (x, y', \beta)$ is an isomorphism as desired. □

0DUD **Lemma 101.45.7.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is étale, f induces an isomorphism between automorphism groups at points (Remark 101.19.5), and for every algebraically closed field k the functor*

$$f : \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y})$$

is an equivalence. Then f is an isomorphism.

Proof. By Lemma 101.14.5 we see that f is universally injective. Combining Lemmas 101.45.1 and 101.45.3 we see that f is representable by algebraic spaces. Hence f is an open immersion by Morphisms of Spaces, Lemma 67.51.2. To finish we remark that the condition in the lemma also guarantees that f is surjective. □

101.46. Normalization

0GMH This section is the analogue of Morphisms of Spaces, Section 67.49.

0GMI **Lemma 101.46.1.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. *there is a surjective smooth morphism $U \to \mathcal{X}$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,*
2. *for every scheme U and every smooth morphism $U \to \mathcal{X}$ every quasi-compact open of U has finitely many irreducible components,*
3. *for every algebraic space Y and smooth morphism $Y \to \mathcal{X}$ the space Y satisfies the equivalent conditions of Morphisms of Spaces, Lemma 67.49.1, and*
4. *for every quasi-compact algebraic stack $\mathcal{Y}$ smooth over $\mathcal{X}$ the space $|\mathcal{Y}|$ has finitely many irreducible components.*

Proof. The equivalence of (1), (2), and (3) follow from Descent, Lemma 35.16.3, Properties of Stacks, Lemma 100.7.1, and Morphisms of Spaces, Lemma 67.49.1. It is also clear from these references that condition (4) implies condition (1). Conversely, assume the equivalent conditions (1), (2), and (3) hold and let $\mathcal{Y} \to \mathcal{X}$ be a smooth morphism of algebraic stacks with $\mathcal{Y}$ quasi-compact. Then we can choose an affine scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ by Properties of Stacks, Lemma 100.6.2. Since V has finitely many irreducible components by (2) and since $|V| \to |\mathcal{Y}|$ is surjective and continuous, we conclude that $|\mathcal{Y}|$ has finitely many irreducible components by Topology, Lemma 5.8.5. □

0GMJ **Lemma 101.46.2.** *Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 101.46.1. Then there exists an integral morphism of algebraic stacks*

$$\mathcal{X}^\nu \longrightarrow \mathcal{X}$$

such that for every scheme U and smooth morphism $U \to \mathcal{X}$ the fibre product $\mathcal{X}^\nu \times_{\mathcal{X}} U$ is the normalization of U.

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Set $R = U \times_{\mathcal{X}} U$. Recall that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces and a presentation $\mathcal{X} = [U/R]$ of $\mathcal{X}$, see Algebraic Stacks, Lemmas 94.16.1 and 94.16.2 and Definition 94.16.5. The assumption on $\mathcal{X}$ means that the normalization U^ν of U is defined, see Morphisms, Definition 29.54.1. By Morphisms of Spaces, Lemma 67.49.5 taking normalization commutes with smooth morphisms of algebraic spaces. Thus we see that the normalization R^ν of R is isomorphic to both $R \times_{s,U} U^\nu$ and $U^\nu \times_{U,t} R$. Thus we obtain two smooth morphisms $s^\nu : R^\nu \to U^\nu$ and $t^\nu : R^\nu \to U^\nu$ of algebraic spaces. A formal computation with fibre products shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the smooth morphism $c : R \times_{s,U,t} R \to R$ extends to c^ν as well. Similarly, the inverse $i : R \to R$ (an isomorphism) induces an isomorphism $i^\nu : R^\nu \to R^\nu$. Finally, the identity $e : U \to R$ lifts to $e^\nu : U^\nu \to R^\nu$ for example because e is a section of s and $R^\nu = R \times_{U,s} U^\nu$. We claim that $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu)$ is a smooth groupoid in algebraic spaces. To see this involves checking the axioms (1), (2)(a), (2)(b), (3)(a), and (3)(b) of Groupoids, Section 39.13 for $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu, e^\nu, i^\nu)$. For example, for (1) we have to see that the two morphisms $a, b : R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \to R^\nu$ we obtain are the same. This holds because we know that the corresponding pair of morphisms $R \times_{s,U,t} R \times_{s,U,t} R \to R$ are the same and the morphisms a and b are the unique extensions of this morphism to the normalizations. Similarly for the other axioms.

Consider the algebraic stack $\mathcal{X}^\nu = [U^\nu/R^\nu]$ (Algebraic Stacks, Theorem 94.17.3). Since we have a morphism $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu) \to (U, R, s, t, c)$ of groupoids in algebraic spaces, we obtain a morphism $\nu : \mathcal{X}^\nu \to \mathcal{X}$ of algebraic stacks. Since $R^\nu = R \times_{s,U} U^\nu$ we see that $U^\nu = \mathcal{X}^\nu \times_{\mathcal{X}} U$ by Groupoids in Spaces, Lemma 78.25.3. In particular, as $U^\nu \to U$ is integral, we see that ν is integral. We omit the verification that the base change property stated in the lemma holds for every smooth morphism from a scheme to $\mathcal{X}$. □

This leads us to the following definition.

0GMK **Definition 101.46.3.** Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 101.46.1. We define the *normalization* of $\mathcal{X}$ as the morphism

$$\nu : \mathcal{X}^\nu \longrightarrow \mathcal{X}$$

constructed in Lemma 101.46.2.

101.47. Points and specializations

0GVY This section is the analogue of Decent Spaces, Section 68.7.

0GVZ **Lemma 101.47.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $f : U \to \mathcal{X}$ be a smooth morphism where U is an algebraic space. Let $x' \rightsquigarrow x$ be a specialization of points of $|\mathcal{X}|$. Let $u \in |U|$ with $f(u) = x$. If $(\mathcal{X}, x')$ satisfy the equivalent conditions of Properties of Stacks, Lemma 100.14.1, then there exists a specialization $u' \rightsquigarrow u$ in $|U|$ with $f(u') = x'$.*

Proof. Choose an étale morphism $(U_1, u_1) \to (U, u)$ where U_1 is an affine scheme. Then we may and do replace U by U_1. Thus we may assume U is an affine scheme.

Consider the algebraic space $R = U \times_{\mathcal{X}} U$ with smooth projections $t, s : R \to U$. Choose a point $w \in U$ mapping to x'; this is possible as $f : |U| \to |\mathcal{X}|$ is open. By our assumption on x' the fibre $F' = t^{-1}(w) = R \times_{t,U} w$ of $t : R \to U$ over w is a quasi-compact algebraic space. Choose an affine scheme T and a surjective étale morphism $T \to F'$. The fact that $x' \rightsquigarrow x$ means that u is in the closure of the image of the morphism

$$T \to F' \to R \xrightarrow{s} U$$

Namely, this image is the fibre of $|U| \to |\mathcal{X}'|$ over x'; if some $u \in V \subset |U|$ open is disjoint from this fibre, then $f(V)$ is an open neighbourhood of x not containing x'; contradiction. Thus by Morphisms, Lemma 29.6.5 we see that there exists $u' \in |U|$ in the fibre of $|U| \to |\mathcal{X}|$ over x' which specializes to u. □

101.48. Decent algebraic stacks

0GW0 This section is the analogue of Decent Spaces, Section 68.6. In particular, the following definition is compatible with the notion of a decent algebraic space defined there.

0GW1 **Definition 101.48.1.** Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *decent* if for every $x \in |\mathcal{X}|$ the equivalent conditions of Properties of Stacks, Lemma 100.14.1 are satisfied.

Some people would rephrase this definition by saying that every point of $\mathcal{X}$ is quasi-compact. A slightly stronger condition would be to ask that any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-separated as well as quasi-compact.

0GW2 **Lemma 101.48.2.** *A quasi-separated algebraic stack $\mathcal{X}$ is decent. More generally, if $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, then $\mathcal{X}$ is decent.*

Proof. Namely, if $\mathcal{X}$ is quasi-separated, then any morphism $f : T \to \mathcal{X}$ whose source is a quasi-compact scheme T, is quasi-compact, see Lemma 101.7.7. If Δ is on known to be quasi-compact, then one uses the description

$$T \times_{f,\mathcal{X},f'} T' = (T \times T') \times_{(f,f'),\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

to prove this. Details omitted. □

0GW3 **Lemma 101.48.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume Y is decent and f is representable (by schemes) or f is representable by algebraic spaces and quasi-separated. Then $\mathcal{X}$ is decent.*

Proof. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Choose a morphism $y : \mathrm{Spec}(k) \to \mathcal{Y}$ in the equivalence class defining y. Set $\mathcal{X}_y = \mathrm{Spec}(k) \times_{y,\mathcal{Y}} \mathcal{X}$. Choose a point $x' \in |\mathcal{X}_y|$ mapping to x, see Properties of Stacks, Lemma 100.4.3. Choose a morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}_y$ in the equivalence class of x'. Diagram

$$\begin{array}{ccccc} \mathrm{Spec}(k') & \xrightarrow[x']{} & \mathcal{X}_y & \longrightarrow & \mathcal{X} \\ & & \downarrow & & \downarrow \\ & & \mathrm{Spec}(k) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

The morphism y is quasi-compact if $\mathcal{Y}$ is decent. Hence $\mathcal{X}_y \to \mathcal{X}$ is quasi-compact as a base change (Lemma 101.7.3). Thus to conclude it suffices to prove that x' is

quasi-compact (Lemma 101.7.4). If f is representable, then $\mathcal{X}_y$ is a scheme and x' is quasi-compact. If f is representable by algebraic spaces and quasi-separated, then $\mathcal{X}_y$ is a quasi-separated algebraic space and hence decent (Decent Spaces, Lemma 68.17.2). □

0GW4 **Lemma 101.48.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is quasi-compact and surjective and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.*

Proof. Let $x : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism where k is a field and denote $y = f \circ x$. Since f is surjective, every point of $|\mathcal{Y}|$ arises in this manner, see Properties of Stacks, Lemma 100.4.4. Consider an affine scheme T and morphism $T \to \mathcal{Y}$. It suffices to show that $T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$ is quasi-compact, see Lemma 101.7.10. We have

$$(T \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},x} \mathrm{Spec}(k) = T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$$

The morphism $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is quasi-compact hence $T \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Since x is a quasi-compact morphism as $\mathcal{X}$ is decent we see that the displayed fibre product is quasi-compact. □

0GW5 **Lemma 101.48.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.*

Proof. Assume $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent. Note that f is a universal homeomorphism by Lemma 101.28.13. Thus the lemma follows from Lemma 101.48.4. □

101.49. Points on decent stacks

0GW6 This section is the analogue of Decent Spaces, Section 68.12. We do not know whether or not the topological space associated to a decent algebraic stack is always sober; see Proposition 101.49.3 for a slightly weaker result.

0GW7 **Lemma 101.49.1.** *Let $\mathcal{X}$ be a decent algebraic stack. Then $|\mathcal{X}|$ is Kolmogorov (see Topology, Definition 5.8.6).*

Proof. Let $x_1, x_2 \in |\mathcal{X}|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|\mathcal{Z}|$ equal to $\overline{\{x_1\}} = \overline{\{x_2\}}$. By Lemma 101.48.3 we see that $\mathcal{Z}$ is decent. After replacing $\mathcal{X}$ by $\mathcal{Z}$ we reduce to the case discussed in the next paragraph.

Assume $|\mathcal{X}|$ is irreducible with generic points x_1 and x_2. Pick an affine scheme U and $u_1, u_2 \in U$ and a smooth morphism $f : U \to \mathcal{X}$ such that $f(u_i) = x_i$. Then we find a third point $u_3 \in U$ which is the generic point of an irreducible component of U whose image $x_3 \in |\mathcal{X}|$ is *also* a generic point of $|\mathcal{X}|$. Namely, we can simply choose u_3 any generic point of an irreducible component passing through u_1 (or u_2 if you like). In the next paragraph we will show that $x_1 = x_3$ and $x_2 = x_3$ which will prove what we want.

By symmetry it suffices to prove that $x_1 = x_3$. Since x_1 is a generic point of $|\mathcal{X}|$ we have a specialization $x_1 \rightsquigarrow x_3$. By Lemma 101.47.1 we can find a specialization $u_1' \rightsquigarrow u_3$ in U (!) mapping to $x_1 \rightsquigarrow x_3$. However, u_3 is the generic point of an irreducible component and hence $u_1' = u_3$ as desired. □

0GW8 **Lemma 101.49.2.** *Let $\mathcal{X}$ be a decent, locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is a sober locally Noetherian topological space.*

Proof. By Lemma 101.8.3 the topological space $|\mathcal{X}|$ is locally Noetherian. By Lemma 101.49.1 the topological space $|\mathcal{X}|$ is Kolmogorov. By Lemma 101.8.4 the topological space $|\mathcal{X}|$ is quasi-sober. This finishes the proof, see Topology, Definition 5.8.6. □

0GW9 **Proposition 101.49.3.** *Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then $|\mathcal{X}|$ is sober.*

Proof. By Lemma 101.49.1 we know that $|\mathcal{X}|$ is Kolmogorov (in fact we will reprove this). Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 100.10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\mathcal{Z}$ is a decent algebraic stack, see Lemma 101.48.3. Also, the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ (Lemma 101.5.6). Hence $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (Lemma 101.7.3). Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a generic point. By Proposition 101.29.1. there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe in addition to all the other properties. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 101.28.13. In particular, X is quasi-compact and $|X|$ is irreducible. Also, by Lemma 101.48.5 we see that X is a decent algebraic space. Then $|\mathcal{X}| = |X|$ is sober by Decent Spaces, Proposition 68.12.4 and hence has a (unique) generic point. □

101.50. Integral algebraic stacks

0GWA This section is the analogue of Spaces over Fields, Section 72.4. Motivated by the considerations in that section and by the result of Proposition 101.49.3 we define an integral algebraic stack as follows (and it does not conflict with the already existing definitions of integral schemes and integral algebraic spaces).

0GWB **Definition 101.50.1.** We say an algebraic stack $\mathcal{X}$ is *integral* if it is reduced, decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, and $|\mathcal{X}|$ is irreducible.

Note that if $\mathcal{X}$ is quasi-separated, then for it to be integral, it suffices that $\mathcal{X}$ is reduced and that $|\mathcal{X}|$ is irreducible, see Lemma 101.50.3.

0GWC **Lemma 101.50.2.** *Let $\mathcal{X}$ be an integral algebraic stack. Then*

(1) *$|\mathcal{X}|$ is sober, irreducible, and has a unique generic point,*
(2) *there exists an open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an integral scheme U.*

Proof. Proposition 101.49.3 tells us that $|\mathcal{X}|$ is sober. Of course it is also irreducible and hence has a unique generic point x (by the definition of sobriety). Proposition 101.29.1 shows the existence of a dense open $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an algebraic space U. Then U is a decent algebraic space by Lemma 101.48.5 (and the fact that $\mathcal{U}$ is decent by Lemma 101.48.3). Since $|U| = |\mathcal{U}|$ we see that $|U|$ is irreducible. Finally, since $\mathcal{U}$ is reduced the morphism $\mathcal{U} \to U$ factors through U_{red}, see Properties of Stacks, Lemma 100.10.3. Now since $\mathcal{U} \to U$ is flat, locally of finite presentation, and surjective (Lemma 101.28.8), this implies that $U = U_{red}$,

i.e., U is reduced (small detail omitted). It follows that U is an integral algebraic space, see Spaces over Fields, Definition 72.4.1. Then finally, we may replace U (and correspondingly $\mathcal{U}$) by an open subspace and assume that U is an integral scheme, see discussion in Spaces over Fields, Section 72.4. □

0GWD **Lemma 101.50.3.** *Let $\mathcal{X}$ be an algebraic stack which is reduced and quasi-separated and whose associated topological space $|\mathcal{X}|$ is irreducible. Then $\mathcal{X}$ is integral.*

Proof. If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is decent by Lemma 101.48.2. If $\mathcal{X}$ is quasi-separated, then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, hence $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact as the base change of Δ by Δ, see Lemma 101.7.3. Thus we see that all the hypotheses of Definition 101.50.1 hold (and we also see that we may replace "quasi-separated" by "$\Delta_\mathcal{X}$ is quasi-compact"). □

0GWE **Lemma 101.50.4.** *Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. There are canonical bijections between the following sets:*

(1) *the set of points of $\mathcal{X}$, i.e., $|\mathcal{X}|$,*
(2) *the set of irreducible closed subsets of $|\mathcal{X}|$,*
(3) *the set of integral closed substacks of $\mathcal{X}$.*

The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends $\mathcal{Z}$ to $|\mathcal{Z}|$.

Proof. Our map defines a bijection between (1) and (2) as $|\mathcal{X}|$ is sober by Proposition 101.49.3. Given $T \subset |\mathcal{X}|$ closed and irreducible, there is a unique reduced closed substack $\mathcal{Z} \subset \mathcal{X}$ such that $|\mathcal{Z}| = T$, namely, $\mathcal{Z}$ is the reduced induced subspace structure on T, see Properties of Stacks, Definition 100.10.4. Then $\mathcal{Z}$ is an integral algebraic stack because it is decent (Lemma 101.48.3), the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (as the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemma 101.5.6), $\mathcal{Z}$ is reduced, and $|\mathcal{Z}|$ is irreducible. □

101.51. Residual gerbes

0H23 This section is the continuation of Properties of Stacks, Section 100.11.

0H24 **Lemma 101.51.1.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbe $\mathcal{Z}_y \subset \mathcal{Y}$ of $\mathcal{Y}$ at y exists and that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. Then $\mathcal{Z}_x = \mathcal{Z}_y \times_\mathcal{Y} \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x.*

Proof. The morphism $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism as the base change of the monomorphism $\mathcal{Z}_y \to \mathcal{Y}$. The morphism π is a universal homeomorphism by Lemma 101.28.13 and hence $|\mathcal{Z}_x| = \{x\}$. Finally, the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ is smooth as a base change of the smooth morphism π, see Lemma 101.33.8. Hence as $\mathcal{Z}_y$ is reduced and locally Noetherian, so is $\mathcal{Z}_x$ (details omitted). Thus $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x by Properties of Stacks, Definition 100.11.8. □

0H25 **Lemma 101.51.2.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. Assume*

(1) *$\mathcal{X}$ is decent or locally Noetherian (or both),*
(2) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact,*
(3) *$|f|(|\mathcal{Y}|)$ is contained in $\{x\} \subset |\mathcal{X}|$, and*
(4) *$\mathcal{Y}$ is reduced.*

Then f factors through the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x (whose existence is guaranteed by Lemma 101.31.1 or 101.31.3).

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 100.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. By Properties of Stacks, Lemma 100.10.3 the morphism f factors through $\mathcal{X}'$. If $\mathcal{X}$ is decent, then by Lemma 101.48.3 the stack $\mathcal{X}'$ is decent. If $\mathcal{X}$ is locally Noetherian, then $\mathcal{X}'$ is locally Noetherian (details omitted). Note that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is the base change of $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ by Lemma 101.5.6 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact by Lemma 101.7.3. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is a generic point. By Proposition 101.29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$. Repeating the arguments above we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X} \to X$ is a gerbe over the algebraic space X. If $\mathcal{X}$ is decent, then by Lemmas 101.28.13 and 101.48.4 the space X is decent. If $\mathcal{X}$ is locally Noetherian, then X is locally Noetherian by fppf descent (details omitted). Hence the corresponding result holds for X, see Decent Spaces, Lemma 68.13.10 or 68.13.9 (small detail omitted). Applying Lemma 101.51.1 we conclude that the result holds for $\mathcal{X}$ as well. □

0H26 **Remark 101.51.3.** We do not know whether Lemma 101.51.2 holds if we only assume $\mathcal{X}$ is locally Noetherian, i.e., we drop the assumption on the inertia being quasi-compact. In this case, if x is a closed point, this is certainly true as follows from the following much simpler lemma.

0H27 **Lemma 101.51.4.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ with residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ (Lemma 101.31.3). Then x is a closed point of $|\mathcal{X}|$ if and only if the morphism $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion.*

Proof. If $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion, then x is a closed point of $|\mathcal{X}|$, see for example Lemma 101.37.4. Conversely, assume x is a closed point of $|\mathcal{X}|$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|Z| = \{x\}$ (Properties of Stacks, Lemma 100.10.1). Then $\mathcal{Z}$ is a locally Noetherian algebraic stack by Lemmas 101.17.4 and 101.17.5. Since also $\mathcal{Z}$ is reduced and $|\mathcal{Z}| = \{x\}$ it follows that $\mathcal{Z} = \mathcal{Z}_x$ is the residual gerbe by definition. □

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CHAPTER 102

Limits of Algebraic Stacks

0CMM

102.1. Introduction

0CMN In this chapter we put material related to limits of algebraic stacks. Many results on limits of algebraic stacks and algebraic spaces have been obtained by David Rydh in [**Ryd08**].

102.2. Conventions

0CMP We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

102.3. Morphisms of finite presentation

0CMQ This section is the analogue of Limits of Spaces, Section 70.3. There we defined what it means for a transformation of functors on Sch to be limit preserving (we suggest looking at the characterization in Limits of Spaces, Lemma 70.3.2). In Criteria for Representability, Section 97.5 we defined the notion "limit preserving on objects". Recall that in Artin's Axioms, Section 98.11 we have defined what it means for a category fibred in groupoids over Sch to be limit preserving. Combining these we get the following notion.

0CMR **Definition 102.3.1.** Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f is *limit preserving* if for every directed limit $U = \lim U_i$ of affine schemes over S the diagram

$$\begin{array}{ccc} \operatorname{colim} \mathcal{X}_{U_i} & \longrightarrow & \mathcal{X}_U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{colim} \mathcal{Y}_{U_i} & \longrightarrow & \mathcal{Y}_U \end{array}$$

of fibre categories is 2-cartesian.

0CMS **Lemma 102.3.2.** *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is limit preserving (Definition 102.3.1), then f is limit preserving on objects (Criteria for Representability, Section 97.5).*

Proof. If for every directed limit $U = \lim U_i$ of affine schemes over U, the functor

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow (\operatorname{colim} \mathcal{Y}_{U_i}) \times_{\mathcal{Y}_U} \mathcal{X}_U$$

is essentially surjective, then f is limit preserving on objects. □

0CMT **Lemma 102.3.3.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. For each i we have

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{Z}_{U_i}$$

Filtered colimits commute with 2-fibre products of categories (details omitted) hence if p is limit preserving we get

$$\begin{aligned} \operatorname{colim}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} &= \operatorname{colim} \mathcal{X}_{U_i} \times_{\operatorname{colim} \mathcal{Y}_{U_i}} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i} \times_{\operatorname{colim} \mathcal{Y}_{U_i}} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Z}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

0CMU **Lemma 102.3.4.** *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. If p and q are limit preserving we get

$$\begin{aligned} \operatorname{colim} \mathcal{X}_{U_i} &= \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Y}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Z}_U} \operatorname{colim} \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

0CMV **Lemma 102.3.5.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

1. *p is limit preserving,*
2. *p is limit preserving on objects, and*
3. *p is locally of finite presentation (see Algebraic Stacks, Definition 94.10.1).*

Proof. In Criteria for Representability, Lemma 97.5.3 we have seen that (2) and (3) are equivalent. Thus it suffices to show that (1) and (2) are equivalent. One direction we saw in Lemma 102.3.2. For the other direction, let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. We have to show that

$$\operatorname{colim} \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \operatorname{colim} \mathcal{Y}_{U_i}$$

is an equivalence. Since we are assuming (2) we know that it is essentially surjective. Hence we need to prove it is fully faithful. Since p is faithful on fibre categories (Algebraic Stacks, Lemma 94.9.2) we see that the functor is faithful. Let x_i and x_i' be objects in the fibre category of $\mathcal{X}$ over U_i. The functor above sends x_i to $(x_i|_U, p(x_i), can)$ where can is the canonical isomorphism $p(x_i|_U) \to p(x_i)|_U$. Thus we assume given a morphism

$$(\alpha, \beta_i) : (x_i|_U, p(x_i), can) \longrightarrow (x_i'|_U, p(x_i'), can)$$

in the category of the right hand side of the first displayed arrow of this proof. Our task is to produce an $i' \geq i$ and a morphism $x_i|_{U_{i'}} \to x_i'|_{U_{i'}}$ which maps to $(\alpha, \beta_i|_{U_{i'}})$.

Set $y_i = p(x_i)$ and $y_i' = p(x_i')$. By (Algebraic Stacks, Lemma 94.9.2) the functor

$$X_{y_i} : (Sch/U_i)^{opp} \to Sets, \quad V/U_i \mapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : f(x) \to y_i|V\}/\cong$$

is an algebraic space over U_i and the same is true for the analogously defined functor $X_{y_i'}$. Since (2) is equivalent to (3) we see that $X_{y_i'}$ is locally of finite presentation over U_i. Observe that (x_i, id) and (x_i', id) define U_i-valued points of X_{y_i} and $X_{y_i'}$. There is a transformation of functors

$$\beta_i : X_{y_i} \to X_{y_i'}, \quad (x/V, \phi) \mapsto (x/V, \beta_i|_V \circ \phi)$$

in other words, this is a morphism of algebraic spaces over U_i. We claim that

$$\begin{array}{ccccc} U & \longrightarrow & & & U_i \\ \downarrow & & & & \downarrow{\scriptstyle (x_i', \mathrm{id})} \\ U_i & \xrightarrow{(x_i, \mathrm{id})} & X_{y_i} & \xrightarrow{\beta_i} & X_{y_i'} \end{array}$$

commutes. Namely, this is equivalent to the condition that the pairs $(x_i|_U, \beta_i|_U)$ and $(x_i'|_U, \mathrm{id})$ as in the definition of the functor $X_{y_i'}$ are isomorphic. And the morphism $\alpha : x_i|_U \to x_i'|_U$ exactly produces such an isomorphism. Arguing backwards the reader sees that if we can find an $i' \geq i$ such that the diagram

$$\begin{array}{ccccc} U_{i'} & \longrightarrow & & & U_i \\ \downarrow & & & & \downarrow{\scriptstyle (x_i', \mathrm{id})} \\ U_i & \xrightarrow{(x_i, \mathrm{id})} & X_{y_i} & \xrightarrow{\beta_i} & X_{y_i'} \end{array}$$

commutes, then we obtain an isomorphism $x_i|_{U_{i'}} \to x_i'|_{U_{i'}}$ which is a solution to the problem posed in the preceding paragraph. However, the diagonal morphism

$$\Delta : X_{y_i'} \to X_{y_i'} \times_{U_i} X_{y_i'}$$

is locally of finite presentation (Morphisms of Spaces, Lemma 67.28.10) hence the fact that $U \to U_i$ equalizes the two morphisms to $X_{y_i'}$, means that for some $i' \geq i$ the morphism $U_{i'} \to U_i$ equalizes the two morphisms, see Limits of Spaces, Proposition 70.3.10. □

0CMW **Lemma 102.3.6.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

(1) *the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is limit preserving, and*
(2) *for every directed limit $U = \lim U_i$ of affine schemes over S the functor*

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \mathrm{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful.

In particular, if p is limit preserving, then Δ is too.

Proof. Let $U = \lim U_i$ be a directed limit of affine schemes over S. We claim that the functor

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \mathrm{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful if and only if the functor

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U} \mathrm{colim}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i}$$

is an equivalence. This will prove the lemma. Since $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U = \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{X}_U$ and $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{X}_{U_i}$ this is a purely category theoretic assertion which we discuss in the next paragraph.

Let $\mathcal{I}$ be a filtered index category. Let $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ be systems of groupoids over $\mathcal{I}$. Let $p : (\mathcal{C}_i) \to (\mathcal{D}_i)$ be a map of systems of groupoids over $\mathcal{I}$. Suppose we have a functor $p : \mathcal{C} \to \mathcal{D}$ of groupoids and functors $f : \operatorname{colim} \mathcal{C}_i \to \mathcal{C}$ and $g : \operatorname{colim} \mathcal{D}_i \to \mathcal{D}$ fitting into a commutative diagram

$$\begin{array}{ccc} \operatorname{colim} \mathcal{C}_i & \xrightarrow[f]{} & \mathcal{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{colim} \mathcal{D}_i & \xrightarrow{g} & \mathcal{D} \end{array}$$

Then we claim that

$$A : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\mathcal{D}} \operatorname{colim} \mathcal{D}_i$$

is fully faithful if and only if the functor

$$B : \operatorname{colim} \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} \operatorname{colim}(\mathcal{C}_i \times_{\mathcal{D}_i} \mathcal{C}_i)$$

is an equivalence. Set $\mathcal{C}' = \operatorname{colim} \mathcal{C}_i$ and $\mathcal{D}' = \operatorname{colim} \mathcal{D}_i$. Since 2-fibre products commute with filtered colimits we see that A and B become the functors

$$A' : \mathcal{C}' \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \quad \text{and} \quad B' : \mathcal{C}' \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}')$$

Thus it suffices to prove that if

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow[f]{} & \mathcal{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

is a commutative diagram of groupoids, then A' is fully faithful if and only if B' is an equivalence. This follows from Categories, Lemma 4.35.10 (with trivial, i.e., punctual, base category) because

$$\mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}') = \mathcal{C}' \times_{A', \mathcal{C} \times_{\mathcal{D}} \mathcal{D}', A'} \mathcal{C}'$$

This finishes the proof. □

0CMX **Lemma 102.3.7.** *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. If $\mathcal{X} \to S$ is locally of finite presentation, then $\mathcal{X}$ is limit preserving in the sense of Artin's Axioms, Definition 98.11.1 (equivalently: the morphism $\mathcal{X} \to S$ is limit preserving).*

Proof. Choose a surjective smooth morphism $U \to \mathcal{X}$ for some scheme U. Then $U \to S$ is locally of finite presentation, see Morphisms of Stacks, Section 101.27. We can write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 94.16.2. Since U is locally of finite presentation over S it follows that the algebraic space R is locally of finite presentation over S. Recall that $[U/R]$ is the stack in groupoids over $(Sch/S)_{fppf}$ obtained by stackyfying the category fibred in groupoids whose fibre category over T is the groupoid $(U(T), R(T), s, t, c)$. Since U and R are limit preserving as functors (Limits of Spaces, Proposition 70.3.10) this category fibred in groupoids is limit preserving. Thus it suffices to show that fppf stackyfication preserves the property of being limit preserving. This is true (hint: use Topologies, Lemma 34.13.2). However, we

give a direct proof below using that in this case we know what the stackyfication amounts to.

Let $T = \lim T_\lambda$ be a directed limit of affine schemes over S. We have to show that the functor

$$\operatorname{colim}[U/R]_{T_\lambda} \longrightarrow [U/R]_T$$

is an equivalence of categories. Let us show this functor is essentially surjective. Let $x \in \mathrm{Ob}([U/R]_T)$. In Groupoids in Spaces, Lemma 78.24.1 the reader finds a description of the category $[U/R]_T$. In particular x corresponds to an fppf covering $\{T_i \to T\}_{i\in I}$ and a $[U/R]$-descent datum (u_i, r_{ij}) relative to this covering. After refining this covering we may assume it is a standard fppf covering of the affine scheme T. By Topologies, Lemma 34.13.2 we may choose a λ and a standard fppf covering $\{T_{\lambda,i} \to T_\lambda\}_{i\in I}$ whose base change to T is equal to $\{T_i \to T\}_{i\in I}$. For each i, after increasing λ, we can find a $u_{\lambda,i} : T_{\lambda,i} \to U$ whose composition with $T_i \to T_{\lambda,i}$ is the given morphism u_i (this is where we use that U is limit preserving). Similarly, for each i, j, after increasing λ, we can find a $r_{\lambda,ij} : T_{\lambda,i} \times_{T_\lambda} T_{\lambda,j} \to R$ whose composition with $T_{ij} \to T_{\lambda,ij}$ is the given morphism r_{ij} (this is where we use that R is limit preserving). After increasing λ we can further assume that

$$s \circ r_{\lambda,ij} = u_{\lambda,i} \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{\lambda,ij} = u_{\lambda,j} \circ \mathrm{pr}_1,$$

and

$$c \circ (r_{\lambda,jk} \circ \mathrm{pr}_{12}, r_{\lambda,ij} \circ \mathrm{pr}_{01}) = r_{\lambda,ik} \circ \mathrm{pr}_{02}.$$

In other words, we may assume that $(u_{\lambda,i}, r_{\lambda,ij})$ is a $[U/R]$-descent datum relative to the covering $\{T_{\lambda,i} \to T_\lambda\}_{i\in I}$. Then we obtain a corresponding object of $[U/R]$ over T_λ whose pullback to T is isomorphic to x as desired. The proof of fully faithfulness works in exactly the same way using the description of morphisms in the fibre categories of $[U/T]$ given in Groupoids in Spaces, Lemma 78.24.1. □

0CMY **Proposition 102.3.8.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is limit preserving,*
(2) *f is limit preserving on objects, and*
(3) *f is locally of finite presentation.*

This is a special case of [**EG15**, Lemma 2.3.15]

Proof. Assume (3). Let $T = \lim T_i$ be a directed limit of affine schemes. Consider the functor

$$\operatorname{colim} \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T \times_{\mathcal{Y}_T} \operatorname{colim} \mathcal{Y}_{T_i}$$

Let (x, y_i, β) be an object on the right hand side, i.e., $x \in \mathrm{Ob}(\mathcal{X}_T)$, $y_i \in \mathrm{Ob}(\mathcal{Y}_{T_i})$, and $\beta : f(x) \to y_i|_T$ in $\mathcal{Y}_T$. Then we can consider (x, y_i, β) as an object of the algebraic stack $\mathcal{X}_{y_i} = \mathcal{X} \times_{\mathcal{Y}, y_i} T_i$ over T. Since $\mathcal{X}_{y_i} \to T_i$ is locally of finite presentation (as a base change of f) we see that it is limit preserving by Lemma 102.3.7. This means that (x, y_i, β) comes from an object over $T_{i'}$ for some $i' \geq i$ and unwinding the definitions we find that (x, y_i, β) is in the essential image of the displayed functor. In other words, the displayed functor is essentially surjective. Another formulation is that this means f is limit preserving on objects. Now we apply this to the diagonal Δ of f. Namely, by Morphisms of Stacks, Lemma 101.27.7 the morphism Δ is locally of finite presentation. Thus the argument above shows that Δ is limit preserving on objects. By Lemma 102.3.5 this implies that Δ is limit preserving. By Lemma 102.3.6 we conclude that the displayed functor above is fully

faithful. Thus it is an equivalence (as we already proved essential surjectivity) and we conclude that (1) holds.

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Criteria for Representability, Lemma 97.5.1 the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is limit preserving on objects. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Since a smooth morphism is locally of finite presentation, we see that $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is limit preserving (first part of the proof). By Criteria for Representability, Lemma 97.5.2 we find that the composition $U \to V$ is limit preserving on objects. We conclude that $U \to V$ is locally of finite presentation, see Criteria for Representability, Lemma 97.5.3. This is exactly the condition that f is locally of finite presentation, see Morphisms of Stacks, Definition 101.27.1. □

102.4. Descending properties

0CPX This section is the analogue of Limits, Section 32.4.

0CPY **Situation 102.4.1.** Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

0CPZ **Lemma 102.4.2.** *In Situation 102.4.1 assume that $\mathcal{X}_0 \to Y_0$ is a morphism from algebraic stack to Y_0. Assume $\mathcal{X}_0$ is quasi-compact and quasi-separated. If $Y \times_{Y_0} \mathcal{X}_0 \to Y$ is separated, then $Y_i \times_{Y_0} \mathcal{X}_0 \to Y_i$ is separated for all sufficiently large $i \in I$.*

Proof. Write $\mathcal{X} = Y \times_{Y_0} \mathcal{X}_0$ and $\mathcal{X}_i = Y_i \times_{Y_0} \mathcal{X}_0$. Choose an affine scheme U_0 and a surjective smooth morphism $U_0 \to \mathcal{X}_0$. Set $U = Y \times_{Y_0} U_0$ and $U_i = Y_i \times_{Y_0} U_0$. Then U and U_i are affine and $U \to \mathcal{X}$ and $U_i \to \mathcal{X}_i$ are smooth and surjective. Set $R_0 = U_0 \times_{\mathcal{X}_0} U_0$. Set $R = Y \times_{Y_0} R_0$ and $R_i = Y_i \times_{Y_0} R_0$. Then $R = U \times_{\mathcal{X}} U$ and $R_i = U_i \times_{\mathcal{X}_i} U_i$.

With this notation note that $\mathcal{X} \to Y$ is separated implies that $R \to U \times_Y U$ is proper as the base change of $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ by $U \times_Y U \to \mathcal{X} \times_Y \mathcal{X}$. Conversely, we see that $\mathcal{X}_i \to Y_i$ is separated if $R_i \to U_i \times_{Y_i} U_i$ is proper because $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth, see Properties of Stacks, Lemma 100.3.3. Observe that $R_0 \to U_0 \times_{Y_0} U_0$ is locally of finite type and that R_0 is quasi-compact and quasi-separated. By Limits of Spaces, Lemma 70.6.13 we see that $R_i \to U_i \times_{Y_i} U_i$ is proper for large enough i which finishes the proof. □

102.5. Descending relative objects

0CN3 This section is the analogue of Limits of Spaces, Section 70.7.

0CN4 **Lemma 102.5.1.** *Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system of algebraic spaces over I. Assume*

(1) *the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
(2) *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim X_i$. If $\mathcal{X}$ is an algebraic stack of finite presentation over X, then there exists an $i \in I$ and an algebraic stack $\mathcal{X}_i$ of finite presentation over X_i with $\mathcal{X} \cong \mathcal{X}_i \times_{X_i} X$ as algebraic stacks over X.

Proof. By Morphisms of Stacks, Definition 101.27.1 the morphism $\mathcal{X} \to X$ is quasi-compact, locally of finite presentation, and quasi-separated. Since X is quasi-compact and $\mathcal{X} \to X$ is quasi-compact, we see that $\mathcal{X}$ is quasi-compact (Morphisms of Stacks, Definition 101.7.2). Hence we can find an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 100.6.2). Set $R = U \times_{\mathcal{X}} U$. We obtain a smooth groupoid in algebraic spaces (U, R, s, t, c) over X such that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 94.16.2. Since $\mathcal{X} \to X$ is quasi-separated and X is quasi-separated we see that $\mathcal{X}$ is quasi-separated (Morphisms of Stacks, Lemma 101.4.10). Thus $R \to U \times U$ is quasi-compact and quasi-separated (Morphisms of Stacks, Lemma 101.4.7) and hence R is a quasi-separated and quasi-compact algebraic space. On the other hand $U \to X$ is locally of finite presentation and hence also $R \to X$ is locally of finite presentation (because $s : R \to U$ is smooth hence locally of finite presentation). Thus (U, R, s, t, c) is a groupoid object in the category of algebraic spaces which are of finite presentation over X. By Limits of Spaces, Lemma 70.7.1 there exists an i and a groupoid in algebraic spaces $(U_i, R_i, s_i, t_i, c_i)$ over X_i whose pullback to X is isomorphic to (U, R, s, t, c). After increasing i we may assume that s_i and t_i are smooth, see Limits of Spaces, Lemma 70.6.3. The quotient stack $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 94.17.3).

There is a morphism $[U/R] \to [U_i/R_i]$, see Groupoids in Spaces, Lemma 78.21.1. We claim that combined with the morphisms $[U/R] \to X$ and $[U_i/R_i] \to X_i$ (Groupoids in Spaces, Lemma 78.20.2) we obtain an isomorphism (i.e., equivalence)

$$[U/R] \longrightarrow [U_i/R_i] \times_{X_i} X$$

The corresponding map

$$[U/_pR] \longrightarrow [U_i/_pR_i] \times_{X_i} X$$

on the level of "presheaves of groupoids" as in Groupoids in Spaces, Equation (78.20.0.1) is an isomorphism. Thus the claim follows from the fact that stackification commutes with fibre products, see Stacks, Lemma 8.8.4. □

102.6. Finite type closed in finite presentation

0CQ0 This section is the analogue of Limits of Spaces, Section 70.11.

0CQ1 **Lemma 102.6.1.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume:*

(1) *f is of finite type and quasi-separated,*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : \mathcal{X}' \to Y$ and a closed immersion $\mathcal{X} \to \mathcal{X}'$ of algebraic stacks over Y.

Proof. Write $Y = \lim_{i \in I} Y_i$ as a limit of algebraic spaces over a directed set I with affine transition morphisms and with Y_i Noetherian, see Limits of Spaces, Proposition 70.8.1. We will use the material from Limits of Spaces, Section 70.23.

Choose a presentation $\mathcal{X} = [U/R]$. Denote (U, R, s, t, c, e, i) the corresponding groupoid in algebraic spaces over Y. We may and do assume U is affine. Then U, R, $R \times_{s,U,t} R$ are quasi-separated algebraic spaces of finite type over Y. We have two morpisms $s, t : R \to U$, three morphisms $c : R \times_{s,U,t} R \to R$, $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$, $\mathrm{pr}_2 : R \times_{s,U,t} R \to R$, a morphism $e : U \to R$, and finally a morphism $i : R \to R$.

These morphisms satisfy a list of axioms which are detailed in Groupoids, Section 39.13.

According to Limits of Spaces, Remark 70.23.5 we can find an $i_0 \in I$ and inverse systems

(1) $(U_i)_{i \geq i_0}$,
(2) $(R_i)_{i \geq i_0}$,
(3) $(T_i)_{i \geq i_0}$

over $(Y_i)_{i \geq i_0}$ such that $U = \lim_{i \geq i_0} U_i$, $R = \lim_{i \geq i_0} R_i$, and $R \times_{s,U,t} R = \lim_{i \geq i_0} T_i$ and such that there exist morphisms of systems

(1) $(s_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(2) $(t_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(3) $(c_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(4) $(p_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(5) $(q_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(6) $(e_i)_{i \geq i_0} : (U_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(7) $(i_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$

with $s = \lim_{i \geq i_0} s_i$, $t = \lim_{i \geq i_0} t_i$, $c = \lim_{i \geq i_0} c_i$, $\mathrm{pr}_1 = \lim_{i \geq i_0} p_i$, $\mathrm{pr}_2 = \lim_{i \geq i_0} q_i$, $e = \lim_{i \geq i_0} e_i$, and $i = \lim_{i \geq i_0} i_i$. By Limits of Spaces, Lemma 70.23.7 we see that we may assume that s_i and t_i are smooth (this may require increasing i_0). By Limits of Spaces, Lemma 70.23.6 we may assume that the maps $R \to U \times_{U_i, s_i} R_i$ given by s and $R \to R_i$ and $R \to U \times_{U_i, t_i} R_i$ given by t and $R \to R_i$ are isomorphisms for all $i \geq i_0$. By Limits of Spaces, Lemma 70.23.9 we see that we may assume that the diagrams

$$\begin{array}{ccc} T_i & \xrightarrow[q_i]{} & R_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle t_i} \\ R_i & \xrightarrow{s_i} & U_i \end{array}$$

are cartesian. The uniqueness of Limits of Spaces, Lemma 70.23.4 then guarantees that for a sufficiently large i the relations between the morphisms s, t, c, e, i mentioned above are satisfied by s_i, t_i, c_i, e_i, i_i. Fix such an i.

It follows that $(U_i, R_i, s_i, t_i, c_i, e_i, i_i)$ is a smooth groupoid in algebraic spaces over Y_i. Hence $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 94.17.3). The morphism of groupoids

$$(U, R, s, t, c, e, i) \to (U_i, R_i, s_i, t_i, c_i, e_i, i_i)$$

over $Y \to Y_i$ determines a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}_i \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_i \end{array}$$

(Groupoids in Spaces, Lemma 78.21.1). We claim that the morphism $\mathcal{X} \to Y \times_{Y_i} \mathcal{X}_i$ is a closed immersion. The claim finishes the proof because the algebraic stack $\mathcal{X}_i \to Y_i$ is of finite presentation by construction. To prove the claim, note that the

left diagram

$$\begin{array}{ccc} U & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}_i \end{array} \qquad \begin{array}{ccc} U & \longrightarrow & Y \times_{Y_i} U_i \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & Y \times_{Y_i} \mathcal{X}_i \end{array}$$

is cartesian by Groupoids in Spaces, Lemma 78.25.3 and the results mentioned above. Hence the right commutative diagram is cartesian too. Then the desired result follows from the fact that $U \to Y \times_{Y_i} U_i$ is a closed immersion by construction of the inverse system (U_i) in Limits of Spaces, Lemma 70.23.3, the fact that $Y \times_{Y_i} U_i \to Y \times_{Y_i} \mathcal{X}_i$ is smooth and surjective, and Properties of Stacks, Lemma 100.9.4. □

There is a version for separated algebraic stacks.

0CQ2 **Lemma 102.6.2.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume:*

(1) *f is of finite type and separated,*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : \mathcal{X}' \to Y$ and a closed immersion $\mathcal{X} \to \mathcal{X}'$ of algebraic stacks over Y.

Proof. First we use exactly the same procedure as in the proof of Lemma 102.6.1 (and we borrow its notation) to construct the embedding $\mathcal{X} \to \mathcal{X}'$ as a morphism $\mathcal{X} \to \mathcal{X}' = Y \times_{Y_i} \mathcal{X}_i$ with $\mathcal{X}_i = [U_i/R_i]$. Thus it is enough to show that $\mathcal{X}_i \to Y_i$ is separated for sufficiently large i. In other words, it is enough to show that $\mathcal{X}_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is proper for i sufficiently large. Since the morphism $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth and since $R_i = \mathcal{X}_i \times_{\mathcal{X}_i \times_{Y_i} \mathcal{X}_i} U_i \times_{Y_i} U_i$ it is enough to show that the morphism $(s_i, t_i) : R_i \to U_i \times_{Y_i} U_i$ is proper for i sufficiently large, see Properties of Stacks, Lemma 100.3.3. We prove this in the next paragraph.

Observe that $U \times_Y U \to Y$ is quasi-separated and of finite type. Hence we can use the construction of Limits of Spaces, Remark 70.23.5 to find an $i_1 \in I$ and an inverse system $(V_i)_{i \geq i_1}$ with $U \times_Y U = \lim_{i \geq i_1} V_i$. By Limits of Spaces, Lemma 70.23.9 for i sufficiently large the functoriality of the construction applied to the projections $U \times_Y U \to U$ gives closed immersions

$$V_i \to U_i \times_{Y_i} U_i$$

(There is a small mismatch here because in truth we should replace Y_i by the scheme theoretic image of $Y \to Y_i$, but clearly this does not change the fibre product.) On the other hand, by Limits of Spaces, Lemma 70.23.8 the functoriality applied to the proper morphism $(s, t) : R \to U \times_Y U$ (here we use that $\mathcal{X}$ is separated) leads to morphisms $R_i \to V_i$ which are proper for large enough i. Composing these morphisms we obtain a proper morphisms $R_i \to U_i \times_{Y_i} U_i$ for all i large enough. The functoriality of the construction of Limits of Spaces, Remark 70.23.5 shows that this is the morphism is the same as (s_i, t_i) for large enough i and the proof is complete. □

102.7. Universally closed morphisms

0H28 This section is the analogue of Limits of Spaces, Section 70.20.

0H29 **Lemma 102.7.1.** *Let $g : Z \to Y$ be a morphism of affine schemes. Let $f : \mathcal{X} \to Y$ be a quasi-compact morphism of algebraic stacks. Let $z \in Z$ and let $T \subset |\mathcal{X} \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If $\mathcal{X}$ is quasi-compact, then there exist an open neighbourhood $V \subset Z$ of z, a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

(1) *Z' is an affine scheme of finite presentation over Y,*
(2) *with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and*
(3) *the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.*

Proof. We will deduce this from the corresponding result for morphisms of schemes. Since $\mathcal{X}$ is quasi-compact, we may choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 32.14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{array}{ccccc} W \times_Y Z & \longleftarrow & W \times_Y V & \xrightarrow{a_1} & W \times_Y Z' \\ \downarrow & & \downarrow c & & \downarrow q \\ \mathcal{X} \times_Y Z & \longleftarrow & \mathcal{X} \times_Y V & \xrightarrow{a_2} & \mathcal{X} \times_Y Z' \end{array}$$

has cartesian squares and the vertical maps are surjective, smooth, and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |\mathcal{X} \times_Y V|$ by c. By Properties of Stacks, Lemma 100.4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 5.6.4 we get

$$q^{-1}\left(\overline{a_2(T_2)}\right) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \to |Z'|$ does not contain z' since the same is true for T'. Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |\mathcal{X} \times_Y Z'|$ as a solution to the problem posed by the lemma. □

0H2A **Lemma 102.7.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. The following are equivalent*

(1) *f is universally closed,*

(2) *for every morphism $Z \to \mathcal{Y}$ which is locally of finite presentation and where Z is an affine scheme the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed, and*
(3) *there exists a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $|\mathbf{A}^n \times (\mathcal{X} \times_{\mathcal{Y}} V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. It is clear that (1) implies (2).

Assume (2). Choose a scheme V which is the disjoint union of affine schemes and a surjective smooth morphism $V \to \mathcal{Y}$. In order to show that f is universally closed, it suffices to show that the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ of f is universally closed, see Morphisms of Stacks, Lemma 101.13.5. Note that property (2) holds for this base change. Hence in order to prove that (2) implies (1) we may assume $Y = \mathcal{Y}$ is an affine scheme.

Assume (2) and assume $\mathcal{Y} = Y$ is an affine scheme. If f is not universally closed, then there exists an affine scheme Z over Y such that $|\mathcal{X} \times_Y Z| \to |Z|$ is not closed, see Morphisms of Stacks, Lemma 101.13.5. This means that there exists some closed subset $T \subset |\mathcal{X} \times_Y Z|$ such that $\mathrm{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 102.7.1. We find an open neighbourhood $V \subset Z$, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |\mathcal{X} \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.

We claim that z' is in the closure of $\mathrm{Im}(T' \to |Z'|)$. This implies that $|\mathcal{X} \times_Y Z'| \to |Z'|$ is not closed and this is absurd as we assumed (2), in other words, the claim shows that (2) implies (1). To see the claim is true we contemplate the following commutative diagram

$$\begin{array}{ccccc} \mathcal{X} \times_Y Z & \longleftarrow & \mathcal{X} \times_Y V & \longrightarrow & \mathcal{X} \times_Y Z' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & V & \xrightarrow{a} & Z' \end{array}$$

Let $T_V \subset |\mathcal{X} \times_Y V|$ be the inverse image of T. By Properties of Stacks, Lemma 100.4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that z is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|\mathcal{X} \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(z)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to \mathcal{Y}$ be as in (3). If we can show that $\mathcal{X} \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Stacks, Lemma 101.13.5. Thus it suffices to show that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2) if f is a quasi-compact morphism of algebraic stacks, $\mathcal{Y} = Y$ is a scheme, and $|\mathbf{A}^n \times \mathcal{X}| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation

where Z is an affine scheme. We have to show the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed. Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 29.39.2. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X} \times_Y Z & \longrightarrow & \mathbf{A}^n \times \mathcal{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbf{A}^n \times Y \end{array} \quad \text{inducing the cartesian square} \quad \begin{array}{ccc} |\mathcal{X} \times_Y Z| & \longrightarrow & |\mathbf{A}^n \times \mathcal{X}| \\ \downarrow & & \downarrow \\ |Z| & \longrightarrow & |\mathbf{A}^n \times Y| \end{array}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Stacks, Lemma 100.9.6). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. □

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CHAPTER 103

Cohomology of Algebraic Stacks

073P

103.1. Introduction

073Q In this chapter we write about cohomology of algebraic stacks. This means in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled "Cohomology of Schemes" and "Cohomology of Algebraic Spaces". The results in this chapter are different from those in [**LMB00**] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapter "Sheaves on Algebraic Stacks" in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 96.1.

103.2. Conventions and abuse of language

073R We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

103.3. Notation

073S Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{étale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 96.4.1. (Think "big site".) Correspondingly the structure sheaf of $\mathcal{X}$ is a sheaf on $\mathcal{X}_{fppf}$. On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z, and in this case $X_{étale}$ indicates the small étale site of X (as defined in Topologies, Definition 34.4.8 or Properties of Spaces, Definition 66.18.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say "sheaf on $\mathcal{X}$" or "sheaf of $\mathcal{O}_\mathcal{X}$-modules" when we mean sheaf on $\mathcal{X}_{fppf}$ or object of $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don't have to mention which topology we are working with. The same isn't true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \to \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Let $\mathcal{G}$ be a sheaf on $\mathcal{Y}_\tau$ for some topology τ. In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on $\mathcal{S}_X$ (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \to \mathcal{Y}$. If $\tau = étale$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{étale}}$ to denote the restriction to the étale site of X, see Sheaves on

Stacks, Section 96.22. If $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}$-module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{étale}}$ instead.

103.4. Pullback of quasi-coherent modules

076W Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_\mathcal{Y}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}),$$

see Sheaves on Stacks, Lemma 96.11.2. In general this functor isn't exact, but if f is flat then it is.

076X **Lemma 103.4.1.** *If $f : \mathcal{X} \to \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ is an exact functor.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_U) & \longleftarrow & QCoh(\mathcal{O}_V) \\ \uparrow & & \uparrow \\ QCoh(\mathcal{O}_\mathcal{X}) & \longleftarrow & QCoh(\mathcal{O}_\mathcal{Y}) \end{array}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 96.15.1). Since f' is a flat morphism of schemes (by our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V. Thus we win. □

0GQF **Lemma 103.4.2.** *Let $\mathcal{X}$ be an algebraic stack. Let I be a set and for $i \in I$ let $x_i : U_i \to \mathcal{X}$ be an object of $\mathcal{X}$. Assume that x_i is flat and $\coprod x_i : \coprod U_i \to \mathcal{X}$ is surjective. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an arrow of $QCoh(\mathcal{O}_\mathcal{X})$. Denote φ_i the restriction of φ to $(U_i)_{étale}$. Then φ is injective, resp. surjective, resp. an isomorphism if and only if each φ_i is so.*

Proof. Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. We may and do think of x as an object of $\mathcal{X}$. This produces a presentation $\mathcal{X} = [U/R]$ for some groupoid in spaces (U, R, s, t, c) and correspondingly an equivalence

$$QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$$

See discussion in Sheaves on Stacks, Section 96.15. The structure of abelian category on the right hand is such that φ is injective, resp. surjective, resp. an isomorphism if and only if the restriction $\varphi|_{U_{étale}}$ is so, see Groupoids in Spaces, Lemma 78.12.6.

For each i we choose an étale covering $\{W_{i,j} \to V \times_{\mathcal{X}} U_i\}_{j \in J_i}$ by schemes. Denote $g_{i,j} : W_{i,j} \to V$ and $h_{i,j} : W_{i,j} \to U_i$ the obvious arrows. Each of the morphisms of schemes $g_{i,j} : W_{i,j} \to U$ is flat and they are jointly surjective. Similarly, for each fixed i the morphisms of schemes $h_{i,j} : W_{i,j} \to U_i$ are flat and jointly surjective. By Sheaves on Stacks, Lemma 96.12.2 the pullback by $(g_{i,j})_{small}$ of the restriction $\varphi|_{U_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$ and the pullback by $(h_{i,j})_{small}$ of the restriction $\varphi|_{(U_i)_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$. Pullback of quasi-coherent modules by a flat morphism of schemes is exact and pullback by a jointly surjective family of flat morphisms of schemes reflects injective, resp. surjective, resp. bijective maps of quasi-coherent modules (in fact this holds for all modules as we can check exactness at stalks). Thus we see

$$\varphi|_{U_{étale}} \text{ injective} \Leftrightarrow \varphi|_{(W_{i,j})_{étale}} \text{ injective for all } i, j \Leftrightarrow \varphi|_{(U_i)_{étale}} \text{ injective for all } i$$

This finishes the proof. □

103.5. Higher direct images of types of modules

076Y The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two versions: one for the étale topology and one for the fppf topology.

076Z **Lemma 103.5.1.** *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_{\mathcal{X}}$ of Mod$(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ such that*

1. *$\mathcal{M}_{\mathcal{X}}$ is a weak Serre subcategory of Mod$(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ (see Homology, Definition 12.10.1) for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_{\mathcal{X}}$ into $\mathcal{M}_{\mathcal{Y}}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of Mod$(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ is in $\mathcal{M}_{\mathcal{X}}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
4. *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$. (Higher direct images computed in étale topology.)

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let $\mathcal{F}$ be an object of $\mathcal{M}_{\mathcal{Y}}$. Choose a surjective smooth morphism $\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is representable by a scheme. By Sheaves on Stacks, Lemma 96.21.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of $\mathcal{F}$ to $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}}$ because the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \to \mathcal{X}$ by the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{U}$. In other words, we may assume that $\mathcal{X}$ is representable by a scheme. Using (3) once more, we see that the question is Zariski local on $\mathcal{X}$, hence we may assume that $\mathcal{X}$ is representable by an affine scheme. Since f is quasi-compact this implies that also $\mathcal{Y}$ is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 96.21.1. Recall that this is a first quadrant spectral sequence hence we may use the last part of Homology, Lemma 12.25.3. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g. Thus the sheaves $g_p^*\mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_{\mathcal{X}}$. The algebraic stacks $\mathcal{V}_p$ are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 101.7.8. Of course each $\mathcal{V}_p$ is representable by an algebraic space (the diagonal of the algebraic stack $\mathcal{Y}$ is representable by algebraic spaces). This reduces us to the case where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme.

In the situation where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \to \mathcal{X}$. But in the current situation the algebraic stacks $\mathcal{V}_p$ are representable by quasi-compact and quasi-separated schemes (because the diagonal of an algebraic space is representable by schemes).

Thus we may assume $\mathcal{Y}$ is representable by a scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume $\mathcal{Y}$ is representable by a separated scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (yet again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. □

Here is the version for the fppf topology.

0770 **Lemma 103.5.2.** *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_{\mathcal{X}}$ of Mod$(\mathcal{O}_{\mathcal{X}})$ such that*

(1) *$\mathcal{O}_{\mathcal{X}}$ is a weak Serre subcategory of Mod$(\mathcal{O}_{\mathcal{X}})$ for all algebraic stacks $\mathcal{X}$,*
(2) *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_{\mathcal{X}}$ into $\mathcal{M}_{\mathcal{Y}}$,*
(3) *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of Mod$(\mathcal{O}_{\mathcal{X}})$ is in $\mathcal{M}_{\mathcal{X}}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
(4) *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$. (Higher direct images computed in fppf topology.)

Proof. Identical to the proof of Lemma 103.5.1. □

103.6. Locally quasi-coherent modules

075X Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We can ask whether $\mathcal{F}$ is *locally quasi-coherent*, see Sheaves on Stacks, Definition 96.12.1. Briefly, this means $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module for the étale topology such that for any morphism $f : U \to \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent on $U_{étale}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$LQCoh(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 96.12.4.

075Y **Lemma 103.6.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 101.33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 96.12.3. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective smooth morphism. Let x be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x, x_i live over schemes V, V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. The restriction $x_i^*\mathcal{F}|_{V_{i,étale}}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_{i,étale}$, see Sheaves on Stacks, Lemma 96.9.3. Hence $x^*\mathcal{F}|_{V_{étale}}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_{i,étale}$ for each i. This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 66.29.6. □

075Z **Lemma 103.6.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$. Then $R^i f_*\mathcal{F}$ (computed in the étale topology) is locally quasi-coherent on $\mathcal{Y}_{étale}$.*

Proof. We will use Lemma 103.5.1 to prove this. We will check its assumptions (1) – (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 96.12.4. Part (3) follows from Lemma 103.6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. Choose any object y of $\mathcal{Y}$ lying over a scheme V. For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

Thus $\mathcal{Z}$ is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 96.22.3 we have

$$R^i f_*\mathcal{F}|_{V_{étale}} = R^i f'_{small,*}\big(g^*\mathcal{F}|_{Z_{étale}}\big)$$

The right hand side is a quasi-coherent sheaf on $V_{étale}$ by Cohomology of Spaces, Lemma 69.3.1. This implies the left hand side is quasi-coherent which is what we had to prove. □

07AP **Lemma 103.6.3.** *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 96.19.7 and Lemma 96.19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 96.12.3). The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent by Lemma 103.6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Lemma 96.12.4.

We are going to reduce the proof of the general case the situation in the first paragraph. Let x be an object of $\mathcal{X}$ lying over the scheme U. We have to show that $\mathcal{F}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. It suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 101.27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence $a_i^*\mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{étale}}$ is quasi-coherent. □

103.7. Flat comparison maps

0760 Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Given an object x of $\mathcal{X}$ lying over the scheme U the restriction $\mathcal{F}|_{U_{étale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U, see Sheaves on Stacks, Definition 96.9.2. Next, let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a morphism of schemes $f : U \to U'$. Thus a 2-commutative diagram

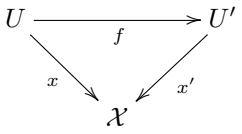

Associated to φ we obtain a comparison map between restrictions

0761 (103.7.0.1) $$c_\varphi : f^*_{small}(\mathcal{F}|_{U'_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}}$$

see Sheaves on Stacks, Equation (96.9.4.1). In this situation we can consider the following property of $\mathcal{F}$.

0762 **Definition 103.7.1.** Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. We say $\mathcal{F}$ has the *flat base change property*[1] if and only if c_φ is an isomorphism whenever f is flat.

[1]This may be nonstandard notation.

Here is a lemma with some properties of this notion.

0764 **Lemma 103.7.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$.*

(1) *If $\mathcal{F}$ has the flat base change property then for any morphism $g : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.*
(2) *The full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of modules with the flat base change property is a weak Serre subcategory.*
(3) *Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does $\mathcal{F}$.*
(4) *The category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$ with the flat base change property has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
(5) *Given $\mathcal{F}$ and $\mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ with the flat base change property then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ has the flat base change property.*
(6) *Given $\mathcal{F}$ and $\mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ having the flat base change property then the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property.*

Proof. Let $g : \mathcal{Y} \to \mathcal{X}$ be as in (1). Let y be an object of $\mathcal{Y}$ lying over a scheme V. By Sheaves on Stacks, Lemma 96.9.3 we have $(g^*\mathcal{F})|_{V_{étale}} = \mathcal{F}|_{V_{étale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on $\mathcal{Y}$ is a special case of a comparison map for the sheaf $\mathcal{F}$ on $\mathcal{X}$, see Sheaves on Stacks, Lemma 96.9.3. In this way (1) is clear.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 12.10.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f_{small}^* is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{étale}}$ are exact (because we are working in the étale topology). Finally, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and the outer two sheaves have the flat base change property then the middle one does as well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 101.33.2) we may assume that each $\mathcal{X}_i$ is representable by a scheme. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a flat morphism $a : U \to U'$ of schemes. By Sheaves on Stacks, Lemma 96.19.10 there exists a jointly surjective family of étale morphisms $U_i' \to U'$ such that $U_i' \to U' \to \mathcal{X}$ factors through $\mathcal{X}_i$. Thus we obtain commutative diagrams

$$\begin{array}{ccccc} U_i = U \times_{U'} U_i' & \xrightarrow{a_i} & U_i' & \xrightarrow{x_i'} & \mathcal{X}_i \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_i} \\ U & \xrightarrow{a} & U' & \xrightarrow{x'} & \mathcal{X} \end{array}$$

Note that each a_i is a flat morphism of schemes as a base change of a. Denote $\psi_i : x_i \to x_i'$ the morphism of $\mathcal{X}_i$ lying over a_i with target x_i'. By assumption the comparison maps $c_{\psi_i} : (a_i)_{small}^*\big(f_i^*\mathcal{F}|_{(U_i')_{étale}}\big) \to f_i^*\mathcal{F}|_{(U_i)_{étale}}$ is an isomorphism. Because the vertical arrows $U_i' \to U'$ and $U_i \to U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U_i')_{étale}}$

and $f_i^*\mathcal{F}|_{(U_i)_{étale}}$ are the restrictions of $\mathcal{F}|_{U'_{étale}}$ and $\mathcal{F}|_{U_{étale}}$ and the map c_{ψ_i} is the restriction of c_φ to $(U_i)_{étale}$, see Sheaves on Stacks, Lemma 96.9.3. Since $\{U_i \to U\}$ is an étale covering, this implies that the comparison map c_φ is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each $\mathcal{F}_i$ has the flat base change property. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. Recall that $\operatorname{colim}_i \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(\operatorname{colim}_i \mathcal{F}_i)|_{U_{étale}} = \operatorname{colim}_i \mathcal{F}_i|_{U_{étale}}$$

and similarly for the restriction to $U'_{étale}$. Hence

$$\begin{aligned} f_{small}^*((\operatorname{colim}_i \mathcal{F}_i)|_{U'_{étale}}) &= f_{small}^*(\operatorname{colim}_i \mathcal{F}_i|_{U'_{étale}}) \\ &= \operatorname{colim}_i f_{small}^*(\mathcal{F}_i|_{U'_{étale}}) \\ &\xrightarrow{\operatorname{colim} c_\varphi} \operatorname{colim}_i \mathcal{F}_i|_{U_{étale}} \\ &= (\operatorname{colim}_i \mathcal{F}_i)|_{U_{étale}} \end{aligned}$$

For the second equality we used that f_{small}^* commutes with colimits (as a left adjoint). The arrow is an isomorphism as each $\mathcal{F}_i$ has the flat base change property. Thus the colimit has the flat base change property and (4) is true.

Part (5) holds because tensor products commute with pullbacks, see Modules on Sites, Lemma 18.26.2. Details omitted.

Let $\mathcal{F}$ and $\mathcal{G}$ be as in (6). Since $\mathcal{F}$ is quasi-coherent it has the flat base change property by Sheaves on Stacks, Lemma 96.12.2. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. As we are using the étale topology, we have

$$\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$$

and similarly for the restriction to $U'_{étale}$ (details omitted). Hence

$$\begin{aligned} f_{small}^*(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U'_{étale}}) &= f_{small}^*(\mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{F}|_{U'_{étale}}, \mathcal{G}|_{U'_{étale}})) \\ &= \mathcal{H}om_{\mathcal{O}_{U'}}(f_{small}^*(\mathcal{F}|_{U'_{étale}}), f_{small}^*(\mathcal{G}|_{U'_{étale}})) \\ &\xrightarrow{c_\varphi} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}}) \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} \end{aligned}$$

Here the second equality is Modules on Sites, Lemma 18.31.4 which uses that $f : U \to U'$ is flat and hence the morphism of ringed sites f_{small} is flat too. The arrow is an isomorphism as both $\mathcal{F}$ and $\mathcal{G}$ have the flat base change property. Thus our $\mathcal{H}om$ has the flat base change property too as desired. □

0765 **Lemma 103.7.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property. Then each $R^i f_*\mathcal{F}$ (computed in the étale topology) has the flat base change property.*

Proof. We will use Lemma 103.5.1 to prove this. For every algebraic stack $\mathcal{X}$ let $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ denote the full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) – (4) of Lemma 103.5.1 the lemma will follow. Properties (1), (2),

and (3) follow from Sheaves on Stacks, Lemmas 96.12.3 and 96.12.4 and Lemmas 103.6.1 and 103.7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. In this case, suppose that $\psi : y \to y'$ is a morphism of $\mathcal{Y}$ lying over a flat morphism $b : V \to V'$ of schemes. For clarity denote $\mathcal{V} = (Sch/V)_{fppf}$ and $\mathcal{V}' = (Sch/V')_{fppf}$ the corresponding algebraic stacks. Consider the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{a} & \mathcal{Z}' & \xrightarrow{x'} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{b} & \mathcal{V}' & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated – even affine) we see that $\mathcal{Z}$ and $\mathcal{Z}'$ are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since $\mathcal{F}$ has the flat base change property we see that

$$a^*_{small}\big(\mathcal{F}|_{Z'_{étale}}\big) \longrightarrow \mathcal{F}|_{Z_{étale}}$$

is an isomorphism. Moreover,

$$R^i f_*\mathcal{F}|_{V'_{étale}} = R^i(f')_{small,*}\big(\mathcal{F}|_{Z'_{étale}}\big)$$

and

$$R^i f_*\mathcal{F}|_{V_{étale}} = R^i(f'')_{small,*}\big(\mathcal{F}|_{Z_{étale}}\big)$$

by Sheaves on Stacks, Lemma 96.22.3. Hence we see that the comparison map

$$c_\psi : b^*_{small}(R^i f_*\mathcal{F}|_{V'_{étale}}) \longrightarrow R^i f_*\mathcal{F}|_{V_{étale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 69.11.2. Thus $R^i f_*\mathcal{F}$ has the flat base change property. Since $R^i f_*\mathcal{F}$ is locally quasi-coherent by Lemma 103.6.2 we win. □

103.8. Locally quasi-coherent modules with the flat base change property

0GQG Let $\mathcal{X}$ be an algebraic stack. We[2] will denote

$$LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

the full subcategory whose objects are étale $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ which are both locally quasi-coherent (Section 103.6) and have the flat base change property (Section 103.7). We have

$$QCoh(\mathcal{O}_\mathcal{X}) \subset LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$$

by Sheaves on Stacks, Lemma 96.12.2.

0771 **Proposition 103.8.1.** *Summary of results on locally quasi-coherent modules having the flat base change property.*

(1) *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, then $\mathcal{F}$ is a sheaf for the fppf topology, i.e., it is an object of $Mod(\mathcal{O}_\mathcal{X})$.*
(2) *The category $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of both $Mod(\mathcal{O}_\mathcal{X})$ and $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*

[2]Apologies for the horrendous notation.

(3) *Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^* : LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
(4) *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and $\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, then*
   (a) *the total direct image $Rf_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and*
   (b) *each $R^if_*\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$.*
(5) *The category $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ as well as in $Mod(\mathcal{O}_\mathcal{X})$.*
(6) *Given $\mathcal{F}$ and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
(7) *Given $\mathcal{F}$ of finite presentation and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) is Sheaves on Stacks, Lemma 96.23.1.

Part (2) for the embedding $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have seen in the proof of Lemma 103.7.3. Let us prove (2) for the embedding $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Since $\mathrm{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\mathrm{Ker}(\varphi)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in the étale topology. However, this étale cokernel is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ hence an fppf sheaf by (1) and we see that the cokernel is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ (i.e., using the fppf topology) with $\mathcal{F}_1$, $\mathcal{F}_2$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. In order to show that $\mathcal{F}_2$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H^1_{fppf}(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of $\mathcal{X}$). This is true because $H^1_{fppf}(x, \mathcal{F}_1) = H^1_{étale}(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 96.23.2 and because of locality of cohomology, see Cohomology on Sites, Lemma 21.7.3. This proves (2).

Part (3) follows from Lemma 103.7.2 and Sheaves on Stacks, Lemma 96.12.3.

Part (4)(b) for $R^if_*\mathcal{F}$ computed in the étale cohomology follows from Lemma 103.7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 96.23.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 96.12.4 and Lemma 103.7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1).

Parts (6) and (7) follow from the corresponding parts of Lemma 103.7.2 and Sheaves on Stacks, Lemma 96.12.4. □

07AQ **Lemma 103.8.2.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is in $LQCoh^{fpc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) follows from a combination of Lemmas 103.6.1 and 103.7.2. The proof of (2) is analogous to the proof of Lemma 103.6.3. Let $\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$.

First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 96.19.7 and Lemma 96.19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 103.8.1. The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 103.8.1. We conclude that $\mathcal{F}$ is locally quasi-coherent and has the flat base change property by Proposition 103.8.1.

Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Again by part (1) it suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 101.27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. □

0GQH **Lemma 103.8.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is quasi-compact, quasi-separated, and representable by algebraic spaces. Let $\mathcal{F}$ be in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Then for an object $y : V \to \mathcal{Y}$ of $\mathcal{Y}$ we have*

$$(R^if_*\mathcal{F})|_{V_{étale}} = R^if'_{small,*}(\mathcal{F}|_{U_{étale}})$$

where $f' : U = V \times_\mathcal{Y} \mathcal{X} \to V$ is the base change of f.

Proof. By Sheaves on Stacks, Lemma 96.21.3 we can reduce to the case where $\mathcal{X}$ is represented by U and $\mathcal{Y}$ is represented by V. Of course this also uses that the pullback of $\mathcal{F}$ to U is in $LQCoh^{fbc}(\mathcal{O}_U)$ by Proposition 103.8.1. Then the result follows from Sheaves on Stacks, Lemma 96.22.2 and the fact that R^if_* may be computed in the étale topology by Proposition 103.8.1. □

0GQI **Lemma 103.8.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functor $f_* : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ is exact and commutes with direct sums. The functors $R^i f_*$ for $i > 0$ vanish on $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. The functors exist by Proposition 103.8.1. By Lemma 103.8.3 this reduces to the case of an affine morphism of algebraic spaces taking higher direct images in the setting of quasi-coherent modules on algebraic spaces. By the discussion in Cohomology of Spaces, Section 69.3 we reduce to the case of an affine morphism of schemes. For affine morphisms of schemes we have the vanishing of higher direct images on quasi-coherent modules by Cohomology of Schemes, Lemma 30.2.3. The vanishing for $R^1 f_*$ implies exactness of f_*. Commuting with direct sums follows from Morphisms, Lemma 29.11.7 for example. □

103.9. Parasitic modules

0772 The following definition is compatible with Descent, Definition 35.12.1.

0773 **Definition 103.9.1.** Let $\mathcal{X}$ be an algebraic stack. A presheaf of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ is *parasitic* if we have $\mathcal{F}(x) = 0$ for any object x of $\mathcal{X}$ which lies over a scheme U such that the corresponding morphism $x : U \to \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

0774 **Lemma 103.9.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules.*

(1) *If $\mathcal{F}$ is parasitic and $g : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.*
(2) *For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have*
   (a) *the τ sheafification of a parasitic presheaf of modules is parasitic, and*
   (b) *the full subcategory of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ consisting of parasitic modules is a Serre subcategory.*
(3) *Suppose $\mathcal{F}$ is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*
(4) *Suppose $\mathcal{F}$ is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*

Proof. To see part (1) let y be an object of $\mathcal{Y}$ which lies over a scheme V such that the corresponding morphism $y : V \to \mathcal{Y}$ is flat. Then $g(y) : V \to \mathcal{Y} \to \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 101.25.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \to x\}$ is a τ-covering of $\mathcal{X}$, then each of the morphisms $x_i \to x$ lies over a flat morphism of schemes. Hence if x lies over a scheme U such that $x : U \to \mathcal{X}$ is flat, so do all of the objects x_i. Hence the presheaf $\mathcal{F}^+$ (see Sites, Section 7.10) is parasitic if the presheaf $\mathcal{F}$ is parasitic. This proves (2)(a) as the sheafification of $\mathcal{F}$ is $(\mathcal{F}^+)^+$.

Let $\mathcal{F}$ be a parasitic τ-module. It is immediate from the definitions that any submodule of $\mathcal{F}$ is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient $\mathcal{F}/\mathcal{F}'$ is a parasitic module by (2)(a). Finally, we have to show that given a short

exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ parasitic, then $\mathcal{F}_2$ is parasitic. This follows immediately on evaluating on x lying over a scheme flat over $\mathcal{X}$. This proves (2)(b), see Homology, Lemma 12.10.2.

Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of $\mathcal{X}$ which lies over a scheme U such that $x : U \to \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \to U \times_{x,\mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \to \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \to x\}$ is a covering for the smooth topology on $\mathcal{X}$. Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \to x\}_{i\in I}$ in $\mathcal{X}$ we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as the étale topology, see More on Morphisms, Lemma 37.38.7. Namely, $\{x_i \to x\}_{i\in I}$ lies over a smooth covering $\{U_i \to U\}_{i\in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \to U\}_{j\in J}$ which refines $\{U_i \to U\}_{i\in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \to x\}$ is an étale covering in $\mathcal{X}$ refining $\{x_i \to x\}_{i\in I}$. This means the map $\mathcal{F}(x) \to \prod_{j\in J} \mathcal{F}(x'_j)$, which is injective as $\mathcal{F}$ is a sheaf in the étale topology, factors through $\mathcal{F}(x) \to \prod_{i\in I} \mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). □

Parasitic modules are preserved under absolutely any pushforward.

0775 **Lemma 103.9.3.** *Let $\tau \in \{étale, fppf\}$. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a parasitic object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

(1) *$H^i_\tau(\mathcal{X}, \mathcal{F}) = 0$ for all i.*
(2) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^if_*\mathcal{F}$ (computed in τ-topology) is a parasitic object of $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$.*

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 96.21.2 we see that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V. By Lemma 103.9.2 it suffices to show that these cohomology groups are zero when $y : V \to \mathcal{Y}$ is flat. Note that $\mathrm{pr} : V \times_{y,\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is flat as a base change of y. Hence by Lemma 103.9.2 we see that $\mathrm{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 96.20.1 to reduce this to the case where $\mathcal{X}$ is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 103.9.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of Lemma 103.5.1) we reduce to the case where the algebraic stack $\mathcal{X}$ is representable by a scheme X. Then $H^i_\tau(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 35.12.2. □

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ aren't exact in general.

0776 **Lemma 103.9.4.** *Let $\mathcal{X}$ be an algebraic stack. Let $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$ be maps in $QCoh(\mathcal{O}_\mathcal{X})$ with $\beta \circ \alpha = 0$. The following are equivalent:*

(1) *in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$ the complex $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact at $\mathcal{G}$,*
(2) $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ *computed in either $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ is parasitic.*

Proof. We have $QCoh(\mathcal{O}_\mathcal{X}) \subset LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Section 103.8. Hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ agree, see Proposition 103.8.1. From now on we will use the étale topology on $\mathcal{X}$.

Let $\mathcal{E}$ be the cohomology of $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ computed in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$. Let $x : U \to \mathcal{X}$ be a flat morphism where U is a scheme. As we are using the étale topology, the restriction functor $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to Mod(U_{étale}, \mathcal{O}_U)$ is exact. On the other hand, by Lemma 103.4.1 and Sheaves on Stacks, Lemma 96.14.2 the restriction functor

$$QCoh(\mathcal{O}_\mathcal{X}) \xrightarrow{x^*} QCoh((Sch/U)_{étale}, \mathcal{O}) \xrightarrow{-|_{U_{étale}}} QCoh(U_{étale}, \mathcal{O}_U)$$

is exact too. We conclude that $\mathcal{E}|_{U_{étale}} = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha))|_{U_{étale}}$.

If (1) holds, then $\mathcal{E} = 0$ hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$ and this is the definition of a parasitic module. If (2) holds, then $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$ hence $\mathcal{E}$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$. This certainly implies that the quasi-coherent module $\mathcal{E}$ is zero, for example apply Lemma 103.4.2 to the map $0 \to \mathcal{E}$. □

103.10. Quasi-coherent modules

0777 We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 96.15. This fact is the basis for the following.

0778 **Lemma 103.10.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ be the category of locally quasi-coherent modules with the flat base change property, see Section 103.8. The inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$Q : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Set $R = U \times_\mathcal{X} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 94.16.2. We may and do replace $\mathcal{X}$ by $[U/R]$. By Sheaves on Stacks, Proposition 96.14.3 there is an equivalence

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

Let us construct a functor

$$q_2 : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U, R, s, t, c)$$

by the following rule: if $\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{étale}}, \alpha)$$

where α is the isomorphism

$$t^*_{small}(f^*\mathcal{F}|_{U_{étale}}) \to t^*f^*\mathcal{F}|_{R_{étale}} \to s^*f^*\mathcal{F}|_{R_{étale}} \to s^*_{small}(f^*\mathcal{F}|_{U_{étale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because $\mathcal{F}$ is locally quasi-coherent and that we used (and needed) the flat base change property in the construction of the descent datum α. We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 78.12.1) holds. Looking at the proof of Sheaves on Stacks, Proposition 96.14.3 we see that $q_2 \circ i$ is the quasi-inverse to q_1. We define $Q = q_1 \circ q_2$. Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and let $\mathcal{G}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. We have

$$\begin{aligned}\mathrm{Mor}_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(i(\mathcal{G}), \mathcal{F}) &= \mathrm{Mor}_{QCoh(U,R,s,t,c)}(q_2(i(\mathcal{G})), q_2(\mathcal{F})) \\ &= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(\mathcal{G}, Q(\mathcal{F}))\end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 96.14.4 and the second equality holds because $q_1 \circ i$ and q_2 are quasi-inverse equivalences of categories. The assertion $Q \circ i \cong \mathrm{id}$ is a formal consequence of the fact that i is fully faithful. □

0779 **Lemma 103.10.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $Q : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$ be the functor constructed in Lemma 103.10.1.*

(1) *The kernel of Q is exactly the collection of parasitic objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
(2) *For any object $\mathcal{F}$ of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic.*
(3) *The functor Q is exact and commutes with all limits and colimits.*

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 103.10.1. Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. It is clear from the proof of Lemma 103.10.1 that $\mathcal{F}$ is in the kernel of Q if and only if $\mathcal{F}|_{U_{étale}} = 0$. In particular, if $\mathcal{F}$ is parasitic then $\mathcal{F}$ is in the kernel. Next, let $x : V \to \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_\mathcal{X} U$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow[q]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q^*_{small} is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p^*_{small}\mathcal{F}|_{U_{étale}} \cong q^*_{small}\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of Q, then $\mathcal{F}|_{V_{étale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \to \mathcal{F}$ becomes an isomorphism after restricting to $U_{étale}$.

To see part (3) note that Q is left exact as a right adjoint. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\mathcal{F}) & \longrightarrow & Q(\mathcal{G}) & \longrightarrow & Q(\mathcal{H}) & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \end{array}$$

Since the kernels and cokernels of a, b, and c are parasitic by part (2) and since the bottom row is a short exact sequence, we see that the top row as a complex of $\mathcal{O}_\mathcal{X}$-modules has parasitic cohomology sheaves (details omitted; this uses that the category of parasitic modules is a Serre subcategory of the category of all modules). By left exactness of Q we see that only exactness at $Q(\mathcal{H})$ is at issue. However, the cokernel $\mathcal{Q}$ of $Q(\mathcal{G}) \to Q(\mathcal{H}))$ may be computed either in $Mod(\mathcal{O}_\mathcal{X})$ or in $QCoh(\mathcal{O}_\mathcal{X})$ with the same result because the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ is a left adjoint and hence right exact. Hence $\mathcal{Q} = Q(\mathcal{Q})$ is both quasi-coherent and parasitic, whence 0 by part (1) as desired.

As a right adjoint Q commutes with all limits. Since Q is exact, to show that Q commutes with all colimits it suffices to show that Q commutes with direct sums, see Categories, Lemma 4.14.12. Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. To see that $Q(\bigoplus \mathcal{F}_i)$ is equal to $\bigoplus Q(\mathcal{F}_i)$ we look at the construction of Q in the proof of Lemma 103.10.1. This uses a presentation $\mathcal{X} = [U/R]$ where U is a scheme. Then $Q(\mathcal{F})$ is computed by first taking the pair $(\mathcal{F}|_{U_{étale}}, \alpha)$ in $QCoh(U, R, s, t, c)$ and then using the equivalence $QCoh(U, R, s, t, c) \cong QCoh(\mathcal{O}_\mathcal{X})$. Since the restriction functor $Mod(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_{U_{étale}})$, $\mathcal{F} \mapsto \mathcal{F}|_{U_{étale}}$ commutes with direct sums, the desired equality is clear. □

0GQJ **Lemma 103.10.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Then $Q_\mathcal{X} \circ f^* = f^* \circ Q_\mathcal{Y}$ where $Q_\mathcal{X}$ and $Q_\mathcal{Y}$ are as in Lemma 103.10.1.*

Proof. Observe that f^* preserves both $QCoh$ and $LQCoh^{fbc}$, see Sheaves on Stacks, Lemma 96.11.2 and Proposition 103.8.1. If $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ then $Q_\mathcal{Y}(\mathcal{F}) \to \mathcal{F}$ has parasitic kernel and cokernel by Lemma 103.10.2. As f is flat we get that $f^*Q_\mathcal{Y}(\mathcal{F}) \to f^*\mathcal{F}$ has parasitic kernel and cokernel by Lemma 103.9.2. Thus the induced map $f^*Q_\mathcal{Y}(\mathcal{F}) \to Q_\mathcal{X}(f^*\mathcal{F})$ has parasitic kernel and cokernel and hence is an isomorphism for example by Lemma 103.9.4. □

0GQK **Lemma 103.10.4.** *Let $\mathcal{X}$ be an algebraic stack. Let x be an object of $\mathcal{X}$ lying over the scheme U such that $x : U \to \mathcal{X}$ is flat. Then for $\mathcal{F}$ in $QCoh^{fbc}(\mathcal{O}_\mathcal{X})$ we have $Q(\mathcal{F})|_{U_{étale}} = \mathcal{F}|_{U_{étale}}$.*

Proof. True because the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic, see Lemma 103.10.2. □

0GQL **Remark 103.10.5.** Let $\mathcal{X}$ be an algebraic stack. The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian, the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is right exact, but not exact in general, see Sheaves on Stacks, Lemma 96.15.1. We can use the functor Q from Lemmas 103.10.1 and 103.10.2 to understand this. Namely, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Then

(1) the cokernel $\mathrm{Coker}(\varphi)$ computed in $Mod(\mathcal{O}_\mathcal{X})$ is quasi-coherent and is the cokernel of φ in $QCoh(\mathcal{O}_\mathcal{X})$,
(2) the image $\mathrm{Im}(\varphi)$ computed in $Mod(\mathcal{O}_\mathcal{X})$ is quasi-coherent and is the image of φ in $QCoh(\mathcal{O}_\mathcal{X})$, and
(3) the kernel $\mathrm{Ker}(\varphi)$ computed in $Mod(\mathcal{O}_\mathcal{X})$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by Proposition 103.8.1 and $Q(\mathrm{Ker}(\varphi))$ is the kernel in $QCoh(\mathcal{O}_\mathcal{X})$.

This follows from the references given.

0GQM **Remark 103.10.6.** Let $\mathcal{X}$ be an algebraic stack. Given two quasi-coherent $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ and $\mathcal{G}$ the tensor product module $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ is quasi-coherent, see Sheaves

on Stacks, Lemma 96.15.1 part (5). Similarly, given two locally quasi-coherent modules with the flat base change property, their tensor product has the same property, see Proposition 103.8.1. Thus the inclusion functors

$$QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$$

are functors of symmetric monoidal categories. What is more interesting is that the functor

$$Q : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

is a functor of symmetric monoidal categories as well. Namely, given $\mathcal{F}$ and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ we obtain

$$\begin{array}{ccc} Q(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{X}} Q(\mathcal{G}) & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G} \\ & \searrow \quad \nearrow & \\ & Q(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) & \end{array}$$

where the south-west arrow comes from the universal property of the north-west arrow (and the fact already mentioned that the object in the upper left corner is quasi-coherent). If we restrict this diagram to $U_{étale}$ for $U \to \mathcal{X}$ flat, then all three arrows become isomorphisms (see Lemmas 103.10.1 and 103.10.2 and Definition 103.9.1). Hence $Q(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{X}} Q(\mathcal{G}) \to Q(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G})$ is an isomorphism, see for example Lemma 103.4.2.

07B2 **Remark 103.10.7.** Let $\mathcal{X}$ be an algebraic stack. Let $Parasitic(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$ denote the full subcategory consisting of parasitic modules. The results of Lemmas 103.10.1 and 103.10.2 imply that

$$QCoh(\mathcal{O}_\mathcal{X}) = LQCoh^{fbc}(\mathcal{O}_\mathcal{X})/Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 12.10.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ which is left adjoint to the quotient functor is a key feature of the situation. In Derived Categories of Stacks, Section 104.5 and especially Lemma 104.5.4 we prove that a similar result holds on the level of derived categories.

0GQN **Lemma 103.10.8.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. The internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. The quasi-coherent module $hom(\mathcal{F}, \mathcal{G}) = Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$ has the following universal property*

$$\mathrm{Hom}_\mathcal{X}(\mathcal{H}, hom(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_\mathcal{X}(\mathcal{H} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}, \mathcal{G})$$

for $\mathcal{H}$ in $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. The construction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 18.27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules; the output $\mathcal{H}om$ is a sheaf for the fppf topology because $\mathcal{F}$ and $\mathcal{G}$ are assumed sheaves in the fppf topology, see Modules on Sites, Lemma 18.27.1. By Sheaves on Stacks, Lemma 96.12.4 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is locally quasi-coherent. By Lemma 103.7.2 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property. Hence $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is an object

of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and it makes sense to apply the functor Q of Lemma 103.10.1 to it. By the universal property of Q we have

$$\mathrm{Hom}_\mathcal{X}(\mathcal{H}, Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) = \mathrm{Hom}_\mathcal{X}(\mathcal{H}, \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$$

for $\mathcal{H}$ quasi-coherent, hence the displayed formula of the lemma follows from Modules on Sites, Lemma 18.27.6. □

0GQP **Lemma 103.10.9.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{Y}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then $f^*hom(\mathcal{F}, \mathcal{G}) = hom(f^*\mathcal{F}, f^*\mathcal{G})$ with notation as in Lemma 103.10.8.*

Proof. We have $f^*\,\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(f^*\mathcal{F}, f^*\mathcal{G})$ by Modules on Sites, Lemma 18.31.4. (Observe that this step is not where the flatness of f is used as the morphism of ringed topoi associated to f is always flat, see Sheaves on Stacks, Remark 96.6.3.) Then apply Lemma 103.10.3 (and here we do use flatness of f). □

103.11. Pushforward of quasi-coherent modules

070A Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}^2_k \setminus \{0\} \longrightarrow \mathbf{A}^2_k = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (Sch/X)_{fppf} \to (Sch/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_\mathcal{X}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_\mathcal{X}$ is quasi-coherent on $\mathcal{Y}$ then we would have $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. However, consider $T = \mathrm{Spec}(k) \to \mathbf{A}^2_k = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = 0$ because $X \times_Y T = \emptyset$ whereas $\Gamma(T, \mathcal{O}_\mathcal{Y}) = k$. On the positive side, for any flat morphism $T \to Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = \Gamma(T, \mathcal{O}_\mathcal{Y})$ as follows from Cohomology of Schemes, Lemma 30.5.2 using that j is quasi-compact and quasi-separated.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

(1) f_* does preserve locally quasi-coherent modules (Lemma 103.6.2),
(2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 103.7.3), and
(3) locally quasi-coherent $\mathcal{O}_\mathcal{Y}$-modules with the flat base change property give rise to quasi-coherent modules on a presentation of $\mathcal{Y}$ and hence quasi-coherent modules on $\mathcal{Y}$, see Sheaves on Stacks, Section 96.15.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in \mathrm{Ob}(\mathcal{Y})$ such that the associated 1-morphism $y : V \to \mathcal{Y}$ is flat, see Lemma 103.11.2. Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$.

However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

077A **Proposition 103.11.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{f_*} LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

where the functors f_* and Q are as in Proposition 103.8.1 and Lemma 103.10.1. Moreover, if we define $R^i f_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{R^i f_*} LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

then the sequence of functors $\{R^i f_{QCoh,*}\}_{i\geq 0}$ forms a cohomological δ-functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then we have

$$\begin{aligned}
\mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}) \\
&= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, Q(f_*\mathcal{F})) \\
&= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_{QCoh,*}\mathcal{F})
\end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 103.8.1 we see that $f_*\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ (and can be computed in either the fppf or étale topology) and we obtain the second equality by Lemma 103.10.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^i f_{QCoh,*}\}_{i\geq 0}$ is a cohomological δ-functor as defined in Homology, Definition 12.12.1 let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $QCoh(\mathcal{O}_\mathcal{X})$. This sequence may not be an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ but we know that it is up to parasitic modules, see Lemma 103.9.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned}
0 \to \mathcal{P}_1 \to \mathcal{F}_1 \to \mathcal{I}_2 \to 0 \\
0 \to \mathcal{I}_2 \to \mathcal{F}_2 \to \mathcal{Q}_2 \to 0 \\
0 \to \mathcal{P}_2 \to \mathcal{Q}_2 \to \mathcal{I}_3 \to 0 \\
0 \to \mathcal{I}_3 \to \mathcal{F}_3 \to \mathcal{P}_3 \to 0
\end{aligned}$$

of $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{P}_i$ parasitic. Note that each of the sheaves $\mathcal{P}_j$, $\mathcal{I}_j$, $\mathcal{Q}_j$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 103.8.1. Applying $R^i f_*$ we obtain long exact sequences

$$\begin{aligned}
0 \to f_*\mathcal{P}_1 \to f_*\mathcal{F}_1 \to f_*\mathcal{I}_2 \to R^1 f_*\mathcal{P}_1 \to \ldots \\
0 \to f_*\mathcal{I}_2 \to f_*\mathcal{F}_2 \to f_*\mathcal{Q}_2 \to R^1 f_*\mathcal{I}_2 \to \ldots \\
0 \to f_*\mathcal{P}_2 \to f_*\mathcal{Q}_2 \to f_*\mathcal{I}_3 \to R^1 f_*\mathcal{P}_2 \to \ldots \\
0 \to f_*\mathcal{I}_3 \to f_*\mathcal{F}_3 \to f_*\mathcal{P}_3 \to R^1 f_*\mathcal{I}_3 \to \ldots
\end{aligned}$$

where are the terms are objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ by Proposition 103.8.1. By Lemma 103.9.3 the sheaves $R^if_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q, see Lemma 103.10.2. Since Q is exact the maps

$$Q(R^if_*\mathcal{F}_3) \cong Q(R^if_*\mathcal{I}_3) \cong Q(R^if_*\mathcal{Q}_2) \to Q(R^{i+1}f_*\mathcal{I}_2) \cong Q(R^{i+1}f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^if_{QCoh,*}\}_{i\geq 0}$ into a cohomological δ-functor. □

0GQQ **Lemma 103.11.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $y : V \to \mathcal{Y}$ in $\mathrm{Ob}(\mathcal{Y})$ with y a flat morphism. Let $\mathcal{F}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then $(f_*\mathcal{F})(y) = (f_{QCoh,*}\mathcal{F})(y)$ and $(R^if_*\mathcal{F})(y) = (R^if_{QCoh,*}\mathcal{F})(y)$ for all $i \in \mathbf{Z}$.*

Proof. This follows from the construction of the functors $R^if_{QCoh,*}$ in Proposition 103.11.1, the definition of parasitic modules in Definition 103.9.1, and Lemma 103.10.2 part (2). □

0GQR **Remark 103.11.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then there is a canonical commutative diagram

$$\begin{array}{ccc} f_{QCoh,*}\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_{QCoh,*}\mathcal{G} & \longrightarrow & f_*\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_*\mathcal{G} \\ \downarrow & & \downarrow c \\ f_{QCoh,*}(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) & \longrightarrow & f_*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) \end{array}$$

The vertical arrow c on the right is the naive relative cup product (in degree 0), see Cohomology on Sites, Section 21.33. The source and target of c are in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 103.8.1. Applying Q to c we obtain the left vertical arrow as Q commutes with tensor products, see Remark 103.10.6. This construction is functorial in $\mathcal{F}$ and $\mathcal{G}$.

0782 **Lemma 103.11.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^qf_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 21.14.5 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^qf_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^qf_{QCoh,*}\mathcal{F} \longrightarrow R^qf_*\mathcal{F}$$

are parasitic modules on $\mathcal{Y}$ (Lemma 103.10.2) hence have vanishing cohomology (Lemma 103.9.3). It follows formally that $H^p(\mathcal{Y}, R^qf_{QCoh,*}\mathcal{F}) = H^p(\mathcal{Y}, R^qf_*\mathcal{F})$ and we win. □

0783 **Lemma 103.11.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = R^pg_{QCoh,*}(R^qf_{QCoh,*}\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 21.14.7 the Leray spectral sequence with

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converges to $R^{p+q}(g \circ f)_*\mathcal{F}$. By the results of Proposition 103.8.1 all the terms of this spectral sequence are objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Z})$. Applying the exact functor $Q_\mathcal{Z} : LQCoh^{fbc}(\mathcal{O}_\mathcal{Z}) \to QCoh(\mathcal{O}_\mathcal{Z})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_\mathcal{Z})$ covering to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$. Hence the result follows if we can show that

$$Q_\mathcal{Z}(R^p g_*(R^q f_*\mathcal{F})) = Q_\mathcal{Z}(R^p g_*(Q_\mathcal{X}(R^q f_*\mathcal{F}))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_\mathcal{X}(R^q f_*\mathcal{F}) \longrightarrow R^q f_*\mathcal{F}$$

are parasitic (Lemma 103.10.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 103.9.3). □

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 96.20 and 96.21.

0784 **Proposition 103.11.6.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Then there is a spectral sequence*

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ ($p+1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 96.20.1. □

0785 **Proposition 103.11.7.** *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that*

1. *f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and*
2. *g is quasi-compact and quasi-separated.*

If $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_{QCoh,*}f_p^*\mathcal{F} \Rightarrow R^{p+q}g_{QCoh,*}\mathcal{F}$$

in $QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q(g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

by Sheaves on Stacks, Proposition 96.21.1. Applying the exact functor $Q_\mathcal{Y} : LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{Y})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_\mathcal{Y})$. □

103.12. Further remarks on quasi-coherent modules

0GQS In this section we collect some results that to help understand how to use quasi-coherent modules on algebraic stacks.

Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{U}$ is represented by the algebraic space U. Consider the functor

$$a : Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow Mod(U_{étale}, \mathcal{O}_U), \quad \mathcal{F} \longmapsto f^*\mathcal{F}|_{U_{étale}}$$

given by pullback (Sheaves on Stacks, Section 96.7) followed by restriction (Sheaves on Stacks, Section 96.10). Applying this functor to locally quasi-coherent modules we obtain a functor

$$b : LQCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

See Sheaves on Stacks, Lemmas 96.12.3 and 96.14.1. We can further limit our functor to even smaller subcategories to obtain

$$c : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

and

$$d : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

About these functors we can say the following:[3]

(1) The functor a is exact. Namely, pullback $f^* = f^{-1}$ is exact (Sheaves on Stacks, Section 96.7) and restriction to $U_{étale}$ is exact, see Sheaves on Stacks, Equation (96.10.2.1).
(2) The functor b is exact. Namely, by Sheaves on Stacks, Lemma 96.12.4 the inclusion $LQCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is exact.
(3) The functor c is exact. Namely, by Proposition 103.8.1 the inclusion functor $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is exact.
(4) The functor d is right exact but not exact in general. Namely, by Sheaves on Stacks, Lemma 96.12.5 the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is right exact. We omit giving an example showing non-exactness.
(5) If f is flat, then d is exact. This follows on combining Lemma 103.4.1 and Sheaves on Stacks, Lemma 96.14.2.
(6) If f is flat, then c kills parasitic objects. Namely, f^* preserves parasitic object by Lemma 103.9.2. Then for any scheme V étale over U and hence flat over $\mathcal{X}$ we see that $0 = f^*\mathcal{F}|_{V_{étale}} = c(\mathcal{F})|_{V_{étale}}$ by the compatibility of restriction with étale localization Sheaves on Stacks, Remark 96.10.2. Hence clearly $c(\mathcal{F}) = 0$.
(7) If f is flat, then $c = d \circ Q$. Namely, the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic by Lemma 103.10.2. Thus, since c is exact (3) and kills parasitic objects (6), we see that c applied to $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.
(8) The functors a, b, c, d commute with colimits and arbitrary direct sums. This is true for f^* and restriction as left adjoints and hence it holds for a. Then it follows for b, c, d by the references given above.
(9) The functors a, b, c, d commute with tensor products.

[3] We suggest working out why these statements are true on a napkin instead of following the references given.

(10) If f is flat and surjective, $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, and $c(\mathcal{F}) = 0$, then $\mathcal{F}$ is parasitic. Namely, by (7) we get $d(Q(\mathcal{F})) = 0$. We may assume U is a scheme by the compatibility of restriction with étale localization (see reference above). Then Lemma 103.4.2 applied to $0 \to Q(\mathcal{F})$ and the morphism $f : U \to \mathcal{X}$ shows that $Q(\mathcal{F}) = 0$. Thus $\mathcal{F}$ is parasitic by Lemma 103.10.2.

(11) If f is flat and surjective, then the functor d reflects exactness. More precisely, let $\mathcal{F}^\bullet$ be a complex in $QCoh(\mathcal{O}_\mathcal{X})$. Then $\mathcal{F}^\bullet$ is exact in $QCoh(\mathcal{O}_\mathcal{X})$ if and only if $d(\mathcal{F}^\bullet)$ is exact. Namely, we have seen one implication in (5). For the other, suppose that $H^i(d(\mathcal{F}^\bullet)) = 0$. Then $\mathcal{G} = H^i(\mathcal{F}^\bullet)$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$ with $d(\mathcal{G}) = 0$. Hence $\mathcal{G}$ is both quasi-coherent and parasitic by (10), whence 0 for example by Remark 103.10.7.

0GQT (12) If f is flat, $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(QCoh(\mathcal{O}_\mathcal{X}))$, and $\mathcal{F}$ of finite presentation and let then we have

$$d(hom(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{O}_U}(d(\mathcal{F}), d(\mathcal{G}))$$

with notation as in Lemma 103.10.8. Perhaps the easiest way to see this is as follows

$$\begin{aligned} d(hom(\mathcal{F}, \mathcal{G})) &= d(Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) \\ &= c(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})) \\ &= f^* \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{U}}(f^*\mathcal{F}, f^*\mathcal{G})|_{U_{étale}} \\ &= \mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{F}|_{U_{étale}}, f^*\mathcal{G}|_{U_{étale}}) \end{aligned}$$

The first equality by construction of hom. The second equality by (7). The third equality by definition of c. The fourth equality by Modules on Sites, Lemma 18.31.4. The final equality by the same reference applied to the flat morphism of ringed topoi $i_U(U_{étale}, \mathcal{O}_U) \to (\mathcal{U}_{étale}, \mathcal{O}_\mathcal{U})$ of Sheaves on Stacks, Lemma 96.10.1.

(13) add more here.

103.13. Colimits and cohomology

0GQU The following lemma in particular applies to diagrams of quasi-coherent sheaves.

0GQV **Lemma 103.13.1.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then*

$$\mathrm{colim}_i H^p(\mathcal{X}, \mathcal{F}_i) \longrightarrow H^p(\mathcal{X}, \mathrm{colim}_i \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $\mathcal{X}$. The same is true for abelian sheaves on $\mathcal{X}_{étale}$ taking cohomology in the étale topology.

Proof. Let $\tau = fppf$, resp. $\tau = étale$. The lemma follows from Cohomology on Sites, Lemma 21.16.2 applied to the site $\mathcal{X}_\tau$. In order to check the assumptions we use Cohomology on Sites, Remark 21.16.3. Namely, let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{X}_\tau)$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (1) – (4) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjetive and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 100.6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_\tau$.
(2) Since coverings in $\mathcal{X}_\tau$ are fppf, resp. étale coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 34.7.4, resp. Lemma 34.4.4.
(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_\tau)$ is equal to the sheaf on $\mathcal{X}_\tau$ determined by the algebraic space $W = U \times_{x,\mathcal{X},x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_\tau$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 101.7.8 for example. Hence we can choose an affine scheme U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^\# \to h_x^\# \times h_{x'}^\#$ is surjective as desired.
(4) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. Let $a, b : U \to U'$ be a morphism over $\mathcal{X}$, i.e., $a, b : x \to x'$ is a morphism in $\mathcal{X}_\tau$. Then the equalizer of h_a and h_b is represented by the equalizer of $a, b : U \to U'$ which is affine scheme over $\mathcal{X}$ and hence in $\mathcal{B}$.

This finished the proof. □

0GQW **Lemma 103.13.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $\mathcal{X}$. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i.$$

The same is true for abelian sheaves on $\mathcal{X}_{étale}$ taking higher direct images in the étale topology.

Proof. We will prove this for the fppf topology; the proof for the étale topology is the same. Recall that $R^i f_* \mathcal{F}$ is the sheaf on $\mathcal{Y}_{fppf}$ associated to the presheaf

$$(y : V \to \mathcal{Y}) \longmapsto H^i(V \times_{y,\mathcal{Y}} \mathcal{X}, \mathrm{pr}^{-1}\mathcal{F})$$

See Sheaves on Stacks, Lemma 96.21.2. Recall that the colimit is the sheaf associated to the presheaf colimit. When V is affine, the fibre product $V \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact and quasi-separated. Hence we can apply Lemma 103.13.1 to $H^p(V \times_{\mathcal{Y}} \mathcal{X}, -)$ where V is affine. Since every V has an fppf covering by affine objects this proves the lemma. Some details omitted. □

0GQX **Lemma 103.13.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f_{QCoh,*}$ and the functors $R^i f_{QCoh,*}$ commute with direct sums and filtered colimits.*

Proof. The functors f_* and $R^i f_*$ commute with direct sums and filtered colimits on all modules by Lemma 103.13.2. The lemma follows as $f_{QCoh,*} = Q \circ f_*$ and $R^i f_{QCoh,*} = Q \circ R^i f_*$ and Q commutes with all colimits, see Lemma 103.10.2. □

0GQY **Lemma 103.13.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functors $R^i f_{QCoh,*}$, $i > 0$ vanish and the functor $f_{QCoh,*}$ is exact and commutes with direct sums and all colimits.*

Proof. Since we have $R^i f_{QCoh,*} = Q \circ R^i f_*$ we obtain the vanishing from Lemma 103.8.4. The vanishing implies that $f_{QCoh,*}$ is exact as $\{R^i f_{QCoh,*}\}_{i \geq 0}$ form a δ-functor, see Proposition 103.11.1. Then $f_{QCoh,*}$ commutes with direct sums for example by Lemma 103.13.3. An exact functor which commutes with direct sums commutes with all colimits. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic stacks.

0GQZ **Lemma 103.13.5.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_\mathcal{X}$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation. Then we have*

$$\operatorname{colim}_i \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

In particular, $\operatorname{Hom}_\mathcal{X}(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 18.27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 7.17.8 part (4) for the site $\mathcal{X}_{fppf}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 7.17.9. Namely, let $\mathcal{B} \subset \operatorname{Ob}(\mathcal{X}_{fppf})$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (2)(a), (2)(b), and (2)(c) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjetive and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 100.6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_{fppf}$.
(2) Since coverings in $\mathcal{X}_{fppf}$ are fppf coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 34.7.4.
(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_{fppf})$ is equal to the sheaf on $\mathcal{X}_{fppf}$ determined by the algebraic space $W = U \times_{x,\mathcal{X},x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_{fppf}$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 101.7.8 for example. Hence we can choose an affine scheme U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^\# \to h_x^\# \times h_{x'}^\#$ is surjective as desired.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Sheaves on Stacks, Lemma 96.15.1. □

103.14. The lisse-étale and the flat-fppf sites

0786 In the book [**LMB00**] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 110.59 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the Stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

0787 **Definition 103.14.1.** Let $\mathcal{X}$ be an algebraic stack.

(1) The *lisse-étale site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{lisse,étale}$[4] of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{lisse,étale}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{lisse,étale}$ which forms a covering of $\mathcal{X}_{étale}$.

(2) The *flat-fppf site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{flat,fppf}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is flat. A covering of $\mathcal{X}_{flat,fppf}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{flat,fppf}$ which forms a covering of $\mathcal{X}_{fppf}$.

We denote $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ the restriction of $\mathcal{O}_{\mathcal{X}}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to "topological" properties in this lemma).

0788 **Lemma 103.14.2.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *The inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{lisse,étale}) \longrightarrow Sh(\mathcal{X}_{étale})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

(2) *The inclusion functor $\mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 7.13.1 and 7.20.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 7.21.5 and 7.21.7. Parts (d), (e) follow from Modules on Sites, Lemmas 18.16.2 and 18.16.4. Part (f) is immediate. □

0GR0 **Lemma 103.14.3.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 103.14.2.*

(1) *For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have*

[4] In the literature the site is denoted Lis-ét($\mathcal{X}$) or Lis-Et($\mathcal{X}$) and the associated topos is denoted $\mathcal{X}_{\text{lis-ét}}$ or $\mathcal{X}_{\text{lis-et}}$. In the Stacks project our convention is to name the site and denote the corresponding topos by $Sh(\mathcal{C})$.

(a) $H^p(\mathcal{X}_{étale}, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}, g^{-1}\mathcal{F})$, *and*
(b) $H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}/x, g^{-1}\mathcal{F})$ *for any object* x *of* $\mathcal{X}_{lisse,étale}$.
The same holds for sheaves of modules.

(2) *For an abelian sheaf* $\mathcal{F}$ *on* $\mathcal{X}_{fppf}$ *we have*
(a) $H^p(\mathcal{X}_{fppf}, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}, g^{-1}\mathcal{F})$, *and*
(b) $H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}/x, g^{-1}\mathcal{F})$ *for any object* x *of* $\mathcal{X}_{flat,fppf}$.
The same holds for sheaves of modules.

Proof. Part (1)(a) follows from Sheaves on Stacks, Lemma 96.23.3 applied to the inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$. Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. Part (2) is proved in the same manner. □

0789 **Lemma 103.14.4.** *Let* $\mathcal{X}$ *be an algebraic stack. Notation as in Lemma 103.14.2.*

(1) *There exists a functor*

$$g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^*. *Moreover it agrees with the functor* $g_!$ *on abelian sheaves and* $g^*g_! = id$.

(2) *There exists a functor*

$$g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^*. *Moreover it agrees with the functor* $g_!$ *on abelian sheaves and* $g^*g_! = id$.

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 18.41.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps Modules on Sites, Equation (18.41.2.1) are isomorphisms.

Lisse-étale case. Let $x \in \mathrm{Ob}(\mathcal{X}_{lisse,étale})$ lying over a scheme U with $x : U \to \mathcal{X}$ smooth. Consider the induced fully faithful functor

$$g' : \mathcal{X}_{lisse,étale}/x \longrightarrow \mathcal{X}_{étale}/x$$

The right hand side is identified with $(Sch/U)_{étale}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 59.49.2 applies.

Flat-fppf case. Let $x \in \mathrm{Ob}(\mathcal{X}_{flat,fppf})$ lying over a scheme U with $x : U \to \mathcal{X}$ flat. Consider the induced fully faithful functor

$$g' : \mathcal{X}_{flat,fppf}/x \longrightarrow \mathcal{X}_{fppf}/x$$

The right hand side is identified with $(Sch/U)_{fppf}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 59.49.2 applies.

In both cases the equality $g^*g_! = \mathrm{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 103.14.2. □

078A **Lemma 103.14.5.** *Let* $\mathcal{X}$ *be an algebraic stack. Notation as in Lemmas 103.14.2 and 103.14.4.*

(1) *We have* $g_!\mathcal{O}_{\mathcal{X}_{lisse,étale}} = \mathcal{O}_{\mathcal{X}}$.
(2) *We have* $g_!\mathcal{O}_{\mathcal{X}_{flat,fppf}} = \mathcal{O}_{\mathcal{X}}$.

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{étale}$ (resp. $\mathcal{C} = \mathcal{X}_{fppf}$) and we denote $\mathcal{C}' = \mathcal{X}_{lisse,étale}$ (resp. $\mathcal{C}' = \mathcal{X}_{flat,fppf}$). Then $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. In this proof we will think of objects V of $\mathcal{C}$ as schemes over $\mathcal{X}$ and objects U of $\mathcal{C}'$ as schemes smooth (resp. flat) over $\mathcal{X}$. Finally, we write $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the $\mathcal{O}$-module homomorphism $g_!\mathcal{O}' \to \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

Recall that $g_!\mathcal{O}'$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}'$ given by the rule

$$V \longmapsto \operatorname{colim}_{V \to U} \mathcal{O}'(U)$$

where the colimit is taken in the category of abelian groups (Modules on Sites, Definition 18.16.1). Below we will use frequently that if

$$V \to U \to U'$$

are morphisms and if $f' \in \mathcal{O}'(U')$ restricts to $f \in \mathcal{O}'(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}' \to \mathcal{O}$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Let us prove that $g_!\mathcal{O}' \to \mathcal{O}$ is surjective. Let $h \in \mathcal{O}(V)$ for some object V of $\mathcal{C}$. It suffices to show that h is locally in the image. Choose an object U of $\mathcal{C}'$ corresponding to a surjective smooth morphism $U \to \mathcal{X}$. Since $U \times_{\mathcal{X}} V \to V$ is surjective smooth, after replacing V by the members of an étale covering of V we may assume there exists a morphism $V \to U$, see Topologies on Spaces, Lemma 73.4.4. Using h we obtain a morphism $V \to U \times \mathbf{A}^1$ such that writing $\mathbf{A}^1 = \operatorname{Spec}(\mathbf{Z}[t])$ the element $t \in \mathcal{O}(U \times \mathbf{A}^1)$ pulls back to h. Since $U \times \mathbf{A}^1$ is an object of $\mathcal{C}'$ we see that $(V \to U \times \mathbf{A}^1, t)$ is an element of the colimit above which maps to $h \in \mathcal{O}(V)$ as desired.

Suppose that $s \in g_!\mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$\operatorname{colim}_{V \to U} \mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \to U_i$, U_i smooth (resp. flat) over $\mathcal{X}$, and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U_n$. Note that W is still smooth (resp. flat) over $\mathcal{X}$, i.e., defines an object of $\mathcal{C}'$. The fibre product

$$V' = V \times_{(\varphi_1, \ldots, \varphi_n), U} W$$

is surjective étale over V, hence it suffices to show that s maps to zero in $g_!\mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W. In other words, we have reduced to the case of (φ, s) where $\varphi : V \to U$ is a morphism with U in $\mathcal{C}'$ and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{(\varphi, 0)} & U \times \mathbf{A}^1 \\ & \searrow^{\varphi} & \uparrow{\scriptstyle (\mathrm{id}, 0)} \\ & & U \end{array}$$

we see that $((\varphi, 0) : V \to U \times \mathbf{A}^1, \mathrm{pr}_2^* x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\mathrm{pr}_1^* s + \mathrm{pr}_2^* x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over $\mathcal{X}$. Then we see that $(V \to Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. □

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

07AR **Lemma 103.14.6.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module with the flat base change property on $\mathcal{X}_{étale}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*
   (b) *$g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{lisse,étale}) \to Sh(\mathcal{X}_{étale})$ is as in Lemma 103.14.2.*

(2) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{fppf}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*
   (b) *$g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ is as in Lemma 103.14.2.*

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) $\Rightarrow$ (1)(b) is immediate as well. To see (1)(b) $\Rightarrow$ (1)(a) let U be a scheme and let $x : U \to \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of $\mathcal{X}$. Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{étale}} = 0$. Let $V \to \mathcal{X}$ be an flat morphism where V is a scheme. Set $W = U \times_\mathcal{X} V$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow[q]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{étale}} \cong q_{small}^*\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of g^*, then $\mathcal{F}|_{V_{étale}} = 0$ as desired. □

103.15. Functoriality of the lisse-étale and flat-fppf sites

0GR1 The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks. We warn the reader that the lisse-étale and flat-fppf topoi are not functorial with respect to all morphisms of algebraic stacks, see Examples, Section 110.59.

07AT **Lemma 103.15.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is smooth, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ which gives a morphism of ringed topoi fitting*

into the following commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{X}_{lisse,étale}) & \xrightarrow{g'} & Sh(\mathcal{X}_{étale}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{lisse,étale}) & \xrightarrow{g} & Sh(\mathcal{Y}_{étale}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ *and* $g'_!(f')^{-1} = f^{-1}g_!$.

(2) *If* f *is flat, then* f *restricts to a continuous and cocontinuous functor* $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$ *which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{flat,fppf}) & \xrightarrow{g'} & Sh(\mathcal{X}_{fppf}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{flat,fppf}) & \xrightarrow{g} & Sh(\mathcal{Y}_{fppf}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ *and* $g'_!(f')^{-1} = f^{-1}g_!$.

Proof. The initial statement comes from the fact that if $x \in \mathrm{Ob}(\mathcal{X})$ lies over a scheme U such that $x : U \to \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \to \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 101.33.2 and 101.25.2. The induced functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 103.14.2) and Sites, Lemma 7.21.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let $\mathcal{F}$ be a sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 7.45. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}f_*\mathcal{F}\right)(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (96.5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \to V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \to \mathcal{X}$ is smooth (resp. flat). Denote

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{lisse,étale} \text{ (resp. } \mathcal{X}_{flat,fppf})$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (96.5.0.1) shows that for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) we have

07AU (103.15.1.1)
$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

Since $(g')^{-1}$ is restriction we see that

$$\left(f'_*(g')^{-1}\mathcal{F}\right)(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'})$$

By Sheaves on Stacks, Lemma 96.23.3 we see that

$$\Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'}) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_*, the adjointness of $g_!$ and g^{-1}, and their "primed" versions. □

0GR2 **Lemma 103.15.2.** *With assumptions and notation as in Lemma 103.15.1. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Then*

07AW (103.15.2.1) $$R^p f'_*\mathcal{H} = \text{sheaf associated to } y \longmapsto H^p((V \times_{y,\mathcal{Y}} \mathcal{X})', (pr')^{-1}\mathcal{H})$$

Here y is an object of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) lying over the scheme V and the notation $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ and pr' are explained in the proof.

Proof. As in the proof of Lemma 103.15.1 let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects (x, φ) where x is an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) and $\varphi : f(x) \to y$ is a morphism in $\mathcal{Y}$. By Equation (103.15.1.1) we have

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

where pr' is the projection. For an object (x, φ) of $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ we can think of φ as a section of $(f')^{-1}h_y$ over x. Thus $(V \times_{\mathcal{Y}} \mathcal{X})'$ is the localization of the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) at the sheaf of sets $(f')^{-1}h_y$, see Sites, Lemma 7.30.3. The morphism

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{lisse,étale} \text{ (resp. } \mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{flat,fppf})$$

is the localization morphism. In particular, the pullback $(\mathrm{pr}')^{-1}$ preserves injective abelian sheaves, see Cohomology on Sites, Lemma 21.13.3.

Choose an injective resolution $\mathcal{H} \to \mathcal{I}^\bullet$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). By the formula for pushforward we see that $R^i f'_*\mathcal{H}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{array}{c} \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i+1}\Big) \end{array}$$

Since $(\mathrm{pr}')^{-1}$ is exact and preserves injectives the complex $(\mathrm{pr}')^{-1}\mathcal{I}^\bullet$ is an injective resolution of $(\mathrm{pr}')^{-1}\mathcal{H}$ and the proof is complete. □

0GR3 **Lemma 103.15.3.** *With assumptions and notation as in Lemma 103.15.1 the canonical (base change) map*

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$).

Proof. Comparing the formula for $g^{-1}R^p f_*\mathcal{F}$ and $R^p f'_*(g')^{-1}\mathcal{F}$ given in Sheaves on Stacks, Lemma 96.21.2 and Lemma 103.15.2 we see that it suffices to show

$$H^p((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'}) = H^p_\tau(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

where $\tau = étale$ (resp. $\tau = fppf$). Here y is an object of $\mathcal{Y}$ lying over a scheme V such that the morphism $y : V \to \mathcal{Y}$ is smooth (resp. flat). This equality follows from Sheaves on Stacks, Lemma 96.23.3. Although we omit the verification of the assumptions of the lemma, we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition. □

103.16. Quasi-coherent modules and the lisse-étale and flat-fppf sites

07AY In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

07AZ **Lemma 103.16.1.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*
(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*

Proof. Proof of (1). We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 101.33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still quasi-coherent, see Modules on Sites, Lemma 18.23.4. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a smooth surjective morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^* \mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 96.11.4 we see that $x^*\mathcal{F}$ is an fppf sheaf and since x was arbitrary we see that $\mathcal{F}$ is a sheaf in the fppf topology. Applying Sheaves on Stacks, Lemma 96.11.3 we see that $\mathcal{F}$ is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 101.25.2 and 101.27.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 96.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 96.19.10 there exists an fppf covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^* \mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 96.11.3 we see that $\mathcal{F}$ is quasi-coherent. □

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 18.23.

07B0 **Lemma 103.16.2.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 103.14.2.*

(1) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*
(2) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 96.9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 103.15.1 we see that $f_i^{-1}g_!\mathcal{H} = g_{i,!}(f_i')^{-1}\mathcal{H}$. Using Lemma 103.16.1 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation. Say we have

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 103.14.4 and Categories, Lemma 4.24.5) we conclude that there exists an exact sequence

$$\bigoplus\nolimits_{j\in J} g_!\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} g_!\mathcal{O} \longrightarrow g_!\mathcal{H} \longrightarrow 0$$

Lemma 103.14.5 shows that $g_!\mathcal{O} = \mathcal{O}_{\mathcal{X}}$. In case (2) we are done. In case (1) we apply Sheaves on Stacks, Lemma 96.11.4 to conclude. □

07B1 **Lemma 103.16.3.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *With g as in Lemma 103.14.2 for the lisse-étale site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

   (b) *if $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 103.10.1.*
(2) *With g as in Lemma 103.14.2 for the flat-fppf site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

   (b) *if $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 103.10.1.*

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 18.23.4. Hence g^{-1} preserves the categories of quasi-coherent modules; here we use that $QCoh(\mathcal{O}_{\mathcal{X}}) = QCoh(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ by Sheaves on Stacks, Lemma 96.11.4. The same is true for $g_!$ by Lemma 103.16.2. We know that $\mathcal{H} \to g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 103.14.2. Conversely, if $\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}})$ then the map $g_!g^{-1}\mathcal{F} \to \mathcal{F}$ is a map of quasi-coherent modules on $\mathcal{X}$ whose restriction to any scheme smooth over $\mathcal{X}$ is an isomorphism. Then the

discussion in Sheaves on Stacks, Sections 96.14 and 96.15 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. By Lemma 103.10.2 the kernel and cokernel of the map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic. Hence by Lemma 103.14.6 and since $g^* = g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F}) \to g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F}) \to Q(\mathcal{F})$ is an isomorphism by our arguments above. □

07B4 **Lemma 103.16.4.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *$QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$.*
(2) *$QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$.*

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 12.10.3.

Since 0 is a quasi-coherent module on any ringed site we see that (1) holds.

By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $Mod(\mathcal{O})$, so (2) holds.

Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{lisse,étale}$ or $\mathcal{X}_{flat,fppf}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for $\mathcal{G}$ and φ, see Lemma 103.14.4. By Lemma 103.16.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. By Sheaves on Stacks, Lemma 96.15.1 we have that $\mathrm{Coker}(g_!\varphi)$ is a quasi-coherent module on $\mathcal{X}$ (and the cokernel in the category of quasi-coherent modules on $\mathcal{X}$). Since g^* is exact (see Lemma 103.14.2) $g^*\mathrm{Coker}(g_!\varphi) = \mathrm{Coker}(g^*g_!\varphi) = \mathrm{Coker}(\varphi)$ is quasi-coherent too (see Lemma 103.16.3). By Proposition 103.8.1 the kernel $\mathrm{Ker}(g_!\varphi)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Since g^* is exact, we have $g^*\mathrm{Ker}(g_!\varphi) = \mathrm{Ker}(g^*g_!\varphi) = \mathrm{Ker}(\varphi)$. Since g^* maps objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ to quasi-coherent modules by Lemma 103.16.3 we conclude that $\mathrm{Ker}(\varphi)$ is quasi-coherent as well. This proves (3).

Finally, suppose that

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is an extension of $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-modules (resp. $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-modules) with $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent. To prove (4) and finish the proof we have to show that $\mathcal{E}$ is quasi-coherent on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Let U be an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$; we think of U as a scheme smooth (resp. flat) over $\mathcal{X}$. We have to show that the restriction of $\mathcal{E}$ to $U_{lisse,étale}$ (resp. $= U_{flat,fppf}$) is quasi-coherent. Thus we may assume that $\mathcal{X} = U$ is a scheme. Because $\mathcal{G}$ is quasi-coherent on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$), we may assume, after replacing U by the members of an étale (resp. fppf) covering, that $\mathcal{G}$ has a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$) where $\mathcal{O}$ is the structure sheaf on the site. We may also assume U is affine. Since $\mathcal{F}$ is quasi-coherent, we have

$$H^1(U_{lisse,étale}, \mathcal{F}) = 0, \quad \text{resp.} \quad H^1(U_{flat,fppf}, \mathcal{F}) = 0$$

Namely, $\mathcal{F}$ is the pullback of a quasi-coherent module $\mathcal{F}'$ on the big site of U (by Lemma 103.16.3), cohomology of $\mathcal{F}$ and $\mathcal{F}'$ agree (by Lemma 103.14.3), and we know that the cohomology of $\mathcal{F}'$ on the big site of the affine scheme U is zero (to get this in the current situation you have to combine Descent, Propositions 35.8.9

and 35.9.3 with Cohomology of Schemes, Lemma 30.2.2). Thus we can lift the map $\bigoplus_{i\in I}\mathcal{O}\to\mathcal{G}$ to $\mathcal{E}$. A diagram chase shows that we obtain an exact sequence

$$\bigoplus\nolimits_{j\in J}\mathcal{O}\to\mathcal{F}\oplus\bigoplus\nolimits_{i\in I}\mathcal{O}\to\mathcal{E}\to 0$$

By (3) proved above, we conclude that $\mathcal{E}$ is quasi-coherent as desired. □

103.17. Coherent sheaves on locally Noetherian stacks

0GR4 This section is the analogue of Cohomology of Spaces, Section 69.12. We have defined the notion of a coherent module on any ringed topos in Modules on Sites, Section 18.23. However, for any algebraic stack $\mathcal{X}$ the category of coherent $\mathcal{O}_\mathcal{X}$-modules is zero, essentially because the site $\mathcal{X}$ contains too many non-Noetherian objects (even if $\mathcal{X}$ is itself locally Noetherian). Instead, we will define coherent modules using the following lemma.

0GR5 **Lemma 103.17.1.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module. The following are equivalent*

1. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_\mathcal{X}$-module,*
2. *$\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module of finite presentation,*
3. *$\mathcal{F}$ is quasi-coherent and for any morphism $f:U\to\mathcal{X}$ where U is a locally Noetherian algebraic space, the pullback $f^*\mathcal{F}|_{U_{étale}}$ is coherent, and*
4. *$\mathcal{F}$ is quasi-coherent and there exists an algebraic space U and a morphism $f:U\to\mathcal{X}$ which is locally of finite type, flat, and surjective, such that the pullback $f^*\mathcal{F}|_{U_{étale}}$ is coherent.*

Proof. Let $f:U\to\mathcal{X}$ be as in (4). Then U is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5) and we see that the statement of the lemma makes sense. Additionally, f is locally of finite presentation by Morphisms of Stacks, Lemma 101.27.5. Let x be an object of $\mathcal{X}$ lying over the scheme V. In order to prove (2) we have to show that, after replacing V by the members of an fppf covering of V, the restriction $x^*\mathcal{F}$ has a global finite presentation on $\mathcal{X}/x\cong(Sch/V)_{fppf}$. The projection $W=U\times_\mathcal{X}V\to V$ is locally of finite presentation, flat, and surjective. Hence we may replace V by the members of an étale covering of W by schemes and assume we have a morphism $h:V\to U$ with $f\circ h=x$. Since $\mathcal{F}$ is quasi-coherent, we see that the restriction $x^*\mathcal{F}$ is the pullback of $h^*_{small}(f^*\mathcal{F})|_{U_{étale}}$ by π_V, see Sheaves on Stacks, Lemma 96.14.2. Since $f^*\mathcal{F}|_{U_{étale}}$ locally in the étale topology has a finite presentation by assumption, we conclude (4) ⇒ (2).

Part (2) implies (1) for any ringed topos (immediate from the definition). The properties "finite type" and "quasi-coherent" are preserved under pullback by any morphism of ringed topoi, see Modules on Sites, Lemma 18.23.4. Hence (1) implies (3), see Cohomology of Spaces, Lemma 69.12.2. Finally, (3) trivially implies (4). □

0GR6 **Definition 103.17.2.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack. An $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ is called *coherent* if $\mathcal{F}$ satisfies one (and hence all) of the equivalent conditions of Lemma 103.17.1. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is denote $Coh(\mathcal{O}_\mathcal{X})$.

0GR7 **Lemma 103.17.3.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The module $\mathcal{O}_\mathcal{X}$ is coherent, any invertible $\mathcal{O}_\mathcal{X}$-module is coherent, and more generally any finite locally free $\mathcal{O}_\mathcal{X}$-module is coherent.*

Proof. Follows from the definition and Cohomology of Spaces, Lemma 69.12.2. □

0GR8 **Lemma 103.17.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally Noetherian algebraic stacks. Then f^* sends coherent modules on $\mathcal{Y}$ to coherent modules on $\mathcal{X}$.*

Proof. Immediate from the definition and the fact that pullback for any morphism of ringed topoi preserves finitely presented modules, see Modules on Sites, Lemma 18.23.4. □

0GR9 **Lemma 103.17.5.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is abelian. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of coherent $\mathcal{O}_\mathcal{X}$-modules, then*

(1) *the cokernel* $\mathrm{Coker}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module,*
(2) *the image* $\mathrm{Im}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module, and*
(3) *the kernel* $\mathrm{Ker}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ may not be coherent, but it is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and $Q(\mathrm{Ker}(\varphi))$ is coherent and is the kernel of φ in Coh$(\mathcal{O}_\mathcal{X})$.*

The inclusion functor Coh$(\mathcal{O}_\mathcal{X}) \to$ QCoh$(\mathcal{O}_\mathcal{X})$ is exact.

Proof. The rules given for taking kernels, images, and cokernels in Coh$(\mathcal{O}_\mathcal{X})$ agree with the prescription for quasi-coherent modules in Remark 103.10.5. Hence the lemma will follow if we can show that the quasi-coherent modules $\mathrm{Coker}(\varphi)$, $\mathrm{Im}(\varphi)$, and $Q(\mathrm{Ker}(\varphi))$ are coherent. By Lemma 103.17.1 it suffices to prove this after restricting to $U_{étale}$ for some surjective smooth morphism $f : U \to \mathcal{X}$. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ is exact. Hence $f^* \mathrm{Coker}(\varphi)$ and $f^* \mathrm{Im}(\varphi)$ are the cokernel and image of a map between coherent $\mathcal{O}_U$-modules hence coherent as desired. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ kills parasitic modules by Lemma 103.9.2. Hence $f^*Q(\mathrm{Ker}(\varphi))|_{U_{étale}} = f^* \mathrm{Ker}(\varphi)|_{U_{étale}}$ by part (2) of Lemma 103.10.2. Thus we conclude that $Q(\mathrm{Ker}(\varphi))$ is coherent in the same way. □

0GRA **Lemma 103.17.6.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in Mod$(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ coherent, then $\mathcal{F}_2$ is coherent.*

Proof. By Sheaves on Stacks, Lemma 96.15.1 part (7) we see that $\mathcal{F}_2$ is quasi-coherent. Then we can check that $\mathcal{F}_2$ is coherent by restricting to $U_{étale}$ for some $U \to \mathcal{X}$ surjective and smooth. This follows from Cohomology of Spaces, Lemma 69.12.3. Some details omitted. □

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. This does not hold for modules on a general ringed topos.

0GRB **Lemma 103.17.7.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then Coh$(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of QCoh$(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. We have*

(1) *if $\mathcal{F}$ is coherent and φ surjective, then $\mathcal{G}$ is coherent,*
(2) *if $\mathcal{F}$ is coherent, then* $\mathrm{Im}(\varphi)$ *is coherent, and*
(3) *if $\mathcal{G}$ coherent and* $\mathrm{Ker}(\varphi)$ *parasitic, then $\mathcal{F}$ is coherent.*

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Then the functor $f^* : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_U)$ is exact (Lemma 103.4.1) and moreover by definition Coh$(\mathcal{O}_\mathcal{X})$ is the full subcategory of QCoh$(\mathcal{O}_\mathcal{X})$ consisting of objects $\mathcal{F}$

such that $f^*\mathcal{F}$ is in $Coh(\mathcal{O}_U)$. The statement that $Coh(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $QCoh(\mathcal{O}_\mathcal{X})$ follows immediately from this and the corresponding fact for U, see Cohomology of Spaces, Lemmas 69.12.3 and 69.12.4. We omit the proof of (1), (2), and (3). Hint: compare with the proof of Lemma 103.17.5. □

Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be surjective, locally of finite presentation, and flat. Observe that U is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5). Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ the isomorphism constructed in Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3. As in Sheaves on Stacks, Section 96.15 we obtain equivalences

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Sheaves on Stacks, Proposition 96.14.3. Recall that in Groupoids in Spaces, Section 78.13 we have defined the full subcategory

$$Coh(U, R, s, t, c) \subset QCoh(U, R, s, t, c)$$

of *coherent modules* as those $(\mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a coherent $\mathcal{O}_U$-module.

0GRC **Lemma 103.17.8.** *In the situation discussed above, the equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$ sends coherent sheaves to coherent sheaves and vice versa, i.e., induces an equivalence $Coh(\mathcal{O}_\mathcal{X}) \cong Coh(U, R, s, t, c)$.*

Proof. This is immediate from the definition of coherent $\mathcal{O}_\mathcal{X}$-modules. For bookkeeping purposes: the material above uses Morphisms of Stacks, Lemma 101.17.5, Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3, Sheaves on Stacks, Section 96.15, Sheaves on Stacks, Proposition 96.14.3, and Groupoids in Spaces, Section 78.13. □

0GRD **Lemma 103.17.9.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent be $\mathcal{O}_\mathcal{X}$-modules. Then the internal hom $hom(\mathcal{F}, \mathcal{G})$ constructed in Lemma 103.10.8 is a coherent $\mathcal{O}_\mathcal{X}$-module.*

Proof. Let $U \to \mathcal{X}$ be a smooth surjective morphism from a scheme. By item (12) in Section 103.12 we see that the restriction of $hom(\mathcal{F}, \mathcal{G})$ to U is the Hom sheaf of the restrictions. Hence this lemma follows from the case of algebraic spaces, see Cohomology of Spaces, Lemma 69.12.5. □

103.18. Coherent sheaves on Noetherian stacks

0GRE This section is the analogue of Cohomology of Spaces, Section 69.13.

0GRF **Lemma 103.18.1.** *Let $\mathcal{X}$ be a Noetherian algebraic stack. Every quasi-coherent $\mathcal{O}_\mathcal{X}$-module is the filtered colimit of its coherent submodules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_\mathcal{X}$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_\mathcal{X}$-submodule which contains both of them, see Lemma 103.17.7. In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to \mathcal{X}$ a surjective smooth morphism (Properties of Stacks, Lemma 100.6.2). Set $R = U \times_\mathcal{X} U$ so that $\mathcal{X} = [U/R]$ as in Algebraic Stacks,

Lemma 94.16.2. By Lemma 103.17.8 we have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $Coh(\mathcal{O}_X) = Coh(U, R, s, t, c)$. In this way we reduce to the problem of proving the corresponding thing for $QCoh(U, R, s, t, c)$. This is Groupoids in Spaces, Lemma 78.13.4; we check its assumptions in the next paragraph.

We urge the reader to skip the rest of the proof. The affine scheme U is Noetherian; this follows from our definition of $\mathcal{X}$ being locally Noetherian, see Properties of Stacks, Definition 100.7.2 and Remark 100.7.3. The projection morphisms $s, t : R \to U$ are smooth (see reference given above) and quasi-separated and quasi-compact (Morphisms of Stacks, Lemma 101.7.8). In particular, R is a quasi-compact and quasi-separated algebraic space smooth over U and hence Noetherian (Morphisms of Spaces, Lemma 67.28.6). □

103.19. Other chapters

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CHAPTER 104

Derived Categories of Stacks

08MW

104.1. Introduction

08MX In this chapter we write about derived categories associated to algebraic stacks. This means in particular derived categories of quasi-coherent sheaves, i.e., we prove analogues of the results on schemes (see Derived Categories of Schemes, Section 36.1) and algebraic spaces (see Derived Categories of Spaces, Section 75.1). The results in this chapter are different from those in [**LMB00**] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapters "Sheaves on Algebraic Stacks" and "Cohomology of Algebraic Stacks" where the terminology we use here is introduced.

104.2. Conventions, notation, and abuse of language

08MY We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. We use notation as explained in Cohomology of Stacks, Section 103.3.

104.3. The lisse-étale and the flat-fppf sites

08MZ The section is the analogue of Cohomology of Stacks, Section 103.14 for derived categories.

07AS **Lemma 104.3.1.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemmas 103.14.2 and 103.14.4.*

(1) *The functor $g_! : Ab(\mathcal{X}_{lisse,étale}) \to Ab(\mathcal{X}_{étale})$ has a left derived functor*
$$Lg_! : D(\mathcal{X}_{lisse,étale}) \longrightarrow D(\mathcal{X}_{étale})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(2) *The functor $g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow D(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^* and such that $g^*Lg_! = id$.

(3) *The functor $g_! : Ab(\mathcal{X}_{flat,fppf}) \to Ab(\mathcal{X}_{fppf})$ has a left derived functor*
$$Lg_! : D(\mathcal{X}_{flat,fppf}) \longrightarrow D(\mathcal{X}_{fppf})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(4) *The functor $g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow D(\mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^* and such that $g^*Lg_! = id$.

Warning: It is not clear (a priori) that $Lg_!$ on modules agrees with $Lg_!$ on abelian sheaves, see Cohomology on Sites, Remark 21.37.3.

Proof. The existence of the functor $Lg_!$ and adjointness to g^* is Cohomology on Sites, Lemma 21.37.2. (For the case of abelian sheaves use the constant sheaf $\mathbf{Z}$ as the structure sheaves.) Moreover, it is computed on a complex $\mathcal{H}^\bullet$ by taking a suitable left resolution $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ and applying the functor $g_!$ to $\mathcal{K}^\bullet$. Since $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$ by Cohomology of Stacks, Lemmas 103.14.4 and 103.14.2 we see that the final assertion holds in each case. □

07AV **Lemma 104.3.2.** *With assumptions and notation as in Cohomology of Stacks, Lemma 103.15.1. We have*

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \text{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

Proof. Let $\tau = étale$ (resp. $\tau = fppf$). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}_\tau$. By Cohomology of Stacks, Lemma 103.15.3 the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that $g^{-1}Rf_*\mathcal{F} = Rf'_*(g')^{-1}\mathcal{F}$ when $\mathcal{F}$ is a sheaf of modules on $\mathcal{X}_\tau$.

Next we show that for $\mathcal{G}$ (either sheaf of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G} \to f^{-1}Lg_!\mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf $\mathcal{I}$ on $\mathcal{X}_\tau$ the induced map

$$\mathrm{Hom}(L(g')_!(f')^{-1}\mathcal{G}, \mathcal{I}[n]) \leftarrow \mathrm{Hom}(f^{-1}Lg_!\mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all $n \in \mathbf{Z}$. (Hom's taken in suitable derived categories.) By the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions this follows from the isomorphism $g^{-1}Rf_*\mathcal{I} \to Rf'_*(g')^{-1}\mathcal{I}$ proved above.

In the case of a bounded complex $\mathcal{G}^\bullet$ (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) the canonical map

07AX (104.3.2.1) $$L(g')_!(f')^{-1}\mathcal{G}^\bullet \to f^{-1}Lg_!\mathcal{G}^\bullet$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors $L(g')_!(f')^{-1}$ and $f^{-1}Lg_!$ are exact functors of triangulated categories.

Suppose that $\mathcal{G}^\bullet$ is a bounded above complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$). The canonical map (104.3.2.1) is an isomorphism because we can use the stupid truncations $\sigma_{\geq -n}$ (see Homology, Section 12.15) to write $\mathcal{G}^\bullet$ as a colimit $\mathcal{G}^\bullet = \operatorname{colim} \mathcal{G}^\bullet_n$ of bounded complexes. This gives a distinguished triangle

$$\bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \mathcal{G}^\bullet \to \ldots$$

and each of the functors $L(g')_!$, $(f')^{-1}$, f^{-1}, $Lg_!$ commutes with direct sums (of complexes).

If $\mathcal{G}^\bullet$ is an arbitrary complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) then we use the canonical truncations $\tau_{\leq n}$ (see Homology, Section 12.15) to write $\mathcal{G}^\bullet$ as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions we conclude that the first identity of the lemma follows from the second in full generality. □

07B3 **Lemma 104.3.3.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemma 103.14.2.*

(1) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*
(2) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 96.9.2 and the discussion following.) After refining the covering we may assume each U_i is an affine scheme. Since each f_i is smooth (resp. flat) by Lemma 104.3.2 we see that $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f_i')^{-1}\mathcal{H}$. Using Cohomology of Stacks, Lemma 103.8.2 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation and where $\mathcal{X} = (Sch/X)_{fppf}$ for some affine scheme $X = \mathrm{Spec}(A)$.

Say our presentation looks like

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Note that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object, namely X/X which is quasi-compact (see Cohomology on Sites, Section 21.16). Hence we have

$$\Gamma(\bigoplus\nolimits_{i\in I} \mathcal{O}) = \bigoplus\nolimits_{i\in I} A$$

by Sites, Lemma 7.17.7. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus\nolimits_{j\in J} A \longrightarrow \bigoplus\nolimits_{i\in I} A \longrightarrow M \longrightarrow 0$$

of an A-module M. Moreover, $\mathcal{H}$ is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf M^a associated to M. Choose a resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

by free A-modules. The complex

$$\ldots \mathcal{O} \otimes_A F_2 \to \mathcal{O} \otimes_A F_1 \to \mathcal{O} \otimes_A F_0 \to \mathcal{H} \to 0$$

is a resolution of $\mathcal{H}$ by free $\mathcal{O}$-modules because for each object U/X of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) the structure morphism $U \to X$ is flat. Hence by construction the value of $Lg_!\mathcal{H}$ is

$$\ldots \to \mathcal{O}_\mathcal{X} \otimes_A F_2 \to \mathcal{O}_\mathcal{X} \otimes_A F_1 \to \mathcal{O}_\mathcal{X} \otimes_A F_0 \to 0 \to \ldots$$

Since this is a complex of quasi-coherent modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) it follows from Cohomology of Stacks, Proposition 103.8.1 that $H^p(Lg_!\mathcal{H})$ is quasi-coherent. □

104.4. Cohomology and the lisse-étale and flat-fppf sites

0H0Y We have already seen that cohomology of a sheaf on an algebraic stack $\mathcal{X}$ can be computed on flat-fppf site. In this section we prove the same is true for (possibly) unbounded objects of the direct category of $\mathcal{X}$.

0H0Z **Lemma 104.4.1.** *Let $\mathcal{X}$ be an algebraic stack. We have $Lg_!\mathbf{Z} = \mathbf{Z}$ for either $Lg_!$ as in Lemma 104.3.1 part (1) or $Lg_!$ as in Lemma 104.3.1 part (3).*

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. We have to show that $H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the canonical map $H^0(Lg_!\mathbf{Z}) \to \mathbf{Z}$ is an isomorphism. Let $f : \mathcal{U} \to \mathcal{X}$ be a surjective, flat morphism where $\mathcal{U}$ is a scheme such that f is also locally of finite presentation. (For example, pick a presentation $U \to \mathcal{X}$ and let $\mathcal{U}$ be the algebraic stack corresponding to U.) By Sheaves on Stacks, Lemmas 96.19.6 and 96.19.10 it suffices to show that the pullback $f^{-1}H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the pullback $H^0(Lg_!\mathbf{Z}) \to f^{-1}\mathbf{Z}$ is an isomorphism. By Lemma 104.3.2 we find $f^{-1}Lg_!\mathbf{Z} = L(g')_!\mathbf{Z}$ where $g' : Sh(\mathcal{U}_{flat,fppf}) \to Sh(\mathcal{U}_{fppf})$ is the corresponding comparison morphism for $\mathcal{U}$. This reduces us to the case studied in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X. In this case the category $\mathcal{X}_{flat,fppf}$ has a final object e, namely X/X, and moreover the functor $u : \mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ sends e to the final object. Since $\mathbf{Z}$ is the free abelian sheaf on the final object (provided the final object exists) we find that $Lg_!\mathbf{Z} = \mathbf{Z}$ by the very construction of $Lg_!$ in Cohomology on Sites, Lemma 21.37.2. □

0H10 **Lemma 104.4.2.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 104.3.1.*

(1) *For K in $D(\mathcal{X}_{étale})$ we have*
   (a) *$R\Gamma(\mathcal{X}_{étale}, K) = R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K)$, and*
   (b) *$R\Gamma(x, K) = R\Gamma(\mathcal{X}_{lisse,étale}/x, g^{-1}K)$ for any object x of $\mathcal{X}_{lisse,étale}$.*
(2) *For K in $D(\mathcal{X}_{fppf})$ we have*
   (a) *$R\Gamma(\mathcal{X}_{fppf}, K) = R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K)$, and*
   (b) *$H^p(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$ for any object x of $\mathcal{X}_{flat,fppf}$.*

In both cases, the same holds for modules, since we have $g^{-1} = g^*$ and there is no difference in computing cohomology by Cohomology on Sites, Lemma 21.20.7.

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. By Lemma 104.4.1 we have

$Lg_!\mathbf{Z} = \mathbf{Z}$. Then we obtain

$$\begin{aligned} R\Gamma(\mathcal{X}_{fppf}, K) &= R\operatorname{Hom}(\mathbf{Z}, K) \\ &= R\operatorname{Hom}(Lg_!\mathbf{Z}, K) \\ &= R\operatorname{Hom}(\mathbf{Z}, g^{-1}K) \\ &= R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K) \end{aligned}$$

This proves (1)(a). Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. □

104.5. Derived categories of quasi-coherent modules

07B5 Let $\mathcal{X}$ be an algebraic stack. As the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ isn't exact, we cannot define $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as the full subcategory of $D(\mathcal{O}_\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. Instead we define the derived category of quasi-coherent modules as a quotient by analogy with Cohomology of Stacks, Remark 103.10.7.

Recall that $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$ denotes the full subcategory of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property, see Cohomology of Stacks, Section 103.8. We will abbreviate

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) = D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

From Derived Categories, Lemma 13.17.1 and Cohomology of Stacks, Proposition 103.8.1 part (2) we deduce that $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$.

Let $Parasitic(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$ denote the full subcategory of parasitic $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Section 103.9. Let us abbreviate

$$D_{Parasitic}(\mathcal{O}_\mathcal{X}) = D_{Parasitic(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$ since $Parasitic(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 103.9.2.

The intersection of the weak Serre subcategories $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ of $Mod(\mathcal{O}_\mathcal{X})$ is another one. Let us similarly abbreviate

$$\begin{aligned} D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) &= D_{Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X}) \\ &= D_{Parasitic}(\mathcal{O}_\mathcal{X}) \cap D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \end{aligned}$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$. Hence a fortiori it is a strictly full, saturated triangulated subcategory of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

07B6 **Definition 104.5.1.** Let $\mathcal{X}$ be an algebraic stack. With notation as above we define the *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves* as the Verdier quotient[1]

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) = D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) / D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$$

[1]This definition is different from the one in the literature, see [**Ols07b**, 6.3], but it agrees with that definition by Lemma 104.5.3.

The Verdier quotient is defined in Derived Categories, Section 13.6. A morphism $a : E \to E'$ of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ becomes an isomorphism in $D_{QCoh}(\mathcal{O}_\mathcal{X})$ if and only if the cone $C(a)$ has parasitic cohomology sheaves, see Derived Categories, Lemma 13.6.10.

Consider the functors

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{H^i} LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{X})$$

Note that Q annihilates the subcategory $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 103.10.2. By Derived Categories, Lemma 13.6.8 we obtain a cohomological functor

07B7 (104.5.1.1) $$H^i : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

Moreover, note that $E \in D_{QCoh}(\mathcal{O}_\mathcal{X})$ is zero if and only if $H^i(E) = 0$ for all $i \in \mathbf{Z}$ since the kernel of Q is exactly equal to $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 103.10.2.

Note that the categories $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ are also weak Serre subcategories of the abelian category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ of modules in the étale topology, see Cohomology of Stacks, Proposition 103.8.1 and Lemma 103.9.2. Hence the statement of the following lemma makes sense.

07B8 **Lemma 104.5.2.** *Let $\mathcal{X}$ be an algebraic stack. Abbreviate $\mathcal{P}_\mathcal{X} = Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. The comparison morphism $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ induces a commutative diagram*

$$\begin{array}{ccccc} D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \uparrow{\scriptstyle \epsilon^*} & & \uparrow{\scriptstyle \epsilon^*} & & \uparrow{\scriptstyle \epsilon^*} \\ D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) & \longrightarrow & D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \end{array}$$

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) / D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

Proof. Since ϵ^* is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 21.29.1 to the following situation: $\mathcal{C} = \mathcal{X}$, $\tau = fppf$, $\tau' = étale$, $\mathcal{O} = \mathcal{O}_\mathcal{X}$, $\mathcal{A} = LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, and $\mathcal{B}$ is the set of objects of $\mathcal{X}$ lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4). Conditions (1) and (2) are clear from the discussion above (explicitly this follows from Cohomology of Stacks, Proposition 103.8.1). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 35.12.4.

We omit the verification that the equivalence of categories $\epsilon^* : D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves. □

Let $\mathcal{X}$ be an algebraic stack. By Cohomology of Stacks, Lemma 103.16.4 the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ forms a weak Serre subcategory

of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ forms a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we can consider

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \subset D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

and similarly

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \subset D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

As above these are strictly full, saturated triangulated subcategories. It turns out that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ is equivalent to either of these.

07B9 **Lemma 104.5.3.** *Let $\mathcal{X}$ be an algebraic stack. Set $\mathcal{P}_\mathcal{X} = Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

(1) *Let $\mathcal{F}^\bullet$ be an object of $D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 103.14.2 for the lisse-étale site we have*
   (a) *$g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,*
   (b) *$g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$, and*
   (d) *the functors g^* and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^*}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

(2) *Let $\mathcal{F}^\bullet$ be an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 103.14.2 for the flat-fppf site we have*
   (a) *$g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$,*
   (b) *$g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$, and*
   (d) *the functors g^* and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^*}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

Proof. The functor $g^* = g^{-1}$ is exact, hence (1)(a), (2)(a), (1)(b), and (2)(b) follow from Cohomology of Stacks, Lemmas 103.16.3 and 103.14.6.

Proof of (1)(c) and (2)(c). The construction of $Lg_!$ in Lemma 104.3.1 (via Cohomology on Sites, Lemma 21.37.2 which in turn uses Derived Categories, Proposition 13.29.2) shows that $Lg_!$ on any object $\mathcal{H}^\bullet$ of $D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is computed as

$$Lg_!\mathcal{H}^\bullet = \operatorname{colim} g_!\mathcal{K}^\bullet_n = g_! \operatorname{colim} \mathcal{K}^\bullet_n$$

(termwise colimits) where the quasi-isomorphism $\operatorname{colim} \mathcal{K}^\bullet_n \to \mathcal{H}^\bullet$ induces quasi-isomorphisms $\mathcal{K}^\bullet_n \to \tau_{\leq n}\mathcal{H}^\bullet$. Since the inclusion functors

$$LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \quad \text{and} \quad LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$$

are compatible with filtered colimits we see that it suffices to prove (c) on bounded above complexes $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. In this case to show that $H^n(Lg_!\mathcal{H}^\bullet)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ we can argue by induction on the integer m such that $\mathcal{H}^i = 0$ for $i > m$. If $m < n$, then $H^n(Lg_!\mathcal{H}^\bullet) = 0$ and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \to \mathcal{H}^\bullet \to H^m(\mathcal{H}^\bullet)[-m] \to \ldots$$

(Derived Categories, Remark 13.12.4) and apply the functor $Lg_!$. Since $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of the module category it suffices to prove (c) for two out of three. We have the result for $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$ by induction and we have the result for $Lg_!H^m(\mathcal{H}^\bullet)[-m]$ by Lemma 104.3.3. Whence (c) holds.

Let us prove (2)(d). By (2)(a) and (2)(b) the functor $g^{-1} = g^*$ induces a functor

$$c : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

see Derived Categories, Lemma 13.6.8. Thus we have the following diagram of triangulated categories

$$\begin{array}{ccc} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \xrightarrow{\quad q \quad} & D_{QCoh}(\mathcal{O}_\mathcal{X}) \\ & {\scriptstyle g^{-1}} \searrow \quad \swarrow {\scriptstyle Lg_!} & \swarrow {\scriptstyle c} \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) & \end{array}$$

where q is the quotient functor, the inner triangle is commutative, and $g^{-1}Lg_! = \mathrm{id}$. For any object of E of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ the map $a : Lg_!g^{-1}E \to E$ maps to a quasi-isomorphism in $D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Hence the cone on a maps to zero under g^{-1} and by (2)(b) we see that $q(a)$ is an isomorphism. Thus $q \circ Lg_!$ is a quasi-inverse to c.

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is equivalent to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$. Applying the last equivalence of Lemma 104.5.2 finishes the proof. □

The following lemma tells us that the quotient functor $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ has a left adjoint. See Remark 104.5.5.

07BA **Lemma 104.5.4.** *Let $\mathcal{X}$ be an algebraic stack. Let E be an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. There exists a canonical distinguished triangle*

$$E' \to E \to P \to E'[1]$$

in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ such that P is in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(E', P') = 0$$

for all P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

Proof. Consider the morphism of ringed topoi $g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$ studied in Cohomology of Stacks, Section 103.14. Set $E' = Lg_!g^*E$ and let P be the cone on the adjunction map $E' \to E$, see Lemma 104.3.1 part (4). By Lemma 104.5.3 parts (2)(a) and (2)(c) we have that E' is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. Hence also P is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. The map $g^*E' \to g^*E$ is an isomorphism as $g^*Lg_! = \mathrm{id}$ by Lemma 104.3.1 part (4). Hence $g^*P = 0$ and whence P is an object of $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ by Lemma 104.5.3 part (2)(b). Finally, for P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ we have

$$\mathrm{Hom}(E', P') = \mathrm{Hom}(Lg_!g^*E, P') = \mathrm{Hom}(g^*E, g^*P') = 0$$

as $g^*P' = 0$ by Lemma 104.5.3 part (2)(b). The distinguished triangle $E' \to E \to P \to E'[1]$ is canonical (more precisely unique up to isomorphism of triangles induces the identity on E) by the discussion in Derived Categories, Section 13.40. □

0H11 **Remark 104.5.5.** The result of Lemma 104.5.4 tells us that

$$D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \subset D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$$

is a left admissible subcategory, see Derived Categories, Section 13.40. In particular, if $\mathcal{A} \subset D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ denotes its left orthogonal, then Derived Categories, Proposition 13.40.10 implies that $\mathcal{A}$ is right admissible in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and that the composition

$$\mathcal{A} \longrightarrow D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is an equivalence. This means that we can view $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as a strictly full saturated triangulated subcategory of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and also of $D(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

104.6. Derived pushforward of quasi-coherent modules

07BB As a first application of the material above we construct the derived pushforward. In Examples, Section 110.61 the reader can find an example of a quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that the direct image functor Rf_* does not induce a functor $D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$. Thus restricting to bounded below complexes is necessary.

07BC **Proposition 104.6.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor Rf_* induces a commutative diagram*

$$\begin{array}{ccccc} D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

and hence induces a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_\mathcal{Y})$$

on quotient categories. Moreover, the functor $R^i f_{QCoh}$ of Cohomology of Stacks, Proposition 103.11.1 are equal to $H^i \circ Rf_{QCoh,*}$ with H^i as in (104.5.1.1).

Proof. We have to show that Rf_*E is an object of $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$ for E in $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. This follows from Cohomology of Stacks, Proposition 103.8.1 and the spectral sequence $R^i f_* H^j(E) \Rightarrow R^{i+j} f_* E$. The case of parasitic modules works the same way using Cohomology of Stacks, Lemma 103.9.3. The final statement is clear from the definition of H^i in (104.5.1.1). □

104.7. Derived pullback of quasi-coherent modules

07BD Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

07BE **Proposition 104.7.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The exact functor f^* induces a commutative diagram*

$$\begin{array}{ccc} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

The composition

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \xrightarrow{f^*} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{q_\mathcal{X}} D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is left derivable with respect to the localization $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and we may define Lf^*_{QCoh} as its left derived functor

$$Lf^*_{QCoh} : D_{QCoh}(\mathcal{O}_\mathcal{Y}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

(see Derived Categories, Definitions 13.14.2 and 13.14.9). If f is quasi-compact and quasi-separated, then Lf^*_{QCoh} and $Rf_{QCoh,*}$ satisfy the following adjointness:

$$\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) = \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)$$

for $A \in D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and $B \in D^+_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. To prove the first statement, we have to show that f^*E is an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ for E in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Since $f^* = f^{-1}$ is exact this follows immediately from the fact that f^* maps $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ into $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Proposition 103.8.1.

Set $\mathcal{D} = D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Let S be the collection of morphisms in $\mathcal{D}$ whose cone is an object of $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Set $\mathcal{D}' = D_{QCoh}(\mathcal{O}_\mathcal{X})$. Set $F = q_\mathcal{X} \circ f^* : \mathcal{D} \to \mathcal{D}'$. Then $\mathcal{D}, S, \mathcal{D}', F$ are as in Derived Categories, Situation 13.14.1 and Definition 13.14.2. Let us prove that $LF(E)$ is defined for any object E of $\mathcal{D}$. Namely, consider the triangle

$$E' \to E \to P \to E'[1]$$

constructed in Lemma 104.5.4. Note that $s : E' \to E$ is an element of S. We claim that E' computes LF. Namely, suppose that $s' : E'' \to E$ is another element of S, i.e., fits into a triangle $E'' \to E \to P' \to E''[1]$ with P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. By Lemma 104.5.4 (and its proof) we see that $E' \to E$ factors through $E'' \to E$. Thus we see that $E' \to E$ is cofinal in the system S/E. Hence it is clear that E' computes LF.

To see the final statement, write $B = q_\mathcal{X}(H)$ and $A = q_\mathcal{Y}(E)$. Choose $E' \to E$ as above. We will use on the one hand that $Rf_{QCoh,*}(B) = q_\mathcal{Y}(Rf_*H)$ and on the other that $Lf^*_{QCoh}(A) = q_\mathcal{X}(f^*E')$.

$$\begin{aligned}
\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(q_\mathcal{X}(f^*E'), q_\mathcal{X}(H)) \\
&= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(f^*E', H') \\
&= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H') \\
&= \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H) \\
&= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)
\end{aligned}$$

Here the colimit is over morphisms $s : H \to H'$ in $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ whose cone $P(s)$ is an object of $D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. The first equality we've seen above. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 21.19.1. Since $Rf_*P(s)$ is an object of $D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$ by Proposition 104.6.1 we see that $\mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*P(s)) = 0$. Thus the fourth equality holds. The final equality holds by construction of E'. $\square$

104.8. Quasi-coherent objects in the derived category

0H12 This section is the continuation of Sheaves on Stacks, Section 96.26. Let $\mathcal{X}$ be an algebraic stack. In that section we defined a triangulated category

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

and we proved that if $\mathcal{X}$ is representable by an algebraic space X then $QC(\mathcal{X})$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. It turns out that we have developed just enough theory to prove the same thing is true for any algebraic stack.

0H13 **Lemma 104.8.1.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ whose cohomology sheaves are parasitic. Then $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 104.3. Let x be an object of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat, i.e., x is an object of $\mathcal{X}_{flat,fppf}$. By Lemma 104.4.2 part (2)(b) we have $R\Gamma(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$. However, our assumption means that the cohomology sheaves of the object $g^{-1}K$ of $D(\mathcal{X}_{flat,fppf})$ are zero, see Cohomology of Stacks, Definition 103.9.1. Hence $g^{-1}K = 0$ and we win. □

0H14 **Lemma 104.8.2.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ such that $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over an affine scheme U such that $U \to \mathcal{X}$ is flat. Then $H^i(\mathcal{X}, K) = 0$ for all i.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 104.3. By Lemma 104.4.2 part (2)(b) our assumption means that $g^{-1}K$ has vanishing cohomology over every object of $\mathcal{X}_{flat,fppf}$ which lies over an affine scheme. Since every object x of $\mathcal{X}_{flat,fppf}$ has a covering by such objects, we conclude that $g^{-1}K$ has vanishing cohomology sheaves, i.e., we conclude $g^{-1}K = 0$. Then of course $R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K) = 0$ which in turn implies what we want by Lemma 104.4.2 part (2)(a). □

0H15 **Lemma 104.8.3.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then $Lg_!K$ satisfies the following property: for any morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map*

$$R\Gamma(x', Lg_!K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, Lg_!K)$$

is a quasi-isomorphism.

Proof. By Lemma 104.5.3 part (2)(c) the object $Lg_!K$ is in $D_{LQCoh^{fbc}}(\mathcal{O}_{\mathcal{X}})$. It follows readily from this that the map displayed in the lemma is an isomorphism if $\mathcal{O}(x') \to \mathcal{O}(x)$ is a flat ring map; we omit the details.

In this paragraph we argue that the question is local for the étale topology. Let $x \to x'$ be a general morphism of $\mathcal{X}_{affine}$. Let $\{x'_i \to x'\}$ be a covering in $\mathcal{X}_{affine,étale}$. Set $x_i = x \times_{x'} x'_i$ so that $\{x_i \to x\}$ is a covering of $\mathcal{X}_{affine,étale}$ too. Then $\mathcal{O}(x') \to \prod \mathcal{O}(x'_i)$ is a faithfully flat étale ring map and

$$\prod \mathcal{O}(x_i) = \mathcal{O}(x) \otimes_{\mathcal{O}(x')} \left(\prod \mathcal{O}(x'_i) \right)$$

Thus a simple algebra argument we omit shows that it suffices to prove the result in the statement of the lemma holds for each of the morphisms $x_i \to x'_i$ in $\mathcal{X}_{affine}$. In other words, the problem is local in the étale topology.

Choose a scheme X and a surjective smooth morphism $f : X \to \mathcal{X}$. We may view f as an object of $\mathcal{X}$ (by our abuse of notation) and then $(Sch/X)_{fppf} = \mathcal{X}/f$, see Sheaves on Stacks, Section 96.9. By Sheaves on Stacks, Lemma 96.19.10 for example, there exist an étale covering $\{x'_i \to x'\}$ such that $x'_i : U'_i = p(x'_i) \to \mathcal{X}$ factors through f. By the result of the previous paragraph, we may assume that $x \to x'$ is a morphism which is the image of a morphism $U \to U'$ of $(Aff/X)_{fppf}$ by the functor $(Sch/X)_{fppf} \to \mathcal{X}$. At this point we see use that the restriction to $(Sch/X)_{fppf}$ of $Lg_!K$ is equal to $f^*Lg_!K = L(g')_!(f')^*K$ by Lemma 104.3.2. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. We may still work étale (or Zariski) locally on U' and hence we may assume $U' \to X$ factors through some affine open of X. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ where $X = \mathrm{Spec}(R)$ is an affine scheme and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. Let $M^\bullet$ be a complex of R-modules representing $R\Gamma(X, K)$. By the construction in More on Algebra, Lemma 15.59.10 we may assume $M^\bullet = \mathrm{colim}\, P_n^\bullet$ where each $P_n^\bullet$ is a bounded above complex of free R-modules. Details omitted; see also More on Algebra, Remark 15.59.11. Consider the complex of modules $M^\bullet_{flat,fppf}$ on $X_{flat,fppf} = (Sch/X)_{flat,fppf}$ given by the rule

$$U \longmapsto \Gamma(U, M^\bullet \otimes_R \mathcal{O}_U)$$

This is a complex of sheaves by the discussion in Descent, Section 35.8. There is a canonical map $M^\bullet_{flat,fppf} \to K$ which by our initial remarks of the proof produces an isomorphism on sections over the affine objects of $X_{flat,fppf}$. Since every object of $X_{flat,fppf}$ has a covering by affine objects we see that $M^\bullet_{flat,fppf}$ agrees with K.

Let $M^\bullet_{fppf}$ be the complex of modules on X_{fppf} given by the same formula as displayed above. Recall that $Lg_!\mathcal{O} = g_!\mathcal{O} = \mathcal{O}$. Since $Lg_!$ is the left derived functor of $g_!$ we conclude that $Lg_!P^\bullet_{n,flat,fppf} = P^\bullet_{n,fppf}$. Since the functor $Lg_!$ commutes with homotopy colimits (or by its construction in Cohomology on Sites, Lemma 21.37.2) and since $M^\bullet = \mathrm{colim}\, P_n^\bullet$ we conclude that $Lg_!M^\bullet_{flat,fppf} = M^\bullet_{fppf}$. Say $U = \mathrm{Spec}(A)$, $U' = \mathrm{Spec}(A')$ and $U \to U'$ corresponds to the ring map $A' \to A$. From the above we see that

$$R\Gamma(U, Lg_!K) = M^\bullet \otimes_R A \quad \text{and} \quad R\Gamma(U', Lg_!K) = M^\bullet \otimes_R A'$$

Since $M^\bullet$ is a K-flat complex of R-modules, by transitivity of tensor product it follows that

$$R\Gamma(U', Lg_!K) \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(U, Lg_!K)$$

is a quasi-isomorphism as desired. □

0H16 **Proposition 104.8.4.** *Let $\mathcal{X}$ be an algebraic stack. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_\mathcal{X})$.*

Proof. By Sheaves on Stacks, Lemma 96.26.6 pullback by the comparison morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ identifies $QC(\mathcal{X})$ with a full subcategory $Q_\mathcal{X} \subset D(\mathcal{X}_{affine,fppf}, \mathcal{O})$. Using the equivalence of ringed topoi in Sheaves on Stacks, Equation (96.24.3.1) we may and do view $Q_\mathcal{X}$ as a full subcategory of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Similarly by Lemma 104.5.4 and Remark 104.5.5 we find that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ may be viewed as the left orthogonal $\mathcal{A}$ of the left admissible subcategory $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

To finish we will show that $Q_\mathcal{X}$ is equal to $\mathcal{A}$ as subcategories of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Step 1: $Q_\mathcal{X}$ is contained in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. An object K of $Q_\mathcal{X}$ is characterized by the property that K, viewed as an object of $D(\mathcal{X}_{affine,fppf}, \mathcal{O})$ satisfies $R\epsilon_* K$ is an object of $QC(\mathcal{X}_{affine}, \mathcal{O})$. This in turn means exactly that for all morphisms $x \to x'$ of $\mathcal{X}_{affine}$ the map

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is an isomorphism, see footnote in statement of Cohomology on Sites, Lemma 21.43.12. Now, if $x' \to x$ lies over a flat morphism of affine schemes, then this means that

$$H^i(x', K) \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \cong H^i(x, K)$$

This clearly means that $H^i(K)$ is a sheaf for the étale topology (Sheaves on Stacks, Lemma 96.25.1) and that it has the flat base change property (small detail omitted).

Step 2: $Q_\mathcal{X}$ is contained in $\mathcal{A}$. To see this it suffices to show that for K in $Q_\mathcal{X}$ we have $\mathrm{Hom}(K, P) = 0$ for all P in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. Consider the object

$$H = R\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(K, P)$$

Let x be an object of $\mathcal{X}$ which lies over an affine scheme $U = p(x)$. By Cohomology on Sites, Lemma 21.35.1 we have the first equality in

$$R\Gamma(x, H) = R\,\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(K|_{\mathcal{X}/x}, P|_{\mathcal{X}/x}) = R\,\mathrm{Hom}_{\mathcal{O}}(K|_{\mathcal{X}_{affine}/x}, P|_{\mathcal{X}_{affine}/x})$$

The second equality stems from the fact that the topos of the site $\mathcal{X}/x$ is equivalent to the topos of the site $\mathcal{X}_{affine}/x$, see Sheaves on Stacks, Equation (96.24.3.1). We may write $K = \epsilon^* N$ for some N in $QC(\mathcal{O})$. Then by Cohomology on Sites, Lemma 21.43.13 we see that

$$R\Gamma(x, H) = R\,\mathrm{Hom}_{D(\mathcal{O}(x))}(R\Gamma(x, N), R\Gamma(x, P))$$

By Lemma 104.8.1 we see that $R\Gamma(x, P) = 0$ if $U \to \mathcal{X}$ is flat and hence $R\Gamma(x, H) = 0$ under the same hypothesis. By Lemma 104.8.2 we conclude that $R\Gamma(\mathcal{X}, H) = 0$ and therefore $\mathrm{Hom}(K, P) = 0$.

Step 3: $\mathcal{A}$ is contained in $Q_\mathcal{X}$. Let K be an object of $\mathcal{A}$ and let $x \to x'$ be a morphism of $\mathcal{X}_{affine}$. We have to show that

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is a quasi-isomorphism, see footnote in statement of Cohomology on Sites, Lemma 21.43.12. By the proof of Lemma 104.5.4 and the discussion in Remark 104.5.5 we see that $\mathcal{A}$ is the image of the restriction of $Lg_!$ to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we may assume $K = Lg_! M$ for some M in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then the desired equality follow from Lemma 104.8.3. □

104.9. Other chapters

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CHAPTER 105

Introducing Algebraic Stacks

072H

105.1. Why read this?

072I We give an informal introduction to algebraic stacks. The goal is to quickly introduce a simple language which you can use to think about local and global properties of your favorite moduli problem. Having done this it should be possible to ask yourself well-posed questions about moduli problems and to start solving them, whilst assuming a general theory exists. If you end up with an interesting result, you can go back to the general theory in the other parts of the stacks project and fill in the gaps as needed.

The point of view we take here is close to the point of view taken in [**KM85**] and [**Mum65**].

105.2. Preliminary

072J Let S be a scheme. An *elliptic curve* over S is a triple $(E, f, 0)$ where E is a scheme and $f : E \to S$ and $0 : S \to E$ are morphisms of schemes such that

(1) $f : E \to S$ is proper, smooth of relative dimension 1,
(2) for every $s \in S$ the fibre E_s is a connected curve of genus 1, i.e., $H^0(E_s, \mathcal{O})$ and $H^1(E_s, \mathcal{O})$ both are 1-dimensional $\kappa(s)$-vector spaces, and
(3) 0 is a section of f.

Given elliptic curves $(E, f, 0)/S$ and $(E', f', 0')/S'$ a *morphism of elliptic curves over* $a : S \to S'$ is a morphism $\alpha : E \to E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ {\scriptstyle 0}\uparrow \downarrow{\scriptstyle f} & & {\scriptstyle f'}\downarrow \uparrow{\scriptstyle 0'} \\ S & \xrightarrow{a} & S' \end{array}$$

is commutative and the inner square is cartesian, in other words the morphism α induces an isomorphism $E \to S \times_{S'} E'$. We are going to define the stack of elliptic curves $\mathcal{M}_{1,1}$. In the rest of the Stacks project we work out the method introduced in Deligne and Mumford's paper [**DM69**] which consists in presenting $\mathcal{M}_{1,1}$ as a category endowed with a functor

$$p : \mathcal{M}_{1,1} \longrightarrow Sch, \quad (E, f, 0)/S \longmapsto S$$

This means you work with fibred categories over the categories of schemes, topologies, stacks fibred in groupoids, coverings, etc, etc. In this chapter we throw all of that out of the window and we think about it a bit differently – probably closer to how the initiators of the theory started thinking about it themselves.

105.3. The moduli stack of elliptic curves

072K Here is what we are going to do:

(1) Start with your favorite category of schemes Sch.
(2) Add a new symbol $\mathcal{M}_{1,1}$.
(3) A morphism $S \to \mathcal{M}_{1,1}$ **is** an elliptic curve $(E, f, 0)$ over S.
(4) A diagram

$$\begin{array}{ccc} S & \xrightarrow{a} & S' \\ & {\scriptstyle (E,f,0)} \searrow \quad \swarrow {\scriptstyle (E',F',0')} & \\ & \mathcal{M}_{1,1} & \end{array}$$

is commutative if and only if there exists a morphism $\alpha : E \to E'$ of elliptic curves over $a : S \to S'$. We say α *witnesses* the commutativity of the diagram.
(5) Note that commutative diagrams glue as follows

$$\begin{array}{ccccc} S & \xrightarrow{a} & S' & \xrightarrow{a'} & S'' \\ & {\scriptstyle (E,f,0)} \searrow & \downarrow {\scriptstyle (E',F',0')} & \swarrow {\scriptstyle (E'',F'',0'')} & \\ & & \mathcal{M}_{1,1} & & \end{array}$$

because $\alpha' \circ \alpha$ witnesses the commutativity of the outer triangle if α and α' witness the commutativity of the left and right triangles.
(6) The composition

$$S \xrightarrow{a} S' \xrightarrow{(E',f',0')} \mathcal{M}_{1,1}$$

is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$.

At the end of this procedure we have enlarged the category Sch of schemes with exactly one object...

Except that we haven't defined what a morphism from $\mathcal{M}_{1,1}$ to a scheme T is. The answer is that it is the weakest possible notion such that compositions make sense. Thus a morphism $F : \mathcal{M}_{1,1} \to T$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates a morphism $F(E, f, 0) : S \to T$ such that given any commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{a} & S' \\ & {\scriptstyle (E,f,0)} \searrow \quad \swarrow {\scriptstyle (E',F',0')} & \\ & \mathcal{M}_{1,1} & \end{array}$$

the diagram

$$\begin{array}{ccc} S & \xrightarrow{a} & S' \\ & {\scriptstyle F(E,f,0)} \searrow \quad \swarrow {\scriptstyle F(E',F',0')} & \\ & T & \end{array}$$

is commutative also. An example is the j-invariant

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$$

which you may have heard of. Aha, so now we're done...

Except, no we're not! We still have to define a notion of morphisms $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$. This we do in exactly the same way as before, i.e., a morphism $F : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates another elliptic curve $F(E, f, 0)$ preserving commutativity of diagrams as above. However, since I don't know of a nontrivial example of such a functor, I'll just define the set of morphisms from $\mathcal{M}_{1,1}$ to itself to consist of the identity for now.

I hope you see how to add other objects to this enlarged category. Somehow it seems intuitively clear that given any "well-behaved" moduli problem we can perform the construction above and add an object to our category. In fact, much of modern day algebraic geometry takes place in such a universe where *Sch* is enlarged with countably many (explicitly constructed) moduli stacks.

You may object that the category we obtain isn't a category because there is a "vagueness" about when diagrams commute and which combinations of diagrams continue to commute as we have to produce a witness to the commutativity. However, it turns out that this, the idea of having witnesses to commutativity, is a valid approach to 2-categories! Thus we stick with it.

105.4. Fibre products

072L The question we pose here is what should be the fibre product

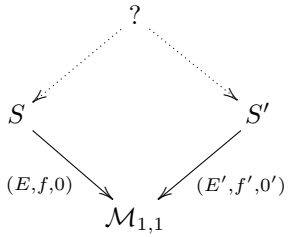

The answer: A morphism from a scheme T into ? should be a triple (a, a', α) where $a : T \to S$, $a' : T \to S'$ are morphisms of schemes and where $\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$ is an isomorphism of elliptic curves over T. This makes sense because of our definition of composition and commutative diagrams earlier in the discussion.

072M **Lemma 105.4.1** (Key fact)**.** *The functor $Sch^{opp} \to Sets$, $T \mapsto \{(a, a', \alpha)$ as above$\}$ is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$.*

Proof. Idea of proof. Relate this functor to

$$Isom_{S \times S'}(E \times S', S \times E')$$

and use Grothendieck's theory of Hilbert schemes. □

072N **Remark 105.4.2.** We have the formula $S \times_{\mathcal{M}_{1,1}} S' = (S \times S') \times_{\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}} \mathcal{M}_{1,1}$. Hence the key fact is a property of the diagonal $\Delta_{\mathcal{M}_{1,1}}$ of $\mathcal{M}_{1,1}$.

In any case the key fact allows us to make the following definition.

072P **Definition 105.4.3.** We say a morphism $S \to \mathcal{M}_{1,1}$ is *smooth* if for every morphism $S' \to \mathcal{M}_{1,1}$ the projection morphism

$$S \times_{\mathcal{M}_{1,1}} S' \longrightarrow S'$$

is smooth.

Note that this is compatible with the notion of a smooth morphism of schemes as the base change of a smooth morphism is smooth. Moreover, it is clear how to extend this definition to other properties of morphisms into $\mathcal{M}_{1,1}$ (or your own favorite moduli stack). In particular we will use it below for *surjective* morphisms.

105.5. The definition

072Q We'll formulate it as a definition and not as a result since we expect the reader to try out other cases (not just the stack $\mathcal{M}_{1,1}$ and not just *Sch* the category of all schemes).

072R **Definition 105.5.1.** We say $\mathcal{M}_{1,1}$ is an *algebraic stack* if and only if

(1) We have descent for objects for the étale topology on *Sch*.
(2) The key fact holds.
(3) there exists a surjective and smooth morphism $S \to \mathcal{M}_{1,1}$.

The first condition is a "sheaf property". We're going to spell it out since there is a technical point we should make. Suppose given a scheme S and an étale covering $\{S_i \to S\}$ and morphisms $e_i : S_i \to \mathcal{M}_{1,1}$ such that the diagrams

$$\begin{array}{ccc} S_i \times_S S_j & \xrightarrow[\text{id}]{} & S_i \times_S S_j \\ & {\scriptstyle e_i \circ \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle e_j \circ \mathrm{pr}_2} & \\ & \mathcal{M}_{1,1} & \end{array}$$

commute. The sheaf condition does *not* guarantee the existence of a morphism $e : S \to \mathcal{M}_{1,1}$ in this situation. Namely, we need to pick witnesses α_{ij} for the diagrams above and require that

$$\mathrm{pr}_{02}^* \alpha_{ik} = \mathrm{pr}_{12}^* \alpha_{jk} \circ \mathrm{pr}_{01}^* \alpha_{ij}$$

as witnesses over $S_i \times_S S_j \times_S S_k$. I think it is clear what this means... If not, then I'm afraid you'll have to read some of the material on categories fibred in groupoids, etc. In any case, the displayed equation is often called the *cocycle condition*. A more precise statement of the "sheaf property" is: given $\{S_i \to S\}$, $e_i : S_i \to \mathcal{M}_{1,1}$ and witnesses α_{ij} satisfying the cocycle condition, there exists a unique (up to unique isomorphism) $e : S \to \mathcal{M}_{1,1}$ with $e_i \cong e|_{S_i}$ recovering the α_{ij}.

As you can see even formulating a precise statement takes a bit of work. The proof of this "sheaf property" relies on a fundamental technique in algebraic geometry, namely descent theory. My suggestion is to initially simply accept the "sheaf property" holds, and see what it implies in practice. In fact, a certain amount of mental agility is required to boil the "sheaf property" down to a manageable statement that you can fit on a napkin. Perhaps the simplest variant which is already a bit interesting is the following: Suppose we have a finite Galois extension L/K of fields with Galois group $G = \mathrm{Gal}(L/K)$. Set $T = \mathrm{Spec}(L)$ and $S = \mathrm{Spec}(K)$. Then $\{T \to S\}$ is an étale covering. Let $(E, f, 0)$ be an elliptic curve over L. (Yes, this just means that $E \subset \mathbf{P}^2_L$ is given by a Weierstrass equation and 0 is the usual point at infinity.) Denote $E_\sigma = E \times_{T,\mathrm{Spec}(\sigma)} T$ the base change. (Yes, this corresponds to applying σ to the coefficients of the Weierstrass equation, or is it σ^{-1}?) Now, suppose moreover that for every $\sigma \in G$ we are given an isomorphism

$$\alpha_\sigma : E \longrightarrow E_\sigma$$

over T. The cocycle condition above means in this situation that

$$(\alpha_\tau)^\sigma \circ \alpha_\sigma = \alpha_{\tau\sigma}$$

for $\sigma, \tau \in G$. If you've ever done any group cohomology then this should be familiar. Anyway, the "glueing" condition on $\mathcal{M}_{1,1}$ says that if you have a solution to this set of equations, then there exists an elliptic curve E' over S such that $E \cong E' \times_S T$ (it says a little bit more because it also tells you how to recover the α_σ).

Challenge: Can you prove this entirely using only elliptic curves defined in terms of Weierstrass equations?

105.6. A smooth cover

072S The last thing we have to do is find a smooth cover of $\mathcal{M}_{1,1}$. In fact, in some sense the existence of a smooth cover *implies*[1] the key fact! In the case of elliptic curves we use the Weierstrass equation to construct one.

Set

$$W = \mathrm{Spec}(\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$$

where $\Delta \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ is a certain polynomial (see below). Set

$$\mathbf{P}^2_W \supset E_W : zy^2 + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

Denote $f_W : E_W \to W$ the projection. Finally, denote $0_W : W \to E_W$ the section of f_W given by $(0 : 1 : 0)$. It turns out that there is a degree 12 homogeneous polynomial Δ in a_i where $\deg(a_i) = i$ such that $E_W \to W$ is smooth. You can find it explicitly by computing partials of the Weierstrass equation – of course you can also look it up. You can also use pari/gp to compute it for you. Here it is

$$\begin{aligned}\Delta = &-a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 + \\ &(8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 + \\ &((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 + \\ &(16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 + \\ &(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 + \\ &- 27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2\end{aligned}$$

You may recognize the last two terms from the case $y^2 = x^3 + Ax + B$ having discriminant $-64A^3 - 432B^2 = -16(4A^3 + 27B^2)$.

072T **Lemma 105.6.1.** *The morphism $W \xrightarrow{(E_W, f_W, 0_W)} \mathcal{M}_{1,1}$ is smooth and surjective.*

Proof. Surjectivity follows from the fact that every elliptic curve over a field has a Weierstrass equation. We give a rough sketch of one way to prove smoothness. Consider the sub group scheme

$$H = \left\{ \left(\begin{matrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{matrix} \right) \middle| \begin{matrix} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{matrix} \right\} \subset \mathrm{GL}_{3,\mathbf{Z}}$$

[1] This is a bit of a cheat because in checking the smoothness you have to prove something close to the key fact – after all smoothness is defined in terms of fibre products. The advantage is that you only have to prove the existence of these fibre products in the case that on one side you have the morphism that you are trying to show provides the smooth cover.

There is an action $H \times W \to W$ of H on the Weierstrass scheme W. To find the equations for this action write out what a coordinate change given by a matrix in H does to the general Weierstrass equation. Then it turns out the following statements hold

(1) any elliptic curve $(E, f, 0)/S$ has Zariski locally on S a Weierstrass equation,
(2) any two Weierstrass equations for $(E, f, 0)$ differ (Zariski locally) by an element of H.

Considering the fibre product $S \times_{\mathcal{M}_{1,1}} W = Isom_{S \times W}(E \times W, S \times E_W)$ we conclude that this means that the morphism $W \to \mathcal{M}_{1,1}$ is an H-torsor. Since $H \to \mathrm{Spec}(\mathbf{Z})$ is smooth, and since torsors over smooth group schemes are smooth we win. □

072U **Remark 105.6.2.** The argument sketched above actually shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack. It is true about 50% of the time that an argument proving a moduli stack is algebraic will show that it is a global quotient stack.

105.7. Properties of algebraic stacks

072V Ok, so now we know that $\mathcal{M}_{1,1}$ is an algebraic stack. What can we do with this? Well, it isn't so much the fact that it is an algebraic stack that helps us here, but more the point of view that properties of $\mathcal{M}_{1,1}$ should be encoded in the properties of morphisms $S \to \mathcal{M}_{1,1}$, i.e., in families of elliptic curves. We list some examples

Local properties:

$$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth} \Leftrightarrow W \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth}$$

Idea. Local properties of an algebraic stack are encoded in the local properties of its smooth cover.

Global properties:

$$\mathcal{M}_{1,1} \text{ is quasi-compact} \Leftarrow W \text{ is quasi-compact}$$
$$\mathcal{M}_{1,1} \text{ is irreducible} \Leftarrow W \text{ is irreducible}$$

Idea. Some global properties of an algebraic stack can be read off from the corresponding property of a *suitable*[2] smooth cover.

Quasi-coherent sheaves:

$$QCoh(\mathcal{O}_{\mathcal{M}_{1,1}}) = H\text{-equivariant quasi-coherent modules on } W$$

Idea. On the one hand a quasi-coherent module on $\mathcal{M}_{1,1}$ should correspond to a quasi-coherent sheaf $\mathcal{F}_{S,e}$ on S for each morphism $e : S \to \mathcal{M}_{1,1}$. In particular for the morphism $(E_W, f_W, 0_W) : W \to \mathcal{M}_{1,1}$. Since this morphism is H-equivariant we see the quasi-coherent module $\mathcal{F}_W$ we obtain is H-equivariant. Conversely, given an H-equivariant module we can recover the sheaves $\mathcal{F}_{S,e}$ by descent theory starting with the observation that $S \times_{e,\mathcal{M}_{1,1}} W$ is an H-torsor.

Picard group:

$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbf{Z}/12\mathbf{Z}$$

[2] I suppose that it is possible an irreducible algebraic stack exists which doesn't have an irreducible smooth cover – but if so it is going to be quite nasty!

Idea. We have seen the first equality above. Note that $\mathrm{Pic}(W) = 0$ because the ring $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ has trivial class group. There is an exact sequence

$$\mathbf{Z}\Delta \to \mathrm{Pic}_H(\mathbf{A}^5_{\mathbf{Z}}) \to \mathrm{Pic}_H(W) \to 0$$

The middle group equals $\mathrm{Hom}(H, \mathbf{G}_m) = \mathbf{Z}$. The image Δ is 12 because Δ has degree 12. This argument is roughly correct, see [**FO10**].

Étale cohomology: Let Λ be a ring. There is a first quadrant spectral sequence converging to $H^{p+q}_{étale}(\mathcal{M}_{1,1}, \Lambda)$ with E_2-page

$$E_2^{p,q} = H^q_{étale}(W \times H \times \ldots \times H, \Lambda) \quad (p \text{ factors } H)$$

Idea. Note that

$$W \times_{\mathcal{M}_{1,1}} W \times_{\mathcal{M}_{1,1}} \ldots \times_{\mathcal{M}_{1,1}} W = W \times H \times \ldots \times H$$

because $W \to \mathcal{M}_{1,1}$ is a H-torsor. The spectral sequence is the Čech-to-cohomology spectral sequence for the smooth cover $\{W \to \mathcal{M}_{1,1}\}$. For example we see that $H^0_{étale}(\mathcal{M}_{1,1}, \Lambda) = \Lambda$ because W is connected, and $H^1_{étale}(\mathcal{M}_{1,1}, \Lambda) = 0$ because $H^1_{étale}(W, \Lambda) = 0$ (of course this requires a proof). Of course, the smooth covering $W \to \mathcal{M}_{1,1}$ may not be "optimal" for the computation of étale cohomology.

105.8. Other chapters

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CHAPTER 106

More on Morphisms of Stacks

0BPK

106.1. Introduction

0BPL In this chapter we continue our study of properties of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00**].

106.2. Conventions and abuse of language

0BPM We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2.

106.3. Thickenings

0BPN The following terminology may not be completely standard, but it is convenient. If $\mathcal{Y}$ is a closed substack of an algebraic stack $\mathcal{X}$, then the morphism $\mathcal{Y} \to \mathcal{X}$ is representable.

0BPP **Definition 106.3.1.** Thickenings.

(1) We say an algebraic stack $\mathcal{X}'$ is a *thickening* of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and the associated topological spaces are equal.
(2) Given two thickenings $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{Y} \subset \mathcal{Y}'$ a *morphism of thickenings* is a morphism $f' : \mathcal{X}' \to \mathcal{Y}'$ of algebraic stacks such that $f'|_{\mathcal{X}}$ factors through the closed substack $\mathcal{Y}$. In this situation we set $f = f'|_{\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$ and we say that $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings.
(3) Let $\mathcal{Z}$ be an algebraic stack. We similarly define *thickenings over* $\mathcal{Z}$ and *morphisms of thickenings over* $\mathcal{Z}$. This means that the algebraic stacks $\mathcal{X}'$ and $\mathcal{Y}'$ are endowed with a structure morphism to $\mathcal{Z}$ and that f' fits into a suitable 2-commutative diagram of algebraic stacks.

Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let U' be a scheme and let $U' \to \mathcal{X}'$ be a surjective smooth morphism. Setting $U = \mathcal{X} \times_{\mathcal{X}'} U'$ we obtain a morphism of thickenings

$$(U \subset U') \longrightarrow (\mathcal{X} \subset \mathcal{X}')$$

and $U \to \mathcal{X}$ is a surjective smooth morphism. We can often deduce properties of the thickening $\mathcal{X} \subset \mathcal{X}'$ from the corresponding properties of the thickening $U \subset U'$. Sometimes, by abuse of language, we say that a morphism $\mathcal{X} \to \mathcal{X}'$ is a thickening if it is a closed immersion inducing a bijection $|\mathcal{X}| \to |\mathcal{X}'|$.

0CJ7 **Lemma 106.3.2.** *Let $i : \mathcal{X} \to \mathcal{X}'$ be a morphism of algebraic stacks. The following are equivalent*

(1) *i is a thickening of algebraic stacks (abuse of language as above), and*

(2) *i is representable by algebraic spaces and is a thickening in the sense of Properties of Stacks, Section 100.3.*

In this case i is a closed immersion and a universal homeomorphism.

Proof. By More on Morphisms of Spaces, Lemmas 76.9.10 and 76.9.8 the property P that a morphism of algebraic spaces is a (first order) thickening is fpqc local on the base and stable under base change. Thus the discussion in Properties of Stacks, Section 100.3 indeed applies. Having said this the equivalence of (1) and (2) follows from the fact that $P = P_1 + P_2$ where P_1 is the property of being a closed immersion and P_2 is the property of being surjective. (Strictly speaking, the reader should also consult More on Morphisms of Spaces, Definition 76.9.1, Properties of Stacks, Definition 100.9.1 and the discussion following, Morphisms of Spaces, Lemma 67.5.1, Properties of Stacks, Section 100.5 to see that all the concepts all match up.) The final assertion is clear from the foregoing. □

We will use the lemma without further mention. Using the same references More on Morphisms of Spaces, Lemmas 76.9.10 and 76.9.8 as used in the lemma, allows us to define a first order thickening as follows.

0BPQ **Definition 106.3.3.** We say an algebraic stack $\mathcal{X}'$ is a *first order thickening* of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and $\mathcal{X} \to \mathcal{X}'$ is a first order thickening in the sense of Properties of Stacks, Section 100.3.

If $(U \subset U') \to (\mathcal{X} \subset \mathcal{X}')$ is a smooth cover by a scheme as above, then this simply means that $U \subset U'$ is a first order thickening. Next we formulate the obligatory lemmas.

0BPR **Lemma 106.3.4.** *Let $\mathcal{Y} \subset \mathcal{Y}'$ be a thickening of algebraic stacks. Let $\mathcal{X}' \to \mathcal{Y}'$ be a morphism of algebraic stacks and set $\mathcal{X} = \mathcal{Y} \times_{\mathcal{Y}'} \mathcal{X}'$. Then $(\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings. If $\mathcal{Y} \subset \mathcal{Y}'$ is a first order thickening, then $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening.*

Proof. See discussion above, Properties of Stacks, Section 100.3, and More on Morphisms of Spaces, Lemma 76.9.8. □

0BPS **Lemma 106.3.5.** *If $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{X}' \subset \mathcal{X}''$ are thickenings of algebraic stacks, then so is $\mathcal{X} \subset \mathcal{X}''$.*

Proof. See discussion above, Properties of Stacks, Section 100.3, and More on Morphisms of Spaces, Lemma 76.9.9 □

0BPT **Example 106.3.6.** Let $\mathcal{X}'$ be an algebraic stack. Then $\mathcal{X}'$ is a thickening of the reduction $\mathcal{X}'_{red}$, see Properties of Stacks, Definition 100.10.4. Moreover, if $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, then $\mathcal{X}'_{red} = \mathcal{X}_{red} \subset \mathcal{X}$. In other words, $\mathcal{X} = \mathcal{X}'_{red}$ if and only if $\mathcal{X}$ is a reduced algebraic stack.

0BPU **Lemma 106.3.7.** *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is a thickening and the canonical diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \xrightarrow{\Delta'} & \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \end{array}$$

is cartesian.

Proof. Since $\mathcal{X} \to \mathcal{Y}'$ factors through the closed substack $\mathcal{Y}$ we see that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} = \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}$. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is isomorphic to the composition

$$\mathcal{X} \times_{\mathcal{Y}'} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$$

both of which are thickenings as base changes of thickenings (Lemma 106.3.4). Hence so is the composition (Lemma 106.3.5). Since $\mathcal{X} \to \mathcal{X}'$ is a monomorphism, the final statement of the lemma follows from Properties of Stacks, Lemma 100.8.6 applied to $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}'$. □

0BPV **Lemma 106.3.8.** *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Let $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ be the corresponding diagonal morphisms. Then each property from the following list is satisfied by Δ if and only if it is satisfied by Δ': (a) representable by schemes, (b) affine, (c) surjective, (d) quasi-compact, (e) universally closed, (f) integral, (g) quasi-separated, (h) separated, (i) universally injective, (j) universally open, (k) locally quasi-finite, (l) finite, (m) unramified, (n) monomorphism, (o) immersion, (p) closed immersion, and (q) proper.*

Proof. Observe that

$$(\Delta, \Delta') : (\mathcal{X} \subset \mathcal{X}') \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \subset \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')$$

is a morphism of thickenings (Lemma 106.3.7). Moreover Δ and Δ' are representable by algebraic spaces by Morphisms of Stacks, Lemma 101.3.3. Hence, via the discussion in Properties of Stacks, Section 100.3 the lemma follows for cases (a), (b), (c), (d), (e), (f), (g), (h), (i), and (j) by using More on Morphisms of Spaces, Lemma 76.10.1.

Lemma 106.3.7 tells us that $\mathcal{X} = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')} \mathcal{X}'$. Moreover, Δ and Δ' are locally of finite type by the aforementioned Morphisms of Stacks, Lemma 101.3.3. Hence the result for cases (k), (l), (m), (n), (o), (p), and (q) by using More on Morphisms of Spaces, Lemma 76.10.3. □

As a consequence we obtain the following pleasing result.

0BPW **Lemma 106.3.9.** *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Then* [**Con07a**, Theorem 2.2.5]

(1) *$\mathcal{X}$ is an algebraic space if and only if $\mathcal{X}'$ is an algebraic space,*
(2) *$\mathcal{X}$ is a scheme if and only if $\mathcal{X}'$ is a scheme,*
(3) *$\mathcal{X}$ is DM if and only if $\mathcal{X}'$ is DM,*
(4) *$\mathcal{X}$ is quasi-DM if and only if $\mathcal{X}'$ is quasi-DM,*
(5) *$\mathcal{X}$ is separated if and only if $\mathcal{X}'$ is separated,*
(6) *$\mathcal{X}$ is quasi-separated if and only if $\mathcal{X}'$ is quasi-separated, and*
(7) *add more here.*

Proof. In each case we reduce to a question about the diagonal and then we use Lemma 106.3.8 applied to the morphism of thickenings

$$(\mathcal{X} \subset \mathcal{X}') \to (\mathrm{Spec}(\mathbf{Z}) \subset \mathrm{Spec}(\mathbf{Z}))$$

We do this after viewing $\mathcal{X} \subset \mathcal{X}'$ as a thickening of algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 94.19.2.

Case (1). An algebraic stack is an algebraic space if and only if its diagonal is a monomorphism, see Morphisms of Stacks, Lemma 101.6.3 (this also follows immediately from Algebraic Stacks, Proposition 94.13.3).

Case (2). By (1) we may assume that $\mathcal{X}$ and $\mathcal{X}'$ are algebraic spaces and then we can use More on Morphisms of Spaces, Lemma 76.9.5.

Case (3) – (6). Each of these cases corresponds to a condition on the diagonal, see Morphisms of Stacks, Definitions 101.4.1 and 101.4.2. □

106.4. Morphisms of thickenings

0CJ8 If $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings of algebraic stacks, then often properties of the morphism f are inherited by f'. There are several variants.

0CJ9 **Lemma 106.4.1.** *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*
(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is universally injective if and only if f' is universally injective,*
(7) *f is universally open if and only if f' is universally open,*
(8) *f is quasi-DM if and only if f' is quasi-DM,*
(9) *f is DM if and only if f' is DM,*
(10) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(11) *f is representable if and only if f' is representable,*
(12) *f is representable by algebraic spaces if and only if f' is representable by algebraic spaces,*
(13) *add more here.*

Proof. By Lemma 106.3.2 the morphisms $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$ are universal homeomorphisms. Thus any condition on $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is equivalent with the corresponding condition on $|f'| : |\mathcal{X}'| \to |\mathcal{Y}'|$ and the same is true after arbitrary base change by a morphism $\mathcal{Z}' \to \mathcal{Y}'$. This proves that (2), (3), (4), (6), (7) hold.

In cases (8), (9), (10), (12) we can translate the conditions on f and f' into conditions on the diagonals Δ and Δ' as in Lemma 106.3.8. See Morphisms of Stacks, Definition 101.4.1 and Lemma 101.6.3. Hence these cases follow from Lemma 106.3.8.

Proof of (11). If f' is representable, then so is f, because for a scheme T and a morphism $T \to \mathcal{Y}$ we have $\mathcal{X} \times_{\mathcal{Y}} T = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T)$ and $\mathcal{X} \to \mathcal{X}'$ is a closed immersion (hence representable). Conversely, assume f is representable, and let $T' \to \mathcal{Y}'$ be a morphism where T' is a scheme. Then

$$\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{Y}'} T') = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T') \to \mathcal{X}' \times_{\mathcal{Y}'} T'$$

is a thickening (by Lemma 106.3.4) and the source is a scheme. Hence the target is a scheme by Lemma 106.3.9.

In cases (1) and (5) if either f or f' has the stated property, then both f and f' are representable by (11). In this case choose an algebraic space V' and a surjective smooth morphism $V' \to \mathcal{Y}'$. Set $V = \mathcal{Y} \times_{\mathcal{Y}'} V'$, $U' = \mathcal{X}' \times_{\mathcal{Y}'} V'$, and $U = \mathcal{X} \times_{\mathcal{Y}'} V'$. Then the desired results follow from the corresponding results for the morphism $(U \subset U') \to (V \subset V')$ of thickenings of algebraic spaces via the principle of

Properties of Stacks, Lemma 100.3.3. See More on Morphisms of Spaces, Lemma 76.10.1 for the corresponding results in the case of algebraic spaces. □

106.5. Infinitesimal deformations of algebraic stacks

0CJA This section is the analogue of More on Morphisms of Spaces, Section 76.18.

0CJB **Lemma 106.5.1.** *Consider a commutative diagram*

$$\begin{array}{ccc} (\mathcal{X} \subset \mathcal{X}') & \xrightarrow[(f,f')]{} & (\mathcal{Y} \subset \mathcal{Y}') \\ & \searrow \quad \swarrow & \\ & (\mathcal{B} \subset \mathcal{B}') & \end{array}$$

of thickenings of algebraic stacks. Assume

(1) $\mathcal{Y}' \to \mathcal{B}'$ *is locally of finite type,*
(2) $\mathcal{X}' \to \mathcal{B}'$ *is flat and locally of finite presentation,*
(3) f *is flat, and*
(4) $\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$ *and* $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$.

Then f' is flat and for all $y' \in |\mathcal{Y}'|$ in the image of $|f'|$ the morphism $\mathcal{Y}' \to \mathcal{B}'$ is flat at y'.

Proof. Choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.4. The statement about flatness of $\mathcal{Y}'/\mathcal{B}'$ at points in the image of $|f'|$ follows in the same manner. □

0CJC **Lemma 106.5.2.** *Consider a commutative diagram*

$$\begin{array}{ccc} (\mathcal{X} \subset \mathcal{X}') & \xrightarrow[(f,f')]{} & (\mathcal{Y} \subset \mathcal{Y}') \\ & \searrow \quad \swarrow & \\ & (\mathcal{B} \subset \mathcal{B}') & \end{array}$$

of thickenings of algebraic stacks. Assume $\mathcal{Y}' \to \mathcal{B}'$ locally of finite type, $\mathcal{X}' \to \mathcal{B}'$ flat and locally of finite presentation, $\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$, and $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$. Then

0CJD (1) *f is flat if and only if f' is flat,*
0CJE (2) *f is an isomorphism if and only if f' is an isomorphism,*
0CJF (3) *f is an open immersion if and only if f' is an open immersion,*
0CJG (4) *f is a monomorphism if and only if f' is a monomorphism,*
0CJH (5) *f is locally quasi-finite if and only if f' is locally quasi-finite,*

0CJI (6) *f is syntomic if and only if f' is syntomic,*
0CJJ (7) *f is smooth if and only if f' is smooth,*
0CJK (8) *f is unramified if and only if f' is unramified,*
0CJL (9) *f is étale if and only if f' is étale,*
0CJM (10) *f is finite if and only if f' is finite, and*
(11) *add more here.*

Proof. In case (1) this follows from Lemma 106.5.1.

In cases (6), (7) this can be proved by the method used in the proof of Lemma 106.5.1. Namely, choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.5.

In cases (8) and (9) we first see that the assumption for f or f' implies that both f and f' are DM morphisms of algebraic stacks, see Lemma 106.4.1. Then we can choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective étale(!) morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective étale as well. Hence the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{array}{ccc} (W \subset W') & \longrightarrow & (V \subset V') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 76.18.5.

In cases (2), (3), (4), (10) we first conclude by Lemma 106.4.1 that f and f' are representable by algebraic spaces. Thus we may choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$, an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$, and then $W' = V' \times_{\mathcal{Y}'} \mathcal{X}'$ will be an algebraic space. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W = V \times_{\mathcal{Y}} \mathcal{X}$ as well. Then we have to see that $W' \to V'$ is an isomorphism, resp. an open immersion, resp. a monomorphism, resp. finite, if and only if $W \to V$ has the same property. See Properties of Stacks, Lemma 100.3.3. Thus we conclude by applying the results for algebraic spaces as above.

In the case (5) we first observe that f and f' are locally of finite type by Morphisms of Stacks, Lemma 101.17.8. On the other hand, the morphism f is quasi-DM if and only if f' is by Lemma 106.4.1. The last thing to check to see if f or f' is locally quasi-finite (Morphisms of Stacks, Definition 101.23.2) is a condition on underlying topological spaces which holds for f if and only if it holds for f' by the discussion in the first paragraph of the proof. □

106.6. Lifting affines

0CJN Consider a solid diagram

$$\begin{array}{ccc} W & \dashrightarrow & W' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an affine scheme and $W \to \mathcal{X}$ is smooth. The question we address in this section is whether we can find W' and the dotted arrows so that the square is cartesian and $W' \to \mathcal{X}'$ is smooth. We do not know the answer in general, but if $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening we will prove the answer is yes.

To study this problem we introduce the following category.

0CJP **Remark 106.6.1** (Category of lifts)**.** Consider a diagram

$$\begin{array}{ccc} W & & \\ {\scriptstyle x}\downarrow & & \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an algebraic space, and $W \to \mathcal{X}$ is smooth. We will construct a category $\mathcal{C}$ and a functor

$$p : \mathcal{C} \longrightarrow W_{spaces,étale}$$

(see Properties of Spaces, Definition 66.18.2 for notation) as follows. An object of $\mathcal{C}$ will be a system $(U, U', a, i, x', \alpha)$ which forms a commutative diagram

0CJQ (106.6.1.1)

$$\begin{array}{ccc} U & \xrightarrow[i]{} & U' \\ {\scriptstyle a}\downarrow & & \\ W & & \downarrow{\scriptstyle x'} \\ {\scriptstyle x}\downarrow & & \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

with commutativity witnessed by the 2-morphism $\alpha : x \circ a \to x' \circ i$ such that U and U' are algebraic spaces, $a : U \to W$ is étale, $x' : U' \to \mathcal{X}'$ is smooth, and such that $U = \mathcal{X} \times_{\mathcal{X}'} U'$. In particular $U \subset U'$ is a thickening. A morphism

$$(U, U', a, i, x', \alpha) \to (V, V', b, j, y', \beta)$$

is given by (f, f', γ) where $f : U \to V$ is a morphism over W, $f' : U' \to V'$ is a morphism whose restriction to U gives f, and $\gamma : x' \circ f' \to y'$ is a 2-morphism

witnessing the commutativity in right triangle of the diagram below

0CJR (106.6.1.2)

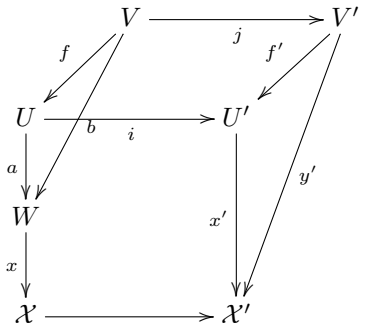

Finally, we require that γ is compatible with α and β: in the calculus of 2-categories of Categories, Sections 4.28 and 4.29 this reads

$$\beta = (\gamma \star \mathrm{id}_j) \circ (\alpha \star \mathrm{id}_f)$$

(more succinctly: $\beta = j^*\gamma \circ f^*\alpha$). Another formulation is that objects are commutative diagrams (106.6.1.1) with some additional properties and morphisms are commutative diagrams (106.6.1.2) in the category $Spaces/\mathcal{X}'$ introduced in Properties of Stacks, Remark 100.3.7. This makes it clear that $\mathcal{C}$ is a category and that the rule $p : \mathcal{C} \to W_{spaces,étale}$ sending $(U, U', a, i, x', \alpha)$ to $a : U \to W$ is a functor.

0CJS **Lemma 106.6.2.** *For any morphism (106.6.1.2) the map $f' : V' \to U'$ is étale.*

Proof. Namely $f : V \to U$ is étale as a morphism in $W_{spaces,étale}$ and we can apply Lemma 106.5.2 because $U' \to \mathcal{X}'$ and $V' \to \mathcal{X}'$ are smooth and $U = \mathcal{X} \times_{\mathcal{X}'} U'$ and $V = \mathcal{X} \times_{\mathcal{X}'} V'$. □

0CJT **Lemma 106.6.3.** *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is fibred in groupoids.*

Proof. We claim the fibre categories of p are groupoids. If (f, f', γ') as in (106.6.1.2) is a morphism such that $f : U \to V$ is an isomorphism, then f' is an isomorphism by Lemma 106.5.2 and hence (f, f', γ') is an isomorphism.

Consider a morphism $f : V \to U$ in $W_{spaces,étale}$ and an object $\xi = (U, U', a, i, x', \alpha)$ of $\mathcal{C}$ over U. We are going to construct the "pullback" $f^*\xi$ over V. Namely, set $b = a \circ f$. Let $f' : V' \to U'$ be the étale morphism whose restriction to V is f (More on Morphisms of Spaces, Lemma 76.8.2). Denote $j : V \to V'$ the corresponding thickening. Let $y' = x' \circ f'$ and $\gamma = \mathrm{id} : x' \circ f' \to y'$. Set

$$\beta = \alpha \star \mathrm{id}_f : x \circ b = x \circ a \circ f \to x' \circ i \circ f = x' \circ f' \circ j = y' \circ j$$

It is clear that $(f, f', \gamma) : (V, V', b, j, y', \beta) \to (U, U', a, i, x', \alpha)$ is a morphism as in (106.6.1.2). The morphisms (f, f', γ) so constructed are strongly cartesian (Categories, Definition 4.33.1). We omit the detailed proof, but essentially the reason is that given a morphism $(g, g', \epsilon) : (Y, Y', c, k, z', \delta) \to (U, U', a, i, x', \alpha)$ in $\mathcal{C}$ such that g factors as $g = f \circ h$ for some $h : Y \to V$, then we get a unique factorization $g' = f' \circ h'$ from More on Morphisms of Spaces, Lemma 76.8.2 and after that one can produce the necessary ζ such that $(h, h', \zeta) : (Y, Y', c, k, z', \delta) \to (V, V', b, j, y', \beta)$ is a morphism of $\mathcal{C}$ with $(g, g', \epsilon) = (f, f', \gamma) \circ (h, h', \zeta)$.

Therefore $p : \mathcal{C} \to W_{étale}$ is a fibred category (Categories, Definition 4.33.5). Combined with the fact that the fibre categories are groupoids seen above we conclude that $p : \mathcal{C} \to W_{étale}$ is fibred in groupoids by Categories, Lemma 4.35.2. □

0CJU **Lemma 106.6.4.** *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is a stack in groupoids.*

Proof. By Lemma 106.6.3 we see the first condition of Stacks, Definition 8.5.1 holds. As is customary we check descent of objects and we leave it to the reader to check descent of morphisms. Thus suppose we have $a : U \to W$ in $W_{spaces,étale}$, a covering $\{U_k \to U\}_{k \in K}$ in $W_{spaces,étale}$, objects $\xi_k = (U_k, U'_k, a_k, i_k, x'_k, \alpha_k)$ of $\mathcal{C}$ over U_k, and morphisms

$$\varphi_{kk'} = (f_{kk'}, f'_{kk'}, \gamma_{kk'}) : \xi_k|_{U_k \times_U U_{k'}} \to \xi_{k'}|_{U_k \times_U U_{k'}}$$

between restrictions satisfying the cocycle condition. In order to prove effectivity we may first refine the covering. Hence we may assume each U_k is a scheme (even an affine scheme if you like). Let us write

$$\xi_k|_{U_k \times_U U_{k'}} = (U_k \times_U U_{k'}, U'_{kk'}, a_{kk'}, x'_{kk'}, \alpha_{kk'})$$

Then we get an étale (by Lemma 106.6.2) morphism $s_{kk'} : U'_{kk'} \to U'_k$ as the second component of the morphism $\xi_k|_{U_k \times_U U_{k'}} \to \xi_k$ of $\mathcal{C}$. Similarly we obtain an étale morphism $t_{kk'} : U'_{kk'} \to U'_{k'}$ by looking at the second component of the composition

$$\xi_k|_{U_k \times_U U_{k'}} \xrightarrow{\varphi_{kk'}} \xi_{k'}|_{U_k \times_U U_{k'}} \to \xi_{k'}$$

We claim that

$$j : \coprod\nolimits_{(k,k') \in K \times K} U'_{kk'} \xrightarrow{(\coprod s_{kk'}, \coprod t_{kk'})} (\coprod\nolimits_{k \in K} U'_k) \times (\coprod\nolimits_{k \in K} U'_k)$$

is an étale equivalence relation. First, we have already seen that the components s, t of the displayed morphism are étale. The base change of the morphism j by $(\coprod U_k) \times (\coprod U_k) \to (\coprod U'_k) \times (\coprod U'_k)$ is a monomorphism because it is the map

$$\coprod\nolimits_{(k,k') \in K \times K} U_k \times_U U_{k'} \longrightarrow (\coprod\nolimits_{k \in K} U_k) \times (\coprod\nolimits_{k \in K} U_k)$$

Hence j is a monomorphism by More on Morphisms, Lemma 37.3.4. Finally, symmetry of the relation j comes from the fact that $\varphi_{kk'}^{-1}$ is the "flip" of $\varphi_{k'k}$ (see Stacks, Remarks 8.3.2) and transitivity comes from the cocycle condition (details omitted). Thus the quotient of $\coprod U'_k$ by j is an algebraic space U' (Spaces, Theorem 65.10.5). Above we have already shown that there is a thickening $i : U \to U'$ as we saw that the restriction of j on $\coprod U_k$ gives $(\coprod U_k) \times_U (\coprod U_k)$. Finally, if we temporarily view the 1-morphisms $x'_k : U'_k \to \mathcal{X}'$ as objects of the stack $\mathcal{X}'$ over U'_k then we see that these come endowed with a descent datum with respect to the étale covering $\{U'_k \to U'\}$ given by the third component $\gamma_{kk'}$ of the morphisms $\varphi_{kk'}$ in $\mathcal{C}$. Since $\mathcal{X}'$ is a stack this descent datum is effective and translating back we obtain a 1-morphism $x' : U' \to \mathcal{X}'$ such that the compositions $U'_k \to U' \to \mathcal{X}'$ come equipped with isomorphisms to x'_k compatible with $\gamma_{kk'}$. This means that the morphisms $\alpha_k : x \circ a_k \to x'_k \circ i_k$ glue to a morphism $\alpha : x \circ a \to x' \circ i$. Then $\xi = (U, U', a, i, x', \alpha)$ is the desired object over U. □

0CJV **Lemma 106.6.5.** *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{X}$ be a smooth morphism. There exists an étale covering $\{W_i \to W\}_{i \in I}$ and for each i a cartesian diagram*

$$\begin{array}{ccc} W_i & \longrightarrow & W_i' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

with $W_i' \to \mathcal{X}'$ smooth.

Proof. Choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. As usual we set $U = \mathcal{X} \times_{\mathcal{X}'} U'$. Then $U \to \mathcal{X}$ is a surjective smooth morphism. Therefore the base change

$$V = W \times_{\mathcal{X}} U \longrightarrow W$$

is a surjective smooth morphism of algebraic spaces. By Topologies on Spaces, Lemma 73.4.4 we can find an étale covering $\{W_i \to W\}$ such that $W_i \to W$ factors through $V \to W$. After covering W_i by affines (Properties of Spaces, Lemma 66.6.1) we may assume each W_i is affine. We may and do replace W by W_i which reduces us to the situation discussed in the next paragraph.

Assume W is affine and the given morphism $W \to \mathcal{X}$ factors through U. Picture

$$W \xrightarrow{i} U \to \mathcal{X}$$

Since W and U are smooth over $\mathcal{X}$ we see that i is locally of finite type (Morphisms of Stacks, Lemma 101.17.8). After replacing U by $\mathbf{A}^n_U$ we may assume that i is an immersion, see Morphisms, Lemma 29.39.2. By Morphisms of Stacks, Lemma 101.44.4 the morphism i is a local complete intersection. Hence i is a Koszul-regular immersion (as defined in Divisors, Definition 31.21.1) by More on Morphisms, Lemma 37.62.3.

We may still replace W by an affine open covering. For every point $w \in W$ we can choose an affine open $U'_w \subset U'$ such that if $U_w \subset U$ is the corresponding affine open, then $w \in i^{-1}(U_w)$ and $i^{-1}(U_w) \to U_w$ is a closed immersion cut out by a Koszul-regular sequence $f_1, \ldots, f_r \in \Gamma(U_w, \mathcal{O}_{U_w})$. This follows from the definition of Koszul-regular immersions and Divisors, Lemma 31.20.7. Set $W_w = i^{-1}(U_w)$; this is an affine open neighbourhood of $w \in W$. Choose lifts $f'_1, \ldots, f'_r \in \Gamma(U'_w, \mathcal{O}_{U'_w})$ of $f_1, \ldots, f_r$. This is possible as $U_w \to U'_w$ is a closed immersion of affine schemes. Let $W'_w \subset U'_w$ be the closed subscheme cut out by $f'_1, \ldots, f'_r$. We claim that $W'_w \to \mathcal{X}'$ is smooth. The claim finishes the proof as $W_w = \mathcal{X} \times_{\mathcal{X}'} W'_w$ by construction.

To check the claim it suffices to check that the base change $W'_w \times_{\mathcal{X}'} X' \to X'$ is smooth for every affine scheme X' smooth over $\mathcal{X}'$. Choose an étale morphism

$$Y' \to U'_w \times_{\mathcal{X}'} X'$$

with Y' affine. Because $U'_w \times_{\mathcal{X}'} X'$ is covered by the images of such morphisms, it is enough to show that the closed subscheme Z' of Y' cut out by $f'_1, \ldots, f'_r$ is

smooth over X'. Picture

$$\begin{array}{ccccc} Z' & \longrightarrow & Y' & & \\ \downarrow & & \downarrow & & \\ W'_w \times_{\mathcal{X}'} X' & \longrightarrow & U'_w \times_{\mathcal{X}'} X' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \\ W'_w = V(f'_1, \ldots, f'_r) & \longrightarrow & U'_w & & \end{array}$$

Set $X = \mathcal{X} \times_{\mathcal{X}'} X'$, $Y = X \times_{X'} Y' = \mathcal{X} \times_{\mathcal{X}'} Y'$, and $Z = Y \times_{Y'} Z' = X \times_{X'} Z' = \mathcal{X} \times_{\mathcal{X}'} Z'$. Then $(Z \subset Z') \to (Y \subset Y') \subset (X \subset X')$ are (cartesian) morphisms of thickenings of affine schemes and we are given that $Z \to X$ and $Y' \to X'$ are smooth. Finally, the sequence of functions $f'_1, \ldots, f'_r$ map to a Koszul-regular sequence in $\Gamma(Y', \mathcal{O}_{Y'})$ by More on Algebra, Lemma 15.30.5 because $Y' \to U'_w$ is smooth and hence flat. By More on Algebra, Lemma 15.31.6 (and the fact that Koszul-regular sequences are quasi-regular sequences by More on Algebra, Lemmas 15.30.2, 15.30.3, and 15.30.6) we conclude that $Z' \to X'$ is smooth as desired. □

0CJW **Lemma 106.6.6.** *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Consider a commutative diagram*

$$\begin{array}{ccccc} W'' & \longleftarrow & W & \longrightarrow & W' \\ {\scriptstyle x''}\downarrow & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ \mathcal{X}' & \longleftarrow & \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

with cartesian squares where W', W, W'' are algebraic spaces and the vertical arrows are smooth. Then there exist

1. *an étale covering $\{f'_k : W'_k \to W'\}_{k \in K}$,*
2. *étale morphisms $f''_k : W'_k \to W''$, and*
3. *2-morphisms $\gamma_k : x'' \circ f''_k \to x' \circ f'_k$*

such that (a) $(f'_k)^{-1}(W) = (f''_k)^{-1}(W)$, (b) $f'_k|_{(f'_k)^{-1}(W)} = f''_k|_{(f''_k)^{-1}(W)}$, and (c) pulling back γ_k to the closed subscheme of (a) agrees with the 2-morphism given by the commutativity of the initial diagram over W.

Proof. Denote $i : W \to W'$ and $i'' : W \to W''$ the given thickenings. The commutativity of the diagram in the statement of the lemma means there is a 2-morphism $\delta : x' \circ i' \to x'' \circ i''$ This is the 2-morphism referred to in part (c) of the statement. Consider the algebraic space

$$I' = W' \times_{x', \mathcal{X}', x''} W''$$

with projections $p' : I' \to W'$ and $q' : I' \to W''$. Observe that there is a "universal" 2-morphism $\gamma : x' \circ p' \to x'' \circ q'$ (we will use this later). The choice of δ defines a morphism

$$\begin{array}{ccccc} W & & \xrightarrow{\quad\delta\quad} & & I' \\ & & & \swarrow^{p'} \quad \searrow_{q'} & \\ & & W' & & W'' \end{array}$$

such that the compositions $W \to I' \to W'$ and $W \to I' \to W''$ are $i : W \to W'$ and $i' : W \to W''$. Since x'' is smooth, the morphism $p' : I' \to W'$ is smooth as a base change of x''.

Suppose we can find an étale covering $\{f'_k : W'_k \to W'\}$ and morphisms $\delta_k : W'_k \to I'$ such that the restriction of δ_k to $W_k = (f'_k)^{-1}$ is equal to $\delta \circ f_k$ where $f_k = f'_k|_{W_k}$. Picture

$$\begin{array}{ccccc} W_k & \xrightarrow{f_k} & W & \xrightarrow{\delta} & I' \\ \downarrow & & & \nearrow_{\delta_k} & \downarrow{\scriptstyle p'} \\ W'_k & & \xrightarrow{f'_k} & & W' \end{array}$$

In other words, we want to be able to extend the given section $\delta : W \to I'$ of p' to a section over W' after possibly replacing W' by an étale covering.

If this is true, then we can set $f''_k = q' \circ \delta_k$ and $\gamma_k = \gamma \star \mathrm{id}_{\delta_k}$ (more succinctly $\gamma_k = \delta_k^* \gamma$). Namely, the only thing left to show at this is that the morphism f''_k is étale. By construction the morphism $x' \circ p'$ is 2-isomorphic to $x'' \circ q'$. Hence $x'' \circ f''_k$ is 2-isomorphic to $x' \circ f'_k$. We conclude that the composition

$$W'_k \xrightarrow{f''_k} W'' \xrightarrow{x''} \mathcal{X}'$$

is smooth because $x' \circ f'_k$ is so. As f_k is étale we conclude f''_k is étale by Lemma 106.5.2.

If the thickening is a first order thickening, then we can choose any étale covering $\{W'_k \to W'\}$ with W'_k affine. Namely, since p' is smooth we see that p' is formally smooth by the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 76.19.6). As W_k is affine and as $W_k \to W'_k$ is a first order thickening (as a base change of $\mathcal{X} \to \mathcal{X}'$, see Lemma 106.3.4) we get δ_k as desired.

In the general case the existence of the covering and the morphisms δ_k follows from More on Morphisms of Spaces, Lemma 76.19.7. □

0CJX **Lemma 106.6.7.** *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 is a gerbe.*

Proof. In Lemma 106.6.4 we have seen that it is a stack in groupoids. Thus it remains to check conditions (2) and (3) of Stacks, Definition 8.11.1. Condition (2) follows from Lemma 106.6.5. Condition (3) follows from Lemma 106.6.6. □

0CKG **Lemma 106.6.8.** *In Remark 106.6.1 assume $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening. Then*

(1) *the automorphism sheaves of objects of the gerbe $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 106.6.1 are abelian, and*
(2) *the sheaf of groups $\mathcal{G}$ constructed in Stacks, Lemma 8.11.8 is a quasi-coherent $\mathcal{O}_W$-module.*

Proof. We will prove both statements at the same time. Namely, given an object $\xi = (U, U', a, i, x', \alpha)$ we will endow $Aut(\xi)$ with the structure of a quasi-coherent $\mathcal{O}_U$-module on $U_{spaces,étale}$ and we will show that this structure is compatible with pullbacks. This will be sufficient by glueing of sheaves (Sites, Section 7.26) and the construction of $\mathcal{G}$ in the proof of Stacks, Lemma 8.11.8 as the glueing of the

automorphism sheaves $Aut(\xi)$ and the fact that it suffices to check a module is quasi-coherent after going to an étale covering (Properties of Spaces, Lemma 66.29.6).

We will describe the sheaf $Aut(\xi)$ using the same method as used in the proof of Lemma 106.6.6. Consider the algebraic space

$$I' = U' \times_{x', \mathcal{X}', x'} U'$$

with projections $p' : I' \to U'$ and $q' : I' \to U'$. Over I' there is a universal 2-morphism $\gamma : x' \circ p' \to x' \circ q'$. The identity $x' \to x'$ defines a diagonal morphism

$$\begin{array}{ccccc} U' & & \xrightarrow{\quad\quad} & & I' \\ & \xrightarrow[\Delta']{} & & \swarrow^{p'} \quad \searrow^{q'} & \\ & & U' & & U' \end{array}$$

such that the compositions $U' \to I' \to U'$ and $U' \to I' \to U'$ are the identity morphisms. We will denote the base change of U', I', p', q', Δ' to $\mathcal{X}$ by U, I, p, q, Δ. Since $W' \to \mathcal{X}'$ is smooth, we see that $p' : I' \to U'$ is smooth as a base change.

A section of $Aut(\xi)$ over U is a morphism $\delta' : U' \to I'$ such that $\delta'|_U = \Delta$ and such that $p' \circ \delta' = \mathrm{id}_{U'}$. To be explicit, $(\mathrm{id}_U, q' \circ \delta', (\delta')^*\gamma) : \xi \to \xi$ is a formula for the corresponding automorphism. More generally, if $f : V \to U$ is an étale morphism, then there is a thickening $j : V \to V'$ and an étale morphism $f' : V' \to U'$ whose restriction to V is f and $f^*\xi$ corresponds to $(V, V', a \circ f, j, x' \circ f', f^*\alpha)$, see proof of Lemma 106.6.3. a section of $Aut(\xi)$ over V is a morphism $\delta' : V' \to I'$ such that $\delta'|_V = \Delta \circ f$ and $p' \circ \delta' = f'$[1].

We conclude that $Aut(\xi)$ as a sheaf of sets agrees with the sheaf defined in More on Morphisms of Spaces, Remark 76.17.7 for the thickenings $(U \subset U')$ and $(I \subset I')$ over $(U \subset U')$ via $\mathrm{id}_{U'}$ and p'. The diagonal Δ' is a section of this sheaf and by acting on this section using More on Morphisms of Spaces, Lemma 76.17.5 we get an isomorphism

0CKH (106.6.8.1) $$\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'}) \longrightarrow Aut(\xi)$$

on $U_{spaces,étale}$. There three things left to check

(1) the construction of (106.6.8.1) commutes with étale localization,
(2) $\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'})$ is a quasi-coherent module on U,
(3) the composition in $Aut(\xi)$ corresponds to addition of sections in this quasi-coherent module.

[1] A formula for the corresponding automorphism is $(\mathrm{id}_V, h', (\delta')^*\gamma)$. Here $h' : V' \to V'$ is the unique (iso)morphism such that $h'|_V = \mathrm{id}_V$ and such that

$$\begin{array}{ccc} V' & \xrightarrow{h'} & V' \\ & \searrow^{q' \circ \delta'} & \downarrow{\scriptstyle f'} \\ & & U' \end{array}$$

commutes. Uniqueness and existence of h' by topological invariance of the étale site, see More on Morphisms of Spaces, Theorem 76.8.1. The reader may feel we should instead look at morphisms $\delta'' : V' \to V' \times_{\mathcal{X}'} V'$ with $\delta'' \circ j = \Delta_{V'/\mathcal{X}'}$ and $\mathrm{pr}_1 \circ \delta'' = \mathrm{id}_{V'}$. This would be fine too: as $V' \times_{\mathcal{X}'} V' \to I'$ is étale, the same topological invariance tells us that sending δ'' to $\delta' = (V' \times_{\mathcal{X}'} V' \to I') \circ \delta''$ is a bijection between the two sets of morphisms.

We will check these in order.

To see (1) we have to show that if $f : V \to U$ is étale, then (106.6.8.1) constructed using ξ over U, restricts to the map (106.6.8.1)

$$\mathcal{H}om_{\mathcal{O}_V}(\Delta_V^*\Omega_{V\times_{\mathcal{X}}V/V}, \mathcal{C}_{V/V'}) \to Aut(\xi|_V)$$

constructed using $\xi|_V$ over V on $V_{spaces,étale}$. This follows from the discussion in the footnote above and More on Morphisms of Spaces, Lemma 76.17.8.

Proof of (2). Since p' is smooth, the morphism $I \to U$ is smooth, and hence the relative module of differentials $\Omega_{I/U}$ is finite locally free (More on Morphisms of Spaces, Lemma 76.7.16). On the other hand, $\mathcal{C}_{U/U'}$ is quasi-coherent (More on Morphisms of Spaces, Definition 76.5.1). By Properties of Spaces, Lemma 66.29.7 we conclude.

Proof of (3). There exists a morphism $c' : I' \times_{p',U',q'} I' \to I'$ such that (U', I', p', q', c') is a groupoid in algebraic spaces with identity Δ'. See Algebraic Stacks, Lemma 94.16.1 for example. Composition in $Aut(\xi)$ is induced by the morphism c' as follows. Suppose we have two morphisms

$$\delta'_1, \delta'_2 : U' \longrightarrow I'$$

corresponding to sections of $Aut(\xi)$ over U as above, in other words, we have $\delta'_i|U = \Delta_U$ and $p' \circ \delta'_i = \mathrm{id}_{U'}$. Then the composition in $Aut(\xi)$ is

$$\delta'_1 \circ \delta'_2 = c'(\delta'_1 \circ q' \circ \delta'_2, \delta'_2)$$

We omit the detailed verification[2]. Thus we are in the situation described in More on Groupoids in Spaces, Section 79.5 and the desired result follows from More on Groupoids in Spaces, Lemma 79.5.2. □

0CKI **Proposition 106.6.9** (Emerton)**.** *Let $\mathcal{X} \subset \mathcal{X}'$ be a first order thickening of algebraic stacks. Let W be an affine scheme and let $W \to \mathcal{X}$ be a smooth morphism. Then there exists a cartesian diagram*

Email of Matthew Emerton dated April 27, 2016.

$$\begin{array}{ccc} W & \longrightarrow & W' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

with $W' \to \mathcal{X}'$ smooth and W' affine.

Proof. Consider the category $p : \mathcal{C} \to W_{spaces,étale}$ introduced in Remark 106.6.1. The proposition states that there exists an object of $\mathcal{C}$ lying over W. Namely, if we have such an object $(W, W', a, i, y', \alpha)$ then $W = \mathcal{X} \times_{\mathcal{X}'} W'$. Hence $W \to W'$ is a thickening of algebraic spaces so W' is affine by More on Morphisms of Spaces, Lemma 76.9.5 and More on Morphisms, Lemma 37.2.3.

Lemma 106.6.7 tells us $\mathcal{C}$ is a gerbe over $W_{spaces,étale}$. This means we can étale locally find a solution and these local solutions are étale locally isomorphic; this part does not require the assumption that the thickening is first order. By Lemma 106.6.8 the automorphism sheaves of objects of our gerbe are abelian and fit together to form a quasi-coherent module $\mathcal{G}$ on $W_{spaces,étale}$. We will verify conditions (1) and (2) of Cohomology on Sites, Lemma 21.11.1 to conclude the existence of an

[2]The reader can see immediately that it is necessary to precompose δ'_1 by $q' \circ \delta'_2$ to get a well defined U'-valued point of the fibre product $I' \times_{p',U',q'} I'$.

object of $\mathcal{C}$ lying over W. Condition (1) is true: the étale coverings $\{W_i \to W\}$ with each W_i affine are cofinal in the collection of all coverings. For such a covering W_i and $W_i \times_W W_j$ are affine and $H^1(W_i, \mathcal{G})$ and $H^1(W_i \times_W W_j, \mathcal{G})$ are zero: the cohomology of a quasi-coherent module over an affine algebraic space is zero for example by Cohomology of Spaces, Proposition 69.7.2. Finally, condition (2) is that $H^2(W, \mathcal{G}) = 0$ for our quasi-coherent sheaf $\mathcal{G}$ which again follows from Cohomology of Spaces, Proposition 69.7.2. This finishes the proof. □

106.7. Infinitesimal deformations

0DNQ We continue the discussion from Artin's Axioms, Section 98.21.

0DNR **Lemma 106.7.1.** *Let $\mathcal{X}$ be an algebraic stack over a scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A and set $y = x|_B$. Then $Inf_x(M) \otimes_A B = Inf_y(M \otimes_A B)$.*

Proof. Recall that $\mathrm{Inf}_x(M)$ is the set of automorphisms of the trivial deformation of x to $A[M]$ which induce the identity automorphism of x over A. The trivial deformation is the pullback of x to $\mathrm{Spec}(A[M])$ via $\mathrm{Spec}(A[M]) \to \mathrm{Spec}(A)$. Let $G \to \mathrm{Spec}(A)$ be the automorphism group algebraic space of x (this exists because $\mathcal{X}$ is an algebraic space). Let $e : \mathrm{Spec}(A) \to G$ be the neutral element. The discussion in More on Morphisms of Spaces, Section 76.17 gives

$$\mathrm{Inf}_x(M) = \mathrm{Hom}_A(e^*\Omega_{G/A}, M)$$

By the same token

$$\mathrm{Inf}_y(M \otimes_A B) = \mathrm{Hom}_B(e_B^*\Omega_{G_B/B}, M \otimes_A B)$$

Since $G \to \mathrm{Spec}(A)$ is locally of finite presentation by assumption, we see that $\Omega_{G/A}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 76.7.15. Hence $e^*\Omega_{G/A}$ is a finitely presented A-module. Moreover, $\Omega_{G_B/B}$ is the pullback of $\Omega_{G/A}$ by More on Morphisms of Spaces, Lemma 76.7.12. Therefore $e_B^*\Omega_{G_B/B} = e^*\Omega_{G/A} \otimes_A B$. we conclude by More on Algebra, Lemma 15.66.4. □

0DNS **Lemma 106.7.2.** *Let $\mathcal{X}$ be an algebraic stack over a base scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. Then the functor*

$$F : B' \longmapsto \{\textit{lifts of } x|_{B' \otimes_{A'} A} \textit{ to } B'\}/\textit{isomorphisms}$$

is a sheaf on the site $(Aff/\mathrm{Spec}(A'))_{fppf}$ of Topologies, Definition 34.7.8.

Proof. Let $\{T_i' \to T'\}_{i=1,\ldots n}$ be a standard fppf covering of affine schemes over A'. Write $T' = \mathrm{Spec}(B')$. As usual denote

$$T'_{i_0 \ldots i_p} = T'_{i_0} \times_{T'} \ldots \times_{T'} T'_{i_p} = \mathrm{Spec}(B'_{i_0 \ldots i_p})$$

where the ring is a suitable tensor product. Set $B = B' \otimes_{A'} A$ and $B_{i_0 \ldots i_p} = B'_{i_0 \ldots i_p} \otimes_{A'} A$. Denote $y = x|_B$ and $y_{i_0 \ldots i_p} = x|_{B_{i_0 \ldots i_p}}$. Let $\gamma_i \in F(B_i')$ such that γ_{i_0} and γ_{i_1} map to the same element of $F(B'_{i_0 i_1})$. We have to find a unique $\gamma \in F(B')$ mapping to γ_i in $F(B_i')$.

Choose an actual object y_i' of $Lift(y_i, B_i')$ in the isomorphism class γ_i. Choose isomorphisms $\varphi_{i_0 i_1} : y'_{i_0}|_{B'_{i_0 i_1}} \to y'_{i_1}|_{B'_{i_0 i_1}}$ in the category $Lift(y_{i_0 i_1}, B'_{i_0 i_1})$. If the

maps $\varphi_{i_0i_1}$ satisfy the cocycle condition, then we obtain our object γ because $\mathcal{X}$ is a stack in the fppf topology. The cocycle condition is that the composition

$$y'_{i_0}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_0i_1}|_{B'_{i_0i_1i_2}}} y'_{i_1}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_1i_2}|_{B'_{i_0i_1i_2}}} y'_{i_2}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_2i_0}|_{B'_{i_0i_1i_2}}} y'_{i_0}|_{B'_{i_0i_1i_2}}$$

is the identity. If not, then these maps give elements

$$\delta_{i_0i_1i_2} \in \mathrm{Inf}_{y_{i_0i_1i_2}}(J_{i_0i_1i_2}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2}$$

Here $J = \mathrm{Ker}(B' \to B)$ and $J_{i_0\ldots i_p} = \mathrm{Ker}(B'_{i_0\ldots i_p} \to B_{i_0\ldots i_p})$. The equality in the displayed equation holds by Lemma 106.7.1 applied to $B' \to B'_{i_0\ldots i_p}$ and y and $y_{i_0\ldots i_p}$, the flatness of the maps $B' \to B'_{i_0\ldots i_p}$ which also guarantees that $J_{i_0\ldots i_p} = J \otimes_{B'} B'_{i_0\ldots i_p}$. A computation (omitted) shows that $\delta_{i_0i_1i_2}$ gives a 2-cocycle in the Čech complex

$$\prod \mathrm{Inf}_y(J) \otimes_B B_{i_0} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2} \to \ldots$$

By Descent, Lemma 35.9.2 this complex is acyclic in positive degrees and has $H^0 = \mathrm{Inf}_y(J)$. Since $\mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1})$ acts on morphisms (Artin's Axioms, Remark 98.21.4) this means we can modify our choice of $\varphi_{i_0i_1}$ to get to the case where $\delta_{i_0i_1i_2} = 0$.

Uniqueness. We still have to show there is at most one γ restricting to γ_i for all i. Suppose we have objects y', z' of $\mathit{Lift}(y, B')$ and isomorphisms $\psi_i : y'|_{B'_i} \to z'|_{B'_i}$ in $\mathit{Lift}(y_i, B'_i)$. Then we can consider

$$\psi_{i_1}^{-1} \circ \psi_{i_0} \in \mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1}$$

Arguing as before, the obstruction to finding an isomorphism between y' and z' over B' is an element in the H^1 of the Čech complex displayed above which is zero. □

0DNT **Lemma 106.7.3.** *Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A. Then $T_x(M) \otimes_A B = T_y(M \otimes_A B)$.*

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. We first reduce the lemma to the case where x lifts to U. Recall that $T_x(M)$ is the set of isomorphism classes of lifts of x to $A[M]$. Therefore Lemma 106.7.2[3] says that the rule

$$A_1 \mapsto T_{x|_{A_1}}(M \otimes_A A_1)$$

is a sheaf on the small étale site of $\mathrm{Spec}(A)$; the tensor product is needed to make $A[M] \to A_1[M \otimes_A A_1]$ a flat ring map. We may choose a faithfully flat étale ring map $A \to A_1$ such that $x|_{A_1}$ lifts to a morphism $u_1 : \mathrm{Spec}(A_1) \to U$, see for example Sheaves on Stacks, Lemma 96.19.10. Write $A_2 = A_1 \otimes_A A_1$ and set $B_1 = B \otimes_A A_1$ and $B_2 = B \otimes_A A_2$. Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_y(M \otimes_A B) & \longrightarrow & T_{y|_{B_1}}(M \otimes_A B_1) & \longrightarrow & T_{y|_{B_2}}(M \otimes_A B_2) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & T_x(M) & \longrightarrow & T_{x|_{A_1}}(M \otimes_A A_1) & \longrightarrow & T_{x|_{A_2}}(M \otimes_A A_2)
\end{array}$$

[3] This lemma applies: $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is locally of finite presentation by Morphisms of Stacks, Lemma 101.27.6 and the assumption that $\mathcal{X} \to S$ is locally of finite presentation. Therefore $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is locally of finite presentation as a base change of Δ.

The rows are exact by the sheaf condition. We have $M \otimes_A B_i = (M \otimes_A A_i) \otimes_{A_i} B_i$. Thus if we prove the result for the middle and right vertical arrow, then the result follows. This reduces us to the case discussed in the next paragraph.

Assume that x is the image of a morphism $u : \mathrm{Spec}(A) \to U$. Observe that $T_u(M) \to T_x(M)$ is surjective since $U \to \mathcal{X}$ is smooth and representable by algebraic spaces, see Criteria for Representability, Lemma 97.6.3 (see discussion preceding it for explanation) and More on Morphisms of Spaces, Lemma 76.19.6. Set $R = U \times_{\mathcal{X}} U$. Recall that we obtain a groupoid (U, R, s, t, c, e, i) in algebraic spaces with $\mathcal{X} = [U/R]$. By Artin's Axioms, Lemma 98.21.6 we have an exact sequence

$$T_{e \circ u}(M) \to T_u(M) \oplus T_u(M) \to T_x(M) \to 0$$

where the zero on the right was shown above. A similar sequence holds for the base change to B. Thus the result we want follows if we can prove the result of the lemma for $T_u(M)$ and $T_{e \circ u}(M)$. This reduces us to the case discussed in the next paragraph.

Assume that $\mathcal{X} = X$ is an algebraic space locally of finite presentation over S. Then we have

$$T_x(M) = \mathrm{Hom}_A(x^*\Omega_{X/S}, M)$$

by the discussion in More on Morphisms of Spaces, Section 76.17. By the same token

$$T_y(M \otimes_A B) = \mathrm{Hom}_B(y^*\Omega_{X/S}, M \otimes_A B)$$

Since $X \to S$ is locally of finite presentation, we see that $\Omega_{X/S}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 76.7.15. Hence $x^*\Omega_{X/S}$ is a finitely presented A-module. Clearly, we have $y^*\Omega_{X/S} = x^*\Omega_{X/S} \otimes_A B$. we conclude by More on Algebra, Lemma 15.66.4. □

0DNU **Lemma 106.7.4.** *Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. If there exists a faithfully flat finitely presented A'-algebra B' and an object y' of $\mathcal{X}$ over B' lifting $x|_{B' \otimes_{A'} A}$, then there exists an object x' over A' lifting x.*

Proof. Let $I = \mathrm{Ker}(A' \to A)$. Set $B'_1 = B' \otimes_{A'} B'$ and $B'_2 = B' \otimes_{A'} B' \otimes_{A'} B'$. Let $J = IB'$, $J_1 = IB'_1$, $J_2 = IB'_2$ and $B = B'/J$, $B_1 = B'_1/J_1$, $B_2 = B'_2/J_2$. Set $y = x|_B$, $y_1 = x|_{B_1}$, $y_2 = x|_{B_2}$. Let F be the fppf sheaf of Lemma 106.7.2 (which applies, see footnote in the proof of Lemma 106.7.3). Thus we have an equalizer diagram

$$F(A') \longrightarrow F(B') \rightrightarrows F(B'_1)$$

On the other hand, we have $F(B') = \mathrm{Lift}(y, B')$, $F(B'_1) = \mathrm{Lift}(y_1, B'_1)$, and $F(B'_2) = \mathrm{Lift}(y_2, B'_2)$ in the terminology from Artin's Axioms, Section 98.21. These sets are nonempty and are (canonically) principal homogeneous spaces for $T_y(J)$, $T_{y_1}(J_1)$, $T_{y_2}(J_2)$, see Artin's Axioms, Lemma 98.21.2. Thus the difference of the two images of y' in $F(B'_1)$ is an element

$$\delta_1 \in T_{y_1}(J_1) = T_x(I) \otimes_A B_1$$

The equality in the displayed equation holds by Lemma 106.7.3 applied to $A' \to B'_1$ and x and y_1, the flatness of $A' \to B'_1$ which also guarantees that $J_1 = I \otimes_{A'} B'_1$.

We have similar equalities for B' and B'_2. A computation (omitted) shows that δ_1 gives a 1-cocycle in the Čech complex

$$T_x(I) \otimes_A B \to T_x(I) \otimes_A B_1 \to T_x(I) \otimes_A B_2 \to \ldots$$

By Descent, Lemma 35.9.2 this complex is acyclic in positive degrees and has $H^0 = T_x(I)$. Thus we may choose an element in $T_x(I) \otimes_A B = T_y(J)$ whose boundary is δ_1. Replacing y' by the result of this element acting on it, we find a new choice y' with $\delta_1 = 0$. Thus y' maps to the same element under the two maps $F(B') \to F(B'_1)$ and we obtain an element o $F(A')$ by the sheaf condition. □

106.8. Formally smooth morphisms

0DNV In this section we introduce the notion of a formally smooth morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks. Such a morphism is characterized by the property that T-valued points of $\mathcal{X}$ lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 106.8.7. It turns out that this criterion is often easier to use than the Jacobian criterion.

0DNW **Definition 106.8.1.** A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be *formally smooth* if it is formally smooth on objects as a 1-morphism in categories fibred in groupoids as explained in Criteria for Representability, Section 97.6.

We translate the condition of the definition into the language we are currently using (see Properties of Stacks, Section 100.2). Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

0DNX (106.8.1.1)

$$\begin{array}{ccc} T & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle i}\downarrow & \nearrow_{y} & \downarrow{\scriptstyle f} \\ T' & \longrightarrow & \mathcal{Y} \end{array}$$

where $i : T \to T'$ is a first order thickening of affine schemes. Let

$$\gamma : y \circ i \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 4.28 and 4.29.) Given (106.8.1.1) and γ a *dotted arrow* is a triple (x', α, β) consisting of a morphism $x' : T' \to \mathcal{X}$ and 2-arrows $\alpha : x' \circ i \to x$, $\beta : y \to f \circ x'$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_i)$, in other words such that

$$\begin{array}{ccccc} & & f \circ x' \circ i & & \\ & {\scriptstyle \beta \star \mathrm{id}_i}\nearrow & & \searrow{\scriptstyle \mathrm{id}_f \star \alpha} & \\ y \circ i & & \xrightarrow{\gamma} & & f \circ x \end{array}$$

is commutative. A *morphism of dotted arrows* $(x'_1, \alpha_1, \beta_1) \to (x'_2, \alpha_2, \beta_2)$ is a 2-arrow $\theta : x'_1 \to x'_2$ such that $\alpha_1 = \alpha_2 \circ (\theta \star \mathrm{id}_i)$ and $\beta_2 = (\mathrm{id}_f \star \theta) \circ \beta_1$.

The category of dotted arrows just described is a special case of Categories, Definition 4.44.1.

0DNY **Lemma 106.8.2.** *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is formally smooth (Definition 106.8.1) if and only if for every diagram (106.8.1.1) and γ the category of dotted arrows is nonempty.*

Proof. Translation between different languages omitted. □

0H1I **Lemma 106.8.3.** *The base change of a formally smooth morphism of algebraic stacks by any morphism of algebraic stacks is formally smooth.*

Proof. Follows from Categories, Lemma 4.44.2 and the definition. □

0H1J **Lemma 106.8.4.** *The composition of formally smooth morphisms of algebraic stacks is formally smooth.*

Proof. Follows from Categories, Lemma 4.44.3 and the definition. □

0H1K **Lemma 106.8.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

(1) *f is formally smooth,*
(2) *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is formally smooth as a morphism of algebraic spaces.*

Proof. Follows from Categories, Lemma 4.44.2 and the definition. □

0DNZ **Lemma 106.8.6.** *Let $T \to T'$ be a first order thickening of affine schemes. Let $\mathcal{X}'$ be an algebraic stack over T' whose structure morphism $\mathcal{X}' \to T'$ is smooth. Let $x : T \to \mathcal{X}'$ be a morphism over T'. Then there exists a morphsm $x' : T' \to \mathcal{X}'$ over T' with $x'|_T = x$.*

Proof. We may apply the result of Lemma 106.7.4. Thus it suffices to construct a smooth surjective morphism $W' \to T'$ with W' affine such that $x|_{T \times_{W'} T'}$ lifts to W'. (We urge the reader to find their own proof of this fact using the analogous result for algebraic spaces already established.) We choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. Observe that $U' \to T'$ is smooth and that the projection $T \times_{\mathcal{X}'} U' \to T$ is surjective smooth. Choose an affine scheme W and an étale morphism $W \to T \times_{\mathcal{X}'} U'$ such that $W \to T$ is surjective. Then $W \to T$ is a smooth morphism of affine schemes. After replacing W by a disjoint union of principal affine opens, we may assume there exists a smooth morphism of affines $W' \to T'$ such that $W = T \times_{T'} W'$, see Algebra, Lemma 10.137.19. By More on Morphisms of Spaces, Lemma 76.19.6 we can find a morphism $W' \to U'$ over T' lifting the given morphism $W \to U'$. This finishes the proof. □

The following lemma is the main result of this section. It implies, combined with Limits of Stacks, Proposition 102.3.8, that we can recognize whether a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is smooth in terms of "simple" properties of the 1-morphism of stacks in groupoids $\mathcal{X} \to \mathcal{Y}$.

0DP0 **Lemma 106.8.7** (Infinitesimal lifting criterion)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation and formally smooth.*

Proof. Assume f is smooth. Then f is locally of finite presentation by Morphisms of Stacks, Lemma 101.33.5. Hence it suffices given a diagram (106.8.1.1) and a $\gamma : y \circ i \to f \circ x$ to find a dotted arrow (see Lemma 106.8.2). Forming fibre

products we obtain

$$\begin{array}{ccccc} T & \longrightarrow & T' \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T' & \longrightarrow & T' & \longrightarrow & \mathcal{Y} \end{array}$$

Thus we see it is sufficient to find a dotted arrow in the left square. Since $T' \times_{\mathcal{Y}} \mathcal{X} \to T'$ is smooth (Morphisms of Stacks, Lemma 101.33.3) existence of a dotted arrow in the left square is guaranteed by Lemma 106.8.6.

Conversely, suppose that f is locally of finite presentation and formally smooth. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $a : U \to \mathcal{X}$ and $b : U \to \mathcal{Y}$ are representable by algebraic spaces and locally of finite presentation (use Morphisms of Stacks, Lemma 101.27.2 and the fact seen above that a smooth morphism is locally of finite presentation). We will apply the general principle of Algebraic Stacks, Lemma 94.10.9 with as input the equivalence of More on Morphisms of Spaces, Lemma 76.19.6 and simultaneously use the translation of Criteria for Representability, Lemma 97.6.3. We first apply this to a to see that a is formally smooth on objects. Next, we use that f is formally smooth on objects by assumption (see Lemma 106.8.2) and Criteria for Representability, Lemma 97.6.2 to see that $b = f \circ a$ is formally smooth on objects. Then we apply the principle once more to conclude that b is smooth. This means that f is smooth by the definition of smoothness for morphisms of algebraic stacks and the proof is complete. □

106.9. Blowing up and flatness

0CQ3 This section quickly discusses what you can deduce from More on Morphisms of Spaces, Sections 76.38 and 76.39 for algebraic stacks over algebraic spaces.

0CQ4 **Lemma 106.9.1.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume*

(1) *Y is quasi-compact and quasi-separated,*
(2) *f is of finite type and quasi-separated,*
(3) *V is quasi-compact, and*
(4) *$\mathcal{X}_V$ is flat and locally of finite presentation over V.*

Then there exists a V-admissible blowup $Y' \to Y$ and a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ with $\mathcal{X}'_V = \mathcal{X}_V$ such that $\mathcal{X}' \to Y'$ is flat and of finite presentation.

Proof. Observe that $\mathcal{X}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Let $R = U \times_{\mathcal{X}} U$ so that we obtain a groupoid (U, R, s, t, c) in algebraic spaces over Y with $\mathcal{X} = [U/R]$ (Algebraic Stacks, Lemma 94.16.2). We may apply More on Morphisms of Spaces, Lemma 76.39.1 to $U \to Y$ and the open $V \subset Y$. Thus we obtain a V-admissible blowup $Y' \to Y$ such that the strict transform $U' \subset U_{Y'}$ is flat and of finite presentation over Y'. Let $R' \subset R_{Y'}$ be the strict transform of R. Since s and t are smooth (and in particular flat) it follows from Divisors on Spaces, Lemma 71.18.4 that we have cartesian diagrams

$$\begin{array}{ccc} R' & \longrightarrow & R_{Y'} \\ \downarrow & & \downarrow{\scriptstyle s_{Y'}} \\ U' & \longrightarrow & U_{Y'} \end{array} \quad \text{and} \quad \begin{array}{ccc} R' & \longrightarrow & R_{Y'} \\ \downarrow & & \downarrow{\scriptstyle t_{Y'}} \\ U' & \longrightarrow & U_{Y'} \end{array}$$

In other words, U' is an $R_{Y'}$-invariant closed subspace of $U_{Y'}$. Thus U' defines a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ by Properties of Stacks, Lemma 100.9.11. The morphism $\mathcal{X}' \to Y'$ is flat and locally of finite presentation because this is true for $U' \to Y'$. On the other hand, we already know $\mathcal{X}' \to Y'$ is quasi-compact and quasi-separated (by our assumptions on f and because this is true for closed immersions). This finishes the proof. □

106.10. Chow's lemma for algebraic stacks

0CQ5 In this section we discuss Chow's lemma for algebraic stacks.

0CQ6 **Lemma 106.10.1.** *Let Y be a quasi-compact and quasi-separated algebraic space. Let $V \subset Y$ be a quasi-compact open. Let $f : \mathcal{X} \to V$ be surjective, flat, and locally of finite presentation. Then there exists a finite surjective morphism $g : Y' \to Y$ such that $V' = g^{-1}(V) \to Y$ factors Zariski locally through f.*

Proof. We first prove this when Y is a scheme. We may choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $\{U \to V\}$ is an fppf covering of schemes. By More on Morphisms, Lemma 37.48.6 there exists a finite surjective morphism $V' \to V$ such that $V' \to V$ factors Zariski locally through U. By More on Morphisms, Lemma 37.48.4 we can find a finite surjective morphism $Y' \to Y$ whose restriction to V is $V' \to V$ as desired.

If Y is an algebraic space, then we see the lemma is true by first doing a finite base change by a finite surjective morphism $Y' \to Y$ where Y' is a scheme. See Limits of Spaces, Proposition 70.16.1. □

0CQ7 **Lemma 106.10.2.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is separated and of finite type,*
(2) *Y is quasi-compact and quasi-separated,*
(3) *V is quasi-compact, and*
(4) *$\mathcal{X}_V$ is a gerbe over V.*

Then there exists a commutative diagram

$$\begin{array}{ccccc} \overline{Z} & \xleftarrow{j} & Z & \xrightarrow{h} & \mathcal{X} \\ & {\scriptstyle \overline{g}}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

with j an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$.

Proof. Suppose we have a commutative diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

and a quasi-compact open $V' \subset Y'$, such that $Y' \to Y$ is a proper morphism of algebraic spaces, $\mathcal{X}' \to \mathcal{X}$ is a proper morphism of algebraic stacks, $V' \subset Y'$ maps surjectively onto V, and $\mathcal{X}'_{V'}$ is a gerbe over V'. Then it suffices to prove the lemma for the pair $(f' : \mathcal{X}' \to Y', V')$. Some details omitted.

Overall strategy of the proof. We will reduce to the case where the image of f is open and f has a section over this open by repeatedly applying the above remark. Each step is straightforward, but there are quite a few of them which makes the proof a bit involved.

Using Limits of Spaces, Proposition 70.16.1 we reduce to the case where Y is a scheme. (Let $Y' \to Y$ be a finite surjective morphism where Y' is a scheme. Set $\mathcal{X}' = \mathcal{X}_{Y'}$ and apply the initial remark of the proof.)

Using Lemma 106.9.1 (and Morphisms of Stacks, Lemma 101.28.8 to see that a gerbe is flat and locally of finite presentation) we reduce to the case where f is flat and of finite presentation.

Since f is flat and locally of finite presentation, we see that the image of $|f|$ is an open $W \subset Y$. Since $\mathcal{X}$ is quasi-compact (as f is of finite type and Y is quasi-compact) we see that W is quasi-compact. By Lemma 106.10.1 we can find a finite surjective morphism $g : Y' \to Y$ such that $g^{-1}(W) \to Y$ factors Zariski locally through $\mathcal{X} \to Y$. After replacing Y by Y' and $\mathcal{X}$ by $\mathcal{X} \times_Y Y'$ we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, and (c) W is the image of $|f|$. We will use induction on n. The base case is $n = 0$: this implies $V = \emptyset$ and in this case we can take $\overline{Z} = \emptyset$. If $n > 0$, then for $i = 1, \ldots, n$ consider the reduced closed subschemes Y_i with underlying topological space $Y \setminus W_i$. Consider the finite morphism

$$Y' = Y \amalg \coprod\nolimits_{i=1,\ldots,n} Y_i \longrightarrow Y$$

and the quasi-compact open

$$V' = (W_1 \cap \ldots \cap W_n \cap V) \amalg \coprod_{i=1,\ldots,n} (V \cap Y_i).$$

By the initial remark of the proof, if we can prove the lemma for the pairs

$$(\mathcal{X} \to Y, W_1 \cap \ldots \cap W_n \cap V) \quad \text{and} \quad (\mathcal{X} \times_Y Y_i \to Y_i, V \cap Y_i), \quad i = 1, \ldots, n$$

then the result is true. Here we use the settheoretic equality $V = (W_1 \cap \ldots \cap W_n \cap V) \cup \bigcup_{i=1,\ldots n}(V \cap Y_i)$. The induction hypothesis applies to the second type of pairs above. Hence we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, and (d) $V \subset W_1 \cap \ldots \cap W_n$. The morphisms

$$T_{ij} = \mathit{Isom}_{\mathcal{X}}(x_i|_{W_i \cap W_j \cap V}, x_j|_{W_i \cap W_j \cap V}) \longrightarrow W_i \cap W_j \cap V$$

are surjective, flat, and locally of finite presentation (Morphisms of Stacks, Lemma 101.28.10). We apply Lemma 106.10.1 to each quasi-compact open $W_i \cap W_j \cap V$ and the morphisms $T_{ij} \to W_i \cap W_j \cap V$ to get finite surjective morphisms $Y'_{ij} \to Y$. After replacing Y by the fibre product of all of the Y'_{ij} over Y we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (e) x_i and x_j are Zariski

locally isomorphic over $W_i \cap W_j \cap V$. Let $y \in V$ be arbitrary. Suppose that we can find a quasi-compact open neighbourhood $y \in V_y \subset V$ such that the lemma is true for the pair $(\mathcal{X} \to Y, V_y)$, say with solution $\overline{Z}_y, Z_y, \overline{g}_y, g_y, h_y$. Since V is quasi-compact, we can find a finite number $y_1, \ldots, y_m$ such that $V = V_{y_1} \cup \ldots \cup V_{y_m}$. Then it follows that setting

$$\overline{Z} = \coprod \overline{Z}_{y_j}, \quad Z = \coprod Z_{y_j}, \quad \overline{g} = \coprod \overline{g}_{y_j}, \quad g = \coprod g_{y_j}, \quad h = \coprod h_{y_j}$$

is a solution for the lemma. Given y by condition (e) we can choose a quasi-compact open neighbourhood $y \in V_y \subset V$ and isomorphisms $\varphi_i : x_1|_{V_y} \to x_i|_{V_y}$ for $i = 2, \ldots, n$. Set $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. This leads us to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (f) there are isomorphisms $\varphi_{ij} : x_i|_V \to x_j|_V$ satisfying $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. The morphisms

$$I_{ij} = Isom_{\mathcal{X}}(x_i|_{W_i \cap W_j}, x_j|_{W_i \cap W_j}) \longrightarrow W_i \cap W_j$$

are proper because f is separated (Morphisms of Stacks, Lemma 101.6.6). Observe that φ_{ij} defines a section $V \to I_{ij}$ of $I_{ij} \to W_i \cap W_j$ over V. By More on Morphisms of Spaces, Lemma 76.39.6 we can find V-admissible blowups $p_{ij} : Y_{ij} \to Y$ such that s_{ij} extends to $p_{ij}^{-1}(W_i \cap W_j)$. After replacing Y by the fibre product of all the Y_{ij} over Y we get to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (g) there are isomorphisms $\varphi_{ij} : x_i|_{W_i \cap W_j} \to x_j|_{W_i \cap W_j}$ satisfying

$$\varphi_{jk}|_V \circ \varphi_{ij}|_V = \varphi_{ik}|_V.$$

After replacing Y by another V-admissible blowup if necessary we may assume that V is dense and scheme theoretically dense in Y and hence in any open subspace of Y containing V. After such a replacement we conclude that

$$\varphi_{jk}|_{W_i \cap W_j \cap W_k} \circ \varphi_{ij}|_{W_i \cap W_j \cap W_k} = \varphi_{ik}|_{W_i \cap W_j \cap W_k}$$

by appealing to Morphisms of Spaces, Lemma 67.17.8 and the fact that $I_{ik} \to W_i \cap W_j$ is proper (hence separated). Of course this means that (x_i, φ_{ij}) is a desent datum and we obtain a morphism $x : W \to \mathcal{X}$ agreeing with x_i over W_i because $\mathcal{X}$ is a stack. Since x is a section of the separated morphism $\mathcal{X} \to W$ we see that x is proper (Morphisms of Stacks, Lemma 101.4.9). Thus the lemma now holds with $\overline{Z} = Y$, $Z = W$, $\overline{g} = \mathrm{id}_Y$, $g = \mathrm{id}_W$, $h = x$. □

0CQ8 **Theorem 106.10.3** (Chow's lemma)**.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume*

(1) *Y is quasi-compact and quasi-separated,*
(2) *f is separated of finite type.*

This is a result due to Ofer Gabber, see [**Ols05**, Theorem 1.1]

Then there exists a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \longleftarrow & X & \longrightarrow & \overline{X} \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X \to \mathcal{X}$ is proper surjective, $X \to \overline{X}$ is an open immersion, and $\overline{X} \to Y$ is proper morphism of algebraic spaces.

Proof. The rough idea is to use that $\mathcal{X}$ has a dense open which is a gerbe (Morphisms of Stacks, Proposition 101.29.1) and appeal to Lemma 106.10.2. The reason this does not work is that the open may not be quasi-compact and one runs into technical problems. Thus we first do a (standard) reduction to the Noetherian case.

First we choose a closed immersion $\mathcal{X} \to \mathcal{X}'$ where $\mathcal{X}'$ is an algebraic stack separated and of finite type over Y. See Limits of Stacks, Lemma 102.6.2. Clearly it suffices to prove the theorem for $\mathcal{X}'$, hence we may assume $\mathcal{X} \to Y$ is separated and of finite presentation.

Assume $\mathcal{X} \to Y$ is separated and of finite presentation. By Limits of Spaces, Proposition 70.8.1 we can write $Y = \lim Y_i$ as the directed limit of a system of Noetherian algebraic spaces with affine transition morphisms. By Limits of Stacks, Lemma 102.5.1 there is an i and a morphism $\mathcal{X}_i \to Y_i$ of finite presentation from an algebraic stack to Y_i such that $\mathcal{X} = Y \times_{Y_i} \mathcal{X}_i$. After increasing i we may assume that $\mathcal{X}_i \to Y_i$ is separated, see Limits of Stacks, Lemma 102.4.2. Then it suffices to prove the theorem for $\mathcal{X}_i \to Y_i$. This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian. We may replace $\mathcal{X}$ by its reduction (Properties of Stacks, Definition 100.10.4). This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian and $\mathcal{X}$ is reduced. Since $\mathcal{X} \to Y$ is separated and Y quasi-separated, we see that $\mathcal{X}$ is quasi-separated as an algebraic stack. Hence the inertia $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Thus by Morphisms of Stacks, Proposition 101.29.1 there exists a dense open substack $\mathcal{V} \subset \mathcal{X}$ which is a gerbe. Let $\mathcal{V} \to V$ be the morphism which expresses $\mathcal{V}$ as a gerbe over the algebraic space V. See Morphisms of Stacks, Lemma 101.28.2 for a construction of $\mathcal{V} \to V$. This construction in particular shows that the morphism $\mathcal{V} \to Y$ factors as $\mathcal{V} \to V \to Y$. Picture

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

Since the morphism $\mathcal{V} \to V$ is surjective, flat, and of finite presentation (Morphisms of Stacks, Lemma 101.28.8) and since $\mathcal{V} \to Y$ is locally of finite presentation, it follows that $V \to Y$ is locally of finite presentation (Morphisms of Stacks, Lemma 101.27.12). Note that $\mathcal{V} \to V$ is a universal homeomorphism (Morphisms of Stacks, Lemma 101.28.13). Since $\mathcal{V}$ is quasi-compact (see Morphisms of Stacks, Lemma 101.8.2) we see that V is quasi-compact. Finally, since $\mathcal{V} \to Y$ is separated the same is true for $V \to Y$ by Morphisms of Stacks, Lemma 101.27.17 applied to $\mathcal{V} \to V \to Y$ (whose assumptions are satisfied as we've already seen).

All of the above means that the assumptions of Limits of Spaces, Lemma 70.13.3 apply to the morphism $V \to Y$. Thus we can find a dense open subspace $V' \subset V$ and an immersion $V' \to \mathbf{P}^n_Y$ over Y. Clearly we may replace V by V' and $\mathcal{V}$ by the inverse image of V' in $\mathcal{V}$ (recall that $|\mathcal{V}| = |V|$ as we've seen above). Thus we may assume we have a diagram

$$\begin{array}{ccccc} \mathcal{V} & & \longrightarrow & & \mathcal{X} \\ \downarrow & & & & \downarrow \\ V & \longrightarrow & \mathbf{P}^n_Y & \longrightarrow & Y \end{array}$$

where the arrow $V \to \mathbf{P}^n_Y$ is an immersion. Let $\mathcal{X}'$ be the scheme theoretic image of the morphism

$$j : \mathcal{V} \longrightarrow \mathbf{P}^n_Y \times_Y \mathcal{X}$$

and let Y' be the scheme theoretic image of the morphism $V \to \mathbf{P}^n_Y$. We obtain a commutative diagram

$$\begin{array}{ccccccc} \mathcal{V} & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathbf{P}^n_Y \times_Y \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & Y' & \longrightarrow & \mathbf{P}^n_Y & \longrightarrow & Y \end{array}$$

(See Morphisms of Stacks, Lemma 101.38.4). We claim that $\mathcal{V} = V \times_{Y'} \mathcal{X}'$ and that Lemma 106.10.2 applies to the morphism $\mathcal{X}' \to Y'$ and the open subspace $V \subset Y'$. If the claim is true, then we obtain

$$\begin{array}{ccccc} \overline{X} & \longleftarrow & X & \longrightarrow & \mathcal{X}' \\ & \searrow^{\overline{g}} & \downarrow{\scriptstyle g} & \swarrow_{f} & \\ & & Y' & & \end{array} \qquad h : X \to \mathcal{X}'$$

with $X \to \overline{X}$ an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$. Then the composition $X \to \mathcal{X}' \to \mathcal{X}$ is proper (as a composition of proper morphisms) and its image contains $|\mathcal{V}|$, hence this composition is surjective. As well, $\overline{X} \to Y' \to Y$ is proper as a composition of proper morphisms.

The last step is to prove the claim. Observe that $\mathcal{X}' \to Y'$ is separated and of finite type, that Y' is quasi-compact and quasi-separated, and that V is quasi-compact (we omit checking all the details completely). Next, we observe that $b : \mathcal{X}' \to \mathcal{X}$ is an isomorphism over $\mathcal{V}$ by Morphisms of Stacks, Lemma 101.38.7. In particular $\mathcal{V}$ is identified with an open substack of $\mathcal{X}'$. The morphism j is quasi-compact (source is quasi-compact and target is quasi-separated), so formation of the scheme theoretic image of j commutes with flat base change by Morphisms of Stacks, Lemma 101.38.5. In particular we see that $V \times_{Y'} \mathcal{X}'$ is the scheme theoretic image of $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$. However, by Morphisms of Stacks, Lemma 101.37.5 the image of $|\mathcal{V}| \to |V \times_{Y'} \mathcal{X}'|$ is closed (use that $\mathcal{V} \to V$ is a universal homeomorphism as we've seen above and hence is universally closed). Also the image is dense (combine what we just said with Morphisms of Stacks, Lemma 101.38.6) we conclude $|\mathcal{V}| = |V \times_{Y'} \mathcal{X}'|$. Thus $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$ is an isomorphism and the proof of the claim is complete. $\square$

106.11. Noetherian valuative criterion

0CQL In this section we will discuss (refined) valuative criteria for morphisms of algebraic stacks using only discrete valuation rings in the Noetherian setting. There are many different variants and we will add more here over time as needed.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks (or algebraic spaces or schemes). A *refined* valuative criterion is one where we are given a morphism $\mathcal{U} \to \mathcal{X}$ (with some properties) and we only look at existence or uniqueness of dotted arrows in solid diagrams of the form

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{X} \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & \mathcal{Y} \end{array}$$

We use this terminology below to describe the results we have obtained sofar.

Non-Noetherian valuative criteria for morphisms of algebraic stacks

(1) Morphisms of Stacks, Section 101.40 (for separatedness of the diagonal),
(2) Morphisms of Stacks, Section 101.41 (for separatedness),
(3) Morphisms of Stacks, Section 101.42 (for universal closedness),
(4) Morphisms of Stacks, Section 101.43 (for properness).

For algebraic spaces we have the following valuative criteria

(1) Morphisms of Spaces, Section 67.42 (for universal closedness),
(2) Morphisms of Spaces, Lemma 67.42.5 (refined for universal closedness)
(3) Morphisms of Spaces, Section 67.43 (for separatedness),
(4) Morphisms of Spaces, Section 67.44 (for properness),
(5) Decent Spaces, Section 68.16 (for universal closedness for decent spaces),
(6) Decent Spaces, Lemma 68.17.11 (for universal closedness for decent morphisms between algebraic spaces),
(7) Cohomology of Spaces, Section 69.19 contains Noetherian valuative criteria
    (a) Cohomology of Spaces, Lemma 69.19.1 (for separatedness using discrete valuation rings),
    (b) Cohomology of Spaces, Lemma 69.19.2 (for properness using discrete valuation rings),
    (c) Cohomology of Spaces, Remark 69.19.3 (discusses how to reduce to complete discrete valuation rings),
(8) Limits of Spaces, Section 70.21 discussing Noetherian valuative criteria
    (a) Limits of Spaces, Lemma 70.21.2 (for separatedness using discrete valuation rings and generic points)
    (b) Limits of Spaces, Lemma 70.21.3 (for properness using discrete valuation rings and generic points)
    (c) Limits of Spaces, Lemma 70.21.4 (for universal closedness using discrete valuation rings).
(9) Limits of Spaces, Section 70.22 discussing refined Noetherian valuative criteria
    (a) Limits of Spaces, Lemmas 70.22.1 and 70.22.3 (refined for properness using discrete valuation rings),

(b) Limits of Spaces, Lemma 70.22.2 (refined for separatedness using discrete valuation rings),

For schemes we have the following valuative criteria

(1) Schemes, Section 26.20 (for universal closedness)
(2) Schemes, Section 26.22 (for separatedness),
(3) Morphisms, Section 29.42 (for properness)
(4) Morphisms, Lemma 29.42.2 (refined for universal closedness),
(5) Limits, Section 32.15 discussing Noetherian valuative criteria
    (a) Limits, Lemma 32.15.2 (for separatedness using discrete valuation rings and generic points)
    (b) Limits, Lemma 32.15.3 (for properness using discrete valuation rings and generic points)
    (c) Limits, Lemma 32.15.4 (for universal closedness using discrete valuation rings).
(6) Limits, Section 32.16 discussing refined Noetherian valuative criteria
    (a) Limits, Lemmas 32.16.1 and 32.16.3 (refined for properness using discrete valuation rings),
    (b) Limits, Lemma 32.16.2 (refined for separatedness using discrete valuation rings),
(7) Limits, Section 32.17 discussing valuative criteria over a Noetherian base where one can get discrete valuation rings essentially of finite type over the base.

This ends our list of previous results.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0H2B **Lemma 106.11.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f finite type and $\mathcal{Y}$ locally Noetherian. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & \mathcal{Y} \end{array}$$

of algebraic stacks where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ *to y.*

Proof. Choose an affine scheme V, a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ mapping v to y. The map $|V| \to |\mathcal{Y}|$ is open and by Properties of Stacks, Lemma 100.4.3 the image of $|\mathcal{X} \times_{\mathcal{Y}} V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|\mathcal{X} \times_{\mathcal{Y}} V| \to |V|$. If we prove the lemma for $\mathcal{X} \times_{\mathcal{Y}} V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : \mathcal{X} \to Y$ and $y \in |Y|$ as in the lemma where Y is a Noetherian affine scheme. Since f is quasi-compact, we conclude that $\mathcal{X}$ is quasi-compact. Hence we can choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the same as the image of $|W| \to |Y|$. In this way we reduce to the case of schemes which is Limits, Lemma 32.15.1. □

0E80 **Lemma 106.11.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *$\mathcal{Y}$ is locally Noetherian,*
(2) *f is locally of finite type and quasi-separated,*
(3) *for every commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where A is a discrete valuation ring and K its fraction field and any 2-arrow $\gamma : y \circ j \to f \circ x$ the category of dotted arrows (Morphisms of Stacks, Definition 101.39.1) is either empty or a setoid with exactly one isomorphism class.

Then f is separated.

Proof. To prove that f is separated we have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper. We already know that Δ is representable by algebraic spaces, locally of finite type (Morphisms of Stacks, Lemma 101.3.3) and quasi-compact and quasi-separated (by definition of f being quasi-separated). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Set

$$V = \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} U$$

It suffices to show that the morphism of algebraic spaces $V \to U$ is proper (Properties of Stacks, Lemma 100.3.3). Observe that U is locally Noetherian (use Morphisms of Stacks, Lemma 101.17.5 and the fact that $U \to \mathcal{Y}$ is locally of finite type) and $V \to U$ is of finite type and quasi-separated (as the base change of Δ and properties of Δ listed above). Applying Cohomology of Spaces, Lemma 69.19.2 it suffices to show: Given a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{v} & V & \longrightarrow & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle g} & \cdots\nearrow & \downarrow{\scriptstyle \Delta} \\ \mathrm{Spec}(A) & \xrightarrow{u} & U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dashed arrow making the diagram commute. By Morphisms of Stacks, Lemma 101.39.4 the categories of dashed and dotted arrows are equivalent. Assumption (3) implies there is a unique dotted arrow up to isomorphism, see Morphisms of Stacks, Lemma 101.41.1. We conclude there is a unique dashed arrow as desired. □

0CQM **Lemma 106.11.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{u} & \mathcal{U} & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle j}\downarrow & & & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \xrightarrow{y} & & \mathcal{Y} \end{array}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathcal{X} \\ {\scriptstyle j'}\downarrow & \overset{x'}{\nearrow} & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A') & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 101.39.1), then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. By Properties of Stacks, Lemma 100.4.3 the image I of $|\mathcal{U} \times_{\mathcal{Y}} V| \to |\mathcal{X} \times_{\mathcal{Y}} V|$ is the inverse image of the image of $|h|$. Since $|\mathcal{X} \times_{\mathcal{Y}} V| \to |\mathcal{X}|$ is open (Morphisms of Stacks, Lemma 101.27.15) we conclude that I is dense in $|\mathcal{X} \times_{\mathcal{Y}} V|$. Also since the category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 101.39.4) the assumption on existence of dotted arrows (after extension) is inherited by the morphisms $\mathcal{U} \times_{\mathcal{Y}} V \to \mathcal{X} \times_{\mathcal{Y}} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphisms $\mathcal{U} \times_{\mathcal{Y}} V \to \mathcal{X} \times_{\mathcal{Y}} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is an affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 101.39.3.) Then $\mathcal{U}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{U}$. Then we may and do replace $\mathcal{U}$ by U. Thus we may assume that $\mathcal{U}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ and $\mathcal{U} = U$ are affine schemes. By Chow's lemma (Theorem 106.10.3) we can choose a surjective proper morphism $X \to \mathcal{X}$ where X is an algebraic space. We will use below that $X \to Y$ is separated as a composition of separated morphisms. Consider the algebraic space $W = X \times_{\mathcal{X}} U$. The projection morphism $W \to X$ is of finite type. We may replace X by the scheme theoretic image of $W \to X$ and hence we may assume that the image of $|W|$ in $|X|$ is dense in $|X|$ (here we use that the image of $|h|$ is dense in $|\mathcal{X}|$, so after this replacement, the morphism $X \to \mathcal{X}$ is still surjective). We claim that for every solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & W & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. First, it is enough to prove there exists a dotted arrow after replacing K by an extension and A by a valuation ring in this extension dominating A, see Morphisms of Spaces, Lemma 67.41.4. By the assumption of the lemma we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and an arrow $\mathrm{Spec}(A') \to \mathcal{X}$ lifting the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to Y$

and compatible with the composition $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \to W \to X$. Because $X \to \mathcal{X}$ is proper, we can use the valuative criterion of properness (Morphisms of Stacks, Lemma 101.43.1) to find an extension K''/K' and a valuation ring $A'' \subset K''$ dominating A' and a morphism $\mathrm{Spec}(A'') \to X$ lifting the composition $\mathrm{Spec}(A'') \to \mathrm{Spec}(A') \to \mathcal{X}$ and compatible with the composition $\mathrm{Spec}(K'') \to \mathrm{Spec}(K') \to \mathrm{Spec}(K) \to X$. Then K''/K and $A'' \subset K''$ and the morphism $\mathrm{Spec}(A'') \to X$ is a solution to the problem posed above and the claim holds.

The claim implies the morphism $X \to Y$ is proper by the case of the lemma for algebraic spaces (Limits of Spaces, Lemma 70.22.1). By Morphisms of Stacks, Lemma 101.37.6 we conclude that $\mathcal{X} \to Y$ is proper as desired. □

0E95 **Lemma 106.11.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{u} & \mathcal{U} & \longrightarrow & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow_{h} & & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & & \xrightarrow{y} & & \mathcal{Y} \end{array}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$, the category of dotted arrows is either empty or a setoid with exactly one isomorphism class, then f is separated.

Proof. We have to prove Δ is a proper morphism. Assume first that Δ is separated. Then we may apply Lemma 106.11.3 to the morphisms $\mathcal{U} \to \mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Observe that Δ is quasi-compact as f is quasi-separated. Of course Δ is locally of finite type (true for any diagonal morphism, see Morphisms of Stacks, Lemma 101.3.3). Finally, suppose given a 2-commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{u} & \mathcal{U} & \longrightarrow & \mathcal{X} \\ {\scriptstyle j}\downarrow & \nearrow_{h} & & & \downarrow{\scriptstyle \Delta} \\ \mathrm{Spec}(A) & & \xrightarrow{y} & & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to \Delta \circ h \circ u$. By Morphisms of Stacks, Lemma 101.41.1 and the assumption in the lemma we find there exists a unique dotted arrow. This proves the last assumption of Lemma 106.11.3 holds and the result follows.

In the general case, it suffices to prove Δ is separated since then we'll be back in the previous case. In fact, we claim that the assumptions of the lemma hold for

$$\mathcal{U} \to \mathcal{X} \quad \text{and} \quad \Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Namely, since Δ is representable by algebraic spaces, the category of dotted arrows for a diagram as in the previous paragraph is a setoid (see for example Morphisms of Stacks, Lemma 101.39.2). The argument in the preceding paragraph shows these categories are either empty or have one isomorphism class. Thus Δ is separated. □

0H2C **Lemma 106.11.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian and that f is of finite type. If given any 2-commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ x$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \xrightarrow{} & \mathcal{X} \\ {\scriptstyle j'}\downarrow & \overset{x'}{\nearrow} & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A') & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 101.39.1), then f is universally closed.

Proof. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. The category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 101.39.4). Hence the assumption on existence of dotted arrows (after extension) is inherited by the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is a Noetherian affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 101.39.3.) To prove that f is universally closed it suffices to show that $|\mathcal{X} \times \mathbf{A}^n| \to |Y \times \mathbf{A}^n|$ is closed for all n by Limits of Stacks, Lemma 102.7.2. Since the assumption in the lemma is inherited by the product morphism $\mathcal{X} \times \mathbf{A}^n \to Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|\mathcal{X}| \to |Y|$ is closed.

Assume Y is a Noetherian affine scheme. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace $\mathcal{X}$ by the reduced induced closed subspace structure on T; we omit the verification that property on the existence of dotted arrows is preserved by this replacement. Thus we reduce to proving that the image of $|\mathcal{X}| \to |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|\mathcal{X}| \to |Y|$. By Lemma 106.11.1 we may choose a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. It follows immediately from the assumption in the lemma that y is in the image of $|\mathcal{X}| \to |Y|$ and the proof is complete. □

106.12. Moduli spaces

0DUF This section discusses morphisms $f : \mathcal{X} \to Y$ from algebraic stacks to algebraic spaces. Under suitable hypotheses Y is called a *moduli space* for $\mathcal{X}$. If $\mathcal{X} = [U/R]$ is a presentation, then we obtain an R-invariant morphism $U \to Y$ and (under suitable hypotheses) Y is a *quotient* of the groupoid (U, R, s, t, c). A discussion of the different types of quotients can be found starting with Quotients of Groupoids, Section 83.1.

0DUG **Definition 106.12.1.** Let $\mathcal{X}$ be an algebraic stack. Let $f : \mathcal{X} \to Y$ be a morphism to an algebraic space Y.

(1) We say f is a *categorical moduli space* if any morphism $\mathcal{X} \to W$ to an algebraic space W factors uniquely through f.
(2) We say f is a *uniform categorical moduli space* if for any flat morphism $Y' \to Y$ of algebraic spaces the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space.

Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces.

(3) We say f is a *categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Ob}(\mathcal{C})$ and any morphism $\mathcal{X} \to W$ with $W \in \mathrm{Ob}(\mathcal{C})$ factors uniquely through f.
(4) We say is a *uniform categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Ob}(\mathcal{C})$ and for every flat morphism $Y' \to Y$ in $\mathcal{C}$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space in $\mathcal{C}$.

By the Yoneda lemma a categorical moduli space, if it exists, is unique. Let us match this with the language introduced for quotients.

0DUH **Lemma 106.12.2.** *Let (U, R, s, t, c) be a groupoid in algebraic spaces with $s, t : R \to U$ flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$. Given an algebraic space Y there is a 1-to-1 correspondence between morphisms $f : \mathcal{X} \to Y$ and R-invariant morphisms $\phi : U \to Y$.*

Proof. Criteria for Representability, Theorem 97.17.2 tells us $\mathcal{X}$ is an algebraic stack. Given a morphism $f : \mathcal{X} \to Y$ we let $\phi : U \to Y$ be the composition $U \to \mathcal{X} \to Y$. Since $R = U \times_{\mathcal{X}} U$ (Groupoids in Spaces, Lemma 78.22.2) it is immediate that ϕ is R-invariant. Conversely, if $\phi : U \to Y$ is an R-invariant morphism towards an algebraic space, we obtain a morphism $f : \mathcal{X} \to Y$ by Groupoids in Spaces, Lemma 78.23.2. You can also construct f from ϕ using the explicit description of the quotient stack in Groupoids in Spaces, Lemma 78.24.1. □

0DUI **Lemma 106.12.3.** *With assumption and notation as in Lemma 106.12.2. Then f is a (uniform) categorical moduli space if and only if ϕ is a (uniform) categorical quotient. Similarly for moduli spaces in a full subcategory.*

Proof. It is immediate from the 1-to-1 correspondence established in Lemma 106.12.2 that f is a categorical moduli space if and only if ϕ is a categorical quotient (Quotients of Groupoids, Definition 83.4.1). If $Y' \to Y$ is a morphism, then $U' = Y' \times_Y U \to Y' \times_Y \mathcal{X} = \mathcal{X}'$ is a surjective, flat, locally finitely presented morphism as a base change of $U \to \mathcal{X}$ (Criteria for Representability, Lemma 97.17.1). And $R' = Y' \times_Y R$ is equal to $U' \times_{\mathcal{X}'} U'$ by transitivity of fibre products. Hence $\mathcal{X}' = [U'/R']$, see Algebraic Stacks, Remark 94.16.3. Thus the base change of our situation to Y' is another situation as in the statement of the lemma. From this it

immediately follows that f is a uniform categorical moduli space if and only if ϕ is a uniform categorical quotient. □

0DUJ **Lemma 106.12.4.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. If for every affine scheme Y' and flat morphism $Y' \to Y$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space, then f is a uniform categorical moduli space.*

Proof. Choose an étale covering $\{Y_i \to Y\}$ where Y_i is an affine scheme. For each i and j choose a affine open covering $Y_i \times_Y Y_j = \bigcup Y_{ijk}$. Set $\mathcal{X}_i = Y_i \times_Y \mathcal{X}$ and $\mathcal{X}_{ijk} = Y_{ijk} \times_Y \mathcal{X}$. Let $g : \mathcal{X} \to W$ be a morphism towards an algebraic space. Then we consider the diagram

$$\begin{array}{ccccc} \mathcal{X}_i & \longrightarrow & \mathcal{X} & \xrightarrow{g} & W \\ \downarrow & & \downarrow & \nearrow & \\ Y_i & \longrightarrow & Y & & \end{array}$$

The assumption that $\mathcal{X}_i \to Y_i$ is a categorical moduli space, produces a unique dotted arrow $h_i : Y_i \to W$. The assumption that $\mathcal{X}_{ijk} \to Y_{ijk}$ is a categorical moduli space, implies the restriction of h_i and h_j to Y_{ijk} are equal. Hence h_i and h_j agree on $Y_i \times_Y Y_j$. Since $Y = \coprod Y_i / \coprod Y_i \times_Y Y_j$ (by Spaces, Section 65.9) we conclude that there is a unique morphism $Y \to W$ through which g factors. Thus f is a categorical moduli space. The same argument applies after a flat base change, hence f is a uniform categorical moduli space. □

106.13. The Keel-Mori theorem

0DUK In this section we start discussing the theorem of Keel and Mori in the setting of algebraic stacks. For a discussion of the literature, please see Guide to Literature, Subsection 112.5.2.

0DUL **Definition 106.13.1.** Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *well-nigh affine* if there exists an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$.

We give this property a somewhat ridiculous name because we do not intend to use it too much.

0DUM **Lemma 106.13.2.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is well-nigh affine, and*
(2) *there exists a groupoid scheme (U, R, s, t, c) with U and R affine and $s, t : R \to U$ finite locally free such that $\mathcal{X} = [U/R]$.*

If true then $\mathcal{X}$ is quasi-compact, quasi-DM, and separated.

Proof. Assume $\mathcal{X}$ is well-nigh affine. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. Then we obtain a groupoid (U, R, s, t, c) in algebraic spaces and an isomorphism $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 94.16.1 and Remark 94.16.3. Since $s, t : R \to U$ are flat, finite, and finitely presented morphisms (as base changes of $U \to \mathcal{X}$) we see that s, t are finite locally free (Morphisms, Lemma 29.48.2). This implies that R is affine (as finite morphisms are affine) and hence (2) holds.

Suppose that we have a groupoid scheme (U, R, s, t, c) with U and R are affine and $s, t : R \to U$ finite locally free. Set $\mathcal{X} = [U/R]$. Then $\mathcal{X}$ is an algebraic stack by Criteria for Representability, Theorem 97.17.2 (strictly speaking we don't need this here, but it can't be stressed enough that this is true). The morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 97.17.1. Thus we can check whether $U \to \mathcal{X}$ is finite by checking whether the projection $U \times_{\mathcal{X}} U \to U$ has this property, see Properties of Stacks, Lemma 100.3.3. Since $U \times_{\mathcal{X}} U = R$ by Groupoids in Spaces, Lemma 78.22.2 we see that this is true. Thus $\mathcal{X}$ is well-nigh affine.

Proof of the final statement. We see that $\mathcal{X}$ is quasi-compact by Properties of Stacks, Lemma 100.6.2. We see that $\mathcal{X} = [U/R]$ is quasi-DM and separated by Morphisms of Stacks, Lemma 101.20.1. □

0DUN **Lemma 106.13.3.** *Let the algebraic stack $\mathcal{X}$ be well-nigh affine.*

(1) *If $\mathcal{X}$ is an algebraic space, then it is affine.*
(2) *If $\mathcal{X}' \to \mathcal{X}$ is an affine morphism of algebraic stacks, then $\mathcal{X}'$ is well-nigh affine.*

Proof. Part (1) follows from immediately from Limits of Spaces, Lemma 70.15.1. However, this is overkill, since (1) also follows from Lemma 106.13.2 combined with Groupoids, Proposition 39.23.9.

To prove (2) we choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_{\mathcal{X}} U$ admits an affine morphism to U (Morphisms of Stacks, Lemma 101.9.2). Therefore U' is an affine scheme. Of course $U' \to \mathcal{X}'$ is surjective, flat, finite, and finitely presented as a base change of $U \to \mathcal{X}$. □

0DUP **Lemma 106.13.4.** *Let the algebraic stack $\mathcal{X}$ be well-nigh affine. There exists a uniform categorical moduli space*

$$f : \mathcal{X} \longrightarrow M$$

in the category of affine schemes. Moreover f is separated, quasi-compact, and a universal homeomorphism.

Proof. Write $\mathcal{X} = [U/R]$ with (U, R, s, t, c) as in Lemma 106.13.2. Let C be the ring of R-invariant functions on U, see Groupoids, Section 39.23. We set $M = \mathrm{Spec}(C)$. The R-invariant morphism $U \to M$ corresponds to a morphism $f : \mathcal{X} \to M$ by Lemma 106.12.2. The characterization of morphisms into affine schemes given in Schemes, Lemma 26.6.4 immediately guarantees that $\phi : U \to M$ is a categorical quotient in the category of affine schemes. Hence f is a categorical moduli space in the category of affine schemes (Lemma 106.12.3).

Since $\mathcal{X}$ is separated by Lemma 106.13.2 we find that f is separated by Morphisms of Stacks, Lemma 101.4.12.

Since $U \to \mathcal{X}$ is surjective and since $U \to M$ is quasi-compact, we see that f is quasi-compact by Morphisms of Stacks, Lemma 101.7.6.

By Groupoids, Lemma 39.23.4 the composition

$$U \to \mathcal{X} \to M$$

is an integral morphism of affine schemes. In particular, it is universally closed (Morphisms, Lemma 29.44.7). Since $U \to \mathcal{X}$ is surjective, it follows that $\mathcal{X} \to M$ is universally closed (Morphisms of Stacks, Lemma 101.37.6). To conclude that $\mathcal{X} \to M$ is a universal homeomorphism, it is enough to show that it is universally bijective, i.e., surjective and universally injective.

We have $|\mathcal{X}| = |U|/|R|$ by Morphisms of Stacks, Lemma 101.20.2. Thus $|f|$ is surjective and even bijective by Groupoids, Lemma 39.23.6.

Let $C \to C'$ be a ring map. Let (U', R', s', t', c') be the base change of (U, R, s, t, c) by $M' = \mathrm{Spec}(C') \to M$. Setting $\mathcal{X}' = [U'/R']$, we observe that $M' \times_M \mathcal{X} = \mathcal{X}'$ by Quotients of Groupoids, Lemma 83.3.6. Let C^1 be the ring of R'-invariant functions on U'. Set $M^1 = \mathrm{Spec}(C^1)$ and consider the diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \\ M^1 & & \downarrow{\scriptstyle f} \\ \downarrow & & \\ M' & \longrightarrow & M \end{array}$$

By Groupoids, Lemma 39.23.5 and Algebra, Lemma 10.46.11 the morphism $M^1 \to M'$ is a homeomorphism. On the other hand, the previous paragraph applied to (U', R', s', t', c') shows that $|f'|$ is bijective. We conclude that f induces a bijection on points after any base change by an affine scheme. Thus f is universally injective by Morphisms of Stacks, Lemma 101.14.7.

Finally, we still have to show that f is a uniform moduli space in the category of affine schemes. This follows from the discussion above and the fact that if the ring map $C \to C'$ is flat, then $C' \to C^1$ is an isomorphism by Groupoids, Lemma 39.23.5. □

0DUQ **Lemma 106.13.5.** *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}'$ and $\mathcal{X}$ are well-nigh affine, h is étale, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Then there exists a cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

where $M' \to M$ is étale and the vertical arrows are the moduli spaces constructed in Lemma 106.13.4.

Proof. Observe that h is representable by algebraic spaces by Morphisms of Stacks, Lemmas 101.45.3 and 101.45.1. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_{\mathcal{X}} U$ is an algebraic space with a finite (in particular affine) morphism $U' \to \mathcal{X}'$. By Lemma 106.13.3 we conclude that U' is affine. Setting $R = U \times_{\mathcal{X}} U$ and $R' = U' \times_{\mathcal{X}'} U'$ we obtain groupoids (U, R, s, t, c) and (U', R', s', t', c') such that $\mathcal{X} = [U/R]$ and $\mathcal{X}' = [U'/R']$,

see proof of Lemma 106.13.2. we see that the diagrams

$$\begin{array}{ccc} R' & \xrightarrow{f} & R \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ U' & \xrightarrow{f} & U \end{array} \qquad \begin{array}{ccc} R' & \xrightarrow{f} & R \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ U' & \xrightarrow{f} & U \end{array} \qquad \begin{array}{ccc} G' & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ U' & \xrightarrow{f} & U \end{array}$$

are cartesian where G and G' are the stabilizer group schemes. This follows for the first two by transitivity of fibre products and for the last one this follows because it is the pullback of the isomorphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$ (by the already used Morphisms of Stacks, Lemma 101.45.3). Recall that M, resp. M' was constructed in Lemma 106.13.4 as the spectrum of the ring of R-invariant functions on U, resp. the ring of R'-invariant functions on U'. Thus $M' \to M$ is étale and $U' = M' \times_M U$ by Groupoids, Lemma 39.23.7. It follows that $R' = M' \times_M U$, in other words the groupoid (U', R', s', t', c') is the base change of (U, R, s, t, c) by $M' \to M$. This implies that the diagram in the lemma is cartesian by Quotients of Groupoids, Lemma 83.3.6. □

0DUR **Lemma 106.13.6.** *Let the algebraic stack $\mathcal{X}$ be well-nigh affine. The morphism*

$$f : \mathcal{X} \longrightarrow M$$

of Lemma 106.13.4 is a uniform categorical moduli space.

Proof. We already know that M is a uniform categorical moduli space in the category of affine schemes. By Lemma 106.12.4 it suffices to show that the base change $f' : M' \times_M \mathcal{X} \to M'$ is a categorical moduli space for any flat morphism $M' \to M$ of affine schemes. Observe that $\mathcal{X}' = M' \times_M \mathcal{X}$ is well-nigh affine by Lemma 106.13.3. This after replacing $\mathcal{X}$ by $\mathcal{X}'$ and M by M', we reduce to proving f is a categorical moduli space.

Let $g : \mathcal{X} \to Y$ be a morphism where Y is an algebraic space. We have to show that $g = h \circ f$ for a unique morphism $h : M \to Y$.

Uniqueness. Suppose we have two morphisms $h_i : M \to Y$ with $g = h_1 \circ f = h_2 \circ f$. Let $M' \subset M$ be the equalizer of h_1 and h_2. Then $M' \to M$ is a monomorphism and $f : \mathcal{X} \to M$ factors through M'. Thus $M' \to M$ is a universal homeomorphism. We conclude M' is affine (Morphisms, Lemma 29.45.5). But then since $f : \mathcal{X} \to M$ is a categorical moduli space in the category of affine schemes, we see $M' = M$.

Existence. Below we will show that for every $p \in M$ there exists a cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

with $M' \to M$ an étale morphism of affines and p in the image such that the composition $\mathcal{X}' \to \mathcal{X} \to Y$ factors through M'. This means we can construct the map $h : M \to Y$ étale locally on M. Since Y is a sheaf for the étale topology and by the uniqueness shown above, this is enough (small detail omitted).

Let $y \in |Y|$ be the image of p. Let $(V, v) \to (Y, y)$ be an étale morphism with V affine. Consider $\mathcal{X}' = V \times_Y \mathcal{X}$. Observe that $\mathcal{X}' \to \mathcal{X}$ is separated and étale as the base change of $V \to Y$. Moreover, $\mathcal{X}' \to \mathcal{X}$ induces isomorphisms on automorphism

groups (Morphisms of Stacks, Remark 101.19.5) as this is true for $V \to Y$, see Morphisms of Stacks, Lemma 101.45.5. Choose a presentation $\mathcal{X} = [U/R]$ as in Lemma 106.13.2. Set $U' = \mathcal{X}' \times_{\mathcal{X}} U = V \times_Y U$ and choose $u' \in U'$ mapping to p and v (possible by Properties of Spaces, Lemma 66.4.3). Since $U' \to U$ is separated and étale we see that every finite set of points of U' is contained in an affine open, see More on Morphisms, Lemma 37.45.1. On the other hand, the morphism $U' \to \mathcal{X}'$ is surjective, finite, flat, and locally of finite presentation. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 39.24.1 there exists an R'-invariant affine open subscheme $U'' \subset U'$ containing u'. Let $\mathcal{X}'' \subset \mathcal{X}'$ be the corresponding open substack. Then $\mathcal{X}''$ is well-nigh affine. By Lemma 106.13.5 we obtain a cartesian square

$$\begin{array}{ccc} \mathcal{X}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M'' & \longrightarrow & M \end{array}$$

with $M'' \to M$ étale. Since $\mathcal{X}'' \to M''$ is a categorical moduli space in the category of affine schemes we obtain a morphism $M'' \to V$ such that the composition $\mathcal{X}'' \to \mathcal{X}' \to V$ is equal to the composition $\mathcal{X}'' \to M'' \to V$. This proves our claim and finishes the proof. □

0DUS **Lemma 106.13.7.** *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}$ is well-nigh affine, h is étale, h is separated, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Then there exists a cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

where $M' \to M$ is a separated étale morphism of schemes and $\mathcal{X} \to M$ is the moduli space constructed in Lemma 106.13.4.

Proof. Choose an affine scheme U and a surjective, flat, finite, and locally finitely presented morphism $U \to \mathcal{X}$. Since h is representable by algebraic spaces (Morphisms of Stacks, Lemmas 101.45.3 and 101.45.1) we see that $U' = \mathcal{X}' \times_{\mathcal{X}} U$ is an algebraic space. Since $U' \to U$ is separated and étale, we see that U' is a scheme and that every finite set of points of U' is contained in an affine open, see Morphisms of Spaces, Lemma 67.51.1 and More on Morphisms, Lemma 37.45.1. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 39.24.1 there exists an open covering $U' = \bigcup U'_i$ by R'-invariant affine open subschemes $U'_i \subset U'$. Let $\mathcal{X}'_i \subset \mathcal{X}'$ be the corresponding open substacks. These are well-nigh affine as $U'_i \to \mathcal{X}'_i$ is surjective, flat, finite and of finite presentation. By Lemma 106.13.5 we obtain cartesian diagrams

$$\begin{array}{ccc} \mathcal{X}'_i & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M'_i & \longrightarrow & M \end{array}$$

with $M_i' \to M$ an étale morphism of affine schemes and vertical arrows as in Lemma 106.13.4. Observe that $\mathcal{X}'_{ij} = \mathcal{X}'_i \cap \mathcal{X}'_j$ is an open subspace of $\mathcal{X}'_i$ and $\mathcal{X}'_j$. Hence we get corresponding open subschemes $V_{ij} \subset M_i'$ and $V_{ji} \subset M_j'$. By the result of Lemma 106.13.6 we see that both $\mathcal{X}'_{ij} \to V_{ij}$ and $\mathcal{X}'_{ji} \to V_{ji}$ are categorical moduli spaces! Thus we get a unique isomorphism $\varphi_{ij} : V_{ij} \to V_{ji}$ such that

$$\begin{array}{ccccccc} \mathcal{X}'_i & \longleftarrow & \mathcal{X}'_i \cap \mathcal{X}'_j & & \longrightarrow & & \mathcal{X}'_j \\ \downarrow & & \swarrow & \searrow & & & \downarrow \\ M'_i & \longleftarrow & V_{ij} & \xrightarrow{\varphi_{ij}} & V_{ji} & \longrightarrow & M'_j \end{array}$$

is commutative. These isomorphisms satisfy the cocyclce condition of Schemes, Section 26.14 by a computation (and another application of the previous lemma) which we omit. Thus we can glue the affine schemes in to scheme M', see Schemes, Lemma 26.14.1. Let us identify the M_i' with their image in M'. We claim there is a morphism $\mathcal{X}' \to M'$ fitting into cartesian diagrams

$$\begin{array}{ccc} \mathcal{X}'_i & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ M'_i & \longrightarrow & M' \end{array}$$

This is clear from the description of the morphisms into the glued scheme M' in Schemes, Lemma 26.14.1 and the fact that to give a morphism $\mathcal{X}' \to M'$ is the same thing as given a morphism $T \to M'$ for any morphism $T \to \mathcal{X}'$. Similarly, there is a morphism $M' \to M$ restricting to the given morphisms $M_i' \to M$ on M_i'. The morphism $M' \to M$ is étale (being étale on the members of an étale covering) and the fibre product property holds as it can be checked on members of the (affine) open covering $M' = \bigcup M_i'$. Finally, $M' \to M$ is separated because the composition $U' \to \mathcal{X}' \to M'$ is surjective and universally closed and we can apply Morphisms, Lemma 29.41.11. □

0DUE **Lemma 106.13.8.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exist a set I and for $i \in I$ a morphism of algebraic stacks*

$$g_i : \mathcal{X}_i \longrightarrow \mathcal{X}$$

with the following properties

(1) $|\mathcal{X}| = \bigcup |g_i|(|\mathcal{X}_i|)$,
(2) *$\mathcal{X}_i$ is well-nigh affine,*
(3) *$\mathcal{I}_{\mathcal{X}_i} \to \mathcal{X}_i \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism, and*
(4) *$g_i : \mathcal{X}_i \to \mathcal{X}$ is representable by algebraic spaces, separated, and étale,*

Proof. For any $x \in |\mathcal{X}|$ we can choose $g : \mathcal{U} \to \mathcal{X}$, $\mathcal{U} = [U/R]$, and u as in Morphisms of Stacks, Lemma 101.32.4. Then by Morphisms of Stacks, Lemma 101.45.4 we see that there exists an open substack $\mathcal{U}' \subset \mathcal{U}$ containing u such that $\mathcal{I}_{\mathcal{U}'} \to \mathcal{U}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Let $U' \subset U$ be the R-invariant open corresponding to the open substack $\mathcal{U}'$. Let $u' \in U'$ be a point of U' mapping to u. Observe that $t(s^{-1}(\{u'\}))$ is finite as $s : R \to U$ is finite. By Properties, Lemma 28.29.5 and Groupoids, Lemma 39.24.1 we can find an R-invariant affine open $U'' \subset U'$ containing u'. Let R'' be the restriction of R to U''. Then $\mathcal{U}'' = [U''/R'']$ is an open substack of $\mathcal{U}'$ containing u, is well-nigh affine, $\mathcal{I}_{\mathcal{U}''} \to \mathcal{U}'' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is

an isomorphism, and $\mathcal{U}'' \to \mathcal{X}$ and is representable by algebraic spaces and étale. Finally, $\mathcal{U}'' \to \mathcal{X}$ is separated as $\mathcal{U}''$ is separated (Lemma 106.13.2) the diagonal of $\mathcal{X}$ is separated (Morphisms of Stacks, Lemma 101.6.1) and separatedness follows from Morphisms of Stacks, Lemma 101.4.12. Since the point $x \in |\mathcal{X}|$ is arbitrary the proof is complete. □

0DUT **Theorem 106.13.9** (Keel-Mori)**.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exists a uniform categorical moduli space*

$$f : \mathcal{X} \longrightarrow M$$

and f is separated, quasi-compact, and a universal homeomorphism.

Proof. We choose a set I[4] and for $i \in I$ a morphism of algebraic stacks $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 106.13.8; we will use all of the properties listed in this lemma without further mention. Let

$$f_i : \mathcal{X}_i \to M_i$$

be as in Lemma 106.13.4. Consider the stacks

$$\mathcal{X}_{ij} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j$$

for $i, j \in I$. The projections $\mathcal{X}_{ij} \to \mathcal{X}_i$ and $\mathcal{X}_{ij} \to \mathcal{X}_j$ are separated by Morphisms of Stacks, Lemma 101.4.4, étale by Morphisms of Stacks, Lemma 101.35.3, and induce isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 101.19.5) by Morphisms of Stacks, Lemma 101.45.5. Thus we may apply Lemma 106.13.7 to find a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}_i & \longleftarrow & \mathcal{X}_{ij} & \longrightarrow & \mathcal{X}_j \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle f_{ij}}\downarrow & & {\scriptstyle f_j}\downarrow \\ M_i & \longleftarrow & M_{ij} & \longrightarrow & M_j \end{array}$$

with cartesian squares where $M_{ij} \to M_i$ and $M_{ij} \to M_j$ are separated étale morphisms of schemes; here we also use that f_i is a uniform categorical quotient by Lemma 106.13.6. Claim:

$$\coprod M_{ij} \longrightarrow \coprod M_i \times \coprod M_i$$

is an étale equivalence relation.

Proof of the claim. Set $R = \coprod M_{ij}$ and $U = \coprod M_i$. We have already seen that $t : R \to U$ and $s : R \to U$ are étale. Let us construct a morphism $c : R \times_{s,U,t} R \to R$ compatible with $\mathrm{pr}_{13} : U \times U \times U \to U \times U$. Namely, for $i, j, k \in I$ we consider

$$\mathcal{X}_{ijk} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j \times_{g_j, \mathcal{X}, g_k} \mathcal{X}_k = \mathcal{X}_{ij} \times_{\mathcal{X}_j} \mathcal{X}_{jk}$$

Arguing exactly as in the previous paragraph, we find that $M_{ijk} = M_{ij} \times_{M_j} M_{jk}$ is a categorical moduli space for $\mathcal{X}_{ijk}$. In particular, there is a canonical morphism $M_{ijk} = M_{ij} \times_{M_j} M_{jk} \to M_{ik}$ coming from the projection $\mathcal{X}_{ijk} \to \mathcal{X}_{ik}$. Putting these morphisms together we obtain the morphism c. In a similar fashion we construct a morphism $e : U \to R$ compatible with $\Delta : U \to U \times U$ and $i : R \to R$ compatible with the flip $U \times U \to U \times U$. Let k be an algebraically closed field. Then

$$\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_i) \to \mathrm{Mor}(\mathrm{Spec}(k), M_i) = M_i(k)$$

[4]The reader who is still keeping track of set theoretic issues should make sure I is not too large.

is bijective on isomorphism classes and the same remains true after any base change by a morphism $M' \to M$. This follows from our choice of f_i and Morphisms of Stacks, Lemmas 101.14.5 and 101.14.6. By construction of 2-fibred products the diagram

$$\begin{array}{ccc} \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_{ij}) & \longrightarrow & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_j) \\ \downarrow & & \downarrow \\ \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_i) & \longrightarrow & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \end{array}$$

is a fibre product of categories. By our choice of g_i the functors in this diagram induce bijections on automorphism groups. It follows that this diagram induces a fibre product diagram on sets of isomorphism classes! Thus we see that

$$R(k) = U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|} U(k)$$

where $|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$ denotes the set of isomorphism classes. In particular, for any algebraically closed field k the map on k-valued point is an equivalence relation. We conclude the claim holds by Groupoids, Lemma 39.3.5.

Let $M = U/R$ be the algebraic space which is the quotient of the above étale equivalence relation, see Spaces, Theorem 65.10.5. There is a canonical morphism $f : \mathcal{X} \to M$ fitting into commutative diagrams

0DUU (106.13.9.1)
$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow[g_i]{} & \mathcal{X} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ M_i & \longrightarrow & M \end{array}$$

Namely, such a morphism f is given by a functor

$$f : \mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, M)$$

for any scheme T compatible with base change. Let $a : T \to \mathcal{X}$ be an object of the left hand side. We obtain an étale covering $\{T_i \to T\}$ with $T_i = \mathcal{X}_i \times_{\mathcal{X}} T$ and morphisms $a_i : T_i \to \mathcal{X}_i$. Then we get $b_i = f_i \circ a_i : T_i \to M_i$. Since $T_i \times_T T_j = \mathcal{X}_{ij} \times_{\mathcal{X}} T$ we moreover get a morphism $a_{ij} : T_i \times_T T_j \to \mathcal{X}_{ij}$. Setting $b_{ij} = f_{ij} \circ a_{ij}$ we find that $b_i \times b_j$ factors through the monomorphism $M_{ij} \to M_i \times M_j$. Hence the morphisms

$$T_i \xrightarrow{b_i} M_i \to M$$

agree on $T_i \times_T T_j$. As M is a sheaf for the étale topology, we see that these morphisms glue to a unique morphism $b = f(a) : T \to M$. We omit the verification that this construction is compatible with base change and we omit the verification that the diagrams (106.13.9.1) commute.

Claim: the diagrams (106.13.9.1) are cartesian. To see this we study the induced morphism

$$h_i : \mathcal{X}_i \longrightarrow M_i \times_M \mathcal{X}$$

This is a morphism of stacks étale over $\mathcal{X}$ and hence h_i is étale (Morphisms of Stacks, Lemma 101.35.6). Since g_i is separated, we see h_i is separated (use Morphisms of Stacks, Lemma 101.4.12 and the fact seen above that the diagonal of $\mathcal{X}$ is separated). The morphism h_i induces isomorphisms on automorphism groups

(Morphisms of Stacks, Remark 101.19.5) as this is true for g_i. For an algebraically closed field k the diagram

$$\begin{array}{ccc} \mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X}) & \longrightarrow & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \\ \downarrow & & \downarrow \\ M_i(k) & \longrightarrow & M(k) \end{array}$$

is a catesian diagram of categories and the top arrow induces bijections on automorphism groups. On the other hand, we have

$$M(k) = U(k)/R(k) = U(k)/U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k),\mathcal{X})|} U(k) = |\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$$

by what we said above. Thus the right vertical arrow in the cartesian diagram above is a bijection on isomorphism classes. We conclude that $|\mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X})| \to M_i(k)$ is bijective. Review: h_i is a separated, étale, induces isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 101.19.5), and induces an equivalence on fibre categories over algebraically closed fields. Hence it is an isomorphism by Morphisms of Stacks, Lemma 101.45.7.

From the claim we get in particular the following: we have a surjective étale morphism $U \to M$ such that the base change of f is separated, quasi-compact, and a universal homeomorphism. It follows that f is separated, quasi-compact, and a universal homeomorphism. See Morphisms of Stacks, Lemma 101.4.5, 101.7.10, and 101.15.5

To finish the proof we have to show that $f : \mathcal{X} \to M$ is a uniform categorical moduli space. To prove this it suffices to show that given a flat morphism $M' \to M$ of algebraic spaces, the base change

$$M' \times_M \mathcal{X} \longrightarrow M'$$

is a categorical moduli space. Thus we consider a morphism

$$\theta : M' \times_M \mathcal{X} \longrightarrow E$$

where E is an algebraic space. For each i we know that f_i is a uniform categorical moduli space. Hence we obtain

$$\begin{array}{ccc} M' \times_M \mathcal{X}_i & \longrightarrow & M' \times_M \mathcal{X} \\ \downarrow & & \downarrow \theta \\ M' \times_M M_i & \xrightarrow{\psi_i} & E \end{array}$$

Since $\{M' \times_M M_i \to M'\}$ is an étale covering, to obtain the desired morphism $\psi : M' \to E$ it suffices to show that ψ_i and ψ_j agree over $M' \times_M M_i \times_M M_j = M' \times_M M_{ij}$. This follows easily from the fact that $f_{ij} : \mathcal{X}_{ij} = \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_j \to M_{ij}$ is a uniform categorical quotient; details omitted. Then finally one shows that ψ fits into the commutative diagram

$$\begin{array}{ccc} M' \times_M \mathcal{X} & & \\ \downarrow & \overset{\theta}{\searrow} & \\ M' & \xrightarrow{\psi} & E \end{array}$$

because "$\{M' \times_M \mathcal{X}_i \to M' \times_M \mathcal{X}\}$ is an étale covering" and the morphisms ψ_i fit into the corresponding commutative diagrams by construction. This finishes the proof of the Keel-Mori theorem. □

The following lemma emphasizes the étale local nature of the construction.

0DUV **Lemma 106.13.10.** *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume*

(1) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
(2) *h is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5).*

Then there exists a cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

where $M' \to M$ is a separated étale morphism of algebraic spaces and the vertical arrows are the moduli spaces constructed in Theorem 106.13.9.

Proof. By Morphisms of Stacks, Lemma 101.45.3 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is finite as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$. Let $f' : \mathcal{X}' \to M'$ and $f : \mathcal{X} \to M$ be as in Theorem 106.13.9. We obtain a commutative diagram as in the lemma because f' is categorical moduli space. Choose I and $g'_i : \mathcal{X}'_i \to \mathcal{X}'$ as in Lemma 106.13.8. Observe that $g_i = h \circ g'_i$ is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 101.19.5). Let $f'_i : \mathcal{X}'_i \to M'_i$ be as in Lemma 106.13.4. In the proof of Theorem 106.13.9 we have seen that the diagrams

$$\begin{array}{ccc} \mathcal{X}'_i & \xrightarrow[g'_i]{} & \mathcal{X}' \\ {\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle f'} \\ M'_i & \longrightarrow & M' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}'_i & \xrightarrow[g_i]{} & \mathcal{X} \\ {\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle f} \\ M'_i & \longrightarrow & M \end{array}$$

are cartesian and that $M'_i \to M'$ and $M'_i \to M$ are étale (this also follows directly from the properties of the morphisms g'_i, g_i, f', f'_i, f listed sofar by arguing in exactly the same way). This first implies that $M' \to M$ is étale and second that the diagram in the lemma is cartesian. We still need to show that $M' \to M$ is separated. To do this we contemplate the diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X}' \times_\mathcal{X} \mathcal{X}' \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M' \times_M M' \end{array}$$

The top horizontal arrow is universally closed as $\mathcal{X}' \to \mathcal{X}$ is separated. The vertical arrows are as in Theorem 106.13.9 (as flat base changes of $\mathcal{X} \to M$) hence universal homeomorphisms. Thus the lower horizontal arrow is universally closed. This (combined with it being an étale monomorphism of algebraic spaces) proves it is a closed immersion as desired. □

106.14. Properties of moduli spaces

0DUW Once the existence of a moduli space has been proven, it becomes possible (and is usually straightforward) to establish properties of these moduli spaces.

0DUX **Lemma 106.14.1.** *Let $p : \mathcal{X} \to Y$ be a morphism of an algebraic stack to an algebraic space. Assume*

(1) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
(2) *Y is locally Noetherian, and*
(3) *p is locally of finite type.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9. Then $M \to Y$ is locally of finite type.

Proof. Since f is a uniform categorical moduli space we obtain the morphism $M \to Y$. It suffices to check that $M \to Y$ is locally of finite type étale locally on M and Y. Since f is a uniform categorical moduli space, we may first replace Y by an affine scheme étale over Y. Next, we may choose I and $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 106.13.8. Then by Lemma 106.13.10 we reduce to the case $\mathcal{X} = \mathcal{X}_i$. In other words, we may assume $\mathcal{X}$ is well-nigh affine. In this case we have $Y = \mathrm{Spec}(A_0)$, we have $\mathcal{X} = [U/R]$ with $U = \mathrm{Spec}(A)$ and $M = \mathrm{Spec}(C)$ where $C \subset A$ is the set of R-invariant functions on U. See Lemmas 106.13.2 and 106.13.4. Then A_0 is Noetherian and $A_0 \to A$ is of finite type. Moreover A is integral over C by Groupoids, Lemma 39.23.4, hence finite over C (being of finite type over A_0). Thus we may finally apply Algebra, Lemma 10.51.7 to conclude. □

0DUY **Lemma 106.14.2.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9.*

(1) *If $\mathcal{X}$ is quasi-separated, then M is quasi-separated.*
(2) *If $\mathcal{X}$ is separated, then M is separated.*
(3) *Add more here, for example relative versions of the above.*

Proof. To prove this consider the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Delta_\mathcal{X}} & \mathcal{X} \times \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times f} \\ M & \xrightarrow{\Delta_M} & M \times M \end{array}$$

Since f is a universal homeomorphism, we see that $f \times f$ is a universal homeomorphism.

If $\mathcal{X}$ is separated, then $\Delta_\mathcal{X}$ is proper, hence $\Delta_\mathcal{X}$ is universally closed, hence Δ_M is universally closed, hence M is separated by Morphisms of Spaces, Lemma 67.40.9.

If $\mathcal{X}$ is quasi-separated, then $\Delta_\mathcal{X}$ is quasi-compact, hence Δ_M is quasi-compact, hence M is quasi-separated. □

0DUZ **Lemma 106.14.3.** *Let $p : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume*

(1) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
(2) *p is proper, and*
(3) *Y is locally Noetherian.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 106.13.9. Then $M \to Y$ is proper.

Proof. By Lemma 106.14.1 we see that $M \to Y$ is locally of finite type. By Lemma 106.14.2 we see that $M \to Y$ is separated. Of course $M \to Y$ is quasi-compact and universally closed as these are topological properties and $\mathcal{X} \to Y$ has these properties and $\mathcal{X} \to M$ is a universal homeomorphism. □

106.15. Stacks and fpqc coverings

0GRG Certain algebraic stacks satisfy fpqc descent. The analogue of this section for algebraic spaces is Properties of Spaces, Section 66.17.

0GRH **Proposition 106.15.1.** *Let $\mathcal{X}$ be an algebraic stack with quasi-affine[5] diagonal. Then $\mathcal{X}$ satisfies descent for fpqc coverings.*

Proposition 3.3.6 of "Intro to Algebraic Stacks" by Anatoly Preygel.

Proof. Our conventions are that $\mathcal{X}$ is a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ over the category of schemes over a base scheme S endowed with the fppf topology. The statement means the following: given an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of schemes over S the functor

$$\mathcal{X}_U \longrightarrow DD(\mathcal{U})$$

is an equivalence. Here on the left we have the category of objects of $\mathcal{X}$ over U and on the right we have the category of descent data in $\mathcal{X}$ relative to $\mathcal{U}$. See discussion in Stacks, Section 8.3.

Fully faithfulness. Suppose we have two objects x, y of $\mathcal{X}$ over U. Then $I = Isom(x, y)$ is an algebraic space over U. Hence a collection of sections of I over U_i whose restrictions to $U_i \times_U U_j$ agree, come from a unique section over U by the analogue of the proposition for algebraic spaces, see Properties of Spaces, Proposition 66.17.1. Thus our functor is fully faithful.

Essential surjectivity. Here we are given objects x_i over U_i and isomorphisms $\varphi_{ij} : \mathrm{pr}_0^* x_i \to \mathrm{pr}_1^* x_j$ over $U_i \times_U U_j$ satisfying the cocyle condition over $U_i \times_U U_j \times_U U_k$.

Let W be an affine scheme and let $W \to \mathcal{X}$ be a morphism. For each i we can form

$$W_i = U_i \times_{x_i, \mathcal{X}} W$$

The projection $W_i \to U_i$ is quasi-affine as the diagonal of $\mathcal{X}$ is quasi-affine. For each pair $i, j \in I$ the isomorphism φ_{ij} induces an isomorphism

$$W_i \times_U U_j = (U_i \times_U U_j) \times_{x_i \circ \mathrm{pr}_0, \mathcal{X}} W \to (U_i \times_U U_j) \times_{x_j \circ \mathrm{pr}_1, \mathcal{X}} W = U_i \times_U W_j$$

Moreover, these isomorphisms satisfy the cocycle condition over $U_i \times_U U_j \times_U U_k$. In other words, these isomorphisms define a descent datum on the schemes W_i/U_i relative to $\mathcal{U}$. By Descent, Lemma 35.38.1 we see that this descent datum is effective[6]. We conclude that there exists a quasi-affine morphism $W' \to U$ and a commutative diagram

$$\begin{array}{ccccc} W' & \longleftarrow & W_i & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & U_i & \xrightarrow{x_i} & \mathcal{X} \end{array}$$

[5] It suffices to assume ind-quasi-affine.

[6] Or use More on Groupoids, Lemma 40.15.3 in the case of ind-quasi-affine diagonal.

whose squares are cartesian. Since $\{W_i \to W'\}_{i \in I}$ is the base change of $\mathcal{U}$ by $W' \to U$ we conclude that it is an fpqc covering. Since W satisfies the sheaf condition for fpqc coverings, we obtain a unique morphism $W' \to W$ such that $W_i \to W' \to W$ is the given morphism $W_i \to W$. In other words, we have the commutative diagrams

$$\begin{array}{ccccc} W_i & \longrightarrow & W' & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ U_i & \longrightarrow & U & & \mathcal{X} \\ & \searrow^{x_i} & & \nearrow & \end{array}$$

compatible with the isomorphisms φ_{ij} and whose square and rectangle are cartesian.

Choose a collection of affine schemes W_α, $\alpha \in A$ and smooth morphisms $W_\alpha \to \mathcal{X}$ such that $\coprod W_\alpha \to \mathcal{X}$ is surjective. By the procedure of the preceding paragraph we produce a diagram

$$\begin{array}{ccccc} W_{\alpha,i} & \longrightarrow & W'_\alpha & \longrightarrow & W_\alpha \\ \downarrow & & \downarrow & & \downarrow \\ U_i & \longrightarrow & U & & \mathcal{X} \\ & \searrow^{x_i} & & \nearrow & \end{array}$$

for each α. Then the morphisms $W'_\alpha \to U$ are smooth and jointly surjective.

Denote x_α the object of $\mathcal{X}$ over W'_α corresponding to $W'_\alpha \to W_\alpha \to \mathcal{X}$. Since $\mathcal{X}$ is an fppf stack and since $\{W'_\alpha \to U\}$ is an fppf covering, it suffices to show that there are isomorphisms $\mathrm{pr}_0^* x_\alpha \to \mathrm{pr}_1^* x_\beta$ over $W'_\alpha \times_U W'_\beta$ satisfying the cocycle condition. However, after pulling back to $W_{\alpha,i}$ we do have such isomorphisms over $W_{\alpha,i} \times_{U_i} W_{\beta,i} = U_i \times_U (W'_\alpha \times_U W'_\beta)$ since the pullback of x_α to $W_{\alpha,i}$ is isomorphic to the pullback of x_i to $W_{\alpha,i}$. Since $\{U_i \times_U (W'_\alpha \times_U W'_\beta) \to W'_\alpha \times_U W'_\beta\}_{i \in I}$ is an fpqc covering and by the aforementioned compatibility of the diagrams above with φ_{ij} these isomorphisms descend to $W'_\alpha \times_U W'_\beta$ and the proof is complete. □

106.16. Tensor functors

0GRI Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Noetherian algebraic stacks. The pullback functor

$$f^* : Coh(\mathcal{O}_\mathcal{X}) \longrightarrow Coh(\mathcal{O}_\mathcal{Y})$$

is a right exact tensor functor: it is additive, right exact, and commutes with tensor products of coherent modules. We can ask to what extent any right exact tensor functor $F : Coh(\mathcal{O}_\mathcal{X}) \to Coh(\mathcal{O}_\mathcal{Y})$ comes from a morphism $f : \mathcal{Y} \to \mathcal{X}$. The reader may consult [**HR19**] for a very general result of this nature. The aim of this section is to give a short proof of Theorem 106.16.8 as an introduction to these ideas.

We begin with some lemmas.

0GRJ **Lemma 106.16.1.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Noetherian algebraic stacks. Any right exact tensor functor $F : Coh(\mathcal{O}_\mathcal{X}) \to Coh(\mathcal{O}_\mathcal{Y})$ extends uniquely to a right exact tensor functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$ commuting with all colimits.*

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 4.26.2. To see that the lemma applies observe that coherent modules

on locally Noetherian algebraic stacks are by definition modules of finite presentation, see Cohomology of Stacks, Definition 103.17.2. Hence a coherent module on $\mathcal{X}$ is a categorically compact object of $QCoh(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 103.13.5. Finally, every quasi-coherent module is a filtered colimit of its coherent submodules by Cohomology of Stacks, Lemma 103.18.1.

Since F is additive, also the extension of F is additive (details omitted). Since F is a tensor functor and since colimits of modules commute with taking tensor products, also the extension of F is a tensor functor (details omitted).

In this paragraph we show the extension commutes with arbitrary direct sums. If $\mathcal{F} = \bigoplus_{j\in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \operatorname{colim}_{J'\subset J \text{ finite}} \bigoplus_{j\in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \operatorname{colim}_{J'\subset J \text{ finite}} F(\bigoplus\nolimits_{j\in J'} \mathcal{H}_j) \\ &= \operatorname{colim}_{J'\subset J \text{ finite}} \bigoplus\nolimits_{j\in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j\in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

In this paragraph we show that the extension is right exact. Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules (see reference given above). Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \operatorname{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \operatorname{colim}_{i\in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \operatorname{colim}_{i\in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \operatorname{colim}_{i\in I} F(\mathcal{F}''_i)$. Since filtered colimits of sheaves of modules is exact we conclude that the extension of F is right exact.

The proof is finished as a right exact functor which commutes with all coproducts commutes with all colimits, see Categories, Lemma 4.14.12. □

0GRK **Lemma 106.16.2.** *Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_\mathcal{X}) \to Mod_B$ be a right exact tensor functor which commutes with direct sums. Let $g : U \to \mathcal{X}$ be a morphism with $U = \operatorname{Spec}(A)$ affine. Then*

1. *$C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative B-algebra and*
2. *there is a ring map $A \to C$*

such that $F \circ g_{QCoh,*} : Mod_A \to Mod_B$ sends M to $M \otimes_A C$ seen as B-module.

Proof. We note that g is quasi-compact and quasi-separated, see Morphisms of Stacks, Lemma 101.7.8. In Cohomology of Stacks, Proposition 103.11.1 we have constructed the functor $g_{QCoh,*} : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_\mathcal{X})$. By Cohomology of Stacks, Remarks 103.11.3 and 103.10.6 we obtain a multiplication

$$\mu : g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \longrightarrow g_{QCoh,*}\mathcal{O}_U$$

which turns $g_{QCoh,*}\mathcal{O}_U$ into a commutative $\mathcal{O}_\mathcal{X}$-algebra. Hence $C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative algebra object in Mod_B, in other words, C is a commutative B-algebra. Observe that we have a map $\kappa : A \to \operatorname{End}_{\mathcal{O}_\mathcal{X}}(g_{QCoh,*}\mathcal{O}_U)$ such that for

any $a \in A$ the diagram

$$\begin{array}{ccc} g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U & \xrightarrow[\mu]{} & g_{QCoh,*}\mathcal{O}_U \\ \downarrow{\scriptstyle \kappa(r)\otimes 1} & & \downarrow{\scriptstyle \kappa(r)} \\ g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U & \xrightarrow{\mu} & g_{QCoh,*}\mathcal{O}_U \end{array}$$

commutes. It follows that we get a map $\kappa' = F(\kappa) : A \to \mathrm{End}_B(C)$ such that $\kappa'(a)(c)c' = \kappa'(a)(cc')$ and of course this means that $a \mapsto \kappa'(a)(1)$ is a ring map $A \to C$.

The morphism $g : U \to \mathcal{X}$ is affine, see Morphisms of Stacks, Lemma 101.9.4. Hence $g_{QCoh,*}$ is exact and commutes with direct sums by Cohomology of Stacks, Lemma 103.13.4. Thus $F \circ g_{QCoh,*} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is a right exact functor which commutes with direct sums and which sends A to C. By Functors and Morphisms, Lemma 56.3.1 we see that the functor $F \circ g_{QCoh,*}$ sends an A-module M to $M \otimes_A C$ viewed as a B-module. □

0GRL **Lemma 106.16.3.** *Notation as in Lemma 106.16.2. Assume $\mathcal{X}$ is Noetherian and g is surjective and flat. Then $B \to C$ is universally injective.*

Proof. Consider the natural map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ in $QCoh(\mathcal{O}_\mathcal{X})$. Pulling back to U and using adjunction we find that the composition

$$\mathcal{O}_U = g^*\mathcal{O}_\mathcal{X} \xrightarrow{g^*1} g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity in $QCoh(\mathcal{O}_U)$. Write $g_{QCoh,*}\mathcal{O}_U = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Lemma 103.18.1. For i large enough the map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ factors through $\mathcal{F}_i$, see Cohomology of Stacks, Lemma 103.13.5. Say $s : \mathcal{O}_\mathcal{X} \to \mathcal{F}_i$ is the factorization. Then

$$\mathcal{O}_U \xrightarrow{g^*s} g^*\mathcal{F}_i \to g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity. In other words, we see that s becomes the inclusion of a direct summand upon pullback to U. Set $\mathcal{F}_i^\vee = hom(\mathcal{F}_i, \mathcal{O}_\mathcal{X})$ with notation as in Cohomology of Stacks, Lemma 103.10.8. In particular there is an evaluation map $ev : \mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Evaluation at s defines a map $s^\vee : \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Dual to the statement about s we see that $g^*(s^\vee)$ is surjective, see see Cohomology of Stacks, Section 103.12 for compatibility of hom and $\otimes$ with restriction to U. Since g is surjective and flat, we conclude that $s^\vee$ is surjective (see locus citatus). Since F is right exact, we conclude that $F(\mathcal{F}_i^\vee) \to F(\mathcal{O}_\mathcal{X}) = B$ is surjective. Choose $\lambda \in F(\mathcal{F}_i^\vee)$ mapping to $1 \in B$. Denote $e = F(s)(1) \in F(\mathcal{F}_i)$ the image of 1 by the map $F(s) : B = F(\mathcal{O}_\mathcal{X}) \to F(\mathcal{F}_i)$. Then the map

$$F(ev) : F(\mathcal{F}_i) \otimes_B F(\mathcal{F}_i^\vee) = F(\mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee) \longrightarrow F(\mathcal{O}_\mathcal{X}) = B$$

sends $e \otimes \lambda$ to 1 by construction. Hence the map $B \to F(\mathcal{F}_i)$, $b \mapsto be$ is universally injective because we have the one-sided inverse $F(\mathcal{F}_i) \to B$, $\xi \mapsto F(ev)(\xi \otimes \lambda)$. Since this is true for all i large enough we conclude. □

0GRM **Lemma 106.16.4.** *Let $B \to C$ be a ring map. If*

(1) *the coprojections $C \to C \otimes_B C$ are flat and*
(2) *$B \to C$ is universally injective,*

then $B \to C$ is faithfully flat.

Proof. The map $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective as $B \to C$ is universally injective. Thus it suffices to show that $B \to C$ is flat which follows from Descent, Theorem 35.4.25. □

The following very simple version of Künneth should become obsoleted when we write a section on Künneth theorems for cohomology of quasi-coherent modules on algebraic stacks.

0GRN **Lemma 106.16.5.** *Let $a : \mathcal{Y} \to \mathcal{X}$ and $b : \mathcal{Z} \to \mathcal{X}$ be representable by schemes, quasi-compact, quasi-separated, and flat. Then $a_{QCoh,*}\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} = f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ where $f : \mathcal{Y} \times_\mathcal{X} \mathcal{Z} \to \mathcal{X}$ is the obvious morphism.*

Proof. We abbreviate $\mathcal{P} = \mathcal{Y} \times_\mathcal{X} \mathcal{Z}$. Since $a \circ \mathrm{pr}_1 = f$ and $b \circ \mathrm{pr}_2 = f$ we obtain maps $a_*\mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{P}$ and $b_*\mathcal{O}_\mathcal{Z} \to f_*\mathcal{O}_\mathcal{P}$ (using relative pullback maps, see Sites, Section 7.45). Hence we obtain a relative cup product

$$\mu : a_*\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_*\mathcal{O}_\mathcal{Z} \longrightarrow f_*\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$$

Applying Q and its compatibility with tensor products (Cohomology of Stacks, Remark 103.10.6) we obtain an arrow $Q(\mu) : a_{QCoh,*}\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} \to f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ in $QCoh(\mathcal{O}_\mathcal{X})$. Next, choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. It suffices to prove the restriction of $Q(\mu)$ to $U_{étale}$ is an isomorphism, see Cohomology of Stacks, Section 103.12. In turn, by the material in the same section, it suffices to prove the restriction of μ to $U_{étale}$ is an isomorphism (this uses that the source and target of μ are locally quasi-coherent modules with the base change property). Moreover, we may compute pushforwards in the étale topology, see Cohomology of Stacks, Proposition 103.8.1. Then finally, we see that $a_*\mathcal{O}_\mathcal{Y}|_{U_{étale}} = (V \to U)_{small,*}\mathcal{O}_V$ where $V = U \times_\mathcal{X} \mathcal{Y}$. Similarly for b_* and f_*. Thus the result follows from the Künneth formula for flat, quasi-compact, quasi-separated morphisms of schemes, see Derived Categories of Schemes, Lemma 36.23.1. □

0GRP **Lemma 106.16.6.** *Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $f_i : \mathrm{Spec}(B) \to \mathcal{X}$, $i = 1, 2$ be two morphisms. Let $t : f_1^* \to f_2^*$ be an isomorphism of the tensor functors $f_i^* : QCoh(\mathcal{O}_\mathcal{X}) \to Mod_B$. Then there is a 2-arrow $f_1 \to f_2$ inducing t.*

Proof. Choose an affine scheme $U = \mathrm{Spec}(A)$ and a surjective smooth morphism $g : U \to \mathcal{X}$, see Properties of Stacks, Lemma 100.6.2. Since the diagonal of $\mathcal{X}$ is affine, we see that $U_i = \mathrm{Spec}(B) \times_{f_i,\mathcal{X},g} U$ is affine. Say $U_i = \mathrm{Spec}(C_i)$. Then C_i is the B-algebra endowed with ring map $A \to C_i$ constructed in Lemma 106.16.2 using the functor $F = f_i^*$. Therefore t induces an isomorphism $C_1 \to C_2$ of B-algebras, compatible with the ring maps $A \to C_1$ and $A \to C_2$. In other words, we have a commutative diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(B) & \xrightarrow{f_i} & \mathcal{X} \end{array} \qquad \begin{array}{ccccc} & & U_2 & & \\ & \swarrow & \downarrow{\scriptstyle \cong} & \searrow & \\ \mathrm{Spec}(B) & \longleftarrow & U_1 & \longrightarrow & U \end{array}$$

This already shows that the objects f_1 and f_2 of $\mathcal{X}$ over $\mathrm{Spec}(B)$ become isomorphic after the smooth covering $\{U_1 \to \mathrm{Spec}(B)\}$. To show that this descends to an isomorphism of f_1 and f_2 over $\mathrm{Spec}(B)$, we have to show that our isomorphism (which comes from the commutative diagrams above) is compatible with the descent

data over $U_1 \times_{\mathrm{Spec}(B)} U_1$. For this we observe that $U \times_{\mathcal{X}} U$ is affine too, that we have the morphism $g' : U \times_{\mathcal{X}} U \to \mathcal{X}$, and that

$$U_i \times_{\mathrm{Spec}(B)} U_i = \mathrm{Spec}(B) \times_{f_i, \mathcal{X}, g'} (U \times_{\mathcal{X}} U)$$

It follows that the isomorphism $C_1 \otimes_B C_1 \to C_2 \otimes_B C_2$ coming from the isomorphism $C_1 \to C_2$ is compatible with the morphisms $U_i \times_{\mathrm{Spec}(B)} U_i \to U \times_{\mathcal{X}} U$. Some details omitted. □

0GRQ **Lemma 106.16.7.** *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_{\mathcal{X}}) \to Mod_B$ be a right exact tensor functor which commutes with direct sums. Then F comes from a unique morphism $\mathrm{Spec}(B) \to \mathcal{X}$.*

Proof. Choose a surjective smooth morphism $g : U \to \mathcal{X}$ with $U = \mathrm{Spec}(A)$ affine, see Properties of Stacks, Lemma 100.6.2. Apply Lemma 106.16.2 to get the finite type commutative B-algebra $C = F(g_{QCoh,*}\mathcal{O}_U)$ and the ring map $A \to C$. By Lemma 106.16.3 the ring map $B \to C$ is universally injective. Consider the algebra

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U)$$

Since g is flat, quasi-compact, and quasi-separated by Lemma 106.16.5 we have the first equality in

$$g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U = f_{QCoh,*}\mathcal{O}_{U \times_{\mathcal{X}} U} = g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_{\mathcal{X}} U})$$

where $f : U \times_{\mathcal{X}} U \to \mathcal{X}$ is the obvious morphism and $\mathrm{pr}_2 : U \times_{\mathcal{X}} U \to U$ is the second projection. The second equality follows from Cohomology of Stacks, Lemma 103.11.5 and $f = g \circ \mathrm{pr}_2$. Since the diagonal of $\mathcal{X}$ is affine, we see that $U \times_{\mathcal{X}} U = \mathrm{Spec}(R)$ is affine. Let us use $\mathrm{pr}_2^\sharp : A \to R$ to view R as an A-algebra. All in all we obtain

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U) = F(g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_{\mathcal{X}} U})) = R \otimes_A C$$

where the final equality follows from the final statement of Lemma 106.16.2. Since $A \to R$ is flat (because pr_2 is flat as a base change of $U \to \mathcal{X}$), we conclude that $C \otimes_B C$ is flat over C. By Lemma 106.16.4 we conclude that $B \to C$ is faithfully flat.

We claim there is a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(C \otimes_B C) & \longrightarrow & U \times_{\mathcal{X}} U \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Spec}(C) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \dashrightarrow & \mathcal{X} \end{array}$$

The arrow $\mathrm{Spec}(C) \to U = \mathrm{Spec}(A)$ comes from the ring map $A \to C$ in the statement of Lemma 106.16.2. The arrow $\mathrm{Spec}(C \otimes_B C) \to U \times_{\mathcal{X}} U$ similarly comes from the ring map $R \to C \otimes_B C$. To verify the top square commutes use Lemma 106.16.6; details omitted. We conclude we get the dotted arrow $\mathrm{Spec}(B) \to \mathcal{X}$ by Proposition 106.15.1.

The statement that F is the functor corresponding to pullback by the dotted arrow is also clear from this and the corresponding statement in Lemma 106.16.2. Details omitted. □

For a ring B let us denote Mod_B^{fg} the category of finitely generated B-modules (AKA finite B-modules).

0GRR **Theorem 106.16.8.** *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a Noetherian ring. Let $F : Coh(\mathcal{O}_\mathcal{X}) \to Mod_B^{fg}$ be a right exact tensor functor. Then F comes from a unique morphism $\mathrm{Spec}(B) \to \mathcal{X}$.*

Proof. By Lemma 106.16.1 we can extend F uniquely to a right exact tensor functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}_B$ commuting with all direct susms. Then we can apply Lemma 106.16.7. □

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CHAPTER 107

The Geometry of Algebraic Stacks

0DQR

107.1. Introduction

0DQS This chapter discusses a few geometric properties of algebraic stacks. The initial versions of Sections 107.3 and 107.5 were written by Matthew Emerton and Toby Gee and can be found in their original form in [**EG17**].

107.2. Versal rings

0DQT In this section we elucidate the relationship between deformation rings and local rings on algebraic stacks of finite type over a locally Noetherian base.

0DQU **Situation 107.2.1.** Here $\mathcal{X}$ is an algebraic stack locally of finite type over a locally Noetherian scheme S.

Here is the definition.

0DQV **Definition 107.2.2.** In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. A *versal ring to $\mathcal{X}$ at x_0* is a complete Noetherian local S-algebra A with residue field k such that there exists a versal formal object (A, ξ_n, f_n) as in Artin's Axioms, Definition 98.12.1 with $\xi_1 \cong x_0$ (a 2-isomorphism).

We want to prove that versal rings exist and are unique up to smooth factors. To do this, we will use the predeformation categories of Artin's Axioms, Section 98.3. These are always deformation categories in our situation.

0DQW **Lemma 107.2.3.** *In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Then $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category and $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $Inf(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k-vector spaces.*

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through it. By Artin's Axioms, Section 98.3 we obtain a predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ over the category $\mathcal{C}_\Lambda$. (As pointed out in locus citatus this category only depends on the morphism $\mathrm{Spec}(k) \to S$ and not on the choice of Λ.) By Artin's Axioms, Lemmas 98.6.1 and 98.5.2 $\mathcal{F}_{\mathcal{X},k,x_0}$ is actually a deformation category. By Artin's Axioms, Lemma 98.8.1 we find that $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k-vector spaces. □

0DQX **Lemma 107.2.4.** *In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Then a versal ring to $\mathcal{X}$ at x_0 exists. Given a pair A, A' of these, then $A \cong A'[[t_1, \ldots, t_r]]$ or $A' \cong A[[t_1, \ldots, t_r]]$ as S-algebras for some r.*

Proof. By Lemma 107.2.3 and Formal Deformation Theory, Lemma 90.13.4 (note that the assumptions of this lemma hold by Formal Deformation Theory, Lemmas 90.16.6 and Definition 90.16.8). By the uniqueness result of Formal Deformation Theory, Lemma 90.14.5 there exists a "minimal" versal ring A of $\mathcal{X}$ at x_0 such that any other versal ring of $\mathcal{X}$ at x_0 is isomorphic to $A[[t_1, \ldots, t_r]]$ for some r. This clearly implies the second statement. □

0DQY **Lemma 107.2.5.** *In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let l/k be a finite extension of fields and denote $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ the induced morphism. Given a versal ring A to $\mathcal{X}$ at x_0 there exists a versal ring A' to $\mathcal{X}$ at $x_{l,0}$ such that there is a S-algebra map $A \to A'$ which induces the given field extension l/k and is formally smooth in the $\mathfrak{m}_{A'}$-adic topology.*

Proof. Follows immediately from Artin's Axioms, Lemma 98.7.1 and Formal Deformation Theory, Lemma 90.29.6. (We also use that $\mathcal{X}$ satisfies (RS) by Artin's Axioms, Lemma 98.5.2.) □

0DQZ **Lemma 107.2.6.** *In Situation 107.2.1 let $x : U \to \mathcal{X}$ be a morphism where U is a scheme locally of finite type over S. Let $u_0 \in U$ be a finite type point. Set $k = \kappa(u_0)$ and denote $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ the induced map. The following are equivalent*

(1) *x is versal at u_0 (Artin's Axioms, Definition 98.12.2),*
(2) *$\hat{x} : \mathcal{F}_{U,k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth,*
(3) *the formal object associated to $x|_{\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0})}$ is versal, and*
(4) *there is an open neighbourhood $U' \subset U$ of x such that $x|_{U'} : U' \to \mathcal{X}$ is smooth.*

Moreover, in this case the completion $\mathcal{O}^\wedge_{U,u_0}$ is a versal ring to $\mathcal{X}$ at x_0.

Proof. Since $U \to S$ is locally of finite type (as a composition of such morphisms), we see that $\mathrm{Spec}(k) \to S$ is of finite type (again as a composition). Thus the statement makes sense. The equivalence of (1) and (2) is the definition of x being versal at u_0. The equivalence of (1) and (3) is Artin's Axioms, Lemma 98.12.3. Thus (1), (2), and (3) are equivalent.

If $x|_{U'}$ is smooth, then the functor $\hat{x} : \mathcal{F}_{U,k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth by Artin's Axioms, Lemma 98.3.2. Thus (4) implies (1), (2), and (3). For the converse, assume x is versal at u_0. Choose a surjective smooth morphism $y : V \to \mathcal{X}$ where V is a scheme. Set $Z = V \times_{\mathcal{X}} U$ and pick a finite type point $z_0 \in |Z|$ lying over u_0 (this is possible by Morphisms of Spaces, Lemma 67.25.5). By Artin's Axioms, Lemma 98.12.6 the morphism $Z \to V$ is smooth at z_0. By definition we can find an open neighbourhood $W \subset Z$ of z_0 such that $W \to V$ is smooth. Since $Z \to U$ is open, let $U' \subset U$ be the image of W. Then we see that $U' \to \mathcal{X}$ is smooth by our definition of smooth morphisms of stacks.

The final statement follows from the definitions as $\mathcal{O}^\wedge_{U,u_0}$ prorepresents $\mathcal{F}_{U,k,u_0}$. □

0DZS **Lemma 107.2.7.** *In Situation 107.2.1. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism such that $\mathrm{Spec}(k) \to S$ is of finite type with image s. Let A be a versal ring to $\mathcal{X}$ at x_0. The following are equivalent*

(1) *x_0 is in the smooth locus of $\mathcal{X} \to S$ (Morphisms of Stacks, Lemma 101.33.6),*

(2) $\mathcal{O}_{S,s} \to A$ *is formally smooth in the* $\mathfrak{m}_A$*-adic topology, and*
(3) $\mathcal{F}_{\mathcal{X},k,x_0}$ *is unobstructed.*

Proof. The equivalence of (2) and (3) follows immediately from Formal Deformation Theory, Lemma 90.9.4.

Note that $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology if and only if $\mathcal{O}_{S,s} \to A' = A[[t_1, \ldots, t_r]]$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology. Hence (2) does not depend on the choice of our versal ring by Lemma 107.2.4. Next, let l/k be a finite extension and choose $A \to A'$ as in Lemma 107.2.5. If $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology, then $\mathcal{O}_{S,s} \to A'$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology, see More on Algebra, Lemma 15.37.7. Conversely, if $\mathcal{O}_{S,s} \to A'$ is formally smooth in the $\mathfrak{m}_{A'}$-adic topology, then $\mathcal{O}_{S,s}^\wedge \to A'$ and $A \to A'$ are regular (More on Algebra, Proposition 15.49.2) and hence $\mathcal{O}_{S,s}^\wedge \to A$ is regular (More on Algebra, Lemma 15.41.7), hence $\mathcal{O}_{S,s} \to A$ is formally smooth in the $\mathfrak{m}_A$-adic topology (same lemma as before). Thus the equivalence of (2) and (1) holds for k and x_0 if and only if it holds for l and $x_{0,l}$.

Choose a scheme U and a smooth morphism $U \to \mathcal{X}$ such that $\mathrm{Spec}(k) \times_{\mathcal{X}} U$ is nonempty. Choose a finite extension l/k and a point $w_0 : \mathrm{Spec}(l) \to \mathrm{Spec}(k) \times_{\mathcal{X}} U$. Let $u_0 \in U$ be the image of w_0. We may apply the above to l/k and to $l/\kappa(u_0)$ to see that we can reduce to u_0. Thus we may assume $A = \mathcal{O}_{U,u_0}^\wedge$, see Lemma 107.2.6. Observe that x_0 is in the smooth locus of $\mathcal{X} \to S$ if and only if u_0 is in the smooth locus of $U \to S$, see for example Morphisms of Stacks, Lemma 101.33.6. Thus the equivalence of (1) and (2) follows from More on Algebra, Lemma 15.38.6. □

We recall a consequence of Artin approximation.

0DR0 **Lemma 107.2.8.** *In Situation 107.2.1. Let* $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ *be a morphism such that* $\mathrm{Spec}(k) \to S$ *is of finite type with image* s*. Let* A *be a versal ring to* $\mathcal{X}$ *at* x_0*. If* $\mathcal{O}_{S,s}$ *is a G-ring, then we may find a smooth morphism* $U \to \mathcal{X}$ *whose source is a scheme and a point* $u_0 \in U$ *with residue field* k*, such that*

(1) $\mathrm{Spec}(k) \to U \to \mathcal{X}$ *coincides with the given morphism* x_0*,*
(2) *there is an isomorphism* $\mathcal{O}_{U,u_0}^\wedge \cong A$*.*

Proof. Let (ξ_n, f_n) be the versal formal object over A. By Artin's Axioms, Lemma 98.9.5 we know that $\xi = (A, \xi_n, f_n)$ is effective. By assumption $\mathcal{X}$ is locally of finite presentation over S (use Morphisms of Stacks, Lemma 101.27.5), and hence limit preserving by Limits of Stacks, Proposition 102.3.8. Thus Artin approximation as in Artin's Axioms, Lemma 98.12.7 shows that we may find a morphism $U \to \mathcal{X}$ with source a finite type S-scheme, containing a point $u_0 \in U$ of residue field k satisfying (1) and (2) such that $U \to \mathcal{X}$ is versal at u_0. By Lemma 107.2.6 after shrinking U we may assume $U \to \mathcal{X}$ is smooth. □

0DR1 **Remark 107.2.9** (Upgrading versal rings)**.** In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A be a versal ring to $\mathcal{X}$ at x_0. By Artin's Axioms, Lemma 98.9.5 our versal formal object in fact comes from a morphism

$$\mathrm{Spec}(A) \longrightarrow \mathcal{X}$$

over S. Moreover, the results above each can be upgraded to be compatible with this morphism. Here is a list:

(1) in Lemma 107.2.4 the isomorphism $A \cong A'[[t_1, \ldots, t_r]]$ or $A' \cong A[[t_1, \ldots, t_r]]$ may be chosen compatible with these morphisms,
(2) in Lemma 107.2.5 the homomorphism $A \to A'$ may be chosen compatible with these morphisms,
(3) in Lemma 107.2.6 the morphism $\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0}) \to \mathcal{X}$ is the composition of the canonical map $\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0}) \to U$ and the given map $U \to \mathcal{X}$,
(4) in Lemma 107.2.8 the isomorphism $\mathcal{O}^\wedge_{U,u_0} \cong A$ may be chosen so $\mathrm{Spec}(A) \to \mathcal{X}$ corresponds to the canonical map in the item above.

In each case the statement follows from the fact that our maps are compatible with versal formal elements; we note however that the implied diagrams are 2-commutative only up to a (noncanonical) choice of a 2-arrow. Still, this means that the implied map $A' \to A$ or $A \to A'$ in (1) is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 90.28.3.

0DR2 **Lemma 107.2.10.** *In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A be a versal ring to $\mathcal{X}$ at x_0. Then the morphism $\mathrm{Spec}(A) \to \mathcal{X}$ of Remark 107.2.9 is flat.*

Proof. If the local ring of S at the image point is a G-ring, then this follows immediately from Lemma 107.2.8 and the fact that the map from a Noetherian local ring to its completion is flat. In general we prove it as follows.

Step I. If A and A' are two versal rings to $\mathcal{X}$ at x_0, then the result is true for A if and only if it is true for A'. Namely, after possible swapping A and A', we may assume there is a formally smooth map $\varphi : A \to A'$ such that the composition

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \to \mathcal{X}$, see Lemma 107.2.4 and Remark 107.2.9. Since $A \to A'$ is faithfully flat we obtain the equivalence from Morphisms of Stacks, Lemmas 101.25.2 and 101.25.5.

Step II. Let l/k be a finite extension of fields. Let $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ be the induced morphism. Let A be a versal ring to $\mathcal{X}$ at x_0 and let $A \to A'$ be as in Lemma 107.2.5. Then again the composition

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \to \mathcal{X}$, see Remark 107.2.9. Arguing as before and using step I to see choice of versal rings is irrelevant, we see that the lemma holds for x_0 if and only if it holds for $x_{l,0}$.

Step III. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then we can choose a finite type point z_0 on $Z = U \times_{\mathcal{X}} x_0$ (this is a nonempty algebraic space). Let $u_0 \in U$ be the image of z_0 in U. Choose a scheme and a surjective étale map $W \to Z$ such that z_0 is the image of a closed point $w_0 \in W$ (see Morphisms of Spaces, Section 67.25). Since $W \to \mathrm{Spec}(k)$ and $W \to U$ are of finite type, we see that $\kappa(w_0)/k$ and $\kappa(w_0)/\kappa(u_0)$ are finite extensions of fields (see Morphisms, Section 29.16). Applying Step II twice we may replace x_0 by $u_0 \to U \to \mathcal{X}$. Then we see our morphism is the composition

$$\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0}) \to U \to \mathcal{X}$$

The first arrow is flat because completion of Noetherian local rings are flat (Algebra, Lemma 10.97.2) and the second arrow is flat as a smooth morphism is flat. The composition is flat as composition preserves flatness. □

0DR3 **Remark 107.2.11.** In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. By Lemma 107.2.3 and Formal Deformation Theory, Theorem 90.26.4 we know that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Unwinding the definitions, this means we can choose

(1) a Noetherian complete local Λ-algebra A with residue field k and a versal formal object ξ of $\mathcal{F}_{\mathcal{X},k,x_0}$ over A,
(2) a Noetherian complete local Λ-algebra B with residue field k and an isomorphism

$$\underline{B}|_{\mathcal{C}_\Lambda} \longrightarrow \underline{A}|_{\mathcal{C}_\Lambda} \times_{\underline{\xi},\mathcal{F}_{\mathcal{X},k,x_0},\underline{\xi}} \underline{A}|_{\mathcal{C}_\Lambda}$$

The projections correspond to formally smooth maps $t : A \to B$ and $s : A \to B$ (because ξ is versal). There is a map $c : B \to B \widehat{\otimes}_{s,A,t} B$ which turns (A, B, s, t, c) into a cogroupoid in the category of Noetherian complete local Λ-algebras with residue field k (on prorepresentable functors this map is constructed in Formal Deformation Theory, Lemma 90.25.2). Finally, the cited theorem tells us that ξ induces an equivalence

$$[\underline{A}|_{\mathcal{C}_\Lambda}/\underline{B}|_{\mathcal{C}_\Lambda}] \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over $\mathcal{C}_\Lambda$. In fact, we also get an equivalence

$$[\underline{A}/\underline{B}] \longrightarrow \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over the completed category $\widehat{\mathcal{C}}_\Lambda$ (see discussion in Formal Deformation Theory, Section 90.22 as to why this works). Of course A is a versal ring to $\mathcal{X}$ at x_0.

107.3. Multiplicities of components of algebraic stacks

0DR4 If X is a locally Noetherian scheme, then we may write X (thought of simply as a topological space) as a union of irreducible components, say $X = \bigcup T_i$. Each irreducible component is the closure of a unique generic point ξ_i, and the local ring $\mathcal{O}_{X,\xi_i}$ is a local Artin ring. We may define the *multiplicity of X along T_i* or the *multiplicity of T_i in X* by

$$m_{T_i,X} = \mathrm{length}_{\mathcal{O}_{X,\xi_i}} \mathcal{O}_{X,\xi_i}$$

In other words, it is the length of the local Artinian ring. Please compare with Chow Homology, Section 42.9.

Our goal here is to generalise this definition to locally Noetherian algebraic stacks. If $\mathcal{X}$ is a stack, then its topological space $|\mathcal{X}|$ (see Properties of Stacks, Definition 100.4.8) is locally Noetherian (Morphisms of Stacks, Lemma 101.8.3). The irreducible components of $|\mathcal{X}|$ are sometimes referred to as the irreducible components of $\mathcal{X}$. If $\mathcal{X}$ is quasi-separated, then $|\mathcal{X}|$ is sober (Morphisms of Stacks, Lemma 101.30.3), but it need not be in the non-quasi-separated case. Consider for example the non-quasi-separated algebraic space $X = \mathbf{A}^1_{\mathbf{C}}/\mathbf{Z}$. Furthermore, there is no structure sheaf on $|\mathcal{X}|$ whose stalks can be used to define multiplicities.

0DR5 **Lemma 107.3.1.** *Let $f : U \to \mathcal{X}$ be a smooth morphism from a scheme to a locally Noetherian algebraic stack. The closure of the image of any irreducible component of $|U|$ is an irreducible component of $|\mathcal{X}|$. If $U \to \mathcal{X}$ is surjective, then all irreducible components of $|\mathcal{X}|$ are obtained in this way.*

Proof. The map $|U| \to |\mathcal{X}|$ is continuous and open by Properties of Stacks, Lemma 100.4.7. Let $T \subset |U|$ be an irreducible component. Since U is locally Noetherian, we can find a nonempty affine open $W \subset U$ contained in T. Then $f(T) \subset |\mathcal{X}|$ is irreducible and contains the nonempty open subset $f(W)$. Thus the closure of $f(T)$ is irreducible and contains a nonempty open. It follows that this closure is an irreducible component.

Assume $U \to \mathcal{X}$ is surjective and let $Z \subset |\mathcal{X}|$ be an irreducible component. Choose a Noetherian open subset V of $|\mathcal{X}|$ meeting Z. After removing the other irreducible components from V we may assume that $V \subset Z$. Take an irreducible component of the nonempty open $f^{-1}(V) \subset |U|$ and let $T \subset |U|$ be its closure. This is an irreducible component of $|U|$ and the closure of $f(T)$ must agree with Z by our choice of T. □

The preceding lemma applies in particular in the case of smooth morphisms between locally Noetherian schemes. This particular case is implicitly invoked in the statement of the following lemma.

0DR6 **Lemma 107.3.2.** *Let $U \to X$ be a smooth morphism of locally Noetherian schemes. Let T' is an irreducible component of U. Let T be the irreducible component of X obtained as the closure of the image of T'. Then $m_{T',U} = m_{T,X}$.*

Proof. Write ξ' for the generic point of T', and ξ for the generic point of T. Let $A = \mathcal{O}_{X,\xi}$ and $B = \mathcal{O}_{U,\xi'}$. We need to show that $\text{length}_A A = \text{length}_B B$. Since $A \to B$ is a flat local homomorphism of rings (since smooth morphisms are flat), we have

$$\text{length}_A(A)\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B)$$

by Algebra, Lemma 10.52.13. Thus it suffices to show $\mathfrak{m}_A B = \mathfrak{m}_B$, or equivalently, that $B/\mathfrak{m}_A B$ is reduced. Since $U \to X$ is smooth, so is its base change $U_\xi \to \text{Spec}\,\kappa(\xi)$. As U_ξ is a smooth scheme over a field, it is reduced, and thus so its local ring at any point (Varieties, Lemma 33.25.4). In particular,

$$B/\mathfrak{m}_A B = \mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'} = \mathcal{O}_{U_\xi,\xi'}$$

is reduced, as required. □

Using this result, we may show that there exists a good notion of multiplicity by looking smooth locally.

0DR7 **Lemma 107.3.3.** *Let $U_1 \to \mathcal{X}$ and $U_2 \to \mathcal{X}$ be two smooth morphisms from schemes to a locally Noetherian algebraic stack $\mathcal{X}$. Let T_1' and T_2' be irreducible components of $|U_1|$ and $|U_2|$ respectively. Assume the closures of the images of T_1' and T_2' are the same irreducible component T of $|\mathcal{X}|$. Then $m_{T_1',U_1} = m_{T_2',U_2}$.*

Proof. Let V_1 and V_2 be dense subsets of T_1' and T_2', respectively, that are open in U_1 and U_2 respectively (see proof of Lemma 107.3.1). The images of $|V_1|$ and $|V_2|$ in $|\mathcal{X}|$ are non-empty open subsets of the irreducible subset T, and therefore have non-empty intersection. By Properties of Stacks, Lemma 100.4.3, the map $|V_1 \times_{\mathcal{X}} V_2| \to |V_1| \times_{|\mathcal{X}|} |V_2|$ is surjective. Consequently $V_1 \times_{\mathcal{X}} V_2$ is a non-empty

algebraic space; we may therefore choose an étale surjection $V \to V_1 \times_{\mathcal{X}} V_2$ whose source is a (non-empty) scheme. If we let T' be any irreducible component of V, then Lemma 107.3.1 shows that the closure of the image of T' in U_1 (respectively U_2) is equal to T'_1 (respectively T'_2).

Applying Lemma 107.3.2 twice we find that

$$m_{T'_1,U_1} = m_{T',V} = m_{T'_2,U_2},$$

as required. □

At this point we have done enough work to show the following definition makes sense.

0DR8 **Definition 107.3.4.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $T \subset |\mathcal{X}|$ be an irreducible component. The *multiplicity* of T in $\mathcal{X}$ is defined as $m_{T,\mathcal{X}} = m_{T',U}$ where $f : U \to \mathcal{X}$ is a smooth morphism from a scheme and $T' \subset |U|$ is an irreducible component with $f(T') \subset T$.

This is independent of the choice of $f : U \to \mathcal{X}$ and the choice of the irreducible component T' mapping to T by Lemmas 107.3.1 and 107.3.3.

As a closing remark, we note that it is sometimes convenient to think of an irreducible component of $\mathcal{X}$ as a closed substack. To this end, if $T \subset |\mathcal{X}|$ is an irreducible component, then we may consider the unique reduced closed substack $\mathcal{T} \subset \mathcal{X}$ with $|\mathcal{T}| = T$, see Properties of Stacks, Definition 100.10.4. If $\mathcal{X}$ is quasi-separated, then an irreducible component is an integral stack; see Morphisms of Stacks, Section 101.50 for further discussion.

107.4. Formal branches and multiplicities

0DR9 It will be convenient to have a comparison between the notion of multiplicity of an irreducible component given by Definition 107.3.4 and the related notion of multiplicities of irreducible components of (the spectra of) versal rings of $\mathcal{X}$ at finite type points.

In Situation 107.2.1 let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism, where k is a finite type field over S. Let A, A' be versal rings to $\mathcal{X}$ at x_0. After possibly swapping A and A', we know there is a formally smooth[1] map $\varphi : A \to A'$ compatible with versal formal objects, see Lemma 107.2.4 and Remark 107.2.9. Moreover, φ is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 90.28.3. In particular, we find that $\varphi(\mathfrak{p})A'$ is a well defined ideal of A' by Formal Deformation Theory, Lemma 90.28.4. Since $A \to A'$ is formally smooth, in fact $\varphi(\mathfrak{p})A'$ is a minimal prime of A' and every minimal prime of A' is of this form for a unique minimal prime $\mathfrak{p} \subset A$ (all of this is easy to prove by writing A' as a power series ring over A). Therefore, recalling that minimal primes correspond to irreducible components, the following definition makes sense.

0DRA **Definition 107.4.1.** Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S. The *formal branches of $\mathcal{X}$ through x_0* is the set of irreducible components of $\mathrm{Spec}(A)$ for any choice of versal ring to $\mathcal{X}$ at x_0 identified for different choices of A by the procedure described above.

[1] In the sense that A' becomes isomorphic to a power series ring over A.

Suppose in the situation of Definition 107.4.1 we are given a finite extension l/k. Set $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ equal to the composition of $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ with x_0. Let $A \to A'$ be as in Lemma 107.2.5. Since $A \to A'$ is faithfully flat, the morphism

$$\mathrm{Spec}(A') \to \mathrm{Spec}(A)$$

sends (generic points of) irreducible components to (generic points of) irreducible components. This will be a surjective map, but in general this map will not be a bijection. In other words, we obtain a surjective map

$$\text{formal branches of } \mathcal{X} \text{ through } x_{l,0} \longrightarrow \text{formal branches of } \mathcal{X} \text{ through } x_0$$

It turns out that if l/k is purely inseparable, then the map is injective as well (we'll add a precise statement and proof here if we ever need this).

0DRB **Lemma 107.4.2.** *In the situation of Definition 107.4.1 there is a canonical surjection from the set of formal branches of $\mathcal{X}$ through x_0 to the set of irreducible components of $|\mathcal{X}|$ containing x_0 in $|\mathcal{X}|$.*

Proof. Let A be as in Definition 107.4.1 and let $\mathrm{Spec}(A) \to \mathcal{X}$ be as in Remark 107.2.9. We claim that the generic point of an irreducible component of $\mathrm{Spec}(A)$ maps to a generic point of an irreducible component of $|\mathcal{X}|$. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) \times_{\mathcal{X}} U & \xrightarrow{q} & U \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{j} & \mathcal{X} \end{array}$$

By Lemma 107.2.10 we see that j is flat. Hence q is flat. On the other hand, f is surjective smooth hence p is surjective smooth. This implies that any generic point $\eta \in \mathrm{Spec}(A)$ of an irreducible component is the image of a codimension 0 point η' of the algebraic space $\mathrm{Spec}(A) \times_{\mathcal{X}} U$ (see Properties of Spaces, Section 66.11 for notation and use going down on étale local rings). Since q is flat, $q(\eta')$ is a codimension 0 point of U (same argument). Since U is a scheme, $q(\eta')$ is the generic point of an irreducible component of U. Thus the closure of the image of $q(\eta')$ in $|\mathcal{X}|$ is an irreducible component by Lemma 107.3.1 as claimed.

Clearly the claim provides a mechanism for defining the desired map. To see that it is surjective, we choose $u_0 \in U$ mapping to x_0 in $|\mathcal{X}|$. Choose an affine open $U' \subset U$ neighbourhood of u_0. After shrinking U' we may assume every irreducible component of U' passes through u_0. Then we may replace $\mathcal{X}$ by the open substack corresponding to the image of $|U'| \to |\mathcal{X}|$. Thus we may assume U is affine has a point u_0 mapping to $x_0 \in |\mathcal{X}|$ and every irreducible component of U passes through u_0. By Properties of Stacks, Lemma 100.4.3 there is a point $t \in |\mathrm{Spec}(A) \times_{\mathcal{X}} U|$ mapping to the closed point of $\mathrm{Spec}(A)$ and to u_0. Using going down for the flat local ring homomorphisms

$$A \longrightarrow \mathcal{O}_{\mathrm{Spec}(A) \times_{\mathcal{X}} U, \bar{t}} \longleftarrow \mathcal{O}_{U,u_0}$$

we see that every minimal prime of $\mathcal{O}_{U,u_0}$ is the image of a minimal prime of the local ring in the middle and such a minimal prime maps to a minimal prime of A. This proves the surjectivity. Some details omitted. $\square$

Let A be a Noetherian complete local ring. Then the irreducible components of $\mathrm{Spec}(A)$ have multiplicities, see introduction to Section 107.3. If $A' = A[[t_1, \ldots, t_r]]$, then the morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ induces a bijection on irreducible components preserving multiplicities (we omit the easy proof). This and the discussion preceding Definition 107.4.1 mean that the following definition makes sense.

0DRC **Definition 107.4.3.** Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S. The *multiplicity of a formal branch of $\mathcal{X}$ through x_0* is the multiplicity of the corresponding irreducible component of $\mathrm{Spec}(A)$ for any choice of versal ring to $\mathcal{X}$ at x_0 (see discussion above).

0DRD **Lemma 107.4.4.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ is a morphism where k is a field of finite type over S with image $s \in S$. If $\mathcal{O}_{S,s}$ is a G-ring, then the map of Lemma 107.4.2 preserves multiplicities.*

Proof. By Lemma 107.2.8 we may assume there is a smooth morphism $U \to \mathcal{X}$ where U is a scheme and a k-valued point u_0 of U such that $\mathcal{O}^\wedge_{U,u_0}$ is a versal ring to $\mathcal{X}$ at x_0. By construction of our map in the proof of Lemma 107.4.2 (which simplifies greatly because $A = \mathcal{O}^\wedge_{U,u_0}$) we find that it suffices to show: the multiplicity of an irreducible component of U passing through u_0 is the same as the multiplicity of any irreducible component of $\mathrm{Spec}(\mathcal{O}^\wedge_{U,u_0})$ mapping into it.

Translated into commutative algebra we find the following: Let $C = \mathcal{O}_{U,u_0}$. This is essentially of finite type over $\mathcal{O}_{S,s}$ and hence is a G-ring (More on Algebra, Proposition 15.50.10). Then $A = C^\wedge$. Therefore $C \to A$ is a regular ring map. Let $\mathfrak{q} \subset C$ be a minimal prime and let $\mathfrak{p} \subset A$ be a minimal prime lying over $\mathfrak{q}$. Then

$$R = C_\mathfrak{p} \longrightarrow A_\mathfrak{p} = R'$$

is a regular ring map of Artinian local rings. For such a ring map it is always the case that

$$\mathrm{length}_R R = \mathrm{length}_{R'} R'$$

This is what we have to show because the left hand side is the multiplicity of our component on U and the right hand side is the multiplicity of our component on $\mathrm{Spec}(A)$. To see the equality, first we use that

$$\mathrm{length}_R(R)\mathrm{length}_{R'}(R'/\mathfrak{m}_R R') = \mathrm{length}_{R'}(R')$$

by Algebra, Lemma 10.52.13. Thus it suffices to show $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$, which is a consequence of being a regular homomorphism of zero dimensional local rings. □

107.5. Dimension theory of algebraic stacks

0DRE The main results on the dimension theory of algebraic stacks in the literature that we are aware of are those of [**Oss15**], which makes a study of the notions of codimension and relative dimension. We make a more detailed examination of the notion of the dimension of an algebraic stack at a point, and prove various results relating the dimension of the fibres of a morphism at a point in the source to the dimension of its source and target. We also prove a result (Lemma 107.6.4 below) which allow us (under suitable hypotheses) to compute the dimension of an algebraic stack at a point in terms of a versal ring.

While we haven't always tried to optimise our results, we have largely tried to avoid making unnecessary hypotheses. However, in some of our results, in which we compare certain properties of an algebraic stack to the properties of a versal ring to this stack at a point, we have restricted our attention to the case of algebraic stacks that are locally finitely presented over a locally Noetherian scheme base, all of whose local rings are G-rings. This gives us the convenience of having Artin approximation available to compare the geometry of the versal ring to the geometry of the stack itself. However, this restrictive hypothesis may not be necessary for the truth of all of the various statements that we prove. Since it is satisfied in the applications that we have in mind, though, we have been content to make it when it helps.

If X is a scheme, then we define the dimension $\dim(X)$ of X to be the Krull dimension of the topological space underlying X, while if x is a point of X, then we define the dimension $\dim_x(X)$ of X at x to be the minimum of the dimensions of the open subsets U of X containing x, see Properties, Definition 28.10.1. One has the relation $\dim(X) = \sup_{x \in X} \dim_x(X)$, see Properties, Lemma 28.10.2. If X is locally Noetherian, then $\dim_x(X)$ coincides with the supremum of the dimensions at x of the irreducible components of X passing through x.

If X is an algebraic space and $x \in |X|$, then we define $\dim_x X = \dim_u U$, where U is any scheme admitting an étale surjection $U \to X$, and $u \in U$ is any point lying over x, see Properties of Spaces, Definition 66.9.1. We set $\dim(X) = \sup_{x \in |X|} \dim_x(X)$, see Properties of Spaces, Definition 66.9.2.

0DRF **Remark 107.5.1.** In general, the dimension of the algebraic space X at a point x may not coincide with the dimension of the underlying topological space $|X|$ at x. E.g. if k is a field of characteristic zero and $X = \mathbf{A}^1_k/\mathbf{Z}$, then X has dimension 1 (the dimension of $\mathbf{A}^1_k$) at each of its points, while $|X|$ has the indiscrete topology, and hence is of Krull dimension zero. On the other hand, in Algebraic Spaces, Example 65.14.9 there is given an example of an algebraic space which is of dimension 0 at each of its points, while $|X|$ is irreducible of Krull dimension 1, and admits a generic point (so that the dimension of $|X|$ at any of its points is 1); see also the discussion of this example in Properties of Spaces, Section 66.9.

On the other hand, if X is a *decent* algebraic space, in the sense of Decent Spaces, Definition 68.6.1 (in particular, if X is quasi-separated; see Decent Spaces, Section 68.6) then in fact the dimension of X at x does coincide with the dimension of $|X|$ at x; see Decent Spaces, Lemma 68.12.5.

In order to define the dimension of an algebraic stack, it will be useful to first have the notion of the relative dimension, at a point in the source, of a morphism whose source is an algebraic space, and whose target is an algebraic stack. The definition is slightly involved, just because (unlike in the case of schemes) the points of an algebraic stack, or an algebraic space, are not describable as morphisms from the spectrum of a field, but only as equivalence classes of such.

0DRG **Definition 107.5.2.** If $f : T \to \mathcal{X}$ is a locally of finite type morphism from an algebraic space to an algebraic stack, and if $t \in |T|$ is a point with image $x \in |\mathcal{X}|$, then we define *the relative dimension* of f at t, denoted $\dim_t(T_x)$, as follows: choose a morphism $\operatorname{Spec} k \to \mathcal{X}$, with source the spectrum of a field, which represents x,

and choose a point $t' \in |T \times_{\mathcal{X}} \operatorname{Spec} k|$ mapping to t under the projection to $|T|$ (such a point t' exists, by Properties of Stacks, Lemma 100.4.3); then

$$\dim_t(T_x) = \dim_{t'}(T \times_{\mathcal{X}} \operatorname{Spec} k).$$

Note that since T is an algebraic space and $\mathcal{X}$ is an algebraic stack, the fibre product $T \times_{\mathcal{X}} \operatorname{Spec} k$ is an algebraic space, and so the quantity on the right hand side of this proposed definition is in fact defined (see discussion above).

0DRH **Remark 107.5.3.** (1) One easily verifies (for example, by using the invariance of the relative dimension of locally of finite type morphisms of schemes under base-change; see for example Morphisms, Lemma 29.28.3) that $\dim_t(T_x)$ is well-defined, independently of the choices used to compute it.

(2) In the case that $\mathcal{X}$ is also an algebraic space, it is straightforward to confirm that this definition agrees with the definition of relative dimension given in Morphisms of Spaces, Definition 67.33.1.

We next recall the following lemma, on which our study of the dimension of a locally Noetherian algebraic stack is founded.

0DRI **Lemma 107.5.4.** *If $f : U \to X$ is a smooth morphism of locally Noetherian algebraic spaces, and if $u \in |U|$ with image $x \in |X|$, then*

$$\dim_u(U) = \dim_x(X) + \dim_u(U_x)$$

where $\dim_u(U_x)$ is defined via Definition 107.5.2.

Proof. See Morphisms of Spaces, Lemma 67.37.10 noting that the definition of $\dim_u(U_x)$ used here coincides with the definition used there, by Remark 107.5.3 (2). □

0DRJ **Lemma 107.5.5.** *If $\mathcal{X}$ is a locally Noetherian algebraic stack and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}$ be a smooth morphism from an algebraic space to $\mathcal{X}$, let u be any point of $|U|$ mapping to x. Then we have*

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_u(U_x)$$

where the relative dimension $\dim_u(U_x)$ is defined by Definition 107.5.2 and the dimension of $\mathcal{X}$ at x is as in Properties of Stacks, Definition 100.12.2.

Proof. Lemma 107.5.4 can be used to verify that the right hand side $\dim_u(U) + \dim_u(U_x)$ is independent of the choice of the smooth morphism $U \to \mathcal{X}$ and $u \in |U|$. We omit the details. In particular, we may assume U is a scheme. In this case we can compute $\dim_u(U_x)$ by choosing the representative of x to be the composite $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$, where the first morphism is the canonical one with image $u \in U$. Then, if we write $R = U \times_{\mathcal{X}} U$, and let $e : U \to R$ denote the diagonal morphism, the invariance of relative dimension under base-change shows that $\dim_u(U_x) = \dim_{e(u)}(R_u)$. Thus we see that the right hand side is equal to $\dim_u(U) - \dim_{e(u)}(R_u) = \dim_x(\mathcal{X})$ as desired. □

0DRK **Remark 107.5.6.** For Deligne–Mumford stacks which are suitably decent (e.g. quasi-separated), it will again be the case that $\dim_x(\mathcal{X})$ coincides with the topologically defined quantity $\dim_x |\mathcal{X}|$. However, for more general Artin stacks, this will typically not be the case. For example, if $\mathcal{X} = [\mathbf{A}^1/\mathbf{G}_m]$ (over some field, with the quotient being taken with respect to the usual multiplication action of $\mathbf{G}_m$ on

$\mathbf{A}^1$), then $|\mathcal{X}|$ has two points, one the specialisation of the other (corresponding to the two orbits of $\mathbf{G}_m$ on $\mathbf{A}^1$), and hence is of dimension 1 as a topological space; but $\dim_x(\mathcal{X}) = 0$ for both points $x \in |\mathcal{X}|$. (An even more extreme example is given by the classifying space $[\operatorname{Spec} k/\mathbf{G}_m]$, whose dimension at its unique point is equal to -1.)

We can now extend Definition 107.5.2 to the context of (locally finite type) morphisms between (locally Noetherian) algebraic stacks.

0DRL **Definition 107.5.7.** If $f : \mathcal{T} \to \mathcal{X}$ is a locally of finite type morphism between locally Noetherian algebraic stacks, and if $t \in |\mathcal{T}|$ is a point with image $x \in |\mathcal{X}|$, then we define the *relative dimension* of f at t, denoted $\dim_t(\mathcal{T}_x)$, as follows: choose a morphism $\operatorname{Spec} k \to \mathcal{X}$, with source the spectrum of a field, which represents x, and choose a point $t' \in |\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k|$ mapping to t under the projection to $|\mathcal{T}|$ (such a point t' exists, by Properties of Stacks, Lemma 100.4.3; then

$$\dim_t(\mathcal{T}_x) = \dim_{t'}(\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k).$$

Note that since $\mathcal{T}$ is an algebraic stack and $\mathcal{X}$ is an algebraic stack, the fibre product $\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k$ is an algebraic stack, which is locally Noetherian by Morphisms of Stacks, Lemma 101.17.5. Thus the quantity on the right side of this proposed definition is defined by Properties of Stacks, Definition 100.12.2.

0DRM **Remark 107.5.8.** Standard manipulations show that $\dim_t(\mathcal{T}_x)$ is well-defined, independently of the choices made to compute it.

We now establish some basic properties of relative dimension, which are obvious generalisations of the corresponding statements in the case of morphisms of schemes.

0DRN **Lemma 107.5.9.** *Suppose given a Cartesian square of morphisms of locally Noetherian stacks*

$$\begin{array}{ccc} \mathcal{T}' & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

in which the vertical morphisms are locally of finite type. If $t' \in |\mathcal{T}'|$, with images t, x', and x in $|\mathcal{T}|$, $|\mathcal{X}'|$, and $|\mathcal{X}|$ respectively, then $\dim_{t'}(\mathcal{T}'_{x'}) = \dim_t(\mathcal{T}_x)$.

Proof. Both sides can (by definition) be computed as the dimension of the same fibre product. □

0DRP **Lemma 107.5.10.** *If $f : \mathcal{U} \to \mathcal{X}$ is a smooth morphism of locally Noetherian algebraic stacks, and if $u \in |\mathcal{U}|$ with image $x \in |\mathcal{X}|$, then*

$$\dim_u(\mathcal{U}) = \dim_x(\mathcal{X}) + \dim_u(\mathcal{U}_x).$$

Proof. Choose a smooth surjective morphism $V \to \mathcal{U}$ whose source is a scheme, and let $v \in |V|$ be a point mapping to u. Then the composite $V \to \mathcal{U} \to \mathcal{X}$ is also smooth, and by Lemma 107.5.4 we have $\dim_x(\mathcal{X}) = \dim_v(V) - \dim_v(V_x)$, while $\dim_u(\mathcal{U}) = \dim_v(V) - \dim_v(V_u)$. Thus

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_v(V_x) - \dim_v(V_u).$$

Choose a representative $\operatorname{Spec} k \to \mathcal{X}$ of x and choose a point $v' \in |V \times_{\mathcal{X}} \operatorname{Spec} k|$ lying over v, with image u' in $|\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k|$; then by definition $\dim_u(\mathcal{U}_x) = \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k)$, and $\dim_v(V_x) = \dim_{v'}(V \times_{\mathcal{X}} \operatorname{Spec} k)$.

Now $V \times_{\mathcal{X}} \operatorname{Spec} k \to \mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k$ is a smooth surjective morphism (being the base-change of such a morphism) whose source is an algebraic space (since V and $\operatorname{Spec} k$ are schemes, and $\mathcal{X}$ is an algebraic stack). Thus, again by definition, we have

$$\begin{aligned}\dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k) &= \dim_{v'}(V \times_{\mathcal{X}} \operatorname{Spec} k) - \dim_{v'}(V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}) \\ &= \dim_v(V_x) - \dim_{v'}((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}).\end{aligned}$$

Now $V \times_{\mathcal{X}} \operatorname{Spec} k \cong V \times_{\mathcal{U}} (\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k)$, and so Lemma 107.5.9 shows that $\dim_{v'}((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}) = \dim_v(V_u)$. Putting everything together, we find that

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_u(\mathcal{U}_x),$$

as required. □

0DRQ **Lemma 107.5.11.** *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of algebraic stacks.*

1. *The function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is upper semi-continuous on $|\mathcal{T}|$.*
2. *If f is smooth, then the function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is locally constant on $|\mathcal{T}|$.*

Proof. Suppose to begin with that $\mathcal{T}$ is a scheme T, let $U \to \mathcal{X}$ be a smooth surjective morphism whose source is a scheme, and let $T' = T \times_{\mathcal{X}} U$. Let $f' : T' \to U$ be the pull-back of f over U, and let $g : T' \to T$ be the projection.

Lemma 107.5.9 shows that $\dim_{t'}(T'_{f'(t')}) = \dim_{g(t')}(T_{f(g(t'))})$, for $t' \in T'$, while, since g is smooth and surjective (being the base-change of a smooth surjective morphism) the map induced by g on underlying topological spaces is continuous and open (by Properties of Spaces, Lemma 66.4.6), and surjective. Thus it suffices to note that part (1) for the morphism f' follows from Morphisms of Spaces, Lemma 67.34.4, and part (2) from either of Morphisms, Lemma 29.29.4 or Morphisms, Lemma 29.34.12 (each of which gives the result for schemes, from which the analogous results for algebraic spaces can be deduced exactly as in Morphisms of Spaces, Lemma 67.34.4.

Now return to the general case, and choose a smooth surjective morphism $h : V \to \mathcal{T}$ whose source is a scheme. If $v \in V$, then, essentially by definition, we have

$$\dim_{h(v)}(\mathcal{T}_{f(h(v))}) = \dim_v(V_{f(h(v))}) - \dim_v(V_{h(v)}).$$

Since V is a scheme, we have proved that the first of the terms on the right hand side of this equality is upper semi-continuous (and even locally constant if f is smooth), while the second term is in fact locally constant. Thus their difference is upper semi-continuous (and locally constant if f is smooth), and hence the function $\dim_{h(v)}(\mathcal{T}_{f(h(v))})$ is upper semi-continuous on $|V|$ (and locally constant if f is smooth). Since the morphism $|V| \to |\mathcal{T}|$ is open and surjective, the lemma follows. □

Before continuing with our development, we prove two lemmas related to the dimension theory of schemes.

To put the first lemma in context, we note that if X is a finite dimensional scheme, then since $\dim X$ is defined to equal the supremum of the dimensions $\dim_x X$, there exists a point $x \in X$ such that $\dim_x X = \dim X$. The following lemma shows that we may furthermore take the point x to be of finite type.

0DRR **Lemma 107.5.12.** *If X is a finite dimensional scheme, then there exists a closed (and hence finite type) point $x \in X$ such that $\dim_x X = \dim X$.*

Proof. Let $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

0DRS (107.5.12.1) $$Z_0 \supset Z_1 \supset \ldots \supset Z_d.$$

The subset Z_d is a minimal irreducible closed subset of X, and thus any point of Z_d is a generic point of Z_d. Since the underlying topological space of the scheme X is sober, we conclude that Z_d is a singleton, consisting of a single closed point $x \in X$. If U is any neighbourhood of x, then the chain

$$U \cap Z_0 \supset U \cap Z_1 \supset \ldots \supset U \cap Z_d = Z_d = \{x\}$$

is then a strictly descending chain of irreducible closed subsets of U, showing that $\dim U \geq d$. Thus we find that $\dim_x X \geq d$. The other inequality being obvious, the lemma is proved. □

The next lemma shows that $\dim_x X$ is a *constant* function on an irreducible scheme satisfying some mild additional hypotheses.

0DRT **Lemma 107.5.13.** *If X is an irreducible, Jacobson, catenary, and locally Noetherian scheme of finite dimension, then $\dim U = \dim X$ for every non-empty open subset U of X. Equivalently, $\dim_x X$ is a constant function on X.*

Proof. The equivalence of the two claims follows directly from the definitions. Suppose, then, that $U \subset X$ is a non-empty open subset. Certainly $\dim U \leq \dim X$, and we have to show that $\dim U \geq \dim X$. Write $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

$$X = Z_0 \supset Z_1 \supset \ldots \supset Z_d.$$

Since X is Jacobson, the minimal irreducible closed subset Z_d is equal to $\{x\}$ for some closed point x.

If $x \in U$, then

$$U = U \cap Z_0 \supset U \cap Z_1 \supset \ldots \supset U \cap Z_d = \{x\}$$

is a strictly decreasing chain of irreducible closed subsets of U, and so we conclude that $\dim U \geq d$, as required. Thus we may suppose that $x \notin U$.

Consider the flat morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$. The non-empty (and hence dense) open subset U of X pulls back to an open subset $V \subset \operatorname{Spec} \mathcal{O}_{X,x}$. Replacing U by a non-empty quasi-compact, and hence Noetherian, open subset, we may assume that the inclusion $U \to X$ is a quasi-compact morphism. Since the formation of scheme-theoretic images of quasi-compact morphisms commutes with flat base-change Morphisms, Lemma 29.25.16 we see that V is dense in $\operatorname{Spec} \mathcal{O}_{X,x}$, and so in particular non-empty, and of course $x \notin V$. (Here we use x also to denote the closed point of $\operatorname{Spec} \mathcal{O}_{X,x}$, since its image is equal to the given point $x \in X$.) Now $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$ is Jacobson Properties, Lemma 28.6.4 and hence V contains a closed point z of $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$. The closure in X of the image of z is then an irreducible closed subset Z of X containing x, whose intersection with U is non-empty, and for which there is no irreducible closed subset properly contained in Z and properly containing $\{x\}$ (because pull-back to $\operatorname{Spec} \mathcal{O}_{X,x}$ induces a bijection between irreducible closed subsets of X containing x and irreducible closed subsets

of $\operatorname{Spec}\mathcal{O}_{X,x}$). Since $U \cap Z$ is a non-empty closed subset of U, it contains a point u that is closed in X (since X is Jacobson), and since $U \cap Z$ is a non-empty (and hence dense) open subset of the irreducible set Z (which contains a point not lying in U, namely x), the inclusion $\{u\} \subset U \cap Z$ is proper.

As X is catenary, the chain

$$X = Z_0 \supset Z \supset \{x\} = Z_d$$

can be refined to a chain of length $d+1$, which must then be of the form

$$X = Z_0 \supset W_1 \supset \ldots \supset W_{d-1} = Z \supset \{x\} = Z_d.$$

Since $U \cap Z$ is non-empty, we then find that

$$U = U \cap Z_0 \supset U \cap W_1 \supset \ldots \supset U \cap W_{d-1} = U \cap Z \supset \{u\}$$

is a strictly decreasing chain of irreducible closed subsets of U of length $d+1$, showing that $\dim U \geq d$, as required. □

We will prove a stack-theoretic analogue of Lemma 107.5.13 in Lemma 107.5.17 below, but before doing so, we have to introduce an additional definition, necessitated by the fact that the notion of a scheme being catenary is not an étale local one (see the example of Algebra, Remark 10.164.8 which makes it difficult to define what it means for an algebraic space or algebraic stack to be catenary (see the discussion of [**Oss15**, page 3]). For certain aspects of dimension theory, the following definition seems to provide a good substitute for the missing notion of a catenary algebraic stack.

0DRU **Definition 107.5.14.** We say that a locally Noetherian algebraic stack $\mathcal{X}$ is *pseudo-catenary* if there exists a smooth and surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme.

0DRV **Example 107.5.15.** If $\mathcal{X}$ is locally of finite type over a universally catenary locally Noetherian scheme S, and $U \to \mathcal{X}$ is a smooth surjective morphism whose source is a scheme, then the composite $U \to \mathcal{X} \to S$ is locally of finite type, and so U is universally catenary Morphisms, Lemma 29.17.2. Thus $\mathcal{X}$ is pseudo-catenary.

The following lemma shows that the property of being pseudo-catenary passes through finite-type morphisms.

0DRW **Lemma 107.5.16.** *If $\mathcal{X}$ is a pseudo-catenary locally Noetherian algebraic stack, and if $\mathcal{Y} \to \mathcal{X}$ is a locally of finite type morphism, then there exists a smooth surjective morphism $V \to \mathcal{Y}$ whose source is a universally catenary scheme; thus $\mathcal{Y}$ is again pseudo-catenary.*

Proof. By assumption we may find a smooth surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme. The base-change $U \times_{\mathcal{X}} \mathcal{Y}$ is then an algebraic stack; let $V \to U \times_{\mathcal{X}} \mathcal{Y}$ be a smooth surjective morphism whose source is a scheme. The composite $V \to U \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is then smooth and surjective (being a composite of smooth and surjective morphisms), while the morphism $V \to U \times_{\mathcal{X}} \mathcal{Y} \to U$ is locally of finite type (being a composite of morphisms that are locally finite type). Since U is universally catenary, we see that V is universally catenary (by Morphisms, Lemma 29.17.2), as claimed. □

We now study the behaviour of the function $\dim_x(\mathcal{X})$ on $|\mathcal{X}|$ (for some locally Noetherian stack $\mathcal{X}$) with respect to the irreducible components of $|\mathcal{X}|$, as well as various related topics.

0DRX **Lemma 107.5.17.** *If $\mathcal{X}$ is a Jacobson, pseudo-catenary, and locally Noetherian algebraic stack for which $|\mathcal{X}|$ is irreducible, then $\dim_x(\mathcal{X})$ is a constant function on $|\mathcal{X}|$.*

Proof. It suffices to show that $\dim_x(\mathcal{X})$ is locally constant on $|\mathcal{X}|$, since it will then necessarily be constant (as $|\mathcal{X}|$ is connected, being irreducible). Since $\mathcal{X}$ is pseudo-catenary, we may find a smooth surjective morphism $U \to \mathcal{X}$ with U being a universally catenary scheme. If $\{U_i\}$ is an cover of U by quasi-compact open subschemes, we may replace U by $\coprod U_i$,, and it suffices to show that the function $u \mapsto \dim_{f(u)}(\mathcal{X})$ is locally constant on U_i. Since we check this for one U_i at a time, we now drop the subscript, and write simply U rather than U_i. Since U is quasi-compact, it is the union of a finite number of irreducible components, say $T_1 \cup \ldots \cup T_n$. Note that each T_i is Jacobson, catenary, and locally Noetherian, being a closed subscheme of the Jacobson, catenary, and locally Noetherian scheme U.

By Lemma 107.5.4, we have $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$. Lemma 107.5.11 (2) shows that the second term in the right hand expression is locally constant on U, as f is smooth, and hence we must show that $\dim_u(U)$ is locally constant on U. Since $\dim_u(U)$ is the maximum of the dimensions $\dim_u T_i$, as T_i ranges over the components of U containing u, it suffices to show that if a point u lies on two distinct components, say T_i and T_j (with $i \neq j$), then $\dim_u T_i = \dim_u T_j$, and then to note that $t \mapsto \dim_t T$ is a constant function on an irreducible Jacobson, catenary, and locally Noetherian scheme T (as follows from Lemma 107.5.13).

Let $V = T_i \setminus (\bigcup_{i' \neq i} T_{i'})$ and $W = T_j \setminus (\bigcup_{i' \neq j} T_{i'})$. Then each of V and W is a non-empty open subset of U, and so each has non-empty open image in $|\mathcal{X}|$. As $|\mathcal{X}|$ is irreducible, these two non-empty open subsets of $|\mathcal{X}|$ have a non-empty intersection. Let x be a point lying in this intersection, and let $v \in V$ and $w \in W$ be points mapping to x. We then find that

$$\dim T_i = \dim V = \dim_v(U) = \dim_x(\mathcal{X}) + \dim_v(U_x)$$

and similarly that

$$\dim T_j = \dim W = \dim_w(U) = \dim_x(\mathcal{X}) + \dim_w(U_x).$$

Since $u \mapsto \dim_u(U_{f(u)})$ is locally constant on U, and since $T_i \cup T_j$ is connected (being the union of two irreducible, hence connected, sets that have non-empty intersection), we see that $\dim_v(U_x) = \dim_w(U_x)$, and hence, comparing the preceding two equations, that $\dim T_i = \dim T_j$, as required. □

0DRY **Lemma 107.5.18.** *If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of locally Noetherian algebraic stacks, and if $z \in |\mathcal{Z}|$ has image $x \in |\mathcal{X}|$, then $\dim_z(\mathcal{Z}) \leq \dim_x(\mathcal{X})$.*

Proof. Choose a smooth surjective morphism $U \to \mathcal{X}$ whose source is a scheme; the base-changed morphism $V = U \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ is then also smooth and surjective, and the projection $V \to U$ is a closed immersion. If $v \in |V|$ maps to $z \in |\mathcal{Z}|$, and

if we let u denote the image of v in $|U|$, then clearly $\dim_v(V) \leq \dim_u(U)$, while $\dim_v(V_z) = \dim_u(U_x)$, by Lemma 107.5.9. Thus

$$\dim_z(\mathcal{Z}) = \dim_v(V) - \dim_v(V_z) \leq \dim_u(U) - \dim_u(U_x) = \dim_x(\mathcal{X}),$$

as claimed. □

0DRZ **Lemma 107.5.19.** *If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then $\dim_x(\mathcal{X}) = \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}$, where $\mathcal{T}$ runs over all the irreducible components of $|\mathcal{X}|$ passing through x (endowed with their induced reduced structure).*

Proof. Lemma 107.5.18 shows that $\dim_x(\mathcal{T}) \leq \dim_x(\mathcal{X})$ for each irreducible component $\mathcal{T}$ passing through the point x. Thus to prove the lemma, it suffices to show that

0DS0 (107.5.19.1) $$\dim_x(\mathcal{X}) \leq \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}.$$

Let $U \to \mathcal{X}$ be a smooth cover by a scheme. If T is an irreducible component of U then we let $\mathcal{T}$ denote the closure of its image in $\mathcal{X}$, which is an irreducible component of $\mathcal{X}$. Let $u \in U$ be a point mapping to x. Then we have $\dim_x(\mathcal{X}) = \dim_u U - \dim_u U_x = \sup_T \dim_u T - \dim_u U_x$, where the supremum is over the irreducible components of U passing through u. Choose a component T for which the supremum is achieved, and note that $\dim_x(\mathcal{T}) = \dim_u T - \dim_u T_x$. The desired inequality (107.5.19.1) now follows from the evident inequality $\dim_u T_x \leq \dim_u U_x$. (Note that if $\operatorname{Spec} k \to \mathcal{X}$ is a representative of x, then $T \times_{\mathcal{X}} \operatorname{Spec} k$ is a closed subspace of $U \times_{\mathcal{X}} \operatorname{Spec} k$.) □

0DS1 **Lemma 107.5.20.** *If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then for any open substack $\mathcal{V}$ of $\mathcal{X}$ containing x, there is a finite type point $x_0 \in |\mathcal{V}|$ such that $\dim_{x_0}(\mathcal{X}) = \dim_x(\mathcal{V})$.*

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ whose source is a scheme, and consider the function $u \mapsto \dim_{f(u)}(\mathcal{X})$; since the morphism $|U| \to |\mathcal{X}|$ induced by f is open (as f is smooth) as well as surjective (by assumption), and takes finite type points to finite type points (by the very definition of the finite type points of $|\mathcal{X}|$), it suffices to show that for any $u \in U$, and any open neighbourhood of u, there is a finite type point u_0 in this neighbourhood such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. Since, with this reformulation of the problem, the surjectivity of f is no longer required, we may replace U by the open neighbourhood of the point u in question, and thus reduce to the problem of showing that for each $u \in U$, there is a finite type point $u_0 \in U$ such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. By Lemma 107.5.4 $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$, while $\dim_{f(u_0)}(\mathcal{X}) = \dim_{u_0}(U) - \dim_{u_0}(U_{f(u_0)})$. Since f is smooth, the expression $\dim_{u_0}(U_{f(u_0)})$ is locally constant as u_0 varies over U (by Lemma 107.5.11 (2)), and so shrinking U further around u if necessary, we may assume it is constant. Thus the problem becomes to show that we may find a finite type point $u_0 \in U$ for which $\dim_{u_0}(U) = \dim_u(U)$. Since by definition $\dim_u U$ is the minimum of the dimensions $\dim V$, as V ranges over the open neighbourhoods V of u in U, we may shrink U down further around u so that $\dim_u U = \dim U$. The existence of desired point u_0 then follows from Lemma 107.5.12. □

0DS2 **Lemma 107.5.21.** *Let $\mathcal{T} \hookrightarrow \mathcal{X}$ be a locally of finite type monomorphism of algebraic stacks, with $\mathcal{X}$ (and thus also $\mathcal{T}$) being Jacobson, pseudo-catenary, and*

locally Noetherian. Suppose further that $\mathcal{T}$ is irreducible of some (finite) dimension d, and that $\mathcal{X}$ is reduced and of dimension less than or equal to d. Then there is a non-empty open substack $\mathcal{V}$ of $\mathcal{T}$ such that the induced monomorphism $\mathcal{V} \hookrightarrow \mathcal{X}$ is an open immersion which identifies $\mathcal{V}$ with an open subset of an irreducible component of $\mathcal{X}$.

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ with source a scheme, necessarily reduced since $\mathcal{X}$ is, and write $U' = \mathcal{T} \times_{\mathcal{X}} U$. The base-changed morphism $U' \to U$ is a monomorphism of algebraic spaces, locally of finite type, and thus representable Morphisms of Spaces, Lemma 67.51.1 and 67.27.10; since U is a scheme, so is U'. The projection $f' : U' \to \mathcal{T}$ is again a smooth surjection. Let $u' \in U'$, with image $u \in U$. Lemma 107.5.9 shows that $\dim_{u'}(U'_{f(u')}) = \dim_u(U_{f(u)})$, while $\dim_{f'(u')}(\mathcal{T}) = d \geq \dim_{f(u)}(\mathcal{X})$ by Lemma 107.5.17 and our assumptions on $\mathcal{T}$ and $\mathcal{X}$. Thus we see that

(107.5.21.1)

0DS3 $$\dim_{u'}(U') = \dim_{u'}(U'_{f(u')}) + \dim_{f'(u')}(\mathcal{T}) \geq \dim_u(U_{f(u)}) + \dim_{f(u)}(\mathcal{X}) = \dim_u(U).$$

Since $U' \to U$ is a monomorphism, locally of finite type, it is in particular unramified, and so by the étale local structure of unramified morphisms Étale Morphisms, Lemma 41.17.3, we may find a commutative diagram

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

in which the scheme V' is non-empty, the vertical arrows are étale, and the upper horizontal arrow is a closed immersion. Replacing V by a quasi-compact open subset whose image has non-empty intersection with the image of U', and replacing V' by the preimage of V, we may further assume that V (and thus V') is quasi-compact. Since V is also locally Noetherian, it is thus Noetherian, and so is the union of finitely many irreducible components.

Since étale morphisms preserve pointwise dimension Descent, Lemma 35.21.2 we deduce from (107.5.21.1) that for any point $v' \in V'$, with image $v \in V$, we have $\dim_{v'}(V') \geq \dim_v(V)$. In particular, the image of V' can't be contained in the intersection of two distinct irreducible components of V, and so we may find at least one irreducible open subset of V which has non-empty intersection with V'; replacing V by this subset, we may assume that V is integral (being both reduced and irreducible). From the preceding inequality on dimensions, we conclude that the closed immersion $V' \hookrightarrow V$ is in fact an isomorphism. If we let W denote the image of V' in U', then W is a non-empty open subset of U' (as étale morphisms are open), and the induced monomorphism $W \to U$ is étale (since it is so étale locally on the source, i.e. after pulling back to V'), and hence is an open immersion (being an étale monomorphism). Thus, if we let $\mathcal{V}$ denote the image of W in $\mathcal{T}$, then $\mathcal{V}$ is a dense (equivalently, non-empty) open substack of $\mathcal{T}$, whose image is dense in an irreducible component of $\mathcal{X}$. Finally, we note that the morphism is $\mathcal{V} \to \mathcal{X}$ is smooth (since its composite with the smooth morphism $W \to \mathcal{V}$ is smooth), and also a monomorphism, and thus is an open immersion. □

0DS4 **Lemma 107.5.22.** *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks, whose source is irreducible*

and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then for all $t \in |T|$, we have that $\dim_t(\mathcal{T}_{f(t)}) \geq \dim \mathcal{T} - \dim \mathcal{Z}$, and there is a non-empty (equivalently, dense) open subset of $|\mathcal{T}|$ over which equality holds.

Proof. Replacing $\mathcal{X}$ by $\mathcal{Z}$, we may and do assume that f is scheme theoretically dominant, and also that $\mathcal{X}$ is irreducible. By the upper semi-continuity of fibre dimensions (Lemma 107.5.11 (1)), it suffices to prove that the equality $\dim_t(\mathcal{T}_{f(t)}) = \dim \mathcal{T} - \dim \mathcal{Z}$ holds for t lying in some non-empty open substack of $\mathcal{T}$. For this reason, in the argument we are always free to replace $\mathcal{T}$ by a non-empty open substack.

Let $T' \to \mathcal{T}$ be a smooth surjective morphism whose source is a scheme, and let T be a non-empty quasi-compact open subset of T'. Since $\mathcal{Y}$ is quasi-separated, we find that $T \to \mathcal{Y}$ is quasi-compact (by Morphisms of Stacks, Lemma 101.7.7, applied to the morphisms $T \to \mathcal{Y} \to \operatorname{Spec} \mathbf{Z}$). Thus, if we replace $\mathcal{T}$ by the image of T in $\mathcal{T}$, then we may assume (appealing to Morphisms of Stacks, Lemma 101.7.6 that the morphism $f : \mathcal{T} \to \mathcal{X}$ is quasi-compact.

If we choose a smooth surjection $U \to \mathcal{X}$ with U a scheme, then Lemma 107.3.1 ensures that we may find an irreducible open subset V of U such that $V \to \mathcal{X}$ is smooth and scheme-theoretically dominant. Since scheme-theoretic dominance for quasi-compact morphisms is preserved by flat base-change, the base-change $\mathcal{T} \times_{\mathcal{X}} V \to V$ of the scheme-theoretically dominant morphism f is again scheme-theoretically dominant. We let Z denote a scheme admitting a smooth surjection onto this fibre product; then $Z \to \mathcal{T} \times_{\mathcal{X}} V \to V$ is again scheme-theoretically dominant. Thus we may find an irreducible component C of Z which scheme-theoretically dominates V. Since the composite $Z \to \mathcal{T} \times_{\mathcal{X}} V \to \mathcal{T}$ is smooth, and since $\mathcal{T}$ is irreducible, Lemma 107.3.1 shows that any irreducible component of the source has dense image in $|\mathcal{T}|$. We now replace C by a non-empty open subset W which is disjoint from every other irreducible component of Z, and then replace $\mathcal{T}$ and $\mathcal{X}$ by the images of W and V (and apply Lemma 107.5.17 to see that this doesn't change the dimension of either $\mathcal{T}$ or $\mathcal{X}$). If we let $\mathcal{W}$ denote the image of the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$, then $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$ (since the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$ is smooth), and is irreducible (being the image of an irreducible scheme). Thus we end up with a commutative diagram

$$\begin{array}{ccccc} W & \longrightarrow & \mathcal{W} & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{T} & \longrightarrow & \mathcal{X} \end{array}$$

in which W and V are schemes, the vertical arrows are smooth and surjective, the diagonal arrows and the left-hand upper horizontal arroware smooth, and the induced morphism $\mathcal{W} \to \mathcal{T} \times_{\mathcal{X}} V$ is an open immersion. Using this diagram, together with the definitions of the various dimensions involved in the statement of the lemma, we will reduce our verification of the lemma to the case of schemes, where it is known.

Fix $w \in |W|$ with image $w' \in |\mathcal{W}|$, image $t \in |\mathcal{T}|$, image v in $|V|$, and image x in $|\mathcal{X}|$. Essentially by definition (using the fact that $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$, and that

the fibre of a base-change is the base-change of the fibre), we obtain the equalities

$$\dim_v V_x = \dim_{w'} \mathcal{W}_t$$

and

$$\dim_t \mathcal{T}_x = \dim_{w'} \mathcal{W}_v.$$

By Lemma 107.5.4 (the diagonal arrow and right-hand vertical arrow in our diagram realise W and V as smooth covers by schemes of the stacks $\mathcal{T}$ and $\mathcal{X}$), we find that

$$\dim_t \mathcal{T} = \dim_w W - \dim_w W_t$$

and

$$\dim_x \mathcal{X} = \dim_v V - \dim_v V_x.$$

Combining the equalities, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_{w'} \mathcal{W}_v - \dim_w W + \dim_w W_t + \dim_v V - \dim_{w'} \mathcal{W}_t$$

Since $W \to \mathcal{W}$ is a smooth surjection, the same is true if we base-change over the morphism $\operatorname{Spec} \kappa(v) \to V$ (thinking of $W \to \mathcal{W}$ as a morphism over V), and from this smooth morphism we obtain the first of the following two equalities

$$\dim_w W_v - \dim_{w'} \mathcal{W}_v = \dim_w (W_v)_{w'} = \dim_w W_{w'};$$

the second equality follows via a direct comparison of the two fibres involved. Similarly, if we think of $W \to \mathcal{W}$ as a morphism of schemes over $\mathcal{T}$, and base-change over some representative of the point $t \in |\mathcal{T}|$, we obtain the equalities

$$\dim_w W_t - \dim_{w'} \mathcal{W}_t = \dim_w (W_t)_{w'} = \dim_w W_{w'}.$$

Putting everything together, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_w W_v - \dim_w W + \dim_v V.$$

Our goal is to show that the left-hand side of this equality vanishes for a non-empty open subset of t. As w varies over a non-empty open subset of W, its image $t \in |\mathcal{T}|$ varies over a non-empty open subset of $|\mathcal{T}|$ (as $W \to \mathcal{T}$ is smooth).

We are therefore reduced to showing that if $W \to V$ is a scheme-theoretically dominant morphism of irreducible locally Noetherian schemes that is locally of finite type, then there is a non-empty open subset of points $w \in W$ such that $\dim_w W_v = \dim_w W - \dim_v V$ (where v denotes the image of w in V). This is a standard fact, whose proof we recall for the convenience of the reader.

We may replace W and V by their underlying reduced subschemes without altering the validity (or not) of this equation, and thus we may assume that they are in fact integral schemes. Since $\dim_w W_v$ is locally constant on W, replacing W by a non-empty open subset if necessary, we may assume that $\dim_w W_v$ is constant, say equal to d. Choosing this open subset to be affine, we may also assume that the morphism $W \to V$ is in fact of finite type. Replacing V by a non-empty open subset if necessary (and then pulling back W over this open subset; the resulting pull-back is non-empty, since the flat base-change of a quasi-compact and scheme-theoretically dominant morphism remains scheme-theoretically dominant), we may furthermore assume that W is flat over V. The morphism $W \to V$ is thus of relative dimension d in the sense of Morphisms, Definition 29.29.1 and it follows from Morphisms, Lemma 29.29.6 that $\dim_w(W) = \dim_v(V) + d$, as required. $\square$

0DS5 **Remark 107.5.23.** We note that in the context of the preceding lemma, it need not be that $\dim \mathcal{T} \geq \dim \mathcal{Z}$; this does not contradict the inequality in the statement of the lemma, because the fibres of the morphism f are again algebraic stacks, and so may have negative dimension. This is illustrated by taking k to be a field, and applying the lemma to the morphism $[\operatorname{Spec} k/\mathbf{G}_m] \to \operatorname{Spec} k$.

If the morphism f in the statement of the lemma is assumed to be quasi-DM (in the sense of Morphisms of Stacks, Definition 101.4.1; e.g. morphisms that are representable by algebraic spaces are quasi-DM), then the fibres of the morphism over points of the target are quasi-DM algebraic stacks, and hence are of non-negative dimension. In this case, the lemma implies that indeed $\dim \mathcal{T} \geq \dim \mathcal{Z}$. In fact, we obtain the following more general result.

0DS6 **Lemma 107.5.24.** *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks which is quasi-DM, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then $\dim \mathcal{Z} \leq \dim \mathcal{T}$, and furthermore, exactly one of the following two conditions holds:*

(1) *for every finite type point $t \in |T|$, we have $\dim_t(\mathcal{T}_{f(t)}) > 0$, in which case $\dim \mathcal{Z} < \dim \mathcal{T}$; or*
(2) *$\mathcal{T}$ and $\mathcal{Z}$ are of the same dimension.*

Proof. As was observed in the preceding remark, the dimension of a quasi-DM stack is always non-negative, from which we conclude that $\dim_t \mathcal{T}_{f(t)} \geq 0$ for all $t \in |\mathcal{T}|$, with the equality

$$\dim_t \mathcal{T}_{f(t)} = \dim_t \mathcal{T} - \dim_{f(t)} \mathcal{Z}$$

holding for a dense open subset of points $t \in |\mathcal{T}|$. □

107.6. The dimension of the local ring

0DS7 An algebraic stack doesn't really have local rings in the usual sense, but we can define the dimension of the local ring as follows.

0DS8 **Lemma 107.6.1.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $U \to \mathcal{X}$ be a smooth morphism and let $u \in U$. Then*

$$\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R_u,e(\overline{u})}) = 2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})})$$

Here $R = U \times_{\mathcal{X}} U$ with projections $s, t : R \to U$ and diagonal $e : U \to R$ and R_u is the fibre of $s : R \to U$ over u.

Proof. This is true because $s : \mathcal{O}_{U,\overline{u}} \to \mathcal{O}_{R,e(\overline{u})}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{R,e(\overline{u})}) = \dim(\mathcal{O}_{U,\overline{u}}) + \dim(\mathcal{O}_{R_u,e(\overline{u})})$$

by Algebra, Lemma 10.112.7. □

0DS9 **Lemma 107.6.2.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point Morphisms of Stacks, Definition 101.18.2). Let $d \in \mathbf{Z}$. The following are equivalent*

(1) *there exists a scheme U, a smooth morphism $U \to \mathcal{X}$, and a finite type point $u \in U$ mapping to x such that $2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})}) = d$, and*

(2) *for any scheme U, a smooth morphism $U \to \mathcal{X}$, and finite type point $u \in U$ mapping to x we have $2\dim(\mathcal{O}_{U,\overline{u}}) - \dim(\mathcal{O}_{R,e(\overline{u})}) = d$.*

Here $R = U \times_{\mathcal{X}} U$ with projections $s, t : R \to U$ and diagonal $e : U \to R$ and R_u is the fibre of $s : R \to U$ over u.

Proof. Suppose we have two smooth neighbourhoods (U, u) and (U', u') of x with u and u' finite type points. After shrinking U and U' we may assume that u and u' are closed points (by definition of finite type points). Then we choose a surjective étale morphism $W \to U \times_{\mathcal{X}} U'$. Let W_u be the fibre of $W \to U$ over u and let $W_{u'}$ be the fibre of $W \to U'$ over u'. Since u and u' map to the same point of $|\mathcal{X}|$ we see that $W_u \cap W_{u'}$ is nonempty. Hence we may choose a closed point $w \in W$ mapping to both u and u'. This reduces us to the discussion in the next paragraph.

Assume $(U', u') \to (U, u)$ is a smooth morphism of smooth neighbourhoods of x with u and u' closed points. Goal: prove the invariant defined for (U, u) is the same as the invariant defined for (U', u'). To see this observe that $\mathcal{O}_{U,u} \to \mathcal{O}_{U',u'}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{U',\overline{u}'}) = \dim(\mathcal{O}_{U,\overline{u}}) + \dim(\mathcal{O}_{U'_u,\overline{u}'})$$

by Algebra, Lemma 10.112.7. (We omit working through all the steps to relate properties of local rings and their strict henselizations, see More on Algebra, Section 15.45). On the other hand we have

$$R' = U' \times_{U,t} R \times_{s,U} U'$$

Thus we see that

$$\dim(\mathcal{O}_{R',e(\overline{u}')}) = \dim(\mathcal{O}_{R,e(\overline{u})}) + \dim(\mathcal{O}_{U'_u \times_u U'_u,(\overline{u}',\overline{u}')})$$

To prove the lemma it suffices to show that

$$\dim(\mathcal{O}_{U'_u \times_u U'_u,(\overline{u}',\overline{u}')}) = 2\dim(\mathcal{O}_{U'_u,\overline{u}'})$$

Observe that this isn't always true (example: if U'_u is a curve and u' is the generic point of this curve). However, we know that u' is a closed point of the algebraic space U'_u locally of finite type over u. In this case the equality holds because, first $\dim_{(u',u')}(U'_u \times_u U'_u) = 2\dim_{u'}(U'_u)$ by Varieties, Lemma 33.20.5 and second the agreement of dimension with dimension of local rings in closed points of locally algebraic schemes, see Varieties, Lemma 33.20.3. We omit the translation of these results for schemes into the language of algebraic spaces. □

0DSA **Definition 107.6.3.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point. The *dimension of the local ring of $\mathcal{X}$ at x* is $d \in \mathbf{Z}$ if the equivalent conditions of Lemma 107.6.2 are satisfied.

To be sure, this is motivated by Lemma 107.6.1 and Properties of Stacks, Definition 100.12.2. We close this section by establishing a formula allowing us to compute $\dim_x(\mathcal{X})$ in terms of properties of the versal ring to $\mathcal{X}$ at x.

0DSB **Lemma 107.6.4.** *Suppose that $\mathcal{X}$ is an algebraic stack, locally of finite type over a locally Noetherian scheme S. Let $x_0 : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism where k is a field of finite type over S. Represent $\mathcal{F}_{\mathcal{X},k,x_0}$ as in Remark 107.2.11 by a cogroupoid (A, B, s, t, c) of Noetherian complete local S-algebras with residue field k. Then*

$$\textit{the dimension of the local ring of } \mathcal{X} \textit{ at } x_0 = 2\dim A - \dim B$$

Proof. Let $s \in S$ be the image of x_0. If $\mathcal{O}_{S,s}$ is a G-ring (a condition that is almost always satisfied in practice), then we can prove the lemma as follows. By Lemma 107.2.8, we may find a smooth morphism $U \to \mathcal{X}$, whose source is a scheme, containing a point $u_0 \in U$ of residue field k, such that induced morphism $\mathrm{Spec}(k) \to U \to \mathcal{X}$ coincides with x_0 and such that $A = \mathcal{O}^\wedge_{U,u_0}$. Write $R = U \times_{\mathcal{X}} U$. Then we may identify $\mathcal{O}^\wedge_{R,e(u_0)}$ with B. Hence the equality follows from the definitions.

In the rest of this proof we explain how to prove the lemma in general, but we urge the reader to skip this.

First let us show that the right hand side is independent of the choice of (A, B, s, t, c). Namely, suppose that (A', B', s', t', c') is a second choice. Since A and A' are versal rings to $\mathcal{X}$ at x_0, we can choose, after possibly switching A and A', a formally smooth map $A \to A'$ compatible with the given versal formal objects ξ and ξ' over A and A'. Recall that $\widehat{\mathcal{C}}_\Lambda$ has coproducts and that these are given by completed tensor product over Λ, see Formal Deformation Theory, Lemma 90.4.4. Then B prorepresents the functor of isomorphisms between the two pushforwards of ξ to $A\widehat{\otimes}_\Lambda A$. Similarly for B'. We conclude that

$$B' = B \otimes_{(A\widehat{\otimes}_\Lambda A)} (A'\widehat{\otimes}_\Lambda A')$$

It is straightforward to see that

$$A\widehat{\otimes}_\Lambda A \longrightarrow A\widehat{\otimes}_\Lambda A' \longrightarrow A'\widehat{\otimes}_\Lambda A'$$

is formally smooth of relative dimension equal to 2 times the relative dimension of the formally smooth map $A \to A'$. (This follows from general principles, but also because in this particular case A' is a power series ring over A in r variables.) Hence $B \to B'$ is formally smooth of relative dimension $2(\dim(A') - \dim(A))$ as desired.

Next, let l/k be a finite extension. let $x_{l,0} : \mathrm{Spec}(l) \to \mathcal{X}$ be the induced point. We claim that the right hand side of the formula is the same for x_0 as it is for $x_{l,0}$. This can be shown by choosing $A \to A'$ as in Lemma 107.2.5 and arguing exactly as in the preceding paragraph. We omit the details.

Finally, arguing as in the proof of Lemma 107.2.10 we can use the compatibilities in the previous two paragraphs to reduce to the case (discussed in the first paragraph) where A is the complete local ring of U at u_0 for some scheme smooth over $\mathcal{X}$ and finite type point u_0. Details omitted. □

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Part 8

Topics in Moduli Theory

CHAPTER 108

Moduli Stacks

0DLT

108.1. Introduction

0DLU In this chapter we verify basic properties of moduli spaces and moduli stacks such as Hom, $Isom$, $\mathcal{C}oh_{X/B}$, $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, $\mathcal{P}ic_{X/B}$, $\mathrm{Pic}_{X/B}$, $Mor_B(Z, X)$, $\mathcal{S}paces'_{fp,flat,proper}$, $\mathcal{P}olarized$, and $\mathcal{C}omplexes_{X/B}$. We have already shown these algebraic spaces or algebraic stacks under suitable hypotheses, see Quot, Sections 99.3, 99.4, 99.5, 99.6, 99.8, 99.9, 99.10, 99.11, 99.12, 99.13, 99.14, and 99.16. The stack of curves, denoted $Curves$ and introduced in Quot, Section 99.15, is discussed in the chapter on moduli of curves, see Moduli of Curves, Section 109.3.

In some sense this chapter is following the footsteps of Grothendieck's lectures [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**].

108.2. Conventions and abuse of language

0DLV We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

108.3. Properties of Hom and Isom

0DLW Let $f : X \to B$ be a morphism of algebraic spaces which is of finite presentation. Assume $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is of finite presentation, flat over B with support proper over B, then the functor $Hom(\mathcal{F}, \mathcal{G})$ defined by

$$T/B \longmapsto \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

is an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $Hom(\mathcal{F}, \mathcal{G}) \to B$ is of finite presentation. See Quot, Proposition 99.3.10.

If both $\mathcal{F}$ and $\mathcal{G}$ are of finite presentation, flat over B with support proper over B, then the subfunctor

$$Isom(\mathcal{F}, \mathcal{G}) \subset Hom(\mathcal{F}, \mathcal{G})$$

is an algebraic space affine of finite presentation over B. See Quot, Proposition 99.4.3.

108.4. Properties of the stack of coherent sheaves

0DLX Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then the stack $\mathcal{C}oh_{X/B}$ parametrizing flat families of coherent modules with proper support is algebraic. See Quot, Theorem 99.6.1.

0DLY **Lemma 108.4.1.** *The diagonal of $\mathcal{C}oh_{X/B}$ over B is affine and of finite presentation.*

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 99.5.3. From the proof we find that we have to show $Isom(\mathcal{F}, \mathcal{G}) \to T$ is affine and of finite presentation for a pair of finitely presented $\mathcal{O}_{X_T}$-modules $\mathcal{F}$, $\mathcal{G}$ flat over T with support proper over T. This was discussed in Section 108.3. □

0DLZ **Lemma 108.4.2.** *The morphism $\mathcal{C}oh_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}oh_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.4.1. To prove that $\mathcal{C}oh_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathcal{C}oh_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This follows from Quot, Lemma 99.5.6 (small detail omitted). □

0DM0 **Lemma 108.4.3.** *Assume $X \to B$ is proper as well as of finite presentation. Then $\mathcal{C}oh_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 101.39.10).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper over R and a coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}_K$, show there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ flat over R whose generic fibre is $\mathcal{F}_K$. Observe that by Flatness on Spaces, Theorem 77.4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. Denote $j : X_K \to X$ the embedding of the generic fibre. As a base change of the affine morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ the morphism j is affine. Thus $j_*\mathcal{F}_K$ is quasi-coherent. Write

$$j_*\mathcal{F}_K = \operatorname{colim} \mathcal{F}_i$$

as a filtered colimit of its finite type quasi-coherent $\mathcal{O}_X$-submodules, see Limits of Spaces, Lemma 70.9.2. Since $j_*\mathcal{F}_K$ is a sheaf of K-vector spaces over X, it is flat over $\mathrm{Spec}(R)$. Thus each $\mathcal{F}_i$ is flat over R as flatness over a valuation ring is the same as being torsion free (More on Algebra, Lemma 15.22.10) and torsion freeness is inherited by submodules. Finally, we have to show that the map $j^*\mathcal{F}_i \to \mathcal{F}_K$ is an isomorphism for some i. Since $j^*j_*\mathcal{F}_K = \mathcal{F}_K$ (small detail omitted) and since j^* is exact, we see that $j^*\mathcal{F}_i \to \mathcal{F}_K$ is injective for all i. Since j^* commutes with colimits, we have $\mathcal{F}_K = j^*j_*\mathcal{F}_K = \operatorname{colim} j^*\mathcal{F}_i$. Since $\mathcal{F}_K$ is coherent (i.e., finitely presented), there is an i such that $j^*\mathcal{F}_i$ contains all the (finitely many) generators over an affine étale cover of X. Thus we get surjectivity of $j^*\mathcal{F}_i \to \mathcal{F}_K$ for i large enough. □

0DN9 **Lemma 108.4.4.** *Let B be an algebraic space. Let $\pi : X \to Y$ be a quasi-finite morphism of algebraic spaces which are separated and of finite presentation over B. Then π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$.*

Proof. Let $(T \to B, \mathcal{F})$ be an object of $\mathcal{C}oh_{X/B}$. We claim

(a) $(T \to B, \pi_{T,*}\mathcal{F})$ is an object of $\mathcal{C}oh_{Y/B}$ and
(b) for $T' \to T$ we have $\pi_{T',*}(X_{T'} \to X_T)^*\mathcal{F} = (Y_{T'} \to Y_T)^*\pi_{T,*}\mathcal{F}$.

Part (b) guarantees that this construction defines a functor $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ as desired.

Let $i : Z \to X_T$ be the closed subspace cut out by the zeroth fitting ideal of $\mathcal{F}$ (Divisors on Spaces, Section 71.5). Then $Z \to B$ is proper by assumption

(see Derived Categories of Spaces, Section 75.7). On the other hand i is of finite presentation (Divisors on Spaces, Lemma 71.5.2 and Morphisms of Spaces, Lemma 67.28.12). There exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type with $i_*\mathcal{G} = \mathcal{F}$ (Divisors on Spaces, Lemma 71.5.3). In fact $\mathcal{G}$ is of finite presentation as an $\mathcal{O}_Z$-module by Descent on Spaces, Lemma 74.6.7. Observe that $\mathcal{G}$ is flat over B, for example because the stalks of $\mathcal{G}$ and $\mathcal{F}$ agree (Morphisms of Spaces, Lemma 67.13.6). Observe that $\pi_T \circ i : Z \to Y_T$ is quasi-finite as a composition of quasi-finite morphisms and that $\pi_{T,*}\mathcal{F} = (\pi_T \circ i)_*\mathcal{G})$. Since i is affine, formation of i_* commutes with base change (Cohomology of Spaces, Lemma 69.11.1). Therefore we may replace B by T, X by Z, $\mathcal{F}$ by $\mathcal{G}$, and Y by Y_T to reduce to the case discussed in the next paragraph.

Assume that $X \to B$ is proper. Then π is proper by Morphisms of Spaces, Lemma 67.40.6 and hence finite by More on Morphisms of Spaces, Lemma 76.35.1. Since a finite morphism is affine we see that (b) holds by Cohomology of Spaces, Lemma 69.11.1. On the other hand, π is of finite presentation by Morphisms of Spaces, Lemma 67.28.9. Thus $\pi_{T,*}\mathcal{F}$ is of finite presentation by Descent on Spaces, Lemma 74.6.7. Finally, $\pi_{T,*}\mathcal{F}$ is flat over B for example by looking at stalks using Cohomology of Spaces, Lemma 69.4.2. □

0DNA **Lemma 108.4.5.** *Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 108.4.4 is an open immersion.*

Proof. Omitted. Hint: If $\mathcal{F}$ is an object of $\mathcal{C}oh_{Y/B}$ over T and for $t \in T$ we have $\text{Supp}(\mathcal{F}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DNB **Lemma 108.4.6.** *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 108.4.4 is a closed immersion.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the sheaf of ideals cutting out X as a closed subspace of Y. Recall that π_* induces an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathcal{O}_Y$-modules annihilated by $\mathcal{I}$, see Morphisms of Spaces, Lemma 67.14.1. The same, mutatis mutandis, is true after base by $T \to B$ with $\mathcal{I}$ replaced by the ideal sheaf $\mathcal{I}_T = \text{Im}((Y_T \to Y)^*\mathcal{I} \to \mathcal{O}_{Y_T})$. Analyzing the proof of Lemma 108.4.4 we find that the essential image of $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ is exactly the objects $\xi = (T \to B, \mathcal{F})$ where $\mathcal{F}$ is annihilated by $\mathcal{I}_T$. In other words, ξ is in the essential image if and only if the multiplication map

$$\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I} \longrightarrow \mathcal{F}$$

is zero and similarly after any further base change $T' \to T$. Note that

$$(Y_{T'} \to Y_T)^*(\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I}) = (Y_{T'} \to Y_T)^*\mathcal{F} \otimes_{\mathcal{O}_{Y_{T'}}} (Y_{T'} \to Y)^*\mathcal{I})$$

Hence the vanishing of the multiplication map on T' is representable by a closed subspace of T by Flatness on Spaces, Lemma 77.8.6. □

0DNC **Situation 108.4.7** (Numerical invariants)**.** Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Given an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ denote $E_{i,T}$ the derived pullback of E_i to X_T. The object

$$K_i = Rf_{T,*}(E_{i,T} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_T}} \mathcal{F})$$

of $D(\mathcal{O}_T)$ is perfect and its formation commutes with base change, see Derived Categories of Spaces, Lemma 75.25.1. Thus the function

$$\chi_i : |T| \longrightarrow \mathbf{Z}, \quad \chi_i(t) = \chi(X_t, E_{i,t} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_t}} \mathcal{F}_t) = \chi(K_i \otimes^{\mathbf{L}}_{\mathcal{O}_T} \kappa(t))$$

is locally constant by Derived Categories of Spaces, Lemma 75.26.3. Let $P : I \to \mathbf{Z}$ be a map. Consider the substack

$$\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P. More precisely, an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ is in $\mathcal{C}oh^P_{X/B}$ if and only if $\chi_i(t) = P(i)$ for all $i \in I$ and $t \in T$.

0DND **Lemma 108.4.8.** *In Situation 108.4.7 the stack $\mathcal{C}oh^P_{X/B}$ is algebraic and*

$$\mathcal{C}oh^P_{X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

is a flat closed immersion. If I is finite or B is locally Noetherian, then $\mathcal{C}oh^P_{X/B}$ is an open and closed substack of $\mathcal{C}oh_{X/B}$.

Proof. This is immediately clear if I is finite, because the functions $t \mapsto \chi_i(t)$ are locally constant. If I is infinite, then we write

$$I = \bigcup\nolimits_{I' \subset I \text{ finite}} I'$$

and we denote $P' = P|_{I'}$. Then we have

$$\mathcal{C}oh^P_{X/B} = \bigcap\nolimits_{I' \subset I \text{ finite}} \mathcal{C}oh^{P'}_{X/B}$$

Therefore, $\mathcal{C}oh^P_{X/B}$ is always an algebraic stack and the morphism $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ is always a flat closed immersion, but it may no longer be an open substack. (We leave it to the reader to make examples). However, if B is locally Noetherian, then so is $\mathcal{C}oh_{X/B}$ by Lemma 108.4.2 and Morphisms of Stacks, Lemma 101.17.5. Hence if $U \to \mathcal{C}oh_{X/B}$ is a smooth surjective morphism where U is a locally Noetherian scheme, then the inverse images of the open and closed substacks $\mathcal{C}oh^{P'}_{X/B}$ have an open intersection in U (because connected components of locally Noetherian topological spaces are open). Thus the result in this case. □

0DNE **Lemma 108.4.9.** *Let $f : X \to B$ be as in the introduction to this section. Let $E_1, \ldots, E_r \in D(\mathcal{O}_X)$ be perfect. Let $I = \mathbf{Z}^{\oplus r}$ and consider the map*

$$I \longrightarrow D(\mathcal{O}_X), \quad (n_1, \ldots, n_r) \longmapsto E_1^{\otimes n_1} \otimes \ldots \otimes E_r^{\otimes n_r}$$

Let $P : I \to \mathbf{Z}$ be a map. Then $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ as defined in Situation 108.4.7 is an open and closed substack.

Proof. We may work étale locally on B, hence we may assume that B is affine. In this case we may perform absolute Noetherian reduction; we suggest the reader skip the proof. Namely, say $B = \operatorname{Spec}(\Lambda)$. Write $\Lambda = \operatorname{colim} \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \operatorname{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 70.7.1 and 70.6.9. Then after increasing i we may assume there exist perfect objects $E_{1,i}, \ldots, E_{r,i}$ in $D(\mathcal{O}_{X_i})$

whose derived pullback to X are isomorphic to $E_1, \ldots, E_r$, see Derived Categories of Spaces, Lemma 75.24.3. Clearly we have a cartesian square

$$\begin{array}{ccc} \mathcal{C}oh^P_{X/B} & \longrightarrow & \mathcal{C}oh_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{C}oh^P_{X_i/\operatorname{Spec}(\Lambda_i)} & \longrightarrow & \mathcal{C}oh_{X_i/\operatorname{Spec}(\Lambda_i)} \end{array}$$

and hence we may appeal to Lemma 108.4.8 to finish the proof. □

0DNF **Example 108.4.10** (Coherent sheaves with fixed Hilbert polynomial)**.** Let $f : X \to B$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $P : \mathbf{Z} \to \mathbf{Z}$ be a numerical polynomial. Then we can consider the open and closed algebraic substack

$$\mathcal{C}oh^P_{X/B} = \mathcal{C}oh^{P,\mathcal{L}}_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P: an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ lies in $\mathcal{C}oh^P_{X/B}$ if and only if

$$P(n) = \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$$

for all $n \in \mathbf{Z}$ and $t \in T$. Of course this is a special case of Situation 108.4.7 where $I = \mathbf{Z} \to D(\mathcal{O}_X)$ is given by $n \mapsto \mathcal{L}^{\otimes n}$. It follows from Lemma 108.4.9 that this is an open and closed substack. Since the functions $n \mapsto \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$ are always numerical polynomials (Spaces over Fields, Lemma 72.18.1) we conclude that

$$\mathcal{C}oh_{X/B} = \coprod\nolimits_{P \text{ numerical polynomial}} \mathcal{C}oh^P_{X/B}$$

is a disjoint union decomposition.

108.5. Properties of Quot

0DM1 Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\operatorname{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\operatorname{Quot}_{\mathcal{F}/X/B} \to B$ is locally of finite presentation. See Quot, Proposition 99.8.4.

0DM2 **Lemma 108.5.1.** *The diagonal of* $\operatorname{Quot}_{\mathcal{F}/X/B} \to B$ *is a closed immersion. If* $\mathcal{F}$ *is of finite type, then the diagonal is a closed immersion of finite presentation.*

Proof. Suppose we have a scheme T/B and two quotients $\mathcal{F}_T \to \mathcal{Q}_i$, $i = 1, 2$ corresponding to T-valued points of $\operatorname{Quot}_{\mathcal{F}/X/B}$ over B. Denote $\mathcal{K}_1$ the kernel of the first one and set $u : \mathcal{K}_1 \to \mathcal{Q}_2$ the composition. By Flatness on Spaces, Lemma 77.8.6 there is a closed subspace of T such that $T' \to T$ factors through it if and only if the pullback $u_{T'}$ is zero. This proves the diagonal is a closed immersion. Moreover, if $\mathcal{F}$ is of finite type, then $\mathcal{K}_1$ is of finite type (Modules on Sites, Lemma 18.24.2) and we see that the diagonal is of finite presentation by the same lemma. □

0DM3 **Lemma 108.5.2.** *The morphism* $\operatorname{Quot}_{\mathcal{F}/X/B} \to B$ *is separated. If* $\mathcal{F}$ *is of finite presentation, then it is also locally of finite presentation.*

Proof. To check $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.5.1. The second statement is part of Quot, Proposition 99.8.4. □

0DM4 **Lemma 108.5.3.** *Assume $X \to B$ is proper as well as of finite presentation and $\mathcal{F}$ quasi-coherent of finite type. Then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 67.41.1).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K, an algebraic space X proper over R, a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, and a coherent quotient $\mathcal{F}_K \to \mathcal{Q}_K$, show there exists a quotient $\mathcal{F} \to \mathcal{Q}$ where $\mathcal{Q}$ is a finitely presented $\mathcal{O}_X$-module flat over R whose generic fibre is $\mathcal{Q}_K$. Observe that by Flatness on Spaces, Theorem 77.4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. We first solve the existence of $\mathcal{Q}$ affine locally.

Affine locally we arrive at the following problem: let $R \to A$ be a finitely presented ring map, let M be a finite A-module, let $\varphi : M_K \to N_K$ be an A_K-quotient module. Then we may consider

$$L = \{x \in M \mid \varphi(x \otimes 1) = 0\}$$

The $M \to M/L$ is an A-module quotient which is torsion free as an R-module. Hence it is flat as an R-module (More on Algebra, Lemma 15.22.10). Since M is finite as an A-module so is L and we conclude that L is of finite presentation as an A-module (by the reference above). Clearly M/L is the unique such quotient with $(M/L)_K = N_K$.

The uniqueness in the construction of the previous paragraph guarantees these quotients glue and give the desired $\mathcal{Q}$. Here is a bit more detail. Choose a surjective étale morphism $U \to X$ where U is an affine scheme. Use the above construction to construct a quotient $\mathcal{F}|_U \to \mathcal{Q}_U$ which is quasi-coherent, is flat over R, and recovers $\mathcal{Q}_K|U$ on the generic fibre. Since X is separated, we see that $U \times_X U$ is an affine scheme étale over X as well. Then $\mathcal{F}|_{U\times_X U} \to \mathrm{pr}_1^*\mathcal{Q}_U$ and $\mathcal{F}|_{U\times_X U} \to \mathrm{pr}_2^*\mathcal{Q}_U$ agree as quotients by the uniquess in the construction. Hence we may descend $\mathcal{F}|_U \to \mathcal{Q}_U$ to a surjection $\mathcal{F} \to \mathcal{Q}$ as desired (Properties of Spaces, Proposition 66.32.1). □

0DP1 **Lemma 108.5.4.** *Let B be an algebraic space. Let $\pi : X \to Y$ be an affine quasi-finite morphism of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then π_* induces a morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$.*

Proof. Set $\mathcal{G} = \pi_*\mathcal{F}$. Since π is affine we see that for any scheme T over B we have $\mathcal{G}_T = \pi_{T,*}\mathcal{F}_T$ by Cohomology of Spaces, Lemma 69.11.1. Moreover π_T is affine, hence $\pi_{T,*}$ is exact and transforms quotients into quotients. Observe that a quasi-coherent quotient $\mathcal{F}_T \to \mathcal{Q}$ defines a point of $\mathrm{Quot}_{X/B}$ if and only if $\mathcal{Q}$ defines an object of $\mathcal{C}oh_{X/B}$ over T (similarly for $\mathcal{G}$ and Y). Since we've seen in Lemma 108.4.4 that π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ we see that if $\mathcal{F}_T \to \mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then $\mathcal{G}_T \to \pi_{T,*}\mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$. □

0DP2 **Lemma 108.5.5.** *Let B be an algebraic space. Let $\pi : X \to Y$ be an affine open immersion of algebraic spaces which are separated and of finite presentation*

over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 108.5.4 is an open immersion.

Proof. Omitted. Hint: If $(\pi_*\mathcal{F})_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DP3 **Lemma 108.5.6.** *Let B be an algebraic space. Let $j : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module and set $\mathcal{F} = j^*\mathcal{G}$. Then there is an open immersion*

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/Y/B}$$

of algebraic spaces over B.

Proof. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ then we can consider $\mathcal{G}_T \to j_{T,*}\mathcal{F}_T \to j_{T,*}\mathcal{Q}$. Looking at stalks one finds that this is surjective. By Lemma 108.4.4 we see that $j_{T,*}\mathcal{Q}$ is finitely presented, flat over B with support proper over B. Thus we obtain a T-valued point of $\mathrm{Quot}_{\mathcal{G}/Y/B}$. This defines the morphism of the lemma. We omit the proof that this is an open immersion. Hint: If $\mathcal{G}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DP4 **Lemma 108.5.7.** *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 108.5.4 is an isomorphism.*

Proof. For every scheme T over B the morphism $\pi_T : X_T \to Y_T$ is a closed immersion. Then $\pi_{T,*}$ is an equivalence of categories between $QCoh(\mathcal{O}_{X_T})$ and the full subcategory of $QCoh(\mathcal{O}_{Y_T})$ whose objects are those quasi-coherent modules annihilated by the ideal sheaf of X_T, see Morphisms of Spaces, Lemma 67.14.1. Since a qotient of $(\pi_*\mathcal{F})_T$ is annihilated by this ideal we obtain the bijectivity of the map $\mathrm{Quot}_{\mathcal{F}/X/B}(T) \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ for all T as desired. □

0DP5 **Lemma 108.5.8.** *Let $X \to B$ be as in the introduction to this section. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical closed immersion $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to \mathcal{G})$. By right exactness of pullbacks we find that $\mathcal{K}_T \to \mathcal{F}_T \to \mathcal{G}_T \to 0$ is an exact sequecnce for all schemes T over B. In particular, a quotient of $\mathcal{G}_T$ determines a quotient of $\mathcal{F}_T$ and we obtain our transformation of functors $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$. This transformation is a closed immersion by Flatness on Spaces, Lemma 77.8.6. Namely, given an element $\mathcal{F}_T \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then we see that the pull back to T'/T is in the image of the transformation if and only if $\mathcal{K}_{T'} \to \mathcal{Q}_{T'}$ is zero. □

0DP6 **Remark 108.5.9** (Numerical invariants)**.** Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that we have a morphism

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

which sends the element $\mathcal{F}_T \to \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ to the object $\mathcal{Q}$ of $\mathcal{C}oh_{X/B}$ over T, see proof of Quot, Proposition 99.8.4. Hence we can form the fibre product diagram

$$\begin{array}{ccc} \mathrm{Quot}^P_{\mathcal{F}/X/B} & \longrightarrow & \mathcal{C}oh^P_{X/B} \\ \downarrow & & \downarrow \\ \mathrm{Quot}_{\mathcal{F}/X/B} & \longrightarrow & \mathcal{C}oh_{X/B} \end{array}$$

This is the defining diagram for the algebraic space in the upper left corner. The left vertical arrow is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$ (in the last case we sometimes use the notation $\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B}$). See Situation 108.4.7 and Lemmas 108.4.8 and 108.4.9 and Example 108.4.10.

0DP7 **Lemma 108.5.10.** *Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then tensoring with $\mathcal{L}$ defines an isomorphism*

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$$

Given a numerical polynomial $P(t)$, then setting $P'(t) = P(t+1)$ this map induces an isomorphism $\mathrm{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}^{P'}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$ of open and closed substacks.

Proof. Set $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$. Observe that $\mathcal{G}_T = \mathcal{F}_T \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then we send it to the element $\mathcal{G}_T \to \mathcal{Q} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$ of $\mathrm{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}(T)$. This is compatible with pullbacks and hence defines a transformation of functors as desired. Since there is an obvious inverse transformation, it is an isomorphism. We omit the proof of the final statement. □

0DP8 **Lemma 108.5.11.** *Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B} = \mathrm{Quot}^{P',\mathcal{L}^{\otimes n}}_{\mathcal{F}/X/B}$$

where $P'(t) = P(nt)$.

Proof. Follows immediately after unwinding all the definitions. □

108.6. Boundedness for Quot

0DP9 Contrary to what happens classically, we already know the Quot functor is an algebraic space, but we don't know that it is ever represented by a finite type algebraic space.

0DPA **Lemma 108.6.1.** *Let $n \geq 0$, $r \geq 1$, $P \in \mathbf{Q}[t]$. The algebraic space*

$$X = \mathrm{Quot}^P_{\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}/\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z}}$$

parametrizing quotients of $\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}$ with Hilbert polynomial P is proper over $\mathrm{Spec}(\mathbf{Z})$.

Proof. We already know that $X \to \mathrm{Spec}(\mathbf{Z})$ is separated and locally of finite presentation (Lemma 108.5.2). We also know that $X \to \mathrm{Spec}(\mathbf{Z})$ satisfies the existence part of the valuative criterion, see Lemma 108.5.3. By the valuative criterion for properness, it suffices to prove our Quot space is quasi-compact, see

Morphisms of Spaces, Lemma 67.44.1. Thus it suffices to find a quasi-compact scheme T and a surjective morphism $T \to X$. Let m be the integer found in Varieties, Lemma 33.35.18. Let

$$N = r\binom{m+n}{n} - P(m)$$

We will write $\mathbf{P}^n$ for $\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ and unadorned products will mean products over $\text{Spec}(\mathbf{Z})$. The idea of the proof is to construct a "universal" map

$$\Psi : \mathcal{O}_{T \times \mathbf{P}^n}(-m)^{\oplus N} \longrightarrow \mathcal{O}_{T \times \mathbf{P}^n}^{\oplus r}$$

over an affine scheme T and show that every point of X corresponds to a cokernel of this in some point of T.

Definition of T and Ψ. We take $T = \text{Spec}(A)$ where

$$A = \mathbf{Z}[a_{i,j,E}]$$

where $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, N\}$ and $E = (e_0, \ldots, e_n)$ runs through the multi-indices of total degree $|E| = \sum_{k=0,\ldots n} e_k = m$. Then we define Ψ to be the map whose (i, j) matrix entry is the map

$$\sum\nolimits_{E=(e_0,\ldots,e_n)} a_{i,j,E} T_0^{e_0} \ldots T_n^{e_n} : \mathcal{O}_{T \times \mathbf{P}^n}(-m) \longrightarrow \mathcal{O}_{T \times \mathbf{P}^n}$$

where the sum is over E as above (but i and j are fixed of course).

Consider the quotient $\mathcal{Q} = \text{Coker}(\Psi)$ on $T \times \mathbf{P}^n$. By More on Morphisms, Lemma 37.54.1 there exists a $t \geq 0$ and closed subschemes

$$T = T_0 \supset T_1 \supset \ldots \supset T_t = \emptyset$$

such that the pullback $\mathcal{Q}_p$ of $\mathcal{Q}$ to $(T_p \setminus T_{p+1}) \times \mathbf{P}^n$ is flat over $T_p \setminus T_{p+1}$. Observe that we have an exact sequence

$$\mathcal{O}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n}(-m)^{\oplus N} \to \mathcal{O}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n}^{\oplus r} \to \mathcal{Q}_p \to 0$$

by pulling back the exact sequence defining $\mathcal{Q} = \text{Coker}(\Psi)$. Therefore we obtain a morphism

$$\coprod (T_p \setminus T_{p+1}) \longrightarrow \text{Quot}_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} \supset \text{Quot}^P_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} = X$$

Since the left hand side is a Noetherian scheme and the inclusion on the right hand side is open, it suffices to show that any point of X is in the image of this morphism.

Let k be a field and let $x \in X(k)$. Then x corresponds to a surjection $\mathcal{O}_{\mathbf{P}^n_k}^{\oplus r} \to \mathcal{F}$ of coherent $\mathcal{O}_{\mathbf{P}^n_k}$-modules such that the Hilbert polynomial of $\mathcal{F}$ is P. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbf{P}^n_k}^{\oplus r} \to \mathcal{F} \to 0$$

By Varieties, Lemma 33.35.18 and our choice of m we see that $\mathcal{K}$ is m-regular. By Varieties, Lemma 33.35.12 we see that $\mathcal{K}(m)$ is globally generated. By Varieties, Lemma 33.35.10 and the definition of m-regularity we see that $H^i(\mathbf{P}^n_k, \mathcal{K}(m)) = 0$ for $i > 0$. Hence we see that

$$\dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m)) = \chi(\mathcal{K}(m)) = \chi(\mathcal{O}_{\mathbf{P}^n_k}(m)^{\oplus r}) - \chi(\mathcal{F}(m)) = N$$

by our choice of N. This gives a surjection

$$\mathcal{O}_{\mathbf{P}^n_k}^{\oplus N} \longrightarrow \mathcal{K}(m)$$

Twisting back down and using the short exact sequence above we see that $\mathcal{F}$ is the cokernel of a map

$$\Psi_x : \mathcal{O}_{\mathbf{P}^n_k}(-m)^{\oplus N} \to \mathcal{O}_{\mathbf{P}^n_k}^{\oplus r}$$

There is a unique ring map $\tau : A \to k$ such that the base change of Ψ by the corresponding morphism $t = \mathrm{Spec}(\tau) : \mathrm{Spec}(k) \to T$ is Ψ_x. This is true because the entries of the $N \times r$ matrix defining Ψ_x are homogeneous polynomials $\sum \lambda_{i,j,E} T_0^{e_0} \ldots T_n^{e_n}$ of degree m in $T_0, \ldots, T_n$ with coefficients $\lambda_{i,j,E} \in k$ and we can set $\tau(a_{i,j,E}) = \lambda_{i,j,E}$. Then $t \in T_p \setminus T_{p+1}$ for some p and the image of t under the morphism above is x as desired. □

0DPB **Lemma 108.6.2.** *Let B be an algebraic space. Let $X = B \times \mathbf{P}^n_{\mathbf{Z}}$. Let $\mathcal{L}$ be the pullback of $\mathcal{O}_{\mathbf{P}^n}(1)$ to X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume B is an affine scheme. In this case $\mathcal{L}$ is an ample invertible module on X (by Constructions, Lemma 27.10.6 and the definition of ample invertible modules in Properties, Definition 28.26.1). Thus we can find $r' \geq 0$ and $r \geq 0$ and a surjection

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$$

by Properties, Proposition 28.26.13. By Lemma 108.5.10 we may replace $\mathcal{F}$ by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$ and $P(t)$ by $P(t+r')$. By Lemma 108.5.8 we obtain a closed immersion

$$\mathrm{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B}$$

Since we've shown that $\mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B} \to B$ is proper in Lemma 108.6.1 we conclude. □

0DPC **Lemma 108.6.3.** *Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 67.40.2. Thus we may assume B is an affine scheme. Then we can find a closed immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n_B}(1) \cong \mathcal{L}^{\otimes d}$ for some $d \geq 1$. See Morphisms, Lemma 29.39.3. Changing $\mathcal{L}$ into $\mathcal{L}^{\otimes d}$ and the numerical polynomial $P(t)$ into $P(dt)$ leaves $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ unaffected; some details omitted. Hence we may assume $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_B}(1)$. Then the isomorphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ of Lemma 108.5.7 induces an isomorphism $\mathrm{Quot}^P_{\mathcal{F}/X/B} \cong \mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$. Since $\mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ is proper over B by Lemma 108.6.2 we conclude. □

0DPD **Lemma 108.6.4.** *Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.*

Proof. We have already seen that $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated and locally of finite presentation, see Lemma 108.5.2. Thus it suffices to show that the open subspace $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ of Remark 108.5.9 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 67.8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, \mathcal{F}_i, \mathcal{L}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, \mathcal{F}, \mathcal{L}$. This follows from Limits of Spaces, Lemmas 70.7.1 (to find X_i), 70.7.2 (to find $\mathcal{F}_i$), 70.7.3 (to find $\mathcal{L}_i$), and 70.5.9 (to make X_i separated). Because

$$\mathrm{Quot}_{\mathcal{F}/X/B} = B \times_{B_i} \mathrm{Quot}_{\mathcal{F}_i/X_i/B_i}$$

and similarly for $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ we reduce to the case discussed in the next paragraph.

Assume B is affine and Noetherian. We may replace $\mathcal{L}$ by a positive power, see Lemma 108.5.11. Thus we may assume there exists an immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{L}$. By Morphisms, Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_B$ such that i factors through an open immersion $j : X \to X'$. By Properties, Lemma 28.22.5 there exists a finitely presented $\mathcal{O}_{X'}$-module $\mathcal{G}$ such that $j^*\mathcal{G} = \mathcal{F}$. Thus we obtain an open immersion

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/X'/B}$$

by Lemma 108.5.6. Clearly this open immersion sends $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ into $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$. Now $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$ is proper over B by Lemma 108.6.3. Therefore it is Noetherian and since any open of a Noetherian algebraic space is quasi-compact we win. □

108.7. Properties of the Hilbert functor

0DM5 Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then $\mathrm{Hilb}_{X/B}$ is an algebraic space locally of finite presentation over B. See Quot, Proposition 99.9.4.

0DM6 **Lemma 108.7.1.** *The diagonal of* $\mathrm{Hilb}_{X/B} \to B$ *is a closed immersion of finite presentation.*

Proof. In Quot, Lemma 99.9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 108.5.1. □

0DM7 **Lemma 108.7.2.** *The morphism* $\mathrm{Hilb}_{X/B} \to B$ *is separated and locally of finite presentation.*

Proof. To check $\mathrm{Hilb}_{X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.7.1. The second statement is part of Quot, Proposition 99.9.4. □

0DM8 **Lemma 108.7.3.** *Assume $X \to B$ is proper as well as of finite presentation. Then $\mathrm{Hilb}_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 67.41.1).*

Proof. In Quot, Lemma 99.9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 108.5.3. □

0DPE **Lemma 108.7.4.** *Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces an open immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.*

Proof. Omitted. Hint: If $Z \subset X_T$ is a closed subscheme which is proper over T, then Z is also closed in Y_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. If $Z \subset Y_T$ is an element of $\mathrm{Hilb}_{Y/B}(T)$ and for $t \in T$ we have $|Z_t| \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. □

0DPF **Lemma 108.7.5.** *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces a closed immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.*

Proof. Since π is a closed immersion, it is immediate that given a closed subscheme $Z \subset X_T$, we can view Z as a closed subscheme of X_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. This transformation is immediately seen to be a monomorphism. To prove that it is a closed immersion, you can use Lemma 108.5.8 for the map $\mathcal{O}_Y \to \mathcal{O}_X$ and the identifications $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, $\mathrm{Hilb}_{Y/B} = \mathrm{Quot}_{\mathcal{O}_Y/Y/B}$ of Quot, Lemma 99.9.2. □

0DPG **Remark 108.7.6** (Numerical invariants)**.** Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus we can define

$$\mathrm{Hilb}^P_{X/B} = \mathrm{Quot}^P_{\mathcal{O}_X/X/B}$$

where $\mathrm{Quot}^P_{\mathcal{O}_X/X/B}$ is as in Remark 108.5.9. The morphism

$$\mathrm{Hilb}^P_{X/B} \longrightarrow \mathrm{Hilb}_{X/B}$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$. In the last case we sometimes use the notation $\mathrm{Hilb}^{P,\mathcal{L}}_{X/B}$.

0DPH **Lemma 108.7.7.** *Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus this lemma is an immediate consequence of Lemma 108.6.3. □

0DPI **Lemma 108.7.8.** *Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 71.14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.*

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 99.9.2. Thus this lemma is an immediate consequence of Lemma 108.6.4. □

108.8. Properties of the Picard stack

0DM9 Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation. Then the stack $\mathcal{P}ic_{X/B}$ parametrizing invertible sheaves on X/B is algebraic, see Quot, Proposition 99.10.2.

0DMA **Lemma 108.8.1.** *The diagonal of $\mathcal{P}ic_{X/B}$ over B is affine and of finite presentation.*

Proof. In Quot, Lemma 99.10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 108.4.1. □

0DMB **Lemma 108.8.2.** *The morphism $\mathcal{P}ic_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. In Quot, Lemma 99.10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 108.4.2. □

0DNG **Lemma 108.8.3.** *Assume $X \to B$ is smooth in addition to being proper. Then $\mathcal{P}ic_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 101.39.10).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper and smooth over R and an invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}_K$, show there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ whose generic fibre is $\mathcal{L}_K$. Observe that X_K is Noetherian, separated, and regular (use Morphisms of Spaces, Lemma 67.28.6 and Spaces over Fields, Lemma 72.16.1). Thus we can write $\mathcal{L}_K$ as the difference in the Picard group of $\mathcal{O}_{X_K}(D_K)$ and $\mathcal{O}_{X_K}(D'_K)$ for two effective Cartier divisors D_K, D'_K in X_K, see Divisors on Spaces, Lemma 71.8.4. Finally, we know that D_K and D'_K are restrictions of effective Cartier divisors $D, D' \subset X$, see Divisors on Spaces, Lemma 71.8.5. □

0DNH **Lemma 108.8.4.** *Assume $f_{T,*}\mathcal{O}_{X_T} \cong \mathcal{O}_T$ for all schemes T over B. Then the inertia stack of $\mathcal{P}ic_{X/B}$ is equal to $\mathbf{G}_m \times \mathcal{P}ic_{X/B}$.*

Proof. This is explained in Examples of Stacks, Example 95.17.2. □

0DPJ **Lemma 108.8.5.** *Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathcal{P}ic_{X/B} \to B$ is smooth.*

Proof. We already know that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Lemma 108.8.2. Thus it suffices to show that $\mathcal{P}ic_{X/B} \to B$ is formally smooth, see More on Morphisms of Stacks, Lemma 106.8.7. Taking base change, this immediately reduces to the following problem: given a first order thickening $T \subset T'$ of affine schemes, given $X' \to T'$ proper, flat, of finite presentation and of relative dimension ≤ 1, and for $X = T \times_{T'} X'$ given an invertible $\mathcal{O}_X$-module $\mathcal{L}$, prove that there exists an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ whose restriction to X is $\mathcal{L}$. Since $T \subset T'$ is a first order thickening, the same is true for $X \subset X'$, see More on Morphisms of Spaces, Lemma 76.9.8. By More on Morphisms of Spaces, Lemma 76.11.1 we see that it suffices to show $H^2(X, \mathcal{I}) = 0$ where $\mathcal{I}$ is the quasi-coherent ideal cutting out X in X'. Denote $f : X \to T$ the structure morphism. By Cohomology of Spaces, Lemma 69.22.9 we see that $R^p f_* \mathcal{I} = 0$ for $p > 1$. Hence we get the desired vanishing by Cohomology of Spaces, Lemma 69.3.2 (here we finally use that T is affine). □

108.9. Properties of the Picard functor

0DMD Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation such that moreover for every T/B the canonical map

$$\mathcal{O}_T \longrightarrow f_{T,*}\mathcal{O}_{X_T}$$

is an isomorphism. Then the Picard functor $\mathrm{Pic}_{X/B}$ is an algebraic space, see Quot, Proposition 99.11.8. There is a closed relationship with the Picard stack.

0DME **Lemma 108.9.1.** *The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ turns the Picard stack into a gerbe over the Picard functor.*

Proof. The definition of $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ being a gerbe is given in Morphisms of Stacks, Definition 101.28.1, which in turn refers to Stacks, Definition 8.11.4. To prove it, we will check conditions (2)(a) and (2)(b) of Stacks, Lemma 8.11.3. This follows immediately from Quot, Lemma 99.11.2; here is a detailed explanation.

Condition (2)(a). Suppose that $\xi \in \mathrm{Pic}_{X/B}(U)$ for some scheme U over B. Since $\mathrm{Pic}_{X/B}$ is the fppf sheafification of the rule $T \mapsto \mathrm{Pic}(X_T)$ on schemes over B (Quot, Situation 99.11.1), we see that there exists an fppf covering $\{U_i \to U\}$ such that $\xi|_{U_i}$ corresponds to some invertible module $\mathcal{L}_i$ on X_{U_i}. Then $(U_i \to B, \mathcal{L}_i)$ is an object of $\mathcal{P}ic_{X/B}$ over U_i mapping to $\xi|_{U_i}$.

Condition (2)(b). Suppose that U is a scheme over B and $\mathcal{L}, \mathcal{N}$ are invertible modules on X_U which map to the same element of $\mathrm{Pic}_{X/B}(U)$. Then there exists an fppf covering $\{U_i \to U\}$ such that $\mathcal{L}|_{X_{U_i}}$ is isomorphic to $\mathcal{N}|_{X_{U_i}}$. Thus we find isomorphisms between $(U \to B, \mathcal{L})|_{U_i} \to (U \to B, \mathcal{N})|_{U_i}$ as desired. □

0DMF **Lemma 108.9.2.** *The diagonal of $\mathrm{Pic}_{X/B}$ over B is a quasi-compact immersion.*

Proof. The diagonal is an immersion by Quot, Lemma 99.11.9. To finish we show that the diagonal is quasi-compact. The diagonal of $\mathcal{P}ic_{X/B}$ is quasi-compact by Lemma 108.8.1 and $\mathcal{P}ic_{X/B}$ is a gerbe over $\mathrm{Pic}_{X/B}$ by Lemma 108.9.1. We conclude by Morphisms of Stacks, Lemma 101.28.14. □

0DNI **Lemma 108.9.3.** *The morphism $\mathrm{Pic}_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathrm{Pic}_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact. This is immediate from Lemma 108.9.2. Since the morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective, flat, and locally of finite presentation (by Lemma 108.9.1 and Morphisms of Stacks, Lemma 101.28.8) it suffices to prove that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Morphisms of Stacks, Lemma 101.27.12. This follows from Lemma 108.8.2. □

0DNJ **Lemma 108.9.4.** *Assume the geometric fibres of $X \to B$ are integral in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is separated.*

Proof. Since $\mathrm{Pic}_{X/B} \to B$ is quasi-separated, it suffices to check the uniqueness part of the valuative criterion, see Morphisms of Spaces, Lemma 67.43.2. This immediately reduces to the following problem: given

(1) a valuation ring R with fraction field K,
(2) an algebraic space X proper and flat over R with integral geometric fibre,
(3) an element $a \in \mathrm{Pic}_{X/R}(R)$ with $a|_{\mathrm{Spec}(K)} = 0$,

then we have to prove $a = 0$. Applying Morphisms of Stacks, Lemma 101.25.6 to the surjective flat morphism $\mathcal{P}ic_{X/R} \to \mathrm{Pic}_{X/R}$ (surjective and flat by Lemma 108.9.1 and Morphisms of Stacks, Lemma 101.28.8) after replacing R by an extension we may assume a is given by an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Since $a|_{\mathrm{Spec}(K)} = 0$ we find $\mathcal{L}_K \cong \mathcal{O}_{X_K}$ by Quot, Lemma 99.11.3.

Denote $f : X \to \mathrm{Spec}(R)$ the structure morphism. Let $\eta, 0 \in \mathrm{Spec}(R)$ be the generic and closed point. Consider the perfect complexes $K = Rf_*\mathcal{L}$ and $M = Rf_*(\mathcal{L}^{\otimes -1})$ on $\mathrm{Spec}(R)$, see Derived Categories of Spaces, Lemma 75.25.4. Consider the functions $\beta_{K,i}, \beta_{M,i} : \mathrm{Spec}(R) \to \mathbf{Z}$ of Derived Categories of Spaces, Lemma 75.26.1 associated to K and M. Since the formation of K amd M commutes with base change (see lemma cited above) we find $\beta_{K,0}(\eta) = \beta_{M,0}(\eta) = 1$ by Spaces over Fields, Lemma 72.14.3 and our assumption on the fibres of f. By upper semi-continuity we find $\beta_{K,0}(0) \geq 1$ and $\beta_{M,0}(0) \geq 1$. By Spaces over Fields, Lemma 72.14.4 we conclude that the restriction of $\mathcal{L}$ to the special fibre X_0 is trivial. In turn this gives $\beta_{K,0}(0) = \beta_{M,0} = 1$ as above. Then by More on Algebra, Lemma 15.76.6 we can represent K by a complex of the form

$$\ldots \to 0 \to R \to R^{\oplus \beta_{K,1}(0)} \to R^{\oplus \beta_{K,2}(0)} \to \ldots$$

Now $R \to R^{\oplus \beta_{K,1}(0)}$ is zero because $\beta_{K,0}(\eta) = 1$. In other words $K = R \oplus \tau_{\geq 1}(K)$ in $D(R)$ where $\tau_{\geq 1}(K)$ has tor amplitude in $[1, b]$ for some $b \in \mathbf{Z}$. Hence there is a global section $s \in H^0(X, \mathcal{L})$ whose restriction s_0 to X_0 is nonvanishing (again because formation of K commutes with base change). Then $s : \mathcal{O}_X \to \mathcal{L}$ is a map of invertible sheaves whose restriction to X_0 is an isomorphism and hence is an isomorphism as desired. □

0DPK **Lemma 108.9.5.** *Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is smooth.*

Proof. By Lemma 108.8.5 we know that $\mathcal{P}ic_{X/B} \to B$ is smooth. The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective and smooth by combining Lemma 108.9.1 with Morphisms of Stacks, Lemma 101.33.8. Thus if U is a scheme and $U \to \mathcal{P}ic_{X/B}$ is surjective and smooth, then $U \to \mathrm{Pic}_{X/B}$ is surjective and smooth and $U \to B$ is surjective and smooth (because these properties are preserved by composition). Thus $\mathrm{Pic}_{X/B} \to B$ is smooth for example by Descent on Spaces, Lemma 74.8.3. □

108.10. Properties of relative morphisms

0DPL Let B be an algebraic space. Let X and Y be algebraic spaces over B such that $Y \to B$ is flat, proper, and of finite presentation and $X \to B$ is separated and of finite presentation. Then the functor $Mor_B(Y, X)$ of relative morphisms is an algebraic space locally of finite presentation over B. See Quot, Proposition 99.12.3.

0DPM **Lemma 108.10.1.** *The diagonal of $Mor_B(Y, X) \to B$ is a closed immersion of finite presentation.*

Proof. There is an open immersion $Mor_B(Y, X) \to \mathrm{Hilb}_{Y \times_B X/B}$, see Quot, Lemma 99.12.2. Thus the lemma follows from Lemma 108.7.1. □

0DPN **Lemma 108.10.2.** *The morphism $Mor_B(Y, X) \to B$ is separated and locally of finite presentation.*

Proof. To check $Mor_B(Y, X) \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 108.10.1. The second statement is part of Quot, Proposition 99.12.3. □

0DPP **Lemma 108.10.3.** *With B, X, Y as in the introduction of this section, in addition assume $X \to B$ is proper. Then the subfunctor $Isom_B(Y, X) \subset Mor_B(Y, X)$ of isomorphisms is an open subspace.*

Proof. Follows immediately from More on Morphisms of Spaces, Lemma 76.49.6. □

0DPQ **Remark 108.10.4** (Numerical invariants)**.** Let B, X, Y be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_{Y \times_B X})$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that

$$Mor_B(Y, X) \subset \mathrm{Hilb}_{Y \times_B X/B}$$

is an open subspace, see Quot, Lemma 99.12.2. Thus we can define

$$Mor_B^P(Y, X) = Mor_B(Y, X) \cap \mathrm{Hilb}^P_{Y \times_B X/B}$$

where $\mathrm{Hilb}^P_{Y \times_B X/B}$ is as in Remark 108.7.6. The morphism

$$Mor_B^P(Y, X) \longrightarrow Mor_B(Y, X)$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$, $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_{Y \times_B X}$-module $\mathcal{L}$. In the last case we sometimes use the notation $Mor_B^{P,\mathcal{L}}(Y, X)$.

0DPR **Lemma 108.10.5.** *With B, X, Y as in the introduction of this section, let $\mathcal{L}$ be ample on X/B and let $\mathcal{N}$ be ample on Y/B. See Divisors on Spaces, Definition 71.14.1. Let P be a numerical polynomial. Then*

$$Mor_B^{P,\mathcal{M}}(Y, X) \longrightarrow B$$

is separated and of finite presentation where $\mathcal{M} = pr_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_B X}} pr_2^*\mathcal{L}$.

Proof. By Lemma 108.10.2 the morphism $Mor_B(Y, X) \to B$ is separated and locally of finite presentation. Thus it suffices to show that the open and closed subspace $Mor_B^{P,\mathcal{M}}(Y, X)$ of Remark 108.10.4 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 67.8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Note that X and Y are schemes and that $\mathcal{L}$ and $\mathcal{N}$ are ample invertible sheaves on X and Y (this follows immediately from the definitions). Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, Y_i, \mathcal{L}_i, \mathcal{N}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, Y, \mathcal{L}, \mathcal{N}$. This follows from Limits, Lemmas 32.10.1 (to find X_i, Y_i), 32.10.3 (to find $\mathcal{L}_i$, $\mathcal{N}_i$), 32.8.6 (to make $X_i \to B_i$ separated), 32.13.1 (to make $Y_i \to B_i$ proper), and 32.4.15 (to make $\mathcal{L}_i$, $\mathcal{N}_i$ ample). Because

$$Mor_B(Y, X) = B \times_{B_i} Mor_{B_i}(Y_i, X_i)$$

and similarly for $Mor_B^P(Y, X)$ we reduce to the case discussed in the next paragraph.

Assume B is a Noetherian affine scheme. By Properties, Lemma 28.26.15 we see that $\mathcal{M}$ is ample. By Lemma 108.7.8 we see that $\mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$ is of finite presentation over B and hence Noetherian. By construction

$$Mor^{P,\mathcal{M}}_B(Y,X) = Mor_B(Y,X) \cap \mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$$

is an open subspace of $\mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$ and hence quasi-compact (as an open of a Noetherian algebraic space is quasi-compact). □

108.11. Properties of the stack of polarized proper schemes

0DPS In this section we discuss properties of the moduli stack

$$\mathcal{P}olarized \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

whose category of sections over a scheme S is the category of proper, flat, finitely presented scheme over S endowed with a relatively ample invertible sheaf. This is an algebraic stack by Quot, Theorem 99.14.15.

0DPT **Lemma 108.11.1.** *The diagonal of $\mathcal{P}olarized$ is separated and of finite presentation.*

Proof. Recall that $\mathcal{P}olarized$ is a limit preserving algebraic stack, see Quot, Lemma 99.14.8. By Limits of Stacks, Lemma 102.3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 102.3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given an affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of $\mathcal{P}olarized$ over U, the algebraic space

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi)$$

is separated. The rule which to an isomorphism $\upsilon_T \to \chi_T$ assigns the underlying isomorphism $Y_T \to X_T$ defines a morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X)$$

Since we have seen in Lemmas 108.10.2 and 108.10.3 that the target is a separated algebraic space, it suffices to prove that this morphism is separated. Given an isomorphism $f : Y_T \to X_T$ over some scheme T/U, then clearly

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \times_{Isom_U(Y,X),[f]} T = Isom(\mathcal{N}_T, f^*\mathcal{L}_T)$$

Here $[f] : T \to Isom_U(Y, X)$ indicates the T-valued point corresponding to f and $Isom(\mathcal{N}_T, f^*\mathcal{L}_T)$ is the algebraic space discussed in Section 108.3. Since this algebraic space is affine over U, the claim implies Δ is separated.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffice to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{P}olarized \times \mathcal{P}olarized$ is quasi-compact (see for example Properties of Stacks, Lemma 100.3.3). We can assume $U = \coprod U_i$ is a disjoint union of affine opens. Since $\mathcal{P}olarized$ is limit preserving (see above), we see that $\mathcal{P}olarized \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 102.3.8 and Morphisms of Stacks, Lemmas 101.27.2 and 101.33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of $\mathcal{P}olarized$ over U, we show the algebraic space

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let P be the Hilbert polynomial $n \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$, see Varieties, Lemma 33.45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 36.32.2) we see that we get the same Hilbert polynomial in every point of U. Set $\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \mathrm{pr}_2^*\mathcal{L}$ on $Y \times_U X$. Given $(f, \varphi) \in Isom_{\mathcal{P}olarized}(\upsilon, \chi)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\chi(Y_t, (\mathrm{id} \times f)^*\mathcal{M}^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes 2n}) = P(2n)$$

where in the middle equality we use the isomorphism $\varphi : f^*\mathcal{L}_T \to \mathcal{N}_T$. Setting $P'(t) = P(2t)$ we find that the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X)$$

(see earlier) has image contained in the intersection

$$Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $Mor_U(Y, X)$ (see Lemma 108.10.3 and Remark 108.10.4). Now $Mor_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Lemma 108.10.5. Thus the intersection is a Noetherian algebraic space too. Since the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

is affine (see above) we conclude. □

0DPU **Lemma 108.11.2.** *The morphism $\mathcal{P}olarized \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{P}olarized \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.11.1. To prove that $\mathcal{P}olarized \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{P}olarized$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This is Quot, Lemma 99.14.8. □

0E96 **Lemma 108.11.3.** *Let $n \geq 1$ be an integer and let P be a numerical polynomial. Let*

$$T \subset |\mathcal{P}olarized|$$

be a subset with the following property: for every $\xi \in T$ there exists a field k and an object $(X, \mathcal{L})$ of $\mathcal{P}olarized$ over k representing ξ such that

(1) *the Hilbert polynomial of $\mathcal{L}$ on X is P, and*
(2) *there exists a closed immersion $i : X \to \mathbf{P}_k^n$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$.*

Then T is a Noetherian topological space, in particular quasi-compact.

Proof. Observe that $|\mathcal{P}olarized|$ is a locally Noetherian topological space, see Morphisms of Stacks, Lemma 101.8.3 (this also uses that $\mathrm{Spec}(\mathbf{Z})$ is Noetherian and hence $\mathcal{P}olarized$ is a locally Noetherian algebraic stack by Lemma 108.11.2 and Morphisms of Stacks, Lemma 101.17.5). Thus any quasi-compact subset of $|\mathcal{P}olarized|$

is a Noetherian topological space and any subset of such is also Noetherian, see Topology, Lemmas 5.9.4 and 5.9.2. Thus all we have to do is a find a quasi-compact subset containing T.

By Lemma 108.7.7 the algebraic space

$$H = \mathrm{Hilb}^{P,\mathcal{O}(1)}_{\mathbf{P}^n_{\mathbf{Z}}/\operatorname{Spec}(\mathbf{Z})}$$

is proper over $\operatorname{Spec}(\mathbf{Z})$. By Quot, Lemma 99.9.3[1] the identity morphism of H corresponds to a closed subspace

$$Z \subset \mathbf{P}^n_H$$

which is proper, flat, and of finite presentation over H and such that the restriction $\mathcal{N} = \mathcal{O}(1)|_Z$ is relatively ample on Z/H and has Hilbert polynomial P on the fibres of $Z \to H$. In particular, the pair $(Z \to H, \mathcal{N})$ defines a morphism

$$H \longrightarrow \mathcal{P}olarized$$

which sends a morphism of schemes $U \to H$ to the classifying morphism of the family $(Z_U \to U, \mathcal{N}_U)$, see Quot, Lemma 99.14.4. Since H is a Noetherian algebraic space (as it is proper over $\mathbf{Z}$)) we see that $|H|$ is Noetherian and hence quasi-compact. The map

$$|H| \longrightarrow |\mathcal{P}olarized|$$

is continuous, hence the image is quasi-compact. Thus it suffices to prove T is contained in the image of $|H| \to |\mathcal{P}olarized|$. However, assumptions (1) and (2) exactly express the fact that this is the case: any choice of a closed immersion $i : X \to \mathbf{P}^n_k$ with $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$ we get a k-valued point of H by the moduli interpretation of H. This finishes the proof of the lemma. □

108.12. Properties of moduli of complexes on a proper morphism

0DPV Let $f : X \to B$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Then the stack $\mathcal{C}omplexes_{X/B}$ parametrizing relatively perfect complexes with vanishing negative self-exts is algebraic. See Quot, Theorem 99.16.12.

0DPW **Lemma 108.12.1.** *The diagonal of $\mathcal{C}omplexes_{X/B}$ over B is affine and of finite presentation.*

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 99.16.5. From the proof we find that we have to show: given a scheme T over B and objects $E, E' \in D(\mathcal{O}_{X_T})$ such that (T, E) and (T, E') are objects of the fibre category of $\mathcal{C}omplexes_{X/B}$ over T, then $Isom(E, E') \to T$ is affine and of finite presentation. Here $Isom(E, E')$ is the functor

$$(Sch/T)^{opp} \to Sets, \quad T' \mapsto \{\varphi : E_{T'} \to E'_{T'} \text{ isomorphism in } D(\mathcal{O}_{X_{T'}})\}$$

where $E_{T'}$ and $E'_{T'}$ are the derived pullbacks of E and E' to $X_{T'}$. Consider the functor $H = \mathcal{H}om(E, E')$ defined by the rule

$$(Sch/T)^{opp} \to Sets, \quad T' \mapsto \mathrm{Hom}_{\mathcal{O}_{X_{T'}}}(E_T, E'_T)$$

[1] We will see later (insert future reference here) that H is a scheme and hence the use of this lemma and Quot, Lemma 99.14.4 isn't necessary.

By Quot, Lemma 99.16.1 this is an algebraic space affine and of finite presentation over T. The same is true for $H' = \mathcal{H}om(E', E)$, $I = \mathcal{H}om(E, E)$, and $I' = \mathcal{H}om(E', E')$. Therefore we see that

$$Isom(E, E') = (H' \times_T H) \times_{c, I \times_T I', \sigma} T$$

where $c(\varphi', \varphi) = (\varphi \circ \varphi', \varphi' \circ \varphi)$ and $\sigma = (\mathrm{id}, \mathrm{id})$ (compare with the proof of Quot, Proposition 99.4.3). Thus $Isom(E, E')$ is affine over T as a fibre product of schemes affine over T. Similarly, $Isom(E, E')$ is of finite presentation over T. □

0DPX **Lemma 108.12.2.** *The morphism $\mathit{Complexes}_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathit{Complexes}_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 108.12.1. To prove that $\mathit{Complexes}_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathit{Complexes}_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This follows from Quot, Lemma 99.16.8 (small detail omitted). □

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CHAPTER 109

Moduli of Curves

0DMG

109.1. Introduction

0DMH In this chapter we discuss some of the familiar moduli stacks of curves. A reference is the celebrated article of Deligne and Mumford, see [**DM69**].

109.2. Conventions and abuse of language

0DMI We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 100.2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

109.3. The stack of curves

0DMJ This section is the continuation of Quot, Section 99.15. Let $\mathcal{C}\mathit{urves}$ be the stack whose category of sections over a scheme S is the category of families of curves over S. It is somewhat important to keep in mind that a *family of curves* is a morphism $f : X \to S$ where X is an algebraic space (!) and f is flat, proper, of finite presentation and of relative dimension ≤ 1. We already know that $\mathcal{C}\mathit{urves}$ is an algebraic stack over $\mathbf{Z}$, see Quot, Theorem 99.15.11. If we did not allow algebraic spaces in the definition of our stack, then this theorem would be false.

Often base change is denoted by a subscript, but we cannot use this notation for $\mathcal{C}\mathit{urves}$ because $\mathcal{C}\mathit{urves}_S$ is our notation for the fibre category over S. This is why in Quot, Remark 99.15.5 we used B-$\mathcal{C}\mathit{urves}$ for the base change

$$B\text{-}\mathcal{C}\mathit{urves} = \mathcal{C}\mathit{urves} \times B$$

to the algebraic space B. The product on the right is over the final object, i.e., over $\mathrm{Spec}(\mathbf{Z})$. The object on the left is the stack classifying families of curves on the category of schemes over B. In particular, if k is a field, then

$$k\text{-}\mathcal{C}\mathit{urves} = \mathcal{C}\mathit{urves} \times \mathrm{Spec}(k)$$

is the moduli stack classifying families of curves on the category of schemes over k. Before we continue, here is a sanity check.

0DMK **Lemma 109.3.1.** *Let $T \to B$ be a morphism of algebraic spaces. The category*

$$\mathrm{Mor}_B(T, B\text{-}\mathcal{C}\mathit{urves}) = \mathrm{Mor}(T, \mathcal{C}\mathit{urves})$$

is the category of families of curves over T.

Proof. A family of curves over T is a morphism $f : X \to T$ of algebraic spaces, which is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 67.33.2). This is exactly the same as the definition in Quot, Situation 99.15.1 except that T the base is allowed to be an algebraic space. Our default base category for algebraic stacks/spaces is the category of schemes,

hence the lemma does not follow immediately from the definitions. Having said this, we encourage the reader to skip the proof.

By the product description of B-$\mathcal{C}urves$ given above, it suffices to prove the lemma in the absolute case. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $s, t : R \to U$.

Let $v : T \to \mathcal{C}urves$ be a morphism. Then $v \circ p$ corresponds to a family of curves $X_U \to U$. The canonical 2-morphism $v \circ p \circ t \to v \circ p \circ s$ is an isomorphism $\varphi : X_U \times_{U,s} R \to X_U \times_{U,t} R$. This isomorphism satisfies the cocycle condition on $R \times_{s,t} R$. By Bootstrap, Lemma 80.11.3 we obtain a morphism of algebraic spaces $X \to T$ whose pullback to U is equal to X_U compatible with φ. Since $\{U \to T\}$ is an étale covering, we see that $X \to T$ is flat, proper, of finite presentation by Descent on Spaces, Lemmas 74.11.13, 74.11.19, and 74.11.12. Also $X \to T$ has relative dimension ≤ 1 because this is an étale local property. Hence $X \to T$ is a family of curves over T.

Conversely, let $X \to T$ be a family of curves. Then the base change X_U determines a morphism $w : U \to \mathcal{C}urves$ and the canonical isomorphism $X_U \times_{U,s} R \to X_U \times_{U,t} R$ determines a 2-arrow $w \circ s \to w \circ t$ satisfying the cocycle condition. Thus a morphism $v : T = [U/R] \to \mathcal{C}urves$ by the universal property of the quotient $[U/R]$, see Groupoids in Spaces, Lemma 78.23.2. (Actually, it is much easier in this case to go back to before we introduced our abuse of language and direct construct the functor $Sch/T \to \mathcal{C}urves$ which "is" the morphism $T \to \mathcal{C}urves$.)

We omit the verification that the constructions given above extend to morphisms between objects and are mutually quasi-inverse. □

109.4. The stack of polarized curves

0DPY In this section we work out some of the material discussed in Quot, Remark 99.15.13. Consider the 2-fibre product

$$\begin{array}{ccc} \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized & \longrightarrow & \mathcal{P}olarized \\ \downarrow & & \downarrow \\ \mathcal{C}urves & \longrightarrow & \mathcal{S}paces'_{fp,flat,proper} \end{array}$$

We denote this 2-fibre product by

$$PolarizedCurves = \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

This fibre product parametrizes polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. More precisely, an object of $PolarizedCurves$ is a pair $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, of finite presentation, and has relative dimension ≤ 1, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S.

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects of $PolarizedCurves$ is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of

schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner.

0DPZ **Lemma 109.4.1.** *The morphism PolarizedCurves $\to$ Polarized is an open and closed immersion.*

Proof. This is true because the 1-morphism $\mathcal{C}urves \to \mathcal{S}paces'_{fp,flat,proper}$ is representable by open and closed immersions, see Quot, Lemma 99.15.12. □

0DQ0 **Lemma 109.4.2.** *The morphism PolarizedCurves $\to$ Curves is smooth and surjective.*

Proof. Surjective. Given a field k and a proper algebraic space X over k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Spaces over Fields, Lemma 72.9.3 the algebraic space X is a scheme. Hence X is a proper scheme of dimension ≤ 1 over k. By Varieties, Lemma 33.43.4 we see that X is H-projective over κ. In particular, there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Then $(X, \mathcal{L})$ is an object of *PolarizedCurves* over k which maps to X.

Smooth. Let $X \to S$ be an object of $\mathcal{C}urves$, i.e., a morphism $S \to \mathcal{C}urves$. It is clear that

$$PolarizedCurves \times_{\mathcal{C}urves} S \subset \mathcal{P}ic_{X/S}$$

is the substack of objects $(T/S, \mathcal{L}/X_T)$ such that $\mathcal{L}$ is ample on X_T/T. This is an open substack by Descent on Spaces, Lemma 74.13.2. Since $\mathcal{P}ic_{X/S} \to S$ is smooth by Moduli Stacks, Lemma 108.8.5 we win. □

0E6F **Lemma 109.4.3.** *Let $X \to S$ be a family of curves. Then there exists an étale covering $\{S_i \to S\}$ such that $X_i = X \times_S S_i$ is a scheme. We may even assume X_i is H-projective over S_i.*

Proof. This is an immediate corollary of Lemma 109.4.2. Namely, unwinding the definitions, this lemma gives there is a surjective smooth morphism $S' \to S$ such that $X' = X \times_S S'$ comes endowed with an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ which is ample on X'/S'. Then we can refine the smooth covering $\{S' \to S\}$ by an étale covering $\{S_i \to S\}$, see More on Morphisms, Lemma 37.38.7. After replacing S_i by a suitable open covering we may assume $X_i \to S_i$ is H-projective, see Morphisms, Lemmas 29.43.6 and 29.43.4 (this is also discussed in detail in More on Morphisms, Section 37.50). □

109.5. Properties of the stack of curves

0DSP The following lemma isn't true for moduli of surfaces, see Remark 109.5.2.

0DSQ **Lemma 109.5.1.** *The diagonal of Curves is separated and of finite presentation.*

Proof. Recall that *Curves* is a limit preserving algebraic stack, see Quot, Lemma 99.15.6. By Limits of Stacks, Lemma 102.3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 102.3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given a scheme U and two objects $Y \to U$ and $X \to U$ of *Curves* over U, the algebraic space

$$Isom_U(Y, X)$$

is separated. This we have seen in Moduli Stacks, Lemmas 108.10.2 and 108.10.3 that the target is a separated algebraic space.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffices to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{C}urves \times \mathcal{C}urves$ is quasi-compact (see for example Properties of Stacks, Lemma 100.3.3). We choose $U = \coprod U_i$ to be a disjoint union of affine opens with a surjective smooth morphism

$$U \longrightarrow PolarizedCurves \times PolarizedCurves$$

Then $U \to \mathcal{C}urves \times \mathcal{C}urves$ will be surjective and smooth since $PolarizedCurves \to \mathcal{C}urves$ is surjective and smooth by Lemma 109.4.2. Since *PolarizedCurves* is limit preserving (by Artin's Axioms, Lemma 98.11.2 and Quot, Lemmas 99.15.6, 99.14.8, and 99.13.6), we see that $PolarizedCurves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 102.3.8 and Morphisms of Stacks, Lemmas 101.27.2 and 101.33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $(Y, \mathcal{N})$ and $(X, \mathcal{L})$ of *PolarizedCurves* over U, we show the algebraic space

$$Isom_U(Y, X)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let Q, P be the Hilbert polynomials of these families, i.e.,

$$Q(n) = \chi(Y_u, \mathcal{N}_u^{\otimes n}) \quad \text{and} \quad P(n) = \chi(X_u, \mathcal{L}_u^{\otimes n})$$

see Varieties, Lemma 33.45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ and $u \mapsto \chi(X_u, \mathcal{L}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 36.32.2) we see that we get the same Hilbert polynomial in every point of U. Set

$$\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \mathrm{pr}_2^*\mathcal{L}$$

on $Y \times_U X$. Given $(f, \varphi) \in Isom_U(Y, X)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\begin{aligned}
\chi(Y_t, (\mathrm{id} \times f)^*\mathcal{M}^{\otimes n}) &= \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) \\
&= n \deg(\mathcal{N}_t) + n \deg(f_t^*\mathcal{L}_t) + \chi(Y_t, \mathcal{O}_{Y_t}) \\
&= Q(n) + n \deg(\mathcal{L}_t) \\
&= Q(n) + P(n) - P(0)
\end{aligned}$$

by Riemann-Roch for proper curves, more precisely by Varieties, Definition 33.44.1 and Lemma 33.44.7 and the fact that f_t is an isomorphism. Setting $P'(t) = Q(t) + P(t) - P(0)$ we find

$$Isom_U(Y, X) = Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $Mor_U(Y, X)$, see Moduli Stacks, Lemma 108.10.3 and Remark 108.10.4. Now $Mor_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Moduli Stacks, Lemma 108.10.5. Thus the intersection is a Noetherian algebraic space too and the proof is finished. □

0DSR **Remark 109.5.2.** The boundedness argument in the proof of Lemma 109.5.1 does not work for moduli of surfaces and in fact, the result is wrong, for example because K3 surfaces over fields can have infinite discrete automorphism groups. The "reason" the argument does not work is that on a projective surface S over a field, given ample invertible sheaves $\mathcal{N}$ and $\mathcal{L}$ with Hilbert polynomials Q and P, there is no a priori bound on the Hilbert polynomial of $\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{L}$. In terms of intersection theory, if H_1, H_2 are ample effective Cartier divisors on S, then there is no (upper) bound on the intersection number $H_1 \cdot H_2$ in terms of $H_1 \cdot H_1$ and $H_2 \cdot H_2$.

0DSS **Lemma 109.5.3.** *The morphism $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 109.5.1. To prove that $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{C}urves$ is limit preserving, see Limits of Stacks, Proposition 102.3.8. This is Quot, Lemma 99.15.6. □

109.6. Open substacks of the stack of curves

0E0E Below we will often characterize an open substack of $\mathcal{C}urves$ by a property P of morphisms of algebraic spaces. To see that P defines an open substack it suffices to check

(o) given a family of curves $f : X \to S$ there exists a largest open subscheme $S' \subset S$ such that $f|_{f^{-1}(S')} : f^{-1}(S') \to S'$ has P and such that formation of S' commutes with arbitrary base change.

Namely, suppose (o) holds. Choose a scheme U and a surjective smooth morphism $m : U \to \mathcal{C}urves$. Let $R = U \times_{\mathcal{C}urves} U$ and denote $t, s : R \to U$ the projections. Recall that $\mathcal{C}urves = [U/R]$ is a presentation, see Algebraic Stacks, Lemma 94.16.2 and Definition 94.16.5. By construction of $\mathcal{C}urves$ as the stack of curves, the morphism m is the classifying morphism for a family of curves $C \to U$. The 2-commutativity of the diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ {\scriptstyle t}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{C}urves \end{array}$$

implies that $C \times_{U,s} R \cong C \times_{U,t} R$ (isomorphism of families of curves over R). Let $W \subset U$ be the largest open subscheme such that $f|_{f^{-1}(W)} : f^{-1}(W) \to W$ has P as

in (o). Since formation of W commutes with base change according to (o) and by the isomorphism above we find that $s^{-1}(W) = t^{-1}(W)$. Thus $W \subset U$ corresponds to an open substack

$$\mathcal{C}urves^P \subset \mathcal{C}urves$$

according to Properties of Stacks, Lemma 100.9.8.

Continuing with the setup of the previous paragrpah, we claim the open substack $\mathcal{C}urves^P$ has the following two universal properties:

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to S$ has P,
(2) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to \mathrm{Spec}(k)$ has P.

This follows by considering the 2-fibre product

$$\begin{array}{ccc} T & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{C}urves \end{array}$$

Observe that $T \to S$ is surjective and smooth as the base change of $U \to \mathcal{C}urves$. Thus the open $S' \subset S$ given by (o) is determined by its inverse image in T. However, by the invariance under base change of these opens in (o) and because $X \times_S T \cong C \times_U T$ by the 2-commutativity, we find $q^{-1}(S') = p^{-1}(W)$ as opens of T. This immediately implies (1). Part (2) is a special case of (1).

Given two properties P and Q of morphisms of algebraic spaces, supposing we already have established $\mathcal{C}urves^Q$ is an open substack of $\mathcal{C}urves$, then we can use exactly the same method to prove openness of $\mathcal{C}urves^{Q,P} \subset \mathcal{C}urves^Q$. We omit a precise explanation.

109.7. Curves with finite reduced automorphism groups

0DST Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Lemma 109.5.1 the automorphism group algebraic space $Aut(X)$ is finite type and separated over k. In particular, $Aut(X)$ is a group scheme, see More on Groupoids in Spaces, Lemma 79.10.2. If the characteristic of k is zero, then $Aut(X)$ is reduced and even smooth over k (Groupoids, Lemma 39.8.2). However, in general $Aut(X)$ is not reduced, even if X is geometrically reduced.

0DSU **Example 109.7.1** (Non-reduced automorphism group)**.** Let k be an algebraically closed field of characteristic 2. Set $Y = Z = \mathbf{P}^1_k$. Choose three pairwise distinct k-valued points a, b, c in $\mathbf{A}^1_k$. Thinking of $\mathbf{A}^1_k \subset \mathbf{P}^1_k = Y = Z$ as an open subschemes, we get a closed immersion

$$T = \mathrm{Spec}(k[t]/(t-a)^2) \amalg \mathrm{Spec}(k[t]/(t-b)^2) \amalg \mathrm{Spec}(k[t]/(t-c)^2) \longrightarrow \mathbf{P}^1_k$$

Let X be the pushout in the diagram

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

Let $U \subset X$ be the affine open part which is the image of $\mathbf{A}^1_k \amalg \mathbf{A}^1_k$. Then we have an equalizer diagram

$$\mathcal{O}_X(U) \longrightarrow k[t] \times k[t] \rightrightarrows k[t]/(t-a)^2 \times k[t]/(t-b)^2 \times k[t]/(t-c)^2$$

Over the dual numbers $A = k[\epsilon]$ we have a nontrivial automorphism of this equalizer diagram sending t to $t + \epsilon$. We leave it to the reader to see that this automorphism extends to an automorphism of X over A. On the other hand, the reader easily shows that the automorphism group of X over k is finite. Thus $Aut(X)$ must be non-reduced.

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. If $Aut(X)$ is geometrically reduced, then it need not be the case that it has dimension 0, even if X is smooth and geometrically connected.

0DSV **Example 109.7.2** (Smooth positive dimensional automorphism group)**.** Let k be an algebraically closed field. If X is a smooth genus 0, resp. 1 curve, then the automorphism group has dimension 3, resp. 1. Namely, in the genus 0 case we have $X \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 53.10.4. Since

$$Aut(\mathbf{P}^1_k) = \mathrm{PGL}_{2,k}$$

as functors we see that the dimension is 3. On the other hand, if the genus of X is 1, then we see that the map $X = \underline{\mathrm{Hilb}}^1_{X/k} \to \underline{\mathrm{Pic}}^1_{X/k}$ is an isomorphism, see Picard Schemes of Curves, Lemma 44.6.7 and Algebraic Curves, Theorem 53.2.6. Thus X has the structure of an abelian variety (since $\underline{\mathrm{Pic}}^1_{X/k} \cong \underline{\mathrm{Pic}}^0_{X/k}$). In particular the (co)tangent bundle of X are trivial (Groupoids, Lemma 39.6.3). We conclude that $\dim_k H^0(X, T_X) = 1$ hence $\dim Aut(X) \leq 1$. On the other hand, the translations (viewing X as a group scheme) provide a 1-dimensional piece of $\mathrm{Aut}(X)$ and we conlude its dimension is indeed 1.

It turns out that there is an open substack of $\mathcal{C}urves$ parametrizing curves whose automorphism group is geometrically reduced and finite. Here is a precise statement.

0DSW **Lemma 109.7.3.** *There exist an open substack $\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ with the following properties*

1. *$\mathcal{C}urves^{DM} \subset \mathcal{C}urves$ is the maximal open substack which is DM,*
2. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$,*
   (b) *the group algebraic space $Aut_S(X)$ is unramified over S,*
3. *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$,*
   (b) *$Aut(X)$ is geometrically reduced over k and has dimension 0,*
   (c) *$Aut(X) \to \mathrm{Spec}(k)$ is unramified.*

Proof. The existence of an open substack with property (1) is Morphisms of Stacks, Lemma 101.22.1. The points of this open substack are characterized by (3)(c) by Morphisms of Stacks, Lemma 101.22.2. The equivalence of (3)(b) and (3)(c) is the statement that an algebraic space G which is locally of finite type, geometrically reduced, and of dimension 0 over a field k, is unramified over k. First, G is a scheme by Spaces over Fields, Lemma 72.9.1. Then we can take an affine open in G and observe that it will be proper over k and apply Varieties, Lemma 33.9.3. Minor details omitted.

Part (2) is true because (3) holds. Namely, the morphism $Aut_S(X) \to S$ is locally of finite type. Thus we can check whether $Aut_S(X) \to S$ is unramified at all points of $Aut_S(X)$ by checking on fibres at points of the scheme S, see Morphisms of Spaces, Lemma 67.38.10. But after base change to a point of S we fall back into the equivalence of (3)(a) and (3)(c). □

0E6G **Lemma 109.7.4.** *Let X be a proper scheme over a field k of dimension ≤ 1. Then properties (3)(a), (b), (c) are also equivalent to $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.*

Proof. In the discussion above we have seen that $G = Aut(X)$ is a group scheme over $\mathrm{Spec}(k)$ which is finite type and separated; this uses Lemma 109.5.1 and More on Groupoids in Spaces, Lemma 79.10.2. Then G is unramified over k if and only if $\Omega_{G/k} = 0$ (Morphisms, Lemma 29.35.2). By Groupoids, Lemma 39.6.3 the vanishing holds if $T_{G/k,e} = 0$, where $T_{G/k,e}$ is the tangent space to G at the identity element $e \in G(k)$, see Varieties, Definition 33.16.3 and the formula in Varieties, Lemma 33.16.4. Since $\kappa(e) = k$ the tangent space is defined in terms of morphisms $\alpha : \mathrm{Spec}(k[\epsilon]) \to G = Aut(X)$ whose restriction to $\mathrm{Spec}(k)$ is e. It follows that it suffices to show any automorphism

$$\alpha : X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \longrightarrow X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

over $\mathrm{Spec}(k[\epsilon])$ whose restriction to $\mathrm{Spec}(k)$ is id_X. Such automorphisms are called infinitesimal automorphisms.

The infinitesimal automorphisms of X correspond 1-to-1 with derivations of $\mathcal{O}_X$ over k. This follows from More on Morphisms, Lemmas 37.9.1 and 37.9.2 (we only need the first one as we don't care about the reverse direction; also, please look at More on Morphisms, Remark 37.9.7 for an elucidation). For a different argument proving this equality we refer the reader to Deformation Problems, Lemma 93.9.3. □

109.8. Cohen-Macaulay curves

0E0H There is an open substack of $\mathcal{C}urves$ parametrizing the Cohen-Macaulay "curves".

0E0I **Lemma 109.8.1.** *There exist an open substack $\mathcal{C}urves^{CM} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM}$,*
   (b) *the morphism $X \to S$ is Cohen-Macaulay,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM}$,*
   (b) *X is Cohen-Macaulay.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.26.7 the set

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E1F **Lemma 109.8.2.** *There exist an open substack $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM,1}$,*
   (b) *the morphism $X \to S$ is Cohen-Macaulay and has relative dimension 1 (Morphisms of Spaces, Definition 67.33.2),*
(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{CM,1}$,*
   (b) *X is Cohen-Macaulay and X is equidimensional of dimension 1.*

Proof. By Lemma 109.8.1 it is clear that we have $\mathcal{C}urves^{CM,1} \subset \mathcal{C}urves^{CM}$ if it exists. Let $f : X \to S$ be a family of curves such that f is a Cohen-Macaulay morphism. By More on Morphisms of Spaces, Lemma 76.26.8 we have a decomposition

$$X = X_0 \amalg X_1$$

by open and closed subspaces such that $X_0 \to S$ has relative dimension 0 and $X_1 \to S$ has relative dimension 1. Since f is proper the subset

$$S' = S \setminus f(|X_0|)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay and has relative dimension 1. Moreover, formation of S' commutes with arbitrary base change because this is true for the decomposition above (as relative dimension behaves well with respect to base change, see Morphisms of Spaces, Lemma 67.34.3). Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

109.9. Curves of a given genus

0E6H The convention in the Stacks project is that the genus g of a proper 1-dimensional scheme X over a field k is defined only if $H^0(X, \mathcal{O}_X) = k$. In this case $g = \dim_k H^1(X, \mathcal{O}_X)$. See Algebraic Curves, Section 53.8. The conditions needed to define the genus define an open substack which is then a disjoint union of open substacks, one for each genus.

0E6I **Lemma 109.9.1.** *There exist an open substack $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$,*
   (b) *$f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after arbitrary base change, and the fibres of f have dimension 1,*

(2) *given a scheme X proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{h0,1}$,
   (b) $H^0(X, \mathcal{O}_X) = k$ *and* $\dim(X) = 1$.

Proof. Given a family of curves $X \to S$ the set of $s \in S$ where $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ is open in S by Derived Categories of Spaces, Lemma 75.26.2. Also, the set of points in S where the fibre has dimension 1 is open by More on Morphisms of Spaces, Lemma 76.31.5. Moreover, if $f : X \to S$ is a family of curves all of whose fibres have dimension 1 (and in particular f is surjective), then condition (1)(b) is equivalent to $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for every $s \in S$, see Derived Categories of Spaces, Lemma 75.26.7. Thus we see that the lemma follows from the general discussion in Section 109.6. □

0E6J **Lemma 109.9.2.** *We have* $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves^{CM,1}$ *as open substacks of* $\mathcal{C}urves$.

Proof. See Algebraic Curves, Lemma 53.6.1 and Lemmas 109.9.1 and 109.8.2. □

0E1J **Lemma 109.9.3.** *Let $f : X \to S$ be a family of curves such that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$, i.e., the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$ (Lemma 109.9.1). Then*
(1) $f_*\mathcal{O}_X = \mathcal{O}_S$ *and this holds universally,*
(2) $R^1 f_*\mathcal{O}_X$ *is a finite locally free $\mathcal{O}_S$-module,*
(3) *for any morphism $h : S' \to S$ if $f' : X' \to S'$ is the base change, then $h^*(R^1 f_*\mathcal{O}_X) = R^1 f'_*\mathcal{O}_{X'}$.*

Proof. We apply Derived Categories of Spaces, Lemma 75.26.7. This proves part (1). It also implies that locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Spaces, Lemma 75.25.4). Thus for $s \in S$ we have

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 15.76.7. Since everything is compatible with base change we also see that (3) holds. □

0E6K **Lemma 109.9.4.** *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{h0,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g$$

where each $\mathcal{C}urves_g$ is characterized as follows:
(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g$,*
   (b) $f_*\mathcal{O}_X = \mathcal{O}_S$, *this holds after arbitrary base change, the fibres of f have dimension* 1, *and $R^1 f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
(2) *given a scheme X proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g$,
   (b) $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, *and the genus of X is g.*

Proof. We already have the existence of $\mathcal{C}urves^{h0,1}$ as an open substack of $\mathcal{C}urves$ characterized by the conditions of the lemma not involving R^1f_* or H^1, see Lemma 109.9.1. The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 109.6 and Lemma 109.9.3. This proves the characterization in (1). The characterization in (2) follows from the definition of the genus in Algebraic Curves, Definition 53.8.1. □

109.10. Geometrically reduced curves

0E0F There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced "curves".

0E0G **Lemma 109.10.1.** *There exist an open substack $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{geomred}$,*
   (b) *the fibres of the morphism $X \to S$ are geometrically reduced (More on Morphisms of Spaces, Definition 76.29.2),*

(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{geomred}$,*
   (b) *X is geometrically reduced over k.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.29.6 the set

$$E = \{s \in S : \text{the fibre of } X \to S \text{ at } s \text{ is geometrically reduced}\}$$

is open in S. Formation of this open commutes with arbitrary base change by More on Morphisms of Spaces, Lemma 76.29.3. Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E1G **Lemma 109.10.2.** *We have $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves^{CM}$ as open substacks of $\mathcal{C}urves$.*

Proof. This is true because a reduced Noetherian scheme of dimension ≤ 1 is Cohen-Macaulay. See Algebra, Lemma 10.157.3. □

109.11. Geometrically reduced and connected curves

0E1H There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced and connected "curves". We will get rid of 0-dimensional objects right away.

0E1I **Lemma 109.11.1.** *There exist an open substack $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}$,*
   (b) *the geometric fibres of the morphism $X \to S$ are reduced, connected, and have dimension 1,*

(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}$,*
   (b) *X is geometrically reduced, geometrically connected, and has dimension 1.*

Proof. By Lemmas 109.10.1, 109.10.2, 109.8.1, and 109.8.2 it is clear that we have

$$\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves^{geomred} \cap \mathcal{C}urves^{CM,1}$$

if it exists. Let $f : X \to S$ be a family of curves such that f is Cohen-Macaulay, has geometrically reduced fibres, and has relative dimension 1. By More on Morphisms of Spaces, Lemma 76.36.9 in the Stein factorization

$$X \to T \to S$$

the morphism $T \to S$ is étale. This implies that there is an open and closed subscheme $S' \subset S$ such that $X \times_S S' \to S'$ has geometrically connected fibres (in the decomposition of Morphisms, Lemma 29.48.5 for the finite locally free morphism $T \to S$ this corresponds to S_1). Formation of this open commutes with arbitrary base change because the number of connected components of geometric fibres is invariant under base change (it is also true that the Stein factorization commutes with base change in our particular case but we don't need this to conclude). Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E6L **Lemma 109.11.2.** *We have $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves^{h0,1}$ as open substacks of $\mathcal{C}urves$. In particular, given a family of curves $f : X \to S$ whose geometric fibres are reduced, connected and of dimension 1, then $R^1 f_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with arbitrary base change.*

Proof. This follows from Varieties, Lemma 33.9.3 and Lemmas 109.9.1 and 109.11.1. The final statement follows from Lemma 109.9.3. □

0E1K **Lemma 109.11.3.** *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{grc,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves^{grc,1}_g$$

where each $\mathcal{C}urves^{grc,1}_g$ is characterized as follows:

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}_g$,*
   (b) *the geometric fibres of the morphism $f : X \to S$ are reduced, connected, of dimension 1 and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{grc,1}_g$,*
   (b) *X is geometrically reduced, geometrically connected, has dimension 1, and has genus g.*

Proof. First proof: set $\mathcal{C}urves^{grc,1}_g = \mathcal{C}urves^{grc,1} \cap \mathcal{C}urves_g$ and combine Lemmas 109.11.2 and 109.9.4. Second proof: The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 109.6 and Lemma 109.11.2. This proves the characterization in (1). The characterization in (2) follows as well since the genus of a geometrically reduced and connected proper 1-dimensional scheme X/k is defined (Algebraic Curves, Definition 53.8.1 and Varieties, Lemma 33.9.3) and is equal to $\dim_k H^1(X, \mathcal{O}_X)$. □

109.12. Gorenstein curves

0E1L There is an open substack of $\mathcal{C}urves$ parametrizing the Gorenstein "curves".

0E1M **Lemma 109.12.1.** *There exist an open substack $\mathcal{C}urves^{Gorenstein} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   (b) *the morphism $X \to S$ is Gorenstein,*
(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   (b) *X is Gorenstein.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 76.27.7 the set

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Gorenstein. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 109.6. □

0E6M **Lemma 109.12.2.** *There exist an open substack $\mathcal{C}urves^{Gorenstein,1} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,*
   (b) *the morphism $X \to S$ is Gorenstein and has relative dimension 1 (Morphisms of Spaces, Definition 67.33.2),*
(2) *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,*
   (b) *X is Gorenstein and X is equidimensional of dimension 1.*

Proof. Recall that a Gorenstein scheme is Cohen-Macaulay (Duality for Schemes, Lemma 48.24.2) and that a Gorenstein morphism is a Cohen-Macaulay morphism (Duality for Schemes, Lemma 48.25.4. Thus we can set $\mathcal{C}urves^{Gorenstein,1}$ equal to the intersection of $\mathcal{C}urves^{Gorenstein}$ and $\mathcal{C}urves^{CM,1}$ inside of $\mathcal{C}urves$ and use Lemmas 109.12.1 and 109.8.2. □

109.13. Local complete intersection curves

0E0J There is an open substack of $\mathcal{C}urves$ parametrizing the local complete intersection "curves".

0DZV **Lemma 109.13.1.** *There exist an open substack $\mathcal{C}urves^{lci} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{lci}$,*
   (b) *$X \to S$ is a local complete intersection morphism, and*

(c) $X \to S$ *is a syntomic morphism.*

(2) *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*

(a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci}$,

(b) *X is a local complete intersection over k.*

Proof. Recall that being a syntomic morphism is the same as being flat and a local complete intersection morphism, see More on Morphisms of Spaces, Lemma 76.48.6. Thus (1)(b) is equivalent to (1)(c). In Section 109.6 we have seen it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is a local complete intersection morphism and such that formation of S' commutes with arbitrary base change. This follows from the more general More on Morphisms of Spaces, Lemma 76.49.7. □

109.14. Curves with isolated singularities

0E0K We can look at the open substack of $\mathcal{C}urves$ parametrizing "curves" with only a finite number of singular points (these may correspond to 0-dimensional components in our setup).

0DZW **Lemma 109.14.1.** *There exist an open substack $\mathcal{C}urves^+ \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $X \to S$ the following are equivalent*

(a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^+$,*

(b) *the singular locus of $X \to S$ endowed with any/some closed subspace structure is finite over S.*

(2) *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*

(a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^+$,

(b) *$X \to \mathrm{Spec}(k)$ is smooth except at finitely many points.*

Proof. To prove the lemma it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that the fibre of $S' \times_S X \to S'$ have property (2). (Formation of the open will automatically commute with base change.) By definition the locus $T \subset |X|$ of points where $X \to S$ is not smooth is closed. Let $Z \subset X$ be the closed subspace given by the reduced induced algebraic space structure on T (Properties of Spaces, Definition 66.12.5). Now if $s \in S$ is a point where Z_s is finite, then there is an open neighbourhood $U_s \subset S$ of s such that $Z \cap f^{-1}(U_s) \to U_s$ is finite, see More on Morphisms of Spaces, Lemma 76.35.2. This proves the lemma. □

109.15. The smooth locus of the stack of curves

0DZT The morphism

$$\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

is smooth over a maximal open substack, see Morphisms of Stacks, Lemma 101.33.6. We want to give a criterion for when a curve is in this locus. We will do this using a bit of deformation theory.

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Choose a Cohen ring Λ for k, see Algebra, Lemma 10.160.6. Then we are in the situation described in Deformation Problems, Example 93.9.1 and Lemma 93.9.2. Thus we

obtain a deformation category $\mathcal{D}ef_X$ on the category $\mathcal{C}_\Lambda$ of Artinian local Λ-algebras with residue field k.

0DZU **Lemma 109.15.1.** *In the situation above the following are equivalent*

(1) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through the open where* $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
(2) *the deformation category* $\mathcal{D}ef_X$ *is unobstructed.*

Proof. Since $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Lemma 109.5.3) formation of the open substack where $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is smooth commutes with flat base change (Morphisms of Stacks, Lemma 101.33.6). Since the Cohen ring Λ is flat over $\mathbf{Z}$, we may work over Λ. In other words, we are trying to prove that

$$\Lambda\text{-}\mathcal{C}urves \longrightarrow \mathrm{Spec}(\Lambda)$$

is smooth in an open neighbourhood of the point $x_0 : \mathrm{Spec}(k) \to \Lambda\text{-}\mathcal{C}urves$ defined by X/k if and only if $\mathcal{D}ef_X$ is unobstructed.

The lemma now follows from Geometry of Stacks, Lemma 107.2.7 and the equality

$$\mathcal{D}ef_X = \mathcal{F}_{\Lambda\text{-}\mathcal{C}urves,k,x_0}$$

This equality is not completely trivial to establish. Namely, on the left hand side we have the deformation category classifying all flat deformations $Y \to \mathrm{Spec}(A)$ of X as a scheme over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. On the right hand side we have the deformation category classifying all flat morphisms $Y \to \mathrm{Spec}(A)$ with special fibre X where Y is an algebraic space and $Y \to \mathrm{Spec}(A)$ is proper, of finite presentation, and of relative dimension ≤ 1. Since A is Artinian, we find that Y is a scheme for example by Spaces over Fields, Lemma 72.9.3. Thus it remains to show: a flat deformation $Y \to \mathrm{Spec}(A)$ of X as a scheme over an Artinian local ring A with residue field k is proper, of finite presentation, and of relative dimension ≤ 1. Relative dimension is defined in terms of fibres and hence holds automatically for Y/A since it holds for X/k. The morphism $Y \to \mathrm{Spec}(A)$ is proper and locally of finite presentation as this is true for $X \to \mathrm{Spec}(k)$, see More on Morphisms, Lemma 37.10.3. □

Here is a "large" open of the stack of curves which is contained in the smooth locus.

0DZX **Lemma 109.15.2.** *The open substack*

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+} \subset \mathcal{C}urves$$

has the following properties

(1) $\mathcal{C}urves^{lci+} \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
(2) *given a family of curves* $X \to S$ *the following are equivalent*
   (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci+}$,
   (b) $X \to S$ *is a local complete intersection morphism and the singular locus of* $X \to S$ *endowed with any/some closed subspace structure is finite over* S,
(3) *given* X *a proper scheme over a field* k *of dimension* ≤ 1 *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci+}$,
   (b) X *is a local complete intersection over* k *and* $X \to \mathrm{Spec}(k)$ *is smooth except at finitely many points.*

Proof. If we can show that there is an open substack $\mathcal{C}urves^{lci+}$ whose points are characterized by (2), then we see that (1) holds by combining Lemma 109.15.1 with Deformation Problems, Lemma 93.16.4. Since

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+}$$

inside $\mathcal{C}urves$, we conclude by Lemmas 109.13.1 and 109.14.1. □

109.16. Smooth curves

0DZY In this section we study open substacks of $\mathcal{C}urves$ parametrizing smooth "curves".

0DZZ **Lemma 109.16.1.** *There exist an open substacks*

$$\mathcal{C}urves^{smooth,1} \subset \mathcal{C}urves^{smooth} \subset \mathcal{C}urves$$

such that

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,*
   (b) *f is smooth, resp. smooth of relative dimension 1,*

(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,*
   (b) *X is smooth over k, resp. X is smooth over k and X is equidimensional of dimension 1.*

Proof. To prove the statements regarding $\mathcal{C}urves^{smooth}$ it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is smooth and such that the formation of this open commutes with base change. We know that there is a maximal open $U \subset X$ such that $U \to S$ is smooth and that formation of U commutes with arbitrary base change, see Morphisms of Spaces, Lemma 67.37.9. If $T = |X| \setminus |U|$ then $f(T)$ is closed in S as f is proper. Setting $S' = S \setminus f(T)$ we obtain the desired open.

Let $f : X \to S$ be a family of curves with f smooth. Then the fibres X_s are smooth over $\kappa(s)$ and hence Cohen-Macaulay (for example you can see this using Algebra, Lemmas 10.137.4 and 10.135.3). Thus we see that we may set

$$\mathcal{C}urves^{smooth,1} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{CM,1}$$

and the desired equivalences follow from what we've already shown for $\mathcal{C}urves^{smooth}$ and Lemma 109.8.2. □

0E1N **Lemma 109.16.2.** *The morphism $\mathcal{C}urves^{smooth} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.*

Proof. Follows immediately from the observation that $\mathcal{C}urves^{smooth} \subset \mathcal{C}urves^{lci+}$ and Lemma 109.15.2. □

0E81 **Lemma 109.16.3.** *There exist an open substack $\mathcal{C}urves^{smooth,h0} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$,*
   (b) *$f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and f is smooth of relative dimension 1,*

(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth,h0}$,*
   (b) *X is smooth, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$,*
   (c) *X is smooth, $\dim(X) = 1$, and X is geometrically connected,*
   (d) *X is smooth, $\dim(X) = 1$, and X is geometrically integral, and*
   (e) *$X_{\overline{k}}$ is a smooth curve.*

Proof. If we set

$$\mathcal{C}urves^{smooth,h0} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{h0,1}$$

then we see that (1) holds by Lemmas 109.9.1 and 109.16.1. In fact, this also gives the equivalence of (2)(a) and (2)(b). To finish the proof we have to show that (2)(b) is equivalent to each of (2)(c), (2)(d), and (2)(e).

A smooth scheme over a field is geometrically normal (Varieties, Lemma 33.25.4), smoothness is preserved under base change (Morphisms, Lemma 29.34.5), and being smooth is fpqc local on the target (Descent, Lemma 35.23.29). Keeping this in mind, the equivalence of (2)(b), (2)(c), 2(d), and (2)(e) follows from Varieties, Lemma 33.10.7. □

0E82 **Definition 109.16.4.** We denote $\mathcal{M}$ and we name it the *moduli stack of smooth proper curves* the algebraic stack $\mathcal{C}urves^{smooth,h0}$ parametrizing families of curves introduced in Lemma 109.16.3. For $g \geq 0$ we denote $\mathcal{M}_g$ and we name it the *moduli stack of smooth proper curves of genus g* the algebraic stack introduced in Lemma 109.16.5. [**DM69**]

Here is the obligatory lemma.

0E83 **Lemma 109.16.5.** *There is a decomposition into open and closed substacks*

$$\mathcal{M} = \coprod\nolimits_{g \geq 0} \mathcal{M}_g$$

where each $\mathcal{M}_g$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{M}_g$,*
   (b) *$X \to S$ is smooth, $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*

(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{M}_g$,*
   (b) *X is smooth, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and X has genus g,*
   (c) *X is smooth, $\dim(X) = 1$, X is geometrically connected, and X has genus g,*
   (d) *X is smooth, $\dim(X) = 1$, X is geometrically integral, and X has genus g, and*
   (e) *$X_{\overline{k}}$ is a smooth curve of genus g.*

Proof. Combine Lemmas 109.16.3 and 109.9.4. You can also use Lemma 109.11.3 instead. □

0E84 **Lemma 109.16.6.** *The morphisms $\mathcal{M} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{M}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\mathcal{M}$ is an open substack of $\mathcal{C}urves^{lci+}$ this follows from Lemma 109.15.2. □

109.17. Density of smooth curves

0E85 The title of this section is misleading as we don't claim $\mathcal{C}urves^{smooth}$ is dense in $\mathcal{C}urves$. In fact, this is false as was shown by Mumford in [**Mum75**]. However, we will see that the smooth "curves" are dense in a large open.

0E86 **Lemma 109.17.1.** *The inclusion*

$$|\mathcal{C}urves^{smooth}| \subset |\mathcal{C}urves^{lci+}|$$

is that of an open dense subset.

Proof. By the very construction of the topology on $|\mathcal{C}urves^{lci+}|$ in Properties of Stacks, Section 100.4 we find that $|\mathcal{C}urves^{smooth}|$ is an open subset. Let $\xi \in |\mathcal{C}urves^{lci+}|$ be a point. Then there exists a field k and a scheme X over k with X proper over k, with $\dim(X) \leq 1$, with X a local complete intersection over k, and with X is smooth over k except at finitely many points, such that ξ is the equivalence class of the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves^{lci+}$ determined by X. See Lemma 109.15.2. By Deformation Problems, Lemma 93.17.6 there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Consider the classifying morphism

$$\mathrm{Spec}(k[[t]]) \longrightarrow \mathcal{C}urves^{lci+}$$

determined by Y. The image of the closed point is ξ and the image of the generic point is in $|\mathcal{C}urves^{smooth}|$. Since the generic point specializes to the closed point in $|\mathrm{Spec}(k[[t]])|$ we conclude that ξ is in the closure of $|\mathcal{C}urves^{smooth}|$ as desired. □

109.18. Nodal curves

0DSX In algebraic geometry a special role is played by nodal curves. We suggest the reader take a brief look at some of the discussion in Algebraic Curves, Sections 53.19 and 53.20 and More on Morphisms of Spaces, Section 76.55.

0DSY **Lemma 109.18.1.** *There exist an open substack $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
   (b) *f is at-worst-nodal of relative dimension 1,*
2. *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
   (b) *the singularities of X are at-worst-nodal and X is equidimensional of dimension 1.*

Proof. In fact, it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is at-worst-nodal of relative dimension 1 and such that formation of S' commutes with arbitrary base change. By More on Morphisms of Spaces, Lemma 76.55.4 there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with base change. Hence we can take

$$S' = S \setminus |f|(|X| \setminus |X'|)$$

This is open because a proper morphism is universally closed by definition. □

0E00 **Lemma 109.18.2.** *The morphism $\mathcal{C}urves^{nodal} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.*

Proof. Follows immediately from the observation that $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves^{lci+}$ and Lemma 109.15.2. □

109.19. The relative dualizing sheaf

0E6N This section serves mainly to introduce notation in the case of families of curves. Most of the work has already been done in the chapter on duality.

Let $f : X \to S$ be a family of curves. There exists an object $\omega^\bullet_{X/S}$ in $D_{QCoh}(\mathcal{O}_X)$, called the *relative dualizing complex*, having the following property: for every base change diagram

$$\begin{array}{ccc} X_U & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & S \end{array}$$

with $U = \mathrm{Spec}(A)$ affine the complex $\omega^\bullet_{X_U/U} = L(g')^*\omega^\bullet_{X/S}$ represents the functor

$$D_{QCoh}(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad K \longmapsto \mathrm{Hom}_U(Rf_*K, \mathcal{O}_U)$$

More precisely, let $(\omega^\bullet_{X/S}, \tau)$ be the relative dualizing complex of the family as defined in Duality for Spaces, Definition 86.9.1. Existence is shown in Duality for Spaces, Lemma 86.9.5. Moreover, formation of $(\omega^\bullet_{X/S}, \tau)$ commutes with arbitrary base change (essentially by definition; a precise reference is Duality for Spaces, Lemma 86.9.6). From now on we will identify the base change of $\omega^\bullet_{X/S}$ with the relative dualizing complex of the base changed family without further mention.

Let $\{S_i \to S\}$ be an étale covering with S_i affine such that $X_i = X \times_S S_i$ is a scheme, see Lemma 109.4.3. By Duality for Spaces, Lemma 86.10.1 we find that $\omega^\bullet_{X_i/S_i}$ agrees with the relative dualizing complex for the proper, flat, and finitely presented morphism $f_i : X_i \to S_i$ of schemes discussed in Duality for Schemes, Remark 48.12.5. Thus to prove a property of $\omega^\bullet_{X/S}$ which is étale local, we may assume $X \to S$ is a morphism of schemes and use the theory developed in the chapter on duality for schemes. More generally, for any base change of X which is a scheme, the relative dualizing complex agrees with the relative dualizing complex of Duality for Schemes, Remark 48.12.5. From now on we will use this identification without further mention.

In particular, let $\mathrm{Spec}(k) \to S$ be a morphism where k is a field. Denote X_k the base change (this is a scheme by Spaces over Fields, Lemma 72.9.3). Then $\omega^\bullet_{X_k/k}$ is isomorphic to the complex $\omega^\bullet_{X_k}$ of Algebraic Curves, Lemma 53.4.1 (both represent the same functor and so we can use the Yoneda lemma, but really this holds because of the remarks above). We conclude that the cohomology sheaves $H^i(\omega^\bullet_{X_k/k})$ are nonzero only for $i = 0, -1$. If X_k is Cohen-Macaulay and equidimensional of dimension 1, then we only have H^{-1} and if X_k is in addition Gorenstein, then $H^{-1}(\omega_{X_k/k})$ is invertible, see Algebraic Curves, Lemmas 53.4.2 and 53.5.2.

0E6P **Lemma 109.19.1.** *Let $X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 109.8.2). Then $\omega^\bullet_{X/S} = \omega_{X/S}[1]$ where*

$\omega_{X/S}$ is a pseudo-coherent $\mathcal{O}_X$-module flat over S whose formation commutes with arbitrary base change.

Proof. We urge the reader to deduce this directly from the discussion above of what happens after base change to a field. Our proof will use a somewhat cumbersome reduction to the Noetherian schemes case.

Once we show $\omega^\bullet_{X/S} = \omega_{X/S}[1]$ with $\omega_{X/S}$ flat over S, the statement on base change will follow as we already know that formation of $\omega^\bullet_{X/S}$ commutes with arbitrary base change. Moreover, the pseudo-coherence will be automatic as $\omega^\bullet_{X/S}$ is pseudo-coherent by definition. Vanishing of the other cohomology sheaves and flatness may be checked étale locally. Thus we may assume $f : X \to S$ is a morphism of schemes with S affine (see discussion above). Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. Since $\mathcal{C}urves^{CM,1}$ is locally of finite presentation over $\mathbf{Z}$ (as an open substack of $\mathcal{C}urves$, see Lemmas 109.8.2 and 109.5.3), we can find an i and a family of curves $X_i \to S_i$ whose pullback is $X \to S$ (Limits of Stacks, Lemma 102.3.5). After increasing i if necessary we may assume X_i is a scheme, see Limits of Spaces, Lemma 70.5.11. Since formation of $\omega^\bullet_{X/S}$ commutes with arbitrary base change, we may replace S by S_i. Doing so we may and do assume S_i is Noetherian. Then f is clearly a Cohen-Macaulay morphism (More on Morphisms, Definition 37.22.1) by our assumption on the fibres. Also then $\omega^\bullet_{X/S} = f^!\mathcal{O}_S$ by the very construction of $f^!$ in Duality for Schemes, Section 48.16. Thus the lemma by Duality for Schemes, Lemma 48.23.3. □

0E6Q **Definition 109.19.2.** Let $f : X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 109.8.2). Then the $\mathcal{O}_X$-module

$$\omega_{X/S} = H^{-1}(\omega^\bullet_{X/S})$$

studied in Lemma 109.19.1 is called the *relative dualizing sheaf* of f.

In the situation of Definition 109.19.2 the relative dualizing sheaf $\omega_{X/S}$ has the following property (which moreover characterizes it locally on S): for every base change diagram

$$\begin{array}{ccc} X_U & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & S \end{array}$$

with $U = \mathrm{Spec}(A)$ affine the module $\omega_{X_U/U} = (g')^*\omega_{X/S}$ represents the functor

$$QCoh(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad \mathcal{F} \longmapsto \mathrm{Hom}_A(H^1(X, \mathcal{F}), A)$$

This follows immediately from the corresponding property of the relative dualizing complex given above. In particular, if $A = k$ is a field, then we recover the dualizing module of X_k as introduced and studied in Algebraic Curves, Lemmas 53.4.1, 53.4.2, and 53.5.2.

0E6R **Lemma 109.19.3.** *Let $X \to S$ be a family of curves with Gorenstein fibres equidimensional of dimension 1 (Lemma 109.12.2). Then the relative dualizing sheaf $\omega_{X/S}$ is an invertible $\mathcal{O}_X$-module whose formation commutes with arbitrary base change.*

Proof. This is true because the pullback of the relative dualizing module to a fibre is invertible by the discussion above. Alternatively, you can argue exactly as in the proof of Lemma 109.19.1 and deduce the result from Duality for Schemes, Lemma 48.25.10. □

109.20. Prestable curves

0E6S The following definition is equivalent to what appears to be the generally accepted notion of a prestable family of curves.

0E6T **Definition 109.20.1.** Let $f : X \to S$ be a family of curves. We say f is a *prestable family of curves* if

(1) f is at-worst-nodal of relative dimension 1, and
(2) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change[1].

Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is prestable[2] in the sense of the definition. Unwinding the definitions we see the following are equivalent

(1) X is prestable,
(2) the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$,
(3) $X_{\overline{k}}$ is connected and it is smooth over $\overline{k}$ apart from a finite number of nodes (Algebraic Curves, Definition 53.16.2).

This shows that our definition agrees with most definitions one finds in the literature.

0E6U **Lemma 109.20.2.** *There exist an open substack $\mathcal{C}urves^{prestable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,*
   (b) *$X \to S$ is a prestable family of curves,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$.*

Proof. Given a family of curves $X \to S$ we see that it is prestable if and only if the classifying morphism factors both through $\mathcal{C}urves^{nodal}$ and $\mathcal{C}urves^{h0,1}$. An alternative is to use $\mathcal{C}urves^{grc,1}$ (since a nodal curve is geometrically reduced hence has H^0 equal to the ground field if and only if it is connected). In a formula

$$\mathcal{C}urves^{prestable} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{h0,1} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{grc,1}$$

Thus the lemma follows from Lemmas 109.9.1 and 109.18.1. □

For each genus $g \geq 0$ we have the algebraic stack classifying the prestable curves of genus g. In fact, from now on we will say that $X \to S$ is a *prestable family of curves*

[1] In fact, it suffices to require $f_*\mathcal{O}_X = \mathcal{O}_S$ because the Stein factorization of f is étale in this case, see More on Morphisms of Spaces, Lemma 76.36.9. The condition may also be replaced by asking the geometric fibres to be connected, see Lemma 109.11.2.

[2] We can't use the term "prestable curve" here because curve implies irreducible. See discussion in Algebraic Curves, Section 53.20.

of genus g if and only if the classifying morphism $S \to \mathcal{C}urves$ factors through the open substack $\mathcal{C}urves_g^{prestable}$ of Lemma 109.20.3.

0E6V **Lemma 109.20.3.** *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{prestable} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g^{prestable}$$

where each $\mathcal{C}urves_g^{prestable}$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,*
   (b) *$X \to S$ is a prestable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*

(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and the genus of X is g.*

Proof. Since we have seen that $\mathcal{C}urves^{prestable}$ is contained in $\mathcal{C}urves^{h0,1}$, this follows from Lemmas 109.20.2 and 109.9.4. □

0E6W **Lemma 109.20.4.** *The morphisms $\mathcal{C}urves^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{C}urves_g^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\mathcal{C}urves^{prestable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. □

109.21. Semistable curves

0E6X The following lemma will help us understand families of semistable curves.

0E6Y **Lemma 109.21.1.** *Let $f : X \to S$ be a prestable family of curves of genus $g \geq 1$. Let $s \in S$ be a point of the base scheme. Let $m \geq 2$. The following are equivalent*

(1) *X_s does not have a rational tail (Algebraic Curves, Example 53.23.1), and*
(2) *$f^* f_* \omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$, is surjective over $f^{-1}(U)$ for some $s \in U \subset S$ open.*

Proof. Assume (2). Using the material in Section 109.19 we conclude that $\omega_{X_s}^{\otimes m}$ is globally generated. However, if $C \subset X_s$ is a rational tail, then $\deg(\omega_{X_s}|_C) < 0$ by Algebraic Curves, Lemma 53.23.2 hence $H^0(C, \omega_{X_s}|_C) = 0$ by Varieties, Lemma 33.44.12 which contradicts the fact that it is globally generated. This proves (1).

Assume (1). First assume that $g \geq 2$. Assumption (1) implies $\omega_{X_s}^{\otimes m}$ is globally generated, see Algebraic Curves, Lemma 53.23.6. Moreover, we have

$$\mathrm{Hom}_{\kappa(s)}(H^1(X_s, \omega_{X_s}^{\otimes m}), \kappa(s)) = H^0(X_s, \omega_{X_s}^{\otimes 1-m})$$

by duality, see Algebraic Curves, Lemma 53.4.2. Since $\omega_{X_s}^{\otimes m}$ is globally generated we find that the restriction to each irreducible component has nonegative degree. Hence the restriction of $\omega_{X_s}^{\otimes 1-m}$ to each irreducible component has nonpositive degree. Since $\deg(\omega_{X_s}^{\otimes 1-m}) = (1-m)(2g-2) < 0$ by Riemann-Roch (Algebraic Curves, Lemma 53.5.2) we conclude that the H^0 is zero by Varieties, Lemma 33.44.13. By cohomology and base change we conclude that

$$E = Rf_* \omega_{X/S}^{\otimes m}$$

is a perfect complex whose formation commutes with arbitrary base change (Derived Categories of Spaces, Lemma 75.25.4). The vanishing proved above tells us that $E \otimes^{\mathbf{L}} \kappa(s)$ is equal to $H^0(X_s, \omega_{X_s}^{\otimes m})$ placed in degree 0. After shrinking S we find $E = f_*\omega_{X/S}^{\otimes m}$ is a locally free $\mathcal{O}_S$-module placed in degree 0 (and its formation commutes with arbitrary base change as we've already said), see Derived Categories of Spaces, Lemma 75.26.5. The map $f^*f_*\omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$ is surjective after restricting to X_s. Thus it is surjective in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S.

Assume (1) and $g = 1$. By Algebraic Curves, Lemma 53.23.6 the assumption (1) means that ω_{X_s} is isomorphic to $\mathcal{O}_{X_s}$. If we can show that after shrinking S the invertible sheaf $\omega_{X/S}$ because trivial, then we are done. We may assume S is affine. After shrinking S further, we can write

$$Rf_*\mathcal{O}_X = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1 compatibly with further base change, see Lemma 109.9.3. By duality this means that

$$Rf_*\omega_{X/S} = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1[3]. In particular we obtain an isomorphism $\mathcal{O}_S \to f_*\omega_{X/S}$ which is compatible with base change since formation of $Rf_*\omega_{X/S}$ is compatible with base change (see reference given above). By adjointness, we get a global section $\sigma \in \Gamma(X, \omega_{X/S})$. The restriction of this section to the fibre X_s is nonzero (a basis element in fact) and as ω_{X_s} is trivial on the fibres, this section is nonwhere zero on X_s. Thus it nowhere zero in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S. □

Motivated by Lemma 109.21.1 we make the following definition.

0E6Z **Definition 109.21.2.** Let $f : X \to S$ be a family of curves. We say f is a *semistable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

In particular, a prestable family of curves of genus 0 is never semistable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is semistable. Unwinding the definitions we see the following are equivalent

(1) X is semistable,
(2) X is prestable, has genus ≥ 1, and does not have a rational tail,
(3) $X_{\overline{k}}$ is connected, is smooth over $\overline{k}$ apart from a finite number of nodes, has genus ≥ 1, and has no irreducible component isomorphic to $\mathbf{P}^1_{\overline{k}}$ which meets the rest of $X_{\overline{k}}$ in only one point.

[3]Use that $Rf_*\omega_{X/S}^\bullet = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_{X/S}^\bullet) = R\mathcal{H}om_{\mathcal{O}_S}(Rf_*\mathcal{O}_X, \mathcal{O}_S)$ by Duality for Spaces, Lemma 86.3.3 and Remark 86.3.5 and then that $\omega_{X/S}^\bullet = \omega_{X/S}[1]$ by our definitions in Section 109.19.

To see the equivalence of (2) and (3) use that X has no rational tails if and only if $X_{\overline{k}}$ has no rational tails by Algebraic Curves, Lemma 53.23.6. This shows that our definition agrees with most definitions one finds in the literature.

0E70 **Lemma 109.21.3.** *There exist an open substack $\mathcal{C}urves^{semistable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{semistable}$,*
   (b) *$X \to S$ is a semistable family of curves,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{semistable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 1, and X has no rational tails,*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and $\omega_{X_s}^{\otimes m}$ is globally generated for $m \geq 2$.*

Proof. The equivalence of (2)(b) and (2)(c) is Algebraic Curves, Lemma 53.23.6. In the rest of the proof we will work with (2)(b) in accordance with Definition 109.21.2.

By the discussion in Section 109.6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 109.21.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, because the (non)existence of rational tails is insensitive to ground field extensions by Algebraic Curves, Lemma 53.23.6. □

0E71 **Lemma 109.21.4.** *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{semistable} = \coprod\nolimits_{g \geq 1} \mathcal{C}urves_g^{semistable}$$

where each $\mathcal{C}urves_g^{semistable}$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{semistable}$,*
   (b) *$X \to S$ is a semistable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{semistable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and X has no rational tail,*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and $\omega_{X_s}^{\otimes m}$ is globally generated for $m \geq 2$.*

Proof. Combine Lemmas 109.21.3 and 109.20.3. □

0E72 **Lemma 109.21.5.** *The morphisms $\mathcal{C}urves^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{C}urves_g^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\mathcal{C}urves^{semistable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. □

109.22. Stable curves

0E73 The following lemma will help us understand families of stable curves.

0E74 **Lemma 109.22.1.** *Let $f : X \to S$ be a prestable family of curves of genus $g \geq 2$. Let $s \in S$ be a point of the base scheme. The following are equivalent*

(1) *X_s does not have a rational tail and does not have a rational bridge (Algebraic Curves, Examples 53.23.1 and 53.24.1), and*
(2) *$\omega_{X/S}$ is ample on $f^{-1}(U)$ for some $s \in U \subset S$ open.*

Proof. Assume (2). Then ω_{X_s} is ample on X_s. By Algebraic Curves, Lemmas 53.23.2 and 53.24.2 we conclude that (1) holds (we also use the characterization of ample invertible sheaves in Varieties, Lemma 33.44.15).

Assume (1). Then ω_{X_s} is ample on X_s by Algebraic Curves, Lemmas 53.24.6. We conclude by Descent on Spaces, Lemma 74.13.2. □

Motivated by Lemma 109.22.1 we make the following definition.

0E75 **Definition 109.22.2.** Let $f : X \to S$ be a family of curves. We say f is a *stable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 2 and does not have a rational tails or bridges for all $s \in S$.

In particular, a prestable family of curves of genus 0 or 1 is never stable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is stable. Unwinding the definitions we see the following are equivalent

(1) X is stable,
(2) X is prestable, has genus ≥ 2, does not have a rational tail, and does not have a rational bridge,
(3) X is geometrically connected, is smooth over k apart from a finite number of nodes, and ω_X is ample.

To see the equivalence of (2) and (3) use Lemma 109.22.1 above. This shows that our definition agrees with most definitions one finds in the literature.

0E76 **Lemma 109.22.3.** *There exist an open substack $\mathcal{C}urves^{stable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,*
   (b) *$X \to S$ is a stable family of curves,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges,*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and ω_{X_s} is ample.*

Proof. By the discussion in Section 109.6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 109.22.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, either

because the (non)existence of rational tails or bridges is insensitive to ground field extensions by Algebraic Curves, Lemmas 53.23.6 and 53.24.6 or because ampleness is insensitive to base field extensions by Descent, Lemma 35.25.6. □

0E77 **Definition 109.22.4.** We denote $\overline{\mathcal{M}}$ and we name the *moduli stack of stable curves* the algebraic stack $\mathcal{C}urves^{stable}$ parametrizing stable families of curves introduced in Lemma 109.22.3. For $g \geq 2$ we denote $\overline{\mathcal{M}}_g$ and we name the *moduli stack of stable curves of genus g* the algebraic stack introduced in Lemma 109.22.5. [**DM69**]

Here is the obligatory lemma.

0E78 **Lemma 109.22.5.** *There is a decomposition into open and closed substacks*

$$\overline{\mathcal{M}} = \coprod\nolimits_{g \geq 2} \overline{\mathcal{M}}_g$$

where each $\overline{\mathcal{M}}_g$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,*
   (b) *$X \to S$ is a stable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*

(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and X has no rational tails or bridges.*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and ω_{X_s} is ample.*

Proof. Combine Lemmas 109.22.3 and 109.20.3. □

0E79 **Lemma 109.22.6.** *The morphisms $\overline{\mathcal{M}} \to \mathrm{Spec}(\mathbf{Z})$ and $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\overline{\mathcal{M}}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 109.18.2. □

0E7A **Lemma 109.22.7.** *The stacks $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are open substacks of $\mathcal{C}urves^{DM}$. In particular, $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are DM (Morphisms of Stacks, Definition 101.4.2) as well as Deligne-Mumford stacks (Algebraic Stacks, Definition 94.12.2).*

Proof. Proof of the first assertion. Let X be a scheme proper over a field k whose singularities are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges. We have to show that the classifying morphism $\mathrm{Spec}(k) \to \overline{\mathcal{M}} \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$. We may first replace k by the algebraic closure (since we already know the relevant stacks are open substacks of the algebraic stack $\mathcal{C}urves$). By Lemmas 109.22.3, 109.7.3, and 109.7.4 it suffices to show that $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. This is proven in Algebraic Curves, Lemma 53.26.3.

Since $\mathcal{C}urves^{DM}$ is the maximal open substack of $\mathcal{C}urves$ which is DM, we see this is true also for the open substack $\overline{\mathcal{M}}$ of $\mathcal{C}urves^{DM}$. Finally, a DM algebraic stack is Deligne-Mumford by Morphisms of Stacks, Theorem 101.21.6. □

0E87 **Lemma 109.22.8.** *Let $g \geq 2$. The inclusion*

$$|\mathcal{M}_g| \subset |\overline{\mathcal{M}}_g|$$

is that of an open dense subset.

Proof. Since $\overline{\mathcal{M}}_g \subset \mathcal{C}urves^{lci+}$ is open and since $\mathcal{C}urves^{smooth} \cap \overline{\mathcal{M}}_g = \mathcal{M}_g$ this follows immediately from Lemma 109.17.1. □

109.23. Contraction morphisms

0E7B We urge the reader to familiarize themselves with Algebraic Curves, Sections 53.23, 53.24, and 53.25 before continuing here. The main result of this section is the existence of a "stabilization" morphism

$$\mathcal{C}urves_g^{prestable} \longrightarrow \overline{\mathcal{M}}_g$$

See Lemma 109.23.5. Loosely speaking, this morphism sends the moduli point of a nodal genus g curve to the moduli point of the associated stable curve constructed in Algebraic Curves, Lemma 53.25.2.

0E88 **Lemma 109.23.1.** *Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g : Y \to S$ be families of curves. Let $c : X \to Y$ be a morphism over S. If $c_{s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$ and $R^1c_{s,*}\mathcal{O}_{X_s} = 0$, then after replacing S by an open neighbourhood of s we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$.*

Proof. Let $(U, u) \to (S, s)$ be an étale neighbourhood such that $\mathcal{O}_{Y_U} = (X_U \to Y_U)_*\mathcal{O}_{X_U}$ and $R^1(X_U \to Y_U)_*\mathcal{O}_{X_U} = 0$ and the same is true after base change by $U' \to U$. Then we replace S by the open image of $U \to S$. Given $S' \to S$ we set $U' = U \times_S S'$ and we obtain étale coverings $\{U' \to S'\}$ and $\{Y_{U'} \to Y_{S'}\}$. Thus the truth of the statement for the base change of c by $S' \to S$ follows from the truth of the statement for the base change of $X_U \to Y_U$ by $U' \to U$. In other words, the question is local in the étale topology on S. Thus by Lemma 109.4.3 we may assume X and Y are schemes. By More on Morphisms, Lemma 37.72.7 there exists an open subscheme $V \subset Y$ containing Y_s such that $c_*\mathcal{O}_X|_V = \mathcal{O}_V$ and $R^1c_*\mathcal{O}_X|_V = 0$ and such that this remains true after any base change by $S' \to S$. Since $g : Y \to S$ is proper, we can find an open neighbourhood $U \subset S$ of s such that $g^{-1}(U) \subset V$. Then U works. □

0E89 **Lemma 109.23.2.** *Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g_i : Y_i \to S$, $i = 1, 2$ be families of curves. Let $c_i : X \to Y_i$ be morphisms over S. Assume there is an isomorphism $Y_{1,s} \cong Y_{2,s}$ of fibres compatible with $c_{1,s}$ and $c_{2,s}$. If $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1c_{1,s,*}\mathcal{O}_{X_s} = 0$, then there exist an open neighbourhood U of s and an isomorphism $Y_{1,U} \cong Y_{2,U}$ of families of curves over U compatible with the given isomorphism of fibres and with c_1 and c_2.*

Proof. Recall that $\mathcal{O}_{S,s} = \operatorname{colim} \mathcal{O}_S(U)$ where the colimit is over the system of affine neighbourhoods U of s. Thus the category of algebraic spaces of finite presentation over the local ring is the colimit of the categories of algebraic spaces of finite presentation over the affine neighbourhoods of s. See Limits of Spaces, Lemma 70.7.1. In this way we reduce to the case where S is the spectrum of a local ring and s is the closed point.

Assume $S = \mathrm{Spec}(A)$ where A is a local ring and s is the closed point. Write $A = \mathrm{colim}\, A_j$ with A_j local Noetherian (say essentially of finite type over $\mathbf{Z}$) and local transition homomorphisms. Set $S_j = \mathrm{Spec}(A_j)$ with closed point s_j. We can find a j and families of curves $X_j \to S_j$, $Y_{j,i} \to S_j$, see Lemma 109.5.3 and Limits of Stacks, Lemma 102.3.5. After possibly increasing j we can find morphisms $c_{j,i} : X_j \to Y_{j,i}$ whose base change to s is c_i, see Limits of Spaces, Lemma 70.7.1. Since $\kappa(s) = \mathrm{colim}\, \kappa(s_j)$ we can similarly assume there is an isomorphism $Y_{j,1,s_j} \cong Y_{j,2,s_j}$ compatible with $c_{j,1,s_j}$ and $c_{j,2,s_j}$. Finally, the assumptions $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1 c_{1,s,*}\mathcal{O}_{X_s} = 0$ are inherited by $c_{j,1,s_j}$ because $\{s_j \to s\}$ is an fpqc covering and $c_{1,s}$ is the base of $c_{j,1,s_j}$ by this covering (details omitted). In this way we reduce the lemma to the case discussed in the next paragraph.

Assume S is the spectrum of a Noetherian local ring Λ and s is the closed point. Consider the scheme theoretic image Z of

$$(c_1, c_2) : X \longrightarrow Y_1 \times_S Y_2$$

The statement of the lemma is equivalent to the assertion that Z maps isomorphically to Y_1 and Y_2 via the projection morphisms. Since taking the scheme theoretic image of this morphism commutes with flat base change (Morphisms of Spaces, Lemma 67.30.12, we may replace Λ by its completion (More on Algebra, Section 15.43).

Assume S is the spectrum of a complete Noetherian local ring Λ. Observe that X, Y_1, Y_2 are schemes in this case (More on Morphisms of Spaces, Lemma 76.43.6). Denote X_n, $Y_{1,n}$, $Y_{2,n}$ the base changes of X, Y_1, Y_2 to $\mathrm{Spec}(\Lambda/\mathfrak{m}^{n+1})$. Recall that the arrow

$$\mathcal{D}ef_{X_s \to Y_{2,s}} \cong \mathcal{D}ef_{X_s \to Y_{1,s}} \longrightarrow \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 93.10.6. Thus there is an isomorphism of formal objects $(X_n \to Y_{1,n}) \cong (X_n \to Y_{2,n})$ of $\mathcal{D}ef_{X_s \to Y_{1,s}}$. Finally, by Grothendieck's algebraization theorem (Cohomology of Schemes, Lemma 30.28.3) this produces an isomorphism $Y_1 \to Y_2$ compatible with c_1 and c_2. □

0E7C **Lemma 109.23.3.** *Let $f : X \to S$ be a family of curves. Let $s \in S$ be a point. Let $h_0 : X_s \to Y_0$ be a morphism to a proper scheme Y_0 over $\kappa(s)$ such that $h_{0,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_0}$ and $R^1 h_{0,*}\mathcal{O}_{X_s} = 0$. Then there exist an elementary étale neighbourhood $(U, u) \to (S, s)$, a family of curves $Y \to U$, and a morphism $h : X_U \to Y$ over U whose fibre in u is isomorphic to h_0.*

Proof. We first do some reductions; we urge the reader to skip ahead. The question is local on S, hence we may assume S is affine. Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. For some i we can find a family of curves $X_i \to S_i$ whose base change is $X \to S$. This follows from Lemma 109.5.3 and Limits of Stacks, Lemma 102.3.5. Let $s_i \in S_i$ be the image of s. Observe that $\kappa(s) = \mathrm{colim}\, \kappa(s_i)$ and that X_s is a scheme (Spaces over Fields, Lemma 72.9.3). After increasing i we may assume there exists a morphism $h_{i,0} : X_{i,s_i} \to Y_i$ of finite type schemes over $\kappa(s_i)$ whose base change to $\kappa(s)$ is h_0, see Limits, Lemma 32.10.1. After increasing i we may assume Y_i is proper over $\kappa(s_i)$, see Limits, Lemma 32.13.1. Let $g_{i,0} : Y_0 \to Y_{i,0}$ be the projection. Observe that this is a faithfully flat morphism as the base change of $\mathrm{Spec}(\kappa(s)) \to \mathrm{Spec}(\kappa(s_i))$. By flat

base change we have

$$h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^* h_{i,0,*}\mathcal{O}_{X_{i,s_i}} \quad \text{and} \quad R^1 h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^* R h_{i,0,*}\mathcal{O}_{X_{i,s_i}}$$

see Cohomology of Schemes, Lemma 30.5.2. By faithful flatness we see that $X_i \to S_i$, $s_i \in S_i$, and $X_{i,s_i} \to Y_i$ satisfies all the assumptions of the lemma. This reduces us to the case discussed in the next paragraph.

Assume S is affine of finite type over $\mathbf{Z}$. Let $\mathcal{O}_{S,s}^h$ be the henselization of the local ring of S at s. Observe that $\mathcal{O}_{S,s}^h$ is a G-ring by More on Algebra, Lemma 15.50.8 and Proposition 15.50.12. Suppose we can construct a family of curves $Y' \to \mathrm{Spec}(\mathcal{O}_{S,s}^h)$ and a morphism

$$h' : X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}^h) \longrightarrow Y'$$

over $\mathrm{Spec}(\mathcal{O}_{S,s}^h)$ whose base change to the closed point is h_0. This will be enough. Namely, first we use that

$$\mathcal{O}_{S,s}^h = \mathrm{colim}_{(U,u)} \mathcal{O}_U(U)$$

where the colimit is over the filtered category of elementary étale neighbourhoods (More on Morphisms, Lemma 37.35.5). Next, we use again that given Y' we can descend it to $Y \to U$ for some U (see references given above). Then we use Limits, Lemma 32.10.1 to descend h' to some h. This reduces us to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(\Lambda)$ where $(\Lambda, \mathfrak{m}, \kappa)$ is a henselian Noetherian local G-ring and s is the closed point of S. Recall that the map

$$\mathcal{D}ef_{X_s \to Y_0} \to \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 93.10.6. (This is the only important step in the proof; everything else is technique.) Denote $\Lambda^\wedge$ the $\mathfrak{m}$-adic completion. The pullbacks X_n of X to $\Lambda/\mathfrak{m}^{n+1}$ define a formal object ξ of $\mathcal{D}ef_{X_s}$ over $\Lambda^\wedge$. From the equivalence we obtain a formal object ξ' of $\mathcal{D}ef_{X_s \to Y_0}$ over $\Lambda^\wedge$. Thus we obtain a huge commutative diagram

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow \ldots \longrightarrow & X_s \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow \ldots \longrightarrow & Y_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & \mathrm{Spec}(\Lambda/\mathfrak{m}^{n+1}) & \longrightarrow & \mathrm{Spec}(\Lambda/\mathfrak{m}^n) & \longrightarrow \ldots \longrightarrow & \mathrm{Spec}(\kappa)
\end{array}$$

The formal object (Y_n) comes from a family of curves $Y' \to \mathrm{Spec}(\Lambda^\wedge)$ by Quot, Lemma 99.15.9. By More on Morphisms of Spaces, Lemma 76.43.3 we get a morphism $h' : X_{\Lambda^\wedge} \to Y'$ inducing the given morphisms $X_n \to Y_n$ for all n and in particular the given morphism $X_s \to Y_0$.

To finish we do a standard algebraization/approximation argument. First, we observe that we can find a finitely generated Λ-subalgebra $\Lambda \subset A \subset \Lambda^\wedge$, a family of curves $Y'' \to \mathrm{Spec}(A)$ and a morphism $h'' : X_A \to Y''$ over A whose base change to $\Lambda^\wedge$ is h'. This is true because $\Lambda^\wedge$ is the filtered colimit of these rings A and we can argue as before using that $\mathcal{C}urves$ is locally of finite presentation (which gives us Y'' over A by Limits of Stacks, Lemma 102.3.5) and using Limits of Spaces, Lemma

70.7.1 to descend h' to some h''. Then we can apply the approximation property for G-rings (in the form of Smoothing Ring Maps, Theorem 16.13.1) to find a map $A \to \Lambda$ which induces the same map $A \to \kappa$ as we obtain from $A \to \Lambda^\wedge$. Base changing h'' to Λ the proof is complete. □

0E8A **Lemma 109.23.4.** *Let $f : X \to S$ be a prestable family of curves of genus $g \geq 2$. There is a factorization $X \to Y \to S$ of f where $g : Y \to S$ is a stable family of curves and $c : X \to Y$ has the following properties*

(1) *$\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$, and*
(2) *for any $s \in S$ the morphism $c_s : X_s \to Y_s$ is the contraction of rational tails and bridges discussed in Algebraic Curves, Section 53.25.*

Moreover $c : X \to Y$ is unique up to unique isomorphism.

Proof. Let $s \in S$. Let $c_0 : X_s \to Y_0$ be the contraction of Algebraic Curves, Section 53.25 (more precisely Algebraic Curves, Lemma 53.25.2). By Lemma 109.23.3 there exists an elementary étale neighbourhood (U, u) and a morphism $c : X_U \to Y$ of families of curves over U which recovers c_0 as the fibre at u. Since ω_{Y_0} is ample, after possibly shrinking U, we see that $Y \to U$ is a stable family of genus g by the openness inherent in Lemmas 109.22.3 and 109.22.5. After possibly shrinking U once more, assertion (1) of the lemma for $c : X_U \to Y$ follows from Lemma 109.23.1. Moreover, part (2) holds by the uniqueness in Algebraic Curves, Lemma 53.25.2. We conclude that a morphism c as in the lemma exists étale locally on S. More precisely, there exists an étale covering $\{U_i \to S\}$ and morphisms $c_i : X_{U_i} \to Y_i$ over U_i where $Y_i \to U_i$ is a stable family of curves having properties (1) and (2) stated in the lemma.

To finish the proof it suffices to prove uniqueness of $c : X \to Y$ (up to unique isomorphism). Namely, once this is done, then we obtain isomorphisms

$$\varphi_{ij} : Y_i \times_{U_i} (U_i \times_S U_j) \longrightarrow Y_i \times_{U_j} (U_i \times_S U_j)$$

satisfying the cocycle condition (by uniqueness) over $U_i \times U_j \times U_k$. Since $\overline{\mathcal{M}_g}$ is an algebraic stack, we have effectiveness of descent data and we obtain $Y \to S$. The morphisms c_i descend to a morphism $c : X \to Y$ over S. Finally, properties (1) and (2) for c are immediate from properties (1) and (2) for c_i.

Finally, if $c_1 : X \to Y_i$, $i = 1, 2$ are two morphisms towards stably families of curves over S satisfying (1) and (2), then we obtain a morphism $Y_1 \to Y_2$ compatible with c_1 and c_2 at least locally on S by Lemma 109.23.2. We omit the verification that these morphisms are unique (hint: this follows from the fact that the scheme theoretic image of c_1 is Y_1). Hence these locally given morphisms glue and the proof is complete. □

0E8B **Lemma 109.23.5.** *Let $g \geq 2$. There is a morphism of algebraic stacks over $\mathbf{Z}$*

$$\textit{stabilization} : \mathcal{C}\textit{urves}_g^{\textit{prestable}} \longrightarrow \overline{\mathcal{M}}_g$$

which sends a prestable family of curves $X \to S$ of genus g to the stable family $Y \to S$ associated to it in Lemma 109.23.4.

Proof. To see this is true, it suffices to check that the construction of Lemma 109.23.4 is compatible with base change (and isomorphisms but that's immediate), see the (abuse of) language for algebraic stacks introduced in Properties of Stacks,

Section 100.2. To see this it suffices to check properties (1) and (2) of Lemma 109.23.4 are stable under base change. This is immediately clear for (1). For (2) this follows either from the fact that the contractions of Algebraic Curves, Lemmas 53.23.6 and 53.24.6 are stable under ground field extensions, or because the conditions characterizing the morphisms on fibres in Algebraic Curves, Lemma 53.25.2 are preserved under ground field extensions. □

109.24. Stable reduction theorem

0E8C In the chapter on semistable reduction we have proved the celebrated theorem on semistable reduction of curves. Let K be the fraction field of a discrete valuation ring R. Let C be a projective smooth curve over K with $K = H^0(C, \mathcal{O}_C)$. According to Semistable Reduction, Definition 55.14.6 we say C has *semistable reduction* if either there is a prestable family of curves over R with generic fibre C, or some (equivalently any) minimal regular model of C over R is prestable. In this section we show that for curves of genus $g \geq 2$ this is also equivalent to stable reduction.

0E8D **Lemma 109.24.1.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $K = H^0(C, \mathcal{O}_C)$ having genus $g \geq 2$. The following are equivalent*

(1) *C has semistable reduction (Semistable Reduction, Definition 55.14.6), or*
(2) *there is a stable family of curves over R with generic fibre C.*

Proof. Since a stable family of curves is also prestable, it is immediate that (2) implies (1). Conversely, given a prestable family of curves over R with generic fibre C, we can contract it to a stable family of curves by Lemma 109.23.4. Since the generic fibre already is stable, it does not get changed by this procedure and the proof is complete. □

The following lemma tells us the stable family of curves over R promised in Lemma 109.24.1 is unique up to unique isomorphism.

0E97 **Lemma 109.24.2.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth proper curve over K with $K = H^0(C, \mathcal{O}_C)$ and genus g. If X and X' are models of C (Semistable Reduction, Section 55.8) and X and X' are stable families of genus g curves over R, then there exists a unique isomorphism $X \to X'$ of models.*

Proof. Let Y be the minimal model for C. Recall that Y exists, is unique, and is at-worst-nodal of relative dimension 1 over R, see Semistable Reduction, Proposition 55.8.6 and Lemmas 55.10.1 and 55.14.5 (applies because we have X). There is a contraction morphism

$$Y \longrightarrow Z$$

such that Z is a stable family of curves of genus g over R (Lemma 109.23.4). We claim there is a unique isomorphism of models $X \to Z$. By symmetry the same is true for X' and this will finish the proof.

By Semistable Reduction, Lemma 55.14.3 there exists a sequence

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that $X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular, $X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1, and X_m is regular. By Semistable Reduction, Lemma 55.8.5 there is a sequence

$$X_m = Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 = Y$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind[4]. By Semistable Reduction, Lemma 55.14.4 each Y_i is at-worst-nodal of relative dimension 1 over R. To prove the claim it suffices to show that there is an isomorphism $X \to Z$ compatible with the morphisms $X_m \to X$ and $X_m = Y_n \to Y \to Z$. Let $s \in \mathrm{Spec}(R)$ be the closed point. By either Lemma 109.23.2 or Lemma 109.23.4 we reduce to proving that the morphisms $X_{m,s} \to X_s$ and $X_{m,s} \to Z_s$ are both equal to the canonical morphism of Algebraic Curves, Lemma 53.25.2.

For a morphism $c : U \to V$ of schemes over $\kappa(s)$ we say c has property (*) if $\dim(U_v) \leq 1$ for $v \in V$, $\mathcal{O}_V = c_*\mathcal{O}_U$, and $R^1c_*\mathcal{O}_U = 0$. This property is stable under composition. Since both X_s and Z_s are stable genus g curves over $\kappa(s)$, it suffices to show that each of the morphisms $Y_s \to Z_s$, $X_{i+1,s} \to X_{i,s}$, and $Y_{i+1,s} \to Y_{i,s}$, satisfy property (*), see Algebraic Curves, Lemma 53.25.2.

Property (*) holds for $Y_s \to Z_s$ by construction.

The morphisms $c : X_{i+1,s} \to X_{i,s}$ are constructed and studied in the proof of Semistable Reduction, Lemma 55.14.3. It suffices to check (*) étale locally on $X_{i,s}$. Hence it suffices to check (*) for the base change of the morphism "$X_1 \to X_0$" in Semistable Reduction, Example 55.14.1 to $R/\pi R$. We leave the explicit calculation to the reader.

The morphism $c : Y_{i+1,s} \to Y_{i,s}$ is the restriction of the blow down of an exceptional curve $E \subset Y_{i+1}$ of the first kind, i.e., $b : Y_{i+1} \to Y_i$ is a contraction of E, i.e., b is a blowing up of a regular point on the surface Y_i (Resolution of Surfaces, Section 54.16). Then $\mathcal{O}_{Y_i} = b_*\mathcal{O}_{Y_{i+1}}$ and $R^1b_*\mathcal{O}_{Y_{i+1}} = 0$, see for example Resolution of Surfaces, Lemma 54.3.4. We conclude that $\mathcal{O}_{Y_{i,s}} = c_*\mathcal{O}_{Y_{i+1,s}}$ and $R^1c_*\mathcal{O}_{Y_{i+1,s}} = 0$ by More on Morphisms, Lemmas 37.72.1, 37.72.2, and 37.72.4 (only gives surjectivity of $\mathcal{O}_{Y_{i,s}} \to c_*\mathcal{O}_{Y_{i+1,s}}$ but injectivity follows easily from the fact that $Y_{i,s}$ is reduced and c changes things only over one closed point). This finishes the proof. □

From Lemma 109.24.1 and Semistable Reduction, Theorem 55.18.1 we immediately deduce the stable reduction theorem.

0E98 **Theorem 109.24.3.** *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus $g \geq 2$. Then* [**DM69**, Corollary 2.7]

(1) *there exists an extension of discrete valuation rings $R \subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable family of curves $Y \to \mathrm{Spec}(R')$ of genus g with $Y_{K'} \cong C_{K'}$ over K', and*
(2) *there exists a finite separable extension L/K and a stable family of curves $Y \to \mathrm{Spec}(A)$ of genus g where $A \subset L$ is the integral closure of R in L such that $Y_L \cong C_L$ over L.*

[4]In fact we have $X_m = Y$, i.e., X_m does not contain any exceptional curves of the first kind. We encourage the reader to think this through as it simplifies the proof somewhat.

Proof. Part (1) is an immediate consequence of Lemma 109.24.1 and Semistable Reduction, Theorem 55.18.1.

Proof of (2). Let L/K be the finite separable extension found in part (3) of Semistable Reduction, Theorem 55.18.1. Let $A \subset L$ be the integral closure of R. Recall that A is a Dedekind domain finite over R with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, see More on Algebra, Remark 15.112.6. Set $S = \mathrm{Spec}(A)$, $S_i = \mathrm{Spec}(A_{\mathfrak{m}_i})$, $U = \mathrm{Spec}(L)$, and $U_i = S_i \setminus \{\mathfrak{m}_i\}$. Observe that $U \cong U_i$ for $i = 1, \ldots, n$. Set $X = C_L$ viewed as a scheme over the open subscheme U of S. By our choice of L and A and Lemma 109.24.1 we have stable families of curves $X_i \to S_i$ and isomorphisms $X \times_U U_i \cong X_i \times_{S_i} U_i$. By Limits of Spaces, Lemma 70.18.4 we can find a finitely presented morphism $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Alternatively, you can use that $S = \bigcup_{i=1,\ldots,n} S_i$ is an open covering of S and $S_i \cap S_j = U$ for $i \neq j$ and use $n - 1$ applications of Limits of Spaces, Lemma 70.18.1 to get $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Clearly $Y \to S$ is the stable family of curves we were looking for. □

109.25. Properties of the stack of stable curves

0E99 In this section we prove the basic structure result for $\overline{\mathcal{M}}_g$ for $g \geq 2$.

0E9A **Lemma 109.25.1.** *Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is separated.*

Proof. The statement means that the morphism $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is separated. We will prove this using the refined Noetherian valuative criterion as stated in More on Morphisms of Stacks, Lemma 106.11.4

Since $\overline{\mathcal{M}}_g$ is an open substack of $\mathcal{C}urves$, we see $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation by Lemma 109.5.3. In particular the stack $\overline{\mathcal{M}}_g$ is locally Noetherian (Morphisms of Stacks, Lemma 101.17.5). By Lemma 109.22.8 the open immersion $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ has dense image. Also, $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ is quasi-compact (Morphisms of Stacks, Lemma 101.8.2), hence of finite type. Thus all the preliminary assumptions of More on Morphisms of Stacks, Lemma 106.11.4 are satisfied for the morphisms

$$\mathcal{M}_g \to \overline{\mathcal{M}}_g \quad \text{and} \quad \overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$$

and it suffices to check the following: given any 2-commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{M}_g & \longrightarrow & \overline{\mathcal{M}}_g \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(R) & & \longrightarrow & & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

where R is a discrete valuation ring with field of fractions K the category of dotted arrows is either empty or a setoid with exactly one isomorphism class. (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 101.39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the uniqueness result stated and proved in Lemma 109.24.2. □

0E9B **Lemma 109.25.2.** *Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is quasi-compact.*

Proof. We will use the notation from Section 109.4. Consider the subset

$$T \subset |PolarizedCurves|$$

of points ξ such that there exists a field k and a pair $(X, \mathcal{L})$ over k representing ξ with the following two properties

(1) X is a stable genus g curve, and
(2) $\mathcal{L} = \omega_X^{\otimes 3}$.

Clearly, under the continuous map

$$|PolarizedCurves| \longrightarrow |\mathcal{C}urves|$$

the image of the set T is exactly the open subset

$$|\overline{\mathcal{M}}_g| \subset |\mathcal{C}urves|$$

Thus it suffices to show that T is quasi-compact. By Lemma 109.4.1 we see that

$$|PolarizedCurves| \subset |\mathcal{P}olarized|$$

is an open and closed immersion. Thus it suffices to prove quasi-compactness of T as a subset of $|\mathcal{P}olarized|$. For this we use the criterion of Moduli Stacks, Lemma 108.11.3. First, we observe that for $(X, \mathcal{L})$ as above the Hilbert polynomial P is the function $P(t) = (6g-6)t+(1-g)$ by Riemann-Roch, see Algebraic Curves, Lemma 53.5.2. Next, we observe that $H^1(X, \mathcal{L}) = 0$ and $\mathcal{L}$ is very ample by Algebraic Curves, Lemma 53.25.3. This means exactly that with $n = P(3) - 1$ there is a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_k$$

such that $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ as desired. □

Here is the main theorem of this section.

0E9C **Theorem 109.25.3.** *Let $g \geq 2$. The algebraic stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack, proper and smooth over* $\mathrm{Spec}(\mathbf{Z})$. *Moreover, the locus $\mathcal{M}_g$ parametrizing smooth curves is a dense open substack.*

Proof. Most of the properties mentioned in the statement have already been shown. Smoothness is Lemma 109.22.6. Deligne-Mumford is Lemma 109.22.7. Openness of $\mathcal{M}_g$ is Lemma 109.22.8. We know that $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is separated by Lemma 109.25.1 and we know that $\overline{\mathcal{M}}_g$ is quasi-compact by Lemma 109.25.2. Thus, to show that $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is proper and finish the proof, we may apply More on Morphisms of Stacks, Lemma 106.11.3 to the morphisms $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$. Thus it suffices to check the following: given any 2-commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{M}_g & \longrightarrow & \overline{\mathcal{M}}_g \\ {\scriptstyle j}\downarrow & & & & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

where A is a discrete valuation ring with field of fractions K, there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of

dotted arrows for the induced diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \overline{\mathcal{M}}_g \\ {\scriptstyle j'}\downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

is nonempty (Morphisms of Stacks, Definition 101.39.1). (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 101.39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the result contained in the stable reduction theorem, i.e., Theorem 109.24.3. □

109.26. Other chapters

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Part 9

Miscellany

CHAPTER 110

Examples

026Z

110.1. Introduction

0270 This chapter will contain examples which illuminate the theory.

110.2. An empty limit

0AKK This example is due to Waterhouse, see [**Wat72**]. Let S be an uncountable set. For every finite subset $T \subset S$ consider the set M_T of injective maps $T \to \mathbf{N}$. For $T \subset T' \subset S$ finite the restriction $M_{T'} \to M_T$ is surjective. Thus we have an inverse system over the directed partially ordered set of finite subsets of S with surjective transition maps. But $\lim M_T = \emptyset$ as an element in the limit would define an injective map $S \to \mathbf{N}$.

110.3. A zero limit

0ANX Let $(S_i)_{i \in I}$ be a directed inverse system of nonempty sets with surjective transition maps and with $\lim S_i = \emptyset$, see Section 110.2. Let K be a field and set

$$V_i = \bigoplus\nolimits_{s \in S_i} K$$

Then the transition maps $V_i \to V_j$ are surjective for $i \geq j$. However, $\lim V_i = 0$. Namely, if $v = (v_i)$ is an element of the limit, then the support of v_i would be a finite subset $T_i \subset S_i$ with $\lim T_i \neq \emptyset$, see Categories, Lemma 4.21.7.

For each i consider the unique K-linear map $V_i \to K$ which sends each basis vector $s \in S_i$ to 1. Let $W_i \subset V_i$ be the kernel. Then

$$0 \to (W_i) \to (V_i) \to (K) \to 0$$

is a nonsplit short exact sequence of inverse systems of vector spaces over the directed set I. Hence W_i is a directed system of K-vector spaces with surjective transition maps, vanishing limit, and nonvanishing $R^1 \lim$.

110.4. Non-quasi-compact inverse limit of quasi-compact spaces

09ZJ Let $\mathbf{N}$ denote the set of natural numbers. For every integer n, let I_n denote the set of all natural numbers $> n$. Define T_n to be the unique topology on $\mathbf{N}$ with basis $\{1\}, \ldots, \{n\}, I_n$. Denote by X_n the topological space $(\mathbf{N}, T_n)$. For each $m < n$, the identity map,

$$f_{n,m} : X_n \longrightarrow X_m$$

is continuous. Obviously for $m < n < p$, the composition $f_{p,n} \circ f_{n,m}$ equals $f_{p,m}$. So $((X_n), (f_{n,m}))$ is a directed inverse system of quasi-compact topological spaces.

Let T be the discrete topology on $\mathbf{N}$, and let X be $(\mathbf{N}, T)$. Then for every integer n, the identity map,

$$f_n : X \longrightarrow X_n$$

is continuous. We claim that this is the inverse limit of the directed system above. Let (Y, S) be any topological space. For every integer n, let

$$g_n : (Y, S) \longrightarrow (\mathbf{N}, T_n)$$

be a continuous map. Assume that for every $m < n$ we have $f_{n,m} \circ g_n = g_m$, i.e., the system (g_n) is compatible with the directed system above. In particular, all of the set maps g_n are equal to a common set map

$$g : Y \longrightarrow \mathbf{N}.$$

Moreover, for every integer n, since $\{n\}$ is open in X_n, also $g^{-1}(\{n\}) = g_n^{-1}(\{n\})$ is open in Y. Therefore the set map g is continuous for the topology S on Y and the topology T on $\mathbf{N}$. Thus $(X, (f_n))$ is the inverse limit of the directed system above.

However, clearly X is not quasi-compact, since the infinite open covering by singleton sets has no inverse limit.

09ZK **Lemma 110.4.1.** *There exists an inverse system of quasi-compact topological spaces over* $\mathbf{N}$ *whose limit is not quasi-compact.*

Proof. See discussion above. □

110.5. The structure sheaf on the fibre product

0FLS Let X, Y, S, a, b, p, q, f be as in the introduction to Derived Categories of Schemes, Section 36.23. Picture:

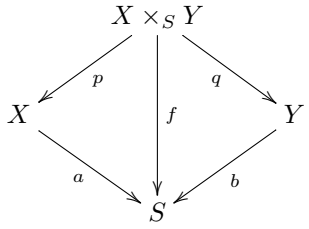

Then we have a canonical map

$$can : p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_{X \times_S Y}$$

which is not an isomorphism in general.

For example, let $S = \mathrm{Spec}(\mathbf{R})$, $X = \mathrm{Spec}(\mathbf{C})$, and $Y = \mathrm{Spec}(\mathbf{C})$. Then $X \times_S Y = \mathrm{Spec}(\mathbf{C}) \amalg \mathrm{Spec}(\mathbf{C})$ is a discrete space with two points and the sheaves $p^{-1}\mathcal{O}_X$, $q^{-1}\mathcal{O}_Y$ and $f^{-1}\mathcal{O}_S$ are the constant sheaves with values $\mathbf{C}$, $\mathbf{C}$, and $\mathbf{R}$. Hence the source of *can* is the constant sheaf with value $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ on the discrete space with two points. Thus its global sections have dimension 8 as an $\mathbf{R}$-vector space whereas taking global sections of the target of *can* we obtain $\mathbf{C} \times \mathbf{C}$ which has dimension 4 as an $\mathbf{R}$-vector space.

Another example is the following. Let k be an algebraically closed field. Consider $S = \mathrm{Spec}(k)$, $X = \mathbf{A}^1_k$, and $Y = \mathbf{A}^1_k$. Then for $U \subset X \times_S Y = \mathbf{A}^2_k$ nonempty open the images $p(U) \subset X = \mathbf{A}^1_k$ and $q(U) \subset \mathbf{A}^1_k$ are open and the reader can show that

$$\left(p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y\right)(U) = \mathcal{O}_X(p(U)) \otimes_k \mathcal{O}_Y(q(U))$$

This is not equal to $\mathcal{O}_{X\times_S Y}(U)$ if U is the complement of an irreducible curve C in $X \times_S Y = \mathbf{A}^2_k$ such that both $p|_C$ and $q|_C$ are nonconstant.

Returning to the general case, let $z = (x, y, s, \mathfrak{p})$ be a point of $X \times_S Y$ as in Schemes, Lemma 26.17.5. Then on stalks at z the map *can* gives the map

$$can_z : \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X\times_S Y,z}$$

This is a flat ring homomorphism as the target is a localization of the source (details omitted; hint reduce to the case that X, Y, and S are affine). Observe that the source is in general not a local ring, and this gives another way to see that *can* is not an isomorphism in general.

More generally, suppose we have an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then there is a canonical map

$$\begin{aligned}
& p^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G} \\
&= p^{-1}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_S} q^{-1}(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}) \\
&= p^{-1}\mathcal{F} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{O}_Y \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
&\xrightarrow{can} p^{-1}\mathcal{F} \otimes_{q^{-1}\mathcal{O}_X} \mathcal{O}_{X\times_S Y} \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
&= p^{-1}\mathcal{F} \otimes_{q^{-1}\mathcal{O}_X} \mathcal{O}_{X\times_S Y} \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{O}_{X\times_S Y} \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{G} \\
&= p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\mathcal{G}
\end{aligned}$$

which is rarely an isomorphism.

110.6. A nonintegral connected scheme whose local rings are domains

0568 We give an example of an affine scheme $X = \mathrm{Spec}(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness of X means A has no nontrivial idempotents, see Algebra, Lemma 10.21.3. The local rings of X are domains if, whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes. As long as A is not a domain, then X is not integral (Properties, Definition 28.3.1).

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blowup of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blowups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = \mathrm{Spec}(A_k)$. Observe that X_k is a union of 2^k+1 affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = \mathrm{Spec}(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each

A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k-1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the $2^k + 1$ lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

110.7. Noncomplete completion

05JA Let R be a ring and let $\mathfrak{m}$ be a maximal ideal. Consider the completion

$$R^\wedge = \lim R/\mathfrak{m}^n.$$

Note that $R^\wedge$ is a local ring with maximal ideal $\mathfrak{m}' = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m})$. Namely, if $x = (x_n) \in R^\wedge$ is not in $\mathfrak{m}'$, then $y = (x_n^{-1}) \in R^\wedge$ satisfies $xy = 1$, whence $R^\wedge$ is local by Algebra, Lemma 10.18.3. Now it is always true that $R^\wedge$ complete in its limit topology (see the discussion in More on Algebra, Section 15.36). But beyond that, we have the following questions:

(1) Is it true that $\mathfrak{m}R^\wedge = \mathfrak{m}'$?
(2) Is $R^\wedge$ viewed as an $R^\wedge$-module $\mathfrak{m}'$-adically complete?
(3) Is $R^\wedge$ viewed as an R-module $\mathfrak{m}$-adically complete?

It turns out that these questions all have a negative answer. The example below was taken from an unpublished note of Bart de Smit and Hendrik Lenstra. See also [**Bou61**, Exercise III.2.12] and [**Yek11**, Example 1.8]

Let k be a field, $R = k[x_1, x_2, x_3, \ldots]$, and $\mathfrak{m} = (x_1, x_2, x_3, \ldots)$. We will think of an element f of $R^\wedge$ as a (possibly) infinite sum

$$f = \sum a_I x^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\mathfrak{m}' \subset R^\wedge$ is the collection of f with zero constant term. In particular, the element

$$f = x_1 + x_2^2 + x_3^3 + \ldots$$

is in $\mathfrak{m}'$ but not in $\mathfrak{m}R^\wedge$ which shows that (1) is false in this example. However, if (1) is false, then (3) is necessarily false because $\mathfrak{m}' = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m})$ and we can apply Algebra, Lemma 10.96.5 with $n = 1$.

To finish we prove that $R^\wedge$ is not $\mathfrak{m}'$-adically complete. For $n \geq 1$ let $K_n = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m}^n)$. Then we have short exact sequences

$$0 \to K_n/(\mathfrak{m}')^n \to R^\wedge/(\mathfrak{m}')^n \to R/\mathfrak{m}^n \to 0$$

The projection map $R^\wedge \to R/\mathfrak{m}^{n+1}$ sends $(\mathfrak{m}')^n$ onto $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. It follows that $K_{n+1} \to K_n/(\mathfrak{m}')^n$ is surjective. Hence the inverse system $(K_n/(\mathfrak{m}')^n)$ has surjective transition maps and taking inverse limits we obtain an exact sequence

$$0 \to \lim K_n/(\mathfrak{m}')^n \to \lim R^\wedge/(\mathfrak{m}')^n \to \lim R/\mathfrak{m}^n \to 0$$

by Algebra, Lemma 10.87.1. Thus we see that $R^\wedge$ is complete with respect to $\mathfrak{m}'$ if and only if $K_n = (\mathfrak{m}')^n$ for all $n \geq 1$.

To show that $R^\wedge$ is not $\mathfrak{m}'$-adically complete in our example we show that $K_2 = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m}^2)$ is not equal to $(\mathfrak{m}')^2$. Note that an element of $(\mathfrak{m}')^2$ can be written as a finite sum

05JB (110.7.0.1) $$\sum\nolimits_{i=1,\ldots,t} f_i g_i$$

with $f_i, g_i \in R^\wedge$ having vanishing constant terms. To get an example we are going to choose an $z \in K_2$ of the form

$$z = z_1 + z_2 + z_3 + \ldots$$

with the following properties

(1) there exist sequences $1 < d_1 < d_2 < d_3 < \ldots$ and $0 < n_1 < n_2 < n_3 < \ldots$ such that $z_i \in k[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]$ homogeneous of degree d_i, and
(2) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ the element z_i cannot be written as a sum (110.7.0.1) with $t \leq i$.

Clearly this implies that z is not in $(\mathfrak{m}')^2$ because the image of the relation (110.7.0.1) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ for i large enough would produce a contradiction. Hence it suffices to prove that for all $t > 0$ there exists a $d \gg 0$ and an integer n such that we can find an homogeneous element $z \in k[x_1, \ldots, x_n]$ of degree d which cannot be written as a sum (110.7.0.1) for the given t in $k[[x_1, \ldots, x_n]]$. Take $n > 2t$ and any $d > 1$ prime to the characteristic of k and set $z = \sum_{i=1,\ldots,n} x_i^d$. Then the vanishing locus of the ideal

$$(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = (dx_1^{d-1}, \ldots, dx_n^{d-1})$$

consists of one point. On the other hand,

$$\frac{\partial(\sum_{i=1,\ldots,t} f_i g_i)}{\partial x_j} \in (f_1, \ldots, f_t, g_1, \ldots, g_t)$$

by the Leibniz rule and hence the vanishing locus of these derivatives contains at least

$$V(f_1, \ldots, f_t, g_1, \ldots, g_t) \subset \mathrm{Spec}(k[[x_1, \ldots, x_n]]).$$

Hence this is a contradiction as the dimension of $V(f_1, \ldots, f_t, g_1, \ldots, g_t)$ is at least $n - 2t \geq 1$.

05JC **Lemma 110.7.1.** *There exists a local ring R and a maximal ideal $\mathfrak{m}$ such that the completion $R^\wedge$ of R with respect to $\mathfrak{m}$ has the following properties*

(1) *$R^\wedge$ is local, but its maximal ideal is not equal to $\mathfrak{m}R^\wedge$,*
(2) *$R^\wedge$ is not a complete local ring, and*
(3) *$R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module.*

Proof. This follows from the discussion above as (with $R = k[x_1, x_2, x_3, \ldots]$) the completion of the localization $R_\mathfrak{m}$ is equal to the completion of R. □

110.8. Noncomplete quotient

05JD Let k be a field. Let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

Note that in particular $z_i z_j t = 0$ in this ring. Any element f of R can be uniquely written as a finite sum

$$f = \sum\nolimits_{i=0,\ldots,d} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Moreover, if f is written in this way, then $f \in (x^n)$ if and only if $f_i = 0$ for $i < n$. So x is a nonzerodivisor and $\bigcap(x^n) = 0$. Let $R^\wedge$ be the completion of R with respect to the ideal (x). Note that $R^\wedge$ is (x)-adically complete, see Algebra, Lemma 10.96.3. By the above we see that an element of $R^\wedge$ can be uniquely written as an infinite sum

$$f = \sum\nolimits_{i=0}^{\infty} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Consider the element

$$f = \sum\nolimits_{i=1}^{\infty} x^i w_i = x w_1 + x^2 w_2 + x^3 w_3 + \ldots$$

i.e., we have $f_n = w_n$. Note that $f \in (t, x^n)$ for every n because $x^m w_m \in (t)$ for all m. We claim that $f \notin (t)$. To prove this assume that $tg = f$ where $g = \sum g_l x^l$ in canonical form as above. Since $t z_i z_j = 0$ we may as well assume that none of the g_l have terms involving the products $z_i z_j$. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

(1) if the monomial m does not involve any z_i, then ctm is a term of f_l, and
(2) if the monomial m does involve a z_i then it is equal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see Algebra, Lemma 10.96.10.

05JE **Lemma 110.8.1.** *There exists a ring R complete with respect to a principal ideal I and a principal ideal J such that R/J is not I-adically complete.*

Proof. See discussion above. □

110.9. Completion is not exact

05JF A quick example is the following. Suppose that $R = k[t]$. Denote $M^\wedge$ the completion of an R-module with respect to $I = (t)$. Let $P = K = \bigoplus_{n \in \mathbf{N}} R$ and $M = \bigoplus_{n \in \mathbf{N}} R/(t^n)$. Then there is a short exact sequence $0 \to K \to P \to M \to 0$ where the first map is given by multiplication by t^n on the nth summand. We claim that $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is not exact in the middle. Namely, $\xi = (t^2, t^3, t^4, \ldots) \in P^\wedge$ maps to zero in $M^\wedge$ but is not in the image of $K^\wedge \to P^\wedge$, because it would be the image of $(t, t, t, \ldots)$ which is not an element of $K^\wedge$.

A "smaller" example is the following. In the situation of Lemma 110.8.1 the short exact sequence $0 \to J \to R \to R/J \to 0$ does not remain exact after completion with respect to I. Namely, if $f \in J$ is a generator, then $f : R \to J$ is surjective,

hence $R \to J^\wedge$ is surjective, hence the image of $J^\wedge \to R$ is $(f) = J$ but the fact that R/J is noncomplete means that the kernel of the surjection $R \to (R/J)^\wedge$ is strictly bigger than J, see Algebra, Lemmas 10.96.1 and 10.96.10. By the same token the sequence $R \to R \to R/(f) \to 0$ does not remain exact on completion.

05JG **Lemma 110.9.1.** *Completion is not an exact functor in general; it is not even right exact in general. This holds even when I is finitely generated on the category of finitely presented modules.*

Proof. See discussion above. □

110.10. The category of complete modules is not abelian

07JQ Let R be a ring and let $I \subset R$ be a finitely generated ideal. Consider the category $\mathcal{A}$ of I-adically complete R-modules, see Algebra, Definition 10.96.2. Let $\varphi : M \to N$ be a morphism of $\mathcal{A}$. The cokernel of φ in $\mathcal{A}$ is the completion $(\mathrm{Coker}(\varphi))^\wedge$ of the usual cokernel (as I is finitely generated this completion is complete, see Algebra, Lemma 10.96.3). Let $K = \mathrm{Ker}(\varphi)$. We claim that K is complete and hence is the kernel of φ in $\mathcal{A}$. Namely, let $K^\wedge$ be the completion. As M is complete we obtain a factorization

$$K \to K^\wedge \to M \xrightarrow{\varphi} N$$

Since φ is continuous for the I-adic topology, $K \to K^\wedge$ has dense image, and $K = \mathrm{Ker}(\varphi)$ we conclude that $K^\wedge$ maps into K. Thus $K^\wedge = K \oplus C$ and K is a direct summand of a complete module, hence complete.

We will give an example that shows that $\mathrm{Im} \neq \mathrm{Coim}$ in general. We take $R = \mathbf{Z}_p = \lim_n \mathbf{Z}/p^n\mathbf{Z}$ to be the ring of p-adic integers and we take $I = (p)$. Consider the map

$$\mathrm{diag}(1, p, p^2, \ldots) : \Big(\bigoplus\nolimits_{n\geq 1} \mathbf{Z}_p\Big)^\wedge \longrightarrow \prod\nolimits_{n\geq 1} \mathbf{Z}_p$$

where the left hand side is the p-adic completion of the direct sum. Hence an element of the left hand side is a vector $(x_1, x_2, x_3, \ldots)$ with $x_i \in \mathbf{Z}_p$ with p-adic valuation $v_p(x_i) \to \infty$ as $i \to \infty$. This maps to $(x_1, px_2, p^2x_3, \ldots)$. Hence we see that $(1, p, p^2, \ldots)$ is in the closure of the image but not in the image. By our description of kernels and cokernels above it is clear that $\mathrm{Im} \neq \mathrm{Coim}$ for this map.

07JR **Lemma 110.10.1.** *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The category of I-adically complete R-modules has kernels and cokernels but is not abelian in general.*

Proof. See above. □

110.11. The category of derived complete modules

0ARC Please read More on Algebra, Section 15.93 before reading this section.

Let A be a ring, let I be an ideal of A, and denote $\mathcal{C}$ the category of derived complete modules as defined in More on Algebra, Definition 15.92.4.

Let T be a set and let M_t, $t \in T$ be a family of derived complete modules. We claim that in general $\bigoplus M_t$ is not a derived complete module. For a specific example, let $A = \mathbf{Z}_p$ and $I = (p)$ and consider $\bigoplus_{n\in\mathbf{N}} \mathbf{Z}_p$. The map from $\bigoplus_{n\in\mathbf{N}} \mathbf{Z}_p$ to its p-adic completion isn't surjective. This means that $\bigoplus_{n\in\mathbf{N}} \mathbf{Z}_p$ cannot be derived complete as this would imply otherwise, see More on Algebra, Lemma 15.92.3. Hence the

inclusion functor $\mathcal{C} \to \mathrm{Mod}_A$ does not commute with either direct sums or (filtered) colimits.

Assume I is finitely generated. By the discussion in More on Algebra, Section 15.93 the category $\mathcal{C}$ has arbitrary colimits. However, we claim that filtered colimits are not exact in the category $\mathcal{C}$. Namely, suppose that $A = \mathbf{Z}_p$ and $I = (p)$. One has inclusions $f_n : \mathbf{Z}_p/p\mathbf{Z}_p \to \mathbf{Z}_p/p^n\mathbf{Z}_p$ of p-adically complete A-modules given by multiplication by p^{n-1}. There are commutative diagrams

$$\begin{array}{ccc} \mathbf{Z}_p/p\mathbf{Z}_p & \xrightarrow[f_n]{} & \mathbf{Z}_p/p^n\mathbf{Z}_p \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p} \\ \mathbf{Z}_p/p\mathbf{Z}_p & \xrightarrow{f_{n+1}} & \mathbf{Z}_p/p^{n+1}\mathbf{Z}_p \end{array}$$

We claim: the colimit of these inclusions in the category $\mathcal{C}$ gives the map $\mathbf{Z}_p/p\mathbf{Z}_p \to 0$. Namely, the colimit in Mod_A of the system on the right is $\mathbf{Q}_p/\mathbf{Z}_p$. Thus the colimit in $\mathcal{C}$ is

$$H^0((\mathbf{Q}_p/\mathbf{Z}_p)^\wedge) = H^0(\mathbf{Z}_p[1]) = 0$$

by More on Algebra, Section 15.93 where $^\wedge$ is derived completion. This proves our claim.

0ARD **Lemma 110.11.1.** *Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian and the inclusion functor $F : \mathcal{C} \to Mod_A$ is exact and commutes with arbitrary limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary direct sums and colimits, but F does not commute with these in general. Finally, filtered colimits are not exact in $\mathcal{C}$ in general, hence $\mathcal{C}$ is not a Grothendieck abelian category.*

Proof. See More on Algebra, Lemma 15.93.1 and discussion above. □

110.12. Nonflat completions

0AL8 The completion of a ring with respect to an ideal isn't always flat, contrary to the Noetherian case. We have seen two examples of this phenomenon in More on Algebra, Example 15.91.9. In this section we give two more examples.

0AL9 **Lemma 110.12.1.** *Let R be a ring. Let M be an R-module which is countable. Then M is a finite R-module if and only if $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is surjective.*

Proof. If M is a finite module, then the map is surjective by Algebra, Proposition 10.89.2. Conversely, assume the map is surjective. Let $m_1, m_2, m_3, \ldots$ be an enumeration of the elements of M. Let $\sum_{j=1,\ldots,m} x_j \otimes a_j$ be an element of the tensor product mapping to the element $(m_n) \in M^{\mathbf{N}}$. Then we see that $x_1, \ldots, x_m$ generate M over R as in the proof of Algebra, Proposition 10.89.2. □

0ALA **Lemma 110.12.2.** *Let R be a countable ring. Let M be a countable R-module. Then M is finitely presented if and only if the canonical map $M \otimes_R R^{\mathbf{N}} \to M^{\mathbf{N}}$ is an isomorphism.*

Proof. If M is a finitely presented module, then the map is an isomorphism by Algebra, Proposition 10.89.3. Conversely, assume the map is an isomorphism. By Lemma 110.12.1 the module M is finite. Choose a surjection $R^{\oplus m} \to M$ with kernel K. Then K is countable as a submodule of $R^{\oplus m}$. Arguing as in the proof

of Algebra, Proposition 10.89.3 we see that $K \otimes_R R^{\mathbf{N}} \to K^{\mathbf{N}}$ is surjective. Hence we conclude that K is a finite R-module by Lemma 110.12.1. Thus M is finitely presented. □

0ALB **Lemma 110.12.3.** *Let R be a countable ring. Then R is coherent if and only if $R^{\mathbf{N}}$ is a flat R-module.*

Proof. If R is coherent, then $R^{\mathbf{N}}$ is a flat module by Algebra, Proposition 10.90.6. Assume $R^{\mathbf{N}}$ is flat. Let $I \subset R$ be a finitely generated ideal. To prove the lemma we show that I is finitely presented as an R-module. Namely, the map $I \otimes_R R^{\mathbf{N}} \to R^{\mathbf{N}}$ is injective as $R^{\mathbf{N}}$ is flat and its image is $I^{\mathbf{N}}$ by Lemma 110.12.1. Thus we conclude by Lemma 110.12.2. □

Let R be a countable ring. Observe that $R[[x]]$ is isomorphic to $R^{\mathbf{N}}$ as an R-module. By Lemma 110.12.3 we see that $R \to R[[x]]$ is flat if and only if R is coherent. There are plenty of noncoherent countable rings, for example

$$R = k[y, z, a_1, b_1, a_2, b_2, a_3, b_3, \ldots]/(a_1 y + b_1 z, a_2 y + b_2 z, a_3 y + b_3 z, \ldots)$$

where k is a countable field. This ring is not coherent because the ideal (y, z) of R is not a finitely presented R-module. Note that $R[[x]]$ is the completion of $R[x]$ by the principal ideal (x).

0ALC **Lemma 110.12.4.** *There exists a ring such that the completion $R[[x]]$ of $R[x]$ at (x) is not flat over R and a fortiori not flat over $R[x]$.*

Proof. See discussion above. □

It turns out there is a ring R such that $R[[x]]$ is flat over R, but $R[[x]]$ is not flat over $R[x]$. See this post by Badam Baplan. Namely, let R be a valuation ring. Then R is coherent (Algebra, Example 10.90.2) and hence $R[[x]]$ is flat over R by Algebra, Proposition 10.90.6. On the other hand, we have the following lemma.

0F1Y **Lemma 110.12.5.** *Let R be a domain with fraction field K. If $R[[x]]$ is flat over $R[x]$, then R is normal if and only if R is completely normal (Algebra, Definition 10.37.3).*

Proof. Suppose we have $\alpha \in K$ and a nonzero $r \in R$ such that $r\alpha^n \in R$ for all $n \geq 1$. Then we consider $f = \sum r\alpha^{n-1}x^n$ in $R[[x]]$. Write $\alpha = a/b$ for $a, b \in R$ with b nonzero. Then we see that $(ax - b)f = -rb$. It follows that rb is in the ideal $(ax - b)R[[x]]$. Let $S = \{h \in R[x] : h(0) = 1\}$. This is a multiplicative subset and flatness of $R[x] \to R[[x]]$ implies that $S^{-1}R[x] \to R[[x]]$ is faithfully flat (details omitted; hint: use Algebra, Lemma 10.39.16). Hence

$$S^{-1}R[x]/(ax - b)S^{-1}R[x] \to R[[x]]/(ax - b)R[[x]]$$

is injective. We conclude that $hrb = (ax - b)g$ for some $h \in S$ and $g \in R[x]$. Writing $h = 1 + h_1 x + \ldots + h_d x^d$ shows that we obtain

$$1 + h_1 x + \ldots + h_d x^d = (1/r)(\alpha x - 1)g$$

This factorization in $K[x]$ gives a corresponding factorization in $K[x^{-1}]$ which shows that α is the root of a monic polynomial with coefficients in R as desired. □

0F1Z **Lemma 110.12.6.** *If R is a valuation ring of dimension > 1, then $R[[x]]$ is flat over R but not flat over $R[x]$.*

Proof. The arguments above show that this is true if we can show that R is not completely normal (valuation rings are normal, see Algebra, Lemma 10.50.3). Let $\mathfrak{p} \subset \mathfrak{m} \subset R$ be a chain of primes. Pick nonzero $x \in \mathfrak{p}$ and $y \in \mathfrak{m} \setminus \mathfrak{p}$. Then $xy^{-n} \in R$ for all $n \geq 1$ (if not then $y^n/x \in R$ which is absurd because $y \notin \mathfrak{p}$). Hence $1/y$ is almost integral over R but not in R. □

Next, we will construct an example where the completion of a localization is nonflat. To do this consider the ring

$$R = k[y, z, a_1, a_2, a_3, \ldots]/(ya_i, a_i a_j)$$

Denote $f \in R$ the residue class of z. We claim the ring map

0ALD (110.12.6.1) $$R[[x]] \longrightarrow R_f[[x]]$$

isn't flat. Let I be the kernel of $y : R[[x]] \to R[[x]]$. A typical element g of I looks like $g = \sum g_{n,m} a_m x^n$ where $g_{n,m} \in k[z]$ and for a given n only a finite number of nonzero $g_{n,m}$. Let J be the kernel of $y : R_f[[x]] \to R_f[[x]]$. We claim that $J \neq IR_f[[x]]$. Namely, if this were true then we would have

$$\sum z^{-n} a_n x^n = \sum\nolimits_{i=1,\ldots,m} h_i g_i$$

for some $m \geq 1$, $g_i \in I$, and $h_i \in R_f[[x]]$. Say $h_i = \bar{h}_i \bmod (y, a_1, a_2, a_3, \ldots)$ with $\bar{h}_i \in k[z, 1/z][[x]]$. Looking at the coefficient of a_n and using the description of the elements g_i above we would get

$$z^{-n} x^n = \sum \bar{h}_i \bar{g}_{i,n}$$

for some $\bar{g}_{i,n} \in k[z][[x]]$. This would mean that all $z^{-n}x^n$ are contained in the finite $k[z][[x]]$-module generated by the elements $\bar{h}_i$. Since $k[z][[x]]$ is Noetherian this implies that the $R[z][[x]]$-submodule of $k[z, 1/z][[x]]$ generated by $1, z^{-1}x, z^{-2}x^2, \ldots$ is finite. By Algebra, Lemma 10.36.2 we would conclude that $z^{-1}x$ is integral over $k[z][[x]]$ which is absurd. On the other hand, if (110.12.6.1) were flat, then we would get $J = IR_f[[x]]$ by tensoring the exact sequence $0 \to I \to R[[x]] \xrightarrow{y} R[[x]]$ with $R_f[[x]]$.

0ALE **Lemma 110.12.7.** *There exists a ring A complete with respect to a principal ideal I and an element $f \in A$ such that the I-adic completion $A_f^\wedge$ of A_f is not flat over A.*

Proof. Set $A = R[[x]]$ and $I = (x)$ and observe that $R_f[[x]]$ is the completion of $R[[x]]_f$. □

110.13. Nonabelian category of quasi-coherent modules

0ALF In Sheaves on Stacks, Section 96.11 we defined the category of quasi-coherent modules on a category fibred in groupoids over Sch. Although we show in Sheaves on Stacks, Section 96.15 that this category is abelian for algebraic stacks, in this section we show that this is not the case for formal algebraic spaces.

Namely, consider $\mathbf{Z}_p$ viewed as topological ring using the p-adic topology. Let $X = \mathrm{Spf}(\mathbf{Z}_p)$, see Formal Spaces, Definition 87.9.9. Then X is a sheaf in sets on $(Sch/\mathbf{Z})_{fppf}$ and gives rise to a stack in setoids $\mathcal{X}$, see Stacks, Lemma 8.6.2. Thus the discussion of Sheaves on Stacks, Section 96.15 applies.

Let $\mathcal{F}$ be a quasi-coherent module on $\mathcal{X}$. Since $X = \operatorname{colim} \operatorname{Spec}(\mathbf{Z}/p^n\mathbf{Z})$ it is clear from Sheaves on Stacks, Lemma 96.12.2 that $\mathcal{F}$ is given by a sequence $(\mathcal{F}_n)$ where

(1) $\mathcal{F}_n$ is a quasi-coherent module on $\operatorname{Spec}(\mathbf{Z}/p^n\mathbf{Z})$, and
(2) the transition maps give isomorphisms $\mathcal{F}_n = \mathcal{F}_{n+1}/p^n\mathcal{F}_{n+1}$.

Converting into modules we see that $\mathcal{F}$ corresponds to a system (M_n) where each M_n is an abelian group annihilated by p^n and the transition maps induce isomorphisms $M_n = M_{n+1}/p^nM_{n+1}$. In this situation the module $M = \lim M_n$ is a p-adically complete module and $M_n = M/p^nM$, see Algebra, Lemma 10.98.2. We conclude that the category of quasi-coherent modules on X is equivalent to the category of p-adically complete abelian groups. This category is not abelian, see Section 110.10.

0ALG **Lemma 110.13.1.** *The category of quasi-coherent*[1] *modules on a formal algebraic space X is not abelian in general, even if X is a Noetherian affine formal algebraic space.*

Proof. See discussion above. □

110.14. Nonsplit locally split sequence

0H7F Consider the short exact sequence

$$0 \to M \to \bigoplus\nolimits_{p \text{ prime}} \mathbf{Z}_{(p)} \to \mathbf{Q} \to 0$$

where M is the kernel of the map on the right which uses the inclusion $\mathbf{Z}_{(p)} \to \mathbf{Q}$ on each summand. This sequence of $\mathbf{Z}$-modules is nonsplit because there are no nonzero homomorphisms $\mathbf{Q} \to \mathbf{Z}_{(p)}$. On the other hand, if we localize at any prime p, then the sequence becomes split.

0H7G **Lemma 110.14.1.** *There exists a ring R and a nonsplit sequence of modules which becomes split Zariski locally.*

Proof. See discussion above. □

110.15. Regular sequences and base change

063Z We are going to construct a ring R with a regular sequence (x, y, z) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

To construct our example we first construct a peculiar module E over the ring $k[x, y, z]$ where k is any field. Namely, E will be a push-out as in the following diagram

$$\begin{array}{ccccc}
\frac{xk[x,y,z,y^{-1}]}{xyk[x,y,z]} & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]} & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \\
\downarrow{\scriptstyle z/x} & & \downarrow & & \downarrow \\
\frac{k[x,y,z,y^{-1}]}{yzk[x,y,z]} & \longrightarrow & E & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]}
\end{array}$$

[1] With quasi-coherent modules as defined above. Due to how things are setup in the Stacks project, this is really the correct definition; as seen above our definition agrees with what one would naively have defined to be quasi-coherent modules on $\operatorname{Spf}(A)$, namely complete A-modules.

where the rows are short exact sequences (we dropped the outer zeros due to typesetting problems). Another way to describe E is as

$$E = \{(f, g) \mid f \in k[x, y, z, x^{-1}, y^{-1}], g \in k[x, y, z, y^{-1}]\}/\sim$$

where $(f, g) \sim (f', g')$ if and only if there exists a $h \in k[x, y, z, y^{-1}]$ such that

$$f = f' + xh \bmod yk[x, y, z, x^{-1}], \quad g = g' - zh \bmod yzk[x, y, z]$$

We claim: (a) $x : E \to E$ is injective, (b) $y : E/xE \to E/xE$ is injective, (c) $E/(x, y)E = 0$, (d) there exists a nonzero element $\delta \in E/zE$ such that $x\delta = y\delta = 0$.

To prove (a) suppose that (f, g) is a pair that gives rise to an element of E and that $(xf, xg) \sim 0$. Then there exists a $h \in k[x, y, z, y^{-1}]$ such that $xf + xh \in yk[x, y, z, x^{-1}]$ and $xg - zh \in yzk[x, y, z]$. We may assume that $h = \sum a_{i,j,k}x^iy^jz^k$ is a sum of monomials where only $j \leq 0$ occurs. Then $xg - zh \in yzk[x, y, z]$ implies that only $i > 0$ occurs, i.e., $h = xh'$ for some $h' \in k[x, y, z, y^{-1}]$. Then $(f, g) \sim (f + xh', g - zh')$ and we see that we may assume that $g = 0$ and $h = 0$. In this case $xf \in yk[x, y, z, x^{-1}]$ implies $f \in yk[x, y, z, x^{-1}]$ and we see that $(f, g) \sim 0$. Thus $x : E \to E$ is injective.

Since multiplication by x is an isomorphism on $\frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]}$ we see that E/xE is isomorphic to

$$\frac{k[x, y, z, y^{-1}]}{yzk[x, y, z] + xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]} = \frac{k[x, y, z, y^{-1}]}{xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]}$$

and hence multiplication by y is an isomorphism on E/xE. This clearly implies (b) and (c).

Let $e \in E$ be the equivalence class of $(1, 0)$. Suppose that $e \in zE$. Then there exist $f \in k[x, y, z, x^{-1}, y^{-1}]$, $g \in k[x, y, z, y^{-1}]$, and $h \in k[x, y, z, y^{-1}]$ such that

$$1 + zf + xh \in yk[x, y, z, x^{-1}], \quad 0 + zg - zh \in yzk[x, y, z].$$

This is impossible: the monomial 1 cannot occur in zf, nor in xh. On the other hand, we have $ye = 0$ and $xe = (x, 0) \sim (0, -z) = z(0, -1)$. Hence setting δ equal to the congruence class of e in E/zE we obtain (d).

0640 **Lemma 110.15.1.** *There exists a local ring R and a regular sequence x, y, z (in the maximal ideal) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.*

Proof. Let $R = k[x, y, z] \oplus E$ where E is the module above considered as a square zero ideal. Then it is clear that x, y, z is a regular sequence in R, and that the element $\delta \in E/zE \subset R/zR$ gives an element with the desired properties. To get a local example we may localize R at the maximal ideal $\mathfrak{m} = (x, y, z, E)$. The sequence x, y, z remains a regular sequence (as localization is exact), and the element δ remains nonzero as it is supported at $\mathfrak{m}$. □

0641 **Lemma 110.15.2.** *There exists a local homomorphism of local rings $A \to B$ and a regular sequence x, y in the maximal ideal of B such that $B/(x, y)$ is flat over A, but such that the images $\overline{x}, \overline{y}$ of x, y in $B/\mathfrak{m}_AB$ do not form a regular sequence, nor even a Koszul-regular sequence.*

Proof. Set $A = k[z]_{(z)}$ and let $B = (k[x, y, z] \oplus E)_{(x,y,z,E)}$. Since x, y, z is a regular sequence in B, see proof of Lemma 110.15.1, we see that x, y is a regular sequence in B and that $B/(x, y)$ is a torsion free A-module, hence flat. On the other hand,

there exists a nonzero element $\delta \in B/\mathfrak{m}_A B = B/zB$ which is annihilated by $\overline{x}, \overline{y}$. Hence $H_2(K_\bullet(B/\mathfrak{m}_A B, \overline{x}, \overline{y})) \neq 0$. Thus $\overline{x}, \overline{y}$ is not Koszul-regular, in particular it is not a regular sequence, see More on Algebra, Lemma 15.30.2. □

110.16. A Noetherian ring of infinite dimension

02JC A Noetherian local ring has finite dimension as we saw in Algebra, Proposition 10.60.9. But there exist Noetherian rings of infinite dimension. See [**Nag62b**, Appendix, Example 1].

Namely, let k be a field, and consider the ring

$$R = k[x_1, x_2, x_3, \ldots].$$

Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1})$ for $i = 1, 2, \ldots$ which are prime ideals of R. Let S be the multiplicative subset

$$S = \bigcap\nolimits_{i \geq 1} (R \setminus \mathfrak{p}_i).$$

Consider the ring $A = S^{-1}R$. We claim that

(1) The maximal ideals of the ring A are the ideals $\mathfrak{m}_i = \mathfrak{p}_i A$.
(2) We have $A_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ which is a Noetherian local ring of dimension 2^i.
(3) The ring A is Noetherian.

Hence it is clear that this is the example we are looking for. Details omitted.

110.17. Local rings with nonreduced completion

02JD In Algebra, Example 10.119.5 we gave an example of a characteristic p Noetherian local domain R of dimension 1 whose completion is nonreduced. In this section we present the example of [**FR70**, Proposition 3.1] which gives a similar ring in characteristic zero.

Let $\mathbf{C}\{x\}$ be the ring of convergent power series over the field $\mathbf{C}$ of complex numbers. The ring of all power series $\mathbf{C}[[x]]$ is its completion. Let $K = \mathbf{C}\{x\}[1/x]$ be the field of convergent Laurent series. The K-module $\Omega_{K/\mathbf{C}}$ of algebraic differentials of K over $\mathbf{C}$ is an infinite dimensional K-vector space (proof omitted). We may choose $f_n \in x\mathbf{C}\{x\}$, $n \geq 1$ such that $\mathrm{d}x, \mathrm{d}f_1, \mathrm{d}f_2, \ldots$ are part of a basis of $\Omega_{K/\mathbf{C}}$. Thus we can find a $\mathbf{C}$-derivation

$$D : \mathbf{C}\{x\} \longrightarrow \mathbf{C}((x))$$

such that $D(x) = 0$ and $D(f_i) = x^{-n}$. Let

$$A = \{f \in \mathbf{C}\{x\} \mid D(f) \in \mathbf{C}[[x]]\}$$

We claim that

(1) $\mathbf{C}\{x\}$ is integral over A,
(2) A is a local domain,
(3) $\dim(A) = 1$,
(4) the maximal ideal of A is generated by x and xf_1,
(5) A is Noetherian, and
(6) the completion of A is equal to the ring of dual numbers over $\mathbf{C}[[x]]$.

Since the dual numbers are nonreduced the ring A gives the example.

Note that if $0 \neq f \in x\mathbf{C}\{x\}$ then we may write $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbf{C}[[x]]$. Hence $D(f^{n+1}/(n+1)) \in \mathbf{C}[[x]]$ and $D(f^{n+2}/(n+2)) \in \mathbf{C}[[x]]$. Thus we see $f^{n+1}, f^{n+2} \in A$! In particular we see (1) holds. We also conclude that the fraction field of A is equal to the fraction field of $\mathbf{C}\{x\}$. It also follows immediately that $A \cap x\mathbf{C}\{x\}$ is the set of nonunits of A, hence A is a local domain of dimension 1. If we can show (4) then it will follow that A is Noetherian (proof omitted). Suppose that $f \in A \cap x\mathbf{C}\{x\}$. Write $D(f) = h$, $h \in \mathbf{C}[[x]]$. Write $h = c + xh'$ with $c \in \mathbf{C}$, $h' \in \mathbf{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - c = xh'$. On the other hand $f - cxf_1 = xg$ with $g \in \mathbf{C}\{x\}$, but by the computation above we have $D(g) = h' \in \mathbf{C}[[x]]$ and hence $g \in A$. Thus $f = cxf_1 + xg \in (x, xf_1)$ as desired.

Finally, why is the completion of A nonreduced? Denote $\hat{A}$ the completion of A. Of course this maps surjectively to the completion $\mathbf{C}[[x]]$ of $\mathbf{C}\{x\}$ because $x \in A$. Denote this map $\psi : \hat{A} \to \mathbf{C}[[x]]$. Above we saw that $\mathfrak{m}_A = (x, xf_1)$ and hence $D(\mathfrak{m}_A^n) \subset (x^{n-1})$ by an easy computation. Thus $D : A \to \mathbf{C}[[x]]$ is continuous and gives rise to a continuous derivation $\hat{D} : \hat{A} \to \mathbf{C}[[x]]$ over ψ. Hence we get a ring map

$$\psi + \epsilon\hat{D} : \hat{A} \longrightarrow \mathbf{C}[[x]][\epsilon].$$

Since $\hat{A}$ is a one dimensional Noetherian complete local ring, if we can show this arrow is surjective then it will follow that $\hat{A}$ is nonreduced. Actually the map is an isomorphism but we omit the verification of this. The subring $\mathbf{C}[x]_{(x)} \subset A$ gives rise to a map $i : \mathbf{C}[[x]] \to \hat{A}$ on completions such that $i \circ \psi = \mathrm{id}$ and such that $D \circ i = 0$ (as $D(x) = 0$ by construction). Consider the elements $x^n f_n \in A$. We have

$$(\psi + \epsilon D)(x^n f_n) = x^n f_n + \epsilon$$

for all $n \geq 1$. Surjectivity easily follows from these remarks.

110.18. Another local ring with nonreduced completion

0GHH In this section we make an example of a Noetherian local domain of dimension 2 complete with respect to a principal ideal such that the recompletion of a localization is nonreduced.

Let p be a prime number. Let k be a field of characteristic p such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \begin{array}{l} \sum a_{i,j} x^i y^j \in k[[x, y]] \text{ such that for all } n \geq 0 \text{ we have} \\ [k^p(a_{n,n}, a_{n,n+1}, a_{n+1,n}, a_{n,n+2}, a_{n+2,n}, \ldots) : k^p] < \infty \end{array} \right\}$$

As a set we have

$$k^p[[x, y]] \subset A \subset k[[x, y]]$$

Every element f of A can be uniquely written as a series

$$f = f_0 + f_1 xy + f_2 (xy)^2 + f_3 (xy)^3 + \ldots$$

with

$$f_n = a_{n,n} + a_{n,n+1} y + a_{n+1,n} x + a_{n,n+2} y^2 + a_{n+2,n} x^2 + \ldots$$

and the condition in the formula defining A means that the coefficients of f_n generate a finite extension of k^p. From this presentation it is clear that A is an

$k^p[[x,y]]$-subalgebra of $k[[x,y]]$ complete with respect to the ideal xy. Moreover, we clearly have

$$A/xyA = C \times_k D$$

where $k^p[[x]] \subset C \subset k[[x]]$ and $k^p[[y]] \subset D \subset k[[y]]$ are the subrings of power series from Algebra, Example 10.119.5. Hence C and D are dvrs and we see that A/xyA is Noetherian. By Algebra, Lemma 10.97.5 we conclude that A is Noetherian. Since $\dim(k[[x,y]]) = 2$ using Algebra, Lemma 10.112.4 we conclude that $\dim(A) = 2$.

Let $f = \sum a_i x^i$ be a power series such that $k^p(a_0, a_1, a_2, \ldots)$ has infinite degree over k^p. Then $f \notin A$ but $f^p \in A$. We set

$$B = A[f] \subset k[[x,y]]$$

Since B is finite over A we see that B is Noetherian. Also, B is complete with respect to the ideal generated by xy, see Algebra, Lemma 10.97.1. In fact B is free over A with basis $1, f, f^2, \ldots, f^{p-1}$; we omit the proof.

We claim the ring

$$(B_y)^\wedge = (B[1/y])^\wedge = \lim B[1/y]/(xy)^n B[1/y] = \lim B[1/y]/x^n B[1/y]$$

is nonreduced. Namely, this ring is free over

$$(A_y)^\wedge = (A[1/y])^\wedge = \lim A[1/y]/(xy)^n A[1/y] = \lim A[1/y]/x^n A[1/y]$$

with basis $1, f, \ldots, f^{p-1}$. However, there is an element $g \in (A_y)^\wedge$ such that $f^p = g^p$. Namely, we can just take $g = \sum a_i x^i$ (the same expression as we used for f) which makes sense in $(A_y)^\wedge$. Hence we see that

$$(B_y)^\wedge = (A_y)^\wedge[f]/(f^p - g^p) \cong (A_y)^\wedge[\tau]/(\tau^p)$$

is nonreduced. In fact, this example shows slightly more. Namely, observe that $(A_y)^\wedge$ is a dvr with uniformizer x and residue field the fraction field of the dvr D given above. Hence we see that even

$$(B_y)^\wedge[1/(xy)] = ((B_y)^\wedge)_{xy}$$

is nonreduced. This produces an example of the following kind.

0GHI **Lemma 110.18.1.** *There exists a local Noetherian 2-dimensional domain $(B, \mathfrak{m})$ complete with respect to a principal ideal $I = (b)$ and an element $f \in \mathfrak{m}$, $f \notin I$ such that the I-adic completion $C = (B_f)^\wedge$ of the principal localization B_f is nonreduced and even such that $C_b = C[1/b] = (B_f)^\wedge[1/b]$ is nonreduced.*

Proof. See discussion above. □

110.19. A non catenary Noetherian local ring

02JE Even though there is a successful dimension theory of Noetherian local rings there are non-catenary Noetherian local rings. An example may be found in [**Nag62b**, Appendix, Example 2]. In fact, we will present this example in the simplest case. Namely, we will construct a local Noetherian domain A of dimension 2 which is not universally catenary. (Note that A is automatically catenary, see Exercises, Exercise 111.18.3.) The existence of a Noetherian local ring which is not universally catenary implies the existence of a Noetherian local ring which is not catenary – and we spell this out at the end of this section in the particular example at hand.

Let k be a field, and consider the formal power series ring $k[[x]]$ in one variable over k. Let

$$z = \sum\nolimits_{i=1}^{\infty} a_i x^i$$

be a formal power series. We assume z as an element of the Laurent series field $k((x)) = k[[x]][1/x]$ is transcendental over $k(x)$. Put

$$z_j = x^{-j}(z - \sum\nolimits_{i=1,\ldots,j-1} a_i x^i) = \sum\nolimits_{i=j}^{\infty} a_i x^{i-j} \in k[[x]].$$

Note that $z = xz_1$. Let R be the subring of $k[[x]]$ generated by x, z and all of the z_j, in other words

$$R = k[x, z_1, z_2, z_3, \ldots] \subset k[[x]].$$

Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x - 1, z_1, z_2 + a_1, z_3 + a_1 + a_2, \ldots)$ of R.

We have $xz_{j+1} + a_j = z_j$. Hence $R/\mathfrak{m} = k$ and $\mathfrak{m}$ is a maximal ideal. Moreover, any element of R not in $\mathfrak{m}$ maps to a unit in $k[[x]]$ and hence $R_{\mathfrak{m}} \subset k[[x]]$. In fact it is easy to deduce that $R_{\mathfrak{m}}$ is a discrete valuation ring and residue field k.

We claim that

$$R/(x-1) = k[x, z_1, z_2, z_3, \ldots]/(x-1) \cong k[z].$$

Namely, the relation above implies that $z_{j+1} = z_j - a_j - (x-1)z_{j+1}$, and hence we may express the class of z_{j+1} in terms of z_j in the quotient $R/(x-1)$. Since the fraction field of R has transcendence degree 2 over k by construction we see that z is transcendental over k in $R/(x-1)$, whence the desired isomorphism. Hence $\mathfrak{n} = (x-1, z)$ and is a maximal ideal. In fact the map

$$k[x, x^{-1}, z]_{(x-1,z)} \longrightarrow R_{\mathfrak{n}}$$

is an isomorphism (since x^{-1} is invertible in $R_{\mathfrak{n}}$ and since $z_{j+1} = x^{-1}z_j - a_j = \ldots = f_j(x, x^{-1}, z)$). This shows that $R_{\mathfrak{n}}$ is a regular local ring of dimension 2 and residue field k.

Let S be the multiplicative subset

$$S = (R \setminus \mathfrak{m}) \cap (R \setminus \mathfrak{n}) = R \setminus (\mathfrak{m} \cup \mathfrak{n})$$

and set $B = S^{-1}R$. We claim that

(1) The ring B is a k-algebra.
(2) The maximal ideals of the ring B are the two ideals $\mathfrak{m}B$ and $\mathfrak{n}B$.
(3) The residue field at these maximal ideals is k.
(4) We have $B_{\mathfrak{m}B} = R_{\mathfrak{m}}$ and $B_{\mathfrak{n}B} = R_{\mathfrak{n}}$ which are Noetherian regular local rings of dimensions 1 and 2.
(5) The ring B is Noetherian.

We omit the details of the verifications.

Whenever given a k-algebra B with the properties listed above we get an example as follows. Take $A = k + \mathrm{rad}(B) \subset B$ with $\mathrm{rad}(B) = \mathfrak{m}B \cap \mathfrak{n}B$ the Jacobson radical. It is easy to see that B is finite over A and hence A is Noetherian by Eakin's theorem (see [**Eak68**], or [**Nag62b**, Appendix A1], or insert future reference here). Also A is a local domain with the same fraction field as B and residue field k. Since the dimension of B is 2 we see that A has dimension 2 as well, by Algebra, Lemma 10.112.4.

If A were universally catenary then the dimension formula, Algebra, Lemma 10.113.1 would give $\dim(B_{\mathfrak{m}B}) = 2$ contradiction.

Note that B is generated by one element over A. Hence $B = A[x]/\mathfrak{p}$ for some prime $\mathfrak{p}$ of $A[x]$. Let $\mathfrak{m}' \subset A[x]$ be the maximal ideal corresponding to $\mathfrak{m}B$. Then on the one hand $\dim(A[x]_{\mathfrak{m}'}) = 3$ and on the other hand

$$(0) \subset \mathfrak{p}A[x]_{\mathfrak{m}'} \subset \mathfrak{m}'A[x]_{\mathfrak{m}'}$$

is a maximal chain of primes. Hence $A[x]_{\mathfrak{m}'}$ is an example of a non catenary Noetherian local ring.

110.20. Existence of bad local Noetherian rings

0AL7 Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. In [**Lec86a**] it was shown that A is the completion of a Noetherian local domain if $\text{depth}(A) \geq 1$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. This produces many examples of Noetherian local domains with "bizarre" properties.

Applying this for example to $A = \mathbf{C}[[x, y]]/(y^2)$ we find a Noetherian local domain whose completion is nonreduced. Please compare with Section 110.17.

In [**LLPY01**] conditions were found that characterize when A is the completion of a reduced local Noetherian ring.

In [**Hei93**] it was shown that A is the completion of a local Noetherian UFD R if $\text{depth}(A) \geq 2$ and A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring, or contains $\mathbf{Z}$ as a subring and A is torsion free as a $\mathbf{Z}$-module. In particular R is normal (Algebra, Lemma 10.120.11) hence the henselization of R is a normal domain too (More on Algebra, Lemma 15.45.6). Thus A as above is the completion of a henselian Noetherian local normal domain (because the completion of R and its henselization agree, see More on Algebra, Lemma 15.45.3).

Apply this to find a Noetherian local UFD R such that $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$. Note that $\text{Spec}(R^\wedge)$ is the union of a regular 2-dimensional and a regular 3-dimensional component. The ring R cannot be universally catenary: Let

$$X \longrightarrow \text{Spec}(R)$$

be the blowing up of the maximal ideal. Then X is an integral scheme. There is a closed point $x \in X$ such that $\dim(\mathcal{O}_{X,x}) = 2$, namely, on the level of the complete local ring we pick x to lie on the strict transform of the 2-dimensional component and not on the strict transform of the 3-dimensional component. By Morphisms, Lemma 29.52.1 we see that R is not universally catenary. Please compare with Section 110.19.

The ring above is catenary (being a 3-dimensional local Noetherian UFD). However, in [**Ogo80**] the author constructs a normal local Noetherian domain R with $R^\wedge \cong \mathbf{C}[[x, y, z, w]]/(wx, wy)$ such that R is not catenary. See also [**Hei82**] and [**Lec86b**].

In [**Hei94**] it was shown that A is the completion of a local Noetherian ring R with an isolated singularity provided A contains either $\mathbf{Q}$ or $\mathbf{F}_p$ as a subring or A has residue characteristic $p > 0$ and p cannot map to a nonzero zerodivisor in any proper localization of A. Here we say a Noetherian local ring R has an isolated singularity if $R_{\mathfrak{p}}$ is a regular local ring for all nonmaximal primes $\mathfrak{p} \subset R$.

The papers [**Nis12**] and [**Nis16**] contain long lists of "bad" Noetherian local rings with given completions. In particular it constructs an example of a 2-dimensional Nagata local normal domain whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and one whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$.

As an aside, in [**Loe03**] it was shown that A is the completion of an excellent Noetherian local domain if A is reduced, equidimensional, and no integer in A is a zero divisor. However, this doesn't lead to "bad" Noetherian local rings as we obtain excellent ones!

110.21. Dimension in Noetherian Jacobson rings

0EEH Let k be the algebraic closure of a finite field. Let $A = k[x, y]$ and $X = \mathrm{Spec}(A)$. Let $C = V(x)$ be the y-axis (this could be any other 1-dimensional integral closed subscheme of X). Let $C_1, C_2, C_3, \ldots$ be an enumeration of the other integral closed subschemes of X of dimension 1. Let $p_1, p_2, p_3, \ldots$ be an enumeration of the closed points of C.

Claim: for every n there exists an irreducible closed $Z_n \subset X$ of dimension 1 such that

$$\{p_n\} = Z_n \cap (C \cup C_1 \cup \ldots \cup C_n)$$

set theoretically. To do this set $Y = C \cup C_1 \cup C_2 \cup \ldots \cup C_n$. This is a reduced affine algebraic scheme of dimension 1 over k. It is enough to find $f \in k[x, y]$ with $V(f) \cap Y = \{p_n\}$ set theoretically because then we can take Z_n to be a suitable irreducible component of $V(f)$. Since the restriction map

$$k[x, y] \longrightarrow \Gamma(Y, \mathcal{O}_Y)$$

is surjective, it suffices to find a regular function g on Y whose zero set is $\{p_n\}$ set theoretically. To see this is possible, we choose an effective Cartier divisor $D \subset Y$ whose support is p_n (this is possible by Varieties, Lemma 33.38.3). Thus it suffices to show that $\mathcal{O}_X(ND) \cong \mathcal{O}_X$ for some $N > 0$. But the Picard group of an affine 1-dimensional algebraic scheme over the algebraic closure of a finite field is torsion (insert future reference here) and we conclude the claim is true.

Choose Z_n as above for all n. Since $k[x, y]$ is a UFD we may write $Z_n = V(f_n)$ for some irreducible element $f_n \in A$. Let $S \subset k[x, y]$ be the multiplicative subset generated by $f_1, f_2, f_3, \ldots$. Consider the Noetherian ring $B = S^{-1}A$.

Obviously, the ring map $A \to B$ identifies local rings and induces an injection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Moreover, looking at the curve C_1 we see that only the points of $C \cap C_1$ are removed when passing from $\mathrm{Spec}(A)$ to $\mathrm{Spec}(B)$. In particular, we see that $\mathrm{Spec}(B)$ has an infinite number of maximal ideals corresponding to maximal ideals of A. On the other hand, xB is a maximal ideal because the spectrum of B/xB consists of a unique prime ideal as we removed all the closed points of $C = V(x)$ (but not the generic point). Finally, for $i \geq 1$ consider the curve C_i. Write $C_i = V(g_i)$ for $g_i \in A$ irreducible. If $C_i = Z_n$ for some n, then $g_i B$ is the unit ideal. If not, then all but finitely many of the closed points of C_i survive the passage from A to B: namely, only the points of $(Z_1 \cup \ldots \cup Z_{i-1} \cup C) \cap C_i$ are removed from C_i.

The structure of the prime spectrum of B given above shows that B is Jacobson by Algebra, Lemma 10.61.4. The maximal ideals are the maximal ideals of A which

are in $\mathrm{Spec}(B)$ (and there an inifinitude of these) together with the maximal ideal xB. Thus we see that we have local rings of dimensions 1 and 2.

0EEI **Lemma 110.21.1.** *There exists a Jacobson, universally catenary, Noetherian domain B with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ such that $\dim(B_{\mathfrak{m}_1}) = 1$ and $\dim(B_{\mathfrak{m}_2}) = 2$.*

Proof. The construction of B is given above. We just point out that B is universally catenary by Algebra, Lemma 10.105.4 and Morphisms, Lemma 29.17.5. □

110.22. Underlying space Noetherian not Noetherian

0G61 We give two examples to show that a scheme whose underlying topological space is Noetherian may not be a Noetherian scheme.

0G62 **Example 110.22.1.** Let k be a field, and let $A = k[x_1, x_2, x_3, \dots]/(x_1^2, x_2^2, x_3^2, \dots)$. Any prime ideal of A contains the nilpotents $x_1, x_2, x_3, \dots$, so $\mathfrak{p} = (x_1, x_2, x_3, \dots)$ is the only prime ideal of A. Therefore the underlying topological space of $\mathrm{Spec}\, A$ is a single point and in particular is Noetherian. However $\mathfrak{p}$ is clearly not finitely generated.

0G63 **Example 110.22.2.** Let k be a field, and let $A \subseteq k[x, y]$ be the subring generated by k and the monomials $\{xy^i\}_{i \geq 0}$. The prime ideals of A that do not contain x are in one-to-one correspondence with the prime ideals of $A_x \cong k[x, x^{-1}, y]$. If $\mathfrak{p}$ is a prime ideal that does contain x, then it contains every xy^i, $i \geq 0$, because $(xy^i)^2 = x(xy^{2i}) \in \mathfrak{p}$ and $\mathfrak{p}$ is radical. Consequently $\mathfrak{p} = (\{xy^i\}_{i \geq 0})$. Therefore the underlying topological space of $\mathrm{Spec}\, A$ is Noetherian, since it consists of the points of the Noetherian scheme $\mathrm{Spec}(A[x, x^{-1}, y])$ and the prime ideal $\mathfrak{p}$. But the ring A is non-Noetherian because $\mathfrak{p}$ is not finitely generated. Note that in this example, A also has the property of being a domain.

110.23. Non-quasi-affine variety with quasi-affine normalization

0271 The existence of an example of this kind is mentioned in [**DG67**, II Remark 6.6.13]. They refer to the fifth volume of EGA for such an example, but the fifth volume did not appear.

Let k be a field. Let $Y = \mathbf{A}^2_k \setminus \{(0, 0)\}$. We are going to construct a finite surjective birational morphism $\pi : Y \longrightarrow X$ with X a variety over k such that X is not quasi-affine. Namely, consider the following curves in Y:

$$\begin{aligned} C_1 &: x = 0 \\ C_2 &: y = 0 \end{aligned}$$

Note that $C_1 \cap C_2 = \emptyset$. We choose the isomorphism $\varphi : C_1 \to C_2$, $(0, y) \mapsto (y^{-1}, 0)$. We claim there is a unique morphism $\pi : Y \to X$ as above such that

$$C_1 \underset{\varphi}{\overset{\mathrm{id}}{\rightrightarrows}} Y \xrightarrow{\pi} X$$

is a coequalizer diagram in the category of varieties (and even in the category of schemes). Accepting this for the moment let us show that such an X cannot be quasi-affine. Namely, it is clear that we would get

$$\Gamma(X, \mathcal{O}_X) = \{f \in k[x, y] \mid f(0, y) = f(y^{-1}, 0)\} = k \oplus (xy) \subset k[x, y].$$

In particular these functions do not separate the points $(1,0)$ and $(-1,0)$ whose images in X (we will see below) are distinct (if the characteristic of k is not 2).

To show that X exists consider the Zariski open $D(x+y) \subset Y$ of Y. This is the spectrum of the ring $k[x,y,1/(x+y)]$ and the curves C_1, C_2 are completely contained in $D(x+y)$. Moreover the morphism

$$C_1 \amalg C_2 \longrightarrow D(x+y) \cap Y = \mathrm{Spec}(k[x,y,1/(x+y)])$$

is a closed immersion. It follows from More on Algebra, Lemma 15.5.1 that the ring

$$A = \{f \in k[x,y,1/(x+y)] \mid f(0,y) = f(y^{-1},0)\}$$

is of finite type over k. On the other hand we have the open $D(xy) \subset Y$ of Y which is disjoint from the curves C_1 and C_2. It is the spectrum of the ring

$$B = k[x,y,1/xy].$$

Note that we have $A_{xy} \cong B_{x+y}$ (since A clearly contains the elements $xyP(x,y)$ any polynomial P and the element $xy/(x+y)$). The scheme X is obtained by glueing the affine schemes $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ using the isomorphism $A_{xy} \cong B_{x+y}$ and hence is clearly of finite type over k. To see that it is separated one has to show that the ring map $A \otimes_k B \to B_{x+y}$ is surjective. To see this use that $A \otimes_k B$ contains the element $xy/(x+y) \otimes 1/xy$ which maps to $1/(x+y)$. The morphism $Y \to X$ is given by the natural maps $D(x+y) \to \mathrm{Spec}(A)$ and $D(xy) \to \mathrm{Spec}(B)$. Since these are both finite we deduce that $Y \to X$ is finite as desired. We omit the verification that X is indeed the coequalizer of the displayed diagram above, however, see (insert future reference for pushouts in the category of schemes here). Note that the morphism $\pi : Y \to X$ does map the points $(1,0)$ and $(-1,0)$ to distinct points in X because the function $(x+y^3)/(x+y)^2 \in A$ has value $1/1$, resp. $-1/(-1)^2 = -1$ which are always distinct (unless the characteristic is 2 – please find your own points for characteristic 2). We summarize this discussion in the form of a lemma.

0272 **Lemma 110.23.1.** *Let k be a field. There exists a variety X whose normalization is quasi-affine but which is itself not quasi-affine.*

Proof. See discussion above and (insert future reference on normalization here). □

110.24. Taking scheme theoretic images

0GIK Let k be a field. Let t be a variable. Let $Y = \mathrm{Spec}(k[t])$ and $X = \coprod_{n \geq 1} \mathrm{Spec}(k[t]/(t^n))$. Denote $f : X \to Y$ the morphism using the closed immersion $\mathrm{Spec}(k[t]/(t^n)) \to \mathrm{Spec}(k[t])$ for each $n \geq 1$. In this case we have

(1) The scheme theoretic image (Morphisms, Definition 29.6.2) of f is Y. On the other hand, the image of f is the closed point $t = 0$ in Y. Thus the underlying closed subset of the scheme theoretic image of f is not equal to the closure of the image of f.
(2) The formation of the scheme theoretic image does not commute with restriction to the open subscheme $V = \mathrm{Spec}(k[t,1/t]) \subset Y$. Namely, the preimage of V in X is empty and hence the scheme theoretic image of $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is the empty scheme. This is not equal to $Y \cap V$.

110.25. Images of locally closed subsets

0GZL Chevalley's theorem says that the image of a constructible set by a finitely presented morphism of affine schemes is constructible, see Algebra, Theorem 10.29.10 and Morphisms, Section 29.22. We will see the same thing does not hold for images of locally closed subsets.

Let k be a field of characteristic 0. Consider the projection morphism

$$f : X = \mathrm{Spec}(k[t, x_1, x_2, \ldots, y_1, y_2, \ldots]) \longrightarrow \mathrm{Spec}(k[x_1, x_2, \ldots, y_1, y_2, \ldots]) = Y$$

This is a morphism of finite presentation. Let Z be the closed subset of X defined by

$$x_1(t-1) = 0, \quad x_2(t-1)(t-2) = 0, \quad x_3(t-1)(t-2)(t-3) = 0, \quad \ldots$$

Let $U = \bigcup_{j \geq 1} U_j$ be the open of X defined by

$$U_j = \text{points where } y_j(t-1)(t-2)...(t-j) \text{ is nonzero}$$

Then we have

$$f(Z \cap U_j) = \text{points where } x_1, \ldots, x_j \text{ are zero and } y_j \text{ is nonzero}$$

We claim that $B = f(Z \cap U) = \bigcup_{j \geq 1} f(Z \cap U_j)$ is not a finite union of locally closed subsets of Y.

Proof of the claim. Suppose that $B = A_1 \cup \cdots \cup A_m$ is a finite cover of B by locally closed subsets of Y. We will show by induction on n that $m \geq n$. The base case $n = 1$ is OK as B is nonempty. Assume $n > 1$ and that the induction hypothesis holds for $n - 1$. Since the closure of B is $(x_1 = 0)$, one of the A_i must contain some nonempty open subset of $(x_1 = 0)$. Then A_i must be open in $(x_1 = 0)$. But any such open subset cannot contain a point with $y_1 = 0$; indeed, for points of B, $y_1 = 0$ forces $x_2 = 0$, and this shows B contains no neighborhood of (x, y) inside $(x_1 = 0)$. Therefore, the remaining $m - 1$ elements restrict to a constructible cover of $B \cap (y_1 = 0)$. However, observe that the right shift map $x_i \mapsto x_{i+1}$, $y_i \mapsto y_{i+1}$ identifies B with $B \cap (y_1 = 0)$! Thus by induction hypothesis, we see that $m - 1 \geq n - 1$ and we conclde $m \geq n$. This finishes the proof of the induction step and thereby establishes the claim.

0GZM **Lemma 110.25.1.** *There exists a morphism $f : X \to Y$ of finite presentation between affine schemes and a locally closed subset T of X such that $f(T)$ is not a finite union of locally closed subsets of Y.*

Proof. See discussion above. □

110.26. A locally closed subscheme which is not open in closed

078B This is a copy of Morphisms, Example 29.3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if

$n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

The morphism $Z \to X$ also gives an example of bad behaviour of scheme theoretic images of immersions. Namely, the arguments above show that the scheme theoretic image of the immersion $Z \to X$ is X. On the other hand, we see

(1) Z is not topologically dense in X, and
(2) the scheme theoretic image of $Z = Z \cap U \to U$ is just Z. This is not equal to $U \cap X = U$ and hence formation of the scheme theoretic image in this case does not commute with restrictions to opens.

110.27. Nonexistence of suitable opens

086G This section complements the results of Properties, Section 28.29.

Let k be a field and let $A = k[z_1, z_2, z_3, \ldots]/I$ where I is the ideal generated by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $S = \mathrm{Spec}(A)$. Let $s \in S$ be the closed point corresponding to the maximal ideal (z_i). We claim there is no quasi-compact open $V \subset S \setminus \{s\}$ which is dense in $S \setminus \{s\}$. Note that $S \setminus \{s\} = \bigcup D(z_i)$. Each $D(z_i)$ is open and irreducible with generic point η_i. We conclude that $\eta_i \in V$ for all i. However, a principal affine open of $S \setminus \{s\}$ is of the form $D(f)$ where $f \in (z_1, z_2, \ldots)$. Then $f \in (z_1, \ldots, z_n)$ for some n and we see that $D(f)$ contains only finitely many of the points η_i. Thus V cannot be quasi-compact.

Let k be a field and let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $T = \mathrm{Spec}(B)$. Consider the principal open $U = D(x)$. We claim there is no quasi-compact open $V \subset T$ such that $V \cap U = \emptyset$ and $V \cup U$ is dense in T. Let $t \in T$ be the closed point corresponding to the maximal ideal (x, z_i). The closure of U in T is $\overline{U} = U \cup \{t\}$. Hence $V \subset \bigcup_i D(z_i)$ is a quasi-compact open. By the arguments of the previous paragraph we see that V cannot be dense in $\bigcup D(z_i)$.

086H **Lemma 110.27.1.** *Nonexistence quasi-compact opens of affines:*

(1) *There exist an affine scheme S and affine open $U \subset S$ such that there is no quasi-compact open $V \subset S$ with $U \cap V = \emptyset$ and $U \cup V$ dense in S.*
(2) *There exists an affine scheme S and a closed point $s \in S$ such that $S \setminus \{s\}$ does not contain a quasi-compact dense open.*

Proof. See discussion above. □

Let X be the glueing of two copies of the affine scheme T (see above) along the affine open U. Thus there is a morphism $\pi : X \to T$ and $X = U_1 \cup U_2$ such that π maps U_i isomorphically to T and $U_1 \cap U_2$ isomorphically to U. Note that X is quasi-separated (by Schemes, Lemma 26.21.6) and quasi-compact. We claim there does not exist a separated, dense, quasi-compact open $W \subset X$. Namely, consider the two closed points $x_1 \in U_1$, $x_2 \in U_2$ mapping to the closed point $t \in T$ introduced above. Let $\tilde{\eta} \in U_1 \cap U_2$ be the generic point mapping to the (unique) generic point η of U. Note that $\tilde{\eta} \rightsquigarrow x_1$ and $\tilde{\eta} \rightsquigarrow x_2$ lying over the specialization

$\eta \rightsquigarrow t$. Since $\pi|_W : W \to T$ is separated we conclude that we cannot have both x_1 and $x_2 \in W$ (by the valuative criterion of separatedness Schemes, Lemma 26.22.2). Say $x_1 \notin W$. Then $W \cap U_1$ is a quasi-compact (as X is quasi-separated) dense open of U_1 which does not contain x_1. Now observe that there exists an isomorphism $(T, t) \cong (S, s)$ of schemes (by sending x to z_1 and z_i to z_{i+1}). Hence by the first paragraph of this section we arrive at a contradiction.

086I **Lemma 110.27.2.** *There exists a quasi-compact and quasi-separated scheme X which does not contain a separated quasi-compact dense open.*

Proof. See discussion above. □

110.28. Nonexistence of quasi-compact dense open subscheme

087H Let X be a quasi-compact and quasi-separated algebraic space over a field k. We know that the schematic locus $X' \subset X$ is a dense open subspace, see Properties of Spaces, Proposition 66.13.3. In fact, this result holds when X is reasonable, see Decent Spaces, Proposition 68.10.1. A natural question is whether one can find a quasi-compact dense open subscheme of X. It turns out this is not possible in general.

Assume the characteristic of k is not 2. Let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $U = \operatorname{Spec}(B)$. Denote $0 \in U$ the closed point all of whose coordinates are zero. Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

where Δ is the image of the diagonal morphism of U over k and

$$\Gamma = \{((x, 0, 0, 0, \ldots), (-x, 0, 0, 0, \ldots)) \mid x \in \mathbf{A}^1_k, x \neq 0\}.$$

It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space (Spaces, Theorem 65.10.5). Note that j is not an immersion because $(0, 0) \in \Delta$ is in the closure of Γ. Hence X is not a scheme. On the other hand, X is quasi-separated as R is quasi-compact. Denote 0_X the image of the point $0 \in U$. We claim that $X \setminus \{0_X\}$ is a scheme, namely

$$X \setminus \{0_X\} = \operatorname{Spec}\left(k[x^2, x^{-2}]\right) \amalg \operatorname{Spec}\left(k[z_1, z_2, z_3, \ldots]/(z_i z_j)\right) \setminus \{0\}$$

(details omitted). On the other hand, we have seen in Section 110.27 that the scheme on the right hand side does not contain a quasi-compact dense open.

087I **Lemma 110.28.1.** *There exists a quasi-compact and quasi-separated algebraic space which does not contain a quasi-compact dense open subscheme.*

Proof. See discussion above. □

Using the construction of Spaces, Example 65.14.2 in the same manner as we used the construction of Spaces, Example 65.14.1 above, one obtains an example of a quasi-compact, quasi-separated, and locally separated algebraic space which does not contain a quasi-compact dense open subscheme.

110.29. Affines over algebraic spaces

088V Suppose that $f : Y \to X$ is a morphism of schemes with f locally of finite type and Y affine. Then there exists an immersion $Y \to \mathbf{A}^n_X$ of Y into affine n-space over X. See the slightly more general Morphisms, Lemma 29.39.2.

Now suppose that $f : Y \to X$ is a morphism of algebraic spaces with f locally of finite type and Y an affine scheme. Then it is not true in general that we can find an immersion of Y into affine n-space over X.

A first (nasty) counter example is $Y = \mathrm{Spec}(k)$ and $X = [\mathbf{A}^1_k/\mathbf{Z}]$ where k is a field of characteristic zero and $\mathbf{Z}$ acts on $\mathbf{A}^1_k$ by translation $(n, t) \mapsto t + n$. Namely, for any morphism $Y \to \mathbf{A}^n_X$ over X we can pullback to the covering $\mathbf{A}^1_k$ of X and we get an infinite disjoint union of $\mathbf{A}^1_k$'s mapping into $\mathbf{A}^{n+1}_k$ which is not an immersion.

A second counter example is $Y = \mathbf{A}^1_k \to X = \mathbf{A}^1_k/R$ with $R = \{(t, t)\} \amalg \{(t, -t), t \neq 0\}$. Namely, in this case the morphism $Y \to \mathbf{A}^n_X$ would be given by some regular functions $f_1, \ldots, f_n$ on Y and hence the fibre product of Y with the covering $\mathbf{A}^{n+1}_k \to \mathbf{A}^n_X$ would be the scheme

$$\{(f_1(t), \ldots, f_n(t), t)\} \amalg \{(f_1(t), \ldots, f_n(t), -t), t \neq 0\}$$

with obvious morphism to $\mathbf{A}^{n+1}_k$ which is not an immersion. Note that this gives a counter example with X quasi-separated.

088W **Lemma 110.29.1.** *There exists a finite type morphism of algebraic spaces $Y \to X$ with Y affine and X quasi-separated, such that there does not exist an immersion $Y \to \mathbf{A}^n_X$ over X.*

Proof. See discussion above. □

110.30. Pushforward of quasi-coherent modules

078C In Schemes, Lemma 26.24.1 we proved that f_* transforms quasi-coherent modules into quasi-coherent modules when f is quasi-compact and quasi-separated. Here are some examples to show that these conditions are both necessary.

Suppose that $Y = \mathrm{Spec}(A)$ is an affine scheme and that $X = \coprod_{n \in \mathbf{N}} Y$. We claim that $f_*\mathcal{O}_X$ is not quasi-coherent where $f : X \to Y$ is the obvious morphism. Namely, for $a \in A$ we have

$$f_*\mathcal{O}_X(D(a)) = \prod\nolimits_{n \in \mathbf{N}} A_a$$

Hence, in order for $f_*\mathcal{O}_X$ to be quasi-coherent we would need

$$\prod\nolimits_{n \in \mathbf{N}} A_a = \left(\prod\nolimits_{n \in \mathbf{N}} A\right)_a$$

for all $a \in A$. This isn't true in general, for example if $A = \mathbf{Z}$ and $a = 2$, then $(1, 1/2, 1/4, 1/8, \ldots)$ is an element of the left hand side which is not in the right hand side. Note that f is a non-quasi-compact separated morphism.

Let k be a field. Set

$$A = k[t, z, x_1, x_2, x_3, \ldots]/(tx_1z, t^2x_2^2z, t^3x_3^3z, \ldots)$$

Let $Y = \mathrm{Spec}(A)$. Let $V \subset Y$ be the open subscheme $V = D(x_1) \cup D(x_2) \cup \ldots$. Let X be two copies of Y glued along V. Let $f : X \to Y$ be the obvious morphism. Then we have an exact sequence

$$0 \to f_*\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(1,-1)} j_*\mathcal{O}_V$$

where $j : V \to Y$ is the inclusion morphism. Since

$$A \longrightarrow \prod A_{x_n}$$

is injective (details omitted) we see that $\Gamma(Y, f_*\mathcal{O}_X) = A$. On the other hand, the kernel of the map

$$A_t \longrightarrow \prod A_{tx_n}$$

is nonzero because it contains the element z. Hence $\Gamma(D(t), f_*\mathcal{O}_X)$ is strictly bigger than A_t because it contains $(z, 0)$. Thus we see that $f_*\mathcal{O}_X$ is not quasi-coherent. Note that f is quasi-compact but non-quasi-separated.

078D **Lemma 110.30.1.** *Schemes, Lemma 26.24.1 is sharp in the sense that one can neither drop the assumption of quasi-compactness nor the assumption of quasi-separatedness.*

Proof. See discussion above. □

110.31. A nonfinite module with finite free rank 1 stalks

065J Let $R = \mathbf{Q}[x]$. Set $M = \sum_{n \in \mathbf{N}} \frac{1}{x-n} R$ as a submodule of the fraction field of R. Then M is not finitely generated, but for every prime $\mathfrak{p}$ of R we have $M_\mathfrak{p} \cong R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.

An example of a similar flavor is $R = \mathbf{Z}$ and $M = \sum_{p \text{ prime}} \frac{1}{p}\mathbf{Z} \subset \mathbf{Q}$, which equals the set of fractions $\frac{a}{b}$ with b nonzero and squarefree.

110.32. A noninvertible ideal invertible in stalks

0CBZ Let A be a domain and let $I \subset A$ be a nonzero ideal. Recall that when we say I is invertible, we mean that I is invertible as an A-module. We are going to make an example of this situation where I is not invertible, yet $I_\mathfrak{q} = (f) \subset A_\mathfrak{q}$ is a (nonzero) principal ideal for every prime ideal $\mathfrak{q} \subset A$. In the literature the property that $I_\mathfrak{q}$ is principal for all primes $\mathfrak{q}$ is sometimes expressed by saying "I is a locally principal ideal". We can't use this terminology as our "local" always means "local in the Zariski topology" (or whatever topology we are currently working with).

Let $R = \mathbf{Q}[x]$ and let $M = \sum \frac{1}{x-n} R$ be the module constructed in Section 110.31. Consider the ring[2]

$$A = \mathrm{Sym}^*_R(M)$$

and the ideal $I = MA = \bigoplus_{d \geq 1} \mathrm{Sym}^d_R(M)$. Since M is not finitely generated as an R-module we see that I cannot be generated by finitely many elements as an ideal in A. Since an invertible module is finitely generated, this means that I is not invertible. On the other hand, let $\mathfrak{p} \subset R$ be a prime ideal. By construction $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence

$$A_\mathfrak{p} = \mathrm{Sym}^*_{R_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{Sym}^*_{R_\mathfrak{p}}(R_\mathfrak{p}) = R_\mathfrak{p}[T]$$

[2]The ring A is an example of a non-Noetherian domain whose local rings are Noetherian.

as a graded $R_{\mathfrak{p}}$-algebra. It follows that $I_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ is generated by the nonzerodivisor T. Thus certainly for any prime ideal $\mathfrak{q} \subset A$ we see that $I_{\mathfrak{q}}$ is generated by a single element.

0CC0 **Lemma 110.32.1.** *There exists a domain A and a nonzero ideal $I \subset A$ such that $I_{\mathfrak{q}} \subset A_{\mathfrak{q}}$ is a principal ideal for all primes $\mathfrak{q} \subset A$ but I is not an invertible A-module.*

Proof. See discussion above. □

110.33. A finite flat module which is not projective

052H This is a copy of Algebra, Remark 10.78.4. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_{\mathfrak{m}} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

The morphism $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is also an example of a flat closed immersion which is not open.

05FY **Lemma 110.33.1.** *Strange flat modules.*

(1) *There exists a ring R and a finite flat R-module M which is not projective.*
(2) *There exists a closed immersion which is flat but not open.*

Proof. See discussion above. □

110.34. A projective module which is not locally free

05WG We give two examples. One where the rank is between 0 and 1 and one where the rank is $\aleph_0$.

05WH **Lemma 110.34.1.** *Let R be a ring. Let $I \subset R$ be an ideal generated by a countable collection of idempotents. Then I is projective as an R-module.*

Proof. Say $I = (e_1, e_2, e_3, \ldots)$ with e_n an idempotent of R. After inductively replacing e_{n+1} by $e_n + (1 - e_n)e_{n+1}$ we may assume that $(e_1) \subset (e_2) \subset (e_3) \subset \ldots$ and hence $I = \bigcup_{n \geq 1}(e_n) = \mathrm{colim}_n\, e_n R$. In this case

$$\mathrm{Hom}_R(I, M) = \mathrm{Hom}_R(\mathrm{colim}_n\, e_n R, M) = \lim_n \mathrm{Hom}_R(e_n R, M) = \lim_n e_n M$$

Note that the transition maps $e_{n+1}M \to e_n M$ are given by multiplication by e_n and are surjective. Hence by Algebra, Lemma 10.86.4 the functor $\mathrm{Hom}_R(I, M)$ is exact, i.e., I is a projective R-module. □

Suppose that $P \subset Q$ is an inclusion of R-modules with Q a finite R-module and P locally free, see Algebra, Definition 10.78.1. Suppose that Q can be generated by N elements as an R-module. Then it follows from Algebra, Lemma 10.15.7 that P is finite locally free (with the free parts having rank at most N). And in this case P is a finite R-module, see Algebra, Lemma 10.78.2.

Combining this with the above we see that a non-finitely-generated ideal which is generated by a countable collection of idempotents is projective but not locally free. An explicit example is $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents

$$e_n = (1, 1, \ldots, 1, 0, \ldots)$$

where the sequence of 1's has length n.

05WJ **Lemma 110.34.2.** *There exists a ring R and an ideal I such that I is projective as an R-module but not locally free as an R-module.*

Proof. See above. □

0H8D **Remark 110.34.3.** The example above where $R = \prod_{n\in\mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents $e_n = (1, 1, \ldots, 1, 0, \ldots)$ also gives us an example $M = I$ of a non-finitely generated projective module whose dual is finitely generated and even finite free. Namely, using the strategy in the proof of Lemma 110.34.1 the reader shows that $\mathrm{Hom}_R(I, R) \cong R$.

05WK **Lemma 110.34.4.** *Let K be a field. Let C_i, $i = 1, \ldots, n$ be smooth, projective, geometrically irreducible curves over K. Let $P_i \in C_i(K)$ be a rational point and let $Q_i \in C_i$ be a point such that $[\kappa(Q_i) : K] = 2$. Then $[P_1 \times \ldots \times P_n]$ is nonzero in $\mathrm{CH}_0(U_1 \times_K \ldots \times_K U_n)$ where $U_i = C_i \setminus \{Q_i\}$.*

Proof. There is a degree map $\deg : \mathrm{CH}_0(C_1 \times_K \ldots \times_K C_n) \to \mathbf{Z}$ Because each Q_i has degree 2 over K we see that any zero cycle supported on the "boundary"

$$C_1 \times_K \ldots \times_K C_n \setminus U_1 \times_K \ldots \times_K U_n$$

has degree divisible by 2. □

We can construct another example of a projective but not locally free module using the lemma above as follows. Let C_n, $n = 1, 2, 3, \ldots$ be smooth, projective, geometrically irreducible curves over $\mathbf{Q}$ each with a pair of points $P_n, Q_n \in C_n$ such that $\kappa(P_n) = \mathbf{Q}$ and $\kappa(Q_n)$ is a quadratic extension of $\mathbf{Q}$. Set $U_n = C_n \setminus \{Q_n\}$; this is an affine curve. Let $\mathcal{L}_n$ be the inverse of the ideal sheaf of P_n on U_n. Note that $c_1(\mathcal{L}_n) = [P_n]$ in the group of zero cycles $\mathrm{CH}_0(U_n)$. Set $A_n = \Gamma(U_n, \mathcal{O}_{U_n})$. Let $L_n = \Gamma(U_n, \mathcal{L}_n)$ which is a locally free module of rank 1 over A_n. Set

$$B_n = A_1 \otimes_{\mathbf{Q}} A_2 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

so that $\mathrm{Spec}(B_n) = U_1 \times \ldots \times U_n$ all products over $\mathrm{Spec}(\mathbf{Q})$. For $i \leq n$ we set

$$L_{n,i} = A_1 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} L_i \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

which is a locally free B_n-module of rank 1. Note that this is also the global sections of $\mathrm{pr}_i^*\mathcal{L}_n$. Set

$$B_\infty = \mathrm{colim}_n B_n \quad \text{and} \quad L_{\infty,i} = \mathrm{colim}_n L_{n,i}$$

Finally, set

$$M = \bigoplus\nolimits_{i\geq 1} L_{\infty,i}.$$

This is a direct sum of finite locally free modules, hence projective. We claim that M is not locally free. Namely, suppose that $f \in B_\infty$ is a nonzero function such that M_f is free over $(B_\infty)_f$. Let $e_1, e_2, \ldots$ be a basis. Choose $n \geq 1$ such that $f \in B_n$. Choose $m \geq n + 1$ such that $e_1, \ldots, e_{n+1}$ are in

$$\bigoplus\nolimits_{1\leq i\leq m} L_{m,i}.$$

Because the elements $e_1, \ldots, e_{n+1}$ are part of a basis after a faithfully flat base change we conclude that the Chern classes

$$c_i(\mathrm{pr}_1^*\mathcal{L}_1 \oplus \ldots \oplus \mathrm{pr}_m^*\mathcal{L}_m), \quad i = m, m-1, \ldots, m-n$$

are zero in the chow group of

$$D(f) \subset U_1 \times \ldots \times U_m$$

Since f is the pullback of a function on $U_1 \times \ldots \times U_n$ this implies in particular that

$$c_{m-n}(\mathcal{O}_W^{\oplus n} \oplus \mathrm{pr}_1^*\mathcal{L}_{n+1} \oplus \ldots \oplus \mathrm{pr}_{m-n}^*\mathcal{L}_m) = 0.$$

on the variety

$$W = (C_{n+1} \times \ldots \times C_m)_K$$

over the field $K = \mathbf{Q}(C_1 \times \ldots \times C_n)$. In other words the cycle

$$[(P_{n+1} \times \ldots \times P_m)_K]$$

is zero in the chow group of zero cycles on W. This contradicts Lemma 110.34.4 above because the points Q_i, $n+1 \leq i \leq m$ induce corresponding points Q_i' on $(C_n)_K$ and as $K/\mathbf{Q}$ is geometrically irreducible we have $[\kappa(Q_i') : K] = 2$.

05WL **Lemma 110.34.5.** *There exists a countable ring R and a projective module M which is a direct sum of countably many locally free rank 1 modules such that M is not locally free.*

Proof. See above. □

110.35. Zero dimensional local ring with nonzero flat ideal

05FZ In [**Laz67**] and [**Laz69**] there is an example of a zero dimensional local ring with a nonzero flat ideal. Here is the construction. Let k be a field. Let X_i, Y_i, $i \geq 1$ be variables. Take $R = k[X_i, Y_i]/(X_i - Y_iX_{i+1}, Y_i^2)$. Denote x_i, resp. y_i the image of X_i, resp. Y_i in this ring. Note that

$$x_i = y_ix_{i+1} = y_iy_{i+1}x_{i+2} = y_iy_{i+1}y_{i+2}x_{i+3} = \ldots$$

in this ring. The ring R has only one prime ideal, namely $\mathfrak{m} = (x_i, y_i)$. We claim that the ideal $I = (x_i)$ is flat as an R-module.

Note that the annihilator of x_i in R is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Consider the R-module M generated by elements e_i, $i \geq 1$ and relations $e_i = y_ie_{i+1}$. Then M is flat as it is the colimit $\mathrm{colim}_i R$ of copies of R with transition maps

$$R \xrightarrow{y_1} R \xrightarrow{y_2} R \xrightarrow{y_3} \ldots$$

Note that the annihilator of e_i in M is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Since every element of M, resp. I can be written as fe_i, resp. hx_i for some $f, h \in R$ we see that the map $M \to I$, $e_i \to x_i$ is an isomorphism and I is flat.

05G0 **Lemma 110.35.1.** *There exists a local ring R with a unique prime ideal and a nonzero ideal $I \subset R$ which is a flat R-module*

Proof. See discussion above. □

110.36. An epimorphism of zero-dimensional rings which is not surjective

06RH In [**Laz68**] and [**Laz69**] one can find the following example. Let k be a field. Consider the ring homomorphism

$$k[x_1, x_2, \ldots, z_1, z_2, \ldots]/(x_i^{4^i}, z_i^{4^i}) \longrightarrow k[x_1, x_2, \ldots, y_1, y_2, \ldots]/(x_i^{4^i}, y_i - x_{i+1}y_{i+1}^2)$$

which maps x_i to x_i and z_i to x_iy_i. Note that $y_i^{4^{i+1}}$ is zero in the right hand side but that y_1 is not zero (details omitted). This map is not surjective: we can think of the above as a map of $\mathbf{Z}$-graded algebras by setting $\deg(x_i) = -1$, $\deg(z_i) = 0$,

and $\deg(y_i) = 1$ and then it is clear that y_1 is not in the image. Finally, the map is an epimorphism because

$$y_{i-1} \otimes 1 = x_i y_i^2 \otimes 1 = y_i \otimes x_i y_i = x_i y_i \otimes y_i = 1 \otimes x_i y_i^2.$$

hence the tensor product of the target over the source is isomorphic to the target.

06RI **Lemma 110.36.1.** *There exists an epimorphism of local rings of dimension* 0 *which is not a surjection.*

Proof. See discussion above. □

110.37. Finite type, not finitely presented, flat at prime

05G1 Let k be a field. Consider the local ring $A_0 = k[x, y]_{(x,y)}$. Denote $\mathfrak{p}_{0,n} = (y + x^n + x^{2n+1})$. This is a prime ideal. Set

$$A = A_0[z_1, z_2, z_3, \ldots]/(z_n z_m, z_n(y + x^n + x^{2n+1}))$$

Note that $A \to A_0$ is a surjection whose kernel is an ideal of square zero. Hence A is also a local ring and the prime ideals of A are in one-to-one correspondence with the prime ideals of A_0. Denote $\mathfrak{p}_n$ the prime ideal of A corresponding to $\mathfrak{p}_{0,n}$. Observe that $\mathfrak{p}_n$ is the annihilator of z_n in A. Let

$$C = A[z]/(xz^2 + z + y)[\frac{1}{2zx + 1}]$$

Note that $A \to C$ is an étale ring map, see Algebra, Example 10.137.7. Let $\mathfrak{q} \subset C$ be the maximal ideal generated by x, y, z and all z_n. As $A \to C$ is flat we see that the annihilator of z_n in C is $\mathfrak{p}_n C$. We compute

$$\begin{aligned} C/\mathfrak{p}_n C &= A_0[z]/(xz^2 + z + y, y + x^n + x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(xz^2 + z - x^n - x^{2n+1})[1/(2zx + 1)] \\ &= k[x]_{(x)}[z]/(z - x^n) \times k[x]_{(x)}[z]/(xz + x^{n+1} + 1)[1/(2zx + 1)] \\ &= k[x]_{(x)} \times k(x) \end{aligned}$$

because $(z - x^n)(xz + x^{n+1} + 1) = xz^2 + z - x^n - x^{2n+1}$. Hence we see that $\mathfrak{p}_n C = \mathfrak{r}_n \cap \mathfrak{q}_n$ with $\mathfrak{r}_n = \mathfrak{p}_n C + (z - x^n)C$ and $\mathfrak{q}_n = \mathfrak{p}_n C + (xz + x^{n+1} + 1)C$. Since $\mathfrak{q}_n + \mathfrak{r}_n = C$ we also get $\mathfrak{p}_n C = \mathfrak{r}_n \mathfrak{q}_n$. It follows that $\mathfrak{q}_n$ is the annihilator of $\xi_n = (z - x^n)z_n$. Observe that on the one hand $\mathfrak{r}_n \subset \mathfrak{q}$, and on the other hand $\mathfrak{q}_n + \mathfrak{q} = C$. This follows for example because $\mathfrak{q}_n$ is a maximal ideal of C distinct from $\mathfrak{q}$. Similarly we have $\mathfrak{q}_n + \mathfrak{q}_m = C$ for $n \neq m$. At this point we let

$$B = \mathrm{Im}(C \longrightarrow C_{\mathfrak{q}})$$

We observe that the elements ξ_n map to zero in B as $xz + x^{n+1} + 1$ is not in $\mathfrak{q}$. Denote $\mathfrak{q}' \subset B$ the image of $\mathfrak{q}$. By construction B is a finite type A-algebra, with $B_{\mathfrak{q}'} \cong C_{\mathfrak{q}}$. In particular we see that $B_{\mathfrak{q}'}$ is flat over A.

We claim there does not exist an element $g' \in B$, $g' \notin \mathfrak{q}'$ such that $B_{g'}$ is of finite presentation over A. We sketch a proof of this claim. Choose an element $g \in C$ which maps to $g' \in B$. Consider the map $C_g \to B_{g'}$. By Algebra, Lemma 10.6.3 we see that B_g is finitely presented over A if and only if the kernel of $C_g \to B_{g'}$ is finitely generated. But the element $g \in C$ is not contained in $\mathfrak{q}$, hence maps to a nonzero element of $A_0[z]/(xz^2 + z + y)$. Hence g can only be contained in finitely many of the prime ideals $\mathfrak{q}_n$, because the primes $(y + x^n + x^{2n+1}, xz + x^{n+1} + 1)$

are an infinite collection of codimension 1 points of the 2-dimensional irreducible Noetherian space $\mathrm{Spec}(k[x,y,z]/(xz^2+z+y))$. The map

$$\bigoplus\nolimits_{g \notin \mathfrak{q}_n} C/\mathfrak{q}_n \longrightarrow C_g, \quad (c_n) \longrightarrow \sum c_n \xi_n$$

is injective and its image is the kernel of $C_g \to B_{g'}$. We omit the proof of this statement. (Hint: Write $A = A_0 \oplus I$ as an A_0-module where I is the kernel of $A \to A_0$. Similarly, write $C = C_0 \oplus IC$. Write $IC = \bigoplus Cz_n \cong \bigoplus (C/\mathfrak{r}_n \oplus C/\mathfrak{q}_n)$ and study the effect of multiplication by g on the summands.) This concludes the sketch of the proof of the claim. This also proves that $B_{g'}$ is not flat over A for any g' as above. Namely, if it were flat, then the annihilator of the image of z_n in $B_{g'}$ would be $\mathfrak{p}_n B_{g'}$, and would not contain $z - x^n$.

As a consequence we can answer (negatively) a question posed in [**GR71**, Part I, Remarques (3.4.7) (v)]. Here is a precise statement.

05G2 **Lemma 110.37.1.** *There exists a local ring A, a finite type ring map $A \to B$ and a prime $\mathfrak{q}$ lying over $\mathfrak{m}_A$ such that $B_\mathfrak{q}$ is flat over A, and for any element $g \in B$, $g \notin \mathfrak{q}$ the ring B_g is neither finitely presented over A nor flat over A.*

Proof. See discussion above. □

110.38. Finite type, flat and not of finite presentation

05LB In this section we give some examples of ring maps and morphisms which are of finite type and flat but not of finite presentation.

Let R be a ring which has an ideal I such that R/I is a finite flat module but not projective, see Section 110.33 for an explicit example. Note that this means that I is not finitely generated, see Algebra, Lemma 10.108.5. Note that $I = I^2$, see Algebra, Lemma 10.108.2. The base ring in our examples will be R and correspondingly the base scheme $S = \mathrm{Spec}(R)$.

Consider the ring map $R \to R \oplus R/I\epsilon$ where $\epsilon^2 = 0$ by convention. This is a finite, flat ring map which is not of finite presentation. All the fibre rings are complete intersections and geometrically irreducible.

Let $A = R[x,y]/(xy, ay; a \in I)$. Note that as an R-module we have $A = \bigoplus_{i \geq 0} Ry^i \oplus \bigoplus_{j>0} R/Ix^j$. Hence $R \to A$ is a flat finite type ring map which is not of finite presentation. Each fibre ring is isomorphic to either $\kappa(\mathfrak{p})[x,y]/(xy)$ or $\kappa(\mathfrak{p})[x]$.

We can turn the previous example into a projective morphism by taking $B = R[X_0, X_1, X_2]/(X_1X_2, aX_2; a \in I)$. In this case $X = \mathrm{Proj}(B) \to S$ is a proper flat morphism which is not of finite presentation such that for each $s \in S$ the fibre X_s is isomorphic either to $\mathbf{P}^1_s$ or to the closed subscheme of $\mathbf{P}^2_s$ defined by the vanishing of X_1X_2 (this is a projective nodal curve of arithmetic genus 0).

Let $M = R \oplus R \oplus R/I$. Set $B = \mathrm{Sym}_R(M)$ the symmetric algebra on M. Set $X = \mathrm{Proj}(B)$. Then $X \to S$ is a proper flat morphism, not of finite presentation such that for $s \in S$ the geometric fibre is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$. In particular these fibres are smooth and geometrically irreducible.

05LC **Lemma 110.38.1.** *There exist examples of*

(1) *a flat finite type ring map with geometrically irreducible complete intersection fibre rings which is not of finite presentation,*

(2) *a flat finite type ring map with geometrically connected, geometrically reduced, dimension 1, complete intersection fibre rings which is not of finite presentation,*
(3) *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or to the vanishing locus of X_1X_2 in $\mathbf{P}^2_s$ which is not of finite presentation, and*
(4) *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$ which is not of finite presentation.*

Proof. See discussion above. □

110.39. Topology of a finite type ring map

05JH Let $A \to B$ be a local map of local domains. If A is Noetherian, $A \to B$ is essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite then there exists a prime $\mathfrak{q} \subset B$, $\mathfrak{q} \neq \mathfrak{m}_B$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. See More on Morphisms, Lemma 37.52.6.

In this section we give an example that shows this result is false A is no longer Noetherian. Namely, let k be a field and set

$$A = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k, a_i \in k((y)) \text{ for } i \geq 1\}$$

and

$$C = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k[y], a_i \in k((y)) \text{ for } i \geq 1\}.$$

The inclusion $A \to C$ is of finite type as C is generated by y over A. We claim that A is a local ring with maximal ideal $\mathfrak{m} = \{a_1x + a_2x^2 + \ldots \in A\}$ and no prime ideals besides (0) and $\mathfrak{m}$. Namely, an element $f = a_0 + a_1x + a_2x^2 + \ldots$ of A is invertible as soon as $a_0 \neq 0$. If $\mathfrak{q} \subset A$ is a nonzero prime ideal, and $f = a_ix^i + \ldots \in \mathfrak{q}$, then using properties of power series one sees that for any $g \in k((y))$ the element $g^{i+1}x^{i+1} \in \mathfrak{q}$, i.e., $gx \in \mathfrak{q}$. This proves that $\mathfrak{q} = \mathfrak{m}$.

As to the spectrum of the ring C, arguing in the same way as above we see that any nonzero prime ideal contains the prime $\mathfrak{p} = \{a_1x + a_2x^2 + \ldots \in C\}$ which lies over $\mathfrak{m}$. Thus the only prime of C which lies over (0) is (0). Set $\mathfrak{m}_C = yC + \mathfrak{p}$ and $B = C_{\mathfrak{m}_C}$. Then $A \to B$ is the desired example.

05JI **Lemma 110.39.1.** *There exists a local homomorphism $A \to B$ of local domains which is essentially of finite type and such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is finite such that for every prime $\mathfrak{q} \neq \mathfrak{m}_B$ of B the ring map $A \to B/\mathfrak{q}$ is not the localization of a quasi-finite ring map.*

Proof. See the discussion above. □

110.40. Pure not universally pure

05JJ Let k be a field. Let

$$R = k[[x, xy, xy^2, \ldots]] \subset k[[x, y]].$$

In other words, a power series $f \in k[[x, y]]$ is in R if and only if $f(0, y)$ is a constant. In particular $R[1/x] = k[[x, y]][1/x]$ and R/xR is a local ring with a maximal ideal whose square is zero. Denote $R[y] \subset k[[x, y]]$ the set of power series $f \in k[[x, y]]$ such that $f(0, y)$ is a polynomial in y. Then $R \to R[y]$ is a finite type but not finitely presented ring map which induces an isomorphism after inverting x. Also

there is a surjection $R[y]/xR[y] \to k[y]$ whose kernel has square zero. Consider the finitely presented ring map $R \to S = R[t]/(xt - xy)$. Again $R[1/x] \to S[1/x]$ is an isomorphism and in this case $S/xS \cong (R/xR)[t]/(xy)$ maps onto $k[t]$ with nilpotent kernel. There is a surjection $S \to R[y]$, $t \longmapsto y$ which induces an isomorphism on inverting x and a surjection with nilpotent kernel modulo x. Hence the kernel of $S \to R[y]$ is locally nilpotent. In particular $S \to R[y]$ induces a universal homeomorphism on spectra.

First we claim that S is an S-module which is relatively pure over R. Since on inverting x we obtain an isomorphism we only need to check this at the maximal ideal $\mathfrak{m} \subset R$. Since R is complete with respect to its maximal ideal it is henselian hence we need only check that every prime $\mathfrak{p} \subset R$, $\mathfrak{p} \neq \mathfrak{m}$, the unique prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ satisfies $\mathfrak{m}S + \mathfrak{q} \neq S$. Since $\mathfrak{p} \neq \mathfrak{m}$ it corresponds to a unique prime ideal of $k[[x, y]][1/x]$. Hence either $\mathfrak{p} = (0)$ or $\mathfrak{p} = (f)$ for some irreducible element $f \in k[[x, y]]$ which is not associated to x (here we use that $k[[x, y]]$ is a UFD – insert future reference here). In the first case $\mathfrak{q} = (0)$ and the result is clear. In the second case we may multiply f by a unit so that $f \in R[y]$ (Weierstrass preparation; details omitted). Then it is easy to see that $R[y]/fR[y] \cong k[[x, y]]/(f)$ hence f defines a prime ideal of $R[y]$ and $\mathfrak{m}R[y] + fR[y] \neq R[y]$. Since $S \to R[y]$ induces a universal homeomorphism on spectra we deduce the desired result for S also.

Second we claim that S is not universally relatively pure over R. Namely, to see this it suffices to find a valuation ring $\mathcal{O}$ and a local ring map $R \to \mathcal{O}$ such that $\mathrm{Spec}(R[y] \otimes_R \mathcal{O}) \to \mathrm{Spec}(\mathcal{O})$ does not hit the closed point of $\mathrm{Spec}(\mathcal{O})$. Equivalently, we have to find $\varphi : R \to \mathcal{O}$ such that $\varphi(x) \neq 0$ and $v(\varphi(x)) > v(\varphi(xy))$ where v is the valuation of $\mathcal{O}$. (Because this means that the valuation of y is negative.) To do this consider the ring map

$$R \longrightarrow \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k[y^{-1}], a_i \in k((y))\}$$

defined in the obvious way. We can find a valuation ring $\mathcal{O}$ dominating the localization of the right hand side at the maximal ideal (y^{-1}, x) and we win.

05JK **Lemma 110.40.1.** *There exists a morphism of affine schemes of finite presentation $X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation such that $\mathcal{F}$ is pure relative to S, but not universally pure relative to S.*

Proof. See discussion above. □

110.41. A formally smooth non-flat ring map

057V Let k be a field. Consider the k-algebra $k[\mathbf{Q}]$. This is the k-algebra with basis $x_\alpha, \alpha \in \mathbf{Q}$ and multiplication determined by $x_\alpha x_\beta = x_{\alpha+\beta}$. (In particular $x_0 = 1$.) Consider the k-algebra homomorphism

$$k[\mathbf{Q}] \longrightarrow k, \quad x_\alpha \longmapsto 1.$$

It is surjective with kernel J generated by the elements $x_\alpha - 1$. Let us compute J/J^2. Note that multiplication by x_α on J/J^2 is the identity map. Denote z_α the class of $x_\alpha - 1$ modulo J^2. These classes generate J/J^2. Since

$$(x_\alpha - 1)(x_\beta - 1) = x_{\alpha+\beta} - x_\alpha - x_\beta + 1 = (x_{\alpha+\beta} - 1) - (x_\alpha - 1) - (x_\beta - 1)$$

we see that $z_{\alpha+\beta} = z_\alpha + z_\beta$ in J/J^2. A general element of J/J^2 is of the form $\sum \lambda_\alpha z_\alpha$ with $\lambda_\alpha \in k$ (only finitely many nonzero). Note that if the characteristic

of k is $p > 0$ then

$$0 = pz_{\alpha/p} = z_{\alpha/p} + \ldots + z_{\alpha/p} = z_\alpha$$

and we see that $J/J^2 = 0$. If the characteristic of k is zero, then

$$J/J^2 = \mathbf{Q} \otimes_{\mathbf{Z}} k \cong k$$

(details omitted) is not zero.

We claim that $k[\mathbf{Q}] \to k$ is a formally smooth ring map if the characteristic of k is positive. Namely, suppose given a solid commutative diagram

$$\begin{array}{ccc} k & \longrightarrow & A \\ \uparrow & \nearrow & \uparrow \\ k[\mathbf{Q}] & \xrightarrow{\varphi} & A' \end{array}$$

with $A' \to A$ a surjection whose kernel I has square zero. To show that $k[\mathbf{Q}] \to k$ is formally smooth we have to prove that φ factors through k. Since $\varphi(x_\alpha - 1)$ maps to zero in A we see that φ induces a map $\overline{\varphi} : J/J^2 \to I$ whose vanishing is the obstruction to the desired factorization. Since $J/J^2 = 0$ if the characteristic is $p > 0$ we get the result we want, i.e., $k[\mathbf{Q}] \to k$ is formally smooth in this case. Finally, this ring map is not flat, for example as the nonzerodivisor $x_2 - 1$ is mapped to zero.

057W **Lemma 110.41.1.** *There exists a formally smooth ring map which is not flat.*

Proof. See discussion above. □

110.42. A formally étale non-flat ring map

060H In this section we give a counterexample to the final sentence in [**DG67**, 0, Example 19.10.3(i)] (this was not one of the items caught in their later errata lists). Consider $A \to A/J$ for a local ring A and a nonzero proper ideal J such that $J^2 = J$ (so J isn't finitely generated); the valuation ring of an algebraically closed non-archimedean field with J its maximal ideal is a source of such (A, J). These non-flat quotient maps are formally étale. Namely, suppose given a commutative diagram

$$\begin{array}{ccc} A/J & \longrightarrow & R/I \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & R \end{array}$$

where I is an ideal of the ring R with $I^2 = 0$. Then $A \to R$ factors uniquely through A/J because

$$\varphi(J) = \varphi(J^2) \subset (\varphi(J)A)^2 \subset I^2 = 0.$$

Hence this also provides a counterexample to the formally étale case of the "structure theorem" for locally finite type and formally étale morphisms in [**DG67**, IV, Theorem 18.4.6(i)] (but not a counterexample to part (ii), which is what people actually use in practice). The error in the proof of the latter is that the very last step of the proof is to invoke the incorrect [**DG67**, 0, Example 19.3.10(i)], which is how the counterexample just mentioned creeps in.

060I **Lemma 110.42.1.** *There exist formally étale nonflat ring maps.*

Proof. See discussion above. □

110.43. A formally étale ring map with nontrivial cotangent complex

06E5 Let k be a field. Consider the ring

$$R = k[\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}]/(x_1y_1, x_{nm}^m - x_n, y_{nm}^m - y_n)$$

Let A be the localization at the maximal ideal generated by all x_n, y_n and denote $J \subset A$ the maximal ideal. Set $B = A/J$. By construction $J^2 = J$ and hence $A \to B$ is formally étale (see Section 110.42). We claim that the element $x_1 \otimes y_1$ is a nonzero element in the kernel of

$$J \otimes_A J \longrightarrow J.$$

Namely, (A, J) is the colimit of the localizations (A_n, J_n) of the rings

$$R_n = k[x_n, y_n]/(x_n^n y_n^n)$$

at their corresponding maximal ideals. Then $x_1 \otimes y_1$ corresponds to the element $x_n^n \otimes y_n^n \in J_n \otimes_{A_n} J_n$ and is nonzero (by an explicit computation which we omit). Since $\otimes$ commutes with colimits we conclude. By [**Ill72**, III Section 3.3] we see that J is not weakly regular. Hence by [**Ill72**, III Proposition 3.3.3] we see that the cotangent complex $L_{B/A}$ is not zero. In fact, we can be more precise. We have $H_0(L_{B/A}) = \Omega_{B/A}$ and $H_1(L_{B/A}) = 0$ because $J/J^2 = 0$. But from the five-term exact sequence of Quillen's fundamental spectral sequence (see Cotangent, Remark 92.12.5 or [**Rei**, Corollary 8.2.6]) and the nonvanishing of $\mathrm{Tor}_2^A(B, B) = \mathrm{Ker}(J \otimes_A J \to J)$ we conclude that $H_2(L_{B/A})$ is nonzero.

06E6 **Lemma 110.43.1.** *There exists a formally étale surjective ring map $A \to B$ with $L_{B/A}$ not equal to zero.*

Proof. See discussion above. □

110.44. Flat and formally unramified is not formally étale

0G64 In More on Morphisms, Lemma 37.8.7 it is shown that an unramified flat morphism of schemes $X \to S$ is formally étale. The goal of this section is to give two examples that illustrate that we cannot replace 'unramified' by 'formally unramified'. The first example exploits special properties of perfect rings, while the second example shows the result fails even for maps of Noetherian regular rings.

0G65 **Lemma 110.44.1.** *Let $A = \mathbb{F}_p[T]$ be the polynomial ring in one variable over $\mathbb{F}_p$. Let A_{perf} denote the perfect closure of A. Then $A \to A_{perf}$ is flat and formally unramified, but not formally étale.*

Proof. Note that under the Frobenius map $F_A : A \to A$, the target copy of A is a free-module over the domain with basis $\{1, T, \ldots, T^{p-1}\}$. Thus, F_A is faithfully flat, and consequently, so is $A \to A_{perf}$ since it is a colimit of faithfully flat maps. Since A_{perf} is a perfect ring, the relative Frobenius $F_{A_{perf}/A}$ is a surjection. In other words, $A_{perf} = A[A_{perf}^p]$, which readily implies $\Omega_{A_{perf}/A} = 0$. Then $A \to A_{perf}$ is formally unramified by More on Morphisms, Lemma 37.6.7

It suffices to show that $A \to A_{perf}$ is not formally smooth. Note that since A is a smooth $\mathbb{F}_p$-algebra, the cotangent complex $L_{A/\mathbb{F}_P} \simeq \Omega_{A/\mathbb{F}_p}[0]$ is concentrated in degree 0, see Cotangent, Lemma 92.9.1. Moreover, $L_{A_{perf}/\mathbb{F}_p} = 0$ in $D(A_{perf})$

by Cotangent, Lemma 92.10.3. Consider the distinguished triangle of cotangent complexes

$$L_{A/\mathbb{F}_p} \otimes_A A_{perf} \to L_{A_{perf}/\mathbb{F}_p} \to L_{A_{perf}/A} \to (L_{A/\mathbb{F}_p} \otimes_A A_{perf})[1]$$

in $D(A_{perf})$, see Cotangent, Section 92.7. We find $L_{A_{perf}/A} = \Omega_{A/\mathbb{F}_p} \otimes_A A_{perf}[1]$, that is, $L_{A_{perf}/A}$ is equal to a free rank 1 A_{perf} module placed in degree -1. Thus $A \to A_{perf}$ is not formally smooth by More on Morphisms, Lemma 37.13.5 and Cotangent, Lemma 92.11.3. □

The next example also involves rings of prime characteristic, but is perhaps a little more surprising. The drawback is that it requires more knowledge of characteristic p phenomena than the previous example. Recall that we say a ring A of prime characteristic is F-finite if the Frobenius map on A is finite.

0G66 **Lemma 110.44.2.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring of prime characteristic $p > 0$ such that $[\kappa : \kappa^p] < \infty$. Then the canonical map $A \to A^\wedge$ to the completion of A is flat and formally unramified. However, if A is regular but not excellent, then this map is not formally étale.*

Proof. Flatness of the completion is Algebra, Lemma 10.97.2. To show that the map is formally unramified, it suffices to show that $\Omega_{A^\wedge/A} = 0$, see Algebra, Lemma 10.148.3.

We sketch a proof. Choose $x_1, \ldots, x_r \in A$ which map to a p-basis $\overline{x}_1, \ldots, \overline{x}_r$ of κ, i.e., such that κ is minimally generated by $\overline{x}_i$ over κ^p. Choose a minimal set of generators $y_1, \ldots, y_s$ of $\mathfrak{m}$. For each n the elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ generate $A/\mathfrak{m}^n$ over $(A/\mathfrak{m}^n)^p$ by Frobenius. Some details omitted. We conclude that $F : A^\wedge \to A^\wedge$ is finite. Hence $\Omega_{A^\wedge/A}$ is a finite $A^\wedge$-module. On the other hand, for any $a \in A^\wedge$ and n we can find $a_0 \in A$ such that $a - a_0 \in \mathfrak{m}^n A^\wedge$. We conclude that $\mathrm{d}(a) \in \bigcap \mathfrak{m}^n \Omega_{A^\wedge/A}$ which implies that $\mathrm{d}(a)$ is zero by Algebra, Lemma 10.51.4. Thus $\Omega_{A^\wedge/A} = 0$.

Suppose A is regular. Then, using the Cohen structure theorem $x_1, \ldots, x_r, y_1, \ldots, y_s$ is a p-basis for the ring $A^\wedge$, i.e., we have

$$A^\wedge = \bigoplus\nolimits_{I,J} (A^\wedge)^p x_1^{i_1} \ldots x_r^{i_r} y_1^{j_1} \ldots y_s^{j_s}$$

with $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ and $0 \leq i_a, j_b \leq p-1$. Details omitted. In particular, we see that $\Omega_{A^\wedge}$ is a free $A^\wedge$-module with basis $\mathrm{d}(x_1), \ldots, \mathrm{d}(x_r), \mathrm{d}(y_1), \ldots, \mathrm{d}(y_s)$.

Now if $A \to A^\wedge$ is formally étale or even just formally smooth, then we see that $NL_{A^\wedge/A}$ has vanishing cohomology in degrees $-1, 0$ by Algebra, Proposition 10.138.8. It follows from the Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) for the ring maps $\mathbf{F}_p \to A \to A^\wedge$ that we get an isomorphism $\Omega_A \otimes_A A^\wedge \cong \Omega_{A^\wedge}$. Hence we find that Ω_A is free on $\mathrm{d}(x_1), \ldots, \mathrm{d}(x_r), \mathrm{d}(y_1), \ldots, \mathrm{d}(y_s)$. Looking at fraction fields and using that A is normal we conclude that $a \in A$ is a pth power if and only if its image in $A^\wedge$ is a pth power (details omitted; use Algebra, Lemma 10.158.2). A second consequence is that the operators $\partial/\partial x_a$ and $\partial/\partial y_b$ are defined on A.

We will show that the above lead to the conclusion that A is finite over A^p with p-basis $x_1, \ldots, x_r, y_1, \ldots, y_s$. This will contradict the non-excellency of A by a result

of Kunz, see [**Kun76**, Corollary 2.6]. Namely, say $a \in A$ and write

$$a = \sum\nolimits_{I,J} (a_{I,J})^p x_1^{i_1} \ldots x_r^{i_r} y_1^{j_1} \ldots y_s^{j_s}$$

with $a_{I,J} \in A^\wedge$. To finish the proof it suffices to show that $a_{I,J} \in A$. Applying the operator

$$(\partial/\partial x_1)^{p-1} \ldots (\partial/\partial x_r)^{p-1} (\partial/\partial y_1)^{p-1} \ldots (\partial/\partial y_s)^{p-1}$$

to both sides we conclude that $a_{I,J}^p \in A$ where $I = (p-1, \ldots, p-1)$ and $J = (p-1, \ldots, p-1)$. By our remark above, this also implies $a_{I,J} \in A$. After replacing a by $a' = a - a_{I,J}^p x^I y^J$ we can use a 1-order lower differential operators to get another coefficient $a_{I,J}$ to be in A. Etc. □

0G67 **Remark 110.44.3.** Non-excellent regular rings whose residue fields have a finite p-basis can be constructed even in the function field of $\mathbb{P}^2_k$, over a characteristic p field $k = \overline{k}$. See [**DS18**, §4.1].

The proof of Lemma 110.44.2 actually shows a little more.

0G68 **Lemma 110.44.4.** *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of characteristic $p > 0$. Suppose $[\kappa : \kappa^p] < \infty$. Then A is excellent if and only if $A \to A^\wedge$ is formally étale.*

Proof. The backward implication follows from Lemma 110.44.2. For the forward implication, note that we already know from Lemma 110.44.2 that $A \to A^\wedge$ is formally unramified or equivalently that $\Omega_{A^\wedge/A}$ is zero. Thus, it suffices to show that the completion map is formally smooth when A is excellent. By Néron-Popescu desingularization $A \to A^\wedge$ can be written as a filtered colimit of smooth A-algebras (Smoothing Ring Maps, Theorem 16.12.1). Hence $NL_{A^\wedge/A}$ has vanishing cohomology in degree -1. Thus $A \to A^\wedge$ is formally smooth by Algebra, Proposition 10.138.8. □

110.45. Ideals generated by sets of idempotents and localization

04QK Let R be a ring. Consider the ring

$$B(R) = R[x_n; n \in \mathbf{Z}]/(x_n(x_n - 1), x_n x_m; n \neq m)$$

It is easy to show that every prime $\mathfrak{q} \subset B(R)$ is either of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n; n \in \mathbf{Z})$$

or of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n - 1) + (x_m; n \neq m, m \in \mathbf{Z}).$$

Hence we see that

$$\mathrm{Spec}(B(R)) = \mathrm{Spec}(R) \amalg \coprod\nolimits_{n \in \mathbf{Z}} \mathrm{Spec}(R)$$

where the topology is not just the disjoint union topology. It has the following properties: Each of the copies indexed by $n \in \mathbf{Z}$ is an open subscheme, namely it is the standard open $D(x_n)$. The "central" copy of $\mathrm{Spec}(R)$ is in the closure of the union of any infinitely many of the other copies of $\mathrm{Spec}(R)$. Note that this last copy of $\mathrm{Spec}(R)$ is cut out by the ideal $(x_n, n \in \mathbf{Z})$ which is generated by the idempotents x_n. Hence we see that if $\mathrm{Spec}(R)$ is connected, then the decomposition above is exactly the decomposition of $\mathrm{Spec}(B(R))$ into connected components.

Next, let $A = \mathbf{C}[x, y]/((y - x^2 + 1)(y + x^2 - 1))$. The spectrum of A consists of two irreducible components $C_1 = \mathrm{Spec}(A_1)$, $C_2 = \mathrm{Spec}(A_2)$ with $A_1 = \mathbf{C}[x, y]/(y -$

$x^2 + 1)$ and $A_2 = \mathbf{C}[x, y]/(y + x^2 - 1)$. Note that these are parametrized by $(x, y) = (t, t^2-1)$ and $(x, y) = (t, -t^2+1)$ which meet in $P = (-1, 0)$ and $Q = (1, 0)$. We can make a twisted version of $B(A)$ where we glue $B(A_1)$ to $B(A_2)$ in the following way: Above P we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_n \in B(A_2) \otimes \kappa(P)$, but above Q we let $x_n \in B(A_1) \otimes \kappa(Q)$ correspond to $x_{n+1} \in B(A_2) \otimes \kappa(Q)$. Let $B^{twist}(A)$ denote the resulting A-algebra. Details omitted. By construction $B^{twist}(A)$ is Zariski locally over A isomorphic to the untwisted version. Namely, this happens over both the principal open $\mathrm{Spec}(A) \setminus \{P\}$ and the principal open $\mathrm{Spec}(A) \setminus \{Q\}$. However, our choice of glueing produces enough "monodromy" such that $\mathrm{Spec}(B^{twist}(A))$ is connected (details omitted). Finally, there is a central copy of $\mathrm{Spec}(A) \to \mathrm{Spec}(B^{twist}(A))$ which gives a closed subscheme whose ideal is Zariski locally on $B^{twist}(A)$ cut out by ideals generated by idempotents, but not globally (as $B^{twist}(A)$ has no nontrivial idempotents).

04QL **Lemma 110.45.1.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ and a closed subscheme $T \subset X$ such that T is Zariski locally on X cut out by ideals generated by idempotents, but T is not cut out by an ideal generated by idempotents.*

Proof. See above. □

110.46. A ring map which identifies local rings which is not ind-étale

09AN Note that the ring map $R \to B(R)$ constructed in Section 110.45 is a colimit of finite products of copies of R. Hence $R \to B(R)$ is ind-Zariski, see Pro-étale Cohomology, Definition 61.4.1. Next, consider the ring map $A \to B^{twist}(A)$ constructed in Section 110.45. Since this ring map is Zariski locally on $\mathrm{Spec}(A)$ isomorphic to an ind-Zariski ring map $R \to B(R)$ we conclude that it identifies local rings (see Pro-étale Cohomology, Lemma 61.4.6). The discussion in Section 110.45 shows there is a section $B^{twist}(A) \to A$ whose kernel is not generated by idempotents. Now, if $A \to B^{twist}(A)$ were ind-étale, i.e., $B^{twist}(A) = \mathrm{colim}\, A_i$ with $A \to A_i$ étale, then the kernel of $A_i \to A$ would be generated by an idempotent (Algebra, Lemmas 10.143.8 and 10.143.9). This would contradict the result mentioned above.

09AP **Lemma 110.46.1.** *There is a ring map $A \to B$ which identifies local rings but which is not ind-étale. A fortiori it is not ind-Zariski.*

Proof. See discussion above. □

110.47. Non flasque quasi-coherent sheaf associated to injective module

0273 For more examples of this type see [**BGI71**, Exposé II, Appendix I] where Illusie explains some examples due to Verdier.

Consider the affine scheme $X = \mathrm{Spec}(A)$ where

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n)$$

is the ring from Properties, Example 28.25.2. Set $I = (x) \subset A$. Consider the quasi-compact open $U = D(x)$ of X. We have seen in loc. cit. that there is a section $s \in \mathcal{O}_X(U)$ which does not come from an A-module map $I^n \to A$ for any $n \geq 0$.

Let $\alpha : A \to J$ be the embedding of A into an injective A-module. Let $Q = J/\alpha(A)$ and denote $\beta : J \to Q$ the quotient map. We claim that the map

$$\Gamma(X, \widetilde{J}) \longrightarrow \Gamma(U, \widetilde{J})$$

is not surjective. Namely, we claim that $\alpha(s)$ is not in the image. To see this, we argue by contradiction. So assume that $x \in J$ is an element which restricts to $\alpha(s)$ over U. Then $\beta(x) \in Q$ is an element which restricts to 0 over U. Hence we know that $I^n\beta(x) = 0$ for some n, see Properties, Lemma 28.25.1. This implies that we get a morphism $\varphi : I^n \to A$, $h \mapsto \alpha^{-1}(hx)$. It is easy to see that this morphism φ gives rise to the section s via the map of Properties, Lemma 28.25.1 which is a contradiction.

0274 **Lemma 110.47.1.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ and an injective A-module J such that $\widetilde{J}$ is not a flasque sheaf on X. Even the restriction $\Gamma(X, \widetilde{J}) \to \Gamma(U, \widetilde{J})$ with U a standard open need not be surjective.*

Proof. See above. □

In fact, we can use a similar construction to get an example of an injective module whose associated quasi-coherent sheaf has nonzero cohomology over a quasi-compact open. Namely, we start with the ring

$$A = k[x, y, w_1, u_1, w_2, u_2, \ldots]/(x^n w_n, y^n u_n, u_n^2, w_n^2)$$

where k is a field. Choose an injective map $A \to I$ where I is an injective A-module. We claim that the element $1/xy$ in $A_{xy} \subset I_{xy}$ is not in the image of $I_x \oplus I_y \to I_{xy}$. Arguing by contradiction, suppose that

$$\frac{1}{xy} = \frac{i}{x^n} + \frac{j}{y^n}$$

for some $n \geq 1$ and $i, j \in I$. Clearing denominators we obtain

$$(xy)^{n+m-1} = x^m y^{n+m} i + x^{n+m} y^m j$$

for some $m \geq 0$. Multiplying with $u_{n+m}w_{n+m}$ we see that $u_{n+m}w_{n+m}(xy)^{n+m-1} = 0$ in A which is the desired contradiction. Let $U = D(x) \cup D(y) \subset X = \mathrm{Spec}(A)$. For any A-module M we have an exact sequence

$$0 \to H^0(U, \widetilde{M}) \to M_x \oplus M_y \to M_{xy} \to H^1(U, \widetilde{M}) \to 0$$

by Mayer-Vietoris. We conclude that $H^1(U, \widetilde{I})$ is nonzero.

0CRZ **Lemma 110.47.2.** *There exists an affine scheme $X = \mathrm{Spec}(A)$ whose underlying topological space is Noetherian and an injective A-module I such that $\widetilde{I}$ has nonvanishing H^1 on some quasi-compact open U of X.*

Proof. See above. Note that $\mathrm{Spec}(A) = \mathrm{Spec}(k[x, y])$ as topological spaces. □

110.48. A non-separated flat group scheme

06E7 Every group scheme over a field is separated, see Groupoids, Lemma 39.7.3. This is not true for group schemes over a base.

Let k be a field. Let $S = \mathrm{Spec}(k[x]) = \mathbf{A}^1_k$. Let G be the affine line with 0 doubled (see Schemes, Example 26.14.3) seen as a scheme over S. Thus a fibre of $G \to S$ is either a singleton or a set with two elements (one in U and one in V). Thus we can endow these fibres with the structure of a group (by letting the element in U be the zero of the group structure). More precisely, G has two opens U, V which map isomorphically to S such that $U \cap V$ is mapped isomorphically to $S \setminus \{0\}$. Then

$$G \times_S G = U \times_S U \cup V \times_S U \cup U \times_S V \cup V \times_S V$$

where each piece is isomorphic to S. Hence we can define a multiplication $m : G \times_S G \to G$ as the unique S-morphism which maps the first and the last piece into U and the two middle pieces into V. This matches the pointwise description given above. We omit the verification that this defines a group scheme structure.

06E8 **Lemma 110.48.1.** *There exists a flat group scheme of finite type over the affine line which is not separated.*

Proof. See the discussion above. □

08IX **Lemma 110.48.2.** *There exists a flat group scheme of finite type over the infinite dimensional affine space which is not quasi-separated.*

Proof. The same construction as above can be carried out with the infinite dimensional affine space $S = \mathbf{A}^\infty_k = \operatorname{Spec} k[x_1, x_2, \ldots]$ as the base and the origin $0 \in S$ corresponding to the maximal ideal $(x_1, x_2, \ldots)$ as the closed point which is doubled in G. The resulting group scheme $G \to S$ is not quasi-separated as explained in Schemes, Example 26.21.4. □

110.49. A non-flat group scheme with flat identity component

06RJ Let $X \to S$ be a monomorphism of schemes. Let $G = S \amalg X$. Let $m : G \times_S G \to G$ be the S-morphism

$$G \times_S G = X \times_S X \amalg X \amalg X \amalg S \longrightarrow G = X \amalg S$$

which maps the summands $X \times_S X$ and S into S and maps the summands X into X by the identity morphism. This defines a group law. To see this we have to show that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$ as maps $G \times_S G \times_S G \to G$. Decomposing $G \times_S G \times_S G$ into components as above, we see that we need to verify this for the restriction to each of the 8-pieces. Each piece is isomorphic to either S, X, $X \times_S X$, or $X \times_S X \times_S X$. Moreover, both maps map these pieces to S, X, S, X respectively. Having said this, the fact that $X \to S$ is a monomorphism implies that $X \times_S X \cong X$ and $X \times_S X \times_S X \cong X$ and that there is in each case exactly one S-morphism $S \to S$ or $X \to X$. Thus we see that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$. Thus taking $X \to S$ to be any nonflat monomorphism of schemes (e.g., a closed immersion) we get an example of a group scheme over a base S whose identity component is S (hence flat) but which is not flat.

06RK **Lemma 110.49.1.** *There exists a group scheme G over a base S whose identity component is flat over S but which is not flat over S.*

Proof. See discussion above. □

110.50. A non-separated group algebraic space over a field

06E9 Every group scheme over a field is separated, see Groupoids, Lemma 39.7.3. This is not true for group algebraic spaces over a field (but see end of this section for positive results).

Let k be a field of characteristic zero. Consider the algebraic space $G = \mathbf{A}^1_k/\mathbf{Z}$ from Spaces, Example 65.14.8. By construction G is the fppf sheaf associated to the presheaf

$$T \longmapsto \Gamma(T, \mathcal{O}_T)/\mathbf{Z}$$

on the category of schemes over k. The obvious addition rule on the presheaf induces an addition $m : G \times G \to G$ which turns G into a group algebraic space over $\mathrm{Spec}(k)$. Note that G is not separated (and not even quasi-separated or locally separated). On the other hand $G \to \mathrm{Spec}(k)$ is of finite type!

06EA **Lemma 110.50.1.** *There exists a group algebraic space of finite type over a field which is not separated (and not even quasi-separated or locally separated).*

Proof. See discussion above. □

Positive results: If the group algebraic space G is either quasi-separated, or locally separated, or more generally a decent algebraic space, then G is in fact separated, see More on Groupoids in Spaces, Lemma 79.9.4. Moreover, a finite type, separated group algebraic space over a field is in fact a scheme by More on Groupoids in Spaces, Lemma 79.10.2. The idea of the proof is that the schematic locus is open dense, see Properties of Spaces, Proposition 66.13.3 or Decent Spaces, Theorem 68.10.2. By translating this open we see that every point of G has an open neighbourhood which is a scheme.

110.51. Specializations between points in fibre étale morphism

06UJ If $f : X \to Y$ is an étale, or more generally a locally quasi-finite morphism of schemes, then there are no specializations between points of fibres, see Morphisms, Lemma 29.20.8. However, for morphisms of algebraic spaces this doesn't hold in general.

To give an example, let k be a field. Set

$$P = k[u, u^{-1}, y, \{x_n\}_{n \in \mathbf{Z}}].$$

Consider the action of $\mathbf{Z}$ on P by k-algebra maps generated by the automorphism τ given by the rules $\tau(u) = u$, $\tau(y) = uy$, and $\tau(x_n) = x_{n+1}$. For $d \geq 1$ set $I_d = ((1 - u^d)y, x_n - x_{n+d}, n \in \mathbf{Z})$. Then $V(I_d) \subset \mathrm{Spec}(P)$ is the fix point locus of τ^d. Let $S \subset P$ be the multiplicative subset generated by y and all $1 - u^d$, $d \in \mathbf{N}$. Then we see that $\mathbf{Z}$ acts freely on $U = \mathrm{Spec}(S^{-1}P)$. Let $X = U/\mathbf{Z}$ be the quotient algebraic space, see Spaces, Definition 65.14.4.

Consider the prime ideals $\mathfrak{p}_n = (x_n, x_{n+1}, \ldots)$ in $S^{-1}P$. Note that $\tau(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$. Hence each of these define point $\xi_n \in U$ whose image in X is the same point x of X. Moreover we have the specializations

$$\ldots \rightsquigarrow \xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots$$

We conclude that $U \to X$ is an example of the promised type.

06UK **Lemma 110.51.1.** *There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization of points $x \rightsquigarrow x'$ in $|X|$ with $f(x) = f(x')$ in $|Y|$.*

Proof. See discussion above. □

110.52. A torsor which is not an fppf torsor

04AF In Groupoids, Remark 39.11.5 we raise the question whether any G-torsor is a G-torsor for the fppf topology. In this section we show that this is not always the case.

Let k be a field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be the group scheme

$$G = (\mu_{2,k})^\infty = \mu_{2,k} \times_k \mu_{2,k} \times_k \mu_{2,k} \times_k \ldots = \lim_n (\mu_{2,k})^n$$

where $\mu_{2,k}$ is the group scheme of second roots of unity over $\mathrm{Spec}(k)$, see Groupoids, Example 39.5.2. As an inverse limit of affine schemes we see that G is an affine group scheme. In fact it is the spectrum of the ring $k[t_1, t_2, t_3, \ldots]/(t_i^2 - 1)$. The multiplication map $m : G \times_k G \to G$ is on the algebra level given by $t_i \mapsto t_i \otimes t_i$.

We claim that any G-torsor over k is of the form

$$P = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots]/(x_i^2 - a_i))$$

for certain $a_i \in k^*$ and with G-action $G \times_k P \to P$ given by $x_i \to t_i \otimes x_i$ on the algebra level. We omit the proof. Actually for the example we only need that P is a G-torsor which is clear since over $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ the scheme P becomes isomorphic to G in a G-equivariant manner. Note that P is trivial if and only if $k' = k$ since if P has a k-rational point then all of the a_i are squares.

We claim that P is an fppf torsor if and only if the field extension $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)/k$ is finite. If k' is finite over k, then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fppf covering which trivializes P and we see that P is indeed an fppf torsor. Conversely, suppose that P is an fppf G-torsor. This means that there exists an fppf covering $\{S_i \to \mathrm{Spec}(k)\}$ such that each P_{S_i} is trivial. Pick an i such that S_i is not empty. Let $s \in S_i$ be a closed point. By Varieties, Lemma 33.14.1 the field extension $\kappa(s)/k$ is finite, and by construction $P_{\kappa(s)}$ has a $\kappa(s)$-rational point. Thus we see that $k \subset k' \subset \kappa(s)$ and k' is finite over k.

To get an explicit example take $k = \mathbf{Q}$ and $a_i = i$ for example (or a_i is the ith prime if you like).

077B **Lemma 110.52.1.** *Let S be a scheme. Let G be a group scheme over S. The stack G-Principal classifying principal homogeneous G-spaces (see Examples of Stacks, Subsection 95.14.5) and the stack G-Torsors classifying fppf G-torsors (see Examples of Stacks, Subsection 95.14.8) are not equivalent in general.*

Proof. The discussion above shows that the functor G-*Torsors* $\to$ G-*Principal* isn't essentially surjective in general. □

110.53. Stack with quasi-compact flat covering which is not algebraic

04AG In this section we briefly describe an example due to Brian Conrad. You can find the example online at this location. Our example is slightly different.

Let k be an algebraically closed field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be an affine group scheme. In Examples of Stacks, Lemma 95.15.4 we have given several different equivalent ways to view $\mathcal{X} = [\mathrm{Spec}(k)/G]$ as a stack in groupoids over $(Sch/\mathrm{Spec}(k))_{fppf}$. In particular $\mathcal{X}$ classifies fppf G-torsors. More precisely, a 1-morphism $T \to \mathcal{X}$ corresponds to an fppf G_T-torsor P over T and 2-arrows correspond to isomorphisms of torsors. It follows that the diagonal 1-morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$$

is representable and affine. Namely, given any pair of fppf G_T-torsors P_1, P_2 over a scheme T/k the scheme $Isom(P_1, P_2)$ is affine over T. The trivial G-torsor over $\mathrm{Spec}(k)$ defines a 1-morphism

$$f : \mathrm{Spec}(k) \longrightarrow \mathcal{X}.$$

We claim that this is a surjective 1-morphism. The reason is simply that by definition for any 1-morphism $T \to \mathcal{X}$ there exists a fppf covering $\{T_i \to T\}$ such that P_{T_i} is isomorphic to the trivial G_{T_i}-torsor. Hence the compositions $T_i \to T \to \mathcal{X}$ factor through f. Thus it is clear that the projection $T \times_{\mathcal{X}} \mathrm{Spec}(k) \to T$ is surjective (which is how we define the property that f is surjective, see Algebraic Stacks, Definition 94.10.1). In a similar way you show that f is quasi-compact and flat (details omitted). We also record here the observation that

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong G$$

as schemes over k.

Suppose there exists a surjective smooth morphism $p : U \to \mathcal{X}$ where U is a scheme. Consider the fibre product

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathcal{X} \end{array}$$

Then we see that W is a nonempty smooth scheme over k which hence has a k-point. This means that we can factor f through U. Hence we obtain

$$G \cong \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong (\mathrm{Spec}(k) \times_k \mathrm{Spec}(k)) \times_{(U \times_k U)} (U \times_{\mathcal{X}} U)$$

and since the projections $U \times_{\mathcal{X}} U \to U$ were assumed smooth we conclude that $U \times_{\mathcal{X}} U \to U \times_k U$ is locally of finite type, see Morphisms, Lemma 29.15.8. It follows that in this case G is locally of finite type over k. Altogether we have proved the following lemma (which can be significantly generalized).

04AH **Lemma 110.53.1.** *Let k be a field. Let G be an affine group scheme over k. If the stack $[\mathrm{Spec}(k)/G]$ has a smooth covering by a scheme, then G is of finite type over k.*

Proof. See discussion above. □

To get an explicit example as in the title of this section, take for example $G = (\mu_{2,k})^\infty$ the group scheme of Section 110.52, which is not locally of finite type over k. By the discussion above we see that $\mathcal{X} = [\mathrm{Spec}(k)/G]$ has properties (1) and (2) of Algebraic Stacks, Definition 94.12.1, but not property (3). Hence $\mathcal{X}$ is not an algebraic stack. On the other hand, there does exist a scheme U and a surjective, flat, quasi-compact morphism $U \to \mathcal{X}$, namely the morphism $f : \mathrm{Spec}(k) \to \mathcal{X}$ we studied above.

110.54. Limit preserving on objects, not limit preserving

07Z0 Let S be a nonempty scheme. Let $\mathcal{G}$ be an injective abelian sheaf on $(Sch/S)_{fppf}$. We obtain a stack in groupoids

$$\mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

over S, see Examples of Stacks, Lemma 95.14.2. This stack is limit preserving on objects over $(Sch/S)_{fppf}$ (see Criteria for Representability, Section 97.5) because every $\mathcal{G}$-torsor is trivial. On the other hand, *$\mathcal{G}$-Torsors* is in general not limit preserving (see Artin's Axioms, Definition 98.11.1) as $\mathcal{G}$ need not be limit preserving as a sheaf. For example, take any nonzero injective sheaf $\mathcal{I}$ and set $\mathcal{G} = \prod_{n \in \mathbf{Z}} \mathcal{I}$ to get an example.

07Z1 **Lemma 110.54.1.** *Let S be a nonempty scheme. There exists a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ such that p is limit preserving on objects, but $\mathcal{X}$ is not limit preserving.*

Proof. See discussion above. □

110.55. A non-algebraic classifying stack

077C Let $S = \mathrm{Spec}(\mathbf{F}_p)$ and let μ_p denote the group scheme of pth roots of unity over S. In Groupoids in Spaces, Section 78.20 we have introduced the quotient stack $[S/\mu_p]$ and in Examples of Stacks, Section 95.15 we have shown $[S/\mu_p]$ is the classifying stack for fppf μ_p-torsors: Given a scheme T over S the category $\mathrm{Mor}_S(T, [S/\mu_p])$ is canonically equivalent to the category of fppf μ_p-torsors over T. Finally, in Criteria for Representability, Theorem 97.17.2 we have seen that $[S/\mu_p]$ is an algebraic stack.

Now we can ask the question: "How about the category fibred in groupoids $\mathcal{S}$ classifying étale μ_p-torsors?" (In other words $\mathcal{S}$ is a category over Sch/S whose fibre category over a scheme T is the category of étale μ_p-torsors over T.)

The first objection is that this isn't a stack for the fppf topology, because descent for objects isn't going to hold. For example the μ_p-torsor $\mathrm{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$ over $T = \mathrm{Spec}(\mathbf{F}_p(T))$ is fppf locally trivial, but not étale locally trivial.

A fix for this first problem is to work with the étale topology and in this case descent for objects does work. Indeed it is true that $\mathcal{S}$ is a stack in groupoids over $(Sch/S)_{étale}$. Moreover, it is also the case that the diagonal $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable (by schemes). This is true because given two μ_p-torsors (whether they be étale locally trivial or not) the sheaf of isomorphisms between them is representable by a scheme.

Thus we can finally ask if there exists a scheme U and a smooth and surjective 1-morphism $U \to \mathcal{S}$. We will show in two ways that this is impossible: by a direct argument (which we advise the reader to skip) and by an argument using a general result.

Direct argument (sketch): Note that the 1-morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{F}_p)$ satisfies the infinitesimal lifting criterion for formal smoothness. This is true because given a first order infinitesimal thickening of schemes $T \to T'$ the kernel of $\mu_p(T') \to \mu_p(T)$ is isomorphic to the sections of the ideal sheaf of T in T', and hence $H^1_{étale}(T, \mu_p) = H^1_{étale}(T', \mu_p)$. Moreover, $\mathcal{S}$ is a limit preserving stack. Hence if $U \to \mathcal{S}$ is smooth, then $U \to \mathrm{Spec}(\mathbf{F}_p)$ is limit preserving and satisfies the infinitesimal lifting criterion for formal smoothness. This implies that U is smooth over $\mathbf{F}_p$. In particular U is reduced, hence $H^1_{étale}(U, \mu_p) = 0$. Thus $U \to \mathcal{S}$ factors as $U \to \mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ and the first arrow is smooth. By descent of smoothness, we see that $U \to \mathcal{S}$ being smooth would imply $\mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ is smooth. However, this is not the case as $\mathrm{Spec}(\mathbf{F}_p) \times_{\mathcal{S}} \mathrm{Spec}(\mathbf{F}_p)$ is μ_p which is not smooth over $\mathrm{Spec}(\mathbf{F}_p)$.

Structural argument: In Criteria for Representability, Section 97.19 we have seen that we can think of algebraic stacks as those stacks in groupoids for the étale topology with diagonal representable by algebraic spaces having a smooth covering. Hence if a smooth surjective $U \to \mathcal{S}$ exists then $\mathcal{S}$ is an algebraic stack, and in particular satisfies descent in the fppf topology. But we've seen above that $\mathcal{S}$ does not satisfies descent in the fppf topology.

Loosely speaking the arguments above show that the classifying stack in the étale topology for étale locally trivial torsors for a group scheme G over a base B is algebraic if and only if G is smooth over B. One of the advantages of working with the fppf topology is that it suffices to assume that $G \to B$ is flat and locally of finite presentation. In fact the quotient stack (for the fppf topology) $[B/G]$ is algebraic if and only if $G \to B$ is flat and locally of finite presentation, see Criteria for Representability, Lemma 97.18.3.

110.56. Sheaf with quasi-compact flat covering which is not algebraic

078E Consider the functor $F = (\mathbf{P}^1)^\infty$, i.e., for a scheme T the value $F(T)$ is the set of $f = (f_1, f_2, f_3, \ldots)$ where each $f_i : T \to \mathbf{P}^1$ is a morphism of schemes. Note that $\mathbf{P}^1$ satisfies the sheaf property for fpqc coverings, see Descent, Lemma 35.13.7. A product of sheaves is a sheaf, so F also satisfies the sheaf property for the fpqc topology. The diagonal of F is representable: if $f : T \to F$ and $g : S \to F$ are morphisms, then $T \times_F S$ is the scheme theoretic intersection of the closed subschemes $T \times_{f_i, \mathbf{P}^1, g_i} S$ inside the scheme $T \times S$. Consider the group scheme SL_2 which comes with a surjective smooth affine morphism $\mathrm{SL}_2 \to \mathbf{P}^1$. Next, consider $U = (\mathrm{SL}_2)^\infty$ with its canonical (product) morphism $U \to F$. Note that U is an affine scheme. We claim the morphism $U \to F$ is flat, surjective, and universally open. Namely, suppose $f : T \to F$ is a morphism. Then $Z = T \times_F U$ is the infinite fibre product of the schemes $Z_i = T \times_{f_i, \mathbf{P}^1} \mathrm{SL}_2$ over T. Each of the morphisms $Z_i \to T$ is surjective smooth and affine which implies that

$$Z = Z_1 \times_T Z_2 \times_T Z_3 \times_T \ldots$$

is a scheme flat and affine over Z. A simple limit argument shows that $Z \to T$ is open as well.

On the other hand, we claim that F isn't an algebraic space. Namely, if F were an algebraic space it would be a quasi-compact and separated (by our description of fibre products over F) algebraic space. Hence cohomology of quasi-coherent sheaves would vanish above a certain cutoff (see Cohomology of Spaces, Proposition 69.7.2 and remarks preceding it). But clearly by taking the pullback of $\mathcal{O}(-2, -2, \ldots, -2)$ under the projection

$$(\mathbf{P}^1)^\infty \longrightarrow (\mathbf{P}^1)^n$$

(which has a section) we can obtain a quasi-coherent sheaf whose cohomology is nonzero in degree n. Altogether we obtain an answer to a question asked by Anton Geraschenko on mathoverflow.

078F **Lemma 110.56.1.** *There exists a functor $F : Sch^{opp} \to Sets$ which satisfies the sheaf condition for the fpqc topology, has representable diagonal $\Delta : F \to F \times F$, and such that there exists a surjective, flat, universally open, quasi-compact morphism $U \to F$ where U is a scheme, but such that F is not an algebraic space.*

Proof. See discussion above. □

110.57. Sheaves and specializations

05LD In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 34.4.6. Moreover, a scheme will be an object of this site. Recall that if x, x' are points of a scheme X we say x is a *specialization* of x' or we write $x' \rightsquigarrow x$ if $x \in \overline{\{x'\}}$. This is true in particular if $x = x'$.

Consider the functor $F : Sch_{étale} \to Ab$ defined by the following rules:

$$F(X) = \prod\nolimits_{x \in X} \prod\nolimits_{x' \in X, x' \rightsquigarrow x, x' \neq x} \mathbf{Z}/2\mathbf{Z}$$

Given a scheme X we denote $|X|$ the underlying set of points. An element $a \in F(X)$ will be viewed as a map of sets $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$, $(x, x') \mapsto a(x, x')$ which is zero if $x = x'$ or if x is not a specialization of x'. Given a morphism of schemes $f : X \to Y$ we define

$$F(f) : F(Y) \longrightarrow F(X)$$

by the rule that for $b \in F(Y)$ we set

$$F(f)(b)(x, x') = \begin{cases} 0 & \text{if } x \text{ is not a specialization of } x' \\ b(f(x), f(x')) & \text{else.} \end{cases}$$

Note that this really does define an element of $F(X)$. We claim that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms then $F(f) \circ F(g) = F(g \circ f)$. Namely, let $c \in F(Z)$ and let $x' \rightsquigarrow x$ be a specialization of points in X, then

$$F(g \circ f)(x, x') = c(g(f(x)), g(f(x'))) = F(g)(F(f)(c))(x, x')$$

because $f(x') \rightsquigarrow f(x)$. (This also works if $f(x) = f(x')$.)

Let G be the sheafification of F in the étale topology.

I claim that if X is a scheme and $x' \rightsquigarrow x$ is a specialization and $x' \neq x$, then $G(X) \neq 0$. Namely, let $a \in F(X)$ be an element such that when we think of a as a function $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$ it is nonzero at (x, x'). Let $\{f_i : U_i \to X\}$ be an étale covering of X. Then we can pick an i and a point $u_i \in U_i$ with $f_i(u_i) = x$. Since generalizations lift along flat morphisms (see Morphisms, Lemma 29.25.9) we can find a specialization $u_i' \rightsquigarrow u_i$ with $f_i(u_i') = x'$. By our construction above we see that $F(f_i)(a) \neq 0$. Hence a determines a nonzero element of $G(X)$.

Note that if $X = \mathrm{Spec}(k)$ where k is a field (or more generally a ring all of whose prime ideals are maximal), then $F(X) = 0$ and for every étale morphism $U \to X$ we have $F(U) = 0$ because there are no specializations between distinct points in fibres of an étale morphism. Hence $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 37.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 59.45.2. Since it is always the case that $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

As a variant we can consider the presheaf F_n which associates to a scheme X the collection of maps $a : |X|^{n+1} \to \mathbf{Z}/2\mathbf{Z}$ where $a(x_0, \ldots, x_n)$ is nonzero only if $x_n \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations and $x_n \neq x_{n-1} \neq \ldots \neq x_0$. Let G_n be the sheaf associated to F_n. In exactly the same way as above one shows that G_n is nonzero if $\dim(X) \geq n$ and is zero if $\dim(X) < n$.

05LE **Lemma 110.57.1.** *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) $G(X) = 0$ *whenever* $\dim(X) < n$,
(2) $G(X)$ *is not zero if* $\dim(X) \geq n$, *and*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$.*

Proof. See the discussion above. □

05LF **Remark 110.57.2.** Here are some remarks:

(1) The presheaves F and F_n are separated presheaves.
(2) It turns out that F, F_n are not sheaves.
(3) One can show that G, G_n is actually a sheaf for the fppf topology.

We will prove these results if we need them.

110.58. Sheaves and constructible functions

05LG In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 34.4.6. Moreover, a scheme will be an object of this site. In this section we say that a *constructible partition* of a scheme X is a locally finite disjoint union decomposition $X = \coprod_{i \in I} X_i$ such that each $X_i \subset X$ is a locally constructible subset of X. Locally finite means that for any quasi-compact open $U \subset X$ there are only finitely many $i \in I$ such that $X_i \cap U$ is not empty. Note that if $f : X \to Y$ is a morphism of schemes and $Y = \coprod Y_j$ is a constructible partition, then $X = \coprod f^{-1}(Y_j)$ is a constructible partition of X. Given a set S and a scheme X a *constructible function* $f : |X| \to S$ is a map such that $X = \coprod_{s \in S} f^{-1}(s)$ is a constructible partition of X. If G is an (abstract group) and $a, b : |X| \to G$ are constructible functions, then $ab : |X| \to G, x \mapsto a(x)b(x)$ is a constructible function too. The reason is that given any two constructible partitions there is a third one refining both.

Let A be any abelian group. For any scheme X we define

$$F(X) = \frac{\{a : |X| \to A \mid a \text{ is a constructible function}\}}{\{\text{locally constant functions } |X| \to A\}}$$

We think of an element a of $F(X)$ simply as a function well defined up to adding a locally constant one. Given a morphism of schemes $f : X \to Y$ and an element $b \in F(Y)$, then we define $F(f)(b) = b \circ f$. Thus F is a presheaf on $Sch_{étale}$.

Note that if $\{f_i : U_i \to X\}$ is an fppf covering, and $a \in F(X)$ is such that $F(f_i)(a) = 0$ in $F(U_i)$, then $a \circ f_i$ is a locally constant function for each i. This means in turn that a is a locally constant function as the morphisms f_i are open. Hence $a = 0$ in $F(X)$. Thus we see that F is a separated presheaf (in the fppf topology hence a fortiori in the étale topology).

Let G be the sheafification of F in the étale topology. Since F is separated, and since $F(X) \neq 0$ for example when X is the spectrum of a discrete valuation ring, we see that G is not zero.

Let $X = \mathrm{Spec}(k)$ where k is a field. Then any étale covering of X can be dominated by a covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ with k'/k a finite separable extension of fields. Since $F(\mathrm{Spec}(k')) = 0$ we see that $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 37.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes

étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 59.45.2. Since $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

The sheaf G is limit preserving, see Limits of Spaces, Definition 70.3.1. Namely, let R be a ring which is written as a directed colimit $R = \operatorname{colim}_i R_i$ of rings. Set $X = \operatorname{Spec}(R)$ and $X_i = \operatorname{Spec}(R_i)$, so that $X = \lim_i X_i$. Then $G(X) = \operatorname{colim}_i G(X_i)$. To prove this one first proves that a constructible partition of $\operatorname{Spec}(R)$ comes from a constructible partitions of some $\operatorname{Spec}(R_i)$. Hence the result for F. To get the result for the sheafification, use that any étale ring map $R \to R'$ comes from an étale ring map $R_i \to R'_i$ for some i. Details omitted.

05LH **Lemma 110.58.1.** *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) *$G(\operatorname{Spec}(k)) = 0$ whenever k is a field,*
(2) *G is limit preserving,*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$, and*
(4) *G is not zero.*

Proof. See discussion above. □

110.59. The lisse-étale site is not functorial

07BF The *lisse-étale* site $X_{lisse,étale}$ of X is the category of schemes smooth over X endowed with (usual) étale coverings, see Cohomology of Stacks, Section 103.14. Let $f : X \to Y$ be a morphism of schemes. There is a functor

$$u : Y_{lisse,étale} \longrightarrow X_{lisse,étale}, \quad V/Y \longmapsto V \times_Y X$$

which is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh(X_{lisse,étale}) \longrightarrow Sh(Y_{lisse,étale}), \quad u_s : Sh(Y_{lisse,étale}) \longrightarrow Sh(X_{lisse,étale}),$$

see Sites, Section 7.13. We claim that, in general, u does **not** define a morphism of sites, see Sites, Definition 7.14.1. In other words, we claim that u_s is not left exact in general. Note that representable presheaves are sheaves on lisse-étale sites. Hence, by Sites, Lemma 7.13.5 we see that $u_s h_V = h_{V \times_Y X}$. Now consider two morphisms

$$\begin{array}{ccc} V_1 & \overset{a}{\underset{b}{\rightrightarrows}} & V_2 \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

of schemes V_1, V_2 smooth over Y. Now if u_s is left exact, then we would have

$$u_s \text{Equalizer}(h_a, h_b : h_{V_1} \to h_{V_2}) = \text{Equalizer}(h_{a \times 1}, h_{b \times 1} : h_{V_1 \times_Y X} \to h_{V_2 \times_Y X})$$

We will take the morphisms $a, b : V_1 \to V_2$ such that there exists no morphism from a scheme smooth over Y into $(a = b) \subset V_1$, i.e., such that the left hand side is the empty sheaf, but such that after base change to X the equalizer is nonempty and smooth over X. A silly example is to take $X = \operatorname{Spec}(\mathbf{F}_p)$, $Y = \operatorname{Spec}(\mathbf{Z})$ and $V_1 = V_2 = \mathbf{A}^1_{\mathbf{Z}}$ with morphisms $a(x) = x$ and $b(x) = x + p$. Note that the equalizer of a and b is the fibre of $\mathbf{A}^1_{\mathbf{Z}}$ over (p).

07BG **Lemma 110.59.1.** *The lisse-étale site is not functorial, even for morphisms of schemes.*

Proof. See discussion above. □

110.60. Sheaves on the category of Noetherian schemes

0GE8 Let S be a locally Noetherian scheme. As in Artin's Axioms, Section 98.25 consider the inclusion functor

$$u : (\mathit{Noetherian}/S)_{fppf} \longrightarrow (\mathit{Sch}/S)_{fppf}$$

of the fppf site of locally Noetherian schemes over S into a big fppf site of S. As explained in the section referenced, this functor is continuous. Hence we obtain an adjoint pair of functors

$$u^s : \mathit{Sh}((\mathit{Sch}/S)_{fppf}) \longrightarrow \mathit{Sh}((\mathit{Noetherian}/S)_{fppf})$$

and

$$u_s : \mathit{Sh}((\mathit{Noetherian}/S)_{fppf}) \longrightarrow \mathit{Sh}((\mathit{Sch}/S)_{fppf})$$

see Sites, Section 7.13. However, we claim that u in general does not define a morphism of sites, see Sites, Definition 7.14.1. In other words, we claim that the functor u_s is not left exact in general.

Let p be a prime number and set $S = \mathrm{Spec}(\mathbf{F}_p)$. Consider the injective map of sheaves

$$a : \mathcal{F} \longrightarrow \mathcal{G}$$

on $(\mathit{Noetherian}/S)_{fppf}$ defined as follows: for U a locally Noetherian scheme over S we define

$$\mathcal{G}(U) = \Gamma(U, \mathcal{O}_U)^* = \mathrm{Mor}_S(U, \mathbf{G}_{m,S})$$

and we take

$$\mathcal{F}(U) = \{f \in \mathcal{G}(U) \mid \text{fppf locally } f \text{ has arbitrary } p\text{-power roots}\}$$

A Noetherian $\mathbf{F}_p$-algebra A has a nilpotent nilradical $I \subset A$, the p-power roots of 1 in A are of the elements of the form $1 + a$, $a \in I$, and hence no-nontrivial p-power root of 1 has arbitrary p-power roots. We conclude that $\mathcal{F}(U)$ is a p-torsion free abelian group for any locally Noetherian scheme U; some details omitted. It follows that $p : \mathcal{F} \to \mathcal{F}$ is an injective map of abelian sheaves on $(\mathit{Noetherian}/S)_{fppf}$.

To get a contradiction, assume u_s is exact. Then $p : u_s\mathcal{F} \to u_s\mathcal{F}$ is injective too and we find that $(u_s\mathcal{F})(V)$ is a p-torsion free abelian group for any V over S. Since representable presheaves are sheaves on fppf sites, by Sites, Lemma 7.13.5, we see that $u_s\mathcal{G}$ is represented by $\mathbf{G}_{m,S}$. Using that $u_s\mathcal{F} \to u_s\mathcal{G}$ is injective, we find a p-torsion free subgroup

$$(u_s\mathcal{F})(V) \subset \Gamma(V, \mathcal{O}_V)^*$$

for every scheme V over S with the following property: for every morphism $V \to U$ of schemes over S with U locally Noetherian the subgroup

$$\mathcal{F}(U) \subset \Gamma(U, \mathcal{O}_U)^*$$

maps into the subgroup $(u_s\mathcal{F})(V)$ by the restriction mapping $\Gamma(U, \mathcal{O}_U)^* \to \Gamma(V, \mathcal{O}_V)^*$.

The actual contradiction now is obtained as follows: let $k = \bigcup_{n \geq 0} \mathbf{F}_p(t^{1/p^n})$ and set

$$B = k \otimes_{\mathbf{F}_p(t)} k$$

and $V = \mathrm{Spec}(B)$. Since we have the two projection morphisms $V \to \mathrm{Spec}(k)$ corresponding to the two coprojections $k \to B$ and since $\mathrm{Spec}(k)$ is Noetherian, we conclude the subgroup

$$(u_s\mathcal{F})(V) \subset B^*$$

contains $k^* \otimes 1$ and $1 \otimes k^*$. This is a contradiction because

$$(t^{1/p} \otimes 1) \cdot (1 \otimes t^{-1/p}) = t^{1/p} \otimes t^{-1/p}$$

is a nontrivial p-torsion unit of B.

0GE9 **Lemma 110.60.1.** *With $S = \mathrm{Spec}(\mathbf{F}_p)$ the inclusion functor $(Noetherian/S)_{fppf} \to (Sch/S)_{fppf}$ does not define a morphism of sites.*

Proof. See discussion above. □

110.61. Derived pushforward of quasi-coherent modules

07DC Let k be a field of characteristic $p > 0$. Let $S = \mathrm{Spec}(k[x])$. Let $G = \mathbf{Z}/p\mathbf{Z}$ viewed either as an abstract group or as a constant group scheme over S. Consider the algebraic stack $\mathcal{X} = [S/G]$ where G acts trivially on S, see Examples of Stacks, Remark 95.15.5 and Criteria for Representability, Lemma 97.18.3. Consider the structure morphism

$$f : \mathcal{X} \longrightarrow S$$

This morphism is quasi-compact and quasi-separated. Hence we get a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_S),$$

see Derived Categories of Stacks, Proposition 104.6.1. Let's compute $Rf_{QCoh,*}\mathcal{O}_\mathcal{X}$. Since $D_{QCoh}(\mathcal{O}_S)$ is equivalent to the derived category of $k[x]$-modules (see Derived Categories of Schemes, Lemma 36.3.5) this is equivalent to computing $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For this we can use the covering $S \to \mathcal{X}$ and the spectral sequence

$$H^q(S \times_\mathcal{X} \ldots \times_\mathcal{X} S, O) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

see Cohomology of Stacks, Proposition 103.11.6. Note that

$$S \times_\mathcal{X} \ldots \times_\mathcal{X} S = S \times G^p$$

which is affine. Thus the complex

$$k[x] \to \mathrm{Map}(G, k[x]) \to \mathrm{Map}(G^2, k[x]) \to \ldots$$

computes $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Here for $\varphi \in \mathrm{Map}(G^{p-1}, k[x])$ its differential is the map which sends $(g_1, \ldots, g_p)$ to

$$\varphi(g_2, \ldots, g_p) + \sum\nolimits_{i=1}^{p-1} (-1)^i \varphi(g_1, \ldots, g_i + g_{i+1}, \ldots, g_p) + (-1)^p \varphi(g_1, \ldots, g_{p-1}).$$

This is just the complex computing the group cohomology of G acting trivially on $k[x]$ (insert future reference here). The cohomology of the cyclic group G on $k[x]$ is exactly one copy of $k[x]$ in each cohomological degree ≥ 0 (insert future reference here). We conclude that

$$Rf_*\mathcal{O}_\mathcal{X} = \bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[-n]$$

Now, consider the complex

$$E = \bigoplus\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

This is an object of $D_{QCoh}(\mathcal{O}_\mathcal{X})$. We interrupt the discussion for a general result.

08IY **Lemma 110.61.1.** *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{O}_\mathcal{X})$ whose cohomology sheaves are locally quasi-coherent (Sheaves on Stacks, Definition 96.12.1) and satisfy the flat base change property (Cohomology of Stacks, Definition 103.7.1). Then there exists a distinguished triangle*

$$K \to \prod_{n\geq 0} \tau_{\geq -n}K \to \prod_{n\geq 0} \tau_{\geq -n}K \to K[1]$$

in $D(\mathcal{O}_\mathcal{X})$. In other words, K is the derived limit of its canonical truncations.

Proof. Recall that we work on the "big fppf site" $\mathcal{X}_{fppf}$ of $\mathcal{X}$ (by our conventions for sheaves of $\mathcal{O}_\mathcal{X}$-modules in the chapters Sheaves on Stacks and Cohomology on Stacks). Let $\mathcal{B}$ be the set of objects x of $\mathcal{X}_{fppf}$ which lie over an affine scheme U. Combining Sheaves on Stacks, Lemmas 96.23.2, 96.16.1, Descent, Lemma 35.12.4, and Cohomology of Schemes, Lemma 30.2.2 we see that $H^p(x, \mathcal{F}) = 0$ if $\mathcal{F}$ is locally quasi-coherent and $x \in \mathcal{B}$. Now the claim follows from Cohomology on Sites, Lemma 21.23.10 with $d = 0$. □

08IZ **Lemma 110.61.2.** *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}_n$ is a collection of locally quasi-coherent sheaves with the flat base change property on $\mathcal{X}$, then $\oplus_n \mathcal{F}_n[n] \to \prod_n \mathcal{F}_n[n]$ is an isomorphism in $D(\mathcal{O}_\mathcal{X})$.*

Proof. This is true because by Lemma 110.61.1 we see that the direct sum is isomorphic to the product. □

We continue our discussion. Since a quasi-coherent module is locally quasi-coherent and satisfies the flat base change property (Sheaves on Stacks, Lemma 96.12.2) we get

$$E = \prod_{m\geq 0} \mathcal{O}_\mathcal{X}[m]$$

Since cohomology commutes with limits we see that

$$Rf_*E = \prod_{m\geq 0} \Big(\bigoplus_{n\geq 0} \mathcal{O}_S[m-n]\Big)$$

Note that this complex is not an object of $D_{QCoh}(\mathcal{O}_S)$ because the cohomology sheaf in degree 0 is an infinite product of copies of $\mathcal{O}_S$ which is not even a locally quasi-coherent $\mathcal{O}_S$-module.

07DD **Lemma 110.61.3.** *A quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks need not induce a functor $Rf_* : D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$.*

Proof. See discussion above. □

110.62. A big abelian category

07JS The purpose of this section is to give an example of a "big" abelian category $\mathcal{A}$ and objects M, N such that the collection of isomorphism classes of extensions $\mathrm{Ext}_\mathcal{A}(M, N)$ is not a set. The example is due to Freyd, see [**Fre64**, page 131, Exercise A].

We define $\mathcal{A}$ as follows. An object of $\mathcal{A}$ consists of a triple (M, α, f) where M is an abelian group and α is an ordinal and $f : \alpha \to \mathrm{End}(M)$ is a map. A morphism $(M, \alpha, f) \to (M', \alpha', f')$ is given by a homomorphism of abelian groups $\varphi : M \to M'$ such that for *any* ordinal β we have

$$\varphi \circ f(\beta) = f'(\beta) \circ \varphi$$

Here the rule is that we set $f(\beta) = 0$ if β is not in α and similarly we set $f'(\beta)$ equal to zero if β is not an element of α'. We omit the verification that the category so defined is abelian.

Consider the object $Z = (\mathbf{Z}, \emptyset, f)$, i.e., all the operators are zero. The observation is that computed in $\mathcal{A}$ the group $\mathrm{Ext}^1_{\mathcal{A}}(Z, Z)$ is a proper class and not a set. Namely, for each ordinal α we can find an extension $(M, \alpha+1, f)$ of Z by Z whose underlying group is $M = \mathbf{Z} \oplus \mathbf{Z}$ and where the value of f is always zero except for

$$f(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This clearly produces a proper class of isomorphism classes of extensions. In particular, the derived category of $\mathcal{A}$ has proper classes for its collections of morphism, see Derived Categories, Lemma 13.27.6. This means that some care has to be exercised when defining Verdier quotients of triangulated categories.

07JT **Lemma 110.62.1.** *There exists a "big" abelian category $\mathcal{A}$ whose* Ext*-groups are proper classes.*

Proof. See discussion above. □

110.63. Weakly associated points and scheme theoretic density

084J Let k be a field. Let $R = k[z, x_i, y_i]/(z^2, zx_iy_i)$ where i runs over the elements of $\mathbf{N}$. Note that $R = R_0 \oplus M_0$ where $R_0 = k[x_i, y_i]$ is a subring and M_0 is an ideal of square zero with $M_0 \cong R_0/(x_iy_i)$ as R_0-module. The prime $\mathfrak{p} = (z, x_i)$ is weakly associated to R as an R-module (Algebra, Definition 10.66.1). Indeed, the element z in $R_\mathfrak{p}$ is nonzero but annihilated by $\mathfrak{p}R_\mathfrak{p}$. On the other hand, consider the open subscheme

$$U = \bigcup D(x_i) \subset \mathrm{Spec}(R) = S$$

We claim that $U \subset S$ is scheme theoretically dense (Morphisms, Definition 29.7.1). To prove this it suffices to show that $\mathcal{O}_S \to j_*\mathcal{O}_U$ is injective where $j : U \to S$ is the inclusion morphism, see Morphisms, Lemma 29.7.5. Translated back into algebra, we have to show that for all $g \in R$ the map

$$R_g \longrightarrow \prod R_{x_i g}$$

is injective. Write $g = g_0 + m_0$ with $g_0 \in R_0$ and $m_0 \in M_0$. Then $R_g = R_{g_0}$ (details omitted). Hence we may assume $g \in R_0$. We may also assume g is not zero. Now $R_g = (R_0)_g \oplus (M_0)_g$. Since R_0 is a domain, the map $(R_0)_g \to \prod (R_0)_{x_i g}$ is injective. If $g \in (x_iy_i)$ then $(M_0)_g = 0$ and there is nothing to prove. If $g \notin (x_iy_i)$ then, since (x_iy_i) is a radical ideal of R_0, we have to show that $M_0 \to \prod (M_0)_{x_i g}$ is injective. The kernel of $R_0 \to M_0 \to (M_0)_{x_n}$ is (x_iy_i, y_n). Since (x_iy_i, y_n) is a radical ideal, if $g \notin (x_iy_i, y_n)$ then the kernel of $R_0 \to M_0 \to (M_0)_{x_n g}$ is (x_iy_i, y_n). As $g \notin (x_iy_i, y_n)$ for all $n \gg 0$ we conclude that the kernel is contained in $\bigcap_{n \gg 0}(x_iy_i, y_n) = (x_iy_i)$ as desired.

Second example due to Ofer Gabber. Let k be a field and let R, resp. R' be the ring of functions $\mathbf{N} \to k$, resp. the ring of eventually constant functions $\mathbf{N} \to k$.

Then $\mathrm{Spec}(R)$, resp. $\mathrm{Spec}(R')$ is the Stone-Čech compactification[3] $\beta\mathbf{N}$, resp. the one point compactification[4] $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. All points are weakly associated since all primes are minimal in the rings R and R'.

084K **Lemma 110.63.1.** *There exists a reduced scheme X and a schematically dense open $U \subset X$ such that some weakly associated point $x \in X$ is not in U.*

Proof. In the first example we have $\mathfrak{p} \notin U$ by construction. In Gabber's examples the schemes $\mathrm{Spec}(R)$ or $\mathrm{Spec}(R')$ are reduced. □

110.64. Example of non-additivity of traces

087J Let k be a field and let $R = k[\epsilon]$ be the ring of dual numbers over k. In other words, $R = k[x]/(x^2)$ and ϵ is the congruence class of x in R. Consider the short exact sequence of complexes

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{1} & R \\ \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow \\ R & \xrightarrow{1} & R & \longrightarrow & 0 \end{array}$$

Here the columns are the complexes, the first row is placed in degree 0, and the second row in degree 1. Denote the first complex (i.e., the left column) by $A^\bullet$, the second by $B^\bullet$ and the third $C^\bullet$. We claim that the diagram

087K (110.64.0.1)
$$\begin{array}{ccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \\ {\scriptstyle 1+\epsilon}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \end{array}$$

commutes in $K(R)$, i.e., is a diagram of complexes commuting up to homotopy. Namely, the square on the right commutes and the one on the left is off by the homotopy $1 : A^1 \to B^0$. On the other hand,

$$\mathrm{Tr}_{A^\bullet}(1+\epsilon) + \mathrm{Tr}_{C^\bullet}(1) \neq \mathrm{Tr}_{B^\bullet}(1).$$

087L **Lemma 110.64.1.** *There exists a ring R, a distinguished triangle $(K, L, M, \alpha, \beta, \gamma)$ in the homotopy category $K(R)$, and an endomorphism (a, b, c) of this distinguished triangle, such that K, L, M are perfect complexes and $Tr_K(a) + Tr_M(c) \neq Tr_L(b)$.*

Proof. Consider the example above. The map $\gamma : C^\bullet \to A^\bullet[1]$ is given by multiplication by ϵ in degree 0, see Derived Categories, Definition 13.10.1. Hence it is also true that

$$\begin{array}{ccc} C^\bullet & \xrightarrow{\gamma} & A^\bullet[1] \\ \downarrow & & \downarrow \\ C^\bullet & \xrightarrow{\gamma} & A^\bullet[1] \end{array}$$

[3] Every element $f \in R$ is of the form ue where u is a unit and e is an idempotent. Then Algebra, Lemma 10.26.5 shows $\mathrm{Spec}(R)$ is Hausdorff. On the other hand, $\mathbf{N}$ with the discrete topology can be viewed as a dense open subset. Given a set map $\mathbf{N} \to X$ to a Hausdorff, quasi-compact topological space X, we obtain a ring map $\mathcal{C}^0(X; k) \to R$ where $\mathcal{C}^0(X; k)$ is the k-algebra of locally constant maps $X \to k$. This gives $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{C}^0(X; k)) = X$ proving the universal property.

[4] Here one argues that there is really only one extra maximal ideal in R'.

commutes in $K(R)$ as $\epsilon(1+\epsilon) = \epsilon$. Thus we indeed have a morphism of distinguished triangles. □

110.65. Being projective is not local on the base

08J0 In the chapter on descent we have seen that many properties of morphisms are local on the base, even in the fpqc topology. See Descent, Sections 35.22, 35.23, and 35.24. This is not true for projectivity of morphisms.

08J1 **Lemma 110.65.1.** *The properties*

$$\mathcal{P}(f) = \text{“}f \text{ is projective”, and}$$
$$\mathcal{P}(f) = \text{“}f \text{ is quasi-projective”}$$

are not Zariski local on the base. A fortiori, they are not fpqc local on the base.

Proof. Following Hironaka [**Har77**, Example B.3.4.1], we define a proper morphism of smooth complex 3-folds $f : V_Y \to Y$ which is Zariski-locally projective, but not projective. Since f is proper and not projective, it is also not quasi-projective.

Let Y be projective 3-space over the complex numbers $\mathbf{C}$. Let C and D be smooth conics in Y such that the closed subscheme $C \cap D$ is reduced and consists of two complex points P and Q. (For example, let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$, $D = \{[x, y, z, w] : xy = w^2, z = 0\}$, $P = [1, 0, 0, 0]$, and $Q = [0, 1, 0, 0]$.) On $Y - Q$, first blow up the curve C, and then blow up the strict transform of the curve D (Divisors, Definition 31.33.1). On $Y - P$, first blow up the curve D, and then blow up the strict transform of the curve C. Over $Y - P - Q$, the two varieties we have constructed are canonically isomorphic, and so we can glue them over $Y - P - Q$. The result is a smooth proper 3-fold V_Y over $\mathbf{C}$. The morphism $f : V_Y \to Y$ is proper and Zariski-locally projective (since it is a blowup over $Y - P$ and over $Y - Q$), by Divisors, Lemma 31.32.13. We will show that V_Y is not projective over $\mathbf{C}$. That will imply that f is not projective.

To do this, let L be the inverse image in V_Y of a complex point of $C - P - Q$, and M the inverse image of a complex point of $D - P - Q$. Then L and M are isomorphic to the projective line $\mathbf{P}^1_{\mathbf{C}}$. Next, let E be the inverse image in V_Y of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0, and we have rational equivalences of cycles $L \sim L_0 + M_0$ and $M \sim M_0$ on E (using that C and D are isomorphic to $\mathbf{P}^1$). Note the asymmetry resulting from the order in which we blew up the two curves. Near Q, the opposite happens. So the inverse image of Q is the union of two lines L'_0 and M'_0, and we have rational equivalences $L \sim L'_0$ and $M \sim L'_0 + M'_0$ on E. Combining these equivalences, we find that $L_0 + M'_0 \sim 0$ on E and hence on V_Y. If V_Y were projective over $\mathbf{C}$, it would have an ample line bundle H, which would have degree > 0 on all curves in V_Y. In particular H would have positive degree on $L_0 + M'_0$, contradicting that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 42.20.3 and Lemma 42.28.2). So V_Y is not projective over $\mathbf{C}$. □

In different terminology, Hironaka's 3-fold V_Y is a small resolution of the blowup Y' of Y along the reduced subscheme $C \cup D$; here Y' has two node singularities. If we define Z by blowing up Y along C and then along the strict transform of D,

then Z is a smooth projective 3-fold, and the non-projective 3-fold V_Y differs from Z by a "flop" over $Y - P$.

110.66. Non-effective descent data for projective schemes

08KE In the chapter on descent we have seen that descent data for schemes relative to an fpqc morphism are effective for several classes of morphisms. In particular, affine morphisms and more generally quasi-affine morphisms satisfy descent for fpqc coverings (Descent, Lemma 35.38.1). This is not true for projective morphisms.

08KF **Lemma 110.66.1.** *There is an etale covering $X \to S$ of schemes and a descent datum $(V/X, \varphi)$ relative to $X \to S$ such that $V \to X$ is projective, but the descent datum is not effective in the category of schemes.*

Proof. We imitate Hironaka's example of a smooth separated complex algebraic space of dimension 3 which is not a scheme [**Har77**, Example B.3.4.2].

Consider the action of the group $G = \mathbf{Z}/2 = \{1, g\}$ on projective 3-space $\mathbf{P}^3$ over the complex numbers by

$$g[x, y, z, w] = [y, x, w, z].$$

The action is free outside the two disjoint lines $L_1 = \{[x, x, z, z]\}$ and $L_2 = \{[x, -x, z, -z]\}$ in $\mathbf{P}^3$. Let $Y = \mathbf{P}^3 - (L_1 \cup L_2)$. There is a smooth quasi-projective scheme $S = Y/G$ over $\mathbf{C}$ such that $Y \to S$ is a G-torsor (Groupoids, Definition 39.11.3). Explicitly, we can define S as the image of the open subset Y in $\mathbf{P}^3$ under the morphism

$$\begin{aligned} \mathbf{P}^3 &\to \text{Proj } \mathbf{C}[x, y, z, w]^G \\ &= \text{Proj } \mathbf{C}[u_0, u_1, v_0, v_1, v_2]/(v_0 v_1 = v_2^2), \end{aligned}$$

where $u_0 = x + y$, $u_1 = z + w$, $v_0 = (x - y)^2$, $v_1 = (z - w)^2$, and $v_2 = (x - y)(z - w)$, and the ring is graded with u_0, u_1 in degree 1 and v_0, v_1, v_2 in degree 2.

Let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$ and $D = \{[x, y, z, w] : xy = w^2, z = 0\}$. These are smooth conic curves in $\mathbf{P}^3$, contained in the G-invariant open subset Y, with $g(C) = D$. Also, $C \cap D$ consists of the two points $P := [1, 0, 0, 0]$ and $Q := [0, 1, 0, 0]$, and these two points are switched by the action of G.

Let $V_Y \to Y$ be the scheme which over $Y - P$ is defined by blowing up D and then the strict transform of C, and over $Y - Q$ is defined by blowing up C and then the strict transform of D. (This is the same construction as in the proof of Lemma 110.65.1, except that Y here denotes an open subset of $\mathbf{P}^3$ rather than all of $\mathbf{P}^3$.) Then the action of G on Y lifts to an action of G on V_Y, which switches the inverse images of $Y - P$ and $Y - Q$. This action of G on V_Y gives a descent datum $(V_Y/Y, \varphi_Y)$ on V_Y relative to the G-torsor $Y \to S$. The morphism $V_Y \to Y$ is proper but not projective, as shown in the proof of Lemma 110.65.1.

Let X be the disjoint union of the open subsets $Y - P$ and $Y - Q$; then we have surjective etale morphisms $X \to Y \to S$. Let V be the pullback of $V_Y \to Y$ to X; then the morphism $V \to X$ is projective, since $V_Y \to Y$ is a blowup over each of the open subsets $Y - P$ and $Y - Q$. Moreover, the descent datum $(V_Y/Y, \varphi_Y)$ pulls back to a descent datum $(V/X, \varphi)$ relative to the etale covering $X \to S$.

Suppose that this descent datum is effective in the category of schemes. That is, there is a scheme $U \to S$ which pulls back to the morphism $V \to X$ together with its descent datum. Then U would be the quotient of V_Y by its G-action.

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

Let E be the inverse image of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0. It follows that the inverse image of $Q = g(P)$ in E is the union of two lines $L_0' = g(M_0)$ and $M_0' = g(L_0)$. As shown in the proof of Lemma 110.65.1, we have a rational equivalence $L_0 + M_0' = L_0 + g(L_0) \sim 0$ on E.

By descent of closed subschemes, there is a curve $L_1 \subset U$ (isomorphic to $\mathbf{P}^1$) whose inverse image in V_Y is $L_0 \cup g(L_0)$. (Use Descent, Lemma 35.37.1, noting that a closed immersion is an affine morphism.) Let R be a complex point of L_1. Since we assumed that U is a scheme, we can choose a function f in the local ring $O_{U,R}$ that vanishes at R but not on the whole curve L_1. Let D_{loc} be an irreducible component of the closed subset $\{f = 0\}$ in $\operatorname{Spec} O_{U,R}$; then D_{loc} has codimension 1. The closure of D_{loc} in U is an irreducible divisor D_U in U which contains the point R but not the whole curve L_1. The inverse image of D_U in V_Y is an effective divisor D which intersects $L_0 \cup g(L_0)$ but does not contain either curve L_0 or $g(L_0)$.

Since the complex 3-fold V_Y is smooth, $O(D)$ is a line bundle on V_Y. We use here that a regular local ring is factorial, or in other words is a UFD, see More on Algebra, Lemma 15.122.2. The restriction of $O(D)$ to the proper surface $E \subset V_Y$ is a line bundle which has positive degree on the 1-cycle $L_0 + g(L_0)$, by our information on D. Since $L_0 + g(L_0) \sim 0$ on E, this contradicts that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 42.20.3 and Lemma 42.28.2). Therefore the descent datum $(V/X, \varphi)$ is in fact not effective; that is, U does not exist as a scheme. □

In this example, the descent datum *is* effective in the category of algebraic spaces. More precisely, U exists as a smooth separated algebraic space of dimension 3 over $\mathbf{C}$, for example by Algebraic Spaces, Lemma 65.14.3. Hironaka's 3-fold U is a small resolution of the blowup S' of the smooth quasi-projective 3-fold S along the irreducible nodal curve $(C \cup D)/G$; the 3-fold S' has a node singularity. The other small resolution of S' (differing from U by a "flop") is again an algebraic space which is not a scheme.

110.67. A family of curves whose total space is not a scheme

0D5D In Quot, Section 99.15 we define a family of curves over a scheme S to be a proper, flat, finitely presented morphism of relative dimension ≤ 1 from an algebraic space X to S. If S is the spectrum of a complete Noetherian local ring, then X is a

scheme, see More on Morphisms of Spaces, Lemma 76.43.6. In this section we show this is not true in general.

Let k be a field. We start with a proper flat morphism

$$Y \longrightarrow \mathbf{A}^1_k$$

and a point $y \in Y(k)$ lying over $0 \in \mathbf{A}^1_k(k)$ with the following properties

(1) the fibre Y_0 is a smooth geometrically irreducible curve over k,
(2) for any proper closed subscheme $T \subset Y$ dominating $\mathbf{A}^1_k$ the intersection $T \cap Y_0$ contains at least one point distinct from y.

Given such a surface we construct our example as follows.

$$\begin{array}{ccccc} Y & \longleftarrow & Z & \longrightarrow & X \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbf{A}^1_k & & \end{array}$$

Here $Z \to Y$ is the blowup of Y in y. Let $E \subset Z$ be the exceptional divisor and let $C \subset Z$ be the strict transform of Y_0. We have $Z_0 = E \cup C$ scheme theoretically (to see this use that Y is smooth at y and moreover $Y \to \mathbf{A}^1_k$ is smooth at y). By Artin's results ([**Art70**]; use Semistable Reduction, Lemma 55.9.7 to see that the normal bundle of C is negative) we can blow down the curve C in Z to obtain an algebraic space X as in the diagram. Let $x \in X(k)$ be the image of C.

We claim that X is not a scheme. Namely, if it were a scheme, then there would be an affine open neighbourhood $U \subset X$ of x. Set $T = X \setminus U$. Then T dominates $\mathbf{A}^1_k$ (as the fibres of $X \to \mathbf{A}^1_k$ are proper of dimension 1 and the fibres of $U \to \mathbf{A}^1_k$ are affine hence different). Let $T' \subset Z$ be the closed subscheme mapping isomorphically to T (as $x \notin T$). Then the image of T' in X contradicts condition (2) above (as $T' \cap Z_0$ is contained in the exceptional divisor E of the blowing up $Z \to Y$).

To finish the discussion we need to construct our Y. We will assume the characteristic of k is not 3. Write $\mathbf{A}^1_k = \mathrm{Spec}(k[t])$ and take

$$Y \quad : \quad T_0^3 + T_1^3 + T_2^3 - tT_0T_1T_2 = 0$$

in $\mathbf{P}^2_{k[t]}$. The fibre of this for $t = 0$ is a smooth projective genus 1 curve. On the affine piece $V_+(T_0)$ we get the affine equation

$$1 + x^3 + y^3 - txy = 0$$

which defines a smooth surface over k. Since the same is true on the other affine pieces by symmetry we see that Y is a smooth surface. Finally, we see from the affine equation also that the fraction field is $k(x, y)$ hence Y is a rational surface. Now the Picard group of a rational surface is finitely generated (insert future reference here). Hence in order to choose $y \in Y_0(k)$ with property (2) it suffices to choose y such that

0DYB (110.67.0.1) $$\mathcal{O}_{Y_0}(ny) \notin \mathrm{Im}(\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_0)) \text{ for all } n > 0$$

Namely, the sum of the 1-dimensional irreducible components of a T contradicting (2) would give an effective Cartier divisor intersection Y_0 in the divisor ny for some

$n \geq 1$ and we would conclude that $\mathcal{O}_{Y_0}(ny)$ is in the image of the restriction map. Observe that since Y_0 has genus ≥ 1 the map

$$Y_0(k) \to \mathrm{Pic}(Y_0), \quad y \mapsto \mathcal{O}_{Y_0}(y)$$

is injective. Now if k is an uncountable algebraically closed field, then using the countability of $\mathrm{Pic}(Y)$ and the remark just made, we can find a $y \in Y_0(k)$ satisfying (110.67.0.1) and hence (2).

0D5E **Lemma 110.67.1.** *There exists a field k and a family of curves $X \to \mathbf{A}^1_k$ such that X is not a scheme.*

Proof. See discussion above. □

110.68. Derived base change

08J2 Let $R \to R'$ be a ring map. In More on Algebra, Section 15.60 we construct a derived base change functor $- \otimes^{\mathbf{L}}_R R' : D(R) \to D(R')$. Next, let $R \to A$ be a second ring map. Picture

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_R R' & = & A' \\ \uparrow & & \uparrow & \nearrow & \\ R & \longrightarrow & R' & & \end{array}$$

Given an A-module M the tensor product $M \otimes_R R'$ is a $A \otimes_R R'$-module, i.e., an A'-module. For the ring map $A \to A'$ there is a derived functor

$$- \otimes^{\mathbf{L}}_A A' : D(A) \longrightarrow D(A')$$

but this functor does not agree with $- \otimes^{\mathbf{L}}_R R'$ in general. More precisely, for $K \in D(A)$ the canonical map

$$K \otimes^{\mathbf{L}}_R R' \longrightarrow K \otimes^{\mathbf{L}}_A A'$$

in $D(R')$ constructed in More on Algebra, Equation (15.61.0.1) isn't an isomorphism in general. Thus one may wonder if there exists a "derived base change functor" $T : D(A) \to D(A')$, i.e., a functor such that $T(K)$ maps to $K \otimes^{\mathbf{L}}_R R'$ in $D(R')$. In this section we show it does not exist in general.

Let k be a field. Set $R = k[x, y]$. Set $R' = R/(xy)$ and $A = R/(x^2)$. The object $A \otimes^{\mathbf{L}}_R R'$ in $D(R')$ is represented by

$$x^2 : R' \longrightarrow R'$$

and we have $H^0(A \otimes^{\mathbf{L}}_R R') = A \otimes_R R'$. We claim that there does not exist an object E of $D(A \otimes_R R')$ mapping to $A \otimes^{\mathbf{L}}_R R'$ in $D(R')$. Namely, for such an E the module $H^0(E)$ would be free, hence E would decompose as $H^0(E)[0] \oplus H^{-1}(E)[1]$. But it is easy to see that $A \otimes^{\mathbf{L}}_R R'$ is not isomorphic to the sum of its cohomology groups in $D(R')$.

08J3 **Lemma 110.68.1.** *Let $R \to R'$ and $R \to A$ be ring maps. In general there does not exist a functor $T : D(A) \to D(A \otimes_R R')$ of triangulated categories such that an A-module M gives an object $T(M)$ of $D(A \otimes_R R')$ which maps to $M \otimes^{\mathbf{L}}_R R'$ under the map $D(A \otimes_R R') \to D(R')$.*

Proof. See discussion above. □

110.69. An interesting compact object

09R4 Let R be a ring. Let (A, d) be a differential graded R-algebra. If $A = R$, then we know that every compact object of $D(A, \mathrm{d}) = D(R)$ is represented by a finite complex of finite projective modules. In other words, compact objects are perfect, see More on Algebra, Proposition 15.79.3. The analogue in the language of differential graded modules would be the question: "Is every compact object of $D(A, \mathrm{d})$ represented by a differential graded A-module P which is finite and graded projective?"

For general differential graded algebras, this is not true. Namely, let k be a field of characteristic 2 (so we don't have to worry about signs). Let $A = k[x, y]/(y^2)$ with

(1) x of degree 0
(2) y of degree -1,
(3) $\mathrm{d}(x) = 0$, and
(4) $\mathrm{d}(y) = x^2 + x$.

Then $x : A \to A$ is a projector in $K(A, \mathrm{d})$. Hence we see that

$$A = \mathrm{Ker}(x) \oplus \mathrm{Im}(1 - x)$$

in $K(A, \mathrm{d})$, see Differential Graded Algebra, Lemma 22.5.4 and Derived Categories, Lemma 13.4.14. It is clear that A is a compact object of $D(A, \mathrm{d})$. Then $\mathrm{Ker}(x)$ is a compact object of $D(A, \mathrm{d})$ as follows from Derived Categories, Lemma 13.37.2.

Next, suppose that M is a differential graded (right) A-module representing $\mathrm{Ker}(x)$ and suppose that M is finite and projective as a graded A-module. Because every finite graded projective module over $k[x, y]/(y^2)$ is graded free, we see that M is finite free as a graded $k[x, y]/(y^2)$-module (i.e., when we forget the differential). We set $N = M/M(x^2 + x)$. Consider the exact sequence

$$0 \to M \xrightarrow{x^2+x} M \to N \to 0$$

Since $x^2 + x$ is of degree 0, in the center of A, and $\mathrm{d}(x^2 + x) = 0$ we see that this is a short exact sequence of differential graded A-modules. Moreover, as $\mathrm{d}(y) = x^2 + x$ we see that the differential on N is linear. The maps

$$H^{-1}(N) \to H^0(M) \quad \text{and} \quad H^0(M) \to H^0(N)$$

are isomorphisms as $H^*(M) = H^0(M) = k$ since $M \cong \mathrm{Ker}(x)$ in $D(A, \mathrm{d})$. A computation of the boundary map shows that $H^*(N) = k[x, y]/(x, y^2)$ as a graded module; we omit the details. Since N is a free $k[x, y]/(y^2, x^2 + x)$-module we have a resolution

$$\ldots \to N[2] \xrightarrow{y} N[1] \xrightarrow{y} N \to N/Ny \to 0$$

compatible with differentials. Since N is bounded and since $H^0(N) = k[x, y]/(x, y^2)$ it follows from Homology, Lemma 12.25.3 that $H^0(N/Ny) = k[x]/(x)$. But as N/Ny is a finite complex of free $k[x]/(x^2 + x) = k \times k$-modules, we see that its cohomology has to have even dimension, a contradiction.

09R5 **Lemma 110.69.1.** *There exists a differential graded algebra (A, d) and a compact object E of $D(A, d)$ such that E cannot be represented by a finite and graded projective differential graded A-module.*

Proof. See discussion above. □

110.70. Two differential graded categories

09R6 In this section we construct two differential graded categories satisfying axioms (A), (B), and (C) as in Differential Graded Algebra, Situation 22.27.2 whose objects do not come with a $\mathbf{Z}$-grading.

Example I. Let X be a topological space. Denote $\underline{\mathbf{Z}}$ the constant sheaf with value $\mathbf{Z}$. Let A be an $\underline{\mathbf{Z}}$-torsor. In this setting we say a sheaf of abelian groups $\mathcal{F}$ is *A-graded* if given a local section $a \in A(U)$ there is a projector $p_a : \mathcal{F}|_U \to \mathcal{F}|_U$ such that whenever we have a local isomorphism $\underline{\mathbf{Z}}|_U \to A|_U$ then $\mathcal{F}|_U = \bigoplus_{n\in\mathbf{Z}} p_n(\mathcal{F})$. Another way to say this is that locally on X the abelian sheaf $\mathcal{F}$ has a $\mathbf{Z}$-grading, but on overlaps the different choices of gradings differ by a shift in degree given by the transition functions for the torsor A. We say that a pair $(\mathcal{F}, \mathrm{d})$ is an *A-graded complex of abelian sheaves*, if $\mathcal{F}$ is an A-graded abelian sheaf and $\mathrm{d} : \mathcal{F} \to \mathcal{F}$ is a differential, i.e., $\mathrm{d}^2 = 0$ such that $p_{a+1} \circ \mathrm{d} = \mathrm{d} \circ p_a$ for every local section a of A. In other words, $\mathrm{d}(p_a(\mathcal{F}))$ is contained in $p_{a+1}(\mathcal{F})$.

Next, consider the category $\mathcal{A}$ with

(1) objects are A-graded complexes of abelian sheaves, and
(2) for objects $(\mathcal{F}, \mathrm{d})$, $(\mathcal{G}, \mathrm{d})$ we set

$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \mathrm{d}), (\mathcal{G}, \mathrm{d})) = \bigoplus \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$$

where $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ is the group of maps of abelian sheaves f such that $f(p_a(\mathcal{F})) \subset p_{a+n}(\mathcal{G})$ for all local sections a of A. As differential we take $\mathrm{d}(f) = \mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}$, see Differential Graded Algebra, Example 22.26.6.

We omit the verification that this is indeed a differential graded category satisfying (A), (B), and (C). All the properties may be verified locally on X where one just recovers the differential graded category of complexes of abelian sheaves. Thus we obtain a triangulated category $K(\mathcal{A})$.

Twisted derived category of X. Observe that given an object $(\mathcal{F}, \mathrm{d})$ of $\mathcal{A}$, there is a well defined A-graded cohomology sheaf $H(\mathcal{F}, \mathrm{d})$. Hence it is clear what is meant by a quasi-isomorphism in $K(\mathcal{A})$. We can invert quasi-isomorphisms to obtain the *derived category $D(\mathcal{A})$ of complexes of A-graded sheaves*. If A is the trivial torsor, then $D(\mathcal{A})$ is equal to $D(X)$, but for nonzero torsors, one obtains a kind of *twisted* derived category of X.

Example II. Let C be a smooth curve over a perfect field k of characteristic 2. Then $\Omega_{C/k}$ comes endowed with a canonical square root. Namely, we can write $\Omega_{C/k} = \mathcal{L}^{\otimes 2}$ such that for every local function f on C the section $\mathrm{d}(f)$ is equal to $s^{\otimes 2}$ for some local section s of $\mathcal{L}$. The "reason" is that

$$\mathrm{d}(a_0 + a_1 t + \ldots + a_d t^d) = (\sum\nolimits_{i \text{ odd}} a_i^{1/2} t^{(i-1)/2})^2 \mathrm{d}t$$

(insert future reference here). This in particular determines a canonical connection

$$\nabla_{can} : \Omega_{C/k} \longrightarrow \Omega_{C/k} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$

whose 2-curvature is zero (namely, the unique connection such that the squares have derivative equal to zero). Observe that the category of vector bundles with connections is a tensor category, hence we also obtain canonical connections ∇_{can} on the invertible sheaves $\Omega_{C/k}^{\otimes n}$ for all $n \in \mathbf{Z}$.

Let $\mathcal{A}$ be the category with

(1) objects are pairs $(\mathcal{F}, \nabla)$ consisting of a finite locally free sheaf $\mathcal{F}$ endowed with a connection
$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$
whose 2-curvature is zero, and

(2) morphisms between $(\mathcal{F}, \nabla_{\mathcal{F}})$ and $(\mathcal{G}, \nabla_{\mathcal{G}})$ are given by
$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}})) = \bigoplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_C} \Omega_{C/k}^{\otimes n})$$
For an element $f : \mathcal{F} \to \mathcal{G} \otimes \Omega_{C/k}^{\otimes n}$ of degree n we set
$$\mathrm{d}(f) = \nabla_{\mathcal{G} \otimes \Omega_{C/k}^{\otimes n}} \circ f + f \circ \nabla_{\mathcal{F}}$$
with suitable identifications.

We omit the verification that this forms a differential graded category with properties (A), (B), (C). Thus we obtain a triangulated homotopy category $K(\mathcal{A})$.

If $C = \mathbf{P}^1_k$, then $K(\mathcal{A})$ is the zero category. However, if C is a smooth proper curve of genus > 1, then $K(\mathcal{A})$ is not zero. Namely, suppose that $\mathcal{N}$ is an invertible sheaf of degree $0 \leq d < g - 1$ with a nonzero section σ. Then set $(\mathcal{F}, \nabla_{\mathcal{F}}) = (\mathcal{O}_C, \mathrm{d})$ and $(\mathcal{G}, \nabla_{\mathcal{G}}) = (\mathcal{N}^{\otimes 2}, \nabla_{can})$. We see that
$$\mathrm{Hom}^n_{\mathcal{A}}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}})) = \begin{cases} 0 & \text{if} \quad n < 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2}) & \text{if} \quad n = 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}) & \text{if} \quad n = 1 \end{cases}$$
The first 0 because the degree of $\mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}^{\otimes -1}$ is negative by the condition $d < g-1$. Now, the section $\sigma^{\otimes 2}$ has derivative equal zero, hence the homomorphism group
$$\mathrm{Hom}_{K(\mathcal{A})}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}}))$$
is nonzero.

110.71. The stack of proper algebraic spaces is not algebraic

0D1Q In Quot, Section 99.13 we introduced and studied the stack in groupoids
$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow \mathcal{S}ch_{fppf}$$
the stack whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S. We proved that this satisfies many of Artin's axioms. In this section we why this stack is not algebraic by showing that formal effectiveness fails in general.

The canonical example uses that the universal deformation space of an abelian variety of dimension g has g^2 formal parameters whereas any effective formal deformation can be defined over a complete local ring of dimension $\leq g(g+1)/2$. Our example will be constructed by writing down a suitable non-effective deformation of a K3 surface. We will only sketch the argument and not give all the details.

Let $k = \mathbf{C}$ be the field of complex numbers. Let $X \subset \mathbf{P}^3_k$ be a smooth degree 4 surface over k. We have $\omega_X \cong \Omega^2_{X/k} \cong \mathcal{O}_X$. Finally, we have $\dim_k H^0(X, T_{X/k}) = 0$, $\dim_k H^1(X, T_{X/k}) = 20$, and $\dim_k H^2(X, T_{X/k}) = 0$. Since $L_{X/k} = \Omega_{X/k}$ because X is smooth over k, and since $\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = H^i(X, T_{X/k})$, and because we

have Cotangent, Lemma 92.23.1 we find that there is a universal deformation of X over

$$k[[x_1, \ldots, x_{20}]]$$

Suppose that this universal deformation is effective (as in Artin's Axioms, Section 98.9). Then we would get a flat, proper morphism

$$f : Y \longrightarrow \mathrm{Spec}(k[[x_1, \ldots, x_{20}]])$$

where Y is an algebraic space recovering the universal deformation. This is impossible for the following reason. Since Y is separated we can find an affine open subscheme $V \subset Y$. Since the special fibre X of Y is smooth, we see that f is smooth. Hence Y is regular being smooth over regular and it follows that the complement D of V in Y is an effective Cartier divisor. Then $\mathcal{O}_Y(D)$ is a nontrivial element of $\mathrm{Pic}(Y)$ (to prove this you show that the complement of a nonempty affine open in a proper smooth algebraic space over a field is always a nontrivial in the Picard group and you apply this to the generic fibre of f). Finally, to get a contradiction, we show that $\mathrm{Pic}(Y) = 0$. Namely, the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective, because $H^1(X, \mathcal{O}_X) = 0$ (hence all deformations of $\mathcal{O}_X$ to $Y \times \mathrm{Spec}(k[[x_i]]/\mathfrak{m}^n)$ are trivial) and Grothendieck's existence theorem (which says that coherent modules giving rise to the same sheaves on thickenings are isomorphic). If X is general enough, then $\mathrm{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(1)$. Hence it suffices to show that $\mathcal{O}_X(n)$, $n > 0$ does not deform to the first order neighbourhood[5]. Consider the cup-product

$$H^1(X, \Omega_{X/k}) \times H^1(X, T_{X/k}) \longrightarrow H^2(X, \mathcal{O}_X)$$

This is a nondegenerate pairing by coherent duality. A computation shows that the Chern class $c_1(\mathcal{O}_X(n)) \in H^1(X, \Omega_{X/k})$ in Hodge cohomology is nonzero. Hence there is a first order deformation whose cup product with $c_1(\mathcal{O}_X(n))$ is nonzero. Then finally, one shows this cup product is the obstruction class to lifting.

0D1R **Lemma 110.71.1.** *The stack in groupoids*

$$p'_{fp,flat,proper} : \mathcal{S}paces'_{fp,flat,proper} \longrightarrow Sch_{fppf}$$

whose category of sections over a scheme S is the category of flat, proper, finitely presented algebraic spaces over S (see Quot, Section 99.13) is not an algebraic stack.

Proof. If it was an algebraic stack, then every formal object would be effective, see Artin's Axioms, Lemma 98.9.5. The discussion above show this is not the case after base change to $\mathrm{Spec}(\mathbf{C})$. Hence the conclusion. □

110.72. An example of a non-algebraic Hom-stack

0AF8 Let $\mathcal{Y}, \mathcal{Z}$ be algebraic stacks over a scheme S. The *Hom-stack* $\underline{\mathrm{Mor}}_S(\mathcal{Y}, \mathcal{Z})$ is the stack in groupoids over S whose category of sections over a scheme T is given by the category

$$\mathrm{Mor}_T(\mathcal{Y} \times_S T, \mathcal{Z} \times_S T)$$

whose objects are 1-morphisms and whose morphisms are 2-morphisms. We omit the proof this is indeed a stack in groupoids over $(Sch/S)_{fppf}$ (insert future reference here). Of course, in general the Hom-stack will not be algebraic. In this

[5]This argument works as long as the map $c_1 : \mathrm{Pic}(X) \to H^1(X, \Omega_{X/k})$ is injective, which is true for k any field of characteristic zero and any smooth hypersurface X of degree 4 in $\mathbf{P}^3_k$.

section we give an example where it is not true and where $\mathcal{Y}$ is representable by a proper flat scheme over S and $\mathcal{Z}$ is smooth and proper over S.

Let k be an algebraically closed field which is not the algebraic closure of a finite field. Let $S = \mathrm{Spec}(k[[t]])$ and $S_n = \mathrm{Spec}(k[t]/(t^n)) \subset S$. Let $f : X \to S$ be a map satisfying the following

(1) f is projective and flat, and its fibres are geometrically connected curves,
(2) the fibre $X_0 = X \times_S S_0$ is a nodal curve with smooth irreducible components whose dual graph has a loop consisting of rational curves,
(3) X is a regular scheme.

To make such a surface X we can take for example

$$X \quad : \quad T_0T_1T_2 - t(T_0^3 + T_1^3 + T_2^3) = 0$$

in $\mathbf{P}^2_{k[[t]]}$. Let A_0 be a non-zero abelian variety over k for example an elliptic curve. Let $A = A_0 \times_{\mathrm{Spec}(k)} S$ be the constant abelian scheme[6] over S associated to A_0. We will show that the stack $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A]))$ is not algebraic.

Recall that $[S/A]$ is on the one hand the quotient stack of A acting trivially on S and on the other hand equal to the stack classifying fppf A-torsors, see Examples of Stacks, Proposition 95.15.3. Observe that $[S/A] = [\mathrm{Spec}(k)/A_0] \times_{\mathrm{Spec}(k)} S$. This allows us to describe the fibre category over a scheme T as follows

$$\begin{aligned} \mathcal{X}_T &= \underline{\mathrm{Mor}}_S(X, [S/A])_T \\ &= \mathrm{Mor}_T(X \times_S T, [S/A] \times_S T) \\ &= \mathrm{Mor}_S(X \times_S T, [S/A]) \\ &= \mathrm{Mor}_{\mathrm{Spec}(k)}(X \times_S T, [\mathrm{Spec}(k)/A_0]) \end{aligned}$$

for any S-scheme T. In other words, the groupoid $\mathcal{X}_T$ is the groupoid of fppf A_0-torsors on $X \times_S T$. Before we discuss why $\mathcal{X}$ is not an algebraic stack, we need a few lemmas.

0AF9 **Lemma 110.72.1.** *Let W be a two dimensional regular integral Noetherian scheme with function field K. Let $G \to W$ be an abelian scheme. Then the map $H^1_{fppf}(W, G) \to H^1_{fppf}(\mathrm{Spec}(K), G)$ is injective.*

Sketch of proof. Let $P \to W$ be an fppf G-torsor which is trivial in the generic point. Then we have a morphism $\mathrm{Spec}(K) \to P$ over W and we can take its scheme theoretic image $Z \subset P$. Since $P \to W$ is proper (as a torsor for a proper group algebraic space over W) we see that $Z \to W$ is a proper birational morphism. By Spaces over Fields, Lemma 72.3.2 the morphism $Z \to W$ is finite away from finitely many closed points of W. By (insert future reference on resolving indeterminacies of morphisms by blowing quadratic transformations for surfaces) the irreducible components of the geometric fibres of $Z \to W$ are rational curves. By More on Groupoids in Spaces, Lemma 79.11.3 there are no nonconstant morphisms from rational curves to group schemes or torsors over such. Hence $Z \to W$ is finite, whence Z is a scheme and $Z \to W$ is an isomorphism by Morphisms, Lemma 29.54.8. In other words, the torsor P is trivial. □

[6] An abelian scheme over a base S is a flat proper group scheme of finite presentation over S all of whose fibres are abelian varieties.

0H8E **Remark 110.72.2.** We can get rid of the 2-dimensional condition on W in Lemma 110.72.1. Namely, if W is noetherian, regular and integral, then (1) $P(W) \to P(K)$ is bijective for every G-torsor P and (2) $H^1(W, G) \to H^1(K, G)$ is injective. A sketch of the proof goes as follows. First (2) clearly follows from (1). For (1), the injectivity is trivial, and surjectivity (as in the current proof) is about proving that the schematic closure of a rational section is a section. This question is fppf-local on W, so after étale base change we may assume $P = G$ (note that étale covers preserve regularity). Then observe that $G = \underline{\mathrm{Pic}}^0_{G'/W}$ where G' is the dual abelian scheme. The conclusion follows from the fact that G' is a regular scheme, so generically defined invertible sheaves extend. In fact, one can weaken "W is regular" to "smooth W-schemes are locally factorial". Laurent Moret-Bailly learned the Picard trick from Raynaud; there should be a reference somewhere in Raynaud's work.

0AFA **Lemma 110.72.3.** *Let G be a smooth commutative group algebraic space over a field K. Then $H^1_{fppf}(\mathrm{Spec}(K), G)$ is torsion.*

Proof. Every G-torsor P over $\mathrm{Spec}(K)$ is smooth over K as a form of G. Hence P has a point over a finite separable extension L/K. Say $[L : K] = n$. Let $[n](P)$ denote the G-torsor whose class is n times the class of P in $H^1_{fppf}(\mathrm{Spec}(K), G)$. There is a canonical morphism

$$P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P \to [n](P)$$

of algebraic spaces over K. This morphism is symmetric as G is abelian. Hence it factors through the quotient

$$(P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P)/S_n$$

On the other hand, the morphism $\mathrm{Spec}(L) \to P$ defines a morphism

$$(\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))/S_n \longrightarrow (P \times_{\mathrm{Spec}(K)} \ldots \times_{\mathrm{Spec}(K)} P)/S_n$$

and the reader can verify that the scheme on the left has a K-rational point. Thus we see that $[n](P)$ is the trivial torsor. □

To prove $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A])$ is not an algebraic stack, by Artin's Axioms, Lemma 98.9.5, it is enough to show the following.

0AFB **Lemma 110.72.4.** *The canonical map $\mathcal{X}(S) \to \lim \mathcal{X}(S_n)$ is not essentially surjective.*

Sketch of proof. Unwinding definitions, it is enough to check that $H^1(X, A_0) \to \lim H^1(X_n, A_0)$ is not surjective. As X is regular and projective, by Lemmas 110.72.3 and 110.72.1 each A_0-torsor over X is torsion. In particular, the group $H^1(X, A_0)$ is torsion. It is thus enough to show: (a) the group $H^1(X_0, A_0)$ is non-torsion, and (b) the maps $H^1(X_{n+1}, A_0) \to H^1(X_n, A_0)$ are surjective for all n.

Ad (a). One constructs a nontorsion A_0-torsor P_0 on the nodal curve X_0 by glueing trivial A_0-torsors on each component of X_0 using non-torsion points on A_0 as the isomorphisms over the nodes. More precisely, let $x \in X_0$ be a node which occurs in a loop consisting of rational curves. Let $X'_0 \to X_0$ be the normalization of X_0 in $X_0 \setminus \{x\}$. Let $x', x'' \in X'_0$ be the two points mapping to x_0. Then we take $A_0 \times_{\mathrm{Spec}(k)} X'_0$ and we identify $A_0 \times x'$ with $A_0 \times \{x''\}$ using translation $A_0 \to A_0$ by

a nontorsion point $a_0 \in A_0(k)$ (there is such a nontorsion point as k is algebraically closed and not the algebraic closure of a finite field – this is actually not trivial to prove). One can show that the glueing is an algebraic space (in fact one can show it is a scheme) and that it is an nontorsion A_0-torsor over X_0. The reason that it is nontorsion is that if $[n](P_0)$ has a section, then that section produces a morphism $s : X_0' \to A_0$ such that $[n](a_0) = s(x') - s(x'')$ in the group law on $A_0(k)$. However, since the irreducible components of the loop are rational to section s is constant on them (More on Groupoids in Spaces, Lemma 79.11.3). Hence $s(x') = s(x'')$ and we obtain a contradiction.

Ad (b). Deformation theory shows that the obstruction to deforming an A_0-torsor $P_n \to X_n$ to an A_0-torsor $P_{n+1} \to X_{n+1}$ lies in $H^2(X_0, \omega)$ for a suitable vector bundle ω on X_0. The latter vanishes as X_0 is a curve, proving the claim. □

0AFC **Proposition 110.72.5.** *The stack $\mathcal{X} = \underline{\mathrm{Mor}}_S(X, [S/A])$ is not algebraic.*

Proof. See discussion above. □

0AFD **Remark 110.72.6.** Proposition 110.72.5 contradicts [**Aok06b**, Theorem 1.1]. The problem is the non-effectivity of formal objects for $\underline{\mathrm{Mor}}_S(X, [S/A])$. The same problem is mentioned in the Erratum [**Aok06a**] to [**Aok06b**]. Unfortunately, the Erratum goes on to assert that $\underline{\mathrm{Mor}}_S(\mathcal{Y}, \mathcal{Z})$ is algebraic if $\mathcal{Z}$ is separated, which also contradicts Proposition 110.72.5 as $[S/A]$ is separated.

110.73. An algebraic stack not satisfying strong formal effectiveness

0CXW This is [**Bha16**, Example 4.12]. Let k be an algebraically closed field. Let A be an abelian variety over k. Assume that $A(k)$ is not torsion (this always holds if k is not the algebraic closure of a finite field). Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. We claim there exists an ideal $I \subset k[x, y]$ such that

$$\mathcal{X}_{\mathrm{Spec}(k[x,y]^\wedge)} \longrightarrow \lim \mathcal{X}_{\mathrm{Spec}(k[x,y]/I^n)}$$

is not essentially surjective. Namely, let I be the ideal generated by $xy(x + y - 1)$. Then $X_0 = V(I)$ consists of three copies of $\mathbf{A}^1_k$ glued into a triangle at three points. Hence we can make an infinite order torsor P_0 for A over X_0 by taking the trivial torsor over the irreducible components of X_0 and glueing using translation by nontorsion points. Exactly as in the proof of Lemma 110.72.4 we can lift P_0 to a torsor P_n over $X_n = \mathrm{Spec}(k[x, y]/I^n)$. Since $k[x, y]^\wedge$ is a two dimensional regular domain we see that any torsor P for A over $\mathrm{Spec}(k[x, y]^\wedge)$ is torsion (Lemmas 110.72.1 and 110.72.3). Hence the system of torsors is not in the image of the displayed functor.

0CXX **Lemma 110.73.1.** *Let k be an algebraically closed field which is not the closure of a finite field. Let A be an abelian variety over k. Let $\mathcal{X} = [\mathrm{Spec}(k)/A]$. There exists an inverse system of k-algebras R_n with surjective transition maps whose kernels are locally nilpotent and a system (ξ_n) of $\mathcal{X}$ lying over the system $(\mathrm{Spec}(R_n))$ such that this system is not effective in the sense of Artin's Axioms, Remark 98.20.2.*

Proof. See discussion above. □

110.74. A counter example to Grothendieck's existence theorem

0ARE Let k be a field and let $A = k[[t]]$. Let X be the glueing of $U = \mathrm{Spec}(A[x])$ and $V = \mathrm{Spec}(A[y])$ by the identification

$$U \setminus \{0_U\} \longrightarrow V \setminus \{0_V\}$$

sending x to y where $0_U \in U$ and $O_V \in V$ are the points corresponding to the maximal ideals (x, t) and (y, t). Set $A_n = A/(t^n)$ and set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$. Let $\mathcal{F}_n$ be the coherent sheaf on X_n corresponding to the $A_n[x]$-module $A_n[x]/(x) \cong A_n$ and the $A_n[y]$ module 0 with obvious glueing. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals generate by t. Then $(\mathcal{F}_n)$ is an object of the category $Coh_{\text{support proper over } A}(X, \mathcal{I})$ defined in Cohomology of Schemes, Section 30.27. On the other hand, this object is not in the image of the functor Cohomology of Schemes, Equation (30.27.0.1). Namely, if it where there would be a finite $A[x]$-module M, a finite $A[y]$-module N and an isomorphism $M[1/t] \cong N[1/t]$ such that $M/t^n M \cong A_n[x]/(x)$ and $N/t^n N = 0$ for all n. It is easy to see that this is impossible.

0ARF **Lemma 110.74.1.** *Counter examples to algebraization of coherent sheaves.*

(1) *Grothendieck's existence theorem as stated in Cohomology of Schemes, Theorem 30.27.1 is false if we drop the assumption that $X \to \mathrm{Spec}(A)$ is separated.*
(2) *The stack of coherent sheaves $\mathcal{C}oh_{X/B}$ of Quot, Theorems 99.6.1 and 99.5.12 is in general not algebraic if we drop the assumption that $X \to S$ is separated*
(3) *The functor $\mathrm{Quot}_{\mathcal{F}/X/B}$ of Quot, Proposition 99.8.4 is not an algebraic space in general if we drop the assumption that $X \to B$ is separated.*

Proof. Part (1) we saw above. This shows that $Coh_{X/A}$ fails axiom [4] of Artin's Axioms, Section 98.14. Hence it cannot be an algebraic stack by Artin's Axioms, Lemma 98.9.5. In this way we see that (2) is true. To see (3), note that there are compatible surjections $\mathcal{O}_{X_n} \to \mathcal{F}_n$ for all n. Thus we see that $\mathrm{Quot}_{\mathcal{O}_X/X/A}$ fails axiom [4] and we see that (3) is true as before. □

110.75. Affine formal algebraic spaces

0ANY Let K be a field and let $(V_i)_{i \in I}$ be a directed inverse system of nonzero vector spaces over K with surjective transition maps and with $\lim V_i = 0$, see Section 110.3. Let $R_i = K \oplus V_i$ as K-algebra where V_i is an ideal of square zero. Then R_i is an inverse system of K-algebras with surjective transition maps with nilpotent kernels and with $\lim R_i = K$. The affine formal algebraic space $X = \mathrm{colim}\, \mathrm{Spec}(R_i)$ is an example of an affine formal algebraic space which is not McQuillan.

0CBC **Lemma 110.75.1.** *There exists an affine formal algebraic space which is not McQuillan.*

Proof. See discussion above. □

Let $0 \to W_i \to V_i \to K \to 0$ be a system of exact sequences as in Section 110.3. Let $A_i = K[V_i]/(ww'; w, w' \in W_i)$. Then there is a compatible system of surjections $A_i \to K[t]$ with nilpotent kernels and the transition maps $A_i \to A_j$ are surjective with nilpotent kernels as well. Recall that V_i is free over K with basis given by $s \in S_i$. Then, if the characteristic of K is zero, the degree d part of A_i is free over K

with basis given by s^d, $s \in S_i$ each of which map to t^d. Hence the inverse system of the degree d parts of the A_i is isomorphic to the inverse system of the vector spaces V_i. As $\lim V_i = 0$ we conclude that $\lim A_i = K$, at least when the characteristic of K is zero. This gives an example of an affine formal algebraic space whose "regular functions" do not separate points.

0CBD **Lemma 110.75.2.** *There exists an affine formal algebraic space X whose regular functions do not separate points, in the following sense: If we write $X = \operatorname{colim} X_\lambda$ as in Formal Spaces, Definition 87.9.1 then $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is a field, but X_{red} has infinitely many points.*

Proof. See discussion above. □

Let K, I, and (V_i) be as above. Consider systems

$$\Phi = (\Lambda, J_i \subset \Lambda, (M_i) \to (V_i))$$

where Λ is an augmented K-algebra, $J_i \subset \Lambda$ for $i \in I$ is an ideal of square zero, $(M_i) \to (V_i)$ is a map of inverse systems of K-vector spaces such that $M_i \to V_i$ is surjective for each i, such that M_i has a Λ-module structure, such that the transition maps $M_i \to M_j$, $i > j$ are Λ-linear, and such that $J_j M_i \subset \operatorname{Ker}(M_i \to M_j)$ for $i > j$. Claim: There exists a system as above such that $M_j = M_i/J_j M_i$ for all $i > j$.

If the claim is true, then we obtain a representable morphism

$$\operatorname{colim}_{i \in I} \operatorname{Spec}(\Lambda/J_i \oplus M_i) \longrightarrow \operatorname{Spf}(\lim \Lambda/J_i)$$

of affine formal algebraic spaces whose source is not McQuillan but the target is. Here $\Lambda/J_i \oplus M_i$ has the usual Λ/J_i-algebra structure where M_i is an ideal of square zero. Representability translates exactly into the condition that $M_i/J_j M_i = M_j$ for $i > j$. The source of the morphism is not McQuillan as the projections $\lim_{i \in I} M_i \to M_i$ are not be surjective. This is true because the maps $\lim V_i \to V_i$ are not surjective and we have the surjection $M_i \to V_i$. Some details omitted.

Proof of the claim. First, note that there exists at least one system, namely

$$\Phi_0 = (K, J_i = (0), (V_i) \xrightarrow{\mathrm{id}} (V_i))$$

Given a system Φ we will prove there exists a morphism of systems $\Phi \to \Phi'$ (morphisms of systems defined in the obvious manner) such that $\operatorname{Ker}(M_i/J_j M_i \to M_j)$ maps to zero in $M_i'/J_j' M_i'$. Once this is done we can do the usual trick of setting $\Phi_n = (\Phi_{n-1})'$ inductively for $n \geq 1$ and taking $\Phi = \operatorname{colim} \Phi_n$ to get a system with the desired properties. Details omitted.

Construction of Φ' given Φ. Consider the set U of triples $u = (i, j, \xi)$ where $i > j$ and $\xi \in \operatorname{Ker}(M_i \to M_j)$. We will let $s, t : U \to I$ denote the maps $s(i, j, \xi) = i$ and $t(i, j, \xi) = j$. Then we set $\xi_u \in M_{s(u)}$ the third component of u. We take

$$\Lambda' = \Lambda[x_u; u \in U]/(x_u x_{u'}; u, u' \in U)$$

with augmentation $\Lambda' \to K$ given by the augmentation of Λ and sending x_u to zero. We take $J_k' = J_k \Lambda' + (x_{u,\ t(u) \geq k})$. We set

$$M_i' = M_i \oplus \bigoplus\nolimits_{s(u) \geq i} K \epsilon_{i,u}$$

As transition maps $M_i' \to M_j'$ for $i > j$ we use the given map $M_i \to M_j$ and we send $\epsilon_{i,u}$ to $\epsilon_{j,u}$. The map $M_i' \to V_i$ induces the given map $M_i \to V_i$ and sends $\epsilon_{i,u}$

to zero. Finally, we let Λ' act on M'_i as follows: for $\lambda \in \Lambda$ we act by the Λ-module structure on M_i and via the augmentation $\Lambda \to K$ on $\epsilon_{i,u}$. The element x_u acts as 0 on M_i for all i. Finally, we define

$$x_u \epsilon_{i,u} = \text{image of } \xi_u \text{ in } M_i$$

and we set all other products $x_{u'}\epsilon_{i,u}$ equal to zero. The displayed formula makes sense because $s(u) \geq i$ and $\xi_u \in M_{s(u)}$. The main things the check are $J'_j M'_i \subset M'_i$ maps to zero in M'_j for $i > j$ and that $\mathrm{Ker}(M_i \to M_j)$ maps to zero in $M'_i/J_j M'_i$. The reason for the last fact is that $\xi = x_{(i,j,\xi)}\epsilon_{i,(i,j,\xi)} \in J'_j M'_i$ for any $\xi \in \mathrm{Ker}(M_i \to M_j)$. We omit the details.

0CBE **Lemma 110.75.3.** *There exists a representable morphism $f : X \to Y$ of affine formal algebraic spaces with Y McQuillan, but X not McQuillan.*

Proof. See discussion above. □

110.76. Flat maps are not directed limits of finitely presented flat maps

0ATE The goal of this section is to give an example of a flat ring map which is not a filtered colimit of flat and finitely presented ring maps. In [**Gab96**] it is shown that if A is a nonexcellent local ring of dimension 1 and residue characteristic zero, then the (flat) ring map $A \to A^\wedge$ to its completion is not a filtered colimit of finite type flat ring maps. The example in this section will have a source which is an excellent ring. We encourage the reader to submit other examples; please email stacks.project@gmail.com if you have one.

For the construction, fix a prime p, and let $A = \mathbf{F}_p[x_1, \ldots, x_n]$. Choose an absolute integral closure A^+ of A, i.e., A^+ is the integral closure of A in an algebraic closure of its fraction field. In [**HH92**, §6.7] it is shown that $A \to A^+$ is flat.

We claim that the A-algebra A^+ is not a filtered colimit of finitely presented flat A-algebras if $n \geq 3$.

We sketch the argument in the case $n = 3$, and we leave the generalization to higher n to the reader. It is enough to prove the analogous statement for the map $R \to R^+$, where R is the strict henselization of A at the origin and R^+ is its absolute integral closure. Observe that R is a henselian regular local ring whose residue field k is an algebraic closure of $\mathbf{F}_p$.

Choose an ordinary abelian surface X over k and a very ample line bundle L on X. The section ring $\Gamma_*(X, L) = \bigoplus_n H^0(X, L^n)$ is the coordinate ring of the affine cone over X with respect to L. It is a normal ring for L sufficiently positive. Let S denote the henselization of $\Gamma_*(X, L)$ at vertex of the cone. Then S is a henselian Noetherian normal domain of dimension 3. We obtain a finite injective map $R \to S$ as the henselization of a Noether normalization for the finite type k-algebra $\Gamma_*(X, L)$. As R^+ is an absolute integral closure of R, we can also fix an embedding $S \to R^+$. Thus R^+ is also the absolute integral closure of S. To show R^+ is not a filtered colimit of flat R-algebras, it suffices to show:

(1) If there exists a factorization $S \to P \to R^+$ with P flat and finite type over R, then there exists a factorization $S \to T \to R^+$ with T finite flat over R.
(2) For any factorization $S \to T \to R^+$ with $S \to T$ finite, the ring T is not R-flat.

Indeed, since S is finitely presented over R, if one could write $R^+ = \mathrm{colim}_i P_i$ as a filtered colimit of finitely presented flat R-algebras P_i, then $S \to R^+$ would factor as $S \to P_i \to R^+$ for $i \gg 0$, which contradicts the above pair of assertions. Assertion (1) follows from the fact that R is henselian and a slicing argument, see More on Morphisms, Lemma 37.23.5. Part (2) was proven in [**Bha12**]; for the convenience of the reader, we recall the argument.

Let $U \subset \mathrm{Spec}(S)$ be the punctured spectrum, so there are natural maps $X \leftarrow U \subset \mathrm{Spec}(S)$. The first map gives an identification $H^1(U, \mathcal{O}_U) \simeq H^1(X, \mathcal{O}_X)$. By passing to the Witt vectors of the perfection and using the Artin-Schreier sequence[7], this gives an identification $H^1_{étale}(U, \mathbf{Z}_p) \simeq H^1_{étale}(X, \mathbf{Z}_p)$. In particular, this group is a finite free $\mathbf{Z}_p$-module of rank 2 (since X is ordinary). To get a contradiction assume there exists an R-flat T as in (2) above. Let $V \subset \mathrm{Spec}(T)$ denote the preimage of U, and write $f : V \to U$ for the induced finite surjective map. Since U is normal, there is a trace map $f_*\mathbf{Z}_p \to \mathbf{Z}_p$ on $U_{étale}$ whose composition with the pullback $\mathbf{Z}_p \to f_*\mathbf{Z}_p$ is multiplication by $d = \deg(f)$. Passing to cohomology, and using that $H^1_{étale}(U, \mathbf{Z}_p)$ is nontorsion, then shows that $H^1_{étale}(V, \mathbf{Z}_p)$ is nonzero. Since $H^1_{étale}(V, \mathbf{Z}_p) \simeq \lim H^1_{étale}(V, \mathbf{Z}/p^n)$ as there is no $R^1 \lim$ interference, the group $H^1(V_{étale}, \mathbf{Z}/p)$ must be non-zero. Since T is R-flat we have $\Gamma(V, \mathcal{O}_V) = T$ which is strictly henselian and the Artin-Schreier sequence shows $H^1(V, \mathcal{O}_V) \neq 0$. This is equivalent to $H^2_{\mathfrak{m}}(T) \neq 0$, where $\mathfrak{m} \subset R$ is the maximal ideal. Thus, we obtain a contradiction since T is finite flat (i.e., finite free) as an R-module and $H^2_{\mathfrak{m}}(R) = 0$. This contradiction proves (2).

0ATF **Lemma 110.76.1.** *There exists a commutative ring A and a flat A-algebra B which cannot be written as a filtered colimit of finitely presented flat A-algebras. In fact, we may either choose A to be a finite type $\mathbf{F}_p$-algebra or a 1-dimensional Noetherian local ring with residue field of characteristic 0.*

Proof. See discussion above. □

110.77. The category of modules modulo torsion modules

0B0J The category of torsion groups is a Serre subcategory (Homology, Definition 12.10.1) of the category of all abelian groups. More generally, for any ring A, the category of torsion A-modules is a Serre subcategory of the category of all A-modules, see More on Algebra, Section 15.53. If A is a domain, then the quotient category (Homology, Lemma 12.10.6) is equivalent to the category of vector spaces over the fraction field. This follows from the following more general proposition.

0EA5 **Proposition 110.77.1.** *Let A be a ring. Let S be a multiplicative subset of A. Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of modules for which any element is annihilated by some element of S. Then there is a canonical equivalence $Mod_A/\mathcal{T} \to Mod_{S^{-1}A}$.*

Proof. The functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ given by $M \mapsto M \otimes_A S^{-1}A$ is exact (by Algebra, Proposition 10.9.12) and maps modules in $\mathcal{T}$ to zero. Thus, by the universal property given in Homology, Lemma 12.10.6, the functor descends to a functor $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{S^{-1}A}$.

[7] Here we use that S is a strictly henselian local ring of characteristic p and hence $S \to S$, $f \mapsto f^p - f$ is surjective. Also S is a normal domain and hence $\Gamma(U, \mathcal{O}_U) = S$. Thus $H^1_{étale}(U, \mathbf{Z}/p)$ is the kernel of the map $H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ induced by $f \mapsto f^p - f$.

Conversely, any A-module M with $M \otimes_A S^{-1}A = 0$ is an object of $\mathcal{T}$, since $M \otimes_A S^{-1}A \cong S^{-1}M$ (Algebra, Lemma 10.12.15). Thus Homology, Lemma 12.10.7 shows that the functor $\mathrm{Mod}_A/\mathcal{T} \to \mathrm{Mod}_{S^{-1}A}$ is faithful.

Furthermore, this embedding is essentially surjective: a preimage to an $S^{-1}A$-module N is N_A, that is N regarded as an A-module, since the canonical map $N_A \otimes_A S^{-1}A \to N$ which maps $x \otimes a/s$ to $(a/s) \cdot x$ is an isomorphism of $S^{-1}A$-modules. □

0B0K **Proposition 110.77.2.** *Let A be a ring. Let $Q(A)$ denote its total quotient ring (as in Algebra, Example 10.9.8). Let Mod_A denote the category of A-modules and $\mathcal{T}$ its Serre subcategory of torsion modules. Let $Mod_{Q(A)}$ denote the category of $Q(A)$-modules. Then there is a canonical equivalence $Mod_A/\mathcal{T} \to Mod_{Q(A)}$.*

Proof. Follows immediately from applying Proposition 110.77.1 to the multiplicative subset $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$, since a module is a torsion module if and only if all of its elements are each annihilated by some element of S. □

0B0L **Proposition 110.77.3.** *Let A be a Noetherian integral domain. Let K denote its field of fractions. Let Mod_A^{fg} denote the category of finitely generated A-modules and $\mathcal{T}^{fg}$ its Serre subcategory of finitely generated torsion modules. Then $Mod_A^{fg}/\mathcal{T}^{fg}$ is canonically equivalent to the category of finite dimensional K-vector spaces.*

Proof. The equivalence given in Proposition 110.77.2 restricts along the embedding $\mathrm{Mod}_A^{fg}/\mathcal{T}^{fg} \to \mathrm{Mod}_A/\mathcal{T}$ to an equivalence $\mathrm{Mod}_A^{fg}/\mathcal{T}^{fg} \to \mathrm{Vect}_K^{fd}$. The Noetherian assumption guarantees that Mod_A^{fg} is an abelian category (see More on Algebra, Section 15.53) and that the canonical functor $\mathrm{Mod}_A^{fg}/\mathcal{T}^{fg} \to \mathrm{Mod}_A/\mathcal{T}$ is full (else torsion submodules of finitely generated modules might not be objects of $\mathcal{T}^{fg}$). □

0B0M **Proposition 110.77.4.** *The quotient of the category of abelian groups modulo its Serre subcategory of torsion groups is the category of $\mathbf{Q}$-vector spaces.*

Proof. The claim follows directly from Proposition 110.77.2. □

110.78. Different colimit topologies

0B2Y This example is [**TSH98**, Example 1.2, page 553]. Let $G_n = \mathbf{Q} \times \mathbf{R}^n$, $n \geq 1$ seen as a topological group for addition endowed with the usual (Euclidean) topology. Consider the closed embeddings $G_n \to G_{n+1}$ mapping $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, 0)$. We claim that $G = \operatorname{colim} G_n$ endowed with the topology

$$U \subset G \text{ open} \Leftrightarrow G_n \cap U \text{ open } \forall n$$

is not a topological group.

To see this we consider the set

$$U = \{(x_0, x_1, x_2, \ldots) \text{ such that } |x_j| < |\cos(jx_0)| \text{ for } j > 0\}$$

Using that jx_0 is never an integral multiple of $\pi/2$ as π is not rational it is easy to show that $U \cap G_n$ is open. Since $0 \in U$, if the topology above made G into a topological group, then there would be an open neighbourhood $V \subset G$ of 0 such

that $V + V \subset U$. Then, for every $j \geq 0$ there would exist $\epsilon_j > 0$ such that $(0, \ldots, 0, x_j, 0, \ldots) \in V$ for $|x_j| < \epsilon_j$. Since $V + V \subset U$ we would have

$$(x_0, 0, \ldots, 0, x_j, 0, \ldots) \in U$$

for $|x_0| < \epsilon_0$ and $|x_j| < \epsilon_j$. However, if we take j large enough such that $j\epsilon_0 > \pi/2$, then we can choose $x_0 \in \mathbf{Q}$ such that $|\cos(jx_0)|$ is smaller than ϵ_j, hence there exists an x_j with $|\cos(jx_0)| < |x_j| < \epsilon_j$. This contradiction proves the claim.

0B2Z **Lemma 110.78.1.** *There exists a system $G_1 \to G_2 \to G_3 \to \ldots$ of (abelian) topological groups such that $\operatorname{colim} G_n$ taken in the category of topological spaces is different from $\operatorname{colim} G_n$ taken in the category of topological groups.*

Proof. See discussion above. □

110.79. Universally submersive but not V covering

0EU8 Let A be a valuation ring. Let $\mathfrak{p} \subset A$ be a prime ideal which is neither the minimal prime nor the maximal ideal. (A good case to keep in mind is when A has three prime ideals and $\mathfrak{p}$ is the one in the "middle".) Consider the morphism of affine schemes

$$\operatorname{Spec}(A_\mathfrak{p}) \amalg \operatorname{Spec}(A/\mathfrak{p}) \longrightarrow \operatorname{Spec}(A)$$

We claim this is universally submersive. In order to prove this, let $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a morphism of affine schemes given by the ring map $A \to B$. Then we have to show that

$$\operatorname{Spec}(B_\mathfrak{p}) \amalg \operatorname{Spec}(B/\mathfrak{p}B) \to \operatorname{Spec}(B)$$

is submersive. First of all it is surjective. Next, suppose that $T \subset \operatorname{Spec}(B)$ is a subset such that $T_1 = \operatorname{Spec}(B_\mathfrak{p}) \cap T$ and $T_2 = \operatorname{Spec}(B/\mathfrak{p}B) \cap T$ are closed. Then we see that T is the image of the spectrum of a B-algebra because both T_1 and T_2 are spectra of B-algebras. Hence to show that T is closed it suffices to show that T is stable under specialization, see Algebra, Lemma 10.41.5. To see this, suppose that $p \rightsquigarrow q$ is a specialization of points in $\operatorname{Spec}(B)$ with $p \in T$. Let A' be a valuation ring and let $\operatorname{Spec}(A') \to \operatorname{Spec}(B)$ be a morphism such that the generic point η of $\operatorname{Spec}(A')$ maps to p and the closed point s of $\operatorname{Spec}(A')$ maps to q, see Schemes, Lemma 26.20.4. Observe that the image of the composition $\gamma : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is exactly the set of points $\xi \in \operatorname{Spec}(A)$ with $\gamma(\eta) \rightsquigarrow \xi \rightsquigarrow \gamma(s)$ (details omitted). If $\mathfrak{p} \notin \operatorname{Im}(\gamma)$, then we see that either both $p, q \in \operatorname{Spec}(B_\mathfrak{p})$ or both $p, q \in \operatorname{Spec}(B/\mathfrak{p}B)$. In this case the fact that T_1, resp. T_2 is closed implies that $q \in T_1$, resp. $q \in T_2$ and hence $q \in T$. Finally, suppose $\mathfrak{p} \in \operatorname{Im}(\gamma)$, say $\mathfrak{p} = \gamma(r)$. Then we have specializations $p \rightsquigarrow r$ and $r \rightsquigarrow q$. In this case $p, r \in \operatorname{Spec}(B_\mathfrak{p})$ and $r, q \in \operatorname{Spec}(B/\mathfrak{p}B)$. Then we fist conclude $r \in T_1 \subset T$, then $r \in T_2$ as r maps to $\mathfrak{p}$, and then $q \in T_2 \subset T$ as desired.

On the other hand, we claim that the singleton family

$$\{\operatorname{Spec}(A_\mathfrak{p}) \amalg \operatorname{Spec}(A/\mathfrak{p}) \longrightarrow \operatorname{Spec}(A)\}$$

is not a V covering. See Topologies, Definition 34.10.7. Namely, if it where a V covering, there would be an extension of valuation ring $A \subset B$ such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors through $\operatorname{Spec}(A_\mathfrak{p}) \amalg \operatorname{Spec}(A/\mathfrak{p})$. This would imply $\operatorname{Spec}(A')$ is disconnected which is absurd.

0EU9 **Lemma 110.79.1.** *There exists a morphism $X \to Y$ of affine schemes which is universally submersive such that $\{X \to Y\}$ is not a V covering.*

Proof. See discussion above. □

110.80. The spectrum of the integers is not quasi-compact

0EUE Of course the title of this section doesn't refer to the spectrum of the integers as a topological space, because any spectrum is quasi-compact as a topological space (Algebra, Lemma 10.17.8). No, it refers to the spectrum of the integers in the canonical topology on the category of schemes, and the definition of a quasi-compact object in a site (Sites, Definition 7.17.1).

Let U be a nonprincipal ultrafilter on the set P of prime numbers. For a subset $T \subset P$ we denote $T^c = P \setminus T$ the complement. For $A \in U$ let $S_A \subset \mathbf{Z}$ be the multiplicative subset generated by $p \in A$. Set

$$\mathbf{Z}_A = S_A^{-1}\mathbf{Z}$$

Observe that $\mathrm{Spec}(\mathbf{Z}_A) = \{(0)\} \cup A^c \subset \mathrm{Spec}(\mathbf{Z})$ if we think of P as the set of closed points of $\mathrm{Spec}(\mathbf{Z})$. If $A, B \in U$, then $A \cap B \in U$ and $A \cup B \in U$ and we have

$$\mathbf{Z}_{A\cap B} = \mathbf{Z}_A \times_{\mathbf{Z}_{A\cup B}} \mathbf{Z}_B$$

(fibre product of rings). In particular, for any integer n and elements $A_1, \ldots, A_n \in U$ the morphisms

$$\mathrm{Spec}(\mathbf{Z}_{A_1}) \amalg \ldots \amalg \mathrm{Spec}(\mathbf{Z}_{A_n}) \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

factors through $\mathrm{Spec}(\mathbf{Z}[1/p])$ for some p (namely for any $p \in A_1 \cap \ldots \cap A_n$). We conclude that the family of flat morphisms $\{\mathrm{Spec}(\mathbf{Z}_A) \to \mathrm{Spec}(\mathbf{Z})\}_{A\in U}$ is jointly surjective, but no finite subset is.

For a $\mathbf{Z}$-module M we set

$$M_A = S_A^{-1}M = M \otimes_{\mathbf{Z}} \mathbf{Z}_A$$

Claim I: for every $\mathbf{Z}$-module M we have

$$M = \mathrm{Equalizer}\Big(\prod\nolimits_{A\in U} M_A \rightrightarrows \prod\nolimits_{A,B\in U} M_{A\cup B} \Big)$$

First, assume M is torsion free. Then $M_A \subset M_P$ for all $A \in U$. Hence we see that we have to prove

$$M = \bigcap\nolimits_{A\in U} M_A \text{ inside } M_P = M \otimes \mathbf{Q}$$

Namely, since U is nonprincipal, for any prime p we have $\{p\}^c \in U$. Also, $M_{\{p\}^c} = M_{(p)}$ is equal to the localization at the prime (p). Thus the above is clear because already $M_{(2)} \cap M_{(3)} = M$. Next, assume M is torsion. Then we have

$$M = \bigoplus\nolimits_{p\in P} M[p^\infty]$$

and correspondingly we have

$$M_A = \bigoplus\nolimits_{p\notin A} M[p^\infty]$$

because we are localizing at the primes in A. Suppose that $(x_A) \in \prod M_A$ is in the equalizer. Denote $x_p = x_{\{p\}^c} \in M[p^\infty]$. Then the equalizer property says

$$x_A = (x_p)_{p\notin A}$$

and in particular it says that x_p is zero for all but a finite number of $p \notin A$. To finish the proof in the torsion case it suffices to show that x_p is zero for all but a finite number of primes p. If not write $\{p \in P \mid x_p \neq 0\} = T \amalg T'$ as the disjoint

union of two infinite sets. Then either $T \notin U$ or $T' \notin U$ because U is an ultrafilter (namely if both T, T' are in U then U contains $T \cap T' = \emptyset$ which is not allowed). Say $T \notin U$. Then $T = A^c$ and this contradicts the finiteness mentioned above. Finally, suppose that M is a general module. Then we look at the short exact sequence

$$0 \to M_{tors} \to M \to M/M_{tors} \to 0$$

and we look at the following large diagram

$$\begin{array}{ccccc}
M_{tors} & \longrightarrow & \prod_{A\in U} M_{tors,A} & \rightrightarrows & \prod_{A,B\in U} M_{tors,A\cup B} \\
\downarrow & & \downarrow & & \downarrow \\
M & \longrightarrow & \prod_{A\in U} M_A & \rightrightarrows & \prod_{A,B\in U} M_{A\cup B} \\
\downarrow & & \downarrow & & \downarrow \\
M/M_{tors} & \longrightarrow & \prod_{A\in U} (M/M_{tors})_A & \rightrightarrows & \prod_{A,B\in U} (M/M_{tors})_{A\cup B}
\end{array}$$

Doing a diagram chase using exactness of the columns and the result for the torsion module M_{tors} and the torsion free module M/M_{tors} proving Claim I for M. This gives an example of the phenomenon in the following lemma.

0EUF **Lemma 110.80.1.** *There exists a ring A and an infinite family of flat ring maps $\{A \to A_i\}_{i\in I}$ such that for every A-module M*

$$M = \mathit{Equalizer}\Big(\prod_{i\in I} M \otimes_A A_i \rightrightarrows \prod_{i,j\in I} M \otimes_A A_i \otimes_A A_j \Big)$$

but there is no finite subfamily where the same thing is true.

Proof. See discussion above. □

We continue working with our nonprincipal ultrafilter U on the set P of prime numbers. Let R be a ring. Denote $R_A = S_A^{-1}R = R \otimes \mathbf{Z}_A$ for $A \in U$. Claim II: given closed subsets $T_A \subset \mathrm{Spec}(R_A)$, $A \in U$ such that

$$(\mathrm{Spec}(R_{A\cup B}) \to \mathrm{Spec}(R_A))^{-1} T_A = (\mathrm{Spec}(R_{A\cup B}) \to \mathrm{Spec}(R_B))^{-1} T_B$$

for all $A, B \in U$, there is a closed subset $T \subset \mathrm{Spec}(R)$ with $T_A = (\mathrm{Spec}(R_A) \to \mathrm{Spec}(R))^{-1}(T)$ for all $A \in U$. Let $I_A \subset R_A$ for $A \in U$ be the radical ideal cutting out T_A. Then the glueing condition implies $S_{A\cup B}^{-1} I_A = S_{A\cup B}^{-1} I_B$ in $R_{A\cup B}$ for all $A, B \in U$ (because localization preserves being a radical ideal). Let $I' \subset R$ be the set of elements mapping into $I_P \subset R_P = R \otimes \mathbf{Q}$. Then we see for $A \in U$ that

(1) $I_A \subset I'_A = S_A^{-1} I'$, and
(2) $M_A = I'_A / I_A$ is a torsion module.

Of course we obtain canonical identifications $S_{A\cup B}^{-1} M_A = S_{A\cup B}^{-1} M_B$ for $A, B \in U$. Decomposing the torsion modules M_A into their p-primary components, the reader easily shows that there exist p-power torsion R-modules M_p such that

$$M_A = \bigoplus\nolimits_{p \notin A} M_p$$

compatible with the canonical identifications given above. Setting $M = \bigoplus_{p\in P} M_p$ we find canonical isomorphisms $M_A = S_A^{-1} M$ compatible with the above canonical identifications. Then we get a canonical map

$$I' \longrightarrow M$$

of R-modules which recovers the map $I_A \to M_A$ for all $A \in U$. This is true by all the compatibilities mentioned above and the claim proved previously that M is the equalizer of the two maps from $\prod_{A \in U} M_A$ to $\prod_{A,B \in U} M_{A \cup B}$. Let $I = \mathrm{Ker}(I' \to M)$. Then I is an ideal and $T = V(I)$ is a closed subset which recovers the closed subsets T_A for all $A \in U$. This proves Claim II.

0EUG **Lemma 110.80.2.** *The scheme* $\mathrm{Spec}(\mathbf{Z})$ *is not quasi-compact in the canonical topology on the category of schemes.*

Proof. With notation as above consider the family of morphisms

$$\mathcal{W} = \{\mathrm{Spec}(\mathbf{Z}_A) \to \mathrm{Spec}(\mathbf{Z})\}_{A \in U}$$

By Descent, Lemma 35.13.5 and the two claims proved above this is a universal effective epimorphism. In any category with fibre products, the universal effective epimorphisms give $\mathcal{C}$ the structure of a site (modulo some set theoretical issues which are easy to fix) defining the canonical topology. Thus $\mathcal{W}$ is a covering for the canonical topology. On the other hand, we have seen above that any finite subfamily

$$\{\mathrm{Spec}(\mathbf{Z}_{A_i}) \to \mathrm{Spec}(\mathbf{Z})\}_{i=1,\ldots,n}, \quad n \in \mathbf{N}, A_1, \ldots, A_n \in U$$

factors through $\mathrm{Spec}(\mathbf{Z}[1/p])$ for some p. Hence this finite family cannot be a universal effective epimorphism and more generally no universal effective epimorphism $\{g_j : T_j \to \mathrm{Spec}(\mathbf{Z})\}$ can refine $\{\mathrm{Spec}(\mathbf{Z}_{A_i}) \to \mathrm{Spec}(\mathbf{Z})\}_{i=1,\ldots,n}$. By Sites, Definition 7.17.1 this means that $\mathrm{Spec}(\mathbf{Z})$ is not quasi-compact in the canonical topology. To see that our notion of quasi-compactness agrees with the usual topos theoretic definition, see Sites, Lemma 7.17.3. □

110.81. Other chapters

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CHAPTER 111

Exercises

0275

111.1. Algebra

0276 This first section just contains some assorted questions.

02CG **Exercise 111.1.1.** Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X - 1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

02CH **Exercise 111.1.2.** Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be *coherent* if the last property holds.)

02CI **Exercise 111.1.3.** Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(1) Show that $r(M \otimes_A N) = r(M)r(N)$.
(2) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.
(3) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

02CJ **Exercise 111.1.4.** Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(1) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(2) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(3) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

02CK **Remark 111.1.5.** Of course the idea of this exercise is to find a simple argument in each case rather than applying a "big" theorem. Nonetheless it is good to be guided by general principles.

02CL **Exercise 111.1.6.** Algebra. (Silly and should be easy.)

(1) Give an example of a ring A and a nonsplit short exact sequence of A-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0.$$
(2) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that
$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$
is split as a sequence of B-modules.

02CM **Exercise 111.1.7.** Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(1) Show that the characteristic of k is prime to n.
(2) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n for $n \geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

02CN **Exercise 111.1.8.** Let $\nu : k[x] \setminus \{0\} \to \mathbf{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f + g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f + g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(1) Show that if f, g, and $f + g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f + g) = min(\nu(f), \nu(g))$.
(2) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?
(3) Show that if ν attains a negative value then $\nu(f) = -n \deg(f)$ for some $n \in \mathbf{N}$.
(4) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0, \text{ or } \nu(f) > 0\}$ is a prime ideal of $k[x]$.
(5) Describe all possible ν.

Let A be a ring. An *idempotent* is an element $e \in A$ such that $e^2 = e$. The elements 1 and 0 are always idempotent. A *nontrivial idempotent* is an idempotent which is not equal to zero. Two idempotents $e, e' \in A$ are called *orthogonal* if $ee' = 0$.

078G **Exercise 111.1.9.** Let A be a ring. Show that A is a product of two nonzero rings if and only if A has a nontrivial idempotent.

078H **Exercise 111.1.10.** Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Show that the map $A \to A/I$ induces a bijection on idempotents. (Hint: It may be easier to prove this when I is nilpotent. Do this first. Then use "absolute Noetherian reduction" to reduce to the nilpotent case.)

111.2. Colimits

0277

078I **Definition 111.2.1.** A *directed set* is a nonempty set I endowed with a preorder $\leq$ such that given any pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A *system of rings* over I is given by a ring A_i for each $i \in I$ and a map of rings $\varphi_{ij} : A_i \to A_j$ whenever $i \leq j$ such that the composition $A_i \to A_j \to A_k$ is equal to $A_i \to A_k$ whenever $i \leq j \leq k$.

One similarly defines systems of groups, modules over a fixed ring, vector spaces over a field, etc.

078J **Exercise 111.2.2.** Let I be a directed set and let (A_i, φ_{ij}) be a system of rings over I. Show that there exists a ring A and maps $\varphi_i : A_i \to A$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for all $i \leq j$ with the following universal property: Given any ring B and maps $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j$, then there exists a unique ring map $\psi : A \to B$ such that $\psi_i = \psi \circ \varphi_i$.

078K **Definition 111.2.3.** The ring A constructed in Exercise 111.2.2 is called the *colimit* of the system. Notation $\operatorname{colim} A_i$.

078L **Exercise 111.2.4.** Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Prove that there is a bijection

$$\operatorname{Spec}(A) = \{(\mathfrak{p}_i)_{i \in I} \mid \mathfrak{p}_i \subset A_i \text{ and } \mathfrak{p}_i = \varphi_{ij}^{-1}(\mathfrak{p}_j) \ \forall i \leq j\} \subset \prod\nolimits_{i \in I} \operatorname{Spec}(A_i)$$

The set on the right hand side of the equality is the limit of the sets $\mathrm{Spec}(A_i)$. Notation $\lim \mathrm{Spec}(A_i)$.

078M **Exercise 111.2.5.** Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Suppose that $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$ is surjective for all $i \leq j$. Show that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ is surjective for all i. (Hint: You can try to use Tychonoff, but there is also a basically trivial direct algebraic proof based on Algebra, Lemma 10.18.6.)

078N **Exercise 111.2.6.** Let $A \subset B$ be an integral ring extension. Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Use the exercises above, the fact that this holds for a finite ring extension (proved in the lectures), and by proving that $B = \mathrm{colim}\, B_i$ is a directed colimit of finite extensions $A \subset B_i$.

02CO **Exercise 111.2.7.** Let $(I, \geq)$ be a directed set. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$\mathrm{colim}_{i \in I}\, M \otimes_A N_i \cong M \otimes_A \Big(\mathrm{colim}_{i \in I}\, N_i \Big).$$

0278 **Definition 111.2.8.** A module M over R is said to be of *finite presentation* over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

02CP **Exercise 111.2.9.** Prove that any module over any ring is

(1) the colimit of its finitely generated submodules, and
(2) in some way a colimit of finitely presented modules.

111.3. Additive and abelian categories

057X

057Y **Exercise 111.3.1.** Let k be a field. Let $\mathcal{C}$ be the category of filtered vector spaces over k, see Homology, Definition 12.19.1 for the definition of a filtered object of any category.

(1) Show that this is an additive category (explain carefully what the direct sum of two objects is).
(2) Let $f : (V, F) \to (W, F)$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (explain precisely what the kernel and cokernel of f are).
(3) Give an example of a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

057Z **Exercise 111.3.2.** Let R be a Noetherian domain. Let $\mathcal{C}$ be the category of finitely generated torsion free R-modules.

(1) Show that this is an additive category.
(2) Let $f : N \to M$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (make sure you define precisely what the kernel and cokernel of f are).
(3) Give an example of a Noetherian domain R and a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

0580 **Exercise 111.3.3.** Give an example of a category which is additive and has kernels and cokernels but which is not as in Exercises 111.3.1 and 111.3.2.

111.4. Tensor product

0CYG Tensor products are introduced in Algebra, Section 10.12. Let R be a ring. Let Mod_R be the category of R-modules. We will say that a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$

(1) is additive if $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F(M), F(N))$ is a homomorphism of abelian groups for any R-modules M, N, see Homology, Definition 12.3.1.
(2) R-linear if $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F(M), F(N))$ is R-linear for any R-modules M, N,
(3) right exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $F(M_1) \to F(M_2) \to F(M_3) \to 0$ is exact,
(4) left exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact,
(5) commutes with direct sums, if given a set I and R-modules M_i the maps $F(M_i) \to F(\bigoplus M_i)$ induce an isomorphism $\bigoplus F(M_i) = F(\bigoplus M_i)$.

0CYH **Exercise 111.4.1.** Let R be a ring. With notation as above.

(1) Give an example of a ring R and an additive functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ which is not R-linear.
(2) Let N be an R-module. Show that the functor $F(M) = M \otimes_R N$ is R-linear, right exact, and commutes with direct sums,
(3) Conversely, show that any functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ which is R-linear, right exact, and commutes with direct sums is of the form $F(M) = M \otimes_R N$ for some R-module N.
(4) Show that if in (3) we drop the assumption that F commutes with direct sums, then the conclusion no longer holds.

111.5. Flat ring maps

0279

02CQ **Exercise 111.5.1.** Let S be a multiplicative subset of the ring A.

(1) For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
(2) Show that $S^{-1}A$ is flat over A.

02CR **Exercise 111.5.2.** Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:

(1) $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
(2) $A = k[x, y]$ and $N = A/(x, y)$.

02CS **Exercise 111.5.3.** Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

02CT **Remark 111.5.4.** If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbf{R}$.

02CU **Exercise 111.5.5.** Find a flat but not free module over $\mathbf{Z}_{(2)}$.

02CV **Exercise 111.5.6.** Flat deformations.

(1) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.

(2) Suppose that $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and

$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p).$$

Show that there exists a flat $\mathbf{Z}/p^2\mathbf{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)

(3) Now let $p = 2$ and consider the same question for $k = \mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ and

$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6).$$

However, in this case show that there does *not* exist a flat $\mathbf{Z}/4\mathbf{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)

02CW **Exercise 111.5.7.** Let $(A, \mathfrak{m}, k)$ be a local ring and let k'/k be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

02CX **Remark 111.5.8.** The same result holds for arbitrary field extensions K/k.

111.6. The Spectrum of a ring

027A

02CY **Exercise 111.6.1.** Compute $\mathrm{Spec}(\mathbf{Z})$ as a set and describe its topology.

02CZ **Exercise 111.6.2.** Let A be any ring. For $f \in A$ we define $D(f) := \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$. Prove that the open subsets $D(f)$ form a basis of the topology of $\mathrm{Spec}(A)$.

02D0 **Exercise 111.6.3.** Prove that the map $I \mapsto V(I)$ defines a natural bijection

$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \mathrm{Spec}(A) \text{ closed}\}$$

027B **Definition 111.6.4.** A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

02D1 **Exercise 111.6.5.** Prove that $\mathrm{Spec}(A)$ is quasi-compact for any ring A.

027C **Definition 111.6.6.** A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

02D2 **Exercise 111.6.7.** Show that $\mathrm{Spec}(A)$ is **not** Hausdorff in general. Prove that $\mathrm{Spec}(A)$ is T_0. Give an example of a topological space X that is not T_0.

02D3 **Remark 111.6.8.** Usually the word compact is reserved for quasi-compact and Hausdorff spaces.

027D **Definition 111.6.9.** A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called *irreducible* if it is an irreducible topological space with the topology induced from X. This definition implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

02D4 **Exercise 111.6.10.** Prove that $\mathrm{Spec}(A)$ is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

02D5 **Exercise 111.6.11.** Prove that a closed subset $T \subset \mathrm{Spec}(A)$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

027E **Definition 111.6.12.** A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

02D6 **Exercise 111.6.13.** Show that in a T_0 space X every irreducible closed subset has at most one generic point.

02D7 **Exercise 111.6.14.** Prove that in $\mathrm{Spec}(A)$ every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of $\mathrm{Spec}(A)$ with the set of irreducible closed subsets of X.

02D8 **Exercise 111.6.15.** Give an example to show that an irreducible subset of $\mathrm{Spec}(\mathbf{Z})$ does not necessarily have a generic point.

027F **Definition 111.6.16.** A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes. (It is called *Artinian* if any increasing sequence of closed subsets stabilizes.)

02D9 **Exercise 111.6.17.** Show that if the ring A is Noetherian then the topological space $\mathrm{Spec}(A)$ is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

027G **Definition 111.6.18.** A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

02DA **Exercise 111.6.19.** Prove that any irreducible subset of X is contained in an irreducible component of X.

02DB **Exercise 111.6.20.** Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbf{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

02DC **Exercise 111.6.21.** Show that irreducible components of $\mathrm{Spec}(A)$ correspond to minimal primes of A.

027H **Definition 111.6.22.** A point $x \in X$ is called *closed* if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a *specialization* of y, or that y is a *generalization* of x if $x \in \overline{\{y\}}$.

02DD **Exercise 111.6.23.** Show that closed points of $\mathrm{Spec}(A)$ correspond to maximal ideals of A.

02DE **Exercise 111.6.24.** Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in $\mathrm{Spec}(A)$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

02DF **Exercise 111.6.25.** Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

027I **Definition 111.6.26.** A topological space X is called *connected* if it is nonempty and not the union of two nonempty disjoint open subsets. A *connected component* of X is a maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

02DG **Exercise 111.6.27.** Let A be a nonzero ring. Show that $\mathrm{Spec}(A)$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

02DH **Exercise 111.6.28.** Let T be a connected component of $\mathrm{Spec}(A)$. Prove that T is stable under generalization. Prove that T is an open subset of $\mathrm{Spec}(A)$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbf{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

02DI **Exercise 111.6.29.** Compute $\mathrm{Spec}(k[x])$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

02DJ **Exercise 111.6.30.** Compute $\mathrm{Spec}(k[x, y])$, where k is algebraically closed. [Hint: use the morphism $\varphi : \mathrm{Spec}(k[x, y]) \to \mathrm{Spec}(k[x])$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

02DK **Exercise 111.6.31.** Compute $\mathrm{Spec}(\mathbf{Z}[y])$. [Hint: as above.] (Affine 1-space over $\mathbf{Z}$.)

111.7. Localization

0766

0767 **Exercise 111.7.1.** Let A be a ring. Let $S \subset A$ be a multiplicative subset. Let M be an A-module. Let $N \subset S^{-1}M$ be an $S^{-1}A$-submodule. Show that there exists an A-submodule $N' \subset M$ such that $N = S^{-1}N'$. (This useful result applies in particular to ideals of $S^{-1}A$.)

0768 **Exercise 111.7.2.** Let A be a ring. Let M be an A-module. Let $m \in M$.

(1) Show that $I = \{a \in A \mid am = 0\}$ is an ideal of A.
(2) For a prime $\mathfrak{p}$ of A show that the image of m in $M_\mathfrak{p}$ is zero if and only if $I \not\subset \mathfrak{p}$.
(3) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A.
(4) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A.
(5) Show that $M = 0$ if and only if $M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$.

0769 **Exercise 111.7.3.** Find a pair (A, f) where A is a domain with three or more pairwise distinct primes and $f \in A$ is an element such that the principal localization $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is a field.

076A **Exercise 111.7.4.** Let A be a ring. Let M be a finite A-module. Let $S \subset A$ be a multiplicative set. Assume that $S^{-1}M = 0$. Show that there exists an $f \in S$ such that the principal localization $M_f = \{1, f, f^2, \ldots\}^{-1}M$ is zero.

076B **Exercise 111.7.5.** Give an example of a triple (A, I, S) where A is a ring, $0 \neq I \neq A$ is a proper nonzero ideal, and $S \subset A$ is a multiplicative subset such that $A/I \cong S^{-1}A$ as A-algebras.

111.8. Nakayama's Lemma

076C

076D **Exercise 111.8.1.** Let A be a ring. Let I be an ideal of A. Let M be an A-module. Let $x_1, \ldots, x_n \in M$. Assume that

(1) M/IM is generated by $x_1, \ldots, x_n$,
(2) M is a finite A-module,
(3) I is contained in every maximal ideal of A.

Show that $x_1, \ldots, x_n$ generate M. (Suggested solution: Reduce to a localization at a maximal ideal of A using Exercise 111.7.2 and exactness of localization. Then reduce to the statement of Nakayama's lemma in the lectures by looking at the quotient of M by the submodule generated by $x_1, \ldots, x_n$.)

111.9. Length

027J

076E **Definition 111.9.1.** Let A be a ring. Let M be an A-module. The *length* of M as an R-module is

$$\text{length}_A(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words, the supremum of the lengths of chains of submodules.

076F **Exercise 111.9.2.** Show that a module M over a ring A has length 1 if and only if it is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

076G **Exercise 111.9.3.** Compute the length of the following modules over the following rings. Briefly(!) explain your answer. (Please feel free to use additivity of the length function in short exact sequences, see Algebra, Lemma 10.52.3).

(1) The length of $\mathbf{Z}/120\mathbf{Z}$ over $\mathbf{Z}$.
(2) The length of $\mathbf{C}[x]/(x^{100} + x + 1)$ over $\mathbf{C}[x]$.
(3) The length of $\mathbf{R}[x]/(x^4 + 2x^2 + 1)$ over $\mathbf{R}[x]$.

02DL **Exercise 111.9.4.** Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)

111.10. Associated primes

0CR7 Associated primes are discussed in Algebra, Section 10.63

0CR8 **Exercise 111.10.1.** Compute the set of associated primes for each of the following modules.

(1) $R = k[x, y]$ and $M = R/(xy(x + y))$,
(2) $R = \mathbf{Z}[x]$ and $M = R/(300x + 75)$, and
(3) $R = k[x, y, z]$ and $M = R/(x^3, x^2y, xz)$.

Here as usual k is a field.

0CR9 **Exercise 111.10.2.** Give an example of a Noetherian ring R and a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ is not the only associated prime of $R/\mathfrak{p}^2$.

0CRA **Exercise 111.10.3.** Let R be a Noetherian ring with incomparable prime ideals $\mathfrak{p}$, $\mathfrak{q}$, i.e., $\mathfrak{p} \not\subset \mathfrak{q}$ and $\mathfrak{q} \not\subset \mathfrak{p}$.

(1) Show that for $N = R/(\mathfrak{p} \cap \mathfrak{q})$ we have $\text{Ass}(N) = \{\mathfrak{p}, \mathfrak{q}\}$.
(2) Show by an example that the module $M = R/\mathfrak{p}\mathfrak{q}$ can have an associated prime not equal to $\mathfrak{p}$ or $\mathfrak{q}$.

111.11. Ext groups

0CRB Ext groups are defined in Algebra, Section 10.71.

0CRC **Exercise 111.11.1.** Compute all the Ext groups $\text{Ext}^i(M, N)$ of the given modules in the category of $\mathbf{Z}$-modules (also known as the category of abelian groups).

(1) $M = \mathbf{Z}$ and $N = \mathbf{Z}$,
(2) $M = \mathbf{Z}/4\mathbf{Z}$ and $N = \mathbf{Z}/8\mathbf{Z}$,
(3) $M = \mathbf{Q}$ and $N = \mathbf{Z}/2\mathbf{Z}$, and
(4) $M = \mathbf{Z}/2\mathbf{Z}$ and $N = \mathbf{Q}/\mathbf{Z}$.

0CRD **Exercise 111.11.2.** Let $R = k[x, y]$ where k is a field.

(1) Show by hand that the Koszul complex
$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(x,y)} R \xrightarrow{f \mapsto f(0,0)} k \to 0$$
is exact.
(2) Compute $\text{Ext}^i_R(k, k)$ where $k = R/(x, y)$ as an R-module.

0CRE **Exercise 111.11.3.** Give an example of a Noetherian ring R and finite modules M, N such that $\text{Ext}^i_R(M, N)$ is nonzero for all $i \geq 0$.

0CRF **Exercise 111.11.4.** Give an example of a ring R and ideal I such that $\text{Ext}^1_R(R/I, R/I)$ is not a finite R-module. (We know this cannot happen if R is Noetherian by Algebra, Lemma 10.71.9.)

111.12. Depth

0CS0 Depth is defined in Algebra, Section 10.72 and further studied in Dualizing Complexes, Section 47.11.

0CS1 **Exercise 111.12.1.** Let R be a ring, $I \subset R$ an ideal, and M an R-module. Compute $\text{depth}_I(M)$ in the following cases.

(1) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}$,
(2) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(300)$,
(3) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(7)$,
(4) $R = k[x, y, z]/(x^2 + y^2 + z^2)$, $I = (x, y, z)$, $M = R$,
(5) $R = k[x, y, z, w]/(xz, xw, yz, yw)$, $I = (x, y, z, w)$, $M = R$.

Here k is a field. In the last two cases feel free to localize at the maximal ideal I.

0CS2 **Exercise 111.12.2.** Give an example of a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ of depth ≥ 1 and a prime ideal $\mathfrak{p}$ such that

(1) $\text{depth}_{\mathfrak{m}}(R) \geq 1$,
(2) $\text{depth}_{\mathfrak{p}}(R_{\mathfrak{p}}) = 0$, and
(3) $\dim(R_{\mathfrak{p}}) \geq 1$.

If we don't ask for (3) then the exercise is too easy. Why?

0CS3 **Exercise 111.12.3.** Let $(R, \mathfrak{m})$ be a local Noetherian domain. Let M be a finite R-module.

(1) If M is torsion free, show that M has depth at least 1 over R.
(2) Give an example with depth equal to 1.

0CS4 **Exercise 111.12.4.** For every $m \geq n \geq 0$ give an example of a Noetherian local ring R with $\dim(R) = m$ and $\text{depth}(R) = n$.

0CSZ **Exercise 111.12.5.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module. Show that there exists a canonical short exact sequence

$$0 \to K \to M \to Q \to 0$$

such that the following are true

(1) $\text{depth}(Q) \geq 1$,
(2) K is zero or $\text{Supp}(K) = \{\mathfrak{m}\}$, and
(3) $\text{length}_R(K) < \infty$.

Hint: using the Noetherian property show that there exists a maximal submodule K as in (2) and then show that $Q = M/K$ satisfies (1) and K satisfies (3).

0CT0 **Exercise 111.12.6.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module of depth ≥ 2. Let $N \subset M$ be a nonzero submodule.

(1) Show that $\text{depth}(N) \geq 1$.
(2) Show that $\text{depth}(N) = 1$ if and only if the quotient module M/N has $\text{depth}(M/N) = 0$.
(3) Show there exists a submodule $N' \subset M$ with $N \subset N'$ of finite colength, i.e., $\text{length}_R(N'/N) < \infty$, such that N' has depth ≥ 2. Hint: Apply Exercise 111.12.5 to M/N and choose N' to be the inverse image of K.

0CT1 **Exercise 111.12.7.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Assume that R is reduced, i.e., R has no nonzero nilpotent elements. Assume moreover that R has two distinct minimal primes $\mathfrak{p}$ and $\mathfrak{q}$.

(1) Show that the sequence of R-modules

$$0 \to R \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/\mathfrak{p} + \mathfrak{q} \to 0$$

is exact (check at all the spots). The maps are $x \mapsto (x \bmod \mathfrak{p}, x \bmod \mathfrak{q})$ and $(y \bmod \mathfrak{p}, z \bmod \mathfrak{q}) \mapsto (y - z \bmod \mathfrak{p} + \mathfrak{q})$.
(2) Show that if $\text{depth}(R) \geq 2$, then $\dim(R/\mathfrak{p} + \mathfrak{q}) \geq 1$.
(3) Show that if $\text{depth}(R) \geq 2$, then $U = \text{Spec}(R) \setminus \{\mathfrak{m}\}$ is a connected topological space.

This proves a very special case of Hartshorne's connectedness theorem which says that the punctured spectrum U of a local Noetherian ring of depth ≥ 2 is connected.

0CT2 **Exercise 111.12.8.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $x, y \in \mathfrak{m}$ be a regular sequence of length 2. For any $n \geq 2$ show that there do not exist $a, b \in R$ with

$$x^{n-1}y^{n-1} = ax^n + by^n$$

Suggestion: First try for $n = 2$ to see how to argue. Remark: There is a vast generalization of this result called the monomial conjecture.

111.13. Cohen-Macaulay modules and rings

0CT3 Cohen-Macaulay modules are studied in Algebra, Section 10.103 and Cohen-Macaulay rings are studied in Algebra, Section 10.104.

0CT4 **Exercise 111.13.1.** In the following cases, please answer yes or no. No explanation or proof necessary.

(1) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a Cohen-Macaulay local ring?
(2) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a regular local ring?
(3) Let k be a field. Is the local ring $k[x]_{(x)}$ a Cohen-Macaulay local ring?
(4) Let k be a field. Is the local ring $k[x]_{(x)}$ a regular local ring?
(5) Let k be a field. Is the local ring $(k[x,y]/(y^2-x^3))_{(x,y)} = k[x,y]_{(x,y)}/(y^2-x^3)$ a Cohen-Macaulay local ring?
(6) Let k be a field. Is the local ring $(k[x,y]/(y^2,xy))_{(x,y)} = k[x,y]_{(x,y)}/(y^2,xy)$ a Cohen-Macaulay local ring?

111.14. Singularities

027K

02DM **Exercise 111.14.1.** Let k be any field. Suppose that $A = k[[x,y]]/(f)$ and $B = k[[u,v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u,v)^3$. Show that A and B are isomorphic rings.

02DN **Remark 111.14.2.** A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x,y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a, b, c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomorphism class of pairs (k', q).

0D1S **Exercise 111.14.3.** Let R be a ring. Let $n \geq 1$. Let A, B be $n \times n$ matrices with coefficients in R such that $AB = f1_{n\times n}$ for some nonzerodivisor f in R. Set $S = R/(f)$. Show that

$$\ldots \to S^{\oplus n} \xrightarrow{B} S^{\oplus n} \xrightarrow{A} S^{\oplus n} \xrightarrow{B} S^{\oplus n} \to \ldots$$

is exact.

111.15. Constructible sets

0FJ3 Let k be an algebraically closed field, for example the field $\mathbf{C}$ of complex numbers. Let $n \geq 0$. A polynomial $f \in k[x_1, \ldots, x_n]$ gives a function $f : k^n \to k$ by evaluation. A subset $Z \subset k^n$ is called an *algebraic set* if it is the common vanishing set of a collection of polynomials.

0FJ4 **Exercise 111.15.1.** Prove that an algebraic set can always be written as the zero locus of finitely many polynomials.

With notation as above a subset $E \subset k^n$ is called *constructible* if it is a finite union of sets of the form $Z \cap \{f \neq 0\}$ where f is a polynomial.

0FJ5 **Exercise 111.15.2.** Show the following

(1) the complement of a constructible set is a constructible set,
(2) a finite union of constructible sets is a constructible set,
(3) a finite intersection of constructible sets is a constructible set, and
(4) any constructible set E can be written as a finite disjoint union $E = \coprod E_i$ with each E_i of the form $Z \cap \{f \neq 0\}$ where Z is an algebraic set and f is a polynomial.

0FJ6 **Exercise 111.15.3.** Let R be a ring. Let $f = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in R[x]$. (As usual this notation means $a_0, \ldots, a_d \in R$.) Let $g \in R[x]$. Prove that we can find $N \geq 0$ and $r, q \in R[x]$ such that

$$a_d^N g = qf + r$$

with $\deg(r) < d$, i.e., for some $c_i \in R$ we have $r = c_0 + c_1 x + \ldots + c_{d-1}x^{d-1}$.

111.16. Hilbert Nullstellensatz

027L

02DO **Exercise 111.16.1.** *A silly argument using the complex numbers!* Let $\mathbf{C}$ be the complex number field. Let V be a vector space over $\mathbf{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda \mathrm{id}_V$ is not invertible.

(1) Show that $\mathbf{C}(X)$ has uncountable dimension over $\mathbf{C}$.
(2) Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
(3) Show that if a finitely generated $\mathbf{C}$-algebra R is a field, then the map $\mathbf{C} \to R$ is an isomorphism.
(4) Show that any maximal ideal $\mathfrak{m}$ of $\mathbf{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbf{C}$.

027M **Remark 111.16.2.** Let k be a field. Then for every integer $n \in \mathbf{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k. This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves it in the case $k = \mathbf{C}$.

02DP **Exercise 111.16.3.** Let k be a field. Please use Remark 111.16.2.

(1) Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
(2) Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \notin \mathfrak{m}$.
(3) Show by an example that this statement fails when R is not of finite type over a field.
(4) Show that any radical ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

02DQ **Remark 111.16.4.** This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $\mathrm{Spec}(k[x_1, \ldots, x_n])$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

02DR **Exercise 111.16.5.** Let $A = \mathbf{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $\mathrm{Spec}(A)$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.

111.17. Dimension

02LT

076H **Exercise 111.17.1.** Construct a ring A with finitely many prime ideals having dimension > 1.

076I **Exercise 111.17.2.** Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that $\mathbf{C}[x, y]/(f)$ has dimension 1.

02LU **Exercise 111.17.3.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n \geq 1$. Let $\mathfrak{m}' = (\mathfrak{m}, x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Show that

$$\dim(R[x_1, \ldots, x_n]_{\mathfrak{m}'}) = \dim(R) + n.$$

111.18. Catenary rings

027N

027O **Definition 111.18.1.** A Noetherian ring A is said to be *catenary* if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

Here $ht(\mathfrak{p}/\mathfrak{q})$ means the height of $\mathfrak{p}/\mathfrak{q}$ in the ring $A/\mathfrak{q}$. In a formula

$$ht(\mathfrak{p}/\mathfrak{q}) = \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \dim((A/\mathfrak{q})_\mathfrak{p}) = \dim((A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}})$$

A topological space X is *catenary*, if given $T \subset T' \subset X$ with T and T' closed and irreducible, then there exists a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_n = T'$$

and every such chain has the same (finite) length.

0CT5 **Exercise 111.18.2.** Show that the notion of catenary defined in Algebra, Definition 10.105.1 agrees with the notion of Definition 111.18.1 for Noetherian rings.

02DS **Exercise 111.18.3.** Show that a Noetherian local domain of dimension 2 is catenary.

077D **Exercise 111.18.4.** Let k be a field. Show that a finite type k-algebra is catenary.

0CT6 **Exercise 111.18.5.** Give an example of a finite, sober, catenary topological space X which does not have a dimension function $\delta : X \to \mathbf{Z}$. Here $\delta : X \to \mathbf{Z}$ is a dimension function if for $x, y \in X$ we have

(1) $x \rightsquigarrow y$ and $x \neq y$ implies $\delta(x) > \delta(y)$,

(2) $x \rightsquigarrow y$ and $\delta(x) \geq \delta(y) + 2$ implies there exists a $z \in X$ with $x \rightsquigarrow z \rightsquigarrow y$ and $\delta(x) > \delta(z) > \delta(y)$.

Describe your space clearly and succinctly explain why there cannot be a dimension function.

111.19. Fraction fields

027P

02DT **Exercise 111.19.1.** Consider the domain

$$\mathbf{Q}[r, s, t]/(s^2 - (r-1)(r-2)(r-3), t^2 - (r+1)(r+2)(r+3)).$$

Find a domain of the form $\mathbf{Q}[x, y]/(f)$ with isomorphic field of fractions.

111.20. Transcendence degree

077E

077F **Exercise 111.20.1.** Let $K'/K/k$ be field extensions with K' algebraic over K. Prove that $\mathrm{trdeg}_k(K) = \mathrm{trdeg}_k(K')$. (Hint: Show that if $x_1, \ldots, x_d \in K$ are algebraically independent over k and $d < \mathrm{trdeg}_k(K')$ then $k(x_1, \ldots, x_d) \subset K$ cannot be algebraic.)

0CVP **Exercise 111.20.2.** Let k be a field. Let K/k be a finitely generated extension of transcendence degree d. If $V, W \subset K$ are finite dimensional k-subvector spaces denote

$$VW = \{f \in K \mid f = \sum\nolimits_{i=1,\ldots,n} v_i w_i \text{ for some } n \text{ and } v_i \in V, w_i \in W\}$$

This is a finite dimensional k-subvector space. Set $V^2 = VV$, $V^3 = VV^2$, etc.

(1) Show you can find $V \subset K$ and $\epsilon > 0$ such that $\dim V^n \geq \epsilon n^d$ for all $n \geq 1$.
(2) Conversely, show that for every finite dimensional $V \subset K$ there exists a $C > 0$ such that $\dim V^n \leq Cn^d$ for all $n \geq 1$. (One possible way to proceed: First do this for subvector spaces of $k[x_1, \ldots, x_d]$. Then do this for subvector spaces of $k(x_1, \ldots, x_d)$. Finally, if $K/k(x_1, \ldots, x_d)$ is a finite extension choose a basis of K over $k(x_1, \ldots, x_d)$ and argue using expansion in terms of this basis.)
(3) Conclude that you can redefine the transcendence degree in terms of growth of powers of finite dimensional subvector spaces of K.

This is related to Gelfand-Kirillov dimension of (noncommutative) algebras over k.

111.21. Dimension of fibres

0CVQ Some questions related to the dimension formula, see Algebra, Section 10.113.

0CVR **Exercise 111.21.1.** Let k be your favorite algebraically closed field. Below $k[x]$ and $k[x, y]$ denote the polynomial rings.

(1) For every integer $n \geq 0$ find a finite type extension $k[x] \subset A$ of domains such that the spectrum of A/xA has exactly n irreducible components.
(2) Make an example of a finite type extension $k[x] \subset A$ of domains such that the spectrum of $A/(x - \alpha)A$ is nonempty and reducible for every $\alpha \in k$.

(3) Make an example of a finite type extension $k[x, y] \subset A$ of domains such that the spectrum of $A/(x - \alpha, y - \beta)A$ is irreducible[1] for all $(\alpha, \beta) \in k^2 \setminus \{(0, 0)\}$ and the spectrum of $A/(x, y)A$ is nonempty and reducible.

0CVS **Exercise 111.21.2.** Let k be your favorite algebraically closed field. Let $n \geq 1$. Let $k[x_1, \ldots, x_n]$ be the polynomial ring. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $k[x_1, \ldots, x_n] \subset A$ be a finite type extension of domains. Set $d = \dim(A)$.

(1) Show that $d - 1 \geq \dim(A/\mathfrak{m}A) \geq d - n$ if $A/\mathfrak{m}A \neq 0$.
(2) Show by example that every value can occur.
(3) Show by example that $\mathrm{Spec}(A/\mathfrak{m}A)$ can have irreducible components of different dimensions.

111.22. Finite locally free modules

027Q

027R **Definition 111.22.1.** Let A be a ring. Recall that a *finite locally free* A-module M is a module such that for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that M_f is a finite free A_f-module. We say M is an *invertible module* if M is finite locally free of rank 1, i.e., for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f \cong A_f$ as an A_f-module.

078P **Exercise 111.22.2.** Prove that the tensor product of finite locally free modules is finite locally free. Prove that the tensor product of two invertible modules is invertible.

078Q **Definition 111.22.3.** Let A be a ring. The *class group of* A, sometimes called the *Picard group of* A is the set $\mathrm{Pic}(A)$ of isomorphism classes of invertible A-modules endowed with a group operation defined by tensor product (see Exercise 111.22.2).

Note that the class group of A is trivial exactly when every invertible module is isomorphic to a free module of rank 1.

078R **Exercise 111.22.4.** Show that the class groups of the following rings are trivial

(1) a polynomial ring $A = k[x]$ where k is a field,
(2) the integers $A = \mathbf{Z}$,
(3) a polynomial ring $A = k[x, y]$ where k is a field, and
(4) the quotient $k[x, y]/(xy)$ where k is a field.

078S **Exercise 111.22.5.** Show that the class group of the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field of characteristic not 2 and where $f(x) = (x - t_1) \ldots (x - t_n)$ with $t_1, \ldots, t_n \in k$ distinct and $n \geq 3$ an odd integer is not trivial. (Hint: Show that the ideal $(y, x - t_1)$ defines a nontrivial element of $\mathrm{Pic}(A)$.)

02DU **Exercise 111.22.6.** Let A be a ring.

(1) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the *trace* and *determinant* of φ and prove that your construction is "functorial in the triple (A, M, φ)".
(2) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $\mathrm{Trace}(\varphi \circ \psi) = \mathrm{Trace}(\psi \circ \varphi)$ and $\det(\varphi \circ \psi) = \det(\psi \circ \varphi)$.
(3) In case M is finite locally free show that Trace defines an A-linear map $\mathrm{End}_A(M) \to A$ and det defines a multiplicative map $\mathrm{End}_A(M) \to A$.

[1] Recall that irreducible implies nonempty.

02DV **Exercise 111.22.7.** Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.

(1) Define $\mathrm{Trace}_{B/A}$ and $\mathrm{Norm}_{B/A}$ using Trace and det from Exercise 111.22.6.
(2) Let $b \in B$ and let $\pi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism. Show that $\pi(V(b)) = V(\mathrm{Norm}_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(3) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(\mathrm{Norm}_{B/A}(b))$ equals $\mathrm{Norm}_{B'/A'}(b \otimes 1)$.
(4) Compute $\mathrm{Norm}_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(5) Compute the norm of $y - y^3$ under the finite flat map $\mathbf{Q}[x] \to \mathbf{Q}[y]$, $x \to y^n$. (Hint: use the "base change" $A = \mathbf{Q}[x] \subset A' = \mathbf{Q}(\zeta_n)(x^{1/n})$.)

111.23. Glueing

027S

02DW **Exercise 111.23.1.** Suppose that A is a ring and M is an A-module. Let f_i, $i \in I$ be a collection of elements of A such that

$$\mathrm{Spec}(A) = \bigcup D(f_i).$$

(1) Show that if M_{f_i} is a finite A_{f_i}-module, then M is a finite A-module.
(2) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

02DX **Remark 111.23.2.** In algebraic geometric language this means that the property of "being finitely generated" or "being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property "being of finite presentation".

078T **Exercise 111.23.3.** Suppose that $A \to B$ is a ring map. Let $f_i \in A$, $i \in I$ and $g_j \in B$, $j \in J$ be collections of elements such that

$$\mathrm{Spec}(A) = \bigcup D(f_i) \quad \text{and} \quad \mathrm{Spec}(B) = \bigcup D(g_j).$$

Show that if $A_{f_i} \to B_{f_i g_j}$ is of finite type for all i, j then $A \to B$ is of finite type.

111.24. Going up and going down

027T

027U **Definition 111.24.1.** Let $\phi : A \to B$ be a homomorphism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

02DY **Exercise 111.24.2.** In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(1) k is a field, $A = k$, $B = k[x]$.
(2) k is a field, $A = k[x]$, $B = k[x, y]$.
(3) $A = \mathbf{Z}$, $B = \mathbf{Z}[1/11]$.

(4) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(5) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(2 + i)]$.
(6) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(14 + 7i)]$.
(7) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

0FKE **Exercise 111.24.3.** Let A be a ring. Let $B = A[x]$ be the polynomial algebra in one variable over A. Let $f = a_0 + a_1x + \ldots + a_rx^r \in B = A[x]$. Prove carefully that the image of $D(f)$ in $\mathrm{Spec}(A)$ is equal to $D(a_0) \cup \ldots \cup D(a_r)$.

02DZ **Exercise 111.24.4.** Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x, y])$ of the following maps:

(1) $\mathrm{Spec}(k[x, yx^{-1}]) \to \mathrm{Spec}(k[x, y])$, where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(2) $\mathrm{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \mathrm{Spec}(k[x, y])$.
(3) $\mathrm{Spec}(k[t, 1/(t - 1)]) \to \mathrm{Spec}(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(4) $k = \mathbf{C}$ (complex numbers), $\mathrm{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

02E0 **Remark 111.24.5.** Finding the image as above usually is done by using elimination theory.

111.25. Fitting ideals

027V

02E1 **Exercise 111.25.1.** Let R be a ring and let M be a finite R-module. Choose a presentation

$$\bigoplus\nolimits_{j \in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are *generators* of M. The map $\bigoplus_{j \in J} R \to R^{\oplus n}$ is given by a $n \times J$ matrix A with coefficients in R. In other words, $A = (a_{ij})_{i=1,\ldots,n, j \in J}$. The columns $(a_{1j}, \ldots, a_{nj})$, $j \in J$ of A are said to be the *relations*. Any vector $(r_i) \in R^{\oplus n}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A. Of course any finite R-module has a lot of different presentations.

(1) Show that the ideal generated by the $(n - k) \times (n - k)$ minors of A is independent of the choice of the presentation. This ideal is the *kth Fitting ideal of* M. Notation $Fit_k(M)$.
(2) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)
(3) Show that the following are equivalent:
   (a) $Fit_{r-1}(M) = (0)$ and $Fit_r(M) = R$, and
   (b) M is locally free of rank r.

111.26. Hilbert functions

027W

027X **Definition 111.26.1.** A *numerical polynomial* is a polynomial $f(x) \in \mathbf{Q}[x]$ such that $f(n) \in \mathbf{Z}$ for every integer n.

027Y **Definition 111.26.2.** A *graded module* M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n \in \mathbf{Z}} M_n$ into A-submodules. We will say that M is *locally finite* if all of the M_n are finite A-modules. Suppose that A

is a Noetherian ring and that φ is a *Euler-Poincaré function* on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbf{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M'')$. The *Hilbert function* of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that M has a *Hilbert polynomial* if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

027Z **Definition 111.26.3.** A *graded A-algebra* is a graded A-module $B = \bigoplus_{n \geq 0} B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B,\ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a *graded module M over a graded A-algebra B* is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*, etc, etc.

02E2 **Exercise 111.26.4.** Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E3 **Exercise 111.26.5.** Let $A = \mathbf{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E4 **Exercise 111.26.6.** Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

02E5 **Exercise 111.26.7.** Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

02E6 **Exercise 111.26.8.** Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

02E7 **Exercise 111.26.9.** Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

02E8 **Exercise 111.26.10.** Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbf{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

111.27. Proj of a ring

0280

0281 **Definition 111.27.1.** Let R be a graded ring. A *homogeneous* ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous pieces in R then $f_i \in I$ for each i.

0282 **Definition 111.27.2.** We define the *homogeneous spectrum Proj*(R) of the graded ring R to be the set of homogeneous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that $\mathrm{Proj}(R)$ is a subset of $\mathrm{Spec}(R)$ and hence has a natural induced topology.

0283 **Definition 111.27.3.** Let $R = \oplus_{d\geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \mathrm{Proj}(R)$.

02E9 **Exercise 111.27.4.** On the topology on $\mathrm{Proj}(R)$. With definitions and notation as above prove the following statements.

(1) Show that $D_+(f)$ is open in $\mathrm{Proj}(R)$.
(2) Show that $D_+(ff') = D_+(f) \cap D_+(f')$.
(3) Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \mathrm{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \mathrm{Proj}(R)$. No proof necessary.
(4) Let $g \in R_0$ be a homogeneous element of degree 0. Express $D(g) \cap \mathrm{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogeneous elements of positive degree.
(5) Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of $\mathrm{Proj}(R)$.
078U (6) Show that there is a canonical bijection $D_+(f) \to \mathrm{Spec}(R_{(f)})$. (Hint: Imitate the proof for Spec but at some point thrown in the radical of an ideal.)
(7) Show that the map from (6) is a homeomorphism.
(8) Give an example of an R such that $\mathrm{Proj}(R)$ is not quasi-compact. No proof necessary.
(9) Show that any closed subset $T \subset \mathrm{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset R$.

02EA **Remark 111.27.5.** There is a continuous map $\mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(R_0)$.

02EB **Exercise 111.27.6.** If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\mathrm{Proj}(R) \to \mathrm{Spec}(A)$ is a bijection and in fact a homeomorphism.

02EC **Exercise 111.27.7.** Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \mathrm{Proj}(R) \to \mathrm{Spec}(A)$. Set $U = \mathrm{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism. Thus we may think of U as an open subset of $\mathrm{Proj}(R)$. Let $Z \subset \mathrm{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the *strict transform* Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in $\mathrm{Proj}(R)$ of the locally closed subset $Z \cap U \subset U \subset \mathrm{Proj}(R)$.

02ED **Exercise 111.27.8.** Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blowup algebra for A and I.

(1) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.
(2) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(3) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ in the blowup along J are disjoint.

02EE **Exercise 111.27.9.** Let R be a graded ring.

(1) Show that $\text{Proj}(R)$ is empty if $R_n = (0)$ for all $n >> 0$.
(2) Show that $\text{Proj}(R)$ is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

02EF **Exercise 111.27.10.** Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(1) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(2) Show that the ring map above induces a bijective map from $\text{Proj}(\bar{R})$ onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)
(3) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogeneous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(4) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogeneous ideal P and Q from part (c) and consider $V(P + Q)$.)

111.28. Cohen-Macaulay rings of dimension 1

0284

0285 **Definition 111.28.1.** A Noetherian local ring A is said to be *Cohen-Macaulay* of dimension d if it has dimension d and there exists a system of parameters $x_1, \ldots, x_d$ for A such that x_i is a nonzerodivisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

02EG **Exercise 111.28.2.** Cohen-Macaulay rings of dimension 1. Part I: Theory.

(1) Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zerodivisor then
   (a) $\dim A/xA = 0$, in other words A/xA is Artinian, in other words $\{x\}$ is a system of parameters for A.
   (b) A is has no embedded prime.
(2) Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if A has no embedded prime then there exists a nonzerodivisor in $\mathfrak{m}$.

02EH **Exercise 111.28.3.** Cohen-Macaulay rings of dimension 1. Part II: Examples.

(1) Let A be the local ring at (x, y) of $k[x, y]/(x^2, xy)$.
   (a) Show that A has dimension 1.
   (b) Prove that every element of $\mathfrak{m} \subset A$ is a zerodivisor.
   (c) Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).
(2) Let A be the local ring at (x, y) of $k[x, y]/(x^2)$. Find a nonzerodivisor in $\mathfrak{m}$ (no proof required).

02EI **Exercise 111.28.4.** Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

02EJ **Exercise 111.28.5.** Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that A has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

(1) Show that for every element y of $\mathfrak{m}$ there exists an integer n such that y can be written as $y = ux^n$ with $u \in A^*$ a unit.
(2) Show that x is a nonzerodivisor in A.
(3) Conclude that A is a domain.

02EK **Exercise 111.28.6.** Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

(1) Suppose that $x \in \mathfrak{m}^d$ maps to a nonzerodivisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$. Show that x is a nonzerodivisor.
(2) Suppose the depth of A is at least 1. Namely, suppose that there exists a nonzerodivisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$ which is a nonzerodivisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map of multiplication by $\bar{x}$

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$

is injective. Then show that x is a nonzerodivisor.

02EL **Exercise 111.28.7.** Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogeneous ideal I.

(1) Show that there exists a homogeneous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
(2) Show that $f(n) \leq n + 1$.
(3) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
(4) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
(5) Show that $f(n) = \deg(F)$ for all $n >> 0$.

02EM **Exercise 111.28.8.** Cohen-Macaulay rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macaulay of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: f, F, $x, y \in \mathfrak{m}$, I as in Exercise 111.28.7. Please use any results from the problems above.

(1) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x} + \beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with F.
   (a) Show that z is a nonzerodivisor on A.
   (b) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large n. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)

(2) What condition on k guarantees the existence of such a z? (No proof required; it's too easy.)
   Now we are going to assume there exists a z as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).

(3) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.

(4) Conclude that $I = (F)$.

(5) Conclude that the function f has values

$$2, 3, 4, \ldots, d-1, d, d, d, d, d, d, d, \ldots$$

02EN **Remark 111.28.9.** This suggests that a local Noetherian Cohen-Macaulay ring of dimension 1 and embedding dimension 2 is of the form B/FB, where B is a 2-dimensional regular local ring. This is more or less true (under suitable "niceness" properties of the ring).

111.29. Infinitely many primes

0286 A section with a collection of strange questions on rings where infinitely many primes are not invertible.

02EO **Exercise 111.29.1.** Give an example of a finite type $\mathbf{Z}$-algebra R with the following two properties:

(1) There is no ring map $R \to \mathbf{Q}$.
(2) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} \cong \mathbf{F}_p$.

02EP **Exercise 111.29.2.** For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. Give an example of an $f \in \mathbf{Z}[x, u]$ such that the following two properties hold:

(1) There exist infinitely many p such that f_p does not have a zero in $\mathbf{F}_p$.
(2) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

02EQ **Exercise 111.29.3.** For $f \in \mathbf{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbf{F}_p[x, y]$. Give an "interesting" example of an f such that f_p is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^p y = xy(x^{p-1} - y^{p-1})$ is "uninteresting"; any f depending only on x, u is "uninteresting", etc.

02ER **Remark 111.29.4.** Let $h \in \mathbf{Z}[y]$ be a monic polynomial of degree d. Then:

(1) The map $A = \mathbf{Z}[x] \to B = \mathbf{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(2) For all primes p the map $A_p = \mathbf{F}_p[x] \to B_p = \mathbf{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

02ES **Exercise 111.29.5.** Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. For $g \in \mathbf{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbf{F}_p[y]$.

(1) Give an example of a h and g such that there does not exist a f with the property
$$f_p = Norm_{B_p/A_p}(g_p).$$
(2) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have
$$Norm_{B_p/A_p}(g_p) \quad \text{divides} \quad f_p.$$
If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.
(3) Discuss the relevance of this to Exercises 6 and 7 of the previous set.

02ET **Exercise 111.29.6.** Unsolved problems. They may be really hard or they may be easy. I don't know.

(1) Is there any $f \in \mathbf{Z}[x, u]$ such that f_p is irreducible for an infinite number of p? (Hint: Yes, this happens for $f(x, u) = u - x - 1$ and also for $f(x, u) = u^2 - x^2 + 1$.)
(2) Let $f \in \mathbf{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization
$$f_p = f_{1,p} f_{2,p}$$
with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)
(3) Analogous question to the one in (b) but now with $f \in \mathbf{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.

111.30. Filtered derived category

0287 In order to do the exercises in this section, please read the material in Homology, Section 12.19. We will say A is a filtered object of $\mathcal{A}$, to mean that A comes endowed with a filtration F which we omit from the notation.

0288 **Exercise 111.30.1.** Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite and that each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$. Show that there exists an isomorphism $I \cong \bigoplus \mathrm{gr}^p(I)$ with filtration $F^p(I)$ corresponding to $\bigoplus_{p' \geq p} \mathrm{gr}^p(I)$.

0289 **Exercise 111.30.2.** Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite. Show the following are equivalent:

(1) For any solid diagram
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & \swarrow & \\ I & & \end{array}$$
of filtered objects with (i) the filtrations on A and B are finite, and (ii) $\mathrm{gr}(\alpha)$ injective the dotted arrow exists making the diagram commute.
(2) Each $\mathrm{gr}^p I$ is injective.

Note that given a morphism $\alpha : A \to B$ of filtered objects with finite filtrations to say that $\mathrm{gr}(\alpha)$ injective is the same thing as saying that α is a *strict monomorphism* in the category $\mathrm{Fil}(\mathcal{A})$. Namely, being a monomorphism means $\mathrm{Ker}(\alpha) = 0$ and strict means that this also implies $\mathrm{Ker}(\mathrm{gr}(\alpha)) = 0$. See Homology, Lemma 12.19.13. (We only use the term "injective" for a morphism in an abelian category, although it makes sense in any additive category having kernels.) The exercises above justifies the following definition.

028A **Definition 111.30.3.** Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume the filtration on I is finite. We say I is *filtered injective* if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

We make the following definition to avoid having to keep saying "with a finite filtration" everywhere.

028B **Definition 111.30.4.** Let $\mathcal{A}$ be an abelian category. We denote $\mathit{Fil}^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects consist of those $A \in \mathrm{Ob}(\mathrm{Fil}(\mathcal{A}))$ whose filtration is finite.

028C **Exercise 111.30.5.** Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show that there exists a strict monomorphism $\alpha : A \to I$ of A into a filtered injective object I of $\mathrm{Fil}^f(\mathcal{A})$.

028D **Definition 111.30.6.** Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathrm{Fil}(\mathcal{A})$. We say that α is a *filtered quasi-isomorphism* if for each $p \in \mathbf{Z}$ the morphism $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$ is a quasi-isomorphism.

028E **Definition 111.30.7.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathrm{Fil}^f(\mathcal{A})$. We say that $K^\bullet$ is *filtered acyclic* if for each $p \in \mathbf{Z}$ the complex $\mathrm{gr}^p(K^\bullet)$ is acyclic.

028F **Exercise 111.30.8.** Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of bounded below complexes of $\mathrm{Fil}^f(\mathcal{A})$. (Note the superscript f.) Show that the following are equivalent:

(1) α is a filtered quasi-isomorphism,
(2) for each $p \in \mathbf{Z}$ the map $\alpha : F^pK^\bullet \to F^pL^\bullet$ is a quasi-isomorphism,
(3) for each $p \in \mathbf{Z}$ the map $\alpha : K^\bullet/F^pK^\bullet \to L^\bullet/F^pL^\bullet$ is a quasi-isomorphism, and
(4) the cone of α (see Derived Categories, Definition 13.9.1) is a filtered acyclic complex.

Moreover, show that if α is a filtered quasi-isomorphism then α is also a usual quasi-isomorphism.

028G **Exercise 111.30.9.** Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a complex $I^\bullet$ of $\mathrm{Fil}^f(\mathcal{A})$, and a morphism $A[0] \to I^\bullet$ such that

(1) each I^p is filtered injective,
(2) $I^p = 0$ for $p < 0$, and
(3) $A[0] \to I^\bullet$ is a filtered quasi-isomorphism.

028H **Exercise 111.30.10.** Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let $K^\bullet$ be a bounded below complex of objects of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists

a filtered quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ a complex of $\mathrm{Fil}^f(\mathcal{A})$ having filtered injective terms I^n, and bounded below. In fact, we may choose α such that each α^n is a strict monomorphism.

028I **Exercise 111.30.11.** Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta} & \\ I^\bullet & & \end{array}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ are bounded below and assume each I^n is a filtered injective object. Also assume that α is a filtered quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is a strict monomorphism in every degree then we can find a β which makes the diagram commute.

028J **Exercise 111.30.12.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$, $K^\bullet$ be complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume

(1) $K^\bullet$ bounded below and filtered acyclic, and
(2) $I^\bullet$ bounded below and consisting of filtered injective objects.

Then any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

028K **Exercise 111.30.13.** Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta_i} & \\ I^\bullet & & \end{array}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ bounded below and each I^n a filtered injective object. Also assume α a filtered quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

111.31. Regular functions

078V

0E9D **Exercise 111.31.1.** Consider the affine curve X given by the equation $t^2 = s^5 + 8$ in $\mathbf{C}^2$ with coordinates s, t. Let $x \in X$ be the point with coordinates $(1, 3)$. Let $U = X \setminus \{x\}$. Prove that there is a regular function on U which is not the restriction of a regular function on $\mathbf{C}^2$, i.e., is not the restriction of a polynomial in s and t to U.

0E9E **Exercise 111.31.2.** Let $n \geq 2$. Let $E \subset \mathbf{C}^n$ be a finite subset. Show that any regular function on $\mathbf{C}^n \setminus E$ is a polynomial.

0E9F **Exercise 111.31.3.** Let $X \subset \mathbf{C}^n$ be an affine variety. Let us say X is a *cone* if $x = (a_1, \ldots, a_n) \in X$ and $\lambda \in \mathbf{C}$ implies $(\lambda a_1, \ldots, \lambda a_n) \in X$. Of course, if $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ is a prime ideal generated by homogeneous polynomials in $x_1, \ldots, x_n$, then the affine variety $X = V(\mathfrak{p}) \subset \mathbf{C}^n$ is a cone. Show that conversely the prime ideal $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ corresponding to a cone can be generated by homogeneous polynomials in $x_1, \ldots, x_n$.

0E9G **Exercise 111.31.4.** Give an example of an affine variety $X \subset \mathbf{C}^n$ which is a cone (see Exercise 111.31.3) and a regular function f on $U = X \setminus \{(0, \ldots, 0)\}$ which is not the restriction of a polynomial function on $\mathbf{C}^n$.

078W **Exercise 111.31.5.** In this exercise we try to see what happens with regular functions over non-algebraically closed fields. Let k be a field. Let $Z \subset k^n$ be a Zariski locally closed subset, i.e., there exist ideals $I \subset J \subset k[x_1, \ldots, x_n]$ such that

$$Z = \{a \in k^n \mid f(a) = 0 \ \forall \ f \in I, \ \exists \ g \in J, \ g(a) \neq 0\}.$$

A function $\varphi : Z \to k$ is said to be *regular* if for every $z \in Z$ there exists a Zariski open neighbourhood $z \in U \subset Z$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(u) \neq 0$ for all $u \in U$ and such that $\varphi(u) = f(u)/g(u)$ for all $u \in U$.

(1) If $k = \bar{k}$ and $Z = k^n$ show that regular functions are given by polynomials. (Only do this if you haven't seen this argument before.)
(2) If k is finite show that (a) every function φ is regular, (b) the ring of regular functions is finite dimensional over k. (If you like you can take $Z = k^n$ and even $n = 1$.)
(3) If $k = \mathbf{R}$ give an example of a regular function on $Z = \mathbf{R}$ which is not given by a polynomial.
(4) If $k = \mathbf{Q}_p$ give an example of a regular function on $Z = \mathbf{Q}_p$ which is not given by a polynomial.

111.32. Sheaves

028L A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *monomorphism* if for every pair of morphisms $a, b : W \to X$ we have $f \circ a = f \circ b \Rightarrow a = b$. A monomorphism in the category of sets is an injective map of sets.

078X **Exercise 111.32.1.** Carefully prove that a map of sheaves of sets is an monomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are injective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. An isomorphism in the category of sets is a bijective map of sets.

078Y **Exercise 111.32.2.** Carefully prove that a map of sheaves of sets is an isomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are bijective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *epimorphism* if for every pair of morphisms $a, b : Y \to Z$ we have $a \circ f = b \circ f \Rightarrow a = b$. An epimorphism in the category of sets is a surjective map of sets.

02EU **Exercise 111.32.3.** Carefully prove that a map of sheaves of sets is an epimorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are surjective.

02EV **Exercise 111.32.4.** Let $f : X \to Y$ be a map of topological spaces. Prove pushforward f_* and pullback f^{-1} for sheaves of **sets** form an adjoint pair of functors.

02EW **Exercise 111.32.5.** Let $j : U \to X$ be an open immersion. Show that

(1) Pullback $j^{-1} : Sh(X) \to Sh(U)$ has a left adjoint $j_! : Sh(U) \to Sh(X)$ called *extension by the empty set*.

(2) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in Sh(U)$.
(3) Pullback $j^{-1} : Ab(X) \to Ab(U)$ has a left adjoint $j_! : Ab(U) \to Ab(X)$ called *extension by zero*.
(4) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in Ab(U)$.

Observe that extension by zero differs from extension by the empty set!

028M **Exercise 111.32.6.** Let $X = \mathbf{R}$ with the usual topology. Let $\mathcal{O}_X = \underline{\mathbf{Z}/2\mathbf{Z}}_X$. Let $i : Z = \{0\} \to X$ be the inclusion and let $\mathcal{O}_Z = \underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Prove the following (the first three follow from the definitions but if you are not clear on the definitions you should elucidate them):

(1) $i_*\mathcal{O}_Z$ is a skyscraper sheaf.
(2) There is a canonical surjective map from $\underline{\mathbf{Z}/2\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Denote the kernel $\mathcal{I} \subset \mathcal{O}_X$.
(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$.
(4) The sheaf $\mathcal{I}$ on X cannot be locally generated by sections (as in Modules, Definition 17.8.1.)

028N **Exercise 111.32.7.** Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Show that $\mathcal{F}$ is the quotient of a (possibly very large) direct sum of sheaves all of whose terms are of the form

$$j_!(\underline{\mathbf{Z}}_U)$$

where $U \subset X$ is open and $\underline{\mathbf{Z}}_U$ denotes the constant sheaf with value $\mathbf{Z}$ on U.

02EX **Remark 111.32.8.** Let X be a topological space. In the category of abelian sheaves the direct sum of a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ is the sheaf associated to the presheaf $U \mapsto \oplus\mathcal{F}_i(U)$. Consequently the stalk of the direct sum at a point x is the direct sum of the stalks of the $\mathcal{F}_i$ at x.

078Z **Exercise 111.32.9.** Let X be a topological space. Suppose we are given a collection of abelian groups A_x indexed by $x \in X$. Show that the rule $U \mapsto \prod_{x\in U} A_x$ with obvious restriction mappings defines a sheaf $\mathcal{G}$ of abelian groups. Show, by an example, that usually it is not the case that $\mathcal{G}_x = A_x$ for $x \in X$.

0790 **Exercise 111.32.10.** Let X, A_x, $\mathcal{G}$ be as in Exercise 111.32.9. Let $\mathcal{B}$ be a basis for the topology of X, see Topology, Definition 5.5.1. For $U \in \mathcal{B}$ let A_U be a subgroup $A_U \subset \mathcal{G}(U) = \prod_{x\in U} A_x$. Assume that for $U \subset V$ with $U, V \in \mathcal{B}$ the restriction maps A_V into A_U. For $U \subset X$ open set

$$\mathcal{F}(U) = \left\{ (s_x)_{x\in U} \middle| \begin{array}{l} \text{for every } x \text{ in } U \text{ there exists } V \in \mathcal{B} \\ x \in V \subset U \text{ such that } (s_y)_{y\in V} \in A_V \end{array} \right\}$$

Show that $\mathcal{F}$ defines a sheaf of abelian groups on X. Show, by an example, that it is usually not the case that $\mathcal{F}(U) = A_U$ for $U \in \mathcal{B}$.

0E9H **Exercise 111.32.11.** Give an example of a topological space X and a functor

$$F : Sh(X) \longrightarrow Sets$$

which is exact (commutes with finite products and equalizers and commutes with finite coproducts and coequalizers, see Categories, Section 4.23), but there is no point $x \in X$ such that F is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$.

111.33. Schemes

028O Let LRS be the category of locally ringed spaces. An affine scheme is an object in LRS isomorphic in LRS to a pair of the form $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. A scheme is an object $(X, \mathcal{O}_X)$ of LRS such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

028P **Exercise 111.33.1.** Find a 1-point locally ringed space which is not a scheme.

028Q **Exercise 111.33.2.** Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.

03KB **Exercise 111.33.3.** Suppose that X is a scheme whose underlying topological space is a finite discrete set. Show that X is an affine scheme.

028R **Exercise 111.33.4.** Show that there exists a non-affine scheme having three points.

028S **Exercise 111.33.5.** Suppose that X is a nonempty quasi-compact scheme. Show that X has a closed point.

02EY **Remark 111.33.6.** When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ and that j is an *open immersion*. More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is *isomorphic* to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.

028T **Exercise 111.33.7.** Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.

028U **Exercise 111.33.8.** Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

028V **Exercise 111.33.9.** (This is pretty hard.) Given an example of a scheme X, and open subscheme $U \subset X$ and a closed subscheme $Z \subset U$ such that Z does not extend to a closed subscheme of X.

028W **Exercise 111.33.10.** Give an example of a scheme X, a field K, and a morphism of ringed spaces $\mathrm{Spec}(K) \to X$ which is NOT a morphism of schemes.

028X **Exercise 111.33.11.** Do all the exercises in [**Har77**, Chapter II], Sections 1 and 2... Just kidding!

028Y **Definition 111.33.12.** A scheme X is called *integral* if X is nonempty and for every nonempty affine open $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ is a domain.

028Z **Exercise 111.33.13.** Give an example of a morphism of *integral* schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

0290 **Exercise 111.33.14.** Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

02EZ **Remark 111.33.15.** It turns out this cannot happen with S separated. Do you know why?

0291 **Exercise 111.33.16.** Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbf{Q}$ such that $\mathrm{Spec}(\mathbf{C}) \times_{\mathrm{Spec}(\mathbf{Q})} V$ is not integral.

0292 **Exercise 111.33.17.** Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} V$ is not reduced for some finite field extension k'/k.

02F0 **Remark 111.33.18.** If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

111.34. Morphisms

0293 An important question is, given a morphism $\pi : X \to S$, whether the morphism has a section or a rational section. Here are some example exercises.

0294 **Exercise 111.34.1.** Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t, 1/xt]) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

(1) Show there does not exist a morphism $\sigma : S \to X$ such that $\pi \circ \sigma = \mathrm{id}_S$.
(2) Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

0295 **Exercise 111.34.2.** Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t]/(x^2 + t)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

0296 **Exercise 111.34.3.** Let $A, B, C \in \mathbf{C}[t]$ be nonzero polynomials. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(A + Bx^2 + Cy^2)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$. (Hint: Symbolically, write $x = X/Z$, $y = Y/Z$ for some $X, Y, Z \in \mathbf{C}[t]$ of degree $\leq d$ for some d, and work out the condition that this solves the equation. Then show, using dimension theory, that if $d >> 0$ you can find nonzero X, Y, Z solving the equation.)

02F1 **Remark 111.34.4.** Exercise 111.34.3 is a special case of "Tsen's theorem". Exercise 111.34.5 shows that the method is limited to low degree equations (conics when the base and fibre have dimension 1).

0297 **Exercise 111.34.5.** Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(1 + tx^3 + t^2y^3)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t])$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

0298 **Exercise 111.34.6.** Consider the schemes

$$X = \mathrm{Spec}(\mathbf{C}[\{x_i\}_{i=1}^8, s, t]/(1+sx_1^3+s^2x_2^3+tx_3^3+stx_4^3+s^2tx_5^3+t^2x_6^3+st^2x_7^3+s^2t^2x_8^3))$$

and

$$S = \mathrm{Spec}(\mathbf{C}[s, t])$$

and the morphism of schemes

$$\pi : X \longrightarrow S$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

0299 **Exercise 111.34.7.** (For the number theorists.) Give an example of a closed subscheme

$$Z \subset \mathrm{Spec}\left(\mathbf{Z}[x, \frac{1}{x(x-1)(2x-1)}]\right)$$

such that the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is finite and surjective.

029A **Exercise 111.34.8.** If you do not like number theory, you can try the variant where you look at

$$\mathrm{Spec}\left(\mathbf{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}]\right) \longrightarrow \mathrm{Spec}(\mathbf{F}_p[t])$$

and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; although it may not be necessary in this particular case.)

02F2 **Remark 111.34.9.** In Exercises 111.34.7 and 111.34.8 we are given a morphism $f : X \to S$ where S is the spectrum of a Dedekind ring A with f flat and surjective with geometrically irreducible generic fibre. In both cases the outcome is that there exists a finite surjective morphism $S' \to S$ such that the base change $X_{S'} \to S'$ does have a section. It turns out this holds if A is excellent, its residue fields at maximal ideals are algebraic extensions of finite fields, and $\mathrm{Pic}(A')$ is torsion when A' is the integral closure of A in a finite extension of its fraction field, see [**MB89**]. For example, if $A = \mathbf{Z}$ or $A = \mathbf{F}_p[x]$ or a finite extension of these. However, it turns out that there exists a Dedekind ring A with finite residue fields at maximal primes whose Picard group has a nontorsion element, see [**Gol64**], and the result is false for the spectrum of such a ring. Exercise 111.34.10 gives a geometric example where it doesn't work.

029B **Exercise 111.34.10.** Prove there exist a $f \in \mathbf{C}[x, t]$ which is not divisible by $t-\alpha$ for any $\alpha \in \mathbf{C}$ such that there does not exist any $Z \subset \mathrm{Spec}(\mathbf{C}[x, t, 1/f])$ which maps finite surjectively to $\mathrm{Spec}(\mathbf{C}[t])$. (I think that $f(x, t) = (xt - 2)(x - t + 3)$ works. To show any candidate has the required property is not so easy I think.)

0EG6 **Exercise 111.34.11.** Let $A \to B$ be a finite type ring map. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors through a closed immersion $\mathrm{Spec}(B) \to \mathbf{P}^n_A$ for some n. Prove that $A \to B$ is a finite ring map, i.e., that B is finite as an A-module. Hint: if A is Noetherian (please just assume this) you can argue using that $H^i(Z, \mathcal{O}_Z)$ for $i \in \mathbf{Z}$ is a finite A-module for every closed subscheme $Z \subset \mathbf{P}^n_A$.

0EG7 **Exercise 111.34.12.** Let k be an algebraically closed field. Let $f : X \to Y$ be a morphism of projective varieties such that $f^{-1}(\{y\})$ is finite for every closed point $y \in Y$. Prove that f is finite as a morphism of schemes. Hints: (a) being finite

is a local property, (b) try to reduce to Exercise 111.34.11, and (c) use a closed immersion $X \to \mathbf{P}^n_k$ to get a closed immersion $X \to \mathbf{P}^n_Y$ over Y.

111.35. Tangent Spaces

029C

029D **Definition 111.35.1.** For any ring R we denote $R[\epsilon]$ the ring of *dual numbers*. As an R-module it is free with basis 1, ϵ. The ring structure comes from setting $\epsilon^2 = 0$.

029E **Exercise 111.35.2.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point, let $s = f(x)$. Consider the solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\kappa(x)[\epsilon]) & \dashrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X. If $\kappa(x) = \kappa(s)$ show that the set of dotted arrows which make the diagram commute are in one to one correspondence with the set of linear maps

$$\mathrm{Hom}_{\kappa(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}, \kappa(x))$$

In other words: describe such a bijection. (This works more generally if $\kappa(x) \supset \kappa(s)$ is a separable algebraic extension.)

029F **Definition 111.35.3.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. We dub the set of dotted arrows of Exercise 111.35.2 the *tangent space of X over S* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

029G **Exercise 111.35.4.** For any field K prove that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(K[\epsilon_1]) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K[\epsilon_2]) & \longrightarrow & \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \end{array}$$

is a pushout diagram in the category of schemes. (Here $\epsilon_i^2 = 0$ as before.)

029H **Exercise 111.35.5.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Define addition of tangent vectors, using Exercise 111.35.4 and a suitable morphism

$$\mathrm{Spec}(K[\epsilon]) \longrightarrow \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)).$$

Similarly, define scalar multiplication of tangent vectors (this is easier). Show that $T_{X/S,x}$ becomes a $\kappa(x)$-vector space with your constructions.

029I **Exercise 111.35.6.** Let k be a field. Consider the structure morphism $f : X = \mathbf{A}^1_k \to \mathrm{Spec}(k) = S$.

(1) Let $x \in X$ be a closed point. What is the dimension of $T_{X/S,x}$?
(2) Let $\eta \in X$ be the generic point. What is the dimension of $T_{X/S,\eta}$?
(3) Consider now X as a scheme over $\mathrm{Spec}(\mathbf{Z})$. What are the dimensions of $T_{X/\mathbf{Z},x}$ and $T_{X/\mathbf{Z},\eta}$?

02F3 **Remark 111.35.7.** Exercise 111.35.6 explains why it is necessary to consider the tangent space of X over S to get a good notion.

029J **Exercise 111.35.8.** Consider the morphism of schemes

$$f : X = \mathrm{Spec}(\mathbf{F}_p(t)) \longrightarrow \mathrm{Spec}(\mathbf{F}_p(t^p)) = S$$

Compute the tangent space of X/S at the unique point of X. Isn't that weird? What do you think happens if you take the morphism of schemes corresponding to $\mathbf{F}_p[t^p] \to \mathbf{F}_p[t]$?

029K **Exercise 111.35.9.** Let k be a field. Compute the tangent space of X/k at the point $x = (0, 0)$ where $X = \mathrm{Spec}(k[x, y]/(x^2 - y^3))$.

029L **Exercise 111.35.10.** Let $f : X \to Y$ be a morphism of schemes over S. Let $x \in X$ be a point. Set $y = f(x)$. Assume that the natural map $\kappa(y) \to \kappa(x)$ is bijective. Show, using the definition, that f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}.$$

Match it with what happens on local rings via Exercise 111.35.2 in case $\kappa(x) = \kappa(s)$.

029M **Exercise 111.35.11.** Let k be an algebraically closed field. Let

$$\begin{array}{rcl} f : \mathbf{A}^n_k & \longrightarrow & \mathbf{A}^m_k \\ (x_1, \ldots, x_n) & \longmapsto & (f_1(x_i), \ldots, f_m(x_i)) \end{array}$$

be a morphism of schemes over k. This is given by m polynomials $f_1, \ldots, f_m$ in n variables. Consider the matrix

$$A = \left(\frac{\partial f_j}{\partial x_i} \right)$$

Let $x \in \mathbf{A}^n_k$ be a closed point. Set $y = f(x)$. Show that the map on tangent spaces $T_{\mathbf{A}^n_k/k,x} \to T_{\mathbf{A}^m_k/k,y}$ is given by the value of the matrix A at the point x.

111.36. Quasi-coherent Sheaves

029N

029O **Definition 111.36.1.** Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *quasi-coherent* if for every affine open $\mathrm{Spec}(R) = U \subset X$ the restriction $\mathcal{F}|_U$ is of the form $\widetilde{M}$ for some R-module M.

It is enough to check this conditions on the members of an affine open covering of X. See Schemes, Section 26.24 for more results.

029P **Definition 111.36.2.** Let X be a topological space. Let $x, x' \in X$. We say x is a *specialization* of x' if and only if $x \in \overline{\{x'\}}$.

029Q **Exercise 111.36.3.** Let X be a scheme. Let $x, x' \in X$. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$. Show that $\mathcal{F}_x \neq 0$.

029R **Exercise 111.36.4.** Find an example of a scheme X, points $x, x' \in X$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$ and $\mathcal{F}_x = 0$.

029S **Definition 111.36.5.** A scheme X is called *locally Noetherian* if and only if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that R is Noetherian. A scheme is *Noetherian* if it is locally Noetherian and quasi-compact.

If X is locally Noetherian then any affine open of X is the spectrum of a Noetherian ring, see Properties, Lemma 28.5.2.

029T **Definition 111.36.6.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *coherent* if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finite R-module M.

029U **Exercise 111.36.7.** Let $X = \mathrm{Spec}(R)$ be an affine scheme.

(1) Let $f \in R$. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.
(2) Let $I \subset R$ be an ideal. Let $i : Z \to X$ be the closed subscheme of X corresponding to I. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathcal{G} = i^*\mathcal{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. (Why is this silly?)
(3) Assume that R is Noetherian. Let $f \in R$. Let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

029V **Remark 111.36.8.** If $U \to X$ is a quasi-compact immersion then any quasi-coherent sheaf on U is the restriction of a quasi-coherent sheaf on X. If X is a Noetherian scheme, and $U \subset X$ is open, then any coherent sheaf on U is the restriction of a coherent sheaf on X. Of course the exercise above is easier, and shouldn't use these general facts.

111.37. Proj and projective schemes

029W

029X **Exercise 111.37.1.** Give examples of graded rings S such that

(1) $\mathrm{Proj}(S)$ is affine and nonempty, and
(2) $\mathrm{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbf{P}^n_A$ for any $n \geq 0$, any ring A.

029Y **Exercise 111.37.2.** Give an example of a nonconstant morphism of schemes $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^5_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$.

029Z **Exercise 111.37.3.** Give an example of an isomorphism of schemes

$$\mathbf{P}^1_{\mathbf{C}} \to \mathrm{Proj}(\mathbf{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$$

02A0 **Exercise 111.37.4.** Give an example of a morphism of schemes $f : X \to \mathbf{A}^1_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[T])$ such that the (scheme theoretic) fibre X_t of f over $t \in \mathbf{A}^1_{\mathbf{C}}$ is (a) isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when $t = 0$. We will call X_0 the *special fibre* of the morphism. This can be done in many, many ways. Try to give examples that satisfy (each of) the following additional restraints (unless it isn't possible):

(1) Can you do it with special fibre projective?
(2) Can you do it with special fibre irreducible and projective?
(3) Can you do it with special fibre integral and projective?
(4) Can you do it with special fibre smooth and projective?

(5) Can you do it with f a flat morphism? This just means that for every affine open $\mathrm{Spec}(A) \subset X$ the induced ring map $\mathbf{C}[t] \to A$ is flat, which in this case means that any nonzero polynomial in t is a nonzerodivisor on A.
(6) Can you do it with f a flat and projective morphism?
(7) Can you do it with f flat, projective and special fibre reduced?
(8) Can you do it with f flat, projective and special fibre irreducible?
(9) Can you do it with f flat, projective and special fibre integral?

What do you think happens when you replace $\mathbf{P}^1_{\mathbf{C}}$ with another variety over $\mathbf{C}$? (This can get very hard depending on which of the variants above you ask for.)

02A1 **Exercise 111.37.5.** Let $n \geq 1$ be any positive integer. Give an example of a surjective morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ with X affine.

02A2 **Exercise 111.37.6.** Maps of Proj. Let R and S be graded rings. Suppose we have a ring map

$$\psi : R \to S$$

and an integer $e \geq 1$ such that $\psi(R_d) \subset S_{de}$ for all $d \geq 0$. (By our conventions this is not a homomorphism of graded rings, unless $e = 1$.)

(1) For which elements $\mathfrak{p} \in \mathrm{Proj}(S)$ is there a well-defined corresponding point in $\mathrm{Proj}(R)$? In other words, find a suitable open $U \subset \mathrm{Proj}(S)$ such that ψ defines a continuous map $r_\psi : U \to \mathrm{Proj}(R)$.
(2) Give an example where $U \neq \mathrm{Proj}(S)$.
(3) Give an example where $U = \mathrm{Proj}(S)$.
(4) (Do not write this down.) Convince yourself that the continuous map $U \to \mathrm{Proj}(R)$ comes canonically with a map on sheaves so that r_ψ is a morphism of schemes:

$$\mathrm{Proj}(S) \supset U \longrightarrow \mathrm{Proj}(R).$$

(5) What can you say about this map if $R = \bigoplus_{d \geq 0} S_{de}$ (as a graded ring with S_e, S_{2e}, etc in degree 1, 2, etc) and ψ is the inclusion mapping?

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \mathrm{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogeneous of degree n in $R_f = R[1/f]$). See [**Har77**, page 116+]. Note that there are natural maps

$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

02A3 **Exercise 111.37.7.** Pathologies in Proj. Give examples of R as above such that

(1) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
(2) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.

02A4 **Exercise 111.37.8.** Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Show that any finite set of points of X is contained in a standard affine open.

02A5 **Exercise 111.37.9.** Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Let $Z, Z' \subset X$ be two closed subschemes. Let $\varphi : Z \to Z'$ be an isomorphism. Assume $Z \cap Z' = \emptyset$. Show that for any $z \in Z$ there exists an affine open $U \subset X$ such that $z \in U$,

$\varphi(z) \in U$ and $\varphi(Z \cap U) = Z' \cap U$. (Hint: Use Exercise 111.37.8 and something akin to Schemes, Lemma 26.11.5.)

111.38. Morphisms from the projective line

0DJ0 In this section we study morphisms from $\mathbf{P}^1$ to projective schemes.

0DJ1 **Exercise 111.38.1.** Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a finite type scheme over k. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow X$$

over k, there exist a nonzero $f \in k[t]$ and a morphism $\psi : \mathrm{Spec}(k[t, 1/f]) \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t, 1/f]) \longrightarrow X$$

0DJ2 **Exercise 111.38.2.** Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow \mathbf{P}^n_k$$

over k, there exists a morphism $\psi : \mathrm{Spec}(k[t]) \to \mathbf{P}^n_k$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t]) \longrightarrow \mathbf{P}^n_k$$

Hint: the image of φ is in a standard open $D_+(T_i)$ for some i; then show that you can "clear denominators".

0DJ3 **Exercise 111.38.3.** Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a projective scheme over k. Show that for any morphism

$$\varphi : \mathrm{Spec}(k(t)) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathrm{Spec}(k[t]) \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t]) \longrightarrow X$$

Hint: use Exercise 111.38.2.

0DJ4 **Exercise 111.38.4.** Let k be a field. Let X be a projective scheme over k. Let K be the function field of $\mathbf{P}^1_k$ (see hint below). Show that for any morphism

$$\varphi : \mathrm{Spec}(K) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathbf{P}^1_k \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathbf{P}^1_k \longrightarrow X$$

Hint: use Exercise 111.38.3 for each of the two pieces of the affine open covering $\mathbf{P}^1_k = D_+(T_0) \cup D_+(T_1)$, use that $D_+(T_0)$ is the spectrum of a polynomial ring and that K is the fraction field of this polynomial ring.

111.39. Morphisms from surfaces to curves

02A6

02A7 **Exercise 111.39.1.** Let R be a ring. Let $R \to k$ be a map from R to a field. Let $n \geq 0$. Show that

$$\mathrm{Mor}_{\mathrm{Spec}(R)}(\mathrm{Spec}(k), \mathbf{P}^n_R) = (k^{n+1} \setminus \{0\})/k^*$$

where k^* acts via scalar multiplication on k^{n+1}. From now on we denote $(x_0 : \ldots : x_n)$ the morphism $\mathrm{Spec}(k) \to \mathbf{P}^n_k$ corresponding to the equivalence class of the element $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$.

02A8 **Exercise 111.39.2.** Let k be a field. Let $Z \subset \mathbf{P}^2_k$ be an irreducible and reduced closed subscheme. Show that either (a) Z is a closed point, or (b) there exists an homogeneous irreducible $F \in k[X_0, X_1, X_2]$ of degree > 0 such that $Z = V_+(F)$, or (c) $Z = \mathbf{P}^2_k$. (Hint: Look on a standard affine open.)

02A9 **Exercise 111.39.3.** Let k be a field. Let $Z_1, Z_2 \subset \mathbf{P}^2_k$ be irreducible closed subschemes of the form $V_+(F)$ for some homogeneous irreducible $F_i \in k[X_0, X_1, X_2]$ of degree > 0. Show that $Z_1 \cap Z_2$ is not empty. (Hint: Use dimension theory to estimate the dimension of the local ring of $k[X_0, X_1, X_2]/(F_1, F_2)$ at 0.)

02AA **Exercise 111.39.4.** Show there does not exist a nonconstant morphism of schemes $\mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$. Here a *constant morphism* is one whose image is a single point. (Hint: If the morphism is not constant consider the fibres over 0 and ∞ and argue that they have to meet to get a contradiction.)

02AB **Exercise 111.39.5.** Let k be a field. Suppose that $X \subset \mathbf{P}^3_k$ is a closed subscheme given by a single homogeneous equation $F \in k[X_0, X_1, X_2, X_3]$. In other words,

$$X = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F)) \subset \mathbf{P}^3_k$$

as explained in the course. Assume that

$$F = X_0 G + X_1 H$$

for some homogeneous polynomials $G, H \in k[X_0, X_1, X_2, X_3]$ of positive degree. Show that if X_0, X_1, G, H have no common zeros then there exists a nonconstant morphism

$$X \longrightarrow \mathbf{P}^1_k$$

of schemes over $\mathrm{Spec}(k)$ which on field points (see Exercise 111.39.1) looks like $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1)$ whenever x_0 or x_1 is not zero.

111.40. Invertible sheaves

02AC

02AD **Definition 111.40.1.** Let X be a locally ringed space. An *invertible $\mathcal{O}_X$-module* on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

02AE **Exercise 111.40.2.** General facts.

(1) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
(2) Suppose $X \to Y$ is a morphism of locally ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

02AF **Exercise 111.40.3.** Algebra.

(1) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $\mathrm{Spec}(A)$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)

(2) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

02AG **Definition 111.40.4.** Let R be a ring. An *invertible module* M is an R-module M such that $\widetilde{M}$ is an invertible sheaf on the spectrum of R. We say M is *trivial* if $M \cong R$ as an R-module.

In other words, M is invertible if and only if it satisfies all of the following conditions: it is flat, of finite presentation, projective, and locally free of rank 1. (Of course it suffices for it to be locally free of rank 1).

02AH **Exercise 111.40.5.** Simple examples.

02AI (1) Let k be a field. Let $A = k[x]$. Show that $X = \mathrm{Spec}(A)$ has only trivial invertible $\mathcal{O}_X$-modules. In other words, show that every invertible A-module is free of rank 1.

02AJ (2) Let A be the ring

$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$

Show there exists a nontrivial invertible A-module, unless $k = \mathbf{F}_2$. (Hint: Think about $\mathrm{Spec}(A)$ as identifying 0 and 1 in $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$.)

02AK (3) Same question as in (2) for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbf{F}_2$ works as well).

02AL **Exercise 111.40.6.** Higher dimensions.

(1) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$ where k is a field. Show that every invertible sheaf on $\mathbf{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use Exercise 111.40.5 (1) to find a generator for this; then you still have to think. Another way to is to use Exercise 111.40.3 and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)

(2) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbf{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbf{A}^2_k$. Not easy; but you can find it in [**Har77**].

(3) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = \mathrm{Spec}(k[x, y, z]/(xy - z^2))$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $\mathrm{Spec}(k[x, y, z, 1/x]/(xy - z^2))$ and $\mathrm{Spec}(k[x, y, z, 1/y]/(xy - z^2))$ which are "easy" to deal with.

02AM **Definition 111.40.7.** Let X be a locally ringed space. The *Picard group of* X is the set $\mathrm{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with addition given by tensor product. See Modules, Definition 17.25.9. For a ring R we set $\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Spec}(R))$.

02AN **Exercise 111.40.8.** Let R be a ring.

(1) Show that if R is a Noetherian normal domain, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[t])$. [Hint: There is a map $R[t] \to R$, $t \mapsto 0$ which is a left inverse to the map $R \to R[t]$. Hence it suffices to show that any invertible $R[t]$-module M such that $M/tM \cong R$ is free of rank 1. Let K be the fraction field of R. Pick a trivialization $K[t] \to M \otimes_{R[t]} K[t]$ which is possible by Exercise 111.40.5 (1). Adjust it so it agrees with the trivialization of M/tM above. Show that it is in fact a trivialization of M over $R[t]$ (this is where normality comes in).]
(2) Let k be a field. Show that $\mathrm{Pic}(k[x^2, x^3, t]) \neq \mathrm{Pic}(k[x^2, x^3])$.

111.41. Čech Cohomology

02AO

02F4 **Exercise 111.41.1.** Čech cohomology. Here k is a field.

(1) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = \mathrm{Spec}(k[x])$, $U_2 = \mathrm{Spec}(k[y])$ with $U_1 \cap U_2 = \mathrm{Spec}(k[z, 1/z])$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line with zero doubled.) Compute $\check{H}^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.
(2) For each element in $\check{H}^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules
$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$
such that the boundary $\delta(1) \in \check{H}^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. See also below. It is also OK to show abstractly such a thing has to exist.)

02AP **Definition 111.41.2.** (Definition of delta.) Suppose that
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to \check{H}^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment
$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$
This is clearly a 1-coboundary in the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

111.42. Cohomology

0D8P

0D8Q **Exercise 111.42.1.** Let $X = \mathbf{R}$ with the usual Euclidean topology. Using only formal properties of cohomology (functoriality and the long exact cohomology sequence) show that there exists a sheaf $\mathcal{F}$ on X with nonzero $H^1(X, \mathcal{F})$.

0D8R **Exercise 111.42.2.** Let $X = U \cup V$ be a topological space written as the union of two opens. Then we have a long exact Mayer-Vietoris sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

What property of injective sheaves is essential for the construction of the Mayer-Vietoris long exact sequence? Why does it hold?

0D8S **Exercise 111.42.3.** Let X be a topological space.

(1) Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ if X has 2 or fewer points.
(2) What if X has 3 points?

0D8T **Exercise 111.42.4.** Let X be the spectrum of a local ring. Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ and any sheaf of abelian groups $\mathcal{F}$.

0D8U **Exercise 111.42.5.** Let $f : X \to Y$ be an affine morphism of schemes. Prove that $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Feel free to impose some further conditions on X and Y and use the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

0D8V **Exercise 111.42.6.** Let A be a ring. Let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $\mathbf{A}^{n+1}_A = \text{Spec}(A[T_0, \ldots, T_n])$ and let

$$U = \bigcup\nolimits_{i=0,\ldots,n} D(T_i) \subset \mathbf{A}^{n+1}_A$$

be the complement of the image of the closed immersion $0 : \text{Spec}(A) \to \mathbf{A}^{n+1}_A$. Construct an affine surjective morphism

$$f : U \longrightarrow \mathbf{P}^n_A$$

and prove that $f_*\mathcal{O}_U = \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\mathbf{P}^n_A}(d)$. More generally, show that for a graded $A[T_0, \ldots, T_n]$-module M one has

$$f_*(\widetilde{M}|_U) = \bigoplus\nolimits_{d \in \mathbf{Z}} \widetilde{M(d)}$$

where on the left hand side we have the quasi-coherent sheaf $\widetilde{M}$ associated to M on $\mathbf{A}^{n+1}_A$ and on the right we have the quasi-coherent sheaves $\widetilde{M(d)}$ associated to the graded module $M(d)$.

0D8W **Exercise 111.42.7.** Let A be a ring and let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Carefully compute the cohomology of the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$ of the structure sheaf on $\mathbf{P}^n_A$. Feel free to use Čech cohomology and the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

0D8X **Exercise 111.42.8.** Let A be a ring and let $\mathbf{P}^n_A = \mathrm{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $F \in A[T_0, \ldots, T_n]$ be homogeneous of degree d. Let $X \subset \mathbf{P}^n_A$ be the closed subscheme corresponding to the graded ideal (F) of $A[T_0, \ldots, T_n]$. What can you say about $H^i(X, \mathcal{O}_X)$?

0D8Y **Exercise 111.42.9.** Let R be a ring such that for any left exact functor $F : \mathrm{Mod}_R \to Ab$ we have $R^iF = 0$ for $i > 0$. Show that R is a finite product of fields.

111.43. More cohomology

0DAI

0DAJ **Exercise 111.43.1.** Let k be a field. Let $X \subset \mathbf{P}^n_k$ be the "coordinate cross". Namely, let X be defined by the homogeneous equations

$$T_iT_j = 0 \text{ for } i > j > 0$$

where as usual we write $\mathbf{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$. In other words, X is the closed subscheme corresponding to the quotient $k[T_0, \ldots, T_n]/(T_iT_j; i > j > 0)$ of the polynomial ring. Compute $H^i(X, \mathcal{O}_X)$ for all i. Hint: use Čech cohomology.

0DAK **Exercise 111.43.2.** Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \mathrm{Spec}(A)$ be the complement of $V(I)$. For any A-module M write down a complex of A-modules (in terms of A, $f_1, \ldots, f_t$, M) whose cohomology groups give $H^n(U, \widetilde{M})$.

0DAL **Exercise 111.43.3.** Let k be a field. Let $U \subset \mathbf{A}^d_k$ be the complement of the closed point 0 of $\mathbf{A}^d_k$. Compute $H^n(U, \mathcal{O}_U)$ for all n.

0DAM **Exercise 111.43.4.** Let k be a field. Find explicitly a scheme X projective over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) = 100$.

0DAN **Exercise 111.43.5.** Let $f : X \to Y$ be a finite locally free morphism of degree 2. Assume that X and Y are integral schemes and that 2 is invertible in the structure sheaf of Y, i.e., $2 \in \Gamma(Y, \mathcal{O}_Y)$ is invertible. Show that the $\mathcal{O}_Y$-module map

$$f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

has a left inverse, i.e., there is an $\mathcal{O}_Y$-module map $\tau : f_*\mathcal{O}_X \to \mathcal{O}_Y$ with $\tau \circ f^\sharp = \mathrm{id}$. Conclude that $H^n(Y, \mathcal{O}_Y) \to H^n(X, \mathcal{O}_X)$ is injective[2].

0DAP **Exercise 111.43.6.** Let X be a scheme (or a locally ringed space). The rule $U \mapsto \mathcal{O}_X(U)^*$ defines a sheaf of groups denoted $\mathcal{O}^*_X$. Briefly explain why the Picard group of X (Definition 111.40.7) is equal to $H^1(X, \mathcal{O}^*_X)$.

0DAQ **Exercise 111.43.7.** Give an example of an affine scheme X with nontrivial $\mathrm{Pic}(X)$. Conclude using Exercise 111.43.6 that $H^1(X, -)$ is not the zero functor for any such X.

0DAR **Exercise 111.43.8.** Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \mathrm{Spec}(A)$ be the complement of $V(I)$. Given a quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module $\mathcal{F}$ and $\xi \in H^p(U, \mathcal{F})$ with $p > 0$, show that there exists $n > 0$ such that $f_i^n\xi = 0$ for $i = 1, \ldots, t$. Hint: One possible way to proceed is to use the complex you found in Exercise 111.43.2.

[2] There does exist a finite locally free morphism $X \to Y$ between integral schemes of degree 2 where the map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ is not injective.

0DAS **Exercise 111.43.9.** Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \mathrm{Spec}(A)$ be the complement of $V(I)$. Let M be an A-module whose I-torsion is zero, i.e., $0 = \mathrm{Ker}((f_1, \ldots, f_t) : M \to M^{\oplus t})$. Show that there is a canonical isomorphism

$$H^0(U, \widetilde{M}) = \mathrm{colim}\, \mathrm{Hom}_A(I^n, M).$$

Warning: this is not trivial.

0DAT **Exercise 111.43.10.** Let A be a Noetherian ring. Let I be an ideal of A. Let M be an A-module. Let $M[I^\infty]$ be the set of I-power torsion elements defined by

$$M[I^\infty] = \{x \in M \mid \text{ there exists an } n \geq 1 \text{ such that } I^n x = 0\}$$

Set $M' = M/M[I^\infty]$. Then the I-power torsion of M' is zero. Show that

$$\mathrm{colim}\, \mathrm{Hom}_A(I^n, M) = \mathrm{colim}\, \mathrm{Hom}_A(I^n, M').$$

Warning: this is not trivial. Hints: (1) try to reduce to M finite, (2) show any element of $\mathrm{Ext}^1_A(I^n, N)$ maps to zero in $\mathrm{Ext}^1_A(I^{n+m}, N)$ for some $m > 0$ if $N = M[I^\infty]$ and M finite, (3) show the same thing as in (2) for $\mathrm{Hom}_A(I^n, N)$, (3) consider the long exact sequence

$$0 \to \mathrm{Hom}_A(I^n, M[I^\infty]) \to \mathrm{Hom}_A(I^n, M) \to \mathrm{Hom}_A(I^n, M') \to \mathrm{Ext}^1_A(I^n, M[I^\infty])$$

for M finite and compare with the sequence for I^{n+m} to conclude.

111.44. Cohomology revisited

0DB3

0DB4 **Exercise 111.44.1.** Make an example of a field k, a curve X over k, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a cohomology class $\xi \in H^1(X, \mathcal{L})$ with the following property: for every surjective finite morphism $\pi : Y \to X$ of schemes the element ξ pulls back to a nonzero element of $H^1(Y, \pi^*\mathcal{L})$. Hint: construct X, k, $\mathcal{L}$ such that there is a short exact sequence $0 \to \mathcal{L} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ where $Z \subset X$ is a closed subscheme consisting of more than 1 closed point. Then look at what happens when you pullback.

0DB5 **Exercise 111.44.2.** Let k be an algebraically closed field. Let X be a projective 1-dimensional scheme. Suppose that X contains a cycle of curves, i.e., suppose there exist an $n \geq 2$ and pairwise distinct 1-dimensional integral closed subschemes $C_1, \ldots, C_n$ and pairwise distinct closed points $x_1, \ldots, x_n \in X$ such that $x_n \in C_n \cap C_1$ and $x_i \in C_i \cap C_{i+1}$ for $i = 1, \ldots, n-1$. Prove that $H^1(X, \mathcal{O}_X)$ is nonzero. Hint: Let $\mathcal{F}$ be the image of the map $\mathcal{O}_X \to \bigoplus \mathcal{O}_{C_i}$, and show $H^1(X, \mathcal{F})$ is nonzero using that $\kappa(x_i) = k$ and $H^0(C_i, \mathcal{O}_{C_i}) = k$. Also use that $H^2(X, -) = 0$ by Grothendieck's theorem.

0DB6 **Exercise 111.44.3.** Let X be a projective surface over an algebraically closed field k. Prove there exists a proper closed subscheme $Z \subset X$ such that $H^1(Z, \mathcal{O}_Z)$ is nonzero. Hint: Use Exercise 111.44.2.

0DB7 **Exercise 111.44.4.** Let X be a projective surface over an algebraically closed field k. Show that for every $n \geq 0$ there exists a proper closed subscheme $Z \subset X$ such that $\dim_k H^1(Z, \mathcal{O}_Z) > n$. Only explain how to do this by modifying the arguments in Exercise 111.44.3 and 111.44.2; don't give all the details.

0DB8 **Exercise 111.44.5.** Let X be a projective surface over an algebraically closed field k. Prove there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $H^2(X, \mathcal{F})$ is nonzero. Hint: Use the result of Exercise 111.44.4 and a cleverly chosen exact sequence.

0DB9 **Exercise 111.44.6.** Let X and Y be schemes over a field k (feel free to assume X and Y are nice, for example qcqs or projective over k). Set $X \times Y = X \times_{\mathrm{Spec}(k)} Y$ with projections $p : X \times Y \to X$ and $q : X \times Y \to Y$. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ prove that

$$H^n(X \times Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G}) = \bigoplus\nolimits_{a+b=n} H^a(X, \mathcal{F}) \otimes_k H^b(Y, \mathcal{G})$$

or just show that this holds when one takes dimensions. Extra points for "clean" solutions.

0DBA **Exercise 111.44.7.** Let k be a field. Let $X = \mathbf{P}|^1 \times \mathbf{P}^1$ be the product of the projective line over k with itself with projections $p : X \to \mathbf{P}^1_k$ and $q : X \to \mathbf{P}^1_k$. Let

$$\mathcal{O}(a, b) = p^*\mathcal{O}_{\mathbf{P}^1_k}(a) \otimes_{\mathcal{O}_X} q^*\mathcal{O}_{\mathbf{P}^1_k}(b)$$

Compute the dimensions of $H^i(X, \mathcal{O}(a, b))$ for all i, a, b. Hint: Use Exercise 111.44.6.

111.45. Cohomology and Hilbert polynomials

0DCD

0DCE **Situation 111.45.1.** Let k be a field. Let $X = \mathbf{P}^n_k$ be n-dimensional projective space. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Recall that

$$\chi(X, \mathcal{F}) = \sum\nolimits_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F})$$

Recall that the *Hilbert polynomial* of $\mathcal{F}$ is the function

$$t \longmapsto \chi(X, \mathcal{F}(t))$$

We also recall that $\mathcal{F}(t) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$ where $\mathcal{O}_X(t)$ is the tth twist of the structure sheaf as in Constructions, Definition 27.10.1. In Varieties, Subsection 33.35.13 we have proved the Hilbert polynomial is a polynomial in t.

0DCF **Exercise 111.45.2.** In Situation 111.45.1.

(1) If $P(t)$ is the Hilbert polynomial of $\mathcal{F}$, what is the Hilbert polynomial of $\mathcal{F}(-13)$.
(2) If P_i is the Hilbert polynomial of $\mathcal{F}_i$, what is the Hilbert polynomial of $\mathcal{F}_1 \oplus \mathcal{F}_2$.
(3) If P_i is the Hilbert polynomial of $\mathcal{F}_i$ and $\mathcal{F}$ is the kernel of a surjective map $\mathcal{F}_1 \to \mathcal{F}_2$, what is the Hilbert polynomial of $\mathcal{F}$?

0DCG **Exercise 111.45.3.** In Situation 111.45.1 assume $n \geq 1$. Find a coherent sheaf whose Hilbert polynomial is $t - 101$.

0DCH **Exercise 111.45.4.** In Situation 111.45.1 assume $n \geq 2$. Find a coherent sheaf whose Hilbert polynomial is $t^2/2 + t/2 - 1$. (This is a bit tricky; it suffices if you just show there is such a coherent sheaf.)

0DCI **Exercise 111.45.5.** In Situation 111.45.1 assume $n \geq 2$ and k algebraically closed. Let $C \subset X$ be an integral closed subscheme of dimension 1. In other words, C is a projective curve. Let $dt + e$ be the Hilbert polynomial of $\mathcal{O}_C$ viewed as a coherent sheaf on X.

(1) Give an upper bound on e. (Hints: Use that $\mathcal{O}_C(t)$ only has cohomology in degrees 0 and 1 and study $H^0(C, \mathcal{O}_C)$.)
(2) Pick a global section s of $\mathcal{O}_X(1)$ which intersects C transversally, i.e., such that there are pairwise distinct closed points $c_1, \ldots, c_r \in C$ and a short exact sequence
$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}_C(1) \to \bigoplus\nolimits_{i=1,\ldots,r} k_{c_i} \to 0$$
where k_{c_i} is the skyscraper sheaf with value k in c_i. (Such an s exists; please just use this.) Show that $r = d$. (Hint: twist the sequence and see what you get.)
(3) Twisting the short exact sequence gives a k-linear map $\varphi_t : \Gamma(C, \mathcal{O}_C(t)) \to \bigoplus_{i=1,\ldots,d} k$ for any t. Show that if this map is surjective for $t \geq d - 1$.
(4) Give a lower bound on e in terms of d. (Hint: show that $H^1(C, \mathcal{O}_C(d - 2)) = 0$ using the result of (3) and use vanishing.)

0DCJ **Exercise 111.45.6.** In Situation 111.45.1 assume $n = 2$. Let $s_1, s_2, s_3 \in \Gamma(X, \mathcal{O}_X(2))$ be three quadric equations. Consider the coherent sheaf
$$\mathcal{F} = \text{Coker}\left(\mathcal{O}_X(-2)^{\oplus 3} \xrightarrow{s_1, s_2, s_3} \mathcal{O}_X\right)$$
List the possible Hilbert polynomials of such $\mathcal{F}$. (Try to visualize intersections of quadrics in the projective plane.)

111.46. Curves

0EG8

0EG9 **Exercise 111.46.1.** Let k be an algebraically closed field. Let X be a projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Prove there is a natural closed subscheme
$$Z \subset \mathbf{P}^n \times X$$
such that the closed point $((\lambda_0 : \ldots : \lambda_n), x)$ is in Z if and only if the section $\lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes at x. (Hint: describe Z affine locally.)

0EGA **Exercise 111.46.2.** Let k be an algebraically closed field. Let X be a smooth curve over k. Let $r \geq 1$. Show that the closed subset
$$D \subset X \times X^r = X^{r+1}$$
whose closed points are the tuples $(x, x_1, \ldots, x_r)$ with $x = x_i$ for some i, has an invertible ideal sheaf. In other words, show that D is an effective Cartier divisor. Hints: reduce to $r = 1$ and use that X is a smooth curves to say something about the diagonal (look in books for this).

0EGB **Exercise 111.46.3.** Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let T be a scheme of finite type over k and let
$$D_1 \subset X \times T \quad \text{and} \quad D_2 \subset X \times T$$
be two effective Cartier divisors such that for $t \in T$ the fibres $D_{i,t} \subset X_t$ are not dense (i.e., do not contain the generic point of the curve X_t). Prove that there is a canonical closed subscheme $Z \subset T$ such that a closed point $t \in T$ is in Z if and only if for the scheme theoretic fibres $D_{1,t}$, $D_{2,t}$ of D_1, D_2 we have
$$D_{1,t} \subset D_{2,t}$$

as closed subschemes of X_t. Hints: Show that, possibly after shrinking T, you may assume $T = \mathrm{Spec}(A)$ is affine and there is an affine open $U \subset X$ such that $D_i \subset U \times T$. Then show that $M_1 = \Gamma(D_1, \mathcal{O}_{D_1})$ is a finite locally free A-module (here you will need some nontrivial algebra — ask your friends). After shrinking T you may assume M_1 is a free A-module. Then look at

$$\Gamma(U \times T, \mathcal{I}_{D_2}) \to M_1 = A^{\oplus N}$$

and you define Z as the closed subscheme cut out by the ideal generated by coefficients of vectors in the image of this map. Explain why this works (this will require perhaps a bit more commutative algebra).

0EGC **Exercise 111.46.4.** Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Let $r \geq 1$. Prove there is a natural closed subscheme

$$Z \subset \mathbf{P}^n \times X \times \ldots \times X = \mathbf{P}^n \times X^r$$

such that the closed point $((\lambda_0 : \ldots : \lambda_n), x_1, \ldots, x_r)$ is in Z if and only if the section $s_\lambda = \lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes on the divisor $D = x_1 + \ldots + x_r$, i.e., the section s_λ is in $\mathcal{L}(-D)$. Hint: explain how this follows by combining then results of Exercises 111.46.2 and 111.46.3.

0EGD **Exercise 111.46.5.** Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that there is a natural closed subset

$$Z \subset X^r$$

such that a closed point $(x_1, \ldots, x_r)$ of X^r is in Z if and only if $\mathcal{L}(-x_1 - \ldots - x_r)$ has a nonzero global section. Hint: use Exercise 111.46.4.

0EGE **Exercise 111.46.6.** Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $r \geq s$ be integers. Show that there is a natural closed subset

$$Z \subset X^r \times X^s$$

such that a closed point $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ of $X^r \times X^s$ is in Z if and only if $x_1 + \ldots + x_r - y_1 - \ldots - y_s$ is linearly equivalent to an effective divisor. Hint: Choose an auxiliary invertible module $\mathcal{L}$ of very high degree so that $\mathcal{L}(-D)$ has a nonvanshing section for any effective divisor D of degree r. Then use the result of Exercise 111.46.5 twice.

0EGR **Exercise 111.46.7.** Choose your favorite algebraically closed field k. As best as you can determine all possible $\mathfrak{g}^r_d$ that can exist on some curve of genus 7. While doing this also try to

(1) determine in which cases the $\mathfrak{g}^r_d$ is base point free, and
(2) determine in which cases the $\mathfrak{g}^r_d$ gives a closed embedding in $\mathbf{P}^r$.

Do the same thing if you assume your curve is "general" (make up your own notion of general – this may be easier than the question above). Do the same thing if you assume your curve is hyperelliptic. Do the same thing if you assume your curve is trigonal (and not hyperelliptic). Etc.

111.47. Moduli

0EGN In this section we consider some naive approaches to moduli of algebraic geometric objects.

Let k be an algebraically closed field. Suppose that M is a moduli problem over k. We won't define exactly what this means here, but in each exercise it should be clear what we mean. To understand the following it suffices to know what the objects of M over k are, what the isomorphisms between objects of M over k are, and what the families of object of M over a variety are. Then we say the *number of moduli of* M is $d \geq 0$ if the following are true

(1) there is a finite number of families $X_i \to V_i$, $i = 1, \ldots, n$ such that every object of M over k is isomorphic to a fibre of one of these and such that $\max \dim(V_i) = d$, and
(2) there is no way to do this with a smaller d.

This is really just a very rough approximation of better notions in the literature.

0EGP **Exercise 111.47.1.** Let k be an algebraically closed field. Let $d \geq 1$ and $n \geq 1$. Let us say the moduli of hypersurfaces of degree d in P^n is given by

(1) an object is a hypersurface $X \subset \mathbf{P}^n_k$ of degree d,
(2) an isomorphism between two objects $X \subset \mathbf{P}^n_k$ and $Y \subset \mathbf{P}^n_k$ is an element $g \in \mathrm{PGL}_n(k)$ such that $g(X) = Y$, and
(3) a family of hypersurfaces over a variety V is a closed subscheme $X \subset \mathbf{P}^n_V$ such that for all $v \in V$ the scheme theoretic fibre X_v of $X \to V$ is a hypersurfaces in $\mathbf{P}^n_v$.

Compute (if you can – these get progressively harder)

(1) the number of moduli when $n = 1$ and d arbitrary,
(2) the number of moduli when $n = 2$ and $d = 1$,
(3) the number of moduli when $n = 2$ and $d = 2$,
(4) the number of moduli when $n \geq 1$ and $d = 2$,
(5) the number of moduli when $n = 2$ and $d = 3$,
(6) the number of moduli when $n = 3$ and $d = 3$, and
(7) the number of moduli when $n = 2$ and $d = 4$.

0EGQ **Exercise 111.47.2.** Let k be an algebraically closed field. Let $g \geq 2$. Let us say the moduli of hyperelliptic curves of genus g is given by

(1) an object is a smooth projective hyperelliptic curve X of genus g,
(2) an isomorphism between two objects X and Y is an isomorphism $X \to Y$ of schemes over k, and
(3) a family of hyperelliptic curves of genus g over a variety V is a proper flat[3] morphism $X \to Y$ such that all scheme theoretic fibres of $X \to V$ are smooth projective hyperelliptic curves of genus g.

Show that the number of moduli is $2g - 1$.

111.48. Global Exts

0DD0

[3]You can drop this assumption without changing the answer to the question.

0DD1 **Exercise 111.48.1.** Let k be a field. Let $X = \mathbf{P}^3_k$. Let $L \subset X$ and $P \subset X$ be a line and a plane, viewed as closed subschemes cut out by 1, resp., 2 linear equations. Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_L, \mathcal{O}_P)$$

for all i. Make sure to do both the case where L is contained in P and the case where L is not contained in P.

0DD2 **Exercise 111.48.2.** Let k be a field. Let $X = \mathbf{P}^n_k$. Let $Z \subset X$ be a closed k-rational point viewed as a closed subscheme. For example the point with homogeneous coordinates $(1 : 0 : \ldots : 0)$. Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_Z, \mathcal{O}_Z)$$

for all i.

0DD3 **Exercise 111.48.3.** Let X be a ringed space. Define cup-product maps

$$\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{H}) \times \mathrm{Ext}^j_X(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^{i+j}_X(\mathcal{F}, \mathcal{H})$$

for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$. (Hint: this is a super general thing.)

0DD4 **Exercise 111.48.4.** Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. Prove the following statements

(1) $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$, and
(2) $\mathrm{Ext}^i_X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$.

Here $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Conclude that

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$$

0DD5 **Exercise 111.48.5.** Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Construct a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E}) \to H^i(X, \mathcal{O}_X)$$

for all i. Generalize to a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \to H^i(X, \mathcal{F})$$

for any $\mathcal{O}_X$-module $\mathcal{F}$.

0DD6 **Exercise 111.48.6.** Let k be a field. Let $X = \mathbf{P}^d_k$. Set $\omega_{X/k} = \mathcal{O}_X(-d-1)$. Prove that for finite locally free modules $\mathcal{E}$, $\mathcal{F}$ the cup product on Ext combined with the trace map on Ext

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \times \mathrm{Ext}^{d-i}_X(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^d_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \to H^d(X, \omega_{X/k}) = k$$

produces a nondegenerate pairing. Hint: you can either reprove duality in this setting or you can reduce to cohomology of sheaves and apply the Serre duality theorem as proved in the lectures.

111.49. Divisors

02AQ We collect all relevant definitions here in one spot for convenience.

02AR **Definition 111.49.1.** Throughout, let S be any scheme and let X be a Noetherian, integral scheme.

(1) A *Weil divisor* on X is a formal linear combination $\Sigma n_i [Z_i]$ of prime divisors Z_i with integer coefficients.

(2) A *prime divisor* is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.

(3) The *Weil divisor associated to a rational function* $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows
   (a) If $f \in \mathcal{O}_{X,Z}^*$ then $v_Z(f) = 0$.
   (b) If $f \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$
   (c) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

(4) An *effective Cartier divisor* on a scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $\text{Spec}(A) = U \subset S$ in S so that $D \cap U = \text{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.

(5) The *Weil divisor* $[D]$ *associated to an effective Cartier divisor* $D \subset X$ of our Noetherian integral scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows
   (a) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.
   (b) If the generic point ξ of Z is in D then
   $$v_Z(D) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$
   where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzerodivisor which defines D in an affine neighbourhood of ξ (as in (4) above).

(6) Let S be a scheme. The *sheaf of total quotient rings* $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzerodivisor for every $u \in U$. In particular the elements of S_U are all nonzerodivisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence
$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{K}_S^*/\mathcal{O}_S^* \to 0.$$

(7) A *Cartier divisor* on a scheme S is a global section of the quotient sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.

(8) The *Weil divisor associated to a Cartier divisor* $\tau \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ over our Noetherian integral scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe
   (a) If the germ of τ at the generic point ξ of Z is zero – in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is "zero" – then $v_Z(\tau) = 0$.
   (b) Find an affine open neighbourhood $\text{Spec}(A) = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set
   $$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

02F5 **Remarks 111.49.2.** Here are some trivial remarks.

(1) On a Noetherian integral scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.

(2) To make sense out of the definitions above one needs to show that

$$\text{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \text{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \text{length}_{\mathcal{O}}(\mathcal{O}/(b))$$

for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

02F6 **Exercise 111.49.3.** (On any scheme.) Describe how to assign a Cartier divisor to an effective Cartier divisor.

02F7 **Exercise 111.49.4.** (On an integral scheme.) Describe how to assign a Cartier divisor D to a rational function f such that the Weil divisor associated to D and to f agree. (This is silly.)

02F8 **Exercise 111.49.5.** Give an example of a Weil divisor on a variety which is not the Weil divisor associated to any Cartier divisor.

02F9 **Exercise 111.49.6.** Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associated to a Cartier divisor for some $n > 1$.

02FA **Exercise 111.49.7.** Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associated to a Cartier divisor for any $n > 1$. (Hint: Consider a cone, for example $X : xy - zw = 0$ in $\mathbf{A}^4_k$. Try to show that $D = [x = 0, z = 0]$ works.)

02FB **Exercise 111.49.8.** On a separated scheme X of finite type over a field: Give an example of a Cartier divisor which is not the difference of two effective Cartier divisors. Hint: Find some X which does not have any nonempty effective Cartier divisors for example the scheme constructed in [**Har77**, III Exercise 5.9]. There is even an example with X a variety – namely the variety of Exercise 111.49.9.

02AS **Exercise 111.49.9.** Example of a nonprojective proper variety. Let k be a field. Let $L \subset \mathbf{P}^3_k$ be a line and let $C \subset \mathbf{P}^3_k$ be a nonsingular conic. Assume that $C \cap L = \emptyset$. Choose an isomorphism $\varphi : L \to C$. Let X be the k-variety obtained by glueing C to L via φ. In other words there is a surjective proper birational morphism

$$\pi : \mathbf{P}^3_k \longrightarrow X$$

and an open $U \subset X$ such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, $\pi^{-1}(U) = \mathbf{P}^3_k \setminus (L \cup C)$ and such that $\pi|_L = \pi|_C \circ \varphi$. (These conditions do not yet uniquely define X. In order to do this you need to specify the structure sheaf of X along points of $Z = X \setminus U$.) Show X exists, is a proper variety, but is not projective. (Hint: For existence use the result of Exercise 111.37.9. For non-projectivity use that $\text{Pic}(\mathbf{P}^3_k) = \mathbf{Z}$ to show that X cannot have an ample invertible sheaf.)

111.50. Differentials

02AT **Definitions and results.** Kähler differentials.

(1) Let $R \to A$ be a ring map. The *module of Kähler differentials of A over R* is denoted $\Omega_{A/R}$. It is generated by the elements $\mathrm{d}a$, $a \in A$ subject to the relations:

$$\mathrm{d}(a_1 + a_2) = \mathrm{d}a_1 + \mathrm{d}a_2, \quad \mathrm{d}(a_1 a_2) = a_1 \mathrm{d}a_2 + a_2 \mathrm{d}a_1, \quad \mathrm{d}r = 0$$

The canonical universal R-derivation $\mathrm{d} : A \to \Omega_{A/R}$ maps $a \mapsto \mathrm{d}a$.

(2) Consider the short exact sequence
$$0 \to I \to A \otimes_R A \to A \to 0$$
which defines the ideal I. There is a canonical derivation $\mathrm{d} : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words
$$(I/I^2, \mathrm{d}) \cong (\Omega_{A/R}, \mathrm{d}).$$
(3) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have
$$\Omega_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega_{A/R}.$$
(4) If A is a finitely presented R-algebra then $\Omega_{A/R}$ is a finitely presented A-module. Hence in this case the *fitting* ideals of $\Omega_{A/R}$ are defined.
(5) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/S}$ and a $\mathcal{O}_S$-linear derivation
$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$
such that for any affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ we have
$$\Gamma(\mathrm{Spec}(A), \Omega_{X/S}) = \Omega_{A/R}$$
compatibly with d.

02FC **Exercise 111.50.1.** Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(1) Consider the ring map
$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$
Show that the Fitting ideals of $\Omega_{A/R}$ are (starting with the zeroth Fitting ideal)
$$(\epsilon), A, A, \ldots$$
(2) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y - t)(y - 1), x(x - t))$. Show that the Fitting ideals of $\Omega_{A/R}$ in A are (assume characteristic k is zero for simplicity)
$$x(2x - t)(2y - t - 1)A,\ (x, y, t) \cap (x, y - 1, t),\ A,\ A, \ldots$$
So the 0-the Fitting ideal is cut out by a single element of A, the 1st Fitting ideal defines two closed points of $\mathrm{Spec}(A)$, and the others are all trivial.
(3) Consider the map $R = k[t] \to A = k[x, y, t]/(xy - t^n)$. Compute the Fitting ideals of $\Omega_{A/R}$.

02FD **Remark 111.50.2.** The kth Fitting ideal of $\Omega_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have "constant" fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't guarantee it works either (and yes this is a flat family). In "good cases" – such as in (c) – for families of curves you expect the 0-th Fitting ideal to be zero and the 1st Fitting ideal to define (scheme-theoretically) the singular locus.

02FE **Exercise 111.50.3.** Suppose that R is a ring and

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives

$$(\partial f_i/\partial x_j)$$

is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

02FF **Exercise 111.50.4.** Find a generalization of the result of Exercise 111.50.3 to the case where $A = R[x, y]/(f)$.

0D1T **Exercise 111.50.5.** Let k be a field, let $f_1, \ldots, f_c \in k[x_1, \ldots, x_n]$, and let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Assume that $f_j(0, \ldots, 0) = 0$. This means that $\mathfrak{m} = (x_1, \ldots, x_n)A$ is a maximal ideal. Prove that the local ring $A_\mathfrak{m}$ is regular if the rank of the matrix

$$(\partial f_j/\partial x_i)|_{(x_1,\ldots,x_n)=(0,\ldots,0)}$$

is c. What is the dimension of $A_\mathfrak{m}$ in this case? Show that the converse is false by giving an example where $A_\mathfrak{m}$ is regular but the rank is less than c; what is the dimension of $A_\mathfrak{m}$ in your example?

111.51. Schemes, Final Exam, Fall 2007

02AU These were the questions in the final exam of a course on Schemes, in the Spring of 2007 at Columbia University.

02FG **Exercise 111.51.1** (Definitions)**.** Provide definitions of the following concepts.

(1) X is a *scheme*
(2) the morphism of schemes $f : X \to Y$ is *finite*
(3) the morphisms of schemes $f : X \to Y$ is *of finite type*
(4) the scheme X is *Noetherian*
(5) the $\mathcal{O}_X$-module $\mathcal{L}$ on the scheme X is *invertible*
(6) the *genus* of a nonsingular projective curve over an algebraically closed field

02FH **Exercise 111.51.2.** Let $X = \mathrm{Spec}(\mathbf{Z}[x, y])$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is zero when restricted to the standard affine open $D(x)$.

(1) Show that every global section s of $\mathcal{F}$ is killed by some power of x, i.e., $x^n s = 0$ for some $n \in \mathbf{N}$.
(2) Do you think the same is true if we do not assume that $\mathcal{F}$ is quasi-coherent?

02FI **Exercise 111.51.3.** Suppose that $X \to \mathrm{Spec}(R)$ is a proper morphism and that R is a discrete valuation ring with residue field k. Suppose that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ is the empty scheme. Show that X is the empty scheme.

02FJ **Exercise 111.51.4.** Consider the projective[4] variety

$$\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}^1_{\mathbf{C}} \times_{\mathrm{Spec}(\mathbf{C})} \mathbf{P}^1_{\mathbf{C}}$$

over the field of complex numbers $\mathbf{C}$. It is covered by four affine pieces, corresponding to pairs of standard affine pieces of $\mathbf{P}^1_{\mathbf{C}}$. For example, suppose we use homogeneous coordinates X_0, X_1 on the first factor and Y_0, Y_1 on the second. Set $x = X_1/X_0$, and $y = Y_1/Y_0$. Then the 4 affine open pieces are the spectra of the rings

$$\mathbf{C}[x, y], \quad \mathbf{C}[x^{-1}, y], \quad \mathbf{C}[x, y^{-1}], \quad \mathbf{C}[x^{-1}, y^{-1}].$$

Let $X \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the closed subscheme which is the closure of the closed subset of the first affine piece given by the equation

$$y^3(x^4 + 1) = x^4 - 1.$$

(1) Show that X is contained in the union of the first and the last of the 4 affine open pieces.
(2) Show that X is a nonsingular projective curve.
(3) Consider the morphism $pr_2 : X \to \mathbf{P}^1$ (projection onto the first factor). On the first affine piece it is the map $(x, y) \mapsto x$. Briefly explain why it has degree 3.
(4) Compute the ramification points and ramification indices for the map $pr_2 : X \to \mathbf{P}^1$.
(5) Compute the genus of X.

02FK **Exercise 111.51.5.** Let $X \to \mathrm{Spec}(\mathbf{Z})$ be a morphism of finite type. Suppose that there is an infinite number of primes p such that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{F}_p)$ is not empty.

(1) Show that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Q})$ is not empty.
(2) Do you think the same is true if we replace the condition "finite type" by the condition "locally of finite type"?

111.52. Schemes, Final Exam, Spring 2009

02AV These were the questions in the final exam of a course on Schemes, in the Spring of 2009 at Columbia University.

02AW **Exercise 111.52.1.** Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $x \in X$ be a point. Assume that $\mathrm{Supp}(\mathcal{F}) = \{x\}$.

(1) Show that x is a closed point of X.
(2) Show that $H^0(X, \mathcal{F})$ is not zero.
(3) Show that $\mathcal{F}$ is generated by global sections.
(4) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

02AX **Remark 111.52.2.** Let k be a field. Let $\mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Any invertible sheaf on $\mathbf{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2_k}(n)$ for some $n \in \mathbf{Z}$. Recall that

$$\Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(n)) = k[X_0, X_1, X_2]_n$$

is the degree n part of the polynomial ring. For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^2_k$ set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^2_k}} \mathcal{O}_{\mathbf{P}^2_k}(n)$ as usual.

[4]The projective embedding is $((X_0, X_1), (Y_0, Y_1)) \mapsto (X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1)$ in other words $(x, y) \mapsto (1, y, x, xy)$.

02AY **Exercise 111.52.3.** Let k be a field. Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^2_k$, i.e., a finite locally free $\mathcal{O}_{\mathbf{P}^2_k}$-module. We say $\mathcal{E}$ is *split* if $\mathcal{E}$ is isomorphic to a direct sum invertible $\mathcal{O}_{\mathbf{P}^2_k}$-modules.

(1) Show that $\mathcal{E}$ is split if and only if $\mathcal{E}(n)$ is split.
(2) Show that if $\mathcal{E}$ is split then $H^1(\mathbf{P}^2_k, \mathcal{E}(n)) = 0$ for all $n \in \mathbf{Z}$.
(3) Let
$$\varphi : \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1)$$
be given by linear forms $L_0, L_1, L_2 \in \Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$. Assume $L_i \neq 0$ for some i. What is the condition on L_0, L_1, L_2 such that the cokernel of φ is a vector bundle? Why?
(4) Given an example of such a φ.
(5) Show that $\mathrm{Coker}(\varphi)$ is not split (if it is a vector bundle).

02AZ **Remark 111.52.4.** Freely use the following facts on dimension theory (and add more if you need more).

(1) The dimension of a scheme is the supremum of the length of chains of irreducible closed subsets.
(2) The dimension of a finite type scheme over a field is the maximum of the dimensions of its affine opens.
(3) The dimension of a Noetherian scheme is the maximum of the dimensions of its irreducible components.
(4) The dimension of an affine scheme coincides with the dimension of the corresponding ring.
(5) Let k be a field and let A be a finite type k-algebra. If A is a domain, and $x \neq 0$, then $\dim(A) = \dim(A/xA) + 1$.

02B0 **Exercise 111.52.5.** Let k be a field. Let X be a projective, reduced scheme over k. Let $f : X \to \mathbf{P}^1_k$ be a morphism of schemes over k. Assume there exists an integer $d \geq 0$ such that for every point $t \in \mathbf{P}^1_k$ the fibre $X_t = f^{-1}(t)$ is irreducible of dimension d. (Recall that an irreducible space is not empty.)

(1) Show that $\dim(X) = d + 1$.
(2) Let $X_0 \subset X$ be an irreducible component of X of dimension $d+1$. Prove that for every $t \in \mathbf{P}^1_k$ the fibre $X_{0,t}$ has dimension d.
(3) What can you conclude about X_t and $X_{0,t}$ from the above?
(4) Show that X is irreducible.

02B1 **Remark 111.52.6.** Given a projective scheme X over a field k and a coherent sheaf $\mathcal{F}$ on X we set
$$\chi(X, \mathcal{F}) = \sum\nolimits_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

02B2 **Exercise 111.52.7.** Let k be a field. Write $\mathbf{P}^3_k = \mathrm{Proj}(k[X_0, X_1, X_2, X_3])$. Let $C \subset \mathbf{P}^3_k$ be a *type* $(5, 6)$ *complete intersection curve*. This means that there exist $F \in k[X_0, X_1, X_2, X_3]_5$ and $G \in k[X_0, X_1, X_2, X_3]_6$ such that
$$C = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F, G))$$
is a variety of dimension 1. (Variety implies reduced and irreducible, but feel free to assume C is nonsingular if you like.) Let $i : C \to \mathbf{P}^3_k$ be the corresponding closed

immersion. Being a complete intersection also implies that

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(-11) \xrightarrow{\begin{pmatrix} -G \\ F \end{pmatrix}} \mathcal{O}_{\mathbf{P}^3_k}(-5) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-6) \xrightarrow{(F,G)} \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

is an exact sequence of sheaves. Please use these facts to:

(1) compute $\chi(C, i^*\mathcal{O}_{\mathbf{P}^3_k}(n))$ for any $n \in \mathbf{Z}$, and
(2) compute the dimension of $H^1(C, \mathcal{O}_C)$.

02B3 **Exercise 111.52.8.** Let k be a field. Consider the rings

$$A = k[x, y]/(xy)$$
$$B = k[u, v]/(uv)$$
$$C = k[t, t^{-1}] \times k[s, s^{-1}]$$

and the k-algebra maps

$$\begin{array}{llll} A \longrightarrow C, & x \mapsto (t, 0), & y \mapsto (0, s) \\ B \longrightarrow C, & u \mapsto (t^{-1}, 0), & v \mapsto (0, s^{-1}) \end{array}$$

It is a true fact that these maps induce isomorphisms $A_{x+y} \to C$ and $B_{u+v} \to C$. Hence the maps $A \to C$ and $B \to C$ identify $\text{Spec}(C)$ with open subsets of $\text{Spec}(A)$ and $\text{Spec}(B)$. Let X be the scheme obtained by glueing $\text{Spec}(A)$ and $\text{Spec}(B)$ along $\text{Spec}(C)$:

$$X = \text{Spec}(A) \amalg_{\text{Spec}(C)} \text{Spec}(B).$$

As we saw in the course such a scheme exists and there are affine opens $\text{Spec}(A) \subset X$ and $\text{Spec}(B) \subset X$ whose overlap is exactly $\text{Spec}(C)$ identified with an open of each of these using the maps above.

(1) Why is X separated?
(2) Why is X of finite type over k?
(3) Compute $H^1(X, \mathcal{O}_X)$, or what is its dimension?
(4) What is a more geometric way to describe X?

111.53. Schemes, Final Exam, Fall 2010

069Q These were the questions in the final exam of a course on Schemes, in the Fall of 2010 at Columbia University.

069R **Exercise 111.53.1** (Definitions)**.** Provide definitions of the following concepts.

(1) a separated scheme,
(2) a quasi-compact morphism of schemes,
(3) an affine morphism of schemes,
(4) a multiplicative subset of a ring,
(5) a Noetherian scheme,
(6) a variety.

069S **Exercise 111.53.2.** Prime avoidance.

(1) Let A be a ring. Let $I \subset A$ be an ideal and let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be prime ideals such that $I \not\subset \mathfrak{q}_i$. Show that $I \not\subset \mathfrak{q}_1 \cup \mathfrak{q}_2$.
(2) What is a geometric interpretation of (1)?
(3) Let $X = \text{Proj}(S)$ for some graded ring S. Let $x_1, x_2 \in X$. Show that there exists a standard open $D_+(F)$ which contains both x_1 and x_2.

069T **Exercise 111.53.3.** Why is a composition of affine morphisms affine?

069U **Exercise 111.53.4** (Examples)**.** Give examples of the following:

(1) A reducible projective scheme over a field k.
(2) A scheme with 100 points.
(3) A non-affine morphism of schemes.

069V **Exercise 111.53.5.** Chevalley's theorem and the Hilbert Nullstellensatz.

(1) Let $\mathfrak{p} \subset \mathbf{Z}[x_1, \ldots, x_n]$ be a maximal ideal. What does Chevalley's theorem imply about $\mathfrak{p} \cap \mathbf{Z}$?
(2) In turn, what does the Hilbert Nullstellensatz imply about $\kappa(\mathfrak{p})$?

069W **Exercise 111.53.6.** Let A be a ring. Let $S = A[X]$ as a graded A-algebra where X has degree 1. Show that $\mathrm{Proj}(S) \cong \mathrm{Spec}(A)$ as schemes over A.

069X **Exercise 111.53.7.** Let $A \to B$ be a finite ring map. Show that $\mathrm{Spec}(B)$ is a H-projective scheme over $\mathrm{Spec}(A)$.

069Y **Exercise 111.53.8.** Give an example of a scheme X over a field k such that X is irreducible and such that for some finite extension k'/k the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is connected but reducible.

111.54. Schemes, Final Exam, Spring 2011

069Z These were the questions in the final exam of a course on Schemes, in the Spring of 2011 at Columbia University.

06A0 **Exercise 111.54.1** (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *separated* scheme,
(2) a *universally closed* morphism of schemes,
(3) *A dominates B* for local rings A, B contained in a common field,
(4) the *dimension* of a scheme X,
(5) the *codimension* of an irreducible closed subscheme Y of a scheme X,

06A1 **Exercise 111.54.2** (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) The valuative criterion of properness for a morphism $X \to Y$ of varieties for example.
(2) The relationship between $\dim(X)$ and the function field $k(X)$ of X for a variety X over a field k.
(3) Fill in the blank: The category of nonsingular projective curves over k and nonconstant morphisms is anti-equivalent to .....
(4) Noether normalization.
(5) Jacobian criterion.

06A2 **Exercise 111.54.3.** Let k be a field. Let $F \in k[X_0, X_1, X_2]$ be a homogeneous form of degree d. Assume that $C = V_+(F) \subset \mathbf{P}^2_k$ is a smooth curve over k. Denote $i : C \to \mathbf{P}^2_k$ the corresponding closed immersion.

(1) Show that there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_C \to 0$$

of coherent sheaves on $\mathbf{P}^2_k$: tell me what the maps are and briefly why it is exact.

(2) Conclude that $H^0(C, \mathcal{O}_C) = k$.
(3) Compute the genus of C.
(4) Assume now that $P = (0 : 0 : 1)$ is not on C. Prove that $\pi : C \to \mathbf{P}^1_k$ given by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ has degree d.
(5) Assume k is algebraically closed, assume all ramification indices (the "e_i") are 1 or 2, and assume the characteristic of k is not equal to 2. How many ramification points does $\pi : C \to \mathbf{P}^1_k$ have?
(6) In terms of F, what do you think is a set of equations of the set of ramification points of π?
(7) Can you guess K_C?

06A3 **Exercise 111.54.4.** Let k be a field. Let X be a "triangle" over k, i.e., you get X by glueing three copies of $\mathbf{A}^1_k$ to each other by identifying 0 on the first copy to 1 on the second copy, 0 on the second copy to 1 on the third copy, and 0 on the third copy to 1 on the first copy. It turns out that X is isomorphic to $\mathrm{Spec}(k[x, y]/(xy(x + y + 1)))$; feel free to use this. Compute the Picard group of X.

06A4 **Exercise 111.54.5.** Let k be a field. Let $\pi : X \to Y$ be a finite birational morphism of curves with X a projective nonsingular curve over k. It follows from the material in the course that Y is a proper curve and that π is the normalization morphism of Y. We have also seen in the course that there exists a dense open $V \subset Y$ such that $U = \pi^{-1}(V)$ is a dense open in X and $\pi : U \to V$ is an isomorphism.

(1) Show that there exists an effective Cartier divisor $D \subset X$ such that $D \subset U$ and such that $\mathcal{O}_X(D)$ is ample on X.
(2) Let D be as in (1). Show that $E = \pi(D)$ is an effective Cartier divisor on Y.
(3) Briefly indicate why
   (a) the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a coherent cokernel Q which is supported in $Y \setminus V$, and
   (b) for every n there is a corresponding map $\mathcal{O}_Y(nE) \to \pi_*\mathcal{O}_X(nD)$ whose cokernel is isomorphic to Q.
(4) Show that $\dim_k H^0(X, \mathcal{O}_X(nD)) - \dim_k H^0(Y, \mathcal{O}_Y(nE))$ is bounded (by what?) and conclude that the invertible sheaf $\mathcal{O}_Y(nE)$ has lots of sections for large n (why?).

111.55. Schemes, Final Exam, Fall 2011

07DE These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2011 at Columbia University.

07DF **Exercise 111.55.1** (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *Noetherian* ring,
(2) a *Noetherian* scheme,
(3) a *finite* ring homomorphism,
(4) a *finite* morphism of schemes,
(5) the *dimension* of a ring.

07DG **Exercise 111.55.2** (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) Zariski's Main Theorem.

(2) Noether normalization.
(3) Chinese remainder theorem.
(4) Going up for finite ring maps.

07DH **Exercise 111.55.3.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring whose residue field has characteristic not 2. Suppose that $\mathfrak{m}$ is generated by three elements x, y, z and that $x^2 + y^2 + z^2 = 0$ in A.

(1) What are the possible values of $\dim(A)$?
(2) Give an example to show that each value is possible.
(3) Show that A is a domain if $\dim(A) = 2$. (Hint: look at $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.)

07DI **Exercise 111.55.4.** Let A be a ring. Let $S \subset T \subset A$ be multiplicative subsets. Assume that

$$\{\mathfrak{q} \mid \mathfrak{q} \cap S = \emptyset\} = \{\mathfrak{q} \mid \mathfrak{q} \cap T = \emptyset\}.$$

Show that $S^{-1}A \to T^{-1}A$ is an isomorphism.

07DJ **Exercise 111.55.5.** Let k be an algebraically closed field. Let

$$V_0 = \{A \in \mathrm{Mat}(3 \times 3, k) \mid \mathrm{rank}(A) = 1\} \subset \mathrm{Mat}(3 \times 3, k) = k^9.$$

(1) Show that V_0 is the set of closed points of a (Zariski) locally closed subset $V \subset \mathbf{A}^9_k$.
(2) Is V irreducible?
(3) What is $\dim(V)$?

07DK **Exercise 111.55.6.** Prove that the ideal (x^2, xy, y^2) in $\mathbf{C}[x, y]$ cannot be generated by 2 elements.

07DL **Exercise 111.55.7.** Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that for some $\alpha, \beta \in \mathbf{C}$ the $\mathbf{C}$-algebra map

$$\mathbf{C}[t] \longrightarrow \mathbf{C}[x, y]/(f), \quad t \longmapsto \alpha x + \beta y$$

is finite.

07DM **Exercise 111.55.8.** Show that given finitely many points $p_1, \ldots, p_n \in \mathbf{C}^2$ the scheme $\mathbf{A}^2_{\mathbf{C}} \setminus \{p_1, \ldots, p_n\}$ is a union of two affine opens.

07DN **Exercise 111.55.9.** Show that there exists a surjective morphism of schemes $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$. (Surjective just means surjective on underlying sets of points.)

07DP **Exercise 111.55.10.** Let k be an algebraically closed field. Let $A \subset B$ be an extension of domains which are both finite type k-algebras. Prove that the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains a nonempty open subset of $\mathrm{Spec}(A)$ using the following steps:

(1) Prove it if $A \to B$ is also finite.
(2) Prove it in case the fraction field of B is a finite extension of the fraction field of A.
(3) Reduce the statement to the previous case.

111.56. Schemes, Final Exam, Fall 2013

09TV These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2013 at Columbia University.

09TW **Exercise 111.56.1** (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *radical ideal* of a ring,
(2) a *finite type* ring homomorphism,
(3) a *differential a la Weil*,
(4) a *scheme*.

09TX **Exercise 111.56.2** (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) result on hilbert polynomials of graded modules.
(2) dimension of a Noetherian local ring $(R, \mathfrak{m})$ and $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.
(3) Riemann-Roch.
(4) Clifford's theorem.
(5) Chevalley's theorem.

09TY **Exercise 111.56.3.** Let $A \to B$ be a ring map. Let $S \subset A$ be a multiplicative subset. Assume that $A \to B$ is of finite type and $S^{-1}A \to S^{-1}B$ is surjective. Show that there exists an $f \in S$ such that $A_f \to B_f$ is surjective.

09TZ **Exercise 111.56.4.** Give an example of an injective local homomorphism $A \to B$ of local rings, such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not surjective.

09U0 **Situation 111.56.5** (Notation plane curve)**.** Let k be an algebraically closed field. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be an irreducible polynomial homogeneous of degree d. We let

$$D = V(F) \subset \mathbf{P}^2$$

be the projective plane curve given by the vanishing of F. Set $x = X_1/X_0$ and $y = X_2/X_0$ and $f(x, y) = X_0^{-d}F(X_0, X_1, X_2) = F(1, x, y)$. We denote K the fraction field of the domain $k[x, y]/(f)$. We let C be the abstract curve corresponding to K. Recall (from the lectures) that there is a surjective map $C \to D$ which is bijective over the nonsingular locus of D and an isomorphism if D is nonsingular. Set $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Finally, we denote $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ the element of $\Omega_{K/k}$ discussed in the lectures. Denote K_C the divisor of zeros and poles of ω.

09U1 **Exercise 111.56.6.** In Situation 111.56.5 assume $d \geq 3$ and that the curve D has exactly one singular point, namely $P = (1 : 0 : 0)$. Assume further that we have the expansion

$$f(x, y) = xy + h.o.t$$

around $P = (0, 0)$. Then C has two points v and w lying over P characterized by

$$v(x) = 1, v(y) > 1 \quad \text{and} \quad w(x) > 1, w(y) = 1$$

(1) Show that the element $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ of $\Omega_{K/k}$ has a first order pole at both v and w. (The behaviour of ω at nonsingular points is as discussed in the lectures.)
(2) In the lectures we have shown that ω vanishes to order $d-3$ at the divisor $X_0 = 0$ pulled back to C under the map $C \to D$. Combined with the information of (1) what is the degree of the divisor of zeros and poles of ω on C?
(3) What is the genus of the curve C?

09U2 **Exercise 111.56.7.** In Situation 111.56.5 assume $d = 5$ and that the curve $C = D$ is nonsingular. In the lectures we have shown that the genus of C is 6 and that the linear system K_C is given by

$$L(K_C) = \{h\omega \mid h \in k[x, y],\ \deg(h) \leq 2\}$$

where deg indicates total degree[5]. Let $P_1, P_2, P_3, P_4, P_5 \in D$ be pairwise distinct points lying in the affine open $X_0 \neq 0$. We denote $\sum P_i = P_1 + P_2 + P_3 + P_4 + P_5$ the corresponding divisor of C.

(1) Describe $L(K_C - \sum P_i)$ in terms of polynomials.
(2) What are the possibilities for $l(\sum P_i)$?

09U3 **Exercise 111.56.8.** Write down an F as in Situation 111.56.5 with $d = 100$ such that the genus of C is 0.

09U4 **Exercise 111.56.9.** Let k be an algebraically closed field. Let K/k be finitely generated field extension of transcendence degree 1. Let C be the abstract curve corresponding to K. Let $V \subset K$ be a g^r_d and let $\Phi : C \to \mathbf{P}^r$ be the corresponding morphism. Show that the image of C is contained in a quadric[6] if V is a complete linear system and d is large enough relative to the genus of C. (Extra credit: good bound on the degree needed.)

09U5 **Exercise 111.56.10.** Notation as in Situation 111.56.5. Let $U \subset \mathbf{P}^2_k$ be the open subscheme whose complement is D. Describe the k-algebra $A = \mathcal{O}_{\mathbf{P}^2_k}(U)$. Give an upper bound for the number of generators of A as a k-algebra.

111.57. Schemes, Final Exam, Spring 2014

0AAL These were the questions in the final exam of a course on Schemes, in the Spring of 2014 at Columbia University.

0AAM **Exercise 111.57.1** (Definitions)**.** Let $(X, \mathcal{O}_X)$ be a scheme. Provide definitions of the italicized concepts.

(1) the *local ring of X at a point x*,
(2) a *quasi-coherent* sheaf of $\mathcal{O}_X$-modules,
(3) a *coherent* sheaf of $\mathcal{O}_X$-modules (please assume X is locally Noetherian,
(4) an *affine open* of X,
(5) a *finite morphism of schemes $X \to Y$*.

0AAN **Exercise 111.57.2** (Theorems)**.** Precisely state a nontrivial fact discussed in the lectures related to each item.

(1) on birational invariance of pluri-genera of varieties,
(2) being an affine morphism is a local property,
(3) the topology of a scheme theoretic fibre of a morphism, and
(4) valuative criterion of properness.

0AAP **Exercise 111.57.3.** Let $X = \mathbf{A}^2_{\mathbf{C}}$ where $\mathbf{C}$ is the field of complex numbers. A *line* will mean a closed subscheme of X defined by one linear equation $ax + by + c = 0$ for some $a, b, c \in \mathbf{C}$ with $(a, b) \neq (0, 0)$. A *curve* will mean an irreducible (so nonempty) closed subscheme $C \subset X$ of dimension 1. A *quadric* will mean a curve

[5] We get ≤ 2 because $d - 3 = 5 - 3 = 2$.

[6] A quadric is a degree 2 hypersurface, i.e., the zero set in $\mathbf{P}^r$ of a degree 2 homogeneous polynomial.

defined by one quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some $a, b, c, d, e, f \in \mathbf{C}$ and $(a, b, c) \neq (0, 0, 0)$.

(1) Find a curve C such that every line has nonempty intersection with C.
(2) Find a curve C such that every line and every quadric has nonempty intersection with C.
(3) Show that for every curve C there exists another curve such that $C \cap C' = \emptyset$.

0AAQ **Exercise 111.57.4.** Let k be a field. Let $b : X \to \mathbf{A}^2_k$ be the blow up of the affine plane in the origin. In other words, if $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$, then $X = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$ where $\mathfrak{m} = (x, y) \subset k[x, y]$. Prove the following statements

(1) the scheme theoretic fibre E of b over the origin is isomorphic to $\mathbf{P}^1_k$,
(2) E is an effective Cartier divisor on X,
(3) the restriction of $\mathcal{O}_X(-E)$ to E is a line bundle of degree 1.

(Recall that $\mathcal{O}_X(-E)$ is the ideal sheaf of E in X.)

0AAR **Exercise 111.57.5.** Let k be a field. Let X be a projective variety over k. Show there exists an affine variety U over k and a surjective morphism of varieties $U \to X$.

0AAS **Exercise 111.57.6.** Let k be a field of characteristic $p > 0$ different from $2, 3$. Consider the closed subscheme X of $\mathbf{P}^n_k$ defined by

$$\sum\nolimits_{i=0,\ldots,n} X_i = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^2 = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^3 = 0$$

For which pairs (n, p) is this variety singular?

111.58. Commutative Algebra, Final Exam, Fall 2016

0D5F These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2016 at Columbia University.

0D5G **Exercise 111.58.1** (Definitions)**.** Let R be a ring. Provide definitions of the italicized concepts.

(1) the *local ring of R at a prime* $\mathfrak{p}$,
(2) a *finite* R-module,
(3) a *finitely presented* R-module,
(4) R is *regular*,
(5) R is *catenary*,
(6) R is *Cohen-Macaulay*.

0D5H **Exercise 111.58.2** (Theorems)**.** Precisely state a nontrivial fact discussed in the lectures related to each item.

(1) regular rings,
(2) associated primes of Cohen-Macaulay modules,
(3) dimension of a finite type domain over a field, and
(4) Chevalley's theorem.

0D5I **Exercise 111.58.3.** Let $A \to B$ be a ring map such that

(1) A is local with maximal ideal $\mathfrak{m}$,
(2) $A \to B$ is a finite[7] ring map,

[7]Recall that this means B is finite as an A-module.

(3) $A \to B$ is injective (we think of A as a subring of B).

Show that there is a prime ideal $\mathfrak{q} \subset B$ with $\mathfrak{m} = A \cap \mathfrak{q}$.

0D5J **Exercise 111.58.4.** Let k be a field. Let $R = k[x, y, z, w]$. Consider the ideal $I = (xy, xz, xw)$. What are the irreducible components of $V(I) \subset \mathrm{Spec}(R)$ and what are their dimensions?

0D5K **Exercise 111.58.5.** Let k be a field. Let $A = k[x, x^{-1}]$ and $B = k[y]$. Show that any k-algebra map $\varphi : A \to B$ maps x to a constant.

0D5L **Exercise 111.58.6.** Consider the ring $R = \mathbf{Z}[x, y]/(xy - 7)$. Prove that R is regular.

Given a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ for $n \geq 0$ we let $\varphi_R(n) = \dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1})$.

0D5M **Exercise 111.58.7.** Does there exist a Noetherian local ring R with $\varphi_R(n) = n+1$ for all $n \geq 0$?

0D5N **Exercise 111.58.8.** Let R be a Noetherian local ring. Suppose that $\varphi_R(0) = 1$, $\varphi_R(1) = 3$, $\varphi_R(2) = 5$. Show that $\varphi_R(3) \leq 7$.

111.59. Schemes, Final Exam, Spring 2017

0DSZ These were the questions in the final exam of a course on schemes, in the Spring of 2017 at Columbia University.

0DT0 **Exercise 111.59.1** (Definitions)**.** Let $f : X \to Y$ be a morphism of schemes. Provide brief definitions of the italicized concepts.

(1) the *scheme theoretic fibre* of f at $y \in Y$,
(2) f is a *finite morphism*,
(3) a *quasi-coherent* $\mathcal{O}_X$-module,
(4) X is *variety*,
(5) f is a *smooth morphism*,
(6) f is a *proper morphism*.

0DT1 **Exercise 111.59.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item.

(1) pushforward of quasi-coherent sheaves,
(2) cohomology of coherent sheaves on projective varieties,
(3) Serre duality for a projective scheme over a field, and
(4) Riemann-Hurwitz.

0DT2 **Exercise 111.59.3.** Let k be an algebraically closed field. Let $\ell > 100$ be a prime number different from the characteristic of k. Let X be the nonsingular projective model of the affine curve given by the equation

$$y^\ell = x(x-1)^3$$

in $\mathbf{A}^2_k$. Answer the following questions:

(1) What is the genus of X?
(2) Give an upper bound for the gonality[8] of X.

[8]The gonality is the smallest degree of a nonconstant morphism from X to $\mathbf{P}^1_k$.

0DT3 **Exercise 111.59.4.** Let k be an algebraically closed field. Let X be a reduced, projective scheme over k all of whose irreducible components have the same dimension 1. Let $\omega_{X/k}$ be the relative dualizing module. Show that if $\dim_k H^1(X, \omega_{X/k}) > 1$, then X is disconnected.

0DT4 **Exercise 111.59.5.** Give an example of a scheme X and a nontrivial invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that both $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are nonzero.

0DT5 **Exercise 111.59.6.** Let k be an algebraically closed field. Let $g \geq 3$. Let X and X' be smooth projective curves over k of genus g and $g+1$. Let $Y \subset X \times X'$ be a curve such that the projections $Y \to X$ and $Y \to X'$ are nonconstant. Prove that the nonsingular projective model of Y has genus $\geq 2g+1$.

0DT6 **Exercise 111.59.7.** Let k be a finite field. Let $g > 1$. Sketch a proof of the following: there are only a finite number of isomorphism classes of smooth projective curves over k of genus g. (You will get credit for even just trying to answer this.)

111.60. Commutative Algebra, Final Exam, Fall 2017

0EEJ These were the questions in the final exam of a course on commutative algebra, in the Fall of 2017 at Columbia University.

0EEK **Exercise 111.60.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) the *left adjoint* of a functor $F : \mathcal{A} \to \mathcal{B}$,
(2) the *transcendence degree* of an extension L/K of fields,
(3) a *regular function* on a classical affine variety $X \subset k^n$,
(4) a *sheaf* on a topological space,
(5) a *local ring*, and
(6) a morphism of schemes $f : X \to Y$ being *affine*.

0EEL **Exercise 111.60.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Yoneda lemma,
(2) Mayer-Vietoris,
(3) dimension and cohomology,
(4) Hilbert polynomial, and
(5) duality for projective space.

0EEM **Exercise 111.60.3.** Let k be an algebraically closed field. Consider the closed subset X of k^5 with Zariski topology and coordinates x_1, x_2, x_3, x_4, x_5 given by the equations

$$x_1^2 - x_4 = 0, \quad x_2^5 - x_5 = 0, \quad x_3^2 + x_3 + x_4 + x_5 = 0$$

What is the dimension of X and why?

0EEN **Exercise 111.60.4.** Let k be a field. Let $X = \mathbf{P}^1_k$ be the projective space of dimension 1 over k. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For $d \in \mathbf{Z}$ denote $\mathcal{E}(d) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ the dth Serre twist of $\mathcal{E}$ and $h^i(X, \mathcal{E}(d)) = \dim_k H^i(X, \mathcal{E}(d))$.

(1) Why is there no $\mathcal{E}$ with $h^0(X, \mathcal{E}) = 5$ and $h^0(X, \mathcal{E}(1)) = 4$?
(2) Why is there no $\mathcal{E}$ with $h^1(X, \mathcal{E}(1)) = 5$ and $h^1(X, \mathcal{E}) = 4$?

(3) For which $a \in \mathbf{Z}$ can there exist a vector bundle $\mathcal{E}$ on X with

$$\begin{array}{ll} h^0(X, \mathcal{E}) = 1 & h^1(X, \mathcal{E}) = 1 \\ h^0(X, \mathcal{E}(1)) = 2 & h^1(X, \mathcal{E}(1)) = 0 \\ h^0(X, \mathcal{E}(2)) = 4 & h^1(X, \mathcal{E}(2)) = a \end{array}$$

Partial answers are welcomed and encouraged.

0EEP **Exercise 111.60.5.** Let X be a topological space which is the union $X = Y \cup Z$ of two closed subsets Y and Z whose intersection is denoted $W = Y \cap Z$. Denote $i : Y \to X$, $j : Z \to X$, and $k : W \to X$ the inclusion maps.

(1) Show that there is a short exact sequence of sheaves

$$0 \to \underline{\mathbf{Z}}_X \to i_*(\underline{\mathbf{Z}}_Y) \oplus j_*(\underline{\mathbf{Z}}_Z) \to k_*(\underline{\mathbf{Z}}_W) \to 0$$

where $\underline{\mathbf{Z}}_X$ denotes the constant sheaf with value $\mathbf{Z}$ on X, etc.

(2) What can you conclude about the relationship between the cohomology groups of X, Y, Z, W with $\mathbf{Z}$-coefficients?

0EEQ **Exercise 111.60.6.** Let k be a field. Let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring in infinitely many variables. Denote $\mathfrak{m}$ the maximal ideal of A generated by all the variables. Let $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

(1) Show $H^1(U, \mathcal{O}_U) = 0$. Hint: Čech cohomology computation.
(2) What is your guess for $H^i(U, \mathcal{O}_U)$ for $i \geq 1$?

0EER **Exercise 111.60.7.** Let A be a local ring. Let $a \in A$ be a nonzerodivisor. Let $I, J \subset A$ be ideals such that $IJ = (a)$. Show that the ideal I is principal, i.e., generated by one element (which will turn out to be a nonzerodivisor).

111.61. Schemes, Final Exam, Spring 2018

0ELF These were the questions in the final exam of a course on schemes, in the Spring of 2018 at Columbia University.

0ELG **Exercise 111.61.1** (Definitions)**.** Provide brief definitions of the italicized concepts. Let k be an algebraically closed field. Let X be a projective curve over k.

(1) a *smooth* algebra over k,
(2) the *degree* of an invertible $\mathcal{O}_X$-module on X,
(3) the *genus* of X,
(4) the *Weil divisor class group* of X,
(5) X is *hyperelliptic*, and
(6) the *intersection number* of two curves on a smooth projective surface over k.

0ELH **Exercise 111.61.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Riemann-Hurwitz theorem,
(2) Clifford's theorem,
(3) factorization of maps between smooth projective surfaces,
(4) Hodge index theorem, and
(5) Riemann hypothesis for curves over finite fields.

0ELI **Exercise 111.61.3.** Let k be an algebraically closed field. Let $X \subset \mathbf{P}^3_k$ be a smooth curve of degree d and genus ≥ 2. Assume X is not contained in a plane and that there is a line ℓ in $\mathbf{P}^3_k$ meeting X in $d-2$ points. Show that X is hyperelliptic.

0ELJ **Exercise 111.61.4.** Let k be an algebraically closed field. Let X be a projective curve with pairwise distinct singular points $p_1, \ldots, p_n$. Explain why the genus of the normalization of X is at most $-n + \dim_k H^1(X, \mathcal{O}_X)$.

0ELK **Exercise 111.61.5.** Let k be a field. Let $X = \mathrm{Spec}(k[x, y])$ be affine 2 space. Let

$$I = (x^3, x^2y, xy^2, y^3) \subset k[x, y].$$

Let $Y \subset X$ be the closed subscheme corresponding to I. Let $b : X' \to X$ be the blowing up of the ideal (x, y), i.e., the blow up of affine space at the origin.

(1) Show that the scheme theoretic inverse image $b^{-1}Y \subset X'$ is an effective Cartier divisor.
(2) Given an example of an ideal $J \subset k[x, y]$ with $I \subset J \subset (x, y)$ such that if $Z \subset X$ is the closed subscheme corresponding to J, then the scheme theoretic inverse image $b^{-1}Z$ is not an effective Cartier divisor.

0ELL **Exercise 111.61.6.** Let k be an algebraically closed field. Consider the following types of surfaces

(1) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves,
(2) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves and the genus of C_1 is > 0,
(3) $S \subset \mathbf{P}^3_k$ is a hypersurface of degree 4, and
(4) $S \subset \mathbf{P}^3_k$ is a smooth hypersurface of degree 4.

For each type briefly indicate why or why not the class of surfaces of this type contains rational surfaces.

0ELM **Exercise 111.61.7.** Let k be an algebraically closed field. Let $S \subset \mathbf{P}^3_k$ be a smooth hypersurface of degree d. Assume that S contains a line ℓ. What is the self square of ℓ viewed as a divisor on S?

111.62. Commutative Algebra, Final Exam, Fall 2019

0FWJ These were the questions in the final exam of a course on commutative algebra, in the Fall of 2019 at Columbia University.

0FWK **Exercise 111.62.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *constructible subset* of a Noetherian topological space,
(2) the *localization* of an R-module M at a prime $\mathfrak{p}$,
(3) the *length* of a module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$,
(4) a *projective module* over a ring R, and
(5) a *Cohen-Macaulay* module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$.

0FWL **Exercise 111.62.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) images of constructible sets,
(2) Hilbert Nullstellensatz,
(3) dimension of finite type algebras over fields,

(4) Noether normalization, and
(5) regular local rings.

For a ring R and an ideal $I \subset R$ recall that $V(I)$ denotes the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $I \subset \mathfrak{p}$.

0FWM **Exercise 111.62.3** (Making primes)**.** Construct infinitely many distinct prime ideals $\mathfrak{p} \subset \mathbf{C}[x, y]$ such that $V(\mathfrak{p})$ contains (x, y) and $(x - 1, y - 1)$.

0FWN **Exercise 111.62.4** (No prime)**.** Let $R = \mathbf{C}[x, y, z]/(xy)$. Argue briefly there does not exist a prime ideal $\mathfrak{p} \subset R$ such that $V(\mathfrak{p})$ contains $(x, y - 1, z - 5)$ and $(x - 1, y, z - 7)$.

0FWP **Exercise 111.62.5** (Frobenius)**.** Let p be a prime number (you may assume $p = 2$ to simplify the formulas). Let R be a ring such that $p = 0$ in R.

(1) Show that the map $F : R \to R$, $x \mapsto x^p$ is a ring homomorphism.
(2) Show that $\mathrm{Spec}(F) : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is the identity map.

Recall that a *specialization* $x \rightsquigarrow y$ of points of a topological space simply means y is in the closure of x. We say $x \rightsquigarrow y$ is an *immediate specialization* if there does not exist a z different from x and y such that $x \rightsquigarrow z$ and $z \rightsquigarrow y$.

0FWQ **Exercise 111.62.6** (Dimension)**.** Suppose we have a sober topological space X containing 5 distinct points x, y, z, u, v having the following specializations

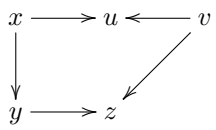

What is the minimal dimension such an X can have? If X is the spectrum of a finite type algebra over a field and $x \rightsquigarrow u$ is an immediate specialization, what can you say about the specialization $v \rightsquigarrow z$?

0FWR **Exercise 111.62.7** (Tor computation)**.** Let $R = \mathbf{C}[x, y, z]$. Let $M = R/(x, z)$ and $N = R/(y, z)$. For which $i \in \mathbf{Z}$ is $\mathrm{Tor}_i^R(M, N)$ nonzero?

0FWS **Exercise 111.62.8.** Let $A \to B$ be a flat local homomorphism of local Noetherian rings. Show that if A has depth k, then B has depth at least k.

111.63. Algebraic Geometry, Final Exam, Spring 2020

0G12 These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2020 at Columbia University.

0G13 **Exercise 111.63.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *scheme*,
(2) a *morphism of schemes*,
(3) a *quasi-coherent module* on a scheme,
(4) a *variety* over a field k,
(5) a *curve* over a field k,
(6) a *finite morphism* of schemes,
(7) the *cohomology* of a sheaf of abelian groups $\mathcal{F}$ over a topological space X,
(8) a *dualizing sheaf* on a scheme X of dimension d proper over a field k, and

(9) a *rational map* from a variety X to a variety Y.

0G14 **Exercise 111.63.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) cohomology of abelian sheaves on a Noetherian topological space X of dimension d,
(2) sheaf of differentials $\Omega^1_{X/k}$ of a smooth variety over a field k,
(3) dualizing sheaf ω_X of a smooth projective variety X over the field k,
(4) a smooth proper genus 0 curve over an algebraically closed field k, and
(5) the genus of a plane curve of degree d.

0G15 **Exercise 111.63.3.** Let k be a field. Let X be a scheme over k. Assume $X = X_1 \cup X_2$ is an open covering with X_1, X_2 both isomorphic to $\mathbf{P}^1_k$ and $X_1 \cap X_2$ isomorphic to $\mathbf{A}^1_k$. (Such a scheme exists, for example you can take $\mathbf{P}^1_k$ with ∞ doubled.) Show that $\dim_k H^1(X, \mathcal{O}_X)$ is infinite.

0G16 **Exercise 111.63.4.** Let k be an algebraically closed field. Let Y be a smooth projective curve of genus 10. Find a good lower bound for the genus of a smooth projective curve X such that there exists a nonconstant morphism $f : X \to Y$ which is not an isomorphism.

0G17 **Exercise 111.63.5.** Let k be an algebraically closed field of characteristic 0. Let

$$X : T_0^d + T_1^d - T_2^d = 0 \subset \mathbf{P}^2_k$$

be the Fermat curve of degree $d \geq 3$. Consider the closed points $p = [1 : 0 : 1]$ and $q = [0 : 1 : 1]$ on X. Set $D = [p] - [q]$.

(1) Show that D is nontrivial in the Weil divisor class group.
(2) Show that dD is trivial in the Weil divisor class group. (Hint: try to show that both $d[p]$ and $d[q]$ are the intersection of X with a line in the plane.)

0G18 **Exercise 111.63.6.** Let k be an algebraically closed field. Consider the 2-uple embedding

$$\varphi : \mathbf{P}^2 \longrightarrow \mathbf{P}^5$$

In terms of the material/notation in the lectures this is the morphism

$$\varphi = \varphi_{\mathcal{O}_{\mathbf{P}^2}(2)} : \mathbf{P}^2 \longrightarrow \mathbf{P}(\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)))$$

In terms of homogeneous coordinates it is given by

$$[a_0 : a_1 : a_2] \longmapsto [a_0^2 : a_0a_1 : a_0a_2 : a_1^2 : a_1a_2 : a_2^2]$$

It is a closed immersion (please just use this). Let $I \subset k[T_0, \ldots, T_5]$ be the homogeneous ideal of $\varphi(\mathbf{P}^2)$, i.e., the elements of the homogeneous part I_d are the homogeneous polynomials $F(T_0, \ldots, T_5)$ of degree d which restrict to zero on the closed subscheme $\varphi(\mathbf{P}^2)$. Compute $\dim_k I_d$ as a function of d.

0G19 **Exercise 111.63.7.** Let k be an algebraically closed field. Let X be a proper scheme of dimension d over k with dualizing module ω_X. You are given the following information:

(1) $\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{d-i}(X, \mathcal{F}) \to H^d(X, \omega_X) \xrightarrow{t} k$ is nondegenerate for all i and for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, and
(2) ω_X is finite locally free of some rank r.

Show that $r = 1$. (Hint: see what happens if you take $\mathcal{F}$ a suitable module supported at a closed point.)

111.64. Commutative Algebra, Final Exam, Fall 2021

0GRS These were the questions in the final exam of a course on commutative algebra, in the Fall of 2021 at Columbia University.

0GRT **Exercise 111.64.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *multiplicative subset* of a ring A,
(2) an *Artinian ring* A,
(3) the *spectrum of a ring* A as a topological space,
(4) a *flat ring map* $A \to B$,
(5) the *height* of a prime ideal $\mathfrak{p}$ in A, and
(6) the functors $Tor_i^A(-,-)$ over a ring A.

0GRU **Exercise 111.64.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Artinian rings,
(2) flatness and prime ideals,
(3) lengths of $A/\mathfrak{m}^n$ for $(A, \mathfrak{m})$ Noetherian local,
(4) the dimension formula for universally catenary Noetherian rings,
(5) completion of a Noetherian local ring, and
(6) Matlis duality for Artinian local rings.

0GRV **Exercise 111.64.3** (Units)**.** What is the structure of the group of units of $\mathbf{Z}[x, 1/x]$ as an abelian group? No explanation necessary.

0GRW **Exercise 111.64.4** (Ideals)**.** Let $A = \mathbf{F}_2[x, y]/(x^2, xy, y^2)$ and denote $\overline{x}$ and $\overline{y}$ the images of x and y in A. List the ideals of A. No explanation necessary.

0GRX **Exercise 111.64.5** (Tor and Ext)**.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Set $\varphi(n) = \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

(1) Show that $\mathrm{Tor}_1^A(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.
(2) Show that $\mathrm{Ext}^1_A(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.

0GRY **Exercise 111.64.6** (Two vectors)**.** Let $A = \mathbf{Z}[a_1, a_2, a_3, b_1, b_2, b3]$. Set $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $A^{\oplus 3}$. Consider the set

$$Z = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid a, b \text{ map to linearly dependent vectors of } \kappa(\mathfrak{p})^{\oplus 3}\}$$

(1) Prove the Z is a closed subset of $\mathrm{Spec}(A)$.
(2) What is the dimension $\dim(Z)$ of Z?
(3) What would happen to $\dim(Z)$ if we replaced $\mathbf{Z}$ by a field?

0GRZ **Exercise 111.64.7** (Injectives)**.** Let $(A, \mathfrak{m}, \kappa)$ be an Artinian local ring. Assume A is injective as an A-module. Show that $\mathrm{Hom}_A(\kappa, A)$ has dimension 1 has a κ-vector space.

111.65. Algebraic Geometry, Final Exam, Spring 2022

0GY8 These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2022 at Columbia University.

0GY9 **Exercise 111.65.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *scheme*,
(2) a *quasi-coherent module* on a scheme X,
(3) a *flat* morphism of schemes $X \to Y$,
(4) a *finite* morphism of schemes $X \to Y$,
(5) a *group scheme* G over a base scheme S,
(6) a *family of varieties* over a base scheme S,
(7) the *degree* of a closed point x on a variety X over the field k,
(8) the usual *logarithmic height* of a point $p = (a_0 : \ldots : a_n)$ in $\mathbf{P}^n(\mathbf{Q})$, and
(9) a C_i *field*.

0GYA **Exercise 111.65.2** (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) morphisms from a scheme X to the affine scheme $\mathrm{Spec}(A)$,
(2) cohomology of a quasi-coherent module $\mathcal{F}$ on an affine scheme X,
(3) the Picard group of $\mathbf{P}^1_k$ where k is a field,
(4) the dimensions of fibres of a flat proper morphism $X \to S$ for S Noetherian,
(5) $\mathbf{G}_m$-equivariant modules on a scheme S, and
(6) Bezout's theorem on intersections (restrict to a special case if you like).

0GYB **Exercise 111.65.3** (Cubic hypersurfaces)**.** Let $F \in \mathbf{C}[T_0, \ldots, T_n]$ be homogeneous of degree 3. Given 3 vectors $x, y, z \in \mathbf{C}^{n+1}$ consider the condition

$$(*) \quad F(\lambda x + \mu y + \nu z) = 0 \text{ in } \mathbf{C}[\lambda, \mu, \nu]$$

(1) What is the dimension of the space of all choices of x, y, z?
(2) How many equations on the coordinates of x, y, and z is condition (*)?
(3) What is the expected dimension of the space of all triples x, y, z such that (*) is true?
(4) What is the dimension of the space of all triples such that x, y, z are linearly dependent?
(5) Conclude that on a hypersurface of degree 3 in $\mathbf{P}^n$ we expect to find a linear subspace of dimension 2 provided $n \geq a$ where it is up to you to find a.

0GYC **Exercise 111.65.4** (Heights)**.** Let K be a field. Let $h_n : \mathbf{P}^n(K) \to \mathbf{R}$, $n \geq 0$ be a collection of functions satisfying the 2 axioms we discussed in the lectures. Let X be a projective variety over K. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and recall that we have constructed in the lectures an associated height function $h_{\mathcal{L}} : X(K) \to \mathbf{R}$. Let $\alpha : X \to X$ be an automorphism of X over K.

(1) Prove that $P \mapsto h_{\mathcal{L}}(\alpha(P))$ differs from the function $h_{\alpha^*\mathcal{L}}$ by a bounded amount. (Hint: recall that if there is a morphism $\varphi : X \to \mathbf{P}^n$ with $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbf{P}^n}(1)$, then by construction $h_{\mathcal{L}}(P) = h_n(\varphi(P))$ and play around with that. In general write $\mathcal{L}$ as a difference of two of these.)

(2) Assume that $h_{\mathcal{L}}(P) - h_{\mathcal{L}}(\alpha(P))$ is unbounded on $X(K)$. Show that $h_{\mathcal{N}}$ with $\mathcal{N} = \mathcal{L} \otimes \alpha^*\mathcal{L}^{\otimes -1}$ is unbounded on $X(K)$.
(3) Assume X is an elliptic curve and that $\mathcal{L}$ is a symmetric ample invertible module on X such that $h_{\mathcal{L}}$ is unbounded on $X(K)$. Show that there exists an invertible module $\mathcal{N}$ of degree 0 such that $h_{\mathcal{N}}$ is unbounded. (Hints: Recall that X is an abelian variety of dimension 1. Thus $h_{\mathcal{L}}$ is quadratic up to a constant by results in the lectures. Choose a suitable point $P_0 \in X(K)$. Let $\alpha : X \to X$ be translation by P_0. Consider $P \mapsto h_{\mathcal{L}}(P) - h_{\mathcal{L}}(P + P_0)$. Apply the results you proved above.)

0GYD **Exercise 111.65.5** (Monomorphisms)**.** Let $f : X \to Y$ be a monomorphism in the category of schemes: for any pair of morphisms $a, b : T \to X$ of schemes if $f \circ a = f \circ b$, then $a = b$. Show that f is injective on points. Does you argument say anything else?

0GYE **Exercise 111.65.6** (Fixed points)**.** Let k be an algebraically closed field.

(1) If $G = \mathbf{G}_{m,k}$ show that if G acts on a projective variety X over k, then the action has a fixed point, i.e., prove there exists a point $x \in X(k)$ such that $a(g, x) = x$ for all $g \in G(k)$.
(2) Same with $G = (\mathbf{G}_{m,k})^n$ equal to the product of $n \geq 1$ copies of the multiplicative group.
(3) Give an example of an action of a connected group scheme G on a smooth projective variety X which does not have a fixed point.

111.66. Algebraic Geometry, Final Exam, Spring 2025

0H8P These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2025 at Columbia University.

0H8Q **Exercise 111.66.1** (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *scheme* X,
(2) a *quasi-coherent module* on a scheme X,
(3) the *Picard group* of a scheme X,
(4) give a morphism $f : X \to Y$ of schemes and an $\mathcal{O}_X$-module $\mathcal{F}$, the *pushforward* $f_*\mathcal{F}$,
(5) a *closed immersion* of schemes, and
(6) a *variety* over a given field k.

0H8R **Exercise 111.66.2** (Theorems)**.** Precisely and succintly state one nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) the topological space of a scheme X,
(2) a representability criterion for functors on the category of schemes,
(3) the Picard functor for a smooth projective curve X over an algebraically closed field k,
(4) torsion in the Picard group of a smooth projective curve X over an algebraically closed field k,
(5) a theorem on the Picard group of a product $X \times Y \times Z$ of smooth projective varieties X, Y, Z over an algebraically closed field k.

0H8S **Exercise 111.66.3.** Let k be a field. Explain why $\mathrm{Spec}(k[x])$ has infinitely many points.

0H8T **Exercise 111.66.4.** Let k be a field and let $x_1, \ldots, x_n$ be variables. Let $f \in k[x_1, \ldots, x_n]$ be a nonconstant polynomial. Let $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/(f))$. What property should f have in order for X to be a variety over k?

0H8U **Exercise 111.66.5.** Find a singular point (you do not need to find all of them) on the spectrum of $\mathbf{C}[x, y]/(x^5 - 5xy^4 + 20y - 16)$ viewed as a variety over $\mathbf{C}$.

0H8V **Exercise 111.66.6.** Let X be a smooth projective curve over an algebraically closed field k. Let $x \in X(k)$ be a closed point on X. Let $U = X \setminus \{x\}$. What can you say about the kernel of the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$?

0H8W **Exercise 111.66.7.** Let X be a smooth projective curve over an algebraically closed field k of genus $g \geq 100$. What values can $h^0(\mathcal{L}) = \dim_k H^0(X, \mathcal{L})$ take for $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module of degree $2g - 4$ on X? Answer as completely as you can.

111.67. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces

(55) Semistable Reduction
(56) Functors and Morphisms
(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
(60) Crystalline Cohomology
(61) Pro-étale Cohomology
(62) Relative Cycles
(63) More Étale Cohomology
(64) The Trace Formula

Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
(93) Deformation Problems

Algebraic Stacks

(94) Algebraic Stacks
(95) Examples of Stacks
(96) Sheaves on Algebraic Stacks
(97) Criteria for Representability
(98) Artin's Axioms
(99) Quot and Hilbert Spaces
(100) Properties of Algebraic Stacks
(101) Morphisms of Algebraic Stacks
(102) Limits of Algebraic Stacks
(103) Cohomology of Algebraic Stacks
(104) Derived Categories of Stacks
(105) Introducing Algebraic Stacks
(106) More on Morphisms of Stacks
(107) The Geometry of Stacks

Topics in Moduli Theory

(108) Moduli Stacks
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CHAPTER 112

A Guide to the Literature

03B0

112.1. Short introductory articles

03B1

- Barbara Fantechi: *Stacks for Everybody* [**Fan01**]
- Dan Edidin: *What is a stack?* [**Edi03**]
- Dan Edidin: *Notes on the construction of the moduli space of curves* [**Edi00**]
- Angelo Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces*, and especially the appendix. [**Vis89**]

112.2. Classic references

03B2

- Mumford: *Picard groups of moduli problems* [**Mum65**]
  Mumford never uses the term "stack" here but the concept is implicit in the paper; he computes the picard group of the moduli stack of elliptic curves.
- Deligne, Mumford: *The irreducibility of the space of curves of given genus* [**DM69**]
  This influential paper introduces "algebraic stacks" in the sense which are now universally called Deligne-Mumford stacks (stacks with representable diagonal which admit étale presentations by schemes). There are many foundational results *without proof*. The paper uses stacks to give two proofs of the irreducibility of the moduli space of curves of genus g.
- Artin: *Versal deformations and algebraic stacks* [**Art74**]
  This paper introduces "algebraic stacks" which generalize Deligne-Mumford stacks and are now commonly referred to as *Artin stacks*, stacks with representable diagonal which admit smooth presentations by schemes. This paper gives deformation-theoretic criterion known as Artin's criterion which allows one to prove that a given moduli stack is an Artin stack without explicitly exhibiting a presentation.

112.3. Books and online notes

03B3

- Laumon, Moret-Bailly: *Champs Algébriques* [**LMB00**]
  This book is currently the most exhaustive reference on stacks containing many foundational results. It assumes the reader is

familiar with algebraic spaces and frequently references Knutson's book [**Knu71**]. There is an error in chapter 12 concerning the functoriality of the lisse-étale site of an algebraic stack. One doesn't need to worry about this as the error has been patched by Martin Olsson (see [**Ols07b**]) and the results in the remaining chapters (after perhaps slight modification) are correct.

- The Stacks Project Authors: *Stacks Project* [**Aut**].
  You are reading it!
- Anton Geraschenko: *Lecture notes for Martin Olsson's class on stacks* [**Ols07a**]
  This course systematically develops the theory of algebraic spaces before introducing algebraic stacks (first defined in Lecture 27!). In addition to basic properties, the course covers the equivalence between being Deligne-Mumford and having unramified diagonal, the lisse-étale site on an Artin stack, the theory of quasi-coherent sheaves, the Keel-Mori theorem, cohomological descent, and gerbes (and their relation to the Brauer group). There are also some exercises.
- Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche, and Kresch: *Algebraic stacks*, online notes for a book being currently written [**BCE$^+$07**]
  The aim of this book is to give a friendly introduction to stacks without assuming a sophisticated background with a focus on examples and applications. Unlike [**LMB00**], it is not assumed that the reader has digested the theory of algebraic spaces. Instead, Deligne-Mumford stacks are introduced with algebraic spaces being a special case with part of the goal being to develop enough theory to prove the assertions in [**DM69**]. The general theory of Artin stacks is to be developed in the second part. Only a fraction of the book is now available on Kresch's website.
- Olsson, Martin: *Algebraic spaces and stacks*, [**Ols16**]
  Highly recommended introduction to algebraic spaces and algebraic stacks starting at the level of somebody who has mastered Hartshorne's book on algebraic geometry.

112.4. Related references on foundations of stacks

03B4

- Vistoli: *Notes on Grothendieck topologies, fibered categories and descent theory* [**Vis05**]
  Contains useful facts on fibered categories, stacks and descent theory in the fpqc topology as well as rigorous proofs.
- Knutson: *Algebraic Spaces* [**Knu71**]
  This book, which evolved from his PhD thesis under Michael Artin, contains the foundations of the theory of algebraic spaces. The book [**LMB00**] frequently references this text. See also Artin's papers on algebraic spaces: [**Art69a**], [**Art69b**], [**Art69c**], [**Art70**], [**Art71b**], [**Art71a**], [**Art73**], and [**Art74**]

- Grothendieck et al, *Théorie des Topos et Cohomologie Étale des Schémas I, II, III* also known as SGA4 [**AGV71**]
  Volume 1 contains many general facts on universes, sites and fibered categories. The word "champ" (French for "stack") appears in Deligne's Exposé XVIII.
- Jean Giraud: *Cohomologie non abélienne* [**Gir65**]
  The book discusses fibered categories, stacks, torsors and gerbes over general sites but does not discuss algebraic stacks. For instance, if G is a sheaf of abelian groups on X, then in the same way $H^1(X, G)$ can be identified with G-torsors, $H^2(X, G)$ can be identified with an appropriately defined set of G-gerbes. When G is not abelian, then $H^2(X, G)$ is defined as the set of G-gerbes.
- Kelly and Street: *Review of the elements of 2-categories* [**KS74**]
  The category of stacks form a 2-category although a simple type of 2-category where are 2-morphisms are invertible. This is a reference on general 2-categories. I have never used this so I cannot say how useful it is. Also note that [**Aut**] contains some basics on 2-categories.

112.5. Papers in the literature

03B6 Below is a list of research papers which contain fundamental results on stacks and algebraic spaces. The intention of the summaries is to indicate only the results of the paper which contribute toward stack theory; in many cases these results are subsidiary to the main goals of the paper. We divide the papers into categories with some papers falling into multiple categories.

04UW **112.5.1. Deformation theory and algebraic stacks.** The first three papers by Artin do not contain anything on stacks but they contain powerful results with the first two papers being essential for [**Art74**].

- Artin: *Algebraic approximation of structures over complete local rings* [**Art69a**]
  It is proved that under mild hypotheses any effective formal deformation can be approximated: if $F : (Sch/S) \to (Sets)$ is a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$, and $\hat{\xi} \in F(\hat{\mathcal{O}}_{S,s})$ is an effective formal deformation, then for any $n > 0$, there exists an residually trivial étale neighborhood $(S', s') \to (S, s)$ and $\xi' \in F(S')$ such that ξ' and $\hat{\xi}$ agree up to order n (ie. have the same restriction in $F(\mathcal{O}_{S,s}/\mathfrak{m}^n)$).
- Artin: *Algebraization of formal moduli I* [**Art69b**]
  It is proved that under mild hypotheses any effective formal versal deformation is algebraizable. Let $F : (Sch/S) \to (Sets)$ be a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$ be a locally closed point, $\hat{A}$ be a complete Noetherian local $\mathcal{O}_S$-algebra with residue field k' a finite extension of $k(s)$, and $\hat{\xi} \in F(\hat{A})$ be an effective formal versal deformation of an element $\xi_0 \in F(k')$.

Then there is a scheme X finite type over S and a closed point $x \in X$ with residue field $k(x) = k'$ and an element $\xi \in F(X)$ such that there is an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{A}$ identifying the restrictions of ξ and $\hat{\xi}$ in each $F(\hat{A}/\mathfrak{m}^n)$. The algebraization is unique if $\hat{\xi}$ is a universal deformation. Applications are given to the representability of the Hilbert and Picard schemes.

- Artin: *Algebraization of formal moduli. II* [**Art70**]
  Vaguely, it is shown that if one can contract a closed subset $Y' \subset X'$ formally locally around Y', then exists a global morphism $X' \to X$ contracting Y with X an algebraic space.
- Artin: *Versal deformations and algebraic stacks* [**Art74**]
  This momentous paper builds on his work in [**Art69a**] and [**Art69b**]. This paper introduces Artin's criterion which allows one to prove algebraicity of a stack by verifying deformation-theoretic properties. More precisely (but not very precisely), Artin constructs a presentation of a limit preserving stack $\mathcal{X}$ locally around a point $x \in \mathcal{X}(k)$ as follows: assuming the stack $\mathcal{X}$ satisfies Schlessinger's criterion([**Sch68**]), there exists a formal versal deformation $\hat{\xi} \in \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ of x. Assuming that formal deformations are effective (i.e., $\mathcal{X}(\hat{A}) \to \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ is bijective), then one obtains an effective formal versal deformation $\xi \in \mathcal{X}(\hat{A})$. Using results in [**Art69b**], one produces a finite type scheme U and an element $\xi_U : U \to \mathcal{X}$ which is formally versal at a point $u \in U$ over x. Then if we assume $\mathcal{X}$ admits a deformation and obstruction theory satisfying certain conditions (ie. compatibility with étale localization and completion as well as constructibility condition), then it is shown in section 4 that formal versality is an open condition so that after shrinking U, $U \to \mathcal{X}$ is smooth. Artin also presents a proof that any stack admitting an fppf presentation by a scheme admits a smooth presentation by a scheme so that in particular one can form quotient stacks by flat, separated, finitely presented group schemes.
- Conrad, de Jong: *Approximation of Versal Deformations* [**CdJ02**]
  This paper offers an approach to Artin's algebraization result by applying Popescu's powerful result: if A is a Noetherian ring and B a Noetherian A-algebra, then the map $A \to B$ is a regular morphism if and only if B is a direct limit of smooth A-algebras. It is not hard to see that Popescu's result implies Artin's approximation over an arbitrary excellent scheme (the excellence hypothesis implies that for a local ring A, the map $A^{\mathrm{h}} \to \hat{A}$ from the henselization to the completion is regular). The paper uses Popescu's result to give a "groupoid" generalization of the main theorem in [**Art69b**] which is valid over arbitrary excellent base schemes and for arbitrary points $s \in S$. In particular, the results in [**Art74**] hold under an arbitrary excellent

base. They discuss the étale-local uniqueness of the algebraization and whether the automorphism group of the object acts naturally on the henselization of the algebraization.

- Jason Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  The paper establishes that Artin's axioms for algebraization are compatible with the composition of 1-morphisms.
- Martin Olsson: *Deformation theory of representable morphism of algebraic stacks* [**Ols06a**]
  This generalizes standard deformation theory results for morphisms of schemes to representable morphisms of algebraic stacks in terms of the cotangent complex. These results cannot be viewed as consequences of Illusie's general theory as the cotangent complex of a representable morphism $X \to \mathcal{X}$ is not defined in terms of cotangent complex of a morphism of ringed topoi (because the lisse-étale site is not functorial).

04UX **112.5.2. Coarse moduli spaces.** Papers discussing coarse moduli spaces.

- Keel, Mori: *Quotients in Groupoids* [**KM97**]
  It had apparently long been "folklore" that separated Deligne-Mumford stacks admitted coarse moduli spaces. A rigorous (although terse) proof of the following theorem is presented here: if $\mathcal{X}$ is an Artin stack locally of finite type over a Noetherian base scheme such that the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ is finite, then there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated and Y an algebraic space locally of finite type over S. The hypothesis that the inertia is finite is precisely the right condition: there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated if and only if the inertia is finite.
- Conrad: *The Keel-Mori Theorem via Stacks* [**Con05b**]
  Keel and Mori's paper [**KM97**] is written in the groupoid language and some find it challenging to grasp. Brian Conrad presents a stack-theoretic version of the proof which is quite transparent although it uses the sophisticated language of stacks. Conrad also removes the Noetherian hypothesis.
- Rydh: *Existence of quotients by finite groups and coarse moduli spaces* [**Ryd07a**]
  Rydh removes the hypothesis from [**KM97**] and [**Con05b**] that $\mathcal{X}$ be finitely presented over some base.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  They define a *tame Artin stack* as an Artin stack with finite inertia such that if $\phi : \mathcal{X} \to Y$ is the coarse moduli space, ϕ_* is exact on quasi-coherent sheaves. They prove that for an Artin stack with finite inertia, the following are equivalent: $\mathcal{X}$ is tame if and only if the stabilizers of $\mathcal{X}$ are linearly reductive if and only if $\mathcal{X}$ is étale locally on the coarse moduli space a quotient of an affine scheme by a linearly reductive group scheme. For a tame Artin stack, the coarse moduli space is particularly nice. For instance, the coarse moduli space commutes with arbitrary base

change while a general coarse moduli space for an Artin stack with finite inertia will only commute with flat base change.

- Alper: *Good moduli spaces for Artin stacks* [**Alp08**]
  For general Artin stacks with infinite affine stabilizer groups (which are necessarily non-separated), coarse moduli spaces often do not exist. The simplest example is $[\mathbf{A}^1/\mathbf{G}_m]$. It is defined here that a quasi-compact morphism $\phi : \mathcal{X} \to Y$ is a *good moduli space* if $\mathcal{O}_Y \to \phi_*\mathcal{O}_\mathcal{X}$ is an isomorphism and ϕ_* is exact on quasi-coherent sheaves. This notion generalizes a tame Artin stack in [**AOV08**] as well as encapsulates Mumford's geometric invariant theory: if G is a reductive group acting linearly on $X \subset \mathbf{P}^n$, then the morphism from the quotient stack of the semi-stable locus to the GIT quotient $[X^{ss}/G] \to X//G$ is a good moduli space. The notion of a good moduli space has many nice geometric properties: (1) ϕ is surjective, universally closed, and universally submersive, (2) ϕ identifies points in Y with points in $\mathcal{X}$ up to closure equivalence, (3) ϕ is universal for maps to algebraic spaces, (4) good moduli spaces are stable under arbitrary base change, and (5) a vector bundle on an Artin stack descends to the good moduli space if and only if the representations are trivial at closed points.

04UY **112.5.3. Intersection theory.** Papers discussing intersection theory on algebraic stacks.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]
  This paper develops the foundations for intersection theory with rational coefficients for Deligne-Mumford stacks. If $\mathcal{X}$ is a separated Deligne-Mumford stack, the chow group $\mathrm{CH}_*(\mathcal{X})$ with rational coefficients is defined as the free abelian group of integral closed substacks of dimension k up to rational equivalence. There is a flat pullback, a proper push-forward and a generalized Gysin homomorphism for regular local embeddings. If $\phi : \mathcal{X} \to Y$ is a moduli space (ie. a proper morphism with is bijective on geometric points), there is an induced push-forward $\mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_k(Y)$ which is an isomorphism.
- Edidin, Graham: *Equivariant Intersection Theory* [**EG98**]
  The purpose of this article is to develop intersection theory with integral coefficients for a quotient stack $[X/G]$ of an action of an algebraic group G on an algebraic space X or, in other words, to develop a G-equivariant intersection theory of X. Equivariant chow groups defined using only invariant cycles does not produce a theory with nice properties. Instead, generalizing Totaro's definition in the case of BG and motivated by the fact that if $V \to X$ is a vector bundle then $\mathrm{CH}_i(X) \cong \mathrm{CH}_i(V)$ naturally, the authors define $\mathrm{CH}_i^G(X)$ as follows: Let $\dim(X) = n$ and $\dim(G) = g$. For each i, choose a l-dimensional G-representation

V where G acts freely on an open subset $U \subset V$ whose complement as codimension $d > n - i$. So $X_G = [X \times U/G]$ is an algebraic space (it can even be chosen to be a scheme). Then they define $\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+l-g}(X_G)$. For the quotient stack, one defines $\mathrm{CH}_i([X/G]) = \mathrm{CH}^G_{i+g}(X) = \mathrm{CH}_{i+l}(X_G)$. In particular, $\mathrm{CH}_i([X/G]) = 0$ for $i > \dim[X/G] = n - g$ but can be non-zero for $i \ll 0$. For example $\mathrm{CH}_i(B\mathbf{G}_m) = \mathbf{Z}$ for $i \leq 0$. They establish that these equivariant Chow groups enjoy the same functorial properties as ordinary Chow groups. Furthermore, they establish that if $[X/G] \cong [Y/H]$ that $\mathrm{CH}_i([X/G]) = \mathrm{CH}_i([Y/H])$ so that the definition is independent on how the stack is presented as a quotient stack.

- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]
  Kresch defines Chow groups for arbitrary Artin stacks agreeing with Edidin and Graham's definition in [**EG98**] in the case of quotient stack. For algebraic stacks with affine stabilizer groups, the theory satisfies the usual properties.
- Behrend and Fantechi: *The intrinsic normal cone* [**BF97**]
  Generalizing a construction due to Li and Tian, Behrend and Fantechi construct a virtual fundamental class for a Deligne-Mumford stack.

04UZ **112.5.4. Quotient stacks.** Quotient stacks[1] form a very important subclass of Artin stacks which include almost all moduli stacks studied by algebraic geometers. The geometry of a quotient stack $[X/G]$ is the G-equivariant geometry of X. It is often easier to show properties are true for quotient stacks and some results are only known to be true for quotient stacks. The following papers address: When is an algebraic stack a global quotient stack? Is an algebraic stack "locally" a quotient stack?

- Laumon, Moret-Bailly: [**LMB00**, Chapter 6]
  Chapter 6 contains several facts about the local and global structure of algebraic stacks. It is proved that an algebraic stack $\mathcal{X}$ over S is a quotient stack $[Y/G]$ with Y an algebraic space (resp. scheme, resp. affine scheme) and G a finite group if and only if there exists an algebraic space (resp. scheme, resp. affine scheme) Y' and an finite étale morphism $Y' \to \mathcal{X}$. It is shown that any Deligne-Mumford stack over S and $x : \mathrm{Spec}(K) \to \mathcal{X}$ admits an representable, étale and separated morphism $\phi : [X/G] \to \mathcal{X}$ where G is a finite group acting on an affine scheme over S such that $\mathrm{Spec}(K) = [X/G] \times_{\mathcal{X}} \mathrm{Spec}(K)$. The existence of presentations with geometrically connected fibers is also discussed in detail.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  First, they establish some fundamental (although not very difficult) facts concerning when a given algebraic stack (always

[1] In the literature, *quotient stack* often means a stack of the form $[X/G]$ with X an algebraic space and G a subgroup scheme of GL_n rather than an arbitrary flat group scheme.

assumed finite type over a Noetherian scheme in this paper) is a quotient stack. For an algebraic stack $\mathcal{X}$: $\mathcal{X}$ is a quotient stack if and only if there exists a vector bundle $V \to \mathcal{X}$ such that for every geometric point, the stabilizer acts faithfully on the fiber if and only if there exists a vector bundle $V \to \mathcal{X}$ and a locally closed substack $V^0 \subset V$ such that V^0 is representable and surjects onto $\mathcal{X}$. They establish that an algebraic stack is a quotient stack if there exists finite flat cover by an algebraic space. Any smooth Deligne-Mumford stack with generically trivial stabilizer is a quotient stack. They show that a $\mathbf{G}_m$-gerbe over a Noetherian scheme X corresponding to $\beta \in H^2(X, \mathbf{G}_m)$ is a quotient stack if and only if β is in the image of the Brauer map $\mathrm{Br}(X) \to \mathrm{Br}'(X)$. They use this to produce a non-separated Deligne-Mumford stack that is not a quotient stack.

- Totaro: *The resolution property for schemes and stacks* [**Tot04**]
  A stack has the resolution property if every coherent sheaf is the quotient of a vector bundle. The first main theorem is that if $\mathcal{X}$ is a normal Noetherian algebraic stack with affine stabilizer groups at closed points, then the following are equivalent: (1) $\mathcal{X}$ has the resolution property and (2) $\mathcal{X} = [Y/\mathrm{GL}_n]$ with Y quasi-affine. In the case $\mathcal{X}$ is finite type over a field, then (1) and (2) are equivalent to: (3) $\mathcal{X} = [\mathrm{Spec}(A)/G]$ with G an affine group scheme finite type over k. The implication that quotient stacks have the resolution property was proven by Thomason. The second main theorem is that if $\mathcal{X}$ is a smooth Deligne-Mumford stack over a field which has a finite and generically trivial stabilizer group $I_{\mathcal{X}} \to \mathcal{X}$ and whose coarse moduli space is a scheme with affine diagonal, then $\mathcal{X}$ has the resolution property. Another cool result states that if $\mathcal{X}$ is a Noetherian algebraic stack satisfying the resolution property, then $\mathcal{X}$ has affine diagonal if and only if the closed points have affine stabilizer.
- Kresch: *On the Geometry of Deligne-Mumford Stacks* [**Kre09**]
  This article summarizes general structure results of Deligne-Mumford stacks (of finite type over a field) and contains some interesting results concerning quotient stacks. It is shown that any smooth, separated, generically tame Deligne-Mumford stack with quasi-projective coarse moduli space is a quotient stack $[Y/G]$ with Y quasi-projective and G an algebraic group. If $\mathcal{X}$ is a Deligne-Mumford stack whose coarse moduli space is a scheme, then $\mathcal{X}$ is Zariski-locally a quotient stack if and only if it admits a Zariski-open covering by stack quotients of schemes by finite groups. If $\mathcal{X}$ is a Deligne-Mumford stack proper over a field of characteristic 0 with coarse moduli space Y, then: Y is projective and $\mathcal{X}$ is a quotient stack if and only if Y is projective and $\mathcal{X}$ possesses a generating sheaf if and only if $\mathcal{X}$ admits a closed embedding into a smooth proper DM stack with projective coarse moduli space. This motivates a definition that a Deligne-Mumford stack is *projective* if there exists a closed

embedding into a smooth, proper Deligne-Mumford stack with projective coarse moduli space.

- Kresch, Vistoli *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is shown that in characteristic 0 and for a fixed n, the following two statements are equivalent: (1) every smooth Deligne-Mumford stack of dimension n is a quotient stack and (2) the Azumaya Brauer group coincides with the cohomological Brauer group for smooth schemes of dimension n.
- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]
  It is shown that a reduced Artin stack finite type over a field with affine stabilizer groups admits a stratification by quotient stacks.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**]
  Lemma 2.2.3 establishes that for any separated Deligne-Mumford stack is étale-locally on the coarse moduli space a quotient stack $[U/G]$ where U affine and G a finite group. [**Ols06b**, Theorem 2.12] shows in this argument G is even the stabilizer group.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  This paper shows that a tame Artin stack is étale locally on the coarse moduli space a quotient stack of an affine by the stabilizer group.
- Alper: *On the local quotient structure of Artin stacks* [**Alp10**]
  It is conjectured that for an Artin stack $\mathcal{X}$ and a closed point $x \in \mathcal{X}$ with linearly reductive stabilizer, then there is an étale morphism $[V/G_x] \to \mathcal{X}$ with V an algebraic space. Some evidence for this conjecture is given. A simple deformation theory argument (based on ideas in [**AOV08**]) shows that it is true formally locally. A stack-theoretic proof of Luna's étale slice theorem is presented proving that for stacks $\mathcal{X} = [\mathrm{Spec}(A)/G]$ with G linearly reductive, then étale locally on the GIT quotient $\mathrm{Spec}(A^G)$, $\mathcal{X}$ is a quotient stack by the stabilizer.

04V0 **112.5.5. Cohomology.** Papers discussing cohomology of sheaves on algebraic stacks.

- Olsson: *Sheaves on Artin stacks* [**Ols07b**]
  This paper develops the theory of quasi-coherent and constructible sheaves proving basic cohomological properties. This paper corrects a mistake in [**LMB00**] in the functoriality of the lisse-étale site. The cotangent complex is constructed. In addition, the following theorems are proved: Grothendieck's Fundamental Theorem for proper morphisms, Grothendieck's Existence Theorem, Zariski's Connectedness Theorem and finiteness theorem for proper pushforwards of coherent and constructible sheaves.
- Behrend: *Derived l-adic categories for algebraic stacks* [**Beh03**]
  Proves the Lefschetz trace formula for algebraic stacks.
- Behrend: *Cohomology of stacks* [**Beh04**]
  Defines the de Rham cohomology for differentiable stacks and singular cohomology for topological stacks.

- Faltings: *Finiteness of coherent cohomology for proper fppf stacks* [**Fal03**]
  Proves coherence for direct images of coherent sheaves for proper morphisms.
- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  The appendix contains the proper base change theorem for étale cohomology for tame Deligne-Mumford stacks.

04V1 **112.5.6. Existence of finite covers by schemes.** The existence of finite covers of Deligne-Mumford stacks by schemes is an important result. In intersection theory on Deligne-Mumford stacks, it is an essential ingredient in defining proper push-forward for non-representable morphisms. There are several results about $\overline{\mathcal{M}}_g$ relying on the existence of a finite cover by a *smooth* scheme which was proven by Looijenga. Perhaps the first result in this direction is [**Ses72**, Theorem 6.1] which treats the equivariant setting.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]
  If $\mathcal{X}$ is a Deligne-Mumford stack with a moduli space (ie. a proper morphism which is bijective on geometric points), then there exists a finite morphism $X \to \mathcal{X}$ from a scheme X.
- Laumon, Moret-Bailly: [**LMB00**, Chapter 16]
  As an application of Zariski's main theorem, Theorem 16.6 establishes: if $\mathcal{X}$ is a Deligne-Mumford stack finite type over a Noetherian scheme, then there exists a finite, surjective, generically étale morphism $Z \to \mathcal{X}$ with Z a scheme. It is also shown in Corollary 16.6.2 that any Noetherian normal algebraic space is isomorphic to the algebraic space quotient X'/G for a finite group G acting a normal scheme X.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  Theorem 2.7 states: if $\mathcal{X}$ is an algebraic stack of finite type over a Noetherian ground scheme S, then the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is quasi-finite if and only if there exists a finite surjective morphism $X \to F$ from a scheme X.
- Kresch, Vistoli: *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is proved here that any smooth, separated Deligne-Mumford stack finite type over a field with quasi-projective coarse moduli space admits a finite, flat cover by a smooth quasi-projective scheme.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Proves that if $\mathcal{X}$ is an Artin stack separated and finite type over S, then there exists a proper surjective morphism $X \to \mathcal{X}$ from a scheme X quasi-projective over S. As an application, Olsson proves coherence and constructibility of direct image sheaves under proper morphisms. As an application, he proves Grothendieck's existence theorem for proper Artin stacks.
- Rydh: *Noetherian approximation of algebraic spaces and stacks* [**Ryd08**]
  Theorem B of this paper is as follows. Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal (resp. a

quasi-compact Deligne-Mumford stack with quasi-compact and separated diagonal). Then there exists a scheme Z and a finite, finitely presented and surjective morphism $Z \to X$ that is flat (resp. étale) over a dense quasi-compact open substack $U \subset X$.

04V2 **112.5.7. Rigidification.** Rigidification is a process for removing a flat subgroup from the inertia. For example, if X is a projective variety, the morphism from the Picard stack to the Picard scheme is a rigidification of the group of automorphism $\mathbf{G}_m$.

- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  Let $\mathcal{X}$ be an algebraic stack over S and H be a flat, finitely presented separated group scheme over S. Assume that for every object $\xi \in \mathcal{X}(T)$ there is an embedding $H(T) \hookrightarrow \mathrm{Aut}_{\mathcal{X}(T)}(\xi)$ which is compatible under pullbacks in the sense that for every arrow $\phi : \xi \to \xi'$ over $f : T \to T'$ and $g \in H(T')$, $g \circ \phi = \phi \circ f^*g$. Then there exists an algebraic stack $\mathcal{X}/H$ and a morphism $\rho : \mathcal{X} \to \mathcal{X}/H$ which is an fppf gerbe such that for every $\xi \in \mathcal{X}(T)$, the morphism $\mathrm{Aut}_{\mathcal{X}(T)}(\xi) \to \mathrm{Aut}_{\mathcal{X}/H(T)}(\xi)$ is surjective with kernel $H(T)$.
- Romagny: *Group actions on stacks and applications* [**Rom05**]
  Discusses how group actions behave with respect to rigidifications.
- Abramovich, Graber, Vistoli: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]
  The appendix gives a summary of rigidification as in [**ACV03**] with two alternative interpretations. This paper also contains constructions for gluing algebraic stacks along closed substacks and for taking roots of line bundles.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* ([**AOV08**])
  The appendix handles the more complicated situation where the flat subgroup stack of the inertia $H \subset I_{\mathcal{X}}$ is normal but not necessarily central.

04V3 **112.5.8. Stacky curves.** Papers discussing stacky curves.

- Abramovich, Vistoli: *Compactifying the space of stable maps* [**AV02**]
  This paper introduces *twisted curves*. The moduli space of stable maps from stable curves into an algebraic stack is typically not compact. By using maps from twisted curves, the authors construct a moduli stack which is proper when the target is a tame Deligne-Mumford stack whose coarse moduli space is projective.
- Behrend, Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**]
  Proves a uniformization theorem of Deligne-Mumford analytic curves.

04V4 **112.5.9. Hilbert, Quot, Hom and branchvariety stacks.** Papers discussing Hilbert schemes and the like.

- Vistoli: *The Hilbert stack and the theory of moduli of families* [**Vis91**]
  If $\mathcal{X}$ is a algebraic stack separated and locally of finite type over a locally Noetherian and locally separated algebraic space

S, Vistoli defines the Hilbert stack $\mathcal{H}\mathrm{ilb}(\mathcal{F}/S)$ parameterizing finite and unramified morphisms from proper schemes. It is claimed without proof that $\mathcal{H}\mathrm{ilb}(\mathcal{F}/S)$ is an algebraic stack. As a consequence, it is proved that with $\mathcal{X}$ as above, the Hom stack $\mathcal{H}\mathrm{om}_S(T, \mathcal{X})$ is an algebraic stack if T is proper and flat over S.

- Olsson, Starr: *Quot functors for Deligne-Mumford stacks* [**OS03**]
  If $\mathcal{X}$ is a Deligne-Mumford stack separated and locally of finite presentation over an algebraic space S and $\mathcal{F}$ is a locally finitely-presented $\mathcal{O}_\mathcal{X}$-module, the quot functor $\mathrm{Quot}(\mathcal{F}/\mathcal{X}/S)$ is represented by an algebraic space separated and locally of finite presentation over S. This paper also defines generating sheaves and proves existence of a generating sheaf for tame, separated Deligne-Mumford stacks which are global quotient stacks of a scheme by a finite group.
- Olsson: *Hom-stacks and Restrictions of Scalars* [**Ols06b**]
  Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Artin stacks locally of finite presentation over an algebraic space S with finite diagonal with $\mathcal{X}$ proper and flat over S such that fppf-locally on S, $\mathcal{X}$ admits a finite finitely presented flat cover by an algebraic space (eg. $\mathcal{X}$ is Deligne-Mumford or a tame Artin stack). Then $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack locally of finite presentation over S.
- Alexeev and Knutson: *Complete moduli spaces of branchvarieties* ([**AK10**])
  They define a branchvariety of $\mathbf{P}^n$ as a finite morphism $X \to \mathbf{P}^n$ from a *reduced* scheme X. They prove that the moduli stack of branchvarieties with fixed Hilbert polynomial and total degrees of i-dimensional components is a proper Artin stack with finite stabilizer. They compare the stack of branchvarieties with the Hilbert scheme, Chow scheme and moduli space of stable maps.
- Lieblich: *Remarks on the stack of coherent algebras* [**Lie06b**]
  This paper constructs a generalization of Alexeev and Knutson's stack of branch-varieties over a scheme Y by building the stack as a stack of algebras over the structure sheaf of Y. Existence proofs of Quot and Hom spaces are given.
- Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  As an application of the main result, a common generalization of Vistoli's Hilbert stack [**Vis91**] and Alexeev and Knutson's stack of branchvarieties [**AK10**] is provided. If $\mathcal{X}$ is an algebraic stack locally of finite type over an excellent scheme S with finite diagonal, then the stack $\mathcal{H}$ parameterizing morphisms $g : T \to \mathcal{X}$ from a proper algebraic space T with a G-ample line bundle L is an Artin stack locally of finite type over S.
- Lundkvist and Skjelnes: *Non-effective deformations of Grothendieck's Hilbert functor* [**LS08**]
  Shows that the Hilbert functor of a non-separated scheme is not represented since there are non-effective deformations.
- Halpern-Leistner and Preygel: *Mapping stacks and categorical notions of properness* [**HLP14**]

This paper gives a proof that the Hom stack is algebraic under some hypotheses on source and target which are more general than, or at least different from, the ones in Olsson's paper.

04V5 **112.5.10. Toric stacks.** Toric stacks provide a great class of examples and a natural testing ground for conjectures due to the dictionary between the geometry of a toric stack and the combinatorics of its stacky fan in a similar way that toric varieties provide examples and counterexamples in scheme theory.

- Borisov, Chen and Smith: *The orbifold Chow ring of toric Deligne-Mumford stacks* [**BCS05**]
  Inspired by Cox's construction for toric varieties, this paper defines smooth toric DM stacks as explicit quotient stacks associated to a combinatorial object called a *stacky fan*.
- Iwanari: *The category of toric stacks* [**Iwa09**]
  This paper defines a *toric triple* as a smooth Deligne-Mumford stack $\mathcal{X}$ with an open immersion $\mathbf{G}_m \hookrightarrow \mathcal{X}$ with dense image (and therefore $\mathcal{X}$ is an orbifold) and an action $\mathcal{X} \times \mathbf{G}_m \to \mathcal{X}$. It is shown that there is an equivalence between the 2-category of toric triples and the 1-category of stacky fans. The relationship between toric triples and the definition of smooth toric DM stacks in [**BCS05**] is discussed.
- Iwanari: *Integral Chow rings for toric stacks* [**Iwa07**]
  Generalizes Cox's Δ-collections for toric varieties to toric orbifolds.
- Perroni: *A note on toric Deligne-Mumford stacks* [**Per08**]
  Generalizes Cox's Δ-collections and Iwanari's paper [**Iwa07**] to general smooth toric DM stacks.
- Fantechi, Mann, and Nironi: *Smooth toric DM stacks* [**FMN07**]
  This paper defines a smooth toric DM stack as a smooth DM stack $\mathcal{X}$ with the action of a DM torus $\mathcal{T}$ (ie. a Picard stack isomorphic to $T \times BG$ with G finite) having an open dense orbit isomorphic to $\mathcal{T}$. They give a "bottom-up description" and prove an equivalence between smooth toric DM stacks and stacky fans.
- Geraschenko and Satriano: *Toric Stacks I and II* [**GS11a**] and [**GS11b**]
  These papers define a toric stack as the stack quotient of a toric variety by a subgroup of its torus. A generically stacky toric stack is defined as a torus invariant substack of a toric stack. This definition encompasses and extends previous definitions of toric stacks. The first paper develops a dictionary between the combinatorics of stacky fans and the geometry of the corresponding stacks. It also gives a moduli interpretation of smooth toric stacks, generalizing the one in [**Per08**]. The second paper proves an intrinsic characterization of toric stacks.

04V6 **112.5.11. Theorem on formal functions and Grothendieck's Existence Theorem.** These papers give generalizations of the theorem on formal functions [**DG67**, III.4.1.5] (sometimes referred to Grothendieck's Fundamental Theorem for proper morphisms) and Grothendieck's Existence Theorem [**DG67**, III.5.1.4].

- Knutson: *Algebraic spaces* [**Knu71**, Chapter V]

Generalizes these theorems to algebraic spaces.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**, A.1.1]
  Generalizes these theorems to tame Deligne-Mumford stacks
- Olsson and Starr: *Quot functors for Deligne-Mumford stacks* [**OS03**]
  Generalizes these theorems to separated Deligne-Mumford stacks.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Provides a generalization to proper Artin stacks.
- Conrad: *Formal GAGA on Artin stacks* [**Con05a**]
  Provides a generalization to proper Artin stacks and proves a formal GAGA theorem.
- Olsson: *Sheaves on Artin stacks* [**Ols07b**]
  Provides another proof for the generalization to proper Artin stacks.

04V7 **112.5.12. Group actions on stacks.** Actions of groups on algebraic stacks naturally appear. For instance, symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ and for an action of a group G on a scheme X, the normalizer of G in $\mathrm{Aut}(X)$ acts on $[X/G]$. Furthermore, torus actions on stacks often appear in Gromov-Witten theory.

- Romagny: *Group actions on stacks and applications* [**Rom05**]
  This paper makes precise what it means for a group to act on an algebraic stack and proves existence of fixed points as well as existence of quotients for actions of group schemes on algebraic stacks. See also Romagny's earlier note [**Rom03**].

04V8 **112.5.13. Taking roots of line bundles.** This useful construction was discovered independently by Cadman and by Abramovich, Graber and Vistoli. Given a scheme X with an effective Cartier divisor D, the rth root stack is an Artin stack branched over X at D with a μ_r stabilizer over D and scheme-like away from D.

- Charles Cadman *Using Stacks to Impose Tangency Conditions on Curves* [**Cad07**]
- Abramovich, Graber, Vistoli: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]

04V9 **112.5.14. Other papers.** Potpourri of other papers.

- Lieblich: *Moduli of twisted sheaves* [**Lie07**]
  This paper contains a summary of gerbes and twisted sheaves. If $\mathcal{X} \to X$ is a μ_n-gerbe with X a projective relative surface with smooth connected geometric fibers, it is shown that the stack of semistable $\mathcal{X}$-twisted sheaves is an Artin stack locally of finite presentation over S. This paper also develops the theory of associated points and purity of sheaves on Artin stacks.
- Lieblich, Osserman: *Functorial reconstruction theorem for stacks* [**LO08b**]
  Proves some surprising and interesting results on when an algebraic stack can be reconstructed from its associated functor.
- David Rydh: *Noetherian approximation of algebraic spaces and stacks* [**Ryd08**]
  This paper shows that every quasi-compact algebraic stack with quasi-finite diagonal can be approximated by a Noetherian stack.

There are applications to removing the Noetherian hypothesis in results of Chevalley, Serre, Zariski and Chow.

112.6. Stacks in other fields

03B5

- Behrend and Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**] Gives an overview and comparison of topological, analytic and algebraic stacks.
- Behrang Noohi: *Foundations of topological stacks I* [**Noo05**]
- David Metzler: *Topological and smooth stacks* [**Met05**]

112.7. Higher stacks

05BF

- Lurie: *Higher topos theory* [**Lur09f**]
- Lurie: *Derived Algebraic Geometry I - V* [**Lur09a**], [**Lur09b**], [**Lur09c**], [**Lur09d**], [**Lur09e**]
- Toën: *Higher and derived stacks: a global overview* [**Toë09**]
- Toën and Vezzosi: *Homotopical algebraic geometry I, II* [**TV05**], [**TV08**]

112.8. Other chapters

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CHAPTER 113

Desirables

02B4

113.1. Introduction

02B5 This is basically just a list of things that we want to put in the stacks project. As we add material to the Stacks project continuously this is always somewhat behind the current state of the Stacks project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Thursday, August 31, 2017.

113.2. Conventions

02B7 We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 2.1, but a lot more could be added there. Especially useful would be to find "hidden" conventions and tacit assumptions and put those there.

113.3. Sites and Topoi

02BA We have a chapter on sites and sheaves, see Sites, Section 7.1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 18.1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 21.1. But a lot more could be added, especially in the chapter on cohomology.

113.4. Stacks

02BB We have a chapter on (abstract) stacks, see Stacks, Section 8.1. It would be nice if

(1) improve the discussion on "stackyfication",
(2) give examples of stackyfication,
(3) more examples in general,
(4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes banded by $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

113.5. Simplicial methods

03MZ We have a chapter on simplicial methods, see Simplicial, Section 14.1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. There is a chapter on simplicial spaces, see Simplicial Spaces, Section 85.1. This chapter briefly discusses simplicial topological spaces, simplicial sites, and simplicial topoi. We can further develop "simplicial algebraic geometry" to discuss simplicial schemes (or simplicial algebraic spaces, or simplicial algebraic stacks) and treat geometric questions, their cohomology, etc.

113.6. Cohomology of schemes

02BE There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 30.1. We have a chapter discussing the derived category of quasi-coherent sheaves on a scheme, see Derived Categories of Schemes, Section 36.1. We have a chapter discussing duality for Noetherian schemes and relative duality for morphisms of schemes, see Duality for Schemes, Section 48.1. We also have chapters on étale cohomology of schemes and on crystalline cohomology of schemes. But most of the material in these chapters is very basic and a lot more could/should be added there.

113.7. Deformation theory à la Schlessinger

02BF We have a chapter on this material, see Formal Deformation Theory, Section 90.1. We have a chapter discussing examples of the general theory, see Deformation Problems, Section 93.1. We have a chapter, see Deformation Theory, Section 91.1 which discusses deformations of rings (and modules), deformations of ringed spaces (and sheaves of modules), deformations of ringed topoi (and sheaves of modules). In this chapter we use the naive cotangent complex to describe obstructions, first order deformations, and infinitesimal automorphisms. This material has found some applications to algebraicity of moduli stacks in later chapters. There is also a chapter discussing the full cotangent complex, see Cotangent, Section 92.1.

113.8. Definition of algebraic stacks

02BK An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. See Algebraic Stacks, Section 94.12. A "Deligne-Mumford stack" is an algebraic stack for which there exists a scheme and a surjective étale morphism from that scheme to it as in the paper [**DM69**] of Deligne and Mumford, see Algebraic Stacks, Definition 94.12.2. We will reserve the term "Artin stack" for a stack such as in the papers by Artin, see [**Art69b**], [**Art70**], and [**Art74**]. A possible definition is that an Artin stack is an algebraic stack $\mathcal{X}$ over a locally Noetherian scheme S such that $\mathcal{X} \to S$ is locally of finite type[1].

113.9. Examples of schemes, algebraic spaces, algebraic stacks

02BL The Stacks project currently contains two chapters discussing moduli stacks and their properties, see Moduli Stacks, Section 108.1 and Moduli of Curves, Section 109.1. Over time we intend to add more, for example:

(1) $\mathcal{A}_g$, i.e., principally polarized abelian schemes of genus g,
(2) $\mathcal{A}_1 = \mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
(3) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g-curves with n pairwise distinct labeled points,
(4) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n-pointed nodal projective genus g-curves,
(5) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks $\mathcal{X}$, $\mathcal{Y}$ and the base scheme S),

[1]Namely, these are exactly the algebraic stacks over S satisfying Artin's axioms [-1], [0], [1], [2], [3], [4], [5] of Artin's Axioms, Section 98.14.

(6) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G-bundles of the geometric Langlands programme (with suitable conditions on the scheme X, the group scheme G, and the base scheme S),
(7) $\mathcal{P}ic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

More generally, the Stacks project is somewhat lacking in geometrically meaningful examples.

113.10. Properties of algebraic stacks

02BM This is perhaps one of the easier projects to work on, as most of the basic theory is there now. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

113.11. Lisse étale site of an algebraic stack

02BN This has been introduced in Cohomology of Stacks, Section 103.14. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 110.59. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the "big" étale site as much as possible.

113.12. Things you always wanted to know but were afraid to ask

02BO There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes.

113.13. Quasi-coherent sheaves on stacks

02BP These are defined and discussed in the chapter Cohomology of Stacks, Section 103.1. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section 104.1. A lot more could be added to these chapters.

113.14. Flat and smooth

02BR Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem 97.16.1.

113.15. Artin's representability theorem

02BS This is discussed in the chapter Artin's Axioms, Section 98.1. We also have an application, see Quot, Theorem 99.5.12. There should be a lot more applications and the chapter itself has to be cleaned up as well.

113.16. DM stacks are finitely covered by schemes

02BT We already have the corresponding result for algebraic spaces, see Limits of Spaces, Section 70.16. What is missing is the result for DM and quasi-DM stacks.

113.17. Martin Olsson's paper on properness

02BU This proves two notions of proper are the same. The first part of this is now available in the form of Chow's lemma for algebraic stacks, see More on Morphisms of Stacks, Theorem 106.10.3. As a consequence we show that it suffices to use DVR's in checking the valuative criterion for properness for algebraic stacks in certain cases, see More on Morphisms of Stacks, Section 106.11.

113.18. Proper pushforward of coherent sheaves

02BV We can start working on this now that we have Chow's lemma for algebraic stacks, see previous section.

113.19. Keel and Mori

02BW See [**KM97**]. Their result has been added in More on Morphisms of Stacks, Section 106.13.

113.20. Add more here

02BX Actually, no we should never have started this list as part of the Stacks project itself! There is a todo list somewhere else which is much easier to update.

113.21. Other chapters

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CHAPTER 114

Coding Style

02BY

114.1. List of style comments

02BZ These will be changed over time, but having some here now will hopefully encourage a consistent LaTeX style. We will call "code[1]" the contents of the source files.

(1) Keep all lines in all tex files to at most 80 characters.

(2) Do not use indentation in the tex file. Use syntax highlighting in your editor, instead of indentation, to visualize environments, etc.

(3) Use

```
\medskip\noindent
```

to start a new paragraph, and use

```
\noindent
```

to start a new paragraph just after an environment.

(4) Do not break the code for mathematical formulas across lines if possible. If the complete code complete with enclosing dollar signs does not fit on the line, then start with the first dollar sign on the first character of the next line. If it still does not fit, find a mathematically reasonable spot to break the code.

(5) Displayed math equations should be coded as follows

```
$$
...
...
$$
```

In other words, start with a double dollar sign on a line by itself and end similarly.

(6) *Do not use any macros.* Rationale: This makes it easier to read the tex file, and start editing an arbitrary part without having to learn innumerable macros. And it doesn't make it harder or more timeconsuming to write. Of course the disadvantage is that the same mathematical object may be TeXed differently in different places in the text, but this should be easy to spot.

(7) The theorem environments we use are: "theorem", "proposition", "lemma" (plain), "definition", "example", "exercise", "situation" (definition), "remark", "remarks" (remark). Of course there is also a "proof" environment.

(8) An environment "foo" should be coded as follows

```
\begin{foo}
...
```

[1] It is all Knuth's fault. See [**Knu79**].

```
...
\end{foo}
```

similarly to the way displayed equations are coded.

(9) Instead of a "corollary", just use "lemma" environment since likely the result will be used to prove the next bigger theorem anyway.

(10) Directly following each lemma, proposition, or theorem is the proof of said lemma, proposition, or theorem. No nested proofs please.

(11) The files preamble.tex, chapters.tex and fdl.tex are special tex files. Apart from these, each tex file has the following structure

```
\input{preamble}
\begin{document}
\title{Title}
\maketitle
\tableofcontents
...
...
\input{chapters}
\bibliography{my}
\bibliographystyle{amsalpha}
\end{document}
```

(12) Try to add labels to lemmas, propositions, theorems, and even remarks, exercise, and other environments. If labelling a lemma use something like

```
\begin{lemma}
\label{lemma-bar}
...
\end{lemma}
```

Similarly for all other environments. In other words, the label of a environment named "foo" starts with "foo-". In addition to this please make all labels consist only of lower case letters, digits, and the symbol "-".

(13) Never refer to "the lemma above" (or proposition, etc). Instead use:

```
Lemma \ref{lemma-bar} above
```

This means that later moving lemmas around is basically harmless.

(14) Cross-file referencing. To reference a lemma labeled "lemma-bar" in the file foo.tex which has title "Foo", please use the following code

```
Foo, Lemma \ref{foo-lemma-bar}
```

If this does not work, then take a look at the file preamble.tex to find the correct expression to use. This will produce the "Foo, Lemma <link>" in the output file so it will be clear that the link points out of the file.

(15) If at all possible avoid forward references in proof environments. (It should be possible to write an automated test for this.)

(16) Do not start any sentence with a mathematical symbol.

(17) Do not have a sentence of the type "This follows from the following" just before a lemma, proposition, or theorem. Every sentence ends with a period.

(18) State all hypotheses in each lemma, proposition, theorem. This makes it easier for readers to see if a given lemma, proposition, or theorem applies to their particular problem.

(19) Keep proofs short; less than 1 page in pdf or dvi. You can always achieve this by splitting out the proof in lemmas etc.
(20) In a defining property foobar use

```
{\it foobar}
```

in the code inside the definition environment. Similarly if the definition occurs in the text of the document. This will make it easier for the reader to see what it is that is being defined.
(21) Put any definition that will be used outside the section it is in, in its own definition environment. Temporary definitions may be made in the text. A tricky case is that of mathematical constructions (which are often definitions involving interrelated lemmas). Maybe a good solution is to have them in their own short section so users can refer to the section instead of a definition.
(22) Do not number equations unless they are actually being referenced somewhere in the text. We can always add labels later.
(23) In statements of lemmas, propositions and theorems and in proofs keep the sentences short. For example, instead of "Let R be a ring and let M be an R-module." write "Let R be a ring. Let M be an R-module.". Rationale: This makes it easier to parse the trickier parts of proofs and statements.
(24) Use the

```
\section
```

command to make sections, but try to avoid using subsections and subsubsections.
(25) Avoid using complicated latex constructions.

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CHAPTER 115

Obsolete

073T

115.1. Introduction

073U In this chapter we put some lemmas that have become "obsolete" (see [**Mil17**]).

115.2. Preliminaries

0G5K

0EGF **Remark 115.2.1.** The information which used to be contained in this remark is now subsumed in the combination of Categories, Lemmas 4.24.4 and 4.24.3.

115.3. Homological algebra

0BFJ

076K **Remark 115.3.1.** The following remarks are obsolete as they are subsumed in Homology, Lemmas 12.24.11 and 12.25.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Homology, Definition 12.10.1). Suppose that $K^{\bullet,\bullet}$ is a double complex to which Homology, Lemma 12.25.3 applies such that for some $r \geq 0$ all the objects ${}'E_r^{p,q}$ belong to $\mathcal{C}$. Then all the cohomology groups $H^n(sK^\bullet)$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of ${}'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each ${}'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each ${}'E_\infty^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^\bullet)$ is in $\mathcal{C}$ as claimed. The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if $(K^\bullet, F)$ is a filtered complex to which Homology, Lemma 12.24.11 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to $\mathcal{C}$, then each $H^n(K^\bullet)$ is an object of $\mathcal{C}$.

115.4. Obsolete algebra lemmas

088X

055Z **Lemma 115.4.1.** *Let M be an R-module of finite presentation. For any surjection $\alpha : R^{\oplus n} \to M$ the kernel of α is a finite R-module.*

Proof. This is a special case of Algebra, Lemma 10.5.3. □

00I5 **Lemma 115.4.2.** *Let $\varphi : R \to S$ be a ring map. If*

(1) *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and*
(2) *for any $x \in \mathrm{Ker}(\varphi)$ there exists $n > 0$ such that $x^n = 0$,*

then φ induces a homeomorphism on spectra. Given a prime number p such that

(a) *S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
(b) *the kernel of φ is generated by nilpotent elements,*

then (1) and (2) hold, and for any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. This is a combination of Algebra, Lemmas 10.46.3 and 10.46.7. □

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

00SX **Lemma 115.4.3.** *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$. Assume $S_\mathfrak{q}$ is essentially of finite type over $R_\mathfrak{p}$. Assume given*

(1) *an integer $n \geq 0$,*
(2) *a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \ldots, x_n]$,*
(3) *a surjective $\kappa(\mathfrak{p})$-homomorphism*
$$\psi : (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_\mathfrak{a} \longrightarrow S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q},$$
and
(4) *elements $\overline{f}_1, \ldots, \overline{f}_e$ in $\mathrm{Ker}(\psi)$.*

Then there exist

(1) *an integer $m \geq 0$,*
(2) *and element $g \in S$, $g \notin \mathfrak{q}$,*
(3) *a map*
$$\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \longrightarrow S_g,$$
and
(4) *elements $f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}$ of $\mathrm{Ker}(\Psi)$*

such that

(1) *the following diagram commutes*
$$\begin{array}{ccc} R[x_1, \ldots, x_{n+m}] & \xrightarrow[x_{n+j} \mapsto 0]{} & (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_\mathfrak{a} \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\ S_g & \longrightarrow & S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \end{array},$$
(2) *the element f_i, $i \leq n$ maps to a unit times $\overline{f}_i$ in the local ring*
$$(\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}])_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})},$$
(3) *the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and*
(4) *the induced map $R[x_1, \ldots, x_{n+m}]_\mathfrak{b} \to S_\mathfrak{q}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.*

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Namely, suppose we have constructed
$$\Psi' : R_\mathfrak{p}[x_1, \ldots, x_{n+m}] \longrightarrow S_\mathfrak{q}$$
and $f'_1, \ldots, f'_{e+m} \in R_\mathfrak{p}[x_1, \ldots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \ldots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R-algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_\mathfrak{q}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Pick $y_1, \ldots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \ldots, y_{n+m} \in S$ be elements which generate an R-subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R. Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \ldots, x_n]$ of g_j and d. Set $y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ generate an R-subalgebra of S whose localization is S. We define

$$\Psi : R[x_1, \ldots, x_{n+m}] \to S$$

to be the map that sends x_i to y_i for $i = 1, \ldots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \ldots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal $\mathfrak{b}$ maps onto the ideal $(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]$.

Denote $J = \mathrm{Ker}(\Psi)$. We have a short exact sequence

$$0 \to J_{\mathfrak{b}} \to R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}} \to 0.$$

The surjectivity comes from our choice of $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \to \kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})} \to S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \to 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. □

05GK **Lemma 115.4.4.** *Let $R \to S$ be a ring map of finite presentation. If for some presentation α of S over R the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.*

Proof. Immediate from Algebra, Lemma 10.134.2. □

01DE **Remark 115.4.5** (Projective resolutions)**.** Let R be a ring. For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

02CA **Lemma 115.4.6.** *Let S be a multiplicative set of A. Then the map*

$$f : \mathrm{Spec}(S^{-1}A) \longrightarrow \mathrm{Spec}(A)$$

induced by the canonical ring map $A \to S^{-1}A$ is a homeomorphism onto its image and $\mathrm{Im}(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. This is a duplicate of Algebra, Lemma 10.17.5. □

05IP **Lemma 115.4.7.** *Let $A \to B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A-algebra.*

Proof. Special case of More on Flatness, Proposition 38.13.10. □

053F **Lemma 115.4.8.** *Let R be a domain with fraction field K. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be a finite S-module. Assume that M is flat over R. If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S.*

Proof. This lemma is true because M is finitely presented even without the assumption that $M \otimes_R R'$ is finitely presented for every R' as in the statement of the lemma. This follows from More on Flatness, Proposition 38.13.10. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case R is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. □

02TQ **Lemma 115.4.9.** *Let $A \to B$ be a ring map. Let $f \in B$. Assume that*

(1) *$A \to B$ is flat,*
(2) *f is a nonzerodivisor, and*
(3) *$A \to B/fB$ is flat.*

Then for every ideal $I \subset A$ the map $f : B/IB \to B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A. In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB. Hence the result follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \end{array}$$

with exact rows. □

051A **Lemma 115.4.10.** *If $R \to S$ is a faithfully flat ring map then for every R-module M the map $M \to S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.*

Proof. This lemma is a duplicate of Algebra, Lemma 10.82.11. □

07C2 **Remark 115.4.11.** This reference/tag used to refer to a Section in the chapter Smoothing Ring Maps, but the material has since been subsumed in Algebra, Section 10.127.

07K3 **Lemma 115.4.12.** *Let $(R, \mathfrak{m})$ be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak{m}$ be a nonzerodivisor. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of R. Then*

$$length_R(R/(x)) = \sum\nolimits_i ord_{R/\mathfrak{q}_i}(x)$$

Proof. Special (very easy) case of Chow Homology, Lemma 42.3.2. □

0AXG **Lemma 115.4.13.** *Let A be a Noetherian local normal domain of dimension 2. For $f \in \mathfrak{m}$ nonzero denote $div(f) = \sum n_i(\mathfrak{p}_i)$ the divisor associated to f on the punctured spectrum of A. We set $|f| = \sum n_i$. There exist integers N and M such that $|f + g| \leq M$ for all $g \in \mathfrak{m}^N$.*

Proof. Pick $h \in \mathfrak{m}$ such that f, h is a regular sequence in A (this follows from Algebra, Lemmas 10.157.4 and 10.72.7). We will prove the lemma with $M = \text{length}_A(A/(f,h))$ and with N any integer such that $\mathfrak{m}^N \subset (f,h)$. Such an integer N exists because $\sqrt{(f,h)} = \mathfrak{m}$. Note that $M = \text{length}_A(A/(f+g,h))$ for all $g \in \mathfrak{m}^N$ because $(f,h) = (f+g,h)$. This moreover implies that $f+g, h$ is a regular sequence in A too, see Algebra, Lemma 10.104.2. Now suppose that $\text{div}(f+g) = \sum m_j(\mathfrak{q}_j)$. Then consider the map

$$c : A/(f+g) \longrightarrow \prod A/\mathfrak{q}_j^{(m_j)}$$

where $\mathfrak{q}_j^{(m_j)}$ is the symbolic power, see Algebra, Section 10.64. Since A is normal, we see that $A_{\mathfrak{q}_i}$ is a discrete valuation ring and hence

$$A_{\mathfrak{q}_i}/(f+g) = A_{\mathfrak{q}_i}/\mathfrak{q}_i^{m_i}A_{\mathfrak{q}_i} = (A/\mathfrak{q}_i^{(m_i)})_{\mathfrak{q}_i}$$

Since $V(f+g,h) = \{\mathfrak{m}\}$ this implies that c becomes an isomorphism on inverting h (small detail omitted). Since h is a nonzerodivisor on $A/(f+g)$ we see that the length of $A/(f+g,h)$ equals the Herbrand quotient $e_A(A/(f+g),0,h)$ as defined in Chow Homology, Section 42.2. Similarly the length of $A/(h,\mathfrak{q}_j^{(m_j)})$ equals $e_A(A/\mathfrak{q}_j^{(m_j)},0,h)$. Then we have

$$\begin{aligned} M &= \text{length}_A(A/(f+g,h) \\ &= e_A(A/(f+g),0,h) \\ &= \sum\nolimits_i e_A(A/\mathfrak{q}_j^{(m_j)},0,h) \\ &= \sum\nolimits_i \sum\nolimits_{m=0,\ldots,m_j-1} e_A(\mathfrak{q}_j^{(m)}/\mathfrak{q}_j^{(m+1)},0,h) \end{aligned}$$

The equalities follow from Chow Homology, Lemmas 42.2.3 and 42.2.4 using in particular that the cokernel of c has finite length as discussed above. It is straightforward to prove that $e_A(\mathfrak{q}^{(m)}/\mathfrak{q}^{(m+1)},0,h)$ is at least 1 by Nakayama's lemma. This finishes the proof of the lemma. □

0BJK **Lemma 115.4.14.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. If A and $B/\mathfrak{m}_AB$ are Gorenstein, then B is Gorenstein.*

Proof. Follows immediately from Dualizing Complexes, Lemma 47.21.8. □

0DXL **Lemma 115.4.15.** *Let $(A,\mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s be an integer. Assume*

(1) *A has a dualizing complex,*
(2) *if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then $\text{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > s$.*

Then there exists an $n > 0$ and an ideal $J \subset A$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ such that JI^n annihilates $H^i_\mathfrak{m}(M)$ for $i \leq s$.

Proof. According to Local Cohomology, Lemma 51.9.4 we have to show this for the finite A-module $E^i = \text{Ext}^{-i}_A(M, \omega_A^\bullet)$ for $i \leq s$. The support Z of $E^0 \oplus \ldots \oplus E^s$ is closed in $\text{Spec}(A)$ and does not contain any prime as in (2). Hence it is contained in $V(JI^n)$ for some J as in the statement of the lemma. □

0EFS **Lemma 115.4.16.** *Let $(A,\mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let s and d be integers. Assume*

(a) *A has a dualizing complex,*

(b) $cd(A, I) \leq d$,
(c) *if* $\mathfrak{p} \notin V(I)$ *then* $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) > s$ *or* $depth_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) > d + s$.

Then the assumptions of Algebraic and Formal Geometry, Lemma 52.10.4 hold for $A, I, \mathfrak{m}, M$ *and* $H^i_\mathfrak{m}(M) \to \lim H^i_\mathfrak{m}(M/I^nM)$ *is an isomorphism for* $i \leq s$ *and these modules are annihilated by a power of* I.

Proof. The assumptions of Algebraic and Formal Geometry, Lemma 52.10.4 by the more general Algebraic and Formal Geometry, Lemma 52.10.5. Then the conclusion of Algebraic and Formal Geometry, Lemma 52.10.4 gives the second statement. □

0EFZ **Lemma 115.4.17.** *In Algebraic and Formal Geometry, Situation 52.10.1 we have* $H^s_\mathfrak{a}(M) = \lim H^s_\mathfrak{a}(M/I^nM)$.

Proof. This is immediate from Algebraic and Formal Geometry, Theorem 52.10.8. The original version of this lemma, which had additional assumptions, was superseded by the this theorem. □

0EKR **Lemma 115.4.18.** *Let* A *be a Noetherian ring. Let* $f \in \mathfrak{a}$ *be an element of an ideal of* A. *Let* $U = \mathrm{Spec}(A) \setminus V(\mathfrak{a})$. *Assume*

(1) A *has a dualizing complex and is complete with respect to* f,
(2) A_f *is* (S_2) *and for every minimal prime* $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ *and* $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ *we have* $\dim((A/\mathfrak{p})_\mathfrak{q}) \geq 3$.

Then the completion functor

$$Coh(\mathcal{O}_U) \longrightarrow Coh(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. This lemma is a special case of Algebraic and Formal Geometry, Lemma 52.15.6. □

115.5. Lemmas related to ZMT

073V The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 10.123.12.

00PU **Lemma 115.5.1.** *Let* R *be a ring and let* $\varphi : R[x] \to S$ *be a ring map. Let* $t \in S$. *If* t *is integral over* $R[x]$, *then there exists an* $\ell \geq 0$ *such that for every* $a \in R$ *the element* $\varphi(a)^\ell t$ *is integral over* $\varphi_a : R[y] \to S$, *defined by* $y \mapsto \varphi(ax)$ *and* $r \mapsto \varphi(r)$ *for* $r \in R$.

Proof. Say $t^d + \sum_{i<d} \varphi(f_i)t^i = 0$ with $f_i \in R[x]$. Let ℓ be the maximum degree in x of all the f_i. Multiply the equation by $\varphi(a)^\ell$ to get $\varphi(a)^\ell t^d + \sum_{i<d} \varphi(a^\ell f_i)t^i = 0$. Note that each $\varphi(a^\ell f_i)$ is in the image of φ_a. The result follows from Algebra, Lemma 10.123.1. □

00PR **Lemma 115.5.2.** *Let* $\varphi : R \to S$ *be a ring map. Suppose* $t \in S$ *satisfies the relation* $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. *Set* $u_n = \varphi(a_n)$, $u_{n-1} = u_nt + \varphi(a_{n-1})$, *and so on till* $u_1 = u_2t + \varphi(a_1)$. *Then all of* $u_n, u_{n-1}, \ldots, u_1$ *and* $u_nt, u_{n-1}t, \ldots, u_1t$ *are integral over* R, *and the ideals* $(\varphi(a_0), \ldots, \varphi(a_n))$ *and* $(u_n, \ldots, u_1)$ *of* S *are equal.*

Proof. We prove this by induction on n. As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 10.123.1 that $u_n t$ is integral over R. Of course $u_n = \varphi(a_n)$ is integral over R. Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 10.36.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \to S$ where S' is the integral closure of R in S and the displayed equation we see that $u_{n-1}, \ldots, u_1$ and $u_{n-1}t, \ldots, u_1 t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. □

00PS **Lemma 115.5.3.** *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R.*

Proof. This is immediate from Lemma 115.5.2 since one of the elements u_i will not be in J. □

The following two lemmas are a way of describing closed subschemes of $\mathbf{P}^1_R$ cut out by one (nondegenerate) equation.

00Q4 **Lemma 115.5.4.** *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Assume that for every prime $\mathfrak{p}$ of R at least one coefficient of F is not in $\mathfrak{p}$. Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R-module S_n is finite locally free of rank d.*

Proof. The R-module S_n has a presentation

$$R[X, Y]_{n-d} \to R[X, Y]_n \to S_n \to 0.$$

Thus by Algebra, Lemma 10.79.4 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \to \kappa(\mathfrak{p})[X, Y]$ for all primes $\mathfrak{p}$. This is clear from the assumption that F does not map to the zero polynomial mod $\mathfrak{p}$. The assertion on ranks is clear from this as well. □

00Q5 **Lemma 115.5.5.** *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e. Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define "relatively prime". If you have another definition, then you can show it is equivalent to this one. □

00Q6 **Lemma 115.5.6.** *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in $\mathfrak{p}$. There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e, and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \to (S_{n+e})_f$ for all $n \geq d$.*

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R. In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 115.5.4. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 10.79.4 to the map induced by multiplication by

G from $S_d \to S_{d+e}$. By our choice of G and Lemma 115.5.5 we see $S_d \otimes \kappa(\mathfrak{p}) \to S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \to S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes $\mathfrak{p}'$ of R. And then by Algebra, Lemma 10.79.4 again we see that all the maps $G : S_d \to S_{d+e}$, $n \geq d$ are isomorphisms. □

00Q7 **Remark 115.5.7.** Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms $S_n \to S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R-algebra A as follows:

(1) as an R-module $A = S_{ed}$, and
(2) multiplication $A \times A \to A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed}.

This makes sense because multiplication by G^d induces a bijective map $S_{de} \to S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A.

00Q3 **Lemma 115.5.8.** *Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings*

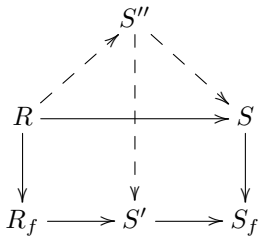

Assume that $R_f \to S'$ is finite. Then we can find a finite ring map $R \to S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f, $i = 1, \ldots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j<d_i} (a_{ij}/f^n) t^j$ for some uniform n. Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i} g_i^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} g_i^j$ of S is killed by a power of f. Hence upon increasing n to n', which replaces g_i by $f^{n'-n} g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} t^j$ with coefficients in R. Also, by construction $Q_i(f^n g_i) = 0$ in S. Thus we get a finite R-algebra $S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \to S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \to S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N. Thus $f^N h$ is an element of the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S''. OK, and it is easy to see that S''/J does the job. □

115.6. Formally smooth ring maps

07GD

00TO **Lemma 115.6.1.** *Let R be a ring. Let S be a R-algebra. If S is of finite presentation and formally smooth over R then S is smooth over R.*

Proof. See Algebra, Proposition 10.138.13. □

0AKC **Remark 115.6.2.** This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKD **Remark 115.6.3.** This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKE **Remark 115.6.4.** This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AKF **Remark 115.6.5.** This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 88.6.3 which became unused because of a rearrangement of the material.

0AM9 **Remark 115.6.6.** This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Lemma 88.9.1 which became unused because of a rearrangement of the material.

0AK9 **Lemma 115.6.7.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let t be the minimal number of generators for I. Let C be a Noetherian I-adically complete A-algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \to C$ with the following property: given*

1. *$c \geq 0$ and B in Algebraization of Formal Spaces, Equation (88.2.0.2) such that for $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,*
2. *an integer $n > 2t\max(c, d)$,*
3. *an A/I^n-algebra map $\psi_n : B/I^nB \to C/I^nC$,*

there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = \lfloor \frac{n}{t} \rfloor$.

Proof. This lemma has been obsoleted by the stronger Algebraization of Formal Spaces, Lemma 88.5.3. In fact, we will deduce the lemma from it.

Let $I \subset A \to C$ be given as in the statement above. Denote $d(\mathrm{Gr}_I(C))$ and $q(\mathrm{Gr}_I(C))$ the integers found in Local Cohomology, Section 51.22. Observe that t is an upper bound for the minimal number of generators of IC and hence we have $d(\mathrm{Gr}_I(C)) + 1 \leq t$, see discussion in Local Cohomology, Section 51.22. We may and do assume $t \geq 1$ since otherwise the lemma does not say anything. We claim that the lemma is true with

$$d = q(\mathrm{Gr}_I(C))$$

Namely, suppose that c, B, n, ψ_n are as in the statement above. Then we see that

$$n > 2t\max(c, d) \Rightarrow n \geq 2tc + 1 \Rightarrow n \geq 2(d(\mathrm{Gr}_I(C)) + 1)c + 1$$

On the other hand, we have

$$n > 2t\max(c, d) \Rightarrow n > t(c + d) \Rightarrow n \geq q(C) + tc \geq q(\mathrm{Gr}_I(C)) + (d(\mathrm{Gr}_I(C)) + 1)c$$

Hence the assumptions of Algebraization of Formal Spaces, Lemma 88.5.3 are satisfied and we obtain an A-algebra homomorphism $\varphi : B \to C$ which is congruent with ψ_n module $I^{n-(d(\mathrm{Gr}_I(C))+1)c}C$. Since

$$\begin{aligned} n - (d(\mathrm{Gr}_I(C)) + 1)c &= \frac{n}{t} + \frac{(t-1)n}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &\geq \frac{n}{t} + \frac{(d(\mathrm{Gr}_I(C))n}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &> \frac{n}{t} + \frac{d(\mathrm{Gr}_I(C))2tc}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &= \frac{n}{t} + 2d(\mathrm{Gr}_I(C))c - (d(\mathrm{Gr}_I(C)) + 1)c \\ &= \frac{n}{t} + d(\mathrm{Gr}_I(C))c - c \\ &\geq m - c \end{aligned}$$

we see that we have the congruence of φ and ψ_n module $I^{m-c}C$ as desired. □

115.7. Sites and sheaves

0EGM

0931 **Remark 115.7.1** (No map from lower shriek to pushforward)**.** Let U be an object of a site $\mathcal{C}$. For any abelian sheaf $\mathcal{G}$ on $\mathcal{C}/U$ one may wonder whether there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

To construct such a thing is the same as constructing a map $j_U^{-1}j_{U!}\mathcal{G} \to \mathcal{G}$. Note that restriction commutes with sheafification. Thus we can use the presheaf of Modules on Sites, Lemma 18.19.2. Hence it suffices to define for V/U a map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{G}(V/U)$$

compatible with restrictions. It looks like we can take the which is zero on all summands except for the one where φ is the structure morphism $\varphi_0 : V \to U$ where we take 1. However, this isn't compatible with restriction mappings: namely, if $\alpha : V' \to V$ is a morphism of $\mathcal{C}$, then denote V'/U the object of $\mathcal{C}/U$ with structure morphism $\varphi_0' = \varphi_0 \circ \alpha$. We need to check that the diagram

$$\begin{array}{ccc} \bigoplus_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) & \longrightarrow & \mathcal{G}(V/U) \\ \downarrow & & \downarrow \\ \bigoplus_{\varphi' \in \mathrm{Mor}_{\mathcal{C}}(V',U)} \mathcal{G}(V' \xrightarrow{\varphi'} U) & \longrightarrow & \mathcal{G}(V'/U) \end{array}$$

commutes. The problem here is that there may be a morphism $\varphi : V \to U$ different from φ_0 such that $\varphi \circ \alpha = \varphi_0'$. Thus the left vertical arrow will send the summand corresponding to φ into the summand on which the lower horizontal arrow is equal to 1 and almost surely the diagram doesn't commute.

115.8. Cohomology

0BM0 Obsolete lemmas about cohomology.

0EH4 **Lemma 115.8.1.** *Let I be an ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^1(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.1. □

0EH5 **Lemma 115.8.2.** *Let I be an ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Given n define

$$H^1_n = \bigcap\nolimits_{m\geq n} \mathrm{Im}\left(H^1(X, I^n\mathcal{F}_{m+1}) \to H^1(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus H^1_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.2. □

0EI7 **Lemma 115.8.3.** *Let I be a finitely generated ideal of a ring A. Let X be a scheme over $\mathrm{Spec}(A)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of $\mathcal{O}_X$-modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^0(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim \Gamma(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. This is a special case of the more general Cohomology, Lemma 20.35.3. □

06YW **Lemma 115.8.4.** *Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. For any complex of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{G}^\bullet$ there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ such that $f^*\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules for any morphism $f : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ of ringed topoi.*

Proof. This follows from Cohomology on Sites, Lemmas 21.17.11 and 21.18.1. □

06YX **Remark 115.8.5.** This remark used to discuss what we know about pullbacks of K-flat complexes being K-flat or not, but is now obsoleted by Cohomology on Sites, Lemma 21.18.1.

The following lemma computes the cohomology sheaves of the derived limit in a special case.

0A08 **Lemma 115.8.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

(1) *K_n is an object of $D^+(\mathcal{O})$ for all n,*

(2) *for $q \in \mathbf{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \to H^q(K_n)$ is an isomorphism for $n \geq n(q)$,*
(3) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(4) *for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all q.*

Then we have $H^m(R\lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. Let $U \in \mathcal{B}$. For each n there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as K_n is bounded below, see Derived Categories, Lemma 13.21.3. If we fix $m \in \mathbf{Z}$, then we see from our assumption (4) that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption (2) this implies that $H^m(U, K_{n+1}) \to H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \ldots, n(m-d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 19.13.6. Thus we have

$$H^m(U, K) = H^m(R\lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 15.87.10 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an n. Finally, recall that $H^m(K)$ is the sheafification of the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of $\mathcal{B}$ these presheaves have the same values. Therefore assumption (3) guarantees that the sheafifications are the same too. The lemma follows. □

0D7P **Lemma 115.8.7.** *In Simplicial Spaces, Situation 85.3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Simplicial Spaces, Remark 85.4.1. Suppose given strictly full weak Serre subcategories*

$$\mathcal{A} \subset Ab(\mathcal{D}), \quad \mathcal{A}_n \subset Ab(\mathcal{C}_n)$$

Then

(1) *the collection of abelian sheaves $\mathcal{F}$ on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_n$ is in $\mathcal{A}_n$ for all n is a strictly full weak Serre subcategory $\mathcal{A}_{total} \subset Ab(\mathcal{C}_{total})$.*

If a_n^{-1} sends $\mathcal{A}$ into $\mathcal{A}_n$ for all n, then

(2) *a^{-1} sends $\mathcal{A}$ into $\mathcal{A}_{total}$ and*
(3) *a^{-1} sends $D_{\mathcal{A}}(\mathcal{D})$ into $D_{\mathcal{A}_{total}}(\mathcal{C}_{total})$.*

If $R^q a_{n,*}$ sends $\mathcal{A}_n$ into $\mathcal{A}$ for all n, q, then

(4) *$R^q a_*$ sends $\mathcal{A}_{total}$ into $\mathcal{A}$ for all q, and*
(5) *Ra_* sends $D^+_{\mathcal{A}_{total}}(\mathcal{C}_{total})$ into $D^+_{\mathcal{A}}(\mathcal{D})$.*

Proof. The only interesting assertions are (4) and (5). Part (4) follows from the spectral sequence in Simplicial Spaces, Lemma 85.9.3 and Homology, Lemma 12.24.11. Then part (5) follows by considering the spectral sequence associated to the canonical filtration on an object K of $D^+_{\mathcal{A}_{total}}(\mathcal{C}_{total})$ given by truncations. We omit the details. □

01DY **Remark 115.8.8.** This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

01FS **Remark 115.8.9.** This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

0DCV **Remark 115.8.10.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DCW **Remark 115.8.11.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DCX **Remark 115.8.12.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Cohomology on Sites, Lemma 21.31.9.

0DDP **Remark 115.8.13.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDQ **Remark 115.8.14.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDR **Remark 115.8.15.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.100.5.

0DDZ **Remark 115.8.16.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE0 **Remark 115.8.17.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE1 **Remark 115.8.18.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.5 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE2 **Remark 115.8.19.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.6 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0DE3 **Remark 115.8.20.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.102.4.

0EWB **Remark 115.8.21.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.3 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWC **Remark 115.8.22.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.4 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWD **Remark 115.8.23.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.5 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

0EWE **Remark 115.8.24.** This tag used to refer to the special case of Cohomology on Sites, Lemma 21.30.7 pertaining to the situation described in Étale Cohomology, Lemma 59.103.4.

03TU **Remark 115.8.25.** This tag used to be in the chapter on étale cohomology, but is no longer suitable there because of a reorganization. The content of the tag was the following: Étale Cohomology, Lemma 59.77.3 can be used to prove that if $f : X \to Y$ is a separated, finite type morphism of schemes and Y is Noetherian, then $Rf_!$ induces a functor $D_{ctf}(X_{étale}, \Lambda) \to D_{ctf}(Y_{étale}, \Lambda)$. An example of this argument, when Y is the spectrum of a field and X is a curve is given in The Trace Formula, Proposition 64.13.1.

0F5D **Lemma 115.8.26.** *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. There exists a unique functor $f^! : Ab(Y_{étale}) \to Ab(X_{étale})$ such that*

(1) *for any open $j : U \to X$ with $f \circ j$ separated there is a canonical isomorphism $j^! \circ f^! = (f \circ j)^!$, and*
(2) *these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 63.6.3.*

Proof. Immediate consequence of More Étale Cohomology, Lemmas 63.6.1 and 63.6.3. □

0F5E **Proposition 115.8.27.** *Let $f : X \to Y$ be a locally quasi-finite morphism. There exist adjoint functors $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ and $f^! : Ab(Y_{étale}) \to Ab(X_{étale})$ with the following properties*

(1) *the functor $f^!$ is the one constructed in More Étale Cohomology, Lemma 63.6.1,*
(2) *for any open $j : U \to X$ with $f \circ j$ separated there is a canonical isomorphism $f_! \circ j_! = (f \circ j)_!$, and*
(3) *these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 63.3.13.*

Proof. See More Étale Cohomology, Sections 63.4 and 63.6. □

0F5G **Lemma 115.8.28.** *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. For an abelian group A and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have $f^!(\overline{y}_* A) = \prod_{f(\overline{x})=\overline{y}} \overline{x}_* A$.*

Proof. Follows from the corresponding statement in More Étale Cohomology, Lemma 63.6.1. □

0F5K **Lemma 115.8.29.** *Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then $g_! \circ f_! = (g \circ f)_!$ and $f^! \circ g^! = (g \circ f)^!$.*

Proof. Combination of More Étale Cohomology, Lemmas 63.4.12 and 63.6.3. □

115.9. Differential graded algebra

0FU5

0BYY **Lemma 115.9.1.** *Let (A, d) and (B, d) be differential graded algebras. Let N be a differential graded (A, B)-bimodule with property (P). Let M be a differential graded A-module with property (P). Then $Q = M \otimes_A N$ is a differential graded B-module which represents $M \otimes_A^{\mathbf{L}} N$ in $D(B)$ and which has a filtration*

$$0 = F_{-1}Q \subset F_0Q \subset F_1Q \subset \ldots \subset Q$$

by differential graded submodules such that $Q = \bigcup F_pQ$, the inclusions $F_iQ \to F_{i+1}Q$ are admissible monomorphisms, the quotients $F_{i+1}Q/F_iQ$ are isomorphic as differential graded B-modules to a direct sum of $(A \otimes_R B)[k]$.

Proof. Choose filtrations $F_\bullet$ on M and N. Then consider the filtration on $Q = M \otimes_A N$ given by

$$F_n(Q) = \sum\nolimits_{i+j=n} F_i(M) \otimes_A F_j(N)$$

This is clearly a differential graded B-submodule. We see that

$$F_n(Q)/F_{n-1}(Q) = \bigoplus\nolimits_{i+j=n} F_i(M)/F_{i-1}(M) \otimes_A F_j(N)/F_{j-1}(N)$$

for example because the filtration of M is split in the category of graded A-modules. Since by assumption the quotients on the right hand side are isomorphic to direct sums of shifts of A and $A \otimes_R B$ and since $A \otimes_A (A \otimes_R B) = A \otimes_R B$, we conclude that the left hand side is a direct sum of shifts of $A \otimes_R B$ as a differential graded B-module. (Warning: Q does not have a structure of (A, B)-bimodule.) This proves the first statement of the lemma. The second statement is immediate from the definition of the functor in Differential Graded Algebra, Lemma 22.33.2. □

115.10. Simplicial methods

08Q0

01AA **Lemma 115.10.1.** *Assumptions and notation as in Simplicial, Lemma 14.32.1. There exists a section $g : U \to V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V. In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Simplicial, Lemmas 14.32.1 and 14.30.8. □

018W **Lemma 115.10.2.** *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X) \longrightarrow cosk_1 sk_1 \mathrm{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} \mathrm{Mor}(V, \mathrm{cosk}_1\mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) &= \mathrm{Mor}(\mathrm{sk}_1 V, \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V, \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk, the second equality by the adjointness of $i_{1!}$ and sk_1, and the first equality by Simplicial, Definition 14.17.1 where the last X denotes the constant simplicial object with value X. By Simplicial,

Lemma 14.20.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!}\mathrm{sk}_1 V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Simplicial, Lemma 14.21.3. Thus the set above is equal to $\mathrm{Mor}(V \times \Delta[k], X) = \mathrm{Mor}(V, \mathrm{Hom}(\Delta[k], X))$. □

018X **Lemma 115.10.3.** *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$ such that the self products $X \times \ldots \times X$ exist. Let $k \geq 0$ and let $C[k]$ be as in Simplicial, Example 14.5.6. With notation as in Simplicial, Lemma 14.15.2 the canonical map*

$$\mathrm{Hom}(C[k], X)_1 \longrightarrow (cosk_0 sk_0 \mathrm{Hom}(C[k], X))_1$$

is identified with the map

$$\prod\nolimits_{\alpha:[k]\to[1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. This is shown in the proof of Hypercoverings, Lemma 25.7.3. □

115.11. Results on schemes

07VA Lemmas that seem superfluous.

03H1 **Lemma 115.11.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}^n_R$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. This is a special case of More on Morphisms, Theorem 37.53.5. □

01YJ **Lemma 115.11.2.** *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
4. *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 30.12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$.

Consider a coherent sheaf $\mathcal{G}$ as in (3). By Cohomology of Schemes, Lemma 30.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z. Since $\mathcal{Q}$ has property $\mathcal{P}$ we conclude that also $((Z_0 \to X)_*\mathcal{I})^{\oplus r}$ has property $\mathcal{P}$. By (2) we deduce property $\mathcal{P}$

for $(Z_0 \to X)_*\mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 30.12.5 at the appropriate point gives the lemma. Some details omitted. □

01YK **Lemma 115.11.3.** *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
(2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
(3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. This follows from Lemma 115.11.2 in exactly the same way that Cohomology of Schemes, Lemma 30.12.6 follows from Cohomology of Schemes, Lemma 30.12.5. □

01XP **Lemma 115.11.4.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-modules. Assume that*

(1) *X is quasi-compact,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on $\mathcal{F}$ factors through $\mathcal{F}'$.

Proof. In other words we claim that $s^n\mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. In other words, we claim that the quotient map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ becomes zero after multiplying by a power of s. This follows from Properties, Lemma 28.17.3. □

0CC3 **Lemma 115.11.5.** *Let $f : X \to Y$ be a morphism schemes. Assume*

(1) *X and Y are integral schemes,*
(2) *f is locally of finite type and dominant,*
(3) *f is either quasi-compact or separated,*
(4) *f is generically finite, i.e., one of (1) – (5) of Morphisms, Lemma 29.51.7 holds.*

Then there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite locally free of degree $\deg(X/Y)$. In particular, the degrees of the fibres of $f^{-1}(V) \to V$ are bounded by $\deg(X/Y)$.

Proof. We may choose V such that $f^{-1}(V) \to V$ is finite. Then we may shrink V and assume that $f^{-1}(V) \to V$ is flat and of finite presentation by generic flatness (Morphisms, Proposition 29.27.1). Then the morphism is finite locally free by Morphisms, Lemma 29.48.2. Since V is irreducible the morphism has a fixed degree. The final statement follows from this and Morphisms, Lemma 29.57.3. □

115.12. Derived categories of varieties

0GXZ Some lemma which were originally part of the chapter on derived categories of varieties but are no longer needed.

0G04 **Lemma 115.12.1.** *Let k be a field. Let X be a separated scheme of finite type over k which is regular. Let $F : D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X)$ be a k-linear exact functor. Assume for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim(Supp(\mathcal{F})) = 0$ there is an isomorphism of k-vector spaces*

$$\mathrm{Hom}_X(\mathcal{F}, M) = \mathrm{Hom}_X(\mathcal{F}, F(M))$$

functorial in M in $D_{perf}(\mathcal{O}_X)$. Then there exists an automorphism $f : X \to X$ over k which induces the identity on the underlying topological space[1] and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that F and $F'(M) = f^*M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}$ are siblings.

Proof. By Derived Categories of Varieties, Lemma 57.11.2 we conclude that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose support is a closed point there are isomorphisms

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = H^0(X, F(M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$$

functorial in M.

Let $x \in X$ be a closed point and apply the above with $\mathcal{F} = \mathcal{O}_x$ the skyscraper sheaf with value $\kappa(x)$ at x. We find

$$\dim_{\kappa(x)} \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(M_x, \kappa(x)) = \dim_{\kappa(x)} \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(F(M)_x, \kappa(x))$$

for all $p \in \mathbf{Z}$. In particular, if $H^i(M) = 0$ for $i > 0$, then $H^i(F(M)) = 0$ for $i > 0$ by Derived Categories of Varieties, Lemma 57.11.3.

If $\mathcal{E}$ is locally free of rank r, then $F(\mathcal{E})$ is locally free of rank r. This is true because a perfect complex K over $\mathcal{O}_{X,x}$ with

$$\dim_{\kappa(x)} \mathrm{Tor}_i^{\mathcal{O}_{X,x}}(K, \kappa(x)) = \begin{cases} r & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 \end{cases}$$

is equal to a free module of rank r placed in degree 0. See for example More on Algebra, Lemma 15.76.7.

If M is supported on a closed subscheme $Z \subset X$, then $F(M)$ is also supported on Z. This is clear because we will have $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_x = 0$ for $x \notin Z$ and hence the same will be true for $F(M)$ and hence we get the conclusion from Derived Categories of Varieties, Lemma 57.11.3.

In particular $F(\mathcal{O}_x)$ is supported at $\{x\}$. Let $i \in \mathbf{Z}$ be the minimal integer such that $H^i(\mathcal{O}_x) \neq 0$. We know that $i \leq 0$. If $i < 0$, then there is a morphism $\mathcal{O}_x[-i] \to F(\mathcal{O}_x)$ which contradicts the fact that all morphisms $\mathcal{O}_x[-i] \to \mathcal{O}_x$ are zero. Thus $F(\mathcal{O}_x) = \mathcal{H}[0]$ where $\mathcal{H}$ is a skyscraper sheaf at x.

Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module with $\dim(\mathrm{Supp}(\mathcal{G})) = 0$. Then there exists a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_n = \mathcal{G}$$

such that for $n \geq i \geq 1$ the quotient $\mathcal{G}_i/\mathcal{G}_{i-1}$ is isomorphic to $\mathcal{O}_{x_i}$ for some closed point $x_i \in X$. Then we get distinguished triangles

$$F(\mathcal{G}_{i-1}) \to F(\mathcal{G}_i) \to F(\mathcal{O}_{x_i})$$

and using induction we find that $F(\mathcal{G}_i)$ is a coherent sheaf placed in degree 0.

Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. We know that $H^i(F(\mathcal{G})) = 0$ for $i > 0$. To get a contradiction assume that $H^i(F(\mathcal{G}))$ is nonzero for some $i < 0$. We choose i

[1] This often forces f to be the identity, see Varieties, Lemma 33.32.1.

minimal with this property so that we have a morphism $H^i(F(\mathcal{G}))[-i] \to F(\mathcal{G})$ in $D_{perf}(\mathcal{O}_X)$. Choose a closed point $x \in X$ in the support of $H^i(F(\mathcal{G}))$. By More on Algebra, Lemma 15.101.2 there exists an $n > 0$ such that

$$H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \longrightarrow \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

is nonzero. Next, we take $m \geq 1$ and we consider the short exact sequence

$$0 \to \mathfrak{m}_x^m\mathcal{G} \to \mathcal{G} \to \mathcal{G}/\mathfrak{m}_x^m\mathcal{G} \to 0$$

By the above we know that $F(\mathcal{G}/\mathfrak{m}_x^m\mathcal{G})$ is a sheaf placed in degree 0. Hence $H^i(F(\mathfrak{m}_x^m\mathcal{G})) \to H^i(F(\mathcal{G}))$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^i(F(\mathfrak{m}_x^m\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathfrak{m}_x^m\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \\ \downarrow & & \downarrow \\ H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \end{array}$$

Since the left vertical arrow is an isomorphism and the bottom arrow is nonzero, we conclude that the right vertical arrow is nonzero for all $m \geq 1$. On the other hand, by the first paragraph of the proof, we know this arrow is isomorphic to the arrow

$$\mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathfrak{m}_x^m\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \longrightarrow \mathrm{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

However, this arrow is zero for $m \gg n$ by More on Algebra, Lemma 15.103.2 which is the contradiction we're looking for.

Thus we know that F preserves coherent modules. By Derived Categories of Varieties, Lemma 57.12.2 we find F is a sibling to the Fourier-Mukai functor F' given by a coherent $\mathcal{O}_{X\times X}$-module $\mathcal{K}$ flat over X via pr_1 and finite over X via pr_2. Since $F(\mathcal{O}_X)$ is an invertible $\mathcal{O}_X$-module $\mathcal{L}$ placed in degree 0 we see that

$$\mathcal{L} \cong F(\mathcal{O}_X) \cong F'(\mathcal{O}_X) \cong \mathrm{pr}_{2,*}\mathcal{K}$$

Thus by Functors and Morphisms, Lemma 56.7.6 there is a morphism $s : X \to X \times X$ with $\mathrm{pr}_2 \circ s = \mathrm{id}_X$ such that $\mathcal{K} = s_*\mathcal{L}$. Set $f = \mathrm{pr}_1 \circ s$. Then we have

$$\begin{aligned} F'(M) &= R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*K \otimes \mathcal{K}) \\ &= R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^*M \otimes s_*\mathcal{L}) \\ &= R\mathrm{pr}_{2,*}(Rs_*(Lf^*M \otimes \mathcal{L})) \\ &= Lf^*M \otimes \mathcal{L} \end{aligned}$$

where we have used Derived Categories of Schemes, Lemma 36.22.1 in the third step. Since for all closed points $x \in X$ the module $F(\mathcal{O}_x)$ is supported at x, we see that f induces the identity on the underlying topological space of X. We still have to show that f is an isomorphism which we will do in the next paragraph.

Let $x \in X$ be a closed point. For $n \geq 1$ denote $\mathcal{O}_{x,n}$ the skyscaper sheaf at x with value $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$. We have

$$\mathrm{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n}) \cong \mathrm{Hom}_X(\mathcal{O}_{x,m}, F(\mathcal{O}_{x,n})) \cong \mathrm{Hom}_X(\mathcal{O}_{x,m}, f^*\mathcal{O}_{x,n} \otimes \mathcal{L})$$

functorially with respect to $\mathcal{O}_X$-module homomorphisms between the $\mathcal{O}_{x,n}$. (The first isomorphism exists by assumption and the second isomorphism because F and F' are siblings.) For $m \geq n$ we have $\mathcal{O}_{X,x}/\mathfrak{m}^n = \mathrm{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n})$ via the action on $\mathcal{O}_{x,n}$ we conclude that $f^\sharp : \mathcal{O}_{X,x}/\mathfrak{m}_x^n \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is bijective for all n.

Thus f induces isomorphisms on complete local rings at closed points and hence is étale (Étale Morphisms, Lemma 41.11.3). Looking at closed points we see that $\Delta_f : X \to X \times_{f,X,f} X$ (which is an open immersion as f is étale) is bijective hence an isomorphism. Hence f is a monomorphism. Finally, we conclude f is an isomorphism as Descent, Lemma 35.25.1 tells us it is an open immersion. □

115.13. Representability in the regular proper case

0FYI This section is obsolete because we improved Derived Categories of Varieties, Theorem 57.6.3 to apply to all proper schemes over a field (whereas before we only proved it for projective schemes over a field).

0FYJ **Lemma 115.13.1.** *Let $f : X' \to X$ be a proper birational morphism of integral Noetherian schemes with X regular. The map $\mathcal{O}_X \to Rf_*\mathcal{O}_{X'}$ canonically splits in $D(\mathcal{O}_X)$.*

The proof given here follows the argument given in [**MS20**, Remark 3.4]

Proof. Set $E = Rf_*\mathcal{O}_{X'}$ in $D(\mathcal{O}_X)$. Observe that E is in $D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 36.11.3. By Derived Categories of Schemes, Lemma 36.11.8 we find that E is a perfect object of $D(\mathcal{O}_X)$. Since $\mathcal{O}_{X'}$ is a sheaf of algebras, we have the relative cup product $\mu : E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to E$ by Cohomology, Remark 20.28.7. Let $\sigma : E \otimes E^\vee \to E^\vee \otimes E$ be the commutativity constraint on the symmetric monoidal category $D(\mathcal{O}_X)$ (Cohomology, Lemma 20.50.6). Denote $\eta : \mathcal{O}_X \to E \otimes E^\vee$ and $\epsilon : E^\vee \otimes E \to \mathcal{O}_X$ the maps constructed in Cohomology, Example 20.50.7. Then we can consider the map

$$E \xrightarrow{\eta\otimes 1} E \otimes E^\vee \otimes E \xrightarrow{\sigma\otimes 1} E^\vee \otimes E \otimes E \xrightarrow{1\otimes\mu} E^\vee \otimes E \xrightarrow{\epsilon} \mathcal{O}_X$$

We claim that this map is a one sided inverse to the map in the statement of the lemma. To see this it suffices to show that the composition $\mathcal{O}_X \to \mathcal{O}_X$ is the identity map. This we may do in the generic point of X (or on an open subscheme of X over which f is an isomorphism). In this case $E = \mathcal{O}_X$ and μ is the usual multiplication map and the result is clear. □

0FYK **Lemma 115.13.2.** *Let X be a proper scheme over a field k which is regular. Let $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent*

1. *$K \in D^b_{Coh}(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$, and*
2. *$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Ext}^i_X(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.*

Proof. The equality in (1) holds by Derived Categories of Schemes, Lemma 36.11.8. The implication (1) ⇒ (2) follows from Derived Categories of Varieties, Lemma 57.5.3. The implication (2) ⇒ (1) follows from More on Morphisms, Lemma 37.69.6. □

0FYL **Lemma 115.13.3.** *Let X be a proper scheme over a field k which is regular.*

1. *Let $F : D_{perf}(\mathcal{O}_X)^{opp} \to Vect_k$ be a k-linear cohomological functor such that*
$$\sum\nolimits_{n\in\mathbf{Z}} \dim_k F(E[n]) < \infty$$
for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(E, K)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

(2) *Let $G : D_{perf}(\mathcal{O}_X) \to Vect_k$ be a k-linear homological functor such that*

$$\sum\nolimits_{n\in\mathbf{Z}} \dim_k G(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \mathrm{Hom}_X(K, E)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

Proof. This follows from Derived Categories of Varieties, Theorem 57.6.3 and Lemma 57.6.4. We also give another proof below.

Proof of (1). The derived category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{perf}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 36.3.1, Theorem 36.15.3, and Proposition 36.17.1. By Derived Categories of Varieties, Lemma 57.6.2 we may assume $F(E) = \mathrm{Hom}_X(E, K)$ for some $K \in \mathrm{Ob}(D_{QCoh}(\mathcal{O}_X))$. Then it follows that K is in $D^b_{Coh}(\mathcal{O}_X)$ by Lemma 115.13.2.

Proof of (2). Consider the contravariant functor $E \mapsto E^\vee$ on $D_{perf}(\mathcal{O}_X)$, see Cohomology, Lemma 20.50.5. This functor is an exact anti-self-equivalence of $D_{perf}(\mathcal{O}_X)$. Hence we may apply part (1) to the functor $F(E) = G(E^\vee)$ to find $K \in D_{perf}(\mathcal{O}_X)$ such that $G(E^\vee) = \mathrm{Hom}_X(E, K)$. It follows that $G(E) = \mathrm{Hom}_X(E^\vee, K) = \mathrm{Hom}_X(K^\vee, E)$ and we conclude that taking $K^\vee$ works. □

115.14. Functor of quotients

08J4

082R **Lemma 115.14.1.** *Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $q_i : \mathcal{F} \to \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{Q}_1$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that there exists a factorization*

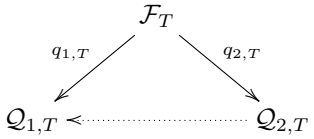

Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\mathrm{Ker}(q_2)$ is a finite type $\mathcal{O}_X$-module and X quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Apply Flatness on Spaces, Lemma 77.8.2 to the map $\mathrm{Ker}(q_2) \to \mathcal{Q}_1$. □

115.15. Spaces and fpqc coverings

0ARG The material here was made obsolete by Gabber's argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 66.17.

03W9 **Lemma 115.15.1.** *Let S be a scheme. Let X be an algebraic space over S. Let $\{f_i : T_i \to T\}_{i\in I}$ be a fpqc covering of schemes over S. Then the map*

$$\mathrm{Mor}_S(T, X) \longrightarrow \prod\nolimits_{i\in I} \mathrm{Mor}_S(T_i, X)$$

is injective.

Proof. Immediate consequence of Properties of Spaces, Proposition 66.17.1. □

03WA **Lemma 115.15.2.** *Let S be a scheme. Let X be an algebraic space over S. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 65.12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 66.17.1. □

03WB **Lemma 115.15.3.** *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated over S, then X satisfies the sheaf condition for the fpqc topology.*

Proof. Immediate consequence of the general Properties of Spaces, Proposition 66.17.1. □

03WC **Remark 115.15.4.** This remark used to discuss to what extend the original proof of Lemma 115.15.3 (of December 18, 2009) generalizes.

115.16. Very reasonable algebraic spaces

07T6 Material that is somewhat obsolete.

03IN **Lemma 115.16.1.** *Let S be a scheme. Let X be a reasonable algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.8.6).*

Proof. Follows from the definitions and Decent Spaces, Lemma 68.12.3. □

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \to X$, then X is very reasonable, see Decent Spaces, Lemma 68.4.7.

03I9 **Lemma 115.16.2.** *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. □

03IA **Lemma 115.16.3.** *Let S be a scheme. Let X be an algebraic space over S. If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 68.5.2. □

03IB **Lemma 115.16.4.** *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Decent Spaces, Lemma 68.5.1. □

03JF **Lemma 115.16.5.** *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Y \to X$ be a representable morphism. If X is very reasonable, so is Y.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 68.5.3. □

03IC **Remark 115.16.6.** Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 65.14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact,

hence X is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 26.21.4. Let $0_1, 0_2$ be the two zeros of U. Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

03K7 **Lemma 115.16.7.** *Let S be a scheme. Let X be a very reasonable algebraic space over S. There exists a set of schemes U_i and morphisms $U_i \to X$ such that*

(1) *each U_i is a quasi-compact scheme,*
(2) *each $U_i \to X$ is étale,*
(3) *both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and*
(4) *the morphism $\coprod U_i \to X$ is surjective (and étale).*

Proof. Decent Spaces, Definition 68.6.1 says that there exist $U_i \to X$ such that (2), (3) and (4) hold. Fix i, and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \to U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i-invariant open (see Groupoids, Lemma 39.19.2). Hence W' is a quasi-compact scheme, $W' \to X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \to W'$ are quasi-compact. This means the family of $W' \to X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. □

115.17. Obsolete lemmas on algebraic spaces

0D45 Lemmas that seem superfluous or are no longer used in the text.

07V2 **Lemma 115.17.1.** *In Cohomology of Spaces, Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is surjective.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 69.16.7 but now is a consequence of it. □

07V3 **Lemma 115.17.2.** *In Cohomology of Spaces, Situation 69.16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally closed.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 69.16.7 but now is a consequence of it. □

0AKU **Remark 115.17.3.** This tag used to refer to an equation in the proof of Formal Spaces, Lemma 87.20.4.

0AKV **Remark 115.17.4.** This tag used to refer to an equation in the proof of Formal Spaces, Lemma 87.20.4.

115.18. Obsolete lemmas on algebraic stacks

0G2T Lemmas that seem superfluous or are no longer used in the text.

0CXS **Lemma 115.18.1.** *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ having (RS*). Let x be an object of $\mathcal{X}$ over an affine scheme U of finite type over S. Let $u_n \in U$, $n \geq 1$ be pairwise distinct*

finite type points such that x is not versal at u_n for all n. After replacing u_n by a subsequence, there exist morphisms

$$x \to x_1 \to x_2 \to \ldots \quad \text{in } \mathcal{X} \text{ lying over } \quad U \to U_1 \to U_2 \to \ldots$$

over S such that

(1) *for each n the morphism $U \to U_n$ is a first order thickening,*
(2) *for each n we have a short exact sequence*

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0$$

with $U_0 = U$ for $n = 1$,
(3) *for each n there does* **not** *exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \to x$ such that the composition*

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \to x_n} x_n|_W \xrightarrow{\alpha} x$$

is the canonical morphism $x|_{U \cap W} \to x$.

Proof. This lemma was originally used in the proof of a criterion for openness of versality (Artin's Axioms, Lemma 98.20.3) but it got replaced by Artin's Axioms, Lemma 98.20.1 from which it readily follows. Namely, after replacing u_n, $n \geq 1$ by a subsequence we may and do assume that there are no specializations among these points, see Properties, Lemma 28.5.11. Then we can apply Artin's Axioms, Lemma 98.20.1 to finish the proof. □

115.19. Variants of cotangent complexes for schemes

08T5 This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 92.25 for applications.

Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

08T6 (115.19.0.1)

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{i} & A \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

of schemes where

(1) U is an open subscheme of X,
(2) V is an open subscheme of Y, and
(3) there exists an isomorphism $A = V \times \mathrm{Spec}(P)$ over V where P is a polynomial algebra over $\mathbf{Z}$ (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V. Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (115.19.0.1), is often denoted $U \to A$ where it is understood that (a) U is an open subscheme of X, (b) $U \to A$ is a morphism over Y, (c) the image of the structure morphism $A \to Y$ is an open $V \subset Y$, and (d) $A \to V$ is an affine space. We'll write $U \to A/V$ to indicate $V \subset Y$

is the image of $A \to Y$. Recall that X_{Zar} denotes the small Zariski site X. There are forgetful functors

$$\mathcal{C}_{X/Y} \to X_{Zar},\ (U \to A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar},\ (U \to A/V) \mapsto V.$$

08T7 **Lemma 115.19.1.** *Let $X \to Y$ be a morphism of schemes.*

(1) *The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar}.*
(2) *The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar}.*
(3) *The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.*

Proof. Ad (1). Given an object $U \to A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \to U$ of X_{Zar} consider the object $i' : U' \to A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \to U$. The morphism $(U' \to A) \to (U \to A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar}.

Ad (2). Given an object $U \to A/V$ and $V' \to V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \to A') \to (U \to A)$ over $V' \to V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \to A/V$ and a morphism $(U', V') \to (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \to A'/V') \to (U \to A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. □

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 8.10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

(1) $U = \bigcup U_i$, and
(2) $A_i = A$ for all i.

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \to Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to A) \mapsto \mathcal{O}(A)$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to A) \mapsto \mathcal{O}(U)$.
(3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \to A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

08T8 (115.19.1.1)
$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) & \xrightarrow{i} & (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{Zar}), \mathcal{O}_X) & & \end{array}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$.

08TA **Remark 115.19.2.** We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar}. There is a third topology $\tau_{X \to Y}$ where a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$

and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 115.19.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \to (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)$$

(recall that τ_X is our "default" topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$\begin{array}{ccccc} Sh(X_{Zar}) & \xleftarrow[\pi]{} & Sh(\mathcal{C}_{X/Y}) & \longleftarrow & Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y}) \\ \downarrow{\scriptstyle f} & & & & \downarrow \\ Sh(Y_{Zar}) & & \longleftarrow & & Sh(\mathcal{C}_{X/Y}, \tau_Y) \end{array}$$

is **not** commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X\to Y})$, but we can certainly find examples where such a sheaf pulls back to zero on X. Note that any presheaf $\mathcal{F}$ on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \to A/V)$ the set $\mathcal{F}(V)$. Even if $\mathcal{F}$ happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of $\mathcal{F}$ to $Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}}_Y$. But what is true is that there is a map $\underline{\mathcal{F}} \to \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

115.20. Deformations and obstructions of flat modules

08VZ In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ. This material is obsolete due to the improved discussion in Quot, Section 99.6.

Our setup will be the following. We assume given

(1) a ring Λ,
(2) an algebraic space X over Λ,
(3) a Λ-algebra A, set $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$, and
(4) a finitely presented $\mathcal{O}_{X_A}$-module $\mathcal{F}$ flat over A.

In this situation we will consider all possible surjections

$$0 \to I \to A' \to A \to 0$$

where A' is a Λ-algebra whose kernel I is an ideal of square zero in A'. Given A' we obtain a first order thickening $X_A \to X_{A'}$ of algebraic spaces over $\mathrm{Spec}(\Lambda)$. For each of these we consider the problem of lifting $\mathcal{F}$ to a finitely presented module $\mathcal{F}'$ on $X_{A'}$ flat over A'. We would like to replicate the results of Deformation Theory, Lemma 91.12.1 in this setting.

To be more precise let $Lift(\mathcal{F}, A')$ denote the category of pairs $(\mathcal{F}', \alpha)$ where $\mathcal{F}'$ is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \to \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}'_1, \alpha_1) \to (\mathcal{F}'_2, \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \to \mathcal{F}'_2$ which are compatible with α_1 and α_2. The set of isomorphism classes of $Lift(\mathcal{F}, A')$ is denoted $\mathrm{Lift}(\mathcal{F}, A')$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{étale}$ flat over A. We introduce the category $Lift(\mathcal{G}, A')$ of pairs $(\mathcal{G}', \beta)$ where $\mathcal{G}'$ is a sheaf of $\mathcal{O}_X \otimes_\Lambda A'$-modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \to \mathcal{G}$.

08W0 **Lemma 115.20.1.** *Notation and assumptions as above. Let $p : X_A \to X$ denote the projection. Given A' denote $p' : X_{A'} \to X$ the projection. The functor p'_* induces an equivalence of categories between*

(1) *the category $Lift(\mathcal{F}, A')$, and*
(2) *the category $Lift(p_*\mathcal{F}, A')$.*

Proof. FIXME. □

Let $\mathcal{H}$ be a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules on $\mathcal{C}_{X/\Lambda}$ flat over A. We introduce the category $Lift_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs $(\mathcal{H}', \gamma)$ where $\mathcal{H}'$ is a sheaf of $\mathcal{O} \otimes_\Lambda A'$-modules flat over A' and $\gamma : \mathcal{H}' \otimes_A A' \to \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_\Lambda A$-modules.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{étale}$ flat over A. Consider the morphisms i and π of Cotangent, Equation (92.27.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_\Lambda A$-modules. Denote $i_*\underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules.

08W1 **Lemma 115.20.2.** *Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between*

(1) *the category $Lift_{\mathcal{O}}(i_*\underline{\mathcal{G}}, A')$, and*
(2) *the category $Lift(\mathcal{G}, A')$.*

Proof. FIXME. □

08W2 **Lemma 115.20.3.** *Notation and assumptions as in Lemma 115.20.2. Consider the object*

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_\Lambda A)$. Given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have

(1) *The category $Lift(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in \mathrm{Ext}^2_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$ is zero.*
(2) *If $Lift(\mathcal{G}, A')$ is nonempty, then $Lift(\mathcal{G}, A')$ is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*
(3) *Given a lift $\mathcal{G}'$, the set of automorphisms of $\mathcal{G}'$ which pull back to $id_{\mathcal{G}}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*

Proof. FIXME. □

Finally, we put everything together as follows.

08W3 **Proposition 115.20.4.** *With Λ, X, A, $\mathcal{F}$ as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have*

(1) *The category $Lift(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in \mathrm{Ext}^2_{X_A}(L, \mathcal{F} \otimes_A I)$ is zero.*
(2) *If $Lift(\mathcal{F}, A')$ is nonempty, then $Lift(\mathcal{F}, A')$ is principal homogeneous under $\mathrm{Ext}^1_{X_A}(L, \mathcal{F} \otimes_A I)$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $\mathrm{Ext}^0_{X_A}(L, \mathcal{F} \otimes_A I)$.*

Proof. FIXME. □

08W4 **Lemma 115.20.5.** *In the situation of Proposition 115.20.4, if $X \to \mathrm{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.*

Proof. FIXME. □

115.21. The stack of coherent sheaves in the non-flat case

0CXY In Quot, Theorem 99.5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we modify the method of proof based on ideas from derived algebraic geometry to get around the flatness hypothesis. An entirely different method is used in Quot, Section 99.6 to get exactly the same result; this is why the method from this section is obsolete.

The only step in the proof of Quot, Theorem 99.5.12 which uses flatness is in the application of Quot, Lemma 99.5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 98.24. The proof of the lemma relies on Deformation Theory, Lemmas 91.12.1 and 91.12.5 from Deformation Theory, Section 91.12. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 91.12.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 91.11.7 and 91.11.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S. Denote $X \otimes^{\mathbf{L}} A$ a "derived base change" of X to A and denote $i : X_A \to X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X \otimes^{\mathbf{L}} A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X \otimes^{\mathbf{L}} A}) = \mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \to \mathrm{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A-flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\mathrm{Ext}^i_{X \otimes^{\mathbf{L}} A}(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where $\mathcal{F}'$ is a flat deformation of $\mathcal{F}$. By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Quot, Lemma 99.5.11 with the only change being that one replaces the first occurrence of $\mathcal{F}$ by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a "direct" construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of $\mathcal{F}$. In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the

ext groups $\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

09DN **Remark 115.21.1** (Direct construction)**.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let U be another algebraic space over B. Denote $q : X \times_B U \to U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \to L_{X \times_B U/B} \to E \to Lq^*L_{U/B}[1]$$

of Cotangent, Section 92.28. For any sheaf $\mathcal{F}$ of $\mathcal{O}_{X \times_B U}$-modules we have the Atiyah class

$$\mathcal{F} \to L_{X \times_B U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

see Cotangent, Section 92.19. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \to \mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} E[1] \to E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow Lq^*L_{U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\ \uparrow{\scriptstyle e_{\mathcal{F}}} & & \uparrow{\scriptstyle Atiyah} & & \uparrow{\scriptstyle =} \\ E(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \end{array}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

09DP **Remark 115.21.2** (Construction of obstruction class)**.** With notation as in Remark 115.21.1 let $i : U \to U'$ be a first order thickening of U over B. Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B'. The fundamental triangle

$$Li^*L_{U'/B} \to L_{U/B} \to L_{U/U'} \to Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \to \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \to \mathcal{I}[1]$. Combined with the map $e_{\mathcal{F}}$ of the previous remark we obtain

$$(\mathrm{id}_{\mathcal{F}} \otimes Lq^*e_{U'}) \cup e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \mathrm{Ext}^2_{\mathcal{O}_{X \times_B U}}(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

09DQ **Lemma 115.21.3.** *In the situation of Remark 115.21.2 assume that $\mathcal{F}$ is flat over U. Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$-module $\mathcal{F}'$ flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.*

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 91.11.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{U'} \to \mathcal{O}_U \to 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to q^*\mathcal{I} \to \mathcal{J} \to 0$$

where the $\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\mathrm{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with $\mathcal{F}$ we obtain the exact sequence

$$\mathcal{F} \otimes_{\mathcal{O}} \mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to \mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J} \to 0$$

(Note that the roles of the letters $\mathcal{I}$ and $\mathcal{J}$ are reversed relative to the notation in Deformation Theory, Lemma 91.11.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\overline{z} = (\overline{x}, \overline{u}) : \mathrm{Spec}(k) \to X \times_B U$. Set $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{U,\overline{u}}$, and $C = \mathcal{O}_{\overline{z}}$. By Cotangent, Lemma 92.28.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\overline{z}} = \mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\overline{z}} \otimes_B \mathcal{I}_{\overline{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 91.11.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \to \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanishing of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J}$. A consideration of the spectral sequence

$$\mathrm{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \mathrm{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 91.11.8 a verification we omit. □

09DR **Lemma 115.21.4.** *In Quot, Situation 99.5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = Coh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 98.13.1).*

Proof (sketch). Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 29.16.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. We will

use Artin's Axioms, Lemma 98.24.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

Choose $E(\mathcal{F})$ and $e_{\mathcal{F}}$ as in Remark 115.21.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\mathrm{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A-module M. Using this and using Deformation Theory, Lemma 91.11.7 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 115.21.3 given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 75.23.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D^-_{Coh}; hint: use Cotangent, Lemma 92.5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 98.24.2. Finally, condition (iv) of Artin's Axioms, Lemma 98.24.3 holds by a variant of Deformation Theory, Lemma 91.12.5 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 98.24.4 applies and the lemma is proved. □

0CXZ **Theorem 115.21.5.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and separated. Then $Coh_{X/B}$ is an algebraic stack over S.*

Proof. This theorem is a copy of Quot, Theorem 99.6.1. The reason we have this copy here is that with the material in this section we get a second proof (as discussed at the beginning of this section). Namely, we argue exactly as in the proof of Quot, Theorem 99.5.12 except that we substitute Lemma 115.21.4 for Quot, Lemma 99.5.11. □

115.22. Modifications

0AS3 Here is a obsolete result on the category of Algebraization of Formal Spaces, Equation (88.30.0.1). Please visit Algebraization of Formal Spaces, Section 88.30 for the current material.

0AE4 **Lemma 115.22.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The category of Algebraization of Formal Spaces, Equation (88.30.0.1) for A is equivalent to the category Algebraization of Formal Spaces, Equation (88.30.0.1) for the henselization A^h of A.*

Proof. This is a special case of Algebraization of Formal Spaces, Lemma 88.30.3. □

The following lemma on rational singularities is no longer needed in the chapter on resolving surface singularities.

0B50 **Lemma 115.22.2.** *In Resolution of Surfaces, Situation 54.9.1. Let M be a finite reflexive A-module. Let $M \otimes_A \mathcal{O}_X$ denote the pullback of the associated $\mathcal{O}_S$-module. Then $M \otimes_A \mathcal{O}_X$ maps onto its double dual.*

Proof. Let $\mathcal{F} = (M \otimes_A \mathcal{O}_X)^{**}$ be the double dual and let $\mathcal{F}' \subset \mathcal{F}$ be the image of the evaluation map $M \otimes_A \mathcal{O}_X \to \mathcal{F}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

Since X is normal, the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma 28.12.5). Hence $\mathcal{Q}_x = 0$ for such points by More on Algebra, Lemma 15.23.3. Thus $\mathcal{Q}$ is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma 30.9.10. We obtain the exact sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{Q}) \to 0$$

because $\mathcal{F}'$ is generated by global sections (Resolution of Surfaces, Lemma 54.9.2). Since $X \to \mathrm{Spec}(A)$ is an isomorphism over the complement of the closed point, and since M is reflexive, we see that the maps

$$M \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F})$$

induce isomorphisms after localization at any nonmaximal prime of A. Hence these maps are isomorphisms by More on Algebra, Lemma 15.23.13 and the fact that reflexive modules over normal rings have property (S_2) (More on Algebra, Lemma 15.23.18). Thus we conclude that $\mathcal{Q} = 0$ as desired. □

115.23. Intersection theory

0AYK

0FIG **Lemma 115.23.1.** *Let $b : X' \to X$ be the blowing up of a smooth projective scheme over a field k in a smooth closed subscheme $Z \subset X$. Picture*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{i} & X \end{array}$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Then $[Lb^*\mathcal{O}_Z] = [\mathcal{O}_E] \cdot \alpha''$ in $K_0(X')$ for some $\alpha'' \in K_0(X')$.

Proof. The schemes X, X', E, Z are smooth and projective over k and hence we have $K'_0(X) = K_0(X) = K_0(\mathit{Vect}(X)) = K_0(D^b_{Coh}(X)))$ and similarly for the other 3. See Derived Categories of Schemes, Lemmas 36.38.1, 36.38.4, and 36.38.5. We will switch between these versions at will in this proof. Consider the short exact sequence

$$0 \to \mathcal{F} \to \pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'} \to 0$$

of finite locally free $\mathcal{O}_E$-modules defining $\mathcal{F}$. Observe that $\mathcal{C}_{E/X'} = \mathcal{O}_{X'}(-E)|_E$ is the restriction of the invertible $\mathcal{O}_X$-module $\mathcal{O}_{X'}(-E)$. Let $\alpha \in K_0(X)$ be an element such that $i^*\alpha = [\mathcal{C}_{Z/X}]$ in $K_0(Z)$. Let $\alpha' = b^*\alpha - [\mathcal{O}_{X'}(-E)]$. Then $j^*\alpha' = [\mathcal{F}]$. We deduce that $j^*\lambda^i(\alpha') = [\wedge^i(\mathcal{F})]$ by Weil Cohomology Theories, Lemma 45.13.1. This means that $[\mathcal{O}_E] \cdot \alpha' = [\wedge^i \mathcal{F}]$ in $K_0(X)$, see Derived Categories of Schemes, Lemma 36.38.8. Let r be the maximum codimension of an irreducible component of Z in X. A computation which we omit shows that $H^{-i}(Lb^*\mathcal{O}_Z) =$

$\wedge^i\mathcal{F}$ for $i \geq 0, 1, \ldots, r-1$ and zero in other degrees. It follows that in $K_0(X)$ we have

$$\begin{aligned}[Lb^*\mathcal{O}_Z] &= \sum\nolimits_{i=0,\ldots,r-1}(-1)^i[\wedge^i\mathcal{F}] \\ &= \sum\nolimits_{i=0,\ldots,r-1}(-1)^i[\mathcal{O}_E]\lambda^i(\alpha') \\ &= [\mathcal{O}_E]\left(\sum\nolimits_{i=0,\ldots,r-1}(-1)^i\lambda^i(\alpha')\right)\end{aligned}$$

This proves the lemma with $\alpha'' = \sum_{i=0,\ldots,r-1}(-1)^i\lambda^i(\alpha')$. □

02TL **Lemma 115.23.2.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of a. Let $f \in R(X)^*$. In this case $i^*div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Special case of Chow Homology, Lemma 42.30.1. □

02SA **Remark 115.23.3.** This remark used to say that it wasn't clear whether the arrows of Chow Homology, Lemma 42.23.2 were isomorphisms in general. However, we've now found a proof of this fact.

02SY **115.23.4. Blowing up lemmas.** In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blowups. These lemmas can be found in [**Ful98**, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 115.23.11 is a composition of infinitely many blowups, but over any given quasi-compact open $W \subset X$ one needs only finitely many blowups (and this is the result of loc. cit.).

02SZ **Lemma 115.23.5.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $D \subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n = \dim_\delta(X) = \dim_\delta(Y)$ and f dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_*[f^{-1}(D)]_{n-1} = [D]_{n-1}$.

Proof. Immediate from Chow Homology, Lemma 42.26.3 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. □

02T0 **Lemma 115.23.6.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral with $\dim_\delta(X) = n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. There exists a projective morphism*

$$\pi : X' \longrightarrow X$$

such that

(1) *X' is integral,*
(2) *$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism,*
(3) *there exist effective Cartier divisors $D, E \subset X'$ such that*

$$\pi^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

(4) *the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D \otimes (1_E)^{-1}$ (see Divisors, Definition 31.14.1),*
(5) *we have*
$$\pi_*([D]_{n-1} - [E]_{n-1}) = div_{\mathcal{L}}(s)$$
in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s, see Divisors, Definition 31.23.10. By Divisors, Lemma 31.34.6 we get (2), (3), and (4). By Divisors, Lemma 31.32.9 we get (1). By Divisors, Lemma 31.32.13 the morphism π is projective. We still have to prove (5). By Chow Homology, Lemma 42.26.3 we have
$$\pi_*(\mathrm{div}_{\mathcal{L}'}(s')) = \mathrm{div}_{\mathcal{L}}(s).$$
Hence it suffices to show that $\mathrm{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Divisors, Lemma 31.27.5. □

02T1 **Definition 115.23.7.** Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = n - 1$. The *ϵ-invariant* of this situation is
$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$
where n_Z, resp. m_Z is the coefficient of Z in the $(n-1)$-cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

02T2 **Lemma 115.23.8.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$. Assume $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$. Then there exists a morphism $b : X' \to X$, and Cartier divisors D_1', D_2', E on X' with the following properties*

(1) *X' is integral,*
(2) *b is projective,*
(3) *b is the blowup of X in the closed subscheme Z,*
(4) *$E = b^{-1}(Z)$,*
(5) *$b^{-1}(D_1) = D_1' + E$, and $b^{-1}D_2 = D_2' + E$,*
(6) *$\dim_\delta(D_1' \cap D_2') \leq n - 2$, and if $Z = D_1 \cap D_2$ then $D_1' \cap D_2' = \emptyset$,*
(7) *for every integral closed subscheme W' with $\dim_\delta(W') = n - 1$ we have*
   (a) *if $\epsilon_{W'}(D_1', E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and*
   $$\epsilon_{W'}(D_1', E) < \epsilon_W(D_1, D_2),$$
   (b) *if $\epsilon_{W'}(D_2', E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and*
   $$\epsilon_{W'}(D_2', E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is equal to $\mathcal{I}_z$. Let $b : X' \to X$ be the blowup of X in Z. (So Zariski locally around Z it is the blowup of X in $\mathcal{I}$.) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 31.32.4. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence

$D_1 = D_1' + E$ for some effective Cartier divisor $D_1' \subset X'$, see Divisors, Lemma 31.13.8. Similarly $D_2 = D_2' + E$. This takes care of assertions (1) – (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z, then b is an isomorphism at the generic point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n-1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X.

Thus we may assume that $X = \mathrm{Spec}(A)$ with A a Noetherian domain, and $D_1 = \mathrm{Spec}(A/a)$, $D_2 = \mathrm{Spec}(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blowup by two patches, namely $U = \mathrm{Spec}(A[s]/(as - b))$ and $V = \mathrm{Spec}(A[t]/(bt - a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$ which maps s to t^{-1}. The effective Cartier divisor E is described by $\mathrm{Spec}(A[s]/(as - b, a)) \subset U$ and $\mathrm{Spec}(A[t]/(bt - a, b)) \subset V$. The closed subscheme D_1' corresponds to $\mathrm{Spec}(A[t]/(bt - a, t)) \subset U$. The closed subscheme D_2' corresponds to $\mathrm{Spec}(A[s]/(as - b, s)) \subset V$. Since "$ts = 1$" we see that $D_1' \cap D_2' = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as - b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A. Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_\mathfrak{p}) = 1$ as well. The final assertion to be shown is that

$$\mathrm{ord}_{A_\mathfrak{p}}(a)\mathrm{ord}_{A_\mathfrak{p}}(b) > \mathrm{ord}_{B_\mathfrak{q}}(a)\mathrm{ord}_{B_\mathfrak{q}}(s)$$

where $B = A[s]/(as - b)$. By Algebra, Lemma 10.124.1 we have $\mathrm{ord}_{A_\mathfrak{p}}(x) \geq \mathrm{ord}_{B_\mathfrak{q}}(x)$ for $x = a, b$. Since $\mathrm{ord}_{B_\mathfrak{q}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. □

02T3 **Definition 115.23.9.** Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given a function $I \to \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The *sum of the effective Cartier divisors* $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 31.13.6.

02T4 **Lemma 115.23.10.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given $n_i \geq 0$ for $i \in I$. Then*

$$[D]_{n-1} = \sum\nolimits_i n_i [D_i]_{n-1}$$

in $Z_{n-1}(X)$.

Proof. Since we are proving an equality of cycles we may work locally on X. Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Chow Homology, Lemma 42.24.2 we see that $D_1 = \mathrm{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D. Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Divisors, Lemma 31.27.5. □

02T5 **Lemma 115.23.11.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist*

(1) *an open subscheme* $U \subset X$ *with* $\dim_\delta(X \setminus U) \leq d-3$,
(2) *a morphism* $b : U' \to U$, *and*
(3) *effective Cartier divisors* $\{D'_j\}_{j\in J}$ *on* U'

with the following properties:

(1) b *is proper morphism* $b : U' \to U$,
(2) U' *is integral,*
(3) b *is an isomorphism over the complement of the union of the pairwise intersections of the* $D_i|_U$,
(4) $\{D'_j\}_{j\in J}$ *is a locally finite collection of effective Cartier divisors on* U',
(5) $\dim_\delta(D'_j \cap D'_{j'}) \leq d-2$ *if* $j \neq j'$, *and*
(6) $b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ *for certain* $n_{ij} \geq 0$.

Moreover, if X *is quasi-compact, then we may assume* $U = X$ *in the above.*

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \ldots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i\in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum\nolimits_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \ldots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i\in I_n})$ let X_{n+1} be the blowup of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\}\in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\}\in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blowup of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 115.23.8 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 31.13.8 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1}. In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 115.23.8. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$, $i, i' \in I_n$ by part (6) of Lemma 115.23.8. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

(115.23.11.1)

02T6 $$\epsilon(X, \{D_i\}_{i\in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d-1, \{i,i'\} \in P(I)\}$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i\in I_n})$ is zero. Namely, if not

then by Lemma 115.23.8 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i\in I}) = \epsilon(X_0, \{D_{0,i}\}_{i\in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i\in I_1}) > \ldots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i\in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$ and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d-2$ for all pairs $\{i, i'\} \in P(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get and infinite sequence of blowups with centers dominating a fixed point of X. In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{array}{ccccccc}
X = X_0 & & & & & & \\
\uparrow{\scriptstyle j_0} & & & & & & \\
U_0 & \xleftarrow{b_0} & X_1 & & & & \\
 & & \uparrow{\scriptstyle j_1} & & & & \\
 & & U_1 & \xleftarrow{b_1} & X_2 & & \\
 & & & & \uparrow{\scriptstyle j_2} & & \\
 & & & & U_2 & \xleftarrow{b_2} & X_3
\end{array}$$

Each of the morphisms $j_n : U_n \to X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \to U_n$ will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i\in I_n}$ on X_n. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum\nolimits_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in P(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has "good" irreducible components which have δ-dimension $d-2$ and "bad" irreducible components which have δ-dimension $d-1$. Let us set

$$\text{Bad}(i, i') = \bigcup\nolimits_{W\subset D_{n,i}\cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-1} W$$

and similarly

$$\text{Good}(i, i') = \bigcup\nolimits_{W\subset D_{n,i}\cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \text{Bad}(i, i') \cup \text{Good}(i, i')$ and moreover we have $\dim_\delta(\text{Bad}(i, i') \cap \text{Good}(i, i')) \leq d-3$. Here is our choice of U_n:

$$U_n = X_n \setminus \bigcup\nolimits_{\{i,i'\}\in P(I_n)} \text{Bad}(i, i') \cap \text{Good}(i, i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \amalg G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ-equidimensional of dimension $d-1$ and $G_{n,i,i'}$ is δ-equidimensional of dimension $d-2$. (So topologically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \to X_n$ be the blowup in Z_n. Note that Lemma 115.23.8 applies to the morphism $b_n : X_{n+1} \to X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \to X$ the morphism obtained as the composition $j_0 \circ \ldots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \ldots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 115.23.8 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (115.23.11.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X. Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \ldots \cup T_n$ is a closed subset of X for each n. Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n. Hence $\{\overline{T_n}\}_{n \geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U. Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X. Hence we can define

$$U' = \lim_n \pi_n^{-1}(U).$$

The induced morphism $b : U' \to U$ is proper since this is local on U, and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n \geq 0} I_n$ using the inclusions $I_n \to I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = \lim_{n \geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X. $\square$

115.24. Commutativity of intersecting divisors

0AYE The results of this section were originally used to provide an alternative proof of the lemmas of Chow Homology, Section 42.28 and a weak version of Chow Homology, Lemma 42.30.5.

02TC **Lemma 115.24.1.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Let $\{i_j : D_j \to X\}_{j\in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j > 0$, $j \in J$. Set $D = \sum_{j\in J} n_j D_j$, and denote $i : D \to X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then*

$$p : \coprod\nolimits_{j\in J} D_j \longrightarrow D$$

is proper and

$$i^*\alpha = p_*\left(\sum n_j i_j^*\alpha\right)$$

in $\mathrm{CH}_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Chow Homology, Lemma 42.24.2 and Divisors, Lemma 31.27.5 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j\in J}$ is locally finite. Write $\alpha = \sum_{a\in A} m_a[W_a]$ with $W_a \subset X$ an integral closed subscheme of δ-dimension $k+1$. Denote $i_a : W_a \to X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a\in A}$ is locally finite on X.

Observe that by Chow Homology, Definition 42.29.1 the class $i^*\alpha$ is the class of a cycle $\sum m_a\beta_a$ for certain $\beta_a \in Z_k(W_a \cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D \cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \amalg J_{a,2} \amalg J_{a,3}$ where

(1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
(2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
(3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

(1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
(2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
(3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^*\mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{rat} \sum\nolimits_{j\in J} n_j\beta_{a,j}$$

in $\mathrm{CH}_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D \cap W_a]_k = \sum n_j[D_j \cap W_a]_k$ as cycles. This is Lemma 115.23.10.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j\in J_{a,s}} n_j D_j$. By Divisors, Lemma 31.27.5

we have

$$\begin{aligned} c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^*\mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^*\mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^*\mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j \in J_{a,3}} n_j c_1(i_a^*\mathcal{L}_j) \cap [W_a]$, see Divisors, Lemma 31.27.5. This is represented by $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^*D \sim_{rat} \sum m_a \beta_a$ by our choice of β_a. For each a we have $\beta_a \sim_{rat} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D, we see that also $\sum m_a \beta_a \sim_{rat} \sum_{a,j} m_a \beta_{a,j}$ on D! (See Chow Homology, Remark 42.19.6.) Ok, and now it is clear that $\sum_a m_a \beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^*\alpha$ and hence $\sum_{a,j} m_a \beta_{a,j}$ represents $p_* \sum_j i_j^* \alpha$ and we win. □

02TD **Lemma 115.24.2.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Assume $\dim_\delta(D \cap D') = n - 2$. Let $i : D \to X$, resp. $i' : D' \to X$ be the corresponding closed immersions. Then*

(1) *there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in \mathrm{CH}_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in \mathrm{CH}_{n-2}(D')$, and*
(2) *we have*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $\mathrm{CH}_{n-2}(X)$.

Proof. Part (2) is a trivial consequence of part (1). Let us write $[D]_{n-1} = \sum n_a [Z_a]$ and $[D']_{n-1} = \sum m_b [Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D'. According to Chow Homology, Definition 42.29.1, we have $i^*D' = \sum m_b i^*[Z_b]$ and $(i')^*D = \sum n_a (i')^*[Z_a]$. By assumption, none of the irreducible components Z_b is contained in D, and hence $i^*[Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n - 2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D, resp. D'. By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f'), let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R$ be the minimal primes over (f). The equality above comes down to the equality

$$\sum_{i=1,\ldots,s} \mathrm{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \mathrm{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1,\ldots,t} \mathrm{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \mathrm{ord}_{R/\mathfrak{q}'_j}(f).$$

By Chow Homology, Lemma 42.3.1 applied with $M = R/(f)$ the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \to R/(f)))$$

OK, and now we note that $\text{Ker}(f' : R/(f) \to R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is

$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f')/(ff'))$$

Since this is symmetric in f and f' we win. □

02TE **Lemma 115.24.3.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n - 2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $\text{CH}_{n-2}(X)$.*

Proof. This lemma is a trivial consequence of Lemmas 115.23.10 and 115.24.2 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \to X$ be the closed immersions Let $p : \coprod D_j \to X$ denote coproduct of the morphisms i_j. Let $\{Z_a\}_{a \in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write

$$[D_j]_{n-1} = \sum d_{j,a}[Z_a].$$

By Lemma 115.23.10 we have

$$[D]_{n-1} = \sum n_j d_{j,a}[Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a}[Z_a].$$

By Lemma 115.24.1 we have

$$D \cdot [D']_{n-1} = p_* \left(\sum n_j i_j^*[D']_{n-1} \right), \quad D' \cdot [D]_{n-1} = p_* \left(\sum m_{j'} i_{j'}^*[D]_{n-1} \right).$$

As in the definition of the Gysin homomorphisms (see Chow Homology, Definition 42.29.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^*[Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j, i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X. Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 115.24.2 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned} D \cdot [D']_{n-1} &= \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j',a}\beta_{a,j}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j,a}\beta_{a,j'}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a}\beta_{a,j'} \\ &= D' \cdot [D]_{n-1} \end{aligned}$$

and we win. □

02TF **Lemma 115.24.4.** *Let (S, δ) be as in Chow Homology, Situation 42.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Then*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $\mathrm{CH}_{n-2}(X)$.

First proof of Lemma 115.24.4. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 115.23.11 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \to X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2, with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D', see Lemma 115.23.5. Hence, by Chow Homology, Lemma 42.26.4 we have

$$D \cdot [D']_{n-1} = b_*\left(b^{-1}(D) \cdot [b^{-1}(D')]_{n-1}\right)$$

in $\mathrm{CH}_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $\mathrm{CH}_{n-2}(X')$ of Lemma 115.24.3.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \to U$ we get from applying Lemma 115.23.11 has as its target an open $U \subset X$ whose complement has codimension ≥ 3. Hence by Chow Homology, Lemma 42.19.3 we see that $\mathrm{CH}_{n-2}(U) = \mathrm{CH}_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. □

Second proof of Lemma 115.24.4. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D'. We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \to \mathcal{O}_X$ to D' to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\mathrm{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \mathrm{Coker}(\psi)$$

and

$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in $K_0(Coh_{\leq n-1}(X))$. On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Chow Homology, Lemma 42.69.7 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \to \mathrm{CH}_{n-2}(X)$ is well defined – which is the key to this proof). □

115.25. Dualizing modules on regular proper models

0C6D In Semistable Reduction, Situation 55.9.3 we let $\omega^\bullet_{X/R} = f^!\mathcal{O}_{\mathrm{Spec}(R)}$ be the relative dualizing complex of $f : X \to \mathrm{Spec}(R)$ as introduced in Duality for Schemes, Remark 48.12.5. Since f is Gorenstein of relative dimension 1 by Semistable Reduction, Lemma 55.9.2 we can use Duality for Schemes, Lemmas 48.25.10, 48.21.7, and 48.25.4 to see that

$$\omega^\bullet_{X/R} = \omega_X[1]$$

for some invertible $\mathcal{O}_X$-module ω_X. This invertible module is often called the *relative dualizing module of X over R*. Since R is regular (hence Gorenstein) of dimension 1 we see that $\omega^\bullet_R = R[1]$ is a normalized dualizing complex for R. Hence $\omega_X = H^{-2}(f^!\omega^\bullet_R)$ and we see that ω_X is not just a relative dualizing module but also a dualizing module, see Duality for Schemes, Example 48.22.1. Thus ω_X represents the functor

$$Coh(\mathcal{O}_X) \to Sets, \quad \mathcal{F} \mapsto \mathrm{Hom}_R(H^1(X, \mathcal{F}), R)$$

by Duality for Schemes, Lemma 48.22.5. This gives an alternative definition of the relative dualizing module in Semistable Reduction, Situation 55.9.3. The formation of ω_X commutes with arbitrary base change (for any proper Gorenstein morphism of given relative dimension); this follows from the corresponding fact for the relative dualizing complex discussed in Duality for Schemes, Remark 48.12.5 which goes back to Duality for Schemes, Lemma 48.12.4. Thus ω_X pulls back to the dualizing module ω_C of C over K discussed in Algebraic Curves, Lemma 53.4.2. Note that ω_C is isomorphic to $\Omega_{C/K}$ by Algebraic Curves, Lemma 53.4.1. Similarly $\omega_X|_{X_k}$ is the dualizing module ω_{X_k} of X_k over k.

0C6E **Lemma 115.25.1.** *In Semistable Reduction, Situation 55.9.3 the dualizing module of C_i over k is*

$$\omega_{C_i} = \omega_X(C_i)|_{C_i}$$

where ω_X is as above.

Proof. Let $t : C_i \to X$ be the closed immersion. Since t is the inclusion of an effective Cartier divisor we conclude from Duality for Schemes, Lemmas 48.9.7 and

48.14.2 that we have $t^!(\mathcal{L}) = \mathcal{L}(C_i)|_{C_i}$ for every invertible $\mathcal{O}_X$-module $\mathcal{L}$. Consider the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{t} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \xrightarrow{s} & \mathrm{Spec}(R) \end{array}$$

Observe that C_i is a Gorenstein curve (Semistable Reduction, Lemma 55.9.2) with invertible dualizing module ω_{C_i} characterized by the property $\omega_{C_i}[0] = g^!\mathcal{O}_{\mathrm{Spec}(k)}$. See Algebraic Curves, Lemma 53.4.1, its proof, and Algebraic Curves, Lemmas 53.4.2 and 53.5.2. On the other hand, $s^!(R[1]) = k$ and hence

$$\omega_{C_i}[0] = g^!s^!(R[1]) = t^!f^!(R[1]) = t^!\omega_X$$

Combining the above we obtain the statement of the lemma. □

115.26. Duplicate and split out references

09AQ This section is a place where we collect duplicates and references which used to say several things at the same time but are now split into their constituent parts.

007V **Lemma 115.26.1.** *Let X be a topological space.*

(1) *Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.*
(2) *Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps which are surjective (resp. injective) on all the stalks.*
(3) *The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).*

Proof. Combination of Sheaves, Lemmas 6.16.3 and 6.17.6. □

05JR **Lemma 115.26.2.** *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. This is a duplicate of Properties, Lemma 28.22.3. □

03IF **Lemma 115.26.3.** *Let S be a scheme. Let X be an algebraic space over S. The map $\{\mathrm{Spec}(k) \to X \text{ monomorphism}\} \to |X|$ is injective.*

Proof. This is a duplicate of Properties of Spaces, Lemma 66.4.12. □

03QZ **Theorem 115.26.4.** *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Then there is an equivalence of categories $Sh(S_{étale}) \to G$-Sets, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$.*

Proof. This is a duplicate of Étale Cohomology, Theorem 59.56.3. □

06IF **Remark 115.26.5.** You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 90.12 for the actual content.

08E6 **Lemma 115.26.6.** *Let X be a locally ringed space. A direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.*

Proof. This is a duplicate of Modules, Lemma 17.14.6. □

08XU **Lemma 115.26.7.** *Let R be a ring. Let E be an R-module. The following are equivalent*

(1) *E is an injective R-module, and*
(2) *given an ideal $I \subset R$ and a module map $\varphi : I \to E$ there exists an extension of φ to an R-module map $R \to E$.*

Proof. This is Baer's criterion, see Injectives, Lemma 19.2.6. □

02PI **Lemma 115.26.8.** *Let R be a local ring.*

(1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = length_R(M) - length_R(N)$.*
(2) *If (M, φ, ψ) is a $(2, 1)$-periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.*
(3) *Suppose that we have a short exact sequence of 2-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. This follows from Chow Homology, Lemmas 42.2.3 and 42.2.4. □

08S8 **Lemma 115.26.9.** *Let A be a ring and let I be an A-module.*

(1) *The set of extensions of rings $0 \to I \to A' \to A \to 0$ where I is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$.*
(2) *Given a ring map $A \to B$, a B-module N, an A-module map $c : I \to N$, and given extensions of rings with square zero kernels:*
   (a) *$0 \to I \to A' \to A \to 0$ corresponding to $\alpha \in \mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$, and*
   (b) *$0 \to N \to B' \to B \to 0$ corresponding to $\beta \in \mathrm{Ext}^1_B(NL_{B/\mathbf{Z}}, N)$*

   then there is a map $A' \to B'$ fitting into Deformation Theory, Equation (91.2.0.1) if and only if β and α map to the same element of $\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, N)$.

Proof. This follows from Deformation Theory, Lemmas 91.2.3 and 91.2.5. □

08UD **Lemma 115.26.10.** *Let $(S, \mathcal{O}_S)$ be a ringed space and let $\mathcal{J}$ be an $\mathcal{O}_S$-module.*

(1) *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$.*
(2) *Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, an f-map $c : \mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:*
   (a) *$0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ corresponding to $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$,*
   (b) *$0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ corresponding to $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G})$*

   then there is a morphism $X' \to S'$ fitting into Deformation Theory, Equation (91.7.0.1) if and only if β and α map to the same element of $\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 91.7.4 and 91.7.6. □

08UL **Lemma 115.26.11.** *Let $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a ringed topos and let $\mathcal{J}$ be an $\mathcal{O}_\mathcal{B}$-module.*

(1) *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$.*

(2) *Given a morphism of ringed topoi* $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$*, an* $\mathcal{O}$*-module* $\mathcal{G}$*, an* $f^{-1}\mathcal{O}_\mathcal{B}$*-module map* $c : f^{-1}\mathcal{J} \to \mathcal{G}$*, and given extensions of sheaves of rings with square zero kernels:*
(a) $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ *corresponding to* $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$,
(b) $0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ *corresponding to* $\beta \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathbf{Z}}, \mathcal{G})$
then there is a morphism $(Sh(\mathcal{C}), \mathcal{O}') \to (Sh(\mathcal{B}, \mathcal{O}_{\mathcal{B}'})$ *fitting into Deformation Theory, Equation (91.13.0.1) if and only if* β *and* α *map to the same element of* $\mathrm{Ext}^1_{\mathcal{O}}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 91.13.4 and 91.13.6. □

0D3H **Remark 115.26.12.** This tag used to point to a section describing several examples of deformation problems. These now each have their own section. See Deformation Problems, Sections 93.4, 93.5, 93.6, and 93.7.

0D3M **Lemma 115.26.13.** *Deformation Problems, Examples 93.4.1, 93.5.1, 93.6.1, and 93.7.1 satisfy the Rim-Schlessinger condition (RS).*

Proof. This follows from Deformation Problems, Lemmas 93.4.2, 93.5.2, 93.6.2, and 93.7.2. □

0D3N **Lemma 115.26.14.** *We have the following canonical* k*-vector space identifications:*

(1) *In Deformation Problems, Example 93.4.1 if* $x_0 = (k, V)$*, then* $T_{x_0}\mathcal{F} = (0)$ *and* $Inf_{x_0}(\mathcal{F}) = End_k(V)$ *are finite dimensional.*
(2) *In Deformation Problems, Example 93.5.1 if* $x_0 = (k, V, \rho_0)$*, then* $T_{x_0}\mathcal{F} = \mathrm{Ext}^1_{k[\Gamma]}(V, V) = H^1(\Gamma, End_k(V))$ *and* $Inf_{x_0}(\mathcal{F}) = H^0(\Gamma, End_k(V))$ *are finite dimensional if* Γ *is finitely generated.*
(3) *In Deformation Problems, Example 93.6.1 if* $x_0 = (k, V, \rho_0)$*, then* $T_{x_0}\mathcal{F} = H^1_{cont}(\Gamma, End_k(V))$ *and* $Inf_{x_0}(\mathcal{F}) = H^0_{cont}(\Gamma, End_k(V))$ *are finite dimensional if* Γ *is topologically finitely generated.*
(4) *In Deformation Problems, Example 93.7.1 if* $x_0 = (k, P)$*, then* $T_{x_0}\mathcal{F}$ *and* $Inf_{x_0}(\mathcal{F}) = Der_k(P, P)$ *are finite dimensional if* P *is finitely generated over* k.

Proof. This follows from Deformation Problems, Lemmas 93.4.3, 93.5.3, 93.6.3, and 93.7.3. □

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CHAPTER 116

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117.2. Definitions listed per chapter

Introduction

Conventions

Set Theory

Categories

In 4.2.1: *category*
In 4.2.4: *isomorphism*
In 4.2.5: *groupoid*
In 4.2.8: *functor*
In 4.2.9: *faithful*, *fully faithful*, *essentially surjective*
In 4.2.10: *subcategory*, *full subcategory*, *strictly full*
In 4.2.15: *natural transformation*, *morphism of functors*
In 4.2.17: *equivalence of categories*, *quasi-inverse*
In 4.2.20: *product category*
In 4.3.1: *opposite category*
In 4.3.2: *contravariant*
In 4.3.3: *presheaf of sets on $\mathcal{C}$*, *presheaf*
In 4.3.6: *representable*
In 4.4.1: *product*
In 4.4.2: *has products of pairs of objects*
In 4.5.1: *coproduct*, *amalgamated sum*
In 4.5.2: *has coproducts of pairs of objects*
In 4.6.1: *fibre product*
In 4.6.2: *cartesian*
In 4.6.3: *has fibre products*
In 4.6.4: *representable*
In 4.8.2: *representable*, *F is relatively representable over G*
In 4.9.1: *pushout*
In 4.9.2: *cocartesian*
In 4.10.1: *equalizer*
In 4.11.1: *coequalizer*
In 4.12.1: *initial*, *final*
In 4.13.1: *monomorphism*, *epimorphism*
In 4.14.1: *limit*
In 4.14.2: *colimit*
In 4.14.6: *product*
In 4.14.7: *coproduct*
In 4.16.1: *connected*
In 4.17.1: *$\mathcal{I}$ is cofinal in $\mathcal{J}$*, *cofinal*
In 4.17.3: *$\mathcal{I}$ is initial in $\mathcal{J}$*, *initial*
In 4.19.1: *directed*, *filtered*, *directed*, *filtered*
In 4.20.1: *codirected*, *cofiltered*, *codirected*, *cofiltered*
In 4.21.1: *preorder*, *preordered set*, *directed set*, *partial order*, *partially ordered set*, *directed partially ordered set*
In 4.21.2: *system over I in $\mathcal{C}$*, *inductive system over I in $\mathcal{C}$*, *inverse system over I in $\mathcal{C}$*, *projective system over I in $\mathcal{C}$*, *transition maps*
In 4.21.4: *directed system*, *directed inverse system*
In 4.22.1: *essentially constant*, *value*, *essentially constant*, *value*
In 4.22.2: *essentially constant system*, *essentially constant inverse system*
In 4.23.1: *left exact*, *right exact*, *exact*
In 4.24.1: *left adjoint*, *right adjoint*
In 4.26.1: *categorically compact*

In 4.27.1: *left multiplicative system*, *right multiplicative system*, *multiplicative system*
In 4.27.4: $s^{-1}f$
In 4.27.12: fs^{-1}
In 4.27.20: *saturated*
In 4.28.1: *horizontal*
In 4.29.1: 2*-category*, 1*-morphisms*, 2*-morphisms*, *vertical*, *composition*, *horizontal*
In 4.29.2: *sub* 2*-category*
In 4.29.4: *equivalent*
In 4.29.5: *functor*, *weak functor*, *pseudo functor*
In 4.30.1: $(2,1)$*-category*
In 4.31.1: *final object*
In 4.31.2: *2-fibre product of* f *and* g
In 4.32.1: 2*-category of categories over* $\mathcal{C}$
In 4.32.2: *fibre category*, *lift*, x *lies over* U, *lift*, ϕ *lies over* f
In 4.33.1: *strongly cartesian morphism*, *strongly* $\mathcal{C}$*-cartesian morphism*
In 4.33.5: *fibred category over* $\mathcal{C}$
In 4.33.6: *choice of pullbacks*, *pullback functor*
In 4.33.9: 2*-category of fibred categories over* $\mathcal{C}$
In 4.34.2: *relative inertia of* $\mathcal{S}$ *over* $\mathcal{S}'$, *inertia fibred category* $\mathcal{I}_{\mathcal{S}}$ *of* $\mathcal{S}$
In 4.35.1: *fibred in groupoids*
In 4.35.6: 2*-category of categories fibred in groupoids over* $\mathcal{C}$
In 4.36.2: *split fibred category*, $\mathcal{S}_F$
In 4.37.2: *split category fibred in groupoids*, $\mathcal{S}_F$
In 4.38.1: *discrete*
In 4.38.2: *category fibred in sets*, *category fibred in discrete categories*
In 4.38.3: 2*-category of categories fibred in sets over* $\mathcal{C}$
In 4.39.1: *setoid*
In 4.39.2: *category fibred in setoids*
In 4.39.3: 2*-category of categories fibred in setoids over* $\mathcal{C}$
In 4.40.1: *representable*
In 4.42.3: *representable*, $\mathcal{X}$ *is relatively representable over* $\mathcal{Y}$
In 4.43.1: *monoidal category*
In 4.43.2: *functor of monoidal categories*
In 4.43.4: *invertible*
In 4.43.5: *left dual*, *right dual*
In 4.43.9: *symmetric monoidal category*
In 4.43.11: *functor of symmetric monoidal categories*
In 4.44.1: *morphism of dotted arrows*

Topology

In 5.4.1: *separated*
In 5.5.1: *base for the topology on* X, *basis for the topology on* X
In 5.5.4: *subbase for the topology on* X, *subbasis for the topology on* X
In 5.6.3: *strict map of topological spaces*, *submersive*
In 5.7.1: *connected*, *connected component*
In 5.7.8: *totally disconnected*
In 5.7.10: *locally connected*
In 5.8.1: *irreducible*, *irreducible component*
In 5.8.6: *generic point*, *Kolmogorov*, *quasi-sober*, *sober*
In 5.9.1: *Noetherian*, *locally Noetherian*
In 5.10.1: *chain of irreducible closed subsets*, *length*, *dimension*, *Krull dimension*, *Krull dimension of* X *at* x
In 5.10.5: *equidimensional*
In 5.11.1: *codimension*
In 5.11.4: *catenary*
In 5.12.1: *quasi-compact*, *quasi-compact*, *retrocompact*
In 5.13.1: *locally quasi-compact*
In 5.15.1: *constructible*, *locally constructible*
In 5.17.2: *closed*, *Bourbaki-proper*, *quasi-proper*, *universally closed*, *proper*
In 5.18.1: *Jacobson*
In 5.19.1: *specialization*, *generalization*, *stable under specialization*, *stable under generalization*
In 5.19.4: *specializations lift along* f, *specializing*, *generalizations lift along* f, *generalizing*
In 5.20.1: *immediate specialization*, *dimension function*
In 5.21.1: *interior*, *nowhere dense*
In 5.22.1: *profinite*
In 5.23.1: *spectral*, *spectral*

In 5.26.1: *extremally disconnected*
In 5.27.2: *isolated point*
In 5.28.1: *partition*, *parts*, *refines*
In 5.28.2: *good stratification*
In 5.28.3: *stratification*, *strata*
In 5.28.4: *locally finite*
In 5.30.1: *topological group*, *homomorphism of topological groups*
In 5.30.5: *profinite group*
In 5.30.7: *topological ring*, *homomorphism of topological rings*
In 5.30.10: *topological module*, *homomorphism of topological modules*

Sheaves on Spaces

In 6.3.1: *presheaf $\mathcal{F}$ of sets on X*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X*
In 6.3.2: *constant presheaf with value A*
In 6.4.4: *presheaf of abelian groups on X*, *abelian presheaf over X*, *morphism of abelian presheaves over X*
In 6.5.1: *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$*
In 6.5.2: *underlying presheaf of sets of $\mathcal{F}$*
In 6.6.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 6.7.1: *sheaf $\mathcal{F}$ of sets on X*, *morphism of sheaves of sets*
In 6.7.4: *constant sheaf with value A*
In 6.8.1: *abelian sheaf on X*, *sheaf of abelian groups on X*
In 6.9.1: *sheaf*
In 6.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 6.11.2: *separated*
In 6.15.1: *type of algebraic structure*
In 6.16.2: *subpresheaf*, *subsheaf*, *injective*, *surjective*, *injective*, *surjective*
In 6.21.7: *f-map $\xi : \mathcal{G} \to \mathcal{F}$*
In 6.21.9: *composition of φ and ψ*
In 6.25.1: *ringed space*, *morphism of ringed spaces*
In 6.25.3: *composition of morphisms of ringed spaces*
In 6.26.1: *pushforward*, *pullback*
In 6.27.1: *skyscraper sheaf at x with value A*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*
In 6.30.1: *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$*
In 6.30.2: *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism of sheaves of sets on $\mathcal{B}$*
In 6.30.8: *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$*, *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*
In 6.30.11: *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$*, *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$*
In 6.31.2: *restriction of $\mathcal{G}$ to U*, *restriction of $\mathcal{G}$ to U*, *open subspace of $(X, \mathcal{O})$ associated to U*, *restriction of $\mathcal{G}$ to U*
In 6.31.3: *extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$*, *extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$*
In 6.31.5: *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e*, *extension by 0*, *extension by 0*

Sites and Sheaves

In 7.2.1: *presheaf of sets*, *Morphisms of presheaves*
In 7.2.2: *presheaf*, *morphism*
In 7.3.1: *injective*, *surjective*
In 7.3.3: *subpresheaf*
In 7.3.5: *image of φ*
In 7.6.1: *family of morphisms with fixed target*
In 7.6.2: *site*, *coverings of $\mathcal{C}$*
In 7.7.1: *sheaf*
In 7.7.5: *$Sh(\mathcal{C})$*
In 7.7.6: *sheaf*
In 7.8.1: *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, *morphism from $\mathcal{U}$ to $\mathcal{V}$*, *refinement*
In 7.8.2: *combinatorially equivalent*, *tautologically equivalent*
In 7.10.9: *separated*
In 7.10.11: *sheaf associated to $\mathcal{F}$*
In 7.11.1: *injective*, *surjective*

In 7.12.1: *effective epimorphism*, *universal effective epimorphism*
In 7.12.2: *weaker than the canonical topology*, *subcanonical*
In 7.12.3: *representable sheaves*, $\underline{U}$
In 7.13.1: *continuous*
In 7.14.1: *morphism of sites*
In 7.14.5: *composition*
In 7.15.1: *topos*, *morphism of topoi*, *composition* $f \circ g$
In 7.17.1: *quasi-compact*
In 7.17.4: *quasi-compact*, *quasi-compact*
In 7.20.1: *cocontinuous*
In 7.25.1: *localization of the site* $\mathcal{C}$ *at the object* U, *localization morphism*, *direct image functor*, *restriction of* $\mathcal{F}$ *to* $\mathcal{C}/U$, *extension of* $\mathcal{G}$ *by the empty set*
In 7.29.2: *special cocontinuous functor* u *from* $\mathcal{C}$ *to* $\mathcal{D}$
In 7.30.4: *localization of the topos* $Sh(\mathcal{C})$ *at* $\mathcal{F}$, *localization morphism*
In 7.32.1: *point of the topos* $Sh(\mathcal{C})$
In 7.32.2: *point* p *of the site* $\mathcal{C}$
In 7.32.6: *skyscraper sheaf*
In 7.36.1: *2-morphism from* f *to* g
In 7.37.2: *morphism* $f : p \to p'$
In 7.38.1: *conservative*, *has enough points*
In 7.40.2: *weakly contractible*, *enough weakly contractible objects*, *enough* P *objects*
In 7.42.1: *sheaf theoretically empty*
In 7.42.3: *almost cocontinuous*
In 7.43.1: *embedding*
In 7.43.2: *subtopos*
In 7.43.4: *open subtopos*
In 7.43.6: *closed subtopos*
In 7.43.7: *open immersion*, *closed immersion*
In 7.44.1: *pushforward*
In 7.45.1: *global sections*
In 7.47.1: *sieve* S *on* U
In 7.47.3: *sieve on* U *generated by the morphisms* f_i
In 7.47.4: *pullback of* S *by* f
In 7.47.6: *topology on* $\mathcal{C}$
In 7.47.8: *finer*, *stronger*, *coarser*, *weaker*
In 7.47.10: *sheaf*
In 7.47.12: *canonical topology*
In 7.48.2: *topology associated to* $\mathcal{C}$
In 7.49.2: *separated*
In 7.49.4: *sheaf associated to* $\mathcal{F}$
In 7.52.1: *point* p

Stacks

In 8.2.2: *presheaf of morphisms from* x *to* y, *presheaf of isomorphisms from* x *to* y
In 8.3.1: *descent datum* (X_i, φ_{ij}) *in* $\mathcal{S}$ *relative to the family* $\{f_i : U_i \to U\}$, *cocycle condition*, *morphism* $\psi : (X_i, \varphi_{ij}) \to (X_i', \varphi_{ij}')$ *of descent data*
In 8.3.4: *pullback functor*
In 8.3.5: *trivial descent datum*, *canonical descent datum*, *effective*
In 8.4.1: *stack*
In 8.4.5: 2*-category of stacks over* $\mathcal{C}$
In 8.5.1: *stack in groupoids*
In 8.5.5: 2*-category of stacks in groupoids over* $\mathcal{C}$
In 8.6.1: *stack in setoids*, *stack in sets*, *stack in discrete categories*
In 8.6.5: 2*-category of stacks in setoids over* $\mathcal{C}$
In 8.10.2: *structure of site on* $\mathcal{S}$ *inherited from* $\mathcal{C}$, $\mathcal{S}$ *is endowed with the topology inherited from* $\mathcal{C}$
In 8.11.1: *gerbe*
In 8.11.4: *gerbe over*
In 8.12.4: $f_*\mathcal{S}$, *pushforward of* $\mathcal{S}$ *along* f
In 8.12.9: $f^{-1}\mathcal{S}$, *pullback of* $\mathcal{S}$ *along* f

Fields

In 9.2.1: *field*, *subfield*
In 9.2.2: *domain*, *integral domain*
In 9.5.1: *characteristic*, *prime subfield of* F
In 9.6.2: *field extension*
In 9.6.3: *tower*
In 9.6.6: *generates the field extension*, *finitely generated field extension*
In 9.7.1: *degree*, *finite*
In 9.7.8: *number field*
In 9.8.1: *algebraic*, *algebraic extension*
In 9.9.1: *minimal polynomial*
In 9.10.1: *algebraically closed*
In 9.10.3: *algebraic closure*

In 9.11.1: *relatively prime*
In 9.12.2: *separable*, *separable*, *separable*
In 9.12.6: *separable degree*
In 9.14.1: *purely inseparable*, *purely inseparable*, *purely inseparable*
In 9.14.7: *separable degree*, *inseparable degree*, *degree of inseparability*
In 9.15.1: *normal*
In 9.15.8: *automorphisms of E over F*, *automorphisms of E/F*
In 9.16.2: *splitting field of P over F*
In 9.16.4: *normal closure E over F*
In 9.20.1: *trace*, *norm*
In 9.20.6: *trace pairing*
In 9.20.8: *discriminant of L/K*
In 9.21.1: *Galois*
In 9.21.3: *Galois group*
In 9.26.1: *algebraically independent*, *purely transcendental extension*, *transcendence basis*
In 9.26.4: *transcendence degree*
In 9.26.9: *algebraic closure of k in K*, *algebraically closed in K*
In 9.27.1: *compositum of K and L in Ω*
In 9.27.2: *linearly disjoint over k in Ω*
In 9.28.1: *algebraic*, *separable*, *purely inseparable*, *normal*, *Galois*

Commutative Algebra

In 10.5.1: *finite R-module*, *finitely generated R-module*, *finitely presented R-module*, *R-module of finite presentation*
In 10.6.1: *finite type*, *S is a finite type R-algebra*, *finite presentation*
In 10.7.1: *finite*
In 10.8.1: *system (M_i, μ_{ij}) of R-modules over I*, *directed system*
In 10.8.6: *homomorphism of systems*
In 10.9.1: *multiplicative subset of R*
In 10.9.2: *localization of A with respect to S*
In 10.9.6: *localization*
In 10.11.2: *relation*
In 10.12.1: *R-bilinear*
In 10.12.6: *(A, B)-bimodule*
In 10.14.1: *base change*, *base change*
In 10.17.1: *spectrum*
In 10.17.3: *Zariski*, *standard opens*
In 10.18.1: *local ring*, *residue field*, *local homomorphism of local rings*, *local ring map $\varphi : R \to S$*
In 10.28.2: *Oka family*
In 10.32.1: *locally nilpotent*, *nilpotent*
In 10.35.1: *Jacobson ring*
In 10.36.1: *integral over R*, *integral*
In 10.36.9: *integral closure*, *integrally closed*
In 10.37.1: *normal*
In 10.37.3: *almost integral over R*, *completely normal*
In 10.37.11: *normal*
In 10.38.1: *integral over I*
In 10.39.1: *flat*, *faithfully flat*, *flat*, *faithfully flat*
In 10.40.1: *support of M*
In 10.40.3: *annihilator of m*, *annihilator of M*
In 10.41.1: *going up*, *going down*
In 10.42.1: *separably generated over k*, *separable over k*
In 10.43.1: *geometrically reduced over k*
In 10.45.1: *perfect*
In 10.45.5: *perfect closure*
In 10.47.4: *geometrically irreducible over k*
In 10.48.3: *geometrically connected over k*
In 10.49.1: *geometrically integral over k*
In 10.50.1: *dominates*, *valuation ring*, *centered*
In 10.50.13: *value group*, *valuation*, *discrete valuation ring*
In 10.52.1: *length*
In 10.52.9: *simple*
In 10.53.1: *Artinian*
In 10.54.1: *essentially of finite type*, *essentially of finite presentation*
In 10.57.1: *homogeneous spectrum*
In 10.58.3: *numerical polynomial*
In 10.59.1: *an ideal of definition of R*
In 10.59.6: *Hilbert polynomial*
In 10.59.8: *$d(M)$*
In 10.60.1: *chain of prime ideals*, *length*
In 10.60.2: *Krull dimension*
In 10.60.3: *height*

In 10.60.10: *system of parameters of R*, *regular local ring*, *regular system of parameters*
In 10.63.1: *associated*
In 10.64.1: *symbolic power*
In 10.65.2: *relative assassin of N over S/R*
In 10.66.1: *weakly associated*
In 10.67.1: *embedded associated primes*, *embedded primes of R*
In 10.68.1: *M-regular sequence*, *M-regular sequence in I*, *regular sequence*
In 10.69.1: *M-quasi-regular*, *quasi-regular sequence*
In 10.70.1: *blowup algebra*, *Rees algebra*, *affine blowup algebra*
In 10.71.2: *resolution*, *resolution of M by free R-modules*, *resolution of M by finite free R-modules*
In 10.72.1: *I-depth*, *depth*
In 10.77.1: *projective*
In 10.78.1: *locally free*, *finite locally free*, *finite locally free of rank r*
In 10.82.1: *universally injective*, *universally exact*
In 10.84.1: *direct sum dévissage*, *Kaplansky dévissage*
In 10.86.1: *Mittag-Leffler*
In 10.88.1: *Mittag-Leffler directed system of modules*
In 10.88.2: *dominates*
In 10.88.7: *Mittag-Leffler*
In 10.90.1: *coherent module*, *coherent ring*
In 10.96.2: *I-adically complete*, *I-adically complete*
In 10.102.5: *rank*
In 10.103.1: *Cohen-Macaulay*
In 10.103.8: *maximal Cohen-Macaulay*
In 10.103.12: *Cohen-Macaulay*
In 10.104.1: *Cohen-Macaulay*
In 10.104.6: *Cohen-Macaulay*
In 10.105.1: *catenary*
In 10.105.3: *universally catenary*
In 10.108.1: *pure*
In 10.109.2: *finite projective dimension*, *projective dimension*
In 10.109.10: *finite global dimension*, *global dimension*
In 10.110.7: *regular*
In 10.112.5: *local ring of the fibre at $\mathfrak{q}$*
In 10.119.8: *uniformizer*
In 10.120.1: *associates*, *irreducible*, *prime*
In 10.120.4: *unique factorization domain*, *UFD*
In 10.120.12: *principal ideal domain*, *PID*
In 10.120.14: *Dedekind domain*
In 10.121.2: *order of vanishing along R*
In 10.121.3: *lattice in V*
In 10.121.5: *distance between M and M'*
In 10.122.3: *quasi-finite at $\mathfrak{q}$*, *quasi-finite*
In 10.123.7: *strongly transcendental over R*
In 10.125.1: *relative dimension of S/R at $\mathfrak{q}$*, *relative dimension of*
In 10.131.1: *derivation*, *R-derivation*, *Leibniz rule*
In 10.131.2: *module of Kähler differentials*, *module of differentials*
In 10.133.1: *differential operator $D : M \to N$ of order k*
In 10.133.4: *module of principal parts of order k*
In 10.134.1: *naive cotangent complex*
In 10.135.1: *global complete intersection over k*, *local complete intersection over k*
In 10.135.5: *complete intersection (over k)*
In 10.136.1: *syntomic*, *flat local complete intersection over R*
In 10.136.5: *relative global complete intersection*
In 10.137.1: *smooth*
In 10.137.5: *standard smooth algebra over R*, *standard smooth*, *standard smooth*
In 10.137.10: *smooth at $\mathfrak{q}$*
In 10.138.1: *formally smooth over R*
In 10.141.1: *small extension*
In 10.143.1: *étale*, *étale at $\mathfrak{q}$*
In 10.144.1: *standard étale*
In 10.148.1: *formally unramified over R*

In 10.149.2: *universal first order thickening*, *conormal module*, $C_{S/R}$
In 10.150.1: *formally étale over* R
In 10.151.1: *unramified*, *G-unramified*, *unramified at* $\mathfrak{q}$, *G-unramified at* $\mathfrak{q}$
In 10.153.1: *henselian*, *strictly henselian*
In 10.155.3: *henselization*, *strict henselization of* R *with respect to* $\kappa \subset \kappa^{sep}$, *strict henselization*
In 10.157.1: (R_k), *regular in codimension* $\leq k$, (S_k)
In 10.160.1: *complete local ring*
In 10.160.4: *coefficient ring*
In 10.160.5: *Cohen ring*
In 10.161.1: *N-1*, *N-2*, *Japanese*
In 10.162.1: *universally Japanese*, *Nagata ring*
In 10.162.9: *analytically unramified*, *analytically unramified*
In 10.165.2: *geometrically normal*
In 10.166.2: *geometrically regular*

Brauer groups

In 11.2.1: *finite*
In 11.2.2: *skew field*
In 11.2.3: *simple*, *simple*
In 11.2.4: *central*
In 11.2.5: *opposite algebra*
In 11.5.2: *Brauer group*
In 11.8.1: *splits*, *splitting field*

Homological Algebra

In 12.3.1: *preadditive*, *additive*
In 12.3.3: *zero object*
In 12.3.5: *direct sum*
In 12.3.8: *additive*
In 12.3.9: *kernel*, *cokernel*, *coimage of* f, *image of* f
In 12.4.1: *Karoubian*
In 12.5.1: *abelian*
In 12.5.3: *injective*, *surjective*, *subobject*, *quotient*
In 12.5.7: *complex*, *exact at* y, *exact at* x_i, *exact*, *exact sequence*, *exact complex*, *short exact sequence*
In 12.5.9: *split*
In 12.6.1: *extension* E *of* B *by* A, *morphism of extensions*
In 12.6.2: Ext*-group*
In 12.9.1: *simple*
In 12.9.2: *Artinian*, *Artinian*
In 12.9.3: *Noetherian*, *Noetherian*
In 12.10.1: *Serre subcategory*, *weak Serre subcategory*
In 12.10.5: *kernel of the functor* F
In 12.11.1: *zeroth* K*-group of* $\mathcal{A}$
In 12.12.1: *cohomological* δ*-functor*, δ*-functor*
In 12.12.2: *morphism of* δ*-functors from* F *to* G
In 12.12.3: *universal* δ*-functor*
In 12.13.2: *homotopy equivalence*, *homotopy equivalent*
In 12.13.4: *quasi-isomorphism*, *acyclic*
In 12.13.8: *homotopy equivalence*, *homotopy equivalent*
In 12.13.10: *quasi-isomorphism*, *acyclic*
In 12.14.1: k*-shifted chain complex* $A[k]_\bullet$
In 12.14.2: $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$
In 12.14.7: k*-shifted cochain complex* $A[k]^\bullet$
In 12.14.8: $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$
In 12.16.1: *category of graded objects of* $\mathcal{A}$
In 12.16.4: *shift*
In 12.17.1: *additive monoidal category*
In 12.18.1: *double complex*
In 12.18.3: *associated simple complex*, *associated total complex*
In 12.19.1: *decreasing filtration*, *filtered object of* $\mathcal{A}$, *morphism* $(A, F) \to (B, F)$ *of filtered objects*, *induced filtration*, *quotient filtration*, *finite*, *separated*, *exhaustive*
In 12.19.3: *strict*
In 12.20.1: *spectral sequence in* $\mathcal{A}$, *morphism of spectral sequences*
In 12.20.2: *limit*, *degenerates at* E_r
In 12.21.1: *exact couple*, *morphism of exact couples*
In 12.21.3: *spectral sequence associated to the exact couple*
In 12.22.1: *differential object*, *morphism of differential objects*
In 12.22.3: *homology*
In 12.22.5: *spectral sequence associated to* (A, d, α)

In 12.23.1: *filtered differential object*
In 12.23.4: *induced filtration*
In 12.23.6: *weakly converges to $H(K)$*, *abuts to $H(K)$*
In 12.24.1: *filtered complex $K^\bullet$ of $\mathcal{A}$*
In 12.24.5: *induced filtration*
In 12.24.7: *regular*, *coregular*, *bounded*, *bounded below*, *bounded above*
In 12.24.9: *weakly converges to $H^*(K^\bullet)$*, *abuts to $H^*(K^\bullet)$*, *converges to $H^*(K^\bullet)$*
In 12.25.2: *weakly converges to $H^n(Tot(K^{\bullet,\bullet}))$*, *abuts to $H^n(Tot(K^{\bullet,\bullet}))$*, *converges to $H^n(Tot(K^{\bullet,\bullet}))$*, *weakly converges to $H^n(Tot(K^{\bullet,\bullet}))$*, *abuts to $H^n(Tot(K^{\bullet,\bullet}))$*, *converges to $H^n(Tot(K^{\bullet,\bullet}))$*
In 12.27.1: *injective*
In 12.27.4: *enough injectives*
In 12.27.5: *functorial injective embeddings*
In 12.28.1: *projective*
In 12.28.4: *enough projectives*
In 12.28.5: *functorial projective surjections*
In 12.31.2: *Mittag-Leffler condition*, *ML*

Derived Categories

In 13.3.1: *triangle*, *morphism of triangles*
In 13.3.2: *triangulated category*, *distinguished triangles*, *pre-triangulated category*
In 13.3.3: *exact functor*, *triangulated functor*
In 13.3.4: *pre-triangulated subcategory*, *triangulated subcategory*
In 13.3.5: *homological*, *cohomological*
In 13.3.6: *δ-functor from $\mathcal{A}$ to $\mathcal{D}$*, *image of the short exact sequence under the given δ-functor*
In 13.5.1: *compatible with the triangulated structure*
In 13.6.1: *saturated*
In 13.6.5: *kernel of F*, *kernel of H*
In 13.6.7: *quotient category $\mathcal{D}/\mathcal{B}$*, *quotient functor*
In 13.8.1: *category of (cochain) complexes*, *bounded below*, *bounded above*, *bounded*
In 13.9.1: *cone*
In 13.9.4: *termwise split injection $\alpha : A^\bullet \to B^\bullet$*, *termwise split surjection $\beta : B^\bullet \to C^\bullet$*
In 13.9.9: *termwise split exact sequence of complexes of $\mathcal{A}$*, *triangle associated to the termwise split sequence of complexes*
In 13.10.1: *distinguished triangle of $K(\mathcal{A})$*
In 13.11.3: *derived category of $\mathcal{A}$*, *bounded derived category*
In 13.13.1: *category of finite filtered objects of $\mathcal{A}$*
In 13.13.2: *filtered quasi-isomorphism*, *filtered acyclic*
In 13.13.5: *filtered derived category of $\mathcal{A}$*
In 13.13.7: *bounded filtered derived category*
In 13.14.2: *right derived functor RF is defined at*, *value of RF at X*, *left derived functor LF is defined at*, *value of LF at X*
In 13.14.9: *right derivable*, *everywhere defined*, *left derivable*, *everywhere defined*
In 13.14.10: *computes*, *computes*
In 13.15.3: *right derived functors of F*, *left derived functors of F*, *right acyclic for F*, *acyclic for RF*, *left acyclic for F*, *acyclic for LF*
In 13.16.2: *ith right derived functor R^iF of F*
In 13.18.1: *injective resolution of A*, *injective resolution of $K^\bullet$*
In 13.19.1: *projective resolution of A*, *projective resolution of $K^\bullet$*
In 13.21.1: *Cartan-Eilenberg resolution*
In 13.23.2: *resolution functor*
In 13.26.1: *filtered injective*
In 13.27.1: *ith extension group*
In 13.27.4: *Yoneda extension*, *equivalent*
In 13.28.1: *zeroth K-group of $\mathcal{D}$*
In 13.31.1: *K-injective*
In 13.33.1: *derived colimit*, *homotopy colimit*
In 13.34.1: *derived limit*, *homotopy limit*
In 13.36.3: *classical generator*, *strong generator*, *weak generator*, *generator*
In 13.37.1: *compact object*
In 13.37.5: *compactly generated*

In 13.40.1: *right orthogonal*, *left orthogonal*
In 13.40.9: *right admissible*, *left admissible*, *two-sided admissible*
In 13.41.1: *Postnikov system*, *morphism of Postnikov systems*

Simplicial Methods

In 14.2.1: $\delta^n_j : [n-1] \to [n]$, $\sigma^n_j : [n+1] \to [n]$
In 14.3.1: *simplicial object U of $\mathcal{C}$*, *simplicial set*, *simplicial abelian group*, *morphism of simplicial objects $U \to U'$*, *category of simplicial objects of $\mathcal{C}$*
In 14.5.1: *cosimplicial object U of $\mathcal{C}$*, *cosimplicial set*, *cosimplicial abelian group*, *morphism of cosimplicial objects $U \to U'$*, *category of cosimplicial objects of $\mathcal{C}$*
In 14.6.1: *product of U and V*
In 14.7.1: *fibre product of V and W over U*
In 14.8.1: *pushout of V and W over U*
In 14.9.1: *product of U and V*
In 14.10.1: *fibre product of V and W over U*
In 14.11.1: *n-simplex of U*, *face of x*, *degeneracy of x*, *degenerate*
In 14.12.1: *n-truncated simplicial object of $\mathcal{C}$*, *morphism of n-truncated simplicial objects*
In 14.13.1: *product $U \times V$ of U and V*, *product $U \times V$ exists*
In 14.14.1: $\mathrm{Hom}(U, V)$
In 14.15.1: $\mathrm{Hom}(U, V)$
In 14.17.1: $\mathrm{Hom}(U, V)$
In 14.18.1: *split*, *split*
In 14.20.1: *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$*
In 14.22.3: *Eilenberg-Maclane object $K(A, k)$*
In 14.26.1: *homotopy from a to b*, *homotopic*, *in the same homotopy class*
In 14.26.6: *homotopy equivalence*, *homotopy equivalent*
In 14.28.1: *homotopy from a to b*, *homotopic*, *in the same homotopy class*
In 14.30.1: *trivial Kan fibration*
In 14.31.1: *Kan fibration*, *Kan complex*

More on Algebra

In 15.3.1: *stably isomorphic*, *stably free*
In 15.8.3: *kth Fitting ideal*
In 15.10.1: *Zariski pair*
In 15.11.1: *henselian pair*
In 15.14.1: *absolutely integrally closed*
In 15.15.1: *auto-associated*
In 15.22.1: *torsion*, *torsion free*, *torsion module*
In 15.23.1: *reflexive*
In 15.23.9: *reflexive hull*
In 15.24.1: *content ideal of x*
In 15.26.1: *strict transform of M along $R \to R[\frac{I}{a}]$*
In 15.28.1: *Koszul complex*
In 15.28.2: *Koszul complex on $f_1, \ldots, f_r$*
In 15.30.1: *M-Koszul-regular*, *M-H_1-regular*, *Koszul-regular*, *H_1-regular*
In 15.32.1: *regular ideal*, *Koszul-regular ideal*, *H_1-regular ideal*, *quasi-regular ideal*
In 15.33.2: *local complete intersection*
In 15.36.1: *topological ring*, *topological module*, *homomorphism of topological modules*, *homomorphism of topological rings*, *linearly topologized*, *linearly topologized*, *ideal of definition*, *pre-admissible*, *admissible*, *pre-adic*, *adic*
In 15.37.1: *formally smooth over R*
In 15.37.3: *formally smooth for the $\mathfrak{n}$-adic topology*
In 15.41.1: *regular*
In 15.46.1: *p-independent over k*, *p-basis of K over k*
In 15.47.1: *J-0*, *J-1*, *J-2*
In 15.50.1: *G-ring*
In 15.52.1: *quasi-excellent*, *excellent*
In 15.55.1: *injective*
In 15.55.5: *$M \mapsto M^\vee$*, *free module*
In 15.59.1: *K-flat*
In 15.59.13: *derived tensor product*
In 15.61.1: *Tor independent over R*
In 15.65.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 15.67.1: *tor-amplitude in $[a, b]$*, *finite tor dimension*, *tor dimension $\leq d$*, *finite tor dimension*

In 15.69.1: *finite projective dimension*, *projective-amplitude in* $[a, b]$
In 15.70.1: *finite injective dimension*, *injective-amplitude in* $[a, b]$
In 15.71.4: I*-projective*
In 15.75.1: *perfect*, *perfect*
In 15.81.2: *finitely presented relative to* R
In 15.82.4: m*-pseudo-coherent relative to* R, *pseudo-coherent relative to* R, m*-pseudo-coherent relative to* R, *pseudo-coherent relative to* R
In 15.83.1: *pseudo-coherent ring map*, *perfect ring map*
In 15.84.1: R*-perfect*, *perfect relative to* R
In 15.89.1: I*-power torsion module*, *an* f*-power torsion module*
In 15.92.4: *derived complete with respect to* I, *derived complete with respect to* I
In 15.105.1: *absolutely flat*, *weakly étale*, *absolutely flat*
In 15.105.3: *weak dimension* $\leq d$
In 15.107.1: *unibranch*, *geometrically unibranch*
In 15.107.6: *number of branches of* A, *number of geometric branches of* A
In 15.110.1: *formally catenary*
In 15.112.1: *extension of discrete valuation rings*, *ramification index*, *weakly unramified*, *residual degree*, *residue degree*
In 15.112.7: *unramified with respect to* A, *tamely ramified with respect to* A, *totally ramified with respect to* A
In 15.113.3: *decomposition group of* $\mathfrak{m}$, *inertia group of* $\mathfrak{m}$
In 15.113.6: *wild inertia group of* $\mathfrak{m}$, *tame inertia group of* $\mathfrak{m}$
In 15.114.3: *mixed characteristic*, *absolute ramification index*
In 15.116.1: *weak solution for* $A \subset B$, *solution for* $A \subset B$, *separable solution*
In 15.118.1: *invertible*, *trivial*
In 15.124.1: *extension of valuation rings*, *weakly unramified*, *residual degree*, *residue degree*
In 15.125.5: *Bézout domain*, *elementary divisor domain*

Smoothing Ring Maps

In 16.2.1: *singular ideal of* A *over* R
In 16.2.3: *elementary standard in* A *over* R, *strictly standard in* A *over* R

Sheaves of Modules

In 17.4.1: *generated by global sections*, *generate*
In 17.4.5: *subsheaf generated by the* s_i
In 17.5.1: *support of* $\mathcal{F}$, *support of* s
In 17.8.1: *locally generated by sections*
In 17.9.1: *finite type*
In 17.10.1: *quasi-coherent sheaf of* $\mathcal{O}_X$*-modules*
In 17.10.6: *sheaf associated to the module* M *and the ring map* α, *sheaf associated to the module* M
In 17.11.1: *finite presentation*
In 17.12.1: *coherent* $\mathcal{O}_X$*-module*
In 17.13.1: *closed immersion of ringed spaces*
In 17.14.1: *locally free*, *finite locally free*, *finite locally free of rank* r
In 17.17.1: *flat*
In 17.17.3: *flat at* x
In 17.20.1: *flat at* x, *flat*
In 17.20.3: *flat over* Y *at a point* $x \in X$, *flat over* Y
In 17.23.1: *annihilator*
In 17.24.1: *Koszul complex*
In 17.24.2: *Koszul complex on* $f_1, \ldots, f_n$
In 17.25.1: *invertible* $\mathcal{O}_X$*-module*, *trivial*
In 17.25.6: *tensor power*
In 17.25.7: *associated graded ring*
In 17.25.9: *Picard group*
In 17.28.1: $\mathcal{O}_1$*-derivation*, φ*-derivation*, *Leibniz rule*
In 17.28.3: *module of differentials*, *universal* φ*-derivation*
In 17.28.10: S*-derivation*, *sheaf of differentials* $\Omega_{X/S}$ *of* X *over* S
In 17.29.1: *differential operator* $D : \mathcal{F} \to \mathcal{G}$ *of order* k
In 17.29.4: *module of principal parts of order* k
In 17.29.8: *differential operator of order* k *on* X/S

In 17.30.1: *de Rham complex of $\mathcal{B}$ over $\mathcal{A}$*
In 17.30.4: *de Rham complex*
In 17.31.1: *naive cotangent complex*
In 17.31.6: *naive cotangent complex*

Modules on Sites

In 18.4.1: *free abelian presheaf*
In 18.5.1: *free abelian sheaf*
In 18.6.1: *ringed site*, *structure sheaf*, *morphism of ringed sites*, *composition of morphisms of ringed sites*
In 18.7.1: *ringed topos*, *structure sheaf*, *morphism of ringed topoi*, *composition of morphisms of ringed topoi*
In 18.8.1: *2-morphism from f to g*
In 18.9.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 18.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 18.13.1: *pushforward*, *pullback*
In 18.16.1: $g_{p!}\mathcal{F}$, $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$
In 18.17.1: *free $\mathcal{O}$-module*, *finite free*, *generated by global sections*, *generated by r global sections*, *generated by finitely many global sections*, *global presentation*, *global finite presentation*
In 18.19.1: *localization of the ringed site $(\mathcal{C}, \mathcal{O})$ at the object U*, *localization morphism*, *direct image functor*, *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*, *extension by zero*
In 18.21.2: *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$*, *localization morphism*
In 18.23.1: *locally free*, *finite locally free*, *locally generated by sections*, *locally generated by r sections*, *of finite type*, *quasi-coherent*, *of finite presentation*, *coherent*
In 18.28.1: *flat*, *flat*, *flat*, *flat*
In 18.31.1: *flat*, *flat*
In 18.31.3: *flat over $(Sh(\mathcal{D}), \mathcal{O}')$*
In 18.32.1: *rank r*, *invertible*, *$\mathcal{O}^*$*
In 18.32.6: *Picard group*
In 18.33.1: *$\mathcal{O}_1$-derivation*, *φ-derivation*, *Leibniz rule*
In 18.33.3: *module of differentials*, *universal φ-derivation*
In 18.33.10: *Y-derivation*, *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 18.34.1: *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k*
In 18.34.4: *module of principal parts of order k*
In 18.35.1: *naive cotangent complex*
In 18.35.4: *naive cotangent complex*
In 18.40.4: *locally ringed site*
In 18.40.6: *locally ringed*
In 18.40.9: *morphism of locally ringed topoi*, *morphism of locally ringed sites*
In 18.43.1: *constant sheaf*, *locally constant*, *finite locally constant*

Injectives

In 19.2.4: *α-small with respect to I*
In 19.10.1: *generator*, *Grothendieck abelian category*
In 19.11.2: *size*

Cohomology of Sheaves

In 20.4.1: *torsor*, *$\mathcal{G}$-torsor*, *morphism of $\mathcal{G}$-torsors*, *trivial $\mathcal{G}$-torsor*
In 20.9.1: *Čech complex*, *Čech cohomology groups*
In 20.12.1: *flasque*, *flabby*
In 20.23.1: *alternating Čech complex*
In 20.23.2: *ordered Čech complex*
In 20.24.2: *locally finite*
In 20.26.2: *K-flat*
In 20.26.14: *derived tensor product*
In 20.26.15: *Tor*
In 20.46.1: *strictly perfect*
In 20.47.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 20.48.1: *tor-amplitude in $[a, b]$*, *finite tor dimension*, *locally has finite tor dimension*, *tor dimension $\leq d$*
In 20.49.1: *perfect*, *perfect*

Cohomology on Sites

In 21.4.1: *pseudo torsor*, *pseudo $\mathcal{G}$-torsor*, *morphism of pseudo $\mathcal{G}$-torsors*, *torsor*, *$\mathcal{G}$-torsor*, *morphism of $\mathcal{G}$-torsors*, *trivial $\mathcal{G}$-torsor*

In 21.8.1: *Čech complex*, *Čech cohomology groups*
In 21.13.4: *totally acyclic*
In 21.17.2: *K-flat*
In 21.17.13: *derived tensor product*
In 21.17.14: *Tor*
In 21.31.2: *qc covering*
In 21.41.1: *simplicial $\mathcal{A}_\bullet$-module*, *simplicial sheaf of $\mathcal{A}_\bullet$-modules*
In 21.43.1: $QC(\mathcal{O})$
In 21.44.1: *strictly perfect*
In 21.45.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 21.46.1: *tor-amplitude in $[a, b]$*, *finite tor dimension*, *locally has finite tor dimension*, *tor dimension $\leq d$*
In 21.47.1: *perfect*, *perfect*

Differential Graded Algebra

In 22.3.1: *differential graded algebra over R*
In 22.3.2: *homomorphism of differential graded algebras*
In 22.3.3: *commutative*, *strictly commutative*
In 22.3.4: *tensor product differential graded algebra*
In 22.4.1: *differential graded module*, *homomorphism of differential graded modules*
In 22.4.3: *k-shifted module*
In 22.5.1: *homotopy between f and g*, *homotopic*
In 22.5.3: *homotopy category*
In 22.6.1: *cone*
In 22.7.1: *admissible monomorphism*, *admissible epimorphism*, *admissible short exact sequence*
In 22.8.2: *triangle associated to $0 \to K \to L \to M \to 0$*, *distinguished triangle*
In 22.11.1: *opposite differential graded algebra*
In 22.11.3: *kth shifted A-module*, *kth shifted A-module*
In 22.22.2: *derived category of (A, d)*
In 22.24.1: *R-linear category $\mathcal{A}$*
In 22.24.2: *functor of R-linear categories*, *R-linear functor*
In 22.25.1: *graded category $\mathcal{A}$ over R*
In 22.25.2: *functor of graded categories over R*, *graded functor*
In 22.25.3: $\mathcal{A}^0$
In 22.25.4: *graded direct sum*
In 22.26.1: *differential graded category $\mathcal{A}$ over R*
In 22.26.2: *functor of differential graded categories over R*
In 22.26.3: *category of complexes of $\mathcal{A}$*, *homotopy category of $\mathcal{A}$*
In 22.26.4: *differential graded direct sum*
In 22.28.1: *(A, B)-bimodule*, *graded (A, B)-bimodule*, *differential graded (A, B)-bimodule*

Divided Power Algebra

In 23.2.1: *divided power structure*
In 23.3.1: *divided power ring*, *homomorphism of divided power rings*
In 23.4.1: *extends*
In 23.6.1: *divided power structure*
In 23.6.5: *compatible with the differential graded structure*
In 23.8.5: *complete intersection*, *local complete intersection*

Differential Graded Sheaves

In 24.3.1: *sheaf of graded $\mathcal{O}$-algebras*, *sheaf of graded algebras*, *homomorphism of graded $\mathcal{O}$-algebras*
In 24.4.1: *graded $\mathcal{A}$-module*, *graded module*, *homomorphism of graded $\mathcal{A}$-modules*
In 24.8.1: *graded $(\mathcal{A}, \mathcal{B})$-bimodule*, *homomorphism of graded $(\mathcal{A}, \mathcal{B})$-bimodules*
In 24.12.1: *sheaf of differential graded $\mathcal{O}$-algebras*, *sheaf of differential graded algebras*, *homomorphism of differential graded $\mathcal{O}$-algebras*
In 24.13.1: *differential graded $\mathcal{A}$-module*, *differential graded module*, *homomorphism of differential graded $\mathcal{A}$-modules*
In 24.17.1: *differential graded $(\mathcal{A}, \mathcal{B})$-bimodule*, *homomorphism of differential graded $(\mathcal{A}, \mathcal{B})$-bimodules*
In 24.21.1: *homotopy between f and g*, *homotopic*

In 24.21.2: *homotopy category*
In 24.22.2: *cone*
In 24.25.2: *graded injective*
In 24.25.7: *K-injective*
In 24.26.4: *derived category of* $(\mathcal{A}, d)$
In 24.28.2: *derived tensor product*, *derived pullback*
In 24.29.2: *derived internal hom*, *derived pushforward*
In 24.33.1: $QC(\mathcal{A}, d)$

Hypercoverings

In 25.2.1: *semi-representable objects*, *semi-representable objects over* X
In 25.2.2: *which associates a presheaf to a semi-representable object*
In 25.3.1: *covering*, *covering*
In 25.3.3: *hypercovering of* X
In 25.4.1: *homology of* K
In 25.6.1: *hypercovering of* $\mathcal{G}$, *hypercovering*

Schemes

In 26.2.1: *locally ringed space* $(X, \mathcal{O}_X)$, *local ring of* X *at* x, *residue field of* X *at* x, *morphism of locally ringed spaces*
In 26.3.1: *open immersion*
In 26.3.3: *open subspace of* X *associated to* U
In 26.4.1: *closed immersion*
In 26.4.4: *closed subspace of* X *associated to the sheaf of ideals* $\mathcal{I}$
In 26.5.2: *standard open covering*, *standard open covering*
In 26.5.3: *structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *of the spectrum of* R, *spectrum*
In 26.5.5: *affine scheme*, *morphism of affine schemes*
In 26.9.1: *scheme*, *morphism of schemes*
In 26.10.2: *open immersion*, *open subscheme*, *closed immersion*, *closed subscheme*, *immersion*, *locally closed immersion*
In 26.12.1: *reduced*
In 26.12.5: *scheme structure on* Z, *reduced induced scheme structure*, *reduction* X_{red} *of* X
In 26.15.1: *representable by a scheme*, *representable*
In 26.15.3: *satisfies the sheaf property for the Zariski topology*, *subfunctor* $H \subset F$, *representable by open immersions*, *covers* F
In 26.17.1: *fibre product*
In 26.17.7: *inverse image* $f^{-1}(Z)$ *of the closed subscheme* Z
In 26.18.1: *scheme over* S, *structure morphism*, *scheme over* R, *morphism* $f : X \to Y$ *of schemes over* S, *base change*, *base change*, *base change*
In 26.18.3: *preserved under arbitrary base change*, *preserved under base change*, *preserved under arbitrary base change*, *preserved under base change*
In 26.18.4: *scheme theoretic fibre* X_s *of* f *over* s, *fibre of* f *over* s
In 26.19.1: *quasi-compact*
In 26.20.1: *universally closed*
In 26.20.3: *satisfies the existence part of the valuative criterion*, *satisfies the uniqueness part of the valuative criterion*
In 26.21.3: *separated*, *quasi-separated*, *separated*, *quasi-separated*
In 26.23.1: *monomorphism*

Constructions of Schemes

In 27.4.5: *relative spectrum of* $\mathcal{A}$ *over* S, *spectrum of* $\mathcal{A}$ *over* S
In 27.5.1: *affine* n*-space over* S, *affine* n*-space over* R
In 27.6.1: *vector bundle associated to* $\mathcal{E}$
In 27.6.2: *vector bundle* $\pi : V \to S$ *over* S, *morphism of vector bundles over* S
In 27.7.1: *cone associated to* $\mathcal{A}$, *affine cone associated to* $\mathcal{A}$
In 27.7.2: *cone* $\pi : C \to S$ *over* S, *morphism of cones*
In 27.8.2: *standard open covering*
In 27.8.3: *structure sheaf* $\mathcal{O}_{Proj(S)}$ *of the homogeneous spectrum of* S, *homogeneous spectrum*
In 27.10.1: *twist of the structure sheaf of* $Proj(S)$
In 27.13.2: *projective* n*-space over* $\mathbf{Z}$, *projective* n*-space over* S, *projective* n*-space over* R

In 27.16.7: *relative homogeneous spectrum of $\mathcal{A}$ over S*, *homogeneous spectrum of $\mathcal{A}$ over S*, *relative Proj of $\mathcal{A}$ over S*
In 27.21.1: *projective bundle associated to $\mathcal{E}$*, *twist of the structure sheaf*
In 27.22.2: *Grassmannian over $\mathbf{Z}$*, *Grassmannian over S*, *Grassmannian over R*

Properties of Schemes

In 28.3.1: *integral*
In 28.4.1: *local*
In 28.4.2: *locally P*
In 28.5.1: *locally Noetherian*, *Noetherian*
In 28.6.1: *Jacobson*
In 28.7.1: *normal*
In 28.8.1: *Cohen-Macaulay*
In 28.9.1: *regular*, *nonsingular*
In 28.10.1: *dimension*, *dimension of X at x*
In 28.11.1: *catenary*
In 28.12.1: *regular in codimension k*, (R_k), (S_k)
In 28.13.1: *Japanese*, *universally Japanese*, *Nagata*
In 28.14.1: *regular locus*, *singular locus*
In 28.15.1: *unibranch at x*, *geometrically unibranch at x*, *unibranch*, *geometrically unibranch*
In 28.15.4: *number of branches of X at x*, *number of geometric branches of X at x*
In 28.18.1: *quasi-affine*
In 28.21.1: *locally projective*
In 28.23.1: *κ-generated*
In 28.24.3: *subsheaf of sections annihilated by $\mathcal{I}$*
In 28.24.6: *subsheaf of sections supported on T*
In 28.26.1: *ample*

Morphisms of Schemes

In 29.4.4: *scheme theoretic intersection*, *scheme theoretic union*
In 29.5.5: *scheme theoretic support of $\mathcal{F}$*
In 29.6.2: *scheme theoretic image*
In 29.7.1: *scheme theoretic closure of U in X*, *scheme theoretically dense in X*
In 29.8.1: *dominant*
In 29.9.1: *surjective*
In 29.10.1: *universally injective*, *radicial*
In 29.11.1: *affine*
In 29.12.1: *ample family of invertible modules on X*
In 29.13.1: *quasi-affine*
In 29.14.1: *local*, *stable under base change*, *stable under composition*
In 29.14.2: *locally of type P*
In 29.15.1: *finite type at $x \in X$*, *locally of finite type*, *finite type*
In 29.16.3: *finite type point*
In 29.17.1: *universally catenary*
In 29.19.1: *J-2*
In 29.20.1: *quasi-finite at a point $x \in X$*, *locally quasi-finite*, *quasi-finite*
In 29.21.1: *finite presentation at $x \in X$*, *locally of finite presentation*, *finite presentation*
In 29.23.1: *open*, *universally open*
In 29.24.1: *submersive*, *universally submersive*
In 29.25.1: *flat at a point $x \in X$*, *flat over S at a point $x \in X$*, *flat*, *flat over S*
In 29.26.3: *canonical scheme structure on T*
In 29.29.1: *relative dimension $\leq d$ at x*, *relative dimension $\leq d$*, *relative dimension d*
In 29.30.1: *syntomic at $x \in X$*, *syntomic*, *local complete intersection over k*, *standard syntomic*
In 29.30.15: *syntomic of relative dimension d*
In 29.31.1: *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X*, *conormal sheaf of i*
In 29.32.1: *sheaf of differentials $\Omega_{X/S}$ of X over S*, *universal S-derivation*
In 29.34.1: *smooth at $x \in X$*, *smooth*, *standard smooth*
In 29.34.13: *smooth of relative dimension d*
In 29.35.1: *unramified at $x \in X$*, *G-unramified at $x \in X$*, *unramified*, *G-unramified*
In 29.36.1: *étale at $x \in X$*, *étale*, *standard étale*

In 29.37.1: *relatively ample*, *f-relatively ample*, *ample on X/S*, *f-ample*
In 29.38.1: *relatively very ample*, *f-relatively very ample*, *very ample on X/S*, *f-very ample*
In 29.40.1: *quasi-projective*, *H-quasi-projective*, *locally quasi-projective*
In 29.41.1: *proper*
In 29.43.1: *projective*, *H-projective*, *locally projective*
In 29.44.1: *integral*, *finite*
In 29.45.1: *universal homeomorphism*
In 29.47.1: *seminormal*, *absolutely weakly normal*
In 29.47.3: *seminormal*, *absolutely weakly normal*
In 29.47.8: *seminormalization*, *absolute weak normalization*
In 29.48.1: *finite locally free*, *rank*, *degree*
In 29.49.1: *equivalent*, *rational map from X to Y*, *S-rational map from X to Y*
In 29.49.3: *rational function on X*
In 29.49.4: *ring of rational functions on X*
In 29.49.6: *function field*, *field of rational functions*
In 29.49.8: *defined in a point $x \in X$*, *domain of definition*
In 29.49.10: *dominant*
In 29.49.11: *birational*, *S-birational*
In 29.50.1: *birational*
In 29.51.8: *degree of X over Y*
In 29.51.11: *modification of X*
In 29.51.12: *alteration of X*
In 29.53.2: *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$*
In 29.53.3: *normalization of X in Y*
In 29.54.1: *normalization*
In 29.55.6: *seminormalization of X in Y*, *weak normalization of X in Y*
In 29.55.8: *weak normalization*
In 29.55.9: *weakly normal*
In 29.57.1: *bounds the degrees of the fibres of f*, *fibres of f are universally bounded*

Cohomology of Schemes

In 30.11.1: *depth k at a point*, *depth k at a point*, *(S_k)*, *(S_k)*
In 30.11.4: *Cohen-Macaulay*
In 30.26.2: *Z is proper over S*

Divisors

In 31.2.1: *associated*, *associated points of X*
In 31.4.1: *embedded associated point*, *embedded point*, *embedded component*
In 31.5.1: *weakly associated*, *weakly associated points of X*
In 31.7.1: *relative assassin of $\mathcal{F}$ in X over S*
In 31.8.1: *relative weak assassin of $\mathcal{F}$ in X over S*
In 31.11.2: *torsion*, *torsion free*
In 31.12.1: *reflexive hull*, *reflexive*
In 31.13.1: *locally principal closed subscheme*, *effective Cartier divisor*
In 31.13.6: *sum of the effective Cartier divisors D_1 and D_2*
In 31.13.12: *pullback of D by f is defined*, *pullback of the effective Cartier divisor*
In 31.14.1: *invertible sheaf $\mathcal{O}_S(D)$ associated to D*, *canonical section*
In 31.14.6: *regular section*
In 31.14.8: *zero scheme*
In 31.18.2: *relative effective Cartier divisor*
In 31.19.1: *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X*, *conormal algebra of f*
In 31.19.5: *normal cone $C_Z X$*, *normal bundle*
In 31.20.2: *regular*, *Koszul-regular*, *H_1-regular*, *quasi-regular*
In 31.21.1: *regular immersion*, *Koszul-regular immersion*, *H_1-regular immersion*, *quasi-regular immersion*
In 31.22.2: *relative quasi-regular immersion*, *relative H_1-regular immersion*
In 31.23.1: *sheaf of meromorphic functions on X*, *$\mathcal{K}_X$*, *meromorphic function*
In 31.23.3: *meromorphic section of $\mathcal{F}$*
In 31.23.4: *pullbacks of meromorphic functions are defined for f*
In 31.23.7: *regular*
In 31.23.10: *ideal sheaf of denominators of s*
In 31.26.2: *prime divisor*, *Weil divisor*
In 31.26.3: *order of vanishing of f along Z*

In 31.26.5: *principal Weil divisor associated to f*
In 31.26.7: *Weil divisor class group*
In 31.27.1: *order of vanishing of s along Z*
In 31.27.4: *Weil divisor associated to s*, *Weil divisor class associated to $\mathcal{L}$*
In 31.32.1: *blowing up of X along Z*, *blowing up of X in the ideal sheaf $\mathcal{I}$*, *exceptional divisor*, *center*
In 31.33.1: *strict transform*, *strict transform*
In 31.34.1: *U-admissible blowup*

Limits of Schemes

Varieties

In 33.3.1: *variety*
In 33.6.1: *geometrically reduced at x*, *geometrically reduced*
In 33.7.1: *geometrically connected*
In 33.8.1: *geometrically irreducible*
In 33.9.1: *geometrically pointwise integral at x*, *geometrically pointwise integral*, *geometrically integral*
In 33.10.1: *geometrically normal at x*, *geometrically normal*
In 33.12.1: *geometrically regular at x*, *geometrically regular over k*
In 33.16.1: *dual numbers*
In 33.16.3: *tangent space of X over S at x*, *tangent vector*
In 33.20.1: *algebraic k-scheme*, *locally algebraic k-scheme*
In 33.26.1: *affine variety*, *projective variety*, *quasi-projective variety*, *proper variety*, *smooth variety*
In 33.33.1: *Euler characteristic of $\mathcal{F}$*
In 33.35.7: *m-regular*
In 33.35.15: *Hilbert polynomial*
In 33.36.1: *absolute frobenius of X*
In 33.36.4: *relative Frobenius morphism of X/S*
In 33.39.3: *δ-invariant of A*
In 33.39.7: *δ-invariant of X at x*
In 33.40.4: *A is a wedge of $A_1, \ldots, A_n$*
In 33.43.1: *curve*
In 33.44.1: *degree*, *degree*
In 33.45.3: *intersection number*
In 33.45.10: *degree of Z with respect to $\mathcal{L}$*
In 33.46.1: *embedding dimension of X at x*
In 33.46.2: *embedding dimension of X/k at x*

Topologies on Schemes

In 34.3.1: *Zariski covering of T*
In 34.3.4: *standard Zariski covering*
In 34.3.5: *big Zariski site*
In 34.3.7: *big Zariski site of S*, *small Zariski site of S*, *big affine Zariski site of S*, *small affine Zariski site of S*
In 34.3.15: *restriction to the small Zariski site*
In 34.4.1: *étale covering of T*
In 34.4.5: *standard étale covering*
In 34.4.6: *big étale site*
In 34.4.8: *big étale site of S*, *small étale site of S*, *big affine étale site of S*, *small affine étale site of S*
In 34.4.15: *restriction to the small étale site*
In 34.5.1: *smooth covering of T*
In 34.5.5: *standard smooth covering*
In 34.5.6: *big smooth site*
In 34.5.8: *big smooth site of S*, *big affine smooth site of S*
In 34.6.1: *syntomic covering of T*
In 34.6.5: *standard syntomic covering*
In 34.6.6: *big syntomic site*
In 34.6.8: *big syntomic site of S*, *big affine syntomic site of S*
In 34.7.1: *fppf covering of T*
In 34.7.5: *standard fppf covering*
In 34.7.6: *big fppf site*
In 34.7.8: *big fppf site of S*, *big affine fppf site of S*
In 34.8.1: *standard ph covering*
In 34.8.4: *ph covering of T*
In 34.8.9: *big ph site*
In 34.8.11: *big ph site of S*, *big affine ph site of S*
In 34.9.1: *fpqc covering of T*
In 34.9.10: *standard fpqc covering*
In 34.9.13: *satisfies the sheaf property for the given family*, *satisfies the sheaf property for the fpqc topology*

In 34.10.1: *standard V covering*
In 34.10.7: *V covering of T*
In 34.10.11: *satisfies the sheaf property for the V topology*

Descent

In 35.2.1: *descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves*, *cocycle condition*, *morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data*
In 35.2.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 35.3.1: *descent datum (N, φ) for modules with respect to $R \to A$*, *cocycle condition*, *morphism $(N, \varphi) \to (N', \varphi')$ of descent data*
In 35.3.4: *effective*
In 35.4.2: *split equalizer*
In 35.4.5: *universally injective*
In 35.4.9: *C*
In 35.4.15: *base extension along f*, *descent morphism for modules*, *effective descent morphism for modules*
In 35.4.19: *f_**
In 35.8.2: *structure sheaf of the big site $(Sch/S)_\tau$*, *structure sheaf of the small site*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*
In 35.12.1: *parasitic*, *parasitic for the τ-topology*
In 35.15.1: *local in the τ-topology*
In 35.20.1: *germ of X at x*, *morphism of germs*, *composition of morphisms of germs*
In 35.20.2: *étale*, *smooth*
In 35.21.1: *étale local*, *smooth local*
In 35.22.1: *τ local on the base*, *τ local on the target*, *local on the base for the τ-topology*
In 35.26.1: *τ local on the source*, *local on the source for the τ-topology*
In 35.32.3: *étale local on source-and-target*
In 35.33.1: *étale local on the source-and-target*
In 35.34.1: *descent datum for $V/X/S$*, *cocycle condition*, *descent datum relative to $X \to S$*, *morphism $g : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$*
In 35.34.3: *descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$*, *morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data*
In 35.34.7: *pullback functor*
In 35.34.9: *pullback functor*
In 35.34.10: *trivial descent datum*, *canonical descent datum*, *effective*
In 35.34.11: *canonical descent datum*, *effective*
In 35.36.1: *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings*

Derived Categories of Schemes

In 36.6.1: *supported on T*
In 36.14.1: *approximation holds for the triple*
In 36.14.2: *approximation by perfect complexes holds*
In 36.22.2: *Tor independent over S*
In 36.35.1: *perfect relative to S*, *S-perfect*
In 36.36.1: *resolution property*
In 36.38.2: *Grothendieck group of X*, *Grothendieck group of coherent sheaves on X*

More on Morphisms

In 37.2.1: *thickening*, *first order thickening*, *finite order thickening*, *nth order thickening*, *morphism of thickenings*, *thickenings over S*, *morphisms of thickenings over S*
In 37.5.1: *first order infinitesimal neighbourhood*, *nth order infinitesimal neighbourhood*
In 37.6.1: *formally unramified*
In 37.7.2: *universal first order thickening*, *conormal sheaf of Z over X*
In 37.8.1: *formally étale*
In 37.11.1: *formally smooth*
In 37.13.1: *naive cotangent complex of f*
In 37.20.1: *normal at x*, *normal morphism*
In 37.21.1: *regular at x*, *regular morphism*
In 37.22.1: *Cohen-Macaulay at x*, *Cohen-Macaulay morphism*

In 37.35.1: *étale neighbourhood of (S, s)*, *morphism of étale neighbourhoods*, *elementary étale neighbourhood*
In 37.58.1: *finitely presented relative to S*, *of finite presentation relative to S*
In 37.59.2: *m-pseudo-coherent relative to S*, *pseudo-coherent relative to S*, *m-pseudo-coherent relative to S*, *pseudo-coherent relative to S*
In 37.60.2: *pseudo-coherent*
In 37.61.2: *perfect*
In 37.62.2: *Koszul at x*, *Koszul morphism*, *local complete intersection morphism*
In 37.64.1: *weakly étale*, *absolutely flat*
In 37.66.1: *ind-quasi-affine*, *ind-quasi-affine*
In 37.73.1: *affine stratification*
In 37.73.4: *affine stratification number*
In 37.75.2: *weighting*, *pondération*
In 37.78.1: *completely decomposed*, *completely decomposed*

More on Flatness

In 38.4.1: *one step dévissage of $\mathcal{F}/X/S$ over s*
In 38.4.2: *one step dévissage of $\mathcal{F}/X/S$ at x*
In 38.4.6: *standard shrinking*
In 38.5.1: *complete dévissage of $\mathcal{F}/X/S$ over s*
In 38.5.2: *complete dévissage of $\mathcal{F}/X/S$ at x*
In 38.5.5: *standard shrinking*
In 38.6.1: *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$*
In 38.6.2: *complete dévissage of $N/S/R$ over $\mathfrak{r}$*
In 38.6.4: *complete dévissage of $N/S/R$ at $\mathfrak{q}$*
In 38.15.2: *impurity of $\mathcal{F}$ above s*
In 38.16.1: *pure along X_s*, *universally pure along X_s*, *pure along X_s*, *universally S-pure*, *universally pure relative to S*, *S-pure*, *pure relative to S*, *S-pure*, *pure relative to S*
In 38.20.10: *$\mathcal{F}$ is flat over S in dimensions $\geq n$*
In 38.21.1: *universal flattening of $\mathcal{F}$ exists*, *universal flattening of X exists*
In 38.21.3: *flattening stratification*, *flattening stratification*
In 38.34.2: *h covering of T*
In 38.34.10: *big h site*
In 38.34.11: *standard h covering*
In 38.34.13: *big h site of S*, *big affine h site of S*

Groupoid Schemes

In 39.3.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on U over S*
In 39.3.3: *restriction*, *pullback*
In 39.4.1: *group scheme over S*, *morphism $\psi : (G, m) \to (G', m')$ of group schemes over S*
In 39.4.3: *closed subgroup scheme*, *open subgroup scheme*
In 39.4.5: *smooth group scheme*, *flat group scheme*, *separated group scheme*
In 39.9.1: *abelian variety*
In 39.10.1: *action of G on the scheme X/S*, *equivariant*, *G-equivariant*
In 39.10.2: *free*
In 39.11.1: *pseudo G-torsor*, *formally principally homogeneous under G*, *trivial*
In 39.11.3: *principal homogeneous space*, *G-torsor*, *G-torsor in the τ topology*, *τ G-torsor*, *τ torsor*, *quasi-isotrivial*, *locally trivial*
In 39.12.1: *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, *equivariant quasi-coherent $\mathcal{O}_X$-module*
In 39.13.1: *groupoid scheme over S*, *groupoid over S*, *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S*
In 39.14.1: *quasi-coherent module on (U, R, s, t, c)*
In 39.17.2: *stabilizer of the groupoid scheme (U, R, s, t, c)*
In 39.18.2: *restriction of (U, R, s, t, c) to U'*
In 39.19.1: *set-theoretically R-invariant*, *R-invariant*, *R-invariant*, *R-invariant*
In 39.20.1: *quotient sheaf U/R*

In 39.20.2: *representable quotient*, *representable quotient*
In 39.21.1: *cartesian*, (U', R', s', t', c') *is cartesian over* (U, R, s, t, c), *morphism of groupoid schemes cartesian over* (U, R, s, t, c)

More on Groupoid Schemes

Étale Morphisms of Schemes

In 41.3.1: *unramified homomorphism of local rings*
In 41.3.5: *unramified at* x, *unramified*
In 41.9.1: *flat*, *faithfully flat*, *flat (resp. faithfully flat)*
In 41.9.3: *flat over* Y *at* $x \in X$, *flat at* $x \in X$, *flat*, *faithfully flat*
In 41.11.1: *étale homomorphism of local rings*
In 41.11.4: *étale at* $x \in X$, *étale*
In 41.21.1: *strict normal crossings divisor*
In 41.21.4: *normal crossings divisor*

Chow Homology and Chern Classes

In 42.2.1: 2*-periodic complex*, *cohomology modules*, *exact*, $(2,1)$*-periodic complex*, *cohomology modules*
In 42.2.2: *multiplicity*, *(additive) Herbrand quotient*
In 42.7.6: δ*-dimension of* Z
In 42.8.1: *cycle on* X, k*-cycle*
In 42.8.3: *support*
In 42.8.4: *effective*
In 42.9.2: *multiplicity of* Z' *in* Z, k*-cycle associated to* Z
In 42.10.2: *multiplicity of* Z' *in* $\mathcal{F}$, k*-cycle associated to* $\mathcal{F}$
In 42.12.1: *pushforward*
In 42.14.1: *flat pullback of* α *by* f
In 42.17.1: *principal divisor associated to* f
In 42.19.1: *rationally equivalent to zero*, *rationally equivalent*, *Chow group of* k*-cycles on* X, *Chow group of* k*-cycles modulo rational equivalence on* X
In 42.22.1: *envelope*
In 42.24.1: *Weil divisor associated to* s, *Weil divisor associated to* $\mathcal{L}$
In 42.25.1: *intersection with the first Chern class of* $\mathcal{L}$
In 42.29.1: *Gysin homomorphism*
In 42.33.1: *bivariant class* c *of degree* p *for* f
In 42.34.1: *Chow cohomology*
In 42.34.4: *first Chern class*
In 42.37.1: *Chern classes of* $\mathcal{E}$ *on* X, *total Chern class of* $\mathcal{E}$ *on* X
In 42.38.1: *intersection with the* j*th Chern class of* $\mathcal{E}$
In 42.38.8: i*th Chern class*, *total Chern class*
In 42.41.1: *degree of a zero cycle*
In 42.46.3: *Chern classes of* E *are defined*
In 42.50.3: *localized Chern character*, *localized* p*th Chern class*
In 42.59.4: *the Gysin map for* f *exists*, *Gysin map*
In 42.68.2: *admissible*, *symbol*, *admissible relation*, *determinant of the finite length* R*-module* M
In 42.68.13: *determinant of* (M, φ, ψ)
In 42.68.29: *symbol associated to* M, a, b
In 42.68.31: *tame symbol*

Intersection Theory

In 43.13.5: *intersect properly*, *intersect properly*
In 43.15.1: *multiplicity of* M *for the ideal of definition* I

Picard Schemes of Curves

In 44.4.1: *Picard functor*
In 44.6.3: *genus*

Weil Cohomology Theories

In 45.5.1: i*th Chow group of* M
In 45.7.3: *classical Weil cohomology theory*
In 45.11.4: *Weil cohomology theory*

Adequate Modules

In 46.3.1: *module-valued functor*, *morphism of module-valued functors*
In 46.3.2: *adequate*, *linearly adequate*
In 46.5.1: *adequate*
In 46.5.7: $Adeq(\mathcal{O})$, $Adeq((Sch/S)_\tau, \mathcal{O})$, $Adeq(S)$

In 46.8.1: *pure projective*, *pure injective*
In 46.8.5: *pure projective resolution*, *pure injective resolution*
In 46.8.8: *pure extension module*

Dualizing Complexes

In 47.2.1: *essential*, *essential extension of*, *essential*
In 47.4.1: *projective cover*, *projective envelope*
In 47.5.1: *injective hull*
In 47.5.5: *indecomposable*
In 47.15.1: *dualizing complex*
In 47.21.1: *Gorenstein*, *Gorenstein*
In 47.27.1: *relative dualizing complex*

Duality for Schemes

In 48.2.2: *dualizing complex*
In 48.20.5: *dualizing complex normalized relative to* $\omega_S^\bullet$
In 48.24.1: *Gorenstein*
In 48.25.2: *Gorenstein at* x, *Gorenstein morphism*
In 48.28.1: *relative dualizing complex*

Discriminants and Differents

In 49.4.1: *trace element*
In 49.7.1: *Kähler different*
In 49.9.1: *different*

de Rham Cohomology

In 50.7.1: *Hodge filtration*
In 50.15.1: *de Rham complex of log poles is defined for* $Y \subset X$ *over* S
In 50.15.3: *de Rham complex of log poles for* $Y \subset X$ *over* S

Local Cohomology

In 51.4.2: *cohomological dimension of* I *in* A
In 51.13.1: I*-depth*, *depth*

Algebraic and Formal Geometry

In 52.6.4: *derived complete with respect to* $\mathcal{I}$
In 52.16.5: $(\mathcal{F}_n)$ *extends to* X
In 52.16.7: $(\mathcal{F}_n)$ *canonically extends to* X
In 52.19.1: $(\mathcal{F}_n)$ *satisfies the* (a, b)*-inequalities*, $(\mathcal{F}_n)$ *satisfies the strict* (a, b)*-inequalities*

Algebraic Curves

In 53.2.7: *nonsingular projective model of* X
In 53.3.1: *linear series of degree* d *and dimension* r, $\mathfrak{g}^r_d$
In 53.8.1: *genus*
In 53.11.1: *geometric genus*
In 53.16.2: *multicross singularity*, *node*, *ordinary double point*, *defines a nodal singularity*
In 53.19.1: *node*, *ordinary double point*, *defines a nodal singularity*, *singularities of* X *are at-worst-nodal*
In 53.19.10: *split node*
In 53.20.2: *at-worst-nodal of relative dimension* 1

Resolution of Surfaces

In 54.5.1: *normalized blowup of* X *at* x
In 54.8.3: *defines a rational singularity*, *reduction to rational singularities is possible for* A
In 54.14.1: *resolution of singularities*
In 54.14.2: *resolution of singularities by normalized blowups*

Semistable Reduction

In 55.3.1: *numerical type*
In 55.3.2: *equivalent types*
In 55.3.4: *numerical type of genus* g
In 55.3.8: (-1)*-index*
In 55.3.11: *topological genus of* T
In 55.3.12: *minimal*
In 55.3.16: (-2)*-index*
In 55.4.1: *Picard group of* T
In 55.8.4: *minimal model*
In 55.11.4: *numerical type associated to* X
In 55.14.6: *semistable reduction*
In 55.14.8: *good reduction*

Functors and Morphisms

Derived Categories of Varieties

In 57.3.2: *a Serre functor exists*, *Serre functor*

In 57.8.1: *Fourier-Mukai functor*, *Fourier-Mukai kernel*
In 57.10.1: *siblings*, *sibling*
In 57.12.1: *siblings*, *sibling*
In 57.15.1: *the Fourier-Mukai kernel of a relative equivalence from X to Y over S*
In 57.18.1: *derived equivalent*

Fundamental Groups of Schemes

In 58.2.1: *G-set*, *discrete G-set*, *morphism of G-sets*, *G-Sets*
In 58.3.6: *Galois category*
In 58.6.1: *fundamental group*, *base point*

Étale Cohomology

In 59.4.1: *étale covering*
In 59.9.1: *presheaf of sets*, *abelian presheaf*
In 59.10.1: *family of morphisms with fixed target*
In 59.10.2: *site*, *coverings*
In 59.11.1: *separated presheaf*, *sheaf*
In 59.11.4: *category of sheaves of sets*, *abelian sheaves*
In 59.13.1: *zeroth Čech cohomology group*
In 59.15.1: *fpqc covering*
In 59.15.5: *satisfies the sheaf property for the fpqc topology*
In 59.16.1: *descent datum*, *cocycle condition*, *effective*
In 59.16.5: *descent datum*
In 59.16.6: *effective*
In 59.17.2: *ringed site*, *quasi-coherent*
In 59.18.1: *Čech complex*, *Čech cohomology groups*
In 59.18.4: *free abelian presheaf on $\mathcal{G}$*
In 59.20.1: *τ-covering*
In 59.20.2: *big τ-site of S*, *small τ-site of S*
In 59.20.4: *standard τ-covering*
In 59.21.1: *étale topos*, *small étale topos*, *Zariski topos*, *small Zariski topos*, *big τ-topos*
In 59.23.1: *constant sheaf*
In 59.23.3: *structure sheaf*
In 59.26.1: *étale*
In 59.26.3: *standard étale*
In 59.27.1: *étale covering*
In 59.27.3: *big étale site over S*, *small étale site over S*, *big*, *small Zariski sites*
In 59.29.1: *geometric point*, *lies over*, *étale neighborhood*, *morphism of étale neighborhoods*
In 59.29.6: *stalk*
In 59.31.3: *support of $\mathcal{F}$*, *support of σ*
In 59.32.2: *henselian*
In 59.32.6: *strictly henselian*
In 59.33.2: *étale local ring of S at $\overline{s}$*, *strict henselization of $\mathcal{O}_{S,s}$*, *henselization of $\mathcal{O}_{S,s}$*, *strict henselization of S at $\overline{s}$*, *henselization of S at s*
In 59.35.1: *direct image*, *pushforward*
In 59.35.3: *direct image*, *pushforward*
In 59.35.4: *higher direct images*
In 59.36.1: *inverse image*, *pullback*
In 59.51.1: *system $(\mathcal{F}_i, \varphi_{i'i})$ of sheaves on $(X_i, f_{i'i})$*
In 59.56.1: *absolute Galois group*, *algebraic*
In 59.57.1: *G-module*, *discrete G-module*, *morphism of G-modules*, *R-G-module*, *morphism of R-G-modules*
In 59.57.2: *continuous group cohomology groups*, *group cohomology groups*, *Galois cohomology groups*, *Galois cohomology groups of K with coefficients in M*
In 59.61.3: *similar*, *equivalent*
In 59.61.4: *Brauer group*
In 59.64.1: *constant sheaf with value E*, *constant sheaf*, *locally constant*, *finite locally constant*, *constant sheaf with value A*, *constant sheaf*, *locally constant*, *finite locally constant*, *constant sheaf with value M*, *constant sheaf*, *locally constant*
In 59.66.1: *trace*
In 59.67.5: *C_r*, *nontrivial solution*
In 59.67.9: *variety*, *curve*
In 59.70.1: *extension by zero*, *extension by zero*
In 59.71.1: *constructible*, *constructible*, *constructible*
In 59.76.1: *$D_c(X_{étale}, \Lambda)$*
In 59.77.1: *$D_{ctf}(X_{étale}, \Lambda)$*

In 59.93.1: *locally acyclic at $\overline{x}$ relative to K*, *locally acyclic relative to K*, *universally locally acyclic relative to K*, *locally acyclic*, *universally locally acyclic*
In 59.95.1: *cohomological dimension of X*
In 59.96.1: *cohomological dimension of f*

Crystalline Cohomology

In 60.2.2: *divided power envelope of J in B relative to (A, I, γ)*
In 60.4.1: *δ is compatible with γ*
In 60.5.2: *divided power thickening*, *homomorphism of divided power thickenings*
In 60.6.1: *divided power A-derivation*
In 60.7.1: *divided power structure γ*
In 60.7.2: *divided power scheme*, *morphism of divided power schemes*
In 60.7.3: *divided power thickening*
In 60.8.1: *divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$*, *morphism of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$*
In 60.8.4: *Zariski, étale, smooth, syntomic, or fppf covering*, *big crystalline site*
In 60.9.1: *crystalline site*
In 60.11.1: *locally quasi-coherent*, *quasi-coherent*, *crystal in $\mathcal{O}_{X/S}$-modules*
In 60.11.3: *crystal in quasi-coherent modules*, *crystal in finite locally free modules*
In 60.12.1: *S-derivation $D : \mathcal{O}_{X/S} \to \mathcal{F}$*
In 60.26.2: *F-crystal on X/S (relative to σ)*, *nondegenerate*

Pro-étale Cohomology

In 61.2.3: *w-local*, *w-local*
In 61.3.1: *local isomorphism*, *identifies local rings*
In 61.4.1: *ind-Zariski*
In 61.7.1: *ind-étale*
In 61.11.1: *w-contractible*
In 61.12.1: *pro-étale covering of T*
In 61.12.6: *standard pro-étale covering*
In 61.12.7: *big pro-étale site*
In 61.12.8: *big pro-étale site of S*, *small pro-étale site of S*, *big affine pro-étale site of S*
In 61.12.14: *restriction to the small pro-étale site*
In 61.26.1: *extension by zero*, *extension by zero*
In 61.27.1: *constructible*
In 61.28.1: *constructible Λ-sheaf*, *lisse*, *adic lisse*, *adic constructible*
In 61.29.1: *constructible*
In 61.29.4: *adic lisse*, *adic constructible*

Relative Cycles

In 62.6.1: *relative r-cycle on X/S*
In 62.7.1: *equidimensional*
In 62.8.1: *effective*
In 62.9.1: *proper relative cycle*

More Étale Cohomology

In 63.3.3: *direct image with compact support*
In 63.3.7: *sections with compact support*
In 63.4.4: *direct image with compact support*
In 63.12.1: *cohomology of K with compact support*, *compactly supported cohomology of K*

The Trace Formula

In 64.3.4: *geometric frobenius*
In 64.3.8: *arithmetic frobenius*
In 64.3.10: *geometric frobenius*
In 64.4.1: *trace*
In 64.6.4: *total right derived functor of F*, *total left derived functor of G*
In 64.7.1: *filtered injective*, *projective*, *filtered quasi-isomorphism*
In 64.8.1: *filtered derived functor*
In 64.10.1: *perfect*
In 64.12.1: *finite Tor-dimension*
In 64.14.1: *global Lefschetz number*
In 64.14.2: *local Lefschetz number*
In 64.15.2: *G-trace of f on P*
In 64.18.1: *$\mathbf{Z}_\ell$-sheaf*, *ℓ-adic sheaf*, *lisse*, *morphism*
In 64.18.6: *torsion*, *stalk*
In 64.18.8: *ℓ-adic cohomology*
In 64.19.1: *L-function of $\mathcal{F}$*

In 64.19.3: *L-function of $\mathcal{F}$*
In 64.27.1: *open*
In 64.31.1: *unramified cusp form on $GL_2(\mathbf{A})$ with values in Λ*

Algebraic Spaces

In 65.5.1: *property $\mathcal{P}$*
In 65.6.1: *algebraic space over S*
In 65.6.3: *morphism $f : F \to F'$ of algebraic spaces over S*
In 65.9.2: *étale equivalence relation*
In 65.9.3: *presentation*
In 65.12.1: *open immersion*, *open subspace*, *closed immersion*, *closed subspace*, *immersion*, *locally closed subspace*
In 65.12.5: *Zariski covering*
In 65.12.6: *small Zariski site F_{Zar}*
In 65.13.2: *separated over S*, *locally separated over S*, *quasi-separated over S*, *Zariski locally quasi-separated over S*
In 65.14.4: *acts freely*, *quotient of U by G*
In 65.16.2: *base change of F' to S*, *viewed as an algebraic space over S'*

Properties of Algebraic Spaces

In 66.3.1: *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*, *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*
In 66.4.1: *point*
In 66.4.7: *topological space*
In 66.5.1: *quasi-compact*
In 66.7.2: *has property $\mathcal{P}$*
In 66.7.5: *has property $\mathcal{P}$ at x*
In 66.8.2: *étale locally constructible*
In 66.9.1: *dimension of X at x*
In 66.9.2: *dimension*
In 66.10.2: *dimension of the local ring of X at x*, *x is a point of codimension d on X*
In 66.12.5: *algebraic space structure on Z*, *reduced induced algebraic space structure*, *reduction X_{red} of X*
In 66.16.2: *étale*
In 66.18.1: *small étale site $X_{étale}$*
In 66.18.2: *$X_{spaces,étale}$*
In 66.18.5: *$X_{affine,étale}$*
In 66.18.7: *étale topos*, *small étale topos*
In 66.18.9: *f-map $\varphi : \mathcal{G} \to \mathcal{F}$*
In 66.19.1: *geometric point*, *geometric point lying over x*
In 66.19.2: *étale neighborhood*, *morphism of étale neighborhoods*
In 66.19.6: *stalk*
In 66.20.3: *support of $\mathcal{F}$*, *support of σ*
In 66.21.2: *structure sheaf*
In 66.22.2: *étale local ring of X at $\overline{x}$*, *strict henselization of X at $\overline{x}$*
In 66.23.2: *geometrically unibranch at x*, *geometrically unibranch*
In 66.23.4: *number of geometric branches of X at x*
In 66.24.1: *Noetherian*
In 66.25.2: *X is regular at x*
In 66.29.1: *quasi-coherent*
In 66.31.2: *locally projective*

Morphisms of Algebraic Spaces

In 67.4.2: *separated*, *locally separated*, *quasi-separated*
In 67.5.2: *surjective*
In 67.6.2: *open*, *universally open*
In 67.7.2: *submersive*, *universally submersive*
In 67.8.2: *quasi-compact*
In 67.9.2: *closed*, *universally closed*
In 67.10.1: *monomorphism*
In 67.13.2: *inverse image $f^{-1}(Z)$ of the closed subspace Z*
In 67.14.4: *scheme theoretic intersection*, *scheme theoretic union*
In 67.15.4: *scheme theoretic support of $\mathcal{F}$*
In 67.16.2: *scheme theoretic image*
In 67.17.3: *scheme theoretic closure of U in X*, *scheme theoretically dense in X*
In 67.18.1: *dominant*
In 67.19.3: *universally injective*
In 67.20.2: *affine*
In 67.20.8: *relative spectrum of $\mathcal{A}$ over X*, *spectrum of $\mathcal{A}$ over X*
In 67.21.2: *quasi-affine*
In 67.22.2: *has property $\mathcal{P}$*
In 67.22.6: *has property $\mathcal{Q}$ at x*
In 67.23.1: *locally of finite type*, *finite type at x*, *of finite type*
In 67.25.2: *finite type point*

In 67.27.1: *locally quasi-finite*, *quasi-finite at x*, *quasi-finite*
In 67.28.1: *locally of finite presentation*, *finite presentation at x*, *of finite presentation*
In 67.30.1: *flat*, *flat at x*
In 67.31.2: *flat at x over Y*, *flat over Y*
In 67.33.1: *dimension of the local ring of the fibre of f at x*, *transcendence degree of $x/f(x)$*, *f has relative dimension d at x*
In 67.33.2: *relative dimension $\leq d$*, *relative dimension d*
In 67.36.1: *syntomic*, *syntomic at x*
In 67.37.1: *smooth*, *smooth at x*
In 67.38.1: *unramified*, *unramified at x*, *G-unramified*, *G-unramified at x*
In 67.39.1: *étale at x*
In 67.40.1: *proper*
In 67.41.1: *satisfies the uniqueness part of the valuative criterion*, *satisfies the existence part of the valuative criterion*, *satisfies the valuative criterion*
In 67.45.2: *integral*, *finite*
In 67.46.2: *finite locally free*, *rank*, *degree*
In 67.47.1: *equivalent*, *rational map from X to Y*, *B-rational map from X to Y*
In 67.47.2: *rational function on X*
In 67.47.3: *ring of rational functions on X*
In 67.47.4: *defined in a point $x \in |X|$*, *domain of definition*
In 67.47.6: *dominant*
In 67.47.7: *birational*
In 67.48.2: *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$*
In 67.48.3: *normalization of X in Y*
In 67.49.6: *normalization*
In 67.53.2: *universal homeomorphism*

Decent Algebraic Spaces

In 68.3.1: *fibres of f are universally bounded*
In 68.6.1: *decent*, *reasonable*, *very reasonable*
In 68.11.2: *residue field of X at x*
In 68.11.5: *elementary étale neighbourhood*, *morphism of elementary étale neighbourhoods*
In 68.11.7: *henselian local ring of X at x*
In 68.13.6: *residual space of X at x*
In 68.17.1: *has property (β)*, *has property (β)*, *decent*, *reasonable*, *very reasonable*
In 68.22.1: *birational*
In 68.24.2: *unibranch at x*, *unibranch*
In 68.24.4: *number of branches of X at x*
In 68.25.1: *catenary*
In 68.25.4: *universally catenary*

Cohomology of Algebraic Spaces

In 69.6.2: *alternating Čech complex*
In 69.12.1: *coherent*

Limits of Algebraic Spaces

In 70.3.1: *limit preserving*, *locally of finite presentation*, *locally of finite presentation over S*, *limit preserving*, *locally of finite presentation*, *relatively limit preserving*
In 70.14.3: *subsheaf of sections annihilated by $\mathcal{I}$*
In 70.14.6: *subsheaf of sections supported on T*

Divisors on Algebraic Spaces

In 71.2.2: *weakly associated*, *weakly associated points of X*, *x is associated to $\mathcal{F}$*, *x is an associated point of X*
In 71.4.2: *the fibre of f over y is locally Noetherian*, *the fibres of f are locally Noetherian*
In 71.4.5: *relative weak assassin of $\mathcal{F}$ in X over Y*
In 71.6.1: *locally principal closed subspace*, *effective Cartier divisor*
In 71.6.6: *sum of the effective Cartier divisors D_1 and D_2*
In 71.6.10: *pullback of D by f is defined*, *pullback of the effective Cartier divisor*
In 71.7.1: *invertible sheaf $\mathcal{O}_X(D)$ associated to D*
In 71.7.4: *regular section*
In 71.7.6: *zero scheme*
In 71.9.2: *relative effective Cartier divisor*
In 71.10.1: *sheaf of meromorphic functions on X*, *$\mathcal{K}_X$*, *meromorphic function*
In 71.10.3: *meromorphic section of $\mathcal{F}$*

In 71.10.6: *pullbacks of meromorphic functions are defined for f*
In 71.10.9: *regular*
In 71.11.3: *relative homogeneous spectrum of $\mathcal{A}$ over X*, *homogeneous spectrum of $\mathcal{A}$ over X*, *relative Proj of $\mathcal{A}$ over X*
In 71.14.1: *relatively ample*, *f-relatively ample*, *ample on X/Y*, *f-ample*
In 71.17.1: *blowing up of X along Z*, *blowing up of X in the ideal sheaf $\mathcal{I}$*, *exceptional divisor*, *center*
In 71.18.1: *strict transform*, *strict transform*
In 71.19.1: *U-admissible blowup*

Algebraic Spaces over Fields

In 72.4.1: *integral*
In 72.4.3: *function field*, *field of rational functions*
In 72.5.2: *degree of X over Y*
In 72.6.2: *prime divisor*, *Weil divisor*
In 72.6.4: *order of vanishing of f along Z*
In 72.6.7: *principal Weil divisor associated to f*
In 72.6.9: *Weil divisor class group*
In 72.7.1: *order of vanishing of s along Z*
In 72.7.4: *Weil divisor associated to s*, *Weil divisor class associated to $\mathcal{L}$*
In 72.8.1: *modification of X*
In 72.8.3: *alteration of X*
In 72.11.1: *geometrically reduced at x*, *geometrically reduced*
In 72.12.1: *geometrically connected*
In 72.13.1: *geometrically irreducible*
In 72.14.1: *geometrically integral*
In 72.17.1: *Euler characteristic of $\mathcal{F}$*
In 72.18.3: *intersection number*

Topologies on Algebraic Spaces

In 73.3.1: *Zariski covering of X*
In 73.4.1: *étale covering of X*
In 73.4.5: $(Spaces/S)_{étale}$
In 73.4.6: $(Spaces/X)_{étale}$
In 73.4.9: *restriction to the small étale site*
In 73.5.1: *smooth covering of X*
In 73.6.1: *syntomic covering of X*
In 73.7.1: *fppf covering of X*
In 73.7.6: $(Spaces/S)_{fppf}$
In 73.7.7: $(Spaces/X)_{fppf}$
In 73.8.1: *ph covering of X*
In 73.8.5: $(Spaces/S)_{ph}$
In 73.8.6: $(Spaces/X)_{ph}$
In 73.9.1: *fpqc covering of X*

Descent and Algebraic Spaces

In 74.3.1: *descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves*, *cocycle condition*, *morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data*
In 74.3.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 74.10.1: *τ local on the base*, *τ local on the target*, *local on the base for the τ-topology*
In 74.14.1: *τ local on the source*, *local on the source for the τ-topology*
In 74.20.1: *smooth local on source-and-target*
In 74.21.1: *étale-smooth local on source-and-target*
In 74.22.1: *descent datum for $V/Y/X$*, *cocycle condition*, *descent datum relative to $Y \to X$*, *morphism $f : (V/Y, \varphi) \to (V'/Y, \varphi')$ of descent data relative to $Y \to X$*
In 74.22.3: *descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to X\}$*, *morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data*
In 74.22.7: *pullback functor*
In 74.22.9: *pullback functor*
In 74.22.10: *trivial descent datum*, *canonical descent datum*, *effective*
In 74.22.11: *canonical descent datum*, *effective*

Derived Categories of Spaces

In 75.3.2: *supported on T*
In 75.5.1: *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves*
In 75.7.2: *T is proper over Y*
In 75.9.1: *elementary distinguished square*

In 75.14.1: *approximation holds for the triple*
In 75.14.2: *approximation by perfect complexes holds*
In 75.20.2: *Tor independent over B*
In 75.28.1: *resolution property*

More on Morphisms of Spaces

In 76.3.1: *radicial*
In 76.5.1: *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X*, *conormal sheaf of i*
In 76.6.1: *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X*, *conormal algebra of i*
In 76.6.5: *normal cone $C_Z X$*, *normal bundle*
In 76.7.2: *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 76.9.1: *thickening*, *first order thickening*, *finite order thickening*, *nth order thickening*, *morphism of thickenings*, *thickenings over B*, *morphisms of thickenings over B*
In 76.12.1: *first order infinitesimal neighbourhood*, *nth order infinitesimal neighbourhood*
In 76.13.1: *formally smooth*, *formally étale*, *formally unramified*
In 76.14.1: *formally unramified*
In 76.15.5: *universal first order thickening*, *conormal sheaf of Z over X*
In 76.16.1: *formally étale*
In 76.19.1: *formally smooth*
In 76.21.1: *naive cotangent complex of f*
In 76.23.2: *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat over the fibre of Y over z*
In 76.26.2: *Cohen-Macaulay at x*, *Cohen-Macaulay morphism*
In 76.27.2: *Gorenstein at x*, *Gorenstein morphism*
In 76.29.2: *the fibre of $f : X \to Y$ at y is geometrically reduced*
In 76.44.2: *Koszul-regular immersion*, *H_1-regular immersion*, *quasi-regular immersion*
In 76.45.3: *m-pseudo-coherent relative to Y*, *pseudo-coherent relative to Y*, *m-pseudo-coherent relative to Y*, *pseudo-coherent relative to Y*
In 76.46.1: *pseudo-coherent*, *pseudo-coherent at x*
In 76.47.1: *perfect*, *perfect at x*
In 76.48.1: *Koszul morphism*, *local complete intersection morphism*, *Koszul at x*
In 76.52.1: *perfect relative to Y*, *Y-perfect*
In 76.55.1: *at-worst-nodal of relative dimension* 1

Flatness on Algebraic Spaces

In 77.2.2: *impurity of $\mathcal{F}$ above y*
In 77.3.1: *pure above y*, *universally pure above y*, *pure above y*, *universally Y-pure*, *universally pure relative to Y*, *Y-pure*, *pure relative to Y*, *Y-pure*, *pure relative to Y*
In 77.11.1: *universal flattening of $\mathcal{F}$ exists*, *universal flattening of X exists*
In 77.11.3: *$\mathcal{F}$ is flat over Y in dimensions $\geq n$*

Groupoids in Algebraic Spaces

In 78.4.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on U over B*
In 78.4.3: *restriction*, *pullback*
In 78.5.1: *group algebraic space over B*, *morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B*
In 78.8.1: *action of G on the algebraic space X/B*, *equivariant*, *G-equivariant*
In 78.8.2: *free*
In 78.9.1: *pseudo G-torsor*, *formally principally homogeneous under G*, *trivial*
In 78.9.3: *principal homogeneous space*, *principal homogeneous G-space over B*, *G-torsor in the τ topology*, *τ G-torsor*, *τ torsor*, *quasi-isotrivial*, *locally trivial*
In 78.10.1: *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, *equivariant quasi-coherent $\mathcal{O}_X$-module*
In 78.11.1: *groupoid in algebraic spaces over B*, *morphism $f : (U, R, s, t, c) \to$*

(U', R', s', t', c') *of groupoids in algebraic spaces over* B
In 78.12.1: *quasi-coherent module on* (U, R, s, t, c)
In 78.16.2: *stabilizer of the groupoid in algebraic spaces* (U, R, s, t, c)
In 78.17.2: *restriction of* (U, R, s, t, c) *to* U'
In 78.18.1: *R-invariant*, *R-invariant*, *R-invariant*
In 78.19.1: *quotient sheaf* U/R
In 78.19.3: *quotient representable by an algebraic space*, *representable quotient*, *representable quotient*, *quotient representable by an algebraic space*
In 78.20.1: *quotient stack*, *quotient stack*

More on Groupoids in Spaces

In 79.15.1: *strongly split over u*, *strong splitting of R over u*, *split over u*, *splitting of R over u*, *quasi-split over u*, *quasi-splitting of R over u*

Bootstrap

In 80.3.1: *representable by algebraic spaces*
In 80.4.1: *property* $\mathcal{P}$

Pushouts of Algebraic Spaces

Chow Groups of Spaces

In 82.2.5: *δ-dimension of T*
In 82.3.1: *cycle on X*, *k-cycle*
In 82.4.2: *$\mathcal{F}$ has length d at x*
In 82.5.2: *multiplicity of Z in Y*, *k-cycle associated to Y*
In 82.6.1: *multiplicity of Z in $\mathcal{F}$*, *k-cycle associated to $\mathcal{F}$*
In 82.8.1: *pushforward*
In 82.10.1: *flat pullback of α by f*
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